P-ADIC FAMILY OF HALF-INTEGRAL WEIGHT MODULAR FORMS VIA
OVERCONVERGENT SHINTANI LIFTING

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Abstract. The classical Shintani map (see [Shn]) is the Hecke-equivariant map from the space of
cusp forms of integral weight to the space of cusp forms of half-integral weight. In this paper, we
will construct a Hecke-equivariant overconvergent Shintani lifting which interpolates the classical
Shintani lifting $p$-adically, following the idea of G. Stevens in [St1]. In consequence, we get a
formal $q$-expansion $\Theta$ whose $q$-coefficients are in an overconvergent distribution ring, which can be
thought of $p$-adic analytic family of overconvergent modular forms of half-integral weight, since the
specializations of $\Theta$ at the arithmetic weights are the classical cusp forms of half-integral weight.

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0.1. Introduction.

The space of modular forms of half-integral weight (see [Shn2] for the precise definition) have
been studied by many people. One of the main reasons for this would be the facts that their
fourier coefficients are rich in arithmetic. For example, the representation numbers of positive
definite integral quadratic forms of odd dimensions arise as fourier coefficients of theta functions
of half-integral weights. H. Cohen showed that his generalized class numbers are the fourier co-
efficients of Eisenstein series of half-integral weight (see [Coh]). Moreover, J. Waldspurger had
shown (see [Wa]) that the square of the fourier coefficients of new cusp forms of half integral weight
(see [Koh] for the theory of new forms in the half-integral case) are closely connected to the special
values of $L$-function of corresponding integral weight cusp forms under the Shimura correspondence.

It was Shimura who initiated the program which relates modular forms of integral weight with
modular forms of half-integral weight. He defined the Hecke actions on half-integral weight modular
forms and constructed integral weight modular forms from half-integral modular forms by using Hecke operators and functional equations of Dirichlet L-functions associated to concerning half-integral and integral weight modular forms, which comes from explicit transformation formulas of some theta functions. After Shimura’s work, Shintani (Shn) gave an inverse construction to the Shimura’s map, where he used the Weil representation to construct the half-integral weight cusp form \( \theta(f) \) from an integral weight cusp form \( f \). Furthermore, he expressed the fourier coefficients of \( \theta(f) \) as some period integrals of \( f \), hence can be described in terms of a certain cohomology class associated to \( f \) as in Shimura’s work (see the chapter 8 of [Shm1]). G. Stevens realized that if we have a \( \Lambda \)-adic cohomology class which interpolates the \( p \)-stabilized ordinary newforms then it gives rise to a \( \Lambda \)-adic version of Shintani lifting for Hida’s universal ordinary \( \Lambda \)-adic modular forms (see [St1]). The main goal of this paper is to generalize his result to the finite slope non-ordinary case, i.e. Robert Coleman’s \( p \)-adic analytic family of overconvergent cuspidal eigenforms of finite slope.

In section 1, we recall the definition of overconvergent modular symbols which is a compact supported group cohomology of a congruence subgroup of \( SL_2(\mathbb{Z}) \) with values in overconvergent distribution and explain how it interpolates the classical cusp forms by the specialization maps at arithmetic weights, i.e. how a cohomological approach to \( p \)-adic family of modular forms works. The more precise dictionary between Robert Coleman’s \( p \)-adic family of overconvergent eigenforms and overconvergent Hecke eigensymbol (see the theorem 3.17) will be given in section 3. Note that before the final section 3, we won’t concentrate on Hecke eigenforms; instead, we deal with modular forms which are not necessarily Hecke eigenforms.

In section 2, we give a purely cohomological formulation of the Shintani lifting following G. Stevens’ work ([St1]), after reviewing the Shimura correspondence, especially the Shintani’s construction of half-integral weight cusp forms from integral weight cusp forms. Then \( p \)-adic lifting of the cohomological Shintani lifting will be given (see the definition 2.10), which we call overconvergent Shintani lifting. Its \( p \)-adic interpolation property essentially asserts that the following diagram commutes:

\[
\begin{array}{ccc}
H^1_c(\Gamma_0(Np), D_N) & \xrightarrow{\theta} & \tilde{D}(\mathbb{Z}_{p,N}^\times)[[q]] \\
\phi_{\kappa,*} & & T_{p^{-1}\kappa} \\
H^1_c(\Gamma_0(Np^m), L_{2k,\chi^2}(R_\kappa)) & \xrightarrow{\Theta_{p,\kappa}} & \mathcal{R}_\kappa[[q]]
\end{array}
\]

where all the notations will be provided in the paper. Furthermore, we describe how the natural Hecke action on overconvergent modular symbol is transformed under the overconvergent Shintani lifting (see the theorem 2.11). The main theorem of section 2 2.11 implies that we can define Hecke actions (see the definitions 2.21 and 2.22) on the space of formal \( q \)-expansions with coefficients in an overconvergent distribution ring so that the overconvergent Shintani lifting is Hecke-equivariant. The important feature of the cohomological (overconvergent) Shintani lifting is that it’s purely cohomological and algebraic which depends only on the arithmetic of integral indefinite binary quadratic forms.

In section 3, we state and prove our main theorem (the theorem 3.28) which asserts the existence of formal \( q \)-expansion \( \Theta \in \tilde{\mathcal{R}}_{h,N}[[q]] \) whose specializations to classical arithmetic points are, up to multiplication by scalars, the \( q \)-expansions of Hecke eigenforms of half-integral weight (in the sense of Shimura [Shm2]), which is obtained via the classical Shintani’s map from the cuspidal Hecke eigenforms of tame conductor \( N \) with finite slope \( \leq h \). In contrast to the ordinary case, the existence of Hecke eigensymbol in \( H^1_c(\Gamma_0(Np), D_N) \) over the whole weight space \( X_N \) is not guaranteed in
the non-ordinary situation. But if we shrink the weight space small enough, then we can prove the existence of Hecke eigensymbol over that small affinoid domain. More precisely, G. Stevens proved that there exists an affinoid subdomain $B_{h,N}$ in $X_N$ such that $H^1_c(\Gamma_0(Np), \mathcal{D}_{B_{h,N}})$ admits a slope $\leq h$ decomposition (see the theorem 3.7) and we prove the existence of Hecke eigensymbol in the slope $\leq h$ part $H^1_c(\Gamma_0(Np), \mathcal{D}_{B_{h,N}})^{(\leq h)}$ (see the theorem 3.17). After introducing overconvergent $p$-adic Hecke algebra for the slope $\leq h$ part $H^1_c(\Gamma_0(Np), \mathcal{D}_{B_{h,N}})^{(\leq h)}$, we define a local overconvergent Shintani lifting over $B_{h,N}$ and prove its Hecke-equivariance and the main theorem as a simple application of tools developed in the paper.

It should be mentioned that H. Hida constructed a $\Lambda$-adic Shimura lifting in \cite{Hi2} which is an inverse to G. Stevens’ $\Lambda$-adic Shintani lifting (in \cite{St1}) which we are generalizing to the finite slope non-ordinary case in our paper. Also, Nick Ramsey constructed half-integral weight eigencurve and studied overconvergent Shintani lifting in \cite{Ram1} and \cite{Ram2}. The overconvergent Shintani lifting here can be thought of an inverse to overconvergent Shimura lifting. It would be worthwhile to write down a local (possibly global) rigid analytic map from Coleman-Mazur integral weight eigencurve (see \cite{Col-Mz} for the tame level 1 and \cite{Buz} for the higher tame level $N$) to half-integral weight eigencurve (see \cite{Ram2}), in the rigid analytic geometric language used in N. Ramsey’s paper \cite{Ram2}, though, in some sense, our overconvergent Shintani lifting answers what it should be.

It’s worthwhile to note that the recent work of Bertolini, Darmon, and Tornaria about Hida families and Shimura lifts (see \cite{BDT}) tells us that the $p$-adic derivative of a certain linear combination of the fourier coefficients of a $\Lambda$-adic cusp form of half-integral weight, evaluated at some arithmetic point, are closely connected to the Stark-Heegner points on the elliptic curve attached to a weight 2 cusp form which moves $p$-adically in the corresponding $\Lambda$-adic cusp form of integral weight under $\Lambda$-adic Shimura lifting. The fourier coefficients of the universal overconvergent half-integral weight modular forms (see the definitions \cite{B1} and \cite{B2}) might subject to the similar story so that the $p$-adic derivative of their certain linear combination could be closely related to the Heegner cycles in the non-ordinary and higher weight $> 2$ case. It would certainly be a good project to develop an analogous theory in the non-ordinary and higher weight $> 2$ case, following them.

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1. Cohomological approach to Coleman’s $p$-adic family of overconvergent modular forms

Throughout the paper, we fix a prime $p \geq 5$ and a positive integer $N$ with $(p, N) = 1$; the "quadratic residue symbol" $(\frac{a}{b})$ has the same meaning as in \cite{Shm2}. We use the notations $\Delta_N$ and $(\mathbb{Z}/N\mathbb{Z})^\times$ interchangeably. Let $K$ be any finite extension field of $\mathbb{Q}_p$ and $| \cdot |$ be the complete non-archimedean absolute value of $K$ normalized by $| p | = p^{-1}$. Let $C_p$ denote the $p$-adic completion of an algebraic closure of $\mathbb{Q}_p$. We fix, once and for all, a field isomorphism between $\mathbb{C}$ and $C_p$. Robert Coleman constructed a $p$-adic family of overconvergent modular forms in \cite{Col1}. It seems to be hard at the moment to define an overconvergent Shintani lifting directly from the Coleman’s way of describing the $p$-adic family. But if we approach Coleman’s $p$-adic family in a cohomological way (using overconvergent modular symbols) as developed by G. Stevens, then there is a natural way to get an overconvergent Shintani lifting which interpolates the classical Shintani $\theta$-lifts in \cite{Shm}. We start from a locally analytic distribution space which will be the coefficient ring of our compact supported cohomology for an arithmetic group and then we describe overconvergent
modular symbols with Hecke action on them and how they interpolate classical modular forms by the specialization maps.

1.1. Overconvergent modular symbols.

The goal of this section is to introduce overconvergent modular symbols and the Hecke action on them. Let $S$ be a compact subset of $\mathbb{Q}_p$ or $\mathbb{Q}_p \times \mathbb{Q}_p$. We define $\mathcal{A}(S)$ as the space of $K$-valued locally analytic functions on $S$. We also define $\mathcal{D}(S)$ as the space of $K$-valued locally analytic distributions on $S$, i.e. $\mathcal{D}(S) := \text{Hom}_{\text{cts}}(\mathcal{A}(S), K)$ with the strong topology. We will use the notation $\mathcal{A}$ and $\mathcal{D}$, when $S = \mathbb{Z}_p^\times \times \mathbb{Z}_p$, i.e. $\mathcal{A} = \mathcal{A}(\mathbb{Z}_p^\times \times \mathbb{Z}_p)$ and $\mathcal{D} = \mathcal{D}(\mathbb{Z}_p^\times \times \mathbb{Z}_p)$. Then $\mathcal{A}$ is a reflexive and complete Hausdorff locally convex $K$-vector space whose strong dual $\mathcal{D}$ is a Fréchet space. We refer to §16 of [Sch] for the detailed definitions.

For any positive integer $M$, let

$$
\Gamma_0(M) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \mid c \equiv 0 \pmod{M} \right\}
$$

be the congruence subgroup of $SL_2(\mathbb{Z})$. Then $\Gamma_0(Np)$ acts on $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ by matrix multiplication on the right, viewing the elements of $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ as row vectors. So it induces a left action on $\mathcal{A}$ by $(\gamma \cdot f)(x, y) := f((x, y) \cdot \gamma)$ where $(x, y) \in \mathbb{Z}_p^\times \times \mathbb{Z}_p$ and $f \in \mathcal{A}$. Consequently it induces a right action of $\Gamma_0(Np)$ on $\mathcal{D}$ which is uniquely characterized by the following integration formula:

$$
(1.1) \quad \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} f(x, y)d(\mu|\gamma)(x, y) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} (\gamma \cdot f)(x, y)d\mu(x, y).
$$

We refer to this action on $\mathcal{D}$ as the dual action induced from action on $\mathcal{A}$. The scalar action of $\mathbb{Z}_p^\times$ on $\mathbb{Z}_p^\times \times \mathbb{Z}_p$ induces a left action of $\mathcal{D}(\mathbb{Z}_p^\times)$ on $\mathcal{A}$ by

$$
(1.2) \quad (\nu \cdot f)(x, y) := \int_{\mathbb{Z}_p^\times} f(x \cdot \lambda, y \cdot \lambda)d\nu(\lambda)
$$

for $\nu \in \mathcal{D}(\mathbb{Z}_p^\times)$ and $f \in \mathcal{A}$.

Hence this induces a right action of $\mathcal{D}(\mathbb{Z}_p^\times)$ on $\mathcal{D}$ by the dual action, i.e. by the following formula:

$$
(1.3) \quad \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} f(x, y)d(\mu|\nu)(x, y) = \int_{\mathbb{Z}_p^\times \times \mathbb{Z}_p} (\nu \cdot f)(x, y)d\mu(x, y)
$$

for $\mu \in \mathcal{D}$ and $\nu \in \mathcal{D}(\mathbb{Z}_p^\times)$. The action of $\Gamma_0(Np)$ clearly commutes with the action of $\mathcal{D}(\mathbb{Z}_p^\times)$. Henceforth we may consider $\mathcal{D}$ as a $\mathcal{D}(\mathbb{Z}_p^\times)[\Gamma_0(Np)]$-module.

Now we define the group of $\mathcal{D}$-valued modular symbols over $\Gamma_0(Np)$ whose elements we will call overconvergent modular symbols (over $\Gamma_0(Np)$).

**Definition 1.1.** We define the group of $\mathcal{D}$-valued modular symbol over $\Gamma_0(Np)$ to be

$$
(1.4) \quad \text{Symb}_{\Gamma_0(Np)}(\mathcal{D}) := \text{Hom}_{\Gamma_0(Np)}(\Delta_0, \mathcal{D})
$$

where $\Delta_0 := \text{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ is the group of divisors of degree 0 supported on the rational cusps $\mathbb{P}^1(\mathbb{Q})$ of the upper half plane.

In other words, $\Phi \in \text{Symb}_{\Gamma_0(Np)}(\mathcal{D})$ is an additive homomorphism $\Phi : \Delta_0 \to \mathcal{D}$ for which $\Phi|\gamma = \Phi$ for all $\gamma \in \Gamma_0(Np)$ where the action of $\gamma \in \Gamma_0(Np)$ on $\text{Hom}(\Delta_0, \mathcal{D})$ is given by

$$
(1.5) \quad (\Phi|\gamma)(D) := \Phi(\gamma \cdot D)|\gamma
$$

for $D \in \Delta_0$.  

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We refer to [Shm1] or the appendix of [Hi] for the definition of compactly supported cohomology $H^1_c(\Gamma_0(Np), D)$.

It turns out there exists a canonical isomorphism (see [A-S1] for the proof)

\[
H^1_c(\Gamma_0(Np), D) = \text{Symbr}_0(Np)(D).
\]

Henceforth we will use the terms $D$-valued degree one compactly supported cohomology and over-convergent modular symbol interchangeably.

Consider the natural map

\[
H^1_c(\Gamma_0(Np), D) \to H^1(\Gamma_0(Np), D)
\]

defined by sending a overconvergent modular symbol $\varphi$ to the cohomology class represented by the 1-cocycle $\gamma \mapsto \varphi(\{ \gamma x \} - \{ x \})$ for any fixed $x \in \mathbb{P}^1(\mathbb{Q}_p)$ (it is independent of the choice of fixed element $x$). Then the parabolic cohomology (see [Shm1] or the appendix of [Hi] for the definition of parabolic cohomology) is canonically isomorphic to the image of the above natural map:

\[
H^1_c(\Gamma_0(Np), D) \to H^1(\Gamma_0(Np), D) \to H^1(\Gamma_0(Np), D).
\]

Remark 1.2. If we replace $Np$ and $D$ by any positive integer $M$ and any right $\mathbb{Z}[\Gamma_0(M)]$-module respectively, (1.4)-(1.7) still remains valid.

We now give the Hecke-module structure of these cohomology groups. Let $S_0(M)$ denote the semi-group

\[
S_0(M) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2^+(\mathbb{Z}) \mid c \equiv 0 \pmod{M}, \ (a, M) = 1 \right\}
\]

where $M$ is a positive integer and $M_2^+(\mathbb{Z})$ is the semi-group of integral $2 \times 2$ matrices with positive determinant. Let $R(\Gamma_0(M), S_0(M))$ be the $\mathbb{Z}$-module generated by the double cosets $\Gamma_0(M)\alpha \Gamma_0(M), \alpha \in S_0(M)$. This can be equipped with a ring structure by defining multiplication between two double cosets. $R(\Gamma_0(M), S_0(M))$ with this multiplication law extended $\mathbb{Z}$-linearly, becomes a commutative ring with $\Gamma_0(M) = \Gamma_0(M) \cdot 1 \cdot \Gamma_0(M)$ as the unit element. It is called a Hecke ring with respect to a Hecke pair $(\Gamma_0(M), S_0(M))$.

For each positive integer $n$, denote by $T_n$ the formal sum of all double cosets $\Gamma_0(M)\alpha \Gamma_0(M)$ with $\alpha \in S_0(M)^n := \{ \alpha \in S_0(M) \mid \det(\alpha) = n \}$ in $R(\Gamma_0(M), S_0(M))$. For example, $T_p = \Gamma_0(M) \left( \begin{array}{cc} 1 & 0 \\ 0 & p \end{array} \right) \Gamma_0(M)$ for every prime $p$. For primes $p$ satisfying $(p, M) = 1$, let

\[
T_{p,p} = \Gamma_0(M) \left( \begin{array}{cc} p & 0 \\ 0 & p \end{array} \right) \Gamma_0(M).
\]

The structure of $R(\Gamma_0(M), S_0(M))$ is given by the following theorem [Shm1, Theorem 3.34]:

**Proposition 1.3.** (1) $R(\Gamma_0(M), S_0(M))$ is a polynomial ring over $\mathbb{Z}$ in the variables $T_{p,p}$ for all primes $p$ not dividing $M$ and $T_p$ for all primes $p$.

(2) $R(\Gamma_0(M), S_0(M)) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ is a polynomial ring over $\mathbb{Q}_p$ in the variables $T_n$ for all $n > 0$, i.e.

\[
R(\Gamma_0(M), S_0(M)) \otimes_{\mathbb{Z}} \mathbb{Q}_p = \mathbb{Q}_p[T_n : n \in \mathbb{N}]
\]
where \( \chi \) is the set of representatives for \( \Gamma_0(M) \backslash \Gamma_0(M) \alpha \Gamma_0(M) \). Since the action of \( \Gamma_0(M) \) on \( D \) extends to an action of \( S_0(M) \) on \( D \) by the same formula (1.1) \( \text{Hom}(\Delta_0, D) \) is a right \( S_0(M) \)-module. Therefore \( H^1_c(\Gamma_0(M), D) = \text{Hom}_{\Gamma_0(M)}(\Delta_0, D) \) can be viewed as \( R_K \)-module by the action described above, where \( R_K = R(\Gamma_0(M), S_0(M)) \otimes_{\mathbb{Z}} K \). In particular, if \( p \mid M \), then \( U_p \) acts on \( H^1_c(\Gamma_0(M), D) \) by

\[
(1.11) \quad \Phi|U_p = \sum_{i=1}^{p} \Phi\left( \frac{1}{p} \begin{pmatrix} i \\ 0 \end{pmatrix} \right),
\]

for \( \Phi \in H^1_c(\Gamma_0(M), D) \).

Because \( D \) has an action of \( S_0(M) \) by the same formula (1.1), we can also define the \( R_K \)-actions on \( H^1(\Gamma_0(M), D) \) and \( H^1_{\text{par}}(\Gamma_0(M), D) \). In fact the maps in the sequence (1.17) are \( R_K \)-module homomorphisms, i.e. Hecke-equivariant homomorphism.

The matrix \( \iota := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) induces natural involutions on \( H^1_c(\Gamma_0(M), D) \) and \( H^1(\Gamma_0(M), D) \). The involution \( \iota \) decomposes each of cohomology groups as \( \pm \) eigenspaces:

\[
(1.12) \quad H = H^+ \oplus H^-.
\]

We know that each cohomology class \( \varphi \) decomposes as \( \varphi = \varphi^+ + \varphi^- \) where \( \varphi^\pm := \frac{1}{2}(\varphi \pm \varphi|\iota) \) and \( \varphi^{\pm}|\iota = \pm \varphi. \)

Let \( R \) be any commutative ring with unity. We recall the definition of \( L_{k,\chi}(R) \) and \( L^*_{k,\chi}(R) \), where \( \chi \) is an \( R \)-valued Dirichlet character of \( (\mathbb{Z}/M\mathbb{Z})^* \) for some positive integer \( M \), from 4.1 of [SSL]. For any integer \( k \geq 0 \) we define \( \text{Sym}^k(R^2) \) (respectively, \( \text{Sym}^k(R^2)^* \)) as the free \( R \)-module generated by the divided powers monomials \( \sum_{n=0}^{\infty} X^Y \cdot Y^{k-n} \) (respectively, the monomials \( X^n \cdot Y^{k-n} \)) for \( 0 \leq k \leq n \). Then \( L_{k,\chi}(R) \) (respectively, \( L^*_{k,\chi}(R) \)) is \( R[\Gamma_0(M)] \)-module whose underlying \( R \)-module is \( \text{Sym}^k(R^2) \) (respectively, \( \text{Sym}^k(R^2)^* \)) equipped with the following action of \( S_0(M) \): for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_0(M), \)

\[
(F|\gamma)(X, Y) := \begin{cases} 
\chi(a) \cdot F((X, Y) \cdot \gamma) & \text{for } F \in L_{k,\chi}(R) \\
\chi(d) \cdot F((X, Y) \cdot \gamma^*) & \text{for } F \in L^*_{k,\chi}(R)
\end{cases}
\]

where \( \gamma^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \). There is a unique perfect \( R \)-linear pairing \( \langle \cdot, \cdot \rangle : \text{Sym}^k(R^2) \times \text{Sym}^k(R^2)^* \to R \) satisfying \( \langle X^Y Y^{k-i} X^i Y^j, X^{k-j} Y^j \rangle = (-1)^j \delta_{ij} \) where \( \delta_{ij} \) is the Kronecker delta. In particular, we have

\[
(1.13) \quad \langle \frac{(aY - bX)^k}{k!}, P(X, Y) \rangle = P(a, b)
\]

for every \( (a, b) \in R^2 \) and \( P \in \text{Sym}^k(R^2)^* \).
Let \( S_{k+2}(\Gamma_0(M), \chi) \) (respectively \( M_{k+2}(\Gamma_0(M), \chi) \)) be the space of weight \( k+2 \) cuspidal (respectively modular) forms of level \( \Gamma_0(M) \) with nebentype character \( \chi \) and \( S_{k+2}(\Gamma_0(M), \chi; R) \) the subspace of \( S_{k+2}(\Gamma_0(M), \chi) \) whose Fourier coefficient lie in \( R \) where \( R \) is a subfield of \( \mathbb{C} \). We recall the Eichler-Shimura isomorphism (see [Shm1])

\[
S_{k+2}(\Gamma_0(M), \chi) \to H^1_{par}(\Gamma_0(M), L_{k,\chi}(\mathbb{C}))^\pm.
\]

for a positive integer \( M \) and Dirichlet character \( \chi \) defined modulo \( M \).

By the Manin-Drinfeld principle, there is a Hecke-equivariant section to the natural map (1.7):

\[
H^1_{par}(\Gamma_0(M), L_{k,\chi}(R)) \to H^1_c(\Gamma_0(M), L_{k,\chi}(R)),
\]

\[
\Psi \mapsto \psi
\]

for a positive integer \( M \) and Dirichlet character \( \chi \) defined modulo \( M \), if \( R \) is the field of characteristic 0. Henceforth we can consider the element of parabolic cohomology \( H^1_{par}(\Gamma_0(M), L_{k,\chi}(R)) \) as a modular symbol with values in \( L_{k,\chi}(R) \).

### 1.2. \( p \)-adic interpolation of overconvergent modular symbols.

In this section we review how overconvergent modular symbols can be understood as \( p \)-adic families of modular forms (see [St3]). For that we recall the definition of (arithmetic) weights and introduce the specialization map for an arithmetic weight.

We refer to an element of \( \text{Hom}_{cts}(\mathbb{Z}_{p,N}^\times, \mathbb{C}_p^\times) \) as a weight. For \( t \in \mathbb{Z}_{p,N}^\times \) we let \( t_p \) and \( t_N \) be the coordinates of \( t \) under the canonical decomposition \( \mathbb{Z}_{p,N}^\times = \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \). Let \( \Lambda_N = \mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]] \).

Then there is a natural identification between

\[
\text{Hom}_{cts}(\Lambda_N, \mathbb{C}_p) \text{ and } \text{Hom}_{cts}(\mathbb{Z}_{p,N}^\times, \mathbb{C}_p^\times),
\]

where the first Hom denotes continuous algebra homomorphisms and second denotes continuous group homomorphisms.

**Definition 1.4.** A character(weight) \( \kappa \in \text{Hom}_{cts}(\mathbb{Z}_{p,N}^\times, \mathbb{C}_p^\times) \) is called arithmetic of signature of \((k,\chi)\), if it satisfies

\[
\kappa(t) = \chi(t) \cdot t_p^k
\]

for some \( \chi \), a finite order character of \( \mathbb{Z}_{p,N}^\times \), and \( k \in \mathbb{Z}_{\geq 0} \).

Note that we have a natural isomorphism between \( \mathcal{D}(\mathbb{Z}_{p,N}^\times) \) and \( \mathcal{D}(\mathbb{Z}_p^\times)[\Lambda_N] \). Accordingly, we refer to them interchangeably. Let’s define

\[
\mathcal{D}_N := \mathcal{D} \otimes_{\mathcal{D}(\mathbb{Z}_p^\times)} \mathcal{D}(\mathbb{Z}_{p,N}^\times),
\]

where \( \gamma \in S_0(Np) \) acts on \( \mathcal{D}_N \) as follows:

\[
(\mu \otimes_{\mathcal{D}(\mathbb{Z}_p^\times)} \nu) \cdot (\gamma) := \mu \cdot \gamma \otimes_{\mathcal{D}(\mathbb{Z}_p^\times)} [a(\gamma)]_N \cdot \nu,
\]

for \( \mu \in \mathcal{D}, \nu \in \mathcal{D}(\mathbb{Z}_{p,N}^\times) \) and \([a(\gamma)]_N \in \Delta_N \) (here \( a(\gamma) \) is the upper left entry of \( \gamma \)). So \( \mathcal{D}_N \) can be thought as \( \mathcal{D}(\mathbb{Z}_{p,N}^\times)[\Gamma_0(Np)] \)-module and \( H^1_c(\Gamma_0(Np), \mathcal{D}_N) \) is defined and has a natural \( \mathcal{D}(\mathbb{Z}_{p,N}^\times) \)-module structure.

**Definition 1.5.** We define the abstract overconvergent Hecke algebra of tame conductor \( N \) to be the free polynomial algebra

\[
\mathcal{H} := \mathcal{D}(\mathbb{Z}_{p,N}^\times)[T_n : n \in \mathbb{N}]
\]
generated by over \( \mathcal{D}(\mathbb{Z}_{p,N}^\times) \) by \( T_n \in R(\Gamma_0(Np), S_0(Np)) \) for \( n \in \mathbb{N} \).

Note that \( \mathcal{H} \cong R(\Gamma_0(Np), S_0(Np)) \otimes \mathbb{Z} \mathcal{D}(\mathbb{Z}_{p,N}^\times) \). If \( \kappa \in \text{Hom}_{\text{cts}}(\mathbb{Z}_{p,N}^\times, K^\times) \) is any arithmetic point of signature \((k, \chi)\), then let \( R_\kappa := \kappa(\mathcal{D}(\mathbb{Z}_{p,N}^\times)) \) where \( \kappa(r) := r(\kappa) \) for \( r \in \mathcal{D}(\mathbb{Z}_{p,N}^\times) \). We can define a \( R_\kappa \)-linear map, called a specialization map for \( \kappa \),

\[
\phi_\kappa : \mathcal{D}_N = \mathcal{D} \otimes \mathcal{D}(\mathbb{Z}_{p,N}^\times) \to L_{k,\chi}(R_\kappa) 
\]

by

\[
\phi_\kappa(\mu \otimes r) := \kappa(r) \cdot \int_{\mathbb{Z}_{p,N}^\times \times \mathbb{Z}_{p,N}} \chi_\mu(x) \frac{(xY - yX)^k}{k!} d\mu(x, y) 
\]

for \( \mu \in \mathcal{D} \) and \( r \in \mathcal{D}(\mathbb{Z}_{p,N}^\times) \), where we factor \( \chi = \chi_N \cdot \chi_p \) with \( \chi_N \) defined modulo \( N \) and \( \chi_p \) defined modulo a power of \( p \) and \( \kappa(r) := r(\kappa) \). A simple computation confirms that if \( \chi \) is defined modulo \( Np^m \) then \( \phi_\kappa \) is a \( \Gamma_0(Np^m) \)-module homomorphism, and hence induces a homomorphism on cohomology groups:

\[
\phi_{\kappa,*} : H^1_c(\Gamma_0(Np), \mathcal{D}_N) \to H^1_c(\Gamma_0(Np^m), L_{k,\chi}(R_\kappa)).
\]

The map \( \phi_{\kappa,*} \) is Hecke-equivariant. Since 1-cocycles in \( H^1_c(\Gamma_0(Np^m), L_{k,\chi}(R_\kappa)) \) can be viewed as classical modular forms via the Manin-Drinfeld principle and the Eichler-Shimura isomorphism, we can interpret an overconvergent modular symbol as a \( p \)-adic analytic family of modular forms by varying \( k \) and \( \chi \).

2. OVERCONVERGENT SHINTANI LIFTING

We first review the original Shintani lifting in [Shm] and its cohomological realization in [St1]. Then we give an overconvergent version of these constructions. We will prove the Hecke-equivariance of our overconvergent Shintani lifting and describe the Hecke action explicitly on formal \( q \)-expansions with coefficient in \( \mathcal{D}(\mathbb{Z}_{p,N}^\times) := \mathcal{D}(\mathbb{Z}_{p,N}^\times) \otimes \mathcal{D}(\mathbb{Z}_{p,N}) \otimes \mathcal{D}(\mathbb{Z}_{p,N}) \) (see [St1]), which is given as the image of the overconvergent Shintani lifting. This formal \( q \)-expansion can be viewed as the \( p \)-adic family of overconvergent half-integral weight modular forms.

2.1. Classical Shintani lifting and its cohomological interpretation.

We briefly summarize Shintani’s construction in [Shm] of a Hecke-equivariant map from integral weight cusp forms to half-integral weight cusp forms. We then review its cohomological version which is well-suited to \( p \)-adic variation of Shintani map, following [St1] whose results we generalize to the non-ordinary case. Let \( Q = Q(X, Y) = aX^2 + bXY + cY^2 \) be an integral (meaning \( a, b, c \in \mathbb{Z} \)) binary quadratic form. We define the discriminant of \( Q \) as \( \delta_Q := b^2 - 4ac \). We call \( Q \) an indefinite quadratic form if \( \delta_Q > 0 \). Let \( \mathcal{F} \) be the space of integral indefinite binary quadratic forms.

**Definition 2.1.** For \( M \in \mathbb{N} \), we define the set \( \mathcal{F}_M \) as follows:

\[
\mathcal{F}_M := \begin{cases} 
\{ Q(X, Y) = aX^2 + bXY + cY^2 \in \mathcal{F} \mid (a, M) = 1, \ M \mid b, M \mid c \} & \text{if } M \text{ is odd} \\
\{ Q(X, Y) = aX^2 + bXY + cY^2 \in \mathcal{F} \mid (a, M) = 1, \ 2M \mid b, M \mid c \} & \text{if } M \text{ is even}
\end{cases}
\]

If \( Q(X, Y) = aX^2 + bXY + cY^2 \in \mathcal{F} \) satisfies \( (a, b, c) = 1 \), we call \( Q \) a primitive quadratic form. The congruence subgroup \( \Gamma_0(M) \) acts on \( \mathcal{F}_M \) by the following formula:

\[
(Q|\gamma)(X, Y) := Q((X, Y) \cdot \gamma^{-1})
\]

for \( Q \in \mathcal{F}_M \) and \( \gamma \in \Gamma_0(M) \). It is easy to check that this action of \( \Gamma_0(M) \) preserves \( \mathcal{F}_M \). If a binary quadratic form \( Q \) is allowed to have rational coefficients, then we define the action of \( GL_2(\mathbb{Q}) \) on
Now we associate a pair of points $\omega_Q, \omega_Q' \in \mathbb{P}^1(\mathbb{R}) = \mathbb{R} \cup \{i\infty\}$ to each integral indefinite binary quadratic form $Q \in \mathcal{F}_M$ following [Shn]:

$$\omega_Q, \omega_Q' := \begin{cases} 
\left(\frac{b+\sqrt{\delta_Q}}{2c}, \frac{b-\sqrt{\delta_Q}}{2c}\right) & \text{if } c \neq 0 \\
(i\infty, \frac{a}{b}) & \text{if } c = 0 \text{ and } b > 0 \\
(\frac{a}{b}, i\infty) & \text{if } c = 0 \text{ and } b < 0 
\end{cases}$$

Definition 2.2. We define the oriented geodesic path $C_Q$ in the upper half plane $\mathfrak{h}$ following [Shn]:

$$C_Q := \begin{cases} 
\text{the oriented geodesic path joining } \omega_Q \text{ to } \omega_Q' & \text{if } \delta_Q \text{ is a perfect square} \\
\text{the oriented geodesic path joining } \omega \text{ to } \gamma_Q(\omega) & \text{otherwise}, 
\end{cases}$$

where $\omega$ is an arbitrary point in $\mathbb{P}^1(\mathbb{Q})$ and $\gamma_Q = \left(\begin{array}{cc} r & s \\ t & u \end{array}\right)$ is the unique generator (satisfying $r - t\omega_Q > 1$) of $\Gamma_0(M)\sqrt{Q}$ which is the index 2 subgroup (consisting of matrices with positive traces) of the stabilizer of $Q \in \mathcal{F}_M$ in $\Gamma_0(M)$. See [Shn] and [St1] for more details.

For a given Dirichlet character $\chi$ defined modulo $M$, we define a new Dirichlet character $\chi'$ modulo $4M$ by

$$\chi'(d) := \chi(d) \cdot \left(\frac{(-1)^{k+1}M}{d}\right), \quad d \in (\mathbb{Z}/4MZ)^\times. \quad (2.2)$$

For each quadratic form $Q = Q(X,Y) = aX^2 + bXY + cY^2 \in \mathcal{F}_M$ and a Dirichlet character $\chi$ defined modulo $M$, we define

$$\chi(Q) := \chi(a). \quad (2.3)$$

Now we’re ready to construct the classical Shintani $\theta$-lifting. We define

$$\theta_{k,\chi}(f) = \theta_{k,\chi}(f, z) := \begin{cases} 
\sum_{Q \in \mathcal{F}_M/\Gamma_0(M)} I_{k,\chi}(f,Q)q^{\delta_Q/M} & \text{if } M \text{ is odd,} \\
\sum_{Q \in \mathcal{F}_M/\Gamma_0(M)} I_{k,\chi}(f,Q)q^{\delta_Q/4M} & \text{if } M \text{ is even,} 
\end{cases}$$

for $f \in S_{2k+2}(\Gamma_0(M), \chi^2)$ and $z \in \mathfrak{h}$ (the upper half plane), where $q := e^{2\pi iz}$ and the coefficient in the $q$-expansion $I_{k,\chi}(f,Q)$ is defined by

$$I_{k,\chi}(f,Q) := \chi(Q) \cdot \int_{C_Q} f(\tau)Q(1,-\tau)^k d\tau. \quad (2.4)$$

The integral $I_{k,\chi}(f,Q)$ converges for cusp forms $f \in S_{2k+2}(\Gamma_0(M), \chi^2)$ and depends only on the $\Gamma_0(M)$-orbit of $Q$ in $\mathcal{F}_M$.

Recall that $S_{k+\frac{3}{2}}(\Gamma_0(4M), \chi')$ is the space of cusp forms of level $4M$, weight $k + \frac{3}{2}$, and nebentype character $\chi'$ (a Dirichlet character modulo $4M$). We refer to [Shn2] for detailed definitions of half-integral weight modular forms. Shintani proved the following theorem (Theorem 2 in [Shn]).

**Theorem 2.3.** Let $k \geq 0$ and $\chi$ be a Dirichlet character defined modulo $M$. Then for each $f \in S_{2k+2}(\Gamma_0(M), \chi^2)$, the series $\theta_{k,\chi}(f, z)$ is the $q$-expansion of a half-integral weight cusp form in...
where
\( S_{k+\frac{1}{2}}(\Gamma_0(4M), \chi') \). Moreover, the map
\[
\theta_{k, \chi} : S_{2k+2}(\Gamma_0(M), \chi^2) \to S_{k+\frac{1}{2}}(\Gamma_0(4M), \chi')
\]
is a Hecke-equivariant \( \mathbb{C} \)-linear map, i.e.
\[
\theta_{k, \chi}(f|T_l) = \theta_{k, \chi}(f)|T_l^2
\]
for any positive odd prime number \( l \).

We recall the definition of the Hecke operator \( T_p \) on \( S_{k+\frac{1}{2}}(\Gamma_0(4M), \chi') \) when \( p|M \). It is given on \( q \)-expansions by
\[
(\sum_{n=1}^{\infty} \beta_0 q^n)|T_p := \sum_{n=1}^{\infty} \beta_p n q^n.
\]

Proposition 1.5 of [Shm2] tells us that \( T_p \) with \( p|M \) induces a map
\[
T_p : S_{k+\frac{1}{2}}(\Gamma_0(4M), \chi') \to S_{k+\frac{1}{2}}(\Gamma_0(4M), \chi' \cdot (\frac{p}{\gamma}))
\]
So \( T_p \) preserves the level but multiplies the nebentype character by the quadratic character \((\frac{p}{\gamma})\). For a rational prime \( l \) which doesn’t divide \( 4M \), the Hecke operator \( T_{l^2} \) on \( S_{k+\frac{1}{2}}(\Gamma_0(4M), \chi') \) preserves the space \( S_{k+\frac{1}{2}}(\Gamma_0(4M), \chi') \). Theorem 1.7 of [Shm2] tells us that
\[
(\sum_{n=1}^{\infty} \beta_n q^n)|T_{l^2} := \sum_{n=1}^{\infty} (\beta_{l^2 n} + \chi'(l)\left(\frac{-1}{l}\right)^{k+1}\left(\frac{l}{T}ight)l^k \beta_n + \chi'(l^2)^{2k+1} \beta_{l^2 n}) q^n.
\]

Now let’s turn to cohomological Shintani liftings. For each \( Q \in \mathcal{F}_M \), let
\[
D_Q := \partial C_Q = \begin{cases} 
\{\omega_Q\} - \{\omega'_Q\} & \text{if } \delta_Q \text{ is a perfect square} \\
\{\gamma_Q(\omega)\} - \{\omega\} & \text{otherwise}.
\end{cases}
\]
Note that \( D_Q \in \Delta_0 \).

**Definition 2.4.** Let \( R \) be a commutative \( \mathbb{Z}[\frac{1}{Q}] \)-algebra. Let \( k \geq 0 \) be an integer and \( \chi \) be an \( R \)-valued Dirichlet character defined modulo \( M \). We define
\[
J_{k, \chi}(\phi, Q) := \chi(Q) \cdot \langle \phi(D_Q), Q^k \rangle \in R
\]
for \( \phi \in H^1_c(\Gamma_0(M), L_{2k, \chi^2}(R)) = \text{Hom}_{\Gamma_0(M)}(\Delta_0, L_{2k, \chi^2}(R)) \) and \( Q \in \mathcal{F}_M \) (so \( Q^k \in \text{Sym}^k(R)^* \)), where \( \phi(D_Q) \) is the value of any 1-cocycle representing \( \phi \). Note that the definition of \( J_{k, \chi}(\phi, Q) \) is independent of the choice of a representative cocycle and depends only on the \( \Gamma_0(M) \)-orbit of \( Q \) in \( \mathcal{F}_M \).

**Definition 2.5.** We define the cohomological Shintani lifting \( \Theta_{k, \chi} : H^1_c(\Gamma_0(M), L_{2k, \chi^2}(R)) \to R[[q]] \) by
\[
\Theta_{k, \chi}(\phi) := \begin{cases} 
\sum_{Q \in \mathcal{F}_M/\Gamma_0(M)} J_{k, \chi}(\phi, Q) q^{\delta_Q/M} \in R[[q]] & \text{if } M \text{ is odd}, \\
\sum_{Q \in \mathcal{F}_M/\Gamma_0(M)} J_{k, \chi}(\phi, Q) q^{\delta_Q/4M} \in R[[q]] & \text{if } M \text{ is even}.
\end{cases}
\]
for each cohomology class \( \phi \in H^1_c(\Gamma_0(M), L_{2k, \chi^2}(R)) \).

Note that \( \Theta_{k, \chi} \) is an \( R \)-linear map. The following result was proved in [Sti] (Proposition (4.3.3)).
Proposition 2.6. Let $R$ be a commutative $\mathbb{Z}[\frac{1}{M}]$-algebra. Let $k \geq 0$ be an integer and $\chi$ be an $R$-valued Dirichlet character defined modulo $M$.

(a) For $\varphi \in H^1_c(\Gamma_0(M), L_{2k,\chi}(R))$ and $t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

\begin{equation}
\Theta_{k,\chi}(\varphi | t) = -\Theta_{k,\chi}(\varphi).
\end{equation}

(b) For $f \in S_{2k+2}(\Gamma_0(M), \chi^2)$ and $\psi_f \in H^1_c(\Gamma_0(M), L_{2k,\chi}(\mathbb{C}))$ which is the image of $f$ under the composition of the Eichler-Shimura map and the Manin-Drinfeld section,

\begin{equation}
\Theta_{k,\chi}(\psi_f) = \Theta_{k,\chi}(\psi_f^r) = \theta_{k,\chi}(f)
\end{equation}

where $\theta_{k,\chi}(f)$ is defined in (2.4) and $\psi_f^r := \frac{1}{2} \cdot (\psi_f - \psi_f | \iota)$.

(c) If $R$ is the field of characteristic 0 (for example $R = K$) and $\varphi \in H^1_{par}(\Gamma_0(M), L_{2k,\chi}(R))$ is the image of $\psi \in H^1_{par}(\Gamma_0(M), L_{2k,\chi}(R))$ under the Manin-Drinfeld section, then

\begin{equation}
\Theta_{k,\chi}(\varphi) \in S_{k+\frac{3}{2}}(\Gamma_0(4M), \chi'; R)
\end{equation}

where $\chi'$ is defined in (2.2).

2.2. Overconvergent Shintani Lifting and a universal overconvergent half-integral weight modular forms.

In this section we define the Hecke-equivariant overconvergent Shintani lifting (see [St1] for $\Lambda$-adic Shintani liftings) and describe the Hecke-action explicitly on the image of the lifting which we define to be the universal overconvergent half-integral weight modular form. Let $\sigma : \mathbb{Z}^{\times}_{p,N} \to \mathbb{Z}^{\times}_{p,N}$ be the ring homomorphism sending $t$ to $t^2$. We use the same notation for the ring homomorphism

\begin{equation}
\sigma : \mathcal{D}(\mathbb{Z}^{\times}_{p,N}) \to \mathcal{D}(\mathbb{Z}^{\times}_{p,N})
\end{equation}

induced from $\sigma : \mathbb{Z}^{\times}_{p,N} \to \mathbb{Z}^{\times}_{p,N}$, i.e. $\sigma(\mu)(f(t)) = \mu(f(t^2))$ for $\mu \in \mathcal{D}(\mathbb{Z}^{\times}_{p,N})$ and $f \in \mathcal{A}(\mathbb{Z}^{\times}_{p,N})$. We define the space of metaplectic locally analytic distributions on $\mathbb{Z}^{\times}_{p,N}$ by

\begin{equation}
\mathcal{D}(\mathbb{Z}^{\times}_{p,N}) := \mathcal{D}(\mathbb{Z}^{\times}_{p,N}) \otimes_{\mathcal{D}(\mathbb{Z}^{\times}_{p,N}), \sigma} \mathcal{D}(\mathbb{Z}^{\times}_{p,N})
\end{equation}

where the tensor product is taken with respect to $\sigma : \mathcal{D}(\mathbb{Z}^{\times}_{p,N}) \to \mathcal{D}(\mathbb{Z}^{\times}_{p,N})$. We regard $\mathcal{D}(\mathbb{Z}^{\times}_{p,N})$ as a $\mathcal{D}(\mathbb{Z}^{\times}_{p,N})$-algebra by the structure homomorphism

\begin{equation}
\mathcal{D}(\mathbb{Z}^{\times}_{p,N}) \to \mathcal{D}(\mathbb{Z}^{\times}_{p,N})
\end{equation}

given by $r \mapsto r \otimes_{\mathcal{D}(\mathbb{Z}^{\times}_{p,N}), \sigma} 1$.

The following definition is the key step to construct overconvergent Shintani liftings.

Definition 2.7. For each $Q \in \mathcal{F}_{p,N}$ we define a map $J_Q : \mathcal{D} \to \mathcal{D}(\mathbb{Z}^{\times}_{p})$ by the following integration formula:

\begin{equation}
\int_{\mathbb{Z}^{\times}_{p}} f(z) d(\mathcal{D}(J_Q(\mu))) = \int_{\mathbb{Z}^{\times}_{p} \times \mathbb{Z}^{\times}_{p}} f(Q(x, y)) d\mu(x, y)
\end{equation}

for a given $\mu \in \mathcal{D}$ and a locally analytic function $f \in \mathcal{A}(\mathbb{Z}^{\times}_{p})$. This is well defined, since $Q(X, Y) \in \mathbb{Z}^{\times}_{p}$ for $Q \in \mathcal{F}_{p,N}$.

We note that $J_Q$ is not a $\mathcal{D}(\mathbb{Z}^{\times}_{p})$-linear map. In order to make it $\mathcal{D}(\mathbb{Z}^{\times}_{p})$-linear, we twist the target space by $\sigma$ (see (2.14)). Then we extend $J_Q$ to $\tilde{J}_Q : \mathcal{D} \to \mathcal{D}(\mathbb{Z}^{\times}_{p,N})$ by

\begin{equation}
\tilde{J}_Q(\nu \otimes_{\mathcal{D}(\mathbb{Z}^{\times}_{p})} r) := 1 \otimes_{\mathcal{D}(\mathbb{Z}^{\times}_{p,N}), \sigma} J_Q(\nu) \cdot \sigma(r) \cdot [Q]_N \in \mathcal{D}(\mathbb{Z}^{\times}_{p,N})
\end{equation}

where \(\sigma \) is defined \((2.2)\).
for \( \nu \otimes D(Q_p) \), \( r \in D \otimes D(Z_p) = D \). Here \([Q]_N := [a]_N \in \Delta_N\) for each quadratic form \( Q(X, Y) = aX^2 + bXY + cY^2 \in F_{N_p} \).

**Proposition 2.8.** Let \( Q \in F_{N_p} \). The map \( \tilde{J}_Q : D \to \tilde{D}(Z_p) \) is the unique \( D(Z_p) \)-module homomorphism such that for all arithmetic \( \tilde{\kappa} \in \text{Hom}_{cts}(Z_p, \mathbb{C}_p) \) of signature \((k, \chi)\) and the associated arithmetic \( \kappa \in \text{Hom}_{cts}(Z_p, \mathbb{C}_p) \) by \( \sigma : D \to Z_p \), i.e. \( \kappa := \tilde{\kappa} \circ \sigma \) (so \( \kappa \) has a signature \((2k, \chi^2)\)), we have

\[
\tilde{\kappa}(\tilde{J}_Q(\mu)) = \chi(Q) \cdot \langle \phi_\kappa(\mu), Q^k \rangle
\]

for \( \mu \in D \). Here \( \phi_\kappa \) is the specialization map in \([1, 14]\) and \( \langle \cdot, \cdot \rangle \) is the pairing defined by \([1, 13]\).

**Proof.** Because of the construction, \( D(Z_p) \)-linearity is clear. If the interpolation property (2.17) at arithmetic points is proven, then the uniqueness follows. So we concentrate on proving (2.17). Let \( \mu = \nu \otimes D(Q_p) \), \( r \in D = D \otimes D(Z_p) = D(Z_p) \), and let \( \tilde{\kappa} \in \text{Hom}_{cts}(Z_p, \mathbb{C}_p) \) of signature \((k, \chi)\) associated to an arithmetic \( \kappa \in \text{Hom}_{cts}(Z_p, \mathbb{C}_p) \) by \( \sigma : D(Z_p) \to D(Z_p) \) (so \( \kappa \) has a signature \((2k, \chi^2)\)). Then we calculate

\[
\tilde{\kappa}(\tilde{J}_Q(\mu)) = \kappa(r) \cdot \tilde{\kappa}([Q]_N \cdot J_Q(\nu))
\]

\[
= \tilde{\kappa}([Q]_N) \cdot \kappa(r) \cdot \int_{Z_p} \tilde{\kappa}(z)dJ_Q(\nu)(z)
\]

\[
= \chi_N(Q) \cdot \kappa(r) \cdot \int_{Z_p} \tilde{\kappa}(Q(x, y))d\nu(x, y)
\]

\[
= \chi_N(Q) \cdot \kappa(r) \cdot \int_{Z_p} \chi_p(Q(x, y)) \cdot Q(x, y)^k d\nu(x, y)
\]

(because \( Q(x, y) \in \mathbb{Z}_p \) for \( (x, y) \in Z_p \times \mathbb{Z}_p \))

\[
= \chi_N(Q) \chi_p(a) \kappa(r) \cdot \int_{Z_p} \chi_p(x^2) \left( \frac{(xY - yX)^{2k}}{(2k)!}, Q(X, Y)^k \right) d\nu(x, y)
\]

(by the equality \([1, 13]\) and \( Q \in F_{N_p} \))

\[
= \chi_N(Q) \chi_p(Q) \cdot \langle \kappa(r) \cdot \int_{Z_p} \chi_p(x) \left( \frac{(xY - yX)^{2k}}{(2k)!} d\nu(x, y), Q^k \right) \rangle
\]

\[
= \chi(Q) \cdot \langle \phi_\kappa(\mu), Q^k \rangle.
\]

This completes the proof.

\[\square\]

Now we are ready to define the overconvergent Shintani lifting following the definition in \([St1]\).

**Definition 2.9.** For each \( \Phi \in H^1_c(\Gamma_0(Np), D_N) = \text{Hom}_{cts}(\Delta_0, D_N) \) and each \( Q \in F_{N_p} \) we define

\[
(2.18) \quad J(\Phi, Q) := \tilde{J}_Q(\Phi(D_Q)) \in \tilde{D}(Z_p^\times)
\]

where \( \Phi(D_Q) \) is the value of the cocycle representing \( \Phi \) on \( D_Q \).

Note that this definition of \( J(\Phi, Q) \) does not depend on the representative cocycle. A simple calculation confirms that

\[
\tilde{J}_Q(\Phi(D_Q)) = \tilde{J}_{Q|\gamma}(\Phi(D_{Q|\gamma})),
\]

for \( \gamma \).
for $\gamma \in \Gamma_0(Np)$ and $D_Q = \partial C_Q$ as in Definition 2.2. Thus $J(\Phi, Q)$ depends only on the $\Gamma_0(Np)$-equivalence class of $Q$ and the following definition makes sense.

**Definition 2.10.** We define the overconvergent Shintani lifting $\Theta : H_c^1(\Gamma_0(Np), D_N) \to \tilde{D}(\mathbb{Z}_{p, N}^\times)[[q]]$ by

$$\Theta(\Phi) := \begin{cases} \sum_{Q \in \mathcal{F}_{Np}/\Gamma_0(Np)} J(\Phi, Q)q^{d_Q/Np} & \text{if } Np \text{ is odd,} \\ \sum_{Q \in \mathcal{F}_{Np}/\Gamma_0(Np)} J(\Phi, Q)q^{d_Q/4Np} & \text{if } Np \text{ is even.} \end{cases}$$

We have seen that $H_c^1(\Gamma_0(Np), D_N)$ is an $\mathcal{H}$-module but $\tilde{D}(\mathbb{Z}_{p, N}^\times)[[q]]$ doesn’t have an $\mathcal{H}$-action a priori. We will put a $\mathcal{H}$-module structure on $\tilde{D}(\mathbb{Z}_{p, N}^\times)[[q]]$ so that $\Theta$ is $\mathcal{H}$-equivariant. In order to do so, we need an explicit description of $\Theta(\Phi|T_l)$ for all primes $l$ and $\Theta(\Phi|T_{l,l})$ for all primes $l \nmid Np$ in terms of the coefficients of the $q$-expansion of $\Theta(\Phi)$ (see the [1,7] for the definition of $T_{l,l}$) since $\mathcal{H}$ is generated by these Hecke operators over $\tilde{D}(\mathbb{Z}_{p, N}^\times)$. For $\Phi \in H_c^1(\Gamma_0(Np), D)$ and $m \in \Delta_N$, we define the notation $\Phi \otimes m \in H_c^1(\Gamma_0(Np), D_N)$ as follows:

$$\Phi \otimes m \in D \otimes_{D(\mathbb{Z}_{p, N}^\times)} D(\mathbb{Z}_{p, N}) =: \mathcal{D}_N$$

for any $D \in \Delta_0$. Even though we state the theorem for $\Phi \otimes 1$ for simplicity, the formula for general $\Phi \in H_c^1(\Gamma_0(Np), D_N)$ is also straightforward as $\Theta$ is $\mathcal{D}(\mathbb{Z}_{p, N}^\times)$-linear map.

**Theorem 2.11.** First, $\Theta$ is a $\mathcal{D}(\mathbb{Z}_{p, N}^\times)$-linear map. Also, if $\tilde{\Phi} = \Phi \otimes 1 \in H_c^1(\Gamma_0(Np), D_N)$ is an overconvergent modular symbol and we write

$$\Theta(\tilde{\Phi}) = \sum_{n=1}^{\infty} (1 \otimes_{\mathcal{D}(\mathbb{Z}_{p, N}^\times), \sigma} a(n))q^n \in \tilde{D}(\mathbb{Z}_{p, N}^\times)[[q]],$$

then we have the following explicit formula for $T_l$ on the $q$-expansion of $\Theta(\tilde{\Phi})$:

$$\Theta(\tilde{\Phi}|T_l) = \sum_{n=1}^{\infty} \left(1 \otimes_{\mathcal{D}(\mathbb{Z}_{p, N}^\times), \sigma} \left(a(nl^2) + \left(\frac{Np \cdot n}{l}\right)[l]_N \delta_1 \ast a(n) + l[l^2]_N \delta_2 \ast a\left(\frac{n}{l^2}\right)\right)\right)q^n$$

for any odd prime $l$. Here $\delta_1 \in \mathcal{D}(\mathbb{Z}_p^\times)$ is the Dirac distribution at $s \in \mathbb{N}$ (we put $\delta_s = 0$, if $p$ divides $s$), $\ast$ is the convolution product in $\mathcal{D}(\mathbb{Z}_p^\times)$, and we put $a(\frac{n}{l}) = 0$ for $n$ not divisible by $l^2$. We also have the formula for $T_{l,l}$ for any prime number $l \nmid Np$

$$\Theta(\tilde{\Phi}|T_{l,l}) = \sum_{n=1}^{\infty} \left(1 \otimes_{\mathcal{D}(\mathbb{Z}_{p, N}^\times), \sigma} [l^2]_N \cdot \delta_2 \ast a(n)\right)q^n$$

with the same notation as above.

**Proof.** The $\mathcal{D}(\mathbb{Z}_{p, N}^\times)$-linearity of $\Theta$ follows from Proposition 2.8. We first concentrate on the explicit description of $\Theta(\tilde{\Phi}|T_l)$. The description of $T_{l,l}$ for $l \nmid Np$ will be much easier. For $Np$ odd (resp. even) denote by $d_{Np}$ the discriminant of the number field $K = \mathbb{Q}(\sqrt{Np \cdot n})$ (resp. $K = \mathbb{Q}(\sqrt{4Np \cdot n})$) and put $Np \cdot n = d_{Np}c_{Np}^2$ (resp. $4Np \cdot n = d_{Np}c_{Np}^2$). The positive number $c_{Np}$ defined by the above equations is a positive integer or a positive half integer but we will consider only the positive integer $c_{Np}$ (if $c_{Np}$ is not an integer, the terms appearing below are 0). We let

$$\mathcal{L}_s(Np) = \begin{cases} \{Q = aX^2 + bXY + cY^2 : Q \in \mathcal{F}_{Np} \} & \text{if } Np \text{ is odd} \\ \{Q = aX^2 + bXY + cY^2 : Q \in \mathcal{F}_{Np} \} & \text{if } Np \text{ is even} \end{cases}$$
for \( s \in \mathbb{N} \). Then, from the construction of \( \Theta \), we have

\[
\Theta(\tilde{\Phi}) = \sum_{n=1}^{\infty} \left( \sum_{m \mid \alpha_n m > 0, (m, N_p) = 1} \sum_{Q \text{ belongs to } \mathcal{L}_{m}^\infty(N_p)/T_0(N_p)} \left( 1 \otimes \mathcal{D}_{\mathbb{Z}_{p, N}^\times} \right) J_m Q(\Phi(D_Q)) \cdot [mQ]_N \right) q^n.
\]

Notice that \( D_Q = D_{mQ} \) for \( m \in \mathbb{N} \) satisfying \((m, N_p) = 1\) (see the Lemma 2.7. (iii) of [Shn]). Put \( \alpha_{N_p}(s, \tilde{\Phi}, m) := \sum_{Q \text{ belongs to } \mathcal{L}_{s}(N_p)/\Gamma_0(N_p)} \left( J_m Q(\Phi(D_Q)) \cdot [mQ]_N \right) \).

In order to get the desired description of \( T_i \) on \( \Theta(\tilde{\Phi}) \), we will first derive a formula for \( \alpha_{N_p}(s, \tilde{\Phi}|T_i, m) \).

**Lemma 2.12.** Notations being as above, for \( s \in \mathbb{N} \), we have

\[
\alpha_{N_p}(s, \tilde{\Phi}|T_i, m) = \begin{cases} 
\alpha_{N_p}(s, \tilde{\Phi}, m) + \left( 1 + \left( \frac{d_i}{l} \right) \right)[l]_N \delta_1 \star \alpha_{N_p}(s, \tilde{\Phi}, m) & \text{if } (l, c_s) = 1 \\
\alpha_{N_p}(s, \tilde{\Phi}, m) + \left( l - \left( \frac{d_i}{l} \right) \right)[l^2]_N \delta_2 \star \alpha_{N_p}(s, \tilde{\Phi}, m) & \text{if } (l^2, c_s) = l \\
\alpha_{N_p}(s, \tilde{\Phi}, m) + l[l]_N \delta_2 \star \alpha_{N_p}(s, \tilde{\Phi}, m) & \text{if } (l^2, c_s) = l^2
\end{cases}
\]

where \( \delta_1 \) and \( \delta_2 \) are defined in the theorem 2.11 and we put \( \alpha_{N_p}(\frac{s}{l}, \tilde{\Phi}, m) = 0 \) for \( s \) not divisible by \( l^2 \).

**Proof.** Let

\[
\alpha_0 = \begin{pmatrix} l & 0 \\
0 & 1 \end{pmatrix} \quad \text{and} \quad \alpha_j = \begin{pmatrix} 1 & j \\
0 & l \end{pmatrix} \quad \text{for } 1 \leq j \leq l.
\]

Then, for \( T_i \in \mathcal{H} \),

\[
\tilde{\Phi}|T_i = \sum_{j=0}^{l} \tilde{\Phi}|\alpha_j.
\]

Having in mind the action of \( \alpha_j \) on \( \mathcal{D} \otimes \mathcal{D}(\mathbb{Z}_{p, N})^\times = \mathcal{D}_N \), we have

\[
\alpha_{N_p}(s, \tilde{\Phi}|T_i, m) = \sum_{Q \text{ belongs to } \mathcal{L}_{s}(N_p)/\Gamma_0(N_p)} \sum_{j=0}^{l} J_m Q|_{\alpha_j^{-1}}(\Phi(\alpha_j \cdot D_Q)) \cdot [mQ]_N[a(\alpha_j)^2]_N,
\]

where \( a(\alpha_j) \) is the upper left entry of \( \alpha_j \).

Assume \( d_s > 1 \) and we have \( K = \mathbb{Q}(\sqrt{d_s}) = \mathbb{Q}(\sqrt{d_sc_s^2}) \) (remember that \( d_s \) is the discriminant of \( K \)). In this case \( Q \in \mathcal{L}_s(N_p) \) has a non-perfect square discriminant and so \( D_Q = \{ \gamma_Q \omega \} \) \( - \{ \omega \} \) for any point \( \omega \in \mathfrak{h} \) by the Lemma 2.7 (i) of [Shn], where \( \gamma_Q \) was given in the Definition 2.2. Let \( Q_1, Q_2, \ldots, Q_h \) be a complete set of representatives of \( \Gamma_0(N_p) \)-equivalence classes of quadratic forms in \( \mathcal{L}_s(N_p) \). Let

\[
Q_i(X, Y) = a^iX^2 + b^iXY + c^iY^2 \quad (1 \leq i \leq h).
\]

Also put \( \omega^i_1 = \frac{b^i + c^i \sqrt{d_s}}{2} \) and \( \omega^i_2 = c^i \) and denote by \( L^i \) the lattice in \( K \) generated by \( \omega^i_1 \) and \( \omega^i_2 \). If we denote by \( \mathcal{L}^i = \mathcal{L}^i(X, Y) \) the primitive integral binary quadratic form with discriminant \( d_sc_s^2 \) associated to \( L^i \) (look at the Lemma 2.2 of [Shn] for the precise dictionary between the lattices in \( K \) and quadratic forms.). In fact it’s easy to see

\[
\mathcal{L}^i(X, Y) = Q^i(X, Y) \quad (1 \leq i \leq h).
\]
It follows from the Lemma 2.2 and Lemma 2.6 of [Shm] that \(\{L^1, L^2, \ldots, L^h\}\) forms a complete set of representatives of equivalence classes of lattices in \(K\) with conductor \(c_s\) (see the page 110 of [Shm] for the definition of the equivalence relation and the conductor of \(L^i\)).

Put
\[
Q_{i,j} := Q_i|\alpha_j^{-1} \quad (1 \leq i \leq h \text{ and } 0 \leq j \leq l)
\]

\(e_1 = [O^X_{c_s} : O^X_{c_s}]\) and \(e_2 = [O^X_{c_s} / (c_s, l) : O^X_{c_s}]\)

where the definition of \(O^X_{c_s}\) is given on the page 111 of [Shm]. Having in mind that
\[
[Q_{i,j}]_N = [Q_i]_N \quad (1 \leq j \leq l) \text{ and } [Q_{i,j}]_N = [l^2]_N \cdot [Q_i]_N \quad (j = 0),
\]
we get
\[
\alpha_{N_p}(s, \tilde{\Phi}|T_l, m) = \sum_{i=0}^h \sum_{j=0}^l J_{mQ_{i,j}} \left( \Phi\left(\{\alpha_j \gamma Q_i \alpha_j^{-1} \cdot \omega\} - \{\omega\}\right) \right) \cdot [mQ_{i,j}]_N
\]

Define
\[
L^{i,j} := \left\{ \begin{array}{ll}
\omega_1^i Z + \omega_2^j Z & \text{if } j = 0 \\
(\omega_1^i + j\omega_2^j)Z + l\omega_3 Z & \text{if } 1 \leq j \leq l.
\end{array} \right.
\]

Then \(L^{i,0}, L^{i,1}, \ldots, L^{i,l}\) are mutually distinct sublattices of \(L^i\) with index \(l\). Let \(\mathcal{L}^{i,j}\) be the quadratic form corresponding to \(L^{i,j}\). Then we have
\[
Q_{i,j}(X, Y) = \left\{ \begin{array}{ll}
\mathcal{L}^{i,j}(X, Y) & \text{if the conductor of } L^{i,j} = c_s l \\
l \mathcal{L}^{i,j}(X, Y) & \text{if the conductor of } L^{i,j} = c_s \\
l^2 \mathcal{L}^{i,j}(X, Y) & \text{if the conductor of } L^{i,j} = \frac{c_s l}{l^2}
\end{array} \right.
\]

and
\[
\alpha_j \gamma_{Q_i} \alpha_j^{-1} = \left\{ \begin{array}{ll}
\gamma_{L^{i,j}} & \text{if the conductor of } L^{i,j} = c_s l \\
\gamma_{L^{i,j}}^e & \text{if the conductor of } L^{i,j} = c_s \\
\gamma_{L^{i,j}}^{e_2} & \text{if the conductor of } L^{i,j} = \frac{c_s l}{l^2}
\end{array} \right.
\]

Now our Lemma 2.12 can be derived from Lemma 2.5, Lemma 2.7 and Lemma 2.2 of [Shm]. For example, if \((l, c_s) = 1\), then Lemma 2.5, Lemma 2.7 and Lemma 2.2 of [Shm] implies the following:

\[
\alpha_{N_p}(s, \tilde{\Phi}|T_l, m) = e_1 e_1^{-1} \sum_{L^{i,j} \text{ belongs to } \mathcal{L}_{d|l}(Np)/T_0(Np)} J_{mL^{i,j}}(\Phi(D_{L^{i,j}})) [mL^{i,j}]_N + \left(1 + \frac{d_s}{l}\right) \sum_{L^{i,j} \text{ belongs to } \mathcal{L}_{s}(Np)/T_0(Np)} J_{lmL^{i,j}}(\Phi(D_{L^{i,j}})) [lmL^{i,j}]_N
\]

\[
= \alpha_{N_p}(s, \tilde{\Phi}, m) + \left(1 + \frac{d_s}{l}\right)[l]_N d_l \ast \alpha_{N_p}(s, \tilde{\Phi}, m).
\]

The cases \((l^2, c_s) = l\) and \((l^2, c_s) = l^2\) also can be checked in the similar way. If \(d_s = 1\), the proof is similar and much simpler. This completes the proof of Lemma 2.12. \(\square\)
Now we will use the Lemma 2.12 to prove the theorem. We can express $\Theta(\tilde{\Phi})$ as follows:

$$\Theta(\tilde{\Phi}) = \sum_{n=1}^{\infty} \left( 1 \otimes_{\mathbb{Z}^2} a(n) \right) q^n = \sum_{n=1}^{\infty} \left( 1 \otimes_{\mathbb{Z}^2} \left( \sum_{m \mid c_n, m > 0} \alpha_{Np}\left( \frac{n}{m^2}, \tilde{\Phi}, m \right) \right) \right) q^n.$$

If we apply $T_l$, then we have

$$\Theta(\tilde{\Phi}\mid T_l) = \sum_{n=1}^{\infty} \left( 1 \otimes_{\mathbb{Z}^2} \left( \sum_{m \mid c_n, m > 0} \alpha_{Np}\left( \frac{n}{m^2}, \tilde{\Phi}\mid T_l, m \right) \right) \right) q^n.$$

Let

$$b(n) := \sum_{m \mid c_n, m > 0} \alpha_{Np}\left( \frac{n}{m^2}, \tilde{\Phi}\mid T_l, m \right).$$

Note that we can assume $c_n$ is an integer. Let’s analyze the $q^n$-coefficient case by case. If $l$ is a factor of $Np$, we have, by Lemma 2.12

$$b(n) = \sum_{m \mid c_n, m > 0} \alpha_{Np}\left( \frac{n}{m^2}, \tilde{\Phi}, m \right) = \sum_{m \mid c_n, m > 0} \alpha_{Np}\left( \frac{n}{m^2}, \tilde{\Phi}, m \right) = a(nl^2),$$

since $l \mid p$ implies $\delta_l = 0$ ($\delta_l = 0$) and $l \mid N$ implies $[l]_N = 0$ ($[l^2]_N = 0$). Next we examine the case that $l$ is prime to both $Np$ and $c_n$. Then Lemma 2.12 implies that

$$b(n) = \sum_{m \mid c_n, m > 0} \left( \alpha_{Np}\left( \frac{nl^2}{m^2}, \tilde{\Phi}, m \right) + \left( 1 + \left( \frac{d_n}{l} \right) \right) [l]_N \delta_l \cdot \alpha_{Np}\left( \frac{n}{m^2}, \tilde{\Phi}, m \right) \right)$$

$$= \sum_{m \mid c_n, m > 0} \alpha_{Np}\left( \frac{nl^2}{m^2}, \tilde{\Phi}, m \right) + \sum_{m \mid c_n, m > 0} \left( 1 + \left( \frac{d_n}{l} \right) \right) [l]_N \delta_l \cdot \alpha_{Np}\left( \frac{n}{m^2}, \tilde{\Phi}, m \right)$$

$$= \sum_{m \mid c_n, m > 0} \alpha_{Np}\left( \frac{nl^2}{m^2}, \tilde{\Phi}, m \right) + \sum_{m \mid c_n, m > 0} \left( \frac{Np \cdot n}{l} \right) [l]_N \delta_l \cdot \alpha_{Np}\left( \frac{n}{m^2}, \tilde{\Phi}, m \right)$$

$$= a(nl^2) + \left( \frac{Np \cdot n}{l} \right) [l]_N \delta_l \cdot a(n).$$

Finally consider the case that $l$ is prime to $Np$ but is a factor of $c_n$. Set $c_n = c'_n l^k$, where $c'_n$ is prime to $l$. Then we get

$$b(n) = \sum_{m \mid c'_n, m^l > 0} \alpha_{Np}\left( \frac{n}{m^2l^2}, \tilde{\Phi}\mid T_l, m^l \right).$$
Since Lemma 2.12 shows that
\[ \alpha_{N_p}(\frac{n}{m^2l^2}, \tilde{\Phi}, t) = \begin{cases} \alpha_{N_p}(\frac{n}{m^2l^2}, \tilde{\Phi}, t) + l[l^2]_N \delta_{l^2} \ast \alpha_{N_p}(\frac{n}{m^2l^2}, \tilde{\Phi}, t) & (0 \leq i \leq k - 2) \\ \alpha_{N_p}(\frac{n}{m^2l^2}, \tilde{\Phi}, t) + (l - \frac{d}{l})(l^2]_N \delta_{l^2} \ast \alpha_{N_p}(\frac{n}{m^2l^2}, \tilde{\Phi}, t) & (i = k - 1) \\ \alpha_{N_p}(\frac{n}{m^2l^2}, \tilde{\Phi}, t) + (1 + \frac{d}{l})(l^2]_N \delta_{l^2} \ast \alpha_{N_p}(\frac{n}{m^2l^2}, \tilde{\Phi}, t) & (i = k) \end{cases} \]
for any \( t \in \mathbb{N} \), we can conclude that
\[ b(n) = \sum_{m \mid n \mid 2} a(\frac{n}{m^2l^2}, \tilde{\Phi}, m) + \sum_{m \mid n \mid 2} l[l^2]_N \delta_{l^2} \ast \alpha_{N_p}(\frac{n}{m^2l^2}, \tilde{\Phi}, m) \]
This completes the proof of the description of \( T_l \)-action in the theorem 2.11. The formula \( T_{l,l} \)-action for \( l \mid Np \) is easily derived, because \( T_{l,l} \)-action on \( \tilde{\Phi} \) is the same as \( \left( \begin{array}{cc} l & 0 \\ 0 & l \end{array} \right) \)-action on it.

Now we define the action of \( T_l \) for any rational prime \( l \) and \( T_{l,l} \) for \( l \mid Np \) on formal \( q \)-expansions
\[ \Theta = \sum_{n=1}^{\infty} (a(n) \otimes_{\mathcal{D}(\mathbb{Z}_p^\times, \sigma)} b(n))q^n \in \hat{D}(\mathbb{Z}_p^\times, [[q]]) \]
as follows:
\[ (2.21) \quad \Theta|_{T_l} = \sum_{n=1}^{\infty} (a(n) \otimes_{\mathcal{D}(\mathbb{Z}_p^\times, \sigma)} b(n^2) + (\frac{Np \cdot \frac{n}{l}}{l})(l^2]_N \delta_{l^2} \ast b(\frac{n}{l^2}))q^n \]
\[ (2.22) \quad \Theta|_{T_{l,l}} = \sum_{n=1}^{\infty} (a(n) \otimes_{\mathcal{D}(\mathbb{Z}_p^\times, \sigma)} l[l^2]_N \cdot \delta_{l^2} \ast b(n))q^n. \]
Then we can conclude that the overconvergent Shintani lifting \( \Theta \) is \( \mathcal{H} \)-linear map by Theorem 2.11 except for operators containing \( T_2 \). But if \( N \) is even, then even \( \Theta(\Phi|T_2) = \Theta(\Phi)|T_2 \) is true so that \( \Theta \) is actually \( \mathcal{H} \)-linear.

Because we will show that our overconvergent Shintani lifting \( \Theta \) interpolates the classical Shintani liftings, it is reasonable to define the image of \( \Theta \) to be a universal overconvergent half-integral weight modular form.

**Definition 2.13.** We say that \( \Theta \in \hat{D}(\mathbb{Z}_p^\times, [[q]]) \) is a universal overconvergent half-integral weight modular form if
\[ \Theta = \Theta(\Phi) \]
for some \( \Phi \in H^1_k(\Gamma_0(Np), \mathcal{D}_N) \).

A universal overconvergent half-integral weight modular form will be viewed as a p-adic family of overconvergent half-integral weight modular forms. Finally we give a definition of an overconvergent half-integral weight modular form of weight \( \kappa \in \text{Hom}_{cts}(\mathbb{Z}_{N,p}^\times, K^\times) \).
Definition 2.14. We say that $\theta \in \mathbb{C}_p[[q]]$ is an overconvergent half-integral weight modular form of weight $\kappa \in \text{Hom}_{cts}(\mathbb{Z}_{N,p}^\times, K^\times)$ if it can be written as

$$\theta = \Theta(\kappa) = \sum_{n=1}^{\infty} (1 \otimes \mathcal{D}_{(p,N),\sigma} \alpha_n(\kappa)) q^n$$

where $\Theta = \sum_{n=1}^{\infty} (1 \otimes \mathcal{D}_{(p,N),\sigma} \alpha_n) q^n \in \mathcal{D}(\mathbb{Z}_{p,N})[[q]]$ is a universal overconvergent half-integral weight modular form.

3. Connection to the eigencurve and $p$-adic family of half-integral weight modular forms

Since points on the (half-integral) eigencurve correspond to (half-integral) overconvergent Hecke eigenforms, the existence of a Hecke eigensymbol in $H^1_c(\Gamma_0(Np), D_N)$ which interpolates overconvergent Hecke eigenforms and a local version of overconvergent Shintani lifting will give us a local piece of the half-integral eigencurve (see [Ram2]). So we will construct a Hecke eigensymbol and a local version of lifting whose meaning will be precise later.

3.1. Slope $\leq h$ decompositions of overconvergent modular symbols.

The goal of this section is to review the result of G. Stevens in [St3] which guarantees the existence of slope $\leq h$ decomposition of overconvergent modular symbols over some $K$-affinoid space.

In order to make a connection to the (half-integral) eigencurve, we need an appropriate $p$-adic Hecke algebra over $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]]$, but unlike the ordinary case (slope 0 case), it is impossible to define a global $p$-adic Hecke algebra finite over $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]]$ which parametrizes all overconvergent Hecke eigenforms of arbitrary finite slope. Instead we can define a local $p$-adic Hecke algebra finite over $A(B)$, where $B$ is some affioid subdomain of the weight space $\mathcal{X}_N$, using slope $\leq h$ decomposition. We first review some background knowledge on rigid spaces, especially weight space, and state the theorem of G. Stevens in [St3].

Let $\Lambda$ be the completed group algebra $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]]$ and let $\Lambda_N$ be the completed group algebra $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]]$ where $\mathbb{Z}_{p,N}^\times := \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times = \mathbb{Z}_p^\times \times \Delta_N$. Then $\mathbb{Z}_p[[\mathbb{Z}_{p,N}^\times]] \cong \mathbb{Z}_p[[\mathbb{Z}_p^\times]]((\mathbb{Z}/N\mathbb{Z})^\times) \cong \bigoplus_p \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ where the direct product is taken over the Dirichlet characters of $(\mathbb{Z}/N\mathbb{Z})^\times$.

Now consider the category $A\text{F}_{\mathbb{Z}_p}$ whose objects are $\text{Spf}(A)$ where $A$ is an adic and noetherian ring which is $\mathbb{Z}_p$-algebra and $A/J$ ($J$ is the biggest ideal of definition) is a finitely generated $\mathbb{F}_p$-algebra and morphisms are locally ringed space morphisms. Let $\text{Rig}_K$ be the category of $K$-rigid analytic varieties. ($K$-rigid analytic variety is a locally $G$-ringed space and a morphism is a locally $G$-ringed morphism.- A locally $G$-ringed space is a pair $(X, \mathcal{O}_X)$, where $X$ is a set equipped with a saturated Grothendieck topology and $\mathcal{O}_X$ is a sheaf of rings such that all stalks are local rings.) P.Berthelot had constructed a functor from $A\text{F}_{\mathbb{Z}_p}$ to $\text{Rig}_K$. We refer to [Col-Mz] and [De] for construction. Note that $\text{Spf}(\mathbb{Z}_p[[1+p\mathbb{Z}_p]])$ belongs to $A\text{F}_{\mathbb{Z}_p}$. So we can construct a $K$-rigid analytic variety associated to $\mathbb{Z}_p[[1+p\mathbb{Z}_p]]$. This turns out to be the open unit disk $B(0,1)_K$ defined over $K$. Since $\Lambda$ and $\Lambda_N$ are finite direct sums of $\mathbb{Z}_p[[1+p\mathbb{Z}_p]]$, we can apply P. Berthelot’s contraction to $\Lambda$ and $\Lambda_N$. We denote the resulting $K$-rigid analytic varieties by $\mathcal{X}$ and $\mathcal{X}_N$ respectively. In this case $\mathcal{X}$ (respectively $\mathcal{X}_N$) is the finite union of $\varphi(p)$ (respectively $\varphi(pN)$) open unit disks and each component corresponds to the Dirichlet character of $(\mathbb{Z}/p\mathbb{Z})^\times$ (respectively $(\mathbb{Z}/pN\mathbb{Z})^\times$). We call $\mathcal{X}$ (respectively $\mathcal{X}_N$) weight space (respectively weight space of tame conductor $N$). So $\mathcal{X} = \mathcal{X}_1$. 

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For any $K$-rigid analytic variety $X$ (respectively any $K$-affinoid variety $X$) we define $A(X)$ to be the ring of $K$-rigid analytic functions on $X$ (respectively $K$-affinoid functions on $X$). Then there is a natural ring homomorphism
\[ \Lambda_N \to A(X_N) \]
for $N \geq 1$. In the case that $X$ is a $K$-affinoid variety over $X_N$, the $R$-valued points of $X$, for any commutative $\mathbb{Q}_p$-algebra $R$, are given by continuous homomorphisms from $A(X)$ to $R$, i.e.
\[ X(R) := \text{Hom}_{cts}(A(X), R). \]
The $R$-valued points of $X_N$ are given by
\[ X_N(R) = \text{Hom}_{cts}(\Lambda_N, R). \]
for $N \geq 1$. Recall that there is a natural bijection between
\[ \text{Hom}_{cts}(\Lambda_N, \mathbb{C}_p) \quad \text{and} \quad \text{Hom}_{cts}(\mathbb{Z}_{p,N}^\times, \mathbb{C}_p^\times). \]
So in particular, the $\mathbb{C}_p$-points of the weight space $X_N$ is $\text{Hom}_{cts}(\mathbb{Z}_{p,N}^\times, \mathbb{C}_p^\times)$.

**Definition 3.1.** A character $\kappa : 1 + p\mathbb{Z}_p \to \mathbb{C}_p^\times$ is called arithmetic if
\[ \kappa(t) = t^k \]
for some $k \in \mathbb{Z}^\geq 0$ and for all $t$ sufficiently close to 1 in $1 + p\mathbb{Z}_p$.

**Definition 3.2.** Let $\kappa \in X(\mathbb{C}_p)$ be a $\mathbb{C}_p$-point of a $K$-affinoid variety $X$ over $X_N$ or $X_N$ itself. We call $\kappa$ an arithmetic point of signature of $(k, \chi)$, if the associated group character $\kappa : \mathbb{Z}_{p,N}^\times \to \mathbb{C}_p^\times$, via the natural homomorphism $\text{(3.1)}$ and the correspondence $\text{(3.2)}$, satisfies
\[ \kappa(t) = \chi(t) \cdot t_p^k \]
where $t = (t_p, t_N) \in \mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times$ and for some $\chi$, a finite character of $\mathbb{Z}_{p,N}^\times$, and $k \in \mathbb{Z}^\geq 0$.

We use the following notation: for any $K$-affinoid variety $X$ (or $X_N$) and any commutative $\mathbb{Q}_p$-algebra $R$,
\[ X_{\text{arith}}(R) := \{ \kappa \in X(R) \mid \kappa \text{ is arithmetic.} \}. \]

Now we state the connection between locally analytic distributions $\mathcal{D}(\mathbb{Z}_{p,N}^\times)$ and $A(X_N)$, the $K$-rigid analytic functions on $X_N$. Recall that there is a topological $K$-algebra isomorphism (as $K$-Fréchet spaces)
\[ F : \mathcal{D}(\mathbb{Z}_{p,N}^\times) \cong A(X_N) \]
where $f_\mu(\kappa) := \int_{\mathbb{Z}_{p,N}^\times} \kappa(x) d\mu(x)$ which is the $p$-adic Fourier transform of Amice (see the $\text{[Ami1]}$ and $\text{[Ami2]}$).

**Definition 3.3.** We define
\[ \mathcal{D}_B := \mathcal{D} \otimes_{A(X)} A(B) \]
where $B$ is any $K$-affinoid subdomain of $X_N$ and $A(B)$ is the associated $K$-affinoid algebra, endowed with the spectral norm $| \cdot |_{A(B)}$. 19
Let $S_0(Np)$ act on $D_B$ by the formula $(\mu \otimes_{A(\mathcal{X})} \lambda) x = \mu x \otimes_{A(\mathcal{X})} a [N] \lambda$ for $B \subseteq \mathcal{X}_N$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $[a]_N$ is the image of $a$ in $(\mathbb{Z}/N\mathbb{Z})^\times \subseteq \Lambda^\times_N$. Since the action of $A(\mathcal{B})$ on

$\mathcal{D}_B$ commutes with the action of $\Gamma_0(Np)$ on $\mathcal{D}_B$, $H^1_c(\Gamma_0(Np), \mathcal{D}_B)$ inherits an action of $A(\mathcal{B})$. Indeed $H^1_c(\Gamma_0(Np), \mathcal{D}_B)$ is equipped with an $A(\mathcal{B})[T]$-module structure. In particular, $U_p$ acts on $H^1_c(\Gamma_0(N), \mathcal{D}_B)$. In fact we have an $A(\mathcal{B})$-linear endomorphism $U_p : H^1_c(\Gamma_0(Np), \mathcal{D}_B) \to H^1_c(\Gamma_0(Np), \mathcal{D}_B)$. G. Stevens proved the following theorem in [St3] which is essential to prove the existence of slope $\leq h$ decomposition.

**Theorem 3.4.**

1. $H^1_c(\Gamma_0(Np), \mathcal{D}_B)$ is orthonormalizable as an $A(\mathcal{B})$-module.
2. The $U_p$-action on $H^1_c(\Gamma_0(Np), \mathcal{D}_B)$ is completely continuous.

The above theorem enables us to apply the Fredholm-Riesz-Serre theory, and in particular, we have the Fredholm series $\det(1 - U_p \cdot T | H^1_c(\Gamma_0(Np), \mathcal{D}_B)) \in A(\mathcal{B})[[T]]$ which is entire in $T$. For details see [St3] and [Ser].

Now we recall the definition of slope $\leq h$ decomposition over $B$ from [A-S2].

**Definition 3.5.** $x \in H^1_c(\Gamma_0(Np), \mathcal{D}_B)$ is said to have slope $\leq h$ with respect to $U_p$ for some $h \in \mathbb{R}$ if there is a polynomial $Q \in A(\mathcal{B})[T]$ with the following properties:

1. $Q^*(U_p) \cdot x = 0$ where $Q^*(T) = T^d \cdot Q(1/T)$ with $d = \deg(Q)$;
2. the leading coefficient of $Q$ is a multiplicative unit with respect to $| \cdot |_{A(\mathcal{B})}$; and
3. every slope of $Q$ is $\leq h$, where the slopes of $Q$ are the slopes of the Newton polygon of $Q$ (see [A-S2] for the definition of the Newton polygon and its slope).

$H^1_c(\Gamma_0(Np), \mathcal{D}_B)^{(\leq h)}$ is defined as the set of all elements of $H^1_c(\Gamma_0(Np), \mathcal{D}_B)$ having slope $\leq h$.

Then $H^1_c(\Gamma_0(Np), \mathcal{D}_B)^{(\leq h)}$ is an $A(\mathcal{B})$-submodule of $H^1_c(\Gamma_0(Np), \mathcal{D}_B)$.

**Definition 3.6.** A slope $\leq h$ decomposition of $H^1_c(\Gamma_0(Np), \mathcal{D}_B)$ over $B$ is an $A(\mathcal{B})[U_p]$-module decomposition

$$H^1_c(\Gamma_0(Np), \mathcal{D}_B) = H^1_c(\Gamma_0(Np), \mathcal{D}_B)^{(\leq h)} \oplus H^1_c(\Gamma_0(Np), \mathcal{D}_B)^{(\leq h)}$$

such that

1. $H^1_c(\Gamma_0(Np), \mathcal{D}_B)^{(\leq h)}$ is a finitely generated $A(\mathcal{B})$-module; and
2. for every polynomial $Q \in A(\mathcal{B})[T]$ of slope $\leq h$, the map

$$Q^*(U_p) : H^1_c(\Gamma_0(Np), \mathcal{D}_B)^{(\leq h)} \to H^1_c(\Gamma_0(Np), \mathcal{D}_B)^{(\leq h)}$$

is an isomorphism of $A(\mathcal{B})$-modules.

Then the following theorem is due to G. Stevens([St3]).

**Theorem 3.7.** Let $x_0 \in \mathcal{X}_N(K)$ be any $K$-point and let $h$ be a fixed nonnegative rational number. Then there exists a $K$-affinoid subdomain $B_{h,N} \subseteq \mathcal{X}_N$ containing $x_0$ such that $H^1_c(\Gamma_0(Np), \mathcal{D}_{B_{h,N}})$ admits a slope $\leq h$ decomposition over $B_{h,N}$.

$H^1_c(\Gamma_0(Np), \mathcal{D}_{B_{h,N}})^{(\leq h)}$ will play a central role to define the $p$-adic overconvergent Hecke algebra with slope $\leq h$ whose maximum spectrum (associated $K$-affinoid variety) can be thought of as a local piece of the eigencurve.
3.2. **Slope \( \leq h \) overconvergent p-adic Hecke algebra and p-adic families of overconvergent Hecke eigenforms.**

Note that the Hecke operators \( T_n \) for all \( n \in \mathbb{N} \) preserves \( H^1_c(\Gamma_0(Np), D_{B_{h,N}})^{(\leq h)} \).

**Definition 3.8.** We define the universal overconvergent p-adic Hecke algebra of tame conductor \( N \) with slope \( \leq h \)

\[
\mathcal{R}_{h,N} := \text{Im}(H_B \xrightarrow{H} \text{End}_{A(B_{h,N})}(H^1_c(\Gamma_0(Np), D_{B_{h,N}})^{(\leq h)}))
\]

where \( H_B := A(B_{h,N})[T_n : n \in \mathbb{N}] \) is the abstract overconvergent Hecke algebra over \( B_{h,N} \) of tame conductor \( N \).

It is known that \( A(B_{h,N}) \) is a principal ideal domain and we have the following proposition.

**Proposition 3.9.** We have that

1. \( \mathcal{R}_{h,N} \) is a finite, flat and torsion-free \( A(B_{h,N}) \)-algebra.
2. \( \mathcal{R}_{h,N} \) is a \( K \)-affinoid algebra.

**Proof.**

1. Since \( H^1_c(\Gamma_0(Np), D_{B_{h}})^{(\leq h)} \) is a finitely generated \( A(B_{h,N}) \)-module, \( \mathcal{R}_{h,N} \) should be a finitely generated \( A(B_{h,N}) \)-module by the definition. The \( A(B_{h,N}) \)-torsion-freeness of \( \mathcal{R}_{h,N} \) is clear from the definition. Since \( A(B_{h,N}) \) is a PID, torsion-freeness implies the freeness. So in particular \( \mathcal{R}_{h,N} \) is flat over \( A(B_{h,N}) \).

2. This part follows from [BGR](Proposition 5 on page 223) since, by (1), \( \mathcal{R}_{h,N} \) is finitely generated as an \( A(B_{h,N}) \)-module.

Let \( \Omega_h := \text{Sp}(\mathcal{R}_h) \) (respectively, \( \Omega_{h,N} := \text{Sp}(\mathcal{R}_{h,N}) \)) be the \( K \)-affinoid space associated to \( \mathcal{R}_h \) (respectively, \( \mathcal{R}_{h,N} \)) - See 7.1 of [BGR]. Then we have \( A(\Omega_h) = \mathcal{R}_h \) and \( A(\Omega_{h,N}) = \mathcal{R}_{h,N} \). So we get the following commutative diagrams:

\[
\begin{array}{ccc}
\mathcal{R}_{h,N} & \xrightarrow{\quad} & \mathcal{R}_h \\
\downarrow & & \downarrow \\
A(B_{h,N}) & \xrightarrow{\quad} & A(B_h) \\
\downarrow & & \downarrow \\
A(\mathcal{X}_N) & \xrightarrow{\quad} & A(\mathcal{X}) \\
\downarrow & & \downarrow \\
\Lambda_N & \xrightarrow{\quad} & \Lambda \\
\end{array}
\quad
\begin{array}{ccc}
\Omega_{h,N} & \xrightarrow{\quad} & \Omega_h \\
\downarrow & & \downarrow \\
B_{h,N} & \xrightarrow{\quad} & B_h \\
\downarrow & & \downarrow \\
\mathcal{X}_N & \xrightarrow{\quad} & \mathcal{X} \\
\downarrow & & \downarrow \\
\Lambda & \xrightarrow{\quad} & \Lambda \\
\end{array}
\]

**Definition 3.10.** We define the universal overconvergent Hecke eigenform of tame conductor \( N \) with slope \( \leq h \) to be

\[
f_{h,N} := \sum_{n=1}^{\infty} \alpha_n q^n \in \mathcal{R}_{h,N}[[q]]
\]

where \( \alpha_n \) is the image of \( T_n \) in \( \text{End}_{A(B_{h,N})}(H^1_c(\Gamma_0(Np), D_{B_{h,N}})^{(\leq h)}) \) under the map \( H \).

We recall that \( S_{k+2}(\Gamma_0(Np^m), \chi; R) \) is the subspace of \( S_{k+2}(\Gamma_0(Np^m), \chi) \) consisting of cusp forms whose Fourier coefficients belong to \( R \) where \( R \) is any subfield of \( \mathbb{C}_p \). The following theorem was proven by R. Coleman [Col2, 0.1].
Theorem 3.11. Let $\kappa \in \Omega_{h,N}^{\text{arith}}(K)$ be an arithmetic $K$-point of signature $(k, \chi)$ with $k \geq 0$. If $h < k + 1$ or if $h = k + 1$, and $f_{h,N}(\kappa) := \sum_{n=1}^{\infty} \alpha_n(\kappa)q^n$ is not in the image of the map $\Theta^{k+1}$ where $\Theta$ is the operator which acts as $q \frac{d}{dq}$ on formal $q$-expansions, then

$$f_{h,N}(\kappa) := \sum_{n=1}^{\infty} \alpha_n(\kappa)q^n \in M_{k+2}(\Gamma_0(Np^m), \chi; K)$$

where $\alpha_n(\kappa) := \kappa(\alpha_n)$ and $\chi$ is defined modulo $Np^m$ and moreover, $f_{h,N}(\kappa)$ is an Hecke eigenform. Furthermore, the $U_p$-eigenvalue of $f_{h,N}(\kappa)$ has $p$-adic valuation (called the slope of $f_{h,N}(\kappa)$) $\leq h$.

We will use the following notations:

$$\Omega_{h,N}^{\text{classical}}(K) := \{ \kappa : \mathcal{R}_{h,N} \to K \mid f_{h,N}(\kappa) \text{ is a classical Hecke eigenform.} \}.$$  

$$B_{h,N}^{\text{classical}}(K) := \{ \hat{\kappa} : A(B_{h,N}) \to K \mid \exists \kappa \in \Omega_{h,N}^{\text{classical}}(K) \text{ such that } \kappa \text{ is lying above } \hat{\kappa} \}.$$  

The elements of $\Omega_{h,N}^{\text{classical}}(K)$ will be referred to as classical $K$-points of $\Omega_{h,N}^{\text{classical}}$. We can view $f_{h,N}$ as an analytic function on $\Omega_{h,N}^{\text{classical}}$ which interpolates the $q$-expansions of Hecke eigenforms at classical points.

3.3. Overconvergent Hecke eigensymbol.

In this section we will prove the existence of an overconvergent Hecke eigensymbol.

Definition 3.12. For $\kappa \in \Omega_{h,N}^{\text{classical}}(K)$ and $\Phi \in H^1_c(\Gamma_0(Np), D_{B_{h,N}})$, we define

$$\Phi_{\kappa} := \phi_{\kappa,*}(\Phi)$$

where the map $\phi_{\kappa,*}$ is given in (1.18). For $\kappa \in \Omega_{h,N}^{\text{classical}}(K)$ of signature $(k, \chi)$, we also define

$$\varphi_{\kappa}^\pm := \varphi_{f_{h,N}(\kappa)} := \frac{1}{\Omega_{f_{h,N}(\kappa)}^\pm} \psi_{f_{h,N}(\kappa)}^\pm$$

where we fix, once and for all, complex periods $\Omega_{f_{h,N}(\kappa)}^\pm$ for each Hecke eigenform $f$ so that

$$\frac{1}{\Omega_{f_{h,N}(\kappa)}^\pm} \cdot \psi_{f_{h,N}(\kappa)}^\pm \in H^1_{par}(\Gamma_0(Np^m), L_{k,\chi}(R_{\kappa}))$$

For details see (4.3.4) of [St1] about the existence of these complex periods.

Once we fix complex periods for each $f$, we can give an algebraic version of the classical Shintani $\theta$-lifting. Define

$$\theta_{k,\chi}^*(f) := \frac{1}{\Omega_f^*} \cdot \theta_{k,\chi}(f)$$

for $f \in S_{2k+2}(\Gamma_0(M), \chi^2)$, a Hecke eigenform. Then, by theorem (4.3.6) of [St1], under the assumption of the proposition 2.6

$$\Theta_{k,\chi}(\varphi_f^-) = \theta_{k,\chi}^*(f)$$

where $\varphi_f^- := \frac{1}{\Omega_f^-} \cdot \psi_f^-$. Note that $\theta_{k,\chi}^*$ depends on the choice of complex periods and is defined only on Hecke eigenforms.
For \( \hat{k} \in B_{h,N}^{\text{classical}}(K) \), we similarly define an \( R_{\hat{k}} \)-linear map
\[
\phi_{\hat{k}} : D_{B_{h,N}} = \mathcal{D} \otimes_{A} A(B_{h,N}) \rightarrow L_{k,\chi}(R_{\hat{k}})
\]
by replacing \( \kappa \) by \( \hat{k} \) in (3.17), where \( R_{\hat{k}} := \hat{k}(A(B_{h,N})) \). Then the following theorem due to G. Stevens (see [St3]) is one of the major steps for the existence of an overconvergent Hecke-eigen symbol.

**Theorem 3.13.** Assume \( \hat{k} \in B_{h,N}^{\text{classical}}(K) \) of signature \( (k, \chi) \) and \( h < k + 1 \). Then
\[
(1) \text{ there is a Hecke-equivariant isomorphism}
\]
\[
H_{1}^{1}(\Gamma_{0}(Np), D_{B_{h,N}} \otimes_{A(B_{h,N})} K)(\leq h) \cong H_{1}^{1}(\Gamma_{0}(Np)^{\kappa}, L_{k,\chi}(K))(\leq h)
\]
induced from (3.14), where \( m \) is the smallest positive integer for which \( \chi \) is defined modulo \( Np^{m} \) and \( \otimes_{A(B_{h,N})} \) is the tensor product taken with respect to \( \hat{k} : A(B_{h,N}) \rightarrow K \).

(2) There is a canonical identification
\[
H_{1}^{1}(\Gamma_{0}(Np), D_{B_{h,N}}) \otimes_{A(B_{h,N})} K \simeq H_{1}^{1}(\Gamma_{0}(Np), D_{B_{h,N}} \otimes_{A(B_{h,N})} K)
\]
in the case \( k > 0 \).

In the remaining of this section, we will use the following abbreviated notations:
\[
H_{c}^{1}(D)^{\ast} := H_{c}^{1}(\Gamma_{0}(Np), D_{B_{h,N}})(\leq h)^{\ast},
\]
\[
H_{c}^{1}(L_{\hat{k}})^{\ast} := H_{c}^{1}(\Gamma_{0}(Np^{m}), L_{k,\chi}(R_{\hat{k}}))(\leq h)^{\ast},
\]
where \( \hat{k} \in B_{h,N}^{\text{classical}}(K) \) is a \( K \)-point of signature \( (k, \chi) \), \( 0 \leq h < k + 1 \) and \( m \) is the smallest positive integer for which \( \chi \) is defined modulo \( Np^{m} \). Let \( m_{\hat{k}} \) be a maximal ideal \( \ker(\hat{k}) \subseteq A(B_{h,N}) \). Robert Coleman proved that there is a point \( \kappa \in \Omega^{\text{classical}}_{h,N}(\mathbb{C}) \) which is unramified over the point \( \hat{k} \in B_{h,N}^{\text{classical}}(K) \) in [Col1]. Put \( m_{\kappa} = \ker(\kappa) \subseteq \mathcal{R} \).

**Corollary 3.14.** There is a Hecke-equivariant \( \mathcal{R}/m_{\hat{k}} \mathcal{R} \)-module isomorphism between
\[
H_{1}^{1}(D)^{\ast} / m_{\hat{k}} H_{c}^{1}(D)^{\ast} \simeq H_{1}^{1}(L_{\hat{k}})^{\ast}
\]
induced from the map (3.13). The right hand side is an \( \mathcal{R}(\hat{k}) \)-module where \( \mathcal{R}(\hat{k}) := \text{Im}(R_{\hat{k}}[T_{n} : n \in \mathbb{N}] \rightarrow \text{End}_{K}(H_{c}^{1}(L_{\hat{k}})^{\ast})) \). We have a \( K \)-algebra isomorphism from \( \mathcal{R}/m_{\hat{k}} \mathcal{R} \) to \( \mathcal{R}(\hat{k}) \) so that we can regard the right hand side a \( \mathcal{R}/m_{\hat{k}} \mathcal{R} \)-module by this \( K \)-algebra isomorphism.

**Proof.** It follows from
\[
H_{c}^{1}(D)^{\ast} / m_{\hat{k}} H_{c}^{1}(D)^{\ast} \simeq H_{1}^{1}(D)^{\ast} \otimes_{A(B_{h,N})} K
\]
\[
\simeq H_{1}^{1}(L_{\hat{k}})^{\ast} \text{ (by (3.14) and (3.15)).}
\]

**Lemma 3.15.** If \( \kappa \in \Omega_{h,N}^{\text{classical}}(\mathbb{C}) \) is unramified over \( \hat{k} \in B_{h,N}^{\text{classical}}(K) \), then \( H_{c}^{1}(D)^{\ast}_{\kappa} \) is a free \( \mathcal{R}_{(\kappa)} \)-module of rank 1, where we use the subscript \( (\kappa_{i}) \) to denote the localization at \( \kappa_{i} \) (using the maximal ideal \( m_{\kappa_{i}} \)).

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Proof. Note that $\mathcal{R}/m_\kappa \mathcal{R}$ is a finite free $K$-algebra and so it is the product of local $K$-algebras $\prod_{i=1}^r \mathcal{R}_{\kappa_i}$ (d is some positive integer) where each $\mathcal{R}_{\kappa_i}$ is isomorphic to $\mathcal{R}(m_\kappa \mathcal{R})_{(\kappa_i)}$-localization at $\kappa_i$ of $\mathcal{R}/m_\kappa \mathcal{R}$, where $\kappa_i$'s are points lying over $\hat{k}$. Then there exists some $i > 0$ such that $\kappa = \kappa_i$ which is unramified over $\hat{k}$. Let $m_{\kappa_i}$ be $\ker(\kappa_i) \subseteq \mathcal{R}$:

$$
\begin{array}{c}
m_{\kappa_i} \rightarrow \mathcal{R} \\
\mathcal{R} \rightarrow \mathcal{R}_{\kappa_i} \\
\mathcal{R}_{\kappa_i} \rightarrow A(B_{h,N})
\end{array}
$$

We denote the localization at $\kappa_i$ (using the maximal ideal $m_{\kappa_i}$) by the subscript $(\kappa_i)$. If $N$ is square-free, then it is known that $H_c^1(L_\kappa)^* \otimes \mathcal{R}(\kappa_i)$-module of rank 1 by a slight extension of Eichler-Shimura theory. If $N$ is not square-free, we still know that $H_c^1(L_\kappa)^*_{(\kappa_i)}$ is a free $\mathcal{R}_{\kappa_i}$-module of rank 1. So we conclude that $(H_c^1(D)^* / m_\kappa H_c^1(D))^*_{(\kappa_i)}$ is a free $\mathcal{R}(m_\kappa \mathcal{R})_{(\kappa_i)}$-module of rank 1 by the above corollary 3.14.

Then we have

$$(H_c^1(D)^* / m_\kappa H_c^1(D))^*_{(\kappa_i)} \simeq H_c^1(D)^*_{(\kappa_i)}/m_{\kappa_i} H_c^1(D)^*_{(\kappa_i)} \simeq H_c^1(D)^*_{(\kappa_i)}/m_{\kappa_i} H_c^1(D)^*_{(\kappa_i)} \text{ (because of unramifiedness).}$$

so we have that $H_c^1(D)^*_{(\kappa_i)}/m_{\kappa_i} H_c^1(D)^*_{(\kappa_i)}$ is a free $\mathcal{R}(m_\kappa \mathcal{R})_{(\kappa_i)}$-module of rank 1. Hence it is a free $\mathcal{R}/m_\kappa \mathcal{R}$-module of rank 1, since $\kappa_i$ is unramified over $\hat{k}$. Now if we apply the Nakayama Lemma, we conclude that $H_c^1(D)^*_{(\kappa_i)}$ is a free $\mathcal{R}_{(\kappa_i)}$-module of rank 1.

Remark 3.16. $H_c^1(D)^*$ doesn’t have to be a free $\mathcal{R}$-module of rank 1. Local free rank 1 property for every localization at prime ideals doesn’t imply the global rank 1 property in general.

Theorem 3.17. Let $\kappa_0 \in \Omega_{h,N}^{\text{classical}}(K)$ be any classical $K$-point which is unramified over $B_{h,N}$. There is an overconvergent Hecke eigensymbol $\Phi \in H_c^1(\Gamma_0(Np), \mathcal{D}_{B_{h,N}}(\Sigma^h))$ and a choice of periods $\Omega_\kappa \in R_\kappa$ for $\kappa \in \Omega_{h,N}^{\text{classical}}(K)$ such that

(1) $\Omega_{\kappa_0} \neq 0$ 
(2) $\Phi_\kappa = \Omega_\kappa \varphi_\kappa$ for any point $\kappa \in \Omega_{h,N}^{\text{classical}}(K)$.

Proof. Having Lemma 3.15 in our hands, the theorem follows from the same argument in the proof of the theorem 5.5 in [St1].

3.4. $p$-adic family of overconvergent half-integral weight Hecke eigenforms via local overconvergent Shintani lifting.

Let $\sigma : \Lambda_N \rightarrow \Lambda_N$ be the ring homomorphism associated to the group homomorphism $t \rightarrow t^2$ on $\mathbb{Z}_{p,N}^\times$. Then the functoriality of the P. Berthelot’s construction of $K$-rigid analytic variety associated to $\Lambda_N$ gives us the map $\sigma : \mathcal{X}_N \rightarrow \mathcal{X}_N$ and consequently we get the ring homomorphism (using the same notation $\sigma$)

$$(1.37) \quad \sigma : A(\mathcal{X}_N) \rightarrow A(\mathcal{X}_N),$$

which commutes with the natural map $\Lambda_N \rightarrow A(\mathcal{X}_N)$ and $\sigma : \Lambda_N \rightarrow \Lambda_N$. If we use the $p$-adic Fourier transform of Amice (3.3), the above $\sigma$ is in fact same as $\sigma : \mathcal{D}(\mathbb{Z}_{p,N}^\times) \rightarrow \mathcal{D}(\mathbb{Z}_{p,N}^\times)$ in (2.13).
**Definition 3.18.** We define

\[(3.18) \quad \tilde{A}_{h,N} := A(B_{h,N}) \otimes_{A(\mathcal{X}_N),\sigma} A(\mathcal{X}_N)\]

where $\otimes_{A(\mathcal{X}_N),\sigma}$ is the completed tensor product taken with respect to $\sigma : A(\mathcal{X}_N) \rightarrow A(\mathcal{X}_N)$ which is the ring homomorphism given by the base change from $\sigma : \Lambda_N \rightarrow \Lambda_N$.

**Definition 3.19.** We define the overconvergent metaplectic $p$-adic Hecke algebra of tame conductor $N$ with slope $\leq h$ by

\[(3.19) \quad \tilde{R}_{h,N} := R_{h,N} \otimes_{A(B_{h,N}),\sigma_h} \tilde{A}_{h,N}\]

where $\otimes_{A(B_{h,N}),\sigma_h}$ is the completed tensor product taken with respect to $\sigma_h : A(B_{h,N}) \rightarrow \tilde{A}_{h,N}$ which is the ring homomorphism given by the base change from $\sigma : A(\mathcal{X}_N) \rightarrow A(\mathcal{X}_N)$.

Now we can formulate the following diagram:

\[
\begin{array}{ccc}
\tilde{R}_{h,N} & \xrightarrow{\sigma_h} & R_{h,N} \\
\downarrow & & \downarrow \\
\tilde{A}_{h,N} & \xrightarrow{\sigma_h} & A(B_{h,N}) \\
\downarrow & & \downarrow \\
A(\mathcal{X}_N) & \xrightarrow{\sigma} & A(\mathcal{X}) \\
\downarrow & & \downarrow \\
\Lambda_N & \xrightarrow{\sigma} & \Lambda
\end{array}
\]

**Proposition 3.20.** $\tilde{R}_{h,N}$ is a $K$-affinoid algebra.

*Proof.* The category of $K$-affinoid algebras is closed under completed tensor product with respect to any contractive $K$-algebra homomorphism (Proposition 10 on the page 225 of [BGR]). So the proposition follows, since $\sigma_h$ is a contractive homomorphism. \hfill $\square$

We regard $\tilde{R}_{h,N}$ as a $\Lambda_N$-algebra by equipping it with the structure homomorphism $\Lambda_N \rightarrow \tilde{R}_{h,N}$ given by $f \mapsto 1 \otimes_{A(B_{h,N}),\sigma_h} f$. Then we can easily check that $\tilde{R}_{h,N} := R_{h,N} \otimes_{A(B_{h,N}),\sigma_h} \tilde{R}_{h,N}$ is isomorphic as an $A(\mathcal{X}_N)$-algebra to $R_{h,N} \otimes_{A(\mathcal{X}_N),\sigma} A(\mathcal{X}_N)$ whose $A(\mathcal{X}_N)$-algebra structure is given by $A(\mathcal{X}_N) \rightarrow \tilde{R}_N$, $\lambda \mapsto 1 \otimes_{A(\mathcal{X}_N),\sigma} \lambda$. So by the $p$-adic Fourier transform $\Phi$, we have

\[(3.20) \quad \tilde{R}_{h,N} \cong R_{h,N} \otimes_{D(\mathbb{Z}_{p,N}),\sigma} \mathcal{D}(\mathbb{Z}_{p,N})\]

as a $D(\mathbb{Z}_{p,N})$-algebra. Note that the ring homomorphism

\[(3.21) \quad R_{h,N} \rightarrow \tilde{R}_{h,N}\]

given by $\alpha \mapsto \alpha \otimes_{A(\mathcal{X}_N),\sigma} 1$ is not a homomorphism of $A(\mathcal{X}_N)$-algebras. This is reflected in the fact that the $K$-rigid analytic map induced by pullback on the corresponding $K$-affinoid spaces

\[(3.22) \quad \tilde{\Omega}_{h,N} \rightarrow \Omega_{h,N}\]

does not preserve the signatures of classical points. Indeed, if $\tilde{\kappa} \in \tilde{\Omega}_{h,N}^{\text{classical}}(K)$ has signature $(k,\chi)$ and lies over $\kappa \in \Omega_{h,N}^{\text{classical}}(K)$, then the signature of $\kappa$ is $(2k,\chi^2)$. If we let $\Omega_h$ (respectively, $\tilde{B}_{h,N}$) be the $K$-affinoid space associated to $\mathcal{R}_h$ (respectively, $\tilde{A}_{h,N}$), then we have the following commutative diagram by pulling back the previous diagram:
Now we prepare for the final notations to define a local overconvergent Shintani lifting over $\mathcal{R}_{h,N}$.

Let

$$\mathcal{D}_{\mathcal{R}_{h,N}} := \mathcal{D}_{B_{h,N}} \otimes_{A(B_{h,N})} \mathcal{R}_{h,N}$$

where $S_0(Np)$ acts on $\mathcal{D}_{\mathcal{R}_{h,N}}$ through the first factor. If $\kappa \in \Omega_{h,N}(K)$ is any $K$-point, then let $R_{\kappa} := \kappa(\mathcal{R}_{h,N})$. For $\kappa \in \Omega_{h,N}^{\text{classical}}(K)$ of signature $(k, \chi)$, we define an $R_{\kappa}$-linear map

$$\phi_\kappa : \mathcal{D}_{\mathcal{R}_{h,N}} = \mathcal{D} \otimes_{A(X)} \mathcal{R}_{h,N} \rightarrow L_{k,\chi}(R_{\kappa})$$

by

$$\phi_\kappa(\mu \otimes r) := \kappa(r) \cdot \int_{\mathbb{Z}_p^2} \chi_p(x) \frac{(xY - yX)^k}{k!} d\mu(x,y)$$

for $\mu \in \mathcal{D}$ and $r \in \mathcal{R}_{h,N}$, where we factor $\chi = \chi_N \cdot \chi_p$ with $\chi_N$ defined modulo $N$ and $\chi_p$ defined modulo a power of $p$. A simple computation confirms that if $\chi$ is defined modulo $Np^m$ then $\phi_\kappa$ is a $\Gamma_0(Np^m)$-module homomorphism, hence induces a homomorphism on the cohomology groups:

$$\phi_{\kappa,*} : H^1_c(\Gamma_0(Np), \mathcal{D}_{\mathcal{R}_{h,N}}) \rightarrow H^1_c(\Gamma_0(Np^m), L_{k,\chi}(R_{\kappa}))$$

which is Hecke-equivariant.

We state the following Lemma which is almost identical to the proposition 2.8 which is used crucially to construct a local overconvergent Shintani lifting.

**Lemma 3.21.** For each $Q \in F_{Np}$ there is a unique $\mathcal{R}_{h,N}$-module homomorphism

$$\tilde{J}_Q^B : \mathcal{D}_{B_{h,N}} \otimes_{A(B_{h,N})} \mathcal{R}_{h,N} \rightarrow \tilde{\mathcal{R}}_{h,N}$$

such that for all $\tilde{\kappa} \in \tilde{\Omega}_{h,N}^{\text{classical}}(K)$ of signature $(k, \chi)$ lying over $\kappa \in \Omega_{h,N}^{\text{classical}}(K)$, we have

$$\tilde{\kappa}(\tilde{J}_Q^B(\mu)) = \chi(Q) \cdot \langle \phi_\kappa(\mu), Q^k \rangle$$

for $\mu \in \mathcal{D}_{B_{h,N}} \otimes_{A(B_{h,N})} \mathcal{R}_{h,N} = \mathcal{D}_{\mathcal{R}_{h,N}}$ and $\phi_\kappa(\mu)$ was defined in 3.23.

**Proof.** Once we define the appropriate map, the proofs of uniqueness and interpolation properties are exactly same as the proposition 2.8. So we concentrate on constructing the appropriate $\mathcal{R}_{h,N}$-linear map. We recall that there is a topological $K$-algebra isomorphism (as $K$-Fréchet space)

$$F : \mathcal{D}(\mathbb{Z}_p^X) \simeq A(X)$$

by the $p$-adic Fourier transform of Amice([Ami1, Ami2]). So using this isomorphism we get a well-defined $K$-linear map $\mathcal{D} \rightarrow A(X)$ by $\nu \mapsto F(J_Q(\nu))$ where $J_Q$ is defined in 2.16.
We have for all $A$.

The interpolation property at classical points follows from the same computation as in the proposition.

We define a local overconvergent Shintani lifting over $B_{h,N}$ by the following:

$$\mathcal{D}_{B_{h,N}} \otimes_{A(B_{h,N})} \mathcal{R}_{h,N} \xrightarrow{\tilde{j}_Q} \tilde{\mathcal{R}}_{h,N}$$

$$(\nu \otimes_{A(X)} f) \otimes_{A(B_{h,N})} r \xrightarrow{r} r \otimes_{A(B_{h,N}),\sigma_h} \left(\sigma_h(f) \cdot (1 \otimes_{A(X),\sigma} [Q]_N \cdot F(j_Q(\nu)))\right)$$

for all $\nu \in \mathcal{D}, f \in A(B_{h,N})$ and $r \in \mathcal{R}_{h,N}$, where $[Q]_N$ and $F(j_Q(\nu))$ are regarded as elements of $A(X)_\nu$ by the natural inclusion $\Lambda_N \rightarrow A(X)_\nu$ and $A(X) \rightarrow A(X)_\nu$ respectively and $\sigma_h(f)$ is the image of $f$ under $\sigma_h : A(B_{h,N}) \rightarrow \tilde{A}_{h,N}$.

Let $[t]$ be the image of $t$ under the natural map $\Lambda \rightarrow A(X)$ and recall $A(X)_\nu$ acts on $\mathcal{D}$. Since we have $F(j_Q([t] \cdot \nu)) = [t]^2 \cdot F(j_Q(\nu))$ and $\sigma([t] \cdot f) = [t]^2 \cdot \sigma(f)$ for all $t \in \mathbb{Z}_p^\times$, the map $\mathcal{D}_{B_{h,N}} \rightarrow \tilde{A}_{h,N}$ by $\nu \otimes_{A(X)} f \mapsto \sigma_h(f) \cdot (1 \otimes_{A(X),\sigma} [Q]_N \cdot F(j_Q(\nu)))$ is $\mathbb{Z}_p$-module homomorphism. Then $\tilde{j}_Q$ is actually an $\mathcal{R}_{h,N}$-linear extension of this map. If we identify $(\mathcal{D} \otimes_{A(X)} A(B_{h,N})) \otimes_{A(B_{h,N})} \mathcal{R}_{h,N}$ with $\mathcal{D} \otimes_{A(X)} \mathcal{R}_{h,N}$, then the map is given by

$$(3.27) \quad \nu \otimes_{A(X)} r \mapsto r \otimes_{A(B_{h,N}),\sigma_h} (1 \otimes_{A(X),\sigma} [Q]_N \cdot F(j_Q(\nu))).$$

The interpolation property at classical points follows from the same computation as in the proposition.

We can define a local overconvergent Shintani lifting over $B_{h,N}$.

**Definition 3.22.** For each $\Phi \in H^1_c(\Gamma_0(Np), \mathcal{D}_{\mathcal{R}_{h,N}})$ and each $Q \in \mathcal{F}$ we define

$$(3.28) \quad J_B(\Phi, Q) := \tilde{j}_Q(\Phi(D_Q)) \in \tilde{\mathcal{R}}_{h,N}$$

where $\Phi(D_Q)$ is the value of the cocycle representing $\Phi$ on $D_Q$. Note that this definition of $J(\Phi, Q)$ does not depend on the representative cocycle, and depends only on the $\Gamma_0(Np)$-equivalence class of $Q$.

**Definition 3.23.** We define a local overconvergent Shintani lifting over $B_{h,N}$ to be a map $\Theta_B : H^1_c(\Gamma_0(Np), \mathcal{D}_{\mathcal{R}_{h,N}}) \rightarrow \tilde{\mathcal{R}}_{h,N}[[q]]$ given by

$$\Theta_B(\Phi) := \begin{cases} \sum_{Q \in \mathcal{F}_{Np}/\Gamma_0(Np)} J_B(\Phi, Q) q^{\delta_Q/Np} & \text{if } Np \text{ is odd}, \\ \sum_{Q \in \mathcal{F}_{Np}/\Gamma_0(Np)} J_B(\Phi, Q) q^{\delta_Q/4Np} & \text{if } Np \text{ is even}. \end{cases}$$

**Definition 3.24.** We say that $\Theta \in \tilde{\mathcal{R}}_{h,N}[[q]]$ is a universal overconvergent half-integral weight modular form over $B_{h,N}$ if

$$\Theta = \Theta_B(\Phi)$$

for some $\Phi \in H^1_c(\Gamma_0(Np), \mathcal{D}_{B_{h,N}})(\leq h)$.

Let me also give a definition of an overconvergent half-integral weight modular form of weight $\kappa \in \text{Hom}_{cts}(A(B_{h,N}), K)$. 

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Definition 3.25. We say that $\theta \in \mathbb{C}_p[[q]]$ is an overconvergent half-integral weight modular form of weight $\kappa \in \text{Hom}_{cts}(A(B_{h,N}), K)$ if it can be written as

$$\theta = \Theta(\kappa) = \sum_{n=1}^{\infty} \left( r_n(\kappa) \otimes \mathcal{D}(\mathbb{Z}_{p,N}^\times), \alpha_n(\kappa) \right) q^n$$

where $\Theta = \sum_{n=1}^{\infty} \left( r_n \otimes \mathcal{D}(\mathbb{Z}_{p,N}^\times), \alpha_n \right) q^n \in \hat{\mathcal{R}}_{h,N}[[q]]$ is a universal overconvergent half-integral weight modular form.

Motivated by the theorem 2.11 it’s tempting to define the actions of $T_l$ for any rational prime $l$ and $T_{l,l}$ for $l \nmid Np$ on a universal overconvergent half-integral weight modular form over $B_{h,N}$

$$\Theta = \sum_{n=1}^{\infty} \left( r(n) \otimes \mathcal{D}(\mathbb{Z}_{p,N}^\times), b(n) \right) q^n \in \hat{\mathcal{R}}_{h,N}[[q]]$$

as follows:

$$(3.29) \quad \Theta|T_l = \sum_{n=1}^{\infty} \left( r(n) \otimes \mathcal{D}(\mathbb{Z}_{p,N}^\times), \left( b(nl^2) + \left( \frac{Np \cdot n}{l} \right)[l]_N \delta_l * b(n) + l[l^2]_N \delta_{l^2} * b(\frac{n}{l^2}) \right) q^n$$

$$(3.30) \quad \Theta|T_{l,l} = \sum_{n=1}^{\infty} \left( r(n) \otimes \mathcal{D}(\mathbb{Z}_{p,N}^\times), [l^2]_N \cdot \delta_{l^2} * b(n) \right) q^n.$$

Henceforth we can view $\hat{\mathcal{R}}_{h,N}[[q]]$ as an $\mathcal{H}$-module where $\mathcal{H}$ was given in the Definition 1.5. We warn you that $T_n$-action on $\hat{\mathcal{R}}_{h,N}[[q]]$ is NOT given by the $\mathcal{R}_{h,N}$-module structure of $\hat{\mathcal{R}}_{h,N}$. Also notice that $\mathcal{D}(\mathbb{Z}_{h,N}^\times)$ act on $\hat{\mathcal{R}}_{h,N}$ through the first factor. Then we can conclude that our local overconvergent Shintani lifting $\Theta_B$ is $\mathcal{H}$-linear map by Theorem 2.11 except for operators containing $T_2$. But if $N$ is even, then $\Theta_B(\Phi|T_2) = \Theta_B(\Phi)|T_2$ will still be true so that $\Theta$ is actually $\mathcal{H}$-linear. Therefore we get the following local version of the theorem 2.11.

Theorem 3.26. The map $\Theta_B$ is an $\mathcal{H}$-module homomorphism except for operators containing $T_2$.

We can talk about a universal overconvergent half-integral weight Hecke eigenform over $B_{h,N}$, since we defined Hecke action on $\hat{\mathcal{R}}_{h,N}[[q]]$. Theorem 3.26 implies that the map $\Theta_B$ sends a Hecke eigensymbol to a universal overconvergent half-integral weight Hecke eigenform over $B_{h,N}$. Now we focus on the interpolation property of $\Theta_B$ and will find the universal overconvergent half-integral weight Hecke eigenform over $B_{h,N}$ corresponding to the Hecke eigensymbol we constructed in the theorem 3.17.

Theorem 3.27. For each $\Phi \in H^1_c(\Gamma_0(Np), \mathcal{D}_{\mathcal{R}_{h,N}})$ and for each classical point $\tilde{\kappa} \in \hat{\Omega}_{h,N}^{\text{classical}}(K)$ of signature $(k, \chi)$ lying over $\kappa \in \Omega_{h,N}^{\text{classical}}(K)$ we have

$$(3.31) \quad \Theta_B(\Phi)(\tilde{\kappa}) \mid T_p^{m-1} = \Theta_{k,\chi}(\Phi_\kappa)$$

where $\Phi_\kappa = \phi_{k,*}(\Phi)$ and $m$ is the smallest positive integer for which $\chi$ is defined modulo $Np^m$. Note that here we understand $T_p$ as the action on formal $q$-expansion given by (2.7).
Proof. For $Np$ odd, this follows from the following calculation:

$$
\Theta_B(\Phi)(\kappa) | T_p^{m-1} = \left( \sum_{Q \in \mathcal{F}_{Np}/\Gamma_0(Np)} \kappa(J(\Phi, Q))q^{\delta_Q/Np} \right) | T_p^{m-1}
$$

$$
= \sum_{Q \in \mathcal{F}_{Np}/\Gamma_0(Np)} \chi(Q)\langle \phi_\kappa(\Phi(D_Q)), Q^k \rangle q^{\delta_Q/Np^m}
$$

$$
= \sum_{Q \in \mathcal{F}_{Np^m}/\Gamma_0(Np^m)} J_{k,\chi}(\Phi_\kappa, Q) q^{\delta_Q/Np^m}
$$

$$
= \Theta_{k,\chi}(\Phi_\kappa).
$$

The same computation applies to the $Np$ even case. □

We state the main theorem on the existence of the universal overconvergent half-integral weight Hecke eigenform over $B_{h,N}$ and prove it.

**Theorem 3.28.** Let $\kappa_0 \in \Omega_{h,N}^{\text{classical}}(K)$ be a fixed classical point unramified over $B_{h,N}$. Then there is a formal $q$-expansion $\Theta = \sum_{n=1}^{\infty} \beta_n q^n \in \tilde{R}_{h,N}[[q]]$ and a choice of periods $\Omega_\kappa \in K$, for $\kappa \in \Omega_{h,N}^{\text{classical}}(K)$, with the following properties:

1. $\Omega_{\kappa_0} \neq 0$
2. For every classical point $\kappa \in \tilde{\Omega}_{h,N}^{\text{classical}}(K)$ of signature $(k, \chi)$,

$$
\Theta(\kappa) := \sum_{n=1}^{\infty} \beta_n(\kappa) q^n \in S_{k+\frac{1}{2}}(\Gamma_0(4Np^m), \chi^*; K)
$$

where $m$ is the smallest positive integer for which $\chi$ is defined modulo $Np^m$ and $\chi^*$ is defined by $\chi^*(d) = \chi(d)(\frac{-1}{d})^{(k+1)(Np)}$.

3. If $\kappa$ is the image of $\kappa \in \tilde{\Omega}_{h,N}^{\text{classical}}(K)$ of signature $(k, \chi)$ under the map $\varphi_{22}$, then

$$
\Theta(\kappa) = \Omega_\kappa \cdot \theta_{k,\chi}^*(f_{h,N}(\kappa)) | T_p^{1-m}
$$

where $\theta_{k,\chi}^*(f_{h,N}(\kappa)) | T_p^{1-m} := \alpha_p(\kappa)^{1-m} \cdot \theta_{k,\chi}^*(f_{h,N}(\kappa)) | T_p^{m-1}$.

Proof. There is an Hecke eigensymbol $\Phi \in H^1_c(\Gamma_0(Np), \mathcal{D}_{B_{h,N}})^{(\leq h)}$ and a choice of periods $\Omega_\kappa \in R_\kappa$ by theorem 3.17. We will use $\Phi \otimes 1$ to prove the existence of a formal $q$-expansion $\Theta$. In order for that, we should make sure that $\Phi \otimes 1 \in H^1_c(\Gamma_0(Np), \mathcal{D}_{R_{h,N}})$. This follows from the flatness of $R_{h,N}$ as $A(B_{h,N})$-module, which was guaranteed by proposition 3.9

$$
H^1_c(\Gamma_0(Np), \mathcal{D}_{B_{h,N}})^{(\leq h)} \rightarrow H^1_c(\Gamma_0(Np), \mathcal{D}_{B_{h,N}})
$$
preserves the injectiveness after tensoring $\otimes A(B_{h,N})\mathcal{R}_{h,N}$

$$H^1_c(\Gamma_0(Np), D_{B_{h,N}}) \otimes A(B_{h,N}) \mathcal{R}_{h,N} \rightarrow H^1_c(\Gamma_0(Np), D_{B_{h,N}}) \otimes A(B_{h,N}) \mathcal{R}_{h,N}$$

where the second map $H^1_c(\Gamma_0(Np), D_{B_{h,N}}) \otimes A(B_{h,N}) \mathcal{R}_{h,N} \rightarrow H^1_c(\Gamma_0(Np), D_{B_{h,N}})$ is given by sending $\phi \otimes A(B_{h,N})r$ to $D \mapsto \phi(D) \otimes A(B_{h,N})r$ for $D \in \Delta_0$, which is easily checked to be a well-defined injective homomorphism.

Now we can define

$$\Theta := \Theta_B(\Phi \otimes 1) = \sum_{n=1}^{\infty} \beta_n q^n \in \hat{\mathcal{R}}_{h,N}[q].$$

Then by (1) of theorem 3.17 (1) follows. Let’s prove (2) and (3). Let $\tilde{\kappa} \in \hat{\Omega}_{h,N}^{\text{classical}}(K)$ be a classical $K$-point of signature $(k, \chi)$ and let $\kappa$ be its image in $\Omega_{h,N}^{\text{classical}}(K)$ under the map $\hat{\Theta}_B$. By the theorem 3.27 and (3.32) we have

$$\Theta(\Phi)(\tilde{\kappa}) \mid T_p^{m-1} = \Theta_{k,\chi}(\Phi_{\kappa}) = \Theta_{k,\chi}(\Omega_{\kappa} : \nu_{\kappa}) = \Omega_{\kappa} \cdot \theta_{k,\chi}^{*}(\mathfrak{f}_{h,N}(\kappa)).$$

By the Shintani’s theorem 2.3 for $m = 1$, it follows that $\Theta(\tilde{\kappa}) \mid T_p^2 = \Omega_{\kappa} \cdot \theta_{k,\chi}^{*}(\mathfrak{f}_{h,N}(\kappa)[T_p]) = \Omega_{\kappa} \cdot \theta_{k,\chi}^{*}(\alpha_{p}(\kappa) \cdot \mathfrak{f}_{h,N}(\kappa)) = \alpha_{p}(\kappa) \cdot \Theta(\tilde{\kappa})$. So we can conclude that $(\beta_{np^2})^{\kappa} = \alpha_{p}(\kappa) \cdot \beta_{n}(\tilde{\kappa})$ for every $n$ and every $\tilde{\kappa} \in \hat{\Omega}_{h,N}^{\text{classical}}(K)$ of signature $(k, \chi)$ with $\chi$ defined modulo $Np$. Hence we have that

$$(3.33) \quad \beta_{np^2} = \alpha_{p} \cdot \beta_{n} \in \hat{\mathcal{R}}_{h,N}$$

for all $n \geq 1$. Now we apply the Hecke operator $T_p^{m-1}$ to $\Theta(\Phi)(\tilde{\kappa}) \mid T_p^{m-1} = \Omega_{\kappa} \cdot \theta_{k,\chi}^{*}(\mathfrak{f}_{h,N}(\kappa))$ and multiply by $\alpha_{p}(\kappa)^{1-m}$ to obtain

$$\Theta(\tilde{\kappa}) = \alpha_{p}(\kappa)^{1-m} \cdot \Theta(\tilde{\kappa}) \mid T_p^{2(m-1)} \quad \text{(by (3.33))}$$

$$= \alpha_{p}(\kappa)^{1-m} \cdot \Omega_{\kappa} \cdot \theta_{k,\chi}^{*}(\mathfrak{f}_{h,N}(\kappa))[T_p]^{m-1}$$

$$= \Omega_{\kappa} \cdot \theta_{k,\chi}^{*}(\mathfrak{f}_{h,N}(\kappa))[T_p]^{1-m}.$$

This proves the part (3). The classical Shintani’s theorem 2.3 tells us that

$$\theta_{k,\chi}^{*}(\mathfrak{f}_{h,N}(\kappa)) \in S_{k+\frac{1}{2}}(\Gamma_0(4Np^m), \chi'; K)$$

where $\chi'$ is the character of $(\mathbb{Z}/4Np^m\mathbb{Z})^\times$ defined by $\chi'(d) := \chi(d) \cdot \left(\frac{(-1)^{k+1}Np}{d}\right).$ (see (2.2)) If we apply $T_p$ to $\theta_{k,\chi}^{*}(\mathfrak{f}_{h,N}(\kappa))$, then it multiplies the Nebentype by $(\mathcal{E})$ by proposition 1.5 of [Shm2]. Therefore we have that

$$\theta_{k,\chi}^{*}(\mathfrak{f}_{h,N}(\kappa))[T_p]_{1-m} \in S_{k+\frac{1}{2}}(\Gamma_0(4Np^m), \chi^*; K).$$

where $\chi^*(d) = \chi(d) \left(\frac{(-1)^{k+1}Np}{d}\right).$

So we conclude that $\Theta(\tilde{\kappa}) \in S_{k+\frac{1}{2}}(\Gamma_0(4Np^m), \chi^*; K)$. (part (2)) This finishes the proof of the main theorem. 

Remark 3.29. $\Omega_{h,N}$ (respectively $\tilde{\Omega}_{h,N}$) can be viewed as a $K$-affinoid subvariety of the eigencurve (respectively half-integral eigencurve). So our construction in fact gives us the local $K$-rigid analytic map from integral eigencurve to half-integral eigencurve.
We finish the paper summarizing the whole picture of the overconvergent Shintani lifting and its interpolation properties by the following commutative diagram:

\[
\begin{array}{ccc}
H^1_c(\Gamma_0(Np), D_N) & \xrightarrow{\Theta} & \tilde{D}(\mathbb{Z}_{p,N}^\times)[[q]] \\
\downarrow & & \downarrow \\
H^1_c(\Gamma_0(Np), D_{Bh,N})^{(\leq h)} & \xrightarrow{\Theta_B} & \tilde{R}_{h,N}[[q]] \\
\phi_{\kappa,*} \downarrow & & \uparrow \kappa \\
H^1_c(\Gamma_0(Np^m), L_{2k,\chi^2}(R_\kappa)) & \xrightarrow{\Theta_{k,\chi}} & \mathbb{C}_p[[q]] \\
\cong \downarrow & & \downarrow T^{m-1}_p \\
S_{2k+2}(\Gamma_0(Np^m), \chi^2) & \xrightarrow{\theta_{k,\chi}} & S_{k+\frac{3}{2}}(\Gamma_0(4Np^m), \chi') \\
\end{array}
\]

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