Multiple periodic solutions to a discrete $p(k)$ - Laplacian problem

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Abstract

We investigate the existence of multiple periodic solutions to the anisotropic discrete system. We apply the linking method and a new three critical point theorem which we provide.

1 Introduction

Difference equations serve as mathematical models in diverse areas, such as economy, biology, physics, mechanics, computer science, finance - see for example [1], [9], [15]. Some of these models are of independent interest since their mathematical structure allows for obtaining new abstract tools. One of models arising in the study of elastic mechanics is the $p(x)$ -Laplacian. In this note we will study its discrete counterpart, namely, we consider the existence of multiple $m$—periodic solutions to the following system

$$\begin{align*}
\Delta \left( |\Delta u(k-1)|^{p(k-1)-2} \Delta u(k-1) \right) + \lambda f(k, u(k+1), u(k), u(k-1)) &= 0, \\
u(k+m) &= u(k), \quad k \in \mathbb{Z}.
\end{align*}$$

The approach we apply is a variational one and concerns investigations of an action functional in a suitable chosen space. Precisely speaking, we are interested in finding at least three critical points to the action functional connected with (1) using linking arguments known in the literature, [21], and a three critical point theorem which we develop in this work as a generalization of the result from [7], which does not apply to our case. The abstract tool which we provide seems to apply also for continuous problems. Since the
setting in which we work is a discrete one, we have a different approach when compared with continuous problems corresponding to a $p(x)$–Laplacian, see for example [13].

For the above system $\lambda > 0$ is fixed and we will determine ranges for parameter $\lambda$ corresponding to the existence of multiple solutions; $m \geq 2$ is a fixed natural number; $(\Delta u)(k - 1) = u(k) - u(k - 1)$ stands for the forward difference operator; $u(k) \in \mathbb{R}^n$ for all $k \in \mathbb{Z}$; $p : \mathbb{Z} \to [1, +\infty)$ is an $m$–periodic function, i.e. $p(k + m) = p(k)$ for all $k \in \mathbb{Z}$; $f : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function $m$–periodic with respect to $k$; i.e. $f(k, u_1, u_2, u_3) = f(k + m, u_1, u_2, u_3)$ for all $(k, u_1, u_2, u_3) \in \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$; continuity means that for any fixed $k \in \mathbb{Z}$ the function $f(k, \cdot, \cdot, \cdot)$ is continuous. We underline that here $p$ need not satisfy $p(k) \geq 2$ as is commonly assumed.

Several authors have investigated discrete BVPs with Dirichlet, periodic and Neumann boundary conditions by the critical point theory. They applied classical variational tools such as direct methods, the mountain geometry, linking arguments, the degree theory. We refer to the following works far from being exhaustive: [2], [3], [12], [14], [17], [19], [20]. Inspiration to our investigations in this note lies in [16], where the discrete $p$–Laplacian is considered in [7], where a new three critical point theorem is developed, which is applicable to problems with the $p$–Laplacian both discrete and continuous. We provide similar results as in the papers mentioned, however, in the setting of the discrete $p(k)$–Laplacian. The discrete $p(k)$–Laplacian operator differs from the classical discrete $p$–Laplacian and in this respect our investigations are new.

Continuous versions of problems like (1) are known to be mathematical models of various phenomena arising in the study of elastic mechanics, see [22], electrorheological fluids, see [18], or image restoration, see [8]. Variational continuous anisotropic problems were started by Fan and Zhang in [10] and later considered by many authors and the use of many methods, see [13] for an extensive survey of such boundary value problems.

Now, we provide some tools which are used throughout the paper.

**Definition 1** Let $X$ be a normed space. We say that a functional $J : X \to \mathbb{R}$ is coercive if

$$\lim_{\|u\| \to \infty} J(u) = +\infty$$
and anti-coercive if
\[ \lim_{\|u\| \to \infty} J(u) = -\infty. \]

**Definition 2** Let \( E \) be a Banach space. We say that a \( C^1 \)-functional \( J : E \to \mathbb{R} \) satisfies the Palais-Smale condition if every sequence \( (u_n) \) in \( E \) for which \( \{J(u_n)\} \) is bounded and \( J'(u_n) \to 0 \), has a convergent subsequence.

**Proposition 3** \(^{[21]} \) Let \( E \) be a Banach space. Assume that \( J \in C^1(E, \mathbb{R}) \) satisfies the Palais-Smale condition and is bounded from below on \( E \). Assume further that \( J \) has a local linking at the origin \( 0 \), namely, there exists a decomposition \( E = Y \oplus W \), where \( Y \) is a finite dimensional subspace of \( E \), and a positive real number \( \rho > 0 \) for which
\[ J(u) < J(0) \text{ for all } u \in Y \text{ with } 0 < \|u\| \leq \rho \]
and
\[ J(u) \geq J(0) \text{ for all } u \in W \text{ with } \|u\| \leq \rho. \]
Then \( J \) has at least three critical points.

The paper is organized as follows. Firstly, we provide some new three critical point theorem applicable to anisotropic problems, then we give variational formulation of problem under consideration. Existence and multiplicity results are finally considered by linking arguments and our multiplicity results. Finally some other recent three critical point theorems are discussed with respect to applicability to our problem. Examples are given throughout the text.

## 2 Some generalization of the three critical point theorem

In this section we follow \([7]\) and \([11]\) in order to derive a type of the three critical points theorem, which would be applicable in our case. The main result from \([11]\) reads

**Theorem 4 (Main result)** Let \( (X, \|\cdot\|) \) be a uniformly convex Banach space with strictly convex dual space, \( J \in C^1(X, \mathbb{R}) \) be a functional with compact derivative, \( \mu \in C^1(X, \mathbb{R}_+) \) be a convex coercive functional such that its
derivative is an operator $\mu' : X \to X^*$ admitting a continuous inverse. Let $\tilde{x} \in X$ and $r > 0$ be fixed. Assume that the following conditions are satisfied:

**B.1** $\liminf_{\|x\| \to \infty} \frac{J(x)}{\mu(x)} \geq 0$;

**B.2** $\inf_{x \in X} J(x) < \inf_{\mu(x) \leq r} J(x)$;

**B.3** $\mu(\tilde{x}) < r$ and $J(\tilde{x}) < \inf_{\mu(x) = r} J(x)$.

Then there exists a nonempty open set $A \subseteq (0, +\infty)$ such that for all $\lambda \in A$ the functional $\mu + \lambda J$ has at least three critical points in $X$.

The Authors in [7] consider the above result in case $\mu(x) = \|x\|^p$ and they suggest to replace condition **B.1** with

**B.4** The functional $J$ is bounded from below on $X$, i.e. there exists $C \in \mathbb{R}$ such that $J(x) \geq C$ for every $x \in X$.

They prove that in this case the three critical points result could also be obtained. We follow their method in order to get the mentioned theorem.

**Theorem 5** Let $(X, \|\cdot\|)$ be a uniformly convex Banach space with strictly convex dual space, $J \in C^1(X, \mathbb{R})$ be a functional with compact derivative, $\mu \in C^1(X, \mathbb{R}_+)$ be a convex coercive functional such that its derivative is an operator $\mu' : X \to X^*$ admitting a continuous inverse. Let $\tilde{x} \in X$ and $r > 0$ be fixed. Assume that conditions **B.2**-**B.4** are satisfied. Then there exists a nonempty open set $A \subseteq (0, +\infty)$ such that for all $\lambda \in A$ the functional $\mu + \lambda J$ has at least three critical points in $X$.

**Proof.** Consider the functional $J_C(x) = J(x) - C$. We will show that the functional $J_C$ satisfies conditions **B.1**-**B.3** of Theorem 4. By **B.4** it follows that $J_C(x) \geq 0$ for every $x \in X$. Hence, **B.1** is satisfied with $J_C$. Since

$$\inf_{x \in X} J_C(x) = \inf_{x \in X} J(x) - C$$

and

$$\inf_{\mu(x) \leq r} J_C(x) = \inf_{\mu(x) \leq r} J(x) - C,$$
thus the functional $J_C$ satisfies conditions (B.2) and (B.3) of Theorem 4.

Then there exists a nonempty open set $A \subseteq (0, +\infty)$ such that for all $\lambda \in A$ the functional $\mu + \lambda J_C$ has at least three critical points in $X$. Since critical points of $\mu + \lambda J_C$ and $\mu + \lambda J$ coincide we have the assertion. ■

Since a finite dimensional Hilbert space is a Banach space with strictly convex dual space, we provide a result which is applicable for the problem under consideration.

**Lemma 6** Let $(X, \|\cdot\|)$ be a finite dimensional Hilbert space, $J \in C^1(X, \mathbb{R})$ be bounded from below on $X$. Let $\mu \in C^1(X, \mathbb{R}^+)$ be a coercive functional whose derivative admits a continuous inverse. Let $\bar{x} \in X$ and $r > 0$ be fixed. Assume that conditions (B.2) and (B.3) are satisfied. Then there exists a nonempty open set $A \subseteq (0, +\infty)$ such that for all $\lambda \in A$ the functional $\mu + \lambda J$ has at least three critical points in $X$.

### 3 Variational framework and auxiliary results

Let $F : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a $C^1-$ function; i.e. $F$ is continuous and has continuous partial derivatives $F'_2, F'_3$ with respect to the second and the third variable, respectively. Assume, apart from growth conditions which will be given further, that $F$ has the following structure properties:

(A.1) $f(k, u_1, u_2, u_3) = F'_2(k-1, u_2, u_3) + F'_3(k, u_1, u_2)$ for all $(k, u_1, u_2) \in \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n$;

(A.2) $F(k, u_1, u_2) = F(k + m, u_1, u_2)$ for all $(k, u_1, u_2) \in \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n$;

(A.3) $F(k, 0, 0) = 0$ for all $k \in \mathbb{Z}$.

From now on, we will use the following notations

$$p^- = \min_{k \in [1, m]} p(k), \quad p^+ = \max_{k \in [1, m]} p(k).$$

Define the space

$$H_m = \{ u = \{ u(k) \}_{k \in \mathbb{Z}} : u(k) \in \mathbb{R}^n, u(k + m) = u(k), k \in \mathbb{Z} \},$$

which equipped with the Euclidean norm
\[ \|u\|_e = \left( \sum_{k=1}^{m} |u(k)|^2 \right)^{1/2} \]

becomes a Hilbert space.

Put

\[
W = \{ u = \{ u(k) \}_{k \in \mathbb{Z}} : u(k) = a \in \mathbb{R}^n, k \in \mathbb{Z} \} \quad \text{and} \quad Y = W^\perp.
\]

Thus \( W \) consists of constant sequences and we have an orthogonal decomposition

\[ H_m = Y \oplus W. \]

With fixed \( \lambda > 0 \) we define the action functional \( J_m : H_m \to \mathbb{R} \) corresponding to (1) by

\[
J_m(u) = \sum_{k=1}^{m} \left( \frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} - \lambda F(k, u(k+1), u(k)) \right).
\]

**Lemma 7** Assume that conditions (A.1)-(A.2) hold and fix \( \lambda > 0 \). Then the functional \( J_m : H_m \to \mathbb{R} \) is continuously differentiable in the sense of Gâteaux on \( H_m \). Moreover, \( u \in H_m \) is a critical point of \( J_m \) if and only if it satisfies (1).

**Proof.** The continuity of \( J_m \) is immediate. Let us take an arbitrary \( u \in H_m \). Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be given by the formula \( \varphi(\varepsilon) = J_m(u + \varepsilon h) \), where \( h \in H_m \) is a fixed non-zero direction. Then

\[
\varphi(\varepsilon) = \sum_{k=1}^{m} \frac{1}{p(k-1)} |\Delta (u + \varepsilon h)(k-1)|^{p(k-1)} - \lambda \sum_{k=1}^{m} F(k, (u + \varepsilon h)(k+1), (u + \varepsilon h)(k)).
\]
Since $\varphi$ is continuously differentiable we have

$$
\varphi'(\varepsilon) = \sum_{k=1}^{m} |\Delta(u + \varepsilon h)(k - 1)|^{p(k-1)-2} \Delta(u + \varepsilon h)(k - 1) \Delta h(k - 1) - \\
\lambda \sum_{k=1}^{m} (F'_2(k, (u + \varepsilon h)(k + 1), (u + \varepsilon h)(k))h(k + 1) + \\
F'_3(k, (u + \varepsilon h)(k + 1), (u + \varepsilon h)(k))h(k)).
$$

Letting $\varepsilon = 0$ and in view of (A.2) we get

$$
\varphi'(0) = \sum_{k=1}^{m} |\Delta u(k - 1)|^{p(k-1)-2} \Delta u(k - 1) \Delta h(k - 1) - \\
\lambda \sum_{k=1}^{m} \left( F'_2(k - 1, u(k), u(k - 1)) + F'_3(k, u(k + 1), u(k)) \right) h(k).
$$

Using Abel’s summation by parts formula and owing to $m-$periodic conditions and (A.1) we obtain

$$
\varphi'(0) = - \sum_{k=1}^{m} \Delta \left( |\Delta u(k - 1)|^{p(k-1)-2} \Delta u(k - 1) \right) + \\
\lambda f(k, u(k + 1), u(k), u(k - 1))h(k).
$$

We have shown that $J_m$ has a continuous Gâteaux derivative. Letting the derivative 0, we see that $u$ is a critical point of $J_m$ if and only if it satisfies (1).

Now, we prove some auxiliary results which we use later on.

**Lemma 8 (C.1)** For every $s > 0$

$$
\sum_{k=1}^{m} |u(k)|^s \leq m \|u\|_{e}^s \text{ for all } u \in H_m.
$$
For every $s \geq 2$

$$\sum_{k=1}^{m} |u(k)|^s \geq m^{2-s} \|u\|_e^s \text{ for all } u \in H_m.$$  \hfill (C.2)

For all $u \in H_m$ we have

$$\sum_{k=1}^{m} |\Delta u(k-1)|^{p(k-1)} \leq m(2^{p^+} \|u\|_{p^+}^+ + 1).$$  \hfill (C.3)

**Proof.** To see (C.1) note that for all $k \in [1, m]$ we have

$$|u(k)|^2 \leq \sum_{i=1}^{m} |\Delta u(i)|^2.$$

Thus

$$|u(k)|^s \leq \left( \left( \sum_{i=1}^{m} |\Delta u(i)|^2 \right)^{\frac{1}{2}} \right)^s,$$

which leads to

$$\sum_{k=1}^{m} |u(k)|^s \leq m \|u\|_e^s.$$

Relation (C.2) follows immediately by Hölder’s inequality.

By (C.1) show we deduce that

$$\sum_{k=1}^{m} |\Delta u(k-1)|^{p(k-1)} \leq$$

$$\sum_{\{k \in [1, m] : |\Delta u(k-1)| \leq 1\}} |\Delta u(k-1)|^{p^-} + \sum_{\{k \in [1, m] : |\Delta u(k-1)| > 1\}} |\Delta u(k-1)|^{p^+} \leq$$

$$\sum_{k=1}^{m} |\Delta u(k-1)|^{p^+} + \sum_{k=1}^{m} 1 \leq 2^{p^+} \sum_{k=1}^{m} |u(k)|^{p^+} + m \leq m \left( 2^{p^+} \|u\|_{p^+}^+ + 1 \right).$$

Thus (C.3) holds.

The proof of Lemma 8 is complete. □
4 Multiple periodic solutions

4.1 Result by the linking method

In this section we investigate the existence $m-$periodic solutions by applying Proposition 3.

Assume that $F$ satisfies additionally:

(A.4) There exist $m-$periodic functions $s, r : \mathbb{Z} \to [2, +\infty), \alpha_1, \alpha_2 : \mathbb{Z} \to (0, +\infty)$ and a function $\alpha_3 : \mathbb{Z} \to \mathbb{R}$ for which

$$F(k, u_1, u_2) \geq \alpha_1(k) |u_1|^s(k) + \alpha_2(k) |u_2|^r(k) + \alpha_3(k)$$

for all $k \in \mathbb{Z}$ and all $u_1, u_2 \in \mathbb{R}^n$ such that $|u_1|, |u_2| \geq M$, where $M \geq 1$ is fixed and sufficiently large.

(A.5) There exists a constant $\eta > 0$ such that

$$F(k, u_1, u_2) \geq 0 \quad \text{for all } k \in \mathbb{Z} \text{ and all } u_1, u_2 \in \mathbb{R}^n \text{ for which } |u_1| + |u_2| \leq 2 \eta;$$

and at least one of the below conditions:

(A.6.1) $\lim_{|u_1| + |u_2| \to 0} \frac{F(k, u_1, u_2)}{|u_1|^s(k) + |u_2|^r(k)} = 0$ uniformly in $k \in \mathbb{Z}$;

(A.6.2) $\lim_{|u_1| + |u_2| \to 0} \frac{F(k, u_1, u_2)}{|u_1|^s(k) + |u_2|^r(k)} = 0$ uniformly in $k \in \mathbb{Z}$;

(A.6.3) $\lim_{|u_1| + |u_2| \to 0} \frac{F(k, u_1, u_2)}{|u_1|^s(k) + |u_2|^r(k)} = 0$ uniformly in $k \in \mathbb{Z}$, where

$$s^- = \min_{k \in [1, m]} s(k), \quad s^+ = \max_{k \in [1, m]} s(k), \quad r^- = \min_{k \in [1, m]} r(k), \quad r^+ = \max_{k \in [1, m]} r(k).$$

Note that both (A.6.2) and (A.6.3) imply (A.6.4). Now, we give some examples of nonlinear terms which can be considered by our approach.

Example 9 Let $m \geq 2$ be a fixed even natural number. Assume that

$$f(k, t_1, t_2, t_3) = 4t_2^3 \left(2 + (-1)^k \left(\cos \left(|t_1|^4 + |t_2|^4\right) - \cos \left(|t_2|^4 + |t_3|^4\right)\right)\right).$$
for \((k, t_1, t_2, t_3) \in \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\). Let us take the function \(F: \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) given by

\[
F(k, t_1, t_2) = |t_1|^4 + |t_2|^4 + (-1)^k \sin \left( |t_1|^4 + |t_2|^4 \right)
\]

and functions \(s, r: \mathbb{Z} \rightarrow [2, +\infty)\) such that

\[
r(k) = s(k) = \begin{cases} 4 & \text{for } k = 2l, \\ 2 & \text{for } k = 2l + 1; l \in \mathbb{Z}. \end{cases}
\]

With such defined functions we see that conditions (A.1)-(A.5) and (A.6.3) are fulfilled with \(M = 1, \eta \in (0, \frac{1}{2}], \alpha_1(k) = \alpha_2(k) = 1, \alpha_3(k) = -1\) for all \(k \in \mathbb{Z}\), but conditions (A.6.1), (A.6.2) are not.

**Example 10** Let \(m \geq 2\) be a fixed natural number. Assume that

\[
f(k, t_1, t_2, t_3) = 4t_2^3 \left( \cos^2 \left( \frac{k - 1}{m} \pi \right) t_3^4 + \cos^2 \left( \frac{k}{m} \pi \right) t_1^4 \right).
\]

for \((k, t_1, t_2, t_3) \in \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n\). Let us take the function \(F: \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}\) given by

\[
F(k, t_1, t_2) = \cos^2 \left( \frac{k}{m} \pi \right) |t_1 t_2|^4
\]

and functions \(s, r: \mathbb{Z} \rightarrow [2, +\infty)\) such that

\[
r(k) = s(k) = \sin \left( \frac{k}{m} \pi \right) + 3.
\]

With such defined functions conditions (A.1)-(A.5) and (A.6.1)-(A.6.3) are fulfilled with \(M = \sqrt{2}, \alpha_1(k) = \alpha_2(k) = \cos^2 \left( \frac{k}{m} \pi \right), \alpha_3(k) = 0\) for all \(k \in \mathbb{Z}\).

Set

\[
\alpha_i^- = \min_{k \in [1, m]} \alpha_i(k) \quad \text{for } i = 1, 2, 3.
\]

**Lemma 11** Suppose that conditions (A.1)-(A.2) are satisfied. Assume that (A.4) holds with either \(s^- > p^+\) or \(r^- > p^+\). Then the functional \(J_m\) is anti-coercive on \(H_m\) for all \(\lambda > 0\).
Proof. Using (A.4), (C.2) and (C.3) we obtain
\[
J_m(u) \leq \frac{1}{p} m \left( 2^{p^+} \|u\|_{p^+}^{p^+} + 1 \right) - \\
\lambda \sum_{k=1}^{m} \left( \alpha_1(k) |u(k+1)|^{s(k)} + \alpha_2(k) |u(k)|^{r(k)} + \alpha_3(k) \right) \leq \\
\frac{1}{p} m \left( 2^{p^+} \|u\|_{p^+}^{p^+} + 1 \right) - \lambda \alpha_1 \sum_{k=1}^{m} |u(k)|^{s^+} - \lambda \alpha_2 \sum_{k=1}^{m} |u(k)|^{r^+} - \lambda m \alpha_3 \leq \\
\frac{1}{p} 2^{p^+} m \|u\|_{p^+}^{p^+} - \lambda \alpha_1 \frac{m^{2-s^+}}{p} \|u\|_{s^+} - \lambda \alpha_2 \frac{m^{2-r^+}}{p} \|u\|_{r^+} + \left( \frac{1}{p} - \lambda \alpha_3 \right) m.
\]
Since $s^- > p^+$ or $r^- > p^+$, so $J_m$ is anti-coercive on $H_m$. □

Put
\[
\lambda_1 = \frac{2^{p^+} m^{2-s^+}}{p \alpha_1^s}; \quad \lambda_2 = \frac{2^{p^+} m^{2-r^+}}{p \alpha_2^r}; \quad \lambda_3 = \frac{2^{p^+} m^{2-r^+}}{p (\alpha_1 + \alpha_2)}.
\]

Lemma 12 Suppose that conditions (A.1)-(A.2) and (A.4) are satisfied. The following assertions are true:
(a) If $s^- = p^+$ and $r^- < p^+$, then the functional $J_m$ is anti-coercive on $H_m$ for any $\lambda \in (\lambda_1, +\infty)$;
(b) If $r^- = p^+$ and $s^- < p^+$, then the functional $J_m$ is anti-coercive on $H_m$ for any $\lambda \in (\lambda_2, +\infty)$;
(c) If $r^- = s^- = p^+$, then the functional $J_m$ is anti-coercive on $H_m$ for any $\lambda \in (\lambda_3, +\infty)$.

Proof. Assume that $s^- = p^+$ and $r^- < p^+$. Let $\lambda \in (\lambda_1, +\infty)$. Arguing as in the proof of Lemma 11 we obtain
\[
J_m(u) \leq \left( \frac{1}{p} 2^{p^+} m - \lambda \alpha_1 m^{2-p^+} \right) \|u\|_{p^+}^{p^+} - \lambda \alpha_2 m^{2-r^-} \|u\|_{r^-} + \left( \frac{1}{p} - \lambda \alpha_3 \right) m.
\]
Hence assertion (a) holds. In the remaining cases we proceed as in the above. □
Remark 13  It easy to verify that
\[ \|u\|_{p^+} = \left( \sum_{k=1}^{m} |\Delta u(k-1)|^{p^+} \right)^{\frac{1}{p^+}} \]

is also a norm on \( Y \), while it is obviously not a norm on \( H_m \). Since all norms on \( Y \) are equivalent, therefore there exists a constant \( \xi > 0 \) such that
\[ \sum_{k=1}^{m} |\Delta u(k-1)|^{p^+} \geq \xi \|u\|_{e}^{p^+}. \]  \hspace{1cm} (2)

Now, We are able to formulate our main result in two cases, depending on relation between functions \( r, s \) and \( p \).

Case I. \( s^- > p^+ \).

Theorem 14  Let \( s^- > p^+ \) and \( \lambda > 0 \) be fixed. Assume that conditions (A.1)-(A.5) are satisfied and that at least one of the following conditions holds:
(a) (A.6.2) with \( s^- \leq r^+ \);
(b) (A.6.3) with \( s^- \leq r^- \).

Then problem (1) has at least three \( m \)-periodic solutions, two of which are nontrivial.

Proof. Assume that (A.1)-(A.5) with (a) hold. Choose a positive real number \( \varepsilon \) satisfying
\[ \varepsilon < \frac{\xi}{2\lambda mp^+}, \]
where \( \xi > 0 \) is a constant defined in Remark 13. By (A.5), (A.6.2) there exists \( \rho \in (0, \rho_0) \), where \( \rho_0 = \min \{1, \eta\} \), such that
\[ F(k, u_1, u_2) \leq \varepsilon (|u_1|^{s^-} + |u_2|^{r^+}) \quad \text{for} \quad |u_1| + |u_2| \leq 2\rho. \]  \hspace{1cm} (3)
If \( u \in Y \) with \( 0 < \|u\|_e \leq \rho \) then \( |u(k)| \leq \rho \) for all \( k \in \mathbb{Z} \). By (2), (3) and (C.1) we obtain

\[
J_m(u) \geq \frac{1}{p^+} \eta \|u\|_e^{p^+} - \lambda \varepsilon \sum_{k=1}^{m} (|u(k+1)|^{s^-} + |u(k)|^{r^+}) \geq \frac{1}{p^+} \eta \|u\|_e^{p^+} - \lambda \varepsilon \sum_{k=1}^{m} (|u(k)|^{s^-} + |u(k)|^{r^+}) \geq \|u\|_e^{s^-} \left( \frac{1}{p^+} \eta - 2\lambda \varepsilon m \right) > 0.
\]

Thus, by above and (A.3) we obtain

\[
J_m(u) > J_m(0) \quad \text{for all} \quad u \in Y \quad \text{with} \quad 0 < \|u\|_e \leq \rho.
\]

Notice that for every \( u \in W \) we have \( \Delta u(k-1) = 0 \) for all \( k \in \mathbb{Z} \), so

\[
J_m(u) = -\lambda \sum_{k=1}^{m} F(k, u(k+1), u(k)).
\]

If \( u \in W \) with \( \|u\|_e \leq \rho \) then

\[
|u(k+1)| + |u(k)| \leq 2\eta
\]

for all \( k \in \mathbb{Z} \). Thus, by (A.5) and (A.3) it follows that

\[
J_m(u) \leq J_m(0) \quad \text{for all} \quad u \in W \quad \text{with} \quad \|u\|_e \leq \rho.
\]

Let \( \Phi_m = -J_m \). We see that \( \Phi_m \) has a local linking at the origin 0 with respect to the decomposition \( H_m = Y \oplus W \).

By Lemma [1] we deduce that \( \Phi_m \) satisfies the Palais-Smale condition. Moreover, \( \Phi_m \) is bounded from below, since as coercive and continuous has a minimizer. We have shown that the assumptions of Lemma [3] are satisfied, so \( \Phi_m \) has at least three critical points, two of them are nonzero critical points. By Lemma [7] these are nontrivial \( m \)-periodic solutions of problem (1). In case (b) we follow analogously.

For \( s^- = p^+ \) with \( r^- < p^+ \) we can observe that

\[ \]
Corollary 15 Let $s^- = p^+$ with $r^- < p^+$ and let $\lambda \in (\lambda_1, +\infty)$. Assume that conditions (A.1)-(A.5) and condition (A.6.2) with $s^- \leq r^+$ are satisfied. Then problem (1) has at least three $m$-periodic solutions, two of which are nontrivial.

Case II. $r^- > p^+$.

Theorem 16 Let $r^- > p^+$ and $\lambda > 0$ be fixed. Assume that conditions (A.1)-(A.5) are satisfied and at least one of the following conditions holds:

(a) (A.6.1) with $r^- \leq s^+$;

(b) (A.6.3) with $r^- \leq s^-.$

Then problem (1) has at least three $m$-periodic solutions, two of which are nontrivial.

For $r^- = p^+$ with $s^- < p^+$ we can observe that

Corollary 17 Let $s^- = r^- = p^+$ with $\lambda \in (\lambda_2, +\infty)$. Assume that conditions (A.1)-(A.5) and condition (A.6.1) with $r^- \leq s^+$ are satisfied. Then problem (1) has at least three $m$-periodic solutions, two of which are nontrivial.

In particular, if $s^- = r^- = p^+$ we obtain

Corollary 18 Let $s^- = r^- = p^+$ and let $\lambda \in (\lambda_3, +\infty)$. Assume that conditions (A.1)-(A.5) and (A.6.3) are satisfied. Then problem (1) has at least three $m$-periodic solutions, two of which are nontrivial.
4.2 Result by the three critical point theorem

In this section we provide multiplicity results for problem (1) using Lemma 6. Assume that $F$ apart from structure conditions (A.1)-(A.3) has the following properties:

(A.7) There exists a constant $C \in \mathbb{R}$ such that

$$F(k,u_1,u_2) \leq C \text{ for all } (k,u_1,u_2) \in \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n;$$

(A.8) There exists a number $\rho_1 > 0$ such that

$$F(k,u_1,u_2) < 0$$

for all $k \in \mathbb{Z}$ and all $u_1,u_2 \in \mathbb{R}^n$ for which $0 < |u_1| \leq \rho_1$, $0 < |u_2| \leq \rho_1$;

(A.9) There exist numbers $\rho_3,\rho_2$ such that $\rho_3 \geq \rho_2 > \rho_1$ and

$$F(k,u_1,u_2) > 0$$

for all $k \in \mathbb{Z}$ and all $u_1,u_2 \in \mathbb{R}^n$ for which $\rho_2 < |u_1| \leq \rho_3$, $\rho_2 < |u_2| \leq \rho_3$.

**Theorem 19** Assume that conditions (A.1)-(A.3) and (A.7)-(A.9) hold. Then there exists a nonempty open set $A \subseteq (0, +\infty)$ such that for all $\lambda \in A$ problem (1) has at least three solutions in $Y$, two of which are necessarily non-zero.

**Proof.** We will consider problem (1) in $Y$ since on $H_m$ functional $\mu$ given by

$$\mu(u) = \sum_{k=1}^{m} \frac{1}{p(k-1)}|\Delta u(k-1)|^{p(k-1)}$$

is not coercive. Thus $\mu : Y \rightarrow \mathbb{R}$ and we define $J : Y \rightarrow \mathbb{R}$ by

$$J(u) = -\sum_{k=1}^{m} F(k,u(k+1),u(k)).$$
Then $J_m(u) = \mu(u) + \lambda J(u)$ on $Y$. We will show that $\mu$ is coercive on $Y$. Let us take $u \in Y$ with $\|u\|_{p^-} > 1$. Recalling Remark [13] we have

$$\mu(u) \geq \frac{1}{p^+} \left( \sum_{\{k \in [1, m]: |\Delta u(k-1)| \leq 1\}} |\Delta u(k-1)|^{p(k-1)} + \right.$$ 

$$\sum_{\{k \in [1, m]: \Delta u(k-1)| > 1\}} |\Delta u(k-1)|^{p(k-1)} \right) \geq$$

$$\frac{1}{p^+} \left( \sum_{k=1}^{m} |\Delta u(k-1)|^{p^-} - \sum_{k=1}^{m} 1 \right) \geq \frac{1}{p^+} \|u\|_{p^-}^{p^-} - \frac{1}{p^+} m.$$

Thus we see that $\mu$ is coercive on $Y$.

By (A.7) we find that $J$ is bounded from below on $Y$. From (A.9) it follows that

$$-F(k, u(k+1), u(k)) < 0$$

for all $u \in Y$ such that $\rho_2 < |u(k)| \leq \rho_3$, for all $k \in \mathbb{Z}$. This means, that there exists a point $u \in Y$ such that $J(u) < 0$. Hence

$$\inf_{u \in Y} J(u) < 0.$$

For all $u \in Y$ satisfying (A.8) it is clear that $|u(k)| \leq \rho_1$ for all $k \in \mathbb{Z}$. So $|\Delta u(k-1)| \leq 2\rho_1$ for all $k \in \mathbb{Z}$ and consequently

$$\frac{1}{p(k-1)} |\Delta u(k-1)|^{p(k-1)} \leq \frac{1}{p(k-1)} (2\rho_1)^{p(k-1)}$$

for all $k \in \mathbb{Z}$.

Thus $\mu(u) \leq r_2$, where $r_2 = \sum_{k=1}^{m} \frac{1}{p(k-1)} (2\rho_1)^{p(k-1)}$. Since $\mu$ is continuous, coercive, non-negative and $\mu(0) = 0$, we get $r \in (0, r_2)$ such that $J(u) \geq 0$ for $\mu(u) \leq r$, by (A.3) and (A.8). Therefore (B.2) is satisfied. Now, putting $\tilde{u} = 0$ we observe that

$$0 = \mu(0) < r \quad \text{and} \quad 0 = J(0) < \inf_{\mu(u)=r} J(u).$$
Hence, condition (B.3) is satisfied. Thus, by Lemma 6 we see that there exists a nonempty open set \( A \subseteq (0, +\infty) \) such that for all \( \lambda \in A \) the functional \( J_m \) has at least three critical points on \( Y \). Since by Lemma 7 critical points of \( J_m \) are solutions of problem (II), we get the assertion.

Since we obtain solutions in \( Y \) we know that these are not constant functions.

Now, we give an example to illustrate Theorem 19.

**Example 20** Let \( m \geq 2 \) be a fixed natural number. Let us consider the function \( F : \mathbb{Z} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) given by the formula

\[
F(k, u_1, u_2) = -\sin(u_1^2 + u_2^2) \left| \sin \left( \frac{k}{m\pi} \right) \right|
\]

which satisfies conditions \((A.1)-(A.3)\) and \((A.7)-(A.9)\).

### 5 Final comments

There are other abstract theorems which pertain to the existence of three critical points requiring different sets of assumptions, see for example [4], Theorem 2.1, [5], Theorem 3.3 and [6], Theorem 2.6. However the applicability of these results due to the structure of discrete \( p(k) \)-Laplacian being different from this of \( p \)-Laplacian seems to be much more difficult.

Much more suitable seems the multiplicity Theorem 2.3 from [4]. We use notation from the proof of Theorem 19.

Put

\[
\lambda^* = \left( \inf_{r > \inf X} \inf_{u \in \mu^{-1}((-\infty, r))} \left( \sup_{u \in \mu^{-1}((-\infty, r))} \frac{J(u)}{r - \mu(u)} \right) \right)^{-1}
\]

where we read \( \frac{1}{0} = +\infty \) if this case occurs.

**Theorem 21** [4] Let \((X, \|\cdot\|)\) be a finite dimensional Banach space and \( \mu, J \in C^1(X, \mathbb{R}) \) with \( \mu \) coercive. Assume that the functional \( \mu - \lambda J \) is anticoercive on \( X \) for all \( \lambda \in (0, \lambda^*) \). Then, for all \( \lambda \in (0, \lambda^*) \) the functional \( \mu - \lambda J \) admits at least three distinct critical points.
Let us compare results from Subsection 4.2 the above result from [4]. Firstly, we seem to have much more precise eigenvalue intervals. Next, these eigenvalue intervals are relatively easy to be determined. We however underline that in the application of Theorem 2.3 from [4] no condition around 0 is required but we are confined to the case when the action functional is anti-coercive for all $\lambda > 0$ since it is rather difficult to calculate $\lambda^*$ directly. So both approaches have their own advantages and disadvantages.

The multiplicity results obtain by using Theorem [21] reads

**Theorem 22** Suppose that conditions (A.1)-(A.2) are satisfied. Assume that (A.4) holds with either $s^- > p^+$ or $r^- > p^+$. Then for all $\lambda \in (0, \lambda^*)$ problem [3] has at least three distinct solutions in $Y$.

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