HARMONIC MAPS BETWEEN TWO CONCENTRIC ANNULI IN $\mathbb{R}^3$

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Abstract. Given two annuli $A(r, R)$ and $A(r_*, R_*)$, in $\mathbb{R}^3$ equipped with the Euclidean metric and the weighted metric $|y|^{-2}$ respectively, we minimize the Dirichlet integral, i.e. the functional $F[f] = \int_{A(r, R)} |Df|^2 |f|^2$, where $f$ is a homeomorphism between $A(r, R)$ and $A(r_*, R_*)$, which belongs to the Sobolev class $W^{1,2}$. The minimizer is a certain generalized radial mapping, i.e. a mapping of the form $f(|x|\eta) = \rho(|x|)T(\eta)$, where $T$ is a conformal mapping of the unit sphere onto itself and $\rho(t) = R_* \left( \frac{R_*}{R_* - r_*} \right)^{\frac{R_*}{R_* - r_*}}$. It should be noticed that in this case no Nitsche phenomenon occur.

1. Introduction and statement of the main result

The general law of hyperelasticity tells us that there exists an energy integral $E[h] = \int_X E(x, h, Dh) dx$ where $E : X \times Y \times \mathbb{R}^{n \times n} \to \mathbb{R}$ is a given stored-energy function characterizing mechanical properties of the material. Here $X$ and $Y$ are nonempty bounded domains in $\mathbb{R}^n$, $n > 2$. The mathematical models of nonlinear elasticity have been firstly studied by Antman [2], Ball [5], and Ciarlet [8]. One of interesting and important problems in nonlinear elasticity is whether the radially symmetric minimizers are indeed global minimizers of the given physically reasonable energy. This leads us to study energy minimal homeomorphisms $h : A^{out} \to A^{out}_*$ of Sobolev class $W^{1,2}$ between annuli $A = A(r, R) = \{ x \in \mathbb{R}^n : r < |x| < R \}$ and $A_* = A(r_*, R_*) = \{ x \in \mathbb{R}^n : r_* < |x| < R_* \}$. Here $0 \leq r < R$ and $0 \leq r_* < R_*$ are the inner and outer radii of $A$ and $A_*$. The variational approach to Geometric Function Theory [3, 4] makes this problem more important. Indeed, several papers are devoted to understand the expected radial symmetric properties see [17] and the references therein. Many times experimentally known answers to practical problems has led us to deeper study of such mathematically challenging problems. We seek to minimize the 2-harmonic energy of mappings between two annuli in $\mathbb{R}^3$. We consider the modified Dirichlet energy $F[f] = \int_A |Dh|^2 |f|^2$ and solve the problem of modified Dirichlet energy in the fourth section. The problem for Dirichlet energy $E[f] = \int_A |Dh|^2$ is considered in the appendix below, but not solved.

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completely. The research is related to the J. C. C. Nitsche conjecture [20]. The conjecture has been solved by Kovalev, Iwaniec and Onninen in [10] after some partial results by Lyzzaik [18], Weitsman [24] and Kalaj [15]. The conjecture raised a very important research in Geometric Function Theory connected to the nonlinear elasticity. See for example the papers [4], [11] and [14].

In order to formulate the main result, let us define the generalized radial mappings.

We say that $f : A \to A^*$ is a generalized radial mapping, if there exists a conformal transformation $T$ of $S$ onto itself, so that $f(x) = \rho(|x|)T \left( \frac{x}{|x|} \right)$. If $T$ is the identity, then we remove the prefix "generalized". For the representation of the class of conformal mappings of the sphere onto itself we refer to the books [1] and [23].

The following is the main result of the paper

**Theorem 1.1.** Assume that $F$ is the family a homeomorphisms between spherical rings $A(r, R)$ and $A(r^*, R^*)$ in $R^3$ that belongs to $W^{1,2}$. Then for the Dirichlet integral of $f \in F$ with respect to the weight $\varphi(w) = |w|^{-2}$, we have

$$\mathcal{F}[f] = \int_{A(r, R)} \frac{\|Df\|^2}{|f|^2} dx \geq 4\pi \left( 2(R-r) + \frac{r R \log \frac{R_r}{r_r}}{R-r} \right)^2,$$

where $dx$ is the Lebesgue measure, and the infimum is achieved for the following generalized radial diffeomorphisms between annuli

$$f_1(x) = r_* \left( \frac{r_*}{R_*} \right)^{R(x-|x|)} T \left( \frac{x}{|x|} \right), \quad f_2(x) = R_* \left( \frac{r_*}{R_*} \right)^{R(|x|-r)} T \left( \frac{x}{|x|} \right).$$

The minimizer is unique up to a conformal change $T$ of $S$.

**Remark 1.2.** If we denote the outer boundary of $A$ by $\partial A$ and consider the subfamily of homomorphisms $\mathcal{F}_0 = \{ f \in \mathcal{F} : f(x) = \frac{R}{R^*} x, \text{ for } x \in \partial A \}$, then the minimizer is the mapping $h(x) = \rho(x) \frac{x}{|x|}$. See the paper by Koski and Onninen [17] where they make this constraint in order to prove that the minimizer is radial but for annuli on the plane, and $p$ energy. On the other hand when $R_* = r_* = 1$, then the result says that the mappings $h(x) = T(x/|x|)$, of the unit sphere onto itself minimize the energy of mappings onto the unit sphere. This is an old problem solved by several authors (see for example [7], [9], [19]). Theorem 1.1 together with its Corollary 5.1 says that the case of degeneric annuli ($r = r_* = 0$) is substantially different from the case of proper annuli concerning the Dirichlet energy. In the case of degeneric annuli, the minimal energy is zero [12].

2. **Harmonic mappings and $p$–harmonic mappings**

In the following we define several classes of mappings which appear as the critical points of various energy integrals. Assume that $h = \varphi^2$ is a
positive smooth real function defined in the domain $\mathcal{A}_\ast$. Then it defines
the Riemannian manifold $(\mathcal{A}_\ast, h)$. Assume that $\mathcal{A}$ is equipped with the
Euclidean metric $g = 1$ and let $f : (\mathcal{A}, g) \to (\mathcal{A}_\ast, g)$ be a $C^1$ map between
manifolds. The energy density is defined \cite[Chapter IX]{[22]} by
\[
e(f) = \operatorname{Tr}_g(f^*h) = \sum_{\alpha,\beta,i,j} g^{\alpha\beta}(x) h_{ij}(f(x)) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.
\]
Thus
\[
e(u) = \varrho^2(f(x)) \sum_{\alpha,\beta,i,j} \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta} = \varrho^2(f(x)) \| Dh \|^2,
\]
where $\| \cdot \|$ is the Gram-Schmidt norm defined by $\| Dh \|^2 = \operatorname{Tr}(Dh^*Dh)$.
Assume that $2 \leq p \leq n$ and let $\varphi \overset{\text{def}}{=} \varrho^p$. The classical Dirichlet problem
concerns the energy minimal mapping $h : \mathcal{A} \to \mathbb{R}^n$ of the Sobolev class $h \in \mathcal{W}^{1,n}_0(\mathcal{A}, \mathbb{R}^n)$ whose boundary values are explicitly prescribed by
means of a given mapping $h_\circ \in \mathcal{W}^{1,n}_0(\mathcal{A}, \mathbb{R}^n)$. More precisely we deal with
the energy integral
\[
\mathcal{E}_p[h] = \mathcal{E}_{p,p}[h] \overset{\text{def}}{=} \int_{\mathcal{A}} e(f(x))^{p/2} dx = \int_{\mathcal{A}} \varphi(h(x)) \| Dh \|^p dx.
\]
Let us consider the variation $h \sim h + \epsilon \eta$, in which $\eta \in C^\infty(\mathcal{A}, \mathbb{R}^n)$ and $\epsilon \to 0$, leads to the integral form of the $p$-harmonic system of equations
\begin{equation}
(2.1)
\int_{\mathcal{A}} \left( (\nabla \rho, \eta) \| Dh \|^p + \langle \varphi(h) \| Dh \|^{p-2} Dh, D\eta \rangle \right) = 0, \quad \text{for every } \eta \in C^\infty_0(\mathcal{A}, \mathbb{R}^n).
\end{equation}
Equivalently
\begin{equation}
(2.2)
\Delta_p h = \operatorname{Div}(\varphi(h) \| Dh \|^{p-2} Dh) - \frac{1}{p} \| Dh \|^p \nabla \varphi = 0,
\end{equation}
in the sense of distributions. The solutions to \cite{[22]} are called $p$-harmonic mappings.
If $p = 2$ the equation is called the harmonic equation, and the solutions are called the harmonic mappings.

Similarly as in in \cite{[11]} (see also \cite{[16]}), it can be derived the general $(\varphi, p)$-harmonic equation which by using a different variation as the following.

The situation is different if we allow $h$ to slip freely along the boundaries. The inner variation come to stage in this case. This is simply a change of the variable; $h_\epsilon = h \circ \eta_\epsilon$, where $\eta_\epsilon : \overset{\text{out}}{\mathcal{A}} \to \mathcal{A}$ is a $C^\infty$-smooth diffeomorphism
of $\mathcal{A}$ onto itself, depending smoothly on a parameter $\epsilon \approx 0$ where $\eta_0 = id : \overset{\text{out}}{\mathcal{A}} \to \mathcal{A}$. Let us take on the inner variation of the form
\begin{equation}
(2.3)
\eta_\epsilon(x) = x + \epsilon \eta(x), \quad \eta \in C^\infty(\mathcal{A}, \mathbb{R}^n).
\end{equation}
By using the notation $y = x + \epsilon \eta(x) \in \mathcal{A}$, we obtain
\[
\varphi(h_\epsilon) Dh_\epsilon(x) = \varphi(h(y)) Dh(y)(I + \epsilon D\eta(x)).
\]
Hence
\[ \varphi(h_\epsilon(x)) \parallel Dh_\epsilon(x) \parallel^p = \varphi(h(y)) \parallel Dh(y) \parallel^p + p\epsilon \varphi(h(y)) \langle \parallel Dh(y) \parallel^{p-2} D^*h(y) \cdot Dh(y), D\eta \rangle + o(\epsilon). \]

Integration with respect to \( x \in A \) we obtain
\[ E^\rho[\epsilon] = \int_A \varphi(h_\epsilon(x)) \parallel Dh_\epsilon(x) \parallel^pdx = \int_A \left[ \varphi(h(y)) \parallel Dh(y) \parallel^p + p\epsilon \varphi(h(y)) \langle \parallel Dh(y) \parallel^{p-2} D^*h(y) \cdot Dh(y), D\eta(y) \rangle \right] dy + o(\epsilon). \]

We now make the substitution \( y = x + \epsilon \eta(x) \), which is a diffeomorphism for small \( \epsilon \), for which we have: \( x = y - \epsilon \eta(y) + o(\epsilon) \), \( D\eta(x) = D\eta(y) + o(1) \), when \( \epsilon \to 0 \), and the change of volume element \( dx = [1 - \epsilon \text{Tr} D\eta(y)] dy + o(\epsilon) \).

Further
\[ \int_A \varphi(h(y)) \parallel Dh(y) \parallel^pdy = \int_A \varphi(h(y)) \parallel Dh(y) \parallel^p[1 - \epsilon \text{Tr} D\eta(y)] dy + o(\epsilon) \]

The so called equilibrium equation for the inner variation is obtained from \( \frac{d}{d\epsilon} E^\rho[\epsilon] = 0 \) at \( \epsilon = 0 \),

\[ \int_A \langle \varphi(h) \parallel Dh \parallel^{p-2} D^*h \cdot Dh - \frac{\varphi(h)}{p} \parallel Dh \parallel^pI, D\eta \rangle dy = 0 \]

or, by using distributions
\[ \text{Div} \left( \varphi(h) \parallel Dh \parallel^{p-2} D^*h \cdot Dh - \frac{\varphi(h)}{p} \parallel Dh \parallel^pI \right) = 0. \]

By putting
\[ h(x) = H(t) \frac{x}{t}, \quad t = |x| \]

we get
\[ Dh(x) = \frac{H(t)}{t} I + \frac{tH'(t) - H(t)}{t} \cdot \frac{x \otimes x}{|x|^2} \]

and
\[ \parallel Dh \parallel^2 = H(t)^2 + (n - 1) \frac{H(t)^2}{t^2} \]

Then we obtain
\[ D^*h \cdot Dh = \frac{H(t)^2}{t^2} I + \frac{t^2 H'(t)^2 - H(t)^2}{t^2} \frac{x \otimes x}{|x|^2} \]

We will focus on a particular problem, i.e. the case \( n = 3, p = n - 1 = 2 \) and \( \varphi(y) = |y|^{-2} \). So we consider the harmonic mappings between threedimensional Riemannian manifolds \((A, g)\) and \((A_*, h)\).
Then we have
\[
\varphi(h) \parallel Dh \parallel^{p-2} D^*h \cdot Dh - \frac{\varphi(h)}{p} \parallel Dh \parallel^p I = \varphi(h) \left( D^*h \cdot Dh - \frac{1}{2} \parallel Dh \parallel^2 I \right)
\]
\[
= \left( -\frac{\dot{H}(t)^2}{2H^2(t)} I + \frac{t^2 \dot{H}(t)^2}{t^2 H^2(t)} \frac{x \otimes x}{|x|^2} \right)
\]
\[
= (M(t) - t^{-2}) \frac{x \otimes x}{|x|^2} - \frac{M(t)}{2} I,
\]
where
\[
(2.6) \quad M(t) = \frac{\dot{H}(t)^2}{H^2(t)}, \quad t = |x|, \quad x = (x_1, x_2, x_3).
\]

Now (2.5) reduces to the differential equation
\[
\left( \frac{2M(t)}{t} + \frac{M'(t)}{2} \right) \frac{x}{|x|} = 0.
\]

By having in the mind the substitution (2.6) we obtain the following equation
\[
(2.7) \quad \left( \frac{2H(t)\dot{H}(t) - tH(t)^2 + tH(t)\ddot{H}(t)}{t^2 H(t)} \right) x \equiv 0.
\]

In order to consider the equation (2.2) for the case \( n - 1 = 2 = p \), we first have
\[
\text{Div} \left( \frac{Dh}{|h|^2} \right) = \frac{1}{2} \parallel Dh \parallel^2 \nabla \rho = -\frac{\parallel Dh \parallel^2}{|h|^4} h.
\]

Then
\[
(2.8) \quad \Delta h = \frac{2}{|h|^2} \sum_{j=1}^3 \sum_{k=1}^3 D_k h_j \langle h, D_k h \rangle e_j - \frac{\parallel Dh \parallel^2}{|h|^2} h.
\]

Put in the previous equation \( h(x) = H(t) \frac{x}{|x|} \), where \( t = |x| \). Then we have
\[
\Delta h = -\frac{2H(t)}{t^3} + \frac{2tH'(t)}{t^2} + \frac{t^2 H''(t)}{t^2} x
\]
and
\[
(2.9) \quad \parallel Dh \parallel^2 = \frac{2H(t)^2}{t^2} + H'(t)^2
\]
and
\[
\frac{2}{|h|^2} \sum_{j=1}^3 \sum_{k=1}^3 D_k h_j \langle h, D_k h \rangle e_j = \frac{2}{H(t)^2} \frac{H(t)H'(t)^2}{t} x.
\]

By plugging the previous three quantities in (2.8) we get again (2.7).
It follows from our main result that if instead of \( h(x) = H(t) \frac{x}{|x|} \), we put the following constraint \( h(x) = H(t)T \left( \frac{x}{|x|} \right) \) in (2.8) we again arrive to the following equation

\[
\left( \frac{2H(t)\dot{H}(t) - t\dot{H}(t)^2 + tH(t)\ddot{H}(t)}{t^2 H(t)} \right) T \left( \frac{x}{|x|} \right) \equiv 0,
\]

which is equivalent to (2.7). We will solve those equations later.

It is easily seen that one of the solutions of (2.7) is induced by the function \( H(t) = 1 \), namely the mapping \( h(x) = \frac{x}{|x|} \). This mapping is harmonic and solves both equations (2.2) and (2.5) but it is not a diffeomorphism. This makes a substantial difference between the corresponding equations in [11], where the authors Iwaniec and Onninen shown that the mapping \( f(x) = \frac{x}{|x|} \) is generalized \( n \)-harmonic but it is not \( n \)-harmonic.

3. Some preliminary results

For a mapping \( f \in \mathcal{F}(A, A_*) \) we put

\[
f(x) = \rho(x) S(x), \quad |S(x)| = 1.
\]

Then

\[
Df(x) = \nabla \rho(x) \otimes S(x) + \rho DS(x).
\]

So for any vector \( k \) we have

\[
Df(x)k = \langle \nabla \rho(x), k \rangle S(x) + \rho DS(x)k.
\]

It follows that

\[
|Df(x)k|^2 = |\nabla \rho(x), k|^2 + \rho^2 |DS(x)k|^2 + 2 \langle \nabla \rho(x), k \rangle \langle S(x), DS(x)e_i \rangle.
\]

Since \( |S(x)|^2 = 1 \), we have \( \langle S(x), DS(x)k \rangle = 0 \). Thus

\[
(3.1) \quad |Df(x)k|^2 = |\nabla \rho(x), k|^2 + \rho^2 |DS(x)k|^2.
\]

So summing for \( k = e_1, \) and \( i = 1, \ldots, n \) we get

\[
(3.2) \quad |Df(x)|^2 = |\nabla \rho(x)|^2 + \rho^2 \|DS\|^2.
\]

Let \( f \) be a function between \( A \) and \( B \). By \( N(y, f) \) we denote the cardinal number of \( f^{-1}(y) \) if the last set is finite and we set \( N(y, f) = +\infty \) in the other case. The function \( y \to N(y, f) \) is defined on \( B \). If \( f \) is surjective then \( N(y, f) \geq 1 \) for every \( y \in B \). The following proposition hold.

Proposition 3.1. [21] Let \( U \) be an open subset of \( \mathbb{R}^n \) and let \( f : U \to \mathbb{R}^n \) be \( C^1 \) mapping. Then the function \( y \to N(y, f) \) is measurable on \( \mathbb{R}^n \) and

\[
(3.3) \quad \int_{\mathbb{R}^n} N(y, f) \, dy = \int_U |J(x, f)| \, dx,
\]

where \( J(x, f) \) is the Jacobian of \( f \).
Further, let $h$ be a $C^1$ surjection from an $n - 1$ dimensional rectangle $K^{n - 1}$ onto the unit sphere $S^{n - 1}$. Let the function $f$ be defined in the $n$ dimensional rectangle $K^n = [0, 1] \times K^{n - 1}$ by $f(t, u) = rh(u)$. Thus $f$ is a $C^1$ surjection from $K^n$ onto the unit ball $B^n$. It is easy to obtain the formula $J(x, f) = t^{n - 1}D_h(u)$, where $x = (t, u) \in K^n$, and $D_h$ denotes the norm of the vector product

$$D_h = \left| \frac{\partial h}{\partial x_1} \times \cdots \times \frac{\partial h}{\partial x_{n-1}} \right|.$$ 

According to Proposition 3.1 it follows that

$$\frac{1}{n} \omega_{n-1} = \mu(B^n) = \int_{B^n} dy \leq \int_{S^n} N(y, f) dy$$

$$= \int_{K^n} |J(x, f)| dx = \int_0^1 t^{n-1} dt \int_{K^{n-1}} D_h(u) du = \frac{1}{n} \int_{K^{n-1}} D_h(u) du.$$ 

Consequently we have

$$(3.4) \quad \int_{K^{n-1}} D_h(u) du \geq \omega_{n-1}.$$ 

Let $x \in A(r, R)$ and define $N = x/|x|$. Then consider the following system of mutually orthogonal vectors $(U_1, \ldots, U_{n-1}, N)$ of the unit norm. The vectors $(U_1, \ldots, U_{n-1})$ are arbitrarily chosen. Then we define the Gram determinant of $S$ at $x$ by

$$D_S(x) = |D_{U_1}S(x) \times \cdots \times D_{U_{n-1}}S(x)|.$$ 

Now we have the following refined version of [15 Proposition 1.6].

**Lemma 3.2.** Let $f$ be a $C^1$ surjection between the spherical rings $A(r, R)$ and $A(r_s, R_s)$, and let $S = f/|f|$. Let $S(t)$ be a sphere of radius $t$ centered at the origin. Then

$$(3.5) \quad \int_{S(t)} D_S(x) d\sigma(x) \geq \omega_{n-1},$$

where $\omega_{n-1}$ denote the measure of $S$.

**Proof.** Let $K^{n-1}$ be an $n - 1$-dimensional rectangle and let $g : K^{n-1} \rightarrow P^{n-1}$ be the spherical coordinates of $S(t)$. Then the function $Sog$ is a differentiable surjection from $K^{n-1}$ onto the unit sphere $S$. Then by (3.3) we have

$$\int_{K^{n-1}} D_{Sog} dK \geq \omega_{n-1}.$$ 

Further we obtain

$$D_{Sog}(x) = \left| S'(g(x)) \frac{\partial g(x)}{\partial x_1} \times \cdots \times S'(g(x)) \frac{\partial g(x)}{\partial x_{n-1}} \right|.$$ 

Hence we obtain

$$\omega_{n-1} \leq \int_{K^{n-1}} D_S(g(x))D_g(x)dK(x) = \int_S D_S(\zeta) d\sigma(\zeta).$$ 

Thus we have proved (3.5). \qed
4. THE PROOF OF THE MAIN RESULT

First we prove the following corollary of Lemma 3.2.

**Lemma 4.1.** Let $f$ be a $C^1$ homeomorphism between the spherical rings $\mathbb{A}(r, R)$ and $\mathbb{A}(r_*, R_*)$ in $\mathbb{R}^3$, and let $S = f/|f|$. Let $\mathbb{S}(t)$ be the sphere centered at 0 with the radius $t \in (r, R)$. Then

\[
\int_{\mathbb{S}(t)} \left( \|DS\|^2 - \left|DS(x) \frac{x}{|x|}\right|^2 \right) d\sigma(x) \geq 8\pi.
\]

The inequality (4.1) is sharp and is attained for the mappings of the form $f(x) = \rho(|x|)T \left( \frac{x}{|x|} \right)$, where $T$ is an arbitrary conformal transformation of the 2-sphere $\mathbb{S}$.

**Proof.** For fixed $x \in \mathbb{A}(r, R)$ let $N = x/|x|$ and assume that $U$, $V$ and $N$ is a system of mutually orthogonal vectors of the unit norm. Then

\[
\|DS\|^2 = \|DU\|^2 + \|DV\|^2 + \|DN\|^2
\]

and so

\[
\|DS\|^2 - \|DN\|^2 = \|DU\|^2 + \|DV\|^2 \geq 2\|DU \times DV\| = 2D_S.
\]

By integrating in $\mathbb{S}(t)$ and using Lemma 4.1 we get (4.1).

Further if $T$ is a conformal mapping of $\mathbb{S}$ onto itself, and $f(x) = \rho(|x|)T \left( \frac{x}{|x|} \right)$ then the mapping $S : \mathbb{S}(t) \to \mathbb{S}$ defined by $S(x) = T \left( \frac{x}{|x|} \right)$ is a conformal diffeomorphism between $\mathbb{S}(t)$ and $\mathbb{S}$. Moreover

\[
2D_S(x) = 2|DU S(x) \times DV S(x)| = |DU|^2 + |DV|^2
\]

(4.3)

\[
= \|DS\|^2 = \|DS\|^2 - \left|DS(x) \frac{x}{|x|}\right|^2.
\]

Thus

\[
\int_{\mathbb{S}(t)} \|DS\|^2 d\sigma(\eta) = 8\pi.
\]

**Proof of Theorem 1.1.** Before we go to the detailed proof let us make one shortcut. For every constant $a > 0$ we have

\[
\mathcal{F} \left[ \frac{af}{|f|^2} \right] = \mathcal{F} \left[ f \right].
\]

In order to prove this statement, by calculations we find that for $g = \frac{af}{|f|^2}$ we have

\[
g_{x_i} = \frac{af_{x_i}}{|f|^2} - \frac{2af \langle f, f_{x_i} \rangle}{|f|^4}, \quad i = 1, \ldots, n.
\]

Thus we obtain

\[
|g_{x_i}|^2 = a^2 \frac{|f_{x_i}|^2}{|f|^4}, \quad i = 1, \ldots, n.
\]
Summing the previous inequalities we get
\[ \| Dg \|^2 = a^2 \frac{\| Df \|^2}{|f|^4}. \]

It follows that
\[ \frac{\| Df \|^2}{|f|^2} = \frac{\| Dg \|^2}{|g|^2}. \]

This implies (4.4).

Thus we can assume that \( f \) maps the inner boundary onto the inner boundary and the outer boundary onto the outer boundary, that means the following:
\[ \lim_{|x| \to r} |f(x)| = r_* \]
and
\[ \lim_{|x| \to R} |f(x)| = R_* \]

By (3.2) and Fubini’s theorem we have
\[ \mathcal{F}[f] = \int_{h(r,R)} \left( \frac{\| \nabla \rho \|^2}{\rho^2} + \| DS \|^2 \right) dx = \int_r^R dt \int_{S(t)} \left( \frac{\| \nabla \rho \|^2}{\rho^2} + \| DS \|^2 \right) d\sigma(\eta) \]

For fixed \( \eta \), consider the curve
\[ \alpha(t) = f(t\eta) = \rho(t\eta)S(t\eta). \]

Then we have
\[ |\alpha'(t)|^2 = |f'(t\eta)\eta|^2 \]
and \( |\alpha(r)| = r_* \) and \( |\alpha(R)| = R_* \).

So
\[ |\alpha'(t)|^2 = \langle \nabla \rho(t\eta), \eta \rangle^2 + \rho^2(t\eta)|DS(t\eta)\eta|^2 \]

Moreover
\[ |\nabla \rho|^2 \geq \langle \nabla \rho(t\eta), \eta \rangle^2 = |\alpha'(t)|^2 - \rho^2(t\eta)|DS(t\eta)\eta|^2 \]

So
\[ A \geq 4\pi \int_r^R t^2 dt \frac{|\alpha'(t)|^2}{\alpha^2(t)} dt + \int_r^R R^2 \int_{S(t)} \| DS(t\eta) \|^2 - |DS(t\eta)\eta|^2 d\sigma(\eta) \]
\[ = 4\pi \int_r^R t^2 dt \frac{|\alpha'(t)|^2}{\alpha^2(t)} dt + \int_r^R \int_{S(t)} \left( \| DS(\zeta) \|^2 - \left| DS(\zeta) \frac{\zeta}{t} \right|^2 \right) d\sigma(\zeta) \]

Further from (4.2) we have
\[ \int_{S(t)} \left( \| DS(\zeta) \|^2 - \left| DS(\zeta) \frac{\zeta}{t} \right|^2 \right) d\sigma(\zeta) \geq 8\pi. \]
Therefore
\[(4.8) \quad A \geq 4\pi \int_{r}^{R} \left( t^2 \frac{|\alpha'(t)|^2}{|\alpha(t)|^2} + 2 \right) dt \geq \int_{r}^{R} \left( t^2 \frac{(|\alpha(t)|')^2}{|\alpha(t)|^2} + 2 \right) dt. \]

If \( f(x) = H(t)T(\frac{x}{|x|}) \) then in view of (3.1) and (4.3) we have
\[ \mathcal{F}[f] = \mathcal{H}[H] = 4\pi \int_{r}^{R} \left( \frac{t^2 \dot{H}^2}{H^2} + 2 \right) dt = 4\pi \int_{r}^{R} \left( t^2 \frac{(|\alpha(t)|')^2}{|\alpha(t)|^2} + 2 \right) dt, \]
where \( \alpha(t) = \rho(t\eta)\eta \), and \( \eta \) is any fixed vector. The Euler-Lagrange equation for the energy integral \( \mathcal{H} \), as in (2.7) reduces to
\[(4.9) \quad 2 \dot{H}(t)H(t) - t \ddot{H}(t)^2 + t \dddot{H}(t)H(t) = 0. \]

By taking the substitution \( H(t) = \exp(K(t)) \) in (4.9) we arrive to the differential equation
\[ e^{K(t)} \left( 2 \dot{K}(t) + t \ddot{K}(t) \right) = 0 \]
whose general solution is
\[ K(t) = c_1 + \frac{c_2}{t}. \]
Thus the general solution of (4.9) is
\[ H(t) = ae^{b/t}, \quad a > 0, \quad b \in \mathbb{R}. \]

The diffeomorphisms
\[ H_1(t) = r_\ast \left( \frac{R_\ast}{r_\ast} \right)^{\frac{R(t-r)}{R-r}} \]
and
\[ H_2(t) = \frac{R_\ast r_\ast}{H_1(t)} = R_\ast \left( \frac{r_\ast}{R_\ast} \right)^{\frac{R(r-t)}{(R-r)t}} \]
map the interval \([r, R]\) onto \([r_\ast, R_\ast]\). The mapping \( H_1 \) preserves the orientation, and \( H_2 \) changes the orientation. The energy of this stationary mappings is
\[ \mathcal{F}[H_1] = \mathcal{F}[H_2] = 4\pi \left( 2(R - r) + \frac{rR \log \left( \frac{R_\ast}{r_\ast} \right)^2}{R - r} \right). \]

To prove that they are minimizers, we need to show that, we only need to show that the given energy integral
\[ \mathcal{H}[H] = 4\pi \int_{r}^{R} \left( \frac{t^2 \dot{H}^2}{H^2} + 2 \right) dt \]
attains its minimum.

Define
\[ \Lambda(t, H, \dot{H}) = \left( \frac{t^2 \dot{H}^2}{H^2} + 2 \right), \]
and show that it is convex in $K = \dot{H}$. For $K = \dot{H}$ we have the following formula

$$\partial_{KK} \Lambda[t, H, K] = \frac{2t^2}{H^2}$$

which is clearly positive. Further since $r \leq t \leq R$ and $r_\ast \leq H(t) \leq R_\ast$, we can find a positive constant $C$ so that

$$(4.10) \quad C|\dot{H}|^2 \leq \Lambda[s, H, \dot{H}],$$

which implies that the function $L$ is coercive.

Let $H_m = H_m(t) : [r, R] \to [r_\ast, R_\ast]$ be a sequence of smooth bijections with $H_m(r) = r_\ast$, $H_m(R) = R_\ast$ and

$$\inf_{H:[r,R] \to [r_\ast,R_\ast]} \mathcal{H}[H] = \lim_{m \to \infty} \mathcal{H}[H_m].$$

Then up to a subsequence it converges to a monotone increasing function $H_\circ$. Moreover, since $H_m$ is a bounded sequence of $W^{1,2}$, it converges, up to a subsequence weakly to a mapping $H_\circ \in W^{1,2}$.

By using the convexity of $\mathcal{L}$ and the fact that $\mathcal{L}$ is coercive, by standard theorem from the calculus of variation (see [9, p. 79]), we obtain that

$$\mathcal{H}[H_\circ] = \lim_{m \to \infty} \mathcal{H}[H_m].$$

Further as $\mathcal{L}[s, H, K] \in C^\infty(\mathbb{R}^3)$, with $\partial_{KK}^2 \mathcal{L}[s, H, K] > 0$, we infer that $H_\circ \in C^\infty[r, R]$ (see [13, p. 17]) and $H_\circ$ is the solution of our Euler-Lagrange equation. Thus it coincides with $H_1$ or $H_2$.

To prove the equali statement, assume that in all inequalities (4.1), (4.7), (4.8) is attained the equality. If (4.8) is an equality, then

$$\langle \alpha'(t), \frac{\alpha(t)}{|\alpha(t)|} \rangle = |\alpha'(t)|$$

for every $t$. This implies that $\alpha'(t) = \varphi(t)\alpha(t)$ for $\varphi(t) > 0$. Thus if $\alpha(t) = (x(t), y(t), z(t))$ we obtain $x(t) = c_1 \exp(\int_r^t \varphi(t)dt)$, $y(t) = c_2 \exp(\int_r^t \varphi(t)dt)$, $z = c_3 \exp(\int_r^t \varphi(t)dt)$. In other words $\alpha$ is the part of the line

$$\frac{x}{c_1} = \frac{y}{c_2} = \frac{z}{c_3},$$

orthogonal to the spheres that connect two points from the sphere. This means that if $\alpha(t) = \rho(t)\eta S(t\eta)$, then $S(t\eta) = S(\eta)$. In particular $D_NS = 0$. If the equality is attained in (4.1), then it is attained in (4.2). Therefore $S_U \perp S_V$ that means the mapping $S(t\eta) = S(\eta)$ conformally maps $S(t)$ onto $S$ and does not depend on $t$. Here the vectors $U$ and $V$ are mutually orthogonal and of unit norm (as in Lemma 3.1). If the equality is attained in (4.3), we get

$$|\nabla \rho(x)| = |D_N \rho(x)|,$$

thus $D_U \rho(x) = 0$ and $D_V \rho(x) = 0$ which implies that $\rho(x) = \rho(|x|)$ (by abusing the notation). Thus we have proved that $u(x) = \rho(|x|)T \left(\frac{x}{|x|}\right)$ where $T$ is a conformal mapping of $S$ onto itself.
5. Appendix

It follows from Theorem 1.1 that

**Corollary 5.1.** Let $f \in W^{1,2}$ be a homeomorphism between $\mathbb{A}(r,R)$ and $\mathbb{A}(r_*,R_*)$. Then

$$\mathcal{E}[f] = \int_{h(r,R)} |Df|^2 dx \geq \frac{4\pi}{R_*^2} \left(2(R-r) + \frac{rR \log \frac{R}{r_*}^2}{R-r}\right).$$

It seems that (5.1) is not sharp, but it shows that the minimizer of Dirichlet energy is not zero for the case of non-degenerated annuli. This is somehow complementary result to result for the case of degenerated annuli, where the infimum of the Dirichlet energy of Sobolev homomorphisms with free boundary condition is zero ([12, Theorem 1.6]). It should be noticed the following, the solution to the equation $\Delta h = 0$, if $h(x) = H_\frac{r}{|x|}$, according to (2) is given by

$$H(t) = ar + \frac{b}{t^2}.$$ 

Now the solution to the boundary value problem

$$\left\{ \begin{array}{ll}
\Delta h = 0, & \text{if } h = H(|x|) \frac{x}{|x|}; \\
H(r) = r_*, H(R) = R_*, & \text{where } 0 < r < R \text{ and } 0 < r_* < R_,
\end{array} \right.$$ 

is given by

$$H(t) = \frac{r^2R^2(-Rr_* + rR_*)}{(r^3 - R^3) t^2} + \frac{(r^2r_* - R^2R_*) t}{r^3 - R^3}.$$ 

Then

$$H'(t) = \frac{r^2r_* - R^2R_*}{r^3 - R^3} + \frac{2 (r^2R^3 r_* - r^3R^2 R_*)}{(r^3 - R^3) t^3}.$$ 

So $H'(t) > 0$ for $t \in [r,R]$ if and only if

$$(-2r^2R^3 r_* + 2r^3R^2 R_*) + (-r^2r_* + R^2 R_*) t^3 \geq 0, \quad t \in [r,R].$$

It follows that

$$r^3 r_* + 2R^3 r_* - 3rR^2 R_* \leq 0$$

i.e. the condition

$$\frac{r_*}{R_*} \leq \frac{3rR^2}{r^3 + 2R^3},$$

is sufficient and necessary for existence of radial Euclidean harmonic mappings between given annuli (the so-called generalized Nitsche condition). In this case the harmonic mapping $h(x) = H(r) \frac{x}{|x|}$ satisfies the equation

$$\mathcal{E}[h] = \frac{4\pi (r^3 + 2R^3) r_*^2 - 6r^2R^2 r_* R + R (2r^3 + R^3)}{R^3 - r^3}.$$
It is clear that the quantity $X$ on the right hand side of (5.3) is bigger than the quantity $Y$ on the right hand side of (5.1). It is also clear that, 

$$Z = \inf \{ \mathcal{E}[h] : h \in W^{1,2}(\mathbb{A}(r, R), \mathbb{A}(r_*, R_*)) \} \in (Y, X],$$

and probably $Z < X$, in view of ([12, Theorem 1.6]), but the right value of $Z$ remains so far un-known.

**Conjecture 5.2.** Assume that $\mathcal{F}$ is the family a homeomorphisms between spherical rings $\mathbb{A}(r, R)$ and $\mathbb{A}(r_*, R_*)$ in $\mathbb{R}^n$ that belongs to $W^{1,n-1}$. Then the Dirichlet integral of $f \in \mathcal{F}$ with respect to the weight $\psi(y) = |y|^{1-n}$ i.e. the integral

$$\mathcal{F}[f] = \int_{\mathbb{A}(r,R)} \frac{\| Df \|^{n-1}}{|f|^{n-1}} dx,$

achieves its minimum for generalised-radial difeomorphisms between annuli.

**References**

[1] **Lars V. Ahlfors**, *Möbius transformations in several dimensions.* (Preobrazovaniya Mëbiusa v mnogomernom prostranstve). Transl. from the English. (Russian) Sovremennaya Matematika. Vvodnye kursy. Moskva: Mir. 112 p. (1986).

[2] **S. S. Antman**, *Nonlinear problems of elasticity.* Applied Mathematical Sciences, 107. Springer-Verlag, New York, 1995.

[3] **K. Astala, T. Iwaniec, and G. Martin**, *Elliptic partial differential equations and quasiconformal mappings in the plane,* Princeton University Press, 2009.

[4] **K. Astala, T. Iwaniec, and G. Martin**, *Deformations of annuli with smallest mean distortion,* Arch. Ration. Mech. Anal. 195 (2010), no. 3, 899–921.

[5] **J. M. Ball**, *Convexity conditions and existence theorems in nonlinear elasticity,* Arch. Rational Mech. Anal. 63 (1976, 77), no. 4, 337–403.

[6] **Jean-Christophe Bourgoin**, *The minimality of the map $x/|x|$ for weighted energy,* Calculus of Variations and Partial Differential Equations April 2006, Volume 25, Issue 4, pp 469–489.

[7] **Haim Brezis, Jean-Michel Coron, Elliott H. Lieb**, *Harmonic Maps with Defects,* Commun. Math. Phys. 107, 649–705 (1986).

[8] **P. G. Ciarlet**, *Mathematical elasticity Vol. I. Three-dimensional elasticity,* Studies in Mathematics and its Applications, 20. North-Holland Publishing Co., Amsterdam, 1988.

[9] **B. Dacorogna**, *Introduction to the calculus of variations,* 2004 by Imperial College Press.

[10] **T. Iwaniec, L. V. Kovalev; J. Onninen**: *The Nitsche conjecture.* J. Amer. Math. Soc. 24 (2011), no. 2, 345–373.

[11] **T. Iwaniec, J. Onninen**: *n-harmonic mappings between annuli: the art of integrating free Lagrangians.* Mem. Amer. Math. Soc. 218 (2012), no. 1023, viii+105 pp.

[12] **T. Iwaniec, J. Onninen**, *p-harmonic energy of deformations between punctured balls.* Adv. Calc. Var. 2, No. 1, 93–107 (2009).

[13] **J. Jost, X. Li-Jost**, *Calculus of variations.* Cambridge Studies in Advanced Mathematics, 64. Cambridge University Press, Cambridge, 1998.

[14] **D. Kalaj**: *Deformations of annuli on Riemann surfaces and the generalization of Nitsche conjecture.* J. Lond. Math. Soc. (2) 93 (2016), no. 3, 683-702.

[15] **D. Kalaj**: *On the Nitsche conjecture for harmonic mappings in $\mathbb{R}^2$ and $\mathbb{R}^3$.* Israel J. Math. 150 (2005), 241–251.

[16] **D. Kalaj**: *n-harmonic energy minimal deformations between annuli, arXiv:1703.06639*
[17] Aleksiš Koski, Jani Onninen: Radial symmetry of $p$-harmonic minimizers, arXiv:1710.01067 to appear in Arch. Rational Mech. Anal.

[18] Lyzzaik, A. The modulus of the image of annuli under univalent harmonic mappings and a conjecture of J. C. C. Nitsche. J. London Math. soc., (2) 64 (2001), pp. 369-384.

[19] Hong, Min-Chun: On the minimality of the $p$-harmonic map $x/|x| : B^n \to S^{n-1}$. Calc. Var. Partial Differ. Equ. 13, No. 4, 459-468 (2001).

[20] J.C.C. Nitsche: On the modulus of doubly connected regions under harmonic mappings, Amer. Math. Monthly, 69, (1962), 781–782.

[21] T. Rado, P.V. Reichelderfer Continuous transformation in analysis with an introduction to algebraic topology. Berlin-Göttingen-Heidelberg; Springer Verlag, 1955.

[22] R. Schoen; S. T. Yau, Lectures on harmonic maps. Conference Proceedings and Lecture Notes in Geometry and Topology, II. International Press, Cambridge, MA, 1997. vi+394 pp.

[23] M. Vuorinen: Conformal geometry and quasiregular mappings. Lecture Notes in Mathematics. 1319. Berlin etc.: Springer-Verlag. xix, 209 p. (1988).

[24] A. Weitsman, Univalent harmonic mappings of Annuli and a conjecture of J.C.C. Nitsche, Israel J. Math. 124, 327–331 (2001).

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