Concentration of positive ground state solutions for critical Kirchhoff equation with competing potentials

Yongpeng Chen\textsuperscript{1}, Zhipeng Yang\textsuperscript{2}\textsuperscript{*}

School of Science, Guangxi University of Science and Technology, Liuzhou 545006. P.R.China.\textsuperscript{1}
Mathematical Institute, Georg-August-University of Göttingen, Göttingen 37073, Germany.\textsuperscript{2}

Abstract

In this paper, we consider the following singularly perturbed Kirchhoff equation

\[-(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \Delta u + V(x)u = P(x)|u|^{p-2}u + Q(x)|u|^4u, \quad x \in \mathbb{R}^3,\]

where \(\varepsilon > 0\) is a small parameter, \(a, b > 0\) are constants, \(p \in (4, 6)\) and \(V, P, Q\) are potential functions satisfying some competing conditions. We prove the existence of a positive ground state solution by using variational methods, and we determine a concrete set related to the potentials \(V, P\) and \(Q\) as the concentration position of these ground state solutions as \(\varepsilon \to 0\).

Keywords: Kirchhoff equation, critical exponent, concentration, competing potentials.
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1 Introduction and Main Results

In this paper, we investigate the existence and concentration behavior of positive ground solutions to the following Kirchhoff type equation with critical exponent

\[-(\varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx) \Delta u + V(x)u = P(x)|u|^{p-2}u + Q(x)|u|^4u, \quad x \in \mathbb{R}^3,\]  \hspace{1cm} (1.1)

where \(\varepsilon > 0\) is a small parameter, \(a, b > 0\) are constants, \(p \in (4, 6)\). This problem motivated by some works related to the following Kirchhoff equation

\[
\begin{aligned}
&\left\{ \begin{array}{ll}
-M\left(\int_{\Omega} |\nabla u|^2 \, dx\right) \Delta u = f(x, u) & \text{in } \Omega, \\
\quad u = 0 & \text{on } \partial\Omega,
\end{array} \right. \\
&\text{where } M(t) = a + bt \quad (a, b > 0) \text{ for all } t \geq 0. \quad \text{(1.2)}
\end{aligned}
\]

Such a problem is often referred to as being nonlocal because of the presence of the term \(M\left(\int_{\Omega} |\nabla u|^2 \, dx\right)\) which implies that the equation (1.2) is no longer a pointwise identity. This phenomenon leads to some mathematical difficulties, which make the study of such a class of problems particularly interesting, see for example \[2\] and \[5\] for more information.

\textsuperscript{*}zhipeng.yang@mathematik.uni-goettingen.de
about (1.2). Recall that (1.1) is called degenerate when \( a = 0 \) and \( b > 0 \), and a nondegenerate one when \( a > 0 \) and \( b > 0 \) (see e.g. [5], [23]).

On one hand, (1.2) is related to the stationary analogue of the Kirchhoff equation

\[
\begin{cases}
  u_{tt} - M \left( \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(x, u) & \text{in } \Omega, \\
  u = 0 & \text{on } \partial \Omega.
\end{cases}
\]  

(1.3)

It was proposed by Kirchhoff in [14] as a generalization of the well-known D’Alembert wave equation

\[
\rho \frac{\partial^2 u}{\partial t^2} - \left( \frac{p_0}{\lambda} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = f(x, u)
\]

for free vibrations of elastic strings. It seems that the first result concerning the global solvability was proved by Bernstein [3]. This result was generalized to the case \( n \geq 1 \) by Pohozaev in [24]. In [19], Lions proposed an abstract framework to the problem. Since then, (1.3) received much more attention. We have to point out that nonlocal problems also appear in other fields as biological systems, where \( u \) describes a process which depends on the average of itself (for example, population density). See, for example, [2] and the references therein. To the best of our knowledge, the variational methods were first involved in [1] and [20]. After that, there have been many works about the existence of nontrivial solutions to (1.2) by using different variational techniques, see e.g. [4, 6, 9, 21, 23, 25, 30, 33] and the references therein.

On the other hand, (1.1) can come back to the following equation

\[
\begin{cases}
  - \left( \varepsilon^2 a + \varepsilon b \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \right) \Delta u + V(x)u = f(u) & x \in \mathbb{R}^3, \\
  u \in H^1(\mathbb{R}^3) 
\end{cases}
\]  

(1.4)

The existence and multiplicity of solutions to (1.4) with \( \varepsilon = 1 \) were studied in some recent works. Li and Ye [16] obtained the existence of a positive ground state of (1.4) with \( f(u) = |u|^{p-1}u \) for \( 2 < p < 5 \). In [6], Deng, Peng and Shuai studied the existence and asymptotical behavior of nodal solutions of (1.4) with \( V \) and \( f \) is radially symmetric in \( x \) as \( b \to 0^+ \). Very recently, Li et al proved that the positive ground state solution of (1.4) with \( V \equiv 1 \) and \( f(u) = |u|^{p-1}u \) (\( 1 < p < 5 \)) is unique and nondegenerate (see [13]).

For the concentration behavior of solutions as \( \varepsilon \to 0^+ \), He and Zou [10] considered the multiplicity and concentration behavior of the positive solutions of (1.4) by using Ljusternik-Schnirelmann theory (see [29]) and minimax methods, the author obtained the multiplicity of positive solutions, which concentrate on the global minima of \( V(x) \) as \( \varepsilon \to 0^+ \). A similar result for the critical case \( f(u) = \lambda g(u) + |u|^4u \) was obtained separately in [11] and [26], where the subcritical term \( g(u) \sim |u|^{p-2}u \) with \( 4 < p < 6 \). In [13], He, Li and Peng constructed a family of positive solutions which concentrates around a local minimum of \( V \) as \( \varepsilon \to 0^+ \) for a critical problem \( f(u) = g(u) + |u|^4u \) with \( g(u) \sim |u|^{p-2}u \) (\( 4 < p < 6 \)). For the more delicate case that \( f(u) = \lambda |u|^{p-2}u + |u|^4u \) with \( 2 < p \leq 4 \) we refer to He and Li [12], where a family of positive solutions which concentrates around a local minimum of \( V \) as \( \varepsilon \to 0^+ \) were obtained.

In this paper, we are concerned with the existence and concentration behavior of ground state solutions for (1.1). We note that (1.1) involves critical exponent and three different potentials which make our problem more complicated. This brings a competition between the potentials \( V, P \) and \( Q \); each one would like to attract ground states to their minimum or maximum points, respectively. It
makes difficulties in determining the concentration position of solutions. This kind of problem can be traced back to [27], and [28] for the semilinear Schrödinger equation. In [7], the authors found new concentration phenomena for Dirac equations with competing potentials and subcritical or critical nonlinearities, respectively. See also [8, 31, 32] for other related results.

We need some notations to help us to determine the concentration set of solutions. Set

\[ 0 < V_{\text{min}} := \min_{x \in \mathbb{R}^3} V(x), \quad V_{\text{max}} := \sup_{x \in \mathbb{R}^3} V(x), \quad V := \{ x \in \mathbb{R}^3 : V(x) = V_{\text{min}} \}, \quad V_{\infty} := \liminf_{|x| \to \infty} V(x), \]

\[ 0 < P_{\text{min}} := \inf_{x \in \mathbb{R}^3} P(x), \quad P_{\text{max}} := \max_{x \in \mathbb{R}^3} P(x), \quad P := \{ x \in \mathbb{R}^3 : P(x) = P_{\text{max}} \}, \quad P_{\infty} := \limsup_{|x| \to \infty} P(x), \]

\[ 0 < Q_{\text{min}} := \inf_{x \in \mathbb{R}^3} Q(x), \quad Q_{\text{max}} := \max_{x \in \mathbb{R}^3} Q(x), \quad Q := \{ x \in \mathbb{R}^3 : Q(x) = Q_{\text{max}} \}, \quad Q_{\infty} := \limsup_{|x| \to \infty} Q(x), \]

Moreover, we assume that \( V, P \) and \( Q \) are three locally Hölder continuous and bounded functions satisfying the following conditions:

(PQ1) \( P \cap Q = \{ x \in \mathbb{R}^3 : P(x) = P_{\text{max}}, Q(x) = Q_{\text{max}} \} \neq \emptyset. \)

(PQ2) \( P_{\text{max}} > P_{\infty} \) and there exist \( R > 0 \) and \( x^* \in P \cap Q \) such that \( V(x^*) \leq V(x) \) for all \( |x| \geq R. \)

(VQ1) \( V \cap Q = \{ x \in \mathbb{R}^3 : V(x) = V_{\text{min}}, Q(x) = Q_{\text{max}} \} \neq \emptyset. \)

(VQ2) \( V_{\infty} > V_{\text{min}} \) and there exist \( R > 0 \) and \( x^* \in V \cap Q \) such that \( P(x^*) \geq P(x) \) for all \( |x| \geq R. \)

Define the following set

\[ A_V = \{ x \in P \cap Q : V(x) = V(x^*) \} \cup \{ x \notin P \cap Q : V(x) < V(x^*) \} \]

and

\[ A_P = \{ x \in V \cap Q : P(x) = P(x^*) \} \cup \{ x \notin V \cap Q : P(x) > P(x^*) \}. \]

Obviously, under the assumptions (PQ1) and (PQ2), the set \( A_V \) is bounded and we can assume \( V(x^*) = \min_{x \in P \cap Q} V(x) \). Similarly, under the assumptions (VQ1) and (VQ2), the set \( A_P \) is bounded and we can assume \( P(x^*) = \max_{x \in V \cap Q} P(x) \).

Now, we can state our main results as follows.

**Theorem 1.1** Suppose that the potentials \( V(x), P(x), Q(x) \) satisfy conditions (PQ1) and (PQ2). Then for any \( \varepsilon > 0 \) small enough, problem (1.1) has at least one positive ground state solution \( u_\varepsilon \). Moreover, if \( V(x), P(x), Q(x) \) are uniformly continuous on \( \mathbb{R}^3 \), then

1. there exists a maximum point \( x_\varepsilon \in \mathbb{R}^3 \) of \( u_\varepsilon \) such that \( \lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, A_V) = 0 \), and there exist some constants \( c, C > 0 \) such that

\[ u_\varepsilon(x) \leq C \exp\left(-\frac{c}{\varepsilon} |x - x_\varepsilon| \right). \]

2. set \( \tilde{u}_\varepsilon(x) := u_\varepsilon(\varepsilon x + \tilde{x}_\varepsilon) \), where \( \tilde{x}_\varepsilon \) is a maximum point of \( \tilde{u}_\varepsilon \). If \( x_\varepsilon \to x_0 \) as \( \varepsilon \to 0 \), then up to a subsequence, \( \tilde{u}_\varepsilon \) converges in \( H^1(\mathbb{R}^3) \) to a positive ground state solution of

\[ -(\varepsilon^2 a + \varepsilon b) \int_{\mathbb{R}^3} |\nabla u|^2 dx \Delta u + V(x_0)u = P(x_0)|u|^{p-2}u + Q(x_0)|u|^4u, \quad x \in \mathbb{R}^3. \]
In particular if $V \cap P \cap Q \neq \emptyset$, then $\lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, V \cap P \cap Q) = 0$, and up to a subsequence, $\tilde{u}_\varepsilon$ converges in $H^1(\mathbb{R}^3)$ to a positive ground state solution of

$$-(\varepsilon^2 a + \varepsilon b) \int_{\mathbb{R}^3} |\nabla u|^2 dx \Delta u + V_{\min} u = P_{\max} |u|^{p-2} u + Q_{\max} |u|^4 u, \quad x \in \mathbb{R}^3.$$  \hfill (1.2)

**Theorem 1.2** Suppose that the potentials $V(x)$, $P(x)$, $Q(x)$ satisfy conditions (VQ1) and (VQ2). Then for any $\varepsilon > 0$ small enough, problem (1.1) has at least one positive ground state solution $u_\varepsilon$. Moreover, if $V(x)$, $P(x)$, $Q(x)$ are uniformly continuous on $\mathbb{R}^3$, then

1. there exists a maximum point $x_\varepsilon \in \mathbb{R}^3$ of $u_\varepsilon$ such that $\lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, A_P) = 0$, and there exist some constants $c, C > 0$ such that

$$u_\varepsilon(x) \leq C \exp\left(-\frac{c}{\varepsilon} |x - x_\varepsilon|\right).$$

2. set $\tilde{u}_\varepsilon(x) := u_\varepsilon(\varepsilon x + \tilde{x}_\varepsilon)$, where $\tilde{x}_\varepsilon$ is a maximum point of $u_\varepsilon$. If $x_\varepsilon \to x_0$ as $\varepsilon \to 0$, then up to a subsequence, $\tilde{u}_\varepsilon$ converges in $H^1(\mathbb{R}^3)$ to a positive ground state solution of

$$-(\varepsilon^2 a + \varepsilon b) \int_{\mathbb{R}^3} |\nabla u|^2 dx \Delta u + V(x_0) u = P(x_0) |u|^{p-2} u + Q(x_0) |u|^4 u, \quad x \in \mathbb{R}^3.$$  \hfill (2.1)

In particular if $V \cap P \cap Q \neq \emptyset$, then $\lim_{\varepsilon \to 0} \text{dist}(x_\varepsilon, V \cap P \cap Q) = 0$, and up to a subsequence, $\tilde{u}_\varepsilon$ converges in $H^1(\mathbb{R}^3)$ to a positive ground state solution of

$$-(\varepsilon^2 a + \varepsilon b) \int_{\mathbb{R}^3} |\nabla u|^2 dx \Delta u + V_{\min} u = P_{\max} |u|^{p-2} u + Q_{\max} |u|^4 u, \quad x \in \mathbb{R}^3.$$  \hfill (2.2)

It is worth to note that we will overcome some difficulties. The first one is the appearance of the nonlocal $(\int_{\mathbb{R}^3} |\nabla u|^2 dx) \Delta u$, one does not know in general

$$\int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u_n \nabla \varphi dx = \int_{\mathbb{R}^3} |\nabla u|^2 dx \int_{\mathbb{R}^3} \nabla u \nabla \varphi dx + o_n(1), \quad \forall \varphi \in H^1(\mathbb{R}^3)$$

and

$$\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 - \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 = \left(\int_{\mathbb{R}^3} |\nabla u_n - \nabla u|^2 dx\right)^2 + o_n(1)$$

from $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$. The second one is that nonlinearity term is critical, then the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^t(\mathbb{R}^3)$ is not compact for any $t \in (2, 6)$ so that the standard variational methods can’t be applied directly. Thus, some new technical analysis need to be established.

This paper is organized as follows. In the forthcoming section we collect some necessary preliminary Lemmas which will be used later. In section 3, we study the auxiliary problem of (1.1). In section 4, We we are devoted to main results associated with (1.1) and some properties as $\varepsilon \to 0^+$.  

**Notation.** In this paper we make use of the following notations.

- For any $R > 0$ and for any $x \in \mathbb{R}^3$, $B_R(x)$ denotes the ball of radius $R$ centered at $x$.
- $L^p(\mathbb{R}^3)$, $1 \leq p < +\infty$ denotes the Lebesgue space with the norm $|u|_p = (\int_{\mathbb{R}^3} |u|^p dx)^\frac{1}{p}$.
- $L^\infty(\mathbb{R}^3)$ denotes the Lebesgue space with the norm $|u|_\infty = \text{ess sup} |f|$.
- The letters $C, C_i$ stand for positive constants (possibly different from line to line).
- "$\rightharpoonup$" for the strong convergence and "$\rightarrow$" for the weak convergence.
- $\mu(A)$ denotes the Lebesgue measure of $A \subset \mathbb{R}^3$. 


2 Preliminaries

Throughout the paper, we consider the Sobolev space \( E = H^1(\mathbb{R}^3) \) with the following standard norm
\[
\|u\| = \left( \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)dx \right)^{\frac{1}{2}}
\]
and denote the norm of \( D^{1,2}(\mathbb{R}^3) \) by
\[
\|u\|_{D^{1,2}} = \left( \int_{\mathbb{R}^3} (|\nabla u|^2)dx \right)^{\frac{1}{2}}.
\]
In the following, we denote by \( S \) the best Sobolev constant:
\[
S|u|^3_0 \leq \int_{\mathbb{R}^3} |\nabla u|^2 dx.
\]
Making the change of variable \( x \mapsto \varepsilon x \) and \( v(x) = u(\varepsilon x) \), problem (1.1) reduces to the equation
\[
-(a + b \int_{\mathbb{R}^3} |\nabla v|^2 dx) \Delta v + V(\varepsilon x)v = P(\varepsilon x)|v|^{p-2}v + Q(\varepsilon x)|v|^4v, \quad x \in \mathbb{R}^3. \tag{2.1}
\]
Thus, it suffices to study (2.1) and the norm
\[
\|v\|_\varepsilon = (a \int_{\mathbb{R}^3} |\nabla v|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)v^2 dx)^{\frac{1}{2}}
\]
is an equivalent norm on \( E \).

The corresponding energy functional
\[
J_\varepsilon(v) = \frac{1}{2} \|v\|_\varepsilon^2 + \frac{b}{4} \int_{\mathbb{R}^3} |\nabla v|^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} P(\varepsilon x)|v|^p dx - \frac{1}{6} \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6 dx.
\]
It is easy to check that \( J_\varepsilon \) is well defined on \( E \) and \( J_\varepsilon \in C^1(E, \mathbb{R}) \).

Let us define the Nehari manifold \([29]\) associated with \( J_\varepsilon \)
\[
\mathcal{N}_\varepsilon := \left\{ u \in E \setminus \{0\} \mid I_\varepsilon(u) = 0 \right\},
\]
where \( I_\varepsilon(u) = \langle J'_\varepsilon(u), u \rangle \).

**Lemma 2.1** There exists \( \sigma > 0 \) which is independent of \( \varepsilon \) such that
\[
\|v\|_\varepsilon > \sigma \quad \text{and} \quad J_\varepsilon(v) \geq \frac{p-2}{2p} \sigma^2 \quad \text{for all} \quad v \in \mathcal{N}_\varepsilon.
\]

**Proof:** For any \( v \in \mathcal{N}_\varepsilon \), we have
\[
0 = \|v\|_\varepsilon^2 + b \int_{\mathbb{R}^3} |\nabla v|^2 dx - \int_{\mathbb{R}^3} P(\varepsilon x)|v|^p dx - \int_{\mathbb{R}^3} Q(\varepsilon x)v^6 dx.
\]
\[
\geq \|v\|_\varepsilon^2 + b \int_{\mathbb{R}^3} |\nabla v|^2 dx - C(\|v\|_\varepsilon^p + \|v\|_\varepsilon^6)
\]

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which implies that there exists $\sigma > 0$ such that $\|v\|_\varepsilon > \sigma > 0$. In the above inequality, we have used the boundness of $P(x)$ and $Q(x)$, and the Sobolev embedding Theorem.

On the other hand, we have

\[
J_\varepsilon(v) = \frac{1}{2}\|v\|_\varepsilon^2 + \frac{b}{4} \int_{\mathbb{R}^3} |\nabla v|^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} P(\varepsilon x)|v|^p dx - \frac{1}{6} \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6 dx,
\]

\[
\geq \frac{1}{2}\|v\|_\varepsilon^2 + \frac{b}{4} \int_{\mathbb{R}^3} |\nabla v|^2 dx - \frac{1}{p} \int_{\mathbb{R}^3} P(\varepsilon x)|v|^p dx - \frac{1}{6} \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6 dx,
\]

\[
= \frac{1}{2}\|v\|_\varepsilon^2 + \frac{b}{4} \int_{\mathbb{R}^3} |\nabla v|^2 dx - \frac{1}{p}(\|v\|_\varepsilon^2 + b(\int_{\mathbb{R}^3} |\nabla v|^2 dx)^2),
\]

\[
\geq \left(\frac{1}{2} - \frac{1}{p}\right)\|v\|_\varepsilon^2
\]

\[
\geq \frac{p - 2}{2p} \sigma^2.
\]

\[\square\]

**Remark 2.2** By a direct computation we have $\langle I_\varepsilon'(u), u \rangle < 0$, which implies that $I_\varepsilon'(u) \neq 0$, $\forall u \in \mathcal{N}_\varepsilon$. It follows from the Implicit Function Theorem that $\mathcal{N}_\varepsilon$ is a $C^1$-manifold.

One can easily check that the functional $J_\varepsilon$ satisfies the mountain-pass geometry, that is the following lemma holds.

**Lemma 2.3** $J_\varepsilon$ has the mountain geometry structure.

1. There exist $a_0, r_0 > 0$ independent of $\varepsilon$, such that $J_\varepsilon(v) \geq a_0$, for all $v \in E$ with $\|v\| = r_0$.

2. For any $v \in E \setminus \{0\}$, $\lim_{t \to \infty} J_\varepsilon(tv) = -\infty$.

**Lemma 2.4** For any $v \in E \setminus \{0\}$, there exists a unique $t(v) > 0$ such that $t(v)v \in \mathcal{N}_\varepsilon$ and

\[
J_\varepsilon(t(v)v) = \max_{t \geq 0} J_\varepsilon(tv).
\]

**Proof:** For any $v \in E \setminus \{0\}$, define $g(t) = J_\varepsilon(tv)$, $t \in [0, +\infty)$. Then

\[
g(t) = \frac{t^2}{2}\|v\|_\varepsilon^2 + \frac{bt^4}{4} \int_{\mathbb{R}^3} |\nabla v|^2 dx - \frac{t^p}{p} \int_{\mathbb{R}^3} P(\varepsilon x)|v|^p dx - \frac{t^6}{6} \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6 dx.
\]

It is easy to see that $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for $t > 0$ large enough, so there exists $t_0 > 0$ such that

\[
g'(t_0) = 0 \quad \text{and} \quad g(t_0) = \max_{t \geq 0} g(t) = \max_{t \geq 0} J_\varepsilon(tv).
\]

It follows from $g'(t_0) = 0$ that $t_0v \in \mathcal{N}_\varepsilon$.

If there exist $0 < t_1 < t_2$ such that $t_1v \in \mathcal{N}_\varepsilon$ and $t_2v \in \mathcal{N}_\varepsilon$. Then

\[
\frac{1}{t_1^2}\|v\|_\varepsilon^2 + b\int_{\mathbb{R}^3} |\nabla v|^2 dx^2 = t_1^{p-4} \int_{\mathbb{R}^3} P(\varepsilon x)|v|^p dx + t_1^2 \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6 dx
\]

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where

\[ \frac{1}{t^2} \|v\|^2 + b(\int_{\mathbb{R}^3} |\nabla v|^2 dx)^2 = t^{p-4} \int_{\mathbb{R}^3} P(\varepsilon x)|v|^p dx + t^2 \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6 dx. \]

It follows that

\[ \left( \frac{1}{t_1^2} - \frac{1}{t_2^2} \right) \|v\|^2_\varepsilon = (t_1^{p-4} - t_2^{p-4}) \int_{\mathbb{R}^3} P(\varepsilon x)|v|^p dx + (t_1^2 - t_2^2) \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6 dx, \]

which is a contradiction. \(\blacksquare\)

**Lemma 2.5** For any \(\varepsilon > 0\), let

\[ c_\varepsilon = \inf_{v \in \mathcal{N}_\varepsilon} J_\varepsilon(v), \quad c_\varepsilon^* = \inf_{v \in \mathcal{E} \setminus \{0\}} \max_{t \geq 0} J_\varepsilon(tv), \quad c_\varepsilon^{**} = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_\varepsilon(\gamma(t)), \]

where

\[ \Gamma = \{\gamma(t) \in C([0,1], \mathcal{E}) : \gamma(0) = 0, J_\varepsilon(\gamma(1)) < 0\}. \]

Then, \(c_\varepsilon = c_\varepsilon^* = c_\varepsilon^{**}\).

**Proof:** We divided the proof into three steps.

**Step1:** \(c_\varepsilon^* = c_\varepsilon\). By Lemma 2.4 we have

\[ c_\varepsilon^* = \inf_{v \in \mathcal{E} \setminus \{0\}} \max_{t \geq 0} J_\varepsilon(tv) = \inf_{v \in \mathcal{E} \setminus \{0\}} J_\varepsilon(t(v)v) = \inf_{v \in \mathcal{N}_\varepsilon} J_\varepsilon(v) = c_\varepsilon. \]

**Step2:** \(c_\varepsilon^* \geq c_\varepsilon^{**}\). From Lemma 2.4, for any \(v \in \mathcal{E} \setminus \{0\}\), there exists \(T\) large enough, such that \(J_\varepsilon(Tv) < 0\). Define \(\gamma(t) = tv, t \in [0,1]\), then \(\gamma(t) \in \Gamma\). Thus

\[ c_\varepsilon^{**} = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_\varepsilon(\gamma(t)) \leq \sup_{t \in [0,1]} J_\varepsilon(\gamma(t)) \leq \max_{t \geq 0} J_\varepsilon(tv). \]

It follows that \(c_\varepsilon^* \geq c_\varepsilon^{**}\).

**Step3:** \(c_\varepsilon^{**} \geq c_\varepsilon\). The manifold \(\mathcal{N}_\varepsilon\) separates \(\mathcal{E}\) into two components. It is easy to know that the component containing the origin also contains a small ball around the origin. It follows from \(g(t) = (J_\varepsilon(Tv), v) \geq 0\) for all \(0 \leq t \leq t(v)\) in Lemma 2.4 that \(J_\varepsilon(v) \geq 0\) in this component. Thus every \(\gamma \in \Gamma\) has to cross \(\mathcal{N}_\varepsilon\). Then \(c_\varepsilon^{**} \geq c_\varepsilon\). \(\blacksquare\)

**Lemma 2.6** Any \((PS)_\varepsilon\) sequence \(\{v_n\}\) for \(J_\varepsilon\) is bounded, and

\[ \limsup_{n \to \infty} \|v_n\| \leq \sqrt{\frac{2p}{p-2}} c. \]

**Proof:** Suppose that \(\{v_n\}\) is a \((PS)_\varepsilon\) sequence of \(J_\varepsilon\), we have

\[ J_\varepsilon(v_n) \to c, \quad J_\varepsilon'(v_n) \to 0. \]

Thus

\[ c + o(1) + o(1)\|v_n\|_\varepsilon = J_\varepsilon(v_n) - \frac{1}{p} \langle J_\varepsilon'(v_n), v_n \rangle \]
\[ = \left( \frac{1}{2} - \frac{1}{p} \right) \|v_n\|^2_\varepsilon + \left( \frac{1}{4} - \frac{1}{p} \right) b(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^2 \]
\[ + \left( \frac{1}{p} - \frac{1}{6} \right) \int_{\mathbb{R}^3} Q(\varepsilon x)v_n^6 dx. \]
It follows that
\[
\left(\frac{1}{2} - \frac{1}{p}\right)\|v_n\|_2^2 \leq c + o(1) + o(1)\|v_n\|_\varepsilon.
\]
Then \(\{v_n\}\) is bounded in \(E\), and
\[
\limsup_{n \to \infty} \|v_n\|_\varepsilon \leq \sqrt{\frac{2p}{p-2}} c.
\]

\[\blacksquare\]

**Lemma 2.7** If \(\{v_n\}\) is a \((PS)_{c_\varepsilon}\) sequence of \(J_\varepsilon\) in \(E\), then there exists \(v \in E\) such that \(v_n \rightharpoonup v\) in \(E\) and \(J'_\varepsilon(v) = 0\).

**Proof:** The proof is similar in [17], we give it for completeness. By Lemma 2.6, we know that \(\{v_n\}\) is bounded in \(E\). Then, up to a subsequence, we have
\[
\begin{align*}
v_n & \rightharpoonup v \quad \text{in } E \\
v_n & \rightarrow v \quad \text{a.e. in } \mathbb{R}^3, \\
v_n & \rightarrow v \quad \text{in } L^q(\mathbb{R}^3), \text{ for } 2 \leq q \leq 6.
\end{align*}
\]
For any function \(\phi \in C_0^\infty(\mathbb{R}^3)\), since \(J'_\varepsilon(v_n) \to 0\), we have
\[
o(1) = \int_{\mathbb{R}^3} (a\nabla v_n \nabla \phi + V(\varepsilon x)v_n \phi)dx + b\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \int_{\mathbb{R}^3} \nabla v_n \nabla \phi dx \\
- \int_{\mathbb{R}^3} P(\varepsilon x)|v_n|^{p-2}v_n \phi dx - \int_{\mathbb{R}^3} Q(\varepsilon x)|v_n|^4v_n \phi dx,
\]
Assume that \(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx \to A^2\), as \(n \to \infty\). Then, we have
\[
0 = \int_{\mathbb{R}^3} (a\nabla v \nabla \phi + V(\varepsilon x)v \phi)dx + bA^2\int_{\mathbb{R}^3} \nabla v \nabla \phi dx \\
- \int_{\mathbb{R}^3} P(\varepsilon x)|v|^{p-2}v \phi dx - \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^4v \phi dx.
\]
Thus, we can get
\[
\int_{\mathbb{R}^3} (a|\nabla v|^2 + V(\varepsilon x)v^2)dx + bA^2\int_{\mathbb{R}^3} |\nabla v|^2 dx = \int_{\mathbb{R}^3} P(\varepsilon x)|v|^p dx + \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6 dx.
\]
It is easy to know that \(\int_{\mathbb{R}^3} |\nabla v|^2 dx \leq A^2\). If \(\int_{\mathbb{R}^3} |\nabla v|^2 dx < A^2\), we have
\[
\int_{\mathbb{R}^3} (a|\nabla v|^2 + V(\varepsilon x)v^2)dx + b(\int_{\mathbb{R}^3} |\nabla v|^2 dx)^2 < \int_{\mathbb{R}^3} P(\varepsilon x)|v|^p dx + \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6 dx.
\]
Then, there exists $0 < t_0 < 1$ such that $t_0 v \in \mathcal{N}_\varepsilon$. Thus, we have

$$c_\varepsilon \leq J'_\varepsilon(t_0v) - \frac{1}{4} (J'_\varepsilon(t_0v), t_0v)$$

$$= \frac{t_0^2}{4} \int_{\mathbb{R}^3} (a|\nabla v|^2 + V(\varepsilon x)v^2)dx + \left(\frac{\beta}{4} - \frac{t_0^p}{p}\right) \int_{\mathbb{R}^3} P(\varepsilon x)|v|^pdx + \frac{t_0^6}{4} - \frac{t_0^6}{6} \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6dx$$

$$\leq \frac{1}{4} \int_{\mathbb{R}^3} (a|\nabla v|^2 + V(\varepsilon x)v^2)dx + \left(\frac{1}{4} - \frac{1}{p}\right) \int_{\mathbb{R}^3} P(\varepsilon x)|v|^pdx + \left(\frac{1}{4} - \frac{1}{6}\right) \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6dx$$

$$\leq \liminf_{n \to \infty} \left[ \frac{1}{4} \int_{\mathbb{R}^3} (a|\nabla v|^2 + V(\varepsilon x)v^2)dx + \left(\frac{1}{4} - \frac{1}{p}\right) \int_{\mathbb{R}^3} P(\varepsilon x)|v|^pdx \right.$$

$$\left. + \left(\frac{1}{4} - \frac{1}{6}\right) \int_{\mathbb{R}^3} Q(\varepsilon x)|v|^6dx \right]$$

$$= \liminf_{n \to \infty} \left[ J'_\varepsilon(v_n) - \frac{1}{4} (J'_\varepsilon(v_n), v_n) \right]$$

$$= c_\varepsilon.$$ 

Then, as $n \to \infty$, we have $\int_{\mathbb{R}^3} |\nabla v_n|^2dx \to \int_{\mathbb{R}^3} |\nabla v|^2dx$. Therefore, $J'_\varepsilon(v) = 0$. $\square$

In order to investigate (2.1), we need some results about (2.1) with constant coefficients. Consider the following problem

$$- (a + b \int_{\mathbb{R}^3} |\nabla v|^2dx) \Delta v + kv = \tau |v|^{p-2}v + \nu |v|^4v, \quad x \in \mathbb{R}^3.$$  

(2.2) where $k, \tau$ and $\nu$ are positive constants. The associated energy functional is

$$\Phi^{*}_{k\tau \nu}(v) = \frac{1}{2}||v||^2_k + \frac{b}{4} \int_{\mathbb{R}^3} |\nabla v|^2dx)^2 - \frac{\tau}{p} \int_{\mathbb{R}^3} |v|^pdx - \frac{\nu}{6} \int_{\mathbb{R}^3} |v|^6dx,$$

where $||v||^2_k := \int_{\mathbb{R}^3} (a|\nabla v|^2 + kv^2)dx$ and $||v||_k$ is an equivalent norm on $E$.

By Lemma 2.5, we have

$$m^{*}_{k\tau \nu} := \inf_{v \in E \setminus \{0\}} \max_{t \geq 0} \Phi^{*}_{k\tau \nu}(tv) = \inf_{v \in \mathcal{N}^{*}_{k\tau \nu}} \Phi^{*}_{k\tau \nu}(tv),$$

where $\mathcal{N}^{*}_{k\tau \nu} = \{v \in E \setminus \{0\} | (\Phi^{*}_{k\tau \nu})'(v), v) = 0\}$. 

Lemma 2.8 [19] For $t, s > 0$ and $\lambda$ is a positive constant, the following system

$$\begin{align*}
\Phi(t, s) &= t - aS\lambda^{-\frac{1}{2}}(t + s)^{\frac{1}{2}} = 0, \\
\Psi(t, s) &= s - bS^2\lambda^{-\frac{3}{2}}(t + s)^{\frac{3}{2}} = 0.
\end{align*}$$

has a unique solution $(t_0, s_0)$. Moreover, if $\Phi(t, s) \geq 0$ and $\Psi(t, s) \geq 0$, then $t \geq t_0, s \geq s_0$.

In order to establish the existence result, we need to show that the mountain pass value is less than the critical level. This result can be found in the following Lemma.

Lemma 2.9 [19] For any $\varepsilon > 0$ and $Q(x) \equiv q > 0$, we have $c_\varepsilon < \frac{ab}{4q} S^3 + \frac{(b^2 S^4 + 4qaS)^{3/2}}{24q^2} + \frac{b^3 S^6}{24q^2}$. 

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Lemma 2.10  Problem (2.2) has at least one positive ground state solution.

Proof: By Lemma 2.3 and Lemma 2.5 there exits a sequence \( \{v_n\} \) which is a \((PS)_{m^*_{kT\nu}}\) sequence of \( \Phi^*_{kT\nu}(v) \). From Lemma 2.6 we know that \( \{v_n\} \) is bounded in \( E \). Hence, up to a subsequence, we have

\[
\begin{align*}
v_n & \to v \quad \text{in } E, \\
v_n & \to v \quad \text{a.e. in } \mathbb{R}^3, \\
v_n & \to v \quad \text{in } L^q(\mathbb{R}^3), \quad \text{for } 2 \leq q \leq 6.
\end{align*}
\]

It follows from Lemma 2.7 that \((\Phi^*_{kT\nu})'(v) = 0\).

Since \( \{v_n\} \) is a \((PS)_{m^*_{kT\nu}}\) sequence of \( \Phi^*_{kT\nu} \), we have

\[
o(1) = \langle (\Phi^*_{kT\nu})'(v_n), v_n \rangle = \|v_n\|^2_k + b \int_{\mathbb{R}^3} |\nabla v_n|^2 dx - \tau \int_{\mathbb{R}^3} |v_n|^p dx - \nu \int_{\mathbb{R}^3} |v_n|^6 dx.
\]

Since \( \{v_n\} \) is bounded in \( E \), as \( n \to \infty \), we can assume

\[
\|v_n\|^2_k \to l_1,
\]

\[
b \left( \int_{\mathbb{R}^3} |\nabla v_n|^2 dx \right) \to l_2,
\]

and

\[
\tau \int_{\mathbb{R}^3} |v_n|^p dx + \nu \int_{\mathbb{R}^3} |v_n|^6 dx \to l_3.
\]

Then we have \( l_3 = l_1 + l_2 \).

If \( l_3 = 0 \), we have \( v_n \to 0 \) in \( E \). Then \( \Phi^*_{kT\nu}(v_n) \to 0 \), which contradicts \( m^*_{kT\nu} > 0 \). Thus, \( l_3 \neq 0 \).

In (2.5), if \( \int_{\mathbb{R}^3} |v_n|^p dx \to 0 \), then we have

\[
\nu \int_{\mathbb{R}^3} |v_n|^6 dx \to l_3.
\]

By the definition of the best constant \( S \), we have

\[
a^3 \nu \int_{\mathbb{R}^3} |v_n|^6 dx \leq a^3 \nu (S^{-1} \int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^3 \leq \nu S^{-3} \|v_n\|^6_k
\]

and

\[
b(\nu \int_{\mathbb{R}^3} |v_n|^6 dx)^{2/3} \leq b(\nu \int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^2 \leq b(\nu \int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^2 \leq b(\nu \int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^2.
\]

Letting \( n \to \infty \) in the above two inequalities, we have

\[
a^3 l_3 \leq \nu S^{-3} l_1^3 \quad \text{and} \quad b l_3^{2/3} \leq \nu S^{-2} l_2.
\]

Therefore,

\[
l_1 \geq aS \nu^{-3/2} (l_1 + l_2)^{1/3} \quad \text{and} \quad l_2 \geq bS^2 \nu^{-2/3} (l_1 + l_2)^{2/3}.
\]

By Lemma 2.8 we have

\[
\frac{1}{3} l_1 + \frac{1}{12} l_2 \geq c^* := \frac{ab}{4\nu S^3} + \frac{(b^2 S^4 + 4\nu aS)^{3/2}}{24\nu^2} + \frac{b^3 S^6}{24\nu^2}.
\]
On the other hand,

\[ m_{k^*}^\nu = \Phi_{k^*}^\nu(v_n) + o(1) \]

\[ = \frac{1}{2} \|v_n\|^2 + \frac{b}{4} \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx - \frac{\tau}{p} \int_{\mathbb{R}^3} |v_n|^p \, dx - \frac{\nu}{6} \int_{\mathbb{R}^3} |v_n|^6 \, dx + o(1) \]

\[ = \frac{1}{2} \|v_n\|^2 + \frac{b}{4} \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx - \frac{\nu}{6} \int_{\mathbb{R}^3} |v_n|^6 \, dx + o(1) \]

\[ = \frac{1}{3} \|v_n\|^2 + \frac{b}{12} \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx + o(1) \]

Using (2.3) and (2.4) in the above expression, we obtain

\[ m_{k^*}^\nu = \frac{1}{3} l_1 + \frac{1}{12} l_2 \geq c^* , \]

which contradicts Lemma 2.9.

Therefore, \( \int_{\mathbb{R}^3} |v_n|^p \to l_4 > 0 , \) as \( n \to \infty \). Then, by Lions’s Lemma, there exists \( (y_n) \subset \mathbb{R}^3 \), \( \rho, \eta > 0 \) such that

\[ \limsup_{n \to \infty} \int_{B_{\rho}(y_n)} |v_n|^2 \, dx \geq \eta. \]  

(2.7)

Let \( \bar{v}_n(x) = v_n(x + y_n) \), then \( \|\bar{v}_n\|_k \leq C \) in \( E \), so there exists \( \bar{v} \in E \) such that \( \bar{v}_n \to \bar{v} \) in \( E \) and \( \bar{v}_n \to \bar{v} \) a.e in \( \mathbb{R}^3 \), by (2.7), we get \( \bar{v} \neq 0 \).

It is easy to prove that

\[ (\Phi_{k^*}^\nu)'(\bar{v}) = m_{k^*}^\nu, \quad (\Phi_{k^*}^\nu)'(\bar{v}_n) \to 0. \]

Then, we have \( (\Phi_{k^*}^\nu)'(\bar{v}) = 0 \) and \( \bar{v} \in N_{k^*}. \)

Moreover,

\[ m_{k^*}^\nu = \lim_{n \to \infty} \left[ (\Phi_{k^*}^\nu)(\bar{v}_n) - \frac{1}{4}(\Phi_{k^*}^\nu)'(\bar{v}_n, \bar{v}_n) \right] \]

\[ = \lim_{n \to \infty} \left[ \frac{1}{4} \|\bar{v}_n\|^2 + \frac{1 - \frac{1}{p}}{p} \int_{\mathbb{R}^3} |\bar{v}_n|^p \, dx + \frac{1 - \frac{1}{6}}{6} \int_{\mathbb{R}^3} |\bar{v}_n|^6 \, dx \right] \]

\[ \geq \frac{1}{4} \|\bar{v}\|^2 + \frac{1 - \frac{1}{p}}{p} \int_{\mathbb{R}^3} |\bar{v}|^p \, dx + \frac{1 - \frac{1}{6}}{6} \int_{\mathbb{R}^3} |\bar{v}|^6 \, dx \]

\[ = (\Phi_{k^*}^\nu)(\bar{v}) - \frac{1}{4}(\Phi_{k^*}^\nu)'(\bar{v}, \bar{v}) \]

\[ = (\Phi_{k^*}^\nu)(\bar{v}), \]

which means \( (\Phi_{k^*}^\nu)'(\bar{v}) = m_{k^*}^\nu \). It is easy to know that \( |\bar{v}| \in N_{k^*} \) and \( (\Phi_{k^*}^\nu)'(\bar{v}) = m_{k^*}^\nu \).

According to the proof of Theorem 4.3 in [29], we can show that \( (\Phi_{k^*}^\nu)'(\bar{v}) = 0 \). Without loss of generality, we can assume \( \bar{v} \geq 0 \). By the theory of elliptic regularity, \( \bar{v} \in C^2(\mathbb{R}^3) \), and by using strong maximum principle, we get \( \bar{v} > 0 \) in \( \mathbb{R}^3 \).

\[ \square \]

**Lemma 2.11** For the positive constants \( k_i, \tau_i \) and \( \nu_i \), \( i = 1, 2 \). If

\[ \min \{ k_2 - k_1, \tau_1 - \tau_2, \nu_1 - \nu_2 \} \geq 0, \]

then \( m_{k_1, \tau_1, \nu_1}^* \leq m_{k_2, \tau_2, \nu_2}^* \). Additionally, if \( \max \{ k_2 - k_1, \tau_1 - \tau_2, \nu_1 - \nu_2 \} > 0, \) then \( m_{k_1, \tau_1, \nu_1}^* < m_{k_2, \tau_2, \nu_2}^*. \)
In this section, we mainly consider an auxiliary problem, for $c \in [V_{min}, V_{max}]$, $d \in [P_{min}, P_{max}]$ and $e \in [Q_{min}, Q_{max}]$, we define

$$V_\epsilon^c(x) = \max\{c, V(\epsilon x)\},$$

$$P_\epsilon^d(x) = \min\{d, P(\epsilon x)\},$$

$$Q_\epsilon^e(x) = \min\{e, Q(\epsilon x)\}.$$  

When $x = 0$, we set $V(0) = \max\{c, V(0)\}$, $P(0) = \min\{d, P(0)\}$ and $Q(0) = \min\{e, Q(0)\}$.

Consider the following equation

$$-(a + b \int_{\mathbb{R}^3} |\nabla v|^2 dx) \Delta v + V_\epsilon^c(x)v = P_\epsilon^d(x)|v|^{p-2}v + Q_\epsilon^e(x)|v|^4v, \quad x \in \mathbb{R}^3. \tag{3.1}$$

whose energy functional is

$$J_\epsilon^{cd}(v) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla v|^2 + V_\epsilon^c(x)v^2) dx + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^2 - \frac{1}{p} \int_{\mathbb{R}^3} P_\epsilon^d(x)|v|^p dx - \frac{1}{6} \int_{\mathbb{R}^3} Q_\epsilon^e(x)|v|^6 dx.$$  

By Lemma 2.5, we have

$$c_\epsilon^{cd} := \inf_{v \in E \setminus \{0\}} \max_{t \geq 0} J_\epsilon^{cd}(tv) = \inf_{v \in \mathcal{N}_\epsilon^{cd}} J_\epsilon^{cd}(v),$$

where $\mathcal{N}_\epsilon^{cd} = \{v \in E \setminus \{0\} \mid (J_\epsilon^{cd})'(v), v) = 0\}$.

**Lemma 3.1** For any $y \in \mathbb{R}^3$, \( \limsup_{\epsilon \to 0} c_\epsilon \leq m_{V(y)}^*P(y)Q(y) \).

**Proof:** Let $v$ be a ground state solution of

$$-(a + b \int_{\mathbb{R}^3} |\nabla v|^2 dx) \Delta v + V(y)v = P(y)|v|^{p-2}v + Q(y)|v|^4v, \quad x \in \mathbb{R}^3.$$  

Then, we have $v \in \mathcal{N}_{V(y)}^*P(y)$ and $\Phi_{V(y)}^*P(y)Q(y)(v) = m_{V(y)}^*P(y)Q(y)$.

Define $v_\epsilon(x) = v(x - \frac{y}{\epsilon})$. From Lemma 2.4, we know that there exists a unique $t_\epsilon > 0$ satisfying

$$t_\epsilon v_\epsilon \in \mathcal{N}_\epsilon \quad \text{and} \quad J_\epsilon(t_\epsilon v_\epsilon) = \sup_{t \geq 0} J_\epsilon(tv_\epsilon).$$

By $t_\epsilon v_\epsilon \in \mathcal{N}_\epsilon$, we can prove $t_\epsilon$ is bounded. It follows from Lemma 2.1 that $t_\epsilon$ has a positive lower bound. Without loss of generality, we assume $t_\epsilon \to t_0 > 0$. 

\[\square\]
By Lemma 2.5, the proof is completed. 

**Proof:** For any \( \varepsilon > 0 \), we have 

\[
\frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} (a|\nabla v_\varepsilon|^2 + V(\varepsilon x)v_\varepsilon^2)dx + b t_\varepsilon^4 \left( \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx \right)^2 = t_\varepsilon^p \int_{\mathbb{R}^3} P(\varepsilon x)|v_\varepsilon|^p dx + t_\varepsilon^6 \int_{\mathbb{R}^3} Q(\varepsilon x)|v_\varepsilon|^6 dx.
\]

Thus, we have 

\[
t_\varepsilon^2 \int_{\mathbb{R}^3} (a|\nabla v|^2 + V(\varepsilon x + y)v^2)dx + b t_\varepsilon^4 \left( \int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^2 = t_\varepsilon^p \int_{\mathbb{R}^3} P(\varepsilon x + y)|v|^p dx + t_\varepsilon^6 \int_{\mathbb{R}^3} Q(\varepsilon x + y)|v|^6 dx.
\]

Letting \( \varepsilon \to 0 \), by Lebesgue Dominated Convergence Theorem, we get 

\[
t_0^2 \int_{\mathbb{R}^3} (a|\nabla v|^2 + V(y)v^2)dx + b t_0^4 \left( \int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^2 = t_0^p \int_{\mathbb{R}^3} P(y)|v|^p dx + t_0^6 \int_{\mathbb{R}^3} Q(y)|v|^6 dx,
\]

which means \( t_0 v \in \mathcal{N}_{V(y)P(y)Q(y)}^* \). By using Lemma 2.4 and \( v \in \mathcal{N}_{V(y)P(y)Q(y)}^* \), we obtain \( t_0 = 1 \). So we get 

\[
c_\varepsilon \leq J_\varepsilon(t_\varepsilon v_\varepsilon)
\]

\[
= \frac{t_\varepsilon^2}{2} \int_{\mathbb{R}^3} (a|\nabla v_\varepsilon|^2 + V(\varepsilon x)v_\varepsilon^2)dx + b t_\varepsilon^4 \left( \int_{\mathbb{R}^3} |\nabla v_\varepsilon|^2 dx \right)^2 - \frac{t_\varepsilon^6}{6} \int_{\mathbb{R}^3} Q(\varepsilon x)|v_\varepsilon|^6 dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla v|^2 + V(\varepsilon x + y)v^2)dx + b t_\varepsilon^4 \left( \int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^2 - \frac{t_\varepsilon^6}{6} \int_{\mathbb{R}^3} Q(\varepsilon x + y)|v|^6 dx
\]

\[
= \frac{1}{2} \int_{\mathbb{R}^3} (a|\nabla v|^2 + V(y)v^2)dx + b t_\varepsilon^4 \left( \int_{\mathbb{R}^3} |\nabla v|^2 dx \right)^2 - \frac{1}{p} \int_{\mathbb{R}^3} P(y)|v|^p dx
\]

\[
= \Phi_{V(y)P(y)Q(y)}^*(v) + o_\varepsilon(1)
\]

\[
= m_{V(y)P(y)Q(y)}^*(v) + o_\varepsilon(1).
\]

Therefore, \( \limsup_{\varepsilon \to 0} c_\varepsilon \leq m_{V(y)P(y)Q(y)}^* \). 

**Lemma 3.2** For any \( \varepsilon > 0 \), we have 

\[
c_\varepsilon \int_{\mathbb{R}^3}^V(0)\,dQ_{\max} \geq m_{V(0)} P_{Q_{\max}}^*.
\]

**Proof:** For any \( v \in E \), we have 

\[
J_\varepsilon^V(0)\,dQ_{\max}(tv) \geq \Phi_{V(0)} P_{Q_{\max}}^*(tv). \]

So 

\[
\inf_{u \in E} \max_{t > 0} J_\varepsilon^V(0)\,dQ_{\max}(tv) \geq \inf_{u \in E} \max_{t > 0} \Phi_{V(0)} P_{Q_{\max}}^*(tv).
\]

By Lemma 3.2, the proof is completed.
4 Proof of the Main Results

Lemma 4.1 Suppose that the potential functions $V(x)$, $P(x)$ and $Q(x)$ satisfy conditions (PQ1) and (PQ2). Then for any $\varepsilon > 0$ small enough, problem (1.1) has at least one positive ground state solution.

Proof: By Lemma 2.5, we can choose a sequence $\{v_n\} \subset N_\varepsilon$ such that $J_\varepsilon(v_n) \to c_\varepsilon$. In view of Ekeland’s variational principle, the sequence can be chosen to be a $(PS)_{c_\varepsilon}$ sequence of $J_\varepsilon(v)$. From Lemma 2.6 we know that $\{v_n\}$ is bounded in $E$. Hence, up to a subsequence, we have

$$v_n \to v \quad \text{in} \quad E$$

$$v_n \to v \quad \text{a.e. in} \quad \mathbb{R}^3$$

$$v_n \to v \quad \text{in} \quad L^q(\mathbb{R}^3), \quad \text{for} \quad 2 \leq q \leq 6.$$ 

It follows from Lemma 2.7 that $J'_\varepsilon(v) = 0$. Now we prove $v_\varepsilon \neq 0$.

Claim 1: there exist $(y_n) \subset \mathbb{R}^3$ and $\rho, \eta > 0$ such that

$$\limsup_{n \to \infty} \int_{B_\rho(y_n)} |v_n|^2 dx \geq \eta. \quad (4.1)$$

Otherwise, we have $v_n \to 0$ in $L^q(\mathbb{R}^3)$, $2 < q < 6$.

Since $\{v_n\}$ is a $(PS)_{c_\varepsilon}$ sequence of $J_\varepsilon$, we have

$$o(1) = \langle J'_\varepsilon(v_n), v_n \rangle$$

$$= ||v_n||^2_\varepsilon + b(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^2 - \int_{\mathbb{R}^3} P(\varepsilon x)|v_n|^p dx - \int_{\mathbb{R}^3} Q(\varepsilon x)|v_n|^6 dx.$$ 

Thus,

$$o(1) = ||v_n||^2_\varepsilon + b(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^2 - \int_{\mathbb{R}^3} Q(\varepsilon x)|v_n|^6 dx.$$ 

Since $\{v_n\}$ is bounded in $E$, as $n \to \infty$, we can assume

$$||v_n||^2_\varepsilon \to l_1, \quad (4.2)$$

$$b(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^2 \to l_2, \quad (4.3)$$

and

$$\int_{\mathbb{R}^3} Q(\varepsilon x)|v_n|^6 dx \to l_3. \quad (4.4)$$

Then we have $l_3 = l_1 + l_2$.

If $l_3 = 0$, we have $v_n \to 0$ in $E$. Then $J_\varepsilon(v_n) \to 0$, which contradicts $c_\varepsilon > 0$. Thus, $l_3 \neq 0$. By the definition of the best constant $S$, we have

$$S \leq \frac{1}{a} \int_{\mathbb{R}^3} a |\nabla v_n|^2 dx \leq \frac{1}{a} ||v_n||^2_\varepsilon \leq \frac{1}{a} (Q^{-1}_{\max} \int_{\mathbb{R}^3} Q(\varepsilon x)|v_n|^6 dx)^{1/3}$$

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and
\[ bS^2 \leq \frac{b\left( \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx \right)^2}{Q_{\max}^{-2/3} \left( \int_{\mathbb{R}^3} Q(\varepsilon x)|v_n|^6 \, dx \right)^{2/3}}. \]

Letting \( n \to \infty \) in the above two inequalities, we have
\[ aS_1^{1/3} \leq Q_{\max}^{1/3} l_1 \quad \text{and} \quad bS_2^{2/3} \leq Q_{\max}^{2/3} l_2. \]

Therefore,
\[ l_1 \geq aS_{\max}^{-1/3} (l_1 + l_2)^{1/3} \quad \text{and} \quad l_2 \geq bS_{\max}^{-2/3} (l_1 + l_2)^{2/3}. \]

By Lemma 2.8 we have
\[ \frac{1}{3} l_1 + \frac{1}{12} l_2 \geq c^* := \frac{ab}{4Q_{\max}^3} S^3 + \frac{(b^2 S^4 + 4Q_{\max}^2 aS)^{3/2}}{24Q_{\max}^2} + \frac{b^3 S^6}{24Q_{\max}^2}. \]

On the other hand,
\[ c_\varepsilon = J_\varepsilon(v_n) + o(1) \]
\[ = \frac{1}{2} \|v_n\|^2 + \frac{b}{4} \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx - \frac{1}{p} \int_{\mathbb{R}^3} P(\varepsilon x)|v_n|^p \, dx - \frac{1}{6} \int_{\mathbb{R}^3} Q(\varepsilon x)|v_n|^6 \, dx + o(1) \]
\[ = \frac{1}{2} \|v_n\|^2 + \frac{b}{4} \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx - \frac{1}{6} \int_{\mathbb{R}^3} Q(\varepsilon x)|v_n|^6 \, dx + o(1) \]
\[ = \frac{1}{2} \|v_n\|^2 + \frac{b}{4} \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx - \frac{1}{6} (\|v_n\|_{\varepsilon}^2 + b(\int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx)) + o(1) \]
\[ = \frac{1}{3} \|v_n\|^2 + \frac{b}{12} \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx + o(1). \]

Using (4.2) and (4.3) in the above expression, we obtain
\[ c_\varepsilon = \frac{1}{3} l_1 + \frac{1}{12} l_2 \geq c^*, \]
which contradicts Lemma 2.9. Therefore, **Claim 1** holds.

Let \( \tilde{v}_n(x) = v_n(x + y_n) \). Then \( \|\tilde{v}_n\|_{\varepsilon} \leq C \) in \( E \). So there exists \( \tilde{v} \in E \) such that
\[ \tilde{v}_n \to \tilde{v} \quad \text{in} \ E \]
\[ \tilde{v}_n \to \tilde{v} \quad \text{a.e. in} \ \mathbb{R}^3. \]

By (4.1), we get \( \tilde{v} \neq 0 \). Thus, there exists \( \delta > 0 \) satisfying \( \mu \{ x : |\tilde{v}(x)| > \delta \} > 0 \). By (PQ2), without loss of generality, we may assume \( x^* = 0 \in P \cap \mathcal{Q} \), such that \( \beta := V(0) \leq V(x) \) for \( |x| \geq R \). Let \( P_\infty < d < P_{\max} \). For \( v_n \), there exists \( t_n > 0 \) such that \( t_nv_n \in N_{\varepsilon dQ_{\max}} \).

**Claim 2:** \( t_n \) is bounded.

From \( t_nv_n \in N_{\varepsilon dQ_{\max}} \), we get
\[ \int_{\mathbb{R}^3} (a|\nabla v_n|^2 + V_\varepsilon^2(x)v_n^2) \, dx + bt_n^2\int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx \]
\[ = t_n^{-p-2} \int_{\mathbb{R}^3} P_\varepsilon^d(x)|v_n|^p \, dx + t_n^4 \int_{\mathbb{R}^3} Q(\varepsilon x)|v_n|^6 \, dx. \]
Then, by the boundness of \( \{v_n\} \) in \( E \), we have
\[
C + t_n^2 C \geq t_n^4 \int_{\mathbb{R}^3} Q(\varepsilon x)|v_n|^p \, dx \geq C t_n^4 \int_{\{x : |\tilde{v}(x)| > \delta\}} |\tilde{v}_n|^6 \, dx.
\]
By Egoroff Theorem, there exists \( E_0 \subset \{ x : |\tilde{v}(x)| > \delta\} \) such that \( \mu\{ x : |\tilde{v}(x)| > \delta\} \setminus E_0 \leq 0 \) and \( \tilde{v}_n \to \tilde{v} \) uniformly in \( \{ x : |\tilde{v}(x)| > \delta\} \setminus E_0 \). Thus, we can obtain
\[
C + t_n^2 C \geq C t_n^4 \int_{\{x : |\tilde{v}(x)| > \delta\} \setminus E_0} |\tilde{v}_n|^6 \, dx \geq C t_n^4.
\]
It follows that \textbf{Claim 2} holds.

\textbf{Claim 3:} \( J_{\varepsilon}^{\beta dQ_{\max}}(t_n v_n) = J_{\varepsilon}(t_n v_n) + o_n(1) \).

First, we note
\[
J_{\varepsilon}^{\beta dQ_{\max}}(t_n v_n) = J_{\varepsilon}(t_n v_n) + \frac{1}{2} \int_{\mathbb{R}^3} (V_{\varepsilon}^\beta(x) - V(\varepsilon x))(t_n v_n)^2 \, dx
+ \frac{1}{p} \int_{\mathbb{R}^3} (P(\varepsilon x) - P_{\varepsilon}^b(x))|t_n v_n|^p \, dx.
\]
For
\[
\int_{\mathbb{R}^3} (V_{\varepsilon}^\beta(x) - V(\varepsilon x))(t_n v_n)^2 \, dx = \int_{\{x : V(\varepsilon x) < \beta\}} (\beta - V(\varepsilon x))(t_n v_n)^2 \, dx.
\]
By \( (PQ2) \), we know \( \{ x : V(\varepsilon x) < \beta\} \) is bounded. If \( v_{\varepsilon} = 0 \), we have \( v_n \to 0 \) in \( L_{\text{loc}}^2(\mathbb{R}^3) \). Then, by Lebesgue Dominated Convergence Theorem, we have
\[
\int_{\mathbb{R}^3} (V_{\varepsilon}^\beta(x) - V(\varepsilon x)) f^2(t_n v_n) \, dx = o_n(1).
\]
For
\[
\int_{\mathbb{R}^3} (P(\varepsilon x) - P_{\varepsilon}^b(x))|t_n v_n|^p \, dx = \int_{\{x : P(\varepsilon x) > b\}} (P(\varepsilon x) - b)|t_n v_n|^p \, dx.
\]
By the choice of \( b \), we know \( \{ x : P(\varepsilon x) > b\} \) is bounded. Similarly, we also have
\[
\int_{\mathbb{R}^3} (P(\varepsilon x) - P_{\varepsilon}^b(x))|t_n v_n|^p \, dx = o_n(1).
\]
Thus we have \textbf{Claim 3}.

Therefore,
\[
c_{\varepsilon}^{\beta dQ_{\max}} \leq J_{\varepsilon}^{\beta dQ_{\max}}(t_n v_n) = J_{\varepsilon}(t_n v_n) + o_n(1) \leq J_{\varepsilon}(v_n) + o_n(1).
\]
Letting \( n \to \infty \), we have \( c_{\varepsilon}^{\beta dQ_{\max}} \leq c_{\varepsilon} \). By Lemma\ref{Lemma3.1} we can obtain
\[
\limsup_{\varepsilon \to 0} c_{\varepsilon} \leq m^*_{V(0)P(0)Q(0)} = m^*_P Q_{\max}.
\]
Then we have \( \limsup_{\varepsilon \to 0} c_{\varepsilon}^{\beta dQ_{\max}} \leq m^*_P Q_{\max} \). It follows from Lemma\ref{Lemma3.2} that
\[
\limsup_{\varepsilon \to 0} c_{\varepsilon}^{\beta dQ_{\max}} = \limsup_{\varepsilon \to 0} c_{\varepsilon}^{V(0)dQ_{\max}} \geq m^*_V(0) Q_{\max} = m^*_P Q_{\max}.
\]
Therefore, $m_{\beta dQ_{\text{max}}}^* \leq m_{\beta P_{\text{max}}Q_{\text{max}}}^*$, which contradicts Lemma 2.11. Thus, we have $v_\varepsilon \neq 0$.

Moreover,

$$c_\varepsilon = \lim_{n \to \infty} \left[ J_\varepsilon(v_n) - \frac{1}{4}(J'_\varepsilon(v_n), v_n) \right]$$

$$= \lim_{n \to \infty} \left[ \frac{1}{4} \|v_n\|^2 + \left( \frac{1}{4} - \frac{1}{p} \right) \int_{\mathbb{R}^3} P(\varepsilon x)|v_n|^p \, dx + \left( \frac{1}{4} - \frac{1}{6} \right) \int_{\mathbb{R}^3} Q(\varepsilon x)|v_n|^6 \, dx \right]$$

$$\geq \frac{1}{4} \|v_\varepsilon\|^2 + \left( \frac{1}{4} - \frac{1}{p} \right) \int_{\mathbb{R}^3} P(\varepsilon x)|v_\varepsilon|^p \, dx + \left( \frac{1}{4} - \frac{1}{6} \right) \int_{\mathbb{R}^3} Q(\varepsilon x)|v_\varepsilon|^6 \, dx$$

$$= J_\varepsilon(v_\varepsilon) - \frac{1}{4}(J'_\varepsilon(v_\varepsilon), v_\varepsilon)$$

$$= J_\varepsilon(v_\varepsilon).$$

Hence, $J_\varepsilon(v_\varepsilon) = c_\varepsilon$. A similar argument to the one used in Lemma 2.10 shows $v_\varepsilon \in C^2(\mathbb{R}^3)$ and $v_\varepsilon > 0$. Then, $v_\varepsilon$ is a positive ground state solution of (2.1). Thus $u_\varepsilon(x) := v_\varepsilon(x)$ a positive ground state solution of (1.1). \hfill \Box

**Lemma 4.2** Assume additionally that the potential functions $V(x)$, $P(x)$ and $Q(x)$ are uniformly continuous on $\mathbb{R}^3$. Let $v_n := v_{\varepsilon_n}$ be the solution obtained in Lemma 4.1 with $\varepsilon_n \to 0$, as $n \to +\infty$. Then

1. there exists $y_n \in \mathbb{R}^3$ satisfying $\lim_{n \to \infty} \text{dist}(\varepsilon_n y_n, A_V) = 0$.

2. up to a subsequence, $\lim_{n \to \infty} \varepsilon_n y_n = y_0$. Set $\tilde{v}_n(x) = v_n(x + y_n)$, then $\tilde{v}_n(x) \to v$ in $E$, where $v$ is a positive ground state solution of

$$-(a + b \int_{\mathbb{R}^3} |\nabla v|^2 \, dx) \Delta v + V(y_0)v = P(y_0)|v|^{p-2}v + Q(y_0)|v|^4v, \quad x \in \mathbb{R}^3.$$  

**Proof:** Let $v_n$ be the positive ground state solution obtained in Lemma 4.1 with $\varepsilon_n \to 0$. We claim that there exist $(y_n) \subset \mathbb{R}^3$ and $\rho, \eta > 0$ such that

$$\lim_{n \to \infty} \sup_{B_{\rho}(y_n)} \int_{B_\rho(x)} |v_n|^2 \, dx \geq \eta. \quad (4.5)$$

Suppose by contradiction that (4.5) does not hold. Then, by Lions’s Lemma, we can obtain

$$v_n \to 0 \text{ in } L^q(\mathbb{R}^3), \text{ for } 2 < q < 6.$$  

From

$$c_{\varepsilon_n} = J_{\varepsilon_n}(v_n) - \frac{1}{p}(J'_{\varepsilon_n}(v_n), v_n)$$

$$= \left( \frac{1}{2} - \frac{1}{p} \right) \|v_n\|_{\varepsilon_n}^2 + \left( \frac{1}{2} - \frac{1}{p} \right) b \int_{\mathbb{R}^3} |\nabla v_n|^2 \, dx + \left( \frac{1}{2} - \frac{1}{6} \right) \int_{\mathbb{R}^3} Q(\varepsilon_n x)|v_n|^6 \, dx,$$

we have

$$\left( \frac{1}{2} - \frac{1}{p} \right) \|v_n\|_{\varepsilon_n}^2 \leq c_{\varepsilon_n}.$$  

Then, by Lemma 3.1 and $V(x)$ has a positive lower bound, it is not difficult to prove that $\{v_n\}$ is bounded in $E$.  

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It follows from $J'_{\epsilon_n}(v_n) = 0$ that

$$0 = \langle J'_{\epsilon_n}(v_n), v_n \rangle = \|v_n\|^2_{\epsilon_n} + b\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2 - \int_{\mathbb{R}^3} P(\epsilon_n x)|v_n|^p dx - \int_{\mathbb{R}^3} Q(\epsilon_n x)|v_n|^6 dx.$$ 

Thus,

$$o(1) = \|v_n\|^2_{\epsilon_n} + b\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2 - \int_{\mathbb{R}^3} Q(\epsilon_n x)|v_n|^6 dx.$$ 

Since $\{v_n\}$ is bounded in $E$, as $n \to \infty$, we can assume

$$\|v_n\|^2_{\epsilon_n} \to l_1, \quad (4.6)$$

$$b\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2 \to l_2, \quad (4.7)$$

and

$$\int_{\mathbb{R}^3} Q(\epsilon_n x)|v_n|^6 dx \to l_3. \quad (4.8)$$

Then we have $l_3 = l_1 + l_2$.

If $l_3 = 0$, we have $v_n \to 0$ in $E$. Then $J_{\epsilon}(v_n) \to 0$, which contradicts Lemma 2.1. Thus, $l_3 \neq 0$.

By the definition of the best constant $S$, we have

$$S \leq \frac{1}{a} \int_{\mathbb{R}^3} a|\nabla v_n|^2 dx \leq \frac{1}{a} \|v_n\|^2_{\epsilon_n}$$

and

$$bS^2 \leq \frac{b\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx\right)^2}{Q_{max}^{-2/3}\left(\int_{\mathbb{R}^3} Q(\epsilon x)|v_n|^6 dx\right)^{2/3}}.$$ 

Letting $n \to \infty$ in the above two inequalities, we have

$$aS l_3^{1/3} \leq Q_{max}^{1/3} l_1 \quad \text{and} \quad bS^2 l_3^{2/3} \leq Q_{max}^{2/3} l_2.$$ 

Therefore,

$$l_1 \geq aS Q_{max}^{-1/3}(l_1 + l_2)^{1/3} \quad \text{and} \quad l_2 \geq bS^2 Q_{max}^{-2/3}(l_1 + l_2)^{2/3}.$$ 

By Lemma 2.8, we have

$$\frac{1}{3} l_1 + \frac{1}{12} l_2 \geq c^* := \frac{ab}{4Q_{max}} S^3 + \frac{(b^2 S^4 + 4Q_{max} aS)^{3/2}}{24Q_{max}^2} + \frac{b^3 S^6}{24Q_{max}^2}.$$ 

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On the other hand,

\[ c_{\varepsilon_n} = J_{\varepsilon_n}(v_n) = \frac{1}{2}\|v_n\|_{\varepsilon_n}^2 + \frac{b}{4}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx - \frac{1}{p}\int_{\mathbb{R}^3} P(\varepsilon_n x)|v_n|^p dx - \frac{1}{6}\int_{\mathbb{R}^3} Q(\varepsilon_n x)|v_n|^6 dx \]

\[ = \frac{1}{2}\|v_n\|_{\varepsilon_n}^2 + \frac{b}{4}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx - \frac{1}{6}\int_{\mathbb{R}^3} Q(\varepsilon_n x)|v_n|^6 dx + o(1) \]

\[ = \frac{1}{2}\|v_n\|_{\varepsilon_n}^2 + \frac{b}{4}\int_{\mathbb{R}^3} |\nabla v_n|^2 dx - \frac{1}{6}(\|v_n\|_{\varepsilon_n}^2 + b(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^2) + o(1) \]

\[ = \frac{1}{3}\|v_n\|_{\varepsilon_n}^2 + \frac{b}{12}(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx)^2 + o(1). \]

Using (4.6) and (4.7) in the above expression, we obtain

\[ \lim_{n \to \infty} c_{\varepsilon_n} = \frac{1}{3}l_1 + \frac{1}{12}l_2 \geq c^*, \]

whichcontradictsLemma[2.9].Therefore,(4.5)holds.

Set \( \tilde{v}_n(x) = v_n(x + y_n) \). Then \( \tilde{v}_n \) is bounded in \( E \). Thus there exists \( \tilde{v} \in E \) such that \( \tilde{v}_n \to \tilde{v} \) in \( E \). By (4.5), we know \( \tilde{v} \not\equiv 0 \). Let \( \tilde{V}_{\varepsilon_n}(x) = V(\varepsilon_n(x + y_n)) \), \( \tilde{P}_{\varepsilon_n}(x) = P(\varepsilon_n(x + y_n)) \) and \( \tilde{Q}_{\varepsilon_n}(x) = Q(\varepsilon_n(x + y_n)) \). Then \( \tilde{v}_n \) solves the following problems separately

\[ -(a + b\int_{\mathbb{R}^3} |\nabla v|^2 dx)\Delta v + \tilde{V}_{\varepsilon_n}(x)v = \tilde{P}_{\varepsilon_n}(x)|v|^{p-2}v + \tilde{Q}_{\varepsilon_n}(x)|v|^4v, \quad x \in \mathbb{R}^3. \tag{4.9} \]

The corresponding energy functional

\[ \tilde{J}_{\varepsilon_n}(v) = \frac{1}{2}\int_{\mathbb{R}^3} \left( a|\nabla v|^2 + \tilde{V}_{\varepsilon_n}(x)v^2 \right) dx + \frac{b}{4}\int_{\mathbb{R}^3} |\nabla v|^2 dx \]

\[ - \frac{1}{p}\int_{\mathbb{R}^3} \tilde{P}_{\varepsilon_n}(x)|v|^p dx - \frac{1}{6}\int_{\mathbb{R}^3} \tilde{Q}_{\varepsilon_n}(x)|v|^6 dx, \]

**Claim 1:** \( \varepsilon_n y_n \) must be bounded.

Otherwise, without loss of generality, we assume \( \varepsilon_n y_n \to \infty \) as \( n \to \infty \). Up to a subsequence, we have

\[ V(\varepsilon_n y_n) \to V_0 \geq \beta, \]

\[ P(\varepsilon_n y_n) \to P_0 < P_{\max}, \]

\[ Q(\varepsilon_n y_n) \to Q_0 \leq Q_{\max}. \]

Then \( \tilde{v} \) is a solution of the following equation

\[ -(a + b\int_{\mathbb{R}^3} |\nabla v|^2 dx)\Delta v + V_0 v = P_0|v|^{p-2}v + Q_0|v|^4v, \quad x \in \mathbb{R}^3. \]

In fact, for any test function \( \phi \in C_0^\infty(\mathbb{R}^3) \), since \( \tilde{v}_n \) is a solution of equation (4.9), we have

\[ 0 = \int_{\mathbb{R}^3} (a\nabla \tilde{v}_n \nabla \phi + \tilde{V}_{\varepsilon_n}(x)\tilde{v}_n \phi) dx + b \int_{\mathbb{R}^3} |\nabla \tilde{v}_n|^2 dx \int_{\mathbb{R}^3} \nabla \tilde{v}_n \nabla \phi dx \]

\[ - \int_{\mathbb{R}^3} \tilde{P}_{\varepsilon_n}(x)|\tilde{v}_n|^{p-2}\tilde{v}_n \phi dx - \int_{\mathbb{R}^3} \tilde{Q}_{\varepsilon_n}(x)|\tilde{v}_n|^4 \tilde{v}_n \phi dx, \]
Assume that \( \int_{\mathbb{R}^3} |\nabla \tilde{v}_n|^2 dx \to A^2 \), as \( n \to \infty \). Note the uniformly continuity of \( V(x), P(x), Q(x) \), we have

\[
0 = \int_{\mathbb{R}^3} (a \nabla \tilde{v} \nabla \phi + V_0 \tilde{v} \phi) dx + b A^2 \int_{\mathbb{R}^3} \nabla \tilde{v} \nabla \phi dx - \int_{\mathbb{R}^3} P_0 |\tilde{v}|^{p-2} \tilde{v} \phi dx - \int_{\mathbb{R}^3} Q_0 |\tilde{v}|^4 \tilde{v} \phi dx.
\]

Thus, we can get

\[
\int_{\mathbb{R}^3} (a|\nabla \tilde{v}|^2 + V_0 \tilde{v}^2) dx + b A^2 \int_{\mathbb{R}^3} |\nabla \tilde{v}|^2 dx = \int_{\mathbb{R}^3} P_0 |\tilde{v}|^p dx + \int_{\mathbb{R}^3} Q_0 |\tilde{v}|^6 dx.
\]

It is easy to know that \( \int_{\mathbb{R}^3} |\nabla \tilde{v}|^2 dx \leq A^2 \). If \( \int_{\mathbb{R}^3} |\nabla \tilde{v}|^2 dx < A^2 \), we have

\[
\int_{\mathbb{R}^3} (a|\nabla \tilde{v}|^2 + V_0 \tilde{v}^2) dx + b(\int_{\mathbb{R}^3} |\nabla \tilde{v}|^2 dx)^2 < \int_{\mathbb{R}^3} P_0 |\tilde{v}|^p dx + \int_{\mathbb{R}^3} Q_0 |\tilde{v}|^6 dx.
\]

Then, there exists \( 0 < \tilde{t} < 1 \) such that \( \tilde{\tilde{v}} \in N_{\tilde{V}_0 P_0 Q_0}^* \). Thus, we have

\[
m_{\tilde{V}_0 P_0 Q_0}^* = \Phi_{\tilde{V}_0 P_0 Q_0}^* (\tilde{\tilde{v}}) - \frac{1}{4} \langle (\Phi_{\tilde{V}_0 P_0 Q_0}^* (\tilde{\tilde{v}}), \tilde{\tilde{v}}) \rangle = \frac{1}{4} \int_{\mathbb{R}^3} (a|\nabla \tilde{v}|^2 + V_0 \tilde{v}^2) dx + \frac{1}{4} \int_{\mathbb{R}^3} P_0 |\tilde{v}|^p dx + \frac{1}{4} \int_{\mathbb{R}^3} Q_0 |\tilde{v}|^6 dx.
\]

Therefore,

\[-(a + b) \int_{\mathbb{R}^3} |\nabla \tilde{v}|^2 dx \Delta \tilde{v} + V_0 \tilde{v} = P_0 |\tilde{v}|^{p-2} \tilde{v} + Q_0 |\tilde{v}|^4 \tilde{v}, \quad x \in \mathbb{R}^3.\]
It follows that
\[ m_{\beta P_{\max}}^{*}Q_{\max} < m_{V_{0}P_{0}Q_{0}}^{*} \leq \Phi_{V_{0}P_{0}Q_{0}}^{*} (\bar{v}) \]
\[ = \Phi_{V_{0}P_{0}Q_{0}}^{*} (\bar{v}) - \frac{1}{4} (\Phi_{V_{0}P_{0}Q_{0}}^{*})'(\bar{v}), \bar{v}) \]
\[ = \frac{1}{4} \int_{\mathbb{R}^3} (a|\nabla \bar{v}|^2 + V_{0\bar{v}}^2)dx + \left( \frac{1}{4} - \frac{1}{p} \right) \int_{\mathbb{R}^3} P_{0}\bar{v}|^p dx + \left( \frac{1}{4} - \frac{1}{6} \right) \int_{\mathbb{R}^3} Q_{0}\bar{v}|^6 dx \]
\[ \leq \liminf_{n \to \infty} \left[ \frac{1}{4} \int_{\mathbb{R}^3} (a|\nabla \bar{v}_n|^2 + \bar{V}_{\varepsilon_n}(x)\bar{v}_n^2)dx + \left( \frac{1}{4} - \frac{1}{p} \right) \int_{\mathbb{R}^3} \bar{P}_{\varepsilon_n}(x)|\bar{v}_n|^p dx \right. \]
\[ \left. + \left( \frac{1}{4} - \frac{1}{6} \right) \int_{\mathbb{R}^3} \bar{Q}_{\varepsilon_n}(x)|\bar{v}_n|^6 dx \right] \]
\[ = \liminf_{n \to \infty} \left[ \bar{J}_{\varepsilon_n}(\bar{v}_n) - \frac{1}{4} \langle \bar{J}'_{\varepsilon_n}(\bar{v}_n), \bar{v}_n \rangle \right] = \liminf_{n \to \infty} J_{\varepsilon_n}(v_n) = \liminf_{n \to \infty} c_{\varepsilon_n}, \]
which contradicts Lemma 3.1. Thus \( \varepsilon_n y_n \) must be bounded. And, up to a subsequence, we can assume \( \varepsilon_n y_n \to y_0 \).

**Claim 2:** \( y_0 \in A_V \).

If \( y_0 \notin A_V \), we have two cases.

1. \( \beta < V(y_0), P(y_0) = P_{\max} \) and \( Q(y_0) = Q_{\max} \), then \( m_{\beta P_{\max}}^{*}Q_{\max} < m^{*}_{V(y_0)P(y_0)Q(y_0)} \).

2. \( \beta \leq V(y_0), P(y_0) < P_{\max} \) or \( Q(y_0) < Q_{\max} \), then \( m_{\beta P_{\max}Q_{\max}}^{*} < m^{*}_{V(y_0)P(y_0)Q(y_0)} \).

From **Claim 1**, we know that \( \bar{v} \) is a solution of the following equation
\[ - (a + b \int_{\mathbb{R}^3} |\nabla v|^2 dx) \Delta v + V(y_0)v = P(y_0) |v|^{p-2} v + Q(y_0) |v|^4 v, \quad x \in \mathbb{R}^3. \] (4.10)

Similar to the arguments in **Claim 1**, we have \( m^{*}_{V(y_0)P(y_0)Q(y_0)} \leq \liminf_{n \to \infty} c_{\varepsilon_n} \).

Applying Lemma 3.1 and Lemma 2.1, we get
\[ \limsup_{n \to \infty} c_{\varepsilon_n} \leq m_{\beta P_{\max}Q_{\max}}^{*} < m^{*}_{V(y_0)P(y_0)Q(y_0)} \leq \liminf_{n \to \infty} c_{\varepsilon_n}, \]
which is absurd. Therefore, \( y_0 \in A_V \), which means that
\[ \lim_{n \to \infty} \text{dist}(\varepsilon_n y_n, A_V) = 0. \]

**Claim 3:** \( \bar{v} \) is a positive ground state solution of (4.10).

Repeating the arguments in **Claim 1** again, we get
\[ \Phi_{V(y_0)P(y_0)Q(y_0)}^{*} (\bar{v}) \leq \liminf_{n \to \infty} J_{\varepsilon_n}(v_n) \leq \liminf_{n \to \infty} c_{\varepsilon_n} \leq \limsup_{n \to \infty} c_{\varepsilon_n} \leq m^{*}_{V(y_0)P(y_0)Q(y_0)}. \]

So we get
\[ \Phi_{V(y_0)P(y_0)Q(y_0)}^{*} (\bar{v}) = m^{*}_{V(y_0)P(y_0)Q(y_0)}. \]

Thus \( \bar{v} \) is a ground state solution. By the theory of elliptic regularity, \( \bar{v} \in C^2(\mathbb{R}^3) \), and by using strong maximum principle, we get \( \bar{v} > 0 \) in \( \mathbb{R}^3 \).

**Claim 4:** \( \bar{v}_n \) converges strongly to \( \bar{v} \) in \( E \).
From Claim 3, we know that

$$\lim_{{n \to \infty}} J_{{\epsilon_n}} (\tilde{v}_n) = \lim_{{n \to \infty}} J_{{\epsilon_n}} (v_n) = \lim_{{n \to \infty}} c_{{\epsilon_n}} = \Phi^*_{{V(y_0)P(y_0)Q(y_0)}} (\tilde{v}).$$

It follows that

$$m^*_{{V(y_0)P(y_0)Q(y_0)}} = \Phi^*_{{V(y_0)P(y_0)Q(y_0)}} (\tilde{v}) - \frac{1}{4} \left( (\Phi^*_{{V(y_0)P(y_0)Q(y_0)}})' (\tilde{v}), \tilde{v} \right)$$

$$= \frac{1}{4} \int_{{\mathbb{R}^3}} (a |\nabla \tilde{v}|^2 + V(y_0) \tilde{v}^2) dx + (\frac{1}{4} - \frac{1}{p}) \int_{{\mathbb{R}^3}} P(y_0) |\tilde{v}|^p dx$$

$$+ (\frac{1}{4} - \frac{1}{6}) \int_{{\mathbb{R}^3}} Q(y_0) |\tilde{v}|^6 dx$$

$$\leq \liminf_{{n \to \infty}} \left[ \frac{1}{4} \int_{{\mathbb{R}^3}} (a |\nabla \tilde{v}_n|^2 + \tilde{V}_{{\epsilon_n}} (x) \tilde{v}_n^2) dx + (\frac{1}{4} - \frac{1}{p}) \int_{{\mathbb{R}^3}} \tilde{P}_{{\epsilon_n}} (x) |\tilde{v}_n|^p dx 

+ (\frac{1}{4} - \frac{1}{6}) \int_{{\mathbb{R}^3}} \tilde{Q}_{{\epsilon_n}} (x) |\tilde{v}_n|^6 dx \right]$$

$$= \liminf_{{n \to \infty}} \left[ \tilde{J}_{{\epsilon_n}} (\tilde{v}_n) - \frac{1}{4} (\tilde{J}'_{{\epsilon_n}} (\tilde{v}_n), \tilde{v}_n) \right]$$

$$= \Phi^*_{{V(y_0)P(y_0)Q(y_0)}} (\tilde{v}).$$

Thus, as \(n \to \infty\), we have

$$\int_{{\mathbb{R}^3}} (a |\nabla \tilde{v}_n|^2 + \tilde{V}_{{\epsilon_n}} (x) \tilde{v}_n^2) dx \to \int_{{\mathbb{R}^3}} (a |\nabla \tilde{v}|^2 + V(y_0) \tilde{v}^2) dx. \quad (4.11)$$

Since

$$\int_{{\mathbb{R}^3}} a |\nabla \tilde{v}|^2 dx \leq \liminf_{{n \to \infty}} \int_{{\mathbb{R}^3}} a |\nabla \tilde{v}_n|^2 dx$$

and

$$\int_{{\mathbb{R}^3}} V(y_0) \tilde{v}^2 dx \leq \liminf_{{n \to \infty}} \int_{{\mathbb{R}^3}} \tilde{V}_{{\epsilon_n}} (x) \tilde{v}_n^2 dx,$$

it follows from (4.11) that

$$\lim_{{n \to \infty}} \int_{{\mathbb{R}^3}} a |\nabla \tilde{v}_n|^2 dx = \int_{{\mathbb{R}^3}} a |\nabla \tilde{v}|^2 dx$$

and

$$\lim_{{n \to \infty}} \int_{{\mathbb{R}^3}} \tilde{V}_{{\epsilon_n}} (x) \tilde{v}_n^2 dx = \int_{{\mathbb{R}^3}} V(y_0) \tilde{v}^2 dx.$$ 

Then, it is easy to prove that

$$\int_{{\mathbb{R}^3}} (a |\nabla \tilde{v}_n|^2 + V(y_0) \tilde{v}_n^2) dx \to \int_{{\mathbb{R}^3}} (a |\nabla \tilde{v}|^2 + V(y_0) \tilde{v}^2) dx.$$

Thus, we have \(\|\tilde{v}_n\| \to \|\tilde{v}\|\), noting that \(\tilde{v}_n \to \tilde{v}\) in \(E\), So \(\tilde{v}_n \to \tilde{v}\) in \(E\) is obtained. \(\square\)

**Remark 4.3** In fact, from the proof of Lemma 4.2, we can get the following results.

1. There exists \(\varepsilon^* > 0\), a family \(\{y_\varepsilon\} \subset \mathbb{R}^3\) and \(\rho, \eta > 0\) such that, for all \(\varepsilon \in (0, \varepsilon^*)\),

$$\int_{{B_\rho(y_\varepsilon)}} |v_\varepsilon|^2 dx \geq \eta. \quad (4.12)$$
Thus, \( \varepsilon \) from Remark 4.5.

Lemma 4.4 There exists \( \varepsilon^* > 0 \) such that

\[
\lim_{|x| \to \infty} \tilde{v}_\varepsilon(x) = 0 \quad \text{uniformly on } \varepsilon \in (0, \varepsilon^*),
\]

and there exists \( C > 0 \) independent of \( \varepsilon \) such that \( |\tilde{v}_\varepsilon|_\infty \leq C \) uniformly on \( \varepsilon \in (0, \varepsilon^*) \), where \( \tilde{v}_\varepsilon \) are obtained in Lemma 4.2. Furthermore, there exist constants \( C, c > 0 \) such that

\[
|\tilde{v}_\varepsilon(x)| \leq C \exp(-c|x|)
\]

for all \( x \in \mathbb{R}^3 \).

Proof: The proof of this lemma can be obtained from Lemma 4.4 and Lemma 4.5 in [19]. \( \square \)

Remark 4.5 From (4.12) and Lemma 4.4, we have

\[
\frac{\eta}{2} \leq \int_{B_\rho(0)} |\tilde{v}_\varepsilon|_2^2 dx \leq C|\tilde{v}_\varepsilon|_\infty.
\]

Thus, there exists \( \eta' > 0 \), such that \( |\tilde{v}_\varepsilon|_\infty \geq \eta' \). If \( b_\varepsilon \) is a maximum point of \( \tilde{v}_\varepsilon \), by \( \lim_{|x| \to \infty} \tilde{v}_\varepsilon(x) = 0 \) uniformly on \( \varepsilon \in (0, \varepsilon^*) \), we can get \( R_0 > 0 \) such that \( |b_\varepsilon| \leq R_0 \).

The proof of Theorem 1.1:

By Lemma 4.1 for \( \varepsilon > 0 \) small enough, problem (1.1) has a positive ground state solution \( u_\varepsilon(x) = v_\varepsilon(x, \varepsilon) \).

(1) From Remark 4.5, \( \tilde{v}_\varepsilon \) has a maximum point \( b_\varepsilon \). Then \( v_\varepsilon \) has a maximum point \( z_\varepsilon := b_\varepsilon + y_\varepsilon \).

Thus, \( u_\varepsilon(x) \) has maximum value at \( x_\varepsilon := \varepsilon z_\varepsilon \). Noting the boundness of \( b_\varepsilon \), by Remark 4.3, we have \( \lim_{\varepsilon \to 0} \inf dist(x_\varepsilon, A_\varepsilon) = 0 \).

Moreover, it follows from Lemma 4.4 that

\[
u_\varepsilon(x) = v_\varepsilon(x, \varepsilon) = \tilde{v}_\varepsilon(x, \varepsilon) - y_\varepsilon \leq C \exp(-c|\varepsilon x - y_\varepsilon|) \leq C \exp(-c|x - x_\varepsilon|).
\]

(2) Since \( \tilde{x}_\varepsilon \) is a maximum point of \( u_\varepsilon \), then \( \tilde{b}_\varepsilon := \tilde{x}_\varepsilon - y_\varepsilon \) is the maximum point of \( \tilde{v}_\varepsilon \). In view of Remark 4.5, we know \( \tilde{b}_\varepsilon \) is bounded. Moreover, \( \varepsilon(\tilde{b}_\varepsilon + y_\varepsilon) = \tilde{x}_\varepsilon \to x_0 \) as \( \varepsilon \to 0 \).

On the other hand, by (4.12), there exist \( \rho, \eta > 0 \) such that

\[
\lim_{\varepsilon \to 0} \sup_{y_\varepsilon} \int_{B_\rho(y_\varepsilon)} |v_\varepsilon|^2 dx \geq \eta.
\]

So we have

\[
\lim_{\varepsilon \to 0} \sup_{y_\varepsilon} \int_{B_\rho+y_\varepsilon} |v_\varepsilon|^2 dx \geq \lim_{\varepsilon \to 0} \sup_{y_\varepsilon} \int_{B_\rho(y_\varepsilon)} |v_\varepsilon|^2 dx \geq \eta.
\]

Then, using the same argument as in the proof of Lemma 4.2, we get \( v_\varepsilon(x + \tilde{b}_\varepsilon + y_\varepsilon) \to v \) in \( E \), as \( \varepsilon \to 0 \), where \( v \) is a positive ground state solution of

\[
-(a + b) \int_{\mathbb{R}^3} |\nabla v|^2 dx \Delta v + V(y_0) v = P(y_0)|v|^{p-2} v + Q(y_0)|v|^4 v, \quad x \in \mathbb{R}^3.
\]

Thus, \( u_\varepsilon(x + \tilde{x}_\varepsilon) = v_\varepsilon(x + \tilde{b}_\varepsilon + y_\varepsilon) \to v \) in \( E \), as \( \varepsilon \to 0 \).

The proof of Theorem 1.2 is similar to Theorem 1.1, so we omit the detail. \( \square \)
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References

[1] C. O. Alves, F. J. S. A. Corrêa, and T. F. Ma. Positive solutions for a quasilinear elliptic equation of Kirchhoff type. *Comput. Math. Appl.*, 49(1):85–93, 2005.

[2] A. Arosio and S. Panizzi. On the well-posedness of the Kirchhoff string. *Trans. Amer. Math. Soc.*, 348(1):305–330, 1996.

[3] S. Bernstein. Sur une classe d’équations fonctionnelles aux dérivées partielles. *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]*, 4:17–26, 1940.

[4] C. Chen, Y. Kuo, and T. Wu. The Nehari manifold for a Kirchhoff type problem involving sign-changing weight functions. *J. Differential Equations*, 250(4):1876–1908, 2011.

[5] P. D’Ancona and S. Spagnolo. Global solvability for the degenerate Kirchhoff equation with real analytic data. *Invent. Math.*, 108(2):247–262, 1992.

[6] Y. Deng, S. Peng, and W. Shuai. Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$. *J. Funct. Anal.*, 269(11):3500–3527, 2015.

[7] Y. Ding and X. Liu. Semi-classical limits of ground states of a nonlinear Dirac equation. *J. Differential Equations*, 252(9):4962–4987, 2012.

[8] Y. Ding and X. Liu. Semiclassical solutions of Schrödinger equations with magnetic fields and critical nonlinearities. *Manuscripta Math.*, 140(1-2):51–82, 2013.

[9] X. He and W. Zou. Infinitely many positive solutions for Kirchhoff-type problems in $\mathbb{R}^3$. *Nonlinear Anal.*, 70(3):1407–1414, 2009.

[10] X. He and W. Zou. Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$. *J. Differential Equations*, 252(2):1813–1834, 2012.

[11] X. He and W. Zou. Ground states for nonlinear Kirchhoff equations with critical growth. *Ann. Mat. Pura Appl. (4)*, 193(2):473–500, 2014.

[12] Y. He and G. Li. Standing waves for a class of Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. *Calc. Var. Partial Differential Equations*, 54(3):3067–3106, 2015.

[13] Y. He, G. Li, and S. Peng. Concentrating bound states for Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. *Adv. Nonlinear Studies*, 14(2):483–510, 2014.

[14] G. Kirchhoff. *Vorlesungen über Mechanik*. Birkhäuser Basel, 1883.

[15] G. Li, P. Luo, S. Peng, C. Wang, and C. Xiang. A singularly perturbed Kirchhoff problem revisited. *J. Differential Equations*, 268(2):541–589, 2020.
[16] G. Li and H. Ye. Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in \( \mathbb{R}^3 \). *J. Differential Equations*, 257(2):566–600, 2014.

[17] G. Li and H. Ye. Existence of positive solutions for nonlinear Kirchhoff type problems in \( \mathbb{R}^3 \) with critical Sobolev exponent. *Math. Methods Appl. Sci.*, 37(16):2570–2584, 2014.

[18] J. L. Lions. On some questions in boundary value problems of mathematical physics. 30:284–346, 1978.

[19] Z. Liu and S. Guo. Existence and concentration of positive ground states for a Kirchhoff equation involving critical Sobolev exponent. *Z. Angew. Math. Phys.*, 66(3):747–769, 2015.

[20] T. F. Ma and J. E. Muñoz Rivera. Positive solutions for a nonlinear nonlocal elliptic transmission problem. *Appl. Math. Lett.*, 16(2):243–248, 2003.

[21] A. Mao and Z. Zhang. Sign-changing and multiple solutions of Kirchhoff type problems without the P.S. condition. *Nonlinear Anal.*, 70(3):1275–1287, 2009.

[22] K. Ono. Global existence, decay, and blowup of solutions for some mildly degenerate nonlinear Kirchhoff strings. *J. Differential Equations*, 137(2):273–301, 1997.

[23] K. Perera and Z. Zhang. Nontrivial solutions of Kirchhoff-type problems via the Yang index. *J. Differential Equations*, 221(1):246–255, 2006.

[24] S. I. Pohozaev. A certain class of quasilinear hyperbolic equations. *Mat. Sb. (N.S.)*, 96(138):152–166, 168, 1975.

[25] W. Shuai. Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains. *J. Differential Equations*, 259(4):1278–1287, 2011.

[26] X. Wang. On concentration of positive bound states of nonlinear Schrödinger equations. *Comm. Math. Phys.*, 153(2):229–244, 1993.

[27] X. Wang and B. Zeng. On concentration of positive bound states of nonlinear Schrödinger equations with competing potential functions. *SIAM J. Math. Anal.*, 28(3):633–655, 1997.

[28] M. Willem. *Minimax theorems*, volume 24 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1996.

[29] X. Wu. Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in \( \mathbb{R}^N \). *Nonlinear Anal. Real World Appl.*, 12(2):1278–1287, 2011.

[30] M. Yang. Concentration of positive ground state solutions for Schrödinger-Maxwell systems with critical growth. *Adv. Nonlinear Stud.*, 16(3):389–408, 2016.

[31] Z. Yang, Y. Yu, and F. Zhao. The concentration behavior of ground state solutions for a critical fractional Schrödinger-Poisson system. *Math. Nachr.*, 292(8):1837–1868, 2019.

[32] Z. Zhang. Existence of nontrivial solutions and high energy solutions for Schrödinger-Kirchhoff-type equations in \( \mathbb{R}^N \).* Nonlinear Anal. Real World Appl.*, 12(2):1278–1287, 2011.

[33] Z. Zhang and K. Perera. Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow. *J. Math. Anal. Appl.*, 317(2):456–463, 2006.