Incompressible Limit of the Compressible Hydrodynamic Flow of Liquid Crystals

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Abstract
This paper is concerned with the incompressible limit of the compressible hydrodynamic flow of liquid crystals with periodic boundary conditions in \(\mathbb{R}^N(N = 2, 3)\). It is rigorously shown that the local (and global) strong solution of the compressible system converges to the local (and global) strong solution of the incompressible system. Furthermore, the convergence rates are also obtained in some sense.

Key Words: compressible flow, incompressible flow, incompressible limit, liquid crystals.

AMS subject classifications: 35Q35, 76N10.

1 Introduction

The hydrodynamic flow of incompressible liquid crystals was first derived by Ericksen [6] and Leslie [16] in 1960s. However, its rigorous mathematical analysis did not take place until 1990s, when Lin [17] and Lin and Liu [18, 19, 20] made some very important progress towards the existence of global weak solutions and partial regularity of the incompressible hydrodynamic flow equation of liquid crystals.

In the context of hydrodynamics, the basic variable is the flow map (particle trajectory) \(x(X, t)\). \(X\) is the original labeling (Lagrangian coordinate) of the particle. It is also referred to as

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to as material coordinate. $x$ is the current (Eulerian) coordinate and referred to as reference coordinate. For a given velocity field $u(x, t)$, the flow map is defined by the following ordinary differential equation:

$$x_t(X, t) = u(x(X, t), t), \quad x(X, 0) = X. \quad (1.1)$$

The deformation tensor $F(X, t)$ is defined as

$$F(X, t) = \frac{\partial x}{\partial X}. \quad (1.2)$$

Applying the chain rule, we see that $F(x, t)$ satisfies the following transport equation:

$$F_t + u \cdot \nabla F = \frac{d}{dt} F = \frac{\partial x_t}{\partial X} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial X} = \nabla u F, \quad (1.3)$$

which stands for $F_{ij,t} + u_k \nabla_k F_{ij} = \nabla_k v_i F_{kj}$.

Let us define the density as

$$\rho(x, t) = \frac{\rho_0(X)}{\det F}. \quad (1.4)$$

By the identity of the variation of the determinant of a tensor

$$\delta \det F = \det F tr(F^{-1} \delta F), \quad (1.5)$$

we have

$$\partial_t \rho + u \cdot \nabla \rho = \frac{d}{dt} \left( \frac{\rho_0(X)}{\det F} \right) = -\frac{\rho_0(X)}{(\det F)^2} \det F tr \left( F^{-1} \frac{d}{dt} F \right) = -\rho \nabla \cdot u, \quad (1.6)$$

then we get the transport equation

$$\partial_t \rho + u \cdot \nabla \rho + \rho \nabla \cdot u = 0, \quad (1.7)$$

which can also be reached by the conservation of mass.

The momentum equation of motion for hydrodynamics flow of liquid crystals can be derived from the least action principle (Hamilton principle). The action functional takes the form:

$$\mathcal{A}(x) = \int_0^t \int_{\Omega_0} \left( \frac{1}{2} \rho_0(X) |x_t|^2 - W^\lambda(F) \right) dX dt, \quad \lambda > 0, \quad (1.8)$$

where $\Omega_0$ is the original domain occupied by the material. The first part of $\mathcal{A}(x)$ denotes the kinetic energy and the second one denotes the elastic energy. By the definition of the density in (1.4), we get

$$\int_0^t \int_{\Omega_0} \rho_0(X) |x_t|^2 dX dt = \int_0^t \int_{\Omega_0} \frac{\rho_0(X)}{\det F} |x_t|^2 \det F dX dt = \int_0^t \int_{\Omega} \rho(x, t) |u(x, t)|^2 dx dt. \quad (1.9)$$
In this paper, we will consider the isotropic energy functions $W^\lambda(F)$ of the form

$$W^\lambda(F) = \left( \lambda^2 \omega \left( \frac{\rho_0(X)}{\det F} \right) + \frac{\nu}{2} |F^{-1} \nabla_X n_0(X)|^2 \right) \det F,$$  \hspace{1cm} (1.10)

where $\omega$ is a $C^\infty$ function and denotes the energy density, $n_0(X) : \Omega_0 \to S^2$ is a unit-vector field that represents the molecular orientation of the liquid crystal material. The first term of the R.H.S. of (1.10) should be regarded as a penalization term which drives the motion toward incompressibility in the limit as the parameter $\lambda$ becomes large. We can use the kinematic assumption to give

$$n_t + u \cdot \nabla n = 0,$$  \hspace{1cm} (1.11)

then we have

$$n(x(X,t),t) = n_0(X),$$  \hspace{1cm} (1.12)

which implies that the center of gravity of the molecules moves along the streamline of the velocity, and the second term of the R.H.S. of (1.10) is equivalent to $\frac{\nu}{2} |\nabla n|^2$ in Eulerian coordinate, where $\nu > 0$ is viscosity of the fluid and denotes the microscopic elastic relaxation time.

For any $y = (y_1, y_2, \ldots, y_N) \in C^1_c(\Omega; \mathbb{R}^N)$, let $x^\epsilon = x + \epsilon y(x)$, $F^\epsilon = \frac{\partial x^\epsilon}{\partial x}$ and $\rho^\epsilon = \rho(x^\epsilon(X,t),t)$. Then we apply $\delta = \frac{d}{d\epsilon}|_{\epsilon=0}$ to (1.8) and get

$$0 = \delta A(x^\epsilon) = \int_0^t \int_{\Omega_0} \left[ \rho_0(X)(x_t, y_t) - \delta W^\lambda(F) \right] dX dt,$$  \hspace{1cm} (1.13)

where we denote the inner product $f \cdot g$ by $(f,g)$ for some $f,g : \Omega \to \mathbb{R}^N$. By using the definition of $\rho(x,t)$, we have

$$\int_0^t \int_{\Omega_0} \rho_0(X)(x_t, y_t) dX dt = -\int_0^t \int_{\Omega_0} \rho_0(X)(x_t, y) dX dt = -\int_0^t \int_{\Omega_0} \frac{\rho_0(X)}{\det F} (x_t, y) dxdt$$

$$= -\int_0^t \int_{\Omega} \rho(x,t)(u_t + u \cdot \nabla u, y) dxdt.$$  \hspace{1cm} (1.14)

Now we turn to show the calculation about the second term of the R.H.S of (1.13).

Noting the facts that

$$\delta \rho^\epsilon = \delta \left( \frac{\rho_0(X)}{\det F^\epsilon} \right) = -\frac{\rho_0(X)}{(\det F)^2} \det F tr (F^{-1} \delta F^\epsilon)$$

$$= -\frac{\rho_0(X)}{\det F} tr \left( \frac{\partial X}{\partial x} \frac{\partial y}{\partial X} \right) = -\frac{\rho_0(X)}{\det F} \nabla \cdot y,$$  \hspace{1cm} (1.15)

and

$$\delta (\det F^\epsilon) = \det F tr (F^{-1} \nabla_X y) = \det F \nabla \cdot y,$$  \hspace{1cm} (1.16)
we use the definition of $\rho$ to give
\[
-\lambda^2 \int_0^t \int_{\Omega_0} \delta \left( \omega \left( \frac{\rho_0(X)}{\det F} \right) \det F^* \right) dX dt
\]
\[
= -\lambda^2 \int_0^t \int_{\Omega_0} \left( -\omega' \left( \frac{\rho_0(X)}{\det F} \right) \nabla \cdot y \det F + \omega \left( \frac{\rho_0(X)}{\det F} \right) \det F \nabla \cdot y \right) dX dt
\]
\[
= -\lambda^2 \int_0^t \int_{\Omega} \left( \omega(\rho) - \rho \omega'(\rho) \right) \nabla \cdot y dx dt
\]
\[
= -\lambda^2 \int_0^t \int_{\Omega} (\nabla P(\rho), y) dx dt,
\] (1.17)
where we define the pressure as
\[
P(\rho) = -\omega(\rho) + \rho \omega'(\rho),
\] (1.18)
which can also be reached by the first law of thermodynamics. Here we show the proof briefly. According to the first law of thermodynamics, we obtain
\[
dW = -PdV - SdT,
\] (1.19)
where $W$, $V$, $S$ and $T$ denote the energy, volume, entropy, and temperature, respectively. Moreover,
\[
W = \omega V, \quad V = \frac{m}{\rho}, \quad \omega = \omega(\rho),
\] (1.20)
where $m$ is the total mass. From (1.19), we have
\[
-\frac{\nu}{2} \int_0^t \int_{\Omega_0} \delta \left( \left| (F^*)^{-1} \nabla \chi n_0(X) \right|^2 \det F^* \right) dX dt
\]
\[
= -\frac{\nu}{2} \int_0^t \int_{\Omega_0} \left[ 2 \left( F^{-1} \nabla \chi n_0(X), \delta(F^*)^{-1} \nabla \chi n_0(X) \right) \det F + \left| F^{-1} \nabla \chi n_0(X) \right|^2 \delta \left( \det F^* \right) \right] dX dt,
\]
\[
= V_1 + V_2,
\] (1.22)
where
\[
V_2 = -\frac{\nu}{2} \int_0^t \int_{\Omega_0} \left| F^{-1} \nabla \chi n_0(X) \right|^2 \det F \text{tr} \left( F^{-1} \delta F^* \right) dX dt
\]
where \( D \) is so-called deformation tensor, \( \mu \) and \( \kappa \) are the shear viscosity and the bulk viscosity coefficients satisfying \( \mu > 0 \) and \( N \kappa + 2 \mu \geq 0 \). Let \( u^\tau = u + \tau v, v \in C^1_c(\Omega; \mathbb{R}^N) \). Then applying \( \bar{\delta} = \frac{d}{d\tau} \bigg|_{\tau=0} \) to \( \triangle \), and by using integration by parts, we have

\[
\bar{\delta} \triangle = \frac{\mu}{2} \int_{\Omega} (\nabla u + \nabla^T u, \nabla v + \nabla^T v) \, dx + \kappa \int_{\Omega} (\nabla \cdot u, \nabla \cdot v) \, dx
\]
\[ \begin{align*}
&= \frac{\mu}{2} \int_{\Omega} \left[ (\nabla u, \nabla v) + (\nabla^T u, \nabla^T v) \right] \, dx + \frac{\mu}{2} \int_{\Omega} \left[ (\nabla u, \nabla T v) + (\nabla^T u, \nabla v) \right] \, dx \\
&\quad + \kappa \int_{\Omega} (\nabla \cdot u, \nabla \cdot v) \, dx \\
&= \int_{\Omega} (\mu \Delta u + (\kappa + \mu) \nabla (\nabla \cdot u), v) \, dx = 0,
\end{align*} \tag{1.29} \]

where the third term of (1.28) denotes the microscopic dissipation, and has been treated as zero when we apply \( \tilde{\delta} \) to (1.28). Clearly, if the system is incompressible, the term \((\kappa + \mu)\nabla (\nabla \cdot u)\) in (1.29) should be zero.

Combining (1.14), (1.17), (1.23), (1.24) and (1.29), we have the momentum equation

\[
\rho u_t + \rho (u \cdot \nabla) u + \lambda^2 \nabla (P(\rho)) = \mu \Delta u + (\kappa + \mu) \nabla (\nabla \cdot u) - \nu \nabla \left( \nabla n \otimes \nabla n - \frac{|\nabla n|^2}{2} I \right), \tag{1.30}
\]

Finally, we deal with the heat flow for harmonic maps. For any \( C^1 \) map \( n : \Omega \rightarrow S^2 \), let

\[
I(n) = \theta \int_{\Omega} |\nabla n|^2 \, dx
\]
denote the energy of \( n \), where \( \theta > 0 \) is the viscosity constant. The critical points of the energy \( I(n) \) are called harmonic maps. To get the critical points of the energy \( I(n) \), let

\[
m(\tau) = \frac{n + \tau \varphi}{|n + \tau \varphi|}, \tag{1.31}
\]

Then by applying \( \tilde{\delta} = \frac{d}{d\tau} \bigg|_{\tau=0} \) to \( I(m(\tau)) \), we have

\[
0 = \frac{\theta}{2} \tilde{\delta} \left( \int_{\Omega} |\nabla m(\tau)|^2 \, dx \right) = \theta \int_{\Omega} \nabla m(0) : \nabla m'(0) \, dx = \theta \int_{\Omega} \nabla n : \nabla m'(0) \, dx. \tag{1.32}
\]

Direct calculation from (1.31) yields

\[
m'(\tau) = \frac{\varphi}{|n + \tau \varphi|} - \frac{(n + \tau \varphi)(n + \tau \varphi, \varphi)}{|n + \tau \varphi|^3}, \tag{1.33}
\]

and by using the fact that \( |n| = 1 \), we obtain \( m'(0) = \varphi - (n, \varphi)n \). Consequently, we have

\[
\begin{align*}
\int_{\Omega} \nabla n : \nabla m'(0) \, dx &= \int_{\Omega} \nabla n : \nabla [\varphi - (n, \varphi)n] \, dx = \int_{\Omega} \nabla n : \nabla \varphi \, dx - \int_{\Omega} \nabla n : \nabla [(n, \varphi)n] \, dx \\
&= - \int_{\Omega} (\Delta n, \varphi) \, dx - \int_{\Omega} \nabla n : \nabla [(n, \varphi)n] \, dx.
\end{align*} \tag{1.34}
\]

Noting that \( \nabla [(n, \varphi)n] = (n, \varphi) \nabla n + (\nabla \varphi \cdot n + \nabla n \cdot \varphi)n \), one can easily deduces that

\[
\nabla n : \nabla [(n, \varphi)n] = |\nabla n|^2 (n, \varphi), \tag{1.35}
\]

where we have used the fact \( \nabla (|n|^2) = 0 \).

Combining (1.32), (1.34) and (1.35), we have

\[
\int_{\Omega} (\theta(\Delta n + |\nabla n|^2 n), \varphi) \, dx = 0, \tag{1.36}
\]
then we get the heat flow for harmonic maps by considering the evolution equation associated with the harmonic maps

\[ n_t + (u \cdot \nabla)n = \theta(\Delta n + |\nabla n|^2 n). \quad (1.37) \]

In conclusion, the compressible hydrodynamic flow equation of liquid crystals in \( \Omega = T^N \subset \mathbb{R}^N \) can be written as follows (the functions and the viscosity constants should depend on the value of the parameter \( \lambda \)):

\[
\begin{align*}
\rho_t^\lambda + \nabla \cdot (\rho^\lambda u^\lambda) &= 0, \\
(\rho^\lambda u^\lambda)_t + \nabla \cdot (\rho^\lambda u^\lambda \otimes u^\lambda) + \lambda^2 \nabla (P(\rho^\lambda)) &= \mu^\lambda \Delta u^\lambda + (\kappa^\lambda + \mu^\lambda) \nabla (\nabla \cdot u^\lambda) - \nu^\lambda \nabla \cdot \left( \nabla n^\lambda \otimes \nabla n^\lambda - \frac{|\nabla n^\lambda|^2}{2} I_N \right), \\
n_t^\lambda + (u^\lambda \cdot \nabla)n^\lambda &= \theta^\lambda (\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda). 
\end{align*}
\]

(1.38)

Here \( x \in T^N \), a torus in \( \mathbb{R}^N \), \( N = 2 \) or \( 3 \), \( t > 0 \), the coefficients \( \mu^\lambda, \nu^\lambda, \theta^\lambda > 0 \), and 
\( 2\mu^\lambda + N\kappa^\lambda > 0 \). The unknown functions are the density \( \rho^\lambda \), the velocity \( u^\lambda \in \mathbb{R}^N \), and the molecular orientation of the liquid crystal material \( n^\lambda : T^N \times (0, +\infty) \rightarrow S^2 \), which is a unit vector, i.e., \( |n^\lambda| = 1 \). \( P(\rho) \) is the smooth pressure-density function with \( P'(\rho) > 0 \) for \( \rho > 0 \). And the symbol \( \otimes \) and \( \circ \) denote the tensor product such that \( u \otimes u = (u^i u^j)_{1 \leq i, j \leq N} \) and \( \nabla n \otimes \nabla n = (n_{x_i} \cdot n_{x_j})_{1 \leq i, j \leq N} \), respectively.

From the mathematical point of view, it is reasonable to expect that, as \( \rho^\lambda \rightarrow 1 \), the first equation in (1.38) yields the incompressible condition \( \nabla \cdot u = 0 \). Suppose that the limits \( u^\lambda \rightarrow u \) and \( n^\lambda \rightarrow n \) exist as \( \lambda \rightarrow \infty \), and the viscosity coefficients satisfy that

\[
\mu^\lambda \rightarrow \mu > 0, \quad \kappa^\lambda \rightarrow \kappa, \quad \nu^\lambda \rightarrow \nu > 0, \quad \theta^\lambda \rightarrow \theta > 0, \quad \text{as} \quad \lambda \rightarrow \infty, \quad (1.39)
\]

then (at least formally) we obtain the following incompressible model by taking \( \lambda \rightarrow \infty \):

\[
\begin{align*}
\nabla \cdot u &= 0, \\
(u_t + (u \cdot \nabla)u + \nabla p) &= \mu \Delta u - \nu \nabla \cdot (\nabla n \otimes \nabla n), \\
n_t + (u \cdot \nabla)n &= \theta(\Delta n + |\nabla n|^2 n),
\end{align*}
\]

(1.40)

where \( p \) is the limit of the term \( \frac{\lambda^2}{\rho} \nabla \cdot \left( P(\rho^\lambda) \right) - \nu^\lambda \frac{|\nabla n|^2}{2} \). For simplicity we assume in the following that \( \kappa^\lambda \equiv \kappa, \mu^\lambda \equiv \mu, \mu^\lambda \equiv \mu \) and \( \theta^\lambda \equiv \theta \) are constants independent of \( \lambda \), satisfying that \( \mu, \nu, \theta > 0 \) and \( 2\mu + N\kappa \geq 0 \).

In a series of papers, Lin [17] and Lin and Liu [18, 19, 20] addressed the existence and partial regularity theory of suitable weak solutions to the incompressible hydrodynamic...
flow of liquid crystals of variable length. More precisely, they considered the approximate
equation of incompressible hydrodynamic flow of liquid crystals (i.e., \(|\nabla n|^2\) is replaced by\(\frac{(1-|n|^2)n}{\epsilon^2}\)), and proved \cite{18}, among many other results, the local existence of classical
solutions and the global existence of weak solutions in dimension two and three. For any
fixed \(\epsilon > 0\), they also showed the existence and uniqueness of global classical solution
either in dimension two or dimension three when the fluid viscosity \(\mu\) is sufficiently large;
in \cite{20}, Lin and Liu extended the classical theorem by Caffarelli-Kohn-Nirenberg \cite{1} on the
Navier-Stokes equation that asserts the one dimensional parabolic Hausdorff measure of the
singular set of any suitable weak solution is zero. See also \cite{22, 23, 28} for relevant results.

The paper is organized as follows. In Section 2, we state the main results of this paper.
In Section 3, we prove that the strong local solutions of (1.38) exist for sufficiently small
disturbances from the general incompressible initial data, meanwhile, we get the uniform
stability of the local solutions family which yields a lifespan of the system (1.38). Based
on it, we show that the local solutions of (1.38) converge to a local solution of the limiting
incompressible system (1.40) by means of compactness arguments. In Section 4, the global
existence of the strong solution to the incompressible system (1.40) is derived provided the
initial data of the incompressible model sufficiently small. In Section 5, we obtain the
convergence rates about \((\rho^\lambda, u^\lambda, n^\lambda) \to (1, u, n)\) in some sense when \(\lambda \to \infty\). These results
depend on some methods modified from \cite{11, 13, 15, 27}.

2 Notations and Statements of Main Results

Noting that

\[
\nabla \cdot \left( \nabla n^\lambda \otimes \nabla n^\lambda - \frac{[\nabla n^\lambda]^2}{2} I_N \right) = \sum_{i=1}^{n} \Delta n_i^\lambda \nabla n_i^\lambda,
\]

we can rewrite (1.38) as follows when \(|\rho^\lambda - 1|\) is small:

\[
\begin{align*}
\rho^\lambda_t + \nabla \cdot (\rho^\lambda u^\lambda) &= 0, \\
\rho^\lambda u^\lambda + \left( u^\lambda \cdot \nabla \right) u^\lambda + \frac{\lambda^2}{\rho^\lambda} \nabla P(\rho^\lambda) &= \frac{\mu}{\rho^\lambda} \Delta u^\lambda + \frac{(\kappa + \mu)}{\rho^\lambda} \nabla (\nabla \cdot u^\lambda) - \frac{\nu}{\rho^\lambda} \sum_{i=1}^{N} \Delta n_i^\lambda \nabla n_i^\lambda, \\

n_i^\lambda + (u^\lambda \cdot \nabla) n^\lambda &= \theta (\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda).
\end{align*}
\]

Throughout this paper, for simplicity, we will denote \(\sum_{i=1}^{N} \Delta n_i^\lambda \nabla n_i^\lambda\) by \(\Delta n^\lambda \cdot \nabla n^\lambda\), the
norms in \(L^2(\mathbb{T}^n)\), \(H^s(\mathbb{T}^n)\) and \(L^\infty(\mathbb{T}^n)\) by \(\| \cdot \|\), \(\| \cdot \|_s\) and \(\| \cdot \|_\infty\) respectively, and especially
where there are some differences between (2.3) and (2.4) in terms of the norms in \((2.5)\) and \((2.6)\).

where \(U = (\rho, u, n)\). We point out that the parameter \(\lambda\) is hidden in the definition of \(E_s(U(t))\) and \(\tilde{E}_s(U(t))\). Similarly, we define

\[
\begin{aligned}
F_s(U(t)) &= \frac{1}{2} \sum_{|\alpha| \leq s} \int_{\mathbb{T}^n} \left[ \lambda^2 |\nabla^\alpha (\rho - 1)|^2 + |\nabla^\alpha u|^2 \right] \, dx + \frac{1}{2} \sum_{|\beta| \leq s-1} \int_{\mathbb{T}^n} |\nabla \nabla^\beta n|^2 \, dx, \\
\tilde{F}_s(U(t)) &= \frac{1}{2} \sum_{|\alpha| \leq s} \int_{\mathbb{T}^n} \left[ \lambda^2 \frac{P'(\rho)}{\rho} |\nabla^\alpha (\rho - 1)|^2 + \rho |\nabla^\alpha u|^2 \right] \, dx + \frac{1}{2} \sum_{|\beta| \leq s-1} \int_{\mathbb{T}^n} |\nabla \nabla^\beta n|^2 \, dx,
\end{aligned}
\tag{2.4}
\]

where there are some differences between (2.3) and (2.4) in terms of \(n\). It is obvious that

\[E_s(U(t)) \sim \tilde{E}_s(U(t)), \quad F_s(U(t)) \sim \tilde{F}_s(U(t)),\]

provided that \(|\rho - 1|\) is sufficiently small.

Now we state the main results of this paper.

**Theorem 2.1** Consider the compressible model \((1.38)\) with the following initial data

\[
\rho^\lambda(x, 0) = 1 + \overline{\rho}_0^\lambda(x), \quad u^\lambda(x, 0) = u_0(x) + \overline{u}_0^\lambda(x), \quad n^\lambda(x, 0) = \frac{n_0(x) + \overline{n}_0^\lambda(x)}{|n_0(x) + \overline{n}_0^\lambda(x)|},
\tag{2.6}
\]

where \(u_0, n_0\) satisfy

\[
u_0(x) \in H^{s+1}(\mathbb{T}^n), \quad \nabla \cdot u_0 = 0, \quad n_0 \in H^{s+1}(\mathbb{T}^n, S^2),
\tag{2.7}
\]

for any \(s \geq \left\lceil \frac{N}{2} \right\rceil + 2\). Moreover, for some small positive constants \(\delta_0\), the functions \(\overline{\rho}_0^\lambda(x), \overline{u}_0^\lambda(x), \overline{n}_0^\lambda(x)\) are assumed to satisfy

\[
\|\overline{\rho}_0^\lambda\|_s \leq \lambda^{-2}\delta_0, \quad \|\overline{u}_0^\lambda(x)\|_{s+1} \leq \lambda^{-1}\delta_0, \quad \|\nabla n^\lambda(x, 0) - \nabla n_0(x)\|_s \leq \lambda^{-1}\delta_0.
\tag{2.8}
\]

Then the following statements hold:

**Uniform stability:** There exist fixed constants \(T_0\) and \(C\) independent of \(\lambda\) such that the unique strong solution \((\rho^\lambda, u^\lambda, n^\lambda)\) of system \((1.38)\) exists for all large \(\lambda\) on the time interval \([0, T_0]\). Furthermore, the solution family satisfy:

\[
\begin{aligned}
&\left\{ \begin{array}{l}
E_s(U^\lambda(t)) + \int_0^t \left[ \mu \|\nabla u^\lambda\|^2_s + (\kappa + \mu) \|\nabla \cdot u^\lambda\|^2_s + \theta \|\nabla n^\lambda\|^2_s \right] \, dt \\
E_{s-1}(\partial_t U^\lambda(t)) + \int_0^t \left[ \mu \|\nabla \partial_t u^\lambda\|^2_{s-1} + (\kappa + \mu) \|\nabla \cdot \partial_t u^\lambda\|^2_{s-1} + \theta \|\nabla \partial_t n^\lambda\|^2_{s-1} \right] \, dt \leq C,
\end{array} \right. \\
&\|n^\lambda\|_s = 1, \quad \text{in} \quad \overline{Q_{T_0}} = \mathbb{T}^N \times [0, T_0],
\end{aligned}
\tag{2.9}
\]
where \( E_{s-1}(\partial_t U(t)) = \frac{1}{2} \sum_{|\beta| \leq s-1} \int_{\Omega} (\lambda^2 |\nabla^\beta \partial_t \rho|^2 + |\nabla^\beta \partial_t u|^2 + |\nabla^\beta \partial_t n|^2) \, dx \).

Local existence of solutions for incompressible system: There exist functions \( u \) and \( n \) such that

\[
\begin{cases}
\rho^\lambda \to 1 & \text{in } L^\infty([0, T_0]; H^s) \cap \text{Lip}([0, T_0]; H^{s-1}), \\
(u^\lambda, n^\lambda) \rightharpoonup (u, n) & \text{weakly} in \ L^\infty([0, T_0]; H^s) \cap \text{Lip}([0, T_0]; H^{s-1}), \\
(u^\lambda, n^\lambda) \to (u, n) & \text{in } C([0, T_0]; H^{s'})
\end{cases}
\]  

(2.10)

for any \( s' \) in \([0, s)\), and the function pair \((u, n)\) is the unique strong solution of the incompressible system of liquid crystal (1.4) with the initial data

\[
u(x, 0) = u_0, \quad n(x, 0) = n_0,
\]  

(2.11)

for some \( p \in L^\infty([0, T_0]; H^{s-1}) \cap L^2([0, T_0]; H^s)\).

**Theorem 2.2** Consider the strong solutions \((\rho^\lambda, u^\lambda, n^\lambda)\) of system (1.38) obtained in Theorem 2.1. Suppose in addition that the initial data satisfies

\[
\|u_0\|^2_s + \|\nabla n_0\|^2_{s-1} \leq \varepsilon_0,
\]  

(2.12)

where \( \varepsilon_0 \) is a positive constant. If \( \varepsilon_0 \) and \( \lambda^{-1} \) are sufficiently small, then for every fixed \( T > 0 \), the strong solution \((\rho^\lambda, u^\lambda, n^\lambda)\) exists on the time interval \([0, T^\lambda)\) with \( T^\lambda > T \), which satisfies

\[
E_s(U^\lambda(t)) + \int_0^t [\mu \|\nabla u\|^2_s + (\kappa + \mu) \|\nabla \cdot u\|^2_s + \theta \|\nabla^2 n\|^2_{s-1}] \leq 4(\varepsilon_0 + \lambda^{-2} \delta_0^2),
\]

(2.13)

and

\[
E_{s-1}(\partial_t U^\lambda(t)) + \int_0^t [\mu \|\nabla \partial_t u\|^2_{s-1} + (\kappa + \mu) \|\nabla \cdot \partial_t u\|^2_{s-1} + \theta \|\nabla \partial_t n\|^2_{s-1}] \leq C \exp C t,
\]

(2.14)

for \( t \in [0, T] \), and \( T^\lambda \to \infty \) as \( \lambda \to \infty \). Furthermore, as \( \lambda \to \infty \), \((\rho^\lambda, u^\lambda, n^\lambda)\) converges to the unique global strong solution \((1, u, n)\) of the incompressible system of liquid crystal (1.40), with \(|n(x, t)| = 1\) in \( \overline{Q_T} \) and

\[
\|u\|^2_s + \|\nabla n\|^2_{s-1} + \int_0^t (\mu \|\nabla u\|^2_s + \theta \|\nabla^2 n\|^2_{s-1}) \leq C_1 \varepsilon_0,
\]

(2.15)

where \( C_1 \) is a uniform constant depending only on \( N, s, \mu, \kappa, \nu, \theta \) and the domain, but independent of \( \varepsilon_0 \) and \( \lambda \).
Theorem 2.3 Under the assumptions of Theorem 2.1, the convergence rate of $\rho^\lambda$, $u^\lambda$ and $n^\lambda$ when $\lambda \to \infty$ is deduced in the following sense that
\[
\lambda\|\rho^\lambda - 1\|_s^2 + \|u^\lambda - u\|_s^2 + \|n^\lambda - n\|_s^2 + \int_0^t \left(\|u^\lambda - u\|_1^2 + \|n^\lambda - n\|_3^2\right) \leq C\lambda^{-1}. \tag{2.16}
\]
Furthermore, we have
\[
\|\nabla (\rho^\lambda - 1)\|_{s-2}^2 \leq C\lambda^{-4}. \tag{2.17}
\]
The statement also holds for the global strong solution given in Theorem 2.2.

Remark 2.1 In Theorem 2.3, we do not give the convergence rates about the higher order derivatives of $u^\lambda$ and $n^\lambda$ because we know little about the convergence rate of the pressure.

3 Local Existence and Uniform Stability

Let $U_0 = (1 + \overline{U}_0, u_0 + \overline{u}_0, \frac{n_0 + \overline{n}_0}{m_0 + \overline{m}_0})$. We consider a set of functions $B^3_{T_0}(U_0) = B_{T_0}^3(U_0, \delta, s, K_1, K_2)$ contained in $L^\infty([0, T_0]; H^s) \cap Lip([0, T_0]; H^{s-1})$ with $s \geq 3$ and defined by
\[
\begin{cases}
|\lambda (\rho - 1)| + |u - u_0| + |\nabla n - \nabla n_0| < \delta, \\
E_s(U(t)) + \int_0^t [\mu \|\nabla u\|_s^2 + (\kappa + \mu)\|\nabla \cdot u\|_s^2 + \theta \|\nabla n\|_s^2] \leq K_1, \\
E_{s-1}(\partial_t U(t)) + \int_0^t [\mu \|\nabla \partial_t u\|_{s-1}^2 + (\kappa + \mu)\|\nabla \cdot \partial_t u\|_{s-1}^2 + \theta \|\nabla \partial_t n\|_{s-1}^2] \leq K_2.
\end{cases} \tag{3.1}
\]
For $V = (\xi^\lambda, v^\lambda, m^\lambda) \in B^3_{T_0}(U_0)$, define $U = (\rho^\lambda, u^\lambda, n^\lambda) = \Lambda(V)$ as the unique solution of the following problem:
\[
\begin{cases}
\rho^\lambda + (v^\lambda \cdot \nabla)\rho^\lambda + \xi^\lambda \nabla \cdot u^\lambda = 0, \\
u^\lambda + (v^\lambda \cdot \nabla)u^\lambda + \lambda^2 \frac{P'(\xi^\lambda)}{\xi^\lambda} \nabla \rho^\lambda = \frac{\mu}{\xi^\lambda} \Delta u^\lambda + \frac{\kappa + \mu}{\xi^\lambda} \nabla (\nabla \cdot u^\lambda) - \frac{\kappa}{\xi^\lambda} \Delta n^\lambda \cdot \nabla n^\lambda, \tag{3.2}
\end{cases}
\]
\[
n^\lambda + (v^\lambda \cdot \nabla) n^\lambda = \theta (\Delta n^\lambda + |\nabla m^\lambda|^2 n^\lambda).
\]
We plan to show that for appropriate choices of $T_0$, $\delta$, $K_1$ and $K_2$ independent of $\lambda$, $\Lambda$ maps $B^3_{T_0}(U_0)$ into itself and it is a contraction in certain function spaces. We emphasize that the solutions will depend on the value of the parameter $\lambda$, but for convenience, the dependence will not always be displayed in this section.

We need the following lemma for the proofs.

Lemma 3.1 Assume that $f, g \in H^k(\mathbb{T}^N)$. Then for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{N}$, $|\alpha| \leq k$, we have
\[
(i) \quad \|D^\alpha (fg)\| \leq C_{N,k}(\|f\|_\infty ||D^k g|| + \|g\|_\infty ||D^k f||),
\]
(ii) $\|D^\alpha (fg) - f D^\alpha g\| \leq C_{N,k}(\|D f\|_\infty \|D^{k-1}g\| + \|g\|_\infty \|D^k f\|)$,

where $C_{N,k}$ is a constant depending only on $k$ and $N$.

Before proceeding any further, we apply $D^\alpha$ to (3.2) and get

$$\begin{align*}
\partial_t D^\alpha \rho + (v \cdot \nabla) D^\alpha \rho + \xi \nabla \cdot D^\alpha u &= \Pi_1, \\
\partial_t D^\alpha u + (v \cdot \nabla) D^\alpha u + \lambda^2 \frac{P(\xi)}{\xi} \nabla D^\alpha \rho &= \frac{\mu}{\xi} \Delta D^\alpha u + \frac{k+\mu}{\xi} \nabla(D^\alpha \nabla \cdot u) - \frac{\nu}{\xi} D^\alpha (\Delta n \cdot \nabla) + \Pi_2,
\end{align*}$$

(3.3)

where

$$\begin{align*}
\Pi_1 &= -[D^\alpha (v \cdot \nabla \rho) - v \cdot \nabla D^\alpha \rho] - [D^\alpha (\xi \nabla \cdot u) - \xi \nabla \cdot D^\alpha u], \\
\Pi_2 &= -[D^\alpha (v \cdot \nabla u) - v \cdot \nabla D^\alpha u] - \lambda^2 \left\{ D^\alpha \left( \frac{P(\xi)}{\xi} \nabla \rho \right) - \frac{P'(\xi)}{\xi} \nabla \rho \right\} \\
&\quad + \left\{ D^\alpha \left( \frac{\mu}{\xi} \Delta u \right) - \frac{\mu}{\xi} \Delta D^\alpha u \right\} + \left\{ D^\alpha \left[ \frac{k+\mu}{\xi} \nabla (\nabla \cdot u) \right] - \frac{k+\mu}{\xi} \nabla D^\alpha (\nabla \cdot u) \right\} \\
&\quad - \left\{ D^\alpha \left[ \frac{\nu}{\xi} (\Delta n \cdot \nabla) \right] - \frac{\nu}{\xi} D^\alpha (\Delta n \cdot \nabla) \right\}.
\end{align*}$$

We will prove that $\Lambda$ maps $B^\lambda_{t_0}(U_0)$ into itself by two steps and denote by $C$ the constants independent of $\lambda$, $K_1$ and $K_2$ in these two steps.

**Step One: Estimates about $n$.**

Taking the $L^2$ inner product of (3.3) with $D^\alpha n$, and using integration by parts, we have

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^n} |D^\alpha n|^2 + \theta \int_{\mathbb{T}^n} |\nabla D^\alpha n|^2 &= - \int_{\mathbb{T}^n} D^\alpha (v \cdot \nabla n) \cdot D^\alpha n + \theta \int_{\mathbb{T}^n} D^\alpha (|\nabla m|^2 n) \cdot D^\alpha n \\
&= N_1 + N_2.
\end{align*}$$

(3.4)

Next, we will give the estimates about $N_1$ and $N_2$ in the following three cases respectively.  

**Case 1: $|\alpha| = 0$ or $D^\alpha = \partial_t$.**

When $|\alpha| = 0$, we use integration by parts and the Sobolev embedding $H^2(\mathbb{T}^N) \hookrightarrow L^\infty(\mathbb{T}^N)$ for $N = 2, 3$ to get

$$|N_1| = \left| \int_{\mathbb{T}^N} (\nabla \cdot v) \frac{|m|^2}{2} \right| \leq \frac{1}{2} \|\nabla \cdot v\|_\infty \|n\|^2 \leq CK_1^\frac{1}{2} \|n\|^2$$

(3.5)

and

$$|N_2| = \theta \int_{\mathbb{T}^N} |\nabla m|^2 |n|^2 dx \leq \theta \|\nabla m\|_\infty^2 \|n\|^2 \leq CK_1 \|n\|^2.$$

(3.6)
On the other hand, when $D^\alpha = \partial_t$, we use the Cauchy inequality to give

$$\begin{align*}
|N_1| &= \left| -\int_{\mathcal{T}^N} (v_t \cdot \nabla n_t) \cdot n_t - \int_{\mathcal{T}^N} (v \cdot \nabla n_t) \cdot n_t \right| \\
&\leq ||v_t||_{\infty} ||\nabla n_t|| ||n_t|| + ||v||_{\infty} ||\nabla n_t|| ||n_t|| \\
&\leq \frac{\theta}{8} ||\nabla n_t||^2 + C (||\nabla n||^2 + ||v||_2^2) ||n_t||^2 + C ||v_t||_2^2 \\
&\leq \frac{\theta}{8} ||\nabla n_t||^2 + C (||\nabla n||^2 + K_1) ||n_t||^2 + CK_2
\end{align*}$$

(3.7)

and

$$\begin{align*}
|N_2| &= \left| 2\theta \int_{\mathcal{T}^N} (\nabla m : \nabla n)(n \cdot n_t) + \theta \int_{\mathcal{T}^N} |\nabla n|^2 |n_t|^2 \right| \\
&\leq 2\theta ||\nabla m||_{\infty} ||\nabla n|| ||n||_{\infty} ||n_t|| + \theta ||\nabla m||_{\infty}^2 ||n_t||^2 \\
&\leq C (||\nabla m||_2^2 ||\nabla n||^2 + ||\nabla m||_2^2) ||n_t||^2 + C ||n||_2^2 \\
&\leq C (K_1 K_2 + K_1) ||n_t||^2 + C ||n||_2^2.
\end{align*}$$

(3.8)

Case 2: $D^\alpha = \nabla_i \nabla^\beta$ with $|\beta| \leq s - 1$.

Integrating by parts and using Lemma 3.1, we have

$$\begin{align*}
|N_1| &= \left| \int_{\mathcal{T}^N} \nabla^\beta (v \cdot \nabla n) \cdot \Delta \nabla^\beta n \right| \\
&\leq C ||\nabla^\beta (v \cdot \nabla n)|| ||\Delta \nabla^\beta n|| \\
&\leq C (||v||_{\infty} ||\nabla n|| + ||n||_{\infty} ||\nabla n||_{\infty} ||\nabla^s v||) ||\nabla^2 \nabla^\beta n|| \\
&\leq \frac{\theta}{4} ||\nabla^2 \nabla^\beta n|| + CK_1 ||\nabla n||_{s-1}^2.
\end{align*}$$

(3.9)

Similarly, we have

$$\begin{align*}
|N_2| &= \left| -\theta \int_{\mathcal{T}^N} \nabla^\beta (|\nabla m|^2 n) \cdot \Delta \nabla^\beta n \right| \\
&\leq C ||\nabla^\beta (|\nabla m|^2 n)|| ||\Delta \nabla^\beta n|| \\
&\leq C (||\nabla m||_2^2 ||\nabla^{s-1} n|| + ||n||_{\infty} ||\nabla^{s-1} (|\nabla m|^2)||) ||\nabla^2 \nabla^\beta n|| \\
&\leq C (||\nabla m||_2^2 ||\nabla^{s-1} n|| + ||n||_{\infty} ||\nabla m||_{\infty} ||\nabla^{s} m||) ||\nabla^2 \nabla^\beta n|| \\
&\leq \frac{\theta}{4} ||\nabla^2 \nabla^\beta n||^2 + CK_1^2 ||n||_s^2.
\end{align*}$$

(3.10)

Case 3: $D^\alpha = \nabla_i \nabla^\gamma \partial_t$ with $|\gamma| \leq s - 2$.

It follows from Lemma 3.1 and the Cauchy inequality that

$$\begin{align*}
|N_1| &= \left| -\int_{\mathcal{T}^N} \nabla^\beta (v_t \cdot \nabla n) \cdot \nabla^\beta n_t - \int_{\mathcal{T}^N} \nabla^\beta (v \cdot \nabla n_t) \cdot \nabla^\beta n_t \right| \\
&\leq C \left[ ||\nabla^\beta (v_t \cdot \nabla n)|| + ||\nabla^\beta (v \cdot \nabla n_t)|| \right] ||\nabla^\beta n_t|| \\
&\leq C (||v_t||_{\infty} ||\nabla^s n|| + ||\nabla n||_{\infty} ||\nabla^{s-1} v|| + ||v||_{\infty} ||\nabla^s n_t|| + ||\nabla n_t||_{\infty} ||\nabla^{s-1} v||) ||n_t||_{s-1}
\end{align*}$$
Indeed, if \( N \) and if

\[ \theta \leq \frac{\theta}{8} \| \nabla n_t \|_{s-1}^2 + C(K_1 + K_2) \| n_t \|_{s-1}^2 + C \| n \|_{s}^2. \tag{3.11} \]

On the other hand, noting that

\[ N_2 = -2\theta \int_{\mathbb{T}^N} \nabla^\gamma [ (\nabla m : \nabla m_t) n ] \cdot \Delta \nabla^\gamma n_t - \int_{\mathbb{T}^N} \nabla^\gamma (|\nabla m|^2 n_t) \cdot \Delta \nabla^\gamma n_t \]

\[ = N_{2,1} + N_{2,2}, \tag{3.12} \]

and using Lemma 3.11 and the Cauchy inequality, we have

\[ \| N_{2,2} \| \leq C \| \nabla^\gamma (|\nabla m|^2 n_t) \| \| \Delta \nabla^\gamma n_t \|
\]

\[ \leq C (\| \nabla m \|_\infty^2 \| \nabla^{s-2} n_t \| + \| n_t \|_\infty \| \nabla^{s-2} (|\nabla m|^2) \|) \| \nabla n_t \|_{s-1}
\]

\[ \leq C (\| \nabla m \|_2^2 \| \nabla^{s-2} n_t \| + \| n_t \|_2 \| \nabla m \|_\infty \| \nabla^{s-1} m_t \|)
\]

\[ + \| n \|_2 \| \nabla m_t \|_\infty \| \nabla^{s-1} m \|) \| \nabla n_t \|_{s-1}
\]

\[ \leq \frac{\theta}{8} \| \nabla n_t \|_{s-1}^2 + C K_2^2 \| n_t \|_{s-2}^2. \tag{3.13} \]

Nevertheless, the estimates of \( N_{2,1} \) are slightly different between the cases \( s = 3 \) and \( s \geq 4 \). Indeed, if \( s \geq 4 \), then

\[ \| N_{2,1} \| \leq C \| \nabla^\gamma [ (\nabla m : \nabla m_t) n ] \| \| \Delta \nabla^\gamma n_t \|
\]

\[ \leq C \{ \| (\nabla m : \nabla m_t) n \| + \| (\nabla m : \nabla m_t) n \| \} \| \nabla n_t \|_{s-1}
\]

\[ \leq C \{ \| (\nabla m : \nabla m_t) n \| + \| (\nabla m : \nabla m_t) n \|
\]

\[ + \| (\nabla m : \nabla m_t) n \| + \| (\nabla m : \nabla m_t) n \| \} \| \nabla n_t \|_{s-1}
\]

\[ \leq C (\| \nabla m \|_{L^1} \| \nabla m_t \|_{L^1} \| n \|_{\infty} + \| \nabla^2 m \|_{L^1} \| \nabla m_t \|_{L^1} \| n \|_{\infty}
\]

\[ + C \| \nabla m \|_{\infty} \| \nabla^2 m_t \|_{\infty} + \| \nabla m \|_{L^4} \| \nabla m_t \|_{L^4} \| n \|_{\infty}) \| \nabla n_t \|_{s-1}
\]

\[ \leq C (\| \nabla m \|_2 \| \nabla m_t \|_1 \| n \|_2 + C \| \nabla m \|_2 \| \nabla^2 m_t \|_1 \| n \|_2 + \| \nabla m \|_1 \| \nabla m_t \|_1 \| n \|_{\infty}) \| \nabla n_t \|_{s-1}
\]

\[ \leq \frac{\theta}{8} \| \nabla n_t \|_{s-1}^2 + C K_1 K_2 \| n \|_s^2. \tag{3.15} \]
In conclusion, substituting the estimates (3.5), (3.6), (3.9) and (3.10) into (3.4), and using the Cauchy inequality, we obtain
\[
\frac{d}{dt} \sum_{|\alpha| \leq s} \| \nabla^{\alpha} n \|^2 + \theta \sum_{|\alpha| \leq s} \| \nabla^{\alpha+1} n \|^2 \leq C(K_1^2 + 1)\|n\|^2_s.
\] (3.16)

Owing to \( |n(x,0)| = |n_0(x)| = 1, \ x \in \mathbb{T}^N \), we have \( \|n(x,0)\| = \|n_0(x)\| \). Together with the Gronwall’s inequality, (2.8) and (3.16), we find that
\[
\|n\|^2_s \leq e^{C(K_1^2 + 1)T_0} \|n(x,0)\|^2_s \leq e^{C(K_1^2 + 1)T_0} (\|n_0\|^2_s + \lambda^{-2}\delta_0^2) \leq C,
\] (3.17)
where we have chosen \( T_0 \) small enough for \( T_0 < T_1 = \frac{1}{K_1+1} \) and \( \lambda > 1 \). Furthermore, integrating (3.16) over \([0,t]\) and using (3.17), we obtain
\[
\theta \int_0^t \| \nabla n \|^2_s \leq C \left( K_1^2 + 1 \right) T_0 + \|n_0\|^2_s + \lambda^{-2}\delta_0^2 \leq C.
\] (3.18)

On the other hand, substituting the estimates (3.7), (3.8) and (3.11)-(3.15) into (3.4), and then using (3.17) and the Cauchy inequality, we obtain
\[
\frac{d}{dt} \sum_{|\beta| \leq s-1} \int_{\mathbb{T}^n} |\nabla^{\beta} n_t|^2 + \theta \sum_{|\beta| \leq s-1} \int_{\mathbb{T}^n} |\nabla^{\beta+1} n_t|^2 \leq C \left( K_1^2 + K_2^2 + 1 \right) \|n_t\|^2_{s-1} + C \left( K_1^2 + K_2^2 + 1 \right).
\] (3.19)

Recalling that the constans on the initial data and (1.38), we use Lemma 3.1 to get
\[
\|n_t(x,0)\|_{s-1} \leq C \left[ \|u(x,0) \cdot \nabla n(x,0)\|_{s-1} + \|\Delta n(x,0)\|^2 + |\nabla n(x,0)|^2 \right]_{s-1} + C \left( \|n_0\|_{s-1} + \lambda^{-1}\delta_0 \right) + C \left( \|n_0\|_s + \lambda^{-1}\delta_0 \right) + C \left( \|n_0\|_{s-1} + \lambda^{-1}\delta_0 \right) + C \left( \|n_0\|_{s-1} + \lambda^{-1}\delta_0 \right)
\]
\[
\leq C.
\] (3.20)

Then by the Gronwall’s inequality , we have
\[
\|n_t\|^2_{s-1} \leq C e^{C(K_1^2 + K_2^2 + 1)T_0} \left[ \|n_t(x,0)\|^2_{s-1} + (K_1^2 + K_2^2 + 1)T_0 \right] \leq C,
\] (3.21)
provided that \( T_0 < T_2 = \frac{1}{K_1+K_2+1} \). Furthermore, integrating (3.19) over \([0,t]\) leads us to
\[
\theta \int_0^t \| \nabla n_t \|^2_{s-1} \leq C.
\] (3.22)

**Step Two: Estimates about \( \rho \) and \( u \).**

Taking the \( L^2 \) inner product of the first two equations of (3.3) with \( \lambda^2 \frac{P'(\xi)}{\xi} D^\alpha (\rho - 1) \) and \( \xi D^\alpha u \) respectively, and using integration by parts, we arrive at
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^n} \left[ \frac{P'(\xi)}{\xi} \lambda D^\alpha (\rho - 1)^2 + \xi |D^\alpha u|^2 \right].
\]
\[ + \mu \int_{\mathbb{T}^n} |\nabla D^\alpha u|^2 + (\kappa + \mu) \int_{\mathbb{T}^n} |\nabla \cdot D^\alpha u|^2 \]

\[ = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8 + I_9, \tag{3.23} \]

where

\[ I_1 = \frac{1}{2} \int_{\mathbb{T}^n} \left[ |\lambda D^\alpha (\rho - 1)|^2 \partial_\xi \left( \frac{P'(\xi)}{\xi} \right) + \xi |D^\alpha u|^2 \right], \]

\[ I_2 = \frac{1}{2} \int_{\mathbb{T}^n} \left[ |\lambda D^\alpha (\rho - 1)|^2 \nabla \cdot \left( \frac{P'(\xi)}{\xi} v \right) + \nabla \cdot (\xi v) |D^\alpha u|^2 \right], \]

\[ I_3 = \lambda^2 \int_{\mathbb{T}^n} P''(\xi) D^\alpha (\rho - 1) D^\alpha u \cdot \nabla \xi, \]

\[ I_4 = -\lambda^2 \int_{\mathbb{T}^n} \frac{P'(\xi)}{\xi} \left[ D^\alpha (v \cdot \nabla \rho) - v \cdot \nabla D^\alpha \rho \right] + \left[ D^\alpha (\xi \nabla \cdot u) - \xi \nabla \cdot D^\alpha u \right], \]

\[ I_5 = -\nu \int_{\mathbb{T}^n} D^\alpha (\Delta n \cdot \nabla n) \cdot D^\alpha u, \]

\[ I_6 = -\int_{\mathbb{T}^n} \xi [D^\alpha (v \cdot \nabla u) - v \cdot \nabla D^\alpha u] \cdot D^\alpha u, \]

\[ I_7 = -\lambda^2 \int_{\mathbb{T}^n} \xi \left[ D^\alpha \left( \frac{P'(\xi)}{\xi} \nabla \rho \right) - \frac{P'(\xi)}{\xi} \nabla D^\alpha \rho \right] \cdot D^\alpha u, \]

\[ I_8 = \int_{\mathbb{T}^n} \xi \left[ D^\alpha \left( \frac{\mu}{\xi} \Delta u \right) - \frac{\mu}{\xi} \Delta D^\alpha u \right] \cdot D^\alpha u \]

\[ + \int_{\mathbb{T}^n} \xi \left\{ D^\alpha \left[ \frac{\kappa + \mu}{\xi} \nabla (\xi \nabla u) \right] - \frac{\kappa + \mu}{\xi} \nabla D^\alpha (\xi \nabla u) \right\} \cdot D^\alpha u, \]

\[ I_9 = -\int_{\mathbb{T}^n} \xi \left\{ D^\alpha \left[ \frac{\nu}{\xi} (\Delta n \cdot \nabla n) \right] - \frac{\nu}{\xi} D^\alpha (\Delta n \cdot \nabla n) \right\} \cdot D^\alpha u. \]

We will estimate all these terms in the three cases as mentioned in Step One.

First of all, we give the estimates of \( I_1 - I_3 \) for all the three cases. Without losing any generality, we assume that \( \lambda \geq 1 \). Furthermore, we choose \( \delta \) small enough so that (2.5) and \( |\xi - 1| \leq \frac{1}{2} \) hold. Then by the smoothness of the pressure \( P(\cdot) \) and the Sobolev embedding \( H^2(\mathbb{T}^N) \hookrightarrow L^\infty(\mathbb{T}^N) \), we have

\[ |I_1| \leq \left\| \frac{\xi P''(\xi) - P'(\xi)}{\xi^2} \right\|_\infty \left\| \xi \right\|_\infty \left\| \lambda D^\alpha (\rho - 1) \right\|^2 + \left\| \xi \right\|_\infty \left\| D^\alpha u \right\|^2 \]

\[ \leq C \left\| \xi \right\|_2 \left( \left\| \lambda D^\alpha (\rho - 1) \right\|^2 + \left\| D^\alpha u \right\|^2 \right) \]

\[ \leq C \lambda^{-1} K_1^\frac{1}{2} \left( \left\| \lambda D^\alpha (\rho - 1) \right\|^2 + \left\| D^\alpha u \right\|^2 \right), \tag{3.24} \]

\[ |I_2| \leq \left( \left\| \frac{\xi P''(\xi) - P'(\xi)}{\xi^2} \right\|_\infty \left\| \nabla \xi \right\|_\infty \left\| v \right\|_\infty + \left\| \frac{P'(\xi)}{\xi} \right\|_\infty \left\| \nabla \cdot v \right\|_\infty \right) \left\| \lambda D^\alpha (\rho - 1) \right\|^2 \]

\[ + \left( \left\| \nabla \xi \right\|_\infty \left\| v \right\|_\infty + \left\| \xi \right\|_\infty \left\| \nabla \cdot v \right\|_\infty \right) \left\| D^\alpha u \right\|^2 \]

\[ \leq C \left( \lambda^{-1} \left\| \lambda (\xi - 1) \right\|_3 \left\| v \right\|_2 + \left\| v \right\|_3 \left( \left\| \lambda D^\alpha (\rho - 1) \right\|^2 + \left\| D^\alpha u \right\|^2 \right) \right) \]

\[ \leq C \left( \lambda^{-1} K_1 + K_1^\frac{1}{2} \right) \left( \left\| \lambda D^\alpha (\rho - 1) \right\|^2 + \left\| D^\alpha u \right\|^2 \right), \tag{3.25} \]
\[ |I_3| \leq \lambda \|P''(\xi)\|_{\infty} \|\nabla \xi\|_{\infty} (\|\lambda D^\alpha (\rho - 1)\|^2 + \|D^\alpha u\|^2) \]
\[ \leq C \|\lambda \nabla \xi\|_2 (\|\lambda D^\alpha (\rho - 1)\|^2 + \|D^\alpha u\|^2) \]
\[ \leq CK_1^2 (\|\lambda D^\alpha (\rho - 1)\|^2 + \|D^\alpha u\|^2). \]  

(3.26)

Secondly, we give the estimates about \( I_4 - I_9 \) when \( |\alpha| = 0 \) or \( D^\alpha = \partial_t \). When \( |\alpha| = 0 \), the quantities \( I_j, 4 \leq j \leq 9, \) are all equal to zero except for \( I_5 \). Thus we need only to estimate \( I_5 \). By using (3.17) and integration by parts, we have

\[ |I_5| \leq C\|u\|\|\nabla n\|_\infty \|\Delta n\| \leq C\|u\|^2 + C. \]  

(3.27)

When \( D^\alpha = \partial_t \), the estimates about \( I_4 - I_9 \) can be shown as follows. By using (3.17), (3.21) and the Cauchy inequality, we have

\[ |I_4| \leq C\|\lambda \rho_t\| (\|v_t\|_\infty \|\lambda \nabla \rho\| + \|\lambda \xi_t\|_\infty \|\nabla \cdot u\|) \leq CK_2^\frac{1}{2} (\|\lambda \nabla \rho\|^2 + \|\lambda \rho_t\|^2 + \|\nabla \cdot u\|^2), \]  

(3.28)

\[ |I_5| \leq C (\|\Delta n_t\| \|\nabla n\|_\infty + \|\Delta n\|_\infty \|\nabla n_t\|) \|u_t\| \]
\[ \leq C (\|\Delta n_t\| \|\nabla n\|_2 + \|\Delta n\|_2 \|\nabla n_t\|) \|u_t\| \]
\[ \leq C\|u_t\|^2 + C\|\nabla^2 n\|^2 + C, \]  

(3.29)

\[ |I_6| \leq C\|v_t \cdot \nabla u\| \|u_t\| \leq C\|v_t\|_\infty \|\nabla u\| \|u_t\| \leq CK_2^\frac{3}{2} (\|\nabla u\|^2 + \|u_t\|^2), \]  

(3.30)

\[ |I_7| \leq C\lambda^2 \left( \left( \frac{P(\xi)}{\xi} \right)_t \nabla \rho \right) \|u_t\| \leq C\|\lambda \xi_t\|_\infty \|\lambda \nabla \rho\| \|u_t\| \leq CK_2^\frac{1}{2} (\|\lambda \nabla \rho\|^2 + \|u_t\|^2). \]  

(3.31)

\[ |I_8| \leq C \left[ \left( \frac{1}{\xi} \right)_t \Delta u \right] + \left( \frac{1}{\xi} \right)_t \nabla (\nabla \cdot u) \|u_t\| \]
\[ \leq C\lambda^{-1} (\|\lambda \xi_t\|_\infty \|\Delta u\| + \|\lambda \xi_t\|_\infty \|\nabla (\nabla \cdot u)\|) \|u_t\| \]
\[ \leq C\lambda^{-1} K_2^\frac{1}{2} (\|u_t\|^2 + \|u_t\|^2), \]  

(3.32)

\[ |I_9| \leq C \left( \frac{1}{\xi} \right)_t (\Delta n \cdot \nabla n) \|u_t\| \leq C\lambda^{-1} \|\lambda \xi_t\|_\infty \|\Delta n\|_\infty \|\nabla n\| \|u_t\| \leq C\lambda^{-2} K_1 \|u_t\|^2 + C. \]  

(3.33)

Thirdly, by using Lemma 3.1, we give the estimates about \( I_4 - I_9 \) when \( D^\alpha = \nabla_1 \nabla^\beta \) for \( |\beta| \leq s - 1, s \geq 3 \).

\[ |I_4| \leq C \left[ \|\nabla v\|_\infty \|\nabla^s \rho\| + \|\nabla \rho\|_\infty \|\nabla^s v\| + \|\nabla \xi\|_\infty \|\nabla^{s-1} (\nabla \cdot u)\| + \|\nabla \cdot u\|_\infty \|\nabla^s \xi\| \right] \]
where we have used integration by parts and (3.17) in (3.35). Similarly as (3.34), one can prove by induction, we have

\[ |I_5| = \left| \nu \int_{\mathbb{T}^n} \nabla^\beta (\Delta n \cdot \nabla n) \cdot \Delta \nabla^\beta u \right| \leq C \|\nabla^2 \nabla^\beta u\| \|\nabla^\beta (\Delta n \cdot \nabla n)\| \leq C \|\nabla^2 \nabla^\beta u\| (\|\Delta n\|_\infty \|\nabla^s n\| + \|\nabla n\|_\infty \|\nabla^{s-1} \Delta n\|) \leq \frac{\mu}{4} \|\nabla u\|^2 + C \|\nabla^2 n\|_{s-1}^2, \tag{3.35} \]

where we have used integration by parts and (3.17) in (3.35). Similarly as (3.34), one obtains

\[ |I_6| \leq C \|u\|_s (\|\nabla v\|_\infty \|\nabla u\|_{s-1} + \|\nabla u\|_\infty \|v\|_s) \leq CK^{\frac{1}{2}} \|u\|^2_s. \tag{3.36} \]

On the other hand, noting the fact that

\[ \left\| \nabla^s \left( \frac{P'(\xi)}{\xi} \right) \right\| \leq C (\|\nabla \xi\|_{s-1} + \|\nabla \xi\|_{s-1}^s) = C \left( \lambda^{-1} \|\lambda \nabla \xi\|_{s-1} + \lambda^{-s} \|\lambda \nabla \xi\|_{s-1}^s \right), \tag{3.37} \]

which can be proved by induction, we have

\[ |I_7| \leq C \lambda^2 \left[ \left\| \frac{\xi P''(\xi) - P'(\xi)}{\xi^2} \nabla \xi \right\| \|\nabla^{s-1} \nabla \rho\| + \|\nabla \rho\|_\infty \left\| \nabla^s \left( \frac{P'(\xi)}{\xi} \right) \right\| \|u\|_s \right. \leq C \left( \|\lambda \nabla \xi\|_2 \|\lambda \nabla^s \rho\| + \|\lambda \nabla \rho\|_2 \left( \|\lambda \nabla \xi\|_{s-1} + \lambda^{-1-s} \|\lambda \nabla \xi\|_{s-1}^s \right) \right) \|u\|_s \right. \leq C \left( K^{\frac{1}{2}} + \lambda^{1-s} K^{\frac{s}{2}} \right) \left( \|\lambda (\rho - 1)\|^2 + \|u\|_2^2 \right), \tag{3.38} \]

\[ |I_8| \leq C \|u\|_s \left[ \left\| \nabla \left( \frac{1}{\xi} \right) \right\|_\infty \|\nabla^{s-1} \Delta u\| + \|\Delta u\|_\infty \left\| \nabla^s \left( \frac{1}{\xi} \right) \right\| + \left\| \nabla \left( \frac{1}{\xi} \right) \right\|_\infty \|\nabla^{s-1} \nabla (\nabla \cdot u)\| + \|\nabla (\nabla \cdot u)\|_\infty \left\| \nabla^s \left( \frac{1}{\xi} \right) \right\| \right. \leq C \left( \lambda^{-1} K^{\frac{1}{2}} + \lambda^{-s} K^{\frac{s}{2}} \right) \|u\|_s (\|\nabla u\|_s + \|\nabla \cdot u\|_s) \leq C \left( \lambda^{-2} K + \lambda^{-2s} K^{s} \right) \|u\|^2 + \frac{\mu}{4} \|\nabla u\|^2 + \frac{K + \mu}{2} \|\nabla \cdot u\|^2, \tag{3.39} \]

\[ |I_9| \leq C \|u\|_s \left[ \left\| \nabla \left( \frac{1}{\xi} \right) \right\|_\infty \|\nabla^{s-1} (\Delta n \cdot \nabla n)\| + \|\Delta n \cdot \nabla n\|_\infty \left\| \nabla^s \left( \frac{1}{\xi} \right) \right\| \right. \leq C \|u\|_s \left[ \|\nabla \xi\|_\infty (\|\Delta n\|_\infty \|\nabla^{s-1} \nabla n\| + \|\nabla n\|_\infty \|\nabla^{s-1} \Delta n\|) + \|\Delta n\|_2 \|\nabla n\|_2 \left\| \nabla^s \left( \frac{1}{\xi} \right) \right\| \right. \leq C \|u\|_s \left[ \lambda^{-1} \|\lambda \nabla \xi\|_2 \|\Delta n\|_2 \|\nabla n\|_{s-1} + \lambda^{-1} \|\lambda \nabla \xi\|_2 \|\nabla n\|_2 \|\Delta n\|_{s-1} + \|\Delta n\|^2 \|\nabla n\|^2 \right. \left( \lambda^{-1} \|\lambda \nabla \xi\|_{s-1} + \lambda^{-s} \|\lambda \nabla \xi\|_{s-1}^s \right) \right] \]
\[
\leq C \left( \lambda^{-2}K_1 + \lambda^{-2s}K_1^s \right) \|u\|_s^2 + C\|\nabla^2 n\|_{s-1}^2, \tag{3.40}
\]

where we have used (3.17) in (3.40).

We now tie the estimates (3.24)–(3.27) and (3.34)–(3.40) together and get

\[
\frac{d}{dt} \sum_{|\alpha| \leq s} \int_{\mathbb{T}^n} \left[ \frac{P^\alpha(x)}{\xi} \lambda^2 |\nabla^\alpha (\rho - 1)|^2 + \xi |\nabla^\alpha u|^2 \right]
\]
\[+ \sum_{|\alpha| \leq s} \mu \int_{\mathbb{T}^n} |\nabla^{\alpha} u|^2 + \sum_{|\alpha| \leq s} \left( \kappa + \mu \right) \int_{\mathbb{T}^n} |\nabla^{\alpha} (\nabla \cdot u)|^2 \leq C \left( K_1^s + K_2 + 1 \right) \left( \|\lambda (\rho - 1)\|_s^2 + \|u\|_s^2 \right) + C\|\nabla^2 n\|_{s-1}^2. \tag{3.41}
\]

Recalling the constraints on the initial data (2.5), we have

\[
\|\lambda \varphi_0\|_s^2 + \|u_0 + \varphi_0\|_s^2 \leq (2\lambda^{-2}\delta_0^2 + \|u_0\|_s^2). \tag{3.42}
\]

Then by (2.5), (3.18), (3.41), (3.42) and the Gronwall’s inequality, we get

\[
\|\lambda (\rho - 1)\|_s^2 + \|u\|_s^2 \leq e^{C(K_1^s + K_2 + 1)} T_0 \left( 2\lambda^{-2}\delta_0^2 + \|u_0\|_s^2 + C \int_0^t \|\nabla^2 n\|_{s-1}^2 \right) \leq C, \tag{3.43}
\]

provided that \(T_0 < T_3 \doteq \min \left\{ T_2, \frac{1}{K_1^s + K_2 + 1} \right\} \). Furthermore, we integrate (3.41) over \([0,t]\) and get

\[
\mu \int_0^t \|\nabla u\|_s^2 + (\kappa + \mu) \int_0^t \|\nabla \cdot u\|_s^2 \leq C. \tag{3.44}
\]

Finally, we estimate \(I_4 - I_9\) when \(D^\alpha = \nabla_i \nabla^\gamma \partial_t\) with \(|\gamma| \leq s - 2\). For \(I_6\), by virtue of (3.43) and the Cauchy inequality, we are led to

\[
|I_6| \leq C \|u_t\|_{s-1} \left( \|\nabla v\|_\infty \|\nabla^{s-2} \nabla u_t\| + \|\nabla u_t\|_\infty \|\nabla^{s-2} v\| + \|v_t\|_\infty \|\nabla^{s-1} \nabla u\| \\
+ \|\nabla u\|_\infty \|\nabla^{s-1} v_t\| \right) \\
\leq C \|u_t\|_{s-1} \left( K_1^\frac{1}{2} \|u_t\|_{s-1} + K_1^\frac{1}{2} \|\nabla u_t\|_2 + K_2^{\frac{1}{2}} \|u_t\|_s \right) \\
\leq C \left( K_1 + K_2 + 1 \right) \|u_t\|_{s-1}^2 + \frac{\mu}{6} \|\nabla u_t\|_s^2 + C, \tag{3.45}
\]

while for the terms \(I_4, I_5, I_7, I_8\) and \(I_9\), some extra discussions have to be given since the methods to estimate these five terms are different between the cases \(s = 3\) and \(s \geq 4\). Indeed, when \(s \geq 4\), we use Lemma 3.1, the Sobolev embedding \(H^2(\mathbb{T}^N) \hookrightarrow L^\infty(\mathbb{T}^N)\), the Cauchy inequality and (3.43) to estimate these terms and get

\[
|I_4| \leq \|\lambda \rho_t\|_{s-1} \left[ \|\nabla v\|_\infty \|\lambda \nabla^{s-2} \nabla \rho_t\| + \|\lambda \nabla \rho_t\|_\infty \|\nabla^{s-1} v\| + \|v_t\|_\infty \|\lambda \nabla^{s-1} \nabla \rho\| \\
+ \|\lambda \nabla \rho\|_\infty \|\nabla^{s-1} v_t\| + \|\lambda \nabla \xi\|_\infty \|\nabla^{s-2} (\nabla \cdot u_t)\| + \|\nabla \cdot u_t\|_\infty \|\lambda \nabla^{s-1} \xi\| \right].
\]
\[
\begin{align*}
& \quad + \| \lambda \xi_t \|_\infty \| \nabla^{s-1}(\nabla \cdot u) \| + \| \nabla \cdot u \|_\infty \| \lambda \nabla^{s-1} \xi_t \| \\
& \leq C \| \lambda \rho_t \|_{s-1} \left( K_1^\frac{1}{2} \| \lambda \rho_t \|_{s-1} + K_2^\frac{1}{2} \| \lambda (\rho - 1) \|_s + K_2^\frac{1}{2} \| u_t \|_{s-1} + K_2^\frac{1}{2} \| u \|_s \right) \\
& \leq C(K_1 + K_2 + 1) \left( \| \lambda \rho_t \|_{s-1}^2 + \| \lambda (\rho - 1) \|_s^2 + \| u_t \|_{s-1}^2 \right) + C,
\end{align*}
\]

(3.46)

\[
|I_5| = \left| -\nu \int_{\mathbb{T}^n} \nabla^\gamma (\Delta n_t \cdot \nabla n_t) \cdot \Delta \nabla^\gamma u_t - \nu \int_{\mathbb{T}^n} \nabla^\gamma (\Delta n \cdot \nabla n_t) \cdot \Delta \nabla^\gamma u_t \right|
\leq C \left( \| \Delta n_t \|_\infty \| \nabla^{s-2} \nabla n_t \| + \| \nabla n_t \|_\infty \| \nabla^{s-2} \Delta n_t \| \right) \| \Delta \nabla^\gamma u_t \|
\quad + C \left( \| \Delta n_t \|_\infty \| \nabla^{s-2} \nabla n_t \| + \| \nabla n_t \|_\infty \| \nabla^{s-2} \Delta n_t \| \right) \| \Delta \nabla^\gamma u_t \|
\leq C \left( \| \nabla n_t \|_3 \| \nabla n \|_{s-2} + \| \nabla n_t \|_2 \| \nabla n_t \|_{s-1} \right) \| \nabla u_t \|_{s-1}
\quad + C \left( \| \nabla n \|_3 \| \nabla n_t \|_{s-2} + \| \nabla n_t \|_2 \| \nabla n_t \|_{s-1} \right) \| \nabla u_t \|_{s-1}
\leq \frac{\mu}{6} \| \nabla u_t \|_{s-1}^2 + C \| \nabla n_t \|_{s-1}^2,
\]

(3.47)

where we have used integration by parts and (3.17) in (3.47). By using a similar induction procedure of (3.37), we have

\[
\left\| \nabla^{s-1} \left( \frac{P'(\xi)}{\xi} \right) \right\| \leq C \left( \| \nabla \xi \|_{s-2} + \| \nabla \xi \|_{s-2}^{s-1} \right).
\]

(3.48)

and

\[
\left\| \nabla^{s-1} \left( \frac{P'(\xi)}{\xi} \right)_t \right\| \leq C \left\| \nabla^{s-1} \left( \frac{P''(\xi) - P'(\xi)}{\xi^2} \right) \right\|
\leq C \left\| \frac{P''(\xi) - P'(\xi)}{\xi^2} \right\|_\infty \| \nabla^{s-1} \xi_t \| + C \| \xi_t \|_\infty \left\| \nabla^{s-1} \left( \frac{P''(\xi) - P'(\xi)}{\xi^2} \right) \right\|
\leq C \| \xi_t \|_{s-1} + C \| \xi_t \|_{s-1} \left( \| \nabla \xi \|_{s-2} + \| \nabla \xi \|_{s-2}^{s-1} \right).
\]

(3.49)

Then we use these two facts to obtain the estimates about \( I_7, I_8 \) and \( I_9 \).

\[
\begin{align*}
|I_7| & \leq \lambda^2 \| u_t \|_{s-1} \left[ \left\| \nabla \left( \frac{P'(\xi)}{\xi} \right) \right\|_\infty \| \nabla^{s-2} \nabla \rho_t \| + \| \nabla \rho_t \|_\infty \left\| \nabla^{s-1} \left( \frac{P'(\xi)}{\xi} \right) \right\| \right. \\
& \quad + \left. \left\| \left( \frac{P'(\xi)}{\xi} \right)_t \right\|_\infty \| \nabla^{s-1} \nabla \rho \| + \| \nabla \rho \|_\infty \right\| \nabla^{s-1} \left( \frac{P'(\xi)}{\xi} \right)_t \right\| \right] \\
& \leq C \| u_t \|_{s-1} \left[ \left( K_1^\frac{3}{2} + \lambda^{2-s} K_1^{\frac{s-1}{2}} \right) \| \lambda \rho_t \|_{s-1} + K_2^\frac{1}{2} \left( \lambda^{-1} K_1^\frac{3}{2} + \lambda^{1-s} K_1^{\frac{s-1}{2}} + 1 \right) \| \lambda (\rho - 1) \|_s \right] \\
& \leq C \left( K_1^\frac{3}{2} + K_2^\frac{1}{2} + 1 \right) \left( \| \lambda \rho_t \|_{s-1}^2 + \| u_t \|_{s-1}^2 \right) + C,
\end{align*}
\]

(3.50)

\[
\begin{align*}
|I_8| & \leq C \| u_t \|_{s-1} \left[ \left\| \nabla \left( \frac{1}{\xi} \right) \right\|_\infty \| \nabla^{s-2} \Delta u_t \| + \| \Delta u_t \|_\infty \left\| \nabla^{s-1} \left( \frac{1}{\xi} \right) \right\| + \left\| \left( \frac{1}{\xi} \right)_t \right\|_\infty \right\| \nabla^{s-1} \Delta u \right\|
\quad + \left. \left\| \Delta u \| \infty \right\| \nabla^{s-1} \left( \frac{1}{\xi} \right)_t \right\| + \left\| \nabla \left( \frac{1}{\xi} \right) \right\|_\infty \| \nabla^{s-2} \nabla (\nabla \cdot u_t) \| + \| \nabla (\nabla \cdot u_t) \|_\infty \left\| \nabla^{s-1} \left( \frac{1}{\xi} \right) \right\| \\
& \quad + \left. \left\| \left( \frac{1}{\xi} \right)_t \right\|_\infty \right\| \nabla^{s-1} \nabla (\nabla \cdot u) \| + \| \nabla (\nabla \cdot u) \|_\infty \left\| \nabla^{s-1} \left( \frac{1}{\xi} \right)_t \right\| \right]
\end{align*}
\]

\[
\end{align*}
\]
\[ \begin{align*}
& \leq C \| u_t \|_{s-1} \left[ \left( \lambda^{-1} K_1^\beta + \lambda^{1-s} K_1^{s-1} \right) \left( \| \nabla u_t \|_{s-1} + \| \nabla \cdot u_t \|_{s-1} \right) \\
& \quad + \lambda^{-1} K_2^\beta \left( \lambda^{-1} K_1^\beta + \lambda^{1-s} K_1^{s-1} + 1 \right) \left( \| \nabla u \|_{s} + \| \nabla \cdot u \|_{s} \right) \right] \\
& \leq \frac{\mu}{6} \| \nabla u_t \|_{s-1}^2 + \frac{k + \lambda}{2} \| \nabla \cdot u_t \|_{s-1}^2 + C \left( K_1^s + K_2^s + 1 \right) \| u_t \|_{s-1}^2 + C \left( \| \nabla u \|_{s}^2 + \| \nabla \cdot u \|_{s}^2 \right).
\end{align*} \]

(3.51)

\[ |I_0| \leq C \| D^2 u_t \| \left[ \left( \left( \frac{1}{\xi} \right)_t \right) \left\| \| \nabla^{s-1} (\Delta n \cdot \nabla n) \| + \| \Delta n \cdot \nabla n \| \right\| \nabla^{s-1} \left( \frac{1}{\xi} \right)_t \right]
\]

\[ + \left\| \nabla \left( \frac{1}{\xi} \right)_t \right\| \| \nabla^{s-2} (\Delta n \cdot \nabla n) \|_{t} \| + \| \Delta n \cdot \nabla n \|_{t} \left\| \nabla^{s-1} \left( \frac{1}{\xi} \right)_t \right\| \right. \]

\[ \leq C \| \nabla^2 u_t \| \left[ \left[ \left( \frac{1}{\xi} \right)_t \right] \| \Delta n \|_{t} \left\| \nabla^{s-1} \nabla n \| + \| \nabla^2 \left( \frac{1}{\xi} \right)_t \right\| \left\| \nabla n \|_{t} \right\| \right]\]

\[ + \| \Delta n \|_{t} \left\| \nabla n \| \right\| \left\| \nabla^{s-1} \left( \frac{1}{\xi} \right)_t \right\| \right. \]

\[ \leq C \| u_t \|_{s-1} \left[ \lambda^{-1} \left( K_1^\beta + K_2^\beta \right) + \left( \lambda^{-1} K_1^\beta + 1 \right) \left( \lambda^{-1} K_1^\beta + \lambda^{1-s} K_1^{s-1} \right) \right. \]

\[ \left. \left( \lambda^{-1} K_1^\beta + \lambda^{1-s} K_1^{s-1} \right) \left[ \| \Delta n \|_{t} \| \Delta n \|_{t} \| \nabla n \|_{t} \| \nabla n \|_{t} \| \nabla n \|_{t} \| \nabla n \|_{t} \| \nabla n \|_{t} \| \nabla n \|_{t} \| \nabla n \|_{t} \| \nabla n \|_{t} \right. \right. \]

\[ \leq C \left( K_1^s + K_2^s + 1 \right) \| u_t \|_{s-1} + C \left( \| \nabla n \|_{t}^2 + \| \nabla^2 n \|_{s-1}^2 + 1 \right). \]

(3.52)

On the other hand, when \( s = 3 \), more refined estimates are needed. We here only deal with the terms with \( |\gamma| = s-2 = 1 \) (\( D^\alpha = \nabla_i \nabla_j \partial_\xi \)) for simplicity, and the case \( |\gamma| = 0 \) can be estimated similarly (actually more easily). We use Lemma 3.1 the Sobolev embedding \( H^1(T^N) \hookrightarrow L^4(T^N), \ H^2(T^N) \hookrightarrow L^\infty(T^N) \) and the Cauchy inequality to give

\[ |I_4| \leq \lambda \| \nabla_i \nabla_j \partial_\xi \| \left\| \| \nabla_i \nabla_j (v \cdot \nabla \rho) \| + \| \nabla_i \nabla_j (v_t \cdot \nabla \rho) \| + \| \nabla_i \nabla_j (\xi_t \cdot \nabla \rho) \| \right\| \]

\[ \leq \lambda \| \nabla_i \nabla_j \partial_\xi \| \left\| \| \nabla_i \nabla_j v \cdot \nabla \rho_t \| + \| \nabla_i \nabla_j v \cdot \nabla \rho \| + \| \nabla_i \nabla_j v_t \cdot \nabla \rho \| + \| \nabla^2 \left( \xi_t \cdot \nabla \rho \right) \| \right\| \]

\[ \leq C \lambda \| \nabla^2 \partial_\xi \| \left\| \| \nabla^2 v \|_{L^4} \| \partial_\xi \|_{L^4} + \| \nabla v \|_{L^4} \| \nabla^2 \partial_\xi \| + \| \nabla v \|_{L^4} \| \nabla \partial_\xi \| + \| \nabla \partial_\xi \|_{L^4} \| \nabla^2 v \| \right\| \]

\[ + \| \nabla^2 \partial_\xi \|_{L^4} \| \nabla u \|_{L^4} + \| \nabla \xi \|_{\infty} \| \nabla (\nabla \partial_\xi) \| + \| \xi_t \|_{\infty} \| \nabla^2 u \| + \| \nabla u \|_{\infty} \| \nabla^2 \partial_\xi \| \right\| \]

\[ \leq C \lambda \| \nabla^2 \partial_\xi \| \left\| \| \nabla^2 v \|_{L^4} \| \nabla \rho_t \| + \| \nabla v \|_{L^4} \| \nabla^2 \partial_\xi \| + \| \nabla v \|_{L^4} \| \nabla \partial_\xi \| + \| \nabla \partial_\xi \|_{L^4} \| \nabla v \| \right\| \]

\[ + \| \nabla^2 \partial_\xi \|_{L^4} \| \nabla u \|_{L^4} + \| \nabla \xi \|_{\infty} \| \nabla (\nabla \partial_\xi) \| + \| \xi_t \|_{\infty} \| \nabla^2 u \| + \| \nabla u \|_{\infty} \| \nabla^2 \partial_\xi \| \right\| \].
\[ \leq CK_1^{\frac{1}{2}} \| \lambda \rho_t \|^2 + CK_2^{\frac{1}{2}} \| \lambda \rho_t \|_2 + CK_1^{\frac{1}{2}} \| \lambda \nabla^2 \rho_t \|_2 + CK_2^{\frac{1}{2}} \| \lambda \nabla^2 \rho_t \|_3 \]
\[ \leq C(K_1 + K_2 + 1) \left( \| \lambda \rho_t \|^2 + \| u_t \|^2 \right) + C. \]  

(3.53)

Noting that
\[
I_5 = \nu \int_{T^n} \left[ \nabla_i \nabla_j (\Delta n_t \cdot \nabla n) + \nabla_i \nabla_j (\Delta n \cdot \nabla n_t) \right] \cdot \nabla_i \nabla_j u_t
\]
\[ = -\nu \int_{T^n} \left[ \nabla_j (\Delta n_t \cdot \nabla n) + \nabla_j (\Delta n \cdot \nabla n_t) \right] \cdot \Delta \nabla_j u_t
\]
\[ = -\nu \int_{T^n} \left[ \nabla_j \Delta n_t \cdot \nabla n + \Delta n_t \cdot \nabla_j n + \nabla_j \Delta n \cdot \nabla n_t + \Delta n \cdot \nabla_j n_t \right] \cdot \Delta \nabla_j u_t, \quad (3.54)
\]
then we obtain
\[
\left| I_5 \right| \leq C \left( \| \nabla \Delta n_t \| \| \nabla n \|_\infty + \| \Delta n_t \| \| \nabla^2 n \|_\infty + \| \Delta \nabla n \| \| \nabla n_t \|_\infty + \| \Delta n \| \| \nabla^2 n_t \|_\infty \right) \| \Delta \nabla u_t \|
\]
\[ \leq C \left( \| \nabla \Delta n_t \| \| \nabla n \|_2 + \| \Delta n_t \| \| \nabla^2 n \|_2 + \| \Delta \nabla n \| \| \nabla n_t \|_2 + \| \Delta n \| \| \nabla^2 n_t \|_2 \right) \| \Delta \nabla u_t \|
\]
\[ \leq \frac{\mu}{6} \| \nabla u_t \|^2 + C \left( \| \nabla n_t \|^2 + \| \nabla^2 n \|^2 \right). \]  

(3.55)

Clearly,
\[
I_7 = \lambda^2 \int_{T^n} \xi \left\{ \nabla_i \nabla_j \left( \frac{P'(\xi)}{\xi} \nabla_i \rho_t \right) - \frac{P'(\xi)}{\xi} \nabla_i \nabla_j \rho_t \right\} + \nabla_i \nabla_j \left[ \left( \frac{P'(\xi)}{\xi} \right)_t \nabla \rho \right] \cdot \nabla_i \nabla_j u_t
\]
\[ = I_{7,1} + I_{7,2}, \]

(3.56)

then we get
\[
\left| I_{7,1} \right| \leq C \lambda^2 \| \nabla^2 u_t \| \left( \| \nabla \xi \|^2 \| \nabla \rho_t \| + \| \nabla^2 \xi \| \| \nabla \rho_t \|_L^4 + \| \nabla \xi \| \| \nabla^2 \rho_t \| \right)
\]
\[ \leq C \| \nabla^2 u_t \| \left( \| \lambda \nabla \xi \|^2 \| \lambda \nabla \rho_t \| + \| \lambda \nabla^2 \xi \| \| \lambda \nabla \rho_t \| + \| \lambda \nabla \xi \| \| \lambda \nabla^2 \rho_t \| \right)
\]
\[ \leq C \left( K_1 + K_1^{\frac{1}{2}} \right) \left( \| \lambda \rho_t \|^2 + \| u_t \|^2 \right), \]  

(3.57)

where we have used the fact that
\[
\nabla_i \nabla_j \left( \frac{P'(\xi)}{\xi} \nabla_i \rho_t \right) - \frac{P'(\xi)}{\xi} \nabla_i \nabla_i \nabla_j \rho_t
\]
\[ = \left( \frac{P''(\xi)}{\xi} - \frac{2P''(\xi)}{\xi^2} + \frac{2P''(\xi)}{\xi^3} \right) \nabla_i \xi \cdot \nabla_j \xi \cdot \nabla_i \rho_t + \frac{\xi P''(\xi) - P'(\xi)}{\xi^2} \nabla_i \nabla_j \xi \cdot \nabla_i \rho_t
\]
\[ + \frac{\xi P''(\xi) - P'(\xi)}{\xi^2} \nabla_i \xi \cdot \nabla_j \rho_t. \]  

(3.58)

We infer from (3.39) and (3.56) that
\[
\left| I_{7,2} \right| \leq C \lambda^2 \| \nabla^2 u_t \| \left[ \left( \frac{P'(\xi)}{\xi} \right)_t \right]_\infty \| \nabla^3 \rho \| + \| \nabla \rho \|_\infty \left\| \nabla^2 \left( \frac{P'(\xi)}{\xi} \right)_t \right\|
\]
\[ \leq C \| u_t \|_2 \| \lambda (\rho - 1) \|_3 \left[ \| \lambda \xi \|_2 + \| \lambda \xi \|_2 \left( \lambda^{-1} \| \lambda \nabla \xi \| + \lambda^{-2} \| \lambda \nabla^2 \xi \|_2 \right) \right].
\]
\[ \leq C \left( K_1^3 + K_2^3 + 1 \right) \| u_t \|_2^2 + C. \]  \hspace{1cm} (3.59)

In conclusion, we have

\[ |I_7| \leq C \left( K_1^3 + K_2^3 + 1 \right) \left( \| \lambda \rho_t \|_2^2 + \| u_t \|_2^2 \right) + C. \]  \hspace{1cm} (3.60)

Next we come to give the estimate of \( I_8 \).

\[
|I_8| \leq C \| \nabla^2 u_t \| \left\{ \left\| \nabla_i \nabla_j \left( \frac{1}{\xi} \Delta u_t \right) - \frac{1}{\xi} \Delta \nabla_i \nabla_j u_t \right\| + \left\| \nabla^2 \left( \frac{1}{\xi} \right) \Delta u \right\| \right. \\
+ \left\| \nabla_i \nabla_j \left[ \frac{1}{\xi} \nabla (\nabla \cdot u_t) \right] - \frac{1}{\xi} \nabla_i \nabla_j \nabla (\nabla \cdot u_t) \right\| + \left\| \nabla^2 \left( \frac{1}{\xi} \right) \nabla (\nabla \cdot u) \right\| \right\} \\
+ \left\| \nabla^2 \left( \frac{1}{\xi} \right) \nabla (\nabla \cdot u) \right\| \right\} \\
\leq C \| \nabla^2 u_t \| \left\{ \| \nabla \xi \|_\infty \| \Delta u_t \| + \| \nabla^2 \xi \|_{L^1} \| \Delta u_t \|_{L^1} + \| \nabla \xi \|_\infty \| \nabla \Delta u_t \| \\
+ \left\| \left( \frac{1}{\xi} \right) \right\| \| \nabla^2 u_t \| + \| \Delta u_t \|_\infty \left\| \nabla^2 \left( \frac{1}{\xi} \right) \right\| + \| \nabla \xi \|_\infty \| \nabla (\nabla \cdot u_t) \| \\
+ \| \nabla (\nabla \cdot u) \| \| \nabla^2 \left( \frac{1}{\xi} \right) \right\| \right\} \\
\leq C \lambda^{-1} \| \nabla^2 u_t \| \left\{ \lambda^{-1} \| \lambda (\xi - 1) \|_2^2 \left( \| \nabla u_t \|_1 + \| \nabla \cdot u_t \|_1 \right) \\
+ \| \lambda (\xi - 1) \|_3 \left( \| \nabla u_t \|_2 + \| \nabla \cdot u_t \|_2 \right) + \| \lambda \xi \|_2 \left( \| \nabla u_t \|_3 + \| \nabla \cdot u_t \|_3 \right) \\
\| \nabla u_t \|_3 + \| \nabla \cdot u_t \|_3 \right\} \left[ \| \lambda \xi \|_2 + \| \lambda \xi \|_2 \left( \lambda^{-1} \| \lambda \nabla \xi \| + \lambda^{-2} \| \lambda \nabla \xi \|_2 \right) \right] \right\} \\
\leq \frac{\mu}{6} \| \nabla u_t \|_2^2 + \frac{\kappa + \mu}{2} \| \nabla \cdot u_t \|_2^2 + C \left( K_1^3 + K_2^3 + 1 \right) \| u_t \|_2^2 + C \left( \| \nabla u_t \|_3 + \| \nabla \cdot u_t \|_3 \right), \hspace{1cm} (3.61)
\]

where we have used the fact that

\[
\nabla_i \nabla_j \left( \frac{1}{\xi} \Delta u_t \right) - \frac{1}{\xi} \Delta \nabla_i \nabla_j u_t \\
= \frac{1}{\xi^3} \nabla_i \xi \nabla_j \xi \Delta u_t - \frac{1}{\xi^2} \nabla_i \nabla_j \xi \Delta u_t - \frac{1}{\xi^2} \nabla_i \xi \nabla_j \Delta u_t, \hspace{1cm} (3.62)
\]

and similar calculations on the term \( \nabla_i \nabla_j \left[ \frac{1}{\xi} \nabla (\nabla \cdot u_t) \right] - \frac{1}{\xi} \nabla (\nabla_i \nabla_j \nabla \cdot u_t). \)

For \( I_9 \), noting that

\[
|I_9| = \int \xi \left\{ \nabla_i \nabla_j \left[ \frac{\nu}{\xi} (\nabla n \cdot \Delta n)_t \right] - \frac{\nu}{\xi} \nabla_i \nabla_j (\Delta n \cdot \nabla n)_t + \nabla_i \nabla_j \left[ \frac{\nu}{\xi} \right] (\nabla n \cdot \Delta n) \right\} \cdot \nabla_i \nabla_j u_t \\
= I_{9,1} + I_{9,2}, \hspace{1cm} (3.63)
\]

one obtains from \( (3.49) \) that

\[
|I_{9,1}| \leq C \left\| \nabla^2 \left[ \left( \frac{\nu}{\xi} \right) (\nabla n \cdot \Delta n) \right] \right\| \| \nabla^2 u_t \|
\]

\]
\[
\begin{align*}
&\leq C \left[ \left\| \left( \frac{\nu}{\xi} \right)_t \right\| \| \nabla^2 (\nabla n - \Delta n) \| + \| \nabla n \|_\infty \| \Delta n \|_\infty \right] \| \nabla^2 u_t \|
\leq C \lambda^{-1} \left\{ \| \lambda \xi_2 \|_2 \| \nabla n \|_\infty \| \nabla^2 n \| + \| \lambda \xi_1 \| \| \Delta n \|_\infty \| \nabla^3 n \|ight.
\left. + \| \nabla n \|_2 \| \Delta n \|_2 \left[ \| \lambda \xi_2 \|_2 + \| \lambda \xi_1 \|_2 \left( \lambda^{-1} \| \lambda \nabla \xi \| + \lambda^{-1} \| \lambda \nabla \xi \| \right) \right] \right\} \| \nabla^2 u_t \|
\leq C \left( K_3^3 + K_2^3 + 1 \right) \| u_t \|_2^3 + C \| \nabla n \|_3^2, \tag{3.64}
\end{align*}
\]

and
\[
\begin{align*}
|I_{9,2}| &\leq C \left( \| \nabla \xi \|_\infty^2 \| (\Delta n \cdot \nabla n)_t \| + \| \nabla^2 \xi \|_L^2 \| (\Delta n \cdot \nabla n)_t \|_L^2 \right.
\left. + \| \nabla \xi \|_\infty \| \nabla (\Delta n \cdot \nabla n)_t \| \right) \| \nabla^2 u_t \|
\leq C \left( \lambda^{-2} \| \lambda \nabla \xi \|_1^2 \| \Delta n_t \| \| \nabla n \|_\infty + \lambda^{-2} \| \lambda \nabla \xi \|_1^2 \| \Delta n_t \| \| \nabla n_t \|_1 \right.
\left. + \lambda^{-1} \| \lambda \nabla \xi \|_2 \| \nabla n_t \| \| \nabla n_t \|_\infty + \lambda^{-1} \| \lambda \nabla \xi \|_2 \| \Delta n_t \| \| \nabla^2 n \|_\infty \right.
\left. + \lambda^{-1} \| \lambda \nabla \xi \|_2 \| \nabla n_t \| \| \nabla n_t \|_\infty + \lambda^{-1} \| \lambda \nabla \xi \|_2 \| \Delta n_t \| \| \nabla^2 n_t \| \right) \| u_t \|_2
\leq C \left( K_1 + K_1 \| \nabla n \|_3 + K_1^\frac{3}{2} \| \nabla n \|_3 + K_1^\frac{3}{2} \| \nabla n_t \|_2 \right) \| u_t \|_2
\leq C \left( K_1^3 + K_1 \right) \| u_t \|_2^3 + C \left( \| \nabla n \|_3^2 + \| \nabla n_t \|_2^2 \right) . \tag{3.65}
\end{align*}
\]

In conclusion, we have
\[
|I_9| \leq C \left( K_1^3 + K_2^3 + 1 \right) \| u_t \|_2^3 + C \left( \| \nabla n \|_3^2 + \| \nabla n_t \|_2^2 \right) . \tag{3.66}
\]

Now combining the estimates \((3.25)\)–\((3.33)\), \((3.45)\)–\((3.47)\), \((3.50)\)–\((3.53)\), \((3.55)\), \((3.60)\), \((3.61)\) and \((3.66)\) together, we have
\[
\begin{align*}
\frac{d}{dt} \sum_{|\beta| \leq s-1} \int_{T^n} \left( \frac{P(\xi)}{\xi} \| \lambda \nabla^\beta \rho t \| + \| \nabla^\beta u_t \|^2 \right)
\leq C \left( K_1^3 + K_2^3 + 1 \right) \left( \| \lambda \rho t \|_s^2 - 1 + \| u_t \|_{s-1}^2 \right)
\nonumber \\
+ C \left( \| \nabla \xi \|_s^2 + \| \nabla \cdot u \|_s^2 + \| \nabla^2 \xi \|_{s-1}^2 + \| \nabla n_t \|_{s-1}^2 + 1 \right) . \tag{3.67}
\end{align*}
\]

Recalling the restraints on the initial data and \((1.38)\), we have
\[
\| \lambda \nabla^\beta \rho t (x, 0) \|^2 + \| \nabla^\beta u_t (x, 0) \|^2
\leq C \left\{ \| \lambda(u_0 + \bar{\nu}_0) \cdot \nabla \bar{\rho}_0 \|_{s-1}^2 + \| \lambda(\bar{\rho}_0 + 1) \nabla \cdot \bar{\nu}_0 \|_{s-1}^2 + \| \lambda^2 \nabla \bar{\rho}_0 \|_{s-1}^2
\right.
\left. + \|[u_0 + \bar{\nu}_0) \cdot \nabla] (u_0 + \bar{\nu}_0) \|_{s-1}^2 + \| \Delta (u_0 + \bar{\nu}_0) + \nabla (\nabla \cdot \bar{\nu}_0) \|_{s-1}^2
\right.
\left. + \| \Delta (n_0 + \bar{\nu}_0) \cdot \nabla (n_0 + \bar{\nu}_0) \|_{s-1}^2 \right\}
\]

24
\[ \leq C. \] (3.68)

Then by the Gronwall’s inequality and the fact following by (3.18), (3.22), (3.44) that
\[
\int_0^t \left( \|\nabla u\|_s^2 + \|\nabla \cdot u\|_s^2 + \|\nabla^2 n\|_{s-1}^2 + \|\nabla n_t\|_{s-1}^2 \right) \leq C,
\]
we get
\[
\left( \|\lambda \rho_t\|_{s-1}^2 + \|u_t\|_{s-1}^2 \right) + \mu \int_0^t \|\nabla u_t\|_{s-1}^2 + (\kappa + \mu) \int_0^t \|\nabla \cdot u_t\|_{s-1}^2 \leq C,
\]
provided that \( T_0 \) is small enough such that \( T_0 < T_4 \equiv \min \{ T_3, \frac{1}{K_1^2 + K_2^2 + 1} \} \).

It remains to show the first inequality of (3.1). It suffices to show \( \|\lambda (\rho - 1)\|_s + \|u - u_0\|_s + \|\nabla n - \nabla n_0\|_{s-1} < c_0^{-1} \delta \) by Sobolev’s inequality, where \( c_0 \) is the Sobolev constant. Let \( \overline{\rho} = \rho - 1, \overline{\pi} = u - u_0, \overline{\nabla n} = \nabla n - \nabla n_0 \). We proceed as (3.1), (3.9) and (3.10), and then use (2.7), (2.8) and (3.17) to give
\[
\frac{d}{dt} \sum_{|\beta| \leq s-1} \|\nabla \nabla^\beta \overline{\pi}\|^2 + \theta \sum_{|\beta| \leq s-1} \|\nabla^2 \nabla^\beta \overline{\pi}\|^2 \leq CK_1 \|\nabla \overline{\pi}\|_{s-1}^2 + CK_1^2 \left( \|n\|_s^2 + \|n_0\|_s^2 \right) + C \left( \|\nabla n_0\|_{s-1}^2 + \|\nabla m\|_{s-1}^2 \right),
\]
where we have dealt with the term such as \( \int_{\mathbb{T}^n} \nabla_i \nabla^\beta (v \cdot \nabla n_0) \cdot \nabla_i \nabla^\beta \Delta \overline{\pi} \) by integration by parts and the Cauchy inequality as follows:
\[
\left| \int_{\mathbb{T}^n} \nabla_i \nabla^\beta (v \cdot \nabla n_0) \cdot \nabla_i \nabla^\beta \Delta \overline{\pi} \right| = \left| - \int_{\mathbb{T}^n} \nabla^\beta (v \cdot \nabla n_0) \cdot \Delta \nabla^\beta \Delta \overline{\pi} \right| \leq \frac{\theta}{12} \|\Delta \nabla^\beta \overline{\pi}\|^2 + C \|\nabla^\beta (v \cdot \nabla n_0)\|^2,
\]
and the terms \( \int_{\mathbb{T}^n} \nabla_i \nabla^\beta (\Delta n_0) \cdot \nabla_i \nabla^\beta \Delta \overline{\pi} \) and \( \int_{\mathbb{T}^n} \nabla_i \nabla^\beta (\|\nabla m\|_{s-1}^2 n_0) \cdot \nabla_i \nabla^\beta \Delta \overline{\pi} \) were treated by the same method shown in (3.72). Since (2.8) implies that \( \|\nabla \overline{\pi}(x,0)\|_s^2 \leq \lambda^{-2} \delta_0^2 \), we conclude
\[
\|\nabla \overline{\pi}\|_{s-1}^2 \leq \left[ \lambda^{-2} \delta_0^2 + C(K_1^2 + 1)T_0 \right] e^{CK_1 T_0} < c_0^{-1} \delta,
\]
provided that \( \lambda^{-1} \) and \( T_0(< T_4) \) are both sufficiently small such that (3.73) holds. And then integrating (3.71) over \([0, \tau]\), we have
\[
\int_0^\tau \|\nabla^2 \overline{\pi}\|_{s-1}^2 d\tau \leq \lambda^{-2} \delta_0^2 + C K_1 \delta T_0 + C(K_1^2 + 1)T_0.
\]
Similarly as (3.41) and (3.71), we have
\[
\frac{d}{dt} \sum_{|\alpha| \leq s} \left( \frac{P^i(\xi)}{\xi} \lambda |\nabla^\alpha \overline{\rho}|^2 + \xi |\nabla^\alpha \overline{\pi}|^2 \right)
\]
$$\begin{align*}
+ \sum_{|\alpha| \leq s} \mu \int_{T^n} |\nabla \nabla^\alpha \pi|^2 + \sum_{|\alpha| \leq s} (\kappa + \mu) \int_{T^n} |\nabla^\alpha (\nabla \cdot \pi)|^2 \\
\leq C (K_1^s + K_2 + 1) (||\lambda p||_s^2 + ||\pi||_s^2) + C ||\nabla^2 n||_{s-1}^2 + C ||(\nu \cdot \nabla) u_0||_s^2 + C ||\nabla u_0||_s^2 \\
+ C \left( \frac{1}{\xi} \nabla^s u_0 - \frac{1}{\xi} \nabla^s \Delta u_0 \right) + C ||\nabla \cdot u_0||_s^2 \\
+ C \left( \frac{1}{\xi} \nabla \left( \nabla \cdot u_0 \right) \right) - \frac{1}{\xi} \nabla^s \nabla \left( \nabla \cdot u_0 \right) \right)^2
\leq C (K_1^s + K_2 + 1) (||\lambda p||_s^2 + ||\pi||_s^2) + C ||\nabla^2 \pi||_{s-1}^2 + CK_1 + C.
\end{align*}$$
(3.75)

Then following by (2.8), (3.74), (3.75), the Gronwall’s inequality and the Cauchy inequality, we get

$$\begin{align*}
||\lambda p||_s^2 + ||\pi||_s^2 &\leq e^{C(K_1^s + K_2 + 1)T_0} \left[ 2\lambda^{-2} \delta_0^2 + C \int_0^t ||\nabla \pi||_s^2 + C(K_1 + 1)T_0 \right] \\
&\leq Ce^{C(K_1^s + K_2 + 1)T_0} \left[ \lambda^{-2} \delta_0^2 + CK_1 \delta T_0 + (K_1^2 + 1)T_0 \right] \\
&\leq c_0^{-1} \delta,
\end{align*}$$
(3.76)

provided that $T_0(< T_4)$ and $\lambda^{-1}$ are both small enough such that (3.73) and (3.76) hold.

As a conclusion, we have the following lemma.

**Lemma 3.2** Under the assumptions of Theorem 2.1 and suppose that $B_{T_0}^\lambda (U_0)$ is defined by (3.1) and $\Lambda : V \to U$ is defined by the system (3.2). Then there exist constants $T_0, \delta, K_1$ and $K_2$ independent of $\lambda$ such that $\Lambda$ maps $B_{T_0}^\lambda (U_0)$ into itself.

Now we plan to show that $\Lambda$ is a contractive map in some sense. In the proofs of this part and the following lemmas in this section, $C$ denotes the constant depending on the initial data, the domain, $N$, $s$ and the viscosity coefficients $\mu, \kappa, \nu$ and $\theta$. Firstly, let $U = \Lambda(V)$, $\hat{U} = \Lambda(\hat{V})$, where $V, \hat{V} \in B_{T_0}^\lambda$. Then by the definition, we have

$$\begin{align*}
\left\{ \begin{array}{l}
(\rho - \hat{\rho})_t + (v \cdot \nabla)(\rho - \hat{\rho}) + \xi \nabla \cdot (u - \hat{u}) + [(v - \hat{v}) \cdot \nabla] \hat{\rho} + (\xi - \hat{\xi}) \nabla \cdot \hat{u} = 0, \\
(u - \hat{u})_t + (v \cdot \nabla)(u - \hat{u}) + [(v - \hat{v}) \cdot \nabla] \hat{u} + \lambda^2 \frac{P'(|\xi|)}{\xi} \nabla(\rho - \hat{\rho}) + \lambda^2 \left( \frac{P'(|\xi|)}{\xi} - \frac{P'(|\xi|)}{\xi} \right) \nabla \hat{\rho}
\end{array} \right.
\end{align*}$$
(3.77)

$$\begin{align*}
\left\{ \begin{array}{l}
\frac{d}{dt} \int_{T^n} |n - \hat{n}|^2 + \int_{T^n} (v \cdot \nabla) \frac{|n - \hat{n}|^2}{2} + [(v - \hat{v}) \cdot \nabla] \hat{n} \cdot (n - \hat{n})
\end{array} \right.
\end{align*}$$
\[
\begin{align*}
\theta \int_{T^n} |\nabla (n - \hat{n})|^2 + \theta \int_{T^n} |\nabla m|^2 |n - \hat{n}|^2 \\
+ \theta \int_{T^n} [(\nabla m + \nabla \hat{m}):(\nabla m - \nabla \hat{m})]n \cdot (n - \hat{n}),
\end{align*}
\]

(3.78)

and

\[
\begin{align*}
-\frac{1}{2} \frac{d}{dt} \int_{T^n} |\nabla (n - \hat{n})|^2 &+ \int_{T^n} (v \cdot \nabla)(n - \hat{n}) \cdot \Delta(n - \hat{n}) + \int_{T^n} [(v - \hat{v})\nabla]n \cdot \Delta(n - \hat{n}) \\
= \theta \int_{T^n} \Delta(n - \hat{n})^2 + \theta \int_{T^n} |\nabla m|^2 |n - \hat{n}| \cdot \Delta(n - \hat{n}) \\
+ \theta \int_{T^n} [(\nabla m - \nabla \hat{m}):(\nabla m + \nabla \hat{m})]n \cdot \Delta(n - \hat{n}).
\end{align*}
\]

(3.79)

Then multiplying (3.77) by \(\lambda^2 \frac{P'(\xi)}{\xi}(\rho - \hat{\rho})\) and \(\xi(u - \hat{u})\) respectively, we have

\[
\begin{align*}
\frac{\lambda^2}{2} \frac{d}{dt} \int_{T^n} \frac{P'(\xi)}{\xi} |\rho - \hat{\rho}|^2 - \frac{\lambda^2}{2} \int_{T^n} |\rho - \hat{\rho}|^2 \left( \frac{P'(\xi)}{\xi} \right) \\
+ \lambda^2 \int_{T^n} \frac{P'(\xi)}{\xi} (\rho - \hat{\rho})(v \cdot \nabla)(\rho - \hat{\rho}) + \lambda^2 \int_{T^n} P'(\xi)(\rho - \hat{\rho})\nabla \cdot (u - \hat{u}) \\
+ \lambda^2 \int_{T^n} \frac{P'(\xi)}{\xi} (\rho - \hat{\rho})[(v - \hat{v}) \cdot \nabla]\hat{\rho} + \lambda^2 \int_{T^n} \frac{P'(\xi)}{\xi} (\rho - \hat{\rho})(\xi - \hat{\xi})\nabla \cdot \hat{u} = 0, \tag{3.80}
\end{align*}
\]

and

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{T^n} \xi |u - \hat{u}|^2 &- \frac{1}{2} \int_{T^n} \xi |u - \hat{u}|^2 + \int_{T^n} \xi (v \cdot \nabla)\frac{|u - \hat{u}|^2}{2} \\
+ \int_{T^n} \xi [(v - \hat{v}) \cdot \nabla]\hat{u} \cdot (u - \hat{u}) + \lambda^2 \int_{T^n} P'(\xi)[(u - \hat{u}) \cdot \nabla](\rho - \hat{\rho}) \\
+ \lambda^2 \int_{T^n} \xi \left( \frac{P'(\xi)}{\xi} - \frac{P'(\hat{\xi})}{\hat{\xi}} \right) [(u - \hat{u}) \cdot \nabla]\hat{\rho} \\
= -\mu \int_{T^n} |\nabla(u - \hat{u})|^2 + \int_{T^n} \xi \left( \frac{\mu}{\xi} - \frac{\mu}{\hat{\xi}} \right) \Delta\hat{u} \cdot (u - \hat{u}) - (\kappa + \mu) \int_{T^n} |\nabla \cdot (u - \hat{u})|^2 \\
+ \int_{T^n} \xi \left( \frac{\kappa + \mu}{\xi} - \frac{\kappa + \hat{\mu}}{\hat{\xi}} \right) [(u - \hat{u}) \cdot \nabla] \hat{\nabla} \\
+ \nu \int_{T^n} |\nabla \cdot \Delta(n - \hat{n}) + \nabla(n - \hat{n}) \cdot \Delta \hat{n}| \cdot (u - \hat{u}) + \int_{T^n} \xi \left( \frac{\nu}{\xi} - \frac{\nu}{\hat{\xi}} \right) [(u - \hat{u}) \cdot \nabla] \hat{\nabla} \cdot \Delta \hat{n}.
\end{align*}
\]

(3.81)

In conclusion, using (3.78)–(3.81) and preceding similarly as before, we obtain

\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{T^n} \left[ \frac{P'(\xi)}{\xi} |\lambda(\rho - \hat{\rho})|^2 + \xi |u - \hat{u}|^2 + |n - \hat{n}|^2 + |\nabla(n - \hat{n})|^2 \right] \\
+ \mu \int_{T^n} |\nabla(u - \hat{u})|^2 + (\kappa + \mu) \int_{T^n} |\nabla \cdot (u - \hat{u})|^2 + \theta \int_{T^n} |\nabla(n - \hat{n})|^2 + \theta \int_{T^n} |\Delta(n - \hat{n})|^2 \\
\leq \quad C \left( \|\lambda(\rho - \hat{\rho})\|^2 + \|u - \hat{u}\|^2 + \|n - \hat{n}\|^2 + \|\lambda(\xi - \hat{\xi})\|^2 + \|v - \hat{v}\|^2 + \|\nabla(m - \hat{m})\|^2 \right) \\
+ \frac{\mu}{2} \|\nabla(u - \hat{u})\|^2 + \frac{\theta}{2} \|\Delta(n - \hat{n})\|^2,
\end{align*}
\]

(3.82)
where we have used the following two estimates different from before and the same discussions also have been used to deal with the last two terms of the R.H.S. of (3.31),

\[
\left| \int_{T_0}^{\xi} \left( \frac{\mu}{\xi} - \frac{\mu}{\xi} \right) \Delta \tilde{u} \cdot (u - \tilde{u}) \right| \leq C \| \xi - \tilde{\xi} \| \| \Delta \tilde{u} \|_1 \| u - \tilde{u} \|_1 \\
\leq C \| \xi - \tilde{\xi} \| \| \tilde{u} \|_1 (\| u - \tilde{u} \| + \| \nabla (u - \tilde{u}) \|) \\
\leq C \| \xi - \tilde{\xi} \| (\| u - \tilde{u} \| + \| \nabla (u - \tilde{u}) \|) \\
\leq \frac{\mu}{8} \| \nabla (u - \tilde{u}) \|^2 + C(\lambda^{-1} \| \lambda (\xi - \tilde{\xi}) \|^2 + \| u - \tilde{u} \|^2), (3.83)
\]

and

\[
\left| \int_{T_0}^{\xi} \left( \frac{K + \mu}{\xi} - \frac{K + \mu}{\xi} \right) \nabla \nabla \cdot (u - \tilde{u}) \right| \leq \frac{\mu}{8} \| \nabla (u - \tilde{u}) \|^2 + C(\| \lambda (\xi - \tilde{\xi}) \|^2 + \| u - \tilde{u} \|^2).
\]

(3.84)

Then noting that \((U - \tilde{U})(0) = 0\), the Gronwall’s inequality, and (3.32), we obtain

\[
\sup_{0 \leq t \leq T_0} \left( \| \lambda (\rho - \tilde{\rho}) \|^2 + \| u - \tilde{u} \|^2 + \| n - \tilde{n} \|_1^2 \right) \\
\leq CT_0 e^{CT_0} \sup_{0 \leq t \leq T_0} \left( \| \lambda (\xi - \tilde{\xi}) \|^2 + \| v - \tilde{v} \|^2 + \| m - \tilde{m} \|_1^2 \right).
\]

(3.85)

Moreover, we integrate (3.32) over \([0, t]\) and get

\[
\int_0^t \left( \| u - \tilde{u} \|_1^2 + \| n - \tilde{n} \|_2^2 \right) \\
\leq C \left( T_0^2 e^{CT_0} + T_0 \right) \sup_{0 \leq t \leq T_0} \left( \| \lambda (\xi - \tilde{\xi}) \|^2 + \| v - \tilde{v} \|^2 + \| m - \tilde{m} \|_1^2 \right).
\]

(3.86)

If we take \(T_0 (< T_1)\) small enough such that (3.73), (3.76) and \(C \left( T_0^2 e^{CT_0} + T_0 e^{CT_0} + T_0 \right) < 1\) hold, then we can prove the contraction.

In conclusion, we have the following lemma.

**Lemma 3.3** Under the assumptions of Theorem 2.1, the maps \(\Lambda : V \to U\) is a contraction in the sense that

\[
\sup_{0 \leq t \leq T_0} \left( \| \lambda (\rho - \tilde{\rho}) \|^2 + \| u - \tilde{u} \|^2 + \| n - \tilde{n} \|_1^2 \right) + \int_0^t \left( \| u - \tilde{u} \|_1^2 + \| n - \tilde{n} \|_2^2 \right) \\
\leq \tau \sup_{0 \leq t \leq T_0} \left( \| \lambda (\xi - \tilde{\xi}) \|^2 + \| v - \tilde{v} \|^2 + \| m - \tilde{m} \|_1^2 \right)
\]

for some \(0 < \tau < 1\), provided that \(T_0\) is small enough.

In order to prove Theorem 2.1 we also need the following lemma.

**Lemma 3.4** Consider the incompressible system of liquid crystal (1.40) with the initial condition (2.7) for \(s \geq 3\). Then there exists at most one strong solution.
Proof. Assume that \((u_1, n_1)\) and \((u_2, n_2)\) are two strong solutions of (1.40) with the same initial data (2.7). Then we have
\[
\begin{aligned}
\nabla \cdot (u_1 - u_2) &= 0, \\
\partial_t (u_1 - u_2) + (u_1 \cdot \nabla)(u_1 - u_2) + [(u_1 - u_2) \cdot \nabla]u_2 + \nabla (p_1 - p_2) \\
&= \mu \Delta (u_1 - u_2) - \nu \nabla \cdot [\nabla (n_1 - n_2) \odot \nabla n_1] - \nu \nabla \cdot [\nabla n_2 \odot \nabla (n_1 - n_2)], \\
\partial_t (n_1 - n_2) + (u_1 \cdot \nabla)(n_1 - n_2) + [(u_1 - u_2) \cdot \nabla]n_2 \\
&= \theta \Delta (n_1 - n_2) + \theta |\nabla n_1|^2 (n_1 - n_2) + \theta [(\nabla n_1 - \nabla n_2) : (\nabla n_1 + \nabla n_2)]n_2.
\end{aligned}
\] (3.88)

Multiplying (3.88)\(_1\) by \((u_1 - u_2)\) and multiplying (3.88)\(_3\) by \((n_1 - n_2)\) and \(\Delta (n_1 - n_2)\) respectively, we have
\[
\frac{1}{2} \frac{d}{dt} \left( \|u_1 - u_2\|^2 + \|n_1 - n_2\|^2 \right) \\
+ \mu \|\nabla (u_1 - u_2)\|^2 + \theta \|\nabla (n_1 - n_2)\|^2 + \theta \|\Delta (n_1 - n_2)\|^2 \\
\leq C (\|u_1 - u_2\|^2 + \|n_1 - n_2\|^2) + \frac{\mu}{2} \|\nabla (u_1 - u_2)\|^2 + \frac{\theta}{2} \|\Delta (n_1 - n_2)\|^2.
\] (3.89)

Noting that \(\|u_1(x, 0) - u_2(x, 0)\|^2 + \|n_1(x, 0) - n_2(x, 0)\|^2_1 = 0\), and by the Gronwall’s inequality, we have
\[
\|u_1 - u_2\|^2 + \|n_1 - n_2\|^2_1 = 0.
\] (3.90)

This completes the proof of Lemma 3.5. \(\square\)

Before proving Theorem 2.1, we give the following lemma.

Lemma 3.5 \((28)\) Assume that \(X \subset E \subset Y\) are Banach spaces and \(X \leftrightarrow E\). Then the following embedding are compact:
\[
\begin{align*}
(i) \quad & \left\{ \varphi : \varphi \in L^q(0, T; X), \frac{\partial \varphi}{\partial t} \in L^1(0, T; Y) \right\} \leftrightarrow L^q(0, T; E), \quad \text{if} \quad 1 \leq q \leq \infty; \\
(ii) \quad & \left\{ \varphi : \varphi \in L^\infty(0, T; X), \frac{\partial \varphi}{\partial t} \in L^r(0, T; Y) \right\} \leftrightarrow C([0, T]; E), \quad \text{if} \quad 1 < r \leq \infty.
\end{align*}
\]

Now we are in a position to give the proof of Theorem 2.1.

Proof of Theorem 2.1. For any fixed \(\lambda\), the standard procedure produces a sequence
\[
\{\rho^\lambda, u^\lambda, n^\lambda\}_{i=1}^\infty
\]
satisfying
\[
\begin{aligned}
\partial_t \rho^\lambda_i + (u^\lambda_i \cdot \nabla) \rho^\lambda_i + \rho^\lambda_i \nabla \cdot u^\lambda_i &= 0, \\
\partial_t u^\lambda_i + (u^\lambda_i \cdot \nabla) u^\lambda_i + \lambda^2 \frac{\rho^\lambda_i}{\rho_i^\lambda} \nabla \rho^\lambda_i &= \frac{\mu}{\rho_i} \Delta u^\lambda_i + \frac{(\kappa + \mu)}{\rho_i} \nabla \cdot u^\lambda_i - \frac{\nu}{\rho_i} (\Delta n^\lambda_i + \nabla n^\lambda_i), \\
\partial_t n^\lambda_i + (u^\lambda_i \cdot \nabla) n^\lambda_i &= \theta (\Delta n^\lambda_i + |\nabla n^\lambda_i|^2 n^\lambda_i).
\end{aligned}
\] (3.91)
as well as the uniform estimates

\[
\begin{aligned}
E_s (U_i^\lambda(t)) &+ \int_0^t \left[ \mu \| \nabla u_i^\lambda(t) \|^2 + (\kappa + \mu) \| \nabla \cdot u_i^\lambda(t) \|^2 + \theta \| \nabla n_i^\lambda(t) \|^2 \right] dt \\
E_{s-1} (\partial_t U_i^\lambda(t)) &+ \int_0^t \left[ \mu \| \nabla \partial_t u_i^\lambda(t) \|^2 + (\kappa + \mu) \| \nabla \cdot \partial_t u_i^\lambda(t) \|^2 + \theta \| \nabla \partial_t n_i^\lambda(t) \|^2 \right] dt 
\end{aligned}
\] (3.92)

Let \( \hat{\rho}_{i+1}^\lambda = \rho_{i+1}^\lambda - \rho_i^\lambda, \hat{u}_{i+1}^\lambda = u_{i+1}^\lambda - u_i^\lambda, \) and \( \hat{n}_{i+1}^\lambda = n_{i+1}^\lambda - n_i^\lambda. \) In view of Lemma 3.3, we have

\[
\sum_{i=2}^{\infty} \| \hat{\rho}_i^\lambda \| < \infty, \quad \sum_{i=2}^{\infty} \left( \| \hat{n}_i^\lambda \|^2 + \int_0^{T_0} \| \nabla \hat{n}_i^\lambda \|^2 \right) < \infty, \quad \sum_{i=2}^{\infty} \left( \| \hat{n}_i^\lambda \|_1 + \int_0^{T_0} \| \Delta \hat{n}_i^\lambda \|^2 \right) < \infty. \tag{3.93}
\]

Let \( \rho^\lambda = \rho_i^\lambda + \sum_{i=2}^{\infty} \hat{\rho}_i^\lambda, \) \( u^\lambda = u_i^\lambda + \sum_{i=2}^{\infty} \hat{u}_i^\lambda \) and \( n^\lambda = n_i^\lambda + \sum_{i=2}^{\infty} \hat{n}_i^\lambda, \) then we have

\[
\rho_i^\lambda \to \rho^\lambda, \quad \text{in} \quad L^\infty([0; T_0]; L^2),
\]

\[
u_i^\lambda \to \nu^\lambda, \quad \text{in} \quad L^\infty([0; T_0]; L^2) \cap L^2([0; T_0]; H^1),
\]

\[
n_i^\lambda \to n^\lambda, \quad \text{in} \quad L^\infty([0; T_0]; H^1) \cap L^2([0; T_0]; H^2).
\]

It follows obviously that

\[
rho^\lambda, u^\lambda, n^\lambda \in L^\infty([0; T_0]; H^s) \cap \text{Lip}([0; T_0]; H^{s-1})
\]

satisfy the estimates (3.92) according to the lower semi-continuity. For any \( s' \in [0; s), \) by the Sobolev interpolation inequalities, we have

\[
\| (\rho_i^\lambda, u_i^\lambda, n_i^\lambda) - (\rho^\lambda, u^\lambda, n^\lambda) \|_{s'} \leq C \| (\rho_i^\lambda, u_i^\lambda, n_i^\lambda) - (\rho^\lambda, u^\lambda, n^\lambda) \|^\theta \| (\rho_i^\lambda, u_i^\lambda, n_i^\lambda) \|_s + \| (\rho^\lambda, u^\lambda, n^\lambda) \|_s \] (3.94)

as \( i \to \infty, \) where we have used lemma 3.3 to get

\[
\| (\rho_i^\lambda, u_i^\lambda, n_i^\lambda) - (\rho^\lambda, u^\lambda, n^\lambda) \| \leq \sum_{j=i+1}^{\infty} \| (\rho_j^\lambda, u_j^\lambda, n_j^\lambda) - (\rho_j^\lambda, u_j^\lambda, n_j^\lambda) \| \leq \frac{C \tau^i}{1 - \tau}. \tag{3.95}
\]

Hence, \( (\rho^\lambda, u^\lambda, n^\lambda) \in C([0; T_0]; H^{s'}). \) Besides, with the aid of Lemma 3.5 and (3.92), one deduces easily that \( (\rho^\lambda, u^\lambda, n^\lambda) \) is a strong solution of compressible Liquid crystal model (3.8). Finally, multiplying (3.8) by \( n^\lambda, \) we get an equation for \( (|n^\lambda|^2 - 1) \) as

\[
\frac{1}{2} \left( |n^\lambda|^2 - 1 \right)_t + \frac{1}{2} \left( u^\lambda \cdot \nabla \right) \left( |n^\lambda|^2 - 1 \right) = \frac{\theta}{2} \Delta \left( |n^\lambda|^2 - 1 \right) + \theta |\nabla n^\lambda|^2 \left( |n^\lambda|^2 - 1 \right). \tag{3.96}
\]

Multiplying (3.96) by \( (|n^\lambda|^2 - 1), \) integrating over \( \mathbb{T}^n, \) and then using the Gronwall’s inequality and the assumption that \( |n^\lambda(x, 0)|^2 = 1, \) we have

\[
\int_{\mathbb{T}^n} |n^\lambda|^2 - 1 |^2 = 0 \quad \text{for all} \quad t \in [0; T_0], \tag{3.97}
\]
which implies \(|n^\lambda| = 1\) a.e. in \(Q_{T_0}\), then from the regularity of \(n^\lambda\), we conclude that

\[
|n^\lambda| = 1, \text{ in } Q_{T_0}. \tag{3.98}
\]

The uniqueness can be proved by a similar argument as Lemma 3.3. This completes the proof of the uniform stability part of Theorem 2.1.

Next, we are going to show that \((\rho^\lambda, u^\lambda, n^\lambda)\) converges to the unique strong solution to the corresponding incompressible system \((1.40)\) as \(\lambda \to \infty\). To see this, note first that \((2.9)\) implies that \(\rho^\lambda \to 1\) in \(L^\infty([0, T_0]; H^s) \cap \text{Lip}([0, T_0]; H^{s-1})\), and there exists a subsequence \(\{(u^{\lambda_j}, n^{\lambda_j})\}_j\) of \(\{(u^\lambda, n^\lambda)\}_\lambda\) with a limit \(u\) and \(n\) such that

\[
\left\{
\begin{array}{l}
(u^{\lambda_j}, n^{\lambda_j}) \to (u, n) \text{ weakly* in } L^\infty([0, T_0]; H^s) \cap \text{Lip}([0, T_0]; H^{s-1}), \\
(u^{\lambda_j}, n^{\lambda_j}) \to (u, n) \text{ in } C([0, T_0]; H^{s'})
\end{array}
\right.
\tag{3.99}
\]

for any \(0 \leq s' < s\), where we have used the fact that the embedding \(H^s \hookrightarrow H^{s'}\) is compact and Lemma 3.3. Now we are to let \(j \to \infty\) (that is \(\lambda_j \to \infty\)) in \((1.38)\).

First of all, multiplying \((1.38)_1\) and \((1.38)_3\) by two smooth test functions \(\psi_1(x, t)\) and \(\psi_3(x, t)\) with compact in \([t \in [0, T_0]\) respectively, we have

\[
\int_0^{T_0} \int_T^n \nabla \cdot u^{\lambda_j} \psi_1 dx dt = \int_0^{T_0} \int_T^n \left[ \rho^{\lambda_j} + \left( u^{\lambda_j} \cdot \nabla \right) \rho^{\lambda_j} + \left( \rho^{\lambda_j} - 1 \right) \nabla \cdot u^{\lambda_j} \right] \psi_1 dx dt, \tag{3.100}
\]

\[
\int_0^{T_0} \int_T^n \left[ n^{\lambda_j} + \left( u^{\lambda_j} \cdot \nabla \right) n^{\lambda_j} - \Theta \left( \Delta n^{\lambda_j} + |\nabla n^{\lambda_j}|^2 n^{\lambda_j} \right) \right] \psi_3 dx dt = 0. \tag{3.101}
\]

Then \((u, n)\) satisfies \((1.40)_1\) and \((1.40)_3\). Let \(\psi(x, t)\) be a smooth test function of \((1.38)_2\) with compact supports in \(t \in [0, T_0]\) and the divergence free condition \(\nabla \cdot \psi = 0\), we have

\[
\int_0^{T_0} \int_T^n \left[ u_t^{\lambda_j} + \left( u^{\lambda_j} \cdot \nabla \right) u^{\lambda_j} - \frac{\mu}{\rho^{\lambda_j}} \Delta u^{\lambda_j} - \frac{\kappa + \mu}{\rho^{\lambda_j}} \nabla \left( \nabla \cdot u^{\lambda_j} \right) \\
+ \frac{\nu}{\rho^{\lambda_j}} \nabla \cdot \left( \nabla n^{\lambda_j} \otimes \nabla n^{\lambda_j} \right) - \frac{\nu}{\rho^{\lambda_j}} \nabla \left( \frac{|\nabla n^{\lambda_j}|^2}{2} \right) \right] \psi dx dt = 0,
\]

\[
= \int_0^{T_0} \int_T^n -\lambda_j^2 \psi \nabla \int_1^{\rho(x)} \frac{P'(\xi)}{\xi} d\xi dx dt = 0, \tag{3.102}
\]

then let \(j \to \infty\) and get

\[
P \left[ u_t + (u \cdot \nabla) u - \mu \Delta u + \nu \nabla \cdot (\nabla n \odot \nabla n) - \nu \nabla \left( \frac{|\nabla n|^2}{2} \right) \right] = 0, \tag{3.103}
\]

where \(P\) is the \(L^2\)-projection on the divergence zero vector fields. If

\[
\quad u_t + (u \cdot \nabla) u - \mu \Delta u + \nu \nabla \cdot (\nabla n \odot \nabla n) - \nu \nabla \left( \frac{|\nabla n|^2}{2} \right) = -\nabla \hat{p} \tag{3.104}
\]
for some $\hat{p} \in L^\infty([0, T_0]; H^{s-1}) \cap L^2([0, T_0]; H^s)$, then we have

$$\frac{\lambda^2}{\rho^s_0} \nabla \left( P(\rho^\lambda) \right) \rightarrow \nabla \hat{p}, \quad \text{weakly}^* \quad \text{in} \quad L^\infty([0, T_0]; H^{s-2}) \cap L^2([0, T_0]; H^{s-1}). \quad (3.105)$$

Taking $p = \hat{p} - \nu |\nabla n|^2_2$, then (1.40) follows from (3.105) directly. Actually, Lemma 3.4 ensures that the sequence $(\rho^\lambda, u^\lambda, n^\lambda)$ itself converges as well. Moreover, similarly as (3.96)–(3.98), we can prove $|n| = 1$ in $Q_{T_0}$. This completes the proof of Theorem 2.1. □

**Remark 3.1** It follows from (3.105) that

$$\|\lambda^2 \nabla \rho^\lambda\|_{s-2} + \int_0^t \|\lambda^2 \nabla \rho^\lambda\|_{s-1}^2 \leq C, \quad t \in [0, T_0]. \quad (3.106)$$

**Remark 3.2** It follows from (1.38) and (2.9) that

$$\|\lambda \nabla \cdot u^\lambda\|_{s-1} \leq \|\lambda \rho^\lambda\|_{s-1} + \|\lambda(u^\lambda \cdot \nabla)\rho^\lambda\|_{s-1} + \|\lambda(\rho^\lambda - 1) \nabla \cdot u^\lambda\|_{s-1} \leq, \quad t \in [0, T_0]. \quad (3.107)$$

### 4 Global existence of strong solutions

In this section, we devote ourselves to getting a priori energy estimates (2.13) and (2.14) for small initial displacements and small initial data, and then together with Theorem 2.1, Theorem 2.2 can be proved by a standard procedure. First of all, we assume that $|n| = 1$ for all $(x, t) \in Q_T$, and

$$F_s(U(t)) + \int_0^t [\mu \|\nabla u\|_s^2 + (\kappa + \mu) \|\nabla \cdot u\|_s^2 + \theta \|\nabla^2 n\|_{s-1}^2] \leq 4 (\varepsilon_0 + \lambda^{-2} \delta_0^2) \quad (4.1)$$

for $t \in [0, T]$, then what we need to do is to prove the following desired estimates:

$$F_s(U(t)) + \int_0^t [\mu \|\nabla u\|_s^2 + (\kappa + \mu) \|\nabla \cdot u\|_s^2 + \theta \|\nabla^2 n\|_{s-1}^2] \leq 2 (\varepsilon_0 + \lambda^{-2} \delta_0^2). \quad (4.2)$$

for $t \in [0, T]$. Then (2.13) follows by the standard continuity argument and the fact $F_s(U(0)) < 4(\varepsilon_0 + \lambda^{-2} \delta_0^2)$.

Now we plan to give (2.14) under the assumptions that $|n| = 1$ and (4.1). We go back to (3.7), (3.8), (3.11)–(3.15), (3.24)–(3.26), (3.28)–(3.33) and (3.45)–(3.66), replace $(\xi, v, m)$ by $(\rho, u, n)$, and then we make a similar (just a little different) argument to give the following fact: when $D^\alpha = \partial_t$, we obtain

$$\sum_{j=1}^{2} |N_j| + \sum_{j=1}^{9} |I_j| \leq C \left( \|\nabla^2 n\|_s^2 + 1 \right) \left( \|\lambda \rho_t\|_s^2 + \|u_t\|_s^2 + \|n_t\|_s^2 \right), \quad (4.3)$$
where we have used \( (1.38) \) to give the following fact and then used it to estimate \( I_1 \):

\[
\|\rho_t\|_\infty \leq \|\nabla \cdot (\rho u)\|_\infty \leq C (\|u\|_2 \|\nabla \rho\|_2 + \|\rho\|_\infty \|\nabla \cdot u\|_2) \leq C \lambda^{-1} + C \leq C. \tag{4.4}
\]

On the other hand, when \( D^\gamma = \nabla_i \nabla^\gamma \partial_t \) with \( |\gamma| \leq s - 2 \), we have

\[
\sum_{j=1}^2 |N_j| + \sum_{j=1}^9 |I_j| \leq \frac{\mu}{2} \|\nabla u_t\|_{s-1}^2 + \frac{\kappa + \mu}{2} \|\nabla \cdot u_t\|_{s-1}^2 + \left[ \frac{\theta}{4} + C(\varepsilon_0 + \lambda^{-2}) \right] \|\nabla n_t\|_{s-1}^2
\]

\[
+ C \left( \|\nabla u_t\|_s^2 + \|\nabla \cdot u_t\|_s^2 + \|\nabla^2 n\|_{s-1}^2 + 1 \right) \left( \|\lambda \rho_t\|_{s-1}^2 + \|u_t\|_{s-1}^2 + \|n_t\|_{s-1}^2 \right), \tag{4.5}
\]

where we have used the fact indicated by \( (4.1) \) that \( \|\nabla n\|_{s-1}^2 \leq C(\varepsilon_0 + \lambda^{-2}) \).

Combining \( (4.3) \) and \( (4.5) \), we have

\[
\frac{d}{dt} \sum_{|\beta| \leq s-1} \int_{T_N} \left( \frac{P(\rho)}{\rho} |\nabla \beta \rho_t|^2 + \rho |\nabla \beta u_t|^2 + |\nabla \beta n_t|^2 \right)
\]

\[
+ \sum_{|\beta| \leq s-1} \mu \int_{T_N} |\nabla \beta u_t|^2 + \sum_{|\beta| \leq s-1} (\kappa + \mu) \int_{T_N} |\nabla \cdot \nabla u_t|^2 + \sum_{|\beta| \leq s-1} \theta \int_{T_N} |\nabla \beta n_t|^2
\]

\[
\leq C \left( \|\nabla u_t\|_s^2 + \|\nabla \cdot u_t\|_s^2 + \|\nabla^2 n\|_{s-1}^2 + 1 \right) \left( \|\lambda \rho_t\|_{s-1}^2 + \|u_t\|_{s-1}^2 + \|n_t\|_{s-1}^2 \right), \tag{4.6}
\]

provided that \( \varepsilon_0 \) and \( \lambda^{-1} \) are both small enough.

Then by the Gronwall’s inequality, \( (2.5) \), \( (3.20) \), \( (3.68) \) and \( (4.1) \), we get

\[
\|\lambda \rho_t\|_{s-1}^2 + \|u_t\|_{s-1}^2 + \|n_t\|_{s-1}^2 + \int_0^t \left[ \|\nabla u_t\|_{s-1}^2 + (\kappa + \mu) \|\nabla \cdot u_t\|_{s-1}^2 + \theta \|\nabla n_t\|_{s-1}^2 \right]
\]

\[
\leq C(1 + t) \exp Ct \leq C \exp Ct, \ t \in [0, T]. \tag{4.7}
\]

And thus we can conclude that Remark \( 3.1 \) and \( 3.2 \) hold for \( t \in [0, T] \) by the same discussion shown in Section 3.

Secondly, we shall turn to prove \( (4.2) \). We go back to \( (3.4) \), \( (3.9) \) and \( (3.10) \), replace \((v, m)\) by \((u, n)\), and then make a different argument by noting that \( |n| = 1 \) to give

\[
\frac{1}{2} \sum_i \int_{T_N} |\nabla \beta n_i|^2 + \theta \sum_i \int_{T_N} |\nabla n_i \nabla \beta n_i|^2
\]

\[
= \int_{T_N} \nabla \beta (u \cdot \nabla n) \cdot \Delta \nabla \beta n - \theta \int_{T_N} \nabla \beta (|\nabla n|_2^2 n) \cdot \Delta \nabla \beta n
\]

\[
\leq C \|\nabla \beta n\| \left( \|\nabla (u \cdot \nabla n)\| + \|\nabla \beta (|\nabla n|_2^2 n)\| \right)
\]

\[
\leq \frac{\theta}{4} \|\nabla \beta n\|_2^2 + C \|\nabla (u \cdot \nabla n)\|_2^2 + C \|\nabla \beta (|\nabla n|_2^2 n)\|_2^2
\]

\[
\leq \frac{\theta}{4} \|\nabla \beta n\|_2^2 + C \left( \|u\|_\infty^2 \|\nabla n\|_2^2 + \|\nabla n\|_\infty^2 \|\nabla^{-1} n\|_2^2 \right)
\]

\[
+ C \left( \|\nabla n\|_\infty^2 \|\nabla^{-1} n\|_2^2 + \|n\|_\infty^2 \|\nabla n\|_\infty^2 \right)
\]
where we have used Poincaré inequality in the last step since
\[
\int_{T^n} \nabla^k n = 0, \quad k \geq 1.
\] (4.9)

We conclude from (4.8) that
\[
\sum_{|\beta| \leq s-1} \frac{d}{dt} \int_{T^n} |\nabla \nabla^\beta n|^2 + \theta \sum_{|\beta| \leq s-1} \int_{T^n} |\nabla^2 \nabla^\beta n|^2 \leq 0,
\] (4.10)
provided that \(\varepsilon_0\) and \(\lambda^{-1}\) are both small enough. Then by integrating (4.10) over \([0, t]\), we have
\[
\|\nabla n\|^2_{s-1} + \theta \int_0^t \|\nabla^2 n\|^2_{s-1} \leq \|\nabla n_0\|^2_{s-1} + \lambda^{-2} \delta_0^2 \leq \varepsilon_0 + \lambda^{-2} \delta_0^2.
\] (4.11)

Thirdly, recalling (3.23) and replacing \((\xi, v)\) by \((\rho, u)\), it yields
\[
\frac{1}{2} \frac{d}{dt} \int_{T^n} \left[ \lambda^2 \frac{P'(\rho)}{\rho} |D^\alpha (\rho - 1)|^2 + \rho |D^\alpha u|^2 \right] + \mu \int_{T^n} |\nabla D^\alpha u|^2 + (\kappa + \mu) \int_{T^n} |\nabla \cdot D^\alpha u|^2
= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_8,
\] (4.12)
where there are some slight changes from \(I_j\) shown as follows:

\[
J_1 = \frac{\lambda^2}{2} \int_{T^n} \frac{2P'(\rho) - P''(\rho)\rho}{\rho} \nabla \cdot \nabla D^\alpha (\rho - 1)|^2;
\]
\[
J_2 = \lambda^2 \int_{T^n} P''(\rho) D^\alpha (\rho - 1) D^\alpha u \cdot \nabla \rho;
\]
\[
J_3 = -\lambda^2 \int_{T^n} \frac{P'(\rho)}{\rho} D^\alpha (\rho - 1)\{[D^\alpha (\nabla \rho) - u \cdot \nabla D^\alpha \rho] + [D^\alpha (\rho \nabla \cdot u) - \rho \nabla \cdot D^\alpha u]\},
\]
\[
J_4 = -\nu \int_{T^n} D^\alpha (\Delta n \cdot \nabla n) \cdot D^\alpha u;
\]
\[
J_5 = -\int_{T^n} \rho [D^\alpha (u \cdot \nabla u) - u \cdot \nabla D^\alpha u] \cdot D^\alpha u,
\]
\[
J_6 = -\lambda^2 \int_{T^n} \rho \left[ D^\alpha \left( \frac{P'(\rho)}{\rho} \nabla \rho \right) - \frac{P'(\rho)}{\rho} \nabla D^\alpha \rho \right] \cdot D^\alpha u,
\]
\[
J_7 = \int_{T^n} \rho \left[ D^\alpha \left( \frac{\mu}{\rho} \Delta u \right) - \frac{\mu}{\rho} \Delta D^\alpha u \right] \cdot D^\alpha u
+ \int_{T^n} \rho \left\{ D^\alpha \left( \frac{\kappa + \mu}{\rho} \nabla (\nabla \cdot u) \right) - \frac{\kappa + \mu}{\rho} \nabla D^\alpha (\nabla \cdot u) \right\} \cdot D^\alpha u.
\]
\[ J_S = - \int_{\mathbb{T}^N} \rho \left\{ D^\alpha \left[ \frac{\nu}{\rho} (\Delta n \cdot \nabla n) \right] - \frac{\nu}{\rho} D^\alpha (\Delta n \cdot \nabla n) \right\} \cdot D^\alpha u, \]

since we have used (1.38) to give

\[
\frac{\lambda^2}{2} \int_{\mathbb{T}^N} \left| D^\alpha (\rho - 1)^2 \partial_t \left( \frac{P'(\rho)}{\rho} \right) + \frac{\lambda^2}{2} \int_{\mathbb{T}^N} |D^\alpha (\rho - 1)|^2 \nabla \cdot \left( \frac{P'(\rho)}{\rho} \nu \right) \right.
\]

\[
= \frac{\lambda^2}{2} \int_{\mathbb{T}^N} \left[ \left( \frac{P'(\rho)}{\rho} \right)' \rho_t + \nabla \cdot \left( \frac{P'(\rho)}{\rho} u \right) \right] \left| D^\alpha (\rho - 1) \right|^2
\]

\[
= \frac{\lambda^2}{2} \int_{\mathbb{T}^N} \left[ \left( \frac{P'(\rho)}{\rho} \right)' \rho_t + \left( \frac{P'(\rho)}{\rho} \right)' \nabla \rho \cdot u + \left( \frac{P'(\rho)}{\rho} \right) \rho \nabla \cdot u \right]
\]

\[
+ \frac{\lambda^2}{2} \int_{\mathbb{T}^N} \left[ \left( \frac{P'(\rho)}{\rho} - \left( \frac{P'(\rho)}{\rho} \right)' \right) \rho \nabla \cdot u \right] |D^\alpha (\rho - 1)|^2
\]

\[
= \frac{\lambda^2}{2} \int_{\mathbb{T}^N} \left[ \frac{2P'(\rho) - P''(\rho)\rho}{\rho} \rho \nabla \cdot u \right] |D^\alpha (\rho - 1)|^2. \quad (4.13)
\]

Now we give the dispersive estimates about \( J_k \) for \( k = 1, 2, \ldots, 8 \), when \( D^\alpha = \nabla^\alpha \) with \( |\alpha| \leq s, s \geq 3 \).

\[
|J_1| \leq C\lambda^{-1} \|
\lambda \nabla \cdot u \|_{\infty} \|
\lambda \nabla^\alpha (\rho - 1) \|^{2} \leq C\lambda^{-1} \|
\lambda \nabla^\alpha (\rho - 1) \|^{2}, \quad (4.14)
\]

where we have used Remark 3.2.

\[
|J_2| \leq C\lambda \|
\nabla \rho \|_{L^4} \|
\nabla^\alpha u \|_{L^4} \|
\lambda \nabla^\alpha (\rho - 1) \|
\]

\[
\leq C\lambda \|
\nabla \rho \|_{L^4} \|
\nabla^\alpha u \| + \|
\nabla \nabla^\alpha u \| \|
\lambda \nabla^\alpha (\rho - 1) \|
\]

\[
\leq C\lambda^{-1} \|
\lambda^2 \nabla \rho \|_{s-2} \|
\nabla^\alpha u \| + \|
\nabla \nabla^\alpha u \| \|
\lambda \nabla^\alpha (\rho - 1) \|
\]

\[
\leq \frac{\mu}{8} \|
\nabla \nabla^\alpha u \|^2 + C(\lambda^{-1} + \lambda^{-2}) (\|
\nabla^\alpha u \|^2 + \|
\lambda \nabla^\alpha (\rho - 1) \|^2), \quad (4.15)
\]

where we have used the Sobolev embedding \( H^1(\mathbb{T}^N) \hookrightarrow L^4(\mathbb{T}^N) \) and Remark 3.2. Then we use Lemma 3.1 to give

\[
|J_3| \leq C\lambda \|
\lambda \nabla^\alpha (\rho - 1) \| \|
\nabla^\alpha (u \cdot \nabla \rho) - u \cdot \nabla \nabla^\alpha \rho \| + \|
\nabla^\alpha (\rho \nabla \cdot u) - \rho \nabla \cdot \nabla^\alpha u \|
\]

\[
\leq C\lambda \|
\lambda \nabla^\alpha (\rho - 1) \| \|
\nabla u \|_{\infty} \|
\nabla \rho \|_{s-1} + \|
\nabla \rho \|_{\infty} \|
\lambda^{s-1} \nabla \cdot u \|
\]

\[
+ \|
\nabla \cdot u \|_{\infty} \|
\Lambda^\sigma \rho \|
\]

\[
\leq C\lambda^{-1} \|
\lambda \nabla^\alpha (\rho - 1) \| \|
\nabla u \|_{s-1} \|
\lambda^2 \nabla \rho \|_{s-1}
\]

\[
\leq C\lambda^{-1} \|
\lambda \nabla^\alpha (\rho - 1) \|^2 + C\lambda^{-1} \|
\lambda^2 \nabla \rho \|_{s-1}^2, \quad (4.16)
\]

\[
|J_5| \leq C \|
\nabla^\alpha u \| \|
\nabla^\alpha (u \cdot \nabla u) - u \cdot \nabla \nabla^\alpha u \| \leq C \|
\nabla^\alpha u \| \|
\nabla u \|_{\infty} \|
\Lambda^\sigma \nabla \rho \|
\]
\[ |J_6| \leq C \lambda^2 \|\nabla u\| \|\nabla^s \left( \frac{P'(\rho)}{\rho} \nabla \rho - \frac{P''(\rho)}{\rho} \nabla \nabla^s \rho \right) \| \]
\[ \leq C \lambda^2 \|\nabla u\| \left[ \|\nabla \left( \frac{P'(\rho)}{\rho} \right) \|_{\infty} \|\nabla^s \rho\| + \|\nabla \rho\|_{\infty} \|\nabla^s \left( \frac{P'(\rho)}{\rho} \right) \| \right] \]
\[ \leq C \lambda^2 \|\nabla u\| \|\nabla \rho\|_2 \|\nabla^s \rho\| \]
\[ \leq C \lambda^{-1} \|\nabla u\| \|\lambda \nabla \rho\|_{s-1} \|\lambda \nabla^s \rho\| \]
\[ \leq C \lambda^{-1} \|\nabla u\|^2 + C \lambda^{-1} \|\lambda \nabla \rho\|_{s-1}^2, \]  
(4.18)

\[ |J_7| \leq C \|\nabla u\| \left[ \|\nabla^s \left( \frac{\mu}{\rho} \Delta u \right) \| + \|\nabla^s \left( \frac{\kappa + \mu}{\rho} \nabla (\nabla \cdot u) \right) \| \right] \]
\[ \leq C \|\nabla u\| \left[ \|\nabla \left( \frac{\mu}{\rho} \right) \|_{\infty} \|\nabla^s-1 \Delta u\| + \|\Delta u\|_{\infty} \|\nabla^s \left( \frac{\mu}{\rho} \right) \| \right] \]
\[ + \|\nabla \left( \frac{\kappa + \mu}{\rho} \right) \|_{\infty} \|\nabla^s (\nabla \cdot u)\| + \|\nabla (\nabla \cdot u)\|_{\infty} \|\nabla^s \left( \frac{\kappa + \mu}{\rho} \right) \| \]
\[ \leq C \lambda^{-1} \|\nabla u\|^2 \|\lambda \nabla \rho\|_{s-1} \|\lambda \nabla^s \rho\| \]
\[ \leq C \lambda^{-1} (\|\nabla u\|^2 + \|\nabla u\|_{s}^2), \]  
(4.19)

\[ |J_8| \leq C \|\nabla u\| \left[ \|\nabla^s \left( \frac{\nu}{\rho} (\Delta n \cdot \nabla n) \right) \| + \|\nabla^s \left( \frac{\nu + \kappa}{\rho} (\nabla \cdot \nabla n) \right) \| \right] \]
\[ \leq C \|\nabla u\| \left[ \|\nabla \left( \frac{\nu}{\rho} \right) \|_{\infty} \|\nabla^s-1 (\Delta n \cdot \nabla n)\| + \|\Delta n\|_{\infty} \|\nabla n\|_{\infty} \|\nabla^s \left( \frac{\nu}{\rho} \right) \| \right] \]
\[ \leq C \|\nabla u\| \left[ \|\nabla \rho\|_2 \|\Delta n\|_{\infty} \|\nabla^s n\| + \|\nabla n\|_{\infty} \|\nabla^s-1 \Delta n\| \right] \]
\[ + \|\nabla \rho\|_2 \|\nabla^s n\| \|\nabla \rho\|_{s-1} \|\nabla^s \rho\|_{s-1} \]
\[ \leq C \lambda^{-1} (\|u\|^2_{s} + \|\nabla^2 u\|_{s-1}^2). \]  
(4.20)

Particularly, we have to make some argument about the estimate of \( J_4 \). When \(|\alpha| = 0 \), the similar estimate shown in (3.27) can be obtained as
\[ |J_4| \leq C \|u\| \|\nabla n\|_{\infty} \|\Delta n\|_2 \leq C \|u\|^2 + C (\varepsilon_0 + \lambda^{-2} \delta_0^2) \|\nabla^2 u\|_{2}^2. \]  
(4.21)

On the other hand, when \( \nabla^\alpha = \nabla_1 \nabla_2, \ |\beta| \leq s - 1 \), we give the following estimate similar as (3.35)
\[ |J_4| = \left| \int_{\mathbb{T}^n} \nabla^\beta (\Delta n \cdot \nabla n) \cdot \Delta \nabla^\beta u \right| \leq \frac{\mu}{8} \|\nabla^2 \nabla^\beta u\|^2 + C \|\nabla^\beta (\Delta n \cdot \nabla n)\|^2 \]
\[ \leq \frac{\mu}{8} \|\nabla^2 \nabla^\beta u\|^2 + C \|\Delta n\|^2 \|\nabla^s n\|^2 + C \|\nabla n\|^2 \|\nabla^{s+1} n\|^2 \]
\[ \leq \frac{\mu}{8} \|\nabla^2 \nabla^\beta u\|^2 + C (\varepsilon_0 + \lambda^{-2} \delta_0^2) \|\nabla^2 n\|^2_{s-1}, \]  
(4.22)
In conclusion, we combine (4.14)–(4.22) to give

\[
\frac{1}{2} \frac{d}{dt} \sum_{|\alpha| \leq s} \int_{T^n} \left[ \lambda^2 \frac{P''(\rho)}{\rho} |\nabla^\alpha (\rho - 1)|^2 + \rho |\nabla^\alpha u|^2 \right] + \mu \sum_{|\alpha| \leq s} \int_{T^n} |\nabla \nabla^\alpha u|^2 + (\kappa + \mu) \sum_{|\alpha| \leq s} \int_{T^n} |\nabla \cdot \nabla^\alpha u|^2 \\
\leq \frac{\mu}{4} \|\nabla u\|_s^2 + C \left( \frac{\lambda}{\sqrt{s}} + \lambda^{-1} \right) \|\nabla u\|_s^2 + C \left( \lambda_0 + \lambda^{-2} \right) \|\nabla^2 u\|_{s-1}^2 \\
C \lambda^{-1} \|\lambda^2 \nabla \rho\|_{s-1}^2 + C \lambda^{-1} \left( \|\nabla u\|_s^2 + \|\lambda (\rho - 1)\|_{s-1}^2 \right). \tag{4.23}
\]

then we can use the Gronwall's inequality, Remark 3.1 and (4.11) to give (4.2), provided that \(\varepsilon_0\) and \(\lambda^{-1}\) are both small enough. Finally, \(|n| = 1\) in \(Q_T\) can be proved by repeating the procedure shown in (3.96)–(3.98).

5 The convergence rates about \(u^\lambda\) and \(n^\lambda\) when \(\lambda \to \infty\)

In this section, we will prove Theorem 2.3 by the modulated energy method with the help of uniform estimates (2.9).

**Proof of Theorem 2.3.** First of all, let's rewrite (1.38) as follows:

\[
\begin{cases}
\rho_t^\lambda + \text{div}(\rho^\lambda u^\lambda) = 0, \\
(\rho^\lambda u^\lambda)_t + \nabla \cdot (\rho^\lambda u^\lambda \otimes u^\lambda) + \lambda^2 \nabla P(\rho^\lambda) = \mu \Delta u^\lambda + (\kappa + \mu) \nabla \text{div} u^\lambda - \nu (\Delta n^\lambda \cdot \nabla n^\lambda), \tag{5.1} \\
n_t^\lambda + (u^\lambda \cdot \nabla) n^\lambda = \theta (\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda) \text{ in } T^n.
\end{cases}
\]

Multiplying (5.1) by \(u^\lambda\) and then integrating over \(T^n\), we have

\[
\frac{d}{dt} \int_{T^n} \left( \frac{1}{2} \rho^\lambda |u^\lambda|^2 + \lambda^2 Q(\rho^\lambda) \right) + \mu \int_{T^n} |\nabla u^\lambda|^2 + (\kappa + \mu) \int_{T^n} |\nabla \cdot u^\lambda|^2 \\
= -\nu \int_{T^n} (u^\lambda \cdot \nabla) n^\lambda \cdot \Delta n^\lambda, \tag{5.2}
\]

where \(Q(\rho) = \rho \int_1^\rho P'(z) \frac{dz}{z} \). Multiplying (5.1) by \(\nu (\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda)\), and then integrating over \(T^n\) and noting that \(|n^\lambda| = 1\), we have

\[
\frac{d}{dt} \int_{T^n} \frac{\nu}{2} |\nabla n^\lambda|^2 + \nu \theta \int_{T^n} |\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda|^2 = \nu \int_{T^n} (u^\lambda \cdot \nabla) n^\lambda \cdot \Delta n^\lambda. \tag{5.3}
\]

Combining (5.2) and (5.3), we obtain

\[
\frac{d}{dt} \int_{T^n} \left( \frac{1}{2} \rho^\lambda |u^\lambda|^2 + \frac{\nu}{2} |\nabla n^\lambda|^2 + \lambda^2 Q(\rho^\lambda) \right) + \mu \int_{T^n} |\nabla u^\lambda|^2 \\
+(\kappa + \mu) \int_{T^n} |\nabla \cdot u^\lambda|^2 + \nu \theta \int_{T^n} |\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda|^2 = 0. \tag{5.4}
\]
Let
\[
\Pi^\lambda(x, t) = \lambda^2 \left[ Q(\rho^\lambda) - P(1)(\rho^\lambda - 1) \right],
\]
then we have \(\int_{T^n} \Pi^\lambda(x, 0) \leq C \lambda^{-2}\) from the Taylor series and (2.8). Integrating (5.4) over \([0, t]\) and using the conservation of mass, we have the basic energy estimates as follows:
\[
\int_{T^n} \left( \frac{1}{2} \rho^\lambda |u^\lambda|^2 + \frac{\nu}{2} |\nabla n^\lambda|^2 + \Pi^\lambda(x, t) \right) + \mu \int_0^t \int_{T^n} |\nabla u^\lambda|^2 + (\kappa + \mu) \int_0^t \int_{T^n} |\nabla \cdot u^\lambda|^2 \\
+ \nu \theta \int_0^t \int_{T^n} |\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda|^2 = \int_{T^n} \left( \frac{1}{2} \rho_0^\lambda |u_0^\lambda|^2 + \frac{\nu}{2} |\nabla n_0^\lambda|^2 + \Pi^\lambda(x, 0) \right). \tag{5.6}
\]

Similarly, from (1.38), we get the basic energy law for the incompressible system as
\[
\int_{T^n} \left( \frac{1}{2} |u|^2 + \frac{\nu}{2} |\nabla n|^2 \right) + \mu \int_0^t \int_{T^n} |\nabla u|^2 + \nu \theta \int_0^t \int_{T^n} |\Delta n + |\nabla n|^2 n|^2 \\
= \int_{T^n} \left( \frac{1}{2} |u_0|^2 + \frac{\nu}{2} |\nabla n_0|^2 \right). \tag{5.7}
\]

Secondly, multiplying (5.1) by \(u\) and then integrating over \(T^n\), we have
\[
\int_{T^n} \rho^\lambda u^\lambda u + \int_0^t \int_{T^n} \rho^\lambda u^\lambda \left[ (u \cdot \nabla)u + \nabla p - \mu \Delta u + \nu \nabla \cdot (\nabla u \otimes \nabla n) \right] \\
- \int_0^t \int_{T^n} (\rho^\lambda u \otimes u^\lambda) \cdot \nabla u + \mu \int_0^t \int_{T^n} \nabla u^\lambda \cdot \nabla u + \nu \int_0^t \int_{T^n} (u \cdot \nabla) n^\lambda \cdot \Delta n^\lambda \\
= \int_{T^n} \rho_0^\lambda u_0^\lambda u_0. \tag{5.8}
\]

Then multiplying (5.1) by \(\nu n\), we obtain
\[
\nu \int_{T^n} n^\lambda \cdot n - \nu \int_{T^n} n_0^\lambda \cdot n_0 + \nu \int_0^t \int_{T^n} n^\lambda \cdot [(u \cdot \nabla)n - \theta (\Delta n + |\nabla n|^2 n)] \\
+ \nu \int_0^t \int_{T^n} (u^\lambda \cdot \nabla)n^\lambda \cdot n = \nu \theta \int_0^t \int_{T^n} (\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda) \cdot n. \tag{5.9}
\]

Similarly, multiplying (5.1) by \(\nu \Delta n\), we have
\[
- \nu \int_{T^n} \nabla n^\lambda \cdot \nabla + \nu \int_0^t \int_{T^n} \nabla n^\lambda \cdot \left\{- \nabla [(u \cdot \nabla)n] + \theta \left[ \nabla \Delta n + \nabla \left( |\nabla n|^2 n \right) \right]\right\} \\
+ \nu \int_0^t \int_{T^n} (u^\lambda \cdot \nabla)n^\lambda \cdot \Delta n - \nu \theta \int_0^t \int_{T^n} (\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda) \cdot \Delta n \\
= - \nu \int_{T^n} \nabla n_0^\lambda \cdot \nabla n_0. \tag{5.10}
\]

In conclusion, combining (5.6)–(5.10), and noting the fact that \(|n^\lambda| = |n| = 1\), we have
\[
\int_{T^n} \left( \frac{1}{2} \sqrt{\rho} |u^\lambda - u|^2 + \frac{\nu}{2} |n^\lambda - n|^2 + \frac{\nu}{2} |\nabla n^\lambda - \nabla n|^2 + \Pi^\lambda(x, t) \right) + \mu \int_0^t \int_{T^n} |\nabla u^\lambda - \nabla u|^2 \\
+ (\kappa + \mu) \int_0^t \int_{T^n} |\nabla \cdot u^\lambda|^2 + \nu \theta \int_0^t \int_{T^n} \left( |\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda| - (\Delta n + |\nabla n|^2 n)^2 \right)
\]
\[
\begin{align*}
R^\Lambda(t) &= \int_{T^n} \left( \frac{1}{2} \left| \sqrt{\rho_0^\Lambda} u_0^\Lambda - u_0 \right|^2 + \frac{\nu}{2} |a_0^\Lambda - n_0|^2 + \frac{\nu}{2} |\nabla a_0^\Lambda - \nabla n_0|^2 + \Pi^\Lambda(x, 0) \right) \\
&- \mu \int_0^t \int_{T^n} \nabla u^\Lambda \cdot \nabla u + \nu \int_0^t \int_{T^n} \sqrt{\rho^\Lambda} (\sqrt{\rho^\Lambda} - 1) u^\Lambda \cdot u - \int_{T^n} \sqrt{\rho_0^\Lambda} \left( \sqrt{\rho_0^\Lambda} - 1 \right) u_0^\Lambda \cdot u_0 \\
&+ \int_0^t \int_{T^n} \rho^\Lambda u^\Lambda \cdot [(u \cdot \nabla) u + \nabla p - \mu \Delta u + \nu \nabla \cdot (\nabla n \otimes \nabla n)] \\
&- \int_0^t \int_{T^n} (\rho^\Lambda u^\Lambda \otimes u^\Lambda) \cdot \nabla u - \nu \int_0^t \int_{T^n} (u^\Lambda \cdot \nabla) n^\Lambda \cdot n + \nu \int_0^t \int_{T^n} (u \cdot \nabla) n^\Lambda \cdot n^\Lambda \\
&- \nu \theta \int_0^t \int_{T^n} (\Delta n + |\nabla n|^2 n^\Lambda) \cdot n^\Lambda - \nu \theta \int_0^t \int_{T^n} (\Delta n^\Lambda + |\nabla n^\Lambda|^2 n^\Lambda) \cdot n^\Lambda \\
&+ \nu \int_0^t \int_{T^n} (u \cdot \nabla) n^\Lambda \cdot \Delta n^\Lambda - \nu \theta \int_0^t \int_{T^n} (u \cdot \nabla) n^\Lambda \cdot \Delta n^\Lambda \\
&- \nu \theta \int_0^t \int_{T^n} (\Delta n + |\nabla n|^2 n^\Lambda) \cdot |\nabla n|^2 n^\Lambda - \nu \theta \int_0^t \int_{T^n} (\Delta n^\Lambda + |\nabla n^\Lambda|^2 n^\Lambda) \cdot |\nabla n|^2 n^\Lambda \\
&= \int_{T^n} \left( \frac{1}{2} \left| \sqrt{\rho_0^\Lambda} u_0^\Lambda - u_0 \right|^2 + \frac{\nu}{2} |a_0^\Lambda - n_0|^2 + \frac{\nu}{2} |\nabla a_0^\Lambda - \nabla n_0|^2 + \Pi^\Lambda(x, 0) \right) + \sum_{1 \leq i \leq 8} R_i^\Lambda(t), (5.11)
\end{align*}
\]

where

\[
\begin{align*}
R_1^\Lambda(t) &= \int_{T^n} \sqrt{\rho^\Lambda} (\sqrt{\rho^\Lambda} - 1) u^\Lambda \cdot u - \int_{T^n} \sqrt{\rho_0^\Lambda} \left( \sqrt{\rho_0^\Lambda} - 1 \right) u_0^\Lambda \cdot u_0, \\
R_2^\Lambda(t) &= \int_0^t \int_{T^n} \rho^\Lambda u^\Lambda \cdot [(u \cdot \nabla) u] - \int_0^t \int_{T^n} (\rho^\Lambda u^\Lambda \otimes u^\Lambda) \cdot \nabla u, \\
R_3^\Lambda(t) &= \int_0^t \int_{T^n} \rho^\Lambda u^\Lambda \cdot \nabla p, \\
R_4^\Lambda(t) &= \int_0^t \int_{T^n} \nabla u^\Lambda \cdot \nabla u - \mu \int_0^t \int_{T^n} \rho^\Lambda u^\Lambda \cdot \Delta u, \\
R_5^\Lambda(t) &= \nu \int_0^t \int_{T^n} \rho^\Lambda u^\Lambda \cdot (\nabla n \otimes \nabla n) + \nu \int_0^t \int_{T^n} (u \cdot \nabla) n^\Lambda \cdot \Delta n^\Lambda \\
&- \nu \theta \int_0^t \int_{T^n} (u^\Lambda \cdot \nabla) n^\Lambda \cdot \Delta n^\Lambda - \nu \theta \int_0^t \int_{T^n} (u \cdot \nabla) n \cdot \Delta n^\Lambda, \\
R_6^\Lambda(t) &= \nu \int_0^t \int_{T^n} (u \cdot \nabla) n^\Lambda \cdot n + \nu \int_0^t \int_{T^n} (u \cdot \nabla) n \cdot n^\Lambda, \\
R_7^\Lambda(t) &= -\nu \theta \int_0^t \int_{T^n} (\Delta n + |\nabla n|^2 n^\Lambda) \cdot |\nabla n|^2 n^\Lambda - \nu \theta \int_0^t \int_{T^n} (\Delta n^\Lambda + |\nabla n^\Lambda|^2 n^\Lambda) \cdot |\nabla n|^2 n^\Lambda, \\
R_8^\Lambda(t) &= -\nu \theta \int_0^t \int_{T^n} (\Delta n + |\nabla n|^2 n^\Lambda) \cdot |\nabla n|^2 n^\Lambda - \nu \theta \int_0^t \int_{T^n} (\Delta n^\Lambda + |\nabla n^\Lambda|^2 n^\Lambda) \cdot |\nabla n|^2 n^\Lambda.
\end{align*}
\]

Now we estimate each term above. Firstly, we have from [229] that

\[
|R_1^\Lambda(t)| \leq \|u\|_\infty \left( \int_{T^n} \rho^\Lambda |u|^2 \right)^{\frac{1}{2}} \left( \int_{T^n} |\sqrt{\rho^\Lambda} - 1|^2 \right)^{\frac{1}{2}} + \|u_0\|_\infty \left( \int_{T^n} \rho^\Lambda |u_0|^2 \right)^{\frac{1}{2}} \left( \int_{T^n} |\sqrt{\rho_0^\Lambda} - 1|^2 \right)^{\frac{1}{2}}
\]
where we have used the incompressibility

Now we turn to estimate the third and forth term and get

\[ R^2 = \left( \int_{T^n} \rho^\lambda |u^\lambda|^2 \right)^\frac{1}{2} \left( \int_{T^n} \left| \frac{\rho^\lambda - 1}{\sqrt{\rho^\lambda} + 1} \right|^2 \right)^\frac{1}{2} + \|u_0\|_2 \left( \int_{T^n} \rho^\lambda |u^\lambda_0|^2 \right)^\frac{1}{2} \left( \int_{T^n} \left| \frac{\rho^\lambda - 1}{\sqrt{\rho^\lambda} + 1} \right|^2 \right)^\frac{1}{2} \]

\[ \leq C \left\| \frac{1}{\sqrt{\rho^\lambda}} + 1 \right\|_\infty \left( \int_{T^n} |\rho^\lambda - 1|^2 \right)^{\frac{1}{2}} + C \left\| \frac{1}{\sqrt{\rho^\lambda}} + 1 \right\|_\infty \left( \int_{T^n} |\rho^\lambda - 1|^2 \right)^{\frac{1}{2}} \]

\[ \leq C \lambda^{-1}. \quad (5.12) \]

Then we estimate the second term \( R_2^\lambda(t) \) and get

\[ |R_2^\lambda(t)| = \left| - \int_0^t \int_{T^n} \left[ (\sqrt{\rho^\lambda} u^\lambda - u) \otimes (\sqrt{\rho^\lambda} u^\lambda - u) \right] \cdot \nabla u + \int_0^t \int_{T^n} (\rho^\lambda - \sqrt{\rho^\lambda}) u^\lambda \cdot [(u \cdot \nabla) u] \right| \]

\[ - \int_0^t \int_{T^n} (\sqrt{\rho^\lambda} u^\lambda - u) \cdot \nabla \left( \frac{|u|^2}{2} \right) \]

\[ \leq C \lambda^{-1} + C \int_0^t \int_{T^n} \sqrt{\rho^\lambda} u^\lambda - u^2 + \int_0^t \int_{T^n} (\sqrt{\rho^\lambda} u^\lambda - u) \cdot \nabla \left( \frac{|u|^2}{2} \right) \]

\[ \leq C \lambda^{-1} + C \int_0^t \int_{T^n} \sqrt{\rho^\lambda} u^\lambda - u^2, \quad (5.13) \]

where we have used the incompressibility \( \nabla \cdot u = 0 \) to give the following fact:

\[ \left| \int_0^t \int_{T^n} (\sqrt{\rho^\lambda} u^\lambda - u) \cdot \nabla \left( \frac{|u|^2}{2} \right) \right| \]

\[ = \left| - \int_0^t \int_{T^n} \sqrt{\rho^\lambda} (\sqrt{\rho^\lambda} - 1) u^\lambda \cdot \nabla \left( \frac{|u|^2}{2} \right) + \int_0^t \int_{T^n} \rho^\lambda u^\lambda \cdot \nabla \left( \frac{|u|^2}{2} \right) \right| \]

\[ = \left| - \int_0^t \int_{T^n} \sqrt{\rho^\lambda} (\sqrt{\rho^\lambda} - 1) u^\lambda \cdot \nabla \left( \frac{|u|^2}{2} \right) + \int_0^t \int_{T^n} \rho^\lambda \cdot \frac{|u|^2}{2} \right| \]

\[ \leq C \left( \int_0^t \int_{T^n} |\rho^\lambda - 1|^2 \right)^{\frac{1}{2}} + C \left( \int_0^t \int_{T^n} |\rho^\lambda|^2 \right)^{\frac{1}{2}} \]

\[ \leq C \lambda^{-1}. \quad (5.14) \]

Next we estimate the third and forth term and get

\[ |R_3^\lambda(t)| = \left| \int_0^t \int_{T^n} \rho^\lambda p \right| \]

\[ \leq C \left( \int_0^t \int_{T^n} |\rho^\lambda|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{T^n} p^2 \right)^{\frac{1}{2}} \leq C \lambda^{-1}, \quad (5.15) \]

\[ |R_4^\lambda(t)| \leq \mu \int_0^t \int_{T^n} |(1 - \rho^\lambda) u^\lambda \cdot \Delta u| \]

\[ \leq C \left( \int_0^t \int_{T^n} |1 - \rho^\lambda|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{T^n} |u^\lambda|^2 \Delta u^2 \right)^{\frac{1}{2}} \leq C \lambda^{-1}. \quad (5.16) \]

Now we turn to estimate \( R_5^\lambda(t) \) as

\[ R_5^\lambda(t) = \nu \int_0^t \int_{T^n} \rho^\lambda (u^\lambda \cdot \nabla) n \cdot \Delta n + \nu \int_0^t \int_{T^n} \rho^\lambda u^\lambda \cdot \nabla \left( \frac{|\nabla n|^2}{2} \right) + \nu \int_0^t \int_{T^n} (u \cdot \nabla)n^\lambda \cdot \Delta n^\lambda \]
\[-\nu \int_0^t \int_{\Omega_T} (u^3 \cdot \nabla)n^3 \cdot \Delta n - \nu \int_0^t \int_{\Omega_T} (u \cdot \nabla)n \cdot \Delta n^3 = \nu \int_0^t \int_{\Omega_T} (\rho^3 - 1)(u^3 \cdot \nabla)n \cdot \Delta n + \nu \int_0^t \int_{\Omega_T} (u^2 \cdot \nabla)n \cdot \Delta n^3 - \nu \int_0^t \int_{\Omega_T} n^3 \cdot \nabla^n \frac{\|n\|^2}{2} \]

\[= \nu \int_0^t \int_{\Omega_T} (\rho^3 - 1)(u^3 \cdot \nabla)n \cdot \Delta n + \nu \int_0^t \int_{\Omega_T} (\rho^3 - 1)\frac{\|n\|^2}{2} \]

\[-\nu \int_0^t \int_{\Omega_T} [u^3 - u \cdot \nabla](n^3 - n) \cdot \Delta n + \nu \int_0^t \int_{\Omega_T} (u \cdot \nabla)(n^3 - n)(\Delta n^3 - \Delta n) \]

\[= \nu \int_0^t \int_{\Omega_T} (\rho^3 - 1)(u^3 \cdot \nabla)n \cdot \Delta n + \nu \int_0^t \int_{\Omega_T} \rho^3 \frac{\|n\|^2}{2} \]

\[-\nu \int_0^t \int_{\Omega_T} [\sqrt{\rho^3} u^3 - u \cdot \nabla](n^3 - n) \cdot \Delta n - \int_0^t \int_{\Omega_T} [(1 - \sqrt{\rho^3})u^3 \cdot \nabla](n^3 - n) \cdot \Delta n \]

\[\quad - \nu \int_0^t \int_{\Omega_T} (\nabla u \cdot \nabla)(n^3 - n) \cdot (\nabla^n - \nabla n) \tag{5.17}, \]

where we have used the incompressibility \(\nabla \cdot u = 0\) in the last inequality. Then by the Cauchy inequality and the uniform estimates (2.49), we get

\[|R_6^3(t)| \leq C \left( \int_0^t \int_{\Omega_T} |\rho^3 - 1|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\Omega_T} |u^3|^2 \|\nabla^n\| |\Delta n|^2 \right)^\frac{1}{2} \]

\[+ C \left( \int_0^t \int_{\Omega_T} |\rho^3|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\Omega_T} \frac{\|n\|^4}{4} \right)^\frac{1}{2} \]

\[+ C \|\Delta n\|_\infty \left( \int_0^t \int_{\Omega_T} |\sqrt{\rho^3} u^3 - u|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\Omega_T} |\nabla^n - \nabla n|^2 \right)^\frac{1}{2} \]

\[+ \|u^3\|_\infty \|\Delta n\|_\infty \left( \int_0^t \int_{\Omega_T} \|\nabla^n - \nabla n\|^2 \right)^\frac{1}{2} \left( \int_0^t \int_{\Omega_T} |1 - \sqrt{\rho^3}|^2 \right)^\frac{1}{2} \]

\[+ C \|\nabla u\|_\infty \int_0^t \int_{\Omega_T} |\nabla^n - \nabla n|^2 \]

\[\leq C(\lambda^{-1} + \lambda^{-2}) + C \int_0^t \int_{\Omega_T} |\sqrt{\rho^3} u^3 - u|^2 + C \left( \|\nabla^n\|_3^2 + 1 \right) \int_0^t \int_{\Omega_T} |\nabla^n - \nabla n|^2. \tag{5.18} \]

Noting that \(|n| = 1\) and \(|n^3| = 1\) in \(\Omega_T\), we have the following two facts:

\[\nabla(|n^3|^2) = \nabla(|n|^2) = 0, \tag{5.19} \]

and

\[(\Delta n + |\nabla n|^2 n) \cdot n = (\Delta n^3 + |\nabla n|^2 n^3) \cdot n^3 = 0. \tag{5.20} \]

With the help of (5.19), we have

\[R_6^3(t) = \nu \int_0^t \int_{\Omega_T} (u^3 \cdot \nabla)n^3 \cdot (n - n^3) + \nu \int_0^t \int_{\Omega_T} (u \cdot \nabla)n \cdot (n^3 - n) \]
By the Cauchy inequality, we have

\[ |R_0^\lambda(t)| = \nu \int_0^t \int_{T^n} [(u^\lambda - u) \cdot \nabla] n^\lambda \cdot (n - n^\lambda) + \nu \int_0^t \int_{T^n} (u \cdot \nabla)(n - n^\lambda) \cdot (n^\lambda - n), \]

(5.21)

then following by the Cauchy inequality, we obtain

\[
|R_0^\lambda(t)| \leq C \|\nabla n^\lambda\|_\infty \left( \int_0^t \int_{T^n} |u^\lambda - u|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{T^n} |n^\lambda - n|^2 \right)^{\frac{1}{2}} \\
+ C \|u\|_\infty \left( \int_0^t \int_{T^n} |n^\lambda - n|^2 \right)^{\frac{1}{2}} \left( \int_0^t \int_{T^n} |\nabla n^\lambda - \nabla n|^2 \right)^{\frac{1}{2}} \\
\leq C \int_0^t \int_{T^n} |u^\lambda - u|^2 + C \int_0^t \int_{T^n} |n^\lambda - n|^2 + C \int_0^t \int_{T^n} |\nabla n^\lambda - \nabla n|^2. \tag{5.22}
\]

According to (5.20), we have

\[
R_7^\lambda(t) = -\nu \theta \int_0^t \int_{T^n} (\Delta n + |\nabla n|^2 n) \cdot (n^\lambda - n) + \nu \theta \int_0^t \int_{T^n} (\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda) \cdot (n - n^\lambda) \\
= -\nu \theta \int_0^t \int_{T^n} [(\Delta n + |\nabla n|^2 n) - (\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda)] \cdot (n^\lambda - n), \tag{5.23}
\]

then we use the Cauchy inequality to get

\[
|R_7^\lambda(t)| \leq \frac{\nu \theta}{4} \int_0^t \int_{T^n} |(\Delta n + |\nabla n|^2 n) - (\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda)|^2 + C \int_0^t \int_{T^n} |n - n^\lambda|^2. \tag{5.24}
\]

Similarly, with the help of (5.20), we derive that

\[
R_8^\lambda(t) = -\nu \theta \int_0^t \int_{T^n} (\Delta n + |\nabla n|^2 n) \cdot |\nabla n^\lambda|^2 n^\lambda - \nu \theta \int_0^t \int_{T^n} (\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda) \cdot |\nabla n|^2 n \\
= -\nu \theta \int_0^t \int_{T^n} (\Delta n + |\nabla n|^2 n) \cdot |\nabla n^\lambda|^2 (n^\lambda - n) \\
- \nu \theta \int_0^t \int_{T^n} (\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda) \cdot |\nabla n|^2 (n - n^\lambda) \\
= \nu \theta \int_0^t \int_{T^n} [(\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda) - (\Delta n + |\nabla n|^2 n)] \cdot |\nabla n|^2 (n^\lambda - n) \\
- \nu \theta \int_0^t \int_{T^n} [(|\nabla n^\lambda - \nabla n) \cdot (\nabla n^\lambda + \nabla n)](\Delta n + |\nabla n|^2 n) \cdot (n^\lambda - n). \tag{5.25}
\]

By the Cauchy inequality, we have

\[
|R_8^\lambda(t)| \leq \frac{\nu \theta}{4} \int_0^t \int_{T^n} |(\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda) - (\Delta n + |\nabla n|^2 n)|^2 + C \|\nabla n\|_\infty^4 \int_0^t \int_{T^n} |n^\lambda - n|^2 \\
+ C \int_0^t \int_{T^n} |n^\lambda - n|^2 \\
\leq \frac{\nu \theta}{4} \int_0^t \int_{T^n} |(\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda) - (\Delta n + |\nabla n|^2 n)|^2
\]

(5.26)
\[ +C \| \nabla n \|^2_0 \int_0^t \int_{T_n^t} |\nabla n - \nabla n^\lambda|^2 + C \int_0^t \int_{T_n^t} |n^\lambda - n|^2. \tag{5.26} \]

In conclusion, combining all the estimates about \( R^\lambda(t) \) for \( i = 1, 2, \ldots, 8 \), we have

\[
\int_{T_n^t} \left( \frac{1}{2} |\sqrt{\rho^\lambda} u^\lambda - u_0|^2 + \frac{1}{2} n^\lambda - n|^2 + \frac{1}{2} |\nabla n^\lambda - \nabla n^\lambda|^2 + \Pi^\lambda(x, t) \right) \\
+ \mu \int_0^t \int_{T_n^t} |\nabla u^\lambda - \nabla u|^2 + (\kappa + \mu) \int_0^t \int_{T_n^t} |\nabla \cdot u^\lambda|^2 \\
+ \frac{1}{2} \nu \int_0^t \int_{T_n^t} |(\Delta n^\lambda + |\nabla n^\lambda|^2 n^\lambda) - (\Delta n + |\nabla n|^2 n)|^2
\leq \int_{T_n^t} \left( \frac{1}{2} |\sqrt{\rho^0_0 u_0^\lambda - u_0|^2} + \frac{1}{2} n^\lambda_0 - n_0|^2 + \Pi^\lambda(x, 0) \right) + C \lambda^{-1} \\
+ C \left( \| \nabla n \|^2_0 + 1 \right) \int_0^t \int_{T_n^t} \left( |\sqrt{\rho^\lambda} u^\lambda - u|^2 + |n^\lambda - n|^2 + |\nabla n^\lambda - \nabla n|^2 \right). \tag{5.27} \]

Noting the following facts that

\[
\int_{T_n^t} |\sqrt{\rho^0_0 u_0^\lambda - u_0|^2} = \int_{T_n^t} |\sqrt{\rho^0_0 u_0^\lambda - u_0|^2} + \int_{T_n^t} |u_0^\lambda - u_0|^2 \leq C(\lambda^{-4} + \lambda^{-2}), \tag{5.28} \]

and

\[
\int_{T_n^t} |u^\lambda - u|^2 \leq \int_{T_n^t} |\sqrt{\rho^\lambda} u^\lambda - u_0|^2 + \int_{T_n^t} \rho^\lambda u^\lambda - u|^2 \\
\leq C \lambda^{-2} + \int_{T_n^t} |\sqrt{\rho^\lambda} u^\lambda - u|^2, \tag{5.29} \]

By (5.27)–(5.29), the Gronwall's inequality and the fact that \( \int_0^t \| \nabla n \|^2_0 \leq C \), we have

\[
\| u^\lambda - u\|^2 + \nu \| \nabla n^\lambda - \nabla n\|^2 \leq C \lambda^{-1}, \tag{5.30} \]

and then integrating (5.27) over \([0, t]\) to get

\[
\int_0^t (\mu \| \nabla u^\lambda - \nabla u\|^2 + \nu \theta) \Delta(n^\lambda + |\nabla n^\lambda|^2 n^\lambda) - (\Delta n + |\nabla n|^2 n)|^2 \leq C \lambda^{-1}. \tag{5.31} \]

Finally, we have from (1.38)_3 and (1.40)_3 that

\[
(n^\lambda - n)_t - \theta \Delta(n^\lambda - n) = -[(u^\lambda - u) \cdot \nabla] n^\lambda - (u \cdot \nabla)(n^\lambda - n) \\
+ \theta [(\nabla n^\lambda - \nabla n) : (\nabla n^\lambda + \nabla n)] \cdot n^\lambda + \theta |\nabla n|^2 (n^\lambda - n), \tag{5.32} \]

then we have by the parabolic theory that

\[
\| n^\lambda - n\|^2_0 \leq C \| \nabla n^\lambda \|^2_\infty \| u^\lambda - u\|^2 + C \| \nabla n^\lambda \|^2_\infty \| \nabla n^\lambda - \nabla n\|^2 \\
+ C \left( \| \nabla n^\lambda \|^2_\infty + \| \nabla n \|^2_\infty \right) \| \nabla n^\lambda - \nabla n\|^2 + C \| \nabla n \|^4_\infty \| n^\lambda - n\|^2 \\
\leq C \lambda^{-1}. \tag{5.33} \]
Applying $\nabla$ to (5.32), we have

\[
\nabla(n^\lambda - n)_t - \theta \Delta \nabla(n^\lambda - n)
\]
\[
= -[\nabla(u^\lambda - u) \cdot \nabla]n^\lambda - [(u^\lambda - u) \cdot \nabla] \nabla n^\lambda - (\nabla u \cdot \nabla)(n^\lambda - n) - (u \cdot \nabla) \nabla(n^\lambda - n)
\]
\[
+ \theta[\nabla^2 n^\lambda - \nabla^2 n] \cdot n^\lambda + \theta[(\nabla n^\lambda - \nabla n) \cdot (\nabla^2 n^\lambda + \nabla^2 n)] \cdot n^\lambda
\]
\[
+ \theta[(\nabla n^\lambda - \nabla n) \cdot (\nabla n^\lambda + \nabla n)] \cdot n^\lambda + 2\theta(\nabla n \cdot \nabla^2 n) \cdot (n^\lambda - n) + \theta |\nabla n|^2 (\nabla n^\lambda - \nabla n),
\]
(5.34)

and then (5.30), (5.31) and (5.33) imply

\[
\int_0^t \|\nabla n^\lambda - \nabla n\|_2^2 \leq C \lambda^{-1}.
\]
(5.35)

Now we complete the proof of Theorem 2.3. \qed

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