Verified Runtime Model Validation for Partially Observable Hybrid Systems

Stefan Mitsch and André Platzer

Computer Science Department
Carnegie Mellon University, Pittsburgh PA 15213, USA
{smitsch,aplatzer}@cs.cmu.edu

Abstract. Formal verification provides strong safety guarantees about models of cyber-physical systems. Hybrid system models describe the required interplay of computation and physical dynamics, which is crucial to guarantee what computations lead to safe physical behavior (e.g., cars should not collide). Control computations that affect physical dynamics must act in advance to avoid possibly unsafe future circumstances. Formal verification then ensures that the controllers correctly identify and provably avoid unsafe future situations under a certain model of physics. But any model of physics necessarily deviates from reality and, moreover, any observation with real sensors and manipulation with real actuators is subject to uncertainty. This makes runtime validation a crucial step to monitor whether the model assumptions hold for the real system implementation. The key question is what property needs to be runtime-monitored and what a satisfied runtime monitor entails about the safety of the system: the observations of a runtime monitor only relate back to the safety of the system if they are themselves accompanied by a proof of correctness. For an unbroken chain of correctness guarantees, we, thus, synthesize runtime monitors in a provably correct way from provably safe hybrid system models. This paper advances these techniques to make the synthesized monitoring conditions robust to partial observability of sensor uncertainty and partial controllability due to actuator disturbance. We show that the monitoring conditions result in provable safety guarantees with fallback controllers that react to monitor violation at runtime. We evaluate the method by synthesizing monitoring conditions from existing case studies on train and road traffic control.

1 Introduction

Correctness arguments for cyber-physical systems (CPSs) crucially depend on models in order to enable predictions about their future behavior. Absent any models, neither tests nor verification provide any correctness results except for the limited amount of concrete (test) cases, because they cannot provide predictions for other cases. Models of physics are crucial in CPSs, but necessarily deviate from reality. Even cyber components may come with surprises when their detailed implementation is more complex
than their verification model. Linking cyber and physical components with sensors and actuators adds another layer of uncertainty. These discrepancies are inevitable and call into question how safety analysis results about models can ever transfer to CPS implementations. Not using models, however, would invalidate all predictions [34].

Safety of the CPS implementation at runtime crucially depends on a faithful implementation of the model. However, even though rigorous correct-by-construction approaches promise provably correct implementation of the software portion of the model, their correctness is still predicated on meeting the assumptions of the model about the physical environment and the physical effects of actuators. Whether or not these environment and actuator effects are faithfully represented in the model can only be checked from actual measurements [27] during system operation at runtime. That way, runtime monitoring can help to check the present run of a CPS implementation.

The key question, however, is what property needs to be runtime-monitored and what a runtime monitor says about the safety of the system. There is a fundamental asymmetry of the power of runtime monitoring in CPS compared to purely discrete software systems. In pure software it may suffice to monitor the critical property itself and suspend the software upon violation, raising an exception to propagate the violation to surrounding code for mitigation. In cyber-physical systems any such attempt would be fundamentally flawed, because there is no way of reverting time and trying something else. A desired property of a self-driving car, e.g., is to always keep at least 1ft distance to other cars. Runtime monitoring this property as stated, however, would raise an alarm when the car is less than 1ft away, which (when going fast) makes a collision inevitable and all runtime monitoring futile. The underlying problem is that this property expresses unrealistic implicit assumptions about the continuous dynamics of a car.

In our approach, i) offline safety proofs ensure that a controller avoids all safety violations with respect to an explicit model of physics, ii) offline monitor synthesis turns these models into runtime monitors that act in advance with provable safety guarantees about the true CPS, and iii) offline proofs justify that all future model behavior will be safe if the monitor is satisfied at runtime. For example, the specification “keep at least 1ft distance to other cars” in our approach considers the dynamics and therefore results in a monitor “always keep sufficient stopping distance” that acts ahead of time as opposed to after a collision is inevitable. Verified machine code implementation of runtime monitors can be obtained with VeriPhy [4].

Fig. 1 illustrates the intuition behind safety proofs and their relation to system execution and monitoring: A system should stay outside unsafe states (¬safe), so it has to avoid all paths that might eventually end up in unsafe states (⟨α⟩¬safe). A safety proof (⟨α⟩safe) shows that a model α of the system indeed avoids such paths, no matter how often it is repeated (α∗). At runtime, we can check implementation correctness with monitors that inspect sensors and influence decisions on each iteration α (every time the controller of the loop α∗ runs). To guarantee safety, monitors must provably detect when the system is about to enter a path to the unsafe states. Such an approach crucially

\footnote{Some implementations deliberately choose to implement the model in a liberal way to interact with unverified components (e.g., use machine learning), or to allow for adaptation at runtime. These implementations require monitoring of the implementation for compliance with its model.}
(a) A proof of $[\alpha^*]_{\text{safe}}$ shows that all runs are safe, which in particular means the model $\alpha^*$ avoids $\langle\alpha^*\rangle_{\neg \text{safe}}$.

(b) Monitors detect when a real system is about to enter paths to the unsafe states.

Fig. 1: Logical characterization of safety and monitors

| Statement | Effect |
|-----------|--------|
| $\alpha;\beta$ | sequential composition, first run hybrid program $\alpha$, then hybrid program $\beta$ |
| $\alpha \cup \beta$ | nondeterministic choice, following either hybrid program $\alpha$ or $\beta$ |
| $\alpha^*$ | nondeterministic repetition, repeats $n \geq 0$ times hybrid program $\alpha$ |
| $x := \theta$ | assign value of term $\theta$ to variable $x$ (discrete jump) |
| $x := \ast$ | assign arbitrary real number to variable $x$ |
| $?F$ | check that formula $F$ holds in current state, and abort if it does not |
| $(x_i' = \theta_1, \ldots, x_n' = \theta_n \& F)$ | evolve $x_i$ along differential equation system $x_i' = \theta_i$ restricted to maximum evolution domain $F$ |

Table 1: Hybrid program representations of hybrid systems.

requires logical foundations for analyzing both $[\alpha]_{\text{safe}}$ safety and $\langle\alpha\rangle_{\neg \text{safe}}$ liveness in the same framework, which our chosen differential dynamic logic [30,32] supports.

This paper advances techniques for these questions from prior work [27] with built-in systematic ways of coping with the inevitable complications of sensor uncertainty and actuator disturbance in partially observable hybrid systems.

2 Differential Dynamic Logic by Example

This section recalls differential dynamic logic $\mathcal{DL}$ [29,33,30], which we use to syntactically characterize the semantic conditions required for correctness of the ModelPlex runtime monitoring approach. The proof calculus of $\mathcal{DL}$ [29,33,30,32] is also exploited to guarantee correctness of the specific ModelPlex monitors produced for concrete CPS models. This section also introduces a simple flight collision avoidance protocol that will be used as a running example to illustrate the concepts throughout.

Syntax summary. Differential dynamic logic uses hybrid programs as a notation for hybrid systems (Table 1). The set of $\mathcal{DL}$ formulas is generated by the following grammar ($\sim \in \{<,\leq,=,\geq,>\}$ and $\theta_1,\theta_2$ are arithmetic expressions in $+,-,\cdot,/\over$ over the reals):

$$
\phi ::= \theta_1 \sim \theta_2 | \neg \phi | \phi \land \psi | \phi \lor \psi | \phi \rightarrow \psi | \forall x \phi | \exists x \phi | [\alpha] \phi | \langle \alpha \rangle \phi
$$
dL allows us to make statements that we want to be true for all runs of a hybrid program (\(\langle \alpha \rangle \phi\)) or for at least one run (\(\langle \alpha \rangle \phi\)). Both constructs are necessary to derive safe monitors. We need proofs of \(\langle \alpha \rangle \phi\) so that we can be sure all behavior of a model are safe. We need proofs of \(\langle \alpha \rangle \phi\) to find monitor specifications that detect whether or not a system execution fits to the verified model. Differential dynamic logic comes with a verification technique to prove such correctness properties of hybrid programs (see [30] for an overview of dL, and [15] for an overview of the KeYmaera X prover we use).

Example: Horizontal flight collision avoidance. We model a simple horizontal collision avoidance protocol for two constant-speed airplanes [16]: our controlled ownship takes angular velocity \(w_o\) as pilot commands and can fly straight (\(w_o := 0\)) or enter a circular wait pattern (\(w_o := 1\)) to avoid collision with a straight-path intruder airplane.

The linear and angular velocities of ownship and intruder are independently controlled, but for position and orientation we use a reference frame relative to the the ownship centered at \((0, 0, 0)\) while the intruder is at \((x, y, \theta)\). The ownship moves with constant linear velocity \(v_o\) and pilot-controlled angular velocity \(w_o\) along a straight line or circle, the intruder with constant linear velocity \(v_i\) on a straight path (\(w_i := 0\)).

\[
\text{flight} \equiv (\text{ctrl}; \text{plant})^* \tag{1}
\]

\[
\text{ctrl} \equiv (w_o := 0; \ ?\mathcal{I}) \cup (w_o := 1; \ ?\mathcal{J}(w_o)) \tag{2}
\]

\[
\mathcal{I} \equiv v_i \sin \theta x - (v_i \cos \theta - v_o)y > v_o + v_i \tag{3}
\]

\[
\mathcal{J}(w) \equiv v_i w \sin \theta x - v_i w \cos \theta y + v_o v_i \cos \theta > v_o v_i + v_i w \tag{4}
\]

\[
\text{plant} \equiv \{ x' = v_i \cos \theta - v_o + w_o y, y' = v_i \sin \theta - w_o x, \theta' = w_i - w_o \} \tag{5}
\]

The flight protocol \(\text{flight}\) describes the pilot and collision avoidance controller \(\text{ctrl}\) of the ownship and flight dynamics \(\text{plant}\) of both airplanes. Controller and flight dynamics are repeated nondeterministically often, indicated by \(^*\) in (1). The pilot has two control choices: The pilot may choose a straight path \(w_o := 0\) if \(\mathcal{I}\) indicates that it is safe, or the pilot may choose a circular evasion maneuver \(w_o := 1\) if allowed by \(\mathcal{J}(w_o)\) in (2).

The flight dynamics (5) keep the (moving) ownship at the origin by combining both ownship and intruder motion in the relative position \((x, y)\); the differential equation \(\theta' = w_i - w_o\) rotates the reference frame with angular velocity \(-w_o\) since \(w_i = 0\).

Formula (6) specifies safety of the flight protocol \(\text{flight}\): all runs that start in states satisfying the assumptions \(A\) must stay in states satisfying the safety condition \(S\).

\[
v_o = 1 \land v_i = 1 \land x^2 + y^2 > 0 \rightarrow [\text{flight}] (x^2 + y^2 > 0) \tag{6}
\]

Semantics. The semantics of dL [29,31,32] is a Kripke semantics in which the states of the Kripke model are the states of the hybrid system. Let \(\mathbb{R}\) denote the set of real numbers and \(\mathcal{V}\) denote the set of variables. A state is a map \(\omega : \mathcal{V} \rightarrow \mathbb{R}\) assigning a real
value $\omega(x)$ to each variable $x \in \mathcal{V}$. We write $\omega_x^\alpha$ to refer to a state $\tilde{\omega}$ that equals $\omega$ except that $\tilde{\omega}(x) = r$. We write $\omega \in [\phi]$ if formula $\phi$ is true in state $\omega$. The semantics of a hybrid program $\alpha$ is a relation $[\alpha]$ between initial and final states. For example $\omega \in [\alpha]\phi$ if $\nu \in [\phi]$ for all $(\omega, \nu) \in [\alpha]$, so all runs of $\alpha$ from $\omega$ are safe.

**Notation.** In the remainder of the paper we use shortcut notation for formulas and hybrid programs. We use $x \in \mathcal{U}_{[l, u]}(y)$ to say that $x$ is in the interval $[l, u]$ around $y$ (so $y + l \leq x \leq y + u$) and $x \in \mathcal{U}_\Delta(y)$ to say $x \in \mathcal{U}_{[-\Delta, \Delta]}(y)$. We use $x \in \mathcal{U}_\Delta(y)$ to refer to a program that nondeterministically picks any value from the interval $[y - \Delta, y + \Delta]$ for $x$, which is the hybrid program $x :=*$; $y - \Delta \leq x \land x \leq y + \Delta$, and $x \in \alpha$ to refer to the effect on the variables $x$ of running program $\alpha$. In monitors, we use $x$ to denote present state variables and $x^+$ for next state variables.

We use the following characteristics of hybrid programs and $\mathcal{DL}$ formulas (see Appendix A [32, Lemmas 9, 11, 12, 17]): (i) Hybrid programs only change their bound variables $\mathcal{BV}(\alpha)$ but not the complement $\mathcal{BV}(\alpha)^c$; (ii) The truth of a formula $\phi$ only depends on its free variables $\mathcal{FV}(\phi)$; (iii) Similar states (that agree on the free variables) have similar transitions according to the transition relation $[\ ]$ of hybrid programs. We use $\mathcal{V}(\alpha) = \mathcal{FV}(\alpha) \cup \mathcal{BV}(\alpha)$.

## 3 Monitor Synthesis for Verified Runtime Validation

CPS are almost impossible to get right without sufficient attention to prior analysis, for instance by formal verification. Performed offline, these approaches result in a verified model of a CPS, i.e. formula (7) is proved valid, for example using the differential dynamic logic proof calculus [30,32] implemented in KeYmaera X [15]:

$$A \rightarrow [\alpha^+]S$$  \hspace{1cm} (7)

Formula (7) expresses that all runs of the hybrid system $\alpha^*$, which start in states that satisfy the precondition $A$ and repeat $\alpha$ arbitrarily many times, only end in states that satisfy the postcondition $S$. The model $\alpha^*$ is a *hybrid system model* of a CPS, which means that it describes both the discrete control actions of the controllers in the system and the continuous physics of the plant and the system’s environment.

Whether or not the control choices, actuator and environment effects are faithfully represented in the model can only be checked from actual measurements [27]. Intuitively, such checks compare the values $x$ in state $\nu_{i-1}$ to the values $x^+$ in state $\nu_i$ taken at successive sample times to determine compatibility of the unknown system behavior $\gamma_{i-1}$ with model $\alpha^*$ (Fig. 2).

For example, values $x = 2$ and $x^+ = 3$ are compliant with the differential equation $x' = 2$, because starting the program at $x$ can produce $x^+$, as witnessed by a proof of the $\mathcal{DL}$ formula $x = 2 \land x^+ = 3 \rightarrow (x' = 2):x^+ = x$. No $x^+ < x$ is compliant with the program, since the program can reach only states where $x^+ \geq x$. For instance, $\gamma_{i-1}$ checked for validation against model $\alpha^*$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{System execution $\gamma_i$ checked for validation against model $\alpha^*$.}
\end{figure}
ModelPlex [27] provides the basis for obtaining such runtime checks from hybrid system models both automatically and in a provably correct manner (see Appendix D for a basic example how a proof synthesizes runtime checks). ModelPlex analyzes hybrid programs $\alpha$, starting from monitor conditions in $\text{dL}$ of the form $\langle \alpha \rangle \mathcal{T}^+$, where $\mathcal{T}^+$ is shortcut notation for $x^+ = x$ for all $x \in \text{BV}(\alpha)$ to collect the effect of executing $\alpha$. We use the satisfaction relation $(\omega, \nu) \models \phi$ [27] to refer to a monitor condition that is satisfied over two states $\omega$ and $\nu$.

**Definition 1 (Transition satisfaction relation).** The satisfaction relation $(\omega, \nu) \models \phi$ of $\text{dL}$ formula $\phi$ for a pair of states $(\omega, \nu)$ evaluates $\phi$ in the state resulting from state $\omega$ by interpreting variable $x^+$ as $\nu(x)$ for all $x \in V$, i.e., $(\omega, \nu) \models \phi$ iff $\omega_{x^+} \nu \in [\phi]$.

The central correctness result about ModelPlex [27, Theorem 1] guarantees that the resulting monitoring formula preserves safety, i.e., if the monitoring formula evaluates to true with current sensor measurements and control choices, then the system is safe. In Appendix B we provide complementary safety guarantees when monitor violation is mitigated with fallback control.

In the following sections, we advance the ModelPlex monitor synthesis techniques to account for actuator disturbance and sensor uncertainty in partially observable cases.

### 4 Robustness to Actuator Disturbance

Actuator disturbance results in discrepancies between the chosen control decisions $u$ and their physical effect $\tilde{u}$ (e.g., wheel slip). A typical pattern to model piecewise constant actuator disturbance [25] chooses a nondeterministic value $\tilde{u}$ in the $\Delta$-neighborhood around the control choice $u$ (but does not affect other state variables) as input to the plant: $u \in \text{ctrl}(x)$; $\tilde{u} \in \mathcal{U}_\Delta(u)$; $x' = f(x, \tilde{u})$.

Actuator disturbance is partially observable by monitoring the difference between the intended effect $x' = f(x, u)$ and the actual effect $x' = f(x, \tilde{u})$ from observable quantities. As a side effect, safety and invariant properties of models do not mention the perturbed actuator effect. For example, a car controller can estimate deviation from its acceleration choice only after the fact by observing the actual resulting speed difference. The safety property of interest is usually about the actual resulting speed and not the perturbed acceleration. For monitor synthesis, we therefore adapt the input-output relation $\mathcal{T}^+$ to conjecture existence of an effective actuator effect $\tilde{u}^+$ that explains all other effects collected in $\mathcal{T}^+$ about a program $\alpha$. The monitor condition (8) can be turned into arithmetical form with the prior synthesis process [27] and preserves safety by Theorem 1.

$$\langle \alpha \rangle \exists \tilde{u}^+ \mathcal{T}^+$$

**Theorem 1 (Monitor with actuator disturbance preserves invariants).** Let $\alpha$ be a hybrid program of the shape $u \in \text{ctrl}(x)$; $\tilde{u} \in \mathcal{U}_\Delta(u)$; $x' = f(x, \tilde{u})$. Let $\alpha^*$ preserve an invariant $J$ where $\tilde{u} \not\in \text{FV}(J)$, so $J \rightarrow [\alpha^*]J$ is valid. Assume the system transitions from state $\omega$ to $\nu$, which agree on $\text{BV}(\alpha)^0$, and assume $\omega \in [J]$. If the monitor condition (8) is satisfied, i.e., $(\omega, \nu) \models \langle \alpha \rangle \exists \tilde{u}^+ \mathcal{T}^+$, then the invariant is preserved, i.e., $\nu \in [J]$. 

$$\langle \alpha \rangle \exists \tilde{u}^+ \mathcal{T}^+$$
Theorem 1 extends to multiple control outputs \( u \) and their effects with disturbance \( \tilde{u} \) in a straightforward way. The controller \( u \in \text{ctrl}(x) \) does not access \( \tilde{u} \) and \( \tilde{u} \notin \text{FV}(J) \) is therefore natural as no information about \( \tilde{u} \) needs to be maintained in the invariant \( J \).

**Example 1.** In the flight protocol example, the ownship pilot may choose angular velocity \( w_o = 0 \) for straight flight or \( w_o = 1 \) for an evasion maneuver. Either decision is followed precisely without disturbance in the plant. Here, we extend the flight protocol with a pilot decision \( w_p \) which is subject to actuator disturbance \( \Delta \geq 0 \) before taking effect to analyze safety for slightly imperfect evasion maneuvers:

\[
\alpha \equiv \left( (w_p := 0; ?I) \cup (w_p := 1; ?J(w_p)) \right); w_o := \ast; ?(0 \leq w_o \leq w_p \Delta); \text{plant}
\]

The true acceleration \( w_o \) is unobservable, so the monitor condition \( (\alpha) \exists w_o^+ (x^+ = x \land y^+ = y \land \theta^+ = \theta \land w_o^+ = w_o \land x_0^+ = x_0 \land y_0^+ = y_0 \land \theta_0^+ = \theta_0 \land w_o^+ = w_o) \) extends the state recall \( T^+ \) with \( w_o^+ = w_p \) and existentially quantifies away the unobservable \( w_o^+ \). The resulting arithmetical model monitor condition is obtained by an automated KeYmaera X tactic [14] to symbolically execute \( \alpha \) with subsequent quantifier elimination to synthesize a quantifier-free model monitor condition in real arithmetic without \( \exists w_o^+ \).

## 5 Partial Observability from Sensor Uncertainty

Monitor correctness [27] requires that all bound variables of a program \( \alpha \) are monitored, i.e., \( \text{BV}(\alpha) \subseteq \text{FV}(T^+) \). This is the appropriate behavior except, of course, for variables that are unobservable in the CPS implementation. For example, when controllers use sensors to obtain information, only the measurement is known, but not the true value that the sensor is measuring. If a variable is unobservable then it cannot be included in a runtime monitor. But there may still be indirect implications about unobservable quantities when they are related to observable quantities, which necessitates monitors that indirectly check the properties of the true quantity from the measured quantity.

![Fig. 3](image-url)

(a) Safety proof: measurements \( \hat{x} \) are taken near true \( x \) (bars indicate uncertainty) but system model behavior follows true \( x \).

(b) Monitor: estimates true \( x \) from sample measurements \( \hat{x} \), considering \( \hat{x} \) plausible if model behavior fits to any possible true \( x \).

(\( x \)) Controllers observe true behavior through sensors. (b) Monitors have to check existence of behavior that fits to the model and explains the measurements.
Unobservability results in a crucial difference between monitoring and control (and its safety proofs). Controllers estimate true behavior by taking measurements and observing the effect of their decisions in the next measurements that are again taken from the true values. Monitors have to decide whether the measurements explain a true behavior that fits to the expected behavior model, see Fig. 3.

The measurements of a controller are taken from points that are linked through the underlying true behavior by assumption. Monitors cannot make this assumption, which results in a number of challenges that we address in this section: (i) check existence of behavior as explanation from a set of potential true values into a set of potential next true values, (ii) link explanations to a full path through sets of potential true values around a history of observations, and (iii) guarantee safety in both cases.

We use $y$ with $y \in x$ to refer to an unobservable state variable and we use $\hat{y}$ to denote a measurement of $y$ with some uncertainty $\Delta$, so $\hat{y} \in \mathcal{U}_\Delta(y)$. We assume non-faulty sensors that function according to their characteristics, i.e., they reliably deliver values that only deviate from the true values by at most some known sensor uncertainty $\Delta$. In an implementation, sensor fusion methods for detecting sensor faults and correcting measurement outliers can satisfy this assumption. Def. 2 captures what it means for states to be similar with respect to a measurement that is subject to uncertainty.

**Definition 2 (Uncertainty similarity).** We use $\omega \approx^\Delta \nu$ to denote the fact that $\omega = \nu$ on $\{y\}^\infty$ with $\omega(y) \in \mathcal{U}_\Delta(\nu(\hat{y}))$ and say that state $\omega$ is $\Delta$-uncertainty-similar to state $\nu$.

In the following subsections, we develop $\Delta$-techniques to synthesize monitors that indicate whether or not there exist states that are $\Delta$-uncertainty-similar to measured states and connected through a program $\alpha$, i.e., for measured states $\omega$ and $\nu$ do there exist uncertainty-similar states $\tilde{\omega}$ and $\tilde{\nu}$ such that $\tilde{\omega} \approx^\Delta \omega$, $\tilde{\nu} \approx^\Delta \nu$ and $(\tilde{\omega}, \tilde{\nu}) \in \llbracket \alpha \rrbracket$.

### 5.1 Model Monitors for Pairwise Consistency of Measurements

Monitoring based on measurements requires us to decide whether the true values $y$ fit to a model $\alpha$ by only looking at the measurements $\hat{y}$. Intuitively, this can be answered by finding an unobservable prior state $y \in \mathcal{U}_\Delta(\hat{y})$ close to the measurement $\hat{y}$, such that running the model $\alpha$ on this possible $y$ predicts a next unobservable $y^+$ that is within measurement uncertainty $y^+ \in \mathcal{U}_\Delta(\hat{y}^+)$ to the next measurement $\hat{y}^+$. This intuition is illustrated in Fig. 4: a pair of two consecutive measurements is plausible if the potential true values of the second measurement overlap with the values predicted by the model $\alpha$ from one of the potential prior $y$ (which is estimated from the previous measurement).

Our goal when formalizing this intuition into monitoring conditions is to shift proof effort offline. Therefore we combine offline
proofs with online monitoring: (i) offline we prove that the modeled dynamics ensure that all potential true values around the measurements satisfy an invariant property for safety (contraction, see Def. 3), and (ii) online we monitor that some potential true values around the measurements can be connected with the modeled dynamics (see monitor condition (9)). Both requirements are expressible by quantifying over potential true values in $\mathcal{dL}$.

**Definition 3 (Contraction).** The contraction $\forall y \in \mathcal{U}_\delta(\hat{y})J$ of margin $\delta \geq 0$ ensures $\forall y \in \mathcal{U}_\delta(\hat{y})J$; i.e. $J$ for all potential values $y$ in the $\delta$-neighborhood of the value $\hat{y}$. Program $\beta$ is contraction-safe for margin $\delta$ with measurement uncertainty $\Delta$ if $\forall y \in \mathcal{U}_\delta(\hat{y})J \rightarrow \{ u \in \beta; x' = f(x, u); ?t = \varepsilon; \hat{y} : \mathcal{U}_\Delta(y) \} \forall y \in \mathcal{U}_\delta(\hat{y})J$ is valid.

Monitor condition (9) answers plausibility of two measurements $\hat{y}$ and $\hat{y}^+$ taken $t = \varepsilon$ time apart according to the model dynamics by asking for existence of true values $y$ and $y^+$.

$$\chi_m \equiv \exists y \in \mathcal{U}_\Delta(\hat{y}) (u \in \text{ctrl}(\hat{y}); x' = f(x, u); ?t = \varepsilon; \hat{y} : \mathcal{U}_\Delta(y)) (\exists y^+ \mathcal{T}^+) \quad (9)$$

To facilitate monitor synthesis, we need both measurements $\hat{y}$ and $\hat{y}^+$ in a single loop iteration; therefore, (9) takes fresh measurements after $x' = f(x, u)$. Now monitor condition (9) tells us that there exist true values close to the measurements; in order for these true values to have the desired properties, the controller $u \in \text{ctrl}(\hat{y})$ must be contraction-safe, see Theorem 2.

**Theorem 2 (Pairwise measurement monitor preserves invariants).** Let $\alpha$ be a program of the shape $u \in \text{ctrl}(\hat{y}); x' = f(x, u); ?t = \varepsilon; \hat{y} : \mathcal{U}_\Delta(\hat{y})$ with contraction-safe controller $u \in \text{ctrl}(\hat{y})$ for margin $\Delta$ and measurement uncertainty $\Delta$, i.e., $\forall y \in \mathcal{U}_\Delta(\hat{y})J$ is valid. Assume the system transitions from $\omega$ to $\nu$, which agree on $\text{BV}(\alpha)$, with non-faulty sensors, so $\omega \in [[y \in \mathcal{U}_\Delta(\hat{y})]]$ and $\nu \in [[y \in \mathcal{U}_\Delta(\hat{y})]]$. If the contraction holds $\omega \in [[y \in \mathcal{U}_\Delta(\hat{y})J]]$ and the pairwise measurement monitor is satisfied $(\omega, \nu) = \exists y \in \mathcal{U}_\Delta(\hat{y}) (\alpha) (\exists y^+ \mathcal{T}^+)$, then $J$ is preserved, i.e., $\nu \in [J]$.

Theorem 2 extends to vectors of several unobservable variables $y$ and their measured variables $\hat{y}$ in a straightforward way. The test $?t = \varepsilon$ is reflected in the monitor condition (9), so the monitor is only satisfied for states $\omega$ and $\nu$ according to the sampling interval $\varepsilon$ (formally: $\nu(t) = \omega(t) + \varepsilon$ for clock $t'$ = 1).

Besides safety, monitoring for existence of unobservable values $y$ that fit to the present measurements $\hat{y}$ also guarantees that the variation between true values is bounded, but not that it forms a linked chain of model executions because true values are freshly estimated from only the last measurement (see Fig. 3b and Proposition 1).

**Proposition 1 (Bounded variation coincidence).** Let $\alpha$ be a hybrid program of the form $u \in \text{ctrl}(\hat{y}); x' = f(x, u); ?t = \varepsilon; \hat{y} : \mathcal{U}_\Delta(\hat{y})$. Assume the system transitions through the sequence of states $\nu_0, \nu_1, \ldots, \nu_n$, which agree on $\text{BV}(\alpha)^\mathcal{dL}$, such that $(\nu_{i-1}, \nu_i) \models \chi_m$ with (9) for all $1 \leq i \leq n$. Then there are $\omega_{i-1} \overset{\Delta}{\approx} y_{i-1}, \mu_i \overset{\Delta}{\approx} y_i$ such that $(\omega_{i-1}, \mu_i) \in [\alpha]$.

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2 Monitor condition (9) is transformed from its $\mathcal{dL}$ representation into a real arithmetic representation by an offline proof using the ModelPlex synthesis process (see Appendix G).
Proposition 2 and Corollary 1 bound the maximum variation between true values \( y \) and measurements \( \hat{y} \) that the monitor condition must allow according to the model.

**Proposition 2 (Single-step variation distance).** Let \( \alpha \) be a hybrid program of the form
\[
u: \in \text{ctrl}(\hat{y}); \; x'=f(x,u); \; ?t = \varepsilon; \; \hat{y}: \in \mathcal{U}_\Delta(y).
\]
Assume the model transitions \((\omega, \nu) \in [\alpha]\) with intermediate states \(\mu\) and \(\hat{\mu}\) such that \((\omega, \mu) \in [u: \in \text{ctrl}(\hat{y})]\), \((\mu, \hat{\mu}) \in [x'=f(x,u)]\), and \((\hat{\mu}, \nu) \in [\hat{y}: \in \mathcal{U}_\Delta(y)]\). If \((\omega, \nu) \models \chi_m\), then the variation distance in a single \(\alpha\)-step is bounded: \(v(y) \in \mathcal{U}_{\Delta}(\omega(y) + \hat{\mu}(y) - \mu(y))\) and \(v(y) \in \mathcal{U}_{\Delta}(\omega(y) + \hat{\mu}(y) - \mu(y))\).

**Corollary 1 (Multi-step distance).** Assume the model transitions through a sequence of states \(\nu_0, \nu_1, \ldots, \nu_n\) with intermediate states \(\mu_i\) and \(\hat{\mu}_i\) s.t. \(\nu_{i-1}, \mu_i) \in [u: \in \text{ctrl}(\hat{y})]\), \((\mu_i, \hat{\mu}_i) \in [x'=f(x,u); ?t = \varepsilon]\), and \((\hat{\mu}_i, \nu_i) \in [\hat{y}: \in \mathcal{U}_\Delta(y)]\). If \((\nu_{i-1}, \nu_i) \models \chi_m\) for all \(1 \leq i \leq n\) then the variation is bounded: \(v_n(y) \in \mathcal{U}_{2\Delta(n+1)}(\nu_0(y) + \sum_{i=1}^n (\hat{\mu}_i(y) - \mu_i(y)))\).

As a consequence of Proposition 2 and Corollary 1, Theorem 2 is useful for single-step consistency checks: such a monitor (i) keeps at least \(\Delta \geq 0\) safety margin with a contraction-safe controller because otherwise freshly estimating true \(y\) from measurements on every iteration does not preserve invariants, and (ii) detects “large” deviations that occur in a single monitoring step. However, pairwise consistency is unable to detect gradual drift, as illustrated in the following example, and therefore has to be safeguarded for its entire \(2\Delta(n+1)\) deviation over \(n\) steps, which inhibits motion, and cannot exploit improving the safety margin \(\Delta\) over a history of measurements.

**Example 2.** The flight protocol so far assumes perfect knowledge about the intruder linear velocity \(v_i\). Here, we extend the protocol when the ownership takes measurements \(v_i\) of the intruder velocity \(v_i\) with a sensor that might be off by uncertainty \(\epsilon\).

\[
\alpha \equiv (\{w_o := 0; \exists \forall v_i \in \mathcal{U}_e(v_i) \mathcal{J}(w_o) := 1; \exists \forall v_i \in \mathcal{U}_e(v_i) \mathcal{J}(w_o)\}); \quad \text{plant}; \quad \hat{v}_i; \in \mathcal{U}_e(v_i)
\]

Now the true linear intruder velocity \(v_i\) is unobservable, so the monitor condition \(\exists v_i \in \mathcal{U}_e(v_i) \{\alpha\} \exists v_i^+ (x^+=x \land y^+=y \land \theta^+ = \theta \land w_o^+=w_o \land x_0^+=x_0 \land y_0^+=y_0 \land \theta_0^+=\theta_0 \land v_i^+=\hat{v}_i)\) extends the state recall \(\mathcal{J}^+\) with \(\hat{v}_i\) and existentially quantifies away both unobservable prior velocity \(v_i\) and current velocity \(v_i^+\). From this monitor condition, the ModelPlex synthesis proof discovers (among other things) that consecutive velocity measurements \(\hat{v}_i\) and \(\hat{v}_i^+\) are at most \(2\epsilon\) apart, because intruder speed in the model is constant.

Fig. 5a illustrates the monitor behavior when measuring the constant linear intruder velocity \(v_i = 0.5\) with uncertainty \(\epsilon = 0.1\): the thick error bars represent the range of possible measurements \(\hat{v}_i\) according to the true value \(v_i\), the thin error bars the measurement range allowed per \(\hat{v}_i^+ \in \mathcal{U}_{2\epsilon}(\hat{v}_i)\) from the previous measurement. The measurements \(\hat{v}_i\) at \(t = 0\) to \(t = 5\) vary around the true velocity as would be expected from a sensor, which includes the worst case of two consecutive measurements hitting opposite bounds of the uncertainty. However, since the monitor does not keep history, the true velocity \(v_i\) is allowed to drift at times \(t = 6\) and \(t = 7\). The monitor detects violations if the \(2\epsilon\) uncertainty is exceeded in a single step, as illustrated with the measurement at \(t = 8\).
Monitoring contractions and single-step conformance guarantees safety up to the current measurement per Theorem 2 and detects large single-step deviations per Proposition 1 (e.g., at $t = 8$ in Fig. 5a), but cannot detect gradual drift early. In order to react to drift in measurements even before contractions are violated, next we extend our monitors with state estimation over the entire measurement history.

5.2 Model Monitors for Rolling Consistency of Measurements

Even if measurement pairs in isolation do not exceed the detectable single-step violation of the previous section, the resulting aggregated drift over multiple measurements is still detectable when we keep a history of the control choices. Instead of an explicit list of measurement and control choice histories, we aggregate the actuation and measurement history into what really matters: acceptable bounds for the upcoming true values (solid blue small range in Fig. 5b) and measurements (solid light-brown large range in Fig. 5b). The bounds are updated on each monitor execution, taking into account the current control choice. The resulting rolling state estimator detects gradual violation over the course of multiple measurements.

**Definition 4 (Non-diverging rolling state estimator).** A rolling state estimator $[l, u] := e(\hat{y}_0, \hat{y}, y - y_0, \Delta, [l_0, u_0])$ updates the estimate $[l, u]$ from the previous measurement $\hat{y}_0$, current measurement $\hat{y}$, the modeled interpolated plant effect $y - y_0$ and the previous estimate $[l_0, u_0]$ with $y_0 \in \mathcal{U}_{[l_0, u_0]}(\hat{y}_0)$ and $\mathcal{U}_{[l_0, u_0]}(\hat{y}_0) \subseteq \mathcal{U}_\Delta(\hat{y})$. The estimator $[l, u] := e(\hat{y}_0, \hat{y}, y - y_0, \Delta, [l_0, u_0])$ is non-diverging if $u - l \leq u_0 - l_0$ for all $\hat{y}_0, \hat{y}, y, \Delta, l, u$ with $y \in \mathcal{U}_{[l, u]}(\hat{y})$ and $\mathcal{U}_{[l, u]}(\hat{y}) \subseteq \mathcal{U}_\Delta(\hat{y})$.

The rolling state estimator updates estimates $y \in \mathcal{U}_{[l, u]}(\hat{y})$ of true $y$ on every measurement such that the history of measurements is preserved in aggregate form. On each step, the monitor checks for existence of a true state $y$ in the estimate, and uses the rolling state estimator to incorporate the current measurement $\hat{y}$ into the estimate for
the next check. This also means that the plant effect $y - y_0$ is existentially quantified in the monitor condition (10) below, so does not need to be directly observable.

**Example 3.** The following basic rolling state estimator is non-diverging:

$$l = \max (-\Delta, \hat{y}_0 - \hat{y} + y - y_0 + l_0) \quad u = \min (\Delta, \hat{y}_0 - \hat{y} + y - y_0 + u_0)$$

For a sequence of $n$ measurements, a non-diverging rolling state estimator keeps tighter bounds compared to the $2\Delta(n + 1)$ bounds of Corollary 1 without measurement history. Note that measurements typically vary due to sensor uncertainty, so the estimate almost surely even improves over time by observing measurements.

The monitor condition (10) checks the plausibility of a history of measurements $\hat{y} \in \mathcal{U}_\Delta(y)$ by estimating the true $y \in \mathcal{U}_{[l, u]}(\hat{y})$ from the observations.

$$\chi_m \equiv \exists y \in \mathcal{U}_{[l, u]}(\hat{y}) \langle y_0 := y; \hat{y}_0 := \hat{y}; u := \text{ctrl}(\hat{y}); x' = f(x, u); \alpha \rangle \in \mathcal{U}_\Delta(y);$$

$$[l, u] := e(y_0, \hat{y}, y - y_0, \Delta, [l_0, u_0]) \left( \exists y^+ \mathcal{T}^+ \right)$$

**Theorem 3 (Monitor with rolling state estimator maintains invariants).** Let $\alpha$ be a program of the shape $y_0 := y; \hat{y}_0 := \hat{y}; u := \text{ctrl}(\hat{y}); x' = f(x, u); \alpha$ with a contraction-safe program $y_0 := y; \hat{y}_0 := \hat{y}; u := \text{ctrl}(\hat{y})$ for margin $[l, u]$ and measurement uncertainty $\Delta$. Assume the system transitions from $\omega$ to $\nu$, which agree on $\text{BV}(\alpha)$, with non-faulty sensors, so both $\omega \in \mathcal{U}_{\Delta}(\hat{y})$ and $\nu \in \mathcal{U}_{\Delta}(\hat{y})$. If $\omega \in \mathcal{U}_{\Delta}(\hat{y})$ and the monitor condition (10) is satisfied, i.e., $(\omega, \nu) \models \exists y \in \mathcal{U}_{[l, u]}(\hat{y}) (\alpha) (\exists y^+ \mathcal{T}^+)$, then the invariant $J$ is maintained: $\nu \in \mathcal{J}$.

Theorem 3 extends to vectors of several unobservable variables $y$ and their measured variables $\hat{y}$ in a straightforward way.

**Example 4.** We extend the flight protocol with a velocity estimator $e(\hat{v}_{i0}, \hat{v}_i, 0, [l_0, u_0])$ that updates lower and upper bounds on the deviation between the next measurement $\hat{v}_i$ and the true velocity $v_i$ from the current measurement $\hat{v}_{i0}$ and the current estimation bounds $[l_0, u_0]$. Since $v_i$ is constant (but not perfectly known), the interpolated plant effect $v_i - \hat{v}_{i0}$ is 0.

$$l = \max (-\Delta, \hat{v}_{i0} - \hat{v}_i + l_0) \quad u = \min (\Delta, \hat{v}_{i0} - \hat{v}_i + u_0)$$

### 6 Evaluation

The process for synthesizing model monitors from hybrid systems models is systematic, implemented as a synthesis tactic in KeYmaera X [15] and analogous to [27], see Appendix G. We ran the synthesis tactic on previous case studies on train control [35], and road traffic control [26]. The automated steps of the synthesis process, its duration,
and the size of the resulting monitor condition in terms of arithmetical and logical operators are summarized in Table 2. The duration measurements were taken on a 2.4 GHz Intel Core i7 with 16GB of memory.

For each case study, we synthesized monitors for the original model and analyzed extended models that include sensor uncertainty and actuator disturbance. The column “Dim.” gives an intuition on the complexity of the case study in terms of the number of model variables. For the monitor conditions, we list the size in terms of the number of arithmetical, comparison, and logical operators in the monitor formula in the intermediate quantified form (column “∀∃”) and the final quantifier-free form (column “∀∃-free”), as well as the duration of the synthesis steps: column “Proof Check” lists the duration of checking the safety proof, column “Discovery” lists the duration of discovering the intermediate quantified monitor form, and column “QE” the duration of obtaining the quantifier-free form. Finally, column “External” lists how much of the total synthesis duration (Proof Check + Discovery + QE) is spent in external solvers.

The main insight is that the synthesis and discovery of monitor conditions with the techniques in this paper processes CPS models with modest computation and time resources. Heuristics for existential and universal quantifiers [27, Opt. 1] improve QE performance significantly, but the remaining complexity and duration still hinges on the performance of external solvers for eliminating the remaining quantifiers that describe all potential phenomena of sensor uncertainty and actuator disturbance. Additional arithmetical simplifications beyond [27, Opt. 1] are needed to make progress despite the limitations [8] of quantifier elimination procedures.

## 7 Related Work

Runtime verification and monitoring for finite state discrete systems has received significant attention (e.g., [7,18,24]). Others monitor continuous-time signals (e.g., [12,28]). We focus on hybrid systems models of CPS to combine both, and our methods are robust to sensor uncertainty and actuator disturbance.
Several tools for formal verification of hybrid systems are actively developed (e.g., SpaceEx [13], dReach [20], and extended NuSMV/MathSat [6]). Provably correct monitor synthesis, however, crucially relies on the rewriting capabilities and flexibility of combining $\cdot$ and $\langle \cdot \rangle$ modalities in $dL$ [30,32] and KeYmaera X [15].

**Combined Offline and Runtime Verification.** In [9], offline model checking is combined with runtime monitoring for path planning of robots. For offline verification, the method assumes that motion of the robot stays inside a tube around the planned path; staying inside the tube is monitored at runtime. This can only be sound when augmented with additional assumptions on the continuous dynamics between sampling points [34], which we handle explicitly in our approach [27]. We do not ignore physics models and environment behavior and therefore, our monitors: (i) detect when uncertainty accumulates to unsafe deviations from the model before becoming unsafe, and (ii) are able to distinguish between violations caused by uncertainty in our own system (sensors, actuators) vs. unsafe environments, which makes our approach better suited to partially known, dynamic environments (as opposed to fully known stationary obstacles).

Reachset conformance testing [37] computes reachable sets of hybrid automata at runtime to transfer safety properties of reachability analysis methods by falsifying simulations or recorded data. The crucial benefit of our methods is to perform expensive computations offline and provably guarantee safety when the monitor conditions are satisfied at runtime of the monitored system from measurements and control decisions that are subject to actuator disturbance.

**Monitoring and Sandboxing.** Specification mining techniques for LTL can be adapted to monitor for safety violations [23] and intervene ahead of time, assuming that the next environment input is available to the monitor. In cyber-physical systems, this is feasible only when the next input can be prevented from becoming actuated, which is the rationale behind our controller monitors [27] safeguarding untrusted control. When the next input is measured through sensors after the fact it may already present a (unpreventable) safety violation; counteracting these requires additional assumptions on the nature of the violation [27] and, as presented in this paper, means to detect gradual deviation from the model that accumulates to violation over time.

Shields [21] and robust reactive system synthesis [3] are approaches to detect and correct erroneous control output in discrete-time models, which however do not address assumptions on the continuous behavior between the sampling points that are explicit in our differential equations. The safety guarantees require that a shield can take corrective actions that satisfy the specification, which corresponds to our controller monitors. Monitoring based on discrete-time models is useful for discrete high-level planning tasks (e.g., waypoint planning), but does not give guarantees about the resulting continuous physical motion and is unable to detect effects related to continuous dynamics, such as gradual sensor drift.

Languages for modeling runtime monitors based on sensor events [19] are purely discrete (e.g., speed lower than threshold), come without correctness guarantees on the mapping between monitor and inputs/outputs and without correctness guarantees on the safety properties and alarms. In contrast, our methods provably guarantee that sat-
isfied monitors at runtime imply system safety (and in particular safety of the resulting physical effects) by relating the observed dynamics to the safe models verified offline.

Robustness estimation methods [38,11,10] measure the degree to which a monitor given as a signal/metric temporal logic specification is satisfied in order to allow bounded perturbation akin to our actuator disturbance, but cannot detect gradual drift in sensor measurements. The methods assume a finite time horizon, compact inputs and outputs, and restrictions on the dynamics (e.g., piecewise constant between sampling points [10]), so are useful for detecting violations after they occur, but for safety of the system at runtime need to be augmented with a predictive model of the continuous dynamics, which we handle explicitly.

**Summary.** In summary, our approach improves over existing runtime monitoring techniques with provably correct monitor conditions, explicit dynamics with sensor uncertainty and actuator disturbance, and shifts expensive computation offline:

**Specification** Other methods [23,21,19,38,11,10,3] start from purely discrete monitor specifications and therefore the continuous dynamics are implicit and unchecked (e.g., a monitor specification “always positive distance to obstacles” has the implicit unrealistic assumption that a car can stop instantaneously from any speed). We, in contrast, start from safety requirements about dynamical hybrid systems models with continuous dynamics and therefore synthesize monitor specifications that are provably correct with respect to the dynamics model (e.g., a specification “always positive distance to obstacles” in our approach results in a monitor “always sufficient stopping distance” that acts ahead of time as opposed to after a collision).

**Assumptions between Sampling Points** Methods that rely on discrete time or combine discrete-time with continuous-time descriptions (e.g., [19,22,2,23,9,21,11,10,3]) require additional assumptions on the continuous dynamics between sampling points in order to be sound [34], which we handle explicitly [27], now in order to characterize partial controllability and partial observability and to distinguish between deviation caused by mere uncertainty vs. actually unsafe environment behavior.

**Offline Computation** Some methods [37] rely on extensive runtime computations (e.g., reachable sets). We, in contrast, perform expensive computations offline. At runtime, we only evaluate the resulting formula in real arithmetic for concrete sensor values and control decisions, which enables fast enough responses.

Crucially, we prove correctness properties that correctly link satisfied monitors to offline safety proofs, result in monitors that warn ahead of time, and account for partial controllability and partial observability so that the monitored system inherits the safety guarantees about the model.

**8 Conclusion**

Provable guarantees about the safety of cyber-physical systems at runtime are crucial as systems become increasingly autonomous. Formal verification techniques provide an important basis by proving safety of CPS models, which then requires transferring the guarantees of offline proofs to system execution. We answer the key question of how
offline proofs transfer by runtime monitoring, and crucially, what property needs to be runtime-monitored to imply safety of the monitored system. Our techniques significantly extend previous methods to models of practical interest by implementing proof tactics for differential dynamic logic that correctly synthesize monitor conditions that are robust to bounded sensor uncertainty and bounded actuator disturbance, which are both fundamental sources of partial observability in models.

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A Supporting Lemmas

The proofs use the following lemmas specialized from [32, Lemmas 9, 11, 12 with 17].

Lemma 1 (Bound effect lemma). If \((\omega, \nu) \in \llbracket \alpha \rrbracket\), then \(\omega = \nu\) on \(BV(\alpha)\).

The truth of a formula \(\phi\) only depends on its free variables \(FV(\phi)\):

Lemma 2 (Coincidence lemma). If \(\omega = \tilde{\omega}\) on \(FV(\phi)\) then \(\omega \in \llbracket \phi \rrbracket\) iff \(\tilde{\omega} \in \llbracket \phi \rrbracket\).

Similar states (that agree on the free variables) have similar transitions according to the transition relation of hybrid programs:

Lemma 3 (Coincidence lemma). If \(\omega = \tilde{\omega}\) on \(V \supseteq FV(\alpha)\) and \((\omega, \nu) \in \llbracket \alpha \rrbracket\), then there is a \(\tilde{\nu}\) such that \((\tilde{\omega}, \tilde{\nu}) \in \llbracket \alpha \rrbracket\) and \(\nu = \tilde{\nu}\) on \(V\).

The logical state relation \(\langle \alpha \rangle^{\mathcal{T}^+}\) from [27, Def. 3] captures runs of hybrid programs \(\alpha\):  

Lemma 4 (Logical state relation). Let \(V\) be the set of all variables. Two states \(\omega, \nu\) that agree on \(V \setminus BV(\alpha)\), i.e., \(\omega(z) = \nu(z)\) for all \(z \in V \setminus BV(\alpha)\), satisfy \((\omega, \nu) \in \llbracket \alpha \rrbracket\) iff \((\omega, \nu) \models (\alpha)\mathcal{T}^+\).

B Monitor Guarantees

The monitor conditions in this paper are proven to preserve invariants from the safety proofs and thus system safety when they are satisfied at runtime. Here, we analyze what a fallback controller ensures about the system when the monitor conditions are not satisfied at runtime.

We complement the ModelPlex safety results [27, Theorems 1 and 2] with recoverability guarantees provided by fallback actions when the monitors are unsatisfied.

If the actions that led to monitor violation do not have permanent physical effect (i.e., they can be discarded or undone instantaneously, such as wrong control decisions before they are handed to actuators), then monitor violation is recoverable with a safe fallback control, see Theorem 4.

Theorem 4 (Control violation recoverability). Let \(J\) be an inductive invariant of program \([w: \inctrl(x); x'=f(x, u)]^*\), i.e., \(J \rightarrow [w: \inctrl(x); x'=f(x, u)]\) is valid. Let \(\omega \in \llbracket J \rrbracket\) and \((\omega, \mu) \not\models \chi_c\) violate monitor condition \(\chi_c\). Let fallback by a hybrid program such that for all states \(\omega, \nu\) with \((\omega, \nu) \in \llbracket \text{fallback} \rrbracket\) we have \((\omega, \nu) \models \chi_c\). Let \(\tilde{\mu}\) be a state recovered from monitor violation by program fallback, i.e., \((\omega, \tilde{\mu}) \in \llbracket \text{fallback} \rrbracket\). Then for all states \(\nu\) with \((\tilde{\mu}, \nu) \in \llbracket x'=f(x, u) \rrbracket\) we have \(\nu \in \llbracket J \rrbracket\).
Proof. From $(\omega, \bar{\mu}) \in \{\text{fallback}\}$ and so $(\omega, \bar{\mu}) \models \chi_\alpha$ by assumption, we get $(\omega, \bar{\mu}) \in \{u \in \text{ctrl}(x)\}$ by [27, Thm. 2]. Now $(\omega, \bar{\mu}) \in \{u \in \text{ctrl}(x)\}$ and in turn $(\omega, \nu) \in \{u \in \text{ctrl}(x); x' = f(x, u)\}$ from the semantics of sequential composition with assumption $(\bar{\mu}, \nu) \in \{x' = f(x, u)\}$. Hence we conclude $\nu \in \{J\}$ by assumption $J \rightarrow \{u \in \text{ctrl}(x); x' = f(x, u)\}J$ with assumption $\omega \in \{J\}$.

By Theorem 4, control violations are recoverable entirely by replacing unsafe actions with safe fallback before they take physical effect. Model violations, which are observed on the physical effects, are detected at the earliest point in time, so would be recoverable when acting one step earlier, see Theorem 5.

Theorem 5 (Model violation recoverability). Let $J$ be an inductive invariant of program $(u \in \text{ctrl}(x); x' = f(x, u))^*$, i.e., $J \rightarrow \{u \in \text{ctrl}(x); x' = f(x, u)\}J$ is valid. Let $\omega \in \{J\}$, and let states $(\omega, \mu) \models \chi_m$ satisfy monitor condition $\chi_m$ and $(\mu, \nu) \not\models \chi_v$ violate it. Let fallback by a hybrid program s.t. for all states $\omega, \nu$ with $(\omega, \nu) \in \{\text{fallback}\}$ we have $(\omega, \nu) \models \chi_v$. Let $\bar{\mu}$ be a state recovered from monitor violation by program fallback, i.e., $(\mu, \bar{\mu}) \in \{\text{fallback}\}$. Then for all states $\bar{\nu}$ with $(\bar{\mu}, \bar{\nu}) \in \{x' = f(x, u)\}$ we have $\bar{\nu} \in \{J\}$.

Proof. From [27, Thm. 1] with $(\omega, \mu) \models \chi_m$ we get $\mu \in \{J\}$ as $\omega \in \{J\}$. Now $(\mu, \bar{\mu}) \in \{\text{fallback}\}$ and $(\mu, \bar{\mu}) \models \chi_v$ by assumption and so we get $(\mu, \bar{\mu}) \in \{u \in \text{ctrl}(x)\}$ by [27, Thm. 2] and in turn $(\mu, \bar{\nu}) \in \{u \in \text{ctrl}(x); x' = f(x, u)\}$ by the semantics of sequential composition with assumption $(\bar{\mu}, \bar{\nu}) \in \{x' = f(x, u)\}$. Now $(\mu, \bar{\nu}) \in \{u \in \text{ctrl}(x); x' = f(x, u)\}$ and so we conclude $\bar{\nu} \in \{J\}$ from assumption $J \rightarrow \{u \in \text{ctrl}(x); x' = f(x, u)\}J$ with $\mu \in \{J\}$.

C Proofs

Proof of Theorem 1. The model monitor $(\omega, \nu) \models \langle \alpha \rangle \exists \bar{s}^+ \bar{y}^+$ is satisfied by assumption, so we get $(\omega, \nu) \models \exists \bar{s}^+ \bar{y}^+ \langle \alpha \rangle$ by Barcan [30] with $\bar{s}^+ \not\in V(\alpha)$. Hence, there exists a state $\nu_{\bar{s}}^+ \in \{\bar{s}^+\}$ for $\bar{s} \in \mathbb{R}$ such that $(\omega, \nu_{\bar{s}}^+) \models \langle \alpha \rangle \bar{y}^+$ and so $(\omega, \nu_{\bar{s}}^+) \in \{\alpha\}$ by Lemma 4. Now $\omega \in \{J\}$ and $\models J \rightarrow \{\alpha\}J$ by assumption and therefore $\omega \in \{\alpha\}J$ and in turn $\nu_{\bar{s}}^+ \in \{J\}$ by $(\omega, \nu_{\bar{s}}^+) \in \{\alpha\}$. Since $\nu = \nu_{\bar{s}}^+$ on $\{\bar{s}\}^c$ we conclude $\nu \in \{J\}$ by Lemma 2 with $\bar{s} \not\in FV(J)$.

Proof of Theorem 2. The model monitor is satisfied $(\omega, \nu) \models \exists y \in \mathcal{U}_{\Delta}(\bar{y}) \langle \alpha \rangle \langle \exists y^+ \bar{y}^+ \rangle$ by assumption, so from $y^+ \not\in V(\alpha)$ we get $(\omega, \nu) \models \exists y \in \mathcal{U}_{\Delta} \langle \bar{y} \rangle \exists y^+ \langle \alpha \rangle \bar{y}^+$ by Barcan [30]. Hence there exist $r \in \mathbb{R}$ for $y$ and $s \in \mathbb{R}$ for $\bar{y}$ as well as states $\omega^+_y$ and $\nu^+_y$ with $\omega^+_y \in \{y \in \mathcal{U}_{\Delta}(\bar{y})\}$ and $\omega^+_y \models \langle \alpha \rangle \bar{y}^+$ and so $(\omega^+_y, \nu^+_y) \in \{\alpha\}$ by Lemma 4. We get $\omega^+_y \in \{\forall y \in \mathcal{U}_{\Delta}(\bar{y})\}J$ from $\omega \in \{\forall y \in \mathcal{U}_{\Delta}(\bar{y})\}J$ by $\omega^+_y = \omega$ on $\{y\}^c$ and $y \not\in FV(\forall y \in \mathcal{U}_{\Delta}(\bar{y})\}J$ with Lemma 2. Since $u \in \text{ctrl}(\bar{y})$ is contraction-safe for margin $\Delta$ and measurement uncertainty $\Delta$ by assumption, we now know $\omega^+_y \in \{\alpha\}J$ by Def. 3 using $\omega^+_y \in \{\forall y \in \mathcal{U}_{\Delta}(\bar{y})\}J$. Hence $\nu^+_y \in \{\forall y \in \mathcal{U}_{\Delta}(\bar{y})\}J$ by $\omega^+_y, \nu^+_y \in \{\alpha\}$. Since $\nu = \nu^+_y$ on $\{y\}^c$ we get $\nu \in \{\forall y \in \mathcal{U}_{\Delta}(\bar{y})\}J$ by Lemma 2 since $y \not\in FV(\forall y \in \mathcal{U}_{\Delta}(\bar{y})\}J$. Therefore from $\nu \in \{y \in \mathcal{U}_{\Delta}(\bar{y})\}J$ by assumption we conclude $\nu \in \{J\}$.  


Proof of Proposition 2. The proof follows the sketch below. Note that the true evolution measured in $\nu_{i-1} \rightarrow \nu_i \rightarrow \nu_{i+1}$ is not necessarily a connected path since the measurements allow jumps: the endpoint $\mu_i$ of the $\alpha$-run $(\omega_{i-1}, \mu_i) \in [\alpha]$ might be different from the start of the next $(\omega_i, \mu_{i+1}) \in [\alpha]$.

$$\begin{array}{cccccc}
\omega_0 & \xrightarrow{\alpha} & \mu_1 & \xrightarrow{\Delta} & \omega_2 & \xrightarrow{\alpha} & \mu_3 \\
\nu_0 & \xrightarrow{\Delta} & \nu_1 & \xrightarrow{\Delta} & \nu_2 & \xrightarrow{\Delta} & \nu_3 & \cdots & \nu_{n-1} & \xrightarrow{\Delta} & \nu_n \\
\omega_1 & \xrightarrow{\alpha} & \mu_2 & \xrightarrow{\Delta} & \omega_2 & \xrightarrow{\Delta} & \omega_3 & \xrightarrow{\Delta} & \omega_n-1 & \xrightarrow{\alpha} & \mu_n \\
\end{array}$$

From $(\nu_{i-1}, \nu_i) \models \chi_m$, i.e., $(\nu_{i-1}, \nu_i) \models \exists y \in \mathcal{U}_\Delta(\bar{y}) \langle \alpha \rangle (\exists y^+ \langle \alpha \rangle T^+) we get $(\nu_{i-1}, \nu_i) \models \exists y \in \mathcal{U}_\Delta(\bar{y}) \exists y^+ \langle \alpha \rangle T^+$ by Barcan [30] with $y^+ \notin \mathcal{V}(\alpha)$. Hence there exist $\omega_{i-1} \approx^\Delta \nu_{i-1}$ and $\mu_i \approx^\Delta \nu_i$ with $(\omega_{i-1}, \mu_i) \models (\alpha) T^+$ and we conclude $(\omega_{i-1}, \mu_i) \in [\alpha]$ by Lemma 4.

Proof of Proposition 2. Since $y \notin \mathcal{B}V(u: \in ctrl(\bar{y}))$ and $y \notin \mathcal{B}V(\bar{y}: \in \mathcal{U}_\Delta(\bar{y}))$ we get $\mu(y) = \omega(y)$ and $\nu(y) = \mu(y)$. Hence, $\nu(y) - \omega(y) = \mu(y) - \mu(y)$. From $(\omega, \nu) \models \chi_m$ and non-faulty measurement we know there exist $\omega(y)$ and $\nu(y)$ such that $\omega(y) \in \mathcal{U}_\Delta(\omega(\bar{y}))$ and $\nu(y) \in \mathcal{U}_\Delta(\nu(\bar{y}))$. Therefore $\nu(y) \in \mathcal{U}_\Delta(\omega(\bar{y}) + \mu(\bar{y}) - \mu(\bar{y}))$ since the plant effect on $y$ is $\mu(\bar{y}) - \mu(\bar{y})$ and in turn $\nu(y) \in \mathcal{U}_\Delta(\omega(\bar{y}) + \mu(\bar{y}) - \mu(\bar{y}))$ by $y \in \mathcal{U}_\Delta(\bar{y})$. Hence we conclude $\nu(y) \in \mathcal{U}_\Delta(\omega(y) + \mu(\bar{y}) - \mu(\bar{y}))$ by $\omega(y) \in \mathcal{U}_\Delta(\omega(\bar{y}))$.

Proof of Corollary 2. Follows from $2\Delta$ variation of consecutive measurements, i.e., $\bar{y} \in \mathcal{U}_\Delta(\bar{y})$ and the additional sensor uncertainty $y \in \mathcal{U}_\Delta(\bar{y})$ that is relevant at the beginning and end of the sequence of states but not at intermediate states since overlapping.

Proof of Theorem 3. The model monitor is satisfied $(\omega, \nu) \models \exists y \in \mathcal{U}_{[l,u]}(\bar{y}) \langle \alpha \rangle (\exists y^+ \langle \alpha \rangle T^+)$ by assumption, so we get $(\omega, \nu) \models \exists y \in \mathcal{U}_{[l,u]}(\bar{y}) \exists y^+ \langle \alpha \rangle T^+$ by Barcan [30] with $y^+ \notin \mathcal{V}(\alpha)$. Hence there exist $r \in \mathbb{R}$ for $y$ and $s \in \mathbb{R}$ for $y^+$ as well as states $\omega^*_y$ and $\nu^*_y$ with $\omega^*_y \in [\forall y \in \mathcal{U}_{[l,u]}(\bar{y})]$ and $\nu^*_y \models (\alpha) T^+$ and so $(\omega^*_y, \nu^*_y) \models [\alpha]$ by Lemma 4. We get $\omega^*_y \in [\forall y \in \mathcal{U}_{[l,u]}(\bar{y}) \langle \alpha \rangle J] from assumption $\omega \in [\forall y \in \mathcal{U}_{[l,u]}(\bar{y}) \langle \alpha \rangle J]$ by Lemma 2 since $\omega^*_y = \omega$ on $\{y\}^G$ and $y \notin \mathcal{FV}(\forall y \in \mathcal{U}_{[l,u]}(\bar{y}) \langle \alpha \rangle J). Program \ y_0 := y; y_0 := \bar{y}; u : \in ctrl(\bar{y})$ is contraction-safe for margin $[l, u]$ and measurement uncertainty $\Delta$ by assumption and therefore by Def. 3 using $\omega^*_y \in [\forall y \in \mathcal{U}_{[l,u]}(\bar{y}) \langle \alpha \rangle J]$ we get

$$\omega^*_y \in \{ y_0 := y; y_0 := \bar{y}; u : \in ctrl(\bar{y}); x' := f(x, u); ?t = \varepsilon; \bar{y} : \in \mathcal{U}_\Delta(\bar{y}) \forall y \in \mathcal{U}_{[l,u]}(\bar{y}) \langle \alpha \rangle J \}.$$ 

Now $\mu \in [\forall y \in \mathcal{U}_{[l,u]}(\bar{y}) \langle \alpha \rangle J]$ for all reachable states $\mu$ such that

$$(\omega^*_y, \mu) \in \{ y_0 := y; y_0 := \bar{y}; u : \in ctrl(\bar{y}); x' := f(x, u); ?t = \varepsilon; \bar{y} : \in \mathcal{U}_\Delta(\bar{y}) \}.$$ 

In particular, since the model monitor is satisfied there exists a state $\mu$ such that also $(\mu, \nu^*_y) \in [[l, u]] := e(\bar{y}, \bar{y}, y - y_0, \Delta, [l_0, u_0])$. The estimator $[l, u] := e(\bar{y}, \bar{y}, y - y_0, \Delta, [l_0, u_0])$ is non-diverging (Def. 4) by assumption, so we get $\nu^*_y \in \mathcal{FV}(\forall y \in \mathcal{U}_{[l,u]}(\bar{y}) J]$ and in turn also $\nu \in \mathcal{FV}(\forall y \in \mathcal{U}_{[l,u]}(\bar{y}) J]$ by Lemma 2 with $\nu = \nu^*_y$ on $\{y\}^G$ and $y \notin \mathcal{FV}(\forall y \in \mathcal{U}_{[l,u]}(\bar{y}) J]$. Hence, from $\nu \in \mathcal{FV}(\forall y \in \mathcal{U}_{[l,u]}(\bar{y}) J]$ by assumption we conclude $\nu \in [J].$
D Derive Monitor Condition in Arithmetical Form

The monitor condition \( \langle a := a + 1 \cup b := *; ?b \leq 3 \rangle \langle a^+ = a \land b^+ = b \rangle \) for a program \( a := a + 1 \cup b := *; ?b \leq 3 \) is turned into arithmetical form with a proof in \( \mathcal{DL} \).

\[
\begin{align*}
&\vdash (a^+ = a + 1 \land b^+ = b) \lor (a^+ = a \land b^+ \leq 3) \\
&\vdash (a^+ = a + 1 \land b^+ = b) \lor \exists b \leq 3 (a^+ = a \land b^+ = b) \\
&\vdash (a^+ = a + 1 \land b^+ = b) \lor \langle b := *; ?b \leq 3 \rangle (a^+ = a \land b^+ = b) \\
&\vdash \langle a := a + 1 \cup b := *; ?b \leq 3 \rangle (a^+ = a \land b^+ = b)
\end{align*}
\]

The resulting monitor formula \( a^+ = a + 1 \land b^+ = b \lor a^+ = a \land b^+ \leq 3 \) of the sequent proof means that either the output \( a^+ \) is the input \( a \) incremented by 1 while \( b \) stayed unchanged, or that the output \( b^+ \leq 3 \) while \( a \) stayed unchanged. The monitor formula is satisfied over states \( \omega \) and \( \nu \) with \( \omega(a) = 2, \omega(b) = 3 \) and \( \nu(a) = 3, \nu(b) = 3 \), i.e. \( \langle \omega, \nu \rangle \models a^+ = a + 1 \land b^+ = b \lor a^+ = a \land b^+ \leq 3 \); it is violated on \( \nu(a) = 2, \nu(b) = 4 \).

E Measurement Modeling Patterns

In safety proofs, a useful modeling pattern for representing measurements takes measurements before the controller \( \hat{y} : \mathcal{U}_\Delta(y) \); \( u : \text{ctrl}(\hat{y}) \); \( x' = f(x, u) \), e.g., as used for modeling speed and position sensors of ground robots in [25]. That way, the loop invariants \( J \) in a safety proof are less cluttered with information about the measurements, which are of temporary nature for making a control decision. For the purpose of deriving monitoring conditions as introduced in Section 5, however, it is beneficial to have access to a pair of measurements in the loop body, so the program shape becomes \( u : \text{ctrl}(\hat{y}) \); \( x' = f(x, u) \); \( \hat{y} : \mathcal{U}_\Delta(y) \). In order to avoid duplicate safety analyses, Lemma 5 provides a way of transferring invariant properties between these two shapes.

**Lemma 5 (Measurement rollover).** Assume formula

\[
A \rightarrow [\langle \hat{y} : \mathcal{U}_\Delta(y) ; u : \text{ctrl}(\hat{y}) ; x' = f(x, u) \rangle] S
\]

is proven with invariant \( J \), i.e., \( A \rightarrow J \); \( J \rightarrow [\langle \hat{y} : \mathcal{U}_\Delta(y) ; u : \text{ctrl}(\hat{y}) ; x' = f(x, u) \rangle] J \), and \( J \rightarrow S \) are valid. The measurement \( \hat{y} \) is bound only in \( \hat{y} : \mathcal{U}_\Delta(y) \) but nowhere else and assume \( \hat{y} \notin \text{FV}(J) \). Then, measurement after \( x' = f(x, u) \) is equivalent to measurement before \( u : \text{ctrl}(\hat{y}) \):

\[
J \rightarrow [\langle \hat{y} : \mathcal{U}_\Delta(y) ; u : \text{ctrl}(\hat{y}) ; x' = f(x, u) \rangle] J
\]

\[
\equiv J \land \hat{y} \in \mathcal{U}_\Delta(y) \rightarrow [\langle u : \text{ctrl}(\hat{y}) ; x' = f(x, u) ; \hat{y} : \mathcal{U}_\Delta(y) \rangle] J
\]

**Proof.** The proof uses the axioms \([*: x := *] P \leftrightarrow \forall x P \) and \([-?Q] P \leftrightarrow (P \rightarrow Q)\) [32] together with propositional reasoning and bound renaming to split off and introduce measurements:
First, the test \( \{?\} \) and random assignment \( [\ast] \) split off the measurement after plant \( x' = f(x, u) \), then \( \{?\} \) and \( [\ast] \) are applied in the inverse direction to introduce the measurement before \( u :\text{ctrl}(\hat{y}) \):

\[
\begin{align*}
\text{id} & \quad \vdash F \land \hat{y} \in \mathcal{U}_\Delta(y) \rightarrow [u :\text{ctrl}(\hat{y}); x' = f(x, u)] J \\
\text{[\ast]} \quad \vdash J \land \hat{y} \in \mathcal{U}_\Delta(y) \rightarrow [u :\text{ctrl}(\hat{y}); x' = f(x, u)] J \\
\text{[\ast]} \quad \vdash J \rightarrow [\hat{y} \in \mathcal{U}_\Delta(y); u :\text{ctrl}(\hat{y}); x' = f(x, u)] J \\
\text{BR} & \quad \vdash G \land \hat{y} \in \mathcal{U}_\Delta(y) \rightarrow [u :\text{ctrl}(\hat{y}); x' = f(x, u); \hat{y} \in \mathcal{U}_\Delta(y)] J \\
\text{\{\ast\}} \quad \vdash [\ast] \rightarrow [\ast] \quad \vdash G \land \hat{y} \in \mathcal{U}_\Delta(y) \rightarrow [u :\text{ctrl}(\hat{y}); x' = f(x, u); \hat{y} \in \mathcal{U}_\Delta(y)] J \\
\text{\{\ast\}} \quad \vdash G \land \hat{y} \in \mathcal{U}_\Delta(y) \rightarrow [u :\text{ctrl}(\hat{y}); x' = f(x, u); \hat{y}_{\text{ghost}} := y] J \\
\text{\{\ast\}} \quad \vdash G \land \hat{y} \in \mathcal{U}_\Delta(y) \rightarrow [u :\text{ctrl}(\hat{y}); x' = f(x, u)] J \\
\text{\{\ast\}} \quad \vdash G \land \hat{y} \in \mathcal{U}_\Delta(y) \rightarrow [u :\text{ctrl}(\hat{y}); x' = f(x, u)] J
\end{align*}
\]

Lemma 5 allows us to switch between the measurement modeling patterns for safety proofs and model monitors.

F Proof-Guided Model Monitors for Nonlinear Dynamics

Prior techniques [27] require symbolic closed-form polynomial solutions to characterize differential equations. Hybrid systems with nonlinear dynamics, however, do not necessarily have symbolic closed-form solutions, or their solutions are not expressible in first-order real arithmetic. In such cases, hybrid systems safety proofs employ a more general technique based on invariant properties of differential equations [36] to abstract away the concrete trajectories of nonlinear differential equations by describing \textit{invariant regions} that confine the trajectories. The challenge with such overapproximations is that they are not enough to conclude the existence of a model run as required for correct monitor synthesis [27].

The crucial insight that we gain from a successful safety proof for monitoring and model validation is that the potentially complicated exact behavior is not relevant in detail for safety, but that it was enough to stay inside a safety-relevant region. We exploit this observation to synthesize monitoring conditions that only check these safety-critical invariant regions and thereby allow for a wider safe variety of actual system behavior instead of insisting on the specific modeled behavior.

In this section, we discuss techniques to translate models to make sure that their differential invariants are expressed explicitly and their specific dynamics are abstracted to
any behavior inside the invariant regions. That way, the invariant conditions are picked up during monitor synthesis in a provably correct way and become represented in the monitoring conditions that are derived from these models. As a main insight, we exploit the fact that the desired monitoring conditions must concisely capture paths to safety violation. Specifically, a model monitor for a model \( \alpha \) of the form \( \text{ctrl}; \text{plant} \) uses a model \( \tilde{\alpha} \) that overapproximates the plant \( \{ x' = \theta \land Q \} \) with nondeterministic assignments \( x := \ast \). These assignments are guarded by the evolution domain \( Q \) and differential invariants \( R(x,x_0) \) before and after the nondeterministic assignments to conservatively preserve the semantics of evolution domain constraints in differential equations, so \( \tilde{\alpha} \) has the shape \( x_0 := x; \ ?Q; \ x := \ast; \ ?(Q \land R(x,x_0)) \).

Theorem 6 guarantees that a satisfied monitor for program \( \tilde{\alpha} \) preserves an inductive safety property for one control step, so can be extended to loops in a straightforward way as in previous work [27]. For simplicity, we assume that the world outside \( \tilde{\alpha} \) is unmodified on \( \text{BV}((\tilde{\alpha})^0) \), which can be lifted easily with previous techniques [27].

**Theorem 6 (Nonlinear model monitor correctness).** Let \( \alpha \) be a hybrid program of the form \( w; \text{ctrl}(x); x' = f(x,u) \) with evolution domain constraint \( Q \). Let \( \alpha^* \) be provably safe with invariant \( J \), so \( A \rightarrow [\alpha^*]S \), \( A \rightarrow J \), \( J \rightarrow [\alpha]J \), and \( J \rightarrow S \) are valid. Let \( x = \text{BV}(x' = f(x,u)) \) and let \( x_0 \) be fresh variables not in \( V(\alpha) \). Let the differential invariants \( R(x,x_0) \) be provable, so \( J \rightarrow [w;\text{ctrl}(x); x_0 := x; x' = f(x,u)]R(x,x_0) \) is valid. Assume the system transitions from state \( \omega \) to state \( \nu \), which agree on \( \text{BV}(\alpha)^0 \), and assume \( \omega \models J \). If the nonlinear model monitor

\[
\chi_{\tilde{\alpha}} \equiv \{(w;\text{ctrl}(x); x_0 := x; \ ?Q; \ x := \ast; \ ?(Q \land R(x,x_0))^\ast \} \}
\]

is satisfied, i.e., \( (\omega,\nu) \models \chi_{\tilde{\alpha}} \), then the invariant \( J \) is preserved, i.e., \( \nu \models [J] \).

**Proof.** Follows from [27, Theorem 1] by reducing the complicated dynamics \( J \rightarrow ([w;\text{ctrl}(x); \{ x' = \theta \land Q \}]J) \) to the safety proof of the nondeterministic approximation, i.e., formula (11) is valid.

\[
\begin{align*}
( J & \rightarrow [w;\text{ctrl}(x); x_0 := x; \ ?Q; \ x := \ast; \ ?Q \land R(x,x_0)]J) \\
& \quad \overset{\text{id}}{\Rightarrow} ( J \rightarrow [w;\text{ctrl}(x); \{ x' = \theta \land Q \}]J)
\end{align*}
\]

We use \( \vdash \text{ghost} \) to introduce assignments to fresh variables \( x_0 \) that store the state of \( x \) before the plant, so that it is available when describing the differential equation with
the differential invariant $R(x, x_0)$ in step $dC$. The side condition closes by assumption $J \rightarrow [u \in \text{ctrl}(x) : x_0 := x; x' = f(x, u)]R(x, x_0)$. Differential invariants hold throughout the entire evolution of a differential equation, from beginning to end: step $DW$, $[?]$ makes it available after the differential equation, step $[?] Q$ at the beginning using the axiom $[?] q(x) \{ x' = f(x) \land q(x) \} \implies \{ x' = f(x) \land q(x) \} p(x)$ derived from DI [32]. Now that the differential equation is safeguarded by its differential invariants, in step GVR and $[\ast]$ we abstract from the concrete dynamics by allowing any evolution that satisfies the differential invariants.

$\square$

**Example 5 (Model monitor).** The flight protocol with a nondeterministic plant turns into the following monitor condition, where $\Upsilon^+$ is $\bigwedge_{z \in \{ x, y, \theta, w_0 \}} z^+ := z$:

\[
\langle \text{ctrl}; x_0 := x; y_0 := y; \theta_0 := \theta; ?T; x := *; y := *; \theta := *; \rangle \land (w_o = 0 \rightarrow S = S(0)) \land (w_o = 1 \rightarrow T = T(0)) \Upsilon^+
\]

The synthesis steps in Section 6 (Fig. 6) produce this arithmetical monitor condition:

\[
\mathcal{I} \land \theta^+ = \theta \land \sin \theta^+ x^+ - (\cos \theta^+ - 1)y^+ = \sin \theta x - (\cos \theta - 1)y \land w_o^+ = 0 \land w_i^+ = 0 \\
\lor \mathcal{J}(w_o^+) \land \sin \theta^+ x^+ - \cos \theta^+ y^+ + \cos \theta^+ = \sin \theta x - \cos \theta y + \cos \theta \land w_o^+ = 1 \land w_i^+ = 0
\]

The model monitors derived by abstracting from the specific continuous dynamics to the invariant regions identified in the safety proof allow monitoring of models without symbolic closed-form solution. The invariant regions allow for some deviation from the specific modeled dynamics, but still assume that control choices are perfectly turned into physical effects and that the real world is perfectly observable. The techniques in the main part of this paper present two fundamental extensions with different flavors of uncertainty: actuator disturbance due to partial controllability and sensor uncertainty due to partial observability.

### G Implementation

The process for synthesizing monitors from hybrid systems models with actuator disturbance and sensor uncertainty is illustrated in Fig. 6, with original model $\alpha$ and overapproximated model $\tilde{\alpha}$. The process is implemented as a tactic in KeYmaera X [15]. The bottom step in Fig. 6 uses $\text{dl}$ automation and quantifier elimination (QE) to synthesize an easily executable quantifier-free arithmetical model monitoring condition $F(x, x^+)$ as illustrated in the following sequent proof. The steps are a straightforward application of $\text{dl}$ axioms from the innermost formula going outwards. The existential quantifier can be instantiated by applying [27, Opt. 1], since the state recall is $\Upsilon^+ \equiv x^+ = x$. Any
verified runtime validation for partially observable hybrid systems

Fig. 6: Proof-guided model transformation for synthesizing a monitor for nonlinear differential equations with sensor uncertainty by monitoring for pairwise existence of unobservable true values \( y, y^+ \). Synthesis correct by a chain of semantical representation, logic characterization, and arithmetical form of a monitor.

remaining quantified sub-formulas are turned into their equivalent quantifier-free form with external solvers for QE, which are connected to KeYmaera X (e.g., Mathematica).

\[
\begin{align*}
\vdash & F(x \setminus y, x^+ \setminus y^+) \\
R & \vdash \exists y \in \mathcal{U}_\Delta(\hat{y}) \left( Q(u) \land Q(x^+) \land R(x^+, u) \land (\exists y^+ \forall^+) \right) \\
\exists R & \vdash \exists y \in \mathcal{U}_\Delta(\hat{y}) \left( Q(u) \land \exists x (Q \land R(x, u) \land (\exists y^+ \forall^+)) \right) \\
(\forall=) & \vdash \exists y \in \mathcal{U}_\Delta(\hat{y})(x_0 := u) \left( Q(u) \land \exists x (Q \land R(x, x_0) \land (\exists y^+ \forall^+)) \right) \\
(?) & \vdash \exists y \in \mathcal{U}_\Delta(\hat{y})(u: \in \text{ctrl}(\hat{y}))(x_0 := x) \left( ?Q \exists x (Q \land R(x, x_0) \land (\exists y^+ \forall^+)) \right) \\
(\forall?) & \vdash \exists y \in \mathcal{U}_\Delta(\hat{y})(u: \in \text{ctrl}(\hat{y}))(x_0 := x) \left( ?Q \forall x (Q \land R(x, x_0) \land (\exists y^+ \forall^+)) \right) \\
(?) & \vdash \exists y \in \mathcal{U}_\Delta(\hat{y})(u: \in \text{ctrl}(\hat{y}))(x_0 := x) \left( ?Q \forall x (Q \land R(x, x_0) \land (\exists y^+ \forall^+)) \right)
\end{align*}
\]

KeYmaera X applies time-bounded heuristics to reduce formula size before calling external solvers, since the performance of external solvers in this final step depends on the number of variables and quantifier alternations in the monitor condition. As a result, the proof step numbers reported here may slightly differ when rerunning the synthesis on other platforms.