Master formula approach to broken chiral $U(3) \times U(3)$ symmetry

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The master formula approach to chiral symmetry breaking proposed by Yamagishi and Zahed is extended to the $U_R(3) \times U_L(3)$ group, in which effects of the $U_A(1)$ anomaly and the flavor symmetry breaking $m_u \neq m_d \neq m_s$ are properly contained. New identities for the gluon topological susceptibility and $\pi^0, \eta, \eta' \rightarrow \gamma(\gamma') \gamma(\gamma')$ decays are derived, which exactly embody the consequence from broken chiral symmetry in QCD without relying on any unphysical limit.

PACS numbers: 11.30.Rd, 11.40.-q

Phenomena of $\eta$ and $\eta'$ mesons have been and are currently an attractive subject both theoretically and experimentally. Various properties of these mesons are closely related to the existence of the $U_A(1)$ anomaly of QCD and flavor mixing due to the mass difference among $u, d, s$ quarks. Since the $U_A(1)$ anomaly is a realization of the nontrivial topology of gluonic configurations, it is one of the most appropriate subjects to access the fundamental aspects of QCD. Several approaches, e.g., based on the chiral perturbation theory [1, 2] and the extension of technique of the zero-momentum Ward identities with PCAC hypothesis to the singlet sector [3, 4], have been developed to analyze reaction processes of $\eta$ and $\eta'$ mesons and a gluonic context of the nonperturbative QCD domain. Most of those discussions, however, rely on a certain unphysical limit such as chiral, soft pion, and large $N_c$.

The master formula approach to chiral symmetry breaking proposed by Yamagishi and Zahed is a powerful tool to analyzing hadronic processes including ground and fully embodies consequences from broken chiral symmetry in QCD without relying on any unphysical limit.

An explicit form of the gluon topological susceptibility $\Omega^a$ associated with the external gauge fields $\phi^a_\mu$ [9]

\[ \Omega^a = \frac{N_c}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left[ \frac{\lambda^a}{2} \left( F^\mu\nu F_{\rho\sigma} + \frac{1}{3} F^A_\mu F^A_{\rho\sigma} \right) 
+ i \frac{2}{3} (a_\mu a_\nu F_{\rho\sigma} + a_\nu F^\mu_\rho a_\sigma + F_{\rho\sigma} a_\mu a_\sigma - \frac{3}{2} a_\mu a_\nu a_\rho a_\sigma) \right], \]

(4)

with $q_T = (u, d, s)$ and $m_u^a \lambda^a = \text{diag}(m_u, m_d, m_s)$. The term $L^0_{\text{QCD}}$ represents the QCD Lagrangian in which the quark masses, the vacuum angle, and all external fields are set to zero. The flavor matrix $\lambda^a$ is taken to be the Gell-Mann matrices for $a = 1, \ldots, 8$ and $\lambda^0 = \sqrt{2/3}$ so that they satisfy Tr[$\lambda^a \lambda^b$] = $2\delta^{ab}$. The vacuum angle $\theta$ and the gluon topological charge density $\omega$ are introduced via $\theta^a = (\Theta^{\mu}(x))^a_{\mu} - \delta^{a0}$ and $\omega^a = \text{Tr}[\lambda^a] = \Theta^{\mu}(x)\delta^{a0}$, respectively. Clearly $\theta^a = \omega^a = 0$ for $a\neq 0$ and $\theta^0 \omega^0 = \theta^0 \omega^0 = \theta^0 \omega^0 = \delta$. The vector, axial-vector, scalar, and pseudoscalar external fields and the vacuum angle, $\phi = (\phi^a_\mu, a^a_\mu, s, p^a, -\theta^a)$, are treated as sources to generate current and density operators $O = (V^\mu_\mu, A^a_\mu, \Sigma^a, \Pi^a, \omega^a)$, which can be defined by $O = \delta(\partial^a x L_{\text{QCD}})/\delta \phi^a$.

The system described by Eq. (1) and its effective theory has an approximate $U(3)$ anomaly of QCD

\[ \nabla^{\mu\nu} V_\mu + a^{0a} A^a_\mu + p^{0a} \Pi^a + (s + m_\eta) a^{ac} \Sigma^c = 0, \]

(2)

\[ \nabla^{\mu} A_\mu + a^{ab} V^b_\mu + p^{0b} \Sigma^b - (s + m_\eta) a^{0c} \Pi^c - \omega^a = \Omega^a, \]

(3)

where $a^{ab} = f^{abc} a^c_\mu$, $\sigma^{abc} = d^{abc} a^c_\mu$, and $\nabla^{ac} = \partial^a a^c_\mu + \epsilon^{abc} \phi^d_\mu$. The structure constants $f^{abc}$ and $d^{abc}$ are defined by $f^{abc} = -(i/4)\text{Tr}[[\lambda^a, \lambda^b] \lambda^c]$ and $d^{abc} = (1/4)\text{Tr}[[\lambda^a, \lambda^b, \lambda^c]]$, respectively. With this definition, we have $f^{abc} = 0$ and $d^{abc} = \sqrt{3} g^{abc}$. An explicit form of the non-abelian anomaly $\Omega^a$ associated with the external gauge fields $\phi^a_\mu$ and $\phi^0_\mu$ is [9]

\[ \Omega^a = \frac{N_c}{16\pi^2} \varepsilon_{\mu\nu\rho\sigma} \text{Tr} \left[ \lambda^a \left( F^\mu_\nu F_{\rho\sigma} + \frac{1}{3} F^A_\mu F^A_{\rho\sigma} \right) 
+ i \frac{2}{3} (a_\mu a_\nu F_{\rho\sigma} + a_\nu F^\mu_\rho a_\sigma + F_{\rho\sigma} a_\mu a_\sigma - \frac{3}{2} a_\mu a_\nu a_\rho a_\sigma) \right], \]

(4)

with $F^\mu_\nu = \partial_\mu a_\nu - \partial_\nu a_\mu - i[v_\mu, a_\nu] - i[a_\mu, v_\nu]$, $F^A_\mu = \partial_\mu F^A - \partial_\nu F^A a_\nu a_\sigma - \partial_\nu a_\mu a_\nu a_\sigma$ and $a_\mu = a^{0a}(\lambda^a/2)$. We take a convention of $\varepsilon_{0123} = +1$. 

Through this paper the term “pion” ("\pi") is used for expressing the nonet ground pseudoscalar mesons generally, and the symbols $\pi^{0, \pm, \mp}, K^{0, +}, K^{0, -}, \eta, \eta'$ are used for referring to the specific mesons.

Consider QCD with massive $u, d, s$ quarks. The Lagrangian is written as

\[ \mathcal{L}_{\text{QCD}} = L^0_{\text{QCD}} + \overline{q} \gamma^\mu (\gamma^\mu_{\mu} + 3m_\eta^2) \lambda^a \frac{\lambda^a}{2} q 
- q(m_\eta^a + s^a - i\bar{p} \gamma_5 \lambda^a q - \theta^a \omega^a), \]

(1)
The VB equations (2) and (3) constitute basic equations for the master formula approach [5].

Introducing the extended S-matrix $S$ being a functional of the external fields $\phi$, $S = S[\phi]$, and using the Peierls-Dyson formula [5, 10, 11],

$$O(x) = -iS^\dagger \frac{\delta}{\delta\phi(x)} S,$$

(Eqs. (2) and (3) can be rewritten as

$$X_\phi^a(x)S = 0, \quad X_\phi^a(x)S = \Omega^aS,$$

where $X_\phi = [\nabla_\mu (\delta/\delta a_\mu) + g_\mu (\delta/\delta a_\mu) + p(\delta/\delta p) + (\mu + m_\phi^2)(\delta/\delta \phi)]$ and $X_\phi^a = [\nabla_\mu (\delta/\delta a_\mu) + g_\mu (\delta/\delta a_\mu) + p(\delta/\delta p) + (\pi + m_\phi^2)(\delta/\delta \phi) + (\delta/\delta \delta)]$. By applying functional derivatives of $\phi$ to Eq. (6) and using the expression of the $T^*$-product [5, 11, 12],

$$T^*[O(x_1) \cdots O(x_n)] = (-i)^n S^\dagger \frac{\delta}{\delta\phi(x_1)} \cdots \frac{\delta}{\delta\phi(x_n)} S,$$

we immediately obtain the vector and axial Ward identities satisfied by the operators $O = (V_\mu^a, A_\mu^a, \Sigma^a, \Pi^a, \omega^a)$.

The VB equations (2) and (3) or Eq. (6) are satisfied by the system both in the Wigner and the Nambu-Goldstone (NG) phases. If the chiral symmetry in the system is spontaneously broken, the NG bosons (pions) appear and couple to the axial current: $\langle 0 | A_\mu^a(x) | Ng(p) \rangle \neq 0$. The master equations proposed in Ref. [5] may be understood as the VB equations incorporating the information on the spontaneous symmetry breaking.1

Now we proceed to the construction of the $U_R(3) \times U_L(3)$ master equations. Besides their key role in the singlet sector, the mass difference among the $u, d, s$ quarks and the $U_A(1)$ anomaly are responsible for the major complication for the construction. Thus we first examine the effects of them carefully.

The mass difference in the quarks violates the flavor symmetry explicitly as well as the chiral symmetry. Thus the pure U(3) base cannot be the mass eigenstates of hadrons and mixing of the U(3) base occurs. Necessary information on the flavor symmetry breaking here is summarized in the following matrix elements,

$$\langle 0 | A_\mu^a(x) | P(p) \rangle |_{\phi=0} = \tilde{f}_\pi^a \pi_\mu e^{-ipx},$$
$$\langle 0 | \Pi^a(x) | P(p) \rangle |_{\phi=0} = G^{a \mu} e^{-ipx},$$
$$\langle 0 | \omega^a(x) | P(p) \rangle |_{\phi=0} = A^{a \mu} e^{-ipx},$$

where $P$ denotes the physical pion state (mass eigenstates): $P = (\pi^\pm, K^{+}, K^{0}, K^{-}, \eta, \eta'\rangle$. (Capital indices are used for representing mass eigenstates.) Because

of the flavor symmetry breaking, the pion decay constant $f_\pi$ and the coupling constants $G^{a \mu}$ and $A^{a \mu}$ become a quantity with two indices which specify the flavor U(3) base and the mass eigenstate.2 Note that $A^{a \mu}$ has nonzero value only for $a = 0$.

The $U_A(1)$ anomaly brings an additional non-conserving contribution, which amounts to $\omega^0$, to the singlet axial VB equation (note again that $\omega^0 = 0$ for $a \neq 0$). Because of this the operators $A_\mu^a$ and $\omega^0$ get renormalized under the change of the QCD scale [1]: $(A_\mu^a)_{\text{ren}} = Z_A A_\mu^a$, $(\omega^0)_{\text{ren}} = \omega^0 + (Z_A - 1) \partial^\mu A^a_{\mu}$, where $Z_A$ is the renormalization factor. Accordingly, the singlet pion decay constant $f_\pi^0$ and $A^{a \mu}$ also get renormalized as

$$(f_\pi^0)_{\text{ren}} = Z_A f_\pi^0,$$
$$(A^{a \mu})_{\text{ren}} = A^{a \mu} + (Z_A - 1) f_\pi^0 (m_\pi^0)_{\text{ren}},$$

where $m_\pi$ is the diagonal mass matrix of the nonet pions. These are thus scale dependent and not being physical constants. This is contrast to the octet pion decay constants $f_\pi^0$ with $a \neq 0$, which are scale independent.

As pointed out by Shore and Veneziano (see e.g., Refs. [4, 13, 14]), the renormalization group (RG) variant $f_\pi^\mu$, which is defined in Eq. (8) as a coupling of the singlet axial current to the physical pions, does not satisfy the Gell-Mann-Oakes-Renner (GOR) type of mass relations. Then it was introduced new singlet pion decay constant $f_\pi^\mu$ being RG invariant and shown that use of the new constant provides a natural extension of the GOR relation to the singlet sector.

According to the Shore-Veneziano observation, let us introduce new pion decay constant $f_\pi^0$ and quantity $A^{a \mu}$, both of which are RG invariant. Introducing $\delta f_\pi^0 = f_\pi^0 - f_\pi^\mu$ and $\delta A^{a \mu} = A^{a \mu} - A^{a \mu}(\pi^\mu)^{e\mu}$, the combination $(f_\pi m_\pi^2)_{\mu}^0 - A^{a \mu}$ can be expressed as

$$(f_\pi m_\pi^2)_{\mu}^0 - A^{a \mu} = (f_\pi m_\pi^2)_{\mu}^0 - A^{a \mu} + (\delta f_\pi m_\pi^2)_{\mu}^0 - \delta A^{a \mu},$$

From Eqs. (11) and (12), we find that $(f_\pi m_\pi^2) - A, f_\pi m_\pi^2, A(f_\pi)^{-1}$, and $\delta f_\pi m_\pi^2 - \delta A$ are RG invariant individually. We choose $f_\pi$ and $A^{a \mu}$ so as to be $f_\pi m_\pi^2 - \delta A = 0$. The value of $f_\pi$ and $A^{a \mu}$ must be extracted from experimental data. By sandwiching the axial VB equation (3) between the vacuum and one-pion states and using Eqs. (8)-(13), we have the following mass relation:

$$(f_\pi m_\pi^2)_{\mu}^{f_\pi} = (m_\pi G f_\pi)^{ab} + A^{ab},$$

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1 Strictly speaking, the ground pseudoscalar meson associated with the singlet current is not the NG boson because of the $U_A(1)$ anomaly. However, we can still treat it with the same footing as the other mesons in the octet sector.

2 Breaking pattern of the flavor symmetry, i.e., $m_\pi^a m_\pi^b = m_\pi^0 m_\pi^0 + m_\pi^3 m_\pi^3 + m_\pi^1 m_\pi^1 + m_\pi^2 m_\pi^2$ implies that $f_\pi^0$ and $G^{a \mu}$ are block diagonal in $(1, 2) \times \{\pi^+, \pi^-\}, (4, 5) \times \{K^+, K^-\}, (6, 7) \times \{K^0, \bar{K}^0\}$, and $(0, 3, 8) \times \{\eta, \eta'\}$. In the isospin limit they become $(1, 2, 3) \times \{\pi^+, \pi^-\}, (4, 5, 6, 7) \times \{K^+, K^-, K^0, \bar{K}^0\}$, and $(0, 8) \times \{\eta, \eta'\}$ and the first two blocks can be diagonalized.
where \( f_\mu^a \equiv f_\mu^a \) for \( a \neq 0 \). The appearance of \( A^a b \) is nothing but the consequence from the \( U_A(1) \) anomaly.

The information on the flavor symmetry breaking and the \( U_A(1) \) anomaly described above can be incorporated into the VB equations by introducing the following two modifications: 1) Introduce new pseudoscalar and scalar external field \( J^P \) and \( Y^P \) defined as,

\[
J^P = (G^{-1}p)^P + (\tilde{f}_\pi^T \nabla \tilde{a})^P - (\tilde{f}_\pi^T \Box + f_\pi^{-1}A)^P a^0 \theta^a, \\
Y^P = (G^{-1}s)^P,
\]

(15)

with \( \tilde{a}_\mu^a = a_\mu^a + \partial_\mu \theta^a \), and consider \( \phi = (a^0, \tilde{a}_\mu^a, Y^P, J^P, -\theta^a) \) as independent external field variables. 2) Introduce new extended S-matrix as

\[
\hat{S} = S \exp \left[-i (\delta \mathcal{I} + \delta \mathcal{I}_\Omega) \right],
\]

(16)

with

\[
\delta \mathcal{I} = \int dx \left( YG^{-1}C + \frac{1}{2} (\tilde{a} \tilde{f}_\pi^T) \cdot (\tilde{a} \tilde{f}_\pi^T) \right)

- \frac{1}{2} \theta(A + f_\pi \tilde{f}_\pi^T \Box) - (\partial^2 f_\pi) \cdot (\tilde{a} \tilde{f}_\pi^T) \),
\]

(17)

where the parameter \( C \) has a flavor index \( C = C^a \) in which only \( (C^0, C^3, C^8) \) have nonzero value, and

\[
\delta \mathcal{I}_\Omega = \frac{N_c}{72 \pi^2} \hat{\epsilon}^{\mu \nu \rho \sigma} \int d^4x \left( a_\mu^a \nabla_\nu a_\rho^C \nabla_\sigma a_0^0 - \frac{1}{2} \partial_a^0 \partial_\sigma a_\rho^0 \right).
\]

(18)

Equations (15) and (17) can be understood as a straightforward extension of the SU(2) case (see Sec. 4.2 in Ref. [5]). A comment on the term \( \delta \mathcal{I}_\Omega \) will be needed. As discussed by Kaiser and Leutwyler [1], the non-abelian anomaly \( \Omega^a \) including the external singlet axial gauge field \( a_\mu^0 \) is not invariant under the change of the QCD scale because of the \( U_A(1) \) anomaly. This leads to an inconsistency with the RG invariance of the VB equations, which results from that of the effectiv action and the extended S-matrix. The term \( \delta \mathcal{I}_\Omega \) is introducted to cure the inconsistency and corresponds to sum of the two contact terms \( P_1 \) and \( P_2 \) of Eq. (78) in Ref. [1]. With this additional term the non-abelian anomaly \( \Omega^a \) is replaced with the RG invariant \( \Omega_\mu^a \), which is the same form as \( \Omega^a \) but \( a_\mu^0 \) is replaced with \( -\partial_\mu \theta^a \).

The new current and density operators \( \hat{O} = (\tilde{a}_\mu^a, J_\mu^a, \tilde{a}_\mu^a, \tilde{\pi}^a, \tilde{\omega}^a) \) defined by \( \hat{O} = -i \hat{S} \delta \mathcal{O} \) with \( \phi = (v^a_\mu, a^0_\mu, Y^P, J^P, -\theta^a) \), are related to the old ones as follows:

\[
V_\mu^a = j_\mu^a + (\tilde{a}_\mu^a, \tilde{f}_\pi^a) + \frac{\delta(\delta \mathcal{I}_\Omega)}{\delta \theta^a},
\]

(19)

\[
A_\mu^a = j_\mu^a - (\tilde{\nu} \tilde{f}_\pi^a),
\]

(20)

\[
\Sigma^a = G_\mu^P \tilde{\sigma}^P + C^a,
\]

(21)

\[
\Pi^a = G_\mu^P \tilde{\pi}^P,
\]

(22)

\[
\omega^a = \omega^a, + [A(f_\pi^{-1})] - f_\pi \tilde{f}_\pi^a \tilde{\sigma}^P + (\delta f_\pi \tilde{f}_\pi^a \tilde{\partial}_\mu)^a + (A + f_\pi \tilde{f}_\pi^a \tilde{\partial}^a) \tilde{\delta}_\theta + \delta(\delta \mathcal{I}_\Omega) \delta(-\theta^a).
\]

(23)

It is found from Eqs. (9) and (22) that the new pseudoscalar density \( \tilde{\pi}^a \) satisfies \( \langle 0 | \tilde{\pi}^a(x) | P(k) \rangle = \delta \tilde{\pi}^a e^{-i k x} \). This fact allows one to identify \( \tilde{\pi}^a \) as the (normalized) interpolating pion field. The change of the field variable \( p^a \to J^P \) in Eq. (15) is responsible for the separation of the one-pion component from \( A^a_\mu \) and \( \omega^a \) [Eqs. (20) and (23)]. Equations (8) and (10) indicates that the new operators \( J_\mu^a_\Pi \) and \( \omega^a \) = \( \tilde{T} [\lambda^a] \tilde{\omega} \) have no one-pion component surviving on the pion mass-shell, \( \langle 0 | j_\mu^a_\Pi | P(p) \rangle = \langle 0 | \omega^a | P(p) \rangle = 0 \) at \( p^2 = (m_\pi^2)^P \), in contrast to the original \( A^a_\mu \) and \( \omega^a \). The modifications arising from \( \delta \mathcal{I}_\Omega \), which generate terms only including external fields in Eqs. (19)-(23) [except \( \delta(\delta \mathcal{I}_\Omega)/\delta \phi \)], play a key role for the two-point functions including \( A^a_\mu \) and \( \omega^a \) [5].

Substituting the above expressions into the VB equations (2) and (3), we have

\[
0 = \left( \nabla_\mu j_\mu^a \right)^a + (a_\mu^a j_\mu^a) + \left( f^a_\mu - f^a_{\mu} \right) + \left( Y + m_\pi G \right)^a \tilde{\sigma}^P
\]

\[ + (\lambda^a \lambda^C)^a \tilde{J}^C + \lambda^a \tilde{V} \]

(24)

and

\[ \left( -[\Box - m_\pi^2 - f_\pi^{-1} K f_\pi] J^P \right) + \left( -J^P - (f_\pi^{-1})^a + \frac{1}{2} \left( \tilde{f}_\pi^a + \tilde{\omega}^a \right) \right) K^P \]

\[ = -J^P - (f_\pi^{-1} K f_\pi)^a \]

(25)

respectively, where \( \tilde{\omega}^a = f_\mu^a (G^{-1})^B \tilde{G}^C J^B \). We have also used the expressions

\[ \chi_{J_1}^{aP} = \left( \nabla_\mu \tilde{a}_\mu^a \right) f_\pi \tilde{f}_\pi^a \tilde{a}_\mu^a + \frac{1}{2} \left( \tilde{f}_\pi^a \tilde{f}_\pi^a \right) \tilde{a}_\mu^a \]

\[ + \left( f_\pi^{-1} \tilde{f}_\pi^a \tilde{f}_\pi^a \right) \tilde{a}_\mu^a \]

(26)

\[ \chi_{J_2}^a = \tilde{a}_\mu^a \tilde{f}_\pi^a \tilde{a}_\mu^a - \tilde{a}_\mu^a \tilde{f}_\pi^a \tilde{a}_\mu^a + \tilde{a}_\mu^a \tilde{f}_\pi^a \tilde{a}_\mu^a \]

(27)

\[ \chi_{J_3}^a = \left( f_\pi^{-1} \tilde{\omega}^a \right) \tilde{a}_\mu^a \tilde{f}_\pi^a \tilde{a}_\mu^a - \tilde{a}_\mu^a \tilde{f}_\pi^a \tilde{a}_\mu^a \]

(28)

\[ \chi_{J_4} = \left( f_\pi^{-1} \tilde{a}_\mu^a \right) + \tilde{a}_\mu^a - \tilde{a}_\mu^a \tilde{a}_\mu^a \]

(29)

\[ \mathcal{J}^P = \left( J - f_\pi^{-1} \tilde{a}_\mu^a \right) \tilde{f}_\pi^a \tilde{f}_\pi^a \tilde{a}_\mu^a \]

(30)
The terms $\chi_{V1}, \chi_{V2}$, and $\chi_A$ are due to the flavor symmetry breaking and thus they vanish in the flavor symmetric limit.

In deriving Eqs. (24) and (25) we have used the relation
\[
(C^a)^{ab} = (Gf^a)^{ab}. 
\]
(31)
This follows from the fact that the $\pi^P$ is the normalized interpolating pion field. Then the mass relation (14) can be rewritten as
\[
(f_x m_x^2 f_x^a)^{ab} = (m_q C)^{ab} + A^{ab}. 
\]
(32)
From $(m_q C)^T = (m_q C)$ and $A^{ab} = 0$ for $a \neq 0$, one can fix $A^{ab}$ up to one constant: $A^{ab} = \text{Tr}[\lambda^a] \text{Tr}[\lambda^b] A$.

Equation (21) indicates that if $\langle \phi \rangle = 0$ then the constant $(C^0, C^3, C^8)$ reduces to $C^0 = -\sqrt{2/3}(\langle uu \rangle + \langle dd \rangle + \langle ss \rangle)$, $C^3 = -\langle uu \rangle - \langle dd \rangle)$, and $C^8 = -\sqrt{1/3}(\langle uu \rangle + \langle dd \rangle - 2s)$), respectively. Thus, in this case Eq. (32) reduces to the generalized GOR relation derived by Shore [4], if the constant $A$ is identified with the non-perturbative coefficient appearing in the gluon topological susceptibility in QCD. The quantity $\langle \phi \rangle$ represents deviation of the mass relation (32) from the GOR. It is noted that the generalized GOR relation is hold at leading order of $1/f_x^2$ expansion proposed in Ref. [5] because of $\langle \phi^P \rangle = 0 + O(f_x^{-2})$.

Introducing retarded and advanced Green’s functions satisfying
\[
[-\Delta - m_x^2 - f_x^{-1}Kf_x]^{PQ} G_{R,A}^{QR}(x, y) = \delta^{PR}(x - y),
\]
Eq. (25) can be formally solved for $\pi^P$ as (suppressing all indices, summations, and integrals)
\[
\hat{\pi} = (1 + G_R f_x^{-1}Kf_x)\pi_{in}
- G_R f_x^{-1}(\nabla j_A + a \hat{j}_V - \hat{\omega}) - G_R f_x^{-1}\mathcal{F} \sigma
- G_R J^P + G_R f_x^{-1} \Omega_0 - G_R f_x^{-1} \chi_A
= (1 + G_A f_x^{-1} K f_x)\pi_{out}
- G_A f_x^{-1}(\nabla j_A + a \hat{j}_V - \hat{\omega}) - G_A f_x^{-1} \mathcal{F} \sigma
- G_A J^P + G_A f_x^{-1} \Omega_0 - G_A f_x^{-1} \chi_A,
\]
(33)
where $\pi_{in}$ ($\pi_{out}$) is the in-state (out-state) asymptotic pion field.

With Eq. (5), Eqs. (24) and (25) can be rewritten as a linear equation satisfied by the extended S-matrix $\hat{S}$:
\[
T_a^\alpha(x) \hat{S} = 0, 
\]
(34)
\[
T_a^\alpha(x) \hat{S} = 0,
\]
(35)
with
\[
T_a^\alpha = \left[ \nabla_\mu \frac{\delta}{\delta \phi_\mu} + a \frac{\delta}{\delta \mu} + \frac{1}{2} f \frac{\delta}{\delta J} + (Y + m_y G) \frac{\delta}{\delta Y} \right]^a
+ \chi V^P \frac{\delta}{\delta J^P} + i \chi V^2, 
\]
(36)
\[
T_a^\alpha(x) = f_x^P \left[ (-\square - m^2_x) \frac{\delta}{\delta J(x)} + R(x) \right]^P, 
\]
(37)
\[
R^P(x) = -[f_x^{-1} K(x) f_x]^{PQ} \frac{\delta}{\delta J^Q(x)} + i J^P(x)
+ (f_x^{-1})^P \left[ \nabla_\mu \frac{\delta}{\delta a_\mu} + a \frac{\delta}{\delta v_\mu} + \frac{\delta}{\delta \bar{\theta}} \right] (x)
+ [f_x^{-1}]^{P} \frac{\delta}{\delta Y^P(x)}
- i(f_x^{-1} P^- \omega_0^P(x)) + i(f_x^{-1} P^- \chi_A^P(x), 
\]
(38)
Similarly, Eq. (33) can be rewritten as
\[
\frac{\delta}{\delta J} \hat{S} = i \hat{S} (1 + G_R f_x^{-1} K f_x) \pi_{in} - G_R \hat{R} \hat{S}
= i (1 + G_A f_x^{-1} K f_x) \pi_{in} \hat{S} - G_A \hat{R} \hat{S}, 
\]
(39)
with
\[
\hat{R}^P(x) = R^P(x) + [f_x^{-1} K(x) f_x]^{PQ} \frac{\delta}{\delta J^Q(x)}, 
\]
(40)
where $\pi_{out} = \hat{S} \pi_{in}$ is used. Equations (34) and (39) are the desired extension of the master equations to the $U_R(3) \times U_L(3)$ chiral symmetry breaking with the finite $u, d, s$ quark masses and the $U_A(1)$ anomaly. The major consequences from the flavor symmetry breaking are the two-index character of the various constants and the appearance of $\chi_{V1}^P, \chi_{V2}^P, \chi_{A}^P$, and terms, and from the $U_A(1)$ anomaly are the mass relation (32) and the appearance of the operator $\hat{\omega}$.

Using Eq. (39), we can directly derive the commutation relations of the creation and annihilation operator of pion with the extended S-matrix $\hat{S}$,
\[
[a^P, \hat{S}] = R^P(\hat{S} = [\hat{S}, a^P] = R^P(-k) \hat{S}, 
\]
(41)
where $R^P(k) = \int dxe^{ikx} R^P(x)$. Iterative use of Eq. (41) results in the $U_R(3) \times U_L(3)$ version of the chiral reduction formula $\chi(R)$ for the scattering amplitudes involving any number of on-shell pions with their physical masses,
\[
(a; P_1(k_1), \cdots, P_n(k_n)) \hat{S} |\beta; Q_1(l_1), \cdots, Q_n(l_n)| = [R^{P_1}(k_1) \cdots R^{P_n}(k_n) R^Q \cdots (-l_1) \cdots (-l_n)]^S 
\times |\alpha; S| /|\beta; S| = 0. 
\]
(42)
where $k_i (l_i)$ is the four momentum of the outgoing (incoming) pion $P_i (Q_i)$, and $a$ and $\beta$ describe states of other particles. Here we consider the case that no two pions have equal momenta. The symbol $[| S]$ means normalized symmetric permutations of the functional derivative operators, $| D_1 \cdots D_n| = (1/n!) \sum_{\text{perms}} D_1 \cdots D_n$. This operation clearly shows the crossing symmetry in Eq. (42). With Eq. (42) together with Eq. (7), scattering amplitudes are expressed in terms of Green’s functions of operators $\hat{O} = (j^{P}_{V\mu}, j^{A}_{\mu}, \sigma^P, \pi^P, \omega^P)$. The $\chi(R$
takes the form of functional derivative, and all constraints which stem from broken chiral symmetry are contained in $R^a(k)$. We can apply the same prescription as proposed in Ref. [15] to obtain the $\chi$RF including off-shell pions.

Here we note that the singlet axial current $j^a_A$ (or its functional derivative form) always appears with the RG invariant combination $\partial^\mu j^a_A - \omega^a$ in the axial master equation (39) and the $\chi$RF (42). [In the absence of the external fields, the RG transformation properties of $j^a_A$ and $\omega^a$ are $(j^a_A)_{\text{ren.}} = Z_A j^a_A$ and $(\omega^a)_{\text{ren.}} = \omega^a + (Z_A - 1) j^a_A$, respectively.] The existence of $\omega$ originating from the $U(1)_A$ anomaly is therefore crucial for ensuring the RG invariance of the master equations and the $\chi$RF.

With Eqs. (7) and (23), we can derive the chiral Ward identity for the gluon topological susceptibility $\chi$ as

$$
\chi = -i \int d^4x \langle T^a [\omega(x) \omega(0)] \rangle
$$

$$
= A - i \int d^4x \langle T^a [\hat{\omega}(x) \hat{\omega}(0)] \rangle
$$

$$
- i6A(f_\pi m_\pi f_\pi^T)^{-1} \int d^4x \langle \hat{\omega}(x) \hat{\omega}(0) \rangle
$$

$$
- i2\sqrt{6}A(f_\pi^{-1})^2 \int d^4x \langle T^a [\hat{\omega}(x) \hat{\pi}^Q(0)] \rangle
$$

(43)

This clearly shows how the pion poles contribute to the gluon topological susceptibility. It is noted that the appearance of the constant $A$, which corresponds to the leading contribution in the large $N_c$ limit of the QCD with massive quarks, $\chi = A + \mathcal{O}(1/N_c)$ [14], is due to the modification $\delta \mathcal{I}$ [Eq. (17)]. Making use of the chiral Ward identities of $\langle T^a [\omega(x) \pi^Q(0)] \rangle$ and $\langle T^a [\pi^P(x) \pi^Q(0)] \rangle$ derived from the axial master equation (39), the above identity can be further written as

$$
\chi = A - 6A^2[(f_\pi m_\pi f_\pi^T)^{-1}]^{100}_1
$$

$$
- i(1 - 6A)[(f_\pi m_\pi f_\pi^T)^{-1}]^{100}_1 \int d^4x \langle \hat{\omega}(x) \hat{\omega}(0) \rangle
$$

$$
+ 6A^2[(f_\pi m_\pi f_\pi^T)^{-1}]^{100}_1 [(m_\pi \tilde{G}(0))]_{ab} \hat{\omega}(x) \hat{\omega}(0)
$$

(44)

Note that this identity is obtained without relying on any unphysical limits, and thus the most general expression constrained only from the broken chiral symmetry. We can see that Eq. (44) contains the results from previous works in some limits. For this, we note that the second term on the right hand side comes from the pion pole contribution in $\langle T^a [\pi^P(x) \pi^Q(0)] \rangle$. With the mass relation (32) we obtain

$$
A - 6A^2[(f_\pi m_\pi f_\pi^T)^{-1}]^{100}_1 = A \left(1 + A \sum_{q=u,d,s} \frac{1}{m_q C_q} \right)^{-1}
$$

(45)

where $C_u = \bar{C} + C_3$, $C_d = \bar{C} - C_3$, and $C_s = (C_0 - \sqrt{2}C_8)/\sqrt{6}$ with $\bar{C} = (2C_0 + C_8)/\sqrt{3}$. At leading order of $1/f_\pi$ expansion [5] $(C_u, C_d, C_s) = (\langle u\bar{u} \rangle, \langle d\bar{d} \rangle, \langle s\bar{s} \rangle)$. Therefore, the first and second terms of our general expression (44), i.e., the contributions from the leading term in the large $N_c$ limit and the pion pole term, consistently reduce to the classic result [4, 16]. Also, our result approaches to zero in the chiral limit, $\chi \to 0$ ($m_q \to 0$), showing the noncommutative character of the $N_c \to \infty$ and $m_q \to 0$ limits [17].

The third and forth terms in Eq. (44) are the new results. The third term shows how the higher meson states or glueballs $X$ satisfying $\langle 0 |\omega | X \rangle \neq 0$ contribute to the gluon topological susceptibility. The broken chiral symmetry indicates that the existence of the forth term proportional to $\langle \hat{\pi}^P \rangle$ characterizing the deviation of the mass relation (32) from the GOR. This term, however, is expected to be a minor effect compared to the second term. Also, it is noted that the RG transformation property of $\omega$ indicates that the RG invariance of the zero-momentum projected two-point function $\int d^4x \langle T^a [\omega(x) \omega(0)] \rangle$ and thus Eq. (44).

Finally, as an illustration of the $\chi$RF (42), consider the two-photon decay of the $a^0$, $\eta$, and $\eta'$ mesons: $P(p) \to \gamma^+ \gamma^- (\eta (\eta'))$ with $P = (\pi^0, \eta, \eta')$. For this purpose, we need to evaluate the amplitude $\int d\tau_{\eta} d\eta [(f_\pi^{-1})^{Pc} \delta \delta \eta$ $\Omega_c^c(\tau)]$, the $\chi$RF gives

$$
\int d\tau_{\eta} d\eta [(f_\pi^{-1})^{Pc} \delta \delta \eta$ $\Omega_c^c(\tau)] = \int d\tau_{\eta} d\eta [(f_\pi^{-1})^{Pc} \delta \delta \eta$ $\Omega_c^c(\tau)] + \int d\tau_{\eta} d\eta [(f_\pi^{-1})^{Pc} \delta \delta \eta$ $\Omega_c^c(\tau)]
$$

(46)

where $W^c = \partial^\mu j^a_{\chi} - \omega^c$. With the expression of the electromagnetic current $j_\mu = j_\mu + (1/\sqrt{3}) j_3$, and of the external electromagnetic field $-A = V = \sqrt{3} e^8$, we can derive the general expression for the $P \to \gamma^+ \gamma^-$ decay amplitude:

$$
g_{\gamma\gamma P}(q_1^2, q_2^2; p^2)
$$

$$
= (f_\pi^{-1})^{Pc} \left[ e^2 N_c \delta \delta \eta \right] A \left( \frac{e^2 N_c}{8\pi} \right) - \delta \delta \eta \sqrt{6} F_{\gamma\gamma P}(q_1^2, q_2^2; p^2)
$$

$$
+ F_{\gamma\gamma A1}(q_1^2, q_2^2; p^2) - F_{\gamma\gamma A2}(q_1^2, q_2^2; p^2)
$$

(47)

with $(e^2 N_c, e^8, c_0) = (2/3, 2/3 \sqrt{3}, 4/\sqrt{9})$. Here we have defined

$$
\int d\tau_{\eta} d\eta [(f_\pi^{-1})^{Pc} \delta \delta \eta$ $\Omega_c^c(\tau)] = \int e^8 \gamma_{\eta\gamma \pi}(q_1^2, q_2^2; p^2)
$$

(48)

$$
\int d\tau_{\eta} d\eta [(f_\pi^{-1})^{Pc} \delta \delta \eta$ $\Omega_c^c(\tau)] = e^8 \gamma_{\eta\gamma \pi}(q_1^2, q_2^2; p^2)
$$

(49)
and
\[
\int dx_{q_1} dy_{q_2} dz_{-p} (T^\ast [j_{\mu_{1}}^p(x)j_{\mu_{2}}^\ast(y)] X_{3}(z)) = F_{\gamma A1}(q_1^2, q_2^2; p^2) - F_{\gamma A2}(q_1^2, q_2^2; p^2),
\]
modulo \((2\pi)^d \delta^4(p - q_1 - q_2).\) In Eq. (50) we have used the general expression of the vector-vector-axial correlation function \([18, 19]\) denoting \(F\) as a general expression of the vector-vector-axial correlation function.

The broken chiral symmetry relates the current and density operators \((49)\) and \((50)\). The correlation functions \(F\) are related to the two-photon decay formula of pions. Using the PCAC hypothesis \((47)\) is applicable both for on- and off-shell mesons. Using the PCAC hypothesis \(g_{\gamma \gamma p}(q_1^2, q_2^2; m_p^2) \sim g_{\gamma \gamma p}(q_1^2, q_2^2; 0)\) and setting \(q_1\) and \(q_2\) to the photon point, \(q_1^2 = q_2^2 = 0,\) our general expression \((47)\) consistently reduces to the two-photon decay formula of \(\pi^0, \eta, \eta'\) mesons derived by Shore \([4, 14]\).

The investigations in this direction will be discussed elsewhere.

To summarize, we have derived an extension of the master equations for chiral symmetry breaking to the \(U_R(3) \times U_L(3)\) chiral group, in which the \(U_A(1)\) anomaly and flavor symmetry breaking are properly contained. With the master equations and the \(\chi RF,\) new identities for the gluon topological susceptibility \(\chi\) and \(P \rightarrow \gamma^{(*)}\gamma^{(*)}\) decay amplitude have been derived, showing how the constraints from the broken chiral symmetry are entered into those quantities without relying on any unphysical limits.

The \(\chi RF\) is applicable to any reaction processes including ground pseudoscalar mesons, e.g., \(\pi, K, \bar{K}, \eta, \eta'\) production reactions on a baryon target and heavy meson decays such as \(J/\psi \rightarrow 3P, \gamma 2P, \gamma 3P, \cdots\) with \(P = \pi, K, \bar{K}, \eta, \eta'.\) The heavy meson decays are interesting in relation with new meson resonance states appearing in the decay processes \([20, 21]\). A proper treatment of the final state interactions in the decay processes will be vital for exploring the properties of those new meson states. Thus the \(\chi RF,\) which enables one to separate details of each reaction mechanism from the general framework required by broken chiral symmetry, will provide an useful theoretical basis for the analysis of such processes.

The investigations in this direction will be discussed elsewhere.

The author would like to thank Dr. C.-H. Lee for sending his note, and Dr. T. Sato for careful reading and useful comments on the manuscript. This work was supported by U.S. Department of Energy, Office of Nuclear Physics Division, under Contract No. DE-AC05-06OR23177.

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