SYMMETRIC FINITE REPRESENTABILITY OF $\ell^p$-SPACES IN
REARRANGEMENT INVARIANT SPACES ON $[0,1]$

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Abstract. For a separable rearrangement invariant space $X$ on $[0,1]$ of fundamental
type we identify the set of all $p \in [1,\infty]$ such that $\ell^p$ is finitely represented in $X$ in such
a way that the unit basis vectors of $\ell^p$ ($c_0$ if $p = \infty$) correspond to pairwise disjoint
and equimeasurable functions. This can be treated as a follow up of a paper by the
first-named author related to separable rearrangement invariant spaces on $(0,\infty)$.

1. Introduction

Given a Banach space $X$, recall that $\ell^p$, for $1 \leq p \leq \infty$, is said to be finitely represented
in $X$ if for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exist $x_1, x_2, \ldots, x_n \in X$ such that for every
$a = (a_k)_{k=1}^n \in \mathbb{R}^n$ we have

$$(1 + \varepsilon)^{-1} \|a\|_p \leq \left\| \sum_{k=1}^n a_k x_k \right\|_X \leq (1 + \varepsilon) \|a\|_p,$$

where $\|a\|_p := \left( \sum_{k=1}^n |a_k|^p \right)^{1/p}$, for $1 \leq p < \infty$, and $\|a\|_\infty := \sup_{1 \leq k \leq n} |a_k|$, for $p = \infty$.
A celebrated result by Dvoretzky showed that $\ell^2$ is finitely represented in an arbitrary
infinite-dimensional Banach space $X$; see [9] or [1, Theorem 11.3.13].

A related, though more restrictive concept, is the block finite representability. Given
a Banach space $X$ and $\{z_i\}_{i=1}^\infty$, a bounded sequence in $X$, the space $\ell^p$, for $1 \leq p \leq \infty$,
is said to be block finitely represented in $\{z_i\}_{i=1}^\infty$ if for every $n \in \mathbb{N}$ and $\varepsilon > 0$ there
exist $0 = m_0 < m_1 < \cdots < m_n$ and $\alpha_i \in \mathbb{R}$ such that the vectors $u_k = \sum_{i=m_{k-1}+1}^{m_k} \alpha_i z_i$,
for arbitrary $a = (a_k)_{k=1}^n \in \mathbb{R}^n$. Krivine proved, for an arbitrary normalized sequence
$\{z_i\}_{i=1}^\infty$ in a Banach space $X$ such that $\{z_i : i \geq 1\}$ is not a relatively compact set, that $\ell^p$
is block finitely represented in $\{z_i\}_{i=1}^\infty$ for some $p$ with $1 \leq p \leq \infty$; see, for example, [13],
[22] and [1] Theorem 11.3.9.

A next important step has been made by Maurey and Pisier showing in their seminal paper [19]
that, for every infinite-dimensional Banach space $X$, the spaces $\ell_{pX}$ and $\ell_{qX}

\begin{equation}
(1 + \varepsilon)^{-1} \|a\|_p \leq \left\| \sum_{k=1}^n a_k u_k \right\|_X \leq (1 + \varepsilon) \|a\|_p
\end{equation}

for arbitrary $a = (a_k)_{k=1}^n \in \mathbb{R}^n$. The work of the first author was completed as a part of the implementation of the development
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are finitely represented in $X$, where $p_x := \sup \{p \in [1, 2] : X \text{ has type } p \}$ and $q_x := \inf \{q \in [2, \infty) : X \text{ has cotype } q \}$. Later on, in [20], Shepp discovered similar connections between the finite representability of $\ell^p$-spaces and the upper and lower estimate notions in the case of Banach lattices (see also [4] and references therein).

In the special case of rearrangement invariant spaces, the study of the finite representability of $\ell^p$-spaces was undertaken in [4]. Recall that measurable functions $x(t)$ and $y(t)$ on the measure space $(I, m)$, where $I = [0, 1]$ or $(0, \infty)$ and $m$ is the Lebesgue measure, are equimeasurable if it holds:

$$m\{s \in I : |x(s)| > \tau \} = m\{s \in I : |y(s)| > \tau \} \text{ for all } \tau > 0.$$  

Let $X$ be a rearrangement invariant space on $I$ (for all undefined notions see Section 2 below). We say that $\ell^p$, for $1 \leq p \leq \infty$, is symmetrically finitely represented in $X$ if for every $n \in \mathbb{N}$ and each $\varepsilon > 0$ there exist equimeasurable functions $x_k \in X$, $k = 1, 2, \ldots, n$, such that $\text{supp } x_i \cap \text{supp } x_j = \emptyset$, $i \neq j$, and for any $a = (a_k)_{k=1}^n$

$$\tag{2} (1 + \varepsilon)^{-1}\|a\|_p \leq \left\| \sum_{k=1}^n a_k x_k \right\|_{X} \leq (1 + \varepsilon)\|a\|_p.$$  

Sometimes the following weaker notion is also considered (see e.g. [11 p. 264]). The space $\ell^p$ is crudely symmetrically finitely represented in a rearrangement invariant space $X$ on $I$ if there exists a constant $C > 0$ such that for every $n \in \mathbb{N}$ we can find equimeasurable functions $x_k \in X$, $k = 1, 2, \ldots, n$, such that $\text{supp } x_i \cap \text{supp } x_j = \emptyset$, $i \neq j$, and for every $a = (a_k)_{k=1}^n$

$$C^{-1}\|a\|_p \leq \left\| \sum_{k=1}^n a_k x_k \right\|_{X} \leq C\|a\|_p.$$  

The set of all $p \in [1, \infty]$ such that $\ell^p$ is symmetrically finitely represented (resp. crudely symmetrically finitely represented) in $X$ we will denote by $\mathcal{F}(X)$ (resp. $\mathcal{F}_c(X)$).

If $\alpha_x$ and $\beta_x$ are the Boyd indices of a rearrangement invariant space $X$, it can be easily shown that $\mathcal{F}(X) \subset [1/\beta_x, 1/\alpha_x]$ (see also the very beginning of the proof of Theorem 2). In the converse direction, in the book [13] (see Theorem 2.b.6) it was stated without proof that $\max \mathcal{F}(X) = 1/\alpha_x$ and $\min \mathcal{F}(X) = 1/\beta_x$ for every rearrangement invariant space $X$. Later on, a full proof of the mentioned result was given in the paper [4]. More unexpected was the fact that it is possible to give even for some classes of r.i. spaces a rather simple complete description of the set $\mathcal{F}(X)$. In [4], it has been established for Lorentz spaces on $(0, \infty)$. More recently, in [5], the latter result has been extended to a wide class of separable rearrangement invariant spaces on $(0, \infty)$ of fundamental type. In particular, the coincidence of the sets $\mathcal{F}(X)$ and $\mathcal{F}_c(X)$ has been shown. More precisely, the main result of [5], identifying the latter sets in terms of the Boyd indices and some other appropriate dilation indices of $X$, can be stated in the following way.

**Theorem 1.** For every separable rearrangement invariant space $X$ on $(0, \infty)$ of fundamental type we have:

(i) If $\alpha_x^\infty \leq \beta_x^0$, then $\mathcal{F}(X) = \mathcal{F}_c(X) = [1/\beta_x, 1/\alpha_x]$.

(ii) If $\alpha_x^\infty > \beta_x^0$, then $\mathcal{F}(X) = \mathcal{F}_c(X) = [1/\beta_x, 1/\alpha_x^\infty] \cup [1/\beta_x^0, 1/\alpha_x]$.

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1The authors intended to present the proof in the next volume of the monograph, which, however, was never published.
It is worth to note that the condition of being a space of fundamental type is not too restrictive as it is satisfied by most of the well-known and important rearrangement invariant spaces (in particular, Orlicz, Lorentz, Marcinkiewicz spaces).

Observe that the method of the proof of Theorem 1 is based on using the so-called spreading sequence spaces (see e.g. [1, Chapter 11]) and so seems to be not applicable in the case of function spaces on \([0, 1]\).

The main aim of this paper is to present similar description of the set \(\mathcal{F}(X)\) for rearrangement invariant spaces on \([0, 1]\). Namely, by applying Theorem 1 to a suitable extension of a rearrangement invariant space on \([0, 1]\) to the semi-axis, we prove the following result.

**Theorem 2.** Let \(X\) be a separable rearrangement invariant space on \([0, 1]\) of fundamental type. Then,

\[
\mathcal{F}(X) = \mathcal{F}_c(X) = \left[1/\beta_X, 1/\alpha_X\right].
\]

As a consequence of Theorem 2 we obtain a complete description of the set of all \(p \in [1, \infty]\) such that \(\ell^p\) is symmetrically finitely represented in a separable Orlicz space and a Lorentz space (see Theorems 8 and 9).

Along the way, we compliment and refine some constructions related to the definition of partial dilation indices of rearrangement invariant spaces on \((0, \infty)\) introduced in [5] (see Proposition 7).

Let us note that the structure of the set \(\mathcal{F}(X)\) is closely connected with the spectral theory of operators and so it plays an important role when studying the normal solvability and invertibility of operators between function spaces, as well, in the theory of functional-differential equations and the theory of dynamical systems, etc. (see e.g. [2] and references therein). Moreover, the properties of the set \(\mathcal{F}(X)\) are used in the study of geometric properties of rearrangement invariant spaces (see, e.g., [3], [6]).

2. Preliminaries

2.1. Banach function and sequence lattices. We will use the standard definitions and results from the theory of Banach function lattices over a \(\sigma\)-finite measure space (see [7, 11, 15]).

Let \((\Omega, \Sigma, \mu)\) be a \(\sigma\)-finite measure space and let \(L^0 := L^0(\Omega, \Sigma, \mu)\) be the linear topological space of all (equivalence classes of) a.e. finite real-valued functions defined on \(\Omega\) with the natural algebraic operations and the topology of convergence in measure \(\mu\) on sets of finite measure. We say that a Banach space \(E \subseteq L^0\) is a Banach lattice on \(\Omega\) if \(E\) satisfies the ideal property, that is, we have \(y \in E\) and \(\|y\|_E \leq \|x\|_E\) whenever \(x \in E\), \(y \in L^0\) and \(|y| \leq |x|\) a.e.

If \(E\) is a Banach function lattice, then the Köthe dual (or associated) function lattice \(E'\) consists of all \(y \in L^0(\Omega, \Sigma, \mu)\) such that

\[
\|y\|_{E'} := \sup \{ \int_{\Omega} x(t)y(t) \, d\mu : \|x\|_E \leq 1 \} < \infty
\]

(in the case when \(E\) is a Banach sequence lattice modelled on \(\mathbb{Z}\) the integral should be replaced with the sum over \(\mathbb{Z}\)).

One can easily check that \(E'\) is complete with respect to the norm \(y \mapsto \|y\|_{E'}\) and \(E\) is continuously embedded into its second Köthe dual \(E''\), with \(\|x\|_{E''} \leq \|x\|_E\) for \(x \in E\). A Banach lattice \(E\) has the Fatou property (or is maximal) if from \(x_n \in E, n = 1, 2, \ldots\),
\[
\sup_{n=1,2,\ldots} \|x_n\|_E < \infty, \ x \in L^0(\Omega, \Sigma, \mu) \text{ and } x_n \to x \text{ a.e. on } \Omega \text{ it follows that } x \in E \text{ and } |x|_E \leq \liminf_{n \to \infty} |x_n|_E. \]

Note that a Banach lattice \( E \) has the Fatou property if and only if the natural inclusion of \( E \) into \( E'' \) is a surjective isometry \([11] \text{ Theorem 6.1.7}\).

A Banach lattice \( E \) is said to have an order continuous norm if for every \( x \in E \) and any decreasing sequence of sets \( A_n \in \Sigma \) with \( \mu(\bigcap_{n=1}^{\infty} A_n) = 0 \) it follows \( \|x_{\chi_{A_n}}\|_E \to 0 \) as \( n \to \infty \).

Any Köthe dual lattice \( E' \) is embedded isometrically into (Banach) dual space \( E^* \) and \( E' = E^* \) if and only if \( E \) has an order continuous norm \([11] \text{ Corollary 6.1.2}\).

2.2. Rearrangement invariant function spaces. A rearrangement invariant (in brief, r.i.) (or symmetric) space \( X \) on the measure space \((I, m)\), where \( I = [0,1] \) or \((0,\infty)\) and \( m \) is the Lebesgue measure, is a Banach function lattice on \( I \) satisfying the following condition: if \( f \in X \), \( g \in L^0(I, m) \) and \( g^* \leq f^* \), then \( g \in X \) and \( \|g\|_X \leq \|f\|_X \). Here and below, \( f^*(t) \) is the right-continuous nonincreasing rearrangement of \( |f(s)| \), i.e.,

\[
f^*(t) := \inf\{\tau \geq 0 : m\{s \in I : |f(s)| > \tau\} \leq t\}, \quad 0 < t < m(I).
\]

Functions \( f^* \) and \( f \) are equimeasurable (see Section [1]). Following \([15] \text{ §2.a}\), in what follows, we assume that a r.i. space is separable or has the Fatou property. Moreover, for a r.i. space \( X \) the normalization condition \( \|\chi_{[0,1]}\|_X = 1 \) will be assumed.

The Köthe dual space \( X' \) for a r.i. space \( X \) is again a r.i. space, and \( X^* = X' \) if and only if \( X \) is separable. Every r.i. space \( X \) on \([0,1]\) (resp. \((0,\infty)\)) satisfies the embeddings

\[
L^\infty[0,1] \subseteq X \subseteq L^1[0,1]
\]

(resp.

\[
(L^1 \cap L^\infty)(0,\infty) \subseteq X \subseteq (L^1 + L^\infty)(0,\infty)
\]

(if \( X, Y \) are r.i. spaces and \( C > 0 \), then the notation \( X \subseteq Y \) means that this embedding is continuous and \( \|x\|_Y \leq C \|x\|_X \), for \( x \in X \)).

The fundamental function of \( X \) is defined by \( \phi_X(t) := \|\chi_A\|_X \), where \( A \) is a measurable set with \( m(A) = t \).

The family of r.i. spaces includes many classical spaces appearing in analysis, in particular, \( L^p \)-spaces, Orlicz spaces, Lorentz spaces and many others.

Let \( N \) be an Orlicz function on \([0,\infty)\), i.e., \( N \) is a convex continuous increasing function on \([0,\infty)\) with \( N(0) = 0 \) and \( N(\infty) = \infty \). The Orlicz space \( L_N(I) \) consists of all measurable functions \( x(t) \) on \( I \) for which the Luxemburg norm

\[
\|x\|_{L_N} := \inf\left\{ u > 0 : \int_I N(|x(t)|/u) \, dt \leq 1 \right\}
\]

is finite (see \([12], [13], [21]\)). In particular, if \( N(s) = s^p, 1 \leq p < \infty \), we obtain the space \( L^p \) with the usual norm. Every Orlicz space \( L_N(I) \) has the Fatou property; \( L_N[0,1] \) (resp. \( L_N(0,\infty) \)) is separable if and only if the function \( N \) satisfies the \( \Delta_2 \)-condition (resp. \( \Delta_2 \)-condition), i.e., \( \sup_{u \geq 1} N(2u)/N(u) < \infty \) (resp. \( \sup_{u > 0} N(2u)/N(u) < \infty \)). The fundamental function of \( L_N(I) \) can be calculated by the formula: \( \phi_{L_N}(t) = 1/N^{-1}(1/t), \ t \in I \), where \( N^{-1} \) is the inverse function for \( N \).

Another important class of r.i. spaces is formed by the Lorentz spaces. Let \( 1 \leq q < \infty \), and let \( \psi \) be an increasing concave function on \( I \) such that \( \psi(0) = 0 \). The Lorentz
space $\Lambda_q(\psi) := \Lambda_q(\psi)(I)$ consists of all functions $x(t)$ measurable on $I$ and satisfying the condition:

\[(3) \quad \|x\|_{\Lambda_q(\psi)} := \left( \int_I x^*(t)^q d\psi(t) \right)^{1/q} < \infty\]

(see [16, 13, 15, p. 121]). For every $1 \leq q < \infty$ and any concave increasing function $\psi$, 

$\Lambda_q(\psi)$ is a separable r.i. space with the Fatou property and $\phi_{\Lambda_q(\psi)}(t) = \psi(t)^{1/q}$, $t \in I$.

2.3. Shift exponents of Banach sequence lattices and dilation indices of r.i. function spaces. Let $E$ be a Banach sequence lattice modelled on $\mathbb{Z}$ such that the shift operator $\tau_n a := (a_{k-n})_{k \in \mathbb{Z}}$, where $a = (a_k)_{k \in \mathbb{Z}}$, is bounded in $E$ for every $n \in \mathbb{Z}$. Then, denoting $\mathbb{Z}_+ = \{k \in \mathbb{Z} : k \geq 0\}$ and $\mathbb{Z}_- = \{k \in \mathbb{Z} : k \leq 0\}$, for each $n \in \mathbb{Z}$ we set $\tau_n^0 a := \chi_{\mathbb{Z}_-} \cdot \tau_n (a \chi_{\mathbb{Z}_-})$ and $\tau_n^\infty a := \chi_{\mathbb{Z}_+} \cdot \tau_n (a \chi_{\mathbb{Z}_+})$. Since the norms $\|\tau_n\|_{E \to E}$, $\|\tau_n^0\|_{E \to E}$ and $\|\tau_n^\infty\|_{E \to E}$ are subadditive in $n$, we can define the shift exponents of $E$ by

$$
\gamma_E := - \lim_{n \to \infty} \frac{1}{n} \log_2 \|\tau_n\|_{E \to E}, \quad \delta_E := \lim_{n \to \infty} \frac{1}{n} \log_2 \|\tau_n\|_{E \to E},
$$

$$
\gamma_E^0 := - \lim_{n \to \infty} \frac{1}{n} \log_2 \|\tau_n^0\|_{E \to E}, \quad \delta_E^0 := \lim_{n \to \infty} \frac{1}{n} \log_2 \|\tau_n^0\|_{E \to E},
$$

$$
\gamma_E^\infty := - \lim_{n \to \infty} \frac{1}{n} \log_2 \|\tau_n^\infty\|_{E \to E}, \quad \delta_E^\infty := \lim_{n \to \infty} \frac{1}{n} \log_2 \|\tau_n^\infty\|_{E \to E}.
$$

Next, we introduce the definitions of the dilation indices of r.i. function spaces. First, consider r.i. spaces on $[0, 1]$. For any $\tau > 0$, the dilation operator $\sigma_\tau x(t) := x(t/\tau)$, for $0 \leq t \leq \min\{1, 1/\tau\}$ and zero elsewhere, is bounded in any r.i. space $X$ on $[0, 1]$ and $\|\sigma_\tau\|_{X \to X} \leq \max(1, \tau)$, $\tau > 0$; see e.g. [13] or [15] Theorem 2.4.4]. The numbers

$$
\alpha_X = \lim_{\tau \to 0^+} \frac{\log_2 \|\sigma_\tau\|_{X \to X}}{\log_2 \tau} \quad \text{and} \quad \beta_X = \lim_{\tau \to \infty} \frac{\log_2 \|\sigma_\tau\|_{X \to X}}{\log_2 \tau}
$$

are called the lower and upper Boyd indices of $X$\textsuperscript{3}. Then, $0 \leq \alpha_X \leq \beta_X \leq 1$ (see, for instance, [13, §II.4]). Equivalently, we have

$$
\alpha_X = - \lim_{n \to \infty} \frac{1}{n} \log_2 \|\sigma_{2^{-n}}\|_{X \to X} \quad \text{and} \quad \beta_X = \lim_{n \to \infty} \frac{1}{n} \log_2 \|\sigma_{2^n}\|_{X \to X}.
$$

In the case of r.i. spaces on $(0, \infty)$ we set $\sigma_\tau x(t) := x(t/\tau)$, $t > 0$. As above, for an arbitrary r.i. space $X$ on $(0, \infty)$, we have $\|\sigma_\tau\|_{X \to X} \leq \max(1, \tau)$, $\tau > 0$, and define the Boyd indices of $X$ by

$$
\alpha_X = - \lim_{n \to \infty} \frac{1}{n} \log_2 \|\sigma_{2^{-n}}\|_{X \to X} \quad \text{and} \quad \beta_X = \lim_{n \to \infty} \frac{1}{n} \log_2 \|\sigma_{2^n}\|_{X \to X}.
$$

Moreover, in the case of r.i. spaces on $(0, \infty)$ we will use also the so-called partial dilation indices. First, for every $\tau > 0$ and $x \in L^0(0, \infty)$ we set $\sigma_\tau^0 x := \chi_{[0, 1]} \sigma_\tau (x \chi_{[0, 1]})$ and

$$
\alpha_X^0 := - \lim_{n \to \infty} \frac{1}{n} \log_2 \|\sigma_{2^{-n}}^0\|_{X \to X} \quad \text{and} \quad \beta_X^0 := \lim_{n \to \infty} \frac{1}{n} \log_2 \|\sigma_{2^n}^0\|_{X \to X}.
$$

It is easy to see that $\alpha_X^0 = \alpha_X[0, 1]$ and $\beta_X^0 = \beta_X[0, 1]$, where $X[0, 1]$ is the r.i. space on $[0, 1]$, obtained by restriction of $X$ to $[0, 1]$, i.e.,

$$
X[0, 1] := \{f \in X : \text{supp} f \subset [0, 1]\}, \quad \text{with} \quad \|f\|_{X[0, 1]} := \|f\|_X.
$$

\textsuperscript{3}In several places in the literature (see e.g. [15, p. 131]) the dilation indices of a r.i. space are taken to be the reciprocals of $\alpha_X$ and $\beta_X$. 
Secondly, for a r.i. space $X$ on $(0, \infty)$ we denote by $G_X$ the set of all functions $f \in X$ of the form $f = c\chi_{[1,2]} + g$, where $c > 0$, $\text{supp} \, g \subset (2, \infty)$ and $|g| \leq c$. Furthermore, for every integer $n \geq 0$ we put $G^n_X := \{f \in X : \sigma_{2^n} f \in G_X\}$. Then, if $\sigma^\infty_{2^n}$ is the restriction of the operator $\sigma_{2^n}$ to the set $G_X$ if $n \leq 0$ and to the set $G_X$ if $n \geq 0$, we define

$$
\alpha^{\infty}_X := - \lim_{n \to \infty} \frac{1}{n} \log_2 \|\sigma_{2^n}\|_{X \to X}, \quad \beta^{\infty}_X := \lim_{n \to \infty} \frac{1}{n} \log_2 \|\sigma^\infty_{2^n}\|_{X \to X}.
$$

Clearly, $0 \leq \alpha_X \leq \alpha^{\infty}_X \leq \beta^{\infty}_X \leq \beta_X \leq 1$ and $0 \leq \alpha_X \leq \alpha^{\infty}_X \leq \beta^{\infty}_X \leq \beta_X \leq 1$.

Let $\psi$ be a positive function on $(0,1]$. Then, the dilation function $M_\psi$ and the dilation indices $\mu_\psi$ and $\nu_\psi$ are defined as follows:

$$
\tilde{M}_\psi(t) := \sup_{0 < s \leq \min(1,1/t)} \frac{\psi(ts)}{\psi(s)}
$$

and

$$
\mu_\psi = - \lim_{n \to \infty} \frac{1}{n} \log_2 \tilde{M}_\psi(2^{-n}), \quad \nu_\psi = \lim_{n \to \infty} \frac{1}{n} \log_2 \tilde{M}_\psi(2^n).
$$

For a positive function $\psi$ on $(0, \infty)$, we define three dilation functions by

$$
M_\psi(t) := \sup_{s > 0} \frac{\psi(ts)}{\psi(s)}, \quad M^0_\psi(t) := \sup_{0 < s \leq \min(1,1/t)} \frac{\psi(ts)}{\psi(s)}, \quad M^\infty_\psi(t) := \sup_{s \geq \max(1,1/t)} \frac{\psi(ts)}{\psi(s)},
$$

and six dilation indices by

$$
\mu_\psi = - \lim_{n \to \infty} \frac{1}{n} \log_2 M_\psi(2^{-n}), \quad \nu_\psi = \lim_{n \to \infty} \frac{1}{n} \log_2 M_\psi(2^n),
$$

$$
\mu^0_\psi = - \lim_{n \to \infty} \frac{1}{n} \log_2 M^0_\psi(2^{-n}), \quad \nu^0_\psi = \lim_{n \to \infty} \frac{1}{n} \log_2 M^0_\psi(2^n),
$$

$$
\mu^\infty_\psi = - \lim_{n \to \infty} \frac{1}{n} \log_2 M^\infty_\psi(2^{-n}), \quad \nu^\infty_\psi = \lim_{n \to \infty} \frac{1}{n} \log_2 M^\infty_\psi(2^n).
$$

In the case when $\psi$ is quasi-concave (that is, $\psi(0) = 0$, $\psi$ is nondecreasing and $\psi(t)/t$ is nonincreasing), we have $0 \leq \mu_\psi \leq \nu_\psi \leq 1$ (in the case of $[0,1]$) and $0 \leq \mu_\psi \leq \mu^0_\psi \leq \nu^0_\psi \leq 1 \leq \nu_\psi \leq 1$ (in the case of $(0, \infty)$). In particular, the fundamental function $\phi_X$ of a r.i. space $X(I)$ is quasi-concave on $I$. One can easily check that from the above definitions it follows that $\alpha_X \leq \mu_{\phi_X}, \nu_{\phi_X} \leq \beta_X$ (in the case of $[0,1]$) and $\alpha_X \leq \mu_{\phi_X}, \alpha^{\infty}_X \leq \mu^{\infty}_{\phi_X}, \nu_{\phi_X} \leq \beta_X, \nu^{\infty}_{\phi_X} \leq \beta_X$ (in the case of $(0, \infty)$).

**Definition 3.** A r.i. space $X$ on $[0,1]$ (resp. on $(0, \infty)$) is said to be of fundamental type whenever $\alpha_X = \mu_{\phi_X}$ and $\beta_X - \nu_{\phi_X}$ (resp.

$$
\alpha_X = \mu_{\phi_X}, \quad \alpha_X = \mu_{\phi_X}, \quad \alpha^\infty_X = \mu^\infty_{\phi_X}, \quad \beta_X = \nu_{\phi_X}, \quad \beta^\infty_X = \nu^\infty_{\phi_X}.
$$

The most known and important r.i. spaces, in particular, all Lorentz and Orlicz spaces, are of fundamental type. The first example of a r.i. space of non-fundamental type has been constructed by Shimogaki, [23].

For a detailed information related to r.i. spaces and their Boyd indices we refer to the books [7, 13, 15].

Given two positive functions (quasinorms) $P$ and $Q$, we write $P \asymp Q$ if there exists a positive constant $C$ that does not depend on the arguments of $P$ and $Q$ such that
\( C^{-1} p \leq Q \leq C P \). Finally, by \( \text{supp} \ f \) we denote the support of a function \( f \), i.e., the set \( \{ t : f(t) \neq 0 \} \).

3. Auxiliary results

**Lemma 4.** Let \( \psi \) be a positive function on \( (0, \infty) \). Then \( \mu_{\psi} = \min(\mu_{\psi}^0, \mu_{\psi}^\infty) \) and \( \nu_{\psi} = \max(\nu_{\psi}^0, \nu_{\psi}^\infty) \).

Hence, if \( X \) is a r.i. space on \( (0, \infty) \) of fundamental type, then \( \alpha_X = \min(\alpha_X^0, \alpha_X^\infty) \) and \( \beta_X = \max(\beta_X^0, \beta_X^\infty) \).

**Proof.** Since the proof of both equalities for the shift exponents is very similar, we prove only the second one, for \( \nu_{\psi} \).

Let \( t > 1 \). Representing the dilation function \( M_{\psi} \) in the following way:

\[
M_{\psi}(t) = \max \left( \sup_{0 < s \leq 1/t} \frac{\psi(st)}{\psi(s)}, \sup_{s \geq 1} \frac{\psi(st)}{\psi(s)}, \sup_{1/t \leq s \leq 1} \frac{\psi(st)}{\psi(s)} \right)
\]

we get

\[
\nu_{\psi} = \lim_{t \to \infty} \frac{\log_2 M_{\psi}(t)}{\log_2 t} = \max \left( \lim_{t \to \infty} \frac{\log_2 M_{\psi}^0(t)}{\log_2 t}, \lim_{t \to \infty} \frac{\log_2 M_{\psi}^\infty(t)}{\log_2 t}, \lim_{t \to \infty} \frac{\log_2 \sup_{1/t \leq s \leq 1} \frac{\psi(st)}{\psi(s)}}{\log_2 t} \right)
\]

Thus, it remains only to verify that

\[
(4) \quad \limsup_{t \to \infty} \frac{\log_2 \sup_{1/t \leq s \leq 1} \frac{\psi(st)}{\psi(s)}}{\log_2 t} \leq \max(\nu_{\psi}^0, \nu_{\psi}^\infty).
\]

We claim that for all \( t > 1 \) and \( s = t^{-\lambda} \), where \( \lambda \in [0, 1] \), the following formula holds:

\[
(5) \quad \frac{\log_2 \left( \frac{\psi(ts)}{\psi(s)} \right)}{\log_2 t} = (1 - \lambda) \frac{\log_2 \left( \frac{\psi(t^{-\lambda})}{\psi(1)} \right)}{\log_2 (t^{1-\lambda})} + \lambda \frac{\log_2 \left( \frac{\psi(t^{\lambda})}{\psi(t)} \right)}{\log_2 (t^{\lambda})}.
\]

In fact,

\[
\frac{\log_2 \left( \frac{\psi(ts)}{\psi(s)} \right)}{\log_2 t} = \frac{\log_2 \left( \frac{\psi(ts)}{\psi(s)} \right)}{\log_2 t} + \frac{\log_2 \left( \frac{\psi(1)}{\psi(s)} \right)}{\log_2 t}
\]

\[
= (1 - \lambda) \frac{\log_2 \left( \frac{\psi(t^{-\lambda})}{\psi(1)} \right)}{\log_2 (t^{1-\lambda})} + \lambda \frac{\log_2 \left( \frac{\psi(t^{\lambda})}{\psi(t)} \right)}{\log_2 (t^{\lambda})},
\]

and [5] is proved.
Next, for every $t > 1$ we choose $s(t) \in [1/t, 1]$ so that
\[ \sup_{s \in [1/t, 1]} \frac{\psi(ts)}{\psi(s)} = \frac{\psi(ts(t))}{\psi(s(t))}. \]
Then $s(t) = t^{-\lambda(t)}$, where $0 \leq \lambda(t) \leq 1$. Let $\{t_n\}$ be any sequence such that $t_n > 1$ and
\[ \lim_{n \to \infty} t_n = \infty. \]
Then, we can assume that the one of the following three conditions is fulfilled:
(a) $\lim_{n \to \infty} t_n^{1 - \lambda(t_n)} = \infty$ and $\lim_{n \to \infty} t_n^{\lambda(t_n)} = \infty$; (b) $\lim_{n \to \infty} t_n^{1 - \lambda(t_n)} = \infty$ and $\{t_n^{\lambda(t_n)}\}$ is bounded from above; (c) $\lim_{n \to \infty} t_n^{1 - \lambda(t_n)} = \infty$ and $\{t_n^{\lambda(t_n)}\}$ is bounded from above.

If (a) holds, then formula (3) implies that
\[
\frac{\log_2(\sup_{1/t \leq s \leq 1} \frac{\psi(t_n s)}{\psi(s)})}{\log_2 t_n} \leq (1 - \lambda(t_n)) \frac{\log_2(\sup_{s \geq 1} \frac{\psi(s t_n^{1 - \lambda(t_n)})}{\psi(s)})}{\log_2(t_n^{1 - \lambda(t_n)})} + \lambda(t_n) \frac{\log_2(\sup_{1/t \leq s \leq 1} \frac{\psi(s t_n^{\lambda(t_n)})}{\psi(s)})}{\log_2(t_n^{\lambda(t_n)})}
\]
and hence
\[
\limsup_{n \to \infty} \frac{\log_2(\sup_{1/t \leq s \leq 1} \frac{\psi(t_n s)}{\psi(s)})}{\log_2 t_n} \leq \max(\nu^0_\psi, \nu^\infty_\psi).
\]
In the case (b) $\lambda(t_n) \to 1$ as $n \to \infty$. Therefore, from (3) it follows that
\[
\limsup_{n \to \infty} \frac{\log_2(\sup_{1/t \leq s \leq 1} \frac{\psi(t_n s)}{\psi(s)})}{\log_2 t_n} \leq \lim_{n \to \infty} \frac{\log_2(\sup_{s \leq 1} \frac{\psi(s t_n^{\lambda(t_n)})}{\psi(s)})}{\log_2(t_n^{\lambda(t_n)})} = \nu^0_\psi.
\]
Similarly, if (c) holds, then $\lambda(t_n) \to 0$ as $n \to \infty$, which implies that
\[
\limsup_{n \to \infty} \frac{\log_2(\sup_{1/t \leq s \leq 1} \frac{\psi(t_n s)}{\psi(s)})}{\log_2 t_n} \leq \lim_{n \to \infty} \frac{\log_2(\sup_{s \geq 1} \frac{\psi(s t_n^{1 - \lambda(t_n)})}{\psi(s)})}{\log_2(t_n^{1 - \lambda(t_n)})} = \nu^\infty_\psi.
\]
Summarizing all, we get inequality (4) and thereby arrive at the desired result.

The second assertion of the lemma related to the dilation indices of a r.i. space of fundamental type is a straightforward consequence of the first one and Definition 3.

\[ \square \]

Lemma 5. Suppose $X$ is a r.i. space on $(0, \infty)$ such that
\[ \|x\|_X \asymp \max(\|x^*\chi_{[0,1]}\|_Y, \|x\|_Z), \]
where $Y$ and $Z$ are r.i. spaces on $[0, 1]$ and $(0, \infty)$, respectively. Then,
(a) $\phi_X(t) \asymp \max(\phi_Y(t), \phi_Z(t))$ for $0 < t \leq 1$ and $\phi_X(t) = \phi_Z(t)$ for $t > 1$;
(b) $\alpha_\infty^0 = \max(\alpha_Y, \alpha_\infty^0_Z)$ and $\beta_\infty^0 = \max(\beta_Y, \beta_\infty^0_Z)$;
(c) $\alpha_\infty = \alpha_\infty^0$ and $\beta_\infty = \beta_\infty^0$.

Proof. Since the assertions (a) and (b) are obvious, it suffices to prove (c).
Let $\tau > 1$ and $x \in \mathcal{G}_X$. By homogeneity, we can assume that $x = \chi_{[1,2]} + y$, where $\text{supp } y \subset (2, \infty)$ and $|y| \leq 1$. Then, by the hypothesis, it follows

$$\|x\|_X \asymp \max(1, \|\chi_{[1,2]} + y\|_Z) = \|x\|_Z$$

and $\|\sigma_\tau x\|_X \asymp \max(1, \|\sigma_\tau x\|_Z) = \|\sigma_\tau x\|_Z$. Consequently,

$$\|\sigma_\tau^{-1}\|_{X \rightarrow X} = \|\sigma_\tau\|_{\mathcal{G}_X \rightarrow X} = \sup_{x \in \mathcal{G}_X} \|\sigma_\tau x\|_X = \sup_{x \in \mathcal{G}_Z} \|\sigma_\tau x\|_Z = \|\sigma_\tau^{-1}\|_{Z \rightarrow Z}.$$ 

Thus, $\beta_X^\infty = \beta_Z^\infty$. The equality $\alpha_X^\infty = \alpha_Z^\infty$ can be obtained in the same way, so we skip this proof. \[\square\]

Let $X$ be a r.i. space on $[0, 1]$. We introduce the r.i. space $X_1$ on $(0, \infty)$ given by the norm

$$\|x\|_{X_1} := \max \left(\|x^*\chi_{[0,1]}\|_X, \|x\|_{L^1(0,\infty)}\right).$$

**Lemma 6.** If $X$ is a r.i. space on $[0, 1]$ of fundamental type, so is $X_1$.

**Proof.** Thanks to Lemma 5, we need to prove only that $\beta_{X_1} = \nu_{\phi_{X_1}}$ and $\alpha_{X_1} = \mu_{\phi_{X_1}}$. The first equality is immediate. Indeed, by the same lemma, we have $\phi_{X_1}(t) \asymp t$, for $t > 1$, and hence $\nu_{\phi_{X_1}} = 1$. It remains to note that, for an arbitrary r.i. space $Y$, we have $\nu_{\phi_Y} \leq \beta_Y \leq 1$.

The equality $\alpha_{X_1} = \mu_{\phi_{X_1}}$ is a little bit more delicate. Let $n > 0$ and $x \in X_1$ be any nonincreasing nonnegative function such that $\|x\|_{X_1} = 1$. Without loss of generality, we may assume that $x(1) > 0$. First, we introduce the function $y$ by

$$y(t) = x(t)\chi_{[0,1]}(t) + x(1)\chi_{[1,a+1]}(t),$$

where $a := \frac{1}{x(1)} \int_1^\infty x(s) \, ds$. Clearly, $y$ is a nonincreasing nonnegative function on $(0, \infty)$ and $\|y\|_{X_1} = \|x\|_{X_1} = 1$. In particular, this implies that $x(1) \leq 1$ and $\int_1^\infty x(s) \, ds \leq 1$. In consequence

$$\|x\|_{X_1} \leq \frac{1}{a} \text{ and } a \geq 1.$$ 

Next, from the definition of $y$ it follows that

$$\int_0^t x(2^{-n}s) \, ds \leq \int_0^t y(2^{-n}s) \, ds, \quad t > 0.$$ 

Consequently, since $X$ is separable or has the Fatou property (see Section 2.2, applying Proposition 2.3), we have

$$\|(\sigma_{2^{-n}} x)\chi_{(0,1]}\|_X \leq \|(\sigma_{2^{-n}} y)\chi_{(0,1]}\|_X.$$ 

Moreover, by using the first inequality from (7), we get

$$\|(\sigma_{2^{-n}} y)\chi_{(0,1]}\|_X \leq \|(\sigma_{2^{-n}} x)\chi_{(0,2^{-n})}\|_X + x(1)\|\chi_{(2^{-n}\min(1,2^{-n}(a+1))}\|_X \leq \|\sigma_{2^{-n}}\|_{X \rightarrow X} \|x\chi_{(0,1]}\|_X + \frac{1}{a} \phi_X(\min(1,2^{-n}a)).$$ 

Observe that $\phi_X(\min(1,2^{-n}a)) = \phi_X(1) = 1$ if $a \geq 2^n$, and, since $\phi_X$ is quasi-concave, by the second inequality from (7), $\phi_X(\min(1,2^{-n}a)) = \phi_X(2^{-n}a) \leq a\phi_X(2^{-n}) \leq aM_{\phi_X}(2^{-n})$.
if $a < 2^n$. Therefore, summing all, we obtain
\[ \| (\sigma_{2^{-n}}) \chi_{[0,1]} \|_X \leq \max \left( \| \sigma_{2^{-n}} \|_{X \to X}, M_{\Phi_X}(2^{-n}), 2^{-n} \right) = \| \sigma_{2^{-n}} \|_{X \to X}. \]
Combining this together with (8) and the equality $\| \sigma_{2^{-n}} \|_{L^1 \to L^1} = 2^{-n}$, we deduce that
\[ \| \sigma_{2^{-n}} \|_{X_1 \to X_1} \leq \| \sigma_{2^{-n}} \|_{X \to X}. \]
Hence, since $X$ is of fundamental type, we conclude
\[ \alpha_{X_1} \geq \alpha_X = \mu_{\Phi_X} \geq \mu_{\Phi_{X_1}}. \]
Since the opposite inequality is immediate, everything is done.

\[ \square \]

4. Connections between of dilation indices of a r.i. space on $(0, \infty)$ and shift exponents of a suitable Banach sequence lattice

Following [5] (see also [10]), we assign to every r.i. function space $X$ on $(0, \infty)$ a certain Banach sequence lattice $E_X$ such that the sequence $\{ \chi_{\Delta_k} \}_{k \in \mathbb{Z}}$, where $\Delta_k := [2^k, 2^{k+1})$, is equivalent in $X$ to the unit vector basis $\{ e_k \}_{k \in \mathbb{Z}}$ in $E_X$.

Let $X$ be a r.i. space on $(0, \infty)$. For an arbitrary sequence $a = (a_k)_{k \in \mathbb{Z}}$ we introduce the following step function
\[ Sa(t) := \sum_{k \in \mathbb{Z}} a_k \chi_{\Delta_k}(t), \quad t > 0. \]
We associate to $X$ the Banach sequence lattice $E_X$ equipped with the norm
\[ \left\| \sum_{k \in \mathbb{Z}} a_k e_k \right\|_{E_X} := \| Sa \|_X. \]
A crucial role in the proof of Theorem 11 is played by properties of the shift exponents of the Banach sequence lattice $E_X$ (see [5]). In this section, we present a full proof of a refined version of Lemma 2 from [5], which was proved there only in part 3.

**Proposition 7.** For every r.i. space $X$ on $(0, \infty)$ and all $n \in \mathbb{Z}$ we have:

(i) $\| \tau_n \|_{E_X \to E_X} \leq \| \sigma_{2^n} \|_{X \to X} \leq 2\| \tau_n \|_{E_X \to E_X}$;

(ii) $\| \tau_n^0 \|_{E_X \to E_X} \leq \| \sigma_{2^n}^0 \|_{X \to X} \leq 2\| \tau_n^0 \|_{E_X \to E_X}$;

(iii) $\frac{1}{2} \| \tau_n^\infty \|_{E_X \to E_X} \leq \| \sigma_{2^n}^\infty \|_{X \to X} \leq 4 \| \tau_n^\infty \|_{E_X \to E_X}$.

Hence, $\alpha_X = \gamma_{E_X}$, $\alpha_X^0 = \gamma_{E_X}^0$, $\alpha_X^\infty = \gamma_{E_X}^\infty$, $\beta_X = \delta_{E_X}$, $\beta_X^0 = \delta_{E_X}^0$ and $\beta_X^\infty = \delta_{E_X}^\infty$.

**Proof.** Part (i) is proved in [5, Lemma 2].

(ii) By the definition of the operator $\tau_n^0$, for every $n \in \mathbb{Z}$ and any $a = (a_k)_{k \in \mathbb{Z}}$, we have $\tau_n^0 a = \sum_{k \leq \min(0,n)} a_k e_k$. Therefore,
\[ S(\tau_n^0 a) = \sum_{k \leq \min(0,n)} a_k \chi_{\Delta_k} = \sum_{j \leq \min(0,-n)} a_j \chi_{\Delta_{n+j}} = \sigma_{2^n}(Sa^{(n,-)}), \]
where
\[ a^{(n,-)} := \sum_{j \leq \min(0,-n)} a_j e_j. \]

\[ ^3 \text{We correct here a certain inaccuracy in the definition of the indices } \alpha_X^\infty \text{ and } \beta_X^\infty \text{ in the paper [5].} \]
Since \( S(a^{n,-}) \cup \sup \sigma_2^n(Sa^{n,-}) \subset [0,1] \), from the definition of the operator \( \sigma_2^0 \) (see Section 2.3) it follows that for every \( n \in \mathbb{Z} \) and any \( a = (a_k)_{k \in \mathbb{Z}} \)

\[
S(\tau_n^0 a) = \sigma_2^0(Sa^{n,-}).
\]

Hence, in view of the inequality \( |a^{(m,-)}| \leq |a| \), we get

\[
\|\tau_n^0 a\|_{E_X} = \|S(\tau_n^0 a)\|_X = \|\sigma_2^n(Sa^{n,-})\|_X \leq \|\sigma_2^n\|_{X \to X} \|Sa^{n,-}\|_X
= \|\sigma_2^n\|_{X \to X} \|a^{(m,-)}\|_{E_X} \leq \|\sigma_2^n\|_{X \to X} |a|_{E_X}.
\]

As a result, we conclude that \( \|\tau_n^0\|_{E_X \to E_X} \leq 2 \). In particular, since \( \|\sigma_2\|_{X \to X} \leq 2 \), the last inequality implies that

\[
\|\tau_n^0\|_{E_X \to E_X} \leq 2.
\]

Before proving the opposite inequality, we define the averaging operator \( Q \) by

\[
Qx(t) := \sum_{k \in \mathbb{Z}} 2^{-k} \int_{\Delta_k} x(s) \, ds \chi_{\Delta_k}(t), \ t > 0.
\]

It is well known that \( Q \) is a one norm projection on each r.i. space \( X \); see e.g. [13, §II.3.2] (recall that \( X \) is assumed to be separable or to have the Fatou property; see Section 2.2).

Let \( x \in X \). Setting

\[
a_x := \left( 2^{-k} \int_{\Delta_k} x(s) \, ds \right)_{k \in \mathbb{Z}},
\]

and comparing the operators \( S \) and \( Q \), one can see that

\[
Sa_x = Qx.
\]

Moreover, we have

\[
a_{\sigma_2x} = \sum_{k \in \mathbb{Z}} 2^{-k} \int_{\Delta_k} x(s/2) \, ds \cdot e_k = \sum_{k \in \mathbb{Z}} 2^{-k} \int_{\Delta_k} x(s) \, ds \cdot e_k = \tau_1 a_x.
\]

Let \( x \in X \), \( \supp x \subset [0,1] \). Without loss of generality, we can assume that \( x = x^* \).

Then, it can be easily checked that \( x(t) \leq Qa_2^n x(t) \), \( t > 0 \). In addition, in this case \( a_x = a_x \chi_{\mathbb{Z}^+} \) and from (14) and (9) it follows that \( a_{\sigma_2^n x} = \tau_1^0 a_x^{(1,-)} \). Thus, by (13),

\[
\|\sigma_2^n x\|_X \leq \|\sigma_2^n (Qa_2^n x)\|_X = \|\sigma_2^n (Sa_{\sigma_2^n x})\|_X = \|\sigma_2^n (S\tau_1^0 a_x^{(1,-)})\|_X.
\]

Furthermore, in view of the definition of the operator \( \sigma_2^n \) and (9), we have

\[
\sigma_2^n (S\tau_1^0 a_x^{(1,-)}) = \sigma_2^n (S\tau_1^0 a_x^{(max(0,n)+1,-)}).
\]

Therefore, by (10), (11) and the inequality \( |a_x^{(m,-)}| \leq |a_x| \), \( m \in \mathbb{Z} \), it holds that

\[
\|\sigma_2^n x\|_X \leq \|\sigma_2^n (S\tau_1^0 a_x^{(max(0,n)+1,-)})\|_X \leq \|S(\tau_1^0 a_x^{(max(0,n)+1,-)})\|_X
\leq \|\tau_1^0 a_x^{(max(0,n)+1,-)}\|_{E_X} \leq \|\tau_1^0\|_{E_X \to E_X} \|Qx\|_X \leq 2 \|\tau_1^0\|_{E_X \to E_X} \|x\|_X.
\]

Thus, \( \|\sigma_2^n\|_{X \to X} \leq 2 \|\tau_1^0\|_{E_X \to E_X} \), and (ii) is proved.

(iii) Let \( n \in \mathbb{Z} \). Given \( a = (a_k)_{k \in \mathbb{Z}} \), we have \( \tau_n^\infty a = \sum_{k \geq \max(0,n)} a_k e_k \). Hence,

\[
S(\tau_n^\infty a) = \sum_{k \geq \max(0,n)} a_{k-n} \chi_{\Delta_k} = \sum_{j \geq \max(0,-n)} a_j \chi_{\Delta_{n+j}} = \sigma_2^n(Sa^{(n,+)}),
\]
where $a^{(n,+)} := \sum_{j \geq \max(0,-n)} a_j e_j$. Observe that $\text{supp} S a^{(n,+)} \cup \text{supp} \sigma_2^n (S a^{(n,+)}) \subset (1, \infty)$.

However, in general, the function $S a^{(n,+)}$ does not belong to the set $\mathcal{G}_X$. So, we need slightly to change it. Let

$$b := \|a^{(n,+)}\|_{\ell^\infty} e_{\max(0,-n)} + a^{(n,+)} \chi_{\{k \geq \max(1,-n+1)\}}.$$ 

Then, $Sb \in \mathcal{G}_X$ if $n \geq 0$ and $Sb \in \mathcal{G}^n_X$ if $n < 0$. Moreover, $|a^{(n,+)}| \leq |b|$ and hence from (13) it follows that

$$S(\tau_n^\infty a) \leq \sigma_2^n (Sb).$$

Combining this estimate with the inequalities

$$\|e_{\max(0,-n)}\|_{E_X} = \|\chi\Delta_{\max(0,-n)}\|_{E_X} \leq \frac{1}{\|a^{(n,+)}\|_{\ell^\infty}} \sum_{j \geq \max(0,-n)} a_j \chi_j \|_{E_X} = \|a^{(n,+)}\|_{E_X},$$

and $|a^{(n,+)}| \leq |a|$, we obtain

$$\|\tau_n^\infty a\|_{E_X} = S(\tau_n^\infty a) \leq \sigma_2^n (Sb) \leq \sigma_2^n \|a^{(n,+)}\|_{\ell^\infty} e_{\max(0,-n)} + a^{(n,+)} \chi_{\{k \geq \max(1,-n+1)\}} \leq 2 \|\sigma_2^n \|_{E_X} \leq 2\|\sigma_2^n\|_{E_X} \leq 2.$$

Thus, $\|\tau_n^\infty\|_{E_X \rightarrow E_X} \leq 2\|\sigma_2^n\|_{E_X} \leq 2$. Since $\|\sigma_2^n\|_{E_X \rightarrow E_X} \leq \|\sigma_2\|_{E_X} \leq 2$, the latter inequality implies that

$$\|\tau_1^\infty\|_{E_X \rightarrow E_X} \leq 2.$$

It remains to prove the opposite inequality. Let $n \in \mathbb{Z}$ and let $x$ be any function from the set $\mathcal{G}_X$ if $n \geq 0$ and $\mathcal{G}^n_X$ if $n \leq 0$, that is, $x = c \chi_{\Delta_{\max(0,-n)}} + y$, where $\text{supp} y \subset (\max(2,2^{-n+1}), \infty)$ and $|y| \leq c$. It can be assumed also that $x$ is nonnegative and nonincreasing for $t \geq \max(1,2^{-n})$, and, by homogeneity, that $c = 1$. Then, setting $x' := \chi_{\Delta_{\max(0,-n)}} + \sigma_2 x$, we have $Q x' \geq x$. Indeed, $Q x' = x(t) = 0$ for $0 < t < \max(1,2^{-n})$ and $Q x'(t) = x(t) = 1$ for $\max(1,2^{-n}) \leq t \leq \max(2,2^{-n+1})$. Finally, if $2^k < t \leq 2^{k+1}$, where $k \geq \max(1,-n+1)$, we have

$$Q x'(t) = 2^{-k} \int_{\Delta_0} \sigma_2 x(s) \, ds = 2^{-k} \int_{2^k}^{2^{k+1}} x(s/2) \, ds \geq x(2^k) \geq x(t).$$

Furthermore, from (14) it follows

$$a_{x'} = e_{\max(0,-n)} + \sum_{k \geq 1+\max(0,-n)} 2^{-(k-1)} \int_{\Delta_{k-1}} x(s) \, ds \cdot e_k = e_{\max(0,-n)} + \tau_1^\infty a_x^{(1,+)}.$$ 

Thus, applying successively (13), (15), (16), the inequality $\chi_{\Delta_{\max(0,-n)}} \leq x$ and the fact that the operator $Q$ defined in (12) is a one norm projection in $X$, we obtain

$$\|\sigma_2^n x\|_{X} \leq \|\sigma_2^n (Q x')\|_{X} = \|\sigma_2^n (S a_{x'})\|_{X} = \|\sigma_2^n (S (e_{\max(0,-n)} + \tau_1^\infty a_x^{(1,+)}))\|_{X} \leq \|\tau_1^\infty\|_{E_X \rightarrow E_X} \|\chi\Delta_{\max(0,-n)}\|_{E_X} \|x\|_{X} + \|Q x\|_{X} \leq 4\|\tau_1^\infty\|_{E_X \rightarrow E_X} \|x\|_{X}.$$

Thus, $\|\sigma_2^n\|_{X \rightarrow X} \leq 4\|\tau_1^\infty\|_{E_X \rightarrow E_X}$, and the proof of (iii) is completed.
It remains to note that all the required equalities for the dilation indices of $X$ and the shift indices of $E_X$ follow immediately from the obtained inequalities for norms of the dilation and shift operators.

\section{Proof of the main results}

\textbf{Proof of Theorem 3} First, it can be easily showed that for every r.i. space $X$ on $[0,1]$ we have the embedding

\begin{equation}
\mathcal{F}(X) \subseteq [1/\beta_X, 1/\alpha_X].
\end{equation}

Indeed, assume that $p \in \mathcal{F}(X)$. Then, as an immediate consequence of the definition of the set $\mathcal{F}(X)$ (see (2)), for every $m \in \mathbb{N}$, we can find functions $u_m,v_m \in X$, $\|u_m\|_X = \|v_m\|_X = 1$, satisfying $\|\tilde{\sigma}_m u_m\|_X \geq \frac{1}{2} m^{1/p}$ and $\|\tilde{\sigma}_1/m v_m\|_X \geq \frac{1}{2} m^{-1/p}$. This implies that $\|\tilde{\sigma}_{m} \|_{X \rightarrow X} \geq \frac{1}{2} m^{1/p}$ and $\|\tilde{\sigma}_{1/m} \|_{X \rightarrow X} \geq \frac{1}{2} m^{-1/p}$. Then, these inequalities and the definition of the Boyd indices of $X$ imply that $1/\beta_X \leq p \leq 1/\alpha_X$. Thus, it remains only to prove the opposite embedding

\begin{equation}
\mathcal{F}(X) \supseteq [1/\beta_X, 1/\alpha_X].
\end{equation}

From now we will assume that $X$ is a separable r.i. space on $[0,1]$ of fundamental type. Let $p \in [1/\beta_X, 1/\alpha_X]$. For every $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$, we need to find equimeasurable functions $x_k \in X$, $k = 1, 2, \ldots, m$, such that $\operatorname{supp} x_i \cap \operatorname{supp} x_j = \emptyset$ for $i \neq j$ and for any $a_k \in \mathbb{R}$

\begin{equation}
(1 + \varepsilon)^{-1} \|(a_k x_k)\|_p \leq \left\| \sum_{k=1}^{m} a_k x_k \right\|_X \leq (1 + \varepsilon) \|(a_k x_k)\|_p.
\end{equation}

Suppose first that $p > 1$. Consider the r.i. space $X_1$ on $(0, \infty)$ defined by formula (6). From Lemmas 5 and 6 it follows that $X_1$ is a separable r.i. space of fundamental type, $\alpha_{X_1}^0 = \alpha_X$, $\beta_{X_1}^0 = \beta_X$, $\alpha_{X_1}^\infty = \beta_{X_1}^\infty = 1$. Moreover, applying Lemma 4 we have that $\alpha_{X_1} = \min(\alpha_X, 1) = \alpha_X$ and $\beta_{X_1} = \max(\beta_X, 1) = 1$. Therefore, by Theorem 1

\begin{equation}
\mathcal{F}(X_1) = \{1\} \cup [1/\beta_X, 1/\alpha_X].
\end{equation}

Let $m \in \mathbb{N}$ and $\varepsilon \in (0,1)$ be fixed. For definiteness, we will further assume that $p < \infty$ (the case when $p = \infty$ can be considered quite similarly). Set $\eta := \frac{\varepsilon}{2(1+\varepsilon)}$ and take $n \in \mathbb{N}$ such that

\begin{equation}
n > \max \left\{ \left( \frac{2m}{1-\eta} \right)^{2p/(p-1)}, \left( \frac{2m}{\eta} \right)^{2p/(p-1)} \right\}.
\end{equation}

By (20), for the given $p \in [1/\beta_X, 1/\alpha_X]$, there exist equimeasurable functions $f_k \in X_1$, $k = 1, 2, \ldots, n$, $\operatorname{supp} f_i \cap \operatorname{supp} f_j = \emptyset$ if $i \neq j$, such that for any $a_k \in \mathbb{R}$ we have

\begin{equation}
(1 - \eta) \|(a_k f_k)\|_p \leq \left\| \sum_{k=1}^{n} a_k f_k \right\|_{X_1} \leq (1 + \eta) \|(a_k f_k)\|_p.
\end{equation}

Since $X_1$ is separable, we can assume that $f_k(t) = f(t - (k - 1)h)$, $k = 1, \ldots, n$, where $f$ is a nonincreasing, nonnegative function on $(0, \infty)$, $\operatorname{supp} f = (0, h)$ for some $h > 0$. Moreover, by (4),

\begin{equation}
\left\| \sum_{k=1}^{n} f_k \right\|_{X_1} = \max \left( \|\sigma_n (f \chi_{[0,1/n]}\|_X, \|\sigma_n f\|_{L^1(0,\infty)} \right).
\end{equation}
Therefore, from (22) it follows that
\[ \| \sigma_n f \|_{L^1(0, \infty)} \leq 2n^{1/p}, \]
and, hence,
\[ (23) \quad \| f_k \|_{L^1(0, \infty)} = \| f \|_{L^1(0, \infty)} = n^{-1} \| \sigma_n f \|_{L^1(0, \infty)} \leq 2n^{(1-p)/p}, \quad k = 1, 2, \ldots, n. \]
In particular, by (23) and Hölder inequality, for all \( a_k \in \mathbb{R}, \ k = 1, \ldots, m, \) not all of which are zero, we have
\[ \left\| \sum_{k=1}^{m} a_k f_k \right\|_{L^1(0, \infty)} \leq 2 \sum_{k=1}^{m} |a_k| n^{(1-p)/p} \leq 2 \left( \frac{m}{n} \right)^{(p-1)/p} \| (a_k)_{k=1}^{m} \|_p. \]
Since the choice of \( n \) (see (21)) ensures that \( 2(m/n)^{(p-1)/p} < 1 - \eta, \) we infer that
\[ \left\| \sum_{k=1}^{m} a_k f_k \right\|_{L^1(0, \infty)} < (1 - \eta) \| (a_k)_{k=1}^{m} \|_p. \]
Thus, from the definition of the norm in \( X_1 \) and inequality (22), for all \( a_k \in \mathbb{R}, \) it follows
\[ (24) \quad (1 - \eta) \| (a_k)_{k=1}^{m} \|_p \leq \left\| \left( \sum_{k=1}^{m} a_k f_k \right)^* \chi_{[0,1]} \right\|_{X} \leq (1 + \eta) \| (a_k)_{k=1}^{m} \|_p. \]

Now, we claim that
\[ (25) \quad f(t) \leq 2n^{(1-p)/(2p)} \quad \text{for} \quad t > \frac{2n^{(1-p)/(2p)}}{1 - \eta}. \]
Indeed, let
\[ m \{ t > 0 : f(t) \geq n^{(1-p)/(2p)} \| f \|_{X_1} \} \geq \delta. \]
Then, by (23) and (22), we have
\[ 2n^{(1-p)/p} \geq \| f \|_{L^1(0, \infty)} \geq \int_{0}^{\delta} f(t) \, dt \geq n^{(1-p)/(2p)} \delta \| f \|_{X_1} \geq n^{(1-p)/(2p)} \delta (1 - \eta), \]
which implies that
\[ \delta \leq \frac{2n^{(1-p)/(2p)}}{1 - \eta}. \]
Since \( f \) is nonincreasing and \( \| f \|_{X_1} \leq 2 \) (see (22)), we obtain (25).

Let the sets \( E_k \subset (0, \infty) \) be chosen in such a way that each function \( f_k \chi_{E_k}, \ k = 1, \ldots, m, \) is equimeasurable with the function \( f \chi_{[0,1]/m]. \) Furthermore, suppose that coefficients \( a_k, \ k = 1, \ldots, m, \) are fixed. Then, there are pairwise disjoint sets \( A_k \subset (0, \infty), \ m(\bigcup_{k=1}^{m} A_k) = 1 \) (they depend on \( a_k \) and some of them may be empty), such that
\[ \left( \sum_{k=1}^{m} a_k f_k \right)^* \chi_{[0,1]} = \left| \sum_{k=1}^{m} a_k f_k \chi_{A_k} \right|. \]
Denoting \( E'_k := A_k \setminus E_k, \ k = 1, \ldots, m, \) we have
\[ \left\| \sum_{k=1}^{m} a_k f_k \chi_{E_k} \right\|_{X_1} \geq \left\| \sum_{k=1}^{m} a_k f_k \chi_{A_k} \right\|_{X_1} - \left\| \sum_{k=1}^{m} a_k f_k \chi_{E'_k} \right\|_{X_1} \]
\[ = \left\| \left( \sum_{k=1}^{m} a_k f_k \right)^* \chi_{[0,1]} \right\|_{X} - \left\| \sum_{k=1}^{m} a_k f_k \chi_{E'_k} \right\|_{X_1}. \]
Observe that from (21) it follows
\[ \frac{1}{m} \geq \frac{2n^{(1-p)/(2p)}}{1 - \eta}. \]
Therefore, in view of (25), \( f_k(t)\chi_{E_k^c}(t) \leq 2n^{(1-p)/(2p)} \) for all \( t > 0 \). Thus, by (26), (24) and (21), we have
\[
\left\| \sum_{k=1}^{m} a_k f_k \chi_{E_k} \right\|_{X_1} \geq (1 - \eta)\| (a_k)_{k=1}^{m} \|_p - 2n^{(1-p)/(2p)} m \max_{k=1,\ldots,m} |a_k| \\
\geq (1 - \eta - 2n^{(1-p)/(2p)}) (a_k)_{k=1}^{m} \|_p \geq (1 - 2\eta) (a_k)_{k=1}^{m} \|_p.
\]
Since
\[
\left\| \sum_{k=1}^{m} a_k f_k \chi_{E_k} \right\|_{X_1} \leq \left\| \sum_{k=1}^{m} a_k f_k \chi_{[0,1]} \right\|_{X},
\]
then, thanks to the choice of \( \eta \), the last inequality combined with (24) implies that
\[
(1 + \varepsilon)^{-1} (a_k)_{k=1}^{m} \|_p \leq \left\| \sum_{k=1}^{m} a_k f_k \chi_{E_k} \right\|_{X_1} \leq (1 + \varepsilon) (a_k)_{k=1}^{m} \|_p.
\]
Recall that the functions \( f_k \chi_{E_k}, k = 1,2,\ldots,m \), are pairwise disjoint and equimeasurable with the function \( f \chi_{[0,1]/m} \). Moreover, their choice does not depend on coefficients \( a_k \), \( k = 1,2,\ldots,m \). Taking for \( x_k \) the translates of \( f \chi_{[0,1]/m} \) to the intervals \( ((k-1)/m, k/m), \) i.e., \( x_k(t) := f(t-(k-1)/m) \chi_{(k-1)/m,k/m} \), \( k = 1,2,\ldots,m \), we see that these are pairwise disjoint and equimeasurable functions from \( X \). Also, since the functions \( \sum_{k=1}^{m} a_k f_k \chi_{E_k} \) and \( \sum_{k=1}^{m} a_k x_k \) are equimeasurable for all \( a_k \) and \( X \subseteq L^1[0,1] \), we have
\[
\left\| \sum_{k=1}^{m} a_k x_k \right\|_X = \left\| \sum_{k=1}^{m} a_k f_k \chi_{E_k} \right\|_{X_1}.
\]
Therefore, from (27) it follows that \( x_k, k = 1,2,\ldots,m \), satisfy inequality (19) for all \( a_k \in \mathbb{R} \). Since \( \varepsilon > 0 \) and \( m \in \mathbb{N} \) are arbitrary, as a result, the embedding (18) is proved when \( \beta_X < 1 \).

It remains to consider the case when \( \beta_X = 1 \) and to show (19) for \( p = 1 \). Note that a sketch of the proof of this partial result is given in [13, Proposition 2.b.7]. For the reader’s convenience, we provide some details.

We will use a simple duality argument. Clearly, we can assume that \( X \neq L^1 \). Therefore, the Köthe dual \( X' \neq L^\infty \) and hence the closure \( (X')_0 \) of \( L^\infty \) in \( X' \) is a separable r.i. space. Moreover, since \( \beta_X = 1 \), we have \( \alpha_{(X')_0} = 0 \) (see e.g. [13, Theorem II.4.11]).

Let \( m \in \mathbb{N} \) and \( \varepsilon \in (0,1) \) be arbitrary. As was proved above, there exist \( y_k \in (X')_0, k = 1,2,\ldots,m \), \( \text{supp} \ y_i \cap \text{supp} \ y_j = \emptyset \) for \( i \neq j \), such that for all \( b_k \in \mathbb{R} \) we have
\[
(1 + \varepsilon)^{-1} \| (b_k)_{k=1}^{m} \|_\infty \leq \left\| \sum_{k=1}^{m} b_k y_k \right\|_{(X')_0} \leq (1 + \varepsilon) \| (b_k)_{k=1}^{m} \|_\infty.
\]
Since \( ((X')_0)^* = ((X')_0)' = X \) (see Section 2.2), we can find pairwise disjoint and equimeasurable functions \( x_k \in X, k = 1,2,\ldots,m \), such that \( \| x_k \|_X = 1/\| y_k \|_{(X')_0} \) and \( \int_{0}^{1} x_k(t)y_k(t) \, dt = 1, k = 1,2,\ldots,m \) (in particular, this implies that \( \int_{0}^{1} x_i(t)y_j(t) \, dt = 0, \)
for \( i \neq j \). Moreover, for given \( a_k \in \mathbb{R}, k = 1, 2, \ldots, m \), we take \( b_k \in \mathbb{R}, k = 1, 2, \ldots, m \), so that \( \|(b_k)_{k=1}^m\|_\infty = 1 \) and \( \|(a_k)_{k=1}^m\|_1 = \sum_{k=1}^m a_k b_k \). Then, from (28) it follows that
\[
\left\| \sum_{k=1}^m a_k x_k \right\|_X \geq \int_0^1 \left( \sum_{k=1}^m a_k x_k(t) \right) \left( \sum_{i=1}^m b_i y_i(t) \right) dt \cdot \left\| \sum_{i=1}^m b_i y_i \right\|_{(X')_0}^{-1} \\
\geq (1 + \varepsilon)^{-1} \sum_{k=1}^m a_k b_k = (1 + \varepsilon)^{-1} \|(a_k)_{k=1}^m\|_1.
\]

In the opposite direction, using (28) once more, we have
\[
\left\| \sum_{k=1}^m a_k x_k \right\|_X \leq \sum_{k=1}^m |a_k| \left\| x_k \right\|_X = \sum_{k=1}^m \frac{|a_k|}{\|y_k\|_{(X')_0}} \leq (1 + \varepsilon) \|(a_k)_{k=1}^m\|_1.
\]

Thus, we get (19) also in the case \( p = 1 \), and hence the proof of the embedding (18) is completed.

In conclusion, we give, as an application of Theorem 2, a description of the set of \( p \) such that \( \ell^p \) is symmetrically finitely represented in Orlicz and Lorentz spaces (for their definition see Section 2.2).

Recall that every Orlicz space \( L_N = L_N[0,1] \) is of fundamental type (see e.g. [8] or [17, Theorem 4.2]) and its fundamental function can be calculated by the formula \( \phi_{L_N}(t) = 1/N^{-1}(1/t), 0 < t \leq 1 \), where \( N^{-1} \) is the inverse function for \( N \). Moreover, an Orlicz space \( L_N \) on \([0,1]\) is separable if and only if the function \( N \) satisfies the \( \Delta_2^\infty \)-condition (see [12] § II.10 or Section 2.2).

**Theorem 8.** Let \( L_N = L_N[0,1] \) be an Orlicz space such that the function \( N \) satisfies the \( \Delta_2^\infty \)-condition. Then, \( \mathcal{F}(L_N) = [1/\beta_N, 1/\alpha_N] \), where
\[
\alpha_N = - \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \leq 0} \frac{N^{-1}(2^{k+n})}{N^{-1}(2^k)}, \quad \beta_N = \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \leq 0} \frac{N^{-1}(2^k)}{N^{-1}(2^{k+n})}.
\]

Now, suppose that \( 1 \leq q < \infty \) and \( \psi \) is an increasing concave function on \([0,1]\) such that \( \psi(0) = 0 \). Then, it is easy to check (see also [13, §II.4.4] or [17, p. 28]) that the Lorentz space \( \Lambda_q(\psi) = \Lambda_q(\psi)[0,1] \) is a separable r.i. space of fundamental type. Since \( \phi_{\Lambda_q(\psi)} = \psi^{1/q} \), applying Theorem 2, we get the following result:

**Theorem 9.** Let \( 1 \leq q < \infty \), and let \( \psi \) be an increasing concave function on \([0,1]\) such that \( \psi(0) = 0 \). Then, \( \mathcal{F}(\Lambda_q(\psi)) = [1/\beta_{\psi,q}, 1/\alpha_{\psi,q}] \), where
\[
\alpha_{\psi,q} = - \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \leq 0} \left( \frac{\psi(2^{k+n})}{\psi(2^k)} \right)^{1/q}, \quad \beta_{\psi,q} = \lim_{n \to \infty} \frac{1}{n} \log_2 \sup_{k \leq 0} \left( \frac{\psi(2^k)}{\psi(2^{k+n})} \right)^{1/q}.
\]

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