CONSTRUCTION OF QUASI-CANONICAL LIFTINGS OF K3 SURFACES OF FINITE HEIGHT IN ODD CHARACTERISTIC

KENTARO INOUE

Abstract. We construct a quasi-canonical lifting of a K3 surface of finite height over a finite field of characteristic \( p \geq 3 \). Such results are previously obtained by Nygaard-Ogus when \( p \geq 5 \). For this purpose, we use the display-theoretic deformation theory developed by Langer, Zink, and Lau. We study the display structure of the crystalline cohomology of deformations of a K3 surface of finite height in terms of the Dieudonné display of the enlarged formal Brauer group.

1. Introduction

The notion of quasi-canonical liftings of varieties over a perfect field \( k \) of positive characteristic is introduced by Nygaard-Ogus to show that Tate’s conjecture holds for K3 surfaces of finite height ([NO, Definition 1.5]). Let \( X_0 \) be a K3 surface of finite height over a finite field \( k \) of characteristic \( p \). They proved that if there is a quasi-canonical lifting of \( X_0 \), the K3 surface \( X_0 \) satisfies Tate’s conjecture [NO, Theorem 2.1 and Remark 2.2.3]. Moreover, they constructed a quasi-canonical lifting of \( X_0 \) under the assumption that \( p \geq 5 \).

In this paper, we prove the following theorem:

**Theorem 1.1** (see Theorem 4.1). Let \( X_0 \) be a K3 surface of finite height \( h < \infty \) over a finite field \( k \) of odd characteristic \( p \). Then there exists a totally ramified finite extension \( V/W(k) \) of degree \( h \) and a quasi-canonical lifting \( X/V \) of \( X_0/k \).

This result is previously obtained by Nygaard-Ogus when \( p \geq 5 \) (see [NO, Theorem 5.6]). We give an alternative proof when \( p \geq 5 \). Our result seems new when \( p = 3 \).

Recall that Nygaard-Ogus established a one-to-one correspondence between deformations of \( X_0 \) over \( k[t]/(t^n) \) and deformations of the crystalline cohomology \( H^2_{\text{crys}}(X_0/W(k)) \) over \( k[t]/(t^n) \) as crystals with additional structure (Frobenius, pairing, and Hodge filtration) under the assumption that \( p \geq 5 \). We need the assumption that \( p \geq 5 \) to justify some calculation for divided power (see [NO, Lemma 4.6]). Furthermore, this correspondence is proved only over the base \( k[t]/(t^n) \) because crystals behave well only over a base whose PD envelope is \( p \)-torsion free (see the proof of [NO, Theorem 4.5]).

In this paper, we use the display-theoretic deformation theory of K3 surfaces developed by Langer, Zink, and Lau ([LZ2], [La2]). When \( p \geq 3 \), the crystalline cohomology of a K3 surface is naturally equipped with a display structure, and there is a one-to-one correspondence between deformations of \( X_0 \) over an arbitrary Artin local ring \( R \) with residue field \( k \) and deformations of the crystalline cohomology \( H^2_{\text{crys}}(X_0/W(k)) \) over \( R \) as displays with additional structure.

The key step to prove Theorem 1.1 is to study a relation between the display associated to the crystalline cohomology of a K3 surface and the Dieudonné display of
the enlarged formal Brauer group. We note that the following Theorem is a display-theoretic analogue of [NO, Theorem 3.20] (for the notation on displays, see Section 2).

**Theorem 1.2** (see Corollary 3.4). Let $k$ be a perfect field of characteristic $p \geq 3$, $R$ be an Artin local ring with residue field $k$, and $X$ be a $K3$ surface of finite height over $R$ (i.e. a proper flat scheme over $R$ whose closed fiber is a $K3$ surface of finite height). Let $\widehat{Br}_{X/R}$ (resp. $\psi_{X/R}$) be the formal Brauer group (resp. the enlarged formal Brauer group) associated to $X/R$. Then there exists the following exact sequence of displays over the small Witt frame $\widehat{W}(R)$:

$$0 \to D(\psi_{X/R}) \to H^2_{\text{cris}}(X/\widehat{W}(R)) \to D(\widehat{Br}^*_X)(-1) \to 0.$$ 

Moreover, this sequence is compatible with base change with respect to $R$.

**Remark 1.3.** From [NO, Theorem 2.1] and Theorem 1.1, it follows that Tate’s conjecture holds for $K3$ surfaces of finite height over finite fields of characteristic $p \geq 3$. Note that Tate’s conjecture for $K3$ surfaces is previously proved by Madapusi Pera [MP1, Theorem 1], [KM, Theorem A.1]. (See also [MP2], [IIK, Section 6.4].)

**Remark 1.4.** If an extension of the base field $k$ is allowed, Theorem 1.1 is previously obtained by Ito-Ito-Koshikawa [IIK, Corollary 9.11] using the étaleness of the Kuga-Satake morphism. However, it seems difficult to control the extension degree by the method of [IIK].

**Remark 1.5.** In characteristic 2, it is not known whether the crystalline cohomology of a $K3$ surface is equipped with a display structure. This is why we assume $p \geq 3$ throughout this paper.

The organization of this paper is as follows. In Section 2 we review the display-theoretic deformation theory of $p$-divisible groups and $K3$-surfaces. In Section 3 we study the relation between the Dieudonné display of the enlarged formal Brauer group and the display of a $K3$ surface. In Section 4 we prove the main theorem of this paper.

## 2. Frames and displays

In this section, we review some results about displays following [La2] (see also [LZ1], [LZ2]).

### 2.1. Frames.

**Definition 2.1** ([La2, Definition 2.0.1]). A frame $S = (S, \sigma, \tau)$ consists of a commutative $\mathbb{Z}$-graded ring

$$S = \bigoplus_{n \in \mathbb{Z}} S_n$$

and ring homomorphisms $\sigma, \tau : S \to S_0$ satisfying the following conditions:

1. $\tau_0 : S_0 \to S_0$ is the identity, and $\tau_{-n} : S_{-n} \to S_0$ is bijective for $n \geq 1$. Let $t \in S_{-1}$ be the unique element such that $\tau_{-1}(t) = 1$.
2. $\sigma_0 : S_0 \to S_0$ is a Frobenius lift (i.e. a ring homomorphism satisfying $\sigma_0(x) - x \in pS_0$ for all $x \in S_0$), and $\sigma_{-1}(t) = p$.
3. $p \in \text{Rad}(S_0)$. Here $\text{Rad}(S_0)$ is the Jacobson radical of $S_0$. 


A morphism of frames is a morphism of graded rings that commutes with $\sigma$ and $\tau$.

**Remark 2.2.** A frame $S$ is uniquely determined by the graded ring $S_{\geq 0} = \bigoplus_{n \geq 0} S_n$ together with the ring homomorphism $\sigma : S_{\geq 0} \to S_0$ and the homomorphism of graded $S_{\geq 0}$-modules $t : S_{\geq 1} \to S_{\geq 0}$ satisfying the following conditions:

1. $\sigma_0 : S_0 \to S_0$ is a Frobenius lift, and $\sigma(t(a)) = p\sigma(a)$ for all $a \in S_{\geq 1}$.
2. $p \in \text{Rad}(S_0)$.

**Example 2.3.** Let $A$ be a $p$-torsion free ring with $p \in \text{Rad}(A)$, and $\sigma_0 : A \to A$ be a Frobenius lift. There is a unique frame $A = (A[t], \sigma, \tau)$ (deg$(t) = -1$) such that $\sigma^{-1}(t) = p$ and $\tau^{-1}(t) = 1$. This frame is called the tautological frame associated to $A$.

**Example 2.4.** Let $R$ be an Artin local ring with residue field of characteristic $p \geq 3$. Let $\widehat{W}(R)$ be the small Witt ring defined in [Zi1, Section 2]. We obtain a frame $(\widehat{W}(R), \sigma, \tau)$ as follows. We set $\widehat{W}(R)_0 = \widehat{W}(R)$ and $\widehat{W}(R)_n = \widehat{I}(R) = V(\widehat{W}(R))$ as a $\widehat{W}(R)_0$-module for $n \geq 1$. For $n, m \geq 1$, a multiplication map $\widehat{W}(R)_n \times \widehat{W}(R)_m \to \widehat{W}(R)_{n+m}$ is given by

$$\widehat{I}(R) \times \widehat{I}(R) \to \widehat{I}(R), \ (V(a), V(b)) \mapsto V(ab).$$

Let $\sigma_0 : \widehat{W}(R) \to \widehat{W}(R)$ be the Witt vector Frobenius, and we put $\sigma_n(V(a)) = a$ for $a \in \widehat{W}(R)$ and $n \geq 1$. The map $t : \widehat{W}(R)_1 \to \widehat{W}(R)_0$ is the inclusion map, and $t : \widehat{W}(R)_{n+1} \to \widehat{W}(R)_n$ is the multiplication map $p : \widehat{I}(R) \to \widehat{I}(R)$ for $n \geq 1$. This determines a frame $(\widehat{W}(R), \sigma, \tau)$ uniquely by Remark 2.2. This frame is denoted simply by $\widehat{W}(R)$ and called the small Witt frame associated to $R$.

**Remark 2.5.** Let $A$ be as in Example 2.3. Let $R$ be as in Example 2.4 and $f : A \to \widehat{W}(R)$ be a ring homomorphism. Then $f$ induces the unique morphism of graded rings $A[t] \to \widehat{W}(R)$ which commutes with $\tau$. This map commutes with $\sigma$, and induces a morphism of frames $A \to \widehat{W}(R)$.

### 2.2. Displays

**Definition 2.6.** ([La2 Definition 3.2.1]). Let $S = (S, \sigma, \tau)$ be a frame. Let $\pi : S \to S_0/(tS_1)$ be the composite of the projection $S \to S_0$ and $S_0 \to S_0/(tS_1)$.

1. An $S$-display (or simply display) $\underline{M} = (M, F)$ consists of a finite projective graded $S$-module $M$ and a $\sigma$-linear map $F : M \to M^\sigma$ which induces an isomorphism of $S_0$-modules $M^\sigma \simeq M^\tau$. Here, for a ring homomorphism $f : A \to B$ and an $A$-module $N$, we put $N' := N \otimes_A f B$.
2. A morphism of displays is a morphism of graded $S$-modules that commutes with $F$.
3. We call a display $(M, F)$ an effective display if $M^\tau := M \otimes_{S_{\geq 0}} S_0/tS_1$ concentrates on non-negative degrees.
4. An effective display $(M, F)$ is called a d-display if $(M^\tau)_i = 0$ for $i \notin [0, d]$. Here $(M^\tau)_i$ denotes the degree $i$ part of the graded $S_0/tS_1$-module $M^\tau$.

**Example 2.7.** Let $S = (S, \sigma, \tau)$ be a frame and $n \in \mathbb{Z}$. Then $(S(n), \sigma)$ is an $S$-display. Here we put $S(n)_i := S_{n+i}$. This display is denoted by $S(n)$.

**Definition 2.8.** Let $\underline{M} = (M, F)$ and $\underline{M}' = (M', F')$ be $S$-displays.

- The direct sum of them is the $S$-display $\underline{M} \oplus \underline{M}' = (M \oplus M', F \oplus F')$. 


• The tensor product of them is the $S$-display $M \otimes_S M' = (M \otimes_S M', F \otimes_S F')$.
• The dual of $M$ is the $S$-display $M^* = (M^*, (F^*)^{-1})$. Here $M^*$ is the dual of the $S$-module $M$.

**Definition 2.9.** Let $M = (M, F)$, $M' = (M', F')$, and $M'' = (M'', F'')$ be $S$-displays. A sequence of morphisms of displays

$$0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$$

is an exact sequence of displays if the underlying sequence of $S$-modules

$$0 \rightarrow M \rightarrow M' \rightarrow M'' \rightarrow 0$$

is exact.

**Definition 2.10.** Let $S \rightarrow S'$ be a morphism of frames, and $M = (M, F)$ be an $S$-display. The base change of $M$ by $S \rightarrow S'$ is the $S'$-display $M \otimes_S S' = (M \otimes_S S', F \otimes_S S')$.

**Remark 2.11.** Let $A$ be a $p$-adic ring (i.e. complete and separated in the $p$-adic topology). Assume that $A$ is $p$-torsion free. Let $\sigma : A \rightarrow A$ be a Frobenius lift. An $A$-display is equivalent to a finite projective $A$-module $M$ with a descending finite filtration by direct summands $(F^i M)_{i \in \mathbb{Z}}$ together with $\sigma$-linear maps $F_i : F^i M \rightarrow M$ satisfying the following conditions ([La2, Example 3.6.2]):

- $pF_{i+1} = F_i |_{F_{i+1}M}$ for all $i \in \mathbb{Z}$.
- $M$ is generated by the union of all $F_i(F^i M)$.

In this equivalence, a $d$-display corresponds to a triple

$$(M, (F^i M)_{i \in \mathbb{Z}}, F_i)$$

such that $F^i M = 0$ for $i \geq d + 1$ and $F^i M = M$ for $i \leq 0$.

**Remark 2.12.** Let $R$ be as in Example 2.11 An effective $\hat{W}(R)$-display is equivalent to a display $(P_i, \imath_i, \alpha_i, F_i)_{i \geq 0}$ over the small Witt frame in [LZ2, Definition 3] (see also [La2, Example 3.6.3]). In this equivalence, a $d$-display corresponds to $(P_i, \imath_i, \alpha_i, F_i)_{i \geq 0}$ such that $\alpha_i : \hat{I}(R) \otimes P_i \rightarrow P_{i+1}$ is surjective (equivalently, $(P_i, \imath_i, \alpha_i, F_i)_{i \geq 0}$ has a normal decomposition $(L_i)_{i \geq 0}$ in the sense of [LZ2, Definition 3] such that $L_i = 0$ for $i \geq d + 1$). In particular, a 1-display over $\hat{W}(R)$ is equivalent to a quadruple

$$(P, Q, F, F_1),$$

where $P$ is a finite projective $\hat{W}(R)$-module, $Q$ is a submodule of $P$ such that $\hat{I}(R) P \subset Q$, the quotient $P/Q$ is projective over $R$, the maps $F : P \rightarrow P$ and $F_1 : Q \rightarrow P$ are $\sigma$-linear morphisms such that $F_1(Q)$ generates $P$ as a $\hat{W}(R)$-module. We note that this is a Dieudonné display over $R$ in the sense of [La1, Definition 2.6].

2.3. Displays and $p$-divisible groups over $p$-torsion free bases with Frobenius lifts. Let $A$, $p$, and $\sigma$ be as in Remark 2.11 Moreover, we assume that $p \geq 3$. Let $G$ be a $p$-divisible group over $A$. We have a covariant Dieudonné crystal $\mathbb{D}(G)$ which is a crystal on the nilpotent crystalline site $NilCrys(Spf(A)/Spf(\mathbb{Z}_p))$ ([Me, Chapter IV]). We set

$$D(G) := \mathbb{D}(G)_A$$

$$F^1 D(G) := \text{Ker}(\mathbb{D}(G)_A \rightarrow \text{Lie}(G)) = \text{Lie}(G^*)^*.$$
Let $G_0 := G \otimes A / pA$. The ring morphism $\sigma$, the Frobenius map $\operatorname{Fr} : A / pA \to A / pA$, and the Verschiebung map $G_0^{(p)} \to G_0$ induce the following morphisms of $A$-modules
\[
D(G)^\sigma = (\mathbb{D}(G_0)_{(A \to A / pA)})^\sigma \\
= (\operatorname{Fr}_{\text{crys}} \mathbb{D}(G_0))_{(A \to A / pA)} \\
= \mathbb{D}(G_0^{(p)})_{(A \to A / pA)} \\
\to \mathbb{D}(G_0)_{(A \to A / pA)} \\
= D(G).
\]
Note that the PD thickening $A \to A / pA$ is nilpotent because $p \geq 3$. The $\sigma$-linear morphism induced by the composite of the above morphisms is denoted by $F : D(G) \to D(G)$.

The following result is presumably well-known to experts (see \cite{La1}, Theorem 3.17). We give the proof for readers’ convenience.

**Proposition 2.13.** The triple $(D(G) \supset F^1 D(G), F)$ is a 1-display over the tautological frame $A$ (see Remark 2.11).

**Proof.** It suffices to prove that $F(F^1 D(G))$ is contained in $pD(G)$, and $D(G)$ is generated by $F(D(G))$ and $p^{-1} F(F^1 D(G))$.

We consider the following $A / pA$-linear map
\[
F_0 : (\mathbb{D}(G_0)_{A / pA})^{(p)} = D(G)^\sigma \otimes A / pA \to D(G) \otimes A / pA = \mathbb{D}(G_0)_{A / pA}.
\]
We consider the following diagram.
\[
\begin{array}{ccc}
\operatorname{Lie}(G_0^{(p)})^* & \xrightarrow{F_0} & F^1 D(G)^\sigma \otimes A / pA \\
\phantom{\operatorname{Lie}(G_0^{(p)})^*} & & \downarrow \\
\operatorname{Lie}(G_0)^* & \xrightarrow{F_0} & F^1 D(G) \otimes A / pA \\
\end{array}
\]
In this diagram, the left vertical map is zero because this map is induced by the relative Frobenius morphism $G_0^* \to (G_0^{(p)})^* = (G_0^{(p)})^*$. Thus $F(F^1 D(G)) \subset pD(G)$.

Let us fix an $A$-submodule $L$ of $D(G)$ such that $D(G) = L \oplus F^1 D(G)$. The $\sigma$-linear maps $F|_L : L \to D(G)$ and $p^{-1} F : F^1 D(G) \to D(G)$ induce a $\sigma$-linear map
\[
\tilde{F} : D(G) = L \oplus F^1 D(G) \to D(G).
\]
It suffices to show that $\tilde{F}$ induces a surjection of $A$-modules $D(G)^\sigma \to D(G)$. By Nakayama’s lemma, it suffices to prove that, for an arbitrary field $k$ and an arbitrary ring morphism $A / pA \to k$, the Frobenius-linear map $\tilde{F} \otimes_A k$ induces a surjective morphism of $k$-vector spaces $D(G)^\sigma \otimes_A k \to D(G) \otimes_A k$. We may assume that $k$ is a perfect field. Since $A$ is $p$-torsion free, the Frobenius lift $\sigma : A \to A$ induces a ring homomorphism $A \to W(A)$ by \cite{La2}, VII, Proposition 4.12. Thus the composite of $A \to A / pA \to k$ induces a ring homomorphism $A \to W(k)$ commuting with Frobenius lifts. Taking base change by $A \to W(k)$, it suffices to consider the case where $A = W(k)$ and $\sigma$ is the Witt vector Frobenius.

In this case, $D(G)$ is the covariant Dieudonné module of the $p$-divisible group $G_0$ over $k$. We have a $\sigma^{-1}$-linear map $V : D(G) \to D(G)$ satisfying $V(D(G)) = F^{-1}(pD(G))$. 

\[\text{QUASI-CANONICAL LIFTINGS OF K3 SURFACES OF FINITE HEIGHT} \]
Therefore $pL \oplus F^1 D(G)$ is contained in $V(D(G))$. Since
\[
\dim_k((pL \oplus F^1 D(G))/pD(G)) = \dim_k(F^1 D(G)/pF^1 D(G)) = \dim_k((\text{Lie}(G^*_0))^*) = \dim G^*
\]
and
\[
\dim_k(V(D(G))/pD(G)) = \dim_k(D(G)/F(D(G))) = \dim G^*
\]
(the last equality follows from [De, Chapter III, Section 9]), we see that $pL \oplus F^1 D(G)$ coincides with $V(D(G))$. Therefore, we have
\[
\text{Im}(\tilde{F}) = F(L) + (p^{-1}F)(F^1 D(G)) = (p^{-1}F)(pL \oplus F^1 D(G)) = (p^{-1}F)(V(D(G))) = D(G).
\]

**Definition 2.14.** The display defined in Proposition 2.13 is called the Dieudonné display associated to $G$ over the tautological frame $A$. It is denoted by $\mathcal{D}(G)$.

**Proposition 2.15.** There exists a functorial isomorphism of displays
\[
(\mathcal{D}(G))^* \simto \mathcal{D}(G^*).
\]

**Proof.** See [BBM, Proposition 5.3.6]. □

2.4. Displays and $p$-divisible groups over Artin local rings. Let $p \geq 3$ and $R$ be as in Example 2.4. Let $G$ be a $p$-divisible group over $R$. We have a co-variant Dieudonné crystal $\mathcal{D}(G)$ which is a crystal on the nilpotent crystalline site $\text{NilCrys}(\text{Spec}(R)/\text{Spf}(\mathbb{Z}_p))$. We set
\[
\begin{align*}
D(G) &:= \mathcal{D}(G)(\widehat{W(R)} \rightarrow R) \\
F^1 D(G) &:= \text{Ker}(\mathcal{D}(G)(\widehat{W(R)} \rightarrow R) \rightarrow \mathcal{D}(G)_R \rightarrow \text{Lie}(G)).
\end{align*}
\]

The Witt vector Frobenius $\sigma : \widehat{W(R)} \rightarrow \widehat{W(R)}$, the Frobenius map $\text{Fr} : R/pR \rightarrow R/pR$, and the Verschiebung map $(G \otimes R/pR)^{(p)} \rightarrow G \otimes R/pR$ induce the following morphisms of $\widehat{W(R)}$-modules:
\[
\begin{align*}
D(G)^\sigma &= (\mathcal{D}(G \otimes R/pR)(\widehat{W(R)} \rightarrow R/pR))^\sigma \\
&= (\text{Fr}^*_{\text{cryst}} \mathcal{D}(G \otimes R/pR))(\widehat{W(R)} \rightarrow R/pR) \\
&= \mathcal{D}((G \otimes R/pR)^{(p)})(\widehat{W(R)} \rightarrow R/pR) \\
&\rightarrow \mathcal{D}(G \otimes R/pR)(\widehat{W(R)} \rightarrow R/pR) \\
&= D(G).
\end{align*}
\]

The $\sigma$-linear map induced by the composite of the above morphisms is denoted by $F : D(G) \rightarrow D(G)$. 
**Proposition 2.16** (Lau). There is a unique $\sigma$-linear morphism $F_1 : F^1D(G) \to D(G)$ which is functorial in $R$ and $G$ such that the quadruple

$$(D(G), F^1D(G), F, F_1)$$

is a 1-display over the small Witt frame $\widehat{W}(R)$ (see Remark 2.12).

**Proof.** See [La1, Proposition 3.17]. □

**Definition 2.17.** The display defined in Proposition 2.16 is called the Dieudonné display associated to $G$ over the small Witt frame $\widehat{W}(R)$. It is denoted by $D(G)$.

**Proposition 2.18.** There exists a functorial isomorphism of $\widehat{W}(R)$-displays

$$(D(G))^* \sim D(G^*)$$

**Proof.** Let $k$ be the residue field of $R$ and $G_k := G \otimes_R k$. By [La1, Proposition 3.11], we have the universal deformation of $G_n := G \otimes_R A_n$ for $n \geq 1$. Let $\sigma : A \to A$ be a Frobenius lift such that $\sigma(t_i) = t_i^p$ for all $i$ and $\sigma$ coincides with the Witt vector Frobenius on $W(k)$. The Frobenius lift $\sigma$ defines tautological frames $A$ and $A_n$. We have the unique morphism of local $W(k)$-algebras $A \to \widehat{W}(R)$ corresponding to $G/R$. The morphism of local $W(k)$-algebras $A \to \widehat{W}(R)$ which sends $t_i$ to $[x_i]$ for all $i$ induces a morphism of frames $A \to \widehat{W}(R)$. Here, $[x_i]$ denotes the Teichmüller lift of $x_i$. Since $R$ is an Artin local ring, the map $A \to R$ (resp. $A \to \widehat{W}(R)$) factors as $A \to A_m \to R$ (resp. $A \to A_m \to \widehat{W}(R)$) for some $m \geq 1$. Then, by taking base change of the isomorphism of displays over $A_m$ in Proposition 2.15

$$D(\mathcal{G} \otimes_A A_m)^* \sim D((\mathcal{G} \otimes_A A_m)^*)$$

by $A_m \to \widehat{W}(R)$, we get the desired isomorphism. □

The following theorem is a main theorem of display-theoretic Dieudonné theory proved by Lau (see [La1] for details).

**Theorem 2.19** (Lau). The functor

$$(p\text{-divisible groups over } R) \to (1\text{-displays over } \widehat{W}(R))$$

defined by

$$G \mapsto D(G)$$

gives an equivalence of categories.

**Proof.** See [La1, Theorem 3.19]. □

2.5. Displays and $p$-adic formal schemes over $p$-torsion free bases with Frobenius lifts. Let $A$ and $\sigma : A \to A$ be as in Remark 2.11. Moreover, we assume that $A$ is Noetherian and $p \geq 3$. Let $\mathcal{X}$ be a $p$-adic proper smooth formal scheme over $\text{Spf}(A)$ satisfying the following conditions:

- The Hodge-de Rham spectral sequence associated to $\mathcal{X}/A$

$$E_1^{i,j} = H^j(\mathcal{X}, \Omega^i_{\mathcal{X}/A}) \Rightarrow H^i_{\text{dR}}(\mathcal{X}/A)$$

degenerates at $E_1$-page.
For \( i, j \geq 0 \), \( H^j(\mathcal{X}, \Omega^i_{\mathcal{X}/A}) \) is a finite projective \( A \)-module. It follows from this that \( H^j(\mathcal{X}, \Omega^i_{\mathcal{X}/A}) \) is compatible with base change with respect to \( A \).

We put \( X_0 := \mathcal{X} \otimes A/pA \).

When \( \mathcal{X} \) is projective over \( A \), the following result is proved by Langer-Zink (see [LZ1, Theorem 5.5]).

**Proposition 2.20.** Let

\[
H^n_{\text{dR}}(\mathcal{X}/A) \supset F^1 H^n_{\text{dR}}(\mathcal{X}/A) \supset F^2 H^n_{\text{dR}}(\mathcal{X}/A) \supset \cdots
\]

be the Hodge filtration. Let

\[
F : H^n_{\text{dR}}(\mathcal{X}/A) \to H^n_{\text{dR}}(\mathcal{X}/A)
\]

be a \( \sigma \)-linear map induced by \( \sigma \) and the identification \( H^n_{\text{dR}}(\mathcal{X}/A) \cong H^n_{\text{crys}}(X_0/A) \). We assume \( p > n \). Then the triple

\[
(H^n_{\text{dR}}(\mathcal{X}/A), F^* H^n_{\text{dR}}(\mathcal{X}/A), F)
\]

is an \( n \)-display over the tautological frame \( A \) (see Remark 2.11).

**Proof.** It suffices to prove that \( F(F^i H^n_{\text{dR}}(\mathcal{X}/A)) \) is contained in \( p^i H^n_{\text{dR}}(\mathcal{X}/A) \) for \( i \geq 1 \), and \( H^n_{\text{dR}}(\mathcal{X}/A) \) is generated by the union of \( (p^{-i} F)(F^i H^n_{\text{dR}}(\mathcal{X}/A)) \) for \( i \geq 0 \).

The former assertion is proved in [LZ1, Lemma 5.4] when \( \mathcal{X} \) is projective over \( A \). In the general case, we use the results of Berthelot-Ogus as follows. By [BO1, Theorem 8.16 and Theorem 8.20], for \( i \in [0, n] \), we get the following commutative diagram in the derived category.

\[
\begin{array}{ccc}
\Omega_{\mathcal{X}^a/A}^{i+1} & \xrightarrow{\epsilon} & \Omega_{\mathcal{X}^a/A}^i \\
\downarrow & & \downarrow \\
\hat{L} \eta \Omega_{\mathcal{X}^a/A}^i & \xrightarrow{p^i A \otimes A} & \Omega_{\mathcal{X}^a/A}^i \\
\end{array}
\]

Here \( \epsilon : \mathbb{N} \to \mathbb{N} \) is defined by \( \epsilon(j) = \max\{j - i, 0\} \), \( \eta : \mathbb{N} \to \mathbb{N} \) is defined by \( \eta(j) = \epsilon(j) + j \), and \( \Omega_{\mathcal{X}^a/A}^j \) (resp. \( \Omega_{\mathcal{X}^a/A}^j \)) is the subcomplex of \( \Omega_{\mathcal{X}^a/A}^j \) whose degree \( j \) part is 0 for \( j \leq i - 1 \) and \( \Omega_{\mathcal{X}^a/A}^j \) for \( j \geq i \) (resp. 0 for \( j \leq -1 \) and \( \Omega_{\mathcal{X}^a/A}^{j+1} \) for \( j \geq 0 \)). Taking hypercohomology, the first assertion is proved.

We prove the latter assertion. By the same argument as the proof of Proposition 2.13, it suffices to consider the case that \( A = W(k) \) where \( k \) is a perfect field and \( \sigma \) is the Witt vector Frobenius. Then the latter assertion follows from the strong divisibility of the crystalline cohomology \( H^n_{\text{crys}}(X_0/W(k)) \) (see [Fo, Proposition in Section 1]).

The \( n \)-display given by Proposition 2.20 is denoted by \( H^n_{\text{dR}}(\mathcal{X}/A) \).

2.6. **Displays and K3 surfaces over Artin local rings.** Let \( R \) be an Artin local ring with perfect residue field \( k \) of characteristic \( p \geq 3 \).

**Definition 2.21.** (La2, Definition 8.0.1). A K3 display \( (M, \langle , \rangle) \) consists of a \( \hat{W}(R) \)-display \( M = (M, F) \) and a symmetric perfect pairing

\[
\langle , \rangle : M \times M \to \hat{W}(R)(-2)
\]
(i.e. a morphism of \( \widehat{W}(R) \)-displays \( M \otimes M \to \widehat{W}(R)(-2) \) which is a symmetric perfect pairing of graded \( \widehat{W}(R) \)-modules) such that
\[
M^\pi \cong R \oplus R(-1)^{\oplus 20} \oplus R(-2)
\]
as graded \( R \)-modules. Here, \( \pi : \widehat{W}(R) \to R \) is the composite of the two projection maps \( \widehat{W}(R) \to \widehat{W}(R) \) and \( \widehat{W}(R) \to R \), and we put \( M^\pi := M \otimes_{\widehat{W}(R),\pi} R \).

The following result is proved by Langer-Zink [LZ2, Proposition 19] (see also [La2, Section 8.1]).

**Proposition 2.22.** Let \( X \) be a K3 surface of over \( R \). Then \( H^2_{\text{crys}}(X/\widehat{W}(R)) \) is canonically equipped with K3 display structure. To be more precise, there exists a canonical K3 display
\[
(H^2_{\text{crys}}(X/\widehat{W}(R)), (, ))
\]
satisfying the following conditions:

1. The degree 0 part \((H^2_{\text{crys}}(X/\widehat{W}(R)))_0\) coincides with \( H^2_{\text{crys}}(X/\widehat{W}(R)) \), and \( (, )_0 : H^2_{\text{crys}}(X/\widehat{W}(R)) \times H^2_{\text{crys}}(X/\widehat{W}(R)) \to \widehat{W}(R) \) is the pairing defined by Poincaré duality.
2. For \( i = 1, 2 \), let \( M^i \) be the image of the composite of the following maps:
\[
(H^2_{\text{crys}}(X/\widehat{W}(R))), \xrightarrow{\sigma^i} H^2_{\text{crys}}(X/\widehat{W}(R)) \to H^2_{\text{dR}}(X/R).
\]
Then \( H^3_{\text{dR}}(X/R) \supset M^1 \supset M^2 \) is the Hodge filtration.
3. The K3 display \((H^2_{\text{crys}}(X/\widehat{W}(R)), (, ))\) is compatible with base change with respect to \( R \).

**Proof.** We briefly give a sketch of the proof for readers’ convenience. Let \( X_0 := X \otimes_R k \). Let \( \mathcal{X} \to \text{Spf}(W(k)[[t_1, \ldots, t_{20}]]) \) be the universal deformation of \( X_0 \). We put \( \mathcal{A} := W(k)[[t_1, \ldots, t_{20}]] \) and \( \mathcal{A}_n := \mathcal{A}/(t_1^n, \ldots, t_{20}^n) \). Let \( \sigma \) be a Frobenius lift such that \( \sigma(t_i) = t_i^n \) for all \( i \) and such that \( \sigma \) coincides with the Witt vector Frobenius on \( W(k) \). The Frobenius lift \( \sigma \) defines the tautological frames \( \mathcal{A} \) and \( \mathcal{A}_n \). We have the unique morphism of local \( W(k) \)-algebras \( A \to R, (t_i \mapsto x_i) \) corresponding to \( X/R \). The morphism of local \( W(k) \)-algebras \( A \to \widehat{W}(R) \) which sends \( t_i \) to \([x_i]\) for all \( i \) induces a morphism of frames \( \mathcal{A} \to \widehat{W}(R) \). Here, \([x_i]\) denotes the Teichmüller lift of \( x_i \). Since \( R \) is an Artin local ring, the ring morphism \( A \to R \) (resp. the morphism of frames \( A \to \widehat{W}(R) \)) factors as \( A \to A_m \to R \) (resp. \( A \to A_m \to \widehat{W}(R) \)) for some \( m \geq 1 \). Then, by taking base change of the \( A_m \)-display \( H^2_{\text{dR}}((\mathcal{X} \otimes_A A_m)/A_m) \) (defined in Subsection 2.2) and a symmetric perfect pairing
\[
H^2_{\text{dR}}((\mathcal{X} \otimes_A A_m)/A_m) \times H^2_{\text{dR}}((\mathcal{X} \otimes_A A_m)/A_m) \to A_m(-2)
\]
by the morphism of frames \( A_m \to \widehat{W}(R) \), we get the desired K3 display. \( \square \)

The following theorem is a display-theoretic deformation theory for K3 surfaces proved by Lau [La2]. (When \( X \) is ordinary, it is proved by Langer and Zink [LZ2].)
Theorem 2.23 (Lau). Let $S \to R$ be a surjection of Artin local rings with perfect residue field of characteristic $p \geq 3$ and $X$ be a $K3$ surface over $R$. Then

$$X'/S \to H^2_{\text{crys}}(X'/\hat{W}(S))$$

gives an equivalence of categories between deformations of $X$ to $K3$ surfaces over $S$ and deformations of $H^2_{\text{crys}}(X/\hat{W}(R))$ to $K3$ displays over $\hat{W}(S)$.

Proof. See [La2, Theorem 8.1.1].

□

3. Crystalline cohomology and the enlarged formal Brauer group

First, we recall some results in [NO, Section 3]. Let $p$ be a prime number, $k$ be a perfect field of characteristic $p$, $R$ be an Artin local ring over $W(k)$ with residue field $k$, and $X/R$ be a $K3$ surface of finite height (i.e. a proper flat scheme over $R$ whose closed fiber $X_0$ is a $K3$ surface of finite height). To this, we can associate a formal $p$-divisible group $\hat{Br}_{X/R}$ which is called the formal Brauer group of $X/R$ and a $p$-divisible group $\psi_{X/R}$ which is called the enlarged formal Brauer group. We note that $\hat{Br}_{X/R}$ is the connected component of the identity of $\psi_{X/R}$. For a $p$-divisible group $G$ over $R$, let $\mathbb{D}(G)$ be the (covariant) Dieudonné crystal on $\text{NilCrys}(\text{Spec}(R)/\text{Spf}(\mathbb{Z}_p))$. Let $\mathbb{H}^2_{\text{crys}}(X/R)$ be the crystal on $\text{Crys}(\text{Spec}(R)/\text{Spf}(\mathbb{Z}_p))$ such that

$$\mathbb{H}^2_{\text{crys}}(X/R)(S \to R) = H^2_{\text{crys}}(X/S)$$

for any $(S \to R) \in \text{Crys}(\text{Spec}(R)/\text{Spf}(\mathbb{Z}_p))$. By [NO, Theorem 3.16], there is a natural morphism of $F$-crystals

$$\rho : \mathbb{D}(\psi_{X/R}) \to \mathbb{H}^2_{\text{crys}}(X/R).$$

By the composition with the inclusion $\mathbb{D}(\hat{Br}_{X/R}) \hookrightarrow \mathbb{D}(\psi_{X/R})$, we get a morphism $\mathbb{D}(\hat{Br}_{X/R}) \to \mathbb{H}^2_{\text{crys}}(X/R)$. Taking the dual, we get a morphism of $F$-crystals

$$\mathbb{H}^2_{\text{crys}}(X/R) \to \mathbb{D}(\hat{Br}_{X/R}^*)(-1)$$

because $\mathbb{H}^2_{\text{crys}}(X/R)^* = \mathbb{H}^2_{\text{crys}}(X/R)(2)$ and $\mathbb{D}(\hat{Br}_{X/R})^* = \mathbb{D}(\hat{Br}_{X/R})(1)$.

Theorem 3.1. There is the following exact sequence of $F$-crystals:

$$0 \to \mathbb{D}(\psi_{X/R}) \xrightarrow{\rho} \mathbb{H}^2_{\text{crys}}(X/R) \to \mathbb{D}(\hat{Br}_{X/R}^*)(-1) \to 0.$$ 

Moreover, by evaluating on the trivial PD thickening $R \xrightarrow{id} R$, we get the following commutative diagram:

$$\begin{array}{cccccc}
0 & \to & 0 & \to & F^2H^2_{\text{dR}}(X/R) & \to & F^1\mathbb{D}(\hat{Br}_{X/R}^*)(-1)_R & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & F^1\mathbb{D}(\psi_{X/R})_R & \to & F^1H^2_{\text{dR}}(X/R) & \to & \mathbb{D}(\hat{Br}_{X/R}^*)(-1)_R & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathbb{D}(\psi_{X/R})_R & \xrightarrow{\rho} & H^2_{\text{dR}}(X/R) & \to & \mathbb{D}(\hat{Br}_{X/R}^*)(-1)_R & \to & 0 \\
\end{array}$$

In this diagram, all horizontal rows are exact.

Proof. See [NO, Theorem 3.20].

□
In this section, we prove a display-theoretic analogue of Theorem 3.1. By evaluating on the PD thickening $\hat{W}(R) \to R$, we get a morphism of $\hat{W}(R)$-modules
\[ \hat{\rho} : D(\psi_{X/R})(\hat{W}(R) \to R) \to H^2_{\text{crys}}(X/\hat{W}(R)). \]
We want to construct a morphism of $\hat{W}(R)$-displays $D(\psi_{X/R}) \to H^2_{\text{crys}}(X/\hat{W}(R))$ which induces $\hat{\rho}$ on the underlying $\hat{W}(R)$-modules. In order to do this, we work on $p$-torsion free bases.

In the following, we assume $p \geq 3$. Let $\mathcal{X} \to \text{Spf } W(k)[[t_1, \ldots, t_{20}]]$ be the universal deformation of $X_0$. We put $A := W(k)[[t_1, \ldots, t_{20}]]$, $A_n := A/(t_1^n, \ldots, t_{20}^n)$, and $\mathcal{X}_n := \mathcal{X} \otimes_A A_n$. Let $\sigma : A \to A$ be the endomorphism such that $\sigma(t_i) = t_i^p$ and such that $\sigma$ is the Frobenius on $W(k)$. By evaluating $\rho$ on the trivial PD thickening $A_n \to A_n$, we get a morphism of $A_n$-modules
\[ \rho_{A_n} : D(\psi_{\mathcal{X}_n/A_n})_{A_n} \to H^2_{\text{dR}}(\mathcal{X}_n/A_n). \]
Here $D(\psi_{\mathcal{X}_n/A_n})_{A_n}$ is the underlying $A_n$-module of the Dieudonné display associated to $\psi_{\mathcal{X}_n/A_n}$, and $H^2_{\text{dR}}(\mathcal{X}_n/A_n)$ is the underlying $A_n$-module of the $K$3 display associated to the $K$3 surface $\mathcal{X}_n/A_n$.

**Proposition 3.2.** The map $\rho_{A_n}$ defines a morphism of $A_n$-displays
\[ \rho_{A_n} : D(\psi_{\mathcal{X}_n/A_n}) \to H^2_{\text{dR}}(\mathcal{X}_n/A_n). \]

**Proof.** It suffices to prove that $\rho_{A_n}$ commutes with $F$ and preserves filtrations. The former is clear because $\rho$ is a morphism of $F$-crystals. We prove the latter assertion. Since $D(\psi_{\mathcal{X}_n/A_n})$ is a 1-display, it suffices to prove that $\rho_{A_n}(F^1 D(\psi_{\mathcal{X}_n/A_n}))$ is contained in $F^1 H^2_{\text{dR}}(\mathcal{X}_n/A_n)$. This assertion follows from the commutative diagram in Theorem 3.1. \[ \square \]

Composing $\rho_{A_n}$ with $D(\hat{\text{Br}}_{\mathcal{X}_n/A_n}) \to D(\psi_{\mathcal{X}_n/A_n})$, we get a morphism of $A_n$-displays $D(\hat{\text{Br}}_{\mathcal{X}_n/A_n}) \to H^2_{\text{dR}}(\mathcal{X}_n/A_n)$. By taking the dual of this map, we get a morphism of $A_n$-displays
\[ H^2_{\text{dR}}(\mathcal{X}_n/A_n) \to D(\hat{\text{Br}}^*_{\mathcal{X}_n/A_n})(-1) \]
because $D(\hat{\text{Br}}_{\mathcal{X}_n/A_n})^* \cong D(\hat{\text{Br}}^*_{\mathcal{X}_n/A_n})(1)$ and $(H^2_{\text{dR}}(\mathcal{X}_n/A_n))^* \cong H^2_{\text{dR}}(\mathcal{X}_n/A_n)(2)$.

**Theorem 3.3.** There is the following exact sequence of $A_n$-displays:
\[ 0 \to D(\psi_{\mathcal{X}_n/A_n}) \xrightarrow{\rho_{A_n}} H^2_{\text{dR}}(\mathcal{X}_n/A_n) \to D(\hat{\text{Br}}^*_{\mathcal{X}_n/A_n})(-1) \to 0 \]

**Proof.** It suffices to show that the following three sequences of $A_n$-modules are exact.
\[ 0 \to D(\psi_{\mathcal{X}_n/A_n}) \xrightarrow{\rho_{A_n}} H^2_{\text{dR}}(\mathcal{X}_n/A_n) \to D(\hat{\text{Br}}^*_{\mathcal{X}_n/A_n}) \to 0 \]
\[ 0 \to F^1 D(\psi_{\mathcal{X}_n/A_n}) \xrightarrow{\rho_{A_n}} F^1 H^2_{\text{dR}}(\mathcal{X}_n/A_n) \to D(\hat{\text{Br}}^*_{\mathcal{X}_n/A_n}) \to 0 \]
\[ 0 \to F^2 D(\psi_{\mathcal{X}_n/A_n}) \xrightarrow{\rho_{A_n}} F^2 H^2_{\text{dR}}(\mathcal{X}_n/A_n) \to F^1 D(\hat{\text{Br}}^*_{\mathcal{X}_n/A_n}) \to 0 \]
The exactness of all sequences follows from Theorem 3.1. \[ \square \]

**Corollary 3.4.** Let $X/R$, $\psi_{X/R}$, and $\hat{\text{Br}}_{X/R}$ be the same as the beginning of this section. Then there is the following exact sequence of $\hat{W}(R)$-displays:
\[ 0 \to D(\psi_{X/R}) \xrightarrow{\rho} H^2_{\text{crys}}(X/\hat{W}(R)) \to D(\hat{\text{Br}}^*_{X/R})(-1) \to 0. \]
Moreover, this sequence is compatible with base change with respect to $R$.

Proof. Let $A \to R, (t_i \mapsto x_i)$ be the morphism which corresponds to the deformation $X/R$. We define $\phi : A \to \hat{W}(R)$ by $\phi(t_i) = [x_i]$, where $[x_i]$ denotes the Teichmüller lift of $x_i$. Then, for a sufficiently large $n$, the map $\phi$ induces a morphism of frames $A_n \to \hat{W}(R)$. Taking the base change of the sequence in Theorem 3.3 by this morphism, we get the desired exact sequence. □

4. A CONSTRUCTION OF QUASI-CANONICAL LIFTINGS

In this section, we prove the main result of this paper.

Theorem 4.1. Let $X_0$ be a $K3$ surface of finite height $h < \infty$ over a finite field $k$ of characteristic $p \geq 3$. Then there exist a totally ramified finite extension $V/W(k)$ of degree $h$ and a quasi-canonical lifting $X/V$ of $X_0/k$.

Proof. We follow the strategy of Nygaard-Ogus [NO]. Instead of the crystalline deformation theory used in [NO], we use the display-theoretic deformation theory of $K3$ surfaces.

We put $k = \mathbb{F}_p$. Let $\hat{\text{Br}}_{X_0/k}$ (resp. $\psi_{X_0/k}$) be the formal Brauer group (resp. the enlarged formal Brauer group) of $X_0/k$. Let $\text{Fr} : \hat{\text{Br}}_{X_0/k} \to \hat{\text{Br}}_{X_0/k}$ be the relative Frobenius morphism. By [Haz, Theorem 24.2.6, Theorem 24.3.4], there exists a totally ramified finite extension $V/W(k)$ of degree $h$, a lifting $G$ of $\hat{\text{Br}}_{X_0/k}$ over $V$, and an endomorphism $F : G \to G$ which lifts $\text{Fr}^m : \hat{\text{Br}}_{X_0/k} \to \hat{\text{Br}}_{X_0/k}$. Let $\pi$ be a prime element of $V$.

For each $n \geq 1$, we put $G_n := G \otimes_V V/\pi^{n+1}$. Let $H_n$ denote the unique lifting of the étale part of $\psi_{X_0/k}$ over $V/\pi^{n+1}$. We define a $\hat{W}(V/\pi^{n+1})$-display $K(G_n)$ as

$$K(G_n) := D(G_n) \oplus D(H_n) \oplus D(G_n^*)(-1).$$

Taking base change by $\hat{W}(V/\pi^{n+1}) \to W(k)$, the right hand side gives the slope decomposition of $H^2_{\text{cris}}(X_0/W(k))$. We have a canonical perfect pairing

$$D(G_n) \times D(G_n^*)(-1) \to \hat{W}(V/\pi^{n+1})(-2)$$

and a perfect pairing

$$D(H_n) \times D(H_n) \to \hat{W}(V/\pi^{n+1})(-2)$$

which is the unique lifting of the pairing on the slope one part of $H^2_{\text{cris}}(X_0/W(k))$ ([Z1] Lemma 42]). These pairings define a $K3$ display structure on $K(G_n)$.

Let $\mathcal{X}$ be a formal lifting which corresponds to the inverse system of $K3$ displays $\{K(G_n)\}$ (Theorem 2.23). Let $X_n := \mathcal{X} \otimes_V V/\pi^{n+1}$. Then $X_n$ is a $K3$ surface over $V/\pi^{n+1}$ corresponding to the $K3$ display $K(G_n)$. We consider the following morphisms of $\hat{W}(V/\pi^{n+1})$-displays:

$$D(G_n^*)(-1) \to K(G_n) = H^2_{\text{cris}}(X_n/\hat{W}(V/\pi^{n+1})) \to D(\hat{\text{Br}}_{X_n/(V/\pi^{n+1})})(-1),$$

where the last morphism is the one in Corollary 3.4. The composite of the above morphisms is an isomorphism of displays because it becomes an isomorphism after the
base change by $\widehat{W}(V/\pi^{n+1}) \to \widehat{W}(k)$. Then we have the following diagram.

$$
\begin{array}{cccccc}
0 & \longrightarrow & D(\psi_{X_n/(V/\pi^{n+1})}) & \longrightarrow & H^2_{\text{crys}}(X_n/\widehat{W}(V/\pi^{n+1})) & \longrightarrow & D(\widehat{Br}^*_{X_n/(V/\pi^{n+1})})(-1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \cong & & \\
0 & \longrightarrow & D(G_n) & \oplus & D(H_n) & \longrightarrow & K(G_n) & \longrightarrow & D(G_n^*)(-1) & \longrightarrow & 0 \\
\end{array}
$$

In this diagram, both rows are exact by Theorem 3.3. So we get a canonical isomorphism of displays

$$
D(\psi_{X_n/(V/\pi^{n+1})}) \sim D(G_n) \oplus D(H_n) = D(G_n \oplus H_n).
$$

By Theorem 2.19, this induces an isomorphism

$$
\psi_{X_n/(V/\pi^{n+1})} \sim G_n \oplus H_n
$$

of $p$-divisible groups over $V/\pi^{n+1}$. In particular, the enlarged formal Brauer group $\psi_{X_n/(V/\pi^{n+1})}$ splits into a direct sum of a connected part and an étale part. Since the map $\text{Pic}(X_n) \to \text{Pic}(X_0)$ is surjective by the argument in the proof of [Ny, Proposition 1.8], there is a proper smooth scheme $X$ over $V$ and a line bundle $\mathcal{L}$ on $X$ such that $X \otimes V/\pi^{n+1} = X_n$ and $\mathcal{L}|_{X_0}$ is ample and primitive.

By [NO, Definition 1.5 and Theorem 1.9], in order to prove that $X/V$ is a quasi-canonical lifting of $X_0/k$, it suffices to prove that there exists $\gamma \in W_{\text{crys}}(\overline{K})$ of non-zero degree such that the action of $\gamma$ on $H^2_{\text{dR}}(X/V) \otimes_V \overline{K}$ preserves the Hodge filtration. Let $H$ denote the unique lifting of the étale part of $\psi_{X_0/k}$. By the definition of $K(G_n)$,

$$
H^2_{\text{dR}}(X/V) = D(G)_V \oplus D(H)_V \oplus D(G^*)(-1)_V
$$

$$
\supseteq F^1 D(G)_V \oplus D(H)_V \oplus D(G^*)(-1)_V
$$

$$
\supseteq F^1 D(G^*)(-1)_V
$$

is the Hodge filtration on $H^2_{\text{dR}}(X/V)$. Here $\overline{K}$ is an algebraic closure of the fraction field of $V$ and $W_{\text{crys}}(\overline{K})$ is the crystalline Weil group (for the definition, see [BO2, Definition 4.1]). By [BO2, Proposition 3.14], we have $W_{\text{crys}}(\overline{K})$-equivariant isomorphisms

$$
H^2_{\text{dR}}(X/V) \otimes \overline{K} = (D(G)_V \oplus D(H)_V \oplus D(G^*)(-1)_V) \otimes_V \overline{K}
$$

$$
\cong (D(\widehat{Br}_{X_0/k})W(k) \oplus D(H)_W(k) \oplus D(\widehat{Br}^*_{X_0/k})(-1)W(k)) \otimes W(k) \overline{K}
$$

$$
\cong H^2_{\text{crys}}(X_0/W(k)) \otimes W(k) \overline{K}.
$$

Since the isomorphism

$$
D(G^*)_V \otimes_V \overline{K} \cong D(G^*_0)W(k) \otimes W(k) \overline{K} = D(\widehat{Br}^*_{X_0/k})W(k) \otimes W(k) \overline{K}
$$

is functorial in $G$, an element $\gamma \in W_{\text{crys}}(\overline{K})$ with $\deg(\gamma) = m$ acts on $D(G^*)_V \otimes_V \overline{K}$ as $D(F^*) \otimes \gamma$. Therefore, the action of such $\gamma \in W_{\text{crys}}(\overline{K})$ on $H^2_{\text{dR}}(X/V) \otimes_V \overline{K}$ preserves the Hodge filtration. This completes the proof.

\textbf{Acknowledgements.} The author would like to thank his advisor, Tetsushi Ito, for useful discussions and warm encouragement. The author would also like to thank Kazuhiro Ito for helpful discussions.
References

[BBM] P. Berthelot, L. Breen, W. Messing: Théorie de Dieudonné Cristalline II, Lecture Notes in Mathematics, 930. Springer-Verlag, Berlin, 1982.

[BO1] P. Berthelot, A. Ogus: Notes on crystalline cohomology, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978.

[BO2] P. Berthelot, A. Ogus: F-isocrystals and de Rham cohomology. I, Invent. Math. 72 (1983), no. 2, 159-199.

[De] M. Demazure: Lectures on p-divisible groups, Reprint of the 1972 original. Lecture Notes in Mathematics, 302. Springer-Verlag, Berlin, 1986.

[Fo] J.-M. Fontaine: Cohomologie de de Rham, cohomologie cristalline et représentations p-adiques, Algebraic geometry (Tokyo/Kyoto, 1982), 86-108, Lecture Notes in Math., 1016, Springer, Berlin, 1983.

[Haz] M. Hazewinkel: Formal groups and applications, Corrected reprint of the 1978 original. AMS Chelsea Publishing, Providence, RI, 2012.

[IIK] K. Ito, T. Ito, T. Koshikawa: CM liftings of K3 surfaces over finite fields and their applications to the Tate conjecture, Forum Math. Sigma (2021), Vol. 9: e29 1-70.

[KM] W. Kim, K. Madapusi Pera: 2-adic integral canonical models, Forum Math. Sigma 4 (2016), Paper No. e28, 34 pp.

[LZ1] A. Langer, T. Zink: De Rham-Witt cohomology and displays, Doc. Math. 12 (2007), 147-191.

[LZ2] A. Langer, T. Zink: Grothendieck-Messing deformation theory for varieties of K3-type, Tunis. J. Math. 1 (2019), no. 4, 455-517.

[La1] E. Lau: Relations between Dieudonné displays and crystalline Dieudonné theory, Algebra Number Theory 8 (2014), no. 9, 2201-2262.

[La2] E. Lau: Higher frames and G-displays, preprint, 2018, https://arxiv.org/abs/1809.09727

[Laz] M. Lazard: Commutative formal groups, Lecture Notes in Mathematics, Vol. 443. Springer-Verlag, Berlin-New York, 1975.

[MP1] K. Madapusi Pera: The Tate conjecture for K3 surfaces in odd characteristic, Invent. Math. 201 (2015), no. 2, 625-668.

[MP2] K. Madapusi Pera: Erratum to appendix to ‘2-adic integral canonical models’, Forum Math. Sigma 8 (2020), Paper No. e14, 11 pp.

[Me] W. Messing: The crystals associated to Barsotti-Tate groups: with applications to abelian schemes, Lecture Notes in Mathematics, Vol. 264. Springer-Verlag, Berlin-New York, 1972.

[Ny] N. Nygaard: The Tate conjecture for ordinary K3 surfaces over finite fields, Invent. Math. 74 (1983), no. 2, 213-237.

[NO] N. Nygaard, A. Ogus: Tate’s conjecture for K3 surfaces of finite height, Ann. of Math. (2) 122 (1985), no. 3, 461-507.

[Zi1] T. Zink: A Dieudonné theory for p-divisible groups, Class field theory—its centenary and prospect (Tokyo, 1998), 139-160, Adv. Stud. Pure Math., 30, Math. Soc. Japan, Tokyo, 2001.

[Zi2] T. Zink: The display of a formal p-divisible group, Cohomologies p-adiques et applications arithmétiques, I. Astérisque No. 278 (2002), 127-248.

Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

Email address: inoue.kentarou.73c@st.kyoto-u.ac.jp