Research Article

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More on \(\mu\)-semi-Lindelöf sets in \(\mu\)-spaces

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Abstract: Sarsak [On \(\mu\)-compact sets in \(\mu\)-spaces, Questions Answers Gen. Topology 31 (2013), no. 1, 49–57] introduced and studied the class of \(\mu\)-Lindelöf sets in \(\mu\)-spaces. Mustafa [\(\mu\)-semi compactness and \(\mu\)-semi Lindelöfness in generalized topological spaces, Int. J. Pure Appl. Math. 78 (2012), no. 4, 535–541] introduced and studied the class of \(\mu\)-semi-Lindelöf sets in generalized topological spaces (GTSs); the primary purpose of this paper is to investigate more properties and mapping properties of \(\mu\)-semi-Lindelöf sets in \(\mu\)-spaces. The class of \(\mu\)-semi-Lindelöf sets in \(\mu\)-spaces is a proper subclass of the class of \(\mu\)-Lindelöf sets in \(\mu\)-spaces. It is shown that the property of being \(\mu\)-semi-Lindelöf is not monotonic, that is, if \((X, \mu)\) is a \(\mu\)-space and \(A \subset B \subset X\), where \(A\) is \(\mu\)-\(\omega\)-semi-Lindelöf, then \(A\) need not be \(\mu\)-semi-Lindelöf. We also introduce and study a new type of generalized open sets in GTSs, called \(\omega\)-\(\mu\)-semi-open sets, and investigate them to obtain new properties and characterizations of \(\mu\)-semi-Lindelöf sets in \(\mu\)-spaces.

Keywords: generalized topology, \(\mu\)-space, \(\mu\)-open, \(\mu\)-semi-open, \(\mu\)-Lindelöf set, \(\mu\)-Lindelöf space, \(\mu\)-semi-Lindelöf set, \(\mu\)-semi-Lindelöf space

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1 Introduction and preliminaries

A topological space \(X\) is said to be Lindelöf [1] if every open cover of \(X\) has a countable subcover. Lindelöf spaces play a vital role in the theory of general topology as a natural generalization of compact spaces. Since then, related concepts had been of special interests to several mathematicians. For instance, nearly Lindelöf spaces [2], strongly Lindelöf spaces [3], almost Lindelöf spaces [4], semi-Lindelöf spaces [5], and rc-Lindelöf spaces [6]. For more related studies, one can see [7–12].

The study of generalized topological spaces (GTSs) was first initiated by Császár [13], which in role, motivated a lot of authors to generalize the topological notions including covering properties to the generalized topological surroundings. For instance, Sarsak [14,15], Mustafa [16], Arar [17,18], Abuage et al. [19], and Roy [20] studied several analogous notions via GTSs. The primary purpose of this paper is to continue the study of \(\mu\)-semi-Lindelöf sets in \(\mu\)-spaces introduced by Mustafa in [16]. In Section 2, we study the relationship between \(\mu\)-semi-Lindelöf sets and \(\mu\)-Lindelöf sets as we also obtain new characterizations of \(\mu\)-semi-Lindelöf sets; in Section 3, we mainly deduce that if \((X, \mu)\) is a \(\mu\)-space and \(A \subset B \subset X\), where \(A\) is \(\mu\)-\(\omega\)-semi-Lindelöf, then \(A\) need not be \(\mu\)-semi-Lindelöf, as we also give corrections to some results in [16]; in Section 4, we introduce and study a new type of generalized open sets in generalized topological spaces, called \(\omega\)-\(\mu\)-semi-open sets, and investigate them in Sections 5 and 6 to obtain more characterizations of \(\mu\)-semi-Lindelöf spaces and to obtain several properties of \(\mu\)-semi-Lindelöf sets related to sums, products, images, and preimages.

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A generalized topology (GT) [13] \( \mu \) on a nonempty set \( X \) is a collection of subsets of \( X \) such that \( \emptyset \in \mu \) and \( \mu \) is closed under arbitrary unions. Elements of \( \mu \) will be called \( \mu \)-open sets, and a subset \( A \) of \((X, \mu)\) will be called \( \mu \)-closed if \( X \setminus A \) is \( \mu \)-open. Clearly, a subset \( A \) of \((X, \mu)\) is \( \mu \)-open if and only if for each \( x \in A \), there exists \( U_t \in \mu \) such that \( x \in U_t \subseteq A \), or equivalently, \( A \) is the union of \( \mu \)-open sets. The pair \((X, \mu)\) will be called GTS. A space \( X \) or \((X, \mu)\) will always mean a GTS. A space \((X, \mu)\) is called a \( \mu \)-space [21] if \( X \in \mu \). \((X, \mu)\) is called a quasi-topological space [22] if \( \mu \) is closed under finite intersections. Clearly, every topological space is a quasi-topological space, every quasi-topological space is a GTS, and a space \((X, \mu)\) is a topological space if and only if \((X, \mu)\) is both \( \mu \)-space and quasi-topological space.

If \( A \) is a subset of a space \((X, \mu)\), then the \( \mu \)-closure of \( A \) [23], \( c_\mu(A) \), is the intersection of all \( \mu \)-closed sets containing \( A \) and the \( \mu \)-interior of \( A \) [23], \( i_\mu(A) \), is the union of all \( \mu \)-open sets contained in \( A \). It was pointed out in [23] that each of the operators \( c_\mu \) and \( i_\mu \) are monotonic [24], i.e., if \( A \subseteq B \subseteq X \), then \( c_\mu(A) \subseteq c_\mu(B) \) and \( i_\mu(A) \subseteq i_\mu(B) \), idempotent [24], i.e., if \( A \subseteq X \), then \( c_\mu(c_\mu(A)) = c_\mu(A) \) and \( i_\mu(i_\mu(A)) = i_\mu(A) \), \( c_\mu \) is enlarging [24], i.e., if \( A \subseteq X \), then \( c_\mu(A) \supseteq A \), \( i_\mu \) is restricting [24], i.e., if \( A \subseteq X \), then \( i_\mu(A) \subseteq A \), \( A \) is \( \mu \)-open if and only if \( A = i_\mu(A) \), and \( c_\mu(A) = X \setminus i_\mu(X \setminus A) \).

Clearly, \( A \) is \( \mu \)-closed if and only if \( A = c_\mu(A) \), \( c_\mu(A) \) is the smallest \( \mu \)-closed set containing \( A \) and \( i_\mu(A) \) is the largest \( \mu \)-open set contained in \( A \), and \( x \in c_\mu(A) \) if and only if any \( \mu \)-open set containing \( x \) intersects \( A \).

A function \( f : (X, \mu) \rightarrow (Y, \kappa) \) is called \((\mu, \kappa)\)-continuous [25] if the inverse image of each \( \kappa \)-open set is \( \mu \)-open, and called \((\mu, \kappa)\)-closed [25] if the image of every \( \mu \)-closed set is \( \kappa \)-closed.

If \((X, \tau)\) is a topological space and \( A \subseteq X \), then \( \tau \) and \( \text{Int} A \) will stand, respectively, for the closure of \( A \) in \( X \) and the interior of \( A \) in \( X \).

A subset \( A \) of a topological space \((X, \tau)\) is called semi-open [26] if \( A \subseteq \text{Int} \overline{A} \); and \( A \) is called semi-closed if \( X \setminus A \) is semi-open. \( A \) is called preopen [27] (resp. \( \alpha \)-open [28]) if \( A \subseteq \overline{\text{Int} A} \) (resp. \( A \subseteq \overline{\text{Int} \overline{A}} \)). It is known that the arbitrary union of semi-open (resp. preopen, \( \alpha \)-open) sets is semi-open (resp. preopen, \( \alpha \)-open).

The families of semi-open (resp. preopen, \( \alpha \)-open) subsets of a topological space \((X, \tau)\) will be denoted by \( \text{SO}(X) \) (resp. \( \text{PO}(X), \alpha(X) \)). Clearly, if \( \mu = \text{SO}(X) \) (resp. \( \text{PO}(X), \alpha(X) \)), then \((X, \mu)\) is a \( \mu \)-space.

For the concepts and terminology not defined here, the reader is referred to [29]. In concluding this section, we recall the following definitions and facts for their importance in the material of our paper.

**Definition 1.1.** [23] Let \( A \) be a subset of a space \((X, \mu)\). Then \( A \) is called

(i) \( \mu \)-semi-open if \( A \subseteq c_\mu(i_\mu(A)) \);
(ii) \( \mu \)-preopen if \( A \subseteq i_\mu(c_\mu(A)) \);
(iii) \( \mu \)-\( \alpha \)-open if \( A \subseteq i_\mu(c_\mu(i_\mu(A))) \).

As in [23], for a space \((X, \mu)\), we will denote the class of \( \mu \)-semi-open sets by \( \sigma(\mu) \) or \( \sigma \), the class of \( \mu \)-preopen sets by \( \pi(\mu) \) or \( \pi \), and the class of \( \mu \)-\( \alpha \)-open sets by \( \alpha(\mu) \) or \( \alpha \).

**Proposition 1.2.** [23] Let \((X, \mu)\) be a space. Then each of the families \( \sigma, \pi, \) and \( \alpha \) is a GT.

**Proposition 1.3.** [23] Let \((X, \mu)\) be a space. Then

(i) \( \mu \subseteq \alpha \);
(ii) \( \alpha = \sigma \cap \pi \);
(iii) \( \sigma(\sigma) = \sigma \);
(iv) \( \sigma(\alpha) = \alpha \).

**Remark 1.4.** If \((X, \mu)\) is a \( \mu \)-space, then it is easy to see that

(i) \( (X, \sigma(\mu)) \) is a \( \sigma(\mu) \)-space;
(ii) \( (X, \alpha(\mu)) \) is an \( \alpha(\mu) \)-space.

**Definition 1.5.** [30] Let \( A \) be a subset of a space \((X, \mu)\). Then \( A \) is called \( \mu \)-semi-closed if \( X \setminus A \) is \( \mu \)-semi-open.
**Definition 1.6.** [31]
(i) A subset \( A \) of a \( \mu \)-space \((X, \mu)\) is called \( \mu \)-Lindelöf if any cover of \( A \) by \( \mu \)-open subsets of \( X \) has a countable subcover.
(ii) A \( \mu \)-space \((X, \mu)\) is called \( \mu \)-Lindelöf if any cover of \( X \) by \( \mu \)-open sets has a countable subcover.

**2 \( \mu \)-semi-Lindelöf sets**

This section is mainly devoted to investigate more properties of \( \mu \)-semi-Lindelöf sets in \( \mu \)-spaces.

**Definition 2.1.**
(i) A subset \( A \) of a topological space \((X, \tau)\) is called semi-compact relative to \( X \) [32] (resp. semi-Lindelöf in \( X \) [33]) if any cover of \( A \) by semi-open subsets of \( X \) has a finite (resp. countable) subcover. We will usually use the term “in \( X \)” to mean “relative to \( X \)”.
(ii) A topological space \((X, \tau)\) is called semi-compact [34] (resp. semi-Lindelöf [5]) if any cover of \( X \) by semi-open subsets of \( X \) has a finite (resp. countable) subcover.

**Definition 2.2.** [16]
(i) A subset \( A \) of a \( \mu \)-space \((X, \mu)\) is called \( \mu \)-semi-compact (resp. \( \mu \)-semi-Lindelöf) relative to \( X \) if any cover of \( A \) by \( \mu \)-open subsets of \( X \) has a finite (resp. countable) subcover. We will say \( A \) is \( \mu \)-semi-compact (resp. \( \mu \)-semi-Lindelöf) to mean \( A \) is \( \mu \)-semi-compact (resp. \( \mu \)-semi-Lindelöf) relative to \( X \).
(ii) A \( \mu \)-space \((X, \mu)\) is called \( \mu \)-semi-compact (resp. \( \mu \)-semi-Lindelöf) if any cover of \( X \) by \( \mu \)-semi-open sets has a finite (resp. countable) subcover.

**Remark 2.3.** Let \( A \) be a subset of a \( \mu \)-space \((X, \mu)\). Then it is clear from Proposition 1.3(iii) and (iv) that
(i) \( A \) is \( \mu \)-semi-Lindelöf if and only if \( A \) is \( \sigma(\mu) \)-semi-Lindelöf.
(ii) \( A \) is \( \mu \)-semi-Lindelöf if and only if \( A \) is \( \alpha(\mu) \)-semi-Lindelöf.

**Remark 2.4.** If \( A \) is a subset of a topological space \((X, \tau)\) and \( \mu = \tau \) (resp. \( SO(X), \alpha(X) \)), then
\[
A \text{ is } \mu \text{-semi-Lindelöf} \iff A \text{ is semi-Lindelöf in } (X, \tau).
\]
In particular,
\[
(X, \mu) \text{ is } \mu \text{-semi-Lindelöf} \iff (X, \tau) \text{ is semi-Lindelöf}.
\]

**Remark 2.5.** If a subset \( A \) of a \( \mu \)-space \((X, \mu)\) is \( \mu \)-semi-Lindelöf, then \( A \) is \( \mu \)-Lindelöf; in particular, if a \( \mu \)-space \((X, \mu)\) is \( \mu \)-semi-Lindelöf, then \( (X, \mu) \) is \( \mu \)-Lindelöf. However, the converse need not be true even for topological spaces as the following example tells.

**Example 2.6.** Let \( X \) be an uncountable set and consider the topology \( \tau = \{X, \emptyset, \{p\}\} \) on \( X \), where \( p \in X \). Now, if \( x \in A = X \setminus \{p\} \), then \( A_x = \{p, x\} \) is \( \tau \)-semi-open because \( \text{Int} A_x = \{p\} \), and thus, \( \text{Int} A_x = \{p\} = X \).
Therefore, \( A_x \subseteq \text{Int} A_x \). Observe that the collection \( \mathcal{A} = \{A_x : x \in A\} \) is a cover of \( X \) by \( \tau \)-semi-open sets, but \( \mathcal{A} \) has no countable subcover, so, \((X, \tau)\) is not semi-Lindelöf. However, \((X, \tau)\) is Lindelöf (even compact).

**Remark 2.7.**
(i) A subset \( A \) of a \( \mu \)-space \((X, \mu)\) is \( \mu \)-semi-Lindelöf if and only if \( A \) is \( \sigma(\mu) \)-Lindelöf.
(ii) A \( \mu \)-space \((X, \mu)\) is \( \mu \)-semi-Lindelöf if and only if \((X, \sigma(\mu))\) is \( \sigma(\mu) \)-Lindelöf.

The proofs of the following three propositions are either straightforward or from [35] and thus omitted.
Proposition 2.8.
(i) A subset $A$ of a $\mu$-space $(X, \mu)$ is $\mu$-semi-Lindelöf if and only if for every family $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ of $\mu$-semi-closed sets having the property that for every countable subfamily $\mathcal{F}^i$ of $\mathcal{F}$, $(\bigcap \mathcal{F}^i) \cap A \neq \emptyset$, then $(\bigcap \mathcal{F}) \cap A \neq \emptyset$.
(ii) A $\mu$-space $(X, \mu)$ is $\mu$-semi-Lindelöf if and only if for every family $\mathcal{F} = \{F_\alpha : \alpha \in \Lambda\}$ of $\mu$-semi-closed sets having the property that for every countable subfamily $\mathcal{F}^i$ of $\mathcal{F}$, $(\bigcap \mathcal{F}^i) \neq \emptyset$, then $(\bigcap \mathcal{F}) \neq \emptyset$.

Definition 2.9. [35] A filter base $\mathcal{F}$ on a nonempty set $X$ is called a strong filter base on $X$ if whenever $\mathcal{F}^i$ is a countable subcollection of $\mathcal{F}$, there exists $F \in \mathcal{F}$ such that $F \subset \bigcap \mathcal{F}^i$.

Definition 2.10. [35] A strong filter base $\mathcal{F}$ on a nonempty set $X$ is called a maximal strong filter base on $X$ if whenever $\mathcal{H}$ is a strong filter base on $X$ with $\mathcal{F} \subset \mathcal{H}$, then $\mathcal{F} = \mathcal{H}$.

Proposition 2.11. [35] Every strong filter base $\mathcal{F}$ on a nonempty set $X$ is contained in a maximal strong filter base on $X$.

Definition 2.12. A filter base $\mathcal{F}$ on a $\mu$-space $(X, \mu)$ is said to $\mu_\alpha$-converge to a point $x \in X$ if for each $\mu$-semi-open subset $U$ of $X$ such that $x \in U$, there exists $F \in \mathcal{F}$ such that $F \subset U$. $\mathcal{F}$ is said to $\mu_\alpha$-accumulate at $x \in X$ if $U \cap F \neq \emptyset$ for every $F \in \mathcal{F}$ and for every $\mu$-semi-open subset $U$ of $X$ such that $x \in U$.

Proposition 2.13. Let $\mathcal{F}$ be a filter base on a $\mu$-space $(X, \mu)$ and $x \in X$. Then
(i) If $\mathcal{F}$ $\mu_\alpha$-converges to $x$, then $\mathcal{F}$ $\mu_\alpha$-accumulates at $x$;
(ii) If $\mathcal{F}$ is a maximal filter base (maximal strong filter base), then $\mathcal{F}$ $\mu_\alpha$-converges to $x$ if and only if $\mathcal{F}$ $\mu_\alpha$-accumulates at $x$.

Proposition 2.14. For a subset $A$ of a $\mu$-space $(X, \mu)$, the following are equivalent:
(i) $A$ is $\mu$-semi-Lindelöf;
(ii) Every maximal strong filter base on $X$, each of whose members meets $A$, $\mu_\alpha$-converges to some point of $A$;
(iii) Every strong filter base on $X$, each of whose members meets $A$, $\mu_\alpha$-accumulates at some point of $A$.

Proof. (i) $\Rightarrow$ (ii): Let $\mathcal{F}$ be a maximal strong filter base on $X$, each of whose members meets $A$, such that $\mathcal{F}$ does not $\mu_\alpha$-converge to any point of $A$. Since $\mathcal{F}$ is maximal, it follows from Proposition 2.13(ii) that $\mathcal{F}$ does not $\mu_\alpha$-accumulate at any point of $A$. Thus, for each $x \in A$, there exist $F_x \in \mathcal{F}$ and a $\mu$-semi-open subset $U_x$ of $X$ such that $x \in U_x$ and $U_x \cap F_x = \emptyset$. But $A$ is $\mu$-semi-Lindelöf, so there exist $x_1, x_2, x_3, \ldots \in X$ such that $A \subset \bigcup_{i=1}^{\infty} U_{x_i}$. Since $\mathcal{F}$ is a strong filter base on $X$, there exists $F \in \mathcal{F}$ such that $F \subset \bigcap_{i=1}^{\infty} F_{x_i}$, but $U_{x_i} \cap F_{x_i} = \emptyset$ for each $i \in \{1, 2, 3, \ldots\}$, so $U_{x_i} \cap F = \emptyset$ for each $i \in \{1, 2, 3, \ldots\}$, i.e., $\emptyset = \left(\bigcup_{i=1}^{\infty} U_{x_i}\right) \cap F \cap A \cap F$, a contradiction.

(ii) $\Rightarrow$ (iii): Let $\mathcal{F}$ be a strong filter base on $X$, each of whose members meets $A$. Then $\mathcal{F}^A = \{F \cap A : F \in \mathcal{F}\}$ is a strong filter base on $X$. Thus by Proposition 2.11, $\mathcal{F}^A$ is contained in a maximal strong filter base $\mathcal{H}$ on $X$, each of whose members meets $A$. By (ii), $\mathcal{H}$ $\mu_\alpha$-converges to some point $x$ of $A$, thus by Proposition 2.13(i), $\mathcal{H}$ $\mu_\alpha$-accumulates at $x$, but $\mathcal{F}^A \subset \mathcal{H}$, so $\mathcal{F}^A$ $\mu_\alpha$-accumulates at $x$. Hence, $\mathcal{F}$ $\mu_\alpha$-accumulates at $x$.

(iii) $\Rightarrow$ (i): Suppose that $A$ is not $\mu$-semi-Lindelöf. Then by Proposition 2.8(i), there exists a cover $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ of $A$ by $\mu$-semi-open subsets of $X$ such that for any countable subset $\Lambda_0$ of $\Lambda$,

$$
(\bigcap \{X \setminus U_\alpha : \alpha \in \Lambda_0\}) \cap A \neq \emptyset.
$$

For each countable subset $\Lambda_0$ of $\Lambda$, let

$$
F_{\Lambda_0} = (\bigcap \{X \setminus U_\alpha : \alpha \in \Lambda_0\}) \cap A.
$$

Then $\mathcal{F} = \{F_{\Lambda_0} : \Lambda_0 \text{ is a countable subset of } \Lambda\}$ is a strong filter base on $X$, each of whose members meets $A$. Thus by (iii), $\mathcal{F}$ $\mu_\alpha$-accumulates at some point $x$ of $A$. Since $\mathcal{U}$ is a cover of $A$, there exists $a_0 \in \Lambda$ such that
x ∈ Uα, but $\mathcal{F}$ $\mu_\alpha$-accumulates at x and $U_0$ is $\mu$-semi-open, so $U_0 \cap F \neq \emptyset$ for every $F \in \mathcal{F}$. Let $F = (X \setminus U_0) \cap A$. Then $F \in \mathcal{F}$ and thus, $U_0 \cap (X \setminus U_0) \cap A \neq \emptyset$, a contradiction. □

Corollary 2.15. For a $\mu$-space $(X, \mu)$, the following are equivalent:
(i) $X$ is $\mu$-semi-Lindelöf;
(ii) Every maximal strong filter base on $X$ $\mu_\sigma$-converges to some point of $X$;
(iii) Every strong filter base on $X$ $\mu_\alpha$-accumulates at some point of $X$.

3 Subspaces

This section is mainly devoted to discuss some statements in [16] and [33] concerning subspaces.

Definition 3.1. [31] Let $A$ be a nonempty subset of a space $(X, \mu)$. The generalized subspace topology on $A$ is the collection $\{U \cap A : U \in \mu\}$ and will be denoted by $\mu_A$. The generalized subspace $A$ is the GTS $(A, \mu_A)$.

Remark 3.2. [31] Let $A$ be a nonempty subset of a space $(X, \mu)$. Then it is easy to see that (i) If $(X, \mu)$ is a $\mu$-space, then $(A, \mu_A)$ is a $\mu_A$-space;
(ii) A subset $B$ of $A$ is $\mu_A$-closed if and only if $B = F \cap A$, for some $\mu$-closed set $F$.

Proposition 3.3. [33] Let $(X, \tau)$ be a topological space, and $A \subset B \subset X$, where $B$ is preopen. Then $A$ is semi-open (resp. semi-closed) in $B$ if and only if $A = S \cap B$, for some semi-open (resp. semi-closed) set $S$.

Proposition 3.4. ([33, Corollary 2.6], sufficiency of [33, Corollary 2.8]) Let $A$ be a preopen subset of a topological space $X$. If $A$ is semi-compact (resp. semi-Lindelöf) in $X$, then $A$ is a semi-compact (resp. semi-Lindelöf) subspace.

Proposition 3.5. [16, Theorem 2.7] Let $A, B$ be subsets of a $\mu$-space $(X, \mu)$ with $A \subset B$. If $A$ is $\mu$-semi-compact (resp. $\mu$-semi-Lindelöf), then $A$ is $\mu_B$-semi-compact (resp. $\mu_B$-semi-Lindelöf).

In the proof of Proposition 3.5 [16, Theorem 2.7], the author used without proof the following: If $(X, \mu)$ is a space, and $A \subset B \subset X$, where $A$ is $\mu_B$-semi-open, then $A = S \cap B$, for some $\mu$-semi-open set $S$. For a matter of convenience and importance, we will prove this fact, to proceed, we introduce the following two lemmas.

Lemma 3.6. [36, Proposition 3.8] Let $(X, \mu)$ be a space, and $A \subset X$. Then $A$ is $\mu$-semi-open if and only if $U \subset A \subset c_\mu(U)$, for some $\mu$-open set $U$.

Lemma 3.7. [30, Lemma 3.11] Let $(X, \mu)$ be a space, and $A \subset B \subset X$. Then $c_\mu(A) = c_\mu(A) \cap B$.

Proposition 3.8. Let $(X, \mu)$ be a space, and $A \subset B \subset X$. Then
(i) If $A$ is $\mu_B$-semi-open (resp. $\mu_B$-semi-closed), then $A = S \cap B$, for some $\mu$-semi-open (resp. $\mu$-semi-closed) set $S$;
(ii) If $A$ is $\mu$-open (resp. $\mu$-closed), then $A$ is $\mu_B$-open (resp. $\mu_B$-closed);
(iii) If $A$ is $\mu$-semi-open, then $A$ is $\mu_B$-semi-open.

Proof.
(i) Let $A$ be $\mu_B$-semi-open. Then it follows from Lemmas 3.6 and 3.7 that there exists a $\mu$-open set $U$ such that $U \cap B \subset A \subset c_\mu(U \cap B) \cap B \subset c_\mu(U)$. 


Let $S = U \cup A$. Since $A \subset c_p(U)$, then clearly,

$$U \subset S \subset c_p(U).$$

Thus by Lemma 3.6, $S$ is $\mu$-semi-open. Since $U \cap B \subset A$,

$$S \cap B = (U \cap B) \cup A = A.$$

Now if $A$ be $\mu_B$-semi-closed, then $B \backslash A$ is $\mu_B$-semi-open, and thus, $B \backslash A = S \cap B$, for some $\mu$-semi-open set $S$. Hence, $A = (X \backslash S) \cap B$, and $X \backslash S$ is $\mu$-semi-closed.

(ii) This is clear from Definition 3.1 and Remark 3.2(ii) since $A = A \cap B$.

(iii) Since $A$ is $\mu$-semi-open, it follows from Lemma 3.6 that there exists a $\mu$-open $U$ such that

$$U \subset A \subset c_p(U).$$

But $A \subset B$, so

$$U \subset A \subset c_p(U) \cap B.$$

Since $U$ is $\mu$-open, and $U \subset B$, so by (ii), $U$ is $\mu_B$-open. Also by Lemma 3.7, $c_p(U) \cap B = c_{\mu_B}(U)$. Thus by Lemma 3.6, $A$ is $\mu_B$-semi-open. \qed

**Remark 3.9.** From Proposition 3.8(i), we observe that the condition “preopen” is not essential for the necessity of Proposition 3.3.

**Corollary 3.10.** [16, Corollary 2.8] Let $A$ be a subset of a $\mu$-space $(X, \mu)$. If $A$ is $\mu$-semi-compact (resp. $\mu$-semi-Lindelöf), then $A$ is $\mu_A$-semi-compact (resp. $\mu_A$-semi-Lindelöf).

**Corollary 3.11.** Let $A$ be a subset of a topological space $X$. If $A$ is semi-compact (resp. semi-Lindelöf) in $X$, then $A$ is a semi-compact (resp. semi-Lindelöf) subspace.

**Remark 3.12.** From Corollary 3.11, we observe that the condition “preopen” of Proposition 3.4 is not essential.

**Proposition 3.13.** (necessity of [33, Corollary 2.8]) Let $A$ be a preopen subset of a topological space $X$. If $A$ is a semi-compact (resp. semi-Lindelöf) subspace, then $A$ is semi-compact (resp. semi-Lindelöf) in $X$.

The following example shows that the condition “preopen” in Proposition 3.13 is essential.

**Example 3.14.** Consider the space of Example 2.6, that is, the space $(X, \tau)$, where $X$ is an uncountable set, and $\tau = \{X, \emptyset, \{p\}\}$, where $p \in X$. Then $A = X \backslash \{p\}$ is not $\tau$-preopen because $\overline{A} = A$, and thus, $\text{Int} A = \text{Int} A = \emptyset$. Therefore, $A \notin \text{Int} A$. Now, if $x \in A$, then $A_x = \{p, x\}$ is $\tau$-semi-open. Thus, the collection $\mathcal{A} = \{A_x : x \in A\}$ is a cover of $A$ by $\tau$-semi-open sets and $\mathcal{A}$ has no countable subcover of $A$. Hence, $A$ is not semi-compact (semi-Lindelöf) in $X$. Now, the subspace topology $\tau_A = \{A, \emptyset\}$, therefore, the only $\tau_A$-semi-open sets are $A, \emptyset$. Hence, $A$ is a semi-compact (semi-Lindelöf) subspace.

(Sufficiency of [16, Theorem 2.9]) Let $A, B$ be subsets of a $\mu$-space $(X, \mu)$ with $A \subset B$. If $A$ is $\mu_B$-semi-compact (resp. $\mu_B$-semi-Lindelöf), then $A$ is $\mu$-semi-compact (resp. $\mu$-semi-Lindelöf).

We point out here to an error in the proof of the sufficiency of [16, Theorem 2.9], that is, the use of that if $(X, \mu)$ is a space, and $S \subset X$, where $S$ is $\mu$-semi-open, then $S \cap B$ is $\mu_B$-semi-open, where $B \subset X$, actually, this need not be true (even for topological spaces) as the following two easy examples tell.

**Example 3.15.** Let $X = \{1, 2, 3, 4\}$ and $\mu = \{X, \emptyset, \{1, 2, 3\}, \{3, 4\}\}$. Then $\mu$ is a $\text{GT}$ on $X$. If $S = \{1, 3, 4\}$, then $S$ is $\mu$-semi-open. Let $B = \{1, 2\}$. Then $\mu_B = \{B, \emptyset\}$, and $S \cap B = \{1\}$ is not $\mu_B$-semi-open.
Example 3.16. Let $X = \{1, 2, 3\}$ and $\tau = \{X, \emptyset, \{1\}\}$. Then $S = \{1, 2\}$ is $\tau$-semi-open. Let $B = \{2, 3\}$. Then $\tau_B = \{B, \emptyset\}$, and $S \cap B = \{2\}$ is not $\tau_B$-semi-open.

Remark 3.17. We observe from Example 3.16 that the condition "preopen" is essential for the sufficiency of Proposition 3.3 (note that $B$ is not preopen as $B \not\subseteq \text{Int } B = \emptyset$).

Concerning the sufficiency of Proposition 3.3, it might be convenient to raise the following question.

**Question.** Let $(X, \mu)$ be a GTS, where $S$ is $\mu$-semi-open and $B$ is $\mu$-preopen. Is $S \cap B$ necessarily $\mu_B$-semi-open?

(Necessity of [16, Corollary 2.10]) Let $A$ be a subset of a $\mu$-space $(X, \mu)$. If $A$ is $\mu_A$-semi-compact (resp. $\mu_A$-semi-Lindelöf), then $A$ is $\mu$-semi-compact (resp. $\mu$-semi-Lindelöf).

We observe that since the condition "preopen" in Proposition 3.13 is essential, the sufficiency of [16, Theorem 2.9] and the necessity of [16, Corollary 2.10] are not correct.

The following proposition includes a correction of the sufficiency of [16, Theorem 2.9].

**Proposition 3.18.** Let $B$ be a nonempty subset of a $\mu$-space $(X, \mu)$ and $A \subseteq B$. Then $A$ is $\mu$-semi-compact (resp. $\mu$-semi-Lindelöf) if and only if $A$ is $(\sigma(\mu))_B$-compact (resp. $(\sigma(\mu))_B$-Lindelöf).

**Proof.** We will see the case of $\mu$-semi-Lindelöf, the other case is similar.

**Necessity.** Observe first by Remark 1.4(i) that since $(X, \mu)$ is a $\mu$-space, $(X, \sigma(\mu))$ is a $\sigma(\mu)$-space, thus by Remark 3.2(i), $(B, (\sigma(\mu))_B)$ is a $(\sigma(\mu))_B$-space. Suppose that $\mathcal{A} = \{A_\alpha : \alpha \in \Lambda\}$ is a cover of $A$ by $(\sigma(\mu))_B$-open sets. Then $A_\alpha = S_\alpha \cap B$, where $S_\alpha$ is $\sigma(\mu)$-open for each $\alpha \in \Lambda$. Thus, $S = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of $A$ by $\mu$-semi-open sets, but $A$ is $\mu$-semi-Lindelöf, so there exist $a_1, a_2, a_3, \ldots \in \Lambda$ such that $A \subseteq \bigcup_1^\infty S_{a_\alpha}$, and thus $A \subseteq \bigcup_1^\infty (S_{a_\alpha} \cap B) = \bigcup_1^\infty A_{a_\alpha}$. Hence, $A$ is $(\sigma(\mu))_B$-Lindelöf.

**Sufficiency.** Suppose that $S = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of $A$ by $\mu$-semi-open sets. Then $\mathcal{A} = \{\emptyset : \alpha \in \Lambda\}$ is a $(\sigma(\mu))_B$-open cover of $A$, but $A$ is $(\sigma(\mu))_B$-Lindelöf, so there exist $a_1, a_2, a_3, \ldots \in \Lambda$ such that $A \subseteq \bigcup_1^\infty (S_{a_\alpha} \cap B) \subset \bigcup_1^\infty S_{a_\alpha}$. Hence, $A$ is $\mu$-semi-Lindelöf.

The following corollary includes a correction of the necessity of [16, Corollary 2.10].

**Corollary 3.19.** Let $A$ be a nonempty subset of a $\mu$-space $(X, \mu)$. Then $A$ is $\mu$-semi-compact (resp. $\mu$-semi-Lindelöf) if and only if $A$ is $(\sigma(\mu))_A$-compact (resp. $(\sigma(\mu))_A$-Lindelöf).

## 4 $\omega_\mu$-semi-open sets

**Definition 4.1.** Let $(X, \mu)$ be a space and $A$ be a subset of $X$. Then $A$ is called $\omega_\mu$-semi-open if whenever $x \in A$, there exists a $\mu$-semi-open set $U_x$ containing $x$ such that $U_x \setminus A$ is countable. $A$ is called $\omega_\mu$-semi-closed if $X \setminus A$ is $\omega_\mu$-semi-open. The collection of all $\omega_\mu$-semi-open subsets of $X$ will be denoted by $\omega_\mu(X)$.

**Proposition 4.2.** Let $(X, \mu)$ be a space and $A$ be a subset of $X$. Then $A$ is $\omega_\mu$-semi-open if and only if whenever $x \in A$, there exists a $\mu$-semi-open set $U_x$ containing $x$ and a countable subset $C_x$ of $X$ such that $U_x \setminus C_x \subset A$.

**Proof.** **Necessity.** Let $A$ be $\omega_\mu$-semi-open and $x \in A$. Then there exists a $\mu$-semi-open set $U_x$ containing $x$ such that $U_x \setminus A$ is countable. Let $C_x = U_x \setminus A$. Then $U_x \setminus C_x \subset A$.

**Sufficiency.** Let $x \in A$. Then by assumption, there exists a $\mu$-semi-open set $U_x$ containing $x$ and a countable subset $C_x$ of $X$ such that $U_x \setminus C_x \subset A$. Thus, $U_x \setminus A \subset C_x$, and therefore, $U_x \setminus A$ is countable. Hence, $A$ is $\omega_\mu$-semi-open.
Corollary 4.3. Let \((X, \mu)\) be a space and \(A\) be an \(\omega_\mu\)-semi-closed subset of \(X\). Then \(A \subset B \cup C\) for some \(\mu\)-semi-closed subset \(B\) of \(X\) and some countable subset \(C\) of \(X\).

Proof. Let \(A\) be an \(\omega_\mu\)-semi-closed subset of \(X\). Then \(X \setminus A\) is \(\omega_\mu\)-open. If \(X \setminus A = \emptyset\), choose \(B = X\) and \(C = \emptyset\). If \(X \setminus A \neq \emptyset\), choose \(x \in X \setminus A\). Thus by Proposition 4.2, there exists a \(\mu\)-semi-open set \(U_x\) containing \(x\) and a countable subset \(C_x\) of \(X\) such that \(U_x \setminus C_x \subset X \setminus A\). Therefore, \(A \subset (X \setminus U_x) \cup C_x\). Let \(B = X \setminus U_x\) and \(C = C_x\). Then \(B\) is \(\mu\)-semi-closed, and \(A \subset B \cup C\). □

Proposition 4.4.
(i) If \((X, \mu)\) is a space, then \((X, \omega_\mu)\) is a space.
(ii) If \((X, \mu)\) is a \(\mu\)-space, then \((X, \omega_\mu)\) is an \(\omega_\mu\)-space.
(iii) If \(A\) is a \(\mu\)-semi-open subset of a space \((X, \mu)\), then \(A\) is \(\omega_\mu\)-semi-open, that is, \(\sigma(\mu) \subset \omega_\mu\).
(iv) Let \(A\) be a subset of a space \((X, \mu)\). Then \(A\) is \(\omega_\mu\)-semi-open if and only if \(A\) is \(\omega_\mu\)-semi-open, that is, \(\omega_\mu = \omega_\mu\).
(v) Let \(A\) be a subset of a space \((X, \mu)\). Then \(A\) is \(\omega_\mu\)-semi-open if and only if \(A\) is \(\omega_\mu\)-semi-open.
(vi) Let \(A\) be a countable subset of a \(\mu\)-space \((X, \mu)\). Then \(A\) is \(\omega_\mu\)-semi-open.

Proof.
(i) Clearly, \(\emptyset\) is \(\omega_\mu\)-semi-open, that is, \(\emptyset \subset \omega_\mu\). Now let \(U_a\) be \(\omega_\mu\)-semi-open for each \(a \in A\) and let \(x \in \bigcup_{a \in A} U_a\). Then \(x \in U_{a_0}\) for some \(a_0 \in A\). Since \(U_{a_0}\) is \(\omega_\mu\)-semi-open, there exists a \(\mu\)-semi-open set \(V_{a_0}\) containing \(x\) such that \(V_{a_0} \setminus U_{a_0}\) is countable. Thus, \(V_{a_0} \setminus \bigcup_{a \in A} U_a\) is countable. Hence, \(V_{a_0} \setminus \bigcup_{a \in A} U_a\) is \(\omega_\mu\)-semi-open, that is, \(\bigcup_{a \in A} U_a \subset \omega_\mu\). Hence, \((X, \omega_\mu)\) is a space.
(ii) As \((X, \mu)\) is a \(\mu\)-space, \(X \in \mu\). Thus, \(X\) is \(\mu\)-semi-open. Let \(x \in X\). Then \(X \setminus x = \emptyset\) is countable. Thus, \(X\) is \(\omega_\mu\)-semi-open, that is, \(X \in \omega_\mu\). Hence, \((X, \omega_\mu)\) is an \(\omega_\mu\)-space.
(iii) Let \(A\) be \(\mu\)-semi-open and \(x \in A\). Then \(A \setminus x = \emptyset\) is countable. Thus, \(A\) is \(\omega_\mu\)-semi-open.
(iv) This follows from Proposition 1.3(iii).
(v) Suppose that \(A\) is \(\omega_\mu\)-semi-open and let \(x \in A\). Then there exists a \(\mu\)-semi-open set \(U\) containing \(x\) such that \(U \setminus A\) is countable. By (iii), \(U\) is \(\omega_\mu\)-semi-open. By (iv), \(U\) is \(\omega_\mu\)-semi-open. Thus, \(A\) is \(\omega_\mu\)-semi-open.
Conversely, suppose that \(A\) is \(\omega_\mu\)-semi-open and let \(x \in A\). Then there exists an \(\omega_\mu\)-semi-open set \(U\) containing \(x\) such that \(U \setminus A\) is countable. By (iv), \(U\) is \(\omega_\mu\)-semi-open. Thus, there exists a \(\mu\)-semi-open set \(V\) containing \(x\) such that \(V \setminus U\) is countable. Now \(V \setminus U = ((V \setminus U) \setminus A) \cup ((U \setminus V) \setminus A)\). As \(U \setminus A\) and \(V \setminus U\) are both countable, \((V \setminus U) \setminus A\) and \((U \setminus V) \setminus A\) are both countable, and thus, \(V \setminus A\) is countable. Hence, \(A\) is \(\omega_\mu\)-semi-open.
(vi) Let \(x \in X \setminus A\). As \((X, \mu)\) is a \(\mu\)-space, \(x \in \mu\), and thus, \(X\) is \(\mu\)-semi-open. Now \(X \setminus (X \setminus A) = A\) is countable by assumption. Thus, \(X \setminus A\) is \(\omega_\mu\)-semi-open, that is, \(A\) is \(\omega_\mu\)-semi-closed. □

The following easy example shows that the converse of Proposition 4.4(iii) need not be correct in general even for topological spaces.

Example 4.5. Let \(X\) be an uncountable set and \(\mu\) be the indiscrete topology on \(X\). If \(A\) is a nonempty countable subset of \(X\), then by Proposition 4.4(vi), \(X \setminus A\) is \(\omega_\mu\)-semi-open; however, \(X \setminus A\) is not \(\mu\)-semi-open.

5 Applications

Proposition 5.1. Let \(A\) be an \(\omega_\mu\)-semi-open subset of a \(\mu\)-space \((X, \mu)\). If \(A\) is \(\mu\)-semi-Lindelöf, then \(A = B \setminus C\) for some \(\mu\)-semi-open subset \(B\) of \(X\) and some countable subset \(C\) of \(X\).
Proof. Since $A$ is $\omega_\mu$-semi-open, it follows that for each $x \in A$ there exists a $\mu$-semi-open set $U_x$ containing $x$ such that $U_x \setminus A$ is countable, but $A$ is $\mu$-semi-Lindelöf, so there exist $x_1, x_2, x_3, \ldots \in A$ such that $A \subseteq \bigcup_{i=1}^{\infty} U_{x_i}$. Thus, $A = \bigcup_{i=1}^{\infty} U_{x_i} \setminus \bigcup_{i=1}^{\infty} (U_{x_i} \setminus A)$. Let $B = \bigcup_{i=1}^{\infty} U_{x_i}$ and $C = \bigcup_{i=1}^{\infty} (U_{x_i} \setminus A)$. Then $B$ is $\mu$-semi-open and $C$ is countable.

**Proposition 5.2.** Let $A$ be a subset of a $\mu$-space $(X, \mu)$. Then $A$ is $\mu$-semi-Lindelöf if and only if $A$ is $\omega_0(\mu)$-Lindelöf.

**Proof. Necessity.** Suppose that $S = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of $A$ by $\omega_0(\mu)$-open sets. Then for each $x \in A$, $x \in S_{\alpha(x)}$, for some $\alpha(x) \in \Lambda$. Thus, there exists a $\mu$-semi-open set $U_x$ containing $x$ such that $U_x \setminus S_{\alpha(x)}$ is countable. Now $\mathcal{A} = \{U_x : x \in A\}$ is a cover of $A$ by $\mu$-semi-open sets, but $A$ is $\mu$-semi-Lindelöf, so there exist $x_1, x_2, x_3, \ldots \in A$ such that $A \subseteq \bigcup_{i=1}^{\infty} (U_{x_i} \setminus A) \subseteq \bigcup_{i=1}^{\infty} (U_{x_i} \setminus S_{\alpha(x_i)}) \subseteq \bigcup_{i=1}^{\infty} S_{\alpha(x_i)}$. Since $\bigcup_{i=1}^{\infty} (U_{x_i} \setminus S_{\alpha(x_i)})$ is a countable subset of $A$, it is covered by a countable subcollection $\mathcal{B}$ of $S$. Thus, $S$ has $\mathcal{B} \cup \{S_{\alpha(x)} : i \in \mathbb{N}\}$ as a countable subcover. Hence, $A$ is $\omega_0(\mu)$-Lindelöf.

**Sufficiency.** Follows from Proposition 4.4(iii).

**Proposition 5.3.** Let $A$ be a $\mu$-semi-Lindelöf subset of a $\mu$-space $(X, \mu)$ and $B$ be an $\omega_{\mu}$-semi-closed subset of $X$. Then $A \cap B$ is $\mu$-semi-Lindelöf. In particular, an $\omega_{\mu}$-semi-closed subset $A$ of a $\mu$-semi-Lindelöf $\mu$-space $(X, \mu)$ is $\mu$-semi-Lindelöf.

**Proof.** Suppose that $S = \{S_\alpha : \alpha \in \Lambda\}$ is a cover of $A \cap B$ by $\omega_{\mu}$-semi-open sets. Then $\mathcal{A} = \{S_\alpha : \alpha \in \Lambda\} \cup \{X \setminus B\}$ is a cover of $A$ by $\omega_{\mu}$-semi-open sets, but $A$ is $\mu$-semi-Lindelöf, so it follows from Proposition 5.2 that there exist $\alpha_1, \alpha_2, \alpha_3, \ldots \in \Lambda$ such that $A \subseteq \bigcup_{\alpha \in \Lambda} S_\alpha \cup (X \setminus B)$. Thus, $A \cap B \subseteq \bigcup_{\alpha \in \Lambda} S_\alpha \cap B \subseteq \bigcup_{\alpha \in \Lambda} S_\alpha$. Hence, $A \cap B$ is $\mu$-semi-Lindelöf.

**Corollary 5.4.** [16, Theorem 2.11] Let $A$ be a $\mu$-semi-Lindelöf subset of a $\mu$-space $(X, \mu)$ and $B$ be a $\mu$-semi-closed subset of $X$. Then $A \cap B$ is $\mu$-semi-Lindelöf. In particular, a $\mu$-semi-closed subset $A$ of a $\mu$-semi-Lindelöf $\mu$-space $(X, \mu)$ is $\mu$-semi-Lindelöf.

**Corollary 5.5.** For a $\mu$-space $(X, \mu)$, the following are equivalent:

(i) $X$ is $\mu$-semi-Lindelöf;

(ii) Every proper $\omega_0(\mu)$-semi-closed subset of $X$ is $\mu$-semi-Lindelöf;

(iii) Every proper $\mu$-semi-closed subset of $X$ is $\mu$-semi-Lindelöf.

**Proof.** (i) $\Rightarrow$ (ii): Follows from Proposition 5.3.

(ii) $\Rightarrow$ (iii): Follows from Proposition 4.4(iii).

(iii) $\Rightarrow$ (i): Let $\mathcal{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of $X$ by $\mu$-semi-open subsets of $X$. Choose $\alpha_0 \in \Lambda$ such that $U_{\alpha_0} \neq \emptyset$. Then $X \setminus U_{\alpha_0}$ is a proper $\mu$-semi-closed subset of $X$, thus by (iii), $X \setminus U_{\alpha_0}$ is $\mu$-semi-Lindelöf, so there exist $\alpha_1, \alpha_2, \alpha_3, \ldots \in \Lambda$ such that $X \setminus U_{\alpha_0} \subseteq \bigcup_{i=1}^{\infty} U_{\alpha_i}$ and thus $X = \bigcup_{i=0}^{\infty} U_{\alpha_i}$. Hence, $X$ is $\mu$-semi-Lindelöf.

**Definition 5.6.** [31] Let $(X_\alpha, \mu_\alpha)$ be a GTS for each $\alpha \in \Lambda$, where $\{X_\alpha : \alpha \in \Lambda\}$ is a disjoint family of sets. The collection $\mu$ of subsets of $\bigcup X_\alpha$ is defined as follows:

$$
\mu = \{ U \subset \bigcup X_\alpha : U \cap X_\alpha \in \mu_\alpha, \ \forall \alpha \in \Lambda \}.
$$

**Proposition 5.7.** [31] Let $(X_\alpha, \mu_\alpha)$ be a GTS for each $\alpha \in \Lambda$, where $\{X_\alpha : \alpha \in \Lambda\}$ is a disjoint family of sets, and let $\mu$ be as in Definition 5.6. Then $\mu$ is a GT on $\bigcup X_\alpha$. The GTS $(\bigcup X_\alpha, \mu)$ will be called the generalized topological sum of $X_\alpha$, $\alpha \in \Lambda$ and will be denoted by $\oplus X_\alpha$. 
Corollary 5.8. [31] Let \((\oplus X_\alpha, \mu)\) be the generalized topological sum of \((X_\alpha, \mu_\alpha), \alpha \in \Lambda\). Then a subset \(A\) of \(\oplus X_\alpha\) is \(\mu\)-closed if and only if \(A \cap X_\alpha\) is \(\mu_\alpha\)-closed for each \(\alpha\).

Remark 5.9. [31] Let \((X_\alpha, \mu_\alpha)\) be a GTS for each \(\alpha \in \Lambda\), and let \((\oplus X_\alpha, \mu)\) be the generalized topological sum of \((X_\alpha, \mu_\alpha), \alpha \in \Lambda\). Then it is easy to see the following:

(i) \(\bigcup \mu_\alpha \subseteq \mu\);

(ii) \(\mu_{X_\alpha} = \mu_\alpha\) for each \(\alpha \in \Lambda\).

Remark 5.10. [31] Let \((X_\alpha, \mu_\alpha)\) be a \(\mu_\alpha\)-space for each \(\alpha \in \Lambda\), and let \((\oplus X_\alpha, \mu)\) be the generalized topological sum of \((X_\alpha, \mu_\alpha), \alpha \in \Lambda\). Then it is easy to see the following:

(i) \((\oplus X_\alpha, \mu)\) is a \(\mu\)-space;

(ii) \(X_\alpha\) is \(\mu\)-open (\(\mu\)-closed) for each \(\alpha \in \Lambda\).

Remark 5.11. We observe from Remark 5.10(ii) that if \((X_\alpha, \mu_\alpha)\) is a \(\mu_\alpha\)-space for each \(\alpha \in \Lambda\), and if \((\oplus X_\alpha, \mu)\) is the generalized topological sum of \((X_\alpha, \mu_\alpha), \alpha \in \Lambda\), then \(X_\alpha\) is \(\mu\)-semi-open (\(\mu\)-semi-closed) for each \(\alpha \in \Lambda\).

Proposition 5.12. [16] Let \((X, \mu)\) be a \(\mu\)-space. Then the countable union of subsets of \(X\), each of which \(\mu\)-semi-Lindelöf, is \(\mu\)-semi-Lindelöf.

Proposition 5.13. Let \((X_\alpha, \mu_\alpha)\) be a \(\mu_\alpha\)-space for each \(\alpha \in \Lambda\), and let \((\oplus X_\alpha, \mu)\) be the generalized topological sum of \((X_\alpha, \mu_\alpha), \alpha \in \Lambda\). Then \(\oplus X_\alpha\) is \(\mu\)-semi-Lindelöf if and only if \(X_\alpha\) is \(\mu\)-semi-Lindelöf for each \(\alpha \in \Lambda\) and \(\Lambda\) is countable.

Proof. Necessity. By Remark 5.11, \(X_\alpha\) is \(\mu\)-semi-closed for each \(\alpha \in \Lambda\). Since \(\oplus X_\alpha\) is \(\mu\)-semi-Lindelöf, so by Corollary 5.4, \(X_\alpha\) is \(\mu\)-semi-Lindelöf for each \(\alpha \in \Lambda\). Also by Remark 5.11, \(X_\alpha\) is \(\mu\)-open for each \(\alpha \in \Lambda\), thus, \(\mathcal{A} = \{X_\alpha : \alpha \in \Lambda\}\) is a cover of \(\oplus X_\alpha\) by \(\mu\)-open sets, but \(\oplus X_\alpha\) is \(\mu\)-semi-Lindelöf, so, \(\Lambda\) is countable.

Sufficiency. Follows from Proposition 5.12. \(\square\)

6 Mapping properties

This section is mainly devoted to study several mapping properties of \(\mu\)-semi-Lindelöf sets in \(\mu\)-spaces.

Proposition 6.1. Let \(f : (X, \mu) \rightarrow (Y, \kappa)\) be a \((\omega_\alpha(\mu), \sigma(\kappa))\)-continuous function, where \((X, \mu)\) is a \(\mu\)-space and \((Y, \kappa)\) is a \(\kappa\)-space. If \(A\) is \(\mu\)-semi-Lindelöf, then \(f(A)\) is \(\kappa\)-semi-Lindelöf.

Proof. Suppose that \(S = \{X_\alpha : \alpha \in \Lambda\}\) is a cover of \(f(A)\) by \(\kappa\)-open sets. Then \(\mathcal{A} = \{f^{-1}(S_\alpha) : \alpha \in \Lambda\}\) is a cover of \(A\), but \(f\) is \((\omega_\alpha(\mu), \sigma(\kappa))\)-continuous, so \(f^{-1}(S_\alpha)\) is \(\omega_\mu\)-open for each \(\alpha \in \Lambda\). Since \(A\) is \(\mu\)-semi-Lindelöf, it follows from Proposition 5.2 that there exist \(a_1, \ldots, a_\lambda \in \Lambda\) such that \(A \subseteq \bigcup_{i=1}^{\lambda} f^{-1}(S_{a_i})\). Thus, \(f(A) \subseteq \bigcup_{i=1}^{\lambda} f^{-1}(S_{a_i}) \subseteq \bigcup_{i=1}^{\lambda} S_{a_i}\). Hence, \(f(A)\) is \(\kappa\)-semi-Lindelöf. \(\square\)

Corollary 6.2. Let \(f : (X, \mu) \rightarrow (Y, \kappa)\) be a \((\sigma(\mu), \sigma(\kappa))\)-continuous function, where \((X, \mu)\) is a \(\mu\)-space and \((Y, \kappa)\) is a \(\kappa\)-space. If \(A\) is \(\mu\)-semi-Lindelöf, then \(f(A)\) is \(\kappa\)-semi-Lindelöf.

Corollary 6.3. Let \(f : (X, \mu) \rightarrow (Y, \kappa)\) be a \((\sigma(\mu), \sigma(\kappa))\)-continuous surjection, where \((X, \mu)\) is a \(\mu\)-space and \((Y, \kappa)\) is a \(\kappa\)-space. If \(X\) is \(\mu\)-semi-Lindelöf, then \(Y\) is \(\kappa\)-semi-Lindelöf.

Proposition 6.4. Let \(f : (X, \mu) \rightarrow (Y, \kappa)\) be a \((\sigma(\mu), \omega_\alpha(\kappa))\)-closed function, where \((X, \mu)\) is a \(\mu\)-space and \((Y, \kappa)\) is a \(\kappa\)-space. If for each \(y \in Y\), \(f^{-1}(y)\) is \(\mu\)-semi-Lindelöf, then \(f^{-1}(A)\) is \(\mu\)-semi-Lindelöf whenever \(A\) is \(\kappa\)-semi-Lindelöf.
Proof. Suppose that $S = \{S_a : a \in A\}$ is a cover of $f^{-1}(A)$ by $\mu$-semi-open sets. Then it follows by assumption that for each $y \in A$, there exists a countable subcollection $S_y$ of $S$ such that $f^{-1}(y) \subset \bigcup S_y$. Let $V_y = \bigcup S_y$. Then $V_y$ is $\mu$-semi-open. Let $H_y = f^{-1}(V_y)$. Then $H_y$ is $\omega_k$-semi-open as $f$ is $(\sigma(\mu), \omega(\kappa))$-closed, also $y \in H_y$ for each $y \in A$ as $f^{-1}(y) \subset V_y$. Thus, $\mathcal{H} = \{H_y : y \in A\}$ is a cover of $A$ by $\omega_k$-semi-open sets, but $A$ is $\kappa$-semi-Lindelöf, so it follows from Proposition 5.2 that there exist $y_1, y_2, y_3, \ldots \in A$ such that $A \subset \bigcup_{i=1}^{\infty} H_{y_i}$. Thus, $f^{-1}(A) \subset \bigcup_{i=1}^{\infty} f^{-1}(H_{y_i}) \subset \bigcup_{i=1}^{\infty} V_{y_i}$. Since $S^\kappa$ is a countable subcollection of $S$ for each $i \in \mathbb{N}$, it follows that $\bigcup_{i=1}^{\infty} S^\kappa$ is a countable subcollection of $S$. Hence, $f^{-1}(A)$ is $\mu$-semi-Lindelöf.

Corollary 6.5. [16, Theorem 2.18] Let $f : (X, \mu) \to (Y, \kappa)$ be a $(\sigma(\mu), \sigma(\kappa))$-closed function, where $(X, \mu)$ is a $\mu$-space and $(Y, \kappa)$ is a $\kappa$-space. If for each $y \in Y$, $f^{-1}(y)$ is $\mu$-semi-Lindelöf, then $f^{-1}(A)$ is $\mu$-semi-Lindelöf whenever $A$ is $\kappa$-semi-Lindelöf.

Corollary 6.6. Let $f : (X, \mu) \to (Y, \kappa)$ be a $(\sigma(\mu), \sigma(\kappa))$-closed function, where $(X, \mu)$ is a $\mu$-space and $(Y, \kappa)$ is a $\kappa$-space. If for each $y \in Y$, $f^{-1}(y)$ is $\mu$-semi-Lindelöf, then $(X, \mu)$ is $\mu$-semi-Lindelöf whenever $(Y, \kappa)$ is $\kappa$-semi-Lindelöf.

Proposition 6.7. [31] Let $(X, \mu)$ and $(Y, \kappa)$ be GTSs, and let $\mathcal{U} = \{U \times V : U \in \mu, V \in \kappa\}$. Then $\mathcal{U}$ generates a $GT$ $\lambda$ on $X \times Y$, called the generalized product topology on $X \times Y$, that is,

$$\lambda = \{\text{all possible unions of members of } \mathcal{U}\}.$$

$\lambda$ will sometimes be denoted by $\mu \times \kappa$.

Remark 6.8. [31] Let $(X, \mu)$ and $(Y, \kappa)$ be GTSs, $\sigma$ be the generalized product topology on $X \times Y$, $A \subset X$, $B \subset Y$, and $K \subset X \times Y$. Then it is easy to see the following:

(i) $K$ is $\lambda$-open if and only if for each $(x, y) \in K$, there exist $U_x \in \mu$ and $V_y \in \kappa$ such that $(x, y) \in U_x \times V_y \subset K$;

(ii) $c_\kappa(A \times B) = c_\kappa(A) \times c_\kappa(B)$;

(iii) $i_\kappa(A \times B) = i_\kappa(A) \times i_\kappa(B)$;

(iv) $\lambda_{\kappa \times \kappa} = \mu_{\kappa} \times \kappa_{\kappa}$.

Remark 6.9. [31] Let $(X, \mu)$ be a $\mu$-space, $(Y, \kappa)$ be a $\kappa$-space, and $\lambda$ be the generalized product topology on $X \times Y$. Then it is clear that $(X \times Y, \lambda)$ is a $\lambda$-space.

Lemma 6.10. Let $(X, \mu)$ be a $\mu$-space, $(Y, \kappa)$ be a $\kappa$-space, and $\lambda$ be the generalized product topology on $X \times Y$. Then the projection $P_X : (X \times Y, \lambda) \to (X, \mu)$ (resp. $P_Y : (X \times Y, \lambda) \to (Y, \kappa)$) is $(\sigma(\lambda), \sigma(\mu))$-continuous (resp. $(\sigma(\lambda), \sigma(\kappa))$-continuous).

Proof. We will show that the projection $P_X : (X \times Y, \lambda) \to (X, \mu)$ is $(\sigma(\lambda), \sigma(\mu))$-continuous, the other case is similar. Let $A$ be a $\mu$-semi-open subset of $X$. Then $(P_X)^{-1}(A) = A \times Y$. We want to show that $A \times Y$ is $\lambda$-semi-open. Now by Remark 6.8, $c_\kappa(i_\kappa(A \times Y)) = c_\kappa(i_\kappa(A)) \times Y \supset A \times Y$. Thus, $A \times Y$ is $\lambda$-semi-open.

Corollary 6.11. Let $(X, \mu)$ be a $\mu$-space, $(Y, \kappa)$ be a $\kappa$-space, and $\lambda$ be the generalized product topology on $X \times Y$. If $X \times Y$ is $\lambda$-semi-Lindelöf, then $(X, \mu)$ is $\mu$-semi-Lindelöf and $(Y, \kappa)$ is $\kappa$-semi-Lindelöf.

Proof. Observe first by Remark 6.9 that since $(X, \mu)$ is a $\mu$-space and $(Y, \kappa)$ is a $\kappa$-space, $(X \times Y, \lambda)$ is a $\lambda$-space. The result follows from Corollary 6.3 and Lemma 6.10.

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