On inverse and right inverse ordered semigroups

A. Jamadar and K. Hansda

Abstract

A regular ordered semigroup $S$ is called right inverse if every principal left ideal of $S$ is generated by an $\mathcal{R}$-unique ordered idempotent. Here we explore the theory of right inverse ordered semigroups. We show that a regular ordered semigroup is right inverse if and only if any two right inverses of an element $a \in S$ are $\mathcal{R}$-related. Furthermore, different characterizations of right Clifford, right group-like, group like ordered semigroups are done by right inverse ordered semigroups. Thus a foundation of right inverse semigroups has been developed.

Key Words and phrases: ordered regular, ordered inverse, ordered idempotent, completely regular, right inverse.

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1 Introduction

Right inverse semigroups are those, every element of which has unique right inverse. Thus naturally it becomes generalization of inverse semigroups. Many extensive studies have been done on right inverse semigroups by P.S. Venkatesan [13], G.L. Bailes [2] and some others. P.S. Venkatesan [13] studied these semigroups under the name of right unipotent semigroups. He showed that a semigroup is right inverse if and only if every right ideal of it generated by an idempotent.

T. Saito [12], studied inverse semigroup by introducing simple ordered on it. Bhuniya and Hansda [1] have deal with ordered semigroups in which any two inverses of an element are $\mathcal{H}$-related. These ordered semigroups are the analogue of inverse semigroups. Hansda and Jamadar [9] named these ordered semigroups inverse ordered semigroups. They gave a detailed exposition on the characterization of these ordered semigroups. Here we generalize such ordered semigroups into right inverse ordered semigroups. This paper is inspired by the works done by P.S.Venkatesan [13], G.L.Bailes [2].
The presentation of this article is as follows: This section is followed by preliminaries. Section 3 is devoted to the right inverse ordered semigroups. Here Clifford ordered semigroups have been characterized by right inverse semigroups.

2 Preliminaries

An ordered semigroup is a partiality ordered set \((S, \cdot, \leq)\), and at the same time a semigroup \((S, \cdot)\) such that for all \(a, b, x \in S\) \(a \leq b\) implies \(xa \leq xb\) and \(ax \leq bx\). It is denoted by \((S, \cdot, \leq)\). For every subset \(H \subseteq S\), denote \(\langle H \rangle = \{ t \in S : t \leq h, \text{ for some } h \in H \}\). Throughout this article, unless stated otherwise, \(S\) stands for an ordered semigroup and We assume that \(S\) does not contain the zero element.

An equivalence relation \(\rho\) is called left (right) congruence if for \(a, b, c \in S\) \(a \rho b\) implies \(ca \rho cb\) \((ac \rho bc)\). By a congruence we mean both left and right congruence. A congruence \(\rho\) is called semilattice congruence on \(S\) if for all \(a, b \in S\), \(a \rho a^2\) and \(ab \rho ba\). By a complete semilattice congruence we mean a semilattice congruence \(\sigma\) on \(S\) such that for \(a, b \in S\), \(a \leq b\) implies that \(a \sigma ab\). The ordered semigroup \(S\) is called complete semilattice of subsemigroups of type \(\tau\) if there exists a complete semilattice congruence \(\rho\) such that \((x)\rho\) is a type \(\tau\) subsemigroup of \(S\). Let \(I\) be a nonempty subset of \(S\). Then \(I\) is called a left (right) ideal of \(S\), if \(SI \subseteq I\) \((IS \subseteq I)\) and \(I\) \(\subseteq I\). If \(I\) is both left and right ideal, then it is called an ideal of \(S\). We call \(S\) a (left, right) simple ordered semigroup if it does not contain any proper (left,right) ideal. For \(a \in S\), the smallest (left, right) ideal of \(S\) that contains \(a\) is denoted by \((L(a), R(a))I(a)\).

\(S\) is said to be regular (resp. Completely regular, right regular) ordered semigroup if for every \(a \in S\), \(a \in (aSa)\) \((a \in a^2Sa^2\), \(a \in (a^2S)\)). Due to Kehayopulu \[6\] Green’s relations on a regular ordered semigroup give as follows:

\[ a \mathcal{L} b \text{ if } L(a) = L(b), a \mathcal{R} b \text{ if } R(a) = R(b), a \mathcal{J} b \text{ if } I(a) = I(b), \mathcal{H} = \mathcal{L} \cap \mathcal{R}. \]

This four relation \(\mathcal{L}, \mathcal{R}, \mathcal{J},\) and \(\mathcal{H}\) are equivalence relation.

A regular ordered semigroup \(S\) is said to be group-like (resp. left group-like) \([\Pi]\) ordered semigroup if for every \(a, b \in S\), \(a \in (Sb)\) and \(b \in (aS)\) (resp. \(a \in (Sb)\)). Right group like ordered semigroup can be defined dually. A regular ordered semigroup \(S\) is called a right (left) Clifford \([\Pi]\) ordered semigroup if for all \(a \in S\), \((Sa) \subseteq (aS)\), \((aS) \subseteq (Sa)\). Every right (left) group like ordered semigroup is a right (left) Clifford ordered semigroup. An element \(b \in S\) is said to be an inverse of \(a \in S\) if \(a \leq aba\) and \(b \leq bab\). The set of all inverses of an element \(a\) is denoted by \(V_{\leq}(a)\).
Theorem 2.1. Let $S$ be a regular ordered semigroup. Then the following statements are equivalent.

1. $S$ is right Clifford ordered semigroup;
2. for all $e \in E(S), (Se) \subseteq (eS)$;
3. for all $a \in S$, and $e \in E(S)$, there is $x \in S$ such that $ea \leq ax$;
4. for all $a, b \in S$, there is $x \in S$ such that $ba \leq ax$;
5. $L \subseteq R$ on $S$.

Lemma 2.2. Let $S$ be a right Clifford ordered semigroup. Then the following conditions hold in $S$.

1. $a \in (a^2Sa)$, for every $a \in S$;
2. $ef \in (feSef)$, for every $e, f \in E(S)$.

Theorem 2.3. Let $S$ be an ordered ordered semigroup. Then $S$ is right (left) Clifford ordered semigroup if and only if $R(L)$ is the least complete semilattice congruence on $S$.

Theorem 2.4. Let $S$ be a regular ordered semigroup. Then $S$ is right (left) Clifford ordered semigroup if and only if it is a complete semilattice of right (left) group like ordered semigroups.

3 Right Inverse ordered semigroup

3.1 Right inverse ordered semigroups

Let $S$ be an ordered semigroup and $ho$ be an equivalence relation on $S$. In broad sense $ho$-unique we shall mean the uniqueness in respect of the relation $ho$. For example consider a subset $T$ of $S$ such that $a, b$ are generators of $T$. Now if $apb$ we say that $T$ is generated by $ho$-unique element $a$.

Definition 3.1. A regular ordered semigroup $S$ is called right inverse if every principal left ideal is generated by an $R$–unique ordered idempotent of $S$.

We now present results on the role of ordered idempotents to characterize right inverse ordered semigroups.

Theorem 3.2. A regular ordered semigroup $S$ is a right inverse if and only if for any two idempotents $e, f \in E(S), eLf$ implies $eHf$.

Theorem 3.3. Let $S$ be a regular ordered semigroup. Then $S$ is left (right) group like ordered semigroup if and only if any two ordered idempotents are $L (R)$–related.
Corollary 3.4. Every right inverse left group like ordered semigroup is a group like ordered semigroup.

Theorem 3.5. Let $S$ be a regular ordered semigroup. Then any two inverses of an element are $L-$related if and only if $ef \in (eSfSe]$; for some $e, f \in E_\leq(S)$.

In the following theorem, we have shown that any two inverses of an element are $R$-related in a right inverse ordered semigroup. So in the broad sense they are $R$-unique.

Theorem 3.6. The following conditions are equivalent on a regular ordered semigroup $S$.

1. $S$ is right inverse;
2. for $a \in S$ and $a', a'' \in V_\leq(a)$, $a'Ra''$;
3. for $e, f \in E_\leq(S)$, $ef \in (fSeSf]$;
4. $(eS] \cap (fS] = (efS]$;
5. for $e \in E_\leq(S)$ and $x \in (Se]$ implies $x' \in (eS]$, where $x \in S$ and $x' \in V_\leq(x)$.

Corollary 3.7. Let $S$ be a right inverse ordered semigroup. Then any two ordered idempotents $e, f \in E_\leq(S)$ are $H$-commutative if and only if $(Se] \cap (Sf] = (Sef]$.

Example 3.8. The ordered semigroup $S = \{a, e, f\}$ defined by multiplication and order below.

\[
\begin{array}{ccc}
  \cdot & a & e & f \\
 a & a & e & f \\
 e & a & e & f \\
 f & a & e & f \\
\end{array}
\]

\[
' \leq' = \{(a, a), (a, e), (a, f), (e, e), (f, f)\}.
\]

From above table it is clear that $a, e, f \in E_\leq(S)$. Here $ae \leq ace = eae = ceeae$. So $ae \in (eSaSe]$. Also $a \leq aa$ implies that $ea \leq eaa = ac_ea = aeeaa$. So $ea \in (aSeSa]$. Similarly $af \in (fSaSf]$ and $fa \in (aSfSa]$. Also $ef = fef = fefe$ that is $ef \in (fSeSf]$. Similarly it can be shown that $fe \in (eSfSe]$. Thus $(S, \cdot, \leq)$ is a right inverse ordered semigroup.

Theorem 3.9. Let $F$ be a semigroup. Then the ordered semigroup $P_f(F)$ of all subsets of $F$ is a right inverse ordered semigroup if and only if $F$ is a right inverse semigroup.

Theorem 3.10. Let $S$ be a regular ordered semigroup. Then $S$ is a right inverse ordered semigroup if and only if $L_e \subseteq (R_e)'$ for any idempotent $e$ in $S$. 

Theorem 3.11. An ordered semigroup $S$ is right Clifford if and only if $S$ is right inverse and for every $a \in S$, $a \in (a^2 Sa]$.

Theorem 3.12. Let $S$ be a right inverse ordered semigroup. If $S$ is left Clifford then $S$ is union of group like ordered semigroups.

In the following we show that in a right inverse ordered semigroup $R$ is a congruence if and only if $L = H$.

Theorem 3.13. Let $S$ be a right inverse ordered semigroup. The following are equivalent:
1. $R$ is a congruence on $S$;
2. $L = H$;
3. $S$ is a complete semilattice of right group like ordered semigroups.

Our paper ends up with the corollary that follows from Theorem 3.13 and Theorem 3.12 and which gives a characterization on right inverse semigroup to become a completely regular ordered semigroup.

Corollary 3.14. Let $S$ be a right inverse and left regular ordered semigroup. Then following conditions are equivalent.
1. $R$ is a congruence on $S$;
2. $L = H$;
3. $S$ is a complete semilattice of right group like ordered semigroups;
4. $S$ is completely regular.

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