Capacity Region of the Symmetric Injective $K$-User Deterministic Interference Channel

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Abstract

We characterize the capacity region of the symmetric injective $K$-user Deterministic Interference Channel (DIC) for all channel parameters. The achievable rate region is derived by first projecting the achievable rate region of Han-Kobayashi (HK) scheme, which is in terms of common and private rates for each user, along the direction of aggregate rates for each user (i.e., the sum of common and private rates). We then show that the projected region is characterized by only the projection of those facets in the HK region for which the coefficient of common rate and private rate are the same for all users, hence simplifying the region. Furthermore, we derive a tight converse for each facet of the simplified achievable rate region.

Index Terms
Deterministic Interference Channel, Capacity Region.

I. INTRODUCTION

The deterministic interference channel (DIC), originally introduced in [1], represents a basic yet fruitful instance of interference channels that effectively captures the broadcast and interference phenomena in multi-user networks. For example, intuitions from the two-user DIC have lead to the capacity approximation of the two-user Gaussian interference channels in [2]. Further operational connections between the Gaussian interference channel and the two-user DIC are also established in [3], [4]. However, despite its simplicity, characterizing the capacity region of the general $K$-user DIC has still remained an unsolved problem.

Our main result in this paper is to characterize the capacity region of the $K$-user DIC in a symmetric injective case. There have been several attempts at this problem in the past. In particular, the capacity region of the symmetric injective 3-user DIC has been characterized [5]. However, extending prior approaches to the general symmetric injective $K$-user DIC becomes extremely cumbersome due to the explosive growth in the number of parameters in both achievable schemes and the converse.

To overcome this challenge, we propose new techniques in both the development of the achievable rate region and the converse. For deriving the achievable rate region, we consider the general Han-Kobayashi (HK) scheme [6], in which the message of each user is into two parts: private message which is supposed to be decoded only at the desired destination and common message which is supposed to be decoded at all destinations. This scheme results in an achievable rate region that is in terms of common and private rates for the users. The challenge is then to eliminate the common and private rates and derive an achievable rate region that is in terms of the aggregate rates for the users (i.e., the sum of common and private rates). While a Fourier-Moutzkin (FM) elimination method can be used to solve this problem for $K$-user DIC with small number of users (e.g. $K \leq 3$ as done in [5]), applying FM method to networks with large number of users becomes extremely cumbersome. We overcome this challenge by directly projecting the achievable rate region of HK scheme along the direction of aggregate rates for the users, and exploiting the algebraic properties of the rate region to remove loose facets of the rate region. In particular, we show that the achievable rate region can be obtained by projecting only those facets of achievable rate region of HK for which the coefficient of common rate and private rate are the same for all users.

We also derive a tight converse for each facet of the achievable rate region. In particular, we use the structure of the facets of the achievable rate region to systematically bound the mutual information between the transmit and receive signal of each user by the corresponding term in each facet.

The rest of the paper is organized as follows. We first explain the system model of the symmetric injective $K$-user DIC and state the main result in Section II. We then elaborate upon the derivation of the achievable rate region in Section III. Finally, in Section IV we provide a tight converse for the achievable rate region.

II. SYSTEM MODEL AND MAIN RESULT

The system model of the symmetric injective $K$-user DIC is shown in Fig.1. In this model, source nodes and destination nodes are represented by $S_i$ and $D_i$ respectively, $\forall i \in [K]$ where $[K] \triangleq \{1, \ldots, K\}$. Furthermore, the received signal at $D_i$,
As a special case, Theorem 1 recovers the capacity region of the 2-user DIC of El Gamal-Costa [1], by choosing Remark 3.

Remark 4. redundant bounds.

choice of and

union injective

The capacity region of the symmetric injective 3-user DIC in [5].

The invertibility restriction of Remark 1. interference channel.

Based on the above model, we state our main result of this paper, which is the capacity characterization of the symmetric injective K-user DIC.

Theorem 1. The capacity region of the symmetric injective K-user deterministic interference channel is characterized as

\[ A_{p(x_1), \ldots, p(x_K)} \triangleq \left\{ \left( R_1, \ldots, R_K \right) | R_i \geq 0, \forall i \in [K], \sum_{i=1}^{K} a_i R_i \leq \sum_{i=1}^{K} \sum_{j=1}^{K} H(Y_i|V_{S_{i,j}}), a_i \in \mathbb{Z}^{\geq 0} \text{ and } S_{i,j} \subseteq [K] \right\}, \]

satisfying

\[ \sum_{i=1}^{K} \sum_{j=1}^{K} a_i \]

for some function \( h_i(\cdot) \), \( \forall i \in [K] \). We refer to interference channel that satisfies this property as injective deterministic interference channel.

**Remark 1.** The invertibility restriction of \( f_i(\cdot) \)'s was originally imposed in [1] for the 2-user DIC and later generalized for the symmetric injective 3-user DIC in [5].

**Remark 2.** The above model is “symmetric” since transmitter \( i \) causes the same interference \( V_i = g_i(X_i) \) to all receivers.

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Proof. As mentioned earlier, region $A_1$ can be obtained by projecting region $A_1$ based on linear transformation $A$. Since region $A_1$ is polyhedra, region $A_2$ can be found by projecting all facets of $A_1$ according to the projection matrix $A$.

Note that the facets of region $A_1$ are obtained by linear combinations of the inequalities characterizing this region. Hence, according to (7), all possible facets of $A_1$ can be written as follows.

$$
\sum_{i=1}^{K} \sum_{j=1}^{K} c_{i,j} R_{ip} + \sum_{k \in M_{i,j}} R_{kc} \leq \sum_{i=1}^{K} \sum_{j=1}^{K} c_{i,j} H(Y_i|V_{M_{i,j}^c}),
$$

(10)
for all $\mathcal{M}_{i,j} \subseteq [K]$ and $c_{i,j} \in \mathbb{R}_{\geq 0}$ ($i = 1, \ldots, K$ and $j = 1, \ldots, 2^K$), where $c_{i,j}$ is the corresponding coefficient of the inequality in (7) with subset $\mathcal{M}_{i,j}$.

By simplifying the left-hand side of (11), we have

$$\sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}(R_{ip} + \sum_{k \in \mathcal{M}_{i,j}} R_{kc})$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}R_{ip} + \sum_{k=1}^{K} \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}I_{\mathcal{M}_{i,j}}(k)R_{kc}$$

$$= \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}R_{ip} + \sum_{k=1}^{K} \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}I_{\mathcal{M}_{i,j}}(k)R_{kc}$$

$$= \sum_{m=1}^{K} \sum_{j=1}^{2^K} c_{m,j}R_{mp} + \sum_{m=1}^{K} \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}I_{\mathcal{M}_{i,j}}(m)R_{mc}$$

$$= \sum_{m=1}^{K} \sum_{j=1}^{2^K} d_{m}R_{mp} + \sum_{m=1}^{K} \sum_{j=1}^{2^K} c_{m,j}R_{mc},$$

where

$$d_{m} \triangleq \sum_{j=1}^{2^K} c_{m,j}, \quad e_{m} \triangleq \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}I_{\mathcal{M}_{i,j}}(m), \quad \forall m \in [K].$$

Thus, all facets of region $A_1$ can now be written as

$$\sum_{i=1}^{K} d_{i}R_{ip} + \sum_{i=1}^{K} e_{i}R_{ic} \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}),$$

for all $\mathcal{M}_{i,j} \subseteq [K]$, $c_{i,j} \in \mathbb{R}_{\geq 0}$, and $d_{i}$ and $e_{i}$ defined in (13). Now note that, according to Lemma 3 proved in Appendix B, the projection of (14) according to the linear transformation matrix $A$ would result in the following bound

$$\sum_{i=1}^{K} \min(d_{i}, e_{i})R_{i} \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}).$$

Therefore, region $A_2$ that is obtained by projection of region $A_1$ based on linear transformation $A$, is characterized as

$$\{(R_1, \ldots, R_K) \mid \sum_{i=1}^{K} a_iR_i \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}), R_i \geq 0 \text{ for all } i \in \{1, \ldots, K\}, c_{i,j} \in \mathbb{Z}_{\geq 0}, \mathcal{M}_{i,j} \subseteq [K],$$

and $a_i = \min(d_{i}, e_{i})$, where $d_{i}$ and $e_{i}$ are defined in (13).

Now, note that region $A_3$ is exactly the same as the above region, except further restricting to the choice of $c_{i,j}$’s to satisfy

$$a_m = d_m = e_m, \quad \forall m \in [K].$$

To complete the proof, we only need to show that any inequality in (16) that does not satisfy the constraint of (17) is redundant, meaning that it can be obtained by linear combination of inequalities already considered in region $A_3$. Let us consider inequalities that can be written as follows

$$\sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}(R_{ip} + \sum_{k \in \mathcal{M}_{i,j}} R_{kc}) \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}),$$

where

$$\sum_{j=1}^{2^K} c_{m,j} \neq \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j}I_{\mathcal{M}_{i,j}}(m) \quad \text{for some } m.$$
coefficient of private rate is greater than or equal to the coefficient of common rate for all users and its projection according to transformation matrix $\mathbf{A}$ results in a tighter bound on $\sum_{k=1}^{K} \min(d_k, e_k) R_k$ compared to the projection of (18). In the second step, we obtain another inequality, based on the resulting inequality of the first step, such that the coefficient of private rate and common rate are the same for all users while its projection leads to a tighter bound on $\sum_{k=1}^{K} \min(d_k, e_k) R_k$ compared to the projection of (18).

**Step 1:** In this step, we aim to find new inequality

$$\sum_{i=1}^{K} \sum_{j=1}^{2^K} \tilde{c}_{i,j} (R_{jp} + \sum_{k \in M_{i,j}} R_{kc}) \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} \tilde{c}_{i,j} H(Y_i | V_{M^c_{i,j}}),$$

(20)

such that $a)$

$$\tilde{d}_m = d_m, \quad \tilde{e}_m = \min(e_m, d_m), \quad \forall m \in [K],$$

(21)

where $\tilde{d}_m$ and $\tilde{e}_m$ are similarly defined as (13) and $b)$ the projection of (20), according to transformation matrix $\mathbf{A}$, results in a tighter bound on $\sum_{k=1}^{K} \min(d_k, e_k) R_k$ compared to the projection of (18).

Let us consider a $m$ where $e_m > d_m$ and define

$$\mathcal{J}_c \triangleq \{(i, j)|i \in [K] \setminus m, j \in [2^K] \text{ and } m \in M_{i,j}\},$$

$$\mathcal{N}_c \triangleq \{(i, j')|M_{i,j'} = M_{i,j} \setminus m, \forall(i, j') \in \mathcal{J}_c\},$$

$$\alpha_{i',j'} \triangleq \beta_{i,j} \text{ for } (i, j') \in \mathcal{N}_c, (i, j) \in \mathcal{J}_c \text{ s.t. } M_{i,j'} = M_{i,j} \setminus m,$$

(22)

for all $\beta_{i,j} \in \mathbb{Z}_{\geq 0}$ satisfying $\beta_{i,j} \leq c_{i,j}, \forall(i, j) \in \mathcal{J}_c$ and

$$\sum_{(i,j) \in \mathcal{J}_c} \beta_{i,j} = e_m - d_m,$$

$$\beta_{i,j} = 0, \forall(i,j) \notin \mathcal{J}_c.$$

(23)

By introducing coefficient $\tilde{c}_{i,j}$ as follows

$$\tilde{c}_{i,j} \triangleq \begin{cases} c_{i,j} & \text{if } (i, j) \notin \mathcal{N}_c, (i, j) \notin \mathcal{J}_c, \\ c_{i,j} + \alpha_{i,j} & \text{if } (i, j) \notin \mathcal{N}_c, \\ c_{i,j} - \beta_{i,j} & \text{if } (i, j) \in \mathcal{J}_c, \end{cases}$$

(24)

we now consider the following inequality in region $A_2$

$$\sum_{i=1}^{K} \sum_{j=1}^{2^K} \tilde{c}_{i,j} (R_{jp} + \sum_{k \in M_{i,j}} R_{kc}) \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} \tilde{c}_{i,j} H(Y_i | V_{M^c_{i,j}}),$$

(25)

and show that

$$(A) \quad \begin{cases} \tilde{d}_m = d_m, \quad \tilde{e}_m = \min(e_m, d_m) \\ \tilde{d}_{m'} = d_{m'}, \quad \tilde{e}_{m'} = e_{m'}, \quad \forall m' \neq m, \\ \sum_{(i,j) \in \mathcal{J}_c} \tilde{c}_{i,j} H(Y_i | V_{M^c_{i,j}}) \leq \sum_{(i,j) \in \mathcal{J}_c} c_{i,j} H(Y_i | V_{M^c_{i,j}}), \end{cases}$$

(26)

$$(B) \quad \begin{cases} \sum_{(i,j) \in \mathcal{J}_c} \tilde{c}_{i,j} H(Y_i | V_{M^c_{i,j}}) \leq \sum_{(i,j) \in \mathcal{J}_c} c_{i,j} H(Y_i | V_{M^c_{i,j}}). \end{cases}$$

(27)

(28)

We first prove the existence of such $\beta_{i,j}$’s satisfying (24) in Claim 1.

**Claim 1.** By considering (22), there exists $\beta_{i,j} \leq c_{i,j}$ for all $(i, j)$ such that (24).

**Proof.** One can easily verify the existence of such $\beta_{i,j} \leq c_{i,j}, \forall(i, j) \in \mathcal{J}_c$ satisfying (24) by showing that $\sum_{(i,j) \in \mathcal{J}_c} \beta_{i,j}$ can take value $e_m - d_m$ as follows

$$\sum_{(i,j) \in \mathcal{J}_c} \beta_{i,j} \leq \sum_{(i,j) \in \mathcal{J}_c} c_{i,j} = \sum_{(i,j) \in \mathcal{J}_c} c_{i,j} - \sum_{j: m \notin M_{m,j}} c_{m,j} \leq e_m - d_m \leq 0$$

(29)

We next present Claim 2 to verify (27).
Claim 2. By considering \([22], [23]\), we have \([27]\).

Proof. We first find \(d_m\) and \(e_m\), then we derive \(d_{m'}\) and \(e_{m'}\) for all \(m' \neq m\) as follows

\[
\hat{d}_m = \sum_{j=1}^{2^K} \hat{c}_{m,j} = \sum_{j=1}^{2^K} e_{m,j} = d_m,
\]

where step (a) follows from the fact \((m,j) \notin N_e\) and \((m,j) \notin J_e\).

\[
\hat{e}_m = \sum_{i=1}^{2^K} \hat{c}_{i,m} = \sum_{i=1}^{2^K} e_{i,m} = \hat{e}_m,
\]

where step (a) follows from \([25]\). Step (b) follows from the fact that \(I_{M_{i,j}}(m) = 0\) for \((i,j') \in N_e\) and \(I_{M_{i,j}}(m) = 1\) for \((i,j) \in J_e\) according to \([22]\). Finally, step (c) follows from \([24]\).

Regarding \(d_{m'}\) and \(e_{m'}\) where \(m' \neq m\), we have

\[
\hat{d}_{m'} = \sum_{j=1}^{2^K} \hat{c}_{m',j} = \sum_{j: (m',j) \notin J_e, (m',j) \notin N_e} \hat{e}_{m',j} = \sum_{j: (m',j) \in J_e} \hat{e}_{m',j} + \sum_{j: (m',j) \in N_e} \hat{e}_{m',j} + \sum_{j: (m',j) \in J_e} \hat{e}_{m',j},
\]

where step (a) and (b) follow from \([25]\) and \([23]\), respectively.

\[
\hat{e}_{m'} = \sum_{i=1}^{2^K} \hat{c}_{i,m'} = \sum_{i: (i,j) \notin N_e, (i,j) \notin J_e} \hat{e}_{i,m'} = \sum_{i: (i,j) \in J_e} \hat{e}_{i,m'} + \sum_{i: (i,j) \in N_e} \hat{e}_{i,m'} + \sum_{i: (i,j) \in J_e} \hat{e}_{i,m'},
\]

where step (a) follows from \([25]\). Step (b) follows from \([23]\) and the fact \(I_{M_{i,j}}(m') = I_{M_{i,j}}(m)\) for \((i,j') \in N_e\) and \((i,j) \in J_e\) and \(m' \neq m\).

\[
\frac{\beta_{m'}}{\min(\hat{d}_k, \hat{e}_k)} = \min(\hat{d}_k, \hat{e}_k) R_k \text{ where } \min(\hat{d}_k, \hat{e}_k) = \min(d_k, e_k), \forall k \in [K].
\]

Based on Claim 2 and Lemma 3, the projection of \([26]\) would result in a bound on \(\sum_{k=1}^{K} \min(\hat{d}_k, \hat{e}_k) R_k\) where \(\min(\hat{d}_k, \hat{e}_k) = \min(d_k, e_k), \forall k \in [K].\)

We now provide the following Claim to prove \([28]\).
Claim 3. By considering \([22]-[23]\), we have \([28]\).

Proof. The proof is as follows

\[
\sum_{i=1}^{K} \sum_{j=1}^{2^K} \hat{c}_{i,j} H(Y_i|V_{M_{i,j}^c}) \overset{(a)}{=} \sum_{(i,j):(i,j) \notin \mathcal{N}_{c}} c_{i,j} H(Y_i|V_{M_{i,j}^c}) + \sum_{(i,j) \notin \mathcal{N}_{c}} \alpha_{i,j} H(Y_i|V_{M_{i,j}^c}) - \sum_{(i,j):(i,j) \notin \mathcal{J}_e} \beta_{i,j} H(Y_i|V_{M_{i,j}^c}) \\
\overset{(b)}{=} \sum_{(i,j) \in [K]\times[j \in [2^K]} c_{i,j} H(Y_i|V_{M_{i,j}^c}) + \sum_{(i,j) \notin \mathcal{J}_e} \beta_{i,j} H(Y_i|V_{m}V_{M_{i,j}^c}) - \sum_{(i,j):(i,j) \notin \mathcal{J}_e} \beta_{i,j} H(Y_i|V_{M_{i,j}^c}) \\
\overset{(c)}{\leq} \sum_{(i,j) \in [K]\times[j \in [2^K]} c_{i,j} H(Y_i|V_{M_{i,j}^c}),
\]

where step (a) follows from \([25]\). Step (b) follows from \([23]\) and the fact \(\mathcal{M}_{i,j} = \mathcal{M}_{i,j} \setminus m\) for \((i,j) \in \mathcal{N}_{c}, (i,j) \in \mathcal{J}_e\). Finally, step (c) follows from the fact \(H(Y_i|V_mV_{M_{i,j}^c}) \leq H(Y_i|V_{M_{i,j}^c})\) for all \((i,j) \in \mathcal{J}_e\) \(\square\)

Therefore, it is obvious that if Claims 1-3 are satisfied, the bound obtained by projecting \([26]\) becomes tighter than the bound found by projecting \([18]\).

By repeating the aforementioned process for all \(m\) where \(e_m > d_m\) and updating the resulting inequality, i.e. replacing coefficients \(c_{i,j}\)’s with \(\hat{c}_{i,j}\)’s, we find inequality \([20]\) which satisfies \(a)\) \([21]\) and \(b)\) its projection leads to a tighter bound compared to the projection of \([18]\).

Step 2: In this step, we aim to find new inequality

\[
\sum_{i=1}^{K} \sum_{j=1}^{2^K} \hat{c}_{i,j} (R_{sp} + \sum_{k \in \mathcal{M}_{i,j}} R_{kc}) \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} \hat{c}_{i,j} H(Y_i|V_{M_{i,j}^c}),
\]

from \([20]\) such that \(a)\) \(\hat{d}_m = \hat{e}_m, \forall m \in [K]\) where \(\hat{d}_m\) and \(\hat{e}_m\) are similarly defined as \([13]\) and \(b)\) the projection of \([25]\), according to transformation matrix \(A\), results in a tighter bound on \(\sum_{k=1}^{K} \min(d_k, e_k)R_k\) compared to the projection of \([20]\). Let us consider a \(m\) where \(d_m > e_m\) and define

\[
\mathcal{J}_p \triangleq \{(i,j)|i = m, m \notin \mathcal{M}_{i,j}\}, \\
\mathcal{N}_{p}^+ \triangleq \{(i,j)|i \neq m, i \notin \mathcal{M}_{i,j}\}, \\
\mathcal{N}_{p}^- \triangleq \{(i,j)|i \neq m, i \notin \mathcal{M}_{i,j}\}, \\
\gamma_{m'} \triangleq \sum_{(k,j) \in \mathcal{J}_p} \alpha_{k,j} I_{M_{k,j}}(m'), \forall m' \neq m, \\
\mu_{i,j} \triangleq \beta_{i,j} \text{ for } (i,j) \in \mathcal{N}_{p}^+, (i,j) \notin \mathcal{N}_{p}^- \text{ s.t. } \mathcal{M}_{i,j} = \mathcal{M}_{i,j} \cup \{i\},
\]

for all \(\alpha_{i,j}, \beta_{i,j} \in \mathbb{Z}_{\geq 0}\) satisfying \(\alpha_{i,j}, \beta_{i,j} \leq c_{i,j}\) and

\[
\sum_{(i,j) \notin \mathcal{J}_p} \alpha_{i,j} = d_m - e_m, \quad \alpha_{i,j} = 0 \land (i,j) \notin \mathcal{J}_p. \\
\sum_{j,(m') \in \mathcal{N}_{p}^-} \beta_{i,j} = \gamma_{m'}, \forall m' \neq m, \quad \beta_{i,j} = 0 \land (i,j) \notin \mathcal{N}_{p}^-.
\]

By introducing coefficient \(\hat{c}_{i,j}\) as follows

\[
\hat{c}_{i,j} \triangleq \begin{cases} 
 c_{i,j} & \text{if } (i,j) \notin \mathcal{J}_p, (i,j) \notin \mathcal{N}_{p}^+, (i,j) \notin \mathcal{N}_{p}^-; \\
 c_{i,j} + \mu_{i,j} & \text{if } (i,j) \notin \mathcal{N}_{p}^+; \\
 c_{i,j} - \beta_{i,j} & \text{if } (i,j) \notin \mathcal{N}_{p}^-; \\
 c_{i,j} - \alpha_{i,j} & \text{if } (i,j) \in \mathcal{J}_p,
\end{cases}
\]

where \(c_{i,j} \triangleq \hat{c}_{i,j}\) for all \(i,j\), we now consider the following inequality in region \(A_2\)

\[
\sum_{i=1}^{K} \sum_{j=1}^{2^K} \hat{c}_{i,j} (R_{sp} + \sum_{k \in \mathcal{M}_{i,j}} R_{kc}) \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} \hat{c}_{i,j} H(Y_i|V_{M_{i,j}^c}),
\]

and show that

\[
\begin{align*}
\sum_{i \in [K], j \in [2^K]} \hat{c}_{i,j} H(Y_i|V_{M_{i,j}^c}) & \leq \sum_{i \in [K], j \in [2^K]} c_{i,j} H(Y_i|V_{M_{i,j}^c}).
\end{align*}
\]
We first present Claim 4 to prove the existence of such \( \alpha_{i,j} \)'s and \( \beta_{i,j} \)'s satisfying (39) and (40), respectively.

**Claim 4.** By considering (36)-(37), there exists \( \alpha_{i,j} \leq c_{i,j} \) for all \((i,j) \in \mathcal{J}_p\) and \( \beta_{i,j} \leq c_{i,j} \) for all \((i,j) \in \mathcal{N}_p^-\) satisfying (39) and (40), respectively.

**Proof.** One can easily prove that there exists such \( \alpha_{i,j} \leq c_{i,j} \) and \( \beta_{i,j} \leq c_{i,j} \) satisfying (39) and (40) by showing that \( \sum_{(i,j) \in \mathcal{J}_p} \alpha_{i,j} \) and \( \sum_{(m',j) \in \mathcal{N}_p^-} \beta_{m',j} \) can respectively take values \( d_m - e_m \) and \( \gamma_m' \) for \( m' \neq m \) as follows

\[
\sum_{(i,j) \in \mathcal{J}_p} \alpha_{i,j} = \sum_{j \in m \in M_{i,j}} \alpha_{m,j} \leq \sum_{j \in m \in M_{i,j}} c_{m,j} = \sum_{j \in m \in M_{i,j}} c_{m,j} - \sum_{(i,j) \in M_{i,j}} c_{i,j} = \sum_{j \in m \in M_{i,j}} c_{m,j} \geq \sum_{(i,j) \in M_{i,j}} c_{i,j} - e_m, \tag{45}
\]

\[
\sum_{(m',j) \in \mathcal{N}_p^-} \beta_{m',j} = \sum_{j \in m' \notin M_{i,j}} \beta_{m',j} \leq \sum_{j \in m' \notin M_{i,j}} c_{m',j} \geq 0. \tag{46}
\]

Since we have

\[
\sum_{m' \notin M_{i,j}} c_{m',j} \geq \sum_{m' \notin M_{i,j}} c_{m',j} \geq \sum_{m' \notin M_{i,j}} c_{m',j} \geq \sum_{m' \notin M_{i,j}} c_{m',j} \geq \gamma_m', \tag{47}
\]

\( \forall m' \neq m \) where step (a) follows from \( d_m \geq e_m \) implying

\[
\sum_{j \in m' \notin M_{i,j}} c_{m',j} + \sum_{j \in m' \notin M_{i,j}} \beta_{m',j} \geq \sum_{j \in m' \notin M_{i,j}} c_{m',j} + \sum_{j \in m' \notin M_{i,j}} \beta_{m',j} - \sum_{(i,j) \in M_{i,j}} c_{i,j}. \tag{48}
\]

Step (b) follows from (37). Based on (45), (46), and (47), the proof of existence of such \( \alpha_{i,j} \leq c_{i,j} \) and \( \beta_{i,j} \leq c_{i,j} \) completes.

We next demonstrate Claim 5 to prove (43).

**Claim 5.** By considering (36)-(41), we have (43).

**Proof.** We first find \( \hat{d}_m \) and \( \hat{e}_m \), then we derive \( \hat{d}_m' \) and \( \hat{e}_m' \) for all \( m' \neq m \) as follows

\[
\hat{d}_m = \sum_{j=1}^{2^K} \hat{c}_{m,j}, \tag{49}
\]

\[
\hat{e}_m = \sum_{i=1}^K \sum_{j=1}^{2^K} \hat{c}_{i,j} I_{M_{i,j}}(m), \tag{50}
\]

where step (a) follows from (41). Step (b) follows from the fact that \( (m, j) \notin \mathcal{N}_p^- \) and \( (m, j) \notin \mathcal{N}_p^- \). Finally, step (c) follows from (39).
where step (a) follows from (41) and step (b) follows from \( m \notin \mathcal{M}_{i,j} \) for \((i,j) \in \mathcal{J}_p \). Finally, step (c) follows from (38), meaning that \( \mu_{i,j} = \beta_{i,j} \) and \( I_{\mathcal{M}_{i,j}}(m) = I_{\mathcal{M}_{i,j}}(m) \) for \((i,j) \in \mathcal{N}_p^+ \) and \((i,j') \in \mathcal{N}_p^+ \).

Regarding \( \ddot{d}_{m'} \) and \( \ddot{e}_{m'} \) where \( m' \neq m \), we have

\[
\ddot{d}_{m'} = \sum_{j=1}^{2^K} \ddot{e}_{m',j} = \sum_{j=1}^{2^K} \sum_{j':(m',j') \in \mathcal{N}_p^+} \beta_{m',j} - \sum_{j':(m',j') \in \mathcal{N}_p^-} \alpha_{m',j} \tag{51}
\]

where steps (a), (b), and (c) follow from (41), \((m',j) \notin \mathcal{J}_p \), and (38), respectively. Steps (d) and (e) follow from (36). Finally, steps (e) and (f) follow from (38) and (40).

Based on Claim 5 and Lemma 3, the projection of (42) would result in a bound on \( \sum_{k=1}^{K} \min(\hat{d}_k, \hat{e}_k)R_k \) where \( \min(\hat{d}_k, \hat{e}_k) = \min(d_k, e_k) \), \( \forall k \in [K] \).

Finally, we prove (44) in the following Claim.

**Claim 6.** By considering (36), (41), we have (44).

**Proof.**

\[
\sum_{i=1}^{K} \sum_{j=1}^{2^K} \hat{c}_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}) \overset{(a)}{=} \quad \sum_{(i,j):i \in [K], j \in [2^K]} c_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}) + \sum_{(i,j):(i,j') \in \mathcal{J}_p} \mu_{i,j'}H(Y_i|V_{\mathcal{M}_{i,j'}}) - \sum_{(i,j):i \in \mathcal{N}_p^-} \beta_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}) + \sum_{(i,j):i \in \mathcal{N}_p^+} \alpha_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}) \\
- \sum_{(i,j):i \in \mathcal{N}_p^-} \alpha_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}) - \sum_{(i,j):i \in \mathcal{N}_p^+} \beta_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}) + \sum_{(i,j):i \in \mathcal{N}_p^+} \mu_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}) - \sum_{(i,j):i \in \mathcal{N}_p^-} \mu_{i,j}H(Y_i|V_{\mathcal{M}_{i,j}}) \tag{52}
\]

.. where steps (a) and (b) follow from (41) and (37), respectively. Steps (c) and (d) follow from (36). Finally, steps (e) and (f) follow from (38) and (40).
where steps (a) and (b) follow from (41) and (38), respectively. Step (c) follows from the independence of \( V_i \)'s and the fact that \( I(Y_i; V_i | V_B) \leq H(V_i | V_B) = H(V_i) \) for \( i \notin \mathcal{B} \), which implies \( H(Y_i | V_i V_{M_{i,j}^c}) - H(Y_i | V_i V_{M_{i,j}}^c) \leq H(V_i) \) where \( i \notin \mathcal{M}_{i,j}^c \). Furthermore, step (d) follows from the fact that \( a_{i,j} = 0 \) for \( \forall i \neq m \) and \( m \notin \mathcal{M}_{i,j} \) for \( (i,j) \in \mathcal{J}_p \).

We can find an upper bound on (42) by using \( H(Y_m | V_m V_{\mathcal{M}_{i,j} \cup \{m\}}) \geq H(Y_m | X_m V_{\mathcal{M}_{i,j} \cup \{m\}}) = H(V_{M_{i,j}}) \) due to \( V_m = g_m(X_m), m \notin \mathcal{M}_{i,j} \) for \( (i,j) \in \mathcal{J}_p \), and (2) as follows

\[
\sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j} H(Y_i | V_{M_{i,j}}^c) \leq \sum_{(i,j) \in [K] \times [2^K]} c_{i,j} H(Y_i | V_{M_{i,j}}^c) + \sum_{(i,j) \in [K] \times [2^K]} \sum_{j \in \mathcal{J}_p \setminus \{i\}} a_{m,j} H(V_{M_{m,j}})
\]

\[
\cong \sum_{(i,j) \in [K] \times [2^K]} c_{i,j} H(Y_i | V_{M_{i,j}}^c) + \sum_{m \neq m'} \mu_{m,j} H(V_{m'}) - \sum_{m \neq m'} \gamma_{m'} H(V_{m'})
\]

\[
= \sum_{(i,j) \in [K] \times [2^K]} c_{i,j} H(Y_i | V_{M_{i,j}}^c) + \sum_{m \neq m'} H(V_{m'}) \left( \sum_{j \in \mathcal{J}_p \setminus \{i\}} \mu_{m',j} - \sum_{m \neq m'} \gamma_{m'} \right)
\]

\[
= \sum_{(i,j) \in [K] \times [2^K]} c_{i,j} H(Y_i | V_{M_{i,j}}^c)
\]

where step (a) follows from the fact that \( H(V_{M_{m,j}}) = \sum_{m \neq m'} I_{\mathcal{M}_{m,j}}(m') H(V_{m'}) \) for \( (m,j) \in \mathcal{J}_p \) and (37). Step (b) follows from (38) and (40).

Therefore, it is easy to verify the projection of (42) leads to a tighter bound compared to the projection of (20) if Claims 4-6 are satisfied.

By repeating the above process for all \( m \) where \( d_m > c_m \) and updating the resulting inequality, i.e. replacing coefficients \( c_{i,j} \)'s with \( \bar{c}_{i,j} \)'s, we find inequality (35) such that (a) \( \bar{c}_m = c_m, \forall m \in [K] \) and (b) its projection leads to a tighter bound compared to the projection of (18). Note that the projection of (35) leads to a tighter bound compared to the projection of (20) and the projection of (20) results in a tighter bound compared to the projection of (18).

We now show that region \( \mathcal{A}_3 \) is matching region \( \mathcal{A} \) according to the following lemma.

**Lemma 2.** Region \( \mathcal{A}_3 \) is the same as region \( \mathcal{A} \), i.e.

\[
\mathcal{A}_3 = \mathcal{A}.
\]

**Proof.** In order to prove this, we need to show that we can express any inequalities of region \( \mathcal{A}_3 \) in terms of inequalities of region \( \mathcal{A} \) and vice versa. First, we consider an inequality \( \sum_{i=1}^{K} a_i R_i \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j} H(Y_i | V_{M_{i,j}}^c) \) in region \( \mathcal{A} \). It is easy to see that this inequality can be found by setting following parameters of region \( \mathcal{A}_3 \)

\[
S_{i,j} = M_{i,j} \quad \text{s.t.} \quad q = \text{argmin}_k \sum_{l=1}^{k} c_{i,l} \geq j.
\]

Similarly, consider an inequality \( \sum_{i=1}^{K} a_i R_i \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j} H(Y_i | V_{M_{i,j}}^c) \) in achievable region \( \mathcal{A}_3 \). This inequality can be obtained by setting following parameters

\[
c_{i,j} = \sum_{q=1}^{a_i} I(S_{i,q} = M_{i,j}),
\]

where

\[
I(\mathcal{A} = \mathcal{B}) \triangleq \begin{cases} 1 & \text{if } \mathcal{A} = \mathcal{B}, \\ 0 & \text{otherwise.} \end{cases}
\]

**IV. CONVERSE**

Consider a coding scheme with rate-tuple \( (R_1, \ldots, R_K) \) in a \( K \)-user DIC, in particular encoder function \( \Phi_i^n : [1 : 2^{nR_i}] \rightarrow \mathcal{X}_{2^n_i}^n \) and decoder function \( \Psi_i^n : Y_{D_i}^n \rightarrow [1 : 2^{nR_i}] \) for \( i = 1, \ldots, K \), with vanishing error probability for sufficiently large \( n \). Our goal is to show that there exists a product distribution \( \prod_{i=1}^{K} p(x_i) \) such that (4) holds for all choices of \( a_i \)'s and \( S_{i,j} \)'s.
According to Fano’s inequality, we have
\[
\sum_{i=1}^{K} n a_i R_i \leq \sum_{i=1}^{K} a_i I(W_i; Y_i^n) + n\epsilon_n \\
\leq \sum_{i=1}^{K} a_i I(X^n_i; Y_i^n) + n\epsilon_n \\
= \sum_{i=1}^{K} (a_i H(Y_i^n) - a_i H(Y_i^n|X_i^n)) + n\epsilon_n \\
\overset{(a)}{=} \sum_{i=1}^{K} (a_i H(Y_i^n) - a_i (\sum_{j \neq i} H(V_j^n))) + n\epsilon_n \\
\overset{(b)}{=} \sum_{i=1}^{K} (a_i H(Y_i^n) - (L - a_i)(H(V_i^n))) + n\epsilon_n \\
\overset{(c)}{=} \sum_{i=1}^{K} \sum_{q=1}^{\phi_i} (H(Y_i^n) - H(V_i^n|\phi_i)) + n\epsilon_n, 
\]
where \(W_i\) represents the message of \(i^{th}\) source and step (a) follows from (2) and the independence of \(V_i^j\)'s. Step (b) can be derived by letting \(L \triangleq \sum_{i=1}^{K} a_i\). Finally, step (c) follows from the independence of \(V_i^j\)'s and introducing all subsets \(\phi_i \subseteq [K]\) such that \(\sum_{i=1}^{K} \sum_{q=1}^{\phi_i} I(\phi_i, q)(m) = L - a_m, \forall m \in [K]\).

By letting \(S_{i,q} = [K] \setminus \phi_i, q\), we can simplify (58) as follows
\[
\sum_{i=1}^{K} \sum_{q=1}^{\phi_i} \sum_{m=1}^{a_m} (H(Y_i^n) - H(V_{i,q}^n)) \leq n\sum_{i=1}^{K} a_i R_i - n\epsilon_n, 
\]
for all subsets \(S_{i,q}\)'s satisfying \(\sum_{i=1}^{K} \sum_{q=1}^{\phi_i} I(S_{i,q})(m) = a_m\) for \(m = 1, \ldots, K\). Note that \(\sum_{i=1}^{K} \sum_{q=1}^{\phi_i} I(S_{i,q})(m) = \sum_{i=1}^{K} \sum_{q=1}^{\phi_i} (1 - I(\phi_i, q)(m)) = a_m, \forall m \in [K]\).

Considering inequality \(H(Y) - H(X) \leq H(Y|X),\) which is due to \(H(Y) - H(Y|X) = I(X; Y) \leq H(X),\) (59) can be simplified more as
\[
\sum_{i=1}^{K} \sum_{q=1}^{\phi_i} \sum_{m=1}^{a_m} (H(Y_i^n) - H(V_{i,q}^n)) \leq n\sum_{i=1}^{K} a_i R_i - n\epsilon_n, 
\]
where \(Y_i^n \triangleq (Y_{i,1}, \ldots, Y_{i,n}), V_{i,q}^n \triangleq (V_{i,q,1}^n, \ldots, V_{i,q,n}^n),\) and all subsets \(S_{i,q}\)'s satisfying \(\sum_{i=1}^{K} \sum_{q=1}^{\phi_i} I(S_{i,q})(m) = a_m\) for \(m = 1, \ldots, K\).

By letting \(n \rightarrow \infty\) and considering (60) for all \(a_i\)'s and \(S_{i,q}\)'s as well as the convexity of \(A\), we can conclude that there exists a product distribution satisfying (4) for all choices of \(a_i\)'s and \(S_{i,q}\)'s.

V. Conclusion

We considered the symmetric injective \(K\)-user deterministic interference channel and characterized the corresponding capacity region. The achievable rate region was obtained by projecting the achievable rate region of Han-Kobayashi scheme along the direction of sum of common and private rates for each user. Furthermore, we derived a tight converse for the achievable rate region. An interesting future direction can be characterizing the capacity region of general \(K\)-user deterministic interference channel.

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APPENDIX A

In this part, we analyze the error probability of $K$-user DHC. Since the analysis is the same for Receiver $i$ for $i = 2, \ldots, K$, we only analyze the error probability at Receiver 1. Furthermore, due to symmetry in generating the codewords, the average error does not depend on which message is sent. Therefore, we can assume message indexed by $(c_1, p_1) = (1, 1)$ is sent by Transmitter 1.

An error occurs if either the wrong codewords of Transmitter 1 are jointly typical with the received sequence or the correct codeword is not jointly typical with the received sequence. Let us define the following events

$$E_{c_1, p_1, e_2, \ldots, e_K} = \{(X^1_1(c_1, p_1), V^m_1(c_1), V^m_2(c_2), V^m_K(c_K), Y^n_i) \in A^n_i\}.$$  \hfill (61)

Therefore, the error probability would be

$$P_e = P(E_{c_1, p_1, e_2, \ldots, e_K}) = \sum_{(c_1, p_1) \neq (1, 1)} P(E_{c_1, p_1, e_2, \ldots, e_K}).$$ \hfill (62)

Let us define function $\eta : N \to \{0, 1\}$ as follows

$$\eta(x) = \begin{cases} 
1 & \text{if } x = 1 \\
0 & \text{otherwise.} 
\end{cases}$$ \hfill (63)

By indexing all possible $(\eta(e_2), \ldots, \eta(e_K))$ and considering the $m$th index, we define $\gamma_m \triangleq (c_2, \ldots, c_K)$ for all $m \in [2^{K-1}]$, hence we rewrite (62) as follows

$$P_e \leq \sum_{c_1 \neq 1, p_1 \neq 1} \sum_{m=1}^{2^{K-1}} P(E_{c_1, p_1, \gamma_m}) + \sum_{c_1 \neq 1, p_1 \neq 1} \sum_{m=1}^{2^{K-1}} P(E_{c_1, p_1, \gamma_m}) + \sum_{c_1 = 1, p_1 \neq 1} \sum_{m=1}^{2^{K-1}} P(E_{c_1, p_1, \gamma_m}).$$ \hfill (64)

As $n$ goes to infinity, the first term goes to zero. Each term of $A$ goes to zero if the following condition is hold

$$R_{1p} + R_{1e} + \sum_{k \in \tau_m} R_{kc} \leq I(X_1, V_1, V_{\tau_m}; Y_i | V_2, \ldots, V_K \setminus \tau_m) \leq H(Y_i | V_2, \ldots, V_K \setminus \tau_m),$$ \hfill (65)

for $m = 1, \ldots, 2^{K-1}$ where $\tau_m = \{i | j_i \neq 1 \text{ for } i = 2, \ldots, K\}$. Similarly, each term of $B$ and $C$ goes to zero if the following constraints are hold

$$R_{1c} + \sum_{k \in \tau_m} R_{kc} \leq I(X_1, V_1, V_{\tau_m}; Y_i | V_2, \ldots, V_K \setminus \tau_m) \leq H(Y_i | V_1, V_2, \ldots, V_K \setminus \tau_m),$$ \hfill (66)

for $m = 1, \ldots, 2^{K-1}$. By removing redundant bounds of (65) and (66), we have

$$R_{1p} + \sum_{k \in \tau} R_{kc} \leq H(Y_1 | V_{\tau}), \quad \forall \tau \subseteq \{1, \ldots, K\},$$

$$R_{kp}, R_{kc} \geq 0, \quad \forall k \in [K].$$ \hfill (67)

By considering the error probability of all receivers and indexing all subsets of $\{1, \ldots, K\}$, we have

$$R_{ip} + \sum_{k \in \mathcal{M}_{i, j}} R_{kc} \leq H(Y_i | V_{\mathcal{M}_{i, j}}), \quad \forall i \in [K], \quad \forall j \in [2^K],$$

$$R_{ip}, R_{kc} \geq 0, \quad \forall k \in [K],$$ \hfill (68)

where $\mathcal{M}_{i, j} \subseteq [K]$ represents the $j$th subset of the corresponding $i$th receiver.

It should be noted one can easily verify that (68) matches the region found in Appendix I of [5] for the case of $K = 3$. 

Lemma 3. If \( \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j} (R_{ip} + \sum_{k \in M_{i,j}} R_{kc}) \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j} H(Y_i | V_{M_{i,j}}) \), then the projection of this inequality according to transformation matrix \( A \) would result in

\[
\sum_{i=1}^{K} \min(d_i, e_i) R_i \leq \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j} H(Y_i | V_{M_{i,j}}). \tag{69}
\]

Proof. The proof is based on Fourier-Motzkin elimination by considering \( \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j} H(Y_i | V_{M_{i,j}}) \) and inequalities \( R_{ip} + R_{ic} \leq R_i, R_{ip} + R_{ic} \geq R_i, R_{ic} \geq 0 \), \( R_{ip} \geq 0 \) for \( \forall i \in [K] \). Without loss of generality, we assume that \( d_j \geq e_j \) for a specific \( j \in [K] \). We first eliminate \( R_{jc} \) as follows

\[
\left\{ \begin{array}{l}
R_j - R_{jp} \\
0
\end{array} \right\} \leq R_{jc} \leq \left\{ \begin{array}{l}
\frac{1}{d_j} (\theta - d_j R_{jp} - \sum_{i \neq j} d_i R_{ip} + e_i R_{ic}) \\
R_j - R_{jp}
\end{array} \right. \tag{70}
\]

where \( \theta \equiv \sum_{i=1}^{K} \sum_{j=1}^{2^K} c_{i,j} H(Y_i | V_{M_{i,j}}) \). We next eliminate \( R_{jp} \) as follows

\[
0 \leq R_{jp} \leq \left\{ \begin{array}{l}
\frac{1}{d_j} (\theta - \sum_{i \neq j} d_i R_{ip} + e_i R_{ic}) \\
R_j - R_{jp}
\end{array} \right. \tag{71}
\]

therefore, after eliminating \( R_{jc} \) and \( R_{jp} \), we obtain inequalities \( \sum_{i \neq j} d_i R_{ip} + e_i R_{ic} \leq \theta - \min(d_j, e_j) R_j \) and \( R_j \geq 0 \). Note that we first eliminate \( R_{jp} \) then \( R_{jc} \) in the case of \( e_j \geq d_j \). By performing the similar technique to the resulting inequalities and eliminating the remaining common and private rates, we obtain \( (69) \). \( \square \)
### APPENDIX C

| $a_i$'s | $S_{i,j}$'s | Bound |
|---------|-------------|-------|
| $a_1 = 1$ | $S_{1,1} = \{1\}$ | (10) of 5 |
| $a_1 = 1, a_2 = 1$ | $S_{1,1} = \{1\}, S_{2,2} = \{1, 2\}$ | (11) of 5 |
| $a_1 = 1, a_2 = 2$ | $S_{1,1} = \{2\}, S_{2,1} = \{1\}$ | (12) of 5 |
| $a_1 = 2, a_2 = 1$ | $S_{1,1} = \{1, 2\}, S_{1,2} = \{1\}, S_{2,1} = \{1\}$ | (13) of 5 |
| $a_1 = 1, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{2, 3\}, S_{2,1} = \{1\}, S_{3,1} = \{1\}$ | (14) of 5 |
| $a_1 = 1, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{2\}, S_{2,1} = \{3\}, S_{3,1} = \{1\}$ | (15) of 5 |
| $a_1 = 1, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{1\}, S_{2,1} = \{2, 3\}, S_{3,1} = \{1\}$ | (16) of 5 |
| $a_1 = 1, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{1\}, S_{2,1} = \{1, 2, 3\}$ | (17) of 5 |
| $a_1 = 2, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{1, 2, 3\}, S_{1,2} = \{1\}, S_{2,1} = \{1\}, S_{3,1} = \{1\}$ | (18) of 5 |
| $a_1 = 2, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{2, 3\}, S_{1,2} = \{1\}, S_{2,1} = \{1\}$ | (19) of 5 |
| $a_1 = 2, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{1, 3\}, S_{2,2} = \{1\}, S_{3,1} = \{1\}$ | (20) of 5 |
| $a_1 = 2, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{1\}, S_{1,2} = \{1, 2\}, S_{2,1} = \{1\}$ | (21) of 5 |
| $a_1 = 2, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{1\}, S_{2,1} = \{1, 2\}$ | (22) of 5 |
| $a_1 = 2, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{1\}, S_{2,2} = \{1, 2\}, S_{3,1} = \{1, 2\}$ | (23) of 5 |
| $a_1 = 2, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{1\}, S_{2,2} = \{1, 2, 3\}, S_{3,1} = \{1\}$ | (24) of 5 |
| $a_1 = 2, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{1\}, S_{2,2} = \{1, 2\}, S_{3,1} = \{1, 2\}$ | (25) of 5 |
| $a_1 = 3, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{1\}, S_{2,1} = \{1, 2, 3\}, S_{2,2} = \{1\}$ | (26) of 5 |
| $a_1 = 3, a_2 = 1, a_3 = 1$ | $S_{1,1} = \{1\}, S_{2,1} = \{1, 2, 3\}, S_{2,2} = \{1\}, S_{3,1} = \{1\}$ | (27) of 5 |
| $a_1 = 2, a_2 = 2, a_3 = 1$ | $S_{1,1} = \{1, 2, 3\}, S_{1,2} = \{2\}$ | (28) of 5 |
| $a_1 = 2, a_2 = 2, a_3 = 1$ | $S_{1,1} = \{1, 2, 3\}, S_{1,2} = \{1\}$ | (29) of 5 |
| $a_1 = 2, a_2 = 2, a_3 = 1$ | $S_{1,1} = \{1, 2, 3\}, S_{1,2} = \{\}$, $S_{2,2} = \{1\}, S_{3,1} = \{1\}$ | (30) of 5 |
| $a_1 = 2, a_2 = 2, a_3 = 1$ | $S_{1,1} = \{2, 3\}, S_{1,2} = \{1\}, S_{2,2} = \{1\}$ | (31) of 5 |
| $a_1 = 2, a_2 = 2, a_3 = 1$ | $S_{1,1} = \{2\}, S_{1,2} = \{2\}$ | (32) of 5 |
| $a_1 = 3, a_2 = 2, a_3 = 1$ | $S_{1,1} = \{1\}, S_{1,2} = \{\}$, $S_{2,1} = \{1, 2, 3\}, S_{2,2} = \{1\}$, $S_{3,1} = \{1\}$ | (33) of 5 |
| $a_1 = 3, a_2 = 2, a_3 = 1$ | $S_{1,1} = \{\}, S_{1,2} = \{\}$, $S_{2,1} = \{1, 2, 3\}, S_{2,2} = \{1\}$, $S_{3,1} = \{1\}$ | (34) of 5 |
| $a_1 = 3, a_2 = 2, a_3 = 1$ | $S_{1,1} = \{\}, S_{1,2} = \{\}$, $S_{2,1} = \{2, 3\}, S_{2,2} = \{1\}$, $S_{3,1} = \{1\}$ | (35) of 5 |
| $a_1 = 3, a_2 = 2, a_3 = 1$ | $S_{1,1} = \{\}, S_{1,2} = \{\}$, $S_{2,1} = \{2\}, S_{2,2} = \{1\}$, $S_{3,1} = \{1\}$ | (36) of 5 |
| $a_1 = 4, a_2 = 2, a_3 = 1$ | $S_{1,1} = \{\}, S_{1,2} = \{\}$, $S_{2,1} = \{1, 2, 3\}, S_{2,2} = \{1\}$, $S_{3,1} = \{1\}$, $S_{3,2} = \{1\}$ | (37) of 5 |

**Table II:** $a_i$'s and $S_{i,j}$'s for the case $K = 3$