Quantum Functions

Andre Kornell

Abstract. Weaver has recently defined the notion of a quantum relation on a von Neumann algebra. We demonstrate that the corresponding notion of a quantum function between two von Neumann algebras coincides with that of a normal unital ∗-homomorphism in the opposite direction.

A relation between sets $X$ and $Y$ is simply a subset of $Y \times X$. Weaver has recently proposed the following generalization of relations to the noncommutative setting:

**Definition (Weaver, [5, Definition 2.1]).** Let $M \subseteq B(H)$ and $N \subseteq B(K)$ be von Neumann algebras. A quantum relation between $M$ and $N$ is an ultraweakly closed subspace $V \subseteq B(H,K)$ such that $N'V'M' \subseteq V$.

This definition reduces to the usual one when $M = \ell^\infty(X)$ and $N = \ell^\infty(Y)$. It has many other virtues. It is simple to state and simple to handle, and many familiar properties of relations have natural analogs.

**Definition (Weaver, [5, Definition 2.4]).** Let $M_0$, $M_1$, and $M_2$ be von Neumann algebras.

- The diagonal quantum relation on $M_0$ is the quantum relation $M_0'$ between $M_0$ and $M_0$.
- If $V$ is a quantum relation between $M_0$ and $M_1$, then the inverse of $V$ is the quantum relation $V^*$ between $M_1$ and $M_0$.
- If $V_0$ is a quantum relation between $M_0$ and $M_1$, and $V_1$ is a quantum relation between $M_1$ and $M_2$, then their composition $V_1 \circ V_0$ is the quantum relation between $M_0$ and $M_2$ defined by $V_1 \circ V_0 = V_1V_0 = \text{span}\{v_1v_0 \mid v_1 \in V_1, v_0 \in V_0\}$ ultraweak.

If we interpret inclusion between quantum relations as the proper generalization of inclusion between classical relations, we arrive immediately at the following definitions.

**Definition (Weaver, [5, Definition 2.4]).** Let $V$ be a quantum relation on a von Neumann algebra $M$. Then $V$ is said to be

- reflexive in case $M' \subseteq V$,
- symmetric in case $V^* = V$,
- antisymmetric in case $V \cap V^* \subseteq M'$, and

The research reported here was supported by National Science Foundation grant DMS-0753228.

1Here and elsewhere, I have reversed the usual order of the Cartesian product so that the composition of two functions coincides with their composition as relations.

2Weaver uses adjoint and product in place of inverse and composition. I prefer the classical terminology.
Theorem 1.1 (Dixmier, [4, Theorem IV.5.5])

Let \( M \subseteq B(H) \) and \( N \subseteq B(K) \) be von Neumann algebras. A quantum function from \( M \) to \( N \) is a quantum relation \( \mathcal{V} \) between \( M \) and \( N \) such that \( M' \subseteq \mathcal{V}' \mathcal{V} \) and \( \mathcal{V}' \mathcal{V}^* \subseteq N' \).

Theorem. Let \( M \subseteq B(H) \) and \( N \subseteq B(K) \) be von Neumann algebras. There is a canonical bijective correspondence between normal unital *-homomorphisms \( N \rightarrow M \), and quantum functions from \( M \) to \( N \). This correspondence is functorial.

Conventions used in this article. Let \( H \) be a Hilbert space. If \( \xi \in H \), then \( \xi \in B(\mathbb{C}, H) \) is defined by \( \xi(\epsilon) = \epsilon \xi \). If \( V \) and \( W \) are ultraweakly closed subspaces of \( B(H) \), then \( VW = \overline{\text{span}\{vw \mid v \in V, w \in W\}} \) and \( V \overline{\mathcal{W}} = \overline{\text{span}\{v \otimes w \mid v \in V, w \in W\}} \). The tensor product of two Hilbert spaces is defined in such a way that \( H \otimes \mathbb{C} = H = \mathbb{C} \otimes H \). The von Neumann algebra of scalar operators on \( H \) is denoted by \( \mathbb{C}_H \).

Definition. Let \( M \) and \( N \) be von Neumann algebras. Then \( vN(M, N) \) denotes the set of normal unital *-homomorphisms \( N \rightarrow M \), and \( qF(M, N) \) denotes the set of quantum functions from \( M \) to \( N \).

1. Functions from Homomorphisms

Let \( M \subseteq B(H) \) and \( N \subseteq B(K) \) be von Neumann algebras, and let \( \pi : N \rightarrow M \) be a normal unital *-homomorphism.

Theorem 1.1 (Dixmier, [4, Theorem IV.5.5]). There is a Hilbert space \( L \) and an isometry \( w \in B(H, K \otimes L) \) such that \( \pi(b) = w^*(b \otimes 1)w \).

Let \( L \) and \( w \) be as in Theorem 1.1 above.

Proposition 1.2. For all \( b \in N \), \((b \otimes 1)w = w\pi(b)\).  

Proof. For all \( b \in N \), 

\[ w^*(b^* \otimes 1)(b \otimes 1)w^* = w\pi(b^*)w^* = w\pi(b)(b^*)w^* = w^*(b^* \otimes 1)w^*(b \otimes 1)w^* \]

so \( w^*(b^* \otimes 1)(1 - ww^*)(b \otimes 1)w^* = 0 \). Since \( 1 - ww^* \) is a projection, we conclude that \( (1 - ww^*)(N \overline{\text{span} C_L})w^* = 0 \), i.e., \( ww^* \in (N \overline{\text{span} C_L})' \). Therefore, for all \( b \in N \), 

\[ (b \otimes 1)w = (b \otimes 1)ww^*w = w^*(b \otimes 1)w = w\pi(b) \]  

Definition 1.3. The set \( E(\pi) = \{ v \in B(H, K) \mid \forall b \in N \, bv = v\pi(b) \} \) is a quantum relation between \( M \) and \( N \).
Proposition 1.4. \( \mathcal{M}' \subseteq \mathfrak{S}(\pi)^* \mathfrak{S}(\pi) \)

Proof. Choose a basis \( \{e_\alpha\}_{\alpha \in I} \) of \( \mathcal{L} \). For each \( \alpha \in I \), \( (1 \otimes \tilde{e}_\alpha)(w) = (1 \otimes \tilde{e}_\alpha)(b \otimes 1) = (1 \otimes \tilde{e}_\alpha)w \pi(b) \) for all \( b \in \mathcal{N} \), i.e., \( (1 \otimes \tilde{e}_\alpha)w \in \mathfrak{S}(\pi) \). It follows that

\[
1 = w^*w = \sum_{\alpha \in I} w^* (1 \otimes \tilde{e}_\alpha \tilde{e}_\alpha^*) w = \sum_{\alpha \in I} ((1 \otimes \tilde{e}_\alpha w)^* ((1 \otimes \tilde{e}_\alpha) w) ,
\]
so \( \mathcal{C}_H \subseteq \mathfrak{S}(\pi)^* \mathfrak{S}(\pi) \). We conclude that \( \mathcal{M}' = \mathcal{C}_H \mathcal{M}' \subseteq \mathfrak{S}(\pi)^* \mathfrak{S}(\pi) \mathcal{M}' = \mathfrak{S}(\pi)^* \mathfrak{S}(\pi) \).

\[\square\]

Proposition 1.5. \( \mathfrak{S}(\pi)^\varepsilon \mathfrak{S}(\pi)^* \subseteq \mathcal{N}' \)

Proof. For all \( v_0, v_1 \in \mathfrak{S}(\pi) \), and all \( b \in \mathcal{N} \), \( bv_0 v_1^* = v_0 \pi(b) v_1^* = v_0 (v_1 \pi(b^*))^* = v_0 v_1^* b \).

\[\square\]

Proposition 1.6. Therefore \( \mathfrak{S}(\pi) \) is a quantum function from \( \mathcal{M} \) to \( \mathcal{N} \).

Thus, we have defined a function \( \mathfrak{S} : \mathfrak{vN}(\mathcal{N}, \mathcal{M}) \rightarrow \mathfrak{qF}(\mathcal{M}, \mathcal{N}) \).

2. Homomorphisms from Functions

Let \( \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}) \) and \( \mathcal{N} \subseteq \mathcal{B}(\mathcal{K}) \) be von Neumann algebras, and \( \mathcal{V} \) a quantum function from \( \mathcal{M} \) to \( \mathcal{N} \).

Lemma 2.1. There exists a family \( \{u_\alpha\}_{\alpha \in I} \) of partial isometries in \( \mathcal{V} \) such that

1. for all distinct \( \alpha, \beta \in I \), \( u_\alpha u_\beta^* = 0 \), and
2. \( \sum_{\alpha} u_\alpha^* u_\alpha = 1 \).

Proof. Let \( \mathcal{F} \) be the collection of all sets \( S \) of partial isometries in \( \mathcal{V} \) such that \( u u^* = 0 \) whenever \( u, \tilde{u} \in S \) are distinct. Applying Zorn’s Lemma, we obtain a maximal such set \( \hat{S} \).

Suppose that \( \sum_{\alpha \in \hat{S}} u_\alpha u_\alpha^* \neq 1 \), and let \( p = 1 - \sum_{\alpha \in \hat{S}} u_\alpha u_\alpha^* \). The subspace \( \mathcal{V}p \) is non-zero because \( 1 \in \mathcal{M}' \subseteq \mathcal{V} \mathcal{V} \). Therefore, pick \( v \neq 0 \) in \( \mathcal{V}p \subseteq \mathcal{V} \). We will now obtain a partial isometry to add to \( \hat{S} \) from the polar decomposition of \( v \).

Let \( W^*(\mathcal{V}) \) be the von Neumann algebra generated by \( v \), which is the ultraweakly closed subspace of \( \mathcal{B}(\mathcal{H} \otimes \mathcal{K}) \) generated by finite words in the elements of \( \mathcal{V} \) and their conjugates. Since \( \mathcal{V} \mathcal{V}^* \mathcal{V} \subseteq \mathcal{N}^* \mathcal{N} \subseteq \mathcal{V}^* \mathcal{V} \), \( W^*(\mathcal{V}) \) is in fact generated by words of one of the following forms: \( 1, v_0, v_0^* v_0, v_1 v_0^* \). We conclude that \( [\mathcal{K}] [W^*(\mathcal{V})][\mathcal{H}] = \mathcal{V} \), where \( [\mathcal{H}] \) and \( [\mathcal{K}] \) denote projections onto \( \mathcal{H} \) and \( \mathcal{K} \) respectively.

Let \( v = u_v |v| \) be the polar decomposition of \( x \). Since \( [\mathcal{K}] v [\mathcal{H}] = v, [\mathcal{K}] u_v [\mathcal{H}] = u_v \), so \( u_v \in \mathcal{V} \). Furthermore, since \( v p = v, u_v p = u_v \), so \( u_v u_v^* = 0 \) for all \( u \in \hat{S} \). Thus, \( \hat{S} \cup \{u_v\} \) is an element of \( \mathcal{F} \) strictly larger than \( \hat{S} \), a contradiction.

\[\square\]

Definition 2.2. Let \( \mathfrak{S}^{-1}(\mathcal{V}) \) be the normal unital *-homomorphism defined by

\[
\mathfrak{S}^{-1}(\mathcal{V})(b) = \sum_{\alpha \in I} u_\alpha^* bu_\alpha = w_1^* (b \otimes 1) w_1 ,
\]
where \( \{u_\alpha\}_{\alpha \in I} \) is any family of partial isometries in \( \mathcal{V} \) such that \( \sum_{\alpha} u_\alpha^* u_\alpha = 1 \) and \( u_\alpha u_\beta^* = 0 \) whenever \( \alpha \neq \beta \), and the isometry \( w_1 \in B(\mathcal{H}, \mathcal{K} \otimes \ell^2(I)) \) is defined by

\[
w_1 \xi = \sum_{\alpha \in I} u_\alpha \xi \otimes e_\alpha.
\]

\( ^3 \)Lemma 2.1 is a special case of Paschke’s structural theorem for self-dual Hilbert \( W^* \)-modules (\( ^3 \) Theorem 3.12).
Proposition 2.3. The normal unital \(*\)-homomorphism $\mathcal{G}^{-1}(\mathcal{V})$ is well defined.

Proof. Apply Lemma 2.1 above to obtain a family $\{u_{\alpha}\}_{\alpha \in I}$ of partial isometries in $\mathcal{V}$ such that $\sum_{\alpha} u_{\alpha}^* u_{\alpha} = 1$ and $u_{\alpha} u_{\beta}^* = 0$ whenever $\alpha \neq \beta$. Let $w_I = \sum_{\alpha} u_{\alpha} \otimes \varepsilon_\alpha \in \mathcal{B}(\mathcal{H}, \mathcal{K} \otimes l^2(I))$. For all $\xi \in \mathcal{H}$,

$$||w_I(\xi)||^2 = \sum_{\alpha, \beta} (u_{\alpha}(\xi) \varepsilon_\alpha u_{\beta}(\xi) \varepsilon_\beta) = \sum_{\alpha} \langle u_{\alpha} \xi | u_{\alpha} \xi \rangle = \langle \xi \parallel \sum_{\alpha} u_{\alpha}^* u_{\alpha} \parallel \xi \rangle = ||\xi||^2.$$  

Thus, $w_I$ is an isometry, and we may now define a normal unital completely positive map $\pi_I : \mathcal{N} \to \mathcal{B}(\mathcal{H})$ by $\pi_I(b) = w_I^* (b \otimes 1) w_I = \sum_{\alpha} u_{\alpha}^* b u_{\alpha}$. For all $b_0, b_1 \in \mathcal{N}$,

$$\pi_I(b_0) \pi_I(b_1) = \left( \sum_{\alpha} u_{\alpha}^* b_0 u_{\alpha} \right) \left( \sum_{\beta} u_{\beta}^* b_1 u_{\beta} \right) = \sum_{\alpha} u_{\alpha}^* b_0 u_{\alpha} \sum_{\beta} u_{\beta}^* b_1 u_{\beta} = \sum_{\alpha} \sum_{\beta} u_{\alpha}^* b_0 b_1 u_{\alpha} u_{\beta} = \sum_{\alpha} \sum_{\beta} u_{\alpha}^* b_0 b_1 u_{\alpha} = \pi_I(b_0 b_1)$$

because $u_{\alpha} u_{\beta}^* \in \mathcal{V} \mathcal{V}^* \subseteq \mathcal{N}'$. We conclude that $\pi_I$ is a normal unital $\ast$-homomorphism.

For all $b \in \mathcal{N}$, $c \in \mathcal{M}'$,

$$c \pi_I(b) = \sum_{\alpha, \beta \in I} u_{\alpha}^* u_{\alpha} c u_{\beta}^* b u_{\beta} = \sum_{\alpha, \beta \in I} u_{\alpha}^* b u_{\alpha} c u_{\beta}^* u_{\beta} = \pi_I(b)c$$

because $u_{\alpha} c u_{\beta}^* \in \mathcal{M}' \mathcal{V}^* \subseteq \mathcal{V} \mathcal{V}^* \subseteq \mathcal{N}'$. By the Double Commutant Theorem, $\pi_I(\mathcal{N}) \subseteq \mathcal{M}$, so $\pi_I$ may be viewed as a normal unital $\ast$-homomorphism $\mathcal{N} \to \mathcal{M}$.

If $\{u_{\alpha}\}_{\alpha \in I}$ is another family of partial isometries that satisfies $\sum_{\alpha \in I} u_{\alpha}^* u_{\alpha} = 1$ and $u_{\alpha} u_{\beta}^* = 0$ for distinct $\alpha, \beta \in J$, we may obtain in the same way a normal unital $\ast$-homomorphism $\pi_J : \mathcal{N} \to \mathcal{M}$. However, for all $b \in \mathcal{N}$,

$$\pi_J(b) = \sum_{\alpha, \beta \in I} u_{\alpha}^* u_{\alpha} u_{\beta}^* b u_{\beta} = \sum_{\alpha, \beta \in I} u_{\alpha}^* b u_{\alpha} u_{\beta}^* u_{\beta} = \pi_I(b)$$

because $u_{\alpha} u_{\beta}^* \in \mathcal{V} \mathcal{V}^* \subseteq \mathcal{N}'$. Thus, $\pi_I$ is independent of our choice of family $\{u_{\alpha}\}_{\alpha \in I}$, and we may define $\mathcal{G}^{-1}(\mathcal{V}) = \pi_I$. □

Thus, we have defined a function $\mathcal{G}^{-1} : \mathcal{qF}(\mathcal{M}, \mathcal{N}) \to \mathcal{vN}(\mathcal{N}, \mathcal{M})$.

3. $\mathcal{G}^{-1}$ is the Inverse of $\mathcal{G}$

Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ be von Neumann algebras.

Proposition 3.1. Let $\pi : \mathcal{N} \to \mathcal{M}$ be a normal unital $\ast$-homomorphism. Then $\mathcal{G}^{-1}(\mathcal{G}(\pi)) = \pi$.

Proof. Let $\{u_{\alpha}\}_{\alpha \in I}$ be a family of partial isometries in $\mathcal{G}(\pi)$ such that $\sum_{\alpha \in I} u_{\alpha}^* u_{\alpha} = 1$ and $u_{\alpha} u_{\beta}^* = 0$ whenever $\alpha \neq \beta$. For all $b \in \mathcal{N}$,

$$\mathcal{G}^{-1}(\mathcal{G}(\pi))(b) = \sum_{\alpha \in I} u_{\alpha}^* b u_{\alpha} = \sum_{\alpha \in I} u_{\alpha}^* u_{\alpha} \pi(b) = \pi(b).$$
Proposition 3.2. Let $L$ be a Hilbert space, and let $w_0, w_1 \in B(H, K \otimes L)$ be isometries such that $\pi_1(b) = w_1^* (b \otimes 1) w_1$ defines a pair of $\ast$-homomorphisms $N \to M$. If $\pi_0 = \pi_1$, then $w_0 w_1^* \in (N \otimes \mathbb{C} L)'$.

Proof. For all $b \in N$, $(b \otimes 1) w_0 w_1^* = w_0 \pi_0(b) w_1^* = w_0 \pi_1(b) w_1^* = w_0 (w_1 \pi_1(b^*)^*) = w_0 ((b^* \otimes 1) w_1)^*$.

\[ \square \]

Lemma 3.3. Let $\nu$ be a quantum function from $M$ to $N$, and let $\{u_\alpha\}_{\alpha \in I}$ and $w_I$ be as in Definition \ref{def:quantum_function}. Then $(N \otimes \mathbb{C} \ell^2(1))^I w_I M' = \mathcal{V} \mathcal{B}(\mathbb{C}, \ell^2(1))$.

Proof. By definition, $w_I = \sum u_\alpha \otimes \hat{e}_\alpha \in N \otimes \mathcal{B}(\mathbb{C}, \ell^2(I))$, so $(N \otimes \mathcal{B}(\ell^2(I))) w_I M' = (N \otimes \mathcal{B}(\ell^2(I))) w_I M' \subseteq (N \otimes \mathcal{B}(\ell^2(I))) w_I M'$.

Let $f \in \ell^2(I)$ and $v \in \nu$. Then for all $\alpha \in I$,

\[ u_\alpha \otimes \hat{f} = \sum_{\beta \in I} (1 \otimes \hat{e}_\alpha^*)(u_\beta \otimes \hat{e}_\beta) = (1 \otimes \hat{e}_\alpha^*) w_I \in (N \otimes \mathcal{B}(\ell^2(I))) w_I M' , \]

so

\[ v \otimes \hat{f} = \sum_{\alpha \in I} (v \otimes \hat{f})(u_\alpha^* u_\alpha) = \sum_{\alpha \in I} (w_\alpha^* \otimes 1)(u_\alpha \otimes \hat{f}) \in (N \otimes \mathcal{B}(\ell^2(I))) w_I M' \]

because $w_\alpha^* \in \mathcal{V} \mathcal{V}^* \subseteq N'$. It follows that $\mathcal{V} \mathcal{B}(\mathbb{C}, \ell^2(I)) \subseteq (N \otimes \mathcal{B}(\ell^2(I))) w_I M'$, concluding the proof. \[ \square \]

Proposition 3.4. The function $\theta^{-1}$ is injective.

Proof. For $k \in \{0, 1\}$, let $\nu_k \in \mathfrak{B}(M, N)$, and let $\{u_\alpha\}_{\alpha \in I_k}$ be a family of partial isometries in $\nu_k$ such that $u_\alpha u_\alpha^* = 0$ for distinct $\alpha, \beta \in I_k$, and $\sum u_\alpha^* u_\alpha = 1$. We may assume that $I_0$ and $I_1$ have equal, non-zero cardinality by throwing in indexed instances of the zero partial isometry where necessary. Thus, we may choose a unitary $s \in B(\ell^2(I_0), \ell^2(I_1))$.

Suppose that $\varphi^{-1}(\nu_0) = \varphi^{-1}(\nu_1)$. For all $b \in N$,

\[ ((1 \otimes s) w_{I_0})^*(b \otimes 1)((1 \otimes s) w_{I_0}) = w_{I_0}^* (b \otimes 1) w_{I_0} = \varphi^{-1}(\nu_0)(b) \]

By Proposition 3.2, $(1 \otimes s) w_{I_0} w_{I_1}^* \in (N \otimes \mathcal{B}(\ell^2(I_1)))'$.

By Lemma 3.3 above,

\[ \nu_1 \mathcal{B}(\mathbb{C}, \ell^2(I_1)) = (N \otimes \mathcal{B}(\ell^2(I_1)))' w_{I_1} M' \]

\[ \supseteq (N \otimes \mathcal{B}(\ell^2(I_1)))' ((1 \otimes s) w_{I_0} w_{I_1}^*) w_{I_1} M' \]

\[ = (N \otimes \mathcal{B}(\ell^2(I_1)))' (1 \otimes s) w_{I_0} M' \]

\[ = (1 \otimes s)(N \otimes \mathcal{B}(\ell^2(I_0))) w_{I_0} M' \]

\[ = (1 \otimes s)(\nu_0 \mathcal{B}(\mathbb{C}, \ell^2(I_0))) \]

\[ = \nu_0 \mathcal{B}(\mathbb{C}, \ell^2(I_1)) . \]

Choosing an arbitrary unit vector $f \in \ell^2(I_1)$, we conclude that

\[ \nu_1 = (1 \otimes \hat{f}^*) (\nu_1 \mathcal{B}(\mathbb{C}, \ell^2(I_1))) \supseteq (1 \otimes \hat{f}^*) (\nu_0 \mathcal{B}(\mathbb{C}, \ell^2(I_1))) = \nu_0 . \]

Similarly, $\nu_1 \subseteq \nu_0$, so $\nu_1 = \nu_0$. \[ \square \]
Theorem 3.5. Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ and $\mathcal{N} \subseteq \mathcal{B}(\mathcal{K})$ be von Neumann algebras. The function
$$\mathcal{G} : vN(\mathcal{N}, \mathcal{M}) \rightarrow qF(\mathcal{M}, \mathcal{N})$$
defined by
$$\mathcal{G}(\pi) = \{ v \in \mathcal{B}(\mathcal{H}, \mathcal{K}) | \forall b \in \mathcal{N} \ bv = \pi(b)v \}$$
is a bijection.

Proof. The theorem follows immediately from Propositions 3.1 and 3.4. □

4. Functoriality

Proposition 4.1. Let $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_2$ be von Neumann algebras. If $V_0 \in qF(\mathcal{M}_0, \mathcal{M}_1)$ and $V_1 \in qF(\mathcal{M}_1, \mathcal{M}_2)$, then $V_1 V_0 \in qF(\mathcal{M}_0, \mathcal{M}_2)$.

Proof. Clearly, $V_0 \subseteq V_0^* V_0 \subseteq V_0^* M_1 V_0 \subseteq V_0^* M_1^* V_1 V_0 = (V_1 V_0)^* (V_1 V_0)$ For any two morphisms
$$V_1 V_0 (V_1 V_0)^* = V_1 V_0 V_0^* V_1 \subseteq V_1 M_1' V_1^* \subseteq V_1 V_1^* \subseteq M_2$$

□

Proposition 4.2. Let $\mathcal{M}_0$, $\mathcal{M}_1$, and $\mathcal{M}_2$ be von Neumann algebras. If $\pi_1 : \mathcal{M}_2 \rightarrow \mathcal{M}_1$ and $\pi_0 : \mathcal{M}_1 \rightarrow \mathcal{M}_0$ are normal unital $*$-homomorphisms, then $\mathcal{G}(\pi_0 \circ \pi_1) = \mathcal{G}(\pi_1) \mathcal{G}(\pi_0)$.

Proof. Clearly, $\mathcal{G}(\pi_1) \mathcal{G}(\pi_0) \subseteq \mathcal{G}(\pi_0 \circ \pi_1)$. By Definition 2.2 \ $\mathcal{G}^{-1}(\mathcal{G}(\pi_1) \mathcal{G}(\pi_0)) = \mathcal{G}^{-1}(\mathcal{G}(\pi_0 \circ \pi_1))$. By Theorem 3.5 we conclude that $\mathcal{G}(\pi_1) \mathcal{G}(\pi_0) = \mathcal{G}(\pi_0 \circ \pi_1)$. □

Proposition 4.3. Let $\mathcal{M}$ be a von Neumann algebra, and let $\iota : \mathcal{M} \rightarrow \mathcal{M}$ be the identity $*$-homomorphism. Then $\mathcal{G}(\iota) = \mathcal{M}'$.

Proof. This is an immediate consequence of Definition 1.3 □

Definition 4.4. Let $vN$ be the category whose objects are von Neumann algebras, and whose morphisms are normal unital $*$-homomorphisms.

Definition 4.5. Let $qF$ be the following category:

- The objects of $qF$ are von Neumann algebras.
- For any two objects $\mathcal{M}$ and $\mathcal{N}$, a morphism from $\mathcal{M}$ to $\mathcal{N}$ is a quantum function from $\mathcal{M}$ to $\mathcal{N}$.
- For any two morphisms $V_0 \in qF(\mathcal{M}_0, \mathcal{M}_1)$ and $V_1 \in qF(\mathcal{M}_1, \mathcal{M}_2)$, $V_1 \circ V_0 = V_1 V_0$.
- For any object $\mathcal{M}$, the identity morphism at $\mathcal{M}$ is the quantum function $\mathcal{M}' \in qF(\mathcal{M}, \mathcal{M})$.

Theorem 4.6. The functor $\mathcal{G} : vN \rightarrow qF$ defined by

- for all objects $\mathcal{M}$ of $vN$, $\mathcal{G}(\mathcal{M}) = \mathcal{M}$, and
- for all morphisms $\pi \in vN(\mathcal{N} \subseteq \mathcal{B}(\mathcal{K}), \mathcal{M} \subseteq \mathcal{B}(\mathcal{H}))$,
$$\mathcal{G}(\pi) = \{ v \in \mathcal{B}(\mathcal{H}, \mathcal{K}) | \forall b \in \mathcal{N} \bv = \nu \pi(b) \}$$
is a coisomorphism of categories.

Proof. This is a straightforward consequence of Proposition 4.2, Proposition 4.3 and Theorem 3.5. □
References

[1] G. Kuperberg and N. Weaver, *A Von Neumann Algebra Approach to Quantum Metrics*, arXiv:1005.0353v2.

[2] V. M. Manuilov and E. V. Troitsky, *Hilbert C*-Modules*, Translations of Mathematical Monographs \textbf{226} (2005).

[3] W. L. Paschke, *Inner Product Modules over B*-algebras*, Transactions of the American Mathematical Society \textbf{182} (1973), 443-468.

[4] M. Takesaki, *Theory of Operator Algebras I*, Springer, 1979.

[5] N. Weaver, *Quantum Relations*, arXiv:1005.0354v1.

Department of Mathematics, University of California, Berkeley, CA 94720-3840
E-mail address: kornell@math.berkeley.edu