Improvement of the method of diagonal Padé approximants for perturbative series in gauge theories

G. Cvetić

Department of Physics, Universität Bielefeld, 33501 Bielefeld, Germany

Abstract

Recently it has been pointed out that diagonal Padé approximants to truncated perturbative series in gauge theories have the remarkable property of being independent of the choice of the renormalization scale as long as the gauge coupling parameter $\alpha(p^2)$ is taken to evolve according to the one-loop renormalization group equation – i.e., in the large-$\beta_0$ approximation. In this letter we propose and describe an improvement to the method of diagonal Padé approximants. The improved method results in approximants which are independent of the chosen renormalization scale when $\alpha(p^2)$ evolves at any chosen (in principle arbitrary) loop-level.

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*e-mail: cvetic@physik.uni-bielefeld.de; or: cvetic@doom.physik.uni-dortmund.de
Padé approximants (PA’s), either diagonal or nondiagonal, when applied to any truncated perturbative series (TPS), possess by construction the same formal accuracy as the original TPS. This means that, when expanding the applied PA in a power series of the expansion parameter of the TPS, we reproduce all the coefficients of the TPS. PA’s, being rational fractions, additionally act as a kind of analytical continuation of the TPS and thus often represent a substantial improvement of the results deduced directly from the TPS. This had been the main motivation for applying PA’s to TPS’s in gauge theories such as QCD.

Recently it has been noted that the diagonal Padé approximants (dPA’s), when applied to TPS’s in gauge theories, have in addition the remarkable property of being independent of the choice of the renormalization scale (RScl) if the gauge coupling parameter $\alpha(p^2)$ is taken to evolve according to its one-loop renormalization group equation:

$$\alpha(p^2) = \frac{\alpha(q^2)}{1 + \beta_0 \ln(p^2/q^2)\alpha(q^2)}.$$  

This is the direct consequence of the mathematical property of dPA’s that they are invariant under the homographic transformations of the argument (Ref. [1], Part I): $z \mapsto az/(1 + bz)$. Since a full observable (formally an infinite perturbation series) must be RScl–independent, the mentioned property of dPA’s suggests that the dPA for an available TPS of an observable in a gauge theory (approximately) sums up a substantial set of diagrams and thus represents a very reasonable resummation method there. The authors of [5] further investigated this dPA method and showed that the resummed diagrams represent systematic approximations to the Neubert’s concept of the distribution of momentum flow through a bubble-dressed gluon propagator. The authors of [5] pointed out the need for further improvements of the method, in particular to obtain RScl-independence beyond the large-$\beta_0$ (one-loop flow) limit.

We present here an algorithm which improves the dPA method in this sense – by constructing approximants which are RScl-independent at any chosen loop-level of evolution of $\alpha(p^2)$ and which, when expanded in power series of $\alpha$, give the same formal accuracy as the TPS’s to which they are applied.

A generic observable $S$ in a gauge theory (e.g., QED or QCD) can in general be redefined so that it has the following form as a formal perturbation series:

$$S \equiv a(q^2)f(q^2) = a(q^2) \left[ 1 + r_1(q^2)a(q^2) + r_2(q^2)a^2(q^2) + \cdots + r_n(q^2)a^n(q^2) + \cdots \right] .$$  

(1)

Here, $a(q^2) \equiv \alpha(q^2)/\pi$ and $q^2$ is a chosen renormalization scale (RScl). The full series (1) is of course independent of $q^2$. In practice we have only a limited number of coefficients $r_j(q^2)$ available ($i=1,\ldots,n$), i.e., we know only a truncated perturbation series (TPS)

$$S_n(q^2) \equiv a(q^2)f^{(n)}(q^2) = a(q^2) \left[ 1 + r_1(q^2)a(q^2) + r_2(q^2)a^2(q^2) + \cdots + r_n(q^2)a^n(q^2) \right] .$$  

(2)

This TPS explicitly depends on the RScl $q^2$ – changing $q^2$ changes the value of $S_n$ in general by a term $\sim a^{n+2}$. The RScl-independence of (1) and the RScl-dependence of (2) suggest

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1 We refer to Ref. [1] (Part I) for the basic theory of PA’s.

2 For a review of methods of dealing with power expansions in quantum field theory, see Ref. [3].
that an approximant that has the same formal accuracy and is RScl-invariant is a good candidate to be closer than (2) to the full answer (1).

We will now construct such approximants. The gauge coupling parameter \( a(p^2) \equiv \alpha(p^2)/\pi \) evolves according to the perturbative renormalization group equation (RGE)

\[
\frac{da(p^2)}{d \ln p^2} = - \sum_{j=0}^{\infty} \beta_j a^{j+2}(p^2), \tag{3}
\]

where \( \beta_j \) are constants if particle threshold effects are ignored. They are positive in QCD and negative in QED. Only a limited number of these perturbative coefficients \( \beta_j (\beta_0, \ldots, \beta_3) \) are known in QCD (cf. [7], in \( \overline{\text{MS}} \) scheme) and in QED (cf. [8], in \( \overline{\text{MS}} \), \( \text{MOM} \) and in on-shell schemes). Hence, in practice the RGE (3) will always be truncated at some level. We now introduce the ratio of gauge coupling parameters at two different renormalization scales \( p^2 \) and \( q^2 \)

\[
k(a_q, u) \equiv \frac{a(p^2)}{a(q^2)} \quad \text{where:} \quad a_q = a(q^2) \, , \, u = \ln(p^2/q^2). \tag{4}
\]

Formally expanding this function in powers of \( u \equiv \ln(p^2/q^2) \) results in the following series:

\[
k(a_q, u) = 1 + \sum_{j=1}^{\infty} u^j k_j(a_q) \, , \quad \text{where:} \quad k_j(a_q) = \frac{1}{j!} \left. \frac{\partial^j}{\partial u^j} k(a_q, u) \right|_{u=0}. \tag{5}
\]

We note that \( k_j(a_q) \sim a^j(q^2) \) since RGE (3) gives the connection

\[
k_j(a_q) = (-1)^j \beta_0^j a^j(q^2) + \mathcal{O} \left( a^{j+1}(q^2) \right), \quad k_0(a_q) = 1, \tag{6}
\]

and the terms of higher orders can also be explicitly obtained from RGE (3). At this point we rearrange the formal power series (4) for \( S/a(q^2) \) into a related series in \( k_j(a(q^2)) \)

\[
S \equiv a(q^2) f(q^2) = a(q^2) \left[ 1 + \sum_{j=1}^{\infty} f_j(q^2) k_j(a(q^2)) \right]. \tag{7}
\]

We note that the coefficient \( f_j(q^2) \) depends solely on the first \( j \) coefficients \( r_1(q^2), \ldots, r_j(q^2) \) of the original series (1), as implied by relations (10). In addition, \( f_j(q^2) \) depends on the first \( j \) coefficients \( \beta_0, \ldots, \beta_{j-1} \) of RGE (3). We then define the corresponding formal series \( F(q^2) \), which is in powers of \((-\beta_0 a(q^2))\)

\[
a(q^2) F(q^2) \equiv a(q^2) \left[ 1 + \sum_{j=1}^{\infty} f_j(q^2) (-1)^j \beta_0^j a^j(q^2) \right]. \tag{8}
\]

We construct for \( a(q^2) F(q^2) \) the diagonal Padé approximants (dPA’s) with argument \( a(q^2) \)

\[
a(q^2) [M - 1/M] F(q^2) = a(q^2) \left[ 1 + \sum_{m=1}^{M-1} \bar{a}_m(q^2) a^m(q^2) \right] \left[ 1 + \sum_{n=1}^{M} \bar{b}_n(q^2) a^n(q^2) \right]^{-1}, \tag{9}
\]

\[
a(q^2) F(q^2) = a(q^2) [M - 1/M] F(q^2) + \mathcal{O} \left( a^{2M+1}(q^2) \right). \tag{10}
\]
The above dPA depends only on the first \((2M-1)\) coefficients \(f_j(q^2)\) \((j=1,\ldots,2M-1)\), due to the standard requirement \((10)\). Since, as mentioned earlier, the coefficient \(f_j(q^2)\) is a unique function of only \(r_1(q^2),\ldots,r_j(q^2)\), the above dPA depends only on the first \(2M-1\) coefficients \(r_1(q^2),\ldots,r_{2M-1}(q^2)\) of the original series \((1)\), i.e., it is uniquely determined once we have the TPS \((3)\) with \(n=2M-1\) available. Unless we have an exceptional situation when the denominator in the dPA \((3)\) has multiple zeros as polynomial of \(a(q^2)\), we can uniquely decompose this dPA into a sum of simple fractions

\[
a(q^2)[M-1/M]_f(q^2) = a(q^2) \sum_{i=1}^{M} \frac{\tilde{\alpha}_i}{[1 + \tilde{u}_i(q^2)\beta_0 a(q^2)]} = \sum_{i=1}^{M} \frac{a(q^2)}{[1 + \tilde{u}_i(q^2)\beta_0 a(q^2)]}. \tag{11}
\]

Here, \([-\tilde{u}_i(q^2)\beta_0]^{-1}\) are the \(M\) zeros of the denominator of the dPA \((3)\) which is regarded as a polynomial of \(a(q^2)\). The above expression \((4)\) is a weighed sum of the one-loop-evolved gauge coupling parameters\(^3\) \(a(p_i^2)\), with generally complex scales \(p_i^2\) determined by the relation \(\tilde{u}_i(q^2) = \ln(p_i^2/q^2)\), and with weights \(\tilde{\alpha}_i\). The approximant that we are looking for is then obtained by replacing in the above weighed sum the one-loop-evolved gauge coupling parameters with those which evolve according to the full RGE \((3)\)

\[
a(q^2)G_f^{[M-1/M]}(q^2) \equiv a(q^2) \sum_{i=1}^{M} \tilde{\alpha}_i k \left( a(q^2), \tilde{u}_i \right). \tag{12}
\]

The functions \(k(a(q^2), \tilde{u}_i)\) appearing here are defined via \((3)\) as ratios of gauge coupling parameters \(a\) at the RScl \(q^2\) and the new scales \(p_i^2 = q^2 \exp[\tilde{u}_i(q^2)]\)

\[
k \left( a(q^2), \tilde{u}_i \right) = a(p_i^2)/a(q^2) \quad \text{where:} \quad \ln(p_i^2/q^2) = \tilde{u}_i(q^2). \tag{13}
\]

We stress that \(a(p_i^2) = \alpha(p_i^2)/\pi\) \((i=1,\ldots,M)\) are the gauge coupling parameters evolved from the RScl \(q^2\) to \(p_i^2\) by the RGE \((3)\) whose loop-level precision (i.e., the number of included coefficients \(\beta_j\)) can be chosen as high as possible\(^4\), independently of the number \(n=2M-1\) of the coefficients of the available TPS \(S_n(q^2)\) of Eq. \((3)\). By \((13)\), the obtained approximants \((12)\) can be written in a somewhat more transparent form

\[
a(q^2)G_f^{[M-1/M]}(q^2) \equiv \sum_{i=1}^{M} \tilde{\alpha}_i a(p_i^2) \quad \text{where:} \quad p_i^2 = q^2 \exp \left[ \tilde{u}_i(q^2) \right]. \tag{14}
\]

Function \(k(a, u)\) in \((12)\), which depends on two (in general complex) arguments, can be called the kernel of the above approximant. We will call the above approximant the modified diagonal Baker-Gammel approximant (modified dBGA) with kernel \(k\), since there exists a certain (but limited) similarity with the diagonal Baker-Gammel approximants as defined in Ref. \((3)\) (Part II, Sec. 1.2). We emphasize again that this modified dBGA of order

\(^3\) Evolved from the RScl \(q^2\) to \(p_i^2\), by the one-loop ("large-\(\beta_0\") version of RGE \((3)\).

\(^4\)In QED and QCD, this would mean inclusion of \(\beta_0,\ldots,\beta_3\) since these coefficients are now available in certain schemes – see earlier discussion.
Having the value \( a(q^2) \), we should stress that exact values \( a(p_i^2) \) can be obtained only if we evolve the RGE (3) \( \text{numerically} \) from \( u = \ln(q^2/q^2) = 0 \) to \( u = \ln(p_i^2/q^2) = \tilde{u}_i(q^2) \). This numerical integration should be performed with additional care when \( \tilde{u}_i(q^2) \)'s are complex. Approximate values for \( a(p_i^2) \) can be obtained by perturbatively expanding the solution \( a(p_i^2)/a(q^2) \) in powers of \( a(q^2) \), but such expansion would be reasonable only if the chosen RScl \( q^2 \) is not far away from the scales \( p_i^2 \). In any case, the resulting approximant (14) does not represent an analytical formula once we go beyond the large-\( \beta_0 \) approximation.

It can be shown that this modified dBGA (14) of order \( 2M-1 \), for observable \( S \) of Eq. (1), fulfills the two requirements that we wanted to achieve:

1. It has the same formal accuracy as the TPS \( S_{2M-1}(q^2) \) of (2):
   \[
   S = a(q^2)G_f^{[M-1/M]}(q^2) + O\left(a^{2M+1}(q^2)\right). \tag{15}
   \]

2. It is fully invariant under the change of the renormalization scale \( q^2 \). In fact, the weights \( \tilde{a}_i \) and the scales \( p_i^2 = q^2 \exp[\tilde{u}_i(q^2)] \) are separately independent of the chosen renormalization scale \( q^2 \).

The formal proofs of these two statements will be given in a longer paper [4]. Furthermore, also the discussion of similarities and differences between the presented modified dBGA’s (14) and the usual dBGA’s of Ref. [1] will be given in that longer paper.

Within the presented algorithm, the case of one-loop evolution of \( \alpha(p^2) \) (the large-\( \beta_0 \) approximation) means: \( k(a, u) = 1/(1 + \beta_0 u a) \), and \( k_j(a) = \beta_0(-a)^j \), when using notation of Eqs. (4)–(6). Therefore, in the one-loop case, expansion (7) for \( f(q^2) \equiv S/a(q^2) \) and (8) for \( F(q^2) \) are identical. The modified dBGA (14) is in this case reduced to the usual dPA (11).

One may worry what happens when the parameters \( \tilde{a}_i \) and \( \tilde{u}_i(q^2) \) in the modified dBGA (12)–(14) are not simultaneously real. In that case, the modified dBGA could be complex. Since relation (13) and the RScl-invariance are valid for the entire modified dBGA’s, they are valid separately for their real and imaginary parts. The observable \( S \) is real, so we then just take the real part of expression (14). Since \( a(q^2) \) and \( S \) are real, relation (15) implies that the imaginary part of the modified dBGA \( a(q^2)G_f^{[M-1/M]}(q^2) \) must be \( \sim a^{2M+1}(q^2) \) or even less.

It may be helpful not to remain at this rather abstract level, but to write more explicit formulas for dBGA’s (12)–(14) in the practically interesting cases of \( M = 1 \) and \( M = 2 \).

In the case \( M = 1 \) (\( n = 2M - 1 = 1 \)) the method gives the same result as the effective charge (ECH) method [11], and the Brodsky-Lepage-Mackenzie (BLM) method [12] in the large-\( \beta_0 \) approximation

\[
S_1(q^2) \equiv a(q^2) f^{(1)}(q^2) = a(q^2) \left[ 1 + r_1(q^2)a(q^2) \right] \Rightarrow a(q^2)G_f^{[0/1]}(q^2) = a(Q^2) \quad \text{where:} \quad Q^2 = q^2 \exp \left[ -r_1(q^2)/\beta_0 \right]. \tag{16}
\]

Here, \( a(p^2) \equiv \alpha(p^2)/\pi \) evolves according to RGE (3) where the chosen loop-level is arbitrary. It is straightforward to check directly that \( a(Q^2) \) is RScl-invariant and that (14) is satisfied.
In the case $M=2$ ($n\equiv 2M-1=3$), parameters of the dBGA (12)-(14) can also be obtained in a straightforward, although algebraically more involved, manner. RGE (3) implies

\begin{align}
 k_1(a) &= -\beta_0 a - \beta_1 a^2 - \beta_2 a^3 - \ldots , \\
 k_2(a) &= +\beta_0^2 a^2 + (5/2) \beta_0 \beta_1 a^3 + \ldots , \\
 k_3(a) &= -\beta_0^3 a^3 - \ldots .
\end{align}

Here we use the short-hand notation $a \equiv a(q^2) \equiv \alpha(q^2)/\pi$. Inverting relations (17)-(18) yields expressions for $a, a^2$ and $a^3$ in terms of $k_1(a), k_2(a)$ and $k_3(a)$. We insert then these expressions into the truncated series $S_3(q^2)$ of Eq. (2) (if it is available) and thus obtain the first three coefficients of the rearranged truncated series for $S_3(q^2)$ of the form [7]

\begin{align}
 f_1(q^2) &= -\frac{r_1(q^2)}{\beta_0}, \\
 f_2(q^2) &= -\frac{\beta_1}{\beta_0^2} r_1(q^2) + \frac{1}{\beta_0^3} r_2(q^2), \\
 f_3(q^2) &= \left(-\frac{5\beta_2^2}{2\beta_0^4} + \frac{\beta_2}{\beta_0^2}\right) r_1(q^2) + \frac{5\beta_1}{2\beta_0} r_2(q^2) - \frac{1}{\beta_0^5} r_3(q^2).
\end{align}

With these coefficients, we form the truncated series for $a(q^2)\mathcal{F}(q^2)$ [8], and the dPA $a(q^2)[1/2]\mathcal{F}(q^2)$ (10) in the form (11). We then obtain expressions for parameters $\tilde{u}_i(q^2)$ and $\tilde{\alpha}_i$ ($i=1, M$) for the case of $M=2$

\begin{align}
 \tilde{u}_{2,1} &= \left[\frac{f_3 - f_1 f_2}{2(f_2 - f_1^2)} \pm \sqrt{\text{det}}\right], \\
 \tilde{\alpha}_1 &= \frac{(\tilde{u}_2 - f_1)}{(\tilde{u}_2 - \tilde{u}_1)}, \\
 \tilde{\alpha}_2 &= 1 - \tilde{\alpha}_1.
\end{align}

where: $\text{det} = \left[f_3 + f_1(2f_1^2 - 3f_2)\right]^2 + 4(f_2 - f_1^3)^3$.

The plus sign in (21) corresponds to $\tilde{u}_2(q^2)$. For simplicity, we omitted notation of the RScl-dependence in the coefficients $f_i(q^2)$ and $\tilde{u}_i(q^2)$. Of course, expressions (19)-(21) should be inserted into (21)-(22) in order to obtain these parameters explicitly in terms of the original coefficients $r_i(q^2)$ ($i=1, 2, 3$). When we insert these obtained parameters into the dBGA expression (14) ($M=2$), we get the RScl-invariant approximation to $S_3(q^2)$, with the RScl-invariant parameters $\tilde{\alpha}_i$ and $p_i^2 = q^2 \exp[\tilde{u}_i(q^2)]$ ($i=1, 2$) explicitly dependent on the original coefficients $r_i(q^2)$ ($i=1, 2, 3$) and on the RGE coefficients $\beta_j$ ($j=0, 1, 2$) of Eq. (3). Although the parameters obtained above for the case of $M=2$ contain dependence on the first three RGE beta-coefficients ($\beta_0, \beta_1, \beta_2$), we should emphasize that the evolution of the gauge coupling parameters $a(p_i^2)$ appearing in the dBGA (14) can be governed by the RGE (3) with a higher chosen loop-level accuracy, e.g., by inclusion of $\beta_3$ there. On the other hand, at least the first three coefficients ($\beta_0, \beta_1, \beta_2$) should be taken into account in the RGE evolution of $a(p_i^2)$ from the RScl $q^2$ to $p_i^2$ since these three coefficients appear in the parameters $\tilde{\alpha}_i$ and $p_i^2 = q^2 \exp[\tilde{u}_i(q^2)]$ ($i=1, 2$).

Concerning relations (21)-(22) (for $M=2$), we can distinguish several cases

- When $(f_2 - f_1^3) > 0$, then: $\tilde{u}_i, \tilde{\alpha}_i$ are real ($i=1, 2$), $\tilde{u}_1 \neq \tilde{u}_2$ and $0 < \tilde{\alpha}_i < 1$.

\footnote{RGE evolution from RScl $q^2$ to the (possibly complex) scale $p_i^2$ is meant here.}
• When \((f_2 - f_1^2) < 0\) and \(|f_3 + f_1(2f_1^2 - 3f_2)| > 2\sqrt{(f_1^2 - f_2)^3}\), then: \(\tilde{u}_i,\tilde{\alpha}_i\) are real \((i = 1, 2)\) and \(\tilde{u}_1 \neq \tilde{u}_2\).

• When \((f_2 - f_1^2) < 0\) and \(|f_3 + f_1(2f_1^2 - 3f_2)| < 2\sqrt{(f_1^2 - f_2)^3}\), then: \(\tilde{u}_i\) are complex, \(\tilde{\alpha}_i\) generally complex \((i = 1, 2)\), and \(\tilde{u}_1 \neq \tilde{u}_2\).

• When \((f_2 - f_1^2) < 0\) and \(|f_3 + f_1(2f_1^2 - 3f_2)| = 2\sqrt{(f_1^2 - f_2)^3}\) [or when \((f_2 - f_1^2) = 0\) and \(f_3 \neq f_1^3\)], then: the system of equations for \(\tilde{u}_i\) and \(\tilde{\alpha}_i\) is not solvable, i.e., form \((\mathbf{1})\) is not valid, the dBGA \((\mathbf{1})\) has a multiple (double) pole.

• When \((f_2 - f_1^2) = 0\) and \(f_3 = f_1^3\), then: \(\tilde{u}_i,\tilde{\alpha}_i\) are real \((i = 1, 2)\) and \(\tilde{u}_1 = \tilde{u}_2 = f_1\).

Even when the parameters \(\tilde{\alpha}_i\) and \(\tilde{u}_i\) are complex [i.e., when ‘det’ in \((\mathbf{2})\) is negative], it can be checked directly from \((\mathbf{2})\) that \(\tilde{\alpha}_2 = (\tilde{\alpha}_1)^*\) and \(\tilde{u}_2 = (\tilde{u}_1)^*\), and thus that the approximant is again real [note: \(a((p_1^2)^* = a(p_2^2)^n)\].

In QCD, presently available results of perturbative calculations contain, for various observables \(S\) and in specific renormalization schemes, the coefficients \(r_1(q^2)\) and \(r_2(q^2)\) of \((\mathbf{1})\), but not yet \(r_3(q^2)\). Hence, the described algorithm still cannot be applied for \(M = 2\) for QCD observables, due to the fact that \(r_{2M-1}(q^2) = r_3(q^2)\) are not yet known. This, however, is in stark contrast with some QED observables for which perturbative coefficients \(r_3(q^2)\) have already been obtained.

The presented algorithm, although being a clear improvement of the method of dPA’s for perturbative series in gauge theories, still has several deficiencies. One of them is that the obtained approximants probably cannot discern in QCD, on the basis of a given TPS, the nonperturbative behavior originating from (ultraviolet und infrared) renormalons – see arguments in Refs. \([2], [3]-[5]\) for the case of the usual (d)PA’s with which the presented modified dBGA approximants are closely related. Another deficiency is that the algorithm can be applied only in the cases when the available TPS \(S_n(q^2)\) of \((\mathbf{2})\) has odd \(n = 2M-1\), i.e., it cannot be applied in the case \(n = 2\) (which is at present the case of many QCD observables). This is so because the algorithm heavily relies on the decomposition \((\mathbf{1})\) which is valid only for diagonal PA’s. This problematic restriction is present also in the large-\(\beta_0\) limit, i.e., in the usual dPA approach. Another problematic point is that the dBGA \((\mathbf{13})-(\mathbf{17})\) becomes in the cases of \(M \geq 2\) also explicitly renormalization scheme (RSch) dependent, because parameters \(\tilde{u}_i\) and \(\tilde{\alpha}_i\) appearing in such dBGA’s also involve some of the \(\beta_j\) \((j \geq 2)\) RGE coefficients which are, in contrast to \(\beta_0\) and \(\beta_1\), RSch-dependent. This contrasts with some other approaches. For example, in the approach of the principle of minimal sensitivity (PMS) \((\mathbf{10})\), both the RScl- and RSch-independence of the approximant are achieved via a local method, while in the present approach the RScl-invariance is ensured via a more global method and RSch-invariance (i.e., independence of \(\beta_2, \beta_3, \ldots\)) is not ensured at all. It would definitely be instructive to compare the efficiency of the presented method (for \(M = 2\), for the time being in QED only) with the PMS method, as well as with other methods, among them: the effective charge (ECH) method \((\mathbf{11})\), the Brodsky-Lepage-Mackenzie (BLM) approach and

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\(^6\) When all effects of \(\beta_j\)’s \((j \geq 2)\) are neglected, changing the renormalization scheme is equivalent to changing the renormalization scale.
its extensions \cite{12}–\cite{14} and \cite{6}, a new approach \cite{15} based partly on the \textit{ECH} methods, and another new approach \cite{16} using a method of analytic continuation. It may be also useful to investigate which classes of Feynman diagrams the presented algorithm approximately sums up\footnote{Under the term “approximately” we understand here a systematic approximation which would converge to the sum of mentioned diagrams when $M \to \infty$.} in analogy with the work \cite{3} for the case of the \textit{dPA} approach.

\textbf{Abbreviations used frequently in the article:}

(d)\textbf{BGA} – (diagonal) Baker-Gammel approximant; (d)\textbf{PA} – (diagonal) Padé approximant; 
\textbf{RSch} – renormalization scheme; \textbf{RScI} – renormalization scale; \textbf{TPS} – truncated perturbation series.

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