THE VELOCITY FUNCTION OF GALAXIES

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ABSTRACT

We present a galaxy circular velocity function, \( \Psi(\log v_c) \), derived from existing luminosity functions and luminosity-velocity relations. Such a velocity function is desirable for several reasons. First, it enables an objective comparison of luminosity functions obtained in different bands and for different galaxy morphologies, with a statistical correction for dust extinction. In addition, the velocity function simplifies comparison of observations with predictions from high-resolution cosmological \( N \)-body simulations.

We derive velocity functions from five different data sets and find rough agreement among them, but about a factor of 2 variation in amplitude. These velocity functions are then compared with \( N \)-body simulations of a CDM model (corrected for baryonic infall) in order to demonstrate both the utility and the current limitations of this approach. The number density of dark matter halos and the slope of the velocity function near \( v_c \), the circular velocity corresponding to an \( L_\ast \) spiral galaxy, are found to be comparable to those of observed galaxies. The primary sources of uncertainty in construction of \( \Psi(\log v) \) from observations and \( N \)-body simulations are discussed, and explanations to account for discrepancies are suggested.

Subject headings: galaxies: distances and redshifts — galaxies: halos — galaxies: luminosity function, mass function — methods: \( n \)-body simulations

1. INTRODUCTION

Galaxy luminosity functions and the Tully-Fisher (TF) relation are key tools for testing models of galaxy formation and incorporating them into a larger picture of gravitational structure formation. An ultimate goal is to be able to reproduce these quantities starting from cosmological \( N \)-body simulations. A significant complication is that observed galaxy luminosities are dependent upon a number of astrophysical processes (e.g., star formation history, gas cooling, internal extinction, supernova feedback, chemical evolution, gas reheating and sharing between galaxies, stellar mass functions, etc.). These factors, which are generally poorly constrained, obscure the connection between formation processes and observable quantities.

These complications have not, however, deterred attempts to bridge the gap between the dark matter halos generated by \( N \)-body simulations and observed galaxies. In the past decade, the use of semianalytic models (SAMs), which create galaxies from dark matter halos by modeling the relevant baryonic physics as global galaxy properties, has become the favored technique for tackling this issue. SAMs have had impressive success in reproducing both observed luminosity functions and TF relations, although not always both at the same time (Kauffmann, White, & Guiderdoni 1993; Cole et al. 1994; Somerville & Primack 1999). A limitation of this approach is that the current models necessarily contain many degrees of freedom, and a number of aspects of the models are oversimplified (Somerville & Primack 1999).

An alternative approach that complements the SAMs is the use of observational data to generate quantities that may be linked more directly with dissipationless \( N \)-body simulations. One such quantity is the galaxy velocity function, \( \Psi(\log v_c) \), which describes the number density of galaxies per unit circular velocity. \( \Psi(\log v_c) \) can be constructed using published luminosity functions and luminosity-velocity (l-v) relations. The velocity function is valuable for several reasons. First, conversion of luminosity functions into velocity functions places surveys obtained in different bands on equal footing (with the caveats discussed in \$4.2\$). This permits direct comparison of the surveys and provides a single target function for which the simulations can aim. Second, by removing the need to model luminosity or understand the physical origin of the TF relation, a number of processes modeled by standard SAMs can be ignored. Only processes that affect baryonic infall, and hence the gravitational potential, impact the velocity function (see \$4.3\$). These include gas cooling and supernova feedback. For these reasons, the velocity function can be a useful tool for probing the connection between large-scale gravitational physics and galaxy formation when coupled with the latest generation of cosmological \( N \)-body simulations.

Construction of a velocity function was suggested by Cole & Kaiser (1989), and an empirical velocity function was created by Shimasaku (1993). The latter work utilized a sample of nearby, bright galaxies from the Third Reference Catalogue of Bright Galaxies (de Vaucouleurs et al. 1991), with velocities derived from a combination of 21 cm observations and l-v relations. Interestingly, Shimasaku also extended this analysis to attempt to include clusters, finding that the galaxy and cluster velocity functions are consistent with being derived from a single dynamical population.

The goal of this paper is to determine \( \Psi(\log v_c) \). The approach taken will be first to use the luminosity function from a single survey, the Southern Sky Redshift Survey (SSRS2), and a single set of l-v relations to create a detailed velocity function. This analysis will be used to assess the importance of correctly accounting for factors that may alter the resultant \( \Psi(\log v_c) \), such as internal galactic extinc-
tion. The results will then be used as a foundation for production of simplified velocity functions for a variety of surveys and $l-v$ relations to assess the robustness of the results. It will be demonstrated that the derived velocity function is robust to within a factor of 2 for $70 \lesssim v_c \lesssim 260$ km s$^{-1}$. To illustrate this we use the results from adaptive refinement tree (ART) simulations (Krivtsov, Klypin, & Khokhlov 1997; Klypin et al. 1999). Comparison with these results shows agreement between observations and theory at $v \sim v_*$, but an excess of dark matter halos at lower velocities.

With the goal of fostering improvement beyond the current work, we also attempt to identify here the primary observational and theoretical sources of uncertainty. Among the observational limitations, uncertainties associated with the spiral and elliptical $l-v$ relations prove to be the most significant factors. Among the theoretical issues, correction of dissipationless models for the uncertain effects of baryonic infall is one of the most significant sources of uncertainty.

In §2 we define the velocity function. The various ingredients necessary to construct this function are detailed in §3. Then we turn to the data analysis in §4, where we also compare the derived velocity functions with the $N$-body simulation and discuss sources of uncertainty. We discuss our results and conclusions in §5. Throughout this paper the Hubble parameter is taken as $H_0 = 100$ $h$ km s$^{-1}$ Mpc$^{-1}$.

2. SCHECTER VELOCITY FUNCTIONS

Galaxy luminosity functions are normally parameterized using a Schechter function (Schechter 1976) of the form

$$\Phi(L) dL = \Phi_*(L/L_*)^{\alpha} \exp \left( -L/L_* \right) dL,$$

where the three observationally determined parameters $\Phi_*$, $\alpha$, and $L_*$ respectively describe the normalization, faint-end slope, and break point of the luminosity function. If the velocity, $v$, is related to $L$ by a simple power law ($L = Av^n$), the number density of galaxies per unit velocity can also be described by a generalized form of the Schechter function,

$$\Psi(v) dv = \Psi_*(v/v_*)^\beta \exp \left( -v/v_* \right) dv,$$

where $v_* = (L_*/A)^{1/\alpha}$, $\beta = \alpha(x + 1) - 1$, and $\Psi_*$ is the normalization. Equivalently, this can be expressed in terms of $\eta = \log v$ as

$$\Psi(\eta) d\eta = \Psi_*(10^{\eta - \Phi} \eta^{\beta + 1} \exp \left[ -(10^{\eta - \Phi})^2 \right] d\eta,$$

where $\Psi_*( \equiv (\log 10)\Psi_*$).

Luminosity-velocity relations exhibit tight correlation, so we choose to construct a circular velocity function for comparison with cosmological models. Specifically, we define $v_c$ to be the circular velocity measured in the flat part of a spiral galaxy’s rotation curve. For spirals $v_c$ can be observed directly; for ellipticals the velocity dispersion, $\sigma$, is observed, and so it is necessary to convert $\sigma$ to $v_c$. In this paper, the simplifying assumption is made that an elliptical galaxy can be modeled as an isothermal sphere, in which case $v_c = \sqrt{2} \sigma$ (Binney & Tremaine 1987). The quantity $v_c$ can also be extracted from very high resolution $N$-body simulations, so a direct comparison of observed and simulated $\Psi(\log v_c)$ is possible.

3. VELOCITY FUNCTION INGREDIENTS

3.1. Survey Luminosity Functions

We impose several criteria on the input luminosity function to simplify the analysis. First, it is preferable that the selected survey contain a large number of galaxies, encompass a large volume, and extend to luminosities well below $L_*$. Second, morphological information is necessary, since spirals, ellipticals, and irregulars are observed to follow different $l-v$ relations. Table 1 lists a number of recent surveys for which luminosity functions have been computed, as well as Schechter parameters and the approximate magnitude range over which the Schechter fit is valid. Only the CfA2, SRS2, and Automatic Plate Measuring Facility (APM) surveys meet the above criteria (Marzke et al. 1994; Marzke et al. 1998). The APM luminosity functions derived for different morphological types have been called into question by several groups, however (Marzke et al. 1994; Zucca, Pozzetti, & Zamorani 1994), and so we refrain from

| Survey     | Band | Number of Galaxies | Type | $M_*$ | $\alpha$ | $\Phi_*$ (h^3 Mpc$^{-3}$) | $M_{\text{low}}$ | $M_{\text{high}}$ |
|------------|------|--------------------|------|-------|---------|----------------------------|-----------------|-----------------|
| SSRS2      | $B_{\text{SSRS2}}$ | 5306               | All  | $-19.43 \pm 0.06$ | $-1.12 \pm 0.05$ | $12.8 \pm 3.0 \times 10^{-3}$ | $-21.1$ | $-15.58$ |
|             |      |                    | E/S0 | $-19.37 \pm 0.11$ | $-1.00 \pm 0.09$ | $4.4 \pm 6.0 \times 10^{-3}$ | $-21.1$ | $-14$   |
|             |      |                    | Sp   | $-19.43 \pm 0.08$ | $-1.11 \pm 0.07$ | $8.0 \pm 3.0 \times 10^{-3}$ | $-21.1$ | $-15.58$ |
|             |      |                    | Irr  | $-19.78 \pm 0.50$ | $-1.81 \pm 0.24$ | $0.20 \pm 6.0 \times 10^{-3}$ | $-21.1$ | $-14$   |
|             |      |                    | UKST | $-19.50 \pm 0.13$ | $-0.97 \pm 0.15$ | $14.0 \pm 6.0 \times 10^{-3}$ | $-21.25$ | $-15.5$ |
|             |      |                    | CfA  | $-19.68 \pm 0.10$ | $-1.04 \pm 0.08$ | $17.0 \pm 6.0 \times 10^{-3}$ | $-21.5$ | $-14.5$ |
|             |      |                    | LCRS | $-19.80 \pm 0.20$ | $-0.70 \pm 0.05$ | $19.0 \pm 6.0 \times 10^{-3}$ | $-22$ | $-18$   |
|             |      |                    | Cl 1 | $-20.29 \pm 0.07$ | $0.51 \pm 0.14$ | $4.0 \pm 0.0 \times 10^{-3}$ | $-22.5$ | $-18$   |
|             |      |                    | Cl 2 | $-20.23 \pm 0.03$ | $-0.12 \pm 0.05$ | $6.9 \pm 6.0 \times 10^{-3}$ | $-22.5$ | $-16.5$ |
|             |      |                    | Cl 3 | $-19.89 \pm 0.04$ | $-0.31 \pm 0.07$ | $8.5 \pm 6.0 \times 10^{-3}$ | $-22.5$ | $-16.5$ |
|             |      |                    | Cl 4 | $-19.86 \pm 0.05$ | $-0.65 \pm 0.08$ | $7.3 \pm 6.0 \times 10^{-3}$ | $-22$ | $-16.5$ |
|             |      |                    | Cl 5 | $-19.95 \pm 0.09$ | $-1.23 \pm 0.10$ | $1.9 \pm 0.5 \times 10^{-3}$ | $-21.5$ | $-16.5$ |
|             |      |                    | Cl 6 | $-20.10 \pm 0.16$ | $-1.93 \pm 0.13$ | $0.7 \pm 0.5 \times 10^{-3}$ | $-21$ | $-17$   |
|             |      |                    | Gard | $-23.30 \pm 0.30$ | $-1.00 \pm 0.20$ | $14.4 \pm 6.0 \times 10^{-3}$ | $-25$ | $-20.5$ |

Note.—For LCRS, the survey data were obtained using $r_0$, but calibrated to $R_C$. $M_{\text{low}}$ and $M_{\text{high}}$ denote the magnitude range over which the given Schechter function is a good fit to the data.
using these morphological data. Also, although it lacks morphological information, the Las Campanas Redshift Survey (LCRS) spectroscopically subdivides the galaxy population in a manner that roughly corresponds to morphological types (Bromley et al. 1998).

There are other practical considerations regarding the $l-v$ relations that impose further constraints on which surveys can be utilized. In particular, the CFA survey uses the Zwicky magnitude system, for which $l-v$ relations are not published for any morphological type.

The situation is slightly better for the $R$-band LCRS; the $R$-band Tully-Fisher relation is well studied and exhibits a tight correlation, but similar information is not available for ellipticals or irregulars. In fact, $l-v$ relations have thus far been published for all morphological types only in the $B$ band. Consequently, for our initial effort at generating $l-v$ relations pose the greatest challenge to construction of the calibrated luminosity function. In fact, $R$-band. Consequently, for our initial effort at generating Table 2 lists TF parameters derived for the SSRS2 survey assuming no Virgocentric infall, and note that Virgocentric infall corrections to the luminosity function have only a modest effect on the results (Marzke et al. 1998).

### 3.2. Luminosity-Velocity Relations

The observational limitations of luminosity-velocity relations pose the greatest challenge to construction of the velocity function. Derivation of $\Psi(\log v_c)$ requires that well-calibrated $l-v$ relations exist for all morphological types that contribute significantly to the luminosity function.

For spirals the forward Tully-Fisher (TF) relation ($M = a - b(\log 2\nu_c - 2.5)$) has been extensively studied (Tully & Fisher 1977). Independent analyses have generated consistent results in the $I$ and $R$ bands and have demonstrated that the intrinsic scatter in the relation is $\sim 0.4$ mag at these wavelengths (Willick et al. 1996). In the $B$ band less effort has been expended toward calibration of the TF relation because the observed scatter is greater than at longer wavelengths. Still, several calibrations have been published. In particular, the TF relation derived in the work of Yasuda, Fukugita, & Okamura (1997) is chosen for construction of the SSRS2 velocity function. Table 2 lists TF parameters derived by various authors in $B$, $R$, and $K$. The different $B$-band relations will be used to evaluate the effect of the choice of TF parameters on the derived $\Psi(\log v_c)$. Also given in Table 2 is the velocity range spanned by the data used to define each relation. It is important to note that in no case has the TF relation been defined above $\sim 350$ km s$^{-1}$. Data are also sparse below $\sim 100$ km s$^{-1}$, but the work that has been done for both spirals and dwarfs at lower circular velocities indicates that there is no dramatic departure from the TF relation down to $v_c \lesssim 20$ km s$^{-1}$ (Hoffman et al. 1996; Richter, Tammann, & Huchtmeier 1987).

For ellipticals the $D_v-\sigma$ relation is the most accurate means of converting luminosity to velocity dispersion (Dressler et al. 1987). However, SSRS2 and other existing large surveys do not publish effective radii and luminosities for individual galaxies. Since the SSRS2 gives only a luminosity function for the $E/S0$ population, the Faber-Jackson (Faber & Jackson 1976) hereafter FJ relation must be employed. Unfortunately, subsequent to the development of the $D_v-\sigma$ relation scant effort has been given to calibration of the FJ relation, and so we refer to early work by de Vaucouleurs & Olson 1982. Using a sample of 86 E and S0 galaxies with recessional velocities greater than 1550 km s$^{-1}$ and $135 < \sigma < 376$ km s$^{-1}$, these authors observe that

$$L_{BT} \propto \sigma^{3.08 \pm 0.28} ,$$

or

$$M_{BT} = (-19.71 \pm 0.08) + (7.7 \pm 0.7)(\log \sigma - 2.3) + 5 \log h \quad \text{(best)} .$$

Although this is the relation used for the SSRS2 analysis, we caution that the statistical error may be a significant underestimate of the uncertainty in this relation. A larger sample of pure ellipticals in the same paper yields

$$M_{BT} = (-19.38 \pm 0.08) + (9.0 \pm 0.7)(\log \sigma - 2.3) + 5 \log h \quad \text{(high)} .$$

For the remainder of the paper these will respectively be denoted as the “best” and “high” FJ relations, as they represent our best estimate of the true relation and the relation with the highest probable slope. The error associated with the slope of the FJ relation has a negligible effect

### Table 2

| Band   | $a$       | $b$       | $v_{low}$ | $v_{high}$ | Reference |
|--------|-----------|-----------|-----------|------------|-----------|
| $B_T$  | $-18.71 \pm 0.11$ | $6.76 \pm 0.63$ | $67$ | $276$ | 1 |
| $B_L$  | $-18.54 \pm 0.39$ | $7.17 \pm 0.20$ | $\sim 10$ | $\sim 280$ | 2 |
| $B_R$  | $-18.13 \pm 0.70$ | $6.50 \pm 0.63$ | $\sim 10$ | $\sim 80$ | 3 |
| $B_{FG}$ | $-20.00 \pm 0.03$ | $6.17 \pm 0.28$ | $\sim 90$ | $\sim 350$ | 4 |
| $K$    | $-21.41 \pm 0.11$ | $8.59 \pm 0.67$ | $117$ | $273$ | 5 |
| $K'$   | $-22.48 \pm 1.58$ | $8.09 \pm 0.52$ | $\sim 90$ | $\sim 310$ | 6 |
| $K''$  | $-22.67$ | $8.73$ | $\sim 50$ | $\sim 280$ | 7 |

**Note:** The listed values are coefficients to the equation $M = a - b(\log 2\nu_c - 2.5)$, and have been normalized to $H_o = 100$ km s$^{-1}$. The value listed for Courteau 1997 corresponds to the determination using $v_c$ at 2.2 optical scale lengths for the Courteau-Faber “quiet Hubble flow” sample. The imaging for Courteau 1997 was obtained with a Spinrad $r$ filter, $r_s$, but calibrated to $r_g$. In the last two columns $v_{low}$ and $v_{high}$ indicate the limits of the velocity range spanned by the data used to construct these TF relations.

**References:**—(1) Yasuda et al. 1997; (2) Richter et al. 1987; (3) Hoffman et al. 1996 (Ir); (4) Courteau 1997; (5) Malhotra et al. 1996; (6) de Grijs & Peletier 1999; (7) Tully et al. 1998.
on the results, but the difference in zero points is a significant source of uncertainty.

For irregulars, the $l-v$ relationship remains poorly constrained. Fortunately, this does not impede the calculation of $\Psi(\log v)$ because irregulars are only a trace population in the velocity regime ($v \gtrsim 100$ km s$^{-1}$) probed by current cosmological simulations. For completeness, we transform the irregular population using the $l-v$ relation recently derived from a sample of 70 dwarf irregulars (Hoffman et al. 1996). This relation has the same form as the spiral TF relation, as well as a similar slope ($6.50 \pm 0.19$ for 1996). This relation has the same form as the spiral TF relation derived from a sample of 70 dwarf irregulars (Ho†man et al. 1995)."
extinction is higher than that for T98. A comparison of the effects of using B95 and T98 is shown in Figure 1a (see § 4). While the correspondence is good, we caution that this topic remains far from settled, and the extinction correction is one of the most significant sources of uncertainty in constructing $\Psi(\log v)$.

When averaged over inclination, assuming randomly distributed inclination angles, the luminosity corrections derived from T98 are

$$M_{B_1}^{\text{cor}} = \frac{1}{0.92} [M_{B_1} + 0.08(15.6 + 5 \log h_{80})], \quad M_{B_1} < -15.6,$$  

$$M_{R_1}^{\text{cor}} = \frac{1}{0.95} [M_{R_1} + 0.05(16.2 + 5 \log h_{80})], \quad M_{R_1} < -16.2,$$  

$$M_{K_1}^{\text{cor}} = \frac{1}{0.99} [M_{K_1} + 0.01(18.3 + 5 \log h_{80})], \quad M_{K_1} < -18.3.$$

The formalism for inclusion of this luminosity-dependent extinction correction within the generalized Schechter function is given in the Appendix.  

### 4. ANALYSIS AND RESULTS

#### 4.1. SSRS2

With all the ingredients assembled, we now construct a velocity function from the SSRS2 luminosity function. A first test is to assess the impact of internal extinction in the spiral population. Figure 1a shows the extinction-corrected and uncorrected spiral velocity functions, $\Psi(\log v_*)$. To illustrate the impact of the choice of extinction correction, both the T98 and B95 extinction laws are applied. The net effect of both corrections is to shift the function to higher velocity by approximately $30 \text{ km s}^{-1}$. The Tully correction is utilized in all subsequent figures. Next, the impact of different choices for TF and FJ relations is assessed. Figure 1b shows that the relations published by Yasuda et al. (1997) and Richter et al. (1987) are consistent, whereas the disparate values for the FJ relation lead to a significant change at high velocities. For the “best” relation, spirals provide the greatest contribution at all velocities; for the “high” relation, ellipticals dominate above $v_\ast$. This is driven by the change in zero point.

The total velocity function can be seen as the heavy solid curve in Figure 1c, with the two light solid lines tracing the central curve indicating the uncertainty due to the formal (1 σ) statistical errors from the luminosity function and TF parameters. Also displayed are the constituent velocity functions for each morphological type, using the Yasuda et al. (1997), de Vaucouleurs & Olson (1982), and Hoffman et al. (1996) $l$-$v$ relations for spirals, ellipticals/S0’s, and irregulars, respectively. Readily apparent is the dominance of the spiral population. Only at velocities well above $v_\ast$, does the elliptical population contribute substantially. Given this dominance, it is of interest to ask how the total velocity function would differ under the assumption that all galaxies are spirals. Namely, how important is the segregation between morphological types in the translation of luminosity to velocity? From Figure 1d it can be seen that this “spiral approximation” is quite good, altering the total velocity function (composite 1) by $\sim 10\%$, less than the formal errors. The validity of such an approximation is important if we wish to compare with LF surveys that lack morphological information.

There are several important notes of caution that should be mentioned. If the zero points of the $l$-$v$ relations are significantly in error, then at high velocities the elliptical population may dominate. To assess the magnitude of this effect, we plot composite 2 in Figure 1d, which uses the “high” FJ relation. Composite 2 has an $\sim 20\%$ higher amplitude than the spiral approximation near $v_\ast$, and has a steeper slope above $v_\ast$. Also, demonstration that the spiral population is dominant in the $B$-band SSRS2 does not ensure that the same is true for galaxy samples selected in other bands, as we may be observing substantially different galaxies (Loveday 1998). For the $R$ band and bluer bands, this should be a mild effect. In comparison to the $B$ band, ellipticals in the $R$ band are $\sim 0.15 \text{ mag}$ brighter relative to spirals (Fukugita et al. 1997). By the $K$ band, however, the spiral approximation should be very poor. As compared to $B$-band, ellipticals in $K$ are $\gtrsim 1 \text{ mag}$ brighter relative to spirals. Consequently, use of the spiral approximation will artificially inflate both the derived $v_\ast$ and $\Psi_\ast$. In the next section we compare velocity functions derived from different surveys in order to provide a lower limit for the systematic errors that are no doubt present. For completeness we include a $K$-band survey, which illustrates the breakdown of the spiral approximation.

#### 4.2. Survey Comparison

SSRS2 is the only survey for which it is possible to generate velocity functions for each morphological type, and so it is necessary to employ the spiral approximation if we wish to compare velocity functions from different surveys. This is done for the SSRS2, APM, UKST/Durham, and LCRS (Marzke et al. 1998; Loveday et al. 1992; Ratcliffe 1998; Bromley et al. 1998), and also for a $K$-band survey by Gardner et al. (1997). For the $B$-band surveys, the TF relation of Yasuda, Fukugita, & Okamura (1997) is utilized. For the $R$ and $K$ bands we use the work of Courteau (1997) and Malhotra et al. (1996), respectively. The values of the parameters in each of these relations are given in Table 2. The resulting values of $\Psi(\log v_\ast)$ are displayed in Figure 2; Figure 3 shows the same data, but only in the regime where the TF relation is also constrained. The generalized Schechter parameters (eq. [2]) corresponding to these velocity functions are given in Table 4.

The $R$- and $B$-band data all agree within the quoted observational errors. This can be seen by comparing the parameter values in Table 4. We also illustrate this in Figure 4 by plotting $\Psi_{240} = \Psi(\log v_\ast)$ at $v_\ast = 240 \text{ km s}^{-1}$ for each survey. One important note is that, for the LCRS catalog, Figure 2 is more indicative of the actual agreement in $\beta$ than is the value in Table 4. This is because the fit in
survey data to the et al. (1998). We also note that a rough conversion of the CfA match to the data at the faint end (Lin et al. 1996; Bromley derived using a single Schechter — which is a visibly bad individual spectroscopic clans, whereas the value in Table 4 is Figure 2 is the sum of Schechter functions used to fit individual spectroscopic clans, whereas the value in Table 4 is derived using a single Schechter fit which is a visibly bad match to the data at the faint end (Lin et al. 1996; Bromley et al. 1998). We also note that a rough conversion of the CfA survey data to the B band is possible via the transformation $M_B = M_Z - 0.45$ (Shanks et al. 1984). We do not plot the CfA velocity function in Figure 2, but find that it is consistent with the B- and R-band velocity functions.

The K-band data primarily serves to emphasize the limitations of this approach. For K, both $v_*$ and $\Psi_*$ are high relative to the other surveys. The quoted errors are large, but the difference in the resulting velocity function is statistically significant. This is not surprising, as there are several reasons to expect the K-band velocity function to be discrepant. First, and most important, the spiral approximation should break down in K, as discussed at the end of § 4.1. Consequently, for future K-band surveys, morphological information will be necessary if they are to be used to generate a velocity function. In addition, the Gardner et al. (1997) K-band luminosity function assumes a value of $q_0 = 0.02$. Recent supernova surveys indicate that a more likely value is $q_0 \approx -0.5$ for a flat universe (Perlmutter et al. 1999), and so there is an additional uncertainty that we have not included in the error budget, which may decrease $\Psi_*$ and increase $v_*$ by $\sim 20\%$.

**TABLE 4**

| Survey   | TF Relation | $v_*$ (km s$^{-1}$) | $\beta$   | $\Psi_*$ (Mpc$^{-3}$ h$^3$) |
|----------|-------------|--------------------|-----------|-----------------------------|
| SSRS2    | Yasuda      | $247 \pm 7$       | $-1.30 \pm 0.13$ | $7.3 \pm 1.4 \times 10^{-2}$ |
| SSRS2    | Richter     | $261 \pm 15$      | $-1.32 \pm 0.13$ | $7.8 \pm 1.2 \times 10^{-2}$ |
| APM      | Yasuda      | $253 \pm 8$       | $-0.93 \pm 0.37$ | $8.0 \pm 1.3 \times 10^{-2}$ |
| UKST     | Yasuda      | $271 \pm 9$       | $-1.10 \pm 0.20$ | $9.7 \pm 2.0 \times 10^{-2}$ |
| LCRS     | Courteau    | $215 \pm 2$       | $-0.30 \pm 0.12$ | $10.3 \pm 0.7 \times 10^{-2}$ |
| Gardner  | Malhotra    | $265 \pm 11$      | $-1.00 \pm 0.68$ | $11.3 \pm 4.8 \times 10^{-2}$ |

**Note.** The listed values are the derived parameters for the velocity function using an assortment of surveys and Tully-Fisher relations. For LCRS, the published single Schechter function fit is used (Lin et al. 1996), whereas in Fig. 2 the fits to the individual clans are used. Also, no error is given for $\Phi_*$ in the Gardner survey; the error given here is an estimate based upon the relative number of galaxies in the Gardner sample compared to other surveys. Even with no error in $\Phi_*$, however, the error in $\Psi_*$ resulting from statistical uncertainty in the K-band TF relation would be $8.7 \times 10^{-3}$ Mpc$^{-3}$ h$^3$. Note that $\Psi_*$ corresponds to the $(\log v_*)$ velocity function; $\Psi(\log v_*)$, the related quantity for $\Psi(v_*)$, is $\Psi_*=\Psi_*/\ln 10$ (see eq. [3]).
Furthermore, of the three bands used, the TF relation is least well established in the K band. Three notable recent determinations of the K-band TF relation are provided by Malhotra et al. (1996), T98, and de Grijs & Peletier (1999). The parameters for each are given in Table 2. Consistent slopes are found by all three groups, but the zero-point variation is large. The zero point from T98 is dependent upon the assumed distance to Ursa Major (Verheijen 1997), and also no errors are quoted. De Grijs & Peletier (1999) quote a 1σ error of 1.58 mag. Malhotra et al. (1996), whose TF relation we employ, have the smallest quoted errors, but their TF relation is based on DIRBE observations of only seven Local Group galaxies including the Milky Way.

4.3. Comparison with Simulations

Although the comparison of the velocity function with a halo velocity function derived from N-body simulations is more straightforward than a corresponding luminosity function comparison, there are a few caveats. The first of these concerns how to assign an appropriate value of \( v_c \) to each simulated dark matter halo, and the second concerns the association of very high and low velocity halos with galaxies.

It has been known for some time that the density profiles of simulated dark matter halos are not well approximated by isothermal spheres (see, e.g., Navarro, Frenk, & White 1996 and references therein). Unlike isothermal spheres, which have flat circular velocity curves, the maximum rotation velocities of halos are not the same as their virial velocities. Galaxy formation also affects dark matter halo velocity curves due to the infall of cool baryons. There are thus at least three possibilities for the \( v_c \) to use in constructing a halo velocity function:

1. \( v_c = v_{\text{vir}} \equiv \left( G M_{\text{vir}}/R_{\text{vir}} \right)^{1/2} \), the circular velocity of the halo at its virial radius, \( R_{\text{vir}} \).
2. \( v_c = v_{\text{max}} \), the maximum rotation velocity of the halo.
3. \( v_c = v_{\text{max}}^{\text{corr}} \), the maximum velocity of the halo after baryonic infall has occurred.

Clearly, option 1 is inappropriate. Recall that \( v_c \) is the circular velocity measured at the flat part of a disk galaxy rotation curve. Halo velocity curves typically flatten at 10%–20% of the virial radius, and \( v_{\text{vir}} \) may be as small as 60% of \( v_{\text{max}} \) (Bullock et al. 1999a). So, although halo \( v_{\text{vir}} \) velocity functions are the most straightforward of the three to estimate, we will not do so here, in order to focus on the more appropriate options. Choice 2 is more sensible, and only slightly more complex to estimate, as long as density profile information is known about the dark matter halos under consideration. Option 3, correcting the halo velocity curve for the effects of baryonic infall, is, in principle, even more appropriate. However, the uncertainties associated with this correction are large. In the discussion that follows, we will explore both options 2 and 3, making use of very high resolution simulation output that supplies the accurate spatial information needed for such an analysis.

The adaptive refinement tree (ART) N-body code (Kravtsov et al. 1997) reaches high force resolution by refining the grid in all high-density regions. It allows the identification of distinct virialized halos as well as halos that exist as substructure within larger halos. Klypin et al. (1999) have used the combined results from two ΛCDM (\( \Omega_0 = 1 - \Omega_{\Lambda} = 0.3, \ h = 0.7, \ \sigma_8 = 1.0 \)) ART simulations to explore the velocity function of halos over a wide range of halo circular velocities (defining \( v_c = v_{\text{max}}^{\text{corr}} \)). The first simulation uses a 60 h\(^{-1}\) Mpc box with a particle mass of \( m_p = 1.1 \times 10^6 \ h^{-1} M_\odot \), and the second simulation uses a 7.5 h\(^{-1}\) Mpc box with \( m_p = 1.7 \times 10^5 \ h^{-1} M_\odot \). They find that the halo circular velocity function over the range \( v_c \approx 20–400 \) km s\(^{-1}\) is well described by a power law: \( \Psi_{\text{halo}}(\log v_c) \approx 0.2 \times 10^{-2.75 \rho_{100}} \), where \( \rho_{100} = \log (v_c/100 \text{ km s}^{-1}) \). This form of \( \Psi_{\text{halo}}(\log v_c) \) is shown by the thin solid line in Figure 2. We see that near \( v_c \), the observations are in reasonable agreement with the simulations, although the density of simulated halos is slightly low. Correcting this relation for baryonic infall will help to alleviate this discrepancy, as we discuss below.

For \( v_c \approx 400 \) km s\(^{-1}\), the slope of \( \Psi_{\text{halo}}(\log v_c) \) is shallower than that observed. This is not of great concern, however, since high-velocity halos correspond to groups and clusters of galaxies and should not be compared directly with the observed galaxy velocity functions. Modeling galaxies in clusters is a difficult problem and is beyond the scope of this paper. However, as a first step in identifying the appropriate halos for the galaxy velocity function comparison, we can restrict ourselves to halos that are “simple” in the sense that they contain no significant substructure. Using the 60 h\(^{-1}\) Mpc ART simulation and methods outlined in Bullock et al. (1999a), we identify halos with significant substructure as those containing at least one subhalo with \( v_c \geq 120 \) km s\(^{-1}\) within the virial radius.

Our simple-halo velocity function is shown by the filled circles connected by the dashed line in Figure 2. The errors on these points reflect Poisson uncertainties. This first-order correction to the all-halo velocity function demonstrates a falloff similar to that observed; however, the slope

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5 We are unable to push this criterion to a lower value of \( v_c \) because of the incompleteness of our halo catalog, reflecting the finite resolution of the simulation.
remains too shallow for $v_c \approx 400 \text{ km s}^{-1}$. It is likely that if the substructure criteria for simple halos were more stringent, for example, if we excluded all halos with substructure with $v_c \approx 70 \text{ km s}^{-1}$, then the simple-halo velocity function might more closely mirror the data at high $v_c$. Higher resolution simulations would be needed to check this.

Below about 120 km s$^{-1}$ the halo number density exceeds the galaxy number density. There are several factors that may contribute to this excess. It is possible that some fraction of lowest velocity halos are not associated with galaxies, as the baryonic material may be ionized and unable to cool and form galaxies (Efstathiou 1992; Weinberg, Hernquist, & Katz 1997). Another factor may be selection effects in luminosity function surveys, the nature of low surface brightness galaxies are systematically missed (Sprayberry et al. 1997; Dalcanton 1998). An example of this is the LCRS, for which a surface brightness limit was imposed upon the spectroscopic sample used to construct the LF (Shectman et al. 1996). Inclusion of these galaxies can act to steepen the faint-end slope of the observed luminosity function, and hence the observed velocity function. One intriguing observation is that the orbits of satellite galaxies exhibit polar anisotropy (Zaritsky et al. 1997), and the suggestion that this may result from the destruction or inhibited formation of a large population of satellites near the plane of the disk of spiral galaxies (Zaritsky & Gonzalez 1999). Such a scenario could help resolve the discrepancy, but currently there exists no known physical process that could accomplish such destruction or inhibition.

Although a direct comparison with the halo velocity function is an interesting first step, for a detailed comparison one must correct the results of the dissipationless halo $v_c$ for the effect of baryonic infall. As a galaxy forms at the center of a halo, the maximum rotation velocity of the system increases as a result both of direct gravitational effects of the disk and of the contraction that infall induces on the halo. The overall shift in the velocity function will depend on the nature of the infall and the processes of disk formation; these are in principle functions of the initial halo $v_c$ and how the galaxy was assembled, including cooling and supernova feedback.

Assuming that the infall of gas is adiabatic and that gas infall is halted because of angular momentum support of the disk (Fall & Efstathiou 1980), Blumenthal et al. (1986) describe a convenient analytic model for calculating the rotation curve redistribution during the process of disk formation (see also Flores et al. 1993; Dalcanton, Spergel & Summers 1997). Mo, Mao, & White (1998) provide a useful fitting function for the infall-corrected maximum rotation velocity of dark matter halos:

$$v_c \approx v_{c,halo} (1 + 4.34 m_d - 3.76 m_d^2) F_v (v_{c,\text{halo}}, \lambda, m_d), \quad (10)$$

where $v_{c,halo}$ is the maximum rotation velocity of the halo before infall and

$$F_v (v_{c,\text{halo}}, \lambda, m_d) = 2.15 \left( \frac{\lambda}{0.1} \right)^{-2.67 m_d - 0.0038 \lambda + 0.2 \lambda} \left[ 1 + \frac{c_{\text{vir}}}{17.5} \right] \left( \frac{c_{\text{halo}}}{\lambda} \right)^2 - \frac{1.54}{c_{\text{halo}}} c_{\text{vir}}^{-1/2}. \quad (11)$$

Here $m_d$ is the fraction of the total halo mass that forms the disk, $\lambda \equiv J/E^{1/2}G^{-1}M^{-5/2}$ is the dimensionless angular momentum parameter (where $J$ and $E$ are the total angular momentum and energy of the halo), and $c_{\text{vir}} = R_{\text{vir}}/R_s$ describes the nature of the dark matter halo density profile, which is assumed to be of the Navarro et al. (1996) form:

$$\rho_{\text{NFW}}(r) = \rho_c (1 + r/R_s)^{-2}. \quad (12)$$

The normalization parameter, $\rho_c$, is determined by $v_{\text{halo}}$ if $R_s$ and $c_{\text{vir}}$ are given. We have implicitly assumed that the fraction of the total halo angular momentum in the disk is equal to the fraction of mass in the disk, $j_d = m_d$. Using equation (10), we may correct the velocity function obtained from the simulations by using appropriate values for $\lambda$, $c_{\text{vir}}$, and $m_d$. This may be done halo by halo, but for simplicity we use the respective averages of these quantities as a function of $v_{c,\text{halo}}$.

By analyzing the 60 $h^{-1}$ Mpc ART simulation, Bullock et al. (1999a) find that the average halo concentration obeys

$$c_{\text{vir}} (v_{c,\text{halo}}) \approx 13 \sqrt{\frac{v_{\text{halo}}}{200 \text{ km s}^{-1}}} \quad (12)$$

and the average spin parameter, $\langle \lambda \rangle \approx 0.04$, is roughly constant as a function of the halo circular velocity (Bullock et al. 1999b). The main uncertainty in this calculation is $m_d$, the fraction of halo mass that ends up in the disk. This quantity depends on the details of galaxy formation, including gas cooling and supernova reheating. Because of the complexity of the problem, we have used the (fiducial) semi-analytic models (SAMs) of galaxy formation developed by Somerville & Primack (1999) in order to determine a reasonable form for $m_d(v_{c,\text{halo}})$. Using the $\Lambda$CDM cosmology described above, we find that the following fitting function,

$$m_d (x) \approx 0.1 \frac{x - 0.25}{1 + x^2}, \quad (13)$$

where $x \equiv v_{c,\text{halo}}/(200 \text{ km s}^{-1})$, does well in reproducing the average $m_d$ of SAM galaxies over the range of velocities $v_c \approx 60–350 \text{ km s}^{-1}$. These models assume $\Omega_m = 0.020 h^{-2}$ (Burles & Tytler 1998). Note that $m_d$ rises with $v_c$ for $v_{c,\text{halo}} \lesssim 200 \text{ km s}^{-1}$; this is a result of supernova explosions, which act to remove gas more effectively from smaller galaxies, as proposed in Dekel & Silk (1986). After reaching a maximum of $\sim 0.04$ near $v_{c,\text{halo}} \sim 200 \text{ km s}^{-1}$, the value of $m_d$ slowly declines because a smaller fraction of gas in large halos has time to cool.

Using equations (10)–(13), we have corrected the halo velocity function for the effect of infall. This correction, which we will refer to as the SAM infall model, is shown by the heavy solid curve in Figures 2 and 3. The curvature in the SAM infall model is caused by the varying behavior of $m_d$ (eq. [13]). For reference, the medium-weight line shows the result of the infall correction when the mass fraction of the disk is held constant at $m_d = 0.04$. The flattening of the SAM-corrected curve at small velocities is due to the inability of small disks to retain their gas; as the disk mass becomes smaller relative to the mass of the halo, the correction to the halo circular velocity becomes negligible. The bend in the SAM-corrected velocity curve at large $v_c$ is because not all of the gas in large halos has time to cool. The SAM infall is truncated at $v_c \approx 350 \text{ km s}^{-1}$ because the infall calculation is inappropriate for group- and cluster-mass halos. We also truncate the curve below $v_c = 100 \text{ km}$.  

\footnote{Although there is disagreement (cf. Kravtsov et al. 1998; Primack et al. 1999; Moore et al. 1999) about the detailed shape of dark matter halo profiles at very small radii, $r \leq 0.02$, these very inner regions are not important for determining $r_{\text{max}}$, so the NFW profile is appropriate for our needs.}
s^{-1}$, where $m_g \lesssim 0.02$. Below this value the infall-correction formula (eq. [10]) ceases to be a good fit (Mo et al. 1998); however, it is likely that galaxies with any smaller amount of gas will be of very low surface brightness and difficult to detect.

It is obvious from this comparison that the halo velocity function differs markedly from the galaxy velocity function, but there are avenues of theoretical exploration that may help in understanding the differences. At velocities well above $v_\ast$, most halos correspond to groups and clusters rather than individual galaxies. Understanding the falloff of the velocity function at high $v_c$ will require more detailed modeling of galaxy formation within clusters, perhaps using both semianalytic and $N$-body techniques. Near $v_\ast$ for the $\Lambda$CDM cosmology we explore, the halo number density and slope are comparable to the galaxy density within observational errors (see Fig. 4). The halo density is slightly low without infall correction, and slightly high with the approximate infall correction we present. Below $v_\ast$ the halo density exceeds the galaxy density, but the tendency for small $v_c$ objects to have small disk mass fractions due to supernova feedback may help explain the discrepancy: because they have low luminosity and low surface brightness, many low-$v_c$ galaxies will be missed in the luminosity functions we started from. If, however, the discrepancy at the low-$v_c$ end is not purely due to selection effects, it may turn out to pose a real challenge for theory. More detailed modeling of redshift survey selection effects and of small-velocity galaxies, including consistent treatments of gas cooling, baryonic infall, supernova feedback, and disk surface brightnesses, will be needed to explore this problem in detail.

5. DISCUSSION AND CONCLUSIONS

A main goal of this work was to evaluate the robustness with which $\Psi(\log v_c)$ can currently be estimated. While morphological information was incorporated in converting the SSRS2 luminosity function to a velocity function, a key result of this detailed analysis is that treating the entire population as spirals does not significantly alter the resulting velocity function. Furthermore, while the normalization of the velocity function remains poorly constrained [\sim 30\% variance in $\Psi_\ast$ among surveys (excluding $K$), and a factor of 2 variance in $\Psi(\log v_c)$ at $v_c = 240$ km s^{-1}], the shape of the velocity function is similar for all input luminosity functions. Both the shape and the normalization are also consistent within the errors with the velocity function derived by Shimasaku (1993).

The key benefit of our approach is that the models needed to connect $N$-body simulations, and observations become much less complex when we use observed TF relations and extinction corrections instead of trying to reproduce these functions via the semianalytic models. This contrasts with SAMs, which output modeled $l-v$ relations that are dependent on tunable model parameters (e.g., star formation timescales, supernova feedback, etc.) for comparison with observations. Another benefit is that, by converting to velocity, we provide a single target function for the models to attempt to reproduce, in contrast to luminosity functions in different bands from various redshift surveys. Hopefully, this will be of value in simplifying comparison with simulations from different groups.

The main sources of uncertainty limiting the precision with which the velocity function can be constructed via this approach are:

1. Large scatter in the reported values of $\Phi_\ast$, possibly due to local deviation from mean density.
2. Selection bias at the faint end of the luminosity function, which may act to flatten the faint-end slope.
3. The limited velocity range over which the TF relation is well calibrated.
4. Uncertainty in the TF relation. Beyond the statistical errors, there are indications of several potentially significant biases in current relations. In particular, J. A. Willick and S. Courteau (1999, private communication) find that using the circular velocity at 2 disk scale lengths (as determined by disk + bulge fitting) reduces scatter in the TF relation and also can significantly alter the slope relative to other methods of determining $v_c$.
5. Uncertainty regarding the extinction correction, and also oversimplified treatment of extinction by averaging over inclination.
6. Uncertainty in the zero point of the FJ relation. A change in the zero point could noticeably alter the velocity function above $v_\ast$.
7. Lack of a detailed understanding of the correspondence between $\sigma$ in ellipticals and $v_c$ in spirals. While $v_c = \sqrt{2\sigma}/m$ may be true on average for bright ellipticals, scatter in this relation can also alter the velocity function above $v_\ast$.
8. Intrinsic scatter in the TF and FJ relations.

Use of next-generation surveys such as the Anglo-Australian 2 degree Field (2dF) and the Sloan Digital Sky Survey (SDSS) will reduce the cosmic variance of $\Phi_\ast$. Further, an alternative approach to this technique would be the direct construction of the velocity function from a galaxy survey designed to obtain both photometry and slit-based spectral line widths. This could be achieved by a slit-mask survey of a volume-limited subset of the SDSS or 2dF samples.\footnote{8} Bypassing the $l-v$ relations would significantly reduce uncertainty in the derived velocity function, although it would be necessary to separate the spiral and elliptical populations so as to treat rotationally and thermally supported systems correctly. Finally, with the multi-color photometry of SDSS, it will be possible to measure inclinations and correct for extinction effects on a galaxy-by-galaxy basis prior to construction of the luminosity function.

In the meantime, we have demonstrated that existing data are sufficient to construct the velocity function accurate to within roughly a factor of 2 at a given velocity, which is suitably accurate for comparison with predictions from cosmological $N$-body simulations. By comparison with one such $\Lambda$CDM simulation, we have illustrated the usefulness of this approach. The main sources of uncertainty limiting the precision with which the velocity function can be estimated theoretically are the following:

1. The degree to which halos with very large and very small $v_c$ should be associated with galaxies.

7 Unless there is a large error in the FJ zero point, in which case it may be necessary to treat ellipticals separately in order to reproduce the high-velocity end accurately.

8 While 2dF and SDSS both obtain spectral information, the fiber-fed spectrographs only collect information on galactic centers, and hence do not provide information on the rotation curves of spiral galaxies.
2. Uncertainties associated with correcting the \( v_c \) of measured halos for the effect of baryonic infall.

There is reasonable agreement between the observations and simulations near \( v_c \), and the exploration of these uncertainties may help explain the large excess of systems in the simulations below \( \sim 120 \) km s\(^{-1}\) and the slope of the velocity function above \( v_c \). This poses an interesting challenge for models of galaxy formation to address. A key test will be to see whether the incorporation of baryonic infall and cooling physics leads to a theoretical galaxy velocity function consistent with those observed.

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APPENDIX

Starting with the Schechter function,

\[
\Phi(L) dL = \Phi_* \left( \frac{L}{L_*} \right)^x \exp \left( -\frac{L}{L_*} \right) d\left( \frac{L}{L_*} \right),
\]

and the relation

\[
L = Ax^n,
\]

where \( A \) is a constant, we define

\[
\Psi(x) dx = \Phi(L) dL = \frac{\Phi_* \left( \frac{x}{x_*} \right)^{nx} \exp \left[-\left(\frac{x}{x_*}\right)^n\right]}{n\Phi_* \left( \frac{x}{x_*} \right)^{nx+n-1} \exp \left[-\left(\frac{x}{x_*}\right)^n\right]} d\left( \frac{x}{x_*} \right).
\]

Thus

\[
\Psi(x) dx = \frac{\Phi_* \left( \frac{x}{x_*} \right)^\beta \exp \left[-\left(\frac{x}{x_*}\right)^n\right]}{n\Phi_*} d\left( \frac{x}{x_*} \right),
\]

where \( \beta \equiv n(x + 1) - 1 \) and \( \Psi_* = n\Phi_* \).

Inclusion of a magnitude-dependent extinction correction of the form

\[
M^{cor} = \gamma^{-1}(M + C),
\]

where \( \gamma \) and \( C \) are constants, is equivalent to modifying the zero and power of the \( L-x \) relation. Specifically, for the case of a TF relation (which in general will be extinction-corrected) of the form

\[
M^{cor} = a - b(\log 2v_c - 2.5),
\]

in order to relate \( v \) to the observed magnitude, \( M \), from a luminosity function survey (which in general will not be extinction-corrected), we substitute

\[
M^{cor} = \gamma^{-1}(M + C) = a - b(\log 2v_c - 2.5),
\]

or

\[
M = (\gamma a - C) - gb(\log 2v_c - 2.5) = a' - b'[\log (\Delta v) - 2.5].
\]

The resulting modified TF relation is thus of the same form as the original, but with a new slope and offset that are related to the old ones by

\[
a' = \gamma a - C
\]

and

\[
b' = gb.
\]

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