A formula for the nonsymmetric Opdam’s hypergeometric function of type $A_2$

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Abstract

The aim of this paper is to give an explicit formula for the nonsymmetric Heckman-Opdam’s hypergeometric function of type $A_2$. This is obtained by differentiating the corresponding symmetric hypergeometric function.

Keywords. Root systems, Cherednik operators, Hypergeometric functions.

Mathematics Subject Classification. Primary 33C67;17B22. Secondary 33D52.

1 Introduction

The theory of the hypergeometric functions associated to root systems started in the 1980s with Heckman and Opdam via a generalization of the spherical functions on Riemannian symmetric spaces of noncompact type. Several important aspects are studied by them in a series of publications [4, 9, 10, 11, 12]. One of the impressive developments came in 1995s with the work of Opdam [11], where he introduced a remarkable family of orthogonal polynomials (the so called Opdam’s nonsymmetric polynomials) as simultaneous eigenfunctions of Cherednik operators. It contains in particular the presentation of the non-symmetric hypergeometric functions where their investigations become an interesting topics in the theory of special functions and in the harmonic analysis. In this paper, we focus on the non-symmetric hypergeometric function associated to root systems of type $A$, for the purpose in finding an explicit formula for it, as it is done in the symmetric case [1, 2, 13, 3]. The setting is that non-symmetric hypergeometric function can be derived from the symmetric ones via application of a suitable polynomial of Cherednik operators ([12], cor. 7.6). This paper deals with the case where the root system is of type $A_2$, using Opdam’s shifted operators and Cherednik operators, so the problem semble to be more robust for others $A_n$.

In order to describe our approach let us be more specific about $A$-type hypergeometric function. We assume that the reader is familiar with root systems and their basic properties. As a general reference, we mention Opdam [11, 12].
Let \((e_1, e_2, \ldots, e_n)\) be the standard basis of \(\mathbb{R}^n\) and \(\langle ., . \rangle\) be the usual inner product for which this basis is orthonormal. We denote by \(\| . \|\) its Euclidean norm. Let \(V\) be the hyperplane orthogonal to the vector \(e = e_1 + \ldots + e_n\). In \(V\) we consider the root system of type \(A_{n-1}\)

\[ R = \{ e_i - e_j, 1 \leq i \neq j \leq n \} \]

with the subsystem of positives roots

\[ R_+ = \{ e_i - e_j, 1 \leq i < j \leq n \}. \]

The associated Weyl group \(W\) is isomorphic to symmetric group \(S_n\), permuting the \(n\) coordinates. We define the positive Weyl chamber

\[ C = \{ x \in V, x_1 > x_2 > \ldots > x_n \}. \]

Denote \(\pi_n\) the orthogonal projection onto \(V\), which is given by

\[ \pi_n(x) = x - \frac{1}{n} \left( \sum_{j=1}^{n} x_j \right) e, \quad x \in \mathbb{R}^n. \]

The cone of dominant weights is the set

\[ P_+ = \sum_{j=1}^{n-1} \mathbb{Z}_+ \beta_j, \quad \beta_j = \pi_n(e_1 + e_2 + \ldots + e_j). \]

For fixed \(k > 0\), the Dunkl-Cherednik operators \(T_\xi, \xi \in \mathbb{R}^n\), is defined by

\[ T_\xi^k = \partial_\xi + k \sum_{i<j}(\xi_i - \xi_j) \frac{1 - s_{i,j}}{1 - e^{x_i - x_j}} - \langle \rho_k, \xi \rangle \] (1.1)

where \(\rho_k = \frac{k}{2} \sum_{j=1}^{n} (n - 2j + 1)e_j\) and \(s_{i,j}\) acts on functions of variables \((x_1, x_2, ..., x_n)\) by interchanging the variables \(x_i\) and \(x_j\). For each \(\lambda \in \mathbb{V}_c\) (the complexification of \(\mathbb{V}\)) there exists a unique holomorphic \(W\)-invariant function \(F_k(\lambda, .)\) in a \(W\)-invariant tubular neighborhood of \(\mathbb{V}\) such that

\[ p(T_{\pi(e_1)}^k, \ldots, T_{\pi(e_n)}^k)F_k(\lambda, .) = p(\lambda)F_k(\lambda, .); \quad F(\lambda, 0) = 1, \]

for all symmetric polynomial \(p \in \mathbb{C}[X_1, \ldots, X_n]\). In particular

\[ \Delta_k F_k(\lambda, .) = \| \lambda \|^2 F_k(\lambda, .) \] (1.2)

where \(\Delta_k = \sum_{i=1}^{n} (T_{\pi(e_i)})^2\) is the Heckman-Opdam Laplacian. Note that the restriction of \(\Delta_k\) to the set of \(W\)-invariant functions is the differential operator

\[ L_k = \Delta + k \sum_{\alpha \in R^+} \coth \frac{\langle \alpha, x \rangle}{2} \partial_\alpha + \langle \rho_k, \rho_k \rangle \]
Nonsymmetric Opdam’s hypergeometric function of type $A_2$

where $\Delta$ is the ordinary Laplace operator. There exists a unique solution $G_k(\lambda, x)$ of the eigenvalue problem

$$T(\lambda)G_k(\lambda, x) = \langle \lambda, \xi \rangle G_k(\lambda, x), \quad \forall \xi \in \mathbb{R}^N, \quad G_k(\lambda, 0) = 1. \quad (1.3)$$

holomorphic for all $\lambda$ and for $x$ in $\mathbb{V} + iU$ for a neighbourhood $U \subset \mathbb{V}$ of zero. The function $G_k$ is the so-called nonsymmetric Opdam’s hypergeometric function. If $x \in \mathbb{V}$ then

$$|G_k(\lambda, x)| \leq \sqrt{n!} e^{\text{max}_{w \in W} \langle \text{Re}(\lambda), w(x) \rangle}. \quad (1.4)$$

Moreover, the Heckman-Opdam hypergeometric function $F_k$ can be written as

$$F_k(\lambda, x) = \frac{1}{n!} \sum_{w \in W} G_k(\lambda, w.x). \quad (1.5)$$

In other words, for $\lambda$ satisfying $\lambda_i - \lambda_j \neq 0; \pm k$ we have

$$G_k(\lambda, x) = D_q F_k(\lambda, x), \quad x \in \mathbb{V} \quad (1.6)$$

where

$$D_q = \prod_{1 \leq i < j \leq n} \left(1 - \frac{k}{\lambda_i - \lambda_j}\right)^{-1} \prod_{w \in W, w \neq \text{id}} \left(\frac{T(\lambda) - \langle w(\lambda), \xi \rangle}{\langle \lambda, \xi \rangle - \langle w(\lambda), \xi \rangle}\right)$$

and $\xi$ is any element in $\mathbb{V}$ satisfying $\langle \lambda, \xi \rangle - \langle w(\lambda), \xi \rangle \neq 0$ for all $w \neq \text{id}$. However, (1.6) is far from being applied to find an expansion of $G_k$ when an explicit formula of $F_k$ is given, so the polynomial $q$ has degree $n! - 1$. It would therefore be desirable to get another polynomial that is of suitable low degree, this will be described in section 3 when the root system is of type $A_2$.

## 2 An integral formula for Heckman-Opdam hypergeometric function of type A

In [1] an explicit and recursive formula on the dimension $n$ for the A-type Heckman-Opdam’s hypergeometric function is obtained as a consequence of similar formula for Jack polynomials. We have for $\lambda \in P_+$ and $x \in C$

$$F_{k,n}(\lambda + \rho_{k,n}, x) = \frac{c_{\lambda}(\lambda + \rho_{k,n})}{c_{\lambda-1}(\lambda + \rho_{k,n-1}) U_n(\lambda) V_n(x)^{2k-1}} \int_{x_2}^{x_1} \cdots \int_{x_n}^{x_{n-1}} F_{k,n-1} \left( \pi_{n-1}(\lambda) + \rho_{k,n-1}, \pi_{n-1}(\nu) \right) e^{\nu(1+\sqrt{n})} V_{n-1}(\nu) W_k(x, \nu) d\nu$$
with the following notations

\[ c_n(\lambda) = \prod_{\alpha \in \mathbb{R}^+} \frac{\Gamma(\langle \lambda, \alpha \rangle + \rho_{k,n}, \alpha) + k}{\Gamma(\langle \lambda, \alpha \rangle + k)}, \quad U_n(\lambda) = \prod_{j=1}^{n-1} \beta(\lambda_j + (n - j)k, k) \]

\[ \tilde{\lambda} = (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, ..., \lambda_{n-1} - \lambda_n, 0), \quad \bar{\lambda} = (\lambda_1 - \lambda_n, \lambda_2 - \lambda_n, ..., \lambda_{n-1} - \lambda_n), \]

\[ |\nu| = \nu_1 + \nu_2 + ... + \nu_{n-1}; \quad V_n(x) = \prod_{1 \leq i < j \leq n} (e^{x_i} - e^{x_j}), \]

\[ W_k(x, \nu) = \prod_{1 \leq i < j \leq n} |e^{x_i} - e^{x_j}|^{k-1}. \]

It can be simplified to

\[ F_{k,n}(\lambda + \rho_{k,n}, x) = \frac{\Gamma(nk)}{V_n(x)^{2k-1} \Gamma(k)^n} \int_{x_2}^{x_1} ... \int_{x_n}^{x_{n-1}} e^{-\langle \lambda, x \rangle + |\nu||\lambda|/(N-1)} F_{k,n-1}\left( \pi_{n-1}(\bar{\lambda}) + \rho_{k,n-1}, \pi_{n-1}(\nu) \right) e^{[\nu]V_{n-1}(\nu)} W_k(x, \nu) \, d\nu \] (2.1)

By analytic continuation, according to Carlson’s theorem (see [13], p. 186), this formula is still valid for all \( \lambda \in \mathbb{V}_\mathbb{C} \). Indeed, for \( x \in C \) define the functions of variable \( \lambda \in \mathbb{V}_\mathbb{C} \)

\[ H_1(\lambda) = e^{-\langle \lambda, x \rangle} F_{k,n}(\lambda + \rho_{k,n}, x) \]

\[ H_2(\lambda) = \frac{\Gamma(nk)}{\Gamma(k)^n V_n(x)^{2k-1}} \int_{x_2}^{x_1} ... \int_{x_n}^{x_{n-1}} e^{-\langle \lambda, x \rangle + |\nu||\lambda|/(N-1)} F_{k,n-1}\left( \pi_{n-1}(\bar{\lambda}) + \rho_{k,n-1}, \pi_{n-1}(\nu) \right) e^{[\nu]V_{n-1}(\nu)} W_k(x, \nu) \, d\nu \]

We have from (1.4)

\[ |H_1(\lambda)| \leq \sqrt{n!} e^{(\rho_{k,n}, x)} \]

for all \( \lambda \in \mathbb{V}_\mathbb{C}, \) \( Re(\lambda) \in C \) and \( x \in C \). In order to estimate \( H_2 \) we note beforehand the following fact, for \( \nu = (\nu_1, \nu_2, ..., \nu_{n-1}) \) and \( x \in C \) with \( x_{i+1} \leq \nu_i \leq x_i \), if we consider \( \nu^* = (\nu_1, \nu_2, ..., \nu_{n-1}, \nu_n) \in \mathbb{V} \) then we have \( \nu^* \leq x \) where \( \leq \) denotes the partial order on \( \mathbb{V} \) associated to the dual cone \( \sum_{i=1}^{n-1} \mathbb{R}_+(e_i - e_{i+1}) \), which implies that \( \langle Re(\lambda), \nu^* \rangle \leq \langle Re(\lambda), x \rangle \), for \( \lambda \in \mathbb{V}_\mathbb{C} \) with \( Re(\lambda) \in C \). Applying this fact and (1.4) it follows that

\[ \left| e^{-\langle \lambda, x \rangle + [\nu||\lambda|/(n-1))} F_{k,n-1}\left( \pi_{n-1}(\bar{\lambda}) + \rho_{k,n-1}, \pi_{n-1}(\nu) \right) \right| \]

\[ \leq \sqrt{(n-1)!} e^{-\langle Re(\lambda), x - \nu^* \rangle + (\rho_{k,n-1}, \pi_{n-1}(\nu))} \]

\[ \leq \sqrt{(n-1)!} e^{(\rho_{k,n-1}, x_{n-1}(\nu))} \]

Hence \( H_1 \) and \( H_2 \) are Holomorphic functions, bounded for \( Re(\lambda) \in C \) and coincide on \( P_+ \simeq \mathbb{Z}_+^{n-1} \), the Carleson’s Theorem yields \( H_1(\lambda) = H_1(\lambda) \) for all \( \lambda \in \mathbb{V}_\mathbb{C}, \) \( Re(\lambda) \in C \) and thus for all \( \lambda \in \mathbb{V}_\mathbb{C} \) by analytic continuation.

Now, using the fact that \( \pi_{n-1}(\bar{\rho}_{k,n}) = \rho_{k,n-1}^k \) and \( |\bar{\rho}_{k,n}| = kn(n - 1)/2 \), we state the following final form of our recursive formula.
Theorem 2.1. For all $\lambda \in \mathbb{V}_\mathbb{C}$ and $x \in C$,

$$F_{k,n}(\lambda, x) = \frac{\Gamma(nk)}{\Gamma(k)^nV_n(x)^{2k-1}} \int_{x_2}^{x_1} \cdots \int_{x_n}^{x_1} F_{k,n-1}(\pi_{n-1}(\lambda), \pi_{n-1}(\nu)) e^{\nu(1-nk/2 + \pi/\nu(n-1))} V_{n-1}(\nu) W_k(x, \nu) d\nu$$

(2.2)

In the rank-one case, which corresponds to take $n = 2$ and $V = \mathbb{R}(e_1 - e_2)$, the formula (2.2) becomes

$$F_{k,2}(\lambda, x) = \frac{\Gamma(2k)}{\Gamma(2)^2(2\sinh x_1)^{2k-1}} \int_{-x_1}^{x_1} e^{\nu(1-k+2\lambda_1)}(e^{x_1} - e^{\nu})^{k-1}(e^{\nu} - e^{-x_1})^{k-1} d\nu$$

$$= \frac{\Gamma(2k)}{2^k\Gamma(2)^2(\sinh x_1)^{2k-1}} \int_{-x_1}^{x_1} e^{2\nu\lambda_1}(\cosh x_1 - \cosh \nu)^{k-1} d\nu$$

$$= \varphi_{2\lambda_1}^{k-1/2, -1/2}(x_1) = {}_2F_1\left(\frac{k}{2} - \lambda_1, \frac{k}{2} + \lambda_1, k + \frac{1}{2}, -\sinh^2 x_1\right)$$

where $\varphi_{2\lambda_1}^{k-1/2, -1/2}$ is a Jacobi function see (1.4), (3.4) and (3.5) of [7]. We recall here various facts about the Jacobi function $\varphi_{in}^{k-1/2, -1/2}$ that we shall need later, we refer to [7] [8].

$$\varphi_{in}^{k-1/2, -1/2}(2t) = \varphi_{2in}^{k-1/2, -1/2}(t)$$

(2.3)

$$\left(\varphi_{in}^{k-1/2, -1/2}\right)'(t) = \frac{1}{(2k + 1)(\nu^2 - k^2)} \sinh(t)\varphi_{in}^{k+1/2, -1/2}(t)$$

(2.4)

$$\left(\varphi_{in}^{k-1/2, -1/2}\right)''(t) + 2k \coth(t) \left(\varphi_{in}^{k-1/2, -1/2}\right)'(t) = (\nu^2 - k^2)\varphi_{in}^{k+1/2, -1/2}(t)$$

(2.5)

In the rank-two case, where $n = 3$ which is our subject in the next section, Heckman-Opdam’s hypergeometric function has the following integral representation (we omit here the dependance on $n$)

$$F_k(\lambda, x) = \frac{\Gamma(3k)}{\Gamma(3)^3V(x)^{-2k+1}} \int_{x_2}^{x_1} \int_{x_3}^{x_2} e^{(1-\frac{3}{2}(\lambda_3+k))(\nu_3+\nu_2)} \varphi_{i(\lambda_1-\lambda_2)}^{k-\frac{3}{2}, -\frac{1}{2}}(\frac{\nu_1 - \nu_2}{2})$$

$$(e^{\nu_1} - e^{\nu_2}) W_k(\nu, x) d\nu.$$}

(2.6)

where

$$W_k(\nu, x) = \left((e^{x_1} - e^{\nu_1})(e^{x_1} - e^{x_2})(e^{\nu_2} - e^{x_2})(e^{\nu_2} - e^{x_3})(e^{\nu_3} - e^{x_3})\right)^{k-1}$$

and

$$V(x) = (e^{x_1} - e^{x_2})(e^{x_1} - e^{x_3})(e^{x_2} - e^{x_3}).$$

In order to find an expression for $F_k$ of Laplace type, we write

$$F_k(\lambda, x) = \frac{\Gamma(3k)}{4\Gamma(3)^3\left(\prod_{1 \leq i < j \leq 3} \sinh(\frac{x_1 - x_j}{2})\right)}^{2k-1} \int_{x_2}^{x_1} \int_{x_3}^{x_2} e^{-3(\nu_1+\nu_2)\lambda_3/2}$$

$$\varphi_{i(\lambda_1-\lambda_2)}^{k-\frac{3}{2}, -\frac{1}{2}}(\frac{\nu_1 - \nu_2}{2}) \sinh(\frac{\nu_1 - \nu_2}{2}) \left(\prod_{1 \leq i \leq 3} \sinh(\frac{|\nu_i - x_j|}{2})\right)^{k-1} d\nu.$$
With the change of variables
\[
y_1 = \frac{\nu_1 + \nu_2}{2}, \quad t = \frac{\nu_1 - \nu_2}{2},
\]
we have that
\[
F_k(\lambda, x) = \frac{\Gamma(3k)}{2^{3k-2} \Gamma(k)^3 \left( \prod_{1 \leq i < j \leq 3} \sinh\left( \frac{x_i - x_j}{2} \right) \right)^{2k-1}} \int_{\mathbb{R}^2} e^{-3\lambda y_1} \varphi_{(\lambda_1 - \lambda_2)}(t) \sinh(t)
\]
\[
\left( (\cosh(x_1 - y_1) - \cosh t)(\cosh t - \cosh(x_2 - y_1))(\cosh(x_3 - y_1) - \cosh t) \right)^{k-1}
\]
\[
\chi_{[x_2, x_1]}(y_1 + t) \chi_{[x_3, x_2]}(y_1 - t) dy_1 dt.
\]
Now inserting
\[
k^{-\frac{3}{2}} \varphi_{(\lambda_1 - \lambda_2)}(t) = \frac{\Gamma(2k)}{2^k \Gamma(k)^2 (\sinh t)^{2k-1}} \int_{\mathbb{R}} e^{y_2(\lambda_1 - \lambda_2)} (\cosh t - \cosh y_2)^{k-1} \chi_{[-1, 1]} \left( \frac{y_2}{t} \right) dy_2,
\]
with the use of Fubini’s Theorem and the fact that
\[
\chi_{[-1, 1]} \left( \frac{y_2}{t} \right) \chi_{[x_2, x_1]}(y_1 + t) \chi_{[x_3, x_2]}(y_1 - t) = \chi_{\max(|y_2|, |y_1 - x_2|) \leq t \leq \min(y_1 - x_3, x_1 - y_1)}
\]
if follows that
\[
F_k(\lambda, x) = \int_{\mathbb{R}} \int_{\mathbb{R}} e^{3(\lambda_1 + \lambda_2)y_1 + (\lambda_1 - \lambda_2)y_2} R_k(x, y_1, y_2) dy_1 dy_2 \quad (2.7)
\]
where
\[
R_k(x, y_1, y_2) = \frac{\Gamma(2k) \Gamma(3k)}{2^{4k-2} \Gamma(k)^5 \left( \prod_{1 \leq i < j \leq 3} \sinh\left( \frac{x_i - x_j}{2} \right) \right)^{2k-1}} \int_{\max(|y_2|, |y_1 - x_2|)}^{\min(y_1 - x_3, x_1 - y_1)} \left( \frac{\cosh t - \cosh y_1}{\sinh^2 t} \right)^{k-1} \left( \prod_{i=1}^{3} |\cosh(x_i - y_1) - \cosh(t)| \right)^{k-1} dt,
\]
if \( \max(|y_2|, |y_1 - x_2|) \leq \min(y_1 - x_3, x_1 - y_1) \) and \( R_k(x, y_1, y_2) = 0 \), otherwise. We should note here that the condition
\[
\max(|y_2|, |y_1 - x_2|) \leq \min(y_1 - x_3, x_1 - y_1)
\]
is equivalent to
\[
x_3 \leq y_1 \pm y_2, -2y_1 \leq x_1
\]
an thus equivalents to \( y = (y_1 + y_2, y_1 - y_2, -2y_1) \in \text{co}(x) \), the convex hull of the orbit \( W.x \), see proposition 3.6 of [1]. Also, we have \( y = y_1 \sqrt{6} \varepsilon_1 + y_2 \sqrt{2} \varepsilon_2 \) in the orthonormal basis of \( V \)
\[
\varepsilon_1 = (e_1 + e_2 - 2e_3)/\sqrt{6}, \quad \varepsilon_2 = (e_1 - e_2)/\sqrt{2}.
\]
Making the change of variables \( z_1 = \sqrt{6}y_1, \ z_2 = \sqrt{2}y_2 \) in the formula \( (2.7) \) and identify \( \mathbb{R}^2 \) with \( V \) via the basis \( (\varepsilon_1, \varepsilon_2) \) we finally write
\[
F_k(\lambda, x) = \int_{\text{co}(x)} e^{\langle \lambda, z \rangle} N_k(x, z) dz. \quad (2.8)
\]
where \( \mathcal{N}_k(x, z) = R_k \left( x, z_1/\sqrt{6}, z_2/\sqrt{2} \right)/\sqrt{12} \).

Next, for \( f \in C^\infty(V) \), \( W \)-invariant, we define \( V_k(f) \) the \( W \)-invariant function on \( V = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \) by

\[
V_k(f)(x) = \int_{co(x)} f(z) \mathcal{N}_k(x, z) dz; \quad x \in C.
\]

**Proposition 2.2.** For \( f \in C^\infty_c(V) \) and \( x \in C \), we have the following intertwining property

\[
\Delta_k(V_k(f)) = V_k(\Delta(f)).
\]

**Proof.** By inversion formula for Fourier transform and Fubini’s Theorem

\[
V_k(f)(x) = \int_{\mathbb{R}^2} \int_{co(x)} \hat{f}(\xi) e^{i\xi \cdot z} \mathcal{N}_k(x, z) dz d\xi = \int_{\mathbb{R}^2} \hat{f}(\xi) F_k(i\xi, x) d\xi, \tag{2.9}
\]

Here we define the Fourier transform of \( f \) by

\[
\hat{f}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} f(t) e^{-i\xi \cdot t} dt.
\]

From a general estimates of Heckman-Opdam’s hypergeometric function (see for example corollary 6.2 in [12]), the last integral of (2.9) is a \( C^\infty \) as a function of \( x \) and then by (1.2) one has

\[
\Delta_k(V_k(f))(x) = -\int_{\mathbb{R}^2} \|\xi\|^2 \hat{f}(\xi) F_k(i\xi, x) d\xi = \int_{\mathbb{R}^2} \Delta f(\xi) F_k(i\xi, x) d\xi = V_k(\Delta(f))(x),
\]

which proves the desired fact. \( \square \)

### 3 Nonsymmetric Opdam’s hypergeometric function of type \( A_2 \)

We begin with some backgrounds from Heckman-Opdam theory of hypergeometric functions associated to a root system \( R \) with Weyl group \( W \) of a finite-dimensional vector space \( \mathfrak{a} \), we refer to [11, 12] for a more detailed treatment. For a regular weight \( \lambda \in P^+ \) (the set of dominant weights) we denote by \( P_k(\lambda, ) \) the Heckman-Opdam Jacobi polynomial and by \( E_k(\lambda, ) \) the non symmetric Opdam polynomial. In the following we collect some properties and relationships.

(i) \( P_k(\lambda, x) = \sum_{w \in W} E_k(\lambda, wx) \)

(ii) \( \sum_{w \in W} \text{det}(w) E_k(\lambda + \delta, wx) = V(x) P_{k+1}(\lambda, x) \), where \( \delta = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \) and

\[
V(x) = \prod_{\alpha \in R^+} \left( e^{\alpha/2} - e^{-\alpha/2} \right)
\]
So we can write

\[(iii) \quad P_k(\lambda, 0) = \frac{c_k(\rho_k)}{c_k(\lambda + \rho_k)}, \text{ where } \rho_k(\lambda) = \frac{1}{2} \sum_{\alpha \in R^+} k_{\alpha} \lambda, \quad c_k(\lambda) = \prod_{\alpha \in R^+} \frac{\Gamma(\lambda, \bar{\alpha})}{\Gamma(\lambda, \bar{\alpha}) + k_{\alpha}}\]

and \(\bar{\alpha} = 2\alpha/|\alpha|^2\).

\[(iv) \quad P_k(\lambda, x) = P_k(\lambda, 0)F_k(\lambda + \rho_k, x)\]

\[(v) \quad E_k(\lambda, x) = \frac{P_k(\lambda, 0)}{|W|} G_k(\lambda + \rho_k, x).\]

From these facts we give the following consequence.

**Proposition 3.1.** For all \(\lambda, x \in \mathfrak{a}\),

\[
\sum_{w \in W} \det(w)G_k(\lambda, w, x) = d_k(\lambda)V(x)F_{k+1}(\lambda, x), \tag{3.1}
\]

where,

\[
d_k(\lambda) = |W| \frac{c_{k+1}(\rho_{k+1})}{c_k(\rho_k)} \frac{c_k(\lambda)}{c_{k+1}(\lambda)}.
\]

**Proof.** This is closely related to (ii), where by using the above relations one can write for regular \(\lambda \in P_+\),

\[
\sum_{w \in W} \det(w)G_k(\lambda + \rho_{k+1}, x) = d_k(\lambda + \rho_{k+1})V(x)F_{k+1}(\lambda + \rho_{k+1}, x),
\]

since we have \(\rho_k + \delta = \rho_{k+1}\). This identity can be extended for all \(\lambda \in \mathfrak{a}\) via analytic continuation by means of Carlson’s theorem. 

\[\square\]

Let us return now to the space \(\mathcal{V}\), for \(n = 3\) and investigate (3.1). In this case we have

\[
d_k(\lambda) = \frac{1}{(2k+1)(3k+1)(3k+2)} (\lambda_1 - \lambda_2 + k)(\lambda_2 - \lambda_3 + k)(\lambda_1 - \lambda_3 + k).
\]

We introduce the antisymmetric function

\[
F_k^*(\lambda, x) = \frac{1}{6} \sum_{w \in S_3} \det(w)G_k(\lambda, x) = \frac{d_k(\lambda)}{6}V(x)F_{k+1}(\lambda, x) \tag{3.2}
\]

and the following notations:

\[
I_k(\lambda, \nu) = \left(\frac{k-\frac{1}{2}}{2\nu_1(\lambda_1-\lambda_2)}\right) e^{-\frac{1}{2}(\lambda_3+k)(\nu_1+\nu_2)}, \quad \gamma_k = \frac{\Gamma(3k)}{\Gamma(k)^3}
\]

\[
L_k(\lambda, \nu) = \left(\frac{k-\frac{1}{2}}{2\nu_1(\lambda_1-\lambda_2)}\right) e^{-\frac{1}{2}(\lambda_3+k)(\nu_1+\nu_2)}.\]

So we can write

\[
F_k(\lambda, x) = \gamma_kV(x)^{-2k+1} \int_{x_2}^{x_1} \int_{x_3}^{x_2} I_k(\lambda, \nu)e^{\nu_1+\nu_2}(e^{\nu_1} - e^{\nu_2})W_k(\nu, x) d\nu. \tag{3.3}
\]

The integral representation for the functions \(F_k^*\) is given in the following.
Proposition 3.2. We have for $\lambda \in \mathbb{V}_C$ and $x \in C$,

$$F_k^*(\lambda, x) = \frac{\gamma_k}{4k^2} V(x)^{-2k} \int_{x_2}^{x_1} \int_{x_3}^{x_2} L_k(\lambda, \nu)p(x, \nu)W_k(\nu, x)d\nu. \quad (3.4)$$

where

$$p(x, \nu) = -2be^{2(\nu_1+\nu_2)} + (ab + 3)e^{\nu_1+\nu_2}(e^{\nu_1} + e^{\nu_2}) - 2a(e^{2\nu_1} + e^{2\nu_2})$$

$$-2(b^2 + a)e^{\nu_1+\nu_2} + 4b(e^{\nu_1} + e^{\nu_2}) - 6$$

and with $a = e^{x_1} + e^{x_2} + e^{x_3}$ and $b = e^{-x_1} + e^{-x_2} + e^{-x_3}$.

Proof. We first write

$$(\lambda_1 - \lambda_2 + k)(\lambda_1 - \lambda_3 + k)(\lambda_2 - \lambda_3 + k)$$

$$= \frac{(\lambda_1 - \lambda_2 + k)}{4}\left((-3\lambda_3 + k)^2 + 2k(-3\lambda_3 + k) - ((\lambda_1 - \lambda_2)^2 - k^2)\right).$$

The formula (2.6) for parameter $k + 1$, together with (2.4) yield

$$(\lambda_1 - \lambda_2 + k)F_{k+1}(\lambda, x) = 2(2k + 1)\gamma_{k+1} V(x)^{-2k-1} \int_{x_2}^{x_1} \int_{x_3}^{x_2} L_k(\lambda, \nu)W_{k+1}(\nu, x)d\nu.$$  

So using integration by parts,

$$(-3\lambda_3 + k)(\lambda_1 - \lambda_2 + k)F_{k+1}(\lambda, x)$$

$$= 2(2k + 1)\gamma_{k+1} V(x)^{-2k-1} \int_{x_2}^{x_1} \int_{x_3}^{x_2} L_k(\lambda, \nu)(4k - (\partial_{\nu_1} + \partial_{\nu_2}))W_{k+1}(\nu, x)d\nu.$$  

and

$$(-3\lambda_3 + k)^2(\lambda_1 - \lambda_2 + k)F_{k+1}(\lambda, x)$$

$$= 2(2k + 1)\gamma_{k+1} V(x)^{-2k-1} \int_{x_2}^{x_1} \int_{x_3}^{x_2} L_k(\lambda, \nu)(\partial_{\nu_1} + \partial_{\nu_2} - 4k)^2W_{k+1}(\nu, x)d\nu.$$
In the same way with the use of (2.5),

\[
((\lambda_1 - \lambda_2)^2 - k^2)(\lambda_1 - \lambda_2 + k)F_{k+1}(\lambda, x)
\]

\[
= -2(2k + 1)\gamma_{k+1}V(x)^{-2k-1} \int_{x_2}^{x_1} \int_{x_3}^{x_2} e^{-\frac{1}{2}(\lambda_3 + k)(\nu_1 + \nu_2)} (\lambda_1 - \lambda_2 + k)
\]

\[
\varphi_{i(\lambda_1 - \lambda_2)} \left( \frac{\nu_1 - \nu_2}{2} \right) (\partial_{\nu_1} - \partial_{\nu_2})W_{k+1}(\nu, x) d\nu
\]

\[
= 2(2k + 1)\gamma_{k+1}V(x)^{-2k-1} \int_{x_2}^{x_1} \int_{x_3}^{x_2} e^{-\frac{1}{2}(\lambda_3 + k)(\nu_1 + \nu_2)} \frac{\varphi_{i(\lambda_1 - \lambda_2)}' \left( \frac{\nu_1 - \nu_2}{2} \right)}{\lambda_1 - \lambda_2 - k} (\partial_{\nu_1} - \partial_{\nu_2})^2 W_{k+1}(\nu, x) d\nu
\]

\[
-4k(2k + 1)\gamma_{k+1}V(x)^{-2k-1} \int_{x_2}^{x_1} \int_{x_3}^{x_2} e^{-\frac{1}{2}(\lambda_3 + k)(\nu_1 + \nu_2)} \frac{\varphi_{i(\lambda_1 - \lambda_2)}' \left( \frac{\nu_1 - \nu_2}{2} \right)}{\lambda_1 - \lambda_2 - k} \left( \frac{e^{\nu_1} + e^{\nu_2}}{e^{\nu_1} - e^{\nu_2}} \right) (\partial_{\nu_1} - \partial_{\nu_2}) W_{k+1}(\nu, x) d\nu
\]

\[
= 2(2k + 1)\gamma_{k+1}V(x)^{-2k-1} \int_{x_2}^{x_1} \int_{x_3}^{x_2} L_k(\lambda, \nu) \left\{ (\partial_{\nu_1} - \partial_{\nu_2})^2 - 2k \left( \frac{e^{\nu_1} + e^{\nu_2}}{e^{\nu_1} - e^{\nu_2}} \right) (\partial_{\nu_1} - \partial_{\nu_2}) \right\} W_{k+1}(\nu, x) d\nu.
\]

Hence it follows that

\[
F_k^*(\lambda, x)
\]

\[
= \frac{\gamma_k}{4k^2} V(x)^{-2k} \int_{x_2}^{x_1} \int_{x_3}^{x_2} L_k(\lambda, \nu) \left\{ 24k^2 + 2k \left( \frac{e^{\nu_1} + e^{\nu_2}}{e^{\nu_1} - e^{\nu_2}} \right) (\partial_{\nu_1} - \partial_{\nu_2}) - 10k(\partial_{\nu_1} + \partial_{\nu_2}) + 4\partial_{\nu_1} \partial_{\nu_2} \right\} W_{k+1}(\nu, x) d\nu.
\]

Now, it is straightforward computation to verify that

\[
p(\nu, x) = \left\{ 24k^2 + 2k \left( \frac{e^{\nu_1} + e^{\nu_2}}{e^{\nu_1} - e^{\nu_2}} \right) (\partial_{\nu_1} - \partial_{\nu_2}) - 10k(\partial_{\nu_1} + \partial_{\nu_2}) + 4\partial_{\nu_1} \partial_{\nu_2} \right\} W_{k+1}(\nu, x)
\]

\[
= 0.
\]

Next, we can provide an integral expansion for \(G_k\) by differentiating (3.3) and (3.4). Let us introduce the operators

\[
D_k = D_k(\lambda) = (\lambda_1 - \lambda_3 + 2k)T_{\pi(e_1)} + (\lambda_2 - \lambda_3 + k)T_{\pi(e_2)} + \tau(\lambda) + k(\lambda_1 - \lambda_3) + k^2
\]

\[
D_k^* = D_k^*(\lambda) = (\lambda_1 - \lambda_3 - 2k)T_{\pi(e_1)} + (\lambda_2 - \lambda_3 - k)T_{\pi(e_2)} + \tau(\lambda) - k(\lambda_1 - \lambda_3) + k^2
\]

where \(\tau(\lambda) = \lambda_1^2 + \lambda_2^2 + \lambda_1 \lambda_2\). Recall that \(\pi(e_1) = (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3})\) and \(\pi(e_2) = (-\frac{1}{3}, \frac{2}{3}, -\frac{1}{3})\).

Our main result from this section is the following.

**Theorem 3.3.** For \(\lambda, x \in \mathbb{V}\) we have

\[
(\tau(\lambda) - k^2)G_k(\lambda, x) = D_k(F_k(\lambda, x)) + D_k^*(F_k^*(\lambda, x)).
\]
Proof. The proof is purely computational. Using the following fact, see (5.1) of [11],

\[ wT_{w^{-1}} = T_{w, z} - \sum_{\alpha \in R_+ \cup w \alpha \in R_-} k(\alpha, \xi) s_{\omega}. \]

we obtain

\[
T_{\pi(e_1)}(G_k(\lambda, x)) = \lambda_1 G_k(\lambda, x), \\
T_{\pi(e_1)}(G_k(\lambda, s_{1,2}x)) = \lambda_2 G_k(\lambda, s_{1,2}x) - k G_k(\lambda, x), \\
T_{\pi(e_1)}(G_k(\lambda, s_{2,3}x)) = \lambda_1 G_k(\lambda, s_{2,3}x), \\
T_{\pi(e_1)}(G_k(\lambda, s_{1,3}x)) = \lambda_3 G_k(\lambda, s_{1,3}x) - kg_k(\lambda, s_{1,3}x), \\
T_{\pi(e_1)}(G_k(\lambda, s_{2,3}x)) = \lambda_2 G_k(\lambda, s_{2,3}x) - kG_k(\lambda, s_{2,3}x), \\
T_{\pi(e_1)}(G_k(\lambda, s^2x)) = \lambda_3 G_k(\lambda, s^2x) - kG_k(\lambda, s_{1,2}x) - kG_k(\lambda, s_{2,3}x),
\]

and

\[
T_{\pi(e_2)}(G_k(\lambda, x)) = \lambda_2 G_k(\lambda, x), \\
T_{\pi(e_2)}(G_k(\lambda, s_{1,2}x)) = \lambda_1 G_k(\lambda, s_{1,2}x) + k G_k(\lambda, x), \\
T_{\pi(e_2)}(G_k(\lambda, s_{2,3}x)) = \lambda_3 G_k(\lambda, s_{2,3}x) - kG_k(\lambda, x), \\
T_{\pi(e_2)}(G_k(\lambda, s_{1,3}x)) = \lambda_2 G_k(\lambda, s_{1,3}x) - kG_k(\lambda, s_{1,3}x) + kG_k(\lambda, s_{1,3}x), \\
T_{\pi(e_2)}(G_k(\lambda, s^2x)) = \lambda_3 G_k(\lambda, s^2x) + kG_k(\lambda, s_{2,3}x),
\]

where \( \sigma = s_{1,3}s_{1,2}. \) Thus we have

\[
T_{\pi(e_1)}(F_1(\lambda, x)) = \frac{1}{6} \left\{ (\lambda_1 - 2k) G_k(\lambda, x) + (\lambda_2 - k) G_k(\lambda, s_{1,2}x) + (\lambda_1 - 2k) G_k(\lambda, s_{2,3}x) + (\lambda_2 - k) G_k(\lambda, s_{1,3}x) + (\lambda_3 - k) G_k(\lambda, s_{1,3}x) \right\},
\]

\[
T_{\pi(e_2)}(F_1(\lambda, x)) = \frac{1}{6} \left\{ \lambda_2 G_k(\lambda, x) + (\lambda_1 - k) G_k(\lambda, s_{1,2}x) + (\lambda_3 + k) G_k(\lambda, s_{2,3}x) + \lambda_2 G_k(\lambda, s_{1,3}x) + (\lambda_3 + k) G_k(\lambda, \sigma x) + (\lambda_1 - k) G_k(\lambda, \sigma^2x) \right\},
\]

\[
T_{\pi(e_1)}(F^*_1(\lambda, x)) = \frac{1}{6} \left\{ (\lambda_1 + 2k) G_k(\lambda, x) - (\lambda_2 + k) G_k(\lambda, s_{1,2}x) - (\lambda_1 + 2k) G_k(\lambda, s_{2,3}x) - \lambda_3 G_k(\lambda, s_{1,3}x) + (\lambda_2 + k) G_k(\lambda, \sigma x) + \lambda_3 G_k(\lambda, \sigma^2x) \right\},
\]

and

\[
T_{\pi(e_2)}(F^*_1(\lambda, x)) = \frac{1}{6} \left\{ \lambda_2 G_k(\lambda, x) - (\lambda_1 + k) G_k(\lambda, s_{1,2}x) - (\lambda_1 + k) G_k(\lambda, s_{2,3}x) - \lambda_3 G_k(\lambda, s_{1,3}x) + (\lambda_3 - k) G_k(\lambda, \sigma x) + (\lambda_1 + k) G_k(\lambda, \sigma^2x) \right\}.
\]

So, formula (3.5) can be checked by a straightforward calculations. \( \square \)
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