Fuzzy topology and fuzzy geometry of the topological concepts

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Abstract. The typical approach to analyze the images in terms of recognition and further interpretation is to obtain an extension of the conventional geometry to the fuzzy case. The fuzzy geometry, which consists of the topological and geometrical structures such as connectedness, adjacency, principle of surrounded-ness (areas seen as surrounded by other areas are perceived as figures), convexity, area, perimeter, height, width, adjacency and etc. are essential to reflect the geometrical ambiguity of an image processing. Moreover, the application of the fuzzy geometry properties is important in soft computing decisions to further analyze and interpret a fuzzy image. The properties of fuzzy geometry based on the concepts of the length, breadth, height and methods of measuring a distance between fuzzy numbers are introduced here due to the numerous of applications in engineering problems and other fields.

1. Fuzzy topology

The geometrical concept such as length, width, distance, area, perimeter, circumference, compactness, adjacent, similarity and others are the categories of the topological fuzzy geometry. These categories are often used to reflect the geometrical spatial uncertainty with the ambiguity of the image apportionment and investigation.

The typical approach to analyze the images in terms of recognition and further interpretation is to obtain an extension of the conventional geometry to the fuzzy case. This extension is called the fuzzy geometry, which consists of the topological concepts such as connectedness, adjacency, principle of surrounded-ness (areas seen as surrounded by other areas are perceived as figures), convexity, area, perimeter, height, width, adjacency and others. These categories of the fuzzy geometry are useful to reflect the geometrical ambiguity by an image processing. Moreover, the application of the fuzzy geometry properties is important in soft computing decisions based on the analyzing a fuzzy image.

If we are performing operation with fuzzy sets, then we assume that there is existed an universal set \( X \subseteq U \) as the fuzzy topological building block for fuzzy sets \( F \). There are applied following initial conditions to imply the fuzzy topological formation here:

The fuzzy set universe (I): \( F \subseteq X, F \cap X \neq \emptyset, x \in X \);

The fuzzy set union (II): \( F_{x_1} \cup F_{x_2} = (x \in F_{x_1}) \lor (x \in F_{x_2}), \forall x \in X \), for the sets \( F_1, F_2, \ldots, F_n \) for \( n > 2 \) the fuzzy set union is: \( F_1 \cup F_2 \cup \ldots \cup F_n = \cup_{j=1}^{n} F_j = \{x: x \in F_j, 1 \leq j \leq n\} \);

The fuzzy set intersection (III): \( F_{x_1} \cap F_{x_2} = (x \in F_{x_1}) \land (x \in F_{x_2}), \forall x \in X \), for the intersection of the \( n - \) sets: \( F_1 \cap F_2 \cap \ldots \cap F_n = \cap_{j=1}^{n} F_j = \{x: x \in F_j, 1 \leq j \leq n\} \);

The fuzzy set complement (IV): \( \overline{F_x} = (x \in X: x \notin F_x), \overline{F_x} \cap F = \emptyset \);

The fuzzy set difference (V): \( F_{x_1} \setminus F_{x_2} = x \in X: (x \in F_{x_1}) \land (x \notin F_{x_2}) \).
Symmetric difference of the fuzzy sets \( F_1, F_2 \) is denoted \( F_1 \Delta F_2 \) to be the set of the elements of the fuzzy sets, which are members of exactly one of \( F_1 \) or \( F_2 \).

The Cartesian product of the fuzzy sets \((x_1,x_2): (x_1 \in F_{x_1}) \land (x_2 \in F_{x_2})\) for the Cartesian product of the \( n \)-fuzzy sets: \( F_1 \times F_2 \times \ldots \times F_n = (x_j \in F_j, 1 \leq j \leq n) \).

The Fuzzy Power set \( \Pi(F_x) = \{F_p : F_p \subseteq F_x\} \), or, alternatively, \( \Pi(F_x) = 2^{F_x} \) is the set of all subsets of \( F_x \).

**Definition 1.** If \( F \subseteq X \), then \( \phi_F(x) \) is the membership grade function of element of \( x \in F \subseteq X : \phi_F(x): X \rightarrow [0,1] \).

**Definition 2.** Fuzzy set \( F \subseteq X \) is based on the fuzzy topological space, and fuzzy set is defined by the ordered pair: \( F = (x, \phi_F(x)|x \in X) \).

There is a discrete form of the fuzzy set given at the universe set of discourse such as: \( F = \sum_x \frac{\phi_F(x)}{x} \).

The elements of the fuzzy set are called members of the open subsets of \( F: x \in F_x \subseteq F \).

**Fuzzy set identities**

The established rules for the identities are commonplace in the set theory. Similarly to the classical set theory common based identities are entrenched in the fuzzy set theory, too.

Let us introduce the theorem, which describes the commonplace set of the identities’ properties for the fuzzy sets.

**Theorem 1.** Let \( F_1, F_2, F_3 \subseteq X \subseteq U \), where \( X \) — the subset of the universal set of \( U \). The following properties hold for the fuzzy set as the subsets of the universe:

1. Commutative Property: for the union: \( F_1 \cup F_2 = F_2 \cup F_1 \), for the intersection: \( F_1 \cap F_2 = F_2 \cap F_1 \);
2. Associative property: for the union: \( F_1 \cup (F_2 \cup F_3) = (F_1 \cup F_2) \cup F_3 \), for the intersection: \( F_1 \cap (F_2 \cap F_3) = (F_1 \cap F_2) \cap F_3 \);
3. Distributive property: for the union: \( F_1 \cup (F_2 \cap F_3) = (F_1 \cup F_2) \cap (F_1 \cup F_3) \), for the intersection: \( F_1 \cap (F_2 \cup F_3) = (F_1 \cap F_2) \cup (F_1 \cap F_3) \);
4. De Morgan’s property for the fuzzy sets:
   (a) \( \overline{F_1 \cup F_2} = \overline{F_1} \cup \overline{F_2} \) and (b) \( \overline{F_1 \cap F_2} = \overline{F_1} \cap \overline{F_2} \).

**Proof.** Let us prove De Morgan’ property part (b).

Let us use the membership table, which is called the Truth Table.

**Table 1.** Truth Table.

| \( x \) | \( F_1 \) | \( F_2 \) | \( F_1 \cup F_2 \) | \( F_1 \cap F_2 \) | \( F_1 \cup F_2 \) | \( F_1 \cap F_2 \) |
|--------|--------|--------|--------|--------|--------|--------|
| \( T \rightarrow True \) | T | T | T | F | F | F |
| \( T \) | F | F | F | T | F | T |
| \( F \rightarrow False \) | T | T | F | T | F | F |
| \( F \) | F | F | T | T | T | T |

We have obtained that the columns headed by \( x \in F_1 \cup F_2 \) and \( x \in F_1 \cap F_2 \) are identical, then the proof is accomplished that the two fuzzy sets are identical or equal.

**Note 1:** The part (a) of the de Morgan property can be proven in similar way by using the Truth Table.

**Note 2:** The standard fuzzy operators such as union and intersection do not satisfy the following properties of the set from the classical set theory: \( F \cup \overline{F} = X \subseteq U \) and \( F \cap \overline{F} = \emptyset \) based on the threshold condition of the characteristic function’s condition: \( \phi_F(0) = 1 \) and \( \phi_F(1) = 1 \) for \( \forall x_1, x_2 \in [0,1] \), if \( x_1 \leq x_2 \), then \( \phi_F(x_1) \geq \phi_F(x_2) \), \( \phi_F(x) \) — continuous function, and \( \phi_F(x) = \begin{cases} 1, & \text{IF } x \leq k, \ k \in [0,1], \\ 0, & \text{IF } x < k. \end{cases} \)

**Note 3:** The fuzzy sets with the characteristic function that qualify for fuzzy intersection and fuzzy union are, indeed, qualified to be included in numerous range of the functions, based on \( t \) — norms and \( s \) — co-norms.
Based on the topology of a fuzzy space, we can define a fuzzy-continuous function.

**Definition 3.** The fuzzy spaces \((X, \phi_X)\) and \((Y, \phi_Y)\) define the function from \(X\) into \(Y\) is called \(F\) – fuzzy continuous function \(\phi(X)\) if \(\phi^{-1}(X)\) from range of \(\phi(X)\) is a function: \(\phi(X) = Y\) if \(\phi^{-1}(Y) = X\).

**The Fuzzy Continuous function theorem.** Let \(\phi: X \rightarrow Y\) is \(F\) – fuzzy function based on the fuzzy spaces \((X, \phi_X)\) \(\rightarrow\) \((Y, \phi_Y)\). Then fuzzy function is \(F\) – fuzzy continuous, if \(\phi(X)\) is one-to-one function.

**Proof.** Let \(\phi^{-1}(X)\) be a function. For each element of the domain \(b \in \text{Dom}(\phi^{-1})\) there is a unique range \(a \in \text{Range}(\phi^{-1})\), such that \((b, a) \in \phi\).

Equivalently, each element of \(b \in \text{Range}(\phi)\) corresponds to unique \(a \in \text{Dom}(\phi)\), such that \((a, b) \in \phi\). Thereafter, function \(\phi(X)\) is \(F\) – fuzzy continuous since it is injective or one-to-one, which satisfy the definition of the fuzzy continuity.

2. **Fuzzy case of the conventional geometry**

The topological space consists of structures, such as length, width, distance, area, perimeter, circumference, compactness, adjacent, similarity and others. These structures are the pivotal categories of the topological fuzzy geometry. These categories are often used to reflect the geometrical spatial uncertainty with the ambiguity of the image apportionment and investigation. There are multiple publication in the field of the fuzzy geometry with generalized standard geometric properties extended to the fuzzy case in [1, 2, 3, 4, 5] there.

The concept of the fuzzy geometry with the applications and image processing was presented in [6]. There is series of research works devoted to the generalized geometric properties of the fuzzy sets in [7, 8, 9, 10, 11, 12, 13] there.

The typical approach to analyze the images in terms of recognition and their further interpretation is to obtain an extension of the conventional geometry to the fuzzy case. This extension according to [10] is called the fuzzy geometry, which consists of the topological concepts such as connectedness, adjacency, principle of surrounded-ness (areas seen as surrounded by other areas are perceived as figures), convexity, area, perimeter, height, width, adjacency and others. These categories of the fuzzy geometry are useful to reflect the geometrical ambiguity of an image processing. Moreover, the application of the fuzzy geometry properties is substantial in soft computing decisions to further analyze and interpret a fuzzy image.

The existing and new geometric properties of the fuzzy sets were introduced in [6].

**Definition 4.** The area of the fuzzy set is the area, which consists of the subset of \(F \subseteq X \subseteq \mathbb{R}^2\) is given by \(A(\phi_F) = \iint \phi_F(x,y)dx\,dy\) , where the integration occurs over a fuzzy region of \(F\): \(F = \{(x,y) | \phi_F(x,y) > 0\}\).

The area of the region represented by the piecewise function is defined by \(A(\phi_F) = \sum_x \phi_F(x,y)\).

**Definition 5.** The perimeter of the fuzzy set \(F\), where the membership function is piecewise constant is given by \(P(\phi_F) = \sum_{k,l,m} |\phi_F(k) - \phi_F(l)| \ast |F(k,l,m)|\).

If \(\phi_F(k), \phi_F(l)\) are two adjacent membership functions, then the perimeter is given by \(P(\phi_F) = \sum_{k,l} |\phi_F(k) - \phi_F(l)|\).

There is example given with the membership function represented by the arcs \((k,l)\) of the regions with corresponding values of \(\phi_F(k), \phi_F(l)\) met at the arcs:

| \(x\) | \(y\) | \(P(\phi_F)\) |
|------|------|-------------|
| 0.1  | 0.2  | 0.6         |
| 0.1  | 0.9  | 0.5         |
| 0.7  | 0.2  | 0.9         |

The area \(A(F) = 4.2\) and the perimeter \(P(F) = 5.4\).

**Definition 6.** The compactness \(C\) of a fuzzy set with the area and perimeter as \(A(\phi_F), P(\phi_F)\) is given by \(C(\phi_F) = \frac{A(\phi_F)}{P(\phi_F)^2}\).
We may interpret the compactness by means of the ratio of the maximum area over the encircled perimeter of the occupied object.

In previous given example the compactness of the fuzzy subsets is: $C(\Phi_F) = 0.144$.

**Definition 7.** The height of the fuzzy sets is defined by $h(\Phi_F) = \int \text{MAX}_x (\Phi_F(x,y)) \, dy$. The width of the fuzzy set is defined by $w(\Phi_F) = \int \text{MAX}_y (\Phi_F(x,y)) \, dx$, where the integration is taken over the region when $\Phi_F(x,y) > 0$.

In case of the piecewise membership function there are the height and width represented by $h(\Phi_F) = \sum_y \text{MAX}_x (\Phi_F(x,y))$ and $w(\Phi_F) = \sum_x \text{MAX}_y (\Phi_F(x,y))$.

In other words, the height and width are interpreted as the summation of the maximum values of membership function for each row and column.

Based in example given the height is $h(\Phi_F) = 2.4$, $w(\Phi_F) = 2.5$.

**Definition 8.** Let $S,T$ be two piecewise constant fuzzy sets of universe $X$. If we partition universe into a finite number of bounded sub-regions, when the membership functions along the arcs $(k,l,m)$ are $\Phi_S(k),\Phi_T(l)$. Then the adjacency between two fuzzy constants is given by $a(\Phi_S,\Phi_T) = \sum_{k,l,m} \Phi_S(k)\Phi_T(l)f(k,l,m)$.

In other words, in the case of the example provided, the adjacency means the product of the membership function with adjacent neighboring value. In our examples given, the adjacency for the center is $a(\Phi_S,\Phi_T) = 0.45$.

There are introduced new geometrical categories of fuzzy image sets such as length, breadth and index of area coverage here:

**Definition 9.** The length of the fuzzy set $S$ is given by $l(S) = \text{MAX}_x \{ \int \Phi_S(x,y) \, dy \}$. In case of the piecewise functions the length is given by $l(S) = \text{MAX}_x \{ \int \Phi_S(x,y) \, dy \}$.

**Definition 10.** The breadth of the fuzzy set is given by $b(S) = \text{MAX}_y \{ \int \Phi_S(x,y) \, dx \}$ and in case of the piecewise functions $b(S) = \text{MAX}_y \{ \int \Phi_S(x,y) \, dx \}$.

In other words, the length and breadth take the summation of the entries in column/row first and maximizes over other columns/rows. The summation is taken over the entries in column/row and then sums over different columns/rows in case of the height/width.

In case of the example given previously the length and breadth are $l(S) = 2.0$, $b(S) = 1.8$.

**Definition 11.** The index of the area coverage of the fuzzy set $S$ is given by $\text{IOAC}(S) = \frac{a(S)}{l(S)b(S)}$.

Based on in previously given example the index is the maximum area of the fuzzy set. The product of length and breadth is $2.0 \times 1.8 = 3.60$ when the real area is 4.2, so the $\text{IOAC}(S) = \frac{4.2}{3.6} = 1.166$.

There is the following inequalities hold for the fuzzy geometrical categories such as: $\frac{l(S)}{h(S)} \leq 1$ and $\frac{b(S)}{w(S)} \leq 1$. If the ratios presented here are equal to 1, then the geometrical image is vertical or horizontally aligned.

**Definition 12.** The center of the gravity of an image of the geometric object is the point of intersection of the major and minor axis.

Major axis is referred to the length of an object, when we rotate the axis through angles between $0^0$ and $90^0$. The corresponding axis which matches to the maximum of the length is called the major axis.

Minor axis is perpendicular to the major axis and matches to the maximum of the breadth.

In our presented example the length is: $l(S) = 2.0$ and breadth is: $b(S) = 1.8$.

If we rotate the object by $45^0$, then we obtain the length and breadth as it is: $l(S)^{45^0} = 0.6 + 0.7 + 0.6 = 1.9$.

Hence, here the object is modified by rotation of an image by $45^0$. The minor axis, which is perpendicular to the major axis is rotated by $45^0$, too.

**Definition 13.** The density of the fuzzy set of $S$ with $U$ support planes is defined by $d(S) = \int_0^S ds = \frac{a(S)}{U}$.
In case of a piecewise function image density is defined by 
\[ d(S) = \sum_{j=1}^{U} \frac{S(j)}{U}. \]

The maximum value of the density is 1. Usually the density is used to find the Center of Gravity of an image of the geometrical object.

3. Fuzzy distance between fuzzy sets

The diverse fields such as remote sensing, data mining, pattern recognition, multivariate data analysis demand to implement methods to precisely measure a distance between fuzzy sets and their corresponding membership functions.

The geometrical and topological concept of the fuzzy number is vital on measuring a distance between fuzzy numbers. The distance concept and definitions of the distance based on metric space are of crucial importance due to the numerous applications in diversified fields of applications of fuzzy sets and soft computing [14, 15, 16, 17, 18, 19].

However, the category of a fuzzy number, especially if a number represents the ambiguously given linguistic form is not well known. Moreover, performing operations with fuzzy numbers to find the distance logically means that the distance between two uncertain points is uncertain, too. Having such retrospective outlook into fuzzy numbers, we are intrinsically predisposed to define mathematically correct distance between two fuzzy numbers. There was introduced the distance between fuzzy sets in [19] by pioneering the concept of the fuzzy distance between two uncertain fuzzy points.

The distance concept of the fuzzy sets are applicable to manifold of the scientific and applied fields. There are many cases of the application of the distance concept to measure the distance in decision analysis, artificial intelligence, pattern recognition, classification problems and others in [20, 21, 22, 23] there.

The distance concept of the fuzzy sets is the application of the classical distance between sets in metric space by extrapolating a metric concept to fuzzy metric space.

The geometrical concepts such as a point, interval, distance in metric space are the building blocks of the application of the conventional geometry to the fuzzy sets. There are given following definitions of the support plane, kernel point, fuzzy numbers here, which are utilized in fuzzy metric space to find the distance between fuzzy membership functions in [24, 25] there.

**Definition 14.** The support of the fuzzy set \( F = (X, \phi_F(x)) \) is given by \( \text{supp} F = \{x \in F \subseteq X, \phi_F(x) > 0\} \).

**Definition 15.** The kernel of a fuzzy set \( F = (X, \phi_F(x)) \) is the fuzzy point, which is given by \( \text{ker} F = \{x_{k_0} \in F \subseteq X: \phi_F(x_{k_0}) = 1\} \).

The kernel \( x_{k_0} \in \mathbb{R} \) is a point, where the membership function is decreasing monotonically in all directions is also called the kernel of the monotonicity.

The Figure 1 presents the geometrical illustration of the fuzzy points with fuzzy pyramidal shape and paraboloidal shape of the membership functions.
It is important to define the geometrical concept of the point in terms of the fuzzy number. Based on [14, 19, 24] the fuzzy number is defined as:

**Definition 16.** A fuzzy number is a fuzzy characteristic function \( \phi_F(x) : F \subseteq X \subseteq \mathbb{R} \rightarrow [0,1] \), with the following criteria applied:

1. \( \phi_F(x) \) – semi-continuous;
2. \( \text{supp } \phi_F(x) \) – closed and bounded interval;
3. if \( \text{supp } \phi_F(x) = [k,l] \), then there \( \exists m, n \) and \( k \leq m \leq n \leq l \);

where \( \phi_F(x) \) is increasing on \( [k,m] \), decreasing on \( [n,l] \) and \( \phi_F(x) = 1 \) at \( [m,n] \).

**Definition 17.** A fuzzy set \( F = (X, \phi_F) \) is normal or normalized if \( \text{ker } F \neq 0 \).

**Definition 18.** The \( \alpha \) – level cut of a fuzzy set \( F_\alpha = (X, \phi_{F_\alpha}) \), \( F_\alpha \subseteq X \) such that the characteristic function cut by level cuts is represented by a function: \( \phi_{F_\alpha}(x) = \left( x \in F_\alpha \subseteq X | \phi_{F_\alpha}(x) \geq \alpha \right), \alpha \in [0,1] \).

**Definition 19.** A fuzzy set, where \( X \) – represents a linear space is called convex if all its \( \alpha \) – level cuts are convex.

**Definition 20.** The height of the fuzzy set \( F = (X, \phi_F) \) is a function of \( h(F) = \sup_x \phi_F(x) \).

**Definition 21.** The cartesian product of the fuzzy sets \( F_1, F_2, ..., F_n \) represented by \( F_1 \otimes F_2 \otimes ... \otimes F_n \).

**Definition 22.** A fuzzy set \( F = (X, \phi_F) \), where \( X \) – represents a linear space is a fuzzy point if fuzzy set is convex.

**Definition 23.** A point \( F = (X, \phi_F) \) is called a fuzzy point, where a fuzzy number is non-negative number and \( \phi_F(x) = 0 \) for \( \forall x < 0 \).

Since we have defined the important foundational structures of the fuzzy sets such as a fuzzy number, a fuzzy point, fuzzy \( \alpha \) level convex cuts, we may reintroduce the Zadeh’s extension principle based on these vital categories, too.
There is presented lemma in [26, 27], which is proved in the article here:

**Lemma 1.** Let a function is given: \( f : X \to Y \). Next let the fuzzy quantity \( A \) about universe \( X \) is given. The fuzzy image over the universe of \( Y \) is the following: \( f(A) = \phi Y \left( f^{-1}(y) \right) \).

If a function is invertible or one-to-one, then there is given one-to-one mapping: Case 1: \( f_B(y) = \phi_A \left( f^{-1}(y) \right) \).

If a function is not bijective and surjective, then by the Extension Principle of Zadeh there is the following statement holds true: Case 2: \( f_B(y) = \sup_{x \in f^{-1}(y)} \phi_A(x) \).

**Proof.** Case 1: If a function is both bijective and surjective, then \( f_B(y) = \phi_A \left( f^{-1}(y) \right) \).

Case 2: If a function is not one-to-one, then that means the height functions \( f_B(y) \) is not surjective – no image of the fuzzy quantity of \( A \) exists. Then the following statement holds based on the definition of the height function of the fuzzy argument: \( h(B) = \sup_{y \in f^{-1}(B)} \phi_A(x) = h(A) \).

**Definition 24.** The fuzzy absolute number \( |F| = \left( X, \phi_{|F|} \right) \subseteq \mathbb{R}^+ \) corresponds to the membership grade function by

\[
\phi_{|F|}(x) = \begin{cases} 
\max(\phi_F(x), \phi_F(-x)), & x \geq 0, \\
0, & x < 0.
\end{cases}
\]

**Definition 25.** Let \( D = \{a, b\} \neq \emptyset \), where \( a, b, c \) are the elements of the set \( a, b, c \in D \). If there is a function exists: \( \delta(a, b) : D \otimes D \to \mathbb{R}^+ \), which is satisfied to the conditions of the metric space by

\[
\delta(a, b) \geq 0, \delta(a, b) = 0 \text{ if } a = b,
\]

\[
\delta(a, b) = \delta(b, a),
\]

\[
\delta(a, b) \leq \delta(a, c) + \delta(c, b), \forall c \in D,
\]

then \( \delta \) is a metric function, which is given over metric space \( (D, \delta) \).

Based on the concept of the metric space, we may generalize it to apply on the definition of the fuzzy distance between fuzzy \( \alpha \) – level cut sets.

**Definition 26.** Let \( (X, \delta) \) define a metric space and the distance between two non-empty sets \( S, T \neq \emptyset \) over metric space is given by non-negative number \( \delta(S, T) = \inf_{x \in S, y \in T} \delta(x, y) \).

**Definition 27.** The fuzzy distance between two non-empty fuzzy sets \( F_1, F_2 \neq \emptyset \) over the metric space \( (X, \delta) \) is defined as the fuzzy number \( \delta_F(F_1, F_2) = \left( X, \phi_{\delta_F} \right) \subseteq \mathbb{R}^+ \), where the corresponding membership function is given by \( \phi_{\delta_F}(F_1, F_2) = \inf_{x \in F_1, y \in F_2} \delta(x, y) \).
Definition 28. $\alpha$ - level cut distance between two fuzzy sets $F_1^\alpha, F_2^\alpha \in [0,1]$ over the metric space is defined by the fuzzy number $\delta_F(F_1^\alpha, F_2^\alpha) = (X, \Phi_{\delta_F})$, $X \in \mathbb{R}^+$ and the corresponding membership function is given by $\Phi_{\delta_F}(y) = \begin{cases} \sup \text{MIN}(\Phi_{F_1^\alpha}, \Phi_{F_2^\alpha}), \delta_F(F_1^\alpha, F_2^\alpha) \leq \alpha, \\ 0, \text{otherwise.} \end{cases}$

There are other definitions of the distances between two crisp fuzzy sets presented in [28] such as:

Definition 29. The distance between two fuzzy sets over the metric space $(X, \delta)$ is the non-negative real number given by $\int_0^1 \rho(F_1, F_2) dx$, $\delta(F, 0) = 0$ for $\forall F \subseteq X$.

There is a theorem presented in [28]. This theorem will let define the distance between two crisp fuzzy sets over the metric space [25].

Theorem 2. Let $(X, \delta)$ be a metric space. Let $S, T \neq \emptyset$ be two non-empty (crisp) sets given on $X$, $S \subseteq X, T \subseteq X$. Then, there is the following conditions hold for the distance between two sets on the metric space:

(1) $\delta(S, T) = \inf_{x \in S, y \in T} \delta(x, y) \in \mathbb{R}^+$ - is non-negative real number and distance $d(S, T) = \delta(x, y)$;
(2) $\Phi_{\delta_F(S, T)}(y) = \begin{cases} 1, y = \delta(S, T), \\ 0, y \neq \delta(S, T). \end{cases}$

Proof. The distance between two crisp sets on the metric space is the following: $d(S, T) = \int_0^1 \delta(S_T, T_T) d\tau = (\tau \delta(S, T)|_0^1 = \delta(S, T)$, and $\text{MIN}(h(S), h(T)) = 1 \Rightarrow \lim_{\tau \rightarrow \text{MIN}(h(S), h(T))} \delta(S_T, T_T) = \delta(S, T)$, $h(S), h(T) - $ are the heights of the of the fuzzy sets of $S, T$.

Next theorem proves that the distance between two crisp fuzzy sets is a convex set over the metric space [25].

Theorem 3. Let $(X, \delta)$ be a metric space. If there is given two non-empty fuzzy sets of $S, T$ over metric space, then the fuzzy distance function $d_F(S, T)$ is a fuzzy convex set given by $d_F(S, T) = \int_0^{h(d_F)} \inf(h_F(S_T, T_T)) d\eta = d(S, T)$, where $h(d_F) - $ is the height of the fuzzy set $d_F$.

Proof. Let height $h(d_F) = \text{MIN}(h(S), h(T)) = H$. Then membership function over the distance $\Phi_{d_F(S, T)}(y)$ is a non-decreasing function in the interval $(-\infty, \lim_{\eta \rightarrow \text{MIN}(h(S), h(T))} \delta(S_T, T_T))$, $\Phi_{d_F(S, T)}(y) = 0$, otherwise. Hence, the fuzzy distance is a convex set. Based on the convexity the distance $d_F(S, T) = \int_0^1 \delta(S_T, T_T) d\tau = \int_0^h \delta(S_T, T_T) d\tau + \int_0^1 \delta(S_T, T_T) d\tau = \int_0^h \delta(S_T, T_T) d\tau + 0$. Since $\inf(d_F(S, T) \tau) = \lim_{\eta \rightarrow \text{MIN}(h(S), h(T))} \delta(S_T, T_T)$, $\forall \tau \in (0, h)$, $\lim_{\eta \rightarrow \text{MIN}(h(S), h(T))} \delta(S_T, T_T) = \sup_{\eta \in (0, h)} \delta(S_T, T_T)$, $\delta(S_T, T_T) < \delta(S_T, T_T)$, then $\delta(S_T, T_T)$ - discontinuous at the left of $\tau^-$.

However, the discontinuity at $(0, h)$ is countable since the distance number $\delta(S_T, T_T)$, $\forall \eta \in (0, h)$ is a non-decreasing function.

The Figure 2 represents the geometry of the distance concept between two fuzzy sets.
Figure 2. Fuzzy set distance \( d_F(S, T) \).

The maximum of the distance function is defined by

**Definition 29.** For the metric space \( (X, \delta) \) the greatest distance between two non-empty sets \( (S, T) \) over metric space is the non-negative number \( \text{MAX} \delta(S, T) = \sup \delta(S, T), x \in S, y \in T \).

**Definition 30.** The greatest distance between two non-empty fuzzy sets over the metric space is defined as the fuzzy number \( \sup \delta(S, T) \) with the corresponding membership function given by

\[
\phi_{\delta(S,T)}(y) = \begin{cases} 
\sup \delta(S, T) \geq y, y = \lim_{\eta \to \text{MIN}(h(S), h(T))} \delta(S, T), \\
0, \text{otherwise.}
\end{cases}
\]

Based on the \( \alpha \) - level cut fuzzy numbers, we may define the fuzzy number by \( \alpha \) – cut represented by the pair of the fuzzy numbers \( \text{INF}(\alpha), \text{SUP}(\alpha) \) with reference to their supremum or lower upper bound of the function and infimum or greatest lower bound defined as it is \([14, 15, 16, 17, 18, 19]\).

**Definition 31.** The \( \alpha \) – level cut of the membership function \( \phi_{F_{\alpha}}(x), 0 < \alpha \leq 1 \) is defined by \( \alpha \) – level cut pair of functions \( \text{INF}(\text{I}(\alpha), \text{S}(\alpha)) \) given by:

\[
\text{I}(\alpha) = \begin{cases} 
\text{inf} \phi_{F_{\alpha}}(x) \text{ if } \alpha > 0, \\
\text{inf} \sup \phi_{F_{\alpha}}(x) \text{ if } \alpha = 0,
\end{cases}
\]

and

\[
\text{S}(\alpha) = \begin{cases} 
\sup \phi_{F_{\alpha}}(x) \text{ if } \alpha > 0, \\
\sup \sup \phi_{F_{\alpha}}(x) \text{ if } \alpha = 0.
\end{cases}
\]

The Hausdorff metric was defined in \([9]\) as it is:

**Definition 32.** If \( C \) is the set of compact subsets in \( \mathbb{R}^2 \), and there are \( P, Q \subseteq C \), then the Hausdorff metric \( h: C \times C \to [0, \infty) \) is given by \( h(P, Q) = \text{MAX} \left( \sup_{Q \in Q} d_{\text{eu}}(q, P) \sup_{P \in P} (p, Q) \right) \), where \( d_{\text{eu}} \) is the Euclidean metric \( \mathbb{R}^2 \).

We define based on the Hausdorff’s metric \( h \) distance for the fuzzy numbers, too:
Definition 33. The metric distance between fuzzy numbers on $F_1 \otimes F_2$ is defined as
\[ d(\Phi_{F_1}^{\alpha_1}(x), \Phi_{F_2}^{\alpha_2}(x)) = \sup_{a \in [0,1]} h(\Phi_{\alpha_1}, \Phi_{\alpha_2}). \]

Next definition of the fuzzy distance was presented in [19].

Definition 34. The fuzzy distance function
\[ d(\Phi_F(x), \Phi_F(y)) = |\Phi_F(x) - \Phi_F(y)| \]

is a function from $F_x \otimes F_y \to F_z$ defined as it is: $d(\Phi_F(x), \Phi_F(y))(z) = \sup_{z=|x-y|} \text{MIN} \left[ \Phi_F(x), \Phi_F(y) \right].$

Based on the $\alpha$ – level cut of the functions $\Phi_F(x), \Phi_F(y)$, we may represent $\alpha$ – level cut form of the distance function as given by the pair of the functions $(I(\alpha), S(\alpha))$.

The $\alpha$ – level cut form of the $\Phi_F(x)$ is $(\Phi_{F_1}^{-1}(\alpha), \Phi_{F_1}^{-1}(\alpha))$ and $\alpha$ – level cut form of the $\Phi_F(y)$ is $(\Phi_{F_2}^{-1}(\alpha), \Phi_{F_2}^{-1}(\alpha))$.

Next definition is the $\alpha$ – level cut form of the distance presented in terms of the pair of functions $(I(\alpha), S(\alpha))$.

Definition 35. The $\alpha$ – level cut distance is defined by
\[ h(I(\alpha) = \left\{ \begin{array}{ll}
\text{MAX}[\Phi_{F_1}^{-1}(\alpha) - \Phi_{F_1}^{-1}(\alpha), 0] & \text{if } \frac{1}{2}\left(\Phi_{F_1}(1) + \Phi_{F_2}(1)\right) \leq \frac{1}{2}\left(\Phi_{F_1}(1) + \Phi_{F_2}(1)\right), \\
\text{MAX}[\Phi_{F_1}^{-1}(\alpha) - \Phi_{F_2}^{-1}(\alpha), 0] & \text{if } \frac{1}{2}\left(\Phi_{F_1}(1) + \Phi_{F_2}(1)\right) \leq \frac{1}{2}\left(\Phi_{F_1}(1) + \Phi_{F_2}(1)\right)
\end{array} \right. \]

and
\[ S(i) = \text{MAX}[\Phi_{F_1}^{-1}(\alpha) - \Phi_{F_1}^{-1}(\alpha), \Phi_{F_2}^{-1}(\alpha) - \Phi_{F_2}^{-1}(\alpha)]. \]

The Hausdorff metric to find a distance between two fuzzy sets uses $\alpha$ – level cuts. The generalization for the distance based on Hausdorff metric was presented in [29]:

Definition 36. The Hausdorff metric for the distance between two non-empty crisp sets is given as the distance between intervals of the $\alpha$ – level cuts of the fuzzy sets $S,T$ as it is: $h(S,T) = \text{MAX}\{|S_1 - T_1|, |S_T - T_R\}$, where $S_1$ and $S_T$ are the left and right point of the $\alpha$ – level cut of $S$, $T_1$ and $T_R$ are the left and right point of the $\alpha$ – level cut of $T$, correspondingly.

There was introduced another generalization of the Hausdorff metric presented in [30]:

Definition 37. The Hausdorff metric to measure a distance between fuzzy sets is defined by
\[ h(S,T) = \int_{\alpha=0}^{1} h(S_\alpha, T_\alpha) d\alpha. \]

If the $y$ – axis is discretized into $n$ – points as $(y_1, y_2, ..., y_n)$ and $S_\alpha, T_\alpha$ are the non-fuzzy $\alpha$ – level cuts of fuzzy sets $S,T$ at $y_\alpha$ respectively, then one another version of the distance measure between two non-empty fuzzy sets by using the Hausdorff metric was presented in [31] and defined by

Definition 38. The Hausdorff metric between fuzzy sets is the metric distance defined by
\[ h(S,T) = \frac{\sum_{\alpha=1}^{n} y_\alpha h(S_\alpha, T_\alpha)}{\sum_{\alpha=1}^{n} y_\alpha}. \]

All of these presented versions of the distance are based on the assumption that the fuzzy sets are normalized such that $\exists x \in X, \phi_X(x) = 1, \phi_T(x) = 1$.

The non-normal cases, where $\phi_X(x) < 1, \phi_T(x) < 1, \forall x \in X$ are represented by the distance:
\[ h(S,T) = \frac{\sum_{\alpha=1}^{n} y_\alpha h(S_\alpha, T_\alpha)}{\sum_{\alpha=1}^{n} y_\alpha} + \zeta \frac{\sum x \phi_X(x) - \phi_T(x)}{\phi_T(x)}, \]

where $\zeta$ – sufficiently small positive value.

The new concept of the distance measure between two non-normal fuzzy sets were introduced in [23]:

Definition 39: The new distance measure between non-normal fuzzy sets is defined by Hausdorff metric distance as it is:
where $\mu = \sup(\alpha \in [0,1], S_\alpha \neq \emptyset \lor T_\alpha \neq \emptyset$ and

$$h(S_\alpha, T_\alpha) = \begin{cases} |T_1 - S_1| & \text{if } |T_1 - S_1| > |T_r - S_r|, \\ T_r - S_r & \text{otherwise}, \end{cases}$$

where $S_1$ and $S_r$ are the left and right point of the $\alpha$ – level cut of $S$, $T_1$ and $T_r$ are the left and right point of the $\alpha$ – level cut of $T$ correspondingly.

4. Conclusion

The geometrical concepts such as compactness, adjacency, height, width, center of gravity, area of coverage are introduced in the article here. These fuzzy geometry structures are playing important role in application problems such as image processing for a soft solution. The newly defined fuzzy geometrical concepts of the fuzzy image such as length, width, breadth, index area of the coverage, support plane, kernel point, fuzzy numbers, fuzzy points were presented for the fuzzy continuous function as well as for the functions , which are piecewise linear and convex in the article here.

There was presented Lemma, which justifies the extension principle by Zadeh. This Lemma was originally proven in the article here. The extended binary operators to perform topological operations with crisp fuzzy sets were presented to reflect the newly expanded version of the Extension Principle. This new version of the extension Principle was generated for the topological operations with crisp fuzzy sets to apply extension principle over union, intersection, summation, difference, product and division of the fuzzy sets. These newly generated binary operations as the expanded versions of the Extension Principle were presented in the article here.

Based on the topological and geometrical structures such as height of the fuzzy set and heights of the cuts of the fuzzy membership grade functions, the Zadeh extension principle was reintroduced and proven for the case, when membership function is not bijective and surjective. There was introduced the definition of the absolute value of the fuzzy numbers. The absolute value of the fuzzy numbers is the essential building material to define the distance between two crisp fuzzy sets. There is newly defined classified versions of the distances between fuzzy sets. The concept of the original metric distance measure is extrapolated to the fuzzy case in the article here.

There were introduced multiple definitions of the fuzzy metric space, which are generated and defined here in the article. Moreover, there was introduced the Hausdorff metric to find the distances between alfa-cuts of the fuzzy sets. There was presented the theorem to establish the conditions for the distance over metric space. This theorem was proven in the article here. There were defined new distances based on the Hausdorff metric. Furthermore, there was established properties of the metric spaces aligned with fuzzy distances here. Moreover, Hausdorff metric was applied to obtain the generalized distance between two non-empty crisp fuzzy sets in the article here. In further progress, the newly defined distance between two non-normal fuzzy sets by using Hausdorff metric was presented here, which expands the fuzzy distance concept to all normalized and non-normal fuzzy cases.

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