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Isometric-path numbers of block graphs

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Abstract

An isometric path between two vertices in a graph $G$ is a shortest path joining them. The isometric-path number of $G$, denoted by $\text{ip}(G)$, is the minimum number of isometric paths required to cover all vertices of $G$. In this paper, we determine exact values of isometric-path numbers of block graphs. We also give a linear-time algorithm for finding the corresponding paths.

Keywords. Isometric path, block graph, cut-vertex, algorithm

1 Introduction

An isometric path between two vertices in a graph $G$ is a shortest path joining them. The isometric-path number of $G$, denoted by $\text{ip}(G)$, is the minimum number of isometric paths required to cover all vertices of $G$. This concept has a close relationship with the game of cops and robbers described as follows. The game is played by two players, the cop and the robber, on a graph. The two players move alternatively, starting with the cop. Each player’s first move consists of choosing a vertex at which to start. At each subsequent move, a player may choose either to stay at the same vertex or to move to an adjacent

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vertex. The object for the cop is to catch the robber, and for the robber is to prevent
this from happening. Nowakowski and Winkler \[7\] and Quilliot \[8\] independently proved
that the cop wins if and only if the graph can be reduced to a single vertex by successively
removing pitfalls, where a *pitfall* is a vertex whose closed neighborhood is a subset of the
closed neighborhood of another vertex. As not all graphs are cop-win graphs, Aigner and
Fromme \[1\] introduced the concept of the *cop-number* of a general graph $G$, denoted by $c(G)$, which is the minimum number of cops needed to put into the graph in order to catch
the robber. On the way to giving an upper bound for the cop-numbers of planar graphs,
they showed that a single cop moving on an isometric path $P$ guarantee that after a finite
number of moves the robber will be immediately caught if he moves onto $P$. Observing
this fact, Fitzpatrick \[4\] then introduced the concept of isometric-path cover and pointed
out that $c(G) \leq \text{ip}(G)$.

The isometric-path number of the Cartesian product $P_{n_1} \times P_{n_2} \times \ldots \times P_{n_d}$ has been
studied in the literature. Fitzpatrick \[5\] gave bounds for the case when $n_1 = n_2 = \ldots = n_d$.
Fisher and Fitzpatrick \[3\] gave exact values for the case $d = 2$. Fitzpatrick et al \[6\] gave
a lower bound, which is in fact the exact value if $d + 1$ is a power of 2, for the case when
$n_1 = n_2 = \ldots = n_d = 2$.

The purpose of this paper is to give exact values of isometric-path numbers of block
graphs. We also give a linear-time algorithm to find the corresponding paths. For technical
reasons, we consider a slightly more general problem as follows. Suppose every vertex $v$
in the graph $G$ is associated with a non-negative integer $f(v)$. We call such function $f$ a
*vertex labeling* of $G$. An *$f$-isometric-path cover* of $G$ is a family $\mathcal{C}$ of isometric paths such
that the following conditions hold.

(C1) If $f(v) = 0$, then $v$ is in an isometric path in $\mathcal{C}$.

(C2) If $f(v) \geq 1$, then $v$ is an end vertex of at least $f(v)$ isometric paths in $\mathcal{C}$, while the
counting is twice if $v$ itself is a path in $\mathcal{C}$. 

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The $f$-isometric-path number of $G$, denoted by $\text{ip}_f(G)$, is the minimum cardinality of an $f$-isometric-path cover of $G$. It is clear that when $f(v) = 0$ for all vertices $v$ in $G$, we have $\text{ip}(G) = \text{ip}_f(G)$. The attempt of this paper is to determine the $f$-isometric-path number of a block graph. Recall that a block graph is a graph in which every block is a complete graph. A cut-vertex of a graph is a vertex whose removal results in a graph with more components than the original graph. It is well-known that in a block graph all internal vertices of an isometric path are cut-vertices.

2 Block graphs

In this section, we determine the $f$-isometric-path numbers for block graphs $G$. Without loss of generality, we may assume that $G$ is connected.

First, a useful lemma.

Lemma 1 Suppose $x$ is a non-cut-vertex of a block graph $G$ with a vertex labeling $f$. If vertex labeling $f'$ is the same as $f$ except that $f'(x) = \max\{1, f(x)\}$, then $\text{ip}_f(G) = \text{ip}_{f'}(G)$.

Proof. As any internal vertex of an isometric path in a block graph is a cut-vertex but $x$ not a cut-vertex, $x$ must be an end vertex of any isometric path. It follows that a collection $C$ is an $f$-isometric-path cover if and only if it is an $f'$-isometric-path cover. The lemma then follows. \hfill \blacksquare

So, now we may assume that $f(v) \geq 1$ for all non-cut-vertices $v$ of $G$, and call such a vertex labeling regular. Now, we have the following theorem for the inductive step.

Theorem 2 Suppose $G$ is a block graph with a regular labeling $f$, and $x$ is a non-cut-vertex in a block $B$ with exactly one cut-vertex $y$ or with no cut-vertex in which case let $y$ be any vertex of $B - \{x\}$. When $f(x) = 1$, let $G' = G - x$ with a regular vertex labeling $f'$ which is the same as $f$ except $f'(y) = f(y) + 1$. When $f(x) \geq 2$, let $G' = G$ with a regular vertex labeling $f'$ which is the same as $f$ except $f'(x) = f(x) - 1$ and $f'(y) = f(y) + 1$. Then $\text{ip}_f(G) = \text{ip}_{f'}(G')$. 

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Proof. We first prove that $\text{ip}_f(G) \geq \text{ip}_{f'}(G')$. Suppose $C$ is an optimal $f$-isometric-path cover of $G$. Choose a path $P$ in $C$ having $x$ as an end vertex. We consider four cases.

**Case 1.**

1. **Case 1.1.** $P = x$ and $f(x) = 1$ (i.e., $G' = G - x$).

   In this case, $C' = (C - \{P\}) \cup \{y\}$ is an $f'$-isometric-path cover of $G'$. Hence, $\text{ip}_f(G) = |C| \geq |C'| \geq \text{ip}_{f'}(G')$.

2. **Case 1.2.** $P = x$ and $f(x) \geq 2$ (i.e., $G' = G$).

   In this case, $C' = (C - \{P\}) \cup \{xy\}$ is an $f'$-isometric-path cover of $G'$. Hence, $\text{ip}_f(G) = |C| \geq |C'| \geq \text{ip}_{f'}(G')$.

3. **Case 1.3.** $P = xz$ for some vertex $z$ in $B - \{x, y\}$.

   In this case, $C' = (C - \{P\}) \cup \{yz\}$ is an $f'$-isometric-path cover of $G'$. Hence, $\text{ip}_f(G) = |C| \geq |C'| \geq \text{ip}_{f'}(G')$.

4. **Case 1.4.** $P = xyQ$, where $Q$ contains no vertices in $B$.

   In this case, $C' = (C - \{P\}) \cup \{yQ\}$ is an $f'$-isometric-path cover of $G'$. Hence, $\text{ip}_f(G) = |C| \geq |C'| \geq \text{ip}_{f'}(G')$.

Next, we prove that $\text{ip}_f(G) \leq \text{ip}_{f'}(G')$. Suppose $C'$ is an optimal $f'$-isometric-path cover of $G'$. Choose a path $P'$ in $C'$ having $y$ as an end vertex. We consider three cases.

**Case 2.**

1. **Case 2.1.** $P' = yx$.

   In this case, $G' = G$ and $C = (C' - \{P'\}) \cup \{x\}$ is an $f$-isometric-path cover of $G$. Hence, $\text{ip}_f(G) \leq |C| \leq |C'| = \text{ip}_{f'}(G')$.

2. **Case 2.2.** $P' = yz$ for some $z$ in $B - \{x, y\}$.

   In this case, $C = (C' - \{P'\}) \cup \{xz\}$ is an $f$-isometric-path cover of $G$. Hence, $\text{ip}_f(G) \leq |C| \leq |C'| = \text{ip}_{f'}(G')$.

3. **Case 2.3.** $P' = yQ$, where $Q$ contains no vertex in $B$.

   In this case, $C = (C' - \{P'\}) \cup \{xyQ\}$ is an $f$-isometric-path cover of $G$. Hence, $\text{ip}_f(G) \leq |C| \leq |C'| = \text{ip}_{f'}(G')$.

Consequently, we have the following result for $f$-isometric-path numbers of connected block graphs.
Theorem 3 If $G$ is a connected block graph with a regular vertex labeling $f$, then $\text{ip}_f(G) = \lceil s(G)/2 \rceil$, where $s(G) = \sum_{v \in V(G)} f(v)$.

Proof. The theorem is obvious when $G$ has only one vertex. For the case when $G$ has more than one vertex, we apply Theorem 2 repeatedly until the graph becomes trivial. Notice that the $s(G') = s(G)$ when apply Theorem 2.

For the isometric-path-cover problem, we have

Corollary 4 If $G$ is a connected block graph, then $\text{ip}(G) = \lceil \text{nc}(G)/2 \rceil$, where $\text{nc}(G)$ is the number of non-cut-vertices of $G$.

Proof. The corollary follows from Theorem 3 and the fact that $\text{ip}(G) = \text{ip}_f(G)$ for the regular vertex labeling $f$ with $f(v) = 1$ if $v$ is a non-cut-vertex and $f(v) = 0$ otherwise.

3 Algorithm

Based on Theorem 2, we are able to design an algorithm for the isometric-path-cover problem in block graphs. Notice that we may only consider connected block graphs with regular vertex labelings. To speed up the algorithm, we may modify Theorem 2 a little bit so that each time a non-cut-vertex is handled.

Theorem 5 Suppose $G$ is a block graph with a regular labeling $f$, and $x$ is a non-cut-vertex in a block $B$ with exactly one cut-vertex $y$ or with no cut-vertex in which let $y$ be any vertex in $B - \{x\}$. Let $G' = G - x$ with a regular vertex labeling $f'$ which is the same as $f$ except $f'(y) = f(y) + f(x)$. Then $\text{ip}_f(G) = \text{ip}_{f'}(G')$.

Proof. The theorem follows from repeatedly applying Theorem 2.

Now, we are ready to give the algorithm.

Algorithm PG Find the $f$-isometric-path number $\text{ip}_f(G)$ of a connected block graph.

Input. A connected block graph $G$ and a regular vertex labeling $f$. 
**Output.** An optimal $f$-isometric-path cover $C$ of $G$ and $\text{ip}_f(G)$.

**Method.**

1. construct a stack $S$ which is empty at the beginning;
2. let $G' \leftarrow G$;
3. **while** ($G'$ has more than one vertex) **do**
4. choose a block $B$ with exactly one cut-vertex $y$ or with no cut-vertex in which case choose any $y \in B$;
5. **for** (all vertices $x$ in $B - \{y\}$) **do**
6. $f(y) \leftarrow f(y) + f(x)$;
7. push $(x, y, f(x))$ into $S$;
8. $G' \leftarrow G' - x$;
9. **end for**;
10. **end while**;
11. $\text{ip}_f(G) \leftarrow \lceil f(r)/2 \rceil$, where $r$ is the only vertex of $G'$;
12. let $C$ be the family of isometric paths containing $\text{ip}(G)$ copies of the path $r$;
13. **while** ($S$ is not empty) **do**
14. pop $(x, y, i)$ from $S$;
15. choose $i$ copies of path $P$ in $C$ using $y$ as an end vertex;
16. **if** ($P = yx$) **then**
17. replace the $i$ copies of $P$ by $i$ copies of $x$ in $C$;
18. **if** ($P = yz$ for some vertex $z$ in the block of $G$ containing $x$) **then**
19. replace the $i$ copies of $P$ by $i$ copies of $xz$ in $C$;
20. **if** ($P = yQ$ where $Q$ has no vertices in the block of $G$ containing $x$) **then**
21. replace the $i$ copies of $P$ by the $i$ copies of $xyQ$ in $C$;
22. **end while**.

Algorithm $\text{PG}$ can be implemented in time linear to the number of vertices and edges.

Notice that we can use the depth-first search to find all blocks and cut-vertices of a graph, see [2].

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