On the determination of curvature and dynamical dark energy

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Abstract. Constraining simultaneously the dark energy (DE) equation of state and the curvature of the universe is difficult due to strong degeneracies. To circumvent this problem when analyzing data it is usual to assume flatness to constrain the DE or, conversely, to assume that the DE is a cosmological constant to constrain the curvature. In this paper, we quantify the impact of such assumptions with an eye to future large surveys. We simulate future data for type Ia supernovae, the cosmic microwave background and baryon acoustic oscillations for a large range of fiducial cosmologies allowing a small spatial curvature. We take into account a possible time evolution of DE through a parameterized equation of state: $w(a) = w_0 + (1-a)w_a$. We then fit the simulated data with a wrong assumption on the curvature or on the DE parameters.

For a fiducial $Λ$CDM cosmology, if flatness is incorrectly assumed in the fit and if the true curvature is within the ranges $0.01 < Ω_k < 0.03$ and $-0.07 < Ω_k < -0.01$, one will be led to conclude erroneously that an evolving DE is present, even with high statistics.

On the other hand, models with curvature and dynamical DE can be confused with a flat $Λ$CDM model when the fit ignores a possible DE evolution. We find that, in the future, with high statistics, such risks of confusion should be limited, but they are still possible, and biases in the cosmological parameters might be important.

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1. Introduction

Since the discovery of the acceleration of the universe [1, 2] many theoretical interpretations have been developed in an effort to find an explanation. These models introduced in general a new component called dark energy (DE) with an unknown nature [3, 4]. The distinction of the various interpretations is critical for future cosmology and for its connections with fundamental physics.

Many studies concentrate on the equation of state (EoS) \( w \), defined as the pressure to density ratio of this new component. This parameter is equal to \(-1\) for a classical cosmological constant but can be different and/or can vary with cosmic time depending on the various DE models (for a review see e.g. [4]). On the experimental side, the determination of the nature of the dark energy by constraining \( w(z) \) is greatly debated and one should use a large variety of probes and methods [5].

The WMAP data combined with the recent SDSS data [6]–[9] are currently in very good agreement with an adiabatic \( \Lambda \)CDM model. However, most of the studies in the literature assume a spatial flatness to constrain the dynamical dark energy parameters or, conversely, assume a cosmological constant to derive constraints on the curvature. At
the current level of statistical precision, this can be justified, but various authors have emphasized the dangers that one can encounter in using these overly simple strategies [10]–[12], and the impact of possible biases, when the statistical errors start to be small, should be investigated.

In general, extracting simultaneously \( w(z) \) and the curvature (and also the matter density) is not possible since these parameters are perfectly degenerate. This is often referred to as the so-called ‘geometrical degeneracy’ [13]–[16]. As almost all inflationary scenarios predict spatial flatness \( (\Omega_k < 10^{-10}) \), this property is often taken for granted in the literature. However the precision is not so compelling. Experimentally, the error on the curvature parameter \( \Omega_k \), as given by the WMAP cosmic microwave background (CMB) data, is \( \pm 0.006(68\%) \); \( \pm 0.013(95\%) \) when adding a supernovae sample, the baryon acoustic oscillations (BAO) from SDSS and assuming a cosmological constant [9]. Assuming a parameterized \( w(a) = w_0 + (1 - a)w_a \) equation of state (see below), the precision is only of \( \pm 0.018(68\%) \); \( \pm 0.04(95\%) \) [9, 17]. Relaxing the combination with SDSS and the supernovae (SNIa: type Ia supernovae) leads even to a precision of \( \pm 0.10 \) [9]. Then, the precision on the cosmic curvature is very dependent on the assumption concerning the dark energy, and is strongly limited today by the number of possible free parameters.

On the other hand, as regards dark energy evaluation, even the more recent standard analyses as the WMAP ones [8, 9] or SNLS [18] give results on the dark energy parameters using the flatness hypothesis and/or no dark energy evolution. These analyses use, as is now common, compressed parameters: \( A \) from BAO [6] and/or of the shift parameter \( R \), to include the WMAP CMB data [8, 9, 17, 19].

Few recent studies [20]–[22] have extracted simultaneously a free curvature and a dynamical dark energy from the present data. Such combined analyses always need some external priors or assumptions (prior on \( h \), the present value of the Hubble constant in units of 100 km s\(^{-1}\) Mpc\(^{-1}\), assuming massless neutrinos, adiabatic perturbations only, ...). It has been known for some time [23] that using incorrect priors can lead to an artificial convergence toward a particular model, e.g. the \( \Lambda \)CDM model (for the consequence of using a strong prior on \( \Omega_M \), see e.g. [24, 25], and [26] for a recent contribution). The situation could be worse when the number of relevant parameters is increased, e.g. when relaxing a constraint on the curvature and/or the nature of dark energy.

In this paper we adopt the same kind of attitude as in [24, 27], focusing on the question of the interplay between the curvature and some DE parameters. We extend the work of previous authors (e.g. [11, 12]), by addressing these questions more quantitatively and taking into account more realistic observational conditions.

To do parameter bias estimation with free curvature, which means comparing the fitted parameter values to fiducial ones, a combined analysis is mandatory. We choose to stay in the framework of three combined probes: type Ia supernovae (SNIa), cosmic microwave background data (CMB) and baryon acoustic oscillations data (BAO). This combination (which is independent of \( H_0 \)) has enough statistical power for us to start to relax the flatness hypothesis. Furthermore, we take advantage of the compressed parameters used in the literature as they contain most of the cosmological information needed to reduce dark energy parameter errors to the same level as from a complete analysis [17]. In a simulation study, where all other parameters are fixed, this will not introduce additional biases, with the main advantage of avoiding a complete CMB and
BAO data analysis. Therefore, we use the $R$ shift parameter [13, 15, 28, 17] for the CMB, and the so-called $A$ parameter for the BAO [6]. This will drastically reduce the computing time, which is useful since we want to simulate a large range of models, and it will not affect the generality of our analysis.

In section 2, we present the simulated sample of SNIa, the $R$ and $A$ parameters, and the statistical method used for minimization. We also introduce a detectability criterion, based on the goodness of the $\chi^2$ value of the fit.

We study in sections 3 and 4 two cases which illustrate the limit of some existing analyses. First, we assume flatness to constrain a dynamical dark energy in a case of a universe with a non-flat curvature and a cosmological constant. Second, we assume that dark energy is a cosmological constant to constrain the curvature in a model of dynamical dark energy. This case is even more complicated, since the number of fiducial parameters has increased as compared to the previous scenario; this is the ‘triple trouble’ situation mentioned in [11].

We conclude in section 5 that to get any strong conclusion on dark energy evolution one will need to fit simultaneously curvature and dynamical dark energy parameters in multi-probe analyses.

2. Cosmological probes; fitting method

2.1. The dark energy parameterization

To simulate and fit the evolving DE models, we need a parameterization of the EoS $w(z)$. We should choose for our study a parameterization that takes into account a possible evolution over the whole redshift range up to the CMB, with a minimum number of free parameters to avoid overly large errors. Indeed it has been shown that an at most two-parameter model can optimally be constrained by future data [29, 25]. We choose the Chevallier–Polarski–Linder (CPL) parameterization [30, 31], which has been extensively used in the literature, in particular by the Dark Energy Task Force (DETF) [5], as a phenomenological benchmark for comparing and contrasting the performances of different DE probes (see e.g. [32]):

$$w(z) = w_0 + w_a z / (1 + z) = w_0 + (1 - a)w_a, \quad (1)$$

$a$ being the scale factor. The free parameters $w_0$ and $w_a$ characterize the DE model. The cosmological constant corresponds to $w_0 = -1$ and $w_a = 0$.

Despite its simplicity, this parameterization, which is bounded for high redshifts ($w(z \to \infty) = w_0 + w_a$), fits the evolution predicted by a large class of quintessence models well (see [33], and also [34] for a recent discussion). Note that the CMB analysis introduces in general a high redshift constraint for the chosen dark energy parameterization [25], namely $w(z_{\text{CMB}}) < 0$ at last scattering. With our parameterization we should impose $w_0 + w_a < 0$ (we refer to this limit as the ‘CMB boundary condition’ in the following). Note however that some DE models can be characterized by an EoS such that $w(z_{\text{CMB}}) \geq 0$. This is the case for the early dark energy models characterized by a non-negligible DE density at last scattering [35] and also for the tracker quintessence models [36] which have null and positive EoS at early time, as shown explicitly in [37]. Therefore we can consider sometimes the part of the parameter space where the ‘CMB boundary condition’ is violated.
As regards our simulations, this particular choice of the parameterization has no strong impact on the result as soon as fiducial models and fitted parameters are using the same modeling. The use of another two-parameter parameterization will not change the main message of this work; we have checked that it can only slightly modify the size of the errors.

2.2. SNIa samples

In the standard Friedmann–Robertson–Walker metric, the apparent magnitude of astrophysical objects can be expressed as a function of the luminosity distance:

\[ m(z) = 5 \log_{10}(D_L) + M_B - 5 \log_{10}(H_0/c) + 25 = M_* + 5 \log_{10}(D_L) \]  

(2)

where \( M_B \) is the absolute magnitude of the SNIa, \( M_* \) may be considered as a normalization parameter and \( D_L(z) \equiv (H_0/c) d_L(z) \) is the \( H_0 \)-independent luminosity distance to an object at redshift \( z \). It is given by

\[ D_L(z) = \left( \frac{1+z}{\sqrt{|\Omega_k|}} \right) S\left( \sqrt{|\Omega_k|} \int_0^z \frac{1}{E(z')} \, dz' \right), \]

(3)

where \( S(x) = \sinh(x), x, \sin(x) \) for \( \Omega_k > 0, =0, <0 \) respectively, with

\[ \Omega_k = 1 - \Omega_M - \Omega_X. \]

(4)

\( \Omega_M \) and \( \Omega_X \) being respectively the reduced matter and DE densities. We have

\[ E(z)^2 = \left( \frac{H(z)}{H_0} \right)^2 = (1+z)^3 \Omega_M + \frac{\rho_X(z)}{\rho_X(0)} \Omega_X + (1+z)^2 \Omega_k, \]

(5)

where

\[ \frac{\rho_X(z)}{\rho_X(0)} = \exp \left[ 3 \int_0^z (1 + w(z')) \, d \ln (1 + z') \right]. \]

(6)

Note that we have neglected the radiation component \( \Omega_R \) and this has no impact on our results. With the parameterization given in equation (1), the last equation becomes

\[ \frac{\rho_X(z)}{\rho_X(0)} = (1+z)^{3(1+w_0+w_a)}e^{-3w_a z/(1+z)}. \]

(7)

The studies presented in this paper are performed with simulated supernovae samples with statistics equivalent to what we expect to have in the future. We concentrate on two sets of data which simulate the statistical power of the forthcoming and future data.

- We simulate forthcoming data from a ground survey such as the large SNLS survey at CFHT [38,18]. This survey started in 2003 and the estimation after five years of running is registering a sample of 700 identified SNIa in the redshift range \( 0.3 < z < 1 \).

- We simulate data from a future space mission like JDEM/SNAP [39], which plans to discover around 2000 identified SNIa at redshift \( 0.2 < z < 1.7 \), with very precise photometry and spectroscopy. The supernovae distribution is given in [40]. This corresponds to our ‘long term’ scenario.
The magnitude dispersion is assumed to be constant and independent of the redshift at the level of 0.15 for all supernovae after correction, for both samples. We have also neglected systematical errors in this study.

A set of very well calibrated SNIa at redshift \( z \leq 0.1 \) should be measured by the SN Factory collaboration [41]. This sample is needed to normalize the Hubble diagram and will be called in the following the ‘Nearby’ sample. A ‘SNAP’ (‘SNLS’) sample means in this paper a simulation of the statistics expected from the JDEM/SNAP like mission (‘SNLS’ survey) combined with the 300 (200) nearby SNIa expected to be measured at the time of SNAP (SNLS).

2.3. The CMB and BAO constraints

We need to add simulated CMB data to take advantage of the current and future constraints on curvature. We have simulated the CMB constraints by using the CMB shift parameter \( R \) [13, 15, 28, 17] which is the scaled distance to recombination:

\[
R = \frac{\sqrt{\Omega_M}}{\sqrt{|\Omega_k|}} S \left( \sqrt{|\Omega_k|} \int_0^{z_{\text{CMB}}} \frac{1}{E(z')} \, dz' \right),
\]

where \( z_{\text{CMB}} = 1089 \) is the redshift at the epoch of recombination. This parameter contains the full geometrical degeneracy inherent to the CMB [15]. It is measured to be \( R_{\text{WMAP}} = 1.71 \pm 0.019 \) from the five-year WMAP data [9].

Using \( R \) we lose some pieces of information which are potentially present in the CMB, but it contains enough statistical power to constrain the dark energy sector at a level comparable to that of a full CMB analysis. To evaluate the relevance of our results and of this assumption, we have made some full CMB calculations and compared the results when evaluating the dark energy parameters in some models. We found a correct agreement between the two methods. For a comparison between the use of the CMB \( R \) parameter and full CMB calculations, one can refer to [28, 17].

We consider BAO measurements to add an additional constraint on the matter density \( \Omega_M \). We use the so-called \( A \) parameter, described in [6], which encodes almost all information on the cosmological parameters from present BAO observations:

\[
A = \frac{\sqrt{\Omega_M}}{E(z_{\text{BAO}})^{1/3}} \left[ \frac{1}{z_{\text{BAO}} \sqrt{|\Omega_k|}} S \left( \sqrt{|\Omega_k|} \int_0^{z_{\text{BAO}}} \frac{1}{E(z')} \, dz' \right) \right]^{2/3},
\]

where \( z_{\text{BAO}} = 0.35 \). It is measured as \( A_{\text{SDSS}} = 0.469(n_s/0.98)^{-0.35} \pm 0.017 \), with \( n_s = 0.95 \) [6]. The definition of \( A \) is an approximation which incorporates a mixture of the transverse and radial information present in the data.

In practice we calculate the values of \( R \) and \( A \) for each simulated fiducial model. These values are used in the fits along with the errors expected for each observational scenario. For the mid-term scenario we take the present errors on \( R \) and \( A \): \( \sigma(R) = 0.019 \) (WMAP) [9] and \( \sigma(A) = 0.017 \) (SDSS) [6]. For the long term scenario we choose \( \sigma(R) = 0.01 \) as roughly expected [42] from the Planck mission [43], and \( \sigma(A) = 0.005 \) as expected for future large surveys like a ‘stage-IV’ space based mission dedicated to BAO as described by the DETF [5, 44].
2.4. The statistical method

To proceed in practice, we choose a fiducial model that is a set of fiducial parameters and simulate data (SNIa magnitudes, $R$ and $A$). In the most general case the fiducial parameters are $\Omega^{F}_M$, $\Omega^{F}_X$, $w^{F}_0$ and $w^{F}_a$. Note that the value of the SNIa normalization parameter $M^F_s$ is taken to $-3.6$: its precise value is not very relevant for the following. For definiteness we will assume $\Omega^{F}_M = 0.3$ throughout the paper except when otherwise specified, but we have checked that reasonable deviations ($\pm 0.2$) from this value have very little influence on our conclusions.

Then, having chosen a fiducial model, we simulate data for this model and we fit the simulated data with a wrong assumption on the curvature or the DE parameters. Finally, we compare the fiducial and fitted values of the various parameters along with the respective errors, to estimate how much our conclusions could be affected by the wrong assumption.

To analyze the (simulated) data, a minimization procedure has been used. A standard Fisher matrix approach allows a fast estimate of the parameter errors. This method is however limited as it does not yield the central values of the fitted parameters. Then we adopt, unless specified, a minimization procedure based on a least squares method. The least squares estimators are determined by the minimum of $\chi^2 = (m - M(z, \Omega, w)^TV^{-1}(m - M(z, \Omega, w)))$, where $m = (m_1, \ldots, m_n)$ is a vector which contains the simulated magnitudes plus $R$ and $A$. The vector $M(z, \Omega, w)$ is the corresponding vector of values for the fitted parameters and $V$ the covariance matrix. The errors on the cosmological parameters are estimated at the minimum by using the first-order error propagation technique: $U = J \cdot V \cdot J^T$ where $U$ is the error matrix on the cosmological parameters and $J$ the Jacobian of the transformation. A full $n$-parameter fit ($n$-fit) of the simulated data gives the central values and errors for the $n$ parameters, along with their correlations which are in general strong.

2.5. Detectability criterion

We want to identify among models the more problematic ones where the fit fails to identify a problem. Indeed, in any data analysis, a wrong assumption can be early detected through a simple $\chi^2$ test: a high $\chi^2$ indicates that the goodness of the fit is bad. In our context, when fitting simulated data with some hypotheses, a high $\chi^2$ will indicate that some of these hypotheses are wrong.

However, rejecting models from a $\chi^2$ test with real data is always tricky; therefore we demand a high level of detectability. Consequently, models where the $\chi^2$ test gives a $5\sigma$ effect ($\chi^2 > 5\sigma(\chi^2)$) where the rms of the $\chi^2$ is $\sigma(\chi^2) = \sqrt{2N_{\text{dof}}}$ and where $N_{\text{dof}}$ is the number of degrees of freedom in the fit, are considered to be easily identified as in conflict with the input hypothesis.

On the other hand some models passing this test can be biased, as the fitted parameters are too far from the fiducial ones. In the following sections, we identify the models which yield a good value of $\chi^2$ even if the fitting hypothesis is incorrect. Careful attention is given to the resulting confusions and biases.

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5 Our simulation tool, the ‘Kosmoshow’, has been developed by A Tilquin and is available upon request or directly at http://marwww.in2p3.fr/renoir/Kosmoshow.html.
3. Reconstruction of DE dynamics with a wrong assumption on curvature

If we consider realistic experimental conditions, the reconstruction of an evolving dark energy is difficult when we leave the curvature free due to a complete degeneracy between the relevant parameters. This comes from the integral relation between cosmological distances and the cosmological parameters [23] and the only way to solve this question is to fix some parameters or/and to combine different information. However, such a strategy can lead to some biased interpretations. This has recently been studied on theoretical grounds in [12]. Following the same approach, we simulate fiducial models such as ΛCDM universes with non-zero curvature and we calculate the errors on the cosmological parameters as expected in future observational conditions. In this way we are able to quantify when some models with curvature, consistent with the CMB constraint, can start to bias the results on the DE EoS when flatness is incorrectly assumed in the fit.

In practice, we fix $\Omega^F_M$ to being in the range 0.1–0.5 (illustrations are given for $\Omega^F_M = 0.3$) and we vary $\Omega^F_X$ to scan $\Omega^F_k$ between $-0.08$ ($-0.07$) and $\Omega^F_k > 0.05$ (0.03) for the short term (long term) scenario. Then we perform the fit of $\Omega_M$, $w_0$ and $w_a$, assuming flatness.

To test the wrong hypothesis we first test the goodness of the fit as described in section 2.5. When fiducial curvature is too far from flatness, the $\chi^2$ is too high and the wrong assumption is detectable. Models with $\Omega_k$ lower than $-0.08$ ($-0.07$) and $\Omega_k$ greater than $0.05$ (0.03) for the short term (long term) scenario are detected by the simple $\chi^2$ test. We study in more detail some models which give good $\chi^2$ fits to determine the risks of a wrong conclusion on the values of the DE parameters.

In figure 1 we present two examples of fits with good $\chi^2$ in the $(w_0, w_a)$ plane. We have chosen two different fiducial models: a ΛCDM open model (upper plots) with $(w_0^F = -1, w_a^F = 0, \Omega^F_M = 0.3, \Omega^F_X = 0.68)$ and a ΛCDM closed model (bottom plots) with $(w_0^F = -1, w_a^F = 0, \Omega^F_M = 0.3, \Omega^F_X = 0.72)$. In each case the full simulated data (SNIa, $R$ and $A$), are fitted with the (wrong) hypothesis of flatness. The 1σ contours correspond to the solid curves and the left (right) plots correspond to the short term (long term) scenario.

One can see that the fitted $w_0$ and $w_a$ central values are far from the fiducial values in each case. We get for positive curvature ($\Omega_k < 0$) a $w_0$ value which is pushed towards the phantom regime $w_0 < -1$ and a $w_a$ value which is positive, and the converse for negative curvature ($\Omega_k > 0$), i.e. that $w_0 > -1$ and $w_a < 0$. Let us stress again that this bias is not detectable with true data since the fit is good (low $\chi^2$). Note that at the same time the $\Omega_M$ value is well reconstructed.

For the long term scenario, the statistical power increases; then the error ellipses are obviously smaller but the bias is higher. For comparison we show the contours resulting from a complete fit where the curvature is a free parameter (dashed contours). These contours are centered on the fiducial values as expected. In the long term scenario the contours from the two approaches are clearly disconnected; hence the problem becomes stronger.

Let us quantify more precisely the biases which are expected on these two DE EoS parameters. We vary the curvature and determine the zones of bias, defined as $|w_0 + 1| > \sigma(w_0)$ and $|w_a| > \sigma(w_a)$. The DE parameter bias means that a cosmological constant with some amount of curvature will be interpreted as a dynamical DE model in a flat universe. The results are summarized in table 1.
Figure 1. Illustration of the bias problem in the \((w_0, w_a)\) plane for a slightly open model (upper plots) with \(\Omega_M^F = 0.3, \Omega_X^F = 0.68 (\Omega_k^F = 0.98)\) and for a slightly closed model (lower plots) with \(\Omega_M^F = 0.3, \Omega_X^F = 0.72 (\Omega_k^F = 1.02)\). Results are given for a short term scenario (left plots) and a long term scenario (right plots). A fiducial \(\Lambda\)CDM model has been assumed. Contours are at 1\(\sigma\) and correspond to some fits with (solid) or without (dashed) the flatness assumption.

- In the positive curvature sector, \(-0.08 < \Omega_k^F < -0.02\) (short term), \(-0.07 < \Omega_k^F < -0.01\) (long term). The upper bound is the bias limit; above this value the results are not biased. The lower bound is the goodness of the fit limit; below this value the \(\chi^2\) explodes.

- In the negative curvature sector, \(0.02 < \Omega_k^F < 0.05\) (short term), \(0.01 < \Omega_k^F < 0.03\) (long term); now the lower bound is the bias limit and the upper bound the goodness of the fit limit.

In other words, if we do not take into account the current and future curvature uncertainties, results on the DE parameters can be biased even in a long term scenario. This implies a potential misinterpretation of the DE dynamics even when using a combination of precise probes both at low and high redshifts, as soon as a spatial flatness is assumed.

We conclude that \(w(z)\) can be reliably measured with the flatness assumption only if the true curvature is \(|\Omega_k| \leq 0.02\) (\(\leq 0.01\)) for a short (long) term scenario.
Table 1. Validity range of the flatness assumption for the determination of the DE EoS parameters $w_0$ and $w_a$.

|       | Bad fit | Biased zone | Validity zone | Biased zone | Bad fit |
|-------|---------|-------------|---------------|-------------|---------|
| Short term | $\Omega^F_k < -0.08$ | $-0.08 < \Omega^F_k < -0.02$ | $-0.02 < \Omega^F_k < 0.02$ | $0.02 < \Omega^F_k < 0.05$ | $\Omega^F_k > 0.05$ |
| Long term | $\Omega^F_k < -0.07$ | $-0.07 < \Omega^F_k < -0.01$ | $-0.01 < \Omega^F_k < 0.01$ | $0.01 < \Omega^F_k < 0.03$ | $\Omega^F_k > 0.03$ |

4. Reconstruction of curvature with a wrong assumption on DE

Another common strategy is to pin down the curvature of the universe when assuming that DE is a cosmological constant (the ΛCDM hypothesis). In this section we will investigate the biases introduced by this assumption.

After a short illustration of the problem, we study in detail the possible confusion of some non-flat dynamical DE models with a flat cosmological constant. Then we quantify the models which can provide a biased estimation of the cosmological parameters in an undetectable way.

4.1. Illustration

In this study, we test the ΛCDM model assumption by means of a fit of $\Omega_M$ and $\Omega_X$. First we perform the fit for two examples which illustrate the situation. We consider two models with a dynamical DE for two different curvatures ($\Omega^F_k = 0.03$ and $\Omega^F_k = -0.02$), still fixing the matter density to $\Omega^F_M = 0.3$.

We show in figure 2 the results of the fits in the $(\Omega_M, \Omega_X)$ plane for the combination of SNIa, $R$ and $A$ in the short term scenario. The straight line corresponds to flat models ($\Omega_M + \Omega_X = 1$). We show as solid curves the contours obtained from a fit assuming that DE is described by a ΛCDM model. We see that the fitted values of the matter and dark energy densities (crosses) are far from the fiducial ones (stars) due to the wrong hypothesis. Therefore the fitted values of curvature are compatible with the flatness at 1σ. For comparison, the fits without the assumption on the DE are performed and the results are plotted as the dotted contours. Note that these ellipses are completely disconnected. These results are given for a short term scenario but this bias still exists for the long term scenario.

With this illustration we have shown that it is possible to interpret a dynamical DE model in a universe with a non-zero spatial curvature as a flat cosmological constant model, for present data as well as for future surveys. In the following we look for the DE models and the possible amount of curvature where such a misinterpretation of combined data is possible.

4.2. Confusion with flatness

We want to identify the DE models with non-zero curvature which can be compatible with a flat ΛCDM model, when assuming $w = -1$. 
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Figure 2. Contours at 1σ in the $(\Omega_M, \Omega_X)$ plane for a short term scenario. The left (right) plot corresponds to the fiducial cosmology $\Omega_F^M = 0.3$, $\Omega_F^X = 0.67$, that is $\Omega_k^F = 0.03$, $w_0^F = -0.9$ and $w_a^F = 0.5$ (same fiducial cosmology $\Omega_F^M = 0.3$, $\Omega_F^X = 0.72$, that is $\Omega_k^F = -0.02$, $w_0^F = -0.9$ and $w_a^F = -1.5$). Plain (dotted) curves are obtained from fits with (without) the assumption that DE is a cosmological constant. The straight line corresponds to flat universes.

For a given set of fiducial parameters $(\Omega_k^F, w_0^F, w_a^F)$ we perform a fit imposing $w = -1$. For each model tested we first verify the goodness of the fit and then we determine whether the fitted value of the curvature is compatible with the flatness, in satisfying $|\Omega_k| < \sigma(\Omega_k)$.

We present in figure 3 the results for four fiducial curvatures $(\Omega_k = 0.06, 0.02, -0.02, -0.03)$ in the short term scenario. Models with a good fit are located between the two solid curves and represent a large area of the plane. This means that many DE models in a curved space may give a correct fit for a cosmological constant. For some of these models there is a confusion with a flat universe, i.e. these non-flat models are interpreted as flat ones. For each fiducial curvature these models form a confusion zone in the plane. Note that the models chosen for the illustration of the problem in figure 2 are marked with stars in figure 3 and are within the confusion zones.

We have performed the same study for the long term scenario. The results for fiducial curvature $(\Omega_k^F)$ between $-0.01$ and $0.03$ are given in figure 4. Thanks to the higher statistics, the area of the $(w_0^F, w_a^F)$ plane where the fit passes the $\chi^2$ test is smaller. Therefore the wrong assumption is detectable for a greater number of models.

As regards models which pass the $\chi^2$ test but may be confused with a flat cosmological constant model, we remark that DE models with positive curvature $(\Omega_k^F < 0)$ are in the lower part of the plane (roughly $w_0^F > -1$ and $w_a^F < 0$) in both figures 3 and 4. With high statistics, most of these problematic models exit the zone of goodness of the fit and can be rejected as soon as $\Omega_k^F < -0.01$. 

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Figure 3. The black contour gives the limit of the goodness of the fits. Other contours correspond to the zones of confusion with a flat ΛCDM model for the following fiducial cosmologies: $(Ω_M^F = 0.3)$: $Ω_X^F = 0.64$ ($Ω_k^F = 0.06$) (plain), $Ω_X^F = 0.68$ ($Ω_k^F = 0.02$) (dotted–dashed), $Ω_X^F = 0.72$ ($Ω_k^F = -0.02$) (long dashed) and $Ω_X^F = 0.73$ ($Ω_k^F = -0.03$) (short dashed). The straight line is the ‘CMB boundary condition’ $w_0 + w_a < 0$. Calculations are done for the short term scenario.

The DE models with negative curvature ($Ω_k^F > 0$) that may be confused with a flat cosmological constant model are in the upper part of the plane (roughly $w_0^F < -1$ and $w_a^F > 0$). Note that some confusion zones are close to, or above, the ‘CMB boundary condition’. If we ignore this limit, some very open models (up to $Ω_k^F = 0.6$) can still give a correct $χ^2$ and be confused with a flat cosmological constant universe. However, for these models, the size of the confusion zone is very tiny and corresponds to a very small phase space volume.

Therefore, our analysis cannot exclude the possibility that some open dynamical DE models with a large departure from flatness can induce a confusion with a flat ΛCDM model. The proximity or violation of the CMB boundary condition indicates that these DE models are rather exotic: of both phantom and early dark energy types. The smallness of the phase space volume for these models shows also that these exotic DE models should be fine-tuned to give a confusion. This situation means that our analysis has reached its limits and that a better description of the DE model (beyond the CPL parameterization) and of the CMB (beyond the use of $R$) should be obtained, to yield a confirmation of these possible confusions.

One can find in the literature constraints on curvature coming from different combined analyses where the DE dynamics is left free [20]–[22]. They give $Ω_k < 0.02$ at 95% CL, concluding that open models are disfavored. Our last result indicates that one has to be
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Figure 4. The black contour gives the limit of the goodness of the fits. Other contours correspond to the zones of confusion with a flat ΛCDM model for the following fiducial cosmologies: (Ω_F M = 0.3): Ω_F k = 0.67 (Ω_F k = 0.03) (plain), Ω_F X = 0.69 (Ω_F k = 0.01) (dotted–dashed) and Ω_F X = 0.71 (Ω_F k = −0.01) (dashed). The straight line is the ‘CMB boundary condition’ w_0 + w_a < 0. Calculations are done for the long term scenario.

Cautious: constraints depend on the statistical methods which are very sensitive to the size of the phase space volume [25] and the current combined analyses of data may have neglected or missed these open exotic DE models.

In summary, even with a large statistical sample, it is still possible to confuse a dynamical DE non-flat model with the flat ΛCDM model when a cosmological constant is assumed. This is due to the geometrical degeneracies between the DE and the CMB parameters and this cannot be completely solved by the use of a combined analysis with SNIa and BAO even at high statistics. To draw any conclusion on the description of dark energy, it is mandatory to fit enough DE parameters with the curvature as a free parameter.

4.3. Bias in the reconstruction of the reduced densities

In this section, we study the biases in the reduced densities Ω_M and Ω_X, still in the context of fiducial curved dynamical DE models analyzed assuming a cosmological constant. For that purpose, we vary Ω_F M in the range 0.1 to 0.5 and Ω_F X in a range such that |Ω_F k| < 0.2. Then, we scan the (w_0^F, w_a^F) plane. We quantify the biases introduced by the wrong hypothesis, when trying to reconstruct the cosmological parameters Ω_M and Ω_X. We define the bias of a parameter p by B_p = |p^F − p| and we say that p is biased (valid) if the bias is larger (smaller) than the error obtained for p, i.e. if B_p > σ(p) (B_p < σ(p)). (One can consult [27] for more details on these definitions.)
Figure 5. The black contour gives the limit of the goodness of the fits. Other contours correspond to the zones of confusion with a flat ΛCDM model (labeled CZ), the validity zone (labeled VZ) and the biased zone (labeled BZ). The fiducial cosmology is $\Omega^F_M = 0.25$ and $\Omega^F_X = 0.7$ ($\Omega^F_k = 0.05$). The straight line is the ‘CMB boundary condition’ $w_0 + w_a < 0$. Calculations are done for the long term scenario.

There are some zones where the $\chi^2$ is too high; then the fit is not good and the problem is detectable. When the $\chi^2$ is good we can define two types of zone.

- The zones where the $\chi^2$ is good and the fitted parameters are well reconstructed even with the assumption of $w = -1$; we call such a zone a validity zone (VZ). Such zones are centered on the Λ point ($w_0^F = -1$, $w_a^F = 0$) where the fit assumption is true.
- The zones where the $\chi^2$ is good but all fitted parameters are wrong. The previously discussed zones of confusion belong to this zone group and correspond to the cases where we obtain erroneously $\Omega_k = 0$.

Figure 5 shows one example where $\Omega^F_M = 0.25$ and $\Omega^F_X = 0.7$ in the long term scenario. The result is generic for all models considered in this study and whatever the statistic is. We see that, with the combined analysis and high statistics, there are large zones in the plane where the cosmological parameters are wrong, if we assume DE to be a cosmological constant. This fact is well known. The new information is that many models will be interpreted as a ΛCDM because of the degeneracy of curvature and dynamical dark energy and that this will not be detectable.

For the short term scenario another situation may appear where $\Omega_k$ is well reconstructed but $\Omega_M$ and $\Omega_X$ are biased, namely the biases compensate to give accidentally the correct curvature. This possibility disappears at high statistics.
The positions of the various zones in the fiducial plane can move when fiducial parameters like $\Omega^F_M$ or $\Omega^F_X$ are changing:

- An increase of $\Omega^F_X$ reinforces the power of the $\chi^2$ test (i.e. the goodness of fit zone decreases) but reinforces also the bias problem (i.e. the VZ decreases).
- Changing $\Omega^F_M$ shifts the location of the various zones in the fiducial plane.

Scanning all possibilities, we can always find some values for the densities where the $\chi^2$ is low, and the fitted results are valid or biased. Our study shows also that the addition of SNIa and BAO reduces the degeneracies but does not eliminate them.

Finally, let us note that we find also that assuming only a constant $w$, different from $-1$, leads to similar confusion and biases.

5. Conclusions

The vicious circle of assuming flatness to constrain dark energy or, conversely, of assuming that dark energy is a cosmological constant (or has a constant EoS) to constrain curvature should be avoided. Indeed, such assumptions may provide very important biases and confusion if they are not true.

We have used a combined analysis using SNIa, BAO and CMB data, taking advantage of the use of the compressed parameter $A$ for BAO and shift parameter $R$ for CMB. The combination will help to break degeneracies, the origin of the biases, which come mainly from the integral relation between cosmological distances and the cosmological parameters [23].

We see that assuming flatness, a cosmological constant with a small amount of curvature can be interpreted as a dynamical DE model in a flat universe. We find that, even using a multi-probe approach and with future high statistics, such hypotheses will bias the interpretation of the data if the true curvature is in the range $0.01 < \Omega_k < 0.03$ or $-0.07 < \Omega_k < -0.01$. This is due to the fact that, even with small statistical errors, this range of non-flat models will give a good $\chi^2$ when assuming flatness because of the degeneracy with the DE sector. We emphasize that these models are always inside the current level of uncertainties of WMAP data (when an evolving DE is assumed in the analysis). Let us also state that the PLANCK mission will not allow us to strongly reduce these uncertainties as the information on the curvature is mainly coming from the first peak which is already quite precisely determined.

Conversely, if we suppose that dark energy is a cosmological constant, a dynamical non-flat DE model may be interpreted as a flat cosmological constant model. We show that for closed models, there is no confusion as soon as $\Omega_k > -0.01$. For open models, there is a greater risk of confusion, independently of the statistical sample, in particular if $0.01 < \Omega_k < 0.05$. However, the confusion may still happen at $\Omega_k > 0.05$, but only for very exotic DE models (of both phantom and early DE types) which seem less favored by the current data.

In addition, we show that in the case of misinterpretation when assuming a cosmological constant, the cosmological parameters $\Omega_M$ and $\Omega_X$ will be biased and this will be undetectable in the fit for a large range of models. In any case, high statistics (long term scenario) help in the sense that they allow us to detect more easily a wrong hypothesis through the $\chi^2$ test. However, there are still a number of DE models where
biases will not be detected and the smaller errors would increase our confidence in a wrong interpretation.

In general, in most of the analyses of real data, two fits are performed independently: one with the assumption of flatness and the other with the assumption of a cosmological constant. Consistency between the two fits is then considered as a proof of reliability. However, this approach is not devoid of drawbacks. For example, in the spirit of our analyses of section 3, if we vary the DE parameters beyond a simple cosmological constant we find some confusion zones which overlap those of figures 3 and 4. This means that these models yield a good $\chi^2$ in all cases and point toward a flat $\Lambda$CDM. In these cases the circular logic [12,10] of the ‘consistency’ test fails.

The strategy which consists of relaxing the curvature and the DE parameters within one global procedure is the one which avoids confusion. In addition, when comparing the errors expected for the cosmological parameters in the fit procedures with or without the wrong assumptions (see the plain and dashed contours of figures 1 and 2), one sees that in a combined analysis using SNIa + CMB + BAO, there is no gain in assuming flatness or/and that DE is a cosmological constant. This fact has already been noticed; see e.g. [11].

In conclusion, our quantitative analysis allows us to strongly support the points already stressed by a few authors: it is imperative to let the curvature and the DE parameters be free parameters. This is already true for the analysis of present data as well as for data expected from the future large surveys.

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References

[1] Riess A G et al., 1998 Astron. J. 116 1009 [SPIRES]
[2] Perlmutter S et al., 1999 Astrophys. J. 517 565 [SPIRES]
[3] Peebles P J E and Ratra B, 2003 Rev. Mod. Phys. 75 559 [SPIRES]
Padmanabhan T, 2003 Phys. Rep. 380 235 [SPIRES]
[4] Copeland E J, Sami M and Tsujikawa S, 2006 Int. J. Mod. Phys. D 15 1753 [SPIRES]
[5] Dark Energy Task Force report to the Astronomy and Astrophysics Advisory Committee
   http://www.nsf.gov/mps/ast/detf.jsp
Albrecht A et al., 2006 arXiv:astro-ph/0609591
[6] Eisenstein D J et al., 2005 Astrophys. J. 633 560 [SPIRES]
[7] Tegmark M et al., 2006 Phys. Rev. D 74 123507 [SPIRES] [arXiv:astro-ph/0608632]
[8] Spergel D N et al., 2007 Astrophys. J. Suppl. 170 377 [arXiv:astro-ph/0603449]
[9] Komatsu E et al., arXiv:0803.0547 http://lambda.gsfc.nasa.gov/product/map/current/parameters.cfm
[10] Wright E L, 2005 Talk given at the Int. Astrophys. Conf. on Relativistic Astrophysics and Cosmology
   (Einstein’s Legacy, Munich, Nov. 2005) [arXiv:astro-ph/0603750]
[11] Linder E V, 2006 Astropart. Phys. 26 102 [SPIRES]
[12] Clarkson C, Cortes M and Bassett B A, 2007 J. Cosmol. Astropart. Phys. JCAP08(2007)011 [SPIRES]
[13] Bond J R, Efsthatiou G and Tegmark M, 1997 Mon. Not. R. Astron Soc. 291 L33
On the determination of curvature and dynamical dark energy

[14] Zaldarriaga M, Spergel D N and Seljak U, 1997 Astrophys. J. 488 1 [SPIRES]
[15] Efstathiou G and Bond J R, 1999 Mon. Not. R. Astron Soc. 304 75
[16] Huey G et al, 1999 Phys. Rev. D 59 063005 [SPIRES]
[17] Wang Y and Mukherjee P, 2007 Phys. Rev. D 76 103533 [SPIRES]
[18] Astier P et al, 2006 Astron. Astrophys. 447 31 [SPIRES]
[19] Kowalski M et al, 2008 arXiv:0804.4142
[20] Zhao G B et al, 2007 Phys. Lett. B 648 8 [SPIRES]
[21] Ichikawa K and Takahashi T, 2007 J. Cosmol. Astropart. Phys. JCAP02(2007)001 [SPIRES]
[22] Wright E L, 2007 Astrophys. J. 664 633 [SPIRES]
[23] Maor I, Brustein R and Steinhardt P J, 2001 Phys. Rev. Lett. 86 6 [SPIRES]
[24] Virey J-M et al, 2004 Phys. Rev. D 70 121301 [SPIRES]
[25] Upadhye A, Ishak M and Steinhardt P J, 2005 Phys. Rev. D 72 063501 [SPIRES]
[26] Sahni V, Shafieloo A and Starobinsky A A, 2008 arXiv:0807.3548 [astro-ph]
[27] Virey J-M et al, 2004 Phys. Rev. D 70 043514 [SPIRES]
[28] Wang Y and Mukherjee P, 2006 Astrophys. J. 650 1 [SPIRES]
[29] Linder E V and Huterer D, 2005 Phys. Rev. Lett. 90 091301 [SPIRES]
[30] Chevallier M and Polarski D, 2001 Int. J. Mod. Phys. D 10 213 [SPIRES]
[31] Linder E V, 2003 Phys. Rev. Lett. 90 091301 [SPIRES]
[32] Virey J-M and Ealet A, 2007 Astron. Astrophys. 464 837 [SPIRES]
[33] Caldwell R R and Linder E V, 2005 Phys. Rev. Lett. 95 141301 [SPIRES]
[34] Linden S and Virey J M, 2008 Phys. Rev. D 78 023526 [SPIRES]
[35] Wetterich C, 2004 Phys. Lett. B 594 17 [SPIRES]
[36] Caldwell R R, Dave R and Steinhardt P J, 1998 Phys. Rev. Lett. 80 1582 [SPIRES]
[37] Corasaniti P S and Copeland E J, 2003 Phys. Rev. D 67 063521 [SPIRES]
[38] see e.g. http://cfht.hawai.edu/Science/CFHTLS-OLD/history_2001.html
[39] http://snap.lbl.gov
[40] Kim A G et al, 2004 Mon. Not. R. Astron Soc. 347 909
[41] Wood-Vasey W M et al, 2004 New Astron. Rev. 48 637
[42] Linder E V, 2008 Gen. Rel. Grav. 40 329 [SPIRES]
[43] http://planck.esa.gov
[44] Cimatti A et al, 2008 arXiv:0804.4433