$r^{w^*}$-closed sets in Alexandroff Spaces

N Bhardwaj$^1$ and P Sharma$^2$

Associate Professor, Department of Mathematics, Lovely Professional University, Punjab, India$^1$
Research Scholar, Department of Mathematics, Lovely Professional University, Punjab, India$^2$
E-mail: nitin.15903@lpu.co.in$^1$
E-mail: pallavi.sharma0303@gmail.com$^2$

Abstract. This paper explained and defined the notion of regular weakly-star closed (briefly known as $rw^*$-closed) sets in alexandroff spaces in which every point has a minimal neighbourhood. We discuss the characterizations and study their properties based on set theory along with the notion of $rw^*$-open sets.

1. Introduction

Alexandroff spaces is a topology which supports the property that an arbitrary intersection of family of open sets is open. It has been named after Russian topologist Pavel Alexandrov and known by the name "Discrete Raume". This kind of topology also states that every point has minimal neighbourhood. This space has all the properties of finite spaces which make relevance to digital topology. Indeed, we can say that Alexandrov discrete spaces is a generalization of finite topological spaces. These spaces are mostly determined by specialization preorders. Francisco [8] studied Alexandroff spaces properly and produced various relevant results related to their topological properties. He produced the characterization in context of sets and neighbourhood in Alexandroff spaces. Alexandroff spaces have applications in Reimannian geometry [15], digital topology [6], path topologies of spacetime and linear orders [14] etc. Generalisation of closed sets constantly assumed a significant role in topological spaces and contributed to the hypothesis of separation axioms, covering lemmas etc. Many mathematicians contributed to the theory of generalisation of closed sets by giving different classes of generalised closed sets. Regular open, $w$-closed, $g$-closed, $rw$-closed sets were introduced by Tong [5], Sundaram [9], Levine [3], Benchalli [12] respt. Pratulanda and Mamun [11] introduced $g^*$-closed sets in Alexandroff spaces and investigate some of its characteristics and showed that $g^*$-closed sets did not have the same kind of results as that of generalised closed sets and obtained a new separation axiom, namely, $T_{w^*}$- axiom. Amar kumar et al [13] studied the notion of $g^*$-closed sets, $g\lambda$-closed and $\lambda^*$ closed sets in Alexandroff spaces and introduced various separation axioms namely $T_{5w/8}$, $T_{3w/8}$ and $T_w$ and showed that $T_{5w/8}$ which can be placed between $T_{3w/8}$ and $T_w$.

This paper developed the notion of $rw^*$-closed sets in Alexandroff spaces and explored various properties of these sets. In the whole paper, a space $(X, \tau_A)$ or simply $X$ represents Alexandroff spaces and $\mathcal{R}$ and $\mathcal{Q}$ denotes the set of real numbers and rational numbers respectively.
2. Preliminaries
In this section, we recall the essential points which can be helpful for subsequent results.

Definition 2.1 [2] A system \( \mathcal{E} \) of subsets together with a set is said to be an Alexandroff space (or \( \sigma \)), if the given below conditions have been fulfilled:
1) An arbitrary intersection of number of sets of \( \mathcal{E} \in \mathcal{E} \).
2) Finite union of number of sets of \( \mathcal{E} \in \mathcal{E} \).
3) \( \emptyset \) and \( X \in \mathcal{E} \).
Components of \( \mathcal{E} \) are known as closed sets. Complement of these sets are known to be open.
One can take open sets instead of closed sets with the conditions of finite intersectability, countable summability and the whole set and non-empty set must be open.

Remark 2.2 \( \tau_{A} \) is not a topology, in general, which can easily be seen when we take \( X = \mathbb{R} \) with \( \tau_{A} \) as a family of all \( \mathcal{E} \) in \( \mathbb{R} \).

Definition 2.3 In \( (X, \tau_{A}) \), a subset \( P \) is known as

2.3.1 Weakly closed sets (w-closed) [4] if \( P \subseteq V \) whenever \( P \subseteq V \) and \( V \) is semiopen in \( X \).

2.3.2 Regular weakly-closed (rw-closed sets) [12] if \( P \subseteq V \) whenever \( P \subseteq V \) and \( V \) is regular semi-open in \( (X, \tau_{A}) \).

2.3.3 Regular open [1] if \( P \) is equal to \( (\overline{P})^{\circ} \) and regular closed if \( P \) is equal to \( (P^{\circ})^{c} \).

2.3.4 Regular generalized (rg-closed) [7] if \( \overline{P} \subseteq V \) whenever \( P \subseteq V \) and \( V \) is regular open in \( X \).

Definition 2.4 [12] If a regular open set \( V \) satisfy the condition \( V \subseteq P \subseteq \text{cl}(V) \) where \( P \subseteq X \), then \( P \) is said to be regular semi-open. The family of regular semi-open is denoted by \( \text{RSO}(X) \).

Lemma 2.5 [10] \( X \setminus P \) is also regular semi-open, if \( P \) is regular semiopen in \( (X, \tau_{A}) \).

3. \( \text{rw}^{*} \)-closed sets in Alexandroff Spaces
This section defines \( \text{rw}^{*} \)-closed sets in alexandroff and study various properties of it.

Definition 3.1 A set \( P \subseteq (X, \tau_{A}) \) is known as regular weakly star-closed (rw*-closed) if there exist \( S \), a closed set such that \( P \subseteq S \subseteq V \) whenever \( P \subseteq V \) and \( V \) is regular semi-open in \( (X, \tau_{A}) \).
\( \text{RW}^{*}C(X) \) represents the collection of all \( \text{rw}^{*} \)-closed sets.

Theorem 3.2 [12] Every w-closed set in \( X \) always implies \( \text{rw}^{*} \)-closed but not conversely.

From the point that each regular semi-open set is semi-open. Conversely, it is not true as seen in the following illustration:

Example 3.3 Suppose \( X = \mathbb{R} \setminus \mathbb{Q} \) and \( \tau_{A} = \{X, \emptyset, H_{1}\} \). Clearly, \( (X, \tau_{A}) \) is not a space in general topology. Let \( P \) be the set containing all irrational numbers of the interval \([0, 1]\). Then \( P \) is not open since the set of irrational numbers neither closed nor open and hence not semi-open which implies it is not w-closed but it is \( \text{rw}^{*} \)-closed set as the only regular semi-open as well as closed set is \( X \) that contains \( P \).

Remark 3.4 Every closed set is \( \text{rw}^{*} \)-closed but the converse of this is not true which can be seen by the given illustration:

Example 3.5 Suppose \( X = \mathbb{R} \setminus \mathbb{Q} \) and \( \tau_{A} = \{X, \emptyset, H_{1}\} \), where \( H_{1} \) is the collection of all countable subsets of \( X \). So, \( (X, \tau_{A}) \) is not a space in general topology. Now, let \( P \) be the set containing all irrational numbers of interval \([0, 1]\) and clearly, it is \( \text{rw}^{*} \)-closed set as the only regular semi-open and closed set that contains \( P \) is \( X \) but \( P \) is not closed.

Theorem 3.6 The subset \( P \) is \( \text{rw}^{*} \)-closed in \( X \) if it is regular generalized closed and regular open.
Let $\mathcal{V}$ be any regular semi-open set s.t $\mathcal{P} \subseteq \mathcal{V}$. Since $\mathcal{P}$ is regular open and regular generalized closed, $\mathcal{P} \subseteq \mathcal{P}$. Thus, there exist closed set $\mathcal{S}$ s.t $\mathcal{P} \subseteq \mathcal{S} \subseteq \mathcal{V}$ wherever $\mathcal{P} \subseteq \mathcal{V}$ and $\mathcal{V}$ is regular semi-open. Hence, $\mathcal{P}$ is $\text{rw}^*$-closed set in $\mathcal{X}$.

**Remark 3.7** In an Alexandroff topological spaces, every regular semi-open sets is not semi-open.

**Example 3.8** Let $\mathcal{X} = \mathbb{R} \setminus \mathcal{Q}$. And topology $\tau_A = \{\mathcal{X}, \emptyset, H_i\}$. Let $\mathcal{P}$ be the set of all irrationals in interval $(0,1)$. Since $\mathcal{P}$ is uncountable, so it is not open and hence not semi-open but clearly, it is regular semi-open.

**Theorem 3.9** If $\mathcal{P}$ and $\mathcal{Q}$ are $\text{rw}^*$-closed sets, then $\mathcal{P} \cup \mathcal{Q}$ is also $\text{rw}^*$-closed set in $\mathcal{X}$.

Let $\mathcal{V}$ be a regular semi-open s.t $\mathcal{P} \cup \mathcal{Q} \subseteq \mathcal{V}$. Then, $\mathcal{P} \subseteq \mathcal{V}$ and $\mathcal{Q} \subseteq \mathcal{V}$. Since $\mathcal{P}$ and $\mathcal{Q}$ are $\text{rw}^*$-closed sets, there exist closed set $\mathcal{S}$ s.t $\mathcal{P} \subseteq \mathcal{S} \subseteq \mathcal{V}$ and $\mathcal{Q} \subseteq \mathcal{L} \subseteq \mathcal{V}$. Hence, $\mathcal{P} \cup \mathcal{Q} \subseteq \mathcal{S} \cup \mathcal{L}$. That is, there exist closed set $\mathcal{W}$ s.t $\mathcal{P} \cup \mathcal{Q} \subseteq \mathcal{S} \cup \mathcal{L} = \mathcal{W} \subseteq \mathcal{V}$ wherever $\mathcal{P} \cup \mathcal{Q} \subseteq \mathcal{V}$ and $\mathcal{V}$ is regular semi-open in $(\mathcal{X}, \tau_A)$.

**Remark 3.10** Generally, the intersection of two $\text{rw}^*$-closed sets is not $\text{rw}^*$ in Alexandroff spaces.

Follows from example 3.5 [12].

**Theorem 3.11** A subset $\mathcal{P}$ of $\mathcal{X}$ is $\text{rw}^*$-closed set, if there exist $\mathcal{S}$ containing $\mathcal{P}$ s.t $\mathcal{S} \setminus \mathcal{P}$ doesn’t contain any non-void regular semi-open set in $\mathcal{X}$.

Suppose $\mathcal{P}$ be an $\text{rw}^*$-closed set. Then, by definition, there exists $\mathcal{S}$ such that $\mathcal{P} \subseteq \mathcal{S} \subseteq \mathcal{V}$ whenever $\mathcal{P} \subseteq \mathcal{V}$ and $\mathcal{V}$ is regular semi-open in $\mathcal{X}$. Let $\mathcal{U}$ be a regular semi-open set contained in $\mathcal{S} \setminus \mathcal{P}$ and $\mathcal{U}$ is non-empty. Now, $\mathcal{U} \subseteq \mathcal{S} \setminus \mathcal{P}$ implies $\mathcal{U} \subseteq \mathcal{X} \setminus \mathcal{P}$. Thus, $\mathcal{P} \subseteq \mathcal{X} \setminus \mathcal{U}$. Since $\mathcal{U}$ is regular semi-open, then $\mathcal{X} \setminus \mathcal{U}$ is also regular semi-open (by lemma 2.5). Also, $\mathcal{P}$ is $\text{rw}^*$-closed set, there exist $\mathcal{S}$ such that $\mathcal{S} \subseteq \mathcal{X} \setminus \mathcal{U}$ whenever $\mathcal{P} \subseteq \mathcal{X} \setminus \mathcal{U}$ and $\mathcal{X} \setminus \mathcal{U}$ is regular semi-open. So, $\mathcal{U} \subseteq \mathcal{X} \setminus \mathcal{S}$. Also, $\mathcal{U} \subseteq \mathcal{X} \setminus \mathcal{P}$. Thus, $\mathcal{U} \subseteq (\mathcal{X} \setminus \mathcal{S}) \cap (\mathcal{S} \setminus \mathcal{P}) = \emptyset$, which is a contradiction to the fact that $\mathcal{U}$ is non-empty. Hence, the proof.

**Theorem 3.12** In $(\mathcal{X}, \tau_A)$, $\mathcal{X} \setminus p$ is regular semi-open or $\text{rw}^*$-closed, for an element $p \in \mathcal{X}$.

Suppose $\mathcal{X} \setminus p$ is not regular semi-open. Thus, the only regular semi-open containing $\mathcal{X} \setminus p$ is $\mathcal{X}$ and there exist a closed set $\mathcal{S}$ s.t $\mathcal{X} \setminus p \subseteq \mathcal{S} \subseteq \mathcal{X}$. Hence, $\mathcal{X} \setminus p$ is an $\text{rw}^*$-closed set in $\mathcal{X}$.

**Theorem 3.13** In $(\mathcal{X}, \tau_A)$, a subset $\mathcal{P}$ is regular closed if $\mathcal{P}$ is regular open and $\text{rw}^*$-closed and hence it is clopen.

It is given that $\mathcal{P}$ is regular open and $\text{rw}^*$ in $\mathcal{X}$. Since $\mathcal{P} \subseteq \mathcal{P}$ and each regular open set is regular semi-open. Hence, there exist $\mathcal{S}$ such that $\mathcal{S} \subseteq \mathcal{P}$ whenever $\mathcal{P} \subseteq \mathcal{P}$ and $\mathcal{P}$ is regular semi-open. Also, $\mathcal{S}$ containing $\mathcal{P}$. Thus, $\mathcal{P} = \mathcal{S}$ which means that $\mathcal{P}$ is closed. Since $\mathcal{P}$ is regular open, then $\mathcal{P}$ is open. Now, $(\mathcal{P}^o)^c = \mathcal{P} = \mathcal{P}$. Therefore, $\mathcal{P}$ is clopen.

**Theorem 3.14** $\mathcal{Q}$ is an $\text{rw}^*$-closed set in $\mathcal{X}$ if $\mathcal{P}$ is an $\text{rw}^*$-closed subset of $\mathcal{X}$ with the condition $\mathcal{P} \subseteq \mathcal{Q} \subseteq \overline{\mathcal{P}}$.

Let $\mathcal{Q} \subseteq \mathcal{V}$ and $\mathcal{V}$ is open. Then, $\mathcal{P} \subseteq \mathcal{V}$. Since, $\mathcal{P}$ is $\text{rw}^*$-closed, there exist $\mathcal{S}$, is a closed set containing $\mathcal{P}$ such that $\mathcal{S} \subseteq \mathcal{V}$. Now, $\overline{\mathcal{Q}} \subseteq \overline{\{\mathcal{P}\}} = \overline{\mathcal{P}} \subseteq \mathcal{V}$ and this shows that $\mathcal{Q}$ is $\text{rw}^*$-closed set in $\mathcal{X}$.

**Remark 3.15** Conversely the above result is not true in general. (Remark 3.4 [12])
Theorem 3.16 Suppose $\mathcal{P}$ is $rw^*$-closed in $(X, \tau_A)$. Then, $\mathcal{P}$ is closed iff there exist closed set $\mathcal{S}$ containing $\mathcal{P}$ s.t $\mathcal{S} \setminus \mathcal{P}$ is regular semi-open.

Suppose $\mathcal{P}$ is closed in $X$. Then, the closure of $\mathcal{P}$ is $\mathcal{P}$ itself and so $\overline{\mathcal{P}} \setminus \mathcal{P} = \emptyset$, which is regular semi-open in $X$. On the other part, suppose there exist closed set $\mathcal{S}$ containing $\mathcal{P}$ s.t $\mathcal{S} \setminus \mathcal{P}$ is regular semi-open. Since $\mathcal{P}$ is $rw^*$-closed set, then, $\mathcal{S} \setminus \mathcal{P}$ doesn’t contain any non-empty regular semi-open set, it follows from theorem 3.11. Hence, $\mathcal{S} \setminus \mathcal{P} = \emptyset$, thus $\mathcal{P}$ is closed in $X$.

Theorem 3.17 If $\mathcal{P} \subseteq (X, \tau_A)$ is regular semi-open and $rw^*$-closed, then $\mathcal{P}$ is closed.

The proof is directly from theorem 3.11 [12].

Corollary 3.18 Let $\mathcal{P}$ is regular semi-open and $rw^*$-closed set and $\mathcal{S}$ be closed in $X$. Then, $\mathcal{P} \cap \mathcal{S}$ is an $rw^*$-closed set in $X$.

Suppose $\mathcal{P}$ be a regular semi-open and $rw^*$-closed set and $\mathcal{S}$ be closed in $X$. By above theorem, $\mathcal{P}$ is also closed and so $\mathcal{P} \cap \mathcal{S}$ is closed and hence $\mathcal{P} \cap \mathcal{S}$ is $rw^*$-closed set.

Theorem 3.19 Suppose $\mathcal{Q} \subseteq \mathcal{P}$ where $\mathcal{P}$ is $rw^*$-closed as well as regular semi-open. Thus, $\mathcal{Q}$ is $rw^*$-closed relative to $\mathcal{P}$ iff $\mathcal{Q}$ is $rw^*$-closed.

Since $\mathcal{P}$ is $rw^*$-closed set, then there exist $\mathcal{S}$, closed set, containing $\mathcal{P}$ s.t $\mathcal{S} \subseteq \mathcal{V}$ wherever $\mathcal{P} \subseteq \mathcal{V}$ and $\mathcal{V}$ is regular semi-open in $X$. Now, $\mathcal{P} \subseteq \mathcal{P}$ and $\mathcal{P}$ is regular semi-open, so $\mathcal{S} \subseteq \mathcal{P}$. That is, $\mathcal{P} = \mathcal{S}$ and so $\mathcal{P}$ is closed. Further, suppose $\mathcal{Q}$ is $rw^*$-closed closed. Then, there exist $\mathcal{S}_1$ which shows the $rw^*$-closedness of $\mathcal{Q}$. Since $\mathcal{P}$ is regular semi-open and $\mathcal{Q} \subseteq \mathcal{V}$ where $\mathcal{V}$ is regular semi-open in $\mathcal{P}$, so $\mathcal{V}$ is regular semi-open in $X$ and hence $\mathcal{S}_1 \subseteq \mathcal{V}$ which implies $\mathcal{Q}$ is $rw^*$-closed in $\mathcal{P}$.

On the other hand, let $\mathcal{Q}$ be a $rw^*$-closed in $\mathcal{P}$. Then, there exist closed set $\mathcal{S}_2$ in $\mathcal{P}$ which shows $\mathcal{Q}$ is $rw^*$ in $\mathcal{P}$. Since, $\mathcal{P}$ is closed, $\mathcal{S}_2$ is closed in $X$. Next, $\mathcal{Q} \subseteq \mathcal{V}_1$, $\mathcal{V}_1$ is regular semi-open in $X$, so $\mathcal{Q} \subseteq \mathcal{V}_1 \cap \mathcal{P}$ where $\mathcal{V}_1 \cap \mathcal{P}$ is regular semi-open and thus, $\mathcal{S}_2 \subseteq \mathcal{V}_1 \cap \mathcal{P} \subseteq \mathcal{V}_1$. Hence proved.

Theorem 3.20 In an Alexandroff space $(X, \tau_A)$, the family of regular semi-open sets $RSO(X, \tau_A) \subseteq \{ S \subset X : S^c \in \tau_A \}$ iff every subset of $(X, \tau_A)$ is $rw^*$-closed.

Let $RSO(X, \tau_A) \subseteq \{ S \subset X : S^c \in \tau_A \}$. Suppose $\mathcal{P}$ be any subset of $(X, \tau_A)$ such that $\mathcal{P} \subseteq \mathcal{V}$ and $\mathcal{V}$ is regular semi-open. Thus, $\mathcal{V} \in RSO(X, \tau_A) \subseteq \{ S \subset X : S^c \in \tau_A \}$ and hence, $\mathcal{V} \in \{ S \subset X : S^c \in \tau_A \}$ this implies $\mathcal{V}$ is closed. Thus, $\mathcal{V}$ is the closed set as well as regular semi-open set such that $\mathcal{P} \subseteq \mathcal{V}$. Hence, $\mathcal{P}$ is $rw^*$-closed set in $X$.

Conversely, suppose each subset in $(X, \tau_A)$ is $rw^*$-closed. Now, let $\mathcal{V} \in RSO(X, \tau_A)$. Since $\mathcal{P} \subseteq \mathcal{P}$ and $\mathcal{P}$ is $rw^*$-closed, then there exist closed set $\mathcal{V}$ such that $\mathcal{P} \subseteq \mathcal{V}$ where $\mathcal{V}$ is regular semi-open in $X$ and it is closed also. Thus, $\mathcal{V} \in \{ S \subset X : S^c \in \tau_A \}$. Therefore, $RSO(X, \tau_A) \subseteq \{ S \subset X : S^c \in \tau_A \}$.

Definition 3.21 Regular star semi-kernel is defined as the intersection of every regular semi-open subsets that contains $\mathcal{P}$. It is denoted by $r^*sker(\mathcal{P})$.

Theorem 3.22 In $(X, \tau_A)$ $\mathcal{P}$ is $rw^*$-closed iff $\mathcal{P} \subseteq S$, a closed set s.t $S \subseteq \mathcal{V}$, wherever $\mathcal{P} \subseteq \mathcal{V}$, and $\mathcal{V}$ is regular semi-open, that is, $S \subseteq r^*sker(\mathcal{P})$.

Firstly, suppose that $\mathcal{P}$ is $rw^*$-closed. Then, $\mathcal{P} \supseteq S$, a closed set s.t $S \subseteq \mathcal{V}$, wherever $\mathcal{P} \subseteq \mathcal{V}$ where $\mathcal{V}$ is regular semi-open. Let $x \in S \subseteq \mathcal{V}$. Let $x \not\in r^*sker(\mathcal{P})$, then there exist regular semi-open set $\mathcal{V}$ containing $\mathcal{P}$ s.t $x \not\in \mathcal{V}$. Since $\mathcal{P}$ is $rw^*$-closed, $\mathcal{V} \subseteq \mathcal{V}$, it implies $x \not\in S \subseteq \mathcal{V}$, which is a contradiction. Hence, $x \in r^*sker(\mathcal{P})$ and thus $S \subseteq r^*sker(\mathcal{P})$.

Conversely, suppose $S \subseteq \mathcal{V}$ and $S \subseteq r^*sker(\mathcal{P})$. Suppose $\mathcal{V}$ is regular semi-open set containing $\mathcal{P}$, then, $r^*sker(\mathcal{P}) \subseteq \mathcal{V}$. Then, $S \subseteq r^*sker(\mathcal{P}) \subseteq \mathcal{V}$ which implies $\mathcal{P}$ is $rw^*$-closed.
Definition 3.23 A subset \( P \) in \((X, \tau_A)\) is known to be regular \( w^* \)-open in \( X \) if the complement of \( P \) is \( rw^* \)-closed in \((X, \tau_A)\).

Theorem 3.24 A set is \( rw^* \)-open iff there exist regular semi-open set \( V \) contained in \( P \) s.t \( S \subseteq V \) and \( S \subseteq P \) wherever \( S \) is closed.

The solution is left to the readers.

References
[1] Stone Marshall Harvey 1937 Transactions of the American Mathematical Society 41.3 pp 375-481.
[2] Alexandroff A. D. 1940 Mathematical collection 8.2 pp 307-348, 8.2 pp 307-348.
[3] Levine Norman 1970 Rendiconti del Circolo Matematico di Palermo 19.1 pp 89-96.
[4] Sivaraj D. 1984 Acta Mathematica Hungarica 44.3-4 pp 207-213.
[5] Tong Jingcheng 1989 Acta Mathematica Hungarica 54.1-2 pp 51-55.
[6] Kong T. Yung, Ralph Kopperman, and Paul R. Meyer 1991 The American mathematical monthly 98.10 pp 901-917.
[7] Palaniappan N. 1993 Kyungpook Math. J. 33.2 pp 211-219.
[8] Arenas Francisco G. 1999 Acta Math. Univ. Comenianae 68.1 pp 17-25.
[9] Sundaram P. and M. Sheik John 2000 Acta Ciencia Indica Mathematics 26.4 pp 389-392.
[10] Pushpalatha A. 2000 Studies on generalizations of mappings in topological spaces.
[11] Das Pratulananda, Md Rashid and Mamun Ar. 2003 Archivum Mathematicum 39.4 pp 299-307.
[12] Benchalli S. S. and R. S. Wali 2007 Bulletin of the Malaysian Mathematical Sciences Society 30.2.
[13] Banerjee Amar Kumar and Jagannath Pal 2016 arXiv preprint arXiv 1609.05150.
[14] Martinez Luis. 2016 A Generalization of the Alexandrov and Path Topologies of Spacetime via Linear Orders.
[15] Harvey John and Catherine Searle 2017 The Journal of Geometric Analysis 27.2 pp 1636-66.