Cahn–Hilliard equation with dynamic boundary conditions and mass constraint on the boundary

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Abstract

The well-known Cahn–Hilliard equation entails mass conservation if a suitable boundary condition is prescribed. In the case when the equation is also coupled with a dynamic boundary condition, including the Laplace–Beltrami operator on the boundary, the total mass on the inside of the domain and its trace on the boundary should be conserved. The new issue of this paper is the setting of a mass constraint on the boundary. The effect of this additional constraint is the appearance of a Lagrange multiplier; in fact, two Lagrange multipliers arise, one for the bulk, the other for the boundary. The well-posedness of the resulting Cahn–Hilliard system with dynamic boundary condition and mass constraint on the boundary is obtained. The theory of evolution equations governed by subdifferentials is exploited and a complete characterization of the solution is given.

Key words: Cahn–Hilliard equation, dynamic boundary condition, mass constraint, variational inequality, Lagrange multipliers.

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1 Introduction

The famous Cahn–Hilliard equation \[ \nabla \cdot (\nabla \phi - \nabla \left( m \frac{\nabla \phi}{|\nabla \phi|} \right)) = \nabla \cdot (m \nabla \phi \nabla \phi) \] offers a realistic description of the evolution phenomena related to solid-solid phase separation processes. In this paper, we are interested to the mathematical investigation of it and aim to analyze questions like existence...
and continuous dependence of solutions for a generalized Cahn–Hilliard equation with dynamic boundary conditions and mass constraints on the boundary. Actually, we can solve the mathematical problem and, in particular, characterize the constraint with the help of a Lagrange multiplier.

Let \( 0 < T < +\infty \) and let \( \Omega \subset \mathbb{R}^d, d = 2 \) or 3, be the bounded smooth domain occupied by the material. Also the boundary \( \Gamma \) of \( \Omega \) is supposed to be smooth enough. We recall the isothermal Cahn–Hilliard equation in the following generalized form:

\[
\frac{\partial u}{\partial t} - \Delta \mu = 0 \quad \text{in } Q := \Omega \times (0,T),
\]

\[
\mu = \tau \frac{\partial u}{\partial t} - \Delta u + \xi + \pi(u) - f, \quad \xi \in \beta(u) \quad \text{in } Q,
\]

where the unknowns \( u := u(x,t) \) and \( \mu := \mu(x,t) \) stand for the order parameter and the chemical potential, respectively. Moreover, \( \tau \) is a viscosity coefficient which can be greater or equal to 0 (we treat both cases); \( \beta \) stands for the subdifferential of the convex part \( \hat{\beta} \) and \( \pi \) stands for the derivative of the concave perturbation \( \hat{\pi} \) of a double well potential \( W = \hat{\beta} + \hat{\pi} \), for example \( W(r) = (r^2 - 1)^2/4 \) with \( \beta(r) = r^3 \) and \( \pi(r) = -r \) for all \( r \in \mathbb{R} \). In general, \( \beta \) is assumed to be a maximal monotone graph in \( \mathbb{R} \times \mathbb{R} \). Recently, this equation was treated in some papers \([11, 12, 16, 17]\) when coupled with a dynamic boundary condition of the following form:

\[
u u \bigg|_\Gamma = u \bigg|_\Gamma \quad \text{on } \Sigma := \Gamma \times (0,T),
\]

\[
\partial_\nu u + \frac{\partial u}{\partial t} - \Delta_\Gamma u + \xi + \pi_\Gamma(u) = f_\Gamma, \quad \xi \in \beta_\Gamma(u) \quad \text{on } \Sigma,
\]

where, \( u \bigg|_\Gamma \) denotes the trace of \( u \) and \( \partial_\nu \) represents the outward normal derivative on \( \Gamma \). \( \Delta_\Gamma \) stands for the Laplace–Beltrami operator on \( \Gamma \) (see, e.g., \([19\text{ Chapter 3}]\)), \( \beta_\Gamma \) and \( \pi_\Gamma \) have the same property as \( \beta \) and \( \pi \), respectively.

About dynamic boundary conditions, let us point out that the mathematical research for the various problem was already running in the 1990’s. For example, the Stefan problem with dynamic boundary conditions was treated in the series of Aiki \([1-3]\). Recent advances in the Cahn–Hilliard equation with the dynamic boundary conditions can be found in \([11, 16, 18, 24]\) and references therein.

As is well known, conservation of \( u \) is required. Therefore, under the homogeneous Neumann boundary condition

\[
\partial_\nu u = 0 \quad \text{on } \Sigma,
\]

we can realize that

\[
\frac{1}{|\Omega|} \int_\Omega u(t)dx = m_0 := \frac{1}{|\Omega|} \int_\Omega u_0dx \quad \text{for all } t \in [0,T],
\]

for a given initial data \( u_0 \). The new issue of this paper is the setting of a mass constraint on the boundary. More precisely, we require that the solution \( u \) satisfies

\[
k_* \leq \int_\Gamma w_\Gamma u_\Gamma(t)d\Gamma \leq k^* \quad \text{for all } t \in [0,T],
\]
where $k_*$ and $k^*$ are fixed constants fulfilling $k_* \leq k^*$ and $w_\Gamma$ is given weight function on $\Gamma$. This kind of problem for the Allen–Cahn equation was treated in [10], by applying the abstract theory developed in [15]. In the case of the Cahn–Hilliard equation, the essential structure of the constraint has been studied in [21, 22]. We can also find a similar treatment for the preservation of the constraint in [3, 9].

A brief outline of the present paper along with a short description of the various items is as follows.

In Section 2, we present the main results, consisting in the well-posedness of the Cahn–Hilliard equation with dynamic boundary conditions and mass constraints on the boundary. We write the system as an evolution inclusion and characterize the solution with the help of the Lagrange multipliers. We also remark that actually there will be two Lagrange multipliers.

In Section 3, we prove the continuous dependence and of course this result entails the uniqueness property.

In Section 4, we prove the existence result. The proof is split in several steps. First, we construct an approximate solution by substituting the maximal monotone graphs with their Moreau–Yosida regularizations, in the case when $\tau > 0$. The solvability of the approximate problem is guaranteed by the abstract theory of doubly nonlinear evolution inclusions [13]. Moreover, arguing in a similar way as in [15], we show that the solution satisfies suitable regularity properties and obtain a strong characterization of the approximate problem by the Lagrange multiplier: in fact, we are able to prove uniform a priori estimates on all the components of the solution. And finally, from these estimates, we can pass to the limit and conclude the existence proof in the case $\tau > 0$. Next, we can proceed by considering the limiting problem as $\tau \to 0$ and derive the well-posedness result in the pure Cahn–Hilliard case as well.

1. Introduction

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   4.4. Passage to the limit as $\tau \to 0$
2 Main results

In this section, we present our main result, which states the well-posedness of the Cahn–Hilliard equation with the dynamic boundary conditions and mass constraints on the boundary. We apply the treatment of the dynamic boundary conditions as in [8, 10] and exploit the abstract theory of the evolution inclusion, essentially referring to [15, 21].

2.1 Definition of the solution by the Lagrange multiplier

Let \( 0 < T < +\infty \) and \( \Omega \subset \mathbb{R}^d, d = 2 \) or \( 3 \), be the bounded domain with smooth boundary \( \Gamma := \partial \Omega \). We use the notation:

\[
\begin{align*}
H_0 &:= L^2(\Omega)_0 := \{ z \in L^2(\Omega) : \int_{\Omega} z \, dx = 0 \}, \\
H_\Gamma &:= L^2(\Gamma), \quad V_0 := H^1(\Omega) \cap H_0, \quad V_\Gamma := H^1(\Gamma),
\end{align*}
\]

with usual norms \( | \cdot |_{H_0}, | \cdot |_{H_\Gamma} \), respectively. Here, \( \nabla_\Gamma \) denotes the surface gradient on \( \Gamma \) (see, e.g., [19, Chapter 3]). Moreover, let \( V_0^* \) be the dual space of \( V_0 \) and \( F : V_0 \to V_0^* \) denote the duality mapping defined by

\[
\langle F y, z \rangle_{V_0^*, V_0} := \int_{\Omega} \nabla y \cdot \nabla z \, dx \quad \text{for all } y, z \in V_0.
\]

Then, the form \( \langle \cdot, \cdot \rangle_{V_0^*, V_0} : V_0^* \times V_0^* \to \mathbb{R} \),

\[
(y^*, z^*)_{V_0^*} := \int_{\Omega} \nabla F^{-1} y^* \cdot \nabla F^{-1} z^* \, dx \quad \text{for all } y^*, z^* \in V_0^*,
\]

yields the inner product in \( V_0^* \). Here, \( F^{-1} \) is the inverse operator of \( F \) and its restriction to \( H_0 \) works as follows: if \( z \in H_0, y = F^{-1} z \) uniquely solves the boundary value problem

\[
\begin{align*}
-\Delta y &= z \quad \text{a.e. in } \Omega, \\
\partial_\nu y &= 0 \quad \text{a.e. on } \Gamma, \\
\int_{\Omega} y \, dx &= 0.
\end{align*}
\]

and consequently lies in \( H^2(\Omega) \), due to well-known elliptic regularity results. The reader can check that testing \( -\Delta y = z \) by some \( \tilde{z} \in V_0 \) leads to

\[
\int_{\Omega} \nabla y \cdot \nabla \tilde{z} \, dx = \int_{\Omega} \tilde{z} \, dx \quad \text{for all } \tilde{z} \in V_0,
\]

that is, \( z = F y \) as expected. Finally, by virtue of the Poincaré–Wirtinger inequality there exists a constant \( C_0 > 0 \) such that

\[
|z|_{H_0}^2 \leq C_0 |z|_{V_0}^2 \quad \text{for all } z \in V_0.
\]

(2.1)
Then, we obtain $V_0 \hookrightarrow \hookrightarrow H_0 \hookrightarrow \hookrightarrow V_0^*$, where “$\hookrightarrow$” stands for the dense and compact embedding, namely $(V_0, H_0, V_0^*)$ is a standard Hilbert triplet. The same considerations hold for $H_\Gamma$ and $V_\Gamma$. Now, we set

$$H_0 := H_0 \times H_\Gamma, \quad V_0 := \{(u, u_\Gamma) \in V_0 \times V_\Gamma : u_{|\Gamma} = u_\Gamma \text{ a.e. on } \Gamma\},$$

where $u_{|\Gamma}$ denotes the trace of $u$. Observe that $H_0$ and $V_0$ are Hilbert spaces with the inner products

$$(u, z)_{H_0} := (u, z)_{H_0} + (u_\Gamma, z_\Gamma)_{H_\Gamma}, \quad \text{for all } u := (u, u_\Gamma), \ z := (z, z_\Gamma) \in H_0,$$

$$(u, z)_{V_0} := (u, z)_{V_0} + (u_\Gamma, z_\Gamma)_{V_\Gamma}, \quad \text{for all } u := (u, u_\Gamma), \ z := (z, z_\Gamma) \in V_0$$

and related norms. Then, we obtain $V_0 \hookrightarrow \hookrightarrow H_0 \hookrightarrow \hookrightarrow V_0^*$ (see, e.g., [10, Appendix]).

As a remark, let us restate that if $u = (u, u_\Gamma) \in V_0$ then $u_\Gamma$ is exactly the trace of $u$ on $\Gamma$, while, if $u = (u, u_\Gamma)$ is just in $H_0$, then $u \in H$ and $u_\Gamma \in H_\Gamma$ are independent.

The initial-value problem for the Cahn–Hilliard equation with dynamic boundary conditions can be set as the following system (2.2)–(2.7):

$$
\frac{\partial u}{\partial t} - \Delta \mu = 0 \quad \text{in } Q,
$$

$$
\mu = \tau \frac{\partial u}{\partial t} - \Delta u + \xi + \pi(u) - f, \quad \xi \in \beta(u) \quad \text{in } Q,
$$

$$
\partial_\nu \mu = 0 \quad \text{on } \Sigma,
$$

$$
u_\Gamma = u_{|\Gamma}, \quad \text{on } \Sigma,
$$

$$
\partial_\nu u + \frac{\partial u_\Gamma}{\partial t} - \Delta_\Gamma u_\Gamma + \xi_\Gamma + \pi_\Gamma(u_\Gamma) = f_\Gamma, \quad \xi_\Gamma \in \beta_\Gamma(u_\Gamma) \quad \text{on } \Sigma,
$$

$$
u(0) = u_0 \quad \text{in } \Omega, \quad u_\Gamma(0) = u_{0\Gamma} \quad \text{on } \Gamma,
$$

where $\tau \geq 0$ is a viscosity coefficient. Testing (2.2) by the constant function 1 and using the boundary condition (2.4), we realize that $\partial u/\partial t$ has zero mean value in $\Omega$. Then, a formal test of (2.2) and (2.3) by an arbitrary element $z \in V_0$ and a subsequent combination produce, with the help of the definition of $F$ and the conditions in (2.4)–(2.6), the variational formulation

$$
\int_\Omega F^{-1} \left( \frac{\partial u}{\partial t} \right) zdx + \tau \int_\Omega \frac{\partial u}{\partial t}zdx + \int_\Gamma \frac{\partial u_\Gamma}{\partial t}z_\Gamma d\Gamma
\]$$

$$+ \int_\Omega \nabla u(t) \cdot \nabla zdx + \int_\Gamma \nabla_\Gamma u_\Gamma(t) \cdot \nabla_\Gamma z_\Gamma d\Gamma + \int_\Omega \xi(t)zdx + \int_\Gamma \xi_\Gamma(t)z_\Gamma d\Gamma
$$

$$+ \int_\Omega \pi(u(t))zdx + \int_\Gamma \pi_\Gamma(u_\Gamma(t))z_\Gamma d\Gamma = \int_\Omega f(t)zdx + \int_\Gamma f_\Gamma(t)z_\Gamma d\Gamma,
$$

for a.e. $t \in (0, T)$, for all $z \in V_0$ with $z_\Gamma = z_{|\Gamma}$. We are now interested to deal not directly with (2.8) but with a variational inequality replacing it, where the solution and the test function vary in a suitable convex set.

Concerning the data, we assume that
(A1) $\beta, \beta_T$, maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$, are the subdifferentials

$$\beta = \partial \widehat{\beta}, \quad \beta_T = \partial \widehat{\beta}_T$$

of some continuous and convex functions

$$\widehat{\beta}, \widehat{\beta}_T : \mathbb{R} \to [0, +\infty) \text{ such that } \widehat{\beta}(0) = \widehat{\beta}_T(0) = 0;$$

(A2) $\pi, \pi_T : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous functions with Lipschitz constants $L$ and $L_T$, respectively;

(A3) $f := (f, f_T) \in L^2(0, T; L^2(\Omega)) \times L^2(0, T; H_T)$ and $u_0 := (u_0, u_{0T}) \in H^1(\Omega) \times V_T$, where $u_{0T} := u_{0T,F}$.

In particular, by (A1) we are asking that $D(\beta) = D(\beta_T) = \mathbb{R}, 0 \in \beta(0)$ and $0 \in \beta_T(0)$.

In this paper, we are interested to the setting of the constraint

$$k_* \leq \int_{\Gamma} w_T u_{\Gamma}(t) d\Gamma \leq k^* \text{ for all } t \in [0, T], \quad (2.9)$$

for the solution to the related variational inequality (cf. (2.8)). Here, $k_*$ and $k^*$ are real constants with $k_* \leq k^*$, and $w := (0, w_T) \in H_0$ is fixed. We require that the weight function $w_T$ satisfies

(A4) $w_T \in H_T, w_T \geq 0$ a.e. on $\Gamma$ and $\sigma_0 := \int_{\Gamma} w_T d\Gamma > 0$.

The last inequality can be seen as a nondegeneracy condition on the weight element $w$.

Hence, let us term (P) the initial-value problem related to the variational inequality and to the constraint in (2.9). Now, we define precisely the notion of solution to the problem (P) by means of a Lagrange multiplier. In order to set $H_0$ as the pivot space, put $m_0 := (1/|\Omega|) \int_{\Omega} u_0 dx$ and let $v(x, t) := u(x, t) - m_0$ be the new unknown function and define analogously $v_0 := u_0 - m_0$ in $\Omega$, $v_{0T} := u_{0T} - m_0$ on $\Gamma$, $h_* := k_* - m_0 \sigma_0$ and $h^* := k^* - m_0 \sigma_0$, respectively.

**Definition 2.1.** The quadruplet $(v, \xi, \omega, \lambda)$ is called the solution of (P) if

$$v = (v, v_T) \text{ with } v \in H^1(0, T; H_0) \cap C([0, T]; V_0) \cap L^2(0, T; H^2(\Omega)), \quad v_T \in H^1(0, T; H_T) \cap C([0, T]; V_T) \cap L^2(0, T; H^2(\Gamma)),$$

$$\xi = (\xi, \xi_T) \in L^2(0, T; H_0), \quad \omega, \lambda \in L^2(0, T),$$
and \( v, v_\Gamma, \xi, \xi_\Gamma, \omega, \lambda \) satisfy

\[
F^{-1}\left( \frac{\partial v}{\partial t} \right) + \tau \frac{\partial v}{\partial t} - \Delta v + \xi + \pi(v + m_0) = f + \omega \quad \text{a.e. in } Q, \tag{2.10}
\]

\[
\xi \in \beta(v + m_0) \quad \text{a.e. in } Q, \tag{2.11}
\]

\[
v_\Gamma = v|_{\Gamma}, \quad \partial_\nu v + \frac{\partial v_\Gamma}{\partial t} - \Delta v_\Gamma + \xi_\Gamma + \pi_\Gamma(v_\Gamma + m_0) + \lambda w_\Gamma = f_\Gamma \quad \text{a.e. on } \Sigma, \tag{2.12}
\]

\[
\xi_\Gamma \in \beta_\Gamma(v_\Gamma + m_0) \quad \text{a.e. on } \Sigma, \tag{2.13}
\]

\[
v(0) = v_0 \quad \text{a.e. in } \Omega, \quad v_\Gamma(0) = v_\Gamma_0 \quad \text{a.e. on } \Gamma, \tag{2.14}
\]

\[
h_* \leq \int_{\Gamma} w_\Gamma v_\Gamma(t)d\Gamma \leq h^* \quad \text{for a.a. } t \in (0, T), \tag{2.15}
\]

\[
\lambda(t) \int_{\Gamma} w_\Gamma(v_\Gamma(t) - z_\Gamma)d\Gamma \geq 0 \quad \text{for a.a. } t \in (0, T)
\]

and for all \( z = (z, z_\Gamma) \in V_0 \) such that \( h_* \leq \int_{\Gamma} w_\Gamma z_\Gamma d\Gamma \leq h^* \). \tag{2.16}

In the case \( \tau = 0 \), the regularity of \( v \) should be modified into

\[
v \in H^1(0, T; V_0^*) \cap L^\infty(0, T; V_0) \cap L^2(0, T; H_0^2(\Omega)).
\]

### 2.2 Remark for the Lagrange multipliers

By comparing (2.3) with (2.10)–(2.11), we realize that

\[
\mu = -F^{-1}\left( \frac{\partial v}{\partial t} \right) + \omega \quad \text{a.e. in } Q,
\]

so that \( \omega \) turns out to be the mean value of the chemical potential \( \mu \)

\[
\omega(t) = \frac{1}{|\Omega|} \int_{\Omega} \mu(t)dx.
\]

On the other hand, \( \lambda \) has the role of a Lagrange multiplier related to the constraint in (2.15) on the boundary. Then, the two Lagrange multipliers \( \omega \) and \( \lambda \) have different meaning; in particular, \( \lambda \) is obtained by solving the problem and it explicitly appears in the variational formulation, while \( \omega \) does not show up in the variational inequality and it can be only identified a posteriori. Indeed, if we test (2.10) by a function \( z \in V_0 \), then \( \omega \) disappears and we obtain (cf. also (2.8))

\[
\int_{\Omega} F^{-1}\left( \frac{\partial v}{\partial t} \right) zdx + \tau \int_{\Omega} \frac{\partial v}{\partial t} zdx + \int_{\Gamma} \frac{\partial v_\Gamma}{\partial t}(t)z_\Gamma d\Gamma
\]

\[
+ \int_{\Omega} \nabla v(t) \cdot \nabla zdx + \int_{\Gamma} \nabla v_\Gamma(t) \cdot \nabla z_\Gamma d\Gamma + \int_{\Omega} \xi(t)zdx + \int_{\Gamma} \xi_\Gamma(t)z_\Gamma d\Gamma
\]

\[
+ \int_{\Omega} \pi(v(t) + m_0)zdx + \int_{\Gamma} \pi_\Gamma(v_\Gamma(t) + m_0)z_\Gamma d\Gamma + \int_{\Gamma} \lambda(t)w_\Gamma z_\Gamma d\Gamma
\]

\[
= \int_{\Omega} f(t)zdx + \int_{\Gamma} f_\Gamma(t)z_\Gamma d\Gamma, \tag{2.17}
\]
Next, we take a general $z_{|\Gamma} = z_\Gamma$, because $(\omega(t), z)_{H_0} = 0$. On the contrary, if we simply integrate (2.10) and set
\[ q := \xi + \pi(v + m_0) - f \quad \text{a.e. in } Q, \quad q_\Gamma := \xi_\Gamma + \pi_\Gamma(v_\Gamma + m_0) - f_\Gamma \quad \text{a.e. on } \Sigma, \] with the help of (2.12) we obtain
\[ \omega(t) = \frac{1}{|\Omega|} \left\{ \int_\Omega q(t)dx + \int_\Gamma \left( \frac{\partial v_\Gamma}{\partial t}(t) + q_\Gamma(t) + \lambda(t)w_\Gamma \right) d\Gamma \right\} \quad \text{for all } t \in [0, T]. \] (2.19)

In the last part of this section, we show how to recover (2.10) and (2.12) from the variational equality (2.17). Define the projection $P_0 : L^2(\Omega) \to H_0$ by
\[ P_0 z := z - \frac{1}{|\Omega|} \int_\Omega z \ dx \quad \text{for all } z \in L^2(\Omega). \]

Take $z \in H^1_0(\Omega)$ (so that $z_{|\Gamma} = 0$ a.e. on $\Gamma$) and use $P_0 z$ as test function in (2.17). We note that $(P_0 z)_{|\Gamma} = -(1/|\Omega|) \int_\Omega z \ dx$ and infer
\[
\int_\Omega F^{-1} \left( \frac{\partial v}{\partial t}(t) \right) z dx + \varphi \frac{\partial v}{\partial t}(t) z dx + \int_\Gamma \frac{\partial v_\Gamma}{\partial t}(t) d\Gamma \left( -\frac{1}{|\Omega|} \int_\Omega z dx \right) + \int_\Omega \nabla t v(t) \cdot \nabla z dx + \int_\Gamma \left( \xi(t) + \pi(v(t) + m_0) - f(t) \right) \left( z - \frac{1}{|\Omega|} \int_\Omega z dx \right) dx
\] + \int_\Gamma \left( \xi_\Gamma(t) + \pi_\Gamma(v_\Gamma(t) + m_0) - f_\Gamma(t) \right) d\Gamma \left( -\frac{1}{|\Omega|} \int_\Omega z dx \right) = 0.

Then, recalling the notation (2.18) we easily obtain the equation in the interior, i.e.,
\[ F^{-1} \left( \frac{\partial v}{\partial t}(t) \right) + \varphi \frac{\partial v}{\partial t}(t) = \Delta v + P_0 q - \frac{1}{|\Omega|} \int_\Gamma \left( \frac{\partial v_\Gamma}{\partial t}(t) + q_\Gamma(t) \right) d\Gamma = 0 \quad \text{a.e. in } Q \]
and, in view of (2.19), we find out that
\[ F^{-1} \left( \frac{\partial v}{\partial t}(t) \right) + \varphi \frac{\partial v}{\partial t}(t) = \Delta v + q = \omega \quad \text{a.e. in } Q. \]

Next, we take a general $z := (z, z_\Gamma) \in V_0$ and note that (2.17) reduces to
\[
\int_\Omega F^{-1} \left( \frac{\partial v}{\partial t}(t) \right) z dx + \varphi \frac{\partial v}{\partial t}(t) z dx + \int_\Gamma \frac{\partial v_\Gamma}{\partial t}(t) z_\Gamma d\Gamma - \int_\Omega \Delta v(t) z dx + \int_\Gamma \partial_x v(t) z_\Gamma d\Gamma
\] + \int_\Omega \nabla_\Gamma v_\Gamma(t) \cdot \nabla_\Gamma z_\Gamma d\Gamma + \int_\Omega q(t) z dx + \int_\Gamma q_\Gamma(t) z_\Gamma d\Gamma + \int_\Gamma \lambda(t) w_\Gamma z_\Gamma d\Gamma = 0,

which means that
\[
\int_\Omega \omega(t) z dx + \int_\Gamma \left( \partial_x v(t) + \frac{\partial v_\Gamma}{\partial t}(t) - \Delta_\Gamma v_\Gamma(t) + q_\Gamma(t) + \lambda(t) w_\Gamma \right) z_\Gamma d\Gamma = 0.
\]
By virtue of the fact that $\int_\Omega \omega(t) z dx = \omega(t) \int_\Omega z dx = 0$, we finally have (cf. (2.12))
\[ \varphi v + \frac{\partial v_\Gamma}{\partial t} - \Delta_\Gamma v_\Gamma + q_\Gamma + \lambda w_\Gamma = 0 \quad \text{a.e. on } \Sigma. \]
2.3 Well-posedness

The first result states the continuous dependence on the data. The uniqueness of the component $v$ of the solution is also guaranteed by this theorem.

**Theorem 2.1.** Let $\tau \geq 0$. Assume (A1)–(A4). For $i = 1, 2$, let $(v^{(i)}, \xi^{(i)}, \omega^{(i)}, \lambda^{(i)})$, with $v^{(i)} = (v^{(i)}_0, v^{(i)}_T)$ and $\xi^{(i)} = (\xi^{(i)}_0, \xi^{(i)}_T)$ be a solution to (P) corresponding to the data $f^{(i)} = (f^{(i)}_0, f^{(i)}_T)$ and $v^{(i)}_0 = (v^{(i)}_0, v^{(i)}_T)$. Then, there exists a positive constant $C > 0$, depending on $L$, $L_T$, and $T$, such that

$$
|v^{(1)}(t) - v^{(2)}(t)|^2_{V_0} + \tau |v^{(1)}(t) - v^{(2)}(t)|^2_{H_0} + |v^{(1)}_T(t) - v^{(2)}_T(t)|^2_{H_T} + \int_0^t |v^{(1)}_t(s) - v^{(2)}_t(s)|^2_{V_0} ds + 2 \int_0^t |\nabla v^{(1)}_T(s) - \nabla v^{(2)}_T(s)|^2_{H_T} ds 
\leq C \left\{ |v^{(1)}_0 - v^{(2)}_0|^2_{V_0} + |v^{(1)}(1) - v^{(2)}(1)|^2_{H_0} + |v^{(1)}_T - v^{(2)}_T|^2_{H_T} + \int_0^T |f^{(1)}(s) - f^{(2)}(s)|^2_{L^2(\Omega)} ds 
+ \int_0^T |f^{(1)}_T(s) - f^{(2)}_T(s)|^2_{H_T} ds \right\} \text{ for all } t \in [0, T].
$$

(2.20)

The second result deals with the existence of the solution. To the aim, we further assume that

(A5) there exist positive constants $c_0$, $\varrho > 0$ such that

$$
|s| \leq c_0 (1 + \hat{\beta}(r)) \text{ for all } r \in \mathbb{R} \text{ and } s \in \beta(r),
$$

(2.21)

$$
|s| \leq c_0 (1 + \hat{\beta}_T(r)) \text{ for all } r \in \mathbb{R} \text{ and } s \in \beta_T(r),
$$

(2.22)

$$
|\beta^o(r)| \leq \varrho |\beta^o_T(r)| + c_0 \text{ for all } r \in \mathbb{R};
$$

(2.23)

(A6) for the initial data $v_0 = (v_0, v_0_T) \in V_0$ the compatibility conditions

$$
h_* \leq \int_\Gamma w_{T_0} v_{0T} d\Gamma \leq h^*, \quad \hat{\beta}(v_0 + m_0) \in L^1(\Omega), \quad \hat{\beta}_T(v_0T + m_0) \in L^1(\Gamma)
$$

(2.24)

must hold.

The minimal section $\beta^o$ of $\beta$ is specified by $\beta^o(r) := \{r^* \in \beta(r) : |r^*| = \min_{s \in \beta(r)} |s|\}$ and the same definition applies to $\beta^o_T$. The reader can compare these assumptions with the analogous ones in [10] (2.17)–(2.21).

We have to distinguish between the cases $\tau > 0$ and $\tau = 0$. To this aim, we introduce the additional regularity assumption for $f$:

(A7) $f \in H^1(0, T; L^2(\Omega))$ or $f \in L^2(0, T; H^1(\Omega))$.

**Theorem 2.2.** Let $\tau > 0$. Then, under the assumptions (A1)–(A6), there exists a unique solution of (P). Moreover, if $\tau = 0$ and (A7) holds, then the problem (P) has a unique solution as well.
2.4 Abstract formulation

In this subsection, an abstract formulation of the problem is given. We can write the problem as an evolution inclusion governed by a subdifferential operator, with essentially the same approach as in [10, 21, 22].

The point of emphasis is that our mass constraint (2.15) reads

\[ h^* \leq (w, v(t))_{H_0} \leq h^* \quad \text{for all } t \in [0, T], \]

with \( w := (0, w_\Gamma) \in H_0 \). Then, by introducing the convex constraint set

\[ K := \{ z \in V_0 : h^* \leq (w, z)_{H_0} \leq h^* \}, \]

let \( I_K : H_0 \rightarrow [0, +\infty] \) denote the indicator function of \( K \). Now, define the proper, lower semicontinuous and convex functional \( \varphi : H_0 \rightarrow [0, +\infty] \) by

\[
\varphi(z) := \begin{cases} 
\frac{1}{2} \int_\Omega |\nabla z|^2 dx + \int_\Omega \beta(z + m_0) dx + \frac{1}{2} \int_\Gamma |\nabla z_\Gamma|^2 d\Gamma + \int_\Gamma \beta_\Gamma(z_\Gamma + m_0) d\Gamma \\
+\infty & \text{if } z \in V_0, \beta(z + m_0) \in L^1(\Omega) \text{ and } \beta_\Gamma(z_\Gamma + m_0) \in L^1(\Gamma),
\end{cases}
\]

Then, the problem (P) can be stated as the Cauchy problem for an evolution inclusion with a perturbation, namely

\[
A_\tau v'(t) + \partial(\varphi + I_K)(v(t)) \ni P\left( f(t) - \Pi_0(v(t)) \right) \quad \text{in } H_0, \quad \text{for a.a. } t \in (0, T), \quad (2.25)
\]

\[
v(0) = v_0 \quad \text{in } H_0, \quad (2.26)
\]

where \( A_\tau z := (F^{-1}z + \tau z_\Gamma, z_\Gamma) \) for \( \tau \geq 0 \), \( Pz := (P_0z, z_\Gamma - (1/|\Omega|) \int_\Omega z dx) \) and \( \Pi_0(z) := (\pi(z + m_0), \pi_\Gamma(z_\Gamma + m_0)) \) for all \( z \in H_0 \).

Hence, let us recall the paper [13] and express our expectation that (2.25)–(2.26) can be solved by the abstract theory of doubly nonlinear evolution inclusions. All this will be discussed in Section 4. On the other hand, Theorem 2.2 allows a characterization in terms of regularity of the solution and presence of the Lagrange multipliers.

We aim to point out that analogous remarks were emphasized in [10] for an Allen–Cahn equation with dynamic boundary conditions and mass constraints; the reader can compare the two problems. In connection with [10], we also quote the abstract approach carried out in [15], which however does not comply here with the structure of (2.25)–(2.26).

3 Continuous dependence

In this section, we prove Theorem 2.1.
Proof of Theorem 2.1. For \( i = 1, 2 \) let \((v^{(i)}, \xi^{(i)}, \omega^{(i)}, \lambda^{(i)})\) be a solution of \((P)\) corresponding to the data \((f^{(i)}, f_{\Gamma}^{(i)}, v_{0}^{(i)}, v_{\Gamma 0}^{(i)})\). We consider the difference between (2.10) written for \( v^{(1)}(s) \) of \( v^{(1)}(s) = (v^{(1)}(s), v_{\Gamma 1}^{(1)}(s)) \) and (2.10) written for \( v^{(2)}(s) \) of \( v^{(2)}(s) = (v^{(2)}(s), v_{\Gamma 2}^{(2)}(s)) \) at the time \( s \in (0, T) \). Then, we take the inner product with \( v^{(1)}(s) - v^{(2)}(s) \) in \( H \). Using the monotonicity of \( \beta \) and the fact \( \int_{\Omega} (v^{(1)}(s) - v^{(2)}(s)) dx = 0 \), we obtain

\[
\frac{1}{2} \frac{d}{ds} \left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{V_{0}} + \tau \frac{d}{ds} \left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{H_{0}} + \left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{H_{0}} - \left( \pi (v^{(1)}(s) + m_{0}) - \pi (v^{(2)}(s) + m_{0}) \right), v^{(1)}(s) - v^{(2)}(s) \right|_{H}, \tag{3.27}
\]

for a.a. \( s \in (0, T) \). Moreover, we take the difference between (2.12) written for \( v_{\Gamma}^{(1)}(s) \) of and (2.12) written for \( v_{\Gamma}^{(2)}(s) \) of at the time \( t = s \), and take the inner product with \( v_{\Gamma}^{(1)}(s) - v_{\Gamma}^{(2)}(s) \) in \( H_{\Gamma} \); hence, we can replace the term

\[- \left( \partial_{\nu} v^{(1)}(s) - \partial_{\nu} v^{(2)}(s), v_{\Gamma}^{(1)}(s) - v_{\Gamma}^{(2)}(s) \right)_{H_{\Gamma}} \]

with the corresponding quantity in (3.27). Then, by exploiting the monotonicity of \( \beta_{\Gamma} \) and the Lipschitz continuities of \( \pi \) and \( \pi_{\Gamma} \), we obtain

\[
\frac{d}{ds} \left\{ \left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{V_{0}} + \tau \left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{H_{0}} + \left| v_{\Gamma}^{(1)}(s) - v_{\Gamma}^{(2)}(s) \right|^{2}_{H_{\Gamma}} \right\} + 2\left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{V_{0}} + 2\left| \nabla_{\Gamma} v_{\Gamma}^{(1)}(s) - \nabla_{\Gamma} v_{\Gamma}^{(2)}(s) \right|^{2}_{H_{\Gamma}} 
\leq \left| f^{(1)}(s) - f^{(2)}(s) \right|^{2}_{H} + (1 + 2L)\left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{H_{0}} + \left| f_{\Gamma}^{(1)}(s) - f_{\Gamma}^{(2)}(s) \right|^{2}_{H_{\Gamma}} 
+ (1 + 2L_{\Gamma})\left| v_{\Gamma}^{(1)}(s) - v_{\Gamma}^{(2)}(s) \right|^{2}_{H_{\Gamma}}.
\]

for a.a. \( s \in (0, T) \). If \( \tau > 0 \), by applying directly the Gronwall lemma, it is straightforward to find a constant \( C > 0 \), depending only on \( L, L_{\Gamma} \) and \( T \), such that the continuous dependence holds. If \( \tau = 0 \), a known compactness inequality (see, e.g., [23 Thm. 16.4, p. 102]) states that for each \( \delta > 0 \) there exists a positive constant \( C_{\delta} \) such that

\[
|z|_{H_{\Gamma}} \leq \delta|z|_{V_{0}} + C_{\delta}|z|_{V_{\Gamma}^{*}} \quad \text{for all } z \in V_{0},
\]

Therefore, taking \( \delta^{2} < 1/(2 + 4L) \) we have

\[
(1 + 2L)\left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{H_{0}} 
\leq (1 + 2L) \left\{ 2\delta^{2}\left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{V_{0}} + 2C_{\delta}^{2}\left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{V_{\Gamma}} \right\} 
\leq \left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{V_{0}} + \tilde{C}\left| v^{(1)}(s) - v^{(2)}(s) \right|^{2}_{V_{\Gamma}^{*}},
\]

for a.a. \( s \in (0, T) \) and some constant \( \tilde{C} \) depending only on \( L \). At this point, we can analogously apply the Gronwall lemma and find a constant \( C > 0 \), with the same dependencies as above, such that (2.20) holds. Thus, Theorem 2.1 is completely proved. \( \square \)
4 Existence

This section is devoted to the proof of Theorem 2.2. We make use of Yosida approximations for the maximal monotone operators \( \beta, \beta_T \) and of well-known results of this theory (see, \[11,5,20\]). For each \( \varepsilon \in (0,1] \), we define \( \beta_{\varepsilon}, \beta_{T,\varepsilon} : \mathbb{R} \rightarrow \mathbb{R} \), along with the associated resolvent operators \( J_{\varepsilon}, J_{T,\varepsilon} : \mathbb{R} \rightarrow \mathbb{R} \) by

\[
\beta_{\varepsilon}(r) := \frac{1}{\varepsilon}(r - J_{\varepsilon}(r)) := \frac{1}{\varepsilon}(r - \frac{1}{\varepsilon}(r + \varepsilon\beta)(r)), \\
\beta_{T,\varepsilon}(r) := \frac{1}{\varepsilon\varrho}(r - J_{T,\varepsilon}(r)) := \frac{1}{\varepsilon\varrho}(r - \frac{1}{\varepsilon\varrho}(r + \varepsilon\varrho\beta_T)(r)) \quad \text{for all } r \in \mathbb{R},
\]

where \( \varrho > 0 \) is the same constant as in \[2.23\]. Note that the two definitions are not symmetric since in the second it is \( \varepsilon\varrho \) and not directly \( \varepsilon \) to be used as approximation parameter. Now, we easily have \( \beta_\varepsilon(0) = \beta_{T,\varepsilon}(0) = 0 \). Moreover, the related Moreau-Yosida regularizations \( \hat{\beta}_\varepsilon, \hat{\beta}_{T,\varepsilon} \) of \( \beta, \beta_T : \mathbb{R} \rightarrow \mathbb{R} \) fulfill

\[
\hat{\beta}_\varepsilon(r) := \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon}|r - s|^2 + \beta(s) \right\} = \frac{1}{2\varepsilon}|r - J_\varepsilon(r)|^2 + \hat{\beta}(J_\varepsilon r) = \int_0^r \beta_\varepsilon(s)ds, \\
\hat{\beta}_{T,\varepsilon}(r) := \inf_{s \in \mathbb{R}} \left\{ \frac{1}{2\varepsilon\varrho}|r - s|^2 + \beta_T(s) \right\} = \int_0^r \beta_{T,\varepsilon}(s)ds \quad \text{for all } r \in \mathbb{R}.
\]

It is well known that \( \beta_\varepsilon \) is Lipschitz continuous with Lipschitz constant \( 1/\varepsilon \) and \( \beta_{T,\varepsilon} \) is also Lipschitz continuous with constant \( 1/(\varepsilon\varrho) \). In addition, we have the standard properties

\[
|\beta_\varepsilon(r)| \leq |\beta_0(r)|, \quad |\beta_{T,\varepsilon}(r)| \leq |\beta_T^r(r)| \quad \text{for all } r \in \mathbb{R},
\]

\[
0 \leq \hat{\beta}_\varepsilon(r) \leq \hat{\beta}(r), \quad 0 \leq \hat{\beta}_{T,\varepsilon}(r) \leq \hat{\beta}_T(r) \quad \text{for all } r \in \mathbb{R}.
\]

Here, we note that from the assumptions \[2.21, 2.22\] and the above properties we also obtain

\[
|\beta_\varepsilon(r)| \leq c_0 \left( 1 + \frac{1}{\varepsilon}\hat{\beta}_\varepsilon(r) \right), \quad \text{(4.28)} \\
|\beta_{T,\varepsilon}(r)| \leq c_0 \left( 1 + \frac{1}{\varepsilon\varrho}\hat{\beta}_{T,\varepsilon}(r) \right) \quad \text{for all } r \in \mathbb{R}, \quad \text{(4.29)}
\]

with the same constant \( c_0 \). Moreover, thanks to \[2.23\] and \[8, Lemma 4.4\], the inequality

\[
|\beta_\varepsilon(r)| \leq \varrho |\beta_{T,\varepsilon}(r)| + c_0 \quad \text{for all } r \in \mathbb{R}, \quad \text{(4.30)}
\]

holds for \( \beta_\varepsilon \) and \( \beta_{T,\varepsilon} \).

4.1 Approximation of the problem

In this subsection, we consider the approximation of problem (P) in the case when \( \tau > 0 \). The limiting case as \( \tau \rightarrow 0 \) will be discussed later. We introduce the following Cauchy
problem: for each $\varepsilon \in (0, 1]$ find $v_\varepsilon$ satisfying

$$A_\varepsilon v_\varepsilon'(t) + \partial(\varphi_\varepsilon + I_K)(v_\varepsilon(t)) \ni P \left( f(t) - \Pi_0(v_\varepsilon(t)) \right)$$

in $H_0$, for a.a. $t \in (0, T)$, \hspace{1cm} (4.31)

$$v_\varepsilon(0) = v_0 \quad \text{in } H_0.$$ \hspace{1cm} (4.32)

with $v_0 = (v_0, v_{0\Gamma}) \in K$ satisfying the compatibility conditions (2.24). Here, $\varphi_\varepsilon : H_0 \to [0, +\infty]$ is defined by

$$\varphi_\varepsilon(z) := \begin{cases} \frac{1}{2} \int_{\Omega} |\nabla z|^2 \, dx + \int_{\Omega} \beta_{\varepsilon}(z + m_0) \, dx \\ + \frac{1}{2} \int_{\Gamma} |\nabla z_{\Gamma}|^2 \, d\Gamma + \int_{\Gamma} \beta_{\Gamma,\varepsilon}(z_{\Gamma} + m_0) \, d\Gamma + \frac{\varepsilon}{2} \int_{\Gamma} |z_{\Gamma}|^2 \, d\Gamma \quad \text{if } z \in V_0, \\ + \infty \quad \text{if } z \in H_0 \setminus V_0. \end{cases}$$

moreover, it is understood that $A_\varepsilon z := (F^{-1}z + \tau z, z_{\Gamma})$, $P z := (P_0 z, z_{\Gamma} - (1/|\Omega|) \int_{\Omega} z \, dx)$ and $\Pi_0(z) := (\pi(z + m_0), \pi_{\Gamma}(z_{\Gamma} + m_0))$ for all $z = (z, z_{\Gamma}) \in H_0$.

As a remark, thanks to the Poincaré–Wirtinger inequality for functions with 0 mean value, there is no need to introduce an approximating term like $(\varepsilon/2) \int_{\Omega} |z|^2 \, dx$ in the expression of $\varphi_\varepsilon$ above. Denote $\partial^* \varphi_\varepsilon$ by the subdifferential of $\varphi_\varepsilon : V_0 \to [0, +\infty]$ from $V_0$ to $V_0^*$. From [10, Lemma 3.1], we obtain the characterization of $\partial^* \varphi_\varepsilon$ by

$$\langle \partial^* \varphi_\varepsilon(z), \tilde{z} \rangle_{V_0^*, V_0} = \langle \nabla z, \nabla \tilde{z} \rangle_{L^2(\Omega)^d} + \langle \beta_{\varepsilon}(z + m_0), \tilde{z} \rangle_{L^2(\Omega)} + \langle \nabla z_{\Gamma}, \nabla \tilde{z}_{\Gamma} \rangle_{H^1_0} + \langle \beta_{\Gamma,\varepsilon}(z_{\Gamma} + m_0), \tilde{z}_{\Gamma} \rangle_{H^1_T} + \varepsilon(z_{\Gamma} + m_0) \rangle_{H^1_T} \quad \text{for all } z = (z, z_{\Gamma}), \tilde{z} = (\tilde{z}, \tilde{z}_{\Gamma}) \in V_0. \hspace{1cm} (4.33)$$

Moreover, there exists a positive constant $C_\varepsilon$ depending on $\varepsilon > 0$ such that

$$|\partial^* \varphi_\varepsilon(z)|_{V_0^*} \leq C_\varepsilon(1 + \varphi_\varepsilon(z)) \quad \text{for all } z \in V_0. \hspace{1cm} (4.34)$$

Now, we recall the fact that the closure $\overline{K}$ of $K$ in $H_0$ is characterized by

$$\overline{K} = \{ z \in H_0 : h_* \leq (w, z)_{H_0} \leq h^* \},$$

which is closed convex subset of $H_0$. Moreover, there exists a function $z_\varepsilon \in C^1(\overline{\Omega})$ such that

$$\int_{\Omega} z_\varepsilon \, dx = 0, \quad z_{\varepsilon|_{\Gamma}} = \frac{1}{\sigma_0},$$

whence $z_\varepsilon := (z_\varepsilon, 1/\sigma_0) \in V_0$. Then, we can deduce the following result.

**Proposition 4.1.** Let $\tau > 0$. For each $\varepsilon \in (0, 1]$, there exist a unique

$$v_\varepsilon \in H^1(0, T; H_0) \cap L^\infty(0, T; V_0)$$

and a pair of functions $v^*_\varepsilon \in L^2(0, T; H_0)$ and $\lambda_\varepsilon \in L^2(0, T)$ such that

$$u_\varepsilon(t) \in \overline{K} \quad \text{for all } t \in [0, T],$$
and

\[ A_\tau \varepsilon'(t) + \varepsilon \lambda (t) w = P \left( f(t) - \Pi_0 (v_\varepsilon(t)) \right) \quad \text{in } H_0, \quad \text{for a.a. } t \in (0, T), \quad (4.35) \]

\[ \varepsilon \lambda (t) w = \varepsilon (t) (0, w_\varepsilon) \in \partial I_{\mathcal{K}} (v_\varepsilon(t)) \quad \text{in } H_0, \quad \text{for a.a. } t \in (0, T), \quad (4.37) \]

\[ v_\varepsilon (0) = v_0 \quad \text{in } H_0. \quad (4.38) \]

**Proof.** We sketch the basic steps of the proof.

1. We claim that for a given \( \bar{v} \in C([0, T]; H_0) \) there exists a unique

\[ v \in H^1(0, T; H_0) \cap L^\infty(0, T; V_0) \subset C([0, T]; H_0) \]

such that

\[ A_\tau v'(t) + \partial (\varepsilon \lambda + I_{\mathcal{K}}) (v(t)) \ni P \left( f(t) - \Pi_0 (\bar{v}(t)) \right) \quad \text{in } H_0, \quad \text{for a.a. } t \in (0, T), \]

\[ v(0) = v_0 \quad \text{in } H_0. \]

Indeed, it suffices to apply the abstract theory of doubly nonlinear evolution inclusions (see, e.g., [13, Thm. 2.1]). We point out that, thanks to \( \tau > 0 \), the operator \( A_\tau \) is coercive in \( H_0 \). Then, we construct the map

\[ \Psi : \bar{u} \mapsto u, \]

from \( C([0, T]; H_0) \) into itself.

2. For given \( \bar{u}^{(i)} \in C([0, T]; H_0) \), put \( u^{(i)} := \Psi \bar{u}^{(i)} \) for \( i = 1, 2 \). Then, using the monotonicity of \( \partial (\varepsilon \lambda + I_{\mathcal{K}}) \) and the special form of \( A_\tau \), it is not difficult to deduce the estimate

\[ |u^{(1)}(t) - u^{(2)}(t)|_{H_0}^2 \leq C_\tau \int_0^t |\bar{u}^{(1)}(s) - \bar{u}^{(2)}(s)|_{H_0}^2 \, ds \quad \text{for all } t \in [0, T], \quad (4.39) \]

where \( C_\tau \) is a constant depending on \( L, L_\Gamma \) and \( \tau \). Owing to \( (4.39) \), we can prove that there exist a suitable \( k \in \mathbb{N} \) such that \( \Psi^k \) is a contraction mapping in \( C([0, T]; H_0) \). Hence, being \( \tau > 0 \) there exists a unique fixed point for \( \Psi \) which yields the unique solution \( v_\varepsilon \) of the problem (4.31)–(4.32).

3. The third step is essentially the same as in the abstract theory developed in [15]. Put

\[ y_\varepsilon(t) := -A_\tau v_\varepsilon'(t) + P \left( f(t) - \Pi_0 (v_\varepsilon(t)) \right) \quad \text{in } H_0, \quad \text{for a.a. } t \in (0, T), \]

and observe that \( y_\varepsilon \in L^2(0, T; H_0) \). In general, for each \( z \in V_0 \) we have that

\[ \partial (\varepsilon \lambda + I_{\mathcal{K}})(z) \subset \partial^* (\varepsilon \lambda + I_{\mathcal{K}}) (z) = \partial^* \varepsilon \lambda (z) + \partial I_{\mathcal{K}} (z). \]
Due to the regularity of the solution, \( v_\varepsilon(t) \) is in \( \mathcal{K} \) for all \( t \in [0,T] \). Another remark is that the last condition (4.47) is equivalent to (see, e.g., [10, Remark 3.2])

\[
\lambda_\varepsilon(t)w \in \partial I_{\mathcal{K}}(v_\varepsilon(t)) \quad \text{in } H_0, \quad \text{for a.a. } t \in (0,T).
\]
4.2 A priori estimates

Let \( \tau > 0 \). In this subsection, we obtain the uniform estimates independent of \( \varepsilon > 0 \). Moreover, our second objective will be to study the limiting behavior as \( \tau \to 0 \). Therefore, under the additional regularity assumption (A7) for \( f \) we also obtain some uniform estimates independent of \( \varepsilon > 0 \) and \( \tau > 0 \).

**Lemma 4.1.** There exist a positive constant \( M_1 \), independent of \( \varepsilon \in (0, 1] \), such that

\[
|v_\varepsilon|_{H^1(0,T; V_0')} + \tau^{1/2} |v_\varepsilon|_{H^1(0,T; H_0)} + |v_\varepsilon|_{L^\infty(0,T; V_0')} + \sup_{t \in (0,T)} \int_\Omega \tilde{\beta}_\varepsilon(v_\varepsilon(t) + m_0) \, dx \\
+ |v_{\Gamma, \varepsilon}|_{H^1(0,T; H_\Gamma)} + |v_{\Gamma, \varepsilon}|_{L^\infty(0,T; V_\Gamma)} + \sup_{t \in (0,T)} \int_\Gamma \tilde{\beta}_{\Gamma, \varepsilon}(v_{\Gamma, \varepsilon}(t) + m_0) \, d\Gamma \leq M_1.
\]

Moreover, if (A7) is assumed, then \( M_1 > 0 \) is obtained independent of \( \varepsilon \in (0, 1] \) and \( \tau > 0 \).

**Proof.** We test (4.43) by \( v'_\varepsilon = \partial v_\varepsilon / \partial t \in L^2(0,T; H) \). Moreover, we add \( v_{\Gamma, \varepsilon} \) to both sides of (4.44) and use it as the boundary condition, obtaining

\[
\int_0^t |v'_\varepsilon(s)|_{V_0'}^2 \, ds + \tau \int_0^t |v'_\varepsilon(s)|_{H_0'}^2 \, ds + \frac{1}{2} |v_\varepsilon(t)|_{V_0'}^2 + \int_\Omega \tilde{\beta}_\varepsilon(v_\varepsilon(t) + m_0) \, dx \\
+ \int_0^t |v'_{\Gamma, \varepsilon}(s)|_{H_\Gamma'}^2 \, ds + \frac{1}{2} |v_{\Gamma, \varepsilon}(t)|_{V_\Gamma'}^2 + \int_\Gamma \tilde{\beta}_{\Gamma, \varepsilon}(v_{\Gamma, \varepsilon}(t) + m_0) \, d\Gamma + \frac{\varepsilon}{2} |v_{\Gamma, \varepsilon}(t)|_{H_\Gamma'}^2 \\
+ \int_0^t \lambda_\varepsilon(s) \left\{ \int_\Gamma w_{\Gamma, v'_{\Gamma, \varepsilon}(s)} \, d\Gamma \right\} \, ds \\
\leq \frac{1}{2} |v_0|_{V_0'}^2 + \int_\Omega \tilde{\beta}_\varepsilon(v_0 + m_0) \, dx + \frac{1}{2} |v_{\Gamma, \varepsilon}(0)|_{V_\Gamma'} \, d\Gamma + \int_\Gamma \tilde{\beta}_{\Gamma, \varepsilon}(v_{\Gamma, \varepsilon}(0) + m_0) \, d\Gamma + \frac{\varepsilon}{2} |v_{\Gamma, \varepsilon}(0)|_{H_\Gamma'}^2 \\
+ \int_0^t \left( f_\varepsilon(s) - \pi(v_\varepsilon(s) + m_0), v'_\varepsilon(s) \right)_{H_\Gamma'} \, ds \\
+ \int_0^t \left( f_{\Gamma}(s) + v_{\Gamma, \varepsilon}(s) - \pi_{\Gamma}(v_{\Gamma, \varepsilon}(s) + m_0), v'_{\Gamma, \varepsilon}(s) \right)_{H_\Gamma'} \, ds \tag{4.49}
\]

for all \( t \in [0,T] \). We note that (cf. (2.24))

\[
\int_\Omega \tilde{\beta}_\varepsilon(v_0 + m_0) \, dx \leq \int_\Omega \tilde{\beta}(v_0 + m_0) \, dx < +\infty, \tag{4.50}
\]

\[
\int_\Gamma \tilde{\beta}_{\Gamma, \varepsilon}(v_{\Gamma, \varepsilon}(0) + m_0) \, d\Gamma \leq \int_\Gamma \tilde{\beta}_{\Gamma}(v_{\Gamma, \varepsilon}(0) + m_0) \, d\Gamma < +\infty. \tag{4.51}
\]

Also by the chain rule differentiation lemma (see, e.g., [4] Lemma 4.4, p. 158 or [5] Lemme 3.3, p. 73) and in view of (4.46)–(4.47), the last term on the left hand side is exactly

\[
\int_0^t \lambda_\varepsilon(s) h'_\varepsilon(s) \, ds = I_{[h_*, h^*]}(h_\varepsilon(t)) - I_{[h_*, h^*]}(h_0) \equiv 0 \quad \text{for all } t \in [0, T], \tag{4.52}
\]
where \( h_0 := (w_\Gamma, v_{0T})_{H^1}. \) We easily see that there exists a positive constant \( \bar{M}_1, \) depending on \( L, L_\Gamma, |\pi(m_0)|, |\pi_\Gamma(m_0)|, |\Omega| \) and \( |\Gamma| \) (but independent of \( \varepsilon \in (0, 1) \) and \( \tau > 0 \), such that

\[
\int_0^t \left( f(s) - \pi(v_\varepsilon(s) + m_0), v'_\varepsilon(s) \right)_{H^1} ds \\
\leq \frac{\tau}{2} \int_0^t \left| v'_\varepsilon(s) \right|^2_{H^1} ds + \frac{1}{\tau} \int_0^t \left( \left| f(s) \right|^2_{H^1} + \left| \pi(v_\varepsilon(s) + m_0) \right|^2_{H^1} \right) ds \\
\leq \frac{\tau}{2} \int_0^t \left| v'_\varepsilon(s) \right|^2_{H^1} ds + \bar{M}_1 \int_0^t \left( 1 + \left| f(s) \right|^2_{H^1} + \left| v_\varepsilon(s) \right|^2_{V_0} \right) ds
\]

(4.53)

and

\[
\int_0^t \left( f_\Gamma(s) + v_{T,\varepsilon}(s) - \pi_\Gamma(v_{T,\varepsilon}(s) + m_0), v'_{T,\varepsilon}(s) \right)_{H^1} ds \\
\leq \frac{1}{2} \int_0^t \left| v'_{T,\varepsilon}(s) \right|^2_{H^1} ds + \bar{M}_1 \int_0^t \left( 1 + \left| f_\Gamma(s) \right|^2_{H^1} + \left| v_{T,\varepsilon}(s) \right|^2_{H^1} \right) ds
\]

(4.54)

for all \( t \in [0, T]. \) Now, we collect the information in (4.50)–(4.54) and then apply the Gronwall lemma to the inequality resulting from (4.49). Hence, we prove the lemma in this case and we see from (4.53) that the constant \( \bar{M}_1 \) depends on \( \tau > 0. \)

On the contrary, if (A7) is assumed, the key estimate (4.53) is modified. Thanks to the growth conditions (2.21)–(2.22) (see also (4.28)–(4.29)), we obtain the following estimate.

Thus, by taking \( \delta < 1 \) we can apply the Gronwall lemma to obtain the estimate with a certain positive constant \( M_1 \) independent of \( \tau > 0. \) On the other hand, if we assume \( f \in H^1(0, T; L^2(\Omega)) \), then we can integrate by parts and use the Young inequality and (2.1), as follows:

\[
\int_0^t \left( f(s), v'_\varepsilon(s) \right)_{H^1} ds \\
= - \int_0^t \left( f'(s), v_\varepsilon(s) \right)_{H^1} ds + \left( f(t), v_\varepsilon(t) \right)_{H^1} - \left( f(0), v_0 \right)_{H^1} \\
\leq \frac{1}{2} \int_0^t \left| f'(s) \right|^2_{H^1} ds + \frac{C_0}{2} \int_0^t \left| v_\varepsilon(s) \right|^2_{V_0} ds + \frac{1}{4} \left| v_\varepsilon(t) \right|^2_{V_0} + \frac{1}{4} \left| v_0 \right|^2_{H^1} + (C_0 + 1) \left| f \right|_{C(0, T; L^2(\Omega))}^2,
\]

for all \( t \in [0, T]. \) Thus, taking \( \delta < 1 \) we can apply the Gronwall lemma to obtain the estimate with a certain positive constant \( M_1 \) independent of \( \tau > 0. \) On the other hand, if we assume \( f \in L^2(0, T; H^1(\Omega)) \), then we have

\[
\int_0^t \left( f(s), v'_\varepsilon(s) \right)_{H^1} ds \leq \frac{\delta}{2} \int_0^t \left| v'_\varepsilon(s) \right|^2_{V_0} ds + \frac{1}{2\delta} \int_0^t \left| f(s) \right|^2_{H^1(\Omega)} ds \quad \text{for all } t \in [0, T].
\]

Thus, by taking \( \delta < 2/3, \) the Gronwall inequality works again to the conclusion. \( \square \)

Thanks to the growth conditions (2.21)–(2.22) (see also (4.28)–(4.29)), we obtain the following estimate.
Lemma 4.2. There exist a positive constant $M_2$, independent of $\varepsilon \in (0, 1]$, such that

$$|\lambda_\varepsilon|_{L^2(0,T)} \leq M_2.$$

Proof. From the expression of $\lambda_\varepsilon$, given by \((4.40)\), we infer that

$$\lambda_\varepsilon(t) = \frac{1}{2} \int_{\Omega} \left\{ F^{-1} \left( \frac{\partial v_\varepsilon}{\partial t} (t) \right) + \tau \frac{\partial v_\varepsilon}{\partial t} (t) + q_\varepsilon(t) \right\} z_\varepsilon dx - \int_{\Omega} \nabla v_\varepsilon(t) \cdot \nabla z_\varepsilon dx$$

for a.a. $t \in (0, T)$. Therefore,

$$|\lambda_\varepsilon|_{L^2(0,T)} \leq 6|z_\varepsilon|_{H_0}^2 \int_0^T \left\{ \left| F^{-1} (v_\varepsilon'(t)) \right|_{H_0}^2 + \tau^2 |v_\varepsilon'(t)|_{H_0}^2 \right\} dt + 6|z_\varepsilon|_{L^1(\Omega)}^2 \int_0^T |q_\varepsilon(t)|_{L^1(\Omega)}^2 dt$$

By virtue of \((4.28) - (4.29)\), there exists a positive constant $\hat{M}_2 > 0$ depending only on $c_0$, $L$, $L_\Gamma$, $|\pi(m_0)|$ and $|\pi_\Gamma(m_0)|$ such that

$$|q_\varepsilon(t)|_{L^1(\Omega)} \leq \int_{\Omega} c_0 \left( 1 + \hat{\beta}_\varepsilon(v_\varepsilon(t) + m_0) \right) dx + \int_{\Omega} \left\{ L |v_\varepsilon(t)| + |\pi(m_0)| \right\} dx + \int_{\Omega} |f(t)| dx$$

$$\leq \hat{M}_2 \left\{ 1 + \int_{\Gamma} \hat{\beta}_\varepsilon(v_\varepsilon(t) + m_0) dx + |v_\varepsilon(t)|_{L^1(\Omega)} + |f(t)|_{L^1(\Omega)} \right\}$$

and

$$|q_\Gamma,\varepsilon(t)|_{L^1(\Gamma)} \leq \int_{\Gamma} \varepsilon |v_\Gamma,\varepsilon(t)| d\Gamma + \int_{\Gamma} c_0 \left( 1 + \hat{\beta}_\varepsilon(v_\Gamma,\varepsilon(t) + m_0) \right) d\Gamma$$

$$+ \int_{\Gamma} \left\{ L |v_\Gamma,\varepsilon(t)| + |\pi(m_0)| \right\} d\Gamma + \int_{\Gamma} |f_\Gamma(t)| d\Gamma$$

$$\leq \hat{M}_2 \left\{ 1 + \int_{\Gamma} \hat{\beta}_\varepsilon(v_\Gamma,\varepsilon(t) + m_0) d\Gamma + |v_\Gamma,\varepsilon(t)|_{L^1(\Gamma)} + |f_\Gamma(t)|_{L^1(\Gamma)} \right\}$$

for a.a. $t \in (0, T)$. Therefore, using Lemma 4.1 and taking into account that

$$|F^{-1}(v_\varepsilon'(t))|_{H_0}^2 \leq C_0 |F^{-1}(v_\varepsilon'(t))|_{V_0}^2 = C_0 |v_\varepsilon'(t)|_{V_0}^2,$$

we can find a positive constant $M_2$, independent of $\varepsilon \in (0, 1]$, to prove the assertion.

\[ \square \]

Lemma 4.3. There exist a positive constant $M_3$, independent of $\varepsilon \in (0, 1]$, such that

$$|\omega_\varepsilon|_{L^2(0,T)} \leq M_3.$$
Thus, Lemmas 4.1 and 4.2 ensure the existence of a positive constant \(M_3\), independent of \(\varepsilon \in (0, 1]\), which yields a bound for \(|\omega_\varepsilon|_{L^2(0,T)}\).

**Lemma 4.4.** There exist two positive constants \(M_4\) and \(M_5\), independent of \(\varepsilon \in (0, 1]\), such that

\[
|\beta_\varepsilon(v_\varepsilon + m_0)|_{L^2(0,T;L^2(\Omega))} + |\beta_\varepsilon(v_{\Gamma,\varepsilon} + m_0)|_{L^2(0,T;H_\Gamma)} \leq M_4, \\
|v_\varepsilon|_{L^2(0,T;H^{3/2}(\Omega))} + |\partial_\nu v_\varepsilon|_{L^2(0,T;H_\Gamma)} \leq M_5.
\]

**Proof.** Testing (4.43) by \(\beta_\varepsilon(v_\varepsilon + m_0) \in L^2(0,T;H^1(\Omega))\) and using (4.44). Then, integrating it over \(\Omega \times (0,t)\) with respect to \((x,s)\), we infer that

\[
\int_0^t \int_\Omega \beta_\varepsilon'(v_\varepsilon(s) + m_0) \left| \nabla v_\varepsilon(s) \right|^2 dx ds + \int_0^t \left| \beta_\varepsilon(v_\varepsilon(s) + m_0) \right|_{L^2(\Omega)}^2 ds \\
+ \int_0^t \int_\Gamma \beta_\varepsilon'(v_{\Gamma,\varepsilon}(s) + m_0) \left| \nabla_{\Gamma} v_{\Gamma,\varepsilon}(s) \right|^2 d\Gamma ds \\
+ \int_0^t \int_\Gamma \beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0) \beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0) d\Gamma ds \\
\leq \int_0^t \left( f(s) - F^{-1}(v_\varepsilon'(s)) - \tau v_\varepsilon'(s) - \pi(v_\varepsilon(s) + m_0) + \omega_\varepsilon(s), \beta_\varepsilon(v_\varepsilon(s) + m_0) \right)_{L^2(\Omega)} ds \\
+ \int_0^t \left( f_{\Gamma}(s) - v_{\Gamma,\varepsilon}'(s) - \pi_{\Gamma}(v_{\Gamma,\varepsilon}(s) + m_0) - \lambda_\varepsilon(s) w_{\Gamma}, \beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0) \right)_{H_\Gamma} ds \\
- \varepsilon \int_0^t \left( v_{\Gamma,\varepsilon}(s), \beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0) \right)_{H_\Gamma} ds 
\text{for all } t \in [0,T],
\]

where we should take care that \((\beta_\varepsilon(v_\varepsilon + m_0))_\Gamma = \beta_\varepsilon(v_{\Gamma,\varepsilon} + m_0) \in L^2(0,T;H^1(\Gamma))\). Here, we use the assumption (4.30) to deduce that

\[
\int_0^t \int_\Gamma \beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0) \beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0) d\Gamma ds \\
= \int_0^t \int_\Gamma \beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0) \left| \beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0) \right|_{H_\Gamma} d\Gamma ds \\
\geq \frac{1}{\varrho} \int_0^t \int_\Gamma \beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0) \left| \beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0) \right|^2 d\Gamma ds - \frac{c_0}{\varrho} \int_0^t \int_\Gamma \beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0) \left| \beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0) \right| d\Gamma ds \\
\geq \frac{1}{2\varrho} \int_0^t \left| \beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0) \right|^2_{H_\Gamma} ds - \frac{c_0^2}{2\varrho} T|\Gamma| 
\text{for all } t \in [0,T],
\]
because $\beta_\varepsilon(r)$ and $\beta_{\Gamma,\varepsilon}(r)$ have the same sign for all $r \in \mathbb{R}$. We also note that
\[
\int_0^t \int_{\Omega} \beta'_\varepsilon(v_\varepsilon(s) + m_0) |\nabla v_\varepsilon(s)|^2 \, dx \, ds \geq 0,
\]
\[
\int_0^t \int_{\Gamma} \beta'(v_{\Gamma,\varepsilon}(s) + m_0) |\nabla v_{\Gamma,\varepsilon}(s)|^2 \, d\Gamma \, ds \geq 0
\]
for all $t \in [0, T]$. Moreover, using the Young inequality and the fact $\varepsilon \leq 1$ we have
\[
-\varepsilon \int_0^t \left( v_{\Gamma,\varepsilon}(s), \beta_\varepsilon(v_{\Gamma,\varepsilon}(s) + m_0) \right)_{H_\Gamma} \, ds
\]
\[
\leq \frac{\delta}{2} \int_0^t \left| \beta_\varepsilon(v_\varepsilon(s) + m_0) \right|^2_{H_\Gamma} \, ds + \frac{1}{2\delta} \int_0^t \left| v_{\Gamma,\varepsilon}(s) \right|^2_{H_\Gamma} d\Gamma
\]
for all $t \in [0, T]$ and $\delta > 0$. Now, there exists a positive constant $\tilde{M}_4$, which depends only on $C_0$, $L$, $L_\Gamma$, $|\pi(m_0)|$, $|\pi_\Gamma(m_0)|$, $|\Omega|$, $|\Gamma|$ and $T$, such that
\[
\int_0^t \left( f(s) - F^{-1}(v'_\varepsilon(s)) - \tau v'_\varepsilon(s) - \tau(v_\varepsilon(s) + m_0) + \omega_\varepsilon(s), \beta_\varepsilon(v_\varepsilon(s) + m_0) \right)_{L^2(\Omega)} \, ds
\]
\[
\leq \frac{1}{2} \int_0^t \left| \beta_\varepsilon(v_\varepsilon(s) + m_0) \right|^2_{L^2(\Omega)} \, ds + \tilde{M}_4 \left( 1 + |f|_{L^2(0,T;L^2(\Omega))}^2 + |v'_\varepsilon|_{L^2(0,T;V_\varepsilon')}^2 + \tau^2 |v'_\varepsilon|_{L^2(0,T;V_\varepsilon')}^2 + |v_\varepsilon|_{L^2(0,T;V_\varepsilon')}^2 + |\omega_\varepsilon|_{L^2(0,T)}^2 \right),
\]
and
\[
\int_0^t \left( f_{\Gamma}(s) - v'_{\Gamma,\varepsilon}(s) - \tau_{\Gamma} v_{\Gamma,\varepsilon}(s) + m_0) - \lambda_\varepsilon(s) w_{\Gamma}, \beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0) \right)_{H_\Gamma} \, ds
\]
\[
\leq \frac{\delta}{2} \int_0^t \left| \beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon}(s) + m_0) \right|^2_{H_\Gamma} \, ds + \frac{1}{2\delta} \int_0^t \left| v_{\Gamma,\varepsilon}(s) \right|^2_{H_\Gamma} d\Gamma
\]
\[
+ \tilde{M}_4 \left( 1 + |f_{\Gamma}|_{L^2(0,T;H_\Gamma^2)}^2 + |v'_{\Gamma,\varepsilon}|_{L^2(0,T;H_\Gamma')}^2 + |v_{\Gamma,\varepsilon}|_{L^2(0,T;H_\Gamma')}^2 + |\lambda_\varepsilon|_{L^2(0,T)}^2 |w_{\Gamma}|_{H_\Gamma}^2 \right),
\]
for all $t \in [0, T]$ and $\delta > 0$, with the help of the Young inequality. Thus, choosing $\delta < 1/(2\theta)$ and recalling Lemmas 4.1–4.3 we deduce that there exist a positive constant $M_4$, independent of $\varepsilon \in (0, 1]$, such that
\[
|\beta_\varepsilon(v_\varepsilon + m_0)|_{L^2(0,T;L^2(\Omega))} + |\beta_{\Gamma,\varepsilon}(v_{\Gamma,\varepsilon} + m_0)|_{L^2(0,T;H_\Gamma)} \leq M_4.
\]
Next, we can compare the terms in (4.43) and conclude that
\[
|\Delta v_\varepsilon|_{L^2(0,T;L^2(\Omega))}
\]
is bounded independently of $\varepsilon$,
whence, taking Lemma 4.1 into account and applying the theory of the elliptic regularity (see, e.g., [6, Thm. 3.2, p. 1.79]), we have that
\[
|v_\varepsilon|_{L^2(0,T;H^{3/2}(\Omega))} \leq \tilde{M}_5,
\]
and, owing to the trace theory (see, e.g., [6, Thm. 2.25, p. 1.62]), that
\[
|\partial_{\nu} v_\varepsilon|_{L^2(0,T;H_\Gamma)} \leq \tilde{M}_5,
\]
for some constant $\tilde{M}_5$ independent of $\varepsilon \in (0, 1]$.
\[\square\]
Lemma 4.5. There exist positive constants $M_6$, $M_7$ and $M_8$, independent of $\varepsilon \in (0, 1]$, such that

$$\left| \beta_{t, \varepsilon}(v_{t, \varepsilon} + m_0) \right|_{L^2(0, T; H^2_\Gamma)} \leq M_6, \quad \left| v_{t, \varepsilon} \right|_{L^2(0, T; H^2(\Omega))} \leq M_7, \quad \left| v_\varepsilon \right|_{L^2(0, T; H^2(\Omega))} \leq M_8.$$

Proof. We test (4.44) by $\beta_{t, \varepsilon}(v_{t, \varepsilon} + m_0) \in L^2(0, T; V_\Gamma)$ and integrate on the boundary, deducing that

$$\int_\Gamma \tilde{\beta}_{t, \varepsilon}(v_{t, \varepsilon}(t) + m_0) d\Gamma + \int_0^t \int_\Gamma \beta'_{t, \varepsilon}(v_{t, \varepsilon}(s) + m_0) \left| \nabla v_{t, \varepsilon}(s) \right|^2 d\Gamma ds + \int_0^t \beta_{t, \varepsilon}(v_{t, \varepsilon}(s) + m_0) d\Gamma ds \leq \int_\Gamma \tilde{\beta}_{t, \varepsilon}(v_0 + m_0) d\Gamma - \int_0^t \left( \varepsilon v_{t, \varepsilon}(s) + \partial_\nu v_{t, \varepsilon}(s), \beta_{t, \varepsilon}(v_{t, \varepsilon}(s) + m_0) \right)_{H_\Gamma} ds$$

for all $t \in [0, T]$. We note that

$$\int_0^t \int_\Gamma \beta'_{t, \varepsilon}(v_{t, \varepsilon}(s) + m_0) \left| \nabla v_{t, \varepsilon}(s) \right|^2 d\Gamma ds \geq 0,$$

due to the properties of $\beta_{t, \varepsilon}$, and

$$\int_\Gamma \tilde{\beta}_{t, \varepsilon}(v_0 + m_0) d\Gamma \leq \int_\Gamma \tilde{\beta}_{t}(v_0 + m_0) d\Gamma < +\infty,$$

by virtue of (2.24). By applying the Young inequality in the last two terms of (4.56), we see that there exist a positive constant $\tilde{M}_6$ independent of $\varepsilon \in (0, 1]$ such that

$$\left| \beta_{t, \varepsilon}(v_{t, \varepsilon} + m_0) \right|_{L^2(0, T; H^2_\Gamma)} \leq \tilde{M}_6.$$

Hence, by comparison in (4.44) we also infer that

$$\left| \Delta v_{t, \varepsilon} \right|_{L^2(0, T; H^2_\Gamma)} \leq \tilde{M}_7$$

and consequently (see, e.g., [19, Section 4.2])

$$\left| v_{t, \varepsilon} \right|_{L^2(0, T; H^2(\Gamma))} \leq \left( \left| v_{t, \varepsilon} \right|_{L^2(0, T; V_\Gamma)}^2 + \left| \Delta v_{t, \varepsilon} \right|_{L^2(0, T; H^2_\Gamma)}^2 \right)^{\frac{1}{2}} \leq (M_4^2 T + \tilde{M}_7^2)^{\frac{1}{2}} =: M_7.$$

Then, in view of Lemma 4.4, using the theory of the elliptic regularity (see, e.g., [6] Thm. 3.2, p. 1.79] along with the boundedness of $\left| v_{t, \varepsilon} \right|_{L^2(0, T; H^3(\Gamma))}$, it turns out that

$$\left| v_\varepsilon \right|_{L^2(0, T; H^2(\Omega))} \leq M_8$$

for some positive constant $M_8$ independent of $\varepsilon \in (0, 1]$. \qed
Remark 4.1. All constants $M_k$, for $k$ from 1 to 8, are obtained independently of $\tau > 0$ provided that (A7) is assumed. Actually, under the additional assumption (A7) the positive constant $M_1$ in Lemma 4.1 is independent of $\tau > 0$.

4.3 Passage to the limit as $\varepsilon \to 0$

In this subsection, we keep $\tau > 0$ fixed and conclude the existence proof by passage to the limit of the approximate solutions as $\varepsilon \to 0$. Indeed, owing to the uniform estimates stated in Lemmas from 4.1 to 4.5, there exist a subsequence of $\varepsilon$ (not relabeled) and some limit functions $v, v_\Gamma, \xi, \xi_\Gamma, \omega, \lambda$ such that

$$v_\varepsilon \rightharpoonup v \quad \text{weakly star in } H^1((0, T; H_0) \cap L^\infty(0, T; V_0) \cap L^2(0, T; H^2(\Omega)), \quad (4.57)$$

$$v_{\Gamma, \varepsilon} \rightharpoonup v_\Gamma \quad \text{weakly star in } H^1((0, T; H_\Gamma) \cap L^\infty(0, T; V_\Gamma) \cap L^2(0, T; H^2(\Gamma)), \quad (4.58)$$

$$\beta_\varepsilon(v_\varepsilon + m_0) \rightharpoonup \xi \quad \text{weakly in } L^2(0, T; L^2(\Omega)), \quad (4.59)$$

$$\beta_{\Gamma, \varepsilon}(v_{\Gamma, \varepsilon} + m_0) \rightharpoonup \xi_\Gamma \quad \text{weakly in } L^2(0, T; H_\Gamma), \quad (4.60)$$

$$\omega_\varepsilon \rightharpoonup \omega \quad \text{weakly in } L^2(0, T), \quad (4.61)$$

$$\lambda_\varepsilon \rightharpoonup \lambda \quad \text{weakly in } L^2(0, T), \quad (4.62)$$

as $\varepsilon \to 0$. From (4.57) and (4.58), due to strong compactness results (see, e.g., [23 Sect. 8, Cor. 4]) we have that

$$v_\varepsilon \to v \quad \text{strongly in } C([0, T]; H_0) \cap L^2(0, T; V_0), \quad (4.63)$$

$$v_{\Gamma, \varepsilon} \to v_\Gamma \quad \text{strongly in } C([0, T]; H_\Gamma) \cap L^2(0, T; V_\Gamma), \quad (4.64)$$

as $\varepsilon \to 0$. Moreover, on account of (4.46) and (4.58) it is a standard matter to deduce that

$$h_\varepsilon \to h \quad \text{weakly in } H^1(0, T) \quad \text{and strongly in } C([0, T]), \quad (4.65)$$

where

$$h_* \leq h(t) := \int_\Gamma w_\Gamma v_\Gamma(t) d\Gamma \leq h^* \quad \text{for all } t \in [0, T].$$

We point out that (4.46), (4.57) and (4.58) imply that $v_\Gamma = v_\Gamma|_\Sigma$ a.e. on $\Sigma$, while (4.45), (4.63), (4.64) entail

$$v(0) = v_0 \quad \text{a.e. in } \Omega, \quad v_{\Gamma}(0) = v_{0\Gamma} \quad \text{a.e. on } \Gamma.$$

Now, (4.62) and (4.65) and the maximal monotonicity of $\partial I_{[h_*, h^*]}$ allow us to conclude that

$$\lambda \in \partial I_{[h_*, h^*]}(h) \quad \text{a.e. in } (0, T),$$

that is equivalent to (2.16). Moreover, (4.63)–(4.64) and the Lipschitz continuity of $\pi, \pi_{\Gamma}$ imply that

$$\pi(v_\varepsilon + m_0) \to \pi(v + m_0) \quad \text{strongly in } C([0, T]; L^2(\Omega)),$$

$$\pi_{\Gamma}(v_{\Gamma, \varepsilon} + m_0) \to \pi_{\Gamma}(v_{\Gamma} + m_0) \quad \text{strongly in } C([0, T]; H_\Gamma),$$
as \( \varepsilon \to 0 \). At this point, we can pass to the limit in (4.43) and (4.44) obtaining (2.10) and (2.12). Moreover, by applying [4, Prop. 2.2, p. 38] and using (4.59)–(4.60) with (4.63)–(4.64), we obtain

\[
\xi \in \beta(v + m_0) \quad \text{a.e. in } Q, \quad \xi_\Gamma \in \beta_\Gamma(v_\Gamma + m_0) \quad \text{a.e. on } \Sigma.
\]

Thus, it turns out that the pair \( \mathbf{v} = (v, v_\Gamma) \) yields, along with \( \xi = (\xi, \xi_\Gamma) \), \( \omega \) and \( \lambda \), a solution of the limit problem, which can be stated exactly as in (2.10)–(2.16). Also, we note the regularities \( v \in C([0, T]; V_0) \) and \( u_\Gamma \in C([0, T]; V_\Gamma) \) for the solution as a consequence of (4.57)–(4.58).

### 4.4 Passage to the limit as \( \tau \to 0 \)

In this subsection, we discuss the limiting problem as \( \tau \to 0 \). We need to assume the additional regularity (A7) for \( f \). For each \( \tau > 0 \), let now \( \mathbf{v}_\tau := (v_\tau, v_{\Gamma, \tau}) \) be the solution to (2.10)–(2.16) with related \( \omega_\tau \), \( \lambda_\tau \) and

\[
h_\tau(t) := \int_{\Gamma} w_{\Gamma, v_{\Gamma, \tau}}(t) d\Gamma \quad \text{for all } t \in [0, T].
\]

On account of Lemma 4.1 with Remark 4.1, we use the uniform estimates in Lemmas 4.1–4.5 to perform the limit procedure as \( \tau \to 0 \).

As in the previous passage to the limit as \( \varepsilon \to 0 \), also in this case a subsequence of \( \tau \) (not relabeled) and some limit functions \( v, v_\Gamma, \xi, \xi_\Gamma, \omega, \lambda \) can be found in order that the same convergences as in (4.58)–(4.62) hold as \( \tau \to 0 \). We can still deduce the same strong convergences as in (4.63)–(4.65) and the passage to the limit can be carried out in a similar way. Of course, here we have to point out that (cf. the estimate in Lemma 4.1)

\[
\tau v_\tau' \to 0 \quad \text{strongly in } L^2(0, T; H_0)
\]

as \( \tau \to 0 \), which is important when we pass to the limit in the equation (2.10), obtaining

\[
F^{-1} \left( \frac{\partial v}{\partial t} \right) - \Delta v + \xi + \pi(v + m_0) = f + \omega \quad \text{a.e. in } Q,
\]

(4.67)

to be coupled with (2.11)–(2.16).

**Remark 4.2.** On the side of the proof, one can make the remark that the solution component \( \mathbf{v} = (v, v_\Gamma) \) of the problem solves the abstract formulation (see Subsections 2.4
and 4.1)

\[ \mathbf{v} \in H^1(0, T; V_0^*) \cap L^\infty(0, T; V_0), \]

\[ \mathbf{v} \in H^1(0, T; H_0) \text{ if } \tau > 0, \]

\[ \mathbf{v}^* := (-\Delta \mathbf{v} + \xi, \partial_\nu \mathbf{v} - \Delta_{\Gamma} \mathbf{v}_{\Gamma} + \xi_{\Gamma}) \in L^2(0, T; H_0), \]

\[ \lambda \in L^2(0, T), \]

\[ A_{\tau} \mathbf{v}'(t) + \mathbf{v}^*(t) + \lambda(t) \mathbf{w} = P \left( \mathbf{f}(t) - \Pi_0(\mathbf{v}(t)) \right) \text{ in } H_0, \text{ for a.a. } t \in (0, T), \]

\[ \mathbf{v}^*(t) \in \partial \varphi(\mathbf{v}(t)) \text{ in } H_0, \text{ for a.a. } t \in (0, T), \]

\[ \lambda(t) \mathbf{w} \in \partial I_K(\mathbf{v}(t)) \text{ in } H_0, \text{ for a.a. } t \in (0, T), \]

\[ \mathbf{v}(0) = \mathbf{v}_0 \text{ in } H_0. \]

Moreover, let us point out that

\[ \mathbf{v}^*(t) + \lambda(t) \mathbf{w} \in \partial(\varphi + I_K)(\mathbf{v}(t)) \text{ in } H_0, \text{ for a.a. } t \in (0, T). \]

Therefore, it is clear that \( \mathbf{v} \) is the solution of the Cauchy problem expressed by (2.25)–(2.26). We note that although the solution \( \mathbf{v} \) of this problem is uniquely determined, the auxiliary quantities \( \mathbf{v}^* \) and \( \lambda \) are not unique in general (cf. [10, Remark 3.3], [15, Remark 2]).

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**References**

[1] T. Aiki, Two-phase Stefan problems with dynamic boundary conditions, Adv. Math. Sci. Appl., 2 (1993), 253–270.

[2] T. Aiki, Multi-dimensional Stefan problems with dynamic boundary conditions, Appl. Anal., 56 (1995), 71–94.

[3] T. Aiki, Periodic stability of solutions to some degenerate parabolic equations with dynamic boundary conditions, J. Math. Soc. Japan, 48 (1996), 37–59.

[4] V. Barbu, *Nonlinear differential equations of monotone types in Banach spaces*, Springer, London, 2010.
[5] H. Brézis, *Opérateurs maximaux monotones et semi-groupes de contractions dans les espèces de Hilbert*, North-Holland, Amsterdam, 1973.

[6] F. Brezzi and G. Gilardi, Partial differential equations, H. Kardestuncer and D. H. Norrie (Eds.), *Finite element handbook*, McGraw-Hill Book Co., New York, 1987, Part 1.

[7] J. W. Cahn and J. E. Hilliard, Free energy of a nonuniform system I. Interfacial free energy, J. Chem. Phys., 2 (1958), 258–267.

[8] L. Calatroni and P. Colli, Global solution to the Allen–Cahn equation with singular potentials and dynamic boundary conditions, Nonlinear Anal., 79 (2013), 12–27.

[9] L. Cherfils, S. Gatti and A. Miranville, A variational approach to a Cahn–Hilliard model in a domain with nonpermeable walls, J. Math. Sci. (N.Y.), 189 (2013), 604–636.

[10] P. Colli and T. Fukao, Allen–Cahn equation with dynamic boundary conditions and mass constraints, Preprint [arXiv:1405.0116] [math.AP] (2014), pp. 1–23, to appear in Math. Methods Appl. Sci.

[11] P. Colli, G. Gilardi and J. Sprekels, On the Cahn–Hilliard equation with dynamic boundary conditions and a dominating boundary potential, J. Math. Anal. Appl. 419 (2014), 972–994.

[12] P. Colli, G. Gilardi and J. Sprekels, A boundary control problem for the viscous Cahn–Hilliard equation with dynamic boundary conditions, Preprint [arXiv:1407.3916] [math.AP] (2014), pp. 1-27.

[13] P. Colli and A. Visintin, On a class of doubly nonlinear evolution equations, Comm. Partial Differential Equations 15 (1990), 737–756.

[14] C. M. Elliott and S. Zheng, On the Cahn–Hilliard equation, Arch. Ration. Mech. Anal., 96 (1986), 339–357.

[15] T. Fukao and N. Kenmochi, Abstract theory of variational inequalities and Lagrange multipliers, pp. 237–246 in *Discrete and continuous dynamical systems, supplement 2013*, 2013.

[16] G. Gilardi, A. Miranville and G. Schimperna, On the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions, Commun. Pure. Appl. Anal., 8 (2009), 881–912.

[17] G. Gilardi, A. Miranville and G. Schimperna, Long-time behavior of the Cahn–Hilliard equation with irregular potentials and dynamic boundary conditions, Chin. Ann. Math. Ser. B, 31 (2010), 679–712.
[18] G. R. Goldstein and A. Miranville, A Cahn–Hilliard–Gurtin model with dynamic boundary conditions, Discrete Contin. Dyn. Syst. Ser. S, 6 (2013), 387–400.

[19] A. Grigor’yan, Heat kernel and analysis on manifolds, American Mathematical Society, International Press, Boston, 2009.

[20] N. Kenmochi, Monotonicity and compactness methods for nonlinear variational inequalities, M. Chipot (Ed.), Handbook of differential equations: Stationary partial differential equations, Vol. 4, North-Holland, Amsterdam (2007), 203–298.

[21] N. Kenmochi and M. Niezgódka, Viscosity approach to modelling non-isothermal diffusive phase separation, Japan J. Indust. Appl. Math., 13 (1996), 135–169.

[22] M. Kubo, The Cahn–Hilliard equation with time-dependent constraint, Nonlinear Anal., 75 (2012), 5672–5685.

[23] J.-L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications, Vol. I, Springer, Berlin, 1972.

[24] R. Racke and S. Zheng, The Cahn–Hilliard equation with dynamic boundary conditions, Adv. Differential Equations, 8 (2003), 83–110.

[25] J. Simon, Compact sets in the spaces $L^p(0,T;B)$, Ann. Mat. Pura. Appl. (4), 146 (1987), 65–96.