RARTS: a Relaxed Architecture Search Method

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Abstract

Differentiable architecture search (DARTS) is an effective method for data-driven neural network design based on solving a bilevel optimization problem. In this paper, we formulate a single level alternative and a relaxed architecture search (RARTS) method that utilizes training and validation datasets in architecture learning without involving mixed second derivatives of the corresponding loss functions. Through weight/architecture variable splitting and Gauss-Seidel iterations, the core algorithm outperforms DARTS significantly in accuracy and search efficiency, as shown in both a solvable model and CIFAR-10 based architecture search. Our model continues to outperform DARTS upon transfer to ImageNet and is on par with recent variants of DARTS even though our innovation is purely on the training algorithm.

1 Introduction

Neural Architecture Search (NAS) is an automated machine learning technique to design an optimal architecture by searching its building blocks from a collection of candidate structures and operations. Despite the success of NAS in several computer vision tasks [24, 25, 9, 14, 21], the search process demands huge computational resources. The current search times have come down considerably from as many as 2000 GPU days in early NAS [25], thanks to subsequent studies [1, 15, 17, 18, 20, 22] among others. Differentiable Architecture Search (DARTS) [16] is an appealing method that avoids searching over all possible combinations by relaxing the categorical architecture indicators to continuous parameters. The higher level architecture can be learned along with lower level weights via stochastic gradient descent by approximately solving a bi-level optimization problem. The 2nd order DARTS is more accurate yet involves a mixed second derivative estimation of loss functions. In spite of the accuracy, it is used less often in practice as it can take much longer search time than the 1st order DARTS. A single level approach (SNAS) based on sampling and reinforcement learning has been proposed in [23]. On CIFAR-10, SNAS is more accurate than the 1st order DARTS yet with 50% more search time than the 2nd order DARTS.

The main contribution of this paper is to introduce a novel Relaxed Architecture Search (RARTS) based on single-level optimization, and only the first order partial derivatives of loss functions. RARTS achieves higher accuracies than the 2nd order
DARTS with shorter search times consistently on the architecture search. To demonstrate and understand the capability of RARTS, we carried out both analytical and experimental studies below.

- Compare RARTS with DARTS directly on the analytical model with quadratic loss functions, and CIFAR-10 based architecture search exactly as conducted in [16]. In case of the analytical model, RARTS iterations approach in a robust fashion the true global minimal point missed by the 1st order DARTS. On the CIFAR-10 architecture search task, the model found by RARTS has smaller size and higher test accuracy than that by the 2nd order DARTS with 65% search time saving.

- Transfer model learned on CIFAR-10 to ImageNet and compare with DARTS and some of its recent variants.

- Prove a convergence theorem for RARTS based on descent of a Lagrangian function, and discover equilibrium equations for the limits.

## 2 Related work

### 2.1 Differentiable Architecture Search

DARTS [16] learns network weights and the architecture parameters simultaneously based on training and validation loss functions. The second order DARTS performs much better than the first order DARTS, however at a considerable overhead of computing mixed second order partial derivatives of the loss functions (see below).

There are a group of DARTS style methods being proposed lately with most improvements gained from modifying search space and training procedures. FairDARTS [4] and P-DARTS [3] improve the search space by reducing the impact of skip-connections. MiLeNAS [11] is a mixed level reformulation of NAS. We shall see that MiLENAS is actually a constrained case of RARTS.

### 2.2 Bilevel optimization

DARTS training relies on an iterative algorithm to solve a bilevel optimization problem [5]:

\[
\min_\alpha L_{\text{val}}(w_0(\alpha), \alpha),
\]

where \( w_0(\alpha) = \arg\min_w L_{\text{train}}(w, \alpha). \) (1)

Here \( w \) denotes the network weights, \( \alpha \) is the architecture parameter, \( L_{\text{train}} \) and \( L_{\text{val}} \) are the training and validation loss functions. DARTS algorithm proceeds as:

- update weight \( w \) by descending along \( \nabla_w L_{\text{train}} \)
- update architecture parameter \( \alpha \) by descending along:

\[
\nabla_\alpha L_{\text{val}}(w - \xi \nabla_w L_{\text{train}}(w, \alpha), \alpha)
\]
where $\xi = 0$ ($\xi > 0$) gives the first (second) order approximation. The second order method requires computing the mixed derivative $\nabla^2_{\alpha, w} L_{\text{train}}$, and is observed to optimize better in a solvable model and through experiments. The bilevel optimization problem also arises in hyper-parameter optimization and meta-learning, see [8] for convergence result on minimizers and a second order algorithm.

3 Methodology

In this section, we introduce RARTS, its iterative algorithm and convergence properties. We then demonstrate advantages of RARTS over DARTS on various datasets.

3.1 Relaxed Architecture Search

As pointed out in [16] and [11], when learning architecture parameter $\alpha$, one should take into account the validation dataset to avoid overfitting. The bi-level formulation (1) is a way to handle both training and validation datasets. However, (1) is solved only approximately by DARTS algorithms whose convergence is not known theoretically. See Theorem 3.2 in (8) for convergence of minimizers, if the $\alpha$-minimization is solved exactly. Even if the weights $w_0$ are learned optimally on the training dataset, or assumption (vi) of Theorem 3.2 [8], it is unclear how optimal $\alpha$ is on the validation dataset. We propose a single level alternative to the bi-level formulation (1) by joint training of an auxiliary network of the same architecture on the validation dataset. The original and the auxiliary networks are related by having their weights in the same tensor shapes, and difference in weight values controlled by a penalty. This way, the training and validation datasets contribute to the search of architecture $\alpha$ via the cooperation of two networks. Specifically, we propose a relaxed architecture search framework through the following relaxed Lagrangian $L = L(y, w, \alpha)$:

$$
L := L_{\text{val}}(y, \alpha) + \lambda L_{\text{train}}(w, \alpha) + \frac{1}{2} \beta \| y - w \|^2_2, \tag{2}
$$

where $w$ and $y$ denote the weights of the original and the auxiliary networks respectively, $\lambda$ and $\beta$ are hyper-parameters controlling the penalty and the learning process.

We minimize the relaxed Lagrangian $L(y, w, \alpha)$ in (2) by iteration on the three variables alternately, because they have different meanings and dynamics. Similar to Gauss-Seidel method in numerical linear algebra [10], we use updated variables immediately in each step and obtain the following three-step iteration:

$$
w^{t+1} = w^t - \eta_w \nabla_w L(y^t, w^t, \alpha^t)
$$

$$
y^{t+1} = y^t - \eta_y \nabla_y L(y^t, w^{t+1}, \alpha^t)
$$

$$
\alpha^{t+1} = \alpha^t - \eta_\alpha \nabla_\alpha L(y^{t+1}, w^{t+1}, \alpha^t). \tag{3}
$$
With explicit gradient $\nabla_{w,y} \|y-w\|_2^2$, we have:

\[
\begin{align*}
    w^{t+1} &= w^t - \lambda \eta_w^t \nabla_w L_{\text{train}}(w^t, \alpha^t) - \beta \eta_w^t (w^t - y^t) \\
y^{t+1} &= y^t - \eta_y^t \nabla_y L_{\text{val}}(y^t, \alpha^t) - \beta \eta_y^t (y^t - w^{t+1}) \\
    \alpha^{t+1} &= \alpha^t - \lambda \eta_\alpha^t \nabla_\alpha L_{\text{train}}(w^{t+1}, \alpha^t) - \beta \eta_\alpha^t \nabla_\alpha L_{\text{val}}(y^{t+1}, \alpha^t) 
\end{align*}
\] (4)

Note that the update of $\alpha$ in Eq. (4) involves both the training loss and the validation loss, which is similar to the second order DARTS but without the mixed second derivatives. The first order DARTS uses $\nabla_\alpha L_{\text{val}}$ only in this step.

If we set $y = w$, remove the $y$ update and the $\beta$ terms in (4), then we recover the first order algorithm of MiLeNAS [11].

### 3.2 Convergence analysis

Suppose that $L_{\text{train}} := L_t$ and $L_{\text{val}} := L_v$ both satisfy Lipschitz gradient property, or there exist positive constants $L_1$ and $L_2$ such that $(z = (y, \alpha), z' = (y', \alpha'))$:

\[
\|\nabla_z L_v(z) - \nabla_z L_v(z')\| \leq L_1 \|z - z'\|, \ \forall (z, z'),
\]

which implies:

\[
L_v(z) - L_v(z') \leq \langle \nabla_z L_v(z'), (z - z') \rangle + \frac{L_1}{2} \|z - z'\|^2,
\]

for any $(z, z')$; similarly $(\zeta = (w, \alpha), \zeta' = (w', \alpha'))$:

\[
\|\nabla_\zeta L_t(\zeta) - \nabla_\zeta L_t(\zeta')\| \leq L_2 \|\zeta - \zeta'\|, \ \forall (\zeta, \zeta'),
\]

which implies:

\[
L_t(\zeta) - L_t(\zeta') \leq \langle \nabla_\zeta L_t(\zeta'), (\zeta - \zeta') \rangle + \frac{L_2}{2} \|\zeta - \zeta'\|^2,
\]

for any $(\zeta, \zeta')$.

**Theorem 1.** Suppose that the loss functions $L_t$ and $L_v$ satisfy Lipschitz gradient property. If the learning rates $\eta_w^t$, $\eta_y^t$ and $\eta_\alpha^t$ are small enough depending only on the Lipschitz constants as well as $(\lambda, \beta)$, and approach nonzero limit at large $t$, the Lagrangian function $L(y, w, \alpha)$ is descending on the iterations of (4). If additionally the Lagrangian $L$ is lower bounded and coercive (its boundedness implies that of its variables), the sequence $(y^t, w^t, \alpha^t)$ converges sub-sequentially to a critical point $(\bar{y}, \bar{w}, \bar{\alpha})$ of $L(y, w, \alpha)$ obeying the equilibrium equations:

\[
\begin{align*}
    \lambda \nabla_w L_t(\bar{w}, \bar{\alpha}) + \beta (\bar{w} - \bar{y}) &= 0, \\
    \nabla_y L_v(\bar{y}, \bar{\alpha}) + \beta (\bar{y} - \bar{w}) &= 0, \\
    \lambda \nabla_\alpha L_t(\bar{w}, \bar{\alpha}) + \nabla_\alpha L_v(\bar{y}, \bar{\alpha}) &= 0.
\end{align*}
\] (5)

The proof is given in the Appendix.
Figure 1: Learning trajectories of RARTS approach the global minimal point \((1, 1)\) of the solvable model at suitable values of \(\lambda\), \(\beta\) and \(y_0\) (\(\lambda = 10\) in middle/right subplots, \(\beta = 10\) in left/right subplots, \(y_0 = 0\) in left/middle subplots), compared with that of the baseline (first order DARTS).

### 3.3 A solvable bilevel model

Consider quadratic \(L_{\text{val}} = \alpha w - 2\alpha + 1, L_{\text{train}} = w^2 - 2\alpha w + \alpha^2\) for problem (1) as in [16]. The model helps compare DARTS and RARTS through bi-level optimization, besides an example for Theorem 1. The learning dynamics start from \((\alpha_0, w_0, y_0) = (2, -2, y_0)\). Clearly, \(w_0(\alpha) = \arg\min_w L_{\text{train}} = \alpha\). Then \(L_{\text{val}}(w_0(\alpha), \alpha) = \alpha^2 - 2\alpha + 1\), the global minimizer of the bilevel problem (1) is \((\alpha^*, w^*) = (1, 1)\), which is approached by the second order DARTS (Fig. 2 of [16]). The learning trajectory of the first order DARTS ends at \((2, 2)\), a spurious minimal point. This is reproduced here in Fig. 1 along with three learning curves from RARTS as the parameters \((\lambda, \beta)\) and the initial value \(y_0\) vary. In Fig. 1a, \(\beta = 10\), \(y_0 = 0\). In Fig. 1b, \(\lambda = 10\), \(y_0 = 0\). In Fig. 1c, \(\lambda = \beta = 10\). In all experiments, the learning rates are fixed at 0.01. For a range of \((\lambda, \beta)\) and \(y_0\), we see that our learning curves enter a small circle around \((1, 1)\). Both loss functions satisfy Lipschitz gradient property, implying descent of Lagrangian \(L\) by the proof of Theorem 1. If \(\lambda > 1/2, \beta > 1\), \(L\) is bounded and coercive as long as \(\alpha^*\) is bounded, which follows from an eigenvalue analysis of linear system (4) and is observed in computation. If \(\lambda \neq 1/4\), there is unique solution to system (5):

\[
(\bar{\alpha}, \bar{w}, \bar{y}) = \left(\frac{4\lambda^4}{2\lambda^4 - 1}, \frac{4\lambda^4 - 2}{2\lambda^4 - 1}, 1 - \frac{1}{2\lambda^4 - 1}\right).
\]

At \(\lambda = 10\), \((\bar{\alpha}, \bar{w}) \approx (1.025, 0.974)\) where global convergence holds for the whole RARTS sequence.

### 4 Experiments

We show by a series of experiments how RARTS works efficiently on various datasets.

#### 4.1 Datasets

**CIFAR-10.** It consists of 50,000 training images and 10,000 test images [13]. Those 3-channel \(32 \times 32\) images are allocated to 10/100 object classes evenly. The train and val data we have used are standard random half splits of training data as in DARTS. The building blocks of the architecture is searched on CIFAR-10.
**ImageNet.** ImageNet [6, 19] is composed of more than 1.2 million training images and 5,000 test images from 1,000 object classes. We train on ImageNet a larger network which is build of the blocks learnt on CIFAR-10.

### 4.2 Results and Discussions

We run RARTS on the CIFAR-10 architecture search task, under the same settings of search space and number of blocks as [16]. We train 50 and 600 epochs in the first and second stages, respectively. The initial learning rate is 0.025 for both stages. Besides the standard $\ell_2$ regularization of the weights, we also adopt the latency penalty [2], which is widely used in many architecture search tasks [12, 22, 21]. The search cost of RARTS is 1.1 GPU days, far less than that of the second order DARTS. The test error of RARTS is 2.65%, outperforming the 3.00% of the first order DARTS and the 2.76% of the second order DARTS. It should also be pointed out that the model found by RARTS has 3.2M parameters, which saves more memory than the 3.3M model found by DARTS. Moreover, RARTS outperforms SNAS in accuracy and search cost at comparable model size. In the Appendix, the architecture found by RARTS is displayed.

The learned building blocks are then transferred to ImageNet, producing the results in Table 2. Our 26.2% accuracy outperforms those of DARTS and SNAS, which is also comparable to those of GDAS and MiLeNAS.

It should be noted that if $y = w$ is enforced e.g. through a multiplier, RARTS essentially becomes that of MiLeNAS. The difference is that MiLeNAS trains a single network on both training and validation datasets, while we train two networks on the two datasets for the same architecture. MiLeNAS seeks a group of models by conducting model size tracking during search, which adds complexity to the search process. Though both methods did away with bi-level optimization, the architecture in our search has more generality and robustness as it is optimized in two networks with different weights.

The improvement of FairDARTS comes mainly from the modification of the search space, by reducing the number of paths (skip-connection). Similarly, P-DARTS also makes non-algorithmic improvements, as it divides search into multiple stages and progressively adds more depth than DARTS. These methods are actually complementary to our approach which is a pure algorithmic advance of DARTS.

### 5 Conclusion

We developed RARTS, a novel relaxed differentiable method for Neural Architecture Search. We proved its convergence theorem and showed how the method works on an analytically solvable model. We demonstrated its high accuracy and search efficiency over the state-of-the-art differentiable methods especially DARTS style algorithms on CIFAR-10 and ImageNet classifications. RARTS is an algorithmic advance of DARTS and a new search tool for various datasets and deep networks. Additional gain can be achieved with search space design (e.g. [4]) for specific data sets. In future work, we shall extend RARTS to other deep learning applications.
Table 1: Comparison of DARTS, RARTS and other methods on CIFAR-10 based network search. DARTS-1/DARTS-2 stands for DARTS 1st/2nd order. SNAS-Mi/SNAS-Mo stands for SNAS plus mild/moderate constraints, trained with TITAN Xp GPUs where DARTS-1/2 takes 0.4/1 GPU Day. All our experiments are conducted on GTX 1080 Ti GPUs. Here: ◆ on resp. authors’ machines, ⋆ on current authors’ machines. Average of 5 runs.

| Method       | Test Error (%) | Parameters (M) | Search GPU Days ◆ | Search GPU Days ⋆ |
|--------------|----------------|----------------|-------------------|-------------------|
| Baseline [16]| 3.29 ± 0.15    | 3.2            | 4                 | -                 |
| AmoebaNet-B  [18]| 2.55 ± 0.05    | 2.8            | 3150              | -                 |
| ENAS [17]    | 2.89           | 4.6            | 0.5               | -                 |
| ENAS [17, 16]| 2.91           | 4.2            | 4                 | -                 |
| SNAS-Mi [23]| 2.98           | 2.9            | 1.5               | -                 |
| SNAS-Mo [23]| 2.85 ± 0.02    | 2.8            | 1.5               | -                 |
| GDAS [7]     | 2.82           | 2.5            | 0.2               | -                 |
| FairDARTS [4]| 2.54 ± 0.05    | 3.32 ± 0.46    | 0.4               | -                 |
| P-DARTS [3]  | 2.50           | 3.4            | 0.3               | -                 |
| DARTS-1 [16]| 3.00 ± 0.14    | 3.3            | 1.5               | 0.7               |
| DARTS-2 [16]| 2.76 ± 0.09    | 3.3            | 4                 | 3.1               |
| MiLeNAS [11]| 2.80 ± 0.04    | 2.9            | 0.3               | -                 |
| RARTS        | 2.65 ± 0.07    | 3.2            | 1.1               | 1.1               |

Table 2: Comparison of DARTS, RARTS and other methods on ImageNet.

| Method   | Top-1 Test Error (%) | Top-5 Test Error (%) | Parameters (M) |
|----------|----------------------|----------------------|----------------|
| SNAS [23]| 27.3                 | 9.2                  | 4.3            |
| DARTS [16]| 26.7                | 8.7                  | 4.7            |
| MiLeNAS [11]| 25.4               | 7.9                  | 4.9            |
| GDAS [7]  | 26.0                 | 8.5                  | 5.3            |
| RARTS    | 26.2                 | 8.5                  | 4.7            |
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A Convergence Proof

Applying Lipschitz gradient inequalities on $L_v$ and $L_t$, we have:

$$L(y^{t+1}, w^{t+1}, \alpha^{t+1}) - L(y^t, w^t, \alpha^t)$$

$$= L_v(y^{t+1}, \alpha^{t+1}) + \lambda L_t(w^{t+1}, \alpha^{t+1}) + \frac{\beta}{2} \|y^{t+1} - w^{t+1}\|^2 - L_v(y^t, \alpha^t) - \lambda L_t(w^t, \alpha^t)$$

$$- \frac{\beta}{2} \|y^t - w^t\|^2$$

$$\leq (\nabla_{y, \alpha} L_v(y^t, \alpha^t), (y^{t+1} - y^t, \alpha^{t+1} - \alpha^t)) + \frac{L_1}{2} \|(y^{t+1} - y^t, \alpha^{t+1} - \alpha^t)\|^2$$

$$+ \lambda (\nabla_{w, \alpha} L_t(w^t, \alpha^t), (w^{t+1} - w^t, \alpha^{t+1} - \alpha^t)) + \frac{L_2}{2} \|(w^{t+1} - w^t, \alpha^{t+1} - \alpha^t)\|^2$$

$$+ \frac{\beta}{2} \|(y^{t+1} - w^{t+1})^2 - \|y^t - w^t\|^2\).$$

Substituting for the $(w, y)$-gradients from the iterations [4], we continue:

$$L(y^{t+1}, w^{t+1}, \alpha^{t+1}) - L(y^t, w^t, \alpha^t)$$

$$\leq (\eta^t)^{-1} (y^{t+1} - y^t + \beta \eta^t (y^t - w^{t+1}), y^{t+1} - y^t)$$

$$+ (\nabla_{\alpha} L_v(y^t, \alpha^t) + \lambda \nabla_{\alpha} L_t(w^t, \alpha^t), \alpha^{t+1} - \alpha^t)$$

$$+ \lambda \eta^t \nabla_{\alpha} L_t(w^t, \alpha^t), (w^{t+1} - w^t + \beta \eta^t (y^t - w^t), w^{t+1} - w^t)$$

$$+ L_1 \|(y^{t+1} - y^t)^2 + \frac{L_1 + L_2}{2} \|\alpha^{t+1} - \alpha^t\|^2 + \frac{L_2}{2} \|(w^{t+1} - w^t)^2\)$$

$$+ \frac{\beta}{2} \|(y^{t+1} - w^{t+1})^2 - \|y^t - w^t\|^2\)$$

$$= (\eta^t)^{-1} + \frac{L_1}{2} \|(y^{t+1} - y^t)^2 + (\eta^t)^{-1} + \frac{L_2}{2} \|(w^{t+1} - w^t\)$$

$$- \beta (y^t - w^{t+1}, y^{t+1} - y^t) + (w^t - y^t, w^{t+1} - w^t))$$

$$+ \frac{\beta}{2} \|(y^{t+1} - w^{t+1})^2 - \|y^t - w^t\|^2\) + \nabla_{\alpha} L_v(y^t, \alpha^t) + \lambda \nabla_{\alpha} L_t(w^t, \alpha^t), \alpha^{t+1} - \alpha^t$$

$$+ \frac{L_1 + L_2}{2} \|\alpha^{t+1} - \alpha^t\|^2\).$$

(6)

We note the following identity

$$\|y^{t+1} - w^{t+1}\|^2$$

$$= \|y^{t+1} - w^t + w^t - w^{t+1}\|^2$$

$$= \|y^{t+1} - w^t\|^2 + 2(y^{t+1} - w^t, w^t - w^{t+1}) + \|w^t - w^{t+1}\|^2,$$

where

$$\|y^{t+1} - w^t\|^2$$

$$= \|w^t + y^t - y^t + y^{t+1}\|^2$$

$$= \|y^t - w^t\|^2 + 2(y^t - w^t, y^{t+1} - y^t) + \|y^{t+1} - y^t\|^2.$$
Upon substitution of the above in the right hand side of (6), we find that:

\[
L(y^{t+1}, w^{t+1}, \alpha^{t+1}) - L(y^t, w^t, \alpha^t) \\
\leq \left( -(\eta^t_y)^{-1} + \frac{L_1}{2} + \frac{\beta}{2} \right) ||y^{t+1} - y^t||^2 + \left( -(\eta^t_w)^{-1} + \frac{L_2}{2} + \frac{\beta}{2} \right) ||w^{t+1} - w^t||^2 \\
+ \beta (w^{t+1} - w^t, y^{t+1} - y^t) + \beta (y^{t+1} - y^t, w^t - w^{t+1}) \\
+ \langle \nabla_\alpha L_v(y^t, \alpha^t) + \lambda \nabla_\alpha L_\ell(w^t, \alpha^t), \alpha^{t+1} - \alpha^t \rangle + \frac{L_1 + L_2}{2} ||\alpha^{t+1} - \alpha^t||^2. \tag{7}
\]

The \(\beta\)-terms cancel out. Substituting for the \(\alpha\)-gradient from the iterations (4), we get:

\[
L(y^{t+1}, w^{t+1}, \alpha^{t+1}) - L(y^t, w^t, \alpha^t) \\
\leq \left( -(\eta^t_y)^{-1} + \frac{L_1}{2} + \frac{\beta}{2} \right) ||y^{t+1} - y^t||^2 + \left( -(\eta^t_w)^{-1} + \frac{L_2}{2} + \frac{\beta}{2} \right) ||w^{t+1} - w^t||^2 \\
- (\eta^t_\alpha)^{-1} ||\alpha^{t+1} - \alpha^t||^2 + \langle \nabla_\alpha L_v(y^t, \alpha^t) - \nabla_\alpha L_v(y^{t+1}, \alpha^t), \alpha^{t+1} - \alpha^t \rangle \\
+ \lambda \langle \nabla_\alpha L_\ell(w^t, \alpha^t) - \nabla_\alpha L_\ell(w^{t+1}, \alpha^t), \alpha^{t+1} - \alpha^t \rangle
\]

where the last two inner product terms are upper bounded by:

\[
(1 + \lambda) L_3 (||y^t - y^{t+1}|| + ||w^t - w^{t+1}||) ||\alpha^{t+1} - \alpha^t||,
\]

for positive constant \(L_3 := \max(L_1, L_2)\). It follows that:

\[
L(y^{t+1}, w^{t+1}, \alpha^{t+1}) - L(y^t, w^t, \alpha^t) \\
\leq \left[ -(\eta^t_y)^{-1} + \frac{L_1}{2} + \frac{\beta}{2} + (1 + \lambda) \frac{L_3}{2} \right] ||y^{t+1} - y^t||^2 \\
+ \left[ -(\eta^t_w)^{-1} + \frac{L_2}{2} + \frac{\beta}{2} + (1 + \lambda) \frac{L_3}{2} \right] ||w^{t+1} - w^t||^2 \\
+ \left[ -(\eta^t_\alpha)^{-1} + (1 + \lambda) \frac{L_3}{2} \right] ||\alpha^{t+1} - \alpha^t||^2. \tag{8}
\]

If

\[
\eta^t_y < \frac{1}{2} \left[ \frac{L_1}{2} + \frac{\beta}{2} + (1 + \lambda) \frac{L_3}{2} \right]^{-1} := c_1, \\
\eta^t_w < \frac{1}{2} \left[ \frac{L_2}{2} + \frac{\beta}{2} + (1 + \lambda) \frac{L_3}{2} \right]^{-1} := c_2, \\
\eta^t_\alpha < \frac{1}{(1 + \lambda)L_3} := c_3,
\]

\(L\) is descending along the sequence \((y^t, w^t, \alpha^t)\). It follows from (8) that:

\[
\frac{1}{2} \min\{c_1^{-1}, c_2^{-1}, c_3^{-1}\} \| (y^{t+1} - y^t, w^{t+1} - w^t, \alpha^{t+1} - \alpha^t) \|^2 \\
\leq L(y^t, w^t, \alpha^t) - L(y^{t+1}, w^{t+1}, \alpha^{t+1}) \to 0
\]
as \(t \to +\infty\), implying that

\[
\lim_{t \to +\infty} \| (y^{t+1} - y^t, w^{t+1} - w^t, \alpha^{t+1} - \alpha^t) \| = 0.
\]
Since $L$ is lower bounded and coercive, $\|(y^t, w^t, \alpha^t)\|$ are uniformly bounded in $t$. Let $(\eta^*_{y^t}, \eta^*_{w^t}, \eta^*_{\alpha^t})$ tend to non-zero limit at large $t$. Then $(y^t, w^t, \alpha^t)$ sub-sequentially converges to a limit point $(\bar{y}, \bar{w}, \bar{\alpha})$ satisfying the equilibrium system (5).

We note that for the solvable model of section 3.3, the equilibrium system (5) reads:

$$\begin{align*}
\lambda(2\bar{w} - 2\bar{\alpha}) + \beta(\bar{w} - \bar{y}) &= 0, \\
\bar{\alpha} + \beta(\bar{y} - \bar{w}) &= 0, \\
\lambda(-2\bar{w} + 2\bar{\alpha}) + \bar{w} - 2 &= 0.
\end{align*}$$

Adding (9) and (10) gives: $\bar{w} = \frac{2\lambda - 1}{2\lambda} \bar{\alpha}$, which together with (11) determines $(\bar{w}, \bar{\alpha})$ uniquely if $\lambda \neq 1/4$. The $\bar{y}$ formula follows readily from (10).
Figure 2: The architecture of the normal cell found by RARTS. The last four edges are simply summed together to construct the next cell. So there is no search along these edges, which follows the convention of architecture search. These figures are plotted with the help of a program posted in public by the authors of DARTS [16].

Figure 3: The architecture of the reduction cell found by RARTS.