Lévy-driven Fluid Queue with Server Breakdowns and Vacations

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Abstract

In this paper, we consider a Lévy-driven fluid queueing system where the server may subject to breakdowns and repairs. In addition, the server will leave for a vacation each time when he finds an empty system. We cast the queueing process as a Lévy process modified to have random jumps at two classes of stopping times. By using the Kella-Whitt martingale method, we obtain the limiting distribution of the virtual waiting time process. Moreover, we investigate the busy period, the correlation structure and the stochastic decomposition properties. These results may be generalized to Lévy processes with multi-class jump inputs or Lévy-driven queues with multiple input classes.

Keywords: Lévy processes; Fluid queues; Server breakdowns and vacations; Kella-Whitt martingale; Stochastic decomposition

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1. Introduction

The class of Lévy processes consisting of all stochastic processes with stationary independent increments is one of the most important family of stochastic processes arising in many areas of applied probability. It covers well-studied processes such as Brownian motion and compound Poisson processes. Lévy processes are used as models in the study of queueing systems, insurance risks, storage systems, mathematical finance and so on. For a comprehensive and outstanding analysis of Lévy processes and their applications, readers can refer to the books by Bertoin [4], Sato [22], Applebaum [1], and Kyprianou [18].

Over recent years it has been a rapid growth in the literature on queues with server breakdowns and vacations due to their widely applications in computer communication networks and manufacturing systems. Li et al. [19] considered an $M/G/1$ queue with Bernoulli vacations and server breakdowns using a supplementary variable method. Gray et al. [12] analyzed a multiple-vacation $M/M/1$ queueing model, where the service station is subject to breakdown while in operation. Ke [15] studied the control policy in $M[X]/M/1$ queue with server breakdowns and multiple vacations. Jain and Jain [14] dealt with a single server working vacation queueing model with multiple types of server breakdowns. Wu and Yin [24] gave a detailed analysis on an $M/G/1$ retrial queue with non-exhaustive random vacations and unreliable server. Wang et al. [23] studied a discrete-time Geo/G/1 queue in which the server operates multiple vacations and may break down while working. Yang and Wu [25] investigated the N-policy $M/M/1$ queueing system with working vacation and server breakdowns. However, most of the papers are focused on queues where customers arrive at the system according to independent Poisson processes or geometrical arrival processes. Queueing systems with Lévy input (or Lévy-driven queues) are still not well investigated in literature. Recently, queueing systems with Lévy input have been attracting increasing attention in the applied probability and stochastic operations research communities. Lévy-driven queues covers the classical $M/G/1$ queue and the reflected Brownian motion as special cases. Kella and Boxma [16] first considered Lévy processes with secondary jump input which were applied to analyze queues with server vacations. Recently, Lieshout and Mandjes [20] analyzed tail asymptotics of a two-node tandem queue with spectrally positive Lévy input. Boxma et al. [5] analyzed a generic class of Lévy-driven queuing systems with server vacation. Dębicki and Mandjes [9] provided a survey on Lévy-driven queues. Dębicki et al. [8] focused on transient analysis of Lévy-driven tandem
queues. Palmowski et al. [21] considered a controlled fluid queuing model with Lévy input. Boxma and Kella [6] generalized known workload decomposition results for Lévy queues with secondary jump inputs and queues with server vacations or service interruptions. However, to the best of our knowledge, no work on Lévy queues with server breakdowns and vacations is found in the queueing literature.

In this paper, we consider a single-server Lévy-driven fluid queue with multiple vacations (exhaustive service) and a server subject to breakdowns and repairs which is motivated by the performance analysis of resources in communication networks. A breakdown at the server is represented by the arrival of a failure. The principal purpose of the present paper is to apply the martingale results which were derived in Kella and Whitt [17] to investigate the stochastic dynamics of the system and realize an extensive analysis of the system from the transient virtual waiting time process to the steady-state distribution of the waiting time process. In addition, we give a detailed analysis of the system busy period and the queue’s correlation structure. Furthermore, we establish decomposition results for the model.

The rest of the paper is organized as follows. In Section 2, we introduce a few basic facts concerning Lévy processes and martingales. In Section 3, we describe the general model and formulate the model as a Lévy process modified to have random jumps at two classes of stopping times. In Section 4, we characterize the steady-state distribution of the virtual waiting time process and the mean length of the busy period. In Section 5, we address the transient distribution of the waiting time process as well as the queue’s correlation structure. In Section 6, we present two stochastic decomposition results.

2. Preliminaries on Lévy processes

In this section, we mainly give several notations, definitions and propositions about Lévy processes. For more details, see [1, 2, 4, 22].

Denote $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\geq 0\})$ be a complete probability space endowed with a $\sigma$-field filtration $\mathcal{F}_t_{\geq 0}$, i.e. an increasing family of sub-fields, which fulfils the usual conditions. That is, each $\mathcal{F}_t$ is $\mathbb{P}$-complete and $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$ for every $t$. Throughout, adapted, stopping times and martingales will be defined with respect to this filtration.
Definition 2.1. Let $X = (X_t, t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if

(i) $X_0 = 0$ (a.s.);

(ii) $X$ has independent and stationary increments;

(iii) $X$ is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$,

$$\lim_{t \rightarrow s} \mathbb{P}(|X_t - X_s| > a) = 0,$$

then $X$ is a Lévy process.

Two central and key results in the foundations of Lévy processes are given by the following propositions (known as the Lévy-Khintchine formula and Lévy-Itô decomposition respectively).

Proposition 2.1. Let $X$ be a Lévy process with Lévy measure $\nu$ ($\int_{\mathbb{R}-\{0\}} (|x|^2 \wedge 1) \nu(dx) < \infty$). Then

$$\mathbb{E}e^{i\theta X_t} = e^{t\psi(\theta)}, \quad t \geq 0,$$

where

$$\psi(\theta) = i\mu \theta - \frac{1}{2} \theta^2 \sigma^2 + \int_{\mathbb{R}-\{0\}} (e^{i\theta x} - 1 - i\theta x 1_{|x|<1}) \nu(dx),$$

$\mu \in \mathbb{R}$ and $\sigma \geq 0$. Moreover, given $\nu$, $\mu$, $\sigma^2$, the corresponding Lévy process is unique in distribution. Furthermore, the jump process of $X$ is a Poisson point process with characteristic measure $\nu$, that is

$$\nu(\Lambda) = \mathbb{E} \left[ \sum_{0<s \leq 1} 1_{\Lambda}(\Delta X_s) \right].$$

The function $\psi(\cdot)$ is called the Lévy exponent of the Lévy process.

Proposition 2.2. Let $X$ be a Lévy process. Then $X$ has a decomposition

$$X_t = B_t + \mu t + \int_{\{x < 1\}} x(N_t(\cdot, dx) - t\nu(dx)) + \int_{\{|x| \geq 1\}} xN_t(\cdot, dx),$$

where $B$ is a Brownian motion and for $0 \notin \Lambda$, $N^\Lambda_t = \int_{\Lambda} N_t(\cdot, dx)$ is a Poisson process independent of $B$.

The following are some known fundamental results about Lévy processes:
(i) Any Lévy process can be represented as an independent sum of a Brownian motion $B$ and a ‘compound Poisson’-like process $X^0$. If $X$ has no negative jumps and the paths of $X^0$ are of bounded variation, then $X^0$ can be a subordinator which can be represented as a nonnegative compound Poisson process or as the limit of a sequence of nonnegative compound Poisson processes \[3\].

(ii) If $\nu((-\infty,0)) = 0$, then the Laplace-Stieltjes transform exists and is given by
$$
E\exp(-\theta X_t) = \exp\left(t \varphi(\theta)\right)
$$
where
$$
\varphi(\theta) = \log E\exp\{-\theta X_1\} = -\mu\theta + \frac{1}{2}\theta^2\sigma^2 + \int_{\mathbb{R}_+} (e^{-\theta x} - 1 + \theta x 1_{|x|<1})\nu(dx),
$$
here the function $\varphi(\cdot)$ is called the Laplace-Stieltjes exponent of the Lévy process. It is easy to get that $\varphi(0) = 0$ and $\varphi(\cdot)$ is convex; If $X$ is not a subordinator, then $\varphi(\theta) \to \infty$ as $\theta \to \infty$; Whenever $EX_1 < 0$, $\varphi(\cdot)$ is strictly increasing. Throughout the paper, exponent will mean the Laplace-Stieltjes exponent $\varphi(\cdot)$ due to the convenience compared to the Lévy exponent $\psi(\cdot)$.

(iii) If $X$ has bounded jumps, then $E|X_1|^n < \infty$ for every $n \geq 0$.

It is known that there always exists a version with sample paths of $X$ that are càdlàg, which is therefore strong Markov. Throughout this paper, every Lévy process mentioned is assumed to be such.

In this paper, we focus only on the spectrally positive Lévy process, i.e. $\nu(-\infty,0) = 0$. For any spectrally positive Lévy process $X$ with $\beta > 0$ and $EX_1 < 0$. Denote by $T^\xi = \inf\{t|X_t = -\xi\}$ for any non-negative random variable $\xi$ which is independent of $X$. Then it follows Theorem 3.12 in [18] that
$$
E\exp\{-\beta T^\xi\} = \exp\{-\varphi^{-1}(\beta)\xi\}, \quad ET^\xi = \frac{E\xi}{-EX_1}.
$$

We now give the generalized Pollaczek-Khinchine formula associated with a reflected Lévy process with no negative jumps which is also a celebrated formula in queueing theory. Given some random variable $0 \leq R_0 \in \mathcal{F}_0$, let
$$
I_t = \left(-\inf_{0\leq s \leq t} X_s - R_0\right)^+, \quad R_t = R_0 + X_t + I_t, \quad t \geq 0.
$$
Then $R$ is a reflected Lévy process with $I$ being its local time at zero. Since $X$ has no negative jumps, $I$ is continuous with $I_0 = 0$ and is the minimal right-continuous nondecreasing process such
that $R_t \geq 0$ for all $t$. The process $\{I_t, t \geq 0\}$ must only increase when $R_t = 0$, so that

$$\int_0^\infty 1_{\{R_t > 0\}} dI_t = 0.$$  

The following result is the famous (generalized) Pollaczek-Khinchine formula in queueing theory. We can refer to the papers [16] and [9] for its proof.

**Proposition 2.3.** If $X$ is a spectrally positive Lévy process such that $EX_1 < 0$, then

$$\lim_{t \to \infty} \mathbb{E} e^{-\theta R_t} = \frac{\theta \varphi'(0)}{\varphi(\theta)}, \quad \theta > 0.$$  

We end this section by presenting the famous Kella-Whitt martingale associated with spectrally positive Lévy process $X$ with exponent $\varphi(\theta)$ that we will apply it to analyze our fluid queue.

**Proposition 2.4.** Let $X$ be a spectrally positive Lévy process with exponent $\varphi(\theta)$. Let $L_t = \int_0^t dL_s^\varsigma + \sum_{0 \leq s \leq t} \Delta L_s$ be an adapted càdlàg process of bounded variation on finite intervals with continuous part $\{L^\varsigma_t\}$ and jumps $\Delta L_s = L_s - L_{s-}$, and define $Z_t = Z_0 + X_t + L_t$, where $Z_0 \in \mathcal{F}_0$. Then

$$M_t = \varphi(\theta) \int_0^t [\exp\{-\theta Z_s\}ds + \exp\{-\theta Z_0\} - \exp\{-\theta Z_t\} - \theta \int_0^t \exp\{-\theta Z_s\}dY^\varsigma_s$$

$$\quad + \sum_{0 \leq s \leq t} [\exp\{-\theta Z_s\} - \exp\{-\theta Z_s - \Delta L_s\}$$

is a local martingale. In addition, if the expected variation of $\{L^\varsigma_t, t \geq 0\}$ and the expected number of jumps of $\{L_t, t \geq 0\}$ are finite on every finite interval and $\{Z_t, t \geq 0\}$ is a non-negative process (or bounded below), then $\{M_t, t \geq 0\}$ is a martingale.

The Kella-Whitt martingale is a stochastic integral with respect to the Wald martingale $N_t = \exp\{-\theta X_t - \varphi(\theta)t\}$ and the readers may refer to the paper [17] for its proof.

3. Model formulation

We consider a single fluid queue with a (nondecreasing) Lévy input and available-processing (or service) processes. For any $t \geq 0$, let $X_t$ be the cumulative input of fluid over the interval $[0, t]$.

We assume that $X$ is a subordinator, that is, a Lévy process, having the exponent

$$\phi(\theta) = -a \theta + \int_{\mathbb{R}^+} (e^{-\theta x} - 1)\nu(dx),$$

where $a > 0$ is a constant and $\nu$ is a Lévy measure.
where \( a \geq 0 \) and \( \int_{\mathbb{R}^+} x \nu(dx) < \infty \). Let \( S(t) \) be the cumulative available processing over the interval \([0,t]\). Here, we assume the server processes the fluid at a constant deterministic rate \( r \) (whenever the fluid level is positive). So that \( S(t) = rt, \quad t \geq 0 \). We assume the storage space is unlimited.

Let \( \{\tau_n, n \geq 1\} \) be the strictly increasing sequence of stopping times at which the server fails. When the server fails, it is repaired immediately and the time required to repair it is a positive random variable \( \xi_n \) which is \( \mathcal{F}_{\tau_n} \)-measurable for \( n \geq 1 \). When the system becomes empty, the server takes a vacation of random length \( \eta_n \) which is \( \mathcal{F}_{\sigma_n} \)-measurable for \( n \geq 1 \), where \( \{\sigma_n, n \geq 1\} \) is a strictly increasing sequence of stopping times at which the server is leaving for a vacation. Vacations continue until, on return from a vacation, the server finds the system is non-empty.

Let \( W_t \) represent the virtual waiting time at time \( t \). Denote the initial workload by \( W_0 \in \mathcal{F}_0 \). Then \( W_t \) can be defined by (the empty sum is zero)

\[
W_t = W_0 + X_t - rt + \sum_{i=1}^{N_t^R} \xi_i + \sum_{i=1}^{N_t^V} \eta_i, \quad t \geq 0,
\]

where

\[
N_t^R = \sup\{n|\tau_n \leq t\}, \quad N_t^V = \sup\{n|\sigma_n \leq t\},
\]

\[
\sigma_n = \inf \left\{ t \geq 0 \left| W_0 + X_t - rt + \sum_{i=1}^{N_t^R} \xi_i + \sum_{i=1}^{n-1} \eta_i = 0 \right. \right\}, \quad n \geq 1.
\]

**Remark 3.1.** It should be noted that here \( W_t, t \geq 0 \), is not the workload process. However, if \( \xi_n \) represents the total arriving workload during the \( n \)th repair period and \( \eta_n \) corresponds to the total arriving workload during the \( n \)th vacation period, then \( W_t \) is the workload in the system conditioned on being in the active periods which are defined as the normal processing fluid periods (excluding the repair periods and vacation periods).

Define \( Y_t = X_t - rt \). Then \( Y_t \) is a spectrally positive Lévy process with exponent \( \varphi(\theta) = \phi(\theta) + r\theta \).

Denote \( \rho = -\phi'(0) \) and impose the conditions \( \rho < r \) and \( \mathbb{E}|Y_t| < \infty \) for all \( t \geq 0 \).

### 4. The steady-state distribution

In this section, we characterize the limiting distribution of the virtual waiting time \( W_t \). Let \( \overset{d}{\to} \) denote convergence in distribution and let \( \overset{p}{\to} \) denote convergence in probability. Under the condition \( \rho < r \), we suppose \( W_t \overset{d}{\to} W \) and \( t^{-1}\mathbb{E}W_t \to 0 \), as \( t \to \infty \), where \( W \) is a random variable.
In order to derive the steady-state distribution of the virtual waiting time \( W_t \), we give the following lemmas.

**Lemma 4.1.** If \( E(N^R_t + N^V_t) < \infty \) for all \( t \), then \( \{M_t, t \geq 0\} \) is a zero-mean real-valued martingale with respect to \( \{\mathcal{F}_t, t \geq 0\} \), where

\[
M_t = \varphi(\theta) \int_0^t e^{-\theta W_s} ds + e^{-\theta W_0} - e^{-\theta W_t} \sum_{k=1}^{N^R_t} \left[ e^{-\theta(W_{r_k} - \xi_k)} - e^{-\theta W_{r_k}} \right] - \sum_{k=1}^{N^V_t} (1 - e^{-\theta \eta_k}). \tag{4.1}
\]

**Proof.** Let \( H_t = \sum_{i=1}^{N^R_t} \eta_i \). Then \( W_t = W_0 + Y_t + H_t \) and \( \{H_t, t \geq 0\} \) is adapted of bounded variation on every finite interval, and \( H_t = 0 \) for all \( t \geq 0 \). Since \( E(N^R_t + N^V_t) < \infty \), by Proposition 2.4 and through a trivial manipulation, we get that (4.1) is a martingale. \( \Box \)

**Lemma 4.2.** Let \( T \) be a nonnegative random variable with possible integer values \( 0, 1, 2, \cdots \) and \( ET < \infty \). If \( \{\zeta_n, n \geq 1\} \) is an i.i.d. sequence with \( \zeta_n \) independent of \( 1_{\{T \geq n\}} \) and \( E\zeta_1 < \infty \), then

\[
E \sum_{n=1}^{T} \zeta_n = (E\zeta_1)(ET).
\]

**Proof.** Since \( \zeta_n \) is independent of \( 1_{\{T \geq n\}} \), the bounded convergence theorem yields

\[
E(\sum_{n=1}^{T} \zeta_n) = E(\sum_{n=1}^{\infty} \zeta_n 1_{\{T \geq n\}}) = \sum_{n=1}^{\infty} (E\zeta_n)(E1_{\{T \geq n\}}) = E\zeta_1 \sum_{n=1}^{\infty} (E1_{\{T \geq n\}}) = E\zeta_1 \sum_{n=1}^{\infty} P(T \geq n) = (E\zeta_1)(ET).
\]

\( \Box \)

With the help of the above two lemmas, we can derive the limiting distribution of the virtual waiting time provided in the following theorem which is the most important result of this paper.

**Theorem 4.1.** Let \( \{\xi_n, n \geq 1\} \) and \( \{\eta_n, n \geq 1\} \) be two positive i.i.d. sequences with \( E\xi_1 < \infty \) and \( E\eta_1 < \infty \), respectively. Suppose that \( n^{-\sigma_n} \xi_n \xrightarrow{p} \lambda^R_1 \) and \( n^{-\sigma_n} \eta_n \xrightarrow{p} \lambda^V_1 \) as \( n \to \infty \) for \( 0 < \lambda^R_1 < \infty \) and \( 0 < \lambda^V_1 < \infty \), respectively. Further, assume that \( \{W_{r_k}, k \geq 1\} \) and \( \{W_{r_k} - \xi_k, k \geq 1\} \) are stationary and ergodic that \( (W_{r_k} - \xi_k, W_{r_k}) \xrightarrow{d} (W^-, W^+) \) where \( W^-, W^+ \) are two proper random variables. If \( \xi_n \) is independent of \( 1_{\{N^R_n \geq n\}} \) and \( \eta_n \) is independent of \( 1_{\{N^V_n \geq n\}} \), then for \( \theta > 0 \),

\[
\lambda_R = \frac{p \varphi'(0)}{E\xi_1}, \quad \lambda_V = \frac{(1-p)\varphi'(0)}{E\eta_1}, \tag{4.2}
\]
\[ \lim_{t \to \infty} t^{-1} \mathbb{E} \sum_{k=1}^{N_t^R} \xi_k = -p \mathbb{E} Y_1, \quad \lim_{t \to \infty} t^{-1} \mathbb{E} \sum_{k=1}^{N_t} \eta_k = -(1-p) \mathbb{E} Y_1, \]  \tag{4.3}

\[ \lim_{t \to \infty} \mathbb{E} e^{-\theta W_t} = \mathbb{E} e^{-\theta W} = \frac{\theta \phi'(0)}{\mathbb{E} \xi_1} \left[ p \mathbb{E} e^{-\theta W} - \mathbb{E} e^{-\theta W^+} + (1-p) \frac{1 - \mathbb{E} e^{-\theta W_1}}{\mathbb{E} \eta_1} \right], \]  \tag{4.4}

where \(0 \leq p \leq 1\).

**Proof.** Dividing (3.1) by \(t\) and taking expectation, we get

\[ t^{-1} \mathbb{E} W_t = t^{-1} \mathbb{E} W_1 + \mathbb{E} Y_1 + t^{-1} \mathbb{E} \sum_{k=1}^{N_t^R} \xi_k + t^{-1} \mathbb{E} \sum_{k=1}^{N_t^V} \eta_k, \]  \tag{4.5}

where \(t^{-1} \mathbb{E} W_t \to 0\) by the assumption. Thus, we have proved (4.3).

Taking \(W_0 = W^*\) where \(W^*\) is an independent copy of \(W\), \(\{W_t, t \geq 0\}\) becomes stationary. By Lemma 4.2 and (3.1), we get

\[ \mathbb{E} N_t^R \mathbb{E} \xi_1 + \mathbb{E} N_t^V \mathbb{E} \eta_1 = -\mathbb{E} Y_1. \]  \tag{4.6}

So that,

\[ \mathbb{E} N_t^R \mathbb{E} \xi_1 = -p \mathbb{E} Y_1, \quad \mathbb{E} N_t^V \mathbb{E} \eta_1 = -(1-p) \mathbb{E} Y_1. \]

Hence,

\[ \mathbb{E} N_t^R = \frac{p \phi'(0)}{\mathbb{E} \xi_1}, \quad \mathbb{E} N_t^V = \frac{(1-p) \phi'(0)}{\mathbb{E} \eta_1}. \]  \tag{4.7}

Conditions \(n^{-1} \tau_n \xrightarrow{p} \lambda_R^{-1}\) and \(n^{-1} \sigma_n \xrightarrow{p} \lambda_V^{-1}\) as \(n \to \infty\) imply that \(t^{-1} N_t^R \xrightarrow{p} \lambda_R\) and \(t^{-1} N_t^V \xrightarrow{p} \lambda_V\). Together with \(t^{-1} N_t^R \xrightarrow{p} \mathbb{E} N_t^R\) and \(t^{-1} N_t^V \xrightarrow{p} \mathbb{E} N_t^V\) w.p.1 as \(t \to \infty\), this implies (4.2).

From Lemma 4.2, optional stopping at \(t = 1\) in (4.1) yields

\[ 0 = \phi(\theta) \mathbb{E} e^{-\theta W} - \mathbb{E} N_t^R \left[ \mathbb{E} e^{-\theta W} - \mathbb{E} e^{-\theta W^+} \right] - \mathbb{E} N_t^V (1 - \mathbb{E} e^{-\theta W_1}) \]  \tag{4.8}

Substituting (4.7) into (4.8), we obtain (4.4).

\[ \square \]

**Remark 4.1.** From [2], we have \(\phi(\theta) = 0\) for some \(\theta > 0\) if and only if \(Y_t\) has a lattice distribution. But for a Lévy process with no negative jumps and \(\mathbb{E} Y_t < 0\), it is not possible. In addition, when
$N_t^R \equiv 0$ for all $t$, i.e., $p = 0$, the present system reduces to the Lévy process-driven queue with server vacations. It may be noted that the equation (4.4) after putting $p = 0$ agrees with the equation (4.6) presented in Kella and Whitt [16]. We should note that when $N_t^Y \equiv 0$ for all $t$, the present system reduces to the Lévy process-driven queue with server breakdowns. However, the equation (4.4) in Theorem 4.1 after putting $p = 1$ cannot characterize the limiting distribution of Lévy-driven queue with server breakdowns due to the fact that $\{W_t, t \geq 0\}$ is not a non-negative process.

**Corollary 4.1.** When the system is stable, we have

$$\omega = \mathbb{E}W = \frac{\varphi''(0)}{2\varphi'(0)} + p \frac{\mathbb{E}(W^+)^2 - \mathbb{E}(W^-)^2}{2\mathbb{E} \xi_1} + (1 - p) \frac{\mathbb{E} \eta_1^2}{2\mathbb{E} \eta_1},$$

$$v = \text{Var}W = \frac{1}{3} \frac{\varphi''(0)}{\varphi'(0)} - \frac{1}{4} \left( \frac{\varphi''(0)}{\varphi'(0)} \right)^2 + (1 - p) \left[ \frac{1}{3} \frac{\mathbb{E} \eta_1^3}{\mathbb{E} \eta_1} - \frac{1}{4} \left( \frac{\mathbb{E} \eta_1^2}{\mathbb{E} \eta_1} \right)^2 \right] + p \left[ \frac{1}{3} \frac{\mathbb{E}(W^+)^3 - \mathbb{E}(W^-)^3}{\mathbb{E} \xi_1} - \frac{1}{4} \left( \frac{\mathbb{E}(W^+)^2 - \mathbb{E}(W^-)^2}{\mathbb{E} \xi_1} \right)^2 \right].$$

Next, we analyze the system busy period which is defined as the time period which starts at the epoch when a vacation is completed and the server begins processing the fluid and ends at the next epoch when the system is empty. Note that a busy period includes the normal processing time of the fluid in the system and some possible repair times of the server due to the server breakdowns.

**Lemma 4.3.** Define $T_n = \inf\{t|Y_t + \sum_{k=1}^{N_t^R} \xi_k + \sum_{i=1}^{n} B_i = 0\}$, $n \geq 1$, where $\{B_i, i \geq 1\}$ is a positive i.i.d. sequence with $\mathbb{E}B_1 < \infty$. If $0 \leq p < 1$, then

$$\mathbb{E}T_n = \frac{n\mathbb{E}B_1}{(1 - p)\varphi'(0)}. \quad (4.9)$$

**Proof.** Let $T_n^b = \inf\{t|Y_t + \sum_{k=1}^{N_t^R} \xi_k + nb = 0\}$, $b > 0$, $n \geq 1$. Since $Y$ is a spectrally positive Lévy process, $\{T_n^b, n \geq 1\}$ is a random walk with $0 < T_n^b < T_{n+1}^b$ w.p.1. Thus, $\mathbb{E}T_n^b = n\mathbb{E}T_1^b$. Since $\mathbb{E}Y_t < 0$ and $0 \leq p < 1$, $\mathbb{E}(Y_t + \sum_{k=1}^{N_t^R} \xi_k) = -(1 - p)\varphi'(0)t < 0$. By Theorem 8.4.4 of Chung (1974), we have $\mathbb{E}T_1^b < \infty$. Since $\mathbb{E}|Y_t| < \infty$, $t^{-1}Y_t \to \mathbb{E}Y_t$ w.p.1 as $t \to \infty$ by the Kolmogorov strong law of large numbers. Taking into account the fact that $Y_{T_n^b} + \sum_{k=1}^{N_t^R} = -nb$, we get

$$-b = n^{-1} \left( Y_{T_n^b} + \sum_{k=1}^{N_t^R} \xi_k \right) = n^{-1} T_n^b \left[ (T_n^b)^{-1} Y_{T_n^b} + (T_n^b)^{-1} \sum_{k=1}^{N_t^R} \xi_k \right],$$

$$\to \mathbb{E}T_1^b[\mathbb{E}Y_1 + (\mathbb{E}N_1^R)(\mathbb{E} \xi_1)] = \mathbb{E}T_1^b[-(1 - p)\varphi'(0)] \text{ w.p.1 as } n \to \infty.$$
So that $E T_n^b = \frac{n b}{(1 - p) \phi'(0)}$. Finally, by conditioning and unconditioning, we obtain (4.9).

By the argument of Lemma 4.3, we obtain the following result.

Theorem 4.2. Let $T = \inf\{t|W_t = 0\}$, where $W_0$ is distributed according to the stationary distribution. Then the mean length of the busy period is given by

$$E T = \frac{E W}{(1 - p) \phi'(0)}.$$ 

Remark 4.2. Note that for the $M/G/1$ queue with server breakdowns and multiple vacations, if $B_i$ is distributed as a service time, then $T_n$ is the $n$-order busy period which is defined as the time period which starts when a vacation is completed and there are $n$ customers in the system and ends at the next departure epoch when the system is empty.

5. The transient distribution

In this section, we focus on analyzing the transient distribution of $W_t$ in terms of Laplace-Stieltjes transform, for some $t > 0$, conditional on $W_0 = x$.

Theorem 5.1. Let $T$ be exponentially distributed with mean $1/\gamma$, independently of $Y$, $\tau_n$ and $\sigma_n$, $n \geq 1$. For $\theta > 0$ and $x \geq 0$,

$$E_x e^{-\theta W_T} = \frac{\gamma}{\phi(\theta)} e^{-\phi^{-1}(\gamma)x} \times \left\{ p \frac{E e^{-\theta W} - E e^{-\theta W^+}}{E e^{-\phi^{-1}(\gamma)W} - E e^{-\phi^{-1}(\gamma)W^+}} + (1 - p) \frac{1 - E e^{-\theta \eta_n}}{1 - E e^{-\phi^{-1}(\gamma)\eta_n}} - e^{-[\theta - \phi^{-1}(\gamma)]x} \right\}. \quad (5.1)$$

Proof. By Lemma 4.1, we have

$$0 = EM_T = \varphi(\theta) \int_0^\infty \int_0^t \gamma e^{-\gamma t} e^{-\theta W} ds dt + e^{-\theta x} - E_x e^{-\theta W_T}$$

$$- \sum_{k=1}^{N_T^R} \left[ e^{-\theta (W_k - \xi_k)} - e^{-\theta W_k} \right] - \sum_{k=1}^{N_T^Y} (1 - e^{-\theta \eta_k})$$

$$= \frac{\varphi(\theta)}{\gamma} \gamma e^{-\theta W_T} + e^{-\theta x} - E_x e^{-\theta W_T} - \sum_{k=1}^{N_T^R} \left[ e^{-\theta W^-} - e^{-\theta W^+} \right]$$

$$- \sum_{k=1}^{N_T^Y} (1 - E e^{-\theta \eta_n}).$$

So that

$$E_x e^{-\theta W_T} = \frac{\gamma}{\varphi(\theta)} e^{-\phi^{-1}(\gamma)x} \left\{ \sum_{k=1}^{N_T^R} [E e^{-\theta W^-} - E e^{-\theta W^+}] + \sum_{k=1}^{N_T^Y} (1 - E e^{-\theta \eta_n}) - e^{-\theta x} \right\}.$$
Note that when $\mathbb{E}Y_1 < 0$, $\varphi(\theta)$ is increasing on $[0, \infty)$. Therefore, the inverse of $\varphi(\theta)$ is well defined on $[0, \infty)$. Hence, the equation $\varphi(\theta) = \gamma$ has exactly one root $\theta = \varphi^{-1}(\gamma)$ on $[0, \infty)$. Finally, using the fact that the root of the denominator should be a root of the numerator as well, we can obtain (5.1).

Next, we derive explicitly the Laplace transform corresponding to the correlation of the virtual waiting time process:

$$c(t) = \frac{\text{Cov}(W_0, W_t)}{\sqrt{\text{Var}W_0 \cdot \text{Var}W_t}} = \frac{\mathbb{E}(W_0W_t) - (\mathbb{E}W_0)^2}{\mathbb{E}(W_0W_t) - \omega^2} = \frac{\mathbb{E}(W_0W_t) - \omega^2}{v}.$$

Here, we assume the system is in steady-state at time 0.

**Theorem 5.2.** For $\theta > 0$ and $\omega$, $v$ as in Corollary 4.1,

$$\int_0^{\infty} c(t)e^{-\theta t} dt = \frac{1}{\theta} - \frac{\varphi'(0)}{v \theta^2} + \frac{\mathbb{E}(W e^{-\varphi^{-1}(\theta)W})}{v \theta} \times \left\{ \frac{pE_1}{\mathbb{E}e^{-\varphi^{-1}(\theta)W} - \mathbb{E}e^{-\varphi^{-1}(\theta)W^+}} + \frac{(1-p)E_1}{1 - \mathbb{E}e^{-\varphi^{-1}(\theta)W^+}} \right\},$$

where

$$\mathbb{E}(W e^{-\varphi^{-1}(\theta)W}) = \left[ \frac{\varphi'(0)}{\varphi(\theta)} - \frac{\theta \varphi''(0)}{2 \varphi^2(\theta)} \right] \left[ \frac{p}{\theta} \frac{\mathbb{E}W^- - \mathbb{E}W^+}{E_1} + \frac{(1-p)}{\theta} \frac{1 - \mathbb{E}W^-}{E_1} \right] + \frac{\varphi'(0)}{\varphi(\theta)} \frac{p}{E_1} \left[ \mathbb{E}(W e^{-\theta W}) - \mathbb{E}(W e^{-\theta W^+}) - \frac{\mathbb{E}W^- - \mathbb{E}W^+}{\theta} \right] + \frac{\varphi'(0)}{\varphi(\theta)} \frac{1-p}{E_1} \left[ \mathbb{E}e^{-\theta W^+} - \frac{1 - \mathbb{E}e^{-\theta W^+}}{\theta} \right].$$

**Proof.** Let $T$ be exponentially distributed with mean $1/\theta$. From (5.1), we get

$$\int_0^{\infty} e^{-\theta t} \mathbb{E}W_t dt = - \frac{\varphi'(0)}{\theta} + e^{-\varphi^{-1}(\theta)x} \left[ \frac{pE_1}{\mathbb{E}e^{-\varphi^{-1}(\theta)W} - \mathbb{E}e^{-\varphi^{-1}(\theta)W^+}} + \frac{(1-p)E_1}{1 - \mathbb{E}e^{-\varphi^{-1}(\theta)W^+}} \right] + x.$$

Straightforward calculus yields

$$\int_0^{\infty} c(t)e^{-\theta t} dt = \frac{1}{v} \int_0^{\infty} [\mathbb{E}(W_0W_t) - \omega^2] e^{-\theta t} dt = \frac{1}{v} \int_0^{\infty} \int_0^{\infty} x \mathbb{E}W_t e^{-\theta t} d\mathbb{P}(W_0 \leq x) dt - \frac{\omega^2}{\theta}. \tag{5.3}$$

Substituting (5.2) into (5.3), we obtain

$$\int_0^{\infty} c(t)e^{-\theta t} dt = \frac{\omega^2}{\theta} +$$
\[
\int_0^\infty \frac{x}{\theta v^2} \left\{ -\frac{\phi'(0)}{\theta} + e^{-\phi^{-1}(\theta)x} \left( \frac{p\xi_1}{Ee^{-\phi^{-1}(\theta)W^+} - Ee^{-\phi^{-1}(\theta)W^-}} + \frac{(1-p)\eta_1}{1 - Ee^{-\phi^{-1}(\theta)\eta_1}} \right) + x \right\} dP(W_0 \leq x) = 1 \theta - \frac{\omega \phi'(0)}{v^2} + \frac{E(W e^{-\phi^{-1}(\theta)W})}{v} \left\{ \frac{p\xi_1}{Ee^{-\phi^{-1}(\theta)W^+} - Ee^{-\phi^{-1}(\theta)W^-}} + \frac{(1-p)\eta_1}{1 - Ee^{-\phi^{-1}(\theta)\eta_1}} \right\},
\]

where \(E(W e^{-\phi^{-1}(\theta)W})\) can be obtained by differentiating \[4.4\]. This completes the proof. □

6. Stochastic decompositions

In this section, we identify some stochastic decomposition properties of the virtual waiting time for our fluid queueing system. Stochastic decomposition properties have been studied in many vacation models. The classical stochastic decomposition property shows that the steady-state system size at an arbitrary point can be represented as the sum of two independent random variable, one of which is the system size of the corresponding standard queueing system without server vacations and the other random variable depends on the meaning of vacations in specific cases (see Fuhrmann and Cooper [11] and Doshi [10]). In addition, stochastic decomposition properties have also been held for Lévy-driven queues with interruptions as well as càdlàg processes with certain secondary jump inputs (see Kella and Whitt [16] and Ivanovs and Kella [13]). It’s worth mentioning that, recently, Boxma and Kella [6] generalize known workload decomposition results for Lévy queues with secondary jump inputs and queues with server vacations or service interruptions. In particular, in the context of our fluid queueing system, we have the following decomposition results.

**Lemma 6.1.** There are two random variables \(U\) and \(V\) such that

\[
\frac{Ee^{-\theta W}}{\theta \xi_1} = Ee^{-\theta U}, \quad (6.1)
\]

\[
\frac{1 - Ee^{-\theta \eta_1}}{\theta \eta_1} = Ee^{-\theta V}. \quad (6.2)
\]

**Proof.** The left hand side of \[6.1\] is the Laplace transformation of the function

\[
f(x) = \frac{P(W^+ > x) - P(W^- > x)}{\xi_1},
\]

which is a bona fide probability density function.
The left hand side of (6.2) is the Laplace transformation of the function
\[ g(x) = \frac{\mathbb{P}(\eta_1 > x)}{\mathbb{E}\eta_1}, \]
which is a bona fide probability density function (stationary residual life density of \( \eta_1 \)).

Applying Lemma 6.1, we can get the following theorem to characterize the stochastic decomposition property.

**Theorem 6.1.** Under the conditions of Theorem 4.1, the distribution of \( W \) is the convolution of two distributions one of which is the distribution of \( R \) in Section 2 and the other is the mixture of the distribution of \( U \) with probability \( p \) and the stationary residual life distribution of \( \eta_1 \) with probability \( 1 - p \).

Furthermore, under some conditions, we have another stochastic decomposition result.

**Lemma 6.2.** Under the conditions of Theorem 4.1, if positive random variables \( \xi_n, n \geq 1 \) are independent of \( W_0, Y_{\tau_n} \) and \( \sum_{i=1}^{N_{\tau_k}} \eta_i \), then we have
\[
\mathbb{E}e^{-\theta(W_{\tau_1} - \xi_1)} = \frac{\theta \xi_1'}{\varphi(\theta)} \left[ \frac{1 - \mathbb{E}e^{-\theta \xi_1}}{\theta \mathbb{E}\xi_1} - \mathbb{E}e^{-\theta W_{\tau_1}} + (1-p) \frac{1 - \mathbb{E}e^{-\theta \eta_1}}{\theta \mathbb{E}\eta_1} \right]. \tag{6.3}
\]

**Proof.** Since \( \xi_k \) is independent of \( W_{\tau_k} - \xi_k = W_0 + Y_{\tau_n} + \sum_{i=1}^{k-1} \xi_i + \sum_{i=1}^{N_{\tau_k}} \eta_i \),
\[
\frac{\mathbb{E}e^{-\theta(W_{\tau_1} - \xi_1)} - \mathbb{E}e^{-\theta W_{\tau_1}}}{\theta \mathbb{E}\xi_1} = \frac{\mathbb{E}e^{-\theta(W_{\tau_1} - \xi_1)}(1 - e^{-\theta \xi_1})}{\theta \mathbb{E}\xi_1} \]
\[
= \mathbb{E}e^{-\theta(W_{\tau_1} - \xi_1)} \frac{1 - \mathbb{E}e^{-\theta \xi_1}}{\theta \mathbb{E}\xi_1} \]
\[
= \mathbb{E}e^{-\theta W_{\tau_1}} \frac{1 - \mathbb{E}e^{-\theta \xi_1}}{\theta \mathbb{E}\xi_1}. \tag{6.4}
\]

Substituting (6.4) into (4.4) yields (6.3).

From Lemma 6.2 we obtain the following theorem which provides another stochastic decomposition.

**Theorem 6.2.** If the assumptions of Lemma 6.2 are satisfied, then the distribution of \( U \) is the convolution of the distribution of \( W^- \) and the stationary residual life distribution of \( \xi_1 \).
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