Turnaround radius in modified gravity

Valerio Faraoni

Physics Department and STAR Research Cluster, Bishop’s University, Sherbrooke, Québec, Canada J1M 1Z7

Abstract

In an accelerating universe in General Relativity there is a maximum radius above which a shell of test particles cannot collapse, but is dispersed by the cosmic expansion. This radius could be used in conjunction with observations of large structures to constrain the equation of state of the universe. We extend the concept of turnaround radius to modified theories of gravity for which the gravitational slip is non-vanishing.

Keywords: modified gravity, dark energy theory, dark energy experiments, galaxy clusters

1. Introduction

Since 1998 cosmologists and theoretical physicists have been trying to explain the present acceleration of the universe discovered with type Ia supernovae. Apart from the problematic cosmological constant, explanations based on a dark energy introduced \textit{ad hoc} (see \cite{1} for a review) are not satisfactory and many researchers have turned to contemplating the possibility of modifying gravity at large scales (\cite{2}, see \cite{3,4,5,6,7} for reviews). Although modifying gravity is a viable possibility, too many dark energy and modified gravity models fit the observational data and it is important to use any test of gravity which may become available, at all scales, to obtain hints on the correct explanation of the cosmic acceleration and, possibly, on the correct theory of gravity.

One possibility which has been pointed out recently is testing theoretical predictions of the turnaround radius with astronomical observations.
In an accelerating Friedmann-Lemaître-Robertson-Walker (FLRW) universe, there is a maximum (areal) radius beyond which a spherical shell of dust cannot collapse but expands forever, driven by the cosmic accelerated expansion. We formulate our final results in terms of the areal radius \( R \) because solutions of modified gravity theories are reported in the literature using various radial coordinates. However, effects peculiar to a particular coordinate system would not be meaningful in relativistic gravity, while effects characterized in a geometric, coordinate-independent, way are physically meaningful. The areal radius separating two points in space is a physical distance identified in a completely geometric way by the area of 2-spheres of symmetry in a spherically symmetric spacetime. In a spatially FLRW universe with line element
\[
ds^2 = -dt^2 + a^2(t) \left( dr^2 + r^2 d\Omega^2 \right),
\]
the physical (areal) radius is \( R(t, r) = a(t)r \) and it expands with the scale factor \( a(t) \), while the comoving radius \( r \) is simply a label attached to elements of the cosmic fluid.

The first comparisons of the prediction for the turnaround radius in the \(^{\Lambda}\)CDM model of General Relativity with objects in the sky have been carried out \(^{\[23, 24, 25, 22\]}\) and, although the precision is still poor, the method holds promise. In General Relativity, the concept of turnaround radius can be made more rigorous by using the Hawking-Hayward quasi-local mass \(^{\[26\]}\). Given the motivation for modified gravity in cosmology, here we propose to extend the scope of studies of the turnaround radius to alternative theories of gravity. We do not commit to any specific modified gravity theory at this stage, but adopt a post-Friedmannian approach \(^{\[27\]}\) in which a post-FLRW metric fits many theories, to lowest order in metric perturbations from an exact FLRW background. Contrary to General Relativity, in which two scalar potentials in the perturbed FLRW metric coincide (apart from the sign), in modified gravity there are two distinct potentials which are not trivially related. We derive the turnaround radius in this scheme and find that the physical (areal) turnaround radius depends on both potentials. We point out that, in order for studies of the turnaround radius to be meaningful, it is not sufficient to pick a theory of modified gravity but efforts must be made to establish which solutions of this theory are generic in some appropriate sense.

The metric signature employed in this paper is \(- + + +\) and we use units in which Newton’s constant \( G \) and the speed of light \( c \) assume the value \( 1 \). Unless that coordinate system is associated with a preferred family of observers.
unity.

2. Turnaround radius in the parametrized post-Friedmannian approach

The parametrized post-Friedmannian approach \[27\] describes perturbations of a FLRW universe in theories of gravity alternative to General Relativity. The line element (1) below is a rather general parametrization of the metric describing perturbed FLRW universes in modified gravity \[28, 29, 30\] (it holds, for example, in $f(R)$ gravity \[31, 32\]). The spacetime metric in the conformal Newtonian gauge is \[33, 34, 35, 36, 37, 38\]

$$ds^2 = a^2(\eta) \left[ - (1 + 2\psi) d\eta^2 + (1 - 2\phi) (dr^2 + r^2 d\Omega^2_{(2)}) \right],$$

where $d\Omega^2_{(2)} = d\theta^2 + \sin^2 \theta d\phi^2$ is the metric on the unit 2-sphere, $\eta$ is the conformal time (related to the comoving time $t$ by $dt = a d\eta$), $a(\eta)$ is the scale factor of the spatially flat FLRW background, and $\phi$ and $\psi$ are two post-Friedmannian scalar potentials. While in General Relativity it is $\psi = -\phi$, in many modified theories of gravity these two potentials do not coincide in absolute value and the gravitational slip $\xi \equiv (\phi - \psi) / \phi$ is used in experiments aiming at detecting deviations from the standard $\Lambda$CDM scenario \[39, 40, 33, 35, 36, 37, 38\]. The ansatz (1) does not include vector and tensor metric perturbations, which is justified at lower order for non-relativistic velocities, and is common practice in the literature on cosmological perturbations \(e.g., \ [41, 42]\).

Following the literature on the turnaround radius in cosmology \[23, 24, 25, 22\], we assume that the FLRW perturbation is spherically symmetric, i.e., $\phi = \phi(r), \psi = \psi(r)$. The numerical importance of deviations from spherical symmetry was discussed in \[43, 23, 24\]. The simplest definition of turnaround radius consists of considering spherical shells of test particles respecting the spacetime symmetry (that is, expanding or contracting but

\[2\] An effect due to a metric of the form (1) would signal modified gravity and, neglecting vector and tensor degrees of freedom in the metric (which is legitimate at this order of approximation), the potentials $\psi$ and $\phi$ are all that remains in the line element. However a better characterization of modified gravity than the metric is, at least in principle, needed and a rigorous derivation of the line element (1) in various classes of modified gravity theories is still missing. Nevertheless, the turnaround radius will probably be unaffected by these theoretical improvements.
not shearing nor rotating) with areal radius $R$, and imposing that they have zero radial acceleration, $\ddot{R} = 0$, where an overdot denotes differentiation with respect to the comoving time $t$ of the FLRW background. We will adopt here this common criterion which, in General Relativity, can be justified rigorously \cite{44} by making use of the Hawking-Hayward quasi-local mass construct \cite{45, 46}. In general, this quasi-local energy construct is not defined in modified gravity and here we will stick to the simple definition $\ddot{R} = 0$ for shells of test particles (dust) in radial motion.

Massive test particles follow timelike geodesics with 4-tangents $u^a$ satisfying $u_c u^c = -1$ and the geodesic equation

$$\frac{du^a}{d\tau} + \Gamma_{bc}^a u^b u^c = 0,$$  

(2)

which we choose to be affinely parametrized by the proper time $\tau$, and where

$$\Gamma_{bc}^a = \frac{1}{2} g^{ad} (g_{db,c} + g_{dc,b} - g_{bc,d})$$  

(3)

are the coefficients of the metric connection. We will perform first order calculations in the metric perturbations $\phi$ and $\psi$, but the density contrast related to the spatial Laplacian of these metric perturbations is allowed to be large \cite{44, 41, 42}. The normalization

$$u_c u^c = g_{00}(u^0)^2 + g_{11}(u^1)^2 = -1$$  

(4)

for massive test particles with purely radial motion ($u^2 = u^3 = 0$) yields

$$(u^0)^2 = \frac{1}{a^2} (1 - 2\psi) + (1 - 2\phi - 2\psi) (u^1)^2$$  

(5)

to first order. Eq. (2) then gives

$$\frac{d u^\mu}{d\tau} + \Gamma_{00}^\mu (u^0)^2 + 2\Gamma_{01}^\mu u^0 u^1 + \Gamma_{11}^\mu (u^1)^2 = 0$$  

(6)

for $\mu = 0, 1$. Eq. (3) provides the only non-vanishing Christoffel symbols to first order

$$\Gamma_{00}^0 = \frac{a_n}{a}, \quad \Gamma_{01}^0 = \Gamma_{10}^0 = \psi', \quad \Gamma_{11}^0 = \frac{a_n}{a} (1 - 2\phi - 2\psi),$$  

(7)

$$\Gamma_{00}^1 = \psi', \quad \Gamma_{01}^1 = \Gamma_{10}^1 = \frac{a_n}{a}, \quad \Gamma_{11}^1 = -\phi',$$  

(8)

4
where a prime denotes differentiation with respect to the radial coordinate $r$ and $a_{\eta} \equiv da/d\eta$. Therefore, it is

$$\frac{du^0}{d\tau} + \frac{a_{\eta}}{a} (u^0)^2 + 2\psi' u^0 u^1 + \frac{a_{\eta}}{a} (1 - 2\phi - 2\psi) (u^1)^2 = 0, \quad (9)$$

$$\frac{du^1}{d\tau} + \psi' (u^0)^2 + \frac{2a_{\eta}}{a} u^0 u^1 - \phi' (u^1)^2 = 0. \quad (10)$$

The areal radius of the spherical spacetime (a geometric and coordinate-independent quantity) is

$$R(t, r) = ar \sqrt{1 - 2\phi} \simeq ar (1 - \phi) \quad (11)$$

to first order. Since

$$\frac{dR}{dt} = (\dot{a}r + a \dot{r}) (1 - \phi) , \quad (12)$$

$$\frac{d^2 R}{dt^2} = (\ddot{a}r + 2\dot{a} \dot{r} + a \ddot{r}) (1 - \phi) , \quad (13)$$

and

$$\dot{r} \equiv \frac{dr}{dt} = \frac{dr}{d\eta} \frac{d\eta}{dt} = \frac{1}{a} \frac{dr}{d\tau} \frac{d\tau}{d\eta} = \frac{u^1}{au^0}, \quad (14)$$

$$\ddot{r} = \frac{d}{dt} \left( \frac{u^1}{au^0} \right) = - \frac{\dot{a}}{a^2} \frac{u^1}{u^0} + \frac{1}{a^2 u^0} \frac{d}{d\tau} \left( \frac{u^1}{u^0} \right), \quad (15)$$

we have

$$\frac{d^2 R}{dt^2} = \left[ \ddot{a}r + \dot{a} \frac{u^1}{au^0} + \frac{1}{au^0} \frac{d}{d\tau} \left( \frac{u^1}{u^0} \right) \right] (1 - \phi) . \quad (16)$$

The criterion $d^2 R/dt^2 = 0$ locating the turnaround radius \[23, 24, 25\] becomes

$$\ddot{a}r + H \frac{u^1}{u^0} + \frac{1}{au^0} \left[ \frac{1}{u^0} \frac{du^1}{d\tau} - \frac{u^1}{(u^0)^2} \frac{du^0}{d\tau} \right] = 0 , \quad (17)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter in comoving time. To zero order we have $R = ar$ and $\dot{R} = \dot{a}r + a \dot{\dot{r}} = HR$ (the Hubble law) for timelike geodesics, so that their 4-tangents have components

$$u^\mu = u^\mu_{(0)} + \delta u^\mu = \left( \frac{1}{a}, 0, 0 \right) + \delta u^\mu \quad (18)$$
in coordinates \((\eta, r, \theta, \varphi)\), where the perturbations \(\delta u^\mu\) are of first order. To zero order it is \(u^0 = d\eta/d\tau = d\eta/dt = 1/a\) and \(u^1 = 0\). Eq. \((\ref{eq:17})\) now yields, to first order,

\[
\ddot{a}r + 2\dot{a}\delta u^1 + a \frac{d(\delta u^1)}{d\tau} = 0.
\]

Eq. \((\ref{eq:18})\) gives \(u^0 = a^{-1}\) and \(u^1 = \delta u^1\) which, substituted into eq. \((\ref{eq:10})\), yields

\[
\frac{d(\delta u^1)}{d\tau} + \frac{\psi'}{a^2} + \frac{2a\eta}{a^2} \delta u^1 = 0.
\]

Inserting this equation into eq. \((\ref{eq:19})\) and using the fact that \(a_\eta = a\dot{a}\) finally gives

\[
\ddot{a}r - \frac{\psi'}{a} = 0
\]

to first order. Eq. \((\ref{eq:21})\) locates the comoving turnaround radius \(r_c\) once the post-Friedmannian potential \(\psi\) is given. In general, this is a transcendental equation\(^3\) and reduces to an algebraic one only for simple forms of \(\psi\). In an accelerating universe \((\ddot{a} > 0)\), and assuming a positive decreasing function \(\psi'(r)\) (as is the case, for example, for a Newtonian point mass \(\psi(r) = -m/r\) and as it is reasonable to expect in general for a single spherical perturbation), there exists a unique intersection between the straight line of positive slope \(y = \ddot{a}r\) and the decreasing function \(y = \psi'(r)\), which defines uniquely the turnaround radius \(r_c\). In a decelerating universe \((\ddot{a} < 0)\), by contrast, there is no intersection between these two curves and there is no turnaround radius.

Eq. \((\ref{eq:21})\) is the main result of this paper. The job is not complete, though, because one would like to know the areal, not the comoving, turnaround radius. The areal turnaround radius is

\[
R_c = a(t)r_c \left[1 - \phi(r_c)\right],
\]

where \(r_c\) is the unique root of eq. \((\ref{eq:21})\). The comoving turnaround radius depends only on the post-Friedmannian potential \(\psi\) but the areal turnaround radius

\(^3\)Often in Brans-Dicke and \(f(R)\) gravity one finds \cite{47, 48, 49, 50} potentials \(\psi \propto (\mu/r)^p\) where \(\mu\) is a constant mass coefficient and the exponent \(p\) is not integer, which makes eq. \((\ref{eq:21})\) non-algebraic.
radius, which is the physical radius, depends also on the second potential \( \phi \). By expanding this equation as

\[
 r \simeq \frac{R}{a} \left[ 1 + \phi \left( \frac{R}{a} \right) \right]
\]

(23)
to first order, eq. (21) turns into the equation satisfied by the areal turnaround radius \( R_c \)

\[
 \dot{a}R_c + \ddot{a} R_c \phi_c - \psi'_c = 0
\]

(24)

where \( \phi_c \equiv \phi (R_c/a) \) and \( \psi'_c \equiv \psi' (R_c/a) \), which makes it clear that \( R_c \) depends on both potentials \( \psi \) and \( \phi \). To make explicit the difference between modified gravity and General Relativity one can use the gravitational slip \( \xi \), which relates the potentials \( \phi \) and \( \psi \) through \( \psi = \phi (1 - \xi) \). Eq. (21) for the turnaround radius \( r_c \) in the geometry (1) can be written as

\[
 \ddot{ar}_c - \frac{\phi'_c (1 - \xi_c)}{a} + \frac{\phi_c \xi'_c}{a} = 0.
\]

(25)

Correspondingly, eq. (24) becomes

\[
 \ddot{a}R_c (1 + \phi_c) - \phi'_c (1 - \xi_c) + \phi_c \xi'_c = 0.
\]

(26)

3. Examples

Here we consider a few examples. Unfortunately, only a handful of analytical solutions of modified gravity theories are known which describe spherically symmetric inhomogeneities embedded in FLRW universes, and for these spacetimes the cosmological “background” is invariably decelerated instead of accelerated (see the recent review [52]), so they are not useful here.

3.1. \( \Lambda \text{CDM} \) model

As a first example, consider the \( \Lambda \text{CDM} \) model of General Relativity and the simple spherical perturbation potentials \( \psi(r) = -\phi(r) = -m/r \), with \( m \) a mass constant. This expression makes sense for sufficiently large values of \( r \) in order to keep the metric perturbations small. Using the FLRW acceleration equation

\[
 \frac{\ddot{a}}{a} = -\frac{4\pi}{3} (3P + \rho)
\]

(27)
where $P$ and $\rho$ are the pressure and energy density of the dark effective fluid dominating the dynamics of the accelerated universe, respectively, eq. (21) provides the comoving turnaround radius

$$r_c = \left[ \frac{3m}{4\pi |\rho + 3P| a^2} \right]^{1/3}. \hspace{1cm} (28)$$

The corresponding areal radius is, to dominant order,

$$R_c = ar_c = \left( \frac{3ma}{4\pi |\rho + 3P|} \right)^{1/3}. \hspace{1cm} (29)$$

where the comoving scale $ma$, rather than the constant $m$, is the physical mass to consider (see the corresponding mass in Refs. [23, 21, 25] and the discussion in [44]). This equation reproduces previous expressions of the turnaround radius found in Refs. [8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25]. In an expanding universe, the quantity $ma$ is identified with the total mass enclosed in a sphere of comoving radius $r$ and areal radius $R = ar$. This mass comprises the contribution of the “local” perturbation plus that of the cosmic fluid enclosed in the sphere [23, 24, 44]. In General Relativity, this mass $ma$ coincides with the Hawking-Hayward quasi-local energy [45, 46] and, because of spherical symmetry, also with the Misner-Sharp-Hernandez mass [46]. In modified gravity, however, these quasi-local energy constructs are not defined.

3.2. Generalized McVittie attractor

Generalized McVittie solutions of the Einstein equations were introduced in [51]. The old McVittie class of solutions of General Relativity [53] describes a central object (under certain conditions, a black hole) embedded in a FLRW universe. The McVittie class of spacetimes has been the subject of much recent attention [54, 53, 55, 56, 57, 58, 59, 60, 61] and it has been shown to solve the equations of cuscuton theory, a special Hořava-Lifschitz theory [62]. Accretion of the surrounding cosmic fluid onto the central McVittie object is artificially forbidden: in [53], McVittie imposed that the time-radius component $G_{tr}$ of the Einstein tensor vanishes, corresponding to zero radial flux of mass-energy onto the central inhomogeneity. The rationale for this assumption was simplifying the search for a solution of the Einstein equations describing a central object (in McVittie’s intentions, a point mass) embedded in a cosmological space. McVittie realized that it was much simpler to forbid
accretion than having to model it. Nonetheless, in General Relativity, one can allow a radial flow of energy, thus obtaining the so-called “generalized McVittie solutions” \[51\]. This radial energy flow is spacelike and rather artificial in this context. All simple models of heat conduction or of viscosity of imperfect fluids in relativity suffer from the inconsistency that heat conduction is instantaneous and obviously contradicts relativity, but imperfect fluids are used routinely as simple models because a proper relativistic thermodynamical treatment increases the complication to the point of rendering simple models impossible. Within the context of Horndeski gravity, the generalized McVittie solutions become more realistic scalar field solutions \[63\], in which the scalar field is susceptible of an effective fluid description. Within the class of generalized McVittie solutions there is a late-time attractor \[64\], with geometry described by

\[
\text{(30)}
\]

where \(m_0\) is a constant. In the approximation \(m_0/r \ll 1\), this line element reduces to

\[
\text{(31)}
\]

with \(\psi = -m/r = -\phi\) and we reduce formally to the previous example. This relation is not surprising since the metric (30) solves both the Einstein and the Horndeski field equations and the ansatz (11) represents a solution of the field equations of a theory, not the theory itself. Therefore, in this case, the turnaround radius does not discriminate between the \(\Lambda\)CDM model based on General Relativity and Horndeski gravity.

4. Discussion and conclusions

In generalized gravity, when the line element assumes the post-Friedmanian form (11) widely used in the literature and with a presently accelerating FLRW background, we have found a formula (eq. (21)) for the turnaround radius \(R_c\). The upper bound to the radius of stable virialized structures that can be obtained by observing the largest bound objects in the universe (those that are on the verge of breaking apart) would constitute an observable to look at in order to see a signature. Indeed the potentials \(\phi\) and \(\psi\) are commonly regarded as observables in modified gravity \[65, 66\].
Several modified gravity theories admit multiple representations, the most well-known situation being that of scalar-tensor gravity which can be analyzed in two conformal frames, the so-called Jordan frame and Einstein frame related by a conformal transformation of the metric plus a non-linear redefinition of the Brans-Dicke-like scalar field present in the theory \cite{67, 68, 69}. These two frames are in principle physically equivalent provided that a rescaling of the units of mass, length, and time (and of all the derived units) is performed \cite{67}, see also \cite{70, 71} and the references therein. However, it may be rather cumbersome to relate physical quantities in the two frames (see, e.g., the recent discussion \cite{72}), which renders the equivalence less useful for practical purposes.

Before eq. (21) can be compared with data on observed structures which are presumably on the verge of virialization (as done in Refs. \cite{23, 24, 25}), some more work needs to be done. Eq. (21) may test how accurately eq. (11) describes the spacetime geometry to first order in the metric perturbations, and it is unlikely that observations will require higher precision than first order. However, the same spacetime metric may be a solution of several theories of gravity, including General Relativity, at once. Thus, the theoretical problem arises of which solutions of a given theory of gravity are significant, in the sense of being generic and of being different from the standard one of Einstein’s theory. There is some degeneracy, in the sense that different theories of gravity, including General Relativity, can admit the same post-Friedmannian metric as a solution to first order in the metric perturbations (the same way that most theories of gravity admit the FLRW or the Schwarzschild metric as an exact solution). If, in addition to the FLRW background, the perturbed solution is the same then observational constraints on the turnaround radius will not detect deviations from General Relativity even if gravity was described by an alternative theory. Work needs to be done to understand which perturbed cosmological solutions are in some sense “generic” and, therefore, expected in various theories of gravity and whether the turnaround radius computed for such theories differs significantly from that of Einstein’s theory. If modified gravity does indeed explain the cosmic acceleration without dark energy, as is hypothesized, then it deviates from Einstein gravity at large scales and the turnaround radius holds promise for an independent test of gravity at the interface between astrophysics and cosmology.

A completely different, statistical mechanics approach to the turnaround radius is contained in Refs. \cite{73, 74}, in which galaxy clusters are studied by
including effective “dark energy particles” forming a perfect fluid and it is found that there is a “reentrant radius” corresponding to our turnaround radius. Beyond this radius thermal equilibrium of the gas is not possible and quintessence (with $P > -\rho$) increases the gravitational instability, while phantom energy (with $P < -\rho$) helps stabilizing against gravitational collapse. A comparison between our macroscopic approach and the statistical mechanics approach of [73, 74] is not straightforward because these authors assume a barotropic equation of state of state for dark energy which is linear and constant. While it is possible to write the field equations of modified gravity in the form of effective Einstein equations by collecting geometric terms into the right hand side as an effective fluid, in general the latter is not a perfect fluid and the effective equation of state is not linear nor constant. Imposing a constant effective equation of state would result in unphysical restrictions on the theory of gravity motivated only by mathematics. Nevertheless, it will be interesting to generalize the approach of [73, 74] to modified gravity in the future.

Acknowledgments

We thank two referees for helpful comments. This work is supported by the Natural Sciences and Engineering Research Council of Canada and by Bishop’s University.

References

References

[1] L. Amendola, S. Tsujikawa, *Dark Energy, Theory and Observations*, Cambridge University Press, Cambridge, 2010

[2] S. Capozziello, S. Carloni, A. Troisi, *Quintessence without scalar fields*, Recent Res. Dev. Astron. Astrophys. *1* (2003) 625.

[3] T.P. Sotiriou, V. Faraoni, *f(R) theories of gravity*, Rev. Mod. Phys. *82* (2010) 451

[4] A. De Felice, S. Tsujikawa, *f(R) theories*, Living Rev. Relat. *13* (2010) 3
[5] S. Nojiri and S.D. Odintsov, *Unified cosmic history in modified gravity: From F(R) theory to Lorentz non-invariant models*, Phys. Repts. **505** (2011) 59

[6] T. Clifton, P.G. Ferreira, A. Padilla, C. Skordis, *Modified gravity and cosmology*, Phys. Repts. **513** (2012) 1

[7] T. Baker, D. Psaltis, C. Skordis, *Linking tests of gravity on all scales: from the strong-field regime to cosmology*, Astrophys. J. **802** (2015) 63

[8] J.M. Souriau, *Un modèle d’univers confronté aux observations*, in *Dynamics and Processes*, Proceedings of the Third Encounter in Mathematics and Physics, Bielefeld, Germany, Nov. 30 - Dec. 4, 1981, Lecture notes in Mathematics vol. 1031, P. Blanchard, W. Streit eds., Springer-Verlag, Berlin (1981), p. 114-160

[9] Z. Stuchlik, *The motion of test particles in black-hole backgrounds with non-zero cosmological constant*, Bull. Astronomical Institutes of Czechoslovakia **34** (1983) 129

[10] Z. Stuchlik, S. Hledik, *Some properties of the Schwarzschild-de Sitter and Schwarzschild-anti-de Sitter spacetimes*, Phys. Rev. D **60** (1999) 044006

[11] Z. Stuchlik, P. Slany, S. Hledik, *Equilibrium configurations of perfect fluid orbiting Schwarzschild-de Sitter black holes*, Astron. Astrophys. **363** (2000) 425

[12] Z. Stuchlik, *Influence of the relict cosmological constant on accretion discs*, Mod. Phys. Lett. A **20** (2005) 561 [arXiv:0804.2266 [astro-ph]]

[13] M. Mizony, M. Lachiéze-Rey, *Cosmological effects in the local static frame*, Astron. Astrophys. **434** (2005) 45 [arXiv:gr-qc/0412084]

[14] Z. Stuchlik, J. Schee, *Influence of the cosmological constant on the motion of Magellanic Clouds in the gravitational field of Milky Way*, JCAP **9** (2011) 018

[15] B.C. Nolan, *Particle and photon orbits in McVittie spacetimes*, Class. Quantum Grav. **31** (2014) 235008 [arXiv:1408.0044 [gr-qc]]
[16] A. Maciel, M. Le Delliou, J.P. Mimoso, A dual null formalism for the collapse of fluids in a cosmological background, arXiv:1506.07122 [gr-qc]

[17] M. Le Delliou, J.P. Mimoso, F.C. Mena, M. Fontanini, D.C. Guariento, E. Abdalla, Separating expansion and collapse in general fluid models with heat flux, Phys. Rev. D 88 (2013) 027301

[18] J.P. Mimoso, M. Le Delliou, F.C. Mena, Local conditions separating expansion from collapse in spherically symmetric models with anisotropic pressures, Phys. Rev. D 88 (2013) 043501

[19] J.P. Mimoso, M. Le Delliou, F.C. Mena, Spherically symmetric models: separating expansion from contraction in models with anisotropic pressures, AIP Conf. Proc. 1458 (2011) 487

[20] J.P. Mimoso, M. Le Delliou, F.C. Mena, Separating expansion from contraction in spherically symmetric models with a perfect-fluid: Generalization of the Tolman-Oppenheimer-Volkoff condition and application to models with a cosmological constant, Phys. Rev. D 81 (2010) 123514

[21] M. Le Delliou, J.P. Mimoso, Separating expansion from contraction and generalizing TOV condition in spherically symmetric models with pressure, AIP Conf. Proc. 1122 (2009) 316

[22] M.T. Busha, F.C. Adams, R.H. Wechsler, A.E. Evrard, Future evolution of structure in an accelerating universe, Astrophys. J. 596 (2003) 713

[23] V. Pavlidou, T.N. Tomaras, Where the world stands still: turnaround as a strong test of ΛCDM cosmology, JCAP 1409 (2014) 020

[24] V. Pavlidou, N. Tetradis, T.N. Tomaras, Constraining dark energy through the stability of cosmic structures, JCAP 1405 (2014) 017

[25] D. Tanoglidis, V. Pavlidou, T.N. Tomaras, Statistics of the end of turnaround-scale structure formation in ΛCDM cosmology, arXiv:1412.6671 [astro-ph.CO]

[26] V. Faraoni, A. Prain, M. Lapierre-Léonard, Turnaround radius with Hawking-Hayward mass, arXiv:1508.01725, to appear in JCAP

[27] P.G. Ferreira, C. Skordis, Linear growth rate of structure in parametrized post-Friedmannian universes, Phys. Rev. D 81 (2010) 104020
[28] E. Bertschinger, *On the growth of perturbations as a test of dark energy and gravity*, Astrophys. J. 648 (2006) 797

[29] A. Joyce, B. Jain, J. Khoury, M. Trodden, *Beyond the standard cosmological model*, Phys. Repts. 568 (2015) 1

[30] C.D. Leonard, T. Baker, P.G. Ferreira, *Exploring dependencies in modified gravity with weak lensing*, Phys. Rev. D 91 (2015) 083504

[31] D. Munshi, G. Pratten, P. Valageas, P. Coles, P. Brax, *Galaxy clustering in 3D and modified gravity theories*, arXiv:1508.00583

[32] M.C. Chiu, A. Taylor, C. Shu, H. Tu, *Cosmological perturbations and quasi-static assumption in f(R) theories*, arXiv:1505.03323

[33] E. Bertschinger, P. Zukin, *Distinguishing modified gravity from dark energy*, Phys. Rev. D D78 (2008) 024015

[34] S.F. Daniel, R.R. Caldwell, *Consequences of a cosmic scalar with kinetic coupling to curvature*, Class. Quantum Grav. 24 (2007) 5573

[35] F.S. Daniel, E.V. Linder, T.L. Smith, R.R. Caldwell, A. Cooray, A. Leauthaud, L. Lombriser, *Testing general relativity with current cosmological data*, Phys. Rev. D 81 (2010) 123508

[36] L. Pogosian, A. Silvestri, K. Koyama, G.B. Zhao, *How to optimally parametrize deviations from general relativity in the evolution of cosmological perturbations?*, Phys. Rev. D 81 (2010) 104023

[37] G.-B. Zhao, T. Giannantonio, L. Pogosian, A. Silvestri, D.J. Bacon, K. Koyama, R.C. Nichol, Y.-S. Song, *Probing modifications of general relativity using current cosmological observations*, Phys. Rev. D 81 (2010) 103510

[38] R. Reyes, R. Mandelbaum, U. Seljak, T. Baldauf, J.E. Gunn, L. Lombriser, R.E. Smith, *Confirmation of general relativity on large scales from weak lensing and galaxy velocities*, Nature 464 (2010), 256

[39] A. Chernin, P. Teerikorpi, Y. Barishev, *Why is the Hubble flow so quiet?*, arXiv:astro-ph/0012021
[40] L. Amendola, M. Kunz, D. Sapone, Measuring the dark side (with weak lensing), *JCAP* **0804** (2008) 013

[41] J. Adamek, D. Daverio, R. Durrer, M. Kunz, General relativistic N-body simulations in the weak field limit, *Phys. Rev. D* **88** (2013) 103527

[42] J. Adamek, R. Durrer, M. Kunz, N-body methods for relativistic cosmology, *Class. Quantum Grav.* **31** (2014) 234006

[43] J.D. Barrow, P. Saich, Growth of large-scale structure with a cosmological constant, *Mon. Not. Roy. Astron. Soc.* **262** (1993) 717

[44] V. Faraoni, M. Lapierre-Léonard and A. Prain, Do Newtonian large-scale structure simulations fail to include relativistic effects?, *Phys. Rev. D* **92** (2015) 023511

[45] S. Hawking, Gravitational radiation in an expanding universe, *J. Math. Phys.* **9** (1968) 598

[46] S.A. Hayward, Quasilocal gravitational energy, *Phys. Rev. D* **49** (1994) 831

[47] C.H. Brans, Mach’s principle and a relativistic theory of gravitation, *Phys. Rev.* **125** (1962) 2194

[48] T. Clifton, J.D. Barrow, The power of general relativity, *Phys. Rev. D* **72** (2005) 103005

[49] T. Clifton, Spherically symmetric solutions to fourth-order theories of gravity, *Class. Quantum Grav.* **23** (2006) 7445 (2006)

[50] T. Clifton, D.F. Mota, J.D. Barrow, Inhomogeneous gravity, *Mont. Not. Roy. Astr. Soc.* **358** (2005) 601

[51] V. Faraoni, A. Jacques, Cosmological expansion and local physics, *Phys. Rev. D* **76** (2007) 063510

[52] V. Faraoni, *Cosmological and Black Hole Apparent Horizons* (Springer, New York, 2015)

[53] G.C. McVittie, The mass-particle in an expanding universe, *Mon. Not. Roy. Astron. Soc.* **93** (1933) 325
[54] N. Kaloper, M. Kleban, D. Martin, McVittie’s legacy: black holes in an expanding universe, Phys. Rev. D 81 (2010) 104044

[55] K. Lake, M. Abdelqader, More on McVittie’s legacy: a Schwarzschild-de Sitter black and white hole embedded in an asymptotically ΛCDM cosmology, Phys. Rev. D 84 (2011) 044045

[56] P. Landry, M. Abdelqader, K. Lake, The McVittie solution with a negative cosmological constant, Phys. Rev. D 86 (2012) 084002

[57] A.M. da Silva, M. Fontanini, D.C. Guariento, How the expansion of the universe determines the causal structure of McVittie spacetimes, Phys. Rev. D 87 (2013) 6, 064030

[58] R. Nandra, A.N. Lasenby, M.P. Hobson, The effect of an expanding universe on massive objects, Mon. Not. Roy. Astron. Soc. 422 (2012) 2945

[59] R. Nandra, A.N. Lasenby, M.P. Hobson, The effect of a massive object on an expanding universe, Mon. Not. Roy. Astron. Soc. 422 (2012) 2931

[60] V. Faraoni, A.F. Zambrano Moreno, R. Nandra, Making sense of the bizarre behaviour of horizons in the McVittie spacetime, Phys.Rev. D 85 (2012) 083526

[61] A.M. Da Silva, D.C. Guariento, C. Molina, Cosmological black holes and white holes with time-dependent mass, Phys. Rev. D 91 (2015) 084043

[62] E. Abdalla, N. Afshordi, M. Fontanini, D.C. Guariento, E. Papantonopoulos, Cosmological black holes from self-gravitating fields, Phys. Rev. D 89 (2014) 104018

[63] N. Afshordi, M. Fontanini, and D.C. Guariento, Horndeski meets McVittie: a scalar field theory for accretion onto cosmological black holes, Phys. Rev. D 90 (2014) 8, 084012

[64] V. Faraoni, C. Gao, X. Chen, Y.-G. Shen, What is the fate of a black hole embedded in an expanding universe?, Phys. Lett. B 671 (2009) 7

[65] M. Motta, I. Sawicky, I.D. Saltas, L. Amendola, M. Kunz, Probing dark energy through scale dependence, Phys. Rev. D 88 (2013) 124035
[66] E. Bellini, I. Sawicky, *Maximal freedom at minimum cost: linear large-scale structure in general modifications of gravity*, JCAP 07 (2014) 050

[67] R.H. Dicke, *Mach’s principle and invariance under transformation of units*, Phys. Rev. 125, 2163 (1962).

[68] Y. Fujii, K. Maeda, *The Scalar-Tensor Theory of Gravitation* (Cambridge University Press, Cambridge, 2003)

[69] V. Faraoni, *Cosmology in Scalar Tensor Gravity* (Kluwer Academic, Dordrecht, 2004)

[70] E.E. Flanagan, *The conformal frame freedom in theories of gravitation*, Class. Quantum Grav. 21 (2004) 3817 (2004)

[71] V. Faraoni, S. Nadeau, *(Pseudo)issue of the conformal frame revisited*, Phys. Rev. D 75 (2007) 023501

[72] V. Faraoni, A. Prain, A.F. Zambrano Moreno, *Black holes and wormholes subject to conformal mappings*, [arXiv:1509.04129](https://arxiv.org/abs/1509.04129)

[73] M. Axenides, G. Georgiu, Z. Roupas, *Gravothermal catastrophe with a cosmological constant*, Phys. Rev. D 86 (2012) 104005

[74] Z. Roupas, M. Axenides, G. Georgiu, E.N. Saridakis, *Galaxy clusters and structure formation in quintessence versus phantom dark energy universe*, Phys. Rev. D 89 (2012) 083002