Dynamic cancellation of a cosmological constant and approach to the Minkowski vacuum

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The \( q \)-theory approach to the cosmological constant problem is reconsidered. The new observation is that the effective classical \( q \)-theory gets modified due to the backreaction of quantum-mechanical particle production by spacetime curvature. Furthermore, a Planck-scale cosmological constant is added to the potential term of the action density, in order to represent the effects from zero-point energies and phase transitions. The resulting dynamical equations of a spatially-flat Friedmann–Robertson–Walker universe are then found to give a steady approach to the Minkowski vacuum, with attractor behavior for a finite domain of initial boundary conditions on the fields. The approach to the Minkowski vacuum is slow and gives rise to an inflation-type increase of the particle horizon.

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1. Introduction

Several years ago, we proposed a particular approach to the cosmological constant problem\(^1\) whose motivation relies on thermodynamics and Lorentz invariance and which goes under the name of \( q \)-theory.\(^2\) The basic idea of \( q \)-theory is to give the proper macroscopic description of the Lorentz-invariant quantum vacuum where the gravitational effects of a (Planck-scale) cosmological constant \( \Lambda \) have been cancelled dynamically by appropriate microscopic degrees of freedom. In general, there are one or more of these vacuum variables (denoted by \( q \), with or without additional suffixes) to characterize the thermodynamics of this static physical system in equilibrium. Several realizations of \( q \)-theory have been given, but the most elegant is the one with
q arising from a four-form field strength $F$ (details and references are given below). The outstanding issue is the dynamics, namely, how the cosmological constant $\Lambda$ is cancelled dynamically and the equilibrium state approached.

In a follow-up paper we established the dynamic relaxation of the vacuum energy density to zero, provided the chemical potential $\mu$ has already the equilibrium value $\mu_0$ corresponding to the Minkowski vacuum. But, then, the cosmological constant problem is replaced by another problem namely, why does the constant $\mu$ have the “right” value $\mu_0$.

Here, we discuss how quantum effects can modify the classical $q$-theory and give rise to the decay of the vacuum energy density (i.e., decay of the effective chemical potential). Related work on vacuum energy decay has been presented in Refs. 6–13, but the feedback on $q$-theory has not been considered in detail.

The present paper is self-contained but somewhat short on the underlying ideas of $q$-theory. More details on the condensed-matter-physics motivation of $q$-theory can be found in a companion paper, which contains a general discussion of the issue of vacuum-energy relaxation.

Throughout, we use natural units with $c = 1$ and $\hbar = 1$, unless stated otherwise.

### 2. Four-form-field-strength realization of classical $q$-theory

Start by neglecting the energy exchange between vacuum and matter. Then, the dynamics is described by the following classical action:

\[
I = \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G(q)} + \epsilon(q) + L_{\text{SM}}(\psi) \right),
\]

\[
q^2 \equiv -\frac{1}{24} F_{\kappa\lambda\mu\nu} F^{\kappa\lambda\mu\nu}, \quad F_{\kappa\lambda\mu\nu} \equiv \nabla_{[\kappa} A_{\lambda\mu\nu]},
\]

\[
F_{\kappa\lambda\mu\nu} = q\sqrt{-g} \epsilon_{\kappa\lambda\mu\nu}, \quad F^{\kappa\lambda\mu\nu} = q\epsilon^{\kappa\lambda\mu\nu}/\sqrt{-g},
\]

where Eqs. (2.1b) and (2.1c) give a particular realization of the vacuum $q$-field in terms of the 4-form field strength $F$ from a three-form gauge field $A^{[3]}$. The symbol $\nabla_\mu$ in (2.1b) denotes the covariant derivative and a pair of square brackets around spacetime indices stands for complete anti-symmetrization. The symbol $\epsilon_{\kappa\lambda\mu\nu}$ in (2.1c) corresponds to the Levi–Civita symbol, which makes $q$ a pseudoscalar. Note that $q$ in the action (2.1a) is a pseudoscalar but not a fundamental pseudoscalar, $q = q(A, g)$. The fundamental fields of the theory considered are the gauge field $A_{\mu\nu}(x)$, the metric $g_{\mu\nu}(x)$, and the generic matter field $\psi(x)$.

In the action (2.1a), $L_{\text{SM}}(\psi)$ is the Lagrange density of the standard-model matter fields $\psi$. The parameters of the matter action may depend, in principle, on the vacuum variable $q$, $L_{\text{SM}} = L_{\text{SM}}(q, \psi)$. But, here, we neglect this $q$ dependence of the standard-model parameters and allow only for a $q$ dependence of the gravitational coupling, $G = G(q)$. 

The variation of the action (2.1a) over the three-form gauge field $A$ gives the generalized Maxwell equation for the four-form field strength $F$,

$$\nabla_\nu \left( \sqrt{-g} \frac{F^{\kappa\lambda\mu\nu}}{q} \left[ \frac{de(q)}{dq} + \frac{R}{16\pi} \frac{dG^{-1}(q)}{dq} \right] \right) = 0.$$  \hspace{1cm} (2.2)

In the spatially-flat ($k = 0$) Friedmann–Robertson–Walker (FRW) universe with comoving coordinates, this can be written as

$$\partial_t \left( \frac{de(q)}{dq} - \frac{3}{8\pi} \left[ \partial_t H + 2H^2 \right] \frac{dG^{-1}(q)}{dq} \right) = 0,$$  \hspace{1cm} (2.3)

for Ricci scalar $R = -6 \left( \partial_t H + 2H^2 \right)$ from our curvature conventions.\textsuperscript{3} Solving (2.3) gives the integration constant $\mu$,

$$\frac{de(q)}{dq} - \frac{3}{8\pi} \left[ \partial_t H + 2H^2 \right] \frac{dG^{-1}(q)}{dq} = \mu.$$  \hspace{1cm} (2.4)

Based on the thermodynamic discussion of Ref.\textsuperscript{2} the integration constant $\mu$ may be called the “chemical potential,” where the chemical potential $\mu$ is conjugate to the conserved quantity $q$ in flat spacetime. See Ref.\textsuperscript{5} for further discussion on the different roles of fundamental and “conserved” scalars for the cosmological constant problem.

3. Energy exchange between matter and vacuum

We are, now, interested in the quantum-dissipative energy exchange between vacuum and matter. In this case, the chemical potential $\mu$ is no longer constant and can relax in the evolving universe. We replace Eq. (2.3) by

$$\partial_t \left( \frac{de(q)}{dq} - \frac{3}{8\pi} \left[ \partial_t H + 2H^2 \right] \frac{dG^{-1}(q)}{dq} \right) = S,$$  \hspace{1cm} (3.1)

where $S$ is a source term. In a companion paper\textsuperscript{14} we use the following Ansatz:

$$S = \Gamma_q \left( \partial_t q \right)^2 + \Gamma_H \left( \partial_t H \right)^2,$$  \hspace{1cm} (3.2)

with nonnegative decay constants $\Gamma_q$ and $\Gamma_H$. This particular Ansatz for $S$ does not discriminate between de-Sitter and Minkowski vacua, and the crucial question is whether or not the Minkowski vacuum is dynamically preferred. In Ref.\textsuperscript{14} the answer is found to be negative, and here we shall use another Ansatz for $S$ which does prefer the Minkowski vacuum; see Sec.\textsuperscript{5}.

The generalized Einstein equation is still valid and is obtained by variation of the action (2.1a) over the metric $g_{\mu\nu}$,

$$\frac{1}{8\pi G(q)} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) + \frac{1}{16\pi} q \frac{dG^{-1}(q)}{dq} R g_{\mu\nu} + \frac{1}{8\pi} \left( \nabla_\mu \nabla_\nu G^{-1}(q) - g_{\mu\nu} \Box G^{-1}(q) \right)$$

$$- \left( \epsilon(q) - q \frac{de(q)}{dq} \right) g_{\mu\nu} + T^{SM}_{\mu\nu}(\psi) = 0,$$  \hspace{1cm} (3.3)
where $\Box$ is the invariant d’Alembertian and $T^{\text{SM}}_{\mu\nu}$ is the energy-momentum tensor of the standard-model matter fields $\psi$ (without dependence on $q$ as discussed in Sec. 2).

4. Constant–$G$ case

For the present article, it suffices to consider the simplest possible Ansatz for the function $G(q)$, namely a constant function,

$$G(q) = G_N,$$  \hspace{1cm} (4.1)

with $G_N$ Newton’s gravitational constant. The generalized Einstein and Maxwell equations from Secs. 2 and 3 become

$$\frac{1}{8\pi G_N} \left( R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \right) = \rho_V(q) g_{\mu\nu} - T^{\text{SM}}_{\mu\nu},$$ \hspace{1cm} (4.2)

$$q \partial_t \left( \frac{d\epsilon(q)}{dq} \right) = -\partial_t \rho_V(q) = q S,$$ \hspace{1cm} (4.3)

where the source term of the gravitational equation (4.2) contains the following vacuum energy density:

$$\rho_V(q) \equiv \epsilon(q) - q \frac{d\epsilon(q)}{dq}. \hspace{1cm} (4.4)$$

Note that the definition (4.4) also explains the first equality in (4.3).

Consider the spatially-flat FRW universe with a single homogeneous perfect-fluid matter component $(M)$ and a homogeneous $q$-field component $(V)$ with pressure $P_V = -\rho_V$. The Einstein equation then gives the following Friedmann equations:

$$3H^2 = 8\pi G_N \left( \rho_V + \rho_M \right),$$ \hspace{1cm} (4.5a)

$$2\partial_t H = -8\pi G_N \left( \rho_M + P_M \right),$$ \hspace{1cm} (4.5b)

where the vacuum contribution to the right-hand side of (4.5b) cancels out, because $\rho_V + P_V = 0$. The evolution equations for the vacuum and matter energy densities are

$$\partial_t \rho_V = -q S,$$ \hspace{1cm} (4.6a)

$$\partial_t \rho_M + 3H \left( P_M + \rho_M \right) = +q S.$$ \hspace{1cm} (4.6b)

Now, assume the matter to have a constant equation-of-state parameter,

$$w_M \equiv \rho_M / P_M = \text{const.} \hspace{1cm} (4.7)$$

Then, the following two ordinary differential equations (ODEs) suffice to determine $q(t)$ and $H(t)$ in a spatially-flat FRW universe:

$$\partial_t \left( \frac{d\epsilon(q)}{dq} \right) = S,$$ \hspace{1cm} (4.8a)
\[ \frac{2}{1 + w_M} \partial_t H + 3 H^2 = 8\pi G_N \rho_V(q), \quad (4.8b) \]

with \( \rho_V(q) \) from Eq. (4.1) and \( S \) from Eq. (3.2) or otherwise.

The companion paper focuses on the source term (3.2), which keeps de-Sitter spacetime stable, whereas the present paper considers another type of source term which distinguishes Minkowski spacetime. Most importantly, we do not wish to take some ad hoc source term but will get a term with a clear physical origin.

5. Particle production and backreaction

Consider the production of massless particles (e.g., gravitons) by the curved spacetime of an expanding spatially-flat FRW universe with appropriate boundary conditions on the matter fields and background. Then, the increase of particle number density is given by

\[ \partial_t n_M \sim R^2 \sim 36 \left( \partial_t H + 2H^2 \right)^2, \quad (5.1a) \]

for Ricci scalar \( R = -6 \left( \partial_t H + 2H^2 \right) \). The typical particle energy \( (E = \hbar \omega \geq 0) \) is determined by the Hubble expansion rate (cf. the discussion on p. 62 of Ref. 24),

\[ E_M \sim \hbar |H|, \quad (5.1b) \]

with \( \hbar \) temporarily reinstated. From (5.1a) and (5.1b), the standard adiabatic change of the particle energy density then gets modified by a source term on the right-hand side of the evolution equation,

\[ \partial_t \rho_M + 3 H (1 + w_M) \rho_M \sim \hbar |H| R^2, \quad (5.2) \]

where \( w_M \) equals 1/3 for the massless particles considered and, from now on, \( \hbar \) will again be set to 1. Introducing a dimensionless constant \( \gamma \), the matter evolution equation reads

\[ \partial_t \rho_M + 4 H \rho_M = \left( \frac{\gamma}{36} \right) |H| R^2 \equiv S_M, \quad (5.3) \]

with a further factor 1/36 inserted for later convenience.

A rough estimate of the coefficient \( \gamma \) in (5.3) is based on the suppressed coefficient in (5.1a) calculated for gravitons and the assumed coefficient of unity in (5.1b), giving

\[ \gamma \big|_{\text{gravitons}} \sim \frac{1}{8\pi}. \quad (5.4) \]

An explicit calculation of the \( \rho_M \) increase by graviton production gives precisely the structure (5.3) with correction terms on the right-hand side and a calculated coefficient equal to

\[ \gamma \big|_{\text{gravitons}} = \frac{1}{32\pi^2}, \quad (5.5) \]

which is a factor 4\( \pi \) smaller than the naive estimate (5.4). Details of this calculations are relegated to Appendix A.
From the Einstein equation (4.2) and the contracted Bianchi identities follows
the covariant conservation of the total energy-momentum tensor of the two components considered, the matter component from the standard-model fields and the vacuum component from the \( q \) field. In the context of a spatially-flat FRW universe with a single perfect-fluid matter component (M) and a vacuum component (V) from the \( q \) field, this energy-momentum conservation implies that the evolution equation of the vacuum energy density must have precisely the opposite source term compared to (5.3),

\[
\partial_t \rho_V = -S_M = - (\gamma/36) |H| R^2 = -\gamma |H|(\partial_t H + 2H^2)^2,
\]

with \( \gamma \approx 3.1663 \times 10^{-3} \) for graviton production according to (5.5). Hence, the right-hand side of (5.1) can be interpreted as describing the backreaction of the particle production given by Eq. (5.2); see below for further discussion. Note that (5.3) and (5.6) are noninvariant under time-reversal, which is appropriate for dissipative processes.

Let us end this section with three general remarks. First, result (5.1) relies on being able to define an adiabatic vacuum, which is possible if the expansion rate vanishes asymptotically in the past and in the future. For a free massless scalar, the imaginary part of the effective action, calculated to quadratic order in the curvature, is given by the spacetime integral of Eq. (29) in Ref. 24. For a spatially-flat \( k = 0 \) FRW universe, this spacetime integral reduces to an integral over the sum of the \( R^2 \) term and the Gauss–Bonnet term, as the term with the square of the Weyl tensor vanishes identically. For an asymptotically-flat \( k = 0 \) FRW universe, the Gauss–Bonnet term integrates to zero, leaving the single \( R^2 \) term. For interacting quantum fields, the complete de-Sitter spacetime may give rise to explosive particle production (used for \( q \)-theory in Ref. 12) but the expanding \( k = 0 \) FRW universe considered here does not. Still, compared to (5.3) for free massless fields in an expanding \( k = 0 \) FRW universe, there may be a somewhat enhanced particle production due to particle self-interactions.

Second, it is well-known that the gravitational backreaction of quantum matter fields is a subtle problem, which is not completely solved. Our description is the simplest possible: keep unchanged the form of the energy-momentum tensor on the right-hand sides of Eqs. (4.5a) and (4.5b) and modify both the evolution equation of the matter component (5.3) and the evolution equation of the vacuum component (5.6). Other changes are certainly to be expected (e.g., vacuum polarization effects), but our minimal description suffices for an exploratory study.

Third, a heuristic explanation of the modified evolution equations is as follows. Start with a spatially-flat FRW universe having a nonvanishing energy density \( \rho_M \neq 0 \) from relativistic particles \( (w_M = 1/3) \) and a vanishing vacuum energy density \( \rho_V(\eta) = 0 \) from an appropriate \( q \)-field value \( \eta \). Then, \( q \) stays at the value \( \eta \) and \( \rho_M \) dilutes by expansion \[\rho_M \propto 1/a(t)^3]\] without extra particle production. Intuitively, it is clear that a Hubble expansion driven solely by massless particles, \( H \equiv (\dot{a}/a) = 1/2 (t-t_0)^{-1}, \) does not create more massless particles. If, now, the Hubble expansion
is modified \( H \neq \frac{1}{2} (t - t_0)^{-1} \) by having \( \rho_V(q) > 0 \), then there will be particle production as \( R \propto (\partial_t H + 2 H^2) \neq 0 \). The “agent” of this particle production is the nontrivial vacuum field \( q - 7 \). Ultimately, the energy produced in particles must come from the agent (here, \( q - 7 \)) responsible for the modification of the Hubble expansion, \( H \neq \frac{1}{2} (t - t_0)^{-1} \). This discussion is analogous to that of the Unruh effect: the energy for the thermal radiation heating up the uniformly-accelerated detector ultimately comes from the agent responsible for the acceleration of the detector (cf. p. 55 of Ref. [25] and p. 108 of Ref. [26]). Another analogy is with Schwinger pair creation in a uniform static electric field: \( 9, 24, 26 \) the energy for the created particles ultimately comes for the agent responsible for the electric field.

6. Q-theory model of vacuum-energy decay

Henceforth, we use the same dimensionless variables as in Ref. [3] effectively obtained by rescaling with appropriate powers of the Planck energy \( E_P \equiv \sqrt{\hbar c^5/\sqrt{G_N}} \approx 1.22 \times 10^{19} \text{ GeV} \) and denoted by lower-case letters. Specifically, we have the dimensionless time \( \tau \) and the dimensionless Hubble parameter \( h \) (taken to be positive). The 4-form field strength \( (2.1c) \) gives rise to the pseudoscalar field \( q \) of mass dimension 2 and rescaling this field \( q \) (denoted \( F \) in Ref. [3]) produces the dimensionless variable \( f \). The overdot in the differential equations below will denote differentiation with respect to \( \tau \) and the prime differentiation with respect to \( f \).

We now present the \( q \)-theory equivalent of the source term found in Sec. 5, which physically corresponds to the backreaction from particle production by the curved spacetime. In addition, we allow for a dimensionless cosmological constant \( \lambda \equiv \Lambda/(E_P)^4 \) in the dimensionless energy density \( \epsilon(f) \). This additional cosmological constant \( \Lambda \) represents the effects from zero-point energies of the matter quantum fields \( 27 \) and cosmic phase transitions \( 27 \).

Specifically, the generalized Maxwell equation \( (4.8a) \) with the specific source term from \( (5.6) \) and the generalized Friedmann equation \( (4.8b) \) for \( w_M = 1/3 \) give

\[
\dot{f} f \epsilon'' = \gamma |h| \left( \dot{h} + 2 h^2 \right)^2, \tag{6.1a}
\]

\[
\dot{h} + 2 h^2 = 2 r_V, \tag{6.1b}
\]

with the dimensionless gravitating vacuum energy density

\[
r_V(f) = \epsilon(f) - f \epsilon'(f) \tag{6.2a}
\]

and the particular Ansatz function

\[
\epsilon(f) = \lambda + f^2 + 1/f^2. \tag{6.2b}
\]

The Ansatz \( (6.2b) \) has two important properties: first, the corresponding values of \( r_V = \lambda - f^2 + 3/f^2 \) range over \((-\infty, \infty)\) for \( f^2 \in (0, \infty) \) and any finite value of \( \lambda \); second, the corresponding vacuum compressibility \( 2 \chi \equiv (f^2 d^2 \epsilon/df^2)^{-1} \) is positive for any \( f^2 \in (0, \infty) \). Other Ansätze for \( \epsilon(f) \) are certainly possible.
Note that, strictly speaking, the ODEs (6.1) can be written solely in terms of $r_V(\tau)$ and $h(\tau)$, without need of the $q$-type field $f(\tau)$. But this is only because of the special case considered, $G(q) = \text{const}$. For generic $G(q)$, the $q$ field appears explicitly on the left-hand side of the generalized Maxwell equation (3.1). Moreover, precisely $q$ theory in the four-form realization gives rise to the $\rho_V$ evolution equation as a field equation, namely, the generalized Maxwell equation (2.3) for the classical theory. For these reasons, it is appropriate to speak of a $q$-theory model of vacuum-energy decay.

The two ODEs (6.1a) and (6.1b) are to be solved simultaneously and the matter energy density is obtained from the solutions $f(\tau)$ and $h(\tau)$ by

$$r_M = h^2 - r_V(f) .$$  \hspace{1cm} (6.3)

The numerical solutions of the ODEs (6.1) will be presented in the next section and some analytic results will be given in Appendix B.

7. Minkowski-vacuum attractor

From the ODEs (6.1a) and (6.1b) follows that the curves for $r_V(\tau)$ and $h(\tau)$ are monotonically decreasing, provided that the decay constant $\gamma$ is positive and that $r_M(\tau) = h(\tau)^2 - r_V[f(\tau)]$ stays positive for all values of $\tau$. Numerical solutions are given in Figs. 1 and 2 for positive and negative cosmological constants, $\lambda = \pm 1$. Similar numerical results have been obtained for $\lambda = 0$. These results show that the $r_V = 0$ Minkowski vacuum is approached without need of fine-tuning: the required asymptotic values of the $q$-type field $f$ are generated dynamically (see the top-left panels of Figs. 1 and 2).

Note that the decay constant used for these numerical results, $\gamma = 10^{-2}$, has a realistic order of magnitude (see Sec. 5). Similar numerical results have been obtained for other values of the decay coupling constant, ranging from $\gamma = 1$ down to $\gamma = 10^{-3}$.

The numerical results establish the existence of the $r_V = 0$ attractor for $\lambda \in \{-1, 0, +1\}$ with a finite domain of boundary conditions $\{h(1), f(1)\} = \{6 \pm 1, 0.6 \pm 0.2\}$; see Fig. 3. The actual domain of attraction for $|\lambda| \leq 1$ may be larger than this rectangle (see also the last paragraph of Appendix B for further discussion of the attractor domain).

The heuristic understanding for the appearance of an attractor is as follows. The left-hand side of (6.1a) equals $-\dot{r}_V$ according to (6.2a). Using (6.1b) for $h^2$, the ODE (6.1a) then reads $\dot{r}_V = -4\gamma |h| (r_V)^2 + \text{(rest-terms)}$. This is not quite the structure relevant for the Poincaré–Lyapunov theorem (given as Theorem 66.2 in Ref. 28 and Theorem 7.1 in Ref. 29), but does indicate a weak approach to the $r_V = 0$ asymptote. In fact, the asymptotic behavior from (6.1) is given by

$$h(\tau) \sim (6\gamma \tau)^{-1/3} , \hspace{1cm} (7.1a)$$

$$r_V(\tau) \sim (6\gamma \tau)^{-2/3} . \hspace{1cm} (7.1b)$$
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Figure 1. Numerical solution of the ODEs 6.1 with auxiliary functions 6.2. The model parameters are \( \{ \lambda, \gamma \} = (1, 1/100) \) and the boundary conditions at \( \tau = 1 \) are \( \{ h(1), f(1) \} = (6, 3/5) \). The initial energy densities are \( \{ r_f(1), r_M(1) \} = \{ 8.97333, 27.0267 \} \). The value of the vacuum energy density at \( \tau = 10^5 \) is \( r_V(10^5) = 3 \times 10^{-3} \).

Figure 2. Same parameters and boundary conditions as in Fig. 1 except for a different value of the cosmological constant, \( \lambda = -1 \). The initial energy densities are \( \{ r_f(1), r_M(1) \} = \{ 6.97333, 29.0267 \} \). The value of the vacuum energy density at \( \tau = 10^5 \) is \( r_V(10^5) = 3 \times 10^{-3} \).

Figure 3. Numerical solutions of the ODEs 6.1 with auxiliary functions 6.2 for \( \lambda = 1 \) (top row) and \( \lambda = -1 \) (bottom row). The numerical value of the decay parameter \( \gamma \) is 1/100. The functions \( f, h \) and \( r_V[f] \) are plotted versus the compactified time coordinate \( \theta \equiv (\tau - 1)/(\tau + \tau_{\text{mid}} - 2) \) with \( \tau_{\text{mid}} = 11/10 \). Four sets of boundary conditions at \( \theta = 0 \) are used: \( \{ h(0), f(0) \} = \{ 6\pm1, 3/5\pm1/5 \} \).
Note that the attractor behavior found here is qualitatively similar to the one of Dolgov theory\cite{30,31} shown numerically in Fig. 2 of Ref. 4 and proven mathematically in App. A of Ref. 32. But the Dolgov theory as such ruins the standard Newtonian dynamics\cite{33} and needs to be modified significantly.\cite{32,34}

Three final remarks are in order. First, the asymptotic decay of the Hubble parameter is slow and gives rise to an inflationary behavior\cite{36–38} of the particle horizon,

$$d_{\text{hor}}(\tau) \equiv d_1 + a(\tau) \int_1^\tau \frac{d\tau'}{a(\tau')} \sim \exp \left[ \frac{3}{2} \frac{(\tau^{2/3} - 1)}{(6 \gamma)^{1/3}} \right], \quad (7.2)$$

where $a(\tau)$ is the scale factor defined by $h = \dot{a}/a$ for $h(\tau)$ given by (7.1a) and $d_1$ is the contribution from times before $\tau = 1$ (a radiation-dominated universe with an initial singularity at $\tau = 0$ gives $d_1 = 2$). Note that this inflationary behavior holds only as long as the particle production is given by the $|H|^2$ term in (5.2), whose dominance over other contributions needs to be verified [see the discussion in Appendix A].

Second, the same type of slow asymptotic decay, $h(\tau) \propto \tau^{-1/3}$ and $r_V(\tau) \propto \tau^{-2/3}$, has also been found in a nonconstant–$G$ model with Ansätze for $G(f)$ and $\epsilon(f)$ from Ref. 3 but now with an arbitrary cosmological constant $\lambda$ added to $\epsilon(f)$.

Third, returning to the constant–$G$ model considered here, it needs to be emphasized that we remain within the framework of standard general relativity (with certain quantum effects of the matter fields included, as discussed in Sec. 5). Moreover, there are essentially no free parameters in the equation system (6.1), as the decay constant $\gamma$ has been calculated to be of order $3 \times 10^{-3}$ for gravitons, according to (5.5) and (A.9).

8. Conclusion

In this article, we have again addressed the cosmological constant problem, which can be formulated as follows: how can it be that the vacuum state does not have an effective cosmological constant $\Lambda$ (or gravitating vacuum energy density $\rho_V = \Lambda$ and pressure $P_V = -\Lambda$) with an energy scale of the order of the known energy scales of elementary particle physics? A particular adjustment-type solution of the cosmological constant problem involves so-called $q$ fields, which are (pseudo-)scalar composites of higher-spin fields (for example, $q$ as a pseudoscalar composite from the field strength $F_{\kappa\lambda\mu\nu}$ and the metric $g_{\mu\nu}$).

The $q$-theory framework serves as the proper tool for studying physical processes related to the quantum vacuum. It describes, in particular, the relaxation of the vacuum energy density (effective cosmological constant) as the backreaction of the deep vacuum to different types of perturbations, such as the Big Bang, inflation, cosmological phase transitions, and vacuum instability in gravitational or other backgrounds.
By considering the energy exchange between this $q$ field in the four-form-field-strength realization and massless particles produced by the spacetime curvature, we have found that a Planck-scale cosmological constant $\Lambda$ of arbitrary sign is cancelled by the $q$-field dynamics without fine-tuning. As mentioned previously, this cancellation occurs within the realm of standard general relativity.

The Minkowski vacuum with $\rho_V = -P_V = 0$ appears as an attractor of the dynamical equations (see Fig. 3 and Appendix B). As the approach to the Minkowski vacuum is rather slow, there occurs an inflationary behavior of the particle horizon, provided the nature of the particle production does not change significantly. The existence of such an inflationary phase (possibly before the start of the “standard” matter-dominated FRW universe) also requires that there are no other, more efficient dissipation processes than particle production by spacetime curvature.

**Appendix A. Energy density of produced particles**

In this appendix, we calculate the energy density of gravitons produced by spacetime curvature, directly following the number-density calculation of Zel’dovich and Starobinsky. We refer to their paper for further details and use the same notation. See also the textbooks for a general discussion of particle production.

From the Bogoliubov coefficient $\beta_k$, the final energy density of produced massless gravitons is given by

$$\rho_{M, \text{gravitons}} = 2 (2\pi)^{-3} a^{-4} \int d^3k \frac{1}{|k|} \beta_k^2 \exp \left[ -\epsilon |k|^2 \right].$$

(A.1)

Compared to the expression for the number density [given by Eq. (6) in Ref. 23 for massless real scalars], there are several different factors in (A.1): the first factor 2 is for the two helicity states of the graviton, the factor $a^{-4}$ contains an extra factor $a^{-1}$ for the redshift of the energy, the integrand of the $k$-integral has the energy factor $|k|$ and, finally, we have added a positive regulator $\epsilon$ which is taken to 0 at the end of the calculation.

The calculation will use the conformal time $\eta$, defined in the standard way by $\eta = \int^t d\tilde{t}/a(\tilde{t})$. The wave equation for gravitons with comoving wave number $k$ is given by

$$\chi_k''(\eta) + k^2 \chi_k(\eta) = V(\eta) \chi_k(\eta),$$

(A.2a)

$$V(\eta) \equiv a''(\eta)/a(\eta),$$

(A.2b)

where the prime denotes differentiation with respect to $\eta$.

Now take the $\beta_k$ expression [given by Eq. (5) in Ref. 23] evaluated for the potential term $V$ from (A.2b) and for the zeroth-order wave function $\chi_k(0) = \exp[-\im k \eta]$. Inserting this expression for $\beta_k$ into (A.1) gives the following triple integral:

$$\rho_M = 2 (2\pi)^{-3} a^{-4} \int_{-\infty}^{\infty} d\eta_1 \int_{-\infty}^{\infty} d\eta_2 \int_0^{\infty} dk \times \pi k \exp[2ik(\eta_1 - \eta_2)] \exp[-\epsilon k^2] V(\eta_1) V(\eta_2).$$

(A.3)
The calculation of this multiple integral is somewhat subtle, as there is a quadratic divergence of the \( k \) integral if the exponentials are omitted.

Introducing new coordinates
\[
\eta_\pm \equiv \frac{\eta_1 \pm \eta_2}{2},
\]
the evaluation of \( \text{(A.3)} \) gives the main result of this appendix,
\[
\rho_{M, \text{gravitons}} = \frac{1}{32 \pi^2} a^{-4} \int_{-\infty}^{\infty} d\eta_+ \int_{-\infty}^{\infty} d\eta_- \times \frac{1}{(\eta_-)^2} \left[ V(\eta_+) V(\eta_+) - V(\eta_- + \eta_+) V(\eta_+ - \eta_-) \right], \tag{A.5}
\]
with \( V \) defined by \( \text{(A.2b)} \). The integral over \( \eta_- \) in \( \text{(A.5)} \) produces a type of damped auto-correlation function of \( V(\eta) \). Equation \( \text{(A.5)} \) has a well-defined integrand (also at \( \eta_- = 0 \)) and improves upon expression (5.122) of Ref. 25, specialized to the isotropic background.

Assuming that the main contribution to the integral over \( \eta_- \) in \( \text{(A.5)} \) comes from an interval \( \Delta \eta > 0 \) and Taylor expanding the second term in the square brackets gives the following rough estimate:
\[
\hat{\gamma} a^{-4} \int_{-\infty}^{\infty} d\eta_+ \left[ \Delta \eta V'(\eta_+) V'(\eta_+) \right], \tag{A.6}
\]
with \( \hat{\gamma} \equiv 1/(32 \pi^2) \). If we now set \( \Delta \eta V'(\eta_+) \sim \Delta V \sim V \), then the result is
\[
\hat{\gamma} a^{-4} \int_{-\infty}^{\infty} d\eta_+ \left| V(\eta_+) V'(\eta_+) \right|. \tag{A.7}
\]
Taking only the second term in \( V' = (a''/a) - (a''/a)(a'/a) \), recalling that \( 6 V \equiv 6 a''/a \subset a^2 R \) and changing back to the coordinate time \( t \), the resulting expression for the energy density of produced gravitons at \( t = \infty \) is
\[
\rho_{M, \text{gravitons}}(\infty) = \frac{1}{1152 \pi^2} \left[ a(\infty) \right]^{-4} \int_{-\infty}^{\infty} dt \ a^4(\tilde{t}) |H(\tilde{t})| R^2(\tilde{t}) + \cdots, \tag{A.8}
\]
with the Hubble parameter \( H = a'/a^2 = (\partial_t a)/a \) and the ellipsis standing for higher-derivative local terms (single integrals) and further nonlocal terms (double integrals). Also note that we get the absolute value \( |H| \) in \( \text{(A.8)} \) because \( \text{(A.6)} \) has a manifestly positive integrand.

For an alternative way to arrive at the local term in \( \text{(A.8)} \) return to the main result \( \text{(A.5)} \). Observe that if the conformal-time correlation length of \( V \) is \( \Delta \eta > 0 \), then the integral over \( \eta_- \) gives approximately \( \Delta \eta \sim 1 V^2 = a^4 (\Delta \eta)^{-1} (a''/a^3) (a''/a^3) \). If we now set \( a \Delta \eta \sim \Delta t \sim |H|^{-1} \), we have for the integrand of the \( \eta_+ \) integral in \( \text{(A.5)} \) approximately \( a^5 |H| R^2 \), which gives \( \text{(A.8)} \).

As the integrand of the integral on the right-hand side of \( \text{(A.8)} \) is nonnegative, we can obtain an equation for the energy density at time \( t \) by taking \( t \) as the upper
limit on the integral and as the argument of the factor \( a^{-4} \),
\[
\rho_{M, \text{gravitons}}(t) = \frac{1}{1152 \pi^2} \left[ a(t) \right]^{-4} \int_{-\infty}^{t} d\tilde{t} \; a^4(\tilde{t}) |H(\tilde{t})| R^2(\tilde{t}) + \cdots. \tag{A.9}
\]
From (A.9) follows that the local change of the energy density is given by (5.3) with additional terms appearing on the right-hand side.

**Appendix B. Analytic results**

The ODEs (6.1) can be solved analytically if the following variable is used:
\[
\xi \equiv \ln[a(\tau)], \tag{B.1}
\]
where ‘ln’ is the natural logarithm and \( a(\tau) \) the FRW cosmic scale factor. Since the Hubble parameter \( h \) (assumed positive) corresponds to the rate of change of the scale factor, \( h(\tau) = \dot{a}/a \), we have that \( h^{-1} d/d\tau \) equals \( d/d\xi \). Furthermore, we take the time derivative of ODE (6.1b) and write the resulting second-order ODE as a pair of first-order ODEs. The three first-order ODEs are then
\[
\frac{d}{d\xi} r_V = -4 \gamma r_V^2, \tag{B.2a}
\]
\[
\frac{d}{d\xi} k = -2 \gamma k^2, \tag{B.2b}
\]
\[
h \frac{d}{d\xi} h + 2 h^2 = k, \tag{B.2c}
\]
where \( k = \dot{h} + 2 h^2 \) is proportional to the Ricci scalar \( R \) of the spatially-flat FRW universe considered.

The solution of the ODEs (B.2) with positive Hubble parameter \( h \) is given by
\[
r_V(\xi) = \frac{1}{4 \gamma (\xi - \xi_0)} , \tag{B.3a}
\]
\[
k(\xi) = \frac{1}{2 \gamma (\xi - \xi_0)} , \tag{B.3b}
\]
\[
h(\xi) = \left( \frac{C_1 + \exp[4 \xi_0] \text{Ei}[4 \xi - 4 \xi_0]}{\gamma \exp[4 \xi]} \right)^{1/2}, \tag{B.3c}
\]
with two integration constants, \( \xi_0 \) and \( C_1 \), and the exponential integral function \( \text{Ei} \[ y \] \), for \( y > 0 \) defined by
\[
\text{Ei} \[ y \] \equiv -\text{P} \int_{-y}^{+\infty} dx \; \frac{\exp[-x]}{x}, \tag{B.4}
\]
where ‘P’ stands for the principal value of the integral. Remark that the solutions of (B.2) have three integration constants, but the original ODE (6.1b) fixes the integration constants in (B.3a) and (B.3b) to be equal.
In principle, it is possible to obtain the function $\xi(\tau)$ by replacing $h(\xi)$ on the left-hand side of (B.3c) by $d\xi/d\tau$ and solving the resulting first-order ODE for $\xi(\tau)$. However, this solution $\xi(\tau)$ is difficult to obtain analytically and numerical methods are more appropriate (cf. Sec. 7).

Regarding the size of the attractor domain of initial boundary conditions \{h(1), f(1)\} as discussed in Sec. 7, the solution (B.3) gives the answer: $f(1)$ must be such as to make $r V(1)$ nonnegative, where $r V[f]$ is given by (6.2a) for the Ansatz (6.2b), and $h(1)$ must also be nonnegative.

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