Energy Error Estimates of Subspace Projection Method and Multigrid Algorithms for Eigenvalue Problems

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Abstract

This paper is to give a new understanding and applications of the subspace projection method for selfadjoint eigenvalue problems. A new error estimate in the energy norm, which is induced by the stiff matrix, of the subspace projection method for eigenvalue problems is given. The relation between error estimates in $L^2$-norm and energy norm is also deduced. Based on this relation, a new type of inverse power method is designed for eigenvalue problems and the corresponding convergence analysis is also provided. Then we present the analysis of the geometric and algebraic multigrid methods for eigenvalue problems based on the convergence result of the new inverse power method.

Keywords. Eigenvalue problem, subspace projection method, energy error estimate, geometric multigrid, algebraic multigrid.

AMS subject classifications. 65N30, 65N25, 65L15, 65B99.

1 Introduction

Large scale eigenvalue problems always occur in discipline of science and engineering such as material science, quantum chemistry or physics, structure mechanics, biological system, data and information fields, etc. With the increasing of the size and complexity of eigenvalue problems, the efficient solvers become very important and there is a strong demand by engineers and scientists for efficient eigenvalue solvers. There exist many numerical methods for solving large scale eigenvalue problems which are based on power iteration, Krylov subspace iteration and so on [4, 28, 29, 30]. The basic idea is the subspace projection method especially for the eigenvalue problems which come from the discretization of differential operators. From this point of view, the Krylov subspace [4, 28, 29, 30] and LOBPCG [18] methods can be seen as providing ways to build the subspace.
subspace projection understanding also provides the idea to find new methods to design efficient
eigenvalue solves which will be discussed in this paper.

It is well known that the algebraic and geometric multigrid methods are efficient solvers for
linear equations which come from the discretization of partial differential equations [6, 7, 9, 15, 25, 26, 29, 32, 33, 40]. So far, the corresponding theory, applications and software have already
been developed very well. A natural topic is to consider the applications of multigrid methods
to eigenvalue problems. In the past three decades, there have appeared many applications of the
multigrid method to eigenvalue problems. The paper [19] gives a very good review of different
multigrid-based approaches for numerical solutions of eigenvalue problems. So far, the designing
of the multigrid-based methods depends strongly on eigenvalue solvers and the multigrid is used
as an inner solver. Recently, we propose a multilevel and multigrid schemes [16, 22, 23, 24, 37, 38, 39] for
eigenvalue problems which is based on a new idea to construct the subspace. This
schemes decompose a large scale eigenvalue problem into standard linear equations plus small scale
eigenvalue problems such that the choices of linear solvers and eigensolvers are both free. We also
find papers [20, 26, 35] also give the similar method for symmetric positive eigenvalue problems
which is only based on the inverse iteration or Rayleigh quotient method. Actually, the method
and results in this paper can also provide a reasonable analysis for their schemes.

The theoretical analysis reveals that the multigrid method depends on the ellipticity which
is included in the matrices since the ellipticity can leads to the famous duality argument (Aubin-
Nitsche technique) [9, 13]. As we know the duality argument deduces the relation between the error
estimates in weak and strong norms, which is the most important basis for the multigrid method
designing. Unfortunately, there is no this type of duality argument in the numerical theory for
algebraic eigenvalue problems. It is well known that there exists the error estimate in $L^2$-norm
for the subspace approximation method [28, 29, 30]. This $L^2$-norm error estimate leads the error
estimates for the Laczos and Arnoldi methods for eigenvalue problems [30]. In order to understand
and design multigrid methods in the sense of subspace method for eigenvalue problems, we give
the error estimate in the energy norm and the relation between error estimates in
$L^2$-norm and energy norm. Furthermore, this relation provides an idea to design and understand the geometric
and algebraic multigrid methods for eigenvalue problems.

An outline of the paper goes as follows. In Section 2, we introduce the subspace method for
solving eigenvalue problem. A new energy error estimate and the relation between error estimates
in $L^2$-norm and energy norm are given in Section 3. In Section 4, we design and analyze a new
type of inverse power method on a special subspace for eigenvalue problem. Based on the results
in Section 4, we will describe the geometric and algebraic multigrid methods for the eigenvalue
problem solving in Sections 5 and 6, respectively. Some concluding remarks are given in the last
section.

2 Subspace projection method

For clarity, we simply introduce some basic knowledge of subspace projection method and some
definitions here. In this paper, we are mainly concerned with the following algebraic eigenvalue
problem: Find $u \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$ such that

$$Au = \lambda u,$$

(2.1)

where $A \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix.

For our description and analysis, we introduce the following $L^2$ and energy inner products

$$(x, y) = x^T y, \quad (x, y)_A = x^T A y.$$

Then the corresponding $L^2$-norm and energy norm can be defined as follows

$$\|x\|_2 := \sqrt{(x, x)}, \quad \|x\|_A := \sqrt{(x, x)_A} = \sqrt{(Ax, x)}.$$
Let $K$ be a $m$-dimensional subspace of $\mathbb{R}^n$. An orthogonal projection technique for the eigenvalue problem onto the subspace $K$ is to seek an approximate eigenpair $(\tilde{\lambda}, \tilde{u}) \in \mathbb{R} \times K$ to the problem \eqref{eq:original_eigenvalue_problem}, such that the following Galerkin condition is satisfied:

$$A\tilde{u} - \tilde{\lambda}\tilde{u} \perp K, \quad \text{(2.2)}$$

or equivalently

$$(A\tilde{u} - \tilde{\lambda}\tilde{u}, v) = 0, \quad \forall v \in K. \quad \text{(2.3)}$$

It is well known that there exists an orthonormal basis \{\(v_1, \cdots, v_m\)\} for $K$ and we denote the matrix with this basis being the column vectors by $V$

$$V \triangleq [v_1, \cdots, v_m] \in \mathbb{R}^{n \times m}. \quad \text{(2.4)}$$

Based on the orthonormal basis $V$ of the subspace $K$, we can transform the original eigenvalue problem \eqref{eq:original_eigenvalue_problem} to a small-scale eigenvalue problem (always it is called Ritz problem). Let $\tilde{u} =Vy$ with $y \in \mathbb{R}^m$. Then the problem \eqref{eq:galerkin_condition} becomes

$$(AVy - \tilde{\lambda}Vy, v_j) = 0, \quad \text{for } j = 1, 2, \cdots, m. \quad \text{(2.5)}$$

So we just need to solve the following small-scale eigenproblem: Find $y \in \mathbb{R}^m$ and $\tilde{\lambda} \in \mathbb{R}$ such that

$$A_m y = \tilde{\lambda} y, \quad \text{(2.6)}$$

where $A_m = V^T A V$.

In order to translate the subspace projection method into the operator form, we define some projection operators.

**Definition 2.1.** ($L^2$-Projector) The $L^2$-projection operator $R_K: \mathbb{R}^n \mapsto K$ is defined as follows

$$(R_K x, y) = (x, y), \quad \forall x \in \mathbb{R}^n \text{ and } \forall y \in K. \quad \text{(2.7)}$$

In order to give the error estimate of the subspace projection method in the norm $\| \cdot \|_A$, we also define the following projection operator by the inner product $\langle \cdot, \cdot \rangle_A$.

**Definition 2.2.** (A-Projector) The Galerkin projection operator $P_K: \mathbb{R}^n \mapsto K$ is defined as follows

$$(P_K x, y)_A = (x, y)_A, \quad \forall x \in \mathbb{R}^n \text{ and } \forall y \in K. \quad \text{(2.8)}$$

Based on the projection operator $R_K$, the Galerkin condition \eqref{eq:galerkin_condition} is equivalent to the following condition

$$R_K(A\tilde{u} - \tilde{\lambda}\tilde{u}) = 0, \quad \text{where } \tilde{\lambda} \in \mathbb{R} \text{ and } \tilde{u} \in K. \quad \text{(2.9)}$$

The above equality can be written as

$$R_K A\tilde{u} = \tilde{\lambda}\tilde{u}, \quad \text{where } \tilde{\lambda} \in \mathbb{R} \text{ and } \tilde{u} \in K. \quad \text{(2.10)}$$

This operator $R_K A$ can be viewed as $R_K A|_K$ from $K$ to $K$.

Now, we state the following error estimate for the subspace projection method which also motivates the analysis in this paper.

**Lemma 2.1.** (\cite[Theorem 4.6]{Ref}) Let $\gamma = \|R_K A(I - R_K)\|_2$, and consider any eigenvalue $\lambda$ of $A$ with associated eigenvector $u$. Let $\tilde{\lambda}$ be the approximate eigenvalue closest to $\lambda$ and $\delta$ the distance between $\lambda$ and the set of approximate eigenvalues other than $\lambda$. Then there exists an approximate eigenvector $\tilde{u}$ associated with $\tilde{\lambda}$ such that

$$\sin[\theta(u, \tilde{u})] \leq \sqrt{1 + \frac{\gamma^2}{\delta^2} \sin[\theta(u, K)]}. \quad \text{(2.11)}$$

Now, we also state some properties about the energy projection operator $P_K$.

**Proposition 2.1.** It is obvious that the following properties hold

$$P_K^2 = P_K \quad \text{and} \quad P_K(I - P_K) = 0. \quad \text{(2.12)}$$
3 Energy error estimate

In this section, we will give some new error estimates of the subspace method for the eigenvalue problem. The new thing in this paper is that we establish error estimates of the eigenfunction approximation in energy norm rather than $L^2$-norm. Furthermore, the relation between the energy norm and $L^2$-norm is also derived here.

We can order the eigenvalues of the matrix $A$ as the following increasing sequence

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n,$$

and the corresponding eigenvectors

$$u_1, u_2, \cdots, u_n,$$

where $(u_i, u_j)_A = \delta_{ij}$ for $i, j = 1, \ldots, n$ and $\delta_{ij}$ is Kronecker notation. Similarly, the eigenpairs of $A_m$ can be ordered as follows

$$\tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_m,$$

and

$$\tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_m,$$

where $(\tilde{u}_i, \tilde{u}_j)_A = \delta_{ij}$ for $i, j = 1, \ldots, m$.

From the min-max principle for eigenvalue problems, the following upper bound property holds.

Proposition 3.1. ([30, Corollary 4.1]) The following inequality holds

$$\lambda_i \leq \tilde{\lambda}_i, \quad i = 1, 2, \cdots, m. \tag{3.2}$$

In this paper, we solve the eigenvalue problem on the subspace $K$ to obtain the eigenpair approximations $(\tilde{\lambda}_1, \tilde{u}_1), \cdots, (\tilde{\lambda}_k, \tilde{u}_k)$ for $k$ exact eigenpairs of eigenvalue problem (2.1). In order to deduce the error estimate in the energy norm, we write the eigenvalue problem (2.1) as the following version: Find $u \in \mathbb{R}^n$ and $\mu \in \mathbb{R}$ such that

$$A^{-1}u = \mu u \tag{3.3}$$

or the following variational form

$$(A^{-1}u, v)_A = \mu(u, v)_A, \quad \forall v \in \mathbb{R}^n. \tag{3.4}$$

It is easy to know that $\mu = 1/\lambda$. For simplicity of notation, we denote $A^{-1}$ by $T$ in this paper. We know that $T$ has the same eigenvector as $A$ with the eigenvalue $\mu$. Now the Galerkin equation for the approximation on the subspace $K$ is defined as follows

$$(\mathcal{P}_K T \tilde{u}, v)_A = \tilde{\mu}(\tilde{u}, v)_A, \quad \forall v \in K. \tag{3.5}$$

The equation (3.5) can also be written as the following operator form

$$\mathcal{P}_K T \tilde{u} = \tilde{\mu} \tilde{u}. \tag{3.6}$$

Obviously, the eigenvalue problem (3.6) has the eigenvalues $\tilde{\mu}_1 \geq \cdots \geq \tilde{\mu}_m$ and the corresponding eigenvectors $\tilde{u}_1, \cdots, \tilde{u}_m$. We can also know that $\tilde{\mu}_i = 1/\tilde{\lambda}_i$.

Before stating the error estimates of the subspace projection method, we introduce a lemma which comes from [32]. For completeness, a proof is also stated here.

Lemma 3.1. ([32, Lemma 6.4]) For any exact eigenpair $(\lambda, u)$ of (2.1), the following equality holds

$$(\tilde{\lambda}_j - \lambda)(\mathcal{P}_K u, \tilde{u}_j) = \lambda(u - \mathcal{P}_K u, \tilde{u}_j), \quad j = 1, \ldots, m. \tag{3.7}$$
Proof. Since $-\lambda (P_K u, \tilde{u}_j)$ appears on both sides, we only need to prove that

$$\tilde{\lambda}_j (P_K u, \tilde{u}_j) = \lambda (u, \tilde{u}_j).$$

From (2.1), (2.3) and (2.7), the following equalities hold

$$\tilde{\lambda}_j (P_K u, \tilde{u}_j) = (P_K u, \tilde{u}_j)_A = (u, \tilde{u}_j)_A = \lambda (u, \tilde{u}_j).$$

Then the proof is complete. \(\square\)

**Theorem 3.1.** Let $(\lambda, u)$ denote an exact eigenpair of the eigenvalue problem (2.1). Assume the eigenpair approximation $(\tilde{\lambda}_i, \tilde{u}_i)$ has the property that $\tilde{\mu}_i = 1/\tilde{\lambda}_i$ is closest to $\mu = 1/\lambda$. The corresponding spectral projection $E^{(i)}_m : \mathbb{R}^n \mapsto \text{span}\{\tilde{u}_i\}$ is defined as follows

$$(E^{(i)}_m w, \tilde{u}_i)_A = (w, \tilde{u}_i)_A, \quad \forall w \in \mathbb{R}^n.$$ Then the following error estimate holds

$$\|u - E^{(i)}_m u\|_A \leq \sqrt{1 + \frac{\tilde{\mu}_i}{\delta_i} \eta_K^2} \|I - P_K\|_A, \quad (3.8)$$

where $\eta_K$ and $\delta_i$ are defined as follows

$$\eta_K := \sup_{\|g\|_A = 1} \|(I - P_K)T g\|_A = \sup_{\|g\|_A = 1} \|(I - P_K)A^{-1} g\|_A, \quad (3.9)$$

$$\delta_i := \min_{j \neq i} |\tilde{\mu}_j - \mu| = \min_{j \neq i} \left| \frac{1}{\tilde{\lambda}_j} - \frac{1}{\lambda} \right|. \quad (3.10)$$

Furthermore, the eigenvector approximation $\tilde{u}_i$ has the following error estimate in the $L^2$-norm

$$\|u - E^{(i)}_m u\|_2 \leq \eta_{K,i} \|u - E^{(i)}_m u\|_A, \quad (3.11)$$

where $\eta_{K,i}$ is defined as follows

$$\eta_{K,i} = \left(1 + \frac{\tilde{\mu}_i}{\delta_i}\right) \eta_K. \quad (3.12)$$

Proof. Similarly to the duality argument in the finite element method, the following inequality holds

$$\|I - P_K\|_2 = \sup_{\|g\|_2 = 1} \|(I - P_K)g\|_2 = \sup_{\|g\|_2 = 1} \|(I - P_K)A^{-1} g\|_A$$

$$= \sup_{\|g\|_A = 1} \|(I - P_K)u, (I - P_K)T g\|_A \leq \eta_K \|(I - P_K)u\|_A. \quad (3.13)$$

Since $(I - E^{(i)}_m)P_K u \in K$ and $(I - E^{(i)}_m)P_K u \perp A \tilde{u}_i$, the following orthogonal expansion holds

$$(I - E^{(i)}_m)P_K u = \sum_{j \neq i} \alpha_j \tilde{u}_j, \quad (3.14)$$

where $\alpha_j = (P_K u, \tilde{u}_j)_A$. From Lemma 3.1, we have

$$\alpha_j = (P_K u, \tilde{u}_j)_A = \tilde{\lambda}_j (P_K u, \tilde{u}_j) = \frac{\tilde{\lambda}_j \lambda}{\tilde{\lambda}_j - \lambda} (u - P_K u, \tilde{u}_j) = \frac{1}{\mu - \tilde{\mu}_j} (u - P_K u, \tilde{u}_j). \quad (3.15)$$
From the property of the eigenvectors \( \tilde{u}_1, ..., \tilde{u}_m \), the following equalities hold
\[
1 = (\tilde{u}_j, \tilde{u}_j)_A = \tilde{\lambda}_j (\tilde{u}_j, \tilde{u}_j) = \tilde{\lambda}_j \|\tilde{u}_j\|^2,
\]
which leads to the following property
\[
\|\tilde{u}_j\|^2 = \frac{1}{\lambda_j} = \tilde{\mu}_j.
\] (3.16)

From (2.3) and the definitions of the eigenvectors \( \tilde{u}_1, ..., \tilde{u}_m \), we have the following equalities
\[
(\tilde{u}_j, \tilde{u}_k)_A = \delta_{jk}, \quad \left( \frac{\tilde{u}_j}{\|\tilde{u}_j\|_2}, \frac{\tilde{u}_k}{\|\tilde{u}_k\|_2} \right) = \delta_{jk}, \quad 1 \leq j, k \leq m.
\] (3.17)

Then from (3.14), (3.15), (3.16) and (3.17), the following estimates hold
\[
\|(I - E_m^{(i)})P_K u\|_A^2 = \left\| \sum_{j \neq i} \alpha_j \tilde{u}_j \right\|_A^2 = \sum_{j \neq i} \alpha_j^2 = \sum_{j \neq i} \left( \frac{1}{\lambda_j - \mu_j} \right)^2 (u - P_K u, \tilde{u}_j)^2
\leq \frac{1}{\delta_i^2} \sum_{j \neq i} \|\tilde{u}_j\|_2^2 (u - P_K u, \tilde{u}_j)^2 = \frac{1}{\delta_i^2} \sum_{j \neq i} \mu_j (u - P_K u, \tilde{u}_j, \tilde{u}_j)^2
\leq \frac{\bar{\mu}_i}{\delta_i^2} \|u - P_K u\|_2^2.\] (3.18)

From (3.13), (3.18) and the orthogonal property \( u - P_K u \perp A (I - E_m^{(i)})P_K u \), we have the following error estimate
\[
\|u - E_m^{(i)} u\|_A^2 = \|u - P_K u\|_A^2 + \|(I - E_m^{(i)})P_K u\|_A^2
\leq \|(I - P_K) u\|_A^2 + \frac{\bar{\mu}_i}{\delta_i^2} \|u - P_K u\|_2^2 \leq \left(1 + \frac{\bar{\mu}_i}{\delta_i^2} \|K\|^2\right) \|(I - P_K) u\|_A^2.
\] (3.19)

This is the desired result (3.18).

Similarly, from (3.14), (3.15), (3.16) and (3.17), the following estimates hold
\[
\|(I - E_m^{(i)})P_K u\|_2^2 = \left\| \sum_{j \neq i} \alpha_j \tilde{u}_j \right\|_2^2 = \sum_{j \neq i} \alpha_j^2 \|\tilde{u}_j\|^2_2
\leq \sum_{j \neq i} \left( \frac{1}{\lambda_j - \mu_j} \right)^2 (u - P_K u, \tilde{u}_j)^2 \|\tilde{u}_j\|_2^2 \leq \left( \frac{1}{\delta_i} \right)^2 \sum_{j \neq i} \|u - P_K u, \tilde{u}_j\|_2^2 \|\tilde{u}_j\|_2^2
\leq \left( \frac{\bar{\mu}_i}{\delta_i} \right)^2 \|u - P_K u\|_2^2.\] (3.20)

Combining (3.13) and (3.20) leads to the following inequalities
\[
\|(I - E_m^{(i)})P_K u\|_2 \leq \frac{\bar{\mu}_i}{\delta_i} \|u - P_K u\|_2 \leq \frac{\bar{\mu}_i}{\delta_i} \eta_K \|(I - P_K) u\|_A.
\] (3.21)

From (3.13), (3.21) and the triangle inequality, we have the following error estimate for the eigenvector approximation in the \( L^2 \)-norm
\[
\|u - E_m^{(i)} u\|_2 \leq \|u - P_K u\|_2 + \|(I - E_m^{(i)})P_K u\|_2
\leq \|u - P_K u\|_2 + \frac{\bar{\mu}_i}{\delta_i} \eta_K \|(I - P_K) u\|_A
\leq \left(1 + \frac{\bar{\mu}_i}{\delta_i}\right) \eta_K \|(I - P_K) u\|_A \leq \eta_{K,i} \|u - E_m^{(i)} u\|_A.
\] (3.22)

This is the second desired result (3.11) and the proof is complete.
In the following analysis, we state the error estimates for multi eigenvalue approximations. For simplicity of notation, we consider the special case that the first \(k\) eigenvalues \(\lambda_1 \leq \cdots \leq \lambda_k\) are closest to the eigenvalue approximations \(\lambda_k \leq \cdots \leq \lambda_k\). Then the corresponding eigenvector approximations \(\tilde{u}_1, \ldots, \tilde{u}_k\) have the error estimates stated in the next theorem.

**Theorem 3.2.** We define spectral projection \(E_m, k : \mathbb{R}^n \rightarrow \text{span}\{\tilde{u}_1, \ldots, \tilde{u}_k\}\) corresponding to the first \(k\) eigenvector approximations \(\tilde{u}_1, \ldots, \tilde{u}_k\) as follows

\[
(E_m, k w, \tilde{u}_i)_A = (w, \tilde{u}_i)_A, \quad \forall w \in \mathbb{R}^n \text{ and } i = 1, \cdots, k.
\]

Then the associated exact eigenvectors \(u_1, \ldots, u_k\) of problem (2.1) have the following error estimate

\[
\|u_i - E_m, k u_i\|_A \leq \sqrt{1 + \frac{\mu_{k+1}}{\delta_{k,i}^2}} \eta_k^2 \|(I - P_K)u_i\|_A, \quad 1 \leq i \leq k,
\]

where \(\delta_{k,i}\) is defined as follows

\[
\delta_{k,i} := \min_{k < j \leq m} |\tilde{\mu}_j - \mu_i| = \min_{k < j \leq m} \left| \frac{1}{\lambda_j} - \frac{1}{\lambda_i} \right|.
\]

Furthermore, these \(k\) exact eigenvectors have the following error estimate in the \(L^2\)-norm

\[
\|u_i - E_m, k u_i\|_2 \leq \eta_{K, k,i} \|u_i - E_m, k u_i\|_A, \quad 1 \leq i \leq k,
\]

where \(\eta_{K, k,i}\) is defined as follows

\[
\eta_{K, k,i} = \left( 1 + \frac{\mu_{k+1}}{\delta_{k,i}} \right) \eta_k.
\]

**Proof.** Since \((I - E_m, k)P_K u_i \in K\) and \((I - E_m, k)P_K u_i \in \text{span}\{\tilde{u}_{k+1}, \ldots, \tilde{u}_m\}\), the following orthogonal expansion holds

\[
(I - E_m, k)P_K u_i = \sum_{j=k+1}^m \alpha_j \tilde{u}_j.
\]

Then from (3.14), (3.16), (3.17) and (3.21), we have the following estimates

\[
\|(I - E_m, k)P_K u_i\|_A^2 = \left\| \sum_{j=k+1}^m \alpha_j \tilde{u}_j \right\|_A^2 = \sum_{j=k+1}^m \alpha_j^2  
\]

\[
= \sum_{j=k+1}^m \left( \frac{1}{\mu_i - \mu_j} \right)^2 \left( u_i - P_K u_i, \tilde{u}_j \right)^2  
\]

\[
\leq \frac{1}{\delta_{k,i}^2} \sum_{j=k+1}^m \|u_j\|_2^2 \left( u_i - P_K u_i, \frac{\tilde{u}_j}{\|u_j\|_2} \right)^2  
\]

\[
= \frac{1}{\delta_{k,i}^2} \sum_{j=k+1}^m \tilde{\mu}_j \left( u_i - P_K u_i, \frac{\tilde{u}_j}{\|u_j\|_2} \right)^2  
\]

\[
\leq \frac{\mu_{k+1}}{\delta_{k,i}^2} \|u_i - P_K u_i\|_2^2.
\]

Similarly combining (3.13) and (3.28) leads to the following inequality

\[
\|(I - E_m, k)P_K u_i\|_A^2 \leq \frac{\mu_{k+1}}{\delta_{k,i}^2} \|(I - P_K)u_i\|_A^2.
\]

From (3.24) and the orthogonal property \(u_i - P_K u_i \perp_A (I - E_m, k)P_K u_i\), we have the following error estimate

\[
\|u_i - E_m, k u_i\|_A^2 = \|u_i - P_K u_i\|_A^2 + \|(I - E_m, k)P_K u_i\|_A^2
\]
which leads to the inequality

\[ (I - \mathcal{P}_K)u_i \leq \left( 1 + \frac{\mu_{k+1}}{\delta_{k,i}} \eta \right) \| \mathcal{P}_K \| A u_i. \]

This is the desired result (3.23).

Similarly, from (3.15), (3.30) and the triangle inequality, we have the following error estimate for the eigenvalue approximation in the \( L^2 \)-norm

\[ \| (I - E_{m,k}) \mathcal{P}_K u_i \|_2^2 = \| \sum_{j=k+1}^m \alpha_j \bar{u}_j \|_2^2 = \sum_{j=k+1}^m \alpha_j^2 \| \bar{u}_j \|_2^2 \]

\[ = \sum_{j=k+1}^m \left( \frac{1}{\mu_i - \mu_j} \right)^2 \| u_i - \mathcal{P}_K u_i, \bar{u}_j \|_2^2 \leq \frac{1}{\delta_{k,i}} \sum_{j=k+1}^m \| \bar{u}_j \|_2 \left( \| u_i - \mathcal{P}_K u_i, \| \bar{u}_j \|_2 \right)^2 \]

\[ = \frac{1}{\delta_{k,i}} \sum_{j=k+1}^m \left( \| u_i - \mathcal{P}_K u_i, \| \bar{u}_j \|_2 \right)^2 \leq \frac{\mu_{k+1}}{\delta_{k,i}} \| u_i - \mathcal{P}_K u_i \|_2, \quad (3.30) \]

which leads to the inequality

\[ \| (I - E_{m,k}) \mathcal{P}_K u_i \|_2 \leq \frac{\mu_{k+1}}{\delta_{k,i}} \| u_i - \mathcal{P}_K u_i \|_2. \]

From (3.13), (3.30) and the triangle inequality, we have the following error estimate for the eigenvector approximation in the \( L^2 \)-norm

\[ \| u_i - E_{m,k} u_i \|_2 \leq \| u_i - \mathcal{P}_K u_i \|_2 + \| (I - E_{m,k}) \mathcal{P}_K u_i \|_2 \]

\[ \leq \left( 1 + \frac{\mu_{k+1}}{\delta_{k,i}} \right) \| (I - \mathcal{P}_K) u_i \|_2 \leq \left( 1 + \frac{\mu_{k+1}}{\delta_{k,i}} \right) \eta \| (I - \mathcal{P}_K) u_i \|_A \]

\[ \leq \left( 1 + \frac{\mu_{k+1}}{\delta_{k,i}} \right) \eta \| u_i - E_{m,k} u_i \|_A. \]

This is the second desired result (3.26) and the proof is complete. \( \square \)

### 4 Inverse power method on a subspace

As an application of the error estimates stated in the previous section, we give an algebraic error estimate for the inverse power method on a special subspace which is constructed by enriching the current eigenspace approximation with a space \( \mathcal{K} \). For more details about this special space, please refer to [24, 37].

For some given eigenvector approximations \( u_1^{(i)}, \ldots, u_k^{(i)} \) which are approximations for the first \( k \) eigenvectors \( u_1, \ldots, u_k \), we do the following inverse power iteration on a subspace:

**Algorithm 4.1. Inverse power method on a subspace**

**For given eigenvector approximations \( u_1^{(i)}, \ldots, u_k^{(i)} \), do the following two steps**

1. **Define the subspace** \( \mathcal{K}_k^{(i+1)} := \mathcal{K} + \text{span}\{u_1^{(i)}, \ldots, u_k^{(i)}\} \) and solve the following eigenvalue problem: Find \( \tilde{u}_i^{(i+1)} \in \mathcal{K}_k^{(i+1)} \) and \( \lambda_i^{(i+1)} \in \mathbb{R} \) such that \( \| \tilde{u}_i^{(i+1)} \|_A = 1 \) and

   \[ (A \tilde{u}_i^{(i+1)}, v) = \lambda_i^{(i+1)} (\tilde{u}_i^{(i+1)}, v), \quad \forall v \in \mathcal{K}_k^{(i+1)}. \quad (4.1) \]

   Solve this eigenvalue problem to obtain the new first \( k \) eigenvector approximations \( \tilde{u}_1^{(i+1)}, \ldots, \tilde{u}_k^{(i+1)} \).

2. **Solve the following \( k \) linear equations:**

   \[ A \tilde{u}_i^{(i+1)} = \lambda_i^{(i+1)} \tilde{u}_i^{(i+1)}, \quad i = 1, \ldots, k. \quad (4.2) \]
We obtain the new eigenvector approximations \( u_1^{(\ell+1)}, \ldots, u_k^{(\ell+1)} \) as the output.

We define spectral projection \( E_{m,k}^{(\ell)} : \mathbb{R}^n \rightarrow \text{span}\{u_1^{(\ell)}, \ldots, u_k^{(\ell)}\} \) corresponding to the eigenvector approximations \( u_1^{(\ell)}, \ldots, u_k^{(\ell)} \) as follows

\[
(E_{m,k}^{(\ell)} w, u_i^{(\ell)})_A = (w, u_i^{(\ell)})_A, \quad \forall w \in \mathbb{R}^n \text{ and } i = 1, \ldots, k.
\]

Then the spectral projections \( E_{m,k}^{(\ell+1)}, \tilde{E}_{m,k}^{(\ell+1)} \) and \( E_k \) can also be defined corresponding to the spaces \( \text{span}\{u_1^{(\ell+1)}, \ldots, u_k^{(\ell+1)}\} \), \( \text{span}\{\tilde{u}_1^{(\ell+1)}, \ldots, \tilde{u}_k^{(\ell+1)}\} \) and \( \text{span}\{u_1, \ldots, u_k\} \), respectively. Based on Theorems 3.1 and 3.2 we give the following error estimate for Algorithm 4.1.

**Theorem 4.1.** There exist exact eigenvectors \( u_1, \ldots, u_k \) such that the resultant eigenvector approximations \( u_1^{(\ell+1)}, \ldots, u_k^{(\ell+1)} \) have the following error estimate

\[
\left( \sum_{i=1}^{k} \|u_i - E_{m,k}^{(\ell+1)} u_i\|^2_A \right)^{1/2} \leq \theta_{K_k^{(\ell+1)}} \sqrt{\frac{\lambda_k}{\lambda_{k+1}}} \sqrt{\left( \sqrt{\lambda_k^{(\ell+1)}} \eta_{K_k^{(\ell+1)},k,k} \right) \left( \sum_{i=1}^{k} \|u_i - E_{m,k}^{(\ell)} u_i\|^2_A \right)^{1/2}},
\]

where \( \theta_{K_k^{(\ell+1)}} \) and \( \eta_{K_k^{(\ell+1)},k,k} \) are defined as follows

\[
\theta_{K_k^{(\ell+1)}} := \sqrt{1 + \frac{\mu_{k+1}^{(\ell+1)} \eta_{k,k}^{(\ell+1)}}{(\delta_{k,k}^{(\ell+1)})^2}},
\]

\[
\eta_{K_k^{(\ell+1)},k,k} := \left( 1 + \frac{\mu_{k+1}^{(\ell+1)}}{\delta_{k,k}^{(\ell+1)}} \right) \eta_{k,k}^{(\ell+1)}, \quad i = 1, \ldots, k;
\]

with

\[
\delta_{k,k}^{(\ell+1)} := \min_{k < j \leq m} |\mu_j^{(\ell+1)} - \mu_k^{(\ell+1)}|, \quad i = 1, \ldots, k \quad \text{and} \quad \mu_i^{(\ell+1)} := \frac{1}{\lambda_i^{(\ell+1)}}, \quad i = 1, \ldots, m.
\]

**Proof.** From Theorem 3.2 there exist exact eigenvectors \( u_1, \ldots, u_k \) such that the following error estimates for the eigenvector approximations \( u_1^{(\ell+1)}, \ldots, u_k^{(\ell+1)} \) hold for \( i = 1, \ldots, k \)

\[
\|u_i - E_{m,k}^{(\ell+1)} u_i\|_A \leq \sqrt{1 + \frac{\lambda_i^{(\ell+1)} \eta_{K_k^{(\ell+1)}}}{(\delta_{k,i}^{(\ell+1)})^2}} \| (I - \mathcal{P}_{K_k^{(\ell+1)}}) u_i \|_A
\]

\[
\leq \sqrt{1 + \frac{\lambda_i^{(\ell+1)} \eta_{K_k^{(\ell+1)}}}{(\delta_{k,i}^{(\ell+1)})^2}} \| u_i - E_{m,k}^{(\ell)} u_i \|_A,
\]

and

\[
\|u_i - \tilde{E}_{m,k}^{(\ell+1)} u_i\|_2 \leq \eta_{K_k^{(\ell+1)},k,i} \| u_i - E_{m,k}^{(\ell+1)} u_i \|_A
\]

\[
\leq \eta_{K_k^{(\ell+1)},k,i} \sqrt{1 + \frac{\lambda_i^{(\ell+1)} \eta_{K_k^{(\ell+1)}}}{(\delta_{k,i}^{(\ell+1)})^2}} \| u_i - E_{m,k}^{(\ell)} u_i \|_A.
\]

Let \( \alpha_i = 1/\|u_i^{(\ell+1)}\|_A \). From (4.11) and (4.12), we have following inequalities

\[
1 = \lambda_i^{(\ell+1)} (u_i^{(\ell+1)}, u_i^{(\ell+1)}) = (A u_i^{(\ell+1)}, u_i^{(\ell+1)}) \leq \|u_i^{(\ell+1)}\|_A \|\tilde{u}_i^{(\ell+1)}\|_A = \frac{1}{\alpha_i}.
\]
Then each $\alpha_i$ has the following estimate

$$\alpha_i \leq 1, \quad \text{for } i = 1, \ldots, k.$$ \hspace{1cm} (4.9)

For the analysis, we define the $L^2$-projections $\pi_k$ and $\tilde{\pi}_{m,k}^{(\ell+1)}$ corresponding to the spaces $\text{span}\{u_1, \ldots, u_k\}$ and $\text{span}\{\tilde{u}_1^{(\ell+1)}, \ldots, \tilde{u}_k^{(\ell+1)}\}$, respectively. Then since $\|u_i\|_A = \|\alpha_i u_i^{(\ell+1)}\|_A = 1$ and $\|\lambda_i u_i\|_2 = \|\sqrt{\Lambda_i} u_i\|_2 = 1$, there exist following equalities

$$\sum_{i=1}^{k} \|u_i - E_{m,k}^{(\ell+1)} u_i\|^2_A = \sum_{i=1}^{k} \|\alpha_i u_i^{(\ell+1)} - E_k(\alpha_i u_i^{(\ell+1)})\|^2_A$$ \hspace{1cm} (4.10)

and

$$\sum_{i=1}^{k} \|\sqrt{\lambda_i^{(\ell+1)} u_i^{(\ell+1)}} - \pi_k(\sqrt{\lambda_i^{(\ell+1)} u_i^{(\ell+1)})\|^2_2 = \sum_{i=1}^{k} \|\sqrt{\Lambda_i} u_i - \tilde{\pi}_{m,k}^{(\ell+1)}(\sqrt{\Lambda_i} u_i)\|^2_2.$$ \hspace{1cm} (4.11)

From the definition of the spectral projection $E_k$, it is easy to know the following property holds

$$\frac{\|u_i^{(\ell+1)}\|_A - E_k u_i^{(\ell+1)}\|_A^2}{\|u_i^{(\ell+1)} - E_k u_i^{(\ell+1)}\|_2^2} \geq \lambda_{k+1}.$$ \hspace{1cm} (4.12)

We define $\mathbf{U} = [u_1, \ldots, u_k] \in \mathbb{R}^{n \times k}$, $\tilde{\mathbf{U}}^{(\ell+1)} = [\tilde{u}_1^{(\ell+1)}, \ldots, \tilde{u}_k^{(\ell+1)}] \in \mathbb{R}^{n \times k}$ and $\mathbf{U}^{(\ell+1)} = [u_1^{(\ell+1)}, \ldots, u_k^{(\ell+1)}] \in \mathbb{R}^{n \times k}$. It is easy to know that there exists a nonsingular matrix $Q \in \mathbb{R}^{k \times k}$ such that

$$\pi_k \tilde{\mathbf{U}}^{(\ell+1)} = [\pi_k \tilde{u}_1^{(\ell+1)}, \ldots, \pi_k \tilde{u}_k^{(\ell+1)}] = \mathbf{U} Q.$$ \hspace{1cm} (4.13)

From the definition of spectral projection $E_k$, the following equation holds

$$(\mathbf{U}^{(\ell+1)} - E_k \mathbf{U}^{(\ell+1)})^T A \mathbf{u}_j = 0, \quad j = 1, \ldots, k.$$ \hspace{1cm} (4.14)

For the following proof, we define three diagonal matrices $D = \text{diag}(\alpha_1, \ldots, \alpha_k)$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_k)$ and $\Lambda^{(\ell+1)} = \text{diag}(\lambda_1^{(\ell+1)}, \ldots, \lambda_k^{(\ell+1)})$.

Combining \(12\), \(13\), \(12\), \(13\) and \(14\) leads to the following estimate

$$\sum_{i=1}^{k} \|u_i - E_{m,k}^{(\ell+1)} u_i\|^2_A = \sum_{i=1}^{k} \|\alpha_i u_i^{(\ell+1)} - E_k(\alpha_i u_i^{(\ell+1)})\|^2_A$$

$$= \text{trace}\left( (\mathbf{U}^{(\ell+1)} - E_k \mathbf{U}^{(\ell+1)})^T A (\mathbf{U}^{(\ell+1)} - E_k \mathbf{U}^{(\ell+1)}) \right)$$

$$= \text{trace}\left( D^T (\mathbf{U}^{(\ell+1)} - E_k \mathbf{U}^{(\ell+1)})^T A (\mathbf{U}^{(\ell+1)} - \pi_k \tilde{\mathbf{U}}^{(\ell+1)} Q^{-1} \Lambda^{-1} Q \Lambda^{(\ell+1)} D) \right)$$

$$= \text{trace}\left( D (\mathbf{U}^{(\ell+1)} - E_k \mathbf{U}^{(\ell+1)})^T A (\mathbf{U}^{(\ell+1)} - \mathbf{U} Q Q^{-1} \Lambda^{-1} Q \Lambda^{(\ell+1)} D) \right)$$

$$= \text{trace}\left( D (\mathbf{U}^{(\ell+1)} - E_k \mathbf{U}^{(\ell+1)})^T (\tilde{\mathbf{U}}^{(\ell+1)} \Lambda^{(\ell+1)} - \mathbf{U} \Lambda Q Q^{-1} \Lambda^{-1} Q \Lambda^{(\ell+1)} D) \right)$$

$$= \text{trace}\left( D (\mathbf{U}^{(\ell+1)} - E_k \mathbf{U}^{(\ell+1)})^T (\tilde{\mathbf{U}}^{(\ell+1)} \Lambda^{(\ell+1)} - \mathbf{U} \Lambda \Lambda^{(\ell+1)} D) \right)$$

$$= \sum_{i=1}^{k} \alpha_i^2 \lambda_i^{(\ell+1)} (u_i^{(\ell+1)} - \pi_k \tilde{u}_i^{(\ell+1)}, u_i^{(\ell+1)} - E_k u_i^{(\ell+1)})$$
For given eigenvector approximation $u_i^{(\ell+1)}$, we have

$$
\sum_{i=1}^{k} \alpha_i \frac{\lambda_{i}^{(\ell+1)}}{\lambda_{k+1}} \left\| u_i^{(\ell+1)} - \pi_k u_i^{(\ell+1)} \right\|_2 \left\| \alpha_i u_i^{(\ell+1)} - E_k(\alpha_i u_i^{(\ell+1)}) \right\|_A.
$$

(4.15)

From (4.8), (4.11) and (4.15), we have

$$
\sum_{i=1}^{k} \left\| u_i - E_{m,k}^{(\ell+1)} u_i \right\|_A^2 = \sum_{i=1}^{k} \left\| \alpha_i u_i^{(\ell+1)} - E_k(\alpha_i u_i^{(\ell+1)}) \right\|_A^2
$$

$$
\leq \sum_{i=1}^{k} \alpha_i^2 \frac{\lambda_{i}^{(\ell+1)}}{\lambda_{k+1}} \left\| \sqrt{\lambda_i^{(\ell+1)}} u_i^{(\ell+1)} - \pi_k \left( \sqrt{\lambda_i^{(\ell+1)}} u_i^{(\ell+1)} \right) \right\|_2^2
$$

$$
\leq \frac{\lambda_{k}^{(\ell+1)}}{\lambda_{k+1}} \sum_{i=1}^{k} \left\| \sqrt{\lambda_i} u_i - \widetilde{\pi}_k^{(\ell+1)} \left( \sqrt{\lambda_i} u_i \right) \right\|_2^2 \leq \frac{\lambda_{k}^{(\ell+1)}}{\lambda_{k+1}} \sum_{i=1}^{k} \lambda_i \left\| u_i - E_{m,k}^{(\ell)} u_i \right\|_A^2
$$

$$
\leq \frac{\lambda_{k}^{(\ell+1)}}{\lambda_{k+1}} \sum_{i=1}^{k} \left( 1 + \frac{\mu_{k+1}^{(\ell+1)} \eta_k^{(\ell+1)}}{\left( \delta_{k,k}^{(\ell+1)} \right)^2} \right) \eta_k^{(\ell+1)} \left( k_i^{(\ell+1)} \right) \left\| u_i - E_{m,k}^{(\ell)} u_i \right\|_A^2
$$

This is the desired result (4.3) and the proof is complete.

**Remark 4.1.** In the convergence result (4.3), the term $\sqrt{\lambda_k/\lambda_{k+1}}$ comes from the inverse power iteration. Different from the normal inverse power method, there exists the term $\sqrt{\lambda_k^{(\ell+1)} \eta_k^{(\ell+1)}}, k, k$ which depends on the subspace $K$. We can accelerate the inverse power iteration largely if the subspace $K$ can make the term $\sqrt{\lambda_k^{(\ell+1)} \eta_k^{(\ell+1)}}, k, k$ be small (less than 1).

**Remark 4.2.** In this paper, we are only concerned with the error estimates for the eigenvector approximation since the error estimates for the eigenvalue approximation can be easily deduced from the following error expansion

$$
0 \leq \lambda_{\hat{i}} - \lambda_i = \frac{(A(u_i - \psi), u_i - \psi)}{(\psi, \psi)} - \lambda_i \left( \frac{(u_i - \psi, u_i - \psi)}{(\psi, \psi)} \right) \leq \left\| u_i - \psi \right\|_A^2 / \left\| \psi \right\|_2^2,
$$

where $\psi$ is the eigenvector approximation for the exact eigenvector $u_i$ and

$$
\hat{\lambda}_i = \frac{(A\psi, \psi)}{(\psi, \psi)}.
$$

It is obvious that the parallel computing method can be used for Step 2 of Algorithm 4.1 since each linear equation can be solved independently. Furthermore, in order to design a complete parallel scheme for eigenvalue problems, we give another version of the inverse power method for only one (may be not the smallest one) eigenpair.

We start from an eigenvector approximation $u_i^{(\ell)}$ which is closest to an exact eigenvector denoted by $u$. Then the new version of the inverse power iteration on a subspace can be defined as follows:

**Algorithm 4.2. Inverse power method on a subspace for one eigenvector**

*For given eigenvector approximation $u_i^{(\ell)}$, do the following two steps*
1. Define the subspace $K^{(t+1)} := K + \text{span}\{u^{(t)}_i\}$ and solve the following eigenvalue problem:

Find $\tilde{u}^{(t+1)}_i \in K^{(t+1)}$ and $\lambda^{(t+1)}_i \in \mathbb{R}$ such that $\|\tilde{u}^{(t+1)}_i\|_A = 1$ and

$$(A\tilde{u}^{(t+1)}_i, v) = \lambda^{(t+1)}_i (\tilde{u}^{(t+1)}_i, v), \quad \forall v \in K^{(t+1)}. \tag{4.16}$$

Solve this eigenvalue problem to obtain a new eigenvector approximation $\tilde{u}^{(t+1)}_i$ which has the biggest orthogonal projection in the direction of $u^{(t)}_i$.

2. Solve the following linear equation:

$$Au^{(t+1)}_i = \lambda^{(t+1)}_i \tilde{u}^{(t+1)}_i. \tag{4.17}$$

We obtain the new eigenvector approximation $u^{(t+1)}_i$ as the output.

We define spectral projection $E_m^{(i,t)} : \mathbb{R}^n \mapsto \text{span}\{u^{(t)}_i\}$ corresponding to the eigenvector approximation $u^{(t)}_i$ as follows

$$(E_m^{(i,t)}w, u^{(t)}_i)_A = (w, u^{(t)}_i)_A, \quad \forall w \in \mathbb{R}^n. \tag{4.18}$$

Then the spectral projections $E_m^{(i,t+1)}$, $\tilde{E}_m^{(i,t+1)}$ and $E$ can also be defined corresponding to eigenvectors $u^{(t+1)}_i$, $\tilde{u}^{(t+1)}_i$ and $u$, respectively. Based on Theorem 3.1 we give the following error estimate for Algorithm 4.2.

**Theorem 4.2.** There exists an exact eigenvector $u$ such that the resultant eigenvector approximation $u^{(t+1)}_i$ has the following error estimate

$$\|u - E_m^{(i,t+1)}u\|_A \leq \theta_{K^{(t+1)}}, \lambda \|u - E_m^{(i,t)}u\|_A, \tag{4.19}$$

where $\theta_{K^{(t+1)}}$ and $\eta_{K^{(t+1)},i}$ are defined as follows

$$\theta_{K^{(t+1)}} := \sqrt{1 + \frac{\mu_1^{(t+1)} \eta_{K^{(t+1)}}^2}{(\delta_i^{(t+1)})^2}}, \quad \eta_{K^{(t+1)},i} := \left(1 + \frac{1}{\delta_i^{(t+1)}}\right) \eta_{K^{(t+1)}}. \tag{4.20}$$

with

$$\delta_i^{(t+1)} := \min_{j \neq i} |\mu_j^{(t+1)} - \mu|, \quad \text{and} \quad \mu_j^{(t+1)} = \frac{1}{\lambda_j^{(t+1)}}, \quad j = 1, ..., m. \tag{4.21}$$

**Proof.** From Theorem 3.1 there exists an exact eigenvector $u$ closest to the eigenvector approximation $u^{(t)}_i$ such that following error estimates for the eigenvector $\tilde{u}^{(t+1)}_i$ hold

$$\|u - \tilde{E}_m^{(i,t+1)}u\|_A \leq \sqrt{1 + \frac{\mu_1^{(t+1)} \eta_{K^{(t+1)}}^2}{(\delta_i^{(t+1)})^2}} \|(I - \mathcal{P}_{K^{(t+1)}})u\|_A$$

$$\leq \sqrt{1 + \frac{\mu_1^{(t+1)} \eta_{K^{(t+1)}}^2}{(\delta_i^{(t+1)})^2}} \|u - E_m^{(i,t)}u\|_A, \tag{4.22}$$

and

$$\|u - \tilde{E}_m^{(i,t+1)}u\|_2 \leq \eta_{K^{(t+1)},i} \|u - \tilde{E}_m^{(i,t+1)}u\|_A$$
which leads to the estimate $\alpha_i \leq 1$. For the analysis, we define the $L^2$-projections $\pi$ and $\tilde{\pi}_m^{(i,\ell+1)}$ corresponding to the spaces $\text{span}\{u\}$ and $\text{span}\{\tilde{u}_i^{(\ell+1)}\}$, respectively. Then from $\|u\|_A = \|\alpha_i u_i^{(\ell+1)}\|_A = 1$ and $\|\sqrt{\lambda} u\|_2 = \|\sqrt{\lambda_i^{(\ell+1)} - \tilde{u}_i^{(\ell+1)}}\|_2 = 1$, we have following equalities

$$
\|u - E_m^{(i,\ell+1)} u\|_A = \|\alpha_i u_i^{(\ell+1)} - E(\alpha_i u_i^{(\ell+1)})\|_A
$$

and

$$
\|\sqrt{\lambda_i^{(\ell+1)} - \tilde{u}_i^{(\ell+1)}} - \pi(\sqrt{\lambda_i^{(\ell+1)}} \tilde{u}_i^{(\ell+1)})\|_2 = \|\sqrt{\lambda} u - \tilde{\pi}_m^{(i,\ell+1)} (\sqrt{\lambda} u)\|_2.
$$

From the definition of the eigenvalue, it is easy to know the following property holds

$$
\frac{|u_i^{(\ell+1)} - E u_i^{(\ell+1)}|^2}{|u_i^{(\ell+1)} - E u_i^{(\ell+1)}|^2} \geq \lambda_1.
$$

Combining (4.17), (4.23) and (4.25) leads to the following estimate

$$
\|u - E_m^{(i,\ell+1)} u\|_A \leq \alpha_i \sqrt{\lambda_i^{(\ell+1)}} \|u - \pi(\sqrt{\lambda_i^{(\ell+1)}} \tilde{u}_i^{(\ell+1)})\|_2
$$

From (4.22), (4.24) and (4.26), we have

$$
\|u - E_m^{(i,\ell+1)} u\|_A \leq \alpha_i \sqrt{\lambda_i^{(\ell+1)}} \|u - \pi(\sqrt{\lambda_i^{(\ell+1)}} \tilde{u}_i^{(\ell+1)})\|_2
$$

This is the desired result (4.18) and the proof is complete.
From the estimate (4.18), in order to guarantee the convergence of Algorithm 4.2, we need to choose the subspace $K$ properly such that the term $\sqrt{\lambda_{i}^{(\ell+1)} / \lambda_{1}} \eta_{K_{i}^{(\ell+1)}}$ is small which is stricter than the condition $\sqrt{\lambda_{i}^{(\ell+1)} / \lambda_{1}} \eta_{K_{i}^{(\ell+1)}}$ is small when $\lambda > \lambda_{1}$. But we can implement Algorithm 4.2 in parallel for different eigenpairs which is the most important advantage of this algorithm.

5 Geometric multigrid method for eigenvalue problem

In this section, we discuss a type of geometric multigrid (GMG) method for the standard elliptic eigenvalue problem [24, 37, 38]. Here, the standard notation for Sobolev spaces $H^{s}(\Omega)$ and their associated norms and semi-norms [1] will be used. We denote $H_{0}^{1}(\Omega) = \{v \in H^{1}(\Omega) : v|_{\partial \Omega} = 0\}$, where $v|_{\partial \Omega} = 0$ is in the sense of trace. The letter $C$ (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences in this section.

The concerned eigenvalue problem in this section is defined as follows: Find $(\lambda, u)$ such that

$$
\begin{cases}
-\Delta u = \lambda u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega.
\end{cases}
$$

(5.1)

In order to use the finite element method to solve the eigenvalue problem (5.1), we need to define the corresponding variational form as follows: Find $(\lambda, u) \in \mathbb{R} \times V$ such that

$$a(u, v) = \lambda b(u, v), \quad \forall v \in V,$n

(5.2)

where $V := H_{0}^{1}(\Omega)$ and

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v d\Omega, \quad b(u, v) = \int_{\Omega} uv d\Omega.$$

(5.3)

The norms $\| \cdot \|_{a}$ and $\| \cdot \|_{b}$ are defined as

$$\|v\|_{a} = \sqrt{a(v, v)} \quad \text{and} \quad \|v\|_{b} = \sqrt{b(v, v)}.$$

Now, we introduce the finite element method for the eigenvalue problem (5.2). First we decompose the computing domain $\Omega \subset \mathbb{R}^{d}$ ($d = 2, 3$) into shape-regular triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$) to produce the mesh $T_{h}$ (cf. [9, 13]). The diameter of a cell $K \in T_{h}$ is denoted by $h_{K}$ and the mesh size $h$ describes the maximum diameter of all cells $K \in T_{h}$. Based on the mesh $T_{h}$, we construct the linear finite element space $V_{h} \subset V$ as follows:

$$V_{h} = \{ v_{h} \in C(\Omega) \mid v_{h}|_{K} \in P_{1}, \forall K \in T_{h} \} \cap H_{0}^{1}(\Omega),$$

(5.4)

where $P_{1}$ denotes the space of polynomials of degree at most 1.

The standard finite element scheme for the eigenvalue problem (5.2) can be defined as follows: Find $(\lambda_{h}, u_{h}) \in \mathbb{R} \times V_{h}$ such that $a(u_{h}, u_{h}) = 1$ and

$$a(u_{h}, v_{h}) = \lambda_{h} b(u_{h}, v_{h}), \quad \forall v_{h} \in V_{h}.$$n

(5.5)

Based on the basis system, the discrete eigenvalue problem (5.5) can be transformed to the following general algebraic eigenvalue problem: Find $(\lambda, u) \in \mathbb{R} \times \mathbb{R}^{n}$ such that

$$Au = \lambda Mu,$n

(5.6)

where $n := \text{dim}(V_{h})$, $A$ and $M$ denote the stiff and mass matrices, respectively, corresponding to the finite element space $V_{h}$.
For simplicity of description in this and next sections, we only consider the applications of Algorithm 4.1. It is not difficult to understand that Algorithm 4.2 can be used similarly. Theorem 4.1 can give the understanding and also a new proof for the GMG method [37] for the eigenvalue problem (5.5). In this GMG method, the basic space \( K \) is chosen as the low dimensional finite element space \( V_H \) which is defined on the coarse mesh \( T_H \) and it is obvious that \( K := V_H \subset V_h \). In order to use Algorithm 4.1, we only need to define the \( L^2 \) inner product in (4.1) with the mass matrix \( M \) by the following way

\[
(v, w) = v^T M w, \quad \forall v \in \mathbb{R}^n \quad \text{and} \quad \forall w \in \mathbb{R}^n
\]

and the linear equations (4.2) are replaced by the following standard linear equations

\[
Au_i^{(\ell + 1)} = \lambda_i^{(\ell + 1)} M u_i^{(\ell + 1)}, \quad i = 1, ..., k,
\]

which can be solved by the well-known GMG method for boundary value problems [6, 7, 9, 15, 25, 27, 31, 33, 34, 40].

With the standard results from the finite element theory [9, 13], we can give the following estimates for the quantities appeared in Theorem 4.1. Combining (4.5) and the well-known Aubin-Nitsche result \( \eta_K^{(\ell + 1)} \leq C H \) (cf. [9, 13, 37]) leads to the following estimate

\[
\eta_{K,(\ell + 1)}^{(\ell + 1), k,k} \leq \left( 1 + \frac{\mu_{k+1}^{(\ell + 1)}}{\delta_{k,k}^{(\ell + 1)}} \right) \eta_K^{(\ell + 1)} \leq \left( 1 + \frac{1}{\delta_{k,k}^{(\ell + 1)} \lambda_{k+1}^{(\ell + 1)}} \right) \eta_K \leq C H,
\]

where we use the property \( \mu_{k+1}^{(\ell + 1)} \leq \mu_{k+1} = 1/\lambda_{k+1} \) and the constant \( C \) depends on the eigenvalue gap \( \delta_{k,k}^{(\ell + 1)} \) and the eigenvalue \( \lambda_{k+1} \). From Theorem 4.1 and (5.8), the following convergence result holds

\[
\left( \sum_{i=1}^k \|u_i - E_{m,k}^{(\ell + 1)} u_i\|_A^2 \right)^{1/2} \leq C \sqrt{\lambda_k^{(\ell + 1)} H} \left( \sum_{i=1}^k \|u_i - E_{m,k} u_i\|_A^2 \right)^{1/2}.
\]

From (5.9), in order to produce the uniform convergence, we only need to choose the size \( H \) of the coarse mesh \( T_H \) small enough such that the following condition holds

\[
C \sqrt{\lambda_k^{(\ell + 1)} H} < 1.
\]

Since the definition \( K = V_H \) results in the sparse matrices for the eigenvalue problem (4.1), the required memory for Algorithm 4.1 is almost optimal and less than mostly existed eigenvalue solvers. For more details, please refer to papers [24, 37, 38] and the numerical examples provided there. Different from the existed GMG methods for eigenvalue problems [5, 8, 10, 12, 14, 15, 31], our method only need to solve standard linear elliptic boundary value problems (5.7) and any efficient linear solvers can be used without any modification. The GMG methods in [8, 10, 14, 15, 31] are designed based on the shift-inverse power method for eigenvalue problem and GMG solver is used as an inner iteration with modifications since there appear singular or nearly singular linear equations. Furthermore, since the scale of eigenvalue problem (4.1) that we need to solve is very small, the choice of the corresponding eigenvalue solver is very free.

6 Algebraic multigrid method for eigenvalue problem

Based on Algorithm 4.1 and the corresponding convergence result in Theorem 4.1 in Section 4, similarly to the idea presented in the previous section, we can design and analyze a type of algebraic
multigrid (AMG) method for the eigenvalue problem (2.1). If we can find a suitable low dimensional subspace $\mathcal{K}$ by some type of coarsening step, a type of inverse power method with a fast convergence rate can be designed based on Algorithm 4.1. Inspired by the AMG method for linear equations [27, 33, 34, 41], the natural low dimensional subspace can be chosen as the coarse space in AMG if we can find a suitable low dimensional method and the AMG method is also an efficient solver for linear equations (4.2).

In this section, we set $n_c = \text{dim}(\mathcal{K}) < n$ as the dimension of the subspace $\mathcal{K}$. First, let us consider a special case that the subspace $\mathcal{K} := \text{span}\{u_j\}_{j=1}^{n_c}$ which is constituted by the eigenvectors corresponding to the smallest $n_c$ eigenvalues of $A$. In the following analysis, we assume any vector $g \in \mathbb{R}^n$ has the expansion $g = \sum_{j=1}^{n} \alpha_j u_j$. Then the following inequalities hold and

$$\eta_{K_k^{(t+1)}} := \sup_{\|g\|_2=1} \left\| (I - \mathcal{P}_{K_k^{(t+1)}}) T g \right\|_A \leq \sup_{\|g\|_2=1} \left\| (I - \mathcal{P}_{K}) \sum_{j=1}^{n} \mu_j \alpha_j u_j \right\|_A$$

$$\leq \sup_{\|g\|_2=1} \left\| \sum_{j=n_c+1}^{n} \mu_j \alpha_j u_j \right\|_A = \sup_{\|g\|_2=1} \left\| \sum_{j=n_c+1}^{n} \sqrt{\lambda_j} \mu_j \alpha_j u_j \right\|_2$$

$$= \sup_{\|g\|_2=1} \left\| \sum_{j=n_c+1}^{n} \sqrt{\mu_j} \alpha_j u_j \right\|_2 \leq \sqrt{\mu_{n_c+1}} \sup_{\|g\|_2=1} \left\| \sum_{j=n_c+1}^{n} \alpha_j u_j \right\|_2 \leq \sqrt{\mu_{n_c+1}}. \quad (6.1)$$

In (4.4) and (4.5), since $\mathcal{K} \subset \mathcal{K}_k^{(t+1)}$, (6.1) and the property $\mu_{k+1} \leq \mu_{k+1} = 1/\lambda_{k+1}$, we have the following estimates

$$\theta_{k+1} \leq \sqrt{1 + \frac{1}{\lambda_{k+1} \lambda_{n_c+1} (\delta_{k,k}^{(t+1)})^2}}, \quad (6.2)$$

$$\eta_{K_k^{(t+1)},k,k} = \left( 1 + \frac{\mu_{k+1}^{(t+1)}}{\delta_{k,k}^{(t+1)}} \right) \eta_{k^{(t+1)}} \leq \left( 1 + \frac{1}{\lambda_{k+1} (\delta_{k,k}^{(t+1)})} \right) \sqrt{\frac{1}{\lambda_{n_c+1}}}. \quad (6.3)$$

Then from (4.3), (6.2) and (6.3), the convergence rate of Algorithm 4.1 has the following estimate

$$\left( \sum_{i=1}^{k} \| u_i - E_{m,k}^{(t+1)} u_i \|_A^2 \right)^{1/2} \leq \sqrt{1 + \frac{1}{\lambda_{k+1} \lambda_{n_c+1} (\delta_{k,k}^{(t+1)})^2}} \sqrt{\frac{\lambda_{k+1}}{\lambda_{k+1}}} \left( 1 + \frac{1}{\lambda_{k+1} (\delta_{k,k}^{(t+1)})} \right) \sqrt{\frac{\lambda_{k+1}}{\lambda_{n_c+1}}} \left( \sum_{i=1}^{k} \| u_i - E_{m,k}^{(t)} u_i \|_A^2 \right)^{1/2}. \quad (6.4)$$

From (6.4), the convergence speed can be improved from $\sqrt{\lambda_k/\lambda_{k+1}}$ if $\lambda_k^{(t+1)} < \lambda_{n_c+1}$ which only need $n_c + 1 > k$ and $\lambda_k^{(t+1)}$ has a coarse accuracy.

If the algebraic eigenvalue problem (2.1) is produced by the discretization of the partial differential operator eigenvalue problem (5.1) with the finite element method, the Weyl's law [21, 30] tells us $\lambda_j$ has the following asymptotic estimate

$$\lambda_j \approx \left( \frac{j}{|\Omega|} \right)^{2/d}, \quad \forall j \in \mathbb{N},$$

which leads to the following estimate

$$\left( \sum_{i=1}^{k} \| u_i - E_{m,k}^{(t+1)} u_i \|_A^2 \right)^{1/2} \leq C \left( \frac{\lambda_{k+1}}{\lambda_{k+1}} \left( \frac{k}{n_c+1} \right)^{1/d} \left( \sum_{i=1}^{k} \| u_i - E_{m,k}^{(t)} u_i \|_A^2 \right)^{1/2} \right), \quad (6.5)$$

where $d$ is the dimension of computing domain $\Omega$ and $|\Omega|$ denotes the volume of $\Omega$. The estimate (6.5) means that we can improve the convergence rate if $k < n_c + 1$, i.e., the dimension of the subspace $\mathcal{K}$ is larger than the number of desired eigenvalues.
Since the eigenvectors of the matrix $A$ are more expensive to compute, the practical value of above estimates is limited. But they provide a useful guidance to design practical AMG method for eigenvalue problems. From (4.3) and (6.1), the first criterion for constructing the subspace $\mathcal{K}$ is that it can approximate the eigenvectors corresponding to small eigenvalues [41]. In order to reduce the computation, the second criterion is that we can use the sparse representation of the subspace $\mathcal{K}$. Fortunately, a suitable coarse space of the AMG method for the linear equations satisfies these two criterions. Thus, we can use the usual coarsening scheme to produce the low dimensional subspace $\mathcal{K}$ for Algorithm [1] which can be called AMG method for eigenvalue problems. For more information, please refer to [17].

7 Concluding remarks

In this paper, we give the energy error estimate of the subspace projection method for eigenvalue problems. Furthermore, the relation between error estimates in $L^2$-norm and energy norm is also provided. Based on the energy error estimate and the relation, a new type of inverse power method based on the subspace projection method is proposed and the convergence analysis is also presented. Then we discuss the geometric and algebraic multigrid methods for eigenvalue problems based on the derived convergence result for the proposed inverse power method on the special subspace.

These analysis and discussion give us a new understanding of the subspace projection method and provide a new idea to design the multigrid method for eigenvalue problems. We would like to point out that the most important aim of this paper is to present the idea and understanding of the application of the coarse subspace, which can be produced by the coarse mesh in GMG and coarsening technique in AMG, to eigenvalue problems. Of course, the idea or tool here can be coupled with other techniques such as shift and inverse, polynomial filtering, restarting (cf. [4, 30]). These will be investigated in our future work.

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