Effective theory for the two–nucleon system

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Abstract

We apply the method of unitary transformations to a model two–nucleon potential and construct from it an effective potential in a subspace of momenta below a given cut–off Λ. The S–matrices in the full space and in the subspace are shown to be identical. We solve numerically the Schrödinger equation in the small momentum space and recover exactly the bound and scattering states of the full theory. We then expand the heavy repulsive meson exchange of the effective potential in a series of local contact terms and discuss the question of naturalness of the corresponding coupling constants. Using our exact effective theory we address further issues related to the chiral perturbation theory approach of the two–nucleon system. The coordinate space representation of the effective potential is also considered.

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1 Introduction

Chiral perturbation theory is hoped to provide insight into the nuclear force problem and possibly even lead to a quantitative framework. Chiral symmetry imposes constraints on possible momentum and spin dependences of the nuclear forces but the framework is restricted to momenta below a certain scale. Only in this regime one can set up a power counting scheme which limits the number of interaction terms in the nuclear Hamiltonian. Whether this scheme works quantitatively and is applicable to interacting nucleons seems not yet fully established, despite a large number of investigations and considerable progress over the last few years (an incomplete list of references is [1]–[9]. For a short state–of–the–art review see e.g. [10]). We therefore think that a model study based on a simplified nuclear force which, however, captures the essential features of the nucleon–nucleon interaction (long/intermediate–range attraction and short–range repulsion) can provide useful insights.

In this study, we investigate the two–nucleon system in a space of momenta whose values are below a given cut–off $\Lambda$. For that, we start with a model nucleon–nucleon (NN) interaction given in full momentum space. The potential consists of two terms, an attractive one due to the exchange of a light meson with mass $\mu_L$ and a repulsive term parametrized in terms of a heavy meson exchange with a mass $\mu_H$. The strengths parameters accompanying these terms are determined by fitting physical properties like e.g. the deuteron binding energy. We then divide the momentum space into two subspaces, spanning the values from zero to $\Lambda$ and from $\Lambda$ to $\infty$, respectively. Consequently, the two–nucleon Hamiltonian can be regarded as a two–by–two matrix connecting the two momentum subspaces. By an unitary transformation it can be block–diagonalized decoupling the two subspaces. In this manner one can construct an effective Hamiltonian acting only in the low momentum subspace. The so constructed effective Hamiltonian comprises the full physics for low–lying bound and scattering states with appropriate boundary conditions. Specifically, for the scattering states the initial momenta should belong to the low momentum subspace. Nevertheless, and this is an important remark, the physics of the high momenta is by the very construction included in the effective low momentum theory. Like in treatments performed in the context of chiral perturbation theory all or some of the resulting effective interaction can be cast into the form of a string of contact interactions of increasing powers in the momenta. For a consistent power counting to emerge, the coefficients accompanying these terms should be of natural size. In other words there should be a momentum scale (the naturalness scale $\Lambda_{\text{scale}}$) such that the properly normalized coefficients are of order one.

In our exact effective theory, we can in fact precisely calculate these coefficients and check the naturalness property. Stated differently, the effective theory obtained by the exact momentum space projection plays the role of QCD and the expansion in terms of contact interactions for the heavy meson exchange is our model of the effective field theory. Note that throughout this investigation, we keep the light meson (pion) exchange explicitly. Further questions that can be addressed are: a) can one find a subdivision of the full momentum range such that the effective interaction is only weakly dependent on the precise choice of the value $\Lambda$, and b) is there relation between $\Lambda$ and the natural momentum scale? We can also investigate the convergence properties of the expansion in terms of local operators since we have the exact solution to the problem under control. The effective interaction is naturally constructed in momentum space, but one can consider it in configuration space as well. Clearly, one has to expect the effective potential to look totally different compared to the original potential and as a consequence, the deuteron wave function will also change. More precisely, the projection of the original potential

\footnote{We stress that the effective theory considered here is not an effective field theory (EFT). This important distinction should be kept in mind.}
into the subspace of small momenta induces non-localities in momentum space which in turn lead to very complicated co-ordinate space expressions. We will show some instructive examples of this phenomenon.

In section II we formulate the model and the way the block-diagonalization is performed. We use an unitary operator proposed long time ago by Okubo \cite{11} (see also ref.\cite{12}), which leads to a nonlinear decoupling equation. We show that the $T$-matrix evaluated with the help of the effective interaction is exactly equal to the original $T$-matrix for the underlying potential. This is related to the fact that the component of the transformed wave function in the subspace of the high momenta above $\Lambda$ is exactly zero for the low energy scattering states in which we are interested. This proof is different from the one given in refs.\cite{11}\cite{12}. An alternative way, which, however, assumes the knowledge of the the scattering states, leads to a linear equation \cite{13}. This is briefly discussed in the end of the section.

The numerical investigations are described in section III. It is shown how the nonlinear decoupling equation can be solved by a suitably chosen iteration. For that, one has to modify the original potential in order to avoid difficulties when the momentum exactly equals the chosen cut-off value. Physical observables, however, do not depend on this modification. We demonstrate (numerically) the exact agreement for the binding energy and scattering phase shifts using the effective and the original Hamiltonian, respectively. We then expand the heavy meson exchange term in a series of local contact terms with increasing dimension. This allows us to extract information on the naturalness property of the expansion coefficients in the effective interaction. As an application we calculate the expectation values of the various terms in the effective potential with respect to the effective deuteron wave function and low energy scattering states. This allows to study the convergence properties of the contact term expansion. We discuss the relevance of these results for a systematic effective field theory treatment of the two-nucleon system. We also regard the configuration space representation of the effective interaction. Note that some of these results were already discussed in the letter \cite{16}.

We summarize and conclude in section IV.

2 Projection formalism for the effective theory

In this section we develop in detail the formalism which allows to study the nucleon–nucleon interaction in a Hilbert space of momenta below a chosen cut-off in momentum space. The starting point is a given potential which involves all momentum scales and reproduces qualitatively the two-nucleon phenomenology (in the S-waves). Besides being interesting in itself, such an effective theory in a space of low momenta can also be used to study various aspects of effective field theory approaches to the nucleon–nucleon interaction. For simplicity, we restrict ourselves to the S–waves. The formalism is, however, more general and can straightforwardly be extended to more complicated potentials.

To be specific, consider a momentum space Hamiltonian for the two–nucleon system of the form

$$ H(\vec{p}, \vec{p}') = H_0(\vec{p}) \delta(\vec{p} - \vec{p}') + V(\vec{p}, \vec{p}') , $$

(2.1)

where $H_0$ stands for the kinetic energy and $V$ for the spin–independent model force. We introduce the projection operators

$$ \eta = \int d^3p |\vec{p}| \langle \vec{p} | , \quad |\vec{p}| \leq \Lambda , $$

(2.2)

$$ \lambda = \int d^3p |\vec{p}| \langle \vec{p} | , \quad |\vec{p}| > \Lambda , $$

(2.3)
where \( \Lambda \) is a momentum cut–off which separates the low from the high momentum region. Its precise value will be given below. Apparently \( \eta^2 = \eta, \lambda^2 = \lambda, \eta \lambda = \lambda \eta = 0 \) and \( \eta + \lambda = 1 \). Using the \( \eta \) and \( \lambda \) projectors, the Schrödinger equation takes the form

\[
\begin{pmatrix}
\eta H\eta & \eta H\lambda \\
\lambda H\eta & \lambda H\lambda
\end{pmatrix}
\begin{pmatrix}
\eta |\Psi\rangle \\
\lambda |\Psi\rangle
\end{pmatrix}
= E
\begin{pmatrix}
\eta |\Psi\rangle \\
\lambda |\Psi\rangle
\end{pmatrix}.
\] (2.4)

Obviously, the low and the high momentum components are coupled. Our aim is to derive an effective Hamiltonian acting on low momentum states only and which furthermore incorporates all the physics related to the possible bound and scattering states with initial asymptotic momenta from the \( \eta \) states. This can be accomplished by an unitary transformation \( U \)

\[
H \rightarrow H' = U^\dagger H U
\] (2.5)
such that

\[
\eta H' \lambda = \lambda H' \eta = 0 .
\] (2.6)

We choose a parametrization of \( U \) given by Okubo [11],

\[
U = \begin{pmatrix}
(1 + A^\dagger A)^{-1/2} & -A^\dagger (1 + AA^\dagger)^{-1/2} \\
A(1 + A^\dagger A)^{-1/2} & (1 + AA^\dagger)^{-1/2}
\end{pmatrix},
\] (2.7)

where \( A \) has to satisfy the condition

\[
A = \lambda A \eta .
\] (2.8)

It is then straightforward to recast the conditions eq.(2.6) in a different form,

\[
\lambda (H - [A, H] - AHA) \eta = 0 .
\] (2.9)

This is a nonlinear equation for the operator \( A \), which takes the explicit form

\[
V(\vec{p}, \vec{q}) = \int d^3 q' A(\vec{p}, \vec{q}') V(\vec{q}', \vec{q}) + \int d^3 p' V(\vec{p}, \vec{p}') A(\vec{p}', \vec{q})
- \int d^3 q' d^3 p' A(\vec{p}, \vec{q}') V(\vec{q}', \vec{p}') A(\vec{p}', \vec{q})
= (E_q - E_p) A(\vec{p}, \vec{q}) .
\] (2.10)

Here we denoted the momenta from the \( \eta \) and \( \lambda \)–spaces by \( \vec{q} \) and \( \vec{p} \), respectively, and \( E_q, E_p \) stand for the corresponding kinetic energies. Once \( A \) and thus \( U \) have been determined, the effective Hamiltonian in the \( \eta \)–space takes the form

\[
\eta H' \eta = \eta (1 + A^\dagger A)^{-1/2} \left( H + A^\dagger H + HA + A^\dagger HA \right) (1 + A^\dagger A)^{-1/2} \eta .
\] (2.11)

This interaction is by its very construction energy–independent and hermitean [11][12][8].

After this block–diagonalization, the Schrödinger equation separates into two effective equations in the respective subspaces. According to eq.(2.5) the connection between the eigenstates of \( H \) and \( H' \) is

\[
|\Psi\rangle = U^\dagger |\Psi\rangle ,
\] (2.12)

so that a priori the transformed problem separates into

\[
\eta H' \eta |\Psi\rangle = E \eta |\Psi\rangle ,
\] (2.13)

\[
\lambda H' \lambda |\Psi\rangle = E \lambda |\Psi\rangle .
\] (2.14)
However, one of the components of $\Psi'$ has to be identically zero [14]. If both $\eta$ and $\lambda$ components of $\Psi'$ would be different from zero, one could form any linear combination thereof and would end up in the original space with an infinite degeneracy, which is incorrect. It remains to be clarified in which space the transformed scattering and bound states reside. Let us first regard scattering states. In which space lies $\Psi'$ if the scattering process is initiated by asymptotic momenta from the low momentum range spanned by $\eta$? For that, we define the effective potential by

$$V' \equiv H' - H_0 \ ,$$

which is, of course, block–diagonal, since $H_0$ and $H'$ have this property. Note further that the resolvent operator of the transformed Hamiltonian $H'$ is block–diagonal as well:

$$U^\dagger (z - H)^{-1} U = (z - H')^{-1} \equiv G'(z) = \begin{pmatrix} (z - \eta H' \eta)^{-1} & 0 \\ 0 & (z - \lambda H' \lambda)^{-1} \end{pmatrix} .$$

(2.16)

Obviously, $z$ should not be in the spectrum of $H$. This block–diagonal form (2.16) has the immediate consequence that the scattering states $|\Psi_q^{(\pm)}\rangle$, defined via

$$|\Psi_q^{(\pm)}\rangle = \lim_{\epsilon \to 0} i\epsilon G'(E_q + i\epsilon) |q\rangle \ ,$$

(2.17)

lie in the $\eta$ ($\lambda$–)space, when the corresponding asymptotic momentum $q$ belongs to the $\eta$ ($\lambda$–)space, respectively. Here, $|q\rangle$ is a momentum eigenstate and $H_0 |q\rangle = E_q |q\rangle$. Immediately, the question arises what is the connection between the scattering states $|\Psi_q^{(\pm)}\rangle \equiv \lim_{\epsilon \to 0} i\epsilon G(E_q + i\epsilon) |q\rangle$ and $|\Psi_q^{(\pm)}\rangle$? It is not obvious that eq. (2.12) holds also for these scattering states, which are defined through specific boundary conditions. We sketch now a proof, showing that eq. (2.12) is indeed valid in this case and that the relation

$$|\Psi_q^{(\pm)}\rangle = U^\dagger |\Psi_q^{(\pm)}\rangle$$

(2.18)

is satisfied. The following steps appear highly plausible but do not replace a mathematically rigorous proof. Consider the left hand side of this equation:

$$|\Psi_q^{(\pm)}\rangle \equiv \lim_{\epsilon \to 0} i\epsilon G'(E_q + i\epsilon) |q\rangle = \lim_{\epsilon \to 0} i\epsilon U^\dagger G(E_q + i\epsilon) U |q\rangle$$

$$= U^\dagger \lim_{\epsilon \to 0} i\epsilon G(E_q + i\epsilon) \left( \eta A + \lambda A^\dagger \right)^{-1/2} \left( \eta + \lambda A A^\dagger \right)^{-1/2} |q\rangle \ .$$

(2.19)

In the last line we used eq. (2.7). Let us define the operators $B$ and $C$ by

$$B \equiv \eta B \eta = \left( \eta + A^\dagger A \right)^{-1/2} - \eta \ ,$$

(2.20)

$$C \equiv \lambda C \lambda = \left( \lambda + A A^\dagger \right)^{-1/2} - \lambda \ .$$

(2.21)

Then eq. (2.19) takes the form

$$|\Psi_q^{(\pm)}\rangle = U^\dagger \lim_{\epsilon \to 0} i\epsilon G(E_q + i\epsilon) \left( |q\rangle + F |q\rangle \right)$$

(2.22)

with

$$F = B + C + A(\eta + B) - A^\dagger (\lambda + C) \ .$$

(2.23)
Now the operator $A$ and as a consequence $B$ and $C$ depend on the interaction $V$ and are linear in $V$ in lowest order. This is obvious from eq. (2.10). Further it can be excluded that the state $F\,|\vec{q}\rangle$ is an exact continuum eigenstate of $H$ to the energy $E_q$, which would be required to generate a pole $\propto 1/(i\epsilon)$ in the application of $G(E_q + i\epsilon)$. Only then the factor $i\epsilon$ could be cancelled. As a consequence one obtains

$$\lim_{\epsilon \to 0} i\epsilon G(E_q + i\epsilon) \, F \, |\vec{q}\rangle = 0 \ ,$$  \hspace{1cm} (2.24)

and thus eq. (2.18) is indeed satisfied. One may also use a perturbative argument and expand the full resolvent operator

$$\lim_{\epsilon \to 0} i\epsilon G(E_q + i\epsilon) \, F \, |\vec{q}\rangle = \lim_{\epsilon \to 0} i\epsilon \left( \sum_{i=0}^{\infty} (G_0(E_q + i\epsilon)V)^i \right) G_0(E_q + i\epsilon) \, F \, |\vec{q}\rangle \ .$$  \hspace{1cm} (2.25)

Because of the above mentioned property of $F$ it is clear that $\langle \vec{p}|F|\vec{q}\rangle$ does not contain a term proportional to $\delta(\vec{p} - \vec{q})$ and therefore one can not generate a pole term $1/(i\epsilon)$ through the application of $G_0(E_q + i\epsilon)$ onto $F|\vec{q}\rangle$. Consequently, each single term of the series on the right hand side of this equation equals to zero, which leads again to (2.24). We thus have shown that the scattering states $\Psi_\vec{q}^{(+)}$ initiated by momenta $\vec{q}$ from the low momentum $\eta$–space are transformed into $\Psi_\vec{q}^{(+)}$, which obey eq. (2.13). The corresponding $\lambda$ components of $\Psi_\vec{q}^{(+)}$ are strictly zero.

After considering the scattering states, we now turn our attention to the bound states. Again, the obvious question is: Where do the transformed bound states of $H$ reside? If the cut–off $\Lambda$ is sufficiently large then $A$ goes to zero. This is a simple consequence of the fact that $V$ is assumed to fall off sufficiently fast for high momenta. Consequently $\lambda H'\lambda$ is approximately equal to $\lambda H\lambda$ which contains only a small portion of $V$ and thus can not support bound states at all. The transformed bound states have therefore to be solutions of eq. (2.13). For the physically reasonable choices we used in section 3 this turns out to be true. It is not known to us of which value of $\Lambda$ this argument breaks down. In section 3 we shall use an iteration procedure to solve the nonlinear equation (2.10). It is known from [13] that the iteration converges only if the lowest eigenvalue of $\lambda H\lambda$ is greater than the largest eigenvalue of the effective $\eta$–space Hamiltonian $\eta H'\eta$. If one chooses a sufficient small cut–off $\Lambda$, than we can expect that $\lambda H\lambda$ will provide a bound state, which of course is not the physical one. In such a case the iteration method has to break down and the nonlinear equation (2.10) has to be solved in another manner. It is conceivable that then the true physical transformed bound state resides in the effective $\lambda$–space.

One can argue just in the same way to see the validity of the equation (2.12) for the states $|\Psi_\vec{q}^{(-)}\rangle \equiv \lim_{\epsilon \to 0} i\epsilon G(E_q - i\epsilon) \, |\vec{q}\rangle$ and $|\Psi_\vec{q}^{(-)}\rangle \equiv \lim_{\epsilon \to 0} i\epsilon G'(E_q - i\epsilon) \, |\vec{q}\rangle$. Therefore, the $S$–matrices in the original and transformed problem are the same:

$$S'_{\vec{q}'\vec{q}} \equiv \langle \Psi_\vec{q}^{(-)}|\Psi_\vec{q}'^{(+)}\rangle = \langle \Psi_\vec{q}^{(-)}|UU^\dagger|\Psi_\vec{q}'^{(+)}\rangle = \langle \Psi_\vec{q}^{(-)}|\Psi_\vec{q}'^{(+)}\rangle = S_{\vec{q}'\vec{q}} \ .$$  \hspace{1cm} (2.26)

As a consequence the on–shell $T$–matrix element evaluated by means of the Lippman–Schwinger (LS) equation

$$T' = V' + V'G_0T' \ ,$$  \hspace{1cm} (2.27)

yields exactly the same matrix elements as gained via the LS equation to the original problem

$$T = V + VG_0T \ .$$  \hspace{1cm} (2.28)

Note that in eq. (2.28) one integrates over the whole (infinite) momentum range whereas in eq. (2.27) only momenta up to the cut–off $\Lambda$ are involved.
An important observation is that in the subspace of momenta below the cut–off most local operators are non–local. For an arbitrary local operator \( O(\vec{p}_1, \vec{p}_2) = O(\vec{p}_1) \delta^3(\vec{p}_1 - \vec{p}_2) \) one obtains in the transformed space

\[
\tilde{O}(\vec{p}_1, \vec{p}_2) = \int d^3p' U^\dagger(\vec{p}_1, \vec{p}') O(\vec{p}') U(\vec{p}', p_2) ,
\]

which in general contains the usual \( \delta \)–function part but in addition also a strong non–local piece. These non–localities, which are easy to handle, are nothing but the trace of the high momentum components from the full space. Note, however, that the free particle Hamiltonian \( H_0 \) and as a consequence the momentum operator \( \vec{p} \) of a particle remain unchanged. Certainly, one could also unitarily transform the operators \( H \) and \( \vec{p} \). This would not yield any new aspects since the original and the unitarily transformed problems would be trivially identical, the unitary transformation only leads to change of basis. We shall not consider this trivial case any further. Physical observables such as the deuteron binding energy and phase shifts are identical in the original and transformed problems as shown before. We remark that for instance the average momentum in the deuteron will change in the transformed problem because the momentum operator is not unitarily transformed. This is not a problem since it is not direct observable. However, all current operators have to be unitarily transformed and this guarantees the equivalence of all observables.

To end this section, we discuss an alternative way of determining the operator \( A \), as exhibited in ref.\[13\]. That method uses the knowledge of the scattering states to the original Hamiltonian \( H \). Let us now look in some more detail at this formalism. As it was already pointed out above, the connection between the scattering states in the original and transformed problem is given by

\[
\left| \Psi^{(+)}_{\vec{q}} \right\rangle = U \left| \Psi'^{(+)}_{\vec{q}} \right\rangle = \left( \eta + \lambda A \eta \right) \left( 1 + A^\dagger A \right)^{-1/2} \eta + \left( \lambda - \eta A^\dagger \lambda \right) \left( 1 + AA^\dagger \right)^{-1/2} \lambda \left| \Psi'^{(+)}_{\vec{q}} \right\rangle .
\]

For momenta \( \vec{q} \) from the \( \eta \)–space this equation takes a simpler form

\[
\left| \Psi^{(+)}_{\vec{q} \in \eta} \right\rangle = \left( \eta + \lambda A \eta \right) \left( 1 + A^\dagger A \right)^{-1/2} \eta \left| \Psi'^{(+)}_{\vec{q} \in \eta} \right\rangle ,
\]

since the resolvent operator of the transformed Hamiltonian \( H' \) is block–diagonal as already pointed out before, cf eq. (2.14). Using the trivial relation \( (\eta + \lambda A \eta)(\eta + \lambda A \eta) = \eta + \lambda A \eta \) we conclude from the eq.\[2.31\] for all momenta \( \vec{q} \in \eta \) that

\[
\lambda \left| \Psi^{(+)}_{\vec{q} \in \eta} \right\rangle = \lambda A \eta \left| \Psi^{(+)}_{\vec{q} \in \eta} \right\rangle .
\]

Projecting this equation on to the state \( \langle \vec{p} |, p > \Lambda \), and making use of the relation

\[
\langle \vec{p}_1 | \Psi^{(+)}_{\vec{p}_2} \rangle = \delta(\vec{p}_1 - \vec{p}_2) + \frac{T(\vec{p}_1, \vec{p}_2, E_{p_2})}{E_{p_2} - E_{p_1} + i\epsilon} ,
\]

where \( T(\vec{p}_1, \vec{p}_2, z) \) is the usual off–shell \( T \)–matrix, we end up with the following linear integral equation for the operator \( A \):

\[
A(\vec{p}, \vec{q}) = \frac{T(\vec{p}, \vec{q}, E_{\vec{q}})}{E_{\vec{q}} - E_{\vec{p}}} \int d^3q' A(\vec{p}, \vec{q}) \frac{T(\vec{q}', \vec{q}, E_{\vec{q}})}{E_{\vec{q}} - E_{\vec{q}'} + i\epsilon} .
\]

Here the integration over \( q' \) goes from 0 to \( \Lambda \). Note this is not a usual Lippmann–Schwinger equation, since the position of the pole, \( E_{\vec{q}} \), in the integration over \( q' \) is not fixed but moves with \( q \). It is the second argument in \( A \) which varies, the first one is a parameter for the integral equation.
3 Applications

This section is split into various paragraphs. First, we discuss in detail how to determine the operator $A$ for a given realistic potential. We then compare the results obtained in the space of momenta below the chosen cut–off with the ones in the unrestricted Hilbert space. Since the projection formalism is exact, we can recover all physical quantities obtained from the full potential to an arbitrary accuracy in the smaller space with low momenta only. This extends the results exhibited in the letter [16]. Third, we use this model to study the expansion of short–range physics in terms of contact terms and draw some conclusions for the application of chiral effective field theories. All these calculations are naturally done in momentum space. For illustration, we finally show some results in coordinate space.

3.1 Basic model and determination of the operator $A$

Our starting point is a model potential which captures the essential features of the nucleon–nucleon (NN) interaction. We choose a momentum space NN potential with an attractive and a repulsive part corresponding to the exchange of a light and a heavy scalar meson, their masses denoted by $\mu_L$ and $\mu_H$, respectively

$$V(\vec{q}_1, \vec{q}_2) = \frac{1}{2\pi^2} \left( \frac{V_H}{t + \mu_H^2} - \frac{V_L}{t + \mu_L^2} \right),$$

(3.1)

with $t = (\vec{q}_1 - \vec{q}_2)^2$. The strengths of the meson exchanges parametrized by $V_L$ and $V_H$, respectively, will be determined later. This potential still contains all partial waves. The numerical investigations will be restricted to NN S–waves. One can work out the corresponding S–wave potential in closed form,

$$V(q_1, q_2) = \frac{1}{2\pi q_1 q_2} \left( V_H \ln \left( \frac{(q_1 + q_2)^2 + \mu_H^2}{(q_1 - q_2)^2 + \mu_H^2} \right) - V_L \ln \left( \frac{(q_1 + q_2)^2 + \mu_L^2}{(q_1 - q_2)^2 + \mu_L^2} \right) \right),$$

(3.2)

with $q_{1,2} = |\vec{q}_{1,2}|$. The nonlinear equation (2.10) simplifies correspondingly. Still, it can only be solved numerically. We do this by iteration starting with

$$A = \frac{V(\vec{p}, \vec{q})}{E_q - E_p}.$$  

(3.3)

After four iterations we perform an average over the values of the operator $A$ with suitably chosen weight factors. The choice of these weight factors is responsible for the quick convergence of this iteration method. A typical total number of iterations is 40 to achieve an accuracy of $0.0001 \text{ GeV}^{-3}$ for the function $A(p,q)$. The function $(E_q - E_p)$ in eq. (2.10) multiplying $A(p,q)$ requires some caution when $p$ and $q$ go to $\Lambda$. We proceed by regularizing the original potential $V(k',k)$. We multiply it with some smooth functions $f(k')$ and $f(k)$ which are zero in a narrow neighborhood of the points $k' = \Lambda$ and $k = \Lambda$ and one elsewhere. The precise form of this regularization does in fact not matter as already argued in [16]. However, for the actual calculations presented here, we choose

$$f(k) = \begin{cases} 1, & \text{for } k \leq \Lambda - a \\ \frac{1}{2} \left( 1 + \cos \left( \frac{\pi(k - \Lambda + a)}{b} \right) \right), & \text{for } \Lambda - a \leq k \leq \Lambda - a + b, \end{cases} \quad \text{(3.4)}$$

$$f(k) = \begin{cases} \frac{1}{2} \left( 1 + \cos \left( \frac{\pi(k - \Lambda - a)}{b} \right) \right), & \text{for } \Lambda + a - b \leq k \leq \Lambda + a, \\ 0, & \text{for } \Lambda - a + b \leq k \leq \Lambda + a - b, \end{cases}$$

for $k \leq \Lambda - a$ and $k \geq \Lambda + a$, respectively.
with $a$ and $b$ parameters of dimension [energy]. This modification of the potential $V$ is depicted in fig. [1]. Now $A(p, q)$ based on that modified potential is well defined for $p, q \to \Lambda$. To further quantify the effect of this regularization, we set for simplicity $b = 0$. Then it is easy to estimate the difference of the regularized on-shell $T$-matrix, $T_{\text{reg}}(q, q)$, to the unregularized (exact) one, $T(q, q)$, by

$$T_{\text{reg}}(q, q) = T(q, q) - a \frac{2\Lambda^2 V(q, \Lambda) T(\Lambda, q)}{E_q - E_\Lambda} + O(a^2),$$

(3.5)

for all $q < \Lambda - a$ and $q > \Lambda + a$. Similarly, the modification of the deuteron binding energy can be obtained by calculating the expectation value of $\langle \Psi_D | V - V_{\text{reg}} | \Psi_D \rangle$:

$$E_{\text{reg}} = E - 4a \Psi_D(\Lambda) \Lambda^2 \int_0^\infty k^2 dk V(\Lambda, k) \Psi_D(k) + O(a^2),$$

(3.6)

where $\Psi_D(k)$ is the momentum space deuteron wave function. Thus for any fixed $q < \Lambda - a$ and infinitesimal $\epsilon > 0$ one can choose $a$ such that $|T_{\text{reg}}(q, q) - T(q, q)| < \epsilon$ and $|E_{\text{reg}} - E| < \epsilon$. Later we shall give numerical examples of the effects caused by that potential modification in some specific cases. We show later numerically that the effective potential $V'(q, q')$ is affected only within the width $a$ for $q, q' \to \Lambda$. There both $V'$ and $V$ go to zero and have no effect on the observables.

We now have to fix the parameters for the potential $V$. We choose these as given in ref.[19]: $V_H = 7.291$, $V_L = 3.177$, $\mu_H = 613.7$ MeV and $\mu_L = 305.9$ MeV. That potential supports one bound state at $E = -2.23$ MeV and leads to S-wave phase shifts in the $^3S_1$ partial wave in fair agreement with results from NN partial wave analysis. Thus this potential captures some essential features of the NN interaction. Next, we have to select a value for the cut–off $\Lambda$. In

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Footnote:

For the light meson, we could have chosen the pion mass. However, since nuclear binding is largely due to correlated two–pion exchange, a somewhat larger value was chosen. All conclusions drawn in what follows are, however, invariant under the precise choice of this number.
principle, $\Lambda$ could take any value, in particular also above the larger of the two effective meson masses in the potential. We shall comment on that below. We are, however, mostly interested in an effective theory with small momenta only and thus we start by setting $\Lambda = 400$ MeV. To be more precise, by “small” we mean a scale which is below the mass of the exchanged heavy particle so that one can consider the situation with a propagating light meson and the heavy meson integrated out and substituted by a string of local contact terms (as will be discussed in more detail below).

The nonlinear equation is discretised using Gauss–Legendre quadrature points. Here we have chosen $a = 20$ keV, $b = 10$ keV and 100 Gauss–Legendre points. The shift in the binding energy due to the regularization is 0.012 KeV, which is a 0.01 permille effect. Note that it can be made smaller if so desired. The resulting $A(p, q)$ is shown in fig. 2. Here we have not shown the function $A(p, q)$ in the region of regularization, i.e. for $p < \Lambda + a$, $q > \Lambda - a$. In fact in this region the function $A(p, q)$ goes smoothly to zero. This is guaranteed by the regularization and can be seen explicitly from eq.(2.10).

3.2 Effective potential, scattering and bound states

Having calculated the operator $A$, we are now in the position to consider observables. First, we need the transformed Hamiltonian $H'$. The determination of $H'$ according to eq.(2.11) requires the calculation of $(q + A^\dagger A)^{-1/2}$. This is done in the following manner: as already described in section 2, we first define the function $B(q, q')$ as given in eq.(2.20). Consequently, this function has to satisfy the following nonlinear equation as follows from simple algebra:

$$
B(q, q') = -\frac{1}{2} \int_{p}^{\infty} p^2 \, dp \, A(p, q) A(p, q') - \frac{1}{2} \int_{0}^{\Lambda} \tilde{q}^2 \, d\tilde{q} \, B(\tilde{q}, q) B(\tilde{q}, q') \\
- \int_{p}^{\infty} p^2 \, dp \, \int_{0}^{\Lambda} \tilde{q}^2 \, d\tilde{q} \, A(p, q) A(p, \tilde{q}) \left( B(\tilde{q}, q') + \int_{0}^{\Lambda} \tilde{q}'^2 \, d\tilde{q}' \, B(\tilde{q}, \tilde{q}') B(\tilde{q}', q') \right). 
$$

3.2 Effective potential, scattering and bound states

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- \int_{p}^{\infty} p^2 \, dp \, \int_{0}^{\Lambda} \tilde{q}^2 \, d\tilde{q} \, A(p, q) A(p, \tilde{q}) \left( B(\tilde{q}, q') + \int_{0}^{\Lambda} \tilde{q}'^2 \, d\tilde{q}' \, B(\tilde{q}, \tilde{q}') B(\tilde{q}', q') \right).
$$
The function $B(q, q')$ defined in eq. (2.20) is obviously symmetric in the arguments $q, q'$: $B(q, q') = B(q', q)$. We have solved equation (3.7) by iteration, using the same Gauss–Legendre quadrature points as discussed before and starting with $B = -(1/2) A^\dagger A$. We have found a very fast convergence of the iteration method in this case. The integrations present in eq.(2.11) to determine $H'$ are performed by standard Gauss–Legendre quadratures and we end up with an effective potential $V'(q', q)$ defined for $q, q' \leq \Lambda$. It is displayed in fig. 3 in comparison to the original underlying potential for $\Lambda = 400$ MeV. Note that the region of the regularization is again not shown to keep the presentation clearer. One finds that the effective and the original potentials have a similar shape for momenta below the cut–off. The main effect of integrating out of high momentum components at the level of the potential seems to be given in this case just by an overall shift.

The solution of the effective LS equation (2.27) is now very simple since the integration is confined to $q \leq \Lambda$. The effective bound state wave function obeys a corresponding homogeneous integral equation

$$\frac{q^2}{m_N} \Psi(q) + \int_0^\Lambda \tilde{q}^2 \, d\tilde{q} \, V'(q, \tilde{q}) \Psi(\tilde{q}) = E \Psi(q), \quad (3.8)$$

with $m_N = 938.9$ MeV the nucleon mass. Using 40 quadrature points the resulting binding energy agrees within 9 digits with the result gained from the corresponding homogeneous equation driven by the original potential $V$ and defined in the whole momentum range. Furthermore, the $S$—wave phase shifts agree perfectly solving either eq.(2.28) in the full momentum space or eq.(2.27) in the space of low momenta only. This is shown in fig. 4. Note that due to the regularization the phase shifts go to zero for $q \to \Lambda$ and that in the figure, the phase shifts are shown as a function of the kinetic energy in the lab frame and the zero occurs at $T_{lab} = 2\Lambda^2/m_N = 341 \, (85)$ MeV for $\Lambda = 400 \, (200)$ MeV.

One can repeat this numerical exercise choosing other cut-off values. Setting for instance
\( \Lambda = 2 \text{ GeV} \), the corresponding function \( A(p, q) \) is shown in fig. 3 and the effective potential \( V'(q', q) \) is rather close to the original one, \( \frac{[V(q', q) - V'(q', q)]}{V(0,0)} \approx 0.02 \). Here we have chosen \( a = 200 \text{ keV} \) and \( b = 100 \text{ keV} \), which leads to the same shift in the deuteron binding energy as in the case with \( \Lambda = 400 \text{ MeV} \). The closeness of the effective potential with the original one for such a large cut-off value had to be expected, since the higher momenta play only a perturbative role. On the other hand for rather small cut-off values of \( \Lambda \), like e.g. 200 MeV, the effective two-nucleon potential is quite different from the original one. This is shown in fig. 6. Again for \( \Lambda = 200 \text{ MeV} \) the binding energy and the phase shifts agree perfectly using the effective and the original formulation, see fig. (4). We have found solutions up to \( \Lambda \approx 100 \text{ MeV} \). For smaller cut-off values, the effective potential develops so much structure that one would have to modify the method of solving the integral equation to determine \( A \). For the reasons mentioned above, we do not pursue this issue here any further. All that demonstrates that the nonlinear decoupling equation can be solved reliably and consequently, the effective Hamiltonian including all the physics can be determined to arbitrary high precision.

Figure 4: Phase shifts from the effective potential (red dashed line) and the original potential (green solid line) as a function of the kinetic energy in the lab frame. Left (right) panel: \( \Lambda = 400 \) (200) MeV.
Figure 5: The function $A(p, q)$ for $\Lambda = 2$ GeV.

The function $A(p, q)$ was expanded into cubic splines $S_k(q)$

$$A(p, q) = \sum_k S_k(q) A(p, q_k)$$

based on a set of suitably chosen gridpoints $\{q_k\}$ in the interval $[0, \Lambda]$. This procedure leads to a system of linear equations for the expansion coefficients $A(p, q_k)$, which is solved by standard methods. Consequently, the dynamical input is the S–wave t–matrix $T(q', q, E_q)$. The resulting $A(p, q)$ agrees perfectly with the one obtained by solving the nonlinear equation.

### 3.3 Implications for chiral effective field theories

Now let us establish some contact to the theory of nuclear forces in chiral perturbation theory. In this approach only the nucleons and lightest mesons (i.e. pions) are typically kept in the theory as explicit degrees of freedom. The structure of the pion–nucleon interaction is constrained by the spontaneously broken chiral symmetry. To take into account the effects of heavier degrees of freedom such as heavier mesons and baryon resonances, contact interactions between the nucleons are introduced in the corresponding Lagrangian. Such contact interactions are not constrained by chiral symmetry. Starting from the most general Lagrangian for pions and nucleons and using power–counting rules one can obtain formal expressions for the effective nucleon–nucleon potential (see for instance [2],[8]), which has to be put into the Lippmann–Schwinger equation to generate the corresponding S–matrix. Note that regularization and renormalization are needed to treat the ultraviolet divergent integrals in the Lippmann–Schwinger equation. Furthermore, the pionic loops in the effective NN–potential should be regularized consistently with the regularization of the Lippmann–Schwinger equation. This regularization is necessarily nonperturbative: the presence of a low–energy bound state signals the failure of perturbation theory. Once the theory is regularized (for instance with some cut–off) all parameters in the Lagrangian may be fixed by fitting to data, like e.g. the low energy NN partial waves and deuteron properties. It is
commonly believed that the coupling constants scaled by some effective mass parameter \( \Lambda_{\text{scale}} \) should have the property of "naturalness", which means they should be of the order of one. Only then this expansion makes sense and the power-counting is self-consistent. The value of the scale \( \Lambda_{\text{scale}} \) is obviously closely connected to the radius of convergence of this expansion. Let us first consider the simpler case when the pionic degrees of freedom are integrated out. In that case, the low-energy NN interaction can be described entirely in terms of contact terms and one expects the scale \( \Lambda_{\text{scale}} \) to be of the order of the pion mass \( m_\pi \). Therefore, all physical parameters which describe the NN phase shifts up to center of mass momenta of the order \( m_\pi \), such as the scattering length \( a \) and the effective range \( r_e \), are expected to scale like appropriate inverse powers of \( m_\pi \). This is, however, not the case in nuclear physics: the NN scattering length in the \(^1S_0\)-channel takes an unnatural large value, \( a = (-23.714 \pm 0.013) \text{ fm} \gg 1/m_\pi \). As a consequence, the expansion in powers of small momenta breaks down at momenta much below the scale \( m_\pi \). The physics of this phenomenon is well understood (there is a virtual bound state very near zero energy) and amounts to a fine tuning between different terms when the corresponding phase shifts are calculated \[4\]. To achieve such cancellations in the effective field theory calculations one has to "fine tune" the parameters. We shall see below how this "fine tuning" works in our model.

The situation is more complicated in the effective theory with pions since different scales appear explicitly. That is why it is not clear a priori what scale enters the coefficients in the Lagrangian. Using a modified dimensional regularization scheme and renormalization group equation arguments, the authors of \[4\] have argued that \( \Lambda_{\text{scale}} \sim 300 \text{ MeV} \). On the other hand it was argued by the Maryland group \[5\] that no useful and systematic effective field theory (EFT) exists for two nucleons with a finite cut-off as a regulator. A systematic EFT is to be understood in the sense that the contributions of the higher-order terms in the effective Lagrangian to the observables at low momenta (i.e. the quantum averages of such operators)
are small and therefore truncation of such operators at some finite order is justified. Within our realistic model, we can address this question in a quantitative manner as shown below.

Let us now apply the concept of the effective theory to our case. The original potential plays the role of the underlying theory of NN interactions in analogy to QCD underlying the true EFT of NN scattering. Integrating out the momenta above some cut–off $\Lambda < \mu_H$, we arrive via the unitary transformation at the effective potential $V'(q, q')$. In analogy to the true EFT we decompose $V'$ into two parts:

$$V'(q', q) = V'_{\text{light}}(q', q) + V'_{\text{contact}}(q', q). \quad (3.10)$$

Here, $V'_{\text{contact}}$ is a string of local contact terms of increasing dimension, which is caused by the heavy mass particle and the high momenta $p > \Lambda$. The piece $V'_{\text{light}}$ is related to the light meson exchange, but again it is modified by integrating out the high momenta and in general it will depend on the order to which we expand $V'_{\text{contact}}$. However, since we work in a model which serves as an illustration, we simplify the procedure and keep $V'_{\text{light}}$ fixed as the light meson exchange: $V'_{\text{light}} = V_{\text{light}}$, where $V_{\text{light}}$ denotes the second term in eq. (3.2). Specifically,

$$V_{\text{contact}} = V^{(0)} + V^{(2)} + V^{(4)} + V^{(6)} + \ldots, \quad (3.11)$$

with

$$
\begin{align*}
V^{(0)} &= C_0, \\
V^{(2)} &= C_2(q'^2 + q^2), \\
V^{(4)} &= C_4(q'^2 + q^2)^2 + C'_4 q'^2 q^2, \\
V^{(6)} &= C_6(q'^2 + q^2)^3 + C'_6(q'^2 + q^2) q'^2 q^2,
\end{align*} \quad (3.12)
$$

and the superscript ’(2n)’ gives the chiral dimension (i.e. the number of derivatives). Thus the first term on the right hand side of eq.(3.10) is the purely attractive part of the original potential due to the light meson exchange. In this way, we have an effective theory for NN interactions, in which the effects of the heavy meson exchange and the high momentum components are approximated by the series of NN contact interactions and the light mesons are treated explicitly. Note that the constants $C_i, C'_i$ in eq. (3.12) correspond to renormalized quantities since the effective potential $V'(q', q)$ is regularized with the sharp cut–off $\Lambda$. The first question we address in our model is: what is the value of the scale $\Lambda_{\text{scale}}$ and its relation to the cut-off $\Lambda$? Of course, a priori these two scales are not related. For the kind of questions we will address in the following, we can, however, derive some lose relation between these two scales as discussed below. Since we know $V'(q', q)$ numerically, we can determine the constants $C_i$ by fitting the eq.(3.11) to $V'(q', q) - V_{\text{light}}(q', q)$. This is done numerically using the standard FORTRAN subroutines for polynomial fits to functions of one variable and taking the corresponding polynoms typically of order ten to eleven. Once this is done for the functions $V(q, 0)$ and $V(q, q)$ all constants $C_i, C'_i$ in eq. (3.12) can be easily evaluated. For the potential parameters used so far and the choices of $\Lambda = 400$ MeV and $\Lambda = 300$ MeV, the resulting constants are given in Table 1:

Naturalness of the coupling constants $C_i, C'_i$ means that

$$\frac{C_{2n}}{C_{2n+2}} = a_n \Lambda_{\text{scale}}^2, \quad (3.13)$$

where the $a_n$ are numbers of order one. Indeed, as one can read off from table [1], such a common scale exists, namely

$$\Lambda_{\text{scale}} = 600 \text{ MeV}. \quad (3.14)$$
Table 1: The values of coupling constants $C_i, C'_i$ in [GeV$^{-2-i}$] for two choices of the cut–off $\Lambda$.

| $\Lambda$ | $C_0$ | $C_2$ | $C_4$ | $C'_4$ | $C_6$ | $C'_6$ |
|----------|-------|-------|-------|--------|-------|--------|
| 400 MeV  | 11.23 | -32.27| 86.73 | 113.9  | -232.6| -913.6 |
| 300 MeV  | 11.15 | -32.93| 85.76 | 115.6  | -265.2| -783.7 |

This is a reasonable value in the sense that it is very close to the mass $\mu_H$ of the heavy mesons, which is integrated out from the theory. Stated differently, the value for $\Lambda_{\text{scale}}$ agrees with naive expectations. In fact, in the full theory (i.e. without the projection into the space of small momenta only), this observation would be trivial since the expansion of the heavy meson propagator gives

$$\frac{1}{t + \mu_H^2} = \frac{1}{\mu_H^2} \left( 1 - \frac{t}{\mu_H^2} + \frac{t^2}{\mu_H^4} + \ldots \right),$$

and thus $\Lambda_{\text{scale}} = \mu_H$. In our approach, however, the large momentum components are mapped

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{Effective two–nucleon potential $V'(q', q)$ (solid green lines) in comparison with the truncated expansion (3.11) (dashed blue lines) for $\Lambda = 300$ MeV.}
\end{figure}

into the Hilbert space of small momenta and thus a priori it is not obvious that $\Lambda_{\text{scale}}$ is indeed given by $\mu_H$. As it turns out, even for cut–off values like $\Lambda = 300$ MeV, there is only a small difference between the heavy meson mass and the ensuing natural mass scale. Another observation is that the values of the $C_i, C'_i$ depend very weakly on the concrete choice of the cut–off $\Lambda$. Clearly, for such an expansion of the heavy mass exchange in terms of contact interactions to make sense, $\Lambda$ has to be chosen below $\mu_H$. Furthermore, since we explicitly keep the light meson, $\Lambda$ should not be smaller than the mass $\mu_L$. If one were to select such a value for the cut–off, one could also consider the possibility of expanding the light meson exchange in a string of contact terms. We do, however, not pursue this option in here. Therefore, we conclude that
one should set $\Lambda < \Lambda_{\text{scale}}$ but it is not possible to find a more precise relation. In fig. 7 we show how well the potential $V'(q', q)$ is reproduced by the truncated expansion eq. (3.11) for $\Lambda = 300$ MeV.

We have also calculated the two-body binding energy and the phase shifts using the form eq. (3.11) with the constants given in table 1. The corresponding results are shown in fig. 8 and in table 2. The agreement with the exact values is good. However, one sees that terms of rather high order should be kept in the effective potential $V_{\text{contact}}$ in order to have the binding energy correct within a few percent. Note, however, that the value of the binding energy is unnaturally small on a typical hadronic scale like the pion mass or the scale of chiral symmetry breaking. The phase shifts are described fairly well for kinetic energies (in the lab) up to about 100 MeV if one retains the first three terms in the expansion eq. (3.11). For $\Lambda = 400$ MeV, one can not expect any reasonably fast convergence for the binding energy any more. This can be traced back to the fact that one is close to the radius of convergence for momenta close to the cut-off. More specifically, with $q = q' = 400$ MeV, the pertinent expansion parameter is $(q'^2 + q^2)/\Lambda_{\text{scale}}^2 \simeq 0.9$. The ensuing very slow convergence is exhibited in table 2.

| $V^{(0)}$ | $V^{(0)} + V^{(2)}$ | $V^{(0)} + V^{(2)} + V^{(4)}$ | $V^{(0)} + V^{(2)} + V^{(4)} + V^{(6)}$ | $V_{\text{contact}}$ |
|---|---|---|---|---|
| E [MeV] | 0.46 | 3.18 | 1.95 | 2.29 | 2.23 |
| E [MeV] | 0.67 | 7.15 | 1.82 | 3.15 | 2.23 |

Table 2: The values of the binding energy calculated with $V_{\text{contact}}$, eq. (3.11), for $\Lambda = 300$ MeV (second row) and $\Lambda = 400$ MeV (third row).

Figure 8: Phase shifts from the effective potential $V'$ (solid line) and the truncated expansion (3.11) as a function of the kinetic energy in the lab frame for $\Lambda = 300$ MeV. Left (right) panel: the constants $C_i, C_i'$ are fitted to the effective potential (to the NN phase shift).

Although we have found that $\Lambda_{\text{scale}} \sim 600$ MeV and therefore the expansion of $V'$ in terms of contact terms seems to converge for the chosen value of the cut-off $\Lambda = 300$ MeV, the ultimative test of convergence of the expansion (3.11) would be to calculate the quantum averages of operators $V^{(0)}, V^{(2)}, V^{(4)}, V^{(6)}, \ldots$, as it was proposed in [5]. This allows to directly draw conclusions about the size of such operators in the effective action. In fact, the potential is not an observable and naturalness of its coefficients can become meaningless when one iterates...
the potential in the LS equation since in general large momenta are involved. In our case, this can not happen and in the worst case, the convergence for observables will be controlled by the parameters $\Lambda/\Lambda_{\text{scale}}$ multiplied by coefficients of order one (with the exception of quantities which require strong fine tuning). These remarks will be substantiated by the results to be presented below. Because of technical difficulties, the authors of ref.\[5\] have used for this check the bound–state wave function in the $^1S_0$–channel, obtained from the zeroth–order potential and without pions (in real world there is, of course, no bound state in the $^1S_0$–channel). This allows only for a very rough estimate of the size of the operators in eq. (3.11). In our model we can perform numerically exact calculations of these quantities using not only the bound–state wave function but also the scattering wave functions. Using the relation eq.(2.33) one obtains for an arbitrary operator $O$

$$
\langle \Psi'_{q'}^{(+)}|O|\Psi'_{q}^{(+)}\rangle = \langle q'|O|q\rangle + I_1 + I_1' + I_2,
$$

(3.16)

with

$$
I_1 = \int q'^2 dq' O(q', q') \frac{T'(q', q, E_q)}{E_q - E_{q'} + i\epsilon},
$$

$$
I_2 = \int q'^2 dq' O(q', q') \frac{T'^*(q', q, E_q)}{E_q - E_{q'} + i\epsilon} \frac{T'(q', q, E_q)}{E_q - E_{q'} + i\epsilon}.
$$

(3.17)

The results for quantum averages of the operators $V^{(2n)}$, ($n = 0, 1, 2, 3$), are shown in table 3. One observes a good convergence in agreement with the naive expectation. Indeed, since the cut–off is choosen to be $\Lambda = 300$ MeV and the scale $\Lambda_{\text{scale}} \sim 600$ MeV, one expects the expansion parameter to be of the order $\Lambda/\Lambda_{\text{scale}} \simeq 1/2$. Such a value agrees well with the one extracted from the results shown in table 3. Note further that for higher energies, the convergence is slower, which is also rather natural.

| $E_{\text{lab}}$ (MeV) | $\langle \Psi|V^{(0)}|\Psi\rangle$ | $\langle \Psi|V^{(2)}|\Psi\rangle$ | $\langle \Psi|V^{(4)}|\Psi\rangle$ | $\langle \Psi|V^{(6)}|\Psi\rangle$ |
|-----------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| deuteron              | 25.94 MeV                       | -4.23 MeV                       | 1.07 MeV                        | -0.38 MeV                       |
| $E_{\text{lab}} = 10$ MeV | 31.34 GeV$^{-2}$               | -6.19 GeV$^{-2}$               | 1.64 GeV$^{-2}$                | -0.58 GeV$^{-2}$               |
| $E_{\text{lab}} = 50$ MeV | 11.54 GeV$^{-2}$               | -3.28 GeV$^{-2}$               | 1.11 GeV$^{-2}$                | -0.44 GeV$^{-2}$               |
| $E_{\text{lab}} = 100$ MeV | 7.79 GeV$^{-2}$                | -3.09 GeV$^{-2}$               | 1.40 GeV$^{-2}$                | -0.71 GeV$^{-2}$               |

Table 3: The quantum averages of the operators $V^{(0)}$, $V^{(2)}$, $V^{(4)}$ and $V^{(6)}$ for the bound (second row) and the scattering states (third to fifth rows) for $\Lambda = 300$ MeV.

In the real world to which one applies the effective field theory, one does not know the true effective potential $V'$ and therefore also the constants $C_i, C'_i$ are unknown. They have to be determined by fitting such a type of effective potential to the NN data, like the low energy phase shifts and/or the deuteron binding energy. We can perform this exercise also in our case. Keeping again $V_{\text{light}}(q', q)$ as a seperate piece we have determined by trial and error the constants $C_0, C_2, \ldots$ by solving the homogeneous and inhomogeneous LS equations and truncating the series at various orders. The results can be summarized as follows: For $\Lambda = 300$ MeV we restrict the fit to the phase shifts to momenta below 260 MeV. In that way, we avoid the distorsion of the phase shifts at the edge of its kinematically allowed region in the model space. A fit including only the first four terms, i.e. the constants $C_0, C_2, C_4$ and $C'_4$ gives an excellent reproduction of the phase shifts as shown in the right panel of fig. 8. The corresponding constants are $C_0 = 10.74,$
$C_2 = -23.96$, $C_4 = 103.7$ and $C'_4 = -288.4$, all (with the exception of $C'_4$) within $25\%$ of the exact values. The deviation of the fitted value for $C'_4$ from the exact one reminds us of the fact that such fine tuning can produce sizeable uncertainties in higher orders in such type of cut–off schemes and thus the interpretation of such values has to be taken with some caution. The bound state energy, which is not fitted, turns out to be $2.27$ MeV, just $2\%$ above the exact value. If one includes the sixth order terms in the fit, the results for the fourth and sixth order coefficients become unstable. That can be traced back to the very small contribution of these terms to the phase shift as long as $q < 260$ MeV. A stable fit can be obtained by demanding that the coefficient obey naturalness within a certain range, say within $20\%$ or so. Clearly, to use polynomials of such high order to fit the smooth phase shifts which are already very well approximated by the first four terms does not make much sense. It is interesting to observe that the fit itself tends to limit the terms in the chiral expansion giving a good description of the phase shifts and binding energy. This behaviour is reminiscent of the one found in the dispersion–theoretical analysis of the nucleon form factors, where one gets a best fit to the hadronic spectral function with a limited number of vector meson poles (for details, see e.g., [21],[22]).

![Effective two–nucleon potential](image)

Figure 9: Effective two–nucleon potential (solid green lines) in comparison with the original potential (dashed blue lines) for momenta less than $300$ MeV in the $^1S_0$ channel.

One can also perform a similar analysis for the two nucleon $^1S_0$ channel, which is expected to be even more troublesome than the $^3S_1$ channel for the effective field theory approach since, as already stated before, the scattering length takes an unnatural large value. The $^1S_0$ two–nucleon phase shift is satisfactorily reproduced by choosing the following set of parameters for the potential eq. (3.1): $V_H = 7.291$, $V_L = 2.605$, $\mu_H = 613.7$ MeV and $\mu_L = 305.9$ MeV. All parameters with the exception of $V_L$ remain the same as for the $^3S_1$ channel. $V_L$ is chosen somewhat smaller to make the attraction weaker. With this parameters one finds for the scattering length $a = -23.6$ fm, which is rather close to the empirical value. The reason for this large value is the virtual bound state of almost zero energy, which we, of course, also recover. We have calculated the effective potential for $A = 300$ MeV, as shown in fig. [3]. Again
the dominant effect of integrating out the higher momenta seems to be given by an overall shift. Using the same decomposition as in eq. (3.10), we have again expanded $V'_{\text{contact}}$ into a string of local contact terms of increasing dimension, cf. eq. (3.12). Keeping only the terms up to fourth order, we have fixed the values of the constants $C_i$ by fitting eq. (3.11) to $V'(q', q) - V_{\text{light}}(q', q)$. The corresponding constants are $C_0 = 10.62 \text{ GeV}^{-2}$, $C_2 = -32.11 \text{ GeV}^{-4}$, $C_4 = 85.8 \text{ GeV}^{-6}$ and $C'_4 = 117 \text{ GeV}^{-6}$, rather close to the constants from table [1]. (The reason for this is that the term in the potential corresponding to the heavy meson exchange is not modified.) Consequently, all conclusions about the size of the scale $\Lambda_{\text{scale}}$ from the analysis of the $^3S_1$ channel are valid in this case as well. The corresponding phase shifts are shown in fig. [10]. Again

![Figure 10: Phase shifts from the effective potential $V'$ (solid line) and the truncated expansion eq. (3.11) as a function of the kinetic energy in the lab frame for $^1S_0$ channel for $\Lambda = 300 \text{ MeV.}$.](image)

As in the case of the $^3S_1$ channel, one sees that the phase shift is described well up to about 100 MeV if the terms up–to–and–including fourth order in the expansion eq. (3.11) are kept explicitly. To further study the convergence of the contact term expansion eq. (3.11) in the $^1S_0$ channel, we have again calculated the expectation values of operators $V^{(0)}$, $V^{(2)}$ and $V^{(4)}$ using the scattering states for different energies. As exhibited in table [4] the values of the quantum averages are slightly different from the ones for the $^1S_0$ channel and the expansion eq. (3.11) is again convergent. However, one observes a much slower convergence for the scattering length,

| $E_{\text{lab}}$ (MeV) | $\langle \Psi | V^{(0)} | \Psi \rangle$ (GeV$^{-2}$) | $\langle \Psi | V^{(2)} | \Psi \rangle$ (GeV$^{-2}$) | $\langle \Psi | V^{(4)} | \Psi \rangle$ (GeV$^{-2}$) |
|------------------------|----------------------|----------------------|----------------------|
| 10                     | 37.42                | -4.68                | 0.90                  |
| 50                     | 11.72                | -2.56                | 0.68                  |
| 100                    | 8.82                 | -2.98                | 1.18                  |

Table 4: Quantum averages of the operators $V^{(0)}$, $V^{(2)}$ and $V^{(4)}$ for the scattering states with $\Lambda = 300 \text{ MeV}$ in the $^1S_0$ channel.

again because of its unnatural large value. Keeping $V^{(0)}$, $V^{(0)} + V^{(2)}$ and $V^{(0)} + V^{(2)} + V^{(4)}$ in the
expansion for $V_{\text{contact}}$ one finds $a = -9.0 \text{ fm}$, $a = -54.5 \text{ fm}$ and $a = -21.2 \text{ fm}$, respectively. This is analogous to the situation with the binding energy in the $^3S_1$ channel, compare table [2] (some recent references dealing with the particular features of this partial wave in the EFT approach are [3],[4],[5]). Certainly, one can achieve a considerably faster convergence with respect to the scattering length by fine tuning the constants $C_i$ (i.e. by fixing them from a direct fit of the effective potential to the phase shift), as it was illustrated for the $^3S_1$ channel.

3.4 Coordinate space representation

As discussed before, applying the method of unitary transformation turns local operators in momentum space into non–local ones. Similar effects happen in the coordinate space representation. To make this transparent, we consider here the configuration space representation of the effective potential $V'$. Denote by $V_l(p', p)$ the effective momentum space potential for angular momentum $l$. The corresponding $r$–space expression is obtained from

$$V_l(x', x) = \frac{2}{\pi} \int p^2 dp p'^2 dp' j_l(px) V_l(p', p) j_l(p'x'), \quad (3.18)$$

with $j_l(y)$ the conventional $l^{th}$ spherical Bessel function. The S–wave potential ($l = 0$) for $\Lambda = 400 \text{ MeV}$ is shown in the left panel of fig. [11]. It is, of course, symmetric under the interchange of $x$ and $x'$, but looks very different from a local potential.

Figure 11: Coordinate space representation of the effective S–wave potential $V_0(x, x')$. Left panel: $\Lambda = 400 \text{ MeV}$ and $0 \leq x, x' \leq 6 \text{ fm}$. Right panel: $\Lambda = 5.5 \text{ GeV}$ and $1 \leq x, x' \leq 2 \text{ fm}$.

Of course, if one increases the value of the the cut-off $\Lambda$, a peak along the diagonal $x = x'$ resembling the delta function should develop and the superposition of the two Yukawa potentials related to the light and heavy meson exchanges, respectively, should appear along the diagonal. This is indeed the case as demonstrated in the right panel of fig. [11] for $\Lambda = 5.5 \text{ GeV}$. Note also that in this case, where $U \sim 1$, the range of the potential is essentially given by the inverse of the light meson mass. One can also construct the momentum and coordinate representations of the deuteron wave function. For a momentum space picture, we refer to ref. [13].
4 Summary and conclusions

In this paper, we have shown how to construct an effective low energy theory for nucleons based on the method of unitary transformations starting from a realistic two–nucleon potential (in momentum space). This unitary transformation can be parametrized by an operator $A$, which obeys a nonlinear integral equation. This equation can be solved numerically and any observable can then be calculated in the space of small momenta only. To the best of our knowledge such an exact momentum space projection has never been done before. While the method is interesting per se, we have also made contact to chiral perturbation theory (CHPT) approaches to the two–nucleon system by studying a series of questions, which can be addressed unambiguously within the framework of our exact low momentum theory. Clearly, this should not be considered a substitute for a realistic CHPT calculation but might be used as a guide. The salient results of this investigation can be summarized as follows:

1) We have demonstrated that the theory projected onto the subspace of momenta below a given momentum space cut–off $\Lambda$ leads to exactly the same S–matrix as the original theory in the full (unrestricted) momentum space provided appropriate boundary conditions for the scattering states are chosen. In particular, the components of the transformed scattering states with initial momenta below the cut–off $\Lambda$ in the subspace of momenta above the cut–off $\Lambda$ are strictly zero. It is important to stress that the exact projection leads to non–localities in momentum space.

2) Starting from an S–wave NN potential with an attractive light ($\mu_L \simeq 300$ MeV) and repulsive heavy meson exchange ($\mu_H \simeq 600$ MeV), we have numerically solved without any approximation the nonlinear equation for the operator $A$ and demonstrated that the bound and scattering state spectrum of the effective and the full theory agree exactly up to the cut–off $\Lambda$. In particular, we have exactly one bound state with a binding energy of 2.23 MeV. These results are independent of the value of the cut–off, which was varied from 200 MeV to 5.5 GeV. We have argued that the most natural choice is $\Lambda$ about 300 MeV. The effective potential can differ substantially from the original one (for values of $\Lambda$ on the small side of the range mentioned before).

3) We have expanded the heavy meson exchange term in a string of local operators with increasing dimension but kept the light meson exchange explicitly. The corresponding coupling constants accompanying these local operators, which are monomials of even power in the momenta, can be determined precisely from the exact solution. We have shown that they are of “natural” size, i.e. of order one, with respect to the mass scale $\Lambda_{\text{scale}} = 600$ MeV. We have also discussed the relation of this scale to the mass of the heavy meson, which is integrated out, and the convergence properties of such type of expansion. In particular, to recover the binding energy within a few percent, one has to retain terms of rather high order in this expansion, cf. eq.(3.11). This is to be expected due to the unnatural smallness of this energy on any hadronic mass scale. The $^3S_1$ scattering phase shift can be well reproduced up to kinetic energies $T_{\text{lab}} \simeq 120$ MeV with the first three terms in the contact term expansion.

4) Based on the expanded heavy meson exchange term, we have also determined the constants $C_i$ directly from a fit to the phase shifts. This is equivalent to the procedure performed in an effective field theory approach. We could show that as long as one does not include polynomials of order six (or higher), the resulting values of these constants are close to their exact ones. Furthermore, the binding energy is reproduced within 2%. Including
dimension six terms, the fits become unstable. This can be traced back to the fact that the contribution of such terms to the phase shifts are very small (at low and moderate energies) and thus can not really be pinned down.

5) We have also studied the quantum averages of the expanded potential in the bound and scattering states. For $\Lambda = 300\,\text{MeV}$, the expansion parameter is of the order of $1/2$ and we find fast convergence for the bound and the low–lying scattering states. As expected, for scattering states with higher energy, the convergence becomes slower.

6) To study the $^1S_0$ channel, we had to slightly readjust the parameters of the model potential. The phase shift can be well reproduced with the terms up–to–and–including fourth order in the contact term expansion of the heavy meson exchange. For the scattering states, the quantum averages of the expanded potential show convergence properties similar to the $^3S_1$ case. There is no bound state in the $^1S_0$, but a virtual one just above threshold. Therefore, the pertinent scattering length is unnaturally large and it shows a similar slow convergence as does the binding energy in the $^3S_1$ channel.

7) In the model space of small momenta only, one can also study the non–localities in the coordinate space representation. We have shown that for typical cut–off values, the effective potential $V(x, x')$ is highly non–local and looks very different from the original one. For very large values of the cut–off, one recovers the original local potential.

We hope that this study might be useful for derivation of NN–forces based on chiral Lagrangians in the low–momentum regime. It should also provide new insights into a consistent and convergent treatment of relativistic effects in few– and many–nucleon systems.

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References

[1] S. Weinberg, Phys. Lett. B251 (1990) 288; Nucl. Phys. B363 (1991) 3.
[2] C. Ordóñez, L. Ray and U. van Kolck, Phys. Rev. Lett. 72 (1994) 1982; Phys. Rev. C53 (1996) 2086; U. van Kolck, Phys. Rev. C49 (1994) 2932.
[3] D.B. Kaplan, M.J. Savage and M.B. Wise, Nucl. Phys. B478 (1996) 629.
[4] D.B. Kaplan, M.J. Savage and M.B. Wise, Phys. Lett. B424 (1998) 390; nucl-th/9802075.
[5] S.R. Beane, T.D. Cohen and D.R. Phillips, Nucl. Phys. A632 (1998) 445.
[6] N. Kaiser, R. Brockmann and W. Weise, Nucl. Phys. A625 (1997) 758; N. Kaiser, S. Gerstendoerfer and W. Weise, nucl-th/9802071.
[7] M.R. Robilotta and C.A. da Rocha, Nucl. Phys. A615 (1997) 391; J.-L. Ballot, M.R. Robilotta and C.A. da Rocha, Phys. Rev. C57 (1998) 1574.
[8] E. Epelbaoum, W. Glöckle and Ulf-G. Meißner, Nucl. Phys. A637 (1998) 107.
[9] T.-S. Park, K. Kubodera, D.-P. Min and M. Rho, nucl-th/9807054.
[10] S.R. Beane, nucl-th/9806070.
[11] S. Okubo, Prog. Theor. Phys. 12 (1954) 603.
[12] N. Fukuda, K. Sawada and M. Taketani, Prog. Theor. Phys. 12 (1954) 156.
[13] K. Suzuki and R. Okamoto, Prog. Theor. Phys. 70 (1983) 439; Prog. Theor. Phys. 92 (1994) 1045.
[14] K. Suzuki and R. Okamoto, private communication.
[15] T. H. Schucan and H. A. Weidenmüller, Ann. Phys. 73 (1972) 108.
[16] E. Epelbaoum, W. Glöckle and Ulf-G. Meißner, nucl-th/9804005, Phys. Lett. B439 (1998) 1.
[17] W. Glöckle, G. Hasberg and A. R. Neghabian, Z. Phys. A305 (1982) 217.
[18] T.T.S. Kuo, Lecture Notes in Physics, Vol. 144, p. 248 (Springer Verlag, 1981).
[19] Ch. Elster, J.H. Thomas and W. Glöckle, Few Body Syst. 24 (1998) 55.
[20] U. van Kolck, hep-ph/9711222.
[21] G. Höhler et al., Nucl. Phys. B114 (1976) 505.
[22] P. Mergell, Ulf-G. Meißner and D. Drechsel, Nucl. Phys. A596 (1996) 367.