Chiral properties of the constituent quark model

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Abstract. We show that, in a model based exclusively on constituent-quark degrees of freedom interacting via a potential, the full axial current is conserved if the spectrum of \(\bar{Q}Q\) states contains a massless pseudoscalar. The current conservation emerges nonperturbatively if the model satisfies certain constraints on (i) the axial coupling \(g_A\) of the constituent quark and (ii) the \(\bar{Q}Q\) potential at large distances. We define the chiral point of the constituent quark model as that set of values of the parameters (such as the masses of the constituent quarks and the couplings in the \(\bar{Q}Q\) potential) for which the mass of the lowest pseudoscalar \(\bar{Q}Q\) bound state vanishes. At the chiral point the main signatures of the spontaneously broken chiral symmetry are shown to be present, namely: the axial current is conserved, the decay constants of the excited pseudoscalar bound states vanish, and the pion decay constant has a nonzero value.

Chiral symmetry is a basic symmetry of massless QCD which, apart from the axial anomaly in the flavor-singlet channel, entails the conservation of the axial-vector current. The masses of the light \(u\) and \(d\) quarks are small compared to the confinement scale, and consequently the chiral limit serves as a good approximation for the light-quark sector of QCD. Chiral symmetry in QCD is spontaneously broken, and is thus not a symmetry of the hadron spectrum: except for the existence of the octet of light pseudoscalar mesons, the lowest-energy part of the hadron spectrum shows no trace of chiral symmetry.

Because of confinement, the calculation of the hadron mass spectrum directly from the QCD Lagrangian is a very challenging task, which requires a nonperturbative approach. QCD-inspired constituent quark models (i.e., models based on constituent-quark degrees of freedom in which mesons appear as \(\bar{Q}Q\) bound states in a potential) proved to be quite successful for the description of the mass spectrum of hadrons and their interactions at low momentum transfers [1, 2, 3]. Because of the proper description of the hadron mass spectrum, the Lagrangian of the constituent quark model cannot be chirally invariant: it would produce a chirally invariant spectrum of hadron states. Consequently, the Noether axial current found in such models is not conserved but satisfies the divergence equation

\[
\partial^\mu [\bar{Q}(x)\gamma_\mu \gamma_5 Q(x)] = 2m_Q \bar{Q}(x)i \gamma_5 Q(x) .
\]  

(1)

In a recent paper [4] we have shown that, nevertheless, taking into account the infinite number of diagrams describing the \(\bar{Q}Q\) interactions, leads to the full axial current of the constituent quarks, which turns out to have the structure

\[
\langle 0 | \bar{q} \gamma_\mu \gamma_5 q | \bar{Q}Q \rangle = g_A (p^2) \left\{ \bar{Q} \gamma_\mu \gamma_5 Q + 2m_Q \frac{p_\mu}{p^2} \bar{Q} \gamma_5 Q \right\} + g_A (p^2) \frac{p_\mu}{p^2} \bar{Q} \gamma_5 Q \left( \frac{2m_Q O(M_\pi^2)}{p^2 - M_\pi^2} \right) .
\]  

(2)
Obviously, the term in curly brackets is transverse by virtue of Eq. (1). Therefore, the full “constituent-quark axial current” is conserved if the mass $M_\pi$ of the pion, the lowest $\bar{Q}Q$ pseudoscalar bound state, vanishes. As has been demonstrated in Ref. [4], to guarantee the axial current conservation up to terms of order $O(M_\pi^2)$ requires that the axial coupling $g_A$ of the constituent quarks is not constant but that it is related to the pion wave function $\Psi_\pi(s)$ by

$$g_A(s) = \eta_A(s - M_\pi^2)\Psi_\pi(s) + O(M_\pi^2), \quad \eta_A = \text{const}. \quad (3)$$

It should be recalled here that the spontaneous breaking of chiral symmetry requires not only the conservation of the axial current, but also the nonvanishing of the coupling of a massive fermion (such as a nucleon or a constituent quark) to the pion, i.e., $g_A(s)$ should be nonzero at $s = M_\pi^2$. Consequently, to be compatible with the spontaneous breaking of chiral symmetry, the potential model should generate a light pseudoscalar bound state for which $\Psi_\pi(s = M_\pi^2)$ has a pole at $s = M_\pi^2$ [4].

In Ref. [4] it is shown that the behavior of $\Psi_\pi(s)$ at $s = M_\pi^2$ is related to the behavior of the potential of the $\bar{Q}Q$ interaction at large separations $r$. More precisely, $\Psi_\pi(s)$ exhibits a pole at $s = M_\pi^2$ only if the potential saturates at large $r$:

$$V(r \to \infty) = \text{const} < \infty. \quad (4)$$

In this case the nearly massless pion is a strongly bound $\bar{Q}Q$ state with binding energy $\varepsilon \approx 2m$.

The observed conservation of the axial current allows us to define the chiral point of the constituent quark model as exactly that set of values of the parameters which leads to a massless lowest pseudoscalar $\bar{Q}Q$ bound state.\(^{1}\) In the following, let us consider certain properties of the constituent quark model at the chiral point.

**Decay constants of pseudoscalar mesons**

Making use of the relation (3) between the axial coupling of the constituent quark, $g_A(s)$, and the pion wave function, the standard quark-model expression for the decay constant of the $n$-th excitation of a pseudoscalar meson [3] takes the form [4]

$$f_P(n) = 2m_Q\eta_A\sqrt{N_c} \int ds \Psi_0(s) \Psi_n(s) \rho(s, m_Q^2, m_Q^2) \frac{s - M_\pi^2}{s} + O(M_\pi^2). \quad (5)$$

The wave functions $\Psi_n(s)$ of the pseudoscalar states satisfy the orthogonality condition

$$\int ds \Psi_n(s) \Psi_m(s) \rho(s, m_Q^2, m_Q^2) = \delta_{nm}. \quad (6)$$

For the ground state, $n = 0$, the decay constant $f_P(0) \equiv f_\pi$ is clearly finite in the chiral limit. With the help of Eq. (6), we obtain the relation [4]

$$f_\pi = 2m_Q\eta_A\sqrt{N_c} + O(M_\pi^2). \quad (7)$$

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\(^{1}\) Precisely, we consider a given potential (i.e., with fixed couplings) and adjust the constituent mass of the light quark until $M_\pi$ vanishes.
For excited states, \( n \neq 0 \), Eq. (5) implies \([4]\), by virtue of the orthogonality condition (6),

\[
f_P(n \neq 0) = -2m_Q\eta AM_\pi^2 \int ds \Psi_0(s)\Psi_n(s) \frac{\rho(s,m_Q^2,m_Q^2)}{s} + O(M_\pi^2).
\]

This decay constant is proportional to \( M_\pi^2 \) and therefore vanishes in the chiral limit, in accordance with the equations of motion in QCD. Also, beyond the chiral limit the decay constants of the excited pseudoscalars are expected to be strongly suppressed compared to the pionic decay constant \( f_\pi \) \([5]\). However, all more accurate predictions for the decay constants of the excited pseudoscalars require a better knowledge of the details of \( g_A(s) \), since in this case the unknown terms of the order \( O(M_\pi^2) \) are of the same order as the contribution given by the main term in \( g_A(s) \).

**Pionic coupling of hadrons**

The result (2) for the full axial current contains an explicit pion pole, thus providing the possibility to extract the amplitude \( A(h_1 \rightarrow h_2 \pi) \) for pionic decays \( h_1 \rightarrow h_2 + \pi \) \([4]\):

\[
p_\mu A(h_1 \rightarrow h_2 \pi) = \left. \lim_{p^2 \rightarrow m_Q^2 \pi^2} \frac{p^2 - M_\pi^2}{f_\pi} \langle h_2 | j_5^\mu | h_1 \rangle \right|_{p^2 = M_\pi^2} = p_\mu \frac{2m_Q}{f_\pi} \langle h_2 | Q\gamma_5 Q | h_1 \rangle.
\]

It is understood that the amplitude \( \langle h_2 | Q\gamma_5 Q | h_1 \rangle \) is calculated in terms of the constituent quark description of the hadrons \( h_1 \) and \( h_2 \). The expression (9) for the amplitude has been successfully applied to pionic decays of charmed mesons \([6]\).

**The chiral constituent quark mass**

Clearly, the constituent quark mass does not vanish in the chiral point. We give now an estimate for the constituent quark mass corresponding to the chiral limit, \( m_Q^0 \), making use of the following relation between the constituent quark mass \( m_Q \) and the current quark mass \( m \) at the chiral-symmetry breaking scale \( \mu_\chi \simeq 1 \text{ GeV} \) \([7]\):

\[
\langle \bar{q}q \rangle = \frac{N_c}{\pi^2} \int_0^\infty dk k^2 \exp(-k^2/\beta_\infty^2) \left\{ \frac{m}{\sqrt{m^2 + k^2}} - \frac{m_Q}{\sqrt{m_Q^2 + k^2}} \right\}, \tag{10}
\]

with \( \beta_\infty \simeq 0.7 \text{ GeV} \) \([7]\). We now have to take into account the dependence of the quark condensate on the value of the current quark mass. For the physical value of the quark condensate, corresponding to the current quark mass \( m = 6 \text{ MeV} \), we use \( \langle \bar{q}q \rangle = -(240 \pm 15 \text{ MeV})^3 \). Eq. (10) then gives \( m_Q = 220 \text{ MeV} \), a typical value of the \( u \) and \( d \) constituent quark mass \([2]\). In order to consider the chiral limit, \( m \rightarrow 0 \), the dependence of the quark condensate on the current quark mass should be taken into account. Setting \( m = 0 \), and making use of the chiral quark condensate \( \langle \bar{q}q \rangle_{m=0} \simeq -(230 \pm 15 \text{ MeV})^3 \), Eq. (10) gives the chiral constituent quark mass \( m_Q^0 = 180 \text{ MeV} \). Let us notice that this is precisely the value of the chiral constituent quark mass of the Godfrey–Isgur model \([2]\).
In summary, we have demonstrated that the relativistic quark picture based exclusively on constituent-quark degrees of freedom is fully compatible with the (well-known) chiral properties of QCD if it encompasses the following features:

- The axial coupling $g_A$ of the constituent quarks is a momentum-dependent quantity, $g_A = g_A(s)$, and is related to the pion $\bar{Q}Q$ wave function.
- The $\bar{Q}Q$ potential $V(r)$ saturates at large interquark separations: $V(r \to \infty) \to \text{const.}$

Under the above conditions, a summation of the infinite number of diagrams describing constituent-quark soft interactions leads to the full axial current of the constituent quarks which is then conserved up to terms of order $O(M_\pi^2)$.

We defined the chiral point of the constituent quark model as that set of values of the parameters of the model (masses of the constituent quarks and couplings in the quark potential) for which the mass of the lowest pseudoscalar $\bar{Q}Q$ bound state, $M_\pi$, vanishes. Although the constituent quark mass clearly does not vanish at the chiral point, we claim that the chiral point of the constituent quark model corresponds to the spontaneously broken chiral limit of QCD for the following three reasons. (i) At the chiral point the full nonperturbative axial current of the constituent quarks is conserved (without the explicit introduction of Goldstone degrees of freedom). (ii) The lowest-energy part of the hadron spectrum has no other traces of chiral symmetry except for a massless pseudoscalar. (iii) Two important signatures of the spontaneously-broken chiral symmetry can be seen: the decay constant $f_\pi$ of the massless pion is finite, that means, nonvanishing, whereas all the decay constants of the excited massive pseudoscalars vanish.

We emphasize that the nonperturbative emergence of chiral symmetry in a model with merely constituent-quark degrees of freedom [4] is qualitatively different from the chiral symmetry of models which explicitly contain Goldstones along with constituent quarks: the latter may be rendered chirally invariant for any value of the constituent quark mass, whereas in our approach chirally symmetry is present only for a definite (nonvanishing) value of the constituent quark mass which leads to a massless ground-state pseudoscalar.

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