ON WEAK CLOSURE OF SOME DIFFUSION EQUATIONS

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Abstract. We study the closure of approximating sequences of some diffusion equations under certain weak convergence. A specific description of the closure under weak $H^1$-convergence is given, which reduces to the original equation when the equation is parabolic. However, the closure under strong $L^2$-convergence may be much larger, even for parabolic equations.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ with Lipschitz boundary, $0 < T < \infty$ a given number, and $\Omega_T = \Omega \times (0, T)$. We study the diffusion equation

\begin{equation}
(1.1) 
\quad u_t = \text{div} \sigma(Du) \quad \text{on} \quad \Omega_T,
\end{equation}

where $\sigma: \mathbb{R}^n \to \mathbb{R}^n$ is a given continuous function representing the diffusion flux, $u = u(x, t)$ is an unknown scalar function, and $Du$ and $u_t$ are the spatial gradient and time derivative of $u$, respectively.

If $\sigma$ is monotone; that is, $(\sigma(q) - \sigma(p)) \cdot (q - p) \geq 0$ for all $p, q \in \mathbb{R}^n$, then the equation (1.1) is called parabolic. For parabolic equations, initial-boundary value problems can be studied as an abstract Cauchy problem of a monotone operator on a Hilbert or Banach space [3, 4]; furthermore, under certain higher smoothness and stronger parabolicity conditions, such problems have been extensively studied in the theory of quasilinear parabolic equation [11, 12].

Recently, for certain non-monotone functions $\sigma$, exact Lipschitz solutions have been constructed for the initial-Neumann problem of (1.1) in [8, 9, 10]; in such cases, solutions can converge weakly to a function that is not a solution of (1.1).

In this note, assume that $\sigma: \mathbb{R}^n \to \mathbb{R}^n$ satisfies

\begin{equation}
(1.2) 
\quad |\sigma(p)| \leq c_1(|p| + 1) \quad (p \in \mathbb{R}^n),
\end{equation}

where $c_1 > 0$ is a constant, and we are interested in approximating sequences of (1.1), especially when $\sigma$ is non-monotone. Here, in general, by an approximating sequence of (1.1) we mean a sequence $w^j$ in $L^2(\Omega_T)$ with

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\(Du^j \in L^2(\Omega_T; \mathbb{R}^n)\) such that

\[
\lim_{j \to \infty} \|u^j_t - \text{div} \sigma(Du^j)\|_{H^{-1}(\Omega_T)} = 0.
\]

We attempt to characterize the limits of approximating sequences under the weak convergence in \(H^1(\Omega_T)\). Let

\[
\Lambda = \{p \in \mathbb{R}^n \mid (\sigma(q) - \sigma(p)) \cdot (q - p) \geq 0 \quad \forall q \in \mathbb{R}^n\}.
\]

Then, \(\sigma\) is monotone if and only if \(\Lambda = \mathbb{R}^n\). Note that \(\Lambda\) may be empty; for example, if \(\sigma\) is the function as shown in Figure 2 below, then for function 
\(-\sigma\) the set \(\Lambda = \emptyset\).

**Lemma 1.1.** For \(p \in \mathbb{R}^n\), let

\[
\Gamma(p) = \{\beta \in \mathbb{R}^n \mid (\beta - \sigma(q)) \cdot (p - q) \geq 0 \quad \forall q \in \Lambda\}.
\]

(If \(\Lambda = \emptyset\), then let \(\Gamma(p) = \mathbb{R}^n\).) Then \(\Gamma(p)\) is a closed convex set in \(\mathbb{R}^n\) containing \(\sigma(p)\), with \(\Gamma(p) = \{\sigma(p)\}\) if \(p \in \Lambda^o\), the interior of \(\Lambda\).

**Proof.** Clearly \(\Gamma(p)\) is closed, convex and contains \(\sigma(p)\). Let \(p \in \Lambda^o\) and \(\beta \in \Gamma(p)\). Let \(q = p - \epsilon \eta\), where \(\epsilon > 0\) and \(\eta \in \mathbb{R}^n\). Then, for all \(\epsilon > 0\) sufficiently small, we have \(q \in \Lambda\) and thus \((\beta - \sigma(q)) \cdot (p - q) = (\beta - \sigma(p - \epsilon \eta)) \cdot \epsilon \eta \geq 0\); hence \((\beta - \sigma(p - \epsilon \eta)) \cdot \eta \geq 0\), which by letting \(\epsilon \to 0^+\) yields \((\beta - \sigma(p)) \cdot \eta \geq 0\). Since this inequality holds for all \(\eta \in \mathbb{R}^n\), it follows that \(\beta = \sigma(p)\). Hence \(\Gamma(p) = \{\sigma(p)\}\) if \(p \in \Lambda^o\). \(\square\)

In what follows, we denote by \(g(p, \beta)\) the convex hull of function \(|\sigma(p) - \beta|^2\) on \(\mathbb{R}^n \times \mathbb{R}^n\). For \(p \in \mathbb{R}^n\), define

\[
Z(p) = \{\beta \in \mathbb{R}^n \mid g(p, \beta) = 0\}, \quad \Sigma(p) = \Gamma(p) \cap Z(p).
\]

Then \(Z(p)\) is a closed convex set in \(\mathbb{R}^n\) containing \(\sigma(p)\), and thus \(\Sigma(p)\) is also a closed convex set in \(\mathbb{R}^n\) containing \(\sigma(p)\), with \(\Sigma(p) = \{\sigma(p)\}\) if \(p \in \Lambda^o\).

**Remark 1.1.** (a) The set \(\Sigma(p)\) can be very complicated, depending on the structure of the function \(\sigma\). In the one spatial dimension, two special functions of diffusion function \(\sigma(p)\) are given as shown in Figures 1 and 2.

(b) Some interesting special structures in the set \(\Sigma(p)\) can be characterized by a variational principle (see Proposition 3.3), which may have some relevance to the existence results in the recent work [8, 9, 10].

**Theorem 1.2.** Let \(u^j\) be an approximating sequence of \((1.1)\) in \(H^1(\Omega_T)\) such that \(u^j\) converges weakly to \(u^*\) in \(H^1(\Omega_T)\). Then

\[
u^*_t \in \text{div} \Sigma(Du^*)
\]

in the sense that \(\nu^*_t \in \text{div} \sigma\) holds in \(H^{-1}(\Omega_T)\) for some \(\sigma \in L^2(\Omega_T; \mathbb{R}^n)\) with \(\sigma(x, t) \in \Sigma(Du^*(x, t))\) for almost every \((x, t) \in \Omega_T\). In particular, if \(\sigma\) is monotone, then \(u^*\) is a solution of \((1.1)\).

The weak convergence of \(u^j\) is necessary in this result. In fact, we have the following result.
\[ \beta = \sigma(p) \]

**Figure 1.** A special function \( \sigma(p) \) in 1-D (including the Höllig function \([7]\)). Here the set \( \Lambda \) is \((-\infty, p_1] \cup [p_2, \infty)\). For all \( q \) the set \( Z(q) \) is the vertical closed half-ray below the graph \( GAEF \). For \( q \in [p_1, p_2] \), the set \( \Gamma(q) \) is the vertical cross-section of the closed rectangle \( ABCD \), and the set \( \Sigma(q) \) is the vertical cross-section of the closed trapezoid \( ABCE \).

\[ \beta = \sigma(p) \]

**Figure 2.** Another special function \( \sigma(p) \) in 1-D with \( \sigma(p) \to 0 \) as \( |p| \to \infty \) (including the Perona-Malik function \([14]\)). Here the set \( \Lambda \) is the point \( \{0\} \). For all \( q \) the set \( Z(q) \) is the vertical cross-section between two horizontal lines \( \beta = A \) and \( \beta = B \). The set \( \Sigma(0) \) is the line segment \( AB \) and, for \( q \neq 0 \), \( \Sigma(q) \) is the vertical line segment as shown in the first or third quadrant.

**Theorem 1.3.** Let \( \bar{u} \in H^1(\Omega_T) \) be such that there exists a function \( \bar{v} \in H^1(\Omega_T; \mathbb{R}^n) \) satisfying \( \bar{u} = \text{div} \, \bar{v} \) and \( \bar{v}_t \in Z(D\bar{u}) \) almost everywhere on \( \Omega_T \). Then there exists an approximating sequence \( u^j \in \bar{u} + H^1_0(\Omega_T) \) such that \( u^j \to \bar{u} \) strongly in \( L^2(\Omega_T) \).
Examples of the functions $\bar{u}$ and $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_n)$ in the theorem are given by

$$\bar{u} = p_1 x_1 + \cdots + p_n x_n, \quad \bar{v}_i = \frac{1}{2} p_i x_i^2 + \beta_i t \quad (i = 1, 2, \cdots, n),$$

where $p = (p_1, \ldots, p_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ are such that $g(p, \beta) = 0$.

The function $\bar{u}$ in Theorem 1.3 satisfies $\bar{u}_t \in \text{div}(D\bar{u})$. In general, the set $Z(p)$ is much larger than the set $\Sigma(p)$ (see Figures 1 and 2). Theorem 1.3 holds even when $\sigma$ is monotone; therefore, even for parabolic equations, there are approximating sequences that converge in the $L^2$-norm but not in the weak $H^1$ convergence.

An interesting problem is to study that for what functions $\bar{u} \in H^1(\Omega_T)$ one can find an exact solution of (1.1) in $\bar{u} + H^1_0(\Omega_T)$; such a function $\bar{u}$ has been called a subsolution of (1.1) in the recent work [9, 10]. From the constructions in [9, 10], Proposition 3.3 below suggests that a subsolution $\bar{u}$ should satisfy a condition $\bar{u}_t \in \text{div} R(D\bar{u})$ for a smaller set $R(p)$ contained in $\Sigma(p)$.

2. Compensated compactness: Proof of Theorem 1.2

Let $N \geq 1$ and $G$ be a bounded domain in $\mathbb{R}^N$. Given a vector function $U = (U_1, \ldots, U_N) \in L^2(G; \mathbb{R}^N)$, let

$$\text{Div} U = \sum_{k=1}^N \frac{\partial U_k}{\partial y_k}, \quad \text{Curl} U = (\frac{\partial U_k}{\partial y_l} - \frac{\partial U_l}{\partial y_k})_{1 \leq k, l \leq N}.$$ 

Then $\text{Div} U \in H^{-1}(G)$ and $\text{Curl} U \in H^{-1}(G; \mathbb{M}^{N \times N})$ are well-defined, where $\mathbb{M}^{N \times N}$ is the space of all $N \times N$ matrices.

We need the following compensated compactness result known as the div-curl lemma [15].

Lemma 2.1 (div-curl lemma). Let $V^j, W^j \in L^2(G; \mathbb{R}^N)$ satisfy $V^j \rightharpoonup V^*$, $W^j \rightharpoonup W^*$ weakly in $L^2(G; \mathbb{R}^N)$. Assume $\text{Div} V^j$ and $\text{Curl} W^j$ converge strongly in $H^{-1}(G)$ and $H^{-1}(G; \mathbb{M}^{N \times N})$, respectively. Then $V^j \cdot W^j$ converges to $V^* \cdot W^*$ in the sense of distributions on $G$; that is, for all $\phi \in C_0^\infty(G)$,

$$\lim_{j \to \infty} \int_G \phi(y) V^j(y) \cdot W^j(y) \, dy = \int_G \phi(y) V^*(y) \cdot W^*(y) \, dy.$$

We refer to [15] for proof and to [5] for some recent developments on the div-curl lemma.

Proof of Theorem 1.2. Let $u^j$ be an approximating sequence of (1.1) in $H^1(\Omega_T)$ such that $u^j$ converges weakly to $u^*$ in $H^1(\Omega_T)$. By selecting a subsequence if necessary, we assume that $u^j \to u^*$ strongly in $L^2(\Omega_T)$ and $\sigma(Du^j) \rightharpoonup \sigma$ weakly in $L^2(\Omega_T; \mathbb{R}^n)$.

Clearly, $u^*_t = \text{div} \sigma$ in $H^{-1}(\Omega_T)$. 
As above, let \( g(p, \beta) \) be the convex hull of \( |\sigma(p) - \beta|^2 \). Then
\[
0 \leq g(p, \beta) \leq c_1(|p|^2 + |\beta|^2 + 1) \quad \forall \ (p, \beta) \in \mathbb{R}^n \times \mathbb{R}^n.
\]

Since \( g(Du^j, \sigma(Du^j)) = 0 \), by convexity and (2.1), we have
\[
0 \leq \int_{\Omega_T} g(Du^j(x,t), \bar{\sigma}(x,t)) \, dx \, dt \leq \liminf_{j \to \infty} \int_{\Omega_T} g(Du^j, \sigma(Du^j)) = 0;
\]
hence, \( g(Du^j(x,t), \bar{\sigma}(x,t)) = 0 \) and
\[
\bar{\sigma}(x,t) \in Z(Du^j(x,t)) \quad \text{a.e. } \Omega_T.
\]

Let \( V^j = (\sigma(Du^j), -u^j) \) and \( W^j = (Du^j, u^j_t) \) be functions in \( L^2(\Omega_T; \mathbb{R}^{n+1}) \). Then \( V^j \rightharpoonup V^* = (\bar{\sigma}, -u^*) \) and \( W^j \rightharpoonup W^* = (Du^*, u^*_t) \) both weakly in \( L^2(\Omega_T; \mathbb{R}^{n+1}) \). Moreover, \( \text{Div} V^j = \text{div} \sigma(Du^j) - u^j_t \to 0 \) strongly in \( H^{-1}(\Omega_T) \) and \( \text{Curl} W^j = 0 \) in \( H^{-1}(\Omega_T; \mathbf{M}^{(n+1)\times(n+1)}) \). Hence, by the div-curl lemma, it follows that \( V^j \cdot W^j \rightharpoonup V^* \cdot W^* \) in the sense of distributions on \( \Omega_T \); that is,
\[
\lim_{j \to \infty} \int_{\Omega_T} \phi V^j \cdot W^j \, dx \, dt = \int_{\Omega_T} \phi V^* \cdot W^* \, dx \, dt \quad \forall \phi \in C_0^\infty(\Omega_T).
\]

Since \( V^j \cdot W^j = \sigma(Du^j) \cdot Du^j - u^j u^j_t \) and \( V^* \cdot W^* = \bar{\sigma} \cdot Du^* - u^* u^*_t \), we thus obtain
\[
\lim_{j \to \infty} \int_{\Omega_T} \phi \sigma(Du^j) \cdot Du^j \, dx \, dt = \int_{\Omega_T} \phi \bar{\sigma} \cdot Du^* \, dx \, dt \quad \forall \phi \in C_0^\infty(\Omega_T).
\]

From (2.2), we have, for all \( q \in \mathbb{R}^n \) and \( \phi \in C_0^\infty(\Omega_T) \),
\[
\lim_{j \to \infty} \int_{\Omega_T} \phi(\sigma(Du^j) - \sigma(q)) \cdot (Du^j - q) \, dx \, dt = \int_{\Omega_T} \phi(\bar{\sigma} - \sigma(q)) \cdot (Du^* - q) \, dx \, dt.
\]
Let \( q \in \Lambda \); then, for all \( \phi \in C_0^\infty(\Omega_T) \) with \( \phi \geq 0 \), we have
\[
\int_{\Omega_T} \phi(\bar{\sigma} - \sigma(q)) \cdot (Du^* - q) \, dx \, dt \geq 0.
\]
This implies that \( (\bar{\sigma} - \sigma(q)) \cdot (Du^* - q) \geq 0 \) almost everywhere in \( \Omega_T \), which is true for all \( q \in \Lambda \), and hence we have
\[
\bar{\sigma}(x,t) \in \Gamma(Du^*(x,t)) \quad \text{a.e. } \Omega_T.
\]

Therefore, \( \bar{\sigma}(x,t) \in \Sigma(Du^*(x,t)) \) almost everywhere in \( \Omega_T \) and \( u^*_t = \text{div} \bar{\sigma} \) in \( H^{-1}(\Omega_T) \). This completes the proof.

3. Associated first-order system and variational problem:

Proof of Theorem 1.3

Let \( u \in L^2(\Omega_T) \) and \( v \in L^2(\Omega_T; \mathbb{R}^n) \) be such that \( Du, v_t, \text{div} v \) are of \( L^2(\Omega_T) \). For such \((u, v)\), we introduce the functional
\[
I(u, v) = \|v_t - \sigma(Du)\|_{L^2(\Omega_T)}^2 + \|u - \text{div} v\|_{L^2(\Omega_T)}^2.
\]
If \((u, v)\) solves the first-order differential system
\[
(3.1) \quad u = \text{div} \, v, \quad v_t = \sigma(Du) \quad \text{on} \quad \Omega_T
\]
in the sense of distribution, then \(u\) is a solution of (1.1) in the same sense.

**Lemma 3.1.** Let \(w = (u, v) \in L^2(\Omega_T; \mathbb{R}^{1+n})\) be such that \(Du, v_t, \text{div} \, v\) are of \(L^2(\Omega_T)\). Then
\[
\|u_t - \text{div} \, \sigma(Du)\|_{H^{-1}(\Omega_T)} \leq \sqrt{I(u, v)}.
\]

Therefore, if a sequence \((w^j, v^j)\) of such functions satisfies \(I(w^j, v^j) \to 0\), then \(w^j\) is an approximating sequence \(w^j\) of (1.1).

**Proof.** If \(\phi \in C^\infty_0(\Omega_T)\), then
\[
\int_{\Omega_T} \phi_t \text{div} \, v \, dx \, dt = \int_{\Omega_T} v_t \cdot D\phi \, dx \, dt.
\]
By density, this equality also holds if \(\phi \in H^1_0(\Omega_T)\). Hence, for all \(\phi \in H^1_0(\Omega_T)\), we have
\[
\langle u_t - \text{div} \, \sigma(Du), \phi \rangle = -\int_{\Omega_T} u \phi_t \, dx \, dt + \int_{\Omega_T} \sigma(Du) \cdot D\phi \, dx \, dt
\]
\[
= -\int_{\Omega_T} (u - \text{div} \, v) \phi_t \, dx \, dt + \int_{\Omega_T} (\sigma(Du) - v_t) \cdot D\phi \, dx \, dt
\]
\[
\leq \|u - \text{div} \, v\|_{L^2(\Omega_T)} \|\phi_t\|_{L^2(\Omega_T)} + \|v_t - \sigma(Du)\|_{L^2(\Omega_T)} \|D\phi\|_{L^2(\Omega_T)}
\]
\[
\leq \sqrt{I(u, v)} (\|\phi_t\|^2_{L^2(\Omega_T)} + \|D\phi\|^2_{L^2(\Omega_T)})^{1/2} \leq \sqrt{I(u, v)} \|\phi\|_{H^1_0(\Omega_T)}.
\]
Therefore,
\[
\|u_t - \text{div} \, \sigma(Du)\|_{H^{-1}(\Omega_T)} = \sup_{\|\phi\|_{H^1_0(\Omega_T)} = 1} \langle u_t - \text{div} \, \sigma(Du), \phi \rangle \leq \sqrt{I(u, v)}.
\]

We now put the functional \(I(u, v)\) in the context of the calculus of variations.

In the following, for a vector function \(w = (u, v) \in W^{1,1}_{\text{loc}}(\Omega_T; \mathbb{R}^{1+n})\), where \(u: \Omega_T \to \mathbb{R}\) and \(v: \Omega_T \to \mathbb{R}^n\), we write its gradient \(\nabla w(x, t)\) as a matrix
\[
\nabla w(x, t) = \begin{pmatrix} Du(x, t) & u_t(x, t) \\ Dv(x, t) & v_t(x, t) \end{pmatrix},
\]
where \(Du\) is considered as a \(1 \times n\) matrix, \(Dv\) as an \(n \times n\) matrix, and \(v_t\) as an \(n \times 1\) matrix.

Accordingly, we denote by \(\mathbf{M}^{(1+n) \times (n+1)}\) the space of \((1+n) \times (n+1)\) matrices \(\xi\) written as \(\xi = \begin{pmatrix} p & \beta \\ B & \beta \end{pmatrix}\), where \(p\) is a \(1 \times n\) matrix viewed as a vector in \(\mathbb{R}^n\), \(\beta\) is an \(n \times 1\) matrix also viewed as a vector in \(\mathbb{R}^n\), \(r \in \mathbb{R}\) is a scalar, and \(B \in \mathbf{M}^{n \times n}\) is an \(n \times n\) matrix.
Then, for \( w = (u, v) \in H^1(\Omega_T; \mathbb{R}^{1+n}) \), the functional \( I(u, v) \) defined above can be written as an integral functional

\[
I(u, v) = I(w) = \int_{\Omega_T} f(w(x, t), \nabla w(x, t)) \, dx \, dt
\]

with the density function \( f : \mathbb{R}^{1+n} \times \mathbb{M}^{1+n} \times (n+1) \to \mathbb{R} \) given by

\[
f(\omega, \xi) = |\sigma(p) - \beta|^2 + (s - \text{tr}B)^2,
\]

where \( \omega = (s, z) \in \mathbb{R}^{1+n} \) and \( \xi = \begin{pmatrix} p & r \\ B & \beta \end{pmatrix} \in \mathbb{M}^{1+n} \times (n+1) \).

Clearly, \( f \) satisfies the growth condition

\[
0 \leq f(\omega, \xi) \leq C(|\omega|^2 + |\xi|^2 + 1)
\]
on \( \mathbb{R}^{1+n} \times \mathbb{M}^{1+n} \times (n+1) \). Define

\[
f^\#(\omega, \xi) = \inf_{\xi \in C_0^\infty(Q; \mathbb{R}^{1+n})} \frac{1}{|Q|} \int Q f(\omega, \xi + \nabla \zeta(x, t)) \, dx \, dt;
\]

here \( Q \subset \mathbb{R}^{n+1} \) is a cube.

Then, for each \( \omega \in \mathbb{R}^{1+n} \), \( f^\#(\omega, \xi) \) is independent of \( Q \) and quasiconvex in \( \xi \) in the sense of Morrey [13] (see also [12, 13]).

**Proposition 3.2.** One has that \( f^\#(\omega, \xi) = g(p, \beta) + (s - \text{tr}B)^2 \), where \( g(p, \beta) \) is the convex hull of \( |\sigma(p) - \beta|^2 \) on \( \mathbb{R}^n \times \mathbb{R}^n \).

**Proof.** Given \( \zeta = (\phi, \psi) \in C_0^\infty(Q; \mathbb{R}^{1+n}) \), we have

\[
f(\omega, \xi + \nabla \zeta) = |\beta + \psi + \sigma(p + D\phi)|^2 + (s - \text{tr}B)^2 - 2(s - \text{tr}B)(\text{div} \psi) + (\text{div} \psi)^2,
\]

and hence \( f^\#(\omega, \xi) = \tilde{g}(p, \beta) + (s - \text{tr}B)^2 \), where

\[
\tilde{g}(p, \beta) = \inf_{(\phi, \psi) \in C_0^\infty(Q; \mathbb{R}^{1+n})} \frac{1}{|Q|} \int Q \left( |\beta + \psi - \sigma(p + D\phi)|^2 + (\text{div} \psi)^2 \right) \, dx \, dt.
\]

Clearly, \( \tilde{g}(p, \beta) \leq |\sigma(p) - \beta|^2 \).

Let \( \rho_\epsilon(\xi) = \tilde{\rho}_\epsilon(|\xi|) \) be a standard mollifying sequence on \( \mathbb{M}^{1+n} \times (n+1) \) and let \( f_\epsilon^\#(\omega, \cdot) = f^\#(\omega, \cdot) * \rho_\epsilon(\cdot) \) be the smoothing sequence of \( f^\# \). Since \( f^\#(\omega, \xi) \) is quasiconvex (and thus continuous) in \( \xi \), it follows that \( f_\epsilon^\#(\omega, \xi) \) is quasiconvex and smooth in \( \xi \) for each \( \epsilon > 0 \). Write \( f_\epsilon^\# = g_\epsilon + h_\epsilon \), where

\[
g_\epsilon = \tilde{g} * \rho_\epsilon = g_\epsilon(p, \beta), \quad h_\epsilon = (s - \text{tr}B)^2 * \rho_\epsilon = (s - \text{tr}B)^2 + c_\epsilon
\]

with \( c_\epsilon \) being a constant such that \( c_\epsilon \to 0 \) as \( \epsilon \to 0^+ \).

From the quasiconvexity of \( f_\epsilon^\#(\omega, \xi) \) in \( \xi \), it follows that \( f_\epsilon^\#(\omega, \xi) \) is rank-one convex in \( \xi \); see [2, 6]. Therefore, for each rank-one matrix \( \eta \) in \( \mathbb{M}^{1+n} \times (n+1) \) of the form

\[
\eta = \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \otimes (q 1) = \begin{pmatrix} q & 1 \\ \alpha \otimes q & \alpha \end{pmatrix} \quad (q \in \mathbb{R}^n, \alpha \in \mathbb{R}^n),
\]
the function \( I(t) = f^\#_t(\omega, \xi + t\eta) = g_e(p + tq, \beta + t\alpha) + (s - \text{tr } B - t\alpha \cdot q)^2 + c_e \) is a convex function of \( t \) on \( \mathbb{R} \). Hence
\[
l''(0) = [(g_e)_{pp}(p, \beta)q] \cdot q + 2[(g_e)_{p\beta}(p, \beta)\alpha] \cdot q + [(g_e)_{\beta\beta}(p, \beta)\alpha] \cdot \alpha + 2(\alpha \cdot q)^2 \geq 0.
\]
In this inequality, replace \((q, \alpha)\) by \((kq, k\alpha)\) with \( k > 0 \), cancel \( k^2 \) and let \( k \to 0 \). Then it follows that
\[
[(g_e)_{pp}(p, \beta)q] \cdot q + 2[(g_e)_{p\beta}(p, \beta)\alpha] \cdot q + [(g_e)_{\beta\beta}(p, \beta)\alpha] \cdot \alpha \geq 0
\]
for all \( p, \beta, q, \alpha \in \mathbb{R}^n \). This proves that \( g_e \) is convex on \( \mathbb{R}^n \times \mathbb{R}^n \) for each \( \epsilon > 0 \); hence, \( \tilde{g} \) is convex on \( \mathbb{R}^n \times \mathbb{R}^n \).

Finally, if \( h(p, \beta) \) is a convex function satisfying \( h(p, \beta) \leq |\sigma(p) - \beta|^2 \) for all \( p, \beta \in \mathbb{R}^n \), then, by Jensen’s inequality, for all \( (\phi, \psi) \in C^0_0(Q; \mathbb{R}^{1+n}) \),
\[
h(p, \beta) \leq \frac{1}{|Q|} \int_Q h(p + D\phi, \beta + \psi t) \mathrm{d}x \mathrm{d}t \leq \frac{1}{|Q|} \int_Q (|\beta + \psi t - \sigma(p + D\phi)|^2 + (\text{div } \psi)^2) \mathrm{d}x \mathrm{d}t.
\]
Hence \( h(p, \beta) \leq \tilde{g}(p, \beta) \); therefore, \( \tilde{g}(p, \beta) \) is the largest convex function below the function \( |\sigma(p) - \beta|^2 \), and thus \( \tilde{g} = g \).

**Remark 3.1.** Given \( \omega \in \mathbb{R}^{1+n} \), let \( Rf(\omega, \cdot) \) be the rank-one convex hull of \( f(\omega, \cdot) \) on \( M^{(1+n)\times(n+1)} \) (see [3]); then, it can be shown that
\[
Rf(\omega, \xi) = h(p, \beta) + (s - \text{tr } B)^2
\]
for some function \( h(p, \beta) \geq g(p, \beta) \). Similarly as in the proof of Proposition 3.2, it follows that \( h \) is convex, and hence \( h = g \) and
\[
Rf(\omega, \xi) = f^\#(\omega, \xi) = g(p, \beta) + (s - \text{tr } B)^2.
\]
Therefore, the rank-one convex hull of \( f \) does not produce more special features than the quasiconvex hull of \( f \).

The following result shows that some special structures in the set \( \Sigma(p) \) defined above have a variational description; this result also shows that the subsolutions \( \bar{u} \) used in [9] [10] automatically satisfy \( \bar{u} \in \text{div}(D\bar{u}) \).

**Proposition 3.3.** For \( \lambda > 0 \), let
\[
g_\lambda(p, \beta) = \inf_{(\phi, \psi) \in C^0_0(Q; \mathbb{R}^{1+n})} \left\{ \frac{1}{|Q|} \int_Q (|\beta + \psi t - \sigma(p + D\phi)|^2 + (\text{div } \psi)^2) \mathrm{d}x \mathrm{d}t \right\}
\]
Then, \( \beta \in \Sigma(p) \) if \( g_\lambda(p, \beta) = 0 \).

**Proof.** Assume \( g_\lambda(p, \beta) = 0 \). From the formula of \( g = \tilde{g} \) in the proof of Proposition 3.2, it follows that \( 0 \leq g \leq g_\lambda \); hence \( \beta \in Z(p) \). To prove \( \beta \in \Gamma(p) \), let \( (\phi^j, \psi^j) \in C^0_0(Q; \mathbb{R}^{1+n}) \) be such that \( \|D\phi^j\|_{L^2(Q)} < \lambda \), \( \text{div } \psi^j = 0 \), and
\[
\lim_{j \to \infty} \int_Q (|\beta + \psi^j - \sigma(p + D\phi^j)|^2 + (\text{div } \psi)^2) = 0.
\]
Let $\eta^j = \beta + \psi_j^t - \sigma(p + D\phi^j)$. Then $\eta^j \to 0$ in $L^2(Q)$. Let $q \in \Lambda$. Then 
$$(\sigma(p + D\phi^j) - \sigma(q)) \cdot (p + D\phi^j - q) \geq 0;$$ hence

$$\int_Q (\beta + \psi^j_t - \eta^j - \sigma(q)) \cdot (p + D\phi^j - q) \, dx \, dt \geq 0.$$ 

Since 
$$\int_Q \psi^j_t \cdot D\phi^j \, dx \, dt = \int_Q \phi^j_t \, \text{div} \psi^j \, dx \, dt = 0,$$

$\eta^j \to 0$ in $L^2(Q)$ and $\|D\phi^j\|_{L^2(Q)} < \lambda$, in (3.7) letting $j \to \infty$, it follows that 
$$(\beta - \sigma(q)) \cdot (p - q) \geq 0.$$ 

Hence $\beta \in \Gamma(p)$; this completes the proof. \(\square\)

**Proof of Theorem 1.3.** Let $\bar{w} = (\bar{u}, \bar{v}) \in H^1(\Omega_T; \mathbb{R}^{1+n})$ satisfy $\bar{u} = \text{div} \bar{v}$ and $\bar{v}_t \in Z(D\bar{u})$ almost everywhere on $\Omega_T$. Consider the relaxed energy (see [1, 6])

$$I^#(w) = \int_{\Omega_T} f^#(w(x,t), \nabla w(x,t)) \, dx \, dt$$

for all $w \in H^1(\Omega_T; \mathbb{R}^{1+n})$. Then $I^#(\bar{w}) = 0$. By [6, Theorem 9.8], there exists a sequence $w^j \in \bar{w} + H^1_0(\Omega_T; \mathbb{R}^{1+n})$ such that $I(w^j) \to 0$ and $w^j \to \bar{w}$ strongly in $L^2(\Omega_T; \mathbb{R}^{1+n})$. If $w^j = (u^j, v^j)$ then, by Lemma 3.1, $u^j \in \bar{u} + H^1_0(\Omega_T)$ is an approximating sequence of (1.1) such that $w^j \to \bar{u}$ in $L^2(\Omega_T)$.

This completes the proof.

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