On solutions for a system of equations of relativistic electrodynamics

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Abstract. The possibility of applying the homotopy (deformation) method to studying the invariance of extremals for a generalized system of equations of relativistic electrodynamics is considered. This approach allows to investigate the stability of extremals for the action functional.

Keywords: relativistic electrodynamics, variational extremal problem, electromagnetic Minkowski potential, deformation method for Hamiltonian systems

1. Introduction

It is known [1,2] that a generalized system of equations of relativistic electrodynamics can be obtained by solving the variational extremal problem for the sum of three action functionals, the first of which determines the freely moving charged mass of a particle in an electromagnetic field. The second functional gives the sum of the contributions of the four-dimensional electromagnetic Minkowski potential \( A = (i\varphi, A_r) \) and the third one takes into account the more complete dynamics of the electromagnetic (or gravity) field. The solution of this variational problem gives a generalized (Hamiltonian) system of equations of the relativistic dynamics [1].

An analysis of the Lyapunov’s stability of solutions of this system is carried out by a deformation (homotopy) method for Hamiltonian systems [3].

2. The variational problem

It is known [1] that a generalized system of equations of relativistic electrodynamics can be obtained through the principle of least action while solving the variational extremum problem (maximization) of the following functional

\[
\max \{ J_{13} \} = \sum_{k=1}^{3} J_k = \int_{t_0}^{t_f} dt \int_V \left[ \rho mc^2 \sqrt{1 - \frac{\dot{v}^2}{c^2}} + \rho \varphi - \frac{\rho}{c} (\vec{A}_r, \vec{v}) - \frac{1}{8\pi} (E^2 - H^2) \right] dV
\]

(1)

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under the constraints
\[ x_i = v_i, \quad x_i(t_0) = x_i^{(0)}, \quad \left| v_i(t) \right| < c, \quad i = 1, 2, 3, \]  
(2)

where \( \rho_m \) and \( \rho \) are the distribution densities of masses and charges in the given geometric volume \( V \) in the three-dimensional space \( R = (x_1, x_2, x_3) \), \( c \) denotes the speed of light and \( \vec{v} \) is the velocity of charged particles.

We note that the variational problem (1)-(2) was only set in [2] but was not considered there any further.

The first functional in (1) defines a freely moving mass \( \rho_m \) in space \( R \) and corresponds to the contribution of the term \( \rho_m c^2 \sqrt{1 - \frac{v^2}{c^2}} \). The contribution of the second integrand to the functional (1) in the Minkowski coordinate system \((ict, x_1, x_2, x_3)\) is affected by the four-dimensional electromagnetic potential \( A(x) = (A_0, A_1, A_2, A_3) = (i\phi, \vec{A}_r) \) which combines the scalar potential \( \phi(x) \) and the vector potential \( \vec{A}_r = (A_1, A_2, A_3) \). The total contribution is equal to \( \rho \phi - \frac{\rho}{c} (\vec{A}_r, \vec{v}) \) where \( \rho \) is the charge density.

More complete dynamics of the electromagnetic (or gravity) field refers to the electromagnetic field tensor \( F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \) [2].

This contribution is represented by the third integral
\[ J_3 = -\frac{1}{8\pi} \int_0^\tau dt \int_V (E^2 - H^2) \, dV. \]

3. The method of obtaining a generalized system of equations of relativistic electrodynamics

Further we briefly repeat the procedure of obtaining a generalized system of equations of relativistic electrodynamics [1].

We introduce the Hamiltonian
\[ H_{13} = \rho_m c^2 \sqrt{1 - \frac{\vec{v}^2}{c^2}} + \rho \phi - \frac{\rho}{c} (\vec{A}_r, \vec{v}) - \frac{1}{8\pi} (E^2 - H^2), \]

and the notation
\[ H^V_{13} = \int_V H_{13} \, dV + \vec{\lambda} \, \vec{v}. \]  
(3)

The necessary optimality condition for the solution of problem (1)-(2) leads to the following vector equations
Calculating the full time derivative of (5) and introducing the notation for the full time derivative of the momentum we obtain

$$\dot{\lambda} = -\frac{\partial H^Y_{13}}{\partial V} - \int_V \left[ c^2 \sqrt{1 - \frac{v^2}{c^2}} \ \text{grad} \ \rho_m + \rho \ \text{grad} \ \varphi + \varphi \ \text{grad} \ \rho - \left( \frac{\partial A_r}{\partial V} \right) \frac{v}{c} - \frac{v}{c} \times \text{rot} \ A_r - \frac{1}{4\pi} \left( (\vec{E} V) \vec{E} + \vec{E} \times \text{rot} \ E - (\vec{H} V) \vec{H} - \vec{H} \times \text{rot} \ H \right) \right] \ dV, \quad (4)$$

$$\frac{\partial H^Y_{13}}{\partial V} = -\int_V \left[ \frac{\dot{v} \rho_m}{c} + \frac{\rho}{c} A_r \right] \ dV + \ddot{\lambda} = 0. \quad (5)$$

Further, eliminating $\ddot{\lambda}$ from equations (4) and (6), we obtain the following integral-differential equation for the motion of charged distributed masses in the volume $V$:

$$\dot{p^V} = \int_V \rho_m \frac{d}{dt} \left( \frac{\dot{v}}{\sqrt{1 - \frac{v^2}{c^2}}} \right) \ dV,$$

$$\ddot{\lambda} = \dot{p^V} + \int_V \left[ \frac{v}{\sqrt{1 - \frac{v^2}{c^2}}} \ \frac{\partial \rho_m}{\partial t} + \frac{\rho}{c} \left( \frac{\partial A_r}{\partial t} + (v \nabla) A_r \right) - \frac{v}{c} A_r \frac{\partial \rho}{\partial t} \right] \ dV. \quad (6)$$

Further, eliminating $\ddot{\lambda}$ from equations (4) and (6), we obtain the following integral-differential equation for the motion of charged distributed masses in the volume $V$:

$$\dot{p^V} = \int_V \left[ \left( \frac{\partial A_r}{\partial t} + \nabla \varphi \right) \rho + \frac{\rho}{c} v \times \text{rot} \ A_r - c^2 \sqrt{1 - \frac{v^2}{c^2}} \ \nabla \rho_m - \left( \frac{1}{c} A_r \right) \frac{\partial v}{\partial t} - \frac{\dot{v}}{c} \frac{\partial \rho_m}{\partial t} \right] \ dV. \quad (7)$$

4. Generalized system of equations of relativistic electrodynamics

If we introduce a “local” impulse $\overrightarrow{p}$ by the integral
\[ \vec{p}' = \int_V \vec{p} \, dV \]

where \( \vec{p} = \rho_m \frac{d}{dt} \left( \sqrt{1 - \frac{\vec{v}^2}{c^2}} \right) \), then equation (7) could be rewritten as an ordinary differential equation which is satisfied for almost every moment \( t \) and almost all points of volume \( V \).

The following result holds true.

**Theorem 1.** Let \( r(t) \in C^{(2)}(V) \), \( \vec{v}(t) \in C^{(1)}(V) \) and also let the integrands of the functional (1) are continuously differentiable. Then the motion in the space \( R \) is governed by the differential equation:

\[
\dot{\vec{p}} = \rho \vec{E} + \frac{\rho}{c} \vec{v} \times \vec{H} + \frac{1}{8\pi} \nabla(E^2 - H^2) - \frac{1}{c} A_r \frac{\partial \rho}{\partial t} - \sqrt{1 - \frac{\vec{v}^2}{c^2}} \nabla \rho_m - \left( \phi - \frac{1}{c}(A_r, \vec{v}) \right) \nabla \rho.
\]

(8)

Following the terminology adopted in [1], we call the systems of equations (7)-(8) the generalized system of equations of relativistic electrodynamics.

**Remark 1.** Note that the necessary condition \( \max_{\vec{v}} H_{13}^c \) is satisfied only when \( |\vec{v}| < c \), where \( c \) is the speed of light, for almost all \( t \in [t_0, T] \), and, therefore, for any optimal trajectory the above inequality \( |\vec{v}| < c \) is also valid.

5. **Stability of the solutions for the system (7)-(8)**

Now we turn to the problem of the stability of solutions to the system (7)-(8) under the conditions (2). It is natural to consider the zero equilibrium position for problem (1)-(2). To this end, we apply the so-called deformation (homotopy) method for investigating the stability of solutions of Hamiltonian systems [3]. The essence of this method is as follows: for the sake of brevity, the system of equations (4)-(5) is considered as a general Hamiltonian system

\[
\begin{align*}
\frac{dy}{dt} &= \frac{\partial H(x, y)}{\partial x} \\
\frac{dx}{dt} &= -\frac{\partial H(x, y)}{\partial y},
\end{align*}
\] 

(9)

where the variables \( x \) and \( y \) of the space \( R^3 \) play the role of variables \( \vec{x} \) and \( \vec{v} \) respectively and the Hamiltonian of system (1)-(2) is defined by (3). Moreover, the Hamiltonian \( H \) of system (9) defines a mapping \( H : R^3 \times R^3 \rightarrow R \) which is smooth with respect to its arguments.

The necessary conditions at the zero equilibrium position (i.e. at the origin \((0,0)\)) have the form
By the Dirichlet theorem [3], this equilibrium state is Lyapunov stable if the point \((0,0)\) is a local minimum point of the Hamiltonian \(H\), and the presence of a minimum of the Hamiltonian \(H\) at the origin \((0,0)\) is only a sufficient condition for the stability of the zero equilibrium state of system (9) (or Eq. (8)).

6. The homotopy method for studying the equilibria of Hamiltonian systems

To study the stability of the equilibrium state of Hamiltonian systems, a homotopy (deformation) method is used [3]. In what follows we give an appropriate reasoning for it.

A one-parameter family of Hamiltonian systems

\[
\begin{align*}
\frac{dy}{dt} & = \frac{\partial H(x,y;\lambda)}{\partial x}, \\
\frac{dx}{dt} & = -\frac{\partial H(x,y;\lambda)}{\partial y},
\end{align*}
\]

(10)

where \(x, y \in \mathbb{R}^3\), \(0 \leq \lambda \leq 1\), \(H : \mathbb{R}^3 \times \mathbb{R}^3 \times [0,1]\) with additional conditions

\[
\frac{\partial H(0,0;\lambda)}{\partial x} = \frac{\partial H(0,0;\lambda)}{\partial y} = 0, \quad 0 \leq \lambda \leq 1,
\]

is called a non-degenerate deformation of the Hamiltonian system

\[
\begin{align*}
\frac{dy}{dt} & = \frac{\partial H_0(x,y)}{\partial x}, \\
\frac{dx}{dt} & = -\frac{\partial H_0(x,y)}{\partial y},
\end{align*}
\]

(11)

to the Hamiltonian system

\[
\begin{align*}
\frac{dy}{dt} & = \frac{\partial H_1(x,y)}{\partial x}, \\
\frac{dx}{dt} & = -\frac{\partial H_1(x,y)}{\partial y},
\end{align*}
\]

(12)

if the Hamiltonian \(H(x,y;\lambda)\) and its partial derivatives \(\frac{\partial H}{\partial x}, \frac{\partial H}{\partial y}\) are continuous in \(\mathbb{R}^3 \times \mathbb{R}^3 \times [0,1]\) and, for each value of the parameter \(\lambda \in [0,1]\), the point \((0,0)\) is the isolated (uniformly by \(\lambda\)) equilibrium state of the system (10) and the equalities

\(H(x,y;0) = H_0(x,y)\), \(H(x,y;1) = H_1(x,y)\)

hold.

The following Theorem 2 is valid [3].
Theorem 2. Let there exist a non-degenerate deformation of the Hamiltonian system (11) to the Hamiltonian system (12). Let the point \((0,0)\) be a minimum point of the Hamiltonian \(H_0\). Then the equilibrium point \((0,0)\) is Lyapunov’s stable equilibrium state of the system (12) (corresponding to the Hamiltonian \(H_1\)).

Comment 2. In the general case (if the system is not Hamiltonian) for solving optimal control problems, one can apply the so-called deformation theorems [3]. The corresponding results will be presented in at the forthcoming papers.

Summing up we can also state the following Theorem 3.

Theorem 3. Let the vector function \(r(t) \in C^2(W)\), \(\tilde{v}(t) \in C^1(W)\) where \(W \subseteq R\) is a bounded region and let the integrands in the functional (1) be continuously differentiable. Then the motion in space \(R\) is determined by equation (8). This solution is stable in Lyapunov’s sense in \(W \subseteq R\).

7. The derivation of general equations of the electromagnetic field

Below we obtain the general equations of the electromagnetic field for moving distributed masses and distributed charges (or electric currents) satisfying the equation (8).

One can apply the differential operators \(\text{rot} = (\nabla \times \cdot)\) and \(\text{div} = (\nabla \cdot \cdot)\) to the equation (8), though, in this case, it is necessary to require more smoothness conditions for the input data, for example, its differentiability in Theorem 1.

In classical field theory it is usually assumed that an arbitrary vector field \(\tilde{W}\) is harmonic (i.e. solenoidal and irrotational). We note that the equation (8) for the motion of charged masses can be written in a general form:

\[
\dot{\tilde{y}} = \tilde{f}(\tilde{y}, t),
\]

which generates a vector field described by the difference

\[
\tilde{w} = \dot{\tilde{y}} - \tilde{f}(\tilde{y}, t).
\]

Therefore, the condition for the harmonicity of the field (13) is not an artificial condition and it is true that

\[
\text{rot} \tilde{w} = 0, \quad \text{div} \tilde{w} = 0
\]

whence it is obvious that \(\tilde{w} = 0\).

Applying the differential operators ‘rot’ and ‘div’ to the equation (8) one comes to the following Theorem 4.

Theorem 4. Let the problem (1)-(2) satisfy the conditions of Theorem 1 and, in addition, the functions in Theorem 1 have continuous second partial derivatives with respect to the coordinates \(\tilde{x} = (x_1, x_2, x_3)\) for all \(t > 0\) while the vector function \(\tilde{v}: |\tilde{v}| \leq c\). Then the application of the differential operators ‘rot’ and ‘div’ to the equation (8) is admissible.

Taking into account the fact that the velocity \(v(t)\) does not depend on the space coordinates \((x_1, x_2, x_3)\) this transform leads to the following equations of the electromagnetic field:
Theorem 4 is proved.

8. A comparison of the generalized system (14)-(15) with the well-known equations of classical electrodynamics

Corollary 1.
Let the motion of a charged particle be governed by its mass $m$ and charge $e$ and, in this case, suppose that the effect of an external electromagnetic field $(\mathbf{E}, \mathbf{H})$ is negligible.

Then the last term in expression (1) vanishes and the integral takes the form $\max \{J_{13}\} = J_1 + J_2$ and thus the equation (8) transforms into the classical form [2]:

$$\mathbf{p} = e\mathbf{E} + \frac{e}{c}\mathbf{v} \times \mathbf{H}.$$  \hspace{1cm} (16)

Applying the operator ‘rot’ to the differential equation (16), we have:
\[
\text{rot } \vec{p} = e \text{rot } \vec{E} + \frac{e}{c} \text{rot}[\vec{v} \times \vec{H}].
\]

Taking into account that the speed \( \vec{v}(t) \) does not depend explicitly on the space variables \((x_1, x_2, x_3)\), i.e. \( \dot{x} \) \( \vec{p} = 0 \), we have that \( e \left( \text{rot } \vec{E} + \frac{1}{c} \vec{v} \text{div } \vec{H} - \frac{1}{c} \vec{v} \nabla \vec{H} \right) = 0 \), or, consequently,

\[
\text{rot } \vec{E} = \frac{1}{c} \left( \vec{v} \nabla \vec{H} - \vec{v} \text{div } \vec{H} \right) = \frac{1}{c} \left( \frac{d \vec{H}}{dt} - \frac{\partial \vec{H}}{\partial t} - \vec{v} \text{div } \vec{H} \right).
\] (17)

Remark 1.
Equation (17) follows as a special case of equation (16) for the motion of a charged particle. It can be shown that the first pair of classical Maxwell-Lorentz equations [2] follows from (17) as a special case which obviously has the form:

\[
\text{rot } \vec{E} = \frac{\partial \vec{H}}{c \partial t}, \quad \text{div } \vec{H} = 0,
\] (18)

in essence, this is the identity for the four-dimensional electromagnetic potential \( \vec{A}_e(\vec{x}) \).

Remark 2.
Note that if \( \frac{d \vec{H}}{dt} = 0 \) then the equation (17) reduces to the following relation:

\[
\text{rot } \vec{E} + \frac{1}{c} \frac{\partial \vec{H}}{\partial t} = \frac{1}{c} \vec{v} \text{div } \vec{H}.
\] (19)

and hence, for the case when one side of (19) vanishes, one again gets the Maxwell-Lorentz equations (18) which are the particular case of the more general equation (19).

Remark 3.
Applying the operator ‘div’ to equation (16) and taking into account \( \text{div } \vec{p} = 0 \) we obtain the relation

\[
e \text{div } \vec{E} + \frac{e}{c} \text{div}[\vec{v} \times \vec{H}] = 0
\]

and, since the velocity \( \vec{v}(t) \) does not depend on the space variables \( \vec{x} \), we get the following equation for the electromagnetic field:

\[
\text{div } \vec{E} = \frac{1}{c} (\vec{v}, \vec{H}).
\] (20)

9. Other corollaries
Corollary 2.
Let $\rho = e$ and $\rho_m = m$ and also the influence of an external electromagnetic field $(\vec{E}, \vec{H})$ on the
particle motion in the equation (1) (via the functional $J_3$) be not negligible.

Then applying the differential operator `$\text{rot}$' to both sides of equation (14) again leads to equation
(17) while applying the differential operator `$\text{div}$' gives the following generalized equation (the
energy of the electromagnetic field is now taken into account through the functional $J_3$):

$$
\Delta(E^2 - H^2) = 8\pi e \left( \frac{1}{c} \text{rot} \vec{H} \cdot \text{div} \vec{E} \right),
$$

hence the equation (20) follows as a special case.

**Corollary 3.**

Suppose that the density of a substance distributed in an arbitrary volume $V$ is constant (i.e.
$\rho_m = \text{const}$) and also the relations $\nabla \rho = 0$ and $\frac{d\vec{H}}{dt} = 0$ hold true.

Then equations (14)-(15) are reduced to the following system which generalizes equations (18)-(19):

$$
\rho \text{rot} \vec{E} + \frac{\rho}{c} \left( \text{v div} \vec{H} + \frac{\partial \vec{H}}{\partial t} \right) - \frac{1}{c} \frac{d\rho}{dt} \vec{H} = 0,
$$

$$
\rho \text{div} \vec{E} - \frac{\rho}{c} (\vec{v}, \text{rot} \vec{H}) - \frac{1}{8\pi} \frac{d\rho}{c} \text{div} \vec{A} + \frac{1}{8\pi} (E^2 - H^2) = 0.
$$

**Corollary 4.**

Let $\rho_m = \text{const}$ and also suppose that the density $\rho$ of the charge distribution in a certain volume
$V$ and the magnetic field $\vec{H}$ satisfy the following conditions:

$$
\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \nabla \rho \equiv 0, \quad \frac{d\vec{H}}{dt} = \frac{\partial \vec{H}}{\partial t} + (\vec{v} \nabla) \vec{H} \equiv 0.
$$

Then equations (14)-(15) take the form:

$$
\rho \text{rot} \vec{E} + \frac{\rho}{c} \left( \text{v div} \vec{H} + \frac{\partial \vec{H}}{\partial t} \right) + \nabla \rho \times \vec{E} - \nabla \varphi \times \nabla \rho - \frac{1}{c} \nabla \rho \times ((\vec{v} \nabla) \vec{A}) = 0,
$$

$$
\rho \text{div} \vec{E} - \frac{\rho}{c} (\vec{v}, \text{rot} \vec{H}) + \frac{1}{8\pi} (E^2 - H^2) + \vec{E} \nabla \rho + \frac{1}{c} (\vec{v} \times \vec{H}) \nabla \rho -
$$

$$
-(\nabla \varphi, \nabla \rho) - \varphi \Delta \rho + \frac{1}{c} (\vec{A}_r, \vec{v}) \Delta \rho + \frac{1}{c} (\nabla \rho, \nabla (\vec{A}_r, \vec{v})) = 0.
$$

**Remark 4.**

It follows from equations (24) and (25) that if the first pair of Maxwell-Lorentz equations (18) is a
particular case of the vector equation (14) under the assumptions:

$$
\nabla \rho = \nabla \rho_m = \frac{d\rho}{dt} = \frac{d\rho}{dt} = \frac{d\vec{H}}{dt} = 0,
$$

then the four scalar equations of the second pair of Maxwell-Lorentz equations:
(27) and (28) are a particular case of equations (14)-(15) under the same restrictions (26), equipped with certain restrictions on partial derivatives of the first and second orders on the fields $\vec{E}$ and $\vec{H}$. These restrictions are discussed in [1-2].

10. Conclusions
In this paper we show how to obtain a generalized system of equations for relativistic electrodynamics and general equations of the electromagnetic field.

The problem of Lyapunov’s stability of solutions of these systems is solved. For this purpose, the deformation method is used for studying the invariance of the extremals of the action functional of the Hamiltonian system (1)-(2).

The system of equations (14)-(15) is the most general system that describes the electromagnetic field and could find further application in solving real problems of the mathematical physics.

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