VANISHING CARLESON MEASURES AND POWER COMPACT WEIGHTED COMPOSITION OPERATORS

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ABSTRACT. In this paper, we characterize Carleson measure and vanishing Carleson measure on Bergman spaces with admissible weights in terms of $t$-Berezin transform and averaging function as key tools. Moreover, power bounded and power compact weighted composition operators are characterized as application of Carleson measure and vanishing Carleson measure respectively on Bergman spaces with admissible weights.

1. INTRODUCTION

Let $\mathcal{H}(\mathbb{D})$ denote the space of analytic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Given a positive integrable function $\sigma \in C^2[0,1]$, we extend it on $\mathbb{D}$ by defining $\sigma(z) = \sigma(|z|)$, $z \in \mathbb{D}$ and call such $\sigma$ a weight function. For $0 < p < \infty$ and a positive Borel measure $\Omega$, the space $L^p(\Omega)$ consists of all measurable functions $f$ on $\mathbb{D}$ for which

$$\|f\|_{L^p(\Omega)} = \int_{\mathbb{D}} |f(z)|^p d\Omega(z) < \infty.$$ 

In the case $p = \infty$, the space of all complex-valued measurable functions $f$ on $\mathbb{D}$ is defined as

$$L^\infty(\Omega) = \{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_\infty = \text{ess sup}_{z \in \mathbb{D}} |f(z)| < \infty \},$$

where the essential supremum is taken with respect to the measure $\Omega$. A sequence $\{f_n\}_{n \in \mathbb{N}}$ is norm bounded in $L^\infty(\Omega)$ if $\sup_{n \in \mathbb{N}} \|f_n\|_\infty$ is finite. Let $dA(z) = \frac{dxdy}{\pi}$ be the normalized Lebesgue area measure on $\mathbb{D}$, we define the weighted Bergman space as

$$A^p_\sigma = \{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{A^p_\sigma} = \int_{\mathbb{D}} |f(z)|^p \sigma(z) dA(z) < \infty \}.$$ 

Note that $A^2_\sigma$ is a closed subspace of $L^2(\sigma dA)$ and hence is a Hilbert space endowed with the inner product

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} \sigma(z) dA(z) \quad f, g \in A^2_\sigma.$$ 

Throughout this paper, we will consider $\sigma$ as admissible weight function. Recall

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that if a weight function \( \sigma \) is non-increasing on \([0, 1)\) and \( \sigma(r)(1 - r)^{-(1 + \delta)} \) is non-decreasing on \([0, 1)\) for some \( \delta > 0 \), then \( \sigma \) is called admissible weight.

We refer the readers [12] for useful facts over pseudohyperbolic metric. The pseudohyperbolic metric is defined as \( \rho(a, z) = |\phi_a(z)| \), where \( \phi_a(z) = \frac{z - a}{1 - az} \) is Möbius transformation. For \( r \) in \((0, 1)\) and \( a \in \mathbb{D} \), \( E(a, r) = \{ z \in \mathbb{D} : \rho(z, a) < r \} \) denotes the pseudohyperbolic disk centre at \( a \) and radius \( r \in (0, 1) \). It turns out \( E(a, r) \) is a Euclidean disk with center \( \frac{1 - r^2}{1 - |z|^2}z \) and radius \( \frac{(1 - |z|^2)r}{1 - |z|^2} \).

For every \( \alpha \), \( |z|^2 \leq r^2 \) is defined as \( K^\alpha(z, w) = \frac{1}{(1 - \bar{w}z)^{\alpha + 2}} \), \( z \in \mathbb{D} \).

is the reproducing kernel in standard weighted Bergman space \( A^p_{\sigma} = A^p_{\sigma} \), where standard weight \( \sigma(z) = (1 - |z|^2)^{\alpha + 2} \), \( \alpha > -1 \). The normalized reproducing kernel of \( A^p_{\sigma} \) is defined as

\[
k^\alpha(z, \cdot) = \frac{K^\alpha(z, \cdot)}{\sqrt{K^\alpha(z, z)}}
\]

where \( K^\alpha(z, z) = \frac{1}{(1 - |z|^2)^{\alpha + 2}} \).

For a finite positive Borel measure \( \Omega \) on \( \mathbb{D} \), the \( t \)-Berezin transform is defined to be

\[
\overline{\Omega}(z) = \int_\mathbb{D} (|k^\alpha_z(w)|)^t \, d\Omega(w), \quad z \in \mathbb{D} 
\]

(1.2)

Note that for \( t = 2 \), the classical Berezin transform is denoted by \( \overline{\Omega}_2 \). Given \( r \) in \((0, 1)\), the Averaging function of \( \Omega \) is defined to be

\[
\overline{\Omega}_r(z) = \frac{\Omega(E(z, r))}{|E(z, r)|}, \quad z \in \mathbb{D} 
\]

(1.3)

If we set \( d\Omega = fdA \), for a Lebesgue measurable function \( f \), then we can write \( \overline{\Omega}_t = \overline{\Omega}_t \) and \( \overline{\Omega}_r = \overline{\Omega}_r \) for simplicity.

Motivated by [21] and [9], in this article we study the power bounded and power compact weighted composition operator on \( A^p_{\sigma} \) by using Carleson measure characterization in terms of \( t \)-Berezin transform and averaging function as a key tool. An operator \( T \) on a normed linear space \((X, \|\cdot\|_X)\) is called power bounded if \( \{T^n\} \) is a bounded sequence in the space of all bounded operators from \( X \) to itself. Also, recall that an operator \( T \) on Banach space \((X, \|\cdot\|_X)\) is said to be power compact, see [2] if there exist some integer \( m > 0 \) such that \( T^m \) is compact from \( X \) to itself.
Denote by $\Lambda_2^2$, the linear space of all double sequences with complex entries. A double sequence $\{\gamma_{j,k}\}_{j,k \in \mathbb{N}}$ of complex numbers is bounded if there exists some $M > 0$ such that $\sup_{j,k} |\gamma_{j,k}| \leq M$. The space $\Lambda_2^\infty$ of all bounded double sequences is defined as

$$\Lambda_2^\infty = \{\gamma_{j,k} = \{\gamma_{j,k}\}_{j,k \in \mathbb{N}} \in \Lambda_2^2 : \|\gamma_{j,k}\|_{\Lambda_2^\infty} = \sup_{j,k} |\gamma_{j,k}| < \infty\}.$$ 

Let $C_{\psi,\phi}$ denoted the well known weighted composition operator on the space $\mathcal{H}(\mathbb{D})$ is defined as

$$C_{\psi,\phi}(f) = \psi(f \circ \phi)$$

where $\psi \in \mathcal{H}(\mathbb{D})$ and $\phi$ is an analytic self map of $\mathbb{D}$. If $\phi(z) = z$ and $\psi = 1$, then $C_{\psi,\phi}$ becomes the multiplication operator $M_\psi$ and the composition operator $C_\phi$ respectively. Denote by $\phi_n$ the $n$th iteration of $\phi$, that is

$$\phi_n = \phi \circ \phi \cdots \phi \text{ \ (n-times)}.$$ 

Note that any power of $C_{\psi,\phi}$ on $\mathcal{H}(\mathbb{D})$ is a weighted composition operators which is defined as

$$C_{\psi,\phi}^n f = \prod_{j=0}^{n-1} (\psi \circ \phi_j) f \circ \phi_n.$$ 

For the sake of simplicity, we set

$$\langle \psi, \phi, n \rangle = \prod_{j=0}^{n-1} \psi \circ \phi_j.$$ 

Thus,

$$\|C_{\psi,\phi}^n f\|_{L^p(\Omega)}^p = \int_{\mathbb{D}} |\langle \psi, \phi, n \rangle(z) f \circ \phi_n(z)|^p d\Omega(z).$$ 

We define $d\Omega_n = |\langle \psi, \phi, n \rangle| d\Omega \circ \phi_n^{-1}$, one can easily see that $\Omega_n$ is a measure and therefore

$$\|C_{\psi,\phi}^n f\|_{L^p(\Omega)}^p = \int_{\mathbb{D}} |f(z)|^p d\Omega_n.$$ 

For $t > 0$, the $t$-Berezin transform and for $0 < r < 1$, the averaging function of $\Omega_n$ are defined as

$$\tilde{\Omega}_{n,t}(z) = \int_{\mathbb{D}} (|k_z^r(w)|)^t d\Omega_n(z), \quad z \in \mathbb{D}$$

and

$$\hat{\Omega}_{n,r}(z) = \frac{\Omega_n(E(z,r))}{|E(z,r)|}, \quad z \in \mathbb{D}$$

respectively. For more about weighted composition operators, Carleson measures and vanishing Carleson measures, we refer to [16]-[12]. Throughout the paper, the expression $E \lesssim F$ means that there exists a constant $C$ such that $E \leq CF$. 

2. Preliminaries

In this section, we prove and collect some useful facts and lemmas that are required for the proof of our main results. The next lemma gives a growth estimate for functions in $A_p^\alpha$ and an asymptotic estimate for norm of $K^\alpha(z,\cdot)$ already proved in [1].

**Lemma 1.** Let $p > 0$, $\sigma$ be an admissible weight and $\alpha > -1$. Then

1. For each $z \in \mathbb{D}$, we have that
   \[
   |f(z)|^p \lesssim \frac{\|f\|_{A_p^\alpha}^p}{\sigma(z)(1-|z|^2)^2} \quad \text{for all } f \in A_p^\alpha.
   \] (2.1)

2. For each $z \in \mathbb{D}$, we have that
   \[
   \|K^\alpha(z,\cdot)\|_{A_p^\alpha} \approx \frac{1}{(\sigma(z))^{\gamma}(1-|z|^2)^{(\alpha+2)}-\theta}.
   \] (2.2)

**Proof.**

Since $\chi_{E(z,r)}(z) = \chi_{E(\xi,r)}(z)$, $z, \xi \in \mathbb{D}$. By Fubini’s theorem for all $f \in L^1(\mathbb{D})$, we have

\[
\|\hat{f}_r\|_1 \leq \int_{\mathbb{D}} \left| \frac{1}{E(z,r)} \int_{E(z,r)} |f(\xi)|dA(\xi) \right| dA(z)
\]
\[
\leq \int_{\mathbb{D}} \frac{dA(z)}{|E(z,r)|} \int_{E(z,r)} |f(\xi)|dA(\xi)
\]
\[
= \int_{\mathbb{D}} |f(\xi)|dA(\xi) \int_{E(z,r)} \frac{dA(z)}{|E(z,r)|}
\]
\[
\leq C\|f\|_1.
\]

The boundedness of the operator $f \mapsto \hat{f}_r$ is trivially holds for $p = \infty$, that is $\|\hat{f}_r\|_\infty \leq \|f\|_\infty$ and also holds for $1 < p < \infty$, by complex interpolation. \(\square\)

**Lemma 2.** Let $1 \leq p \leq \infty$ and $r \in (0,1)$. Then the operator $f \mapsto \hat{f}_r$ is bounded from $L^p(\mathbb{D})$ to $L^p(\mathbb{D})$.

**Proof.** Since $\chi_{E(z,r)}(\xi) = \chi_{E(\xi,r)}(z)$, $z, \xi \in \mathbb{D}$. By Fubini’s theorem for all $f \in L^1(\mathbb{D})$, we have

\[
\int_{\mathbb{D}} h(z)d\Omega(z) \leq C \int_{\mathbb{D}} h(z)\hat{\Omega}_r(z)dA(z)
\]

for all non-negative subharmonic function $h : \mathbb{D} \rightarrow [0, \infty)$. \(\square\)

**Lemma 3.** Suppose $0 < p < \infty$, $r \in (0,1)$ and $\Omega$ be a finite positive Borel measure. Then there exists some constant $C$ such that

\[
\int_{\mathbb{D}} h(z)d\Omega(z) \leq C \int_{\mathbb{D}} h(z)\hat{\Omega}_r(z)dA(z)
\]

for all non-negative subharmonic function $h : \mathbb{D} \rightarrow [0, \infty)$. \(\square\)
Lemma 4. Suppose \( \Omega \) be a positive borel measure on \( \mathbb{D} \), then there exists a constant \( C \) such that

\[
\Omega(E(a, R)) \leq \frac{C}{|E(a, R)|} \int_{E(a, R)} \Omega(E(z, r))dA(z)
\]

(2.3)

Proof. For \( r, R > 0 \), we have

\[
\int_{E(a, R)} \Omega(E(z, r))dA(z) = \int_{\mathbb{D}} \chi_{E(a, R)}(z)d\hat{\Omega}(w) \int_{E(a, R)} \chi_{E(z, r)}(w)d\Omega(w)
\]

\[
= \int_{\mathbb{D}} d\Omega(w) \int_{E(a, R)} \chi_{E(z, r)}(w)dA(w)
\]

Since \( \chi_{E(z, r)}(w) = \chi_{E(w, r)}(z) \) for all \( z, w \in \mathbb{D} \), then

\[
\int_{E(a, R)} \Omega(E(z, r))dA(z) = \int_{\mathbb{D}} d\Omega(w) \int_{E(a, R)} \chi_{E(w, r)}(z)dA(z)
\]

\[
\geq \int_{E(a, R)} \{(E(a, R) \cap E(w, r))d\Omega(w)
\]

\[
\geq \Omega(E(a, R)) \inf_{w \in E(a, R)} \{|(E(a, R) \cap E(w, r))|\}
\]

For \( w \in (E(a, R) \cap E(w, r)) \), there exists a Euclidean disk with diameter \( \frac{1}{2} \min\{r, R\} \) contained in \( (E(a, R) \cap E(w, r)) \). Therefore (2.3) holds.

\[ \Box \]

Lemma 5. Suppose \( 1 \leq p \leq \infty \) and \( s \in \mathbb{R} \). Then \( \frac{\widehat{\Omega}_r(z)}{\sigma^s(z)(1 - |z|^2)^{2(s-1)}} \in \mathcal{L}^p(\mathbb{D}) \) for some \( r, 0 < r < 1 \) if and only if \( \frac{\widehat{\Omega}_R(z)}{\sigma^s(z)(1 - |z|^2)^{2(s-1)}} \in \mathcal{L}^p(\mathbb{D}) \) for all \( 0 < R < 1 \).

Moreover,

\[
\left\| \frac{\widehat{\Omega}_r(z)}{\sigma^s(z)(1 - |z|^2)^{2(s-1)}} \right\|_p \asymp \left\| \frac{\widehat{\Omega}_R(z)}{\sigma^s(z)(1 - |z|^2)^{2(s-1)}} \right\|_p.
\]

Proof. For \( r \) and \( R \), by lemma \[ \Box \] there exists a constant \( C \) such that

\[
\widehat{\Omega}_R(z) \leq \frac{C}{|E(z, R)|} \int_{E(z, R)} \widehat{\Omega}_r(\xi)dA(\xi)
\]

for all \( z \in \mathbb{D} \). Above inequality and \[ \Box \] implies that,

\[
G_R(z) \leq \frac{C}{|E(z, R)|} \int_{E(z, R)} G_r(\xi)dA(\xi)
\]

\( \asymp G_r(z) \),

This accomplished the result.
Lemma 7. Let $G_t(z) = \frac{\hat{G}_t(z)}{\sigma(z)(1 - |z|^2)^{p}}$. Combining this with Lemma \[3\] we find that
\[
\|\frac{\hat{G}_t(z)}{\sigma(z)(1 - |z|^2)^{2(s-1)}}\|_p \leq \|\hat{G}_t(z)\|_p \leq \|G_t(z)\|_p \quad \text{for} \quad 1 \leq p \leq \infty.
\]
Interchanging the role of $r$ and $R$, we get the desired result. \hfill \Box

Lemma 8. Suppose $1 < p < \infty$ and $t > 0$. Then the integral operator
\[
T_{t,s}f(z) = \sigma^{s}(z)(1 - |z|^2)^{(\alpha+2)t-2s}\int_{D} \frac{|K^\alpha(z,\xi)|^t}{\sigma^{s}(\xi)(1 - |\xi|^2)^{2(s-1)}}f(\xi)dA(\xi)
\]
is bounded on $L^p(D)$ whenever $\frac{1}{p} < s < (\alpha + 2)\frac{k}{2} + \frac{1}{p}$.

Proof. Let $P = \frac{1-s}{p}(s+1)$, $Q = \frac{1-s}{p}((\alpha + 2)t - 2s)$, $U = \frac{\alpha+1}{p}(-\alpha+2)$ and $V = \frac{2}{p}$.

Then claim that the intervals $(P,Q)$ and $(U,V)$ are non-empty. By using hypothesis one can easily find that
\[
Q - P = (\alpha + 2)(1 - \frac{1}{p})(t - \frac{1}{\alpha + 2}) > 0
\]
and
\[
V - U = (\alpha + 2) \left( t - \frac{1}{\alpha + 2} \right) > 0
\]
Implies that the intervals $(P,Q)$ and $(U,V)$ are non-empty. Also,
\[
V - P = s - \frac{1-s}{p} > 0
\]
and
\[
Q - U = (\alpha + 2)t - 2 + \frac{1}{p} - s > 0
\]
implies that $P < V$ and $U < Q$. Thus, $(P,Q) \cap (U,V)$ is non-empty. For some $m \in (P,Q) \cap (U,V)$ and take $h(\xi) = (\sigma(\xi)(1 - |\xi|^2))^{m}$ and $\frac{k}{2} + \frac{1}{p} = 1$. From Lemma \[1\] there is a positive constant $C$ such that
\[
\sigma^{s}(z)(1 - |z|^2)^{(\alpha+2)t-2s}\int_{D} \frac{|K^\alpha(z,\xi)|^t}{\sigma^{s}(\xi)(1 - |\xi|^2)^{2(s-1)}}h^p(\xi)dA(\xi) \leq Ch^p(\xi), \quad z \in D
\]
and
\[
\sigma^{-s}(z)(1 - |z|^2)^{(2(1-s))} \int_{D} |K^\alpha(z,\xi)|^t\sigma^{s}(\xi)(1 - |\xi|^2)^{(\alpha+2)t-2s}h^p(\xi)dA(\xi) \leq Ch^p(\xi)
\]
z $\in D$. By Schur’s test, boundedness of the operator $T_{t,s}$ on $L^p(D)$ holds. \hfill \Box

Lemma 7. Let $\{z_k\}_{k=1}^{\infty}$ be an r- lattice. For $1 < p < \infty$ and $\{\lambda_k\}_{k=1}^{\infty} \in L^p$, let
\[
f(z) = \sum_{k=1}^{\infty} \lambda_k \frac{K^\alpha(z_k,z)}{\sigma(z)(1 - |z_k|^2)^{\frac{1}{p}-(\alpha+2)p}} \quad \text{(2.4)}
\]
where $\alpha > -1$. Then $f \in A^p(\mathbb{D})$ and $\|f\|_{p,\sigma} \leq C\|\{\lambda_k\}_{k=1}^{\infty}\|_{L^p}$.

The proof is an easy modification of arguments in Theorem 4.1 in \[1\]. We omit the details.

Lemma 8. Suppose $\Omega \geq 0$, $1 \leq p \leq \infty$, $t > 0$ and $s \in \mathbb{R}$ satisfies $t < s + \frac{1}{p} < 1$.

Then the following are equivalent
Moreover, we have

\[ \|\tilde{M}_{t,s}(z)\|_p \gtrsim \|\tilde{M}_{R,s}(z)\|_p \cong \left\| \left\{ \frac{\Omega_r(z)}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha}{p})t}} \right\}_{k=1}^\infty \right\|_p. \]

**Proof.** We will prove the result in the order: \((a) \iff (b) \iff (c)\).

\((a) \Rightarrow (b).\) For any \(R \in (0,1),\) there exist positive constant \(C_R\) such that for any \(z \in \mathbb{D}\) kernel estimate holds. Thus for \(s \in \mathbb{R},\) we have

\[
\tilde{M}_{t,s}(z) = \frac{\Omega(E(z,R))}{\sigma^s(z)(1-|z|^2)^{2s}} \leq C_R \frac{1}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha}{p})t}} \int_{E(z,R)} |k_z^\alpha(\xi)|^t d\Omega(\xi) \leq C_R \frac{1}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha}{p})t}} \int_{\mathbb{D}} |k_z^\alpha(\xi)|^t d\Omega(\xi) \leq C_R \tilde{M}_{t,s}(z)
\]

Above implies that, \(\|\tilde{M}_{R,s}\|_p \leq C\|\tilde{M}_{t,s}\|_p.\)

\((b) \Rightarrow (a).\) By Lemma 2 and Lemma 3 there is a positive constant \(C\) such that for any \(z \in \mathbb{D}\) and \(s \in \mathbb{R},\) we have

\[
\tilde{M}_{t,s}(z) = \frac{\Omega_t(z)}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha}{p})t}} \leq \frac{1}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha}{p})t}} \int_{\mathbb{D}} |k_z^\alpha(\xi)|^t d\Omega(\xi) \leq \frac{C}{\sigma^s(z)(1-|z|^2)^{2s-(\frac{\alpha}{p})t}} \int_{\mathbb{D}} |k_z^\alpha(\xi)|^t \hat{\Omega}_R(\xi)dA(\xi) \leq \frac{C}{\sigma^s(z)(1-|z|^2)^{2s-(\alpha+2)t}} \int_{\mathbb{D}} |K(z,\xi)|^t \sigma^s(\xi)(1-|\xi|^2)^{2(s-1)} \tilde{M}_{R,s}(\xi)dA(\xi) \leq C T_{t,s-}(\tilde{M}_{R,s})(z)
\]

Since \(t < s + \frac{\alpha}{p} < 1\) and Lemma 4 implies that

\[ \|\tilde{M}_{t,s}\|_p \leq \|T_{t,s-}(\tilde{M}_{R,s})\|_p \leq C\|\tilde{M}_{R,s}\|_p. \]

\((b) \Rightarrow (c).\) Assume that \(\tilde{M}_{R,s} \in L^p(\mathbb{D})\) for some \(R, 0 < R < 1.\) Let \(\{z_k\}_{k=1}^\infty\) be any \(r\)-lattice. By Lemma 4 we may assume \(R < r.\) By triangle inequality, we have \(E(z_k,r) \subset E(z,2r)\) for \(z \in E(z_k,r)\) and for all \(k.\) Thus, we have

\[
\frac{\tilde{\Omega}_r(z_k)}{\sigma^s(z_k)(1-|z_k|^2)^{2s-1}} \leq C \frac{\Omega(E(z,2r))}{\sigma^s(z)(1-|z|^2)^{2s}}
\]
In this section, we are using averaging function and vanishing Carleson measure. We divide our result into two cases: \( 0 < \infty \) and \( 0 < p, q, \sigma \) tools to characterize the \( (p, t) > 0 \). Let \( \Omega \) be a finite positive Borel measure. Recall that, for any \( \Omega \) is a \((p, q, \sigma)\)-Bergman Carleson measure if there exists a finite constant \( C > 0 \) for all \( f \in A_p^p \). Similar to (2.5) \( \Omega_r(z) \leq C \Omega_{2r}(z) \) for \( z \in E(z_k, r) \). Therefore, we have

\[
\int_D \frac{\tilde{\Omega}^p_r(z)}{\sigma^p(z) (1 - |z|^2)^{2(s-1)p}} dA(z) \leq \sum_{k=1}^\infty \int_{E(z_k, r)} \frac{\tilde{\Omega}^p_{2r}(z)}{\sigma^p(z) (1 - |z|^2)^{2(s-1)p}} dA(z)
\]

Above inequality and Lemma \( \ref{lem:meas} \) implies that

\[
\| M_{R,s} \|_p \approx \| M_{R,s} \|_p \leq C \left\| \left\{ \frac{\tilde{\Omega}^p_r(z_k)}{\sigma^p(z_k) (1 - |z_k|^2)^{2(s-1)p}} \right\}_{k} \right\|_p
\]

for any \( R \in (0, 1) \).

3. Carleson measure characterizations

In this section, we are using averaging function and t-Berezin transform as our main tools to characterize the \((p, q, \sigma)\)-Bergman Carleson measure for \( 0 < p, q < \infty \) and \( t > 0 \). Let \( \Omega \) be a finite positive Borel measure. Recall that,

- \( \Omega \) is a \((p, q, \sigma)\)-Bergman Carleson measure if the embedding \( i : A_p^p \rightarrow L^q(\Omega) \) is bounded. In other words, we can say \( \Omega \) is a \((p, q, \sigma)\)-Bergman Carleson measure if there exists a finite constant \( C > 0 \) such that

\[
\int_D |f|^q d\Omega \leq C \| f \|_{A_p^p}^q
\]

for all \( f \in A_p^p \).

- \( \Omega \) is a vanishing \((p, q, \sigma)\)-Bergman Carleson measure if \( \int_D |f_n|^q d\Omega \rightarrow 0 \) as \( n \rightarrow \infty \) whenever \( \{f_n\} \) is a bounded sequence in \( A_p^p \) which converges to 0 uniformly on any compact subset of \( D \).

Note that, by taking \( p = q \) and \( \sigma(z) = 1 \), \( \Omega \) becomes a Bergman Carleson measure and vanishing Carleson measure. We divide our result into two cases: \( 0 < p \leq q < \infty \) and \( 0 < q < p < \infty \).
Theorem 1. Let $\Omega$ be a finite positive Borel measure and $0 < p \leq q < \infty$. Then the following statements are equivalent:

(a) $\Omega$ is a $(p, q, \sigma)$–Bergman Carleson measure.

(b) The function $\frac{\Omega_t(z)}{\sigma^{\frac{2}{p}}(z)(1 - |z|^2)^{2(\frac{2}{p} - \frac{1}{q})}}$ is bounded on $\mathbb{D}$ for $t > \frac{2q}{p(\alpha + 2)}$.

(c) The function $\frac{\Omega_R(z)}{\sigma^{\frac{2}{p}}(z)(1 - |z|^2)^{2(\frac{2}{p} - \frac{1}{q})}}$ is bounded on $\mathbb{D}$ for some (or any) $R \in (0, 1)$.

(d) The sequence $\left\{\frac{\Omega_t(a_k)}{\sigma^{\frac{2}{p}}(z_k)(1 - |z_k|^2)^{2(\frac{2}{p} - \frac{1}{q})}}\right\}_{k=1}^{\infty}$ is bounded for some (or any) $r$-lattice $\{z_k\}_{k=1}^{\infty}$ with $0 < r < 1$.

Furthermore,

$$||i||_{A^p_{\sigma} \rightarrow L^p(\Omega)} \preceq \sup_{\Omega_t(z) \neq 0} \frac{\Omega_t(z)}{\sigma^{\frac{2}{p}}(z)(1 - |z|^2)^{2\left(\frac{2}{p} - \frac{1}{q}\right)}} \preceq \sup_{\Omega_R(z) \neq 0} \frac{\Omega_R(z)}{\sigma^{\frac{2}{p}}(z)(1 - |z|^2)^{2\left(\frac{2}{p} - \frac{1}{q}\right)}} \preceq \sup_k \frac{\hat{\Omega}_r(z_k)}{\sigma^{\frac{2}{p}}(z_k)(1 - |z_k|^2)^{2\left(\frac{2}{p} - \frac{1}{q}\right)}} \quad (3.1)$$

Proof. $(d) \Rightarrow (a)$ We assume that $\{z_k\}_{k=1}^{\infty}$ be an $r$-lattice. We use the elementary inequality $\sum_{k=1}^{\infty} u_k^2 \leq \left(\sum_{k=1}^{\infty} u_k\right)^2$, $u_k \geq 0$, $k = 1, 2, \cdots$, by taking $l = \frac{p}{q} \geq 1$, using Lemma 1 and 12, we obtain

$$\int_{\mathbb{D}} |f(z)|^q d\Omega = \sum_{k=1}^{\infty} \int_{E(z_k, r)} |f(z)|^q d\Omega$$

$$\leq \sum_{k=1}^{\infty} \hat{\Omega}_r(z_k)|E(z_k, r)|\left(\sup_{z \in E(z_k, r)} |f(z)|^p\right)^{\frac{q}{p}}$$

$$\leq C \sum_{k=1}^{\infty} \hat{\Omega}_r(z_k)\left(\frac{\hat{\Omega}_r(z_k)}{\sigma^{\frac{2}{p}}(z_k)(1 - |z_k|^2)^{2\left(\frac{2}{p} - \frac{1}{q}\right)}}\left(\int_{E(z_k, \frac{1}{r})} |f(\xi)|^p \sigma(\xi) dA(\xi)\right)^{\frac{q}{p}}\right)^{\frac{2}{q}}$$

$$\leq C \sup_k \frac{\hat{\Omega}_r(z_k)}{\sigma^{\frac{2}{p}}(z_k)(1 - |z_k|^2)^{2\left(\frac{2}{p} - \frac{1}{q}\right)}} \left(\sum_{k=1}^{\infty} \int_{E(z_k, \frac{1}{r})} |f(\xi)|^p \sigma(\xi) dA(\xi)\right)^{\frac{2}{q}}$$

$$\leq CM^{\frac{q}{2}} \sup_k \frac{\hat{\Omega}_r(z_k)}{\sigma^{\frac{2}{p}}(z_k)(1 - |z_k|^2)^{2\left(\frac{2}{p} - \frac{1}{q}\right)}} ||f||_{p, \sigma}^{\frac{q}{2}} \quad (3.2)$$

Above inequality reveals that

$$||i||_{A^p_{\sigma} \rightarrow L^p(\Omega)} \leq C \sup_k \frac{\hat{\Omega}_r(z_k)}{\sigma^{\frac{2}{p}}(z_k)(1 - |z_k|^2)^{2\left(\frac{2}{p} - \frac{1}{q}\right)}} \cdot$$

$(a) \Rightarrow (c)$. Set $f_z(w) = K^\alpha(z, w)$, $w \in \mathbb{D}$. By Lemma 1 and statement(a), we have

$$\frac{\hat{\Omega}_R(z)}{\sigma^{\frac{2}{p}}(z)(1 - |z|^2)^{2\left(\frac{2}{p} - \frac{1}{q}\right)}} \frac{\Omega(E(z, R))}{\sigma^{\frac{2}{p}}(z)(1 - |z|^2)^{2\left(\frac{2}{p} - \frac{1}{q}\right)}}$$


\[
\begin{align*}
&\leq C \frac{(1 - |z|^2)^{q(\alpha + 2 - \frac{1}{2})}}{\sigma(\tau)(z)} \int_{E(z, R)} |f_\tau(\xi)|^q d\Omega(\xi) \\
&\leq C \frac{(1 - |z|^2)^{q(\alpha + 2 - \frac{1}{2})}}{\sigma(\tau)(z)} \int_{\mathbb{D}} |f_\tau(\xi)|^q d\Omega(\xi) \\
&\leq C \frac{(1 - |z|^2)^{q(\alpha + 2 - \frac{1}{2})}}{\sigma(\tau)(z)} \|i\|_{A^p_q - L^q_\sigma}(\tau) \|f_\tau\|_{A^p_q}^q \\
&\leq C \|i\|_{A^p_q - L^q_\sigma}(\tau)
\end{align*}
\]

Above inequality reveals that
\[
\sup_{z \in \mathbb{D}} \frac{\hat{\Omega}_R(z)}{\sigma(\tau)(z)(1 - |z|^2)^{2(\frac{1}{4p} - \frac{1}{2})}} \leq \|i\|_{A^p_q - L^q_\sigma}(\tau). \quad (3.4)
\]

The equivalence of (a),(c) and (d) follows from above proof of implications. Moreover,
\[
\|i\|_{A^p_q - L^q_\sigma}(\tau) \asymp \sup_{z \in \mathbb{D}} \frac{\hat{\Omega}_R(z)}{\sigma(\tau)(z)(1 - |z|^2)^{2(\frac{1}{4p} - \frac{1}{2})}} \asymp \sup_{k} \frac{\hat{\Omega}_{s_k}(z_k)}{\sigma(\tau)(z_k)(1 - |z_k|^2)^{2(\frac{1}{4p} - \frac{1}{2})}} \quad (3.5)
\]

(b) \Rightarrow (c). For any \( R, 0 < R < 1, \) Lemma 8 yields
\[
\frac{\hat{\Omega}_R(z)}{\sigma(\tau)(z)(1 - |z|^2)^{2(\frac{1}{4p} - \frac{1}{2})}} \leq C \frac{\hat{\Omega}_R(z)}{\sigma(\tau)(z)(1 - |z|^2)^{2(\frac{1}{4p} - \frac{1}{2})}} \quad (3.6)
\]

(a) \Rightarrow (b). The estimate \( C \) reveals that the embedding operator \( i : A^p_q \rightarrow L^q(\Omega) \) is bounded for some \( 0 < p < q < \infty \) if and only if \( i : A^p_{\infty} \rightarrow L^q(\Omega) \) is bounded for some \( 0 < p_1 \leq q_1 < \infty \) with \( \frac{q_1}{p_1} = \frac{q}{p} \). Since \( \Omega \) is a \((p, q, \sigma)\)-carleson measure \( i : A^p_{\infty} \rightarrow L^{q_1}(\Omega) \) where \( N \) is some integer with \( Np > \frac{1}{(\alpha + 2)} \). Let \( f_\tau(\cdot) = K^\sigma_{\tau}(\cdot), z \in \mathbb{D} \) and \( (\cdot) \) tells us that
\[
\frac{\hat{\Omega}_{Nq}(z)}{\sigma(\tau)(z)(1 - |z|^2)^{2(\frac{1}{4p} - \frac{1}{2})Nq}} \asymp \frac{1}{\sigma(\tau)(z)(1 - |z|^2)^{2(\frac{1}{4p} - \frac{1}{2})Nq}} \int_{\mathbb{D}} |f_\tau(\xi)|^{Nq} d\Omega(\xi) \quad (3.7)
\]

\[
\leq \\frac{\|i\|_{A^p_q - L^{Nq}}(\tau)}{\sigma(\tau)(z_k)(1 - |z_k|^2)^{2(\frac{1}{4p} - \frac{1}{2})Nq}} \leq C \sup_{k} \frac{\hat{\Omega}_{s_k}(z_k)}{\sigma(\tau)(z_k)(1 - |z_k|^2)^{2(\frac{1}{4p} - \frac{1}{2})}}
\]

Hence, for any \( z \in \mathbb{D} \)
\[
\frac{\hat{\Omega}_{Nq}(z)}{\sigma(\tau)(z)(1 - |z|^2)^{2(\frac{1}{4p} - \frac{1}{2})Nq}} \leq C \|i\|_{A^p_q - L^{Nq}}(\tau)
\]

\[
\asymp C \sup_{k} \frac{\hat{\Omega}_{s_k}(z_k)}{\sigma(\tau)(z_k)(1 - |z_k|^2)^{2(\frac{1}{4p} - \frac{1}{2})}}
\]
Statement (a) shows that the operator $i : A^p_{\sigma} \to L^q(\Omega)$ is bounded. Since $t > \frac{2q}{p(\alpha+2)}$, we have $\frac{tp(\alpha+2)}{q} > 2$. The above calculations show that

$$
\sup_{z \in \mathbb{D}} \frac{\widetilde{\Omega}_t(z)}{\sigma^p(z)(1-|z|^2)^{\frac{2q}{p(\alpha+2)}-t}} \leq C \|i\|_{A^p_{\sigma} \rightarrow L^q(\Omega)}^{N_q}
$$

$$
\asymp C \sup_k \frac{\hat{\Omega}_t(z_k)}{\sigma^p(z_k)(1-|z_k|^2)^{\frac{2q}{p(\alpha+2)}-t}}.
$$

This completes the proof. \hfill \Box

**Corollary 1.** Let $t > 0$, $0 < p < \infty$, $d\Omega = \sigma dA$ is a positive measure and $C_{\psi, \varphi} : A^p_{\sigma} \rightarrow L^p(\Omega)$ be a bounded operator. Then the following statements are equivalent.

(a) $C_{\psi, \varphi}$ is power bounded, that is,

$$
Q_1 = \sup_{n \in \mathbb{N}} \|C^n_{\psi, \varphi}\|^p < \infty.
$$

(b) The sequence of functions $\{f_n\}_{n=1}^{\infty}$ defined on $\mathbb{D}$ as

$$
f_n(z) = \frac{\widetilde{\Omega}_n(z)}{\sigma(z)^{1/(\alpha+2)}} , \ z \in \mathbb{D}
$$

is a norm bounded family in $L^\infty(\mathbb{D})$ for $t > 2/(\alpha+2)$, that is,

$$
Q_2 = \sup_{n \in \mathbb{N}} \|f_n\|_\infty = \sup_{n \in \mathbb{N}} \sup_{z \in \mathbb{D}} \frac{\widetilde{\Omega}_n(z)}{\sigma(z)^{1/(\alpha+2)}} < \infty.
$$

(c) The sequence of functions $\{g_n\}_{n=1}^{\infty}$ defined on $\mathbb{D}$ as

$$
g_n(z) = \frac{\hat{\Omega}_n(z)}{\sigma(z)} , \ z \in \mathbb{D}
$$

is a norm bounded family in $L^\infty(\mathbb{D})$ for some (or any) $R \in (0, 1)$, that is,

$$
Q_3 = \sup_{n \in \mathbb{N}} \|g_n\|_\infty = \sup_{n \in \mathbb{N}} \sup_{z \in \mathbb{D}} \frac{\hat{\Omega}_n(z)}{\sigma(z)} < \infty.
$$

(d) The double sequence $\gamma_{nk} = \{\gamma_{n,k}\}_{n,k}$, where

$$
\gamma_{n,k} = \frac{\hat{\Omega}_{n,R}(z_k)}{\sigma(z_k)}
$$

is bounded for some (any) $r$-lattice $\{z_k\}_{k=1}^{\infty}$ with fixed $r$, $0 < r < 1$, that is,

$$
Q_4 = \|\gamma_{nk}\|_{A^p_{\sigma}} = \sup_{n,k \in \mathbb{N}} \frac{\hat{\Omega}_{n,R}(z_k)}{\sigma(z_k)} < \infty.
$$

Moreover, $Q_1 \asymp Q_2 \asymp Q_3 \asymp Q_4$.

**Theorem 2.** Let $\Omega \geq 0$ and $0 < p \leq q < \infty$. Then the following statement are equivalent:

(a) $\Omega$ is a vanishing $(p,q,\sigma)$-Bergman Carleson measure.
Proof. The implication (c) ⇒ (d) is trivial because \( z_k \to \partial \mathbb{D} \) as \( k \to \infty \) whenever \( \{z_k\}_{k=1}^\infty \) is an r-lattice. It follows from (3.10) that (b) ⇒ (c).

(a) ⇒ (c) Given \( 0 < R < 1 \). For \( z \in \mathbb{D} \), we set \( f_z(\xi) = \frac{K_\alpha^\ast(\xi)}{\sigma^\ast(z)(1 - |z|^2)^{\frac{\alpha}{2} - \frac{R}{2}}} \), \( \xi \in \mathbb{D} \). One can easily find that \( f_z \in A^p_\sigma \), \( \|f_z\|_{p,\sigma} \leq C \) and \( f_z \to 0 \) uniformly on any compact subset of \( \mathbb{D} \) as \( z \to \partial \mathbb{D} \). Since \( \Omega \) is a vanishing \( (p,q,\sigma) \)-Bergman Carleson measure, it follows from (3.9) that

\[
\frac{\hat{\Omega}_R(z)}{\sigma^\ast(z)(1 - |z|^2)^{\frac{\alpha}{2} + \frac{R}{2}}} \leq C \frac{(1 - |z|^2)^{q(\alpha + 2 - \frac{R}{2})}}{\sigma^\ast(z)} \int_\mathbb{D} |f_z(\xi)|^q d\Omega(\xi) \to 0
\]

as \( k \to 0 \).

(d) ⇒ (a). Suppose (d) holds. For any \( \epsilon > 0 \), there exists a positive integer \( k_0 \) such that

\[
\frac{\hat{\Omega}_R(z_k)}{\sigma^\ast(z_k)(1 - |z_k|^2)^{\frac{\alpha}{2} + \frac{R}{2}}} < \epsilon, \text{ whenever } k > k_0.
\]

Notice that \( \bigcup_{k=1}^{k_0} E(z_k, r) \) is relatively compact in \( \mathbb{D} \). Let us consider a bounded sequence \( \{f_j\}_{j=1}^\infty \) in \( A^p_\sigma \) such that \( f_j \to 0 \) uniformly on any compact subset of \( \mathbb{D} \) as \( j \to \infty \). Similar to the proof of (3.2) and if \( j \) is large enough, we have that

\[
\int_\mathbb{D} |f_j(z)|^q d\Omega(z) \leq \int_{\bigcup_{k=1}^{k_0} E(z_k, r)} |f_j(z)|^q d\Omega(z) + \sum_{k=k_0+1} \int_{E(z_k, r)} |f_j(z)|^q d\Omega(z)
\]

\[
\leq C\epsilon ||f_j||^q_{p,\sigma}
\]

\[
\leq C\epsilon.
\]

(3.8)

where \( C \) is independent of \( \epsilon \).

(a) ⇒ (b) The equivalence of (a), (c) and (d) shows that the measure \( \Omega \) is a vanishing \( (Np, Nq, \sigma) \)-Bergman Carleson measure if \( \Omega \) is a vanishing \( (p,q,\sigma) \)-Bergman Carleson measure. For \( z \in \mathbb{D} \), set \( f_z(\xi) = \frac{K_\alpha^\ast(\xi)}{\sigma^\ast(z)(1 - |z|^2)^{\frac{\alpha}{2} - \frac{R}{2}}} \), \( \xi \in \mathbb{D} \).

One can easily find that \( f_z \in A^p_\sigma \), \( \|f_z\|_{p,\sigma} \leq C \) and \( f_z \to 0 \) uniformly on any
compact subset of $\mathbb{D}$ as $z \to \partial \mathbb{D}$. Since $Np > \frac{4}{(q+2)}$ and it follows from [3,7], we have that

$$\overline{\Omega}_{Nq}(z) \lesssim \frac{1}{\sigma^p(\mathbb{D})} \int_{\partial \mathbb{D}} |f_z(\xi)|^{Nq} d\Omega(\xi).$$

Statement (a) yields that $\Omega$ is a vanishing $(\frac{4p}{Nq}, \frac{q}{4q}, \sigma)$–Bergman Carleson measure. Therefore

$$\lim_{z \to \partial \mathbb{D}} \frac{\overline{\Omega}_{Nq}(z)}{\sigma^p(\mathbb{D})} = 0.$$ 

The proof is completed. \hfill \Box

**Corollary 2.** Let $t > 0$, $0 < p < \infty$, $d\Omega = \sigma dA$ is a positive measure and $C_{\psi, \varphi} : A^p_{\varphi} \to L^p(\Omega)$ be a bounded operator. Then the following statements are equivalent.

(a) $C_{\psi, \varphi}$ is power compact, that is, $C^n_{\psi, \varphi}$ is compact, for some $n \in \mathbb{N}$.

(b) For $t > \frac{2q}{p(n+2)}$, we have

$$\lim_{z \to \partial \mathbb{D}} \frac{\overline{\Omega}_{n, t}(z)}{\sigma^p(\mathbb{D})} = 0,$$

as $z \to \partial \mathbb{D}$.

(c) For some (or any) $R \in (0, 1)$, we have

$$\lim_{z \to \partial \mathbb{D}} \frac{\overline{\Omega}_{n, R}(z)}{\sigma^p(\mathbb{D})} = 0,$$

as $z \to \partial \mathbb{D}$.

(d) For some (or any) $r$-lattice $\{z_k\}_{k=1}^{\infty}$ with $r \in (0, 1)$, we have

$$\lim_{z \to \partial \mathbb{D}} \frac{\overline{\Omega}_{n, r}(z_k)}{\sigma^p(\mathbb{D})} = 0,$$

as $k \to 0$.

**Theorem 3.** Let $\Omega$ be a finite positive Borel measure and $0 < q < p < \infty$. Then the following statements are equivalent:

(a) $\Omega$ is a $(p, q, \sigma)$–Bergman Carleson measure.

(b) $\Omega$ is a vanishing $(p, q, \sigma)$–Bergman Carleson measure.

(c) For $t > \frac{2(q+2)}{p(n+2)}$, we have

$$\widehat{M}_t(z) = \frac{\overline{\Omega}_t(z)}{\sigma^p(\mathbb{D})} \in L^p(\mathbb{D}).$$

(d) For some (or any) $R \in (0, 1)$, we have

$$\widehat{M}_R(z) = \frac{\overline{\Omega}_R(z)}{\sigma^p(\mathbb{D})} \in L^p(\mathbb{D}).$$

(e) For some (or any) $r$-lattice $\{z_k\}_{k=1}^{\infty}$ with $r \in (0, 1)$, we have

$$\widehat{M}_r(z_k) = \frac{\overline{\Omega}_r(z_k)}{\sigma^p(\mathbb{D})} \in L^p(\mathbb{D}).$$


Moreover, we have

\[ \|i\|^q_{A^p_{\mathcal{K}} \rightarrow L^p(\Omega)} \geq \|\hat{M}_t\|_{\frac{p}{q}} \times \|\hat{M}_R\|_{\frac{p}{q}} = \left\| \left\{ \frac{\hat{\Omega}_t(z_k)}{\sigma^\frac{\alpha}{2}(z_k)(1 - |z_k|^2)^{2(\frac{\alpha}{2} - \frac{\beta}{p})}} \right\}_{k=1}^\infty \right\|_{\frac{p}{q}}, \]

(3.9)

**Proof.** Since Lemma 3 implies that the statements (c), (d) and (e) are equivalent with the corresponding norm estimate (3.9) and the implication (b) \( \Rightarrow (a) \) is trivially true, it is sufficient to prove that (d) \( \Rightarrow (a) \), (a) \( \Rightarrow (e) \) and (a) \( \Rightarrow (b) \).

(d) \( \Rightarrow (a) \). Since \( 0 < q < p < \infty \) implies \( \frac{p}{q} > 1 \), therefore the conjugate exponent of \( \frac{p}{q} \) is \( \frac{p}{p-q} \). For \( f \in A^p_{\mathcal{K}} \), we have

\[
\int_D |f(\xi)|^q d\Omega(\xi) \leq C \int_D |f(\xi)|^q \hat{\Omega}_R(\xi) dA(\xi) \\
\leq C \left\| \hat{M}_R \right\|_{\frac{p}{q}} \|f\|_{p,\sigma}^q.
\]

Above inequality follows from Lemma 3 and Holder’s inequality shows that \( \Omega \) is a \((p,q,\sigma)\)-Bergman Carleson measure and \( \|i\|^q_{A^p_{\mathcal{K}} \rightarrow L^p(\Omega)} \leq C \|\hat{M}_R\|_{\frac{p}{q}}. \)

(a) \( \Rightarrow (e) \). Let \( \{\lambda_k\}_{k=1}^\infty \in l^p \) and set \( f \) as in Lemma 7. Statement (a) and Lemma 7 implies that

\[
\int_D \left| \sum_{k=1}^\infty \lambda_k \frac{K^\alpha_{z_k}(z)}{\sigma^\frac{\alpha}{2}(z_k)(1 - |z_k|^2)^{2(\frac{\alpha}{2} - \frac{\beta}{p})}} \right|^q d\Omega(z) \leq C \|i\|^q_{A^p_{\mathcal{K}} \rightarrow L^p(\Omega)} \|\{\lambda_k\}_k\|_{l^p}^q.
\]

(3.10)

Recall that Rademacher functions \( \psi_k \) are defined by

\[
\psi_0(t) = \begin{cases} 1, & \text{if } 0 \leq t - [t] < 1/2 \\ -1, & \text{if } 1/2 \leq t - [t] < 1 \end{cases}
\]

and \( \psi_k(t) = \psi_0(2^k t) \) for \( k = 1, 2, \cdots \), where \( [t] \) denotes the greatest integer less than or equal to \( t \). For \( 0 < q < \infty \), Khinchine’s inequality is given as

\[
C_1 \left( \sum_{k=1}^m |b_k|^2 \right)^{q/2} \leq \int_0^1 \left| \sum_{k=1}^m b_k \psi_k(t) \right|^q dt \leq C_2 \left( \sum_{k=1}^m |b_k|^2 \right)^{q/2},
\]

which holds for all \( m \geq 1 \) and all complex numbers \( b_1, b_2, \cdots b_m \). Let \( \psi_k(t) \) be the \( k \)th Rademacher function on \([0,1]\). Replacing \( \lambda_k \) with \( \psi_k(t)\lambda_k \), integrating w.r.t \( t \) from \( 0 \) to \( 1 \) and applying Khinchine’s inequality in (3.10), we see that

\[
\int_D \left( \sum_{k=1}^\infty |\lambda_k|^2 \frac{|K^\alpha_{z_k}(z)|^2}{\sigma^\frac{\alpha}{2}(z_k)(1 - |z_k|^2)^{2(\frac{\alpha}{2} - \frac{\beta}{p})}} \right)^\frac{q}{2} d\Omega(z) \leq C \|i\|^q_{A^p_{\mathcal{K}} \rightarrow L^p(\Omega)} \|\{\lambda_k\}_k\|_{l^p}^q.
\]

Thus, we have

\[
\sum_{k > k_0} |\lambda_k|^q \frac{\hat{\Omega}_t(z_k)}{\sigma^\frac{\alpha}{2}(z_k)(1 - |z_k|^2)^{2(\frac{\alpha}{2} - \frac{\beta}{p})}} \\
\leq C \sum_{k > k_0} \int_{E(z_k,r)} |\lambda_k|^q |K^\alpha_{z_k}(z)|^q \sigma^\frac{\alpha}{2}(z)(1 - |z|^2)^{2(\frac{\alpha}{2} - \frac{\beta}{p})} d\Omega(z) \\
\leq C_1 \left( \sum_{k=1}^\infty |\lambda_k|^2 \frac{|K^\alpha_{z_k}(z)|^2}{\sigma^\frac{\alpha}{2}(z_k)(1 - |z_k|^2)^{2(\frac{\alpha}{2} - \frac{\beta}{p})}} \right)^\frac{q}{2} d\Omega(z)
\]

where \( C_1 \) is a constant and \( k_0 \) is a large constant.
Fix $R > 0$ and consider $z \in A_k$, let $\Omega(z) = \Omega(z) = \{z \in \overline{D} : \sigma(z) \geq \tau(z)^{1/2} \}$, we have

$$
\frac{\Omega(z)}{\sigma(z)} \leq C
$$

Hence by duality argument, we have

$$
\left\| \frac{\hat{\Omega}_L(z)}{\sigma(z)} \right\|_{L^\infty(\Omega)} \leq C\|i\|_{A^\infty L^\infty} \|\lambda_k\|_{L^1(\Omega)}.
$$

Finally, we will prove the implication $(a) \Rightarrow (b)$. Let us consider a bounded sequence \( \{f_n\}_{n=1}^\infty \) in $A^p$ such that $f_n \to 0$ uniformly on each compact subset of $D$. Let $F$ be any compact subset of $D$ and $\Omega_F$ be the restriction of $\Omega$ to $F$. Then we have

$$
\int_D |f_n(z)|^q d\Omega(z) = \int_F |f_n(z)|^q d\Omega(z) + \int_{D\setminus F} |f_n(z)|^q d\Omega(z)
$$

$$
= I_1 + I_2.
$$

Since $f_n \to 0$ uniformly on $F$ as $n \to \infty$, we have

$$
I_1 = \int_F |f_n(z)|^q d\Omega(z)
$$

$$
\leq C \sup_{z \in F} |f_n(z)|^q \to 0.
$$

Fix $R \in (0, 1)$ and $z \in D$, and we obtain that

$$
\frac{\hat{\Omega}_F(z)}{\sigma(z)} \to 0
$$

as $F$ extended to $D$. By the equivalence of $(a)$ and $(d)$, we have

$$
\frac{\hat{\Omega}_F(z)}{\sigma(z)} \leq \frac{\hat{\Omega}_F(z)}{\sigma(z)} \in L^1(D).
$$

Therefore,

$$
I_2 = \int_D |f_n(z)|^q d\Omega(z)
$$

$$
\leq C \sup_n \|f_n\|^q_{L^p} \left\| \frac{\hat{\Omega}_F(z)}{\sigma(z)} \right\|_{L^\infty(\Omega_F)} \to 0
$$

as $F$ extended to $D$,

$$
\text{above follows from (3.11) and the dominated convergence theorem. Hence}
$$

$$
\lim_{n \to \infty} \int_D |f_n(z)|^q d\Omega(z) = 0
$$

$$
\leq C \sup_n \|f_n\|^q_{L^p} \left\| \frac{\hat{\Omega}_F(z)}{\sigma(z)} \right\|_{L^\infty(\Omega_F)} \to 0
$$

as $F$ extended to $D$.
implies that $\Omega$ is a vanishing $(p, q, \sigma)-$Bergman Carleson measure. This completes the proof. \qed

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