A generalisation of Seymour’s second neighbourhood conjecture

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Abstract

In this note we propose a generalisation of Seymour’s Second Neighbourhood Conjecture to two directed graphs on a vertex set. We prove that this generalisation holds in the case of tournaments, and we show that a natural strengthening of this conjecture does not hold.

1 Introduction

In this note, all graphs are finite. Directed graphs (or digraphs) do not contain parallel edges, but may contain self-loops. Oriented graphs are directed graphs with no self-loops and directed 2-cycles. A digraph $G$ on a vertex set $V = V(G)$ can be represented by its set of edges, $E(G)$, a subset of $V \times V$. The first neighbourhood or out-neighbourhood of a vertex $v$ in $G$ is the set \(\{u \in V(G) \mid (v, u) \in E(G)\}\), denoted by $N^+_G(v)$. The in-neighbourhood of $v$ is the set \(\{u \in V(G) \mid (u, v) \in E(G)\}\), denoted by $N^-_G(v)$. We set $d^+_G(v) = |N^+_G(v)|$ and $d^-_G(v) = |N^-_G(v)|$. The second neighbourhood or second out-neighbourhood of a vertex $v \in V(G)$ is the set $(\bigcup_{u \in N^+_G(v)} N^+_G(u)) \setminus N^+_G(v)$, denoted by $N^{++}_G(v)$. We also set $d^{++}_G(v) = |N^{++}_G(v)|$. Similarly define the second in-neighbourhood $N^{--}_G(v)$ and $d^{--}_G(v)$. We omit the subscripts $G$ if the context is clear. In 1990, Seymour conjectured the following statement:

**Conjecture 1** (Seymour, see [1]). Every oriented graph has a vertex $v$ satisfying $d^{++}(v) \geq d^+(v)$.

A vertex $v$ of $G$ is said to satisfy the second neighbourhood property (SNP) if $d^{++}(v) \geq d^+(v)$. $G$ is said to satisfy the second neighbourhood conjecture (SNC) if it has a vertex satisfying SNP.

In 1996, Fisher [2] proved the conjecture for tournaments, i.e. oriented graphs with an edge between every pair of vertices.

It will be convenient to consider also a weighted version of SNC. Suppose $G$ is weighted by a non-negative real-valued function $\omega : V(G) \to \mathbb{R}_{\geq 0}$. The weight of a set of vertices is the sum of the weights of its members. We say that a vertex $v$ of $G$ satisfies the weighted second neighbourhood property (WSNP) if $\omega(N^{++}(v)) \geq \omega(N^+(v))$. We say $G$ satisfies the weighted second neighbourhood conjecture (WSNC) if for every such function $\omega$, there is a vertex $v$ satisfying WSNP.

**Theorem 2.** The following are equivalent:

1. Every oriented graph satisfies the second neighbourhood conjecture.
2. Every oriented graph satisfies the weighted second neighbourhood conjecture.

Proof. Clearly (2) implies (1). Supposing (1), we first show WSNC holds for positive integer weights \( \omega : V(G) \rightarrow \mathbb{N} \), hence holds for positive rational weights. By continuity, WSNC holds for all non-negative real weights.

Given a weight \( \omega : V(G) \rightarrow \mathbb{N} \), consider the graph \( G' \) formed by duplicating each vertex \( v \) \( \omega(v) \)-many times to get vertices \( v_1, \ldots, v_{\omega(v)} \), with edges \((u_i, v_j)\) whenever \((u, v)\) is an edge of \( G \), over all possible \( i, j \). We call this process blowing up vertex \( v \) with weight \( \omega(v) \). Then \( G \) having the WSNP is equivalent to \( G' \) having the SNP, hence \( G \) satisfies the WSNC.

\[ \square \]

2 A generalisation

We start with a generalisation which turns out to be false and give a counterexample with 36 vertices. Then we give a modification of the generalisation which we believe is true.

Let \( A, B \) be digraphs on the same vertex set \( V \). Recall that a digraph can be viewed as a subset of \( V \times V \), so standard set operations can be performed on them. Let \( I = \{(v, v) \mid v \in V\} \) be the identity graph. The transpose (or inverse) of \( A \) is defined by \( A^T = \{(v, u) \mid (v, u) \in A\} \), i.e. the graph with all edges of \( A \) reversed. We define the product graph \( AB \) to be the subset \( \{(u, v) \mid \exists w \in V, (u, w) \in A, (w, v) \in B\} \). Set \( A(v) = \{u \in V \mid (v, u) \in A\} \) for \( v \in V \).

The second neighbourhood conjecture can be reformulated in the following way: \( A \) is a directed graph with \( A \cap A^T = I \). Then there is a vertex \( v \) such that \(|AA(v)| \geq 2|A(v)| - 1\). This leads to a natural generalisation: Let \( A, B \) be directed graphs on a vertex set \( V \) such that \( A \cap B^T = I \). Is there a vertex \( v \) such that \(|AB(v)| \geq |A(v)| + |B(v)| - 1\)? The original SNC is just with \( A = B \).

Similar to SNC, this has a weighted version \( \omega(AB(v)) \geq \omega(A(v)) + \omega(B(v)) - \omega(v) \), which is equivalent to the unweighted one. However, the weighted version turns out to be false. We provide 2 counterexamples, both with \( A, B \) having no 2-cycles.

Our first counterexample has \( A \subset B \). We give the weights of each vertex and out-neighbours of \( A, B, AB \) in the table below:

| V | weight | \( A \) | \( B \) | \( AB \) |
|---|---|---|---|---|
| 1 | 7 | 1,2,5,6 | 1,2,5,6 | 1,2,3,4,5,6 |
| 2 | 3 | 2,3 | 2,3,4 | 1,2,3,4,5 |
| 3 | 11 | 1,3,4,5 | 1,3,4,5 | 1,2,3,4,5,6 |
| 4 | 3 | 1,4 | 1,4,6 | 1,2,4,5,6 |
| 5 | 3 | 2,5,6 | 2,4,5,6 | 2,3,4,5,6 |
| 6 | 9 | 2,3,6 | 2,3,6 | 1,2,3,4,5,6 |

Thus by blowing up the vertices with the appropriate weights, we obtain a counterexample to the unweighted generalisation with 36 vertices. Our second counterexample has \( B \subset A \), described below:

| V | weight | \( A \) | \( B \) | \( AB \) |
|---|---|---|---|---|
| 1 | 17 | 1,2,5,6 | 1,2,5,6 | 1,2,3,4,5,6 |
| 2 | 11 | 2,3,4 | 2,3,4 | 1,2,3,4,5 |
| 3 | 15 | 1,3,6 | 1,3 | 1,2,3,5,6 |
| 4 | 8 | 1,3,4,5 | 3,4,5 | 1,2,3,4,5,6 |
| 5 | 5 | 2,3,5 | 2,3,5 | 1,2,3,4,5 |
| 6 | 8 | 2,4,5,6 | 2,5,6 | 2,3,4,5,6 |

Thus this yields an unweighted counterexample with 64 vertices after blowing up.

By replacing \( AB \) with \( AB \cup BA \), we obtain a weaker generalised conjecture:
Conjecture 3. Let $A, B$ be directed graphs on a vertex set $V$ such that $A \cap B^T = I$. Then there is a vertex $v$ such that $|AB \cup BA|(v) \geq |A(v)| + |B(v)| - 1$.

The weighted version is as follows:

Conjecture 4. Let $A, B$ be directed graphs on a vertex set $V$ such that $A \cap B^T = I$. Then for any non-negative weight function $\omega$, there is a vertex $v$ such that $\omega((AB \cup BA)(v)) \geq \omega(A(v)) + \omega(B(v)) - \omega(v)$.

This generalises SNC by setting $A = B$. By replacing $A, B$ with $A \cap B, A \cup B$, we may assume that $A \subset B$. The above counterexamples show that the union $AB \cup BA$ is required, even when $A \subset B$ or $B \subset A$. The tournament version of the above conjecture holds; we give a proof based on the technique of winning and losing densities of Fisher [2]:

Theorem 5. Let $A, B$ be directed graphs on a vertex set $V$ such that $A \cap B^T = I$ and $A \cup B^T = V \times V$. Let $\omega : V \rightarrow \mathbb{R}_{\geq 0}$ be a weight function. Then there is a vertex $v$ such that $\omega((AB \cup BA)(v)) \geq \omega(A(v)) + \omega(B(v)) - \omega(v)$.

Proof. We first assume that $A \subset B$. Set $C = AB \cup BA$. Consider the oriented graph $G$ with edges $\{(u, v) \mid u \neq v, (u, v) \in A\}$. By Theorem 1 of [4], $G$ has a losing density. A losing density $l$ is a weight function satisfying $l(N^+_G(v)) \geq l(N^-_G(v))$ for all vertex $v$. Further, if $l(v) > 0$, then $l(N^+_G(v)) = l(N^-_G(v))$. Fix any vertex $v$, set $S_1 = N^-_G(v) = A^T(v) \setminus \{v\}, S_2 = C(v) \cup B^T(v)$.

We show that $l(S_2) \geq l(S_1)$. Let $Q$ be the subgraph $V \setminus C^T(v) \cup \{v\}$. This partitions $V$ into the sets $S_1 \cup S_2 \cup Q \cup (B^T(v) \setminus A^T(v))$. If $l(Q) = 0$, then $S_1 = N^-_G(v)$ and $N^+_G(v) \subset S_2 \cup Q$, thus $l(S_1) \leq l(S_2 \cup Q) = l(S_2)$. Otherwise, we have $l(Q) > 0$. Within $Q$, we have

$$
\sum_{v \in V(Q)} l(v)l(N^+_G(v)) = \sum_{v \in V(Q)} l(v)l(u) = \sum_{v \in V(Q)} l(v)l(N^+_G(v)).
$$

Thus we have $l(N^+_G(v)) \geq l(N^+_G(u))$ for some $u \in Q$ with $l(u) > 0$. For each $w \in S_1$, we cannot have $(w, u) \in B^T$, since otherwise we have $u \in A^T B^T(v)$. Since $A \cup B^T = V \times V$, we must have $(w, u) \in A^T$, hence $N^-_G(u) \supseteq N^+_G(u) \cup S_1$. For each $x \in B^T(v) \setminus A^T(v)$, we cannot have $(x, u) \in A^T$, since otherwise $u \in B^T A^T(v)$. Thus we get $N^+_G(u) \subset N^+_Q(u) \cup S_2$. Since $l(u) > 0$, $l(N^+_G(u)) = l(N^+_Q(u))$. From $l(N^+_Q(u)) \geq l(N^+_Q(u))$, we deduce that $l(S_2) \geq l(S_1)$.

We want to show there is a $v$ such that $\omega(C(v) \setminus B(v)) \geq \omega(A(v) \setminus \{v\})$. In fact, we show stronger statement that

$$\sum_{v \in V} l(v)(\omega(C(v) \setminus B(v)) - \omega(A(v) \setminus \{v\})) \geq 0,$$

then we are done. The above sum is equal to

$$\sum_{v \in V} \omega(v)(l(C^T(v) \setminus B^T(v)) - l(A^T(v) \setminus \{v\})).$$

But the sets $C^T(v) \setminus B^T(v)$ and $A^T(v) \setminus \{v\}$ are just $S_2$ and $S_1$ defined above corresponding to $v$, and $l(S_2) \geq l(S_1)$, thus the above sum is non-negative. □

We remark that, setting $A = B$, we recover the usual proof by Fisher [2] of the SNC for tournaments.

3
References

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