Fully Dynamic \(c\)-Edge Connectivity in Subpolynomial Time

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Abstract

We present a deterministic fully dynamic algorithm for \(c\)-edge connectivity problem with \(n^{o(1)}\) worst case update and query time for any positive integer \(c = (\log n)^{o(1)}\) for a graph with \(n\) vertices. Previously, only polylogarithmic, \(O(\sqrt{n})\), and \(O(n^{2/3})\) worst case update time algorithms were known for fully dynamic 1, 2 and 3-edge connectivity problems respectively.

Our techniques include a multi-level \(c\)-edge connectivity sparsifier, an online-batch update algorithm for the sparsifier, and a general approach to turn an online-batch dynamic algorithm with small amortized update time into a fully dynamic algorithm with worst case update time.
1 Introduction

In the general dynamic graph setting, a fully dynamic algorithm for some property $\mathcal{P}$ supports the following operations on a given graph $G$:

1. \textbf{preprocess}(\(G\)): initialize the algorithm with input graph $G$
2. \textbf{insert}(\(u, v\)): insert edge \((u, v)\) to the graph
3. \textbf{delete}(\(u, v\)): delete edge \((u, v)\) from the graph
4. \textbf{query}(\(\mathcal{P}\)): answer if property $\mathcal{P}$ holds for the graph

The goal is to process graph update operations (edge insertions and deletions) and query operations as efficiently as possible. Update and query time can be categorized into two types: worst case, i.e., the upper bound of running time of any update or query operation, and amortized, i.e., the running time amortized over a sequence of operations.

In this paper, we study the fully dynamic $c$-edge connectivity problem. For an undirected graph $G$, two vertices $x$ and $y$ of $G$ are $c$-edge connected for an integer $c \geq 1$ if $x$ and $y$ cannot be disconnected by removing less than $c$ edges from the graph.

Fully dynamic $c$-edge connectivity has been studied for more than three decades [Fre85, GI91b, GI91a, WT92, EGIN97, Fre97, HK97, HT97, HK99, Tho00, HdT01, KKM13, Wu13, KRKPT16, NS17, NSW17, Wu17, HRT18, CGL+19]. For $c = 1$, it is the classic fully dynamic connectivity problem. For a graph of $n$ vertices and $m$ edges, Frederickson [Fre85] gave the first nontrivial fully dynamic connectivity algorithm with $O(\sqrt{m})$ worst case update time and $O(1)$ query time. The update time was improved to $O(\sqrt{n})$ by Eppstein et al. [EGIN97] using the general sparsification technique. Polylogarithmic amortized update time algorithms were presented in [HT97, HK99, Tho00, HdT01, Wu13, HIKP17]. Kapron et al. [KKM13] gave a Monte Carlo randomized polylogarithmic worst case update time algorithm, and Nanongkai et al. [NS17, Wu17] gave a Las Vegas randomized $n^{o(1)}$ worst case update time algorithm based on [NS17, Wu17]. Recently, Chuzhoy et al. [CGL+19] presented a deterministic algorithm with $n^{o(1)}$ worst case update time.

The study of fully dynamic 2-edge connectivity dates back to the work by Westbrook and Tarjan [WT92] in a context of maintaining 2-edge connected components. Galil and Italiano [GI91b] obtained a fully dynamic 2-edge connectivity algorithm with $O(m^{2/3})$ update time and $O(1)$ query time. The update time was improved to $O(\sqrt{m})$ by Frederickson [Fre97], and $O(\sqrt{n})$ by Eppstein et al. [EGIN97]. All these running times are worst case. To date, the best known worst case update time is still $O(\sqrt{n})$. For the amortized case, Henzinger and King [HK97] proposed the first randomized algorithm for fully dynamic 2-edge connectivity problem with $O(\log^5 n)$ amortized expected time per update and $O(\log n)$ time per query. This bound was further improved in [Tho00, HdT01, HRT18], and the best known result has an $O(\log^2 n)$ amortized update time and $O(\log n)$ query time.

For fully dynamic 3-edge connectivity, the best known result is an $O(n^{2/3})$ worst case update time and $O(\log n)$ query time algorithm by Galil and Italiano [GI91a], combining the sparsification technique by Eppstein et al. [EGIN97].

To our knowledge, for fully dynamic $c$-edge connectivity with $c > 3$, no algorithm with $o(n)$ update and query time was known, even for the amortized case. [DV94, DV95, DW98] studied the incremental case (only edge insertions are allowed). [MS18] gave an offline fully dynamic algorithm for $c = 4.5$ with $O(\sqrt{n})$ time per query. Recently, Liu et al. [LPS19] presented an offline fully dynamic algorithm for $c$-edge connectivity with $c^{O(c)}$ time per query.

1.1 Our contribution

We obtain a deterministic fully dynamic $c$-edge connectivity algorithm with subpolynomial worst case update and query time for any $c = (\log n)^{o(1)}$. 

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**Theorem 1.1.** There is a deterministic fully dynamic $c$-edge connectivity algorithm on a graph of $n$ vertices and $m$ edges with $m^{1+o(1)}$ preprocessing time and $n^{o(1)}$ update and query time for any positive integer $c = (\log n)^{o(1)}$.

Our result is based on the following new ideas, combining the recent developments of vertex sparsifier for $c$-edge connectivity [LPS19], fully dynamic minimum spanning forest [NS17, NSW17, Wu17, CGL19], expander decomposition [CGL19, SW19], and expander pruning [SW19].

- A multi-level $c$-edge connectivity sparsifier such that the sparsifier at each level is a $c$-edge connectivity equivalent graph of the input graph (Section 1.4.1).
- A general approach to turn dynamic algorithms in the online-batch setting with small amortized update time to dynamic algorithms with worst case update time (Section 1.4.2). Based on this reduction, we believe the online-batch setting is a natural abstraction for designing worst case dynamic algorithms.
- Algorithms to preprocess and maintain the multi-level $c$-edge connectivity sparsifier in the online-batch setting with subpolynomial amortized update time (Section 1.4.3 and 1.4.4).

As a corollary, our result implies a fully dynamic algorithm for expander decomposition with update time proportional to the maximum degree of the graph, which might be of independent interest.

For a simple graph $G = (V, E)$, the conductance of $G$ is defined as

$$\min_{\emptyset \subseteq S \subseteq V} \frac{|\partial G(S)|}{\min\{\text{vol}_G(S), \text{vol}_G(V \setminus S)\}},$$

where $\partial G(S)$ denotes the set of edges with one endpoint in $S$ and another endpoint in $V \setminus S$, and $\text{vol}_G(S)$ is the volume of $S$ in $G$, i.e., the sum of degrees of vertices in $S$. A simple graph $G$ is a $\phi$-expander if its conductance is at least $\phi$. A $\phi$-expander decomposition of a simple graph $G = (V, E)$ is a vertex partition $P$ of $V$ such that for each $P \in P$, the induced subgraph $G[P]$ is a $\phi$-expander. A $(\phi, \epsilon)$-expander decomposition of $G$ is a $\phi$-expander decomposition of $G$, and the number of intercluster edges (edges between two vertices from different clusters in $P$) is at most an $\epsilon$ factor of the total number of edges in $G$.

Our algorithm maintains a set of vertex partitions of the graph such that after obtaining each update, the algorithm outputs the access to one of the maintained vertex partitions that is an expander decomposition of the up-to-date graph.

**Corollary 1.2.** Given a conductance parameter $\phi \in (0, 1]$ and a graph $G$ undergoing updates, there is a deterministic fully dynamic algorithm to maintain a set of vertex partitions of $G$ such that after each update, the algorithm specifies one of the vertex partitions which is a $(\phi/n^{o(1)}, \phi n^{o(1)})$-expander decomposition of the up-to-date graph with $\hat{O}(m/\phi^2)$ preprocessing time and $\hat{O}(\Delta m^{o(1)}/\phi^3)$ worst case update time, where $m$ is the maximum number of edges and $\Delta$ is the maximum degree of the graph throughout the updates.

### 1.2 Other related work

The intermediate dynamic graph settings between amortized update time and worst case update time have been proposed and studied. Most notably, emergency planning (a.k.a fault-tolerant or sensitivity setting) considers the scenario of handling a single update batch before answering queries [PT07, DTCR08, BK09, DP09, CLPR10, KB10, DZ16, HN16, DP17, HLNW17, BS19].

Online-batch setting, which allows multiple update batches, has been studied in the parallel dynamic setting [ACHT11, STTW16, AAW17, AABD19, TDB19, DDK+19]. This work studies the

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1We denote $O(f \cdot \text{polylog}(f))$ by $\hat{O}(f)$ and $O(f^{1+o(1)})$ by $\tilde{O}(f)$.
online-batch setting within the classic dynamic graph model, and investigates its connection with dynamic algorithms with worst case update time.

Fully dynamic spanning forest and minimum spanning forest have been studied in \cite{Fre85, EGIN97, HK97, H197, HK99, Th00, HdlT01, Wul13, HRW15, HHKP17}. Recently, the worst case update time has been reduced from \( O(\sqrt{n}) \) \cite{Fre85, EGIN97} to \( n^{o(1)} \) in \cite{NS17, NSW17, Wul17, CGL+19}. This work uses the most recent deterministic fully dynamic spanning forest algorithm by Chuzhoy et al. \cite{CGL+19}.

Expander decomposition was first proposed by Kannan et al. \cite{KVVO04}, and has been used for fully dynamic minimum spanning forest algorithms \cite{NSW17}. This paper makes use of the recent results on deterministic expander decomposition \cite{CGL+19} and expander pruning \cite{NSW17, SW19}.

Our work also uses the “contraction technique” developed by Henzinger and King \cite{HK97} and Holm et al. \cite{HdlT01}. This technique has also been used for the \( n^{o(1)} \) worst case update time algorithm for fully dynamic minimum spanning forest \cite{NSW17}.

Sparsifiers of graphs that preserve certain properties have been extensively studied \cite{AGU72, ADD+93, ST11, AKP19}, and have been applied to dynamic graph algorithms \cite{GHP17, GHP18, DGGP19}. Recently, c-edge connectivity vertex sparsifiers have been proposed in \cite{CDLV19, LPS19}. However, it was unclear how to make use of such sparsifiers to design fully dynamic algorithms.

### 1.3 Notations

In the rest of this paper, unless specified, we assume the graphs are multigraphs. A multigraph \( G = (V,E) \) is defined by a vertex set \( V \) and an edge multiset \( E \). The multiplicity of an edge is the number of appearances of the edge in \( E \). We use \( \text{Simple}(G) \) to denote the simple graph derived from \( G \), i.e., graph on the same set of vertices and edges with all edges having multiplicity 1. For a subset \( S \subseteq V \), we use \( G[S] \) to denote the induced sub-multigraph of \( G \) on \( S \) and \( \partial_G(S) \) to denote the multiset of edges with one endpoint in \( S \) and another endpoint in \( V \setminus S \). For a multiset of edges \( E' \subseteq E \), we use \( \text{End}(E') \) to denote the set of all endpoints of edges in \( E' \) and \( G \setminus E' \) to denote the graph \( (V,E \setminus E') \). We also use \( E'|_{G[S]} \) to denote the multiset of edges in \( E' \) that are in \( G[S] \), i.e., \( E'|_{G[S]} = \{(x,y) \in E' : x,y \in S\} \).

For a vertex partition \( \mathcal{P} \) of graph \( G \), a cluster is a vertex set in \( \mathcal{P} \). An edge of \( G \) is an intercluster edge with respect to \( \mathcal{P} \) if the endpoints of the edge are in different clusters, otherwise, the edge is an inner edge. A vertex of \( G \) is a boundary vertex with respect to \( \mathcal{P} \) if it is an endpoint of an intercluster edge. We use \( G[\mathcal{P}] \) to denote the multigraph that only contains inner edges with respect to \( \mathcal{P} \), i.e., the union of the induced sub-multigraphs on all the clusters in \( \mathcal{P} \). We use \( \partial_G(\mathcal{P}) \) to denote the set of intercluster edges with respect to \( \mathcal{P} \). For an \( S \subseteq V \), we use \( \mathcal{P}|_{G[S]} \) to denote the vertex partition of \( G[S] \) induced by \( \mathcal{P} \), i.e., for every \( P' \in \mathcal{P}|_{G[S]} \), there is a \( P \in \mathcal{P} \) such that \( P' = P \cap S \).

The cut-set of a cut is the multiset of edges that have one endpoint on each side of the cut, and the size of a cut is the cardinality of the cut-set.

For an update sequence \( \text{UpdateSeq} \) on a dynamic graph, we use \( |\text{UpdateSeq}| \) to denote the number of updates in the update sequence.

### 1.4 Our technique

We use the standard degree reduction technique \cite{Har69} to turn the input simple graph with \( n \) vertices and \( m \) edges with arbitrary maximum degree into a multigraph with \( O(n+m) \) vertices and distinct edges\footnote{Two edges are distinct if they connect different pairs of vertices.} of multiplicity at least 1. Every vertex in the input graph corresponds to a vertex in the multigraph so that the \( c \)-edge connectivity between any pair of vertices in the input graph
is the same as the c-edge connectivity of their corresponding vertices in the multigraph. Without loss of generality, we work on the multigraph for the updates and queries in the rest of this section.

1.4.1 Multi-level c-edge connectivity sparsifier

The vertex sparsifier for c-edge connectivity proposed in [CDLV19, LPS19] is a c-edge connectivity equivalent graph constructed by shrinking “equivalent” vertices into a single vertex.

**Definition 1.3.** A multigraph $H = (V_H, E_H)$ is a c-edge connectivity equivalent graph of multigraph $G = (V, E)$ with respect to a set of vertices $T$ if $T$ is a subset of both $V$ and $V_H$ such that the c-edge connectivity in graph $H$ of any pair of vertices in $T$ is the same as it is in $G$.

By Definition 1.3, a c-edge connectivity equivalent graph with respect to $T$ can be used to answer a c-edge connectivity query if the two queried vertices are in $T$.

Our sparsifier, for the purpose of efficient update, is different from the sparsifiers of [CDLV19, LPS19] in two aspects: 1) our sparsifier is multi-level, as opposed to the one-level constructions in [CDLV19, LPS19]; 2) our sparsifier uses the “contraction technique” to contract “equivalent” vertices, instead of shrinking “equivalent” vertices into a single vertex as [CDLV19, LPS19] did.

**One-level c-edge connectivity sparsifier**

We first define c-edge connectivity equivalent partitions.

**Definition 1.4.** Let $G = (V, E)$ be a connected graph and $T$ be a set of vertices, a vertex partition $Q = \{Q_i\}$ of $V$ is a c-edge connectivity equivalent partition of $G$ with respect to $T$ if every $Q_i \in Q$ induces a connected subgraph of $G$, and for any $\emptyset \subseteq T' \subseteq T$ such that the size of a minimum cut partitioning $T$ into $T'$ and $T \setminus T'$ on $G$ is at most $c$, $\partial_G(Q)$ contains the cut-set of a minimum cut on $G$ partitioning $T$ into $T'$ and $T \setminus T'$.

For a vertex partition $P$ of graph $G$ such that $G[P]$ is connected for every $P \in P$, a vertex partition $Q$ is a $(P, c)$-edge connectivity equivalent partition of $G$ if the following conditions hold

1. $Q$ is a refinement of $P$, i.e., for every $Q \in Q$, there is a $P \in P$ such that $Q \subseteq P$.
2. For every $P \in P$, $\{Q \in Q : Q \subseteq P\}$ is a c-edge connectivity equivalent partition of $G[P]$ with respect to $\text{End}(\partial_G(P)) \cap P$.

Our one-level sparsifier construction also uses the “contraction technique” from [HK97, HdLT01, NSW17]. For a simple forest $F = (V_F, E_F)$ and a subset of terminals $S \subseteq V_F$, the contraction of $F$ with respect to $S$ is a set of “superedges” constructed in the following way: First collect a minimal set of edge-disjoint paths in $F$ that connect terminals in the same tree; then contract each collected path into a “superedge”. The contracted graph of a simple graph $G$ with respect to a partition $P$ and a spanning forest $F$ of $G[P]$, denoted by $\text{Contract}_{P,F}(G)$, is the union of $\partial_G(P)$ and the contraction of $F$ with $\text{End}(\partial_G(P))$ as terminals.

The sparsifier of a multigraph $G = (V, E)$ with respect to a $(P, c)$-edge connectivity equivalent partition $Q$ and a parameter $\gamma > c$, denoted by $\text{Sparsifier}(G, P, Q, \gamma)$, is obtained by assigning edge multiplicities to the simple graph $\text{Contract}_{Q,F}(\text{Simple}(G))$ in the following way: For every edge $(x, y)$ of $\text{Contract}_{Q,F}(\text{Simple}(G))$, if $x$ and $y$ are in the same cluster of $Q$, then the multiplicity of $(x, y)$ in $\text{Sparsifier}(G, P, Q, \gamma)$ is $\gamma$, otherwise, the multiplicity of $(x, y)$ in $\text{Sparsifier}(G, P, Q, \gamma)$ is the same as the multiplicity of $(x, y)$ in $G ((x, y)$ is also an edge of $G$ for this case). By the definition of $\text{Sparsifier}(G, P, Q, \gamma)$, we have the following properties:

- The number of distinct edges of $\text{Sparsifier}(G, P, Q, \gamma)$ is at most three times the number of distinct edges of $\partial_G(Q)$.
- $\text{Sparsifier}(G, P, Q, \gamma)$ is a c-edge connectivity equivalent graph of $G$ with respect to $\text{End}(\partial_G(P))$. 

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It remains to specify the vertex partition $\mathcal{P}$. A good partition would satisfy two properties: 
1) The number of distinct intercluster edges is much smaller than the number of distinct edges of the input graph. 2) For each $P \in \mathcal{P}$, $\text{Simple}(G[P])$ is a good expander. To this end, it is natural to use the expander decomposition of a graph for any $\phi \in (0, 1]$ for a graph of $m$ edges.

Multi-level $c$-edge connectivity sparsifier As our final goal, for a graph $G$ with at most $m$ vertices and distinct edges, we want to have a $c$-edge connectivity equivalent graph $H$ such that $\text{Simple}(H)$ is a $\phi$-expander for some $\phi = 1/m^{\Theta(1)}$. The one-level sparsifier construction is not enough for this purpose yet: If the input graph is not a good expander, then it is possible that the one-level sparsifier of the input graph is not a good expander unless the sparsifier contains at most $n^{\Theta(1)}$ distinct edges. In this case, the conductance parameter used in the expander decomposition to construct the one-level sparsifier needs to be at most $1/m^{1-o(1)}$. But with such a small conductance parameter, a large number of edges need to be updated in the sparsifier even when only one edge is changed in the original input graph, resulting in a slow update algorithm. Hence, we define the multi-level $c$-edge connectivity sparsifier.

Definition 1.5. A $(\phi, \eta, \gamma)$ multi-level $c$-edge connectivity sparsifier for a graph $G$ of at most $m$ distinct edges and parameters $0 < \phi, \eta \leq 1$, $\gamma \geq c + 1$ is a set of tuples $\{(G^{(i)}, \mathcal{P}^{(i)}, Q^{(i)})\}_{i=0}^{\ell}$ such that the following conditions hold:

1) $G^{(0)} = G$; for $i > 0$, $G^{(i)} = \text{Sparsifier}(G^{(i-1)}, \mathcal{P}^{(i-1)}, Q^{(i-1)}, \gamma)$ such that $G^{(i)}$ contains at most $m^{\eta^i}$ distinct edges.
2) $\mathcal{P}^{(i)}$ is a $\phi$-expander decomposition of $\text{Simple}(G^{(i)})$.
3) $Q^{(i)}$ is a $\{\mathcal{P}^{(i)}, c\}$-edge connectivity equivalent partition of $G^{(i)}$.
4) $\text{Simple}(G^{(\ell)})$ is a $\phi$-expander.

By definition, suppose $\{(G^{(i)}, \mathcal{P}^{(i)}, Q^{(i)})\}_{i=0}^{\ell}$ is a $(\phi, \eta, \gamma)$ multi-level $c$-edge connectivity sparsifier of $G$ with parameters $0 < \phi, \eta \leq 1$ and $\gamma \geq c + 1$. If $x$ and $y$ are two vertices in $\text{End}(\partial G^{(0)}(\mathcal{P}^{(i)}))$ for every $0 \leq i < \ell$, then the $c$-edge connectivity between $x$ and $y$ in $G$ is the same as it is in $G^{(\ell)}$.

1.4.2 Online-batch dynamic problem

The online-batch dynamic graph setting, motivated by parallelization, has been studied in parallel dynamic algorithms [ACHT11, STTW16, AAW17, AABD19, TDB19, DDK+19].

In this work, we consider the online-batch setting for sequential dynamic graph problems. The setting has two parameters: batch number $\zeta$ and sensitivity $w$. Let $\mathcal{D}$ be a data structure we would like to maintain for an input graph undergoing updates. The dynamic algorithm for data structure $\mathcal{D}$ in online-batch setting has $\zeta + 1$ phases: a preprocessing phase and $\zeta$ update phases.

In the preprocessing phase, the preprocessing algorithm initializes an instance of data structure $\mathcal{D}$ for the given input graph. In the $i$-th update phase for $1 \leq i \leq \zeta$, the update algorithm is given the $i$-th update batch, which is a sequence of at most $w$ updates. The goal of the update algorithm is to maintain the data structure $\mathcal{D}$ according to the $i$-th update batch such that at the end of this phase, $\mathcal{D}$ is updated for the resulted graph after applying the first $i$ update batches sequentially to graph $G$. We say an update algorithm in the online-batch setting has amortized update time $t$ if for each update batch of size at most $w$, the running time of the update algorithm is upper bounded by $t$ times the batch size. The goal of the online-batch setting is to minimize the amortized update time.
The online-batch setting is a generalization of emergency planning (a.k.a sensitivity setting and fault-tolerant). The difference is that the update algorithm in the online-batch setting needs to handle multiple update batches, but only one update batch in emergency planning.

The lemma below shows that an online-batch dynamic algorithm with bounded amortized update time can be turned into a dynamic algorithm with worst case update time in a blackbox way. We remark that such a blackbox reduction does not work for classic amortized dynamic algorithms. To see this, consider the following scenario: There are two update batches where the first batch is large, and the second batch only contains one update. For an amortized dynamic algorithm in the classic setting, it is entirely possible that the total running time of handling the two update batches concentrates on the second batch, so that the worst case update time for one update is equivalent to the total running time of all the batches.

**Lemma 1.6.** Let $G$ be a graph undergoing batch updates. Assume for two parameters $\zeta$ and $w$, there is a preprocessing algorithm with preprocessing time $t_{\text{preprocess}}$ and an update algorithm with amortized update time $t_{\text{amortized}}$ for a data structure $\mathcal{D}$ in the online-batch setting with batch number $\zeta$ and sensitivity $w$, where $t_{\text{preprocess}}$ and $t_{\text{amortized}}$ are functions that map the upper bounds of some graph measures throughout the update, e.g. maximum number of edges, to non-negative numbers.

Then for any $\xi \leq \zeta$ satisfying $w \geq 2 \cdot 6^\xi$, there is a fully dynamic algorithm with preprocessing time $O(2^\xi \cdot t_{\text{preprocess}})$ and worst case update time $O(4^\xi \cdot (t_{\text{preprocess}}/w + w^{(1/\xi)} t_{\text{amortized}}))$ to maintain a set of $O(2^\xi)$ instances of the data structure $\mathcal{D}$ such that after each update, the update algorithm specifies one of the maintained data structure instances satisfying the following conditions

1. The specified data structure instance is for the up-to-date graph.
2. The online-batch update algorithm is executed for at most $\xi$ times on the specified data structure instance with each update batch of size at most $w$.

To see the efficiency of this reduction, consider the following scenario: There is a dynamic algorithm with $m^{1+o(1)}$ preprocessing time and $m^{o(1)}$ amortized update time for some data structure in the online-batch setting of batch number $\zeta = \omega(1)$ and sensitivity $w = \omega(m^{1-o(1)})$, where $m$ is the upper bound on the number of edges throughout the updates. Then Lemma 1.6 implies a fully dynamic algorithm for the data structure with $m^{1+o(1)}$ preprocessing time and $m^{o(1)}$ worst case update time by choosing $\xi = \min\{\zeta, \sqrt{\log n}\}$.

The algorithmic idea of Lemma 1.6 is not new. For example, our algorithm has a framework similar to the dynamic expander pruning algorithm of [NSW17]. In this work, we apply the algorithmic framework from previous works to the online-batch setting to turn an online-batch algorithm with amortized update time into a dynamic algorithm with worst case update time in a blackbox way.

### 1.4.3 Decremental $c$-edge connectivity equivalent partition update

As our main technical contribution of this paper, we give an efficient update algorithm for the multi-level sparsifier update algorithm in the online-batch setting. The multi-level sparsifier update algorithm builds on an update algorithm for $c$-edge connectivity equivalent partitions in the following decremental update scenario: Let $G_0 = (V_0, E_0)$ be a graph, $T_0 \subseteq V_0$ be a subset of vertices, and $Q_0$ be a $c'$-edge connectivity equivalent partition of $G_0$ with respect to $T_0$ for some positive integer $c'$. Now assume some vertices of $G_0$ are removed, and only $V \subseteq V_0$ is left (without loss of generality, assume $G_0[V]$ is a connected induced subgraph). For a positive integer $c$, we want to obtain a $c$-edge connectivity equivalent partition $Q$ of $G = G_0[V]$ with respect to $S \cup T$ based on $Q_0$, where $S = \text{End}((\partial G_0(V)) \cap V)$ and $T = (T_0 \cap V) \setminus S$. More specifically, we want $Q$ to be a refinement of $Q_0|G$, and the number of distinct new intercluster edges of $Q$ with respect to $G$ compared with $Q_0|G$ is $|S|(10c)^O(c)$ (i.e., $\partial_G(Q) \setminus \partial_G(Q_0|G)$ contains at most $|S|(10c)^O(c)$ distinct edges).
We observe that \( c' \) has to be greater than \( c \), otherwise it is impossible to achieve the goal, even when \( G_0 \) is a good expander. To see this, consider the following construction of simple graph \( G_0 \) (Figure 1): For an arbitrary conductance parameter \( \phi \in (0,1) \), let \( \alpha = \lfloor 1/(2c+1)^2 \phi \rfloor \). \( V_0 \) is partitioned into \( U_0, U_1, \ldots, U_{\alpha-1} \) such that each \( U_i \) is a set of \( 2c+1 \) vertices that contains a vertex \( u_i \) in \( T_0 \). The induced subgraph of \( G_0 \) on each \( U_i \) is a clique of size \( 2c+1 \). For every \( 0 \leq i \leq \alpha - 1 \), there are \( c \) edges connecting \( U_i \) and \( U_{(i+1) \mod \alpha} \). By our choice of \( \alpha \), \( G_0 \) is a connected graph of at most \( (2c+1)^2 \alpha/2 \) edges, and thus is a \( \phi \)-expander. Moreover, \( G_0 \) does not have a cut of size less than or equal to \( c \). Hence, \( \{V_0\} \) is a \( c \)-edge connectivity equivalent partition of \( G_0 \).

Assume \( G = G_0[V_0 \setminus U_{\alpha-1}] \). Then \( S \) contains \( 2c \) vertices, and \( T = \{u_0, \ldots, u_{\alpha-2}\} \setminus S \). In graph \( G \), for any \( 0 \leq i < \alpha - 2 \), the cut partitioning \( \bigcup_{j=0}^{\alpha-2} U_j \) and \( \bigcup_{j=i+1}^{\alpha-1} U_j \) is of size \( c \). Thus, for an arbitrary \( c \)-edge connectivity equivalent partition \( Q \) of \( G \), any \( u_i, u_j \in T \) do not belong to same cluster of \( Q \), and the number of intercluster edges of \( Q \) is at least \( c(\alpha - 2) = \Omega(|T_0|c) \), as opposed to \( |S|(10c)^O(c) = (10c)^O(c) \) as needed.

On the other hand, from the construction of the example in Figure 1, we observe that even though \( G_0 \) does not have any cut of size \( c \), there are a lot of cuts of size \( 2c \). Hence, if \( Q_0 \) is a \( 2c \)-edge connectivity equivalent partition of \( G_0 \), then it is easy to obtain the required \( Q \).

The case analysis above suggests that to construct a \( c \)-edge connectivity equivalent partition in the decremental update setting, it is helpful to start with a \( c' \)-edge connectivity equivalent partition of the original graph with parameter \( c' > c \). We formalize this idea as the lemma below.

**Lemma 1.7** (informal, cf. Lemma 6.2 and Lemma 6.3). Let \( G_0 = (V_0, E_0) \) be a graph, \( T_0, V \subseteq V_0 \) be two subsets of vertices such that \( G_0[V] \) is a connected graph. Denote \( S = \text{End}(\partial G_0(V)) \cap V \) and \( T = (T_0 \cap V) \setminus S \). For any positive integer \( c \) and a \((c^2 + 2c)\)-edge connectivity equivalent partition \( Q_0 \) of \( G_0 \) with respect to \( T_0 \), there is a \( c \)-edge connectivity equivalent partition \( Q \) of graph \( G \) with respect to \( S \cup T \) that is a refinement of \( Q_0|_G \) such that \( \partial_G(Q) \setminus (\partial_G(Q_0|_G)) \) contains at most \( |S|(10c)^O(c^2) \) distinct edges. Furthermore, if \( \text{Simple}(G) \) is a \( \phi \)-expander, there is an algorithm with \( |S|(c/\phi)^O(c^2) \) running time to compute \( \partial_G(Q) \setminus (\partial_G(Q_0|_G)) \).

We assume the \((c^2 + 2c)\)-edge connectivity equivalent partition of \( G_0 \) is constructed from a sequence of “IA sets”. For a graph \( G \) and a set of vertices \( T \), we say \( C \) of graph \( G \) is a \((T', T \setminus T', t, \beta)\)-cut for \( T' \subseteq T \) and two integers \( t, \beta \) if the size of cut \( C \) is smaller than or equal to \( \beta \), and \( C \) partitions \( T \) into \( T' \) and \( T \setminus T' \) such that the side containing \( T' \) has at most \( t \) vertices. Inspired by \cite{LPS10}, a set of edges \( E' \) is an IA\(_G(T, t, q, d, d') \) set for some integers \( q \geq t, d \geq d' \) if \( E' \) is the set of intercluster edges of a vertex partition of \( G \), and for every \( \emptyset \subseteq T' \subseteq T \) and \( \beta \leq d \) such that a \((T', T \setminus T', t, \beta)\)-cut of \( G \) exists, there is a \((T', T \setminus T', q, \beta)\)-cut \( C \) such that no connected
component of $G \setminus E'$ contains more than $\max\{0, \beta - d'\}$ edges of $C$’s cut-set. By the definition of IA set, for a graph $G$ such that Simple$(G)$ is a $\phi$-expander, any $\text{IA}_G(T, [d/\phi], q, d, d)$ set with $q \geq \lceil d/\phi \rceil$ is the set of intercluster edges of a $d$-edge connectivity equivalent partition with respect to $T$.

Furthermore, $\text{IA}_G(T, t, q, d, d)$ can be constructed by composition: for $t_1, q_1, \ldots, t_d, q_d$ satisfying

$$t \leq t_1, t_i \leq q_i \text{ for any } 1 \leq i \leq d, \text{ and } (d + 1) \cdot q_i \leq t_{i+1} \text{ for any } 1 \leq i \leq d - 1,$$

if we have $E_i$ is an

$$\text{IA}_G(\bigcup_{j=1}^{d-i-1} E_j \cup \text{End} \left( \bigcup_{j=1}^{d-i} E_j \right), t_i, q_i, d - i + 1, 2),$$

set for any $1 \leq i \leq d$, then $\bigcup_{j=1}^{d-i} E_j$ is an $\text{IA}_G(T, t, q_d \cdot (d + 1), d, d)$ set.

As our key observation, we show that for an $\text{IA}_{G_0}(T_0, t, q, d, 2c + 1)$ set obtained by composition, there is an $\text{IA}_G(S \cup T, t, q + t, c, 1)$ set obtained by adding $O(|S|c^3)$ edges to $\text{IA}_{G_0}(T_0, t, q, d, 2c + 1)|_G$, and the set of edges added to $\text{IA}_{G_0}(T_0, t, q, d, 2c + 1)|_G$ can be efficiently computed. By applying this observation recursively, we can obtain an $\text{IA}_G(S \cup T, t, 2q(c + 1), c, c)$ set by adding a set of $|S|(10c)^{O(c)}$ edges to an $\text{IA}_{G_0}(T_0, t, q, c^2 + 2c, c^2 + 2c)|_G$ set that is obtained by composition. Then we have Lemma [7] by choosing parameters appropriately.

Now we give a high level idea of proving the key observation. First, we compute $F$, a set of cuts which contains all the cuts that are not handled by the given $\text{IA}_{G_0}(T_0, t, q, d, 2c + 1)|_G$ set, but need to be considered by an $\text{IA}_G(S \cup T, t, q + t, c, 1)$ set. We characterize the conditions that these cuts need to satisfy based on the properties of $\text{IA}_{G_0}(T_0, t, q, d, 2c + 1)|_G$. In particular, we show that for a fixed $\text{IA}_{G_0}(T_0, t, r, d, 2c)$ set such that the given $\text{IA}_{G_0}(T_0, t, q, d, 2c + 1)$ set is derived from the $\text{IA}_{G_0}(T_0, t, r, d, 2c)$ set by composition, if a cut $C$ needs to be considered, then the cut-set of $C$ is contained in a connected component of $G' \setminus \text{IA}_{G_0}(T_0, t, r, d, 2c)|_G$ containing at least one vertex from $S$. With this characterization, we show that the number of cuts in $F$ is at most $|S|\ell^{O(c)}$ (as opposed to the $(|S| + |T|)\ell^{O(c)}$ trivial bound), and $F$ can be computed efficiently.

Second, all the cuts in $F$ can be grouped into $O(|S|)$ families $F_1, \ldots, F_{O(|S|)}$ such that in each family $F_i$, all the cuts are “similar” in the following sense:

1. The cut-sets of all cuts in $F_i$ are in the same connected component of $G' \setminus \text{IA}_{G_0}(T_0, t, r, d, 2c)|_G$.
2. The cut-set of every cut in $F_i$ has a subset which induces a cut such that all these induced cuts partition $S$ in the same way.

Finally, we show that no matter how many cuts are in each family $F_i$, there always exists a set $R_i$ of $O(c^3)$ edges such that the cut-set of each cut (or an equivalent cut in terms of cut size and partition of $S \cup T$) in $F_i$ is not contained in a single connected component of $G \setminus R_i$. To prove this, we design an elimination procedure with $F_i$ as input that iteratively chooses a cut from $F_i$ and puts the chosen cut’s cut-set into $R_i$ until for every cut in $F_i$, one of its equivalent cuts does not have its cut-set contained in one connected component of $G \setminus R_i$. The number of cuts chosen by the elimination procedure can be large for an arbitrary set of cuts, but for every family $F_i$, the elimination procedure terminates after choosing at most $O(c^3)$ cuts. This is proved by investigating the relations between cuts induced by subsets of edges in $R_i$ based on the aforementioned properties of $F_i$ and the properties of $\text{IA}_{G_0}(T_0, t, q, d, 2c + 1)$. As a result, $(\bigcup_i R_i) \cup (\text{IA}_{G_0}(T_0, t, q, d, 2c + 1)|_G)$ forms an $\text{IA}_G(S \cup T, t, q + t, c, 1)$ set as wanted.

1.4.4 Online-batch algorithm for multi-level sparsifier

In this section, we give a high level idea of our dynamic algorithm for multi-level sparsifier with subpolynomial amortized update time in the online-batch setting. For the purpose of multigraph
updates and multi-level sparsifier construction, we consider the following update operations for multigraphs:

- **insert**\((u, v, \alpha)\): insert edge \((u, v)\) with edge multiplicity \(\alpha\) to the graph
- **delete**\((u, v)\): delete edge \((u, v)\) from the graph (no matter what the edge multiplicity is)
- **insert**\((v)\): insert a new vertex \(v\) to the graph
- **delete**\((v)\): delete isolated vertex \(v\) from the graph

The following is our main lemma for maintaining a multi-level \(c\)-edge connectivity sparsifier.

**Lemma 1.8.** Let \(G\) be a dynamic multigraph such that throughout the update process, the number of vertices and distinct edges of \(G\) is at most \(m\), and every vertex has at most a constant number of distinct neighbors. For any \(c = (\log m)^{o(1)}\) and some \(\phi = 1/m^{o(1)}\), there is a deterministic dynamic algorithm of \(m^{1+o(1)}\) preprocessing time and \(m^{o(1)}\) amortized update time in the online-batch setting with batch number \(O((\log \log m) / (\log (4c)))\) and sensitivity \(O(m\phi / \log m)\) to maintain a \((\phi, c^{1-o(1)}, \gamma)\) multi-level \(c\)-edge connectivity sparsifier.

The preprocessing algorithm initializes necessary data structures with \(m^{1+o(1)}\) time. In the rest of Section 1.4.4, we focus on the update algorithm.

**One-level sparsifier update algorithm** We prove the following lemma for one-level sparsifier.

**Lemma 1.9.** Let \(G\) be a multigraph with at most \(m\) vertices and distinct edges such that every vertex has a constant number of distinct neighbors. Suppose we have a \(\phi\)-expander decomposition \(\mathcal{P}\) of \(\text{Simple}(G)\), a \((\mathcal{P}, c^2 + 2c)\)-edge connectivity equivalent partition \(Q\) for \(G\), and a multigraph update sequence \(\text{UpdateSeq}\). There is a deterministic algorithm with running time \(\tilde{O}(\text{UpdateSeq} m^{o(1)}/\phi^3)\) to update \(G\) to \(G'\), \(\mathcal{P}\) to \(\mathcal{P}'\), \(Q\) to \(Q'\), and output a multigraph update sequence \(\text{UpdateSeq}'\) of length \(\text{UpdateSeq}'(10c)^O(c)\) such that the following conditions hold

1. \(G'\) is the resulted graph of applying \(\text{UpdateSeq}\) to \(G\).
2. \(\mathcal{P}'\) is a \(\phi/2O(\log^{1/3} m \log^{2/3} \log m)\)-expander decomposition of \(\text{Simple}(G')\) such that \(\mathcal{P}'\) is a refinement of \(\mathcal{P}\), and every vertex involved in the update sequence \(\text{UpdateSeq}\) becomes a singleton of \(\mathcal{P}'\) (if the vertex is in \(G'\)), where a singleton is a cluster of one vertex.
3. \(Q'\) is a \((\mathcal{P}', c)\)-edge connectivity equivalent partition of \(G'\) such that \(Q'\) is a refinement of \(Q\), and the number of distinct new intercluster edges is at most \(\text{UpdateSeq}'(10c)^O(c)\).
4. \(\text{UpdateSeq}'\) updates \(\text{Sparsifier}(G, \mathcal{P}, Q, \gamma)\) to \(\text{Sparsifier}(G', \mathcal{P}', Q', \gamma)\).

We first update \(\mathcal{P}\) to a refined partition \(\mathcal{P}^*\) of \(G\) satisfying the following three properties: 1) Any vertex involved in the multigraph update sequence \(\text{UpdateSeq}\) is a singleton of \(\mathcal{P}^*\) if the vertex is already in \(G\). 2) \(\mathcal{P}^*\) is a \(\phi/2O(\log^{1/3} m \log^{2/3} \log m)\)-expander decomposition of \(\text{Simple}(G)\). 3) The number of distinct new intercluster edges is no more than \(O(|\text{UpdateSeq}|)\).

To achieve this goal, we first remove from \(G\) all the incident edges to the vertices involved in \(\text{UpdateSeq}\), and then update the expander decomposition for the the resulted graph. Because vertices involved in the multigraph update sequence become isolated vertices in the resulted graph, these vertices are singletons in the expander decomposition. In the end, we add all the removed edges back. Since every removed edge becomes an intercluster edge with respect to the new partition, adding edges back does not affect the conductance of each cluster.

We show that if \(k\) edges are removed from a constant degree simple \(\phi\)-expander, then there is a \(\phi/2O(\log^{1/3} m \log^{2/3} \log m)\)-expander decomposition of the resulted graph with \(O(k)\) intercluster edges. The idea is to first use the expander pruning algorithm \[SW19\] to remove a set of vertices with volume at most \(O(k/\phi)\) from the expander such that the remaining vertices form an \(O(\phi)\)-expander, and the number of edges between the removed vertices and the remaining vertices is \(O(k)\). Then
we run the deterministic expander decomposition \cite{CGL+19} with \( \phi/2^{O(\log^{1/3} m \log^{2/3} \log m)} \) being the conductance parameter on each connected component of the induced subgraph of the removed vertices.

With the updated partition \( \mathcal{P}^* \), we further update \( \mathcal{Q} \) to \( \mathcal{Q}^* \) that is a \((\mathcal{P}^*, c)\)-edge connectivity equivalent partition by iteratively applying Lemma 1.7 on every affected cluster of \( \mathcal{P} \), making use of the condition that \( \mathcal{P}^* \) is a refinement of \( \mathcal{P} \). By Lemma 1.7, \( \partial_G(\mathcal{Q}^*) \setminus \partial_G(\mathcal{Q}) \) contains at most \(|\text{UpdateSeq}|(10c)^{O(c)}\) distinct edges. By the properties of contracted graph, the total number of vertex and edge insertions and deletions that transform \( \text{Sparsifier}(G, \mathcal{P^*}, \mathcal{Q^*}, \gamma) \) into \( \text{Sparsifier}(G, \mathcal{P}, \mathcal{Q}, \gamma) \) is at most \( O(|\text{UpdateSeq}|(10c)^{O(c)}) \).

At the end, we apply the update sequence \( \text{UpdateSeq} \) to \( G \) to obtain the updated graph \( G' \), and update \( \mathcal{P}^* \) to \( \mathcal{P}' \) and \( \mathcal{Q}^* \) to \( \mathcal{Q}' \) accordingly. Since all the vertices involved in the update sequence \( \text{UpdateSeq} \) are singletons in \( \mathcal{P}^* \) and \( \mathcal{Q}^* \), the corresponding updates from \( \mathcal{P}^* \) to \( \mathcal{P}' \) and from \( \mathcal{Q}^* \) to \( \mathcal{Q}' \) are to add or delete singletons. Consequently, to update the one-level sparsifier from \( \text{Sparsifier}(G, \mathcal{P^*}, \mathcal{Q^*}, \gamma) \) to \( \text{Sparsifier}(G', \mathcal{P}', \mathcal{Q}', \gamma) \), we only need to apply \( \text{UpdateSeq} \) to \( \text{Sparsifier}(G, \mathcal{P^*}, \mathcal{Q^*}, \gamma) \). So the length of overall \( \text{UpdateSeq}' \) is \(|\text{UpdateSeq}|(10c)^{O(c)}\).

**Online-batch algorithm for multi-level sparsifier** We construct an online-batch algorithm for the multi-level \( c \)-edge connectivity sparsifier by applying the one-level update algorithm iteratively. Let \( \text{UpdateSeq}^{(0)} = \text{UpdateSeq} \). We run batch update algorithm on tuple \((G^{(i)}, \mathcal{P}^{(i)}, \mathcal{Q}^{(i)})\) with update sequence \( \text{UpdateSeq}^{(i)} \), and use the returned update sequence \( \text{UpdateSeq}^{(i+1)} \) as the update sequence for the one-level sparsifier at level \( i+1 \). Note that if the length of \( \text{UpdateSeq} \) is large, e.g., a polynomial of \( m \), the number of distinct edges of some one-level sparsifiers in the multi-level construction can exceed the limit we want to keep. In this case, we directly reconstruct the expander decompositions, edge connectivity equivalent partitions, and one-level edge connectivity sparsifiers for these levels. We show that the running time of such a reconstruction is still at most a subpolynomial factor of the length of the given update sequence.

### 1.4.5 Fully dynamic algorithm for \( c \)-edge connectivity

Now we are ready to discuss our fully dynamic algorithm for \( c \)-edge connectivity. The goal of our update algorithm is to provide access to a \((1/m^{o(1)}, 1/m^{o(1)}, c + 1)\) multi-level \( c \)-edge connectivity sparsifier of the up-to-date graph after each update. Combining Lemma 1.8 and Lemma 1.6 this goal can be achieved by maintaining a set of \( m^{o(1)} \) multi-level sparsifiers with \( m^{1+o(1)} \) preprocessing time and \( m^{o(1)} \) update time.

For a \( c \)-edge connectivity query on two given vertices, we first obtain access to the multi-level sparsifier of the up-to-date input graph. Then we generate an update sequence of constant length that involves the two queried vertices, such that after applying the update sequence to the up-to-date input graph, the \( c \)-edge connectivity of the queried vertices does not change. Instead of applying the online-batch update algorithm for the multi-level sparsifier, we make use of Lemma 1.9 to update the one-level sparsifiers one-by-one from the bottom level to the top level. After updating the top level sparsifier, we use the returned update sequence to create a new graph that is \( c \)-edge connectivity equivalent to the input graph. By Lemma 1.9 every vertex involved in the update sequence forms a singleton in the expander decomposition, so the queried vertices are in the new \( c \)-edge connectivity equivalent graph, and the \( c \)-edge connectivity between the two queried vertices in the new graph is the same as it is in the input graph. The new \( c \)-edge connectivity equivalent graph has a subpolynomial number of vertices and edges because the length of the update sequence for each level is subpolynomial, so we can use the offline \( c \)-edge connectivity algorithm on this new graph to answer the query in subpolynomial time.
1.5 Organization

Section 2 covers preliminaries and facts about fully dynamic minimum spanning forest, expander decomposition, expander pruning and contraction technique that we use. Section 3 gives some useful properties of our multi-level $c$-edge connectivity sparsifier. Section 4 presents the general algorithm to turn an online-batch update algorithm with amortized update time into a fully dynamic algorithm with worst case update time. Section 5 defines cut-partitions and IA sets. Section 6 gives an update algorithm for IA sets in the decremental update setting. Section 7 presents the batch update algorithm for cut partitions, the main subroutine used for maintaining $c$-edge connectivity sparsifiers. Section 8 presents the online-batch algorithm for multi-level $c$-edge connectivity sparsifiers. Section 9 proves Theorem 1.1 and Corollary 1.2.

2 Preliminaries

Unless specified, we assume the graphs are multigraphs. A multigraph $G = (V, E)$ is defined by a vertex set $V$ and an edge multiset $E$. The multiplicity of an edge is the number of appearances of the edge in $E$. We denote the simple version of multigraph $G$ with $\text{Simple}(G)$, i.e., graph on the same set of vertices and edges with edge multiplicity 1 for all the edges.

For a subset $V' \subseteq V$, we denote the induced sub-multigraph of $G$ on $V'$ by $G[V']$. For a vertex partition $\mathcal{P}$ of graph $G$, a cluster is a vertex set in $\mathcal{P}$. An edge of $G$ is an intercluster edge with respect to $\mathcal{P}$ if the two endpoints of the edge are in different clusters, otherwise, the edge is an inner edge (with respect to $\mathcal{P}$). A vertex of $G$ is a boundary vertex with respect to $\mathcal{P}$, i.e., the union of the induced sub-multigraphs on all the clusters of $\mathcal{P}$. We let $\partial_G(\mathcal{P})$ denote the multiset of intercluster edges of $G$ with respect to $\mathcal{P}$, i.e.,

$$\partial_G(\mathcal{P}) = \{(x, y) \in E : \exists P_x, P_y \in \mathcal{P} \text{ s.t. } P_x \neq P_y, x \in P_x, \text{ and } y \in P_y\}.$$  

For a multiset of edges $E' \subseteq E$, let $\text{End}(E')$ denote the set of vertices that is an endpoint of an edge of $E'$, and $G \setminus E'$ denote the graph with $V$ as vertices and $E \setminus E'$ as edges. For a subset $V' \subseteq V$, we let

$$E'|_{G[V']} = \{(x, y) \in E' : x, y \in V'\},$$

i.e., the edges of $E'$ that are in graph $G[V']$.

For the purpose of multigraph updates and multi-level sparsifier construction, a multigraph update sequence consists the following update operations.

- **insert**($u, v, \alpha$): insert edge $(u, v)$ with multiplicity $\alpha$ to the graph
- **delete**($u, v$): delete all the multiple edges $(u, v)$ from the graph
- **insert**($v$): insert a new vertex $v$ to the graph
- **delete**($v$): delete isolated vertex $v$ from the graph

Figure 2: Multigraph update operations

For a multigraph update sequence $\text{UpdateSeq}$, let $|\text{UpdateSeq}|$ denote the number of update operations in the sequence.

Throughout the paper, the logarithm base is 2.
2.1 Cut

We give definitions related to cut.

**Definition 2.1.** For a connected graph $G = (V, E)$, a cut on $G$ is a bipartition of vertices $(V_1, V \setminus V_1)$ of the graph.

The cut-set of a cut is the multiset of edges that has one endpoint in each subset of the bipartition. For cut $(V_1, V \setminus V_1)$, let $\partial_G(V_1)$ denote the cut-set of the cut, i.e.

$$\partial_G(V_1) = \{e = (u, v) \in E : u \in V_1, v \in V \setminus V_1\}.$$

The size of a cut is the number of edges in the multiset $\partial_G(V_1)$, i.e. $|\partial_G(V_1)|$.

We say a set of edges $E'$ induces a cut if $E'$ is the cut-set of a cut, i.e., there exists $V' \subseteq V$ such that $E' = \partial_G(V')$.

**Definition 2.2.** For a connected graph $G = (V, E)$, a cut $(V_1, V \setminus V_1)$ is an atomic cut if both $G[V_1]$ and $G[V \setminus V_1]$ are connected, otherwise, it is non-atomic. A cut $(V', V \setminus V')$ is a simple cut if $G[V']$ is connected.

We remark that the definition of simple cut is not symmetric, i.e. $(V', V \setminus V')$ might not be a simple cut if $(V', V \setminus V')$ is a simple cut, though $(V', V \setminus V')$ and $(V \setminus V', V')$ have the same cut-set.

For a set $T$, $T'$ is a nontrivial subset of $T$ if $\emptyset \subsetneq T' \subsetneq T$. A bipartition $(T', T \setminus T')$ of $T$ is a nontrivial bipartition if $T'$ is a nontrivial subset of $T$.

**Definition 2.3.** Let $T \subseteq V$ be a set of vertices. A cut $(V_1, V \setminus V_1)$ is a $(T', T \setminus T')$-cut if $V_1 \cap T = T'$. Cut $(V_1, V \setminus V_1)$ is a minimum $(T_1, T \setminus T_1)$-cut if $(V_1, V \setminus V_1)$ partitions $T$ into $T_1$ and $T \setminus T_1$, and the number of edges in the cut-set of $(V_1, V \setminus V_1)$ is minimum among all the cuts that partition $T$ into $T_1$ and $T \setminus T_1$.

We use the idea of “intersect” in [LPS19]. But to avoid potential confusion with set intersection, we use “intercept” to replace the original “intersect” in [LPS19].

**Definition 2.4.** For a connected graph $G = (V, E)$, a set of edges $F \subseteq E$ intercepts a cut $C = (V', V \setminus V')$ if no connected component of $G \setminus F$ contains all edges of $\partial_G(V')$.

**Definition 2.5.** Let $C_1 = (V_1, V \setminus V_1)$ and $C_2 = (V_2, V \setminus V_2)$ be two cuts. for a connected graph $G = (V, E)$. Two cuts $C_1 = (V_1, V \setminus V_1)$ and $C_2 = (V_2, V \setminus V_2)$ are parallel if one of $V_1$ and $V \setminus V_1$ is a subset of one of $V_2$ and $V \setminus V_2$.

We prove some useful properties of cuts in Appendix A.

2.2 Expander and expander decomposition

For a graph $G = (V, E)$, the conductance of $G$ is defined as

$$\min_{S \subseteq V : S \neq \emptyset} \frac{|\partial_G(S)|}{\min\{\text{vol}_G(S), \text{vol}_G(V \setminus S)\}},$$

where $\partial_G(S)$ denotes the multiset of edges with one endpoint in $S$ and another endpoint in $V \setminus S$, and $\text{vol}_G(S)$ is the volume of $S$, i.e., the sum of degrees of vertices of $S$ in $G$.

A graph $G$ is a $\phi$-expander if its conductance is at least $\phi$. A $\phi$-expander decomposition of a graph $G = (V, E)$ is a vertex partition $\mathcal{P}$ such that for each $P \in \mathcal{P}$, the induced subgraph $G[P]$ is a $\phi$-expander. A $(\phi, \epsilon)$-expander decomposition of graph $G$ is a $\phi$-expander decomposition of $G$, and the number of intercluster edges is at most an $\epsilon$ factor of the total volume of the graph.

Recently, Chuzhoy et al. [CGL+19] give the following deterministic expander decomposition algorithm.
Theorem 2.6 (cf. Corollary 7.1 of [CGL+19]). Given a simple graph \( G = (V, E) \) of \( m \) edges and a parameter \( \phi \), there is a constant \( \delta > 0 \) and a deterministic algorithm \text{Expander-Decomposition} to compute a \((\phi, \phi/2^{\log^{1/3} m \log^{2/3} m})\)-expander decomposition of \( G \) in time \( \hat{O}(m/\phi^2) \).

We also make use of the dynamic expander pruning algorithm by Saranurak and Wang in [SW19] in an offline way.

Theorem 2.7 (rephrased, cf. Theorem 1.3 of [SW19]). Let \( G = (V, E) \) be a simple \( \phi \)-expander with \( m \) edges. Given access to adjacency lists of \( G \) and a set \( D \) of \( k \leq \phi m/10 \) edges, there is a deterministic algorithm \text{Pruning} to find a pruned set \( P \subseteq V \) in time \( O(k \log m/\phi^2) \) such that all of the following conditions hold:

1. \( \text{vol}_G(P) = 8k/\phi. \)
2. \( |E_G(P, V \setminus P)| \leq 4k. \)
3. \( G'[V \setminus P] \) is a \( \phi/6 \) expander, where \( G' = (V, E \setminus D) \).

2.3 Fully dynamic spanning forest

Combining the development on fully dynamic minimum spanning forest [NS17, NSW17, Wul17], recently Chuzhoy et al. [CGL+19] gave a deterministic fully dynamic algorithm for minimum spanning forest in subpolynomial update time.

Theorem 2.8 (cf. Corollary 7.2 of [CGL+19]). There is a deterministic fully dynamic minimum spanning forest algorithm on a \( n \)-vertex \( m \)-edge graph with \( \hat{O}(m) \) preprocessing time and \( 2^{O((\log n \log \log n)^{2/3})} \) worst case update time.

As a corollary, there is a deterministic fully dynamic spanning forest algorithm for a simple graph such that every edge deletion results at most one edge insertion in the spanning forest.

Corollary 2.9. There is a deterministic fully dynamic spanning forest algorithm on a \( n \)-vertex \( m \)-edge simple graph with \( \hat{O}(m) \) preprocessing time and \( 2^{O((\log n \log \log n)^{2/3})} \) worst case update time such that for every update, the update algorithm makes a change of at most two edges to the spanning forest, and the update algorithm returns a simple graph update sequence of length \( O(1) \) regarding the change of the spanning forest.

2.4 Contracted graph

The contraction technique developed by Henzinger and King [HK97] and Holm et al. [HdLT01] are widely used in dynamic graph algorithms.

Definition 2.10. Given a simple forest \( F = (V, E) \) and a set of terminals \( S \subseteq V \), the set of connecting paths of \( F \) with respect to \( S \), denoted by \text{ConnectingPath}_S(F) \), is defined as follows:

1. \text{ConnectingPath}_S(F) \) is a set of edge disjoint paths of \( F \).
2. For any two terminals \( u, v \in S \) that belong to the same tree of \( F \), the path between \( u \) and \( v \) in \( F \) is partitioned into several paths in \text{ConnectingPath}_S(F) \).
3. For any terminal \( v \in S \) such that the tree of \( F \) containing \( v \) has at least two terminals, \( v \) is an endpoint of some path in \text{ConnectingPath}_S(F) \).
4. For any endpoint \( v \) of some path in \text{ConnectingPath}_S(F) \), either \( v \in S \), or \( v \) is an end point of at least three paths in \text{ConnectingPath}_S(F) \).
It is easy to verify that for fixed \( F \) and \( S \), \( \text{ConnectingPath}_S(F) \) is unique.

A pair of vertices \((u, v)\) is a superedge of \( F \) with respect to \( S \) if \( u \) and \( v \) are two endpoints of some path in \( \text{ConnectingPath}_S(F) \). We denote the set of superedges of \( F \) with respect to \( S \) by \( \text{SuperEdge}_S(F) \). Let \( \text{Contract}_S(F) \) be the simple graph with the endpoints of \( \text{SuperEdge}_S(F) \) as vertices and \( \text{SuperEdge}_S(F) \) as edges.

For an edge \((x, y)\) in \( F \) and an edge \((u, v) \in \text{Contract}_S(F)\), we say \((u, v)\) is the superedge in \( \text{Contract}_S(F) \) covering \((x, y)\) if the path between \( u \) and \( v \) in \( \text{ConnectingPath}_S(F) \) contains edge \((x, y)\).

The contracted graph is defined by a simple graph \( G \), a vertex partition \( P \) of \( G \), and a simple spanning forest of \( G[P] \).

**Definition 2.11** (Contracted graph). Let \( G = (V, E) \) be a graph, \( P \) be a partition of \( V \) such that \( G[P] \) is connected for every \( P \in P \), and \( F \) be a spanning forest of \( G[P] \). The contracted graph of \( G \) with respect to \( P \) and \( F \), denoted by \( \text{Contract}_{P,F}(G) \), is a simple graph with the endpoints of \( \text{SuperEdge}_{\text{End}}(\partial_G(P))(F) \) as vertices, and \( \partial_G(P) \cup \text{SuperEdge}_{\text{End}}(\partial_G(P))(F) \) as edges.

The following properties hold for a contracted graph.

**Claim 2.12.** For any simple graph \( G = (V, E) \), vertex partition \( P \), and spanning forest \( F \) of \( G[P] \), the following conditions hold

1. For any two vertices \( u \) and \( v \) of \( \text{Contract}_{P,F}(G) \), \( u \) and \( v \) are connected in \( \text{Contract}_{P,F}(G) \) if and only if they are connected in \( G \).
2. The number of vertices and edges of \( \text{Contract}_{P,F}(G) \) is at most \( 3|\partial_G(P)| \).

The following lemma is implicit in Section 5 of [HdLT01] by Holm, de Lichtenberg and Thorup.

**Lemma 2.13.** Let \( F = (V, E) \) be a simple forest and \( S \subseteq V \). Assume \( F \) and \( S \) undergo the following updates

- \( \text{Contraction-Insert}(v) \): insert a vertex \( v \) to \( F \)
- \( \text{Contraction-Delete}(v) \): delete an isolated vertex \( v \) from \( F \)
- \( \text{Contraction-Insert}(u, v) \): insert edge \((u, v)\) to \( F \) (if \( u \) and \( v \) are in different trees)
- \( \text{Contraction-Delete}(u, v) \): delete edge \((u, v)\) from \( F \) (if edge \((u, v)\) is in \( F \))
- \( \text{Contraction-Insert-Terminal}(v) \): add \( v \) to terminal set \( S \) (if \( v \) is not in \( S \))
- \( \text{Contraction-Insert-Terminal}(v) \): remove \( v \) from terminal set \( S \) (if \( v \) is in \( S \))

and query

- \( \text{Contraction-Covering-Edge}(u, v) \): return the superedge of \( \text{Contract}_S(F) \) that covers edge \((u, v)\) of \( F \) if such a superedge exists

such that after every update, \( F \) contains at most \( n \) vertices, and \( S \) is a subset of \( V \).

There exist a preprocessing algorithm with \( O(n \text{polylog}(n)) \) running time and another algorithm to handle update and query operations with \( O(\text{polylog}(n)) \) worst case update and query time such that for each update, the update-and-query algorithm maintains \( F, S \) accordingly, and outputs an \( O(1) \)-length simple graph update sequence that updates \( \text{Contract}_S(F) \) accordingly.
2.5 Graph data structure

Let $G$ be a graph and $T$ be a set of vertices of $G$. We use $\mathcal{DS}(G,T)$ to represent a data structure that contains

1. a copy of the graph $G = (V,E)$ represented by adjacency list such that multiple edges connecting same pair of vertices are represented by an edge and its multiplicity
2. a terminal set $T$
3. a simple spanning forest $F$ of graph $G$
4. the simple contracted graph $\text{Contract}_T(F)$

and supports the query operations and update operations as defined in Figure 3 such that such that after each update, the data structure also returns an update sequence for $\text{Contract}_T(F)$.

Query operations:

- $\text{VertexNumber}(x)$: return the number of vertices of the connected component containing vertex $x$ in $G$
- $\text{DistinctEdgeNumber}(x)$: return the number of distinct edges of the connected component containing vertex $x$ in $G$ (parallel edges are only counted once)
- $\text{TerminalNumber}(x)$: return the number of terminals of the connected component containing vertex $x$ in $G$
- $\text{ID}(x)$: return the id of the connected component containing vertex $x$ in $G$
- $\text{OneTerminal}(x)$: return an arbitrary terminal in the connected component containing vertex $x$ in $G$ if exists

Update operations:

- $\text{Insert}(x)$: insert vertex $x$ to the graph
- $\text{Delete}(x)$: delete isolated vertex $x$ from the graph (if the vertex is a terminal of $T$, also delete it from the $T$)
- $\text{Insert}(x, y, \alpha)$: insert edge $(x, y)$ with multiplicity $\alpha$ to the graph
- $\text{Delete}(x, y)$: delete all the $(x, y)$ multiple edges from the graph
- $\text{Insert-Terminal}(x)$: insert vertex $x$ to the terminal set
- $\text{Delete-Terminal}(x)$: delete vertex $x$ from the terminal set

Figure 3: Graph data structure query and update operations

When the terminal set and the corresponding contracted graph are irrelevant, we use $\mathcal{DS}(G)$ to denote a data structure for an arbitrary terminal set.

By Corollary 2.9 and Lemma 2.13, we have the following lemma for the graph data structure.

**Lemma 2.14.** For a dynamic multigraph $G = (V,E)$ with at most $m$ vertices and distinct edges through the updates, there is a preprocessing algorithm to construct $\mathcal{DS}(G,T)$ for an arbitrary $T \subseteq V$ in time $O(m \text{polylog } m)$ and an algorithm supporting all the operations defined in Figure 3 in time $O(\text{polylog } m)$ such that for each update operation, the algorithm outputs an update sequence of length $O(1)$ to update the corresponding $\text{Contract}_T(F)$ in $\mathcal{DS}(G,T)$. 

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3 \(c\)-edge connectivity sparsifier properties

We have the following properties of a one-level sparsifier.

**Lemma 3.1.** Given a graph \(G = (V, E)\), a vertex partition \(P\) of \(V\) such that \(G[P]\) is connected for every \(P \in P\), a \((P, c)\)-edge connectivity equivalent partition \(Q\) for \(G\), and parameter \(\gamma > c\), we have

1. Simple(Sparsifier\((G, P, Q, \gamma)\)) contains at most \(3|\partial_G(Q)|\) edges.
2. Sparsifier\((G, P, Q, \gamma)\) is a \(c\)-edge connectivity equivalent graph of \(G\) with respect to \(End(\partial_G(P))\).

**Proof.** The first property is implied by Claim 2.12. We prove the second property in the rest of this proof.

Let \(x, y\) be two distinct vertices in \(End(\partial_G(P))\). By Definition 1.4 and Sparsifier\((G, P, Q, \gamma)\), \(x\) and \(y\) are in the same connected component of \(G\) if and only if \(x\) and \(y\) are in the same connected component of Sparsifier\((G, P, Q, \gamma)\). In the rest of this proof, we assume \(x\) and \(y\) are in the same connected component of \(G\).

Let \(H = (V_H, E_H)\) denote Sparsifier\((G, P, Q, \gamma)\). Let \(V'\) be the vertex set of the connected component containing \(x\) and \(y\) in \(G\), and \(V'\) be the vertex set of the connected component containing \(x\) and \(y\) in \(H\).

We show that for any \(\alpha \leq c\), the size of minimum cut separating \(x\) and \(y\) in \(G[V']\) is \(\alpha\) if and only if the size of minimum cut separating \(x\) and \(y\) in \(H[V']\) is \(\alpha\).

Consider the case that the size of minimum cut separating \(x\) and \(y\) in \(G[V']\) is \(\alpha\). Let \(C_1 = (V_1, V' \setminus V_1)\) be such a minimum cut. \(C_1\) must be an atomic cut, otherwise it is not minimum. Let \(F = \partial_G[V_1] \cap \partial_G(P)\),

\[\mathcal{P}' = \{P \in P : P \cap V_1 \neq \emptyset \text{ and } P \cap (V' \setminus V_1) \neq \emptyset\},\]

and \(F_P = \partial_G[V_1]|_{G[P]}\) for every \(P \in \mathcal{P}'\).

For any \(P \in \mathcal{P}'\), let \(T_P = End(\partial_G(P)) \cap P\). Since \(C_1\) is an atomic cut for \(G[V']\), both \(T_P \cap V_1\) and \(T_P \cap (V' \setminus V_1)\) are not empty sets. Hence, the size of the minimum cut that partitions \(T_P\) into \(T_P \cap V_1\) and \(T_P \cap (V' \setminus V_1)\) is \(|F_P|\), otherwise \(C_1\) is not the minimum cut separating \(x\) and \(y\). By Definition 1.4, there is a set \(F_P \subseteq \partial_G(Q)\) of size \(|F_P|\) that is the cut-set of a minimum cut \(G[P]\) partitioning \(T_P\) into \(T_P \cap V_1\) and \(T_P \cap (V' \setminus V_1)\). Hence, \(F \cup \left(\bigcup_{P \in \mathcal{P}'} F_P\right)\) induces a cut separating \(x\) and \(y\) for \(G[V']\) of size \(\alpha\), and \(F \cup \left(\bigcup_{P \in \mathcal{P}'} F_P\right)\) is a subset of \(\partial_G(Q)\). By the definition of Sparsifier\((G, P, Q, \gamma)\), \(F \cup \left(\bigcup_{P \in \mathcal{P}'} F_P\right)\) induces a cut separating \(x\) and \(y\) in \(H[V']\) of size \(\alpha\).

On the other hand, by the definition of Sparsifier\((G, P, Q, \gamma)\), for any cut of size at most \(c\) separating \(x\) and \(y\) in \(H[V']\), its cut-set also induces a cut of same size for \(G[V']\) separating \(x\) and \(y\).

Hence, the size of minimum cut separating \(x\) and \(y\) in \(G[V']\) is \(\alpha\) if and only if the size of minimum cut separating \(x\) and \(y\) in \(H[V']\) is \(\alpha\) for any \(\alpha \leq c\). Then the second property holds by Definition 1.3.

By Lemma 3.1 and Definition 1.5, we have the following corollary.

**Corollary 3.2.** For a \((\phi, \eta, \gamma)\) multi-level \(c\)-edge connectivity sparsifier \(\{(G^{(i)}, P^{(i)}, Q^{(i)})\}_{i=0}^{\ell}\) for \(\gamma \geq c + 1\), if \(x, y\) are vertices in \(End(\partial_G(\mathcal{P}^{(i)}))\) for every \(0 \leq i < \ell\), then the \(c\)-edge connectivity between \(x\) and \(y\) in \(G\) is the same as that of \(G^{(\ell)}\).
4 Fully dynamic algorithm from online-batch amortized algorithm

We prove Lemma 1.6 in this section. Assume we have the following algorithms for data structure $D$:

1. **DS-Initialize** ($G$): a preprocessing algorithm with running time $t_{\text{preprocess}}$ that initializes an instance of data structure $D$ for the input graph $G$,
2. **DS-Update** ($G, D, \text{UpdateSeq}$): an update algorithm with amortized update time $t_{\text{amortized}}$ in the online-batch setting with batch number $\xi$ and sensitivity $w$. The input of the algorithm is the access to graph $G$ and data structure $D$ for graph $G$, and an update batch, which is a sequence of updates $\text{UpdateSeq}$ for $G$. The algorithm updates $D$ so that the resulted data structure is for the resulted graph after applying $\text{UpdateSeq}$ on $G$.

We assume $t_{\text{preprocess}}$ and $t_{\text{amortized}}$ are functions that map the upper bounds of some graph measures throughout the update, e.g. maximum number of edges, to non-negative numbers.

We define a multi-level structure. The total number of levels is $\xi + 1$. Let $s = \lfloor (w/2)^{1/\xi} \rfloor$. For $0 \leq i \leq \xi$ and $j \geq 0$, let

$$d_i = s^{\xi-i} \quad \text{and} \quad t_{i,j} = j \cdot d_i.$$

Without loss of generality, assume $t_{i,j} = 0$ if $j < 0$.

**Definition 4.1.** Let $G_{i,j}$ denote the resulted graph of $G$ after applying first $j \cdot d_i$ updates. For a dynamic data structure $D$, let $D_{i,j}$ for every $0 \leq i \leq \xi$ and $j \geq 0$ as follows:

- For $i = 0$ or $j = 0$, $D_{i,j} = \text{DS-Initialize}(G_0,j)$
- For $1 \leq i < \xi$, $j > 0$
  
  $$D_{i,j} = \text{DS-Update} \left( G_{i-1,\lceil j/s \rceil-2}, D_{i-1,\lceil j/s \rceil-2}, \text{UpdateSeq}_{\lceil t_{i-1,\lceil j/s \rceil-2}+1,t_{i,j} \rceil} \right),$$

  where $\text{UpdateSeq}_{[a,b]}$ denotes the update sequence that contains $a$-th, $(a+1)$-th, ..., $b$-th update of the entire update sequence.

We say $D_{i,j}$ depends on $D_{i',j'}$ for some $i' < i$ if there exist $j_{i'}, \ldots, j_i$ such that $j_{i'} = j'$, $j_i = j$ and $D_{h,j_h}$ is obtained by running $\text{DS-Update}$ on $D_{h-1,j_{h-1}}$ for all the $i' < h \leq i$.

By Definition 4.1 and induction, we have the following observation.

**Claim 4.2.** Assume $s \geq 6$. If $D_{i,j}$ depends on $D_{i',j'}$, then $t_{i,j} \leq t_{i',j'} + 2 + d_{i'}/2$.

The goal of the update algorithm is to provide access to $D_{\xi,j}$ at time $j$ (after obtaining the $j$-th update). Ideally, the algorithm below is enough for our purpose, because if we only consider the running time of $\text{DS-Initialize}$ and $\text{DS-Update}$ of the algorithm, then the overall running time is desirable. However, we cannot afford to have multiple instances of each $D_{i,j}$.

(1) For $0 \leq i \leq \xi - 1, j > 0$, compute $D_{i,j}$ by Definition 4.1 with work evenly distributed within time interval $[t_{i,j} + (d_i/2) + 1, t_{i,j+1}]$.
(2) Compute $D_{\xi,j}$ by Definition 4.1 at time $j$.
Let \( \mathcal{P} \) denote the data structure that contains a graph \( G \) and its corresponding data structure \( \mathcal{D} \). The fully dynamic algorithm maintains \( O(2^\xi) \) copies of \( \mathcal{P} \), such that at time \( \tau \), there is a maintained data structure \( \mathcal{P} \) corresponding to \((G_{\xi,\tau}, \mathcal{D}_{\xi,\tau})\).

For every \( 0 \leq i \leq \xi \), we maintain \( 2^{i+1} \) instances of data structure \( \mathcal{P} \) such that all the instances are indexed by an \((i+1)\)-bit binary number. Let \( b_{0,j} = j \mod 2 \) for all the integer\( j \), and

\[
b_{i,j} = b_{i-1,\lfloor j/s\rfloor-2} \circ (j \mod 2)
\]

for all \( 0 < i \leq \xi \) and integer \( j \), where \( \circ \) is the concatenation of two binary strings.

For \( 0 \leq i \leq \xi - 1 \) and integer \( j \), we update \( \mathcal{P}_{b_{i,j}} \) to be \((G_{i,j}, \mathcal{D}_{i,j})\) within the time period \([t_{i,j} + (d_i/2) + 1, t_{i,j+1}]\), and keep \( \mathcal{P}_{b_{i,j}} = (G_{i,j}, \mathcal{D}_{i,j}) \) within the time period \([t_{i,j+1} + 1, t_{i,j+2} + (d_i/2)]\).

Now we are ready to give our preprocessing algorithm and update algorithm. In the preprocessing algorithm, we initialize \( \mathcal{P}_\beta \) to be \((G, \mathcal{D}_{0,0})\) for all the binary number \( \beta \) of length at most \( \xi + 1 \). In the update algorithm, we amortize the updates for each \( \mathcal{P}_{b_{i,j}} \) within the time interval as described above.

**Algorithm 1: Fully-Dynamic-Preprocessing(G, \xi)**

**Input:** access to adjacency list of graph \( G \), parameter \( \xi \)

**Output:** \( \mathcal{P}_\beta \) for all the binary number \( \beta \) of length at most \( \xi + 1 \)

1. for every \( 0 \leq i \leq \xi \) and every \((i+1)\)-bit binary number \( \beta \) do
2. \( G_\beta \leftarrow G \)
3. \( \mathcal{D}_\beta \leftarrow \text{DS-Initialize}(G) \)
4. \( \mathcal{P}_\beta \leftarrow (G_\beta, \mathcal{D}_\beta) \)

We use Fully-Dynamic-Preprocessing and Fully-Dynamic-Update to prove Lemma 1.6.

**Proof of Lemma 1.6.** By Claim 1.2 and induction, for \( 0 \leq i \leq \xi - 1 \), the pair \((G_{b_{i,j}}, \mathcal{D}_{b_{i,j}})\) at level \( i \) is the same as \((G_{i,j}, \mathcal{D}_{i,j})\) within the time period \([t_{i,j} + 1, t_{i,j+2} + (d_i/2)]\). Hence, the algorithm is correct.

We prove the running time of the preprocessing algorithm and the update algorithm. Since the running time of DS-Initialize is \( t_{\text{preprocess}} \), the running time of Fully-Dynamic-PREPROCESSING is \( O(2^\xi \cdot t_{\text{preprocess}}) \), and within the time period \([t_{o,j} + d_0/2 + 1, t_{o,j+1}]\) for every \( j > 0 \), the total work of line 1-4 of Fully-Dynamic-UPDATE for every \( j > 0 \) is \( O(2^\xi \cdot t_{\text{preprocess}}) \). Since

\[
d_0 = s^\xi = \lfloor (w/2)^{1/\xi} \rfloor ^\xi = \Omega(w/2^\xi),
\]

the amortized running time for line 1-4 of Fully-Dynamic-Update is

\[
O \left( \frac{2^\xi t_{\text{preprocess}}}{d_0} \right) = O \left( 4^\xi t_{\text{preprocess}} w^{-1} \right).
\]

For any \( 1 \leq i \leq \xi - 1 \), the total work of line 5-10 of Fully-Dynamic-Update for level \( i \) within time period \([t_{i,j} + d_i/2 + 1, t_{i,j+1}]\) for any \( j > 0 \) is \( O(2^\xi t_{\text{amortized}} d_{i-1}) \). Since the total work is evenly distributed to a time period of length \( \Omega(d_i) \), the amortized running time of line 5-10 of Fully-Dynamic-UPDATE for any \( 1 \leq i \leq \xi - 1 \), \( j > 0 \) is

\[
O \left( \frac{2^\xi t_{\text{amortized}} d_{i-1}}{d_i} \right) = O(2^\xi t_{\text{amortized}} s) = O(2^\xi t_{\text{amortized}} w^{1/\xi}).
\]

The overall amortized running time of line 5-10 of Fully-Dynamic-UPDATE is \( O(2^\xi t_{\text{amortized}} w^{1/\xi}) \).
Algorithm 2: Fully-Dynamic-Update(β), UpdateSeq

Input: access to \(\mathcal{P}_\beta = (G_\beta, \mathcal{D}_\beta)\) for every binary number \(\beta\) of at most \(\xi + 1\) bits; access to the first \(k\) multigraph updates of the graph update sequence \(\text{UpdateSeq}\) at time \(k\).

Output: access to \(\mathcal{D}_\beta\) for some \(\xi + 1\) bit binary string \(\beta\) such that \(\mathcal{D}_\beta = \mathcal{D}_{\xi,k}\) at time \(k\).

1. distribute work of the following process evenly within time interval \([t_0,j + d_0/2 + 1, t_0,j+1]\) for every \(j > 0\):
   2. for every pair \((G_\beta, \mathcal{D}_\beta)\) with first bit of \(\beta\) equal to \((j \mod 2)\) do
   3. apply \(\text{UpdateSeq}_{t_0,j-2+1, t_0,j}\) to \(G_\beta\)
   4. \(\mathcal{D}_\beta \leftarrow \text{DS-Initialize}(G_\beta)\)

5. distribute work of the following process evenly within time interval \([t_{i,j} + d_i/2 + 1, t_{i,j+1}]\) for every \(1 \leq i \leq \xi - 1, j > 0\):
   6. for every binary string \(\beta\) of length at least \(i + 1\) such that the first \(i + 1\) bits of \(\beta\) equals to \(b_{i,j}\) do
   7. \(\beta' \leftarrow \) the first \(i\) bits binary number of \(b_{i,j}\)
   8. reverse \((G_\beta, \mathcal{D}_\beta)\) back to status that is the same as \((G_{i-1, [j/s]-2}, \mathcal{D}_{i-1, [j/s]-2})\)
   9. run \(\text{DS-Update}(G_\beta, \mathcal{D}_\beta, \text{UpdateSeq}_{t_{i-1, [j/s]-2}+1, t_{i,j}})\) to update \(\mathcal{D}_\beta\) to be the same as \(\mathcal{D}_{i,j}\)
   10. apply \(\text{UpdateSeq}_{t_{i-1, [j/s]-2}+1, t_{i,j}}\) to \(G_\beta\)

11. distribute work of the following process at time \(j\):
   12. \(\beta' \leftarrow \) the first \(\xi\) bits of \(b_{\xi,j}\)
   13. reverse \((G_{b_{\xi,j}}, \mathcal{D}_{b_{\xi,j}})\) back to status that is the same as \((G_{\beta'}, \mathcal{D}_{\beta'})\)
   14. run \(\text{DS-Update}(G_{b_{\xi,j}}, \mathcal{D}_{b_{\xi,j}}, \text{UpdateSeq}_{t_{i-1, [j/s]-2}+1, j})\) to update \(\mathcal{D}_{b_{\xi,j}}\) to be the same as \(\mathcal{D}_{\xi,j}\)
   15. apply \(\text{UpdateSeq}_{t_{i-1, [j/s]-2}+1, j}\) to \(G_{b_{\xi,j}}\)
   16. output the access of \(\mathcal{D}_{b_{\xi,j}}\)

The running time of line 11-16 of Fully-Dynamic-Update for every update is

\[ O(t_{\text{amortized}}d_{\xi-1}) = O(t_{\text{amortized}}s) = O(t_{\text{amortized}}w^{1/\xi}). \]

Hence, the overall amortized running time of Fully-Dynamic-Update per update is

\[ O(4\xi t_{\text{preprocess}}w^{-1} + 2\xi t_{\text{amortized}}w^{1/\xi}) = O(4\xi(t_{\text{preprocess}}w^{-1} + t_{\text{amortized}}w^{1/\xi})). \]

\[ \square \]

5 Cut partition and \(\mathbf{IA}\) notation

We focus on cuts with one side containing a small number of vertices. In Section 3 with the expander decomposition, we will show that this is without loss of generality. We start by defining \((t, c)\)-cut for a connected graph.

**Definition 5.1.** For a connected graph \(G = (V, E)\), a cut \(C = (V', V \setminus V')\) is a \((t, c)\)-cut if the cut is of size at most \(c\), and \(|V'| \leq t\).
For a set of vertices $T$ and a $T' \subseteq T$ satisfying $\emptyset \subseteq T'$, a cut $C = (V', V \setminus V')$ for graph $G$ is a $(T', T \setminus T', t, c)$-cut if $C$ is a $(t, c)$-cut such that $T \cap V' = T'$ and $T \cap (V \setminus V') = T \setminus T'$. Cut $C$ is a minimum $(T', T \setminus T', t, c)$-cut if it has the smallest cut size among all the $(T', T \setminus T', t, c)$-cuts.

Cut $C$ for graph $G$ is a simple $(T', T \setminus T', t, c)$-cut if $C$ is a simple cut and a $(T', T \setminus T', t, c)$-cut. Cut $C$ is a minimum simple $(T', T \setminus T', t, c)$-cut if $C$ is a simple $(T', T \setminus T', t, c)$-cut, and the size of cut $C$ is the smallest among all the simple $(T', T \setminus T', t, c)$-cuts.

Let $H$ be an induced subgraph of $G$. A cut $C$ is a minimum simple $(T', T \setminus T', t, c)$-cut with cut-set in $H$ if $C$ is a simple $(T', T \setminus T', t, c)$-cut on $G$ with cut-set contained in $H$, and the size of cut $C$ is the smallest among all simple $(T', T \setminus T', t, c)$-cuts with cut-set in $H$.

We remark that the definition of $(t, c)$-cut is not symmetric. $(V \setminus V', V')$ might not be a $(t, c)$-cut if $(V', V \setminus V')$ is a $(t, c)$-cut.

**Definition 5.2.** Let $G = (V, E)$ be a connected graph, and $T$ be a subset of $V$. For two positive integers $t$ and $c$, a partition $Q$ of $V$ is a $(t, c)$-cut partition with respect to $T$ if all of the following conditions hold:

1. $G[Q]$ is connected for every $Q \in Q$.
2. For every $\emptyset \subseteq T' \subseteq T$ such that the minimum $(T', T \setminus T', t, c)$-cut is of size $\alpha \leq c$, there is a set of edges $E' \subseteq \partial_G(Q)$ such that $E'$ is the cut-set of a cut (not necessarily a $(t, c)$-cut) partitioning $T$ into $T'$ and $T \setminus T'$ of size at most $\alpha$.

For graph $G = (V, E)$ that is not necessarily connected, let $T$ be a subset of $V$. For two positive integers $t, c$, a partition $Q$ of $V$ is a $(t, c)$-cut partition with respect to $T$ if for every $V'$ that forms a connected component of $G$,

1. For every $Q \in Q$, either $Q \cap V' = \emptyset$ or $Q \subseteq V'$ holds.
2. $\{Q \in Q : Q \cap V' \neq \emptyset\}$ is a $(t, c)$-cut partition of $G[V']$ with respect to $T \cap V'$.

To construct and maintain a $(t, c)$-cut partition, we make use of the following IA notation, which is inspired by the “intersect all” idea from [LPS19] by Liu et al. The difference between Definition 3.1 of [LPS19] and our definition below is that Definition 3.1 of [LPS19] considered all the cuts of size at most $c$ while our definition only considers $(t, c)$-cuts for the purpose of efficient update algorithm.

**Definition 5.3.** For a connected graph $G = (V, E)$, $T \subseteq V$ and positive integers $d \geq c > 0$, $q \geq t > 0$, a set of edges $E' \subseteq E$ is an IA$_G(T, t, q, d, c)$ set if the following conditions hold:

1. $E' = \partial_G(P)$ for some vertex partition $P$ of $V$ such that $G[P]$ is connected for every $P \in P$.
2. For every subset $\emptyset \subseteq T' \subseteq T$ such that there is a $(T', T \setminus T', t, d)$-cut of size $\alpha \leq d$, there is a $(T', T \setminus T', t, \alpha)$-cut $(V', V \setminus V')$ such that for every $P \in P$, $G[P]$ contains at most $\max\{\alpha - c, 0\}$ edges of the cut-set of $(V', V \setminus V')$.

For a graph $G = (V, E)$ that is not necessarily connected, $T \subseteq V$, and positive integers $d \geq c > 0$ and $q \geq t > 0$, a set of edges $E' \subseteq E$ is an IA$_G(T, t, q, d, c)$ set if for every $Q \subseteq V$ such that $Q$ forms a connected component of $G$, $E'[G[Q]]$ is an IA$_{G[Q]}(T \cap V', t, q, d, c)$ set.

For simplicity, we use IA$_G(T, t, q, d, c)$ to denote an IA$_G(T, t, q, d, c)$ set when there is no ambiguity.

By Definition 5.2 and Definition 5.3, an IA set with appropriate parameters for a graph induces a $(t, c)$-cut partition.
Claim 5.4. Let $G = (V,E)$ be a connected graph, and $T$ be a subset of $V$. For any $q \geq t > 0$, $d \geq c > 0$ and an $I_{AG}(T,t,q,d,c)$ set, the following properties hold:

1. For any $\emptyset \subset T' \subset T$, if there is a $(T',T \setminus T',t,c)$-cut in $G$ of size $\alpha$ for some $\alpha \leq c$, then there is a $(T',T \setminus T',q,\alpha)$-cut $(V',V \setminus V')$ such that the cut-set of $(V',V \setminus V')$ is a subset of the $I_{AG}(T,t,q,d,c)$ set.

2. The $I_{AG}(T,t,q,d,c)$ set is the set of intercluster edges of a $(t,c)$-cut partition with respect to $T$ for graph $G$.

3. Let $x,y$ be two different vertices of $T$. If $x$ and $y$ are in the same connected component of $G \setminus I_{AG}(T,t,q,d,c)$, then for any $T' \subset T$ satisfying $|T' \cap \{x,y\}| = 1$, there is no $(T',T \setminus T',t,c)$-cut of graph $G$.

In the rest of this section, we prove a few useful lemmas for the later sections.

Lemma 5.5. Let $G = (V,E)$ be a connected graph and $T \subset V$ be a set of vertices. Let $E'$ be an $I_{AG}(T,t_1,q_1,d_1)$ set, and $E''$ be an edge set of $E \setminus E'$ such that $E''$ is an $I_{AG}(E',t_2,q_2,d - c_1,c_2)$ set. If $q_2 \geq t_2 \geq q_1 \geq t_1$, then $E' \cup E''$ is an $I_{AG}(T,t_1,q_2 \cdot (d + 1),d,c_1 + c_2)$ set.

Proof. For any $\emptyset \subset T' \subset T$ such that there is a $(T',T \setminus T',t_1,d)$-cut on $G$, let $\alpha$ denote the size of minimum $(T',T \setminus T',t_1,d)$-cut. We show that there exists a $(T',T \setminus T',q_2 \cdot (d + 1),\alpha)$-cut $(V',V \setminus V')$ such that every connected component of $G \setminus (E' \cup E'')$ contains at most $\max\{\alpha - c_1 - c_2,0\}$ edges of $\partial_G(V')$. Then by Definition 5.3 the lemma holds.

If $\alpha \leq c_1$, then by definition, there is a $(T',T \setminus T',q_1,\alpha)$-cut whose cut-set is in $E'$, and we are done. In the rest of this proof, we consider the case of $\alpha > c_1$.

By definition, there is a $(T',T \setminus T',q_1,\alpha)$-cut $(V',V \setminus V')$ such that in $G \setminus E'$, every connected component contains at most $\max\{\alpha - c_1,0\}$ edges of $\partial_G(V')$. Let $P$ be the vertex partition induced by connected components of $G \setminus E'$. Fix a $P_i \in P$ such that $G[P_i]$ contains at least one edge of $\partial_G(V')$. Let $V' = V' \cap P_i$, $F = \partial_G(V'\mid_{G[P_i]})$, $T_i = (T \cup \text{End}(\partial_G(P_i))) \cap P_i$ and $T_0 = T_i \cap V'$. $F$ is the cut-set of a $(T_i,T \setminus T_i \cap V',q_1,|F|)$-cut on $G[P_i]$ such that $|F| \leq \alpha - c_1$.

Since $t_2 \geq q_1$, by definition of $E''$, there exists a $(T_i,T \setminus T_i \cap V',q_2,|F|)$-cut on $G[P_i]$ such that every connected component of $(G[P_i] \setminus E'')$ contains at most $\max\{|F| - c_2,0\}$ edges of $\partial_G(P_i)$. Now we look at $G \setminus (E' \cup E'')$. Each connected component of $G \setminus (E' \cup E'')$ is a connected component of $G[P_i] \setminus (E''\mid_{G[P_i]})$ for some $P_i \in P$. Let

$$V^\dagger = \left( \bigcup_{P_i : V' \cap P_i \neq \emptyset \text{ and } P_i \not\subset V'} P_i' \right) \cup \left( \bigcup_{P_i : P_i \not\subset V'} P_i \right).$$

By Lemma 5.6 $(V^\dagger,V \setminus V^\dagger)$ is a $(T',T \setminus T')$-cut such that each connected component of $G \setminus (E' \cup E'')$ contains at most $\max\{\alpha - c_1 - c_2,0\}$ edges of $\partial_G(V')$. Since $|\partial_G(V')| \leq \alpha$, there are at most $\alpha$ different $P_i$'s that contain edges of $\partial_G(V')$. Since each $|P_i'| \leq q_2$ for any $P_i \in P$ satisfying $V' \cap P_i \neq \emptyset$ and $P_i \not\subset V'$, and $\bigcup_{P_i : P_i \not\subset V'} P_i \leq q_1$, we have

$$|V^\dagger| \leq q_2 \cdot \alpha + q_1 \leq q_2 \cdot (d + 1).$$

Hence, $(V^\dagger,V \setminus V^\dagger)$ is a $(T',T \setminus T',q_2 \cdot (d + 1),\alpha)$-cut on $G$ such that each connected component of $G \setminus (E' \cup E'')$ contains at most $\max\{\alpha - c_1 - c_2,0\}$ edges of $\partial_G(V^\dagger)$.\]
By Lemma 5.5, we have the following corollary.

**Corollary 5.6.** Let $G = (V, E)$ be a connected graph and $T \subseteq V$ be a set of vertices. For two integers $d \geq c$, let $t_1, q_1, t_2, t_2, \ldots, t_c, q_c$ be $2c$ integers satisfying

$$t_i \leq q_i \text{ for any } 1 \leq i \leq c, \text{ and } (d + 1) \cdot q_i \leq t_{i+1} \text{ for any } 1 \leq i \leq c - 1,$$

and $E_1, \ldots, E_c$ be edge sets such that for any $1 \leq i \leq c$, $E_i$ is an

$$IA_{G \setminus (\bigcup_{j=1}^{c-1} E_j)} \left( T \cup \text{End}(\bigcup_{j=1}^{c-1} E_j), t_i, q_i, d - i + 1, 1 \right)$$

set, then $\bigcup_{j=1}^{c} E_j$ is an $IA_{G}(t_1, q_1 \cdot (d + 1), d, c)$ set.

We say an $IA_{G}(T, t, q, d, c)$ set is derived from an $IA_{G}(T, t, q', d, c')$ set for some $q \geq q'(d+1), c > c'$ if

$$IA_{G}(T, t, q', d, c') \subseteq IA_{G}(T, t, q, d, c),$$

and $IA_{G}(T, t, q, d, c) \setminus IA_{G}(T, t, q', d, c')$ is an

$$IA_{G \setminus IA_{G}(T, t, q', d, c')}(T \cup \text{End}(IA_{G}(T, t, q', d, c')), q', q'(d+1), d - c', c - c')$$

set.

**Lemma 5.7.** Let $G = (V, E)$ be a connected graph, and $T \subseteq V$ be a set of vertices. Let $E'$ be a set of edges satisfying the following two conditions:

1. $E'$ is the set of intercluster edges of a vertex partition of graph $G$.
2. For every $\emptyset \subseteq T' \subseteq T$ satisfying that the minimum $(T', T \setminus T', t, d)$-cut is of size $\alpha \leq d$, and there is a minimum $(T', T \setminus T', t, d)$-cut being a simple cut, there is a $(T', T \setminus T', q, \alpha)$-cut $(V', V \setminus V')$ such that every connected component of $G \setminus E'$ contains at most $\alpha - 1$ edges of $\partial G(V')$.

Then $E'$ is an $IA_{G}(T, t, q + t, d, 1)$ set.

**Proof.** For a $\emptyset \subseteq T_1 \subseteq T$ such that there is a $(T_1, T \setminus T_1, t, d)$-cut, let $(V_1, V \setminus V_1)$ be a minimum $(T_1, T \setminus T_1, t, d)$-cut.

Let $V'$ be a subset of $V_1$ that forms a connected component of $G[V_1]$. $(V' \setminus V \setminus V')$ is a simple cut, and is also a minimum $(V' \cap T, T \setminus (V' \cap T), \partial G(V'))$-cut satisfying $V' \cap T \neq \emptyset$, otherwise, there is a $(T_1, T \setminus T_1, t, d)$-cut with size smaller than $|\partial G(V')|$. By the definition of $E'$, there is a $(V' \cap T, (V \setminus V' \cap T, q, |\partial G(V')|)$-cut $(V', V \setminus V')$ such that every connected component of $G \setminus E'$ contains at most $|\partial G(V')| - 1$ elements of $\partial G(V')$.

Since every connected component of $G \setminus ((\partial G(V_1) \setminus \partial G(V')) \cup \partial G(V''))$ is impossible to contain a vertex from $T_1$, and another vertex from $T \setminus T_1$, there is a subset $E'$ of $(\partial G(V_1) \setminus \partial G(V')) \cup \partial G(V'')$ corresponding to the cut-set of a $(T_1, T \setminus T_1, q + t, |\partial G(V_1)|)$-cut.

If $|E'| < |\partial G(V_1)|$, then there is a $(T_1, T \setminus T_1, q + t, |E'|)$-cut in graph $G$, and thus there is a $(T_1, T \setminus T_1, q + t, |\partial G(V_1)|)$-cut such that every connected component of $G \setminus E'$ contains at most $|\partial G(V_1)| - 1$ elements of the cut-set. Otherwise, $|\partial G(V')| = |\partial G(V'')|$ and $E'$ contains all the edges of $(\partial G(V_1) \setminus \partial G(V')) \cup \partial G(V'')$. Since no connected component of $G \setminus E'$ contains all the edges of $\partial G(V')$, no connected component of $G \setminus E'$ contains all the edges of $E'$. $\square$

We remark that if the second condition of Lemma 5.7 change to “for every $\emptyset \subseteq T' \subseteq T$ satisfying that all the minimum $(T', T \setminus T', t, d)$-cut are simple cuts of size $\alpha \leq d$, there is a $(T', T \setminus T', q, \alpha)$-cut $(V', V \setminus V')$ such that every connected component of $G \setminus E'$ contains at most $\alpha - 1$ edges of $\partial G(V')$", the lemma still holds. But we prefer the current version of Lemma 5.7 because it is less convenient to use an algorithm to check whether all the minimum $(T', T \setminus T', t, d)$-cuts are simple cuts.
6 Decremental update algorithm

In this section, we let \( \mathcal{G}_0 = (V_0, E_0) \) be a connected graph, \( T_0 \subseteq V_0 \) be a set of vertices, and \( V \subseteq V_0 \) be a set of vertices such that \( \mathcal{G}_0[V] \) is a connected graph. Suppose \( \mathcal{G} = (V, E) \) denotes \( \mathcal{G}_0[V] \), \( S = \text{End}(\partial_{\mathcal{G}_0}(V)) \cap V \), and \( T = (T_0 \cap V) \setminus S \).

Our goal is to obtain an \( \mathcal{IA}_G(S \cup T, t, q + t, c, 1) \) set by adding a small number of edges to \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q, d, 2c + 1) \) for an \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q, d, 2c + 1) \) set with respect to parameters \( q \geq t \) and \( d \geq 2c + 1 \).

**Definition 6.1.** For a fixed \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q, d, 2c + 1) \) set, a set of edges \( E' \subseteq E \) is a repair set of \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q, d, 2c + 1) \) with respect to \( G \) if

\[
(\mathcal{IA}_{\mathcal{G}_0}(T_0, t, q, d, 2c + 1)|_{(G)} \cup E')
\]

is an \( \mathcal{IA}_G(S \cup T, t, q + t, c, 1) \) set.

The following lemma shows that there is always a repair set of size linear with respect to \( |S| \).

**Lemma 6.2.** For an \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q, d, 2c) \) set and an \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q', d, 2c + 1) \) set such that \( q' \geq 2t \) and the \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q', d, 2c + 1) \) set is derived from the \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q, d, 2c) \) set, there is a repair set of \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q', d, 2c + 1) \) with respect to \( G \) of size \( |S| (24c^3 + 24c^2 + 4c) \).

The following lemma shows that there is an efficient algorithm to compute a repair set as Lemma 6.2.

**Lemma 6.3.** Let \( G = (V, E) \) be a graph, \( T_0 \subseteq V_0 \) be a set of vertices, and \( V \subseteq V_0 \) be a set of vertices such that \( G_0[V] \) is a connected graph of at most \( m \) vertices and distinct edges. Let \( \mathcal{G} = (V, E) \) denote \( G_0[V] \), \( S = \text{End}(\partial_{\mathcal{G}_0}(V)) \cap V \) and \( T = (T_0 \cap V) \setminus S \). For an \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q', d, 2c + 1) \) set such that \( q' \geq 2t \), and the \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q', d, 2c + 1) \) set is derived from an \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q, d, 2c) \) set, given access to

\[
\mathcal{DG}(G, \emptyset), \mathcal{DG}(G, T_0 \cap V) \text{ and } \mathcal{DG} \left( G \setminus \left( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q, d, 2c)|_{(G)} \right) \cup \emptyset \right),
\]

there is an algorithm \( \text{REPAIR-SET} \) with running time \( O(|S|^2(2q)^{2c+5c+3} \text{poly}(c) \text{polylog}(m)) \) to find a repair set for the \( \mathcal{IA}_{\mathcal{G}_0}(T_0, t, q', d, 2c + 1) \) set with respect to \( G \) of size \( |S| (24c^3 + 24c^2 + 4c) \).

Lemma 6.3 also implies an algorithm to construct an \( \mathcal{IA}_G(T, t, 3t, d, 1) \) set for a given graph \( G \) and a set of vertices \( T \subseteq V \), because an empty set is an \( \mathcal{IA}_G(\emptyset, t, q, d, c) \) set for any parameters \( t, q, d, c \) by Definition 6.3.

**Corollary 6.4.** Let \( G = (V, E) \) be a connected graph of at most \( m \) vertices and distinct edges, \( t, q, d \) be three positive integers such that \( d \geq 1 \), \( q \geq 3t \). Given access to \( \mathcal{DG}(G, \emptyset) \) and a set of vertices \( T \subseteq V \), there is an algorithm to construct an \( \mathcal{IA}_G(T, t, q, d, 1) \) set in time \( O(|T| \cdot (2q)^{d+5d+3} \text{poly}(d) \text{polylog}(m)) \) such that the \( \mathcal{IA}_G(T, t, q, d, 1) \) set contains at most \( |T| (24d^3 + 24d^2 + 4d) \) edges.

In the rest of this section, we prove Lemma 6.2 and Lemma 6.3.
6.1 Proof of Lemma 6.2

We characterize all the bipartitions \((S' \cup T', (S \cup T) \setminus (S' \cup T'))\) for \(\emptyset \subseteq S' \cup T' \subseteq S \cup T\) of \(G\) by three types:

**Definition 6.5.** For \(S' \subseteq S, T' \subseteq T\) satisfying \(\emptyset \subseteq S' \cup T' \subseteq S \cup T\), a bipartition \((S' \cup T', (S \cup T) \setminus (S' \cup T'))\) is

1. a type 1 bipartition if \(\emptyset \subset S' \subset S\),
2. a type 2 bipartition if \(S' = \emptyset\),
3. a type 3 bipartition if \(S' = S\).

**Definition 6.6.** For an IA \(IA\) \((T, V, W, S, \beta)\) is the set of intercluster edges of a vertex partition of \(V\).

For an IA \(IA\) \((T, V, W, S, \beta)\), any vertex partition \(V\) is a type \(\alpha\) repair set for \(IA\) \((T, V, W, S, \beta)\) if \(\emptyset \subset S' \subset S\), a set satisfying one of the following conditions hold:

1. \(W\) is the set of intercluster edges of a vertex partition of \(V\).
2. For every type \(\alpha\) bipartition \((S' \cup T', (S \cup T) \setminus (S' \cup T'))\) such that there is a minimum \((S' \cup T', (S \cup T) \setminus (S' \cup T'), t, c)\)-cut that is simple, let \(\beta\) be the size of the minimum \((S' \cup T', (S \cup T) \setminus (S' \cup T'), t, c)\)-cut. There is a \((S' \cup T', (S \cup T) \setminus (S' \cup T'), t, c)\)-cut \((V', V \setminus V')\) such that every connected component of \(G \setminus (W \cup IA_{G_0}(T_0, t, q, d, 2c + 1))\) contains at most \(\beta - 1\) edges of \(\partial_G(V')\).

By Definition 6.3, Definition 6.6 and Lemma 5.7, we have the following claim.

**Claim 6.7.** For any \(IA_{G_0}(T_0, t, q, d, 2c + 1)\) set satisfying \(q' \geq 2t\), the union of a type 1 repair set, a type 2 repair set and a type 3 repair set for the \(IA_{G_0}(T_0, t, q, d, 2c + 1)\) set with respect to \(G\) is a repair set of the \(IA_{G_0}(T_0, t, q', d, 2c + 1)\) set with respect to \(G\).

**Lemma 6.8.** For an \(IA_{G_0}(T_0, t, q, d, 2c)\) set, an \(IA_{G_0}(T_0, t, q', d, 2c + 1)\) set for some \(q' \geq q \cdot (d + 1)\) that is derived from the \(IA_{G_0}(T_0, t, q, d, 2c)\) set, and \(S' \subset S, T' \subset T\) satisfying \(\emptyset \subset S' \cup T' \subset S \cup T\) such that there is a minimum \((S' \cup T', (S \cup T) \setminus (S' \cup T'), t, c)\)-cut \((V', V \setminus V')\) satisfying one of the following two conditions

1. \((V', V \setminus V')\) is intercepted by \(IA_{G_0}(T_0, t, q, d, 2c)\).
2. \(\partial_G(V')\) is contained in a connected component of \(G \setminus (IA_{G_0}(T_0, t, q, d, 2c))\) that does not have any vertex from \(S\).

there is a \((S' \cup T', (S \cup T) \setminus (S' \cup T'), q', |\partial_G(V')|)\)-cut \(C\) such that every connected component of \(G \setminus IA_{G_0}(T_0, t, q', d, 2c + 1))\) contains at most \(|\partial_G(V')| - 1\) edges of the cut-set of \(C\).

**Proof.** Since the \(IA_{G_0}(T_0, t, q', d, 2c + 1)\) set is derived from the \(IA_{G_0}(T_0, t, q, d, 2c)\) set, we have

\[
IA_{G_0}(T_0, t, q, d, 2c) \subseteq IA_{G_0}(T_0, t, q', d, 2c + 1).
\]

The first condition implies that \(IA_{G_0}(T_0, t, q', d, 2c + 1))\) intercepts \((V', V \setminus V')\), and thus the lemma holds.

Now consider the second condition of the lemma. Suppose \(\partial_G(V')\) is contained in a connected component of \(G \setminus IA_{G_0}(T_0, t, q, d, 2c)\), denoted by \(G[P]\), satisfying \(P \cap S = \emptyset\). \(\partial_G(V')\) induces a \((t, c)\)-cut on \(G[P]\) that partition \(T^+ = (T \cup End(\partial_G(P))) \cap P\) into \(T = T^+ \cap (V' \cap P)\) and \(T^+ \setminus T = T^+ \cap ((V \setminus V') \cap P)\). Note that \(T^+\) is not an empty set, otherwise \(V' \cap (S \cup T) = \emptyset\). Hence, \(\partial_G(V')\) induces a \((T^+, T^+ \setminus T^+, t, |\partial_G(V')|)\)-cut for \(G[P]\).
On the other hand, since the $\text{IA}_{G_0}(T_0, t, q', d, 2c + 1)$ set is derived from the $\text{IA}_{G_0}(T_0, t, q, d, 2c)$ set, we have that $\text{IA}_{G_0}(T_0, t, q', d, 2c + 1)|_{G[\Gamma]}$ is an $\text{IA}_{G[\Gamma]}(T^\dagger, q, q'/(d + 1), d - 2c, 1)$ set. By Definition 6.9 there is a $(T^\circ, T^\dagger \setminus T^\circ, q'/(d + 1), |\partial_G(V')|)$-cut $C'$ of graph $G[\Gamma]$ such that every connected component of $G[\Gamma] \setminus (\text{IA}_{G_0}(T_0, t, q', d, 2c + 1)|_{G[\Gamma]})$ contains at most $|\partial_G(V')| - 1$ edges of the cut-set of $C'$. By Lemma A.6, the cut-set of $C'$ for graph $G[\Gamma]$ induces a $(S' \cup T', (S \cup T) \setminus (S' \cup T'), t + q'/(d + 1), |\partial_G(V')|)$-cut for graph $G$. Since $t + q'/(d + 1) \leq q'$, $C'$ is a $(S' \cup T', (S \cup T) \setminus (S' \cup T'), q', |\partial_G(V')|)$-cut such that every connected component of $G \setminus (\text{IA}_{G_0}(T_0, t, q', d, 2c + 1)|_{G})$ contains at most $|\partial_G(V')| - 1$ edges of the cut-set of $C'$.

Together with Lemma 5.7 we only need to consider bipartitions $(S' \cup T', (S \cup T) \setminus (S' \cup T'), t, c)$-cuts that is a simple cut with cut-set contained in a connected component of $G \setminus (\text{IA}_{G_0}(T_0, t, q, d, 2c)|_{G})$ containing at least one vertex from $S$.

### 6.1.1 Elimination procedure

We first define $(t, c)$-realizable pair.

**Definition 6.9.** For a connected graph $G = (V, E)$, a pair $(E', V')$ for some $V' \subseteq V$, $E' \subseteq E$ is a $(t, c)$-realizable pair if

1. $(V', V \setminus V')$ is a simple $(t, c)$-cut.
2. $E' \subseteq \partial_G(V')$ is the cut-set of an atomic cut.

Two $(t, c)$-realizable pairs $(E', V'), (E'', V'')$ are $S$-equivalent if $L' \cap S = L'' \cap S$, where $(L', V \setminus L')$ is the cut induced by $E'$ such that $V' \subseteq L'$, and $(L'', V \setminus L'')$ is the cut induced by $E''$ such that $V'' \subseteq L''$.

As one of the main tools to construct repair sets, we consider the following elimination procedure for a given set $\Gamma$ of $S$-equivalent $(t, c)$-realizable pairs.

**Input:** A set $\Gamma$ of $S$-equivalent $(t, c)$-realizable pairs satisfying the condition that there is a vertex $x$ such that for every $(E', V') \in \Gamma$, the atomic cut induced by $E'$ separates $V'$ and $x$.

**Output:** A set of edges $W$ that is the set of intercluster edges of a vertex partition.

Let $W = \emptyset$ initially. Repeat the following process until $\Gamma$ is an empty set:

1. Take a pair $(E', V') \in \Gamma$ such that no $(E'', V'') \in \Gamma$ satisfies $\text{End}(E'') \subseteq V'$.
2. Put $\partial_G(V')$ into $W$.
3. Remove all the $(E'', V'')$ from $\Gamma$ if $W$ intercepts a $(V'' \cap (S \cup T), (V \setminus V'') \cap (S \cup T), |V''|, |\partial_G(V'')|)$-cut.

**Figure 4:** Elimination procedure

Since there is a vertex $x$ such that for every $(E', V') \in \Gamma$, the atomic cut induced by $E'$ separates $V'$ and $x$, for $(E', V'), (E'', V'') \in \Gamma$, if $E''$ is in $G[V']$, then $L''$ is a strict subset of $L'$, where $(L', V \setminus L')$ is the cut induced by $E'$ such that $L'$ and $V'$ are on the same side, and $(L'', V \setminus L'')$
is the cut induced by $E''$ such that $L''$ and $V''$ are on the same side. So step (1) of the elimination procedure always finds a $(t, c)$-realizable pair in $\Gamma$ if $\Gamma \neq \emptyset$. Thus, the elimination procedure always stops.

We show that if $\Gamma$ satisfies certain conditions, then the elimination procedure outputs a small set of edges that intercepts a $(V', (S \cup T), (V' \setminus (S \cup T)), |V'|, |\partial_G(V')|)$-cut for each $(E', V') \in \Gamma$.

**Lemma 6.10.** For any $\text{IA}_G(T_0, t, q, d, 2c)$ set and a set of vertices $V^* \subseteq V$ that forms a connected component of $G \setminus \text{IA}_G(T_0, t, q, d, 2c)|_G$, let $\Gamma$ be a set of $(t, c)$-realizable pairs $\{(E_1, V_1), \ldots, (E_k, V_k)\}$ of graph $G$ satisfying the following conditions: $(S_i$ denotes $V_i \cap S$, $T_i$ denotes $V_i \cap T$, and $(L_i, V \setminus L_i)$ denotes the atomic cut induced by $E_i$ satisfying $V_i \subseteq L_i$ for every $1 \leq i \leq k$.)

1. The cut-set of $(V_i, V \setminus V_i)$ is in $G[V^*]$.
2. One of the following two conditions hold:
   - (a) $L_i \cap S = L_j \cap S$ for any $i, j$.
   - (b) $V_i \cap S = V_j \cap S$ for any $i, j$
3. There is a vertex $x \in V$ such that for every $i$, $x \in V \setminus L_i$.
4. $V_i \cap V_j \neq \emptyset$ for any $i \neq j$.

Then the elimination procedure on $\Gamma$ outputs a set $W$ of $4c^3 + 4c^2$ edges intercepting a $(V_i \cap (S \cup T), (V \setminus V_i) \cap (S \cup T), |V_i|, |\partial_G(V_i)|)$-cut for every $(E_i, V_i) \in \Gamma$.

To prove Lemma 6.10, we first prove the following lemma.

**Lemma 6.11.** For any $\text{IA}_G(T_0, t, q, d, 2c)$ set and a set of vertices $V^* \subseteq V$ that forms a connected component of $G \setminus \text{IA}_G(T_0, t, q, d, 2c)|_G$, a set of $(t, c)$-realizable pairs $\{(E_1, V_1), \ldots, (E_k, V_k)\}$ of graph $G$ satisfying the conditions below contains at most $4c^2 + 4c$ pairs. $(S_i$ denotes $V_i \cap S$, $T_i$ denotes $V_i \cap T$, and $(L_i, V \setminus L_i)$ denotes the cut induced by $E_i$ such that $V_i \subseteq L_i$ for any $1 \leq i \leq k$.)

1. $V_i \subseteq V_2 \subseteq \ldots \subseteq V_k$.
2. $L_1 \subseteq L_2 \subseteq \ldots \subseteq L_k$.
3. $V_j \setminus L_i \cap S = \emptyset$ for any $1 \leq i < j \leq k$.
4. $\partial_G(V_i)$ are in $G[V^*]$ for any $1 \leq i \leq k$.
5. For every $i > 1$, $\bigcup_{j=1}^{i-1} \partial_G(V_j)$ does not intercept any $(S_i \cup T_i, (S \cup T) \setminus (S_i \cup T_i), |V_i|, |\partial_G(V_i)|)$-cut.

**Proof.** For $j > i$, let $V_{i,j} = V_j \setminus L_i$, and $T_{i,j} = V_{i,j} \cap T$. We show the following properties:

1. $|V_{i,j}| \leq t$.
2. $\text{End}(\partial_G(V_i)) \subseteq V_j$ and $V_{i,j} \neq \emptyset$.
3. $\partial_G(V_{i,j}) = E_i \cup \left(\partial_G(V_j)|_{G[V \setminus L_i]}\right)$.
4. $|\partial_G(V_{i,j})| \leq 2c$.
5. $S \cap V_{i,j} = \emptyset$.
6. One of the following two conditions hold: (i) $T_{i,j} \neq \emptyset$; (ii) $T_{i,j} = \emptyset$ and $|E_i| > |E_j|$.
7. For any $i \leq i' < j \leq j$, $T_{i',j'} \subseteq T_{i,j}$.

Property (a) is obtained from the fact that $V_{i,j} \subseteq V_j$ and $|V_j| \leq t$ by the fact that $(E_j, V_j)$ is a $(t, c)$-realizable pair.

Property (b) is obtained from the fact that $V_i \subseteq V_j$ by condition (1) and $\partial_G(V_i)$ intercepts $(V_j, V \setminus V_j)$ if $\partial_G(V_i) \cap \partial_G(V_j) \neq \emptyset$, violating condition (5).
For property (c), by property (b), $E_i$ is a subset of $\partial_G(V_{i,j})$, and by the definition of $V_{i,j}$, $\partial_G(V_j)|_{G[V \setminus L_i]}$ is a subset of $\partial_G(V_{i,j})$. Now we show that every edge of $\partial_G(V_{i,j})$ is an edge in either $E_i$ or $\partial_G(V_j)|_{G[V \setminus L_i]}$. Let $(x, y)$ be an edge in $\partial_G(V_{i,j}) = \partial_G(V_j \setminus L_i)$. Without loss of generality, assume $x \in V_{i,j} = V_j \setminus L_i$ and

$$y \in V \setminus V_{i,j} = V \setminus (V_j \setminus L_i) = (V \setminus V_j) \cup L_i = ((V \setminus V_j) \setminus L_i) \cup L_i.$$  

If $y \in (V \setminus V_j) \setminus L_i$, then $(x, y) \in \partial_G(V_j)|_{G[V \setminus L_i]}$. If $y \in L_i$, then $(x, y) \in E_i$.

Property (d) is implied by property (c) and the fact that both $(E_i, V_i)$ and $(E_j, V_j)$ are $(t, c)$-realizable pairs.

Property (e) is obtained from condition (3) of the lemma.

Now we prove property (f). Suppose $T_{i,j} = \emptyset$. We show $|E_i| > |E_j|$ by contradiction. If

$$|E_i| \leq |\partial_G(V_j)|_{G[V \setminus L_i]}|,$$

then cut $(V_j \setminus V_{i,j}, V \setminus (V_j \setminus V_{i,j}))$ satisfies the following conditions

- $|V_j \setminus V_{i,j}| < |V_j| \leq t$.
- $(V_j \setminus V_{i,j}, V \setminus (V_j \setminus V_{i,j}))$ and $(V_j, V \setminus V_j)$ induce the same partition on $S \cup T$ by $T_{i,j} = V_{i,j} \cap T = \emptyset$ and property (e).
- The size of cut $(V_j \setminus V_{i,j}, V \setminus (V_j \setminus V_{i,j}))$ is smaller than or equal to the size of cut $(V_j, V \setminus V_j)$ by properties (b) and (c).

Thus $\partial_G(V_i)$ intercepts a $(S_j \cup T_j, (S \cup T) \setminus (S_j \cup T_j), |V_j|, |\partial_G(V_j)|)$-cut by Lemma A.5 contradicting condition (5) of the lemma. Hence, we have

$$|E_i| > |\partial_G(V_j)|_{G[V \setminus L_i]}|.$$  

Note that $E_j \subseteq \partial_G(V_j)$. Fix a $(x, y) \in E_j$. One of $x$ and $y$ is in $V \setminus L_j$. Suppose $x \in (V \setminus L_j) \subseteq (V \setminus L_i)$. Since $(L_i, V \setminus L_i)$ is atomic, $y \in V \setminus L_i$. Therefore, $E_j \subseteq \partial_G(V_j)|_{G[V \setminus L_i]}$ and

$$|E_j| \leq |\partial_G(V_j)|_{G[V \setminus L_i]}| < |E_i|.$$  

So property (f) holds.

Property (g) is obtained by the definition of $V_{i,j}$ and the conditions (1), (2) of the lemma.

Let $w_1 < w_2 < \cdots < w_\ell$ be all the integers in $[k]$ such that $w_i = 1$ or $T_{w_i-1,w_i} \neq \emptyset$. In the rest of this proof, we prove $\ell \leq 4c + 4$ by contradiction. Then together with properties (f) and (g), $k \leq 4c^2 + 4c$.

Properties (a), (d), (e) and the construction of $w_1, \ldots, w_\ell$ imply that there is a $(T_{w_i,w_j}, T_0 \setminus T_{w_i,w_j}, t, 2c)$-cut on $G_0$ for any $1 \leq i < j \leq \ell$.

By Claim 5.4, there is a $(T_{w_i,w_j}, T_0 \setminus T_{w_i,w_j}, q, 2c)$-cut of $G_0$ whose cut-set is a subset of the $\text{IA}_{G_0}(T_0, t, q, d, 2c)$ set. Let $(Q_{w_i,w_j}, V_0 \setminus Q_{w_i,w_j})$ be such a cut for graph $G_0$. By Definition 5.3 since the $\text{IA}_{G_0}(T_0, t, q, d, 2c)$ set is the set of intercluster edges of a vertex partition, $Q_{w_i,w_j}$ is a union of some connected components of $G_0 \setminus \text{IA}_{G_0}(T_0, t, q, d, 2c)$.

We show that $V^*$ is not a subset of $Q_{w_2c+3,w_\ell}$ by contradiction. Assume $V^*$ is a subset of $Q_{w_2c+3,w_\ell}$. Let $T^* = T \cap V^*$. Since $V^*$ is a vertex subset from a connected component of $G_0 \setminus \text{IA}_{G_0}(T_0, t, q, d, 2c)$, by Claim 5.4, $T^*$ has a non-empty intersection with at most one of $T_{w_i,w_i+1}$ for all the $1 \leq i < \ell$. For an arbitrary $1 \leq i \leq 2c + 2$ such that $T_{w_i,w_i+1}$ does not contain any
vertex from $T^*$, we have $\partial_G(V_{w_i,w_{i+1}}) \subseteq \partial_G(V_{w_i})$ and $\partial_G(V_{w_{i+1}})$ by property (c). Note that $T_{w_i,w_{i+1}}$ is not an empty set, and $V^*$ is a subset of $Q_{w_2c+3,w_2\ell}$. Since $Q_{w_2c+3,w_2\ell}$ does not contain $T_{w_i,w_{i+1}}$ and $\partial_G(V_{w_i}) \cup \partial_G(V_{w_{i+1}})$ are edges in $G[V^*]$ by condition (4), there must be an edge within $G[V_{w_i,w_{i+1}}]$ that is in $\partial_G(Q_{w_2c+3,w_2\ell})$. Since $V_{w_i,w_{i+1}} \cap V_{w_j,w_{j+1}} = \emptyset$ for any $i \neq j$, $\partial_G(Q_{w_2c+3,w_2\ell})$ contains at least $2c + 1$ different edges, contradicting the fact that $(Q_{w_2c+3,w_2\ell}, V_0 \setminus Q_{w_2c+3,w_2\ell})$ is a cut of size at most $2c$ for $G_0$. 

Hence, $V^*$ is not a subset of $Q_{w_2c+3,w_2\ell}$. On the other hand, since $V^*$ is a connected component of $G \setminus (\text{IA}_{G_0}(T_0,t,q,d,2c))_{G}$ and $Q_{w_2c+3,w_2\ell}$ is a union of connected components of $G_0 \setminus \text{IA}_{G_0}(T_0,t,q,d,2c)$, $V^* \cap Q_{w_2c+3,w_2\ell} = \emptyset$.

Meanwhile, since $(Q_{w_2c+3,w_2\ell}, V_0 \setminus Q_{w_2c+3,w_2\ell})$ is a cut for $G_0$ of size at most $2c$, $(Q_{w_2c+3,w_2\ell} \cap V, V \setminus (Q_{w_2c+3,w_2\ell} \cap V))$ is a cut of graph $G$, and all the edges of $\partial_G(Q_{w_2c+3,w_2\ell} \cap V)$ are in $\text{IA}_{G_0}(T_0,t,q,d,2c)_{G}$, and thus $G[Q_{w_2c+3,w_2\ell} \cap V]$ has at most $2c$ connected components by Claim A.3. Since $Q_{w_2c+3,w_2\ell} \cap V$ has no intersection with $V^*$, no edge of $E_i = \partial_G(L_i)$ is in $(Q_{w_2c+3,w_2\ell} \cap V)$ for any $1 \leq i \leq \ell$.

Let $R_1, R_2, \ldots, R_h$ for some $0 \leq h \leq 2c$ be all the non-empty intersections of $T$ and a connected component of $G(Q_{w_2c+3,w_2\ell} \cap V)$. We have $\bigcup_{i=1}^h R_i = T_{w_2c+3,w_2\ell}$.

For every $2c + 3 < i < \ell$, cut $(L_{w_i}, V \setminus L_{w_i})$ satisfies the following conditions:

(i) $T_{w_{i-1},w_i} \subseteq \bigcup_{j=1}^h R_j$ is a subset of $T \cap L_{w_i}$, but not $T \cap L_{w_{i-1}}$.

(ii) If $T \cap L_{w_i}$ has a non-empty intersection with $R_j$ for some $1 \leq j \leq h$, then $R_j \subseteq T \cap L_{w_i}$.

Condition (i) is obtained by property (g) and the fact that each $(L_{w_i}, V \setminus L_{w_i})$ is an atomic cut. Condition (ii) is obtained by the following facts: (1) No edge of $E_{w_i} = \partial_G(L_{w_i})$ is in $G(Q_{w_2c+3,w_2\ell} \cap V)$; (2) Vertices of $R_j$ are connected in $G[V \setminus V^*]$.

Thus, $\ell \leq 2c + 3 + h + 1 \leq 4c + 4$.

**Proof of Lemma 6.11**. By the definition of the elimination procedure, for every $(E',V') \in \Gamma$, $W$ intercepts a $(V' \cap (S \cup T), (V \setminus V') \cap (S \cup T), |V'|, |\partial_G(V')|)$-cut. In the rest of this proof, we show that $|W| \leq 4c^3 + 4c^2$ by showing that all the pairs used to construct $W$ satisfy the conditions of Lemma 6.11 and then the lemma follows.

Let $(E_i, V_i)$ be the $i$-th pair selected by the elimination procedure on $\Gamma$ to construct $W$.

For the first condition of Lemma 6.11, we show that $V_i \subseteq V_j$ for any $i < j$. By the description of the elimination procedure, $\partial_G(V_i) = \emptyset$, and it is impossible to have an edge of $\partial_G(V_j)$ in $G[V_i]$, and another edge of $\partial_G(V_j)$ in $G[V \setminus V_i]$. Hence, $\partial_G(V_i) \cap \partial_G(V_j) = \emptyset$. By the elimination procedure, it is also impossible that $\partial_G(V_j)$ are in $G[V_i]$. Hence, $\partial_G(V_j)$ are in $G[V \setminus V_i]$. With the condition (4) of the Lemma 6.10 and the fact that $(V_i, V \setminus V_i)$ and $(V_j, V \setminus V_j)$ are simple cuts, $V_i \subseteq V_j$.

For the second condition of Lemma 6.11, we show that $L_i \subseteq L_j$ for any $i < j$. By the third condition of the Lemma 6.10, we have $x \in V \setminus L_i$ and $x \in V \setminus L_j$. If $E_i = \partial_G(L_i)$ intersects $(L_j, V \setminus L_j)$, then $(E_j, V_j)$ is not selected to construct $W$, hence $E_i$ does not intersect $(L_j, V \setminus L_j)$. By the first condition of Lemma 6.11 and the elimination procedure, $E_j$ does not intersect $(L_i, V \setminus L_i)$. By Lemma A.2, $(L_i, V \setminus L_i)$ and $(L_i, V \setminus L_i)$ are parallel, and thus $V \setminus L_i$ is a subset or a superset of $V \setminus L_j$. Since $V_i \subseteq V_j$, $E_j$ is not in $G[V_i]$, and thus $L_i \subseteq L_j$.

Now we prove the third condition of Lemma 6.11. If $L_i \cap S = L_j \cap S$ for any $i, j$, then for any $i < j$, since $V_j \subseteq L_j$ we have $(V_j \setminus L_j) \cap S \subseteq (L_j \setminus L_i) \cap S = \emptyset$. If $V_i \cap S = V_j \cap S$ for any $i, j$, we have $(V_j \setminus L_i) \cap S \subseteq (V_j \setminus V_i) \cap S = \emptyset$. Hence, for either cases of the second condition of the current lemma, the third condition of Lemma 6.11 holds.

The forth condition of Lemma 6.11 is implied by the first condition of the current lemma.

The fifth condition of Lemma 6.11 is implied by the definition of elimination procedure.
6.1.2 Type one repair set

We show that there is a type one repair set of size $O(|S|c^3)$.

**Definition 6.12.** An $(S,t,c)$-bipartition system $B$ is a set of $(t,c)$-realizable pairs

$$\{(E_1,Q_1),(E_2,Q_2),\ldots,(E_k,Q_k)\}$$

for graph $G$ satisfying the following conditions

1. For every $(E_i,Q_i) \in B$, $(Q_i, V \setminus Q_i)$ is a simple $(t,c)$-cut such that $\emptyset \subsetneq Q_i \cap S \subseteq S$. $E_i \subseteq \partial_G(Q_i)$ induces an atomic cut $(L_i,V \setminus L_i)$ satisfying $Q_i \subseteq L_i$ such that $\emptyset \subsetneq S_i = L_i \cap S \subseteq S$.
2. For any $(E_i,Q_i)$ and $(E_j,Q_j)$ in $B$, $S_i \neq S_j$, and one of the following two conditions hold:
   a. $S_i = S \setminus S_j$
   b. One of $S_i$ and $S \setminus S_i$ is a strict subset of either $S_j$ or $S \setminus S_j$.
3. A $(E',Q')$ satisfying (1) is not in $B$ if one of the following conditions hold:
   a. Adding $(E',Q')$ to $B$ violates (2).
   b. The union of $\partial_G(Q_i)$ for all $(E_i,Q_i) \in B$ intercepts cut $(Q',V \setminus Q')$.

**Claim 6.13.** An $(S,t,c)$-bipartition system contains at most $2(|S| - 1)$ $(t,c)$-realizable pairs.

**Proof.** Let $B$ be an $(S,t,c)$-bipartition system. For each $(E_i,Q_i) \in B$, let $S_i$ be the subset of $S$ defined as the first condition of Definition 6.12. By Definition 6.12, for any $(E_i,Q_i) \in B$, another $(E_j,Q_j)$ in $B$ satisfies one of the following three conditions:

1. $S_i = S \setminus S_j$
2. One of $S_i$ and $S \setminus S_i$ is a strict subset of $S_j$;
3. One of $S_j$ and $S \setminus S_j$ is a strict subset of $S \setminus S_i$.

The number of $(E_j,Q_j)$ in $B$ satisfying the first condition is at most one. By induction, the number of $(E_j,Q_j)$ in the bipartition system satisfying the second condition is at most $2|S_i| - 2$, and the number of $(E_j,Q_j)$ in the bipartition system satisfying the third condition is at most $2|S \setminus S_i| - 2$. Hence, the total number of realizable pairs in the bipartition system is at most $1 + 1 + 2(|S_i| - 1) + 2(|S \setminus S_i| - 1) = 2(|S| - 1)$.

**Lemma 6.14.** Let $G = (V,E)$ be a connected graph, and $S \subseteq V$ be a set of vertices. Let $S'$ be a subset of $S$ such that $\emptyset \subsetneq S' \subseteq S$. For a partition $\mathcal{P}$ of $V$ such that for every $P \in \mathcal{P}$, $G[P]$ is connected, the number of $P \in \mathcal{P}$ satisfying all of the following conditions is at most two:

1. $S \cap P \neq \emptyset$.
2. There is a cut $(V',V \setminus V)$ of graph $G$ that partitions $S$ into $S'$ and $S \setminus S'$ such that $\partial_G(V')$ are edges of $G[P]$.

**Proof.** Let $G[P_1]$ and $G[P_2]$ be two sets of $\mathcal{P}$ satisfying the two conditions. By the second condition, there exist cuts $C_1 = (V_1,V \setminus V_1)$ and $C_2 = (V_2,V \setminus V_2)$ such that $\partial_G(V_1) \subseteq G[P_1]$, $\partial_G(V_2) \subseteq G[P_2]$, $S' \subseteq V_1$, and $S' \subseteq V_2$ hold.

Since $G[P_1]$ and $G[P_2]$ are connected and disjoint, $\partial_G(V_1)$ does not intercept $C_2$ and $\partial_G(V_2)$ does not intercept $C_1$. By the contrapositive of Lemma 6.12, $C_1$ and $C_2$ are parallel.

Therefore, if there exists another $P_3 \in \mathcal{P}$ satisfying the two conditions with cut $C_3 = (V_3,V \setminus V_3)$ and $S' \subseteq V_3$, then $C_1$, $C_2$, and $C_3$ are pairwise parallel.

Without loss of generality, assume $V_1 \subseteq V_2 \subseteq V_3$. Since $C_1$, $C_2$, and $C_3$ induce the same partition on $S$, $S \cap P_2 \cap V_2 = \emptyset$ and $S \cap P_2 \cap (V \setminus V_2) = \emptyset$. This contradicts the first condition.
Claim 6.15. Let $G = (V, E)$ be a connected graph, $S \subseteq V$ be a set of vertices. If $(E_1, V_1)$ and $(E_2, V_2)$ are two $S$-equivalent $(t,c)$-realizable pairs satisfying the following conditions, then $V_1 \cap V_2 \neq \emptyset$.

1) $V_1 \cap S \neq \emptyset$ and $V_2 \cap S \neq \emptyset$.
2) The induced cuts by $E_1$ and $E_2$ partition $S$ into $S'$ and $S \setminus S'$ for some $\emptyset \subseteq S' \subseteq S$.

Proof. We prove by contradiction. Assume $V_1 \cap V_2 = \emptyset$. Let $(V'_1, V \setminus V'_1)$ be the cut induced by $E_i$ satisfying $V'_1 \cap S = S'$. Since $G[V_1]$ and $G[V_2]$ are connected, $V_2$ belongs to one connected component of $G[V \setminus V_1]$, and this connected component is a subset of $V'_1$ because $\emptyset \subseteq V_2 \cap S \subseteq V'_2 \cap S = V'_1 \cap S$. Hence, in $G[V \setminus V_2]$, $V_1$ and $V \setminus V_1$ are in the same connected component. Thus, no subset of $\partial_G(V_2)$ forms a cut that partitions $S$ into $S'$ and $S \setminus S'$. Contradiction.

By Lemma 6.10, Lemma 6.14 and Claim 6.15, we have the following corollary.

Corollary 6.16. For any $IA_{G_0}(T_0, t, q, d, 2c)$ set, and a $(t,c)$-realizable pair $(E^\dagger, V^\dagger)$ such that both cut $(V^\dagger, V \setminus V^\dagger)$ and the cut induced by $E^\dagger$ partition $S$ nontrivially, there is a set $W_{(E^\dagger, V^\dagger)}$ of at most $8c^3 + 8c^2$ edges such that $W_{(E^\dagger, V^\dagger)}$ is the set of intercluster edges of a vertex partition of $G$, and for each $(t,c)$-realizable pair $(E', V')$ satisfying conditions (a), (b) and (c), $W_{(E^\dagger, V^\dagger)}$ intercepts a $(V' \cap (S \cup T), (V \setminus V') \cap (S \cup T), |V'\setminus |\partial_G(V')|)$-cut.

(a) $(E', V')$ and $(E^\dagger, V^\dagger)$ are $S$-equivalent.
(b) $(V', V \setminus V')$ partitions $S$ nontrivially.
(c) $\partial_G(V')$ is in a connected component of $G \setminus (IA_{G_0}(T_0, t, q, d, 2c)|_G)$ containing a vertex from $S$.

Moreover, $W_{(E^\dagger, V^\dagger)}$ can be constructed as follows:

1) For each $(t,c)$-realizable pair $(E', V')$ satisfying (a), (b) and (c), put $(E', V')$ into $\Gamma_{V^\dagger}$, where $V^\dagger$ is the vertex set of the connected component of $G \setminus (IA_{G_0}(T_0, t, q, d, 2c)|_G)$ containing $\partial_G(V')$.
2) For each $\Gamma_{V^\dagger}$ such that $\Gamma_{V^\dagger} \neq \emptyset$, run the elimination procedure on $\Gamma_{V^\dagger}$, and put the output of the procedure to $W_{(E^\dagger, V^\dagger)}$.

Lemma 6.17. For any $IA_{G_0}(T_0, t, q', d, 2c+1)$ set satisfying the condition that there is an $IA_{G_0}(T_0, t, q, d, 2c)$ set such that the $IA_{G_0}(T_0, t, q', d, 2c+1)$ set is derived from the $IA_{G_0}(T_0, t, q, d, 2c)$ set, there is a type 1 repair set $W_1$ for $IA_{G_0}(T_0, t, q', d, 2c+1)$ with respect to $G$ of at most $|S|(16c^3 + 16c^2 + 2c)$ edges constructed as follows:

1) Find a $(S, t, c)$-bipartition system $B$ for $G$, and put $\partial_G(Q^\dagger)$ into $W_1$ for every $(E^\dagger, Q^\dagger) \in B$.
2) For each $(E^\dagger, Q^\dagger)$ in $B$, put $W_{(E^\dagger, Q^\dagger)}$ obtained as Corollary 6.16 into $W_1$.

Proof. Consider a type 1 bipartition $(S' \cup T', (S' \cup T') \setminus (S' \cup T'))$ such that there is a minimum $(S' \cup T', (S' \cup T') \setminus (S' \cup T'), t, c)$-cut in graph $G$ that is a simple cut. Let $C = (V', V \setminus V')$ be such a minimum simple $(S' \cup T', (S' \cup T') \setminus (S' \cup T'), t, c)$-cut. We only consider the case that $\partial_G(V')$ belongs to a connected component of $G \setminus (IA_{G_0}(T_0, t, q, d, 2c)|_G)$ containing a vertex from $S$, otherwise, there is a $(S' \cup T', (S' \cup T') \setminus (S' \cup T'), |\partial_G(V')|-1)$-cut $C'$ such that every connected component of $G \setminus (IA_{G_0}(T_0, t, q, d, 2c)|_G)$ contains most $|\partial_G(V')|-1$ edges of the cut-set of $C'$ by Lemma 6.8.

Since $S' \neq S$, there must be an $E' \subseteq \partial_G(V')$ such that $E'$ induces an atomic cut that partitions $S$ nontrivially. Thus, $(E', V')$ is a $(t, c)$-realizable pair satisfying that $V' \cap S \neq \emptyset$ and $E'$ partitions $S$ nontrivially.
Consider the case that there is a \((E^\dagger, Q^\dagger) \in \mathcal{B}\) such that \((E', V')\) is \(S\)-equivalent to \((E^\dagger, Q^\dagger)\).

By Corollary 6.16 \(W_{(E', Q^\dagger)}\) intercepts a \((S' \cup T', (S \cup T) \setminus (S' \cup T'), |V'|, |\partial_G(V')|)\)-cut.

Now consider the case that \((E', V')\) is not \(S\)-equivalent to any \((E^\dagger, Q^\dagger) \in \mathcal{B}\). By the definition of partition system, one of the two conditions hold:

1. Adding \((E', V')\) to \(\mathcal{B}\) violates the second condition of Definition 6.12. For this case, since \((E', V')\) is not \(S\)-equivalent to any pair of \(\mathcal{B}\), there is a \((E^\dagger, Q^\dagger)\) in the bipartition system such that all of the following conditions hold:

   \[
   \begin{align*}
   S^\dagger & \cap S^\circ \neq \emptyset, (S \setminus S^\dagger) \cap S^\circ \neq \emptyset, S^\dagger \cap (S \setminus S^\circ) \neq \emptyset \quad \text{and} \quad (S \setminus S^\dagger) \cap (S \setminus S^\circ) \neq \emptyset.
   \end{align*}
   \]

   Hence, the cut induced by \(E'\) and the cut induced by \(E^\dagger\) are not parallel. Since both \(E^\dagger\) and \(E'\) induce atomic cuts, by Lemma A.1 \(\partial_G(Q^\dagger)\) intercepts \((V', V \setminus V')\), which implies \(W_1\) intercepts \((V', V \setminus V')\).

2. \(\bigcup_{(E_i, Q_i) \in \mathcal{B}} \partial_G(Q_i)\) intercepts \((V', V \setminus V')\). In this case, \(W_1\) intercepts \((V', V \setminus V')\).

By Definition 2.4 \(W_1\) is a type 1 repair set. By Claim 6.13 and Corollary 6.16 \(|W_1| \leq |S|(16e^3 + 16c^2 + 2c)\) edges.

### 6.1.3 Type two repair set

For any fixed \(\text{IA}_{G_0}(T_0, t, q, d, 2c)\) set and a set of vertices \(V^* \subseteq V\) that forms a connected component of \(G \setminus (\text{IA}_{G_0}(T_0, t, q, d, 2c)|_G)\), let \(\mathcal{T}_{2,V^*}\) be the set of type 2 bipartitions \((T', (T \setminus T') \cup S)\) satisfying

1. There is a simple \((T', S \cup (T \setminus T'), t, c)\)-cut of \(G\) with cut-set in \(G[V^*]\). Let \(\alpha\) denote the minimum of sizes of simple \((T', S \cup (T \setminus T'), t, c)\)-cuts of \(G\) with cut-set in \(G[V^*]\).

2. \(\text{IA}_{G_0}(T, t, q, d, 2c)|_G\) does not intercept any \((T', S \cup (T \setminus T'), q, \alpha)\)-cut of graph \(G\).

We prove the following lemma for \(\mathcal{T}_{2,V^*}\).

**Lemma 6.18.** For any \(\text{IA}_{G_0}(T_0, t, q, d, 2c)\) set and a set of vertices \(V^* \subseteq V\) that forms a connected component of \(G \setminus (\text{IA}_{G_0}(T_0, t, q, d, 2c)|_G)\) satisfying \(V^* \cap S \neq \emptyset\), we have

1. If \((T', S \cup (T \setminus T')) \in \mathcal{T}_{2,V^*}\), then there exists a \((T', T \setminus T', q, \alpha)\)-cut for graph \(G_0\) with cut-set contained in \(\text{IA}_{G_0}(T, t, q, d, 2c)|_G\), where \(\alpha\) is the minimum of sizes of all the simple \((T', S \cup (T \setminus T'), t, c)\)-cuts of \(G\) with cut-set in \(G[V^*]\). And for each of such a cut with smallest cut size, the side containing \(T'\) contains \(V^*\).

2. If \((T', S \cup (T \setminus T')), (T'', S \cup (T \setminus T'')) \in \mathcal{T}_{2,V^*}\), then \(T' \cap T'' \neq \emptyset\).

**Proof.** For the first property, by Claim 6.13 there is a \(\emptyset \subseteq S' \subseteq S\) such that there is a \((T' \cup S', (S \cup T) \setminus (S' \cup T'), q, \alpha)\)-cut on graph \(G_0\) whose cut-set is in \(\text{IA}_{G_0}(T_0, t, q, d, 2c)\). Let \((V'_0, V_0 / V'_0)\) be such a cut with smallest size. Without loss of generality, assume \(V'_0 \cap (S \cup T) = S' \cup T'\). There must be a connected component of \(G_0[V'_0]\), denoted by \(V'_1\), that contains vertices from both \(S'\) and \(T'\), otherwise, there is a \((T', (S \cup T_0) \setminus T', q, \alpha)\)-cut on graph \(G_0\) whose cut-set is in \(\text{IA}_{G_0}(T_0, t, q, d, 2c)\), and thus a \((T', (S \cup T) \setminus T', q, \alpha)\)-cut on graph \(G\) whose cut-set is in \(\text{IA}_{G_0}(T_0, t, q, d, 2c)|_G\), contradicting \((T', S \cup (T \setminus T')) \in \mathcal{T}_{2,V^*}\).

On the other hand, since there is a simple \((T', S \cup (T \setminus T'), t, \alpha)\)-cut on graph \(G\) whose cut-set is a subset of \(G[V^*]\) and this cut is also a simple \((T', (S \cup T_0) \setminus T')\)-cut on graph \(G_0\), any path between a vertex from \(S\) and a vertex from \(T'\) in \(G_0\) contains an edge of the cut-set of this cut. Since all
vertices of $V^*$ are in the same connected component of $G_0 \setminus IA_{G_0}(T_0, t, q, d, 2c)$, we have $V^* \subseteq V^\dagger$. Then the first property hold.

To prove the second property, let $(V'_0, V_0 \setminus V'_0)$ be a minimum $(T', T \setminus T', q, c)$-cut of $G_0$ whose cut-set is a subset of $IA_{G_0}(T_0, t, q, d, 2c)$, and $(V''_0, V \setminus V''_0)$ be a minimum $(T'', T \setminus T'', q, c)$-cut of $G_0$ whose cut-set is in $IA_{G_0}(T_0, t, q, d, 2c)$. Without loss of generality, assume $V'_0 \cap T = T'$ and $V''_0 \cap T = T''$. By the first property, $V^*$ is a subset of $V'_0$, and is also a subset of $V''_0$. If $T' \cap T'' = \emptyset$, then by Lemma A.3 one of the following conditions hold:

- $(V'_0, V_0 \setminus V'_0)$ is not a minimum $(T', T \setminus T', q, c)$-cut of $G_0$ with cut-set in $IA_{G_0}(T_0, t, q, d, 2c)$.
- $(V''_0, V_0 \setminus V''_0)$ is not a minimum $(T'', T \setminus T'', q, c)$-cut of $G_0$ with cut-set in $IA_{G_0}(T_0, t, q, d, 2c)$.
- There is a $(T', T \setminus T', V'_0, |\partial_{G_0}(V'_0)|)$-cut of $G_0$ with cut-set in $IA_{G_0}(T_0, t, q, d, 2c)$ such that the side containing $T'$ does not contain $V^*$, and a $(T'', T \setminus T'', V''_0, |\partial_{G_0}(V''_0)|)$-cut of $G_0$ with cut-set in $IA_{G_0}(T_0, t, q, d, 2c)$ such that the side containing $T''$ does not containing $V^*$.

The first two cases contradict the definitions of $(V'_0, V \setminus V'_0)$ and $(V''_0, V \setminus V''_0)$. The third case contradicts the first property. Hence $T' \cap T'' \neq \emptyset$.

By Lemma 6.10 and Lemma 6.18 we have the following corollary.

**Corollary 6.19.** For any $IA_{G_0}(T_0, t, q, d, 2c)$ set and a set of vertices $V^* \subseteq V$ that forms a component of $G \setminus (IA_{G_0}(T_0, t, q, d, 2c)|_G)$ satisfying $V^* \cap S \neq \emptyset$, there is a set $W_{2.V^*}$ of $4c^3 + 4c^2$ edges from $G$ such that $W_{2.V^*}$ is the set of intercluster edges of a vertex partition of $G$. For every $(T', S \cup (T \setminus T')) \in \mathcal{T}_{2.V^*}$, and a $(T', S \cup (T \setminus T'), t, c)$-cut $(V^\dagger, V \setminus V^\dagger)$ whose cut-set is in $G[V^*]$, $W_{2.V^*}$ intercepts a $(T', S \cup (T \setminus T'), V^\dagger, |\partial_G(V^\dagger)|)$-cut.

Moreover, $W_{2.V^*}$ can be constructed as follows: Let $s$ be an arbitrary vertex in $S \cap V^*$. For every $(T', S \cup (T \setminus T')) \in \mathcal{T}_{2.V^*}$, put all $(E', V')$ into $\Gamma_{2.V^*}$, where $(V', V \setminus V')$ is a simple $(T', S \cup (T \setminus T'), t, c)$-cut with cut-set in $G[V^*]$, and $E' \subseteq \partial_G(V)$ induces an atomic cut separating $s$ and $V'$. $W_{2.V^*}$ is obtained by running the elimination procedure on $\Gamma_{2.V^*}$.

**Lemma 6.20.** For any $IA_{G_0}(T_0, t, q', d, 2c+1)$ set satisfying the condition that there is an $IA_{G_0}(T_0, t, q, d, 2c)$ set such that the $IA_{G_0}(T_0, t, q', d, 2c+1)$ set is derived from the $IA_{G_0}(T_0, t, q, d, 2c)$ set, there is a type 2 repair set $W_2$ for $IA_{G_0}(T_0, t, q', d, 2c+1)$ with respect to $G$ of at most $|S|(4c^3 + 4c^2)$ edges constructed by letting $W_2 = \bigcup_{V^*} W_{2.V^*}$ for every $V^*$ that forms a connected component of $G \setminus (IA_{G_0}(T_0, t, q, d, 2c)|_G)$ satisfying $V^* \cap S \neq \emptyset$.

**Proof.** Consider a type 2 bipartition $(T', (S \cup T) \setminus T')$ for some $\emptyset \subseteq T' \subseteq T$ such that there is a minimum $(T', (S \cup T) \setminus T', t, c)$-cut in graph $G$ that is a simple cut. Let $C = (V', V \setminus V')$ be such a minimum simple $(T', (S \cup T) \setminus T', t, c)$-cut. We only consider the case that $\partial_G(V')$ belongs to a connected component of $G \setminus (IA_{G_0}(T_0, t, q, d, 2c)|_G)$ containing a vertex from $S$, otherwise, there is a $(T', (S \cup T) \setminus T', q', |\partial_G(V')|)$-cut $(V^\dagger, V \setminus V^\dagger)$ such that every connected component of $G \setminus (IA_{G_0}(T_0, t, q', d, 2c+1)|_G)$ contains at most $|\partial_G(V')| - 1$ edges of $\partial_G(V^\dagger)$ by Lemma 6.8.

Let $V^*$ be the vertex set corresponding to the connected component of $G \setminus (IA_{G_0}(T_0, t, q, d, 2c)|_G)$ containing $\partial_G(V')$. $(T', S \cup (T \setminus T'))$ belongs to $\mathcal{T}_{2.V^*}$. By Corollary 6.19, $W_{2.V^*}$ intercepts a $(T', S \cup (T \setminus T'), |\partial_G(V')|)$-cut with cut-set in $G[V^*]$.

By Definition 2.23, $W_2$ is a type 2 repair set. By Corollary 6.19, $|W_2| \leq |S|(4c^3 + 4c^2)$ edges.
Lemma 6.21. For a \( T \) \( \subseteq \) \( \Gamma \) \( \leq \) \( \beta \) \( G \) \( \leq \) \( \alpha \) \( G \), let \( \mathcal{T}_{3,V^*} \) be the set of type 3 bipartitions \( (S \cup T', T \setminus T') \) for some \( T' \subseteq T \) satisfying

1. There is a simple \((S \cup T', T' \setminus t, c)\)-cut of \( G \) with cut-set in \( G[V^*] \). Let \( \alpha \) denote the minimum of sizes of simple \((S \cup T', T \setminus T', c)\)-cuts of \( G \) with cut-set in \( G[V^*] \).

2. \( \mathcal{I}_{\mathcal{G}_0}(t, q, d, 2c) \) does not intercept any \((S \cup T', T' \setminus t, q, \alpha)\)-cut of \( G \).

Lemma 6.22. For any \( \mathcal{I}_{\mathcal{G}_0}(T_0, t, q, d, 2c) \) set satisfying the condition that there is an \( \mathcal{I}_{\mathcal{G}_0}(T_0, t, q, d, 2c) \) set derived from the \( \mathcal{I}_{\mathcal{G}_0}(T_0, t, q, d, 2c) \) set, a type 2 repair set \( W_2 \) of \( \mathcal{I}_{\mathcal{G}_0}(T_0, t, q, d, 2c + 1) \) with respect to \( G \), and a set of vertices \( V^* \subseteq V \) that forms a connected component of \( G \setminus \{ \mathcal{I}_{\mathcal{G}_0}(T_0, t, q, d, 2c) \} \) such that \( V^* \) \( \cap \) \( S \) \( \neq \emptyset \), there is a set \( W_{3,V^*} \) of at most \( 4c^3 + 4c^2 + c \) edges from \( G \) such that \( W_{3,V^*} \) is the set of intercluster edges of a vertex partition of \( G \), and for every \((S \cup T', T \setminus T') \in \mathcal{T}_{3,V^*} \), there is a \((S \cup T', T \setminus T', 2t, \beta)\)-cut satisfying that every connected component of \( G \setminus (W_{3,V^*} \cup W_2 \cup \mathcal{I}_{\mathcal{G}_0}(T_0, t, q, d, 2c + 1)) \) contains at most \( \beta - 1 \) edges of the cut-set, where \( \beta \) is the size of minimum simple \((S \cup T', T \setminus T', t, c)\)-cut with cut-set in \( G[V^*] \).

Moreover, \( W_{3,V^*} \) can be constructed as follows: Find \( V^\uparrow \) satisfying the following conditions with \( |V^\uparrow| \) minimized:

1. There is a simple cut \((V^\uparrow, V \setminus V^\uparrow)\) with cut-set in \( G[V^*] \) such that \( V^\uparrow \) forms a connected component of \( G[V \setminus V^\uparrow] \).

2. \( V^\uparrow \cap S = S \), \( V^\uparrow \cap T \neq \emptyset \) and \( |V^\uparrow| > t \).

Let \( x \) be any arbitrary vertex in \( T \cap V^\uparrow \), and \( \Gamma_{3,V^*} \) be the set of all the \((t, c)\)-realizable pairs \((E', V')\) satisfying \( V^\uparrow \cap S = S \) and \( E' \subseteq \partial G(V') \) inducing an atomic cut separating \( x \) and \( V' \). \( W_{3,V^*} \) is obtained by \( \partial G(V^\uparrow) \) union the result of running elimination procedure on \( \Gamma_{3,V^*} \).
Proof. The size of $W_{3,V^*}$ is obtained by Lemma 6.10. We show that for every $(S \cup T', T \setminus T') \in T_{3,V^*}$, there is a $(S \cup T', T \setminus T', 2t, \beta)$-cut such that every connected component of $G \setminus (W_{3,V^*} \cup W_2 \cup IA_{G_0}(T_0, t, q', d, 2c + 1))$ contains at most $\beta - 1$ edges of the cut-set of the cut, where $\beta$ is the minimum of sizes of all the simple $(S \cup T', T \setminus T', t, c)$-cuts with cut-set in $G[V^*]$. Let $(V', V \setminus V')$ be an arbitrary minimum simple $(S \cup T', T \setminus T', t, c)$-cut with cut-set in $G[V^*]$ for graph $G$. We consider the following three cases:

**Case 1.** There is a connected component of $G[V \setminus V']$ containing at most $t$ vertices and having a non-empty intersection with $T$. By Lemma 6.21, there is a $(S \cup T', T \setminus T', 2t, \beta)$-cut such that every connected component of $G \setminus (W_2 \cup (IA_{G_0}(T_0, t, q', d, 2c + 1))$ contains at most $\beta - 1$ edges of the cut-set of the cut.

**Case 2.** $x \notin V'$. By the construction of $\Gamma_{3,V^*}$ and Lemma 6.10, $W_{3,V^*}$ intercepts a $(S \cup T', T \setminus T', |V'|, |\partial_G(V')|)$-cut.

**Case 3.** The first two cases do not hold. We show that $\partial_G(V^\uparrow)$ intercepts $(V', V \setminus V')$ for this case. Since $|V^\uparrow| > t$, $V'$ does not contain all the vertices of $V^\uparrow$. But since $x \in V'$, there must be a subset of $\partial_G(V^\uparrow)$ inducing an atomic cut separating $V'$ and some vertex in $V^\uparrow$. If $V^\uparrow$ intercepts such an atomic cut-set, then we are done. Otherwise, all the subsets of $\partial_G(V^\uparrow)$ that induce atomic cuts separating $V'$ and some vertex in $V^\uparrow$ are in $G[V^\uparrow]$. Since Case 1 does not hold, by our choice of $V'$, $S \cup T$ is on one side of these induced cuts. Thus, there must be an $E'' \subseteq \partial_G(V^\uparrow)$ inducing a nontrivial bipartition on $S \cup T$ and belonging to $G[V \setminus V^\uparrow]$. Hence, $\partial_G(V^\uparrow)$ also intercepts $(V', V \setminus V')$.

The lemma follows by the definition of type 3 repair set.

**Lemma 6.23.** For any $IA_{G_0}(T_0, t, q', d, 2c + 1)$ set satisfying the condition that there is an $IA_{G_0}(T_0, t, q, d, 2c)$ set such that the $IA_{G_0}(T_0, t, q', d, 2c + 1)$ set is derived from the $IA_{G_0}(T_0, t, q, d, 2c)$ set, and a type 2 repair set $W_2$ for $IA_{G_0}(T_0, t, q', d, 2c + 1)$ with respect to $G$, if $q' \geq 2t$, there is a set $W_3$ of at most $|S|(4c^3 + 4c^2 + 2c)$ edges such that $W_3 \cup W_2$ is a type 3 repair set for $IA_{G_0}(T_0, t, q', d, 2c + 1)$ with respect to $G$.

Moreover, $W_3$ is obtained by taking the union of the cut-set of an arbitrary minimum $(S \cup T, \emptyset, t, c)$-cut (if exists) and $W_{3,V^*}$ for all the $V^*$ such that $V^*$ is the vertex set of a connected component of $G \setminus (IA_{G_0}(T_0, q, t, d, 2c))$ satisfying $V^* \cap S \neq \emptyset$.

**Proof.** Consider a type 3 bipartition $(S \cup T', T \setminus T')$ for some $T' \subseteq T$ such that there is a minimum $(S \cup T', T \setminus T', t, c)$-cut in graph $G$ that is a simple cut. Let $C = (V', V \setminus V')$ be such a minimum simple $(S \cup T', T \setminus T', t, c)$-cut. We only consider the case that $\partial_G(V^\uparrow)$ belongs to a connected component of $G \setminus (IA_{G_0}(T_0, t, q, d, 2c))$ containing a vertex from $S$, otherwise, there is a $(S \cup T', T \setminus T', q', |\partial_G(V^\uparrow)|)$-cut $(V^\uparrow, V \setminus V^\uparrow)$ such that every connected component of $G \setminus (IA_{G_0}(T_0, t, q', d, 2c + 1))$ contains at most $|\partial_G(V^\uparrow)| - 1$ edges of $\partial_G(V^\uparrow)$ by Lemma 6.8.

Let $V^*$ be the vertex set corresponding to the connected component of $G \setminus (IA_{G_0}(T_0, t, q, d, 2c))$ containing $\partial_G(V^\uparrow)$. $(S \cup T', T \setminus T')$ belongs to $T_{3,V^*}$. By Lemma 6.22 there is a $(S \cup T', T \setminus T', 2t, |\partial_G(V^\uparrow)|)$-cut such that every connected component of $G \setminus (W_{3,V^*} \cup W_2 \cup IA_{G_0}(T_0, t, q', d, 2c + 1))$ contains at most $|\partial_G(V^\uparrow)| - 1$ edges of the cut-set.

Hence, $W_2 \cup W_3$ is a type 3 repair set. By Lemma 6.22, $|W_3| \leq |S|(4c^3 + 4c^2 + 2c)$ edges.

By Lemma 5.7, Claim 6.7, Lemma 6.17, Lemma 6.20, and Lemma 6.23, Lemma 6.2 follows.

### 6.2 Proof of Lemma 6.3

We prove Lemma 6.3 in this subsection. We start by giving a few useful subroutines.
The following algorithm determines if a subset of edges of the graph is the cut-set of an atomic cut.

**Algorithm 3: Atomic-Cut-Verification(_DS(G), E_0)**

**Input:** DS(G): dynamic data structure of graph G  
E_0: a non-empty set of edges  
**Output:** true if E_0 induces an atomic cut; false otherwise

1. U ← ∅, W ← ∅
2. for every edge (x, y) ∈ E_0 do
   3. id_x ← DS(G).ID(x), id_y ← DS(G).ID(y), W ← W ∪ {id_x, id_y}
4. if |W| > 1 then return false
5. for every edge (x, y) ∈ E_0 do run DS(G).Delete(x, y)
6. for every edge (x, y) ∈ E_0 do
   7. id_x ← DS(G).ID(x), id_y ← DS(G).ID(y)
   8. if id_x = id_y then reverse all the changes made on DS(G) on line 5 and return false
   9. U ← U ∪ {id_x, id_y}
10. reverse all the changes made on DS(G) on line 5
11. return true if |U| = 2, otherwise return false

**Lemma 6.24.** Given access to the graph data structure for graph G = (V, E) of at most m vertices and distinct edges, a set of edges E_0 ⊆ E, algorithm Atomic-Cut-Verification determines whether E_0 forms an atomic cut of some connected component of G with running time \( O(|E_0| \text{ polylog}(m)) \).

**Proof.** By the definition of atomic cut, E_0 forms an atomic cut for some connected component of G if and only if the following conditions hold

1. The endpoints of edges in E_0 are in same connected component of G.
2. After removing E_0 from G, for any edge (x, y) ∈ E_0, x and y are in different connected components.
3. After removing E_0 from G, all the endpoints of E_0 belong to two connected components.

By the description of the algorithm and Lemma 2.11, the lemma holds. \( \square \)

The following algorithm enumerates all the simple (t, c)-cuts such that the side containing vertex x has at most t vertices for a given vertex x.

**Algorithm 4: Enumerate-Simple-Cuts(_DS(G), x, c, t)**

**Input:** DS(G): dynamic data structure of graph G  
x: a vertex of G  
c, t: parameters  
**Output:** H = \{V' ⊆ V\}: a set of vertex sets of G such that for each V' ∈ H, x ∈ V', and (V', V \ V') is a simple (t, c)-cut

1. H ← ∅  
2. run FIND-CUT(_DS(G), x, c, t, ∅)  
3. return H
Function \textsc{Find-Cut}(DS(G), x, t, F):

\begin{enumerate}
\item \( V' \leftarrow \emptyset \)
\item run BFS from vertex \( x \) on \( G \setminus F \) and put all the visited vertices to \( V' \), until BFS stops or at least \( t + 1 \) vertices are put into \( V' \)
\item if \( |V'| \leq t \) and \( \partial_G(V') = F \) then
\begin{enumerate}
\item \( H \leftarrow \{V'\} \cup H \) and \textbf{return} \end{enumerate}
\item if \( i = 0 \) then \textbf{return} \end{enumerate}
\begin{enumerate}
\item \( T \leftarrow \) a BFS tree of \( G \setminus F \) rooted at \( x \) until BFS stops or the BFS tree contains \( t + 1 \) vertices
\item for every edge \((x, y)\) of \( T \) do \( \textbf{Find-Cut}(DS(G), x, i - 1, t, F \cup \{(x, y)\}) \)
\end{enumerate}

Lemma 6.25. For a connected graph \( G = (V, E) \) of at most \( m \) vertices and distinct edges, two integers \( c, t \), and a vertex \( x \) of \( G \), there are at most \( t \cdot c \) simple \((t, c)\)-cuts \((V', V \setminus V')\) satisfying \( x \in V' \).

Moreover, given access to the graph data structure of \( G \), algorithm \textsc{Enumerate-Simple-Cuts} outputs all these cuts that are represented by vertex sets of the side containing \( x \) with \( O(t \cdot c + 2 \cdot \text{poly}(c)) \) running time.

\begin{proof}
Let \((V', V \setminus V')\) be a simple \((t, c)\) cut such that \( x \in V' \). Let \( U \) be a proper subset of edges of \( \partial_G(V') \). If we run BFS on \( x \) for graph \( G \setminus U \) until BFS stops or the number of vertices visited is \( t + 1 \), at least one of the edge in the corresponding BFS tree belong to \( \partial_G(V') \setminus U \). Hence, all the required cuts are enumerated. The number of cuts and the running time of the algorithm are obtained by the algorithm. \end{proof}

Algorithm 5: \textsc{Enumerate-Cuts}(DS(G, \( T_1 \)), DS(G, \( T_2 \)), \( T' \), \( c, t \))

\begin{enumerate}
\item if \( |T'| > t \) then \textbf{return} \( \emptyset \)
\item \( H \leftarrow \emptyset \), \( U \leftarrow \emptyset \)
\item for every \( x \in T' \) do \( U \leftarrow U \cup \textsc{Enumerate-Simple-Cuts}(DS(G, T_1), x, c, t) \)
\item for every \( k \leq c \) do
\begin{enumerate}
\item if there exist \( V_1, \ldots, V_k \in U \) such that \((\bigcup_{i=1}^{k}V_i) \cap (T_1 \cup T_2) = T' \) and \( \sum_{i=1}^{k} |V_i| \leq t \) and \( V_i \cap V_j = \emptyset \) for every \( 1 \leq i < j \leq k \) then
\item \( H \leftarrow H \cup \left\{ \bigcup_{i=1}^{k} V_i \right\} \)
\end{enumerate}
\end{enumerate}
\textbf{return} \( H \)
Lemma 6.26. For a connected graph \( G = (V, E) \) of at most \( m \) vertices and distinct edges, two integers \( c, t \), and three vertex sets \( T_1, T_2, T' \subseteq V \) satisfying \( T' \subseteq T_1 \cup T_2 \), there are at most \( O(t^{c+1}) \) \((T', (T_1 \cup T_2) \setminus T', t, c)\)-cuts \((V', V \setminus V')\) such that every connected component of \( G[V'] \) contains at least one vertex from \( T' \).

Moreover, given access to the graph data structures \( \mathcal{DS}(G, T_1), \mathcal{DS}(G, T_2), \) and \( c, t, T' \), there is an algorithm \( \text{Enumerate-Cuts} \) to output all these cuts that are represented by vertex sets of the side containing \( T' \) in \( O(t^{c+1}) \) time.

Proof. If \( |T'| > t \), there is no \((T', (T_1 \cup T_2) \setminus T', t, c)\)-cut. It is easy to verify that for a \( V' \in H \), \((V', V \setminus V')\) is a \((T', (T_1 \cup T_2) \setminus T', t, c)\)-cut satisfying that every connected component of \( V' \) contains at least one vertex from \( T' \).

Let \((V', V \setminus V')\) be a \((T', (T_1 \cup T_2) \setminus T', t, c)\)-cut such that every connected component of \( V' \) contains at least one vertex from \( T' \). Every \( V^* \) that forms a connected component of \( G[V'] \) is a simple \((t, c)\)-cut with \( V^* \cap T' \neq \emptyset \). Hence, \( V' \) is in \( H \). So \( H \) contains all \( V' \) such that \((V', V \setminus V')\) is a \((T', (T_1 \cup T_2) \setminus T', t, c)\)-cut satisfying that every connected component of \( G[V'] \) contains at least one vertex from \( T' \).

By Lemma 6.25, the number of vertex sets in \( H \) is at most \( O(t^{c+1}) \), and running time of the algorithm is \( O(t^{c+1}) \) time.

Algorithm 6: Elimination(\( \mathcal{DS}(G, T), \mathcal{DS}(G, S), \Gamma, c, t \))

**Input:** \( \mathcal{DS}(G, T), \mathcal{DS}(G, S) \): dynamic data structures of graph \( G \) with \( T \) and \( S \) as terminals.
\( \Gamma \): a set of \((t, c)\)-realizable pairs
\( c, t \): parameters

**Output:** \( W \): A set of edges

1. \( W \leftarrow \emptyset \)
2. **for** every \((E^\uparrow, V^\uparrow) \in \Gamma\) **do**
   3. \( H_{V^\uparrow} \leftarrow \text{Enumerate-Cuts}(\mathcal{DS}(G, T), \mathcal{DS}(G, S), V^\uparrow \cap (T \cup S), |V^\uparrow|, |\partial_G(V^\uparrow)|)\)
4. **repeat**
   5. let \((E', V') \in \Gamma \) be an arbitrary pair such that there is no \((E'', V'') \in \Gamma \) satisfying \( \text{End}(E'') \subseteq V' \)
   6. \( W \leftarrow W \cup E' \)
   7. **for** every \((x, y) \in E'\) **do**
      8. \( \mathcal{DS}(G, T).\text{Delete}(x, y) \)
   9. **for** every \((E^\uparrow, V^\uparrow) \in \Gamma\) **do**
      10. if there is a \( V^\circ \in H_{V^\uparrow} \) s.t. it is not the case that all of \( \text{End}(\partial_G(V^\circ)) \) belong to the same connected component of \( \mathcal{DS}(G, T) \) **then**
      11. **remove** \((E^\uparrow, V^\uparrow)\) from \( \Gamma \)
12. **until** \( \Gamma = \emptyset \)
13. reverse all the changes made in line 8
14. return \( W \)

Lemma 6.27. Let \( G = (V, E) \) be a connected graph with at most \( m \) vertices and distinct edges, and a set \( \Gamma \) of \((t, c)\)-realizable pair. Given access to \( \mathcal{DS}(G, T), \mathcal{DS}(G, S), \Gamma, t, c \), there is an algorithm Elimination to implement the elimination procedure with running time

\[
O \left( |\Gamma|^2 \cdot t^{c+1} \cdot \text{poly}(c) \cdot \text{polylog}(m) \right).
\]

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Proof. By the definition of elimination procedure and the description of algorithm \texttt{ENUMERATE-CUTS}, algorithm \texttt{ELIMINATION} implements the elimination procedure. By Lemma \texttt{2.14} and Lemma \texttt{6.26}, the running time of algorithm \texttt{ELIMINATION} is $|\Gamma|^2 \cdot t^{(c+1)+1} \text{poly}(c) \text{polylog}(m)$.

\begin{algorithm}
\textbf{Algorithm 7: EQUIVALENT(}D\check{G}(G, S), E_1, V_1, E_2, V_2, c, t\textbf{)}
\begin{tabbing}
\textbf{Input:} \= $D\check{G}(G, S)$: dynamic data structure of graph $G$ with $S$ as terminals $E_1, V_1, E_2, V_2$: \(t, c\)-realizable pairs $(E_1, V_1)$ and $(E_2, V_2)$\text{,}\= $c, t$: parameters\,
\textbf{Output:} $\text{true}$ if $(E_1, V_1)$ and $(E_2, V_2)$ are $S$-equivalent, or $\text{false}$ if not
\end{tabbing}
\begin{tabbing}
1 $A_1 \leftarrow \emptyset$, $B_1 \leftarrow \emptyset$, $A_2 \leftarrow \emptyset$, $B_2 \leftarrow \emptyset$, $b \leftarrow \text{true}$\text{,}
\begin{tabbing}
2 \hspace{1cm} run $D\check{G}(G, S)$.\text{Delete}(x, y)$ for every $(x, y) \in E_1$
\end{tabbing}
\begin{tabbing}
3 \hspace{1cm} for every $(x, y) \in E_1$ do
\begin{tabbing}
4 \hspace{2cm} $id_1 \leftarrow D\check{G}(G, S).\text{ID}(x)$, $id_2 \leftarrow D\check{G}(G, S).\text{ID}(y)$, $id \leftarrow D\check{G}(G, S).\text{ID}(z)$ for an arbitrary $z \in V_1$
\end{tabbing}
\begin{tabbing}
5 \hspace{2cm} if $id = id_1$ then
\begin{tabbing}
6 \hspace{3cm} $A_1 \leftarrow A_1 \cup \{x\}$, $B_1 \leftarrow B_1 \cup \{y\}$
\end{tabbing}
\begin{tabbing}
7 \hspace{2cm} else
\begin{tabbing}
8 \hspace{3cm} $A_1 \leftarrow A_1 \cup \{y\}$, $B_1 \leftarrow B_1 \cup \{x\}$
\end{tabbing}
\end{tabbing}
\end{tabbing}
\begin{tabbing}
9 reverse all the change made on $D\check{G}(G, S)$ in line 2\text{,}
\begin{tabbing}
10 run $D\check{G}(G, S)$.\text{Delete}(x, y)$ for every $(x, y) \in E_2$
\end{tabbing}
\begin{tabbing}
11 for every $(x, y) \in E_2$ do
\begin{tabbing}
12 \hspace{2cm} $id_1 \leftarrow D\check{G}(G, S).\text{ID}(x)$, $id_2 \leftarrow D\check{G}(G, S).\text{ID}(y)$, $id \leftarrow D\check{G}(G, S).\text{ID}(z)$ for an arbitrary $z \in V_2$
\end{tabbing}
\begin{tabbing}
13 \hspace{2cm} if $id = id_1$ then
\begin{tabbing}
14 \hspace{3cm} $A_2 \leftarrow A_2 \cup \{x\}$, $B_2 \leftarrow B_2 \cup \{y\}$
\end{tabbing}
\begin{tabbing}
15 \hspace{2cm} else
\begin{tabbing}
16 \hspace{3cm} $A_2 \leftarrow A_2 \cup \{y\}$, $B_2 \leftarrow B_2 \cup \{x\}$
\end{tabbing}
\end{tabbing}
\end{tabbing}
\begin{tabbing}
17 run $D\check{G}(G, S)$.\text{Delete}(x, y)$ for every $(x, y) \in E_1$
18 for every $x \in A_1, y \in B_2$ or $x \in B_1, y \in A_2$ such that $D\check{G}(G, S).\text{ID}(x) = D\check{G}(G, S).\text{ID}(y)$ do
19 \hspace{2cm} if $D\check{G}(G, S).\text{TerminalNumber}(x) > 0$ then
20 \hspace{3cm} $b \leftarrow \text{false}$
\end{tabbing}
\begin{tabbing}
21 reverse all the change made on $D\check{G}(G, S)$ at line 10 and 17
22 \text{return } b
\end{tabbing}
\end{algorithm}

Lemma 6.28. Given access to $D\check{G}(G, S)$ for a connected graph $G$ of at most $m$ vertices and distinct edges, two parameters $t, c$, and two $(t, c)$-realizable pairs $(E_1, V_1), (E_2, V_2)$, algorithm \texttt{EQUIVALENT} determines if $(E_1, V_1)$ and $(E_2, V_2)$ are $S$-equivalent in $O(\text{poly}(c) \text{polylog}(m))$ time.

Proof. By Definition \texttt{6.9}, $(E_1, V_1)$ and $(E_2, V_2)$ are $S$-equivalent if and only if the following two conditions hold: (Let $(L_1, V \setminus L_1)$ denote the cut induced by $E_1$ with $V_1 \subseteq L_1$, and $(L_2, V \setminus L_2)$ denote the cut induced by $E_2$ with $V_2 \subseteq L_2$)

1. Every connected component of $G \setminus (E_1 \cup E_2)$ that contains vertices from both $\text{End}(\partial_G(E_1)) \cap L_1$ and $\text{End}(\partial_G(E_2)) \cap (V \setminus L_2)$ does not have vertices from $S$.
2. Every connected component of $G \setminus (E_1 \cup E_2)$ that contains vertices from both $\text{End}(\partial_G(E_1)) \cap (V \setminus L_1)$ and $\text{End}(\partial_G(E_2)) \cap L_2$ does not have vertices from $S$. 38
Hence, algorithm Equivalent outputs true if and only if \((E_1, V_1)\) and \((E_2, V_2)\) are \(S\)-equivalent.

The running time is obtained by Lemma 2.14 and the description of the algorithm.

The following algorithm generates a \((S, t, c)\)-bipartition system for a given vertex set \(S\).

**Algorithm 8: Bipartition-System**

**Input:** \(\mathcal{DS}(G, S)\): dynamic data structure of graph \(G\) with \(S\) as terminals

**Output:** \(B\): a \((S, t, c)\)-bipartition system as defined in Definition 6.12

1. \(U \leftarrow \emptyset\)
2. for \(x \in S\) do
   3. for every \(V'\) in Enumerate-Simple-Cuts(\(\mathcal{DS}(G, S), x, c, t\)) and every \(E' \subseteq \partial_G(V')\) do
      4. if Equivalent(\(\mathcal{DS}(G, E), E', V', \partial_G(V'), c, t\)) = true and \(E'\) partitions \(S\) nontrivially then
         5. \(U \leftarrow U \cup \{(E', V', \partial_G(V'))\}\)
   6. initiate \(A\): \(A[E', V'] \leftarrow \emptyset\) for every \((E', V', \partial_G(V'))\) \(\in U\)
   7. for \((E', V', \partial_G(V')) \in U\) do
      8. for every \(v \in V'\) and every \((V^\dagger, \partial_G(V^\dagger))\) \(\in U\) such that \(v \in V^\dagger\) do
         9. if Equivalent(\(\mathcal{DS}(G, E), E', V', \partial_G(V'), c, t\)) = true then
            10. \(A[E', V'] \leftarrow A[E', V'] \cup \{(V^\dagger, \partial_G(V'))\}\)
   11. for every \((E', V', \partial_G(V')) \in U\) do
      12. if vertices of End(\(\partial_G(V')\)) belong to at least two connected components of \(\mathcal{DS}(G, S)\) then
         13. remove \((E', V')\) from \(U\), and continue
      14. \(B \leftarrow B \cup \{(E', V')\}\)
      15. run \(\mathcal{DS}(G, S)\).Delete\((x, y)\) for every \((x, y) \in \partial_G(V')\)
      16. remove \((E^\dagger, V^\dagger)\) from \(U\) for every \((E^\dagger, V^\dagger) \in A[E', V']\)
   17. return \(B\)

**Lemma 6.29.** Given access to \(\mathcal{DS}(G, S)\) for a connected graph \(G = (V, E)\) of at most \(m\) vertices and distinct edges and a set of vertices \(S\), and two integers \(c, t\), algorithm Bipartition-System computes a \((S, t, c)\)-bipartition system as Definition 6.12 in \(O(|S|(2t)^{2c+1}\text{poly}(c)\text{polylog}(n))\) time.

**Proof.** We show that the output of the algorithm satisfies the definition of an \((S, t, c)\)-bipartition system as Definition 6.12.

By the description of the algorithm, \(U\) contains all the triples \((E', V', \partial_G(V'))\) such that \((E', V')\) is a \((t, c)\)-realizable pair such that \(V' \cap S \neq \emptyset\) and \(E'\) partitions \(S\) nontrivially. By Claim 6.15, \(A[E', V']\) contains all the \((t, c)\)-realizable pairs in \(U\) that are \(S\)-equivalent to \((E', V')\).

By the description of the algorithm, for any \((E', V', \partial_G(V')) \in U\), \((E', V')\) is not in \(B\) if and only if one of the following two conditions hold:

1. \((E', V', \partial_G(V'))\) is removed from \(U\) when executes line 13. For this case, \(\bigcup_{(E'', V'') \in B} \partial_G(V'')\) intercepts \(\partial_G(V')\).
2. \((E', V', \partial_G(V'))\) is removed from \(U\) when executes line 16. For this case, there is a \((E'', V'') \in B\) that is \(S\)-equivalent to \((E', V')\).
By Definition 6.12 $\mathcal{B}$ is a $(S, t, c)$-bipartition system for graph $G$.

Now we analyze the running time of the algorithm. By Lemma 6.24 and Lemma 6.25, the running time of line 2-5 is $O(\left|S\right| (t^{c+1} \cdot 2^c \cdot \text{poly}(c) \cdot \text{polylog}(n)))$, and $U$ contains at most $\left|S\right|(2t)^c$ pairs.

By Lemma 6.28, for every $(E', V') \in U$, line 8-10 takes $O(t \cdot (2t)^c \text{poly}(c) \text{polylog}(m))$ time. Hence, the overall running time for line 7-10 is $O(\left|S\right|(2t)^{2c+1} \text{poly}(c) \text{polylog}(m))$, and for every $(E', V') \in U$, $A[(E', V')]$ contains at most $(2t)^{c+1}$ pairs.

The running time of line 11-16 is $O(\left|U\right|(2t)^{c+1} \text{poly}(c) \text{polylog}(m))$. Hence, the overall running time of the algorithm is $O(\left|S\right|(2t)^{2c+1} \text{poly}(c) \text{polylog}(m))$.

\begin{algorithm}
\begin{algorithmic}
\State \textbf{Input:} $\mathcal{D}_1 = \mathcal{D}(G, S)$, $\mathcal{D}_2 = \mathcal{D}(G, T)$, $\mathcal{D}_3 = \mathcal{D}(G \setminus (\text{IA}_{G_0}(T_0, t, q, d, 2c)|_C), S)$ for some $q \geq t, d \geq 2c$
\State \text{c, t: parameters}
\State \textbf{Output:} $W_1$: a set of edges
\State $B \leftarrow \text{Bipartition-System}(\mathcal{D}_1, c, t)$
\State $W_1 \leftarrow \emptyset$
\For {each $(E^+, V^+) \in B$}
\State $U[] \leftarrow \emptyset$, $CC \leftarrow \emptyset$
\For {every $x \in V^+$}
\For {every $V'$ in \text{Enumerate-Simple-Cuts}(\mathcal{D}_1, x, c, t) and every $E'$ subset of $\partial_G(V')$}
\If {$V' \cap S \neq \emptyset$ and $\text{Equivalent}(\mathcal{D}_1, E', V', E^+, V^+) = \text{true}$ and $\text{End}(\partial_G(V'))$
\State put $\mathcal{D}_3, \text{ID}(x)$ for any $x \in \text{End}(E')$ to $CC$
\State put $(V', E')$ into $U[\mathcal{D}_3, \text{ID}(x)]$
\EndIf
\EndFor
\EndFor
\State $W_1 \leftarrow W_1 \cup \partial_G(V^+)$
\For {every id $\in CC$}
\State $W_1 \leftarrow W_1 \cup \text{Elimination}(\mathcal{D}_2, \mathcal{D}_1, U[id], c, t)$
\EndFor
\State \textbf{return} $W_1$
\end{algorithmic}
\end{algorithm}

\textbf{Lemma 6.30.} Let $G = (V, E)$ be a connected graph of at most $m$ vertices and distinct edges, four parameters $c, d, t, q > 0$ such that $d \geq 2c + 1$ and $q \geq t$, and two vertex sets $T, S \subseteq V$.

Given access to $\mathcal{D}_1 = \mathcal{D}(G, S)$, $\mathcal{D}_2 = \mathcal{D}(G, T)$, $\mathcal{D}_3 = \mathcal{D}(G \setminus (\text{IA}_{G_0}(T_0, t, q, d, 2c)|_C), S)$ for an $\text{IA}_{G_0}(T_0, t, q, d, 2c)$ set, there is a \text{Type-One-Repair-Set} algorithm with

$$O(\left|S\right|(2t)^{2c+1} \text{poly}(c) \text{polylog}(m))$$

running time to output a type one repair set $W_1$ of $\left|S\right|(16^c + 16c^2 + 2c)$ edges for any $\text{IA}_{G_0}(T_0, t, q', d, 2c+1)$ set such that the $\text{IA}_{G_0}(T_0, t, q', d, 2c+1)$ set is derived from the $\text{IA}_{G_0}(T_0, t, q, d, 2c)$ set.

\textbf{Proof.} By the definition of algorithm \text{Type-One-Repair-Set} and Lemma 6.17, the output of algorithm \text{Type-One-Repair-Set} is a repair set of size $\left|S\right|(16^c + 16c^2 + 2c)$ for any $\text{IA}_{G_0}(T_0, t, q', d, 2c+1)$ set such that the $\text{IA}_{G_0}(T_0, t, q', d, 2c+1)$ set is derived from the $\text{IA}_{G_0}(T_0, t, q, d, 2c)$ set.

Now we bound the running time of the algorithm.

By Lemma 6.29, the running time of line 1 is $O(\left|S\right|(2t)^{2c+1} \text{poly}(c) \text{polylog}(n))$. By Claim 6.13, $B$ contains less than $2\left|S\right| (t, c)$-realizable pairs. For each $(E^+, V^+) \in B$, the running time of line 5-9
is $O((2t)^c + 1 \text{poly}(c) \text{polylog}(m))$ by Lemma 6.24 and Lemma 6.28 and $|CC| \leq 2, |U[id]| \leq (2t)^c$ for each $id \in CC$. So, the running time of line 11-12 is $O((2t)^{c+3} + 3 \text{poly}(c) \text{polylog}(m))$ by Lemma 6.27.

Hence, the overall running time of the algorithm is $O(|S|((2t)^c + 3) + \text{poly}(c) \text{polylog}(m))$. \hfill \Box

The following algorithm finds a type 2 repair set.

**Algorithm 10: Type-Two-Repair-Set(ΔS1, ΔS2, ΔS3, c, t, q)**

**Input:** ΔS1 = ΔS(G, S), ΔS2 = ΔS(G, T), ΔS3 = ΔS(G \ (IA_{G_0}(T_0, t, q, d, 2c), S)) for some $q \geq t$, $d \geq 2c$

c, t, q: parameters

**Output:** W2: a set of edges

1. $W_2 \leftarrow \emptyset$
2. for each $s \in S$ do
3.  \hspace{1em} $I_s \leftarrow \emptyset$, $\Gamma_s \leftarrow \emptyset$
4.  \hspace{1em} for every $V'$ in Enumerate-Simple-Cuts(ΔS1, s, c, q) do
5.  \hspace{2em} $I_s \leftarrow I_s \cup (V' \cap T)$
6.  \hspace{1em} for every $x \in I_s$ and every $V' \in \text{Enumerate-Simple-Cuts}(\Delta S_1, x, c, t)$ do
7.  \hspace{3em} if $V' \cap S = \emptyset$ and $\text{End}(\partial_G(V'))$ are in a connected component of $\Delta S_3$ containing s then
8.  \hspace{4em} $b \leftarrow$ true
9.  \hspace{4em} for $V' \in \text{Enumerate-Cuts}(\Delta S_1, \Delta S_2, V' \cap T, |\partial_G(V')|, q)$ do
10.  \hspace{5em} if $\partial_G(V')$ does not belong to same connected component of the graph of $\Delta S_3$ then
11.  \hspace{6em} $b \leftarrow$ false
12.  \hspace{4em} if $b = \text{true}$ then
13.  \hspace{5em} for every $E' \subseteq \partial_G(V')$ do
14.  \hspace{6em} if Atomic-Cut-Verification(ΔS1, E') = true and $E'$ partitions $s$ and $V'$ then
15.  \hspace{7em} put $(V', E')$ into $\Gamma_s$
16. \hspace{1em} $W_2 \leftarrow W_2 \cup \partial_G(V') \cup \text{Elimination}(\Delta S_1, \Gamma_s, c, t)$
17. return $W_2$

**Lemma 6.31.** Let $G = (V, E)$ be a connected graph of at most $m$ vertices and distinct edges, four parameters $c, d, t, q > 0$ such that $d \geq 2c + 1$ and $q \geq t$, and two vertex sets $T, S \subseteq V$.

Given access to $\Delta S_1 = \Delta S(G, S)$, $\Delta S_2 = \Delta S(G, T)$, $\Delta S_3 = \Delta S(G \setminus (\text{IA}_{G_0}(T_0, t, q, d, 2c) \setminus G), S)$ for an $\text{IA}_{G_0}(T_0, t, q, d, 2c)$ set, and parameters $c, t, q$, there is a Type-Two-Repair-Set algorithm with

$$O(|S|((2t)^{c+5c+3}) \text{poly}(c) \text{polylog}(m))$$

running time to output a type two repair set $W$ of $|S|((4c^3 + 4c^2))$ edges for any $\text{IA}_{G_0}(T_0, t, q', d, 2c+1)$ set such that the $\text{IA}_{G_0}(T_0, t, q', d, 2c+1)$ set is derived from the $\text{IA}_{G_0}(T_0, t, q, d, 2c)$ set.

**Proof.** For any $s \in S$, let $V_s$ denote the vertex set corresponding to the connected component of $G \setminus (\text{IA}_{G_0}(T_0, t, q, d, 2c) \setminus G)$ containing vertex $s$.

By the definition of $\mathcal{T}_{2,V_s}$ as defined in Section 6.1.3 for every $(T', S \cup (T \setminus T')) \in \mathcal{T}_{2,V_s}$, there is a vertex $x \in T'$ such that $x \in I_s$ by Lemma 6.18. Hence, by Lemma 6.18 for every
(T', S \cup (T \setminus T')) \in T_2 V^*$, and any minimum simple (T', S \cup (T \setminus T'), t, c)-cut (V', V \setminus V') with cut-set in G[V_s], there is an (E', V') in \Gamma_s, where E' \subseteq \partial_G(V') is the atomic cut separating s and V'.

By Lemma 6.20, W_2 is a type two repair set for any IA_{G_0}(T_0, t, q', d, 2c + 1) set with respect to G such that the IA_{G_0}(T_0, t, q', d, 2c + 1) set is derived from the IA_{G_0}(T_0, t, q, d, 2c) set, and W_2 contains at most \(|S|(4c^2 + 4c^2)\) edges.

Now we bound the running time. For a fixed s \in S, by Lemma 6.25, I_s contains at most \(q^{c+1}\) vertices. For fixed x \in I_s and V', line 9-11 takes
\[
O(q^{(c+1)+1} \cdot \text{poly}(c) \text{polylog}(m)) = O(q^{c+1} \cdot \text{poly}(c) \text{polylog}(m))
\]
time by Lemma 6.26, and line 12-15 takes \(O(2^c \text{poly}(m))\) time. Hence, for a fixed s \in S the running time of line 6-15 is
\[
O(q^{c+1} \cdot t^c \cdot (2q)^{c+1} \cdot \text{poly}(c) \text{polylog}(m)) = O((2q)^{c^2+3c+2} \cdot \text{poly}(c) \text{polylog}(m)),
\]
and \(\Gamma_s\) contains at most \(q^{2c+1}\) \((t, c)\)-realizable pairs. By Lemma 6.27 line 16 takes
\[
O(q^{c^2+5c+3} \text{poly}(c) \text{polylog}(m))
\]
time.

Hence, the overall running time is \(O(|S|(2q)^{c^2+5c+3} \text{poly}(c) \text{polylog}(m))\).

The following algorithm finds a type 3 repair set.

**Algorithm 11: Type-Three-Repair-Set**

**Input:** \(\mathcal{D}_1 = \mathcal{D}(G, S), \mathcal{D}_2 = \mathcal{D}(G, T), \mathcal{D}_3 = \mathcal{D}(G \setminus (\text{IA}_{G_0}(T_0, t, q, d, 2c), S)\) for some \(q \geq t, d \geq 2c\), \(c, t, q, d\) parameters

**Output:** \(W_3\): a set of edges

1. \(W \leftarrow \emptyset\)
2. \(U \leftarrow \emptyset\)
3. \(CC \leftarrow \emptyset\)
4. if \(|S| + |T| \leq t\) then
5.  find the cut with smallest size in Enumerate-Cuts(\(\mathcal{D}_1, \mathcal{D}_2, S \cup T, c, t\)), and put the cut-set in \(W\)
6. for every \(s \in S\) do
7.  \(CC \leftarrow CC \cup \{\mathcal{D}_3.\text{ID}(s)\}\)
8. for every cut \(V'\) in Enumerate-Simple-Cuts(\(\mathcal{D}_1, s, c, t\)) for an arbitrary \(s \in S\) do
9.  if \(V' \cap S = S\) and \(V' \cap T \neq T\) and \(\mathcal{D}_3.\text{ID}(y) \in CC\) for any \(y \in \text{End}(\partial_G(V'))\) and \(\mathcal{D}_3.\text{ID}(y) = \mathcal{D}_3.\text{ID}(z)\) for any \(y, z \in \text{End}(\partial_G(V'))\) then
10. put \(V'\) into \(U[\mathcal{D}_3.\text{ID}(x)]\)

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for $id \in CC$ do
\begin{align*}
\Gamma &\leftarrow \emptyset, E^i \leftarrow \emptyset, \beta^i \leftarrow +\infty \\
\text{for every } V' \in U[id] \text{ do} \\
\text{for every } E' \subseteq \partial_G(V') \text{ s.t. } \text{Atomic-Cut-Verification}(\mathcal{DS}_1, E') = \text{true} \text{ do} \\
&\text{let } x \text{ be an arbitrary vertex in } \text{End}(E') \setminus V' \\
&\text{for every } (y, z) \in E' \text{ do} \\
&D\mathcal{S}_2.\text{Delete}(y, z) \\
&\text{if } D\mathcal{S}_2.\text{TerminalNumber}(x) > 0 \text{ and } t < D\mathcal{S}_2.\text{VertexNumber}(x) < \beta^i \text{ then} \\
&\beta^i \leftarrow D\mathcal{S}_2.\text{VertexNumber}(x), E^i \leftarrow E', r \leftarrow D\mathcal{S}_2.\text{OneTerminal}(x) \\
&\text{reverse all the changes made on } D\mathcal{S}_2 \text{ on line 17} \\
\text{for every cut } V' \in U[id] \text{ s.t. } r \notin V' \text{ do} \\
\text{for every } E' \subseteq \partial_G(V') \text{ s.t. } \text{Atomic-Cut-Verification}(\mathcal{DS}_1, E') = \text{true and } E' \text{ separates } r \text{ and } V' \text{ do} \\
&\Gamma \leftarrow \Gamma \cup (E', V') \\
W_3 &\leftarrow W_3 \cup E^i \cup \text{Elimination}(\mathcal{DS}(G), \Gamma, c, t) \\
\end{align*}
return $W_3$

Lemma 6.32. Let $G = (V, E)$ be a connected graph of at most $m$ vertices and distinct edges, four parameters $c, d, t, q > 0$ such that $d \geq 2c + 1$ and $q \geq t$, and two vertex sets $T, S \subseteq V$.

Given access to $\mathcal{DS}_1 = \mathcal{DS}(G, S), \mathcal{DS}_2 = \mathcal{DS}(G, T) \mathcal{DS}_3 = \mathcal{DS}(G \setminus (\text{IA}_{G_0}(T_0, t, q, d, 2c)|_c), S)$ for an $\text{IA}_{G_0}(T_0, t, q, d, 2c)$ set, there is a TYPE-THREE-REPAIR-SET algorithm with

$$O(|S| (c^2 + 3c + 1) \text{poly}(c) \text{polylog}(m))$$

running time to output a set $W_3$ of at most $|S| (4c^3 + 4c^2 + 2c)$ edges such that for any $\text{IA}_{G_0}(T_0, t, q', d, 2c + 1)$ set that is derived from the $\text{IA}_{G_0}(T_0, t, q, d, 2c)$ set, and a type two repair set $W_2$ of $\text{IA}_{G_0}(T_0, t, q', d, 2c + 1)$ with respect to $G$, $W_3 \cup W_2$ is a type three repair set of $\text{IA}_{G_0}(T_0, t, q', d, 2c + 1)$ with respect to $G$.

Proof. By Lemma 6.23 and the definition of algorithm TYPE-THREE-REPAIR-SET, $W_3$ is a set of $|S| (4c^3 + 4c^2 + 2c)$ edges such that for any $\text{IA}_{G_0}(T_0, t, q', d, 2c + 1)$ set that is derived from the $\text{IA}_{G_0}(T_0, t, q, d, 2c)$ set and a type two repair set $W_2$, $W_3 \cup W_2$ is a type three repair set of $\text{IA}_{G_0}(T_0, t, q', d, 2c + 1)$ with respect to $G$.

Now we bound the running time. By Lemma 6.26, the running time of line 5 is $O(t^3(c+1)^{c+1})$. By Lemma 6.25, the running time of line 8-10 is $O(t^{c+1} \text{poly}(c) \text{polylog}(m))$, and the total number of vertex sets in $U[id]$ for all the ids in CC is $t^c$.

By Lemma 6.14, Lemma 6.24 and Lemma 6.27 the total running time of line 12-24 is

$$O(t^2 + 3c + 1) \text{poly}(c) \text{polylog}(m)$$

for a fixed $id \in CC$. Hence, the running time of line 11-24 is $O(|S| t^2 + 3c + 1) \text{poly}(c) \text{polylog}(m))$.

Hence, the overall running time of algorithm TYPE-THREE-REPAIR-SET is

$$O(|S| t^2 + 3c + 1) \text{poly}(c) \text{polylog}(m)$$

□
Finally, we present the algorithm for Lemma 6.3.

**Algorithm 12: Repair-Set(DES₁, DES₂, DES₃, S, c, t, q)**

| Parameter | Description |
|-----------|-------------|
| S         | a set of terminals |
| c, t, q   | parameters |

**Input:** DES₁ = DES(G, ∅), DES₂ = DES(G, T₀ ∩ V), DES₃ = DES(G \ (IA₀(T₀, t, q, d, 2c)ₐ₀(G), ∅)),
S: a set of terminals

**Output:** W: a set of edges that is a cut repair set

1. for s ∈ S do
2. DES₁.InsertTerminal(s)
3. DES₃.InsertTerminal(s)
4. DES₂.Delete − Terminal(s)
5. W ← TYPE-ONE-REPAIR-SET(DES₁, DES₂, DES₃, c, t)
6. W ← W ∪ TYPE-TWO-REPAIR-SET(DES₁, DES₂, DES₃, c, t, q)
7. W ← W ∪ TYPE-THREE-REPAIR-SET(DES₁, DES₂, DES₃, c, t)
8. reverse all the changes made at line 2, 3 and 4
9. return W

**Proof of Lemma 6.3.** By the description of the algorithm, before execution of line 5, DES₁ = DES(G, S), DES₂ = DES(G, T), and DES₃ = DES(G \ (IA₀(T₀, t, q, d, 2c)ₐ₀(G), S)). By Claim 6.7, Lemma 6.30, Lemma 6.31 and Lemma 6.32 the algorithm returns a repair set of size at most |S|(24c³ + 24c² + 4c).

The running time of the algorithm is obtained by Lemma 6.30, Lemma 6.31 and Lemma 6.32.

7 Cut partition update algorithm

In this section, we give preprocessing and update algorithms for cut partition.

**Definition 7.1.** Let G = (V, E) be a graph, and P is a partition of vertices such that for every P ∈ P, G[P] is connected. For parameters t, c, we say a partition Q of V is a (P, t, c)-cut partition if

1. Q is a refinement of P,
2. For any P ∈ P, {Q ∈ Q : Q ⊆ P} is a (t, c)-cut partition of G[P] with respect to End(∂G(P)) ∩ P.

Throughout the paper, a (P, t, c)-cut partition for graph G is obtained from an IAₐ₀(P)|P|(End(∂G(P))), t, q, c, c)

set for some q ≥ t, and the IAₐ₀(P)|P|(End(∂G(P)), t, q, c, c) set is constructed based on a family of IA sets:

IAₐ₀(P₀)|P₀|(End(∂G(P₀)), t₁, q₁, c, 1), IAₐ₀(P₁)|P₁|(End(∂G(P₁)), t₂, q₂, c − 1, 1),...
IAₐ₀(Pₑ−₁)|Pₑ−₁|(End(∂G(Pₑ−₁)), tₑ, qₑ, 1, 1),

where P₀ = P and Pᵢ is induced by the connected components of

G[Pₑ−₁] \ IAₐ₀(Pₑ−₁)|Pₑ−₁|(End(∂G(Pₑ−₁)), tᵢ, qᵢ, c − i + 1, 1)

with parameters t₁, q₁, \ldots, tₑ, qₑ satisfying

\[ t ≤ t₁, q₁ ≤ tᵢ ≤ qᵢ \text{ for all } 1 ≤ i ≤ c, \text{ and } qᵢ \cdot (c + 1) ≤ tᵢ₊₁ \text{ for all } 1 ≤ i ≤ c − 1 \]
Then by Corollary 5.6 for any $0 \leq i < j \leq c$, $\partial_G(\mathcal{P}_j) \setminus \partial_G(\mathcal{P}_i)$ is an

$$\text{IA}_G[\mathcal{P}_j](\text{End}(\partial_G(\mathcal{P}_i)), t_{i+1}, q_j(c + 1), c - i, j - i)$$

set, and $\partial_G(\mathcal{P}_c) \setminus \partial_G(\mathcal{P}_0)$ is an the $\text{IA}_G[\mathcal{P}](\text{End}(\partial_G(\mathcal{P})), t, q, c)$ set.

Our $(\mathcal{P}, t, c)$-cut partition data structure $\mathcal{DOS}$ of graph $G$ contains the following graph data structures.

1. $\mathcal{DOS} = \mathcal{DOS}(G, \emptyset)$
2. $\mathcal{DOS}_i = \mathcal{DOS}(G[\mathcal{P}_i], \text{End}(\partial_G(\mathcal{P}_i))), \mathcal{DOS}_i' = \mathcal{DOS}(G[\mathcal{P}_i], \emptyset)$ for all $0 \leq i \leq c$.

By Corollary 5.6 for any $0 \leq i < j \leq c$, we have

$$\mathcal{DOS}_j = \mathcal{DOS}(G[\mathcal{P}_i] \setminus \text{IA}_G[\mathcal{P}_i](\text{End}(\partial_G(\mathcal{P}_i)), t_{i+1}, q_j \cdot (c + 1), c - i, j - i), \text{End}(\partial_G(\mathcal{P}_i)) \cup \text{End}(\text{IA}_G[\mathcal{P}_i](\text{End}(\partial_G(\mathcal{P}_i)), t_{i+1}, q_j \cdot (c + 1), c - i, j - i))),$$

and

$$\mathcal{DOS}_j' = \mathcal{DOS}(G[\mathcal{P}_i] \setminus \text{IA}_G[\mathcal{P}_i](\text{End}(\partial_G(\mathcal{P}_i)), t_{i+1}, q_j \cdot (c + 1), c - i, j - i), \emptyset).$$

And thus

$$\mathcal{DOS}_c = \mathcal{DOS}(G[\mathcal{P}] \setminus \text{IA}_G[\mathcal{P}](\text{End}(\partial_G(\mathcal{P})), t, q, c), \text{End}(\partial_G(\mathcal{P})) \cup \text{End}(\text{IA}_G[\mathcal{P}](\text{End}(\partial_G(\mathcal{P})), t, q, c)))$$

for an $\text{IA}_G[\mathcal{P}](\text{End}(\partial_G(\mathcal{P})), t, q, c)$ set.

The $(\mathcal{P}, t, c)$-cut partition derived from $\mathcal{DOS}$ corresponds to the connected components of the graph of $\mathcal{DOS}_c$, i.e.,

$$Q = \{V' \subseteq V : V' \text{ is a connected component of the graph of } \mathcal{DOS}_c\} \quad (2)$$

For a $(\mathcal{P}, t, c)$-cut partition data structure $\mathcal{DOS}$ and a parameter $\gamma$, we use $\text{Sparsifier}(\mathcal{DOS}, \gamma)$ to denote the $\text{Sparsifier}(G, \mathcal{P}, Q, \gamma)$, where $Q$ is defined as Equation 2.

We say a $(\mathcal{P}, t, c)$-cut partition data structure $\mathcal{DOS}$ is with respect to parameters $t_1, q_1, \ldots, t_c, q_c$ if $t_1, q_1, \ldots, t_c, q_c$ satisfy Equation 1 and the edges that are in the graph of $\mathcal{DOS}_i$ but not in the graph of $\mathcal{DOS}_i-1$ is an $\text{IA}_G[\mathcal{P}_i-1](\text{End}(\partial_G(\mathcal{P}_i-1)), t_i, q_i, d - i + 1, 1)$ set for any $1 \leq i \leq c$.

By Theorem 2.6 and Corollary 6.4 we have the following lemma to initialize a cut partition data structure.

**Lemma 7.2.** Given a graph $G = (V, E)$ of at most $m$ vertices and distinct edges, a parameter $\phi \in (0, 1]$, positive integers $c, t$ and $t_1, q_1, \ldots, t_c, q_c$ satisfying Equation 1 there is an algorithm Cut-Partition-Preprocessing, with running time

$$\hat{O}(m/\phi^2) + O(m \phi^{2\delta \log^{1/3} m \log^{2/3} \log m} (20 \cdot c \cdot q_c)^{2 + 5c + 3} \text{poly}(c) \text{polylog}(m)),$$

to initialize a $(\mathcal{P}, t, c)$-cut partition data structure $\mathcal{DOS}$ for graph $G$ with respect to parameters $t_1, q_1, \ldots, t_c, q_c$ such that $\mathcal{P}$ is a $(\phi, \phi^{2\delta \log^{1/3} m \log^{2/3} \log m})$-expander decomposition of $\text{Simple}(G)$, and there are at most $m \phi^{2\delta \log^{1/3} m \log^{2/3} \log m} (10c)^{3c}$ distinct intercluster edges of the $(\mathcal{P}, t, c)$-cut partition with respect to $G$, where $\delta$ is a constant as Theorem 2.6.

In the rest of this section, we prove the following lemma for cut partition update. Lemma 7.3 also implies Lemma 1.9 in Section 1.4.4.
Lemma 7.3. Let $G = (V, E)$ be a graph and $\text{UpdateSeq}$ be a multigraph update sequence that updates $G$ to $G' = (V', E')$ such that both $G$ and $G'$ contains at most $m$ vertices and distinct edges, and every vertex of $G$ or $G'$ has at most $\Delta$ distinct neighbors.

Given access to a $(P, t, c^2 + 2c)$-cut partition data structure $\mathcal{DGS}$ for graph $G$ with respect to parameters $t_1, q_1, \ldots, t_{c^2+2c}, q_{c^2+2c}$ such that $P$ is a $\phi$-expander decomposition of $\text{Simple}(G)$ and parameters $\gamma > c, t_1, q_1, \ldots, t_{c^2+2c}, q_{c^2+2c}$ satisfying

$$t \leq t_1, t_i \leq q_i \text{ for all } 1 \leq i \leq c^2 + 2c, \text{ and } q_i \cdot ((c^2 + 2c) + 2) \leq t_{i+1} \text{ for all } 1 \leq i \leq c^2 + 2c - 1,$$

and $\text{UpdateSeq}$, there is a deterministic algorithm $	ext{Cut-Partition-Update}$ with running time

$$O(\|\text{UpdateSeq}\| \Delta (20c q_{c^2+2c})^2 + 5c^3 + 3 \text{poly}(c) \text{polylog}(m)) + \tilde{O}(\|\text{UpdateSeq}\| \Delta \log m/\phi^3)$$

to update $\mathcal{DGS}$ to a $(P', t, c)$-cut partition data structure $\mathcal{DGS}'$ for graph $G'$ with respect to parameters $t', q_1', \ldots, t_c', q_c'$ and to output an update sequence $\text{UpdateSeq}'$ of

$$O(\Delta \|\text{UpdateSeq}\|(10c)^3c)$$

multigraph update operations such that the following properties hold (let $Q$ denote the cut partition of $\mathcal{DGS}$ for $G$, and $Q'$ denote the cut partition of $\mathcal{DGS}'$)

1. $P'$ is a $(\phi/2^k \log^{1/3} m \log^{2/3} \log m)$-expander decomposition of $\text{Simple}(G')$ such that
   
   a. Every vertex of $G'$ that is involved in the update sequence $\text{UpdateSeq}$ is a singleton in $P'$
   
   b. Every $P \in P'$ which contains at least two vertices is a subset of some vertex set in $P$
   
   c. $|\partial \text{Simple}(G')(P')| \leq |\partial \text{Simple}(G)(P)| + O(\|\text{UpdateSeq}\| \Delta)$.

2. $|\partial \text{Simple}(G') (Q')| \leq |\partial \text{Simple}(G')(Q)| + O(\|\text{UpdateSeq}\| \Delta (10c)^3c)$.

3. $\text{UpdateSeq}'$ updates $\text{Sparsifier}(\mathcal{DGS}, \gamma)$ to $\text{Sparsifier}(\mathcal{DGS}', \gamma)$.

4. $t'_i = t_{w_{i-1}+1}, q'_i = q_{w_{i}}((c^2 + 2c) + 2)$, where $w_0 = 0, w_i = w_{i-1} + 2(c - i) + 3$ for $1 \leq i \leq c$.

We first give a cut partition data structure update algorithm in the following decremental update setting: Given access to $\mathcal{DGS}$ that is a $(P, t, c^2 + 2c)$-cut partition data structure for graph $G = (V, E)$, and a set of edges $R \subseteq E$ such that $\partial_G(P) \cup R = \partial_G(P^*)$ for a vertex partition $P^*$ that is a refinement of $P$, we want to update $\mathcal{DGS}$ to a $(P^*, t, c)$-cut partition data structure $\mathcal{DGS}'$ of $G$.

**Algorithm 13: Update-Partition(\mathcal{DGS}, R, t, c, \gamma, t_1, q_1, \ldots, t_{c^2+2c}, q_{c^2+2c})**

**Input:** $\mathcal{DGS} = \{\mathcal{DGS}, \mathcal{DGS}_0, \mathcal{DGS}_0', \ldots, \mathcal{DGS}_{c^2+2c}, \mathcal{DGS}_{c^2+2c}'\}$ with respect to parameters $t_i, q_i$

$R$: a set of edges

$c, t, \gamma, t_i, q_i$: parameters

**Output:** $\mathcal{DGS}'$: updated from $\mathcal{DGS}$

**NewUpdateSeq:** multigraph update sequence

1. $R[c] \leftarrow R$
2. $h[c] \leftarrow 0$
for $i = c$ down to 1 do
  $S[] \leftarrow \emptyset$, $R[i-1] \leftarrow \{(x, y) \in R[i] : (x, y) \text{ is in the graph of } \mathcal{DS}_{h[i]}\}$
  for $(x, y) \in R[i-1]$ do
    $\mathcal{DS}_{h[i]}$.Delete($x, y$), $\mathcal{DS}_{h[i]}'$.Delete($x, y$), $\mathcal{DS}_{h[i]+2i}$.Delete($x, y$)
  for $(x, y) \in R[i-1]$ do
    $S[\mathcal{DS}_{h[i]}].1D(x) \leftarrow S[\mathcal{DS}_{h[i]}].1D(x) \cup \{x\}$, $S[\mathcal{DS}_{h[i]}].1D(y) \leftarrow S[\mathcal{DS}_{h[i]}].1D(y) \cup \{y\}$
  for every id $S[id] \neq \emptyset$ do
    put $\text{REPAIR-SET}(\mathcal{DS}_{h[i]}'_{G'}, \mathcal{DS}_{h[i]}_{G'}, \mathcal{DS}_{h[i]+2i}_{G'}, S[id], i, t_{h[i]}, q_{h[i]}+2(c+1))$ to $R[i-1]$, where $G'$ is the connected component with identification $id$ in $\mathcal{DS}_{h[i]}$.
  for $(x, y) \in R[i]$ do
    $\mathcal{DS}_{h[i]}$.Insert-Terminal($x$), $\mathcal{DS}_{h[i]}$.Insert-Terminal($y$)
    $h[i-1] \leftarrow h[i]+2i+1$
    NewUpdateSeq $\leftarrow \emptyset$, Seq $\leftarrow \emptyset$
  for every $(x, y) \in R[0]$ do
    if edge $(x, y)$ is in the graph of $\mathcal{DS}_{h[0]}$ then
      $\mathcal{DS}_{h[0]}$.Delete($x, y$)
      Seq $\leftarrow Seq \circ \mathcal{DS}_{h[0]}$.Insert-Terminal($x$) $\circ \mathcal{DS}_{h[0]}$.Insert-Terminal($y$) $\circ \mathcal{DS}_{h[0]}$.Delete($x, y$)
      // $\circ$ denotes sequence concatenation
    else
      remove $(x, y)$ from $R[0]$
  for every operation $op \in Seq$ do
    if $op$ is an edge insertion then
      append $op$ to the end of $\text{NewUpdateSeq}$ with edge multiplicity $\gamma$
    else
      append $op$ to the end of $\text{NewUpdateSeq}$
  for every edge $(x, y) \in R[0]$ do
    append insert($x, y, \alpha$) to the end of $\text{NewUpdateSeq}$, where $\alpha$ is the edge multiplicity of $(x, y)$ in the graph of $\mathcal{DS}$
  return $\mathcal{DS}' = \{\mathcal{DS}, \mathcal{DS}_{h[c]}, \mathcal{DS}_{h[c]+1}, \ldots, \mathcal{DS}_{h[0]}, \mathcal{DS}_{h[0]}\}$, NewUpdateSeq

Lemma 7.4. Let $G = (V, E)$ be a graph with at most $m$ vertices and distinct edges, and $\mathcal{DS}$ be a $(P, t, c^2 + 2c)$-cut partition data structure of $G$ with respect to parameters $t_1, q_1, \ldots, t_{c^2+2c}, q_{c^2+2c}$ satisfying Equation 3. Given access to $\mathcal{DS}$, a set $R \subseteq E$ of $k$ distinct edges such that $\partial_G(P) \cup R$ is $\partial_G(P^*)$ for a vertex partition $P^*$ that is a refinement of $P$, and a parameter $\gamma > c$, algorithm Update-Partition, with running time $O(k(20c \cdot q_{c^2+2c})^{c^2+5c+3} \cdot poly(c) \cdot polylog(m))$, updates $\mathcal{DS}$ to a $(P^*, t, c)$-cut partition data structure $\mathcal{DS}'$ of $G$ with respect parameters

$$t'_i = t_{w_i-1} + 1, q'_i = q_{w_i}((c^2 + 2c) + 2) \text{ for all } 1 \leq i \leq c,$$

where $w_0 = 0, w_i = w_{i-1} + 2(c - i) + 3 \text{ for } 1 \leq i \leq c$, and outputs an update sequence of length $O(k(10c)^3m^3)$ to update $\text{Sparsifier}(\mathcal{DS}, \gamma)$ to $\text{Sparsifier}(\mathcal{DS}', \gamma)$.

$\mathcal{DS}_{h[i]}'_{G'}$ and $\mathcal{DS}_{h[i]}_{G'}$, are specified by the access of the connected components in $\mathcal{DS}_{h[i]}$ and $\mathcal{DS}_{h[i]}$ corresponding to $G'$ respectively. $\mathcal{DS}'{h[i] + 2i}_{G'}$ is specified by $\mathcal{DS}_{h[i] + 2i}$ using $\mathcal{DS}'{h[i]}_{G'}$ to verify if vertices and edges are in $G'$. 

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Let $P_i^*$ be the vertex partition induced by connected components of $D\mathcal{S}_i$ of $D\mathcal{D}\mathcal{S}'$. By Lemma 6.3 in $D\mathcal{D}\mathcal{S}'$, $P_0^* = P^*$, $D\mathcal{S}_0 = D\mathcal{S}(G(P_0^*], \text{End}(|\partial_G(P_0^*)|))$ and $D\mathcal{S}'_0 = D\mathcal{S}(G[P_0^*], \emptyset)$. For $i \geq 1$,

$$D\mathcal{S}_i = D\mathcal{S}(G[P_{i-1}^*] \setminus \text{IA}_{G[P_{i-1}^*]}(\text{End}(|\partial_G(P_{i-1}^*)|)), t_i', q_i', c - i + 1, 1),$$

$$\text{End}(\text{IA}_{G[P_{i-1}^*]}(\text{End}(|\partial_G(P_{i-1}^*)|)), t_i', q_i', c - i + 1, 1))$$

and

$$D\mathcal{S}'_i = D\mathcal{S}(G[P_{i-1}^*] \setminus \text{IA}_{G[P_{i-1}^*]}(\text{End}(|\partial_G(P_{i-1}^*)|)), t_i', q_i', c - i + 1, 1), \emptyset).$$

By Corollary 5.6 $D\mathcal{D}\mathcal{S}'$ is a $(P^*, t, c)$-cut partition data structure of $G$ with respect to parameters $t_i', q_i', \ldots, t_c', q_c'$.

Let $k_i$ denote the number of distinct edges of $R[i]$ for $0 \leq i \leq c$. By Lemma 6.3 and the algorithm, $k_c = k$, and for $0 \leq i < c$,

$$k_i \leq |k_{i+1}| + 2 \cdot |k_{i+1}| \cdot (24c^3 + 24c^2 + 4c) < |k_{i+1}| \cdot (10c)^3$$

edges. By induction, $R[0]$ contains at most $k(10c)^3c$.

By Lemma 2.14 NewUpdateSeq updates Sparsifier($D\mathcal{D}\mathcal{S}, \gamma$) to Sparsifier($D\mathcal{D}\mathcal{S}'$, $\gamma$), and the number of operations of NewUpdateSeq is $O(|R|(10c)^3c)$.

The running time of the algorithm is obtained by Lemma 6.3 and Lemma 2.14.

**Algorithm 14: Decremental-Single-Expander($D\mathcal{S}(G), \phi, D$)**

**Input:** $D\mathcal{S}(G)$: dynamic data structure for simple graph $G$

$\phi$: parameter

$D$: a set of edges

**Output:** $R$: a set of edges

1. $R \leftarrow \emptyset, P \leftarrow \emptyset$
2. $m \leftarrow D\mathcal{S}(G).\text{DistinctEdgeNumber}(x)$ for an arbitrary vertex $x$ in $G$
3. if $|D| \leq m\phi/10$ then
4.   $P \leftarrow \text{Pruning}(G, D), R \leftarrow \partial_G(P) \setminus D$
5. else
6.   $P \leftarrow V$
7. compute $H = G[P] \setminus D$
8. $\mathcal{P} \leftarrow \mathcal{P} \cup \text{Expander-Decomposition}(H[V'], \phi/2O(\log^{1/3} m \log^{2/3} \log m))$ for every $V'$ that forms a connected component of $H$
9. return $R \cup \partial_H(\mathcal{P})$

**Lemma 7.5.** Given access to $D\mathcal{S}(G)$ for a simple graph $G = (V, E)$ such that $G$ is a $\phi$-expander with $m$ edges, and the maximum degree of $G$ is at most $\Delta$, and an edge set $D \subseteq E$ of $k$ edges, there is a deterministic algorithm Decremental-Single-Expander with running time $\tilde{O}(|D| \log m/\phi^3)$ to output an edge set $R$ from graph $G \setminus D$ such that $|R| = O(k)$ and $R$ is the set of intercluster edges of a vertex partition $\mathcal{R}$ for graph $G \setminus D$ such that $\mathcal{R}$ is a $(\phi/2^6 \log^{1/3} m \log^{2/3} \log m)$-expander decomposition of $G \setminus D$, where $\delta$ is the constant as Theorem 2.6.

**Proof.** Let $G' = (V, E \setminus D)$. If $k \leq m\phi/10$, by Theorem 2.7, we have

1. Every connected component of $G'[V \setminus P]$ is a $\phi/6$ expander;
(2) \( |\partial_G(P)| \leq 8k \);
(3) \( \text{vol}_G(P) \leq 8k/\phi \).

If \( k > m\phi/10 \), we have \( P = V \), and \( \text{vol}_G(P) = \text{vol}(G) \leq 2m < 20k/\phi \). Since \( H = G'[P] \), for either case, \( \text{vol}(H) \leq 20k/\phi \).

For a \( V' \) that forms a connected component of \( H \), let \( \alpha \) denote the number of edges in \( H[V'] \).

The output of

\[
\text{Expander-Decomposition}(H[V'], \phi/2^8 \log^{1/3} m \log^{2/3} \log m)
\]

is a \( \phi/2^8 \log^{1/3} m \log^{2/3} \log m \)-expander decomposition of \( H[V'] \) with \( O(\phi \alpha) \) intercluster edges by Theorem 2.6. Since the total number of edges in \( H \) is at most \( 20k/\phi \), sum over all the connected components of \( H \), we obtain a \( (\phi/2^8 \log^{1/3} m \log^{2/3} \log m) \)-expander decomposition of \( H \) with \( O(k) \) intercluster edges.

By the definition of algorithm \text{Decremental-Single-Expander}, \( R \) contains all the intercluster edges for a \( (\phi/2^8 \log^{1/3} m \log^{2/3} \log m) \)-expander decomposition of \( G \setminus D \) with \( O(k) \) intercluster edges.

The running time is obtained by Theorem 2.6 and Theorem 2.7. □

**Algorithm 15: Cut-Partition-Update(\( \mathcal{DS}, \text{UpdateSeq} \))**

| Input: \( \mathcal{DS} = \{\mathcal{DS}, \mathcal{DS}_0, \mathcal{DS}_1, \ldots, \mathcal{DS}_c, \mathcal{DS}_c^2 + 2c, \mathcal{DS}_c^2 + 2c \} \) with respect to parameters \( t_i, q_i \)
| UpdateSeq: a multigraph update sequence \( \phi, c, t, \gamma, t_1, q_1, \ldots, t_c + 2c, q_c + 2c \): parameters
| Output: \( \mathcal{DS}' \): updated from \( \mathcal{DS} \)
| NewUpdateSeq: multigraph update sequence
| 1 \( R \gets \emptyset, I \gets \emptyset \)
| 2 for vertex \( x \) involved in \( \text{UpdateSeq} \) do
| 3 \( I \gets I \cup \{\mathcal{DS}_0, \text{ID}(x)\} \)
| 4 for every \( id \in I \) do
| 5 \( W_{id} \leftarrow \{v : v \text{ is involved in } \text{UpdateSeq} \text{ and } \mathcal{DS}_0, \text{ID}(v) = id\} \)
| 6 let \( D_{id} \) be the distinct edges of \( \{\{u, v\} : u \in W_{id} \text{ and } \mathcal{DS}_0, \text{ID}(v) = id\} \)
| 7 \( R_{id} \leftarrow \text{Decremental-Single-Expander}(\text{Simple}(\mathcal{DS}_0|_{G'}), \phi, D_{id}) \), where \( G' \) is the connected component with identification \( id \) in \( \mathcal{DS}_0 \), and \( \text{Simple}(\mathcal{DS}_0|_{G'}) \) means edge multiplicities are ignored in the subroutine for \( \mathcal{DS}_0|_{G'} \)
| 8 \( R \leftarrow R \cup \{(x, y) \text{ in graph of } \mathcal{DS}_0 : R_i \cup D_i\} \)
| 9 \( \mathcal{DS}', \text{NewUpdateSeq} \leftarrow \text{Update-Partition}(\mathcal{DS}, R, c, t, \gamma, t_1, q_1, \ldots, t_c + 2c, q_c + 2c) \)
| 10 for every vertex \( x \) involved in \( \text{UpdateSeq} \) that is an isolated vertex in the graph of \( \mathcal{DS}' \) do
| 11 \( \text{NewUpdateSeq} \leftarrow \text{NewUpdateSeq} \circ \text{insert}(x) \)
| 12 run \( \mathcal{DS}_i, \text{Insert-Terminal}(x) \) for every \( \mathcal{DS}_i \text{ in } \mathcal{DS}' \)
| 13 \( \text{NewUpdateSeq} \leftarrow \text{NewUpdateSeq} \circ \text{UpdateSeq} \)
| 14 apply \( \text{UpdateSeq} \) to \( \mathcal{DS} \) in \( \mathcal{DS}' \)
| 15 for every vertex \( x \) involved in \( \text{UpdateSeq} \) that is an isolated vertex in the graph of \( \mathcal{DS}' \) do
| 16 \( \text{NewUpdateSeq} \leftarrow \text{NewUpdateSeq} \circ \text{delete}(x) \)
| 17 run \( \mathcal{DS}_i, \text{Delete-Terminal}(x) \) for every \( \mathcal{DS}_i \text{ in } \mathcal{DS}' \)
| 18 return \( \mathcal{DS}' \) and \( \text{NewUpdateSeq} \)
Proof of Lemma 7.3. We make the following observations for each $id \in I$: (Let $G_{id}$ denote the connected component for the graph of $\mathcal{DS}_0$ with connected component identification $id$.)

(a). The number of distinct edges of $D_{id}$ is at most $\Delta |W_{id}|$.
(b). $R_{id}$ is the set of intercluster edges for a $(c/\phi)\log^{1/3}m \log^{2/3}m$-expander decomposition of $\text{Simple}(G_{id})$ with at most $O(|D_{id}|)$ intercluster edges.
(c). For every vertex $v \in W_{id}$, $v$ is an isolated vertex in $G_{id}$ if all the edges of $R_{G_{id}}$ are removed.

Property (a) is obtained by the definition of the algorithm and the fact that the maximum degree of $\text{Simple}(G)$ is at most $\Delta$. Property (b) is obtained by Lemma 7.6. Property (c) is obtained by the fact that $D_{id}$ contains all the distinct edges incident to $v$ in $G_{id}$ for every $v \in W_{id}$.

Hence, $\partial_G(\mathcal{P}) \cup R$ is the set of intercluster edges of a $(c/\phi)\log^{1/3}m \log^{2/3}m$-expander decomposition for graph $G$ such that every vertex involved in the update sequence is a singleton in the new partition. Let $\mathcal{P}^*$ denote this new partition.

By Lemma 7.4 after line 9, the $(\mathcal{P}, t, c^2 + 2c)$-cut partition data structure $\mathcal{DS}$ for graph $G$ is updated to a $(\mathcal{P}^*, t, c)$-cut partition data structure $\mathcal{DS}'$ for graph $G$, and update sequence $\text{NewUpdateSeq}$ updates $\text{Sparsifier}(\mathcal{DS}, \gamma)$ to $\text{Sparsifier}(\mathcal{DS}', \gamma)$ for $\mathcal{DS}'$. The length of $\text{NewUpdateSeq}$ is at most $O(\Delta |\text{UpdateSeq}| (10c)^{\mathcal{P}})$.

Since every vertex involved in the update sequence $\text{UpdateSeq}$ is a singleton in $\mathcal{P}^*$, applying update sequence $\text{UpdateSeq}$ to $\mathcal{DS}$ of $\mathcal{DS}'$ updates the $(\mathcal{P}^*, t, c)$-cut partition data structures for $G'$. After adding necessary isolated vertices to the sparsifier (by appending vertex insertions for these vertices to the end of $\text{NewUpdateSeq}$), $\text{UpdateSeq}$ can also be applied to the sparsifier to obtain the sparsifier of graph $G'$.

The running time is obtained by Lemma 7.4 and Lemma 7.5.

8 Online-batch algorithm for multi-level $c$-edge connectivity sparsifier

We give the online-batch preprocessing and update algorithms for the multi-level $c$-edge connectivity sparsifier in this section.

We first show that cut partition is an edge connectivity equivalent partition using appropriate parameters.

Lemma 8.1. Let $G = (V, E)$ be a graph, and $\mathcal{P}$ be a $\phi$-expander decomposition of $\text{Simple}(G)$ for some $0 < \phi < 1$. For any positive integer $c$, a $(\mathcal{P}, [c/\phi], c)$-cut partition $\mathcal{Q}$ for $G$ is a $(\mathcal{P}, c)$-edge connectivity equivalent partition for graph $G$.

Proof. Let $P$ be an arbitrary cluster in $\mathcal{P}$, and $T$ denote $\text{End}(\partial_G(P)) \cap P$. We show that for any $0 \subseteq T' \subseteq T$, if the minimum cut of $G[P]$ partitioning $T$ into $T'$ and $T \setminus T'$ is of size at most $c$, then there is a subset of edges $E'$ of $G[P]$ such that $E'$ is the cut-set of a minimum cut of $G[P]$ partitioning $T$ into $T'$ and $T \setminus T'$, and $E'$ is a subset of $\partial_G(\mathcal{Q})$.

Let $\alpha$ denote the size of minimum cut of $G[P]$ partitioning $T$ into $T'$ and $T \setminus T'$. Let $(V', P \setminus V')$ be a minimum cut of $G[P]$ partitioning $T$ into $T'$ and $T \setminus T'$. Since the size of cut $(V', P \setminus V')$ on $G[P]$ is at most $c$, $\partial_{G[P]}(V')$ contains at most $c$ edges. Hence, $(V', P \setminus V')$ is a cut of $\text{Simple}(G[P])$ of size at most $c$. Since $\text{Simple}(G[P])$ is a $\phi$-expander, one of $\text{vol}_{\text{simple}(G[P])}(V')$ and $\text{vol}_{\text{simple}(G[P])}(P \setminus V')$ is at most $c/\phi$. Hence, $(V', P \setminus V')$ or $(P \setminus V', V')$ is a $([c/\phi], c)$-cut for graph $G[P]$. By Definition 5.2 and Definition 7.4, there is a $E'$ satisfying the following conditions:
(1) \( E' \subseteq \partial_G(\mathcal{Q}) \).
(2) \( E' \) is in \( G[P] \).
(3) \( E' \) induces a cut of \( G[P] \) with size at most \( \alpha \) that partitions \( T \) into \( T' \) and \( T \setminus T' \).

Hence, \( E' \) induces a minimum cut of \( G[P] \) partitioning \( T \) into \( T' \) and \( T \setminus T' \). \( \square \)

Our multi-level \( c \)-edge connectivity sparsifier data structure is parameterized by \( \phi, \gamma, t, t_1, q_1, \ldots, t_c, q_c \) satisfying \( t \geq \lceil c/\phi \rceil \), \( \gamma \geq c + 1 \), and Equation 1 for \( t, t_1, q_1, \ldots, t_c, q_c \). A multi-level \( c \)-edge connectivity sparsifier data structure with respect to parameters \( \phi, \gamma, t, t_1, q_1, \ldots, t_c, q_c \) of graph \( G \), denoted by \( \mathcal{DS} = \{ \mathcal{DS}^{(i)} \}_{i=0}^\ell \), satisfying the following conditions: (For every \( \mathcal{DS}^{(i)} \), let \( G^{(i)} \) denote the graph of \( \mathcal{DS} \) of \( \mathcal{DS}^{(i)} \), \( \mathcal{P}^{(i)} \) denote the partition induced by the connected components of \( \mathcal{DS} \) of \( \mathcal{DS}^{(i)} \), and \( Q^{(i)} \) denote the partition induced by the connected components of \( \mathcal{DS} \) of \( \mathcal{DS}^{(i)} \).)

(1) \( G^{(0)} \) is the same to \( G \).
(2) For every \( \mathcal{DS}^{(i)} \), \( \mathcal{P}^{(i)} \) is a \( \phi \)-expander decomposition of \( \text{Simple}(G^{(i)}) \), and \( \mathcal{DS}^{(i)} \) is a \( (\mathcal{P}^{(i)}, t, c) \)-cut partition data structure with respect to parameters \( t_1, q_1, \ldots, t_c, q_c \).
(3) For \( 0 < i \leq \ell \), \( G^{(i)} = \text{Sparsifier}(\mathcal{DS}^{(i-1)}, \gamma) \).
(4) \( \text{Sparsifier}(\mathcal{DS}^{(\ell)}, \gamma) \) is an empty graph (for every edge \((x, y)\) of graph \( G^{(\ell)} \), \( x \) and \( y \) are in the same cluster of \( P^{(\ell)} \)).

We say a multi-level \( c \)-edge connectivity sparsifier data structure with respect to \( \phi, \gamma, t, t_1, q_1, \ldots, t_c, q_c \) is a \( (\phi, \eta, \gamma) \) multi-level \( c \)-edge connectivity sparsifier data structure if \( \{ G^{(i)}, \mathcal{P}^{(i)}, Q^{(i)} \}_{i=1}^\ell \) is a \( (\phi, \eta, \gamma) \) multi-level \( c \)-edge connectivity sparsifier as defined in Definition 1.5.

We set the parameters as Figure 5 for a given \( c = (\log m)^{o(1)} \):

\[
\begin{align*}
\phi & \overset{\text{def}}{=} \frac{1}{2 \log^{3/4} m}, \\
\phi_0 & \overset{\text{def}}{=} \phi, \\
\phi_i & \overset{\text{def}}{=} \frac{\phi_{i-1}}{2^6 \log^{1/3} m \log^{2/3} \log m} \quad \text{for } 1 \leq i \leq \zeta + 1, \text{ where } \delta \text{ is the constant of Theorem } 2.6, \\
c_{\zeta+1} & \overset{\text{def}}{=} c \text{ and } c_i \overset{\text{def}}{=} c_{i+1}(c_{i+1} + 2) \text{ for } 0 \leq i \leq \zeta, \\
\gamma & \overset{\text{def}}{=} c_0 + 1, \\
\eta_i & \overset{\text{def}}{=} 4\phi(2^6 \log^{1/3} m \log^{2/3} \log m)^{c_0 + 1} (10c_0)^{3c_0} \quad \text{for } 0 \leq i \leq \zeta + 1, \\
t & \overset{\text{def}}{=} \left\lfloor \frac{c_0}{\phi_{\zeta+1}} \right\rfloor, \\
t_{0,1} & \overset{\text{def}}{=} t, \\
t_{0,j} & \overset{\text{def}}{=} t_{0,j-1} \cdot (c_0 + 2)^{2(i+3)} \quad \text{for } 1 \leq j \leq c_0, \text{ and } q_{0,j} \overset{\text{def}}{=} t_{0,j} \cdot (c_0 + 2)^{\zeta+3} \quad \text{for } 0 \leq j \leq c_0, \\
t_i & \overset{\text{def}}{=} t_{i-1, w_{i,j-1}+1}, \\
q_{i,j} & \overset{\text{def}}{=} q_{i-1, w_{i,j}} \cdot (c_0 + 2) \quad \text{for } 0 \leq i \leq \zeta + 1, 0 \leq j \leq c_i, \text{ where } w_{i,0} \overset{\text{def}}{=} 0, \\
w_{i,j} & \overset{\text{def}}{=} w_{i,j-1} + 2(c_i - j) + 3.
\end{align*}
\]

Figure 5: Parameter setting
Claim 8.2. Given $c = (\log m)^{o(1)}$, and parameters defined as Figure 5, the following conditions hold:

1. $\zeta = \omega(1)$ and $\zeta = O(\log \log \log m)$.
2. $c_i < (4c)^{2^{i+1}} = \log^{1/10} m$ for any $0 \leq i \leq \zeta + 1$.
3. $\phi^{1.5} \leq \phi_i \leq \phi$ for any $0 \leq i \leq \zeta + 1$.
4. $\phi \leq \eta_i \leq \sqrt{\phi}$ for any $0 \leq i \leq \zeta + 1$.
5. $t_{i,j}(c_0 + 2) \leq q_{i,j}$ and $q_{i,j}(c_0 + 2) \leq t_{i,j+1}$ for any $i, j$.
6. $t_{i,j}^{O(c_i^2)}, q_{i,j}^{O(c_i^2)} = m^{o(1)}$ for any $i, j$.

Proof. For the first condition, since $c = (\log m)^{o(1)}$, we have

$$\log \left( \frac{\log m/10}{\log(4c)} \right) = \log \left( \frac{\log m/10}{o(1)\log m + 2} \right) = \log(\omega(1)) = \omega(1),$$

and thus $\zeta = \omega(1)$. On the other hand,

$$\zeta = O \left( \log \left( \frac{\log m/10}{\log(4c)} \right) \right) = O(\log(\log m/10)) = O(\log \log m).$$

For the second condition, we have $c_i = c_{i+1}(c_{i+1} + 2) < 4c_{i+1}^2$ for any $i \leq \zeta$. By induction,

$$c_i \leq 4^{2^{i+1-i} - 1} c^{2^{i+1-i}} < (4c)^{2^{i+1}} \leq (4c)^{\log(\log m/10)} = 2^{\log \log m/10} = \log^{1/10} m.$$  

The third and forth condition hold by

$$\left( 2^{4\log^{1/3} m \log^{2/3} \log m} \right)^{\zeta+1} = 2^{O(\log^{1/3} m \log^{2/3} \log m \log \log m)} = (1/\phi)^{o(1)},$$

and $(10c_0)^{2\zeta} < (10 \log^{1/10} m)^{2(\log^{1/10} m)} = (1/\phi)^{o(1)}$.

The fifth condition holds for $i = 0$ by definition. For $i > 0$, $t_{i,j}(c_0 + 2) \leq q_{i,j}$ is obtained by the definition of $t_{i,j}, q_{i,j}$ and induction. $q_{i,j}(c_0 + 2) \leq t_{i,j+1}$ is proved by the induction of $t_{i,j+1} = q_{i,j}(c_0 + 2)^{\zeta+3-i}$, and the fact that $i \leq \zeta + 1$.

For the six condition, we have

$$t \leq \frac{c_0}{\phi^{1.5}} + 1 = O \left( \frac{\log^{1/10} m}{1/2^{\log^{3/4} m}} \right) \leq 2^{1.6 \log^{3/4} m} = m^{o(1)}.$$  

By the definition of $t_{i,j}$ and $q_{i,j}$, we have

$$t_{i,j}, q_{i,j} \leq 2^{1.6 \log^{3/4} m} \cdot (c_0 + 2)^{2(\zeta+3) - c_0} \cdot (c_0 + 2)^{2(\zeta+3)} = 2^{1.6 \log^{3/4} m} \cdot (\log^{1/10} m)^{O(\log \log m \log^{1/10} m)} < 2^{1.7 \log^{3/4} m},$$

and thus

$$t_{i,j}^{O(c_i^2)}, q_{i,j}^{O(c_i^2)} = 2^{1.7 \log^{3/4} m \cdot O(\log^{1/5} m)} = 2^{O(\log^{0.95} m)} = m^{o(1)}.$$
By Lemma 7.2, we have the following lemma to initialize the data structure for a multi-level 
$c_0$-edge connectivity sparsifier data structure for a given multigraph $G$ with respect to parameters 
$\phi_0, \gamma, t, t_{0,1}, q_{0,1}, \ldots, t_{0,c_0}, q_{0,c_0}$.

**Lemma 8.3.** Given a graph $G$ with at most $m$ vertices and distinct edges, a positive integer 
c = $\log^{o(1)} m$, algorithm Preprocess-Multi-Level-Sparsifier outputs a $(\phi_0, \eta_0, \gamma)$ multi-level 
c_0-edge connectivity sparsifier data structure $\mathcal{MDS} = \{\mathcal{ODS}^{(i)}\}_{i=0}^{\ell}$ with \( \ell = O(\log 1/\eta_0 m) \) for graph 
$G$ with respect to parameters $\phi_0, \gamma, t, t_{0,1}, q_{0,1}, \ldots, t_{0,c_0}, q_{0,c_0}$ defined as Figure 5 in $O(m)$ time.

**Algorithm 16:** Preprocess-Multi-Level-Sparsifier$(G, c)$

**Input:** $G$: a multigraph 
c: a positive integer 
**Output:** $\mathcal{MDS}$: multi-level $c_0$-edge connectivity sparsifier data structure

1. $\ell \leftarrow 0$
2. repeat
3. if $\ell = 0$ then
4. $G^{(0)} \leftarrow G$
5. else
6. $G^{(\ell)} \leftarrow $ Sparsifier$(\mathcal{ODS}^{(\ell-1)}, \gamma)$
7. $\mathcal{ODS}^{(\ell)} \leftarrow $ Cut-Partition-Preprocessing$(G^{(\ell)}, \phi_0, c_0, t, t_{0,1}, q_{0,1}, \ldots, t_{0,c_0}, q_{0,c_0})$
8. $\ell \leftarrow \ell + 1$
9. until Sparsifier$(\mathcal{ODS}^{(\ell-1)}, \gamma)$ is an empty graph
10. return $\mathcal{MDS} = \{\mathcal{ODS}^{(i)}\}_{i=0}^{(\ell-1)}$

Now we prove the following lemma for multi-level sparsifier update.

**Lemma 8.4.** Let $G$ be a graph undergoing multigraph update operations (vertex and edge insertions 
and deletions) such that throughout the updates, the graph has at most $m$ vertices and distinct edges, 
and every vertex of the graph has at most $\Delta = O(1)$ distinct neighbors. Given $c = (\log m)^{o(1)}$, let 
parameters be set as Figure 5. Assume the multigraph updates are partitioned into $\zeta$ multigraph 
update sequences $\text{UpdateSeq}_1, \text{UpdateSeq}_2, \ldots, \text{UpdateSeq}_\zeta$ such that $|\text{UpdateSeq}_j| \leq \frac{\zeta m}{\Delta \log n}$ for every $1 \leq j \leq \zeta$.

Let $\mathcal{MDS}$ be a multi-level sparsifier constructed by Preprocess-Multi-Level-Sparsifier 
with input graph $G$, parameter $c$, and other parameters as defined in Figure 5. If we run algorithm 
Update-Multi-Level-Sparsifier on $\mathcal{MDS}$ with update sequences

$$\text{UpdateSeq}_1, \text{UpdateSeq}_2, \ldots, \text{UpdateSeq}_\zeta$$

sequentially, then the following conditions hold

1. After $j$-th execution of the update algorithm Update-Multi-Level-Sparsifier, $\mathcal{MDS}$ is 
a $(\phi_j, \eta_j, \gamma)$ multi-level $c_j$-edge connectivity sparsifier with respect to parameters $\phi_j, \gamma, t, t_{j,1}, 
q_{j,1}, \ldots, t_{j,c_j}, q_{j,c_j}$.

2. The amortized running time of Update-Multi-Level-Sparsifier is $m^{o(1)}$ for each updated sequence.
We use the following algorithm to prove Lemma 8.4.

**Algorithm 17: Update-Multi-Level-Sparsifier(ODS, c, k, UpdateSeq)**

**Input:** $\text{ODS}$: multi-level $c_{k-1}$-edge connectivity sparsifier data structure \( \{\text{ODS}^{(i)}\}^\ell_{i=1} \)

\( c, k \): parameters

UpdateSeq: multigraph update sequence

**Output:** \( \text{ODS}' \): multi-level $c_k$-edge connectivity sparsifier data structure

1. \( \text{UpdateSeq}^{(0)} \leftarrow \text{UpdateSeq}, r \leftarrow \text{false}, i \leftarrow 0 \)
2. repeat
3. \( \text{if } i > \ell \text{ then} \)
4. \( r \leftarrow \text{true} \)
5. \( \text{if } r = \text{false} \text{ and } |\text{UpdateSeq}^{(i)}| \leq m\phi^{i+1} \text{ then} \)
6. \( \text{run } \text{Cut-Partition-Update}(\text{ODS}^{(i)}, \phi_{k-1}, c_k, t, \gamma, \text{UpdateSeq}^{(i)}, t_{k-1}, \ldots) \text{ to obtain the updated } \text{ODS}^{(i+1)} \text{ and update sequences } \text{UpdateSeq}^{(i+1)} \)
7. \( i \leftarrow i + 1 \)
8. continue
9. \( r \leftarrow \text{true} \)
10. \( G^{(i)} \leftarrow \text{Sparsifier}(\text{ODS}^{(i-1)}, \gamma) \)
11. \( \text{ODS}^{(i)} \leftarrow \text{Cut-Partition-Preprocessing}(G^{(i)}, \phi_k, c_k, t, q_k, \ldots) \)
12. \( i \leftarrow i + 1. \)
13. until \( r = \text{true} \) and \( \text{Sparsifier}(\text{ODS}^{(i-1)}, \gamma) \) is an empty graph
14. return updated \( \{\text{ODS}^{(j)}\}^i_{j=0} \)

We show that Update-Multi-Level-Sparsifier is an online-batch update algorithm for multi-level sparsifier with batch number $\zeta$ as defined in Figure 5 and sensitivity $O(\phi m / \log n)$.

**Proof of Lemma 8.4.** Let $\text{ODS}_0$ be the multi-level sparsifier obtained by running the algorithm Preprocess-Multi-Level-Sparsifier on the input graph with parameters as defined in Figure 4 and $\text{ODS}_j$ be the output of $j$-th execution of Update-Multi-Level-Sparsifier. Denote

\[
\text{ODS}_j = \{\text{ODS}^{(i)}_j\}^\ell_{i=0}.
\]

Let \( \text{UpdateSeq}^{(i)} \) be the multigraph update sequence that is used to update $\text{ODS}^{(j-1)}_i$ to $\text{ODS}^{(j)}_i$ (if exists). By Lemma 8.3 \( \text{ODS}_0 \) is a \((\phi_0, \eta_0, \gamma)\) multi-level $c_0$-edge connectivity sparsifier data structure with respect to parameters $\phi_0, \gamma, t, t_0, \ldots, t_0, c_0, q_0, s_0$ as defined in Figure 5.

By algorithm Update-Multi-Level-Sparsifier, Lemma 7.2 Lemma 7.3 and induction, the following conditions hold: for any $1 \leq j \leq \zeta$ and $0 \leq i \leq \ell_j$,

1. The graph of $\text{DS}$ in $\text{ODS}^{(i)}_j$ is the resulted graph of applying $\text{UpdateSeq}^{(0)}, \ldots, \text{UpdateSeq}^{(j)}$ sequentially to \( G \). And the graph of $\text{DS}$ in $\text{ODS}^{(i)}_j$ is $\text{Sparsifier}(\text{ODS}^{(i-1)}_j, \gamma)$ for any $1 \leq i \leq \ell_j$.
2. $\text{ODS}^{(i)}_j$ is a $(P^{(i)}_j, t, c_j)$-cut partition data structure with respect to parameters $t_{j,1}, q_{j,2}, \ldots, t_{j,c_j}$, $q_j, c_j$, where $P^{(i)}_j$ is the partition induced by connected components of $\text{DS}_0$ in $\text{ODS}^{(i)}_j$.
3. $P^{(i)}_j$ is a $\phi_j$-expander decomposition of the simple version of $\text{DS}_0$ in $\text{ODS}^{(i)}_j$.
For $j = 0$ and $0 \leq i \leq \ell_0$, by Lemma 8.3, the graph of $\mathcal{G}$ in $\mathcal{ODS}_j^{(i)}$ has at most $mn_j^i$ edges. Consider the case of $j > 0$. For $i = 0$, the graph of $\mathcal{G}$ in $\mathcal{ODS}_j^{(i)}$ is same to the original graph after applying the first $j$ update sequences, hence has at most $m$ vertices and distinct edges. For $i > 0$, assume the graph of $\mathcal{G}$ in $\mathcal{ODS}_j^{(i-1)}$ has at most $mn_j^{i-1}$ distinct edges, and the graph of $\mathcal{G}$ in $\mathcal{ODS}_j^{(i)}$ has at most $mn_j^i$ distinct edges. If $\mathcal{ODS}_j^{(i)}$ is obtained by running CUT-PARTITION-PREPROCESSING on $\mathcal{ODS}_j^{(i-1)}$, then the graph of $\mathcal{G}$ in $\mathcal{ODS}_j^{(i)}$ has at most $mn_j^i$ distinct edges by Lemma 7.2. If $\mathcal{ODS}_j^{(i)}$ is obtained by running CUT-PARTITION-UPDATE on $\mathcal{ODS}_j^{(i-1)}$, the graph of $\mathcal{G}$ in $\mathcal{ODS}_j^{(i)}$ has at most

$$mn_j^{i-1} + m\phi^{i+1} < mn_j^i$$

distinct edges. By induction, $\mathcal{ODS}_j$ is a $(\phi_j, \eta_j, \gamma)$ multi-level $c_j$-edge connectivity sparsifier with respect to parameters $\phi_j, \gamma, t, \ell_j, j, q_j, 1, \ldots, t_j, c_j, q_j, c_j$ for every $0 \leq j \leq \zeta$. By Definition 1.5, $\ell_j = O(\log_{1/\eta_j} m)$.

Now we bound the running time. By Claim 8.2, Lemma 7.3 and induction, if $\text{UpdateSeq}_j^{(i)}$ is computed by running UPDATE-MULTI-LEVEL-SPARSIFIER with respect to $\text{UpdateSeq}_j$, then there exists some constant $c' > 1$ such that

$$|\text{UpdateSeq}_j^{(i)}| = O(|\text{UpdateSeq}_j| (c' \cdot \Delta \cdot (10c_0)^{3\alpha_0})^i)$$

$$= |\text{UpdateSeq}_j| (10c_0)^{O(\log^{1/4} m)}$$

$$= O(|\text{UpdateSeq}_j|/\phi_\zeta).$$

(4)

Let $\alpha_j$ be the integer such that for $i \leq \alpha_j$, $\mathcal{ODS}_j^{(i)}$ is obtained by CUT-PARTITION-UPDATE, and for $j > \alpha_j$, $\mathcal{ODS}_j^{(i)}$ is obtained by CUT-PARTITION-PREPROCESSING.

By Lemma 7.3 and Equation 4 for $i \leq \alpha_j$, the running time to obtain $\mathcal{ODS}_j^{(i)}$ is

$$O(|\text{UpdateSeq}_j^{(i)}| \Delta (20c_j q_j, c_j)^{c_j^2 + 5c_j + 3} \text{poly}(c_j) \text{polylog}(m)) + \tilde{O}(|\text{UpdateSeq}_j^{(i)}| \Delta \log m / \phi_\zeta^4)$$

$$= \tilde{O}(|\text{UpdateSeq}_j^{(i)}| (q_j, c_j, \phi_\zeta)^{c_j^2 + 5c_j + 3})$$

$$= \tilde{O}(|\text{UpdateSeq}_j^{(i)}| (q_j, c_j, \phi_\zeta)^{c_j^2 + 5c_j + 4}).$$

(5)

For $i > \alpha_j$, since $|\text{UpdateSeq}_j^{(\alpha_j+1)}| > m\phi^{\alpha_j+2}$ according to the algorithm, by Equation 4 we have

$$m\phi^{\alpha_j+2} = O(|\text{UpdateSeq}_j^{(i)}|/\phi_\zeta),$$

and thus $m = O(|\text{UpdateSeq}_j|/\phi_\zeta^{\alpha_j+3})$. On the other hand, the graph of $\mathcal{ODS}_j^{(\alpha_j)}$ contains at most

$$mn_j^{\alpha_j} = O(|\text{UpdateSeq}_j| n_j^{\alpha_j} / \phi_\zeta^{\alpha_j + 3})$$

$$= O(|\text{UpdateSeq}_j| (2^{3 \log^{1/3} m \log^{2/3} \log m (10c_0)^{3\alpha_0}} O(\log^{1/4} m) / \phi_\zeta^3)$$

(6)

distinct edges.
Hence, by Lemma 7.2 and Equation 6, the running time to obtain ODS\(^{(i)}\) for all the \(i \geq \alpha_j + 1\) is
\[
\tilde{O}(mr_j^{\alpha_j}/\phi_j^2) + O(mr_j^{\alpha_j} 2^{\delta \log^{1/3} m \log^{2/3} m} (20c_j q_j c_j c_j^2 + 5c_j + 3) \text{poly}(c) \text{polylog}(m))
\]
\[
\tilde{O}(mr_j^{\alpha_j} (20c_j q_j c_j c_j^2 + 5c_j + 3))
\]
(7)

Combining Equation 5 and Equation 7, the amortized running time of the algorithm is \(m^{o(1)}\) by Claim 8.2.

9 Fully dynamic algorithms for \(c\)-edge connectivity and expander decomposition

In this section, we prove Theorem 1.1 and Corollary 1.2.

9.1 Fully dynamic algorithm for \(c\)-edge connectivity

Let \(G = (V, E)\) be the original dynamic simple graph with arbitrary degree. We use the degree reduction technique \cite{Har69} to transform \(G\) to a multigraph \(G = (V, E)\) such that every vertex has at most constant number of distinct neighbors as follows:

1. The vertex set \(V\) of \(G\) is
   \[
   V = \{v_{u,w} : (u, w) \in E\} \cup \{v_{u,u} : u \in V\}.
   \]
2. For any edge \((u, w) \in E\), add edge \((v_{u,w}, v_{w,u})\) to \(E\) with edge multiplicity 1.
3. For every vertex \(u\) of \(G\) with degree at least 1, let \(w_{u,0} = u\) and \(w_{u,1}, w_{u,2}, \ldots, w_{u,\text{deg}(u)}\) be the neighbors of \(u\) in \(G\). Add edge \((v_{u,w_{u,i}}, v_{u,w_{u,i+1}})\) to \(E\) with edge multiplicity \(c + 1\) for every \(0 \leq i < \text{deg}(u)\).

To maintain the correspondence between \(G\) and \(G\), for every \(u \in V\), we maintain the list of \(w_{u,0}, \ldots, w_{u,\text{deg}(u)}\).

It is easy to verify that for any \(u, w \in V\), the \(c\)-edge connectivity for \(u\) and \(w\) in \(G\) is the same as the \(c\)-edge connectivity of \(v_{u,u}\) and \(v_{w,w}\) in \(G\).

9.1.1 Update algorithm

To insert or delete an edge for \(G\), we can generate a multigraph update sequence for \(G\) of \(O(1)\) length to maintain the following invariant:

1. Vertex \(v_{u,w}\) is in graph \(G\) iff \(u = w\) or \((u, w)\) is an edge of graph \(G\).
2. For every edge \((u, w)\) of \(G\), there is an edge \((v_{u,w}, v_{w,u})\) with multiplicity 1 in \(G\).
3. For any vertex \(u\) of \(G\), the induced subgraph of \(G\) on all the \(v_{u,w}\) that are in \(G\) forms a path with edge multiplicity \(c + 1\) for each edge on the path.

By Lemma 1.6, Lemma 8.3 and Lemma 8.4, we have the following corollary, which also implies Lemma 1.8.
Corollary 9.1. Let $\overline{G}$ be a simple graph of $n$ vertices undergoing edge insertions and deletions. For any $c = (\log n)^{o(1)}$, there is a fully dynamic algorithm with $m^{1+o(1)}$ preprocessing time and $n^{o(1)}$ worst case update time to maintain a set of $n^{o(1)}$ multi-level sparsifier data structures for the corresponding multigraph such that after obtaining each update of $\overline{G}$, the update algorithm outputs the access to one maintained multi-level sparsifier data structure that is a $(\phi_\zeta, \eta_\zeta, \gamma)$ multi-level $c_\zeta$-edge connectivity sparsifier data structure for the multigraph that corresponds to the up-to-date $\overline{G}$, where $c_\zeta, \phi_\zeta, \eta_\zeta, \gamma$ are defined as Figure 5.

9.1.2 Query algorithm

We use the following query algorithm to answer $c$-edge connectivity queries.

**Algorithm 18: Fully-Dynamic-Query($\text{MDS}, c, u, w$)**

| Input: $\text{MDS} = \{\text{ODS}^{(i)}\}_{i=0}^\ell$: multi-level sparsifier data structure such that the graph of $\text{DS}$ in $\text{ODS}^{(0)}$ is the multigraph $G$ corresponding to the up-to-date $\overline{G}$
| $c$: a positive integer
| $u, v$: two vertices in $\overline{G}$

| Output: true or false

1. let $\text{UpdateSeq}^{(0)}$ be the following multigraph update sequence

   $$(\text{insert}(v'), \text{insert}(v''), \text{insert}(v_{u,u}, v', c+1), \text{insert}(v_{w,w}, v'', c+1)),$$

   where $v', v''$ are two new vertices of $G$

2. for $i = 0$ to $\ell$ do

   3. run

   $\text{Cut-Partition-Update}(\text{ODS}^{(i)}, \text{UpdateSeq}^{(i)}, \phi_\zeta, c_{\zeta+1}, t, \gamma, t_{\zeta,1}, q_{\zeta,1}, \ldots, t_{\zeta,c_{\zeta}}, q_{\zeta,c_{\zeta}})$

   and denote the resulted sequence by $\text{UpdateSeq}^{(i+1)}$, where the parameters are defined as Figure 5 with respect to $c$

4. use $\text{UpdateSeq}^{(\ell+1)}$ to construct a multigraph $H$ containing $v_{u,u}$ and $v_{w,w}$

5. run standard offline $c$-edge connectivity algorithm for $v_{u,u}$ and $v_{w,w}$ on graph $H$, and return the result (reverse the change made on $\text{MDS}$ at line 3 before return)

**Lemma 9.2.** Let $c$ be a positive integer such that $c = (\log n)^{o(1)}$. Given access to $(\phi_\zeta, \eta_\zeta, \gamma)$ multi-level $c_\zeta$-edge connectivity data structure $\text{MDS}$ for multigraph $G$ corresponding to $\overline{G}$, where $\phi_\zeta, \eta_\zeta, c_\zeta, \gamma$ are defined as Figure 5 for a query query($u, w$) for any two vertices $u, w \in \overline{G}$, algorithm Fully-Dynamic-Query($\text{MDS}, c, u, w$) returns true if and only if $u$ and $w$ are $c$-edge connected with running time $n^{o(1)}$.

**Proof.** Since applying $\text{UpdateSeq}^{(0)}$ to the graph of $\text{DS}$ in $\text{ODS}^{(0)}$ does not change the $c$-edge connectivity between $v_{u,u}$ and $v_{w,w}$, the $c$-edge connectivity between $v_{u,u}$ and $v_{w,w}$ for the graph of $\text{DS}$ in $\text{ODS}^{(0)}$ is the same as the $c$-edge connectivity of $u$ and $w$ in $\overline{G}$.

By Lemma 7.3 there exists a constant $c'$ such that for $0 \leq i \leq \ell + 1$

$$|\text{UpdateSeq}^{(i)}| = |\text{UpdateSeq}|(c'\Delta(10c)^{3c})^i.$$ 

Since $\ell = O(\log_{1/\eta_\zeta} n)$, $H$ is a multigraph of $n^{o(1)}$ vertices and edges. For $\text{ODS}^{(i)}$ of $\text{MDS}$, let $G^{(i)}$ denote the graph of $\text{DS}$ in $\text{ODS}^{(i)}$, and $P^{(i)}$ denote the vertex partition induced by connected components of graph of $\text{DS}_0$ in $\text{ODS}^{(i)}$. By Lemma 7.3 for every $0 \leq i \leq \ell$, $v_{u,u}$ and $v_{w,w}$ are in $\text{End}(\partial_G^{(i)}(P^{(i)}))$. Hence $H$ contains $v_{u,u}$ and $v_{w,w}$, and the $c$-edge connectivity between $v_{u,u}$ and
$v_{w,w}$ in $H$ is same to the $c$-edge connectivity for $v_{u,u}$ and $v_{w,w}$ in the graph of $\mathcal{DS}$ in $\mathcal{DS}^{(0)}$. Hence, the algorithm outputs $c$-edge connectivity of $u$ and $w$ correctly.

The running time of the algorithm is obtained by Lemma 7.3 and Claim 8.2.

Theorem 1.1 follows by Corollary 9.1 and Lemma 9.2.

### 9.2 Fully dynamic algorithm for expander decomposition

By Lemma 9.3, we have the following corollary.

**Corollary 9.3.** Let $G$ be a simple graph of $n$ vertices undergoing edge insertions and deletions with at most $m$ edges and $\Delta$ maximum degree throughout the updates. For a $\phi \in (0, 1]$, there is an online-batch update algorithm with $m^{1+o(1)}$ preprocessing time and $\tilde{O}(\Delta \cdot n^{o(1)}/\phi^3)$ amortized update time with batch number $O(\log^{1/4} n)$ and sensitivity $m\phi/(\Delta \log n)$ to maintain a

$$\left(\frac{\phi}{2^O(\log^{2/3} n)}, \phi 2^{O(\log^{2/3} n)}\right)$$

expander decomposition of $G$.

Corollary 1.2 is obtained by Lemma 1.6 and Corollary 9.3.

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Lemma A.1. Let $C_1 = (V_1, V \setminus V_1)$ and $C_2 = (V_2, V \setminus V_2)$ be two cuts for a connected graph $G = (V, E)$ that are not parallel. If $C_1$ is an atomic cut, then the cut-set of $C_1$ intercepts $C_2$.

Proof. By the definition of parallel cuts, both $V_1$ and $V \setminus V_1$ have nontrivial intersections with both $V_2$ and $V \setminus V_2$. Since $C_1$ is an atomic cut, both $G[V_1]$ and $G[V \setminus V_1]$ are connected. On the other hand, $C_1$ is a cut, so there is an edge from the cut-set of $C_2$ in $G[V_1]$, and another edge from the cut-set of $C_2$ in $G[V \setminus V_1]$. Hence the cut-set of $C_1$ intercepts $C_2$. \hfill \Box

Lemma A.2. Let $G = (V, E)$ be a connected graph. Suppose $C_1 = (V_1, V \setminus V_1)$ and $C_2 = (V_2, V \setminus V_2)$ are two cuts on $G$ that are not parallel. Either $\partial_G(V_1)$ intercepts $C_2$ or $\partial_G(V_2)$ intercepts $C_1$.

Proof. If $\partial_G(V_1) \cap \partial_G(V_2) \neq \emptyset$, then we have $\partial_G(V_1)$ intercepts $C_2$, and $\partial_G(V_2)$ intercepts $C_1$ by Definition 2.4. In the rest of the proof, we assume $\partial_G(V_1) \cap \partial_G(V_2) = \emptyset$.

Suppose $\partial_G(V_2)$ does not intercept $C_1$. By Lemma A.1, $C_2$ is a non-atomic cut. On the other hand, $\partial_G(V_1)$ is in a connected component $V'$ of $G \setminus \partial_G(V_2)$, and we have $V' \subseteq V_2$ or $V' \subseteq (V \setminus V_2)$. Since $\text{End}(\partial_G(V_1)) \subseteq V'$, for every edge $(x, y) \in \partial_G(V') \subseteq \partial_G(V_2)$, $\{x, y\}$ is a subset of $V_1$ or is a subset of $V \setminus V_1$. By Definition 2.5 $V_1 \not\subseteq V'$ and $(V \setminus V_1) \not\subseteq V'$. Hence, there are edges $(x, y), (x', y') \in \partial_G(V')$ such that $x, y \in V_1$ and $x', y' \in V \setminus V_1$. Therefore, $\partial_G(V_1)$ intercepts $C_2$. \hfill \Box

Lemma A.3. Let $G = (V, E)$ be a connected graph and $T$ be a set of terminals. Let $C_1 = (V_1, V \setminus V_1)$ be a cut that partitions $T$ into $T_1$ and $T \setminus T_1$, and $C_2 = (V_2, V \setminus V_2)$ be a cut that partitions $T$ into $T_2$ and $T \setminus T_2$. If $T_1 \cap T_1' = \emptyset$, then let $V_1' = V_1 \setminus (V_1 \cap V_2)$, and $V_2' = V_2 \setminus (V_1 \cap V_2)$. $(V_1', V \setminus V_1')$ is a cut partitions $T$ into $T_1$ and $T \setminus T_1$, and $(V_2', V \setminus V_2')$ is a cut partitions $T$ into $T_2$ and $T \setminus T_2$, such that $\partial_G(V_1'), \partial_G(V_2') \subseteq \partial_G(V_1) \cup \partial_G(V_2)$, and one of the following conditions hold:

1. the size of cut $(V_1', V \setminus V_1')$ is smaller than the size of cut $(V_1, V \setminus V_1)$.
2. the size of cut $(V_2', V \setminus V_2')$ is smaller than the size of cut $(V_2, V \setminus V_2)$.
3. the size of cut $(V_1', V \setminus V_1')$ is the same to that of $(V_1, V \setminus V_1)$, and the size of cut $(V_2', V \setminus V_2')$ is the same to that of $(V_2, V \setminus V_2)$.

Proof. Since $T_1 \cap T_2 = \emptyset$, we have $T \cap (V_1 \cap V_2) = \emptyset$. \hfill (8)

If $V_1 \cap V_2 = \emptyset$, then we have $V_1' = V_1$, $V_2' = V_2$ and thus the third condition holds. In the rest of this proof, we consider the case of $V_1 \cap V_2 \neq \emptyset$.

By the definition of $V_1'$ and $V_2'$, for each $(x, y) \in \partial_G(V_1 \cap V_2)$, the following conditions hold:

(a) If $(x, y)$ is in both $\partial_G(V_1)$ and $\partial_G(V_2)$, then $(x, y)$ is in both $\partial_G(V_1')$ and $\partial_G(V_2')$. 

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(b) If \((x, y)\) is \(\partial_G(V_1) \setminus \partial_G(V_2)\), then \((x, y)\) is in \(\partial_G(V'_2)\) but not in \(\partial_G(V'')\).

c) If \((x, y)\) is \(\partial_G(V_2) \setminus \partial_G(V_1)\), then \((x, y)\) is in \(\partial_G(V''_1)\) but not in \(\partial_G(V''_2)\).

Hence, we have

\[|\partial_G(V'_1)| + |\partial_G(V'_2)| = |\partial_G(V_1)| + |\partial_G(V_2)|.\]

So, one of the three conditions holds.

\[\square\]

**Claim A.4.** Let \(G = (V, E)\) be a connected graph and \((V', V \setminus V')\) be a cut of graph \(G\) with size \(c\). Then \(G[V']\) contains at most \(c\) connected components.

**Proof.** Since the size of cut \((V', V \setminus V')\) is \(c\), \(\partial_G(V')\) contains at most \(c\) edges, and thus \(\text{End}(\partial_G(V')) \cap V' \leq c\). Any connected component of \(G[V']\) must contain one vertex in \(\text{End}(\partial_G(V')) \cap V'\), otherwise \(G\) is not connected. Hence, \(G[V']\) contains at most \(c\) connected components.

\[\square\]

**Lemma A.5.** Let \(G = (V, E)\) be a connected graph, and \(T \subseteq T\) be a set of vertices. Let \((V', V \setminus V')\) be a simple cut of graph \(G\), \(E'\) be a subset of \(\partial_G(V')\) such that \(E'\) induces a cut \((V', V \setminus V')\) of graph \(G\) separating \(T\) nontrivially such that \(V' \subseteq V^\dagger\). Let \(E''\) be another set of edges such that \(E''\) induces a cut that partitions \(T\) the same as the cut induced by \(E'\) does. Then there exists a subset of \((\partial_G(V') \setminus E') \cup E''\) induces a cut \((Q, V \setminus Q)\) that partitions \(T\) the same as \((V', V \setminus V')\) does satisfying \(Q \subseteq V' \cup (V \setminus V')\).

**Proof.** Since \(G[V']\) is connected, and \((V' \setminus V')\) induces a cut, \(V \setminus V^\dagger\) forms the union of several connected components of \(G[V \setminus V']\). Let \(L = V' \cup (V \setminus V^\dagger)\). \(G[L]\) is connected, and \(\partial_G(L)\) is \(\partial_G(V') \setminus E'\).

Let \((V^\ddagger, V \setminus V^\ddagger)\) be the cut induced by \(E''\) satisfying \(V^\dagger \cap T = V^\ddagger \cap T\). Since a cut-set restricted on an induced subgraph induces a cut on the induced subgraph, \((L \cap V^\ddagger, L \cap (V \setminus V^\ddagger))\) is a cut of \(G[L]\) with cut-set in \(E''\). Since

\[L \cap T = (V' \cup (V \setminus V^\dagger)) \cap T = (V' \cap T) \cup ((V \setminus V^\dagger) \cap T) = (V' \cap T) \cup ((V \setminus V^\dagger) \cap T),\]

by the definition of \(E''\), \((L \cap V^\ddagger) \cap T = V' \cap T\). Hence \((L \cap V^\ddagger, V \setminus (L \cap V^\ddagger))\) is a cut of \(G\) with cut-set in \((\partial_G(V') \setminus E') \cup E''\) that partitions \(T\) the same as \((V', V \setminus V')\).

\[\square\]

**Lemma A.6.** Let \(G = (V, E)\) be a connected graph, and \(T \subseteq V\) be a set of vertices. Let \((V', V \setminus V)\) be a cut of graph \(G\), and \(V^*\) be a set of vertices such that \(G[V^*]\) is a connected graph with \(E' = \partial_G(V')|_{G[V^*]} \neq \emptyset\). Let \(E''\) be another set of edges on \(G[V^*]\) such that \(E''\) induces a cut that partitions \((T \cup \text{End}(\partial_G(V^*)) \cap V^*)\) the same way that the cut induced by \(E'\) on \(G[V^*]\) does. Then \((\partial_G(V') \setminus E') \cup E''\) induces a cut that partitions \(T\) the same as \((V', V \setminus V')\) does.

**Proof.** We know \(E'\) induces cut \((V' \cap V^*, (V \setminus V') \cap V^*)\) on \(G[V^*]\). Suppose \(E''\) induces cut \((V^\dagger, V^* \setminus V^\ddagger)\) on \(G[V^*]\).

\(\partial_G(V') \setminus E'\) induces a cut on \(G[V \setminus V^*]\) because a cut-set restricted on any induced subgraph induces a cut on that induced subgraph that partitions \(\text{End}(\partial_G(V[V^*])) \cap (V \setminus V^*)\) into \(\text{End}(\partial_G(V[V^*])) \cap V^*\) and \(\text{End}(\partial_G(V[V^*])) \cap (V \setminus V^*)\cap V'\). Since both \(E''\) and \(E'\) induce cuts on \(G[V^*]\) that partition \(\text{End}(\partial_G(V[V^*])) \cap V^*\) into \(\text{End}(\partial_G(V[V^*])) \cap V^* \cap V'\) and \(\text{End}(\partial_G(V[V^*])) \cap V^* \cap V'\) \(\cap V'\cap (V \setminus V')\), \(\partial_G(V') \setminus E' \cup E''\) induces a cut \(((V' \setminus V^*) \cup V^\dagger, (V \setminus V') \setminus V^*) \cup (V^* \setminus V^\ddagger)\) on \(G\).

Then it suffices show that the cut induced by \(\partial_G(V') \setminus E' \cup E''\) partitions \(T\) the same way that \((V', V \setminus V')\) does.

For any \(t \in T\), one of the following holds:

...
\begin{itemize}
  \item \( t \in V' \setminus V^* \).
  \item \( t \in (V \setminus V') \setminus V^* \).
  \item \( t \in V' \cap V^* \). Then \( t \in V^\dagger \).
  \item \( t \in (V \setminus V') \cap V^* \). Then \( t \in V^* \setminus V^\dagger \).
\end{itemize}

Therefore, \((V' \setminus V^* \cup V^\dagger, (V \setminus V') \setminus V^* \cup (V^* \setminus V^\dagger))\) and \((V', V \setminus V')\) partition \( T \) the same way. \( \square \)