MAPPING CLASS ACTION ON $SU(3)$-CHARACTER VARIETIES

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Abstract. Let $\Sigma$ be a compact orientable surface of genus $g = 1$ with $n = 1$ boundary component. The mapping class group $\Gamma$ of $\Sigma$ acts on the $SU(3)$-character variety of $\Sigma$. We show that the action is ergodic with respect to the natural symplectic measure on the character variety.

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Date: October 11, 2019.
2010 Mathematics Subject Classification. 22F50; 22D40; 37A25; 57N05.
Key words and phrases. Character variety, ergodicity, simple closed curves.
1. Introduction

Let $K = \text{SU}(3)$ and consider the commutator map
\[ \kappa : K \times K \to K \]
\[ (a, b) \mapsto [a, b] := aba^{-1}b^{-1}. \]

Let $F_2$ be the free group of two generators. Then $\text{Aut}(F_2)$ acts on the set $\text{Hom}(F_2, K) \approx K \times K$. The quotient of $\text{Out}(F_2) \cong \text{GL}(2, \mathbb{Z})$ acts on the quotient $\text{Hom}(F_2, K)/\text{Inn}(K)$, where $\text{Inn}(K)$ is the group of inner automorphisms of $K$. The index two subgroup $\text{Out}^+(F_2)$ corresponding to $\text{SL}(2, \mathbb{Z}) \subset \text{GL}(2, \mathbb{Z})$ acts on $\mathcal{M}_C := \kappa^{-1}(C)/\text{Inn}(K)$, for each conjugacy class $C \subset K$.

This paper is part of the general program to understand the dynamics of the action of mapping class groups and automorphism groups of free groups on moduli spaces of flat bundles (or, equivalently, character varieties). Suppose that $K$ is a compact Lie group. Suppose that $\Sigma := \Sigma_{g,n}$ is an orientable surface of genus $g > 0$ with boundary of $n$ disjoint circles. Let $\mathcal{M}_C(K)$ be its $K$-character variety, with the symplectic structure $\Omega$ as defined in [Gol84, Gol97], where $C$ is a collection of $n$ conjugation classes. Let $\Gamma$, the mapping class group of $\Sigma$, acts on $\mathcal{M}_C(K)$ preserving $\Omega$. Pickrell-Xia [PX02, PX03] established $\Gamma$-ergodicity of the symplectic measure $\mu$ of $\Omega$ for $g > 1$ or $n > 2$ with $g > 0$. This was previously proved by Goldman [Gol97] when the simple factors of $K$ are locally isomorphic to $\text{SU}(2)$. In this paper, we study the case of $\Sigma := \Sigma_{1,1}$. Then $\Gamma \cong \text{SL}(2, \mathbb{Z})$ [FM12].

**Theorem 1.1.** Let $K = \text{SU}(3)$ and $\Sigma = \Sigma_{1,1}$. The $\Gamma$-action is ergodic on $\mathcal{M}_C(K)$ with respect to the measure $\mu$.

We prove this theorem along the lines of the main results in [GX11], showing that the traces of simple closed curves generate the coordinate ring of $\mathcal{M}_C$. This latter ring-theoretic statement for $K = \text{SU}(3)$ follows from results in [Law07].

**Acknowledgements.** Goldman and Lawton were partially supported by U.S. National Science Foundation grants: DMS 1107452, 1107263, 1107367 1309376 “RNMS: Geometric structures And Representation varieties” (the GEAR Network). Lawton acknowledges support from U.S. National Science Foundation grants: DMS 1309376, and also DMS 0932078000 while he was in residence at the Mathematical Sciences Research Institute in Berkeley, California during the Spring 2015 semester. He was also partially supported by a Simons Foundation Collaboration grant. Xia was partially supported by the Ministry of Science and
Technology Taiwan with grants 103-2115-M-006-007-MY2, 105-2115-M-006-006, 106-2115-M-006-008 and 107-2115-M-006-009 and the National Center for Theoretical Sciences, Taiwan.

**Notation and terminology**

Denote \( \omega := e^{\frac{2\pi i}{3}} \) and the identity transformation by \( \mathbb{I} \). Let \( G \) be a group. Denote the inner automorphism induced by \( A \in G \) by:

\[
G \xrightarrow{\text{Inn}(A)} G \quad \quad B \mapsto ABA^{-1}
\]

The set of conjugacy classes in \( G \) equals the quotient \( G/\text{Inn}(G) \), and we denote the image of a subset \( S \subseteq G \) under the quotient map by \( [S] \).

Denote the centralizer of \( A \) in \( G \) by:

\[
G_A := \text{Fix}(\text{Inn}(A)) < G,
\]

where \( \text{Fix}(S) \) is the set of fixed points of \( S \subseteq G \).

Define the commutator map of \( G \):

\[
G \times G \xrightarrow{\kappa} G \quad \quad (A, B) \mapsto [A, B] := ABA^{-1}B^{-1}.
\]

Suppose further that \( G \) is a Lie group with Lie algebra \( g \), and denote the adjoint representation of \( G \) on \( g \) by \( \text{Ad} \). Identify \( g \) with the Lie algebra of *right-invariant vector fields* on \( G \); then for any right-invariant vector field \( X \in g \), the element \( \text{Ad}(a)(X) \) equals the image of \( X \) under left-multiplication by \( a \). Denote the centralizer of \( a \) in \( g \) by:

\[
g_a := \text{Fix}(\text{Ad}(a)) \subseteq g.
\]

Denote the trace of a matrix \( a \) by \( \text{Tr}(a) \) and the \( \lambda \)-eigenspace of a matrix \( a \) by \( \text{Eig}_\lambda(a) \), for a scalar \( \lambda \in \mathbb{C} \).

If \( (M, \Omega) \) is a symplectic manifold, and \( M \xrightarrow{f} \mathbb{R} \) is a smooth function, denote its *Hamiltonian vector field* by \( \text{Ham}(f) \). We denote the tangent space to a smooth manifold \( M \) at a point \( p \in M \) by \( T_pM \).

When we say a set is *closed*, we mean it to be closed in the classical topology.

**2. Character varieties and the mapping class group**

We fix a base point on the boundary of \( \Sigma := \Sigma_{1,1} \). The fundamental group \( \pi = \pi_1(\Sigma) \cong \mathbb{F}_2 \) is generated by homotopy classes of oriented based loops \( \alpha \) and \( \beta \) and they (could) intersect at the basepoint. We often do not distinguish elements in \( \pi \) and the oriented pointed-loops on \( \Sigma \).
We write
\[ \pi = \langle \alpha, \beta, \sigma | \kappa(\alpha, \beta) = \sigma \rangle, \]
where \( \sigma \) is the boundary element. In this way, we have
\[ R := \text{Hom}(\pi, K) \cong K \times K \quad \text{and} \quad \mathcal{M} := R/K, \]
where the \( K \)-action is by conjugation.

In our case, we have only one boundary circle and we let \( C \subseteq K \) be a conjugacy class and \( c \in C \). Then the relative representation variety and character variety are
\[ R_c := \text{Hom}_C(\pi, K) := \kappa^{-1}(c) \quad \text{and} \quad \mathcal{M}_c := R_c(K)/K_c. \]
Again, the \( K_c \)-action is by conjugation. In this way, a representation \( \rho \in R_c \) corresponds to \((a, b) \in K \times K\) such that \( \kappa(a, b) = c \). Notice that \( \mathcal{M}_c \) is usually and equivalently defined as
\[ \mathcal{M}_c = \kappa^{-1}(C)/K. \]

The space \( \mathcal{M}_c \) has a natural symplectic structure \( \Omega \) [Gol84, Gol97].

The diffeomorphism group of \( \Sigma \) (fixing the boundary, hence, also the base point) acts on \( \pi \) and this action descends to a \( \Gamma \)-action on \( \pi \), fixing the conjugacy class of \( \sigma \). This further induces an action:
\[ \mathcal{M}_c \times \Gamma \longrightarrow \mathcal{M}_c \]
\[ ([\rho], \gamma) \longmapsto [\rho \circ \gamma]. \]
The \( \Gamma \)-action leaves \( \Omega \) and \( \mu \) invariant [Gol97]. For any oriented simple closed curve \( \alpha \) on \( \Sigma \), denote by \( \tau_\alpha \) the Dehn twist along \( \alpha \). The mapping class group \( \Gamma \) contains all Dehn twists; indeed the Dehn twists \textit{generate} \( \Gamma \) (although we do not need this fact). Denote by \( S \) the set of homotopy classes of oriented simple closed curves on \( \Sigma \).

3. Compact Lie groups

This section reviews well known facts that are used in the proofs. \textit{Generic elements} are introduced; these are regular elements which are dense in their maximal tori, and provide nontrivial dynamics.

3.1. Regularity. Suppose \( M \) is an irreducible algebraic set over \( \mathbb{R} \) or \( \mathbb{C} \) and \( M^s \subseteq M \) its singular locus. Then \( U = M \setminus M^s \) is a smooth manifold and Zariski dense in \( M \). The smooth structure on \( U \) gives rise to the Lebesgue measure class on \( U \) and on \( M \), by assigning \( M^s \) to be a null set. We shall always mean this class, which coincides with the measure class discussed in the introduction [Hue95].
Let $G$ be a linear semisimple algebraic group over $\mathbb{C}$ of rank $r$ and $K < G$ a maximal compact subgroup. The corresponding Lie algebras are denoted by $\mathfrak{k}$ and $\mathfrak{g}$, respectively.

Recall that an element $a \in K$ is regular if $K_a$ has dimension $r$. In general, $\dim(K_a) \geq r$, and $K_a$ will contain a maximal torus of $K$. An element $a \in K$ is regular if and only if $K_a$ is a maximal torus (that is, a Cartan subgroup) in $K$.

Recall that an action on a topological space is minimal if every orbit is dense. If $a \in K$, denote the Haar measure on $K_a$ by $\mu_{K_a}$ and the pushforward $L_b \cdot \mu_a$ under left-multiplication $K_a \xrightarrow{L_b} bK_a$ by $\mu_{ba}$.

### 3.2. Genericity

In general regularity is too weak a notion for dynamical complexity. We introduce a notion of genericity which is more useful for constructing nontrivial dynamics.

Let $(a, b) \in K \times K$. Then the cyclic group $\langle a \rangle$ acts on the left-coset $bK_a$ by:

$$b\zeta \mapsto b\zeta a^n$$

where $n \in \mathbb{Z}$ and $\zeta \in K_a$.

**Proposition 3.1.** Let $a \in K$ be a regular element. For any $b \in K$, the following conditions are equivalent:

- the cyclic group $\langle a \rangle < K_a$ is Zariski dense in $K_a$;
- the cyclic group $\langle a \rangle < K_a$ is dense in $K_a$;
- the action of $\langle a \rangle$ on $bK_a$ is minimal;
- the action of $\langle a \rangle$ on $(bK_a, \mu_{ba})$ is ergodic.

In this case, we say that $a$ is generic. The proof of Proposition 3.1 uses standard facts about compact abelian Lie groups, such as:

**Lemma 3.2.** A cyclic subgroup of $K_a$ is dense in the classical topology if and only if it is dense in the Zariski topology.

**Proof.** Any set that is classically dense is also Zariski dense since the Zariski topology is coarser than the classical topology. We now show the converse. Clearly the cyclic group $\langle a \rangle \subset K_a$, and its Zariski-closure is also an abelian subgroup. Its closure in the classical topology

$$H := \langle a \rangle$$

is a closed abelian subgroup of $K_a$. Now every compact linear group is Zariski-closed (See Onishchik-Vinberg [OV90], §4.4, Theorem 5, pp.133–134). Hence $H$ is Zariski closed in $K_a$. Since $\langle a \rangle$ is Zariski dense in $K_a$ and $H \supseteq \langle a \rangle$, $H = K_a$. 

$\square$
Proof of Proposition 3.1. The proof now follows from Lemma 3.2 and the fact that dense subgroups of the torus act minimally (see Katok-Hasselblatt [KH95, §1.4 (p.28)]) and ergodically ([KH95, Proposition 4.2.2 (p.147)], or Walters [Wal82, Theorem 1.9 (p.30)]. □

4. Infinitesimal transitivity and Hamiltonian flows

Let $M$ be a symplectic manifold and $f : M \to \mathbb{R}$ a smooth function. Denote by $\text{Ham}(f)$ the associated Hamiltonian vector field.

Definition 4.1. Let $M$ be a manifold and $\mathcal{F}$ be a set of real smooth $\mathbb{R}$-functions on $M$ such that at $x \in M$, the differentials $df(x)$, for $f \in \mathcal{F}$, span the cotangent space $T_x^*(M)$. Then $\mathcal{F}$ is said to be infinitesimally transitive at $x$. $\mathcal{F}$ is infinitesimally transitive on $M$ if $\mathcal{F}$ is infinitesimally transitive at all $x \in M$.

Proposition 4.2. Let $M$ be a connected symplectic manifold and $\mathcal{F}$ be infinitesimally transitive on $M$. Then the group $\mathcal{H}$ generated by the Hamiltonian flows $\text{Ham}(f)$ of the vector fields $\text{Ham}(f)$, for $f \in \mathcal{F}$, acts transitively on $M$.

Proof. See Lemma 3.2 in [GX11]. □

We now briefly review results of Goldman [Gol86], describing the flows generated by the Hamiltonian vector fields by simple closed curves on $\Sigma$. In this case, the local flow of this vector field on $\mathcal{M}_c$ lifts to a flow on the representation variety $R_c$. Furthermore this flow admits a simple description [Gol86] as follows.

4.1. Invariant functions and flows on groups. We suppose that the adjoint representation $\text{Ad}$ preserves a nondegenerate symmetric bilinear form $\langle \cdot , \cdot \rangle$ on $g$. In the case $G = \text{SU}(n)$, this will be $\langle X,Y \rangle := \text{Tr}(XY)$.

Let $G \to \mathbb{R}$ be a function invariant under the inner automorphisms $\text{Inn}(G)$. Following [Gol86], we describe how $t$ determines a way to associate to every element $x \in G$ a one-parameter subgroup

$$\zeta^t(x) = \exp(tF(x))$$

centralizing $x$. Given $t$, define its variation function $G \xrightarrow{F} g$ by:

$$\langle F(x), v \rangle = \left. \frac{d}{dt} \right|_{t=0} t(x \exp(tv))$$

for all $v \in g$. Invariance of $t$ under $\text{Inn}(G)$ implies that $F$ is $G$-equivariant:

$$F(gxg^{-1}) = \text{Ad}(g)F(x).$$
Taking \( g = x \) implies that the one-parameter subgroup
\[
(1) \quad \zeta^t(x) := \exp(tF(x))
\]
lies in the centralizer of \( x \in G \).

Intrinsically, \( F(x) \in \mathfrak{g} \) is dual (by \( \langle \cdot, \cdot \rangle \)) to the element of \( \mathfrak{g}^* \) corresponding to the left-invariant 1-form on \( G \) extending the covector \( df(x) \in T_x^*(G) \).

4.2. Invariant functions and centralizing one-parameter subgroups. If \( \alpha \in S \) is an oriented homotopy class of based loops, then \( t_\alpha \), the trace function of \( \alpha \) on \( \mathcal{M}_c \), is defined as:

\[
\text{Hom}(\pi, G) \xrightarrow{t_\alpha} \mathbb{C} \\
\rho \mapsto \text{Tr}\left(\rho(\alpha)\right).
\]

Since the function \( G \xrightarrow{\text{Tr}} \mathbb{C} \) is \( \text{Inn}(G) \)-invariant, \( t_\alpha \) defines a function (also denoted by \( t_\alpha \)) on \( \mathcal{M}_c \).

Denote by \( \mathcal{R} = \mathbb{C}[\mathcal{M}] \) and \( \mathcal{R}_c = \mathbb{C}[\mathcal{M}_c] \) the coordinate rings of \( \mathcal{M} \) and \( \mathcal{M}_c \), respectively. Let

\[
\mathcal{S} = \{ t_\alpha \in \mathcal{R} : \alpha \in S \}, \quad \mathcal{S}_c = \{ t_\alpha \in \mathcal{R}_c : \alpha \in S \}.
\]

Let \( \alpha \in S \) and \( \Sigma|\alpha \) denote the surface-with-boundary obtained by splitting \( \Sigma \) along \( \alpha \). The boundary of \( \Sigma|\alpha \) has two components, denoted by \( \alpha_{\pm} \), corresponding to \( \alpha \). The original surface \( \Sigma \) may be reconstructed as a quotient space under the identification of \( \alpha_- \) with \( \alpha_+ \).

The fundamental group \( \pi \) can be reconstructed from the fundamental group \( \pi_1(\Sigma|\alpha) \) as an HNN-extension:

\[
(2) \quad \pi \cong \left( \pi_1(\Sigma|\alpha) \ast \langle \beta \rangle \right) / \left( \beta \alpha_+ \beta^{-1} = \alpha_+ \right).
\]

A representation \( \rho \) of \( \pi \) is determined by:

- the restriction \( \rho' \) of \( \rho \) to the subgroup \( \pi_1(\Sigma|\alpha) \subset \pi \), and
- the value \( \beta' = \rho(\beta) \)

which satisfies:

\[
(3) \quad \beta' \rho'(\alpha_-) \beta'^{-1} = \rho'(\alpha_+).
\]

Furthermore any pair \( (\rho', \beta') \) where \( \rho' \) is a representation of \( \pi_1(\Sigma|\alpha) \) and \( \beta' \in G \) satisfies (3) determines a representation \( \rho \) of \( \pi \).

The twist flow \( \xi^t_\alpha \), for \( t \in \mathbb{R} \) on \( \text{Hom}(\pi, \text{SU}(3)) \), is then defined as follows:

\[
(4) \quad \xi^t_\alpha(\rho) : \gamma \mapsto \begin{cases} 
\rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\
\rho(\beta)\zeta^t(\rho(\alpha_-)) & \text{if } \gamma = \beta.
\end{cases}
\]
where $\zeta^t$ is defined in (1). This flow covers the flow generated by $\text{Ham}(t_\alpha)$ on $\mathcal{M}_c$ (See [Gol86]).

4.3. Infinitesimal transitivity. Let $G = \text{SL}(3, \mathbb{C})$. Choosing $\alpha, \beta \in S$ as in Section 4.2, the structure of the $\text{SL}(3, \mathbb{C})$-character variety of the 1-holed torus is determined in [Law06, Law07].

**Theorem 4.3 ([Law07]).** Let $\mathcal{M} = \text{SL}(3, \mathbb{C})^2/\text{SL}(3, \mathbb{C})$. Then $\mathbb{C}[\mathcal{M}]$ is generated by

$$\mathcal{G} = \{t_\alpha, t_\beta, t_{\alpha\beta}, t_{\alpha^{-1}}, t_{\beta^{-1}}, t_{\beta\alpha^{-1}}, t_{\alpha^{-1}\beta^{-1}}, t_{\kappa(\alpha, \beta)}\}.$$ 

**Corollary 4.4.** For the case of $\Sigma_{1,1}$, $\mathcal{S}_c$ is infinitesimally transitive on $\mathcal{M}_c$. In other words, $\mathbb{C}[\mathcal{M}_c]$ is generated by

$$\mathcal{G} = \{t_\alpha, t_\beta, t_{\alpha\beta}, t_{\alpha^{-1}}, t_{\beta^{-1}}, t_{\beta\alpha^{-1}}, t_{\alpha^{-1}\beta^{-1}}\}.$$ 

**Proof.** Let $\alpha, \beta \in S$. Then $\alpha^i \beta^j \in S$ for $i, j \in \{-1, 0, 1\}$. Notice that $t_{\kappa(\alpha, \beta)}$ is a function of $c$ which is fixed for $\mathcal{M}_c$. The result then follows from Theorem 4.3. \qed

5. The Dehn Twists

Let $\alpha \in S$ and $\tau_\alpha \in \Gamma$ be the corresponding Dehn twist. The fundamental group $\pi$ can be reconstructed from the fundamental group $\pi_1(\Sigma|\alpha)$ as an HNN-extension as in (2). Then $\tau_\alpha$ induces the automorphism $(\tau_\alpha)_*: \text{Aut}(\pi)$ defined by:

$$(\tau_\alpha)_*: \gamma \mapsto \begin{cases} \gamma & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\ \gamma \alpha & \text{if } \gamma = \beta. \end{cases}$$

This further induces the map $(\tau_\alpha)^*$ on $\text{Hom}(\pi, G)$ mapping $\rho$ to:

$$(\tau_\alpha)^*(\rho): \gamma \mapsto \begin{cases} \rho(\gamma) & \text{if } \gamma \in \pi_1(\Sigma|\alpha) \\ \rho(\gamma)\rho(\alpha)^{-1} & \text{if } \gamma = \beta. \end{cases}$$

(See [Gol86]).

5.1. Dehn twists and Hamiltonian twist flows. Let $a := \rho(\alpha)$ and $b = \rho(\beta)$. Then

$$R_c = \{\rho \in \text{Hom}(\pi, K) : \kappa(\rho(\alpha), \rho(\beta)) = c\} = \{(a, b) \in K \times K : \kappa(a, b) = c\}.$$ 

Let

$$H(a, b) := \{a\} \times bK_a, \quad H'(a, b) := aK_b \times \{b\}.$$ 

**Proposition 5.1.** If $(a, b) \in R_c$, then $H(a, b), H'(a, b) \subseteq R_c$. 

Proof. Suppose \( t \in K \). Then \( at = ta \) and \( t^{-1}a^{-1} = a^{-1}t^{-1} \). Then
\[
 k(a, bt) = a(bt)a^{-1}(bt)^{-1} = abta^{-1}t^{-1}b^{-1} = aba^{-1}b^{-1} = k(a, b) = c.
\]
Hence \( (a, bt) \in R_c \). The proof of \( aK_b \times \{b\} \subseteq R_c \) is similar. \( \square \)

**Proposition 5.2.** If \( (a, b) \in R_c \), then the Hamiltonian flows of the vector fields \( \text{Ham}(t_\alpha) \) and \( \text{Ham}(t_\beta) \) preserve \( [H(a, b)] \) and \( [H'(a, b)] \), respectively.

Proof. This follows from (4) and by exchanging \( \alpha \) and \( \beta \). \( \square \)

**Corollary 5.3.** If \( a \) is generic, then \( \langle \tau_\alpha \rangle \) acts ergodically on \( H(a, b) \). If \( b \) is generic, then \( \langle \tau_\beta \rangle \) acts ergodically on \( H'(a, b) \).

Proof. By (5), \( \tau_\alpha(a, b) \in H(a, b) \) and \( \tau_\beta(a, b) \in H'(a, b) \) The corollary then follows from Proposition 3.1. \( \square \)

6. The case of \( K = SU(3) \)

Let \( K = SU(3) \). The classification of conjugacy classes of \( K \) can be described in terms of the trace function
\[
 K \xrightarrow{\text{Tr}} \mathbb{C}.
\]

Let \( \Delta = \text{Tr}(K) \).

If \( \zeta_1, \zeta_2, \zeta_3 \in \mathbb{C} \) are the eigenvalues of \( a \in K \), then they satisfy
\[
|\zeta_1| = |\zeta_2| = |\zeta_3| = 1 \text{ and } \zeta_1\zeta_2\zeta_3 = 1.
\]
The coefficients of the character polynomial \( \chi_a \) are:
\[
1 = 1
\]
\[
\zeta_1 + \zeta_2 + \zeta_3 = \text{Tr}(a)
\]
\[
\zeta_2\zeta_3 + \zeta_3\zeta_1 + \zeta_1\zeta_2 = \overline{\text{Tr}(a)}
\]
\[
\zeta_1\zeta_2\zeta_3 = 1.
\]

Therefore the characteristic polynomial is:
\[
\chi_A(\lambda) = \lambda^3 - z\lambda^2 + \bar{z}\lambda - 1
\]
where \( z = \text{Tr}(a) \in \mathbb{C} \). Furthermore (6) is equivalent to the condition:
\[
|z|^2 - 8\Re(z^3) + 18|z|^2 - 27 \leq 0
\]
and this real polynomial condition exactly describes the image \( \Delta \subseteq \mathbb{C} \).

The traces of central elements are the vertices \( 3, 3\omega, 3\bar{\omega} \) of \( \Delta \). The trace of a regular element of order three is the center 0 of \( \Delta \). The traces of elements of order two are \(-1, -\omega, -\bar{\omega} \), the midpoints of the edges of \( \partial \Delta \).
Proposition 6.1. The map $\text{Tr}$ is a local submersion at almost all points of $\Delta$.

Proof. The map $\text{Tr}$ is smooth. Hence, by Sard’s theorem, almost all points of $\mathbb{C}$ are regular values of $\text{Tr}$ [GP10, §1.7]. Hence $\text{Tr}$ is a local submersion at almost all points of $\Delta$. \hfill \Box

Remark 6.2. It is not difficult to show that $\text{Tr}$ has full rank in the interior of $\Delta$, but we only need Proposition 6.1. For a general discussion of $\Delta$ and Weyl chambers, see [DK00].

Proposition 6.3. The image $\text{Tr}(N)$ of the subset $N \subseteq K$ of generic elements is conull in $\Delta$.

Proof. Let

$$U = \{ (\alpha, \beta) \in \mathbb{R} \times \mathbb{R} : \alpha, \beta, \alpha/\beta \in \mathbb{R} \setminus \mathbb{Q} \}$$

Let $\Delta' \subset \Delta$ be the image of $U$ under the mapping

$$\mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{C}$$

$$(\alpha, \beta) \longmapsto e^{2\pi i \alpha} + e^{2\pi i \beta} + e^{-2\pi i (\alpha + \beta)}$$
Then $\Delta'$ is conull in $\Delta$ and $\text{Tr}^{-1}(\Delta') \subseteq N$. \hfill \Box

7. Central fibers of $\kappa$

7.1. The abelian representations. The fiber $R_{\kappa} = \kappa^{-1}(I)$ consists of commuting pairs $(a, b)$. In this case, $a$ and $b$ lie in a maximal torus $T$. Hence $\mathcal{M}_{k} \cong (T^2 \times T^2)/W$, where $W$ is the Weyl group of $K$ acting diagonally (see [FL14] for more discussion of these abelian character varieties).

7.2. The non-abelian cases.

Proposition 7.1. If $k = \omega I$ where $\omega \neq 1$, then $\mathcal{M}_{k}$ consists of a single point. Specifically, if $(a, b) \in R_{k}$, then there exists $g \in K$ such that
\[
a = ga_0g^{-1}, \quad b = gb_0g^{-1}
\]
where
\[
a_0 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{bmatrix}, \quad b_0 := \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.
\]

Proof. Suppose $(a, b) \in R_{k}$, that is,
\[
aba^{-1}b^{-1} = \omega I.
\]
We first prove:

Lemma 7.2. $a^3 = b^3 = I$

Proof of Lemma 7.2. By (8) and taking traces,
\[
\text{Tr}(a) = \text{Tr}(\omega bab^{-1}) = \omega \text{Tr}(a) \quad \text{and} \quad \text{Tr}(a^{-1}) = \text{Tr}(\omega b^{-1}a^{-1}b) = \omega \text{Tr}(a^{-1}).
\]
Hence $\omega \neq 1$ implies that $\text{Tr}(a) = \text{Tr}(a^{-1}) = 0$. Now apply the Cayley-Hamilton theorem:
\[
a^3 - I = a^3 - \text{Tr}(a)a^2 + \text{Tr}(a^{-1})a - \text{Det}(a)I = 0.
\]
The same argument applied to $b = \omega a^{-1}ba$ implies $b^3 = I$, as claimed. \hfill \Box

Returning to the proof of Proposition 7.1, Lemma 7.2 implies that $a = \text{Inn}(g)a_0$ for some $g \in K$, since $a \neq I$.

We claim that $b = \text{Inn}(g)b_0$. For convenience assume that $a = a_0$; then we show that (8) implies that $b$ is the permutation matrix $b_0$ defined in (7).

Recall that $\text{Eig}_\lambda(a)$ is the $\lambda$-eigenspace of $a$. We claim that (8) implies that
\[
b(\text{Eig}_\lambda(a)) = \text{Eig}_{\omega \lambda}(a).
\]
To see this, rewrite (8) as:

\[(10) \quad ab = \omega ba\]

Suppose that \(v \in \text{Eig}_\lambda(a)\), that is,

\[av = \lambda v\]

so applying (10),

\[a(bv) = \omega bav = \omega \lambda (bv)\]

whence \(bv \in \text{Eig}_{\omega \lambda}(a)\), as claimed.

Since \(a\) is the diagonal matrix \(a_0\) defined by (7), the lines \(\text{Eig}_1(a), \text{Eig}_\omega(a),\) and \(\text{Eig}_{\bar{\omega}}(a)\) are the three coordinate lines in \(\mathbb{C}^3\). Thus (9) implies that \(b = b_0\), concluding the proof of Proposition 7.1. \(\square\)

In particular, both \(a\) and \(b\) have orders 3. Since \(\kappa(a, b)\) has order 3 and is central, the subgroup \((a, b) \subset K\) is a nonabelian group of order 27, a nontrivial central extension of \(\mathbb{Z}/3 \oplus \mathbb{Z}/3\) by \(\mathbb{Z}/3\).

8. Ergodicity

Let \(K\) be any compact Lie group. Each tangent space \(T_a K\) identifies with the Lie algebra \(\mathfrak{k}\) of right-invariant vector fields: namely, a tangent vector \(v \in T_a K\) identifies with the right-invariant vector \(X \in \mathfrak{k}\) such that \(X(a) = v\). In this way, the differential of the commutator map \(\kappa\) at \((a, b) \in K \times K\) identifies with the linear map (see \([Gol84, PX02]\)):

\[
D\kappa_{(a,b)} : \mathfrak{k} \oplus \mathfrak{k} \longrightarrow \mathfrak{k} \\
(X, Y) \longmapsto \text{Ad}(ba)((\text{Ad}(b^{-1}) - \mathbb{I})X \\
+ (\mathbb{I} - \text{Ad}(a^{-1}))Y).
\]

From this formula, the following proposition holds:

**Proposition 8.1.** \(\kappa\) is smooth at \((a, b)\) if and only if \(\mathfrak{k}_a \cap \mathfrak{k}_b = 0\).

For the rest of the section, let \(K = \text{SU}(3)\) which comes with the standard representation \(\Pi\) on \(\mathbb{C}^3\). An element in \((a, b) \in R_c\) corresponds to a representation \(\rho\) of \(\pi\) (see Section 2). Hence \(\rho \circ \Pi\) is a representation of \(\pi\). Denote by \(\mathcal{M}_c^i \subseteq \mathcal{M}_c\) the subspace of irreducible representation classes.

8.1. Relation to the moduli space of \(K\)-bundles. If we fix a complex structure on \(\Sigma\), then we can construct the coarse moduli space \(\mathcal{N}^{ss}\) of semi-stable parabolic \(K\)-bundles [BR89] on \(\Sigma\), endowed with a complex structure. \(\mathcal{N}^{ss}\) contains the subspace \(\mathcal{N}^s\) of stable parabolic \(K\)-bundles.
Proposition 8.2. The set of smooth points of $\mathcal{M}_c$ is a connected manifold.

Proof. There is a homeomorphism $\mathcal{M}_c \cong \mathcal{N}^{ss}$, restricting to a diffeomorphism $\mathcal{M}_c^i \cong \mathcal{N}^s$ (Theorem 1, [BR89]). The moduli space $\mathcal{N}^{ss}$ is irreducible and contains $\mathcal{N}^s$ as an open subvariety (Theorem II, $\mathcal{M}_c$ [BR89]). Hence $\mathcal{N}^s$ is open and connected. Hence $\mathcal{M}_c^i$ is open and connected. Proposition 8.1 implies that a point $[\rho] = [(a,b)] \in \mathcal{M}_c$ is a smooth point if and only if $\rho$ is irreducible, i.e. $\mathcal{M}_c^i$ is also the set of smooth points of $\mathcal{M}_c$. The proposition follows. $\square$

For almost every conjugacy class $c$, the action $\mathcal{M}_c \times \Gamma \rightarrow \mathcal{M}_c$ is ergodic [PX02]. This section proves that this is true for all $c$.

Let $c \in K$. Up to conjugation,

$$c = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix}. $$

For $(a,b) \in R_c$, we have the natural map

$$\iota : K_a \rightarrow H(a,b), \quad \iota(t) = (a, bt).$$

Let $P_2 : K \times K \rightarrow K$ be the projection to the second factor. An element $\rho \in R_c$ corresponds to a pair $(a,b) \in K \times K$. Let

$$T = \text{Tr} \circ P_2 : R_c \rightarrow \Delta \quad \text{and} \quad T := \iota \circ T : K_a \rightarrow \Delta.$$

Proposition 8.3. Let $c \notin Z(K)$. Then $(a,b) \in R_c$ exists such that:

1. $(a,b)$ is a smooth point;
2. $T$ is a submersion at $(a,b)$.

Proof. Let $(a,b) \in R_c$ such that

$$(11) \quad a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix}. $$

Then

$$c = \kappa(a,b) = \begin{bmatrix} b_2 & 0 & 0 \\ 0 & b_1 & 0 \\ 0 & 0 & b_2 \end{bmatrix}. $$

This formula for $c$ implies that $\kappa$ is onto $K$.

Remark 8.4. A result of Goto [Got49] states that for any compact semi-simple Lie group $K$, $K \times K \xrightarrow{\kappa} K$ is surjective. Hence $R_c \neq \emptyset$ for all $c \in K$. 
Note that $a$ is regular. For (1), there are three cases for $b \in Z(K)$, $b_1 = b_2 \neq b_3$ (and its permutation variation) and $b$ being regular.

If $b \in Z(K)$, then $\kappa(a, b) = I = c$ and this violates our hypothesis of $c \neq Z(K)$.

If $b$ is regular, then $t \in \mathfrak{t}_b$ implies

$$t = \begin{bmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{bmatrix}, \quad t_1 + t_2 + t_3 = 0. \quad (13)$$

Then

$$(\text{Ad}(a) - I)t = \begin{bmatrix} t_2 - t_1 & 0 & 0 \\ 0 & t_3 - t_2 & 0 \\ 0 & 0 & t_1 - t_3 \end{bmatrix}.$$}

Hence if $t \in \mathfrak{t}_a$, then $(\text{Ad}(a) - I)t = 0$. This implies $\mathfrak{t}_a \cap \mathfrak{t}_b = 0$. Hence, by Proposition 8.1, we conclude that $\kappa$ is regular at $(a, b)$.

If $b_1 = b_2 \neq b_3$ and $t \in \mathfrak{t}_b$, then

$$t = \begin{bmatrix} t_{11} & t_{12} & 0 \\ t_{21} & t_{22} & 0 \\ 0 & 0 & t_{33} \end{bmatrix}, \quad t_1 + t_2 + t_3 = 0.$$}

Then

$$(\text{Ad}(a) - I)t = \begin{bmatrix} t_{22} - t_{11} & -t_{12} & t_{21} \\ -t_{21} & t_{33} - t_{22} & 0 \\ t_{12} & 0 & t_{11} - t_{33} \end{bmatrix}.$$}

Hence if $t \in \mathfrak{t}_a$, then $(\text{Ad}(a) - I)t = 0$. This implies $t = 0$. Hence $\mathfrak{t}_a \cap \mathfrak{t}_b = 0$. By Proposition 8.1, $\kappa$ is regular at $(a, b)$. We conclude that in all cases $(a, b) \in R_c$ is a smooth point.

Notice that $H(a, b) \subseteq R_c$. We consider $T$ restricted to $H(a, b)$. Let

$$p = \begin{bmatrix} 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{bmatrix}. $$}

Then $a = pqp^{-1}$. The element $t \in K_a$ has the form $t = pt_b p^{-1}$, where $t_b \in K_b$ is diagonal. Then

$$T(t) = \text{Tr}(bt) = \frac{1}{3} \text{Tr}(b) \text{Tr}(t).$$

By Proposition 6.1, $T$ is a local submersion for all almost all $t \in \Delta$ unless $\text{Tr}(b) = 0$. However $\text{Tr}(b) = 0$ implies $b \in Z(K)$ and $c = I$, a contradiction. Hence $\text{Tr}(b) \neq 0$ and $T$ is a local submersion for almost all $t$. \qed
Corollary 8.5. \( T \) is a local submersion for almost all points in \( H(a, b) \).

Proof. Since \( DT = DT \circ D_t \), \( DT \) being surjective implies \( DT_{(a, b)} \) is surjective. \( \square \)

Corollary 8.6. There is a conull set \( V \subset R_c \) such that \( b \) is generic for almost all \((a, b) \in V\).

Proof. The subset containing points at which a map is locally submersive is Zariski open. Hence, by Proposition 8.3, there exists smooth and Zariski open \( V \subset R_c \) such that \( T|_V \) is a submersion. Let \( Q \subset R_c \) be the smooth part. By Proposition 8.2, \( Q \) is connected, hence, irreducible. Since Zariski open subset of a smooth irreducible variety is conull in the Lebesgue class, \( V \) is conull. The map \( T|_V \) is a fibration over an open domain of \( \Delta \). The corollary follows from Proposition 6.3. \( \square \)

Corollary 8.7. Suppose \( c \notin Z(K) \). Suppose \( \beta \in S \) and \( \phi : M_c \rightarrow \mathbb{R} \) is a \( \mu \)-measurable function. If \( \phi \) is \( \tau_\beta \)-invariant, then \( \phi \) is \( \text{Ham}(t_\beta) \)-invariant.

Proof. By Proposition 8.3, \( R_c \) contains a smooth point \( (a, b) \) with \( b \) generic. By Proposition 8.2 and Corollary 8.6, \( b \in K \) is generic for almost all \((a, b) \in R_c \). Hence \( b \in K \) is generic for almost all \([(a, b)] \in M_c \). By Proposition 3.1, 5.1 and Corollary 5.3, \( \tau_\beta \)-orbit is dense in \( H'(a, b) \) and \( \phi \) is \( \text{Ham}(t_\beta) \)-invariant. \( \square \)

With the notations we have adopted, we restate Theorem 1.1 as

Theorem 8.8. The action \( M_c \times \Gamma \rightarrow M_c \) is ergodic.

Proof. Suppose that \( c = I \). Identifying \( \mathbb{R}^2 \) with its dual \( (\mathbb{R}^2)^* \), the group \( SL(2, \mathbb{Z}) \) has the standard dual linear action on \( \mathbb{R}^2 \) which induces the diagonal action on \( T^2 \times T^2 \). This \( SL(2, \mathbb{Z}) \)-action is known to be ergodic because \( SL(2, \mathbb{Z}) \) contains hyperbolic elements, meaning that the eigenvalues of these elements do not have absolute value 1 [BS15, §4].

There is an isomorphism [FM12] \( \iota : \Gamma \rightarrow SL(2, \mathbb{Z}) \). By Section 7.1, \( M_c \cong (T^2 \times T^2)/W \). The \( \Gamma \)-action on \( M_c \) lifts to an action on \( T^2 \times T^2 \). Moreover, this \( \Gamma \)-action is equivariant with respect to \( \iota \). Hence the \( \Gamma \)-action on \( M_c \) is ergodic.

Suppose that \( c \in Z(K) \) and \( c \neq I \), then \( M_c \) is a single point by Proposition 7.1 and the statement is trivially true.

Suppose \( c \notin Z(K) \). Recall that \( H \) is the group generated by all Hamiltonian flows \( t_\beta \) where \( \beta \in S \). Let \( \phi : M_c \rightarrow \mathbb{R} \) be a \( \Gamma \)-invariant \( \mu \)-measurable function. By Corollary 8.7, \( \phi \) is \( H \)-invariant. By Corollary 4.4 and Proposition 4.2, \( \phi \) is constant almost everywhere. Our theorem follows. \( \square \)
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