Abstract

We study the problem of sparsity constrained $M$-estimation with arbitrary corruptions to both explanatory and response variables in the high-dimensional regime, where the number of variables $d$ is larger than the sample size $n$. Our main contribution is a highly efficient gradient-based optimization algorithm that we call Trimmed Hard Thresholding – a robust variant of Iterative Hard Thresholding (IHT) by using trimmed mean in gradient computations. Our algorithm can deal with a wide class of sparsity constrained $M$-estimation problems, and we can tolerate a nearly dimension independent fraction of arbitrarily corrupted samples. More specifically, when the corrupted fraction satisfies $\epsilon \lesssim 1/\left(\sqrt{k \log(nd)}\right)$, where $k$ is the sparsity of the parameter, we obtain accurate estimation and model selection guarantees with optimal sample complexity. Furthermore, we extend our algorithm to sparse Gaussian graphical model (precision matrix) estimation via a neighborhood selection approach. We demonstrate the effectiveness of robust estimation in sparse linear, logistic regression, and sparse precision matrix estimation on synthetic and real-world US equities data.

1 Introduction

Statistical estimation with heavy tailed outliers or even arbitrary corruptions has long been a focus in robust statistics [Box53, Tuk75, Hub11, HRRS11]. In many practical applications, such as gene expression analysis [LW01] and financial forecasting [dP18], it is important to control the impact of outliers in the dataset in order to obtain meaningful results.

To introduce our robust $M$-estimation procedure, we first recap $M$-estimation in the setting with no outliers. $M$-estimation is a standard technique for statistical estimation [vdV00], such as linear regression, logistic regression, and more generally maximum likelihood estimation. Moreover, empirical risk minimization (ERM) [Vap13, DGL13] is the optimal algorithm for many learning problems. These have been extended successfully to the high dimensional setting with sparsity (or other low-dimensional structure), e.g., using Lasso [Tib96, BvdG11, HTW15, Wai19].

Huber’s seminal work [Hub64] proposed $M$-estimation methods for robust regression, where a fraction of response variables are outliers. For these types of robust $M$-estimators, they replace the classical least squared risk minimization objective with a robust counterpart (e.g., Huber loss). However, when there are outliers in the explanatory variables (covariates), there is little advantage over the square loss.

Sparse estimation is also important for estimating Gaussian graphical model structures in the high dimensional setting. It is well known that the conditional independence relationships of Gaussian random variables correspond to the sparsity pattern of its precision (inverse covariance) matrix [KFB09, WJ08]. A well known work [MB06] developed a sparse-regression based neighborhood selection approach for recovering this sparsity pattern by regressing each variable against its neighbors using Lasso. As we discuss below, a key aspect of our contributions is that (unlike previous high
dimensional robustness results) our approach is flexible enough to apply to robust sparse graphical model estimation.

In this paper, we study the problem of sparsity constrained $M$-estimation with arbitrary corruptions to both explanatory and response variables in the high-dimensional regime ($n \ll d$). To model the corruptions, we assume that the adversary replaces an arbitrary $\epsilon$-fraction of the authentic samples with arbitrary values (Definition 2.1). This is stronger than Huber’s contamination model [Hub64], which only allows the adversary to independently corrupt each sample with some probability.

1.1 Related works

Robust sparse regression. Earlier works in robustness of sparse regression problems consider heavy tailed distributions or arbitrary corruptions only in the response variables [Li13, BJK15, BJK17, Loh17, KP18]. Yet these algorithms cannot be trivially extended to the setting with corrupted explanatory variables, which is the main focus of this work.

Another line of recent research [ACG13, VMX17, YLA18, SS18] focus on alternating minimization approaches which extend Least Trimmed Squares [Rou84]. However, these methods only have local convergence guarantees, and cannot handle arbitrary corruptions.

[CCM13] was one of the first papers to provide guarantees for sparse regression with arbitrary outliers in both response and explanatory variables. Those results are specific to sparse regression, however, and cannot be trivially extended to general $M$-estimation problems. Moreover, even for linear regression with covariance matrix $\Sigma = I_d$, the statistical rates are not minimax optimal. [Gao17] optimizes Tukey depth [Tuk75, CGR18] for robust sparse regression under the Huber $\epsilon$-contamination model, and their algorithm is minimax optimal and can handle a constant fraction of outliers ($\epsilon = \text{const.}$). However, it is intractable to compute the Tukey depth [JP78]. Recent results [BDLS17, LSLC18] leverage robust sparse mean estimation in robust sparse regression. Their algorithms are computationally tractable, and can tolerate $\epsilon = \text{const.}$, but they require restrictive assumptions on the covariance matrix ($\Sigma = I_d$), which, in particular, precludes their use in applications such as graphical model estimation.

Robust $M$-estimation via robust gradient descent. Works in [CSX17, HI17] and later [YCRB18a] first leveraged the idea of using robust mean estimation in each step of gradient descent, using a subroutine such as geometric median, and applied this to defend against Byzantine gradient attacks. A similar approach using more sophisticated robust mean estimation methods was later proposed in [PSBR18, DKK+18, YCRB18b, SX18] for robust gradient descent. These methods all focused on low dimensional robust $M$-estimation. Work in [LSLC18] extended the approach to the high-dimensional setting (though is limited to $\Sigma = I_d$).

Even though the corrupted fraction $\epsilon$ can be independent of the ambient dimension $d$ by using sophisticated robust mean estimation algorithms [DKK+16, LRV16, SCV17], or the sum-of-squares framework [KKM18], these algorithms (except [LSLC18]) are not applicable to the high dimensional setting ($n \ll d$), as they require at least $\Omega(d)$ samples.

Robust estimation of graphical models. A line of research using a robustified covariance matrix to substitute for the sample covariance matrix in Gaussian graphical models [LHY+12b, WG17, LT18] leverages GLasso [FHT08] or CLIME [CLL11] to estimate the sparse precision matrix. These robust methods are restricted to Gaussian graphical model estimation, and their techniques cannot be generalized to other $M$-estimation problems.

1.2 Main contributions

- We propose Trimmed Hard Thresholding (Algorithm 1) for sparsity constrained $M$-estimation with arbitrary corruptions in both explanatory and response variables. This algorithm has global
linear convergence rate, and is computationally efficient – we incur little overhead compared with vanilla gradient descent. This is in particular much faster than algorithms relying on sparse PCA relaxations as subroutines ([BDLS17, LSLC18]).

• With corruptions in both response and explanatory variables, we provide theoretical guarantees on statistical estimation in sparse linear and logistic regression. Our algorithm has minimax-optimal statistical error, and tolerates $O(1/(\sqrt{K}\log(nd)))$-fraction of outliers. This fraction is nearly independent of the $d$, which is important in the high dimension regime.

• We extend Trimmed Hard Thresholding to neighborhood selection [MB06] for estimating Gaussian graphical models. We provide sparsity recovery guarantees for model selection in sparse precision matrix estimation, and this result shares similar robustness guarantees with sparse linear regression.

• We demonstrate the effectiveness of Trimmed Hard Thresholding on both synthetic data and (unmodified) real data. We also empirically show the performance of our algorithm under model misspecification.

Notations. We denote the Hard Thresholding operator of sparsity $k'$ by $P_{k'}$, and denote the Euclidean projection onto the $\ell_2$ ball $B$ by $\Pi_B$. We use $E_i \in u_S$ to denote the expectation operator obtained by the uniform distribution over all samples $\{i \in S\}$.

2 Problem formulation

We now formally define the corruption model and the sparsity constrained $M$-estimation. We first introduce the corruption model described above:

Definition 2.1 ($\epsilon$-corrupted samples). Let $\{z_i, i \in G\}$ be i.i.d. observations with distribution $P$. We say that a collection of samples $\{z_i, i \in S\}$ is $\epsilon$-corrupted if an adversary chooses an arbitrary $\epsilon$-fraction of the samples in $G$ and modifies them with arbitrary values.

This corruption model allows corruptions in both explanatory and response variables in regression problems where we observe $z_i = (y_i, x_i)$. Definition 2.1 also allows the adversary to select an $\epsilon$-fraction of samples to delete and corrupt, hence it is stronger than Huber’s $\epsilon$-contamination model [Hub64], where the corrupted samples are only added independently with probability $\epsilon$.

Next, we introduce a classical sparsity constrained $M$-estimation which minimizes the empirical risk only on i.i.d. observations in the set $G$. Suppose that we are interested in estimating some parameter $\beta$ of $P$. Let $\ell : \mathbb{R}^d \times \mathcal{Z} \to \mathbb{R}$ be a convex and differentiable loss function. Our target is the unknown population minimizer $\beta^* = \arg \min_{\beta \in \mathbb{R}^d} E_{z_i \sim P} \ell_i(\beta; z_i)$. Note that $\beta^*$’s definition allows model misspecification. And we solve a sparsity constrained ERM to estimate $\beta^*$

\[
\min_{\beta \in B} f(\beta) = E_{\beta \in u_G} \ell_i(\beta; z_i), \quad \text{s.t. } ||\beta||_0 \leq k,
\]

where, $f(\cdot)$ is defined as the empirical average on $G$, and $B$ is the $\ell_2$ ball including $\beta^*$.

For example, consider linear regression on samples $z_i = (y_i, x_i) \in \mathbb{R} \times \mathbb{R}^d$, with the squared loss function eq. (2a). For logistic regression, we require $Y = \{-1, +1\}$ and use the logistic loss eq. (2b).

\[
\ell_i(\beta; z_i) = \begin{cases} (y_i - x_i^\top \beta)^2, \\
\log(1 + \exp(-y_i x_i^\top \beta)).
\end{cases}
\]

1While our results focus on the squared and logistic loss, our experiments indicate that our approach also works for other loss functions in high dimensional robust $M$-estimation. We also demonstrate experimental results for other $M$-estimators (e.g., Huber loss [Hub64, Loh17]).
To solve eq. (1), under restricted strong convexity (RSC) and restricted smoothness (RSM) conditions (Section 4), a well known result [NRWY12] shows statistical results based on a convex relaxation from $\|\beta\|_0$ to $\|\beta\|_1$. From an optimization viewpoint, existing results reveal that gradient descent algorithms equipped with soft-thresholding [ANW12] or hard-thresholding [BD09, JTK14, SL17, YLZ18, LB18] have linear convergence rate, and achieve known minimax lower bounds in statistical estimation [RWHY11, ZWJ14].

3 Robust sparse estimation via Trimmed Hard Thresholding

Given $\epsilon$-corrupted set $S$ (Definition 2.1), directly running ERM on the entire input dataset with corruptions
\[
\min_{\beta \in \mathbb{R}^d} \mathbb{E}_{i \in \epsilon S} \ell_i(\beta; z_i), \quad \text{s.t.} \quad \|\beta\|_0 \leq k,
\] (3)
can be arbitrarily corrupted. In this case, we are interested in estimating the unknown parameter $\beta^*$ from $\epsilon$-corrupted samples $S$. Classical robust statistics seeks finding a robust counterpart to the loss function in eq. (3), yet this approach cannot deal with the contamination model (Definition 2.1), where $(y_i, x_i)$ can both be arbitrarily corrupted. To address this challenge, recent works use robust mean estimation in each step of gradient step to make eq. (3) robust, even with an $\epsilon$-fraction of $\{z_i, i \in S\}$ being arbitrary outliers.

3.1 Robust gradient estimators

Here, we use a robust mean estimator in each step of gradient descent. We robustify gradient descent by a robust mean estimation over each sample’s gradient $\nabla \ell_i(\beta; z_i)$. Although different robust mean estimators can be used, we use $\hat{G}(\beta)$ to denote the robust gradient estimate at $\beta$ (shorthand as $\hat{G}$ without abuse of notation). We write $G$ as the gradient on the good samples $\mathcal{G}$, and $G_{\text{pop}}$ as the population gradient
\[
G = \mathbb{E}_{i \in \mathcal{G}} \nabla \ell_i(\beta; z_i), \quad \text{and} \quad G_{\text{pop}} = \mathbb{E}_{i \sim P} \nabla \ell_i(\beta; z_i). \quad (4)
\]
To guarantee that the final output $\hat{\beta}$ of robust gradient descent can recover $\beta^*$ with $\|\hat{\beta} - \beta^*\|_2 = O(\delta_2)$, one way is to use a robust mean estimator on gradients in each iteration to estimate the population gradient with $\|\hat{G} - G_{\text{pop}}\|_2 = O(\delta_1 \|\beta - \beta^*\|_2 + \delta_2)$, where positive quantities $\delta_1$ and $\delta_2$ are independent of gradient descent iterations. Note that different robust mean estimators have different $\delta_1$ and $\delta_2$ for different statistical models, and they are functions of $\epsilon$.

In the high dimensional setting, we would require a high dimensional robust mean estimator in each gradient descent step. Previous methods such as [CSX17, HI17, PSBR18, DKK+18, YCRB18a, YCRB18b, SX18] cannot be used because they all require $n = \Omega(d)$ to bound $\|\hat{G} - G_{\text{pop}}\|_2$. A recent work [LSLC18] on robust sparse linear regression uses a robust sparse mean estimator [BDLS17] to guarantee $\|\hat{G} - G_{\text{pop}}\|_2 = O(\delta_1 \|\beta - \beta^*\|_2 + \delta_2)$ with sample complexity $\Omega(k^2 \log(d))$. However, their algorithm can only deal with sparse linear regression under the restricted assumption $\Sigma = I_d$, and cannot be extended to our general $M$-estimation problems. Our algorithm provides a robust gradient estimate during each iteration in IHT, thus can be used for general sparse $M$-estimation problems.

3.2 Trimmed Hard Thresholding

In our algorithm, we use a dimensional trimmed gradient estimate $\hat{G}$ for $G$ – the gradient defined on the original authentic data set $\mathcal{G}$ (eq. (4)). We show that hard thresholding combined with dimensional trimmed gradient estimator with $\|\hat{G} - G\|_\infty$ bound (instead of $\|\hat{G} - G_{\text{pop}}\|_2$) is the key ingredient for the robustness in sparsity constrained statistical estimation.

Let us define the dimensional trimmed gradient estimator:
in IHT is at most perturbation due to outliers $d \epsilon$ contamination fraction $d$

We prove that the perturbation eq. (6) is nearly independent of dimension

by taking a hard thresholding $\hat{\beta}$ (Definition 3.1) to obtain the robust gradient estimate

of corruptions it can tolerate is less than $O(\alpha)$ optimal (in fact trivial) result for robust gradient descent with trimmed gradients, namely, the fraction $2^{-k' n}$-trimmed gradient estimator (Definition 3.1) in linear and logistic regression.

Algorithm 1 Trimmed Hard Thresholding

| Line | Description |
|------|-------------|
| 1    | **Input:** Data samples $\{y_i, x_i\}_{i=1}^N$. |
| 2    | **Output:** The estimation $\hat{\beta}$. |
| 3    | **Parameters:** Hard thresholding parameter $k' = 4\rho^2 k$. |
| 4    | Split samples into $T$ subsets each of size $n$. |
| 5    | Initialize with $\beta^0 = 0_d$. |
| 6    | **for** $t = 0$ to $T - 1$, **do** |
| 7    | At current $\beta^t$, calculate all gradients for current $n$ samples: $g_i^t = \nabla \ell_i(\beta^t)$, $i \in [n]$. |
| 8    | For $(g_i^t)_{i=1}^n$, we use a dimensional $\alpha$-trimmed gradient estimator (Definition 3.1) to get $\hat{G}^t$. |
| 9    | Update the parameter: $\beta^{t+1} = P_{\rho}(\beta^t - \eta \hat{G}^t)$. |
| 10   | Projection: $\beta^{t+1} = \Pi_B(\beta^{t+1})$. |
| 11   | **end for** |
| 12   | Output the estimation $\hat{\beta} = \beta^T$. |

Definition 3.1 (Dimensional $\alpha$-trimmed gradient estimator). Given a set of $\epsilon$-corrupted samples of stochastic gradients $\{\nabla \ell_i(\beta; z_i) \in \mathbb{R}^d, i \in S\}$, for each dimension $j \in [d]$, the dimensional $\alpha$-trimmed gradient estimator removes the largest and smallest $\alpha$ fraction of elements in $\{[\nabla \ell_i(\beta; z_i)]_j \in \mathbb{R}, i \in S\}$, and calculates the mean of the remaining terms. We choose $\alpha = c_0 \epsilon$ for some constant $c_0 \geq 1$, and require $\alpha \leq 1/2 - c_1$ for a small constant $c_1 > 0$.

We show in Proposition 3.1 below, that this dimensional trimmed gradient estimator can guarantee $\|\hat{G} - G\|_\infty = O(\delta_1\|\beta - \beta^\star\|_2 + \delta_2)$ with high probability, in a gradient type optimization algorithm. However, a naive application of trimmed gradient estimator only gives the bound $\|\hat{G} - G\|_2 \leq \|G - G\|_\infty \sqrt{d}$, where perturbation due to outliers scales with the ambient dimension $d$. This gives a sub-optimal (in fact trivial) result for robust gradient descent with trimmed gradients, namely, the fraction of corruptions it can tolerate is less than $O(n/\sqrt{d})$, which tends to 0 in the high dimensional regime.

To address this issue, we propose Trimmed Hard Thresholding (Algorithm 1), which uses hard thresholding after each trimmed gradient update\(^2\). In line 8, we use the trimmed gradient estimator (Definition 3.1) to obtain the robust gradient estimate $\hat{G}^t$. In line 9, we update the parameter by taking a hard thresholding $\beta^{t+1} = P_{\rho}(\beta^t - \eta \hat{G}^t)$. The hard thresholding operator $P_{\rho}(\cdot)$, can be implemented by selecting the $k'$ largest elements in magnitude. Here the hyper-parameter $k'$ is proportional to $k$ (eq. (7)).

A key observation in line 9 is that, in each step of IHT, the iterate $\beta^t$ is sparse, and this enforces the perturbation from outliers to only depend on IHT’s sparsity $k'$ instead of the ambient dimension $d$. Based on a careful analysis of hard thresholding operator in each iteration, we show that the perturbation due to outliers in IHT is at most

\[ \sqrt{k' + k} \|G - G\|_\infty. \]  \hspace{1cm} (6)

We prove that the perturbation eq. (6) is nearly independent of dimension $d$ (eq. (8)). Hence the contamination fraction $\epsilon$ we can tolerate is nearly independent of the ambient dimension. This follows from trimmed gradient estimator guarantees on $\|G - G\|_\infty$ (Proposition 3.1 and 3.2).

3.3 Gradient estimation guarantees

In this section, we show gradient estimation guarantees using the trimmed gradient estimator (Definition 3.1) in linear and logistic regression.\(^2\)Our theory requires splitting samples across different iterations to maintain independence between iterations. We believe this is an artifact of the analysis, and do not use this our experiments. \cite{BWY17, PSBR18} use a similar approach for theoretical analysis.
Suppose that we observe $c\alpha$ samples $\mathbf{z}_i = (y_i, \mathbf{x}_i)$ are drawn from a standard sub-Gaussian design linear model $P$: $y_i = \mathbf{x}_i^T \mathbf{\beta}^* + \xi_i$, with $\mathbf{\beta}^* \in \mathbb{R}^d$ being $k$-sparse. We assume that $\mathbf{x}$‘s are i.i.d. samples from a sub-Gaussian distribution with normalized covariance matrix $\mathbf{\Sigma}$, where $\mathbf{\Sigma}_{jj} \leq 1$ for all $j$, and the stochastic sub-Gaussian noise $\xi$ has mean 0 and variance $\sigma^2$.

Model 3.2 (Sparse logistic regression). Under the contamination model Definition 2.1, authentic samples $\mathbf{z}_i = (y_i, \mathbf{x}_i)$ are drawn from a binary classification model $P$, where the binary label $y_i \in \{-1, +1\}$ follows the conditional probability distribution $\Pr(y_i|\mathbf{x}_i) = 1/(1 + \exp(-y_i\mathbf{x}_i^T \mathbf{\beta}^*))$, with $\mathbf{\beta}^* \in \mathbb{R}^d$ being $k$-sparse. We assume that $\mathbf{x}$‘s are i.i.d. samples from a sub-Gaussian distribution with normalized covariance matrix $\mathbf{\Sigma}$, where $\mathbf{\Sigma}_{jj} \leq 1$ for all $j$.

In both linear and logistic regression, we provide detailed analysis of $\|\hat{\mathbf{G}} - \mathbf{G}\|_\infty$. Compared to previous analysis of trimmed mean estimation [CCM13, YCRB18a], our results hold even if an adversary deletes an $\epsilon$-fraction of samples from $\mathcal{G}$ as defined in Definition 2.1.

**Proposition 3.1.** Suppose that we observe $n = \Omega(\log d)$ $\epsilon$-corrupted (Definition 2.1) samples from sparse linear regression model (Model 3.1). The dimensional $\alpha$-trimmed gradient estimator with $\alpha = c_0 \epsilon$ can guarantees that

$$\|\hat{\mathbf{G}} - \mathbf{G}\|_\infty = O\left(\sqrt{\|\Delta\|^2 + \sigma^2} (\epsilon \log(nd) + \sqrt{\frac{\log d}{n}})\right),$$

with probability at least $1 - d^{-3}$, where $\Delta := \mathbf{\beta} - \mathbf{\beta}^*$, and $c_0 \geq 1$ is a universal constant.

**Proposition 3.2.** Suppose that we observe $n = \Omega(\log d)$ $\epsilon$-corrupted (Definition 2.1) samples from the sparse logistic regression model (Model 3.2). The dimensional $\alpha$-trimmed gradient estimator with $\alpha = c_0 \epsilon$ guarantees that

$$\|\hat{\mathbf{G}} - \mathbf{G}\|_\infty = O\left(\epsilon \log(nd) + \sqrt{\frac{\log d}{n}}\right),$$

with probability at least $1 - d^{-3}$, where $c_0 \geq 1$ is a universal constant.

The proofs are given in Appendix A, where we prove guarantees on recovering the true gradient from $\epsilon$-corrupted sub-exponential gradient samples using the trimmed mean estimator.\(^3\)

**Model misspecification.** Our algorithm is not limited to true linear models, although results are established under linear assumptions. In Section 6, we show that, in a nonlinear model, results of Algorithm 1 on $\epsilon$-corrupted input is comparable to sparse linear regression on uncorrupted input.

## 4 Global linear convergence and statistical guarantees

In this section, we provide statistical guarantees under the models in Section 3.3. Using Proposition 3.1 and Proposition 3.2, we prove Algorithm 1’s global convergence, and study its statistical guarantees. Recall that $f(\cdot)$ is the empirical risk defined on i.i.d. samples $\mathcal{G}$, our analysis depends on RSC/RSM conditions of the curvature of $f(\cdot)$ [NRWY12].

**Definition 4.1 (RSC/RSM).** A differentiable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies $(\mu_{\alpha}, k')$-restricted strong convexity (RSC) with parameter $\mu_{\alpha}$, and $(\mu_{\Psi}, k')$-restricted smoothness (RSM) with parameter $\mu_{\Psi}$ at sparsity level $k'$, if

$$f(\mathbf{\beta}_1) \geq f(\mathbf{\beta}_0) + \langle \mathbf{G}(\mathbf{\beta}_0), \mathbf{\beta}_1 - \mathbf{\beta}_0 \rangle + \frac{\mu_{\alpha}}{2} \|\mathbf{\beta}_1 - \mathbf{\beta}_0\|_2^2,$$

$$f(\mathbf{\beta}_1) \leq f(\mathbf{\beta}_0) + \langle \mathbf{G}(\mathbf{\beta}_0), \mathbf{\beta}_1 - \mathbf{\beta}_0 \rangle + \frac{\mu_{\Psi}}{2} \|\mathbf{\beta}_1 - \mathbf{\beta}_0\|_2^2.$$

\(^3\)The sub-exponential assumption is necessary. In sparse linear regression, the gradient’s distribution is indeed sub-exponential.
hold respectively, for all \( \beta_0, \beta_1 \in \mathbb{R}^d \) with \( \|\beta_0\|_0 \leq k' \) and \( \|\beta_1\|_0 \leq k' \). We define the condition number \( \rho = \mu_L / \mu_\alpha \geq 1 \). And we set the hyper-parameter \( k' \) as

\[
k' = 4\rho^2k. \tag{7}
\]

### 4.1 Theoretical Results

For sparse linear and logistic regression, we present global convergence and statistical guarantees in Theorem 4.1 and Theorem 4.2, respectively. The proofs are collected in Appendix B.

**Sparse linear regression.** By [RWY10, ANW12], when \( n = \Omega(\rho^4k\log d) \), we RSC/RSM condition is satisfied with \( \mu_\alpha = \Omega(\sigma_{\min}(\Sigma)) \), and \( \mu_L = O(\sigma_{\max}(\Sigma)) \).

**Theorem 4.1.** Suppose that we observe \( N = nT \epsilon \)-corrupted samples (Definition 2.1) from the sparse linear regression model (Model 3.1). Under the condition \( n = \Omega(\rho^4k\log d) \), and \( \epsilon = O\left(\frac{1}{\rho^2 \sqrt{k\log(nd)}}\right) \), Algorithm 1 with \( \eta = 1/\mu_L \) outputs \( \hat{\beta} \) satisfying

\[
\|\hat{\beta} - \beta^*\|_2 = O\left(\rho^2\sigma\left(\epsilon\sqrt{k\log(nd)} + \sqrt{\frac{k\log d}{n}}\right)\right),
\]

with probability at least \( 1 - d^{-2} \), where we set \( T = O\left(\rho \log\left(\frac{\|\beta^*\|_2}{\|\hat{\beta} - \beta^*\|_2}\right)\right) \).

**Time complexity.** Algorithm 1 has a global linear convergence rate. In each iteration, we only use \( O(nd\log n) \) operations complexity to calculate trimmed mean. We incur logarithmic overhead compared to normal gradient descent [Bub15].

**Statistical accuracy and robustness.** Compared with [CCM13, BDLS17], our statistical error rate is minimax optimal [RWY11, ZWJ14], and has no dependencies on \( \|\beta^*\|_2 \). Furthermore, the upper bound on \( \epsilon \) is nearly independent of the ambient dimension \( d \), which guarantees Algorithm 1’s robustness in high dimensions.

**Sparse logistic regression.** Because in Algorithm 1, we project each iterate onto a \( \ell_2 \) ball independent of \( n \) and \( d \) (line 10), we have strictly positive \( \mu_\alpha \). It is well known that, when \( n \geq C_1k\log d \) for some problem dependent constant \( C_1 \), the RSC and RSM parameters \( \mu_\alpha, \mu_L \) are strictly positive constants depending on the underlying distribution [NRWY12, LPR15].

**Theorem 4.2.** Suppose that we observe \( N = nT \epsilon \)-corrupted samples (Definition 2.1) the sparse logistic regression model (Model 3.2). Algorithm 1 with \( \eta = 1/\mu_L \) outputs \( \hat{\beta} \), such that

\[
\|\hat{\beta} - \beta^*\|_2 = O\left(\rho^2\left(\epsilon\sqrt{k\log(nd)} + \sqrt{\frac{k\log d}{n}}\right)\right),
\]

with probability at least \( 1 - d^{-2} \), when we set \( T = O\left(\rho \log\left(\frac{\|\beta^*\|_2}{\|\hat{\beta} - \beta^*\|_2}\right)\right) \).

**Statistical accuracy and robustness.** Under the sparse Gaussian linear discriminant analysis (LDA) model, which is a special case of Model 3.2, Algorithm 1 achieves the statistical minimax rate [LPR15, LYCR17].
4.2 Proof outline

Here, we give a proof outline for sparse linear regression (Theorem 4.1), and the technique for Theorem 4.2 is similar. The detailed proofs are given in Appendix B.

Since \( \beta^{t-1} \) and \( \overline{\beta}^t \) are enforced to be \( k' \)-sparse, and \( f \) (the empirical risk defined on \( G \)) satisfies \((\mu, k')\)-RSC and \((\mu_L, k')\)-RSM. Combining the two inequalities, we obtain

\[
f(\overline{\beta}^t) - f(\beta^*) \leq (G(\beta^{t-1}), \overline{\beta}^t - \beta^*) - \frac{\mu_L}{2} \|\beta^* - \beta^{t-1}\|_2^2 + \frac{\mu_L}{2} \|\overline{\beta}^t - \beta^{t-1}\|_2^2.
\]

Expanding the term \((\diamondsuit)\), we have

\[
(\diamondsuit) = \frac{\mu_L}{2} \|\beta^{t-1} - \beta^*\|_2^2 - \frac{\mu_L}{2} \|\overline{\beta}^t - \beta^{t-1}\|_2^2
\]

\[-\left( G(\beta^{t-1}), \overline{\beta}^t - \beta^* \right) + \mu_L \left( \left( \beta^{t-1} - \eta \hat{G}(\beta^{t-1}) \right) - \overline{\beta}^t, \beta^* - \overline{\beta}^t \right) \]

Recall that \( \overline{\beta}^t \) is directly obtained from hard thresholding, and because \( k' = 4\rho^2k \), by applying Lemma 1 of [LB18] in the last term (more details in Appendix B), we have

\[
\left( \left( \beta^{t-1} - \eta \hat{G}(\beta^{t-1}) \right) - \overline{\beta}^t, \beta^* - \overline{\beta}^t \right) \leq \frac{1}{4\rho} \|\overline{\beta}^t - \beta^*\|_2^2.
\]

Putting together the pieces, and using \( \Delta^t = \beta^t - \beta^* \), \( \overline{\Delta}^t = \overline{\beta}^t - \beta^* \), we obtain the key inequality

\[
f(\overline{\beta}^t) - f(\beta^*) \leq \frac{\mu_L}{2} \left( \frac{\rho}{\rho} \|\Delta^{t-1}\|_2^2 - \frac{2\rho}{2\rho} \|\overline{\Delta}^t\|_2^2 \right) + \left( G(\beta^{t-1}) - \hat{G}(\beta^{t-1}) \right) \cdot \overline{\Delta}^t.
\]

Next, we use the crucial fact that the perturbation due to outliers (the inner product term \((\bigstar)\)) is nearly independent of the ambient dimension \( d \), because hard thresholding enforces \( \overline{\Delta}^t \) to be \( k + k' \) sparse, resulting in the bound eq. (6). And this implies

\[
(\bigstar) \leq \sqrt{k' + k} \left\| G(\beta^{t-1}) - \hat{G}(\beta^{t-1}) \right\|_\infty \|\overline{\Delta}^t\|_2.
\]

Applying Proposition 3.1, and using convexity to obtain a lower bound on \( f(\overline{\beta}^t) - f(\beta^*) \), we can solve a quadratic inequality to obtain the following recursion

\[
\|\overline{\Delta}^t\|_2 \leq \left( 1 - \frac{1}{8\rho} \right) \|\Delta^{t-1}\|_2 + \frac{4\Gamma \sigma}{\mu_L},
\]

which holds with high probability. Here \( \Gamma = O\left( \sqrt{k' + k} \left( \epsilon \log(nd) + \sqrt{\frac{\log d}{n}} \right) \right) \). Since Euclidean projection guarantees \( \|\Delta^t\|_2 \leq \|\overline{\Delta}^t\|_2 \) [Bub15], we have established the global linear convergence rate for Algorithm 1. Plugging in \( \Gamma \), we can obtain the following statistical guarantee

\[
\|\hat{\beta} - \beta^*\|_2 = O\left( \rho \sigma \left( \epsilon \sqrt{k \log(nd)} + \sqrt{\frac{k \log d}{n}} \right) \right).
\]

5 Sparsity recovery and its application in Gaussian graphical model estimation

In this section, we demonstrate the sparsity recovery performance of Algorithm 1, which is of great importance in sparse \( M \)-estimation and graphical model learning [MB06, Wai09, RWL10, RWRY11, BvdG11, HTW15].
5.1 Sparsity recovery guarantees

We use \( \text{supp}(v, k) \) to denote top \( k \) indexes of \( v \) with the largest magnitude. Let \( v_{\text{min}} \) denote the smallest absolute value of nonzero element of \( v \). To control the false negative rate, Theorem 5.1 shows that under the \( \beta_{\text{min}} \)-condition, \( \text{supp}(\hat{\beta}, k) \) is exactly \( \text{supp}(\beta^*) \). The proofs are given in Appendix C. Sparsity recovery guarantee for sparse logistic regression is similar, and is omitted due to space constraint.

**Theorem 5.1.** Under the same condition as in Theorem 4.1, and a \( \beta_{\text{min}} \)-condition

\[
\beta_{\text{min}}^* = \Omega\left(\rho^2 \sigma \left(\epsilon \sqrt{k \log(nd)} + \sqrt{\frac{k \log d}{n}}\right)\right),
\]

Algorithm 1 guarantees that \( \text{supp}(\hat{\beta}, k) = \text{supp}(\beta^*) \), with probability at least \( 1 - d^{-2} \).

Existing results on sparsity recovery for \( \ell_1 \) regularized estimators \([Wai09, LSRC15]\) do not require the RSC condition Definition 4.1, but instead require an irrepresentability condition, which is stronger. If \( \epsilon \to 0 \), eq. (9) has the same \( \beta_{\text{min}} \)-condition as IHT for sparsity recovery \([YLZ18]\).

5.2 Robust Gaussian graphical model selection

Here, we consider sparse precision matrix estimation for Gaussian graphical models. It is well known that the sparsity pattern of its precision matrix \( \Theta = \Sigma^{-1} \) matches the conditional independence relationships of Gaussian random variables \([KFB09, WJ08]\).

**Model 5.1 (Sparse precision matrix estimation).** Under the contamination model Definition 2.1, authentic samples \( \{x_i\}_{i=1}^m \) are drawn from a multivariate Gaussian distribution \( \mathcal{N}(0, \Sigma) \). We assume that each row of the precision matrix \( \Theta = \Sigma^{-1} \) (excluding diagonal entry) is \( k \)-sparse – each node has at most \( k \) edges.

For the uncorrupted samples drawn from the Gaussian graphical model, the neighborhood selection (NS) algorithm \([MB06]\) solves a convex relaxation of the following sparsity constrained optimization to regress each variable against its neighbors

\[
\hat{\beta}_j = \arg \min_{\beta \in \mathbb{R}^{d-1}} \frac{1}{m} \sum_{i=1}^m (x_{ij} - x_{i(j)}^T \beta)^2, \quad \text{s.t.} \quad \|\beta\|_0 \leq k, \quad \text{for each } j \in [d],
\]

where \( x_{ij} \) denotes the \( j \)-th coordinate of \( x_i \in \mathbb{R}^d \), and \( (j) \) denotes the index set \( \{1, \ldots, j-1, j+1, \ldots, d\} \) (with \( j \) removed). Let \( \theta_{(j)} \in \mathbb{R}^{d-1} \) denote \( \Theta \)'s \( j \)-th column with the diagonal entry removed, and \( \Theta_{j,j} \in \mathbb{R} \) denote the \( j \)-th diagonal element of \( \Theta \). Then, the sparsity pattern of \( \theta_{(j)} \) can be estimated through \( \hat{\beta}_j \). Details on the connection between \( \theta_{(j)} \) and \( \hat{\beta}_j \) are given in Appendix C.

However, given \( \epsilon \)-corrupted samples from the Gaussian graphical model, this procedure will fail \([LHY+12b, WG17]\). To address this issue, we propose Robust NS (Algorithm 2 in Appendix C), which robustifies Neighborhood Selection \([MB06]\) by using Trimmed Hard Thresholding (with loss function eq. (2a)) to robustify eq. (10). Similar to Theorem 5.1, a \( \theta_{\text{min}} \)-condition guarantees consistent edge selection.

**Corollary 5.1.** Under the same condition as in Theorem 4.1, and a \( \theta_{\text{min}} \)-condition for \( \theta_{(j)} \),

\[
\theta_{(j), \text{min}} = \Omega\left(\theta_{\text{min}}^{1/2} \rho^2 \left(\epsilon \sqrt{k \log(nd)} + \sqrt{\frac{k \log d}{n}}\right)\right),
\]

Robust NS (Algorithm 2) achieves consistent edge selection, with probability at least \( 1 - d^{-1} \).
Similar to Theorem 4.1, the fraction $\epsilon$ is nearly independent of dimension $d$, which provides guarantees of Robust NS in high dimensions. Other Gaussian graphical model selection algorithms include GLasso [FHT08], CLIME[CLL11]. Experiments comparing robustified versions of these algorithms are given in Section 6.

6 Experiments

We demonstrate empirical performance of Trimmed Hard Thresholding (Algorithm 1 and Algorithm 2), and the complete experiment setup is in Appendix D.

6.1 Synthetic data – sparse linear models.

Sparse linear regression. In the first experiment, we generate samples from an exact sparse linear regression model (Model 3.1) with a Toeplitz covariance $\Sigma$. Here, the stochastic noise $\xi \sim N(0, \sigma^2)$, and we vary the noise level $\sigma^2$ in different simulations ④.

We fix $\epsilon = 0.1$, and track the parameter error $\|\beta^t - \beta^*\|_2$ in each iteration. Left plot of Figure 1 shows Algorithm 1’s linear convergence, and the error curves flatten out at the final error level. Furthermore, Algorithm 1 can achieve machine precision when $\sigma^2 = 0$, which means exactly recovering of $\beta^*$.

Misspecified model. Here we consider the best sparse linear approximation under model misspecification. We use the same Toeplitz covariates and true parameter $\beta^*$ as before, but $y_i = \sum_{j=1}^{d} x_{ij} \beta^*_j$. Although this is a non-linear function, the best sparse linear approximation can select relevant features because $\beta^*$ is sparse.

For simplicity, we track the function evaluated on all authentic samples $F(\beta) = \sum_{i \in G} (y_i - x_i^T \beta)^2$. In the right plot of Figure 1, we compare the performance of Algorithm 1 under different $\epsilon$ against the oracle curve (IHT only on authentic samples). The right plot has similar convergence under different $\epsilon$, and shows the robustness of Algorithm 1 without assuming an underlying linear model.

④ Beyond sub-Gaussian error distributions, our algorithm naturally extends to other robust loss functions (e.g., Huber loss). In Appendix D, we study the empirical performance of Algorithm 1 with Huber loss where the stochastic noise follows a heavy-tailed Cauchy distribution.
Sparsity regression. For binary classification problem, we generate data using a sparse LDA model. We run Algorithm 1 with logistic loss under different levels of corruption. In the left plot of Figure 2, we observe similar linear convergence as sparse linear regression problem, and this is consistent with Theorem 4.2.

We then compare Algorithm 1 with Trimmed Lasso estimator [YLA18], which uses alternating minimization for sparse logistic regression. Under different dimensions $d$, the right plot of Figure 2 shows classification error evaluated on authentic test set for $\epsilon = 0.1, 0.2$, which demonstrates that Trimmed Hard Thresholding is better than Trimmed Lasso.

6.2 Synthetic data – Gaussian graphical model.

We draw samples from a Gaussian graphical model with a sparse precision matrix using huge [ZLR+12], and add corruptions. We compare Algorithm 2 with other robust graphical model estimators, which include Trimmed GLasso [YLA18], RCLIME [WG17], Skeptic [LHY+12b], and Spearman [LT18]. We fix $\epsilon = 0.1$, and vary $(n, d)$. We show results for different off-diagonal values 0.3 (Low SNR) and 0.6 (High SNR), with High SNR plots in Appendix D. Figure 3 shows model selection performance measured by receiver operating characteristic (ROC) curves. For the whole regularization path, our algorithm (denoted as Robust NS) has a better ROC compared to other algorithms.

In particular, Robust NS outperforms other methods with higher true positive rate when the false positive rate is small. This validates our theory in Corollary 5.1, guaranteeing sparsity recovery when hard thresholding hyper-parameter $k'$ is suitably chosen to match $\beta^*$’s sparsity $k$.

6.3 Real data experiments.

To demonstrate the efficacy of Algorithm 2, we apply it to a US equities dataset [LHY+12a, ZLR+12], which is heavy-tailed and has many outliers [dP18]. The dataset contains 1,257 daily closing prices of 452 stocks (variables).

It is well known that stocks from the same sector tend to be clustered together [Kin66]. Therefore, we use Robust NS (Algorithm 2) to construct an undirected graph among stocks. Graphs estimated by different algorithms are shown in Figure 4. We can see that stocks from the same sector are clustered together, and these clustering centers can be easily identified. We also compare Algorithm 2 to the
baseline NS approach (without no consideration for corruptions or outliers). We can observe that stocks from Information Technology (colored by purple) are much better clustered by Algorithm 2.

Figure 3: ROC curves of different methods on cluster graphs with arbitrary corruptions. The curve Robust NS denotes Algorithm 2, and Oracle NS denotes the neighborhood selection Lasso only on authentic data with side information.

References

[ACG13] Andreas Alfons, Christophe Croux, and Sarah Gelper. Sparse least trimmed squares regression for analyzing high-dimensional large data sets. The Annals of Applied Statistics, pages 226–248, 2013.

[ANW12] Alekh Agarwal, Sahand Negahban, and Martin J. Wainwright. Fast global convergence of gradient methods for high-dimensional statistical recovery. Ann. Statist., 40(5):2452–2482, 10 2012.

[BD09] Thomas Blumensath and Mike E Davies. Iterative hard thresholding for compressed sensing. Applied and computational harmonic analysis, 27(3):265–274, 2009.

[BDLS17] Sivaraman Balakrishnan, Simon S. Du, Jerry Li, and Aarti Singh. Computationally efficient robust sparse estimation in high dimensions. In Proceedings of the 2017 Conference on Learning Theory, 2017.

[BJK15] Kush Bhatia, Prateek Jain, and Purushottam Kar. Robust regression via hard thresholding. In Advances in Neural Information Processing Systems, pages 721–729, 2015.

[BJK17] Kush Bhatia, Prateek Jain, and Purushottam Kar. Consistent robust regression. In Advances in Neural Information Processing Systems, pages 2107–2116, 2017.

[Box53] George EP Box. Non-normality and tests on variances. Biometrika, 40(3/4):318–335, 1953.

[Bub15] Sébastien Bubeck. Convex optimization: Algorithms and complexity. Foundations and Trends® in Machine Learning, 8(3-4):231–357, 2015.

[BvdG11] Peter Bühlmann and Sara van de Geer. Statistics for high-dimensional data: methods, theory and applications. Springer Science & Business Media, 2011.

[BWY17] Sivaraman Balakrishnan, Martin J Wainwright, and Bin Yu. Statistical guarantees for the em algorithm: From population to sample-based analysis. The Annals of Statistics, 45(1):77–120, 2017.

[CCM13] Yudong Chen, Constantine Caramanis, and Shie Mannor. Robust sparse regression under adversarial corruption. In International Conference on Machine Learning, pages 774–782, 2013.
Figure 4: Graph estimated from the S&P 500 stock data by Algorithm 2 and Vanilla NS approach. Variables are colored according to their sectors. In particular, the stocks from sector Information Technology are colored as purple.

[CGR18] Mengjie Chen, Chao Gao, and Zhao Ren. Robust covariance and scatter matrix estimation under hubers contamination model. *Ann. Statist.*, 46(5):1932–1960, 10 2018.

[CLL11] Tony Cai, Weidong Liu, and Xi Luo. A constrained $\ell_1$ minimization approach to sparse precision matrix estimation. *Journal of the American Statistical Association*, 106(494):594–607, 2011.

[CSX17] Yudong Chen, Lili Su, and Jiaming Xu. Distributed statistical machine learning in adversarial settings: Byzantine gradient descent. *Proceedings of the ACM on Measurement and Analysis of Computing Systems*, 1(2):44, 2017.

[DGL13] Luc Devroye, László Györfi, and Gábor Lugosi. *A probabilistic theory of pattern recognition*, volume 31. Springer Science & Business Media, 2013.

[DKK+16] Ilias Diakonikolas, Gautam Kamath, Daniel M Kane, Jerry Li, Ankur Moitra, and Alistair Stewart. Robust estimators in high dimensions without the computational intractability. In *Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on*, pages 655–664. IEEE, 2016.

[DKK+18] Ilias Diakonikolas, Gautam Kamath, Daniel M Kane, Jerry Li, Jacob Steinhardt, and Alistair Stewart. Sever: A robust meta-algorithm for stochastic optimization. *arXiv preprint arXiv:1803.02815*, 2018.

[dP18] M. L. de Prado. *Advances in Financial Machine Learning*. Wiley, 2018.

[FHT08] Jerome Friedman, Trevor Hastie, and Robert Tibshirani. Sparse inverse covariance estimation with the graphical lasso. *Biostatistics*, 9(3):432–441, 2008.

[Gao17] Chao Gao. Robust regression via mutivariate regression depth. *arXiv preprint arXiv:1702.04656*, 2017.

[HI17] Matthew J Holland and Kazushi Ikeda. Efficient learning with robust gradient descent. *arXiv preprint arXiv:1706.00182*, 2017.

[HRRS11] Frank R Hampel, Elvezio M Ronchetti, Peter J Rousseeuw, and Werner A Stahel. *Robust statistics: the approach based on influence functions*, volume 196. John Wiley & Sons, 2011.

[HTW15] Trevor Hastie, Robert Tibshirani, and Martin Wainwright. *Statistical learning with sparsity: the lasso and generalizations*. CRC press, 2015.

[Hub64] Peter J Huber. Robust estimation of a location parameter. *The annals of mathematical statistics*, pages 73–101, 1964.
[Hub11] Peter J Huber. Robust statistics. In *International Encyclopedia of Statistical Science*, pages 1248–1251. Springer, 2011.

[JP78] David Johnson and Franco Preparata. The densest hemisphere problem. *Theoretical Computer Science*, 6(1):93–107, 1978.

[JTK14] Prateek Jain, Ambuj Tewari, and Purushottam Kar. On iterative hard thresholding methods for high-dimensional m-estimation. In *Advances in Neural Information Processing Systems*, pages 685–693, 2014.

[KFB09] Daphne Koller, Nir Friedman, and Francis Bach. *Probabilistic graphical models: principles and techniques*. MIT press, 2009.

[Kin66] Benjamin King. Market and industry factors in stock price behavior. *the Journal of Business*, 39(1):139–190, 1966.

[KKM18] Adam Klivans, Pravesh K. Kothari, and Raghu Meka. Efficient Algorithms for Outlier-Robust Regression. *arXiv preprint arXiv:1803.03241*, 2018.

[KP18] Sushrut Karmalkar and Eric Price. Compressed sensing with adversarial sparse noise via l1 regression. *arXiv preprint arXiv:1809.08055*, 2018.

[LB18] Haoyang Liu and Rina Foygel Barber. Between hard and soft thresholding: optimal iterative thresholding algorithms. *arXiv preprint arXiv:1804.08841*, 2018.

[LHY+12a] Han Liu, Fang Han, Ming Yuan, John Lafferty, and Larry Wasserman. The nonparanormal skeptic. *arXiv preprint arXiv:1206.6488*, 2012.

[LHY+12b] Han Liu, Fang Han, Ming Yuan, John Lafferty, Larry Wasserman, et al. High-dimensional semiparametric gaussian copula graphical models. *The Annals of Statistics*, 40(4):2293–2326, 2012.

[Li13] Xiaodong Li. Compressed sensing and matrix completion with constant proportion of corruptions. *Constructive Approximation*, 37(1):73–99, 2013.

[Loh17] Po-Ling Loh. Statistical consistency and asymptotic normality for high-dimensional robust m-estimators. *The Annals of Statistics*, 45(2):866–896, 2017.

[LPR15] Tianyang Li, Adarsh Prasad, and Pradeep K Ravikumar. Fast classification rates for high-dimensional gaussian generative models. In *Advances in Neural Information Processing Systems*, pages 1054–1062, 2015.

[LRV16] Kevin A Lai, Anup B Rao, and Santosh Vempala. Agnostic estimation of mean and covariance. In *Foundations of Computer Science (FOCS), 2016 IEEE 57th Annual Symposium on*, pages 665–674. IEEE, 2016.

[LSLC18] Liu Liu, Yanyao Shen, Tianyang Li, and Constantine Caramanis. High dimensional robust sparse regression. *arXiv preprint arXiv:1805.11643*, 2018.

[LSRC15] Yen-Huan Li, Jonathan Scarlett, Pradeep Ravikumar, and Volkan Cevher. Sparsistency of l-regularized m-estimators. In *AISTATS*, 2015.

[LT18] Po-Ling Loh and Xin Lu Tan. High-dimensional robust precision matrix estimation: Cellwise corruption under epsilon-contamination. *Electronic Journal of Statistics*, 12(1):1429–1467, 2018.

[LW01] Cheng Li and Wing Hung Wong. Model-based analysis of oligonucleotide arrays: expression index computation and outlier detection. *Proceedings of the National Academy of Sciences*, 98(1):31–36, 2001.

[LYCR17] Tianyang Li, Xinyang Yi, Constantine Caramanis, and Pradeep Ravikumar. Minimax gaussian classification & clustering. In *Artificial Intelligence and Statistics*, pages 1–9, 2017.

[MB06] Nicolai Meinshausen and Peter Bühlmann. High-dimensional graphs and variable selection with the lasso. *The annals of statistics*, 34(3):1436–1462, 2006.

[NRWY12] Sahand N Negahban, Pradeep Ravikumar, Martin J Wainwright, and Bin Yu. A unified framework for high-dimensional analysis of m-estimators with decomposable regularizers. *Statistical Science*, 27(4):538–557, 2012.

[PSBR18] Adarsh Prasad, Arun Sai Suggala, Sivaraman Balakrishnan, and Pradeep Ravikumar. Robust estimation via robust gradient estimation. *arXiv preprint arXiv:1802.06485*, 2018.
[Rou84] Peter J Rousseeuw. Least median of squares regression. *Journal of the American statistical association*, 79(388):871–880, 1984.

[RWL10] Pradeep Ravikumar, Martin J Wainwright, and John D Lafferty. High-dimensional ising model selection using ℓ₁-regularized logistic regression. *The Annals of Statistics*, 38(3):1287–1319, 2010.

[RWRY11] Pradeep Ravikumar, Martin J Wainwright, Garvesh Raskutti, and Bin Yu. High-dimensional covariance estimation by minimizing ℓ₁-penalized log-determinant divergence. *Electronic Journal of Statistics*, 5:935–980, 2011.

[RWY10] Garvesh Raskutti, Martin J Wainwright, and Bin Yu. Restricted eigenvalue properties for correlated gaussian designs. *Journal of Machine Learning Research*, 11(Aug):2241–2259, 2010.

[RWY11] Garvesh Raskutti, Martin J Wainwright, and Bin Yu. Minimax rates of estimation for high-dimensional linear regression over ℓ_q-balls. *IEEE transactions on information theory*, 57(10):6976–6994, 2011.

[SCV17] Jacob Steinhardt, Moses Charikar, and Gregory Valiant. Resilience: A criterion for learning in the presence of arbitrary outliers. *arXiv preprint arXiv:1703.04940*, 2017.

[SL17] Jie Shen and Ping Li. A tight bound of hard thresholding. *The Journal of Machine Learning Research*, 18(1):7650–7691, 2017.

[SS18] Yanyao Shen and Sujay Sanghavi. Iteratively learning from the best. *arXiv preprint arXiv:1810.11874*, 2018.

[SX18] Lili Su and Jiaming Xu. Securing distributed machine learning in high dimensions. *arXiv preprint arXiv:1804.10140*, 2018.

[Tib96] Robert Tibshirani. Regression shrinkage and selection via the lasso. *Journal of the Royal Statistical Society. Series B (Methodological)*, pages 267–288, 1996.

[Tuk75] John W Tukey. Mathematics and the picturing of data. In *Proceedings of the International Congress of Mathematicians, Vancouver, 1975*, volume 2, pages 523–531, 1975.

[Vap13] Vladimir Vapnik. *The nature of statistical learning theory*. Springer Science & Business media, 2013.

[vdV00] Aad van der Vaart. *Asymptotic statistics*. Cambridge University Press, 2000.

[VMX17] Daniel Vainsencher, Shie Mannor, and Huan Xu. Ignoring is a bliss: Learning with large noise through reweighting-minimization. In *Conference on Learning Theory*, pages 1849–1881, 2017.

[Wai09] Martin J Wainwright. Sharp thresholds for high-dimensional and noisy sparsity recovery using ℓ₁-constrained quadratic programming (lasso). *IEEE transactions on information theory*, 55(5):2183–2202, 2009.

[Wai19] Martin Wainwright. *High-dimensional statistics: A non-asymptotic viewpoint*. Cambridge University Press, 2019.

[WG17] Lingxiao Wang and Quanquan Gu. Robust gaussian graphical model estimation with arbitrary corruption. In *International Conference on Machine Learning*, pages 3617–3626, 2017.

[WJ08] Martin Wainwright and Michael Jordan. Graphical models, exponential families, and variational inference. *Foundations and Trends in Machine Learning*, 1(1–2):1–305, 2008.

[YCRB18a] Dong Yin, Yudong Chen, Kannan Ramchandran, and Peter Bartlett. Byzantine-robust distributed learning: Towards optimal statistical rates. In *Proceedings of the 35th International Conference on Machine Learning*, volume 80 of *Proceedings of Machine Learning Research*, pages 5650–5659. PMLR, 10–15 Jul 2018.

[YCRB18b] Dong Yin, Yudong Chen, Kannan Ramchandran, and Peter Bartlett. Defending against saddle point attack in byzantine-robust distributed learning. *arXiv preprint arXiv:1806.05358*, 2018.

[YL15] Eunho Yang and Aurélie C Lozano. Robust gaussian graphical modeling with the trimmed graphical lasso. In *Advances in Neural Information Processing Systems*, pages 2602–2610, 2015.

[YLA18] Eunho Yang, Aurélie C Lozano, and Aleksandr Aravkin. A general family of trimmed estimators for robust high-dimensional data analysis. *Electronic Journal of Statistics*, 12(2):3519–3553, 2018.
[YLZ18] Xiao-Tong Yuan, Ping Li, and Tong Zhang. Gradient hard thresholding pursuit. *Journal of Machine Learning Research*, 18(166):1–43, 2018.

[ZLR+12] Tuo Zhao, Han Liu, Kathryn Roeder, John Lafferty, and Larry Wasserman. The huge package for high-dimensional undirected graph estimation in r. *Journal of Machine Learning Research*, 13(Apr):1059–1062, 2012.

[ZWJ14] Yuchen Zhang, Martin J Wainwright, and Michael I Jordan. Lower bounds on the performance of polynomial-time algorithms for sparse linear regression. In *Conference on Learning Theory*, pages 921–948, 2014.
Notations. In our proofs, the exponent $-10$ in tail bounds is arbitrary, and can be changed to
other larger constant without affecting the results. \( \{c_j\}_{j=0}^3 \) denote universal constants, and they may
change line by line.

A Proofs for the trimmed gradient estimator

Let us first visit the definition and tail bounds of sub-exponential random variable, as it will be used
in sparse linear regression, where the gradient’s distribution is indeed sub-exponential.

We first present standard concentration inequalities ([Wai19]).

Definition A.1 (Sub-exponential random variables). A random variable \( X \) with mean \( \mu \) is sub-
exponential if there are non-negative parameters \( \nu \) such that
\[
\mathbb{E}[\exp (t (X - \mu))] \leq \exp \left( \frac{\nu^2 t^2}{2} \right), \quad \text{for all } |t| < \frac{1}{\nu}.
\]

Lemma A.1 (Bernstein’s inequality). Suppose that \( X_i, i = 1, \cdots, n \), are i.i.d. sub-exponential ran-
dom variables with parameters \( \nu \). Then
\[
\Pr \left( \frac{1}{n} \sum_{i=1}^n X_i \geq \mu + t \right) \leq \begin{cases} 
\exp \left( -\frac{nt^2}{2\nu^2} \right) & \text{if } 0 \leq t \leq \nu, \text{ and} \\
\exp \left( -\frac{nt}{2\nu} \right) & \text{for } t > \nu.
\end{cases}
\]

We also have a two-sided tail bound
\[
\Pr \left( \left| \frac{1}{n} \sum_{i=1}^n X_i - \mu \right| \geq t \right) \leq 2 \exp \left( -n \min \left( \frac{t^2}{2\nu^2}, \frac{t}{\nu} \right) \right).
\]

Similar to Definition 3.1, we define \( \alpha \)-trimmed mean estimator for one dimensional samples, and
denote it as \( \text{trmean}_\alpha (\cdot) \).

Definition A.2 (\( \alpha \)-trimmed mean estimator). Given a set of \( \epsilon \)-corrupted samples \( \{z_i \in \mathbb{R}, i \in S\} \),
the dimensional trimmed mean estimator \( \text{trmean}_\alpha (\cdot) \) removes the largest and smallest \( \alpha \) fraction of
elements in \( \{z_i \in \mathbb{R}, i \in S\} \), and calculate the mean of the remaining terms. We choose \( \alpha = c_\epsilon \epsilon \), for
a constant \( c_\epsilon \geq 1 \). We also require that \( \alpha \leq 1/2 - c_1 \), for some small constant \( c_1 > 0 \).

In Trimmed Hard Thresholding (Algorithm 1), we first use trimmed mean estimator for each
coordinate of gradients. Lemma A.2 shows the guarantees for this robust gradient estimator in each
coordinate. We note that Lemma A.2 is stronger than guarantees for trimmed mean estimator (Lemma 3) in [YCRB18a].

In our contamination model Definition 2.1, the adversary may delete \( \epsilon \)-fraction of authentic sam-
ple, and then add arbitrary outliers. And Lemma A.2 provides guarantees for trimmed mean estimator
on sub-exponential random variables. The trimmed mean estimator is robust enough, that it allows the adversary to arbitrarily remove \( \epsilon \)-fraction of data points. We use \( G^j \) to denote the \( \mathbb{R}^1 \) samples at the \( j \)-th coordinate of \( G \). We can also define \( S^j \) in the same way.

Lemma A.2. Suppose we observe \( n = \Omega(\log d) \) \( \epsilon \)-corrupted samples from Definition 2.1. For each
dimension \( j \in \{1, 2, \cdots, d\} \), we assume the samples in \( G^j \) are i.i.d. \( \nu \)-sub-exponential with mean \( \mu^j \).
After the contamination, we have the $j$-th $\mathbb{R}^1$ samples as $S^j$. Then, we can guarantee the trimmed mean estimator on $j$-th dimension that

$$|\text{trmean}_\alpha \{ x_i : i \in S^j \} - \mu_j | = O \left( \nu \left( \epsilon \log(nd) + \sqrt{\frac{\log d}{n}} \right) \right)$$

with probability at least $1 - d^{-4}$.

We leave the proof of Lemma A.2 at the end of this section. Then, we present separate analysis of trimmed gradient estimator for sparse linear regression and sparse logistic regression by using Lemma A.2.

### A.1 Sparse linear regression

In this subsection, we use Lemma A.2 to bound $\| \hat{G} - G \|_\infty$. We will show that, in the sparse linear regression model with sub-Gaussian covariates, the distribution of authentic gradients are sub-exponential instead of sub-Gaussian.

In Proposition A.1, we first prove that when the current parameter iterate is $\beta$, the sub-exponential parameter of all authentic gradient is $O \left( \| \Delta \|_2^2 + \sigma^2 \right)$, where $\Delta := \beta - \beta^*$.

To gain some intuition for this, we can consider the sparse linear equation problem, where $\sigma^2 = 0$. When $\beta = \beta^* (\| \Delta \|_2^2 = 0)$, we exactly recover $\beta^*$, and all stochastic gradients of authentic samples are actually zero vectors, as all observations are noiseless. It is clear that we will have sub-exponential parameter as $0$.

**Proposition A.1 (Proposition 3.1).** Suppose we observe $n = \Omega(\log d)$ $\epsilon$-corrupted samples from Definition 2.1, where $P$ follows the sparse linear regression model (Model 3.1). The dimensional $\alpha$-trimmed gradient estimator with $\alpha = c_0 \epsilon$ for some universal constant $c_0 \geq 1$ can guarantee that

$$\| \hat{G} - G \|_\infty = O \left( \sqrt{\| \Delta \|_2^2 + \sigma^2} \left( \epsilon \log(nd) + \sqrt{\frac{\log d}{n}} \right) \right)$$

with probability at least $1 - d^{-3}$, where $\Delta := \beta - \beta^*$.

**Proof of Proposition A.1.** For any $\beta$, the gradient for one sample can be written as

$$g = x (x^\top \beta - y), \quad \text{and} \quad G_{\text{pop}} = \mathbb{E}(g) = \Sigma (\beta - \beta^*),$$

where we omit the subscript $i$ in the proof. For any fixed standard basis vector $v \in \mathbb{S}^{d-1}$, and define $\Delta = \beta - \beta^*$, we have

$$v^\top g = v^\top xx^\top \Delta - v^\top x \xi, \quad \text{and} \quad v^\top G_{\text{pop}} = v^\top \Sigma \Delta. \quad (12)$$

To characterize the tail bounds of $v^\top g$, we study the moment generating function:

$$\mathbb{E}[\exp \left( t (v^\top g - v^\top G_{\text{pop}}) \right)] = \mathbb{E}[\exp \left( t (v^\top (xx^\top - \Sigma) \Delta - v^\top x \xi) \right)].$$

We denote $\gamma \in \{-1, +1\}$ as a Rademacher random variable, which is independent of $x$ and $\xi$. Then
we can use a standard symmetrization technique [Wai19],

\[
E_{x,v}\{\exp \left( t \left( v^\top (xx^\top - \Sigma) \Delta - v^\top x \xi \right) \right) \} \leq E_{x,v,\gamma}\{\exp \left( 2t\gamma \left( v^\top xx^\top \Delta - v^\top x \xi \right) \right) \}
\]

\[
\begin{align*}
&= (i) \sum_{k=0}^{\infty} \frac{1}{k!} (2t)^k E[\gamma^k (v^\top xx^\top \Delta - v^\top x \xi)^k] \\
&= (ii) 1 + \sum_{i=1}^{\infty} \frac{1}{(2i)!} (2t)^{2i} E[(v^\top x)^{2i} (x^\top \Delta - \xi)^{2i}], \\
\end{align*}
\]

where \((i)\) follows from the exponential function’s power series expansion, and \((ii)\) follows from the independence of \(\gamma\), together with the fact that all odd moments of the \(\gamma\) terms have zeros means.

By the Cauchy-Schwarz inequality, we have

\[
E[(v^\top x)^{2i} (x^\top \Delta - \xi)^{2i}] \leq \sqrt{E[(v^\top x)^{4i}] E[(x^\top \Delta - \xi)^{4i}]].
\]

It is clear that \(\xi\) is a sub-Gaussian random variable with parameter \(\sigma\). Since \(x \sim \mathcal{N}(0, \Sigma)\), we have \(v^\top x \sim \mathcal{N}(0, v^\top \Sigma v)\). For any fixed standard basis vector \(v \in \mathbb{S}^{d-1}\), we can conclude that \(v^\top x\) is sub-Gaussian with parameter at most 1 based on Model 3.1. By basic properties of sub-Gaussian random variables [Wai19], we have

\[
\sqrt{E[(v^\top x)^{4i}]} \leq \sqrt{\frac{(4l)!}{2^{2l} (2l)!}} \left( \sqrt{8\epsilon} \right)^{2l} \\
\sqrt{E[(x^\top \Delta - \xi)^{4i}]} \leq (i) \sqrt{\frac{(4l)!}{2^{2l} (2l)!}} \left( 8\epsilon^2 \left( \|\Delta\|_2^2 + \sigma^2 \right) \right)^l,
\]

where \((i)\) follows from the fact that \(x^\top \Delta - \xi\) is the weighted summation of two independent sub-Gaussian random variables. Hence, we have

\[
E[\exp \left( t \left( v^\top g - v^\top G^{pop} \right) \right) ] \leq 1 + \sum_{i=1}^{\infty} \frac{1}{(2i)!} (2t)^{2i} \frac{(4l)!}{2^{2l} (2l)!} \left( \sqrt{8\epsilon} \right)^{4l} \left( \|\Delta\|_2^2 + \sigma^2 \right)^l
\]

\[
\leq (i) 1 + \sum_{i=1}^{\infty} (4t)^{2i} \left( \sqrt{8\epsilon} \right)^{4i} \left( \|\Delta\|_2^2 + \sigma^2 \right)^l
\]

\[
= 1 + \sum_{i=1}^{\infty} (4t)^{2i} \left( 8\epsilon^2 \right)^{2i} \left( \|\Delta\|_2^2 + \sigma^2 \right)^{2l}, \tag{13}
\]

where \((i)\) follows from \((4l)! \leq 2^{4l} ((2l)!)^2 \) (proof by mathematical induction).

When \(f(t) = 32t\epsilon^2 \sqrt{\|\Delta\|_2^2 + \sigma^2} < 1\), eq. (13) converges to \(\frac{1}{1 - f^2(t)}\). Hence,

\[
E[\exp \left( t \left( v^\top g - v^\top G^{pop} \right) \right) ] \leq \frac{1}{1 - f^2(t)} \leq \exp \left( f^2(t) \right).
\]

That being said, \(v^\top g\) is a sub-exponential random variable. By choosing \(v\) as each coordinate in \(\mathbb{R}^d\), each coordinate of gradient has sub-exponential parameter as \(32\sqrt{3\epsilon^2 \sqrt{\|\Delta\|_2^2 + \sigma^2}}\).

Then, applying Lemma A.2 on this collection of corrupted sub-exponential random variables, we
have

$$|\text{trmean}_\alpha \{ x_i : i \in S^j \} - \mu^j | = O \left( \sqrt{\| \Delta \|^2 + \sigma^2 \left( \epsilon \log(nd) + \sqrt{\log d \over n} \right)} \right), \quad (14)$$

with probability at least $1 - d^{-4}$.

Since $G^j$ is the sample average of i.i.d. sub-exponential random variables in $\mathbb{R}$. By Lemma A.1, we directly have

$$\Pr \left( |G^j - \mu^j| \geq c_0 \left( \sqrt{\| \Delta \|^2 + \sigma^2 \sqrt{\log d \over n}} \right) \right) \leq 2 \exp \left( -c_1 n \min \left( \sqrt{\log d \over n}, \log d \over n \right) \right) \leq c_2 d^{-10} \leq d^{-4}. \quad (15)$$

Putting together eq. (14) and eq. (15), and applying union bounds on all $d$ indexes, we have

$$\left\| \hat{G} - G \right\|_\infty = O \left( \sqrt{\| \Delta \|^2 + \sigma^2 \left( \epsilon \log(nd) + \sqrt{\log d \over n} \right)} \right),$$

with probability at least $1 - d^{-3}$. \hfill \Box

### A.2 Sparse logistic regression

In this subsection, we use Lemma A.2 to bound $\| \hat{G} - G \|_\infty$ for sparse logistic regression. The technique for sparse logistic regression is similar to linear regression. Since we can directly show the sub-Gaussian distribution of gradient in this case, applying Lemma A.2 leads to the bound for $\| \hat{G} - G \|_\infty$.

**Proposition A.2 (Proposition 3.2).** Suppose we observe $n = \Omega(\log d)$ $\epsilon$-corrupted samples from Definition 2.1, where $P$ follows the sparse logistic regression model (Model 3.2). The dimensional $\alpha$-trimmed gradient estimator with $\alpha = c_0 \epsilon$ for some universal constant $c_0 \geq 1$ can guarantee that

$$\left\| \hat{G} - G \right\|_\infty = O \left( \epsilon \log(nd) + \sqrt{\log d \over n} \right)$$

with probability at least $1 - d^{-3}$.

**Proof of Proposition A.2.** Under the statistical model of sparse logistic regression, the gradient can be computed as:

$$g = -y x \left( 1 + \exp(y x^\top \beta) \right)^{-1},$$

where we omit the subscript $i$ in the proof.

Since $y \in \{-1, +1\}$, and $1 + \exp(y x^\top \beta) \geq 1$, then for any fixed standard basis vector $v \in \mathbb{S}^{d-1}$, $v^\top g$ is sub-Gaussian with parameter at most 1 based on Model 3.2. Notice that $\nu$-sub-Gaussian random variables are still $\nu$-sub-exponential. Applying Lemma A.2 again, we have

$$\left| \text{trmean}_\alpha \{ x_i : i \in S^j \} - \mu \right| = O \left( \epsilon \log(nd) + \sqrt{\log d \over n} \right) \quad (16)$$

with probability at least $1 - d^{-4}$.  

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Similarly to the sparse linear regression case, we use Lemma A.1

\[
\Pr \left( \left| G' - \mu \right| \geq c_0 \sqrt{\frac{\log d}{n}} \right) \leq 2 \exp \left( -c_1 n \min \left( \sqrt{\frac{\log d}{n}}, \frac{\log d}{n} \right) \right) \\
\leq c_2 d^{-10} \\
\leq d^{-4}. \tag{17}
\]

Putting together eq. (16) and eq. (17), and applying union bounds on all \( d \) indexes, we have

\[
\| \hat{G} - G \|_{\infty} = O \left( \epsilon \log(n) + \sqrt{\frac{\log d}{n}} \right)
\]

with probability at least \( 1 - d^{-3} \). \ \Box

### A.3 Trimmed mean estimator for strong contamination model

Now, it only remains to prove Lemma A.2. The proof technique is as follow: even though an adversary may delete samples from \( G' \), we can still show the concentration inequalities for remaining authentic \( \mathbb{R}^1 \) samples (denoting as \( \tilde{G}' \) in the proof). Then, we show that by using trimmed mean estimator, either the abnormal outliers will be removed, or their effect is controlled.

**Proof of Lemma A.2.** Without loss of generality, we assume \( \mu = 0 \) throughout the proof.

For each dimension \( j \in \{1, 2, \cdots, d\} \), we can split the \( j \)-th one-dimensional samples as \( S_j = \tilde{G}_j \cup B_j \). To study the performance of \( \mathrm{trmean}_\alpha \{ x_i : i \in S_j \} \), we first show a concentration inequality of the sub-exponential variables in \( \tilde{G}_j \), without worrying about removing points from \( G_j \). This part of our proof is similar to Lemma 4.5 in [DKK+16].

**Concentration inequality for \( \tilde{G}_j \)** We consider the set \( \{ x_i : i \in \tilde{G}_j \} \) in \( \mathbb{R}^1 \). Since \( \tilde{G}_j \) is a subset of \( G_j \), by triangle inequality we have,

\[
\left| \mathbb{E}_{x_i, \tilde{G}_j} x_i \right| = \left\| \sum_{i \in \tilde{G}_j} \frac{x_i}{(1 - \epsilon) n} \right\| \leq \left\| \sum_{i \in \tilde{G}_j} \frac{x_i}{(1 - \epsilon) n} \right\|_{A_1} + \left\| \sum_{i \in \tilde{G}_j \setminus \tilde{G}_j} \frac{x_i}{(1 - \epsilon) n} \right\|_{A_2}.
\]

The first term \( A_1 \) is simply the average of i.i.d. sub-exponential random variables. By Lemma A.1, we have

\[
\Pr \left( \left| \sum_{i \in \tilde{G}_j} \frac{x_i}{(1 - \epsilon) n} \right| \geq c_0 \nu \sqrt{\frac{\log d}{n}} \right) \leq 2 \exp \left( -c_1 n \min \left( \sqrt{\frac{\log d}{n}}, \frac{\log d}{n} \right) \right)
\]

\[
\leq c_2 d^{-10}. \tag{18}
\]

For the second term \( A_2 \), we now wish to show that with probability \( 1 - \tau \), there does not exist a subset \( \tilde{G}_j \) so that the \( A_2 \) is more than \( \delta_0 \). This event is equivalent to

\[
\left| \sum_{i \in \tilde{G}_j \setminus \tilde{G}_j} \frac{x_i}{(1 - \epsilon) n} \right| = \left| \sum_{i \in \tilde{G}_j \setminus \tilde{G}_j} \frac{x_i}{\epsilon n} \right| \geq \delta_0
\]

\[
\frac{\epsilon}{1 - \epsilon} \geq \delta_0
\]

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Let $\delta_1 = \frac{1-\epsilon}{\epsilon} \delta_0$. For one subset $\mathcal{G}^j \setminus \tilde{\mathcal{G}}^j$, by Lemma A.1, we have

$$
\Pr \left( \left| \frac{\sum_{i \in \mathcal{G}^j \setminus \tilde{\mathcal{G}}^j} x_i}{en} \right| \geq \delta_1 \right) \leq 2 \exp \left( -en \min \left( \frac{\delta_1^2}{4\nu^2}, \frac{\delta_1}{2\nu} \right) \right).
$$

Then, we take union bounds over all possible $\mathcal{G}^j \setminus \tilde{\mathcal{G}}^j$, which have $\binom{n}{en}$ events. Hence, the tail probability of $A_2$ can be bounded as

$$
\tau \leq 2 \binom{n}{en} \exp \left( -en \min \left( \frac{\delta_1^2}{4\nu^2}, \frac{\delta_1}{2\nu} \right) \right) \leq c_0 \exp \left( nH(\epsilon) - en \min \left( \frac{\delta_1^2}{4\nu^2}, \frac{\delta_1}{2\nu} \right) \right),
$$

where (i) follows from the fact that $\log \binom{n}{en} = O(nH(\epsilon))$ for $n$ large enough, and $H(\cdot)$ is the binary entropy function. Choosing $\delta_1 = c_1 \nu \log(nd)$, and hence $\delta_0 = c_1 \nu \log(nd)$, we have $\tau \leq c_0 \exp(-c_2 \nu \log(nd)) \leq c_3 d^{-10}$.

Combining the analysis on $A_1$ and $A_2$ (eq. (19) and eq. (20)), we have

$$
\Pr \left( \left| \frac{\sum_{i \in \tilde{\mathcal{G}}^j} x_i}{en} \right| \geq \nu \left( c_0 \sqrt{\frac{\log d}{n}} + c_1 \epsilon \log(nd) \right) \right) \leq c_2 d^{-10}.
$$

This completes the concentration bounds on $\left| \frac{\sum_{i \in \tilde{\mathcal{G}}^j} x_i}{en} \right|$ for all possible samples in $\tilde{\mathcal{G}}^j$ without worrying about sample removing.

**Trimmed mean estimator for $\mathcal{S}^j$** Then, we can consider the contribution of each part in $\mathcal{S}^j = \tilde{\mathcal{G}}^j \cup \mathcal{B}^j$. We denote the remaining set after trimming as $\mathcal{R}^j$, and the trimmed set as $\mathcal{T}^j$. Recall that we assume $\mu = 0$, we only need to bound $|\text{trmean}_\alpha \{x_i : i \in \mathcal{S}^j\}|$, which is the empirical average of all samples in the remaining set $\{x_i : i \in \mathcal{R}^j\}$.

As $\mathcal{R}^j$ can be easily separated by the union of two distinct set $\mathcal{B}^j \cap \mathcal{R}^j$ and $\tilde{\mathcal{G}}^j \cap \mathcal{R}^j$, we have the following inequalities,

$$
|\text{trmean}_\alpha \{x_i : i \in \mathcal{S}^j\}| = \left| \frac{1}{(1-2\alpha)n} \sum_{i \in \mathcal{R}^j} x_i \right| \leq \frac{1}{(1-2\alpha)n} \left| \sum_{i \in \tilde{\mathcal{G}}^j} x_i - \sum_{i \in \tilde{\mathcal{G}}^j \cap \mathcal{T}^j} x_i + \sum_{i \in \mathcal{B}^j \cap \mathcal{R}^j} x_i \right| \leq \frac{1}{(1-2\alpha)n} \left( \sum_{i \in \tilde{\mathcal{G}}^j \setminus \mathcal{B}_1} x_i + \sum_{i \in \tilde{\mathcal{G}}^j \cap \mathcal{T}^j} x_i + \sum_{i \in \mathcal{B}^j \cap \mathcal{R}^j} x_i \right).
$$

For any $i \in \tilde{\mathcal{G}}^j$, by Lemma A.1, we have

$$
\Pr \left( |x_i| \geq c_0 \nu \log(nd) \right) \leq 2 \exp \left( -c_1 \min \left( \log(nd), \log^2(nd) \right) \right).
$$
Applying a union bound for all samples, we can control the maximum magnitude for any \( i \in \mathcal{G}^j \),
\[
\Pr\left( \max_{i \in \mathcal{G}^j} |x_i| \geq c_0 \nu \log(nd) \right) \leq 2(1 - c)n \exp\left( -c_1 \min\left( \log(nd), \log^2(nd) \right) \right),
\]
\[
\leq c_1 d^{-10}.
\]

We can bound \( B_1 \) by applying eq. (21). For the trimmed good samples \( \{ i \in \mathcal{G}^j \cap \mathcal{T}^j \} \), we have
\[
B_2 \leq 2\alpha n \max_{i \in \mathcal{G}^j} |x_i|.
\]
Since we choose \( \alpha \geq \epsilon \), we have \( B_3 \leq \epsilon n \max_{i \in \mathcal{G}^j} |x_i| \).

Putting together the pieces, and choosing \( \alpha = \epsilon c \) for some universal constant \( c \geq 1 \), we have
\[
|\text{trmean}_n \{ x_i : i \in \mathcal{S}^j \} - \mu^j | = O \left( \nu \left( \epsilon \log(nd) + \sqrt{\frac{\log d}{n}} \right) \right),
\]
with probability at least \( 1 - d^{-4} \). This completes the proof for Lemma A.2. \( \square \)

## B Statistical estimation via Trimmed Hard Thresholding

Here, we provide statistical estimation performance of Trimmed Hard Thresholding under statistical models. With Proposition 3.1 and Proposition 3.2 in hand, we will show the main theorems for Trimmed Hard Thresholding – the global linear convergence, and its statistical guarantee.

We first introduce a supporting Lemma on the property of hard thresholding operator.

**Lemma B.1** (Lemma 1 in [LB18]). We set \( k' \) in hard thresholding operator as \( k' = kc_p^2 \), where \( c_p \geq 1 \), then we have
\[
\sup \left\{ \frac{|(\beta^* - P_{k'}(z), z - P_{k'}(z))|}{\|\beta^* - P_{k'}(z)\|_2^2} : \beta^*, z \in \mathbb{R}^d, \|\beta^*\|_0 \leq k, \beta^* \neq P_{k'}(z) \right\} = \frac{1}{2} \sqrt{\frac{k}{k'}} = \frac{1}{2c_p}.
\]

Note that \( c_p \) in Lemma B.1 will be specified as \( 2p \) later in the proof, as we choose \( k' = 4p^2 k \) as in eq. (7).

We first study the relationship between the objective function gap \( f(\bar{\beta}^t) - f(\beta^*) \) and the perturbation of robust gradient estimator \( \| G(\beta^{t-1}) - \tilde{G}(\beta^{t-1}) \|_\infty \). Sparse linear regression and logistic regression share this part of the proof, hence we do not distinguish these two models currently.

Since \( \beta^{t-1} \) and \( \bar{\beta}^t \) are \( k' \)-sparse by definition of the algorithm, and \( f \) satisfies \((\mu_\alpha, k')\)-RSC and \((\mu_L, k')\)-RSM, we have
\[
f(\beta^*) \geq f(\beta^{t-1}) + \langle G(\beta^{t-1}), \beta^* - \beta^{t-1} \rangle + \frac{\mu_\alpha}{2} \| \beta^* - \beta^{t-1} \|_2^2, \tag{22}
\]
\[
f(\bar{\beta}^t) \leq f(\beta^{t-1}) + \langle G(\beta^{t-1}), \bar{\beta}^t - \beta^{t-1} \rangle + \frac{\mu_L}{2} \| \bar{\beta}^t - \beta^{t-1} \|_2^2, \tag{23}
\]
where (i) follows from the fact that \( \beta^*, \beta^{t-1} \in B \), and \((\mu_\alpha, k')\)-RSC holds.

Combining these two inequalities, we obtain
\[
f(\bar{\beta}^t) - f(\beta^*) \leq \langle G(\beta^{t-1}), \bar{\beta}^t - \beta^* \rangle - \frac{\mu_\alpha}{2} \| \beta^* - \beta^{t-1} \|_2^2 + \frac{\mu_L}{2} \| \bar{\beta}^t - \beta^{t-1} \|_2^2.
\]
Expanding the last term, we also have
\[
\frac{\mu_L}{2} \| \beta^t - \beta^* \|^2 \leq \frac{\mu_L}{2} \| \beta^{t-1} - \beta^* \|^2 - \frac{\mu_L}{2} \| \beta^t - \beta^{t-1} \|^2 + \mu_L (\beta^{t-1} - \beta^t, \beta^* - \beta^t)
\]
\[
= \frac{\mu_L}{2} \| \beta^{t-1} - \beta^* \|^2 - \frac{\mu_L}{2} \| \beta^t - \beta^{t-1} \|^2 + \mu_L \left( \left( \beta^{t-1} - \eta \hat{G} (\beta^{t-1}) \right) - \beta^t, \beta^* - \beta^t \right)
\]
\[
= \frac{\mu_L}{2} \| \beta^{t-1} - \beta^* \|^2 - \frac{\mu_L}{2} \| \beta^t - \beta^{t-1} \|^2 + \mu_L \left( \left( \beta^{t-1} - \eta \hat{G} (\beta^{t-1}) \right) - \beta^t, \beta^* - \beta^t \right)
\]
\[
+ \mu_L \left( \left( \beta^{t-1} - \eta \hat{G} (\beta^{t-1}) \right) - \beta^t, \beta^* - \beta^t \right)
\]
\[
+ \mu_L \left( \left( \hat{G} (\beta^{t-1}) - \beta^t, \beta^* - \beta^t \right) \right)
\]
\[
T_1
\]
\[
T_2
\]

For the term \( T_1 \), recall that \( \beta^t \) is obtained from hard thresholding, and \( \beta^t = P_{k^t} \left( \beta^{t-1} - \eta \hat{G} (\beta^{t-1}) \right) \), we apply Lemma B.1 with \( z = \beta^{t-1} - \eta \hat{G} (\beta^{t-1}) \):
\[
\left( \beta^{t-1} - \eta \hat{G} (\beta^{t-1}) \right) - \beta^t, \beta^* - \beta^t \right) \leq \frac{1}{2} \| \beta^t - \beta^* \|^2.
\]
\[
The term \( T_2 \) can be bounded by Holder inequality,
\[
\left( \beta^t, \beta^* - \beta^t \right) \geq \left( G (\beta^{t-1}), \beta^t - \beta^* \right) - \left( \beta^t, \beta^* - \beta^t \right) \geq \left( G (\beta^{t-1}), \beta^t - \beta^* \right) - \left( G (\beta^{t-1}), \beta^t - \beta^t \right) \geq \left( G (\beta^{t-1}), \beta^t - \beta^* \right) - \sqrt{k' + k} \left( G (\beta^{t-1}), \beta^t - \beta^t \right).
\]

We denote \( \Xi^t = \beta^t - \beta^* \) and \( \Delta^t = \beta^t - \beta^* \). Putting together the pieces, we have
\[
f (\beta^t) - f (\beta^*) \leq \frac{\mu_L}{2} \left( \left( 1 - \frac{1}{\rho} \right) \| \Delta^{t-1} \|^2 - \left( 1 - \frac{1}{c_{\rho}} \right) \| \Xi^t \|^2 \right) + \sqrt{k' + k} \left( G (\beta^{t-1}) - \hat{G} (\beta^{t-1}) \right) \| \beta^t - \beta^* \|_2^2.
\]
\[
(24)
\]

Now, we separate the analysis of eq. (24) by the case of linear regression and logistic regression and obtain the global linear convergence of \( \| \beta^t - \beta^* \|_2 \).

### B.1 Sparse linear regression

**Theorem B.1 (Theorem 4.1).** Suppose that we observe \( N = nT \) \( \epsilon \)-corrupted samples (Definition 2.1) from the sparse linear regression model (Model 3.1). Under the condition \( n = \Omega \left( \rho^4 k \log d \right) \), Algorithm 1 with \( \eta = 1/\mu_L \) outputs \( \hat{\beta} \) satisfying
\[
\| \hat{\beta} - \beta^* \|_2 = \Omega \left( \rho^2 \sigma \left( \epsilon \sqrt{k} \log (nd) \right) + \sqrt{\frac{k \log d}{n}} \right),
\]
with probability at least \( 1 - d^{-2} \), where we set \( T = O \left( \rho \log \left( \| \beta^* \|_2 / \| \hat{\beta} - \beta^* \|_2 \right) \right) \).

**Proof of Theorem B.1.** In eq. (24), the term \( \left( G (\beta^{t-1}) - \hat{G} (\beta^{t-1}) \right) \| \beta^t - \beta^* \|_2 \) can be controlled by trimmed
mean estimator by a union bound over all indexes. By Proposition A.1 for the gradients, we have

$$\sqrt{k' + k} \left\| G(\beta^{t-1}) - \bar{G}(\beta^{t-1}) \right\|_\infty = O \left( \sqrt{k' + k} \sqrt{\sigma^2 + \|\beta^{t-1} - \beta^*\|_2^2} \left( \epsilon \log(nd) + \sqrt{\frac{\log d}{n}} \right) \right)$$

$$= O \left( \sqrt{k' + k} (\sigma + \|\Delta^{t-1}\|_2) \left( \epsilon \log(nd) + \sqrt{\frac{\log d}{n}} \right) \right)$$

with probability at least $1 - d^{-3}$, where we use the shorthand $\Delta^t = \beta^t - \beta^*$ and $\bar{\Delta}^t = \bar{\beta}^t - \beta^*$.

Let $\Gamma = O \left( \sqrt{k' + k} \left( \epsilon \log(nd) + \sqrt{\frac{\log d}{n}} \right) \right)$. Since $\eta \mu^* \geq \frac{1}{\mu L} \cdot \mu^* = \frac{1}{\rho}$, this implies

$$f(\bar{\beta}^t) - f(\beta^*) \leq \frac{\mu L}{2} \left[ \left( 1 - \frac{1}{\rho} \right) \|\Delta^{t-1}\|_2^2 - \left( 1 - \frac{1}{c_p} \right) \|\bar{\Delta}^t\|_2^2 \right] + \Gamma (\sigma + \|\Delta^{t-1}\|_2) \|\bar{\Delta}^t\|_2.$$  \hspace{1cm} (25)

with probability at least $1 - d^{-3}$.

Applying convexity, we have

$$f(\bar{\beta}^t) - f(\beta^*) \geq \langle G(\beta^*), \bar{\Delta}^t - \beta^* \rangle$$

\hspace{2cm}\begin{align*}
&\geq (i) - \|G(\beta^*)\|_\infty \|\bar{\Delta}^t\|_1, \\
&\geq (ii) - \sqrt{k' + k} \|G(\beta^*)\|_\infty \|\bar{\Delta}^t\|_2,
\end{align*}$$ \hspace{1cm} (26)

where $(i)$ follows from Holder inequality, and $(ii)$ follows from the fact that $\bar{\Delta}^t$ is $(k' + k)$-sparse.

Under the statistical model of sparse linear regression, $\|G(\beta^*)\|_\infty$ can be bounded by controlling the maximum of Gaussian random variables. According to [NRWY12], we have $\|G(\beta^*)\|_\infty = O \left( \sigma \sqrt{\log d/n} \right)$ with probability at least $1 - d^{-3}$.

Combining the eq. (25) and eq. (26), we have

$$0 \leq \frac{\mu L}{2} \left[ \left( 1 - \frac{1}{\rho} \right) \|\Delta^{t-1}\|_2^2 - \left( 1 - \frac{1}{c_p} \right) \|\bar{\Delta}^t\|_2^2 \right] + \Gamma (\sigma + \|\Delta^{t-1}\|_2) \|\bar{\Delta}^t\|_2,$$  \hspace{1cm} (27)

with probability at least $1 - d^{-3}$.

Notice that eq. (27) is a quadratic inequality for $\|\bar{\Delta}^t\|_2$, and we can use the root of eq. (27) to upper bound $\|\bar{\Delta}^t\|_2$:

$$\|\bar{\Delta}^t\|_2 \leq \frac{\Gamma (\sigma + \|\Delta^{t-1}\|_2) + \sqrt{2\Gamma^2 (\sigma + \|\Delta^{t-1}\|_2)^2 + \left( \mu L \left( 1 - \frac{1}{c_p} \right) \right) \cdot \mu L \left( 1 - \frac{1}{\rho} \right) \|\Delta^{t-1}\|_2^2}}{\mu L \left( 1 - \frac{1}{\rho} \right)}$$

\hspace{2cm}\begin{align*}
&\leq (i) \frac{2\Gamma (\sigma + \|\Delta^{t-1}\|_2) + \|\Delta^{t-1}\|_2 \sqrt{\left( \mu L \left( 1 - \frac{1}{c_p} \right) \right) \cdot \mu L \left( 1 - \frac{1}{\rho} \right)}}{\mu L \left( 1 - \frac{1}{\rho} \right)} \\
&= \|\Delta^{t-1}\|_2 \frac{1 - \frac{1}{\rho}}{1 - \frac{1}{c_p}} + \frac{2\Gamma \|\Delta^{t-1}\|_2}{\mu L \left( 1 - \frac{1}{c_p} \right)} + \frac{2\Gamma \sigma}{\mu L \left( 1 - \frac{1}{c_p} \right)}
\end{align*}
where (i) follows from the basic inequality $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$ for non-negative $a, b$.

We choose $c_\rho = 2\rho$, and this leads to $\sqrt{(1 - \frac{1}{\mu}) / (1 - \frac{1}{\rho})} \leq 1 - \frac{1}{4\rho}$. Under the condition $\Gamma \leq \frac{1}{16\mu_\alpha}$, we have $\frac{2\Gamma}{\mu_L (1 - \frac{1}{c_\rho})} \leq \frac{1}{8\rho}$. Then,

$$\|\Sigma\|_2 \leq \left(1 - \frac{1}{8\rho}\right)\|\Delta^{t-1}\|_2 + \frac{4\Gamma \sigma}{\mu_L}$$ (28)

Since $\beta^{t+1} = \Pi_B (\mathcal{F}^{t+1})$ is projection onto a convex set, by the property of Euclidean projection [Bub15], we have

$$\|\Delta^t\|_2 = \|\beta^t - \beta^*\|_2 \leq \|\mathcal{F}^t - \beta^*\|_2 = \|\Sigma\|_2.$$ (29)

Together with eq. (29), eq. (28) establishes global linear convergence of $\Delta^t$.

We apply a union bound on $T$ iterates. Since $1 - T d^{-3} \geq 1 - d^{-2}$ for sufficiently large $d$, we have

$$\|\Delta^T\|_2 \leq \left(1 - \frac{1}{8\rho}\right)^T \|\beta^*\|_2 + \frac{32\Gamma \sigma}{\mu_\alpha}$$

with probability at least $1 - d^{-2}$. Plugging in $\Gamma = O \left(\sqrt{k' + k} \left(\epsilon \log(nd) + \sqrt{\frac{\log d}{n}}\right)\right)$, we can achieve the final error

$$\psi = O \left(\frac{\rho \sigma}{\mu_\alpha} \sqrt{k} \left(\epsilon \log(nd) + \sqrt{\frac{\log d}{n}}\right)\right)$$

by setting $T = O \left(\rho \log \left(\frac{\|\beta^*\|_2}{\psi}\right)\right)$. The condition $\Gamma = O \left(\sqrt{k' + k} \left(\epsilon \log(nd) + \sqrt{\frac{\log d}{n}}\right)\right) \leq \frac{1}{16\mu_\alpha}$ can be achieved if

$$n = \Omega \left(\frac{\rho^2 k \log d}{\mu_\alpha^2}\right), \text{ and } \epsilon = O \left(\frac{\mu_\alpha}{\rho \sqrt{k} \log(nd)}\right).$$

Since $\rho = \mu_L / \mu_\alpha$, and $\mu_L \geq 1$, these conditions can be expressed as

$$n = \Omega \left(\rho^2 k \log d\right), \text{ and } \epsilon = O \left(\frac{1}{\rho^2 \sqrt{k} \log(nd)}\right).$$

The final error can be expressed as

$$\|\hat{\beta} - \beta^*\|_2 = O \left(\rho^2 \sigma \left(\epsilon \sqrt{k} \log(nd) + \sqrt{\frac{k \log d}{n}}\right)\right).$$
B.2 Sparse logistic regression

Theorem B.2 (Theorem 4.2). Suppose that we observe \( N = nT \) \( \epsilon \)-corrupted samples (Definition 2.1) the sparse logistic regression model (Model 3.2). Algorithm 1 with \( \eta = 1/\mu L \) outputs \( \hat{\beta} \), such that

\[
\| \hat{\beta} - \beta^* \|_2 = O\left( \rho^2 \left( \epsilon \sqrt{k \log(nd)} + \sqrt{\frac{k \log d}{n}} \right) \right),
\]

with probability at least \( 1 - d^{-2} \), when we set \( T = O\left( \rho \log \left( \| \beta^* \|_2/\| \hat{\beta} - \beta^* \|_2 \right) \right) \).

Proof of Theorem B.2. For sparse logistic regression with constraint in \( B \), we still have strong convexity for \( \beta^t - 1 \in B \). Hence, we have the same results as in eq. (24)

\[
f(\beta^t) - f(\beta^*) \leq \frac{\mu L}{2} \left[ \left( 1 - \frac{1}{\rho} \right) \| \Delta^{t-1} \|_2^2 - \left( 1 - \frac{1}{c_\rho} \right) \| \Delta^t \|_2^2 \right] + \sqrt{k' + k} \| G(\beta^{t-1}) - \hat{G}(\beta^{t-1}) \|_\infty \| \Xi' \|_2.
\]

(30)

In sparse logistic regression, the term \( \| G(\beta^{t-1}) - \hat{G}(\beta^{t-1}) \|_\infty \) can be controlled by trimmed mean estimator by a union bound on all indexes. By Proposition A.2, we have

\[
\sqrt{k' + k} \| G(\beta^{t-1}) - \hat{G}(\beta^{t-1}) \|_\infty = O\left( \sqrt{k' + k} \left( \epsilon \log(nd) + \sqrt{\frac{\log d}{n}} \right) \right),
\]

with probability at least \( 1 - d^{-3} \).

Let \( \Gamma = O\left( \sqrt{k' + k} \left( \epsilon \log(nd) + \sqrt{\frac{\log d}{n}} \right) \right) \). Since \( \eta \mu_\alpha \geq \frac{1}{\mu L} \cdot \mu_\alpha = \frac{1}{\rho} \), this implies

\[
f(\beta^t) - f(\beta^*) \leq \frac{\mu L}{2} \left[ \left( 1 - \frac{1}{\rho} \right) \| \Delta^{t-1} \|_2^2 - \left( 1 - \frac{1}{c_\rho} \right) \| \Delta^t \|_2^2 \right] + \Gamma \| \Xi' \|_2.
\]

(31)

For \( \beta^t \), it enjoys convexity which implies

\[
f(\beta^t) - f(\beta^*) \geq G(\beta^*) \cdot \beta^t - \beta^*
\]

\[
\geq -\| G(\beta^*) \|_\infty \| \Xi' \|_1,
\]

\[
\geq -\sqrt{k' + k} \| G(\beta^*) \|_\infty \| \Xi' \|_2.
\]

(32)

Under the statistical model of sparse logistic regression, we have \( \| G(\beta^*) \|_\infty = 4 \sqrt{\frac{\log d}{n}} \) (Proposition 12 in [YLZ18]). Then, combining eq. (31) and eq. (32), we have

\[
0 \leq \frac{\mu L}{2} \left[ \left( 1 - \frac{1}{\rho} \right) \| \Delta^{t-1} \|_2^2 - \left( 1 - \frac{1}{c_\rho} \right) \| \Delta^t \|_2^2 \right] + \Gamma \| \Xi' \|_2.
\]

(33)
Solving eq. (33), we have
\[
\|\Sigma^t\|_2 \leq \frac{\Gamma + \sqrt{\Gamma^2 + \left(\mu_L \left(1 - \frac{1}{c_p}\right)\right) \cdot \mu_L \left(1 - \frac{1}{\rho}\right) \|\Delta^t-1\|^2_2}}{\mu_L \left(1 - \frac{1}{c_p}\right)} \\
\leq \frac{2\Gamma + \|\Delta^t-1\|_2 \sqrt{\left(\mu_L \left(1 - \frac{1}{c_p}\right)\right) \cdot \mu_L \left(1 - \frac{1}{\rho}\right)}}{\mu_L \left(1 - \frac{1}{c_p}\right)} \\
= \|\Delta^t-1\|_2 \sqrt{\frac{1 - \frac{1}{\rho}}{1 - \frac{1}{c_p}}} + \frac{2\Gamma}{\mu_L \left(1 - \frac{1}{c_p}\right)}
\]
We choose \(c_p = 2\rho\), and this leads to \(\sqrt{\left(1 - \frac{1}{\rho}\right) / \left(1 - \frac{1}{c_p}\right)} \leq 1 - \frac{1}{4\rho}\).

\[
\|\Sigma^t\|_2 \leq \left(1 - \frac{1}{4\rho}\right) \|\Delta^t-1\|_2 + \frac{4\Gamma}{\mu_L}
\]
Similar to the property of Euclidean projection [Bub15] in linear regression, we have global linear convergence of \(\Delta^t\).

We apply a union bound on \(T\) iterates. Since \(1 - Td^{-3} \geq 1 - d^{-2}\) for sufficiently large \(d\), we have
\[
\|\Delta^T\|_2 \leq \left(1 - \frac{1}{4\rho}\right)^T \|\beta^*\|_2 + \frac{16\Gamma}{\mu_\alpha}
\]
with probability at least \(1 - d^{-2}\). Plugging in \(\Gamma = O\left(\sqrt{k'} + k \left(\epsilon \log(n) + \sqrt{\log d / n}\right)\right)\), we can achieve the final error
\[
\psi = O\left(\frac{\rho \sqrt{k}}{\mu_\alpha} \left(\epsilon \log(n) + \sqrt{\log d / n}\right)\right)
\]
by setting \(T = O\left(\rho \log \left(\frac{\|\beta^*\|_2}{\psi}\right)\right)\). Similar to the proof in sparse linear regression, this final error can be expressed as
\[
\left\|\hat{\beta} - \beta^*\right\|_2 = O\left(\rho^2 \left(\frac{\epsilon \sqrt{k} \log(n)d}{\text{robustness error}} + \sqrt{\frac{k \log d}{n}} \text{statistical error}\right)\right).
\]

\section{Sparsity recovery and sparse precision matrix estimation}

\subsection{Sparsity recovery guarantee}

The same as the main text, we use \(\text{supp}(v, k)\) to denote top \(k\) indexes of \(v\) with the largest magnitude. Let \(v_{\text{min}}\) denote the smallest absolute value of nonzero elements of \(v\).
Theorem C.1 (Theorem 5.1). Under the same condition in Theorem 4.1, and a \( \beta_{\min} \)-condition on \( \beta^* \)
\[
\beta_{\min}^* = \Omega \left( \rho^2 \sigma \left( \epsilon \sqrt{k \log(nd)} + \sqrt{\frac{k \log d}{n}} \right) \right),
\]
Algorithm 1 guarantees \( \text{supp}(\hat{\beta}, k) = \text{supp}(\beta^*) \), with probability at least \( 1 - d^{-2} \).

Proof of Theorem C.1. The sparsity recovery guarantee is similar to [YLZ18]. Since \( \hat{\beta} \) is \( k' \) sparse \((k' \geq k)\) by the definition of hard thresholding operator, we use \( \hat{\beta}_k \) to denote \( P_k(\hat{\beta}) \). We use the technique proof by contradiction. If \( \text{supp}(\hat{\beta}, k) \neq \text{supp}(\beta^*) \), we at least have \( \ell_2 \) error as \( \beta_{\min}^* \). Hence,
\[
\beta_{\min}^* \leq \| \hat{\beta}_k - \beta^* \|_2 \overset{(i)}{\leq} 2 \| \hat{\beta} - \beta^* \|_2 \overset{(ii)}{=} O \left( \rho^2 \sigma \left( \epsilon \sqrt{k \log(nd)} + \sqrt{\frac{k \log d}{n}} \right) \right),
\]
where (i) follows from the triangle inequality and definition of hard thresholding \( \| \hat{\beta}_k - \beta^* \|_2 \leq \| \hat{\beta}_k - \hat{\beta} \|_2 + \| \hat{\beta} - \beta^* \|_2 \leq 2 \| \hat{\beta} - \beta^* \|_2 \), and (ii) follows from the statistical guarantee in Theorem 4.1. This contradicts with the condition eq. (34), and hence we have the result in Theorem C.1. \( \square \)

C.2 Model selection for Gaussian graphical models

We then start to consider the sparsity recovery results for sparse precision matrix estimation – this is the part of Corollary 5.1. We first use following notations for a Gaussian graphical model.

We use \( x_i \) to denote the \( i \)-th samples of Gaussian graphical model, and \( X_j \) to denote the \( j \)-th random variable. Let \( (j) \) be the index set \( \{ 1, \cdots, j - 1, j + 1, \cdots, d \} \). We use \( \Sigma_{(j)} = \Sigma_{(j),(j)} \in \mathbb{R}^{(d-1) \times (d-1)} \) to denote the sub-matrix of covariance matrix \( \Sigma \) with both \( j \)-th row and \( j \)-th column removed, and use \( \sigma_{(j)} \in \mathbb{R}^{d-1} \) to denote \( \Sigma \)'s \( j \)-th column with the diagonal entry removed. Also, we use \( \theta_{(j)} \in \mathbb{R}^{d-1} \) to denote \( \Theta \)'s \( j \)-th column with the diagonal entry removed. and \( \Theta_{j,j} \in \mathbb{R} \) to denote the \( j \)-th diagonal element of \( \Theta \).

By basic probability computation, for each \( j = 1, \cdots, d \), the variable \( X_j \) conditioning \( X_{(j)} \) follows from a Gaussian distribution \( \mathcal{N}(X_j^\top \Sigma_{(j)}^{-1} \sigma_{(j)}, 1 - \sigma_{(j)}^\top \Sigma_{(j)}^{-1} \sigma_{(j)}) \). Then we have the linear regression formulation
\[
X_j = X_j^\top \beta_j + \xi_j,
\]
where \( \beta_j = \Sigma_{(j)}^{-1} \sigma_{(j)} \) and \( \xi_j \sim \mathcal{N}(0, 1 - \sigma_{(j)}^\top \Sigma_{(j)}^{-1} \sigma_{(j)}) \). Notice the definition of precision matrix \( \Theta \), we have \( \beta_j = -\theta_{(j)}/\Theta_{j,j} \), and \( \Theta_{j,j} = 1/\text{Var}(\xi_j) \). Thus for the \( j \)-th variable, \( \theta_{(j)} \) and \( \beta_j \) have the same sparsity pattern. Hence, the sparsity pattern of \( \theta_{(j)} \) can be estimated through \( \hat{\beta}_j \) via solving the optimization eq. (10) (Neighborhood Selection in [MB06]).

In Algorithm 2, we robustify Neighborhood Selection by using Trimmed Hard Thresholding (with loss function eq. (2a)) to robustify eq. (10). In line 6, we use Trimmed Hard Thresholding to regress each variable against its neighbors. In line 9, the sparsity pattern of \( \Theta \) can be estimated by aggregating the neighborhood support set of \( \{ \hat{\beta}_j \}_{j=1}^d \) via intersection or union. Similar to Theorem 5.1, a \( \theta_{\min} \)-condition guarantees consistent edge selection.

Corollary C.1 (Corollary 5.1). Under the same condition in Theorem 4.1, and a \( \theta_{\min} \)-condition for \( \theta_{(j)} \),
\[
\theta_{(j), \min} = \Omega \left( \Theta_{j,j}^{1/2} \rho^2 \left( \epsilon \sqrt{k \log(nd)} + \sqrt{\frac{k \log d}{n}} \right) \right),
\]
Algorithm 2 Neighborhood Selection via Trimmed Hard Thresholding (Robust NS)

1: **Input:** Data samples \( \{x_i\}_{i=1}^m \).
2: **Output:** The sparsity pattern estimation of \( \Theta \).
3: **Parameters:** Hard thresholding parameter \( k' \).
4: **for** each variable \( j \), **do**
5: Use \( X_j \) as response variable, and \( X_{(j)} \) as covariates.
6: Run Algorithm 1 with input \( \{x_{ij}, x_{i(j)}\}_{i=1}^m \). We set the parameter as \( k' \), and the loss function as least square loss eq. (2a).
7: The output of Algorithm 1 is denoted as \( \hat{\beta}_j \in \mathbb{R}^{d-1} \).
8: **end for**
9: Aggregate the neighborhood support set of \( \{\hat{\beta}_j\}_{j=1}^d \) via intersection or union.

Algorithm 2 is consistent in edge selection, with probability at least \( 1 - d^{-1} \).

**Proof of Corollary C.1.** Algorithm 2 iteratively uses Algorithm 1 as a Neighborhood Selection approach for each variable. Hence, we can apply Theorem 5.1 for each variable, and the sparsity patterns are the same according to \( \theta_j^* = -\beta_j / \text{Var}(\xi_j) \). The stochastic noise term \( \sigma \) in sparse linear regression can be expressed as \( 1 / \sqrt{\Theta_{j,j}} \). Hence, under the same condition as Theorem 4.1, for each \( j \in [d] \), we require a \( \theta_{\min} \)-condition for \( \theta_j \),

\[
\theta_{(j),\min} = \Omega \left( \Theta_{j,j}^{1/2} \rho^2 \left( \epsilon \sqrt{k \log(n)}d + \sqrt{\frac{k \log d}{n}} \right) \right),
\]

Using a union bound, we conclude that Algorithm 2 is consistent in edge selection, with probability at least \( 1 - d^{-1} \). \( \square \)

D Full experiments details

We study empirical performance of Trimmed Hard Thresholding (Algorithm 1 and Algorithm 2). And we present the complete details of experimental setup in Section 6.

D.1 Synthetic data – sparse linear models

We first consider the performance of Algorithm 1 under (generalized) linear models with \( \epsilon \)-corrupted samples.

**Sparse linear regression.** In the first experiment, we consider an exact sparse linear regression model (Model 3.1). In this model, the stochastic noise \( \xi \sim \mathcal{N}(0, \sigma^2) \), and we vary the noise level \( \sigma^2 \) in different simulations. We first generate authentic explanatory variables with parameters \( k = 5, d = 1000, n = 300 \), from a Gaussian distribution \( \mathcal{N}(\mathbf{0}_d, \Sigma) \), where the covariance matrix \( \Sigma \) is a Toeplitz matrix with an exponential decay \( \Sigma_{ij} = \exp^{-|i-j|} \). This design matrix is known to enjoy the RSC-condition \[RWY10\], which meets the requirement of Theorem 4.1. The entries of the \( k \)-sparse true parameter \( \beta^* \) are set to either +1 or −1. Fixing the contamination level at \( \epsilon = 0.1 \), we set the covariates of the outliers as \( A \), where \( A \) is a random \( \pm 1 \) matrix of dimension \( \epsilon \times \frac{n}{1-\epsilon} \times d \), and the responses of outliers to \( -A\beta^* \).

To show the performance of Algorithm 1 under different noise levels determined by \( \sigma^2 \), we track the parameter error \( \|\beta^t - \beta^*\|_2 \) in each iteration. In the left plot of Figure 5, Algorithm 1 shows
linear convergence, and the error curves flatten out at the level of the final error, which is consistent with our theory. Furthermore, Algorithm 1 can achieve machine precision when $\sigma^2 = 0$, which means exact recovery of $\beta^*$. 

**Misspecified model.** For the second experiment, we use a sparse linear regression with model misspecification—the underlying authentic samples do not follow a linear model. We use the same Toeplitz covariates and true parameter $\beta^*$, but and corresponding $y_i$’s are calculated as $y_i = \sum_{j=1}^{d} x_{ij}^3 \beta_j^*$. Although this is a non-linear function, sparse linear regression on these authentic samples can still recover the support, as the cubic function is monotone and $\beta^*$ is sparse. We generate outliers using the same distribution as the first experiment, but with a different fraction of corruptions $\epsilon$.

For simplicity, we track the function evaluated on all authentic samples $F(\beta) = \sum_{i \in G} (y_i - x_i^T \beta)^2$. In the right plot of Figure 5, we show the performance of Algorithm 1 under different $\epsilon$, and the oracle curve means using IHT only on authentic samples. The right plot has similar convergence under different values of corrupted fraction $\epsilon$, and shows the robustness of Algorithm 1 without assuming an underlying linear model.

### D.2 Robust M-estimators via Trimmed Hard Thresholding

Classical robust $M$-estimators [Loh17] (such as empirical risk minimization using Huber loss) are widely used in robust statistics in the case where the error distribution is heavy tailed or when there are arbitrary outliers only in the response variables. In the high dimensional setting, given $\epsilon$-corrupted samples Definition 2.1, we can use

$$\min_{\beta \in \mathcal{B}} \mathbb{E}_{i \in S} \ell_i(\beta; z_i), \quad \text{s.t.} \|\beta\|_0 \leq k,$$
where \( \ell_i(\beta; z_i) \) can be chosen as Huber loss with parameter \( \delta \):

\[
H_\delta(\beta; z_i) = \begin{cases} 
\frac{1}{2} (y_i - x_i^T \beta)^2 & \text{for } |y_i - x_i^T \beta| \leq \delta, \\
\delta |y_i - x_i^T \beta| - \frac{1}{2} \delta^2 & \text{otherwise.}
\end{cases}
\]

[Loh17] studied robust M-estimators in high dimensions, and proposed a composite optimization using \( \|\beta\|_1 \) instead of \( \|\beta\|_0 \). They established local convergence guarantee for this composite optimization procedure, using a local RSC condition (Definition 4.1) in a neighborhood around \( \beta^* \). Yet their results do not trivially extend to settings with arbitrarily corrupted covariates.

In our experiments, we use Huber loss in Trimmed Hard Thresholding to deal with heavy-tailed error distribution. In addition to heavy-tailed noise, \( \epsilon \)-fraction of \( \{y_i, x_i\}_{i=1}^n \) are still arbitrarily corrupted.

For the experiments, we use the same Toeplitz covariates and true parameter \( \beta^* \) as in previous experiments on sparse linear models with fixed dimension parameters \( k = 5, d = 1000, n = 300 \). The error distribution is a Cauchy distribution, which is a special case model misspecification, as it doesn’t meet the sub-Gaussian requirement in Model 3.1. For different contamination levels, we set the covariates of the outliers as \( A \), where \( A \) is a random \( \pm 1 \) matrix of dimension \( \epsilon \times d \), and the responses of outliers to \( -A\beta^* \).

Empirically, we observe linear convergence, and this is shown in Figure 6. This linear convergence results validates the local RSC condition proposed in [Loh17], and we can still achieve this even with \( \epsilon \)-fraction of corrupted covariates.

### D.3 Sparse logistic regression

For binary classification problem, we generate samples from a sparse LDA problem, where the distributions of the explanatory variables conditioned on the response variables follow multivariate Gaussian distributions with the same covariance matrix but different means.

We generate authentic samples \( x_i \) from a Gaussian distribution \( \mathcal{N}(\mu_+, I_d) \) if \( y_i = +1 \), and another distribution \( \mathcal{N}(\mu_-, I_d) \) if \( y_i = -1 \). The parameters are fixed \( k = 5, d = 1000, n = 300 \). We set \( \mu_+ = 1_d + v \), where \( v \) is \( k \)-sparse and its entries are set to be either \( +1/\sqrt{k} \) or \( -1/\sqrt{k} \). And we set
µ∗ = 1d − v. The Bayes classifier is β∗ = 2v. This is a special case of Model 3.2, and it is known that sparse logistic regression attains fast classification error rates [LPR15]. We then set the covariates of the outliers as A, where A is a matrix of dimension 1−ε × d, where the entries are random ±3. The responses of outliers follow the distribution Pr(yi|xi) = 1/(1 + exp(yi xi β∗)), which is exactly the opposite of Model 3.2.

We run Algorithm 1 with logistic loss under different levels of outlier fraction ε. In the left plot of Figure 7, we observe similar linear convergence as sparse linear regression This is consistent with Theorem 4.2 for sparse logistic regression, and it is clear that we cannot exactly recover β∗ unless the number of samples n is infinite.

We then compare Algorithm 1 with the Trimmed Lasso estimator for sparse logistic regression [YLA18]. Although they also use a trimming technique, their algorithm is totally different from Algorithm 1, as we use dimensional trimmed mean estimator for gradients in hard thresholding, but they trim samples in each iteration according to the each sample’s loss. Under the same sparse LDA model, we set k = √d, n = 15k. In simulation, we increase d, and plot classification error (averaged over 50 trials on authentic test set) for different ε = 0.1, 0.2. The right plot of Figure 7 shows that Trimmed Hard Thresholding is better than Trimmed Lasso.

D.4 Synthetic data – Gaussian graphical model

We generate Gaussian graphical model samples by huge [ZLR+12]. We choose the “cluster” sparsity pattern, where the clustering parameters are default values in the package where the number of clusters in the graph is d/20, the probability that a pair of nodes within a cluster are connected is 0.3, and there are no edges between nodes within different clusters. The off-diagonal elements of the precision matrix is denoted as v, which is an experiment parameter for SNR.

We then add an additional 1−ε fraction of samples sampled from another distribution. Following the experimental design in [YL15, WG17], each outlier is generated by a mixture of d-dimensional Gaussian distributions ½N(μo, Σo) + ½N(−μo, Σo), where μo = (1.5, 1.5, · · ·, 1.5)T, and Σo = Id. We compare Algorithm 2 with other existing methods: Trimmed GLasso [YLA18], RCLIME [WG17],
Figure 8: ROC curves of different methods on cluster graphs with arbitrary corruptions. The curve Robust NS denotes Algorithm 2, and Oracle NS denotes the neighborhood selection Lasso only on authentic data.

Skeptic [LHY12b], and Spearman [LT18]. The latter two are based on robustifying the covariance matrix, and then using standard graphical model selection algorithms such as GLasso or CLIME. To directly compare these methods, we use CLIME for both of them.

To evaluate model selection performance, we use receiver operating characteristic (ROC) curves to compare our method to others over the full regularization paths. We generate regularization paths for other robust algorithms by tuning the $\lambda$ in CLIME and GLasso. For Algorithm 2, we explicitly tune different sparsity level $k'$ to generate the regularization path.

We set $\epsilon = 0.1$, and vary $(n, d)$, and the SNR parameter $v$ for off-diagonal elements. We use different $(n, d) = (100, 100), (200, 200)$. For different off-diagonal values, we set $v = 0.3$ (Low SNR), and $v = 0.6$ (High SNR). We show ROC curves to demonstrate model selection performance in Figure 8. For the entire regularization path, our algorithm (denoted as Robust NS) has a better ROC compared to other algorithms.

In particular, Robust NS outperforms other methods with higher true positive rate when the false positive rate is small. This is the case where we use smaller hard thresholding sparsity in Algorithm 2, and larger regularization parameter for $\|\Theta\|_1$ other methods based on GLasso and CLIME. This
validates our theory in Corollary 5.1, which guarantees sparsity recovery when hard thresholding hyper-parameter $k'$ is suitably chosen to match $\beta^*$’s sparsity $k$.

### D.5 Real data experiments

Here, we present details of the experiment using US equities data [ZLR+12]. We preprocess it by taking log-transformation and calculate the corresponding daily returns. Obvious outliers are removed by winsorizing each variable so that all samples are within five times the winsorized standard deviation from the winsorized mean. After preprocessing, we present example histograms and QQ plots from the Information Technology sector. In Figure 9, we list the histograms of two typical companies in this sector. As we can see from Figure 10, even after preprocessing on these stock prices, they are still highly non-normal and heavy tailed. We do not add any manual outliers as financial data is already heavy tailed and have many outliers [dP18]. We also compare Algorithm 2 with the baseline NS approach (without consideration for corruptions or outliers).

We limit the number of edges to 2,000 for both methods. The cluster colored by purple denotes the Information Technology sector. In Figure 11, we can easily separate different clusters by using Robust NS. However, the Vanilla NS approach cannot distinguish the sector Information Technology (purple). Furthermore, we can observe that stocks from Information Technology (colored by purple) are much better clustered by Algorithm 2.
(a) Adobe Systems Inc from sector Information Technology.
(b) Agilent Technologies Inc from sector Information Technology.

Figure 10: After the same preprocessing, we present the QQ plot of the daily returns versus standard normal.

(a) Graph estimated by Robust NS (Algorithm 2).
(b) Graph estimated by Vanilla NS approach.

Figure 11: Graph estimated from the S&P 500 stock data by Algorithm 2 and Vanilla NS approach. Variables are colored according to their sector. In particular, the stocks from sector Information Technology are colored as purple.