Exp-function method for Klein–Gordon equation with quadratic nonlinearity

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Abstract. In this paper, the Exp-function method is used to obtain generalized solitonary solutions and periodic solutions of Klein–Gordon equation with quadratic nonlinearity. It is shown that the Exp-function method with the help of symbolic computation provides a straightforward and very effective mathematical tool for solving nonlinear evolution equations in mathematical physics.

1. Introduction
The investigation of exact solutions of nonlinear evolution equations (NLEEs) plays an important role in the study of nonlinear physical phenomena. In the past several decades, many effective methods [1-20] for obtaining exact solutions of NLEEs have been presented such as Bäcklund transformation [2], homogenous balance method [4], homotopy perturbation method [7], variational method [9], asymptotic methods [12], tanh-function method [15], F-expansion method [20] and so on.

Recently, He and Wu [21] proposed a straightforward and concise method called Exp-function method to obtain exact solutions of NLEEs. The Exp-function method leads to both generalized solitonary solutions and periodic solutions. The solution procedure of this method, by the help of Matlab or Mathematica, is of utter simplicity and this method has been successfully applied to many kinds of equations [22-26]. Furthermore this method is also valid for difference equations [27, 28].

The present paper is motivated by the desire to extend the Exp-function method to a Klein–Gordon equation with quadratic nonlinearity, which reads

\[ u_{tt} - \alpha^2 u_{xx} + \beta u - \gamma u^2 = 0, \]  

where \( \alpha, \beta \) and \( \gamma \) are known constants. Jacobi elliptic function solutions and solitary wave solutions of equation (1) can be found in [29-31].

2. Application to the Klein–Gordon equation
Using the transformation

\[ u = U(\eta), \ \eta = kx + \omega t, \]

where \( k \) and \( \omega \) are constants to be determined later, equation (1) becomes

\[ U''(\eta) - \alpha^2 U''(\eta) + \beta U(\eta) - \gamma U(\eta)^2 = 0, \]
where prime denotes the derivative with respect to $\eta$.

According to the Exp-function method [21], we assume that the solution of equation (3) can be expressed in the following form

$$U(\eta) = \frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}, \quad \text{ (4)}$$

where $c$, $d$, $p$ and $q$ are positive integers which are unknown to be further determined, $a_n$ and $b_m$ are unknown constants. Equation (4) can be re-written in an alternative form [22] as follows

$$U(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)}. \quad \text{ (5)}$$

In order to determine values of $c$ and $p$, we balance the linear term of highest order in equation (3) with the highest order nonlinear term [21, 22]. By simple calculation, we have

$$U^n = \frac{c_1 \exp((3p+c)\eta) + \cdots}{c_2 \exp(4p\eta) + \cdots} \quad \text{ (6)}$$

and

$$U^2 = \frac{c_3 \exp(2c\eta) + \cdots}{c_4 \exp(2p\eta) + \cdots} = \frac{c_3 \exp(2(p+c)\eta) + \cdots}{c_4 \exp(4p\eta) + \cdots}, \quad \text{ (7)}$$

where $c_i$ are determined coefficients only for simplicity.

Balancing highest order of Exp-function in equations (6) and (7), we have

$$3p + c = 2(p+c), \quad \text{ (8)}$$

which leads to the result

$$p = c. \quad \text{ (9)}$$

Similarly we balance the linear term of lowest order in equation (3) to determine values of $d$ and $q$, we obtain

$$U^n = \frac{\cdots + d_c \exp(-(3q+d)\eta)}{\cdots + d_c \exp(-4q\eta)} \quad \text{ (10)}$$

and

$$U^2 = \frac{\cdots + d_c \exp(-2d\eta) + \cdots}{\cdots + d_c \exp(-2q\eta) + \cdots} = \frac{\cdots + d_c \exp(-(2q+d)\eta)}{\cdots + d_c \exp(-4q\eta)}, \quad \text{ (11)}$$

where $d_c$ are determined coefficients only for simplicity. Balancing lowest order of Exp-function in equations (10) and (11), we have

$$-(3q+d) = -2(q+d), \quad \text{ (12)}$$

which leads to the result

$$q = d. \quad \text{ (13)}$$
2.1. Case 1: $p=c=1, \ d=q=1$

We can freely choose the values of $c$ and $d$, but we will illustrate that the final solution does not strongly depend upon the choice of values of $c$ and $d$ [21, 22]. For simplicity, we set $p=c=1$ and $d=q=1$, the trial function, equation (5) becomes

$$U(\eta) = \frac{a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \tag{14}$$

Substituting equation (14) into (3) and equating the coefficients of all powers of $\exp(n\eta)$ to zero yields a set of algebraic equations for $a_1, a_0, a_{-1}, b_1, b_0, b_{-1}, k$ and $\omega$ as follows:

$$a_b b^2_\beta - a^2_1 b^2_\gamma = 0,$$

$$-a_{-1} b_1 \omega^2 + a_{-1} b^2_\gamma = 0,$$

$$a_{-1} b^2_\gamma - a_{-1} b_1 \omega^2 + 4 a_{-1} b^2_\gamma = 0,$$

$$a_{-1} b^2_\gamma - a_{-1} b_1 \omega^2 + 4 a_{-1} b^2_\gamma = 0,$$

$$+a_{-1} b^2_\beta + 2 a_{-1} b_1 \beta + a_{-1} b^2_\gamma - 2 a_{-1} a_{-1} \gamma - 2 a_{-1} a_{-1} \gamma - 2 a_{-1} a_{-1} \gamma = 0,$$

$$3 a_{-1} b_1 \omega^2 + 3 a_{-1} b_1 \omega^2 - 6 a_{-1} b_1 \omega^2 - 3 a_{-1} b_1 \omega^2 - 3 a_{-1} b_1 \omega^2 + 6 a_{-1} b_1 \omega^2 + a_{-1} b^2_\beta$$

$$+2 a_{-1} b_1 \beta + 2 a_{-1} b_1 \beta - a_{-1} b^2_\gamma - 2 a_{-1} a_{-1} \gamma - 2 a_{-1} a_{-1} \gamma - 2 a_{-1} a_{-1} \gamma = 0,$$

$$a_{-1} b^2_\gamma - a_{-1} b_1 \omega^2 - 4 a_{-1} b_1 \omega^2 + 4 a_{-1} b_1 \omega^2 = 0,$$

$$a_{-1} b^2_\gamma - a_{-1} b_1 \omega^2 + 4 a_{-1} b^2_\gamma = 0,$$

$$+a_{-1} b^2_\beta + 2 a_{-1} b_1 \beta + 2 a_{-1} b_1 \beta - 2 a_{-1} a_{-1} \gamma - a_{-1} b^2_\gamma - 2 a_{-1} a_{-1} \gamma = 0,$$

$$-a_{-1} b^2_\gamma = 0,$$

$$a_{-1} b^2_\gamma = 0.$$

Solving the system of algebraic equations with the aid of Mathematica, we obtain

$$a_1 = \frac{b_1 \beta}{\gamma}, \ a_0 = \frac{-2 b_1 \beta}{\gamma}, \ a_{-1} = \frac{b^2_0 \beta}{4 b_1 \gamma}, \ b_1 = b_1,$$

$$b_0 = b_0, \ b_{-1} = \frac{b^2_0}{4 b_1}, \ k = k, \ \omega = \pm k \sqrt{\alpha^2 + \frac{\beta}{k^2}}. \tag{15}$$

We, therefore, obtain the following generalized solitaury solution of equation (1)

$$u = \frac{b_1 \beta}{\gamma} \exp \left( kx \pm k \sqrt{\alpha^2 + \frac{\beta}{k^2} t} \right) - \frac{2 b_1 \beta}{\gamma} + \frac{b^2_0 \beta}{4 b_1 \gamma} \exp \left( -kx \mp k \sqrt{\alpha^2 + \frac{\beta}{k^2} t} \right)$$

$$b_1 \exp \left( kx \pm k \sqrt{\alpha^2 + \frac{\beta}{k^2} t} \right) + b_0 + \frac{b^2_0}{4 b_1} \exp \left( -kx \mp k \sqrt{\alpha^2 + \frac{\beta}{k^2} t} \right).$$
\[
\frac{\beta}{\gamma} - \frac{3b_1\beta}{\gamma} \exp\left( kx \pm k\sqrt{\alpha^2 + \frac{\beta}{k^2}t} \right) + b_0 + \frac{b_0^2}{4b_1} \exp\left( -kx \mp k\sqrt{\alpha^2 + \frac{\beta}{k^2}t} \right).
\]  
(17)

To compare equation (17) with that in open literature, we write down Sirendaoreji's solution [31]
\[
u = \frac{\beta}{2\gamma} \left\{ 2 - 3\text{sech}^2 \left[ \frac{1}{2} \sqrt{\frac{\beta}{\omega^2 - \alpha^2}} (x + \omega t) \right] \right\}.
\]  
(18)

We re-write equation (18) in the form
\[
u = \frac{\beta}{\gamma} - \frac{6\beta}{\gamma} \exp\left[ \frac{\beta}{\omega^2 - \alpha^2} (x + \omega t) \right] + 2 + \exp\left[ -\frac{\beta}{\omega^2 - \alpha^2} (x + \omega t) \right].
\]  
(19)

It is obvious that if we set \( b_1 = 1 \), \( b_0 = 2 \) and \( k = \sqrt{\beta/(\omega^2 - \alpha^2)} \), equation (17) turns out to be Sirendaoreji's solution as expressed in (19).

When \( k \) is an imaginary number, the obtained solitary solution can be converted into periodic solution [21, 22]. We write \( k = iK \). Using the transformation
\[
\exp\left[ \pm \left( kx \pm k\sqrt{\alpha^2 + \frac{\beta}{k^2}t} \right) \right] = \cos\left( Kx \pm K\sqrt{\alpha^2 - \frac{\beta}{K^2}t} \right) \pm i \sin\left( Kx \pm K\sqrt{\alpha^2 - \frac{\beta}{K^2}t} \right),
\]
then equation (17) becomes
\[
u = \frac{\beta}{\gamma} - \frac{3b_1\beta}{\gamma} \cos\left( Kx \pm K\sqrt{\alpha^2 - \frac{\beta}{K^2}t} \right) + b_0 + \frac{b_0^2}{4b_1} \cos\left( Kx \pm K\sqrt{\alpha^2 - \frac{\beta}{K^2}t} \right). \]  
(20)

If we search for a periodic solution or compact-like solution, the imaginary part in equation (20) must be zero [21, 22], that requires
\[
b_1 - \frac{b_0^2}{4b_1} = 0. \]  
(21)

From equation (21) we obtain
\[
b_0 = \pm 2b_1. \]  
(22)

Substituting equation (22) into (20) yields two periodic solutions
\[
u = \frac{\beta}{\gamma} - \frac{3\beta}{\gamma} \cos\left( Kx \pm K\sqrt{\alpha^2 - \frac{\beta}{K^2}t} \right) + 1 \]  
(23)

and
\[ u = \beta + \frac{3\beta}{\gamma} \cos \left( Kx \pm K \sqrt{k^2 + \frac{\beta^2}{K^2}} \right) - 1. \]  

(24)

2.2. Case 2: \( p = c = 2, \ d = q = 2 \)

As mentioned above the values of \( c \) and \( d \) can be freely chosen, we set \( p = c = 2 \) and \( d = q = 2 \), then the trial function, equation (5) becomes

\[ U(\eta) = \frac{a_2 \exp(2\eta) + a_2 \exp(\eta) + a_0 + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{b_2 \exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta) + b_{-2} \exp(-2\eta)}. \]  

(25)

There are some free parameters in (25), we set \( b_2 = 1, \ b_1 = 0 \) and \( b_{-1} = 0 \) for simplicity, the trial function, equation (25) is simplified as follows

\[ U(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta) + a_{-2} \exp(-2\eta)}{\exp(2\eta) + b_0 + b_{-2} \exp(-2\eta)}. \]  

(26)

By the same manipulation as illustrated above, we obtain

\[ a_2 = \frac{\beta}{\gamma}, \ a_1 = a_1, \ a_0 = \frac{5a_1^2}{18\beta}, \ a_{-1} = \frac{a_1^2}{36\beta}, \ a_{-2} = \frac{a_1^2}{1296\beta}. \]  

(27)

\[ b_0 = -\frac{a_1^2}{18\beta}, \ b_2 = \frac{a_1^2}{1296\beta}, \ k = k, \ \omega = \pm k \sqrt{\alpha^2 + \beta^2}. \]  

(28)

Substituting equations (27) and (28) into (26) yields the following solution

\[ u = \beta + \frac{a_1}{\gamma} \exp \left( kx \pm k \sqrt{\alpha^2 + \beta^2} \right) - \frac{a_2}{3\beta} \exp \left( -kx \pm k \sqrt{\alpha^2 + \beta^2} \right) \cos \left( kx \pm k \sqrt{\alpha^2 + \beta^2} \right). \]  

(29)

If set \( b_1 = 1 \) and \( b_0 = -a_1^2 / 3\beta \) in equation (17), we can recover the solution (29).

2.3. Case 2: \( p = c = 2, \ d = q = 2 \)

We consider the case \( p = c = 2 \) and \( d = q = 1 \), equation (5) can be expressed as

\[ U(\eta) = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_2 \exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}. \]  

(30)

Setting \( b_2 = 1 \) in (30) for simplicity and using the same manipulation as illustrated above, we obtain

\[ a_2 = \frac{\beta}{\gamma}, \ a_1 = a_1, \ a_0 = -\frac{(b_1 \beta - a_1 \gamma)(5b_2 \beta + 3a_1 \gamma)}{12\beta}, \]  

(31)

\[ a_{-1} = -\frac{(b_1 \beta - a_1 \gamma)(2b_1 \beta + a_1 \gamma)}{108\beta^2 \gamma}, \ b_1 = b_1, \ b_0 = -\frac{(b_1 \beta - a_1 \gamma)(3b_1 \beta + a_1 \gamma)}{12\beta^2}. \]  

(32)
Substituting equations (31)-(33) along with \( b_1 = 1 \) into (30) yields the following solution

\[
\begin{align*}
\beta - a_1 \gamma \\
\beta - a_1 \gamma \\
\end{align*}
\]

(34)

If set \( b_1 = 1 \) and \( b_0 = (\beta - a_1 \gamma) / 3 \beta \) in equation (17), we can recover the solution (34).

3. Conclusion

The Exp-function method has been used to obtain generalized solitary solutions and periodic solutions of a Klein–Gordon equation with quadratic nonlinearity. The results show that the Exp-function method is a promising and powerful new method for NLEEs arising in mathematical physics.

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