ESSENTIAL SPECTRAL SINGULARITIES AND THE SPECTRAL EXPANSION FOR THE HILL OPERATOR

O. A. Veliev
Depart. of Math., Dogus University
Acıbadem, 34722, Kadıköy, Istanbul, Turkey

Abstract. In this paper we investigate the spectral expansion for the one-dimensional Schrödinger operator with a periodic complex-valued potential. For this we consider in detail the spectral singularities and introduce new concepts as essential spectral singularities and singular quasimomenta.

1. Introduction and preliminary facts. In this paper we investigate the one dimensional Schrödinger operator $L(q)$ generated in $L^2(-\infty, \infty)$ by the differential expression

$$l(y) = -y''(x) + q(x)y(x),$$

where $q$ is 1-periodic, Lebesgue integrable on $[0, 1]$ and complex-valued potential. Without loss of generality, we assume that the integral of $q$ over $[0, 1]$ is 0. It is well-known [1, 7, 8] that the spectrum $\sigma(L)$ of the operator $L$ is the union of the spectra $\sigma(L_t)$ of the operators $L_t(q)$ for $t \in (-\pi, \pi]$ generated in $L^2[0, 1]$ by (1) and the boundary conditions

$$y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(0).$$

The eigenvalues of $L_t$ are the roots of the characteristic equation

$$\Delta(\lambda, t) := \begin{vmatrix} \theta(1, \lambda) - e^{it} & \varphi(1, \lambda) \\ \theta'(1, \lambda) & \varphi'(1, \lambda) - e^{it} \end{vmatrix} = 0$$

of the operator $L_t$ which equivalent to

$$F(\lambda) = 2 \cos t,$$

where $F(\lambda) := \varphi'(1, \lambda) + \theta(1, \lambda)$ is the Hill discriminant, $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ are the solutions of the equation $l(y) = \lambda y$ satisfying the following initial conditions

$$\theta(0, \lambda) = \varphi'(0, \lambda) = 1, \quad \theta'(0, \lambda) = \varphi(0, \lambda) = 0.$$
proved that \( L(q) \) is a spectral operator if and only if the projections of the operators \( L_t(q) \) are bounded uniformly with respect to \( t \) in \((-\pi, \pi]\). Tkachenko \[12\] proved that the non-self-adjoint operator \( L \) can be reduced to triangular form if all eigenvalues of the operators \( L_t \) for \( t \in (-\pi, \pi] \) are simple. However, in general, the eigenvalues are not simple and the projections of the operators \( L_t \) are not uniformly bounded. Indeed, Gasymov’s paper \[2\] shows that the operators \( L_t(q) \) with the potential \( q \) of the form
\[
q(x) = \sum_{n=1}^{\infty} q_n e^{inx}, \quad \sum_n |q_n| < \infty
\] (6)
have infinitely many multiple eigenvalues, their projections are not uniformly bounded and no one operator \( L_t(q) \) with nonzero potential of type (6) is spectral, since they have, in general, infinitely many spectral singularities. Gasymov in \[2\] investigated the direct and inverse problems of the operator \( L(q) \) with potential (6) and derived a regularized spectral expansion. The method of \[2\] is applicable only for the potentials of type (6). Gesztezy and Tkachenko \[4\] proved two versions of a criterion for the operator \( L(q) \) with \( q \in L_2[0,1] \) to be a spectral operator of scalar type, in sense of Dunford, one analytic and one geometric. The analytic version was stated in term of the solutions of Hill equation. The geometric version of the criterion uses algebraic and geometric properties of the spectra of periodic/antiperiodic and Dirichlet boundary value problems.

The problem of describing explicitly, for which potentials \( q \) the Hill operators \( L(q) \) are spectral operators appears to have been open for about 50 years. Moreover, the discussed papers show that the set of potentials \( q \) for which \( L(q) \) is spectral is a small subset of the periodic functions and it is very hard to describe explicitly the required subset. In paper \[19\] we found the explicit conditions on the potential \( q \) such that \( L(q) \) is an asymptotically spectral operator and in \[20\] we constructed the spectral expansion for the asymptotically spectral operator. However, the set of the potentials constructed in \[19\] is also a small subset of the periodic functions. Thus the theory of spectral operators is ineffective for the construction of the spectral expansion for the nonself-adjoint periodic differential operators. It is connected with the complicated picture of the projections of the Hill operator with general complex potential. In this paper, we construct the spectral expansion for the operator \( L(q) \) with arbitrary complex-valued locally integrable and periodic potential \( q \). In other word, we investigate in detail the spectral expansion for the general and frequent case when the operator \( L \) is not an asymptotically spectral and hence is not a spectral operator.

Note that the construction of the complete spectral decomposition appears to have been open for about 60 years. In this paper we give a complete spectral decomposition. For this we introduce new concepts as essential spectral singularities (ESS) and singular quasimomenta defined in Definitions 3 and Definition 4.

To discuss more precisely the obtained results and to give the brief scheme of this paper we need some preliminary facts about:
(a) the eigenvalues of \( L_t(q) \) and spectrum of \( L(q) \),
(b) the eigenfunction of \( L_t(q) \),
(c) the spectral singularities of \( L(q) \),
(d) the problems of the spectral expansion of \( L \).

Note that there are a lot of papers about the spectra of \( L_t \) and \( L \) (see \[1, 4, 18\] and references on them). Here we introduce the facts which are used essentially for the construction of the spectral expansion. In this section, after introducing
the preliminary facts we give the definition of ESS and discuss its importance in the construction of the spectral expansion. In section 2 we investigate the spectral singularities and ESS. In Section 3 we construct spectral expansion for the operator $L(q)$ in term of the improper integrals by using the ESS, singular quasimomenta and some parenthesis. Finally, we explain (see Conclusion 1) why it is necessary to use the improper integrals and parenthesis.

(a) On the eigenvalues of $L_t(q)$ and spectrum of $L(q)$.

In the case $q = 0$ the eigenvalues and eigenfunctions of $L_t(q)$ are $(2\pi n + t)^2$ and $e^{i(2\pi n + t)x}$ for $n \in \mathbb{Z}$ respectively. In [17] we proved that the large eigenvalues of the operators $L_t(q)$ for $t \neq 0$, $\pi$ consist of the sequence $\{\lambda_n(t) : n \gg 1\}$ satisfying

$$\lambda_n(t) = (2\pi n + t)^2 + O(n^{-1} \ln |n|)$$

as $n \to \infty$ and the formula (7) is uniform with respect to $t$ in $Q$, where

$$Q_h = \{ t \in Q, |t - \pi k| \geq h, k = 0, \pm 1\},$$

$h \in (0, 1)$ and

$$Q = \{ z \in \mathbb{C} : \text{Im} z < 1, -\pi < \text{Re} z < \pi + 1\}. \quad (9)$$

Note that, the formula $f(n,t) = O(g(n))$ as $n \to \infty$ is said to be uniform with respect to $t$ in a set $I$ if there exist positive constants $M$ and $N$, independent of $t$, such that $|f(n,t)| < M |g(n)|$ for all $t \in I$ and $|n| \geq N$. Moreover, it follows from (7) that for any fixed $h (h \in (0, 1))$ there exists an integer $N(h)$ and positive constant $M(h)$ such that for $|n| > N(h)$ and $t \in Q_h$ there exists unique eigenvalue $\lambda_n(t)$, counting multiplicity, satisfying

$$|\lambda_n(t) - (2\pi n + t)^2| \leq M(h)n^{-1} \ln |n|.$$ \quad (10)

Thus $\lambda_n(t)$ is simple for all $|n| > N(h)$ and $t \in Q_h$.

Besides, as it was shown in [19, 20], the integer $N(h)$ can be chosen so that for $|t| \leq h$ and $|n| > N(h)$ there exist two eigenvalues, counting multiplicity, denoted by $\lambda_n(t)$ and $\lambda_{-n}(t)$ and satisfying

$$|\lambda_{\pm n}(t) - (2\pi n + t)^2| \leq 15\pi nh. \quad (11)$$

Similarly, for $|t - \pi| \leq h$ and $|n| > N(h)$ there exist two eigenvalues, counting multiplicity, denoted by $\lambda_n(t)$ and $\lambda_{-(n+1)}(t)$ such that

$$|\lambda_n(t) - (2\pi n + t)^2| \leq 15\pi nh, \quad |\lambda_{-(n+1)}(t) - (2\pi n + t)^2| \leq 15\pi nh. \quad (12)$$

As we noted above the spectrum $\sigma(L(q))$ of $L(q)$ is the union of the eigenvalues of $L_t$ for all $t \in (-\pi, \pi]$. In [20] we proved that the eigenvalues of $L_t$ can be numbered (counting the multiplicity) by elements of $\mathbb{Z}$ such that, for each $n$ the function $\lambda_n(t)$ is continuous on $[0, \pi]$ and for $|n| > N(h)$ the inequalities (10)-(12) hold. The eigenvalues of $L_{-t}(q)$ coincides with the eigenvalues of $L_t(q)$, because they are roots of equation (4) and $\cos(-t) = \cos t$. We define the eigenvalue $\lambda_n(-t)$ of $L_{-t}(q)$ by $\lambda_n(-t) = \lambda_n(t)$ for all $t \in (0, \pi)$. Thus

$$\sigma(L(q)) = \bigcup_{n \in \mathbb{Z}} \Gamma_n,$$ \quad (13)

where

$$\Gamma_n = \{ \lambda_n(t) : t \in [0, \pi]\} \quad (14)$$

is a continuous curve.

The multiple eigenvalues of $L_t$ are the common roots of (4) and $F'(\lambda) = 0$. Since the Hill discriminant $F(\lambda)$ is a nonzero entire function, the set of zeros of $F'(\lambda)$ is
at most countable and can have no finite limit point. Let \( \mu_1, \mu_2, \ldots \), be the roots of \( F'(\lambda) = 0 \) and

\[
A = \{ \pm t_k : k = 1, 2, \ldots \}, \quad A_n = \{ \pm t_k : \mu_k \in \Gamma_n \}, \quad (15)
\]

where \( t_k = \arccos \frac{1}{2} F(\mu_k) \). Here the real part of the range of usual principal value of \( \arccos z \) is in \([0, \pi]\) and the imaginary part is nonnegative and the quasimomenta \( t + 2\pi n \) for \( n \in \mathbb{Z} \) also are denoted by \( t \). In these notations and by (4) we have

\[
F'(\mu_k) = 0, \quad \mu_k \in \sigma(L_{t_k}) = \sigma(L_{-t_k}), \quad \mu_k \notin \sigma(L_t), \quad \forall t \neq \pm t_k. \quad (16)
\]

Thus, if \( t \notin A \), then all eigenvalues of \( L_t \) are simple eigenvalues. It follows from the well known asymptotic formulas for \( F(\lambda) \) that (see [6, Chap. 1, Sec. 3]) the accumulation points of the set \( A \cap Q \) are 0 and \( \pi \). Therefore \( A = A \cup \{0, \pi\} \).

Suppose that the multiplicity of the eigenvalue \( \mu_k \) is \( j \). If \( t_k \in (-\pi, \pi) \) then \( j \) components (14) of the spectrum of \( L(q) \) meet at the point \( \mu_k \). If \( \mu_k \) is a large number then \( j \leq 2 \). Therefore if \( \lambda_n(0) \) for large \( n \) is the double eigenvalue of \( L_0 \), then it readily follows from the numerations of the eigenvalues and (11) that the components \( \Gamma_n \) and \( \Gamma_{-n} \) are joined. Similarly if \( \lambda_n(\pi) \) for large \( n \) is the double eigenvalue of \( L_\pi \), then by (12) the components \( \Gamma_n \) and \( \Gamma_{-(n+1)} \) are joined.

(b) On the eigenfunctions of \( L_t \).

In [17] we proved that the normalized eigenfunction \( \Psi_{n,t}(x) \) corresponding to the eigenvalue \( \lambda_n(t) \) satisfies

\[
\Psi_{n,t}(x) = \frac{1}{\| e^{itx} \|} e^{it(2\pi n + t)x} + h_{n,t}, \quad \| h_{n,t} \| = O(n^{-1}) \quad (17)
\]

and the formula (17) is uniform with respect to \( t \) in \( Q_h \).

Let \( \Psi_{n,t}^* \) be the normalized eigenfunction of \( (L_t(q))^* \) corresponding to \( \overline{\lambda}_n(t) \). The boundary condition adjoint to (2) is

\[
y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(0). \quad (18)
\]

Therefore, \( (L_t(q))^* = L_{t}^*(\overline{q}) \) and by (17), we have the following uniform with respect to \( t \) in \( Q_h \) asymptotic formula

\[
\Psi_{n,t}^*(x) = \frac{1}{\| e^{itx} \|} e^{it(2\pi n + t)x} + h_{n,t}^*(x), \quad \| h_{n,t}^* \| = O(n^{-1}). \quad (19)
\]

Replacing first and second row of the characteristic determinant in (3) by the row vector \((\theta(x, \lambda), \varphi(x, \lambda))\) we obtain the functions

\[
G_t(x, \lambda) = \theta' \varphi(x, \lambda) + (e^{it} - \varphi')\theta(x, \lambda) \quad (20)
\]

and

\[
\Phi_t(x, \lambda) = \varphi \theta(x, \lambda) + (e^{it} - \theta)\varphi(x, \lambda) \quad (21)
\]

which for \( \lambda = \lambda_n(t) \) are the eigenfunctions (if they are not the zero functions) of \( L_t \) corresponding to the eigenvalue \( \lambda_n(t) \), where for simplicity of the notations \( \varphi(1, \lambda), \theta(1, \lambda), \varphi'(1, \lambda) \) and \( \theta'(1, \lambda) \) are denoted by \( \varphi, \theta, \varphi' \) and \( \theta' \) respectively. Then the normalized eigenfunctions \( \Psi_{n,t}(x) \) and \( \Psi_{n,t}^*(x) \) for \( t \in (-\pi, \pi] \) can be written in the form

\[
\Psi_{n,t}(x) = \frac{\Phi_t(x, \lambda_n(t))}{\| \Phi_t(\cdot, \lambda_n(t)) \|}, \quad \Psi_{n,t}^* = \frac{\Phi_{-t}(x, \lambda_n(t))}{\| \Phi_{-t}(\cdot, \lambda_n(t)) \|} \quad (22)
\]

or

\[
\Psi_{n,t}(x) = \frac{G_t(x, \lambda_n(t))}{\| G_t(\cdot, \lambda_n(t)) \|}, \quad \Psi_{n,t}^* = \frac{G_{-t}(x, \lambda_n(t))}{\| G_{-t}(\cdot, \lambda_n(t)) \|} \quad (23)
\]
It is well known that [9, 10] for each \( t \notin \overline{A} \) the system \( \{ \Psi_{n,t} : n \in \mathbb{Z} \} \) is a Reiz basis of \( L_2[0, 1] \) and \( \{ X_{n,t} : n \in \mathbb{Z} \} \). Defined by

\[
X_{n,t} = \frac{1}{\alpha_n(t)} \Psi_{n,t}^*, \quad \alpha_n(t) = (\Psi_{n,t}, \Psi_{n,t}^*),
\]

is the biorthogonal system, where \((\cdot, \cdot)\) is the inner product in \( L_2[0, 1] \).

(c) On the spectral singularities of \( L \).

Since the spectral singularities of the operator \( L(q) \) are the points of its spectrum in neighborhoods of which the projections of the operators \( L(q) \) are not uniformly bounded, to consider the spectral singularities, first we need to discuss the projections of the operators \( L_t(q) \) and \( L(q) \). It is well-known that (see p. 39 of [10]) if \( \lambda_n(t) \) is a simple eigenvalue of \( L_t \), then the spectral projection \( e(t, \gamma) \) defined by contour integration of the resolvent of \( L_t(q) \), where \( \gamma \) is the closed contour containing only the eigenvalue \( \lambda_n(t) \), has the form

\[
e(t, \gamma)f = \frac{1}{\alpha_n(t)} (f, \Psi_{n,t}^*) \Psi_{n,t}.
\]

One can easily verify that,

\[
\|e(t, \gamma)\| = \left| \frac{1}{\alpha_n(t)} \right| .
\]

In [15] we defined projection \( P(\gamma) \) of \( L \) for the arc \( \gamma \subset \Gamma_n \) which does not contain the multiple eigenvalues of the operators \( L_t \), as follows

\[
P(\gamma) = \lim_{\varepsilon \to 0} \int_{\gamma \cup \gamma_i} (L - \lambda I)^{-1} d\lambda,
\]

where \( \gamma_i^1 \subset \rho(L) \) and \( \gamma_i^2 \subset \rho(L) \) are the connected curves lying in opposite sides of \( \gamma \) and

\[
\lim_{\varepsilon \to 0} \gamma_i^\varepsilon = \gamma, \quad \forall \ i = 1, 2.
\]

Here \( \rho(L) \) denotes the resolvent set of \( L \). Moreover, we proved that if additionally the derivative of the characteristic determinant with respect to the quasimomentum \( t \) is nonzero, which equivalent to the condition \( \lambda_n'(t) \neq 0 \) and holds for \( \lambda_n(t) \in \gamma, \ t \neq 0, \pi \), then

\[
P(\gamma)f = \frac{1}{2\pi i} \int_{\delta} \frac{1}{\alpha_n(t)} (f, \Psi_{t}^*) \Psi_{t} dt, \quad \|P(\gamma)\| = \sup_{t \in \delta} \frac{1}{|\alpha_n(t)|},
\]

where \( \delta = \{ t \in (-\pi, \pi) : \lambda_n(t) \in \gamma \} \) and \((\cdot, \cdot)_I\) for any set \( I \) denotes the inner product in \( L_2(I) \). Thus the uniform boundedness of the projections \( P(\gamma) \) and hence the existence of the spectral singularities depend on the behavior of \( \alpha_n(t) \). To investigate \( \alpha_n(t) \) we use the formula

\[
\alpha_n(t) = -\frac{\varphi(1, \lambda_n(t)) F'(\lambda_n(t))}{\|\Phi_t(\cdot, \lambda_n(t))\|/ \|\Phi_{-t}(\cdot, \lambda_n(t))\|},
\]

which immediately follows from (24), (22), (21), (4), the Wronskian equality

\[
\theta \varphi' - \varphi \theta' = 1
\]

and the formula

\[
F'(\lambda) = \int_0^1 \theta' \varphi^2(x, \lambda) + (\theta - \varphi') \varphi(x, \lambda) - \varphi \theta^2(x, \lambda) dx
\]
obtained in [11] (see (21.4.5) in Section 21 of [11]). Instead of (22) and (21) using (23) and (20) we obtain
\[
\alpha_n(t) = -\frac{\theta'(1, \lambda_n(t)) F'(\lambda_n(t))}{\|G_1(\cdot, \lambda_n(t))\|\|G_{-1}(\cdot, \lambda_n(t))\|}, \quad (32)
\]

In [4], the projections were defined as follows. By Definition 2.4 of [4], a closed arc \( \gamma = \{ z \in \mathbb{C} : z = \lambda(t), t \in [\alpha, \beta] \} \) with \( \lambda(t) \) analytic in an open neighborhood of \([\alpha, \beta]\) and
\[
F'(\lambda(t)) = 2 \cos t, \quad F'(\lambda(t)) \neq 0, \quad \forall t \in [\alpha, \beta], \quad \lambda'(t) \neq 0, \quad \forall t \in (\alpha, \beta)
\]
is called a regular spectral arc of \( L(q) \). The projection \( \tilde{P}(\gamma) \) corresponding to the regular spectral arc \( \gamma \) was defined by
\[
\tilde{P}(\gamma) = \frac{1}{2\pi} \int_{\gamma} (\Phi_+(x, \lambda) F_-(\lambda, f) + \Phi_-(x, \lambda) F_+(\lambda, f)) \frac{1}{\varphi(p(\lambda))} d\lambda, \quad (33)
\]

where
\[
\Phi_\pm(x, \lambda) = \varphi \theta(x, \lambda) + \frac{1}{2} (\varphi' - \theta \pm ip(\lambda)) \varphi(x, \lambda), \quad (34)
\]
\[
p(\lambda) = \sqrt{4 - F'^2(\lambda)}, \quad F_\pm(\lambda, f) = \int f(x) \Phi_\pm(x, \lambda) dx. \quad (35)
\]

Using (4) one can readily see that
\[
\Phi_\pm(x, \lambda_n(t)) = \Phi_\pm(t, x, \lambda_n(t)), \quad (36)
\]
where \( \Phi_\pm \) is defined in (21). If \( \gamma \subset \Gamma_n \) then changing the variable \( t \) to the variable \( \lambda \) in the integral (28), using the formulas (29), (22), (21) and (36) and taking into account the equalities \( \lambda_n(-t) = \lambda_n(t) \), \( \frac{dt}{d\lambda} = -\frac{F'(\lambda)}{p(\lambda)} \) which follows from (4) we obtain
\[
\int \frac{1}{\alpha_n(t)} (f(\Psi^*_{n,t})_{R} \Psi_{n,t}(x)) dt = \int_{\gamma} (\Phi_+(x, \lambda) F_-(\lambda, f) + \Phi_-(x, \lambda) F_+(\lambda, f)) \frac{1}{\varphi(p(\lambda))} d\lambda.
\]
(37)

Therefore, by (28) and (33) we have \( P(\gamma) = \tilde{P}(\gamma) \). Hence in the both cases the projection of \( L(q) \) and its norm are defined by (28). Moreover, one can readily see that the curve \( \gamma \) used in (28) is the same with the regular spectral arc defined in [4]. Thus in [15] and [4] the spectral singularities was defined as follows.

**Definition 1.** We say that \( \lambda \in \sigma(L(q)) \) is a spectral singularity of \( L(q) \) if for all \( \varepsilon > 0 \) there exists a sequence \( \{ \gamma_n \} \) of the regular spectral arcs \( \gamma_n \subset \{ z \in \mathbb{C} : |z - \lambda| < \varepsilon \} \) such that
\[
\lim_{n \to \infty} || P(\gamma_n) || = \infty. \quad (38)
\]

In the similar way, we defined in [19] the spectral singularity at infinity.

**Definition 2.** We say that the operator \( L \) has a spectral singularity at infinity if there exists a sequence \( \{ \gamma_n \} \) of the regular spectral arcs such that \( d(0, \gamma_n) \to \infty \) as \( n \to \infty \) and (38) holds, where \( d(0, \gamma_n) \) is the distance from the point \((0,0)\) to the arc \( \gamma_n \).

The following proposition follows immediately from (28) and the definitions 1 and 2.
Proposition 1. (a) \( \lambda \in \sigma(L(q)) \) is a spectral singularity of \( L(q) \) if and only if there exist \( n \in \mathbb{Z} \) and sequence \( \{t_k\} \subset (-\pi, \pi] \setminus A \) such that \( \lambda_n(t_k) \rightarrow \lambda \) and \( \alpha_n(t_k) \rightarrow 0 \) as \( k \rightarrow \infty \).

(b) The operator \( L \) has a spectral singularity at infinity if and only if there exist sequences \( \{n_k\} \in \mathbb{Z} \) and \( \{t_k\} \subset (-\pi, \pi] \setminus A \) such that \( \alpha_{n_k}(t_k) \rightarrow 0 \) as \( k \rightarrow \infty \).

Thus the spectral singularities and hence the spectrality of \( L(q) \) is connected with the uniform boundedness of \( \frac{1}{\alpha_n} \), while as we see below the spectral expansion of \( L(q) \) is essentially connected with the integrability of this function.

(d) On the problems of the spectral expansion of \( L \).

By Gelfand’s Lemma (see [3]) for every \( f \in L_2(-\infty, \infty) \) there exists \( f_t(x) \) such that

\[
f(x) = \frac{1}{2\pi} \int_0^{2\pi} f_t(x) dt, \tag{39}
\]

\[
f_t(x) = \sum_{k=-\infty}^{\infty} f(x + k)e^{ikt}, \quad \int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 |f_t(x)|^2 dx dt \tag{40}
\]

and

\[
f_t(x + 1) = e^{it}f_t(x). \tag{41}
\]

Let \( h \in (0, 1) \setminus A \), and let \( l \) be a continuous curve joining the points \(-\pi + h \) and \( \pi + h \) and satisfying

\[
l \subset Q_h \setminus A, \tag{42}
\]

where \( Q_h \) and \( A \) are defined in (8) and (15) respectively. If \( f \) is a compactly supposed and continuous function, then \( f_t(x) \) is an analytic function of \( t \) in a neighborhood of \( \mathcal{D} \) for each \( x \), where \( \mathcal{D} \) is the closure of the domain enclosed by \( l \cup [-\pi + h, \pi + h] \).

Hence the Cauchy’s theorem and (39), (41) give

\[
f(x) = \frac{1}{2\pi} \int_l f_t(x) dt. \tag{43}
\]

On the other hand, for each \( t \in l \) we have a decomposition

\[
f_t(x) = \sum_{n \in \mathbb{Z}} a_n(t) \Psi_{n,t}(x) \tag{44}
\]

of \( f_t(x) \) by the basis \( \{\Psi_{n,t} : n \in \mathbb{Z}\} \), where

\[
a_n(t) = \int_0^1 f_t(x) \overline{X_{n,t}(x)} dx = \frac{1}{\alpha_n(t)}(f_t, \Psi_{n,t})^*
\]

(see (b)). Here \( \Psi_{n,t} \) and \( X_{n,t} \) can be extended to \((-\infty, \infty)\) by

\[
\Psi_{n,t}(x + 1) = e^{it}\Psi_{n,t}(x) \quad \& \quad X_{n,t}(x + 1) = e^{it}X_{n,t}(x). \tag{45}
\]

Then the following equality holds

\[
\int_0^1 f_t(x) \overline{X_{n,t}(x)} dx = \int_{-\infty}^{\infty} f(x) \overline{X_{n,t}(x)} dx \tag{46}
\]

(see [3]). Using (44) in (43), we get

\[
f(x) = \frac{1}{2\pi} \int_l f_t(x) dt = \frac{1}{2\pi} \int_l \sum_{n \in \mathbb{Z}} a_n(t) \Psi_{n,t}(x) dt. \tag{47}
\]
In [14, 16, 18] we proved that for the continuous curve \( l \subset Q_h \setminus A \) the series in (47) can be integrated term by term:

\[
\int \sum_{n \in \mathbb{Z}} a_n(t) \Psi_{n,t}(x) dt = \sum_{n \in \mathbb{Z}} \int_{l} a_n(t) \Psi_{n,t}(x) dt.
\]

Therefore we have

\[
f(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{l} a_n(t) \Psi_{n,t}(x) dt,
\]

where the series converges in the norm of \( L_2(a,b) \) for every \( a, b \in \mathbb{R} \).

To get the spectral expansion in the term of \( t \) from (49) we need to replace the integrals over \( l \) by the integral over \(( -\pi, \pi) \). As we see in the next section (see Lemma 1), expressions \( a_n(t) \Psi_{n,t} \) and \( \frac{1}{\alpha_n(t)} \) are piecewise continuous on \(( -\pi, \pi) \). If \( \alpha_n(t) \to 0 \) as \( t \to c \) for some \( c \in (-\pi, \pi) \) then \( \frac{1}{\alpha_n(t)} \to \infty \) and \( a_n(t) \Psi_{n,t}(x) \to \infty \) for some \( f \) and \( x \). By Proposition 1 the boundlessness of \( \frac{1}{\alpha_n} \) is the characterization of the spectral singularities. Moreover, the considerations of the spectral singularities, that is, the consideration of the boundlessness of \( \frac{1}{\alpha_n} \) play only the crucial rule for the investigations of the spectrality of \( L \). On the other hand, the papers [2, 4, 19] show that, in general, the Hill operator \( L \) is not a spectral operator. Since \( \frac{1}{\alpha_n} \) may have an integrable boundlessness, its boundlessness is not a criterion for the nonexistence of the integrals

\[
\int_{\delta} \frac{1}{\alpha_n(t)} (f, \Psi_{n,t}^*) \Psi_{n,t}(x) dt
\]

for \( \delta \subset (-\pi, \pi) \). Hence to construct the spectral expansion for the operator \( L \) we need to introduce a new concept connected with the existence of the integrals (50) for \( \delta \subset (-\pi, \pi) \) which can be reduced to the investigation of the integrability of \( \frac{1}{\alpha_n} \) (see Remark 1 in Section 3). Therefore we introduce the following notions, independent of the choice of \( f \), for the construction of the spectral expansion.

**Definition 3.** We say that a point \( \lambda_0 \in \sigma(L_{\pm t_0}) \subset \sigma(L) \) is an essential spectral singularity (ESS) of the operator \( L \) if there exists \( n \in \mathbb{Z} \) such that \( \lambda_0 = \lambda_n(t_0) \) and for each \( \varepsilon \) the function \( \frac{1}{\alpha_n} \) is not integrable on \( ((t_0 - \varepsilon, t_0 + \varepsilon) \cup (-t_0 - \varepsilon, -t_0 + \varepsilon)) \setminus A_n \).

In this paper we investigate the spectral expansion by using the concept ESS. First (in Section 2) we consider the concept ESS. Then, in Section 3, we construct the spectral expansion for the Hill operator.

2. Spectral singularity and ESS. By (24), (28) and the definitions 1 and 3 to consider the spectral singularities and ESS we need to investigate the dependence on \( t \) of \( \Psi_{n,t}(x) \) and \( \Psi_{n,t}^*(x) \) and \( \alpha_n(t) \). Therefore, first we prove the following lemma.

**Lemma 1.** (a) For each fixed \( x \), the functions \( |\Psi_{n,t}(x)|, |\Psi_{n,t}^*(x)| \) and \( |\alpha_n(t)| \), where \( \Psi_{n,t}(x), \Psi_{n,t}^*(x) \) and \( \alpha_n \) are defined in (22) and (24), continuously depend on \( t \) at \(( -\pi, 0) \cup (0, \pi) \).

(b) If the geometric multiplicity of the eigenvalues \( \lambda_n(0) \) and \( \lambda_n(\pi) \) is 1, then for each fixed \( x \), \( \Psi_{n,t}(x) \), \( \Psi_{n,t}^*(x) \), \( \alpha_n \) are continuous with respect to \( t \) at 0 and \( \pi \) respectively.

(c) For each fixed \( x \), \( \Psi_{n,t}(x) \) and \( \Psi_{n,t}^*(x) \) are bounded functions, with respect to \( t \), at \(( -\pi, 0) \cup (0, \pi) \).
The function $\frac{1}{|\alpha_n|}$ is continuous in $((-\pi, 0) \cup (0, \pi)) \setminus A_n$, where $A_n$ is defined in (15). For each fixed $x$, $\Psi_{n,t}(x)$, $\Psi^*_{n,t}(x)$ and $\alpha_n$, $\frac{1}{\alpha_n}$ are piecewise continuous, with respect to $t$, at $(-\pi, 0) \cup (0, \pi)$.

Proof. (a) It is well-known that [1] for $t \in (-\pi, 0) \cup (0, \pi)$, the operator $L_t$ cannot have two linearly independent eigenfunctions corresponding to one eigenvalue $\lambda_n(t)$. Indeed, otherwise, both solutions $\varphi(x, \lambda_n(t))$ and $\theta(x, \lambda_n(t))$ satisfy the boundary condition (2). But, it implies that $\varphi'(1, \lambda_n(t)) = e^{it}$, $\theta(1, \lambda_n(t)) = e^{it}$

which contradicts (4) for $t \neq 0, \pi$.

As we noted in introduction (see (a)) $\lambda_n(t)$ is a continuous function. Therefore it follows from (21) that $\Phi_t(x, \lambda_n(t))$ for each fixed $x$ depend continuously on $t$. Moreover, by the uniform boundedness theorem $\|\Phi_t(\cdot, \lambda_n(t))\|$ continuously depend on $t$. Since $\theta(x, \lambda)$ and $\varphi(x, \lambda)$ are linearly independent solution, it follows from (21) that $\|\Phi_t(\cdot, \lambda_n(t))\| = 0$ if and only if $\varphi(\lambda_n(t)) = 0$ and $e^{it} - \theta(\lambda_n(t)) = 0$, which is possible for at most finite number of $t$. Thus there may exists a finite set $B = \{u_1, u_2, ..., u_k\} \subset ((-\pi, 0) \cup (0, \pi))$ such that $\|\Phi_t(\cdot, \lambda_n(t))\| = 0$ for $t \in B$. Hence

$$\frac{\Phi_t(x, \lambda_n(t))}{\|\Phi_t(\cdot, \lambda_n(t))\|}$$

is continuous, with respect to $t$, at $((-\pi, 0) \cup (0, \pi)) \setminus B$. In the same way we prove that there may exists a finite set $C = \{v_1, v_2, ..., v_m\}$ such that $\|G_t(\cdot, \lambda_n(t))\| = 0$ for $t \in C$ and

$$\frac{G_t(x, \lambda_n(t))}{\|G_t(\cdot, \lambda_n(t))\|}$$

is continuous, with respect to $t$, at $((-\pi, 0) \cup (0, \pi)) \setminus C$, where $G_t$ is defined in (20). For $t \in ((-\pi, 0) \cup (0, \pi)) \setminus (C \cup B)$ the operator $L_t$ has unique linearly independent eigenfunction. Hence there exists a function $c(t)$ such that $|c(t)| = 1$ and

$$\frac{\Phi_t(x, \lambda_n(t))}{\|\Phi_t(\cdot, \lambda_n(t))\|} = c(t) \frac{G_t(x, \lambda_n(t))}{\|G_t(\cdot, \lambda_n(t))\|}, \forall t \in ((-\pi, 0) \cup (0, \pi)) \setminus (C \cup B).$$

On the other hand if $t \in B \cap C$, then (51) holds which is impossible for $t \neq 0, \pi$. It means that $((-\pi, 0) \cup (0, \pi)) \cap (B \cap C)$ is an empty set. Therefore, it follows from (54) that for each fixed $x$ the absolute value of (52), that is, $|\Psi_{n,t}(x)|$ is continuous, with respect to $t$, at $(-\pi, 0) \cup (0, \pi)$. In the same way, the same statements can be proved for $\Psi^*_{n,t}(x)$.

To prove the continuity of $|\alpha_n|$ at $(-\pi, 0) \cup (0, \pi)$ we use (22)-(24). By (4) if $e^{it} - \theta(\lambda_n(t)) = 0$ then $e^{-it} - \varphi'(\lambda_n(t)) = 0$. Therefore

$$\left( \frac{\Phi_t(\cdot, \lambda_n(t))}{\|\Phi_t(\cdot, \lambda_n(t))\|}, \frac{G_{-t}(\cdot, \lambda_n(t))}{\|G_{-t}(\cdot, \lambda_n(t))\|} \right) \& \left( \frac{G_t(\cdot, \lambda_n(t))}{\|G_t(\cdot, \lambda_n(t))\|}, \frac{\Phi_{-t}(\cdot, \lambda_n(t))}{\|\Phi_{-t}(\cdot, \lambda_n(t))\|} \right)$$

are continuous at $((-\pi, 0) \cup (0, \pi)) \setminus B_1$ and $((-\pi, 0) \cup (0, \pi)) \setminus C_1$ respectively, where $B_1$ and $C_1$ are finite sets and $B_1 \cap C_1 = \emptyset$. Thus arguing as above, we see that $|\alpha_n|$ is continuous at $(-\pi, 0) \cup (0, \pi)$.

(b) Now suppose that the geometric multiplicity of $\lambda_n(0)$ is 1. Then at least one of the entry of characteristic determinant (see (3))

$$\left| \begin{array}{cc} \theta(\lambda_n(0)) - 1 & \varphi(\lambda_n(0)) \\ \theta'(\lambda_n(0)) & \varphi'(\lambda_n(0)) - e^{it} \end{array} \right|$$

would
is not zero, that is, at least one of \( \Phi_0(\cdot, \lambda_n(0)) \) and \( G_0(\cdot, \lambda_n(0)) \) is not zero function. Without loss of generality, assume that \( \Phi_0(\cdot, \lambda_n(0)) \) is not zero function. Then by (22) for each \( x \), \( \Psi_{n,t}(x) \) and \( \Psi^*_{n,t}(x) \) is continuous, with respect to \( t \), at 0. The continuity of \( \alpha_n \) follows from (24). In the same way we prove that they are continuous at \( \pi \) if the geometric multiplicity of \( \lambda_n(\pi) \) is 1.

(c) Now we prove that for each \( x \) the function \( \Psi_{n,t}(x) \) is bounded at \((-\pi,0) \cup (0,\pi)\). By (a) and (b) it is enough to show that it is bounded in some deleted neighborhoods of 0 and \( \pi \), if the geometric multiplicity of \( \lambda_n(0) \) and \( \lambda_n(\pi) \) is 2 respectively. We prove it for \( t = 0 \). The proof for \( t = \pi \) is similar. If the geometric multiplicity of \( \lambda_n(0) \) is 2 then all entries of (55) are zero and hence \( \varphi(\lambda_n(0)) = 0 \). Then it is clear that there exists \( \delta_1 > 0 \) such that

\[
\varphi(\lambda_n(t)) \neq 0
\]

for \( 0 < |t| < \delta_1 \). Since \( \theta(x,\lambda) \) and \( \varphi(x,\lambda) \) are continuous with respect to \((x,\lambda)\) and nonzero functions and \( \lambda_n(t) \) is continuous at \( t = 0 \), there exist constants \( M, \varepsilon \) and \( \delta_2 \) such that

\[
|\theta(x,\lambda_n(t))| < M, \quad |\varphi(x,\lambda_n(t))| < M, \quad ||\varphi(\cdot,\lambda_n(t))|| > \varepsilon, \quad ||\theta(\cdot,\lambda_n(t))|| > \varepsilon
\]

for \( x \in [0,1] \) and \( |t| < \delta_2 \). On the other hand, \( \theta(x,\lambda) \) and \( \varphi(x,\lambda) \) are linearly independent solutions and hence they are linearly independent elements of \( L_2(0,1) \) which implies that there exist positive constants \( c < 1 \) and \( \delta_3 \) such that

\[
|\varphi(\cdot,\lambda_n(t)),\theta(\cdot,\lambda_n(t))| < c ||\varphi(\cdot,\lambda_n(t))|| \quad ||\theta(\cdot,\lambda_n(t))||
\]

for \( |t| < \delta_3 \). Using these inequalities one can easily verify that there exist positive constants \( M_1, \varepsilon \) and \( \delta \) such that

\[
|\Phi_t(x,\lambda_n(t))|^2 < M_1(|\varphi(\lambda_n(t))|^2 + |e^{it}\varphi(\lambda_n(t))|^2),
\]

\[
||\Phi_t(\cdot,\lambda_n(t))||^2 > \varepsilon(|\varphi(\lambda_n(t))|^2 + |e^{it}\varphi(\lambda_n(t))|^2)
\]

for \( x \in [0,1] \) and \( 0 < |t| < \delta \). It with (56) implies that \( \Psi_{n,t}(x) \) is bounded in some deleted neighborhood of 0. In the same way we prove it for \( \Psi^*_{n,t}(x) \).

(d) Since for \( t \in (\{(-\pi,0) \cup (0,\pi)\} \setminus A_n \), the system \( \{\Psi_{n,t} : n \in \mathbb{Z}\} \) is complete we have \( \alpha_n(t) \neq 0 \). Hence \( \frac{1}{\alpha_n} \) is continuous at \((\{(-\pi,0) \cup (0,\pi)\} \setminus A_n \). The last statement of the lemma follows from the fact that the sets \( B, C, B_1, C_1 \) and \( A_n \) are finite.

Using Lemma 1 we prove the following

**Proposition 2.** Let \( \mathbb{E}, \mathbb{S}, \) and \( \mathbb{M} \) be respectively the sets of ESS, spectral singularities and multiple eigenvalues of \( L_t(q) \) for \( t \in (-\pi,\pi) \). Then \( \mathbb{E} \subset \mathbb{S} \subset \mathbb{M} \).

**Proof.** If \( \lambda \in \sigma(L_0(q)) \) is not a spectral singularity then by Proposition 1(a), \( \frac{1}{\alpha_k} \) is bounded in some deleted neighborhood \( D(t_0,\varepsilon) \) of \( t_0 \) for all indices \( k \) such that \( \lambda_k(t_0) = \lambda \), where

\[
D(t_0,\varepsilon) = [t_0 - \varepsilon, t_0] \cup (t_0, t_0 + \varepsilon],
\]

and by Lemma 1, it is piecewise continuous. Therefore \( \frac{1}{\alpha_k} \) is integrable in \( D(t_0,\varepsilon) \), and hence, by Definition 3, \( \lambda \notin \mathbb{E} \). The inclusion \( \mathbb{S} \subset \mathbb{M} \) is well-known (see [4, 15]).
Let $\lambda_k(t)$ be a multiple eigenvalue of multiplicity $m > 1$. Then, using Proposition 3(a) and taking into account that $\lambda_k(t)$ is a real number, we obtain

$$F(\lambda) = 2 \cos t_0 + F^{(m)}(\lambda_0)(\lambda - \lambda_0)^m (1 + o(1))$$

as $\lambda \to \lambda_0$ and

$$2 \cos t = 2 \cos t_0 - (\sin t_0) (t - t_0) - (t - t_0)^2 \left( \frac{1}{2} + o(1) \right)$$

as $t \to t_0$. These equalities with the equality $F(\lambda_k(t)) = 2 \cos t$ and the continuity of $\lambda_k$ give the proof of (a). The proof of (b) follows from the definitions 1 and 3. 

**Theorem 1.** If $t_0 \in (0, \pi)$ and $\lambda_0$ is a multiple eigenvalue of $L(t_0)$, then $\lambda_0$ is a spectral singularity of $L$ and is not an ESS.

**Proof.** Let $\lambda_0$ be a multiple eigenvalue of multiplicity $m > 1$. Then $F'(\lambda) \sim (\lambda_0 - \lambda)^{m-1}$ as $\lambda \to \lambda_0$. Hence, using Proposition 3(a) and taking into account that $\lambda_k$ for $k \in \mathbb{T}(\lambda_0)$ is continuous at $t_0$, that is, for each neighborhood $U$ of $\lambda_0$ there exist a neighborhood $\delta \subset (-\pi, 0) \cup (0, \pi)$ of $t_0$ such that $\lambda_k(\delta) \subset U$ we obtain

$$F'(\lambda_k(t)) \sim (t_0 - t)^{\frac{m-1}{2}}, \forall k \in \mathbb{T}(\lambda_0)$$

as $t \to t_0$. Now to prove the theorem we use Proposition 3(b) and the formula

$$\alpha_k(t) = \frac{-\varphi F'(\lambda)}{\|\varphi_\theta(\cdot, \lambda) + \frac{1}{2}(\varphi' - \theta - ip(\lambda))\varphi(\cdot, \lambda)\| \|\varphi_\theta(\cdot, \lambda) + \frac{1}{2}(\varphi' - \theta + ip(\lambda))\varphi(\cdot, \lambda)\|}$$

(63)
for \( \lambda = \lambda_k(t) \) obtained from (29), (36) and (34). Consider two cases:

Case 1: \( \varphi(\lambda_0) \neq 0 \). Then \( \varphi(\lambda) \sim 1 \) as \( \lambda \to \lambda_0 \), since \( \varphi \) is an entire function. On the other hand \( \varphi' - \theta \pm ip = O(1) \) as \( \lambda \to \lambda_0 \). Using this and taking into account that \( \theta(\cdot, \lambda) \) and \( \varphi(\cdot, \lambda) \) are linearly independent elements of \( L_2(0, 1) \) (see the proof of Lemma 1(c)) we obtain

\[
\left\| \varphi\theta(x, \lambda) + \frac{1}{2} (\varphi' - \theta \pm ip(\lambda)) \varphi(x, \lambda) \right\| \sim 1
\]  

as \( \lambda \to \lambda_0 \). Therefore using (62) in (63) we obtain

\[
\alpha_k(t) \sim (t_0 - t)^{\frac{|m|}{2}}
\]  

as \( t \to t_0 \) for all \( k \in \mathbb{T}(\lambda_0) \). Thus, by Proposition 3(b), \( \lambda_0 \) is a spectral singularity of \( \alpha \) and is not an ESS.

Case 2: \( \varphi(\lambda_0) = 0 \). Then there exists a positive integer \( s \) such that

\[
\varphi(\lambda) \sim (\lambda - \lambda_0)^s
\]  

as \( \lambda \to \lambda_0 \). On the other hand, by (35) and (30) we have

\[
(\varphi'(\lambda) - \theta(\lambda) + ip(\lambda))(\varphi'(\lambda) - \theta(\lambda) - ip(\lambda)) = (\varphi'(\lambda) - \theta(\lambda))^2 + 4 - (\varphi'(\lambda) + \theta(\lambda))^2 = 4 - 4\varphi'(\lambda)\theta(\lambda) = -4\varphi(\lambda)\varphi'(\lambda)
\]  

Since \( p(\lambda_0) = \sin t_0 \neq 0 \), at least one the numbers \( \varphi'(\lambda_0) - \theta(\lambda_0) + ip(\lambda_0) \) and \( \varphi'(\lambda_0) - \theta(\lambda_0) - ip(\lambda_0) \) is not zero. Suppose, without loss of generality, the first of them is not zero. Then using (67) and (66) and arguing as in the proof of (64) we get

\[
\varphi'(\lambda) - \theta(\lambda) - ip(\lambda) = O((\lambda - \lambda_0)^s),
\]

\[
\left\| \varphi\theta(x, \lambda) + \frac{1}{2} (\varphi' - \theta \pm ip(\lambda)) \varphi(x, \lambda) \right\| \sim (\lambda - \lambda_0)^s,
\]

\[
\left\| \varphi\theta(x, \lambda) + \frac{1}{2} (\varphi' - \theta \mp ip(\lambda)) \varphi(x, \lambda) \right\| \sim 1
\]

as \( \lambda \to \lambda_0 \). Now, from (63), (62) and (66) we obtain (65). Therefore the proof of the theorem follows from Proposition 3(b).

By the similar arguments one can find conditions on \( \lambda_n(t) \) for \( t = 0, \pi \) to be or not to be the ESS. Here we prove only one criterion for large value of \( n \) which will be used essentially for the spectral expansion. For this we use the following well-known statements (see [6]). The large eigenvalues of the Dirichlet and Neumann boundary value problems are simple, that is, the multiplicities of the large roots \( \lambda_0 \) and \( \mu_0 \) of \( \varphi(\lambda) = 0 \) and \( \varphi'(\lambda) = 0 \) is 1. It mean that

\[
\varphi(\lambda) \sim (\lambda - \lambda_0) \quad \& \quad \varphi'(\lambda) \sim (\lambda - \mu_0)
\]  

as \( \lambda \to \lambda_0 \) and \( \lambda \to \mu_0 \) respectively.

Similarly, if \( |k| \gg 1 \) and \( \lambda_0 = \lambda_k(0) \) is the multiple eigenvalue of \( L_0 \) then it is double eigenvalue and hence \( F(\lambda_0) = 2, F'(\lambda_0) = 0, F''(\lambda_0) \neq 0 \) which implies that

\[
F'(\lambda) \sim (\lambda - \lambda_0) \quad \& \quad p(\lambda) \sim (\lambda - \lambda_0)
\]  

as \( \lambda \to \lambda_0 \). Therefore by (61) we have

\[
\lambda_k(t) - \lambda_0 \sim t, \quad F'(\lambda_k(t)) \sim t, \quad \forall k \in \mathbb{T}(\lambda_0)
\]  

as \( t \to 0 \). Now using (70) we prove the following main result of this section
Theorem 2. Let \( \lambda_0 \) be a large and multiple eigenvalue of \( L_0 \). Then the following statements are equivalent:

(a) The eigenvalue \( \lambda_0 \) of \( L_0 \) is an ESS of \( L \).
(b) The eigenvalue \( \lambda_0 \) is a spectral singularity of \( L \).
(c) The geometric multiplicity of the eigenvalue \( \lambda_0 \) is 1, that is, there exist one eigenfunction and one associated function corresponding to \( \lambda_0 \).
(d) \( \lambda_0 \) is neither Dirichlet nor Naimann eigenvalue, that is,

\[
\varphi(\lambda_0) \neq 0 \quad \text{and} \quad \theta'(\lambda_0) \neq 0.
\]

The theorem continues to hold if \( L_0 \) is replaced by \( L_\pi \).

Proof. We prove the theorem for \( L_0 \). The proof of the case \( L_\pi \) is the same. First let us prove that (a) and (b) hold if and only if \( \varphi(\lambda_0) \neq 0 \). If the last inequality holds then (64) holds too. Therefore using (64) and (70) in (63) we obtain that

\[
\alpha_k(t) \sim t
\]

as \( t \to 0 \) for all \( k \in T(\lambda_0) \) and hence, by propositions 3(b) and 2, (a) and (b) hold.

Now suppose that \( \varphi(\lambda_0) = 0 \). Then using (30) and the equality \( F(\lambda_0) = 2 \) by direct calculation we obtain \( \theta(\lambda_0) = 1 = \varphi'(\lambda_0) = 1 \). Therefore, the first relation of (68) and the second relation of (69) imply that

\[
\|\varphi(x,\lambda) + \frac{1}{2}(\varphi' - \theta \pm ip(\lambda))\varphi(x,\lambda)\| \sim (\lambda - \lambda_0)
\]

and by (63), (68)-(70) we have \( \alpha_k(t) \sim 1 \) as \( t \to 0 \) for all \( k \in T(\lambda_0) \), that is, (a) and (b) does not hold.

Instead of (29) using (32) in the same way we prove that (a) and (b) hold if and only if \( \theta'(\lambda_0) \neq 0 \). Thus we proved that (a), (b) and (d) are equivalent.

To complete the proof of the theorem we prove that (d) \( \implies \) (c) and (c) \( \implies \) (b). Suppose that (d) and hence (71) holds. If (c) does not hold then both solution \( \varphi(x,\lambda_0) \) and \( \theta(x,\lambda_0) \) are periodic function. In this case by (5) we have \( \varphi(\lambda) = 0 \) which contradicts (71). Thus (d) \( \implies \) (c).

If (c) holds, then, there is one eigenfunction \( \Psi_{n,0} \) corresponding to the eigenvalue \( \lambda_n(0) = \lambda_0 \) and an associated function \( \phi \), satisfying

\[
(L_0 - \lambda_n(0))\phi = \Psi_{n,0}.
\]

Multiplying both sides of (74) by \( \Psi_{n,0}^* \) we obtain \( \alpha_n(0) = 0 \). On the other hand, if the geometric multiplicity of \( \lambda_n(0) \) is 1 then by Lemma 1(b) \( \alpha_n \) is continuous at 0. Hence \( \alpha_n(t) \to 0 \) as \( t \to 0 \) and by Proposition 1(a) \( \lambda_0 \) is a spectral singularity, that is, (b) holds.

If the geometric multiplicity of \( \lambda_n(0) \) is 1, then at least one of \( \varphi(\lambda_n(0)) \) and \( \theta'(\lambda_n(0)) \) is not zero. Indeed if both are zero, then it follows from (30), (4) and (5) that both \( \varphi(x,\lambda_n(0)) \) and \( \theta(x,\lambda_n(0)) \) are eigenfunctions which contradicts the assumption. Without loss of generality, assume that \( \varphi(\lambda_n(0)) \) is not zero. Then (64) holds. Therefore from (63) and (61) we obtain that, if \( \lambda_n(0) \) is an eigenvalue of algebraic multiplicity \( m \) and geometric multiplicity 1, then

\[
\alpha_n(t) \sim t^{m-1}
\]

which implies the following

**Proposition 4.** If \( \lambda_n(0) \), where \( n \in \mathbb{Z} \), is a multiple eigenvalue with geometric multiplicity 1, then it is an ESS. The statement continuous to hold if \( \lambda_n(0) \) is replaced by \( \lambda_n(\pi) \).
Now we use the following classical result (see p.8-9 of [5] and p.34-35 of [1]):

If $q$ is an even function, then the eigenvalues of $L_0$ and $L_\pi$ are either Dirichlet or Naimann eigenvalues.

Note that in [1, 5] this result were proved for the real-valued potentials. However, the proof pass through for the complex-valued potentials without any change. Therefore, Theorem 2, Proposition 2 and Theorem 1 immediately imply the following:

**Corollary 1.** If the potential $q$ is an even function, then

(a) The operators $L_0(q)$ and $L_\pi(q)$ have no associated functions corresponding to the large eigenvalues.

(b) The large eigenvalues of $L_0(q)$ and $L_\pi(q)$ are not spectral singularities.

(c) The operator $L(q)$ may have only finite number of ESS.

### 3. Spectral expansion.

In this section we construct spectral expansion by using (47). The term by term integration in (47) was proved in the papers [14, 16, 18] for the curve $l$ satisfying (42). Here we prove it for a little different curve and by the other method for the independence of this paper. For this first let us construct the suitable curve of integration by taking into account the results of Section 2. Since, only the eigenvalues $\lambda_n(0)$ and $\lambda_n(\pi)$ may become the ESS, we choose the curve of integration so that it only pass over the points 0 and $\pi$. Namely, we construct the curve of integration as follows. Let $h$ be positive number such that $F'(\lambda_n(t)) \neq 0$, $\varphi(\lambda_n(t)) \neq 0$, $\forall t \in (\gamma(0, h) \cup \gamma(\pi, h))$, $\forall n \in \mathbb{Z}$, (76)

where $\gamma(0, h)$ and $\gamma(\pi, h)$ are the semicircles

\[
\gamma(0, h) = \{|t| = h, \text{Im} t \geq 0\}, \quad \gamma(\pi, h) = \{|t - \pi| = h, \text{Im} t \geq 0\}.
\]

(77)

Since the accumulation points of the roots of the equations $F'(\lambda_n(t)) = 0$, $\varphi(\lambda_n(t)) = 0$ are 0 and $\pi$ there exist $\gamma(0, h)$ and $\gamma(\pi, h)$ satisfying (76). Define $l(h)$ by

\[
l(h) = B(h) \cup \gamma(0, h) \cup \gamma(\pi, h),
\]

(78)

where $B(h) = [h, \pi - h] \cup [\pi + h, 2\pi - h]$. Thus $l(h)$ consist of the intervals $[h, \pi - h]$ and $[\pi + h, 2\pi - h]$ and semicircles (77). Denote the points of $A \cap B(h)$ by $t_1, t_2, ..., t_s$ and put

\[
E(h) = B(h) \setminus \{t_1, t_2, ..., t_s\},
\]

(79)

where $A$ is defined in (15) and $A \cap B(h)$ is a finite set because the accumulation points of $A$ are 0 and $\pi$. In (47) instead of $l$ using $l(h) = B(h) \cup \gamma(0, h) \cup \gamma(\pi, h)$ and taking into account that integral over $B(h)$ is equal to the integral over $E(h)$ we obtain

\[
f = \frac{1}{2\pi} \left( \int_{E(h)} f_t(x) dx + \int_{\gamma(0, h)} f_t(x) dx + \int_{\gamma(\pi, h)} f_t(x) dx \right)
\]

(80)
and by (44)

\[
\int_{E(h)} f_t(x) dt = \int_{E(h)} \sum_{k \in \mathbb{Z}} a_k(t) \Psi_{k,t}(x) dt,
\]

(81)

\[
\int_{\gamma(0,h)} f_t(x) dt = \int_{\gamma(0,h)} \sum_{k \in \mathbb{Z}} a_k(t) \Psi_{k,t}(x) dt, \int_{\gamma(h)} f_t(x) dt,
\]

\[
= \int_{\gamma(h)} \sum_{k \in \mathbb{Z}} a_k(t) \Psi_{k,t}(x) dt.
\]

Now we prove that the series in (81) can be integrated term by term. First we prove it for the first integral (see Theorem 3) and then for the second and third integral (see Theorem 4), that is, first we prove the following

\[
\int_{E(h)} \sum_{k \in \mathbb{Z}} a_k(t) \Psi_{k,t}(x) dt = \sum_{n \in \mathbb{Z}} \int_{E(h)} a_n(t) \Psi_{n,t}(x) dt.
\]

(82)

For this we show that the integrals in the right-hand side of (82) exists (see Proposition 5) and then estimate the remainders

\[
R_n(x,t) = \sum_{k > n} a_k(t) \Psi_{k,t}(x), R_{-n}(x,t) = \sum_{k < -n} a_k(t) \Psi_{k,t}(x).
\]

(83)

(see Lemma 2) of the series

\[
\sum_{k \in \mathbb{Z}} a_k(t) \Psi_{k,t}(x).
\]

(84)

**Proposition 5.** Let \( f \) be continuous and compactly supported function and \( h \in (0,1) \).

(a) For each \( n \in \mathbb{Z} \) the integral

\[
\int_{E(h)} a_n(t) \Psi_{n,t}(x) dt
\]

exists, where \( E(h) \) and \( a_n(t) \) are defined in (79) and (44).

(b) If \( \lambda_n(0) \) and \( \lambda_n(\pi) \) are not ESS then the integrals

\[
\int_{(-h,h)} a_n(t) \Psi_{n,t}(x) dt \quad \& \quad \int_{(\pi-h,\pi+h)} a_n(t) \Psi_{n,t}(x) dt
\]

(86)

exist respectively.

**Proof.** (a) Theorem 1 with the Definition 3 implies that \( \frac{1}{\alpha_n} \) is integrable on \( E(h) \). Using the definitions of \( a_n(t) \) and \( f_t \) (see (44) and (40)) and Schwarz inequality and taking into account that \( f \) is a continuous and compactly supported function we obtain that there exists a number \( M \) such that

\[
|a_n(t)| \leq M \left| \frac{1}{\alpha_n(t)} \right|, \forall t \in (-\pi,0) \cup (0,\pi).
\]

(87)

On the other hand, it follows from Lemma 1 that \( a_n(t) \) is a piecewise continuous function and for each fixed \( x \), \( \Psi_{n,t}(x) \) is a piecewise continuous and bounded function on \( E(h) \). Therefore the integral (85) exists.

(b) If \( \lambda_n(0) \) is not ESS then by Definition 3, \( \frac{1}{\alpha_n} \) is integrable on \((-\varepsilon,\varepsilon)\) for some \( \varepsilon > 0 \). Therefore using (87) and arguing as in the proof of (a) we see that the first integral in (86) exists. In the same way we prove that the second integral exists too. \( \square \)
Remark 1. Let $E$ be a subset of $L_2(-\infty, \infty)$ such that if $f \in E$, then the norm $\|f_i\|$ of the Gelfand transform $f_i(x) = \mathcal{T} f(t)$, defined by (40), is bounded in $(-\pi, \pi)$ almost everywhere. Then, by the Schwarz inequality, (87) holds almost everywhere. Therefore the proof of the Proposition 5 shows that if $\alpha_n(t)$ is integrable over the measurable subset $I$ of $(-\pi, \pi)$ then $a_n(t)\Psi_{n,t}(x)$ is also integrable in $I$ for each $x \in [0,1]$.

Now, conversely, suppose that $1/\alpha_n(t)$ is not integrable over $I$. By the definition of $E$, the equality $\|\Psi_{n,t}\| = 1$ implies that $\mathcal{Y}^{-1}\Psi_{n,t} \in E$, where $\mathcal{Y}^{-1}$ is the inverse Gelfand transform. Let $f = \mathcal{Y}^{-1}\Psi_{n,t}$. Then $a_n(t) = 1/\alpha_n(t)$. Therefore using Lemma 1 one can easily show that $a_n(t)\Psi_{n,t}(x)$ is not integrable on $I$ for some $x \in [0,1]$.

Now we estimate (83), by using the following uniform with respect to $t$ in $E(h)$ asymptotic formulas

$$\Psi_{n,t}(x) = e^{i(2\pi n + 1)x} + h_{n,t}(x), \quad \|h_{n,t}\| = O(n^{-1}), \quad (88)$$

$$\Psi^*_{n,t}(x) = e^{i(2\pi n + 1)x} + h^*_{n,t}(x), \quad \|h^*_{n,t}\| = O(n^{-1}), \quad \frac{1}{\alpha_n(t)} = 1 + O(n^{-1}) \quad (89)$$

(see (17), (19) and (24)).

Lemma 2. There exist a positive constants $N$ and $c$, independent of $t$, such that

$$\|R_n(., t)\|^2 \leq c \left( \sum_{k>n} |(f_t, e^{i(2\pi k + t)x})|^2 + \frac{1}{n} \right) \quad (90)$$

for $n > N$ and $t \in E(h)$.

Proof. During the proof of the lemma we denote by $c_1, c_2, ...$ the positive constants that do not depend on $t$. They will be used in the sense that there exists $c_i$ such that the inequality holds. To prove (90) first we prove the inequality

$$\sum_{k>n} |a_k(t)|^2 \leq c_1 \left( \sum_{k>n} |(f_t, e^{i(2\pi k + t)x})|^2 + \frac{1}{n} \right), \quad (91)$$

where $a_k(t)$ is defined in (44), and then the equality

$$\|R_n(., t)\|^2 \leq (1 + O(n^{-1})) \sum_{k>n} |a_k(t)|^2. \quad (92)$$

It follows from (89) that

$$|a_k(t)|^2 \leq 8 |(f_t, e^{i(2\pi k + t)x})|^2 + 8 |(f_t, h^*_{n,t})|^2. \quad (93)$$

Since $f$ is a compactly supported and continuous function we have

$$\|f_t\|^2 < c_2. \quad (94)$$

It with the Schwarz inequality and second equality of (89) implies that

$$|(f_t, h^*_{n,t})|^2 < c_3 n^{-2}. \quad (95)$$

Therefore (91) follows from (93).

Now we prove (92). Since $\{e^{i2\pi kx} : k \in \mathbb{Z}\}$ is an orthonormal basis, using the Bessel inequality and (94) we obtain

$$\sum_{k : |k| > N} |(f_t, e^{i(2\pi k + t)x})|^2 \leq \|f_t\|^2 < c_2. \quad (96)$$
Hence, it follows from (91) that
\[ \sum_{k: |k| > n} |a_k(t)|^2 \leq c_4 \]
and by (88), \(\|a_k(t)h_{k,t}(x)\| \leq |a_k(t)|^2 + c_5 n^{-2}\). Therefore the series
\[ \sum_{k>n} a_k(t)e^{i(2\pi k+t)x} \quad \& \quad \sum_{k>n} a_k(t)h_{k,t}(x) \]
converge in the norm of \(L_2(0,1)\) and we have
\[ \| R_n(\cdot, t) \|^2 = \left\| \sum_{k>n} a_k(t)e^{i(2\pi k+t)x} + \sum_{k>n} a_k(t)h_{k,t}(x) \right\|^2 \leq 2S_1 + 2S_2^2 \]
(96)
where
\[ S_2 = \left\| \sum_{k>n} a_k(t)h_{k,t} \right\| \]
and
\[ S_1 = \left\| \sum_{k>n} a_k(t)e^{i(2\pi k+t)x} \right\|^2 = \sum_{k>n} |a_k(t)|^2 \]
(97)
(98)
Now let us estimate \(S_2\). It follows from the second equality of (88) that
\[ S_2 \leq c_6 \sum_{k>n} \frac{|a_k(t)|}{n} \]
Therefore using the Schwarz inequality for \(l_2\) we obtain
\[ S_2^2 = \left( \sum_{k>n} |a_k(t)|^2 \right) O(n^{-1}). \]
(99)
Thus (92) follows from (96), (98) and (99). It with (91) yields the proof of the lemma.

Now we are ready to prove the following

**Theorem 3.** For every compactly supported and continuous function \(f\) the equality
\[ \int_{E(h)} f_t(x)dt = \sum_{k \in \mathbb{Z}} \int_{E(h)} a_k(t)\Psi_{k,t}(x)dt \]
(100)
holds, where \(0 < h < \frac{1}{15\pi}\). The series in (100) converges in the norm of \(L_2(a,b)\) for every \(a, b \in \mathbb{R}\).

**Proof.** By (41) and (45) we have \(R_n(x+1,t) = e^{it}R_n(x,t)\). Therefore it follows from (90) that
\[ \| R_n(\cdot, t) \|_{(-m,m)}^2 \leq 2mc \left( \sum_{k>n} |(f_t,e^{i(2\pi k+t)x})|^2 + \frac{1}{n} \right), \]
(101)
where \(\| f \|_{(-m,m)}\) is the \(L_2(-m,m)\) norm of \(f\). Since the sequence
\[ \left\{ \sum_{k>n} |(f_t,e^{i(2\pi k+t)x})|^2 : n = 1,2,\ldots \right\} \]
of the continuous nonincreasing functions converges uniformly to zero on \([-\pi, \pi]\) it follows from (101) that \(\| R_n(\cdot, t) \|_{L^2(-m, m)} \) also converges to zero uniformly on \(E(h)\) as \(n \to \infty\). It implies that

\[
\int_{E(h)} \int_{(-m, m)} | R_n(x, t) |^2 \, dx \, dt \to 0 \tag{102}
\]

as \(n \to \infty\). Now using the obvious inequality \(\int_E |f(t)|^2 \, dt \leq 2\pi \int_E |f(t)|^2 \, dt\) and (83), (102), we obtain

\[
\left\| \int_E \sum_{k>n} a_k(t) \Psi_{k,t} \, dt \right\|^2_{L^2(-m, m)} \leq 2\pi \int_{(-m, m)} \int_E \left| \sum_{k>n} a_k(t) \Psi_{k,t}(x) \right|^2 \, dt \, dx \to 0 \tag{103}
\]

as \(n \to \infty\). Thus we have

\[
\int_E \sum_{k>N} a_k(t) \Psi_{k,t}(x) \, dt = \sum_{k>N} \int_E a_k(t) \Psi_{k,t}(x) \, dt, \tag{104}
\]

where the last series converges in the norm of \(L^2(-m, m)\) for every \(m \in \mathbb{N}\). In the same way we prove that

\[
\int_E \sum_{k<-N} a_k(t) \Psi_{k,t}(x) \, dt = \sum_{k<-N} \int_E a_k(t) \Psi_{k,t}(x) \, dt. \tag{105}
\]

Therefore using (104), (105) and Proposition 5(a) we get the proof of the theorem. \(\square\)

Now let us consider the term by term integration of the second and third integral in (81). Using the conditions in (76) and arguing as in the proof of Lemma 1 we see that for each \(x \in [0, 1]\), \(\Psi_{n,t}(x)\) and \(\Psi_{n,t}(x)\) are continuous and bounded on \(\gamma(0, h) \cup \gamma(\pi, h)\). Therefore, Proposition 5(a) continues to hold if we replace \(E(\delta)\) by \(\gamma(0, h)\) and \(\gamma(\pi, h)\). Similarly, Proposition 5(a) continues to hold if we replace \(E(h)\) by \(\gamma(h)\) and \(\gamma(h)\). In the same way we prove (102) when \(E(h)\) is replaced by \(\gamma(0, h)\) and \(\gamma(\pi, h)\). Thus repeating the proof of Theorem 3 we obtain

**Theorem 4.** For every compactly supported and continuous function \(f\) the equalities

\[
\int_{\gamma(0, h)} f(x) \, dt = \sum_{k \in \mathbb{Z}} \int_{\gamma(0, h)} a_k(t) \Psi_{k,t}(x) \, dt \tag{106}
\]

and

\[
\int_{\gamma(\pi, h)} f(x) \, dt = \sum_{k \in \mathbb{Z}} \int_{\gamma(\pi, h)} a_k(t) \Psi_{k,t}(x) \, dt \tag{107}
\]

hold, where \(0 < h < \frac{1}{15\pi}\) and (76) holds. The series in (106) and (107) converges in the norm of \(L^2(a, b)\) for every \(a, b \in \mathbb{R}\).

Now to prove the expansion theorem we try to replace \(\gamma(0, h)\) and \(\gamma(\pi, h)\) by \([-h, h]\) and \([\pi - h, \pi + h]\) in the right hand sides of (106) and (107) respectively.
Theorem 5. Let \( f \) be continuous and compactly supported function, \( 0 < h < \frac{1}{15\pi} \) and (76) holds. Then the following equalities hold

\[
\int_{\gamma(0,h)} f_t(x) dt = \int_{[-h,h]} \left( \sum_{n \leq N(h)} a_n(t) \Psi_{n,t}(x) \right) dt + \quad (108)
\]

\[
\sum_{n > N(h)} \int_{[-h,h]} (a_n(t) \Psi_{n,t}(x) + a_{-n}(t) \Psi_{-n,t}(x)) dt,
\]

\[
\int_{[-h,h]} \left( \sum_{n \leq N(h)} a_n(t) \Psi_{n,t}(x) \right) dt = \lim_{\delta \to 0} \left( \sum_{n \leq N(h)} \int_{\delta < |t| \leq h} a_n(t) \Psi_{n,t}(x) dt \right),
\]

where \( N(h) \) is defined in introduction (see (a)). Moreover, if \( \lambda_n(0) \), where \( n > N(h) \), is not an ESS then

\[
\int_{[-h,h]} (a_n(t) \Psi_{n,t} + a_{-n}(t) \Psi_{-n,t}) dt
\]

\[
= \lim_{\delta \to 0} \left( \int_{\delta < |t| \leq h} a_n(t) \Psi_{n,t} dt + \int_{\delta < |t| \leq h} a_{-n}(t) \Psi_{-n,t} dt \right),
\]

The series in (108) converges in the norm of \( L_2(a,b) \) for every \( a, b \in \mathbb{R} \).

Proof. It follows from (11) that for \( n > N(h) \) the circle

\( C(n) = \{ z \in \mathbb{C} : |z - (2n\pi)^2| = 2n \} \) contains inside only two eigenvalues (counting multiplicities) denoted by \( \lambda_n(t) \) and \( \lambda_{-n}(t) \) of the operators \( L_t \) for \( |t| \leq h \). Moreover, \( C(n) \) lies in the resolvent set of \( L_t \) for \( |t| \leq h \). Consider the total projections

\[
T_n(x,t) = \int_{C(n)} A(x,\lambda,t) d\lambda,
\]

where

\[
A(x,\lambda,t) = \int_0^1 G(x,\xi,\lambda,t) f_t(\xi) d\xi
\]

and \( G(x,\xi,\lambda,t) \) is the Green function of the operator \( L_t \). It is well-known that the Green function \( G(x,\xi,\lambda,t) \) of \( L_t \) is defined by formulas (see [10] pages 36 and 37)

\[
G(x,\xi,\lambda,t) = \frac{H(x,\xi,\lambda,t)}{\Delta(\lambda,t)},
\]

where

\[
H(x,\xi,\lambda,t) = \begin{vmatrix}
\theta(x,\lambda) & \varphi(x,\lambda) & g(x,\xi) \\
\theta - e^{it} & \varphi & g(1,\xi) - e^{it}g(0,\xi) \\
\theta' & \varphi' - e^{it} & g'(1,\xi) - e^{it}g'(0,\xi)
\end{vmatrix},
\]

\[
g(x,\xi) = \frac{1}{2} \begin{vmatrix}
\theta(x,\lambda) & \varphi(x,\lambda) \\
\theta(\xi,\lambda) & \varphi(\xi,\lambda)
\end{vmatrix},
\]

and \( \Delta(\lambda,t) \) is defined in (3). In (116) the positive sign being taken if \( x > \xi \), and the negative sign if \( x < \xi \).
Since $\Delta(\lambda, t)$ is continuous in the compact $C(n) \times U(h)$, where $U(h) = \{ t \in \mathbb{C} : |t| \leq h \}$, there exists a positive constant $c_7$ such that

$$|\Delta(\lambda, t)| \geq c_7, \quad \forall (\lambda, t) \in C(n) \times U(h).$$

(117)

Therefore using (112)-(116) and taking into account that $f_t(x)$ is the sum of finite number of summands (see (40)), we obtain that for any $x \in [0, 1]$ the function $T_n(x, t)$ is analytic in $U(h)$ and there exist $c_8$ such that

$$|T_n(x, t)| \leq c_8$$

for all $(x, t) \in [0, 1] \times U(h)$. It implies that

$$\int_{\gamma(0, h)} T_n(x, t) dt = \int_{[-h, h]} T_n(x, t) dt.$$  

(119)

On the other hand, inside of the circle $C(n)$ the operator $L_t$ for $t \in U(h) \setminus (A_n \cup A_{-n})$ has 2 simple eigenvalues $\lambda_n(t)$ and $\lambda_{-n}(t)$, where $A_n \cup A_{-n}$ is a finite set (see (15)). Therefore

$$T_n(x, t) = a_n(t)\Psi_{n,t} + a_{-n}(t)\Psi_{-n,t}, \quad \forall t \in U(h) \setminus (A_n \cup A_{-n}).$$

(120)

Besides, by (76), for $t \in \gamma(0, h)$ the eigenvalues are simple and hence $(f_t, X_{k,t})$ is continuous function on $\gamma(0, h)$ for each $x$ which implies that

$$\int_{\gamma(0, h)} a_n(t)\Psi_{n,t} + a_{-n}(t)\Psi_{-n,t} dt = \int_{\gamma(0, h)} a_n(t)\Psi_{n,t} dt + \int_{\gamma(0, h)} a_{-n}(t)\Psi_{-n,t} dt.$$  

(121)

Thus it follows from (119)-(121) that

$$\int_{\gamma(0, h)} a_n(t)\Psi_{n,t} dt + \int_{\gamma(0, h)} a_{-n}(t)\Psi_{-n,t} dt = \int_{[-h, h]} (a_n(t)\Psi_{n,t} + a_{-n}(t)\Psi_{-n,t}) dt.$$  

(122)

Now one can readily see that the equalities (110) and (111) follows from (118), (120) and Proposition 5(b) respectively.

It is clear that there exists a closed curve $\Gamma(0)$ such that the curve $\Gamma(0)$ lies in the resolvent set of the operator $L_t$ for $|t| \leq h$ and all eigenvalues of $L_t$ for $|t| \leq h$ that do not lie in $C(n)$ for $n > N(h)$ belong to the set enclosed by $\Gamma(0)$. Therefore instead of $C(n)$ using $\Gamma(0)$ and repeating the above arguments we obtain that the function

$$S_N(x, t) = \sum_{|n| \leq N(h)} a_n(t)\Psi_{n,t}(x)$$

(123)

is analytic in $U(h)$ and there exist $c_9$ such that

$$|S_N(x, t)| \leq c_9$$

(124)

for all $(x, t) \in [0, 1] \times U(h)$ and

$$\int_{\gamma(0, h)} \sum_{|n| \leq N(h)} a_n(t)\Psi_{n,t}(x) = \int_{[-h, h]} \left( \sum_{|n| \leq N(h)} a_n(t)\Psi_{n,t}(x) \right).$$

Now using (106) and taking into account that the integrals of $S_N(x, t)$ over $[-\delta, \delta]$ tend to zero as $\delta \to 0$ (see (124)) we get the proof of the theorem \[\square\]

In the same way we obtain.
The series in (125) converge in the norm of Theorem 7. For each continuous and compactly supported function \( f \) and (76) holds. Then the following equalities hold

\[
\int_{\gamma(\pi, h)} f_t(x) dt = \int_{[\pi - h, \pi + h]} \sum_{n = -N(h) - 1}^{N(h)} a_n(t) \Psi_{n,t}(x) dt + \tag{125}
\]

\[
\sum_{n > N(h)} \int_{[\pi - h, \pi + h]} (a_n(t) \Psi_{n,t}(x) + a_{-(n+1)}(t) \Psi_{-(n+1),t}(x)) dt,
\]

\[
\int_{[\pi - h, \pi + h]} \left( \sum_{n = -N(h) - 1}^{N(h)} a_n(t) \Psi_{n,t}(x) \right) dt
\]

\[
= \lim_{\delta \to 0} \left( \sum_{n = -N(h) - 1}^{N(h)} a_n(t) \Psi_{n,t}(x) dt \right),
\]

\[
\int_{[\pi - h, \pi + h]} \sum_{k = -n - (n+1)} \Psi_{k,t} dt = \lim_{\delta \to 0} \left( \sum_{k = -n - (n+1)} \int_{[\pi - h, \pi + h]} a_k(t) \Psi_{k,t} dt \right).
\]

Moreover, if \( \lambda_n(\pi) \) is not an ESS then

\[
\int_{[\pi - h, \pi + h]} \sum_{k = -n - (n+1)} \Psi_{k,t} dt = \sum_{k = -n - (n+1)} \int_{[\pi - h, \pi + h]} a_k(t) \Psi_{k,t} dt.
\]

The series in (125) converge in the norm of \( L_2(a, b) \) for every \( a, b \in \mathbb{R} \).

Thus by (80) and theorems 3, 5, 6 we have the following spectral expansion theorem

**Theorem 7.** For each continuous and compactly supported function \( f \) the spectral expansion given by the equalities (80) and (100), (108), (125) holds.

In the Conclusion 1 we discuss in detail the necessity of the parenthesis (the handling of the terms \( a_n(t) \Psi_{n,t}(x) \) and \( a_{-(n+1)}(t) \Psi_{-(n+1),t}(x) \)) in the second row of (108) and the convergence of the series with parenthesis. Now in the following remark we discuss the parenthesis in the first row of (108).

**Remark 2.** On the parenthesis in (108). We say that the set

\[
\{ a_k(t) \Psi_{k,t}(x) : k \in T(\Lambda) \},
\]

where \( T(\Lambda) \) is defined in (59), is a bundle corresponding to the multiple eigenvalue \( \Lambda \). If \( \Lambda \) is not ESS of the operator \( L \) then it follows from Definition 3 and Remark 1 that all elements \( a_k(t) \Psi_{k,t}(x) \) of the bundle (126) are integrable functions on \([0, \varepsilon) \cup (0, \varepsilon]\) for all \( x \) and for some \( \varepsilon \). If \( \Lambda \) is an ESS of the operator \( L \) then for some values of \( k \in T(\Lambda) \) the function \( a_k(t) \Psi_{k,t}(x) \) for almost all \( x \) is nonintegrable on \([-\varepsilon, 0) \cup (0, \varepsilon]\), while some of elements of the bundle (126) may be integrable. Instead of \( C(n) \) using a small circle enclosing \( \Lambda \) and repeating the proof of (118) we see that the total sum of elements of (126) is bounded due to the cancellations of the nonintegrable terms of (126). At least two element of the bundle must be nonintegrable in order to do the cancellations. In fact, we may and must to huddle together only the nonintegrable elements of the bundle (126). In case \( \Lambda = \lambda_n(0) \) and \( n \gg 1 \) the bundle (126) consist of \( a_n(t) \Psi_{n,t} \) and \( a_{-(n+1)}(t) \Psi_{-(n+1),t} \) and both of them are nonintegrable. That is why we must to handle they together.
Let \( \lambda_{n,j}(0) \) for \( j = 1, 2, \ldots, s \) be ESS, where \( |n_j| \leq N(h) \). Then the set 
\[ \{ n \in \mathbb{Z} : |n| \leq N(h) \} \]
can be divided into subsets \( T(\lambda_{n,j}(0)) \) for \( j = 1, 2, \ldots, s \) and
\[ \mathcal{K} = \{ n \in \mathbb{Z} : |n| \leq N(h) \} \setminus \bigcup_{j=1,2,\ldots,s} T(\lambda_{n,j}(0)). \]

Therefore the summations over \( \{ n \in \mathbb{Z} : |n| \leq N(h) \} \) in (108) and (109) can be written as the sum of summations over \( T(\lambda_{n_1}(0)), T(\lambda_{n_2}(0)), \ldots, T(\lambda_{n_s}(0)) \) and \( \mathcal{K} \).

In Theorem 5 to avoid the complicated notations the summations over 
\( \{ n \in \mathbb{Z} : |n| \leq N(h) \} \) is taken. We have the same situation with Theorem 6.

Now to write the spectral expansion theorems in a compact form we introduce
some notations and definition. For this we parameterize the Bloch eigenvalues \( \lambda_n(t) \) and Bloch functions \( \Psi_{n,t}(x) \) by quasimomentum \( t \) changing in all \( \mathbb{R} \).

**Notation 1.** Define \( \lambda : \mathbb{R} \to \mathbb{C} \) by \( \lambda(t) = \lambda_{n,t}(t-2\pi n) \) for \( t \in (2\pi n-h, 2\pi(n+1)-h] \), where \( n \in \mathbb{Z} \). Similarly, let \( \Psi(x,t) \) and \( \Psi^*(x,t) \), denotes respectively \( \Psi_{n,t-2\pi n}(x) \) and \( \Psi_{n,t-2\pi n}^*(x) \) if \( t \in (2\pi n-h, 2\pi(n+1)-h) \). Let \( \alpha(t) = (\Psi(\cdot,t), \Psi^*(\cdot,t)) \) and \( a(t) = (f, \Psi(\cdot,t))_\mathbb{R} \)

**Definition 4.** A quasimomentum \( t \) is said to be singular quasimomentum if \( \lambda(t) \) is ESS. By Theorem 1 the set of singular quasimomenta is the subset of \( \{ \pi n : n \in \mathbb{Z} \} \).

Therefore the definition of the singular quasimomenta can also be given as follows: \( \pi n \) is called a singular quasimomentum if \( \lambda(\pi n) \) is ESS.

Let \( \Lambda = \lambda_{n}(0) \) be ESS. It means that:

Case 1. If \( |n| > N(h) \) then \( \lambda_n(0) = \lambda_{-n}(0) \).

Case 2. If \( |n| \leq N(h) \) then \( \lambda_{j}(0) = \Lambda \) for all \( j \in \mathbb{T}(\Lambda) \).

Then in Case 1 the quasimomenta \( \pm 2\pi n \), and in Case 2 the quasimomenta \( 2\pi j \)
for \( j \in \mathbb{T}(\Lambda) \) are the singular quasimomenta corresponding to the ESS \( \lambda_{n}(0) \). In the
same way we define the singular quasimomenta corresponding to the ESS \( \lambda_{n}(\pi) \).

As we noted in Remark 2, if \( \lambda_{n}(0) \) for \( |n| > N(h) \) is ESS then both \( a_n(t)\Psi_{n,t} \) and \( a_{-n}(t)\Psi_{-n,t} \) are nonintegrable in neighborhoods of 0 and we must to handle
them together. In the language of Notation 1, it means that if \( \lambda(2\pi n) \) for \( |n| > N(h) \) is ESS then \( a(t)\Psi(x,t) \) is nonintegrable in the neighborhoods of the singular quasimomenta \( 2\pi n \) and \( -2\pi n \) corresponding to the ESS \( \lambda(2\pi n) \). That is why, the handling \( a_n(t)\Psi_{n,t} \) and \( a_{-n}(t)\Psi_{-n,t} \) in (108) now corresponds to the handling of the neighborhoods of \( 2\pi n \) and \( -2\pi n \) together. Therefore we divide the set \( \mathbb{R} \) of quasimomenta \( t \) into two parts: the set of neighborhoods of singular quasimomenta and the other part of \( \mathbb{R} \).

Similarly, we divide the spectrum \( \sigma(L) \) into two part: the set of neighborhood of ESS and the other part of \( \sigma(L) \). For this introduce the notations.

**Notation 2.** Let \( \{ \pi n_j : j = 1, 2, \ldots, \} \) be the set of the singular quasimomenta. By
Definition 4, \( \Lambda(\pi n_j) \) is an ESS and
\[ \mathbb{E} = \{ \Lambda(\pi n_j) : j = 1, 2, \ldots, \}, \]
where \( \mathbb{E} \) is the set of ESS. For \( |n_j| > N(h) \) define \( B_j(h) \) and \( B_j(h,\delta) \) by
\[ B_j(h) = (\pi n_j - h, \pi n_j + h) \cup (-\pi n_j - h, -\pi n_j + h), \quad B_j(h,\delta) = B_j(h) \setminus B_j(\delta), \]
where \( 0 < h < \frac{1}{2\pi} \) and \( 0 < \delta < h \). For \( |n_j| \leq N(h) \) define \( B_j(h) \) and \( B_j(h,\delta) \) by
\[ B_j(h) = \bigcup_{n \in \mathbb{T}(\pi n - h, \pi n + h)} (\pi n - h, \pi n + h), \quad B_j(h,\delta) = B_j(h) \setminus B_j(\delta), \]
where \( T_j := \mathbb{T}(\lambda(\pi n_j)) \) and \( T(\Lambda) \) is defined in (59). The set \( \lambda(B_j(h)) \) is the part of the spectrum \( \sigma(L) \) of \( L \) lying in the neighborhood of the ESS \( \lambda(\pi n_j) \), where \( \lambda(C) = \{ \lambda(t) : t \in C \} \) for \( C \in \mathbb{R} \). Finally let
\[
B(h) = \bigcup_j B_j(h).
\]

Using this notation and Theorems 3, 5 and 6 we obtain

**Theorem 8.** For each continuous and compactly supported function \( f \) the following expansion holds
\[
f(x) = \frac{1}{2\pi} \int_{\mathbb{R} \setminus B(h)} a(\lambda(t))\Psi(x, \lambda(t))dt + \frac{1}{2\pi} \sum_j \text{p.v.} \int_{B_j(h)} a(\lambda(t))\Psi(x, \lambda(t))dt
\]
(127)
where the p.v. integral over \( B_j(h) \) is the limit as \( \delta \to 0 \) of the integral over \( B_j(h, \delta) \). The first integral and the series in (127) converge in the norm of \( L_2(a, b) \) for every \( a, b \in \mathbb{R} \).

Now changing the variable to \( \lambda \) in (127) as was done in (37) and using Notation 2 we obtain the following spectral expansion.

**Theorem 9.** For each continuous and compactly supported function \( f \) the following spectral expansion holds
\[
f(x) = \frac{1}{2\pi} \int_{\sigma(L) \setminus \lambda(B(h))} (\Phi_+(x, \lambda)F_-(\lambda, f) + \Phi_-(x, \lambda)F_+(\lambda, f)) \frac{1}{\varphi_p(\lambda)}d\lambda + \frac{1}{2\pi} \sum_j \text{p.v.} \int_{\lambda(B_j(h))} (\Phi_+(x, \lambda)F_-(\lambda, f) + \Phi_-(x, \lambda)F_+(\lambda, f)) \frac{1}{\varphi_p(\lambda)}d\lambda
\]
(128)
where the p.v. integral over \( \lambda(B_j(h)) \) is the limit as \( \delta \to 0 \) of the integral over \( \lambda(B_j(h, \delta)) \), the functions \( \Phi_\pm(x, \lambda) \) and \( F_\pm(\lambda, f) \) are defined in (34) and (35). The first integral and the series in (128) converge in the norm of \( L_2(a, b) \) for every \( a, b \in \mathbb{R} \).

Now let us do some conclusion about the obtained spectral expansions.

**Conclusion 1.** At first glance it seems that the obtained spectral expansions have a complicated form, since the series (108) and (125) converge with parenthesis (see Theorem 5 and Theorem 6) and in (127) and (128) the p.v. integrals are used. However, it is only connected with a complicated picture of the spectrum and projections and nature of the Hill operator with complex periodic potential. To confirm it, we now explain the necessity of the parenthesis and p.v. integrals and try to show that the all factors that effect to the spectral expansion are taken into account. First, note that it follows from the Notation 2 that the integrals in (127) and (128) are taking over all \( \mathbb{R} \) and \( \sigma(L) \) except the discrete sets
\[
\{ \pi n_j : j = 1, 2, \ldots \} \quad \& \quad \mathbb{E} = \{ \lambda(\pi n_j) : j = 1, 2, \ldots \}
\]
respectively. Since the corresponding integrals about the points of those sets do not exist, we use the p.v. integral, that is, the limit as \( \delta \to 0 \). Moreover, the sets \( B_j(h) \) are constructed in the way which takes into account the requisite parenthesis in (108) and (125). Let us explain, in detail, why the parenthesis and limits as \( \delta \to 0 \) are necessary for the spectral expansion for the general complex-valued periodic potentials:
Necessity of the parenthesis in (108) and (125) and p.v. integrals in (127) and (128). The series in (108) and (125) converge with parenthesis and in parenthesis is included only the integrals of the functions corresponding to splitting eigenvalues. The parenthesis is necessary, due to the following. If \( n \gg 1 \) and \( \lambda_n(0) \) is ESS, then \( \lambda_n(0) \) is a double eigenvalue, \( \lambda_n(0) = \lambda_{-n}(0) \) and both of the functions \( a_n(t)\Psi_{n,t} \) and \( a_{-n}(t)\Psi_{-n,t} \) has nonintegrable singularities (see (72) and the definition of \( a_n(t) \) in (44)), that is, their integrals do not exist. However, the integral

\[
\int_{[-h,h]} (a_n(t)\Psi_{n,t} + a_{-n}(t)\Psi_{-n,t}) dt
\]

(129) exists. Moreover, even if \( \lambda_n(0) \) and \( \lambda_n(\pi) \) are not ESS respectively, then it is possible that the norm of

\[
\int_{[-h,h]} a_n(t)\Psi_{n,t}(x) dt \quad \& \quad \int_{[\pi-h,\pi+h]} a_n(t)\Psi_{n,t}(x) dt
\]

(130) do not tend to zero as \( n \to \infty \). Therefore the series in (108) and (125) do not converge without parenthesis. More precisely, the series (108) and (125) converge without parenthesis if and only if there exist \( h > 0 \) such that the first and second integrals in (130) respectively exist and tend to zero as \( n \to \pm \infty \). (see Theorem 10).

Note that this situation agree with the well-known result [13] that the root functions of the operators generated by a ordinary differential expression in \([0,1]\) with regular boundary conditions, in general, form a Riesz basis with parenthesis and in parenthesis should be included only the functions corresponding to the splitting eigenvalues. In particular, the periodic \((t=0)\) and antiperiodic \((t=\pi)\) boundary conditions require the parenthesis. It is natural that in the case of the operator \( L \) generated by an ordinary differential expression in \((-\infty,\infty)\) we included in parenthesis the Bloch functions \( \Psi_{n,t}(x) \) near two \( t = 0 \) (see (108)) and \( t = \pi \) (see (125)).

The using of the p.v. integral about singular quasumomenta and ESS in (127) and (128) respectively is necessary, since the integrals about those points do not exist. We do not need the p.v. integral if and only if the operator \( L \) has no ESS.

Thus in the general case we should use the parenthesis and p.v. integrals and one can obtain a spectral expansion without parenthesis and p.v. integrals if and only if \( L(q) \) has no ESS and the integrals in (130) tend to zero as \( |n| \to \infty \). Namely, we have the following.

**Theorem 10.** For each continuous and compactly supported function \( f \) we have the following spectral decompositions

\[
f(x) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_0^{2\pi} a_k(t)\Psi_{k,t}(x) dt
\]

\[
= \frac{1}{2\pi} \int_{\sigma(L)} (\Phi_{+}(x,\lambda)F_{-}(\lambda,f) + \Phi_{-}(x,\lambda)F_{+}(\lambda,f)) \frac{1}{\varphi p(\lambda)} d\lambda
\]

if and only if \( L(q) \) has no ESS and there exists \( h > 0 \) such that the integrals in (130) tend to zero as \( |n| \to \infty \).

**REFERENCES**

[1] M. S. P. Eastham, *The Spectral Theory of Periodic Differential Operators*, New York: Hafner, 1974.

[2] M. G. Gasymov, Spectral analysis of a class of second-order nonself-adjoint differential operators, *Funkts. Anal. Prilozhen*, 14 (1980), 14–19.
[3] I. M. Gelfand, Expansion in series of eigenfunctions of an equation with periodic coefficients, *Sov. Math. Dokl.*, 73 (1950), 1117–1120.

[4] F. Gesztesy and V. Tkachenko, A criterion for Hill’s operators to be spectral operators of scalar type, *J. Analyse Math.*, 107 (2009), 287–353.

[5] W. Magnus and S. Winkler, *Hill’s Equation*, New York: Inter. Publ., 1966.

[6] V. A. Marchenko, *Sturm-Liouville Operators and Applications*, Birkhauser Verlag, Basel, 1986.

[7] D. C. McGarvey, Differential operators with periodic coefficients in $L_p(-\infty, \infty)$, *Journal of Mathematical Analysis and Applications*, 11 (1965), 564–596.

[8] D. C. McGarvey, Perturbation results for periodic differential operators, *Journal of Mathematical Analysis and Applications*, 12 (1965), 187–234.

[9] V. P. Mikhailov, On the Riesz bases in $L_2(0, 1)$, *Sov. Math. Dokl.*, 25 (1962), 981–984.

[10] M. A. Naimark, *Linear Differential Operators*, George G. Harrap, London, 1967.

[11] E. C. Titchmarsh, *Eigenfunction Expansion (Part II)*, Oxford Univ. Press, 1958.

[12] V. A. Tkachenko, Spectral analysis of nonself-adjoint Schrodinger operator with a periodic complex potential, *Sov. Math. Dokl.*, 5 (1964), 413–415.

[13] A. A. Shkalikov, On the Riesz basis property of the root vectors of ordinary differential operators, *Russian Math. Surveys*, 34 (1979), 249–250.

[14] O. A. Veliev, The one dimensional Schrodinger operator with a periodic complex-valued potential, *Sov. Math. Dokl.*, 250 (1980), 1292–1296.

[15] O. A. Veliev, The spectrum and spectral singularities of differential operators with complex-valued periodic coefficients, *Differential Cprime Nye Uravneniya*, 19 (1983), 1316–1324.

[16] O. A. Veliev, The spectral resolution of the nonself-adjoint differential operators with periodic coefficients, *Differential Cprime Nye Uravneniya*, 22 (1986), 2052–2059.

[17] O. A. Veliev, M. Toppamuk Duman, The spectral expansion for a nonself-adjoint Hill operators with a locally integrable potential, *J. Math. Anal. Appl.*, 265 (2002), 76–90.

[18] O. A. Veliev, Uniform convergence of the spectral expansion for a differential operator with periodic matrix coefficients, *Boundary Value Problems*, Volume 2008, Article ID 628973, 22 pp. (2008).

[19] O. A. Veliev, Asymptotic analysis of non-self-adjoint Hill’s operators, *Central European Journal of Mathematics*, 11 (2013), 2234–2256.

[20] O. A. Veliev, On the spectral singularities and spectrality of the Hill’s Operator, *Operators and Matrices*, 10 (2016), 57–71.

Received February 2017; revised May 2017.

E-mail address: oveliev@dogus.edu.tr