The waiting-time distribution of Liffe bond futures

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Abstract.

We apply the Continuous Time Random Walk (CTRW) framework, introduced in finance by Scalas et al [3], to the analysis of the probability distribution of time intervals between two consecutive trades in the case of BTP futures prices traded at LIFFE in 1997. Results corroborate the validity of the CTRW approach for the description of the temporal evolution of financial time series.

PACS numbers: 02.50-r, 02.50.Ey, 02.50.Wp, 89.90.+n
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1. Introduction

In financial markets, the time interval between two consecutive transactions (the so-called waiting time) varies stochastically. There have been various studies on the nature of the stochastic processes generating the sequences of waiting times between two consecutive trades [1, 2]. In two recent papers, Scalas et al [3] and Mainardi et al [4] proposed the Continuous Time Random Walk (CTRW) model [5] as a phenomenological description of tick-by-tick dynamics in financial markets. In the hydrodynamic limit, the model yields a general scaling form [3, 6] for the probability density function of finding the log-price $x$ at time $t$. Scaling is a consequence of the equivalence between CTRWs and fractional diffusion equations [7, 8]. The reader can also consult refs [9, 10]. Here, as in [4], we will not focus our attention on scaling, but on a different property: the waiting-time distribution.

This paper is divided as follows. Section 2 provides a brief summary of the theoretical framework of CTRW. In Section 3 CTRW assumptions are tested on market data. Conclusions are drawn in Section 4.

2. Theory summary

Let us denote by $x$ the logarithm of an asset price. We assume that both the log-price jumps $\xi_i = x(t_i) - x(t_{i-1})$ and the waiting times between two consecutive trades $\tau_i = t_i - t_{i-1}$ are i.i.d. stochastic variables. We also assume that these variables are characterized by the two probability density functions: $\lambda(\xi)$ and $\psi(\tau)$.

It turns out that the evolution equation for $p(x,t)$, the probability of having the log-price $x$ at time $t$, can be written as follows [4]:

$$
\int_0^t \Phi(t-t') \frac{\partial}{\partial t'} p(x,t') \, dt' = -p(x,t) + \int_{-\infty}^{+\infty} \lambda(x-x') p(x',t) \, dx',
$$

where $\Phi$ is a suitable memory kernel related in Laplace space to $\Psi(\tau)$ - the probability that a given waiting interval is greater or equal to $\tau$ - by:

$$
\tilde{\Phi}(s) = \frac{\tilde{\Psi}(s)}{1 - s \tilde{\Psi}(s)}.
$$

In its turn, the probability $\Psi$ is given by:

$$
\Psi(\tau) = \int_{-\infty}^{\infty} \psi(t') \, dt'.
$$

The probability $\Psi(\tau)$ can be estimated from empirical data, and can thus be used to test hypotheses on the memory kernel $\Phi$. In particular, if $\Phi(t)$ exhibits a power-law time decay $\Phi \propto t^{-\beta}$ with $0 < \beta \leq 1$, one can show that [4]:

$$
\Psi(t) \propto E_{\beta}(-t^\beta),
$$

where $E_{\beta}$ is the Mittag-Leffler function of order $\beta$. In the following, we shall discuss this particular hypothesis.
3. Empirical analysis

We examined the waiting-time distribution of BTP futures traded at LIFFE in 1997. Throughout the year, there were four future contracts on BTP bonds, depending on delivery dates: March, June, September, or December. We considered two delivery dates: June (Figure 1) and September (Figure 2). Usually, for a future with a certain maturity, transactions begin some months before the delivery date. At the beginning, there are few trades a day, but closer to the delivery there may be more than 1000 transactions a day. The total number of transactions was about 140,000 for the delivery date of June and 170,000 for September.

Figure 1 shows a comparison between the function $\Psi(\tau)$ estimated for the empirical waiting times and two theoretical functions. The circles refer to market data (delivery date of June 1997) and represent the probability of a waiting time greater than the abscissa $\tau$. We have determined 494 values of $\Psi(\tau)$ for $\tau$ in a interval between 1 s and about 50,000 s, neglecting the intervals of market closure. The solid line is a two-parameter fit obtained by using the Mittag-Leffler type function:

$$\Psi(\tau) = E_\beta[-(\gamma \tau)^\beta],$$

where $\beta$ is the index of the Mittag-Leffler function and $\gamma$ is a time-scale factor, depending on the time unit.

We get an index $\beta = 0.96$ and a scale factor $\gamma = 1/13$. The fit of Figure 1 has a reduced chi-square $\tilde{\chi}^2 \simeq 0.2$. The chi-square values have been computed considering all the 494 values. The dash-dotted line represents the stretched exponential function $\exp\{-{(\gamma \tau)^\beta}/\Gamma(1 + \beta)\}$, whereas the dashed line is the power law function $({\gamma \tau})^{-\beta}/\Gamma(1 - \beta)$. The Mittag-Leffler function interpolates between these two limiting behaviours: the stretched exponential for small time intervals, and the power law for large ones.

Figure 2 shows the results for the delivery date of September 1997. In this case, we have 442 values of $\Psi(\tau)$ for $\tau$ in the interval between 1 s and about 50,000 s; as for data in Figure 1, we get $\beta = 0.96$ and $\gamma = 1/13$; the reduced chi-square, computed considering all the 442 values, is $\simeq 0.2$.

4. Conclusions and outlook

Figures 1, 2 and the preliminary empirical analysis provided in [4] show a satisfactory agreement between the empirical distributions of market data and theoretical predictions drawn from the CTRW hypothesis with additional assumptions. CTRW is thus likely to be a reasonable phenomenological description of the tick-by-tick dynamics for a

BTP stands for *Buoni del Tesoro Poliennali*, i.e., middle and long term Italian Government bonds with fixed interest rates.

† LIFFE stands for *London International Financial Futures and Options Exchange*. It is a London-based derivative market.
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Figure 1. Survival probability for BTP futures with delivery date: June 1997. The Mittag-Leffler function (solid line) of index \( \beta = 0.96 \) and scale factor \( \gamma = 1/13 \) is compared to the stretched exponential (dash-dotted line) and the power (dashed line) functions.

financial market, as it also takes into account both the non-Markovian and the non-local characters of the time evolution in financial time series.

The fitting procedure used in this paper differs from that in [4]. Indeed we found the estimate of the function \( \Psi(\tau) \) and the fitting procedures far from trivial. These points deserve a separate and thorough discussion which will be the subject of a future paper.

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Figure 2. Survival probability for BTP futures with delivery date: September 1997. The Mittag-Leffler function (solid line) of index $\beta = 0.96$ and scale factor $\gamma = 1/13$ is compared with the stretched exponential (dash-dotted line) and the power (dashed line) functions.