Dynamical Determination of the Metric Signature in Spacetime of Nontrivial Topology

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Abstract

The formalism of Greensite for treating the spacetime signature as a dynamical degree of freedom induced by quantum fields is considered, for spacetimes with nontrivial topology of the kind $\mathbb{R}^{D-1} \times T^1$, for varying $D$. It is shown that a dynamical origin for the Lorentzian signature is possible in the five-dimensional space $\mathbb{R}^4 \times T^1$ with small torus radius (periodic boundary conditions), as well as in four-dimensional space with trivial topology. Hence, the possibility exists that the early universe might have been of the Kaluza-Klein type, i.e. multidimensional and of Lorentzian signature.

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It is well-known that the field equations of general relativity do not fix the space-time signature. However, there exist attempts to understand the signature dynamics or, more specifically, possible signature transitions from an Euclidean signature spacetime to a Lorentzian signature spacetime, and vice-versa, both at a classical and quantum level [1]-[7]. Notwithstanding these efforts, the explanation of the origin of the Lorentzian signature of our physical spacetime is still missing.

Recently, a very interesting attempt to devise a quantum mechanism for the dynamical origin of the Lorentzian signature has been made [8, 9]. Let us briefly describe this formalism. Considering the flat-space metric in the following form:

\[ \eta_{ab} = \text{diag}(e^{i\theta}, 1, 1, \ldots, 1), \tag{1} \]

where \( \theta \in [-\pi, \pi] \), one can easily see that the Euclidean path-integral theory is obtained for the Wick angle \( \theta = 0 \), while the Lorentzian signature corresponds to \( \theta = \pi \). Starting from this simple setting, it was suggested in ref. [8] that the Wick angle \( \theta \) in (1) could be treated as a dynamical degree of freedom, which can fluctuate in the interval \( \theta \in [-\pi, \pi] \). Then, in order to fix the value of this degree of freedom and to show that the Lorentzian signature is in fact the preferred choice, the effective potential \( V(\theta) \) for \( \theta \) has been calculated in [8, 9] at one-loop level, under the following assumptions:

(i) For free fields of equal mass, the contributions to the whole path integral from any propagating bosonic degree of freedom is equal and inverse to the contribution of the corresponding fermionic degree of freedom. (ii) For scalars, the real-valued invariant volume (De Witt measure) of integration is used.

Under these conditions, the one-loop potential \( V(\theta) \) induced by a massless scalar in flat spacetime (i.e. \( g_{\mu\nu} = \epsilon^a_\mu \eta_{ab} \epsilon^b_\mu = \eta_{\mu\nu} \)) is given by [8, 9]

\[ V(\theta) = -\frac{\log \det^{-1/2}(-\sqrt{\eta} \eta^{ab} \partial_a \partial_b)}{\int d^Dx}, \tag{2} \]

and use of heat-kernel regularization yields

\[ V(\theta) = -\frac{1}{2} \int_\Lambda^{\infty} \frac{ds}{s} \int \frac{d^Dp}{(2\pi)^D} \exp \left\{ -s[a p_0^2 + \beta (p_1^2 + \ldots + p_{D-1}^2)] \right\}, \tag{3} \]
where $\Lambda$ is a cutoff, $\alpha = e^{-i\frac{\theta}{2}}$, $\beta = e^{i\frac{\theta}{2}}$. In [9] it is explained why heat-kernel regularization is the best one to use in this context. Taking into account assumption (i), the multiplier $(n_B - n_F)$ —where $n_F(n_B)$ is the fermionic (bosonic) number— appears in front of (3). Finally, using the above formalism it was shown in Refs. [8, 9] that the Lorentzian signature is uniquely connected with $D = 4$ dimensions, as it is given by the stationary point of the potential.

Now, our point is the following. Let us start discussing some modifications of the above formalism, for it is quite reasonable that the picture above described may be valid in the early universe, perhaps between the Quantum Gravity and the GUTs epochs. However, at this stage of the evolution of the early universe, its curvature and temperature were still very strong, and an external electromagnetic field might have existed. Moreover, most probably the topology of the universe at this epoch was highly nontrivial. Surely, all these effects (and combinations thereof) may change the picture described in [8, 9], even qualitatively. In this letter we will restrict ourselves to consider only the influence of nontrivial topology on the above formalism.

So, we will suppose that some massless fields, which have induced the $V(\theta)$ of eq. (3), live in a flat spacetime of topology $\mathbb{R}^{D-1} \times T^1$ (for a general discussion of QFT on topologically nontrivial backgrounds, see [12]). Then, expressions (2) and (3) still conserve the same form. However, now one of the space coordinates, $x_1$, for example, is compactified, so that the corresponding momentum component $-p_1$ is discretized and one has to do replacements of the type

$$\int \frac{dp_1}{2\pi} \to \frac{1}{2\pi R} \sum_{n=-\infty}^{\infty} .$$

(4)

The specific discrete values of $p_1$ will depend on whether the fields are subject to periodic or antiperiodic boundary conditions.

**(a) Periodic boundary conditions:** $p_1^2 = \frac{n^2}{R^2}$, $n \in \mathbb{Z}$.

After integrating the non-compactified components of $p$, we find

$$V(\theta) = -\frac{\pi^{D-1}}{2R(2\pi)^{D/2} \beta^{D-1}} \int_{\Lambda} ds \, s^{\frac{D-1}{2}} \theta_3 \left( 0 \left| \frac{s\beta}{\pi R^2} \right. \right) ,$$

(5)
where
\[ \theta_3(0|z) \equiv \sum_{n=-\infty}^{\infty} e^{-\pi n^2} = 1 + 2 \sum_{n=1}^{\infty} e^{-\pi n^2}. \tag{6} \]

Now, let us consider the different limits of expr. (5).

(a1) \( R \to 0 \), more specifically, \( |\frac{\beta \Lambda}{R^2}| \gg 1 \). That limit corresponds to very strong nontrivial topology. Thus, we take the expression (6) itself as a large-\( z \) expansion. Integration proceeds term by term with the help of
\[ \int_{z}^{\infty} dt \, t^{a-1} e^{-t} = \Gamma(a, z), \tag{7} \]
where \( \Gamma(a, z) \) is an incomplete gamma function \([11]\), whose asymptotic behaviour for large arguments is given by
\[ \Gamma(a, z) \sim z^{a-1} e^{-z} \left[ 1 + O\left(\frac{1}{z}\right) \right], \quad |z| \gg 1. \tag{8} \]
As a result, the expansion
\[ V(\theta) = -\frac{2}{(4\pi)^{D/2} \alpha^2 \beta^{D/2} - 1} \left[ \frac{1}{D-1} + \frac{1}{R^2} e^{-\frac{\beta \Lambda}{R^2}} + O\left( \frac{1}{R^2}\right)^2 e^{-\frac{2\beta \Lambda}{R^2}}, \ldots, \frac{1}{R^2} e^{-\frac{4\beta \Lambda}{R^2}}, \ldots \right] \tag{9} \]
follows. Note that we have assumed \( \text{Re} \alpha > 0 \), \( \text{Re} \beta > 0 \).

(a2) \( R \to \infty \), more precisely \( |\frac{\pi^2 R^2}{\beta \Lambda}| \gg 1 \). Such a limit corresponds to “switching off” the compactification. First, a small-\( z \) expansion of \( \theta_3 \) must be performed. It can be obtained by the reciprocal transformation
\[ \theta_3(0|z) = \frac{1}{\sqrt{z}} \theta_3 \left( 0 \mid \frac{1}{z} \right). \tag{10} \]
Then, we can proceed similarly to the previous case, with the difference that now we are led to make variable changes of the type \( u = \frac{\pi^2 R^2 n^2}{\beta s} \) and split the integration domains in the way
\[ \int_{0}^{\frac{\pi^2 R^2 n^2}{\beta \Lambda}} du = \int_{0}^{\infty} du - \int_{\frac{\pi^2 R^2 n^2}{\beta \Lambda}}^{\infty} du. \]
Doing so, incomplete \( \Gamma \) functions appear again. After using the asymptotic expansion (8) for large values of their arguments, one arrives at
\[ V(\theta) = -\frac{1}{(4\pi)^{D/2} \alpha^2 \beta^{D/2} - 1} \left[ \frac{1}{D} + \frac{1}{(\frac{\pi^2 R^2}{\beta \Lambda})^{D/2}} \Gamma\left( \frac{D}{2} \right) \zeta(D) \right] \]
\[-\frac{1}{\pi^2 R^2} e^{-\frac{2 \mu^2}{\beta \Lambda}} + O \left( \frac{1}{(\frac{\pi^2 R^2}{\beta \Lambda})^2} e^{-\frac{2 \mu^2}{\beta \Lambda}}, \ldots, \frac{1}{4 \pi^2 R^2} e^{-\frac{4 \mu^2}{\beta \Lambda}}, \ldots \right) \right].

(11)

\(\zeta\) being the Riemann zeta function. The term where it occurs yields precisely the result which one would obtain after removing the \(n = 0\) piece and performing zeta-function regularization of the rest. As one can notice, its contribution is —of course— independent of the cutoff \(\Lambda\).

(b) Antiperiodic boundary conditions: \(p_1^2 = \left( \frac{n+\frac{1}{2}}{R^2} \right)^2, \quad n \in \mathbb{Z}\).

Now, the integration of the non-compactified \(p\)-components yields

\[
V(\theta) = -\frac{\pi^D}{2R(2\pi)^D} \beta^{D-1} \int_\Lambda \frac{ds}{s} s^{-\frac{D-1}{2}} \theta_2 \left( 0 \left| \frac{s\beta}{\pi R^2} \right. \right),
\]

where

\[
\theta_2(0|\beta) \equiv \sum_{n=-\infty}^{\infty} e^{-\pi \beta (n+\frac{1}{2})^2} = 2 \sum_{n=0}^{\infty} e^{-\pi \beta (n+\frac{1}{2})^2}.
\]

This last equality can be viewed as a large-\(z\) expansion, and will therefore be used for calculating the small-\(R\) expression in a way analogous to the corresponding periodic case.

(b1) For \(R \to 0\), that is for \(\left| \frac{\beta \Lambda}{R^2} \right| \gg 1\), we obtain

\[
V(\theta) = -\frac{2}{(4\pi)^{D-1} \beta^{D-1} \Lambda^{D-1}} \left[ \frac{1}{\beta \Lambda} e^{-\frac{\beta \Lambda}{4R^2}} + O \left( \frac{1}{(\frac{\beta \Lambda}{4R^2})^2} e^{-\frac{\beta \Lambda}{4R^2}}, \ldots, \frac{1}{9 \beta \Lambda} e^{-\frac{9 \beta \Lambda}{4R^2}}, \ldots \right) \right].
\]

(14)

(b2) For \(R \to \infty\), in order to obtain a small-\(z\) expansion from (13) —which is just the opposite— this time we take advantage of the transformation

\[
\theta_2 (0 | z) = \frac{1}{\sqrt{z}} \theta_4 \left( 0 \left| \frac{1}{z} \right. \right),
\]

\[
\theta_4(0|z) \equiv \sum_{n=-\infty}^{\infty} (-1)^n e^{-\pi n^2} = 1 + 2 \sum_{n=1}^{\infty} (-1)^n e^{-\pi n^2}.
\]

This is a useful relation, as well as its counterpart (14) for the analogous periodic case, and both can be regarded as special forms of the general reciprocal transformation for
After integrating, one gets

\[ V(\theta) = -\frac{1}{(4\pi)^{D/2}\alpha^{\frac{D}{2}}\beta^{\frac{D}{2}}\Lambda^{\frac{D}{2}}} \left[ \frac{1}{D} - \frac{1}{(\frac{\pi^2 R^2}{\beta \Lambda})^{D/2}} \Gamma \left( \frac{D}{2} \right) \eta(D) \right. \]

\[ + \frac{1}{\frac{\pi^2 R^2}{\beta \Lambda}} e^{-\frac{\pi^2 R^2}{\beta \Lambda}} + O \left( \frac{1}{(\frac{\pi^2 R^2}{\beta \Lambda})^{\frac{D}{2}}} e^{-\frac{\pi^2 R^2}{\beta \Lambda}}, \ldots, \frac{1}{\frac{4\pi^2 R^2}{\beta \Lambda}} e^{-\frac{4\pi^2 R^2}{\beta \Lambda}}, \ldots \right) \right], \]

which is valid for \( \frac{\pi^2 R^2}{\beta \Lambda} \gg 1 \). Here \( \eta \) is the well-known Dirichlet series

\[ \eta(z) \equiv \sum_{n=0}^{\infty} (-1)^{n-1} n^{-z} = (1 - 2^{1-z})\zeta(z). \]  

Now we discuss the physical consequences of the results just obtained. As we could see, the potential \( V(\theta) \) is complex, so we will look for the value of \( \theta \) determined by the following conditions [8]:

\[ (i) \quad \text{Im} \, V \quad \text{must be stationary, and} \]

\[ (ii) \quad \text{Re} \, V \quad \text{must be a minimum.} \]  

(18)

Note also that, in accordance with the first assumption of [8], our results (5), (9), (11), (12), (14) and (16) should be multiplied by \((n_B - n_F)\).

We are ready to start the analysis of the effective potential. For the \( R \rightarrow \infty \) limit (trivial topology), the analysis made in [8] shows that there exists a unique solution

\[ n_F > n_B, \quad \theta = \pm \pi, \quad D = 4. \]  

(19)

(Note that for \( D = 2 \) or \( n_F = n_B \) there is no preferable choice of \( \theta \), as \( V \) ceases to depend on it. We shall call these cases ‘special’). The solution (19) leads the authors of [8] to conclude that the Lorentzian signature is chosen by quantum dynamics only in \( D = 4 \).

Now we may consider the case when the radius of the compactified dimension is small, and quantum fields satisfy periodic boundary conditions. Then, using the leading term of (13) (with the multiplier \((n_B - n_F)\)) we find the following unique solution
satisfying the requirements (18) (apart from the ‘special’ cases, which are here \( D = 3 \) or \( n_F = n_B \))

\[ n_F > n_B, \quad \theta = \pm \pi, \quad D = 5. \]

Thus, the Lorentzian signature is singled out also in \( D = 5 \), but only if the fifth dimension is compactified with a very small radius. Therefore, the formalism of the dynamical degree of freedom associated to the Wick angle provides a window for Kaluza-Klein-type theories. It is not difficult to show that, had we started from the topology \( \mathbb{R}^{D-n} \times T^n (n < D - 1) \), and did a similar small-\( R \) expansion (taking all the torus radii to be equal) we would have found the following result:

\[ n_F > n_B, \quad \theta = \pm \pi, \quad D = 4 + n. \] (21)

Hence, an early universe with a nontrivial topology could have been multidimensional, and this is compatible with the Lorentzian signature.

As a last example, let us consider the potential (14) corresponding to small radius for the compactified dimension and antiperiodic boundary conditions for the quantum fields. Then, from (14) we have

\[ V(\theta) \sim \frac{8(n_F - n_B)R e^{-\frac{\Lambda}{4R^2}}}{(4\pi)^{\frac{D+1}{2}} \alpha^\frac{D}{2} \beta^\frac{D}{2} \Lambda^\frac{D+1}{2}}. \] (22)

The conditions (18) for the \( V(\theta) \) in (22) mean that there should be some value of \( \theta \), say \( \bar{\theta} \), simultaneously satisfying

\[
(n_B - n_F) e^{-\frac{\Lambda}{4R^2} \cos^2 \frac{\theta}{2}} \left[ \frac{\Lambda}{8R^2} \sin \frac{\theta}{2} \sin \left( \frac{D - 1}{4} \theta + \frac{\Lambda}{4R^2} \sin \frac{\theta}{2} \right) \right. \\
+ \left( \frac{D - 1}{4} + \frac{\Lambda}{8R^2} \cos \frac{\theta}{2} \right) \cos \left( \frac{D - 1}{4} \theta + \frac{\Lambda}{4R^2} \sin \frac{\theta}{2} \right) \left. \right] \bigg|_{\theta = \bar{\theta}} = 0,
\]

\[
(n_F - n_B) e^{-\frac{\Lambda}{4R^2} \cos^2 \frac{\theta}{2}} \cos \left( \frac{D - 1}{4} \theta + \frac{\Lambda}{4R^2} \sin \frac{\theta}{2} \right) \text{ has a minimum at } \theta = \bar{\theta}.
\] (23)

In other words, one takes the potential (22) and requires the coincidence of stationary points of its imaginary part with minima of its real part, bearing in mind all the time that we are restricted to \( \theta \in [-\pi, \pi] \). Now, this situation differs from the previous
cases in the variable nature of this $V(\theta)$ as $\frac{\Lambda}{4R^2}$ varies. We have plotted the curves representing its rescaled real and imaginary parts for $D = 4, 5$ and for different values of $b \equiv \frac{\Lambda}{4R^2}$ (see Fig. 1). The behaviour observed is drastically modified as this parameter increases, going from a regime of noticeable oscillation to one in which a plateau around the origin is formed by both $\text{Re} \ V$ and $\text{Im} \ V$. After studying $V$ for other values of $b$ not shown in the figure, we have detected that the width of this plateau increases as $b$ grows. The flatness of this region indicates an angular range where $V$ becomes practically independent of $\theta$, and therefore, within this range no particular spacetime signature is preferred. (Notice that the situation where $b \gg 1$ is precisely the one in which (22) is an acceptable approximation for $V(\theta)$). Apart from this, and generally speaking, there is no genuine coincidence of stationary points and minima, unless very specific values of $b$ and $D$ are deliberately chosen for that purpose.

Summing up, we have discussed the possibility of a dynamical origin of the Lorentzian signature in a universe with nontrivial topology. Using very precise mathematical techniques, we have shown that the Lorentzian signature can actually be dynamically induced in a multidimensional universe with the topology $\mathbb{R}^4 \times \mathbb{T}^n$, where the radii of the torus are small. It would certainly be of interest to estimate the mass effects on the above results, since it was already shown in [3] that considering massive fields might increase the resulting $D$ until $D = 6$. Hence, the combination of both effects (topology and nonzero mass) may lead to changes in the above results. Finally, let us note that it would also be of great interest to understand the influence of quantum gravity on the dynamical origin of the spacetime signature. This subject surely deserves further study.

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Figure captions

Figure 1. Curves representing Re $v(\theta)$ —solid line— and Im $v(\theta)$ —dashed line— with $v(\theta)$ denoting the rescaled potential $e^{-b\cos \frac{\theta}{2} + i(a\theta + b\sin \frac{\theta}{2})}$, with $a \equiv \frac{D-1}{4}$, $b \equiv \frac{A}{4R^2}$ for $-\pi \leq \theta \leq \pi$. The plots correspond to $D = 4, D = 5$ and to two different values of $b$: (a) $D = 4, b = 1$, (b) $D = 4, b = 10$, (c) $D = 5, b = 1$, (d) $D = 5, b = 10$. In (b) and (d) the formation of a wide plateau around the origin for large values of $b$ is already apparent.
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