RELATION BETWEEN ANDERSON- AND INVARIANT IMBEDDING MODELS FOR TRANSPORT IN FINITE CHAINS AND STUDY OF PHASE- AND DELAY TIME DISTRIBUTIONS FOR STRONG DISORDER

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The invariant imbedding evolution equations for the amplitude reflection and transmission coefficients of a disordered 1D chain are shown to follow from the continuum limit, for weak disorder, of recursion relations between reflection (transmission) coefficients of Anderson chains of $N$ and $N-1$ atoms embedded in infinite non-disordered chains. We show that various available analytical and numerical results for reflection phase distributions in the Anderson- and invariant imbedding models are in good qualitative agreement, as expected from the above equivalence of these models. We also discuss the distribution of reflection phases and of reflection delay times for an Anderson model with a simple distribution of on-site potentials characterizing extreme strong disorder.

1. INTRODUCTION

In recent years a number of studies of phase- and delay time distributions \cite{1,2} for waves reflected by a finite 1D Anderson chain have been published in parallel with other studies of phase- \cite{3,4,5,6,7} and delay time distributions \cite{8,9,10} for continuous disordered chains. In particular, an important result which has emerged from the latest studies of delay-time distributions \cite{4,5,6} for different 1D systems, is the universality of their asymptotic form,

$$P_\tau(\tau, L \to \infty) = \frac{L_c}{kT^2} e^{-\frac{\tau}{L_c}}, \tau \to \infty,$$  \hspace{1cm} (1)

which was first derived in Refs \cite{1,2} and \cite{3} (see, in particular, the footnote 14 in Ref \cite{3} and noting that the continuum localization length is actually twice the value listed in \cite{3} and \cite{1} (see Sect.II.C below)). Here $\tau$ is the time delay, $L_c$ the localization length and $k$ the wavenumber of a reflected wave. Furthermore, the fact that the equation (1), which coincides with the distribution predicted by random matrix theory (RMT) \cite{10}, has been reobtained by Comtet and Texier \cite{11} for two different kinds of disorder in the continuum case and by Ossipov \textit{et al.} \cite{12} for the Anderson model, strongly supports the conclusion that the universality of RMT also extends to strictly 1D random systems where Anderson localization dominates.

On the other hand, the earlier continuous space treatments \cite{13,14,15,16} of phase- and delay time distributions are based on an invariant imbedding description \cite{17,18} of reflection and transmission properties of a 1D disordered system. We recall that the invariant imbedding method, developed originally by S. Chandrasekhar, leads to stochastic differential equations describing the evolution of the amplitude reflection (transmission) coefficient as a function of length of a disordered sample, viewed as a macroscopic continuum \cite{17}. The relationship between this method and corresponding studies for tight-binding chains described by the Anderson model has remained somewhat elusive.

In view of the extensive use which has been made of both models of 1D chains, it is useful to examine their possible connection in more detail. In part \textsection I.A of Section \textsection I we present an exact derivation of the invariant imbedding evolution equations from recursion relations between amplitude reflection (transmission) coefficients of tight-binding Anderson chains of $N$ and $N-1$ atoms \cite{3}, respectively, for weak disorder.

Next, in part \textsection I.B we compare available analytical and/or numerical results for reflection phase distributions for the Anderson- \cite{15,16} and invariant imbedding \cite{13,14,15,16} models of disordered chains, in order to see to what extent they reflect the relationship between these models established in \textsection I.A. In a similar spirit we briefly discuss in part \textsection I.C the effect of discreteness of the Anderson model on the statistical properties of the reflection coefficient in a precise way, for weak disorder.

In the final section \textsection I we discuss the reflection phase- and delay-time distributions analytically for a special case of a strongly disordered one-dimensional Anderson model. Our specific assumption is that the random site energies $\varepsilon_i$ follow a doubly peaked distribution composed of narrow peaks centered at large positive and negative values $\varepsilon_i \equiv \pm c$. This distribution corresponds to a simple tractable case of extreme strong disorder for a tight-binding chain. The phase- and delay-time distributions obtained in this case differ qualitatively from results for the more familiar strong disorder case with a wide symmetric distribution of site energies centered at $\varepsilon_i = 0$. 

II. TRANSPORT IN FINITE CHAINS FOR WEAK DISORDER

A. Derivation of invariant embedding from Anderson model description

Here we first recall the derivation of Barnes and Luck of a recursion relation for the amplitude reflection coefficient of a disordered chain of \( N \) atoms described by the Anderson model, in terms of the reflection coefficient of a chain with one less disordered atom. We then extend this discussion for obtaining a similar recursion relation for amplitude transmission coefficients of disordered chains. These Anderson model recursion relations are the starting point for deriving the invariant imbedding stochastic equation for weak disorder. We also apply them in the study of reflection phase- and time-delay distributions in the special case of strong disorder considered in Section III.

The Anderson model for a chain of \( N \) disordered atomic sites (of spacing \( a = 1 \)), \( 1 \leq m \leq N \), embedded in an infinite non-disordered chain is defined by the tight-binding Schrödinger equation

\[
\varphi_{n+1} + \varphi_{n-1} + \varepsilon_n \varphi_n = E \varphi_n ,
\]

where the \( \varphi_m \) denote the wavefunction amplitudes at sites \( m \) and the sites energies \( \varepsilon_m \) are random variables associated with the disordered sites \( 1 \leq m \leq N \). Finally, \( \varepsilon_m = 0 \) on the two semi-infinite non-disordered chains defined by the sites \( m > N \) and \( m < 1 \), respectively.

Let \( R_N \) and \( T_N \) denote the reflection and transmission amplitudes for an electron wave incident from the right with wavenumber \( -k \), and hence energy

\[
E = 2 \cos k \quad \text{for} \quad a \leq k \leq \pi .
\]

Thus, by definition,

\[
\varphi_n = e^{-ik(n-N)} + R_N e^{ik(n-N)}, \quad \text{for} \quad n \geq N ,
\]

and

\[
\varphi_n = T_N e^{-ikn}, \quad \text{for} \quad n \leq 1 .
\]

We then define the so-called Riccati ratios

\[
Y_n = \frac{\varphi_{n+1}}{\varphi_n} ,
\]

which reduce the Schrödinger equation (2) to the two-point recursion relation

\[
Y_n = E - \varepsilon_n - \frac{1}{Y_{n-1}} ,
\]

where, from (4), the variable \( Y_N \) at the last site \( N \) of the disordered section is related to \( R_N \) by

\[
Y_N = \frac{e^{-ik} + R_N e^{ik}}{1 + R_N} . \tag{8}
\]

By comparing now the equations (4-7) for the chain with \( N \) disordered sites with the corresponding equations for a chain with \( N - 1 \) sites obtained by removing the last disordered site (\( \varepsilon_N = 0 \)), and noting that in both cases \( Y_0 = e^{-ik} \), it follows that both chains have the same \( Y_1, Y_2, \ldots, Y_{N-1} \) but different \( Y_N \). This immediately leads to a recursion relation between \( R_N \) and \( R_{N-1} \), which is given by equation (2) for \( n = N \), where for \( Y_N \) we insert (3), and for \( Y_{N-1} \) the analog of (3) for the chain of \( N - 1 \) sites since \( Y_{N-1} \) is the same for the chain with \( N \) sites and for the chain with \( N - 1 \) sites.

\[
R_N = \frac{e^{2ik} R_{N-1} + i \nu_N (1 + e^{2ik} R_{N-1})}{1 - i \nu_N (1 + e^{2ik} R_{N-1})} , \tag{9}
\]

where

\[
\nu_N = \frac{\varepsilon_N}{2 \sin k} . \tag{10}
\]

We now extend the discussion of Barnes and Luck to obtain an analogous recursion relation for the transmission amplitude \( T_N \). This relation is found by expressing \( T_N = e^{ik} \varphi_1 \), for the chain of length \( N \) in terms of the Riccati ratios \( Y_n \) using the identity \( \varphi_1 = \prod_{n=1}^{N-1} \frac{1}{Y_n} (1 + R_N) \), and \( T_{N-1} = e^{ik} \varphi_1 \), for the chain of length \( N - 1 \) in terms of a similar identity. Then, by using the fact that the Riccati variables \( Y_1, \ldots, Y_{N-1} \) are the same for the chains of lengths \( N \) and \( N - 1 \) and taking the ratio \( T_N/T_{N-1} \) we finally get

\[
T_N = \frac{(1 + R_N) T_{N-1}}{e^{-ik} + e^{ik} R_{N-1}} . \tag{11}
\]

By using (5), this expression is conveniently rewritten in the form

\[
T_N = \frac{e^{ik} T_{N-1}}{1 - i \nu_N (1 + e^{2ik} R_{N-1})} . \tag{11-a}
\]

The equations (3) and (11-a) describe the variation of the reflection and transmission coefficients of a disordered chain when an extra disordered site is added to it. The initial conditions for these recursions are obviously \( R_0 = 0 \) and \( T_0 = 1 \). The unitarity relation for the reflection and transmission coefficients,

\[
|R_N|^2 + |T_N|^2 = 1 , \tag{12}
\]

may be explicitly checked for a small \( N \) e.g. \( N = 1, 2, \ldots \)

For example, for \( N = 1 \) one has, from (3) and (11-a),

\[
R_1 = \frac{i \nu_1}{1 - i \nu_1} , \quad T_1 = \frac{e^{ik}}{1 - i \delta_1} , \tag{13}
\]

which clearly obey (12) and illustrate the growth of the reflection coefficient and the corresponding decrease of
the transmission coefficient of an incident wave, due to scattering by a single random site potential.

In order to derive the invariant imbedding equations for reflection and transmission amplitudes from Eqs (17-18), we first take the continuum limit. To this end we reinstate the lattice constant \( a \) and redefine the chain length \( L = Na \), the end-site energy \( \varepsilon(L) \equiv \varepsilon_N \) and the related variable \( \nu(L) \equiv \nu_N \), as well as the reflection and transmission amplitudes \( R(L) \equiv R_N \) and \( T(L) \equiv T_N \) and, finally, we define the finite differences \( R'(L) = \frac{1}{a} [R(L) - R(L - a)] \) and \( T'(L) = \frac{1}{a} [T(L) - T(L - a)] \). The reinstatement of \( a \) leads furthermore to the replacement of solutions \( \varphi_n = e^{ika} \) for a non-disordered chain by \( \varphi_n \equiv \varphi(na) = e^{ika} \) and hence to the replacement of \( k \) by \( ka \) in (8) and in (11). In the continuum limit, \( a \to 0 \), \( R'(L) \) and \( T'(L) \) reduce to the corresponding derivatives \( dR(L)/dL \) and \( dT(L)/dL \), so that e.g. \( R(L - a) = R(L) - a \, dR(L)/dL \). Now since the invariant imbedding equations are first order differential equations it follows that we have to restrict to contributions up to order \( a \) in the expansion of (8) and (11), where we thus approximate \( e^{ika} \) by \( 1 + ika \). Using these results the continuum limits of (8) and (11) become

\[
a \frac{dR(L)}{dL} = \frac{2ikaR(L)+iv(L)(1+R(L))(1+R(L)+2ikaR(L))}{1+iv(L)(1+R(L))},
\]

and

\[
a \frac{dT(L)}{dL} = \left[ ika + iv(L) \left( 1 + (1 + 2ika)R(L) - a \frac{dR(L)}{dL} \right) \right]
\]

Next, since the invariant imbedding equations are linear in the random potential \( \varepsilon(N) \equiv \varepsilon(L) \) in the present case) we wish to expand the r.h.s. of (14,15) to linear order in \( \nu_L \). For this we require

\[
|\varepsilon(L)| < 2 |\sin ka|
\]

or, within the approximation of (14,15), \( |\varepsilon(L)| << 2ka \). Clearly, this condition is not obeyed near the band edges. By expanding the r.h.s. of (14,15) to linear order in \( \varepsilon(L) \) these equations may finally be reduced to the form

\[
i k \frac{dR(L)}{dL} = -2k^2R(L) - \frac{\varepsilon(L)}{2a^2}(1 + R(L))^2,
\]

and

\[
i k \frac{dT(L)}{dL} = -k^2T(L) - \frac{\varepsilon(L)}{2a^2}(1 + R(L))T(L)
\]

which coincide with the invariant imbedding stochastic equations (17-18) for the amplitude reflection and transmission coefficients for a 1D chain with a random potential

\[
V(L) = -\frac{\varepsilon(L)}{2a^2}.
\]

We emphasize that the validity of Eqs (17-18) is subject to the weak disorder condition (16) for wavenumbers \( k \) different from the band edge values. On the other hand, we recall that previous derivations of these equations (17-18) view the disordered chain as an inhomogeneous continuum and assume that the system of length \( L \) is imbedded invariantly in its extension of length \( L + \Delta L \). Also, these derivations do not involve an explicit restriction to a weak random potential.

B. Analysis of reflection phase distributions in 1D

A variety of detailed numerical and/or analytical results for distributions of reflection phases for weakly disordered Anderson- or invariant imbedding models have been discussed in the literature [2,4,8,10,12]. Here we wish to present a more complete comparison of these results in the perspective of the relation between these models established in (A). Our analysis shows that the results for phase-distributions for chains described by the Anderson- and invariant imbedding models, respectively, are in good qualitative agreement, as expected.

For the purpose of our discussion we refer to a series of figures in the literature where phase distributions for the above models have been plotted over a phase interval of \( 2\pi \): we find it useful to classify these figures in tables (a) and (b) according to whether they correspond to low- or strong reflection systems, respectively. A weakly reflecting chain of length \( L \) (which for the Anderson model is identified with the number of atomic sites, \( N \)) has a localization length \( L_c \) larger than \( L \) while a strongly reflecting one has a localization length smaller than \( L \). Our assignment of the numerical results of Refs [2,4,12] to the \( L << L_c \) or \( L >> L_c \) categories is based on estimates of the localization length using the familiar energy dependent expression for weak disorder, as given in these references. We recall that the curves of fig. 1 of Ref. [6] for low reflection systems and those of fig. 1 of Ref. [10] for strong reflection systems are based on explicit analytic expressions for the phase distribution while the other figures referred to in the tables are obtained from numerical calculations (3,12).

In the low reflection case the uniform phase distributions in figs 2(b) and 2(d), for large \( kL \), of Ref. [6] for the Anderson model are in good agreement with corresponding nearly uniform distributions of the phase of fig. 1 of Ref. [6] for \( kL = 10^2 \) and \( kL = 10^3 \), for the invariant imbedding model. Similarly, the figs 2(a) and 2(c) of Ref. [6] and the fig. 1(b) of Ref. [10] for the Anderson model, for lower \( kL \) values, agree well with the analytic results of fig. 1 of Ref. [6] for the invariant imbedding model, for \( kL = 5 \) and for \( kL = 10 \), which indicate a symmetric double peak phase distribution in intervals of \( 2\pi \). On the other hand, in the case of strong reflection (for weak disorder)
the nearly uniform distributions in the fig. 1(b) of Ref. [3] and in the fig. 1(a) of Ref. [4] for the Anderson model, for large \( kL_c \), are in good agreement with the distributions of fig. 1 of Ref. [16] for \( kL_c = 100 \) and \( kL_c = 200 \) and with the distribution of fig. 6(c) of Ref. [12] for \( kL_c = 100 \) and \( kL_c = 1000 \), for the invariant imbedding model.

In the limit \( kL_c << 1 \) the general expression for the stationary phase distribution obtained in [4] has to be evaluated numerically. In this limit both the results of fig. 1(d) of Ref. [4] for the Anderson model and of fig. 6(a) of Ref. [12] for the invariant imbedding model indicate that the phase distribution \( P_0(\theta) \) is a symmetric function which is strongly peaked about \( \theta = \pi \). Furthermore, as \( kL_c \) is increased from zero the peak at \( \theta = \pi \) shifts progressively to larger \( \theta \) and a small secondary peak develops at \( \theta < \pi \), so that the overall shape of the phase distribution becomes broader, before tending, for \( kL_c >> 1 \), to the results discussed above. We note, in particular, the similarity between the results of fig. 1 of Ref. [16] for moderately large \( kL_c \) (\( kL_c = 10 \) and \( kL_c = 20 \)) and the fig. 6(c) of Ref. [12] for \( kL_c = 10 \).

In conclusion, the above discussion reveals excellent overall agreement between reflection phase distributions obtained, respectively, form the Anderson model for weak disorder and from the invariant imbedding model. This is expected to some extent from the equivalence of these two models in the continuum limit demonstrated in [1 A]. We find, in particular, that the discreteness of the Anderson tight-binding lattice does not significantly influence the qualitative form of the phase distribution for weak disorder.

C. Reflection coefficient in the Anderson model

In order to illustrate more precisely the effect of the discreteness of the Anderson model for weak disorder in a simple typical case, we calculate the first and second moments, \( \langle |R_N|^2 \rangle \) and \( \langle |R_N|^4 \rangle \) of the reflection coefficient. These moments have previously been discussed within the invariant imbedding model [8]. In order to find moments of \( |R_N|^2 \) for weak disorder we first obtain the exact solution for \( R_N \) linear order in the reduced site energies \( \{ \epsilon \} \) by iterating [9] with the initial condition \( R_0 = 0 \). This yields

\[
R_N = \frac{i}{2 \sin k} \sum_{m=1}^{N} e^{2ik(N-m)\epsilon_m}. \tag{20}
\]

Assuming the \( \epsilon_m \) to be independent gaussian variables with mean zero and correlation

\[
\langle \epsilon_m \epsilon_n \rangle = \frac{\epsilon_0^2}{4} \delta_{m,n}, \tag{21}
\]

we obtain successively from (20):

\[
\langle |R_N|^2 \rangle = \frac{\epsilon_0^2 N}{4 \sin^2 k}, \tag{22}
\]

and

\[
\langle |R_N|^4 \rangle = \frac{\epsilon_0^4}{16 \sin^4 k} \left[ 2N^2 + \left( \frac{\sin 2kN}{\sin 2k} \right)^2 \right]. \tag{23}
\]

In obtaining (23) we have expressed averages of products of four random energies in terms of averages of the form (21) using a well-known property of mutually independent gaussian variables [13]. The corresponding results for a rectangular distribution of site energies between values \(-W\) and \( W \) are obtained by replacing \( \epsilon_0^2/3 \).

The Eq. (22) is identified as usual with Ohm’s law for the Landauer resistance for weak disorder, \( \rho \approx \langle |R_N|^2 \rangle = 2N/L_c \), whose expression in terms of the localization length follows from (12) and the definition \( |T_N| \sim e^{-N/L_c} \), in the limit \( N << L_c \). From (22) we have

\[
L_c = \frac{8 \sin^2 k}{\epsilon_0^2}, \quad E = 2 \cos k, \tag{24}
\]

which coincides with the well-known expression obtained by Thouless [23]. Its continuum limit, \( L_c = 8k^2/\epsilon_0^2 \) reduces to the localization length \( L_c = 2k^2/V_0^2 \) in a continuous gaussian potential with correlation parameter \( V_0 = -\epsilon_0/2 \) (equation 14). In fact, in the continuum limit (\( ka \to 0 \)) the Eqs. (22)(23) coincide with the results obtained previously in an invariant- imbedding treatment [8]. As noted above the continuum limit breaks down near the band edges, \( ka = 0, \pi \). Finally, we recall that the second term on the r.h.s. of (23) describes the effect of the inhomogeneity of the reflection phase distribution [8].

III. PHASE AND DELAY TIME RANDOMNESS FOR STRONG DISORDER

In this Section we discuss a simple limiting case of strongly disordered site energies in the Anderson model. We consider a random tight-binding chain where at every site the potential takes large values close to a fixed magnitude, which are positive or negative at random. This model corresponds to an extreme case of strong disorder since small random site energies are a priori excluded. Thus we assume the independently distributed site energies \( \epsilon_n \) to be described by identical doubly peaked distributions with peaks centered at \( \epsilon_n = \pm \epsilon, \epsilon >> 1 \), with small standard deviation, \( \sigma \ll 1 \). Typical cases of such continuous distributions are the doubly-peaked gaussian

\[
p_c(\epsilon_n) = \frac{1}{2\sigma \sqrt{2\pi}} \left[ e^{-\frac{(\epsilon_n-\epsilon)^2}{2\sigma^2}} + e^{-\frac{(\epsilon_n+\epsilon)^2}{2\sigma^2}} \right], \tag{25}
\]

and the doubly peaked rectangular distribution (within flat peaks centered at \( \epsilon_n = \pm \epsilon \) and standard deviation \( \frac{\epsilon}{\sqrt{3}} \ll 1 \))
where the distribution of $z = 1/\varepsilon_N$ is given in terms of the site energy distributions (23) or (26), by

$$p_z(z) = \frac{1}{z^2} p_z \left( \frac{1}{z} \right) .$$

Finally, from Eqs (23, 31) we readily obtain the phase- and delay time distributions in the form

$$P_\theta(\theta_N) = \frac{2|\sin k|}{\theta_N^3} p_z \left( -\frac{2\sin k}{\theta_N} \right) ,$$

and

$$P_\tau(\tau_N) = \frac{\cot k}{\tau_N^3} p_z \left( \frac{\cot k}{\tau_N} \right) .$$

From Eqs (22) and (23, 24) it follows that only small phases in the neighbourhood of the peak positions, $\theta_N \approx \pm \frac{\sin \varepsilon_N}{\varepsilon_N}$, have appreciable probability in our model of extreme strong disorder. We note that the Anderson model with an ordinary wide single-peak distribution of site energies also leads to a characteristic double peak structure for the phase distribution (see the numerical results in fig. 1(c) of Ref. [4]). However, in contrast to (32), the peak positions in the latter case are close to $\pi$ and their separation is of order unity [4].

Turning to the delay time distribution (33) we observe that delay time probabilities are small except near the short-time maximum $\tau_N \sim \frac{\cot k}{\tau_N}$. In particular, the absence of an asymptotic power law tail, $P_\tau \sim 1/\tau^2$, as in [4], is due to the fact that in the present case an incoming particle (wave) does not penetrate into the disordered sample beyond the outermost site at which it is reflected (equation (28-a)).

Ossipov et al. [4] have also studied numerically the delay time distribution for the Anderson model with strong disorder i.e. for a wide single-peak distribution of site energies. They found a long time tail $P_\tau(\tau) \sim 1/\tau^2$, $\tau \to \infty$, which is believed to be associated with Azbel resonances. From this they concluded that the long-time tail exists regardless of the strength of disorder. This conclusion is partly disproved, however, by our study of the above model of extreme strong disorder. Indeed we find that the long-time probability is strongly reduced by an exponentially small factor, $\exp(-c^2/2\sigma^2) << 1$, due to the fact that the incoming particle is reflected right at the edge of the disordered sample.

Note that, to second order in $1/\nu_N$, $R_N$ has unit modulus, which implies that there is only backscattering to this order. Furthermore, to order $1/\nu_N^2$, $R_N$ is independent of scattering by interior sites $n < N$ and hence independent of the length of the disordered chain. In other words, the second order expression (22) describes reflection in an asymptotic stationary regime, $L >> L_c$.

For simplicity, we restrict to the linear order approximation

$$R_N \simeq -1 - \frac{1}{\nu_N} ,$$

for studying the phase- and delay time distribution, $P_\theta(\theta_N)$ and $P_\tau(\tau_N)$, respectively. From (28-a), (22) and (21) we obtain

$$\theta_N \simeq -\frac{2\sin k}{\varepsilon_N} ,$$

and

$$\tau_N \simeq \frac{\cot k}{\varepsilon_N} .$$

Note that the variances, $\text{var} \varepsilon_n = \sigma^2 + c^2$, $\text{var} \varepsilon_n = W^2/3 + c^2$, $c >> 1$, of (23) and (24), respectively, characterize the limit of strong disorder in the present case as do large variances in the corresponding single-peak distributions (obtained by letting $c = 0$), $\text{var} \varepsilon_n = \sigma^2$, with $\sigma >> 1$, and $\text{var} \varepsilon_n = W^2/3$, with $W/\sqrt{3} >> 1$, in the usual Anderson model. The advantage of the above model of strong disorder is that, due to the fact that typical values of the site energies $\varepsilon_n$ are large, we may use perturbation theory in $1/\varepsilon_n$ for studying effects of the disorder.

We now analyse the reflection phase- and delay time distributions for the strongly disordered Anderson model with the distributions (23) of site energies. We shall discuss our analytical expressions in relation to recent numerical results for these distributions in the ordinary Anderson model for strong disorder [3]. The probability distribution of the phase of the reflection amplitude, $R_n = |R_n|e^{i\theta_n}$ and the corresponding distribution of the Wigner delay time, $\tau_n = d\theta_n/dE$, will be found from the explicit solution of the stochastic recursion relation (3). For strong disorder ($c >> 1, \sigma << 1$ or $W/\sqrt{3} << 1$) we obtain by successive iteration of (3)

$$R_N = \frac{1}{\nu_N^2} - \frac{1}{\nu_N} + \frac{1}{\nu_N^2} R_{N-1}^2 + \theta(\nu_N^3) ,$$

and

$$R_N = \frac{1}{\nu_N} - \frac{1}{2\nu_N^3} \sin k + \theta(\nu_N^3, \nu_N^2, \nu_N^{-1}) .$$

Note that, to second order in $1/\nu_N$, $R_N$ has unit modulus, which implies that there is only backscattering to this order. Furthermore, to order $1/\nu_N^2$, $R_N$ is independent of scattering by interior sites $n < N$ and hence independent of the length of the disordered chain. In other words, the second order expression (22) describes reflection in an asymptotic stationary regime, $L >> L_c$.

For simplicity, we restrict to the linear order approximation

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$$\theta_N \simeq -\frac{2\sin k}{\varepsilon_N} ,$$

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$$\tau_N \simeq \frac{\cot k}{\varepsilon_N} .$$

Turning to the delay time distribution (33) we observe that delay time probabilities are small except near the short-time maximum $\tau_N \sim \frac{\cot k}{\tau_N}$. In particular, the absence of an asymptotic power law tail, $P_\tau \sim 1/\tau^2$, as in [4], is due to the fact that in the present case an incoming particle (wave) does not penetrate into the disordered sample beyond the outermost site at which it is reflected (equation (28-a)).

Ossipov et al. [4] have also studied numerically the delay time distribution for the Anderson model with strong disorder i.e. for a wide single-peak distribution of site energies. They found a long time tail $P_\tau(\tau) \sim 1/\tau^2$, $\tau \to \infty$, which is believed to be associated with Azbel resonances. From this they concluded that the long-time tail exists regardless of the strength of disorder. This conclusion is partly disproved, however, by our study of the above model of extreme strong disorder. Indeed we find that the long-time probability is strongly reduced by an exponentially small factor, $\exp(-c^2/2\sigma^2) << 1$, due to the fact that the incoming particle is reflected right at the edge of the disordered sample.
TABLE I. Classification of literature results for $P_d(\theta)$ pertaining (a) to the low reflection regime, $L(N) \ll L_c$, and (b) to the strong reflection regime, $L(N) \gg L_c$, in the invariant imbedding model and in the Anderson model for weak disorder.

(a) $L(N) \ll L_c$

| Anderson Model | Reference | Invariant imbedding | Reference |
|----------------|-----------|---------------------|-----------|
| fig. 2(a), $N = 10$, $k = 0.46\pi$, $L_c \approx 6.4 \times 10^4$ | 2 | fig. 1, $kL = 5$ | 8 |
| fig. 2(b), $N = 50$, $k = 0.46\pi$, $L_c \approx 2.4 \times 10^7$ | 2 | fig. 1, $kL = 10$ | 8 |
| fig. 2(c), $N = 10$, $k = 0.46\pi$, $L_c \approx 2.4 \times 10^3$ | 2 | fig. 1, $kL = 100$ | 8 |
| fig. 2(d), $N = 50$, $k = 0.46\pi$, $L_c \approx 6.4 \times 10^3$ | 2 | fig. 1, $kL = 1000$ | 8 |
| fig. 1(b), $k = \pi/2$, $L_c \approx 2.4 \times 10^3$ | 4 | | |

(b) $L(N) >> L_c$

| Anderson Model | Reference | Invariant imbedding | Reference |
|----------------|-----------|---------------------|-----------|
| fig. 1(a), $N = 10^4$, $k = 2\pi/3$, $L_c = 290$ | 2 | fig. 1, $kL_c = 10$ | 10 |
| fig. 1(b), $N = 10^4$, $k = \pi/2$, $L_c = 390$ | 2 | fig. 1, $kL_c = 20$ | 10 |
| fig. 1(a), $k = \sqrt{\pi}$, $L_c \approx 60$ | 4 | fig. 1, $kL_c = 100$ | 10 |
| fig. 1(d), $k = 0.001\sqrt{\pi}$, $L_c < 10^{-6}$ | 4 | fig. 1, $kL_c = 200$ | 10 |
| fig. 6(a), $kL_c = 0.00001$ | 12 | | |
| fig. 6(a), $kL_c = 0.0001$ | 12 | | |
| fig. 6(b), $kL_c = 0.01$ | 12 | | |
| fig. 6(c), $kL_c = 10$ | 12 | | |
| fig. 6(c), $kL_c = 100$ | 12 | | |
| fig. 6(c), $kL_c = 1000$ | 12 | | |