Beyond Pointwise Submodularity: Non-Monotone Adaptive Submodular Maximization in Linear Time

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In this paper, we study the non-monotone adaptive submodular maximization problem subject to a cardinality constraint. We first revisit the adaptive random greedy algorithm proposed in (Gotovos et al. 2015), where they show that this algorithm achieves a $1/e$ approximation ratio if the objective function is adaptive submodular and pointwise submodular. It is not clear whether the same guarantee holds under adaptive submodularity (without resorting to pointwise submodularity) or not. Our first contribution is to show that the adaptive random greedy algorithm achieves a $1/e$ approximation ratio under adaptive submodularity. One limitation of the adaptive random greedy algorithm is that it requires $O(n \times k)$ value oracle queries, where $n$ is the size of the ground set and $k$ is the cardinality constraint. Our second contribution is to develop the first linear-time algorithm for the non-monotone adaptive submodular maximization problem. Our algorithm achieves a $1/e - \epsilon$ approximation ratio (this bound is improved to $1 - 1/e - \epsilon$ for monotone case), using only $O(n \epsilon^{-2} \log \epsilon^{-1})$ value oracle queries. Notably, $O(n \epsilon^{-2} \log \epsilon^{-1})$ is independent of the cardinality constraint.

1. Introduction

Maximizing a submodular function subject to practical constraints has been extensively studied in the literature (Golovin and Krause 2011, Tang and Yuan 2020, Yuan and Tang 2017b, Krause and Guestrin 2007). For the non-adaptive setting where all items must be selected at once, Nemhauser et al. (1978) show that the greedy algorithm achieves a $1 - 1/e$ approximation ratio when maximizing a monotone submodular function subject to a cardinality constraint. Their algorithms performs $O(n \times k)$ value oracle queries, where $n$ is the size of the ground set and $k$ is the cardinality constraint. Much research has focused on developing fast algorithms for submodular maximization recently (Leskovec et al. 2007, Badanidiyuru and Vondrak 2014, Mirzasoleiman et al. 2016, Ene and Nguyen 2018, Mirzasoleiman et al. 2015). Mirzasoleiman et al. (2015) propose the first linear-time algorithm that achieves a $1 - 1/e - \epsilon$ approximation, using only $O(n \log \frac{1}{\epsilon})$ value oracle queries. Their algorithm performs $k$ rounds: at each round, it draws a small random sample of items, and selects the item with the largest marginal utility from the random sample. They
show that their algorithm can achieve linear time complexity by carefully choosing the size of the random sample. Recently, Buchbinder et al. (2017) extends the previous studies to non-monotone submodular maximization and they develop the first linear-time algorithm for this problem under non-adaptive setting.

Our study focuses on the adaptive submodular maximization problem, a stochastic variant of the classical non-adaptive submodular maximization problem. The input of our problem is a set of items, each item is in a particular state drawn from a known prior distribution. One must select an item before observing its actual state. An adaptive policy specifies which item to pick next based on the current observation. Golovin and Krause (2011) generalize the classical notion of submodularity and monotonicity by introducing adaptive submodularity, whose formal definition is listed in Definition 5, and adaptive monotonicity, whose formal definition is listed in Definition 4. They show that a simple adaptive greedy algorithm achieves a $1 - \frac{1}{e}$ approximation for maximizing a monotone adaptive submodular function subject to a cardinality constraint. Their algorithm requires $O(n \times k)$ value oracle queries. Only recently, Tang (2020a) develops the first linear-time algorithm for the above problem. While the literature on adaptive submodular maximization (Chen and Krause 2013, Tang and Yuan 2020, Tang 2020b, Yuan and Tang 2017a, Fujii and Sakaue 2019, Gabillon et al. 2013, Golovin et al. 2010) typically assumes adaptive monotonicity, the more general problem of non-monotone adaptive submodular maximization is first studied in Gotovos et al. (2015). They propose an adaptive random greedy algorithm that achieves a $1/e$ approximation ratio. However, their result relies on the assumption that the objective function is adaptive submodular and pointwise submodular, whose formal definition is listed in Definition 6. Note that adaptive submodularity does not imply pointwise submodular (Guillory and Bilmes 2010, Golovin and Krause 2011), it is not clear whether the same guarantee holds without resorting to pointwise submodularity or not. Moreover, their algorithm requires $O(n \times k)$ value oracle queries.

Our contributions. Our first contribution is to present an enhanced analytical result by showing that the adaptive random greedy algorithm achieves a $1/e$ approximation ratio under adaptive submodularity (without resorting to pointwise submodularity). Our second contribution is to propose the first linear-time algorithm for the non-monotone adaptive submodular maximization problem subject to a cardinality constraint. Our algorithm achieves a $1/e - \epsilon$ approximation ratio, using only $O(ne^{-2}\log\epsilon^{-1})$ value oracle queries. For
monotone case, our algorithm achieves a $1 - 1/e - \epsilon$ approximation ratio. Our proposed approach generalizes the non-adaptive linear-time algorithm proposed in (Buchbinder et al. 2017) to the adaptive setting.

2. Preliminaries

In the rest of this paper, we use $[m]$ to denote the set $\{1, 2, \cdots, m\}$, and use $|S|$ to denote the cardinality of a set $S$.

2.1. Items and States

The input of our problem is set $E$ of $n$ items. Each item $e \in E$ has a particular state from $O$. We use $\phi : E \rightarrow O$ to denote a realization and let $U$ denote the set of all realizations. Let $\Phi = \{\Phi_e \mid e \in E\}$ denote a random realization drawn from a known prior probability distribution $p(\phi) = \{\Pr[\Phi = \phi] : \phi \in U\}$, where $\Phi_e \in O$ denotes a random realization of $e$. One must select an item before observing its state. After selecting a subset of items, we observe the partial realization $\psi$ of those items. The domain of $\psi$, which is denoted by $\text{dom}(\psi)$, is defined as the subset of items involved in $\psi$. For any realization $\phi$ and any partial realization $\psi$, we say $\psi$ is consistent with $\phi$ if they are equal everywhere in $\text{dom}(\phi)$. In this case, we write $\psi \sim \phi$. We say that $\psi$ is a subrealization of $\psi'$ if $\text{dom}(\psi) \subseteq \text{dom}(\psi')$ and they are equal everywhere in $\text{dom}(\psi)$. In this case, we write $\psi \subseteq \psi'$. We use $p(\phi \mid \psi)$ to denote the conditional distribution over realizations conditioned on a partial realization $\psi$: $p(\phi \mid \psi) = \Pr[\Phi = \phi \mid \psi \sim \Phi]$. There is a utility function $f$ from a subset of items and their states to a non-negative real number: $f : 2^E \times 2^O \rightarrow \mathbb{R}_{\geq 0}$.

2.2. Policies and Problem Formulation

Based on the above notations, we can encode any adaptive policy using a function $\pi$ from a set of partial realizations to $E$, e.g, $\pi$ takes as input the partial realizations of selected items and outputs which item to select next.

Definition 1 (Policy Concatenation). Given two policies $\pi$ and $\pi'$, we define $\pi @ \pi'$ as a policy that runs $\pi$ first, and then runs $\pi'$. Note that running $\pi'$ does not rely on the observation obtained from running $\pi$.

The expected utility $f_{\text{avg}}(\pi)$ of a policy $\pi$ is

$$f_{\text{avg}}(\pi) = \mathbb{E}_{\Phi \sim p(\phi)} f(E(\pi, \Phi), \Phi) \quad (1)$$
where $E(\pi, \phi)$ denotes the subset of items selected by $\pi$ under realization $\phi$. We say a policy $\pi$ is feasible if it selects at most $k$ items for all realizations, that is, $|E(\pi, \Phi)| \leq k$ for all $\phi \in U$.

Our goal is to find a feasible policy $\pi^{opt}$ that maximizes the expected utility:

$$\pi^{opt} \in \arg\max_{\pi \in \Omega} f_{avg}(\pi)$$

where $\Omega$ denotes the set of all feasible policies.

### 2.3. Adaptive Submodularity and Monotonicity

We next introduce the concept of adaptive submodularity. We start by introducing two notations.

**Definition 2 (Conditional Expected Marginal Utility of a Set of Items).** Given any partial realization $\psi$ and any set of items $S$, the conditional expected marginal utility $\Delta(e \mid \psi)$ of $S$ conditioned on $\psi$ is

$$\Delta(S \mid \psi) = \mathbb{E}_\Phi[f(\text{dom}(\psi) \cup S, \Phi) - f(\text{dom}(\psi), \Phi) \mid \Phi \sim \psi]$$

where the expectation is taken over $\Phi$ with respect to $p(\phi \mid \psi) = \Pr(\Phi = \phi \mid \Phi \sim \psi)$.

The algorithm is assumed to access the objective function via a value oracle that returns $\Delta(e \mid \psi)$ given any input element $e \in E$ and partial realization $\psi$.

**Definition 3 (Conditional Expected Marginal Utility of a Policy).** Given any partial realization $\psi$ and a policy $\pi$, the conditional expected marginal utility $\Delta(\pi \mid \psi)$ of $\pi$ conditioned on $\psi$ is

$$\Delta(\pi \mid \psi) = \mathbb{E}_\Phi[f(\text{dom}(\psi) \cup E(\pi, \Phi), \Phi) - f(\text{dom}(\psi), \Phi) \mid \Phi \sim \psi]$$

where the expectation is taken over $\Phi$ with respect to $p(\phi \mid \psi) = \Pr(\Phi = \phi \mid \Phi \sim \psi)$.

We next introduce the adaptive monotonicity and adaptive submodularity.

**Definition 4.** [Golovin and Krause 2011] [Adaptive Monotonicity] A function $f : 2^E \times O^E \rightarrow \mathbb{R}_{\geq 0}$ is adaptive monotone with respect to a prior distribution $p(\phi)$, if for any partial realization $\psi$, it holds that

$$\Delta(e \mid \psi) \geq 0$$

**Definition 5.** [Golovin and Krause 2011] [Adaptive Submodularity] A function $f : 2^E \times O^E \rightarrow \mathbb{R}_{\geq 0}$ is adaptive submodular with respect to a prior distribution $p(\phi)$, if for any two partial realizations $\psi$ and $\psi'$ such that $\psi \subseteq \psi'$, the following holds:

$$\Delta(e \mid \psi) \geq \Delta(e \mid \psi')$$
For comparison purpose, we further introduce the pointwise submodularity.

**Definition 6.** (Golovin and Krause 2011) [Pointwise Submodularity] A function \( f : 2^E \times O^E \to \mathbb{R}_{\geq 0} \) is pointwise submodular if \( f(S, \phi) \) is submodular in terms of \( S \subseteq E \) for all \( \phi \in U \). That is, for any \( \phi \in U \), any two sets \( S_1 \subseteq E \) and \( S_2 \subseteq E \) such that \( S_1 \subseteq S_2 \), and any item \( e / \notin S_2 \), we have

\[
    f(S_1 \cup \{e\}, \phi) - f(S_1, \phi) \geq f(S_2 \cup \{e\}, \phi) - f(S_2, \phi).
\]

Note that adaptive submodularity does not imply pointwise submodularity and vice versa.

### 3. Revisiting Adaptive Random Greedy Policy

We first revisit a simple random greedy policy, called Adaptive Random Greedy Policy \( \pi^{\text{arg}} \), that is proposed in (Gotovos et al. 2015). Gotovos et al. (2015) show that \( \pi^{\text{arg}} \) achieves a \( \frac{1}{e} \) approximation ratio under the condition that \( f \) is adaptive submodular and pointwise submodular. Because adaptive submodularity does not imply pointwise submodularity and vice versa, it is not clear whether adaptive submodularity is a sufficient condition for achieving the above guarantee. We provide a positive answer to this question by showing that \( \pi^{\text{arg}} \) achieves a \( \frac{1}{e} \) approximation ratio under adaptive submodularity (without resorting to pointwise submodularity).

We first explain the idea of \( \pi^{\text{arg}} \) (Algorithm 1). We first add a set \( D \) of \( 2k - 1 \) dummy items to the ground set, such that, for any \( d \in D \), and any partial realization \( \psi \), we have \( \Delta(d | \psi) = 0 \). Let \( E' = E \cup D \). We add these dummy items to ensure that \( \pi^{\text{arg}} \) never selects an item with negative marginal utility. Clearly, these dummy items can be safely removed from the solution returned from any policy without affecting its utility. \( \pi^{\text{arg}} \) runs round by round: Starting with an empty set and at each round \( r \in [k] \), \( \pi^{\text{arg}} \) randomly selects an item from the set \( M(\psi^{r-1}) \), which contains the \( k \) items with the largest marginal utility to the current partial realization \( \psi^{r-1} \).

We next present the main theorem.

**Theorem 1.** If \( f \) is adaptive submodular, then the Adaptive Random Greedy Policy \( \pi^{\text{arg}} \) achieves a \( \frac{1}{e} \) approximation ratio in expectation with \( O(nk) \) value oracle queries.

**Proof:** We first prove the time complexity of \( \pi^{\text{arg}} \). As \( \pi^{\text{arg}} \) performs \( k \) rounds and each round performs \( O(n) \) value oracle queries, the time complexity of \( \pi^{\text{arg}} \) is \( O(nk) \). We next prove the approximation ratio of \( \pi^{\text{arg}} \).

For every \( r \in [k] \), let \( \pi_r^{\text{arg}} \) denote the policy that runs \( \pi^{\text{arg}} \) for \( r \) rounds. We first provide a preparation lemma as follows.
Algorithm 1 Adaptive Random Greedy Policy $\pi^{arg}$

1: $A = \emptyset$; $r = 1.$

2: while $r \leq k$ do
3:   observe $\psi^{r-1};$
4:   $M(\psi^{r-1}) \leftarrow \arg \max_{M \subseteq E' \mid |M| \leq k} \sum_{e \in E'} \Delta(e \mid \psi^{r-1});$
5:   sample $e_r$ uniformly at random from $M(\psi^{r-1});$
6:   $A \leftarrow A \cup \{e_r\}$; $r \leftarrow r + 1;$
7: return $A$

**Lemma 1.** When $f$ is adaptive submodular, for every $r \in [k],$

$$f_{avg}(\pi^{opt} \oplus \pi^{arg}_r) \geq (1 - \frac{1}{k})^r f_{avg}(\pi^{opt})$$

**Proof:** Fix $r \in [k]$ and a partial realization $\psi^{opt|arg(r-1)}$ that is observed after running $\pi^{opt} \oplus \pi^{arg}_{r-1}.$ Observe that,

$$E_{e_r}[\Delta(e_r \mid \psi^{opt|arg(r-1)})] = \frac{1}{k} \sum_{e \in M(\psi^{r-1})} \Delta(e \mid \psi^{opt|arg(r-1)})$$

$$\geq \frac{1}{k} \Delta(M(\psi^{r-1}) \mid \psi^{opt|arg(r-1)})$$

$$= \frac{1}{k} (E_{\Phi}[f_{avg}(\text{dom}(\psi^{opt|arg(r-1)} \cup M(\psi^{r-1}), \Phi) \mid \Phi \sim \psi^{opt|arg(r-1)}] - E_{\Phi}[f(\text{dom}(\psi^{opt|arg(r-1)}, \Phi) \mid \Phi \sim \psi^{opt|arg(r-1)}])$$

$$\geq - \frac{E_{\Phi}[f(\text{dom}(\psi^{opt|arg(r-1)}, \Phi) \mid \Phi \sim \psi^{opt|arg(r-1)}]}{k}$$

where the first inequality is due to $f$ is adaptive submodular. Unfixing $\psi^{opt|arg(r-1)},$ taking the expectation over $\Psi^{opt|arg(r-1)},$ we have

$$E_{\psi^{opt|arg(r-1)}}[E_{e_r}[\Delta(e_r \mid \Psi^{opt|arg(r-1)})]] = f_{avg}(\pi^{opt} \oplus \pi^{arg}_r) - f_{avg}(\pi^{opt} \oplus \pi^{arg}_{r-1}) \geq - \frac{f_{avg}(\pi^{opt} \oplus \pi^{arg}_{r-1})}{k}$$

We are now ready to prove this lemma by induction on $r.$ When $r = 1,$ this lemma holds because $f_{avg}(\pi^{opt} \oplus \pi^{arg}_0) = f_{avg}(\pi^{opt}) \geq (1 - \frac{1}{k})^0 f_{avg}(\pi^{opt}).$ Assume this lemma holds for $r' < r,$ we next prove it for $r > 0.$

$$f_{avg}(\pi^{opt} \oplus \pi^{arg}_r) = f_{avg}(\pi^{opt} \oplus \pi^{arg}_{r-1}) + (f_{avg}(\pi^{opt} \oplus \pi^{arg}_r) - f_{avg}(\pi^{opt} \oplus \pi^{arg}_{r-1}))$$

$$\geq f_{avg}(\pi^{opt} \oplus \pi^{arg}_{r-1}) - \frac{f_{avg}(\pi^{opt} \oplus \pi^{arg}_{r-1})}{k}$$
\[ f_{avg}(\pi_{r}^{arg}) = (1 - \frac{1}{k}) f_{avg}(\pi_{r-1}^{arg}) \]  \hspace{1cm} (11)
\[ \geq (1 - \frac{1}{k})(1 - \frac{1}{k})^{r-1} f_{avg}(\pi^{opt}) \]  \hspace{1cm} (12)
\[ = (1 - \frac{1}{k})^{r} f_{avg}(\pi^{opt}) \]  \hspace{1cm} (13)

The first inequality is due to Eq. (8) and the second inequality follows from the inductive assumption. \( \square \)

Now we are ready to prove the theorem. Let \( \Psi^{r-1} \) denote a random partial realization after running \( \pi_{r-1}^{arg} \). The expectation \( E_{\Psi^{r-1}}[\cdot] \) is taken over all such partial realizations \( \Psi^{r-1} \). Then we have

\[ f_{avg}(\pi_{r}^{arg}) - f_{avg}(\pi_{r-1}^{arg}) = E_{\Psi^{r-1}}[E_{e_{r-1}}[\Delta(e_{r-1} | \Psi^{r-1})]] \]  \hspace{1cm} (14)
\[ = \frac{1}{k} E_{\Psi^{r-1}}[\sum_{e \in M(\Psi^{r-1})} \Delta(e | \Psi^{r-1})] \]  \hspace{1cm} (15)
\[ \geq \frac{1}{k} E_{\Psi^{r-1}}[\Delta(\pi^{opt} | \Psi^{r-1})] \]  \hspace{1cm} (16)
\[ = \frac{f_{avg}(\pi_{opt}^{@\pi_{r-1}^{arg}}) - f_{avg}(\pi_{r-1}^{arg})}{k} \]  \hspace{1cm} (17)
\[ \geq (1 - \frac{1}{k}) f_{avg}(\pi_{opt}) - f_{avg}(\pi_{r-1}^{arg}) \]  \hspace{1cm} (18)

The second equality is due to the design of \( \pi^{arg} \), the first inequality is due to \( f \) is adaptive submodular and Lemma 1 in (Gotovos et al. 2015), and the second inequality is due to Lemma \[ \square \]

We next prove

\[ f_{avg}(\pi_{r}^{arg}) \geq \frac{r}{k} (1 - \frac{1}{k})^{r-1} f_{avg}(\pi^{opt}) \]  \hspace{1cm} (19)

by induction on \( r \). For \( r = 0 \), \( f_{avg}(\pi_{0}^{arg}) \geq 0 \geq 0 (1 - \frac{1}{k})^{0-1} f_{avg}(\pi^{opt}) \). Assume Eq. (19) is true for \( r ' < r \), let us prove it for \( r \).

\[ f_{avg}(\pi_{r}^{arg}) \geq f_{avg}(\pi_{r-1}^{arg}) + (1 - \frac{1}{k})^{r-1} f_{avg}(\pi^{opt}) - f_{avg}(\pi_{r-1}^{arg}) \]  \hspace{1cm} (20)
\[ = (1 - 1/k) f_{avg}(\pi_{r-1}^{arg}) + (1 - \frac{1}{k})^{r-1} f_{avg}(\pi^{opt}) - f_{avg}(\pi_{r-1}^{arg}) \]  \hspace{1cm} (21)
\[ \geq (1 - 1/k) \cdot ((r-1)/k) \cdot (1 - 1/k)^{r-2} \cdot f(\pi^{opt}) + \frac{(1 - \frac{1}{k})^{r-1} f_{avg}(\pi^{opt})}{k} \]  \hspace{1cm} (22)
\[ \geq \frac{(r/k) \cdot (1 - 1/k)^{r-1} \cdot f(\pi^{opt}) \hspace{1cm} (23)}{k} \]
The first equality is due to Eq. (18), the second inequality is due to the inductive assumption. When \( r = k \), we have \( f_{\text{avg}}(\pi_{\text{opt}}) \geq (1 - 1/k)^{k-1} \cdot f(\pi_{\text{opt}}) \geq (1/e) f(\pi_{\text{opt}}) \). This finishes the proof of the theorem. \( \square \)

For monotone case, Gotovos et al. (2015) show that \( \pi_{\text{arg}} \) achieves a \( 1 - 1/e \) approximation ratio.

**Theorem 2.** (Gotovos et al. 2015) If \( f \) is adaptive submodular and adaptive monotone, then the Adaptive Random Greedy Policy \( \pi_{\text{arg}} \) achieves a \( 1 - 1/e \) approximation ratio in expectation with \( O(nk) \) value oracle queries.

### 4. Linear-Time Adaptive Policy

We next present the Linear-Time Adaptive Policy \( \pi_{lt} \) (Algorithm 2). Later we show that it achieves a \( 1/e - \epsilon \) approximation ratio, using only \( O(ne^{-2} \log \epsilon^{-1}) \) value oracle queries.

We first explain the idea of \( \pi_{lt} \) which generalizes the non-adaptive linear-time algorithm (Buchbinder et al. 2017) to the adaptive setting. \( \pi_{lt} \) has two parameters \( q \) and \( s \) and it works for \( k \) rounds: Starting with an empty set. At each round \( r \in [k] \), \( \pi_{lt} \) first samples a random set \( S_r \) of size \( \lceil qn \rceil \) from \( E \). Let \( e_r \) be the item of \( S_r \) that has the \( \lceil d_r \rceil \)-th largest marginal utility to the current partial realization \( \psi^{r-1} \) (by abuse of notation), where \( d_r \) is a random value sampled from range \((0,s]\). \( \pi_{lt} \) adds \( e_r \) to the solution if \( \Delta(e_r | \psi^{r-1}) \geq 0 \).

**Algorithm 2** Linear-Time Adaptive Policy \( \pi_{lt} \)

1. \( A = \emptyset; r = 1 \).
2. \textbf{while} \( r \leq k \) \textbf{do} 
3. \hspace{1em} observe \( \psi^{r-1} \),
4. \hspace{1em} let \( S_r \) be a random set of size \( \min\{\lceil qn \rceil, n\} \) sampled uniformly at random from \( E \);
5. \hspace{1em} let \( d_r \) be a random value sampled from range \((0,s]\);
6. \hspace{1em} let \( e_r \) be the item of \( S_r \) that has the \( \lceil d_r \rceil \)-th largest marginal utility to \( \psi^{r-1} \);
7. \hspace{1em} if \( \Delta(e_r | \psi^{r-1}) \geq 0 \) \textbf{then} 
8. \hspace{2em} \( A \leftarrow A \cup \{e_r\}; r \leftarrow r + 1 \);
9. \hspace{1em} \textbf{return} \( A \)

By setting \( q = 8k^{-1} \epsilon^{-2} \cdot \ln(2\epsilon)^{-1} \) and \( s = k \min\{\lceil qn \rceil, n\}/n \), we have the following theorem.
Theorem 3. If \( f \) is adaptive submodular, then the linear-Time Adaptive Policy \( \pi^{lt} \) achieves a \( (1/e - \epsilon) \) approximation ratio in expectation with \( O(n\epsilon^{-2}\log\epsilon^{-1}) \) value oracle queries.

**Proof:** We first prove the time complexity of \( \pi^{lt} \). Observe that \( \pi^{lt} \) performs \( k \) rounds, and each rounds performs \( \min\{\lceil qn \rceil, n \} \) value oracle queries. Thus the total number of value oracle queries is bounded by \( \min\{\lceil qn \rceil, n \} \times k \leq (qn + 1) \times k \). Since we set \( q = 8k^{-1}\epsilon^{-2} \cdot \ln(2\epsilon)^{-1} \), we have \( \min\{\lceil qn \rceil, n \} \times k \leq (qn + 1) \times k = O(n\epsilon^{-2}\log\epsilon^{-1}) \).

We next prove the approximation ratio of \( \pi^{lt} \). Note that when \( \min\{\lceil qn \rceil, n \} = n \), we have \( s = k \), and \( \pi^{lt} \) is reduced to \( \pi^{lt}_{r-1} \). It follows that \( \pi^{lt} \) achieves a \( 1/e \) approximation ratio due to Theorem 1. In the rest of the proof, we assume that \( \lceil qn \rceil < n \).

We first provide two preparation lemmas: Lemma 2 and Lemma 3. For any \( r \in [k] \), let \( \pi^{lt}_{r-1} \) denote the policy that runs \( \pi^{lt} \) for \( r \) rounds.

**Lemma 2.** When \( f \) is adaptive submodular, for every \( r \in [k] \),

\[
\frac{f_{avg}(\pi^{opt} \@ \pi^{lt}_{r-1})}{f_{avg}(\pi^{opt})} \geq (1 - \frac{1}{k})^r f_{avg}(\pi^{opt})
\]  

(24)

**Proof:** Fix \( r \in [k] \) and a partial realization \( \psi^{opt}_{(r-1)} \) that is observed after running \( \pi^{opt} \@ \pi^{lt}_{r-1} \). Observe that,

\[
\mathbb{E}_{e_r}[\Delta(e_r \mid \psi^{opt}_{(r-1)})] = \sum_{e \in E} \Pr[e \text{ is selected at round } r] \Delta(e \mid \psi^{opt}_{(r-1)}) \]  

(25)

\[
\geq \sum_{e \in E \setminus E^+} \Pr[e \text{ is selected at round } r] \Delta(e \mid \psi^{opt}_{(r-1)})
\]  

(26)

\[
\geq \sum_{e \in E \setminus E^+} \frac{\lceil qn \rceil}{s} \Delta(e \mid \psi^{opt}_{(r-1)})
\]  

(27)

\[
\geq \frac{1}{k} \sum_{e \in E \setminus E^+} \Delta(e \mid \psi^{opt}_{(r-1)})
\]  

(28)

\[
\geq \frac{1}{k} \Delta(E \setminus E^+ \mid \psi^{opt}_{(r-1)})
\]  

(29)

\[
= \frac{1}{k}(\mathbb{E}_{\Phi}[f_{avg}(\text{dom}(\psi^{opt}_{(r-1)}) \cup (E \setminus E^+), \Phi) \mid \Phi \sim \psi^{opt}_{(r-1)}])
\]  

(30)

\[
\geq -\frac{\mathbb{E}_{\Phi}[f(\text{dom}(\psi^{opt}_{(r-1)}), \Phi) \mid \Phi \sim \psi^{opt}_{(r-1)}]}{k}
\]  

(31)

where \( E^+ = \{e \mid e \in E; \Delta(e \mid \psi^{opt}_{(r-1)}) \geq 0 \} \). The first inequality is due to the definition of \( E^+ \), the second inequality is due to \( \Pr[e \text{ is selected at round } r] \leq \frac{\lceil qn \rceil}{s} \) for all \( e \in E \) and
\( \Delta(e) < 0 \) for all \( e \in E \setminus E^+ \), the third inequality is due to \( q = 8k^{-1}e^{-2} \cdot \ln(2e)^{-1} \) and \( s = k[qn]/n \), the forth inequality is due to \( f \) is adaptive submodular.

Unfixing \( \psi_{(r-1)} \), taking the expectation over \( \Psi_{(r-1)} \), we have

\[
\mathbb{E}_{\psi_{(r-1)}} \left[ \mathbb{E}_{e_r} \left[ \Delta(e_r | \Psi_{(r-1)}) \right] \right] = f_{\text{avg}}(\pi_{(r)}^{\text{opt}}@\pi_{(r)}^{lt})-f_{\text{avg}}(\pi_{(r-1)}^{\text{opt}}@\pi_{(r-1)}^{lt}) \geq \frac{f_{\text{avg}}(\pi_{(r)}^{\text{opt}}@\pi_{(r-1)}^{lt})}{k} \tag{33}
\]

The rest of the proof is similar to the proof of Lemma 1, thus omitted here. \( \square \)

The proof of the following lemma is provided in appendix.

**Lemma 3.** \( f_{\text{avg}}(\pi_{(r)}^{lt})-f_{\text{avg}}(\pi_{(r-1)}^{lt}) \geq (1-\epsilon)\frac{f_{\text{avg}}(\pi_{(r)}^{\text{opt}}@\pi_{(r-1)}^{lt})-f_{\text{avg}}(\pi_{(r-1)}^{lt})}{k} \)

Lemma 2 and Lemma 3 imply that

\[
f_{\text{avg}}(\pi_{(r)}^{lt})-f_{\text{avg}}(\pi_{(r-1)}^{lt}) \geq (1-\epsilon)\frac{(1-\epsilon)^{r-1}f_{\text{avg}}(\pi_{(r)}^{\text{opt}})-f_{\text{avg}}(\pi_{(r-1)}^{lt})}{k} \geq \frac{[(1-\epsilon)^{r-1}-\epsilon]f_{\text{avg}}(\pi_{(r)}^{\text{opt}})-f_{\text{avg}}(\pi_{(r-1)}^{lt})}{k} \tag{34}
\]

To prove this theorem, it suffice to show that

\[
f_{\text{avg}}(\pi_{(r)}^{lt}) \geq (r/k) \cdot [(1-1/k)^{r-1}-\epsilon] \cdot f(\pi_{(r)}^{\text{opt}}) \tag{36}
\]

for all \( r \in [k] \). Similar to (Buchbinder et al. 2017) (the proof of Theorem 4.2), we prove Eq. (36) by induction on \( r \). For \( r = 0 \), \( f_{\text{avg}}(\pi_{(0)}^{lt}) \geq 0 \geq (0/k) \cdot [(1-1/k)^{0-1}-\epsilon] \cdot f(\pi_{(0)}^{\text{opt}}) \). Assume Eq. (36) is true for \( r' < r \), let us prove it for \( r \).

\[
f_{\text{avg}}(\pi_{(r)}^{lt}) \geq f_{\text{avg}}(\pi_{(r-1)}^{lt}) + \frac{[(1-\epsilon)^{r-1}-\epsilon]f_{\text{avg}}(\pi_{(r)}^{\text{opt}})-f_{\text{avg}}(\pi_{(r-1)}^{lt})}{k} \tag{37}
\]

\[
= (1-1/k)f_{\text{avg}}(\pi_{(r-1)}^{lt}) + \frac{[(1-\epsilon)^{r-1}-\epsilon]f_{\text{avg}}(\pi_{(r)}^{\text{opt}})}{k} \tag{38}
\]

\[
\geq (1-1/k) \cdot ((r-1)/k) \cdot [(1-1/k)^{r-2}-\epsilon] \cdot f(\pi_{(r)}^{\text{opt}}) + \frac{[(1-\epsilon)^{r-1}-\epsilon]f_{\text{avg}}(\pi_{(r)}^{\text{opt}})}{k} \tag{39}
\]

\[
\geq (r/k) \cdot [(1-1/k)^{r-1}-\epsilon] \cdot f(\pi_{(r)}^{\text{opt}}) \tag{40}
\]

The first inequality is due to (35), the second inequality is due to the inductive assumption. When \( r = k \), we have \( f_{\text{avg}}(\pi_{(k)}^{lt}) \geq [(1-1/k)^{k-1}-\epsilon] \cdot f(\pi_{(k)}^{\text{opt}}) \geq (1/e-\epsilon) f(\pi_{(k)}^{\text{opt}}) \). This finishes the proof of the theorem. \( \square \)
Performance bound for monotone case. We next show that if \( f \) is adaptive monotone, the performance bound of \( \pi^{lt} \) is improved to \( 1 - 1/e - \epsilon \).

**Theorem 4.** If \( f \) is adaptive submodular and adaptive monotone, then the linear-Time Adaptive Policy \( \pi^{lt} \) achieves a \( 1 - 1/e - \epsilon \) approximation ratio in expectation with \( O(n\epsilon^{-2}\log\epsilon^{-1}) \) value oracle queries.

**Proof:** The time complexity result inherits from the Theorem 3. We next prove the performance bound of \( \pi^{lt} \). When \( f \) is adaptive submodular and adaptive monotone, Lemma 3 still holds. Thus, \( f_{\text{avg}}(\pi^{lt}_r) - f_{\text{avg}}(\pi^{lt}_{r-1}) \geq (1 - \epsilon) \frac{f_{\text{avg}}(\pi^{opt}) - f_{\text{avg}}(\pi^{lt}_{r}) - f_{\text{avg}}(\pi^{lt}_{r-1})}{k} \geq (1 - \epsilon) \frac{f_{\text{avg}}(\pi^{opt}) - f_{\text{avg}}(\pi^{lt}_{r-1})}{k} \), where the second inequality is due to \( f \) is adaptive monotone. It follows that \( f_{\text{avg}}(\pi^{lt}_k) \geq (1 - (1 - \frac{1}{e})^k) f_{\text{avg}}(\pi^{opt}) \geq (1 - (1/e)^{1-\epsilon}) f_{\text{avg}}(\pi^{opt}) \geq (1 - 1/e - \epsilon) f_{\text{avg}}(\pi^{opt}) \) through induction on \( r \). \( \square \)

5. Conclusion

In this paper, we study the non-monotone adaptive submodular maximization problem subject to a cardinality constraint. We first revisit the adaptive random algorithm and show that it achieves a \( 1/e \) approximation ratio under adaptive submodularity. Then we propose a linear-time adaptive policy that achieves a \( 1/e - \epsilon \) approximation ratio, using only \( O(n\epsilon^{-2}\log\epsilon^{-1}) \) value oracle queries. In the future, we would like to consider other constraints such as knapsack constraints and general matroid constraints.

6. Appendix

6.1. Proof of Lemma 3

Let \( v_1(\psi^{r-1}), v_2(\psi^{r-1}), \ldots, v_k(\psi^{r-1}) \) be the \( k \) items with the largest marginal contribution to \( \psi^{r-1} \), sorted in a non-increasing order of their marginal contributions. We use \( X_j \) to denote an indicator for the event \( e_r = v_j(\psi^{r-1}) \). The following two lemmas are proved in (Buchbinder et al. 2017).

**Lemma 4.** \( \mathbb{E}[\sum_{j \in [k]} X_j] \geq 1 - \epsilon. \)

**Lemma 5.** \( \mathbb{E}[X_j] \) is a non-increasing function of \( j \).

Now we are ready to prove the lemma. Our proof generalizes the proof of Lemma 4.5 in (Buchbinder et al. 2017) to the adaptive setting. Let \( \Psi^{r-1} \) denote a random partial
realization observed after running $\pi_{r-1}^t$. The expectation $E_{\Psi^{r-1}}[\cdot]$ is taken over all such partial realizations $\Psi^{r-1}$. Then we have

$$f_{avg}(\pi_t^{r}) - f_{avg}(\pi_{r-1}^t) = E_{\Psi^{r-1}}[\mathbb{E}_{e_r}[\max\{\Delta(e_r | \Psi^{r-1}), 0\}]]$$

(41)

$$\geq E_{\Psi^{r-1}}\left[\sum_{j \in [k]} \mathbb{E}[X_j] \max\{\Delta(v_j(\Psi^{r-1}) | \Psi^{r-1}), 0\}\right]$$

(42)

$$\geq E_{\Psi^{r-1}}\left[\frac{\sum_{j \in [k]} \mathbb{E}[X_j] \max\{\Delta(v_j(\Psi^{r-1}) | \Psi^{r-1}), 0\}}{k}\right]$$

(43)

$$\geq (1 - \epsilon)E_{\Psi^{r-1}}[\Delta(\pi^{opt} | \Psi^{r-1})]$$

(44)

$$= (1 - \epsilon)\frac{f_{avg}(\pi^{opt} @ \pi_{r-1}^t) - f_{avg}(\pi_{r-1}^t)}{k}$$

(45)

The second inequality is due to Chebyshev’s sum inequality because $\max\{\Delta(v_j(\Psi^{r-1}) | \Psi^{r-1}), 0\}$ is nonincreasing in $j$ by definition and $\mathbb{E}[X_j]$ is a non-increasing function of $j$ by Lemma 5, the third inequality is due to $f$ is adaptive submodular and Lemma 1 in (Gotovos et al. 2015), the forth inequality is due to Lemma 4.

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