A paradox about an atom and a photon

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In this article we propose a new relativistic paradox concerning the absorption of a photon by a hydrogen atom. We show that the actual cause of the paradox is one of the hypotheses of Bohr model; therefore, in order to solve the paradox, we have to move away from Bohr model. Our analysis is carried out only in the special relativistic framework, so we are not interested in giving a full quantum mechanical treatment of the problem. We derive some expressions for emission and absorption of photons by atoms, which are in perfect agreement with special relativity, although comparable to the classical Bohr formula with an excellent degree of approximation. Quite interestingly, these expressions are no more invariant under a global shift of energy levels, showing a breaking of classical “gauge invariance” of energy. We stress that, to the best of our knowledge, the present approach has never been considered in literature. At the end we will be able to solve the proposed paradox.

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I. INTRODUCTION

So far the special theory of relativity has proven to be one of the most fruitful theories both for the physical consequences and the great conceptual depth, which has allowed us to get a better insight about Nature. Moreover, special relativity is often counter-intuitive to common sense, given our experience in a world in which Galilean relativity is in force up to an excellent degree of approximation.

Just because of the conceptual relevance of special relativity and of its counter-intuitiveness, the use of paradoxes is one of the essential tools to gain a deeper comprehension of the theory itself and of its applications. It is enough to recall the famous twin paradox and ladder paradox, see [1].

In every special relativistic paradox there is a physical phenomenon described by (at least) two different observers, linked to as many inertial reference frames. The paradox consists in the fact that these different observers give contradictory descriptions of the phenomenon, and this fact is in contrast with relativity principle. Actually, at least one of the observers makes a mistake in his assumptions or when trying to apply physical laws to the phenomenon in his reference frame, so the violation of relativity principle is only seeming.

Therefore, when solving a paradox, there are some goals to achieve. In particular we must understand

1. which of the observers is (are) wrong
2. what actually happens
3. why that (those) observer(s) is (are) wrong

According to this working scheme, the great conceptual importance of solving a paradox is apparent. In particular, through the treatment of point 3 one has the opportunity to detect some insidious errors in the application of the theory, thus allowing a deeper and deeper understanding of the theory. Wheeler [2] himself stated the importance of paradoxes in physical progress and understanding.

In view of the importance of the paradox tool, in this article we will start from a new relativistic paradox, which, from what we know, has never been presented before in literature. In this paradox, a hydrogen atom interacts with a photon, and different observers come to different conclusions about the absorption of the photon.

Emission and absorption of photons by atoms are well known phenomena and they may look rather simple. However, in this article we will show that they are not as trivial as they might appear at first glance.

In particular, Bohr model of hydrogen atom [3–5] establishes that an atom can absorb or emit photons with energies equal to energy gaps between atomic levels. We will show that this assumption, which is the core of this paradox, leads to inconsistencies from a relativistic perspective.

In this way, we will be able to show once more that Bohr model of hydrogen atom is definitely ruled out because of its incompatibility with special relativity. This conclusion should not surprise, because it is a well-known and established fact, but it is interesting to explore the origin of this contradiction, and we will do that by means of a paradox. We would like to stress that such an approach to the incompatibility of Bohr model with special relativity has never been considered in literature for all we know.

Therefore this paradox will be an opportunity to discuss the role of photons in atomic excitation and disexcitation from a purely relativistic perspective.

Since we are mainly interested in special relativity, we will not deal with the problem from a quantum mechanical point of view.

II. STATEMENT OF THE PARADOX

Now we are ready to state the paradox, which is the core of this article.

Let $K$ be an inertial frame (the laboratory frame). A hydrogen atom in its ground state is moving in $K$ along the positive direction of the $x$ axis with speed $v$. A photon with energy $E_0$ is moving in the opposite direction. Let $E_0$ be less than the energy necessary in order to get the atom to its first excited state. An observer in $K$ states that the photon will not be absorbed by the atom.

Let $K'$ be a reference frame which is stationary relative to the hydrogen atom, viz. it is moving with speed $v$ relative to $K$. Due to Doppler effect, an observer in $K'$ sees the photon with a higher frequency, and therefore with a higher energy. For a suitable speed $v$, it is possible in $K'$ to see the photon just with the energy of the first quantum jump. In this case, an observer in $K'$ will state that the photon will be absorbed.

Let us assume, for the sake of simplicity, that the spatial axes of $K'$ are parallel to those of $K$ and that time flows in the same direction both in $K$ and $K'$.

With a careful analysis we can isolate two tacit assumptions in this paradox:
1. the energies of the various quantum jumps are the same in every inertial frame
2. a photon can be absorbed only if it has the energy of a quantum jump (Bohr hypothesis)

Now we will examine these two assumptions critically. Throughout this article we will adopt the metric signature $+−−−$ for Minkowski metric tensor $\eta_{\mu\nu}$.

### III. ARE ENERGY LEVELS INVARIANT?

For the purposes of this article, we will not need to deal with the relativistic quantum mechanical treatment of the hydrogen atom, even though we will examine the paradox in the special relativistic framework. For the sake of simplicity, we will follow the “classical” quantum mechanical paradigm.

From this treatment of the hydrogen atom, it is well known that the Hamiltonian eigenvalues (from now on called simply “energy levels”) are

$$E_n = -\frac{1}{2} \mu c^2 \alpha^2 \frac{1}{n^2}, \quad (1)$$

where $\mu$ is the reduced mass of the system, $\alpha$ is the fine-structure constant, and $n$ is a strictly positive integer. A so-called “quantum jump” is nothing but the energy gap between two energy levels of the atom.

Now we have to deal with the energy levels of eq. (1) in the relativistic framework. It is then convenient to introduce the effective mass $m_n$ of the hydrogen atom in its $n$-th energy level. The effective mass is defined as

$$m_n := M + \frac{E_n}{c^2} = M - \frac{1}{2} \mu \alpha^2 \frac{1}{n^2}, \quad (2)$$

where $M$ is the total mass of the electron-proton system. A quick check assures us that $m_n > 0$ for every positive integer $n$; so this quantity is always well-defined.

A question now arises: are the energy levels of eq. (1) invariant? Or are they related to a particular reference frame?

From the quantum mechanical treatment of the hydrogen atom, energy levels are naturally related to the inertial frame which is stationary with respect to the center of mass of the atom. That frame is $K'$.

In order to find the energy levels of the hydrogen atom in another frame $K'$, moving with constant speed $v$ with respect to $K'$, we have to transform the 4-momentum of the atom, where the effective mass behaves for all practical purposes exactly as a rest mass.

Thus, we have that the total energy of the hydrogen atom in $K$ is

$$E = \gamma M c^2 - \frac{1}{2} \gamma \mu c^2 \alpha^2 \frac{1}{n^2},$$

where, as usual, $\gamma = \left(1 - \beta^2\right)^{-1/2}$ and $\beta = v/c$.

There is a correction term $-\gamma \mu c^2 \alpha^2 / (2n^2)$ to the energy $\gamma M c^2$ of the compound system given by proton and electron, because of the binding energy of the electron. Therefore, the energy levels of the moving atom are not the same as those of the stationary atom; more precisely they are

$$E_n (v) = -\frac{1}{2} \gamma \mu c^2 \alpha^2 \frac{1}{n^2}. \quad (3)$$

We notice the presence of the multiplicative corrective term $\gamma$ to the energy levels of eq. (1), which are seen in the center of mass reference frame ($K'$). Since the correction is multiplicative, and not additive, quantum jumps in $K$ are not the same as in $K'$.

We have just got the main result of this section: the energy levels of a hydrogen atom observed in a given inertial frame depend on the motion state of the atom with respect to the given inertial frame. Therefore, the assumption of invariance of energy levels and quantum jumps is wrong.

### IV. ARE PHOTON ENERGIES THE SAME AS THOSE OF QUANTUM JUMPS?

Now let us analyze the issue of photon absorption and emission by atoms. From Bohr model Ritz-Rydberg formula follows easily, see [4, 5]. Ritz-Rydberg formula relates the emitted wavelengths $\lambda$ to the quantum transition between two different energy levels.

$$\frac{1}{\lambda} = \frac{\mu e^4}{8 \pi^2 \hbar^3 c} \left( \frac{1}{n^2} - \frac{1}{m^2} \right),$$

where $m$ is the mass of the electron in the ground state of the atom.
where $m > n$. Here $m$ and $n$ are positive integers respectively labeling the energy levels $E_n$ and $E_m$ of eq. (1). The energy $E_{m,n}$ of the emitted photons in the transition from the $m$-th to the $n$-th energy level is given by $E_{m,n} = \frac{hc}{\lambda}$, whence

$$E_{m,n} = E_m - E_n.$$  (4)

In this way we are naturally led to assume that emitted (and likewise absorbed) photons have energies corresponding to those of quantum jumps.

Nevertheless, in this treatment we have completely neglected linear momentum and energy conservation laws. Actually, we have to take into account 4-momentum conservation for the system made up of the atom and the photon, since the interaction of the hydrogen atom with the photon does not involve external 4-forces. Total 4-momentum conservation law thus holds both for emission and absorption of photons.

However, the analysis of photon emission is perhaps conceptually simpler than the absorption case, so we will begin analyzing the emission case.

A. The emission case

1. Emission by an atom at rest

Let us consider a hydrogen atom in its $m$-th energy level, and let us suppose that the atom is stationary relative to the laboratory reference frame $K$. From now on, we will analyze the physical situation in this frame. The atom returns to a lower energy level $E_n$ ($n < m$) through the emission of a photon.

Now we will apply 4-momentum conservation law to the emission process, using the effective mass, defined in eq. (2). Let us suppose that the photon is emitted in the laboratory frame along the direction defined by the unit vector $\vec{n}$.

From 4-momentum conservation, and from the mass-shell relation with the effective mass, we get an equation for $E_{m,n}$, which yields (cf. [7])

$$E_{m,n} = \left[ 1 - \frac{E_m - E_n}{2(Mc^2 + E_m)} \right] (E_m - E_n).$$  (5)

Notice that the result of eq. (5) does not depend on the direction of emission $\vec{n}$ in the laboratory frame, according to space isotropy.

However, the most important fact is that there is a corrective term, so the square bracket is always strictly less than 1, provided $m \neq n$, as it is the case. We conclude that $E_{m,n} < E_m - E_n$.

Remark 1. From 4-momentum conservation it easy to see that a hydrogen atom cannot jump to a lower energy level without the emission of a photon.

In fact, from 3-momentum conservation, if a disexcitation without a photon emission were possible, the atom would be at rest both in the initial and in the final state. Therefore, from energy conservation, we would have $m_n = m_m$, which is impossible because $m \neq n$.

Since 4-momentum conservation is a covariant law, this result, derived for an atom at rest, holds also for an atom moving at constant velocity.

Now we want to investigate if 4-momentum conservation provides us with some selection rule on $m$ and $n$. Therefore we ask ourselves if the square bracket of eq. (5) may vanish. In that case there would be no emission at all. However, it is not difficult to show that this fact can never happen, so 4-momentum conservation imposes no selection rule.

Let us evaluate the result of eq. (5) also from a numerical point of view. In a hydrogen atom, $Mc^2 \sim 1$ GeV, whereas $|E_n| < 13.6$ eV, so the second term in the square bracket is negligible. Therefore, up to an excellent degree of approximation, eq. (5) drops to eq. (4). Nevertheless, from a rigorous point of view, the emitted photon does not have the energy of the quantum jump, but a lower energy, due to the fact that the atom recoils. In the continuation of this article, we will always proceed in a rigorous manner, so we will not neglect the atom recoil, and we will always use eq. (5), which is in agreement with 4-momentum conservation, unlike eq. (4).

So far we have analyzed the emission of a photon in the laboratory reference frame. Now let us move to consider the frame which is stationary relative to the hydrogen atom after the emission of the photon. What energy of the photon will an observer in this frame see?

Let $u^\mu$ be the recoil 4-velocity of the atom; and here we take $u^\mu$ to be dimensionless. Using tetrad formalism, it is now easy to calculate the energy $E'_{m,n}$ of the emitted photon according to an observer co-moving with the recoiling atom. $E'_{m,n}$ is given by

$$E'_{m,n} = cp_0^\mu u_\mu,$$
where $p^0_\mu$ is the emitted photon 4-momentum. After some passages, we come to an equation very close to eq. (5), which is

$$E'_{m,n} = \left[ 1 + \frac{E_m - E_n}{2(Mc^2 + E_n)} \right] (E_m - E_n).$$

In this equation there is a plus sign, so, according to the co-moving observer, the energy of the emitted photon is greater than the energy gap between $E_m$ and $E_n$.

We conclude that neither in the recoiling atom co-moving frame the photon is emitted with the same energy as the energy gap between the two energy levels.

Therefore, at least in the emission case from an atom at rest, the assumption that an emitted photon has an energy equal to that of the quantum jump proves to be wrong.

2. Emission by a moving atom

Let us examine the case of photon emission by a hydrogen atom moving at constant speed $v$ along the $x$ axis of the laboratory reference frame $K$.

We might think that the relation given by eq. (5) holds in this case too, provided we replace $E_n$ and $E_m$ with the energy levels $E_n(v)$ and $E_m(v)$ of a moving atom, given by eq. (3), and we put $\gamma M$ in place of $M$. In this way we would get

$$E_{m,n} = \gamma E_{m,n}^*,$$

where $E_{m,n}^*$ is the energy of the emitted photon when the atom is at rest, see eq. (5). Nevertheless, we can immediately rule out this result by a general argument. The fact that the atom is moving along the $x$ axis breaks space isotropy, so an equation describing photon emission has to depend on the direction of emission of the photon. The result in eq. (6) is independent of the direction of emission, therefore it has to be ruled out.

Let us find the correct expression. Let $\theta$ be the angle between the direction of emission, given by the unit vector $\vec{n}$, and the $x$ axis of $K$. We can transform the photon 4-momentum from the laboratory frame to the co-moving frame. We obtain

$$E_{m,n} = \frac{E_{m,n}^*}{\gamma (1 - \beta \cos \theta)},$$

where $E_{m,n}^*$ is again given by eq. (5). Notice that eq. (7) correctly depends on the direction of the emission of the photon through the angle $\theta$.

In particular, one has the minimum energy for the emitted photon when $\theta = \pi$, that is when the photon is emitted in the opposite direction to the direction of motion; the maximum energy $E_{\text{max}}$ of the emitted photon is instead attained when $\theta = 0$, that is when the photon is emitted in a parallel direction to the direction of motion.

It is not difficult to prove that for a speed $v$ such that

$$\frac{1 - A^2}{1 + A^2} < \beta < 1,$$

where $A$ is the square bracket of eq. (5), we have $E_{\text{max}} > E_m - E_n$. In this way the energy of the emitted photon is greater than the energy gap between $E_m$ and $E_n$. In particular, for $\beta = (1 - A^2) / (1 + A^2)$, we drop again to eq. (4) and Ritz-Rydberg formula: in this particular case the photon has the same energy as the gap $E_m - E_n$.

B. The absorption case

1. Absorption by an atom at rest

Now, let us move to examine a situation which is closer to our paradox. Let us consider a hydrogen atom in its $n$-th energy level at rest in the laboratory frame $K$. A photon with energy $E_0$ in $K$ is moving towards the atom along the direction given by the unit vector $\vec{n}$ in the laboratory frame. What energy does the photon need to have in order to get the atom to its $m$-th energy level ($m > n$)?

From the treatment of the emission case, we may figure that this energy is not equal to the energy gap between the $m$-th and the $n$-th energy level. Now we will analyze the absorption case in a similar fashion to what we have done for the emission case.
By means of a perfectly analogous procedure, we finally get (cf. [7])

$$E_0 = \left[ 1 + \frac{E_m - E_n}{2(Mc^2 + E_n)} \right] (E_m - E_n).$$

(8)

Again the result is independent of the direction of the incident photon, due to space isotropy.

In the absorption case, however, the energy of the absorbed photon is higher than the energy gap of the corresponding quantum transition, due to the plus sign in the square bracket.

In this case, an analysis of atomic excitation without photon absorption is not as trivial as atomic disexcitation without photons, thus we will defer it until section [V].

It is, instead, obvious that 4-momentum conservation does not impose any selection rule also in the absorption process.

Finally, since the estimates in section [IV A 1] still hold, eq. (8) drops to eq. (4) up to an excellent degree of approximation.

2. Absorption by a moving atom

Now, let us analyze the case of photon absorption by a hydrogen atom moving with speed $v$ along the $x$ axis of the laboratory frame $K$. As in the preceding section, a photon is moving along the direction given by the unit vector $\vec{n}$.

The motion of the atom breaks space isotropy, so, as in the case of photon emission by a moving atom, an equation like eq. (6) is not allowed.

Transforming the photon 4-momentum, we get exactly eq. (7), but now $E_{m,n}^*$ is given by eq. (8).

$$E_{m,n} = \frac{E_{m,n}^*}{\gamma (1 - \beta \cos \vartheta)}$$

(9)

It is possible to show that the minimum energy $E_{\text{min}}$ (when $\vartheta = \pi$), for a speed $v$ such that

$$\frac{B^2 - 1}{B^2 + 1} < \beta < 1,$$

where $B$ is the square bracket of eq. (8), is lower than the energy gap of the quantum transition, $E_m - E_n$. In particular, for $\beta = (B^2 - 1) / (B^2 + 1)$, we drop again to eq. (4) and to Ritz-Rydberg formula: in this particular case the photon energy is the same as $E_m - E_n$.

V. WHEN THE PHOTON IS NOT ABSORBED...

In the previous section we analyzed the case when the photon has the right energy to be absorbed. What happens if the photon is not absorbed?

We wonder whether a hydrogen atom in its energy level $E_n$ can jump to a higher level $E_m$ without a photon absorption. We will carry out this analysis in the atom rest frame $K$, for the sake of simplicity, then the result will extend to every inertial frame, due to covariance.

Let us, then, consider a photon with energy $E_0$ moving along the $x$ axis towards the atom at rest. Let $\varphi$ be the angle between the scattered photon direction (after interaction) and the $x$ axis.

In this case, the calculations are not so straightforward, thus we will give more details in Appendix [A]. Instead, the treatment in this section will present only the results.

Working out Mandelstam $t$ invariant and imposing energy conservation, we have that, if excitation takes place, the scattered photon must have an energy $E_{0,\text{fin}}$, given by

$$E_{0,\text{fin}} = \frac{(m_n^2 - m_m^2) c^2 + 2m_m E_0}{2 [E_0 (1 - \cos \varphi) + m_n c^2]} c^2,$$

(10)

where $m_n$ and $m_m$ are the effective masses related to the energy levels $E_n$ and $E_m$ respectively.

Clearly it must be $E_{0,\text{fin}} > 0$. Imposing this inequality, we have that the case $E_0 < E_{\text{abs}}$, where $E_{\text{abs}}$ is the energy a photon must have to be absorbed (see eq. (8)), is ruled out. We conclude that when the photon has a lower energy than the energy necessary to be absorbed, it cannot excite the atom.

What happens if $E_0 > E_{\text{abs}}$?
This case seems to be not ruled out, because it implies \( E_{0,\text{fin}} > 0 \) and there is no apparent contradiction. However, we must require that the recoiling atom has \( \gamma \geq 1 \). Anyway, it is not hard to prove that this inequality is always fulfilled.

It seems, then, that if \( E_0 > E_{\text{abs}} \), there is no contradiction in assuming that the atom can jump to an excited state without the absorption of the photon.

Actually, our treatment cannot exclude absorption completely. In fact, using 4-momentum conservation, we abandon the attempt of an accurate description of the interaction between the atom and the photon, treating the actual interaction as if it were in a sort of “black box”. The 4-momentum conservation method is well suited only to describe the initial state of the system before going into the “black box” and the final state of the system after it has come out from the “black box”.

So, if we know that in the final state there is an excited atom and a photon, we cannot conclude that the excitation has occurred without the intervention of the photon. In fact, in the “black box” the photon can be absorbed, getting the atom from the energy level \( E_n \) to a higher energy level \( E_{m'} \) than \( E_m (m' > m) \), and then the photon is emitted, getting the atom to the energy level \( E_m \). Perhaps several of these absorption-emission processes may take place in the “black box”.

To sum up what we have obtained in this section, we can distinguish two cases when the photon does not have the right energy \( E_{\text{abs}} \) to be absorbed.

1. \( E_0 < E_{\text{abs}} \): it is impossible to obtain an atomic excitation to the \( m \)-th energy level;

2. \( E_0 > E_{\text{abs}} \): an atomic excitation to the \( m \)-th energy level is possible.

## VI. SOLUTION OF THE PARADOX

Now it is time to give the solution to the paradox stated in section [II](#).

In the paradox we are in the absorption case. In the inertial frame \( K \), the hydrogen atom in its ground state is moving with speed \( v \), and the photon is moving in the opposite direction towards the atom. Therefore, here \( \vartheta = \pi \).

The core of this paradox is the fact that it is stated in a vague way. In fact, the two observers tacitly assume that photons are absorbed only if they have an energy equal to the energy gap between two energy levels, independently of the motion state of the atom. Therefore, in this solution we will interpret the statement “in \( K \) the photon energy is less than the energy necessary to get the atom to its first excited state” as “in \( K \) the photon has an energy which is less than the energy gap of the first quantum jump”. Likewise, we will interpret “in \( K' \) the photon has the right energy to be absorbed” as “in \( K' \) the photon has the energy of the first quantum jump”.

According to the working scheme outlined in the introduction, now we will give a solution for each of the three points. So, first of all, let us describe what actually happens.

Let \( E_0 \) be the energy of the incoming photon in \( K \), such that \( E_0 < \Delta E \), where \( \Delta E \) is the energy gap between the first excited level and the ground level.

\( E_0 \) is such that \( E_0', \) the energy in the co-moving frame \( K' \), is just \( \Delta E \). Transforming the photon energy, we have

\[
E_0 = \frac{\Delta E}{\gamma (1 + \beta)}
\]

and this expression is quite different from the minimum energy expression in eq. (9), because \( E_{m,n}^* \neq \Delta E \). Therefore, the photon cannot be absorbed in \( K \).

The photon cannot be absorbed also in \( K' \), because it should have a greater energy than \( \Delta E \), as prescribed by eq. (8). Thus, the photon cannot be absorbed in both the frames and a photon detector will detect the photon after the atomic interaction both in \( K \) and \( K' \).

In addition, we can say that the hydrogen atom will not get excited because of the interaction with the photon, since the energy of the photon in \( K' \), where the atom is at rest, is less than the absorption energy \( E_{\text{abs}} \). This fact, according to section [V](#), rules out an atomic excitation to the first excited level.

So, we have understood that the photon is not absorbed both in \( K \) and \( K' \), so \( K \) is right, although this is a mere coincidence, since both their lines of reasoning are incorrect. In fact, they are based on two wrong assumptions, those outlined at the end of section [II](#).

Finally, we can explain the error made by \( K' \) (and also by \( K \)) by his faith in Bohr model, and in the two tacit assumptions of section [II](#).
VII. CONCLUSIONS

In their book [8], Aharonov and Rohrlich try to classify the different kinds of paradoxes in physics. They identify three different classes of paradoxes.

**Errors:** The paradox arises from an error in logic or in the understanding of a particular physical theory.

**Gaps:** The paradox arises by a flaw in the theory, albeit not a fatal flaw. It is simply due to a “gap” in the theory.

**Contradictions:** The paradox arises from a fatal flaw, it indicates that the theory is wrong and it has to be changed.

According to this classification scheme, the paradox presented in this article places itself between the error and the gap class. In fact, the paradox arises from an incorrect interpretation of the absorption process, but yet the error is due to the use of Bohr model.

In this way, Bohr model, with its assumption that emitted and absorbed photons have the same energies as those of quantum jumps, gives rise to a contradiction. Therefore, we must rule out this model.

Instead, special relativity yields a formula for emitted and absorbed photons between the energy levels $E_{\text{fin}}$ and $E_{\text{in}}$, which is, for an atom at rest,

$$E_{\text{ph}} = \left[1 + \frac{E_{\text{fin}} - E_{\text{in}}}{2(Mc^2 + E_{\text{in}})}\right] \Delta E,$$

(11)

where $E_{\text{ph}}$ is the photon energy and $\Delta E = |E_{\text{fin}} - E_{\text{in}}|$. This formula correctly predicts no emission or absorption if $\Delta E = 0$, i.e. the final and the initial states coincide.

It is useful to distinguish the two cases.

- In the emission case $E_{\text{fin}} < E_{\text{in}}$, so $E_{\text{ph}} < \Delta E$.
- In the absorption case $E_{\text{fin}} > E_{\text{in}}$, so $E_{\text{ph}} > \Delta E$.

We can explain these two results as follows.

- In the emission case, part of the energy gap $\Delta E$, which is a sort of energy “at disposal”, is taken by the recoiling atom.
- In the absorption case, the photon has to supply an extra energy in order to make the atom recoil after the absorption.

Notice that the photon energy is different between the emission and the absorption case. In both cases, $E_{\text{ph}} \neq \Delta E$, as opposed to Bohr model prediction.

Let us evaluate the magnitude of the corrections to Ritz-Rydberg formula, giving a numerical estimate for transitions between $E_1$ and $E_2$ for a hydrogen atom at rest.

$$\Delta E = E_2 - E_1 = \frac{3}{8} \mu c^2 \alpha^2 = 10.198 \, 719 \, 16 \, \text{eV}$$

$$E_{\text{em}} = \frac{2Mc^2 - \frac{5}{8} \mu c^2 \alpha^2}{2(Mc^2 - \frac{1}{8} \mu c^2 \alpha^2)} \Delta E = 10.198 \, 719 \, 11 \, \text{eV}$$

$$E_{\text{abs}} = \frac{2Mc^2 - \frac{5}{8} \mu c^2 \alpha^2}{2(Mc^2 - \frac{1}{8} \mu c^2 \alpha^2)} \Delta E = 10.198 \, 719 \, 22 \, \text{eV}$$

As one can see, the differences are very small ($\sim 10^{-8} \, \text{eV}$), so they are extremely difficult to detect with an experimental apparatus. In addition, probing orders of magnitude of about $10^{-8} \, \text{eV}$, we have also to take into account fine and hyperfine corrections, so this preliminary analysis is largely invalidated at these energies. Nevertheless we can conclude that, neglecting fine and hyperfine structure, empirical observations are in agreement with Bohr model up to an excellent degree of approximation. However, those tiny differences played a fundamental role in the paradox we have examined.

The formula in eq. (11) has also an interesting feature. In classical mechanics, adding a constant term $a$ to the hydrogen atom Hamiltonian does not change the dynamics of the system. This is a sort of “gauge invariance”. This fact prevents us from knowing what constant $a$ we have chosen to define the Hamiltonian.
In quantum mechanics, adding a constant term $a$ to the Hamiltonian operator shifts its eigenvalues of a quantity $a$. Since one of the probes of energy levels of atoms are emitted and absorbed photons, knowing photon energies allows us to infer atomic energy levels. According to Bohr model, $E_{\text{ph}} = \Delta E$, so a hypothetical shift in the energy levels is not detectable. Again, this fact guarantees a sort of “gauge invariance” and prevents us from knowing the choice of the constant term $a$.

Instead, if in eq. (11) we shift the energy levels of a quantity $a$, the photon energy does not remain invariant. Therefore, in principle, from eq. (11) it would be possible to infer the constant added to the Hamiltonian operator, breaking classical “gauge invariance”.

When the atom is moving, eq. (11) holds no more. We have a slightly more complex formula, given by

$$E_{\text{ph},\text{moving}} = \frac{1}{\gamma (1 - \beta \cos \vartheta)} E_{\text{ph},\text{rest}},$$

where $E_{\text{ph},\text{rest}}$ is given by eq. (11) and $\vartheta$ is the angle between the atomic motion and the direction of the emitted or absorbed photon. In general, $E_{\text{ph},\text{moving}} \neq E_{\text{ph},\text{rest}}$, due to the presence of the kinematic factor $1/ [\gamma (1 - \beta \cos \vartheta)]$. This factor enables a richer case study of photon energies than the simple rest situation.

Finally, we have also investigated if photons are really responsible for atomic transitions between different energy levels, at least from a special relativistic point of view.

In the emission case, we have proven that a disexcitation cannot take place without a photon emission: it would imply a 4-momentum conservation violation. In this case, the 4-momentum conservation method has enabled us to obtain this fundamental result.

Similarly, one would expect that atomic excitations are impossible without photon absorption. The first conceptual problem has been how to characterize the fact that the photon has not been absorbed by the atom. From an operative point of view, we can see that the photon has not been absorbed if we find the photon somewhere after the interaction with the atom by means of an appropriate detector. From a theoretical point of view, this operative layout is given by the 4-momentum conservation method, in which we treat only the initial and final states, abandoning the proposal of a detailed description of the interaction between the atom and the photon.

Nevertheless, the 4-momentum conservation method is able to exclude atomic excitation only for photon energies which are lower than the absorption energy. This method does not rule out atomic excitations for photon energies which are higher than the absorption energy. Actually, the matter is that the 4-momentum conservation method does not examine the actual interaction between the atom and the photon. In this way, nobody assures us that the photon detected in the final state has not undertaken an absorption-emission process as described in section V. Therefore, a photon detected in the final state does not grant us that no photon has ever been absorbed by the atom, whence the 4-momentum conservation formalism is not well suited to describe the situation of absorption without photons.

ACKNOWLEDGMENTS

I am particularly grateful to Dr. Jean-Pierre Zendri, because this paradox was inspired by one of his lectures on laser cooling.

Special thanks to Prof. Francesco Sorge for his suggestions during the writing of this article.

Appendix A: Some calculations

In this appendix we present a more mathematically detailed treatment of the physical situation described in section V.

If in the laboratory frame $K$ the atom is at rest in its energy level $E_n$, and a photon with energy $E_0$ is moving along the $x$ axis, their 4-momenta before interaction are

$$p_\alpha^\mu = \begin{pmatrix} m_n c \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad p_0^\mu = \begin{pmatrix} E_0/c \\ E_0/c \\ 0 \\ 0 \end{pmatrix},$$

where $p_\alpha^\mu$ is the atom 4-momentum and $p_0^\mu$ is the photon 4-momentum. After interaction, if we suppose that the atom jumps to $E_m$, we have

$$\mathbf{p}_\alpha^\mu = \begin{pmatrix} \gamma m_n c \\ p_n \hat{n}_\alpha \end{pmatrix}, \quad \mathbf{p}_0^\mu = \begin{pmatrix} E_{0,\text{fin}}/c \\ (E_{0,\text{fin}}/c) \hat{n}_0 \end{pmatrix}.$$
with \( \vec{n}_a \) and \( \vec{n}_0 \) two unit vectors.

Let \( \varphi \) be the angle between \( \vec{n}_0 \) and the \( x \) axis. Computing Mandelstam \( t \) invariant before and after interaction, and equating the two expressions, one has

\[
-2 \frac{E_0 E_{0,\text{fin}}}{c^2} + 2 \frac{E_0 E_{0,\text{fin}}}{c^2} \cos \varphi = m_m c^2 + m_n c^2 - 2 \gamma m_m m_n c^2. \tag{A1}
\]

From energy conservation

\[
\gamma m_m c^2 = m_n c^2 + E_0 - E_{0,\text{fin}}; \tag{A2}
\]

thus, substituting this expression of \( \gamma m_m c^2 \) into eq. (A1), we have, after some passages, (cf. eq. (10))

\[
E_{0,\text{fin}} = \left( \frac{m_n^2 - m_m^2}{2(E_0 (1 - \cos \varphi) + m_n c^2)} \right) c^2. \tag{A3}
\]

Clearly it must be \( E_{0,\text{fin}} > 0 \). The denominator is always positive, in fact

\[
m_n c^2 > -E_0 (1 - \cos \varphi),
\]

and the right-hand side is always non-positive. Hence, \( E_{0,\text{fin}} > 0 \) is equivalent to

\[
(m_n^2 - m_m^2) c^2 + 2m_n E_0 > 0,
\]

which yields \( E_0 > E_{\text{abs}} \), where \( E_{\text{abs}} \) is given by \( E_{\text{abs}} = (m_n^2 - m_m^2) c^2 / (2m_n) \), which corresponds to eq. (13).

Therefore, the case \( E_0 < E_{\text{abs}} \) is ruled out, because it would imply \( E_{0,\text{fin}} < 0 \).

In addition we have to check whether \( \gamma \geq 1 \) in eq. (A2), otherwise we have an unphysical situation. Since \( \gamma \) is given by (cf. eq. (A2))

\[
\gamma = \frac{m_m c^2 + E_0 - E_{0,\text{fin}}}{m_n c^2},
\]

we must have

\[
E_{0,\text{fin}} \leq E_0 - (m_m - m_n) c^2. \tag{A4}
\]

Let us impose inequality (A4) to the result of eq. (A3). We find an inequality \( E_0 \) has to fulfill.

\[
2E_0^2 (1 - \cos \varphi) - 2E_0 c^2 (m_m - m_n) (1 - \cos \varphi) + (m_m - m_n)^2 c^4 \geq 0
\]

Its reduced discriminant is

\[
\frac{\Delta}{4} = -(m_m - m_n)^2 c^4 \sin^2 \varphi,
\]

which is always non-positive, so inequality (A4) is always fulfilled.

[1] W. Rindler, *Relativity*, 2nd edition (Oxford University Press, Oxford, 2006).
[2] J. A. Wheeler, “From Mendeleev’s atom to the collapsing star,” Trans. N. Y. Acad. Sci. 33 (8), 745–779 (1971).
[3] N. Bohr, “On the constitution of atoms and molecules,” Philos. Mag. 26 (151), 1–24 (1913).
[4] D. Bohm, *Quantum theory* (Dover Publications, New York, 1989), pp. 39–43.
[5] A. Beiser, *Concepts of Modern Physics*, 6th edition (McGraw-Hill, New York, 2003).
[6] C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* (Wiley-VCH, Weinheim, 2005), pp. 790–802.
[7] V. Barone, *Relatività* (Bollati Boringhieri, Torino, 2004), pp. 296–300.
[8] Y. Aharonov and D. Rohrlich, *Quantum Paradoxes* (Wiley-VCH, Weinheim, 2005), pp. 2–8.
[9] J. B. Hartle, *Gravity* (Addison Wesley, San Francisco, 2003).