Six-dimensional heavenly equation. Dressing scheme and the hierarchy.

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Abstract

We consider six-dimensional heavenly equation as a reduction in the framework of general six-dimensional linearly degenerate dispersionless hierarchy. We characterise the reduction in terms of wave functions, introduce generating relation, Lax-Sato equations and develop the dressing scheme for the reduced hierarchy. Using the dressing scheme, we construct a class of solutions for six-dimensional heavenly equation in terms of implicit functions.

1 Introduction

Six-dimensional heavenly equation \([1,2]\)

$$\Theta_{xw} - \Theta_{yz} - \{\Theta_x, \Theta_y\}_{(q,p)} = 0,$$

where \(\Theta = \theta(x,y,z,w,p,q)\), \(\{\cdot,\cdot\}_{(q,p)}\) is the Poisson bracket \(\{f_1, f_2\}_{(q,p)} := \frac{\partial f_1}{\partial q} \frac{\partial f_2}{\partial p} - \frac{\partial f_1}{\partial p} \frac{\partial f_2}{\partial q}\), belongs to the class of quasiclassical self-dual Yang-Mills equations (SDYM equations for the Lie algebra of vector fields) and corresponds to the case of two-dimensional Hamiltonian vector fields. Four-dimensional reductions of this equation include different versions of heavenly equation and related equations \([1,2,3]\).

It can be obtained from a standard SDYM type Lax pair (taken in a special gauge)

$$L = \partial_z - \lambda \partial_x + A_1,$$
$$M = \partial_w - \lambda \partial_y + A_2,$$

where \(A_1, A_2\) (gauge field components) generally belong to some Lie algebra, \(\lambda\) is a complex variable (spectral parameter). In our case \(A_1, A_2\) are two-dimensional Hamiltonian vector fields.

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Commutativity condition for operators (2) implies the existence of potential $F$ (belonging to Lie algebra), $A_1 = \partial_x F$, $A_2 = \partial_y F$, satisfying the equation

$$\partial_z \partial_y F - \partial_w \partial_x F - [\partial_x F, \partial_y F] = 0.$$  

(3)

Six-dimensional heavenly equation case corresponds to two-dimensional Hamiltonian vector fields $F$,

$$F = \{\Theta, \ldots\}_{(q,p)} := \Theta_q \partial_p - \Theta_p \partial_q,$$

and Lax pair (2) takes the form

$$L = \partial_z - \lambda \partial_x + \{\Theta_x, \ldots\}_{(q,p)},$$

$$M = \partial_w - \lambda \partial_y + \{\Theta_y, \ldots\}_{(q,p)}.$$  

(4)

General properties of Lax pairs (more generally, involutive distributions) of this type were discussed in [4] (see also [5], [6] and references therein, [7]). According to Frobenius theorem for vector fields, linear equations

$$L \Psi = (\partial_z - \lambda \partial_x) \Psi + \{\Theta_x, \Psi\}_{(q,p)} = 0,$$

$$M \Psi = (\partial_w - \lambda \partial_y) \Psi + \{\Theta_y, \Psi\}_{(q,p)} = 0$$  

(5)

have four functionally independent solutions. Due to the special structure of vector fields, two of them are trivial,

$$\phi^1 = x + \lambda z, \quad \phi^2 = y + \lambda w,$$

(6)

and two others can be found in generic form of series in $\lambda$,

$$\Psi^1 = q + \sum_{n=1}^{\infty} \Psi^1_n(p, q, x, y, z, w) \lambda^{-n},$$

$$\Psi^2 = p + \sum_{n=1}^{\infty} \Psi^2_n(p, q, x, y, z, w) \lambda^{-n},$$

(7)

which for the case of Hamiltonian vector fields are canonically conjugate,

$$\{\Psi^1, \Psi^2\}_{(q,p)} = 1.$$

This is not a unique admissible form of series for solutions of linear equations [5], they can be also constructed as series in nonnegative powers of $\lambda$. Different basic sets of solutions of linear equations [5] should be connected by diffeomorphism, and this observation leads to formulation of the dressing scheme based on Riemann-Hilbert problem with relation between holomorphic components defined by diffeomorphism (see below).

It is possible to introduce higher times and extend the Lax pair (8) to the hierarchy of the form

$$L_n = \partial_z^n - \lambda^n \partial_x + \{(H_{n-1})_x, \ldots\}_{(q,p)},$$

$$M_n = \partial_w^n - \lambda^n \partial_y + \{(H_{n-1})_y, \ldots\}_{(q,p)}.$$  

(8)
where $H_{n-1}$ are polynomials in $\lambda$ of the order $n-1$. The involutive distribution with the basis (4), (8) is of codimension four and the number of independent solutions of linear equations of the form (5) defined by this distribution remains equal to four; functions (7) retain their form, and (6), (7) are slightly modified to take into account the higher times,

$$\phi^1 = x + \sum_{n=1}^{\infty} \lambda^n z_n, \quad \phi^2 = y + \sum_{n=1}^{\infty} \lambda^n w_n. \quad (9)$$

The coefficients of the polynomials $H_{n-1}$ are connected with $\Theta$ by commutativity conditions of the higher flow with initial $L, M$ operators. These conditions provide closed systems of equations, the systems $[M_n, L], [L_n, M]$ are six-dimensional, and the systems $[L_n, L], [M_n, M]$ are five-dimensional. This fact is connected with the degeneracy and special structure of wave functions (9) for linearly degenerate quasiclassical YM type hierarchies [4], leading to simultaneous appearance of systems of different dimensionalities in the same framework. If we consider generic linearly degenerate six-dimensional dispersionless hierarchy [8], [9], from which six-dimensional heavenly equation case can be obtained as a reduction (see below), this degeneracy disappears.

A general feature of quasiclassical self-dual YM type hierarchies [4] is that the basis of linear operators for the higher flows of the hierarchy (8) can be represented in compact recursive form,

$$\tilde{L}_n := L_{n+1} - \lambda L_n = \partial_{z_{n+1}} - \lambda \partial z_n + \{\Theta_{z_n}, \ldots\} (q,p),$$

$$\tilde{M}_n := M_{n+1} - \lambda M_n = \partial_{w_{n+1}} - \lambda \partial w_n + \{\Theta_{w_n}, \ldots\} (q,p). \quad (10)$$

It is interesting to note that each of the operators $\tilde{L}_k, \tilde{M}_m$ is exactly of the same form as $L, M$ operators (4) (for another set of variables), and a commutator of arbitrary pair of operators $\tilde{L}_k, \tilde{M}_m$ gives heavenly equation (11) for the respective set of variables,

$$\Theta_{z_n w_{n+1}} - \Theta_{w_n z_{n+1}} - \{\Theta_{z_n}, \Theta_{w_n}\} (q,p) = 0$$

representing kind of intertwining equation for higher flows of the hierarchy. This phenomenon is known for Yang-Mills type hierarchies both in the standard [10] and quasiclassical [4] case.

### 2 The hierarchy. General six-dimensional case

We will describe six-dimensional heavenly equation hierarchy as a reduction of general six-dimensional linearly degenerate hierarchy and construct generating equations and Lax-Sato equations for 6D heavenly equation hierarchy.

First we will briefly outline the picture of linearly degenerate hierarchy developed in [8], [9] (see also [4]). This picture starts from introducing the formal series for the wave functions, defining a Plücker form which is a dual object to
the distribution of vector fields, and formulating a generating equation for the hierarchy through the holomorphic properties of this form.

Let us consider the series

$$\Psi^k = \Psi^k_0 + \tilde{\Psi}^k,$$

where

$$\Psi^k_0 = \sum_{n=0}^{\infty} t_n^k \lambda^n, \quad \tilde{\Psi}^k = \sum_{n=1}^{\infty} \Psi^{k_1, \ldots, t_4^k} \lambda^{-n},$$

(11)

where \(1 \leq k \leq 4\) (for six-dimensional hierarchy case), depending on four infinite sequences of independent variables \(t^k = (t^k_0, \ldots, t^k_n, \ldots)\), \(t^k_0 = x^k\).

The hierarchy is generated by the relation

$$\left( J^{-1} d\Psi^1 \wedge d\Psi^2 \wedge d\Psi^3 \wedge d\Psi^4 \right)_- = 0,$$

(12)

where for linearly degenerate case the differentials do not take into account \(\lambda\) (considered as a parameter), and vector fields of respective distribution do not contain derivative over \(\lambda\). Here \((\cdots)_-\) denotes the projection to the part of \((\cdots)\) with negative powers in \(\lambda\) (respectively \((\cdots)_+\) projects to nonnegative powers) and \(J = \det(D_{x^i\Psi^j})_{i,j=1,\ldots,4}\). Relation (12) implies Lax-Sato equations defining the dynamics of wave functions over higher times, moreover, it is equivalent to the set of Lax-Sato equations [9].

Introducing the Jacobian matrix

$$\text{Jac}_0 = \left( \frac{D(\Psi^1, \ldots, \Psi^4)}{D(x^1, \ldots, x^4)} \right), \quad \det(\text{Jac}_0) = J,$$

it is possible to write the hierarchy in the Lax-Sato form,

$$\partial_k \Psi = \sum_{i=1}^{4} \left( (\text{Jac}_0)^{-1} \right)_{ik} \lambda^n \partial_i \Psi, \quad 1 \leq k \leq m,$$

(13)

where \(1 \leq n < \infty\), \(\Psi = (\Psi^1, \ldots, \Psi^4)\), \(\partial_i = \partial_{x^i}\).

First flows of the hierarchy (lowest level integrable distribution) read

$$\partial_{t^k} \Psi = (\lambda \partial_k - \sum_{p=1}^{4} (\partial_p u_p) \partial_p) \Psi, \quad 1 \leq k \leq 4,$$

(14)

where \(u_k = \Psi^k_1\).

To proceed to the case of six-dimensional heavenly equation hierarchy, we will need a reduction \(J = 1\) corresponding to volume-preserving (divergence-free vector fields) case, generating relation in this case is

$$\left( d\Psi^1 \wedge d\Psi^2 \wedge d\Psi^3 \wedge d\Psi^4 \right)_- = 0,$$

(15)

vector fields (11) are divergence-free, \(\sum \partial_p u_p = 0\).
3 The hierarchy. Description of reduction

To obtain six-dimensional heavenly equation hierarchy, we consider a reduction of the hierarchy ([13]) characterised by the condition that two of the series $\Psi^k$ are equal to ‘vacuum’ functions $\Psi^k_0 = \sum_{n=0}^{\infty} t_n^k \lambda^n$ (for finite subsets of times they are polynomial)

$$\left(\Psi^3\right)_- = 0, \quad \Psi^3 = \Psi^3_0 = \sum_{n=0}^{\infty} t_n^3 \lambda^n, \quad \left(\Psi^4\right)_- = 0, \quad \Psi^3 = \Psi^3_0 = \sum_{n=0}^{\infty} t_n^3 \lambda^n, \quad (16)$$

Generating relation for six-dimensional heavenly equation hierarchy case is

$$(d\Psi^1 \wedge d\Psi^2 \wedge d\Psi^3_0 \wedge d\Psi^4_0)_- = 0,$$

We will restrict ourselves to the higher flows of six-dimensional heavenly equation type and drop higher times in $\Psi^1$, $\Psi^2$, also introducing new notations for functions $\Psi^3$, $\Psi^4$ and corresponding times to make the calculations more transparent. Thus we consider the wave functions of the form

$$\Psi^1 = q + \sum_{n=1}^{\infty} \Psi^1_n \lambda^{-n}, \quad \Psi^2 = p + \sum_{n=1}^{\infty} \Psi^2_n \lambda^{-n}, \quad (17)$$

$$\phi^1 := \Psi^3_0 = \sum_{n=0}^{\infty} z_n \lambda^n, \quad \phi^1 := \Psi^3_0 = \sum_{n=0}^{\infty} w_n \lambda^n, \quad x := z_0, \quad z := z_1 \quad (18)$$

$$\phi^2 := \Psi^4_0 = \sum_{n=0}^{\infty} w_n \lambda^n, \quad y := w_0, \quad w := w_1 \quad (19)$$

where $q$, $p$, $z_n$, $w_n$ are independent variables and coefficients $\Psi^1_n$, $\Psi^2_n$ are considered as dependent variables, generating relation for six-dimensional heavenly equation hierarchy reads

$$(d\Psi^1 \wedge d\Psi^2 \wedge d\phi^1 \wedge d\phi^2)_- = 0. \quad (20)$$

Lax-Sato equations ([13]) have rather special structure in this case, taking into account that the only nonzero entries of last two lines of the Jacobian form the unity matrix:

$$\partial_{x_n} \Psi = (\lambda^n \partial_x + (\lambda^n \{\Psi^1, \Psi^2\}_{(x,y)} + \partial_q - (\lambda^n \{\Psi^1, \Psi^2\}_{(q,x)}) \partial_p) \Psi, \quad (21)$$

$$\partial_{y_n} \Psi = (\lambda^n \partial_y + (\lambda^n \{\Psi^1, \Psi^2\}_{(y,z)} + \partial_q - (\lambda^n \{\Psi^1, \Psi^2\}_{(q,y)}) \partial_p) \Psi, \quad \{\Psi^1, \Psi^2\}_{(q,p)} = 1,$$

here $\Psi = (\Psi^1, \Psi^2), \{f, g\}_{(x,y)} := f_x g_y - f_y g_x$. Vector fields are Hamiltonian due to the condition $\{\Psi^1, \Psi^2\}_{(q,p)} = 1$. Lax-Sato equations define the evolution of the series $\Psi^1, \Psi^2$ with the coefficients considered as functions of four variables $(q, p, x, y)$ with respect to the higher times. The first two flows of the hierarchy read

$$\partial_x \Psi = (\lambda \partial_x - (\partial_x \Psi^1) \partial_q - (\partial_x \Psi^2) \partial_p) \Psi, \quad (22)$$

$$\partial_y \Psi = (\lambda \partial_y - (\partial_y \Psi^1) \partial_q - (\partial_y \Psi^2) \partial_p) \Psi.$$
and the first nontrivial order of expansion in $\lambda$ of the condition $\{\Psi^1, \Psi^2\}_{(q,p)} = 1$ gives $\partial_q \Psi^1 + \partial_p \Psi^2 = 0$, thus implying the existence of the potential $\Theta$, $\Psi^1 = -\Theta q$, $\Psi^2 = \Theta p$, transforming equations (22) to the form corresponding to the Lax pair of six-dimensional heavenly equation (4),

$$(\partial_z - \lambda \partial_x) \Psi = \{\Psi, \Theta_x\}_{(q,p)},$$
$$(\partial_w - \lambda \partial_y) \Psi = \{\Psi, \Theta_y\}_{(q,p)}.$$ 

Higher flows (21) can be written in a simple recursive form, which can be also obtained directly from the generating relation (20) (see [4] for more detail)

$$(\partial_{z_{n+1}} - \lambda \partial_{z_{n}}) \Psi = \{\Psi, \Theta_{z_n}\}_{(q,p)},$$
$$(\partial_{w_{n+1}} - \lambda \partial_{w_{n}}) \Psi = \{\Psi, \Theta_{w_n}\}_{(q,p)}. (23)$$

A comparison between Lax-Sato equations (21) and recursive relations (23) provides useful expressions of coefficients of expansion of Poisson brackets in $\lambda$ through the derivatives of potential $\Theta$.

**Remark 1.** 6D heavenly equation and the hierarchy can be easily generalized to the case of multidimensional Poisson bracket and Hamiltonian vector fields, in which the basic equation (connected to hyper-Kähler equations [11]) reads

$$\Theta_{xw} - \Theta_{yz} - \{\Theta_x, \Theta_y\}_{(q,p)} = 0, (24)$$

where

$$\{f, g\}_{(q, p)} := \sum_{i=1}^{N} \frac{\partial f_1}{\partial q_i} \frac{\partial f_2}{\partial p_i} - \frac{\partial f_1}{\partial p_i} \frac{\partial f_2}{\partial q_i},$$

and the generating relation for the hierarchy is

$$\left( \left( \sum_{k=0}^{N-1} d\Psi^{2k+1} \wedge d\Psi^{2k+2} \right) \wedge d\phi^1 \wedge d\phi^2 \right)_- = 0,$$

the generalization of Lax-Sato equations is straightforward.

**Remark 2.** We have already mentioned (see also [4]) that Yang-Mills type linearly degenerate dispersionless hierarchies are rather special, and, if we take into account different sets of times (submanifolds in the space of independent variables) in the generating relation, may contain equations of different dimensionalities. Let us consider a generating relation

$$\left( (d\Psi^1 \wedge d\Psi^2) \wedge (d\phi^1 \wedge d\phi^2 \wedge \cdots \wedge d\phi^N) \right)_- = 0, (25)$$

where $N$ is arbitrary (may be infinite) and $\Psi^1$, $\Psi^2$ are of general form (11), taking into higher times. Evidently, this relation contains several copies of 6DHE hierarchy considered in this work. It also contains copies of standard 4-dimensional heavenly equation hierarchy. Moreover, any solution of generating relation (25) for $N = P - 1$ evidently gives a solution for $N = P$. Involutivity
conditions for the distribution corresponding to relation (25) after restriction to some submanifolds of independent variables lead to equations of higher dimensionalities up to \( N + 4 \) with the Lax pairs of the type

\[
L = \partial_z - \lambda \partial_x + \{\Theta, \ldots\}(q,p),
\]

\[
\tilde{M} = \partial_{\tilde{w}_N} + \sum_{n=1}^{N-1} \lambda^n \partial_{\tilde{w}_n} + \{\tilde{H}_N(\lambda), \ldots\}(q,p),
\]

where \( H_N(\lambda) \) is a polynomial of the order \( N - 2 \) and the set of independent variables consists of \( z, x, p, q, \tilde{w}_n, 1 \leq n \leq N \). We believe that the hierarchy (25) containing subhierarchies of different dimensionalities may have an interesting geometric interpretation, probably in the language of exotic cohomologies developed in [11].

### 4 Dressing scheme and solutions

Starting from the dressing scheme for general six-dimensional hierarchy [12], [8], [9], by the reduction to the 6D heavenly equation hierarchy (20) (see also [4]) we obtain Riemann-Hilbert problem on the unit circle (or the boundary of some region \( G \))

\[
\Psi^1_{in} = F^1(\lambda, \Psi^1, \Psi^2; \phi^1, \phi^2)_{out},
\]

\[
\Psi^2_{in} = F^2(\lambda, \Psi^1, \Psi^2; \phi^1, \phi^2)_{out},
\]

where the diffeomorphism defined by \( F_1, F_2 \) for the case of Hamiltonian reduction should be area-preserving with respect to the variables \( \Psi^1, \Psi^2 \), or the \( \bar{\partial} \) problem in the unit disk (or some region \( G \))

\[
\bar{\partial}\Psi^1 = W_{,2}(\lambda, \tilde{\lambda}, \Psi^1, \Psi^2; \phi^1, \phi^2), \quad W_{,2} := \frac{\partial W}{\partial \Psi^2},
\]

\[
\bar{\partial}\Psi^2 = -W_{,1}(\lambda, \tilde{\lambda}, \Psi^1, \Psi^2; \phi^1, \phi^2), \quad W_{,1} := \frac{\partial W}{\partial \Psi^1}. \tag{26}
\]

Here the Hamiltonian reduction is taken into account explicitly.

The functional freedom of the dressing data consists of functions of 5 variables, that indicates that reduced equations are generically 6-dimensional, like equations of the unreduced linearly-degenerate dispersionless hierarchy.

We search for the solutions of the form

\[
\Psi^1 = q + \tilde{\Psi}^1, \quad \Psi^2 = p + \tilde{\Psi}^2
\]

where \( \tilde{\Psi}^1, \tilde{\Psi}^2 \) are analytic outside \( G \) and go to zero at infinity.

The \( \bar{\partial} \) problem can be obtained by the variation of the action

\[
f = \frac{1}{2\pi i} \int_G \left( \bar{\partial}\tilde{\Psi}^2 \bar{\partial}\tilde{\Psi}^1 - W(\lambda, \tilde{\lambda}, \Psi^1, \Psi^2; \phi^1, \phi^2) \right) d\lambda \wedge d\tilde{\lambda}, \tag{27}
\]
where one should consider independent variations of $\bar{\Psi}^1$, $\bar{\Psi}^2$ possessing required analytic properties, keeping times $q, p, \ldots$ fixed. Using the results of the work [12] in our setting, we come to the following statement:

**Proposition 1.** The function

$$\Theta(q, p, z, w) = \frac{1}{2\pi i} \int_G \left( \bar{\Psi}^2 \bar{\partial} \bar{\Psi}^1 - W(\lambda, \bar{\lambda}, \Psi^1, \Psi^2; \phi^1, \phi^2) \right) d\lambda \wedge d\bar{\lambda},$$

i.e., the action (27) evaluated on the solution of the $\bar{\partial}$ problem (26), gives the potential for 6D heavenly equation hierarchy and satisfies 6D heavenly equation (1).

To prove this proposition, it is enough to check the relations $\Psi^1_1 = -\partial_p \Theta$, $\Psi^2_1 = \partial_q \Theta$.

**A class of solutions**

Below we will construct a class of solutions for the 6DHE hierarchy. We will go along the lines of similar calculation for general heavenly equation presented in [13].

A class of solutions for the 6D heavenly equation hierarchy (1) in terms of implicit functions (similar to to solutions of hyper-Kähler hierarchy presented in [14], [1]) can be constructed using the choice

$$W(\lambda, \bar{\lambda}, \Psi^1, \Psi^2; \phi^1, \phi^2) =
= 2\pi i \left( \sum_{i=1}^M \delta(\lambda - \mu_i)F_i(\Psi^1; \phi^1, \phi^2) + \sum_{i=1}^M \delta(\lambda - \nu_i)G_i(\Psi^2; \phi^1, \phi^2) \right),$$

where $\delta(\lambda - \mu_i)$, $\delta(\lambda - \nu_i)$ are two-dimensional delta functions in the complex plane, and $F_i, G_i$ are some (complex-analytic) functions of three variables. The $\bar{\partial}$ problem (26) in this case reads

$$\bar{\partial} \bar{\Psi}^1 = 2\pi i \sum_{i=1}^M \delta(\lambda - \nu_i)G_i'(\Psi^2; \phi^1, \phi^2),$$
$$\bar{\partial} \bar{\Psi}^2 = -2\pi i \sum_{i=1}^M \delta(\lambda - \mu_i)F_i'(\Psi^1; \phi^1, \phi^2).$$

(29)

The solutions of the $\bar{\partial}$ problem are then of the form

$$\Psi^1 = q + \sum_{i=1}^M \frac{f_i}{\lambda - \nu_i}, \quad \Psi^2 = p + \sum_{i=1}^M \frac{g_i}{\lambda - \mu_i},$$

(30)
and from (26) the functions $f_i, g_i$ are defined as implicit functions,

$$f_i(q, p, z, w) = G_i' \left( p + \sum_{k=1}^{M} \frac{g_k(q, p, z, w)}{\nu_i - \mu_k}; \sum_{n=0}^{\infty} z_n \nu_i^n, \sum_{n=0}^{\infty} w_n \nu_i^n \right),$$

$$g_i(q, p, z, w) = -F_i' \left( q + \sum_{k=1}^{M} \frac{f_k(q, p, z, w)}{\mu_i - \nu_k}; \sum_{n=0}^{\infty} z_n \nu_i^n, \sum_{n=0}^{\infty} w_n \mu_i^n \right).$$  (31)

The potential $\Theta$ solving the general heavenly equation hierarchy is then given by the formula (28), it depends on the set of arbitrary functions of three variables $F_i, G_i,$

$$\Theta(x) = \sum_{i=1}^{M} F_i(\Psi^1(\mu_i); \phi^1(\mu_i), \phi^2(\mu_i)) + \sum_{i=1}^{M} G_i(\Psi^2(\nu_i); \phi^1(\nu_i), \phi^2(\nu_i)) +$$

$$+ \sum_{i=1}^{M} \sum_{j=1}^{M} f_i g_j, \quad (32)$$

where $\Psi^1, \Psi^2$ are given by (30), $\phi^1, \phi^2$ have the form (19), and the functions $f_i, g_i$ are defined as implicit functions by equations (31). Formula (32) corresponds to the special solution of hyper-Kähler hierarchies presented in [14], [1], however, it is important to note that in our case the solution depends on the set of arbitrary functions of three variables, in contrast to the set of functions of one variable in [14], [1].

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References

[1] K. Takasaki, An infinite number of hidden variables in hyper-Kähler metrics, J. Math. Phys. 30(7), 1515–1521 (1989)
[2] Jerzy F. Plebański and Maciej Przanowski, The Lagrangian of a self-dual gravitational field as a limit of the SDYM lagrangian, Phys. Lett. A 212 (1–2), 22–28 (1996)
[3] B.Doubrov and E.V.Ferapontov, On the integrability of symplectic Monge-Ampère equations, Journal of Geometry and Physics 60(10), 1604–1616 (2010)
[4] L.V. Bogdanov and M.V. Pavlov, Linearly degenerate hierarchies of quasiclassical SDYM type, Journal of Mathematical Physics 58 (9), 093505 (2017)
[5] S.V. Manakov, P.M. Santini, Inverse scattering problem for vector fields and the Cauchy problem for the heavenly equation, Phys. Lett. A 359(6), 613–619 (2006)

[6] S.V. Manakov and P.M. Santini, Integrable dispersionless PDEs arising as commutation condition of pairs of vector fields, Journal of Physics: Conference Series 482(1), 012029 (2014)

[7] M. Marvan and A. Sergeyev, Recursion operators for dispersionless integrable systems in any dimension, Inverse Problems 28(2) 025011 (2012)

[8] L.V. Bogdanov, V.S. Dryuma and S.V. Manakov, Dunajski generalization of the second heavenly equation: dressing method and the hierarchy, Journal of Physics A: Mathematical and Theoretical 40 (48), 14383 (2007)

[9] L.V. Bogdanov, A class of multidimensional integrable hierarchies and their reductions, Theoretical and Mathematical Physics 160 (1), 887-893 (2009)

[10] Mark J. Ablowitz, Sarbarish Chakravarty and Leon A. Takhtajan, A self-dual Yang-Mills hierarchy and its reductions to integrable systems in 1+1 and 2+1 dimensions, Communications in Mathematical Physics 158(2), 289–314 (1993)

[11] O.I. Morozov, Deformations of infinite-dimensional Lie algebras, exotic cohomology and integrable nonlinear partial differential equations. II, arXiv:1805.00319 [nlin.SI] (2018)

[12] L.V. Bogdanov and B.G. Konopelchenko, On the $\partial$-dressing method applicable to heavenly equation, Phys. Lett. A 345(1-3), 137-143 (2005)

[13] L.V. Bogdanov, Doubrov-Ferapontov general heavenly equation and the hyper-Kähler hierarchy, Journal of Physics A: Mathematical and Theoretical 48 (23), 235202 (2015)

[14] S.G. Gindikin, A construction of hyper-Kähler metrics, Funktsional. Anal. i Prilozhen., 20(3) 82–83 (1986); Funct. Anal. Appl., 20(3) 238–240 (1986)