Derivations of the Planck Blackbody Spectrum from Thermodynamic Ideas in Classical Physics with Classical Zero-Point Radiation

Timothy H. Boyer

Department of Physics, City College of the City University of New York, New York, New York 10031

Abstract

Based upon thermodynamic ideas, two new derivations of the Planck blackbody spectrum are given within classical physics which includes classical zero-point radiation. The first and second laws of thermodynamics, applied to a harmonic oscillator or a radiation normal mode, require that the canonical potential $\phi(\omega/T)$ is a function of a single variable corresponding to the ratio of the oscillation frequency to the temperature. The second law of thermodynamics involves extremum ideas which may be applied to thermal radiation. Our first derivation of the Planck spectrum is based upon the idea that the canonical potential $\phi(\omega/T)$ is a monotonic function and all its derivatives are monotonic when interpolating between zero-point energy at low temperature and energy equipartition at high temperature; the monotonic behavior precludes the canonical potential from giving a preferred value for the ratio $\omega/T$. Our second derivation of the Planck spectrum is based upon the requirement that the change in the Helmholtz free energy of the radiation in a partitioned box held at constant temperature should be a minimum at thermal equilibrium. Finally, the change in Casimir energy with change in partition position for the radiation in a partitioned box is shown to correspond at high temperature to the absence of zero-point energy when the spectral energy per normal mode is chosen as the traditional Planck spectrum which omits zero-point energy at low temperature; thus the idea of zero-point energy is embedded in the traditional Planck spectrum. It is emphasized that thermal radiation is intimately connected with zero-point radiation and the structure of spacetime in classical physics.
I. INTRODUCTION

A. False Claims in the Physics Literature

The physics literature claims that attempts to explain the blackbody spectrum within classical physics illustrate the breakdown of classical physics. However, this claim that the blackbody spectrum cannot be explained within classical physics is simply erroneous. There have been a number of valid derivations of the blackbody radiation spectrum within classical physics. In the present article, we present two new derivations from thermodynamic points of view.

The claims in the physics literature that classical physics cannot explain the blackbody spectrum are a century out of date because they fail to consider the two crucial aspects needed for understanding the phenomenon. These missing aspects include: 1) the presence of classical electromagnetic zero-point radiation, and 2) the importance of special relativity. The experimentally observed Casimir forces between conducting parallel plates indicate unambiguously the presence of classical electromagnetic zero-point radiation with a Lorentz-invariant spectrum. Of course, those physicists who prefer to discuss physics within the context of quantum theory will describe the Casimir forces in terms of quantum zero-point radiation. However, if one is working within classical theory, then the presence of Casimir forces requires the presence of classical electromagnetic zero-point radiation. In order to fit the experimental data on Casimir forces, the spectrum of classical electromagnetic zero-point radiation must be Lorentz-invariant, scale invariant, and indeed conformal invariant.

The one free parameter regarding classical zero-point radiation is the multiplicative scale factor which is chosen to fit the experimental data; the scale factor gives an energy per normal mode of \((1/2)\hbar \omega\) where \(\omega\) is the angular frequency of the mode and \(\hbar\) is a numerical constant which takes the same value as Planck’s constant.

B. The Influence of Classical Zero-Point Radiation

The presence of classical electromagnetic zero-point radiation will influence all phenomena to a greater or lesser extent. Since there is classical electromagnetic zero-point radiation present in the universe according to classical theory, then classical statistical mechanics (with its assumption that all motion stops at the absolute zero of temperature) is no longer
valid because zero-point radiation drives all electromagnetic systems into random oscillation; rather, classical statistical mechanics can be regarded as simply a large-mass-low-velocity approximation to thermal behavior where the influence of classical zero-point radiation is small. Accordingly, and contrary to what is claimed in the physics literature, the Rayleigh-Jeans law for thermal radiation is not the unique result of classical theory, but is merely the spectrum holding at long wavelength and low temperatures where zero-point radiation has small influence.

Starting from the presence of classical electromagnetic zero-point radiation, several derivations of Planck’s spectrum for blackbody radiation have been given. These include discussion of the motion of a dipole oscillator in a box (analogous to the discussion of Einstein and Stern), the treatment of thermal fluctuations above the zero-point fluctuations, the use of free-particle diamagnetism in the large-mass-low-velocity limit when classical zero-point radiation is present, and use of a time-dilating conformal transformation of classical zero-point radiation in a Rindler frame.

C. Two New Derivations of the Blackbody Spectrum within Classical Physics

In the present article, we offer two new derivations of the blackbody radiation spectrum within classical physics based upon thermodynamic ideas. Both derivations depend upon the presence of classical electromagnetic zero-point radiation.

The first part of the thermodynamic analysis is used subsequently in both derivations. We start by applying the first two laws of thermodynamics to a harmonic oscillator system and find that all the thermodynamic functions for the oscillator depend upon one unknown canonical potential function \( \phi(\omega/T) \) depending upon the single variable \( \omega/T \) corresponding to the ratio of the harmonic oscillator frequency \( \omega \) to the temperature \( T \). The energy \( U(\omega, T) = -\omega \phi'(\omega/T) \) of the oscillator has zero-point energy and energy equipartition as its asymptotic limits, and the full thermal behavior corresponds to the interpolating function between these two limits. The interpolating function must be determined in connection with the extremum ideas of the second law of thermodynamics.

Our first derivation of the Planck spectrum for a harmonic oscillator is based upon the assumption of “thermodynamic smoothness,” that the canonical potential function \( \phi(\omega/T) \) is monotonic and that all its derivatives are monotonic, so as to remove the possibility of a
preferred value for \( \omega/T \). The assumption about monotonic behavior sharply restricts the class of functions allowed in the interpolation, and it is possible to pick out the interpolation function from the restricted class of functions. The Planck spectrum indeed satisfies the required condition.

Our second derivation applies the minimum principle for the Helmholtz free energy to the thermal scalar radiation trapped in a one-dimensional box with a partition. If we choose a test interpolating function \( \phi_t(\omega/T) \) for a single radiation mode, then we can calculate the functional dependence for the change in the Helmholtz free energy (involving infinitely many modes in the box) for fixed temperature as the position of the partition is altered. We see that most interpolation functions \( \phi_t(\omega/T) \) for a mode do not satisfy the minimum principle for the Helmholtz free energy for the partition at the center of the box for a box of arbitrary length. However, we find that indeed the Planck spectrum satisfies the required minimum principle for all box lengths.

Finally, we calculate the change in Casimir energy for a partitioned box. We show that the traditional Planck spectrum which omits zero-point radiation at low temperature does not go over at high temperature to the expected Rayleigh-Jeans result. On the other hand, the full Planck spectrum which includes zero-point radiation at low temperature does indeed go to the Rayleigh-Jeans result at high temperature.

II. SECOND LAW OF THERMODYNAMICS AND ZERO-POINT ENERGY

A. The First and Second Laws of Thermodynamics Applied to the Harmonic Oscillator

We start by considering the thermodynamics of the harmonic oscillator, since a small harmonic oscillator comes to equilibrium with thermal radiation at the same average energy as the radiation normal mode at the same frequency as the oscillator.\footnote{11} Alternatively, we can think of a radiation mode as behaving like a harmonic oscillator.

Now the thermodynamics of a harmonic oscillator has only two thermodynamic variables \( T \) and \( \omega \), and takes a particularly simple form.\footnote{12} In thermal equilibrium with a bath, the average oscillator energy \( \langle \mathcal{E} \rangle \) is denoted by \( U = \langle \mathcal{E} \rangle = \langle J \rangle \omega \), and satisfies \( dQ = dU + dW \) with the entropy \( S \) satisfying \( dS = dQ/T \). Since \( J \) is an adiabatic invariant
for the oscillator, the work done by the system is given by \( dW = - \langle J \rangle d\omega = -(U/\omega)d\omega \). Combing these equations, we have \( dS = dQ/T = [dU - (U/\omega)d\omega]/T \). Writing the differentials in terms of \( T \) and \( \omega \), we have \( dS = (\partial S/\partial T)_{\omega}dT + (\partial S/\partial \omega)_{T}d\omega \) and \( dU = (\partial U/\partial T)_{\omega}dT + (\partial U/\partial \omega)_{T}d\omega \). Therefore \( (\partial S/\partial T)_{\omega} = (\partial U/\partial T)_{\omega}/T \) and \( (\partial S/\partial \omega)_{T} = [(\partial U/\partial \omega)_{T} - (U/\omega)]/T \). Now equating the mixed second partial derivatives \( \partial^{2}S/\partial T \partial \omega = \partial^{2}S/\partial \omega \partial T \), we have \( (\partial^{2}U/\partial \omega \partial T)/T = (\partial^{2}U/\partial T \partial \omega)/T - (\partial U/\partial T)_{\omega}/(T\omega) + [(U/\omega) - (\partial U/\partial \omega)_{T}]/T^{2} \) or \( 0 = -(\partial U/\partial T)_{\omega}/(T\omega) + [(U/\omega) - (\partial U/\partial \omega)_{T}]/T^{2} \). The general solution of this equation is \( U = \omega f(\omega/T) = \omega \langle J \rangle \) where \( f(\omega/T) \) is an unknown function which corresponds to the average value \( \langle J \rangle \) of the action variable of the oscillator. If we had equated the mixed partial derivatives of the energy, then we find the equation \( (\partial S/\partial \omega)_{T} = -(T/\omega)(\partial S/\partial T)_{\omega} \), which has the general solution \( S(\omega, T) = g(\omega/T) \) where \( g \) is an arbitrary function. The information provided by the second law of thermodynamics is that there is a single function \( \phi(\omega/T) \) corresponding to the canonical potential function which gives the Helmholtz free energy as

\[
F(\omega, T) = -T\phi(\omega/T),
\]

the average oscillator energy as

\[
U(\omega, T) = T^{2}\left( \frac{\partial \phi}{\partial T} \right)_{\omega} = -\omega\phi'(\omega/T),
\]

and the entropy as

\[
S(\omega/T) = \phi(\omega/T) + U(\omega, T)/T = \phi(\omega/T) - (\omega/T)\phi'(\omega/T).
\]

Thus the thermodynamics of the harmonic oscillator is determined by one unknown function \( \phi(\omega/T) \). When applied to thermal radiation, the result obtained here purely from the first and second laws of thermodynamics corresponds to the familiar Wien displacement law of classical physics.

### B. Possibility of Zero-Point Energy and Zero-Point Radiation

The energy expression \( \text{(2)} \) for a harmonic oscillator (or an electromagnetic radiation mode) in thermal equilibrium allows two limits which make the energy independent from one of its two thermodynamic variables. When the temperature \( T \) becomes very large
so that the ratio \((\omega/T)\) is small, the average energy \(U\) of the mode in Eq. (2) becomes independent of the frequency \(\omega\) provided \(\phi'(\omega/T) \to -\text{const}_1 \times (\omega/T)^{-1}\) so that

\[
U = -\omega \phi'(\omega/T) \to -\omega \times [-\text{const}_1 \times T/\omega] = \text{const}_1 \times T \quad \text{for} \quad \omega/T << 1. \tag{4}
\]

This is the familiar high-temperature limit where we expect to recover the Rayleigh-Jeans equipartition limit. Therefore we choose this constant as \(\text{const}_1 = k_B\) corresponding to Boltzmann’s constant. With this choice, our thermal radiation now goes over to the Rayleigh-Jeans limit for high temperature or low frequency.

In the other limit of small temperature where the ratio \(\omega/T\) is large, the dependence on temperature is eliminated provided \(\phi'(\omega/T) \to -\text{const}_2\), so that

\[
U = -\omega \phi'(\omega/T) \to -\omega \times [-\text{const}_2] = \text{const}_2 \times \omega \quad \text{for} \quad \omega/T >> 1. \tag{5}
\]

At this point, any theoretical description of thermal radiation involves a choice, which should be based on experimental observation. If we choose this second constant to vanish, \(\text{const}_2 = 0\), then this limit does not force us to introduce any constant beyond Boltzmann’s constant, which entered for the high-temperature limit of thermal radiation. On the other hand, if we choose a non-zero value for this constant, \(\text{const}_2 \neq 0\), then we are introducing a second constant into the theory of thermal radiation, which constant has different dimensions from those of Boltzmann’s constant. The units of this new constant \(\text{const}_2\) correspond to energy times time. Furthermore, the choice of a non-zero value for this constant means that at temperature \(T = 0\), there is random, temperature-independent energy present in the harmonic oscillator. If this harmonic oscillator has electromagnetic interactions, it must be in equilibrium with the radiation in the thermal bath, and therefore random zero-point radiation must be present in the system. This random radiation which exists at temperature \(T = 0\) is classical electromagnetic zero-point radiation.

We emphasize that thermodynamics allows classical zero-point radiation within classical physics. The physicists of the early 20th century were not familiar with the idea of classical zero-point radiation, and so they made the assumption \(\text{const}_2 = 0\) which excluded the possibility of classical zero-point radiation. In his monograph on classical electron theory, Lorentz\(^\text{[14]}\) makes the explicit assumption that there is no radiation present at \(T = 0\). Today, we know that the exclusion of classical zero-point radiation is a poor choice. However, the
current textbooks of modern physics continue to present only the outdated, century-old view.\[1\]

Once the possibility of classical zero-point radiation is introduced into classical theory, one looks for other phenomena where the zero-point radiation will play a crucial role. In particular, the Casimir force\[6\] between two uncharged conducting parallel plates will be influenced by the presence of classical electromagnetic zero-point radiation. By comparing theoretical calculations with experiments, one finds that the scale constant for classical zero-point radiation appearing in Eq. (5) must take the value $const_2 = 1.05 \times 10^{-34}$ Joule-sec. However, this value corresponds to the value of a familiar constant in physics; it corresponds to the value $h/2$ where $h$ is Planck’s constant. Thus in order to account for the experimentally observed Casimir forces between parallel plates, the scale of classical zero-point radiation must be such that $const_2 = h/2$, and for each normal mode, the average energy becomes

$$U = -\omega \phi'(\omega/T) \rightarrow (h/2)\omega \text{ for } T \rightarrow 0. \quad (6)$$

We emphasize that Planck’s constant enters classical electromagnetic theory as the scale factor in classical electromagnetic zero-point radiation. There is no connection whatsoever to any idea of quanta. Many physicists are misled by the textbooks of modern physics and regard Planck’s constant as a “quantum constant.”\[15\] This is a completely misleading idea. A physical constant is a numerical value associated with certain aspects of nature; the constant may appear in several different theories, just as Cavendish’s constant $G$ appears in both Newtonian physics and also in general relativity. Indeed, Planck’s constant $h = 2\pi \hbar$ was introduced into physics in 1899 before the advent of quantum theory.\[16\] Planck’s constant can appear in both classical and quantum theories.

III. DERIVATION OF THE PLANCK SPECTRUM BASED UPON THE IDEA OF THERMODYNAMIC SMOOTHNESS

A. Choosing Constants Such that $k_B = 1$ and $h = 1$

When dealing with the thermodynamics of the harmonic oscillator, it is convenient to absorb Boltzmann’s constant $k_B$ into the definition of temperature and to absorb Planck’s constant into the definition of frequency.\[17\] In this convention, the two constants become
\( const_1 = 1 \) and \( const_2 = 1/2 \). In the thermodynamic review above, we see that the thermodynamics of the harmonic oscillator, and therefore of the blackbody radiation spectrum is determined by one unknown function \( \phi(z) \) where \( z = \omega/T \) which has the asymptotic limits for its derivative given by

\[
\phi'(z) \to -z^{-1} \text{ for } z \to 0 \quad \text{and} \quad \phi'(z) \to -1/2 \text{ for } z \to \infty. \tag{7}
\]

The function \( \phi \) itself then has the asymptotic limits determined by integrating once, giving for \( -\phi(z) \)

\[
-\phi(z) \to \ln z \text{ for } z \to 0 \quad \text{and} \quad -\phi(z) \to z/2 \text{ for } z \to \infty \tag{8}
\]

plus possible constants.

**B. Thermodynamic Smoothness Applied to the Harmonic Oscillator**

In obtaining the results in Eqs. (7) and (8), we have used the first and second laws of thermodynamics including the idea of an entropy function \( S \) which is a state function. However, the analysis does not include the concept that the entropy function assumes a maximum value associated with stability. This stability idea includes the notion of thermodynamic smoothness which demands that the canonical potential for the oscillator does not distinguish any frequency \( \omega \) at a given temperature \( T \), nor any temperature at a given frequency. At a minimum, the notion of smoothness demands that any interpolation function \( \phi(\omega/T) = \phi(z) \) for the canonical potential of the oscillator is monotonic and all its derivatives are monotonic; the monotonic behavior prevents a single value for \( \omega/T \) from being distinguished by the canonical potential.

Now the set of functions which are monotonic and all of whose derivatives are monotonic is extremely limited. The set includes \( x, e^x, \sinh x, \cosh x, \tanh x \), their inverses and powers. In particular, we notice that the hyperbolic sine function has the asymptotic limits

\[
2\sinh(z/2) \to z \text{ for } z \to 0 \quad \text{while} \quad 2\sinh(z/2) \to e^{z/2} \text{ for } z \to \infty \tag{9}
\]

But this looks like exactly the exponentiation of the interpolation limits in Eq. (8) which we required for the canonical potential function \( \phi(z) \). This suggests that the needed smooth interpolation function \( \phi_{\text{cp}} \) is given by

\[
\phi_{\text{cp}}(z) = -\ln[2\sinh(z/2)] \tag{10}
\]
We can check that $\phi_{Pz}(z)$ given in Eq. (10) is indeed monotonic and all its derivatives are monotonic. The function $\phi_{Pz}(z)$ is clearly monotonic since both $\ln x$ and $\sinh x$ are monotonic for $0 < x$. The first derivative of $\phi_{Pz}(z)$ is

$$\phi'_{Pz}(z) = -(1/2) \coth(z/2)$$

(11)

This function also is monotonic, and, since $\coth x = 1/x + x/3 - x^3/45 + 2x^5/495 - ... \to 1/x$ for $x \to 0$, while $\coth x \to 1$, for $x \to \infty$; thus we see that $\phi'_{Pz}(x)$ in Eq. (11) has asymptotic limits in agreement with Eq. (7).

It is clear that we need to prove that all the derivatives of $\coth x$ are monotonic. One method of proof uses an exponential expansion,

$$-\phi'_{Pz}(z) = \frac{1}{2} \coth \left( \frac{z}{2} \right) = \frac{1}{2} + \frac{1}{\exp(z) - 1} = \frac{1}{2} + \exp(-z) \frac{1}{1 - \exp(-z)}$$

$$= \frac{1}{2} + e^{-z} + e^{-2z} + e^{-3z} + ...$$

(12)

which involves a constant function and then a sum of functions all of which are monotonically decreasing, so that the function is monotonically decreasing in $z$. By the ratio test, the series is absolutely convergent since $0 < e^{-z} < 1$ for $0 < z$. The second derivative gives

$$-\phi''_{Pz}(z) = -e^{-z} - 2e^{-2z} - 3e^{-3z} - ...$$

(13)

and again all of the terms are of the same (negative) sign, and all are monotonically decreasing in magnitude so the that function is monotonically increasing. Again by the ratio test, the series is absolutely convergent since $0 < [(n + 1)/n]e^{-z} < 1$ for sufficiently large $n$ for fixed $z$, $0 < z$. Indeed, it is easy to see that the pattern is repeated upon further differentiation, so that the series expansion in terms of exponentials is absolutely convergent by the ratio test and all derivatives of the canonical potential $-\phi_{Pz}(z)$ are monotonic.

C. Smooth Interpolation Gives the Planck Function

Now the canonical potential $\phi_{Pz}$ for the harmonic oscillator which we have obtained in Eq. (10) by assuming monotonic behavior between the asymptotic limits is exactly that corresponding to the Planck formula including zero-point energy. The Helmholtz free energy
corresponding to the Planck spectrum with zero-point energy for a harmonic oscillator or a radiation mode of frequency $\omega$ is given by

$$F_{Pzp}(\omega, T) = -T \phi_{Pzp}(\omega/T) = T \ln\{2 \sinh[\omega/(2T)]\} \quad (14)$$

The associated energy $U_{Pzp}(\omega, T)$ follows as

$$U_{Pzp}(\omega, T) = T^2 \left( \frac{\partial \phi_{Pzp}}{\partial T} \right)_\omega = -\omega \phi'_{Pzp}(\omega/T) = \frac{\omega}{2} \coth \left( \frac{\omega}{2T} \right) = \frac{\omega}{2} + \frac{\omega}{\exp(\omega/T) - 1} \quad (15)$$

and the entropy $S_{Pzp}(\omega/T)$ as

$$S_{Pzp}(\omega/T) = \phi_{Pzp}(\omega/T) + U(\omega, T)/T = \phi_{Pzp}(\omega/T) - (\omega/T)\phi'_{Pzp}(\omega/T)$$

$$= -\ln \left[ 2 \sinh \left( \frac{\omega}{2T} \right) \right] + \frac{\omega}{2T} \coth \left( \frac{\omega}{2T} \right) \quad (16)$$

The zero-point energy actually makes no contribution to the entropy $S_{Pzp}$ since in the limit of large $z$, $\phi_{Pzp}(z) \to \phi_{zp}(z) = -z/2$ while

$$S_{zp}(z) = \phi_{zp}(z) - z\phi'_{zp}(z) = -z/2 - z(-1/2) = 0. \quad (17)$$

D. Entropy as a Monotonic Function of $U_{Pzp}/(\hbar \omega/2)$

Since the first law of thermodynamics requires that the entropy $S(\omega/T)$ for a harmonic oscillator is a function of the single variable $\omega/T$, and also the energy of the oscillator is given by $U = -\omega \phi'(\omega/T)$ as in Eq. (2), it follows (by using the inverse function of $\phi'(\omega/T)$) that the oscillator entropy $S$ can also be regarded as a function of the single variable $U/\omega$; The variable $U/\omega$ runs from the constant value $U/\omega \to 1/2$ when $\omega >> T$, to the value $U/\omega \to T/\omega$ when $T >> \omega$. We expect the oscillator entropy to be a monotonically increasing function of temperature. Furthermore, the entropy should not distinguish any preferred value of $U/\omega$. Thus we expect that the oscillator entropy should be a monotonic function of $U/\omega$ and all its derivative should be monotonic functions of $U/\omega$. Indeed for the Planck relation given in Eq. (10), we can use the fact that the inverse function for $y = \coth(z/2)$ is

$$\frac{z}{2} = \text{arc} \coth(y) = \frac{1}{2} \ln \left( \frac{y + 1}{y - 1} \right) \quad \text{for} \quad y^2 > 1 \quad (18)$$

where $y = U/(\omega/2)$ and $z = \omega/T$ to obtain

$$\frac{S_{Pzp}(y)}{\hbar k_B} = \frac{1}{2} (y + 1) \ln (y + 1) - \frac{1}{2} (y - 1) \ln (y - 1) - \ln 2 \quad \text{where} \quad y = U_{Pzp}/(\hbar \omega/2) \quad (19)$$
By direct differentiation of the expression in Eq. (19), it is easy to show that the Planck oscillator entropy $S_{Pzp}$ in Eq. (19) is indeed a monotonic function of $U/\omega$ and all the derivatives are monotonic functions.

This concludes our first derivation of the Planck blackbody spectrum based upon thermodynamic ideas. For this derivation, we have discussed merely the thermodynamics of a harmonic oscillator. For the second derivation of the blackbody spectrum, we need to discuss radiation explicitly.

IV. RELATIVISTIC SCALAR FIELD THEORY IN ONE SPATIAL DIMENSION

A. Valid Thermodynamic Systems

Blackbody radiation is the random radiation in an enclosure which is stable under scattering. For a system involving radiation, there are infinitely many normal modes of oscillation for the radiation so that the radiation must be treated in the context of a relativistic field theory.

In order to understand blackbody radiation, we must choose models which do not violate the principles of thermodynamics. There are several models mentioned in the literature which obviously do violate the laws of thermodynamics. The use of a Maxwell demon is the most famous example. However, the use of nonrelativistic statistical mechanics with its energy equipartition for each harmonic oscillator mode obviously violates the laws of thermodynamics when applied to thermal radiation since it gives an ultraviolet divergence for the energy. Similarly, using a nonrelativistic nonlinear dipole oscillator as a radiation scatterer also violates the laws of thermodynamics when applied to thermal radiation since the oscillator scatters the radiation toward the equipartition result.\[19\] Although most physicists repeat the claim that these thermodynamic failures arise due to the use of classical rather than quantum physics, it has been suggested repeatedly and with ever more convincing evidence that, insofar as classical physics is concerned, the failure involves the invalid use of nonrelativistic physics together with a relativistic radiation system.\[20\] The naive combination of nonrelativistic and relativistic physics leads to systems which violate the laws of physics. Indeed, if we consider lifting a nonrelativistic harmonic oscillator system within a relativistic accelerating Rindler frame, it is easy to show that this mixture
of nonrelativistic and relativistic physics violates the laws of thermodynamics. In order to have a valid thermodynamic system for relativistic radiation, we must insist that the interactions of the radiation system do not violate any aspects of relativity.

In our next derivation of the blackbody radiation spectrum, we will use ideas which are usually associated with Casimir forces. We will consider the thermodynamic system involving relativistic radiation in one spatial dimension in a box which contains a partition. The radiation and boundary conditions provide a fully relativistic system.

B. Scalar Field Theory

For simplicity of calculation, we will use relativistic scalar radiation in one spatial dimension. The Lorentz-invariant spacetime interval is $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ with indices $\mu = 0, 1$, and $x^0 = ct, \ x^1 = x$, so that $ds^2 = c^2dt^2 - dx^2$ The Lagrangian density for the massless scalar field $\phi$ is given by

$$L = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi = \frac{1}{2} \left[ \left( \frac{\partial \phi}{\partial ct} \right)^2 - \left( \frac{\partial \phi}{\partial x} \right)^2 \right]$$

(20)

the stress-energy-momentum tensor density is

$$T^{\mu\nu} = \partial^\mu \phi \frac{\partial L}{\partial (\partial_\nu \phi)} - g^{\mu\nu} L$$

(21)

giving energy density

$$u = T^{00} = \frac{1}{2} \left[ \frac{1}{c^2} \left( \frac{\partial \phi}{\partial t} \right)^2 + \left( \frac{\partial \phi}{\partial x} \right)^2 \right]$$

(22)

and momentum density

$$T^{01} = T^{10} = -\frac{1}{c} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x}$$

(23)

The equation of motion for the field corresponds to $\partial_\mu [\partial \phi / (\partial_\nu \phi)] = 0$

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} = 0.$$ 

(24)

If we choose to express the field $\phi(ct, x)$ in a box running from $x = a$ to $x = b$ as a sum over normal modes which vanish at the ends (Dirichlet boundary conditions), then

$$\phi(ct, x) = \sum_{n=1}^{\infty} \varphi_n(ct, x) = \sum_{n=1}^{\infty} q_n(t) \left( \frac{2}{b - a} \right)^{1/2} \sin \left[ \frac{n\pi}{b - a} (x - a) \right]$$ 

(25)
where (using the orthogonality of the spatial normal mode functions) the amplitude $q_n$ of the $n$th normal mode satisfies the differential equation

$$\ddot{q}_n + \omega_n^2 q_n = 0$$

with

$$\omega_n = \frac{n\pi c}{b - a}$$

This same frequency relation (27) arises if we require that the first spatial derivatives of the field vanish at the ends of the box (Neumann boundary conditions), so that a cosine function replaces the sine function in Eq. (25). On the other hand, if we choose to express the field $\varphi(ct, x)$ in terms of normal modes vanishing at $x = a$ (Dirichlet boundary conditions) but with first spatial derivative vanishing at $x = b$, (Neumann boundary conditions) then

$$\varphi(ct, x) = \sum_{n=1}^{\infty} \varphi_n(ct, x) = \sum_{n=1}^{\infty} q_n(t) \left( \frac{2}{b - a} \right)^{1/2} \sin \left( \frac{(n - 1/2)\pi}{b - a} (x - a) \right)$$

where the amplitude $q_n$ again satisfies the harmonic oscillator differential equation (26), but the frequency is now

$$\omega_n = \frac{(n - 1/2)\pi c}{b - a}.$$ 

When the boundary conditions (Dirichlet or Neumann) are the same at both ends of the box, we speak of “like boundary conditions;” if the Dirichlet boundary conditions are used at one end of the box and Neumann boundary conditions at the other, we speak of “unlike boundary conditions.” The energy of the radiation in the box is given by

$$U = \int_{x=a}^{x=b} dx \frac{1}{2} \left[ \frac{1}{c^2} \left( \frac{1}{c^2} \frac{\partial \varphi}{\partial t} \right)^2 + \left( \frac{\partial \varphi}{\partial x} \right)^2 \right]$$

$$= \sum_{n=1}^{\infty} \mathcal{E}_n = \sum_{n=1}^{\infty} \frac{1}{2} (q_n^2 + \omega_n^2 q_n^2).$$

Thus each normal mode of the radiation field behaves like a harmonic oscillator.

V. DERIVATION OF THE PLANCK SPECTRUM BASED UPON THE HELMHOLTZ FREE ENERGY FOR RADIATION IN A PARTITIONED BOX

A. Thermal Radiation in a Box

The simple harmonic oscillator equation of motion in (26) can be solved as $q_n(t) = f_n \cos(\omega_n t - \theta_n)$ where $f_n$ gives the amplitude of the oscillation and $\theta_n$ gives the phase. In
the case of thermal radiation in a box, the phases $\theta_n$ of the normal modes of oscillation are completely uncorrelated so that we may write the radiation field as

$$\varphi(ct, x) = \sum_{n=1}^{\infty} \varphi_n(ct, x) = \sum_{n=1}^{\infty} f_n \left( \frac{2}{b-a} \right)^{1/2} \sin \left[ \frac{\omega_n}{c} (x-a) \right] \cos [\omega_n t - \theta_n]$$  

(31)

where the phases $\theta_n$ are random variables which are independently distributed for each normal mode $n$. Thus when averaged in time or averaged over the random phases, the averages involve

$$\langle \sin(\omega_n t + \theta_n) \sin(\omega_n' t + \theta_n') \rangle = \langle \cos(\omega_n t + \theta_n) \cos(\omega_n' t + \theta_n') \rangle = (1/2)\delta_{nn'}$$  

(32)

while

$$\langle \sin(\omega_n t + \theta_n) \cos(\omega_n' t + \theta_n') \rangle = 0$$  

(33)

The amplitudes $f_n$ of the normal modes take values which are characteristic of the frequency $\omega_n$ of the mode and the temperature $T$ of the box. Thus for thermal radiation, the average energy density $u_n(x)$ of each normal mode $\varphi_n(cx, t)$ contributes separately to the average energy density in the box, corresponding to

$$\langle u_n(x) \rangle = \frac{1}{2} \left\langle \frac{1}{c^2} \left( \frac{\partial \varphi_n}{\partial t} \right)^2 + \left( \frac{\partial \varphi_n}{\partial x} \right)^2 \right\rangle$$

$$= \frac{1}{2} f_n^2 \left( \frac{\omega_n}{c} \right)^2 \left( \frac{2}{b-a} \right) \frac{1}{2} \left\{ \sin^2 \left[ \frac{\omega_n}{c} (x-a) \right] + \cos^2 \left[ \frac{\omega_n}{c} (x-a) \right] \right\}$$

$$= \frac{1}{2} f_n^2 \left( \frac{\omega_n}{c} \right)^2 \left( \frac{2}{b-a} \right) \frac{1}{2}$$  

(34)

Thus the average energy density is uniform across the box for each normal mode, and the average total energy is given by

$$U = \sum_{n=1}^{\infty} \langle E_n \rangle = (b-a) \sum_{n=1}^{\infty} \frac{1}{2} f_n^2 \left( \frac{\omega_n}{c} \right)^2 \left( \frac{2}{b-a} \right) \frac{1}{2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{2} f_n^2 \left( \frac{\omega_n}{c} \right)^2$$  

(35)

B. Classical Zero-Point Radiation in a Partitioned Box

The spectrum of zero-point energy corresponds to an average energy per normal mode given by $U_{zp}(\omega) = (1/2)\hbar\omega$. Since we are using units where $\hbar = 1$, the zero-point energy
simplifies to $U_{zp}(\omega) = \omega/2$. Classical electrodynamics, which involves a field theory in three spatial dimensions, is invariant under Lorentz transformations and under a single scale transformation characterized as $\sigma_{lU^{-1}}$ where the scale parameter $\sigma$ ranges over all positive values. The scale transformation $\sigma_{lU^{-1}}$ carries all lengths $l$ into $l' = \sigma l$, all times $t$ into $t' = \sigma t$, and all energies $U$ into energies $U' = U/\sigma$. Such a scale transformation preserves the values of the fundamental constants $c$ (the speed of light in vacuum) having units of length/time, $e$ (the charge of the electron) whose square has units of energy times length, and $\hbar$ (the scale factor for zero-point radiation) having units of energy times time. The spectrum of zero-point radiation is Lorentz invariant and also scale invariant.\[21\] It is easy to exhibit the scale invariance. Under a scale transformation by a factor $\sigma$, the relationship $U_{zp} = \omega/2$ becomes $U_{zp}/\sigma = (\omega/\sigma)/2$ since the frequency $\omega$ has units of $1/time$. But then we see that the new relationship involves $U'_{zp} = \omega'/2$ which is the same as the original relationship.

The total zero-point energy of the radiation in a box clearly diverges since there are infinitely many normal modes with ever-increasing frequency in the box. However, as realized by Casimir in 1948, the change in zero-point energy associated with a shift in the position of a partition in a box of fixed length is indeed finite.\[6\] Thus we will consider a box of total length $L$ containing a partition located at a distance $x$ from one wall. The partition splits the original box of length $L$ into two boxes, one of length $x$ and one of length $L-x$. We consider the zero-point energy $U_{zp}(x) + U_{zp}(L-x)$ in this partitioned box, and compare it with the energy $2U_{zp}(L/2)$ in the partitioned box when the partition is half-way across the box of length $L$

$$\Delta U_{zp} = U_{zp}(x) + U_{zp}(L-x) - 2U_{zp}(L/2) \quad (36)$$

Since we are dealing with divergent quantities, we introduce a temporary high-frequency cut-off; after finding the change in zero-point energy for the situation including the cut-off, we then take the no-cut-off limit. In our calculation, we also introduce a parameter $s$ so as to treat both the like- and unlike-boundary conditions at the same time. Specifically, the
change in zero-point energy for the partitioned box with a cut-off parameter $\Lambda$ is

$$
\Delta U_{zp}(x, L, \Lambda) = \sum_{n=1}^{\infty} \frac{1}{2} \frac{(n-s)\pi c}{x} \exp \left[ -\Lambda \frac{(n-s)}{x} \right] + \sum_{n=1}^{\infty} \frac{1}{2} \frac{(n-s)\pi c}{L-x} \exp \left[ -\Lambda \frac{(n-s)}{L-x} \right] 
$$

$$
- 2 \sum_{n=1}^{\infty} \frac{1}{2} \frac{(n-s)\pi c}{L/2} \exp \left[ -\Lambda \frac{(n-s)}{L/2} \right]
$$

$$
= -\frac{\pi c}{2} \frac{\partial}{\partial \Lambda} \left\{ \sum_{n=1}^{\infty} \exp \left[ -\Lambda \frac{(n-s)}{x} \right] + \left[ x \to (L-x) \right] - 2\left[ x \to L/2 \right] \right\}
$$

$$
= -\frac{\pi c}{2} \frac{\partial}{\partial \Lambda} \left\{ \exp \left[ \frac{\Lambda s}{x} \right] \frac{1}{\exp[\Lambda/x] - 1} \right\} + \left[ x \to (L-x) \right] - 2\left[ x \to L/2 \right]
$$

$$
= -\frac{\pi c}{2} \frac{\partial}{\partial \Lambda} \left\{ \left[ \frac{x}{\Lambda} + \left( \frac{1}{12} + \frac{s^2}{2} - \frac{s}{2} \right) \right] \frac{\Lambda}{x} + \ldots \right\} + \left[ x \to (L-x), L/2 \right]
$$

$$
= -\frac{\pi c}{2} \left( \frac{1}{12} + \frac{s^2}{2} - \frac{s}{2} \right) \left[ \frac{1}{x} + \frac{1}{L-x} - 4 \right] + O(\Lambda/x)
$$

(37)

where the parameter $s = 0$ or $s = 1/2$. In the no-cut-off limit $\Lambda \to 0$, the change in zero-point energy becomes

$$
\Delta U_{zp}(x, L) = -\frac{\pi c}{2} \left( \frac{1}{12} + \frac{s^2}{2} - \frac{s}{2} \right) \left[ \frac{1}{x} + \frac{1}{L-x} - \frac{4}{L} \right]
$$

(38)

If both the walls and partition require Dirichlet boundary conditions for the field, (corresponding to the frequency in Eq. (27) and $s = 0$ in Eqs. (37) and (38)), then the initial coefficient in Eq. (38) is negative

$$
\Delta U_{zp}^{DD}(x, L) = -\frac{\pi c}{24} \left[ \frac{1}{x} + \frac{1}{L-x} - \frac{4}{L} \right]
$$

(39)

In this case, the partition is attracted to the ends of the large box. This same zero-point energy arises when Neumann boundary conditions are applied for both the ends of the box and the partition, $\Delta U_{zp}^{DD}(x, L) = \Delta U_{zp}^{NN}(x, L)$. On the other hand, if the partition requires Neumann boundary conditions while the ends of the large box require Dirichlet boundary conditions (corresponding to the frequency in Eq. (29) and $s = 1/2$ in Eqs. (37) and (38), then the initial coefficient in Eq. (38) is positive,

$$
\Delta U_{zp}^{DN}(x, L) = +\frac{\pi c}{48} \left[ \frac{1}{x} + \frac{1}{L-x} - \frac{4}{L} \right]
$$

(40)

and the partition is repelled by the ends of the large box. We can also take the limit $L \to \infty$ as the size of the large box becomes infinitely long. Then for the situation of like boundary conditions between the partition and the wall, we have from Eq. (39)

$$
\Delta U_{zp}^{DD}(x, L) = -\frac{\pi c}{24x}
$$

(41)
while for unlike boundary conditions between the partition and the wall, we have from Eq. (40)

$$\Delta U_{zp}^{DN}(x, L) = \frac{\pi c}{48x}$$  \hspace{1cm} (42)

C. Change in Energy for the Rayleigh-Jeans Spectrum

We can also calculate the change in the radiation energy stored in the box for the limit of high temperature where the thermal spectrum is expected to approach energy equipartition $U(\omega, T) \rightarrow k_B T$. Since we are using units where $k_B = 1$, this corresponds to $U(\omega, T) \rightarrow T$. The calculation is carried out in the same style as for zero-point radiation except that the energy per normal mode is different. Thus analogous to Eq. (37), we have

$$\Delta U_{RJ}(x, L, T, \Lambda) = \sum_{n=1}^{\infty} T \exp \left[ -\Lambda \left( \frac{n-s}{x} \right) \right] + [x \to (L - x)] - 2[x \to L/2]$$

$$\sum_{n=1}^{\infty} T \exp \left[ -\Lambda \left( \frac{n-s}{x} \right) \right] + [x \to (L - x)] - 2[x \to L/2]$$

$$\sum_{n=1}^{\infty} T \left\{ \left[ x \Lambda + \left( s - \frac{1}{2} \right) + \left( \frac{1}{12} + \frac{s^2}{2} - \frac{s}{2} \right) \frac{\Lambda}{x} + \cdots \right] + [x \to (L - x), L/2] \right\}$$

$$= 0 + O(\Lambda/x)$$  \hspace{1cm} (43)

In the no-cut-off limit $\Lambda \rightarrow 0$, we find

$$\Delta U_{RJ}(x, L, T) = 0$$  \hspace{1cm} (44)

irrespective of the boundary conditions.

We notice that the basic form of the energy changes in these two limiting situations of zero-point radiation (Eqs. (39)-(42)) and of the Rayleigh-Jeans spectrum (Eq. (44)) might be suggested from scaling or dimensional considerations alone. Each piece $U(x, T)$, $U(L-x, T)$, $U(L/2, T)$ in the energy change $\Delta U(x, L, T)$ involves a single length. Thus there is no dependence upon the ratio of lengths $x/L$. The energy must scale as an inverse length. For the Rayleigh-Jeans spectrum where $U_{RJ}(\omega, T) = T$, the energy must depend upon the temperature $T$, but there is no connection between length and temperature. Furthermore, when calculating the change of energy $\Delta U$, we must introduce a cut-off $\omega_{cut-off}$ in frequency and make the subtractions before taking the cut-off frequency to infinity. However, if $\omega_{cut-off}$ is the cut-off frequency, then the number $n_x$ of modes of frequency lower than the cut-off
frequency for a box of length $x$ is $n_x \frac{\pi c}{x} = \omega_{\text{cut-off}}$, while the number of radiation modes below the cut-off frequency for the part of the box on the other side of the partition is such that $n_{L-x} \frac{\pi c}{(L - x)} = \omega_{\text{cut-off}}$. Therefore the total number of modes entering when the partition is located at position $x$ is

$$n_x + n_{L-x} = \frac{x}{\pi c} \omega_{\text{cut-off}} + \frac{L - x}{\pi c} \omega_{\text{cut-off}} = \frac{L}{\pi c} \omega_{\text{cut-off}} \tag{45}$$

which is independent of the position $x$ of the partition. If each mode makes the same energy contribution (as is the case for the Rayleigh-Jeans spectrum), then on subtraction of the energy when the partition is half-way across the box, the change of energy $\Delta U_{\text{RJ}}$ vanishes. In the case of zero-point radiation, the normal mode energies do indeed depend upon the frequencies $\omega_n$ of the normal modes, and so the energy change is nonvanishing, and indeed scales as an inverse distance as seen in Eqs. (39)-(42).

D. Thermodynamic Minimum Principle for the Helmholtz Free Energy

The radiation in a partitioned box can be regarded as the working substance for a thermodynamic system involving length parameters $x$ and $L$ (analogous to volume) and temperature $T$. For a fixed length $L$ and temperature $T$, the Helmholtz free energy achieves its minimum value at thermal equilibrium. For the case of unlike boundary conditions (where the radiation satisfies Dirichlet boundary conditions at the ends of the box but Neumann boundaries at the partition), the situation at zero temperature involves a repulsion of the partition from the walls so that thermal equilibrium corresponds to the partition being located in the middle of the box at $x = L/2$. In this situation of zero-temperature, the Helmholtz free energy $F = U - TS$ equals the energy since the entropy vanishes at zero temperature. From Eq. (40), we see that the energy is indeed a minimum for the partition in the middle of the box. For finite non-zero temperature, we expect that the equilibrium position is still in the center of the box, but the change in the Helmholtz free energy at other positions will be modified compared to that for zero temperature. In every position $x$ and for any total box length $L$, the change in the Helmholtz free energy for constant temperature must provide the pressure which moves the partition invariably toward its thermal equilibrium position at the center of the box. This requirement places an enormous restriction on the functional form of the energy $U_n(\omega_n, T)$ per normal mode of frequency $\omega_n$. Taken together with the
asymptotic limits $U(\omega, T) \to \omega/2$ for $\omega >> T$ and $U(\omega, T) \to T$ for $T >> \omega$, the minimum Helmholtz free energy condition is sufficient to determine the allowed spectrum of blackbody radiation. We simply assume a test functional form $\phi_t(\omega/T)$ for the canonical potential of a radiation normal mode (which is the same as the canonical potential for a harmonic oscillator discussed above) which satisfies the asymptotic limits in Eq. (8). This test potential $\phi_t(\omega/T)$ then determines the Helmholtz free energy $F_{tn}(\omega_n, T)$ of each normal mode as $F_{tn}(\omega_n, T) = -T \phi_t(\omega_n/T)$, where the frequency $\omega_n$ is related to the length of the box as in Eq. (29). Then we proceed to calculate the change in Helmholtz free energy $\Delta F_t(x, L, T)$ as a function of $x$ for the radiation in the partitioned box of length $L$,

$$\Delta F_t(x, L, T) = \sum_{n=1}^{\infty} F_{tn}(\omega_n(x)) + \sum_{n=1}^{\infty} F_{tn}(\omega_n(L - x)) - 2 \sum_{n=1}^{\infty} F_{tn}(\omega_n(L/2))$$

(46)

Only if the Helmholtz free energy $\Delta F_t(x, L, T)$ so obtained is a smooth monotonic function which reaches its minimum at $x = L/2$ for all box lengths $L$, do we have a possible choice for the spectrum of blackbody radiation.

In the calculations, it seems easiest to first separate off the divergent zero-point energy, calculate the change in the remaining convergent series, and then add back the change in Helmholtz free energy which is associated with the zero-point energy. Numerical calculation easily shows that the Planck formula including zero-point radiation given in Eq. (10) leads to a change in Helmholtz free energy at fixed temperature which meets all the required conditions. In Fig. 1, we give a graph of the change in the Helmholtz free energy $\Delta F_{zp}(x, L, T)$ for a box of total length $L = 5$, and temperatures $T = 0, 1, 3$. The Planck formula indeed satisfies the “minimum” behavior for the Helmholtz free energy which is required by thermodynamics.

On the other hand, for all the test functions $\phi_t(\omega/T)$ which met the asymptotic conditions on the canonical potential but which departed from the Planck formula, it was easy to show that the change in Helmholtz free energy associated with these functions did not provide a monotonic function of $x$ for some total box length $L$. In Fig. 2, we give a graph of the test canonical function $\phi_t(\omega/T) = \{-\ln[2 \sinh(\omega/2T)] + \omega/(2T)\} \exp[-\omega/T] - \omega/(2T)$ which meets the asymptotic conditions (8) for the thermodynamics of the harmonic oscillator but does not match the Planck function, and clearly does not meet the thermodynamic requirements for the change in the Helmholtz free energy for the partitioned box.

It is our conclusion, that thermodynamic arguments provide a basis for the derivation of
the blackbody radiation spectrum.

VI. ZERO-POINT ENERGY IS EMBEDDED IN THE TRADITIONAL PLANCK SPECTRUM

A. The Traditional Planck Spectrum Omits Zero-Point Radiation

All textbooks of modern physics and most physicists present the Planck spectrum without including the zero-point radiation part.\[1\] Thus the Planck energy for a harmonic oscillator is usually given as

\[ U_P(\omega, T) = \frac{\hbar \omega}{\exp[\hbar \omega/(k_B T)] - 1} = \frac{\hbar \omega}{2} \coth \left( \frac{\hbar \omega}{2 k_B T} \right) - \frac{\hbar \omega}{2} \quad (47) \]

Planck’s determination of the blackbody spectrum followed the experimental work of Lummer and Pringsheim\[24\] which measured the random radiation of a source which was above the random radiation surrounding the detector; the zero-point radiation which surrounded a source also surrounded the detector and so was not measured. It is only recently that we have experimental measurements of Casimir forces\[7\] which measure all the radiation surrounding the parallel plates.

Because the traditional Planck formula in (47) omits the zero-point radiation, discussions of the blackbody radiation usually make no reference to zero-point radiation. On the other hand, all the derivations of the blackbody radiation spectrum within classical physics depend crucially upon the presence of zero-point radiation. In the present article, we have given the basis for two derivations of the blackbody spectrum making use of thermodynamic ideas, and the existence of zero-point radiation in the low-temperature asymptotic limit is crucial to the discussions.

B. The Change in Radiation Energy in a Partitioned Box Reveals the Zero-Point Energy Hidden in the Traditional Planck Spectrum

According to the traditional Planck formula in Eq. (47), the high-temperature limit for the energy of a harmonic oscillator of frequency \( \omega \) does not go over fully to the equipartition value \( k_B T \), but rather retains a finite correction \((1/2)\hbar \omega\) associated with the absence of the
zero-point energy contribution. Thus for \( k_B T >> \hbar \omega \), we have from Eq. (47)

\[
U_P(\omega, T) \rightarrow \hbar \omega \left[ \frac{k_B T}{\hbar \omega} - \frac{1}{2} + \frac{1}{12} \left( \frac{k_B T}{\hbar \omega} \right)^2 - \ldots \right] = k_B T - \frac{1}{2} \hbar \omega + O(\omega/T)
\]  

(48)

This failure of the equipartition limit does not seem to bother physicists. However, the failure of this limit becomes glaringly obvious if we calculate the change in Casimir energy \( \Delta U_P \) associated with the use of the traditional Planck formula; the large equipartition \( k_B T \) in Eq. (48) makes no contribution to \( \Delta U_P \) leaving only the negative zero-point result.

The Casimir energy change takes its simplest form when the total length \( L \) of the box goes to infinity, \( L \to \infty \). This is seen for the zero-temperature case in the transition from Eqs. (39) and (40) over to Eqs. (41) and (42). We will take the case where Dirichlet boundary conditions are applied at both the walls and the partition so that the frequencies of the radiation normal modes are given in (27).

The thermal energy \( U_P(x, T) \) follows from the Euler-Maclaurin summation formula,

\[
\sum_{k=1}^{n-1} f_k = \int_0^n f(k)dk - \frac{1}{2}[f(0) + f(n)] + \frac{1}{12}[f'(n) - f'(0)] - \frac{1}{720}[f'''(n) - f'''(0)] + R
\]  

(49)

where \( R \) is a remainder term and the coefficient terms involve the same Bernoulli numbers \( B_n \) as appear in a power-series expansion of the Planck formula.[27] For the traditional Planck formula in Eq. (47), we have for the thermal energy \( U_P(x, T) \) in the box of length \( x \)[28]

\[
U_P(x, T) = \sum_{n=1}^{\infty} \frac{\hbar \omega_n}{\exp[\hbar \omega_n/(k_B T)] - 1} = \sum_{n=1}^{\infty} \frac{\hbar (n\pi c/x)}{\exp[\hbar n\pi c/(k_B T x)] - 1}
\]

\[
= \int_0^{\infty} \frac{dn}{\exp[\hbar n\pi c/(k_B T x)] - 1} - \frac{k_B T}{2} + \frac{\pi c}{24x} + O(1/T^2 x^2)
\]

\[
= \frac{\pi (k_B T)^2}{6 \hbar c} x - \frac{k_B T}{2} + \frac{\pi c}{24x} + O(1/T^2 x^2)
\]  

(50)

Thus the change in the thermal energy \( \Delta U_P(x, T) \) associated with the partition position \( x \)
in the limit $L \to \infty$ becomes

$$
\Delta U_P(x, T) = \lim_{L \to \infty} \Delta U_T(x, L, T) = \lim_{L \to \infty} [U_T(x, T) + U_T(L - x, T) - 2U_T(L/2T)]
$$

\[
= U_T(x, T) + \lim_{L \to \infty} \left[ \frac{\pi (k_B T)^2}{6 hc} (L - x) - \frac{k_B T}{2} + \frac{\pi c}{24(L - x)} - \ldots \right]
\]

\[
- \lim_{L \to \infty} 2 \left[ \frac{\pi (k_B T)^2}{6 hc} \frac{L}{2} - \frac{k_B T}{2} + \frac{\pi c}{24(L/2)} - \ldots \right]
\]

\[
= U_P(x, T) - \left( \frac{\pi x (k_B T)^2}{6 hc} \frac{x}{x - \frac{k_B T}{2}} \right)
\]

(51)

Now introducing the Euler-Maclaurin expansion for a finite-length box given in Eq. (50) into Eq. (51), we have

$$
\Delta U_P(x, T) = \frac{\pi c}{24x}
$$

(52)

provided that the remainder $R$ in the Euler-Maclaurin summation formula is small. But $\pi c/(24x)$ is exactly the negative of the change of zero-point energy in Eq. (41) for the box of length $x$. Thus the traditional Planck expression in Eq. (47) which has no zero-point energy at low temperature betrays its connection to zero-point energy by giving at high temperatures a change in energy $\Delta U_P(x, T)$ which is the negative of the change in zero-point energy.

The evaluation in Eq. (52) giving the connection to zero-point energy holds for situations $xT >> 1$ where the remainder term $R$ for Euler-Maclaurin summation formula has only a small value in Eq. (50). For values of $xT \lesssim 1$, the remainder $R$ becomes relatively large and the Euler-Maclaurin summation formula in (49) does not give a good approximation to the change in the thermal radiation energy in the region of length $x$.

Figure 3 shows the changes in Casimir energy $\Delta U_P^{DD}(x, T)$ when Dirichlet boundary conditions are applied at both the end of the box and at the partition, for three different functions in the limit of an infinitely long box $L \to \infty$. The three functions involve a) the Planck spectrum including zero-point energy $\Delta U_{Pzp}(x, T)$, b) the traditional Planck spectrum without zero-point energy $\Delta U_P(x, T)$, and c) zero-point energy $\Delta U_{zp}(x)$. It is clear that the energy change for the traditional Planck formula which omits zero-point energy goes over to the negative of the change of zero-point energy at large values of $xT$. On the other hand, the Planck formula which includes zero-point energy approaches zero extremely rapidly at large values of $xT$. This vanishing change of energy is consistent with the Rayleigh-Jeans spectrum as the high-temperature limit of the blackbody spectrum where
\( \Delta U_{RJ}(x, T) = 0 \). In the high-temperature Casimir energy changes, we see clear evidence that the idea of zero-point energy is embedded in the traditional Planck formula despite the explicit removal of the zero-point energy from the low-temperature limit.

**VII. DISCUSSION: CONNECTIONS BETWEEN THERMAL RADIATION, ZERO-POINT-RADIATION, AND SPACETIME STRUCTURE**

The Planck spectrum of thermal radiation within classical physics is intimately connected with the spectrum of classical zero-point radiation and with the structure of spacetime. This striking idea seems rarely appreciated among physicists today. In the past, this connection has been derived using non-inertial coordinate frames. Specifically, it has been shown that the correlation function for the zero-point radiation fields depends only upon the geodesic separation between the spacetime points where the correlation function is evaluated.\(^{25}\)

In Minkowski spacetime,\(^9\) for example, the field correlation functions depend upon the Lorentz-invariant spacetime interval \( (ct - ct')^2 - (r - r')^2 \). Furthermore, thermal radiation can be derived from zero-point radiation by the use of a time-dilating conformal transformation in a non-inertial frame.\(^9\)

In the present work, we point out that the use of thermodynamic ideas in connection with classical zero-point radiation leads naturally to the Planck spectrum for blackbody radiation, for both an individual radiation mode and for the Casimir energy change for radiation in a partitioned box. The Planck spectrum appears if one requires that the interpolation between the zero-point energy at low temperature and the equipartition energy at high temperature is thermodynamically smooth in the sense that the canonical potential function \( \phi(\omega/T) \) (which for an oscillator depends upon one variable) is monotonic and all its derivatives are monotonic so that no preferred value of \( \omega/T \) is singled out. Also, the Planck spectrum appears if one uses these same asymptotic limits for a single radiation mode but requires that at fixed temperature the Helmholtz free energy in a partitioned box assumes its minimum value at thermal equilibrium. In addition, the zero-point energy is embedded even in the traditional Planck spectrum which omits zero-point energy as the low temperature limit; the zero-point energy reappears in the high-temperature limit for both a single oscillator (or radiation mode) and also for the change in Casimir energy for radiation in a partitioned box. Indeed, thermal radiation, zero-point radiation, and spacetime structure are all related.
VIII. ACKNOWLEDGEMENTS

I wish to thank Professors Nicholas Giovambattista, V. Paramewaran Nair, and Joel Gersten for helpful discussions.

[1] See the discussion of any textbook of modern physics. For example, R. Eisberg and R. Resnick, Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles 2nd ed. (Wiley, New York 1985) or K. S. Krane, Modern Physics 2nd ed. (Wiley, New York 1996) or J. R. Taylor, C. D. Zafiratos, and M. A. Dubson, Modern Physics for Scientists and Engineers 2nd ed. (Pearson, New York, 2003) or S. T. Thornton and A. Rex, Modern Physics for Scientists and Engineers (Brooks/Cole, Cengage Learning, Boston, MA 2013).

[2] T. H. Boyer, “Derivation of the Blackbody Radiation Spectrum without Quantum Assumptions,” Phys. Rev. 182, 1374-1383 (1969).

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[5] T. H. Boyer, “Derivation of the Planck spectrum for relativistic classical scalar radiation from thermal equilibrium in an accelerating frame,” Phys. Rev. D 81, 105024 (2010).

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[8] T. H. Boyer, “Any classical description of nature requires classical electromagnetic zero-point radiation,” Am. J. Phys. 79, 1163-1167 (2011).

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[10] C. Garrod, Statistical Mechanics and Thermodynamics (Oxford U.P., New York, 1995), p. 128.
[11] See, for example, B. H. Lavenda, *Statistical Physics: A Probabilistic Approach* (Wiley, New York, 1991), p. 74.

[12] T. H. Boyer, “Thermodynamics of the harmonic oscillator: Wien’s displacement law and the Planck spectrum,” Am. J. Phys. 71, 866-870 (2003).

[13] Action-angle variables are discussed by, for example, H. Goldstein, *Classical Mechanics* 2nd ed. (Addison-Wesley, Reading, MA 1981), Sections 10-5 through 10-7, pp. 457-484. The action variables are adiabatic invariants.

[14] Traditional classical electron theory is described by H. A. Lorentz, *The Theory of Electrons* (Dover, New York 1952). This volume is a republication of the second edition of 1915 based on Lorentz’s Columbia University lectures of 1909. On page 20 and on page 240, note 6, Lorentz gives his explicit assumption on the boundary conditions for Maxwell’s equations; the assumption excludes the possibility of classical zero-point radiation.

[15] See the essay by T. H. Boyer, “Is Planck’s Constant a ‘Quantum’ Constant? An Alternative Classical Interpretation,” arXiv 1301.6043.

[16] *Dictionary of Scientific Biography*, edited by C. C. Gillispie (Scribners, New York 1975), Vol. 11, p. 11.

[17] A discussion of natural units is given by C. Garrod, Ref. 10, p. 120. The choice \( \hbar = 1 \) is familiar to particle physicists. The measurement of temperature in energy units is familiar in thermodynamics where the temperature is sometimes denoted by \( \tau \) rather than \( T \). See also, for example, C. Kittel, *Elementary Statistical Physics* (Wiley, New York, 1958), p.27.

[18] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965), p.87, #4.6.25.

[19] J. H. Van Vleck intended to publish this result in the mid 1920s but was diverted by the appearance of Schroedinger’s quantum mechanics. (Private communication to the author from J. H. Van Vleck.) T. H. Boyer, “Equilibrium of random classical electromagnetic radiation in the presence of a nonrelativistic nonlinear electric dipole oscillator,” Phys. Rev. D 13, 2832-2845 (1976) and “Statistical equilibrium of nonrelativistic multiply periodic classical systems and random classical electromagnetic radiation,” Phys. Rev. A 18, 1228-1237 (1978).

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radiation,” Found. Phys. 19, 1371-1383 (1989).

[22] T. H. Boyer, ”Casimir forces and boundary conditions in one dimension: Attraction, repulsion, Planck spectrum, and entropy,” Am. J. Phys. 71, 990-998 (2003).

[23] See, for example, P. M. Morse, Thermal Physics (Benjamin-Cummings, Reading, MA 1969), pp. 101-106.

[24] O. Lummer and E. Pringsheim, “Die Vertheilung der Energie im Spectrum des schwarzen Köpers und des blanken Platins,” Verhandlungen der Deutschen Physikalischen Gesellschaft 1 215-235 (1899), presented 3 November 1899.

[25] T. H. Boyer, “The blackbody radiation spectrum follows from zero-point radiation and the structure of relativistic spacetime in classical physics,” Found. Phys. 42, 595-614 (2012).

[26] See Abramowitz and Stegun in ref. 17, p. 16, #3.6.26 and p. 886, #25.4.7.

[27] The Planck expression without zero-point energy in Eq. (17) can be expanded as \( t/(e^t - 1) = \sum_{n=0}^{\infty} B_n t^n/n! = 1 - t/2 + t^2/12 - t^4/720 + ... \) for \(|t| < 2\pi\). See Abramowitz and Stegun in ref. 17, p. 804, #23.1.1.

[28] Here we have used the integral \( \int_0^\infty dx x^{\nu-1}/(\exp[\mu x]-1) = \mu^{-\nu}\Gamma(\nu)\zeta(\nu) \) from I. S. Gradshteyn and I. M. Ryzhik, Tables of Integrals, Series, and Products (Academic, New York, 1965), p. 325, #23.2.24. For our case, \( \mu = 1, \nu = 2 \), where \( \Gamma(2) = 1 \) and \( \zeta(2) = \pi^2/6 \). See Abramowitz and Stegun in ref. 17, p.255, #6.1.6 and p.807, #23.2.24.

Figure Captions

Fig. 1. Change in Helmholtz Free Energy with Partition Position at Constant Temperature

The change in Helmholtz free energy \( \Delta F_{Pzp}(x, L, T) \) is plotted as a function of partition position \( x \) at three different temperatures for \( 0 < x \leq 2.5 \) in a box of length \( L = 5 \) (for unlike boundary conditions). The Helmholtz free energy change \( \Delta F_{Pzp}(x, L, T) \) is obtained from the Planck expression (with zero-point energy) for the canonical potential \( \phi_{Pzp}(\omega/T) = -\ln[2\sinh(\omega/2T)] \) given in Eq. (10) for each radiation mode of frequency \( \omega \).

a) The solid curve corresponds to temperature \( T = 0 \). b) The dashed curve is for \( T = 1 \).

c) The dotted curve is for \( T = 3 \). The curves show a monotonic decrease toward 0 at the middle of the box \( x = 2.5 \), consistent with thermodynamic requirements.

Fig. 2. Change in Helmholtz Free Energy Assuming a Test Radiation Spectrum Different
from the Planck Spectrum

The change in Helmholtz free energy $\Delta F_t(x, L, T)$ is plotted as a function of partition position $x$ when we consider a test spectrum different from the Planck spectrum (for unlike boundary conditions). Here the canonical potential for each radiation mode of frequency $\omega$ at temperature $T > 0$ is chosen as $\phi_t(\omega/T) = \{-\ln[2 \sinh(\omega/2T)] + \omega/(2T)\} \exp(-\omega/T) - \omega/(2T)$. The box has total length $L = 5$, and the plot shows half the box, $0 < x \leq 2.5$.

a) The solid curve corresponds to the energy change for zero-point energy at zero temperature. b) The dashed curve is for $T = 1$. c) The dotted curve is for $T = 3$.

The curves for temperature $T > 0$ do not show monotonic behavior and so violate the thermodynamic requirements. Therefore the assumed test canonical potential $\phi_t$ cannot correspond to thermal radiation.

Fig. 3. Change in Energy with Partition Position at Constant Temperature

The change in energy $\Delta U(x, T)$ is plotted as a function of partition position $x$ for constant temperature for an infinitely long box, $L \rightarrow \infty$ (for like boundary conditions). a) The solid curve gives the energy change $\Delta U_{zp}(x, T) = \Delta U_P(x, T) + \Delta U_{zp}(x)$ at $T = 1$ following from the full Planck spectrum (which includes zero-point radiation) given in Eq. (15). b) The dashed curve gives the thermal energy change $\Delta U_P(x, T)$ at $T = 1$ following from the traditional Planck spectrum in Eq. (47) (which omits zero-point energy) for each mode. c) The dotted curve gives the zero-point energy change $\Delta U_{zp}(x)$ in Eq. (41). For $xT >> 1$, the traditional Planck spectrum without zero-point energy gives a change in energy $\Delta U_P$ which is the negative of the zero-point energy change $\Delta U_{zp}$. For $xT >> 1$, only the full Planck spectrum with zero-point energy goes over to the expected energy change $\Delta U_{RJ}(x, T) = 0$ holding for the Rayleigh-Jeans spectrum.