A criterion for covariance in complex sequential growth models

Sumati Surya\(^1\) and Stav Zalel\(^2,3\)

\(^1\) Raman Research Institute, CV Raman Ave, Sadashivanagar, Bangalore, 560080, India
\(^2\) Blackett Laboratory, Imperial College, London, SW7 2AZ, United Kingdom

E-mail: stav.zalel11@imperial.ac.uk

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Abstract

The classical sequential growth model for causal sets provides a template for the dynamics in the deep quantum regime. This growth dynamics is intrinsically temporal and causal, with each new element being added to the existing causal set without disturbing its past. In the quantum version, the probability measure on the event algebra is replaced by a quantum measure, which is Hilbert space valued. Because of the temporality of the growth process, in this approach, covariant events (or observables) are measurable only if the quantum measure extends to the associated sigma algebra of events. This is not always guaranteed. In this work we find a criterion for extension (and hence covariance) in complex sequential growth models for causal sets. We find a large family of models in which the measure extends, so that all covariant events/observables are measurable.

Keywords: quantum gravity, discrete spacetime, measure theory, covariant observables

1. Introduction

One of the most challenging quests in any approach to non-perturbative quantum gravity is in finding a consistent dynamics for the full theory. Within each approach the formulation of the dynamics acquires specific features, not all of which can be translated to other approaches. In causal set quantum gravity \([1]\), the emphasis is on the space of discrete histories or causal sets, with the dynamics given by a Hilbert space valued measure or equivalently a decoherence functional. As in the continuum path integral, where each (fixed dimensional) Lorentzian spacetime appears with a complex weight, in causal set theory (CST) each countable causal set appears in the path sum with a complex weight. In continuum-inspired models, the measure is...
given in terms of the discrete Einstein–Hilbert or Benincasa–Dowker action [2–7], but this is not the most natural choice from a fundamental, order theoretic perspective.

One such ‘bottom-up’ approach to CST dynamics is the sequential growth paradigm, the classical version of which serves as a template for the quantum dynamics [8–10]. In this paradigm, the causal set is grown element by element, starting with an initial element. At every stage of the growth the new element can be added to the future of an existing element or left unrelated to it, with some transition probability or amplitude (depending on the case at hand), so that the past of the existing elements is not changed. In the classical growth models, this generates a probability measure space \((\Omega, \mathcal{Z}, \mu)\) where \(\Omega\) is the space of all past finite labelled causal sets, \(\mathcal{Z}\) is an event algebra (or collection of all measurable sets) closed under finite set operations over \(\Omega\) and \(\mu\) is a probability measure.

As in any other approach to quantum gravity, one must look for the appropriate analogues of covariant observables in CST. In the growth paradigm, each element is born at a given stage or ‘coordinate time’ and hence is labelled. Covariance thus translates in CST into label invariance which must be satisfied by the measure. Covariant events (see section 2) moreover, are those events in \(\mathcal{Z}\) which are invariant under relabellings [11]. In a histories framework, which does not give primacy to spatial hypersurfaces, these covariant events are the appropriate analogues of covariant observables\(^4\). Examples of covariant events are (a) the ‘originary’ event which is the collection of causal sets with a single element to the past of all other elements: this is the analogue of a ‘big bang’ (b) the post event which is the collection of causal sets each containing at least one element such that all other elements are either to its past or its future: this is the analogue of a ‘bounce’. Finding the measure of these covariant events is therefore clearly of physical interest.

The classical growth dynamics studied in [8] is Markovian, covariant (in the sense described above) and causal. This reduces the space of possible probability measures drastically, each characterised by a single transition probability per stage of the growth. While these probabilities themselves are covariant, the events in \(\mathcal{Z}\) are not, since they are generated by finite stage events in \(\Omega\) and hence are labelled. Covariant events in \(\mathcal{Z}\) can only be defined after generating the infinite stage events. This means that in order to construct all possible covariant events from \(\mathcal{Z}\), one has to go to the full sigma-algebra \(\mathcal{S}_\mathcal{Z}\) generated by \(\mathcal{Z}\). The covariant events are given by the quotient-sigma-algebra \(\tilde{\mathcal{S}} = \mathcal{S}_\mathcal{Z}/\sim\) where the equivalence relation \(\sim\) is over relabelings of causal sets in \(\Omega\) [11] \(^5\). Because \(\mu\) is a probability measure, by the Kolmogorov–Caratheodory–Hahn extension theorem [15], it possesses a unique extension to \(\mathcal{S}_\mathcal{Z}\) and hence one can in principle calculate the measure of covariant events.

In quantum sequential growth models, the idea is to replace the probability measure by a ‘quantum measure’, which can be realised as a finitely additive vector measure \(\mu_v\) valued in a ‘histories’ Hilbert space \(\mathcal{H}\) [16, 17]. As in the classical growth models, the quantum dynamics is then characterised by the quantum triple \((\Omega, \mathcal{Z}, \mu_v)\). The simplest quantum version of the growth models is obtained by complexifying the classical probability measure, so that \(\mu_v\) is valued in \(\mathbb{C}\). This is the complex sequential growth or CSG dynamics that is the focus of this present work. Given that the ultimate goal is to construct a full theory of quantum gravity, which has the right classical limit, it is likely that the right dynamics is defined over an infinite dimensional \(\mathcal{H}\). Nevertheless, as we will see, even the simple choice \(\mathcal{H} \simeq \mathbb{C}\) possesses non-trivial features.

\(^4\)A more appropriate term, perhaps, is ‘beables’ as used by Bell [12]. For a measure theoretic formulation of standard quantum theory and the role of events in the path integral framework, we refer the reader to [13].

\(^5\)A formulation of the growth dynamics generated by covariant events was adopted in [14] using ‘stem events’. We will not pursue this approach here.
In particular, the extension of the complex valued vector measure $\mu$ to the full sigma algebra $\mathcal{S}_3$ is not guaranteed; it must additionally satisfy certain boundedness conditions [18]. As shown in [17], for complex percolation (CP), where the dynamics is characterised by a single complex number $q$, the measure does not extend and hence cannot be defined for covariant events, unless $q \in [0, 1]$, i.e., for ‘real’ CP ($RCP$). While the latter is not in itself strictly classical, it is a fairly trivial example of CSG. It is therefore of interest to find a larger class of CSG models in which $\mu$ can be extended to $\mathcal{S}_3$.

In [19] it was argued that not all covariant events may be physically relevant and that it would be sufficient for the measure to extend to a subclass of covariant events via some conditional convergence conditions. It can be shown that one such condition is satisfied by the measure of the originary event in the CP model [20]. However, apart from a simple class of covariant events, which includes the originary event, setting up a conditional convergence protocol for other covariant events like the post event becomes rapidly more cumbersome. It is therefore desirable to look for quantum measures $\mu$ that extend to the full-sigma algebra $\mathcal{S}_3$, so that every covariant event is measurable. Such models thus define a consistent covariant dynamics.

In this work we find criteria for $\mu$ to extend to $\mathcal{S}_3$ in CSG models. We find by explicit construction large classes of CSG models that admit an extension and hence define consistent covariant dynamics, as well as those which do not. Our methods follow the spirit of the analysis of the CP dynamics in [17], where the extension of the measure is related to a colinearity criterion.

In section 2 we review the sequential growth paradigm, where we define the event algebra $\mathcal{F}$ generated from finite labelled causal sets and the associated cylinder sets in $\Omega$. We then review the CSG models of [8, 21] in section 2.1 which serve as a template for the quantum dynamics. Next, we define QSG models broadly and the subclass of CSG dynamics in section 2.2. In section 2.3 we use a version of the Caratheodory–Hahn–Kluvnek(CHK) theorem suited to complex measures on $\mathcal{F}$ (proved in appendix B), which states that bounded variation is a necessary and sufficient condition for the extension of $\mathcal{F}$ to $\mathcal{S}_3$. Section 3 contains our main results. In section 3.1 we find criteria for bounded variation, summarised in theorem 3.1. In section 3.2 we translate these criteria to the specific case of CSG by proving two lemmas 3.4 and 3.5 which gives us a useful corollary 3.6 to theorem 3.1. Since bounded variation corresponds to absolute convergence in $C$, this criteria is related to an asymptotic colinearity condition on the $C$-valued amplitudes, given in terms of the constants $\zeta_{\min}^{\mu}$, $\zeta_{\max}^{\mu}$.

Finally in section 3.3 we give explicit examples of CSG models that extend and some that do not. In section 4 we discuss how these results can be used to make predictive statements about covariant observables in quantum gravity. Appendix A lists a few of the standard definitions from causal set theory. The list is not exhaustive and we refer the reader to the literature [8, 22]. In appendix B we show how the CHK theorem implies theorem 2.1 for a complex measure over $\mathcal{F}$.

2. The sequential growth paradigm

In CST there is a natural correspondence between the cardinality $n$ of spacetime regions and the continuum spacetime volume. In the unimodular approach to gravity, the latter appears as a natural ‘time-parameter’. Hence evolution corresponds to increasing spacetime volume (normalised appropriately). This translates in CST to an increase in the cardinality of the causal network.
First three stages of the sequential growth process, which correspond to a portion of \( P \). The three-element causal sets that are order-isomorphic to each other are marked.

set so that the causal set ‘grows’ element by element. This motivation is at the heart of the sequential growth paradigm.

A natural starting point for the growth process is therefore at \( n = 1 \), where, with certainty, a single element \( e_1 \) is born. At stage \( n = 2 \), the new element \( e_2 \) can be added either to the future of \( e_1 \) to form a two-element chain, or left unrelated to it, to form a two-element anti-chain. However, it cannot be added to the past of \( e_1 \). At every stage \( n \), the new causal set element \( e_{n+1} \) is ‘added’ to the existing causal set \( c_n \) so that it is either to the future of some of the elements or left unrelated to them. Importantly, it does not change the past of any of the elements in \( c_n \) [8]. In [8] this is referred to as internal temporality.

The growth process generates a poset of labelled causal sets, termed labelled poscau \( P \). Figure 1 is an illustration of the growth process up to stage \( n = 3 \), and represents the beginning portion of \( P \). Because the elements are labelled by the stage at which they are added, \( P \) is a directed ‘tree’, as evident from figure 1. Each node in the tree is therefore a finite labelled poset. Because of the tree structure, any two nodes are either unrelated (in the sense of the partial order on \( P \)) or are related via a unique set of intermediate nodes. We will refer to the unique set of nodes from \( e_0 \) to an \( n \)-element node as an \( n+1 \)-jointed branch associated with this node.

As \( n \to \infty \), this growth process generates the sample space \( \Omega \) of countable labelled past finite causal sets. The labelling is evident from figure 1, which shows that in some instances the new element at stage \( n \) could have been added at an earlier stage to get the same unlabelled causal set at stage \( n \). As an example, consider the three labelled \( n = 3 \)-element causal sets marked in figure 1. These are all the same unlabelled causal set, but with different time labels corresponding to how they were created. (i) At stage \( n = 1 \) the element \( e_1 \) is either unrelated to \( e_0 \) (in the left two cases) or is to its future (in the third case) (ii) At stage \( n = 2 \) the element \( e_2 \) is added to the future of either \( e_0 \) or \( e_1 \) giving rise to the two figures on the left, or is unrelated to them as in the third figure. Again, what is evident is that the labelling must satisfy the order relation \( e_i \preceq e_j \Rightarrow i < j \). This is referred to as a natural labelling or a linear extension.

We will henceforth call two distinct labelled causal sets \( c, c' \) order-isomorphic to each other (denoted by \( c \sim c' \)) if they are labelings of the same unlabelled causal set. We refer the reader to the literature [8, 11, 14] for a more detailed discussion of this terminology.
Next, one must define the measurable sets which constitute the event algebra, which is a field of subsets of $\Omega$ closed under finite set operations and includes $\Omega$ and $\emptyset$. The event algebra naturally associated with the above growth process is generated by the nodes in $\mathcal{P}$. Let $\Omega_n$ denote the set of $n$-element labelled causal sets, which is of finite cardinality $\Omega_n \equiv |\Omega_n|$ for finite $n$. For example, using figure 1 we find that $|\Omega_2| = 2$ and $|\Omega_3| = 7$, while for large $n$ the growth is super-exponential, with $|\Omega_n| \sim 2^{n^2/4}$, to leading order [23]. Each finite labelled causal set $c'_n \in \Omega_n$, $i \in \mathcal{I}(n) = \{1, \ldots, \Omega_n\}$ is a node in $\mathcal{P}$ and, being labelled, also represents its history of formation, i.e., the unique $(n + 1)$-jointed branch in $\mathcal{P}$, starting from $e_0$. Thus, for each node $c'_i$ we can associate a cylinder set

$$\text{cyl}(c'_i) \equiv \{c \in \Omega \mid c|_n = c'_n\}, \text{ cyl}(c'_i) \subset \Omega$$

(1)

where $c|_n$ denotes the first $n$ elements of the labelled causal set $c \in \Omega$. Because $\mathcal{P}$ is a tree, cylinder sets satisfy the nesting property

$$\text{cyl}(c'_m) \cap \text{cyl}(c'_n) \neq \emptyset \Rightarrow \text{cyl}(c'_m) \subset \text{cyl}(c'_n), \text{ for } m < n.$$  

(2)

In other words, a non-trivial intersection between two distinct cylinder sets is possible only if one is a proper subset of the other.

Such cylinder sets are also naturally defined in the discrete random walk. Consider the random walk on a regular $1 + 1$ dimensional lattice, where the walker starts at some fixed position $x_0$ at $t = 0$. At any discrete time step $t > 0$, there are a finite number of possible paths that the walker could have taken. Associated with any of these paths, $\gamma_t = \{x_0, x_1, \ldots, x_t\}$ is the cylinder set $\text{cyl}(\gamma_t)$ which is a subset of the sample space of infinite time paths, with the property that the first $t$-steps coincide with $\gamma_t$. As in the growth model, such cylinder sets satisfy the nesting property, equation (2).

Because $\mathcal{P}$ is a tree, for any $c'_n \in \Omega_n$,

$$\text{cyl}(c'_n) = \bigsqcup_{\ell(n)} \text{cyl}(c'^{\ell(n)}_{n+1}).$$

(3)

where $\mathcal{C}(c'_n) \equiv \{c'^{\ell(n)}_{n+1}\}$ denotes the set of children of $c'_n$ in $\mathcal{P}$, i.e. the set of $n + 1$ element causal sets emanating from the $c'_n$ node in $\mathcal{P}$. We use the functional notation $\ell(i)$ to denote that $\ell$ is valued in an index set $\mathcal{I}(i, n) \subset \mathcal{I}(n)$ of cardinality $|\mathcal{C}(c'_n)|$, which depends on $n$, or equivalently, $c'_n$. For example, from figure 1 we see that the $n = 2$ antichain $c^2_1$ has 4 children, while the $n = 2$ chain $c^2_2$ has 3 children.

Let $\mathcal{Z}_n$ denote the collection of cylinder sets at level $n$ and $\mathcal{Z}$ the collection of all cylinder sets. The event algebra $\mathcal{E}$ is then generated by taking finite unions, intersections and complements of the elements of $\mathcal{Z}$. The nesting property, equation (2), then implies that for any $\alpha \in \mathcal{E}$, there exists a smallest integer $n_0 < \infty$ and a subset $S_0 \subset \{1, \ldots, \Omega_n\}$ such that $\alpha = \bigsqcup_{k \in S_0} \text{cyl}(c^k_{n_0})$. We define the fine partition of an event $\alpha \in \mathcal{E}$ as $\mathcal{N}_\alpha = \{\text{cyl}(c^k_{n_0})\}, k \in S_0$, of $n_0$-element nodes in $\mathcal{P}$.

Our interest is in events that are covariant. Following [11] we define a covariant set $\alpha \subseteq \Omega$ as

$$\alpha = \{c | c' \sim c \Rightarrow c' \in \alpha\}.$$  

(4)

If $\alpha$ belongs to an event algebra, then we call it a covariant event. In the language of observables, or beables, we will also refer to these as covariant observables.

Using the nesting property, we see that no event $\alpha \in \mathcal{E}$ can be covariant unless $\alpha = \Omega$. Consider the fine partition $\mathcal{N}_\alpha$ (defined above) for any $\alpha \subseteq \Omega$, so that $\alpha = \bigsqcup_{k \in S_0} \text{cyl}(c^k_{n_0})$. Let
\( c'_{\alpha_n} \) be a node in \( N_\alpha \) with the largest number of minimal elements \( m_\alpha \). (i) Assume \( n_\alpha > m_\alpha \), i.e., the \( n_\alpha \)-element antichain \( c'_{\alpha_n} \) does not belong to \( N_\alpha \). Let \( c^{(\alpha)}_{\alpha_{n+1}} \) denote the gregarious child of \( c'_{\alpha_n} \), i.e., one in which the new element \( c_{\alpha_{n+1}} \) is unrelated to all the elements in \( c'_{\alpha_n} \). Thus, there exists an \((n_\alpha + 1)\)-element node \( c^{(\alpha)}_{\alpha_{n+1}} \sim c^{(\alpha)}_{\alpha_{n+1}} \) such that the first \( m_\alpha + 1 \) elements in \( c^{(\alpha)}_{\alpha_{n+1}} \) are the antichain \( c^{(\alpha)}_{\alpha_{n+1}} \). But \( c^{(\alpha)}_{\alpha_{n+1}} \notin N_\alpha \), since otherwise \( m_\alpha \) would not be the largest number of minimal elements for the set of nodes \( N_\alpha \). This means that for every \( c \in \text{cyl}(c^{(\alpha)}_{\alpha_{n+1}}) \), there exists an order-isomorphic \( c' \in \text{cyl}(c^{(\alpha)}_{\alpha_{n+1}}) \). Because of the nested property of cylinder sets, while \( \text{cyl}(c^{(\alpha)}_{\alpha_{n+1}}) \cap \alpha, \text{cyl}(c^{(\alpha)}_{\alpha_{n+1}}) \notin \alpha \), and hence \( \alpha \) is not covariant. (ii) If \( m_\alpha = n_\alpha \), \( c'_{\alpha_n} \in N_\alpha \). Let \( N'_\alpha \) denote the (non-empty) complement of \( N_\alpha \) in the set of all possible \( n_\alpha \) nodes and \( m_\alpha \) the largest number of minimal elements for any node in \( S_\alpha \). The argument (i) then tells us that \( \alpha' \in Y \) is not covariant. Hence \( \alpha \) is not covariant.

This means that the event algebra \( Y \) does not suffice to be able to define covariant observables. In order to do so, one needs to include events obtained from countable set operations on \( Y \). An example of a covariant event is the originary event \( a_{\text{orig}} \) (mentioned earlier) where there is a single element to the past of all the other elements in the causal set, analogous to a big bang. \( a_{\text{orig}} \) is invariant under natural relabellings since the initial element must always come at stage \( n = 0 \). In the sequential growth process, at any finite stage \( n \), the gregarious child is not originary and hence every \( \text{cyl}(c'_{\alpha_n}) \in \alpha \). This means that for every \( c \in \text{cyl}(c'_{\alpha_n}) \), there exists an order-isomorphic \( c' \in \text{cyl}(c'_{\alpha_n}) \). Because of the nested property of cylinder sets, while \( \text{cyl}(c'_{\alpha_n}) \cap \alpha, \text{cyl}(c'_{\alpha_n}) \notin \alpha \), and hence \( \alpha \) is not covariant. (ii) If \( m_\alpha = n_\alpha \), \( c'_{\alpha_n} \in N_\alpha \). Let \( N'_\alpha \) denote the (non-empty) complement of \( N_\alpha \) in the set of all possible \( n_\alpha \) nodes and \( m_\alpha \) the largest number of minimal elements for any node in \( S_\alpha \). The argument (i) then tells us that \( \alpha' \in Y \) is not covariant. Hence \( \alpha \) is not covariant.

The smallest algebra that includes events generated by countable set operations on \( Y \) is its associated sigma-algebra \( \mathcal{S}_Y \). The set of covariant events themselves form a sigma-algebra which is a sub-sigma-algebra of \( \mathcal{S}_Y \) [24]. Equivalently, one can build covariant events from \( \mathcal{S}_Y \) by taking equivalence classes of causal sets under relabellings. In the latter approach, if \( \sim \) denotes equivalence under relabellings, the sigma-algebra of covariant events is the quotient sigma-algebra \( \mathcal{S}_Y / \sim \).

We note that this is not the only way to construct covariant events. In the approach of [14] instead of \( Y \), one considers an event algebra that is generated from covariant ‘stem’ events. The dynamics is defined as a random walk on the associated covariant tree of posets.

### 2.1 Classical sequential growth

We begin by describing the classical sequential growth process of [8]. The dynamics on \( \mathcal{P} \) is a specification of the measure over \( Y \). As in the random walk, one can assign a measure to \( Y \) by letting \( \mu(\text{cyl}(c'_{\alpha_n})) = 1 \), where \( \mathcal{P}(c'_{\alpha_n}) \) is the probability that a directed random walk from the origin in \( \mathcal{P} \) reaches the node \( c'_{\alpha_n} \) by stage \( n \) and is determined by the particular growth process. This choice of measure ensures that \( \mu \) is a finitely additive probability measure, i.e., \( \mu : Y \rightarrow [0, 1] \) and \( \mu(\Omega) = 1 \). By the Kolmogorov–Caratheodory–Hahn extension theorem, \( \mu \) extends to \( \mathcal{S}_Y \), and hence to the sigma-algebra of covariant events.

As discussed in [8] there are certain natural conditions to impose on the measure for the classical sequential growth. The first is (a) covariance, i.e., the measure is the same for
order-isomorphic causal sets. In figure 1 there are three \( n = 3 \)-element order-isomorphic causal sets whose associated cylinder sets must therefore have the same measure. The second is that the transition probabilities satisfy a (b) Markovian sum rule

\[
\sum_{j(i)} \mathbb{P}(c_n^j \to c_n^{(i)}) = 1,
\]

where \( j(i) \) is valued in an index set \( \mathcal{J}(i, n) \) of cardinality \( |\mathcal{E}(c_n^j)| \), for all nodes in \( \mathcal{P} \).

Finally, there is the dynamical causality rule which we term (c) spectator independence\(^8\), which needs a little more terminology to define. Let \( c_n^i \to c_{n+1}^{(j)} \) be a transition and define the associated precursor set to be the past of the new element \( e_{n+1} \). If the precursor set is all of \( c_n^i \), this transition is described as timid and if it is the empty set, it is described as gregarious, introduced previously. Those elements in \( c_n^i \) not in the precursor set of \( e_{n+1} \) are then termed spectators. The idea of condition (c) is that the transition cannot depend explicitly on the spectators, and is hence intrinsically causal.

Consider two non-timid transitions \( c_n^j \to c_{n+1}^{l_1} \) and \( c_n^j \to c_{n+1}^{l_2} \), with \( j_1, j_2 \in \mathcal{J}(i, n) \), and with spectator sets \( P_1, P_2 \) respectively, and consider an \( m \)-element causal set, \( c_n^m \), in \( \mathcal{P} \), which is order-isomorphic to \( P_1 \cup P_2 \). Then there exists children \( c_{m+1}^{l_1}, c_{m+1}^{l_2} \) of \( c_n^m \), with \( l_1, l_2 \in \mathcal{J}(k, m) \) such that the precursor set of the new element in \( c_{m+1}^{l_1} \) is order-isomorphic to \( P_1 \), and that of the new element in \( c_{m+1}^{l_2} \) is order-isomorphic to \( P_2 \).

The requirement (c) can then be expressed as

\[
\frac{\mathbb{P}(c_n^j \to c_{n+1}^{l_1})}{\mathbb{P}(c_n^j \to c_{n+1}^{l_2})} = \frac{\mathbb{P}(c_n^m \to c_{m+1}^{l_1})}{\mathbb{P}(c_n^m \to c_{m+1}^{l_2})}.
\]

This condition can be reformulated as a product rule, which holds even when some of the transition probabilities are set to zero\([25, 26]\).

These three conditions on the transition probabilities simplify the dynamics drastically so that at every stage one has a single independent coupling constant. It is convenient to take this to be the transition probability \( q_n \) from \( c_n^m \) to \( c_{n+1}^m \) [8]. For a generic transition at stage \( n \), \( c_n^i \to c_{n+1}^{(j)} \), the transition probability is given by

\[
\mathbb{P}(c_n^i \to c_{n+1}^{(j)}) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \frac{q_n}{q_k},
\]

where \( \varpi \) is the cardinality of the precursor set and \( m \) denotes the number of maximal elements in the precursor set. Alternatively, one can use the coupling constants \( t_n \),

\[
t_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{1}{q_k},
\]

in terms of which the transition probabilities are

\[
\mathbb{P}(c_n^i \to c_{n+1}^{(j)}) = \frac{\lambda(\varpi, m)}{\lambda(n, 0)} t_k,
\]

where \( \lambda(\alpha, \beta) = \sum_{k=\alpha}^{\beta} \binom{\alpha - \beta}{k - \beta} t_k \).

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\(7\) It is important to note that if there are \( k \) order-isomorphic children in a given transition, then the measure not only counts each equally, but the multiplicity \( k \) appears in the Markovian sum. In this sense the measure does not treat order-isomorphism as ‘gauge’.

\(8\) This is referred to in [8] as ‘Bell causality’. The reason to shy away from this terminology in the present work is its implications for quantum entanglement, which we will not discuss.
One of the simplest growth models is transitive percolation, where \( q_n = q^n \) and \( 0 < q < 1 \), or equivalently \( t_n = t^k \) and \( t > 0 \), so that there is a single parameter that governs the growth. One also has the deterministic dust universe with \( q_n = 1 \), or \( t_0 = 1, t_k = 0, k \geq 1 \), so that only the antichain is generated and the forest universe in which all transition probabilities are equal and are given by \( \mathbb{P}(c_n^i \to c_{n+1}^j) = q_n = (1 + \nu)^{-1} \), or equivalently, \( t_0 = t_1 = 1, t_k = 0, k \geq 2 \).

The forest universe generates, with unit probability, a causal set which is tree-like, with each element in the causal set having a single past link, which is a relation that cannot be inferred from transitivity.

### 2.2. Quantum sequential growth

We wish to construct a quantum dynamics on the tree, \( \mathcal{P} \). To do so, we will follow the method of [17, 27]. The growth paradigm describes the kinematics, while the dynamics is encoded in the measure. This means that both \( \Omega \) and \( \mathfrak{A} \), generated by the collection of cylinder sets \( \mathcal{Z} \) remain as in section 2, but the probability measure is replaced by a quantum measure, which we define as follows. A quantum measure is a Hilbert space \( \mathcal{H} \) valued vector measure \( \mu_x \) on an event algebra \( \mathfrak{A} \), which is finitely additive, i.e., for any finite collection of pairwise disjoint events \( \{\alpha_i\}, \alpha_i \in \mathfrak{A} \),

\[
|\cup\alpha_i| = \sum_i |\alpha_i|
\]

where \( |\alpha_i| \equiv \mu_x(\alpha). \) If \( \mathfrak{A} \) is also a sigma algebra, then the vector measure is also required to be countably additive. In either case, the norm squared of \( \mu_x \) is not additive (finitely or countably, as the case may be) since in general

\[
\langle \alpha|\alpha \rangle = \langle \cup_j \alpha_j|\cup_i \alpha_i \rangle = \sum_j \sum_i \langle \alpha_j|\alpha_i \rangle \neq \sum_i \langle \alpha_i|\alpha_i \rangle,
\]

with the non-vanishing cross terms encoding the pairwise interference of events.

The quantum vector measure can be constructed from a strongly positive decoherence functional \( D: \mathfrak{G} \times \mathfrak{G} \to \mathbb{R}^+ \) where \( \mathcal{H} \) is the histories Hilbert space of [16], with inner product \( \langle \alpha|\beta \rangle = D(\alpha, \beta) \). In this work we will not use the decoherence functional explicitly, but refer the reader to the constructions in [16, 17].

Since the growth process generates cylinder sets, as in the classical case, we start with defining a vector measure \( \mu_x \) on \( \mathfrak{A} \), which must at the very least satisfy the analogues of conditions (a)–(c) discussed in section 2.1. Since \( \mathfrak{A} \) is closed under finite set operations and \( \mu_x \) is additive, we need consider only the measure on cylinder sets \( \mathcal{Z} \). For any \( \text{cyl}(c_n^i) \in \mathcal{Z} \) we denote the associated state \( |c_n^i\rangle \in \mathcal{H} \) labeled by the node \( c_n^i \) in \( \mathcal{P} \).

Condition (a) is straightforward to implement since it requires that \( |c_n^i\rangle = |c_n^j\rangle \) whenever \( c_n^i \sim c_n^j \), i.e., they are order-isomorphic.

For condition (b) we need to use the appropriate analogue of the total probability summing to 1. Because we want to construct a Markovian quantum process on \( \mathcal{P} \), the vector measure of a node should be related to that of its parent node via a linear transformation on \( \mathcal{H} \). Thus for every child \( c_{n+1}^0 \) of \( c_n^i \) we require that there exists a transition matrix \( \tilde{\mathcal{O}}(c_n^i \to c_{n+1}^0) \) such that

\[
|c_{n+1}^0\rangle = \tilde{\mathcal{O}}(c_n^i \to c_{n+1}^0)|c_n^i\rangle.
\]

Since \( \mu_x \) is finitely additive on \( \mathfrak{A} \),

\[
\text{cyl}(c_n^i) = \bigcup_{j\in\mathcal{J}} \text{cyl}(c_n^{j0}) = \sum_{j\in\mathcal{J}} \tilde{\mathcal{O}}(c_n^i \to c_{n+1}^{j0}) = 1
\]
where \( j(i) \) is valued in \( \mathcal{J}(i, n) \) and \( 1 \) denotes the identity operator on \( \mathcal{H} \).

What is much more subtle to implement, is condition (c). Setting aside the conceptual challenges in implementing quantum non-locality, i.e., the Bell inequalities [28], even the straightforward implementation of spectator independence poses a challenge in general. However, when \( \mathcal{H} \cong \mathbb{C} \), i.e., the Hilbert space is just the complex plane, condition (c) or its product form can be unambiguously implemented, since the transition operators simplify to transition amplitudes valued in \( \mathbb{C} \).

It is relatively straightforward to show that arguments of [8] generalise to this complex case, so that again, the complex growth models can be characterised in terms of the \( \{q_n\} \) or the \( \{t_n\} \), with \( q_n, t_n \in \mathbb{C} \), for which the transition amplitudes \( A(c_n \rightarrow c_{n+1}) \) are given by

\[
A(c_n \rightarrow c_{n+1}) = \frac{\lambda(c, m)}{\lambda(n, 0)}, \quad \text{where } \lambda(a, b) = \sum_{k=b}^{a} (a - b) t_k.
\]  

(15)

The quantum measure \( \mu_v(cyl(c_n')) \) is then given by

\[
\mu_v(c_n') \equiv |c_n'| = \prod A(c_m \rightarrow c_{m+1}).
\]  

(16)

where the product is over transitions along the \( (n + 1) \)-jointed branch of \( \mathcal{P} \) connecting \( c_1 = e_0 \) to the node \( c_n' \). We refer to this class of quantum measures as complex sequential growth (CSG) models.

### 2.3. Extension of complex measures on \( \mathcal{Z} \)

The quantum measure space we begin with is \((\Omega, \mathcal{Z}, \mu_v)\), where \( \mu_v \) is constructed from the complex constants \( \{t_0, \ldots, t_n, \ldots\} \), given by equations (15) and (16). As in the classical case, the measure of an arbitrary covariant event is defined only if the measure extends to \( \mathcal{S}_\mathcal{Z} \). However, while the extension of any probability measure on \( \mathcal{Z} \) to \( \mathcal{S}_\mathcal{Z} \) is guaranteed by Kolmogorov’s extension theorem [15], the extension of a vector measure \( \mu_v \) from \( \mathcal{Z} \) to \( \mathcal{S}_\mathcal{Z} \) exists only if \( \mu_v \) satisfies the conditions of the Caratheodory–Hahn–Kluvnek (CHK) extension theorem [18]. Importantly, not every \( \mu_v \) given by equation (16) can be extended to \( \mathcal{S}_\mathcal{Z} \).

The convergence condition most relevant to complex measures is that of bounded variation. The variation of \( \mu_v \) is defined as

\[
|\mu_v|_v(\alpha) \equiv \sup_{\pi} \sum_{\alpha_i \in \pi} \|\langle \alpha_i \rangle\|, \quad \forall \alpha \in A
\]  

(17)

where \( \pi \) is a finite partition of \( \alpha \), i.e., \( \pi = \{\alpha_1, \ldots, \alpha_k\}, k < \infty, \alpha_i \cap \alpha_j = \emptyset, \forall i \neq j \) and \( \alpha = \bigsqcup_{i=1}^{k} \alpha_i \). The measure is said to be of bounded variation if

\[
|\mu_v|_v(\Omega) < \infty.
\]  

(18)

In appendix B, we put together existing results in the literature, to show that the CHK extension theorem for the complex measure space \((\Omega, \mathcal{Z}, \mu_v)\) of interest to us simplifies to the following statement:

**Theorem 2.1.** For a complex measure space \((\Omega, \mathcal{Z}, \mu_v)\), where \( \mathcal{Z} \) is the event algebra generated from finite set operations on cylinder sets and \( \mu_v : \mathcal{Z} \rightarrow \mathbb{C} \), \( \mu_v \) has a unique extension to \( \mathcal{S}_\mathcal{Z} \) iff it is of bounded variation.

Thus bounded variation is both a necessary and a sufficient condition for complex measures to extend to \( \mathcal{S}_\mathcal{Z} \).
In [17] it was shown that complex percolation (CP) is of bounded variation iff it is real and non-negative i.e., \( q \in [0, 1] \)\(^9\). The proof makes crucial use of the Markovian sum rule equation (14). If \( A(c_i^j \rightarrow c_{i+1}^{j+1}) \in \mathbb{C} \) denotes the transition amplitude (which is a special case of the transition matrix of equation (14)),

\[
\sum_{j=0}^{\infty} |A(c_i^j \rightarrow c_{i+1}^{j+1})| \geq 1 \Rightarrow \sum_{j=0}^{\infty} |A(c_i^j \rightarrow c_{i+1}^{j+1})| = 1 + c_i^j, \quad c_i^j \geq 0.
\] (19)

This inequality is saturated (\( c_i^j = 0 \)) iff the \( A(c_i^j \rightarrow c_{i+1}^{j+1}) \) are colinear in \( \mathbb{C} \) for all \( j(i) \in \mathbb{J}(i,n) \).

Since the cylinder sets generate \( \mathcal{F}_3 \), the boundedness (or lack thereof) of the total variation of \( \Omega \) can be characterised completely by the convergence properties of the constants \( \zeta_n^\max \) as one goes to finer partitions. At every stage \( n \), the finiteness of \( \Omega_n \) allows one to define

\[
\zeta_n^\max := \max_{c_i^j \in \Omega_n} c_i^j, \quad \zeta_n^\min := \min_{c_i^j \in \Omega_n} c_i^j.
\] (20)

As we will see in the following section, these constants can be used to give criteria for bounded variation.

3. Extension of the quantum measure in CSG

We present our new results in this section.

Our first result, theorem 3.1, gives a sufficiency condition for bounded variation of a complex measure \( \mu_v \) on \( \mathcal{F}_3 \), and another for determining when it is not, in terms of the constants \( \zeta_n^\max \) and \( \zeta_n^\min \).

Subsequently, we show in lemmas 3.4 and 3.5 that \( \zeta_n^\max \) and \( \zeta_n^\min \) are determined entirely by transitions from the \( n \)-antichain \( c_n^a \) and \( n \)-chain \( c_n^c \) nodes, respectively. In equation (45) we express \( \zeta_n^\max \) and \( \zeta_n^\min \) in terms of the CSG constants \( t_n \), which gives us a useful corollary to theorem 3.1. We then find a large class of non-trivial examples of models in which \( \mu_v \) admits an extension to \( \mathcal{G}_3 \) as also classes in which such an extension is not possible.

3.1. Criteria for bounded variation

**Theorem 3.1.** \( \mu_v \) is of bounded variation if \( \sum_{n=1}^{\infty} \zeta_n^\max \) converges. \( \mu_v \) is not of bounded variation if \( \sum_{n=1}^{\infty} \zeta_n^\min \) diverges.

We find it useful to parse the proof into a set of smaller results.

We start by noting that for any integer \( n > 0 \), \( Z_n \) forms a partition of \( \Omega \), \( \Omega = \bigsqcup_{i=1}^{n} \text{cyl}(c_i^a) \), and therefore by finite additivity we have \( |\Omega| = \sum_{i=1}^{n} |c_i^a| \).

Define

\[
S_n \equiv \sum_{i=1}^{n} \|c_i^a\|.
\] (21)

Since \( \|\Omega\| = 1 \), \( S_n \geq 1 \).

---

\(^9\) Real non-negative CP is however not a classical measure since the quantum measure is the norm or \( \|\mu_v(\alpha)\| = A(\alpha)^2 \), which is non-additive.
Claim 3.2. \( S_n \) is a non-decreasing function of \( n \) and satisfies the inequalities
\[
\prod_{r=1}^{n-1} (1 + \zeta_r^{\min}) \leq S_n \leq \prod_{r=1}^{n-1} (1 + \zeta_r^{\max}). \tag{22}
\]
Therefore, (i) \( \lim_{n \to \infty} S_n < \infty \) if \( \sum_{r=1}^\infty \zeta_r^{\max} < \infty \), and (ii) \( \lim_{n \to \infty} S_n \to \infty \) if \( \sum_{r=1}^\infty \zeta_r^{\min} \to \infty \).

Proof.
\[
S_{n+1} = \sum_{k=1}^{\mathcal{N}_{n+1}} \| c_r^{k} \| = \sum_{i=1}^{\mathcal{N}_n} \sum_{j(i)} \| c_r^{(i)} \| = \sum_{i=1}^{\mathcal{N}_n} \sum_{j(i)} |A(c_r^{i}) \to c_r^{(i+1)}| \| c_r^{i} \| = \sum_{i}(1 + \zeta_r^{i}) \| c_r^{i} \|,
\]
where we have relabelled the \( k = \{1, \ldots, \mathcal{N}_{n+1}\} \) nodes in the second equality in terms of the parent nodes \( i = \{1, \ldots, \mathcal{N}_n\} \), and \( j(i) \in \mathcal{J}(i, n) \), the index set of cardinality \( |\mathcal{C}(c_r^{i})| \) (as in equation (3)). Since \( \zeta_r^{\min} \leq \zeta_r^{i} \leq \zeta_r^{\max} \), we see that
\[
(1 + \zeta_r^{\min})S_n \leq S_{n+1} \leq (1 + \zeta_r^{\max})S_n.
\tag{24}
\]
This proves that \( S_n \) is a non-decreasing function of \( n \). Applying these inequalities recursively and noting that \( S_1 = 1 \) gives us equation (22). Finally, note that for \( a_r \geq 0 \), \( \prod_{r=1}^{\infty} (1 + a_r) \), converges if\( \sum_{r=1}^{\infty} a_r \) converges [29]. This completes the proof. \( \square \)

The following inequalities come in handy to prove the next claim.
\[
|c_r^{i}| = \sum_{j(i)} |c_r^{(i+1)}| \Rightarrow \| |c_r^{i}| \| \leq \sum_{j(i)} \| |c_r^{(i)}| \|, \tag{25}
\]
for a node \( c_r^{i} \) and its children \( \mathcal{C}(c_r^{i}) = \{ c_r^{(i+1)} \} \). Because of the nesting property of cylinder sets, moreover, for any \( m > n \),
\[
cyl(c_r^{i}) = \bigsqcup_{j(i, m)} cyl(c_r^{i, j}) \Rightarrow \| |c_r^{i}| \| \leq \sum_{j(i, m)} \| |c_r^{i, j}| \|, \tag{26}
\]
where \( j(i, m) \) takes values in \( \mathcal{J}(i, n, m) \), which label the set of \( m \)-element descendants of \( c_r^{i} \). (In this notation, \( \mathcal{J}(i, n, m) = \mathcal{J}(i, n, n+1) \)).

Claim 3.3. \( |\mu_\pi(\Omega) = \sup_n S_n \).

Proof. Consider any finite partition \( \pi \) of \( \Omega \). For each \( \alpha \in \pi \) consider its fine partition \( \mathcal{N}_\alpha \) into \( n_\alpha \)-element nodes in \( \mathcal{P} \) so that \( \alpha = \bigsqcup_{k \in \mathcal{S}_n} cyl(c_k^{\alpha}) \). Then from equation (26)
\[
\| |\alpha| \| \leq \sum_{k \in \mathcal{S}_n} \| |c_k^{\alpha}| \|, \tag{27}
\]
Moreover if \( m \) is the largest of the \( n_\alpha \) for the partition \( \pi \), for any \( \alpha \) with \( n_\alpha < m \) we have the additional inequality
\[
\| |\alpha| \| \leq \sum_{k \in \mathcal{S}_n} \| |c_k^{\alpha}| \| \leq \sum_{k \in \mathcal{S}_n, j(k, m)} \sum_{l \in \mathcal{S}_m} \| |c_l^{(k, m)}| \|. \tag{28}
\]
Since \( \{ \text{cyl}(c^k_{m}(\Omega)) \} \) is an \( m \)-level cylinder set partition of \( \Omega \) for each \( \alpha \in \pi \), the union of these partitions provides an \( m \)-level cylinder set partition \( Z_m \) of \( \Omega \), so that
\[
\| \Omega \| = 1 \leq \sum_{\alpha \in \pi} \| \alpha \| \leq \sum_{j=1}^{\| \Omega \|} \| c^j_m \| = S_m. \tag{29}
\]

In other words, for any partition \( \pi \) of \( \Omega \) there exists an \( m \) such that
\[
S_m \supseteq \sum_{\alpha \in \pi} \| \alpha \|. \tag{30}
\]

Since, \( Z_m \) is itself a partition of \( \Omega \), \( |\mu_\pi(\Omega)| \geq S_m \), for every integer \( m \). This proves the claim. \( \square \)

**Proof to theorem 3.1.** Since from claim 3.3 the variation of \( \mu_\pi \) depends only on the \( S_n \), along with claim 3.2, this completes the proof. \( \square \)

### 3.2. Criteria for bounded variation in CSG

We now translate the convergence criterion theorem 3.1 to requirements on the coupling constants \( t_k \) for CSG. We find the important result that transitions from the \( n \)-antichain node \( c^k_n \) determines \( \zeta^{\max}_n \) while the \( n \)-chain node \( c^k_n \) determines \( \zeta^{\min}_n \). This gives an explicit functional form for \( \zeta^{\max}_n, \zeta^{\min}_n \) in terms of the \( t_k \).

Let us first define some notation. Consider the set of possible transitions from a node \( c^j_n \) and let \( \mathcal{T}(c^j_n) \) denote the list of the (possibly repeated) \((\varpi, m)\) values for these transitions. Then by the Markov sum rule,
\[
\sum_{(\varpi,m)\in \mathcal{T}(c^j_n)} \lambda(\varpi, m) = 1 \Rightarrow c^j_n = \sum_{(\varpi,m)\in \mathcal{T}(c^j_n)} \frac{\lambda(\varpi, m)}{\lambda(n, 0)} - 1 \geq 0. \tag{31}
\]

For \( m < n \) we say that \( c^j_n \) is a partial stem in \( c^j_n \) if (i) \( e^j_n \subset c^j_n \) and (ii) for all \( e \in c^j_n \), \( \text{past}(e) \subseteq c^j_n \). Let \( P_m(c^j_n) \) denote the set of all \( m \)-element partial stems in \( c^j_n \). For \( m = n - 1 \), we note that the parent node of \( c^j_n \) in \( \mathcal{P} \) is one of the partial stems in \( P_{n-1}(c^j_n) \). While the rest of the partial stems in \( P_{n-1}(c^j_n) \) are all order-isomorphic to some \((n-1)\)-element node in \( \mathcal{P} \) they are not themselves nodes, since they are not naturally labelled. Moreover, every partial stem \( c^j_{n-1} \in P_{n-1}(c^j_n) \), is associated with a unique element \( e_\pi \equiv c^j_n \cap c^j_{n-1} \) which must be maximal in \( c^j_n \).

For any given \( c^j_{n-1} \in P_{n-1}(c^j_n) \), we can therefore parse the transitions from \( c^j_n \) into \( (A) \) the set of transitions which only involve \( c^j_{n-1} \), so that \( e_\pi = c^j_n \cap c^j_{n-1} \) is always in the spectator set, plus \( (B) \) the set of transitions that always include \( e_\pi \) in the precursor set. In doing this one can relate transition amplitudes from \( c^j_n \) to those from \( c^j_{n-1} \). Let \( l_k(j) \) label the type \( (A) \) children of \( c^j_n \), and similarly let \( l_k(j) \) label the type \( (B) \) children of \( c^j_n \). For any transition of type \( (A) \), \( c^j_n \rightarrow c^j_{n+1} \), there exists a child \( c^j_{n+1} \) of \( c^j_{n-1} \) (where \( j,k \in \Omega(k, n-1) \)) such that \( c^j_{n+1} \sim c^j_{n+1} \cap e_\pi \). This allows us to re-express the transition amplitude as
\[
A(c^j_n \rightarrow c^j_{n+1}) = A(c^j_{n-1} \rightarrow c^j_{n+1}) \times \frac{\lambda(n, 0)}{\lambda(n, 0)}. \tag{32}
\]

Summing over all the transitions from \( c^j_n \), for the given choice of partial stem \( c^j_{n-1} \) we find
\[
\sum_{k,j} A(c^k_n \rightarrow c^{(j)(k)}_{n+1}) = \sum_{A} A(c^A_n \rightarrow c^{A(n)}_{n+1}) + \sum_{B} A(c^B_n \rightarrow c^{B(j)}_{n+1}).
\]

\[
= \left( \sum_{i,k} A(c^i_{n-1} \rightarrow c^{(i)(k)}_n) \right) \frac{\lambda(n-1,0)}{\lambda(n,0)} + \sum_{B} A(c^B_n \rightarrow c^{B(j)}_{n+1}),
\]

(33)

where \(i(j) \in \mathcal{J}(j,n)\), and \(i(k) \in \mathcal{J}(k,n-1)\). Applying the Markov sum rule to the LHS as well as the term in brackets we see that

\[
\sum_{B} A(c^B_n \rightarrow c^{B(j)}_{n+1}) = \left( \frac{\lambda(n,1)}{\lambda(n,0)} \right) + \sum_{B} A(c^B_n \rightarrow c^{B(j)}_{n+1}) \geq \frac{\lambda(n,1)}{\lambda(n,0)},
\]

(34)

Defining \(Q^f_n \equiv c^f_n + 1 \geq 0\), equations (33) and (34) give the useful identities

\[
Q^f_n = Q^{f(j)}_{n-1} \frac{\lambda(n-1,0)}{\lambda(n,0)} + \sum_{B} A(c^B_n \rightarrow c^{B(j)}_{n+1})
\]

(35)

\[
\Rightarrow Q^f_n \geq Q^{f(j)}_{n-1} \frac{\lambda(n-1,0)}{\lambda(n,0)} + \frac{\lambda(n,1)}{\lambda(n,0)}.
\]

(36)

For the \(n\)-antichain node \(c^a_n\), for each transition, \(m = \varpi\), i.e., the number of maximal elements is equal to the cardinality of the precursor set. Hence

\[
A(c^a_n \rightarrow c^{a(j)}_{n+1}) = \frac{t_n}{\lambda(n,0)},
\]

(37)

where \(f(a)\) labels the set of children of \(c^a_n\). For fixed \(m\) there are \(\binom{n}{m}\) possible choices of precursor sets for the new element \(e_{n+1}\). Hence

\[
Q^e_n = \sum_{a=0}^{n} \binom{n}{a} \frac{|t_a|}{\lambda(n,0)}
\]

(38)

Inserting this into equation (35) we find that for the antichain

\[
\sum_{B} A(c^a_n \rightarrow c^{a(j)}_{n+1}) = \sum_{k=1}^{n-1} \binom{n-1}{k-1} |t_k| \geq \frac{\lambda(n,1)}{\lambda(n,0)},
\]

(39)

where \(I_B(a)\) labels the set of type \(B\) children of \(c^a_n\), so that

\[
Q^e_n = Q^{e(j)}_{n-1} \frac{\lambda(n-1,0)}{\lambda(n,0)} + \sum_{k=1}^{n-1} \binom{n-1}{k-1} |t_k| \frac{\lambda(n,1)}{\lambda(n,0)}.
\]

(40)

For the \(n\)-chain node \(c^e_n\), there is a unique \((n-1)\)-element partial stem, the \((n-1)\)-chain \(e_{n-1}\), with \(e_1 = e_n\). For this node, the only possible transition of type \((B)\) is that with \(e_n\) as the (unique) maximal element of the precursor set, i.e., \(c^e_n \rightarrow c^{e(n)}_{n+1}\). In this case, equation (35) reduces to

\[
Q^e_n = Q^{e(j)}_{n-1} \frac{\lambda(n-1,0)}{\lambda(n,0)} + \frac{\lambda(n,1)}{\lambda(n,0)}.
\]

(41)

We are now equipped to prove the main results of this section.
**Corollary 3.7.** For the CSG dynamics $\mu_n$ is of bounded variation if $U_n \equiv \sum_{n=1}^{\infty} \zeta_n^a$ converges and $\mu_n$ is not of bounded variation if $U_n \equiv \sum_{n=1}^{\infty} \zeta_n^c$ does not converge, where $\zeta_n^a, \zeta_n^c$ are given by equation (45).

3.3. Existence and non-trivial examples

From equation (45) it is clear that $\zeta_n^c = 0$ for all $n$ iff the $t_k$ are all colinear. Since $t_0 = 1$ this means that the $t_k$ must all lie on $\mathbb{R}^+$. For such CSG or $\mathbb{R}^+$ SG dynamics, convergence is trivially satisfied, so that we have

**Corollary 3.7.** For $\mathbb{R}^+$ SG dynamics (i.e., with all $t_k \in \mathbb{R}^+$) $\mu_n$ is of bounded variation.

While this establishes the existence of covariant CSG dynamics, $\mathbb{R}^+$ SG is too restricted a subclass and it is therefore of interest to look for non-trivial examples of complex covariant dynamics, i.e., with non-vanishing phases.
We compare $U_a$ and $U_c$ (defined in corollary 3.6) term by term with the series $U_k \equiv \sum_{n=1}^{\infty} \frac{1}{n^k}$, which converges for $x > 1$ and diverges otherwise. Thus, our requirement for convergence of $U_a$ is that there exists an $n_0 < \infty$ and an $x > 1$, such that for all $n > n_0$, $\zeta_n^a < \frac{1}{n^x}$. This means that the complex measure extends. Conversely, if for any $x > 1$, there exists an $n_0 < \infty$ such that $\zeta_n^a > \frac{1}{n^x}$ for all $n > n_0$, then $U_c$ diverges. This means that the complex measure does not extend. It will be useful to define the expression

$$L_n^{a,c}(x) \equiv \zeta_n^{a,c} - \frac{1}{n^x}. \quad (46)$$

to check for convergence or divergence.

### 3.3.1 Finite number of non-zero couplings.

The simplest non-trivial case is $t_k \neq 0$ for some $k > 0$ and $t_{k'} = 0, \forall k' \neq k, k' > 0$. Let $t_k = s e^{i \phi}, s \in \mathbb{R}^+$. Then

$$\zeta_k^a = \frac{1 + R_k(n)s}{\sqrt{1 + 2sR_k(n)\cos \phi + s^2R_k(n)^2}} - 1, \quad (47)$$

where we use the shortform $R_k(n) \equiv \binom{n}{k}$.

We now look for conditions on $s$, $k$, and $\phi$ such that $L_n^{a}(x) < 0$ for large $n$ and $x > 1$. Since $\zeta_n^a \geq 0$, $L_n^{a}(x) < 0$ implies that

$$\left( -\frac{2}{n^x} - \frac{1}{n^{2x}} \right) (1 + s^2R_k(n)^2) + 2sR_k(n) \left( 1 - \cos \phi \right) - \left( -\frac{2}{n^x} - \frac{1}{n^{2x}} \right) \cos \phi < 0. \quad (48)$$

For $n \gg k$ we can use the asymptotic form $\binom{n}{k} \sim \frac{n^k}{k!}$ to show that the dominant contribution to the LHS is

$$\approx \frac{2s}{k!} n^k \left( \frac{s}{k!} n^{k-x} + (1 - \cos \phi) \right). \quad (49)$$

For this to be negative in the large $n$ limit, the first term must dominate, or $k > x > 1$, with no restrictions on $s, \phi$. Thus, we see that the measure is of bounded variation for all choices of $t_k \in \mathbb{C}$ as long as $k \gg 2$.

When $k = 1$,

$$\zeta^c = \zeta^a = \frac{ns + 1}{\sqrt{1 + n^2s^2 + 2ns \cos(\phi)}} - 1 = \frac{1}{ns} + O \left( \frac{1}{n^3s^3} \right), \quad (50)$$

which means that the measure is not of bounded variation.

This simple example can be easily generalised to include an arbitrary but finite number of couplings.

Let $\{t_0, t_1, t_2, \ldots, t_m\}$ be a finite set of non-zero coupling constants where wlog we take $k_m > k_{m-1} > \ldots > k_1 > 0$. Let $t_k = s_i e^{i \phi_i}, s_i \in \mathbb{R}^+$ and $R_i = \binom{n}{k_i}$. Then

$$\zeta^a = \frac{1 + \sum_{i=1}^{m} R_i s_i}{\left| 1 + \sum_{i=1}^{m} R_i s_i e^{i \phi_i} \right|} - 1. \quad (51)$$
Requiring that $\zeta^a_n \leq \frac{1}{n^x}$ for some $x > 1$ leads to the inequality

$$
\left( -\frac{2}{n^x} - \frac{1}{n^{2x}} \right) \left( 1 + \sum_i R_i^2 \Phi_i \right) + 2 \sum_i R_i \Phi_i \left( 1 - \cos \phi_i + \left( -\frac{2}{n^x} - \frac{1}{n^{2x}} \right) \cos \phi_i \right) \\
+ 2 \sum_{i,j \neq j} R_i R_j \Phi_i \Phi_j \left( 1 - \cos(\phi_i - \phi_j) + \left( -\frac{2}{n^x} - \frac{1}{n^{2x}} \right) \cos(\phi_i - \phi_j) \right) < 0. \quad (52)
$$

For $m > 1$ the dominant contributions to the LHS for large $n$, arising from the $k_m$ and $k_{m-1}$ terms are

$$
-\frac{2 \zeta^a_n}{(k_m)^2} n^{2m-x} + \frac{2 \zeta^a_n k_{m-1}}{k_m k_{m-1}} n^{k_m+k_{m-1}} (1 - \cos(\phi_m - \phi_{m-1})). \quad (53)
$$

For this to be negative, $2 \zeta^a_n - x > k_m + k_{m-1} \Rightarrow k_m - k_{m-1} > x$, which implies bounded variation whenever $k_m - k_{m-1} > 1$, with no restrictions on the $s_i, \phi_i$.

On the other hand, if $k_m - k_{m-1} = 1$, then the second term in equation (53) dominates which means that $L_n^m(x) > 0$. Unlike the $m = 1$ case, however this is not sufficient to prove divergence.

Combining these results we have proved the following

**Claim 3.8.** Let $\{t_0, t_1, \ldots, t_m\}$ be the only non-zero CSG coupling constants.

The CSG dynamics is of bounded variation if any one of the following is true

(a) $t_i \in \mathbb{R}^+, i \in \{0, \ldots, m\}$.
(b) $m = 1$ and $k_1 > 1$.
(c) $1 < m < \infty$, $k_m - k_{m-1} > 1$.

It is not of bounded variation if $t_i \notin \mathbb{R}^+$ and $m = 1, k_1 = 1$.

3.3.2. **Countable number of non-zero couplings.** For a countable number of couplings we cannot use the above approximations, and we turn to more general arguments to show existence for non-real $t_k$.

The criterion for convergence is roughly that the $\zeta^a_n$ become sufficiently small as $n$ increases. This in turn means that the amplitudes in the transition at stage $n$ become increasingly colinear according to equation (19).

Let us examine this using an explicit example. Consider a set of countable couplings such that for $k > k_0 > 0$, $t_k = s_k e^{i \phi_k}$, i.e., the $t_k$ become colinear for $k > k_0 > 0$. Then we can express

$$
\zeta^a_n = \frac{\sum_{k < k_0} \binom{n}{k} |t_k| + |L_n^0|}{\sum_{k < k_0} \binom{n}{k} t_k + |L_n^0|} - 1, \quad (54)
$$

where $L_n^0 \equiv \sum_{k > k_0} \binom{n}{k} t_k = e^{i \phi_0} \sum_{k > k_0} \binom{n}{k} s_k$, so that $|L_n^0| \equiv \sum_{k > k_0} \binom{n}{k} |s_k|$.

As in the finite coupling case, the requirement that $\zeta^a_n < \frac{1}{n^x}$ for all $x > 1$ simplifies to
The largest possible contribution from the $R_i$ goes like $\frac{\alpha}{k_0}$. If $s_k$ is a growing function of $k$, then $|I^0_k|$ grows at least as fast as $\sim \left(\frac{\alpha}{\delta}\right)^s 2^{n^{s-1}} s_k$ and hence dominates the contribution from the $R_i$. Thus the dominant contribution to the LHS is

$$\approx -2|I^0_k|^2 n^{-x} + \frac{2}{k_0!} i^{k_0} |I^0_k| s_k (1 - \cos(\phi_k - \phi_0)).$$

(56)

This is negative for large $n$ if

$$|I^0_k| > n^{k_0+x}.$$  

(57)

Let us consider a couple of specific examples. (i) $s_k = s^k$, $k > k_0$, for any $s$, since for large enough $n$, $|I^0_k| \approx (1 + s)^n$ which clearly satisfies this condition. (ii) $s_k = 2^{2^k}$, for which $|I^0_k| \approx 2^{2^n}$.

We have thus shown that

**Claim 3.9.** The complex measure of the CSG dynamics given by the countable set of coupling constants

$$\{t_0, t_1, \ldots, t_{k_0} + 1, t_{k_0} + 2, \ldots, s_k t_{k_0} + \ldots\}$$

(58)

is extendible for $k_0 < \infty$ for $s > 0$ and (i) $s_k = s^k$ or (ii) $s_k = 2^{2^k}$.

Our analysis makes it possible to find other, less simplistic, dynamics for which the complex measure extends to $\mathcal{G}_3$, but we will not explore these further in this work.

The example of (CP) examined in [17], on the other hand, does not satisfy this asymptotic colinearity condition for $0 < \phi < 2\pi$ since $t_k = t_s e^{i\theta k \phi}$. Thus, as $k$ increases, the phase does not stabilise. We discuss this case briefly using the perspective we have gained in our analysis.

In CP, $t_k = t^k$, $q_k = q^k$ and $t = \frac{t_{k-1} \ldots t_0}{q}$. Note that $t$ is real and positive if and only if $q$ is real and $0 < q \leq 1$. Using

$$\lambda(\omega, 1) = \frac{1 - q}{q^\omega}, \quad \lambda(n, 0) = \frac{1}{q^n},$$

(59)

we see that

$$\zeta^n = |1 - q| \sum_{\omega=1}^{n} |q|^n \omega + |q|^n - 1.$$  

(60)

For $|q| = 1$, $q \neq 1$,

$$\zeta^n = n \times |1 - q|$$  

(61)
and hence the sum $S_c \equiv \sum_\infty \zeta_n^c$ is explicitly divergent.

If $|q| > 1$, the $|q|^n$ term in equation (60) dominates and again leads to a divergence in the sum $S_c$. If $|q| < 1$, $q \not\in \mathbb{R}^+$,

$$\zeta_n^c = (1 - |q|^n) \left( \frac{1 - q}{1 - |q|} - 1 \right) \Rightarrow S_c = (\frac{1 - q}{1 - |q|} - 1) \sum_\infty (1 - |q|^n) \quad (62)$$

which is again divergent.

This gives us an alternate proof that $\mathcal{CP}$ is not of bounded variation unless $q \in [0, 1]$.

4. Discussion

In this work we have shown that the quantum measure extends from the event algebra $\mathfrak{Z}$ to $S\mathfrak{Z}$ for several classes of $\mathbb{C}SG$ models. We also find new classes of $\mathbb{C}SG$ models in which it does not extend. Importantly, for the former class of dynamics, this implies that every covariant event in $\mathfrak{Z}$ is measurable. Thus, one may attempt to answer physically interesting questions in these models.

The simplest question to ask is whether the dynamics is originary. As discussed in the introduction, the originary event $\alpha_{\text{orig}}$ is the set of all causal sets for which there is an element $e_0$ to the past of all other elements. As shown in [11, 14] the stem event associated with every node $c_i^j$

$$\text{stem}(c_i^j) = \{ c \in \Omega | c_i^j \text{ is a partial stem in } c \} \quad (63)$$

is itself covariant and hence belongs to $\mathfrak{S}_3$ but not $\mathfrak{Z}$. The originary event of section 2 is then simply $\alpha_{\text{orig}} = \text{stem}(c_0^2)^\circ$, where

$$\text{stem}(c_0^2) = \bigsqcup_{n>0} \bigcup_{i \in I_n} \text{cyl}(c_n^j) \quad (64)$$

over all $n > 0$ and where $I_n$ labels the nodes for which the $n$th element is the only gregarious one. Thus when the measure on $\mathfrak{Z}$ extends to $\mathfrak{S}_3$,

$$|\text{orig} \rangle = |\Omega \rangle - |\text{stem}(c_0^2)\rangle = 1 - \sum_{n>0} \sum_{i \in I_n} |c_n^j\rangle \quad (65)$$

At each stage, the factorisation of the amplitude allows us to express

$$\sum_{i \in I_n} |c_n^j\rangle = \sum_{j \notin I_{n-1}} |c_{n-1}^j\rangle \hat{q}_a = \left( \hat{1} - \sum_{k=0}^{n-1} \sum_{i \in I_k} |c_k^j\rangle \right) \hat{q}_n \quad (66)$$

where $\hat{q}_a$ is the amplitude for the gregarious transition. Simplifying we see that

$$|\text{orig} \rangle = \Pi_{n=1}^\infty (1 - \hat{q}_n) \quad (67)$$

This expression can now be evaluated for each of the possible extendible $\mathbb{C}SG$ dynamics we have considered.

The evaluation becomes trivial for any dynamics in which $t_1 = 0$, since $q_1 = 1$. For the class of $\mathbb{C}SG$ measures that do extend (see claims 3.8 and 3.9) we conclude that $|\text{orig} \rangle = 0$ whenever $t_1 = 0$. Using the principal of preclusion which states that (covariant) sets of quantum
measure zero do not happen, we see that for this class of dynamics we can make the somewhat trivial, but predictive statement that the originary event never happens. It is expected that such preclusions can also occur when \( q_1 \neq 0 \), when there are subtle phase cancellations. We leave such an investigation to future work.

For \( \mathbb{C}P \), which we have seen does not extend, the expression on the RHS has the simple form of the Euler Totient function [17, 20] and is finite for \(|q| \leq 1\). We expect that the measure will depend on this function for the class of dynamics which converges to \( \mathbb{C}P \) at larger \( k \). We postpone a detailed analysis of this to future work, as also explicit calculations of the measure of other covariant observables.

Finally it is worth reiterating that any quantum sequential growth with \( \mathcal{H} \simeq \mathbb{C} \) is unlikely to correspond to full quantum gravity with the right classical limit; the quantum gravity measure would likely live on an infinite dimensional \( \mathcal{H} \). Nevertheless, we believe the techniques presented above will be useful in the ongoing quest for physically relevant, covariant quantum measures.

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Appendix A. Some basic definitions in causal set theory

This section contains the definitions of various standard terms in CST that have appeared in the preceding sections.

- A causal set \textit{sample space} is a collection of causal sets. For sequential growth, this is the collection \( \Omega \) of countable, labelled, past finite causal sets, i.e.,

\[
\Omega \equiv \{ c | \forall e \in c, |\text{Past}(e)| < \infty \}
\]  

(68)

- An \textit{event} is a measurable subset of \( \Omega \).
- A \textit{covariant observable} \( O \subset \Omega \) is a measurable subset of \( \Omega \) such that if \( c \in O \), then so is every relabelling of \( c \).
- An \textit{n element chain} is a completely ordered \( n \)-element set \( c \), i.e., for every \( e_i, e_j \in c \), either \( e_i \prec e_j \) or \( e_j \prec e_i \). An \textit{n-element antichain} is a set of mutually unrelated elements: \( e_i \not\prec e_j, e_j \not\prec e_i \forall e_i, e_j \in c \).
- \textit{Labelled Poscau} \( \mathcal{P} \) refers to the tree of labelled causal sets. A \textit{node} in \( \mathcal{P} \) is a finite element labelled causal set. \textit{Poscau} \( \mathcal{\tilde{P}} \) is obtained from \( \mathcal{P} \) by identifying the nodes which are order-isomorphic. While \( \mathcal{P} \) is a tree, \( \mathcal{\tilde{P}} \) is not.
- A \textit{cylinder set} \( \text{cyl}(c^n) \subseteq \Omega \) such that

\[
\text{cyl}(c^n) \equiv \{ c | c|_n = c^n \}
\]  

(69)

where \( c|_n \) denotes the first \( n \) elements of \( c \).
Appendix B. CHK for $\mathcal{H} \simeq \mathbb{C}$

We now state the relevant parts of the Caratheodory–Hahn–Kluvnek theorem\(^{10}\) [18].

**Theorem B.1.** Let $\mathfrak{A}$ be a field of subsets of $\Omega$ and $\mathfrak{S}_{\mathfrak{A}}$ be the $\sigma$-field generated by $\mathfrak{A}$. Then if $\mu_\nu$ is a (i) bounded, (ii) weakly countably additive vector measure over $\mathfrak{A}$ then the following are equivalent.

(a) $\exists$ countably additive extension of $\mu_\nu$ to $\mathfrak{S}_{\mathfrak{A}}$.

(b) $\mu_\nu$ is (iii) strongly additive.

We define the terminology used in the theorem below.

(a) The semi-variation $\|\mu_\nu\|(\alpha)$ of a vector measure $\mu_\nu$ is defined as

\[
\|\mu_\nu\|(\alpha) = \sup\{\|x^* \mu_\nu\|(\alpha); x^* \in \mathcal{H}^*, \|x^*\| \leq 1\}, \tag{70}
\]

where $\mathcal{H}^*$ is the dual space. Note that $x^* \mu_\nu$ is an inner product measure, itself valued in $\mathbb{C}$. $\mu_\nu$ is said to be bounded if $\|\mu_\nu\|(\Omega) < \infty$.

(b) If for every infinite sequence $\{\alpha_1, \ldots, \alpha_n, \ldots\}$ of pairwise disjoint members of $\mathfrak{A}$ such that $\bigcup \alpha_i \in \mathfrak{A}$, $\mu_\nu(\bigcup \alpha_i) = \sum \mu_\nu(\alpha_i)$, then $\mu_\nu$ is countably additive.

(c) $\mu_\nu$ is weakly countably additive if $x^* \mu_\nu$ is countably additive for every $x^* \in \mathcal{H}^*$.

(d) $\mu_\nu$ is strongly additive if for every sequence $\{\alpha_n\}$ of pairwise disjoint element of $\mathfrak{A}$, $\sum_{n=1}^{\infty} \|\alpha_n\|$ converges in the norm.

We now show how the CHK theorem simplifies to theorem 2.1.

**Proof of theorem 2.1.** From [18] if $\mu_\nu$ is of bounded variation, then it is strongly additive, which in turn implies that it is bounded. For $\mathcal{H} \simeq \mathbb{C}$, the converse can be proved, i.e., boundedness implies bounded variation. Since the former implies that $\|x^* \mu_\nu\|(\Omega) < \infty$ for all $x^* \in \mathcal{H}^*$, by putting $x^* = 1$ we see that $\mu_\nu(\Omega) < \infty$. Thus bounded variation is equivalent to the conditions of boundedness and strong additivity.

Since $\mathfrak{A} = \mathfrak{Z}$, for every $\alpha \in \mathfrak{Z}$ there exists a smallest $n < \infty$ and a subset $S \subset \{1, \ldots, N_n\}$ such that $\alpha = \bigcup_{k \in S} \text{cyl}(c_k)$. Thus, $\mu_\nu$ is trivially countably and weakly countably additive.

Using the CHK theorem, this means that bounded variation of $\mu_\nu$ is sufficient for it to extend to $\mathfrak{S}_{\mathfrak{Z}}$.

That it is also necessary, comes from theorem 6.4 in [30], which states that a complex measure on any $\sigma$-algebra is of bounded variation. This completes the proof. \(\square\)

**ORCID iDs**

Sumati Surya \(\text{https://orcid.org/0000-0002-3735-617X}\)

Stav Zalel \(\text{https://orcid.org/0000-0003-2029-5785}\)

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\(^{10}\) The theorem as stated in [18] has two more equivalent conditions but they are not of direct relevance to this work, so we omit them.
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