Non-local imprints of gravity on quantum theory

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Abstract
During the last two decades or so much effort has been devoted to the discussion of quantum mechanics (QM) that in some way incorporates the notion of a minimum length. This upsurge of research has been prompted by the modified uncertainty relation brought about in the framework of string theory. In general, the implementation of minimum length in QM can be done either by modification of position and momentum operators or by restriction of their domains. In the former case we have the so called soccer-ball problem when the naive classical limit appears to be drastically different from the usual one. Starting with the latter possibility, an alternative approach was suggested in the form of a band-limited QM. However, applying momentum cutoff to the wave-function, one faces the problem of incompatibility with the Schrödinger equation. One can overcome this problem in a natural fashion by appropriately modifying Schrödinger equation. But incompatibility takes place for boundary conditions as well. Such wave-function cannot have any more a finite support in the coordinate space as it simply follows from the Paley–Wiener theorem. Treating, for instance, the simplest quantum-mechanical problem of a particle in an infinite potential well, one can no longer impose box boundary conditions. In such cases, further modification of the theory is in order. We propose a non-local modification of QM, which has close ties to the band-limited QM, but does not require a hard momentum cutoff. In the framework of this model, one can easily work out the corrections to various processes and discuss further the semi-classical limit of the theory.

Keywords Quantum gravity · Quantum mechanics · Minimum length
1 Introduction

General relativity and quantum mechanics are two pillars of modern physics. However, in their foundational concepts, the two theories differ significantly, and it is not an easy task to find a unifying framework for them. In general relativity, basic observables are space-time distances between events, the events being intersection points of the world-lines of two objects. Therefore, it is assumed that such crossing points are accurately localized in space-time. This sharp localization is not possible in quantum theory and thus quantum mechanics denies the observability of basic events of general relativity [1].

This observation by Wigner shows the subtlety of the coexistence of quantum mechanics and general (or even special) theory of relativity, but does not exclude such a possibility. After all, relativistic quantum field theory, which combines the principles of special relativity and quantum mechanics, is one of our best scientific theories. In fact “the framework of special relativity plus quantum mechanics is so rigid that it practically forces quantum field theory upon us” [2].

In the case of gravity (general relativity) it is believed that the best candidate for combining it with quantum mechanics is string theory [3,4]. String theory is a generalization of quantum field theory in which space-time emerges from something deeper and more fundamental. However, at present, “we still don’t know where all these ideas are coming from—or heading to ... without anyone really understanding what is behind it” [2].

Another approach that tries to combine quantum mechanics and general relativity is loop quantum gravity [5,6]. String theory and loop quantum gravity are often viewed as competing, mutually exclusive theories. However, it may happen that they are simply highlighting different aspects of quantum theories of gravity [6], and therefore it is worthwhile to pursue both of these paths in the enigmatic and mysterious land of quantum gravity.
Since a definitive theory of quantum gravity is still unknown, many phenomeno-
logical models have been proposed that incorporate the minimum observable length
predicted by both string theory and quantum gravity models. Such models include
doubly special relativity [7] and models with the generalized uncertainty principle
(GUP) [8–11].

A priory, one cannot exclude the possibility that the gravitational field will remain
classical even at the fundamental level, and only the matter fields are quantized. In this
case, gravity induces a natural nonlinear and non-local modification of Schrödinger
equation [12,13]. The starting point is to replace the classical energy-momentum tensor
in Einstein’s equations by the expectation value of the corresponding quantum energy-
momentum tensor in a given quantum state $\Psi$:

$$ R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle. \tag{1} $$

In the non-relativistic Newtonian limit $\hat{T}_{00} = c^2 \hat{\rho} = mc^2 \hat{\psi}^+ \hat{\psi}$ gives the dominant
contribution to the energy-momentum tensor, and (1) becomes the Poisson equation
for the potential $V(r, t)$ [12,13]:

$$ \Delta V = 4\pi Gm \langle \hat{\psi}^+ \hat{\psi} \rangle, \tag{2} $$

which can be solved using the well-known expression for the corresponding Green’s
function. If now this Newtonian potential energy $mV(r, t)$ is introduced in the
Schrödinger equation, we end up by a nonlinear integro-differential equation (the
Schrödinger–Newton equation), which for a one free point-like object takes the form
[12,13]

$$ i\hbar \partial_t \psi(r, t) = -\frac{\hbar^2}{2m} \Delta \psi(r, t) - \left( Gm^2 \int \frac{|\psi(\cdot, t)|^2}{|\cdot - r|} d^3 \xi \right) \psi(r, t). \tag{3} $$

Therefore, it is expected that gravity will induce non-local and nonlinear modification
of the Schrödinger equation. However, Schrödinger-Newton equation suffers
from superluminal effects [13], as all nonlinear deterministic generalizations of the
Schrödinger equation do [13,14]. It, at the most, can pretend to be a low-energy effect-
tive approximation of a more fundamental theory. Since the exact form of this more
fundamental theory (quantum gravity) is not known at present, it seems appropriate
to explore other approaches also to the question of how the Schrödinger equation is
modified in the presence of gravity.

With the introduction of gravity, one way or another, into quantum theory, it becomes
fairly clear that at distances comparable to the quantum gravity length scale, Planck
length, $l_P = (\hbar G_N / c^3)^{1/2} \approx 10^{-33}$ cm, one must drop the standard picture of space-
time as a continuum endowed with a certain intrinsic geometric structure [15]. These
ideas have been around for a long time now. Already in 1950s, Wheeler observed that
the scale dependence of the gravitational action implies large fluctuations of the metric
and even of the topology on Planck length scale [16–18].
One is thus led to a picture of foamy space-time implying that space-time is basically flat on large length scales but is highly curved with all possible topologies on the Planck length scale. For instance, the foamy space-time can be described in terms of a gas consisting of virtual (Planck size) black holes—continually appearing and disappearing [19].

Apart from this approach, the micro-structure of space-time can be modeled in a number of ways: one can represent space-time coordinates by the non-commuting operators [20], or, equally well, one can assume some sort of discrete structure from the very outset [21,22]. Keeping aside details and proper mathematical structures related to various approaches, the effect of quantum fluctuations of the gravitational field (or of space-time geometry) is conventionally summarized either as a source of absolute minimum uncertainty in length, which implies smoothing out of point-like objects, or as a mechanism for providing momentum cutoff of the order of $\hbar/l_P$ to regularize the ultraviolet divergences. Of course, this may not be a hard momentum cutoff but rather certain modifications of dispersion relations of particles that may render loop Feynman graphs convergent.

In view of the sampling theorem in information theory [25], which tells one how to digitize an analog signal in a precise way, it was noticed in [26] that the application of hard momentum cutoff to the fields implies their representation on the lattice and thus may be considered as one of the simplest ways for introducing discrete space. Inspired by this idea, some basic features of QM with hard momentum cutoff has been worked out in [27,35]. So far fairly little attention has been paid to this discussion. First we attempt further elaboration of physical and mathematical aspects of such theory. As far as the classical limit is concerned, the result, if the limit is naively applied, turns out to be incompatible with reality.

Next we proceed to propose somewhat similar non-local model, which, however, may be adjusted in such a way as to offer a trivial solution of the soccer-ball problem. Corrections due to non-locality are of the same order as obtained in various minimum-length deformed models of QM and can be worked out without much trouble.

2 QM in Hilbert space with hard momentum cutoff

2.1 Setting up the basic formalism

We first put the wave function into a momentum cutoff representation

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\beta}^{\beta} dp \chi(p) e^{ipx/\hbar},$$

(4)

where it is understood that the scale $\hbar/\beta$ is related to the minimum length. In QM the mean square deviation of coordinate, that is the position uncertainty, is usually considered as a standard measure of the spread of a wave function in position space. It can be shown rigorously that the position uncertainty of a wave function with momentum cutoff, Eq. (4), is bounded from below by $\hbar/4\beta$. To see it, we first assume (with no loss of generality) that $\langle x \rangle = 0$. By using the result

$$\langle x^2 \rangle \geq \frac{\hbar^2}{4\beta^2}.$$
\[
\int_{-\beta}^{\beta} dp \left| \frac{d\chi^*(p)}{dp} \right|^2 = \frac{1}{\hbar^2} \int_{-\infty}^{\infty} dx \, x^2 |\psi(x)|^2 ,
\]

which readily follows from the Parseval’s formula
\[
\int_{-\infty}^{\infty} dx \, |\psi(x)|^2 = \int_{-\beta}^{\beta} dp \, |\chi(p)|^2 ,
\]

and the following inequalities (the latter one is the Schwarz’s inequality)
\[
2 \left| \chi(p) \frac{d\chi^*(p)}{dp} \right| \geq \chi(p) \frac{d\chi^*(p)}{dp} + \chi^*(p) \frac{d\chi(p)}{dp} = \frac{d|\chi(p)|^2}{dp} ,
\]
\[
\left( \int_{-\beta}^{\beta} dp \left| \chi(p) \frac{d\chi^*(p)}{dp} \right| \right)^2 \leq \int_{-\beta}^{\beta} dp \, |\chi(p)|^2 \int_{-\beta}^{\beta} dp \left| \frac{d\chi^*(p)}{dp} \right|^2 ,
\]

one obtains at once [36]
\[
1 = \int_{-\beta}^{\beta} dp \, |\chi(p)|^2 = \int_{-\beta}^{\beta} dp \int_{-\beta}^{p} \frac{d|\chi(\xi)|^2}{d\xi} \frac{d\xi}{dp} \leq 2 \int_{-\beta}^{\beta} dp \int_{-\beta}^{p} \frac{d|\chi(\xi)|^2}{d\xi} \frac{d\xi}{dp} \leq 2 \int_{-\beta}^{\beta} dp \int_{-\beta}^{\beta} \left| \chi(\xi) \frac{d\chi^*(\xi)}{d\xi} \right| d\xi \leq 2 \int_{-\beta}^{\beta} dp \left( \int_{-\beta}^{\beta} d\xi \, |\chi(\xi)|^2 \int_{-\beta}^{\beta} \frac{d\chi^*(\xi)}{d\xi} \right)^{1/2} = \frac{4\beta}{\hbar} \left( \int_{-\infty}^{\infty} dx \, x^2 |\psi(x)|^2 \right)^{1/2} .
\]

Let us note that the wave functions of the form (4) obey the integral equation
\[
\psi(x) = \frac{1}{2\pi \hbar} \int_{-\beta}^{\beta} dp \, e^{ipx/\hbar} \int_{-\infty}^{\infty} dy \, e^{-ipy/\hbar} \psi(y) = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \, \psi(y) \frac{\sin[\beta(x-y)/\hbar]}{x-y} .
\]
(5)

Thus the Schrödinger equation is now supplemented by this integral one. But it is plain to note that this system of equations does not always admit a solution even in simple cases. One may easily observe that in general the initial state given by Eq. (4) will evolve into the function which does not admit this sort of representation. It suffices to consider an infinitesimal time development
\[
\psi(t, x) = \psi_0(x) - \frac{it}{\hbar} \hat{H} \psi_0(x) + O \left( t^2 \right) = \psi_0(x) + \frac{it\hbar}{2m} \psi_0''(x) - \frac{1}{\hbar} V(x) \psi_0(x) + O \left( t^2 \right) .
\]
(6)
The problem in Eq. (6) arises because of the term \(V(x)\psi_0(x)\). Even if both \(V(x)\) and \(\psi_0(x)\) were taken to have the same compact support in momentum space, the product
\( V(x) \psi_0(x) \) will not have the same support in general. The significance of this fact in the context similar to our discussion was emphasized in [37].

Thus, one needs to reformulate the set up in a more consistent way. The operator entering the Eq. (5)

\[
\tilde{\psi}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \, \psi(y) \frac{\sin \left[ \beta (x-y)/\hbar \right]}{x-y},
\]

has the property that it projects out of \( \psi(x) \) the part \( \tilde{\psi}(x) \)—the Fourier transform of which coincides with that of \( \psi \) in \( |p| < \beta \) and vanishes elsewhere. By using the integral representation

\[
\frac{1}{2\pi \hbar} \int_{-\beta}^{\beta} dp \, e^{ip(x-y)/\hbar} = \frac{1}{\pi} \frac{\sin \left[ \beta (x-y)/\hbar \right]}{x-y},
\]

it is easy to verify that

\[
\frac{1}{\pi^2} \int_{-\infty}^{\infty} dy \, \frac{\sin \left[ \beta (x-y)/\hbar \right]}{x-y} \times \frac{\sin \left[ \beta (y-z)/\hbar \right]}{y-z} = \frac{1}{\pi} \frac{\sin \left[ \beta (x-z)/\hbar \right]}{x-z}.
\]

This property is typical for projector operators: \( \hat{\Pi}^2 = \hat{\Pi} \). A self-consistent treatment of the problem can be obtained by considering a non-local modification of the Schrödinger equation

\[
i \hbar \partial_t \psi(t,x) = \frac{1}{\pi} \int_{-\infty}^{\infty} dy \, \frac{\sin \left[ \beta (x-y)/\hbar \right]}{x-y} \left\{ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(y) \right\} \psi(t,y),
\]

or more minimalistic version

\[
i \hbar \partial_t \psi(t,x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(t,x) + \frac{1}{\pi} \int_{-\infty}^{\infty} dy \, \frac{\sin \left[ \beta (x-y)/\hbar \right]}{x-y} V(y) \psi(t,y).
\]

In itself, the purpose of modification of the kinetic term is not clear, as the minimalistic version, Eq. (9), already guaranties that the solution will be of the form (4) under assumption that initial state has such a form. It is plain to see that if the scalar product is defined in the standard manner

\[
\langle \psi_1 | \psi_2 \rangle = \int_{-\infty}^{\infty} dx \, \psi_1^* (x) \psi_2(x),
\]

then in Eq. (8) the modified kinetic term is Hermitian while the potential one is not. The potential term in Eq. (9) has the same problem. Accordingly, the potential term should be modified in such a way as to make the Hamiltonian Hermitian. The Hermiticity is recovered by the following redefinition of this term [27]

\[
-\frac{1}{\pi} \int_{-\infty}^{\infty} dy \, \frac{\sin \left[ \beta (x-y)/\hbar \right]}{x-y} \frac{V(y) + V(x)}{2} \psi(y).
\]
Equally well one could use the following redefinition

$$\frac{1}{\pi} \int_{-\infty}^{\infty} dy \frac{\sin \left( \frac{\beta(x-y)}{\hbar} \right)}{x-y} V \left( \frac{x+y}{2} \right) \psi(y) .$$  \hspace{1cm} (11)

Before discussing some more technical aspects, let us work out the classical limits.

### 2.2 Classical limit

By using the momentum operator, one can write the Eq. (10) as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \frac{\sin (\beta \xi/\hbar)}{\xi} \left\{ V(x) + V(x-\xi) \right\} \psi(x-\xi)$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \frac{\sin (\beta \xi/\hbar)}{\xi} \left\{ V(x)e^{-i\hat{p}\xi/\hbar} + e^{-i\hat{p}\xi/\hbar}V(x) \right\} \psi(x) .$$

In the case of Eq. (11) one obtains

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\xi \frac{\sin (\beta \xi/\hbar)}{\xi} e^{-i\hat{p}\xi/2\hbar}V(x)e^{-i\hat{p}\xi/2\hbar}\psi(x) .$$

In the classical regime, when the momentum and position operators commute, the potentials in both cases get modified as follows

$$V(x) \to V(x)\theta(\beta - |p|) ,$$  \hspace{1cm} (12)

where $\theta$ is a Heaviside function. Thus, the classical motion with $|p| > \beta$ becomes essentially free. Basically the same result is obtained in Sect. 3.6. If one applies the Eq. (8) in which either Eq. (10) or Eq. (11) is used for the potential term, then the classical limit will be represented by a modified Hamiltonian

$$\mathcal{H} \to \mathcal{H}\theta(\beta - |p|) .$$  \hspace{1cm} (13)

In both cases the result seems obviously incompatible with the reality. This so called “Soccer-ball problem” is common to many quantum-gravity inspired modifications of quantum mechanics [28], and, perhaps, indicates non-triviality of considering multi-particle states and corresponding macroscopic bodies in such theories [29,30]. Therefore, perhaps “composition law problem” would be more appropriate terminology. Relativity brings another twist in this composition law problem [31–34], which, in our opinion, is currently far from a final resolution.

### 2.3 Incompatibility with box-boundary conditions: an “infinite” potential well

One more technical issue that deserves attention is the potential cutoff. To elucidate this point let us consider the problem of infinite potential well. The equation determining
energy levels can be written as

\[ E_\psi = -\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \frac{\sin(2\beta(x - \xi)/\hbar)}{x - \xi} V(\xi)\psi(2\xi - x), \]

where we have used Eq. (11) and instead of variable \( y \) introduced a new variable \( \xi \): \( y = 2\xi - x \). One can consider this equation for a finite well

\[
V(x) = \begin{cases} 
0 & \text{for } -l < x < l \\
V_0 & \text{for } |x| \geq l
\end{cases}
\]  

and then let \( V_0 \to \infty \). Hence, one obtains

\[
-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} + \frac{V_0}{2\pi} \int_{-l}^{-\infty} d\xi \frac{\sin(2\beta(x - \xi)/\hbar)}{x - \xi} \psi(2\xi - x) \\
+ \frac{V_0}{2\pi} \int_{l}^{\infty} d\xi \frac{\sin(2\beta(x - \xi)/\hbar)}{x - \xi} \psi(2\xi - x) = E_\psi. 
\]  

Letting \( V_0 \to \infty \), one has to impose that the wave-function vanishes outside the well - in order to avoid infinite terms. That implies the equation

\[
-\frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} = E_\psi, \quad \psi = 0 \text{ for } |x| \geq l ,
\]

which can be solved in a standard manner. But, of course, in this case the solution will not have the form (4) since the momentum cutoff prevents the function from having a finite support in the position space. In view of the Paley–Wiener theorem, such functions are analytic in the entire complex plane. As a result, it follows that the non-trivial function of the form (4) can not vanish on any interval of the \( x \)-axis [36,38,39]. Because of this, one can not impose zero boundary conditions outside the well when dealing with an infinite well problem. Then it is clear that the limit \( V_0 \to \infty \) cannot be taken in Eq. (15) and, therefore, one has to set the value of \( V_0 \) somehow. The only reasonable way for doing this seems to be the use of scale \( \beta \). That is, the potential of an infinite well could be modified as

\[
V(x) = \begin{cases} 
0 & \text{for } -l < x < l \\
V_0 = \beta^2/2m & \text{for } |x| \geq l
\end{cases}
\]  

Let us note in passing that the corrections to the low-lying energy levels can be found by exploiting the standard perturbation theory, see Sect. 3.4. On the other hand, for energy levels which are high enough (close to \( \beta^2/2m \)) one may neglect the second
derivative in Eq. (15). Under this assumption, one arrives at the equation

\[
\frac{V_0}{2\pi} \left[ \int_{-\infty}^{l} d\xi \frac{\sin(2\beta(x - \xi)/\hbar)}{x - \xi} \psi(2\xi - x) + \int_{l}^{\infty} d\xi \frac{\sin(2\beta(x - \xi)/\hbar)}{x - \xi} \psi(2\xi - x) \right] = E \psi,
\]

which looks somewhat like the Eigenwert problem addressed in [38]. One could try to use the solution from the cited paper and look for the approximate solutions of Eq. (15). Certainly, in the end one has to check the validity of the applied approximation.

Guided by the example of infinite well, one may loosely argue that the cutoff on the potential should be applied as a general rule. This way, the harmonic oscillator gets modified as

\[
V(x) = \begin{cases} 
  m\omega^2 x^2/2 & \text{for } -\beta/m\omega < x < \beta/m\omega, \\
  \beta^2/2m & \text{for } |x| \geq \beta/m\omega.
\end{cases}
\]  

(17)

While in elementary particle physics one may consider various thought experiments for arguing that the laws of physics forbid us from reaching Planck energy scale [40,41], from the point of view of classical physics this sort of modification of the potential is hard to understand.

In itself, the cutoff on the potential has the following useful role in accordance with the the context of our discussion. The standard Schrödinger equation with the cutoff potential implies that the spread of bound states cannot be made smaller than \( \hbar/\beta \). One can easily verify this statement by using simple examples of “infinite” well and harmonic oscillator.

### 2.4 Relation to the deformed Weyl–Heisenberg algebra

Instead of \( p \) one can introduce a new variable \( P \), which covers the whole axis

\[
P = \frac{2\beta}{\pi} \tan \left( \frac{\pi p}{2\beta} \right).
\]

The Hamiltonian takes the form

\[
\hat{H} = \frac{2\beta^2 \arctan^2 \left( \frac{\pi \hat{P}/2\beta}{\pi^2 m} \right)}{\pi^2 m} + V(\hat{X}),
\]

where \( \hat{X} \) is the position operator with the commutation relation

\[
[\hat{X}, \hat{P}] = i\hbar \left( 1 + \frac{\pi^2 \hat{P}^2}{4\beta^2} \right).
\]
Certainly the above change of variables is not the only admissible one. Equally well one could consider

\[ p = \beta \tanh \left( \frac{\mathcal{P}}{\beta} \right), \]

which leads to

\[ \hat{H} = \frac{\beta^2 \tanh^2 \left( \frac{\mathcal{P}}{\beta} \right)}{2m} + V(\hat{\mathcal{X}}), \]

with the commutation relations

\[ [\hat{\mathcal{X}}, \hat{\mathcal{P}}] = \frac{i\hbar}{1 - \tanh^2 \left( \frac{\mathcal{P}}{\beta} \right)}. \]

One can construct many other examples as well.

In passing, let us mention that it is straightforward to construct Hilbert space representation of the deformed Weyl–Heisenberg algebra by using the mapping: \( \mathcal{P} = f(p) \). Namely, in momentum representation the integration measure is altered as

\[ dp \to d\mathcal{P} \frac{df^{-1}(\mathcal{P})}{d\mathcal{P}}, \]

and the scalar product takes the form

\[ \langle \psi_1(\mathcal{P}) | \psi_2(\mathcal{P}) \rangle = \int_{-\infty}^{\infty} d\mathcal{P} \frac{df^{-1}(\mathcal{P})}{d\mathcal{P}} \psi_1^*(\mathcal{P}) \psi_2(\mathcal{P}). \]

The momentum operator multiplies a state by \( \mathcal{P} \), while the position operator is defined by the replacement

\[ \hat{x} = i\hbar \frac{d}{dp} \to i\hbar f' \left( f^{-1}(\mathcal{P}) \right) \frac{d}{d\mathcal{P}}. \]

Let us note that in the above examples one could leave the Hamiltonian unaltered. Then it would mean in \( \hat{x}, \hat{p} \) variables that not only Hilbert space is restricted but Hamiltonian also is modified. More precisely, in the first example it amounts to the deformation of momentum operator

\[ \hat{p} \to \frac{2\beta}{\pi} \tan \left( \frac{\pi \mathcal{P}}{2\beta} \right), \]

and to the deformation

\[ \hat{p} \to \beta \tanh^{-1} \left( \frac{\mathcal{P}}{\beta} \right), \]
in the second example. In both cases the deformed momentum (momentum at the high energy) is not bounded. This is inconsistent with double special relativity, in which momentum has an invariant maximum called Planck’s momentum. The construction of a generalized uncertainty principle with both minimum length and maximum momentum is considered, for example, in [23], while in [24] a generalization of the uncertainty principle is presented that introduces the existence of a maximal observable momentum, but does not entail the minimum length.

We note in passing that one could introduce somewhat different deformed momentum-operator on the basis of sampling theorem. Namely, the cutoff representation of the wave-function (4) implies that [25]

\[\psi(x) = \sum_{j=-\infty}^{\infty} \psi \left( \frac{\hbar \pi j}{\beta} \right) \sin \left( \frac{\beta x}{\hbar} - \frac{\pi j}{\beta} \right).\]

In other words, such wave-function is determined by knowing its values at the points \(x_j = \hbar \pi j / \beta\). It “suggests” to replace the derivative with some approximate expression. For instance, by using the translation operator \(\hat{U}(\delta x)\psi(x) = \psi(x + \delta x)\), one could introduce the deformed momentum as

\[\hat{P} = \frac{\beta}{i \pi} (\hat{U}(\hbar \pi/2\beta) - \hat{U}(-\hbar \pi/2\beta)) = \frac{2\beta}{\pi} \sin \left( \frac{\pi \hat{P}}{2\beta} \right).\]

Somewhat similar discussion can be found in [42,43].

Correspondingly, the deformed Weyl–Heisenberg algebra will take the form

\[ [\hat{X}, \hat{P}] = i \hbar \sqrt{1 - \frac{\pi^2 \hat{P}^2}{\beta^2}} = i \hbar \left( 1 - \frac{\pi^2 \hat{P}^2}{2\beta^2} - \frac{\pi^4 \hat{P}^4}{8\beta^4} - \cdots \right), \tag{18} \]

where \(\hat{X} = \hat{x}\). In momentum representation one obtains

\[\hat{X} = i \hbar \sqrt{1 - \frac{\pi^2 \hat{P}^2}{\beta^2}} \frac{d}{d\hat{P}}, \quad \hat{P} = \hat{p},\]

where it is understood that \(\hat{P}^2 < \beta^2 / \pi^2\).

2.5 Brief summary

Let us briefly summarize the key points concerning band-limited QM. As it was suggested in [27,35], by using the projection operator (7), one can easily find the modified Schrödinger equation compatible with the momentum cutoff of the wave function. But the problem of compatibility still persists as far as we are concerned with the boundary conditions. An important point here is that if momentum cutoff is imposed on \(\psi(x)\), then it cannot be supported on a finite interval in the coordinate space. Thus, if we make the potential walls impenetrable, then the Schrödinger equation will not have
solution of the form (4). A characteristic difficulty of minimum-length deformed QM is that in the semi-classical limit one is not lead to well defined classical picture, without a proper treatment of multi-particle macroscopic objects. The band limited QM also does not automatically go over to the well defined classical theory.

3 Non-local QM without hard momentum cutoff

3.1 Some introductory remarks

In the sequel we shall outline a possible non-local generalization of quantum mechanics, which has simple logical connections to the band-limited QM and to the microstructure of the background space. First we observe that the UV cutoff of the wavefunction can be understood as a spatial averaging

$$\tilde{\Psi}(r) = \int d^3 \xi \ f(\xi) \Psi(r - \xi),$$

(19)

where the characteristic size of the test function $f(\xi)$ is assumed to be of the order of $l_P$. Namely, using for the sake of simplicity a Gaussian test function

$$f(\xi) = \left( \frac{\pi l_P^2}{3} \right)^{-3/2} e^{-\xi^2/l_P^2},$$

(20)

the Fourier transform of (19) will take the form

$$\tilde{\chi}(k) \propto e^{-k^2 l_P^2/4} \chi(k),$$

clearly indicating the (exponential) suppression of the Fourier modes: $k^2 l_P^2 \gg 1$. Next we observe that the averaging of the wave-function can naturally be understood as a coarse graining due to grainy structure of the space or as a result of background space fluctuations. For instance, one may bear the following simple picture in mind. Various thought experiments for measuring a background space show that its resolution is limited by the Planck length: $l_P = (\hbar G N / c^3)^{1/2} \approx 10^{-33}$ cm [15]. This fact might be taken to suggest that background space undergoes fluctuations in the sense that a position of point can not be known precisely but rather with some probability. This feature of the background space can be described effectively by specifying a distribution function $f(\xi)$, so that the integral $\int f(\xi) d^3 \xi$ over some region, $l^3$, in the vicinity of any point, can be interpreted as the probability that a position of this point is known with the precision $l^3$. In other words, that is the probability that a given point lies (in the operational sense) within this volume. For we are dealing with isotropic and homogeneous background space, it is naturally assumed that $f$ depends just on $\xi$ and does not depend on $r$.

Physically, an introduction of the above distribution function implies that one can measure only averaged quantities over a space region. But the averaging must be done with same care. In particular, as the Schrödinger equation involves the product
of a potential and a wave function at the same point, some care is needed to define the average value of this product “properly” in order to ensure the Hermiticity of Hamiltonian. We shall discuss these questions in what follows.

### 3.2 Modified Schrödinger equation

Let us start by considering an averaged wave-function (19). If \( \Psi(r) \) satisfies the Schrödinger equation, then the equation for \( \tilde{\Psi} \) takes the form

\[
i \hbar \partial_t \tilde{\Psi}(r) = -\frac{\hbar^2}{2m} \Delta \tilde{\Psi}(r) + \int d^3 \xi \ f(\xi) V(r - \xi) \tilde{\Psi}(r - \xi).
\]  

(21)

Replacing in the integrand \( \Psi \) with \( \tilde{\Psi} \), one may naturally interpret this integral as an average value of \( V \tilde{\Psi} \). If we define the scalar product in a standard way

\[
\langle \tilde{\Psi}_1 | \tilde{\Psi}_2 \rangle = \int d^3 \tilde{\Psi}_1^*(r) \tilde{\Psi}_2(r),
\]

then the Hamiltonian in Eq. (21) (in which \( \Psi \) is replaced by \( \tilde{\Psi} \)) is clearly non-Hermitian. One can, however, easily modify the Eq. (21) in such a way as to render the Hamilton operator Hermitian. For instance, one can put the modified equation in the form (from now on we omit the tilde)

\[
i \hbar \partial_t \Psi(r) = -\frac{\hbar^2}{2m} \Delta \Psi(r) + \int d^3 x' f(|r - r'|) V \left( \frac{r + r'}{2} \right) \Psi(r') .
\]  

(22)

In the limit \( l_P \to 0 \), \( f(|r - r'|) \) tends to \( \delta(r - r') \) and one arrives at the standard Schrödinger equation. The last term in Eq. (22) is just a smeared out version of the product \( V(r) \Psi(r) \). Let us note that this sort of equations have been discussed extensively in the context of nuclear physics [51].

One more relatively simple modification of the Schrödinger equation that follows from the above discussion might be

\[
i \hbar \partial_t \Psi(r) = -\frac{\hbar^2}{2m} \Delta \Psi(r) + \Psi(r) \int d^3 \xi f(\xi) V(r - \xi) .
\]  

(23)

In fact, one could use the Eq. (23) for estimating gravitational corrections to the quantum mechanics, but as it is almost trivial generalization, we will mainly focus on Eq. (22).
3.3 Digression on the averaging as a similarity transformation

This may be of some conceptual interest to note that the averaging given by Eqs. (19) and (20) can be viewed as the similarity transformation [52]

\[ \tilde{\Psi} = \hat{B} \Psi, \quad \hat{H} = \hat{B} \hat{H} \hat{B}^{-1}, \]

where \( \hat{B} = e^{\frac{i}{\hbar lP} \nabla^2} \) and \( f(r) = e^{\frac{i}{\hbar lP} \nabla^2} \delta(r). \)

(24)

This transformation, which can be viewed as a formal analog of the Kadanoff–Wilson blocking procedure in renormalization theory [53,54], is obviously non-unitary and therefore the new Hamiltonian is non-Hermitian. The lack of Hermiticity can be interpreted in physical terms as a result of high-frequency modes cutoff. The modified Schrödinger equation obtained by the transformation (24) is non-local

\[ i\hbar \frac{\partial}{\partial t} \tilde{\Psi}(r) = -\frac{\hbar^2}{2m} \Delta \tilde{\Psi}(r) + e^{\frac{i}{\hbar lP} \nabla^2} V(r) e^{-\frac{i}{\hbar lP} \nabla^2} \tilde{\Psi}(r) = -\frac{\hbar^2}{2m} \Delta \tilde{\Psi}(r) \]

\[ + \int d^3 \xi \ f(r - \xi) V(\xi) \sum_{n=0}^{\infty} \frac{(-i l_P/4)^n}{n!} \Delta^n \tilde{\Psi}(\xi). \]

(25)

Retaining the scalar product in its standard form

\[ \langle \tilde{\Psi}_1 | \tilde{\Psi}_2 \rangle = \int d^3 \tilde{\Psi}_1^*(r) \tilde{\Psi}_2(r), \]

if we want to regain a well defined quantum-mechanical picture, we have to modify the Hamiltonian (24) in such a way that its Hermiticity is restored. We shall not pursue the general consideration further, but instead restrict ourselves to the limiting case when average momentum is much smaller than \( \hbar/lP \). Under this assumption, in (25) one can discard first and higher order terms in \( p^2/l_P^2 \) retaining only zeroth order term. This way one arrives at a non-Hermitian Schrödinger equation, which is a predecessor of Eq. (22).

One could provide some further technical details concerning the blocking transformation [(19), (20), (24)]. For inverting this transformation, one usually uses the solution of the Fredholm-type integral Eq. (19) by a Fourier transform method. For the Gaussian kernels this may imply ill-posed problems due to the presence of a fast growing Gaussian function in the deconvolution integral [55]. However, there exist alternative methods of deconvolution of Gaussian kernels, avoiding ill-posed problems [55,56]. One of such methods is what follows. Let us first note that

\[ \Psi(r) = e^{-\frac{i}{\hbar lP} \nabla^2} \tilde{\Psi}(r) = \int d^3 \xi \ \delta(r - \xi) e^{-\frac{i}{\hbar lP} \nabla^2} \tilde{\Psi}(\xi) \]

\[ = \int d^3 \xi \ \tilde{\Psi}(\xi) e^{-\frac{i}{\hbar lP} \nabla^2} \delta(r - \xi). \]
On the other hand,
\[ e^{-\frac{l^2_P}{4} \nabla^2} \delta(r - \xi) = \delta(r - \xi) + \left( e^{-\frac{l^2_P}{4} \nabla^2} - e^{-\frac{l^2_P}{4} \nabla^2} \right) e^{-\frac{l^2_P}{4} \nabla^2} \delta(r - \xi) , \]
and recalling (24) we get
\[ e^{-\frac{l^2_P}{4} \nabla^2} \delta(r - \xi) = \delta(r - \xi) + \sum_{n=1}^{\infty} \frac{(-l^2_P/4)^n}{n!} (2^n - 1) \Delta^n f(r - \xi) . \] (26)

Therefore, the inversion of (19) takes the form
\[ \Psi(r) = \tilde{\Psi}(r) + \sum_{n=1}^{\infty} \frac{(-l^2_P/4)^n}{n!} (2^n - 1) \int d^3\xi \tilde{\Psi}(r - \xi) \Delta^n f(\xi) . \] (27)

Derivatives of the Gaussian function (20) can be expressed through the multivariate Hermite polynomials introduced by Grad [57]. One can use the definition
\[ \tilde{H}^{(n)}_{i_1i_2...i_n}(r; l_P) = (-l^2_P)^n f^{-1}(r) \nabla_{i_1} \nabla_{i_2} ... \nabla_{i_n} f(r) , \] (28)
which generalizes the Rodrigues formula for the univariate Hermite polynomials [58] and simultaneously make multivariate Hermite polynomials dimensionless. Then (27) takes the form
\[ \Psi(r) = \tilde{\Psi}(r) + \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n!} (2^n - 1) \times \int d^3\xi f(\xi) H_{2n}(\xi^2/l_P^2) \tilde{\Psi}(r - \xi) , \] (29)
where
\[ H_{2n}(\xi^2/l_P^2) = \delta_{i_1i_2} \delta_{i_3i_4} ... \delta_{i_{2n-1}i_{2n}} \tilde{H}^{(n)}_{i_1i_2...i_{2n-1}i_{2n}}(\xi; l_P) , \] (30)
is completely contracted version of the multivariate Hermite polynomials (a so called scalar irreducible Hermite polynomials [59]).

Therefore, instead of (25), the modified Schrödinger equation can be written as
\[ i \hbar \partial_t \tilde{\Psi}(r) = -\frac{\hbar^2}{2m} \Delta \tilde{\Psi}(r) + \int d^3\xi f(r - \xi) V(\xi) \tilde{\Psi}(\xi) + \sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n!} (2^n - 1) \times \int \int d^3\xi_1 d^3\xi_2 f(\xi_1) f(\xi_2) V(r - \xi_1) H_{2n}(\xi_2^2/l_P^2) \tilde{\Psi}(r - \xi_1 - \xi_2) . \] (31)
Note that the scalar irreducible Hermite polynomials here can be expressed through Laguerre polynomials [59,60]. Next point is to restore the Hermiticity of the Hamiltonian. However, we will not delve into these issues, despite the fact that the approach presented in this section is somewhat more general, since it is less important for our purposes.

### 3.4 Perturbative corrections

Let us list a few facts that immediately follow from the above discussion. First of all let us see how does a free particle wave-packet get modified

\[
\int d^3k \, e^{-i(\omega(k)t - k \cdot r)} g(k) \rightarrow \int d^3k \, e^{-i(\omega(k)t - k \cdot r)} g(k) \int d^3\xi \, f(\xi) e^{-i k \cdot \xi},
\]

where \( \omega(k) = \hbar k^2/2m \). Denoting by \( \tilde{f}(k) \) the Fourier transform of \( f(\xi) \), one sees that the above modification amounts to replacing \( g(k) \) by the product \( g(k) \tilde{f}(k) \equiv \tilde{g}(k) \).

As the function \( \tilde{f}(k) \) decays fast for \( k \gtrsim k_P \), so does \( \tilde{g}(k) \). Thus, the result is that the wave-function can not be localized beneath the Planck length.

When the particle moves in a potential field, for \( V(r) \) and \( \Psi(r) \) that vary negligibly over the Planck length, one can safely use the decompositions

\[
V(r - \xi) = \sum_j \frac{(-\xi \cdot \nabla)^j}{j!} V(r), \quad \Psi(r - \xi) = \sum_j \frac{(-\xi \cdot \nabla)^j}{j!} \Psi(r),
\]

and treat the Eqs. [(22), (23)] perturbatively. For Eq. (22) one obtains

\[
i \hbar \partial_t \Psi(r) = -\frac{\hbar^2}{2m} \Delta \Psi(r) + V(r) \Psi(r)
+ \left( \Psi \Delta V/4 + \nabla \Psi \cdot \nabla V + V \Delta \Psi \right) \frac{1}{6} \int d^3\xi \, f(\xi)\xi^2 + \text{higher order terms}.
\]

It is plain to see that \( \int d^3\xi \, f(\xi)\xi^2 \) is of the order of \( l_P^2 \). Correspondingly, the energy perturbations read

\[
\delta \mathcal{E}_j \propto l_P^2 \int d^3x \, \Psi_j^* \left( \psi_j \Delta V/4 + \nabla \psi_j \cdot \nabla V + V \Delta \psi_j \right)
= l_P^2 \int d^3x \, |\psi_j|^2 \Delta V/4 - l_P^2 \int d^3x \, V |\nabla \psi_j|^2.
\]

As usual, \( \psi_j \) functions are assumed to be normalized and orthogonal to one another. In the case of Eq.(23), the energy corrections take the form

\[
\delta \mathcal{E}_j \propto l_P^2 \int d^3x \, |\psi_j|^2 \Delta V.
\]

One sees that, in general, the corrections to the energy eigenvalues are real.
3.5 Modified Schrödinger equation and equivalence principle

It is interesting to consider a perturbative analysis of the modified Schrödinger equation in the case of a one-dimensional linear potential \( V(x) = mgx \), assuming that the smearing function is Gaussian and is given by the formula \( f(\xi) = (\pi l^2_p)^{-1/2} e^{-\xi^2/l^2_p} \).

It is clear that for Eq. (23) there are no corrections at all, so we will concentrate on the case of Eq. (22). For the one-dimensional linear potential, it has the form

\[
i\hbar \frac{\partial \Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + mg \int_{-\infty}^{\infty} d\xi \ f(\xi) \left( x - \frac{\xi}{2} \right) \Psi(x - \xi, t). \tag{32}\]

Due to the presence of the smearing function \( f(\xi) \), essentially only the region \(|x-\xi| \leq l_P\) contributes to the integral. Hence we expand

\[
\Psi(x - \xi, t) \approx \Psi(x, t) - \frac{\partial \Psi(x, t)}{\partial x} \xi + \frac{1}{2} \frac{\partial^2 \Psi(x, t)}{\partial x^2} \xi^2,
\]

and taking into account that \( \int_{-\infty}^{\infty} d\xi \ f(\xi) = 1 \) and \( \int_{-\infty}^{\infty} d\xi \ f(\xi) \xi^2 = l_P^2/2 \), we end up with the equation

\[
i\hbar \frac{\partial \Psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi(x, t)}{\partial x^2} + mgx \Psi(x, t) + \frac{mgl^2_p}{4} \left[ x \frac{\partial^2 \Psi(x, t)}{\partial x^2} + \frac{\partial \Psi(x, t)}{\partial x} \right]. \tag{33}\]

It is convenient to rewrite Eq. (33) in the form

\[
i\hbar \frac{\partial \Psi(x, t)}{\partial t} = \left[ \hat{p}^2 + mgx - \frac{mgl^2_p}{4\hbar^2} \left( x \hat{p}^2 - i\hbar \hat{p} \right) \right] \Psi(x, t), \tag{34}\]

where \( \hat{p} = -i\hbar \frac{\partial}{\partial x} \) is the momentum operator in the coordinate representation. But

\[
i\hbar \hat{p} = \frac{1}{2} \left( \hat{p} \ [x, \hat{p}] + [x, \hat{p}] \hat{p} \right) = \frac{1}{2} \left( x \hat{p}^2 - \hat{p}^2 x \right),
\]

and (34) is equivalent to

\[
i\hbar \frac{\partial \Psi(x, t)}{\partial t} = \left[ \hat{p}^2 + mgx - \frac{mgl^2_p}{8\hbar^2} \left( x \hat{p}^2 + \hat{p}^2 x \right) \right] \Psi(x, t), \tag{35}\]

from which it is clear that the Hamiltonian is Hermitian.
The Eq. (35) has a formal solution

\[
\Psi(x, t) = e^{\frac{\mu t}{\hbar^2} \left( \frac{\mu^2}{2m} + mgx - \frac{mg^2l_p^2}{8\hbar^2} (x p^2 + \hat{p}^2 x) \right)} \Psi(x, 0). \tag{36}
\]

To turn this formal solution into a true solution, we must disentangle non-commutative operators in this formal solution. This can be done using the left-oriented Zassenhaus formula [44–46]

\[
e^{\lambda (\hat{X} + \hat{Y})} = \ldots e^{\lambda^5 \hat{C}_5(\hat{X}, \hat{Y})} e^{\lambda^4 \hat{C}_4(\hat{X}, \hat{Y})} e^{\lambda^3 \hat{C}_3(\hat{X}, \hat{Y})} e^{\lambda^2 \hat{C}_2(\hat{X}, \hat{Y})} e^{\lambda \hat{C}_1(\hat{X}, \hat{Y})} e^{\hat{X} \hat{Y} \hat{X} \hat{Y} \hat{X}} \tag{37}
\]

where [45,46]

\[
\hat{C}_2(\hat{X}, \hat{Y}) = \frac{1}{2} [\hat{X}, \hat{Y}], \quad \hat{C}_3(\hat{X}, \hat{Y}) = \frac{1}{3} [\hat{Y}, [\hat{X}, \hat{Y}]] + \frac{1}{6} [\hat{X}, [\hat{X}, \hat{Y}]],
\]

\[
\hat{C}_4(\hat{X}, \hat{Y}) = \frac{1}{8} \left( [[\hat{Y}, [\hat{X}, \hat{Y}]]] + [\hat{Y}, [\hat{X}, [\hat{X}, \hat{Y}]]] \right) + \frac{1}{24} [\hat{X}, [\hat{X}, [\hat{X}, \hat{Y}]]],
\]

\[
\hat{C}_5(\hat{X}, \hat{Y}) = \frac{1}{30} \left( [[\hat{Y}, [\hat{X}, [\hat{X}, \hat{Y}]]]] + \frac{1}{10} [[[\hat{X}, \hat{Y}], \hat{Y}], [\hat{X}, \hat{Y}]]
\]

\[
+ \frac{1}{20} \left( [[[\hat{Y}, [\hat{X}, [\hat{X}, \hat{Y}]]]], [[[\hat{X}, \hat{Y}], \hat{X}], [\hat{X}, \hat{Y}]]] + \frac{1}{120} [\hat{X}, [\hat{X}, [\hat{X}, \hat{Y}]]]]. \tag{38}
\]

In our case we can take

\[
\lambda = -\frac{it}{2\hbar}, \quad \hat{X} = \hat{p}^2 + 2m^2g x, \quad \hat{Y} = -\frac{m^2gl_p^2}{4\hbar^2} \left( x \hat{p}^2 + \hat{p}^2 x \right). \tag{39}
\]

and calculate non-zero nested commutators (only terms \( \sim l_p^2 \) are retained)

\[
\hat{C}_2(\hat{X}, \hat{Y}) = \frac{1}{2} [\hat{X}, \hat{Y}] = \frac{im^2gl_p^2}{2\hbar} \left[ \hat{p}^3 - m^2g \left( x \hat{p}^2 + \hat{p}^2 x \right) \right],
\]

\[
\hat{C}_3(\hat{X}, \hat{Y}) \approx \frac{1}{6} [\hat{X}, [\hat{X}, \hat{Y}]] = -\frac{1}{3} m^4g^2l_p^2 \left( 5 \hat{p}^2 - 2m^2g x \right),
\]

\[
\hat{C}_4(\hat{X}, \hat{Y}) \approx \frac{1}{24} [\hat{X}, [\hat{X}, [\hat{X}, \hat{Y}]]] = -2i\hbar m^6g^3l_p^2 \hat{p},
\]

\[
\hat{C}_5(\hat{X}, \hat{Y}) \approx \frac{1}{120} [\hat{X}, [\hat{X}, [\hat{X}, [\hat{X}, \hat{Y}]]]] = \frac{4}{5} \hbar^2m^8g^4l_p^2. \tag{40}
\]
Then
\[
e^{\lambda \hat{C}_3(\hat{x}, \hat{y})} e^{\lambda^2 \hat{C}_3(\hat{x}, \hat{y})} e^{\lambda^3 \hat{C}_3(\hat{x}, \hat{y})} e^{\lambda^4 \hat{C}_3(\hat{x}, \hat{y})} e^{\lambda^5 \hat{C}_3(\hat{x}, \hat{y})} e^{\lambda^6 \hat{C}_3(\hat{x}, \hat{y})} \approx 1 + \lambda \hat{Y} + \lambda^2 \hat{C}_2(\hat{x}, \hat{Y}) + \lambda^3 \hat{C}_3(\hat{X}, \hat{Y}) + \lambda^4 \hat{C}_4(\hat{X}, \hat{Y}) + \lambda^5 \hat{C}_5(\hat{X}, \hat{Y}) = 1 + \hat{F}(x, \hat{p}), \tag{41}\]

where
\[
\hat{F}(x, \hat{p}) = \frac{i g t l_p^2}{8 \hbar^3} \left[ m \left( \hat{x} \hat{p}^2 + \hat{p}^2 x \right) - \frac{t^2}{3} m g \left( 5 \hat{p}^2 - 2 m^2 g x \right) - \frac{t^2}{3} m^2 g^2 \hat{p} - \frac{t^4}{5} m^3 g^3 \right]. \tag{42}\]

On the other hand, \(\hat{X} = \hat{A} + \hat{B}\) with \(\hat{A} = \hat{p}^2, \hat{B} = 2m^2 gx\), and with the following non-zero nested commutators:
\[
[\hat{A}, \hat{B}] = -4 i \hbar m^2 g \hat{p}, \quad [\hat{A}, [\hat{A}, \hat{B}]] = 0, \quad [\hat{B}, [\hat{A}, \hat{B}]] = 8 \hbar^2 m^4 g^2. \tag{43}\]

Therefore, again using the Zassenhaus formula, we get
\[
e^{\lambda \hat{X}} \Psi(x, 0) = e^{i \frac{mg^2 t^3}{3}} e^{i \frac{gt^2}{2} \hat{p}} e^{-\frac{i}{\hbar} mg \hat{x} t} e^{-i \frac{\hat{p}^2}{2 \hbar} t} \Psi(x, 0). \tag{44}\]

But \(e^{-i \frac{\hat{p}^2}{2 \hbar} t} \Psi(x, 0) = \Psi_{\text{free}}(x, t)\) gives the wave function of a free particle—a solution of the Schrödinger equation with zero potential, and \(e^{\frac{t}{\hbar} a \hat{p}}\) for any real number \(a\) is a spatial translation operator:
\[
e^{i \frac{gt^2}{2} \hat{p}} e^{-\frac{i}{\hbar} mg \hat{x} t} \Psi_{\text{free}}(x, t) = e^{-\frac{i}{\hbar} mg \left( x + \frac{gt^2}{2} \right) t} \Psi_{\text{free}} \left( x + \frac{gt^2}{2}, t \right).
\]

Therefore,
\[
e^{\lambda \hat{X}} \Psi(x, 0) = e^{-\frac{imgt}{\hbar} \left( x + \frac{gt^2}{2} \right)} \Psi_{\text{free}} \left( x + \frac{gt^2}{2}, t \right), \tag{45}\]

and finally (36) takes the form
\[
\Psi(x, t) = \left[ 1 + \hat{F}(x, \hat{p}) \right] e^{-\frac{imgt}{\hbar} \left( x + \frac{gt^2}{2} \right)} \Psi_{\text{free}} \left( x + \frac{gt^2}{2}, t \right). \tag{46}\]

As explained in [47,48], without \(\hat{F}(x, \hat{p})\) term, this relation constitutes a quantum-mechanical embodiment of Einstein’s principle of equivalence. The presence of the \(\hat{F}(x, \hat{p})\) term indicates that the equivalence principle is violated. However, the violation is minuscule and beyond the experimental reach, for example, in neutron quantum
bouncing experiments in the Earth’s gravitational field [49]. It is convenient to introduce the time \( \tau = c / g \approx 1 \text{year} \) and express \( \hat{F}(x, \hat{p}) \) in dimensionless units:

\[
\hat{F}(x, \hat{p}) = i l_P^2 m^3 c^5 \left[ \frac{g(x \hat{p}^2 + \hat{p}^2 x)}{m^2 c^4} \frac{t}{\tau} - \frac{\hat{p}^3 - m^2 g (x \hat{p} + \hat{p} x)}{m^3 c^3} \left( \frac{t}{\tau} \right)^2 \right. \\
- \left. \frac{1}{3} \frac{5 \hat{p}^2 - 2m^2 g x}{m^2 c^2} \left( \frac{t}{\tau} \right)^3 - \frac{\hat{p}}{mc} \left( \frac{t}{\tau} \right)^4 - \frac{1}{5} \left( \frac{t}{\tau} \right)^5 \right].
\]

(47)

For neutron, \( \frac{l_P^2 m^3 c^5}{8 \hbar^2 g} \approx 3 \cdot 10^{-8} \).

The non-local corrections of the Schrödinger equation are intrinsically linked to the Planck length similarly to GUP. It is not surprising, therefore, that similar conclusions about a slight violation of the equivalence principle were made in [50] considering GUP corrections to the geodesic equation, and in [34] when considering Lorentz invariant length scale.

### 3.6 Semi-classical limit

In the case of Eq. (23), the discussion of the semi-classical limit is straightforward. From now on let us assume Gaussian fluctuations for the background space (20). Then one can evaluate the integral defining a non-local term in Eq. (22) as follows (see ref. [51])

\[
i \hbar \partial_t \Psi(r) = -\frac{\hbar^2}{2m} \Delta \Psi(r) + \exp \left( \frac{l_P^2}{4} [\nabla_1 / 2 + \nabla_2]^2 \right) V(r) \Psi(r),
\]

where \( \nabla_1 \) acts on \( V \) and \( \nabla_2 \) on \( \Psi \), respectively. Once again, one sees that if \( V \) and \( \Psi \) vary slowly over the distance \( l_P \), the corrections are strongly suppressed. The WKB approximation to the integro-differential Eq. (22) has been discussed in [61]. For our purposes it is expedient to write the Eq. (22) in the form [61]

\[
i \hbar \partial_t \Psi = \left\{ \hat{\nabla}^2 + \int d^3 \xi \ f(\xi) e^{-i \xi \hat{\nabla}/2\hbar} V(r) e^{-i \xi \hat{\nabla}/2\hbar} \right\} \Psi.
\]

Derivation of the Heisenberg equations can be safely accomplished by allowing operators to act on a wave-function, which is removed at the end of calculation. Doing it
in the coordinate representation, one obtains

\[
\hat{\mathbf{r}} \Psi(r) = i \left[ \hat{H}, \hat{\mathbf{r}} \right] \Psi(r) = i \left[ \frac{\hat{p}^2}{2m}, \hat{\mathbf{r}} \right] \Psi(r) + i \int d^3x' (r' - r) f(|r - r'|)
\]

\[
\times V \left( \frac{r + r'}{2} \right) \Psi(r')
\]

\[
= \frac{\hat{p}}{m} \Psi(r) - i \int d^3\xi \xi f(\xi) V(r - \xi/2) \Psi(r - \xi)
\]

\[
= \frac{\hat{p}}{m} \Psi(r) - i \int d^3\xi \xi f(\xi) e^{-i\xi \hat{p}/2\hbar} V(r) e^{-i\xi \hat{p}/2\hbar} \Psi(r),
\]

(48)

and

\[
\hat{\mathbf{p}} \Psi(r) = \int d^3x' f(|r - r'|) V \left( \frac{r + r'}{2} \right) \nabla r' \Psi(r') - \nabla r \int d^3x' f(|r - r'|)
\]

\[
\times V \left( \frac{r + r'}{2} \right) \Psi(r')
\]

\[
= -2 \int d^3x' f(|r - r'|) \nabla r \left( \frac{r + r'}{2} \right) \Psi(r')
\]

\[
= - \int d^3\xi \xi f(\xi) e^{-i\xi \hat{p}/2\hbar} \nabla r V(r) e^{-i\xi \hat{p}/2\hbar} \Psi(r).
\]

(49)

Thus, the Heisenberg equations [(48), (49)] read

\[
\hat{p} = - \int d^3\xi \xi f(\xi) e^{-i\xi \hat{p}/2\hbar} \nabla V(r) e^{-i\xi \hat{p}/2\hbar},
\]

\[
\hat{r} = \frac{\hat{p}}{m} - i \int d^3\xi \xi f(\xi) e^{-i\xi \hat{p}/2\hbar} V(r) e^{-i\xi \hat{p}/2\hbar}.
\]

As to the equations of classical motion, they can be written immediately by using the modified Hamiltonian

\[
\mathcal{H} = \frac{\hat{p}^2}{2m} + \int d^3\xi \xi f(\xi) e^{-i\xi \hat{p}/2\hbar} V(r) e^{-i\xi \hat{p}/2\hbar}
\]

\[
= \frac{\hat{p}^2}{2m} + V(r) \exp \left( -\frac{l^2 \hat{p}^2}{4\hbar^2} \right),
\]

(50)

which gives (these equations have already been discussed in [62,63])

\[
\dot{r} = \frac{\hat{p}}{m} - \frac{l^2 V(r) \hat{p}}{2\hbar^2} \exp \left( -\frac{l^2 \hat{p}^2}{4\hbar^2} \right), \quad \dot{p} = - \nabla V(r) \exp \left( -\frac{l^2 \hat{p}^2}{4\hbar^2} \right).
\]
The deviation from the standard dynamics disappears as long as the condition $p^2 \ll \hbar^2/l_P^2$ is fulfilled. That means that one should require

$$\mathcal{E} - V(\mathbf{r}) \ll \frac{\hbar^2}{ml_P^2},$$

where $\mathcal{E}$ stands for energy. When we are dealing with the classical motion of macroscopic objects, this requirement is often broken and we face the above mentioned soccer-ball problem. For example the earth has average orbital speed $\approx 30$ km/s and the mass $\approx 6 \times 10^{24}$ kg while $\hbar/l_P \approx 6.5$ kg m/s. In this particular case the condition $p^2 \gg \hbar^2/l_P^2$ is satisfied extremely well. In view of the modified dynamics, it implies that with a great accuracy $\dot{\mathbf{r}} = p/m$ and

$$\dot{p} = -\nabla V(\mathbf{r}) \exp \left( -\frac{l_P^2 p^2}{4\hbar^2} \right).$$

Taking into account that the exponential factor is in this case of the order of $\exp(-10^{55})$, the motion of earth around the sun should be drastically altered. One possible solution to this dramatic puzzle was considered in [29,30,64] and suggests that the effective Planck length for composite objects is many orders of magnitude less than $l_P$.

Corrections to the classical dynamics implied by the Eq. (23) is of course harmless. Namely, in this case the corrections arise due to modification of the potential

$$V(\mathbf{r}) \rightarrow \int d^3 \xi \frac{e^{-\xi^2/l_P^2}}{\pi^{3/2}l_P^3} V(\mathbf{r} - \xi) = \exp \left( \frac{l_P^2 \Delta}{4} \right) V(\mathbf{r})$$

$$= V(\mathbf{r}) + \frac{l_P^2 \Delta}{4} V(\mathbf{r}) + \cdots.$$ 

One could again consider an orbit of the earth and calculate in particular a perihelion shift but for the potential $\propto r^{-1}$ there are no corrections as $\Delta r^{-1} = 0$ for $\mathbf{r} \neq 0$. Moreover, one can claim that, in general, the modified theory given by Eq. (23) should not affect the classical regime. To see it, let us note that the Hamiltonian in this case can be written as

$$\mathcal{H} = \frac{p^2}{2m} + \int d^3 \xi \frac{f(\xi)}{\pi^{3/2}l_P^3} \hat{p}/\hbar V(\mathbf{r}) e^{i\xi \hat{p}/\hbar},$$

and, therefore, one arrives at the standard Hamiltonian in classical regime, since in this regime the momentum and position operators do commute.
4 Concluding remarks

The general reasoning so far given can readily be compared with the momentum cut-off approach for implementing the concept of minimum length into QM [27,35]. This approach implies to restrict the Hilbert space of state vectors to the cut-off functions

\[ \Psi(r) = \int_{k<k_P} d^3k \ e^{-i\mathbf{k}\cdot\mathbf{r}}(\mathbf{k}), \]

where \( k_P \) stands for the Planck momentum: \( k_P = \sqrt{c^3/\hbar G N} \). The averaging in Eq. (19) does basically the same job. The approach based on Eq. (19) for deriving the modified Schrödinger equation may be somewhat advantageous in treating the product \( V(r)\Psi(r) \). An advantage of the approach based on Eq. (19) is that it guides logically in treating the product \( V(r)\Psi(r) \). Also it makes easy to work out the corrections to QM and address the question of a classical limit.

Apart from the trivial generalization given by Eq. (23), we see that the non-local theory, when naively applied to composite objects, leads to unacceptably large effects in the classical limit. Thus, we face the same impasse as in the case of deformed Weyl–Heisenberg algebra [65,66]. As we have seen, the classical limit of band-limited QM also suffers from this soccer-ball problem. To date, this problem has not been satisfactorily resolved, and there is no universally accepted solution to this problem [28].

Interestingly, we are now in a position to write down the minimum-length modified QM that in principle has good semi-classical behavior. Indeed, one may attempt to restore the standard classical picture for non-local theory by incorporating both above considered modifications in a single equation

\[ i\hbar \partial_t \Psi(r) = -\frac{\hbar^2}{2m} \Delta \Psi(r) + \int d^3x' f(|r-r'|) V\left(\frac{r+r'}{2}\right) \left\{ w_1 \Psi(r') + w_2 \Psi(r) \right\}, \]

where the weights, \( 0 \leq w_j \leq 1 \), obey the relation \( w_1 + w_2 = 1 \). That is, \( \Psi(r') \) and \( \Psi(r) \) do not necessarily enter this equation with equal weights. In view of our conceptual framework given in Sect. 3.1, it is natural to assume that the effect of background space fluctuations should depend on the breadth of a wave-function as it determines the length scale probed by the particle. Similar considerations for the harmonic oscillator can indeed be used for estimating the rate of effect [67]. Following this reasoning, by introducing

\[ l^2 = \langle \Psi | \left( \mathbf{\hat{r}} - \langle \Psi | \mathbf{\hat{r}} | \Psi \rangle \right)^2 | \Psi \rangle, \]

as a standard measure of the spread of the wave-function, one could set the weights as \( w_1 = l_P/l \) to some power and \( w_2 = 1 - w_1 \). Equally well, for setting the weights one
could use some other effective scale instead of $l_P$. If the breadth of the initial state is macroscopic, then $w_1 \ll 1$, and one can safely omit the corresponding term, which will lead to the good classical behavior. However, we are afraid, it is difficult to see how this rather ad-hoc solution of the soccer-ball problem could follow from the more fundamental theory.

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