Abstract

In 1945 Einstein concluded that [1]: “The present theory of relativity is based on a division of physical reality into a metric field (gravitation) on the one hand, and into an electromagnetic field and matter on the other hand. In reality space will probably be of a uniform character and the present theory be valid only as a limiting case. For large densities of field and of matter, the field equations and even the field variables which enter into them will have no real significance.”. The dichotomy is resolved by introducing a complex Randers metric with a real valued scalar field and complex valued vector field, providing a unified mathematical framework for gravitation & electromagnetism for which the resulting theory’s predictions agree with General Relativity; to leading order in the gravitational constant. Hence, the related experimental results validate both theories; and the former theory’s metric solutions are free of spurious singularities, because its stress-energy tensor includes the energy & momentum for the gravitational field; like e.g. Maxwell’s stress-energy tensor contains the electromagnetic field.
1 Introduction

1.1 Outline

The outline of the paper is:

1. The exponential metric introduced in the first part, ref. [2], is generalized to a complex Randers metric.
2. Matter is introduced by a real valued matter and complex valued charge density.
3. The field equations for the gravitational and electromagnetic fields are obtained by Geometric Calculus.
4. The equations of motion for point particles are obtained by Lagrangian mechanics.
5. The corresponding known metric solutions for a:
   - spherically symmetric body (Schwarzschild),
   - static charged spherically symmetric body (Reissner-Nordström),
   - rotating uncharged axisymmetric body (Kerr),
   - rotating charged axisymmetric body (Kerr-Newman)

are obtained; which are free of spurious singularities.

2 Gravitation & Electromagnetism: Mathematical Framework

2.1 Complex Randers Metric and Lagrangian Density

Already 1918 Weyl attempted to include electromagnetism to General Relativity (GR) by generalizing to the conformal group [3, 4]; but Einstein pointed out a fundamental flaw (sharpness of spectral lines) for the resulting classical theory (non-integrable gravitational scale factor) [5]. It is avoided in quantum mechanics (wave-mechanical phase factor) [6]. Randers 1941 obtained a geodesic with the Lorentz force from the metric and action [7]

\[
\begin{align*}
    ds^2 &= \sqrt{g_{rs} dx^r dx^s} + A_r dx^r, \\
    S &= \int ds.
\end{align*}
\]

(1)

Soh 1933 made an attempt with a complex metric [8]. Similarly, Einstein 1945 also considered a complex metric in his quest for a unified field theory [9–11].

Starting from the modified theory of General Relativity outlined in the first part, ref. [2], electromagnetism can be included by introducing:

1. A complex Randers metric [7]

\[
    ds = \sqrt{-g_{rs} dx^r dx^s} + i \sqrt{\frac{\kappa}{2\mu_0}} A_r dx^r = \left( \sqrt{-u_r u^r} + i \sqrt{\frac{\kappa}{2\mu_0}} u_r A^r \right) d\tau
\]

(2)

where \( A_j(x) = [\phi_j, A] \) is the electromagnetic 4-potential, with the norm

\[
    ds^2 = -ds^2 = \left[ u_r u^r - \frac{\kappa}{2\mu_0} (u_r A^r)^2 \right] d\tau^2 = g_{rs} dx^r dx^s - \frac{\kappa \varepsilon_0}{2} (v_r A^r)^2 c_0^2 dt^2
\]

(3)

where

\[
    u^i = \frac{dx^i}{d\tau}, \quad u_r u^r = -c_0^2, \quad \kappa \equiv \frac{8\pi G}{c_0^2}, \quad c_0 = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}
\]

(4)
and $g_{ij}$ the metric tensor

$$g_{rs}dx^r dx^s = -\frac{c_0^2 dt^2}{\gamma_{\phi_0}^2} = -e^{2\phi_0/c_0^2}\left[1 - \left(\frac{v}{c_0 e^{2\phi_0/c_0^2}}\right)^2\right]c_0^2 dt^2$$
$$= -e^{2\phi_0/c_0^2}c_0^2 dt^2 + e^{-2\phi_0/c_0^2} (dx^2 + dy^2 + dz^2)$$
$$= e^{-2\phi_0/c_0^2}\left[-e^{4\phi_0/c_0^2}c_0^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\phi^2\right]$$

with the determinant (Cartesian coordinates)

$$\sqrt{-g} = e^{-2\phi_0/c_0^2}$$

where

$$\gamma_{\phi_0} = \frac{1}{e^{2\phi_0/c_0^2} \sqrt{1 - \left(\frac{v}{c_0 e^{2\phi_0/c_0^2}}\right)^2}} = \frac{1}{e^{\phi_0/c_0^2} \sqrt{1 - \beta_\phi^2}} \beta_{\phi_0} = \frac{v}{c_0 e^{2\phi_0/c_0^2}}$$.

The origin of the conformal factor $e^{-2\phi_0/c_0^2}$ and conformal symmetry breaking factor $c_0 \to c_0 e^{2\phi_0/c_0^2}$ is provided in ref. [2].

2. Matter by a real valued mass and complex valued charge density

$$\varepsilon_m = \sqrt{-g} \left(\rho_m c_0 + i \sqrt{\frac{2H_0}{c} \rho_q}\right)$$

3. The Lagrangian density and action

$$\mathcal{L} = \sqrt{-g} \left(-\rho_m c_0 \sqrt{-g} u^r u^r + \rho_q v r A^r\right), \quad S = -\text{Re} \left\{\int \varepsilon_m ds\right\} = \int \mathcal{L} d^4 x.$$ (9)

For a gravitational point source

$$\phi_0 = -\frac{GM}{r}$$

and no electromagnetic field, $A^i = 0$, the exponential metric Eq. (5) satisfies the algebraic relation [2] (Einstein’s equations [13])

$$R^i_j - \frac{1}{2} g^i_j \kappa = T^i_j$$

with (Riemann tensor)

$$R^i_j = -\frac{2}{c_0^2} \beta_\phi^i \phi_j$$

and (stress-energy tensor density)

$$\mathcal{T}^i_j = \sqrt{-g} R^i_j = -\frac{\sqrt{-g}}{4\pi G} \left(\beta_\phi^i \phi_j - \frac{1}{2} \delta^i_j \phi \phi_r\right) = \frac{GM^2}{8\pi} \left[\begin{array}{cccc}
\frac{1}{r} & 0 & 0 & 0 \\
0 & -\frac{1}{r} & 0 & 0 \\
0 & 0 & \frac{1}{r} & 0 \\
0 & 0 & 0 & \frac{1}{r}
\end{array}\right], \quad \mathcal{T}^r_r = \frac{GM^2}{4\pi r^4}$$

which includes the gravitational field; whereas for GR (vacuum)

$$\mathcal{T}^i_j = 0.$$ (14)

Hence, not only the covariant but also the ordinary divergence is zero

$$\mathcal{T}^r_i;_r = 0, \quad \mathcal{T}^r_i;_r = 0.$$ (15)
and the energy-momentum for the gravitational field and sources are conserved. In other words, like for an electrostatic point charge

\[ \phi_q = \frac{Q}{4\pi \varepsilon_0 r} \]  

(16)

and the Maxwell stress-energy tensor for Minkowski metric

\[ T^i_j = \frac{1}{\mu_0} \left( F^{ir} F^r_j - \frac{1}{4} \delta^i_j F^{rs} F_{rs} \right) = \frac{Q^2}{32\pi^2 \varepsilon_0} \begin{bmatrix} -\frac{1}{r^2} & 0 & 0 & 0 \\ 0 & -\frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2} \end{bmatrix}, \quad T^r_r = 0. \]  

(17)

\[ 2.2 \text{ Field Equations: Geometric Calculus} \]

Based on Grassmann’s exterior product [13]

\[ a \wedge b = -b \wedge a. \]  

(18)

Cartan 1899 generalized vector calculus by introducing exterior calculus [15]. The exterior derivative of a smooth function \( f \) is the differential

\[ df \]  

(19)

such that

\[ d(df) = 0. \]  

(20)

More generally

\[ d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta \]  

(21)

and

\[ d \circ d = 0 \]  

(22)

where \( \alpha \) is a \( p \)-form.

In local coordinates, the exterior derivative of a simple \( k \)-form

\[ \varphi = adx^i = adx^{i_1} \wedge dx^{i_2} \ldots \wedge dx^{i_k} \]  

(23)

is

\[ d\varphi \equiv dx^i \wedge \partial_i \varphi = \partial_i adx^i \wedge dx^i \]  

(24)

and for a general \( k \)-form

\[ \omega = \omega_i dx^i, \quad d\omega = dx^i \wedge \partial_i \omega_i = \partial_i \omega_i dx^i \wedge dx^i. \]  

(25)

For example, for a 1-form \( \omega \)

\[ (d\omega)_{ij} = \partial_i \omega_j - \partial_j \omega_i. \]  

(26)

Hence, the framework provides for a concise summary of Maxwell’s equations [10]

\[ \begin{array}{cccc} \text{gauge} & \rightarrow & \text{vector potential} & \rightarrow & \text{field tensor} & \rightarrow & \text{field equations} \\ \varphi & \rightarrow & A = d\varphi & \rightarrow & F = dA & \rightarrow & dF = 0 \end{array} \]  

(27)

Similarly, Geometric Algebra (GA) can be utilized to generate the field equations for the complex Randers metric. Starting from the Geometric Product [14,17,22]

\[ AB \equiv A \cdot B + A \wedge B \]  

(28)

calculus is introduced by

\[ \nabla A \equiv \nabla \cdot A + \nabla \wedge A \]  

(29)

with

\[ \nabla \cdot A_k \equiv (\nabla A_k)_{k-1} = e^r \cdot \partial_r A_k, \quad \nabla \wedge A_k \equiv (\nabla A_k)_{k+1} = e^r \wedge \partial_r A_k \]  

(30)

for an \( k \)-grade multivector field. It follows that

\[ \nabla^2 A = \nabla (\nabla A) = \nabla \cdot (\nabla \wedge A) + \nabla \wedge (\nabla \cdot A) \]  

(31)
since
\[ \nabla \wedge (\nabla \phi) = 0, \quad \nabla \cdot (\nabla \cdot A) = 0, \quad \nabla \wedge (\nabla \wedge A) = 0. \] (32)

Space-Time Algebra (STA) is generated by introducing the four vectors \( \{\gamma_i\} \) satisfying
\[ \gamma_i \cdot \gamma_j = \frac{1}{2} (\gamma_i \gamma_j + \gamma_j \gamma_i) = \eta_{ij} \] (33)
with (pseudoscalar)
\[ I \equiv \gamma_0 \gamma_1 \gamma_2 \gamma_3. \] (34)

Hence, the algebraic properties of STA are those of the Dirac matrices.

In the rest frame defined by the \( \gamma_0 \) vector one introduces
\[ \sigma_i \equiv \gamma_i \gamma_0, \quad \frac{1}{2} (\sigma_i \sigma_j + \sigma_j \sigma_i) = \delta_{ij}, \quad \sigma_i \sigma_i = I \] (35)
i.e., the algebra properties of space are those of the Pauli matrices; the even subalgebra of STA.

In Euclidian 3-space for \( A, B \) 1-grade (vector valued functions), the mathematical framework simplifies to the traditional vector calculus (introduced by Gibbs [23]; vs. Hamilton, Grassman, Peano, etc. [14, 17, 18])
\[ A \wedge B = IA \times B, \quad \bar{\nabla} \wedge A = I \bar{\nabla} \times A \] (36)
with
\[ \nabla^2 = \Box, \quad \gamma_0 \cdot \nabla = \frac{1}{c_0} \partial_t, \quad \gamma_0 \wedge \nabla = \bar{\nabla}. \] (37)

The line element is
\[ ds = \sqrt{-g_{ss}} \, dx^s \, dx^s + i \sqrt{\frac{\kappa}{2 \mu_0}} A_r \, dx^r = \frac{1}{\gamma_0} c_0 dt + i \sqrt{\frac{\kappa}{2 \mu_0}} A_r \, dx^r \]
\[ = \sqrt{e^{2 \omega / c_0^2} - e^{-2 \omega / c_0^2}} \left( \frac{v}{c_0} \right)^2 c_0 dt + i \sqrt{\frac{\kappa}{2 \mu_0}} A_r \, dx^r \]
\[ = \left\{ 1 - \frac{1}{2} \left( \frac{v}{c_0} \right)^2 \right\} + \left[ 1 + \left( \frac{v}{c_0} \right)^2 \right] \phi_s \frac{\partial_s c_0}{c_0} dt + i \sqrt{\frac{\kappa}{2 \mu_0}} A + \ldots, \] (38)
where \( A \) is a 1-form. Prolonging twice gives
\[ \partial ds = \left[ 1 + \left( \frac{v}{c_0} \right)^2 \right] \frac{\nabla \phi_s c_0}{c_0} dt + i \sqrt{\frac{\kappa}{2 \mu_0}} (\nabla \cdot A + \nabla \wedge A) \]
\[ = \left[ 1 + \left( \frac{v}{c_0} \right)^2 \right] \frac{\nabla \phi_s c_0}{c_0} dt + i \sqrt{\frac{\kappa}{2 \mu_0}} (\nabla \cdot A + \nabla F), \]
\[ \partial^2 ds = \left[ 1 + \left( \frac{v}{c_0} \right)^2 \right] \frac{\nabla \phi_s c_0}{c_0} dt + i \sqrt{\frac{\kappa}{2 \mu_0}} (\nabla \cdot A + \nabla F) \] (39)
and setting the R.H.S. to zero one obtains
\[ \Box \phi_\xi = 0, \quad \nabla \wedge (\nabla \cdot A) + \nabla F = 0 \] (40)
i.e., the wave equation for the gravitational field and Maxwell’s equations for vacuum. For the Lorentz gauge
\[ \nabla \cdot A = 0. \] (41)

By introducing the electromagnetic bivector (Riemann–Silberstein vector [24, 25])
\[ F \equiv \nabla \wedge A = \sqrt{\varepsilon_0} \left( \vec{E} + Ic_0 \vec{B} \right) = \sqrt{\varepsilon_0} (E^r \sigma_r + Ic_0 B^r \sigma_r), \] (42)
and contrarily
\[ \sqrt{\varepsilon_0} \vec{E} = \frac{1}{2} (F - \gamma_0 F_{\gamma_0}), \quad Ic_0 \vec{B} = \frac{1}{2} (F + \gamma_0 F_{\gamma_0}), \] (43)
the Maxwell stress-energy tensor is
\[ T = FF = F \cdot F + F \wedge F. \] (44)

Similarly, the energy density and Poynting vector are
\[ \frac{1}{2} |F|^2 = \frac{1}{2} FF^\dagger = \left[ \frac{1}{2} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) + \frac{1}{\mu_0 c_0} \right] = \left[ \frac{1}{2} \left( \varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \frac{1}{\mu_0 c_0} E \times B \right]. \] (45)

### 2.3 Equations of Motion for Point Particles: Lagrangian Mechanics

The Lagrangian density Eq. (9) for a point particle is
\[ \mathcal{L} = \left( -\rho_m c_0 \sqrt{-u^r u_r} + \rho_q u^r A_r \right) \delta^3 (x) \] (46)
with the Lagrangian and action
\[ L = \int \mathcal{L} d^3 x = -m_0 c_0 \sqrt{-u^r u_r} + qu^r A_r = -m_0 c_0 \sqrt{-g_{rs} u^r u^s} + g_{rs} u^r A^s, \quad S = \int L d\tau. \] (47)

The equations of motion are (covariant Euler-Lagrange equations)
\[ \frac{d}{d\tau} \left( \partial_{u^r} L \right) - \partial_t L = 0 \] (48)
which leads to
\[ \frac{d}{d\tau} \left( \partial_{u^r} L \right) = \frac{d}{d\tau} \left( m_0 g_{ir} u^r + q A_i \right) = m_0 \left( g_{ir} \frac{du^r}{d\tau} + u^r \frac{dg_{ir}}{d\tau} \right) + qu^r \partial_r A_i \]
\[ = m_0 \left( g_{ir} \frac{du^r}{d\tau} + u^r u^s \partial_s g_{ir} \right) + qu^r \partial_r A_i \]
\[ = m_0 \left[ g_{ir} \frac{du^r}{d\tau} + \frac{1}{2} u^r u^s \left( \partial_s g_{ir} + \partial_r g_{si} \right) \right] + qu^r \partial_r A_i, \]
\[ -\partial_t L = -m_0 u^r u^s \frac{1}{2} \partial_r g_{rs} - qu^r \partial_t A_r \] (49)
where we have used
\[ u_r u^r = -c_0^2. \] (50)

It follows that
\[ \frac{d}{d\tau} \left( \partial_{u^r} L \right) - \partial_t L = m_0 g_{ir} \frac{du^r}{d\tau} + m_0 u^r u^s \frac{1}{2} \left( \partial_s g_{ir} + \partial_r g_{si} - \partial_t g_{rs} \right) - qu^r F_{ir} = 0 \] (51)
where (electromagnetic field tensor)
\[ F_{ij} \equiv \partial_i A_j - \partial_j A_i = A_{j;i} - A_{i;j} \] (52)
and by raising the index \( i \), it can be written (geodesic with Lorentz force)
\[ \frac{du^i}{d\tau} + \Gamma^i_{r s} u^r u^s = \frac{q}{m_0} u^r F^i_r \] (53)
where (Christoffel symbols [26])
\[ \Gamma^i_{jk} = \frac{1}{2} g^{ir} \left( \partial_k g_{rj} + \partial_r g_{jk} - \partial_r g_{jk} \right). \] (54)

The conjugate momentum is introduced by
\[ p_i \equiv \partial_{u^i} L = m_0 u_i + q A_i \] (55)
and the Hamiltonian is generated by (Lagrange transformation)
\[ H = u_r p^r - L = \frac{m_0}{2} u_r u^r = \frac{1}{2m_0} g_{rs} \left( p^r - q A^r \right) \left( p^s - q A^s \right). \] (56)
and the equations of motion are (Hamilton’s equations)

\[
\frac{dx_i}{d\tau} = \partial_{p_i} H, \quad \frac{dp_i}{d\tau} = -\partial_i H.
\]  

(57)

In conclusion, the outlined mathematical framework generates the equations of motion for GR with the Lorentz force Eq. (53) and the corresponding covariant Hamiltonian Eq. (56).

3 Solutions

Since the metric Eqs. (3) and (5) is a functional for the gravitational and electromagnetic potentials, solving for the latter provides the metric solution directly. Besides, after a static solution has been found, related dynamic solutions can be found by coordinate transformations; e.g. to a rotating co-moving frame.

3.1 Static Spherically Symmetric Body

For a spherically symmetric body with mass \( M \)

\[
\phi_g = -\frac{GM}{r}
\]

(58)

the exponential metric Eq. (5) is

\[
ds^2 = -e^{-\frac{rS}{r}} c_0^2 dt^2 + e^{\frac{rS}{r}} \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 \right)
\]

\[
= -\left( 1 - \frac{rS}{r} + \ldots \right) c_0^2 dt^2 + \left( 1 + \frac{rS}{r} + \ldots \right) \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 \right)
\]

(59)

where

\[
r_S = \frac{2GM}{c_0^2}.
\]

(60)

The Schwarzschild metric for GR in isotropic form is [27]

\[
ds^2 = -\left( 1 - \frac{rS}{4r} \right)^2 c_0^2 dt^2 + \left( 1 + \frac{rS}{4r} \right)^4 \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 \right)
\]

\[
= -\left( 1 - \frac{rS}{r} + \ldots \right) c_0^2 dt^2 + \left( 1 + \frac{rS}{r} + \ldots \right) \left( dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 \right)
\]

(61)

which agrees to leading order.

3.2 Static Charged Spherically Symmetric Body

The solutions to Poisson’s equations

\[
\nabla \phi_g = 4\pi G \rho_m, \quad \nabla \phi_q = -\frac{\rho_l}{\varepsilon_0}
\]

(62)

for a charged spherically symmetric body with mass \( M \) and charge \( Q \) are

\[
\phi_m = -\frac{GM}{r}, \quad \phi_q = \frac{Q}{4\pi\varepsilon_0 r}.
\]

(63)

The metric is obtained from Eqs. (3) and (5) which gives

\[
ds^2 = g_{rs} dx^r dx^s - \frac{\kappa \varepsilon_0}{2} (v_r A^r)^2 c_0^2 dt^2 = -\left[ \frac{1}{\nabla^2 \phi_g} + \frac{\kappa \varepsilon_0}{2} (v_r A^r)^2 \right] c_0^2 dt^2
\]

\[
= -\left( e^{-\frac{rS}{r}} + \frac{r^2}{r^2} \right) c_0^2 dt^2 + e^{\frac{rS}{r}} \left[ dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\varphi^2) \right]
\]

\[
= -\left( 1 - \frac{rS}{r} + \frac{r^2}{r^2} \right) c_0^2 dt^2 + \left( 1 + \frac{rS}{r} \right) \left[ dr^2 + r^2 (d\theta^2 + \sin^2(\theta) d\varphi^2) \right] + \ldots
\]

(64)
The metric for a rotating frame

It becomes singular for

Replacing \( \rho \)

where

The Reissner-Nordström metric for GR is \([28, 29]\)

\[
\begin{align*}
    ds^2 &= -\left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right) c_0^2 dt^2 + \frac{1}{1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 \\
    &= -\left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right) c_0^2 dt^2 + \left(1 + \frac{r_S}{r} - \frac{r_Q^2}{r^2}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2 + \ldots
\end{align*}
\]

It becomes singular for

\[
r = \frac{1}{2} \left( r_S \pm \sqrt{r_S^2 - 4r_Q^2} \right).
\]

It can be transformed into isotropic form by a coordinate transformation \( r \to \rho \)

\[
ds^2 = -A^2(\rho) c_0^2 dt^2 + B^2(\rho) \left[ d\rho^2 + \rho^2 \left( d\theta^2 + \sin^2(\theta) d\varphi^2 \right) \right]
\]

where

\[
    A^2(\rho) = 1 - \frac{r_S}{\rho} + \frac{r_Q^2}{\rho^2}
\]

which gives

\[
    B^2(\rho) d\rho^2 = \frac{dr^2}{1 - \frac{r_S}{\rho} + \frac{r_Q^2}{\rho^2}}, \quad B^2(\rho) \rho^2 = r^2
\]

with

\[
    \frac{d\rho}{\rho} = \frac{dr}{\sqrt{r^2 - r_Sr + r_Q^2}}
\]

and by integrating

\[
    \rho = \frac{r}{2} \left(1 - \frac{r_S}{2r} + \sqrt{1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}}\right)
\]

and solving for \( r \)

\[
r = \rho \left[1 + \frac{r_S}{4\rho} - \frac{r_Q^2}{4\rho^2}\right].
\]

Replacing \( \rho \) with \( r \), the Reissner-Nordström metric in isotropic form is

\[
ds^2 = -\left(1 - \frac{r_S^2}{4r^2} + \frac{r_Q^2}{4r^4}\right) c_0^2 dt^2 + \left[1 + \frac{r_S}{4r} - \frac{r_Q^2}{4r^2}\right]^2 \left[dr^2 + r^2 \left( d\theta^2 + \sin^2(\theta) d\varphi^2 \right) \right]
\]

which agrees to leading order with Eq. (64); but for the latter the electromagnetic energy only contributes to the timelike part of the metric.

### 3.3 Rotating Uncharged Axisymmetric Body

The potential for an axisymmetric body is independent of \( \varphi \), i.e., depends only on \([r, \theta]\)

\[
    \phi \equiv \phi(r, \theta).
\]

The metric for a rotating frame

\[
    \varphi \to \varphi \pm \omega t, \quad d\varphi \to d\varphi \pm \omega c_0 dt
\]
Similarly, for a central mass $M$ rotating with $\omega$ around the $z$-axis, in the co-moving frame a local observer at rest in the plane at distance $r$ observes a mass with angular momentum

$$L_z = M\omega r^2$$

(78)

and the metric is given by Eq. (51)

$$ds^2 = -c^2_0dt^2 + dr^2 + r^2 \left[ d\theta^2 + \sin^2(\theta) \left( d\varphi - \frac{a}{r^2 c_0} c_0 dt \right)^2 \right]$$

$$= - \left( 1 - \frac{\omega^2 r^2}{c_0^2} \sin^2(\theta) \right) c^2_0 dt^2 + \frac{2\omega r^2}{c_0} \sin^2(\theta) d\varphi c_0 dt + dr^2 + r^2 \left( d\theta^2 + \sin^2(\theta) d\varphi^2 \right)$$

(77)

By using Eq. (77) to transform back to the rest frame for the central mass one obtains

$$ds^2 = -e^{\omega r^2/c_0^2} c^2_0 dt^2 + e^{-\omega r^2/c_0^2} \left\{ dr^2 + r^2 \left[ d\theta^2 + \sin^2(\theta) \left( d\varphi - \frac{a}{r^2 c_0} c_0 dt \right)^2 \right] \right\}$$

$$= - \left( e^{\omega r^2/c_0^2} - \frac{a^2}{r^2 c_0^2} e^{-\omega r^2/c_0^2} \sin^2(\theta) \right) c^2_0 dt^2 - \frac{2a}{c_0} e^{-\omega r^2/c_0^2} \sin^2(\theta) d\varphi c_0 dt$$

$$+ e^{-\omega r^2/c_0^2} \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2(\theta) d\varphi^2 \right) \right]$$

(79)

where

$$a = \frac{\omega r^2}{c_0} = \frac{L_z}{Mc_0}$$

(80)

Because the Schwarzschild solution for GR is for a spherically symmetric body $\phi \equiv \phi(r)$, the solution for an axisymmetric body $\phi \equiv \phi(r, \theta)$ can not be obtained by transforming to a rotating system. The solution is the Kerr metric (Boyter-Lindquist form $\Sigma arranging$

$$ds^2 = - \left( 1 - \frac{rsr}{\Sigma} \right) c^2_0 dt^2 - \frac{2arsr}{\Sigma} \sin^2(\theta) d\varphi c_0 dt + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left( r^2 + a^2 + \frac{a^2 rsr}{\rho^2} \right) \sin^2(\theta) d\varphi^2$$

(81)

where

$$\Sigma \equiv r^2 \left( 1 + \frac{a^2}{r^2} \cos^2(\theta) \right), \quad \Delta \equiv r^2 \left( 1 - \frac{rs}{r} + \frac{a^2}{r^2} \right).$$

(84)

It can be written in quasi-isotropic coordinates by introducing $\Sigma$ arranging

$$r \rightarrow r \left( 1 + \frac{rs + 2a}{4r} \right) \left( 1 + \frac{rs - 2a}{4r} \right)$$

(85)

which leads to

$$ds^2 = -c^2_0 dt^2 + \psi^4 \left[ e^{2\psi/3} \left( dr^2 + r^2 d\theta^2 \right) + r^2 \sin^2(\theta) e^{-4\psi/3} \left( d\varphi + \beta^2 dt \right)^2 \right]$$

$$= - \left( 1 - \frac{rs}{r} \right) c^2_0 dt^2 - \frac{2ars}{r} \sin^2(\theta) d\varphi c_0 dt + \left( 1 + \frac{rs}{r} \right) \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2(\theta) d\varphi^2 \right) \right] + \ldots$$

(86)
where
\[
\alpha^2 = \frac{\rho^2 \Delta}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 (\theta)}, \quad \beta^\phi = -\frac{r^2 + a^2 - \Delta}{(r^2 + a^2)^2 - \Delta a^2 \sin^2 (\theta)},
\]
\[
\psi^4 = \frac{\rho^{2/3} \left[(r^2 + a^2)^2 - \Delta a^2 \sin^2 (\theta)\right]^{1/3}}{r^2}.
\]
which agrees to leading order with Eq. (81).

### 3.4 Rotating Charged Axisymmetric Body

The metric Eq. (3) is
\[
ds^2 = \left(g_{rs} - \frac{\kappa e_0}{2} A_r A_s\right) dx^r dx^s
\]  
with the vector potential 1
\[
A^i = \frac{\mu_0 c_0}{4\pi} \int_V \frac{\rho u^i}{r} d^3 x.
\]

For a moving point charge
\[
A^i = \left[\frac{\phi_0}{c_0}, A\right] = \frac{Q}{4\pi c_0 r} [1, \vec{v}], \quad A_i = g_{ir} A^r = \frac{Q}{4\pi c_0} \left[ c_0, \frac{\vec{v}}{c_0} \right]
\]
and the rotating frame metric in section 3.3
\[
A_r dx^r = \frac{Q}{4\pi c_0 r} \left(c_0 dt + \frac{\kappa e_0}{r^2} r \sin (\theta) d\varphi\right) = \frac{Q}{4\pi c_0 r} \left(c_0 dt - \frac{\omega r^2 \sin^2 (\theta)}{c_0} d\varphi\right) = \frac{Q}{4\pi c_0 r} \left(c_0 dt - a \sin^2 (\theta) d\varphi\right)
\]
one obtains
\[
-\frac{\kappa e_0}{2} A_r A_s dx^r dx^s = -\frac{r_Q^2}{r^2} \left(c_0^2 dt^2 - 2a \sin^2 (\theta) d\varphi c_0 dt + a^2 \sin^2 (\theta) d\varphi^2\right)
\]  
which when added to the metric for a static charged body in section 3.2 gives
\[
ds^2 = -\left[ e^{-\frac{\phi_0}{c_0}} + \frac{r_Q^2}{r^2} - a \left(e^{-\frac{\phi_0}{c_0}} - 1\right) \sin^2 (\theta) \right] c_0^2 dt^2 - 2a \left(e^{-\frac{\phi_0}{c_0}} - 1 - \frac{r_Q^2}{r^2}\right) \sin^2 (\theta) d\varphi c_0 dt
\]
\[
+ e^{\frac{\phi_0}{c_0}} \left\{ dr^2 + r^2 \left[d\theta^2 + \left(1 - \frac{r_Q^2 a^2 \sin^2 (\theta)}{r^4}\right) \sin^2 (\theta) d\varphi^2\right]\right\}
\]
\[
= -\left(1 - \frac{r_S}{r} + \frac{r_Q^2}{r^2}\right) c_0^2 dt^2 - 2a \left(\frac{r_S}{r} - \frac{r_Q^2}{r^2}\right) \sin^2 (\theta) d\varphi c_0 dt
\]
\[
+ \left(1 + \frac{r_S}{r}\right) \left[dr^2 + r^2 (d\theta^2 + \sin^2 (\theta) d\varphi^2)\right] + \ldots
\]
\]
where
\[
a = \frac{L_z}{Mc_0}, \quad r_S = \frac{2GM}{c_0^2}, \quad r_Q = \frac{GQ^2}{4\pi c_0 r_0^3}.
\]

The electromagnetic field tensor is
\[
F_{ij} = \partial_i A_j - \partial_j A_i = \begin{bmatrix}
0 & \frac{Q}{4\pi c_0 r^2} & 0 & 0 \\
-\frac{Q}{4\pi c_0 r^2} & 0 & 0 & \frac{Qa \sin^2 (\theta)}{4\pi c_0 r} \\
0 & 0 & 0 & \frac{2Qa \sin (\theta) \cos (\theta)}{4\pi c_0 r} \\
0 & -\frac{Qa \sin^2 (\theta)}{4\pi c_0 r^2} & \frac{2Qa \sin (\theta) \cos (\theta)}{4\pi c_0 r} & 0
\end{bmatrix},
\]
\footnote{Retarded potentials are not required for the stationary case.}
The corresponding Kerr-Newman metric for GR is (Boyer-Lindquist form) \[33-35\]

\[
ds^2 = -\frac{\Delta - \frac{2a^2}{r_0} \sin^2(\theta)}{\rho^2} c_0^2 dt^2 + \frac{2a(\Delta - r^2 - a^2)}{\rho^2} \sin^2(\theta) \, d\varphi c_0 dt
+ \frac{\rho^2 (\Delta - r^2 - a^2)}{\rho^2} d\theta^2 - \frac{\rho^2}{\Delta} d\varphi^2
\]

\[
= -\left(1 - \frac{r S}{r} + \frac{r_0^2}{r^2}\right) c_0^2 dt^2 - \frac{2arS}{r} \sin^2(\theta) \, d\varphi c_0 dt + \left(1 + \frac{r S}{r}ight) \left[d\varphi^2 + r^2 \left(d\theta^2 + \sin^2(\theta) \, d\varphi^2\right)\right] + \ldots \tag{96}
\]

where

\[
\rho^2 = r^2 \left(1 + \frac{a^2}{r^2} \cos^2(\theta)\right), \quad \Delta = r^2 \left(1 - \frac{r S}{r} + \frac{r_0^2}{r^2}\right). \tag{97}
\]

As for the Kerr metric, it can be written in quasi-isotropic coordinates by introducing \[32\]

\[
r \rightarrow r \left(1 + \frac{r_0^2}{r^2} \sqrt{\frac{a^2 + r_0^2}{r^2}}\right) \left(1 + \frac{r S}{r} - \sqrt{\frac{a^2 + r_0^2}{r^2}}\right) \tag{98}
\]

which gives

\[
ds^2 = -\alpha^2 c_0^2 dt^2 + \psi^4 \left[ e^{2\psi} \left( d\varphi^2 + r^2 d\theta^2 \right) + r^2 \sin^2(\theta) \, d\varphi^2 \right]
= -\left(1 - \frac{r S}{r} + \frac{r_0^2}{r^2}\right) c_0^2 dt^2 - \frac{2arS}{r} \sin^2(\theta) \, d\varphi c_0 dt
\]

\[
+ \left(1 + \frac{r S}{r} - \frac{r_0^2}{r^2}\right) \left[d\varphi^2 + r^2 \left(d\theta^2 + \sin^2(\theta) \, d\varphi^2\right)\right] + \ldots \tag{99}
\]

where

\[
\alpha^2 = \frac{\rho^2 \Delta}{r^2 + a^2} - \Delta a^2 \sin^2(\theta), \quad \beta^\phi = -a \frac{\rho^2}{r^2 + a^2} - \Delta a^2 \sin^2(\theta),
\]

\[
\psi^4 = \frac{\psi^2}{\rho^2} \left[\frac{\rho^2}{r^2} - \Delta a^2 \sin^2(\theta)\right]^{1/2}, \quad e^{2\mu} = \frac{\rho^4}{(r^2 + a^2)^2 - \Delta a^2 \sin^2(\theta)}. \tag{100}
\]

The electromagnetic potential is

\[
A_r dx^r = \frac{Q r}{4\pi \varepsilon_0 \rho^2} \left(c_0 dt - a \sin^2(\theta) \, d\varphi\right) \tag{101}
\]

with the field tensor

\[
F_{ij} = \begin{bmatrix}
0 & -Q(r^2 - a^2 \cos^2(\theta)) & 0 \\
(2Qr^2 \sin(\theta)) & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & -2Qa r \sin(\theta) \cos(\theta) \\
0 & 0 & -2Qar (r^2 + a^2) \sin(\theta) \, d\varphi c_0 dt \\
0 & Qa \sin^2(\theta) & 0
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & -Q \frac{r^2}{4\pi \varepsilon_0 \rho^2} & 0 \\
0 & 0 & 0 \\
0 & -Qa \sin^2(\theta) & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & Qa \sin(\theta) \cos(\theta) \\
0 & 0 & 0
\end{bmatrix}
\]

The Kerr-Newman solution agrees to leading order with Eq. \[83\]; but for the latter, as in section \[32\] the electromagnetic energy does not contribute to the spacelike component of the metric.
4 Conclusion

In 1945 Einstein concluded that [1]:

“The present theory of relativity is based on a division of physical reality into a metric field (gravitation) on the one hand, and into an electromagnetic field and matter on the other hand. In reality space will probably be of a uniform character and the present theory be valid only as a limiting case. For large densities of field and of matter, the field equations and even the field variables which enter into them will have no real significance.”

The dichotomy can be resolved by introducing a complex Randers metric with a real valued scalar (gravitational) and complex valued vector (electromagnetic) field for which Geometric Algebra and Lagrangian mechanics provides a unified mathematical framework for: Electromagnetism, Electrodynamics, Special and General Relativity (GR), and Gravitation. The theory’s predictions agree with GR; to leading order in the gravitational constant. So the experimental results validate the presented framework & theory as well. Besides, because the metric is a functional of the scalar and vector fields, solving for the latter provides the metric solution directly. Hence, it is straightforward to obtain the corresponding known GR metric solutions for a:

- spherically symmetric body (Schwarzschild),
- static charged spherically symmetric body (Reissner-Nordström),
- rotating uncharged axisymmetric body (Kerr),
- rotating charged axisymmetric body (Kerr-Newman).

which agree to leading order in the gravitational constant; and are free of spurious singularities. However, the electromagnetic energy only contributes to the timelike component of the metric.

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