REMARKS ON “SINGULARITIES”

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Abstract. We present herewith certain thoughts on the important subject of nowadays physics, pertaining to the so-called “singularities”, that emanated from looking at the theme in terms of ADG (: abstract differential geometry). Thus, according to the latter perspective, we can involve “singularities” in our arguments, while still employing fundamental differential-geometric notions such as connections, curvature, metric and the like, retaining also the form of standard important relations of the classical theory (e.g. Einstein and/or Yang-Mills equations, in vacuum), even within that generalized context of ADG. To wind up, we can extend (in point of fact, calculate) over singularities classical differential-geometric relations/equations, without altering their forms and/or changing the standard arguments; the change concerns thus only the way, we employ the usual differential geometry of smooth manifolds, so that the base “space” acquires now quite a secondary rôle, not contributing at all (!) to the differential-geometric technique/mechanism that we apply. Thus, the latter by definition refers directly to the objects being involved—the objects that “live on that space”, which by themselves are not, of course, ipso facto “singular”!

0. According to the Principle of General Relativity, physical objects are, so to say, “differentiable objects”, in the sense that General Relativity is rooted, roughly speaking, as a mathematical-physical theory, on the classical differential geometry of differential (so-called “smooth”) manifolds. Thus, in view of the above principle, (see, for instance, M. Nakahara [28: p. 28]),

\begin{equation}
(0.1)
\text{“all laws in physics take the same form in any coordinate system”}
\end{equation}
Yet, in accord with (0.1), it is also the so-called "gauge principle" (ibid., p. 10), viz. we still adopt that;

\begin{equation}
\text{"physics should not depend on how we describe it".}
\end{equation}

see also e.g. R. Torretti [37: p. 65]. Thus, in other words, the laws of Nature are independent of any particular coordinate system (: yet, "laboratory", or even "local gauge"), by means of which we virtually effectuate, each time (yet, make computations about) these laws!

Now, classical differential geometry (CDG) cannot be applied, by its very definition, on non-smooth objects, so that, if we are still going to employ methods of CDG to "non-smooth situations", we are thus compelled to concoct other means to overcome such type of impediments of the theory concerned. So it was exactly here that the mathematicians started to feel the need of developing techniques, similar to those of CDG (it seems they were already convinced of the effectiveness of the latter theory, as it actually happened, of course (!)), which, however, now should be able to cope with the so-called "singularities" phenomena.

In this context, we see thus two main tendencies, during the last few decades: The first (older) one imitates methods of CDG, that now could be applied already on not necessarily smooth functions, sticking, however, more or less, at notions of the classical theory, as, for instance, tangent vectors, hence, tangent spaces, vector fields, therefore, differential forms, as well, and the like. We can call this point of view Polish-German school (thus, R. Sikorski, M. Heller, W. Sasin et al., and K. Spallek, Buchner et al.). The other (more recent) one, presumably influenced, by quantum theory problems, is mainly concentrated on "non-commutative" notions, calculations etc, hence, forced also to develop a "non-commutative differential geometry" (French school, A. Connes et al.). In this concern, we further remark that, both of the previous perspectives try
to apply extended (generalized) methods of CDG to non-classical (differential-geometric) frameworks, as, for example, in a “general-relativistic set-up with singularities”.

Within the same context, we should mention here what we may call the American school, appeared about the same period, with the first one, as above, started by N. Aronszjan and further continued by C.D. Marshall, J.W. Smith, K.T. Chen, M.A. Mostow et al. In this regard, we should also notice an early appearance (1956) of such a perspective, as before, of generalizing the notion of a differential manifold, by the Japanese mathematician I. Satake (therefore, the so-called “Satake manifolds”, or even “V-manifolds”).

On the other hand, we have the recently developed “abstract differential geometry” (ADG). Here, quite independently of all the previous methods, this technique started, at a first stage, from topological algebra theory notions, entangled already in an intense geometrical context (we also remark here that the totality of scalar-valued differentiable functions on a given smooth manifold is, in effect, an important (non-normed!) topological algebra), and finally concluded to entirely algebraic (no more topological-algebraic) concepts; the crucial instrument herewith was sheaf theory (of course, the methods of the latter theory are also algebraic), as well as, sheaf cohomology. In point of fact, one aims here at developing classical differential geometry, in a abstracto, where no Calculus, hence, (local) smoothness of the standard theory is needed at all! So a general idea herewith is that;

\[ (0.3) \]

whenever we try to abstract a notion of CDG, we have first to find the pertinent function, that may represent (be connected with) it, and then translate it into the appropriate sheaf morphism.

The big surprise and vindication, as well, of the methods of ADG happened quite recently, when realizing that one could use, as “domain of numbers”, alias “(extended) arithmetics” (yet, “sheaf of coefficients”) of ADG the strange (!), however, quite efficient, indeed, algebra of “generalized functions” and even, more generally, “multi-foam algebra” of such functions, initiated by
E.E. Rosinger, in effect, the respective sheaves of these algebras (see e.g. E.E. Rosinger [33], or even A. Mallios-E.E. Rosinger [25]. The important thing here is that the previous algebras of (generalized) functions contain, by their very definition, a tremendous, in point of fact, by simply referring to multi-foam algebras, as above, the biggest, thus far, amount of singularities (of any “type”), that one can consider. In this context, it is still to be noticed that the aforesaid algebras have already a successful career in problems, pertaining to non-linear PDE’s. So, in other words,

\[(0.4)\]

by applying the technique of ADG, when using Rosinger’s algebra sheaf, as our “arithmetics”, we actually derive, by looking at the corresponding equations (of the theory), solutions, which are free of singularities. Namely, those singularities that are already contained in our (Rosinger’s) algebra (sheaf) of coefficients.

One can remark here that something like this was already a demand of A. Einstein. That is, we do have now, in that context, such a

\[(0.5)\]

“method ... to derive ... solutions ... free of singularities ...”,

as he actually was looking for (see [6: p. 165]). Furthermore, another moral of the preceding is that,

\[(0.6)\]

whenever we meet a “singularity”, in the classical sense of the word, one has just to find the appropriate algebra of coefficients, that contains it, provided, of course, the algebra at issue fits in also well, within the framework of ADG (take, for instance, the pertinent Rosinger’s algebra (sheaf)).

As an anticipation to our last proposition above, one may certainly look at the argument of D. Finkelstein, referring to the “past-future asymmetry of the gravitational field” (: “Eddington-Finkelstein coordinates”, see e.g. [27: p. 828]; my thanks are due here to I. Raptis, who brought to my attention the
relevant work of Finkelstein). Within the same framework, concerning the potential significance of incorporating singularities into the standard equations (of general relativity), see, for instances, M. Heller [12: p. 924], along with the relevant Refs therein.

Now, the aforementioned algebras are, by definition, commutative, while, of course, it is actually through them that we always make our particular calculations. In this concern, we should also notice that calculations, according to a famous apostrophe of N. Bohr, even when referred to a quantum-theoretic framework, have to be commutative (!). Yet, by still paraphrasing the same motto, we can further say that “our measuring apparatus is a classical object, giving classical results, hence, commutative ones (in point of fact, eigenvalues, viz. c-numbers, of the “observable operator”; see also, for instance, R. Gilmore [11: p. 71]).

Yet, as a further corroboration of the methods of abstract differential geometry, that might also be construed, as being in accord with the aforementioned “gauge principle” (cf. (0.2)), one might refer to the fact that the Einstein equation (in vacuo) has exactly the same form in the classical theory and in ADG. (See A Mallios [19: p. 89, (3.11), or even the same author [21: Vol. II; Chapt. IX, Section 3, rel. (3.11)]).

1. By commenting upon (0.3), as above, we can still remark that Leibniz, already at his time, was looking for a “geometric calculus”; that is, for a device, “acting directly on the geometric objects, without [at all] the intervening of coordinates” (commutative or not!). We recall here that the same great scientist wrote once to de L'Hôpital that “the secret of Analysis lies [exactly] in an apt combination of symbols” (!).

In this context, we can even say that the power of differential geometry, the latter being, in effect, an application of “differential analysis” (Calculus) in studying the “geometry” (viz. the inner structure) of some “space”, is finally proven to be the result of an inherent mechanism, supplied by the said device (discipline), being, in point of fact, independent of any notion of “space”. Therefore, a mechanism (: “calculus”) referred to the objects themselves, that
fill up the “space”. This, of course, still justifies our endeavor of today to employ that same mechanism, even in the quantum world (see below “gauge theories”), notwithstanding at that deep there is no, in effect, any notion of “space”, in the usual sense of the term (see also, for instance, C.J. Isham [15: p. 400]). However, to be more precise, the “space” is also here. Namely, even at the quantum deep, as well, the same, as anywhere else; that is,

\[(1.1) \text{the totality of the (“geometrical”) objects (for the case at hand, the elementary particles) themselves (that fill “it” up).}\]

Accordingly, these same objects, viewed, by virtue of the preceding, as “geometrical” ones, can further be treated, as such, too, given that, to this end, we do not actually need any space to refer to, apart from the objects themselves, that we are observing (detecting). The above is still another crucial outcome of ADG. So, practically speaking, we can say that,

\[(1.2) \text{to perform ADG, one does not actually need any “space” at all!}\]

Accordingly, one can also look at the above issue, as a post-anticipation or, at least, as a response to the aforementioned demand of Leibniz.

Here it is also worthwhile to recall an utterance of B. Riemann in his famous Habilitationsschrift (: “Über die Hypothesen, welche der Geometrie zu Grunde liegen”, 1854), in that

\[(1.3) \text{“Maßbestimmungen erfordern eine Unabhängigkeit der Größen vom Ort, die in mehr als einer Weise stattfinden kann.”: Specifications of mass [: measurements] require an independence of quantity from position, which can happen in more than one way.}\]

See also, for instance, M. Spivak [35: p. 140]. The above phraseology of Riemann may certainly be considered, as further supporting the aspect (cf. also (1.1) in the preceding) that the
“space” consists actually of the objects that fill “it” up, so that the \((differential)\) geometry on it virtually \textit{refers to these \textit{same objects}}; in other words, to the \textit{interrelations} among, or \textit{evolutions} of, them.

Indeed, this very type of \((differential)\) geometry, as initiated by K.F. Gauss and then extended by B.G. Riemann to any \(n\)-dimensional \textquote{manifold}\(^{(}(:\text{Mannigfaltigkeit}),\text{concerns} \text{again} \text{the object (\textit{manifold}) itself, irrespectively of any surrounding \textit{space}}.\) The above dictum of Riemann, may also be related, of course, with the \textit{gauge principle} (cf. (0.2)), the latter providing thus still another support of (1.4), or even of (1.2), as above.

Moreover, as further potential applications of ADG, and in conjunction with the above type of \textit{generalized functions}, à la Rosinger, we can still mention, as already hinted at in the preceding, the nowadays \textit{gauge theories}, being (F.M. Atiyah) \textit{physical theories of a geometrical character}, like Yang-Mills theory, for instance, as well as \textit{geometric (pre)quantization}.

2. Now, by coming back to our previous comments, pertaining to \textit{Leibniz’s “geometric calculus” (or “\textit{ars combinatoria}” in his own words, something that he was still attributing to Analysis too, see the preceding Section 1)}, we can further remark that, according to Leibniz,

\begin{equation}
\text{“geometric objects” do or, at least, should exist, by themselves, independently of any supporting or surrounding \textit{“space”}, the latter being thus simply used to provide us with the corresponding coordinates.}
\end{equation}

Consequently, one has to find, as already discussed in the foregoing, a \textit{mechanism (alias \textit{calculus})}, pertaining directly to such objects, without the intervening of any space, providing the \textit{coordinates}, that is, to say, \textit{“location of the objects in the space”}. Of course, even then we may resort to a certain \textit{“reference point”}, that, however, finally disappears from our conclusions (equations), something like, for instance, we effectuate in \textit{affine geometry}. This reminds us here of the famous apostrophe of Archimedes, which goes
(2.2)\[\text{"Δός μοι πᾶ στῶ καὶ τὰν γὰρ κινήσω"} (: \text{give me somewhere to stand and I shall move the earth}).

Cf. Simplicius [34]. We may still understand, in that very same way, the rôle of the \textit{"base-space"} $X$, used in sheaf theory, as applied in ADG (see also (2.6) in the sequel). Hence, by considering here our experience of today, we can certainly anticipate that such a perspective on \textit{"differential geometry"} should naturally have potential applications in the régime of nowadays \textit{quantum theory}, yet, in particular, in \textit{quantum relativity}.

Now, to be fair, we still notice herewith, that the so-called today \textit{"coordinate-free"} differential geometry did exactly the aforesaid job, already (!), concerning the entanglement of coordinates in our calculations (arguments), referring to (differential-)geometric questions, so that, finally, our \textit{conclusions} (formulas) being possible to be \textit{stated in a coordinate-free manner}; however,

the problem was still with the \textit{(algebra of) functions}, used to do the job. As we shall perceive, through the subsequent discussion, the same job could (and, indeed, \textit{can}) be done by other, \textit{more convenient}, means! (See e.g. (0.4), along with the comments following (2.8) in the sequel).

Thus, classically speaking, one employs here again \textit{smooth functions} on a \textit{smooth manifold}, hence, the appearance of \textit{"singularities"}, where the functions involved loose their meaning (viz. their calculational power), notwithstanding, \textit{the (intrinsic !) mechanism} of the method applied (: differential geometry) \textit{is still present, very likely of help (!)}, according to our classical (in the absence of \textit{"singularities"}) previous experience, but, \textit{we cannot read it}, due to our apparatus employed (smooth functions!).

\footnote{\text{In this connection, we can further refer to A. Einstein, saying (ibid., p. 165) that; \"... we cannot judge in what manner and how strongly the exclusion of singularities reduces the manifold of solutions\" [the emphasis here is ours], viz. the amount of information, we get out from (or, which is included in) there (i.e., in the (set of) \textit{"singularities"}, cf., for instance, Finkelstein, as before).}}
Of course, the aforementioned development of recent (classical) differential geometry (viz. that one on smooth manifolds) greatly supported and contributed in the formulation of abstract (viz. axiomatic) differential geometry, while it was underscored and further brought on the stage, by ADG. Thus, what amounts to the same thing,

\[(2.4)\] in that “coordinate-free” treatment of CDG it was virtually hidden the overall power of the inherent mechanism of (classical) differential geometry.

That is, in a sense, the much sought after, already by Leibniz, “geometric calculus”. Indeed, and this is the most fundamental moral, which one gets out from ADG; namely, the fact that:

\[(2.5)\] that inherent powerful mechanism of CDG is, in point of fact, independent of any notion of Calculus, in the classical sense of differential analysis, hence, and this is here still worth mentioning, of any notion of smooth manifold (providing virtually that Calculus, see (2.6) below), whatsoever!

Consequently, within the same context, we further remark that,

\[(2.6)\] the only contribution, within the preceding framework, of the smooth surrounding space was simply to supply the corresponding (smooth) Calculus, thus, in turn, the smooth algebra of coefficients, along with the pertinent, very instrumental, indeed, “de Rham exact sequence”. (For brevity’s sake, we abuse here terminology, just hinting at the (exact) resolution, in effect, which is provided, by the corresponding de Rham complex of the usual differential forms, as an outcome of the classical, for the case at issue, Poincaré’s Lemma).

Indeed, the next really fundamental inference, one gets out from the abstract (: axiomatic) treatment of differential geometry is that,
the aforesaid inherent mechanism ("geometric calculus", à la Leibniz) of CDG is absolutely rooted on the type of "generalized numbers", thus, in the case of CDG, on the algebra (in effect, on the algebra sheaf) of scalar-valued differentiable (: smooth) functions, along with the concomitant ("exact", cf. (2.6)) differential de Rham complex.

Thus, by referring to "Differential Geometry", in the classical sense (of $C^\infty$-manifolds), one actually means the study of the structure of a "locally Euclidean space", through the "geometric calculus" (in the sense of Leibniz, as above); that is, to say, in terms of the "differential geometric mechanism" (we might call it ADG), being, anyway, independent of the particular "space" at issue, the same being virtually based, for the case under consideration, on the classical "de Rham differential triad" (cf., for instance, concerning the last term, [VS: Chapt. X, p. 278; (1.1)])

Therefore, in that sense, we can very likely say that this type of differential geometry cannot be applied in a "true quantum gravity" (see, for instance, C.J. Isham [15: p. 400]), however, by no means (just, because of the "space") its mechanism, as well; cf. also the ensuing remarks in the present section, in particular, Section 5 in the sequel.

In point of fact, in contrast to the above, and in full generality, one realizes, as a moral of ADG, that;

in the general (: abstract) case, any appropriate algebra (sheaf), not even a functional one (!), that is still accompanied with a suitable ("exact differential") de Rham complex, can do the same job (: ADG).

As a result, we come thus to the conclusion that,
Remarks on “singularities”

it would be, of course, of paramount importance, any time we could afford a “mechanism”, hence, at the very end, an “algebra of coefficients”, incorporating previously appeared disturbances (‘singularities’), being, however, still able to provide the pertinent “differential” set-up!

As already said in the preceding, the previous data, as, for instance, in (2.9), are exactly provided by Rosinger’s algebra sheaf, incorporating, lately, the so-called “multi-foam algebras”. This certainly constitutes, so far, an extremely non-trivial corroboration of the abstract method, being, moreover, quite sensible to “analytic” questions, in the classical sense, e.g. applications in PDEs. Yet, it is a hunch that, very likely, the same abstract method, as above, will have further potential applications in problems connected with quantum gravity. Thus, see e.g. [24; 25], as well as, the ensuing Sections 3 and 4 below.

3. By commenting further upon our previous argument in Section 1 (see, for instance, (1.1)), we can still refer to some relevant thoughts of V.I. Denisov and A.A. Logunov [2], where they remark that;

“Minkowski was the first to discover that the space-time, in which all physical processes occur, is unified and has a pseudo-Euclidean geometry. Subsequent study of strong, electromagnetic, and weak interactions has demonstrated that the pseudo-Euclidean geometry is inherent in the fields associated with these interactions” [the underline here is ours].

Yet, they also remark that,

“... for an equation to be covariant it is necessary that it is transformed according to a tensor law for any arbitrary, admissible coordinate transformation”.

The same authors attribute the above to V.A. Fock [10]. Now, according to the relevant set-up of ADG,
“equations” are expressed by (sections of) sheaf morphisms, in effect, by morphisms of vector sheaves, hence, in other words, by “\(A\)-morphisms” (where “\(A\)” stands here for the “structural (algebra) sheaf”, alias “generalized numbers” of the theory). Therefore, by their very definition, in terms of tensorial morphisms (!), hence, in accord with the so-called “principle of general covariance”; see, for instance, Yang-Mills equations, in terms of the curvature, still, within the framework of ADG. Cf. [17: p. 167; (2.3)], or even [21: Vol. II; Chapt. VI, (4.76)/(4.77)].

This, of course, constitutes further another vindication of the naturalness of ADG (viz., in effect, of the sheaf-theoretic treatment of the same). It seems that everything is inherent there, alias “innate” (I owe the latter suggestive (synonymous) expression to I. Raptis).

Within the same vein of ideas concerning (3.1), about the same time, T.H. Parker [29], remarks that,

(3.4) 
“... the topology is inherent in the field” (!);  

still the exclamation sign here is ours. Yet, we also remark that the notion of topology in field theory is, in point of fact, a matter of homotopy, and at the very end, of algebraic topology, as e.g. cohomology classes, Poincaré Lemma, de Rham complex, characteristic classes and the like. Thus, finally, we may say that:

the notion of “field” seems to be herewith predominant and overwhelming, being further inextricably connected with that one of an “(elementary) particle”. Indeed, according to the technical part of ADG,

(3.5)  
the two notions, as above, may be viewed, as identical.
In this concern, the notion of “field” appears thus, as a fundamental one, and, as in the classical case, “not further reducible” (A. Einstein); on the other hand, within the context of ADG, this now is independent of any “surrounding space”, while we are still able to employ directly on the “fields” the whole machinery of ADG, to the extent, at least, that this is feasible, thus far.

Furthermore, the above deliver us from the classical “drawback that the continuum brings” (A. Einstein, again). As a matter of fact, we are trapped here into the latter notion, as a result, in effect, of our adherence to the concept of “space-time continuum”, as an appropriate \( (C^\infty-\text{manifold}) \). However, as already pointed out in the preceding (cf., for instance, (2.4)), this is no more necessary, the machinery of ADG being still in force, without it; in other words, we can still say (see also, for instance, (0.3) in the preceding), that,

\[ (3.6) \]

we are thus able to “formulate statements about a discontinuum without calling upon a continuum space-time”,

something that also provides us with the possibility of thinking of the “real”, without the need of resorting, inevitably, to the “continuum”, a disputable, at the very end, point of view, apart, of course, from its own (mathematical) definition.

Moreover, within the same context, we can still refer here, once more, to A. Einstein himself, by saying that (emphasizing is ours);

\[ (3.7) \]

“... Adhering to the continuum originates with me not in a prejudice, but arises out of the fact that I have been unable to think up anything organic to take its place ...”

See A. Einstein [5: Vol. 2, p. 686]. Yet, we can also refer at this place to R.P. Feynman’s criticism, by referring to the “continuum in the quantum deep”, thus saying, among other things, that;
"... the theory that space is continuous is wrong, because we
get ... infinities [viz. "singularities"] and other similar diffi-
culties ... [while] the simple ideas of geometry, extended down
to infinitely small are wrong!"

Consequently, we thus realize here too, either the lack of an "organic theory",
which would be able to cope with the absence of an a p r i o r i "continuum",
or even, and more so, with the occasional appearances of infinities. However,
we can still remind us here that, avoidance of "infinities" (: "something too
great") does not pertain, anyhow, to "sensible mathematics". That is,

"Sensible mathematics involves neglecting a quantity when it
turns out to be small — not neglecting it just because it is
infinitely great and you do not want it".

See P.A.M. Dirac [3: p. 36].

Yet, when referring, just before (cf. (3.8)), to "simple ideas of geometry",
one rather means those (simple, viz. fundamental) principles of "differential
geometry", that we wanted to be applicable in the quantum deep, as well. But,
as already hinted at in the preceding,

the power and effectiveness of "differential geometry" rests,
in effect, in its inherent (: innate) mechanism, being of an
algebraic (: operational) character, independently of any sur-
rounding space, the same mechanism referring, in point of
fact, directly, to the "objects", we are dealing with.

In this context, and in close connection with our last comments in (3.10)
above, which further point to the aptitude of ADG for confronting with recent
perspectives on the subject, concerning the correspondence,

\[ (3.11) \]

\[
\textit{differential geometry} \text{ (in effect, its mechanism) — "space"},
\]

one can also mention Finkelstein’s apostrophe, in that,
"... we take acts as basic instead of points".

See D.R. Finkelstein [9: p. 425]. Indeed, we can still say that we have again here, once more, another variant of "Klein’s principle" ("space" is determined by (the group of) its automorphisms). Furthermore, as Denisov and Logunov remark (loc. cit.), see also (1.1) in the preceding,

(Pseudo-Euclidean space-time is not a priori, i.e., given from the start, or having an independent existence. It is an integral part of the existence of matter, ... it is [always] the geometry by which matter is transformed.)

In this concern, we can also remark here that,

moS is transformed, according to the (dynamics of the) physical law.

Within the same point of view, we further note that one can still understand the classical, “matter tells space how to curve” (cf. [27: p. 5]), exactly in the sense of (3.14), that is,

space (: matter) is curved, according to (the dynamics of the) physical law.

Yet, by also referring to (3.14), we can actually say that,

the variation (transformation) of the matter is equivalent with the existence of an $A$-connection (viz., physically speaking, with the physical law itself).

Indeed, by restricting ourselves to a (free) boson, for instance, we can associate (3.15) with the basic relation of ADG,

(3.17) \[ \delta(\theta_\alpha) = \delta(g_{\alpha\beta}), \]

so that our assertion in (3.15) is just a consequence of the so-called Atiyah’s criterion (for the existence of an $A$-connection: see [VS: Chapt. VII, p. 115]).
Anastasios Mallios

As a matter of fact, an analogous formula to (3.16) holds true, for any vector sheaf $\mathcal{E}$, in general; viz. one has

$$\delta(\omega^{(\alpha)}) = \tilde{\delta}(g_{\alpha\beta}),$$  

"transformation law of potentials", where one sets

$$\delta(\omega^{(\alpha)}) := \omega^{(\beta)} - Ad(g_{\alpha\beta})\omega^{(\alpha)},$$

such that

$$\omega^{(\alpha)} \in C^0(\mathcal{U}, M_n(\Omega))$$

stands for the corresponding $\mathcal{A}$-connection 0-cochain matrix form of $\mathcal{E}$, associated with a given local frame $\mathcal{U}$ of it: loc. cit., p. 113, Theorem 2.1.

Yet, within the same context, as above one still realizes that; (3.14) implies, in effect, (3.16), in the sense that:

the variation of the matter is actually the way the physical law itself is revealed to us; that is, the “causality” itself, or else (in technical, viz. mathematical, terms) the $\mathcal{A}$-connection (differential equation), detected, finally, through its strength (: curvature/solution of the corresponding differential equation at issue). See also the same rel. (3.19), as above.

Now, looking at the things locally (hence, what we actually observe/measure), we can say that, the whole story is virtually reduced to

$$\text{Aut}\mathcal{L},$$

viz. to the group sheaf of automorphisms of $\mathcal{L}$, the carrier (: “space”) of the (bare) boson at issue. Thus, in general, by considering a vector sheaf $\mathcal{E}$, of rank $n \in \mathbb{N}$, over $X$, carrier of a (bare) fermion (see also A. Mallios [21: Vol. I; Chapt. II, (6.29)], concerning the terminology employed herewith), one looks at the corresponding group sheaf of automorphisms of $\mathcal{E}$,

$$\text{Aut}\mathcal{E}.$$
Therefore, by taking the things locally (on a local gauge $U$ of $E$), one gets:

$$\left(\text{Aut}(E)|_U\right) = \text{Aut}(E|_U) = \text{Aut}(\mathcal{A}^n|_U) = \mathcal{GL}(n, \mathcal{A}|_U)$$

(3.24)

$$= (\text{Aut}(\mathcal{A}^n)|_U) = \mathcal{GL}(n, \mathcal{A}|_U),$$

so that one still obtains,

$$\left(\text{Aut}(E)(U) = \text{Aut}(E|_U) = \mathcal{GL}(n, \mathcal{A})(U) = \mathcal{GL}(n, \mathcal{A}(U)).\right)$$

(3.25)

See also loc. cit., Chapt. I; Section 6, in particular, (6.31) and (6.36).

Thus, in view of (3.22) and (3.23), as above, one further has herewith another instance of Klein's point of view, as before, so that one effectuates, once more (see also (1.1) in the preceding), that, in our case,

(3.26)

“space” is $E$,

in point of fact, it is represented by $E$, or even, equivalently, according to the Kleinian perspective, as before, by its (group sheaf of) automorphisms, $\text{Aut}(E)$.

This latter aspect might also be related, pretty well, with nowadays tendencies to look, namely, at the “quantum deep”, as something of a foamy, fuzzy, and the like, nature (cf., for instance, “quantum soup”(!)). Furthermore, the same point of view, as well, may be connected when arguing, at least, within a “vacuum”, with a quite recent theoreis of the above, in terms of “quantum causal sets”; see thus, for instance, I. Raptis [31]. In this connection, we can still remark that the relevant here “inner product problem” and the “problem of time” (ibid.), are naturally transferred to the respective sheaves, that, within ADG represent the objects (e.g. elementary particles) at issue, in particular to the group sheaf of automorphism, $\text{Aut}E$, as above, this being, in effect, through its action, the “dynamics” itself of the corresponding (vector sheaf) $E$ (: “problem of time” ). Yet, by defining an “inner product” on $E$, we still require this to be unaltered, under the action of $\text{Aut}E$ (: “inner product problem” ). So the previous two fundamental problems of the standard theory referred, in that classical context, to $\text{Diff}(X)$, group of automorphisms (: diffeomorphisms) of the base (space-time) manifold $X$, hence the occasionally concomitant “singularities” (!), are thus reduced here to $E$, therefore, locally to $\mathcal{A}$, our “arithmetics”, which,
as already said, can just now incorporate singularities; see also (3.28) below, as well as, (6.1) in the sequel, or even A. Mallios [21: Vol. II; Chapt. IX].

Yet, by still commenting on the above “Kleinian perspective”, we can further set out the following meditations (argument):

\[
\text{Evolution is an algebraic automorphism (Feynman), expressed analytically, via the Hamiltonian (Schrödinger, viz. } \partial_t = H, \text{ “evolution”); this, in turn, entails again, still algebraically (Heisenberg-Prigogine-Kähler-Hiley), the “time operator” (description, in time, of the physical system).}
\]

(3.27)

The previous conclusion has been motivated mainly from a quite recent account thereof of B.I. Hiley [14]. (I am indebted to Prof. Hiley for kindly giving to me access to his relevant manuscript). Still, the sheaf-theoretic character” of (3.23) may also be construed, as being in accord with the above “evolutionary” point of view, advocated by (3.27).

On the other hand, the previous aspect, concerning Aut(\mathcal{E}), still falls in with the standard “locality principle” (see also (3.24), along with (3.29) below, or even [22: p. 1896]). That is, in other words,

\[
\text{the same “principle of locality” stands in complete agreement with the sheaf-theoretic flavor of the present treatment, in that, both aspects lead naturally from local information to global perspective, while the very notion of a sheaf is still quite akin to the “relativistic” (varying) aspect, yet, equivalently, in terms of sections (our calculations!).}
\]

(3.28)

On the other hand, the preceding point out, once more, to the significance, as well, of the “structure sheaf, or else “sheaf of coefficients”, \mathcal{A}, for the whole subject, in such a manner that, roughly speaking, one can say that;

\[
\text{everything (concerning our calculations) is locally reduced to } \mathcal{A}, \text{ equivalently, to sections of the same sheaf.}
\]

(3.29)
An analogous conclusion is still valid for the space of $A$-connections of $\mathcal{E}$,

\[(\text{3.30}) \quad \text{Conn}_A(\mathcal{E}),\]

hence, for the corresponding moduli space of $\mathcal{E}$,

\[(\text{3.31}) \quad \mathcal{M}(\mathcal{E}) \equiv \text{Conn}_A(\mathcal{E})/\text{Aut}\mathcal{E},\]

as well, where too the things are locally reduced to $n \times n$ matrices (of sections) of $\mathcal{A}$, and/or similar ones of (sections of) “1-forms”, viz. of the $\mathcal{A}$-module $\Omega^1$, hence, finally, of $\mathcal{A}$ again, if, as is usually the case, $\Omega^1$ is still a vector sheaf on $X$. (In this regard, cf. also [VS: Chapts VI, VII]).

Of course, once more, it goes always without saying that,

\[(\text{3.32}) \quad \text{we gain very much in insight, any time we are able to free our conception, we have about a specific physical problem, from any reference to some particular coordinate system (: “space”).}\]

In this context, we can also remark that

\[(\text{3.33}) \quad \text{“local gauges” (viz. “coordinates”, or even “space”) are used here for our calculations, yet, to detect the (independently of the former existing) physical law, while to understand the latter one has to free his conception of the former (viz. of the means applied).}\]

So, within the same vein of ideas, and, technically speaking, as it concerns the operational (alias, litourgical) part of ADG, we can further say that,

\[(\text{3.34}) \quad \text{algebraic topology can be conceived, as a “relational hence, algebraic, in nature way of looking at the topology [: space], therefore, as more akin to physics.}\]

Thus, we come still here to the relevant point of view, emphasized by C. von Westenholz [39: p. 323], when saying that;
“The mathematical structure underlying field quantities ... is essentially de Rham cohomology”.

(3.35)

4. Accordingly, the problem, as well as, our own progress in confronting with it, is thus, very likely, lying, to quote here W. Stevens [36: p. 184], in a “movement through changes in terminology”, which we finally apply, when look at the particular problem, we are interested in.

We wish to terminate the previous discussion by just referring, as an example of the viability of the preceding thoughts, to other authors of the very recent past (the “Polish school” mentioned in the above, see Section 0), in whose work one also finds similar considerations to the foregoing material; thus, we read, for instance, the utterance that,

(4.1)

“... the imagination is very often restricted by constraints coming from the language we use”.

See M. Heller et al. [13: p. 54]. Yet, we still recall, in that context, Einstein’s own motto that, “imagination is more important than knowledge”. So, everything that could affect our imagination (the language we use, for example), affects, in point of fact, our ability of knowing (: describing) the reality.

Finally, as an overall moral of the preceding, we still conclude the following two-way (: “amphidromous”) relation,

(4.2)

\[ \text{physics} \leftrightarrow \text{(differential) geometry}, \]

the second member of the above diagram hinting actually at the innate mechanism, viz. the “ars combinatoria”, or even “geometric calculus”, à la Leibniz, of the mathematical discipline at issue, as explained in the foregoing. Accordingly, by simply paraphrasing herewith S. Mac Lane [16: p. 257], we realize that;

(4.3)

“geometry is not just a subdivision or a subset within Mathematics, but a means ...” to get out of phenomena (: physics) formal rules (: physical laws),
that also fits in well with the preceding, justifying further (4.2). In that sense a formal rule, as before, that is, in other words, a physical law cannot be dependent on (or even restricted by) a “singularity”. Yet, to paraphrase here A. Einstein [6: p. 164], “singularities must be excluded” from a procedure, whose function can be described, according to (4.3).

Thus, what we actually perceive, as “laws of Physis” is given, in the sense that these “laws” are there. On the other hand, it is we, who do not provide the proper theories to follow (: “understand”, or even better, to describe) these laws, yet to further predict their evolution. So it is in that very sense that;

\[
\text{“the “laws” of Nature cannot have anomalies, alias “singularities”, an attribute that is virtually ours, since here again, the manner in which these laws function is, certainly, still, given!”}
\]

Finally, within this same vein of ideas, we can further remark that;

\[
\text{“we are virtually part of what we understand as Nature, not the creators of the latter. Thus, what we ascribe, as anomalies, “singularities” etc., are just our own verdict (interpretation) about Nature and not “particular instances” of it.}
\]

5. Now, let us come back again to the aforementioned remarks of R.P. Feynman (see (3.8)), according to which,

\[
\text{“the simple ideas of geometry, extended down to infinitely small are wrong!”}
\]

Thus, by what has been said in the preceding, concerning the above apostrophe of Feynman, one may remark, that we can virtually have here a quite different situation, pertaining, at least, to the differential geometry, in point of fact, to its inherent (differential-geometric) mechanism, yet, in other words, the implemented thereof “ars combinatoria” à la Leibniz. That is, although we cannot speak of the space (roughly speaking, the “continuum”) that entails, alias supports, the standard differential geometry of smooth manifolds,
notwithstanding, we can still apply, even to that extended deep the accompanied (innate) mechanism of that geometry, that is, in other words, its differentiable functions”, suitably generalized, along with the attached to them differential-geometric technique that still seems to work, even in that régime!; indeed, this is actually due, to the very nature of the generalized functions (in effect, sections of appropriate sheaves), which, thus, are involved.

So, in this context, we can virtually assert that,

\[
\text{“the simple ideas of [differential] geometry”, [at least], are not wrong!}
\]

(5.2)

Indeed, as already explained in the foregoing, the essence (: inherent mechanism) of (differential) geometry may still be applied in that deep, in spite of the presence of an extremely “anomalous”, in the classical sense, carrying space (if any!). Yet, once more, the impediment here is with the classical notion of the supporting space, providing, in effect, the standard “differential triad”, and not with the latter perse, which is virtually independent of the former, that in the quantum deep seems (!) to be different from the usual one. Presumably, what was for Feynman in conflict with the geometry in the quantum domain, was the way of applying it therein; accordingly, to paraphrase him, we can say that,

\[
\text{if we are going to apply any “geometrical reasoning” within the quantum context, this should be done not in a geometrical (classical) way (!), but in an analytic (algebraic) one with symbols.}
\]

(5.3)

See R.P. Feynman [7: begin of p. 44]. Now, it is, actually,
Remarks on “singularities”

Accordingly, and expressed differently, the apparent disagreement of general relativity with quantum theory, the former being rooted on classical differential geometry of the so-called “smooth” manifolds, therefore, the difficulties, as well, when trying to apply the latter to problems in quantum gravity, appear, as a result of the preceding, to be due, very likely, not thus much to the mechanism (idea) p e r s e of differential geometry, as to the type of the same, that is, to say, to the corresponding sort of “differential triad”, in the sense of ADG, that we usually employ in that context!

That is, to put it yet in another way, the fault is with the way we understand resulting the classical differential-geometric mechanism; thus, in that point of view, this mechanism is just an outcome of the standard notion of a (differential-smooth) manifold. Nevertheless, this is simply a misinterpretation, while the essence of the matter is much more deeper, referring, in effect, to an extremely inherent feature of this mechanism, being, in principle, independent of any intervening space. Indeed, one can still apply the abstract differential-geometric technique even in much more general situations, see e.g. [22], [23], [24], [25].

6. On the other hand, by looking at potential physical applications of ADG, the base space of the sheaves involved does not necessarily have the properties we usually ascribe to the analogous “base space” (: space-time) of the classical theory. So, by contrast with what we usually do classically, in a sheaf-theoretic treatment of the situation, in terms of ADG, by considering the properties we want to (or, at least, feel that we should) have,
we virtually transfer all the desired properties, as above, to (the stalks of) the sheaves, that, according to our theory, for that matter, represent the objects, we wish to study (as, for instance, elementary particles). Of course, this is also in accord with our present-day conception of physics, the latter being actually concerned with relations rather (i.e., equations/laws), yet, with functions/sections, that “live” on a given “space”, than this “space” itself (!), whose existence p e r s e is, for that matter, quite disputable, anyway.

Precisely speaking, all the desired properties, that are classically addressed to the underlying (space-time) manifold, are here, that is, within the framework of ADG, transferred to the corresponding “arithmetics”, alias “sheaf of coefficients” $\mathcal{A}$, and finally, in an appropriate manner (here paracompactness of the underlying topological space, base space of the sheaves involved, and fineness of $\mathcal{A}$ are applied), to the $\mathcal{A}$-modules (yet, vector sheaves) concerned.

In this connection, we can still say that, as a general moral of nowadays quantum theory, we usually try to

\begin{equation}
\text{abandon “space” and look, instead, at the objects which “live”}
\end{equation}

on the space and their interrelations, directly, viz. without referring and/or being influenced by “properties” of the space.

Therefore, in that perspective,

\begin{equation}
\text{we can further apply the mechanism of (abstract) differential geometry to the pertinent objects, p e r s e, without thus referring to any supporting space, exactly, because ADG does not depend on the latter.}
\end{equation}

Plus, the very nature of the “arithmetics” applied (!), according to which, we realize that,
based further on ADG, we are thus no more compelled to find “solutions free of singularities” (A. Einstein [6: p. 165]), but (to afford that particular “arithmetics”, so that to be able) to state “equations”, whose solutions can engulf the “singularities”.

In this connection, one can consider the aforementioned “Eddington-Finkelstein coordinates”, simply, as an anticipating example of the above, as already hinted at by (0.6) in the preceding. Now, the same is still suggesting that we might have just here a machinery, suitable to a “relativistic quantum mechanics”, to quote also P.A.M. Dirac [4: p. 85], “in which we will not have ... infinities occurring at all” [the emphasis is ours]. Thus, in view of the foregoing, this will be the new rôle of the “arithmetics (in point of fact, of the appropriate “differential triad”), we would choose, each time, depending on the particular problem at issue.

Finally, by further looking at the two fundamental principles of nowadays physics, as stated in (0.1) and (0.2) at the beginning of the present discussion, we still remark that even by the very formulation of these two principles, we implicitly assume in effect that the rôle of the “space”, involved therein is quite a secondary one (if any, at all!). This, somehow obscured situation due, in point of fact, to the way we were employing hitherto the (classical) differential geometry, is well pointed out, exactly, by the whole technique of Abstract Differential Geometry, as this has been succinctly indicated in the preceding discussion.

Yet, we can still say here that the potential value of the above perspective, as this is exhibited by ADG, lies exactly in its generality and presentation of general fundamental principles, that actually govern CDG (which may thus be regarded as a ‘subtheory’ of ADG); we are thus, in that sense, in accord with a relevant aphorism of G. Darboux, saying that (the emphasis below is ours);
“Le caractère propre des méthodes ... consiste dans l' emploi d’ un petit nombre de principes généraux ...; et les conséquences sont d’ autant plus étendues que les principes eux-mêmes ont plus de généralité.”

(6.5)

See G. Darboux [1: Introd., p. 3]

7. We close the present discussion, by considering, as an example of the preceding, in particular of that part of it, concerning Rosinger’s algebra sheaf, the well-known Uhlenbeck’s theorem on “removable singularities”, pertaining thus to a (smooth) extension of a Yang-Mills field, defined on $S^4$, modulo the north pole (see [38]; yet, T.H. Parker [29] gave a generalization of the same result.

Now, by considering the sheaf of generalized functions à la Rosinger, we can further look at Uhlenbeck’s Yang-Mills field, as defined on

(7.1) $S^4 \setminus \{\infty\} \equiv U \subseteq S^4 \equiv X$

where the closed set (in effect, a nowhere dense set in $S^4$), $\{\infty\} \subseteq S^4$ (: “north pole”), is a “singularity” (set), in the classical sense. Thus, since Rosinger’s algebra sheaf (i.e., our “structural sheaf” $\mathcal{A}$ is flabby, (see e.g. A. Mallios-E.E. Rosinger [24], and/or [25]), one gets that the canonical map (cf. (7.1)),

(7.2) $\Gamma(X, \mathcal{A}) \rightarrow \Gamma(U, \mathcal{A})$

is surjective, by the very definition of the flabbiness of $\mathcal{A}$, so that any (continuous) local section of $\mathcal{A}$ over an open set $U \subseteq S^4$ is extended to a (continuous) global section of $\mathcal{A}$ over $S^4$.

Therefore, by considering a Yang-Mills field

(7.3) $(\mathcal{E}, D)$

on $S^4 \equiv X$, the $\mathcal{A}$-connection $D$ is locally expressed, via a local gauge, say,

(7.4) $U_0 \subseteq U$
of $E$, through a so-called \textit{local $A$-connection matrix of $D$}, associated with $U_0$,
\begin{equation}
\omega \equiv (\omega_{ij}) \in M_n(\Omega(U_0)) = M_n(\Omega)(U_0),
\end{equation}
with $1 \leq i, j \leq n \equiv \text{rk}E$; yet, the $A$-module $\Omega$ of 1-forms is, for the case at issue, also a \textit{vector sheaf} on $X$, so that
\begin{equation}
\omega_{ij} \in \Omega(U_0) \cong \mathcal{A}(U_0)^m,
\end{equation}
with $m = \text{rk}\Omega$. Therefore, based on (7.2),
\begin{equation}
\text{any continuous (local) section of $\Omega$ on $U_0$, hence, in view of (7.5), a local realization, in effect, of $D$ on $U_0$, that is, actually, of $(E, D)$ itself, can be extended to the whole space $S^4$, as well.}
\end{equation}

The above constitutes, virtually, an \textit{ample generalization} of the classical result of K. Uhlenbeck (loc. cit.; p. 24, Theorem 4.1). Yet, it is worth remarking herewith that in our argument, employed in (7.7),
\begin{equation}
\text{\textit{no a priori restriction on the “energy”} (\textit{field strength = curvature of $D$}), as one assumes in the classical case (ibid.), is actually required. Moreover, \textit{no restriction on the dimension} of the “space” is necessary either.}
\end{equation}

On the other hand, it is still worth mentioning here, that the functions, in point of fact, \textit{sections}, which are involved in the previous account of the classical result under discussion, may have a far bigger amount of \textit{“singularities”}, that is, to say, \textit{prohibiting places, concerning the standard theory}; see, for instance, A. Mallios-E.E. Rosinger [24], or even [25]. In conclusion, one thus obtains, within the general setting of ADG (by contrast, cf. also K.K. Uhlenbeck [38: p. 11, Abstract]), that;
\begin{equation}
\text{one can consider \textit{solutions of the Yang-Mills equations over any “space” with “singularities”}, that, for instance, can be engulfed in a Rosinger multi-foam algebra sheaf (cf. [25]), the \textit{8.}}
\end{equation}

latter being viewed, as a \textit{“sheaf of coefficients”}, in the sense of ADG.
Appendix: Quantum gravity. – By taking *second quantization* into account (see, for instance, [20] or even [21: Chapt. II]), one concludes that:

\[(8.1)\]

an equation expressed *in terms of sheaf-morphisms* (take e.g. the Yang-Mills equation, referring to a given Yang-Mills field) *is*, in effect, by definition, *relativistic, with the field itself being the variable*. Furthermore, *the same equation* refers, in point of fact, already, *to the quantum of the field*, *this still being*, by the very definitions (: *second quantization*, again!), just *the elementary particle* in terms of which the equation in focus has been formulated (see also A. Mallios [21: Chapt. VII; (5.8), (5.11)]).

Now, in this context, we further note that:

\[(8.2)\]

the presence of the *field* (: *elementary particle*, see also e.g. loc. cit. (1.5)) in an equation, as before, is virtually exhibited, through the corresponding *field strength* (ibid. (1.9)). Hence, in terms of a "*structure sheaf invariant*" (alias, in the terminology employed herewith, by means of an "\(A\)-invariant", or even, classically speaking, in terms of a "*tensor"), that is, in other words, via an entirely *physical* notion.

As a result, though it might probably sound a bit strange, and paraphrasing actually Leibniz, herewith (cf. (2.1) in the preceding), we can say that:

\[(8.3)\]

*space-independent notions are the real physical ones!* (This might also be related with (1.1)).

So, by saying here "physical", we can also refer to "geometrical", this being actually the connection with Leibniz, as alluded to above, an aspect that might further be connected with the famous maxim of Plato, in that context, namely, in the sense that;
Remarks on “singularities”

physical/geometrical is the “relational” not the “spatial”! In this concern, we thus conclude that “space” is the result of laws (: relations/equations), not the other way around, the latter being thus space-independent (invariant), which, of course, still delineates (8.3).

Therefore, technically speaking now, the above accentuation of “$A$” points out, time and again, the significance of the “sheaf of coefficients”, or even “structure sheaf”, for the “geometry” in general! Indeed, the latter is thus here a spin-off, in effect, of $A$ (: structure sheaf), the quality/effectiveness of our “geometry” being, in that sense, directly dependent on that particular sheaf $A$. [Thus, take, for instance, the so-called “Euclidean spaces”, globally or not, being virtually outcomes, speaking analytically (Descartes), of our classical number fields (here, constant structural sheaves!)].

Consequently, the “geometry”, we actually employ, is “cartesian”, in spirit, that is, “analytical” (consider “$A$” for instance), not “Euclidean”, that might be viewed, as the “physical/relational” one (see (8.3), (8.4), as above). Thus, we are, in effect, in accord with the very etymology of the Greek word “γεωμετρία” (: earth measuring), which, of course, is analytical in the previous sense; yet, the result of measuring is a number (!) (that entails physics, according to Feynman). So it is simply here, where an intrinsic conceptual perplexity/entanglement is actually entailed, when employing the notion of “geometry”, in the aforementioned point of view; that is, one is thus confronted with the correspondence (technical, in point of fact, identification),

$$\text{euclidean} \leftrightarrow \text{cartesian}$$

As a result, one thus arrives at the Einsteinian, so to say, point of view (cf. [6: p. 164, B.]), in that:

$$\text{laws/equations should be universal, not depending on the particular type of the “space” considered,}$$

(8.6)
the latter being, in view of the preceding, virtually due, for that matter, entirely, to our own description (viz. the “cartesian”, as above, aspect) of the space!

Accordingly, we are tempting to say, once more (see thus also (1.1) in the foregoing), that;

(8.7) (any!) “space”, at all, is delimited, by the particles ( : quanta) themselves, to which the laws of Physis actually refer.

Therefore, it would of course be of paramount importance, if any time we could afford a mechanism (yet, “ars combinatoria”, à la Leibniz) that could deal with them (the particles) directly, without thus the intervention of any supporting/surrounding “space” (as it is, for instance, the case, instead, in the classical theory: CDG).

Now, based on the preceding, we are still led to say that;

(8.8) a similar situation as above holds also true, which pertains to the so-called “quantum deep”, the “discrepancies” revealed, when dealing with it, being due, of course, not to the (“Euclidean” (cf. (8.5)) aspect of the) “space”, this being actually the same overall (!) (Physis is united), but simply to the “cartesian” (: analytical, see e.g. A, yet out laboratory) manner of looking at it.

Consequently, we are thus trapped here, time and again, by the above (fictitious) “identification”, still ours(!), for that matter, as it is indicated in the foregoing by (8.5).

On the other hand, classically speaking, the general theory of relativity is (“can be conceived only as”, A. Einstein, loc. cit.) a field theory, in particular, a “continuum theory” (ibid., pp. 140, 164). Here, “continuum theory” is conceived, as that one, that rests on the “continuum”, and this is still in the standard parlance, the so-called “space-time [smooth (!)] continuum”. Thus, Einstein himself was looking at the “quanta” as “singularities” of (viz. something inconsistent with) the previous theory, in the sense, in effect, that the
relevant (quantum) theory did not function, within the initial ("continuous") framework.

Thus, here too, one is actually confronted with the aforesaid *entanglement of the perspectives of the notion of “space”*, as in (8.5), concerning, at least, the employed herewith *mechanism of “differential geometry”*; that is, in point of fact, and *this is here of importance* (!),

\[ (8.9) \]

one should, instead, refer actually, by the above last term (viz. "differential geometry"), to the intrinsic "*ars combinatoria*", à la Leibniz, this being virtually *independent of any* (smooth) "continuum" (main moral of ADG, in that context, see also concluding remarks of the present Section). That is, in other words, one should refrain here from employing the, so to say, "Leibnizian mechanism", *within the wrong scaffolding*, viz. the "continuum"(!), as it concerns, in particular, the quantum theory.

Yet, to repeat it, due to its particular significance, what amounts to the same thing, one is thus led to conclude that;

\[ (8.10) \]

the so-called (smooth) "continuum" is, in effect, *the wrong scaffolding, as it concerns, at least, the quantum theory*, in order to employ what we may call "Leibnizian mechanism" (of differential geometry), the latter *being, anyhow, independent of the former*!

In conclusion, one thus realizes that:
even though Nature is united, we employ for its description, depending on the particular case at issue, different techniques, since we are actually trapped by the success of our method, referring to the classical differential geometry (CDG), in the case of the “continuum”, trying thus to alter it, when confronting with the “discrete” (quantum régime), while still retaining the “Euclidean”, in the sense of the “continuum” aspect of the “space”. Hence, the apparent differences (: “singularities” of the latter), are actually due to the fact that,

\[(8.11)\]

the “cartesian” perspective of the “space” is, in effect, different in nature, from that one of the “Euclidean” point of view, in the sense of (8.5), though both aspects can be, technically speaking, identified (à la Descartes), viz. to the extent that CDG is applicable!

Notwithstanding,

\[(8.11.1)\]

the innate mechanism (viz. the aforementioned “Leibnizian” one) of our method (: differential geometry) permeates both aspects, as they are given by (8.5).

The content of (8.11.2), as above, is an outcome of ADG. Indeed, the same follows from several relevant remarks, throughout the preceding of the present discussion. On the other hand, this point of view has been recently further corroborated, by two particular instances of the manner that one can apply ADG, in that context, which also might, very likely, explain the apparent conflict between, generally speaking, the “continuum” and the “discrete”, pertaining thus to the way of applying thereon the standard differential geometry (of smooth manifolds); thus, in point of fact, the power of the latter is actually the upshot of it, viz. the intrinsic (“Leibnizian”) mechanism, not the particular source of it, that is, for the case in focus, the smooth manifold. So exactly here
appeared lately the aforesaid vindication/application of the previous claim, by two particular examples, as mentioned above, viz. that of “Rosinger’s algebra sheaves” [31, 24, 25], on the one hand, along with those of “Raptis’ reticular (incidence) algebras” [30, 22, 23], on the other.

9. Field quantization, in terms of ADG.— According to the preceding, the “space” on which the (physical) objects (elementary particles) live is united (cf. (8.11), along with (8.3), (8.4)); notwithstanding that its presence, as a notion, is, technically speaking, quite secondary, given that on the one hand,

\[ \text{the (physical) objects themselves are the “variables” of the equations of (the physical laws represented by) the theory (cf. (8.6), (8.7)), and yet, when referring to an elementary particle as above, we actually mean, by virtue of the very definitions, the “quantum” itself of the relevant “field” that is associated with the particle at issue (see also (8.1), as well as, A. Mallios [20; (1.5)] and [21]),} \]

while on the other, that the (“Leibnizian”) mechanism employed (viz. ADG itself) is quite independent of any preexistent “smooth” structure on the “space” in focus.

Within the same vein of ideas, we can still remark here that, as already mentioned above,

\[ \text{our equations representing the laws of Nature, even in the classical theory (CDG), refer to the (physical) objects themselves and are virtually independent of any supporting space (see, for instance, the equations of nowadays “coordinate free” (classical) differential geometry).} \]

Accordingly,

\[ \text{even in the classical theory, the intervention of “space” in our equations is absent.} \]
Of course, this is certainly extremely important, the same equations (laws) being, otherwise deprived of their “geometrical” (: physical/natural) substance (!). Indeed, the only participation here of the (cartesian) “space” (: smooth manifold of CDG) is to supplying us with the Calculus, in point of fact, the “raison d’ être”, and quintessence, as well, of the same Differential Geometry, concerning the classical theory. We thus have here, simply,

another justification, from CDG itself, that “space” is actually irrelevant to the real substance of our equations, the latter referring directly to the laws that govern (relations between) the physical objects themselves.

Now, by looking again at the framework of ADG, thus, in view of our description therein of elementary particles (fields), as pairs

\[(9.4)\]

\[(\mathcal{E}, D)\]

(see, for instance, [20: (1.8), (1.9)]), in terms of sheaf theory, and based further on the respective, within the same set-up, treatment of “second quantization” (see [20], [21]), one can still remark that,

\[(9.6)\]

our equations within the setting of ADG, are, in effect, relativistic, in character (viz. “gauge independent”); thus, in point of fact, they are “space-invariant”, or even, in ADG-parlance, “\(A\)-invariant”.

On the other hand,

\[(9.7)\]

\[(9.7.1)\]

based further on (the sheaf-theoretic, cf., for instance, [20], interpretation of) second quantization,

the same equations, as above, might be construed as being already quantized ones.

We also noted above that our equations are \(A\)-invariant, hence, relativistic (see (9.6)), alias, by employing herewith a classical terminology, “gauge invariant/independent”; yet, in this concern, we should still remark that;
the above “invariance” does not necessarily refer to any surrounding smooth (e.g. space-time) manifold, as it is usually the case in the classical theory (e.g. general relativity, in terms of CDG), but to something, according to the preceding (i.e., to the point of view of ADG), more innate, viz. to our own “arithmetics” $\mathcal{A}$, the latter being for that matter responsible for the whole edifice, as this is presented, thus far. So, in effect, to something more natural, pertaining thus directly to the physical law/relations of the objects at issue, the only parameter here being “$\mathcal{A}$”, with respect exactly to which the equations are invariant. To repeat it, once more:

(9.8) 

our equations are invariant, with respect to our own “arithmetics” $\mathcal{A}$, viz., so to say, relative to the same way of our “measurement”, this being still independent of any space (Riemann, cf. (1.3)), referring instead directly to the (physical) objects (e.g. particles) themselves.

Indeed, all the equations considered hereto are always expressed, exclusively, in terms of sheaf morphisms between the objects (sheaves) involved, the physical objects (particles) at issue being identified (e.g. according to second quantization) with appropriate sheaves.

On the other hand, paraphrasing I. Raptis [32], we can ask,

(9.9) 

“... whether quantizing ... is physically meaningful at all ..”,

(our emphasis) which might also be related with C.v. Westenholz’s aspect that,

(9.10) 

“... Quantization is provided by the Physical law itself”

(emphasis is ours above; see [39: p. 323]). As a result, and in connection with our claim in (9.7.1), we can also remark that;
the aforementioned equations, as in (9.6) and (9.7), might further be viewed, as already quantized ones, to the extent that they prove to be, this depending thus on the type of the theory employed to their deduction, more akin to Physis, a natural criterion thereof, being, of course, dependent on several types of occasional inconsistencies appeared, e.g. “singularities” (always, a spin-off of our theory, used in that context).

Now a given equation may be viewed, as

\[ (9.12) \]

“more akin to Physis”, viz. more natural,

whenever it concerns the particle itself (in vacuo), viz., in terms of ADG, the pair,

\[ (9.13) \quad (E, D), \]

without the intervening of any “space” (other, of course, than the particle at issue), the same being the only variable in the equation, that should also be $A$-invariant (hence, the appearance herewith, as already explained in the preceding, in place of $D$, actually of $R(D)$).

Thus, an equation, as above, may be construed, in effect, as the

\[ (9.14) \]

quantum field theory (alias, quantum relativistic) - equation, of the particle/quantum, under consideration.

Furthermore, within the same vein of ideas, we still remark that;
by considering the aforesaid equations pertaining to a given Yang-Mills field \((\mathcal{E}, D)\) as before, these are actually concerned with what may be construed as the even "formally quantized" version of the previous field, viz. with the following Yang-Mills field,

\[(9.15.1) \quad (\text{End}\mathcal{E}, D_{\text{End}\mathcal{E}}),\]

which can further be viewed, as a "matrix representation" of the given one (see e.g [21]).

Consequently, the resulting correspondence, as above,

\[(9.16) \quad (\mathcal{E}, D) \rightsquigarrow (\text{End}\mathcal{E}, D_{\text{End}\mathcal{E}})\]

might be considered, as the result of the final action of a "second-quantization functor", acting within the "Yang-Mills category" (see also [21], concerning the terminology applied herewith). In this concern, we also note that;

the above transition (9.16) is still encoded in the moduli space of \(\mathcal{E}\),

\[(9.17.1) \quad M(\mathcal{E}) := \text{Conn}_A(\mathcal{E})/\text{Aut}\mathcal{E} = \text{Conn}_A(\mathcal{E})/(\text{End}\mathcal{E})',\]

the inner structure of which is further investigated, in terms of the same "matrix representation" of \(\mathcal{E}\), that is, by means of the pair (9.15.1).

Of course this holds true, in particular for the Einstein equation in vacuo [19], that is, concerning the relation,

\[(9.18) \quad \mathcal{R}ic(\mathcal{E}) = 0,\]

for a given Yang-Mills field,

\[(9.19) \quad (\mathcal{E}, D)\]

on \(X\), the latter being an (abstract) Einstein space. See loc. cit., or even A. Mallios [21], for full details of the terminology applied herewith.
On the other hand, the whole set-up is, in point of fact, extremely “non-linear” and even “relativistic”: Yet, the “second quantization functor”,

\[(\mathcal{E}, D) \leadsto (\text{End} \mathcal{E}, D_{\text{End} \mathcal{E}})\]

supplies a, so to say, “canonical matrix representation” of the initial object, viz. of the (free) elementary particle

\[(\mathcal{E}, D).\]

Now, by looking at (9.18), we see that the same relation is, in effect, the Yang-Mills equation for the curvature tensor. On the other hand, the same meaning of this equation is actually Machado’s (poetic) verse:

\[
\text{... paths are made by walking”, viz. the “traveller [himself] makes the path”, who, in point of fact, does not feel anything on his trace (Spur), viz. he feels curvature zero, during his travel (in vacuo (!), of course).}
\]

Therefore, it would be, at least strange to think that this fundamental law of Physis is valid macroscopically only, and not in the “quantum deep”, as well (!). Accordingly, we are led to claim that:

\[(9.18) \text{ is Einstein’s quantum-equation too, viz. e o i p s o the quantized Einstein’s equation (in vacuum).}\]

Finally, the relation,

\[(9.24) \quad \mathcal{R}ic := tr \circ R(., s)t\]

is an \(\mathcal{A}\)-morphism of \(\Omega^2 \equiv \wedge^2 \mathcal{E} \text{ in } \mathcal{A}\), so that, according to (9.18),

Einstein’s equation (in vacuum) is just the kernel of the same operator (\(\mathcal{A}\)-morphism, as before), that also delimits the corresponding Einstein (\(\mathcal{A}\)-solution) space of the equation at issue (see [21]).

On the other hand, due to the gauge invariance of the trace operator, one further concludes that:
Remarks on “singularities”

(9.26) *Einstein’s equation* (in vacuo) retains its meaning still on the moduli space of a given solution of it.

Yet, concluding the present account, we should like to underscore, once more, a mathematical-physical correspondence, that actually permeated our whole discussion herewith, mainly motivated by a relevant terminology originally employed by Yu.I. Manin [26], and further justified by what we may call “Selesnick’s correspondence principle” [21: Chapt. II], in effect, a formulation of second quantization; so based on the aforesaid correspondence, we can still say that, in the same way that,

\[(9.27.1) \text{ a sheaf is its sections} \]

see [18: Chapt. I; Section 3], thus, also;

\[(9.27.2) \text{ an elementary particle is its states,} \]

viz. the sections (in view of (9.27.1), as above) of the sheaf, that is associated with it, according to the aforementioned correspondence.

Thus, still finally, we are led, time and again, to emphasize that;

locally our descriptions (of the physical events) are made through sections of our (generalized) “arithmetics”, or of what we have called in the preceding “structure sheaf” \( \mathcal{A} \). So it is, via this same structure sheaf \( \mathcal{A} \), that one should also describe the “space”, as we actually did, strongly motivated, in effect, thereto, by ADG itself, by identifying (cf. (1.1)) the former with the (physical) objects \( \leftrightarrow \) (vector) sheaves (cf. (9.27)), that live on it.

In this concern, we should still note that;
even in the classical theory the description of the ("physical") space is actually made (à la Descartes), via the corresponding therein structure sheaf

\[(9.29)\]
\[\mathcal{A} \equiv \mathcal{C}^\infty_X.\]

However, the extremely nice properties of the latter sheaf in the classical case are actually due to the

algebraic-topological (hence, even differential analytic-Hilbert) consequences of the basic arithmetics, viz of our standard numerical fields, we employ therein, hence, the analogous properties of the basic (\(\mathbb{R}\)-, or \(\mathbb{C}\)-) algebra (of global sections)

\[(9.30)\]
\[\mathfrak{A} \equiv \mathcal{A}(X) \equiv \mathcal{C}^\infty_X(X).\]

As a result, one thus comes to the tempting conclusion that, it seems as that,

\[(9.31)\]

*it is more natural to think*: make arguments, ideas, in terms of algebra, rather than geometry.

The above reminds us of a relevant statement of C.v. Westenholz [39: p. 180], in the sense that;

\[(9.32)\]

"... most physical observables must adequately be described in terms of exterior forms and not by vectors".

Yet, even of Einstein’s [6: p. 166],

\[(9.33)\]

"... find a purely algebraic theory for the description of reality".

The reward or the advantage from such a choice, as in (9.31), could be, at least, that, by in reference for instance to ADG;
the algebra (-way of thinking) might contain (viz. engulf) geometrical discrepancies, alias “singularities”, by remaining, notwithstanding, still operative—that is, still work(!) In other words, *algebra proves thus to be more flexible*, being, therefore, able to permeate, go through, or even circumvent, difficulties (: “singularities”), that the standard point of view of *geometry simply cannot*!

So we can think here of the famous “Plank scale”, that, of course, it is *not*, a matter of analysis/algebra (think of Dirac’s aphorism, see thus (3.9)), but rather of the particular manner (viz. still classical differential geometry), that it is here undertaken to exploit “geometry”.

All in all, we are very tempted to say that;

> any time we have an extremely efficacious “geometrical mechanism” (see e.g. classical differential geometry), it is very likely that there will be hidden therein (or even, it would at least be worthwhile to look for) an inherent (deeper) algebraic one that should actually be the “catalyst” to the former, making thus the things to work without necessarily the presence of the initial “geometrical scaffolding/panoply”. (Compare, for instance, the numerously mentioned in the preceding Leibniz’s “ars combinatoria”, or even the same framework of ADG).

Nevertheless, one should then further look for the appropriate, very likely lurking therein, “representation theorem”, alias, “Gel’fand duality”(!), a fact, that it would then actually entail, when properly envisaged, a converse to the preceding statement, as in (9.35) (!) (see, for instance, what happened in nowadays Algebraic Geometry).

Thus, one is virtually led, at the very end, back again to Sophie Germain, in that;

> “L’ algèbre n’ est qu’ une géométrie écrite, la géométrie n’ est qu’ une algèbre figurée.”
Anastasios Mallios

(emphasis above is ours; cf. S. Germain in “Pensées”).

Accordingly, based finally on (9.35)/(9.36), we may wind up, by still saying that:

\[
\text{an established “geometrical theory” entertaining, however, “singularities”, might be effectively viewed just as an insufficiently rendered, an ill-represented in that respect, “algebraic mechanism”.}
\]

(9.37)

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