Quasi-perfect state transfer in a bosonic dissipative network

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Abstract

In this paper we propose a scheme for quasi-perfect state transfer in a network of dissipative harmonic oscillators. We consider ideal sender and receiver oscillators connected by a chain of nonideal transmitter oscillators coupled by nearest-neighbour resonances. From the algebraic properties of the dynamical quantities describing the evolution of the network state, we derive a criterion, fixing the coupling strengths between all the oscillators, apart from their natural frequencies, enabling perfect state transfer in the particular case of ideal transmitter oscillators. Our criterion provides an easily manipulated formula enabling perfect state transfer in the special case where the network nonidealities are disregarded. We also extend such a criterion to dissipative networks where the fidelity of the transferred state decreases due to the loss mechanisms. To circumvent almost completely the adverse effect of decoherence, we propose a protocol to achieve quasi-perfect state transfer in nonideal networks. By adjusting the common frequency of the sender and the receiver oscillators to be out of resonance with that of the transmitters, we demonstrate that the sender's state tunnels to the receiver oscillator by virtually exciting the nonideal transmitter chain. This virtual process makes negligible the decay rate associated with the transmitter line at the expense of delaying the time interval for the state transfer process. Apart from our analytical results, numerical computations are presented to illustrate our protocol.

1. Introduction

A great deal of attention has been devoted recently to the subject of perfect state transfer (PST) in quantum networks. Since an actual quantum processor would require, in fact, the ability to transfer quantum information between spatially separated interacting systems composing a network, protocols have been established for PST in many different and general contexts. Among the several interesting theoretical contributions aiming to advance the understanding and eventual implementation of a quantum processor, the construction of effective two-qubit gates from PST between distant nodes in engineered bosonic and fermionic networks was proposed [1]. A general formalism of the problem of PST in networks of any topology and coupling configuration was also developed [2]. Focusing on spin chains, PST has been pursued in networks extending beyond the nearest-neighbour couplings [3], and a class of qubit networks allowing PST of any state in a fixed period of time has been devised [4]. The problem of the scaling of errors (arising from network nonidealities) with the length of the channel connecting the nodes has been discussed [5]. In this connection, a protocol for arbitrary PST in the presence of random fluctuations in the coupling strengths of a spin chain was reported [6]. The entanglement dynamics in spin chains subject to noise and disorder has also been analysed in [7] and, more recently, percolation strategies based on multipartite measurements have been presented to propagate entanglement in quantum networks [8]. Regarding PST in networks of harmonic oscillators, a comprehensive analysis of this subject has been...
presented in [9], and a protocol for high-efficiency transfer of quantum entanglements in translation-invariant quantum chains has been proposed [10]. In contrast to the achievements with spin chains, there have been no proposals, until this paper, for arbitrary PST through a network of nonideal harmonic oscillators. However, it must be observed that there are classes of noise, such as dephasing or even relaxation, that should map between spin chains and bosonic networks.

As well as in spin chains, significant advances have been made recently in bosonic networks [11–14]. A general treatment of a network of coupled dissipative quantum harmonic oscillators has been presented recently, for an arbitrary topology, i.e. irrespective of the way the oscillators are coupled together, the strength of their couplings and their natural frequencies [15]. Regarding the dissipative mechanism, two different scenarios are considered in [15]: a more realistic one in which each oscillator is coupled to its own reservoir and another with all the network oscillators coupled to a common reservoir. Within such a general treatment of dissipation, the emergence of relaxation- and decoherence-free subspaces in networks of weakly and strongly coupled resonators has also been addressed [16]. We finally mention the proposition of a quantum memory for the preservation of superposition states against decoherence by their evolution in appropriate topologies of such dissipative bosonic networks [17].

Since PST is achieved by appropriate tuning of the intermode coupling and the frequencies of the systems composing the network, the search for the general rules governing these adjustments is of crucial interest. In this regard, it is worth noting that the general recipe presented in [2] contrasts with the procedure in [10] where a degenerate chain of oscillators is considered: all oscillators having the same frequency and interacting with each other with the same coupling strength. The lack of PST in [10] is compensated for by a more realistic possibility of implementation of a translation-invariant network, apart from the high-efficiency transfer operation reported. In the present contribution we also derive general conditions for PST in ideal networks, alternative to that outlined in [2]. Moreover, we demonstrate that the general conditions (or criterion) for PST in ideal networks can be extended to nonideal ones, where dissipation is assumed. Evidently, in this case the fidelity of the transferred state decreases due to environmental effects. However, building on the achievements for bosonic networks mentioned above, in the present study we develop a protocol for weakening the undesired environmental effects, leading to quasi-perfect state transfer (QPST) in a linear network of dissipative oscillators. More specifically, we envisage the transfer of a state between ideal sender and receiver oscillators through a linear chain of nonideal transmitter oscillators. By analogy with [15, 16], here we again adopt the general and more realistic scenario where each transmitter oscillator is coupled to a distinct reservoir. Therefore, our goal resembles that of arbitrary PST in [6]; however, instead of fluctuations in the couplings of a spin chain, we deal with the fluctuations injected by each of the reservoirs coupled to the transmitter oscillators. Anticipating our strategy to achieve QPST despite these sources of nonideality, we adjust the frequency of the sender and the receiver oscillators to be significantly out of resonance with that of the transmitters; within such an arrangement the state to be transferred occupies the transmitter oscillators only virtually, weakening the undesired effects of their decay mechanisms.

Our tunnelling-like mechanism for QPST is similar to that presented in [10] and other related developments [19, 20], in that our network acts as a quantum bus connecting the sender to the receiver systems. However, in [10, 19, 20] the authors consider disorder and random coupling strengths in the chain instead of considering the reservoirs as we have done. We also note that a completely different way to approach the problem of independent baths, developed in the context of a spin chain, is given in [18]. Moreover, it has to be emphasized that our development bears similarities with the inverse eigenvalue problems and mirror conditions already discussed in [1, 3]. Here we add that the inverse eigenvalue problem turns out to be a sufficient condition of our criterion enabling perfect state transfer in the particular case of ideal transmitter oscillators; our necessary condition thus is a simplification of the inverse eigenvalue problem. In addition, we remark that the inverse eigenvalue problems are treated in [1, 3] only in the context of ideal networks, whereas here we also extend our criterion to nonideal networks. We finally stress that the state transfer protocol that is going to be examined, based on networks of nonideal quantum oscillators, is not limited to the single excitation subspace as those based on spin chains.

We point out that the recent proposal of a variety of resources for wiring up quantum systems [21] lends a strongly realistic bias to the possibility of controlling the transfer of information in quantum networks. Following the mastering of the manipulation of the interaction between single atoms and vibrating modes of high-Q cavities [22] and trapped ions [23], circuit cavity quantum electrodynamics and photonic crystals seem to enhance the ability to transfer quantum information to a level enabling the implementation of a logic processor [24]. In this connection, the elaboration of schemes for weakening the noise injection in the processes of state transfer in networks of coupled nonideal quantum systems is indispensable, enabling protocols for fault-tolerant information transfer and enhancing our understanding of fundamental quantum phenomena such as entanglement and decoherence.

The plan of the paper is as follows. In section 2, by analogy with the developments presented in [15], we revisit the treatment of nonideal bosonic networks, introducing the Hamiltonian model. The master equation describing the network dynamics is presented, together with its solution. In section 3, we present a criterion for PST in ideal networks of harmonic oscillators, together with a particular application where the sets of parameters \( \{ \omega_n \} \) and \( \{ \lambda_{mn} \} \) ensuring PST are derived. Our criterion relies on the definition of the matrix \( \Theta(t) \), describing the evolution of the network state. Therefore, the matrix \( \Theta(t) \) serves two purposes: to derive a criterion ensuring PST and to compute the exchange time \( t_{\text{ex}} \).

By exchange time we mean the time at which the state prepared in the sender reaches the receiver (with the fidelity being unity
for ideal networks and less than unity for nonideal ones). Still in section 3, we extend our criterion for PST in ideal networks to the nonideal case, where dissipation is considered, affecting the fidelity of the transferred state. In section 4 we present our main contribution: a protocol for weakening the environmental effects. We introduce a model in which two ideal oscillators (the sender and the receiver) are connected by a transmission line of nonideal oscillators (the transmitters). As anticipated above, we adjust the frequency of the sender and the receiver oscillators to be significantly out of resonance with that of the transmitters. Inasmuch as the transferred states occupy the nonideal transmitter oscillators only virtually, the undesired environmental effects are weakened, thus enabling QPST from the sender to the receiver. The analytical treatment of QPST in small nonideal linear networks of 3, 4, 5 and 6 oscillators is also given in section 4. In section 5 we apply numerical procedures to extend our results to large numbers of nonideal transmitters, providing a comprehensive analysis of all the network parameters involved and demonstrating the robustness of our protocol. Finally, we present our concluding remarks in section 6.

2. The model, the corresponding master equation and its solution

2.1. General bosonic dissipative network

Before introducing our model for PST, we first revisit the developments in [15], considering a network of $N$ interacting dissipative oscillators from the general perspective where oscillators interact with each other. Any particular network topology (or graph) follows from this general approach, with an appropriate choice of the parameters defining the Hamiltonian modelling the network. As stressed above, by topology is meant (i) which resonators are coupled together, (ii) their coupling strengths and (iii) their natural frequencies. Moreover, in a more realistic approach for most physical systems, it is assumed that each network oscillator interacts with its own reservoir, instead of the special case where all the oscillators interact with a common reservoir. Therefore, assuming from here on that the indices $m$, $n$ and $\ell$, labelling the oscillators, run from 1 to $N$, we start from the general Hamiltonian

$$\mathcal{H} = \hbar \sum_{m} \left[ \sum_{n} a_{m}^{\dagger} H_{mn} a_{n} + \sum_{k} \sigma_{mk} b_{mk}\right] + \sum_{k} V_{mk}\left(b_{mk}^{\dagger} a_{m} + b_{mk} a_{m}^{\dagger}\right), \quad (1)$$

where $b_{mk}^{\dagger}$ ($b_{mk}$) is the creation (annihilation) operator for the $k$th bath mode $\omega_{mk}$ coupled to the $m$th network oscillator $\omega_{m}$, whose creation (annihilation) operator is $a_{m}^{\dagger}$ ($a_{m}$). The coupling strengths between the oscillators are given by the set $\{\lambda_{mn}\}$, while those between the oscillators and their reservoirs by $\{V_{mk}\}$. The reservoirs are modelled as a set of $k = 1, \ldots, \infty$ modes, and the elements $H_{mn}$ defining the network topology compose the matrix

$$\mathbf{H} = \begin{pmatrix} \omega_{1} & \lambda_{12} & \cdots & \lambda_{1N} \\ \lambda_{12} & \omega_{2} & \cdots & \lambda_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{1N} & \lambda_{2N} & \cdots & \omega_{N} \end{pmatrix}. \quad (2)$$

To obtain the master equation of the network, we first diagonalize the Hamiltonian $\mathbf{H}$ through a canonical transformation, $A_{m} = \sum_{n} C_{mn}\phi_{n}$, where the coefficients of the $n$th line of the matrix $C$ define the eigenvectors associated with the eigenvalues $\phi_{n}$ of (2). The commutation relations $[A_{m}, A_{n}^{\dagger}] = \delta_{mn}$ and $[A_{m}, A_{n}] = 0$, following from the orthogonality of the matrix $C$, in that $C^{\dagger} = C^{-1}$, enable us to rewrite Hamiltonian (1) in terms of decoupled normal-mode oscillators $\phi_{n}$, each of them interacting, however, with all the $N$ reservoirs. The diagonalized $\mathbf{H}$ helps us to introduce the interaction picture, within which a set of assumptions leads to the master equation describing the network evolution. We first assume that the couplings between the normal-mode oscillators and the reservoirs are weak enough to allow a second-order perturbation approximation. Moreover, under Markovian white noise, the evolved reduced density operator of the network, $\rho(t)$, is assumed to be factorized from the stationary density operator of the reservoirs. Finally, the reservoir frequencies are assumed to be closely spaced enough to allow a continuum summation, the spectral density $\sigma_{m}(\omega_{0})$ and coupling parameter $V_{m}(\omega_{0})$ being slowly varying functions. Thus, after tracing out the degrees of freedom of the absolute zero reservoirs, we obtain the generalized Lindblad form

$$\frac{d \rho(t)}{dt} = \sum_{m,n} \left\{ \frac{i}{\hbar} [\rho(t), a_{m} H_{mn} a_{n}^{\dagger}] + \frac{\Gamma_{mn}}{2} \left( [a_{m} \rho(t), a_{n}^{\dagger}] + [a_{m}^{\dagger} \rho(t), a_{n}] \right) \right\} \equiv \sum_{m,n} \left\{ \frac{i}{\hbar} [\rho(t), a_{m} H_{mn} a_{n}^{\dagger}] + \mathcal{L}_{mn}\rho(t) \right\}, \quad (3)$$

where we have defined the effective damping matrix $\mathcal{G}$, whose elements are

$$\Gamma_{mn} = N \sum_{\ell} C_{\ell m} C_{\ell n},$$

with $\gamma_{m}(\omega_{0}) = \frac{1}{N} \left[ V_{m}(\omega_{0}) \sigma_{m}(\omega_{0}) \right]^{2} \int_{0}^{\infty} \delta(\epsilon) d\epsilon$. In equation (3), the Liouville operators $\mathcal{L}_{mn}\rho(t)$ account for both the direct ($m = n$) and indirect ($m \neq n$) dissipative channels. Through the direct dissipative channels, the oscillators lose excitation to their own reservoirs, at a damping rate $\Gamma_{mm}$, whereas through the indirect channels they lose excitation to all the other reservoirs but not to their own. For Markovian white noise reservoirs, the indirect channels disappear since the spectral densities of the reservoirs are invariant over translation in frequency space, rendering $\gamma_{m}(\omega_{0}) = \gamma_{m}$, and, consequently, $\Gamma_{mn} = N \gamma_{m} \delta_{mn}$ [25, 26].

It is worth stressing that the whole of the derivation presented in the next subsection, where a solution to the master equation is derived, applies to the case of non-Markovian
reservoirs, thus including the indirect dissipative channels. Markovian reservoirs are only assumed in section 4, where the topology of our framework is finally defined as a linear chain of dissipative oscillators.

### 2.2. Solution of the master equation

To obtain a solution of equation (3), we shall employ it to derive the Glauber–Sudarshan P function for the network dP(⟨ηₙ⟩, t) = \( \sum_m \left( \frac{\Gamma_{mm}}{2} + \sum_n H^D_{mn} \frac{\partial}{\partial \eta_m} + \text{c.c.} \right) P(⟨\eta_m⟩, t), \) (4)

where we have defined the matrix elements \( H^D_{mn} = \text{i}H_{mn} + \Gamma_{mn}/2 \), extending the former \( H_{mn} \), to take into account the dissipative \( D \) process. Assuming that the general network described by matrix (2) is composed entirely of dissipative oscillators, the matrix \( H^D \) assumes the form

\[
H^D = \text{i}H + \frac{1}{2} \left( \begin{array}{cccc}
\Gamma_{11} & \Gamma_{12} & \cdots & \Gamma_{1N} \\
\Gamma_{21} & \Gamma_{22} & \cdots & \Gamma_{2N} \\
\vdots & \vdots & \ddots & \vdots \\
\Gamma_{N1} & \Gamma_{N2} & \cdots & \Gamma_{NN}
\end{array} \right).
\]

(5)

For the initial state of the network, we consider the general pure superpositions of coherent states:

\[
\rho(0) = N^2 \sum_{r,s=1}^Q \Lambda_r \Lambda^*_s \left| \beta_r \right\rangle \left\langle \beta_s \right|,
\]

(6)

where \( N \) is the normalization factor, \( \Lambda_r \) is the probability amplitude of the product state \( \left| \beta_r \right\rangle \left\langle \beta_s \right| \) (a non-entangled network state) and the labels \( r \) and \( s \) run from 1 to the integer \( Q \). The superscript \( r \) stands for the \( r \)th state of the superposition while the subscript \( m \) stands for the coherent state of the \( m \)th oscillator. (We stress that the discrete sum of product states in equation (6) can be substituted by the continuum sum \( |\psi(0)⟩ = N^2 \int d\Lambda(\theta) |\beta_r(\theta)\rangle |\beta_s(\theta)\rangle \) with no further complication.) From equations (4) and (6), it is straightforward to show that the network density operator evolves as

\[
\rho(t) = N^2 \sum_{r,s=1}^Q \Lambda_r \Lambda^*_s \left| \beta_r \right\rangle \left\langle \beta_s \right| e^{-\text{i} \sum_{m} \lambda_{m} t} e^{-\sum_{m} \lambda_{m} t} \left| \beta_r \right\rangle \left\langle \beta_s \right|.
\]

(7)

The excitation of the \( m \)th oscillator, given by

\[
\zeta^m_m(t) = \sum_n \Theta_{mn}(t) \beta^*_n,
\]

(8)

follows from the time-dependent matrix elements

\[
\Theta_{mn}(t) = \sum_{m'} D_{mn} \exp(-\sum_{m'} \text{im}_m t) D^{-1}_{m'n},
\]

(9)

where the \( m \)th column of the matrix \( D \) defines the \( m \)th eigenvector associated with the eigenvalue \( \text{im}_m \) of the matrix \( H^D \).

For the reduced density operator of the \( m \)th oscillator, we obtain

\[
\rho_m(t) = N^2 \sum_{r,s} \frac{1}{2} \Lambda_r \Lambda^*_s \left| \beta_r \right\rangle \left\langle \beta_s \right| e^{-\sum_{m} \lambda_{m} t} \left| \beta_r \right\rangle \left\langle \beta_s \right|.
\]

(10)

where the influence of all the other oscillators of the network is present explicitly in the product \( \left| \beta_r \right\rangle \left\langle \beta_s \right| \) and implicitly in the states \( \left| \zeta^m_m(t) \right\rangle \).

### 3. A criterion for PST in ideal networks of harmonic oscillators and its extension to state transfer in the nonideal case

#### 3.1. PST in ideal networks

On the basis of the model developed above, in this section we derive a general criterion for PST, whatever the topology of an ideal \( (\Gamma = 0) \) network of harmonic oscillators. Starting from equation (8) written as the matrix product \( \zeta^m(t) = \Theta(t) \cdot \beta^m \), it is straightforward to conclude that the condition for transferring the state of the first oscillator to that of the \( N \)th one is given by the matrix structure

\[
\Theta(t_{ex}) = \left( \begin{array}{cccc}
0 & 0 & \cdots & 1 \\
0 & \vdots & \ddots & \vdots \\
0 & \vdots & \ddots & \vdots \\
1 & 0 & \cdots & 0
\end{array} \right).
\]

(11)

which reversely ensures that the state of the \( N \)th oscillator can equally be transferred to the first one, no matter what happens to the states of the intervening \( N-2 \) transmitting oscillators, from the second to the last but one. That is why the submatrix formed by removing the first and last rows and columns of \( \Theta \) is left undefined; for any submatrix \( \{\Xi\} \), PST is achieved as long as the states of the first and the last oscillators are interchanged. The specification of the submatrix \( \{\Xi\} \) defines only the state interchange among the transmitter oscillators. Since we are not interested in this internal dynamics, we leave the submatrix \( \{\Xi\} \) undefined. It is worth noting that the criterion for state exchange fixed by the structure of the matrix in equation (11) does not depend on the network state. Evidently, the imposition of this matrix form can be used to compute the exchange time \( t_{ex} \), as is clarified in section 4 where the particular case of a nonideal linear network with \( N = 4 \) oscillators is treated analytically.

Since the evolution of the matrix \( \Theta(t) \), given by

\[
\Theta(t) = \mathbf{C} \cdot \exp(-\text{i}\Phi t) \cdot \mathbf{C}^{-1}
\]

\[
= \exp[-(\mathbf{C} \cdot \text{i}\Phi \cdot \mathbf{C}^{-1}) t] = \exp[-\text{i}\mathbf{H} t],
\]

(12)

governed by the Hamiltonian \( \mathbf{H} \) (whose eigenvalues \( \phi_m \) compose the diagonal matrix \( \Phi \), the sets of parameters \( \{\omega_m\} \) and \( \{\lambda_{mn}\} \) complying with a necessary condition for PST in a particular network topology follow from the commutation relation

\[
[\Theta(t_{ex}), \mathbf{H}] = 0,
\]

(13)

which holds at any time, including the state exchange time \( t_{ex} \). We observe that not all sets \( \{\omega_m\} \) and \( \{\lambda_{mn}\} \) derived from condition (13) ensure PST. For a necessary and sufficient condition we must choose, among the sets of parameters \( \{\omega_m\} \) and \( \{\lambda_{mn}\} \), only those ensuring the reduction, at the exchange time, of matrix (12) to (11), the former being defined by the network topology and the latter by our initial premise. To better clarify our criterion for PST in ideal networks, we
stress that: when PST occurs, the commutation relation (13) is always satisfied (i.e. (13) is a necessary condition for PST); however, when (13) is satisfied, we cannot ensure that PST occurs (and that is why equation (13) by itself is not a sufficient condition). For a sufficient condition, we must guarantee the existence of a time instant \( t \), named exchange time \( t_{\text{ex}} \), where the evolution of the matrix \( \Theta(t) \)—built from the sets of parameters \( \{\omega_m\} \) and \( \{\lambda_{mn}\} \) satisfying (13)—leads to \( \Theta(t_{\text{ex}}) \).

We emphasize that this necessary and sufficient condition applies only after the specification of the submatrix \( [V] \), i.e. only after the choice of the state interchange to occur among the transmitters. As a matter of fact, any mathematical development—the computation of the commutation relation (13)—must start from the specification of the submatrix \( [V] \) and, consequently, the specification of \( \Theta(t_{\text{ex}}) \). As stressed above, we leave the submatrix \( [V] \) in equation (11) undefined only to illustrate that we are not interested at all in the state interchange among the transmitters. Thus, we conclude that the matrix \( \Theta(t) \) can be used for two purposes: to derive a criterion ensuring PST and to compute the exchange time \( t_{\text{ex}} \).

We note again that in section 4 we demonstrate how to obtain the exchange time from the imposition of the reduction of the matrix \( \Theta(t) \) to \( \Theta(t_{\text{ex}}) \), considering, evidently, a specified submatrix \( [V] \).

As already mentioned in the introduction, we observe that our sufficient condition turns out to be the inverse eigenvalue problem addressed in [1, 3], whereas our necessary condition consists in a simplification of this problem since it previously select the sets of parameters \( \{\omega_m\} \) and \( \{\lambda_{mn}\} \) enabling, through the sufficient condition, the state transfer process.

It must be stressed that, for the state to be transferred by tunnelling, we only have to ensure that the common frequency (\( \omega \)) of the sender and the receiver oscillators is distinct from those of the transmitter ones. Under this condition, we can approximately verify relation (13) independently of the choice set of coupling strengths \( \lambda_{mn} \), and these all may have the same value \( \lambda \). Evidently, as discussed below, the magnitude of \( \lambda_{mn} \) will certainly govern the exchange time.

In what follows, we present an application of equation (13) to compute, for a specific choice of \( [V] \), the set of parameters \( \{\omega_m\} \) and \( \{\lambda_{mn}\} \) ensuring PST. We also present another way to compute the exchange time \( t_{\text{ex}} \), based on the time profile of the probability of a successful transfer of the desired state, which is an alternative to the matrix \( \Theta(t_{\text{ex}}) \).

### 3.2. An application of the commutation relation \( \Theta(t_{\text{ex}}), [H] = 0 \)

Considering the Hamiltonian \( [H] \) in its general form given by equation (2), and specifying the particular choice

\[
\Theta(t_{\text{ex}}) = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & \cdots & \cdots & 0 \\
1 & 0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}
\]  

(14)

allowing the transfer of the initial state of the \( m \)th oscillator to the \( [N - (m - 1)] \)th one, we obtain, from commutation (13), the relations

\[
\omega_m = \omega_{N-(m-1)},
\]

\[
\lambda_{mn} = \lambda_{N-(m-1),N-(n-1)},
\]

(15a, 15b)

which ensure PST only under the condition that, at the exchange time, the matrix \( \Theta(t) \) reduces to \( \Theta(t_{\text{ex}}) \). The above relations (25) generalize the one introduced in [9], given by

\[
\omega_m = \omega,
\]

\[
\lambda_{m,m+1} = \lambda_{m+1,m} = \lambda \sqrt{m(N-m)}.
\]

(16a, 16b)

Evidently, as discussed above, each choice of the submatrix \( [V] \) in the general form (11) prompts a different set of parameters \( \{\omega_m\} \) and \( \{\lambda_{mn}\} \) ensuring PST. We also note that different choices of \( [V] \) also prompts different values of the exchange times.

### 3.3. An alternative way to compute \( t_{\text{ex}} \)

Here we present another way to compute the exchange time \( t_{\text{ex}} \), instead of using the condition for transferring the state of the first oscillator to that of the \( N \)th one, given by the matrix structure (11). We stress that this computation of \( t_{\text{ex}} \) applies to both cases of ideal and nonideal networks. In fact, to compute \( t_{\text{ex}} \) in the case of nonideal networks, we disregard dissipation, assuming \( \Gamma = 0 \); although dissipation brings about the decoherence of the transferred state, it does not affect the exchange time, as to be made clear in section 5.

Therefore, focusing on an ideal network, the computation of the exchange time \( t_{\text{ex}} \) follows from the time profile of the probability of a successful transfer of the desired state—or equivalently, the fidelity of the transfer process—given by

\[
P(t) = \text{Tr} \left[ \rho_1(0) \rho_N(t) \right],
\]

(17)

where \( \rho_1(0) \) and \( \rho_N(t) \) stand for the reduced density operators of the sender (at \( t = 0 \)) and the receiver (at \( t \)), respectively. Evidently, to obtain \( t_{\text{ex}} \) from equation (17), we must take into account the set of parameters \( \{\omega_m\} \) and \( \{\lambda_{mn}\} \) ensuring PST, derived from the commutation relation (13) under the condition of the reduction of the matrix \( \Theta(t) \) to \( \Theta(t_{\text{ex}}) \).

A further condition is that the ideal case \( \Gamma = 0 \) must render \( P(t_{\text{ex}}) = 1 \). Although a rather demanding task, the maximization of the probability \( P(t_{\text{ex}}) \) yields, by itself, a necessary and sufficient condition for the derivation of the sets of parameters \( \{\omega_m\} \) and \( \{\lambda_{mn}\} \) ensuring PST. The commutation relation (13), plus the condition of the reduction, at the exchange time, of matrix (12) to (11), provides a shortcut for this task.

We assume, as usual, that the initial density operator of the whole network \( \rho(0) \) factorizes as \( \rho_1(0) \otimes \rho_{N-1}(0) \) with \( \rho_1(0) \) representing, as defined above, the state of the first oscillator, to be transferred to the \( N \)th one, and \( \rho_{N-1}(0) \) standing for the initial states of the transmitter and the receiver oscillators. (The product state \( \rho_1(0) \otimes \rho_{N-1}(0) \) means that the initial state of the sender is disentangled from those of the receiver and the transmitting channel.) In turn, the density operator at the exchange time \( \rho(t_{\text{ex}}) \) factorizes as \( \rho_N(t_{\text{ex}}) \otimes \rho_{N-1}(t_{\text{ex}}) \), where \( \rho_N(t_{\text{ex}}) \) represents the transferred state of the \( N \)th oscillator and \( \rho_{N-1}(t_{\text{ex}}) \) the final state of the
transmitter oscillators plus the sender. Evidently, the state of the \( N \)th oscillator at \( t_{tx} \) must be, in the ideal case where \( \Gamma = 0 \), exactly that prepared in the first oscillator. With this assumption, by substituting equation (10) into \( P(t_{tx}) \), we obtain

\[
P(t_{tx}) = \mathcal{N}^4 \sum_{x,x',x'',x'''} \Lambda_x \Lambda_{x'} \Lambda_{x''} \Lambda_{x'''} \left[ |\beta_n'\rangle |\beta_n''\rangle |\beta_n''\rangle |\beta_n''\rangle \right] \times \exp \left[ -\left[ \zeta_{yy}(t_{tx}) - \beta_n' \right] \left[ \zeta_{yy}(t_{tx}) - \beta_n'' \right] \right].
\]

As an illustrative application of equation (18), we consider the specific case where the state to be transferred to the \( N \)th oscillator is prepared in the first one as the Schrödinger cat-like superposition \( \mathcal{N}(|\alpha\rangle_1 + |\beta\rangle_1) \), while all other oscillators are in the vacuum state. We obtain from equation (18), for the ideal case, the relations

\[
\sum_m C_{Nm} \cos(\phi_m t_{tx}) C_{m1}^{-1} = \pm 1, \quad (19a)
\]
\[
\sum_m C_{Nm} \sin(\phi_m t_{tx}) C_{m1}^{-1} = 0, \quad (19b)
\]

which enable us to determine \( t_{tx} \), remembering that \( \phi_m \) stands for the eigenvalues of the free Hamiltonian \( H \). Therefore, the exchange time \( t_{tx} \) follows from the computation of the eigenstates and the associated eigenvectors of \( H \).

We call the attention to the fact that, although we use the matrix \( \Theta(t) \) to obtain the evolved excitation of the \( N \)th oscillator, we do not impose here the condition for transferring the state of the first oscillator to the \( N \)th, given by (11). Therefore, in this subsection we have indeed presented an alternative way to compute \( t_{tx} \), other than that coming from (11).

Before analysing the transfer of states under noise effects, we again stress that equation (18) may also be applied to calculate \( t_{tx} \) in nonideal networks. In fact, despite inducing decoherence, dissipation does not affect the exchange time, which can be concluded from the time profile of the probability \( P(t_{tx}) \) which reaches local maxima where PST were supposed to occur in the ideal case. Therefore, for nonideal networks we have the same \( t_{tx} \) as for ideal networks, the difference being that in the later case \( P(t_{tx}) \) reaches unity, whereas in the former \( P(t_{tx}) \) reaches local maxima which are less than unity. This very fact motivates us to extend the commutation relation in equation (13) from the ideal to the nonideal case, aiming to achieve (nonperfect) state transfer in nonideal networks despite decoherence.

3.4. State transfer in nonideal networks: environmental effects

We start this section noting that the environmental effects evidently prevent the occurrence of PST in nonideal networks. What we envisage in this section is only to extend the treatment presented above from the ideal to the nonideal case. Such an extension of the above criterion to state transfer in nonideal networks follows directly by assuming a finite value of \( \Gamma \). The damping rates due to dissipation are thus introduced into equation (12), generalizing the relation (13) to

\[
[\Theta(t_{tx}), H^0] = 0.
\]

It can be demonstrated that the above generalized equation (20) provides the same sets of parameters \(|\alpha_m\rangle\) and \(|\lambda_{mn}\rangle\) derived from the particular equation (13), apart from sets of decay rates \( \Gamma_{mn} \) which must also be satisfied. As an illustrative example, let us consider the Hamiltonian \( H^0 \) in its general form given by equation (5) and specify the particular choice of \( \Theta(t_{tx}) \) given by (14). In this case, we obtain from equation (20), the relations

\[
\begin{align*}
\omega_{0m} &= \omega_N (m-1), \quad (21a) \\
\lambda_{mn} &= \lambda_{N-(m-1),N-(n-1)}, \quad (21b) \\
\Gamma_{mn} &= \Gamma_{N-(m-1),N-(n-1)}, \quad (21c)
\end{align*}
\]

where (21a) and (21b) have already been derived above (subsection 3.2) for an ideal network. Therefore, apart from those sets \(|\alpha_m\rangle\) and \(|\lambda_{mn}\rangle\) ensuring PST are those leading to the reduction of matrix (12) to (11), here they are those giving the local maxima of the component \( \Theta_{1N}(t) (= \Theta_{12}(t)) \) of (9). Evidently, such an involved maximization procedure must also take into account the parameter \( t \), thus giving, apart from the desired sets \(|\alpha_m\rangle\) and \(|\lambda_{mn}\rangle\), the exchange time \( t_{tx} \). We note that, under dissipation, the component \( \Theta_{1N}(t) \) oscillates while decreasing steadily from unity as time goes on, and, consequently, \( \Theta_{1N}(t) \) will never reach unity, as it does in the ideal case.

Therefore, we have extended our criterion for PST in ideal networks to nonideal ones; such an extension is given by the extended commutation relation (20) plus the requirement of the maximization of the component \( \Theta_{1N}(t) \) from which we obtain, among the sets \(|\alpha_m\rangle\) and \(|\lambda_{mn}\rangle\) simultaneously satisfying equations (20) and (13), for nonideal networks we obtain an additional relation between the decay rates, as in (21c). When this additional relation is satisfied, we thus end up with those, for \(|\alpha_m\rangle\) and \(|\lambda_{mn}\rangle\), already taking place in the ideal case. However, while in the ideal case the sets \(|\alpha_m\rangle\) and \(|\lambda_{mn}\rangle\) ensuring PST are those leading to the reduction of matrix (12) to (11), here they are those giving the local maxima of the component \( \Theta_{1N}(t) \) of (9). Evidently, such an involved maximization procedure must also take into account the parameter \( t \), thus giving, apart from the desired sets \(|\alpha_m\rangle\) and \(|\lambda_{mn}\rangle\), the exchange time \( t_{tx} \). We note that, under dissipation, the component \( \Theta_{1N}(t) \) oscillates while decreasing steadily from unity as time goes on, and, consequently, \( \Theta_{1N}(t) \) will never reach unity, as it does in the ideal case.

It is worth noting again that, confining ourselves to a linear dissipative network, the adopted strategy of considering the ideal sender and receiver oscillators significantly out of resonance with the nonideal transmitters ensures QPST despite the nonidealities of the transmitter line. In fact, as anticipated in the introduction and demonstrated below, the virtual occupation of the transmitter line protects the transferred state almost perfectly from the damping mechanisms.
4. Quasi-perfect state transfer in a linear dissipative network: our framework and protocol

To circumvent in large part the effects of losses in nonideal networks, next we present a protocol ensuring QPST in a linear (lin) network of nearest-neighbour interacting nonideal harmonic oscillators. This network is built up by coupling the kth oscillator with the (k ± 1)st oscillators, leaving the first oscillator (m = 1) uncoupled from the last one (m = N). The matrix $H_{lin}^D$ obtained for this case has the tri-diagonal form

$$H_{lin}^D = i \begin{pmatrix}
\omega_1 & \lambda_{12} & 0 & \cdots & 0 & 0 & 0 \\
\lambda_{12} & \omega_2 & \lambda_{23} & \cdots & 0 & 0 & 0 \\
0 & \lambda_{23} & \omega_3 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \omega_{N-2} & \lambda_{N-2,N-1} & 0 \\
0 & 0 & 0 & \cdots & \lambda_{N-2,N-1} & \omega_{N-1} & \lambda_{N-1,N} \\
0 & 0 & 0 & \cdots & 0 & \omega_N & \lambda_{N-1,N}
\end{pmatrix}$$

As already made clear, we focus on the case of ideal sender and receiver oscillators, here assumed to be the first and the last, respectively, both with the same frequency $\omega$. All the transmitter oscillators, from the second to the (N − 1)st, are assumed from here on to decay at the same rate $\Gamma_m = \Gamma$, and to be tuned out of resonance with the sender and receiver, to frequency $\Omega$. Regarding the coupling between the oscillators, we assume that the sender and the receiver are connected with their transmitter neighbours with the same strength $\lambda$, whereas the transmitters are connected to each other with the strength $\varepsilon\lambda$, $\varepsilon$ being a dimensionless parameter allowing the couplings within the transmitting channel to be controlled. Finally, assuming Markovian white noise reservoirs, to eliminate the indirect dissipative channels, the above matrix simplifies to

$$H_{lin}^D = i \begin{pmatrix}
\omega & 0 & \cdots & 0 & 0 & 0 \\
\lambda & \Omega - i\Gamma/2 & \varepsilon\lambda & \cdots & 0 & 0 \\
0 & \varepsilon\lambda & \Omega - i\Gamma/2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \Omega - i\Gamma/2 & \varepsilon\lambda \\
0 & 0 & 0 & \cdots & \varepsilon\lambda & \Omega - i\Gamma/2 & \lambda
\end{pmatrix}$$

To illustrate the effectiveness of our protocol for QPST in nonideal networks, we present below an analytical treatment of small linear networks, from 3 to 6 oscillators, apart from a detailed numerical study of large networks with up to 100 oscillators.

4.1. Analytical treatment of QPST in small nonideal linear networks

Now, focusing on our scheme for the transfer of states by tunnelling, we present an analytical treatment of small nonideal linear networks, using the particular case of (14) to compute the exchange time $t_{ex}$. We start by considering the case of QPST in a network with $N = 4$, whose dissipative matrix, following from equation (22), is

$$H_{lin(4)}^D = \begin{pmatrix}
i\omega & i\lambda & 0 & 0 \\
i\lambda & i\Omega + \Gamma/2 & i\lambda & 0 \\
o & i\lambda & i\Omega + \Gamma/2 & i\lambda \\
o & o & o & i\omega
\end{pmatrix}$$

The eigenvalues and eigenvectors of $H_{lin(4)}^D$ define the matrix $\Theta(t)$ and, consequently, the evolved excitation of the oscillators $\zeta'(t) = \Theta(t) \cdot \zeta'$. Next, we introduce the scaled damping rate $\eta = \Gamma/\lambda$, frequency $\sigma = \omega/\lambda$ and time $\tau = \lambda t$, apart from the detunings $\Delta = \Omega = \omega$ and the effective coupling $\mu = \lambda/\Delta$, to stress that we must focus on the regime where $\mu \ll 1$ and $\eta \ll 1$. These parameters ensure the weak coupling between the transmitter oscillators and their respective reservoirs—justifying the master equation derived above—apart from enabling the expansion of the matrix $\Theta(t)$ to second order in $\mu$, giving

$$\Theta(\tau) = e^{-i(\sigma - \mu \eta)\tau} e^{-\eta\mu^2\tau}$$

where we have defined the functions

$$g_\epsilon(\tau) = \mu[\cos(\epsilon\mu^2\tau) - h_\epsilon(\tau)],$$

$$g_\eta(\tau) = \mu[\sin(\epsilon\mu^2\tau) - h_\eta(\tau)],$$

$$h_\epsilon(\tau) = \exp[-(\eta + \mu^2\tau)] \cos(\epsilon\tau),$$

$$h_\eta(\tau) = \exp[-(\eta + \mu^2\tau)] \sin(\epsilon\tau).$$

We stress that matrix (23) follows from equation (9) which, for the case of $N = 4$, may be rewritten in the form (12) as $\Theta(\tau) = \exp[-H_{lin(4)}^D \tau]$. Therefore, matrix (23) was derived from the eigenvalues $\lambda_m$ and corresponding eigenvectors (composing the nth column of matrix $D$) of the dissipative Hamiltonian $H_{lin(4)}^D$. From the condition for the transfer of the state of the first oscillator to the last one, fixed by the matrix structure (14), and assuming the additional restriction $(\epsilon\mu)^2 \ll 1$, we obtain from equation (23) the scaled exchange time

$$t_{ex}^{(N=4)} \approx \frac{\pi}{2\epsilon\mu^2} \sqrt{1 + O(\mu^2)}$$

which implies the relations

$$\zeta_\epsilon'(t_{ex}^{(N=4)}) \approx e^{-\eta\mu^2 t_{ex}^{(N=4)}} \times \left[ e^{i\epsilon t_{ex}^{(N=4)} \lambda} \beta_\epsilon' + (i\gamma_\epsilon(t_{ex}^{(N=4)}) + i\delta_\epsilon(t_{ex}^{(N=4)}) \beta_\epsilon' \right] + O(\mu^2).$$

$$\zeta_\epsilon(t_{ex}^{(N=4)}) \approx e^{-\eta\mu^2 t_{ex}^{(N=4)} \lambda} \times \left[ e^{i\epsilon t_{ex}^{(N=4)} \lambda} \beta_\epsilon' + (i\gamma_\epsilon(t_{ex}^{(N=4)}) + i\delta_\epsilon(t_{ex}^{(N=4)}) \beta_\epsilon' \right] + O(\mu^2).$$
As is evident from the above expressions, the relaxation process represented by the scaled damping rate $\eta$ prohibits a perfect state transfer, attenuating the value of the excitations $\beta_1^4$ and $\beta_2^4$. Moreover, the reservoirs also spoil the desired relation $\zeta_4^{(N=4)}(\tau_4^{(N=4)}) = \beta_4^4$, by mixing it with the excitations of the nonideal transmitter oscillators. However, and this is the core of our technique, the tunnelling mechanism of state transfer prompts the decay function $e^{-\eta_\mu \tau_4^{(N=4)}} = e^{-\pi \Gamma/2\lambda}$, which approaches unity—within the ranges of the parameters outlined above, i.e. $\mu, \eta, (\epsilon \mu)^2 \ll 1$—as the coupling $\epsilon \lambda$ between the transmitting oscillators is increased. We note that the increase of $\epsilon \lambda$ decreases the exchange time $\tau_4^{(N=4)} \sim \pi \Delta_\mu/2\mu \lambda$, and that is the reason for the choice of a strong coupling strength $\epsilon \lambda$ between the transmitter oscillators: to enable the control of the exchange time $\tau_{ex}$, decreasing it as the fidelity of the transfer process increases. Otherwise, without the dimensionless parameter $\epsilon$, increasing the detuning $\Delta_\mu$ (i.e. the process allowing a significant fidelity of the transferred state) would result in an uncontrolled rise in the exchange time $\tau_4^{(N=4)} \sim \pi \Delta_\mu/2\mu \lambda$. From equation (4.1), we conclude that the fidelity of the state transfer mechanism is maximized when the transmitter oscillators are prepared in the vacuum state, apart from weakening the inevitable system–reservoir coupling. In fact, the excitation of the dissipative transmission channel is directly proportional to the intensity of noise injected into the transferred state. We finally note that for the case of an ideal transmission channel, i.e. $\eta = 0$, we obtain—up to first-order corrections in $\mu$ carried out in $g_4(\tau_4^{(N=4)})$ and $g_4(\tau_4^{(N=4)})$ in the general case of an excited transmission channel—the desired relations $\zeta_4^{(N=4)}(\tau_4^{(N=4)}) \approx \beta_4^4$ and $\zeta_4^{(N=4)}(\tau_4^{(N=4)}) \approx \beta_4^4$.

On the basis of the exchange time $\tau_{ex} = \pi/2$ for the simplest network, composed of two oscillators with the coupling strength $\lambda$ [26], it is useful to assign an effective coupling strength to the network which, for the case $N = 4$, turns out to be $\lambda_{eff}^{(N=4)} \approx \epsilon \mu^2 \lambda$.

4.2. Nonideal linear network of $N = 3, 5,$ and $6$ oscillators

Following the steps described above for the case $N = 4$ and adopting the same regime of parameters $\mu, \eta, (\epsilon \mu)^2 \ll 1$, we obtain for $N = 3$ the result

$$\tau_3^{(N=3)} \simeq \frac{\pi}{2\mu} [1 + O(\mu^2)].$$
giving the expected effective coupling strength \( \lambda^{(N=3)} \approx \mu \lambda \) between the sender and receiver oscillators. Evidently, it is also possible to derive analytical expressions for the scaled exchange time for networks with \( N > 4 \) whenever the diagonalization of the associated matrix \( H^{(N)}_{\text{lin}} \) generates a characteristic polynomial that factorizes into parts of degree \( \leq 4 \). We have found that \( N = 8 \) is the limiting case permitting analytical solution, with a characteristic polynomial that factorizes into two parts of degree 4. For \( N = 9 \), the characteristic polynomial factorizes into two parts of degrees \( 4 \times 5 \). Analysing the cases \( N = 5 \) and \( N = 6 \), which factorize into a polynomial of degrees \( 2 \times 3 \) and \( 3 \times 3 \), respectively, we obtain the results

\[
\begin{align*}
\tau_{\text{ex}}^{(N=5)} & \approx \frac{\pi}{2 \epsilon^2 \mu^2} [1 + O(\mu^2)], \\
\tau_{\text{ex}}^{(N=6)} & \approx \frac{\pi}{2 \epsilon^2 \mu^2} [1 + O(\mu^2)].
\end{align*}
\]

We finally note that, without the imposition of the restriction \( (\epsilon \mu)^2 \ll 1 \), the above expressions for \( \tau_{\text{ex}} \) would be approximately rewritten, for small values of \( N \), as

\[
\tau_{\text{ex}}^{(N)} \approx \frac{\pi}{2 \epsilon^2 \mu^2} [1 + \left(A + B \eta^2 - C \epsilon^2\right) \mu^2],
\]

where \( A = N - 1 \), \( B = \sum_{m=1}^{N-1} m \) and \( C = N - 3 \). Evidently, for small \( N \) and \( (\epsilon \mu)^2 \ll 1 \), the above expression is equivalent to the results derived from \( N = 3 \) to \( N = 6 \). The derivation of an analytical expression for \( \tau_{\text{ex}}^{(N)} \) in the general case of any \( N \) is not an easy task. As \( N \) increases, we find that an involved dependence of the second-order correction \( O(\mu^2) \) on \( N \) begins to play a significant role. Although the task of identifying this dependence is still a challenging one, in the present paper we analyse QPST numerically for large values of \( N \), using expression (17) to obtain the fidelity of the transfer process.

5. Numerical treatment of QPST in large nonideal linear networks

Now, still focusing on our scheme for QPST by tunnelling, and using the probability of a successful transfer of the desired state, given by equation (17), we analyse the case of large nonideal linear networks. Consider the transfer of the state \( \mathcal{N}(|\alpha\rangle_i + |-\alpha\rangle_i) \) to the \( n \)th oscillator, with \( \alpha = 5 \) and all other oscillators in the vacuum state. In figures 1(a)–(d) we plot the numerical curves for the exchange probability \( P_{\text{ex}}(\tau) \) against \( \tau \) for the cases \( N = 5, 10, 50, \) and 100, respectively. (We have written the exchange probability \( P_{\text{ex}}(\tau) \) with the subscript \( \text{ex} \) to differentiate it from the recurrence probability \( P(\tau_{\text{rec}}) \) to be defined below.) Considering the above regime of parameters, \( \mu, \eta \ll 1 \)—without requiring the additional restriction \( (\epsilon \mu)^2 \ll 1 \)—we assume, in units of the coupling strength \( \lambda \), the fictitious value \( \sigma^2 = \omega = 10 \), giving \( \Delta_\alpha = \mu^{-1} = 10^2 \) and \( \epsilon = 5 \times 10^2 \), apart from \( \eta = \Gamma = 10^{-3} \). We first observe, as expected, that the exchange time \( \tau_{\text{ex}}^{(N)} \) increases proportionally to \( N \), being around \( \tau_{\text{ex}}^{(N)} \approx \pi \times 10^2 \) for the case of \( N = 5 \), in agreement with the analytical result computed in equation (25). Moreover, as the state to be transferred occupies the virtual nonideal channel for a time interval proportional to \( N \), the fidelity of the transfer process decreases with \( N \), as displayed in figure 1. In fact, for the choice of the decay rate outlined above, we show that the fidelity of the transfer process is about unity for the cases \( N = 5 \) and 10, beginning to exhibit a significant decrease from \( N = 50 \). We have not found any sensible decrease of the fidelity for the cases \( N = 5 \) and 10, even for time intervals many orders of magnitude longer than the exchange time. The shaded regions in the figures follow from the strong oscillations of the probability \( P_{\text{ex}}(\tau) \), coming from the natural frequencies of the oscillators.

Focusing on the case \( N = 10 \), in figure 2(a) we plot the exchange probability \( P_{\text{ex}}(\tau) \) against \( \tau \), considering the same parameters as in figure 1, except for the excitation \( \beta = 5 \) of the coherent states populating the oscillators of the transmitter channel. (b) Probability of recurrence \( P_{\text{rec}}(\tau) \) of the initial superposition back to the first oscillator, plotted against \( \tau \).
to the sender oscillator. The magnitudes of these secondary peaks follow from the probability of finding the superposition state in the receiver oscillator at the recurrence time where $\mathcal{P}_{\text{rec}}(\tau) = \text{Tr}[\rho_1(0)\rho_1(\tau)] = 1$. Since in the recurrence times the sender and the receiver oscillators are approximately in the states $N(|\alpha_1\rangle + | - \alpha_1\rangle)$ and $|\beta\rangle_N$, respectively, it follows that $\mathcal{P}_{\text{rec}}(\tau) = \text{Tr}[\rho_1(0)\rho_N(\tau_{\text{rec}})] = 1/2$ for the values assumed above. In figure 2(b), we assume the same parameters as in figure 2(a) to plot $\mathcal{P}_{\text{rec}}(\tau)$, the above-defined probability of recurrence of the superposition state back to the first oscillator. As expected, in the recurrence time the superposition state recurs back to the sender oscillator.

In figure 3(a), we adopt the same parameters as in figure 1(b), except for the smaller value $\varepsilon = 8 \times 10^2$, to illustrate the expected increase in the exchange time. An interesting feature of this figure is that the fidelity of the transfer process does not depend on $\varepsilon$, despite the increase in the exchange time, which is around seven orders of magnitude greater than that in figure 1(b). These features are reinforced in figure 3(b) where, again with the same parameters as in figure 1(b), we consider the limiting case $\varepsilon = 1$, which results in a formidable increase in the exchange time of around 28 orders of magnitude, still preserving the fidelity.

With the same parameters as in figure 3(a), except for the smaller value of $\Delta_1 = 2 \times 10^3$, in figure 4(a) we first see that the smaller detuning pulls the exchange time back, to about the order of magnitude found in figure 1(b), in spite of the quantity $\varepsilon = 8 \times 10^3$. Moreover, we verify the expected continuous decrease of the process due to such a small detuning, which compels the state to populate more effectively the virtual transmitter channel. In figure 4(b) we use the same parameters as in figure 4(a), except for the coherent state $\alpha = 10$, to show that a larger excitation of the state to be transferred results, as expected, in a smaller fidelity of the process. In fact, the decoherence time of a quantum state varies inversely with its excitation, in accordance with the correspondence principle.

Finally, to illustrate the advantage of our tunnelling-based scheme for state transfer over those where the transfer proceeds non-virtually through all the transmitter oscillators, in figure 5 we plot the exchange probability $\mathcal{P}_{\text{ex}}(\tau)$ against $\tau$ for the case where the sets of parameters $\{\omega_m\}$ and $\{\lambda_{mn}\}$ in equation (3.2) are utilized. Assuming, as in figure 1(a), $\omega = 10$ and $\eta = \Gamma = 10^{-3}$, we find that, whereas the fidelity of our tunnelling scheme is about unity up to at least ten times the exchange time $\tau_{\text{ex}}^{(N-5)} \approx 7 \times 10^2$, that for the case of figure 5 decays to around zero for $\tau \approx 2 \times 10^3$.
6. Concluding remarks

In this work, we have discussed the problem of state transfer in a linear chain of quantum dissipative harmonic oscillators. Assuming the sender and the receiver oscillators to be on-resonant with each other and significantly off-resonant with the transmitter channel—running from the second to the last-but-one oscillator—we take advantage of the tunnelling effect to circumvent almost completely the decoherence during the transmission process. We have assumed ideal sender and receiver oscillators connected by nonideal transmitters. As a matter of fact, a large detuning between the on-resonant and the off-resonant oscillators ensures a high fidelity for the transfer process at the expense of a significant prolongation of the exchange time. In this connection, we show that an increase in the coupling between the transmitter oscillators acts to shorten such a delay. The role played by each of the network parameters is analysed in detail in a set of figures presented after the formal development of our work which is presently under our consideration.

We have also derived a general criterion for PST, whatever the topology of the network, which is applied to the particular case of an ideal linear chain. Such a criterion for PST in ideal networks was also extended to the case of nonideal networks, considering dissipative transmitter oscillators, where the transferred state is adversely affected by decoherence. By extending the criterion we mean that we have only pursued a particular way (among others) to achieve state transfer (and not perfect or quasi-perfect state transfer) in nonideal networks. This particular way is the one that exactly recovers the PST occurring in the ideal case, when disregarding the network nonidealities. And here is our main point: aiming for weakening the noise effects, we proposed the protocol mentioned above, enabling us to achieve QPST in nonideal linear networks by taking advantage of the tunnelling effect.

We hope that the tunnelling scheme presented here can be useful for transferring quantum states between distant nodes of a quantum circuit without undergoing a significant coherence decay. Evidently, the control of the network parameters, such as the natural frequencies and coupling strengths of the oscillators, still represents a sensitive issue to be overcome experimentally. We also point out that recent developments in circuit QED and photonic crystals signal realistic platforms for the experimental implementation of our proposed scheme. In this regard, we note the recent demonstration of the collective interaction of a discrete number of quantum two-state systems mediated by an individual photon in circuit QED [27]. We also mention the observations of deterministic phase- and resonance-controlled all-optical electromagnetically induced transparency in multiple coupled photonic crystal cavities [28].

It is worth mentioning that higher-dimensional transmission channels, instead of the linear one-dimensional chain considered in the present study, could be helpful for the transmission of entangled states along the channel, with transmitters and receivers composed of more than one entangled oscillators. This constitutes an interesting extension of our work which is presently under our consideration together with the study of the evolution of an initial entanglement between the sender and the receiver through the transmission channel. Under our tunnelling-based scheme, it is reasonable to expect the initial entanglement between the sender and the receiver not to evolve to the transmitting channel.

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