SEMI-INFINITE VARIATIONS OF HODGE STRUCTURES AND INTEGRABLE HIERARCHIES OF KDV-TYPE

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ABSTRACT. We introduce integrable KdV-type hierarchy associated naturally with arbitrary semi-simple Frobenius manifold. We present hierarchy in a Lax form and show that it admits bihamiltonian description. The hierarchy allows to extend corresponding semi-infinite variation of Hodge structures by including variation along higher times \( \{ x^{\alpha,r} \} \) satisfying

\[
\frac{\partial}{\partial x^{\alpha,r}} L(x) \subseteq \hbar^{-r} L(x)
\]

1. Introduction

Geometry of families of semi-infinite subspaces plays central role in elegant approach to integrable hierarchies in \[SW\]. Similar geometry can be used to describe Frobenius manifold structures on parameter space of any abstract semi-infinite variation of Hodge structures of Calabi-Yau type and conversely one can associate such semi-infinite variation of Hodge structures (\(\mathbb{F} \)-VHS for short) with arbitrary Frobenius manifold (we recall this in section 2 below). In this note we pursue analysis of relations between \(\mathbb{F} \)-VHS and integrable hierarchies and introduce integrable KdV type hierarchy which is associated with \(\mathbb{F} \)-VHS corresponding to an arbitrary semisimple Frobenius manifold. This note arose from an attempt to understand relation between problem of extension of abstract \(\mathbb{F} \)-VHS of Calabi-Yau type and dressing method from theory of integrable hierarchies (see \[DS\] §1 and references therein).

1.1. Some notations: \(Mat(n, \mathbb{C})\) denotes algebra of \((n \times n)\) matrices, \(Diag \subseteq Mat(n, \mathbb{C})\) denotes subalgebra of matrices whose entries are zero apart from diagonal, for \(A \in Mat(n, \mathbb{C})\) we denote via \(A^\top\) transposed matrix, for a vector space \(V\) we denote via \(V((\hbar))\) the space of Laurent series with values in \(V\), for \(v \in V((\hbar))\),

\[
v = \sum_{i=-m}^{+\infty} v_i \hbar^i\]

we set \(v_{\geq k} = \sum_{i=k}^{+\infty} v_i \hbar^i, v_{<k} = \sum_{i=-m}^{i=k-1} v_i \hbar^i\).

2. Semi-infinite VHS and Frobenius manifolds.

In this section I recall construction of family of Frobenius manifolds associated with abstract \(\mathbb{F} \)-VHS of Calabi-Yau type described in \[B1\] and give also inverse construction. In particular it provides a nice mathematical explanation to universality of WDVV-equations. In this section we work in analytic category and pure even case leaving appropriate adjustments for other categories (formal, \(\mathbb{Z}/2\mathbb{Z}\)-graded etc.) to an interested reader.

Let \(Gr^{(n)}_{\mathbb{F}}\) denotes affine grassmanian (see \[PS\], §8). Recall that it is defined as quotient \(Gr^{(n)}_{\mathbb{F}} := LGL(n, \mathbb{C})/L^+GL(n, \mathbb{C})\) where \(LGL(n, \mathbb{C})\) denotes group of...
maps from circle $S^1 = \{ \hbar \in \mathbb{C} : |\hbar| = 1 \}$ to $GL(n, \mathbb{C})$ which are analytic in some neighborhood of $S^1$ and $L^+ GL(n, \mathbb{C})$ denotes subgroup of elements which are boundary values of analytic maps from disk $\{ \hbar \in \mathbb{C} : |\hbar| \leq 1 \}$ to $GL(n, \mathbb{C})$.

Let $H^{(n)}$ denotes Hilbert space of all square-integrable functions on circle $S^1 = \{ \hbar \in \mathbb{C} : |\hbar| = 1 \}$ with values in $\mathbb{C}^n$, $H^{(n)} = L^2(S^1, \mathbb{C}^n)$. Let us denote also via $H_+^{(n)}$ (resp. $H_-^{(n)}$) closed subspace of $H^{(n)}$ generated by elements of form $v \cdot h^k$, $v \in \mathbb{C}^n$, $k \geq 0$ (resp. $k < 0$), so that $H^{(n)} = H_+^{(n)} \oplus H_-^{(n)}$, and by $pr_+$, (resp. $pr_-$) the orthogonal projection $H^{(n)} \rightarrow H_+^{(n)}$ (resp. $H^{(n)} \rightarrow H_-^{(n)}$) along $H_-^{(n)}$ (resp. $H_+^{(n)}$).

**Lemma 1.** ([PS], §8) Grassmanian $Gr^{(n)}$ can be defined alternatively as set of all closed subspaces $\mathcal{L} \subset H^{(n)}$ having everywhere dense subset consisting of analytic functions and such that:

1. $pr_{+,\mathcal{L}}$ is a Fredholm operator (recall that operator $T$ is Fredholm iff $\dim \ker T, \dim \text{coker } T < \infty$), i.e. $\mathcal{L}$ is in a sense "comparable" with $H_+^{(n)}$,

2. $h\mathcal{L} \subset \mathcal{L}$.

**Proof.** To a class $[\varphi] \in LGL(n, \mathbb{C})/L^+ GL(n, \mathbb{C})$ one can associate subspace $\varphi \cdot H^{(n)}_+ \subset H^{(n)}$. The Fredholm property follows from ([PS], proposition 6.3.1). Conversely, the factorspace $\mathcal{L}/h\mathcal{L}$ for a subspace $\mathcal{L}$ satisfying the above properties is an $n$-dimensional vector space, since inclusion $h\mathcal{L} \subset \mathcal{L}$ is a Fredholm operator of index equal to the index of inclusion $hH^{(n)}_+ \subset H^{(n)}$, i.e. $n$; therefore if $\varphi_i \in \mathcal{L}, i = 1, \ldots, n$, are such that $\{ \varphi_i \mod h\mathcal{L} \}_{i=1..n}$ is a basis for $\mathcal{L}/h\mathcal{L}$, then matrix with columns $\varphi_i$ defines the corresponding element from quotient $LGL(n, \mathbb{C})/L^+ GL(n, \mathbb{C})$. □

Let $\mathcal{L}(x) \in Gr^{(n)}_\mathcal{U}, x \in \mathcal{U}$, be a family of subspaces from $Gr^{(n)}$ parametrized by $\mathcal{U}$.

Our first assumption on family $\mathcal{L}(x)$ is

1) $\frac{\partial}{\partial x} - \text{Griffiths transversality: } \frac{\partial}{\partial x} \mathcal{L}(x) \subset h^{-1} \mathcal{L}(x)$

For family of subspaces having such property one has "symbol of $\frac{\partial}{\partial x}$" map:

$$Symbol(\frac{\partial}{\partial x}) : T_x \mathcal{U} \otimes \mathcal{L}/h\mathcal{L} \rightarrow h^{-1} \mathcal{L}/\mathcal{L}$$

where $T_x \mathcal{U}$ denotes tangent space to $\mathcal{U}$ at a point $x$. Our next assumption on family $\mathcal{L}(x)$ is

2) "Calabi-Yau type": there exists one-dimensional subspace $\{ \lambda[\Omega] \}_{\lambda \in \mathbb{C}}$ $\subset \mathcal{L}/h\mathcal{L}(x)$ for any $x \in \mathcal{U}$, such that map $Symbol(\frac{\partial}{\partial x})(\lambda[\Omega]) : T_x \mathcal{U} \otimes \{ \lambda[\Omega] \} \rightarrow h^{-1} \mathcal{L}/\mathcal{L}$ is an isomorphism.

In particular, $\dim_{\mathbb{C}} \mathcal{U} = \dim_{\mathbb{C}} \mathcal{L}/h\mathcal{L}$ and also the map $T_x \mathcal{U} \rightarrow Hom(\mathcal{L}/h\mathcal{L}, h^{-1} \mathcal{L}/\mathcal{L})$ induced by $Symbol(\frac{\partial}{\partial x})$ is an embedding. Next we need to incorporate a semi-infinite analog of Poincare pairing into our assumptions. Let $G : (H^{(n)})^{\otimes 2} \rightarrow H^{(1)}$ be a linear nondegenerate pairing, which is symmetric in the following sense: $G(a, b)(h) = (-1)^n G(b, a)(-\hbar)$ and has the following property of linearity with respect to multiplication by $h$: $G(ha, b) = G(a, -h b) = hG(a, b)$. Such pairing is uniquely defined by restriction $G|_{(H^{(n)})^{\otimes 2}} = \sum_{i=-\infty}^{\infty} \hbar^i g^{(i)}$ where $g^{(i)} : (\mathbb{C}^n)^{\otimes 2} \rightarrow \mathbb{C}$ is $(-1)^{n+i-1}$-symmetric. Our last assumption on family $\mathcal{L}(x)$ is

3) Isotropy with respect to pairing: $G|_{L^{\otimes 2}} \in h^n H^{(1)}_+$. 


Examples of families of $\mathbb{F}$-subspaces satisfying these three conditions arise naturally in context of noncommutative algebraic geometry (see [B1] where also relation with standard variations of Hodge structures is explained). In a sense one can say that such family of semi-infinite subspaces indicates presence of a non-commutative complex Calabi-Yau manifold.

Let $Gr^{(n)}_\mathbb{F}$ denotes "opposite" grassmanian consisting of closed subspaces of $H^{(n)}$ such that for $S \in Gr^{(n)}_\mathbb{F}$ restriction on $S$ of projection $pr_{-}|_{S}$ to $H^{(n)}$ is a Fredholm operator, that analytic functions are dense in $S$ and which satisfies in addition the following isotropy condition

$$G_{\mid S_{\mathbb{F}} \mathbb{F}} \in h^{n-1}H^{-1}(\mathbb{F}).$$

Transversality implies that intersection $L(x) \cap hS$ is an $n$-dimensional vector space.

There are natural isomorphisms: $i_S : L(x) \cap hS \simeq hS/S$ and $i_L : L(x) \cap hS \simeq L/hL$. Let $\omega \in hS/S$. Constraint imposed on element $[\Omega]$ from the second condition is of presence of a non-commutative algebraic geometry (see [B1] where also relation with standard variations of Hodge structures is explained). In a sense one can say that such family of semi-infinite subspaces indicates presence of a non-commutative complex Calabi-Yau manifold.

$$\text{Proof.} \text{ See proofs of propositions 6.5 from [B1], 4.1, 4.4, 4.8 from [B2].}$$

Now same arguments as in [B2], §4 give the following corollary (we refer reader to [B1] for a definition of Frobenius manifold)

**Corollary 1.** Tensors $C^\gamma_{\alpha\beta}(x^\alpha)$, $\eta_{\alpha\beta}$, $\frac{\partial}{\partial x^\alpha}$ define Frobenius manifold structure on $\mathcal{U}$. Remark that conformal (i.e. equipped with Euler vector field) Frobenius manifold corresponds to the data as above equipped with constant first order differential operator $D_{\partial/\partial h}$ acting on elements of $H^{(n)}$ such that $D_{\partial/\partial h}(\mathcal{L}) \subseteq h^{-2}\mathcal{L}$, $D_{\partial/\partial h}(S) \subseteq h^{-1}S$ and $D_{\partial/\partial h}G = 0$.

Conversely let we are given Frobenius manifold structure ($C^\gamma_{\alpha\beta}(x^\alpha)_{x\in\mathcal{U}}$, $\eta_{\alpha\beta}$, $e = \frac{\partial}{\partial x^\gamma}$) on $\mathcal{U}$. Then operators $\frac{\partial}{\partial x^\alpha} - h^{-1}C_{\alpha\beta}(x)$, $\alpha \in [1, \ldots, n]$, commute with each other and therefore there exists fundamental solution $\phi^\alpha_i$, $i \in [1, \ldots, n]$, to

$$\frac{\partial}{\partial x^\alpha}\phi^\beta(x, h) = h^{-1}\sum_\gamma C^\gamma_{\alpha\beta}(x)\phi^\beta(x, h), \quad \alpha, \beta \in [1, \ldots, n].$$
so that \( \det \phi_i^j(x, \hbar) \neq 0 \). Such solution is unique up to multiplication by an element from \( \text{LGL}(n, \mathbb{C}) \). We have \( h^{-1} \phi_0^\alpha = \frac{\partial}{\partial x_\alpha} \phi_1^\alpha \). We define corresponding family of semi-infinite subspaces as \( \mathcal{L}(x) := \phi(x)H_{n}^{(n)} \), i.e., it is the subspace generated by vectors \( h^k \phi_\alpha, \phi_\alpha = \sum_i \phi_i^j \rho_j \) where \( \rho_j \) is the standard basis in \( \mathbb{C}^n \) and \( \alpha \in [1, \ldots, n] \), \( k \geq 0 \). It follows from (2.2) that it satisfies \( \frac{\partial}{\partial x_\alpha} \)–Griffiths transversality condition. If \( \text{Symbol}(\frac{\partial}{\partial x_\alpha}) \) is written using basis \( \frac{\partial}{\partial x_\alpha}, [h^{-1} \phi_\alpha] \) in \( T_\mathcal{U}, \mathcal{L}/h\mathcal{L}, h^{-1}\mathcal{L}/\mathcal{L} \) respectively then \( \text{Symbol}(\frac{\partial}{\partial x_\alpha}) \) coincides with tensor \( C^\alpha_{\alpha\beta}(x) \) of multiplication. Therefore the second condition is satisfied as one can take for \( \{\Omega\} \) any element corresponding to an invertible element in the algebra defined by \( C^\alpha_{\alpha\beta} \).

Next we define pairing \( G(a, b) := h^{n-2} \eta(h\phi^{-1}a, h\phi^{-1}b) \) where we set \( \eta(h^k v, h^l u) := h^{k(1)} \sum_{\alpha, \beta} \eta_{\alpha\beta} v^n u^\beta \) for \( v = (v^1, \ldots, v^n) \), \( u = (u^1, \ldots, u^n) \). It follows from definition that \( G|_{\mathcal{L} \otimes \mathcal{L}} \in h^n H_{n}^{(1)} \). Notice that because of compatibility of \( \eta \) with multiplication defined by \( C^\alpha_{\alpha\beta} \) the value of \( G(a, b) \) does not depend on \( x \). It follows that for any choice of opposite isotropic semi-infinite subspace \( S \in \text{Gr}^{(n)}_{r} \) and an element \( \omega \) from an open cone in \( hS/S \) we get some Frobenius manifold structure on \( \mathcal{U} \).

Let us specify \( S \) and \( \omega \) giving rise to initial Frobenius manifold.

We set \( S := \phi(x)H_{n}^{(n)} \), \( \omega = [\phi_1] \in hS/S \) where \( \phi_1 = \sum_i \phi_i^j \rho_i \). It follows from definition that subspace \( S \) is opposite to \( \mathcal{L}(x) \) and isotropic. Notice that because of equation (2.2) \( S \) and \( \omega \) do not depend on \( x \in \mathcal{U} \). Also we have \( \phi_1 \in \mathcal{L}(x) \cap hS \), \( i_S(\phi_1) = \omega \). Restriction of \( \text{Symbol}(\frac{\partial}{\partial x_\alpha}) \) on \( i_S(\phi_1) \) is an isomorphism because \( C^\alpha_{\gamma\beta}(x) \) is \( \delta^\alpha_\gamma \).

**Proposition 2.** Applying the construction from corollary [4] to the data \( \mathcal{L}(x), S, \omega \) one gets back the initial Frobenius manifold structure on \( \mathcal{U} \).

**Proof.** We have \( \psi(x) = \phi_1(x) \). Notice that \( [h^{-1} \phi_\alpha], \phi_\alpha = \sum_i \phi_i^j \rho_i \) are constant as elements of \( S/h^{-1}S \) because \( \frac{\partial}{\partial x_\alpha} \phi_1 \in h^{-1}S \). The set \( \{[h^{-1} \phi_\alpha] \}_{\alpha \in [1, \ldots, n]} \) is a basis in \( S/h^{-1}S \). Recall that \( \frac{\partial}{\partial x_\alpha} \psi = h^{-1} \phi_\alpha \). Therefore \( \frac{\partial}{\partial x_\alpha} \psi \in S/h^{-1}S \) are constant. Therefore affine coordinates induced on \( \mathcal{U} \) via the map \( [\psi(x) - \omega]} : \mathcal{U} \to S/h^{-1}S \) coincide with \( x^\alpha \). Also it follows that equations (2.1) hold with tensors \( C^\alpha_{\alpha\beta}(x^\alpha), \eta_{\alpha\beta}, \frac{\partial}{\partial x_\alpha} \) which coincide with tensors of the initial Frobenius manifold.

It is interesting to note that element \( \psi(x) = \mathcal{L}(x) \cap (S + \omega) \) has nice meaning in all situations where Frobenius manifolds appear. That is depending on the context it can be Baker function over small phase space of \( n \)KdV hierarchy, Gromov-Witten 2-point descendent correlator, Saito primitive form, solution to Riemann-Hilbert problem, period vector of Calabi-Yau manifold (eventually of non-commutative deformation of it) and so on.

### 3. Lax operators.

We saw above that Frobenius manifolds can be described in terms of geometry of semi-infinite subspaces similar to the one arising in approach to integrable hierarchies from [SW]. Recall that in the latter context equations of KP hierarchy are described in terms of flow given by multiplication by \( \exp(\sum_{r=1}^{\infty} h^{-r} x^r) \) which acts on set of semi-infinite subspaces. Pursuing such analogy one can conjecture that at least for certain classes of Frobenius manifolds there exists natural *enlarged* family of semi-infinite subspaces \( \mathcal{L}(x) \in \text{Gr}^{(n)}_{r} \) depending on infinite number of parameters.
\{x^{\alpha,r}\}$, $\alpha \in \{1, \ldots, n\}$, $r \in \mathbb{N}$, (higher times) such that
\begin{equation}
\frac{\partial}{\partial x^{\alpha,r}} L(x) \subseteq h^{-r} L(x)
\end{equation}

Intersecting $L(x)$ with $hS$ for some opposite subspace $S$ we see that such family is the same as infinite set of commuting operators of form $\frac{\partial}{\partial x^{\alpha,r}} - \sum_{j=1}^{n} h^{-j} A_j(\alpha,r)$ extending the initial set of $n$ commuting operators $\frac{\partial}{\partial x^{\alpha}} - h^{-1} C_\alpha$. We will show below how dressing method can be used in order to construct and classify such sets of operators in the case of semi-simple Frobenius manifolds.

Let $(C^\gamma_{\alpha\beta}(x))_{x \in U}$, $\eta_{\alpha\beta}$, $\epsilon$ define a semi-simple Frobenius manifold structure on an open domain $U \subset \mathbb{C}^n$. Recall that unless the converse is explicitly mentioned we do not assume it to be conformal, i.e. we do not assume existence of an Euler vector field. Three-tensor $C^\gamma_{\alpha\beta}(x)$ defines structure constants of commutative associative multiplication on tangent space $T_x U$ for any $x \in U$. We denote this multiplication via " $\circ$ ". Multiplication operators $" \circ " \in \text{End}(T_x U)$, $\eta \in T_x U$ form commutative $n$-dimensional subalgebra in $\text{End}(T_x U)$. Our basic assumption (semisimplicity) is that for any $x \in U$ we have decomposition $T_x U = \oplus_{i=1}^{n} \theta_i$ where $\theta_i$ are eigenspaces of all operators $" \circ \"$, $\epsilon \in T_x U$ simultaneously.

Let us introduce our Lax operator. Contrary to the usual case it is in fact a set of $n$ commuting operators in $n$ variables:
\begin{equation}
L = \{L_\alpha(x)\}_{\alpha \in \{1, \ldots, n\}} \quad L_\alpha = \frac{\partial}{\partial x^{\alpha}} - h^{-1} C_\alpha(x)
\end{equation}

$[L_\alpha, L_\beta] = 0$, where $C_\alpha$ are functions of $x = \{x^\alpha\}_{\alpha \in \{1, \ldots, n\}}$ values in $\text{Mat}(n, \mathbb{C})$ whose entries are $(C_\alpha)^\beta_\gamma = C^\gamma_{\alpha\beta}(x)$. We first apply some transformations to reduce the set of operators $\{L_\alpha(x)\}$ to a simpler form. Although it might look strange in the view of standard definition of Frobenius manifold, it is natural from the point of view described in previous section to consider separately changes of coordinates on $U$ acting on $\{L_\alpha(x)\}$ as on components of $\text{Mat}(n, \mathbb{C})$-connection in trivial bundle (and not as on components of a connection on $TU$) and gauge transformations changing the choice of frame in the trivial bundle.

Let us choose nonzero vectors $v_i \in \theta_i$ where $\theta_i \subset T_x U$ are eigenspaces of multiplication operators, $v_i = \sum v_i^\alpha x^\alpha$ and put $(T_0(x))^{i\beta} = v_i^\beta(x)$, $T_0(x) \in GL(n, \mathbb{C})$. Then for all $\alpha$ simultaneously we have: $T_0^{-1} C_\alpha T_0 = a_\alpha(x)$ where $a_\alpha(x) = \text{diag}(a_\alpha^1, \ldots, a_\alpha^n)$ is a function of $x$ with values in $\text{Diag}$. Note that linear combinations of $a_\alpha(x)$ span $\text{Diag}$ as linear space for any $x \in U$. Transformation $T_0(x)$ is determined uniquely up to right multiplication by function with values in $\text{Diag}$. Then transformed Lax operator $\tilde{L} = \{\tilde{L}_\alpha\}$, $\tilde{L}_\alpha := T_0^{-1} L_\alpha T_0$ has the form
\begin{equation}
\tilde{L}_\alpha = \frac{\partial}{\partial x^{\alpha}} + q_\alpha - h^{-1} a_\alpha(x)
\end{equation}

with $q_\alpha = T_0^{-1} (\frac{\partial}{\partial x^{\alpha}} T_0)$. Vectors $v_i$ satisfy $\eta(v_i, v_j) = 0$ for $i \neq j$ where $\eta$ is the pairing defined by tensor $\eta_{\alpha\beta}$. We can normalize $v_i$ so that $\eta(v_i, v_i) = 1$. Then $T_0 T_0^\top = \text{Id}$ and $(T_0^{-1} (\frac{\partial}{\partial x^{\alpha}} T_0))^\top = -T_0^{-1} (\frac{\partial}{\partial x^{\alpha}} T_0)$. We denote transformed Lax operator written using such orthonormal normalization $\tilde{L}_\alpha^{\text{orth}} = \{L_\alpha^{\text{orth}}\}$.

We can simplify further the form of our Lax operators by appropriate change of coordinates on $U$. The equation $[\tilde{L}_\alpha, \tilde{L}_\beta] = 0$ implies that $\frac{\partial}{\partial x^{\alpha}} a_\alpha(x) - \frac{\partial}{\partial x^{\beta}} a_\beta(x) = [T_0^{-1} (\frac{\partial}{\partial x^{\alpha}} T_0), a_\alpha] + [T_0^{-1} (\frac{\partial}{\partial x^{\beta}} T_0), a_\beta]$. Notice that diagonal entries of $[B, a]$ are zero for $a \in \text{Diag}$. Therefore $\frac{\partial}{\partial x^{\alpha}} a_\alpha(x) - \frac{\partial}{\partial x^{\alpha}} a_\beta(x) = 0$ and $a_\alpha(x) = \frac{\partial}{\partial x^{\alpha}} u(x)$ for some
where $e_i$ is the constant matrix whose entries are $(e_i)_{jk} = \delta_{ij}\delta_{ik}$. We have $[q_k(u), e_j] - [q_j(u), e_i] = 0$ as a consequence of $[[L_i, \bar{L}_j], e_j] = 0$. It follows that $q_i(x)$ may have non-zero entries on $i$-th row, $i$-th column and diagonal only and that for some matrix $Q_{kl}(u)$ we have $q_i = [e_i, Q] + \sum_k(q_k)_{jk} e_k$ simultaneously for all $q_i$. Matrix $Q_{kl}$ is determined uniquely if one requires that its diagonal entries are zero. For $L^{\mathrm{orth}}$ we have $(q_i(u))^\top = -q_i(u)$ and $q_i(u) = [e_i, Q]$ for $Q_{kl}(u), Q^\top = Q, Q_{kk} = 0$.

Notice that in order to reduce commuting operators of form $\frac{\partial}{\partial \alpha} - h^{-1} C_\alpha(x)$ to operators of form $[\bar{L}_i, 1]$ and $[\bar{L}_i, x]$ we have used only that $C_\alpha(x)$ are simultaneously diagonalizable and that the vector space spanned by their spectra coincides with the space of all diagonal matrices at any $x \in U$.

4. Dressing Transformations.

**Proposition 3.** There exists formal power series $\bar{T}(h, u) = Id + \sum_{k=1}^\infty h^k T_k(u)$ such that for all $i \in [1, \ldots, n]$ simultaneously

\begin{equation}
\bar{T}^{-1}\bar{L}_i \bar{T} = \frac{\partial}{\partial u^i} - e_i h^{-1} + \sum_{k=0} h^k h_{k,i}(u)
\end{equation}

where $h_{k,i}(u)$ are functions with values in $\Diag$. $\bar{T}$ is determined up to right multiplication by function with values in $Id + (\Diag)h[[h]]$. If $\bar{T}$ is represented in the form $\bar{T} = (id + h S_1(u)) \circ \ldots \circ (id + h^k S_k(u)) \circ \ldots$ then $\bar{T}$ is determined uniquely by requiring that diagonal entries of $S_k$ are zero. In this case entries of $S_k$ (consequently of $\bar{T}_k$) can be found recursively and they are given by certain differential polynomials in $(q_i)_{kl}$ with zero free terms. For $\bar{L}^{\mathrm{orth}}$ we have $\bar{T}(h)\bar{T}(-h)^\top = Id$, $h_{k,2i}(u) = 0$.

**Proof.** We look for $\bar{T}$ in the form $\bar{T} = (id + h S_1(u)) \circ \ldots \circ (id + h^k S_k(u)) \circ \ldots$ so that

\[ [(id + h S_1(u)) \circ \ldots \circ (id + h^k S_k(u))]^{-1} \bar{L}_i [(id + h S_1(u)) \circ \ldots \circ (id + h^k S_k(u))] = \frac{\partial}{\partial u^i} - e_i h^{-1} + \sum_{j=0}^k h^j h_{j,i}(u) + \sum_{j=k+1}^\infty h^j H_{j,i}(u) \]

where $h_{j,i}(u) \in \Diag$. Then equations for $S_{k+1}(u), h_{k,i}(u)$ are

\[-e_i \circ S_{k+1} + H_{k,i} = -S_{k+1} \circ e_i + h_{k,i} \]

Notice that $[\bar{L}_i, \bar{L}_j] = 0$ implies same relation for Lax operators conjugated by $(id + h S_1(u)) \circ \ldots \circ (id + h^k S_k(u))$. Therefore $[e_i, H_{k,j}] - [e_j, H_{k,i}] \in \Diag$. It follows that $[e_i, H_{k,j}] - [e_j, H_{k,i}] = 0$. Therefore matrix $H_{k,i}$ may have non-zero entries on $i$-th row, $i$-th column and diagonal only and for some matrix $S_{k+1}$ and some matrices $h_{k,i} \in \Diag$ we have $H_{k,i} = [e_i, S_{k+1}] + h_{k,i}$. Such matrix $S_{k+1}$ is determined uniquely by requirement that its diagonal entries are zero. It is easy to see by induction that entries of $H_{k,i}$ and $S_k$ are then differential polynomials in...
Proposition 4. If one sets \( T(h, x) = T_0(x) \circ \widetilde{T}(u(x)) \) then for all \( \alpha \in [1, \ldots, n] \) simultaneously

\[
T^{-1}L_\alpha T = \widetilde{T}^{-1}(u(x))\tilde{L}_\alpha \tilde{T}(u(x)) = \frac{\partial}{\partial x^\alpha} a_\alpha(x)h^{-1} + \sum_{k=0}^{+\infty} h^k h_{k, \alpha}(x)
\]

where \( h_{k, \alpha}(x) := \sum_i a_i^k h_i(x,u(x)) \) and \( a_\alpha(x) \) are functions with values in \( \text{Diag} \). Transformation \( T(x) \in \text{Mat}(n, \mathbb{C})[[h]] \) satisfying \( T^{-1}L_\alpha T = \frac{\partial}{\partial x^\alpha} + h_\alpha \), \( h_\alpha \in h^{-1}\text{Diag}[[h]] \) for all \( \alpha \in [1, \ldots, n] \) simultaneously is defined uniquely up to right multiplication by a function with values in \( \text{Diag}[[h]] \).

Proof. The first statement is a consequence of the previous proposition. The proof of the second statement is the same as proof of analogous statement from the previous proposition.

Below we set \( \tilde{L}_i^{\text{norm}} := \widetilde{T}^{-1}L_i \tilde{T}, \tilde{L}_a^{\text{norm}} := T^{-1}L_\alpha T \). We also set \( h_{-1, \alpha} = -a_\alpha \).

We adopt terminology which calls such transformations reducing Lax operators to diagonal form dressing transformations.

5. LAX EQUATIONS.

The construction proceeds quite analogously to the standard case of Lax operator in one variable (see [DS, §1]).

Notice that if \( b \in \text{Diag}(h) \) then \( [\tilde{L}_i^{\text{norm}}, b] = 0 \), \( [\tilde{L}_\alpha^{\text{norm}}, b] = 0 \). Therefore \( [\tilde{L}_i, \tilde{T}b\tilde{T}^{-1}] = 0 \), \( [\tilde{L}_\alpha, \tilde{T}b\tilde{T}^{-1}] = 0 \). For any \( b \in \text{Diag}(h) \) we set \( \tilde{\varphi}(b) = \tilde{T}b\tilde{T}^{-1}, \tilde{\varphi}(b) = TbT^{-1} \).

Proposition 5. For any \( b \in \text{Diag}(h) \) we have

\[
[L_\alpha, \varphi(b)_{\leq -1}] = h^{-1}|C_\alpha, \varphi(b)_0|
\]

\[
[\tilde{L}_i, \tilde{\varphi}(b)_{\leq 0}] = h^0|\epsilon_i, \tilde{\varphi}(b)_1|
\]

Proof. \( [\tilde{L}_i, \tilde{\varphi}(b)_{\leq 0}] \in \text{Mat}(n, \mathbb{C})[[h^{-1}]], \) on the other hand \( [\tilde{L}_i, \tilde{\varphi}(b)_{\leq 0}] = -[\tilde{L}_i, \tilde{\varphi}(b)_{> 0}] = h^0|\epsilon_i, \tilde{\varphi}(b)_1| + s, s \in \text{Mat}(n, \mathbb{C})h[[h]] \). It follows that \( s = 0 \). The proof in the case of \( L_\alpha \) is the same.

Our basic set of equations is

\[
\frac{\partial L_\alpha}{\partial t} = \left[ \sum_{j=1}^{m} h^{-j} A_j, L_\alpha \right] \tag{5.1}
\]

where \( L_\alpha = \frac{\partial}{\partial x^\alpha} - h^{-1}C_\alpha, \alpha \in [1, \ldots, n], [L_\alpha, L_\beta] = 0, \) where \( C_\alpha \) and \( A_j \) are functions of \( x^\alpha, t \) with values in \( \text{Mat}(n, \mathbb{C}) \) and \( \sum_{j=1}^{m} h^{-j} A_j = \varphi(b)_{\leq -1} \) for some
fixed \( b \in \text{Diag}(\langle h \rangle) \). Recall that \( \varphi(b) := T(h, x, t) b T^{-1}(h, x, t) \) where \( T \) is dressing transformation from proposition 4 (with \( t \) considered as a parameter). Notice that in spite of non-uniqueness in the choice of \( T \) (recall that it is defined up to right multiplication by \( R \in \text{Diag}(\langle h \rangle) \)) \( \varphi(b) \) is well-defined. In fact \( \varphi(b) \) for \( b = \sum_{j=-\infty}^{m} h^{-j} b_j \) does not depend on \( b_j \) with \( j \leq 0 \). Notice also that \([\sum_{j=1}^{m} h^{-j} A_j, L_\alpha], L_\beta] = 0\) which implies that it is enough to impose condition \([L_\alpha, L_\beta] = 0\) at some initial value of \( t \) only.

We also consider analogous equations for \( \{\tilde{L}_i\} \) :

\[
\frac{\partial \tilde{L}_i}{\partial t} = \left[ \sum_{j=0}^{m} h^{-j} A_j, \tilde{L}_i \right]
\]

where \( \tilde{L}_i = \frac{\partial}{\partial x_i} + q_i(u, t) - h^{-1} e_i \), \( i = 1, \ldots, n \), \([\tilde{L}_i, \tilde{L}_j] = 0\), \( q_i(u, t) \) and \( \tilde{A}_j(u, t) \) are functions of \( u^t \), \( t \) with values in \( \text{Mat}(n, \mathbb{C}) \); \( \sum_{j=0}^{m} h^{-j} \tilde{A}_j(u, t) = \varphi(b)_{\leq 0} \) for some fixed \( b \in \text{Diag}(\langle h \rangle) \), \( \varphi(b) = \tilde{T}(h, u, t)b\tilde{T}^{-1}(h, u, t) \) where \( \tilde{T}(h, u, t) \) is dressing transformation for \( \tilde{L}_i \) defined by equation (4.1) with \( t \) as a parameter. Also \( \varphi(b) \) for \( b = \sum_{j=-\infty}^{m} h^{-j} b_j \) depends only on \( b_j \) with \( j \geq 0 \).

Similarly we have for \( \{\tilde{L}_i\} \) the following equations:

\[
\frac{\partial \tilde{L}_i}{\partial t} = \left[ \sum_{j=0}^{m} h^{-j} A_j, \tilde{L}_i \right]
\]

where \( \tilde{L}_\alpha = \frac{\partial}{\partial x_\alpha} + q_\alpha(x, t) - h^{-1} a_\alpha(x) \), \( \alpha \in [1, \ldots, n] \), \([\tilde{L}_\alpha, \tilde{L}_\beta] = 0\), \( a_\alpha(x) \in \text{Diag} \), \( q_\alpha(x, t) \) and \( \tilde{A}_j(x, t) \) are functions of \( x^\alpha \), \( t \) with values in \( \text{Mat}(n, \mathbb{C}) \); \( \sum_{j=0}^{m} h^{-j} \tilde{A}_j(x, t) = \varphi(b)_{\leq 0} \) for some fixed \( b \in \text{Diag}(\langle h \rangle) \), \( \varphi(b) = \tilde{T}(h, x, t)b\tilde{T}^{-1}(h, x, t) \) and \( \tilde{T}(h, x, t) \) is dressing transformation defined by equation (4.3) with \( t \) as a parameter. As above \( \varphi(b) \) for \( b = \sum_{j=-\infty}^{m} h^{-j} b_j \) depends only on \( b_j \) with \( j \geq 0 \).

These three sets of equations are in fact closely related.

**Proposition 6.** Let operators \( \{\tilde{L}_\alpha(x, t_1)\} \) of form (3.3) are related with operators \( \{\tilde{L}_i(u, t)\} \) of form (3.4) by a change of coordinates \( u^t = u^x \alpha \) so that \( \tilde{L}_\alpha(x, t_1) = \sum_i \frac{\partial u^\alpha}{\partial x^\alpha} \tilde{L}_i(u(x), t_1) \), then \( \{\tilde{L}_i(u, t)\} \) satisfies equations (5.2) for some \( b \in \text{Diag}(h^{-1}) \) iff \( \{\tilde{L}_\alpha(x, t) = \sum_i \frac{\partial u^\alpha}{\partial x^\alpha} \tilde{L}_i(u(x), t)\} \) satisfies equations (5.3) with the same \( b \in \text{Diag}(h^{-1}) \).

**Proof.** If operators \( \tilde{L}_i(u, t) \) satisfy equations (5.2) then their linear combinations

\[
\sum_i \frac{\partial u^\alpha}{\partial x^\alpha} \tilde{L}_i(u(x), t) \]

also if \( \tilde{T}(h, u, t) \) is dressing transformation for \( \tilde{L}_i(u, t) \): \( \tilde{T}^{-1} \tilde{L}_i \tilde{T} = \frac{\partial}{\partial x^\alpha} - e_i h^{-1} + h_i \), \( h_i \in \text{Diag}(h) \), then \( \tilde{T}(h, u, x, t) = \tilde{T}(h, u(x), t) \) is dressing transformation for \( \tilde{L}_\alpha(x, t) \):

\[
\tilde{T}^{-1}\sum_i \frac{\partial u^\alpha}{\partial x^\alpha} \tilde{L}_i(u(x), t) \tilde{T} = \frac{\partial}{\partial x^\alpha} - a_\alpha(x) h^{-1} + h_\alpha(x, t), h_\alpha(x, t) \in \text{Diag}(h) \]
Conversely if $\tilde{L}_\alpha(x,t)$ satisfy equations (3.4) then $[\tilde{\varphi}(b)_{\leq 0}, \tilde{L}_\alpha(x,t)] = [\tilde{\varphi}(b), a_\alpha]$ and $\partial_{x^\alpha} a_\alpha = 0$. Thus $\{\tilde{L}_\alpha(u,t) := \sum_\alpha \frac{\partial}{\partial x^\alpha} \tilde{L}_\alpha(x(u),t)\}$ has the form (3.4) for any $t$ and satisfies equations (3.5).

Proof. We have

\[
\frac{\partial}{\partial t} \begin{cases} 
\tilde{\varphi}(b)_{\leq 0}, \tilde{L}_\alpha(x,t) \end{cases} = [\tilde{\varphi}(b), a_\alpha]
\]

and consequently $S_0(\partial_{x^\alpha} + q_\alpha, \partial_{x^\alpha} + q_\beta) = 0$. Therefore there exists a gauge transformation $S_0(x)$ such that $S_0(\partial_{x^\alpha} + q_\alpha) S_0^{-1} = \partial_{x^\alpha}$ and consequently $S_0 \tilde{L}_\alpha(x) S_0^{-1}$ is in the form (3.2).

**Proposition 7.** Let operators $\{L_\alpha(x)\}_{\alpha \in [1, \ldots, n]}$ of form (3.2) are related with operators $\{\tilde{L}_\alpha(x)\}$ of form (3.3) by a gauge transformation: $\tilde{L}_\alpha(x) = T_0^{-1}(x) L_\alpha(x) T_0(x)$. Let $\{L_\alpha(x,t)\}_{\alpha \in [1, \ldots, n]}$, $L_\alpha(x,t_1) = L_\alpha(x)$, satisfy equations (3.4) for some $b \in \text{Diag}[\hbar^{-1}]$. If one sets $R(x,t)$ to be family of gauge transformations $R(x,t)$ such that $\partial R/\partial t = -\tilde{\varphi}(b)_0 R, R(x,t_1) = T_0(x)$, then $\tilde{L}_\alpha(x,t) := R^{-1}(x,t) L_\alpha(x,t) R(x,t)$ have the form (3.2) for any $t$ and satisfy equations (3.4) with the same $b \in \text{Diag}[\hbar^{-1}]$.

Conversely, if $\{L_\alpha(x,t)\}_{\alpha \in [1, \ldots, n]}$, $L_\alpha(x,t_1) = L_\alpha(x)$, satisfy equations (3.5) for some $b \in \text{Diag}[\hbar^{-1}]$, then for $R(x,t)$ such that $\partial R/\partial t = -R[\tilde{\varphi}(b)_0, R(x,t_1) = T_0(x)$, operators $L_\alpha(x,t) := R(x,t) L_\alpha(x,t) R^{-1}(x,t)$ are of form (3.4) and satisfy equations (3.4) with the same $b \in \text{Diag}[\hbar^{-1}]$.

Proof. We have $T_0^{-1} C_\alpha(x,t_1) T_0 = a_\alpha(x), q_\alpha(t_1) = T_0^{-1}(\partial_{x^\alpha} T_0)$. If $\partial L_\alpha/\partial t = [\tilde{\varphi}(b)_{\leq -1}, L_\alpha]$ then

\[
\frac{\partial}{\partial t} \begin{cases} 
C_\alpha \end{cases} = -[\tilde{\varphi}(b)_0, C_\alpha]
\]

and therefore

\[
\frac{\partial}{\partial t} (R^{-1} C_\alpha R) = -R^{-1} \frac{\partial}{\partial t} R^{-1} C_\alpha R + R^{-1} \frac{\partial}{\partial t} C_\alpha R + R^{-1} C_\alpha \frac{\partial}{\partial t} R =
\]

\[
= R^{-1} \tilde{\varphi}(b)_0 C_\alpha R - R^{-1} \tilde{\varphi}(b)_0 C_\alpha R + R^{-1} C_\alpha \tilde{\varphi}(b)_0 R - R^{-1} C_\alpha \tilde{\varphi}(b)_0 R = 0
\]

Therefore $R^{-1} C_\alpha(x,t) R = a_\alpha(x)$ does not depend on $t$ and $R^{-1} L_\alpha(x,t) R = \partial_{x^\alpha} + q_\alpha(x,t) - h^{-1} a_\alpha(x), q_\alpha = R^{-1} \frac{\partial}{\partial x^\alpha} R$. $\alpha \in [1, \ldots, n]$, are of the form (3.3) for any $t$.

If $T(x,t)$ is the dressing transformation for $\partial_{x^\alpha} + q_\alpha(x,t) - h^{-1} a_\alpha(x)$ then $R \circ T$ is dressing transformation for $L_\alpha(x,t)$ and $R^{-1} \tilde{\varphi}(b)_0 R = -T b T^{-1} = \tilde{\varphi}(b)$. We have

\[
R^{-1} \circ \left( \frac{\partial}{\partial t} \right) R = \frac{\partial}{\partial t} + R^{-1} \frac{\partial R}{\partial t} = \frac{\partial}{\partial t} - \tilde{\varphi}(b)_0
\]

and

\[
0 = [R^{-1} \circ \left( \frac{\partial}{\partial t} \right) R, R^{-1} \circ \left( \frac{\partial}{\partial x^\alpha} \right) R] = \left[ \frac{\partial}{\partial t} - \tilde{\varphi}(b)_0, \frac{\partial}{\partial x^\alpha} + q_\alpha(x,t) \right]
\]

Therefore $\partial q_\alpha/\partial t = [\tilde{\varphi}(b)_0, \partial_{x^\alpha} + q_\alpha(x,t)] = [\tilde{\varphi}(b)_{\leq 0}, \partial_{x^\alpha} + q_\alpha(x,t) - h^{-1} a_\alpha(x)]$.

The proof of converse statement uses similar arguments.

Equations (3.4) describe essentially the largest possible extension of our family of semi-infinite subspaces to the family satisfying (3.3) because of the following proposition.
Proposition 8. Let \( \sum_{j=1}^{m} h^{-j} A_j(x), L_\alpha \in h^{-1} \text{Mat}(n, \mathbb{C}) \) for all \( \alpha \in [1, \ldots, n] \), then \( \sum_{j=1}^{m} h^{-j} A_j(x) = \varphi(b)_{\leq -1} \) for some \( b = \sum_{j=1}^{m} b_j h^{-j} \in \text{Diag}[h^{-1}] \) and for \( j > 1 \): \( b_j = \text{const.} \)

Proof. Let us set \( T^\alpha \sum_{j=1}^{m} h^{-j} A_j(x) T^{-1} = \sum_{j=1}^{m} h^{-j} B_j(x) \). We have

\[
(5.4) \quad [ \sum_{j=-\infty}^{m} h^{-j} B_j(x) - \frac{\partial}{\partial x^\alpha} h^{-1} a_\alpha + h_\alpha ] = \text{Diag}[h^{-1}] \text{Mat}(n, \mathbb{C}) \text{ where } h_\alpha \in \text{Diag}[h^{-1}]
\]

Let \( B_l \in \text{Diag} \) for all \( l > j > 0 \). Then \( [B_j, a_\alpha] \in \text{Diag} \) and therefore \( [B_j, a_\alpha] = 0 \). It follows that \( B_j \in \text{Diag} \) and the same is true for all \( B_j \) with \( j > 0 \). Therefore using again (5.4) \( \frac{\partial}{\partial x^\alpha} B_j = 0 \) for all \( j > 1 \).

Below we will mainly consider equations (5.1).

5.1. Integrals of motion. Let \( L_\alpha(x, t) \) satisfy (5.1) and \( T^{-1}(x, t)L_\alpha T(x, t) = \frac{\partial}{\partial x^\alpha} + \sum_{k=-\infty}^{+\infty} h^{-k} h_{k, \alpha}(x, t), h_{k, \alpha}(x, t) \in \text{Diag} \).

Proposition 9. \( h_{k, \alpha}(x, t) \) satisfy \( \partial h_{k, \alpha}(x, t)/\partial t + \partial B_k/\partial x^\alpha \) for some \( B_k(x, t) \in \text{Diag} \).

Proof. Equations (5.1) can be written as \( [\frac{\partial}{\partial t} - \sum_{k=1}^{m} h^{-k} A_k(x), L_\alpha] = 0, \alpha \in [1, \ldots, n] \). It follows that

\[
(5.5) \quad [T^{-1}(\frac{\partial}{\partial t} - \sum_{j=1}^{m} h^{-j} A_j(x)) T, \frac{\partial}{\partial x^\alpha} + \sum_{k=-\infty}^{+\infty} h^{-k} h_{k, \alpha}(x, t)] = 0
\]

Let \( T^{-1}(\frac{\partial}{\partial t} - \sum_{j=0}^{m} h^{-j} A_j(x)) T = \frac{\partial}{\partial x^\alpha} - \sum_{j=-\infty}^{m} h^{-j} B_j(x) \). Notice that \( \{ a_\alpha = h_{-1, \alpha} \}_{\alpha \in [1, \ldots, n]} \) generate \( \text{Diag} \). Let for all \( j > l \) \( B_j \in \text{Diag} \). Then \( [B_l, a_\alpha] \in \text{Diag} \) and therefore \( [B_l, a_\alpha] = 0 \) for all \( \alpha \in [1, \ldots, n] \) and consequently \( B_l \in \text{Diag} \). We see that \( B_j \in \text{Diag} \). Therefore setting \( B := \sum_{j=-\infty}^{m} h^{-j} B_j(x, t) \) equation (5.5) can be written

\[
\frac{\partial h_{\alpha}}{\partial t} + \frac{\partial B}{\partial x^\alpha} = 0
\]

Let \( x^\alpha = x^\alpha(s) \) be a curve on \( \mathcal{U} \) and let us consider evolution of \( L|_{x(s)} = L(s, t) \) which is restriction on \( x^\alpha(s) \) of the flat connection corresponding to \( \{ L_\alpha(t) \}_{\alpha \in [1, \ldots, n]} \). \( L(s, t) := \sum_{\alpha}(\partial x^\alpha/\partial s)L_\alpha(x, t) \). If \( \{ L_\alpha(x, t) \}_{\alpha \in [1, \ldots, n]} \) satisfy equation (5.1) for some \( b \in \text{Diag}[h^{-1}] \), then we have analogous equation for \( L(s, t) : \frac{\partial L}{\partial t} = [\varphi(b)_{\leq -1}, L] \). Proposition 3 implies that \( \sum_{\alpha} h_{k, \alpha}^i(x, t)dx^\alpha \) are densities of conservation laws for this equation. Notice also that \( \sum_{\alpha} h_{k, \alpha}^i(x, t)dx^\alpha \) is a closed form because of flatness of \( L \). Therefore for a closed curve \( x(s) \) the integrals \( \int_{x(s)} \sum_{\alpha} h_{k, \alpha}^i(x, t)dx^\alpha \) do not change under deformations of \( x(s) \). The same is true for a curve satisfying periodic or appropriate asymptotic boundary conditions. Analogous results hold for equations (5.2), (5.3).

5.2. Commutativity of flows. Let us consider two sets of equations:

\[
(5.6) \quad \frac{\partial L_\alpha}{\partial t^i} = [M^i_{\leq -1}(L_\alpha) \ L_\alpha, \ M^i = T b^i T^{-1}, \ b^i \in \text{Diag}(h), i = 1, 2]
\]

Proposition 10. Flows defined by \( b^i \in \text{Diag}(h), i = 1, 2 \) in (5.4) commute.
Proposition 11. Equations 5.1 are Hamiltonian with respect to both following brackets on the space of matrix-valued functions:

\[ \{ \cdot, \cdot \} \]

Proof. We must prove that \( \frac{\partial^2 L}{\partial t \partial s} = \frac{\partial^2 L}{\partial s \partial t} \) where derivatives are computed by means of equation (5.1). The proof is parallel to the standard case of Lax operator in one variable (proposition 1.7 from [DS]). We have \( \frac{\partial}{\partial t} \frac{\partial L}{\partial s} = \left( \frac{\partial}{\partial t} (M^{(2)} \leq -1), L_0 \right) + [M^{(2)} \leq -1, [M^{(1)} \leq -1, L_0]]. \) By the same arguments as in proof of proposition 3 we have \( T^{-1}(\frac{\partial}{\partial t} - M^{(i)}) T \in \text{Diag}((\hbar^i)). \) Therefore \( [T^{-1}(\frac{\partial}{\partial t} - M^{(i)}) T, b^{(j)}] = 0 \) and \( [\frac{\partial}{\partial t} - M^{(i)}, b^{(j)}] = 0. \) Therefore \( \frac{\partial}{\partial t} M^{(i)} \leq -1 = [M^{(i)} \leq -1, M^{(i)} \leq -1] \) and

\[
\begin{aligned}
\frac{\partial}{\partial t_1} \frac{\partial L_0}{\partial t} - \frac{\partial}{\partial t_2} \frac{\partial L_0}{\partial s} &= \left( [M^{(2)} \leq -1, M^{(1)} \leq -1], L_0 \right) + [M^{(2)} \leq -1, [M^{(1)} \leq -1, L_0]] \\
&\quad - [[M^{(1)} \leq -1, M^{(2)} \leq -1], L_0] - [M^{(1)} \leq -1, [M^{(2)} \leq -1, L_0]]
\end{aligned}
\]

We have \( [M^{(2)} \leq -1, [M^{(1)} \leq -1, L_0]] - [M^{(1)} \leq -1, [M^{(2)} \leq -1, L_0]] = [[M^{(2)} \leq -1, M^{(1)} \leq -1], L_0]. \) Also \( [M^{(2)}, M^{(1)}] = 0 \) and therefore

\[
[M^{(2)} \leq -1, M^{(1)}] \leq -1 = -[M^{(2)} \leq -1, M^{(1)}] \leq -1 = -[M^{(2)} \leq -1, M^{(1)}] \leq -1 = [M^{(1)} \leq -1, M^{(2)}] \leq -1
\]

Therefore \( [M^{(2)} \leq -1, M^{(1)}] \leq -1 - [M^{(2)} \leq -1, M^{(2)}] \leq -1 + [M^{(2)} \leq -1, M^{(1)}] = 0. \)

5.3. Bi-hamiltonian structure. For any closed curve \( x^\alpha = x^\alpha(s), s \in S^1 \) in \( U \) restriction on \( x(s) \) of flat connection corresponding to \( \{ L_0 \} \) has form \( L(s) = \partial/\partial s - h^{-1} C \), where \( C \) is a function with values in \( \text{Mat}(n, \mathbb{C}). \) We consider the following brackets on the space of matrix-valued functions:

\[
\{ C^i_l(s), C^k_l(\bar{s}) \}_0 = \delta(s - \bar{s}) (\delta^i_k C^l_l(s) - \delta^i_k C^l_l(\bar{s})) \\
\{ C^i_l(s), C^k_l(\bar{s}) \}_1 = \delta^j_i (s - \bar{s}) \delta^j_k \delta^j_l
\]

Adaptation of standard arguments (see proof of proposition 1.8 from [DS]) shows that \( \{ \cdot, \cdot \}_0 - h \{ \cdot, \cdot \}_1 \) is a Poisson brackets for any \( h \in \mathbb{C}. \) If \( \{ L_0(t) \} \) satisfies equations 5.1 then its restriction on \( x(s) \) satisfies the same equation: \( \partial L(s, t)/\partial t = [\varphi(b) \leq -1, L(s, t)]. \) Recall that an equation is called Hamiltonian if for any functional \( f(C) \) we have \( df(C(t))/dt = \{ f, H \}. \) The same arguments as in propositions 3, 4 show that for a deformation \( \tilde{C} \) of \( C \) corresponding to \( L(t)|_{x(s)} \) there exists deformed dressing transformation \( T(s) \in \text{Mat}(n, \mathbb{C})[[\hbar]] \) such that \( T^{-1}(s)(\partial/\partial s - h^{-1} \tilde{C})T(s) = \partial/\partial s + \sum_{k=1}^{+\infty} h^k h_k(s), h_k(s) \in \text{Diag}. \)

Proposition 11. Equations 5.1 are Hamiltonian with respect to both \( \{ \cdot, \cdot \}_0 \) and \( \{ \cdot, \cdot \}_1, \) the corresponding Hamiltonians are \( H_b = \text{Tr} \sum_{k=0}^{m} b_{k+1} \int_{x(s)} h_k ds \) and \( \tilde{H}_b = \text{Tr} \sum_{k=0}^{m-2} b_{k+2} \int_{x(s)} h_k ds \) respectively.

Proof. For a pair of functionals \( f(C), g(C): \)

\[
\{ f, g \} := \int \int \sum_{i,j,k,l} \frac{\delta f(C)}{\delta C^j_i(s)} \frac{\delta g(C)}{\delta C^k_l(s)} (\{ C^j_i(s), C^k_l(\bar{s}) \}_0 - \{ C^j_i(s), C^k_l(\bar{s}) \}_1) d\bar{s}ds
\]
which gives

\[
\{f, g\}_0 = \int \sum_{i,j,k} \frac{\delta f(C)}{\delta C_i^j(s)} \frac{\delta g(C)}{\delta C_k^j(s)} C_k^i(s) - \frac{\delta g(C)}{\delta C_j^i(s)} C_k^j(s))ds
\]

\[
\{f, g\}_1 = -\int \sum_{i,j} \frac{\delta f(C)}{\delta C_i^j(s)} \frac{\partial}{\partial s} \frac{\delta g(C)}{\delta C_i^j(s)} ds
\]

Equation for \(C(t)\) reads as \(\frac{dC}{dt} = -[\varphi(b)_0, C]\) and we have also \([\varphi(b)_0, C] = [\varphi(b)_{(-1)}] = 1\).

We have \(\frac{\partial [\varphi(b)_0]}{\partial t} = \int \sum_{i,j} \frac{\delta f(C)}{\delta C_i^j(s)} \frac{\delta g(C)}{\delta C_j^i(s)} \frac{\partial}{\partial s} \frac{\delta g(C)}{\delta C_j^i(s)} ds\). Therefore we must show that \(\delta H_b/\delta C_j^i = -[\varphi(b)_{(-1)}] = 0\) and that \(\delta H_b/\delta C_j^i = -[\varphi(b)_{(-1)}] = 0\). Recall that \(T^{-1}(\partial/\partial s - h^{-1}C)T = \partial/\partial s + h, h = \sum_{k=0}^{m_1} h^k b_k, h_k \in \text{Diag}\) and that \(b\) denotes \(\sum_{k=1}^{m_1} h^{-k} b_k\).

We have

\[
\frac{\delta H_b(C)}{\delta C_j^i(s)} = \text{Tr}(\sum_{k=0}^{m_1} b_{k+1} \frac{\delta h_k}{\delta C_i^j} + \text{Tr}(b \frac{\delta h}{\delta C_i^j}(-1)) = \text{Tr}(b T^{-1} \frac{\delta(T^{-1}(\partial/\partial s - h^{-1}C)T)}{\delta C_j^i}(-1) = -\text{Tr}(b T^{-1} \frac{\delta h}{\delta C_i^j} T(-1)(s) - \text{Tr}(b[T^{-1} \frac{\delta T}{\delta C_j^i}, \partial/\partial s + h])(-1)(s)
\]

Notice that \([b, \partial/\partial s + h] = 0\) since \(b, h \in \text{Diag}\) and therefore \(\text{Tr}(b[T^{-1} \frac{\delta T}{\delta C_i^j}, \partial/\partial s + h]) = 0\). We have also

\[
\text{Tr}(b T^{-1} \frac{\delta h}{\delta C_i^j} T(-1) = \text{Tr}(b T^{-1} \frac{\delta C}{\delta C_i^j}) = 0 = (\varphi(b)_0)_{(-1)}^i
\]

Therefore \(\delta H_b/\delta C_j^i = -[\varphi(b)_0]_{(-1)}^i\). The proof in the case of \(\tilde{H}_b\) is the same.

6. Concluding remarks.

It is important to understand how to generalize above hierarchies to the case of Frobenius manifolds which are not semi-simple. It is necessary for example for applications to theory of Gromov-Witten invariants where except for rare cases like projective spaces and some other homogenous spaces Frobenius manifolds defined by quantum cohomologies of Kahler manifolds are not semi-simple.

I planned initially to include a section describing the extension of variations of \(\frac{d}{dt}\) – Hodge structures over higher times in the framework of [B1]. However this would increase significantly the volume of this note. I plan to return to it in one of subsequent publications. I plan also to write down some applications including formulas relating partition functions of massive 2D topological field theories paired with gravity (see [N] for conjectures about properties of such partition functions) with \(\tau\) – functions of above hierarchies.

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