ELASTIC LIMIT AND VANISHING EXTERNAL FORCE FOR GRANULAR SYSTEMS

FEI MENG*
School of Science
Nanjing University of Posts and Telecommunications
Nanjing 210003, China

XIAO-PING YANG
Department of Mathematics
Nanjing University
Nanjing 210093, China

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Abstract. We consider two popular models derived from the theory of granular gases. The first model is the inelastic Boltzmann equation with a diffusion term representing the heat bath, the second model is obtained by a self-similar transformation for the inelastic Boltzmann equation in the homogeneous cooling problem. We prove that the steady states of the two models converge to a Maxwellian equilibrium or a Dirac distribution in the elastic limit and the vanishing external force, respectively. Our results show that the limits of the steady states depend on the ratio of external energy and dissipated energy due to inelastic collision. These results provide a partial answer to a question proposed by Gamba, Panferov and Villani (Comm. Math. Phys. 246, 503-541. 2004).

1. Introduction and main results. The granular flows have become a subject of physical research in the last decades, because of their growing importance in the applications. Many kinetic models [1, 4, 7, 16] have been proposed to describe rapid, dilute flows of granular gases. In this case, the binary collisions between particles are often assumed to be the main mechanism of inner-particle interactions in the system. The model for the spatially homogeneous granular gases is given as follows:

$$\frac{\partial f}{\partial t} = Q(f, f),$$

where $f(t, v)$ is a nonnegative particle density function of the microscopic velocity $v$ and the time $t$, while the quadratic operator $Q(f, f)$ models the interaction of particles by means of inelastic binary collisions.

Two different collision operators are widely used in the study of granular gases. Let us first introduce the inelastic pseudo-Maxwell operator [2, 3, 4, 6, 8, 9]. We
call it the pseudo-Maxwell operator because of the analogy to the Maxwell model in the classical elastic case. The corresponding strong form of $Q(f, f)$ is given by the integral

$$Q(f, f) = \frac{\sqrt{\theta(t)}}{4\pi} \int_{R^3 \times S^2} (f'(v)f'(v_\ast) J - f(v)f(v_\ast)) d\sigma dv_\ast,$$

where $v, v_\ast$ are the post-collisional velocities, and $v', v'_\ast$ are the pre-collisional velocities.

The function $\theta(t) = \frac{1}{3} \int_{R^3} f(v, t) |v - u|^2 dv$, where $u = \int_{R^3} vf(v, t) dv$.

From the physical and mathematical viewpoint, the study on the inelastic Boltzmann models was first restricted to above inelastic Maxwell model [2, 4, 5, 6, 8, 9]. The inelastic Maxwell model is important because of its analytic simplification allowing to apply the Fourier transform tools (see Section 2).

The second operator is the so-called inelastic hard-sphere operator:

$$Q(f, f) = \int_{R^3 \times S^2} \frac{1}{e^2} f'(v)f'(v_\ast) - f(v)f(v_\ast)|v - v_\ast| b(\cos \theta) d\sigma dv_\ast.$$ 

The function $b(\cos \theta)$ is the differential collision cross section. We refer to [7, 10, 11, 12, 13] for the discussion about the relation of this operator to the inelastic-Maxwell operator. Roughly speaking, one should consider the inelastic-Maxwell model as an analytical simplification for the hard-sphere model. Let us mention that the temperature $\sqrt{\theta(t)}$ in the Maxwell model is chosen for having the same temperature decay law as its hard-sphere counterpart.

It is important to emphasize that contrary to the case of elastic collisions, the total kinetic energy is not preserved in the inelastic collisions. Therefore the only functions on which the collision operator vanishes are Dirac delta functions corresponding to all particles at rest. In order to obtain nontrivial steady states in granular systems, a certain mechanism of the energy is required. We will consider two different examples of forcing. The first model is the pure diffusion heat bath, and described by the following equation [2, 5, 7, 10, 14]:

$$\frac{\partial f}{\partial t} = Q(f, f) + \mu \triangle f,$$

where the diffusion term represents the effect of the heat bath which models particles uncorrelated random accelerations between collisions.

The second one relates to self-similar solutions, in this case, after an appropriate change of variables, one is led to consider the inelastic Boltzmann equation with an
additional anti-drift term [3, 6, 12, 13].

$$\frac{\partial f}{\partial t} = Q(f, f) - \mu \text{div}(vf).$$  \hspace{1cm} (2)

In equations (1) and (2), \( \mu \) is a constant.

The steady states of above two models have been extensively studied in the past ten years. Many problems such as existence, uniqueness, regularity and moment tail have been discussed. One of the interesting features of the steady states is the non-Maxwellian behavior, see [5, 7, 10, 12]. It is well known that the Maxwellian distribution is a steady state for the elastic Boltzmann equation. Therefore, deviation of the steady states of granular systems from the Maxwellian distribution is an object of intensive study. Indeed, a question has been raised at the end of [10]:

Whether the steady states of equations (1) and (2) converge to some Maxwellian distribution when \( e \to 1 \) and \( \mu \to 0 \)?

In this paper, we give an answer to above question for equations (1) and (2). Our results can be summarized as follows:

**Theorem 1.1.** For the pseudo-Maxwell operator, let \( f \) be a steady state of equations (1) or (2).

(i) Assume that

$$\lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = C \neq 0,$$

then \( f \) converges to \( \frac{1}{(2\pi(8C)^{\frac{3}{2}})^2} \exp(-\frac{|v|^2}{2(8C)^{\frac{3}{2}}}) \) as \( e \to 1 \) and \( \mu \to 0 \) in the sense of probability measures if \( f \) solves equation (1), and \( f \) converges to \( \frac{1}{(256\pi C^2)^{\frac{3}{2}}} \exp(-\frac{|v|^2}{256C^2}) \) as \( e \to 1 \) and \( \mu \to 0 \) in the sense of probability measures if \( f \) solves equation (2).

(ii) Assume that

$$\lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = 0,$$

then \( f \) converges to Dirac delta function as \( e \to 1 \) and \( \mu \to 0 \) in the sense of probability measures.

(iii) Assume that

$$\lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = \infty,$$

then \( \int_{\mathbb{R}^3} |v|^2 \, dv \) (i.e the energy of \( f \)) does blow up as \( e \to 1 \) and \( \mu \to 0 \).

The next Theorem is concerned with hard-sphere operator.

**Theorem 1.2.** For the hard-sphere operator, let \( f \) be a steady state of equation (1) or (2).

(i) Assume that

$$\lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = C \neq 0,$$

then

$$\|f - \frac{1}{(2\pi \theta)^{\frac{3}{2}}} \exp(-\frac{|v|^2}{2\theta})\|_{L^1} \to 0$$

as \( e \to 1 \) and \( \mu \to 0 \) if \( f \) solves equation (1), where

$$\theta = \left( \frac{6C}{b_1^{\frac{3}{2}} \int_{\mathbb{R}^3} M_{1,0,1} |v|^2 \, dv} \right)^{\frac{3}{2}}$$
and } \mathcal{M}_{1,0,1} \text{ is a normalized Maxwellian function, } b_1 = \frac{1}{8} \int_{S^2} (1 - \cos \theta) b(\cos \theta) d\sigma, \text{ as } e \to 1 \text{ and } \mu \to 0 \text{ if } f \text{ solves equation (2), where }
\begin{align*}
\theta &= \frac{9 C^2}{2 b_1 (\int_{R^3} M_{1,0,1} |v|^3 dv)^2},
\end{align*}
and } \mathcal{M}_{1,0,1} \text{ is a normalized Maxwellian function, } b_1 = \frac{1}{8} \int_{S^2} (1 - \cos \theta) b(\cos \theta) d\sigma.

(ii) If
\begin{align*}
\lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = 0,
\end{align*}
then } f \text{ converges to Dirac delta function as } e \to 1 \text{ and } \mu \to 0 \text{ in the sense of probability measures.}

(iii) If
\begin{align*}
\lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = \infty,
\end{align*}
then } \int_{R^3} f |v|^2 dv \text{ does blow up as } e \to 1 \text{ and } \mu \to 0.

**Remark 1.** The exponent in the Maxwellian looks strange at first sight, but it will be shown in Section 3 and 4 that the exponent is related to the energy of the steady states.

**Remark 2.** Let us explain the results from the physical point of view. Three different cases are considered in our results. These cases correspond to the ratio of the external energy and dissipated energy due to inelastic collision. The ratio determines the balance of the external energy and dissipated energy. For example, the case
\begin{align*}
\lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = C
\end{align*}
implies that the dissipated energy is compensated by the external energy, the steady states should converge to some Maxwellian as } e \to 1 \text{ and } \mu \to 0.

We first investigate the pseudo-Maxwell model, our method is new and based on the explicit form of the steady solutions for equations (1) and (2) constructed in [5]. Thanks to this explicit form of the steady state, we are able to find solutions depending on the parameters } \mu \text{ and } e. \text{ For our purpose, we need to take the limit for the solution as } \mu \text{ goes to } 0 \text{ and } e \text{ goes to } 1. \text{ To this end, the limit of coefficients in the explicit form of the steady solutions is studied. We also discover the following fact: the limit of the steady solutions depends on the ratio of } \mu \to 0 \text{ and } e \to 1, \text{ this is the reason why there are three different cases in our results.}

For the hard sphere model, the main difficulty is that we do not get an explicit solution for the hard sphere model. Therefore, the method used for pseudo-Maxwell model does not work. Instead, we follow the same spirit as in [13] and [14], where a special case } \mu = 1 - e \text{ is considered. The key point in our paper is to establish energy estimates, regularity estimates and decay estimates for the steady states. Comparing to the previous work [13] and [14], we shall give more delicate estimates, which depend on the parameters } \mu \text{ and } e. \text{ These estimates will play important roles in our proof. More details can be found in Section 4. Similar to the pseudo-Maxwell model, three different cases are discussed. Thus, the results in [13] and [14] can be considered as a particular case of our results.
The paper is organized as follows. In Section 2, we recall the main results in [5, 10, 12], where the existence and qualitative properties of the steady solutions to the associated equations are given. In Section 3, we prove the steady solution converges to the Maxwellian function or Dirac function for the Maxwell model. In Section 4, we prove the steady solution converges to the Maxwellian or the Dirac function for the hard sphere model.

2. Preliminaries. In this Section, we recall the main results in [5, 10, 12]. Existence and qualitative properties of the steady solutions for pseudo-Maxwell model and hard-sphere model have been studied in these papers.

For the pseudo-Maxwell model, in order to study how the steady states depend on the parameters $e$ and $\mu$ and give an explicit form of the steady states, we review the main steps in [5]. It is obvious that the steady states satisfy the following equations:

\[ Q(f, f) + \mu \Delta f = 0, \]  
\[ Q(f, f) - \mu \text{div}(vf) = 0. \]

Let us first recall the Fourier transform for above equations, we introduce the Fourier-transform of $f$:

\[ \phi(k) = \int_{\mathbb{R}^3} f(v)e^{-ik \cdot v} dv. \]

Then the equation for $\phi(k)$ reads as follows:

\[ \Xi(\phi, \phi) + \Lambda \phi = 0, \]

where

\[ \Xi(\phi, \phi) = \sqrt{\theta} \int_{S^2} (\phi(k_-)\phi(k_+) - \phi(0)\phi(k))dn, \]

here $\Xi(\phi, \phi)$ is the Fourier transform of $Q(f, f)$, and

\[ k_- = \frac{1 + e}{4}(k - |k|n), \quad k_+ = k - k_. \]

$\theta$ is the temperature of the steady states, $\Lambda \phi$ is the Fourier transform of the diffusion term or anti-drift term. In the case of heat bath:

\[ \Lambda \phi = -\mu |k|^2 \phi. \]

In the self-similar solution case:

\[ \Lambda \phi = \mu k \cdot \nabla_k \phi. \]

Next, if $f$ is an isotropic function, then $\phi$ can be written as

\[ \phi = \phi(x), \quad x = \frac{|k|^2}{2}. \]

This leads to

\[ \Xi(\phi, \phi) = \int_{0}^{1} [\phi(z^2sx)\phi((1-\beta)s)x - \phi(0)\phi(x)]ds, \]

where

\[ z = \frac{1 + e}{2}, \quad \beta = 1 - \frac{(1 - e)^2}{4}. \]
In the case of heat bath:
\[ \Lambda \phi = -2 \mu x \phi. \]
In the case of self-similar solutions:
\[ \Lambda \phi = 2 \mu x \frac{\partial \phi}{\partial x}. \]
Now the steady states of equation (3) and (4) can be constructed in a form of a power series, more precisely,

**Theorem 2.1.** [5] The equation
\[ \Xi(\phi, \phi) + \Lambda \phi = 0 \]
has a unique isotropic solution \( \phi(x) \) analytic at \( x = 0 \) for any \( 0 < e < 1 \) and \( \mu > 0 \). Moreover, the function \( \phi(x) \) reads as follows:
\[ \phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \phi_n x^n. \]
The coefficients in the series \( \phi_n \) are given by the following recurrence relations.

In the case of heat bath,
\[ \lambda_1 \theta^2 = 2 \mu, \quad \phi_1 = \theta, \]
\[ \phi_n = \frac{1}{\lambda_n} \sum_{k=1}^{n-1} H(k, n-k) \phi_k \phi_{n-k} + 2 \frac{\mu n}{\lambda_n \theta^2} \phi_{n-1}, n = 2, 3, \ldots. \]

In the case of self-similar solutions,
\[ \lambda_1 \theta^2 = 2 \mu \theta, \quad \phi_1 = \theta, \]
\[ \phi_n = \frac{\theta^2}{\lambda_n \theta^2 - 2 \mu} \sum_{k=1}^{n-1} H(k, n-k) \phi_k \phi_{n-k} + 2 \frac{\mu n}{\lambda_n \theta^2} \phi_{n-1}, n = 2, 3, \ldots. \]

where
\[ \lambda_n = \int_0^1 [1 - z^{2n}s^n - (1 - \beta s)^n] ds, \]
\[ H(k, n-k) = z^{2n} C_n^k \int_0^1 s^k (1 - \beta s)^{n-k} ds. \]

**Remark 3.** In [5], the steady state is constructed in the case of heat bath with liner friction, and the proof can be applied to the self-similar case with minor modifications.

At last, we need to prove the solution \( \phi(x) \) is a characteristic function, to show \( \phi(x) \) corresponding to a positive function after the reverse Fourier transform. We refer to [5] for more details of the proof.

For the hard-sphere model, the existence and qualitative properties of the solutions to equations (3) and (4) were done in [10, 12].

**Theorem 2.2.** [10, 12] For any given inelastic restitution coefficient \( e \in (0, 1) \), and \( \mu > 0 \), there exists at least one positive solution to the equation (3) or (4). Moreover, the solution \( f \) belongs to the Schwartz space of \( C^\infty \) functions decreasing at infinity, and \( \int_{R^3} f(v) dv = 1, \int_{R^3} f(v) dv = 0. \)
3. Convergence to the Maxwellian or Dirac function for the pseudo-Maxwell model. In this Section, we shall prove our main results for the pseudo-Maxwell model. Thanks to Theorem 2.1, we find that how the solution $\phi_n$ depends on the restitution coefficient $e$ and $\mu$, this fact leads us to study the limit of $\phi_n$ when $e \to 1$ and $\mu \to 0$. We first consider the limit of $\lambda_n$ and $H(k,n-k)$ as $e \to 1$.

Let us recall $\lambda_n$ and $H(k,n-k)$ in Theorem 2.1. The result can be formulated as follows.

**Lemma 3.1.** For all $n \geq 1$ and $1 \leq k \leq n$,

$$\lim_{e \to 1} \lambda_n = \frac{n-1}{n+1}, \quad \lim_{e \to 1} H(n-k,k) = \frac{1}{n+1}.$$ 

**Proof.** We first consider $\lambda_n$, since

$$\lambda_n = \int_0^1 [1 - z^{2n}s^n - (1 - \beta s)^n] ds$$

$$\quad = \frac{s - z^{2n}s^{n+1}}{n+1} \bigg|_0^1 + \frac{1}{\beta} \left( \frac{(1 - \beta s)^{n+1}}{n+1} \right) \bigg|_0^1$$

$$\quad = 1 - \frac{z^{2n}}{n+1} + \frac{1}{\beta} \left( \frac{(1 - \beta)^{n+1} - 1}{n+1} \right).$$

Let us notice that $z = \frac{1+e}{2}$, $\beta = 1 - \frac{(1-e)^2}{4}$, then

$$\lim_{e \to 1} z = 1, \quad \lim_{e \to 1} \beta = 1.$$ 

Therefore, we get

$$\lim_{e \to 1} \lambda_n = 1 - \frac{1}{n+1} - \frac{1}{n+1} = \frac{n-1}{n+1}.$$ 

Next, recall that $H(k,n-k) = z^{2n}C_n^k \int_0^1 s^k (1 - \beta s)^{n-k} ds$.

The evaluation of the integrals yields

$$H(k,n-k) = z^{2k} C_n^k \int_0^1 s^k (1 - \beta s)^{n-k} ds.$$ 

Taking the limit, we have

$$\lim_{e \to 1} H(k,n-k) = \frac{1}{n+1}$$

which concludes the proof. \hfill \Box

Now we are in a position to study the asymptotic behavior of $\phi_n$ when $e \to 1$ and $\mu \to 0$.

**Lemma 3.2.** (i) Assume that

$$\lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = C \neq 0.$$ 

In the case of heat bath, it holds:

$$\lim_{e \to 1, \mu \to 0} \phi_n = (8C)^{\frac{n-k}{2}}$$
for all $n \geq 1$.

In the case of self-similar solutions, it holds:

$$\lim_{e \to 1, \mu \to 0} \phi_n = (64C^2)^n$$

for all $n \geq 1$.

(ii) Assume that

$$\lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = 0,$$

it holds

$$\lim_{e \to 1, \mu \to 0} \phi_n = 0$$

for all $n \geq 1$ in either case of heat bath or case of self-similar solutions.

(ii) Assume that

$$\lim_{e \to 1, \mu \to 0} \mu \left(1 - e^2\right) = \infty,$$

it holds

$$\lim_{e \to 1, \mu \to 0} \phi_1 = \infty.$$

for all $n \geq 1$ in either case of heat bath or case of self-similar solutions.

Proof. We will prove the statement by induction argument. For the case of heat bath, when

$$\lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = C\neq 0,$$

since $\lambda_1 \theta^2 = 2\mu$, $\theta = \phi_1$ and $\lambda_1 = \frac{1 - e^2}{\mu}$, it follows that

$$\lim_{e \to 1, \mu \to 0} \phi_1 = (8C)^{\frac{3}{2}}.$$

We conclude the proof by induction. Assume that for all $k < n$,

$$\lim_{e \to 1, \mu \to 0} \phi_k = (8C)^{\frac{k}{2}}.$$

It remains to prove that

$$\lim_{e \to 1, \mu \to 0} \phi_n = (8C)^{\frac{n}{2}}.$$

Note that

$$\phi_n = \frac{1}{\lambda_n} \sum_{k=1}^{n-1} H(k, n-k) \phi_k \phi_{n-k} + 2\frac{\mu n}{\lambda_n \theta^2} \phi_{n-1}.$$

Applying Lemma 3.1 and induction hypothesis, we get

$$\lim_{e \to 1, \mu \to 0} \phi_n = \frac{n + 1}{n - 1} \left[ \frac{1}{(8C)^{\frac{3}{2}}} \cdot (8C)^{\frac{2(n-1)}{2}} + \frac{1}{n + 1} (8C)^{\frac{3}{2}} \cdot (8C)^{\frac{2(n-2)}{2}} + \cdots \right].$$

Since the number of the sum in the bracket is $n - 1$, one has

$$\lim_{e \to 1, \mu \to 0} \phi_n = (8C)^{\frac{n}{2}}.$$

In the case of

$$\lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = 0,$$

We again assume that for all $k < n$,

$$\lim_{e \to 1, \mu \to 0} \phi_k = 0.$$
By the recurrence relations for $\phi_n$, we have
$$\lim_{\epsilon \to 1, \mu \to 0} \phi_n = 0.$$  

In the case of
$$\lim_{\epsilon \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = \infty,$$
we deduce that
$$\lim_{\epsilon \to 1, \mu \to 0} \phi_1 = \infty,$$
which concludes the proof. The proof for the self-similar case is similar to that of the heat bath case, so we omit it. \( \square \)

Taking advantage of the expression of steady states in Theorem 2.1, we finally can prove Theorem 1.1.

**Proof of Theorem 1.1.** For the heat bath case, from Theorem 2.1,
$$\phi(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \phi_n x^n.$$  

We will take the limit for above expression of $\phi(x)$.

In the case of
$$\lim_{\epsilon \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = C \neq 0,$$
applying Lemma 3.2,
$$\lim_{\epsilon \to 1, \mu \to 0} \phi(x) = \lim_{\epsilon \to 1, \mu \to 0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \phi_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (8C)^{\frac{n}{2}} x^n.$$  

We emphasize that the series is convergent [5], this fact allows us to exchange the order of the sum and the limit.

Notice that $x = \frac{|k|}{2}$, and $\exp(-C|k|^2)$ has an expansion:
$$\exp(-\frac{(8C)^{\frac{n}{2}}|k|^2}{2}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (8C)^{\frac{n}{2}} x^n.$$  

This shows that
$$\lim_{\epsilon \to 1, \mu \to 0} \phi(x) = \exp(-(8C)^{\frac{n}{2}} x)$$  

in the sense of point-wise. The proof follows from the fact that the inverse Fourier transform of $\exp(-\frac{(8C)^{\frac{n}{2}}|k|^2}{2})$ is $\frac{1}{(2\pi)^{\frac{3}{2}} (8C)^{\frac{3}{2}}} \exp(-\frac{|v|^2}{2(8C)^{\frac{3}{2}}})$, and the point-wise convergence of the characteristic function implies the convergence of the corresponding distribution functions in the sense of probability measures.

In the case of
$$\lim_{\epsilon \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = 0,$$
by Lemma 3.2, we get
$$\lim_{\epsilon \to 1, \mu \to 0} \phi(x) = 1.$$
The proof follows from the fact that the inverse Fourier transform of 1 is Dirac function, and the point-wise convergence of characteristic function implies the convergence of the corresponding distribution functions in the sense of probability measures.

In the case of
\[ \lim_{\epsilon \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = 0, \]
noticing that \( \phi_1 = \theta \), the proof is obvious.

The proof of the self-similar case is similar. This completes the proof. \( \square \)

4. **Convergence to the Maxwellian or Dirac function for the hard sphere model.** In this Section, we shall consider the hard-sphere model which has been introduced in Section 1. As we mentioned in the introduction, we follow the path in [13], but we will give more precise estimates on the steady states. Therefore, our results recover the result in [13] in some sense.

Although that the solutions of equation (3) and (4) depend on the parameters \( \mu \) and \( e \), however, we omit the subscript \( \mu \) and \( e \) in the sequel when there is no confusion. We start by establishing the upper bound and the lower bound on the energy of the solutions of equations (3) and (4). On one hand, the upper bound of the energy relies on the energy dissipation term, and on the other hand, the key argument for establishing the lower bound is the entropy functional.

In the case of the heat bath, we have the following Lemma.

**Lemma 4.1.** For the hard sphere model, if \( f \) is a positive solution of equation (3), and satisfying \( \int_{R^3} f dv = 1, \int_{R^3} vfv = 0 \), then the following inequalities holds:

\[ \left[ \frac{9 \mu e^2}{2(1 - e^2)b_2} \right]^3 \leq \int_{R^3} |v|^2 dv \leq \left[ \frac{6 \mu}{(1 - e^2)b_1} \right]^3 \]  
where \( b_1 = \frac{1}{8} \int_{S^2} (1 - \cos \theta)b(\cos \theta) d\sigma, \quad b_2 = \int_{S^2} b(\cos \theta) d\sigma, \) and

\[ \int_{R^3} |v|^3 dv \leq \frac{6 \mu}{(1 - e^2)b_1}. \]  
As a by-product,

\[ D_e(f) \leq \frac{\sqrt{2}(1 - e^2)b_2}{2e^2} \left( \frac{6 \mu}{(1 - e^2)b_1} \right)^{\frac{1}{3}}. \]  
where

\[ D_e(f) = \frac{1}{2} \iint_{R^3 \times R^3 \times S^2} f f_* \left( \frac{ff_*}{f f_*} - \ln \frac{ff_*}{f f_*} - 1 \right) |v - v_*| b(\cos \theta) d\sigma dv d v_* \]

**Proof.** As in [11], the energy dissipation functional is given by

\[ \int_{R^3} Q(f, f) |v|^2 dv = -(1 - e^2)b_1 \iint_{R^3 \times R^3} f f_* |v - v_*|^3 dv d v_* , \]

where

\[ b_1 = \frac{1}{8} \int_{S^2} (1 - \cos \theta)b(\cos \theta) d\sigma. \]

Thanks to Theorem 2.2, multiplying the equation (3) by \( |v|^2 \) and integrating by parts, it yields:

\[ (1 - e^2)b_1 \iint_{R^3 \times R^3} f f_* |v - v_*|^3 dv d v_* = 6 \mu. \]
From the Jensen’s inequality, \( \int_{R^3} f dv = 1 \) and \( \int_{R^3} f v dv = 0 \),
\[
\int_{R^3} f_* |v - v_*|^3 dv_* \geq |v|^3.
\]
We get:
\[
\int_{R^3} f |v|^3 dv \leq \frac{6\mu}{(1 - e^2)b_1},
\]
which completes the proof of (6). By Hölder’s inequality and \( \int_{R^3} f dv = 1 \),
\[
\int_{R^3} f |v|^3 dv \geq (\int_{R^3} f |v|^2 dv)^{\frac{3}{2}}.
\]
We obtain
\[
\int_{R^3} f |v|^2 dv \leq \left[ \frac{6\mu}{(1 - e^2)b_1} \right]^{\frac{2}{3}}.
\]
from which the upper bound of (5) follows.

We next study the lower bound on the energy by using the entropy functional. Multiplying the equation (3) by \( \ln f \) and integrating by parts,
\[
\int_{R^3} Q(f, f) \ln f dv = \mu \int_{R^3} \frac{\nabla f}{f} dv.
\]
The term \( \int_{R^3} Q(f, f) \ln f dv \) can be rewritten into two terms, as in [10]
\[
\int_{R^3} Q(f, f) \ln f dv = \frac{1}{2} \int \int \int_{R^3 \times R^3 \times S^2} f f_* (\ln f^* f'_* - \frac{f^* f'_*}{f_*} + 1) |v - v_*| b(\cos \theta) d\sigma dv_*
\]
\[
+ \frac{1}{2} \int \int \int_{R^3 \times R^3 \times S^2} (f^* f'_* - f f_*) |v - v_*| b(\cos \theta) d\sigma dv_*.
\]
If we denote
\[
D_e(f) = \frac{1}{2} \int \int \int_{R^3 \times R^3 \times S^2} f f_* (\frac{f^* f'_*}{f_*} - \frac{f^* f'_*}{f_*} - 1) |v - v_*| b(\cos \theta) d\sigma dv_*.
\]
Then the first term is \(-D_e(f)\). We mention that if \( e = 1 \),
\[
D_1(f) = \frac{1}{2} \int \int \int_{R^3 \times R^3 \times S^2} f f_* \ln \frac{f^*}{f_*} |v - v_*| b(\cos \theta) d\sigma dv_*.
\]
corresponds to the so-called entropy production in the elastic case. Moreover, since the inequality \( x - \ln x - 1 > 0 \) holds, it follows that \( D_e(f) \geq 0 \).

For the second term, passing from the variables \( v' \), \( v'_* \) to \( v \) and \( v_* \), we can write
\[
\int \int \int_{R^3 \times R^3 \times S^2} f^* f'_* |v - v_*| b(\cos \theta) d\sigma dv_* = \frac{1}{e^2} b_2 \int \int_{R^3 \times R^3} f f_* |v - v_*| dv dv_*,
\]
where \( b_2 = \int_{S^2} b(\cos \theta) d\sigma \).
Collecting above estimates, we get
\[
-D_e(f) + \frac{1 - e^2}{2e^2} b_2 \int \int_{R^3 \times R^3} f f_* |v - v_*| dv dv_* = \mu \int_{R^3} \frac{\nabla f}{f} dv.
\]
Since \( D_e(f) \) and \( \int_{R^3} \frac{\nabla f}{f} dv \) are positive, it yields
\[
\mu \int_{R^3} \frac{\nabla f}{f} dv \leq \frac{1 - e^2}{2e^2} b_2 \int \int_{R^3 \times R^3} f f_* |v - v_*| dv dv_*.
\]
and

\[ D_e(f) \leq \frac{1 - e^2}{2e^2b_2} \int_{R^3 \times R^3} f f_\ast |v - v_\ast| dv dv_\ast. \]

On one hand, from Cauchy-Schwarz’s inequality and \( \int_{R^3} f dv = 1 \),

\[
\int_{R^3 \times R^3} f f_\ast |v - v_\ast| dv dv_\ast \leq \left( \int_{R^3 \times R^3} f f_\ast dv dv_\ast \right)^{\frac{1}{2}} \left( \int_{R^3} f f_\ast |v - v_\ast|^2 dv dv_\ast \right)^{\frac{1}{2}}
\]

\[ = \sqrt{2} \left( \int_{R^3} f |v|^2 dv \right)^{\frac{1}{2}}. \]

On the other hand,

\[
0 \leq \int_{R^3} |2\nabla \sqrt{f} + \frac{3v}{f} f_\ast f |v|^2 dv
\]

\[ = \int_{R^3} (4|\nabla \sqrt{f}|^2 + 6v \nabla f) f + \frac{9}{(\int_{R^3} f |v|^2 dv)^2} f |v|^2 dv
\]

\[ = \int_{R^3} \frac{|\nabla f|^2}{f} dv - \frac{18}{\int_{R^3} f |v|^2 dv} + \frac{9}{\int_{R^3} f |v|^2 dv^2}
\]

\[ = \int_{R^3} \frac{|\nabla f|^2}{f} dv - \frac{9}{\int_{R^3} f |v|^2 dv}. \]

Gathering above estimates, we obtain

\[
\mu \int_{R^3} \frac{9}{f |v|^2 dv} \leq \frac{1 - e^2}{2e^2b_2} \sqrt{2} \left( \int_{R^3} f |v|^2 dv \right)^{\frac{1}{2}},
\]

which gives

\[
\int_{R^3} f |v|^2 dv \geq \left( \frac{9\mu e^2}{\sqrt{2}(1 - e^2)b_2} \right)^{\frac{1}{2}}.
\]

By using the upper bound of the energy:

\[
\int_{R^3} f |v|^2 dv \leq \left[ \frac{6\mu}{(1 - e^2)b_1} \right]^{\frac{1}{2}},
\]

we also get the upper bound for \( D_e(f) \):

\[
D_e(f) \leq \frac{1 - e^2}{2e^2b_2} \sqrt{2} \left( \int_{R^3} f |v|^2 dv \right)^{\frac{1}{2}}
\]

\[ \leq \frac{\sqrt{2}(1 - e^2)b_2}{2e^2b_1} \left[ \frac{6\mu}{(1 - e^2)b_1} \right]^{\frac{1}{2}}. \]

This completes the proof. \( \Box \)

For the case of self-similar solutions, we have similar result.

**Lemma 4.2.** Let \( f \) be a positive solution of equation (4) and satisfying \( \int_{R^3} f dv = 1 \), \( \int_{R^3} f dv = 0 \), then the following inequalities hold:

\[
\frac{3\sqrt{2}\mu e^2}{(1 - e^2)b_2} \leq \int_{R^3} f |v|^2 dv \leq \left[ \frac{2\mu}{(1 - e^2)b_1} \right]^2,
\]

\[
\int_{R^3} f |v|^3 dv \leq \left[ \frac{2\mu}{(1 - e^2)b_1} \right]^3,
\]

and

\[
D_e(f) \leq \frac{\sqrt{2}\mu b_2}{e^2b_1}.
\]
where \(b_1, b_2\) and \(D_e(f)\) are given in Lemma 4.1.

**Proof.** By Theorem 2.2, multiplying the equation (4) by \(|v|^2\) and integrating by parts, we find:

\[
(1 - e^2)b_1 \int_{\mathbb{R}^3 \times \mathbb{R}^3} f f_* |v - v_*|^3 dv dv_* = 2\mu \int_{\mathbb{R}^3} |v|^2 dv.
\]

The left term can be estimated as in Lemma 4.1, therefore,

\[
(1 - e^2)b_1(\int_{\mathbb{R}^3} |v|^2)^\frac{3}{2} \leq 2\mu \int_{\mathbb{R}^3} |v|^2 dv.
\]

We have

\[
\int_{\mathbb{R}^3} |v|^2 dv \leq \left[ \frac{2\mu}{(1 - e^2)b_1} \right]^2,
\]

and

\[
\int_{\mathbb{R}^3} |v|^3 dv \leq \left[ \frac{2\mu}{(1 - e^2)b_1} \right]^3.
\]

For the lower bound of the energy, we again use the entropy functional, multiplying the equation (4) by \(\ln f\) and integrating by parts, it follows that

\[
\frac{1}{2} \int \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} f f_* (\ln f_*' f_*' - f_*' f_*' + 1)|v - v_*| b(\cos \theta) d\sigma dv dv_*
\]

\[
+ \frac{1}{2} \int \int \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times S^2} f f_*' |v - v_*| b(\cos \theta) d\sigma dv dv_* - 3\mu = 0.
\]

The term \(3\mu\) is due to the anti-drift term. Using the notation in Lemma 4.1,

\[
-D_e(f) + \frac{1 - e^2}{2e^2} b_2 \int_{\mathbb{R}^3 \times \mathbb{R}^3} f f_* |v - v_*| dv dv_* = 3\mu.
\]

Applying the following inequality used in Lemma 4.1,

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^3} f f_* |v - v_*| dv dv_* \leq \sqrt{2}(\int_{\mathbb{R}^3} |v|^2 dv)^{\frac{3}{2}}.
\]

As a consequence,

\[
(\int_{\mathbb{R}^3} |v|^2 dv)^{\frac{3}{2}} \frac{1 - e^2}{\sqrt{2e^2}} b_2 \geq 3\mu.
\]

which gives

\[
\int_{\mathbb{R}^3} |v|^2 dv \geq \left[ \frac{3\sqrt{2}\mu e^2}{(1 - e^2)b_2} \right]^2.
\]

From the entropy functional and the upper bound of the energy, we also get

\[
D_e(f) \leq \frac{1 - e^2}{2e^2} b_2 \sqrt{2}(\int_{\mathbb{R}^3} |v|^2 dv)^{\frac{3}{2}} \leq \frac{\sqrt{2}\mu b_2}{e^2 b_1}.
\]

This ends the proof. \(\square\)

We further look for moment estimates, regularity estimates and pointwise lower bounds for solutions of equations (3) and (4). Such estimates will play very important roles in our study. We shall use the following notations:

\[
m_k(f) = \int_{\mathbb{R}^3} f(v)|v|^k dv, \quad \|f\|_{H^k} = \left( \sum_{|s| \leq k} \|\partial^s f\|_{L^2}^2 \right)^{\frac{1}{2}}.
\]
Proposition 1. Let $f$ denote the solution of equation (3) or equation (4) and satisfying $\int_{R^3} f dv = 1$, $\int_{R^3} f dv = 0$. Then for any $k \in \mathbb{N}$, there exists $a_1, a_2, a_3 \geq 0$, and $\alpha, \beta, \gamma > 0$, these constants may be different for different $k$, such that

$$m_k \leq a_1 \mu \left( \frac{\mu}{1 - \varepsilon^2} \right)^\alpha + a_2 \left( \frac{\mu}{1 - \varepsilon^2} \right)^\beta,$$

$$\|f\|_{H^k} \leq a_3 \left( \frac{\mu}{1 - \varepsilon^2} \right)^\gamma.$$  

Moreover, if we fix $\varepsilon_0 \in (0,1)$, for $\varepsilon > \varepsilon_0$, and

$$\lim_{\varepsilon \rightarrow 1, \mu \rightarrow 0} \frac{\mu}{1 - \varepsilon^2} = C,$$

there exists $a_4, a_5, a_6 > 0$, such that

$$\|f\|_{H^k \cap L^1} \leq a_4, \quad f \geq a_5 e^{-a_6 |v|^6}.$$

Proof. Since the proof has a bit long, we split it into several steps.

Step 1. Moment bounds. Thanks to Lemma 4.1, we have already established the second and third moment estimates for the solution of equation (3). The first moment can be estimated by H"older’s inequality and $\int_{R^3} f dv = 1$,

$$\int_{R^3} f |v| dv \leq \int_{R^3} f dv \left( \int_{R^3} |v|^2 dv \right)^{\frac{1}{2}} = \left( \int_{R^3} f |v|^2 dv \right)^{\frac{1}{2}} \leq \left[ \frac{6\mu}{(1 - \varepsilon^2)b_1} \right]^\frac{1}{2}.$$

We next study the forth moment for the solution. Multiplying the equation (3) by $|v|^3$ to obtain that

$$\int_{R^3} Q(f,f)|v|^3 dv + \mu \int_{R^3} \Delta f |v|^3 dv = 0.$$

From ([7], Lemma 2, Lemma 3), there holds

$$\int_{R^3} Q(f,f)|v|^3 dv \leq -(1 - \gamma_3) m_4 + \gamma_3 S_3,$$

where $\gamma_3 < 1$ and $S_3 \leq m_3 m_1 + m_2 m_2$.

The moment of Laplacian term can be estimated as follows:

$$\mu \int_{R^3} \Delta f |v|^3 dv = \mu \int_{R^3} f \Delta |v|^3 = 12 \mu m_1.$$

Combining above estimates, we get

$$m_4 \leq \frac{12 \mu m_1}{1 - \gamma_3} + \frac{\gamma_3 S_3}{1 - \gamma_3}.$$

By the upper bound of the second moment, third moment in Lemma 4.1, and the first moment estimate, then there exists $a_1 > 0$ and $a_2 > 0$,

$$m_4 \leq a_1 \mu \left( \frac{\mu}{1 - \varepsilon^2} \right)^\alpha + a_2 \left( \frac{\mu}{1 - \varepsilon^2} \right)^\beta,$$

where $a_1$ and $a_2$ are constants not depending on $\mu$ and $\varepsilon$.

For general $k$, we multiply the equation by $|v|^k$, the term $\int_{R^3} Q(f,f)|v|^k dv$ can be estimated as in [7]. Indeed, from [7], the moment of the solution can be estimated recursively. In other words, the high order moments of the solution can be estimated by lower order moments. By induction argument, we arrive at our goal.

Step 2. $L^2$ and $H^k$ bounds. It is custom to split $Q(f,f)$ as follows:

$$Q(f,f) = Q^+(f,f) - Q^-(f,f).$$
\[ Q^+(f, f) = \int_{R^3 \times S^2} \left( \frac{1}{e^2} f(v_*) f(w_*) |v - v_* b(\cos \theta) d\sigma v_* \right), \]

and

\[ Q^-(f, f) = \int_{R^3 \times S^2} f(v)f(v_*) |v - v_* b(\cos \theta) d\sigma v_* =: fL(f), \]

where

\[ L(f) = b_2 \int_{R^3} f(v_*) |v - v_* | dv_* . \]

We multiplying the equation (3) by \( f \),
\[ \int_{R^3} Q(f, f) f dv + \mu \int_{R^3} \Delta f f dv = 0. \]

By dropping the Laplacian term, we obtain
\[ \int_{R^3} f^2 L f dv = \int_{R^3} Q^-(f, f) f dv \leq \int_{R^3} Q^+(f, f) f dv. \]

For the term \( \int_{R^3} Q^+(f, f) f dv \), from [12], Proposition 2.6, for any \( \epsilon > 0 \), there exists a constant \( C_{\epsilon} > 0 \) and \( \theta > 0 \),
\[ \int_{R^3} Q^+(f, f) f dv \leq C_{\epsilon} \| f \|_{L^1}^{1+2\theta} \| f \|_{L^2}^{2(1-\theta)} + \epsilon \| f \|_{L^1} \| f \|_{L^2}^2. \]

We next estimate \( L(f) \) from below, on one hand, by Jensen’s inequality,
\[ L(f) = b_2 \int_{R^3} f(v_*) |v - v_* | dv_* \geq b_2 |v|. \]

On the other hand, by the triangular inequality,
\[ L(f) \geq b_2 (m_1 - |v|). \]

Thanks to the Hölder’s inequality \( m_1 \geq \frac{m_2}{m_3} \), we have
\[ L(f) \geq b_2 (\frac{m_2}{m_3} - |v|). \]

These two lower bounds together imply that
\[ L(f) \geq A_1 \frac{m_2^2}{m_3} (1 + |v|), \]

where \( A_1 \) is a constant.

By the estimates of \( \int_{R^3} Q^+(f, f) f dv \) and \( L(f) \), we get
\[ A_1 \frac{m_2^2}{m_3} \int_{R^3} f^2 (1 + |v|) dv \leq C_{\epsilon} \| f \|_{L^1}^{1+2\theta} \| f \|_{L^2}^{2(1-\theta)} + \epsilon \| f \|_{L^1} \| f \|_{L^2}^2. \]

If we take \( \epsilon \) small enough, the second term on the right hand side can be absorbed by the left term, we finally obtain
\[ \| f \|_{L^3}^{2\theta} \leq A_2 \frac{m_3^3}{m_2^2}, \]

where \( A_2 \) is a constant. Then the \( L^2 \) bound follows from (5) and (6). The proof of the \( H^k \) bound can be done exactly in [13] by induction argument.

**Step 3.** From the explicit estimates in above steps, it is easy to get uniform \( H^k \) bounds in parameters \( e \) and \( \mu \) when \( e \rightarrow e_0 \) and
\[ \lim_{e \rightarrow e_0, \mu \rightarrow 0} \frac{\mu}{1 - e^2} = C. \]
For the pointwise lower bound, it is a straightforward consequence of Proposition 2.1 in [13].

This ends the proof.

Remark 4. We expect to obtain an explicit estimate for pointwise lower bound, just like moment bounds and $H^k$ bounds, which depend on the parameters $\mu$ and $e$. Unfortunately, it is difficult to track the constants in pointwise estimate. However, for our purpose, the uniform bound for $e_0 < e < 1$ is enough.

We finally state two Lemmas, which will be useful in our proof. The first Lemma deals with the difference between $D_e(f)$ and $D_1(f)$, where we recall that $D_e(f)$ and $D_1(f)$ are defined in Lemma (4.1).

**Lemma 4.3.** [13] For any function satisfying
\[ \|f\|_{H^k \cap L^1} \leq a_4, \quad f \geq a_5 e^{-a_6|v|^8}, \]
there exist a constant $C_1$ depending on $a_4, a_5, a_6$, such that
\[ |D_e(f) - D_1(f)| \leq C_1(1 - e). \]

The second Lemma is the so-called “entropy-entropy production inequality”.

**Lemma 4.4.** [15] For a given function $f \in L^1_2$, we denote by $M(f)$ the Maxwellian function with the same mass, momentum and temperature as $f$. If
\[ \|f\|_{H^2 \cap L^1} \leq a_4, \quad f \geq a_5 e^{-a_6|v|^8}, \]
then for any $\epsilon > 0$, there exists a constant $C_2$, depending on $a_4, a_5, a_6$ and $\epsilon$, such that
\[ (\int_{\mathbb{R}^3} f \ln \frac{f}{M(f)} dv)^{1+\epsilon} \leq C_2 D_1(f). \]

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** In the case of
\[ \lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = 0, \]
by the upper bound of the energy established in Lemma 4.1 and 4.2, we find
\[ \lim_{e \to 1, \mu \to 0} \int_{\mathbb{R}^3} f |v|^2 dv = 0. \]

Using the arguments of [11], we can prove that $f$ converges to the Dirac function as $e \to 1$ and $\mu \to 0$ in the sense of probability measures.

In the case of
\[ \lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = \infty, \]
by the lower bound of the energy established in Lemma 4.1 and 4.2, we have
\[ \lim_{e \to 1, \mu \to 0} \int_{\mathbb{R}^3} f |v|^2 dv = \infty. \]

It remains to treat the case
\[ \lim_{e \to 1, \mu \to 0} \frac{\mu}{1 - e^2} = C. \]

The main idea is the decoupling of the variation $f - M_1$ between the “energy direction” and its “orthogonal direction”. We only consider the case of heat bath, the proof of the self-similar solutions is similar, therefore we omit it.
We first introduce the Maxwellian function $M_{e,\mu}$ with the same mass, momentum, and the temperature as $f$. Recall that $\int_{\mathbb{R}^3} f(v)dv = 1$, and $\int_{\mathbb{R}^3} f(v)vdv = 0$, it is assumed that

$$M_{e,\mu} = \frac{1}{(2\pi\theta_{e,\mu})^{\frac{3}{2}}} \exp\left(-\frac{|v|^2}{2\theta_{e,\mu}}\right).$$

Next, from the energy equation in Lemma 4.1,

$$(1-e^2)b_1 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} ff^*|v-v^*|^3dvdv = 6\mu.$$  

Dividing the above equation by $1-e^2$ and passing to the limit, one obtains

$$b_1 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} ff^*|v-v^*|^3dvdv = 6C.$$  

The only Maxwellian function satisfying $\int_{\mathbb{R}^3} f(v)dv = 1$, $\int_{\mathbb{R}^3} f(v)vdv = 0$, and above equation is

$$M = \frac{1}{(2\pi\theta)^{\frac{3}{2}}} \exp\left(-\frac{|v|^2}{2\theta}\right),$$

where $\theta = \left(\frac{6C}{b_1\int_{\mathbb{R}^3} M_{1,0,1}|v|^3dv}\right)^{\frac{3}{2}}$, and $M_{1,0,1}$ is a normalized Maxwellian function.

We shall show that $f$ converges to $M$ as $\mu \to 0$ and $e \to 1$. On one hand, by Proposition 1, Lemma 4.3 and Lemma 4.4, together with the well known Csiszar-Kullback inequality

$$\|f - M_{e,\mu}\|_{L^1}^2 \leq 2 \int_{\mathbb{R}^3} fln\frac{f}{M_{e,\mu}} dv.$$  

We obtain that there exists $C_3$ and $C_4$ such that

$$\|f - M_{e,\mu}\|_{L^2}^2 + 2\|f - M_{e,\mu}\|_{L^2} \leq C_3(1-e) + C_4D_e(f).$$

On the other hand, we need to prove $M_{e,\mu}$ converges to $M$ when $\mu \to 0$ and $e \to 1$. By the method developed in [13] and [14], we can prove that there exists a constant $C_5$, such that

$$\|M_{e,\mu} - M\|_{L^2} \leq C_5(1-e).$$

Therefore, collecting above estimates, using Cauchy-Schwarz’s inequality,

$$\|f - M\|_{L^2} \leq \|f - M_{e,\mu}\|_{L^2} + \|M_{e,\mu} - M\|_{L^2} \leq C_6\|M_{e,\mu} - M\|_{L^2} + C_7(1-e).$$

Recall (7) in Lemma 4.1, we complete the proof of Theorem 1.2.  

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E-mail address: mengfei@njupt.edu.cn
E-mail address: xpyang@nju.edu.cn