DIOID PARTITIONS OF GROUPS

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Abstract. A partition of a group is a dioid partition if the following three conditions are met: The setwise product of any two parts is a union of parts, there is a part that multiplies as an identity element, and the inverse of a part is a part. This kind of a group partition was first introduced by Tamaschke in 1968. We show that a dioid partition defines a dioid structure over the group, analogously to the way a Schur ring over a group is defined. After proving fundamental properties of dioid partitions, we focus on three part dioid partitions of cyclic groups of prime order. We provide classification results for their isomorphism types as well as for the partitions themselves.

1. Introduction

A dioid is a triple \((D, \oplus, \otimes)\), where \(D\) is a set and \(\oplus\) and \(\otimes\) are two binary operations over \(D\), such that \((D, \oplus, \otimes)\) is a semiring and \(D\) is canonically ordered by \(\oplus\) (see Definition 2.1). Dioids and rings share most of their axioms, except that in the ring case, the additive substructure is a commutative group (hence cannot be canonically ordered by \(\oplus\) - see Remark 2.3). In addition to the inherent algebraic interest in a structure which is both similar to and clearly distinct from a ring, dioids have attracted much attention over the last 30 years due to their interesting practical applications. These include the solution of a variety of optimal path problems in graph theory, extensions of classical algorithms for solving shortest paths problems with time constraints, data analysis techniques, hierarchical clustering and preference analysis, algebraic modeling of fuzziness and uncertainty, and discrete event systems in automation (see [1, 4, 6, 10, 11]).

As is well known, groups give rise in a natural way to rings via the construction of group rings and its generalization to Schur rings. The aim of the present paper is to point out a similar connection between groups and dioids. We consider the following definition, first introduced by Tamaschke [24] (see Remark 1.3).

Definition 1.1. A dioid partition (d-partition for short) of a group \(G\) is a partition \(\Pi\) of \(G\) which satisfies the following:

a. The closure property: \(\forall \pi_1, \pi_2 \in \Pi\), the setwise product \(\pi_1 \pi_2\) is a union of parts of \(\Pi\).

b. Existence of an identity: There exists \(e \in \Pi\) satisfying \(e \pi = \pi e = \pi\) for every \(\pi \in \Pi\). We will also write \(e_{\Pi}\).

c. The inverse property: \(\Pi\) is invariant under the \(G \to G\) mapping defined by \(g \mapsto g^{-1}\) for all \(g \in G\).

When \(e = \{1_G\}\) we shall say that \(\Pi\) is a 1d-partition.
Note that certain classical group partitions satisfy the above conditions and thus form d-partitions.

**Example 1.2.** 1. If $A$ is a subgroup of the group $G$ then the set $\Pi = \{AxA|x \in G\}$ of all double cosets of $A$ in $G$ is a d-partition of $G$ with identity $A$ and $(AxA)^{-1} = Ax^{-1}A$. The case $A = \{1_G\}$ yields the singleton partition $\Pi = \{\{g\}|g \in G\}$ of $G$.

2. Let $\Pi$ be the set of all conjugacy classes of the group $G$. Here the identity element is $Cl(1_G) = \{1_G\}$, and $Cl(x)^{-1} = Cl(x^{-1})$.

The following theorem shows that a d-partition of a group $G$ gives rise to a dioid.

**Theorem 1.3.** Let $G$ be a group and $\Pi$ a d-partition of $G$. Let $D_{\Pi}$ be the set of all possible unions of parts of $\Pi$. Denote set union by $\oplus$ and setwise product of subsets of $G$ by $\otimes$. Then $(D_{\Pi}, \oplus, \otimes)$ is a dioid.

A dioid which is constructed from a d-partition $\Pi$ of $G$ as in Theorem 1.3 will be called a Schur dioid over $G$ (induced by $\Pi$). The reason for this terminology is the strong resemblance of Schur dioids to Schur rings (see Section 3.2). In fact, every Schur ring defines a Schur dioid in a natural way (see Proposition 3.14). The study of Schur rings (and the more general structures of association schemes and coherent configurations) is a well-developed research area which goes back to the work of Issai Schur [22]. Its first systematic treatment was carried by H. Wielandt [28], and since then it has been further developed and found fruitful applications (see [18, 19]). Our interest in studying Schur dioids was motivated by an observation in [7].

In the first part of the paper we consider general properties of d-partitions. We prove several basic results and discuss the analogies and direct relations between Schur dioids and Schur rings. We then present several constructions of d-partitions from other d-partitions. The following theorem provides conditions for refining d-partitions and for making them coarser, and shows that every d-partition can be refined to a 1d-partition. For part (c) see Remark 1.1.

**Theorem 1.4.** Let $G$ be a group. For a d-partition $\Pi$ of $G$ and $A \leq G$ set

\[
\Pi_{<A} := \{\pi \in \Pi|\pi A = A\pi = A\},
\]

\[
\Pi_{>A} := \{\pi \in \Pi|\pi A = A\pi = \pi \setminus \{A\}\}.
\]

a. Let $\Pi$ be a d-partition of $G$, let $A \leq G$, and let $\Pi' := \Pi_{>A} \cup \{A\}$. Then $\Pi'$ is a d-partition of $G$ with $e_{\Pi'} = A$ if and only if $\{\Pi_{<A}, \Pi_{>A}\}$ is a partition of $\Pi$.

b. Let $A \leq G$, let $\Pi_A$ be a d-partition of $A$, and let $\Pi'$ be a d-partition of $G$ with $e_{\Pi'} = A$. Then $\Pi := \Pi_A \cup (\Pi' \setminus \{A\})$ is a d-partition of $G$ with $e_{\Pi} = e_{\Pi_A}$. In particular, for any d-partition $\Pi'$ of $G$ with $e_{\Pi'} = A$ there exists a 1d-partition $\Pi$ of $G$ such that $\Pi_{>A} \subseteq \Pi$.

c. Let $\Pi$ be a d-partition of $G$, and let $A \leq G$ be such that $\Pi_{<A}$ is a partition of $A$. Then $\Pi_{<A}$ is a d-partition of $A$ with $e_{\Pi_{<A}} = e_{\Pi}$ and

\[
\Pi' := \{A\pi A|\pi \in \Pi \setminus \Pi_{<A}\} \cup \{A\}
\]

is a d-partition of $G$ with $e_{\Pi'} = A$.

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1In [2] it is proved that any group with a BN-pair and a finite Weyl group is the product of three conjugates of the Borel subgroup $B$ [2 Theorem 1.5]. The calculations needed for the proof take place in the Schur dioid induced by the double cosets of $B$. 
d. Let $\Pi$ be a $d$-partition of $G$, and let $A < G$ be such that $\Pi_{<A}$ is a partition of $A$. Set $\pi_c := G \backslash A$. Then $\Pi_C := \Pi_{<A} \cup \{\pi_c\}$ is a $d$-partition of $G$ with $e_{\Pi_C} = e_{\Pi_{<A}}$.

The next theorem shows how automorphisms of $G$ can induce new, coarser, $d$-partitions of $G$.

**Theorem 1.5.** Let $G$ be a group and let $A$ be a group that acts on $G$ as automorphisms. Let $\Pi$ be a $d$-partition of $G$. Suppose that the action of $A$ on $G$ induces an action of $A$ on $\Pi$, namely, that for any $a \in A$ and any $\pi \in \Pi$ we have $\pi^a \in \Pi$. Denote by $U_{[\pi]}$ the union of the parts of $\Pi$ contained in the orbit of $\pi \in \Pi$ under the induced action, that is, $U_{[\pi]} := \bigcup_{a \in A} \pi^a$, and set $\Pi' := \{U_{[\pi]} | \pi \in \Pi\}$. Then $\Pi'$ is a $d$-partition of $G$, with $e_{\Pi'} = e_{\Pi}$.

One natural example where the action of $A$ on $G$ induces an action of $A$ on $\Pi$ is provided by taking $\Pi$ to be a double coset partition of an $A$-invariant subgroup of $G$. This idea is elaborated in Section 4 (see Corollary 4.3, Example 4.4 and Corollary 4.5).

Finally we show that $d$-partitions lift from a quotient group to the group.

**Theorem 1.6.** Let $G$ be a group, $N \unlhd G$, and $\overline{\Pi}$ a $d$-partition of $G := G/N$. Then the set $\Pi$ of all of the full preimages in $G$ of the parts of $\overline{\Pi}$ is a $d$-partition of $G$.

In the second part of the paper we study $d$-partitions of finite cyclic groups of prime order. While the Schur partitions of such groups were fully classified by Gordon in [12] (see Theorem 2.9), the possibilities for $d$-partitions are significantly richer. We focus on the case of 3-part $d$-partitions which turn out to be related to natural questions in additive combinatorics. Although the 3-part case is still quite rich, we are able to provide the following classification result.

**Theorem 1.7.** Let $p$ be a prime, and let $G = (\mathbb{Z}_p, +)$. Then the 3-part $d$-partitions of $G$ are all the partitions $\Pi = \{\pi_0, \pi_1, \pi_2\}$ of $G$ with $\pi_0 = \{0\}$ and $|\pi_1| \leq |\pi_2|$, for which exactly one of the following holds:

1. $p = 3$ and $\Pi$ is the singleton partition.
2. $p = 5$ and $\Pi = \{\{0\}, \{1, 4\}, \{2, 3\}\}$.
3. $p > 5$, $\pi_1 = -\pi_1$ and $\pi_1 + \pi_1 = \pi_0 \cup \pi_2$.
4. $p > 5$, $\pi_1 = -\pi_2$, $\pi_1 + \pi_1 = G$ and $\pi_2$ is not an arithmetic progression of size $\frac{p-1}{2}$.
5. $p > 5$, $\pi_1 = -\pi_2$ and $\pi_1 + \pi_1 = \pi_1 \cup \pi_2$.

We note that Theorem 1.7 classifies all of the distinct isomorphism types of 3-part $d$-partitions of $\mathbb{Z}_p$, as each of the cases (1)-(5) defines a distinct isomorphism type.

For cases (4) and (5) of Theorem 1.7 we provide a full and explicit characterization of the corresponding $d$-partitions and use this characterization to estimate their numbers (see Sections 5.2.2 and 5.2.3). In case (3) of the theorem, $\pi_1$ is a symmetric, complete sum-free subset of $G$ (see Section 5.2.1). Such sets received a considerable amount of attention in the literature and found interesting applications (e.g., [21]).

\[\text{2Informally, an isomorphism between two d-partitions is a bijection between the two partitions that respects all of the algebraic content of Definition 1.1. For the precise definition see Section 3.4, Lemma 3.6 and Definition 3.8.}\]
Remark 1.8. During the review process of the current paper, we have learned from the referees that the notion of d-partitions was previously studied by Tamaschke (see [24]). Tamaschke introduces the definition of the object we call a d-partition $\Pi$ of a group $G$, as part of his definition of an $S$-semigroup $T$ on $G$ (which is the semigroup generated by the elements of $\Pi$), and offers interesting extensions of basic concepts in group theory to $S$-semigroups. He also identifies and uses the dioid structure induced by d-partitions, however, and perhaps not surprisingly, without referring to the term dioid or to its general definition. Additional points of overlap between Tamaschke’s discussion and ours are given in Remarks 3.3, 3.9, 4.1, 4.7.

Interestingly, our study of 3-part d-partitions of cyclic groups allows us to settle a question raised by Tamaschke in [25, Problem 1.19] (see Remark 5.5). We believe that reinterpreting Tamaschke’s theory of $S$-semigroups from the point of view of dioids might be of interest in future research.

The paper is organized as follows. Section 2 summarizes some background concepts and results. Section 3 presents several results on d-partitions, including the proof of Theorem 1.3, and analogies between Schur dioids and Schur rings. Section 4 contains the proofs of theorems 1.4, 1.5 and 1.6. Finally, Section 5 presents our results on 3-part d-partitions of cyclic groups of prime order, including the proof of Theorem 1.7.

2. Background Concepts and Results

2.1. Semirings and Dioids.

Definition 2.1 ([11] Chapter 1). A triple $(D, \oplus, \otimes)$, where $D$ is a set and $\oplus, \otimes$ are two binary operations over $D$, is a semiring if the following conditions (a) and (b) hold.

(a) $\oplus$ is commutative and associative, $\otimes$ is associative, and $\otimes$ is distributive over $\oplus$.

(b) $\oplus$ has a neutral element $\varepsilon$, $\otimes$ has a neutral element $e$, and $\varepsilon$ is absorbing for $\otimes$, namely, for any $a \in D$, $a \otimes \varepsilon = \varepsilon \otimes a = \varepsilon$.

We shall assume $\varepsilon \neq e$, and when convenient use $0_D$ for $\varepsilon$ and $1_D$ for $e$, write $+$ for $\oplus$, use the sum symbol $\Sigma$ and omit the $\otimes$ symbol.

The triple $(D, \oplus, \otimes)$ is a dioid if in addition to (a) and (b) the following condition (c) is satisfied:

(c) Existence of order relation condition: The binary relation $\leq_D$ defined over $D$ via:

$$\forall a, b \in D, (a \leq_D b \iff \exists c \in D, a \oplus c = b),$$

is an order relation. In this case we say that $D$ is canonically ordered with respect to $\oplus$.

Furthermore, we say that $\leq_D$ is complete if every subset of elements of $D$ has a supremum in $D$, and that $\leq_D$ is dually complete if every subset of elements of $D$ has an infimum in $D$. The dioid $(D, \oplus, \otimes)$ is complete (dually complete) if $\leq_D$ is complete (dually complete) and the sum operation $\oplus$ can be extended to arbitrary multisets of arguments such that $\otimes$ distributes over the extended sums.

Remark 2.2. If $(D, \oplus, \otimes)$ is a semiring then the relation $\leq_D$ over $D$ is reflexive and transitive. Thus, we can replace condition (c) by the condition that $\leq_D$ is anti-symmetric. Note that some authors reserve the term dioid to idempotent semirings,
namely, to semirings \((D, \oplus, \otimes)\) which satisfy the additional condition \(a \oplus a = a\) for every \(a \in D\). One can easily check that this condition implies that \(\leq_D\) is an order relation. We will refer to these dioids as idempotent dioids.

Remark 2.3. A ring \((D, \oplus, \otimes)\) is a semiring which satisfies the extra condition \((c')\): For any \(a \in D\) there exists \(b \in D\) such that \(a \oplus b = \varepsilon\) (i.e., \((D, \oplus)\) is a commutative group). Thus, both rings and dioids are special semirings. However, using \(\varepsilon \neq e\) assumed in Definition 2.1 \((c)\) and \((c')\) are mutually exclusive [1] Section 1.3.4, Theorem 1).

The Boolean dioid \(B := (\{\varepsilon, e\}, \oplus, \otimes)\) is the smallest dioid. It consists of just the two neutral elements \(\varepsilon\) and \(e\) where \(e \oplus e = e\). It is idempotent and complete.

2.2. Group Semirings. Consider the following natural generalization of the familiar concept of a group ring over a commutative ring.

Definition 2.4. Let \(G\) be a group with identity element \(1_G\) and let \(K\) be a commutative semiring with identity \(1_K \neq 0_K\). Assume further that either \(G\) is finite or, if \(G\) is infinite then \(K\) is a complete dioid. We define the group semiring \(K[G]\) of \(G\) to be the set of all formal sums \(\sum a_g g\) where \(a_g \in K\) for all \(g \in G\). We define a pointwise addition over \(K[G]\) by

\[
\sum_{g \in G} a_g g + \sum_{g \in G} b_g g := \sum_{g \in G} (a_g + b_g) g,
\]

and a convolution product over \(K[G]\) by

\[
\left( \sum_{g \in G} a_g g \right) \left( \sum_{g \in G} b_g g \right) := \sum_{g \in G} \sum_{h \in G} (a_h b_{h^{-1} g}) g = \sum_{g \in G} \sum_{h \in G} (a_{g h^{-1}} b_h) g.
\]

Proposition 2.5. Let \(G\) and \(K\) be as in Definition 2.1 Then \(K[G]\) is a semiring. Moreover, if \(K\) is a dioid then \(K[G]\) is a dioid, and it is complete if \(K\) is complete.

Proof. It is routine to check that \(K[G]\) satisfies the semiring axioms with respect to the two operations defined. The element \(\sum_{g \in G} 0_K g\) is the neutral element with respect to the addition operation and the element \(1_K 1_G + \sum_{g \in G \setminus \{1_G\}} 0_K g\) is the neutral element with respect to the multiplication operation. Of course, one can omit terms with \(0_K\) and identify the subset \(\{s 1_G | s \in K\} \subseteq K[G]\) with \(K\) and the subset \(\{1_K g | g \in G\} \subseteq K[G]\) with \(G\).

In order to prove that \(K[G]\) is a dioid whenever \(K\) is, it remains to prove the existence of a canonical order relation (condition \((c)\) of Definition 2.1). We abuse notation and write \(+\) for the addition operation over \(K\) and over \(K[G]\). Define a binary relation \(\leq_{K[G]}\) over \(K[G]\) by

\[
\forall a, b \in K[G], \ (a \leq_{K[G]} b \iff \exists c \in K[G], \ a + c = b).
\]

By Remark 2.2 it suffices to show that \(\leq_{K[G]}\) is anti-symmetric. Let:

\[
a := \sum_{g \in G} a_g g, \ b := \sum_{g \in G} b_g g, \ c := \sum_{g \in G} c_g g, \ d := \sum_{g \in G} d_g g.
\]

We have to prove that if \(a + c = b\) and \(b + d = a\) then \(a = b\). From \(a + c = b\) we get \(a_g + c_g = b_g, \forall g \in G\). Similarly, \(b + d = a\) implies \(b_g + d_g = a_g, \forall g \in G\). Since \(K\) is a dioid, the first equality implies \(a_g \leq_K b_g, \forall g \in G\), and the second equality implies
can be expressed as a linear combination of the elements of $k$ with (a'): For any $g, \forall g \in G$ and this implies $a = b$. We leave the verification of the completeness claim to the reader.\qed

**Notation 1.** Let $G$ be a group and $K$ be a commutative semiring which is a complete dioid if $G$ is not finite. For any non-empty subset $S$ of $G$ define $\mathbb{S} := \sum_{g \in S} g$, and for any collection $\mathcal{S}$ of non-empty subsets of $G$ define $\mathcal{S} := \{\mathbb{S} | S \in \mathcal{S}\}$.

**Definition 2.6.** Let $G$ be a group and $K$ be a commutative semiring which is a complete dioid if $G$ is not finite. Let $\mathcal{A}$ be a subsemiring of the group semiring $K[G]$. We shall say that $B \subseteq \mathcal{A}$ is a basis of $\mathcal{A}$ over $K$ if each element of $\mathcal{A}$ can be uniquely written as $\sum b \in B$ where $a_b \in K$ for all $b \in B$. For example, $G$ is a basis of $K[G]$ over $K$.\footnote{A basis of $\mathcal{A}$ over $K$ is a subset $B \subseteq \mathcal{A}$ such that each element of $\mathcal{A}$ can be uniquely written as $\sum b \in B$ where $a_b \in K$ for all $b \in B$.}

2.3. **Schur Rings and S-partitions.** In the special case of Definition 2.4 where $K$ is a ring and $G$ is a finite group, we obtain the usual group ring $K[G]$. Schur rings, introduced by Schur in 1933 [22], are special subrings of the group ring $K[G]$, defined as follows.

**Definition 2.7** ([19] Section 2, Definition 2.1). Let $K$ be a ring and let $G$ be a finite group. A subring $\mathcal{A}$ of $K[G]$ is a Schur ring over $G$ if there exists a partition $S$ of $G$ which has the following properties:

- a. The set $S$ is a basis of $\mathcal{A}$ over $K$.
- b. $\{1_G\} \in S$
- c. $X^{-1} \in S$ for all $X \in S$.

We shall call a partition $S$ which has the properties (a)-(c), an S-partition.

Note that in the original Definition 2.1 of [19] $K = \mathbb{C}$.

Let $\mathcal{A}$ be a Schur ring over a finite group $G$, with an associated S-partition $S = \{S_1, ..., S_h\}$ (Definition 2.7 with $K = \mathbb{C}$ for definiteness). For any $1 \leq i, j \leq h$ the product $S_i S_j$ can be expressed as a linear combination of the elements of $S$, as the latter is a basis of $\mathcal{A}$. It follows, from the group multiplication law, that the coefficients of the basis elements are non-negative integers. Writing

$$S_i S_j = \sum_{k=1}^{h} s_{ij}^k S_k,$$

we refer to the $h^3$ coefficients $s_{ij}^k$ as the structure constants of the Schur ring. Note that since $\mathbb{Z} \subseteq \mathbb{C} \subseteq \mathcal{A}$, we can think of the structure constants as elements of the Schur ring.

**Remark 2.8.** One can reformulate the definition of an S-partition in terms of a key property of the structure constants $s_{ij}^k$ by replacing condition (a) of Definition 2.7 with (a’): For any $1 \leq i, j, k \leq h$ and any $z \in S_k$, the number of distinct pairs $(x, y) \in S_i \times S_j$ such that $xy = z$ depends only on $(i, j, k)$ and is independent of the specific choice of $z$. Note that this number equals $s_{ij}^k$, and that this version of the definition does not involve the explicit introduction of a group ring.

2.4. **Finite Cyclic Groups.** Let $n$ be a positive integer. We will work with the realization of a cyclic group of order $n$ as $G := (\mathbb{Z}_n, +_n)$, where $\mathbb{Z}_n := \{0, 1, ..., n-1\}$ and $+_n$ denotes addition modulo $n$. Frequently we write $+$ for $+_n$. Recall that $Aut(G) = (\mathbb{Z}_n^*, \cdot_n)$, where $\mathbb{Z}_n^* = \{a \in \mathbb{Z}_n | \gcd(a, n) = 1\}$, and $\cdot_n$ denotes multiplication modulo $n$. Using this presentation, any $A \leq Aut(G)$ acts on $G$ by $\cdot_n$ multiplication.
For any prime $p$ and any multiplicative subgroup $A$ of $(\mathbb{Z}_p^*, \cdot)$ we set $\Pi(A) := \{\{0\}\} \cup \{Ax | x \in \mathbb{Z}_p^*\}$ - the partition of the additive group $G := (\mathbb{Z}_p, +)$ whose non-trivial parts are all the cosets of the multiplicative subgroup $A \leq (\mathbb{Z}_p^*, \cdot)$. Note that $\Pi(A)$ is not a coset partition of $G$ in the sense of the Example 1.2. We prove that $\Pi(A)$ is an $S$-partition of $G$. By Example 1.2, $\Pi(A)$ is a $d$-partition of $G$, and hence it remains to verify property (a') of Remark 2.8. Let $k$ be the number of distinct cosets of $A$ in $\mathbb{Z}_p^*$. Let $i, j, l \in \{1, \ldots, k\}$ and let $h, g \in Ax_i$. Note that there exists a unique $a \in A$ such that $h = ag$. Suppose that there are exactly $m_g$ distinct pairs $(u, v) \in Ax_i \times Ax_j$ such that $u + v = g$. Then, the $m_g$ distinct pairs $(au, av) \in Ax_i \times Ax_j$ satisfy $au + av = a(u + v) = ag = h$. It follows that $m_g \leq m_h$ and by symmetry $m_g = m_h$.

Gordon gave in [14] an elegant proof that the partitions $\Pi(A)$ exhaust all of the $S$-partitions of $(\mathbb{Z}_p^*, +)$. Thus we have a complete classification of $S$-partitions of cyclic groups of prime order (see [19] for generalizations):

**Theorem 2.9** [12]. Let $p$ be a prime. If $A$ is a multiplicative subgroup of $(\mathbb{Z}_p^*, \cdot)$, then $\Pi(A)$ is an $S$-partition of $(\mathbb{Z}_p^*, +)$. Conversely, if $\Pi$ is an $S$-partition of $(\mathbb{Z}_p^*, +)$ then $\Pi = \Pi(A)$ for some $A \leq \mathbb{Z}_p^*$.

The following lemma provides a useful sufficient condition for a $d$-partition of a prime order group to be an $S$-partition.

**Lemma 2.10.** Let $p$ be a prime and let $G := (\mathbb{Z}_p^*, +)$. Let $\Pi$ be a $d$-partition of $G$, with $e_\Pi = \{0\}$. The following two statements hold:

1. If there exists $\pi \in \Pi \setminus \{e_\Pi\}$ such that $|\pi| = 1$ then $\Pi = \Pi(\{1\})$ (the singleton partition of $G$).
2. If for all $\pi \in \Pi \setminus \{e_\Pi\}$ we have $|\pi| \geq 2$, and there exists $\pi \in \Pi$ such that $|\pi| = 2$, then $\Pi = \Pi(\{-1, 1\})$.

**Proof.** 1. Assume that $\pi \in \Pi \setminus \{e_\Pi\}$ satisfies $|\pi| = 1$. Then $\pi = \{x\}$ for some $x \in \mathbb{Z}_p^*$, and we have, for any positive integer $k$,

$$\pi + \cdots + \pi = \{k \cdot_p x\},$$

$k$ summands

Since $x$ generates $G$ the claim follows.

2. Let $\pi \in \Pi$ be such that $|\pi| = 2$. Then $\pi = \{x_1, x_2\}$ with $x_1 \neq x_2 \in \mathbb{Z}_p^*$. Since $\Pi$ satisfies the inverse property, either $\pi = -\pi$, in which case $x_2 = -x_1$, or $-\pi \in \Pi$ is disjoint from $\pi$. In this second case, we get $\pi + (-\pi) = e_\Pi \cup \{- (x_2 - x_1), x_2 - x_1\}$, and by our assumptions we must have $\{- (x_2 - x_1), x_2 - x_1\} \in \Pi$. Hence we can assume, without loss of generality, that $\pi = \{-x, x\}$ for some $x \in \mathbb{Z}_p^*$. Now set $\pi_k := k \cdot_p \pi$ for $k \in \mathbb{Z}_p^*$, and $\pi_0 := e_\Pi$. Observe that $\pi_1 = \pi$ and that for any $k > 1$, we have $\pi + \pi_{k-1} = \pi_k \cup \pi_{k-1}$. This yields, by an easy induction, that every non-identity part of $\Pi$ is of the form $\{- (k \cdot_p x), k \cdot_p x\}$ which gives our claim. □

We need the following standard lemma, whose easy proof is omitted, for the ensuing corollary.

**Lemma 2.11.** Let $p$ be a prime and let $A \leq \mathbb{Z}_p^*$. Then:

a. Either $A = -A$, or $-A$ is some non-trivial multiplicative coset of $A$.

b. The following conditions are equivalent:

(b1) $A = -A$. 


Corollary 2.12. Let $\pi \in \Pi(A)$, and if $A \neq -A$ then $\pi = -\pi$ for all $\pi \in \Pi(A)$, and if $A \neq -A$ then for every $\pi \in \Pi(A) \setminus \{\{0\}\}$ there exists $\pi' \neq \pi$ such that $\pi = -\pi'$. In particular, if $\Pi(A) = \{\pi_0, \pi_1, \pi_2\}$ then $p \equiv 1 \pmod{4}$ implies $\pi = -\pi$ for all $\pi \in \Pi(A)$ and $p \equiv 3 \pmod{4}$ implies $\pi_1 = -\pi_2$.

2.5. Additive Combinatorics. Following the terminology of [26], an additive set is a non-empty subset $S$ of some abelian group $(G, +)$ written additively. Given two additive sets $A$ and $B$ with the same underlying group, define $A + B := \{a + b | a \in A, b \in B\}$, and $-A := \{-a | a \in A\}$. We shall say that $A$ is symmetric if $A = -A$. The set difference of two sets $A$ and $B$ will be denoted $A \setminus B$ and should not be confused with the arithmetic difference $A - B := A + (-B)$. If $A$ is a subset of $G$ and $G$ is clear from context, we write $A$ for $A \setminus \{0\}$. For any integers $a < b$, the (discrete) closed interval $[a, b]$ is the integer subset $\{a \leq x \leq b | x \in \mathbb{Z}\}$, and we use this notation also for $a, b \in \mathbb{Z}_n$ when $0 \leq a < b \leq n - 1$. An additive subset $S$ of an additive group $(G, +)$ is called an arithmetic progression with step $\delta$ (or, in short, a $\delta$-progression) if there exist $a_0, \delta \in G$ and $N \in \mathbb{N}$ such that $S = \{a_0 + j \cdot \delta | j \in \mathbb{N}_0, 0 \leq j \leq N\} = a_0 + [0, N] \cdot \delta$. Note that we have the easy addition formula for two $\delta$-progressions:

$$((a_0 + [0, N_1] \cdot \delta) + (b_0 + [0, N_2] \cdot \delta)) = (a_0 + b_0 + [0, N_1 + N_2] \cdot \delta).$$

Theorem 2.13 (Cauchy-Davenport Inequality, [3], [5], [26]). Let $p$ be a prime and let $G = (\mathbb{Z}_p, +_p)$. For any two additive subsets $A, B \subseteq G$ we have

$$|A + B| \geq \min(|A| + |B| - 1, p).$$

Theorem 2.14 (Vosper’s Theorem, [27], [26]). Let $p$ be a prime and let $A$ and $B$ be additive subsets of $\mathbb{Z}_p$ such that $|A|, |B| \geq 2$ and $|A + B| \leq p - 2$. Then $|A + B| = |A| + |B| - 1$ if and only if $A$ and $B$ are arithmetic progressions with the same step.

3. D-partitions of Groups

In this section we prove several general properties of d-partitions. We begin with a proof of Theorem 1.3 which claims that if $\Pi$ is a d-partition of $G$, then the set $D_{\Pi}$ of all possible unions of parts of $\Pi$ is a dioid where $\oplus$ and $\otimes$ are taken to be, respectively, set union and setwise product.

Proof of Theorem 1.3 It is straightforward to check that the conditions in Definition 2.7 are satisfied. In particular, the empty set $\emptyset$ is a neutral element with respect to $\oplus$, we have $\emptyset \otimes \pi = \pi \otimes \emptyset = \emptyset$ for any $\pi \in \Pi$, and $e_\Pi$ is a neutral element with respect to $\otimes$. Since set union is an idempotent binary operation, $(D_{\Pi}, \oplus, \otimes)$ is canonically ordered with respect to $\oplus$ (see Remark 2.7, where the canonical order relation $\leq_{D_\Pi}(\text{see Definition 2.11c})$ is set inclusion. \qed

Remark 3.1. Keeping the notation and assumptions of Theorem 1.3, we claim that $D_{\Pi}$ is a complete and dually complete dioid. To see this observe that the supremum of a subset of $D_{\Pi}$ is the union of its elements, and that $G = \bigcup \pi$ is the unique maximal element with respect to $\leq_{D_{\Pi}}$ (G is absorbing with respect to $\otimes$ and all the elements of $D_{\Pi}$ except for $\emptyset$). The infimum of a subset of $D_{\Pi}$ is the intersection
of all of its elements. To see this use the fact that each element of $D_1$ is a union of some elements of $\Pi$, and the intersection of any two distinct elements of $\Pi$ is empty. Finally, unions of infinite families of subsets of $G$ are well defined and the setwise product distributes over such unions. Also note that since $\Pi$ is a partition, $D_\Pi$ is in bijection with the power set of $\Pi$.

3.1. Basic Properties of $d$-partitions. The following lemma summarizes basic properties of the identity element of a $d$-partition.

**Lemma 3.2.** Let $G$ be a group and let $\Pi$ be a partition of $G$ satisfying the closure and existence of an identity element properties (Definition 1.1 (a) and (b)). Then:

a. $e$ is unique.

b. $e$ is a subgroup of $G$.

c. Each $\pi \in \Pi$ is a union of double cosets of $e$.

**Proof.**
a. Standard.

b. Since $e$ is a non-empty subset of $G$, and $e^2 = e$, we get that $e$ is closed under group multiplication. Let $\pi \in \Pi$ be the unique part containing $1_G$. Then, $\pi \subseteq e\pi = \pi$ whence, $e = \pi$ and $1_G \in e$. Let $g \in e$ and let $\sigma$ be the unique part containing $g^{-1}$. Then, $1_G = gg^{-1} \in e\sigma = \sigma$ whence $e = \sigma$. It follows that $e$ is a subgroup of $G$.

c. For any $\pi \in \Pi$ we have

$$\pi = e\pi e = \bigcup_{x \in \pi} exe,$$

and therefore $\pi$ is a union of double cosets of $e$. $\square$

**Remark 3.3.** *Lemma 3.2 is the same as [24, Lemma 1.2].*

The following concept can be used to give an alternative characterization of $d$-partitions.

**Definition 3.4.** Let $G$ be a group and let $\Pi$ be a partition of $G$. We say that $\Pi$ satisfies the intersection property if for all $\pi_1, \pi_2, \pi \in \Pi$ such that $\pi \subseteq \pi_1 \pi_2$, for all $x \in \pi_1$ and for all $y \in \pi_2$ it holds that

$$(x\pi_2) \cap \pi \neq \emptyset, \ (\pi_1y) \cap \pi \neq \emptyset.$$ 

**Theorem 3.5.** Let $\Pi$ be a partition of the group $G$ satisfying the closure and existence of an identity element properties (Definition 1.1 (a) and (b)). Then $\Pi$ satisfies the intersection property if and only if $\Pi$ satisfies the inverse property (Definition 1.2 (c)), equivalently, if and only if $\Pi$ is a $d$-partition.

**Proof.**

1. We suppose that $\Pi$ is a $d$-partition, and prove that it satisfies the intersection property. Let $\pi_1, \pi_2, \pi \in \Pi$ be such that $\pi \subseteq \pi_1 \pi_2$. Since $\pi \subseteq \pi_1 \pi_2$, there exist $\gamma \in \pi$, $\alpha \in \pi_1$, $\beta \in \pi_2$ such that $\gamma = \alpha \beta$. It follows that $\alpha = \gamma \beta^{-1}$. Since $\beta^{-1} \in \pi_2^{-1}$, this shows that $\pi_1 \cap \pi_2^{-1} \neq \emptyset$. By (c) of Definition 1.1, $\pi_2^{-1} \in \Pi$. Hence, $\pi_1 \cap \pi_2^{-1} \neq \emptyset$ implies $\pi_1 \subseteq \pi_2^{-1}$. Hence, for every $x \in \pi_1$ there exist $u \in \pi_2$ and $z \in \pi$ such that $x = zu^{-1}$ which is equivalent to $xu = z$. Thus $z \in (x\pi_2) \cap \pi$ and $(x\pi_2) \cap \pi \neq \emptyset$. The proof that $(\pi_1y) \cap \pi \neq \emptyset$ for all $y \in \pi_2$ is similar.

2. We suppose that $\Pi$ satisfies conditions (a) and (b) of Definition 1.1 and the intersection property, and prove that it satisfies the inverse property. The proof borrows from the ideas of the proof of [12, Lemma 2]. Let $\pi \in \Pi$ be arbitrary. We wish to prove that $\pi^{-1} \in \Pi$. Fix $y \in \pi$ and let $C$ be the unique part in $\Pi$ to which
have proved that \( \pi \) using Lemma 3.2(b) this gives \((\pi) := \{x \mid \pi \} \).

Remark 3.9. Tamaschke defines a homomorphism from one S-semigroup into another (see [24, Definition 2.1]) which extends in a natural way to the associated Schur rings. For bijective homomorphisms, this definition coincides with Definition 3.8 above.

The analogy between Schur rings (see Section 2.3) and Schur dioids suggests the introduction of structure constants for Schur dioids. Their existence follows from the following lemma, whose proof is immediate from Definition 1.1.

**Lemma 3.6.** Let \( G \) be a group, \( \Pi \) a d-partition of \( G \), and \((D_\Pi, \oplus, \otimes)\) the associated Schur dioid over \( G \). Then there exists a function \( d : \Pi^3 \to \{\emptyset, e_\Pi\} \) whose values \( d_{\pi, \sigma, \tau} \in \{\emptyset, e_\Pi\} \), where \( \pi, \sigma, \tau \in \Pi \) are arbitrary, satisfy

\[
\pi \otimes \sigma = \oplus_{\tau \in \Pi} (d_{\pi, \sigma, \tau} \otimes \tau),
\]

for all \( \pi, \sigma \in \Pi \). The \( d_{\pi, \sigma, \tau} \) will be called the structure constants of \((D_\Pi, \oplus, \otimes)\).

**Remark 3.7.** For notational convenience we will write \( 0 \) and \( 1 \) for the values of the dioid structure constants, instead of, respectively, \( \emptyset \) and \( e_\Pi \).

Now we can define isomorphic d-partitions.

**Definition 3.8.** Let \( \Pi_1 \) and \( \Pi_2 \) be two d-partitions with structure constants \((d_1)^{\tau_1}_{\pi_1, \pi_2}\), \( \tau_1, \pi_1, \pi_2 \in \Pi_1 \) and \((d_2)^{\tau_2}_{\sigma_2, \sigma_1}\), \( \tau_2, \sigma_2, \sigma_1 \in \Pi_2 \). Then \( \Pi_1 \) and \( \Pi_2 \) are isomorphic if there exists a bijection \( f : \Pi_1 \to \Pi_2 \) such that for any \( \tau_1, \pi_1, \pi_2 \in \Pi_1 \) we have \((d_1)^{f(\tau_1)}_{f(\pi_1), f(\pi_2)} = (d_2)^{\tau_2}_{\sigma_2, \sigma_1}\).

**Remark 3.9.** Tamaschke defines a homomorphism from one S-semigroup into another (see [24, Definition 2.1]) which extends in a natural way to the associated dioid structures. For bijective homomorphisms, this definition coincides with Definition 3.8 above.

**Remark 3.10.** If \( \Pi \) is a d-partition of a group \( G \) and \( \iota : G \to H \) is a group isomorphism, then \( \iota(\Pi) := \{\iota(\pi) \mid \pi \in \Pi\} \) is a d-partition of \( H \) which is isomorphic to \( \Pi \). However, non-isomorphic groups might still have isomorphic d-partitions. See for instance Corollary 4.5.

### 3.2. S-partitions vs. d-partitions.

Using the concept of a group semiring (Definition 2.4 and Proposition 2.5), we present in this section another approach for constructing a dioid from a suitable group partition, which is on the same footing as the construction of a Schur ring from an S-partition.

**Definition 3.11.** Let \( G \) be a group and \( K \) be a commutative dioid which is a complete dioid if \( G \) is not finite. A subdioid \( A \) of \( K[G] \) is called a generalized \( t \)-Schur dioid over \( G \) if there exists a partition \( \Pi \) of \( G \) which has the following properties:

a. The set \( \Pi \) is a basis of \( A \) over \( K \).
b. \( \{1_G\} \in \Pi \).
c. \( X^{-1} \in \Pi \) for all \( X \in \Pi \).
We shall call a partition $\Pi$ which has the properties (a)-(c), a generalized 1d-partition.

The similarity between Definitions 3.11 and 2.7 is evident. Indeed, it can be verified that 1d-partitions by Definition 3.11 arise as a special case of Definition 3.11 as stated in the following lemma.

**Lemma 3.12.** Definition 3.7 with $e_\Pi = \{1_G\}$ and Definition 3.11 with $K = \mathbb{B}$ (the Boolean dioid) coincide.

For the rest of the paper we work with Definition 3.11. It would be interesting to study dioids which arise from other choices of $K$. Here we just comment on one possible direction. As Schur rings are mainly studied over fields, it makes sense to look for an analogue structure in the case of commutative dioids. An obvious idea is to look at commutative dioids $K$ for which every element besides $\varepsilon$ is invertible with respect to $\otimes$. We call such dioids d-fields (see also [11, Definition 1.5.2.3]). For example, $\mathbb{B}$ is a d-field. However, we are unaware of a systematic study of d-fields.

We offer the following observation whose proof is given in the appendix (Section 6).

**Proposition 3.13.** Let $(D, \oplus, \otimes)$ be a d-field. Then either, for each $d \in D \setminus \{\varepsilon\}$ all finite sums of the form $d, d \oplus d, d \oplus d \oplus d, \ldots$ are distinct, or $D$ is an idempotent dioid. Furthermore, if $D$ is an idempotent dioid then either $D = \mathbb{B} = \{(\varepsilon, e), \oplus, \otimes\}$ or $D$ has no largest element, and, in particular, $D$ is infinite and is not complete.

Our last result in this section is the following direct connection between $S$-partitions and d-partitions.

**Proposition 3.14.** Every $S$-partition $S$ of a finite group $G$ over a commutative ring $R$ of characteristic zero is a 1d-partition. Furthermore, choosing some numbering of the parts of $S$, and denoting the structure constants of the associated Schur dioid by $s_{ij}^{k}$, and those of the associated Schur dioid by $d_{ij}^{k}$, we have that $d_{ij}^{k} = 0$ if and only if $s_{ij}^{k} = 0$ and $d_{ij}^{k} = 1$ if and only if $s_{ij}^{k} > 0$.

**Proof.** Let $S$ be an $S$-partition of a finite group $G$ over $R$ and let $A$ be the associated Schur ring. The existence of an identity, and the inverse property (Definition 1.1 (b) and (c)) follow immediately from (b) and (c) of Definition 2.7. For the closure property (Definition 1.1 (a)), let $S_1$ and $S_2$ be any two elements of $S$. We have to prove that $S_1 S_2$ is a union of elements of $S$. Since $\mathcal{S}$ is a basis of $A$ over $R$, and $S_1, S_2 \in A$, we have $S_1 S_2 = \sum_{S \in S} b_S S$ where $b_S \in R$ are uniquely determined for all $S \in S$. Using the definition of the product of the formal sums $S_1$ and $S_2$, and the assumption that $R$ is of characteristic zero, we get that $b_S = n_S 1_R$, where, for any $s \in S$, the non-negative integer $n_S$ counts the number of distinct pairs $(s_1, s_2)$ with $s_1 \in S_1$ and $s_2 \in S_2$ such that $s_1 s_2 = s$. Note that $n_S$ does not depend on the specific choice of $s \in S$ (see Remark 2.8). We show that if $(S_1 S_2) \cap S_3 \neq \emptyset$ then $S_3 \subseteq S_1 S_2$. Suppose that $x \in S_1$ and $y \in S_2$ are such that $xy \in S_3$. Then $n_{S_3} \geq 1$, and hence, for every $s \in S_3$ there exists a pair $(s_1, s_2)$ with $s_1 \in S_1$ and $s_2 \in S_2$ such that $s_1 s_2 = s$, proving $S_3 \subseteq S_1 S_2$. The fact that $(S_1 S_2) \cap S_3 \neq \emptyset$ implies $S_3 \subseteq S_1 S_2$ proves that $S_1 S_2$ is a union of elements of $S$. The relation between the structure constants follows easily. \[\square\]
4. INDUCING D-PARTITIONS FROM GIVEN ONES

In this section we prove several results which share the following feature: A d-partition $\Pi$ of a group $G$ induces a d-partition $\Pi'$ of a related group. We start with the proof of Theorem 1.4.

Proof of Theorem 1.4. Observe that any $\pi \in \Pi_{<A}$ is contained in $A$ while any $\pi \in \Pi_{>A}$ is a union of non-trivial double cosets of $A$. Hence $\Pi_{<A} \cap \Pi_{>A} = \emptyset$.

a. Suppose that $\{\Pi_{<A}, \Pi_{>A}\}$ is a partition of $\Pi$. We have to prove that $\Pi' = \Pi_{>A} \cup \{A\}$ is a d-partition of $G$ with $e_{\Pi'} = A$. We have

\[ (4.1) \quad \bigcup_{\pi \in \Pi_{<A}} \pi \bigcup \bigcup_{\pi \in \Pi_{>A}} \pi = \bigcup_{\pi \in \Pi} \pi = G. \]

Since $\bigcup_{\pi \in \Pi_{<A}} \pi \subseteq A$ it follows that the set $\bigcup_{\pi \in \Pi_{>A}} \pi$ contains each non-trivial double coset of $A$. Since each part in $\Pi_{>A}$ is a union of non-trivial double cosets of $A$, we can conclude that $\Pi_{>A}$ is a partition of $G \setminus A$ whose parts are unions of double cosets of $A$. Hence, by Equation (4.1), $\bigcup_{\pi \in \Pi_{<A}} \pi = A$. The claim that $\Pi'$ is a d-partition with $e_{\Pi'} = A$ follows easily.

Suppose that $\{\Pi_{<A}, \Pi_{>A}\}$ is not a partition of $\Pi$. We prove that $\Pi' := \Pi_{>A} \cup \{A\}$ is not a partition of $G$. Since $\Pi_{<A} \subseteq \Pi$, $\Pi_{>A} \subseteq \Pi$ and $\Pi_{<A} \cap \Pi_{>A} = \emptyset$, there exists some $\pi \in \Pi$ such that $\pi \notin \Pi_{<A} \cup \Pi_{>A}$. This implies $\pi \cap A \subseteq \pi$, and $\pi \cap \bigcup_{\sigma \in \Pi_{>A}} \sigma = \emptyset$. Consequently, $\pi \cap \bigcup_{\sigma \in \Pi_{>A}} \sigma = \pi \cap A \subset \pi$, proving that $\Pi'$ is not a partition of $G$.

b. Clearly $\Pi$ is a partition of $G$. We prove that it has the closure property. Let $\pi_1, \pi_2 \in \Pi$. If $\pi_1, \pi_2 \in \Pi_A$ then $\pi_1 \pi_2$ is a union of parts from $\Pi_A \subseteq \Pi$. If $\pi_1, \pi_2 \in \Pi' \setminus \{A\}$ then $\pi_1 \pi_2$ is a union of parts from $\Pi' \setminus \{A\}$ and possibly also $A$ (which is the union of all parts in $\Pi_A$). If $\pi_1 \in \Pi_A$ and $\pi_2 \in \Pi' \setminus \{A\}$, then $\pi_1 \subseteq A$ and $\pi_2$ is a union of double cosets of $A$, and so $\pi_1 \pi_2 = \pi_2$. A similar argument applies if $\pi_1 \in \Pi' \setminus \{A\}$ and $\pi_2 \in \Pi_A$. The claim $e_{\Pi} = e_{\Pi_A}$ and the inverse property are easy to verify. Finally, taking $\Pi_A$ to be the partition of $A$ whose parts are the conjugacy classes of $A$ proves the last assertion of (b).

c. We prove that $\Pi_{<A}$ is a d-partition of $A$. By assumption $\Pi_{<A}$ is a partition of $A$. For any $\pi_1, \pi_2 \in \Pi_{<A}$ we have that $\pi_1 \pi_2 \subseteq A$ because of the closure of $A$ under multiplication. On the other hand, since $\Pi$ is a d-partition of $G$, $\pi_1 \pi_2$ is a union of parts of $\Pi$. Hence $\pi_1 \pi_2$ is a union of parts of $\Pi_{<A}$. Moreover, $1_G \in A$ implies $e_{\Pi} \subseteq A$, and since $A^{-1} = A$ we get that $\Pi_{<A}$ has the inverse property.

In order to prove that $\Pi'$ is a d-partition of $G$, we first prove that it is a partition of $G$. It is easily seen that the elements of $\Pi'$ are non-empty and that their union is equal to $G$. We prove that distinct elements of $\Pi'$ are disjoint. For this it suffices to show that if $\pi_1, \pi_2 \in \Pi' \setminus \{A\}$, and $(\pi_1 A) \cap (\pi_2 A) \not= \emptyset$, then $A \pi_1 A = A \pi_2 A$. Assuming $(\pi_1 A) \cap (\pi_2 A) \not= \emptyset$, there exist $a_i \in A$ where $i = 1, 2, 3, 4$ and $x \in \pi_1$, $y \in \pi_2$ such that $a_1 x a_2 = a_3 y a_4$. This implies $x = a_1^{-1} a_3 y (a_4 a_2^{-1}) \in A \pi_2 A$. Since $A$ is a union of parts of $\Pi$, which is a d-partition, $A \pi_2 A$ is a union of parts of $\Pi$ and hence, since $x \in \pi_1 \in \Pi$ satisfies $x \in A \pi_2 A$ we get that $\pi_1 \subseteq A \pi_2 A$. Multiplying by $A$ from each side gives $A \pi_1 A \subseteq A \pi_2 A$. The inclusion $A \pi_2 A \subseteq A \pi_1 A$ is proved similarly and hence $A \pi_1 A = A \pi_2 A$.

Now we prove that the partition $\Pi'$ is a d-partition of $G$. Clearly, $A \tau = \tau A = \tau$ for any $\tau \in \Pi'$ and hence $A$ acts as an identity. In order to complete the proof
of the closure property of $\Pi'$ it suffices to show that if $\pi_1, \pi_2 \in \Pi \setminus \Pi_{<A}$ then $(A\pi_1A)(A\pi_2A)$ is a union of parts of $\Pi'$. We have $(A\pi_1A)(A\pi_2A) = A(\pi_1A\pi_2A)$.

Since $\Pi$ is a d-partition and $A$ is a union of parts of $\Pi$, we get that $\pi_1A\pi_2$ is a union of parts of $\Pi$. Let $\pi \in \Pi$ be such a part. If $\pi \in \Pi_{<A}$ then $A\pi A = A \in \Pi'$. Otherwise $\pi \in \Pi \setminus \Pi_{<A}$ and again $A\pi A \in \Pi'$.

Finally, $(A\pi A)^{-1} = A\pi^{-1} A$, and therefore the inverse property of $\Pi$ implies that $\Pi'$ has the inverse property.

d. Note that by (c), $\Pi_{<A}$ is a d-partition of $A$, and hence, by (b), we can assume $\Pi_{<A} = \{A\}$. We prove the closure property of $\Pi_C$ (the existence of an identity and the inverse property also follow easily from the following arguments). We have $A^2 = A$, and since $\pi_c$ is a union of double cosets of $A$, we have $A\pi_c = \pi_c A = \pi_c$.

It remains to consider $\pi_c^2$. If $|G : A| = 2$ then $\pi_c^2 = A$, since $A \leq G$ and $\pi_c$ is the single non-trivial coset of $A$. Otherwise $|G : A| > 2$. We prove that $\pi_c^2 = G$. Notice that $\pi_c^{-1} = \pi_c$, and that for any $x \in G$ it holds that $x \in \pi_c$ if and only if $Ax \subseteq \pi_c$.

Let $h \in \pi_c$ be arbitrary. Since $h^{-1} \in \pi_c$ and $Ah \subseteq \pi_c$, we have $A = (Ah) h^{-1} \subseteq \pi_c^2$. Furthermore, since $|G : A| > 2$, there exists $g \in \pi_c$ such that $Ag^{-1} \neq Ah^{-1}$ which is equivalent to $g^{-1}h \in \pi_c$. Therefore $Ah = (Ag) (g^{-1}h) \subseteq \pi_c^2$. As $Ah$ is an arbitrary coset of $A$ contained in $\pi_c$ we get $\pi_c \subseteq \pi_c^2$, and therefore $\pi_c^2 = G$. □

**Remark 4.1.** Part (c) of Theorem 1.4 is the same as [24] Proposition 1.4. The translation of our presentation in (c) to Tamaschke’s terminology goes as follows.

Let $T$ be the S-semigroup associated with $\Pi$. The assumption that $\Pi_{<A}$ is a partition of $A$ is equivalent in Tamaschke’s terminology to the assumption that $A$ is a $T$-subgroup of $G$ (see [24] Definition 1.3), and the conclusion that $\Pi_{<A}$ is a d-partition is equivalent to [24] Proposition 1.4 (1)]. The claim concerning $\Pi'$ is equivalent to [24] Proposition 1.4 (2)].

**Remark 4.2.** As a by-product of the proof of part (d) of Theorem 1.4 we find that there are exactly two isomorphism types (see Definition 3.3) of 2-part d-partitions.

**Proof of Theorem 1.3.** It is immediate to see that $\Pi'$ is a partition of $G$. Set $e := e_1$. Let $a \in A$. We have $1_g \in e^a \in \Pi$. This implies $e^a = e$ for all $a \in A$, and hence $e = U_{[e]} \in \Pi'$. Since every $U_{[e]}$ is a union of parts of $\Pi$, we get $e U_{[e]} = U_{[e]} e = U_{[e]}$. This proves $e_{\Pi'} = e$.

Now let $\pi_1, \pi_2 \in \Pi$ and consider the setwise product $U_{[\pi_1]} U_{[\pi_2]}$. We have:

$$U_{[\pi_1]} U_{[\pi_2]} = \bigcup_{a_1, a_2 \in A} \pi_1^{a_1} \pi_2^{a_2} = \bigcup_{a_1, a_2 \in A} \left( \pi_1^{a_2 a_1^{-1}} \right)^{a_1} = \bigcup_{a \in A} \bigcup_{g \in A} (\pi_1 g \pi_2)^a = \bigcup_{a \in A} \bigcup_{g \in A} (\pi_1 g \pi_2)^a.$$

Since $\pi_1, \pi_2 \in \Pi$ we have $\pi_1 \pi_2^g = \bigcup_{i \in I_g} \pi_i$, where $I_g$ is some indexing set. It follows that

$$U_{[\pi_1]} U_{[\pi_2]} = \bigcup_{g \in A} \bigcup_{a \in A} \left( \bigcup_{i \in I_g} \pi_i \right)^a = \bigcup_{g \in A} \bigcup_{a \in A} \left( \bigcup_{i \in I_g} \pi_i^a \right) = \bigcup_{g \in A} \bigcup_{a \in A} \left( \bigcup_{i \in I_g} \pi_i^a \right) = \bigcup_{g \in A} U_{[\pi_i]}.$$

Thus we have shown that $U_{[\pi_1]} U_{[\pi_2]}$ can be written as a union of elements of $\Pi'$. The inverse property follows immediately from $(\pi^a)^{-1} = (\pi^{-1})^a$. □
Corollary 4.3. Let $G$ be a group and let $A$ be a group that acts on $G$ as automorphisms. Let $H \subseteq G$ and $\Pi := \{HxH | x \in G\}$. Then $A$ acts on $\Pi$ if and only if $H$ is $A$-invariant, and in this case $U_{[HxH]} = HO_A(x)H$, where $O_A(x) := \bigcup_{a \in A} x^a$ is the orbit of $x$ under the action of $A$.

Proof. Suppose that $A$ acts on $\Pi$. Then, by the proof of Theorem 1.5, $U_{[H]} = H$ and hence $H$ is invariant under the action of $A$. Conversely, suppose that $A$ acts on $H$. Then, for any $x \in G$ and $a \in A$ we get, using $H^a = H$, that $(HxH)^a = Hx^aH \subseteq \Pi$, and hence $A$ acts on $\Pi$. In this case:

$$U_{[HxH]} = \bigcup_{a \in A} (HxH)^a = \bigcup_{a \in A} Hx^aH = H \left( \bigcup_{a \in A} x^a \right) H = HO_A(x)H.$$ 

Example 4.4. Let $n > 0$ be an integer, $G = (\mathbb{Z}_n, +_n)$, $H \subseteq G$ and $A \subseteq \text{Aut}(G) = (\mathbb{Z}_n, \cdot_n)$. Since $G$ is abelian, each double coset of $H$ in $G$ is a single coset of the form $H + x$ for some $x \in G$, and since $G$ is cyclic, $H$ is a characteristic subgroup of $G$ and hence it is invariant under $A$. Therefore, by Corollary 4.3, $A$ acts on the $d$-partition $\Pi = \{H + x | x \in G\}$ and the orbits of this action form the $d$-partition $\Pi' = \{H + A \cdot_n x | x \in G\}$. In the special case where $n = p$ is a prime, non-trivial examples arise from the choice $H = \{0\}$. For this choice $\Pi$ is the singleton partition, and $\Pi' = \{\{0\}\} \cup \{A \cdot_p x | x \in \mathbb{Z}_p^\times\}$, i.e., the non-identity parts of $\Pi'$ are the cosets of the multiplicative subgroup $A \subseteq \mathbb{Z}_p^\times$, all of which have size $|A|$. The following is a consequence of Corollary 4.3.

Corollary 4.5. Let $G$ be a group, $N \subseteq G$ and $A \subseteq G$ such that $AN = G$. Then:

(a) $\Pi_N := \{(A \cap N)O_A(n) | n \in N\}$ is a $d$-partition of $N$ with $e_{\Pi_N} = A \cap N$.

(b) Let $\Pi_G := \{AyA | y \in G\}$. Then $f : \Pi_G \to \Pi_N$ defined by $f(AyA) = (A \cap N)O_A(n)$, where $n$ is any element of $N$ satisfying $yn^{-1} \in A$, is an isomorphism of $d$-partitions.

In order to prove Corollary 4.5 we begin with a lemma.

Lemma 4.6. Let $G$ be a group, $N \subseteq G$ and $A \subseteq G$ such that $AN = G$. Let $y \in G$. Then there exists $n \in N$ such that $yn^{-1} \in A$ and for any such $n$ it holds that

$$(AyA) \cap N = (A \cap N)O_A(n) = O_A(n)A \cap N.$$

Proof. The existence of $n$ is clear from $AN = G$. Fix $n \in N$ for which $yn^{-1} \in A$. The equality $(A \cap N)O_A(n) = O_A(n)(A \cap N)$ follows from the fact that $O_A(n)$ is normalized by $A$. It remains to prove $(AyA) \cap N = O_A(n)(A \cap N)$. First we prove $(AyA) \cap N \subseteq O_A(n)(A \cap N)$. Let $g \in (AyA) \cap N \subseteq (A \cap N)$. Then there exist $a_1, a_2 \in A$ such that $g = a_1^{-1}na_2 = n^{a_1}(a_1^{-1}a_2)$. Note that $a_1^{-1}a_2 \in A$ and $n^{a_1} \in N$. Since $g \in N$ we get $a_1^{-1}a_2 = (n^{a_1})^{-1}g \in N$. Therefore $a_1^{-1}a_2 \in A \cap N$ and

$$g = n^{a_1}(a_1^{-1}a_2) \in n^{a_1}(A \cap N) \subseteq O_A(n)(A \cap N).$$

Now we prove the reverse inclusion. Let $g \in O_A(n)(A \cap N)$. Since both $O_A(n)$ and $(A \cap N)$ are subsets of $N$ we get $g \in N$, and it remains to prove $g \in AyA$. There exist $a \in A$ and $n' \in A \cap N$ such that

$$g = n'a = a^{-1}nan'.$$

But $an' \in A$ since $n' \in A \cap N$ so $g \in AnA = AyA$ as required. □
Hence, since A Remark 4.7. Corollary 4.5 can also be proved using Tamaschke’s second isomorphism theorem 2.13 in [24] as follows (we use the notation of [24]). Denote by $N$ the set of double cosets of $A \cap N \leq N$. Let $\pi : (A \cap N) n (A \cap N) | n \in N$ be the set of double cosets of $A \cap N \leq N$. Then $\pi$ is a $d$-partition of $N$, and since $A$ leaves $A \cap N$ invariant, $A$ acts on $\pi$ by Corollary 4.3. Let $n \in N$ be arbitrary and set $\pi := (A \cap N) n (A \cap N)$. By Corollary 4.3 and Lemma 4.6 we get

$$U_{[\pi]} = (A \cap N) O_A (n) (A \cap N) = (A \cap N)^2 O_A (n) = (A \cap N) O_A (n),$$

and now claim (a) follows from Theorem 1.5.

(b) The map defined by $A y A \mapsto (A A) \cap N$ from $\Pi_G$ into the power set of $N$ is clearly well-defined. By Lemma 4.6 it is equal to $f$. Hence $f$ is a well-defined $\Pi_G \rightarrow \Pi_N$ map. Its surjectivity follows from $G = \bigcup y \in G$ and its injectivity follows from the fact that distinct double cosets of $A$ have empty intersection. It remains to prove that $f$ preserves the structure constants. Let $y, y' \in G$ and $n, n' \in N$ be arbitrary such that $y n^{-1}, y' (n')^{-1} \in A$. It would suffice to prove that

$$N \cap ((A y A) (A y')) = ((A \cap N) O_A (n)) ((A \cap N) O_A (n')).$$

We have

$$(A y A) (A y') = A n A = A \left( \bigcup_{a \in A} a n^{-1} n' \right) = \bigcup_{a \in A} An a.'$$

By Lemma 4.6 we have $N \cap (A n A) = O_A (n) (A \cap N)$. Therefore,

$$N \cap ((A y A) (A y')) = N \cap \bigcup_{a \in A} A n A = \bigcup_{a \in A} N \cap (A n A') =$$

$$= \bigcup_{a \in A} O_A (n) (A \cap N) = (A \cap N) \bigcup_{a \in A} O_A (n a').$$

Now

$$\bigcup_{a \in A} O_A (n a') = \bigcup_{a \in A} \bigcup_{b \in A} (n a') b = \bigcup_{b \in A} \bigcup_{a \in A} n a b (n') b$$

$$= \bigcup_{a \in A} \bigcup_{a \in A} n a (n') a = O_A (n) O_A (n').$$

Hence, since $A \cap N = (A \cap N)^2$ commutes with $O_A (n)$ and $O_A (n')$ we get

$$N \cap ((A y A) (A y')) = ((A \cap N) O_A (n)) ((A \cap N) O_A (n')).$$

$\square$

**Remark 4.7.** Corollary 4.3 can also be proved using Tamaschke’s second isomorphism theorem 2.13 in [24] as follows (we use the notation of [24]). Denote by $T^{(G)}$ and by $T^{(N)}$ the $S$-semigroups on $G$ and on $N$ associated with $\Pi_G$ and $\Pi_N$ respectively. Since $A \cap N \subseteq A$, we can apply Corollary 4.3 with $H = A \cap N$ and obtain that $\Pi := \{(A \cap N) O_A (x) (A \cap N) | x \in G\}$ is a $d$-partition of $G$. Let $T$ be the corresponding $S$-semigroup. We observe that $A$ and $N$ are $T$-subgroups of $G$ with

$$T_A = [(A \cap N) O_A (a) (A \cap N) | a \in A]$$

$$T_N = [(A \cap N) O_A (n) (A \cap N) | n \in N] = T^{(N)}.$$

Furthermore, $N$ is $T$-normal since $N \subseteq G$, and $A$ is $T$-normal since it normalizes both $A \cap N$ and $O_A (x)$ for any $x \in G$. Now we can apply [24] Theorem 2.13, with
K = A and H = N. By [24, Theorem 2.13] (3) we get
\[ T_H/T_{H \cap K} = T_N/T_{N \cap A} = T_N = T(G) \]
\[ T_{HK}/T_K = T/T_A = [AyA|y \in A] = T(G), \]
and the isomorphism claim follows.

Next we prove that d-partitions lift from quotient groups.

**Proof of Theorem 1.6.** Let \( \nu : G \rightarrow \overline{G} \) be the natural projection and for any \( \overline{S} \subseteq \overline{G} \) let \( \nu^{-1}(\overline{S}) \) be the full preimage of \( \overline{S} \) in \( G \). Recall that \( S \subseteq G \) is the full preimage of \( \overline{S} \) in \( G \) if and only if \( \nu(S) = \overline{S} \) and \( NS = S \). For any \( \overline{\pi} \in \overline{\Pi} \), denote \( \pi := \nu^{-1}(\overline{\pi}) \).

Now let \( \pi \in \Pi \). Then \( N\pi = \pi \) implies that \( \pi \) is a union of cosets of \( N \). Since two distinct cosets of \( N \) are disjoint, and since \( \overline{\pi_1} \neq \overline{\pi_2} \) implies \( \overline{\pi_1} \cap \overline{\pi_2} = \emptyset \), we get that \( \overline{\pi_1} \neq \overline{\pi_2} \) implies \( \pi_1 \cap \pi_2 = \emptyset \). Since \( \pi \) is the full preimage of \( \overline{\pi}, G = \bigcup_{\pi \in \Pi} \pi \) implies \n
\[ G = \bigcup_{\pi \in \Pi} \pi . \]

Therefore \( \Pi \) is a partition of \( G \).

Let \( \pi_1, \pi_2 \in \Pi \). Then \( \nu(\pi_1 \pi_2) = \nu(\pi_1) \nu(\pi_2) \) is a union of parts of \( \Pi \). Hence \( \nu^{-1}(\nu(\pi_1 \pi_2)) = \nu^{-1}(\nu(\pi_1)) \nu^{-1}(\nu(\pi_2)) \) is a union of parts of \( \Pi \). But \( N\pi_1 = \pi_1 \) implies \( N(\pi_1 \pi_2) = \pi_1 \pi_2 \) so \( \pi_1 \pi_2 = \nu^{-1}(\nu(\pi_1 \pi_2)) \). This proves that \( \Pi \) is closed under multiplication. Furthermore, \( e_\Pi = \nu^{-1}(e_\Pi) \) and since for any \( \pi \in \Pi, \pi^{-1} = \nu^{-1}(\nu^{-1}(\overline{\pi}) \overline{\pi}) \), we get the inverse property of \( \Pi \) from that of \( \overline{\Pi} \).

We note that the converse of Theorem 1.6 is not true, namely, if \( G \) is a group, \( N \subseteq G, \Pi \) is a d-partition of \( G \), and \( \nu : G \rightarrow \overline{G} := G/N \) is the natural projection, then \{\( \nu(\pi)|\pi \in \Pi \)\} is not necessarily a d-partition. An easy counterexample is provided by \( G = (\mathbb{Z}, +), N = 2\mathbb{Z} \) and \( \Pi := \{\{0\}, \mathbb{Z} \setminus \{0\}\} \). Clearly, \( \nu(\{0\}) = \{0\} \) and \( \nu(\mathbb{Z} \setminus \{0\}) = \{0, 1\} \) which are distinct but have a non-empty intersection. Nevertheless, there are examples of d-partitions of groups which are mapped nicely to quotients, e.g., double coset partitions.

5. Three part d-partitions of \((\mathbb{Z}_p, +_p)\)

In this section we study d-partitions of the form \( \Pi = \{\pi_0, \pi_1, \pi_2\} \), of \((\mathbb{Z}_p, +_p)\), where \( p \) is a prime. Note that one can assume, without loss of generality, \( \pi_0 := \{0\} \), and \( |\pi_1| \leq |\pi_2| \). Our results indicate that these d-partitions involve interesting structures related to fundamental questions in additive combinatorics (see Section 2.5 for the concepts used in the analysis). Recall, in comparison, that the S-partitions of these groups were fully classified in [12] (see Theorem 2.9).

5.1. Isomorphism Types of 3-part d-partitions of \((\mathbb{Z}_p, +_p)\). We begin with determining the isomorphism types of the 3-part d-partitions of \((\mathbb{Z}_p, +_p)\). Two ”small \( p \)” cases are summarized in the following remark.

**Remark 5.1.** One can check that the only possibilities for \( \Pi \) for the primes 3 and 5 are the singleton partition if \( p = 3 \) and \( \Pi = \{\{0\}, \{1, 4\}, \{2, 3\}\} \) if \( p = 5 \). It turns out that the isomorphism types of these two d-partitions do not repeat for larger primes.

From now on we assume \( p > 5 \).
Lemma 5.2. Let \( p > 5 \) be a prime, \( G = (\mathbb{Z}_p, +) \), and \( \Pi = \{\pi_0, \pi_1, \pi_2\} \) a 3-part \( d \)-partition of \( G \), where \( \pi_0 = \{0\} \) and \( |\pi_1| \leq |\pi_2| \). Then one of the following (a) and (b) holds true:

(a) \( \pi_k = -\pi_k \) for \( k = 1, 2 \), \( \pi_1 + \pi_2 = \pi_1 \cup \pi_2 \) and \( \pi_2 + \pi_2 = G \). In addition, either:

(i) \( \pi_1 + \pi_1 = \pi_0 \cup \pi_2 \), or:

(ii) \( \pi_1 + \pi_1 = G \).

(b) \( \pi_1 = -\pi_2 \), \( \pi_1 + \pi_2 = G \), and \( \pi_1 + \pi_1 = \pi_2 + \pi_2 = \pi_1 \cup \pi_2 \).

Proof. (i) Let \( i, j \in \{1, 2\} \). Theorem 2.13 gives

\[
|\pi_i + \pi_j| \geq \min(|\pi_i| + |\pi_j| - 1, p) \geq 2|\pi_1| - 1,
\]

where for the second inequality we used \( |\pi_1| \leq |\pi_2| \).

(ii) We prove \( \pi_2 \subseteq \pi_1 + \pi_j \). Assume by contradiction that this is not the case. Since by the closure property of a \( d \)-partition \( \pi_i + \pi_j \) is a union of parts, we get \( \pi_i + \pi_j \subseteq \{0\} \cup \pi_1 \) which gives, using Equation (5.1), \( 2|\pi_1| - 1 \leq |\pi_1| + 1 \). This implies \( |\pi_1| \leq 2 \), and hence, by Lemma 2.10 \( |\pi_1| = |\pi_2| \leq 2 \) in contradiction to \( p > 5 \).

(iii) Suppose that at least one of \( i \) and \( j \) is not equal to 1. Note that under this assumption the first inequality in (5.1) gives \( |\pi_i + \pi_j| \geq p - 2 \). We prove that \( \pi_1 \subseteq \pi_i + \pi_j \). Assume by contradiction that \( \pi_1 \not\subseteq \pi_i + \pi_j \). Then \( \pi_1 + \pi_j \subseteq \{0\} \cup \pi_2 \) and we get \( p - 2 \leq |\pi_i + \pi_j| \leq |\pi_2| + 1 \) implying \( |\pi_1| \leq 2 \). Applying Lemma 2.10 we obtain a contradiction as in (ii).

(iv) Now we are ready to prove the claim of the lemma. By the inverse property of a \( d \)-partition and \( |\Pi| = 3 \), either \( \pi_k = -\pi_k \) for \( k = 1, 2 \) or \( \pi_1 = -\pi_2 \). Suppose \( \pi_k = -\pi_k \) for \( k = 1, 2 \). For any \( x \in \pi_1 \) and \( y \in \pi_2 \), \( y \neq -x \) and hence \( x + y \neq 0 \). Hence \( 0 \not\in \pi_1 + \pi_2 \) and \( \pi_1 + \pi_2 = \pi_1 \cup \pi_2 \) by (ii) and (iii). On the other hand, \( 0 \in \pi_2 + \pi_2 \) and \( \pi_2 + \pi_2 = G \). Similarly, \( 0 \in \pi_1 + \pi_1 \) and hence either \( \pi_1 + \pi_1 = \pi_0 \cup \pi_2 \), which gives \( a1 \), or \( \pi_1 + \pi_1 = G \), which gives \( a2 \). Now suppose that \( \pi_1 = -\pi_2 \). Here \( 0 \not\in \pi_1 + \pi_2 \) and hence \( \pi_1 + \pi_2 = G \). On the other hand, \( 0 \not\in \pi_1 + \pi_1 \), \( i = 1, 2 \). Since \( \pi_1 = -\pi_2 \) we have \( |\pi_1| = |\pi_2| \), so in this case we can prove \( \pi_2 \subseteq \pi_1 + \pi_1 \) using the same reasoning as in (iii). Hence we get \( \pi_1 + \pi_1 = \pi_2 + \pi_2 = \pi_1 \cup \pi_2 \). This gives case (b) of the lemma.

We can recast Lemma 5.2 in terms of the structure constants \( d_{i,j}^k \) of the induced dioids (see Lemma 3.6 and Remark 3.7). We have \( \pi_i + \pi_j = \bigcup_{k=0}^{2} d_{i,j}^k \pi_k \), where \( i, j \in \{0, 1, 2\} \). It is sufficient to consider \( i \leq j \in \{1, 2\} \).

Corollary 5.3. Under the assumptions and notation of Lemma 5.2:

(a) If \( \pi_1 = -\pi_1 \) then

\[
\begin{align*}
d_{1,1}^0 &= d_{1,2}^0 = d_{2,1}^2 = d_{2,2}^2 = d_{1,2}^2 = d_{2,2}^2 = 1, \\
d_{1,2}^0 &= 0,
\end{align*}
\]

while \( d_{1,1}^1 \) is either 0 or 1.

(b) If \( \pi_1 = -\pi_2 \) then

\[
\begin{align*}
d_{1,1}^1 &= d_{1,2}^1 = d_{1,2}^0 = d_{1,2}^2 = d_{1,2}^1 = d_{2,2}^2 = d_{2,2}^2 = 1, \\
d_{1,2}^0 &= d_{2,2}^0 = 0.
\end{align*}
\]
We are ready to prove Theorem 1.7 which classifies the isomorphism types of the 3-part d-partitions of a cyclic group of prime order.

**Proof of Theorem 1.7.** Let \( \Pi = \{\pi_0, \pi_1, \pi_2\} \), with \( \pi_0 = \{0\} \) and \( |\pi_1| \leq |\pi_2| \), be a 3-part partition of \( G \). By Remark 5.4 we may assume \( p > 5 \). If \( \Pi \) is a \( d \)-partition then, by Lemma 5.2 exactly one of (3)-(5) holds (note that if \( \pi_2 \) is an arithmetic progression of size \( \frac{p-1}{2} \) then \( |\pi_2 + \pi_2| = p - 2 \) and hence \( \pi_2 + \pi_2 \neq G \)).

We now prove that the conditions stated in (3)-(5) are sufficient for \( \Pi \) to be a \( d \)-partition. Clearly \( \pi_0 \) is an identity element and \( \Pi \) has the inversion property since \( \pi_1 = -\pi_1 \) implies \( \pi_2 = -\pi_2 \) and \( \pi_1 = -\pi_2 \) implies \( \pi_2 = -\pi_1 \). We now prove the closure property. Note that \( |\pi_1| \leq |\pi_2| \) implies

\[
2|\pi_2| - 1 \geq |\pi_1| + |\pi_2| - 1 = p - 2.
\]

Hence, by Theorem 2.13 we have:

(1) Suppose that either condition (3) or (4) of the theorem holds. Since \( \pi_1 = -\pi_1 \) we have that \( \pi_1 + \pi_2 \) is a symmetric set, and also \( 0 \notin \pi_1 + \pi_2 \), which implies by (1) that \( |\pi_1 + \pi_2| \in \{p - 2, p - 1\} \). Since \( p \) is odd, \( x \neq -x \) for any \( 0 \neq x \in G \). Hence, any symmetric subset of \( G \) which does not contain 0 has even cardinality, and this forces \( |\pi_1 + \pi_2| = p - 1 \). Hence \( \pi_1 + \pi_2 = \pi_1 \cup \pi_2 \). It remains to prove that \( \pi_2 - \pi_2 \subset G \). Assume by contradiction that \( \pi_2 - \pi_2 \subset G \). By (**) this implies \( |\pi_2 - \pi_2| \in \{p - 2, p - 1\} \). Since \( \pi_2 = -\pi_2 \), we get that \( \pi_2 - \pi_2 \) is a symmetric set, and that \( 0 \notin \pi_2 - \pi_2 \). Therefore \( |\pi_2 - \pi_2| \) is odd, so \( |\pi_2 - \pi_2| = p - 2 \). Now (**) gives \( 2|\pi_2| - 1 = p - 2 \), and \( |\pi_1| = |\pi_2| = \frac{p - 1}{2} \). Applying Theorem 2.13 with \( A = B = \pi_2 \) we get that \( \pi_2 \) is an arithmetic progression of size \( \frac{p-1}{2} \). Hence, condition (3) of the theorem holds, giving \( \pi_1 + \pi_1 = \pi_0 \cup \pi_2 \), implying \( |\pi_1 + \pi_1| = |\pi_2| + 1 \), but, by Theorem 2.13 \( |\pi_1 + \pi_1| \geq 2|\pi_1| - 1 \). Hence, using \( |\pi_1| = |\pi_2| \) we get \( |\pi_2| \leq 2 \) in contradiction to \( p > 5 \).

(2) Suppose that condition (5) holds, namely, \( \pi_1 = -\pi_2 \) and \( \pi_1 + \pi_1 = \pi_1 \cup \pi_2 \). By negating both sides we get \( \pi_2 + \pi_2 = \pi_1 \cup \pi_2 \). We now prove \( \pi_1 + \pi_2 = G \). Note that \( \pi_1 = -\pi_2 \) implies \( 0 \in \pi_1 + \pi_2 \) and also that \( \pi_1 + \pi_2 \) is symmetric. This implies that \( |\pi_1 + \pi_2| \) is odd. Thus, by (1), \( |\pi_1 + \pi_2| \in \{p - 2, p\} \). Suppose \( |\pi_1 + \pi_2| = p - 2 \). Then, by (1), \( |\pi_1 + \pi_2| = |\pi_1| + |\pi_2| - 1 \). By Theorem 2.13 with \( A = \pi_1 \) and \( B = \pi_2 \), we get that \( \pi_1 \) is an arithmetic progression of length \( |\pi_1| = \frac{p-1}{2} \). Hence \( |\pi_1 + \pi_1| = 2|\pi_1| - 1 \leq |\pi_1| + |\pi_2| \), in contradiction to \( \pi_1 + \pi_1 = \pi_1 \cup \pi_2 \). Therefore \( |\pi_1 + \pi_2| = p \), implying \( \pi_1 + \pi_2 = G \).

\( \square \)

**Remark 5.4.** Using Corollary 2.12 it is easy to check that the 3-part S-partitions of \( (\mathbb{Z}_p, +) \), where \( p \) is a prime, fit in the classification described in Theorem 1.7 as follows.

1: Types (1) and (2) are S-partitions.
2: For \( p \equiv 1 \pmod{4} \), \( p > 5 \), a 3-part S-partition is of type (4).
3: For \( p \equiv 3 \pmod{4} \), \( p > 5 \), a 3-part S-partition is of type (5).
Remark 5.5. For every prime $p \leq 7$, every 3-part $d$-partition is an $S$-partition. This can be verified using Lemma 5.4 and some further explicit, easy calculations. On the other hand, for every prime $p \geq 11$, there exists a 3-part $d$-partition which is not an $S$-partition. To see this note that by Remark 5.4 there is no 3-part $S$-partition of type (3), while the discussion in Section 5.2.1 shows that 3-part $d$-partitions of type (3) exist for every $p \geq 11$. Notice that this observation answers (in the negative) Tamashke’s Problem 1.19 in [25].

5.2. Constructions of 3-part $d$-partitions of $(\mathbb{Z}_p, +)$. Let $p$ be a prime and $G = (\mathbb{Z}_p, +)$. In this section we consider explicit constructions of 3-part $d$-partitions of $G$.

By Theorem 1.7, every 3-part $d$-partition of $G$, where $p > 5$ is a prime, is of the form $\Pi = \{0\}, \mathcal{S}, \mathcal{S} \cup \{0\}$ where $\mathcal{S}$, $0 < |\mathcal{S}| \leq \frac{p-1}{2}$, is a solution to one of the following three equations:

\begin{align}
(5.2) \quad & S + S = \mathcal{S}, \ S = -S \\
(5.3) \quad & S + S = G, \ S = -S, \ 0 \notin S, \text{ and} \\
(5.4) \quad & S + S = \{0\}, \ S = -\mathcal{S} \cup \{0\}. 
\end{align}

In the following we discuss separately the solutions to each of these equations. Equation (5.2) had already been studied in the literature in other contexts. Hence, in Section 5.2.1 (to follow) we review the relevant results and explain their connection to our settings. In sections 5.2.2 and 5.2.3 we provide a full classification of the solutions to Equations (5.3) and (5.4).

5.2.1. Solutions to Equation (5.2). Equation (5.2) fits into a well-studied topic in additive combinatorics - the theory of sum-free sets. An additive set $S$ is called sum-free if $S + S \subseteq \mathcal{S}$, and complete if $S \subseteq S + S$. Thus, $S$ is a solution to Equation (5.2) if and only if $S$ is symmetric, sum-free and complete. Note that all of these properties are invariant under group automorphisms. This part briefly reviews some known results which are relevant to the description of the associated 3-part $d$-partitions.

Let $p \geq 5$ be a prime. The symmetric, sum-free and complete subsets of $G = (\mathbb{Z}_p, +)$ of largest possible cardinality, can be deduced from the classification of the largest cardinality sum-free sets by Diananda and Yap [29, 30] for $p = 3k + 2$ and by Rhemtulla and Street [20] for $p = 3k + 1$. They are uniquely given, up to an automorphism, as follows: For $p = 3k + 2$ it is the interval $[k+1, 2k+1]$ of cardinality $k+1$ and for $p = 3k + 1$ with $k \geq 4$ it is the set $\{k\} \cup [k+2, 2k-1] \cup \{2k+1\}$ of cardinality $k$. In a recent paper [15], whose original motivation was the understanding of 3-part $d$-partitions, we extended this result and characterized all symmetric complete sum-free sets of size at least $c \cdot p$ where $c = 0.318$ and $p$ is a sufficiently large prime, and proved that their number grows exponentially in $p$. Moreover, we have shown that there exist constants $c_1, c_2, c_3 > 0$ such that for every sufficiently large integer $n$ there exists a collection of symmetric complete sum-free subsets of $\mathbb{Z}_n$ whose sizes form an arithmetic progression with first element at most $c_1 \sqrt{n}$, step at most $c_2 \sqrt{n}$, and last element at least $\frac{4}{3} - c_3 \sqrt{n}$ (see [15] for more
5.2.2. Solutions to Equation (5.3). Let $p > 5$ be a prime. We characterize the solutions $S$ of Equation (5.3) with $0 < |S| \leq \frac{p-1}{2}$. It suffices to characterize the symmetric sets $S \subseteq \mathbb{Z}_p^*$ such that $0 < |S| \leq \frac{p-1}{2}$ and $S + S \neq \mathbb{Z}_p$. For this we need the following definition.

**Definition 5.6.** Let $p$ be a prime and let $S \subseteq \mathbb{Z}_p^*$ be a non-empty set of size $s := |S| \leq \frac{p-1}{2}$. We say that $S$ is avoiding if $S$ is symmetric and can be written in the form

$$(5.5) \quad S = \{ \pm i_1, \pm i_2, \ldots, \pm i_{s/2} \},$$

where $1 \leq i_1 < i_2 < \cdots < i_s \leq \frac{p-1}{2}$, and $i_{j+1} \geq i_j + 2$ for all $1 \leq j \leq \frac{s}{2}$, where $i_{\frac{s}{2}+1} := p - i_{\frac{s}{2}}$.

Note that an avoiding set must have a positive even size, and hence such sets exist only for primes $p \geq 5$.

**Theorem 5.7.** Let $p$ be a prime. Let $S \subseteq \mathbb{Z}_p^*$ be a non-empty symmetric set of size $s := |S| \leq \frac{p-1}{2}$. Then $S$ is avoiding, up to an automorphism, if and only if $S + S \neq \mathbb{Z}_p$.

**Proof.** 

a. Assume that, up to an automorphism, $S$ is avoiding. Since the conditions $S = -S$ and $S + S \neq \mathbb{Z}_p$ are invariant under automorphisms, we may assume that $S$ is avoiding. By Definition 5.6 no two elements of $S$ are consecutive mod $p$, and hence $1 \notin S - S = S + S$. Therefore $S + S \neq \mathbb{Z}_p$.

b. Assume that $S + S \neq \mathbb{Z}_p$. Since $S \neq \emptyset$ and $S$ is symmetric, we have $0 \in S + S$. Hence, $S + S \neq \mathbb{Z}_p$ implies the existence of $x \in \mathbb{Z}_p^*$ such that $x \notin S + S$. Then $S = x \cdot (x^{-1}S)$ where $x^{-1}S$ is a symmetric subset of $\mathbb{Z}_p^*$ of cardinality $s$. Hence $x^{-1}S = \{ \pm i_1, \pm i_2, \ldots, \pm i_{s/2} \}$, where $1 \leq i_1 < i_2 < \cdots < i_s \leq \frac{p-1}{2}$. Suppose that $i_{j+1} = i_j + 1$, for some $1 \leq j \leq \frac{s}{2}$, where $i_{\frac{s}{2}+1} := p - i_{\frac{s}{2}}$. Then $x = xi_{j+1} + x(-i_j) \in S + S$ - a contradiction. Hence $i_{j+1} \geq i_j + 2$, for all $1 \leq j \leq \frac{s}{2}$, and $x^{-1}S$ is avoiding.

The following proposition uses elementary combinatorics for counting avoiding sets of a given size.

**Proposition 5.8.** For a prime $p \geq 5$, the number of avoiding subsets of $\mathbb{Z}_p$ of even positive size $s \leq \frac{p-1}{2}$ is $(\frac{(p-1)/2-s/2}{s/2})$.

**Proof.** We have $s \geq 2$ and even. Every set as in Equation (5.3) can be represented by a characteristic vector $\xi \in \{0,1\}^{(p+1)/2}$, indexed by $\{0,1,2,\ldots,(p-1)/2\}$, where $c_j = 1$ if and only if $j \in \{i_1, i_2, \ldots, i_{s/2}\}$. The condition $i_{j+1} \geq i_j + 2$ for all $1 \leq j \leq \frac{s}{2}$, where $i_{\frac{s}{2}+1} := p - i_{\frac{s}{2}}$, translates to the condition that $\xi$ starts and ends with 0, and no two of the $s/2$ entries which are equal to 1 are consecutive. Hence, the number of such distinct vectors $\xi$ is equal to the number of ways to distribute $(p+1)/2 - s/2$ identical balls into $s/2 + 1$ bins so that none of the bins is left empty. This number is precisely $(\frac{(p-1)/2-s/2}{s/2})$.

**Corollary 5.9.** For a prime $p$, the total number of avoiding subsets of $\mathbb{Z}_p$ is bounded above by $2^{p(c+o(1))}$, for $c \approx 0.3471$. 


Proof. It is well known (see, e.g., [16, Section 22.5]) that for every positive integer $n$ and real number $\alpha \in [0, 1]$ (here $[0, 1]$ stands for the real interval),

$$\left(\frac{n}{\alpha n}\right) \leq 2^{n \cdot H(\alpha)},$$

where $H : [0, 1] \to [0, 1]$ is the binary entropy function defined by

$$H(\alpha) = -\alpha \log_2 \alpha - (1 - \alpha) \log_2 (1 - \alpha), \forall \alpha \in [0, 1].$$

We apply this to derive an upper bound on \((p^{-1}/2 - s/2)/s/2\), which, by Proposition \[5.3\] is the number of avoiding sets of positive size $s \leq \frac{p-1}{2}$. Define $\beta := s/p$ and take $n := (p - 1)/2 - s/2 = p(1 - \beta)/2 - (1/2)$. Then $0 < \beta \leq 1/2$, and defining the real parameter $\alpha$ by $s/2 = \alpha n$, we get

$$\alpha = \frac{\beta p/2}{p(1 - \beta)/2 - (1/2)} = \frac{\beta}{1 - \beta} (1 + \Theta(1/p)).$$

Therefore \((p^{-1}/2 - s/2)/s/2 \leq 2^{\beta(1 - \beta) H(\beta^{-1} + o(1))}\). Let $c$ denote the maximum of $\frac{1 - \beta}{1 - \beta}$. \(H(\frac{\beta}{1 - \beta})\) for $\beta$ in the real interval $[0, 1/2]$. One can verify that this maximum is attained for $\beta \approx 0.276$. Summing over all positive even $s \leq \frac{p-1}{2}$ gives the claim of the corollary. \(\square\)

Remark 5.10. Note that by the extra condition following Equation \[5.3\] we have to rule out $S \subseteq \mathbb{Z}_p$ for which $S + S = \mathbb{Z}_p$ and $S_1 := S \cup \{0\}$ is an arithmetic progression of size $\frac{p-1}{2}$. Since $S_1$ is a symmetric set which does not contain 0 we must have that $|S_1| = \frac{p-1}{2}$ is even. This implies $p \equiv 1 \pmod{4}$. Furthermore, up to an automorphism, $S_1 = [\frac{p+1}{2}, \frac{3p-1}{2}]$. The reader can check that

$$S = S_1 \cup \{0\} = [1, \frac{p-1}{2}] \cup \left[\frac{3p+1}{2}, p-1\right]$$

satisfies $|S| = \frac{p-1}{2}$ and $S + S = \mathbb{Z}_p$ and hence should be excluded, together with all of its automorphic images, from the set of solutions of Equation \[5.3\].

5.2.3. Solutions to Equation \[5.4\]. The following theorem characterizes the solutions to Equation \[5.4\].

Theorem 5.11. Let $G = (\mathbb{Z}_p, +)$, where $p > 5$ is a prime. The solutions of Equation \[5.4\] are all the subsets $S \subseteq G$ which satisfy: $|S| = \frac{p-1}{2}$, $S \cap (-S) = \emptyset$ and $S$ is not an arithmetic progression. The number of $d$-partitions of $G$ defined by these solutions is $2^{\frac{p-1}{2} - 1 - \frac{p-1}{2}}$.

Proof. 1. Suppose that $S \subseteq G$ satisfies $|S| = \frac{p-1}{2}$, $S \cap (-S) = \emptyset$ and that $S$ is not an arithmetic progression. We will prove that $S$ is a solution to Equation \[5.4\]. The condition $S \cap (-S) = \emptyset$ implies that for any $x, y \in S$ we have $x + y \neq 0$, and hence $0 \notin S + S$. Since $|S| = \frac{p-1}{2}$, we have, by Theorem 2.18 that $|S + S| \geq 2|S| - 1 = p - 2$. Thus $p - 2 \leq |S + S| \leq p - 1$. However, if $p - 2 = |S + S|$ then $S$ is an arithmetic progression by Theorem 2.13 - a contradiction. Therefore $|S + S| = p - 1$, which together with $0 \notin S + S$ implies $S + S = \{0\}$. Furthermore, $S \cap (-S) = \emptyset$ implies $0 \notin S$, and so $S$, $-S$ and $\{0\}$ are mutually disjoint. Since $|S| = \frac{p-1}{2}$ we get $S = -S \cup \{0\}$.

2. Suppose that $S$ is a solution to Equation \[5.4\]. We will prove $|S| = \frac{p-1}{2}$, $S \cap (-S) = \emptyset$ and that $S$ is not an arithmetic progression. Since $S = -S \cup \{0\}$ we
have that $0 \notin S$, so $G = S \cup (-S) \cup \{0\}$ is a disjoint union, and hence $S \cap (-S) = \emptyset$ and $|S| = \frac{p-1}{2}$. Finally note that $S$ cannot be an arithmetic progression, since, in that case, $|S + S| = 2|S| - 1 = p - 2$ while $S + S = \{0\}$ implies $|S + S| = p - 1$ - a contradiction.

3. It remains to count the number of $d$-partitions defined by the solutions of Equation (5.4). The number of sets $S \subseteq G$ which satisfy $|S| = \frac{p-1}{2}$ and $S \cap (-S) = \emptyset$ is $2^\frac{p-1}{2}$. From this number we have to subtract the number of arithmetic progressions of size $\frac{p-1}{2}$ which satisfy $S \cap (-S) = \emptyset$. Observe that $S \subseteq G$ is an arithmetic progression of size $\frac{p-1}{2}$ which satisfies $S \cap (-S) = \emptyset$, if and only if for any $\delta \in \mathbb{Z}_p$, $\delta \cdot p S$ is also an arithmetic progression with the same properties. Let $S$ be an arithmetic progression such that $|S| = \frac{p-1}{2}$ and $S \cap (-S) = \emptyset$. Let $\delta$ be the step of $S$. Then $\delta \in \mathbb{Z}_p^*$ and $\delta^{-1} \cdot p S$ is an arithmetic progression of step 1 such that $|\delta^{-1} \cdot p S| = \frac{p-1}{2}$ and $(\delta^{-1} \cdot p S) \cap (-\delta^{-1} \cdot p S) = \emptyset$. But one can easily check that this implies $\delta^{-1} \cdot p S = [1, \frac{p-1}{2}]$. Therefore, the number of solutions to Equation (5.4) is $2^\frac{p-1}{2} - (p - 1)$. Finally notice that the solutions to Equation (5.4) divide into pairs $(S, -S)$ where $S$ and $-S$ define the same $d$-partition. This proves the claim of the theorem. 

\[\square\]

6. Appendix

**Proof of Proposition 3.15** We replace $\oplus$ by $+$ and write $xy$ for $x \otimes y$. Let $d \in D \setminus \{\varepsilon\}$, and consider the infinite sequence of sums, $d, d + d, d + d + d, \ldots$. By definition of the canonical order $\leq_D$, any two successive terms in the sequence, with $n$ and $n + 1$ summands ($n \geq 1$ an integer), are either equal or satisfy

\[\underbrace{d + d + \cdots + d}_{n \text{ summands}} <_D \underbrace{d + d + \cdots + d}_{n+1 \text{ summands}}.\]

Thus, if no two successive terms of the sequence are equal, the sequence is strictly increasing with respect to $\leq_D$, and hence, its terms are pairwise distinct. Otherwise, let $n$ be the smallest natural number such that

\[\underbrace{d + d + \cdots + d}_{n \text{ summands}} = \underbrace{d + d + \cdots + d}_{n+1 \text{ summands}}.\]

Then the sequence of sums is strictly increasing up to the $n$-fold sum and, as can be proven by easy induction, any $m$-fold sum with $m > n$ is equal to the $n$-fold sum. In such a case we say that $d$ is $n$-idempotent ([11] p. 15). Using distributivity we get:

\[(\underbrace{e + e + \cdots + e}_{n \text{ summands}}) d = (\underbrace{e + e + \cdots + e}_{n+1 \text{ summands}}) d.\]

Since $d \neq \varepsilon$ it is invertible and we get:

\[(\underbrace{e + e + \cdots + e}_{n \text{ summands}}) = (\underbrace{e + e + \cdots + e}_{n+1 \text{ summands}}).\]

Multiplying by any $d' \in D \setminus \{\varepsilon\}$ we obtain

\[(\underbrace{d' + d' + \cdots + d'}_{n \text{ summands}}) = (\underbrace{d' + d' + \cdots + d'}_{n+1 \text{ summands}}).\]
It follows that either for each \( d \in D \setminus \{ \varepsilon \} \) all finite sums of the form \( d, d + d, d + d + d, \ldots \) are distinct or there exists a unique minimal \( n \) such that each \( d \in D \setminus \{ \varepsilon \} \) is \( n \)-idempotent. Furthermore, assume that each \( d \in D \setminus \{ \varepsilon \} \) is \( n \)-idempotent, where \( n \geq 1 \). Consider
\[
x := e + e + \cdots + e.
\]
Computing \( x^2 \) using the distributive law gives
\[
x^2 = e + e + \cdots + e = e + e + \cdots + e.
\]
Hence \( x^2 = x \). Since \( x \) is invertible this immediately implies \( x = e \). Thus we are forced to have \( n = 1 \), which is equivalent to \( D \) being an idempotent \( d \)-field.

The last part of the proof is based on [14] (Proposition 2.7 and its proof). Let \( D \) be an idempotent \( d \)-field. Suppose that \( D \) has a largest element \( g \). Note that this assumption holds true if \( D \) is finite since in such a case we can simply take \( g \) to be the sum of all elements of \( D \). By assumption, \( e \leq_D g \) which gives, when multiplied by \( g \), \( g \leq_D g^2 \) ([11] Proposition 1.6.1.7]). Since \( g \) is the largest element this forces \( g^2 = g \) and hence, since \( g \) is invertible, \( g = e \). Thus \( e \) is the greatest element of \( D \). Let \( d \in D \setminus \{ \varepsilon \} \). Then \( d \) is invertible and \( d^{-1} \leq e \) since \( e \) is the greatest element of \( D \). Multiplying by \( d \) gives \( e \leq d \) which forces \( d = e \). This proves that \( D = \mathbb{B} \). \( \square \)

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