Sharp Bounds for the Harmonic Numbers

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Abstract

We obtain best upper and lower bounds for the Lodge-Ramanujan and DeTemple-Wang approximations to the nth Harmonic Number.

1 Introduction

For every natural number $n \geq 1$ the Harmonic Number, $H_n$, is the nth partial sum of the harmonic series:

$$H_n := 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}. \quad (1)$$

Although the asymptotics of $H_n$ were determined by Euler, (see [4]), in his famous formula:

$$H_n \sim \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \cdots, \quad (2)$$

where $\gamma = 0.57721 \cdots$ is Euler’s constant and each summand in the asymptotic expansion is of the form $\frac{B_k}{n^k}$, where $B_k$ denotes the kth Bernoulli number, mathematicians have continued to offer alternate approximative formulas to Euler’s. We cite the following formulas, which appear in order of increasing accuracy:

$$H_n \approx \ln n + \gamma + \frac{1}{2n + \frac{1}{2}}. \quad (3)$$

$$\approx \ln \sqrt{n(n+1)} + \gamma + \frac{1}{6n(n+1) + \frac{6}{5}}. \quad (4)$$

$$\approx \ln \left( n + \frac{1}{2} \right) + \gamma + \frac{1}{24 \left( n + \frac{1}{2} \right)^2 + \frac{21}{5}}. \quad (5)$$
The formula (3) is the Tóth-Mare approximation, (see [5]), and it *underestimates* the true value of $H_n$ by terms of order $\frac{1}{72n^3}$; the second, (4), is the Lodge-Ramanujan approximation, and it *overestimates* the true value of $H_n$ by terms of order $\frac{19}{3150 [n(n+1)]^3}$, (see [6]); and the last, (5), is the DeTemple-Wang approximation, and it *overestimates* the true value of $H_n$ by terms of order $\frac{2071}{806400 (n + \frac{1}{3})^6}$, (see [2]).

In 2003, Chao-Ping Chen and Feng Qi, (see [1]), published a proof of the following sharp form of the Tóth-Mare approximation:

**Theorem 1.** For any natural number $n \geq 1$, the following inequality is valid:

\[
\frac{1}{2n + \frac{1}{1-\gamma} - 2} \leq H_n - \ln n - \gamma < \frac{1}{2n + \frac{1}{3}}.
\]  

(6)

The constants $\frac{1}{1-\gamma} - 2 = .3652721 \cdots$ and $\frac{1}{3}$ are the best possible, and equality holds only for $n = 1$.

The first statement of this theorem had been announced ten years earlier by the editors of the “Problems” section of the *American Mathematical Monthly*, Vol 99, No. 7, (Jul-Aug, 1992), p 685, as part of a commentary on the solution of Problem 3432, but they did not publish the proof. So, the first published proof is apparently that of Chen and Qi.

In this paper we will prove sharp forms of the Lodge-Ramanujan approximation and the DeTemple-Wang approximation.

**Theorem 2.** For any natural number $n \geq 1$, the following inequality is valid:

\[
\frac{1}{6n(n+1) + \frac{6}{5}} < H_n - \ln \sqrt{n(n+1)} - \gamma \leq \frac{1}{6n(n+1) + \frac{12\gamma-11-12 \ln 2}{1-\gamma-\ln \sqrt{2}}}.
\]  

(7)

The constants $\frac{12\gamma-11-12 \ln 2}{1-\gamma-\ln \sqrt{2}} = 1.12150934 \cdots$ and $\frac{6}{5}$ are the best possible, and equality holds only for $n = 1$.

and

**Theorem 3.** For any natural number $n \geq 1$, the following inequality is valid:

\[
\frac{1}{24 \left(n + \frac{1}{2}\right)^2 + \frac{21}{5}} \leq H_n - \ln n - \gamma < \frac{1}{24 \left(n + \frac{1}{2}\right)^2 + \frac{54\ln \frac{3}{2} + 54\gamma - 53}{1-\ln \frac{3}{2} - \gamma}}.
\]  

(8)

The constants $\frac{54\ln \frac{3}{2} + 54\gamma - 53}{1-\ln \frac{3}{2} - \gamma} = 3.73929752 \cdots \cdots$ and $\frac{21}{5}$ are the best possible, and equality holds only for $n = 1$. 

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All three theorems are corollaries of the following stronger theorem:

**Theorem 4.** For any natural number \( n \geq 1 \), define \( f_n, \lambda_n, \) and \( d_n \) by:

\[
H_n := \ln n + \gamma + \frac{1}{2n + f_n}
\]

\[
:= \ln \sqrt{n(n+1)} + \gamma + \frac{1}{6n(n+1) + \lambda_n}
\]

\[
:= \ln \left( n + \frac{1}{2} \right) + \gamma + \frac{1}{24 \left( n + \frac{1}{2} \right)^2 + d_n},
\]

respectively. Then for any natural number \( n \geq 1 \) the sequence \( \{f_n\} \) is **monotonically decreasing** while the sequences \( \{\lambda_n\} \) and \( \{d_n\} \) are **monotonically increasing**.

Chen and Qi, (see [1]), proved that the sequence \( \{f_n\} \) decreases monotonically. In this paper we will prove the monotonicity of the sequences \( \{\lambda_n\} \) and \( \{d_n\} \).

## 2 Lemmas

Our proof is based on inequalities satisfied by the **digamma** function, \( \Psi(x) \):

\[
\Psi(x) := \frac{d}{dx} \ln \Gamma(x) \equiv \frac{\Gamma'(x)}{\Gamma(x)} \equiv -\gamma - \frac{1}{x} + x \sum_{n=1}^{\infty} \frac{1}{n(x+n)},
\]

which is the generalization of \( H_n \) to the real variable \( x \) since \( \Psi(x) \) and \( H_n \) satisfy the equation:

\[
\Psi(n+1) = H_n - \gamma.
\]

**Lemma 1.** For every \( x > 0 \) there exist numbers \( \theta_x \) and \( \Theta_x \), with \( 0 < \theta_x < 1 \) and \( 0 < \Theta_x < 1 \), for which the following equations are true:

\[
\Psi(x+1) = \ln x + \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} - \frac{1}{252x^6} + \frac{1}{240x^8}\theta_x,
\]

\[
\Psi'(x+1) = \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}\Theta_x.
\]

**Proof.** Both formulas are well-known. See, for example, [3], pp 124-125.

**Lemma 2.** The following inequalities are true for \( x > 0 \):

\[
\frac{1}{3x(x+1)} - \frac{1}{15x^2(x+1)^2} < 2\Psi(x+1) - \ln \{x(x+1)\}
\]

\[
< \frac{1}{3x(x+1)} - \frac{1}{15x^2(x+1)^2} + \frac{8}{315x^3(x+1)^3},
\]
\[
\frac{1}{x^2} - \frac{1}{x(x+1)} - \frac{1}{3x^3} + \frac{1}{15x^5} - \frac{1}{18x^7} < \frac{1}{x} + \frac{1}{x+1} - 2\Psi'(x+1)
\]

\[
< \frac{1}{x^2} - \frac{1}{x(x+1)} - \frac{1}{3x^3} + \frac{1}{15x^5}. \quad (18)
\]

**Proof.** The inequalities (17) were proved in our paper, (see[6]), for integers \(n\) instead of the real variable \(x\). But the proofs are valid for real \(x\).

For (18) we start with (15) of **Lemma 1**. We conclude that

\[
\frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} < \frac{1}{x} - \Psi'(x+1) < \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5}.
\]

Now we multiply to all three components of the inequality by 2 and add \(\frac{1}{x+1} - \frac{1}{x}\) to them.

**Lemma 3.** The following inequalities are true for \(x > 0\):

\[
\frac{1}{(x+\frac{1}{2})} - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - \frac{1}{42x^7} < \frac{1}{x+\frac{1}{2}} - \Psi'(x+1)
\]

\[
< \frac{1}{(x+\frac{1}{2})} - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5}. \quad (19)
\]

\[
\frac{1}{24x^2} - \frac{1}{24x^3} + \frac{23}{960x^4} - \frac{1}{160x^5} - \frac{11}{8064x^6} - \frac{1}{896x^7} < \Psi(x+1) - \ln \left( \frac{x + \frac{1}{2}}{x} \right)
\]

\[
< \frac{1}{24x^2} - \frac{1}{24x^3} + \frac{23}{960x^4} - \frac{1}{160x^5} - \frac{11}{8064x^6} - \frac{1}{896x^7} + \frac{143}{30720x^8}. \quad (20)
\]

**Proof.** Similar to the proof of **Lemma 2**.

**3 Proof for the Lodge-Ramanujan approximation**

**Proof.** We solve (10) for \(\lambda_n\) and use (13) to obtain

\[
\lambda_n = \frac{1}{\Psi(n+1) - \ln \sqrt{n(n+1)}} - 6n(n+1).
\]

Define

\[
\Lambda_x := \frac{1}{2\Psi(x+1) - \ln x(x+1)} - 3x(x+1).
\]

for all \(x > 0\). Observe that \(2\Lambda_n = \lambda_n\).

We will show that \(\Lambda'_x > 0\) for \(x > 5\). Computing the derivative we obtain

\[
\Lambda'_x = \frac{\frac{1}{x} + \frac{1}{x+1} - \Psi'(x+1)}{\{2\Psi(x+1) - \ln\{x(x+1)\}\}^2} - (6x + 3)
\]
and therefore
\[
\{2\Psi(x + 1) - \ln\{x(x + 1)\}\}^2 \Lambda_x = \frac{1}{x} + \frac{1}{x + 1} - \Psi'(x + 1) - (6x + 3)\{2\Psi(x + 1) - \ln\{x(x + 1)\}\}^2.
\]
By Lemma 2, this is greater than
\[
\frac{1}{x^2} - \frac{1}{x(x + 1)} - \frac{1}{3x^3} + \frac{1}{15x^5} - \frac{1}{18x^7} - (6x + 3) \left\{ \frac{1}{3x(x + 1)} - \frac{1}{15x^2(x + 1)^2} + \frac{8}{315x^3(x + 1)^3} \right\}^2
\]
\[
= 1071x^6 + 840x^5 - 17829x^4 - 49266x^3 - 502999x^2 - 22178x - 3675
\]
\[
= \frac{(x - 5) \left( x^5 + \frac{295}{51} x^4 + \frac{628}{51} x^3 + \frac{784}{1071} x^2 + \frac{32021}{1071} x + \frac{137927}{1071} \right) + 685960}{1051x^7(x + 1)^6}
\]
which is obviously positive for \(x > 5\).

For \(x = 1, 2, 3, 4, 5\), we compute directly:
\[
\Lambda_1 = 0.56075467 \cdots
\]
\[
\Lambda_2 = 0.58418229 \cdots
\]
\[
\Lambda_3 = 0.59158588 \cdots
\]
\[
\Lambda_4 = 0.59481086 \cdots
\]
\[
\Lambda_5 = 0.59649019 \cdots
\]

Therefore, the sequence \(\{\Lambda_n\}, n \geq 1\), is a strictly increasing sequence, and therefore so is the sequence \(\{\lambda_n\}\).

Moreover, in [6], we proved that
\[
\lambda_n = \frac{6}{5} - \Delta_n,
\]
where \(0 < \Delta_n < \frac{38}{175n(n + 1)}\). Therefore
\[
\lim_{n \to \infty} \lambda_n = \frac{6}{5}.
\]
This completes the proof.

4 Proof for the DeTemple-Wang Approximation

Proof. Following the idea in the proof of the LODGE-RAMANUJAN approximation we solve (11) for \(d_n\) and define the corresponding real-variable version. Let
\[
d_x := \frac{1}{\Psi(x + 1) - \ln\left(\frac{x + 1}{2}\right) - 24 \left(\frac{x + 1}{2}\right)^2}
\]
We compute the derivative, ask *when it is positive*, clear the denominator and observe that we have to solve the inequality:

\[
\left\{ \frac{1}{x + \frac{1}{2}} - \Psi'(x + 1) \right\} - 48 \left( x + \frac{1}{2} \right) \left\{ \Psi(x + 1) - \ln \left( x + \frac{1}{2} \right) \right\}^2 > 0.
\]

By Lemma 3, the left hand side of this inequality is

\[
> \frac{1}{(x + \frac{1}{2})} - \frac{1}{x} + \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - \frac{1}{42x^7} - 48 \left( x + \frac{1}{2} \right) \left( \frac{1}{24x^2} - \frac{1}{24x^3} + \frac{23}{960x^4} - \frac{1}{160x^5} - \frac{11}{8064x^6} - \frac{1}{896x^7} + \frac{143}{30720x^8} \right)^2
\]

for all \( x > 0 \). This last quantity is equal to

\[
(-9018009 - 31747716x - 14007876x^2 + 59313792x^3 + 11454272x^4 - 129239296x^5 + 119566592x^6 + 65630208x^7 - 701008896x^8 - 534417408x^9 + 178139136x^{10})/(17340825600x^{16}(1 + 2x))
\]

The denominator,

\[
17340825600x^{16}(1 + 2x),
\]

is evidently *positive* for \( x > 0 \) and the numerator can be written in the form

\[ p(x)(x - 4) + r \]

where

\[
p(x) = 548963242092 + 137248747452x + 34315688832x^2 + 8564093760x^3 + 2138159872x^4 + 566849792x^5 + 111820800x^6 + 11547648x^7 + 178139136x^8 + 178139136x^9
\]

with remainder \( r \) equal to

\[
r = 2195843950359.
\]

Therefore, the numerator is clearly *positive* for \( x > 4 \), and therefore, the derivative, \( d_x' \), too, is *positive* for \( x > 4 \). Finally

\[
d_1 = 3.73929752 \cdots \\
d_2 = 4.08925414 \cdots \\
d_3 = 4.13081174 \cdots \\
d_4 = 4.15288035 \cdots 
\]

Therefore \( \{d_n\} \) is an *increasing* sequence for \( n \geq 1 \).
Now, if we expand the formula for $d_n$ into an asymptotic series in powers of $\frac{1}{(n + \frac{1}{2})}$, we obtain

$$d_n \sim \frac{21}{5} - \frac{1400}{2071 (n + \frac{1}{2})} + \cdots$$

and we conclude that

$$\lim_{n \to \infty} d_n = \frac{21}{5}.$$

This completes the proof. \qed

References

[1] Ch.-P. Chen and F. Qi, The best bounds of the n-th harmonic number, *Global Journal of Mathematics and Mathematical Sciences* 2 (2006), accepted. The best lower and upper bounds of harmonic sequence, *RGMIA Research Report Collection* 6 (2003), no. 2, Article 14. The best bounds of harmonic sequence, available online at [http://front.math.ucdavis.edu/math.CA/0306233](http://front.math.ucdavis.edu/math.CA/0306233).

[2] D. DeTemple and S-H Wang “Half-integer Approximations for the Partial Sums of the Harmonic Series” *Journal of Mathematical Analysis and Applications*, 160 (1991), 149-156.

[3] J. Edwards *A Treatise on the Integral Calculus, Vol II*, Chelsea, New York, 1955.

[4] K. Knopp *Theory and Application of Infinite Series*, Dover, New York, 1990.

[5] L. Tóth, and S. Mare “E 3432” *American Mathematical Monthly*, 98 (1991), no 3, 264.

[6] M. Villarino, “Ramanujan’s Approximation to the nth Partial Sum of the Harmonic Series”, preprint, arXiv [math.CA/0402354](http://arxiv.org/abs/math.CA/0402354).