On Inhibition of Rayleigh–Taylor Instability by Horizontal Magnetic Field in an Inviscid MHD Fluid with Velocity Damping

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Abstract

It is still an open problem whether the inhibition phenomenon of Rayleigh–Taylor (RT) instability by horizontal magnetic field can be mathematically proved in a non-resistive magnetohydrodynamic (MHD) fluid in a two-dimensional (2D) horizontal slab domain, since it had been roughly verified by a 2D linearized motion equations in 2012 \cite{43}. In this paper, we find that this inhibition phenomenon can be rigorously verified in the inhomogeneous, incompressible, inviscid case with velocity damping. More precisely, there exists a critical number $m_C$ such that if the strength $|m|$ of horizontal magnetic field is bigger than $m_C$, then the small perturbation solution around the magnetic RT equilibrium state is exponentially stable in time. Our result is also the first mathematical one based on the nonlinear motion equations for the proof of inhibition of flow instabilities by a horizontal magnetic field in a horizontal slab domain. In addition, we also provide a nonlinear instability result for the case $|m| \in [0, m_C)$. Our instability result presents that horizontal magnetic field can not inhibit the RT instability, if its strength is too small.

Keywords: non-resistive MHD fluids; inviscid fluid; damping; Rayleigh–Taylor instability; exponential stability.

1. Introduction

The equilibrium of the heavier fluid on top of the lighter one and both subject to the gravity is unstable. In this case, the equilibrium state is unstable to sustain small disturbances, and this unstable disturbance will grow and lead to the release of potential energy, as the heavier fluid moves down under the gravity force, and the lighter one is displaced upwards. This phenomenon was first studied by Rayleigh \cite{37} and then Taylor \cite{40}, is called therefore the RT instability. In the last decades, this phenomenon has been extensively investigated from mathematical, physical and numerical aspects, see \cite{3, 41} for examples. It has been also widely investigated how other physical factors, such as elasticity \cite{29, 31}, rotation \cite{2, 3}, internal surface tension \cite{14, 19, 45}, magnetic field \cite{21, 22, 27, 28, 43, 44} and so on, influence the dynamics of RT instability.

In this paper, we are interested in the phenomenon of inhibition of RT instability by magnetic field. This topic goes back to the theoretical work of Kruskal and Schwarzschild \cite{32}. They analyzed the effect of an impressed horizontal magnetic field on the growth of RT instability in the horizontally periodic motion of a completely ionized plasma (with zero resistance) in three dimensions in 1954, and pointed out that the curvature of the magnetic lines can influence the
development of instability, but can not inhibit the growth of RT instability. The inhibition of RT instability by an impressed vertical magnetic field was first verified for inhomogeneous, incompressible, non-resistive magnetohydrodynamic (MHD) fluids in three dimensions by Hide [3, 16]. In 2012, Wang noticed that an impressed horizontal magnetic field can inhibit RT instability in a non-resistive MHD fluid in two dimensions [43]. Later Jiang–Jiang further found that magnetic field always inhibit RT instability, if the condition is satisfied that the non-slip velocity boundary-value condition is imposed in the direction of impressed magnetic field. In this paper, we call such condition the “fixed condition” for the sake of simplicity. It should be noted that all the results stated previously are based on the linearized non-resistive MHD equations. For reader’s convenience, we also summarize the known linear results as follows:

|          | horizontal | vertical |
|----------|------------|----------|
| 2D       | Yes        | Yes      |
| 3D       | No         | Yes      |

Recently, Jiang–Jiang further established a so-called magnetic inhibition theory in viscous non-resistive MHD fluids, which reveals the physical effect of fixed condition in magnetic inhibition phenomena [25]. Roughly speaking, let us consider an element line along an impressed field in the rest state of a MHD fluid, then the element line can be regarded as an elastic string. Thus, the bent element line will restore to its initial location under the magnetic tension, the fixed condition, as well as viscosity. By the magnetic inhibition mechanism of a non-resistive MHD fluid, the assertions in the table above seem to be obvious. However, rigorous mathematical proofs are not easy.

Thanks to the multi-layers method developed in the proof of well-posed problem of surface waves [15], recently the inhibition phenomenon of RT instability by a magnetic field had been rigorously proved based on the (nonlinear) non-resistive MHD equations under a fixed condition, for example, Wang verified the inhibition phenomenon by an impressed non-horizontal magnetic field in the stratified incompressible viscous MHD fluid in a 3D slab domain [44]; moreover, he also proved that the impressed horizontal magnetic field can not inhibit the RT instability for the horizontally periodic motion. Similar results can be also found in other magnetic inhibition phenomena, see [23] for Parker instability and [26] for thermal instability.

However at present it is still an open problem whether the phenomenon of inhibition of RT instability by a horizontal magnetic field can be rigorously proved in a non-resistive MHD fluid in a 2D slab domain. To our knowledge, there are also not any available mathematical proof for the inhibition of other flow instabilities by a horizontal magnetic field in a horizontal slab domain in the both 2D and 3D cases. The purpose of this paper is to move a step in this direction. Fortunately we find that this inhibition phenomenon can be mathematically verified in the inhomogeneous, incompressible, inviscid, non-resistive MHD fluid with velocity damping in two-dimensions. More precisely, there exists a critical number \( m_c \) such that if the strength \( |m| \) of a horizontal magnetic field is bigger than \( m_c \), then the small perturbation solution around the magnetic RT equilibrium state is exponentially stable in time, i.e., RT instability can be inhibited by a horizontal magnetic field in a 2D slab domain. Next we mathematically formulate our result.

1.1. Mathematical formulation for the magnetic RT problem

The system of motion equations of an inhomogeneous, incompressible, inviscid, non-resistive MHD fluid with velocity damping in the presence of a gravitational field in a two-dimensional
domain \( \Omega \) reads as follows:

\[
\begin{align*}
\rho_t + v \cdot \nabla \rho &= 0, \\
\rho v_t + \rho v \cdot \nabla v + \nabla (P + \lambda |M|^2/2) + a \rho v &= \lambda M \cdot \nabla M - \rho ge_2, \\
M_t + v \cdot \nabla M &= M \cdot \nabla v, \\
\text{div} v &= \text{div} M = 0.
\end{align*}
\] (1.1)

Next we shall explain the mathematical notations in system (1.1).

The unknowns \( \rho := \rho(x,t) \), \( v := v(x,t) \), \( M := M(x,t) \) and \( P := P(x,t) \) denote the density, velocity, magnetic field and kinetic pressure of incompressible MHD fluids, resp.. \( x \in \Omega \subset \mathbb{R}^2 \) and \( t > 0 \) are spacial variables and time variables, resp.. The constants \( \lambda > 0 \), \( g > 0 \) and \( a \geq 0 \) stand for the permeability of vacuum, the gravitational constant and the velocity damping coefficient, resp.. \( e_2 = (0,1)^T \) represents the vertical unit vector, and \( -\rho ge_2 \) the gravitational force, where the superscript \( T \) denotes the transposition.

Since we consider the horizontally periodic motion solution of (1.1), we define the horizontally periodic domain

\[
\Omega := 2\pi T \times (0,h),
\] (1.2)

where \( T = \mathbb{R}/\mathbb{Z} \). For the horizontally periodic domain \( \Omega \), the 1D periodic domain \( 2\pi T \times \{0,h\} \), denoted by \( \partial \Omega \), customarily is regarded as the boundary of \( \Omega \). For the well-posedness of the system (1.1), we shall pose the following initial-boundary value conditions:

\[
\begin{align*}
(r, v, M)|_{t=0} &= (r^0, v^0, M^0), \\
v|_{\partial \Omega} \cdot \bar{n} &= 0,
\end{align*}
\] (1.3, 1.4)

where \( \bar{n} \) denotes the outward unit normal vector on \( \partial \Omega \). Here and in what follows, we always use the superscript \( 0 \) to emphasize the initial data.

Now we choose a RT density profile \( \bar{\rho} := \bar{\rho}(x_2) \), which is independent of \( x_1 \) and satisfies

\[
\begin{align*}
\bar{\rho} &\in C^4(\Omega), \quad \inf_{x \in \Omega} \bar{\rho} > 0, \\
\bar{\rho}'|_{x_2=y_2} &> 0 \text{ for some } y_2 \in \{x_2 \mid (x_1, x_2)^T \in \Omega\},
\end{align*}
\] (1.5, 1.6)

where \( \bar{\rho}' := d\bar{\rho}/dx_2 \) and \( \Omega := \mathbb{R} \times [0,h] \). We remark that the second condition in (1.5) prevents us from treating vacuum, while the condition in (1.6) is called RT condition, which assures that there is at least a region in which the density is larger with increasing height \( x_2 \) and leads to the classical RT instability, see [20, Theorem 1.2].

With RT density profile in hand, we further define a magnetic RT equilibria \( \bar{r}_M := (\bar{\rho}, 0, \bar{M}) \), where \( \bar{M} = (m, 0)^T \) and \( m \) is a constant. We often call \( \bar{M} \) an impressive horizontal magnetic field, while the pressure profile \( \bar{P} \) under the equilibrium state is determined by the relation

\[
\nabla \bar{P} = -\bar{\rho} g e_2 \text{ in } \Omega.
\] (1.7)

Denoting the perturbation around the magnetic RT equilibria by

\[
g = \rho - \bar{\rho}, \quad v = v - 0, \quad N = M - \bar{M},
\]
and then using the relation (1.7), we obtain the system of perturbation equations from (1.1):

\[
\begin{align*}
\rho_t + v \cdot \nabla (\rho + \bar{\rho}) &= 0, \\
(\rho + \bar{\rho})v_t + (\rho + \bar{\rho})v \cdot \nabla v + \nabla \beta + a \rho v &= \lambda (N + \bar{M}) \cdot \nabla N - \rho g e_2, \\
N_t + v \cdot \nabla N &= (N + \bar{M}) \cdot \nabla v, \\
\text{div} v &= \text{div} N = 0,
\end{align*}
\]

(1.8)

where \( \beta := P - \bar{P} + \lambda (|M|^2 - |\bar{M}|^2)/2 \), and we call \( \beta \) the total perturbation pressure. The corresponding initial-boundary value conditions read as follows:

\[
\begin{align*}
(\rho, v, N)|_{t=0} &= (\rho^0, v^0, N^0), \\
v|_{\partial \Omega} \cdot \vec{n} &= 0.
\end{align*}
\]

(1.9) (1.10)

We call the initial-boundary value problem (1.8)–(1.10) magnetic RT problem for the sake of simplicity. Obviously, to mathematically prove the inhibition of RT instability by a horizontal magnetic field in a 2D slab domain, it suffices to verify the stability in time of magnetic RT problem with some non-trivial initial datum.

We mention that the well-posedness problem of inviscid fluids with velocity damping had been widely investigated, see [17, 33, 35, 38, 39, 42, 47, 48] for examples. Recently, some authors had further studied the well-posedness problem of the motion equations of incompressible inviscid, non-resistive MHD fluids, i.e., taking \( \rho = 1 \) and \( g = 0 \) in (1.1). For examples, Wu–Wu–Xu first given the existence of unique global(-in-time) solutions with algebraic decay-in-time for the 2D Cauchy problem with small initial perturbation [46], and Du–Yang–Zhu obtained the existence of unique global solutions with exponential decay-in-time for the initial-boundary value problem in a 2D slab domain with small initial perturbation around some non-trivial equilibria [6]. It should be noted that the mathematical methods adopted in [6, 46] for the well-posedness problem not applied to our stability problem, therefore next we shall reformulate magnetic RT problem in Lagrangian coordinates as in [23, 26, 44].

1.2. Reformulation in Lagrangian coordinates

Let the flow map \( \zeta \) be the solution to the initial-value problem

\[
\begin{align*}
\partial_t \zeta(y, t) &= v(\zeta(y, t), t) \quad \text{in } \Omega, \\
\zeta(y, 0) &= \zeta^0(y) \quad \text{in } \Omega,
\end{align*}
\]

(1.11)

where the invertible mapping \( \zeta^0 := \zeta^0(y) \) maps \( \Omega \) to \( \Omega \), and satisfies

\[
\begin{align*}
J^0 := \det \nabla \zeta^0 &= 1 \quad \text{in } \Omega, \\
\zeta^0 \cdot \vec{n} &= y \cdot \vec{n} \quad \text{on } \partial \Omega.
\end{align*}
\]

(1.12) (1.13)

Here and in what follows, “det” denotes a determinant of matrix.

We denote the Eulerian coordinates by \((x, t)\) with \( x = \zeta(y, t) \) and the Lagrangian coordinates by \((y, t) \in \Omega \times \mathbb{R}^+_0\), where \( \mathbb{R}^+_0 := [0, \infty) \). We further assume that, for each fixed \( t > 0 \),

\[
\begin{align*}
\zeta|_{y_2=i} : \mathbb{R} &\rightarrow \mathbb{R} \text{ is a } C^2(\mathbb{R})\text{-diffeomorphism mapping for } i = 0, h, \\
\zeta : \overline{\Omega} &\rightarrow \overline{\Omega} \text{ is a } C^2(\overline{\Omega})\text{-diffeomorphism mapping.}
\end{align*}
\]

(1.14) (1.15)
Since \( v \) satisfies the divergence-free condition, and non-slip boundary-value condition (1.10), we can deduce from (1.11)–(1.13) that

\[
J := \det \nabla \zeta = 1 \text{ in } \Omega,
\]

\[
\zeta \cdot \vec{n} = y \cdot \vec{n} \text{ on } \partial \Omega.
\]

We define the matrix \( A := (A_{ij})_{2 \times 2} \) via

\[
A^T = (\nabla \zeta)^{-1} = (\partial_j \zeta_i)_{2 \times 2}.
\]

Then we further define the differential operators \( \nabla_A, \text{div}_A \) and \( \text{curl}_A \) as follows: for a scalar function \( f \) and a vector function \( X := (X_1, X_2)^T \),

\[
\nabla_A f := (A_{1k} \partial_k f, A_{2k} \partial_k f)^T, \quad \text{div}_A(X_1, X_2)^T := A_{1k} \partial_k X_l, \quad \text{curl}_A X := A_{lk} \partial_k X_l - A_{2k} \partial_k X_1,
\]

where we have used the Einstein convention of summation over repeated indices, and \( \partial_k := \partial_{y_k} \). In particular, \( \text{curl}_I X := \text{curl}_I X \), where \( I \) represents an identity matrix. In addition, we will denote \( (\text{curl}_A X^1, \ldots, \text{curl}_A X^n)^T \) by \( \text{curl}_A (X^1, \ldots, X^n)^T \) for simplicity, where \( X^i = (X^i_1, X^i_2)^T \) is a vector function for \( 1 \leq i \leq n \).

Defining the Lagrangian unknowns

\[
(\vartheta, u, Q, B)(y, t) = (\rho, v, P + \lambda |M|^2/2, M)(\zeta(y, t), t) \quad \text{for } (y, t) \in \Omega \times \mathbb{R}_+^+, 
\]

then in Lagrangian coordinates, the initial-boundary value problem of (1.1), (1.3) and (1.4) is rewritten as follows:

\[
\left\{ \begin{array}{l}
\zeta_t = u, \quad \vartheta_t = 0, \quad \text{div}_A u = 0, \\
\vartheta u_t + \nabla_A Q + a \vartheta u = \lambda B \cdot \nabla_A B - \vartheta g e_2, \\
B_t = B \cdot \nabla_A u, \quad \text{div}_A B = 0,
\end{array} \right. \quad (1.16)
\]

where \((\vartheta^0, u^0, B^0) = (\rho^0(\zeta^0), v^0(\zeta^0), M^0(\zeta^0))\). In addition, the relation (1.7) in Lagrangian coordinates reads as follows:

\[
\nabla_A \bar{P}(\zeta_2) = -\bar{\rho}(\zeta_2) g e_2. \quad (1.17)
\]

Let \( \eta = \zeta - y, \eta^0 = \zeta^0 - y, q = Q - \bar{P}(\zeta_2) - \lambda |M|^2/2, \quad A = (\nabla \eta + I)^T \) and

\[
G_\eta := \bar{\rho}(\eta_2(y, t) + y_2) - \bar{\rho}(y_2).
\]

If \( \zeta^0, \vartheta^0 \) and \( B^0 \) satisfy

\[
B^0 = m \partial_1 \zeta^0 \quad \text{and} \quad \vartheta^0 = \bar{\rho}(y_2),
\]

then the initial-boundary value problem (1.16), together with the relation (1.17), implies that

\[
\left\{ \begin{array}{l}
\eta_t = u, \\
\bar{\rho} u_t + \nabla_A q + a \bar{\rho} u = \lambda m^2 \partial_1^2 \eta + g G_\eta e_2, \\
\text{div}_A u = 0, \\
(\eta, u)|_{t=0} = (\eta^0, u^0), \\
(\eta, u)|_{\partial \Omega} \cdot \vec{n} = 0
\end{array} \right. \quad (1.19)
\]


\begin{equation}
\vartheta = \bar{\rho}(y_2), \quad B = m\partial_1 \zeta,
\end{equation}

please refer to [22] for the derivation.

It should be noted that (1.19), together with (1.20), also implies (1.16). In addition, noting that \( q \) is the sum of the perturbation pressure and perturbation magnetic pressure in Lagrangian coordinates, however we still call \( q \) the perturbation pressure for the sake of simplicity. From now on, we call the initial-boundary value problem (1.19) the transformed MRT problem. Obviously the stability problem of magnetic RT problem reduces to investigate the stability of the transformed MRT problem.

1.3. Notations

Before stating our main results on the transformed MRT problem, we shall introduce simplified notations throughout this paper.

(1) Simplified basic notations: \( I_a := (0, a) \) denotes a time interval, in particular, \( I_\infty = \mathbb{R}^+ \). \( \overline{S} \) denotes the closure of the set \( S \subset \mathbb{R}^n \) with \( n \geq 1 \), in particular, \( \overline{I_T} := [0, T] \). \( \int := \int_\Omega = \int_{(0,2\pi)\times(0,b)} \) denotes the mean value of \( u \) in a periodic box. \( a \lesssim b \) means that \( a \leq cb \).

If not stated explicitly, the positive constant \( c \) may depend on \( g, a, \lambda, m, \bar{\rho} \) and \( \Omega \) in the transformed MRT problem, and may vary from one place to other place. Sometimes we use \( c_i \) for \( i \geq 1 \) to replace \( c \), and to emphasize that \( c_i \) is fixed value. \( \alpha \) always denotes the multiindex with respect to the variable \( y \), \( |\alpha| = \alpha_1 + \alpha_2 \) is called the order of multiindex, \( \partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \), \( [\partial^\alpha, \phi] \varphi := \partial^\alpha (\phi \varphi) - \phi \partial^\alpha \varphi \) and

\[ [\partial^\alpha \text{curl}_{\partial_1^i A}, \phi] \chi := \partial^\alpha \text{curl}_{\partial_1^i A}(\phi \chi) - \phi \partial^\alpha \text{curl}_{\partial_1^i A} \chi \quad \text{for} \quad j = 0, 1. \]

\( \nabla^i f \in \mathbb{X} \) represents that \( \partial^\alpha f \in \mathbb{X} \) for any multiindex \( \alpha \) satisfying \( |\alpha| = i \), where \( \mathbb{X} \) denotes some set of functions.

(2) Simplified Banach spaces, norms and semi-norms:

\[ L^p := L^p(\Omega) = W^{0,p}(\Omega), \quad H^i := W^{i,2}(\Omega), \quad H^i := \{ w \in H^i \mid (w)_{\partial \Omega} = 0 \}, \]

\[ H^{1,i} := \{ w \in H^i \mid \partial_1 w \in H^1 \}, \quad H^{j} := \{ w \in H^j \mid w|_{\partial \Omega} \cdot \bar{n} = 0 \}, \]

\[ H^{1, j}_s := \{ w \in H^j_s \mid \text{div} w = 0 \}, \quad H^{1, j}_s := H^j_s \cap H^{1, j}, \quad L_T^p H^i := L^p(I_T, H^i), \quad \| \cdot \|_{i,j} := \| \partial_1^i \|, \quad \| \cdot \|_{k,i} := \sqrt{\sum_{0 \leq l \leq k} \| \cdot \|_{l,i}^2}, \]

where \( 1 \leq p \leq \infty \), and \( i, k \geq 0 \), \( j \geq 1 \).

In addition, for simplicity, we denote \( \sqrt{\sum_{1 \leq l \leq j} \| f^k \|_{\mathbb{X}}^2 } \) by \( \| (f^1, \ldots, f^j) \|_{\mathbb{X}} \), where \( \| \cdot \|_{\mathbb{X}} \) represents a norm or a semi-norm, and \( f^k \) may be a scalar function, a vector or a matrix for \( 1 \leq k \leq j \).
(3) simplified function classes: for integer \( j \geq 1 \),

\[
H^j_1 := \{ w \in H^j \mid \det(\nabla w + I) = 1 \}, \quad H^{1,j}_{1,s} := H^{1,j}_s \cap H^j_1,
\]

\[
C^0_{B,\text{weak}}(\overline{T}_T, L^2) := L^\infty L^2 \cap C^0(\overline{T}_T, L^2_{\text{weak}}),
\]

\[
H^j_s := \{ \xi \in H^j \mid \xi(y) + y \text{satisfies the diffeomorphism properties as } \zeta \text{ in (1.14) and (1.15)} \},
\]

\[
\mathcal{C}^0(\overline{T}_T, H^{1,j}_s) := \{ \eta \in C(\overline{T}_T, H^j_s) \mid \nabla^j \partial_t \eta \in C^0_{B,\text{weak}}(\overline{T}_T, L^2) \},
\]

\[
\mathcal{S}^{1,4}_{1,s,T} := \{ \eta \in \mathcal{C}^0(\overline{T}_T, H^{1,4}_s) \mid \eta(t) \in H^j_s \text{ for each } t \in \overline{T}_T \},
\]

\[
\mathcal{U}^4_T := \{ u \in C^0(\overline{T}_T, H^3_s) \mid \nabla^4 u \in C^0_{B,\text{weak}}(\overline{T}_T, L^2), \quad u_t \in C^0(\overline{T}_T, H^2_s), \quad u_t \in L^\infty H^3 \},
\]

\[
\Omega^4_T := \{ q \in C(\overline{T}_T, H^3) \cap L^\infty H^4 \mid q_t \in L^\infty H^4 \}.
\]

(4) Energy integral: for any given \( w \in H^1 \),

\[
E(w) := \int g \rho' w^2 dy - \lambda \| \partial_t w \|_0^2.
\]

(5) Energy and dissipation functionals,

\[
\mathcal{E} := \|(\eta, \partial_t \eta, u)\|_{4}^2 + \|(u_t, \nabla q)\|_{3}^2, \quad \mathcal{E}_p := \|(\eta, \partial_t \eta, u)\|_{1,3}^2 + \|(u, u_t, \nabla q)\|_{3}^2,
\]

\[
\mathcal{D} := \|(u, \partial_t \eta)\|_{4}^2 + \|(u_t, \nabla q)\|_{3}^2, \quad \mathcal{D}_p := \|(u, \partial_t \eta)\|_{1,3}^2 + \|(u, u_t, \nabla q)\|_{3}^2.
\]

We call \( \mathcal{E} \), resp. \( \mathcal{D} \) the total energy, resp. dissipation functionals, and \( \mathcal{E}_p \), resp. \( \mathcal{D}_p \) the partial energy, resp. dissipation functionals. In addition, we use the following notation for simplicity

\[
I^0 := \|(\eta^0, \partial_t \eta^0, u^0)\|_{4}^2.
\]

1.4. Main results

Now we introduce the stability result for the transformed MRT problem.

**Theorem 1.1 (Stability).** Assume \( \rho \) satisfies (1.5) – (1.6) and

\[
|m| > m_C := \sqrt{\sup_{w \in H^2_s} \frac{g \int \rho' w^2 dy}{\lambda \| \partial_t w \|_0^2}}.
\]

We further assume \( a > 0, (\eta^0, u^0) \in (H^{1,4}_{1,s} \cap H^4_s) \times H^4_s \) and \( \text{div}_{x_0} u^0 = 0 \), where \( A^0 := (\nabla \eta^0 + I)^{-T} \). Then there exist a sufficiently small constant \( \delta > 0 \) such that, for any \( (\eta^0, u^0) \) satisfying \( I^0 \leq \delta^2 \), the transformed MRT problem (1.19) admits a unique global classical solution \( (\eta, u, q) \) in the function class \( \mathcal{S}^{1,4}_{1,s,\infty} \times \mathcal{U}^{4}_{\infty} \times \Omega^{4}_{\infty} \). Moreover, the solution enjoys the estimate (2.14) and the following properties:

(1) the energy inequality in differential form: for a.e. \( t > 0 \),

\[
\frac{d}{dt} \tilde{\mathcal{E}} + \mathcal{D} \leq 0
\]

for some functional \( \tilde{\mathcal{E}} \), which belongs to \( W^{1,\infty}(\mathbb{R}^+) \) and is equivalent to \( \mathcal{E} \) a.e. \( t > 0 \).

\( ^1 \)Since \( \tilde{\mathcal{E}} \in W^{1,\infty}(\mathbb{R}^+) \), there exists a function \( \tilde{\tilde{\mathcal{E}}} \in W^{1,\infty}(\mathbb{R}^+) \cap AC(\mathbb{R}^+) \) such that \( \tilde{\mathcal{E}} = \tilde{\mathcal{E}} \) and \( \tilde{\mathcal{E}}' = \tilde{\tilde{\mathcal{E}}}' \) for a.e. \( t \in \mathbb{R}^+ \).
(2) the stability estimate of total energy: for a.e. $t > 0$,
\[ E(t) + \int_0^t D(\tau) d\tau \lesssim I^0. \] (1.24)

(3) the exponential stability estimates: for a.e. $t > 0$,
\[ e^{c_1 t} (\|\eta_2(t)\|_2^2 + \mathcal{E}_p(t)) + \int_0^t e^{c_1 \tau} D_p(\tau) d\tau \lesssim I^0, \] (1.25)
\[ e^{c_1 t} \|\eta_1(t) - \eta_1^\infty\|_2 \lesssim \sqrt{I^0}. \] (1.26)

for some positive constant $c_1$, where $\eta_1^\infty \in H^2$ only depends on $y_2$.

**Remark 1.1.** If the assumptions of (1.6) and (1.22) are replaced by
\[ \bar{\rho}' \leq 0 \text{ in } \Omega \text{ and } |m| > 0, \]
then the conclusions in Theorem 1.1 also hold. In addition, Theorem 1.1 also holds for the case $g = 0$.

**Remark 1.2.** By the assumptions of $\bar{\rho}$, it is easy to check that
\[ 0 < m_c \leq \frac{h}{\pi} \sqrt{\frac{g\|\bar{\rho}'\|_{L^\infty}}{\lambda}}, \]
please refer to (4.25) and Lemma 4.6 in [25]. Thus, in view of Theorem 1.1 we see that a horizontal magnetic field can inhibit the RT instability, if the strength of magnetic field is properly large. However it is not clear to the authors that whether non-horizontal magnetic fields can also inhibit the RT instability based on the system (1.1).

**Remark 1.3.** For the case $\Omega = \mathbb{R} \times (0, h)$, we also obtain a similar result, where the exponential stability estimates in (1.25) and (1.26) should be replaced by algebraic stability estimates. We will verify this assertion in a forthcoming paper.

**Remark 1.4.** For each fixed $t \in \mathbb{R}_0^+$, the solution $\eta(y,t)$ in Theorem 1.1 belongs to $\in H^3$. Let $\zeta = \eta + y$, then $\zeta$ satisfies (1.14) and (1.15) for each $t \in \mathbb{R}_0^+$. We denote the inverse transformation of $\zeta$ by $\zeta^{-1}$, and then define that
\[ (\rho, v, N, \beta)(x,t) := (\bar{\rho}(y_2) - \bar{\rho}(\zeta_2), u(y,t), m\partial_1 \eta(y,t), q(y,t))_{y=\zeta^{-1}(x,t)}. \]
Consequently $(\rho, v, N, \beta)$ is a classical solution of the magnetic RT problem (1.8)–(1.10) and enjoys stability estimates, which are similar to (1.24)–(1.25) for sufficiently small $\delta$.

Next we roughly sketch the proof of Theorem 1.1 and the details will be presented in Section 2. The key proof for the existence of global small solutions is to derive an $a \text{ priori}$ energy inequality of differential form (1.23) for some energy functional $\tilde{\mathcal{E}}$, which is equivalent to $\mathcal{E}$. To this purpose, let $(\eta, u)$ be a solution to (1.19), and satisfy, for some $T > 0$,
\[ \det(\nabla \eta + I) = 1 \text{ in } \Omega \times T_T, \] (1.27)
\[ \sup_{t \in T_T} \| (\eta, \partial_1 \eta, u)(t) \|_4 \leq \delta \in (0, 1]. \] (1.28)
For sufficiently small $\delta$, we first derive the horizontal-type energy inequality, and then the cur-type energy inequality, see (2.44) and (2.58). Summing up the both energy inequalities of horizontal-type and cur-type, we can arrive at the total energy inequality

$$
\frac{d}{dt} \tilde{\mathcal{E}} + \mathcal{D} \lesssim \sqrt{\mathcal{E}} \mathcal{D}
$$

(1.29)

for some (total) energy functional $\tilde{\mathcal{E}}$, which is equivalent to $\mathcal{E}$ under the stability condition (1.22). In particular, (1.29) further implies (1.23), which yields the priori stability estimate (1.24).

In addition, similarly to (1.23), we can also establish the partial energy inequality

$$
\frac{d}{dt} \tilde{\mathcal{E}}_p + \mathcal{D}_p \leq 0,
$$

(1.30)

where $\tilde{\mathcal{E}}_p$ is equivalent to $\mathcal{E}_p$. Thanks to the observation that $\|\eta\|_{1,3} \lesssim \|\eta\|_{2,3}$ by the horizontal periodicity, we immediately see that $\mathcal{E}_p$ is equivalent to $\mathcal{D}_p$.

Thus the partial energy inequality (1.30), together with the above equivalence, immediately implies the exponential stability of partial energy (1.25). Finally, (1.26) can be easily deduced from (1.25) by an asymptotic analysis method.

Recently Pan–Zhou–Zhu investigated the well-posedness of the equations of a viscous MHD fluid in a 3D periodic domain [36] under the assumption of that the initial data satisfies some odevity condition. Motivated by Pan–Zhou–Zhu’s result, we have the following exponential stability of total energy.

**Corollary 1.1.** If additionally the initial data $(\eta^0, u^0)$ in Theorem 1.1 satisfies the odevity conditions

$$
(\eta^0_1, u^0_1)(y_1, y_2) = -(\eta^0_1, u^0_1)(-y_1, y_2),
$$

(1.31)

$$
(\eta^0_2, u^0_2)(y_1, y_2) = (\eta^0_2, u^0_2)(-y_1, y_2),
$$

(1.32)

then the solution $(\eta, u, q)$ in Theorem 1.1 also satisfies the odevity conditions

$$
(\eta, u_1)(y_1, y_2, t) = -(\eta, u_1)(-y_1, y_2, t), \quad (\eta_2, u_2)(y_1, y_2, t) = (\eta_2, u_2)(-y_1, y_2, t),
$$

(1.33)

and enjoys the following exponential stability of total energy: for a.e. $t > 0$,

$$
e^{c_2 t} \mathcal{E}(t) + \int_0^t e^{c_2 \tau} \mathcal{D}(	au) d\tau \lesssim I^0
$$

(1.34)

for some positive constant $c_2$.

The key idea to prove Corollary 1.1 is that $\mathcal{E}$ is equivalent to $\mathcal{D}$ under the odevity conditions, and thus we immediately get the exponential stability (1.34). The detailed derivation will be provided in Section 3.

We can not expect the stability result for transformed MRT problem under the condition $|m| \in (0, m_C)$. In fact, this condition results in RT instability.
Theorem 1.2 (Instability). Let \( \bar{\rho} \) satisfy (1.5)–(1.6) and \( a \geq 0 \). If \(|m| \in (0, m_C)\), the equilibria \( (\bar{\rho}, 0, \bar{M}) \) is unstable in the Hadamard sense, that is, there are positive constants \( m_0, \epsilon, \delta_0, \) and \( ((\bar{\eta}^0, \eta^i), (\bar{u}^0, u^i)) \in H^4_s \) such that for any \( \delta \in (0, \delta_0] \) and the initial data

\[
(\eta^0, u^0) := \delta(\bar{\eta}^0, \bar{u}^0) + \delta^2(\eta^i, u^i) \in (H^5_s \cap H^3_1 \cap H^5_5) \times H^5_5,
\]

there is a unique solution \( (\eta, u, q) \) to the transformed MRT problem (1.19), where \( (\eta, u, q) \in S_1^{1,4} \times U_4^1 \times \Omega_4^1 \) for any \( \tau \in I^\max, \) \( T^\max \) denotes the maximal time of existence of the solution, and \( \text{div}_\Omega u^0 = 0 \) with \( A^0 := (\nabla \eta^0 + I)^{-1} \). However, for \( 1 \leq i, j \leq 2, \) and \( k = 0, 1, \)

\[
\| \partial^k_x \chi_i (T^\delta) \|_{L^1} \geq \epsilon
\]

for some escape time \( T^\delta := \frac{1}{\Lambda} \ln \frac{\epsilon}{m_0 \delta} \in I^\max, \) where \( \chi \) can be taken by \( \eta \) or \( u. \)

The proof of Theorem 1.2 is based on a so-called bootstrap instability method. The bootstrap instability method has its origin in [12, 13]. Later, various versions of the bootstrap approach were presented by many authors, see [8, 11, 30] for examples. In particular, recently Jiang–Jiang–Zhan proved the existence of the RT instability solution under \( L^1 \)-norm for the stratified viscous, non-resistive MHD fluids [30]. In this paper, we adapt the version of the bootstrap instability method in [30] to prove Theorem 1.2. It should be noted that the authors in [30] considered the RT instability in viscous fluids. However our problem is the inviscid case, thus there exist some troubles, which are different to the Ref. [30], in the proof of Theorem 1.2. In particular, the absence of the strong continuity of highest order of spatial derivatives of \((\eta, u)\) results in some troubles. Fortunately, these troubles can be overcome by making use of the stability of local(-in-time) solutions (2.80) and the weak continuity, i.e. \( \nabla^4 (\partial_t \eta, u) \in C^0_B, \text{weak}(T_T, L^2) \) for any \( \tau \in I^\max \).

We mention that Jang–Guo ever proved the RT instability of inviscid fluids in a 2D periodic domain [18]. It is not clear to authors that whether Jang–Guo’s result’s can be extended to the slab domain. In other word, it is not clear that whether Theorem 1.2 also holds for the case \( m = 0, \) such case will be further investigated in future.

The rest of this paper is organized as follows. In Sections 2–4, we provide the proof of Theorem 1.1, Corollary 1.1 and Theorem 1.2 in sequence. In Section 5, we will establish the local well-posedness result for the transformed MRT problem (1.19). Finally, in Appendix A we list some well-known mathematical results, which will be used in Sections 2–5.

2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The key step is to derive the total energy inequality (1.23) and the partial energy inequality (1.30) for the transformed MRT problem (1.19) by \textit{a priori} estimates. To this end, let \((\eta, u, q)\) be a solution to (1.19) and satisfy (1.27) and (1.28), where \( \delta \) is sufficiently small, and the smallness of \( \delta \) depends on \( g, a, \lambda, m, \bar{\rho} \) and \( \Omega. \) It should be noted that \( a, m \) and \( \bar{\rho} \) satisfy the assumptions in Theorem 1.1. Next we proceed with \textit{a priori} estimates.

2.1. Preliminary estimates

To be withing, we shall establish some preliminary estimates involving \((\eta, u)\).

Lemma 2.1. For any given \( t \in T_T, \) we have
(1) the estimates for $A$ and $A_t$:

$$
\|A\|_{C^0(\overline{\Omega})} + \|A\|_3 \lesssim 1, \\
\|A_t\|_{i,j} \lesssim \|u\|_{i,j+1} \text{ for } 0 \leq i, j \leq 3.
$$

(2.1)

(2) the estimates $\tilde{A}$:

$$
\|\tilde{A}\|_i \lesssim \|\eta\|_{i+1} \text{ for } 0 \leq i \leq 3, \\
\|\tilde{A}\|_{i,j} \lesssim \|\eta\|_{i,j+1} \text{ for } 1 \leq i + j \leq 4 \text{ and } i \geq 1.
$$

(2.3)

Here and in what follows $\tilde{A} := A - I$.

(3) the estimate of $\text{div} u$: for $0 \leq i \leq 3$,

$$
\|\text{div} u\|_i \lesssim \|(\eta, u)\|_4 \|(\eta, u)\|_{1,i}.
$$

(2.5)

(4) the estimates involving gravity term: for sufficiently small,

$$
\|G\eta\|_3 \lesssim \|\eta\|_3,
$$

(2.6)

$$
\|\mathcal{G}\|_{k,0} \lesssim \begin{cases} 
\|\eta\|_2 \|\eta\|_0 & \text{for } k = 0; \\
\|\eta\|_3 \|\partial_1 \eta\|_{k-1,0} & \text{for } 1 \leq k \leq 4,
\end{cases}
$$

(2.7)

where $\mathcal{G} := g (G\eta - \bar{\rho}^t \eta)$.

**Proof.** (1)–(3) Recalling (1.27) and the definitions of $A$ and $\tilde{A}$, we can compute out that

$$
A = \begin{pmatrix}
\partial_2 \eta + 1 - \partial_1 \eta \\
-\partial_2 \eta_1 & \partial_1 \eta_1
\end{pmatrix}
$$

and thus

$$
\tilde{A} = \begin{pmatrix}
\partial_2 \eta - \partial_1 \eta \\
-\partial_2 \eta_1 & \partial_1 \eta_1
\end{pmatrix}.
$$

(2.8)

Making use of (1.19), (1.28), the embedding inequality (A.1) and the product estimate (A.3), we can easily deduce (2.1)–(2.5) from (1.19) and the expressions of $A$ and $\tilde{A}$.

(4) We turn to verifying (2.6) and (2.7). By (1.28) and Lemma A.8, $\zeta := \eta + y$ satisfies the diffeomorphism properties (1.14) and (1.15) for sufficiently small $\delta$. Thus $\bar{\rho}^{(l)}(y_2 + \eta_2)$ for any $y \in \overline{\Omega}$ makes sense, and

$$
\bar{\rho}^{(l)}(y_2 + \eta_2) - \bar{\rho}^{(l)}(y_2) = \int_0^{\eta_2} \bar{\rho}^{(l+1)}(y_2 + z)dz
$$

(2.9)

for $0 \leq l \leq 3$. Moreover, for any given $t \in \overline{T_T},$

$$
\sup_{y \in \overline{\Omega}} \left| \bar{\rho}^{(l+1)} \right|_{y_2 = y_2 + \eta_2} \lesssim 1,
$$

(2.10)

$$
\sup_{y \in \overline{\Omega}} \sup_{z \in \Psi} \left| \bar{\rho}^{(l+1)}(y_2 + z) \right| \lesssim 1,
$$

(2.11)
where \( \Psi := \{ \tau \mid 0 \leq \tau \leq \eta_2 \} \) for \( \eta_2 \geq 0 \) and := \((\eta_2,0]\) for \( \eta_2 < 0 \). Making use of (1.28), (2.9)–(2.11) and the embedding inequality (A.1), we easily deduce (2.6) from the definition of \( G_\eta \).

Since
\[
\bar{\rho}^{(m)}(y_2 + \eta_2) - \bar{\rho}^{(m)}(y_2) = \bar{\rho}^{(m+1)}(y_2)\eta_2 + \int_0^{\eta_2} (\eta_2(y,t) - z) \bar{\rho}^{(m+2)}(y_2 + z)dz
\]
for \( 0 \leq m \leq 2 \), it is easy to see that
\[
\|G\|_0 \lesssim \|\eta_2\|_2\|\eta_2\|_0
\]
and
\[
\|\partial_k G\|_0 = \|\partial_k^{-1}(\partial_1(y_2 + \eta_2) + \partial_1\eta_2) - \bar{\rho}'\partial_k \eta_2\|_0
\lesssim \|\partial_1(y_2 + \eta_2) - \bar{\rho}'\partial_1\eta_2\|_0 + \||\partial_k^{-1}(\partial_1(y_2 + \eta_2))\partial_1\eta_2\|_0
\lesssim \|\eta_2\|_3\|\partial_1\eta_2\|_{k-1,0} \text{ for } 1 \leq k \leq 4.
\]
Putting the above two estimates together yields (2.7).

\[\Box\]

**Lemma 2.2.** We have
(1) the estimate of \( \text{div}\eta \): for \( 0 \leq i \leq 3 \),
\[
\|\text{div}\eta\|_{j,i} \lesssim \begin{cases} \|\eta\|_3\|\eta\|_{1,i} & \text{for } j = 0; \\ \|\eta\|_3\|\eta\|_{1,i+1} & \text{for } j = 1, \end{cases}
\]
(2.12)
\[
\|\text{div}\eta\|_{j,i} \lesssim \|\eta\|_3\|\eta\|_{2,2} \text{ for } i + j = 4 \text{ and } j \geq 2.
\]
(2.13)

(2) the estimate of \( \eta_2 \):
\[
\|\eta_2\|_j \lesssim \begin{cases} \|\eta\|_{1,0} & \text{for } j = 0, 1; \\ \|\eta\|_{1,j-1} & \text{for } 2 \leq j \leq 4. \end{cases}
\]
(2.14)

**Proof.** (1) Recalling (1.27), we can compute out that
\[
\text{div}\eta = \partial_1\eta_2\partial_2\eta_1 - \partial_1\eta_1\partial_2\eta_2.
\]
(2.15)

Exploiting the product estimate (A.3), we can easily deduce (2.12)–(2.13) from the relation above.

(2) Noting that
\[
\eta_2|_{\partial\Omega} = 0 \text{ and } \partial_2\eta_2 = \text{div}\eta - \partial_1\eta_1,
\]
(2.16)

thus, using (2.16) and (A.3), we get
\[
\|\eta_2\|_0 \lesssim \| (\partial_1\eta_1, \text{div}\eta) \|_0,
\]
\[
\|\eta_2\|_j \lesssim \|\eta_2\|_0 + \|\nabla\eta_2\|_{j-1} \lesssim \| (\partial_1\eta, \text{div}\eta) \|_{j-1} \text{ for } 1 \leq j \leq 4.
\]
We immediately get (2.14) from the two estimates above and (2.12).  \[\Box\]
Lemma 2.3. Let the multiindex $\alpha$ satisfy $|\alpha| \leq 3$ and

$$W^{\alpha} = \partial^\alpha (\text{curl}_A (\bar{\rho}u) - \lambda m^2 \text{curl}_{\partial_1A} \partial_1 \eta).$$

(1) Then we have

$$\begin{aligned}
\|W^{\alpha}\|_0 &\lesssim \|(\partial_1 \eta, u)\|_4, \\
\|\partial^\alpha (\text{curl}_A(\bar{\rho}\chi) - \text{curl}(\bar{\rho}\chi))\|_0 &\lesssim \|\eta\|_4 \|\chi\|_4.
\end{aligned}$$

(2.18)

(2.19)

where $\chi = \eta$, $\partial_1 \eta$ and $u$.

(2) For $\alpha_2 \neq 2$, we have

$$\begin{aligned}
\|W^{\alpha}\|_0 &\lesssim \|(\partial_1 \eta, u)\|_4 \|(\partial_1 \eta, u)\|_{L^3}, \\
\|\partial^\alpha (\text{curl}_A(\bar{\rho}\eta) - \text{curl}(\bar{\rho}\eta))\|_0 &\lesssim \|\eta\|_4 \|\eta\|_2 \text{ for } i = 0, 1.
\end{aligned}$$

(2.20)

(2.21)

(2.22)

PROOF. The estimates (2.18) and (2.20) can be easily derived by using (2.2), (2.4) and the product estimate (A.3). Noting that

$$\text{curl}_A w - \text{curl} w = \bar{A}_{1k} \partial_k w_2 - \bar{A}_{2k} \partial_k w_1,$$

then (2.19) and (2.21) – (2.22) can be easily derived by using (2.3), (2.8), (A.3) and (A.7). □

2.2. Horizontal-type energy inequalities

This section is devoted to establishing the total/partial horizontal-type energy inequalities. Let $0 \leq k \leq 4$, then we apply $\partial^k_1$ to (1.19) to get that

$$\begin{aligned}
\partial^k_1 \eta_1 &= \partial^k_1 u, \\
\bar{\rho} \partial^k_1 u_t + \partial^k_1 \nabla_A q + a \bar{\rho} \partial^k_1 u &= \lambda m^2 \partial^{k+2}_1 \eta + g \bar{\rho} \partial^k_1 \eta_2 \mathbf{e}_2 + \partial^k_1 \mathcal{G} \mathbf{e}_2, \\
\partial^k_1 \text{div}_A u &= 0, \\
(\partial^k_1 \eta, \partial^k_1 u)|_{\partial \Omega} \cdot \bar{n} &= 0.
\end{aligned}$$

(2.23)

Thus we can derive from (2.23) the following horizontal spatial estimates of $(\eta, u)$.

Lemma 2.4. For $0 \leq k \leq 4$,

$$\begin{aligned}
\frac{d}{dt} \left( \int \bar{\rho} \partial^k_1 \eta \cdot \partial^k_1 u dy + \frac{a}{2} \|\sqrt{\bar{\rho}} \partial^k_1 \eta\|_0^2 \right) - E(\partial^k_1 \eta) &\lesssim \|\sqrt{\bar{\rho}} \partial^k_1 u\|^2_0 + \sqrt{\mathcal{E}} D_p, \\
\frac{d}{dt} \left( \|\sqrt{\bar{\rho}} u\|^2_{k,0} - E(\partial^k_1 \eta) \right) + c \|\sqrt{\bar{\rho}} u\|^2_{k,0} &\lesssim \sqrt{\mathcal{E}} D_p.
\end{aligned}$$

(2.24)

(2.25)

PROOF. Multiplying (2.23) by $\partial^k_1 \eta$, resp. $\partial^k_1 u$ in $L^2$, then, using the integrating by parts and (2.23), we have that

$$\begin{aligned}
\frac{d}{dt} \left( \int \bar{\rho} \partial^k_1 \eta \cdot \partial^k_1 u dy + \frac{a}{2} \|\sqrt{\bar{\rho}} \partial^k_1 \eta\|_0^2 \right) - E(\partial^k_1 \eta)
&= \|\sqrt{\bar{\rho}} \partial^k_1 u\|^2_0 + \int \partial^k_1 \mathcal{G} \partial^k_1 \eta_2 dy - \int \partial^k_1 \nabla_A q \cdot \partial^k_1 \eta dy,
\end{aligned}$$

(2.26)
resp.
\[
\frac{1}{2} \frac{d}{dt} \left( \| \sqrt{\rho} u \|^2_{k,0} - E(\partial^k_1 \eta) \right) + a \| \sqrt{\rho} u \|^2_{k,0} = \int \partial^k_1 \mathcal{G} \partial^k_1 u_2 dy - \int \partial^k_1 \nabla \mathcal{A} \cdot \partial^k_1 u dy.
\]
(2.27)

By (2.7), (2.14) and (A.7), we can estimate that
\[
\int \partial^k_1 \mathcal{G} \partial^k_1 \eta_2 dy \leq \| \mathcal{G} \|_{k,0} \| \eta_2 \|_{k,0} \lesssim \sqrt{E} D_p,
\]
(2.28)
\[
\int \partial^k_1 \mathcal{G} \partial^k_1 u_2 dy \leq \| \mathcal{G} \|_{k,0} \| u_2 \|_{k,0} \lesssim \sqrt{E} D_p.
\]
(2.29)

Next we estimate for the integrals involving the pressure in (2.26)–(2.27) by two cases.

(1) We first consider the case \( k = 0 \).

By the integration by parts and the boundary-value condition (2.23) with \( k = 0 \),
\[
- \int \nabla \mathcal{A} q \cdot \eta dy = \int q \mathrm{div} \eta dy - \int \nabla \tilde{\mathcal{A}} q \cdot \eta dy.
\]

Next we estimate for the two integrals on the right hand of the above identity.

Let \( K := \eta_1 ( - \partial_2 \eta_2, \partial_1 \eta_2 )^\top \). Then the identity (2.15) can be rewritten as follows
\[
\mathrm{div} \eta = \mathrm{div} K.
\]
(2.30)

By (2.14), the embedding inequality (A.1) and (A.7), we have
\[
\| K \|_0 \lesssim \| \eta \|_2 \| \eta_2 \|_1 \lesssim \| \eta \|_2 \| \eta \|_{2,1},
\]
(2.31)

Exploiting (2.30) and (2.31), we have
\[
\int q \mathrm{div} \eta dy = - \int K \cdot \nabla q dy \leq \| K \|_0 \| \nabla q \|_0 \lesssim \sqrt{E} D_p.
\]
(2.32)

Noting that
\[
\nabla \tilde{\mathcal{A}} q \cdot \eta = \eta_1 ( - \partial_2 \eta_2 \partial_1 q - \partial_1 \eta_2 \partial_2 q ) + \eta_2 ( \partial_1 \eta_1 \partial_2 q - \partial_2 \eta_1 \partial_1 q ),
\]

thus, using (2.14), (A.3) and (A.7), we deduce that
\[
\left| \int \nabla \tilde{\mathcal{A}} q \cdot \eta dy \right| \lesssim \| \eta \|_2 \| \eta \|_{1,1} \| \nabla q \|_0 \lesssim \sqrt{E} D_p.
\]
(2.33)

Combining with (2.32) and (2.33), we get
\[
\left| \int \nabla \mathcal{A} q \cdot \eta dy \right| \lesssim \sqrt{E} D_p.
\]
(2.34)

Putting (2.28) with \( k = 0 \) and (2.34) into (2.26) yields (2.24) with \( k = 0 \).

In addition, by the integration by parts, (2.23) \(_3\) and (2.23) \(_4\) with \( k = 0 \), we have
\[
- \int \nabla \mathcal{A} q \cdot u dy = \int q \mathrm{div} \mathcal{A} u dy = 0.
\]
(2.35)
Thus putting (2.29) with \(k = 0\) and (2.35) into (2.27) with \(k = 0\) yields (2.25) with \(k = 0\).

(2) Now we further consider the case \(k \neq 0\).

Making use of (2.12), (2.13) and (A.7), we can estimate that

\[
- \int \partial^k_i \nabla A q \cdot \partial^k_i \eta \, dy = \int \partial^k_i q \mathrm{div} A \partial^k_i \eta \, dy - \int [\partial^k_i, A] \nabla q \cdot \partial^k_i \eta \, dy
\]

\[
\leq \|\partial^k_i \nabla A q\|_0 \|\partial^{k+1}_i \eta\|_0 + \|\partial^k_i q\|_0 \|\partial^k_i \mathrm{div} \eta\|_0
\]

\[
\lesssim \sqrt{\mathcal{E}} D_p.
\]  

Putting (2.28) and (2.36) into (2.26) yields (2.24) for \(k \neq 0\).

Exploiting (2.23), we have

\[
- \int \partial^k_i \nabla A q \cdot \partial^k_i u \, dy = \int \partial^k_i q \mathrm{div} A \partial^k_i u \, dy - \int [\partial^k_i, A] \nabla q \cdot \partial^k_i u \, dy
\]

\[
= \int \partial^k_i q \partial^k_i \mathrm{div} A u \, dy - \int \partial^k_i q [\partial^k_i, A^T] : \nabla u \, dy - \int [\partial^k_i, A] \nabla q \cdot \partial^k_i u \, dy
\]

\[
= - \int \partial^k_i q [\partial^k_i, A^T] : \nabla u \, dy - \int [\partial^k_i, A] \nabla q \cdot \partial^k_i u \, dy \lesssim \sqrt{\mathcal{E}} D_p.
\]  

Consequently, putting (2.29) and (2.37) into (2.27) yields (2.25) with \(k \neq 0\). The proof is completed. □

Now we shall establish stabilizing estimates for \(E(\partial^k_i \eta)\) appearing in (2.24) and (2.25).

**Lemma 2.5.** It holds that

\[
\|\eta\|_{i+1,0}^2 \lesssim -E(\partial^k_i \eta) + \|\eta\|_4 \|\eta\|_{2,3}^2 \text{ for any } 0 \leq i \leq 4.
\]  

**Proof.** By the definition of \(m_C\), it is easy to see that

\[
- \int g \rho' w^2 \, dy \geq -\lambda m_C^2 \|\partial^1 w\|_0^2 \text{ for any } w \in H^1_\sigma,
\]

which, together with the stability condition \(|m| > m_C\), implies that

\[
\|w\|_{1,0}^2 \lesssim \lambda (m^2 - m_C^2) \|\partial^1 w\|_0^2 \lesssim -E(w) \text{ for any } w \in H^1_\sigma.
\]  

Let us consider the Stokes problem

\[
\begin{aligned}
-\Delta \tilde{\eta} + \nabla \varpi &= 0, \quad \mathrm{div} \tilde{\eta} = \mathrm{div} \eta \quad \text{in } \Omega, \\
\tilde{\eta} &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(2.40)

By the existence theory of Stokes problem, there exist a unique solution \((\tilde{\eta}, \varpi) \in H^4 \times H^3\) to (2.40). Moreover, \(\partial^k_i (\tilde{\eta}, \varpi)\) is also the solution of Stokes problem for \(1 \leq j \leq 3\) and

\[
\|\tilde{\eta}\|_{j,2} \lesssim \|\mathrm{div} \eta\|_{j,1} \lesssim \|\eta\|_4 \|\eta\|_{2,3},
\]  

(2.41)

where we have used (2.12), (2.13) and (A.7) in the last inequality above.
Now we use $\partial^i_1(\eta - \tilde{\eta})$ to rewrite $E(\partial^i_1\eta)$ as follows:

$$E(\partial^i_1\eta) = E(\partial^i_1(\eta - \tilde{\eta})) + E(\partial^i_1\tilde{\eta}) - I_i,$$

(2.42)

where

$$I_i := 2\lambda m^2 \int \partial^{i+1}_1\eta \cdot \partial^{i+1}_1\tilde{\eta} dy - 2g \int \rho' \partial^{i}_1\eta_2 \partial^{i}_2\tilde{\eta}_2 dy.$$

Note that $\partial^i_1(\eta - \tilde{\eta}) \in H^1_\sigma$, thus, we use (2.39) to get

$$\|\eta - \tilde{\eta}\|^2_{i+1,0} \lesssim -E(\partial^i_1(\eta - \tilde{\eta})).$$

which, together with (2.42) and Young’s inequality, yields

$$\|\eta\|^2_{i+1,0} \lesssim E(\partial^i_1\tilde{\eta}) - E(\partial^i_1\eta) - I_i + \|\tilde{\eta}\|^2_{i+1,0}. \quad (2.43)$$

Making use of (2.14), (2.41) and (A.7), we can estimate that

$$E(\partial^i_1\tilde{\eta}) - I_i + \|\tilde{\eta}\|^2_{i+1,0} \lesssim \|\eta\|_4 \|	ilde{\eta}\|_{2,3}^2 \text{ for } 0 \leq i \leq 4.$$

Finally, putting the above estimate into (2.43) yields (2.38). This completes the proof. \hfill $\square$

Thanks to Lemmas 2.4, 2.5 and (2.14), we easily get the total/partial horizontal-type energy inequalities.

Lemma 2.6. There exist two functionals $E$ and $\tilde{E}$ of $(\eta, u)$ such that

$$\frac{d}{dt} E + c\|u, \partial_1 \eta\|_{L^1_0}^2 \lesssim \sqrt{E} D_p, \quad (2.44)$$

$$\frac{d}{dt} \tilde{E} + c\|u\|_0^2 + \|\partial_1(u, \partial_1 \eta)\|_{L^1_0}^2 \lesssim \sqrt{\tilde{E}} D_p, \quad (2.45)$$

$$\|\tilde{\eta}, \partial_1 \eta, u\|_{L^1_0}^2 - c\|\tilde{\eta}\|_3 \|	ilde{\eta}\|_{L^1_0}^2 \lesssim \tilde{E} \lesssim \|\tilde{\eta}, \partial_1 \eta, u\|_{L^1_0}^2; \quad (2.46)$$

$$\|u\|_0^2 + \|\tilde{\partial}_1(\tilde{\eta}, \partial_1 \eta, u)\|_{L^1_0}^2 - c\|\tilde{\eta}\|_4 \|	ilde{\eta}\|_{L^1_0}^2 \lesssim \mathcal{E} \lesssim \|u\|_0^2 + \|\partial_1(\eta, \partial_1 \eta, u)\|_{L^1_0}^2. \quad (2.47)$$

2.3. Curl-type energy inequality

This section is devoted to establishing curl-type energy inequalities. By (1.17), we have

$$\text{curl}_A(gG_\eta e_2) = \text{curl}_A(-g\rho e_2) = \mathcal{A}_{1j} \partial_j(-g\rho) = g\partial_1 \eta_2.$$

Thus applying curl$_A$ to (1.19)$_2$ yields that

$$\partial_1 \text{curl}_A(\tilde{\rho}u) + a\text{curl}_A(\tilde{\rho}u) = \lambda m^2 \text{curl}_A(\tilde{\rho}^{-1} \partial^i_1(\tilde{\rho} \eta)) + g\tilde{\rho} \partial_1 \eta_2 + \text{curl}_A(\tilde{\rho}u). \quad (2.48)$$

Let the multiindex $\alpha$ satisfy $|\alpha| \leq 2$. Recalling the definition of $W^\alpha$ in (2.17), thus applying $\partial^\alpha$ to (2.48) yields

$$\partial_1 \partial^\alpha \text{curl}_A(\tilde{\rho}u) + a\partial^\alpha \text{curl}_A(\tilde{\rho}u)$$

$$= \partial_1(\lambda m^2(\tilde{\rho}^{-1} \partial^\alpha \text{curl}_A(\tilde{\rho} \partial_1 \eta)) + [\partial^\alpha \text{curl}_A, \tilde{\rho}^{-1}](\tilde{\rho} \partial_1 \eta)) + \partial^\alpha(\tilde{\rho} \partial_1 \eta_2) + W^\alpha. \quad (2.49)$$

Now we derive the following energy estimates for curl$u$ and curl$\partial_1 \eta$.\hfill 16
Lemma 2.7. For multiindex $\alpha$ satisfying $|\alpha| \leq 3$, we have

$$
\frac{d}{dt} \int (2\partial^\alpha \text{curl}_A(\bar{\rho}\eta) \partial^\alpha \text{curl}_A(\bar{\rho}u) + a|\partial^\alpha \text{curl}_A(\bar{\rho}u)|^2) \, dy + c\|\partial^\alpha \text{curl}_A(\bar{\rho}\partial_1\eta)\|^2_0 \\
\lesssim \|\partial^\alpha \text{curl}_A(\bar{\rho}u)\|^2_0 + \|\lambda m^2[\partial^\alpha \text{curl}_A, \bar{\rho}^{-1}](\bar{\rho}\partial_1\eta) + \partial^\alpha (g\bar{\rho}'\eta_2)\|^2_0 \\
+ \left\{ \begin{array}{l}
\sqrt{\mathcal{E}}D, \\
\sqrt{\mathcal{E}}D_p \text{ for } \alpha_1 \geq 1,
\end{array} \right. \tag{2.50}
$$

Proof. (1) Multiplying (2.49) by $\partial^\alpha \text{curl}_A(\bar{\rho}\eta)$ in $L^2$ yields that

$$
\frac{d}{dt} \int \left( \partial^\alpha \text{curl}_A(\bar{\rho}\eta) \partial^\alpha \text{curl}_A(\bar{\rho}u) + \frac{a}{2}|\partial^\alpha \text{curl}_A(\bar{\rho}u)|^2 \right) \, dy + \lambda m^2 \|\bar{\rho}^{-1}\partial^\alpha \text{curl}_A(\bar{\rho}\partial_1\eta)\|^2_0 \\
= I^\alpha_1 + I^\alpha_2 + \|\partial^\alpha \text{curl}_A(\bar{\rho}u)\|^2_0 \\
- \int (\lambda m^2[\partial^\alpha \text{curl}_A, \bar{\rho}^{-1}](\bar{\rho}\partial_1\eta) + \partial^\alpha (g\bar{\rho}'\eta_2)) \partial^\alpha \text{curl}_A(\bar{\rho}\partial_1\eta) \, dy, \tag{2.52}
$$

where

$$
I^\alpha_1 := \int (\partial^\alpha \text{curl}_A(\bar{\rho}\eta) \partial^\alpha \text{curl}_A(\bar{\rho}u) + W^\alpha \partial^\alpha \text{curl}_A(\bar{\rho}\eta)) \, dy \\
- \int (\partial^\alpha (g\bar{\rho}'\eta_2) + \lambda m^2 \partial^\alpha \text{curl}_A\partial_1\eta) \partial^\alpha \text{curl}_A(\bar{\rho}\eta) \, dy,
$$

$$
I^\alpha_2 := a \int \partial^\alpha \text{curl}_A(\bar{\rho}\eta) \partial^\alpha \text{curl}_A(\bar{\rho}\eta) \, dy.
$$

Making use of (2.1), (2.2), (2.14), (2.18), (2.20), (A.3) and (A.7), we can estimate that

$$
I^\alpha_1 \lesssim \left\{ \begin{array}{l}
\sqrt{\mathcal{E}}D, \\
\sqrt{\mathcal{E}}D_p \text{ for } \alpha_1 \geq 1
\end{array} \right. \tag{2.53}
$$

and

$$
I^\alpha_2 \lesssim \sqrt{\mathcal{E}}D_p \text{ for } \alpha_1 \geq 1. \tag{2.54}
$$

In addition, noting the structure

$$
\text{curl}_A(\bar{\rho}\eta) = \partial_i \mathcal{A}_{1j} \partial_j (\bar{\rho}\eta_2) - \partial_i \mathcal{A}_{2j} \partial_j (\bar{\rho}\eta_1)
$$

thus, similarly to (2.53) with further using integral formula by parts, (1.19) and (2.8), we easily estimate that

$$
I^\alpha_2 \lesssim \sqrt{\mathcal{E}}D. \tag{2.55}
$$

Finally, putting (2.53)–(2.55) into (2.52), and then using Young’s inequality and the lower-bound condition $\inf_{\bar{\rho} \in \Pi} \bar{\rho} > 0$, we get (2.50) immediately.
(2) Multiplying (2.49) by $\partial^{\alpha} \text{curl}_A (\bar{\rho}u)$ in $L^2$ yields that

$$\frac{1}{2} \frac{d}{dt} (\|\partial^{\alpha} \text{curl}_A (\bar{\rho}u) \|^2_0 + \lambda \|m \sqrt{\bar{\rho}^{-1}} \partial^{\alpha} \text{curl}_A (\bar{\rho} \partial_1 \eta) \|^2_0) + a \|\partial^{\alpha} \text{curl}_A (\bar{\rho}u) \|^2_0$$

$$= \int \left( \lambda m^2 [\partial^{\alpha} \text{curl}_A, \bar{\rho}^{-1}] (\rho \partial^2 \eta) + \partial^{\alpha} (g \partial \partial_1 \eta_2) \right) \partial^{\alpha} \text{curl}_A (\bar{\rho}u) \, dy + I_3^\alpha, \quad (2.56)$$

where we have defined that

$$I_3^\alpha := \lambda m^2 \int \bar{\rho}^{-1} \partial^{\alpha} \text{curl}_A (\bar{\rho} \partial_1 \eta) \partial^{\alpha} (\text{curl}_A, (\bar{\rho} \partial_1 \eta) - \text{curl}_{\partial A} (\bar{\rho}u)) \, dy$$

$$+ \int (\lambda m^2 [\partial^{\alpha} \text{curl}_{\partial A}, \bar{\rho}^{-1}] (\bar{\rho} \partial_1 \eta) + W^{\alpha}) \partial^{\alpha} \text{curl}_A (\bar{\rho}u) \, dy.$$

Similarly to (2.53), it is easy to estimate that

$$I_3^\alpha \lesssim \begin{cases} \sqrt{\mathcal{E}D}; \\ \sqrt{\mathcal{E}D}_p \text{ for } |\alpha| \leq 1 \text{ or } \alpha_1 \geq 1. \end{cases} \quad (2.57)$$

Plugging (2.57) into (2.56) and then using Young’s inequalities, we obtain (2.51). This completes the proof. \hfill \Box

Now we use Lemmas 2.4 to further derive the total/partial vorticity-type energy inequalities.

**Proposition 2.1.** There exist two functionals $\mathcal{E}^\text{cul}_p$ and $\mathcal{E}^\text{cul}$ of $(\eta, u)$ such that

$$\frac{d}{dt} \mathcal{E}^\text{cul} + c \|(u, \partial_1 \eta)\|^2_4 \lesssim \|(u, \partial^3_1 u, \partial^5_1 \eta)\|^2_0 + \sqrt{\mathcal{E}D}, \quad (2.58)$$

$$\frac{d}{dt} \mathcal{E}^\text{cul}_p + c (\|(u, \partial_1 \eta)\|_1^2 + \|u\|^2_3) \lesssim \|(u, \partial^3_1 u, \partial^5_1 \eta)\|^2_0 + \sqrt{\mathcal{E}D}_p, \quad (2.59)$$

$$\|\text{curl}(\bar{\rho} \partial_1 \eta, \bar{\rho}u)\|^2_3 + c \|\eta\|^2_4 \lesssim \|(u, \partial_1 \eta, u)\|^2_1 \lesssim \mathcal{E}^\text{cul} \lesssim \|(u, \partial_1 \eta, u)\|^2_1, \quad (2.60)$$

$$\|\text{curl}(\bar{\rho}u)\|^2_2 + \|\text{curl}(\bar{\rho} \eta, \bar{\rho} \partial_1 \eta)\|^2_2 - c \|\eta\|^2_4 \|(u, \partial_1 \eta, u)\|^2_1 \lesssim \mathcal{E}^\text{cul}_p \lesssim \|(u, \partial_1 \eta, \partial_1 \eta, u)\|^2_3. \quad (2.61)$$

**Proof.** Let $1 \leq i \leq 3$ and $f = \partial_i \eta$ or $\partial^2_1 \eta$. Then, for any multiindex $\beta$ satisfying $|\beta| = 3 - i$, it is easy to check

$$\| [\partial^i_1 \partial^{\beta} \text{curl}_A, \bar{\rho}^{-1}] (\bar{\rho} f) \|_0 \lesssim \|f\|_{2-i}^2 + \begin{cases} 0 & \text{for } i = 1; \\ \|\eta\|^2_{3,2} \|f\|_1 & \text{for } i = 2, 3. \end{cases}$$

In addition, for any multiindex $\alpha$ satisfying $|\alpha| \leq 3$, it obviously holds that

$$\| [\partial^{\alpha} \text{curl}_A, \bar{\rho}^{-1}] (\bar{\rho} f) \|_0 \lesssim \|f\|_{|\alpha|}.$$
for some functionals $\mathcal{E}_{p,1}^{c\text{ul}}$ and $\mathcal{E}_{p,2}^{c\text{ul}}$, which are equivalent to $\|\text{curl}_A(\hat{\rho}\eta, \hat{\rho\partial_1\eta}, \hat{\rho}u)\|_{i,2-i}^2$ and $\|\text{curl}_A(\hat{\rho}u, \hat{\rho\partial_1\eta})\|_{i,2}^2$, resp.

Similarly, we can easily derive from (2.14), (2.50) and (2.51) that
\[
\frac{d}{dt} \mathcal{E}^{\text{cul}} + c \|\text{curl}_A(\hat{\rho}u, \hat{\rho\partial_1\eta})\|_{i,2}^2 \lesssim \|\eta\|_{2,3}^2 + \sqrt{\mathcal{E}\mathcal{D}},
\]
for some functional $\mathcal{E}^{\text{cul}}$, which is equivalent to $\|\text{curl}_A(\hat{\rho}\eta, \hat{\rho\partial_1\eta}, \hat{\rho}u)\|_{i,2}^2$.

Making use of (2.5), (2.12), (2.13) and Hodge-type elliptic estimate (A.9), we can deduce that, for $0 \leq j \leq 3$,
\[
\|(u, \partial_1\eta)\|_{j,4-j} \lesssim \|(u, \partial_1\eta)\|_{j,0} + \|\text{div} u, \text{curl} u, \text{div} \partial_1\eta, \text{curl} \partial_1\eta\|_{j,3-j}
\lesssim \|(u, \partial_1\eta, \text{curl}(\hat{\rho}u), \text{curl}(\hat{\rho\partial_1\eta}))\|_{j,2-j}
+ \|(\eta, u)\|_4 \|(\eta, u)\|_{1,3} + \begin{cases} \|\eta\|_3 \|\eta\|_{1,2} & \text{for } j = 0, \\ \|\eta\|_3 \|\eta\|_{2,2} & \text{for } 1 \leq j \leq 3 \end{cases}
\]
and
\[
\|u\|_3^2 \lesssim \|(u, \text{curl}(\hat{\rho}u))\|_2 + \|(\eta, u)\|_4 \|(\eta, u)\|_{1,2}.
\]

Making use of (2.21), (2.22), (2.65), (2.66), the interpolation inequality (A.2) and (A.7), we further derive from (2.62)–(2.64) that
\[
\frac{d}{dt} \mathcal{E}_{p,i}^{\text{cul}} + c \|(u, \partial_1\eta)\|_{i,4-i}^2 \lesssim \|(u, \partial_1^2\eta)\|_{i,3-i}^2 + \sqrt{\mathcal{E}\mathcal{D}_p} \text{ for } 1 \leq i \leq 3,
\]
\[
\frac{d}{dt} \mathcal{E}_{p,4}^{\text{cul}} + c \|u\|_3^2 \lesssim \|u\|_2^4 + \|\eta\|_{2,2}^2 + \sqrt{\mathcal{E}\mathcal{D}_p}
\]
\[
\frac{d}{dt} \mathcal{E}^{\text{cul}} + c \|(u, \partial_1\eta)\|_4^2 \lesssim \|(u, \partial_1^2\eta)\|_3^2 + \sqrt{\mathcal{E}\mathcal{D}}.
\]

Consequently, we immediately get (2.58) and (2.59) from the three estimates above by using (A.2) and (A.7), where $\mathcal{E}^{\text{cul}} := \mathcal{E}^{\text{cul}} + d\mathcal{E}_{p,1}^{\text{cul}}$ and $\mathcal{E}_{p,4}^{\text{cul}} = \mathcal{E}_{p,1}^{\text{cul}} + c \sum_{j=2}^4 \mathcal{E}_{p,j}^{\text{cul}}$ for some constants $c$; moreover, exploiting (2.19), (2.21), (2.22) and (A.3), we easily see that (2.60) and (2.61) hold. This completes the proof. \hfill \Box

2.4. Equivalent estimates

This section is devoted to establishing the following equivalent estimates.

Lemma 2.8. For sufficiently small $\delta$, we have
\[
\mathcal{E} \text{ is equivalent to } \|(\eta, \partial_1\eta, u)\|_2^2,
\]
\[
\mathcal{D} \text{ is equivalent to } \|(u, \partial_1\eta)\|_2^4,
\]
\[
\mathcal{E}_{p,4}, \mathcal{D}_p \text{ and } \|(u, \partial_1(u, \partial_1\eta))\|_3^2 \text{ are equivalent},
\]
where the equivalent coefficients in (2.67)–(2.69) are independent of $\delta$.

Proof. To obtain (2.67)–(2.69), the key step is to establish the estimates of $\nabla q$ and $u_t$. By (1.19)$_2$, we have
\[
\begin{cases}
-\text{div} (\nabla q/\hat{\rho}) = f^1 & \text{in } \Omega, \\
\nabla q/\hat{\rho} \cdot \vec{n} = f^2 \cdot \vec{n} & \text{on } \partial\Omega,
\end{cases}
\]
where
\[
f_1 := \text{div} \left( u_t + au - \lambda m^2 \bar{\rho}^{-1} \partial_t^2 \eta + \bar{\rho}^{-1} (\nabla \bar{A} q - g G_{\eta} e_2) \right),
\]
\[
f_2 := - \nabla \bar{A} q / \bar{\rho}.
\]

Note that
\[
\int f_1 \, dy + \int_{\partial \Omega} f_2 \cdot \vec{n} \, dy = 0,
\]
thus applying the elliptic estimate (A.22) yields
\[
\| \nabla q \|_3 \lesssim \| f_1 \|_2 + \| f_2 \|_3 
\lesssim \| (\text{div}_{A_t} u, \text{div}_{A_t} u, \text{div}_{A_t} u) \|_2 + \| (\partial_t^2 \eta, \nabla \bar{A} q, G_{\eta}) \|_3.
\]

Making use of (2.2), (2.3), (2.6) and the product estimate (A.3), we further derive that, for sufficiently small \( \delta \),
\[
\| \nabla q \|_3 \lesssim \| (\eta_2, \partial_t^2 \eta) \|_3 + \| (\eta, u) \|_4 \| (u, u_t) \|_3.
\]

Similarly, dividing (1.19) by \( \bar{\rho} \), and then applying \( \| \cdot \|_3 \) to the resulting identity, we get
\[
\| u_t \|_3 = \| (\lambda m^2 \partial_t^2 \eta + g G_{\eta} e_2 - \nabla \bar{A} q - a \bar{\rho} u) / \bar{\rho} \|_3 
\lesssim \| (\eta_2, \partial_t^2 \eta, u, \nabla q) \|_3,
\]
which, together with (2.70), implies, for sufficiently small \( \delta \),
\[
\| (u_t, \nabla q) \|_3 \lesssim \| (\eta_2, \partial_t^2 \eta, u) \|_3.
\]

Thanks to (2.14), (2.71) and (A.4), we easily see that (2.67)–(2.69) hold. \( \square \)

2.5. A priori stability estimate

Now we are in a position to establish the a priori stability estimates under the a priori assumptions (1.27) and (1.28).

Similarly to (2.65), we have, for sufficiently small \( \delta \),
\[
\| \eta \|_4 \lesssim \| (\eta, \text{curl}(\bar{\rho} \eta)) \|_3.
\]
Making use of (2.46), (2.60), (2.65) with \( i = 0 \), (2.68), (2.72) and the interpolation inequality (A.2), we can derive from (2.44) and (2.58) that, for sufficiently small \( \delta \),
\[
\frac{d}{dt} \tilde{\mathcal{E}} + cD \lesssim \sqrt{\mathcal{E}} D,
\]
where \( \tilde{\mathcal{E}} := \mathcal{E}^{\text{curl}} + c \mathcal{E} \) for some constant \( c \) and
\[
\tilde{\mathcal{E}} \text{ is equivalent to } \| (\eta, \partial_t \eta, u) \|_4^2.
\]

Exploiting (1.28) and (2.67), we further deduce from (2.73) that there exists a positive constant \( \delta_1 \), such that, for any \( \delta \in (0, \delta_1] \),
\[
\frac{d}{dt} \tilde{\mathcal{E}} + cD \leq 0.
\]
In particular, by (2.74), we easily get from (2.75) that, for some 
\( c_3 \geq 1 \),

\[ \mathcal{E}(t) + \int_0^t \mathcal{D}^p(\tau) d\tau \leq c_3 \| (\eta^0, \partial_1 \eta, u^0) \|^2. \]  
(2.76)

Similarly to (2.75), making use of (2.47), (2.61), (2.65), (2.66), (2.69), the interpolation inequality (A.2) and (A.7), we derive from (2.45) and (2.59) that, for sufficiently small \( \delta \),

\[ \frac{d}{dt} \tilde{\mathcal{E}}^p + c \mathcal{D}^p \leq \sqrt{\mathcal{E} \mathcal{D}^p}, \]  
(2.77)

where \( \tilde{\mathcal{E}} := \mathcal{E}^{cul} + c \mathcal{E} \) for some constant \( c \) and

\[ \tilde{\mathcal{E}}^p, \mathcal{E}^p, \mathcal{D}^p \text{ and } \| (u, \partial_1(\eta, \partial_1 \eta, u)) \|^2 \]  
are equivalent.  
(2.78)

Thus, exploiting (2.67) and (2.78), we further deduce from (2.77) that

\[ \frac{d}{dt} \tilde{\mathcal{E}}^p + 2c_1 \tilde{\mathcal{E}}^p \leq 0, \]

which, together with (2.14) and (2.78), further implies that

\[ e^{c_1 t}(\| \eta_2(t) \|^2 + \mathcal{E}^p(t)) + \int_0^t e^{c_1 \tau} \mathcal{D}^p(\tau) d\tau \leq \| (u^0, \partial_1(\eta^0, \partial_1 \eta^0, u^0)) \|^2_{1,3}. \]  
(2.79)

This completes the derivation of the a priori stability estimates (1.24) and (1.25).

2.6. Proof of Theorem 1.1

Now we state the local well-posedness result for the transformed MRT problem (1.19).

**Proposition 2.2.** Let \( b > 0, a \geq 0 \) be constants and \( \iota > 0 \) be the constant in Lemma A.8. We assume that \( \tilde{\rho} \) satisfy (1.3). \( (\eta^0, u^0) \in (H^1_{s,4} \cap H^4_{s}) \times H^4_s, \| (u^0, \partial_1 \eta^0) \|_4 \leq b \) and \( \text{div} \mathcal{A}^0 u^0 = 0 \), where \( \mathcal{A}^0 := (\nabla \zeta^0)^{-T} \) and \( \zeta^0 = \eta^0 + y \). Then there exist a sufficiently small constant \( \delta_2 \leq \iota/2 \), such that if \( \eta^0 \) satisfies

\[ \| \eta^0 \|_4 \leq \delta_2, \]

the transformed MRT problem (1.19) admits a unique local-in-time classical solution \( (\eta, u, q) \in \mathcal{C}^0(I_T, H^3_{s,4}) \times \mathcal{U}^3 \times \mathcal{Q}^3_I \) for some local existence time \( T > 0 \). Moreover, \( \eta \) satisfies

\[ \sup_{t \in I_T} \| \eta \|_4 \leq 2\delta_2 \sqrt{T} \]  
and

\[ \sup_{t \in I_T} \| (\eta, \partial_1 \eta, u) \|_4 \leq c_4 \sqrt{T^0} \]  
(2.80)

for some positive constant \( c_4 \geq 1 \). It should be noted that \( \delta_2 \) and \( T \) may depend on \( g, a, \lambda, m, \tilde{\rho} \) and \( \Omega \); moreover \( T \) further depends on \( b \).

\[ \text{Since } \sup_{t \in I_T} \| \eta \|_4 \leq \iota, \text{ we have, by Lemma A.8} \]

\[ \inf_{(y, t) \in \mathbb{R}^2 \times I_T} \text{det}(\nabla \eta + I) \geq 1/4. \]
Proof. The proof of Proposition 2.2 will be provided in Section 5.

Thanks to the priori estimate (2.76) and Proposition 2.2, we can easily establish Theorem 1.1. Next we briefly give the proof.

Let \((\eta^0, u^0)\) satisfies the assumptions in Theorem 1.1 and

\[ I^0 \leq \delta^2, \quad \text{where} \quad \delta := \min \{\delta_1, \delta_2\} / \sqrt{c_3c_4^2}, \]

where \(c_3\) and \(c_4\) are the constants in (2.76) and (2.80). By Proposition 2.2 and Lemma A.8, there exists a unique local solution \((\eta, u, q)\) to the transformed MRT problem (1.19) with a maximal existence time \(T_{\text{max}}\), which satisfies that

- for any \(\tau \in I_{T_{\text{max}}}\), the solution \((\eta, u, q)\) belongs to \(S_{1,s,\tau}^{1,4} \times \Omega^4_{\tau} \times \Omega^4_{\tau}\) and

\[ \sup_{t \in I_{\tau}} \|\eta\|_4 \leq 2\delta_2; \]

- \(\limsup_{t \to T_{\text{max}}} \|\eta(t)\|_4 > \delta_2\) or \(\limsup_{t \to T_{\text{max}}} \|(u, \partial_1 \eta)(t)\|_4 = \infty\), if \(T_{\text{max}} < \infty\).

Let

\[ T^* := \sup \{\tau \in I_{T_{\text{max}}} \mid \|\eta, \partial_1 \eta, u\|_4 \leq \sqrt{c_3c_4^2} \delta \text{ for any } t \leq \tau\}. \]

We easily see that the definition of \(T^* > 0\) makes sense and \(T^* > 0\). Thus, to obtain the existence of global solutions, next it suffices to verify \(T^* = \infty\). Next we shall prove this fact by contradiction.

We assume that \(T^* < \infty\). Then, for any given \(T^{**} \in I_{T^*}\),

\[ \sup_{T_{T^{**}}} \|\eta, \partial_1 \eta, u\|_4 \leq \sqrt{c_3c_4^2} \delta \leq \delta_1. \]  \hfill (2.81)

Thanks to (2.81), we can follow the argument of (2.73) by further using a regularity method as in the derivation of (5.71) to verify that

\[ \frac{d}{dt} \tilde{E} + cD \leq 0 \text{ for a.e. } t \in I_{T^*}, \text{ where } \tilde{E} \in W^{1,\infty}(I_{T^*}). \]  \hfill (2.82)

Referring to (2.76), we can further derive from (2.80) and (2.82) that

\[ \|\mathcal{E}(t)\|_{L^\infty(I_{T^*})} + \int_0^t \mathcal{D}(\tau) d\tau \leq c_3 \lim_{t \to 0} \sup_{\tau \in I_t} \|\eta, \partial_1 \eta, u\|_4^2 \leq c_3c_4^2 \delta^2. \]  \hfill (2.83)

By the continuity of \((\eta, u)\) and the fact

\[ \sup_{I_{T^*}} \|f\|_0 = \|f\|_{L^\infty L^2} \text{ for any } f \in C^0_{B,\text{weak}}(I_{T^*}, L^2) \text{ with } \tau > 0, \]

de further derive from (2.83) that

\[ \sup_{T_{T^{**}}} \|\eta, \partial_1 \eta, u\|_4 \leq \sqrt{c_3c_4} \delta. \]  \hfill (2.84)

We take \((\eta(T^{**}), u(T^{**}))\) as a initial data. Noting that, by (2.84) and the definition of \(\delta\),

\[ \|\eta, \partial_1 \eta, u\|_4 \leq B := \sqrt{c_3c_4} \delta \text{ and } \|\eta(T^{**})\|_4 \leq \delta_2, \]
then, by Proposition 2.2 there exists a unique local-in-time classical solution, denoted by \((\eta^*, u^*, q^*)\), to the transformed MRT problem (1.19) with \((\eta(T^*), u(T^*))\) in place of \((\eta^0, u^0)\); moreover
\[
\sup_{t \in [T^*, T]} \| (\eta^*, \partial_t \eta^*, u^*) \|_4 \leq c_4 B \leq \sqrt{c_3 c_4^2} \delta \quad \text{and} \quad \sup_{t \in [T^*, T]} \| \eta^* \|_4 \leq 2\delta_2,
\]
where the local existence time \(T > 0\) depends on \(b, g, a, \lambda, m, \rho\) and \(\Omega\).

In view of the existence result of \((\eta^*, u^*, q^*)\), the uniqueness conclusion in Proposition 2.2 and the fact that \(T_{max}\) denotes the maximal existence time, we immediately see that \(T_{max} > T^* + T/2\)
and \(\sup_{t \in [0, T^* + T/2]} \| (\eta, \partial_t \eta, u) \|_4 \leq \sqrt{c_3 c_4^2} \delta\). This contradicts with the definition of \(T^*\). Hence \(T^* = \infty\) and thus \(T_{max} = \infty\). This completes the proof of the existence of global solutions. The uniqueness of global solution is obvious due to the uniqueness result of local solutions in Proposition 2.2 and the fact \(\sup_{t \geq 0} \| \eta \|_4 \leq 2\delta_2\). To complete the proof of Theorem 1.1 we shall verify that the solution \((\eta, u, q)\) satisfies the properties (1.23)–(1.26).

Referring to (2.82) and (2.83), we see that the global solution \((\eta, u)\) enjoys (1.23) and (1.24). Similarly, we also verify that the global solution also satisfies (1.25) by referring to the derivation of (2.75) and (2.83).

Finally, we shall derive (1.26). By (1.25), we easily see that
\[
\partial_t \eta(t) \to 0 \quad \text{in} \quad H^2 \quad \text{as} \quad t \to \infty
\]
and
\[
\left\| \int_0^t u d\tau \right\|_3 \lesssim \int_0^t \| u \|_3 d\tau \lesssim \sqrt{T_0} \quad \text{for any} \quad t > 0
\]
(2.86)
Thanks to (2.86), there exist a subsequence \(\{t_n\}_{n=1}^\infty\) and some function \(\eta^\infty_1 \in H^3\) such that
\[
\int_0^{t_n} u_1 d\tau \to \eta^\infty_1 - \eta^0_1 \quad \text{weakly in} \quad H^3.
\]
Exploiting (1.19), (1.25) and weakly lower semi-continuity, we have
\[
\| \eta_1(t) - \eta^\infty_1 \|_3 \leq \liminf_{t_n \to \infty} \left\| \int_t^{t_n} u_1 d\tau \right\|_3 \lesssim \sqrt{T_0} \liminf_{t_n \to \infty} \int_t^{t_n} e^{-c_1 \tau} d\tau \lesssim \sqrt{T_0} e^{-c_1 t},
\]
which, together with (2.88), yields that (1.26) holds and \(\eta^\infty_1\) only depends on \(y_2\). This completes the proof of Theorem 1.1

3. Proof of Corollary 1.1

This section is devoted to the proof of Corollary 1.1. Let \((\eta, u, q)\) be the classical solution constructed by Theorem 1.1 with initial data \((\eta^0, u^0)\) further satisfying the odevity conditions (1.31) and (1.32). Next we first verify the solution preserves the odevity conditions in (1.33).

Let \(\psi = (-\eta_1, \eta_2)(-y_1, y_2, t), w = (-u_1, u_2)(-y_1, y_2, t)\) and \(p = (q_1, q_2)(-y_1, y_2, t)\). Since the classical solution of the transformed MRT problem (1.19) is unique, thus, to get the relation (1.33), it obviously suffices to verify that \((\psi, w, p)\) is also the classical solution of (1.19). It is obvious that \((\psi, w)\) satisfies (1.19) \(_1\). Next we shall verify that \((\psi, w)\) also satisfies (1.19) \(_2\) and (1.19) \(_3\).

Defining
\[
A = \begin{pmatrix}
\partial_2 \eta_2 + 1 & -\partial_1 \eta_2 \\
-\partial_2 \eta_1 & \partial_1 \eta_1 + 1
\end{pmatrix}
\quad \text{and} \quad
B = \begin{pmatrix}
\partial_2 \psi_2 + 1 & -\partial_1 \psi_2 \\
-\partial_2 \psi_1 & \partial_1 \psi_1 + 1
\end{pmatrix},
\]

where...
then
\begin{equation}
(B_{11}, B_{22}) = (A_{11}, A_{22})|_{y_1 = -y_1} \text{ and } (B_{12}, B_{21}) = -(A_{12}, A_{21})|_{y_1 = -y_1}.
\end{equation}

By (3.1),
\begin{equation}
\nabla B p = (-A_{1i}, A_{2i})^T \partial_i q|_{y_1 = -y_1}
\end{equation}
and
\begin{equation}
\text{div}_B w = \text{div}_A u|_{y_1 = -y_1} = 0.
\end{equation}

By (3.3), we see that \((\psi, w)\) satisfies (1.19) as \((\eta, u)\).

Thanks to (3.2), we have
\begin{equation}
\bar{\rho} \partial_t w_i + \nabla B p + a \bar{\rho} w_i = \begin{cases}
-(\bar{\rho} \partial_t u_1 + A_{1i} \partial_i q + \bar{\rho} u_1)|_{y_1 = -y_1} & \text{for } i = 1; \\
(\bar{\rho} \partial_t u_2 + A_{2i} \partial_i q + a \bar{\rho} u_2)|_{y_1 = -y_1} & \text{for } i = 2,
\end{cases}
\end{equation}
\begin{equation}
\partial_1^2(\psi_1, \psi_2) = \partial_1^2(-\eta_1, \eta_2)|_{y_1 = -y_1} \text{ and } g G_\psi e_2 = g G_\eta e_2|_{y_1 = -y_1},
\end{equation}
where \(G_{\psi}\) is defined in (1.18) with \(\phi_2\) in place of \(\eta_2\). Hence we see that \((\psi, w)\) also satisfies (1.19) by the three identity above. This completes the proof of the verification of preserving oddity of solutions.

Thanks to (1.33) and (A.8), we easily get, for \(0 \leq i \leq 3\),
\begin{equation}
\|\partial_2^i \eta_1\|_0 \lesssim \|\partial_1 \partial_2^i \eta_1\|_0,
\end{equation}
which, together the estimate (2.14) satisfied by \(\eta\), implies that
\(E\) is equivalent to \(D\).

Thus we can further derive (1.34) from (1.23) and (1.24). This completes the proof of Corollary 1.1.

4. Proof of Theorem 1.2

This section is devoted to the proof of instability of transformed MRT problem in Theorem 1.2. We will complete the proof by five subsections. In what follows, the fixed positive constant \(c_i^f\) for \(i \geq 1\) may depend on \(g, a, \lambda, m, \bar{\rho}\) and \(\Omega\).

4.1. Linear instability

To begin with, we exploit modified variational method of ODE as in [14, 29] to prove the existence of unstable solutions of the following linearized MRT problem under the instability condition \(|m| \in (0, m_C)\):
\begin{equation}
\begin{cases}
\eta_t = u, \\
\bar{\rho} u_t + \nabla q + a \bar{\rho} u = \lambda m^2 \partial_1^2 \eta + g \bar{\rho} \eta_2 e_2, \\
div u = 0, \\
(\eta, u)|_{\partial \Omega} \cdot \bar{n} = 0.
\end{cases}
\end{equation}
Proposition 4.1. Let $a \geq 0$ and $\bar{\rho}$ satisfy (1.5) and (1.6). If $|m| \in [0, m_C)$, then the zero solution is unstable to the linearized MRT problem. That is, there is an unstable solution $(\eta, u, q) := e^{t\Upsilon}(w/\Upsilon, w, \beta)$ to the above problem, where

$$(w, \beta) \in H^5_\sigma \times H^4$$

solves the boundary-value problem

$$
\begin{cases}
\Upsilon^2 \bar{\rho}w + \Upsilon \nabla \beta + a \Upsilon \bar{\rho}w = m^2 \partial_1^2 w + g \bar{\rho}' w_2 e_2 & \text{in } \Omega, \\
\text{div} w = 0 & \text{in } \Omega, \\
w \cdot \mathbf{n} = 0 & \text{on } \partial \Omega.
\end{cases}
$$

with some growth rate $\Upsilon \in (2\Lambda/3, \Lambda]$, where $\Lambda$ satisfies

$$E(v) \leq (\Lambda^2 + a\Lambda)\|\sqrt{\bar{\rho}}v\|_0^2$$

for any $v \in H^1_\sigma$. (4.3)

In addition,

$$\|w_i\|_0 \|\partial_1 w_i\|_0 \|\partial_2 w_i\|_0 \neq 0 \text{ for } i = 1, 2.$$ (4.4)

Proof. Similarly to [14, 29], next we divide the proof of Proposition 4.1 into five steps.

1. Let

$$\tilde{E}(\psi, \xi) = \int_0^h \left( g \bar{\rho}'|\psi|^2 - \lambda m^2 \left( \xi^2 |\psi|^2 + |\psi'|^2 \right) \right) dy_2$$

and

$$\mathbb{F} := \{ \xi \in \mathbb{Z}/\{0\} \mid \tilde{E}(\psi, \xi) > 0 \text{ for some } \psi \in H^1_\sigma(0, h) \}.$$ (4.5)

Next we prove the first assertion that the set of instability frequencies $\mathbb{F}$ is not empty.

Recalling the definition of $m_C$ and the condition $|m| \in [0, m_C)$, we see that there exists a function $\omega := (\omega_1, \omega_2)^T \in H^1_\sigma$, such that

$$\int g \bar{\rho}' \omega_2^2 dy - \lambda m^2 \|\partial_1 \omega\|_0^2 > 0.$$ (4.5)

Let $\hat{\omega}(\xi, y_2)$ be the Fourier coefficient of $\omega(y_1, y_2)$ for fixed $y_2$, i.e.,

$$\hat{\omega}(\xi, y_2) = \int_0^{2\pi} \omega(y_1, y_2) e^{-i\xi y_1} dy_1.$$ (4.5)

We define the functions $\varphi$ and $\psi$ by the following relations

$$\hat{\omega}_1(\xi, y_2) = i \varphi(\xi, y_2) \text{ and } \hat{\omega}_2(\xi, y_2) = -\psi(\xi, y_2),$$

where $(\hat{\omega}_1, \hat{\omega}_2)^T = \hat{\omega}$ and $\xi \in \mathbb{Z}$. Obviously,

$$\overline{\partial_1 \omega_1} = \xi \varphi \text{ and } \overline{\partial_1 \omega_2} = -i \xi \psi.$$ (4.6)

Since $\omega_2|_{\partial \Omega} = 0$, then $\psi \in H^1_0(0, h)$. By $\text{div} \omega = 0$, we have

$$\xi \varphi + \psi' = 0.$$ (4.6)
which, together with the fact \( \psi|_{y_2=0, \ h = 0} \), implies that
\[
\psi(0, y_2) = 0 \text{ for } \xi = 0. \tag{4.7}
\]

Exploiting (4.6), (4.7), and Fubini and Parseval Theorems, we have
\[
E(\omega) = \frac{1}{2\pi} \int_0^h \sum_{\xi \in \mathbb{Z}/\{0\}} \left( g \hat{\rho} |\psi|^2 - \lambda m^2 \left( |\xi\psi|^2 + |\psi'|^2 \right) \right) dy_2.
\]

The above identity together with (4.5) imply that there is a \( \xi \in \mathbb{Z}/\{0\} \) and \( \psi \in H^1_0(0, h) \), such that
\[
\tilde{E}(\psi, \xi) > 0.
\]

Hence the set of instability frequencies \( \mathbb{F} \) is not empty.

(2) We define that
\[
J(\psi, \xi) := \int_0^h \hat{\rho} \left( |\xi\psi|^2 + |\psi'|^2 \right) dy_2, \quad H := \{ \psi \in H^1_0(0, h) \mid J(\psi, \xi) = 1 \},
\]
\[
\Xi(\psi, \xi, s) := \xi^2 \tilde{E}(\psi, \xi) - asJ(\psi, \xi).
\]

For any given \( \xi \in \mathbb{F} \), we define that
\[
\mathcal{C}_\xi := \sup \{ s \in \mathbb{R} \mid \Xi(\psi, \xi, s) > 0 \text{ for some } \psi \in H^1_0(0, h) \}.
\]

Recalling the definition of \( \mathbb{F} \), we easily see that
\[
0 < \mathcal{C}_\xi, \quad \mathcal{C}_\xi = \infty \text{ for } a = 0 \text{ and } \mathcal{C}_\xi < \infty \text{ for } a > 0.
\]

Next we prove the second assertion that, for any given \( (\xi, s) \in \mathbb{F} \times [0, \mathcal{C}_\xi] \), there exist \( \alpha > 0 \) and a classical solution \( \psi_0 \in H^1_0(0, h) \cap H^2(0, h) \), which satisfies the following modified boundary-value problem:
\[
\begin{aligned}
& (\alpha + as) \left( \hat{\rho} \xi^2 \psi_0 - (\hat{\rho} \psi_0')' \right) = g \xi^2 \hat{\rho} \psi_0 - \lambda m^2 \xi^2 \left( \xi^2 \psi_0 - \psi_0'' \right) \text{ in } (0, h), \\
& \psi_0(0) = \psi_0(h) = 0.
\end{aligned} \tag{4.8}
\]

Moreover, \( \sup_{\psi \in H} \Xi(\psi, \xi, s) = \Xi(\psi_0, \xi, s) \).

To being with, we shall consider the following variational problem:
\[
\alpha := \sup_{\psi \in H} \Xi(\psi, \xi, s), \tag{4.9}
\]

where \( \xi \in \mathbb{F} \) and \( s \in \mathbb{R}^+ \). Since \( \alpha \) depends on \( s \), sometimes we denote \( \alpha \) by \( \alpha(s) \). Obviously,
\[
\begin{aligned}
\alpha(s_2) - \alpha(s_1) &= a(s_1 - s_2) \text{ for any } s_1, s_2 \geq 0, \\
g \| \hat{\rho}' / \hat{\rho} \|_{L^\infty} &\geq \alpha > 0 \text{ for any } 0 \leq s < \mathcal{C}_\xi, \\
\alpha(\mathcal{C}_\xi) &= 0 \text{ if } \mathcal{C}_\xi \in \mathbb{R}^+, \text{ i.e., the case } a > 0.
\end{aligned} \tag{4.10-4.12}
\]

Since \( 0 \leq \alpha < \infty \), there exists a maximizing sequence \( \{ \psi_n \}_{n=1}^\infty \subset H \) such that \( \psi_n \to \psi_0 \) weakly in \( H^1_0(0, h) \) and strongly in \( L^2(0, h) \). By the convergence results of \( \psi_n \) and the weakly lower semi-continuity, we easily get
\[
J(\psi_0, \xi) \leq \lim_{n \to \infty} \| \sqrt{\hat{\rho}} \psi_n \|^2_{L^2(0, h)} + \liminf_{n \to \infty} \| \psi_n' \|^2_{L^2(0, h)} \leq \liminf_{n \to \infty} J(\psi_n, \xi) = 1 \tag{4.13}
\]
\[ \alpha = \lim_{n \to \infty} \Xi(\psi_n, \xi, s) \leq \varrho \lim_{n \to \infty} \int_0^h \rho' |\psi_n|^2 dy_2 - \lambda m^2 \lim_{n \to \infty} \int_0^h \xi^2 |\psi_n|^2 dy_2 - \lambda m^2 \liminf_{n \to \infty} \int_0^h |\psi'_n|^2 dy_2 - 2as \leq \Xi(\psi_0, \xi). \tag{4.14} \]

*From now on, we consider the case \( s \in [0, \mathcal{C} \xi]. \) Now we prove \( \psi_0 \neq 0 \) by contradiction. We assume that \( \psi_0 = 0, \) then there exists a subsequence \( \{\psi_n\}_{n=1}^\infty \) (still denoted by \( \psi_n \) for simplicity), such that \( \|\psi'_n\|_{L^2(0, h)} \to b > 0 \) and \( \|\sqrt{\varrho} \rho'\|_{L^2(0, h)} \to 1. \) Thus we immediately see that

\[ \Xi(\psi_n, \xi, s) \to \alpha = -(\lambda b m^2 \xi^2 + 2as) \leq 0 \text{ for } s \in [0, \mathcal{C} \xi] \text{ and } a \geq 0, \]

which contradicts with (4.11). Thus we get \( \psi_0 \neq 0. \)

Since \( \psi_0 \neq 0, \) we see from (4.13) that

\[ 0 < \mathfrak{J}(\psi_0, \xi) \leq 1. \]

Thus we derive from (4.14) and the relation above that

\[ \alpha \leq \alpha \mathfrak{J}^{-1}(\psi_0, \xi) \leq \Xi(\psi_0, \xi) \mathfrak{J}^{-1}(\psi_0, \xi) \leq \alpha, \]

which yields \( \mathfrak{J}^{-1}(\psi_0, \xi) = 1. \) This means that \( \psi_0 \) is an achievable point of the variational problem (4.9). In particular, we have

\[ \alpha = \frac{\Xi(\psi_0, \xi, s)}{\mathfrak{J}(\psi_0, \xi)} \geq \frac{\Xi(\psi, \xi, s)}{\mathfrak{J}(\psi, \xi)} \text{ for any } \psi \in H^1_0(0, h). \tag{4.15} \]

For any given \( \phi \in H^1_0(0, h), \) let

\[ \mathfrak{F}(\tau) = \Xi((\psi_0 + \tau \phi), \xi, s) - (\alpha + as)\mathfrak{J}((\psi_0 + \tau \phi), \xi) \text{ for any } \tau \in \mathbb{R}. \]

Then \( \mathfrak{F}(\tau) \in C^\infty(\mathbb{R}). \) By (4.15), \( \mathfrak{F}(0) = 0 \) and \( \mathfrak{F}(\tau) \leq 0. \) Thus we have \( \mathfrak{F}'(0) = 0. \) In particular, we have

\[ (\alpha + as) \int_0^h \bar{\rho} (\xi^2 \psi_0 \phi + \psi'_0 \phi') dy_2 = \int_0^h \xi^2 (g \bar{\rho} \psi_0 \phi - \lambda m^2 (\xi^2 \psi_0 \phi + \psi'_0 \phi')) dy_2. \]

Noting that \( \bar{\rho}(\alpha + as) + \lambda m^2 \xi^2 \neq 0 \) for any \( y_2 \in (0, h), \) thus we equivalently rewrite the above identity as follows:

\[ \int_0^h (\bar{\rho}(\alpha + as) + \lambda m^2 \xi^2)^{-1} (\xi^2 (g \bar{\rho}' \psi_0 \phi - \lambda m^2 \xi^2 - (\alpha + as) \bar{\rho}) \psi_0' + (\alpha + as) \bar{\rho}' \psi'_0) \chi dy_2 = \int_0^h \psi'_0 \chi' dy_2, \tag{4.16} \]

where \( \chi := (\bar{\rho}(\alpha + as) + \lambda m^2 \xi^2) \phi. \) It is easy to see from (4.16) that \( \psi_0 \in H^4(0, h) \) and \( \psi_0 \) is just the classical solution of the modified boundary-value problem (4.8). This completes the proof of the second assertion.
(3) Next we further prove the third assertion that, for any given \( \xi \in \mathbb{F} \), there exist \( \gamma > 0 \) and \( \psi \in H^1_0(0, h) \cap H^4(0, h) \), such that

\[
\begin{aligned}
\gamma^2 (\tilde{\rho} \xi^2 \psi - (\tilde{\rho} \psi')') &= g \xi^2 \tilde{\rho} \psi - \lambda m^2 \xi^2 (\xi^2 \psi - \psi') - a\gamma (\tilde{\rho} \xi^2 \psi - (\tilde{\rho} \psi')') \quad \text{in} \ (0, h), \\
\psi(0) &= \psi(h) = 0.
\end{aligned}
\]  

(4.17)

Moreover,

\[
\sup_{\chi \in \mathcal{H}} \Xi(\chi, \xi, \gamma) = \Xi(\psi, \xi, \gamma) = \gamma > 0.
\]  

(4.18)

The above assertion obviously holds for \( a = 0 \) by the second assertion, thus it suffices to consider the case \( a > 0 \).

Thanks to (4.10)–(4.12), it is easy to see that there exists a fixed point \( \gamma \) satisfying (4.18). Moreover, by the second assertion, there exists \( \psi \in H^1_0(0, h) \cap H^5(0, h) \) satisfying (4.17) with \( \gamma \). This completes the proof of the third assertion.

(4) Now we are in the position to the proof of existence of a solution \((w, \beta)\) to the boundary-value problem (4.2) with some \( \Upsilon > 0 \).

Let \( \psi \) and \( \gamma \) be constructed in the third assertion for any given \( \xi \in \mathbb{F} \). From now on, we denote \((\psi, \gamma)\) by \((\psi_{\xi}, \gamma_{\xi})\) to emphasize the dependence of \( \xi \). We define \( \Lambda := \sup_{\xi \in \mathbb{F}} \gamma_{\xi} \). It is easy to see that

\[-\xi \in \mathbb{F}, \ (\psi_{\xi}, \gamma_{\xi}) = (\psi_{-\xi}, \gamma_{-\xi}) \quad \text{for any} \ \xi \in \mathbb{F} \quad \text{and} \ 0 < \Lambda < \infty.\]

Denoting that

\[
\varphi_{\xi} := -\xi^{-1} \psi'_{\xi} \quad \text{and} \quad \vartheta_{\xi} := (\gamma^2 \tilde{\rho} \varphi_{\xi} + a\gamma \tilde{\rho} \varphi_{\xi} + \lambda m^2 \xi^2 \varphi_{\xi})/\gamma_{\xi},
\]

then it is easy to check that

\[
\begin{aligned}
\gamma_{\xi}^2 \tilde{\rho} \varphi_{\xi} - \gamma_{\xi} \xi \vartheta_{\xi} + a\gamma_{\xi} \tilde{\rho} \varphi_{\xi} &= -\lambda m^2 \xi^2 \varphi_{\xi} \quad \text{in} \ (0, h), \\
\gamma_{\xi}^2 \tilde{\rho} \psi_{\xi} + \gamma_{\xi} \vartheta_{\xi}' + a\gamma_{\xi} \tilde{\rho} \psi_{\xi} &= -\lambda m^2 \xi^2 \psi_{\xi} + g \tilde{\rho} \psi_{\xi} \quad \text{in} \ (0, h), \\
\xi \varphi_{\xi} + \psi_{\xi}' &= 0 \quad \text{in} \ (0, h), \\
\psi_{\xi}(0) &= \psi_{\xi}(h) = 0.
\end{aligned}
\]  

(4.19)

Let

\[
w_1(y) = -i \varphi_{\xi}(y_2) (e^{iy_{1}\xi} - e^{-iy_{1}\xi}) \quad \text{and} \quad (w_2, \theta)(y) = (\psi_{\xi}(y_2), \vartheta_{\xi}(y_2))(e^{iy_{1}\xi} + e^{-iy_{1}\xi}),
\]

then \( w_1 \), \( w_2 \) and \( \theta \) are real-value functions. Recalling (4.13) and (4.19), we easily see that \( \psi_{\xi}, \varphi_{\xi}, \psi'_{\xi} \) and \( \varphi'_{\xi} \) are non-zero function. Thus we have \( \|w_i\|_0 \|\partial_i w_i\|_0 \|\partial_2 w_i\|_0 \neq 0 \) for \( i = 1 \) and \( 2 \). Let \( w = (w_1, w_2)^T \) and \( \beta := \theta - (\theta)_{\Omega} \), then \((w, \beta) \in H^2_0 \times H^4_{\Omega} \). By (4.19), we easily see that \((w, \beta)\) satisfies (4.2) with \( \gamma_{\xi} \) in place of \( \Upsilon \). By the definition of \( \Lambda \), there exists an \( \xi_0 \in \mathbb{F} \) such that \( \gamma_{\xi_0} \in (2\Lambda/3, \Lambda] \). Now we take \( \xi = \xi_0 \) and \( \Upsilon = \gamma_{\xi_0} \). Consequently, we see that the solution \((w, \beta, \Upsilon)\) satisfies (4.4) and the boundary-value problem (4.2) with \( \Upsilon \in (2\Lambda/3, \Lambda] \). In addition, it is easy to see that \((\eta, u, q) := e^{Tt}(w, \Upsilon, w, \beta)\) is a solution of (4.1).

(5) To complete the proof of Proposition 4.4, we shall verifies (4.3).

Let \( \xi \in \mathbb{Z} \), and the two real-valued functions \( \tilde{\varphi}, \tilde{\psi} \in H^1_0(0, h) \) satisfy

\[
\xi \tilde{\varphi} + \tilde{\psi}' = 0.
\]  

(4.20)
In particular, we have
\[ \tilde{\psi} = \tilde{\psi}' = 0 \text{ if } \xi = 0. \tag{4.21} \]
Thanks to (4.18) and the fact \( \gamma \leq \Lambda \), we see that
\[ \tilde{E}(\tilde{\psi}, \xi) \leq \xi^{-2} (\Lambda^2 + a\Lambda) J(\tilde{\psi}, \xi), \text{ if } \xi \in \mathbb{F}. \tag{4.22} \]
In addition, using the definition of \( \mathbb{F} \) and (4.21), we have
\[ \tilde{E}(\tilde{\psi}, \xi) \leq 0 \text{ if } \xi \in \mathbb{Z}/\mathbb{F}. \tag{4.23} \]
Exploiting (4.22) and (4.23), we obtain that, for any \( \xi \in \mathbb{Z} \), and for any \( \tilde{\varphi}, \tilde{\psi} \in H^1_0(0, h) \) satisfying (4.20),
\[ \tilde{E}(\tilde{\psi}, \xi) \leq (\Lambda^2 + a\Lambda) \int_0^h \bar{\rho} \left( |\tilde{\varphi}|^2 + |\tilde{\psi}|^2 \right) dy_2. \tag{4.24} \]
Let \( v \in H^1_1, \hat{v}(\xi, y_2) \) be the Fourier coefficient of \( v(y_1, y_2) \) for fixed \( y_2 \), \( \varphi(\xi, y_2) = i\hat{v}_1(\xi, y_2) \) and \( \psi(\xi, y_2) = \hat{v}_2(\xi, y_2) \). Then \( (\varphi, \psi) \) satisfies \( \xi \varphi + \psi' = 0 \). Consequently, making use of (4.24), and the Fubini’s and Parseval’s theorems, we have
\[ E(v) = \frac{1}{2\pi} \sum_{\xi \in \mathbb{Z}} \tilde{E}(\psi, \xi) \leq \frac{1}{2\pi} (\Lambda^2 + a\Lambda) \sum_{\xi \in \mathbb{Z}} \int_0^h \bar{\rho} \left( |\varphi|^2 + |\psi|^2 \right) dy_2 \]
\[ \leq (\Lambda^2 + a\Lambda) \|\sqrt{\bar{\rho}}v\|^2_0, \]
which yields (4.3). This completes the proof. \( \square \)

4.2. Nonlinear energy estimates

This section is devoted to establishing the following Gronwall-type energy inequality for the solutions of the transformed MRT problem.

**Proposition 4.2.** Let \( \Upsilon > 0 \) be provided by Proposition 4.1 and \( (\eta, u, q) \) be the local solution constructed by Proposition 2.2. There exist \( \delta_1^* \) and \( c_1^* > 0 \) such that, if \( \|(\eta, \partial_1\eta, u)\|_4 \leq \delta_1^* \) in some time interval \( I_{\tilde{T}} \subset I_T \), where \( I_T \) is the existence time interval of \( (\eta, u, q) \), then there exists a functional \( \mathcal{E}(t) \) of \( (\eta, u, q) \) satisfies the Gronwall-type energy inequality
\[ \mathcal{E}(t) \leq c_1^* \left( I^0 + \int_0^t \|\eta_2(\tau)\|^2_0 d\tau \right) + \Upsilon \int_0^t \mathcal{E}(\tau) d\tau \]
for a.e. \( t \in I_{\tilde{T}} \), where the constants \( \delta_1^* \) may depend on \( g, a, \lambda, m, \bar{\rho} \) and \( \Omega \), and \( \mathcal{E}(t) \in W^{1,\infty}(I_{\tilde{T}}) \) satisfies, for a.e. \( t \in I_{\tilde{T}} \),
\[ \mathcal{E}, \mathcal{E} \text{ and } \|(\eta, \partial_1\eta, u)\|^2_4 \text{ are equivalent}. \tag{4.26} \]

**Proof.** Let \( (\eta, u, q) \) be the local solution constructed by Proposition 2.2. We further assume that
\[ \mathcal{E}(t) \leq \delta \in (0, 1] \text{ for any } t \in I_{\tilde{T}} \subset I_T. \tag{4.27} \]
Similarly to Lemma 2.4 for sufficiently small \( \delta \), we can easily derive that, for a.e. \( t \in I_T \):

\[
\frac{d}{dt} \left( \int \tilde{\rho} \partial_i^k \eta \cdot \partial_i^k u dy + \frac{a}{2} \| \sqrt{\tilde{\rho}} \eta \|_{k,0}^2 \right) + c \| \eta \|_{k+1,0}^2 \lesssim \| (\eta, u) \|_{k,0}^2 + \sqrt{\mathcal{E}} \mathcal{D},
\]

\[
\frac{d}{dt} \| (\sqrt{\tilde{\rho}} u, \sqrt{\tilde{\lambda}} m \partial_1 \eta) \|_{k,0}^2 + c \| u \|_{k,0}^2 \lesssim \| \eta_2 \|_{k,0}^2 + \sqrt{\mathcal{E}} \mathcal{D},
\]

(4.28)

(4.29)

where \( 0 \leq k \leq 4 \).

We also verify that \((\eta, u)\) satisfies (2.58) for a.e. \( t \in I_T \). Thus we derive from (2.58) satisfied by \((\eta, u), (4.28)\) and (4.29) that, for some sufficiently small \( \zeta \in (0, 1] \),

\[
\frac{d}{dt} \mathcal{E} + c \| (u, \partial_1 \eta) \|_{3}^2 \lesssim (1 + \zeta^{-1}) \| \eta_2 \|_{3,0}^2 + (1 + \zeta + \zeta^{-1}) \sqrt{\mathcal{E}} \mathcal{D},
\]

(4.30)

where

\[
\mathcal{E} := \zeta \mathcal{E}^{\text{cul}} + \zeta^{-1} \| (\sqrt{\tilde{\rho}} u, \sqrt{\tilde{\lambda}} m \partial_1 \eta) \|_{3,0}^2 + \frac{a}{2} \| \sqrt{\tilde{\rho}} \eta \|_{3,0}^2 + \sum_{k=0}^{3} \int \tilde{\rho} \partial_i^k \eta \cdot \partial_i^k u dy \in W^{1,\infty}(I_T).
\]

Similarly to (2.67), (2.68) and (2.74), for sufficiently small \( \delta \), we have, for a.e. \( t \in I_T \),

\[
\mathcal{E}, \mathcal{E} \text{ and } \| (\eta, \partial_1 \eta, u) \|_4^4 \text{ are equivalent,}
\]

\[
\mathcal{D} \text{ is equivalent to } \| (u, \partial_1 \eta) \|_4^4,
\]

(4.31)

(4.32)

where the equivalent coefficients in (4.31) are independent of \( \delta \).

Exploiting the interpolation inequality (A.2) yields, for any \( \varepsilon \in (0, 1] \),

\[
\| \eta_2 \|_{k,0} \lesssim \begin{cases} 
\varepsilon^{-1} \| \eta_2 \|_0 + \varepsilon \| \eta_2 \|_2 & \text{for } k = 1; \\
\varepsilon^{-(k-1)/(4-k)} \| \eta_2 \|_{1,0} + \varepsilon \| \partial_1 \eta_2 \|_3 & \text{for } 2 \leq k \leq 4.
\end{cases}
\]

(4.33)

Consequently, making use of (4.31)–(4.33), we easily derive (4.25) from (2.80) and (4.30) for sufficiently small \( \delta \). \( \square \)

4.3. Construction of nonlinear solutions

For any given \( \delta > 0 \), let

\[
(\eta^a, u^a, q^a) = \delta e^{\Gamma t} (\tilde{\eta}^0, \tilde{u}^0, \tilde{q}^0),
\]

(4.34)

where \((\tilde{\eta}^0, \tilde{u}^0, \tilde{q}^0) := (w/\bar{\Upsilon}, w, \beta)\) and \((w, \beta, \bar{\Upsilon})\) is provided by Proposition 4.1. Then \((\eta^a, u^a, q^a)\) is also a solution to the linearized MRT problem (4.1), and enjoys the estimate, for any \( i \geq 0 \),

\[
\| \partial_i^j (\eta^a, u^a) \|_5 + \| \partial_i^j q^a \|_4 = \bar{\Upsilon}^i \delta e^{\bar{\Upsilon} t} (\| (\tilde{\eta}^0, \tilde{u}^0) \|_5 + \| \tilde{q}^0 \|_4) \lesssim \bar{\Upsilon}^i \delta e^{\bar{\Upsilon} t}.
\]

(4.35)

In addition, we have by (4.13) that

\[
\| \chi_j \|_0 \| \partial_1 \chi_j \|_0 \| \partial_2 \chi_j \|_0 > 0,
\]

(4.36)

where \( \chi = \tilde{\eta}^0 \) or \( \tilde{u}^0 \), and \( j = 1, 2 \).

Since the initial data of linear solution \((\eta^a, u^a, q^a)\) does not satisfy the necessary compatibility conditions in the transformed MRT problem in general. Therefore, we shall modify the initial data of the linear solution.
Proposition 4.3. Let \((\eta^0, \bar{u}^0) := (w/\Upsilon, w)\) be provided by (4.34), then there exists a constant \(\delta_2 \in (0, 1]\), such that for any \(\delta \in (0, \delta_2]\), there exists \((\eta^\delta, u^\delta)\) enjoying the following properties:

1. The modified initial data

\[
(\eta_0^\delta, u_0^\delta) := \delta(\bar{\eta}^0, \bar{u}^0) + \delta^2(\eta^r, u^r)
\]

belongs to \((H_1^5 \cap H_s^5) \times H_s^5\) and satisfies the compatibility condition

\[
\text{div}_{(\delta^0)} u_0^\delta = 0 \text{ in } \Omega,
\]

where \(\delta^0\) is defined as \(\delta^0\) in place of \(\eta\).

2. Uniform estimate:

\[
\| (\eta^r, u^r) \|_5 \leq c_2^I \tag{4.37}
\]

where the positive constant \(c_2^I\) is independent of \(\delta\).

PROOF. Please refer to [25, Lemma 4.2] or [30, Proposition 5.1]. \(\square\)

Now we define that

\[
c_3^I := \| (\bar{\eta}^0, \partial_1 \bar{\eta}^0, \bar{u}^0) \|_4 + c_2^I > 0, \tag{4.38}
\]

\[
\delta_0 := \min \left\{ \frac{\delta_2}{2c_3^Ic_4}, \frac{\delta^I}{2c_3^Ic_4} \right\} \leq 1, \tag{4.39}
\]

where \(c_4 \geq 1\) is the constant in (2.80).

Let \(\delta \leq \delta_0\). Since \(\delta \leq \delta^I\), we can use Proposition 4.3 to construct \((\eta_0^\delta, u_0^\delta)\), which satisfies

\[
\| (\eta_0^\delta, \partial_1 \eta_0^\delta, u_0^\delta) \|_4 \leq c_3^I \delta \leq \delta_2.
\]

By Proposition 2.2, there exists a (nonlinear) unique solution \((\eta, u, q)\) of the transformed MRT problem (1.19) with initial value \((\eta_0^\delta, u_0^\delta)\) in place of \((\bar{\eta}^0, \bar{u}^0)\), where \((\eta, u, q) \in \mathcal{H}_{1,s}^{1,4} \times \mathcal{U}^I \times \mathcal{Q}^I\) for any \(\tau \in I_{\text{max}}\) and \(T^\text{max}\) denotes the maximal time of existence.

Let \(\epsilon_0 \in (0, 1]\) be a constant, which will be given in (4.65). We define

\[
T^\delta := \Upsilon^{-1} \ln(\epsilon_0/\delta) > 0, \quad \text{i.e., } \delta e^{TT^\delta} = \epsilon_0, \tag{4.40}
\]

\[
T^* := \sup \left\{ t \in I_{\text{max}} \mid \sup_{\tau \in [0,t]} \| (\eta, \partial_1 \eta, u)(\tau) \|_4 \leq 2c_3^Ic_4 \delta_0 \right\}, \tag{4.41}
\]

\[
T^{**} := \sup \left\{ t \in I_{\text{max}} \mid \sup_{\tau \in [0,t]} \| \eta(\tau) \|_5 \leq 2c_3^I \delta e^{TT^\delta} \right\}. \tag{4.42}
\]

Since \((\eta, u)\) satisfies (2.80) with \((\eta_0^\delta, u_0^\delta)\) in place of \((\bar{\eta}^0, \bar{u}^0)\) for some \(T \in I_{\text{max}},\)

\[
c_4 \| (\eta, \partial_1 \eta, u)(t) \|_4^2 |_{t=0} = c_4 \| (\eta_0^\delta, \partial_1 \eta_0^\delta, u_0^\delta) \|_4 \leq c_3^I c_4 \delta < 2c_3^I c_4 \delta, \tag{4.43}
\]

and

\[
\| \eta(t) \|_3 |_{t=0} = \| \eta_0^\delta \|_3 \leq c_3^I \delta < 2c_3^I \delta,
\]

thus \(T^* > 0, T^{**} > 0\) and

\[
\| \eta(T^{**}) \|_0 = 2c_3^I \delta e^{TT^{**}}, \text{ if } T^{**} < T^\text{max}. \tag{4.44}
\]
Noting that $2c_3'c_4\delta \leq \delta_2$, thus by Proposition 2.2 we can check that if $T^* < \infty$, then there exists a $T$ (independent of $\delta$) such that
\[
T^*_T := T^* + T/2 < T^{max}, \quad \sup_{\tau \in [0, T^*_T)} \| (\eta, \partial_t \eta, u)(\tau) \|_3 \leq 2c_3'c_4^2\delta_0,
\]
and, for any $\varsigma \in (T^*, T^*_T)$, there always exists a $t_0 \in [T^*, \varsigma)$ such that
\[
\| (\eta, \partial_t \eta, u)(t_0) \|_4 \geq 2c_3'c_4\delta_0.
\]
From now on, we denote
\[
T^*_T := \begin{cases} T^{**} & \text{for } T^{**} \leq T^*; \\ T^* + \min\{T^{**} - T^*, T\}/2 & \text{for } T^* < T^{**}, \end{cases}
\]
where $T$ comes from (4.35).
By (4.41) and (4.45), $(\eta, u)$ satisfies
\[
\sup_{t \in I_{T^*_T}} \| (\eta, \partial_t \eta, u)(t) \|_4 \leq 2c_3'c_4^2\delta_0 \leq \delta_1^1.
\]
Thus, by Proposition 4.2, $(\eta, u)$ enjoys the Gronwall-type energy inequality (4.25) for a.e. $t \in I_{T^*_T}$.
Using this fact, (4.42) and (4.43), we further have, for a.e. $t \in I_{T^*_T}$,
\[
\mathcal{E}(t) \leq c\delta^2 e^{2\Upsilon t} + \Upsilon \int_0^t \mathcal{E}(\tau) d\tau
\]
for some positive constant $c$. Making use of the fact $\mathcal{E}(t) \in W^{1,\infty}(I_{T^*_T})$, Gronwall’s lemma, and the equivalence (4.25), we further deduce from the above estimate that, for a.e. $t \in I_{T^*_T}$,
\[
\mathcal{E}(t) \lesssim \delta^2 e^{2\Upsilon t},
\]
which implies that, for sufficiently small $\delta$,
\[
\sup_{t \in [0, T^*_T]} \left( \| (\eta, \partial_t \eta, u) \|_4 + \| q \|_3 + \| u_t \|_2 \right) \leq c_4'\delta e^{\Upsilon t}.
\]

4.4. Error estimates

This subsection is devoted to the derivation of error estimates between $(\eta, u)$ and $(\eta^a, u^a)$, where the (nonlinear) solution $(\eta, u, q)$ and the linear solution $(\eta^a, u^a, q^a)$ are constructed in Section 4.3.

**Proposition 4.4.** Let $(\eta^d, u^d, q^d) = (\eta, u, q) - (\eta^a, u^a, q^a)$, then there exists a constant $c_3''$ such that, for any $\delta \in (0, \delta_0]$,
\[
\sup_{t \in I_{T^*_T}} \| (\eta^d, u^d)(t) \|_{W^{1,1}} \leq c_3''\delta e^{3\Upsilon t} \quad \text{for } i = 1 \text{ and } 2,
\]
where $\delta_0$ is defined by (1.39), $c_3''$ only depends on $g, a, \lambda, m, \rho$ and $\Omega$.

**Proof.** Subtracting the both transformed MRT problem (1.19) and the linearized problem (4.1) with $(\eta^a, u^a, q^a)$ in place of $(\eta, u, q)$, we get
\[
\begin{cases}
\eta^d = u^d, \\
\bar{\rho}u^d + \nabla A q^d + a\bar{\rho}u^d - \Upsilon m^2\partial_t^2 \eta^d - g\rho'\eta^d e_2 = g\Re e_2 - \nabla \tilde{A} q^a, \\
\text{div} A u^d = -\text{div} \tilde{A} u^a, \\
(u^d, \eta^d) |_{\partial \Omega} \cdot \tilde{n} = 0, \\
(\eta^d, u^d)|_{t=0} = \delta^2 (\eta^r, u^r),
\end{cases}
\]

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where $\mathcal{R} := \int_0^{\eta_z} (\eta_2(y, t) - z) \tilde{\rho}''(y_2 + z) dz$, $\tilde{A}$ is given by (2.8) and $\mathcal{A} = \tilde{A} + I$.

Differentiating (4.49) with respect to time $t$ and then using (4.49), we get

$$\begin{align*}
\frac{d}{dt} \left( \int \tilde{\rho} u_t^4 + \nabla A q_t^4 + a \tilde{\rho} u_t^4 - \chi m^2 \partial_t^2 u_t^4 - g \tilde{\rho} u_t^2 e_2 \right) \\
= g R_t e_2 - \nabla A q - \nabla \tilde{A} q_t^4 =: N,
\end{align*}$$

$$\text{div}_A u_t^4 = -\text{div}_A u - \text{div}_A u_t^2,$$

$$u_t^4 |_{\partial \Omega} \cdot \mathbf{n} = 0. \tag{4.50}$$

Taking the inner product of (4.50) with $u_t^4$ in $L^2$, we obtain, for a.e. $t \in I_{T^*}$,

$$\frac{1}{2} \frac{d}{dt} \left( \| \tilde{\rho} u_t^4 \|_0^2 - E(u_t^4) \right) + a \| \tilde{\rho} u_t^4 \|_0^2 = \int \mathcal{N} \cdot u_t^4 dy - \int \nabla A q_t^4 \cdot u_t^4 dy. \tag{4.51}$$

Exploiting the integral by parts, the boundary-value conditions of $(\eta, u, u^a)$, (4.50) and the relation

$$\partial_j A_{ij} = 0 \text{ for } j = 1, 2,$$

it is easy to compute out that

$$- \int \nabla A q_t^4 \cdot u_t^4 dy = \frac{d}{dt} \int \nabla q_t^4 \cdot (A_t^T u + \tilde{A}_t^T u_t^a) dy - \int \nabla q_t^4 \cdot \partial_t (A_t^T u + \tilde{A}_t^T u_t^a) dy.$$

Putting the above estimate into (4.51), we conclude that

$$\frac{1}{2} \frac{d}{dt} \left( \| \tilde{\rho} u_t^4 \|_0^2 - E(u_t^4) - I_4 \right) + a \| \tilde{\rho} u_t^4 \|_0^2 = I_5, \tag{4.52}$$

where we have defined that

$$I_4 := 2 \int \nabla q_t^4 \cdot \left( A_t^T u + \tilde{A}_t^T u_t^a \right) dy,$$

$$I_5 := \int \mathcal{N} \cdot u_t^4 dy - \int \nabla q_t^4 \cdot \partial_t (A_t^T u + \tilde{A}_t^T u_t^a) dy.$$

Thanks to (4.35), (4.37), (4.47) and (4.49), it is easy to estimate that, for all $t \in [0, T^*]$,

$$I_5(t) + I_4(t) - I_4(0) - E(u_t^4 |_{t=0}) \lesssim \delta^3 e^{3T_t}. \tag{4.53}$$

Taking the inner product of (4.49) with $t = 0$ and $u_t^4 |_{t=0}$ in $L^2$, and using the integration by parts and (4.50), we have

$$\int \tilde{\rho} |u_t^4 |_{t=0}|^2 dy = \int \left( g R e_2 - \nabla \tilde{\rho} q^a + \chi m^2 \partial_t^2 \eta - g \tilde{\rho} \eta_2 e_2 - a \tilde{\rho} u_t^4 \right) \cdot u_t^4 dy \bigg|_{t=0}
+ \int \nabla q^4 \cdot A_t^T (u + u^a) dy \bigg|_{t=0}.$$

Similarly to (4.53), we easily further deduce from the above identity that

$$\| \tilde{\rho} u_t^4 |_{t=0} \|_0^2 \lesssim \delta^3 e^{3T_t}. \tag{4.54}$$
Using \( (4.53) \) and \( (4.54) \), we easily deduce from \( (4.52) \) that, for all \( t \in [0, T^*_d] \),
\[
\left\| \sqrt{\rho} u^d_t \right\|_0^2 + 2a \int_0^t \left\| \sqrt{\rho} u^d_t(\tau) \right\|_0^2 d\tau \leq E(u^d) + c\delta^3 e^{3\Upsilon t}.
\] (4.55)

Next we shall deal with the term \( E(u^d) \) in right hand side of \( (4.55) \).

By the existence theory of Stokes problem, for any given \( t \in [0, T^*_d] \), there exists a unique \( (\bar{u}, \bar{w}) \in H^3 \times H^2 \) such that
\[
\begin{align*}
\nabla \bar{w} - \Delta \bar{u} &= 0, \quad \text{div} \bar{u} = -\text{div} A \hat{u} \quad \text{in} \, \Omega, \\
\bar{u} &= 0 \quad \text{on} \, \partial \Omega.
\end{align*}
\]
Moreover, the solution \( (\bar{u}, \bar{w}) \) enjoys
\[
\left\| \bar{u} \right\|_3^2 \lesssim \left\| -\text{div} A \hat{u} \right\|_2^2 \lesssim \delta^4 e^{4\Upsilon t}.
\] (4.56)

It is easy to see that \( v^d := u^d - \bar{u} \in H^1_{\sigma} \). Then we derive from \( (4.3) \) that
\[
E(v^d) \leq (\Lambda^2 + a\Lambda) \left\| \sqrt{\rho} v^d \right\|_0^2,
\]
which, together with \( (4.56) \), implies that
\[
E(u^d) \leq (\Lambda^2 + a\Lambda) \left\| \sqrt{\rho} u^d \right\|_0^2 + c\delta^3 e^{3\Upsilon t}.
\]

Putting the above estimate into \( (4.55) \) yields
\[
\left\| \sqrt{\rho} u^d_t \right\|_0^2 + 2a \int_0^t \left\| \sqrt{\rho} u^d_t(\tau) \right\|_0^2 d\tau \leq (\Lambda^2 + a\Lambda) \left\| \sqrt{\rho} u^d(t) \right\|_0^2 + c\delta^3 e^{3\Upsilon t}.
\] (4.57)

Using Newton–Leibniz’s formula and Young’s inequality, we have
\[
a\Lambda \left\| \sqrt{\rho} u^d(t) \right\|_0^2 \leq a\Lambda \left\| \sqrt{\rho} u^d(0) \right\|_0^2 + \int_0^t \left( \sqrt{\rho} u^d_t, \Lambda \sqrt{\rho} u^d \right)(\tau) \right\|_0^2 d\tau.
\] (4.58)

Thus we deduce from \( (4.57) \)–\( (4.58) \) that
\[
\left\| \left( \sqrt{\rho} u^d_t, \Lambda \sqrt{\rho} u^d \right) \right\|_0^2 \leq \Lambda \left\| \sqrt{\rho} u^d(t) \right\|_0^2 + 2a\Lambda \int_0^t \left\| \sqrt{\rho} u^d(\tau) \right\|_0^2 d\tau + c\delta^3 e^{3\Upsilon t}.
\] (4.59)

In addition,
\[
\frac{d}{dt} \left\| \sqrt{\rho} u^d \right\|_0^2 \leq \Lambda^{-1} \left\| \sqrt{\rho} u^d_t \right\|_0^2 + \Lambda \left\| \sqrt{\rho} u^d \right\|_0^2.
\] (4.60)

Combining \( (4.60) \) with \( (4.59) \), we obtain the following differential inequality
\[
\frac{d}{dt} \left\| \sqrt{\rho} u^d \right\|_0^2 + a \left\| \sqrt{\rho} u^d(t) \right\|_0^2 \leq 2\Lambda \left( \left\| \sqrt{\rho} u^d(t) \right\|_0^2 + a \int_0^t \left\| \sqrt{\rho} u^d(\tau) \right\|_0^2 d\tau \right) + c\delta^3 e^{3\Upsilon t}.
\] (4.61)

Applying Gronwall’s inequality to \( (4.61) \) then yields
\[
\left\| u^d(t) \right\|_0^2 + \int_0^t \left\| u^d(\tau) \right\|_0^2 d\tau \lesssim \delta^3 e^{3\Upsilon t}.
\]
We further derive from (4.49) and the above estimate that
\[ \| \eta^d(t) \|_0 \lesssim \sqrt{\delta^3 e^{3\Upsilon t}}. \] (4.62)

Putting the above two estimates and (4.55) together yields that
\[ \| (\eta^d, \partial_1 u^d, u^d, u^d_t)(t) \|_0 \lesssim \sqrt{\delta^3 e^{3\Upsilon t}}. \] (4.63)

Similarly to (4.62), we also derive from (4.49) and the above estimate
\[ \| \partial_1 \eta^d(t) \|_0 \lesssim \sqrt{\delta^3 e^{3\Upsilon t}}. \] (4.64)

Applying curl to (4.49), and then multiplying the resulting identity by curl $u^d$ in $L^2$, we get
\[
\int (\text{curl}(g R e_2 - \nabla \tilde{\eta}^d) + \tilde{\rho} \partial_t u^d + a \tilde{\rho} u^d + g \tilde{\rho} \partial_t \eta^d) \text{curl} u^d dy,
\]

Thanks to (4.63) and (4.64), we easily derive from the above identity that
\[ \| \text{curl} u^d \|_0^2 + \| \text{curl} \eta^d \|_1^2 \lesssim \delta^3 e^{3\Upsilon t}. \]

In addition, by (4.49)3, we have
\[ \| \text{div} u^d \|_0^2 \lesssim \delta^3 e^{3\Upsilon t}. \]

Thus we further derive from the Hodge-type elliptic estimate (A.9) and the above two estimates that
\[ \| \nabla u^d \|_0^2 \lesssim \delta^3 e^{3\Upsilon t}. \]

Similarly to (4.62), we also derive from (4.49) and the above estimate
\[ \| \nabla \eta^d \|_0^2 \lesssim \delta^3 e^{3\Upsilon t}. \]

Putting (4.63) and the above two estimates together, and then using Hölder’s inequality, we get (4.48). This completes the proof. \[ \square \]

4.5. Existence of escape times

Let
\[ m_0 = \min_{\chi_{i=1}^{\infty}, \rho, \rho_0, \epsilon_0} \{ \| \chi_j \|_{L^1}, \| \partial_1 \chi_j \|_{L^1}, \| \partial_2 \chi_j \|_{L^1} \}. \]

Then $m_0 > 0$ by (4.36).

Now we define that
\[ \epsilon_0 = \min \left\{ \left( \frac{c_3^l}{2c_5^l} \right)^2, \frac{c_3^l c_4 \delta_0}{c_4^l}, \frac{m_0^2}{4|c_3^l|^2} \right\} > 0. \] (4.65)

We claim that
\[ T^\delta = T^{\min} = \min \{ T^\delta, T^*, T^{**} \} \neq T^* \text{ or } T^{**}, \] (4.66)

which can be showed by contradiction as follows:
(1) If $T^{\text{min}} = T^\ast$, then $T^\ast \leq T^\ast \leq T^{\text{max}}$ or $T^{\ast\ast} = T^\ast < +\infty$. Noting that $\sqrt{\epsilon_0} \leq c_3/2c_5$, then by (4.34), (4.38), (4.40) and (4.48) that

$$
\|\eta(T^{\ast\ast})\|_0 \leq \|\eta^a(T^{\ast\ast})\|_0 + \|\eta^\ast(T^{\ast\ast})\|_0 \\
\leq \delta e^{TT\ast\ast} (c_3^f + c_5^f \sqrt{\delta e^{TT\ast\ast}}) \leq \delta e^{TT\ast\ast} (c_3^f + c_5^f \sqrt{\epsilon_0}) \\
\leq 3c_3^f \delta e^{TT\ast\ast}/2 < 2c_3^f \delta e^{TT\ast\ast},
$$

which contradicts (4.42).

(2) If $T^{\text{min}} = T^\ast$, then $T^\ast < T^{\ast\ast}$. Recalling $\epsilon_0 \leq c_3^f c_4^\delta_0 / c_4^f$, then we deduce from (4.47) that, for any $t \in I_{T^\ast}$,

$$
\|\eta, \partial_t \eta, \eta(t)\|_3 \leq c_3^f \delta e^{TT\ast\delta} \leq c_3^f c_4^\delta_0 < 2c_3^f c_4^\delta_0,
$$

which contradicts (4.46). Hence $T^{\text{min}} \neq T^\ast$.

Since $T^\delta$ satisfies (4.66), then (4.48) holds to $t = T^\delta$. Using this fact, (4.34) and the condition $\epsilon_0 \leq m_0^2/4|c_5^f|^2$, we find that

$$
\|\partial_j^k \chi_i(T^\delta)\|_{L^1} \geq \|\partial_j^k \chi_i^a(T^\delta)\|_{L^1} - \|\partial_j^k \chi_i^d(T^\delta)\|_{L^1} \\
\geq \delta e^{TT\delta} (\|\partial_j^k \chi_i^0\|_{L^1} - c_5^f \sqrt{\delta e^{TT\ast\ast}}) \geq (m_0 - c_5^f \sqrt{\epsilon_0}) \epsilon_0 \geq m_0 \epsilon_0/2,
$$

where $\chi$ represents $\eta$ or $u$, $1 \leq i, j \leq 2$ and $k = 0, 1$. This completes the proof of Theorem 1.2 by taking $\epsilon := m_0 \epsilon_0/2$.

5. Local well-posedness

This section is devoted to the proof of local well-posedness of the transformed MRT problem (1.19). Recalling (1.18), the transformed MRT problem (1.19) is equivalent to the following initial-boundary value problem

$$
\begin{cases}
\eta_t = u, \\
\bar{\rho} u_t + \nabla_A Q + a \bar{\rho} u = \lambda m^2 \partial_1^2 \eta - g \bar{\rho} e_2, \\
\text{div}_A u = 0, \\
(\eta, u)|_{t=0} = (\eta^0, u^0), \\
(\eta, u)|_{\partial\Omega} \cdot \vec{n} = 0,
\end{cases}
$$

(5.1)

where $Q = q + \bar{P}(\zeta_2) + \lambda |\bar{M}|^2/2$. Hence it suffices to establish the local well-posedness result of the initial-boundary value problem above. Next we roughly sketch the proof frame to establish the local well-posedness result.

Similarly to [10], in which Gu–Wang investigated the well-posedness problem of incompressible inviscid MHD fluid equations with free boundary, we first alternatively use an iteration method to establish the existence result of unique local solutions of the linear $\kappa$-approximate problem

$$
\begin{cases}
\eta_t - \kappa \partial_1^2 \eta = u, \\
\bar{\rho} u_t + \nabla_B Q + a \bar{\rho} u = \lambda m^2 \partial_1^2 \eta - g \bar{\rho} e_2, \\
\text{div}_B u = 0, \\
(\eta, u)|_{t=0} = (\eta^0, u^0), \\
(\eta, u)|_{\partial\Omega} \cdot \vec{n} = 0, \\
\nabla_B Q|_{\partial\Omega} \cdot \vec{n} = -g \bar{\rho} e_2 \cdot \vec{n},
\end{cases}
$$

(5.2)
where $\mathcal{B} := (\nabla \varsigma + I)^{-T}$, and $\varsigma$ is given and belongs to some proper function space, see Proposition 5.4 in Section 5.1 for details. It should be noted that the Neumann boundary-value condition (5.2) makes sure the solvability of $Q$.

Then we derive the $\kappa$-independent estimates of the solutions, denoted by $(\eta^\kappa, u^\kappa, Q^\kappa)$, of the $\kappa$-approximate problem above, and then take the limit of $(\eta^\kappa, u^\kappa, Q^\kappa)$ with respect to $\kappa \to 0$ in some common definition domain. In particular, the obtained limit function, denoted by $(\eta, u, Q)$, is the unique local solution to the linearized problem

$$
\begin{aligned}
\eta_t &= u, \\
\bar{\rho} u_t + \nabla_\mathcal{B} Q + a \bar{\rho} u &= \lambda m^2 \partial_t^2 \eta - g \bar{\rho} e_2, \\
\text{div}_\mathcal{B} u &= 0, \\
(\eta, u)|_{t=0} &= (\eta^0, u^0), \\
(\eta, u)|_{\partial \Omega} \cdot \vec{n} &= 0,
\end{aligned}
$$

(5.3)

see Proposition 5.5 in Section 5.2 for details. It should be noted that the solution $Q$ of (5.3) automatically satisfies (5.2) due to $u_t|_{\partial \Omega} \cdot \vec{n} = 0$.

Finally, since the linearized problem (5.3) admits a unique local solution for any given $\varsigma$, thus we easily arrive at Proposition 2.2 by a standard iteration method as in [4, 5], in which Coutand–Shkoller investigated the well-posedness problem of incompressible Euler equations with free boundary.

Now we turn to introducing some new notations appearing in this section. We denote $\Omega \times I_d$ by $\Omega_d$, for some constant $d > 0$. The notations $P(x_1, \ldots, x_n)$ and $\hat{P}(x_1, \ldots, x_n)$ represent the generic polynomials with respect to the parameters $x_1, \ldots, x_n$, where all the coefficients in $P$ and $\hat{P}$ are equal one, and $\hat{P}$ further satisfies $\hat{P}(0, \ldots, 0) = 0$. It should be noted that $P(x_1, \ldots, x_n)$, $\hat{P}(x_1, \ldots, x_n)$ and $c^\kappa$ may vary from line to line. $a \lesssim b$ means that $a \leq c^\kappa b$, where $c^\kappa$ denotes a generic positive constant, which may depend on $\kappa$, $a$, $g$, $\lambda$, $m$, $\bar{\rho}$ and $\Omega$.

We always use the notations $\mathcal{B}$, resp. $\mathcal{J}$ to represent $(\nabla \varsigma + I)^{-T}$, resp. $\text{det}(\nabla \varsigma + I)$, where $\varsigma$ at least satisfies

$$
\varsigma \in C^0(T_T, H^4) \text{ and } \inf_{(y,t) \in \Omega_T} \text{det}(\nabla \varsigma + I) \geq 1/4 \text{ for some } T > 0.
$$

(5.4)

In addition, we define that

$$
\mathbb{A}_{T,\iota}^{4,1/4} := \{ \psi \in C^0(T_T, H^4) \mid \|\psi\|_3 \leq \iota \}
$$

(5.5)

and

$$
\begin{aligned}
\mathcal{S}_T := \{& (\xi, w, \beta) \in C(T_T, H^4) \times C^0(T_T, H^4) \times (C^0(T_T, H^3) \cap L^\infty_T H^4) \mid \\
& \partial_\xi^2 \xi \in L^2_T H^4, \quad \nabla^4_\mathcal{B} \partial_\xi \xi \in C_{\text{B,weak}}(T_T, L^2), \quad \nabla_\mathcal{B} \beta \in L^2_T H^4, \\
& (\nabla_\mathcal{B} \beta/\bar{\rho})|_{\partial \Omega} \cdot \vec{n} = -g e_2 \cdot \vec{n} \},
\end{aligned}
$$

(5.6)

for some constant $T > 0$, $\iota$ is the positive constant provided in Lemma $\mathbb{A}_{T,\iota}^{4,1/4}$, $\bar{\rho}$ satisfies (1.5) and $g$ is the gravity constant. It should be noted that the function $\varsigma$, which belongs to $\mathbb{A}_{T,\iota}^{4,1/4}$, automatically satisfies (5.4) by Lemma $\mathbb{A}_{T,\iota}^{4,1/4}$.

Finally, some preliminary estimates for $\mathcal{B}$ and $\mathcal{J}$ are collected as follows.
Lemma 5.1. Let $\varsigma$ satisfy (5.1) with $T > 0$. Then
\begin{align}
\|B - I\|_3 &\lesssim \hat{P}(\|\varsigma\|_4), \quad (5.7) \\
\|B\|_{C^0(T)} + \|J\|_{C^0(T)} + \|(J, J^{-1})\|_3 &\lesssim P(\|\varsigma\|_4) \text{ for any } t \in \overline{T}. \quad (5.8)
\end{align}

(1) If additionally $\varsigma$ further satisfies $$(\partial_t \varsigma, \varsigma) \in L_T^\infty H^4$$, then
\begin{align}
\|\partial_t (B, J)\|_3 &\lesssim \hat{P}(\|\varsigma, \varsigma\|_4), \quad (5.9) \\
\|\partial_t B\|_3 &\lesssim \hat{P}(\|\varsigma\|_4) \text{ for a.e. } t \in I_T. \quad (5.10)
\end{align}

(2) If additionally $\varsigma$ further satisfies $\varsigma \in L_T^\infty H^3$ and $\varsigma_t \in L_T^\infty H^2$, then
\begin{align}
\|\partial_t (B, J)\|_1 &\lesssim \hat{P}(\|\varsigma\|_4, \|\varsigma_t\|_3, \|\varsigma_{tt}\|_2) \text{ for a.e. } t \in I_T. \quad (5.11)
\end{align}

**Proof.** Since the derivations of the estimates (5.7)–(5.11) are very elementary, we omit it. \(\square\)

5.1. Solvability of the linear $\kappa$-approximate problem (5.2)

In this section, we construct the unique local solution of the the linear $\kappa$-approximate problem (5.2). To this purpose, we shall rewrite the linear $\kappa$-approximate problem (5.2) as the following equivalent problem (in the sense of classical solutions):
\begin{align}
\begin{cases}
\eta_t - \kappa \partial_1^2 \eta = u, \\
\rho_t + \nabla_B Q + a \rho u = \lambda m^2 \partial_1^2 \eta - g \rho e_2, \\
-\text{div}_B (\nabla_B Q/\rho) = K^1, \\
(\eta, u)|_{t=0} = (\eta^0, u^0), \\
(\eta, u)|_{\partial \Omega} = 0, \quad \nabla_B Q|_{\partial \Omega} = -g \rho e_2 \cdot \mathbf{n},
\end{cases} \quad (5.12)
\end{align}

where $\varsigma|_{t=0} = \eta^0$, $\eta^0$ satisfies
\[
\text{div}_{A^0} u^0 = 0, \quad A^0 = (\nabla \eta^0 + I)^{-T}, \quad \inf_{y \in \overline{T}} (\det A^0) > 0,
\]
and
\[
K^1 := a \text{div}_B u - \text{div}_B u - J_0 \text{div}_B u / J - \lambda m^2 \text{div}_B (\partial_1^2 \eta / \rho).
\]

Then the solvability of the linear $\kappa$-approximate problem reduces to the solvability of the initial-boundary value problem (5.12).

We want to establish the local well-posedness result for (5.12) by an iteration method. To begin with, we shall investigate the solvability of the linear problem
\begin{align}
\begin{cases}
\eta_t - \kappa \partial_1^2 \eta = w, \\
u_t + au = K^2, \\
-\text{div}_B (\nabla_B Q / \rho) = K^1, \\
(\eta, u)|_{t=0} = (\eta^0, u^0), \\
(\eta, u)|_{\partial \Omega} = 0, \quad \nabla_B Q \cdot \mathbf{n}|_{\partial \Omega} = -g \rho e_2 \cdot \mathbf{n},
\end{cases} \quad (5.14)
\end{align}
where \((\varsigma, w, \theta)\) is given, and
\[
K^2 := (\lambda m^2 \partial_1^2 \xi - \nabla_B \theta)/\bar{\rho} - g e_2.
\]

It is easy to see that the above linear problem is equivalent to the following three sub-problems: the initial-boundary value problem of partly dissipative equation for \(\eta\)

\[
\begin{cases}
\eta_t - \kappa \partial_1^2 \eta = w, \\
\eta|_{t=0} = \eta^0, \\
\eta|_{\partial \Omega} \cdot \vec{n} = 0,
\end{cases}
(5.15)
\]

the initial-value problem of ODE for \(u\)

\[
\begin{cases}
u_t + au = K^2, \\
u|_{t=0} = u^0, \\
u|_{\partial \Omega} \cdot \vec{n} = 0.
\end{cases}
(5.16)
\]

and the Neumann boundary-value problem of elliptic equation for \(q\)

\[
\begin{cases}
-\text{div}_B (\nabla_B Q/\bar{\rho}) = K^1 \quad \text{in } \Omega, \\
\nabla_B Q/\bar{\rho} \cdot \vec{n} = -g e_2 \cdot \vec{n} \quad \text{on } \partial \Omega.
\end{cases}
(5.17)
\]

Thus the solvability of the linear problem \((5.14)\) reduces to the solvability of the three sub-problems above. Next we establish the global well-posedness results for the above three sub-problems in sequence.

**Proposition 5.1.** Let \(T > 0, w \in L^2_T H^4_s\) and \(\eta^0 \in H^1_s\). Then the initial-boundary value problem \((5.15)\) admits a unique solution \(\eta \in C(I_T, H^4_s)\), which satisfies \(\nabla^4 \partial_1 \eta \in C^0(I_T, L^2)\) and the estimates

\[
\sup_{t \in I_T} \|\eta\|_4 + \sqrt{\kappa} \|\partial_1 \eta\|_4 + \kappa \|\partial_1^2 \eta\|_{L^2_T H^4_s} \lesssim \|\eta^0\|_4 + \sqrt{\kappa} \|\eta^0\|_{1,4} + (1 + \sqrt{T}) \|w\|_{L^2_T H^4}.
(5.18)
\]

**Proof.** We define the difference quotient with respect variable \(y_1\) as follows:

\[
D^\tau f(y_1, y_2) = (f(y_1 + \tau, y_2) - f(y_1, y_2))/\tau \quad \text{for } \tau \text{ satisfying } |\tau| \in (0, 1).
\]

Let \(\varepsilon \in (0, 1), \chi\) be a 1D standard mollifier (see [34, pp. 38] for the definition), and \(\chi(\varepsilon) := \chi(s/\varepsilon)/\varepsilon\). Let \(\bar{w} = w\) in \(\Omega_T\) and \(\bar{w} = 0\) outside \(\Omega \times (\mathbb{R} \setminus I_T)\). We define the mollification of \(\bar{w}\) with respect to \(t\):

\[
S^t_\varepsilon(\bar{w}) := \chi(\varepsilon) \ast \bar{w}.
(5.19)
\]

Then \(S^t_\varepsilon(\bar{w}) \in C^\infty(\mathbb{R}, H^1_s)\). We can check that

\[
\|S^t_\varepsilon(\bar{w})\|_{L^2_T H^4_s} \lesssim \|w\|_{L^2_T H^4},
(5.20)
\]

\(S^t_\varepsilon(\bar{w}) \to w\) strongly in \(L^2_T H^4\).
Now we consider the $\tau$-approximate problem for (5.15):

\[
\begin{aligned}
\eta^\tau_t &= L(\eta^\tau) + S^\tau_t(\bar{w}) \quad \text{in } \Omega_T, \\
\eta^\tau|_{t=0} &= \eta^0 \quad \text{in } \Omega, \\
\eta^\tau \cdot \bar{n} &= 0 \quad \text{on } \partial\Omega,
\end{aligned}
\]  

(5.21)

where $L : H^4_s \rightarrow H^4_s$ by the rule $L(f) = \kappa D_1^{-\tau} D_1^t f$ for $f \in H^4_s$.

It is easy to see that $L(\varphi) \in C^0(\overline{\Omega_T}, H^4_s)$ for $\varphi \in C^0(\overline{\Omega_T}, H^4_s)$ and $L \in \mathcal{L}(H^4_s)$, where $\mathcal{L}(H^4_s)$ is a set of all linear bounded operators of $H^4_s$. In particular, $L \in C^0(\overline{\Omega_T}, \mathcal{L}(H^4_s))$. By existence theory of the initial-value problem of an abstract ODE equation (see [34, Proposition 2.17]), there exists a unique solution $\eta^\tau \in C^0(\overline{\Omega_T}, H^4_s) \cap C^1(\overline{\Omega_T}, H^4_s)$ to (5.21). Obviously $\eta_t^\tau$ automatically belong to $L^2_T H^4_s$ by the second conclusion in Lemma A.10 and the separability of $H^4_s$.

Let $\alpha$ satisfy $0 \leq |\alpha| \leq 4$. Applying $\partial^\alpha$ to (5.21), and then multiplying the resulting identity $\partial^\alpha \eta^\tau$ in $L^2$, we have, for a.e. $t \in I_T$,

\[
\frac{d}{dt} \int \partial^\alpha \eta^\tau |^2 \, dy + \kappa \int |D_1^t \partial^\alpha \eta^\tau|^2 \, dy = \int S^\tau_t(\partial^\alpha \bar{w}) \cdot \partial^\alpha \eta^\tau \, dy.
\]  

(5.22)

Making use of (5.20), we easily deduce from (5.22) that,

\[
\|\eta^\tau\|_{C^0(\overline{\Omega_T}, H^4_s)} \lesssim \|\eta^0\|_4 + \sqrt{T}\|w\|_{L^2_T H^4_s}.
\]  

(5.23)

In addition, we easily deduce from (5.21) that,

\[
\frac{\kappa}{2} \frac{d}{dt} \|D_1^t \partial^\alpha \eta^\tau\|_0^2 + \int |\partial^\alpha \eta^\tau_t|^2 \, dy = \int \partial^\alpha (S^\tau_t(\partial^\alpha \bar{w})) \cdot \partial^\alpha \eta^\tau_t \, dy.
\]  

(5.24)

Noting that

\[
\|D_1^t \partial^\alpha \eta^\tau\|_{t=0} \lesssim \|\partial^\alpha \eta^0\|_{1,0},
\]  

(5.25)

thus, making use of Young’s inequality, (5.20) and (5.21), we deduce from (5.24) and (5.25) that

\[
\kappa \|D_1^t \eta^\tau\|^2_{C^0(\overline{\Omega_T}, H^4_s)} + \|(L(\eta^\tau), \eta^\tau_t)\|^2_{L^2_T H^4_s} \lesssim \kappa \|\eta^0\|^2_{1,4} + \|w\|^2_{L^2_T H^4_s}.
\]  

(5.26)

Thanks to the regularity of $\eta^\tau$, we can we easily derive (5.21) that, for any $\varphi \in H^1$,

\[
\int D_1^t \partial^\alpha (\eta^\tau(y, t) - \eta^\tau(y, s)) \cdot \varphi \, dy \\
= - \int_s^t \int (\partial^\alpha S^\tau_t(\bar{w}) + L(\partial^\alpha \eta^\tau)) \cdot D_1^t \varphi \, dy \, d\tau.
\]  

(5.27)

Exploiting (5.20), the uniform estimate of $L(\eta^\tau)$ in (5.26) and the fact

\[
\|D_1^\tau \varphi\|_0 \lesssim \|\varphi\|_{1,0},
\]  

we easily derive from the identity (5.21) that

\[
D_1^\tau \nabla^4 \eta^\tau \text{ is uniformly continuous in } H^{-1}.
\]  

(5.28)

Here and in what follows, $H^{-1}$ denotes the dual space of $H^1_0 := \{w \in H^1 \mid w|_{\partial\Omega} = 0\}$.  

40
Proposition 5.2. Let \( \text{(5.16)} \) satisfies the desired conclusion in Proposition 5.1, by taking limit of some sequence of \( \varsigma \) and \( \eta \). Moreover, the limit function \( \eta^\varepsilon \) is just the unique solution to the problem

\[
\begin{cases}
\eta^\varepsilon_t = \kappa \partial_1^2 \eta^\varepsilon + S_T^i(\tilde{w}) & \text{in } \Omega_T, \\
\eta^\varepsilon|_{t=0} = \eta^0 & \text{in } \Omega, \\
\eta^\varepsilon \cdot n = 0 & \text{on } \partial\Omega
\end{cases}
\]

and satisfies

\[
\sup_{t \in \Omega_T} (\|\eta^\varepsilon\|_4 + \sqrt{\kappa}\|\partial_1 \eta^\varepsilon\|_4) + \kappa \|\partial_1^2 \eta^\varepsilon\|_{L^2 T H^4} 
\leq \|\eta^0\|_4 + \sqrt{\kappa}\|\eta^0\|_{1,4} + (1 + \sqrt{T})\|w\|_{L^2 T H^4}.
\]

Thus we further have \( \eta^\varepsilon \in C^0(\overline{T}_T, H^4_s) \).

Noting that \( \eta^\varepsilon \) enjoys the uniform estimate above, thus we again get a limit function \( \eta \), which satisfies the desired conclusion in Proposition 5.1 by taking limit of some sequence of \( \{\eta^\varepsilon\}_{\varepsilon > 0} \). We omit such limit process, since it is very similar to the argument of obtaining \( \eta^\varepsilon \).

**Proposition 5.2.** Let \( a \in \mathbb{R} \) and \( K^2 \in L^1_T H^4_s \) and \( u^0 \in H^4_s \), then the initial-boundary value problem \( \text{(5.16)} \) admits a unique solution \( u \in C^0(\overline{T}_T, H^4_s) \), which satisfies \( u_t \in L^1_T H^4_s \) and

\[
\|u(t)\|_4 \lesssim \|u^0\|_4 + \|K^2\|_{L^1_T H^4}.
\]

**Proof.** Let

\[
u = u^0 e^{-at} + \int_0^t K^2(\tau) e^{-a(t-\tau)} d\tau.
\]

Since \( K^2 \in L^1_T H^4_s \) and \( u^0 \in H^4_s \), it is easy to check that \( u \) given by (5.30) is the unique solution of (5.16) and satisfies (5.29).

**Proposition 5.3.** Let \( T > 0 \), \( \tau \) be the positive constant provided in Lemma A.8 \( \bar{p} \) satisfy (1.5) and \( \xi \in \mathbb{A}_{T, \tau}^{41/4} \) defined by (5.5). If \( K^1 \in C^0(\overline{T}_T, H^1) \) and \( L^\infty_T H^2 \) and satisfies \( \int K^1 J dy = 0 \) for each \( t \in \overline{T}_T \), then there exists a unique solution \( Q \in C^0(\overline{T}_T, H^3) \), which satisfies the boundary-value problem (5.17) for each \( t \in I_T \) and enjoys the estimate

\[
\|Q\|_{L^\infty_F H^4} \lesssim P(\|\xi\|_{L^\infty_F H^4}) \left( \|g\|_3 + \|K^1\|_{L^\infty F H^2} \right).
\]

(1) If we additionally assume that \( K^1 \in L^2_T H^3 \), then

\[
\|\nabla B Q\|_{L^2 F H^4} \lesssim P(\|\xi\|_{L^\infty F H^4}) \left( \sqrt{T}\|g\|_4 + \|K^1\|_{L^2 F H^3} \right).
\]

(2) If we additionally assume that \( \varsigma_t \in L^\infty_T H^3 \) and \( K^1_t \in L^\infty_T H^1 \), then

\[
\|Q_t\|_{L^\infty_F H^3} \lesssim P(\|\xi\|_{L^\infty_F H^4}, \|\varsigma_t\|_{L^\infty T H^3}) \left( \|g\|_3 + \|K^1\|_{L^\infty F H^2} + \|K^1_t\|_{L^\infty F H^1} \right).
\]
Proof. Let \( \varphi = \zeta + y \), then \( \varphi \) is a \( C^1 \)-diffeomorphism mapping by Lemma \( \text{A.8} \). We denote \( \varphi^{-1} \) the inverse mapping of \( \varphi \) with respect to variable \( y \), and then define \( \tilde{K}^1 := K^1(\varphi^{-1}, t) \). By \( \text{[A.46]} \), \( \tilde{K}^1 \in C^0(T_T, H^1) \cap L_T^\infty H^2 \).

Now we consider the following Neumann boundary-value problem of elliptic equation

\[
\begin{align*}
\begin{cases}
-\text{div} (\nabla p/\tilde{\rho}) = \tilde{K}^1 & \text{in } \Omega, \\
\nabla p/\tilde{\rho} \cdot \vec{n} = -g e_2 \cdot \vec{n} & \text{on } \partial\Omega,
\end{cases}
\end{align*}
\]

where \( \tilde{\rho} = \tilde{\rho}(\varphi_2^{-1}) \). It is easy to see that

\[
\int \tilde{K}^1 \, dx = \int K^1 \, dy = 0 = -\int_{\partial\Omega} g e_2 \cdot \vec{n} \, dy.
\]

By Lemma \( \text{A.7} \) the second assertion in Lemma \( \text{A.10} \) \( \text{[A.41]} \), \( \text{[A.48]} \) and the separability of \( H^3 \), there exists a unique solution \( p \in C^0(T_T, H^3) \cap L_T^\infty H^4 \) of the boundary-value problem (5.34) such that

\[
\|p\|_{L_T^\infty H^4} \lesssim P(\|\varsigma\|_{L_T^\infty H^3})(\|g\|_3 + \|K^1\|_{L_T^\infty H^2}).
\]

Let \( \tilde{Q} = p(\varphi) \), then \( \tilde{Q} \in C^0(T_T, H^3) \cap L_T^\infty H^4 \) by \( \text{[A.45]} \). We further define that \( Q := \tilde{Q} - (\tilde{Q})_\Omega \), then \( Q \in C^0(T_T, H^3) \cap L_T^\infty H^4 \) and satisfies (5.31) by \( \text{[A.47]} \). Since \( p \) satisfies (5.34), we have

\[
\begin{align*}
\begin{cases}
-\text{div}_B (\nabla_B Q/\tilde{\rho}) = K^1 & \text{in } \Omega, \\
\nabla_B Q/\tilde{\rho} \cdot \vec{n} = -g e_2 \cdot \vec{n} & \text{on } \partial\Omega,
\end{cases}
\end{align*}
\]

In addition, the uniqueness of \( Q \) is obvious in the class \( C^0(T_T, H^3) \cap L_T^\infty H^4 \).

1) If we additionally assume that \( K^1 \in L_T^2 H^2 \), then \( \tilde{K}^1 \in L_T^2 H^2 \). By Lemma \( \text{A.10} \) we have \( p \in L_T^2 H^4 \) and

\[
\|p\|_{L_T^2 H^4} \lesssim P(\|\varsigma\|_{L_T^\infty H^3})(\sqrt{T}\|g\|_3 + \|\tilde{K}^1\|_{L_T^\infty H^2}).
\]

By (5.36), \( \text{[A.47]} \) and (5.48),

\[
\|\nabla_B Q\|_{L_T^2 H^4} = \|\nabla p(x, t)|_{x = \varphi(y, t)}\|_{L_T^2 H^4} \lesssim P(\|\varsigma\|_{L_T^\infty H^3})\|\nabla p\|_{L_T^2 H^4} \lesssim P(\|\varsigma\|_{L_T^\infty H^3})(\sqrt{T}\|g\|_4 + \|K^1\|_{L_T^\infty H^2}),
\]

which yields (5.32).

2) By (5.50), it is easy to see that

\[
\tilde{K}^1 \in L_T^\infty H^1 \text{ and } \partial_t(\tilde{\rho}, (1/\tilde{\rho})) \in L_T^\infty H^3.
\]

Thanks to Lemma \( \text{A.7} \) it is easy to verify that there exists a unique solution \( \chi \in L_T^\infty H^3 \) to

\[
\begin{align*}
\begin{cases}
-\text{div} (\nabla \chi/\tilde{\rho}) = \tilde{K}^1 + \text{div}_B (\partial_t(1/\tilde{\rho}) \nabla p) & \text{in } \Omega, \\
\nabla \chi/\tilde{\rho} \cdot \vec{n} = \partial_t(1/\tilde{\rho}) \nabla p \cdot \vec{n} & \text{on } \partial\Omega.
\end{cases}
\end{align*}
\]

Moreover, by (5.35), (2.22), (5.48) and (5.50),

\[
\begin{align*}
\|p\|_{L_T^\infty H^4} + \|\chi\|_{L_T^\infty H^3} \\
\lesssim P(\|\varsigma\|_{L_T^\infty H^3}, \|\varsigma\|_{L_T^\infty H^3})(\|g\|_3 + \|K^1\|_{L_T^\infty H^2} + \|K_t\|_{L_T^\infty H^1}).
\end{align*}
\]
Let \( t \in I_T \) and \( D_s \vartheta = (\vartheta(y, t + s) - \vartheta(y, t))/s \) where \( t + s \in I_T \). Since \( p \) satisfies (5.34), thus

\[
\begin{align*}
-\text{div} (\nabla D_s p/\bar{\rho}(x, t + s)) &= D_s \bar{K}^1 + \text{div} (D_s(1/\bar{\rho}) \nabla p) \quad \text{in } \Omega, \\
\nabla D_s p/\bar{\rho}(x, t + s) \cdot \bar{n} &= -D_s(1/\bar{\rho}) \nabla p \cdot \bar{n} \quad \text{on } \partial \Omega.
\end{align*}
\] (5.40)

Subtracting (5.38) from (5.40) yields that

\[
\begin{align*}
-\text{div} (\nabla (\chi - D_s p)/\bar{\rho}(x, t + s)) &= \bar{K}^1_t - D_s \bar{K}^1 + \text{div}((\partial_t(1/\bar{\rho}) - D_s(1/\bar{\rho})) \nabla p) \\
&\quad + (1/\bar{\rho}(x, t) - 1/\bar{\rho}(x, t + s)) \nabla \chi) \quad \text{in } \Omega, \\
(\nabla (\chi - D_s p)/\bar{\rho}(x, t + s)) \cdot \bar{n} &= ((\partial_t(1/\bar{\rho}) - D_s(1/\bar{\rho})) \nabla p(x, t) \\
&\quad + (1/\bar{\rho}(x, t + s)) - 1/\bar{\rho}(x, s)) \nabla \chi) \cdot \bar{n} \quad \text{on } \partial \Omega.
\end{align*}
\] (5.41)

Applying the estimate (A.22) to (5.41), we have, for a.e. \( t \in I_T \),

\[
\|\chi - D_s p\|_3 \lesssim \|\bar{K}^1_t - D_s \bar{K}^1\|_1 + \|((\partial_t(1/\bar{\rho}) - D_s(1/\bar{\rho})) \nabla p, (1/\bar{\rho}(x, t + s) - 1/\bar{\rho}(x, s)) \nabla \chi)\|_2.
\] (5.42)

Noting that the generalized derivative with respect to \( t \) is automatically strong derivative, we easily see that the two terms on the right hand of the inequality (5.42) converge to 0 for a.e. \( t \in I_T \) by (5.37). So, \( \|D_s p - \chi\|_3^2 \to 0 \) as \( s \to 0 \) for a.e. \( t \in I_T \). This means that the strong derivative of \( p \) with respect to \( t \) is equal to that of \( \chi \). In addition, it is easy to check that \( p \in AC^0(T_T, H^3) \), thus \( p_t = \chi \), where \( p_t \) denotes the generalized derivative of \( p \). Hence, \( p_t \in L^\infty(I_T H^3) \) satisfies (5.39) with \( p_t \) in place of \( \chi \). Thanks to (A.47) and (A.49), we immediately get (5.33) from (5.39). This completes the proof of Proposition 5.3. \( \square \)

With Propositions 5.1, 5.3 in hand, next we will use an iteration method to establish an existence result of a unique local solution to the linearized \( \kappa \)-problem (5.2).

**Proposition 5.4.** Let \( A_{\alpha, 1}^{4, 1/4} \) and \( S_\alpha \) are defined by (5.5) and (5.6) with some positive constant \( \alpha \) in place of \( T \), resp.. We assume that \( \alpha \geq 0, \rho \) satisfy (1.5), \( (\eta^0, u^0) \in H^1_s \times H^4_s \) and \( \zeta \in A_{\alpha, 1}^{4, 1/4} \) satisfies \( \zeta \in C^0(T_\alpha, H^3_s) \cap L^\infty_T H^4 \), then the \( \kappa \)-approximate problem (5.2) defined on \( \Omega_\alpha \) admits a unique solution, denoted by \( (\eta^\kappa, u^\kappa, Q^\kappa) \), which belongs to \( S_\alpha \).

**Proof.** Let \( T \ll \alpha \). Thanks to the regularity of \( (\zeta, \zeta_t) \) and the relation

\[
\partial_j(J B_{ij}) = 0 \quad \text{for } i = 1, 2,
\] (5.43)

it is easy to check that, for any \( (\xi, w, \beta) \in S_T \),

\[
K^1(\xi, w) \in C^0(T_T, H^1) \cap L^\infty_T H^1 \cap L^2_T H^3 \quad \text{and} \quad \int K^1(\xi, w) J dy = 0 \quad \text{for each } t \in T_T,
\]

where \( K^1(\xi, w) \) is defined by (5.13) with \( (\xi, w) \) in place of \( (\eta, u) \).

By Proposition 5.3 there exists a function \( Q^1 \in C^0(T_T, H^3) \cap L^\infty_T H^4 \) such that \( \nabla_B Q^1 \in L^2_T H^4 \) and

\[
\begin{align*}
-\text{div}_B (\nabla_B Q^1/\bar{\rho}) &= 0 \quad \text{in } \Omega, \\
\nabla_B Q^1/\bar{\rho} \cdot \bar{n} &= -g e_2 \cdot \bar{n} \quad \text{on } \partial \Omega.
\end{align*}
\]
In view of Propositions 5.1–5.3 and the facts above, we easily see that there exist a solution sequence \( \{(\eta^n, u^n, Q^n)\}_{n=1}^{\infty} \) defined on \( I_T \) and the solutions \( (\eta^n, u^n, Q^n) \) enjoy the following properties:

1. \( (\eta^1, u^1) = 0 \), and \( Q^1 \) satisfies \( \nabla_B Q^1|_{\partial \Omega} \cdot \bar{n} = -g\mathbf{e}_2 \cdot \bar{n} \).
2. \( (\eta^n, u^n, Q^n) \in \mathcal{S}_T \) for \( n \geq 1 \).
3. For \( n \geq 2 \), \( (\eta^n, u^n, Q^n) \) satisfies the following relations:

\[
\begin{cases}
\eta^n_t - \kappa \partial^2_t \eta^n = u^{n-1}, \\
u^n_t + au^n = (\lambda m^2 \partial^2_t \eta^{n-1} - \nabla_B Q^{n-1})/\bar{\rho} - g\mathbf{e}_2 =: K^2, \\
-\text{div}_B (\nabla_B Q^n/\bar{\rho}) = K^1(\eta^n, u^n), \\
(\eta^n, u^n)|_{t=0} = (\eta^0, u^0), \\
(\eta^n, u^n)|_{\partial \Omega \setminus \Gamma^*} = 0, \quad \nabla_B Q^n|_{\partial \Omega} \cdot \bar{n} = -g\bar{\rho}\mathbf{e}_2 \cdot \bar{n},
\end{cases}
\]

where \( K^1(\eta^n, u^n) \) is defined by (5.13) with \( (\eta^n, u^n) \) in place of \( (\eta, u) \), \( K^{1,n} \in C^0(\bar{T}_t, H^1) \cap L^\infty_T H^2 \cap L^2_T H^3 \), \( K^{2,n} \in L^2_T H^4 \), and

\[ \int K^{1,n} J \, dy = 0 \text{ for each } t \in \bar{T}_t. \]

We further define \( (\bar{\eta}^n, \bar{u}^n, \bar{Q}^n) := (\eta^n - \eta^{n-1}, u^n - u^{n-1}, Q^n - Q^{n-1}) \) for \( n \geq 3 \). Then

\[
\begin{cases}
\bar{\eta}^n_t - \kappa \partial^2_t \bar{\eta}^n = \bar{u}^{n-1}, \\
\bar{u}^n_t + a\bar{u}^n = (\lambda m^2 \partial^2_t \eta^{n-1} - \nabla_B \bar{Q}^{n-1})/\bar{\rho}, \\
-\text{div}_B (\nabla_B \bar{Q}^n/\bar{\rho}) = \mathcal{J}_t \text{div}_B \bar{u}^n/\mathcal{J} - \lambda m^2 \text{div}_B (\partial^2_t \eta^n/\bar{\rho}), \\
(\bar{\eta}^n, \bar{u}^n)|_{t=0} = (0, 0), \\
(\bar{\eta}^n, \bar{u}^n)|_{\partial \Omega \setminus \Gamma^*} = 0, \quad \nabla_B \bar{Q}^n|_{\partial \Omega} \cdot \bar{n} = 0.
\end{cases}
\]

Thanks to Propositions 5.1–5.3 (5.8) and (5.9), we easily estimate that, for \( n \geq 3 \),

\[
\begin{align*}
\|\bar{\eta}^n\|_{C^0(\bar{T}_t, H^4)} + \|\partial_t \bar{\eta}^n\|_{L^\infty_T H^4} + \|\partial^2_t \bar{\eta}^n\|_{L^2_T H^4} & \lesssim_T (1 + \sqrt{T})\|\bar{u}^{n-1}\|_{L^2_T H^4} \lesssim_T T(1 + \sqrt{T})\|\bar{u}^{n-1}\|_{C^0(\bar{T}_t, H^4)}, \\
\|\bar{\eta}^n\|_{C^0(\bar{T}_t, H^2)} & \lesssim_T \sqrt{T}\|\nabla_B \bar{Q}^{n-1}, \partial^2_t \eta^{n-1}\|_{L^2_T H^4}, \\
\|\bar{Q}^n\|_{L^\infty_T H^4} & \lesssim P(\|\bar{\xi}\|_{H^1})\left(\|\bar{u}^n\|_{C^0(\bar{T}_t, H^4)} + \|\partial_t \bar{\eta}^n\|_{L^2_T H^4}\right), \\
\|\nabla_B \bar{Q}^n\|_{L^2_T H^4} & \lesssim P(\|\bar{\xi}\|_{H^1})\left(\sqrt{T}\|\bar{u}^n\|_{C^0(\bar{T}_t, H^4)} + \|\partial^2_t \bar{\eta}^n\|_{L^2_T H^4}\right).
\end{align*}
\]

We immediately see from the above four estimates that there exists a sufficiently small \( T_1 \) (the smallness depends on \( \kappa, g, \lambda, m, \bar{\rho}, \Omega \) and the norm \( \|\bar{\xi}\|_{H^4} \)) such that, for \( T = \min\{T_1, \alpha\} \),

\[ \{(\eta^n, \partial_t \eta^n, \partial^2_t \eta^n, u^n, Q^n, \nabla_B Q^n)\}_{n=1}^{\infty} \]

is a Cauchy sequence in \( C^0(\bar{T}_t, H^4) \times L^\infty_T H^4 \times L^2_T H^4 \times C^0(\bar{T}_t, H^4) \times L^\infty_T H^4 \times L^2_T H^4 \). Thus we can get one limit function \( (\eta, u, Q) \), which is the unique local solution of the initial-boundary value
problem (5.12) and also the unique local solution of the linear \( \kappa \)-approximate problem (5.2). In addition, it is easy to see that \((\eta, u, Q) \in S_T\) by further using Propositions 5.1 5.3 and trace theorem.

Noting that the local time \( T_1 \) is independent of the initial data and the local solution constructed above satisfies \((\eta, u)_{|t=0} \in H^{1,4}_0 \times H^{1,4}_0\), thus, if \( T < \alpha \), we can further extend the local solution to be a global solution defined on \( T_0 \) by finite steps; moreover the obtained global solution is the unique solution of of the linearized \( \kappa \)-approximate problem (5.2) and belongs to \( S_\alpha \). This completes the proof of Proposition 5.4. \( \Box \)

5.2. Solvability of linearized problem (5.3)

To investigate the solvability of linearized problem (5.3), we shall first derive \( \kappa \)-independent estimates of the solutions of the linear problem (5.2).

Lemma 5.2. Under the assumptions of Proposition 5.4 we further assume that \( \kappa \in (0, 1] \), \( \|\eta\|_4 \leq \delta \in \mathbb{R}^+ \) and \( \varsigma \) satisfies

\[
\partial_t \varsigma \in L^\infty_\alpha H^4, \quad \varsigma|_{t=0} = \eta^0.
\]

Then there exist polynomials \( \hat{P}(\|\varsigma\|_{L^\infty\alpha H^4}), P(\|\varsigma, \partial_\varsigma\|_{L^\infty\alpha H^4}), P(I^0, \|\varsigma, \partial_\varsigma\|_{L^\infty\alpha H^4}) \) and positive constants \( c, c_5 \geq 1, \delta_3 \) (may depending on \( g, a, \lambda, \mu, \rho \) and \( \Omega \)), such that, for any \( \delta \leq \delta_3 \), the local solution \((\eta^\kappa, u^\kappa, Q^\kappa)\) provided by Proposition 5.4 enjoys the following estimates:

\[
\sup_{t \in I_{T_1}} \mathcal{E}^\kappa(t) \leq 2c_5(I^0 + T\hat{P}(\|\varsigma\|_{L^\infty\alpha H^4})),
\]

\[
\sup_{t \in I_{T_1}} \|\varphi^\kappa(t) + \kappa\|\partial^2 \eta^\kappa\|_{L^2_\alpha H^4} \leq 2c_5I^0 + 1,
\]

\[
\|Q^\kappa\|_{L^\infty\alpha H^4} + \|u^\kappa\|_{L^\infty\alpha H^3} \lesssim P(I^0, \|\varsigma, \partial_\varsigma\|_{L^\infty\alpha H^4}),
\]

where \( \mathcal{E}^\kappa(t) := \|\eta^\kappa, \partial_\eta^\kappa, u^\kappa(t)\|_4^2 \), \( I^0 \) is defined by (1.21), and

\[
T_1 := \min\{1/3c_5P(\|\varsigma, \partial_\varsigma\|_{L^\infty\alpha H^4}), \alpha, 1\}.
\]

Moreover, for \((\eta^\kappa, u^\kappa)\) restricted in \( I_{T_1} \),

\[
\nabla^4 \partial_\eta^\kappa \text{ and } \nabla^4 u^\kappa \text{ are uniformly continuous in } H^{-1}.
\]

If additionally \( \varsigma_0 \in L^\infty_\alpha H^2 \), then

\[
\|Q^\kappa\|_{L^\infty\alpha H^3} \lesssim P(I^0, \|\varsigma, \partial_\varsigma\|_{L^\infty\alpha H^4}, \|\varsigma_0\|_{L^\infty\alpha H^2}).
\]

Proof. From now on we denote \((\eta^\kappa, u^\kappa, Q^\kappa)\) by \((\eta, u, Q)\), and let \( T \leq \min\{\alpha, 1\} \) and \( \|\eta^0\|_4 \leq \delta \in (0, 1] \).

Next we establish the desired uniform estimates for \((\eta, u, Q)\) by eight steps.

1. Estimate of \( \eta \).

Recalling (5.12), we immediately get

\[
\sup_{t \in I_T} \|\eta(t)\|_4^2 + \kappa^2\|\partial^2 \eta\|_{L^2_\alpha H^4} \lesssim \|\eta^0\|_{L^4}^2 + \|u\|_{L^4_\alpha H^4} \lesssim I^0 + T \sup_{t \in I_T} \mathcal{E}^\kappa(t).
\]

2. Estimate of \( Q \).
Noting that $Q$ satisfies

$$\begin{cases}
-\text{div}_B (\nabla_B Q/\bar{\rho}) = K^1 \quad \text{in } \Omega, \\
\nabla_B Q/\bar{\rho} \cdot \bar{n} = -ge_2 \cdot \bar{n} \quad \text{on } \partial \Omega,
\end{cases} \tag{5.51}$$

where $K^1$ is defined by (5.13). By (5.8), (5.9), (5.31) and (5.50), we have

$$\|Q\|_{L^\infty_t H^4} \lesssim P(\|\zeta\|_{L^\infty_t H^4}) \left(\|g\|_3 + \|K^1\|_{L^\infty_t H^2}\right) \lesssim P \left(\|\zeta, \zeta_t\|_{L^\infty_t H^4}\right) \left(1 + \sqrt{\mathcal{E}(t)}\right). \tag{5.52}$$

(3) $L^2$-norm energy estimate of $u$.

Let $q_\zeta = Q - \bar{P} (\zeta_2 + y_2)$, and $G_\zeta := \bar{\rho} (\zeta_2 (y,t) + y_2) - \bar{\rho} (y_2)$. Then (5.2) can be rewritten as follows:

$$\bar{\rho} u_t + \nabla_B g_\zeta + a \bar{\rho} u = \lambda m^2 \partial^2_1 \eta + g G_\zeta e_2. \tag{5.53}$$

Multiplying (5.53) with $J u$ in $L^2$ then yields that

$$\frac{1}{2} \frac{d}{dt} \int \bar{\rho} |u|^2 J dy + a \int \bar{\rho} |u|^2 J dy = \frac{1}{2} \int \bar{\rho} |u|^2 J dy + \lambda m^2 \int \partial^2_1 \eta \cdot u J dy + \int (g G_\zeta - \nabla_B g_\zeta) \cdot u J dy. \tag{5.54}$$

Using the boundary-value condition of $(\zeta, u)$, the integration by parts, (5.2), (5.8) and the integration by parts, we get

$$\int (g G_\zeta - \nabla_B g_\zeta) \cdot u J dy = g \int G_\zeta u_2 J dy \lesssim \bar{P} (\|\zeta_3\|) \|u_2\|_0. \tag{5.55}$$

In addition, making use of (5.2), (5.8), (5.9) and the integration by parts, we get

$$\begin{align*}
\lambda m^2 \int \partial^2_1 \eta \cdot u J dy &= -\lambda m^2 \frac{d}{dt} \int |\partial_1 \eta|^2 J dy + \lambda m^2 \int (|\partial_1 \eta|^2 J_t/2 - \partial_1 \eta \cdot \eta \partial_1 J - \kappa |\partial^2_1 \eta|^2 J) dy \\
&\leq -\lambda m^2 \frac{d}{dt} \int |\partial_1 \eta|^2 J dy - \kappa \lambda m^2 \int |\partial^2_1 \eta|^2 J dy + c P(\|\zeta, \zeta_t\|_3) \|\eta\|_2^2. \tag{5.56}
\end{align*}$$

Noting that $1/4 \leq J \leq 1$, thus, plugging (5.55), (5.56) into (5.54) and then integrating the resulting inequality over $(0, T)$, we immediately get, for any $t \in I_T$,

$$\| (u, \partial_1 \eta) (t) \|_0^2 + \kappa \int_0^t \| \partial^2_1 \eta (\tau) \|_0^2 d \tau \lesssim I^0 + T \bar{P} (\|\zeta\|_{L^\infty_t H^3}) + T P \left(\|\zeta, \zeta_t\|_{L^\infty_t H^3}\right) \sup_{t \in I_T} \| (\eta, u) (t) \|_2^2. \tag{5.57}$$

(4) Curl estimate of $(\eta, u)$.

Applying curl$_B$ to (5.2) yields that

$$\begin{align*}
\partial_t \text{curl}_B (\bar{\rho} u) + a \text{curl}_B (\bar{\rho} u) &= \lambda m^2 \text{curl}_B (\bar{\rho}^{-1} \partial^2_1 (\bar{\rho} \eta)) + \text{curl}_B (\bar{\rho} u) - g \rho B_{12}. \tag{5.58}
\end{align*}$$

Let the multiindex $\alpha$ satisfy $|\alpha| \leq 2$. Applying $\partial^\alpha$ to (5.58) yields

$$\begin{align*}
\partial_t \partial^\alpha \text{curl}_B (\bar{\rho} u) + a \partial^\alpha \text{curl}_B (\bar{\rho} u) &= \lambda m^2 \bar{\rho}^{-1} \partial_1 \partial^\alpha \text{curl}_B (\bar{\rho} \partial_1 \eta) + K_3^\alpha + K_4^\alpha, \tag{5.59}
\end{align*}$$

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where we have defined that
\[K_3^α := \lambda m^2 \partial_1 \left( [\partial^α \text{curl}_B, \tilde{\rho}] (\tilde{\rho} \partial_1 \eta) \right) - \partial^α (g\tilde{B}_{12}) ;\]
\[K_4^α := \partial^α \text{curl}_B (\tilde{\rho} u) - \lambda m^2 \partial^α \text{curl}_B (\partial_1 \eta) .\]

Multiplying (5.59) by \(\partial^α \text{curl}_B (\tilde{\rho} u)\) in \(L^2\), we obtain, for a.e. \(t \in I_T\),
\[
\frac{1}{2} \frac{d}{dt} \int |\partial^α \text{curl}_B (\tilde{\rho} u)|^2 \, dy + a \int |\partial^α \text{curl}_B (\tilde{\rho} u)|^2 \, dy
- \lambda m^2 \int \tilde{\rho}^{-1} \partial_1 \partial^α \text{curl}_B (\tilde{\rho} \partial_1 \eta) \partial^α \text{curl}_B (\tilde{\rho} u) \, dy
= \int K_3^α \partial^α \text{curl}_B (\tilde{\rho} u) \, dy + \int K_4^α \partial^α \text{curl}_B (\tilde{\rho} u) \, dy =: I_6 + I_7. \tag{5.60}
\]

Using the integral by parts and (5.2) again, we obtain
\[
- \lambda m^2 \int \tilde{\rho}^{-1} \partial_1 \partial^α \text{curl}_B (\tilde{\rho} \partial_1 \eta) \partial^α \text{curl}_B (\tilde{\rho} u) \, dy
= \frac{\lambda}{2} \frac{d}{dt} \int \tilde{\rho}^{-1} |m \partial^α \text{curl}_B (\tilde{\rho} \partial_1 \eta)|^2 \, dy + \kappa \lambda m^2 \int \tilde{\rho}^{-1} |\partial^α \text{curl} (\tilde{\rho} \tilde{\partial} \eta)|^2 \, dy + I_8 + I_9, \tag{5.61}
\]
where we have defined that
\[I_8 := \lambda m^2 \int \tilde{\rho}^{-1} \partial^α \text{curl}_B (\tilde{\rho} \partial_1 \eta) \partial^α (\text{curl}_B (\tilde{\rho} u) - \text{curl}_B (\tilde{\rho} \partial_1 \eta)) \, dy ,
I_9 := \kappa \lambda m^2 \int \left( \tilde{\rho}^{-1} \partial^α \text{curl}_B (\tilde{\rho} \partial_1 \eta) \partial^α \text{curl}_B (\tilde{\rho} \tilde{\partial} \eta) + |\partial^α \text{curl}_B (\tilde{\rho} \tilde{\partial} \eta)|^2 \right.
+ \left. 2 \partial^α \text{curl}_B (\tilde{\rho} \tilde{\partial} \eta) \partial^α \text{curl} (\tilde{\rho} \tilde{\partial} \eta) \right) dy.
\]

Thanks to the estimates (5.7)–(5.10), we can estimate that
\[I_6 + I_7 + I_8 \lesssim \hat{P}(\|\varsigma\|_4) + P(\|\varsigma_\perp, \partial_1 \varsigma\|_4) \|\varsigma_\perp, \partial_1 \eta\|_2^2 ,
I_9 \lesssim \kappa P(\|\varsigma_\perp, \partial_1 \varsigma\|_4) \|\eta\|_4 \|\text{curl} (\tilde{\rho} \eta)\|_{2,3} + \|\eta\|_4 \|	ext{curl}_B (\tilde{\rho} \eta)\|_{2,3} .\]

Noting that \(\varsigma|_{t=0} = \eta_0\), by Newton–Leibniz formula and (5.7) with \(t = 0\), we have
\[
\|B(t) - I\|_3 \leq \left\| B^0 - I \int_0^t B_\tau (\tau) \, d\tau \right\|_3 \lesssim \|\eta_0\|_4 + TP(\|\varsigma_\perp\|_{L^\infty_\perp H^3}) , \tag{5.62}
\]
where \(B^0 := B|_{t=0}\). Making use of the above three estimates, we deduce from (5.60)–(5.61) that
\[
\frac{d}{dt} \left( \left\| \text{curl}_B (\tilde{\rho} u) \right\|_3 \right)^2 + c_k \|\text{curl} (\tilde{\rho} \eta)\|_{2,3}^2
\lesssim \hat{P}(\|\varsigma\|_4) + P(\|\varsigma_\perp, \partial_1 \varsigma\|_4) \|\varsigma_\perp, \partial_1 \eta\|_2^2 + \kappa (\|\eta_0\|_4 + TP(\|\varsigma_\perp\|_{L^\infty_\perp H^3})) \|\eta\|_2^2 . \tag{5.63}
\]
In addition, similarly to (5.62), we have
\[
\|\text{curl} f (t)\|_3 \lesssim \left\| \left( \text{curl}_B (\tilde{\rho} u), \sqrt{\lambda \tilde{\rho}} m \text{curl}_B (\tilde{\rho} \partial_1 \eta) \right) \right\|_3^2 + c_k \|\text{curl} (\tilde{\rho} \eta)\|_{2,3}^2
\lesssim \|\eta_0\|_4 \|f(t)\|_4 + TP(\|\varsigma_\perp\|_4) \|f(t)\|_4 + \|\text{curl}_B (\tilde{\rho} u)\|_3 . \tag{5.64}
\]

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Integrating (5.63) over \((0, t)\), and then using Hölder’s inequality and the above estimate and (5.62), we conclude that

\[
\|\text{curl}(u, \partial_1 \eta)(t)\|_3^2 + \kappa \int_0^t \|\text{curl}(\tilde{\rho} \eta)(\tau)\|_2^2 d\tau \\
\lesssim T^0 + T \dot{P}(\|\eta\|_{L^\infty_T H^4}) + (\|\eta\|_4 + T P(\|\langle \xi, \partial_1 \xi, \xi \rangle\|_{L^\infty_T H^4})) \sup_{\tau \in [0, T]} \|(u, \partial_1 \eta)\|_4^2 \\
+ \kappa (\|\eta\|_4 + T P(\|\langle \xi, \xi \rangle\|_{L^\infty_T H^4})) \int_0^t \|\eta(\tau)\|_{2,4}^2 d\tau + \|(u, \partial_1 \eta)(t)\|_3^2.
\]

(5) Divergence estimate of \((\eta, u)\).

Similarly to (5.64), we have

\[
\|\text{div} f(t)\|_3 \lesssim \left\|\left(\text{div}_{B^0} f(t), \text{div}_B f(t), \text{div}_{f_0} B, \text{div}_B f(t)\right)\right\|_3.
\]

Noting that \(\text{div}_B u = 0\) for any \(t \in T\), taking \(f = u\) in the above estimate yields

\[
\|\text{div} u\|_3 \lesssim (\|\eta\|_4 + T P(\|\langle \xi, \xi \rangle\|_4))\|u\|_4.
\]

Applying \(\text{div}_B \partial_1\) to (5.2)_1 and then using (5.2)_3, we have

\[
\partial_1(\text{div}_B \partial_1 \eta) - \kappa \partial_1 \text{div}_B \partial_1^2 \eta = \text{div}_B \partial_1 \eta - \text{div}_{\partial_1 B} u - \kappa \text{div}_{\partial_1 B} \partial_1^2 \eta =: K_5.
\]

We define the mollification of \(f \in L^2_T L^2\) with respect to \(y_1\) as follows:

\[
S^1_\varepsilon(f) := \chi^\varepsilon * \eta.
\]

It is easy to check that

\[
S^1_\varepsilon(f) \to f \text{ in } L^2_T L^2.
\]

Then we can derive from (5.67) that

\[
\partial_1 S^1_\varepsilon(\partial^\alpha \text{div}_B \partial_1 \eta) - \kappa \partial_1 S^1_\varepsilon(\partial^\alpha \text{div}_B \partial_1^2 \eta) = S^1_\varepsilon(\partial^\alpha K_5),
\]

where the multiindex \(\alpha\) satisfies \(|\alpha| \leq 3\).

For \(\phi \in C^\infty_0(I_T)\), we multiply the above identity by \(S^1_\varepsilon(\partial^\alpha \text{div}_B \partial_1 \eta)\phi\) in \(L^2(\Omega_T)\) to get that

\[
\int_0^T \left(\frac{1}{2} \|S^1_\varepsilon(\partial^\alpha \text{div}_B \partial_1 \eta)\|_0^2 \phi_t + \kappa \|S^1_\varepsilon(\partial^\alpha \text{div}_B \partial_1^2 \eta)\|_0^2 \phi\right) d\tau \\
= \int_0^T \int (S^1_\varepsilon(\partial^\alpha K_5) S^1_\varepsilon(\partial^\alpha \text{div}_B \partial_1 \eta) - \kappa S^1_\varepsilon(\partial^\alpha \text{div}_B \partial_1^2 \eta) S^1_\varepsilon(\partial^\alpha \text{div}_{\partial_1 B} \partial_1 \eta)) dy \phi d\tau.
\]

Thanks to (5.69), we can take limits by \(\varepsilon \to 0\) in the above identity to get that

\[
\int_0^T \left(\frac{1}{2} \|\partial^\alpha \text{div}_B \partial_1 \eta\|_0^2 \phi_t + \kappa \|\partial^\alpha \text{div}_B \partial_1^2 \eta\|_0^2 \phi\right) d\tau \\
= \int_0^T \int (\partial^\alpha K_5 \partial^\alpha \text{div}_B \partial_1 \eta - \kappa \partial^\alpha \text{div}_B \partial_1^2 \eta \partial^\alpha \text{div}_{\partial_1 B} \partial_1 \eta) dy \phi d\tau.
\]
In particular, we have, for a.e. $t \in I_T$,
\[
\frac{1}{2} \frac{d}{dt} \| \partial^\alpha \text{div}_B \partial_1 \eta \|_0^2 + \kappa \| \partial^\alpha \text{div}_B \partial_1^2 \eta \|_2^2 = \int (\partial^\alpha K_5 \partial^\alpha \text{div}_B \partial_1 \eta - \kappa \partial^\alpha \text{div}_B \partial_1^2 \eta \partial^\alpha \text{div}_B \partial_1 \eta) dy.
\] (5.71)

Follow the argument of (5.65), we can derive from (5.71) that
\[
\| \partial_1 \eta(t) \|_3^2 + \kappa \int_0^t \| \partial_1 \eta(\tau) \|_2^2 d\tau 
\lesssim I^0 + (\| \eta \|_4^2 + TP(\| (\eta, \partial_1 \eta, \zeta) \|_L^\infty H^1)) \sup_{t \in T} \| (u, \partial_1 \eta) \|_4^2 
+ \kappa (\| \eta \|_4^2 + TP(\| (\eta, \partial_1 \eta, \zeta) \|_L^\infty H^1)) \int_0^t \| \eta(\tau) \|_2^2 d\tau 
+ \kappa P(\| \zeta \|_L^\infty H^{1,4}) \int_0^t \| \eta(\tau) \|_2 \| \partial_1 \eta(\tau) \| d\tau.
\] (5.72)

Consequently, we immediately deduce from (5.65), (5.66) and (5.72) that
\[
\| (\text{div} u, \text{curl} u, \text{div} \partial_1 \eta, \text{curl} \partial_1 \eta, \partial_1 \eta(t)) \|_3^2 + \kappa \int_0^t \| (\text{div} \eta, \text{curl} (\partial_1 \eta)) \|_2^2 d\tau 
\lesssim I^0 + \tilde{P}(\| \zeta \|_L^\infty H^1) + (\| \eta \|_4^2 + TP(\| (\eta, \partial_1 \eta, \zeta) \|_L^\infty H^1)) \sup_{t \in T} \| (u, \partial_1 \eta) \|_4^2 
+ \kappa (\| \eta \|_4^2 + TP(\| (\eta, \partial_1 \eta, \zeta) \|_L^\infty H^1)) \int_0^t \| \eta(\tau) \|_2^2 d\tau 
+ \kappa P(\| \zeta \|_L^\infty H^{1,4}) \int_0^t \| \eta(\tau) \|_2 \| \partial_1 \eta(\tau) \| d\tau + \| (u, \partial_1 \eta) \|_3^2
\] (5.73)

(6) Summing up the estimates $(\eta, u, Q)$.

Thanks to the estimates (5.50), (5.51), (5.73), the interpolation inequality (A.2) and the Hodge-type elliptic estimate (A.9), we have, for sufficiently small $\delta$,
\[
\sup_{t \in T} \mathcal{E}(t) + \kappa \| \partial_1^2 \eta \|_L^2 H^4 
\leq c_5 (I^0 + \tilde{P}(\| \zeta \|_L^\infty H^1)) 
+ c_5 TP(\| (\eta, \partial_1 \eta, \zeta) \|_L^\infty H^1) \left( \sup_{t \in T} \mathcal{E}(t) + \kappa \| \partial_1^2 \eta \|_L^2 H^4 \right)
\] (5.74)

and
\[
\sup_{t \in T} \mathcal{E}(t) + \kappa \| \partial_1^2 \eta \|_L^2 H^4 
\leq c_5 I^0 + c_5 TP(\| (\eta, \partial_1 \eta, \zeta) \|_L^\infty H^1) \left( 1 + \sup_{t \in T} \mathcal{E}(t) + \kappa \| \partial_1^2 \eta \|_L^2 H^4 \right).
\] (5.75)

It should be noted that the two polynomials above are same.

Now we use $c_5$ and $P$ in (5.75) to define $T_1$ by (5.47), and thus get (5.44), resp. (5.45) from (5.74), resp. (5.75) by taking $T = T_1$. Finally, making use of (5.2), satisfied by $(\eta, u, Q)$, (5.45) and (5.52), we easily get (5.46).

(7) Let the multiindex $\beta$ satisfy $|\beta| = 4$. Obviously, there exists $i$ such that $1 \leq i \leq 4$ and $\beta_i \neq 0$. Let $\beta^-$ satisfy $\beta_i^- = \beta_i - 1$ and $\beta^-_j = \beta_j$ for $j \neq i$, and $\beta^+$ satisfy $\beta_i^+ = 1$ and $\beta^+_j = 0$. 

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for \(j \neq i\). Similarly to (5.2), we can deduce from (5.2) and (5.2) that, for any \(\varphi \in H^1_0\) and for any \(s, t \in I_T\),

\[
\int \partial^\alpha \partial_t (\eta(t) - \eta(s)) \cdot \varphi dy = -\int_s^t \int \partial^\beta (u + \kappa \partial^2_1 \eta) \cdot \partial_\alpha \varphi dy d\tau, \\
\int \partial^\beta (\rho u(t) - \rho u(s)) \cdot \varphi dy \\
= \int_s^t \int (\partial^\beta \nabla B Q \cdot \partial^\beta \varphi - \lambda m^2 \partial^\beta \partial_1 \eta \cdot \partial_1 \varphi - \partial^\beta (g \rho e_2 + a \rho u) \cdot \varphi) dy d\tau. 
\]  

(5.77)

Making use of the uniform estimates (5.45) and (5.46), we easily deduce the assertion in (5.48) from the two identities above.

(8) If additionally \(\zeta_u \in L^2_0 H^2\), we can apply the second conclusion in Proposition 5.3 to (5.51). Then, by further using the estimates (5.11), (5.33), (5.45) and (5.46), we can easily get (5.49). This completes the proof.

Thanks to Lemma 5.2, we establish the unique local solvability of the \(\kappa\)-approximate problem (5.3) by a compactness argument.

**Proposition 5.5.** Let the assumptions of Lemma 5.2 be satisfied, \(\delta_0, T_1\) be provided by Lemma 5.2 and \(\zeta_u \in L^2_0 H^2\), then, for any \(\delta \leq \delta_3\), the linearized problem (5.3) defined on \(\Omega_{T_1}\) admits a unique solution \((\eta^L, u^L, Q^L) \in \mathcal{C}^0 (\overline{I_{T_1}}, H^4_{s, 1}) \times \mathcal{I}_{I_{T_1}}^1 \times \mathcal{I}_{I_{T_1}}^4\); moreover the solution satisfies

\[
\sup_{t \in I_{T_1}} \| \eta(t) \|_4 \leq \| \eta^0 \|_4 + \sqrt{T (2 c_5 I^0 + 1)}, \\
\sup_{t \in I_{T_1}} \mathcal{E}^L(t) \leq 2 c_5 (I^0 + T \hat{P} (\| \zeta \|_{L^\infty} H^4)), \\
\sup_{t \in I_{T_1}} \mathcal{E}^L(t) \leq 2 c_5 I^0 + 1, \\
\| Q^L \|_{L^2_{T_1} H^4} + \| u^L \|_{L^\infty_{T_1} H^3} \lesssim P \left( I^0, \| (\zeta, \dot{\zeta}) \|_{L^\infty_{T_1}} H^4 \right), \\
\| Q^L \|_{H^4_{s, 1}} \lesssim P \left( I^0, \| (\zeta, \dot{\zeta}) \|_{L^\infty_{T_1}} H^4, \| \dot{\zeta}_u \|_{L^\infty_{T_1} H^2} \right),
\]  

(5.78)

where \(\mathcal{E}^L(t) := \sup_{t \in I_{T_1}} \| (\eta^L, \partial_1 \eta^L, u^L)(t) \|_4^2\).

**Proof.** Let \((\eta^\kappa, u^\kappa, Q^\kappa) \in \mathcal{S}_T\) be the solution of the linearized problem (5.2) stated as in Lemma 5.2. Thanks to the \(\kappa\)-independent estimates (5.44)–(5.46), (5.48) and (5.49), we can easily follow the compactness argument as in the proof of Proposition 5.1 to obtain a limit function \((\eta^L, u^L, Q^L)\), which is the solution of the linearized problem (5.3), satisfies (5.79)–(5.82) and belongs to \(\mathcal{C}^0 (\overline{I_{T_1}}, H^4_{s, 1}) \times \mathcal{I}_{I_{T_1}}^1 \times \mathcal{I}_{I_{T_1}}^4\).

The estimate (5.78) is obvious by (5.3), satisfied by \((\eta, u)\) and (5.80). In addition, the uniqueness can be easily verified by a standard energy method. The proof of Proposition 5.5 is complete.

5.3. Proof of Proposition 2.2

Let \((\eta^0, u^0)\) satisfy the assumptions in Proposition 2.2, \(\| \eta^0 \|_4 \leq \delta \leq \max \{ \delta_3, \epsilon \} / 2\), and

\[
T_2 := \min \{ 1/3 c_5 P(c_4^2 (b + \delta_3)^2), \delta^2 / (2 c_5 (b + \delta_3) + 1), 1/4 c_5 \} < 1,
\]

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where the positive constant $c_5 \geq 1$ and the polynomial are provided by (5.47) and $\delta_3$ is the constant in Lemma 5.2 with $\alpha = T_2$. By Lemma 5.2, Proposition 5.4 and Lemma A.8 for any $T \leq T_2$, we can construct a solution sequence

$$\{(\eta^n, u^n, Q^n)\}_{n=1}^{\infty} \subset C^0(\overline{T_T}, H^{1,4}_s) \times \Omega_T^1 \times \Omega_T^1$$

such that

1. $$(\eta^1, u^1, Q^1) = (\eta^0, u^0, 0).$$
2. $$(\eta^{n+1}, u^{n+1}, Q^{n+1})$$ satisfies

$$\begin{cases}
\eta_t^{n+1} = u^{n+1}, \\
\rho u_t^{n+1} + \nabla A\eta^{n+1} + a\rho u^{n+1} = \lambda n^2 \partial_1^2 \eta^{n+1} - g \rho e_2, \\
\text{div} A\eta^{n+1} = 0, \\
(\eta^{n+1}, u^{n+1})_{t=0} = (\eta^0, u^0), \\
(\eta^{n+1}, u^{n+1}) = -g e_2 \rho \cdot \bar{n} \text{ on } \partial \Omega
\end{cases} \quad (5.83)$$

for any $n \geq 1$, where $A^n = (\nabla (\eta^n + I))^T$.
3. For $n \geq 2$, $(\eta^n, u^n, Q^n)$ enjoys the following estimates:

$$\sup_{t \in I_T} \|\eta^n(t)\|_4 \leq \|\eta^n\|_4 + \sqrt{T(2c_5 T_0 + 1)} \leq 2\delta \leq \min\{\delta_3, t\} \leq 1, \quad (5.84)$$

$$\sup_{t \in I_T} \mathcal{E}^n(t) \leq c_4^2 T^0, \quad (5.85)$$

$$\sup_{t \in I_T} \mathcal{E}^n(t) + \|Q^n\|_{L^\infty(\mathbb{R}^4)} + \|\partial_t(u^n, Q^n)\|_{L^\infty(\mathbb{R}^4)} \lesssim P(T^0), \quad (5.86)$$

where $\mathcal{E}^n(t) := \|((\eta^n, \partial_t \eta^n, u^n)(t))\|^2_3$.

Similarly to (5.8)–(5.9), by using (5.84) and (5.86), we have, for any $n \geq 1$,

$$\sup_{t \in I_T} \|(A^n, (J^n)^{-1}, A_t^n, J^n)\|_3 \lesssim P(T^0), \quad (5.87)$$

where $J^n := \det(\nabla \eta^n + I)$. In addition, similarly to (5.76) and (5.77), by a regularity method, we can easily derive from (5.83)1 and (5.83)2 that, for any $\varphi \in H^1_0$, $(\eta^n, u^n)$ satisfies

$$\int \partial^3 \partial_1(\eta^n(t) - \eta^n(s)) \cdot \varphi dy = -\int_s^t \int \partial^3 u^n \cdot \partial_1 \varphi dy dt,$$

$$\int \partial^3(\rho u^n(t) - \rho u^n(s)) \cdot \varphi dy$$

$$= \int_s^t \int (\partial^3 - \nabla A_{n-1} Q^n \cdot \partial^3 + \varphi - \lambda n^2 \partial^3 \partial_1 \eta^n \cdot \partial_1 \varphi - \partial^3 (g \rho e_2 + a \rho u^n) \cdot \varphi) dy dt.$$

Thus we further derive from the above two identities, (5.86) and (5.87) that

$$\nabla^3 \partial_1 \eta^n \text{ and } \nabla^3 u^n \text{ are uniformly continuous in } H^{-1}, \quad (5.88)$$

where $n \geq 2$. Next we further prove that $\{(\eta^n, u^n, Q^n)\}_{n=1}^{\infty}$ is a Cauchy sequence.

From now on, we always assume $n \geq 2$. We define that

$$\eta^{n+1} := \eta^{n+1} - \eta^n, \quad u^{n+1} := u^{n+1} - u^n, \quad Q^{n+1} := Q^{n+1} - Q^n \text{ and } A^n := A^n - A^{n-1}.$$
Then it follows from (5.83) that
\[
\begin{aligned}
& \tilde{\eta}_t^{n+1} = \tilde{u}^{n+1}, \\
& \tilde{\rho} \tilde{u}_t^{n+1} + \nabla_{A^n} \tilde{Q}^{n+1} + a \tilde{u}_u^{n+1} = \lambda m^2 \partial^2_{tt} \tilde{\eta}^{n+1} - \nabla_{A^n} Q^n, \\
& \text{div}_{A^n} \tilde{u}^{n+1} = -\text{div}_{A^n} u^n, \\
& (\tilde{\eta}^{n+1}, \tilde{u}^{n+1})|_{t=0} = (0, 0), \\
& (\tilde{\eta}^{n+1}, \tilde{u}^{n+1})|_{\partial \Omega} = 0.
\end{aligned}
\tag{5.89}
\]

It follows from (5.89) and (5.90) that, for \(0 \leq i \leq 4\),
\[
\sup_{t \in I_T} \|\tilde{\eta}^{n+1}(t)\|_i + \sqrt{n} \|\partial_t \tilde{\eta}^{n+1}\|_{L^2_{\infty} H^i} \lesssim T \|\tilde{u}^{n+1}\|_{L^\infty_{\infty} H^i}. \tag{5.91}
\]

By (5.86), (5.87), (5.89), we can estimate that, for \(0 \leq i \leq 3\),
\[
\begin{aligned}
& \sup_{t \in I_T} \|\tilde{A}^n\|_i \lesssim TP(I^0) \|\tilde{u}^n\|_{L^\infty_{\infty} H^{i+1}}, \\
& \sup_{t \in I_T} \|\tilde{A}^n_t\|_i \lesssim P(I^0) \|\tilde{u}^n\|_{L^\infty_{\infty} H^{i+1}}.
\end{aligned} \tag{5.92}
\]

Let \(\zeta^n = \eta^n + y\). \((\zeta^n)^{-1}\) denotes the inverse function of \(\zeta^n\) with respect to the variable \(y\). We define that
\[
K^5 := (\lambda m^2 \partial^2_{tt} \tilde{\eta}^{n+1} - \nabla_{A^n} Q^n) / \tilde{\rho}, \quad K^6 := K^5 - \tilde{u}_t^{n+1} - a \tilde{u}_u^{n+1}, \\
(\beta, \varrho, \tilde{K}^5, \tilde{K}^6) := (\tilde{Q}^{n+1}, \tilde{\rho}, K^5, K^6)|_{y=(\zeta^n)^{-1}(x,t)},
\]
then, by (5.89)_2,
\[
\begin{aligned}
\text{div} (\nabla \beta / \varrho) &= \text{div} \tilde{K}^6 \quad \text{in } \Omega, \\
(\nabla \beta / \varrho) \cdot \tilde{n} &= \tilde{K}^5 \cdot \tilde{n} \quad \text{on } \partial \Omega.
\end{aligned}
\]

Applying the elliptic estimate (A.22) to the above boundary-value problem and then making use of (A.47) and (A.48), we have
\[
\begin{aligned}
\|\tilde{Q}^{n+1}\|_2 &\lesssim P(\|\eta^n\|_3) \|\beta\|_2 \lesssim P(\|\eta^n\|_3) (\|\tilde{K}^5\|_1 + \|\text{div} \tilde{K}^6\|_1) \\
&\lesssim P(\|\eta^n\|_3) (\|K^5\|_1 + \|\text{div}_A K^6\|_0).
\end{aligned} \tag{5.93}
\]

Noting that
\[
\text{div}_{A^n} K^6 = \text{div}_A ((\lambda m^2 \partial^2_{tt} \tilde{\eta}^{n+1} - \nabla_{A^n} Q^n) / \tilde{\rho}) + \text{div}_{A^n} u^n + \text{div}_{A^n} \tilde{u}_u^{n+1} + \partial_t \text{div}_{A^n} u^n,
\]
thus, by using (5.86), (5.87), (5.91) and (5.92), we easily estimate that
\[
\|K^5\|_1 + \|\text{div}_A K^6\|_0 \lesssim P(I^0) \|\tilde{u}^n, \tilde{u}_u^{n+1}, \partial_1 \tilde{\eta}^{n+1}\|_{L^\infty_{\infty} H^2}.
\]

Putting the above estimate into (5.93) yields
\[
\|\tilde{Q}^{n+1}\|_{c^0(\Gamma_f, H^2)} \lesssim P(b) \|\tilde{u}^n, \tilde{u}_u^{n+1}, \partial_1 \tilde{\eta}^{n+1}\|_{L^\infty_{\infty} H^2}. \tag{5.94}
\]

Similarly to the derivation of (5.57), we have that
\[
\sup_{t \in I_T} \|(\tilde{u}^{n+1}, \partial_1 \tilde{\eta}^{n+1})(t)\|_0^2 \lesssim TP(I^0) \|\tilde{u}^n, \tilde{u}_u^{n+1}, \partial_1 \tilde{\eta}^{n+1}\|_{L^\infty_{\infty} H^2}^2. \tag{5.95}
\]
By (5.89)_1 – (5.89)_3, we have
\[
\begin{align*}
\partial_t \left( \text{div}_{A^n} \partial_t \bar{\eta}^{n+1} \right) &= \text{div}_{A^n} \partial_t \bar{\eta}^{n+1} - \partial_t \left( \text{div}_{A^n} u^n \right) - \text{div}_{\partial_t A^n} \bar{u}^{n+1}, \\
\partial_t \left( \text{curl}_{A^n} \left( \bar{\rho} \bar{u}^{n+1} \right) \right) + a \text{curl}_{A^n} \left( \bar{\rho} \bar{u}^{n+1} \right) &= \lambda m^2 (\bar{\rho}^{-1} \partial_t \left( \text{curl}_{A^n} \left( \bar{\rho} \bar{\eta}^{n+1} \right) \right) + \partial_t \left[ \text{curl}_{A^n}, \bar{\rho}^{-1} \right] (\bar{\rho} \partial_t \bar{\eta}^{n+1}) \\
&- \text{curl}_{\partial_t A^n} \partial_t \bar{\eta}^{n+1} + \text{curl}_{A^n} (\bar{\rho} \bar{u}^{n+1}) + \text{curl}_{A^n} (\nabla_{A^n} - 1) Q^n.
\end{align*}
\] (5.96)

Making use of (5.86), (5.87) and (5.91), we can follow the argument of (5.73) to derive from (5.89)_3, (5.96) and (5.97) that, for a.e. \( t \in I_T \),
\[
\begin{align*}
\left\| \left( \text{div}_{A^n} \bar{u}^{n+1}, \text{curl}_{A^n} \bar{u}^{n+1}, \text{div}_{\partial_t A^n} \bar{\eta}^{n+1}, \text{curl}_{\partial_t A^n} \bar{\eta}^{n+1} \right) \right\|_2^2 \\
\lesssim \left( \| \eta^0 \|_3 + TP(I^0) \right) \left( \| \bar{u}^n, \partial_t \bar{\eta}^n, \bar{u}^{n+1}, \partial_t \bar{\eta}^{n+1} \|_{L^\infty_T H^2} + \| \bar{u}^{n+1}, \partial_t \bar{\eta}^{n+1} \|_{L^1_T H^1} \right)^2. \tag{5.98}
\end{align*}
\]

Consequently, summarizing up the estimates (5.90), (5.95) and (5.98) and then using the interpolation inequality (A.2), Hodge-type elliptic estimate (A.9) and Young’s inequality, we obtain, for sufficiently small \( \delta \),
\[
\sup_{t \in I_T} \left\| \left( \bar{\eta}^{n+1}, \partial_t \bar{\eta}^{n+1}, \bar{u}^{n+1} \right) \right\|_2 \lesssim TP(b) \left( \| \bar{u}^n, \partial_t \bar{\eta}^n, \bar{u}^{n+1}, \partial_t \bar{\eta}^{n+1} \|_{L^\infty_T H^2} \right). \tag{5.99}
\]
In addition, by (5.86), (5.87) and (5.89)_2, we get that
\[
\| \bar{u}^{n+1} \|_{C^0(\mathcal{T}_T, H^1)} \lesssim P(b) \left( \| \bar{u}^n, \partial_t \bar{\eta}^{n+1} \|_{C^0(\mathcal{T}_T, H^2)} + \| \bar{Q}^{n+1} \|_{C^0(\mathcal{T}_T, H^2)} \right). \tag{5.100}
\]

By (5.94), (5.99) and (5.100), we immediately see that the sequence \( \{ (\eta^n, u^n, u^n_t, Q^n) \}_{n=1}^\infty \) is a Cauchy sequence in \( C^0(\mathcal{T}_T, H^{1.2} \times H^2 \times H^1 \times H^2) \) for sufficiently small \( T \in (0, T_2] \), where the smallness depends on \( b, g, a, \lambda, m, \bar{\rho} \) and \( \Omega \). Hence
\[
(\eta^n, u^n, u^n_t, Q^n) \to (\eta, u, u_t, Q) \text{ strongly in } C^0(\mathcal{T}_T, H^{1.2} \times H^2 \times H^1 \times H^2).
\]

Thanks to the strong convergence of \( (\eta^n, u^n, u^n_t, Q^n) \), we can take the limit in (5.83), and then get a local classical solution \( (\eta, u, Q) \) to the problem (5.1). Let \( q = Q - \bar{P}(\zeta_2) - \lambda |M|^2/2 \), then \( (\eta, u, q) \) is also a local classical solution to the transformed MRT problem (1.19) by (1.17) and (1.18). In addition, thanks to (5.83), (5.86) and (5.88), we easily follow the compactness argument as in the proof of Proposition 5.5 to further derive that
\[
(\eta, u, q) \in C^0(\mathcal{T}_T, H^{1.4}_s) \times \Omega^4_T \times \Omega^4_T,
\]
\[
\sup_{t \in I_T} \| \eta^n(t) \|_4^2 \leq 2 \delta \leq \iota, \tag{5.101}
\]
\[
\sup_{t \in I_T} \| (\eta, \partial_t \eta, u) \|_4^2 \leq c_4 \iota^0.
\]

In addition, it is easy to check that \( (\eta, u, q) \) is the unique solution of the transformed MRT problem (1.19) in \( C^0(\mathcal{T}_T, H^{1.4}_s) \times \Omega^4_T \times \Omega^4_T \) due to (5.101). This completes the proof of Proposition 2.2.

\textbf{Appendix A. Analytic tools}

This appendix is devoted to providing some mathematical results, which have been used in previous sections. It should be noted that \( \Omega \) appearing in what follows is still defined by (1.2) and we will also use the simplified notations appearing in Section 1.3. In addition, \( \Omega_T := \Omega \times I_T \) and \( a \lesssim b \) still denotes \( a \leq cb \), however the positive constant \( c \) depends on the parameters and domain in the lemma, in which \( c \) appears.
Lemma A.1. (1) Embedding inequality (see [4, 4.12 Theorem]): Let \( D \subset \mathbb{R}^2 \) be a domain satisfying the cone condition, then
\[
\|f\|_{C^0(\overline{D})} = \|f\|_{L^\infty(D)} \lesssim \|f\|_{H^2(D)}.
\] (A.1)

(2) Interpolation inequality in \( H^j \) (see [4, 5.2 Theorem]): Let \( D \) be a domain in \( \mathbb{R}^2 \) satisfying the cone condition, then, for any given \( 0 \leq j < i \),
\[
\|f\|_{H^j(D)} \lesssim \|f\|_{L^2(D)}^{(1-j)/i} \|f\|_{H^i(D)}^{j/i} \lesssim \varepsilon^{-j/(i-j)} \|f\|_{L^2(D)} + \varepsilon\|f\|_{H^i(D)},
\] (A.2)
where the two estimate constants in (A.2) are independent of \( \varepsilon \), the positive constant \( \varepsilon \) is arbitrary, and we have used Young’s inequality in the last inequality above.

(3) Product estimates (see Section 4.1 in [24]): Let \( D \) be a domain satisfying the cone condition, and the functions \( f, g \) are defined in \( D \). Then
\[
\|fg\|_{H^i(D)} \lesssim \begin{cases} \|f\|_{H^j(D)}\|g\|_{H^i(D)} & \text{for } i = 0; \\ \|f\|_{H^j(D)}\|g\|_{H^j(D)} & \text{for } 0 \leq i \leq 2. \end{cases}
\] (A.3)

Lemma A.2. Friedrich’s inequality (see [34, Lemma 1.42]): Let \( 1 \leq p < \infty \), \( n \geq 2 \) and \( D \subset \mathbb{R}^n \) be a bounded Lipchitz domain. Let a set \( \Gamma \subset \partial D \) be measurable with respect to the \( (n-1) \)-dimensional measure \( \mu := \text{meas}_{n-1} \) defined on \( \partial D \) and let \( \text{meas}_{n-1}(\Gamma) > 0 \). Then
\[
\|w\|_{W^{1,p}(D)} \lesssim \|\nabla w\|_{L^p(D)}
\] (A.4)
for any \( w \in W^{1,p}(D) \) with \( u|_{\Gamma} = 0 \) in the sense of trace.

Remark A.1. By Lemma A.2 we easily deduce that
\[
\|w\|_{W^{1,p}(0,a)} \lesssim \|w\|_{L^p(0,a)}
\]
for any \( w \in W^{1,p}(0,a) \) with \( w(0) = 0 \) or \( w(a) = 0 \). Thus we further have
\[
\|w\|_0 \lesssim \|\partial_2 w\|_0 \text{ for any } w \in H^1_0 := \{v \in H^1 \mid v|_{\partial \Omega} = 0\}. \] (A.5)

Lemma A.3. Poincaré inequality (see [34, Lemma 1.43]): Let \( 1 \leq p < \infty \), and \( D \) be a bounded Lipchitz domain in \( \mathbb{R}^n \) for \( n \geq 2 \) or a finite interval in \( \mathbb{R} \). Then for any \( w \in W^{1,p}(D) \),
\[
\|w\|_{L^p(D)} \lesssim \|\nabla w\|_{L^p(D)}^p + \left| \int_D w\mathrm{d}y \right|^p.
\] (A.6)

Remark A.2. By Poincaré inequality, we have, for any given \( i \geq 0 \),
\[
\|w\|_{1,i} \lesssim \|w\|_{2,i} \text{ for any } w \text{ satisfying } \partial_1 w, \partial_1^2 w \in H^i.
\] (A.7)

Remark A.3. By Poincaré inequality, we also have
\[
\|w\|_0 \lesssim \|\partial_1 w\|_0 \text{ for any } w \in H^1 \text{ satisfying } w(y_1, y_2) = -w(-y_1, y_2). \] (A.8)

Lemma A.4. Hodge-type elliptic estimates: If \( w \in H^i_\Gamma \) with \( i \geq 1 \), then
\[
\|\nabla w\|_{i-1} \lesssim \|(\text{div} w, \text{curl} w)\|_{i-1}.
\] (A.9)
Proof. By a regularity method, we can verify that, for \( i \leq j \leq i-1 \),
\[
\| \partial^j_i \nabla \partial w \|_0^2 = \| \partial^j_i \text{div} w \|_0^2 + \| \partial^j_i \text{curl} \partial w \|_0^2,
\]  
which yields (A.9) for \( i = 1 \).

Next we further consider the case \( i \geq 2 \). Since
\[
\Delta w = \nabla \text{div} w + \nabla^\perp \text{curl} w,
\]  
where \( \nabla^\perp := (-\partial_2, \partial_1)^T \), we derive from (A.10) and (A.11) that, for any \( 1 \leq l + k \leq i-1, 1 \leq k \),
\[
\| \partial^{k+1}_l \partial^i w \|_0 = \| \partial^{k-1}_l \partial^i (\Delta w - \partial^2_l w) \|_0 \\
\leq \| \partial^{k-1}_l \partial^i (\nabla \text{div} w, \nabla^\perp \text{curl} w) \|_0 + \| \partial^{k+1}_l \partial^i \partial^2 w \|_0,
\]  
which further yields that
\[
\| \partial^{k+1}_l \partial^i w \|_0 \lesssim \|(\text{div} w, \text{curl} w)\|_{i-1} + \| \partial^{k+1}_l \partial^i \partial^2 w \|_0.
\]  

By an induction method, we easily derive from (A.10) and (A.12) that
\[
\| \partial^{k+1}_l \partial^i w \|_0 \lesssim \|(\text{div} w, \text{curl} w)\|_{i-1},
\]  
which, together with (A.10), yields
\[
\| \nabla w \|_{i-1} \lesssim \|(\text{div} w, \text{curl} w)\|_{i-1}.
\]  

This completes the proof of Lemma [A.4].

Lemma A.5. Extension theorem: Let \( i \geq 0, h > 0, \delta = h/(i+1) \), and \( f \in H^i \), then there exists an extension operator \( E_i^h \) such that \( E_i^h: f \in H^i \rightarrow H^i(\mathbb{T} \times \mathbb{R}) \) such that
\[
E_i^h(f) = 0 \text{ for } y_2 < -\delta/2, \ h + \delta/2 < y_2, \quad \text{(A.13)}
\]
\[
E_i^h(f)|_{\Omega} = f \text{ and } \|E_i^h(f)\|_{H^i(\mathbb{T} \times \mathbb{R})} \lesssim \|f\|_i. \quad \text{(A.14)}
\]

Proof. Let
\[
\Omega_- := \mathbb{T} \times (-\infty, 0), \ \Omega_+ := \mathbb{T} \times (h, +\infty). \quad \text{(A.15)}
\]

Let \( \chi \in C^\infty_0[0, \delta] \) with \( \chi(y_2) = 1 \) for \( y_2 \in (0, \delta/2) \) and \( \chi(y_2) = 0 \) for \( y_2 < -\delta/2, \ h + \delta/2 < y_2 \),
\[
\tilde{f} = \begin{cases}
\chi(y_2) \sum_{j=1}^{i+1} \lambda_j f(y_1, -jy_2) & \text{for } y \in \Omega_-, \\
f & \text{for } y \in \Omega, \\
0 & \text{for } y \in \partial \Omega, \\
\chi(y_2 - h) \sum_{j=1}^{i+1} \lambda_j f(y_1, h - j(y_2 - h)) & \text{for } y \in \Omega_+, \end{cases}
\]
where \( \sum_{j=1}^{i+1} (-j)^k \lambda_j = 1 \) for \( 0 \leq k \leq i \).

By the definition of \( H^i \) and the facts, for \( i \geq 1 \),
\[
\nabla^{i-1} \tilde{f}|_{\Omega_-} = \nabla^{i-1} \tilde{f}|_{\Omega} \text{ on } \mathbb{T} \times \{0\}, \ \nabla^{i-1} \tilde{f}|_{\Omega_+} = \nabla^{i-1} \tilde{f}|_{\Omega} \text{ on } \mathbb{T} \times \{h\},
\]
in the sense of trace by a density argument [1 5.19 Theorem], we can easily check that \( \tilde{f} \) constructed by above belongs to \( H^i(\mathbb{T} \times \mathbb{R}) \) and satisfies (A.13) and (A.14). Consequently, Lemma [A.5] holds. This completes the proof.
Lemma A.6. Dual estimates: Let $\tau$ satisfying $|\tau| \in (0, 1)$, $\varphi, \psi \in H^1$, and $D^1_\tau \varphi = (\varphi(y_1 + \tau, y_2) - \varphi(y_1, y_2))/\tau$. Then

$$\left| \sum_{s=0}^{h} \int_0^{2\pi} (D^1_\tau \varphi \psi)_{y_2=s} dy_1 \right| \lesssim \|\varphi\|_1 \|\psi\|_1. \quad (A.16)$$

Proof. Let $f, \omega \in \mathcal{Y} := \{w \in H^1 \cap C^1(\Omega) | w(y_1, 0) = 0\}$. Defining the Fourier coefficient $f$ by $\tilde{f}(\xi, y_2) := \int_0^{2\pi} f(s, y_2)e^{-i\xi s} ds$, then, using Parseval’s relation on the Torus (see Proposition 3.1.16), Plancherel’s identity, Hölder’s inequality in $l^2$, the horizontal periodicity, and Newton–Leibnitz’s formula, we have, for any $y_2 \in (0, h)$,

$$\left| \int_0^{2\pi} D^1_\tau f \omega(y_1, y_2) dy_1 \right| = \frac{1}{2\pi} \sum_{\xi} \left| \tilde{D}^1_\tau \tilde{f} \tilde{\omega}(y_2) \right| = \frac{1}{2\pi} \sum_{\xi} (e^{i\xi \tau} - 1) \tilde{f} \tilde{\omega}(y_2)$$

$$\lesssim \sum_{\xi} |\xi| \left| \tilde{f} \right|_{L^2(0, \tau)} \left| \partial_2 \tilde{f} \right|_{L^2(0, \tau)} \left| \tilde{\omega} \right|_{L^2(0, \tau)} \left| \partial_2 \tilde{\omega} \right|_{L^2(0, \tau)}$$

$$\lesssim \left( \sum_{\xi} (\xi \left| \tilde{f} \right|_{L^2(0, \tau)})^2 \sum_{\xi} \left| \partial_2 \tilde{f} \right|_{L^2(0, \tau)}^2 \sum_{\xi} (\xi \left| \tilde{\omega} \right|_{L^2(0, \tau)})^2 \sum_{\xi} \left| \partial_2 \tilde{\omega} \right|_{L^2(0, \tau)}^2 \right)^{1/4}$$

$$\lesssim \|f\|_1 \|\omega\|_1. \quad (A.17)$$

Let the extension operator $\mathbb{E}^1_0$ be defined in Lemma A.5 with $i = 1$ and $\delta = h/2$. For simplicity, we denote $\mathbb{E}^1_0(\chi)$ by $\tilde{\chi}$, where $\chi$ represents $\varphi$ or $\psi$. Let $\varepsilon \in (0, 1)$, then we denote by $S^\varepsilon(\tilde{\chi})$ the mollification of $\tilde{f}$ with respect to the 2D variable $(y_1, y_2)$. Exploiting trace theorem and the properties of mollification, we have, for any $\tau \in (0, 1)$,

$$\|(S^\varepsilon(\tilde{\chi}) - \chi)(y_1 + \tau, y_2)\|_{H^{1/2}(0, 2\pi)} \lesssim \|S^\varepsilon(\tilde{\chi}) - \chi\|_1 \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (A.18)$$

Let $\sigma(y_2) \in C^\infty_0(0, h]$ satisfy $\sigma(h) = 1$, and $\tilde{\chi}^\varepsilon := S^\varepsilon(\tilde{\chi}) \sigma(y_2)$. Then $\chi^\varepsilon \in \mathcal{Y}$ by (A.17), we have

$$\left| \int_0^{2\pi} (D^1_\tau (\tilde{\varphi}^\varepsilon)|_{y_2=h} \tilde{\psi}^\varepsilon|_{y_2=h} dy_1 \right| \lesssim \|\tilde{\varphi}^\varepsilon\|_1 \|\tilde{\psi}^\varepsilon\|_1 \lesssim \|S^\varepsilon(\tilde{\varphi})\|_1 \|S^\varepsilon(\tilde{\psi})\|_1. \quad (A.19)$$

Thanks to (A.18), we easily derive from the above estimate that

$$\left| \int_0^{2\pi} (D^1_\tau \varphi \psi)_{y_2=h} dy_1 \right| \lesssim \|\varphi\|_1 \|\psi\|_1. \quad (A.19)$$

Similarly, we also have

$$\left| \int_0^{2\pi} (D^1_\tau \varphi \psi)_{y_2=0} dy_1 \right| \lesssim \|\varphi\|_1 \|\psi\|_1,$$

which, together with (A.19), yields (A.16). \qed

Lemma A.7. Elliptic estimates: Let $k \geq 2$. Assume $A$ is a $2 \times 2$ symmetric matrix function, each element $A_{ij}$ of which belongs to $W^{k, \infty}(\Omega)$, and positivity condition, i.e., there exists a positive
constant $\theta$ such that $(A\xi) \cdot \xi \geq \theta|\xi|$ for a.e. $y \in \Omega$ and all $\xi \in \mathbb{R}^2$. If $f^1 \in H^{k-2}$ and $f^2 \in H^{k-1}$ satisfy the following compatibility condition

$$
\int f^1 \text{d}y + \int_{\partial \Omega} f^2 \tilde{n}_2 \text{d}y_1 = 0,
$$

(A.20)

where $\tilde{n}$ denotes the outward unit normal vector on the boundary $\partial \Omega$, there exists a unique strong solution $p \in H^k$ to the Neumann boundary-value problem of elliptic equation:

$$
\begin{align*}
-\text{div} (A\nabla p) &= f^1 \quad \text{in } \Omega, \\
(A\nabla p) \cdot \tilde{n} &= f^2 \tilde{n}_2 \quad \text{on } \partial \Omega.
\end{align*}
$$

(A.21)

Moreover $p$ satisfies

$$
\|p\|_k \lesssim (1 + \|A\|_{W^{k-1,\infty}(\Omega)}) (\|f^1\|_{k-2} + \|f^2\|_{k-1}),
$$

(A.22)

**Proof.** Noting that $A$ is a symmetry matrix function, we define an inner-product of $H^1$ by

$$
(\varphi, \phi)_{H^1} := \int (A\nabla \varphi) \cdot \nabla \phi \text{d}y \text{ for } \varphi, \phi \in H^1,
$$

and then the corresponding norm by $\|\varphi\|_H := \sqrt{(\varphi, \varphi)_{H^1}}$. Obviously, by Poincaré inequality (A.6) and the positivity condition, we have

$$
\|\varphi\|_1 \lesssim \|\varphi\|_H \lesssim \|A\|_{L^\infty} \|\varphi\|_1.
$$

We define the functional

$$
F(\varphi) := \int f^1 \varphi \text{d}y + \int_{\partial \Omega} f^2 \tilde{n}_2 \varphi \text{d}y_1 \text{ for } \varphi \in H^1.
$$

Then it is easy to check that the functional $F$ is a bounded linear functional on $H^1$. By Riesz representation theorem, there exists a unique $p \in H^1$, such that

$$
(p, \varphi)_{H^1} = F(\varphi) \text{ for any } \varphi \in H^1
$$

(A.23)

and

$$
\|p\|_1 \lesssim \|p\|_H \lesssim \|f^1\|_{k-2} + \|f^2\|_{k-1}.
$$

(A.24)

For any given $\psi \in H^1$, we denote $\varphi = \psi - (\psi)_{\Omega}$. Then $\varphi \in H^1$. Putting it in (A.23), we obtain

$$
\int (A\nabla p) \cdot \nabla \psi \text{d}y = \int f^1 \psi \text{d}y + \int_{\partial \Omega} f^2 \tilde{n}_2 \psi \text{d}y_1 - (\psi)_{\Omega} \left( \int f^1 \text{d}y + \int_{\partial \Omega} f^2 \tilde{n}_2 \text{d}y_1 \right),
$$

which, together with the compatibility condition (A.20), yields

$$
\int (A\nabla p) \cdot \nabla \psi \text{d}y = \int f^1 \psi \text{d}y + \int_{\partial \Omega} f^2 \tilde{n}_2 \psi \text{d}y_1 \text{ for any } \psi \in H^1.
$$

(A.25)

Next we further improve the regularity of $p$. We assert that
\* for any $0 \leq l \leq k - 1$,

$$p \in H^{l+1}, \|p\|_{l+1} \lesssim (1 + \|A\|_{W^{k-1,\infty}(\Omega)}) (\|f^1\|_{k-2} + \|f^2\|_{k-1}), \quad (A.26)$$

\* for any $0 \leq l \leq k - 2$,

$$\int (A\nabla \partial_1^l p) \cdot \nabla \psi dy = \int (\partial_1^l f^1 \psi - [\partial_1^l, A] \nabla p) \cdot \nabla \psi dy + \int_{\partial \Omega} \partial_1^l f^2 \bar{n}_2 \psi dy_1 \text{ for any } \psi \in H^1, \quad (A.27)$$

Noting that (A.26)–(A.27) hold for $l = 0$ due to (A.24) and (A.25), thus, to obtain the above assertion, it suffices to prove that if (A.26)–(A.27) hold for $0 \leq l \leq k - 2$, then

\* (A.27) holds with $l + 1$ in place of $l$, if $l + 1 \leq k - 2$.

\* (A.26) also holds with $l + 1$ in place of $l$.

Next we prove these two facts.

(1) Let $D_1^\tau w = (w(y_1 + \tau, y_2) - w(y_1, y_2))/\tau$ with $|\tau| \in (0, 1)$. Since (A.27) hold for $0 \leq l \leq k - 2$, we can take $\psi = D_1^{-\tau} D_1^\tau \partial_1^l p$ in (A.27) to get

$$\int (A\nabla \partial_1^l p) \cdot \nabla (D_1^{-\tau} D_1^\tau \partial_1^l p) dy = \int (\partial_1^l f^1 D_1^{-\tau} D_1^\tau \partial_1^l p + [\partial_1^l, A] \nabla p) \cdot \nabla D_1^{-\tau} D_1^\tau \partial_1^l p dy + \int_{\partial \Omega} \partial_1^l f^2 D_1^{-\tau} D_1^\tau \partial_1^l p dy_1. \quad (A.28)$$

By the properties of difference quotient, we derive from (A.28) that

$$\int (A(y_1 + \tau)\nabla (D_1^{-\tau} D_1^\tau \partial_1^l p)) \cdot \nabla D_1^{-\tau} D_1^\tau \partial_1^l p dy$$

$$= \int ((D_1^{-\tau}([\partial_1^l, A] \nabla p) - D_1^{-\tau} A\nabla \partial_1^l p) \cdot \nabla D_1^{-\tau} D_1^\tau \partial_1^l p - \partial_1^l f^1 D_1^{-\tau} D_1^\tau \partial_1^l p) dy$$

$$+ \int_{\partial \Omega} D_1^{-\tau} \partial_1^l f^2 \bar{n}_2 (D_1^{-\tau} \partial_1^l p) dy_1 =: I_{10}. \quad (A.29)$$

Thanks to Lemma [A.6] we can estimate that

$$I_{10} \lesssim (\|A\|_{W^{k-1,\infty}(\Omega)}) (\|\nabla \partial_1^l p\|_0 + \|\partial_1^l f^1\|_0 + \|\partial_1^l f^2\|_0) \|D_1^{-\tau} D_1^\tau \partial_1^l p\|_1.$$

Making use of the above estimate, Poincaré inequality [A.6] and the assumption that (A.26) holds for $0 \leq l \leq k - 2$, we can deduce from (A.29) that

$$\|D_1^l \partial_1^l p\|_1 \lesssim (1 + \|A\|_{W^{k-1,\infty}(\Omega)}) (\|f^1\|_{k-2} + \|f^2\|_{k-1}),$$

which implies

$$\|\partial_1^{l+1} p\|_1 \lesssim (1 + \|A\|_{W^{k-1,\infty}(\Omega)}) (\|f^1\|_{k-2} + \|f^2\|_{k-1}). \quad (A.30)$$

Thanks to the regularity $\nabla \partial_1^{l+1} p \in L^2$, we further see from (A.27) for $0 \leq l \leq k - 2$ that (A.27) holds with $l + 1$ in place of $l$, if $l + 1 \leq k - 2$. 

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Next we shall further derive that \( \partial^m_{x} \partial^{l+1-m}_{y} p \in L^2 \) for \( 1 \leq m \leq l + 1 \). Since \( p \) satisfies (A.25), thus

\[
\int A_{22} \partial_2 p \partial_2 \psi dy = \int \left( f^1 \psi - \sum_{i,j \neq 2} A_{ij} \partial_j p \partial_i \psi \right) dy + \int_{\partial \Omega} f^2 \bar{n} \psi dy_1. \tag{A.31}
\]

Let \( \varphi \in C_0^\infty(\Omega) \). Noting that \( A_{22} \geq \theta \), we can take \( \psi = A_{22}^{-1} \varphi \) in (A.31) to get that

\[
\int \partial_2 p \partial_2 \varphi dy = \int_{\Omega} A_{22}^{-1} \left( f^1 + \sum_{i,j \neq 2} \partial_j (A_{ij} \partial_j p) + \partial_2 A_{22} \partial_2 p \right) \varphi dy, \tag{A.32}
\]

which implies

\[
\partial^2_{y} p = -A_{22}^{-1} \left( f^1 + \sum_{i,j \neq 2} \partial_j (A_{ij} \partial_j p) + \partial_2 A_{22} \partial_2 p \right). \tag{A.33}
\]

Since we have assumed that (A.26) holds for \( 0 \leq l \leq k - 2 \), we easily see from (A.30) and (A.33) that (A.26) also holds with \( l + 1 \) in place of \( l \).

Finally, to complete the proof of Lemma A.7, next it suffices to verify (A.21). By the relation (A.33), we can compute out that

\[
- \text{div} (A \nabla p) = f^1 \text{ a.e. in } \Omega. \tag{A.34}
\]

Multiplying the above identity (A.34) by \( \psi \in H^1 \) in \( L^2 \), and then using the integral by parts, we get

\[
- \int (A \nabla p) \cdot \nabla \psi dy + \int_{\partial \Omega} (A \nabla p) \cdot \bar{n} \psi dy_1 = \int f^1 \psi dy. \tag{A.35}
\]

Comparing (A.25) with (A.35), we further have

\[
\int_{\partial \Omega} ((A \nabla p) - f^2) \cdot \bar{n} \psi dy_1 = 0 \text{ for any } \psi \in H^1,
\]

which implies

\[
(A \nabla p) \cdot \bar{n} = f^2 \bar{n}_2 \text{ on } \partial \Omega
\]

in the sense of trace. Hence \( p \) solves problem (A.21). The proof of Lemma A.7 is complete. \( \square \)

**Lemma A.8.** Diffeomorphism mapping theorem: There exists a sufficiently small constant \( \iota \in (0, 1] \), depending on \( \Omega \), such that, for any \( \varsigma \in H^3_\varsigma \) satisfying \( \| \varsigma \|_4 \leq \iota \), \( \psi := \varsigma + y \) (after possibly being redefined on a set of measure zero with respect to variable \( y \)) satisfies the diffeomorphism properties as \( \varsigma \) in (1.14), (1.15) and \( \inf_{y \in \Omega} \det(\nabla \varsigma + I) \geq 1/4 \).

**Proof.** In [25], the authors had proved that Lemma A.8 holds for \( \varsigma \in H^4_\varsigma \), where \( H^4_\varsigma := \{ w \in H^4 \mid w|_{\partial \Omega} = 0 \} \), see [25, Lemma 4.2]. Next we further verify Lemma A.8 based on [25, Lemma 4.2].

Let \( \delta = h/5 \) and \( \Omega_\delta = T \times \delta \). By Lemma A.5, there exists an extension function \( \tilde{\varsigma} \) such that

\[
\tilde{\varsigma}|_{\Omega} = \varsigma, \quad \tilde{\varsigma}|_{\partial \Omega_\delta} = 0 \text{ and } \| \tilde{\varsigma} \|_{H^4(\Omega_\delta)} \lesssim \| \varsigma \|_4.
\]
Thus, thanks to [25, Lemma 4.2], we have

\[ \tilde{\psi} := \zeta + y : \overline{\Omega}_\delta \rightarrow \overline{\Omega}_\delta \text{ is a } C^2\text{-diffeomorphism mapping,} \]

\[ \tilde{\psi} \text{ is an identity map on the boundary } \partial \Omega_\delta. \]  

Noting that \( \zeta = \zeta|_\Omega \), thus

\[ \psi := \zeta + y : \Omega \rightarrow \Omega \text{ is injective.} \]

To complete the proof of Lemma A.8, next it suffices to verify that \( \psi : \Omega \rightarrow \Omega \) is surjective.

To this purpose, we let \( x^0 \in \overline{\Omega} \), then there exists \( y^0 \in \overline{\Omega}_\delta \) such that \( \tilde{\psi}(y^0) = x^0 \) by (A.36). Since \( \zeta|_\partial \Omega = 0 \), it is easy to see that, for sufficiently small \( \iota \), \( \tilde{\psi} := \zeta + y : \Xi_i \rightarrow \Xi_i \) is bijective,

\[ \text{(A.37)} \]

where \( \Xi_i := \mathbb{T} \times \{i\} \) and \( i = 0, h \). By (A.36) and (A.37), we further see that \( y^0 \in \Omega_\delta \setminus \partial \Omega \).

Now we claim that \( y^0 \in \Omega \).

In fact, if (A.38) fails, then \( y^0 \in \Omega_- \) or \( \Omega_+ \), see (A.15) for the definition \( \Omega_- \) and \( \Omega_+ \). We assume that \( y^0 \in \Omega_- \), then \( \tilde{\psi} \) maps the segment \( L_- := \{ y \in \Omega_- \mid y_1 = y^0_1, -\delta \leq y_2 \leq y^0_2 \} \) to a continuous curve, which lies in \( \Omega_\delta \). Noting that \( \tilde{\psi}(y^0_1, -\delta) = (y^0_1, -\delta) \) and \( \tilde{\psi}(y^0_1, y^0_2) \in \Omega \), thus there exists a unique point \( y^c \in L_- \), such that \( \psi(y^c) \in \Xi_0 \), which contradicts with (A.37). Hence \( y^0 \notin \Omega_- \). Similarly \( y^0 \notin \Omega_+ \). Consequently, (A.38) holds. This completes the proof.

Lemma A.9. New version of theorem of continuity in the mean of integral: Let \( f \in C^0(\mathbb{T}_T, L^2) \) and \( \zeta \in C^0(\overline{\Omega}_T) \) satisfies the diffeomorphism properties as (1.14) and (1.15). Then for any given \( \varepsilon > 0 \) and for any given \( s \in \mathbb{T}_T \), there exists a \( \delta > 0 \) such that for any \( t \in \mathbb{T}_T \) satisfying \( |t - s| < \delta \),

\[ \|f(\varphi(y,t), s) - f(\varphi(y,s), s)\|_0 \leq \varepsilon. \]

PROOF. It is well-know that, for any \( \chi \in L^2(\mathbb{R}^n) \) with \( n \geq 1 \),

\[ \lim_{h \to 0} \int_{\mathbb{R}^n} |\chi(x + h) - \chi(x)|dx = 0, \]

see the theorem of continuity in the mean of integral in [49, Theorem 4.21].

Recalling the proof of the above assertion, we easily see that, for any \( \chi \in L^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \) with \( n \geq 1 \),

\[ \lim_{h \to 0} \int_{\mathbb{R}^n} |\chi(x + h) - \chi(x)|^2dx = 0. \]

Then following the argument of (A.39) with further using the assumptions of \( \zeta \) and the embedding inequality (A.1), we easily get desired conclusion in Lemma A.9. □
Lemma A.10. Some results for functions with values in Banach spaces: Let $T > 0$, integers $i$, $j \geq 0$ be given and $1 \leq p \leq \infty$.

1. Assume $f \in L^p_T H^i$, $\partial^k_t \nabla f \in L^p_T L^2$ for any $1 \leq k \leq j$, then $f \in L^p_T H^{j,i}$, where $H^{j,i} := \{ w \in H^i \mid \partial^k_t w \in H^i \text{ for any } 1 \leq k \leq j \}$.

2. Let $X$ be a separable Banach space and $T > 0$. If $w \in C^0(\overline{I_T}, X)$, then $w : I_T \to X$ is a strongly measurable function and

$$\|w\|_X \in C^0(\overline{I_T}).$$  \hspace{1cm} (A.40)

3. Let $j \geq i + 1$ and $c$ be a constant. If $f \in L^p(I_T, H^i)$ with $1 < p \leq \infty$, $\|f\|_j \leq c \|g\|_j$ holds for a.e. $t \in I_T$ and $g \in L^p(I_T, H^j)$, then

$$f \in L^p_T H^i.$$  \hspace{1cm} (A.41)

4. Assume $f \in L^\infty_T H^i$ and $\{D^r_1 f\}_{|r| \in (0,1)}$ is uniformly bounded in $L^\infty_T H^i$.

(a) There exists a subsequence (still denoted by $\{D^r_1 f\}_{|r| \in (0,1)}$) of $\{D^r_1 f\}_{|r| \in (0,1)}$ such that

$$D^r_1 f \to \partial_1 f \text{ weakly-* in } L^\infty_T H^i \text{ as } \tau \to 0.$$  \hspace{1cm} (A.42)

Moreover, $f \in L^\infty_T H^{1,i}$.

(b) If additionally $f \in L^p_T H^i$ and $\{D^r_1 D^s_1 f\}_{|r| \in (0,1)}$ is uniformly bounded in $L^p_T H^i$ with $1 < p < \infty$, then (a subsequence)

$$D^r_1 D^s_1 f \to \partial_1^2 f \text{ weakly in } L^p_T H^i \text{ as } \tau \to 0 \text{ (a subsequence).}$$  \hspace{1cm} (A.43)

Moreover, $f \in L^\infty_T H^{2,i}$.

(c) If additionally $D^r_1 \partial^\alpha f$ in $C^0(\overline{I_T}, L^2_{\text{weak}})$ and $D^r_1 \partial^\alpha f$ is uniformly continuous in $H^{-1}$, where the multiindex $\alpha$ satisfying $|\alpha| = i$, then

$$D^r_1 \partial^\alpha f \to \partial^\alpha \partial_1 f \text{ in } C^0(\overline{I_T}, L^2_{\text{weak}}) \text{ (a subsequence).}$$  \hspace{1cm} (A.44)

5. Let $\varphi = \zeta + \gamma$ and $\zeta \in L^\infty_T H^i$ defined by (5.5).

(a) If $f \in C^0(\overline{I_T}, H^i)$ or $f \in L^p_T H^i$ with $0 \leq i \leq 4$, then

$$F := f(\varphi, t) \in C^0(\overline{I_T}, H^i) \text{ or } L^p_T H^i$$  \hspace{1cm} (A.45)

and

$$\mathcal{F} := f(\varphi^{-1}, t) \in C^0(\overline{I_T}, H^i) \text{ or } L^p_T H^i.$$  \hspace{1cm} (A.46)

Moreover,

$$\|F\|_{L^p_T H^i} \lesssim P(\|\zeta\|_{L^\infty_T H^i}) \|f\|_{L^p_T H^i},$$  \hspace{1cm} (A.47)

$$\|\mathcal{F}\|_{L^p_T H^i} \lesssim P(\|\zeta\|_{L^\infty_T H^i}) \|f\|_{L^p_T H^i}.$$  \hspace{1cm} (A.48)

(b) If $\zeta$ additionally satisfies $\zeta_t \in L^\infty_T H^3$, then, for any $f \in L^p(I_T, H^i)$ satisfying $f_t \in L^p(I_T, H^{i-1})$ with $1 \leq i \leq 4$,

$$F_t = (f_t(x, t) + \zeta_t \cdot \nabla f(y, t))|_{x = \varphi} \in L^p_T H^{i-1}$$  \hspace{1cm} (A.49)

and

$$\mathcal{F}_t = (f_t(y, t) - (\nabla \varphi)^{-1} \zeta_t \cdot \nabla f(y, t))|_{y = \varphi^{-1}} \in L^p_T H^{i-1}.$$  \hspace{1cm} (A.50)
Proof. (1) Since $f$ is a Bochner integrable by $f \in L^p_T H^j$, thus there exists a sequence $\{f^n\}_{n=1}^\infty$ of simple functions such that
\[
\lim_{n \to \infty} \int_0^T \|f^n(t) - f(t)\|_i dt = 0. \tag{A.51}
\]
Similarly to (5.68), we define the mollifications of $f^n$, resp. $f$ with respect to $y_1$ as follows:
\[
S^1_\varepsilon(f^n) := \chi^\varepsilon * f^n, \quad \text{resp.} \quad S^1_\varepsilon(f) := \chi^\varepsilon * f,
\]
where $\varepsilon \in (0, 1)$. Then $S^1_\varepsilon(f^n), S^1_\varepsilon(f) \in L^p_T H^j$. By (A.51), we have, for given $\varepsilon$,
\[
\lim_{n \to \infty} \int_0^T \|S^1_\varepsilon(f^n(t) - f(t))\|_{H^j} dt = 0. \tag{A.52}
\]
By the regularity of $f$, we see that
\[
\int_0^T \|f(t)\|_{H^j} dt < \infty.
\]
Thus
\[
\lim_{\varepsilon \to 0} \int_0^T \|S^1_\varepsilon(f(t)) - f(t)\|_{H^j} dt = 0. \tag{A.53}
\]
Exploiting (A.52) and (A.53), there exists a sequence $\{S^1_{1/m}(f^{nm}(t))\}_{m=1}^\infty$ of simple functions such that
\[
\int_0^T \|S^1_{1/m}(f^{nm}(t)) - f(t)\|_{H^j} dt \leq 1/m,
\]
which yields that
\[
\lim_{m \to \infty} \int_0^T \|S^1_{1/m}(f^{nm}(t)) - f(t)\|_{H^j} dt = 0.
\]
Thus
\[
\|S^1_{1/m}(f^{nm}(t)) - f(t)\|_{H^j} \to 0 \text{ for a.a. } t \in I_T \text{ (a subsequence)}.\]
Hence $f(t): I_T \to H^j$ is also strongly measurable. Thanks to this fact and the regularity of $f$, i.e.,
\[
\infty > \begin{cases} 
\int_0^T \|f(t)\|_{H^j}^p dt & \text{for } p \in [1, \infty); \\
\text{ess sup}_{t \in I_T} \|f\|_{H^j} & \text{for } p = \infty,
\end{cases}
\]
we immediately get $f \in L^p_T H^j$.

(2) The conclusion is obvious by Pettis theorem (see Theorem 7 in APPENDIX E in [7]) and the separability of $X$. In addition, (A.40) is obvious due to the triangle inequality of norm.

(3) By Lemma [A.5] there exists a function $\tilde{f} \in H^i(\mathbb{T} \times \mathbb{R})$ such that
\[
\tilde{f}|_\Omega = f, \quad \|\tilde{f}\|_{H^i(\mathbb{T} \times \mathbb{R})} \lesssim \|f\|_i.
\]
Let $\varepsilon \in (0,1)$. Then we denote $S_{\varepsilon}(\tilde{f})$ the mollifications of $\tilde{f}$ with respect to the 2D variable $(y_1,y_2)$. Obviously, $S_{\varepsilon}(\tilde{f}) \in L^p_T H^j$

$$S_{\varepsilon}(\tilde{f}) \to f \text{ strongly in } L^p_T H^i \text{ for } p > 1, \quad (A.54)$$

$$\|S_{\varepsilon}(\tilde{f})\|_i \lesssim \|f\|_j \text{ for any } t \in I_T,$$

which implies that

$$\|S_{\varepsilon}(\tilde{f})\|_{L^p_T H^j} \lesssim \|g\|_{L^p_T H^j}.$$ 

Thus there exists $\chi \in L^p_T H^j$ such that

$$S_{\varepsilon}(\tilde{f}) \to f \text{ weakly in } L^p_T H^j \text{ for } p > 1, \quad (A.55)$$

which, together with (A.54), yields (A.41) for $p > 1$.

Thanks to the above result for $p > 1$, we easily see that (A.41) also holds for $p = \infty$.

(4)–(a) Since $\{D^h y f\}_{h \in (0,1)}$ is uniformly bounded in $L^\infty_T H^i$, then, for any multindex $\alpha$ satisfying $|\alpha| = i$,

$$D^i_1 \partial^\alpha f \to \omega \text{ weakly-* in } L^\infty_T L^2 \text{ (a subsequence)}.$$ 

This mean that, for any $\chi \in H^1$ and for any $\phi \in C_0^\infty(I_T)$,

$$- \int_0^T \int \partial^\alpha f \partial_1 \chi dy dt = - \lim_{\tau \to 0} \int_0^T \int \partial^\alpha f D^h_1 \chi dy dt$$

$$\quad = \lim_{\tau \to 0} \int_0^T \int D^h_1 \partial^\alpha f \chi dy dt = \int_0^T \int \omega \chi dy dt \text{ (a subsequence).} \quad (A.56)$$

Since $H^1$ is a separable space, thus we further derive from (A.56) that $\omega = \partial_1 \partial^\alpha f = \partial^\alpha \partial_1 f$. Thus, by the first assertion in Lemma A.10 we get $f \in L^\infty_T H^{1,i}$.

(4)–(b) If additionally $\{D^{-h}_1 D^h_1 f\}_{|h| \in (0,1)}$ further is uniformly bounded in $L^2_T H^i$ with respect to $\tau \in (0,1)$, then, for any multiindex $\alpha$ satisfying $|\alpha| = i$,

$$D^{-h}_1 D^h_1 \partial^\alpha f \to \psi^\alpha \text{ in } L^2_T L^2 \text{ and } D^h_1 \nabla^i f \to \nabla^i \partial_1 f \text{ in } L^2_T L^2 \text{ (a subsequence).} \quad (A.57)$$

By (A.57), for any multiindex $\alpha$ satisfying $|\alpha| = i$, for any $\chi \in C^2(\Omega)$, and for any $\phi \in C_0^\infty(I_T)$,

$$- \int_0^T \int \partial^\alpha \partial_1 f \partial_1 \chi dy dt = - \lim_{h \to 0} \int_0^T \int D^h_1 \partial^\alpha f D^h_1 \chi dy dt$$

$$\quad = \lim_{h \to 0} \int_0^T \int D^{-h}_1 D^h_1 \partial^\alpha f \chi dy dt = \int_0^T \int \psi^\alpha \chi dy dt \text{ (a subsequence).}$$

By a density argument, we further derive from the above identity that, for any $\chi \in H^1$ and for any $\phi \in C_0^\infty(I_T)$,

$$- \int_0^T \int \partial^\alpha \partial_1 f \partial_1 \chi dy dt = \int_0^T \int \psi^\alpha \chi dy dt,$$

which implies that $\partial^2_1 \nabla^i f = \nabla^i \partial^2_1 f \in L^2_T L^2$. Moreover $f \in L^2_T H^{2,i}$. 

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(4)–(c) If additionally $D_1^\tau \partial^\alpha f^\varepsilon$ in $C^0(\overline{T_T}, L^2_{\text{weak}})$ and $D_1^\tau \partial^\alpha f^\varepsilon$ is uniformly continuous in $H^{-1}$, then we have

$$D_1^\tau \partial^\alpha f^\varepsilon \to \vartheta \text{ in } C^0(\overline{T_T}, L^2_{\text{weak}}) \text{ (a subsequence)}$$

which implies that, for any $\chi \in L^2$ and for any $\phi \in C_0^\infty(I_T)$,

$$\lim_{\tau \to 0} \int_0^T \int D_1^\tau \partial^\alpha f^\varepsilon \chi \, dy \, dt = \int_0^T \int \vartheta \chi \, dy \, dt \text{ (a subsequence)} \quad (A.58)$$

Thus we get $\vartheta = \partial^\alpha \partial_1 f$ from (A.56) and (A.58).

(5)–(a) Let us first consider the case $f \in C^0(\overline{T_T}, H^i)$. Let $s \in I_T$ be any given. Thanks to (A.9), it is easy to check that for any $\varepsilon > 0$, there exists a $\delta$ such that for any $t \in I_T$ satisfying $|t - s| < \delta$,

$$\|F(y, t) - F(y, s)\|_i = \|f(\varphi(y, t), t) - f(\varphi(y, s), s)\|_i \leq \varepsilon.$$

Thus

$$F := f(\varphi, t) \in C^0(\overline{T_T}, H^i). \quad (A.59)$$

Now we further consider the case $f \in L^p_T H^i$. Let $\tilde{f} = f$ in $\Omega_T$ and $\tilde{f} = 0$ outside $\Omega \times (\mathbb{R} \setminus I_T)$. Let $S^i_\varepsilon(\tilde{f})$ be defined as in (5.19). We can check that $S^i_\varepsilon(\tilde{f}) \in C^0(\mathbb{R}, H^i)$ and

$$S^i_\varepsilon(\tilde{f}(x, t))|_{x=\varphi} \to F(y, s) \text{ strongly in } L^p_T H^i \text{ for } p > 1. \quad (A.60)$$

In addition, it is easy to verify that, for a.e. $t \in I_T$,

$$\|F\|_i \lesssim P(\|\varsigma\|_3)\|f\|_i. \quad (A.61)$$

Thus we easily see from (A.59), (A.60) and (A.61) that (A.45) and (A.47) hold. Next we turn to deriving (A.46) and (A.48).

Thanks to the regularity $\varsigma \in C^0(\overline{T_T}, H^3)$, we have (after possibly being redefined on a set of measure zero)

$$\tilde{\varphi}(\tilde{y}) : \overline{\Omega_T} \to \varphi(\Omega_T) \text{ is a homeomorphism mapping}, \quad (A.62)$$

where $\tilde{\varphi}(y, t) := (\varphi(y, t), t)$, please refer to (8.12) and (8.13) in [30]. In particular, for given $t$,

$$\varphi^{-1}(y, t) \in C^0(\overline{\Omega}) \text{ and } \varphi^{-1}(y, t) : \Omega \to \Omega \text{ is a homeomorphism mapping} \quad (A.63)$$

where $\varphi^{-1}(y, t)$ denotes the inverse mapping of $\varphi$ with respect to $y$.

It is easy to check that

$$\nabla(\varphi^{-1}) = (\nabla \varphi)^{-T}|_{y=\varphi^{-1}}. \quad (A.64)$$

Thanks to Lemma [A.9] [A.62], (A.63) and (A.64), similarly to (A.59), we can verify that

$$\varphi^{-1} \in C^0(\overline{T_T}, H^3) \quad (A.65)$$

Thus (A.46) obviously holds by following the argument of (A.59) again.
(5)–(b) Let \( \phi \in C_0^\infty(I_T) \) and \( \psi \in C_0^\infty(\Omega) \). Let \( S_\delta^1 \), resp. \( S_\delta \) denote the 1D, resp. 2D mollifiers with respect to variables \( t \), resp. \( (y_1,y_2) \). Let \( S_\delta^f \) is defined as \( S_\delta^f \) with \( \nu \) in place of \( \delta \). Then we can compute out that, for sufficiently small \( \delta, \varepsilon \), and \( \nu \),

\[
- \int_0^T \int S_\delta^f(S_\delta(f(x,t)))|_{x=y+S_\delta^f(c)} \psi \phi_t dy dt \\
= \int_0^T \int (S_\delta^f(S_\delta^f(f_t(x,t)))|_{x=y+S_\delta^f(c)} + S_\nu^f(S_\delta^f(\nabla f(x,t)))|_{x=y+S_\delta^f(c)}) \psi \phi dy dt,
\]

which implies that

\[
- \int_0^T \int F(\phi_t) dy dt = \int_0^T \int (f_t + S_\delta^f(\nabla f)|_{x=\nu} \psi \phi dy dt.
\]

Noting that \( C_0^\infty(\Omega) \) is density in \( L^2 \), thus we have, for any \( \varphi \in L^2 \) and for any \( \phi \in C_0^\infty(I_T) \),

\[
- \int_0^T \int F(\phi_t) dy dt = \int_0^T \int (f_t + S_\delta^f(\nabla f)|_{x=\nu} \psi \phi dy dt,
\]

which immediately implies that \( F_t = (f_t(x,t) + \zeta_t \cdot \nabla f(x,t))|_{x=\nu} \in L_T^p L^2 \). Exploiting the third assertion in Lemma A.10 and (A.47), we further have \( F_t \in L_T^p H^1 \).

Thanks to the regularity \( (\zeta, \zeta_t) \), we have (after possibly being redefined on a set of measure zero)

\[
\bar{\varphi}(\bar{y}) : \Omega_T \to \bar{\varphi}(\Omega_T) \text{ is a homeomorphism mapping,} \\
\tilde{\varphi}(\tilde{y}) : \Omega_T \to \tilde{\varphi}(\Omega_T) \text{ is a } C^1 \text{-diffeomorphic mapping},
\]

where \( \bar{\varphi}(y,t) := (\varphi(y,t),t) \), please refer to (8.12) and (8.13) in [30]. Moreover,

\[
\tilde{\varphi}^{-1}(\tilde{x}) = (\varphi^{-1}(y,t),t), \quad \nabla \tilde{\varphi}^{-1} = (\nabla \bar{\varphi})^{-1}|_{\bar{y}=\varphi^{-1},}
\]

where \( \tilde{x} = (x,t) \). In particular, we compute out that

\[
\partial_t \varphi^{-1} = -((\nabla \varphi)^{-1})|_{y=\varphi^{-1}}, \quad (A.66)
\]

Thus we immediately get (A.50) by (A.65) and (A.66). This completes the proof. \( \square \)

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