A study on summation-integral type operators

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\textbf{Abstract.} In this paper, we investigate the approximation properties of the summation-integral type operators as defined by Mishra et al. (Boll. Unione Mat. Ital. (2016) 8:297-305) and determine the local results as well as prove the convergence theorem of the defined operators. Further check the asymptotic behaviour of the operators and obtain the asymptotic formula for the operators, moreover, the quantitative means of Voronovskaja type theorem is also discussed for an upper bound of the pointwise convergent, as well as obtain the Grüss Voronovskaya-type theorem. Finally, the graphical representation is given to support the approximation results of the operators.

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1. Introduction

In 2016, Mishra et al. \cite{10} carried out their works on approximation properties for the operators defined by:

\begin{equation}
S_n^*(g; x) = u_n \sum_{j=0}^{\infty} s_{u_n,j}(x) \int_0^{\infty} s_{u_n,j}(t)g(t)dt,
\end{equation}

where $s_{u_n,j}(x) = e^{-u_n x} \left(\frac{u_n x}{j!}\right)^j$ and they consider $u_n \to \infty$ as $n \to \infty$ with condition $u_1 = 1$, here functions considered to be Lebesgue integrable. By simple calculation, if we take $u_n = n$ then the above operators (1.1), reduced into Szász-Mirakjan Durremeyr operators defined by Mazhar and Totik \cite{8}. The important properties of the defined operators are studied by Mishra et al. which can be applied to the operators defined Mazhar and Totik. Regarding approximation of the function by Durrmeyer type operators, many works have been done in this direction \cite{11–13}.

Also the discussion regarding Durrmeyer-type modification of Szász-Mirakjan operators is seen in \cite{8} where the authors gave an important result for the Durrmeyer type operators, which are defined on $[0, \infty)$ as:

\begin{equation}
A_n(f; x) = f(0)s_{n,0}(x) + n \sum_{j=1}^{\infty} s_{n,j}(x) \int_0^{\infty} s_{n,j-1}(t)g(t)dt.
\end{equation}

All the above operators (1.1, 1.2) are generalized form of the Szász-Mirakjan operators \cite{9,15} define by:

\begin{equation}
SM_n(f; x) = \sum_{j=0}^{\infty} s_{n,j}(x)g \left(\frac{j}{n}\right),
\end{equation}

where $s_{n,j} = e^{-nx} \left(\frac{nx}{j}\right)^j$ is the Szász-Mirakjan basis function. And the above operators \{$SM_n$} were studied by Szász \cite{15}. Also a natural generalization of the Szász-Mirakjan operators can be seen (presented in \cite{3})
in the form of strictly increasing sequence as a simple replacement of $n$ by $u_n$ such that $u_1 = 1$ and $u_n \to \infty$ as $n \to \infty$ in the above operators (1.3) and hence the modification is seen in the form of:

\begin{equation}
S_n^*(f;x) = \sum_{j=0}^{\infty} s_{u_n,j}(x)g\left(\frac{j}{u_n}\right).
\end{equation}

Thus, the works done by Mishra et al. [10] on the natural modification of the operators defined in [8] called as Szász-Mirakjan Durremeyer operators that put a crucial impact. For further proceed, we need some basic lemma.

**Lemma 1.1.** Consider the function $g$ is integrable, continuous and bounded on the given interval $[0, \infty)$, then the central moments can be obtained as:

\begin{equation}
\Theta_{n,m} = u_n \sum_{j=0}^{\infty} s_{u_n,j}(x) \int_{0}^{\infty} s_{u_n,j}(t)(t-x)^m dt,
\end{equation}

where $m = 0, 1, 2, \ldots$. So for $m = 0, 1$, we can get the the central moments as follows:

\begin{equation}
\Theta_{n,0} = 1, \Theta_{n,1} = \frac{1}{u_n},
\end{equation}

in general, we have

\begin{equation}
u_n \Theta_{n,m+1} = x \left(\Theta_{n,m} + 2m\Theta_{n,m-1} + (1 + m)\Theta_{n,m}\right),
\end{equation}

this lead to

\begin{equation}
\Theta_{n,m} = O\left(\frac{1}{u_n^{\frac{1}{m+1}}}\right).
\end{equation}

**Remark 1.1.** The limit of central moment can be written as:

\begin{equation}
\lim_{n \to \infty} \frac{u_n \Theta_{n,2}}{u_n} = \frac{2(1 + u_n x)}{u_n} = 2x.
\end{equation}

**Remark 1.2.** Second order central moment can be written as:

\begin{equation}
\Theta_{n,2} = \frac{2(1 + u_n x)}{u_n^2} = \frac{2}{u_n} \left(x + \frac{1}{u_n}\right) = \frac{2}{u_n} \zeta_n(x) \ (say).
\end{equation}

2. Local Approximation properties

Next, we estimate the approximation of the defined operators (1.1), by a new type of Lipschitz maximal function with order $r \in (0, 1]$, defined by Lenze [7] as

\begin{equation}
\kappa_r(f;x) = \sup_{x, s \geq 0} \frac{|f(u) - f(v)|}{|u - v|^r}, \ u \neq v.
\end{equation}

Using the definition Lipschitz maximal function, we have the following theorem.

**Theorem 2.1.** For any $g \in C_B[0, \infty)$ with $r \in (0, 1]$ then one can obtain

\begin{equation}
|S_n^*(g;x)(g;x) - g(x)| \leq \kappa_r(g, x) (\Theta_{n,2})^\frac{1}{r}.
\end{equation}

**Proof.** By equation (2.1), we can write

\begin{equation}
|S_n^*(g;x) - g(x)| \leq \kappa_r(g, x) S_n^*(|u - v|^r; x).
\end{equation}

Using, Hölder’s inequality with $j = \frac{2}{r}$, $l = \frac{2}{2-r}$, one can get

\begin{equation}
|S_n^*(g;x)(g;x) - g(x)| \leq \kappa_r(g, x) \left(S_n^*(g;x)(|u - v|^r; x)\right)^\frac{1}{2} = \kappa_r(f, x) (\Theta_{n,2})^\frac{1}{2}.
\end{equation}

Next theorem is based on modified Lipschitz type spaces [14] and this spaces is defined by

$$Lip_M^{a_1,a_2}(s) = \left\{ g \in C_B[0, \infty) : |g(u) - g(v)| \leq M \frac{|u - v|^s}{(u + v^{a_1} + va_2)^s}, \text{ where } u, v \geq 0 \text{ are variables, } s \in (0, 1) \right\},$$

here, $a_1, a_2$ are the fixed numbers and $M > 0$ is constant.
Theorem 2.2. For $g \in Lip_{M}^{s_0,s_2}(s)$ and $0 < s \leq 1$, an inequality holds:

$$|S_n^*(g; x) - g(x)| \leq M \left(\frac{\Theta_{n,2}}{x(xa_1 + a_2)}\right)^{\frac{s}{2}}, M > 0.$$ 

Proof. Since $s \in (0, 1]$. To prove the above result, one can discuss by the cases on $s$. Case 1. For $s = 1$, it can be observed that $\frac{1}{(t + x^2a_1 + xa_2)} \leq \frac{1}{x(xa_1 + a_2)}$ and it implies

$$|S_n^*(g; x) - g(x)| \leq S_n^*((g(t) - g(x)); x) \leq MS_n^*\left(\frac{|t - x|}{(t + xa_1 + xa_2)} ; x\right) \leq \frac{M}{(xa_1 + a_2)} S_n^*((t - x); x) \leq \frac{M}{(xa_1 + a_2)} (\Theta_{n,2})^{\frac{1}{2}} \leq M \left(\frac{\Theta_{n,2}}{x(xa_1 + a_2)}\right)^{\frac{s}{2}}.$$ 

Case 2. for $s \in (0, 1)$ and using Hölder inequality with $l = \frac{2}{s}$, $m = \frac{2}{1-s}$, we get

$$|S_n^*(g; x) - g(x)| \leq \left(S_n^*((g(t) - g(x)); x) \right)^{\frac{s}{2}} \leq MS_n^*\left(\frac{|t - x|^2}{(t + x^2a_1 + xa_2)} ; x\right)^{\frac{s}{2}} \leq MS_n^*\left(\frac{|t - x|^2}{(xa_1 + a_2)} ; x\right)^{\frac{s}{2}} \leq M \left(\frac{\Theta_{n,2}}{x(xa_1 + a_2)}\right)^{\frac{s}{2}}.$$ 

Thus, the proof is completed. 

Theorem 2.3. For the continuous and bounded function $g$ defined on $[0, \infty)$, the convergence of the operators can be obtained as:

$$\lim_{n \to \infty} S_n^*(g; x) = g(x),$$

uniformly on any compact interval of $[0, \infty)$.

Proof. Using Bohman-Korovkin theorem, we can get our required result. Since $\lim_{n \to \infty} S_n^*(1; x) \to 1$, $\lim_{n \to \infty} S_n^*(t; x) \to x$, $\lim_{n \to \infty} S_n^*(t^2; x) \to x^2$ and hence the proposed operators $S_n^*(g; x)$ converge uniformly to the function $g(x)$ on any compact interval of $[0, \infty)$. 

3. Asymptotic behaviour of the operators

To check the asymptotic behavior of the operators, we shall prove the Voronovskaya type theorem.

Theorem 3.1. Let us consider the function $g$ is integrable, continuous and bounded on $[0, \infty)$ as well as the second derivative of the function exists at a point $x \in [0, \infty)$, then the convergence of the operators can be obtained as:

$$\lim_{n \to \infty} u_n (S_n^*(g; x) g(t) - g(x)) = g'(x) + xg''(x).$$

Proof. Using the Taylor’s series expansion, one can write

$$g(t) - g(x) = (t - x)g'(x) + \frac{1}{2}(t - x)^2g''(x) + \zeta(t, x)(t - x)^2,$$

where $\zeta(t, x)$ be such that $\lim_{t \to x}\zeta(t, x) = 0$. On applying the proposed operators on the above equation (3.2), we get

$$S_n^*(g; x) g(t) - g(x) = g'(x)S_n^*(t - x; x) + \frac{g''(x)}{2}S_n^*((t - x)^2; x) + S_n^*(\zeta(t, x)(t - x)^2)$$
Here
\[ S_n^* (\zeta(t,x)(t-x)^2) \leq \sqrt{S_n^* (\zeta^2(t,x))} \frac{(t-x)^4}{S_n^* (\zeta^2(t,x))} \]

Using Theorem 2.3, we get
\[ \lim_{n \to \infty} S_n^* (\zeta^2(t,x)) = \zeta^2(t,x) = 0. \]

And using Lemma 1.1, we can have
\[ S_n^* ((t-x)^4) = O(u_n^{-2}), \]
thus
\[ \lim_{n \to \infty} S_n^* (\zeta(t,x)(t-x)^2) = 0 \]

Therefore, from equation (3.3) and Lemma 1.1, one can write
\[ \lim_{n \to \infty} u_n (S_n^*(g(t)g(t)-g(t)) = g'(x) + xg''(x). \]

Hence, the required result. \(\square\)

4. Quantitative approximation

Generally, we check the pointwise convergence of the operators in the form of Voronovskaya-type theorem but in the quantitative means, we determine an upper bound of this convergence. So, here we describe the quantitative means of Voronovskaya type theorem for the proposed operators. Before, proceeding on the main results, we need some functions classes, which are defined below:
\[ B_w(0, \infty) = \{ g : [0, \infty) \to \mathbb{R} \mid |g(x)| \leq Mw(x) \} \]
where \( M > 0 \) is a constant depending on \( f \) and the spaces
\[ C_w[0, \infty) = \{ g \in B_w[0, \infty), g \text{ is continuous} \}, \]
\[ C^k_w[0, \infty) = \{ g \in C_w[0, \infty), \lim_{x \to \infty} \frac{|f(x)|}{w(x)} = k, w(x) = 1 + x^2 \} \]
where \( w(x) = 1 + x^2 \) is a weight function. Here, the weighted modulus of smoothness is defined in [6] and is denoted by \( \Delta(g; \xi) \), given as:
\[ \Delta(g; \xi) = \sup_{0 \leq h \leq \xi, 0 \leq x \leq \infty} \frac{|g(x+h)-g(x)|}{(1+h^2)(1+x^2)}, g \in C^k_w[0, \infty), \xi > 0. \]

The properties of the weighted modulus of smoothness are as:
\[ \lim_{\xi \to 0} \Delta(g; \xi) = 0 \]
and
\[ \Delta(g; \eta \xi) \leq 2(1+\eta)(1+\xi^2)\Delta(g; \xi), \eta > 0. \]

**Remark 4.1.** By the above relation 4.3 and (4.1), one can write:
\[ |g(t)-g(x)| \leq (1+(t-x)^2)(1+x^2)\Delta(g; |t-x|) \leq 2 \left(1 + \frac{|t-x|}{\xi}\right)(1+\xi^2)\Delta(g; \xi)(1+(t-x)^2)(1+x^2). \]

**Theorem 4.1.** For the function \( g \in C^k_w[0, \infty) \) and assuming \( g''(x) \) exists at a point \( x \), the following inequality holds:
\[ u_n \left| S_n^*(g; x)-g(x) - \frac{g'(x)}{u_n} - \frac{g''(x)}{u_n} \left(x + \frac{1}{u_n}\right) \right| = O(1)\Delta \left(g'', \sqrt{\frac{1}{u_n}}\right). \]
Thus, applying the proposed operators (1.1) to the both sides and using the Lemma 1.1, we get

\[
Hence, S_n^*(|\zeta(t, x); x|) \leq 8(1 + x^2) \Delta \left( g'', \delta \right),
\]

and it can be written as:

\[
|g''(\theta) - g''(x)| \leq \begin{cases} 
2(1 + \delta^2)(1 + x^2) \Delta(f'', \delta), & |t - x| < \delta, \\
2(1 + \delta^2)(1 + x^2) \frac{(t-x)^2}{\delta^4} \Delta(g'', \delta), & |t - x| \geq \delta.
\end{cases}
\]

So, for \( \delta \in (0, 1) \), we get

\[
|g''(\theta) - g''(x)| \leq 8(1 + x^2) \left(1 + \frac{(t-x)^4}{\delta^4}\right) \Delta(g'', \delta).
\]

Hence,

\[
S_n^*(|\zeta(t, x); x|) \leq 8(1 + x^2) \left( (t-x)^2 + \frac{(t-x)^6}{\delta^4} \right) \Delta(g'', \delta).
\]

Thus, applying the proposed operators (1.1) to the both sides and using the Lemma 1.1, we get

\[
S_n^*(|\zeta(t, x); x|) \leq 8(1 + x^2) \Delta \left( g'', \delta \right) \left( \frac{1}{u_n} \right) + \frac{8}{\delta^4} \Delta \left( g'', \delta \right), \quad \text{as } u_n \to \infty.
\]

Choose, \( \delta = \sqrt{\frac{1}{u_n}} \), then

\[
S_n^*(|\zeta(t, x); x|) \leq 8O \left( \frac{1}{u_n} \right) (1 + x^2) \Delta \left( g'', \sqrt{\frac{1}{u_n}} \right).
\]

Thus, it yields as:

\[
u_nS_n^*(|\zeta(t, x); x|) = O(1) \Delta \left( g'', \sqrt{\frac{1}{u_n}} \right).
\]

By (4.6) and (4.9), we obtain the required result. \( \square \)

4.1. Grüss Voronovskaya-type Theorem. This type of theorem plays an important role in the theory of approximation. First of all, in 1935, Grüss [5] defined an inequality, known as Grüss inequality after his name and it estimate with a relation between the integral of a product and product of integrals of the two functions. Its interest increased after its publication, now a days, the importance of this inequality is being usually seen in many research articles. Firstly, Gal and Gonska [2] determin the Grüss Voronovskaya-type theorem with the aid of Grüss inequality for the Bernstein’s polynomials and after that many researchers contributed their effort in this regard and effective research is being done in this direction, this type of research put a crucial impact for the linear positive operators in approximation theory. In a note [4], the
authors obtained a new approach with the help of the least concave majorant by using Grüss inequality to the operators on a compact interval. Some research regarding Grüss Voronovskaya can be seen in [1,16–20].

**Theorem 4.2.** Let \( f, g \in C_{w}^{k}[0, \infty) \) for which \( f', f'', g', g'' \in C_{w}^{k}[0, \infty) \) then for each \( x \geq 0 \), an expression can be obtained, which is:

\[
\lim_{n \to \infty} u_{n} (S_{n}^{*}(f; g; x) - S_{n}^{*}(f; x)S_{n}^{*}(g; x)) = 2xf'(x)g'(x).
\]

**Proof.** For the function \( f, g \in C_{w}^{k}[0, \infty) \) with \( f', f'', g', g'' \in C_{w}^{k}[0, \infty) \), we can write:

\[
n (S_{n}^{*}(f; g; x) - S_{n}^{*}(f; x)S_{n}^{*}(g; x)) = n \left\{ (S_{n}^{*}(f; g; x) - f(x)g(x) - (fg)')\Theta_{n,1} \right\}
- \frac{(fg)''}{2!}\Theta_{n,2} - g(x) \left( S_{n}^{*}(f; x) - f(x) \right)
- f'(x)\Theta_{n,1} - \frac{f''(x)}{2!}\Theta_{n,2}
- S_{n}^{*}(f; x) \left( S_{n}^{*}(g; x) - g(x) - g'(x)\Theta_{n,1} \right)
- \frac{g''(x)}{2!}\Theta_{n,2} + \frac{g''(x)}{2!} S_{n}^{*}((t - x)^2; x)
\times (f - S_{n}^{*}(f; x)) + f'(x)g'(x)\Theta_{n,2}
+ g'(x)\Theta_{n,1} (f - S_{n}^{*}(f; x)) \right\}.
\]

For sufficiently large value of \( n \), applying the Theorem 2.3 and with the help of Theorem 4.1 as well as applying the Remark 1.1, we get our desired result.

\[
\lim_{n \to \infty} u_{n} \left( S_{n}^{*}(f; g; x) - U_{n}^{[\alpha]}(f; x)S_{n}^{*}(g; x) \right) = 2xf'(x)g'(x).
\]

**Example 4.1.** For the approximation by the operators defined by (1.1) to the given function, here, we consider the function \( g : [0, 4] \to [0, \infty) \) such that \( g(x) = e^{x} \) for all \( x \in [0, 4] \). Taking \( n = 25, 50, 100 \) and corresponding operators are as \( S_{25}^{*} \)(green), \( S_{50}^{*} \)(red), \( S_{100}^{*} \)(black). Here the convergence can be seen by observing through the given Figure 1. As the value of \( n \) is increased, the approximation is going to be good.

![Figure 1](image_url)

**Figure 1.** The convergence of the operators \( S_{n}^{*}(g; x) \) to the function \( g(x) \)(blue).
Example 4.2. Let the function $g = x^2 \sin 2\pi x$ (blue) and for the values of $n$, equals to 50, 100, 150, 200, 300 then the convergence of the corresponding operators $S_n^{50}$, $S_n^{100}$, $S_n^{150}$, $S_n^{200}$, $S_n^{300}$, represented pink, red, magenta, black, green colors respectively to the function is good as the value of $n$ is large as given in Figure 2. The approximation can also be seen by observing from the given figure.

Figure 2. The convergence of the operators $S_n^g(x)$ to the function $g(x)$ (blue).

**Conclusion:** Here we have determined the approximation properties for the functions belonging to different spaces and moreover, the order of approximation of the operators has been discussed. The asymptotic behaviour of the operator is discussed in the form of Voronovskaja type theorem and in this regard, we determined the upper bound of the pointwise convergent of the asymptotic formula for the operators with the prove of Grüss Voronovskaya-type theorem. Finally, the approximation results of the operator are through graphically to justify the approximation properties of the operators.

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