Implementation of Differential Transform Method (DTM) for Large Deformation Analysis of Cantilever Beam

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Abstract. Large deformation of a cantilever beam is an eminent structural analysis and engineering problem. A non-linear differential equation governs the problem of large deformation of beams. It has many important applications scientific and engineering fields. It is quite intricate and complex to find a closed-form or an exact solution for such a problem. It is relatively easier to use analytic expressions for calculations than numerical or experimental analysis. Large deformation analysis has always been an active area of research for researchers. The main focus of this research is to implement a new method namely Differential Transformation Method (DTM) for non-linear beam problem analysis. DTM is quite reliable and efficient in giving an approximate analytical solution to beam problems. Moreover, it is a new application for DTM. Hence, the primary purpose of this work is to evaluate the utility of the method for solution of beam problem. DTM yields a 2-point boundary value problem. It is based Taylor’s series expansion. DTM helps to calculate the deflection angle and horizontal as well as vertical displacements of a cantilever beam, experiencing large deformation, in an explicit analytical form. Mathematical formulation code will be developed using MATLAB. Variation of rotation angle at free end will be analysed using different number of terms used and values of non-dimensional load “β”. Results of DTM prove the method to be quite effective and suitable for predicting solutions to non-linear beam problems. Therefore, DTM can be used for widespread applications for new, emerging and complex engineering problems.

Index Terms— Cantilever beam, Large deformation, Differential Transformation method (DTM).

1. Introduction
Mechanical systems which involve non-linearity due to large deflection of the members continue to be a thought-provoking problem for engineers. As a result, quite many research works have been stated on the issue of the non-linear large deformation of beams. Chebyshev’s polynomial method was used for large deformation analysis by Schmidt and Dadeppo [1]. Large deformation of a spring-hinged beam was investigated by Nageswara Rao et al. [2] the with a tip concentrated load. Nageswara Rao and Venkateswara Rao [3] studied the large deformations of a non-uniform beam with a rotational load at end. Wang et al. [4] investigated large deformation beam problems in which one end of the beam was able to slide freely while the other end was fixed. The beam with a point load was solved using Elliptic integral method and the shooting-optimization technique.

Lee [5] studied large deformation of Ludwick made cantilever beams with a combined loading effect of a concentrated load at tip and uniformly distributed load over length of beam. Chucheepakul and Phunpaigram [6] studied closed-form solutions with the help of elliptic integrals for large deformation of an elastic beam. Dado and Al- Sadder [7] studied prismatic as well as non-prismatic beams for very large deformations for various loading conditions. It was based on representation of rotation angle by a polynomial of the position along the deformed axis of beam. Integral of the residual was minimized and beam boundary conditions were applied for determining polynomial co-efficients. Wang et al. [8] employed Homotopy Analysis Method (HAM) for analysis of large deflection of a cantilever beam subjected to point load. Tolou and Herder [9] studied the feasibility of the Adomian decomposition
method (ADM) for analysis of complex mechanical systems. Mutyalara et al. [10] studied a uniform cantilever beam for large deformation, subjected to a concentrated load at tip and having an inclination normal to the deflected beam axis.

It is often easier to use analytic expressions for calculations than to employ experimental or numerical techniques. It is also a clear preliminary point for a better comprehension of the association between the physical parameters of beam as well as slope at any point along the beam arc length. Besides, the quest for an analytical solution for the complex non-linear differential equation of large deformation of beams subjected to a concentrated load is of great scientific and engineering concentration. The key resolve of this research work is to establish the efficacy of the DTM for solution of the above-mentioned complex problem. Large deformation of beams is a novel application for the DTM and was previously implemented for other engineering problems.

2. Problem Statement
The curvature $\rho$ of a beam with uniform cross-section is given as: -

$$\rho = \frac{d\theta}{ds} = \frac{M}{EI}$$  \hspace{1cm} (1)

where:
$M$: Flexural moment
$E$: Young’s or Elastic modulus
$I$: Beam moment of inertia
$EI$: Beam Flexural stiffness
$\theta(s)$: slope with respect to the horizontal axis
$s$: coordinate of arc along beam neutral axis

As shown in Fig1, a beam is under deflection due to a point load at tip. Moment at any point in beam is given as:-

$$M = F(L - \delta_h - x)$$ \hspace{1cm} (2)

$F$ being concentrated load at tip. Hence, equation of a prismatic beam for large deformation, is: -

$$\frac{d\theta}{ds} = \frac{F}{EI}(L - \delta_h - x) ; \theta(0) = 0 ; \theta'(L) = 0$$ \hspace{1cm} (3)

$\delta_h$ being horizontal deflection of beam. Since the lateral deflection is much larger than axial elongation at tip, therefore, axial elongation of the beam has been neglected. Differentiation of equation (1) gives:-

$$\frac{d^2\theta}{ds^2} = \frac{dM}{ds} = \frac{d^2\theta}{ds^2} = \frac{M}{EI}$$ \hspace{1cm} (4)
Let $\eta = s/L$ be a dimensionless parameter for position on arc length of beam. Differentiation of equation (2) with respect to $s$, and substituting in equation (4), following form of governing equation and boundary conditions for large deflection of beam is yielded:

$$
\frac{d^2 \theta}{d\eta^2} + \beta \cos \theta = 0 \quad (5a) \\
\theta(0) = 0; \quad (5b) \\
\theta'(1) = 0 \quad (5c)
$$

Where $\beta = FL^2/EI$ being the tip point load in dimensionless form. $\theta_B = \theta(1)$ is the beam rotation angle at free end. Non-dimensional horizontal displacement $\delta_h$ of tip is given by [10, 30, 31]:

$$
\frac{L - \delta_h}{L} = \sqrt{\frac{2EI\sin \theta_B}{FL^2}} = \sqrt{\frac{2\sin \theta_B}{\beta}} \quad (6)
$$

$$
\frac{\delta_h}{L} = 1 - \sqrt{\frac{2\sin \theta_B}{\beta}} \quad (7)
$$

Equation (7) shows that non-dimensional horizontal displacement $\delta_h$ depends upon rotation angle at tip $\theta_B$ and non-dimensional end point load $\beta$. For an infinitesimal deformation, equation (5a) may be written as:

$$
\frac{d^2 \theta}{d\eta^2} + \beta = 0; \quad \theta(0) = 0; \quad \theta'(1) = 0 \quad (8)
$$

And the solution is given as: -

$$
\theta(\eta) = \frac{\beta}{2} (2 - \eta) \eta \quad (9)
$$

For tip $\eta = 1$ and: -

$$
\theta_B = \frac{\beta}{2} \quad (10)
$$

In order to consider large deformation, a non-linear equation has to be solved [10]:
\[ \sqrt{\beta} = K(\mu) - F(\varphi, \mu) \quad (11) \]

Where:
\[ \mu = \frac{1 + \sin \theta_B}{2} \quad ; \quad \varphi = \arcsin \left( \frac{1}{\sqrt{2\mu}} \right) \quad (12) \]

\( K(\mu) \) and \( F(\varphi, \mu) \) being complete elliptic integral of the first kind and elliptic integral of the first kind respectively.

3. Analysis of methods

3.1. Differential Transformation Method (DTM) [32]

Differential transformation of a function \( x(t) \) is given by [33]:
\[ X(k) = \left( \frac{H}{k!} \right)^k \left[ \frac{d^k x(t)}{dt^k} \right]_{t=0} \quad (13) \]

where
- \( x(t) \): Function to be transformed
- \( X(k) \): Transformed function

Similarly, inverse transform of \( X(k) \) is given by:
\[ x(t) = \sum_{k=0}^{\infty} \left( \frac{t}{H} \right)^k X(k) \quad (14) \]

Differential transformation is actually founded upon expansion of Taylor series. As the number of discretes (values of \( X(k) \)) is increased, it becomes more precisely possible to restore the function to be determined. The function \( x(t) \) can be represented as a finite series and equation (18) can be re-written as:
\[ x(t) = \sum_{k=0}^{N} \left( \frac{t}{H} \right)^k X(k) \quad (15) \]

Basic mathematical operations of DTM are listed in table in figure 2.

| Original function          | Transformed function                                      |
|----------------------------|-----------------------------------------------------------|
| \( x(t) = \alpha f(t) \pm \beta g(t) \) | \( X(k) = \alpha F(k) \pm \beta G(k) \)                  |
| \( x(t) = \frac{df(t)}{dt} \)            | \( X(k) = (k+1)F(k+1) \)                                  |
| \( x(t) = \frac{d^2 f(t)}{dt^2} \)      | \( X(k) = (k+1)(k+2)F(k+2) \)                            |
| \( x(t) = t^m \)                     | \( X(k) = \delta(k-m) = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases} \) |
| \( x(t) = \exp(\lambda t) \)           | \( X(k) = \frac{\lambda^k}{k!} \)                        |
| \( x(t) = f(t)g(t) \)                  | \( X(k) = \sum_{l=0}^{k} F(l) G(k-l) \)                  |

Figure 2: Fundamental operations of DTM [32].
3.1.1. Solution with DTM
Applying differential transformation to equation (5a) with respect to $\eta$, and taking $H=1$, gives:

\[(k + 2)(k + 1)\theta(k + 2) - \frac{1}{2} \beta \sum_{l=0}^{k} \theta(l) \theta(k - l) + \frac{1}{24} \beta \left( \sum_{m=0}^{k} \theta(m - v) \left( \sum_{w=0}^{m} \theta(w - v) \theta(v) \right) \right) + \beta \times \delta(k) = 0 \quad (16)\]

Applying transformation on boundary conditions, we get:

\[\theta(0) = 0 \quad (17)\]
\[\theta(1) = C \quad (18)\]

where constant $C$ is determined using the boundary condition in equation (5c) for point $\eta=1$. Using, inverse differential transformation, $\theta(k + 2)$ is calculated from equation (20) as follows:

\[\theta \left( \frac{1}{2} \right) = -0.5 \beta \quad (19a)\]
\[\theta(3) = 0 \quad (19b)\]
\[\theta(4) = \frac{1}{24} \beta C^2 \quad (19c)\]
\[\theta(5) = -\frac{1}{40} \beta^2 C \quad (19d)\]
\[\theta(6) = \frac{1}{240} \beta^3 - \frac{1}{720} \beta C^4 \quad (19e)\]
\[\theta(7) = \frac{1}{336} \beta^2 C^3 \quad (19f)\]
\[\theta(8) = -\frac{13}{6720} \beta^3 C^2 \quad (19g)\]
\[\theta(9) = \frac{1}{1920} \beta^4 C - \frac{1}{8640} \beta^2 C^5 \quad (19h)\]

Substituting equations (19 a-h) into equation (15), the closed-form solution is given as:

\[\theta(\eta) = C + \theta(2)\eta^2 + \theta(4)\eta^4 + \theta(5)\eta^5 + \theta(6)\eta^6 + \theta(7)\eta^7 + \theta(8)\eta^8 + \theta(9)\eta^9 + \theta(10)\eta^{10} + \theta(11)\eta^{11} + \theta(12)\eta^{12} \ldots \ldots \ldots \quad (20)\]

Substitution of the boundary condition from equation (5c) into above equation, in order to obtain the value of $C$, yields:

\[\theta'(1) = 0 = -\beta + \frac{1}{6} \beta C^2 - \frac{1}{8} \beta^2 C + \left( \frac{1}{40} \beta^3 - \frac{1}{120} \beta C^4 \right) + \frac{7}{336} \beta^2 C^3 - \frac{13}{840} \beta^3 C^2 + \ldots \quad (21)\]
Now $\theta(\eta)$ can be determined by substitution of $C$ into equation (20). N=12 was found to be enough to provide an accurate solution. Therefore, equation (5) involves only a limited number of terms and the solution can be calculated without unnecessary computational effort.

4. Results and discussion

Using equations of DTM described above, deflection angle at tip $\theta_{\text{tip}}$ and dimensionless horizontal displacement $\delta_h/L$ were found by varying dimensionless point load $\beta$ and number of terms $N$. Calculations were made using MATLAB (code appended at the end of paper).

Table 1 shows convergence of $\theta_{\text{tip}}$ for different values of non-dimensional point load $\beta$ by increasing number of terms used for transformation. Bold figures indicate the values, which do not change up to fourth place by increasing $N$ any further. For $\beta=0.5$, value of $\theta_{\text{tip}}$ doesn’t change after $N=8$, whereas for other greater values of $\alpha$, value doesn’t change up to fourth place decimal after $N=12$.

| N  | $\theta_{\text{tip}}$ | $\beta=0.5$ | $\beta=1.0$ | $\beta=1.5$ | $\beta=2.0$ |
|----|------------------------|-------------|-------------|-------------|-------------|
| 4  | 0.249479               | 0.495833    | 0.735937    | 0.966666    |
| 6  | 0.249479               | 0.495833    | 0.735937    | 0.966666    |
| 8  | 0.24948                | 0.495885    | 0.736333    | 0.968333    |
| 10 | 0.24948                | 0.495885    | 0.736333    | 0.968333    |
| 12 | 0.24948                | 0.495884    | 0.736321    | 0.968247    |
| 14 | 0.24948                | 0.495884    | 0.736321    | 0.968247    |
| 16 | 0.24948                | 0.495884    | 0.736321    | 0.968247    |

Table 2 shows convergence of $\delta_h/L$ for different values of non-dimensional point load $\beta$ by increasing number of terms. Since $\delta_h/L$ depends on $\theta_{\text{tip}}$ and $\beta$, it follows same trend as does $\theta_{\text{tip}}$.

| N  | $\delta_h/L$ | $\beta=0.5$ | $\beta=1.0$ | $\beta=1.5$ | $\beta=2.0$ |
|----|--------------|-------------|-------------|-------------|-------------|
| 4  | 0.006221     | 0.024537    | 0.053933    | 0.092808    |
| 6  | 0.006221     | 0.024537    | 0.053933    | 0.092808    |
| 8  | 0.006219     | 0.02449    | 0.053726    | 0.092287    |
| 10 | 0.006219     | 0.02449    | 0.053726    | 0.092287    |
| 12 | 0.006219     | 0.024491   | 0.053733    | 0.092314    |
| 14 | 0.006219     | 0.024491   | 0.053733    | 0.092314    |
| 16 | 0.006219     | 0.024491   | 0.053733    | 0.092314    |

Figure 3: $\theta$ vs $\eta$ for different values of $\beta$. 

Now $\theta(\eta)$ can be determined by substitution of $C$ into equation (20). N=12 was found to be enough to provide an accurate solution. Therefore, equation (5) involves only a limited number of terms and the solution can be calculated without unnecessary computational effort.
Figure 3 illustrates variation of deflection angle $\theta$ at different cross-section locations of beam $\eta$ for various point load values of $\beta$. It is clearly depicted that increasing $\beta$ as well as $\eta$, increases deflection angle at a particular cross-section. Furthermore, as $\beta$ is increased beyond 1, increase in $\theta$ becomes more and more significant for a particular value of $\eta$. For values of $\beta$ less than 1, increase in deflection angle is not that much substantial.

![Figure 4: Variation of $\theta_{\text{tip}}$ with $\beta$.](image)

![Figure 5: Variation of $\delta_h/L$ vs $\beta$.](image)

Figures 4 and 5 illustrate variation of tip deflection angle $\theta$ and non-dimensional tip horizontal displacement $\delta_h/L$, with dimensionless point load $\beta$, respectively. Figure 4 compares the results obtained from DTM and actual analytical results. It can be seen that DTM provides results with quite great accuracy. For values of $\beta$ greater than 1, difference between two values increases, as it is evident from Eq. (24) that effect of $\beta$ will be greater for larger values of $\beta$ and also by increasing number of terms $N$. Figure 5 shows that for $\beta=2$, tip horizontal displacement is almost one-tenth of the total beam length. Also, for $\beta>1$, variation of $\delta_h/L$ with $\beta$ is linear.

the paper is optional and left as a decision for the author. Where the author wishes to divide the paper into sections the formatting shown in table 2 should be used.

5. Conclusion

DTM provides a reliable approximate explicit analytical solution to large deformation of cantilever beam under point load. Results obtained from DTM are quite accurate up-to fourth decimal place. Deflection angle, horizontal displacement as well as vertical displacement of beam cross-sections can be determined with the help of DTM. Hence, DTM offers an accurate approximate solution non-linear beam problem considering large deformation. DTM may also be applied to other such non-linear complex engineering problems to determine explicit approximate solutions.
6. Recommendations

Following are the recommendations for further research and study:

(a) Efficacy of DTM for other boundary conditions and loading conditions may be studies in order to determine its accuracy for implementation in those cases.

(b) Results may be compared with other methods like Variational Iteration Method (VIM) for same boundary and loading conditions.

7. Appendix A

Appended below is the MATLAB code used for calculations using DTM:

```matlab
% Code for Differential Transformation Method for large Deformation analysis
% of a cantilever beam
clc
clear all
close all
syms a c z
t1=c;
t2=-0.5*a*z^2;
t3=0;
t4=1/24*a*c^2*z^4;
t5=-1/40*a^2*c*z^5;
t6=((1/240*a^3)-(1/720*a*c^4))*z^6;
t7=1/336*a^2*c^3*z^7;
t8=-13/6720*a^3*c^2*z^8;
t9=((1/1920*a^4*c)-(1/8640*a^2*c^5))*z^9;
t10=((257/1209600*a^3*c^4)-(1/19200*a^5))*z^10;
t11=((67/443520*a^4*c^3)+(1/475200*a^2*c^7))*z^11;
t12=((157/2956800*a^5*c^2)-(13/1182720*a^3*c^6))*z^12;
t13=((529/31449600*a^4*c^5)-(7/748800*a^6*c))*z^13;
t14=((120383/9686476800*a^5*c^4)+(1/3548160*a^3*c^8)+(1/1497600*a^7))*z^14;
t15=((547/106444800*a^6*c^3)-(391/399168000*a^4*c^7))*z^15;
t16=((25469/18598035456*a^5*c^6)-(15107/12300288000*a^7*c^2)-(19/5474304000*a^3*c^10))*z^16;
ThetaZ1=sum(ThetaZ(1:N),1);
prompt='Please enter value of alpha (Non-dimensional load parameter)';
a=input(prompt);
prompt='Please enter value of Eeta (Non-dimensional position along beam)';
z=input(prompt);
c=0;% real values of c are very very small and can be neglected
ThetaZ1=subs(ThetaZ1)
```

8
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