An inverse problem of ocean acoustics *†

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Abstract

Let

\[ \Delta u + k^2 n(z)u = -\frac{\delta(r)}{2\pi r}f(z) \text{ in } \mathbb{R}^2 \times [0,1], \]

\[ u(x^1,0) = 0, \quad u'(x^1,1) = 0, \] (1)

where \( u = u(x^1, z), \quad x^1 := (x_1, x_2), \quad r := |x^1|, \quad x_3 := z, \quad u' = \frac{\partial u}{\partial z}, \quad \delta(r) \) is the delta-function, \( n(z) \) is the refraction coefficient, which is assumed to be a real-valued integrable function, \( k > 0 \) is a fixed wavenumber. The solution to (1)-(2) is selected by the limiting absorption principle.

It is proved that if \( f(z) = \delta(z-1) \), then \( n(z) \) is uniquely determined by the data \( u(x^1,1) \) known \( \forall x^1 \in \mathbb{R}^2 \). Comments are made concerning the earlier study of a similar problem in the literature.

1 Introduction

In [4] the following inverse problem is studied:

\[ [\Delta + k^2 n(z)]u = -\frac{\delta(r)}{2\pi r}f(z), \text{ in } \mathbb{R}^2 \times [0,1], \] (1.1)

\[ u(x^1,0) = u'(x^1,1) = 0, \quad x^1 := (x_1, x_2), \quad x_3 := z, \quad u' := \frac{\partial u}{\partial z}. \] (1.2)

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Here \( k > 0 \) is a fixed wavenumber, \( n(z) > 0 \) is the refraction coefficient, which is assumed in [1] to be a continuous real-valued function satisfying the condition \( 0 \leq n(z) < 1 \), the layer \( \mathbb{R}^2 \times [0, 1] \) models shallow ocean, \( r \ := |x^1| = \sqrt{x_1^2 + x_2^2} \). \( \delta(r) \) is the delta-function, \( \frac{\delta(r)}{2\pi r} = \delta'(x^1) \). \( f(z) \in C^2[0, 1] \) is a function satisfying the following conditions [1], p.127:

\[
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} = -\frac{\delta(r)}{2\pi r} f(z), \quad \text{in } \mathbb{R}^2 \times [0, 1], \quad \varepsilon > 0.
\]

One defines the differential operator corresponding to differential expression (1.1) and the boundary conditions (1.2) in \( L^2(\mathbb{R}^2 \times [0, 1]) \) as a selfadjoint operator (for example, as the Friedrichs extension of the symmetric operator with the domain consisting of \( H^2(\mathbb{R}^2 \times [0, 1]) \) functions vanishing near infinity and satisfying conditions (1.2)), and then the function \( u_\varepsilon(x) \) is uniquely defined. By \( H^m \) we mean the usual Sobolev space. One can prove that the limit of this function \( u(x) = \lim_{\varepsilon \downarrow 0} u_\varepsilon(x) \) does exist globally in the weighted space \( L^2(\mathbb{R}^2 \times [0, 1], \frac{1}{(1+r)^a}) \), \( a > 1 \), and locally in \( H^2(\mathbb{R}^2 \times [0, 1]) \) outside a neighborhood of the set \( \{ r = 0, 0 \leq z \leq 1 \} \), provided \( \lambda_j \neq 0 \ \forall j \), where \( \lambda_j \) are defined in (1.7) below. This limit defines the unique solution to problem (1.1)–(1.2) satisfying the limiting absorption principle if \( \lambda_j \neq 0 \ \forall j \). If \( f(z) = \delta(z-1) \), where \( \delta(z-1) \) is the delta-function, then an analytical formula for \( u_\varepsilon(x) \) can be written:

\[
u_\varepsilon(x) = \sum_{j=1}^{\infty} \psi_j(z) f_j \frac{1}{\sqrt{2\pi}} K_0(r \sqrt{\lambda_j^2 + i\varepsilon}),
\]

where \( K_0(r) \) is the modified Bessel function (the Macdonald function), and \( f_j = \psi_j(1) \) are defined in (1.6) below, and \( \psi_j(z) \) and \( \lambda_j^2 \) are defined in formula (1.7) below. This formula can be checked by direct calculation and is obtained by the separation of variables. The known formula \( F^{-1} \frac{1}{\lambda_j^2 + \alpha^2} = \frac{1}{2\pi} K_0(\alpha r) \) was used, and \( F u := \hat{u} \) is the Fourier transform defined above formula (1.3).

From the formula for \( u_\varepsilon(x) \), the known asymptotics \( K_0(r) = \sqrt{\frac{2}{\pi r}} e^{-r} [1 + O(r^{-1})] \) for large values of \( r \), the boundedness of \( |\psi_j(z)| \) as \( j \to \infty \) and formula (1.8) below, one can see that the limit of \( u_\varepsilon(x) \) as \( \varepsilon \to 0 \) does exist for any \( r > 0 \) and \( z \in [0, 1] \), if and only if \( \lambda_j \neq 0 \). If \( \lambda_j = 0 \) for some \( j = j_0 \), then the limiting absorption principle holds if and only if \( f_{j_0} \neq 0 \). If \( \lambda_j \neq 0 \ \forall j \), then the limiting absorption principle holds and the solution to
problem (1.1)-(1.2) is well defined. If \( \lambda_j = 0 \) for some \( j = j_0 \), then we define the solution to problem (1.1)-(1.2) with \( f(z) = \delta(z - 1) \) by the formula:

\[
    u(x) = \psi_{j_0}(z)\psi_{j_0}(1) \frac{1}{2\pi} \log(\frac{1}{r}) + \sum_{j=1, j \neq j_0}^{\infty} \psi_j(z)\psi_j(1) \frac{1}{2\pi} K_0(r\lambda_j), \quad r := |x^1|.
\]

This solution is unique in the class of functions of the form \( u(x) = \sum_{j=1}^{\infty} u_j(x^1)\psi_j(z) \), where \( \Delta_1 u_j - \lambda_j^2 u_j = -\delta(x^1) \) in \( \mathbb{R}^2 \), \( \Delta_1 w := w_{x_1x_1} + w_{x_2x_2}, \ u_j \in L^2(\mathbb{R}^2) \) if \( \lambda_j^2 > 0 \); if \( \lambda_j^2 < 0 \) then \( u_j \) satisfies the radiation condition \( r^{1/2}(\frac{\partial u_j}{\partial r} - i|\lambda_j| u_j) \to 0 \) as \( r \to \infty \), uniformly in directions \( \hat{x}^1 \); and if \( \lambda_j^2 = 0 \) then \( u_j = \frac{1}{2\pi} \log(\frac{1}{r}) + o(1) \) as \( r \to \infty \).

The inverse problem (IP) consists of finding \( n(z) \) given \( g(x^1) := u(x^1, 1) \) and assuming that \( f(z) = \delta(z - 1) \) in (1.1).

By the cylindrical symmetry one has \( g(x^1) = g(r) \).

It is claimed in [1, p. 137] that the above inverse problem has not more than one solution, and a method for finding this solution is proposed. The arguments in [1] are not satisfactory (see Remark 2.1 below, where some of the incorrect statements from [1], which invalidate the approach in [1], are pointed out).

The aim of our paper is to prove that if \( f(z) = \delta(z - 1) \), then \( n(z) \) can be uniquely and constructively determined from the data \( g(r) \) known for all \( r > 0 \). It is an open problem to find all such \( f(z) \) for which the IP has at most one solution.

The method we use is developed in [5] (see also [8]). Properties of the operator \( \Delta + k^2 n(z) \) in a layer were studied in [6]. In [8] an inverse problem for an inhomogeneous Schrödinger equation on the full axis was investigated.

Let us outline our approach to IP.

Take the Fourier transform of (1.1)-(1.2) with respect to \( x^1 \) and let

\[
    v := v(z, \lambda) := \hat{u} := \int_{\mathbb{R}^2} u(x^1, z)e^{ix^1 \cdot \zeta} dx^1, \quad |\zeta| := \lambda, \quad \zeta \in \mathbb{R}^2,
\]

and

\[
    G(\lambda) := \hat{g}(r).
\]

Then

\[
    \ell v := v'' - \lambda^2 v + q(z)v = -f(z), \quad q(z) := k^2 n(z), \quad v = v(z, \lambda), \quad (1.3)
\]

\[
    v(0, \lambda) = v'(1, \lambda) = 0, \quad (1.4)
\]

\[
    v(1, \lambda) = G(\lambda). \quad (1.5)
\]

IP: The inverse problem is: given \( G(\lambda) \), for all \( \lambda > 0 \) and a fixed \( f(z) = \delta(z - 1) \), find \( q(z) \).
The solution to (1.3)-(1.4) is:

\[ v(z, \lambda) = \sum_{j=1}^{\infty} \frac{\psi_j(z)f_j}{\lambda^2 + \lambda_j^2}, \quad f_j := (f, \psi_j) := \int_0^1 f(z)\psi_j(z)dz, \]  

(1.6)

where \( \psi_j(z) \) are the real-valued normalized eigenfunctions of the operator \( L := -\frac{d^2}{dz^2} - q(z) \):

\[ L\psi_j = \lambda_j^2 \psi_j, \quad \psi_j(0) = \psi'_j(1) = 0, \quad ||\psi_j(z)|| = 1. \]  

(1.7)

We can choose the eigenfunctions \( \psi_j(z) \) real-valued since the function \( q(z) = k^2 n(z) \) is assumed real-valued. One can check that all the eigenvalues are simple, that is, there is just one eigenfunction \( \psi_j \) corresponding to the eigenvalue \( \lambda_j^2 \) (up to a constant factor, which for real-valued normalized eigenfunctions can be either 1 or \(-1\)).

It is known (see e.g. [4, p.71]) that

\[ \lambda_j^2 = \pi^2(j - \frac{1}{2})^2[1 + O(\frac{1}{j^2})] \text{ as } j \to +\infty. \]  

(1.8)

The data can be written as

\[ G(\lambda) = \sum_{j=1}^{\infty} \frac{\psi_j(1)f_j}{\lambda^2 + \lambda_j^2}, \]  

(1.9)

where \( f_j \) are defined in (1.6). The series (1.9) converges absolutely and uniformly on compact sets of the complex plane \( \lambda \) outside the union of small discs centered at the points \( \pm i\lambda_j \). Thus, \( G(\lambda) \) is a meromorphic function on the whole complex \( \lambda \)-plane with simple poles at the points \( \pm i\lambda_j \). Its residue at \( \lambda = i\lambda_j \) equals \( \frac{\psi_j(1)f_j}{2i\lambda_j} \).

If \( f(z) = \delta(z-1) \), then \( f_j = \psi_j(1) \neq 0 \ \forall \ j = 1, 2, ..., \) (see section 2 for a proof of the inequality \( \psi_j(1) \neq 0 \ \forall \ j = 1, 2, .......) \) and the data (1.9) determine uniquely the set

\[ \{\lambda_j^2, \ \psi_j^2(1)\}_{j=1,2,...} \]  

(1.10)

In section 2 we prove the basic result:

**Theorem 1.1.** If \( f(z) = \delta(z-1) \) then the data (1.5) determine \( q(z) \in L^1(0,1) \) uniquely.

An algorithm for calculation of \( q(z) \) from the data is described in section 2.

**Remark 1.2.** The proof and the conclusion of Theorem 1.1 remain valid for other boundary conditions, for example, \( u'(x^1, 0) = u(x^1, 1) = 0 \) with the data \( u(x^1, 0) \) known for all \( x^1 \in \mathbb{R}^2 \).
2 Proofs: uniqueness theorem and inversion algorithm

Proof of Theorem 1.1. The data (1.9) with \( f(z) = \delta(z - 1) \), that is, with \( f_j = \psi_j(1) \), determine uniquely \( \{\lambda_j^2\}_{j=1,2,...} \) since \( \pm i\lambda_j \) are the poles of the meromorphic function \( G(\lambda) \) which is uniquely determined for all \( \lambda \in \mathbb{C} \) by its values for all \( \lambda > 0 \) (in fact, by its values at any infinite sequence of \( \lambda > 0 \) which has a finite limit point on the real axis).

The residues \( \psi_j^2(1) \) of \( G(\lambda) \) at \( \lambda = i\lambda_j \) are also uniquely determined.

Let us show that:

i) \( \psi_j(1) \neq 0 \quad \forall \, j = 1, 2, \ldots \)

ii) The set (1.10) determines \( q(z) \in L^1(0, 1) \) uniquely.

Let us prove i):

If \( \psi_j(1) = 0 \) then equation (1.7) and the Cauchy data \( \psi_j(1) = \psi_j'(1) = 0 \) imply that \( \psi_j(z) \equiv 0 \) which is impossible since \( \| \psi_j(z) \| = 1 \), where \( \|u\|^2 := \int_0^1 |u|^2 \, dx \).

Let us prove ii):

It is sufficient to prove that the set (1.10) determines the norming constants

\[
\alpha_j := \| \Psi_j(z) \|^2
\]

and therefore the set

\[
\{\lambda_j^2, \alpha_j\}_{j=1,2,...},
\]

where the eigenvalues \( \lambda_j^2 \) are defined in (1.7), \( \Psi_j = \Psi(z, \lambda_j) \), \( \psi_j(z) := \frac{\Psi(z, \lambda_j)}{\| \Psi_j \|} \),

\[
-\Psi'' - s^2 \Psi - q(z) \Psi = 0, \quad \Psi(0, s) = 0, \quad \Psi'(0, s) = 1,
\]

and \( \lambda_j \) are the zeros of the equation

\[
\Psi'(1, s) = 0, \quad s = \lambda_j, \quad j = 1, 2, \ldots
\]

The function \( \Psi'(1, s) \) is an entire function of \( \nu = s^2 \) of order \( \frac{1}{2} \), so that (see [4]):

\[
\Psi'(1, s) = \gamma \prod_{j=1}^{\infty} \left( 1 - \frac{s^2}{\lambda_j^2} \right), \quad \gamma = \text{const.}
\]

(2.3)

From the Hadamard factorization theorem for entire functions of order < 1 formula (2.3) follows but the constant factor \( \gamma \) remains undetermined. This factor is determined by the data \( \{\lambda_j^2\}_{\forall j} \) because the main term of the asymptotics of function (2.3) for large positive \( s \) is \( \cos(s) \), and the result in [4], p.243, (see Claim 1 below) implies that the constant \( \gamma \) in formula (2.3) can be computed explicitly:

\[
\gamma = \prod_{j=1}^{\infty} \frac{\lambda_j^2}{(\lambda_j^2)^2}, \quad (2.3')
\]
where \( \lambda_j^0 \) are the roots of the equation \( \cos(s) = 0, \lambda_j^0 = \frac{(2j-1)\pi}{2}, j = 1, 2, ..., \) and the infinite product in (2.3') converges because of (1.8).

A simple derivation of (2.3'), independent of the result formulated in Claim 1 below, is based on the formula:

\[
1 = \lim_{y \to +\infty} \frac{\Psi'(1, iy)}{\cos(iy)} = \gamma \prod_{j=1}^{\infty} \frac{(\lambda_j^0)^2}{\lambda_j^2}. 
\]

For convenience of the reader let us formulate the result from [4], p.243, which yields formula (2.3') as well:

**Claim 1:** The function \( w(\lambda) \) admits the representation

\[
w(\lambda) = \cos(\lambda) - B \frac{\sin(\lambda)}{\lambda} + \frac{h(\lambda)}{\lambda},
\]

where \( B = \text{const}, h(\lambda) = \int_0^1 H(t) \sin(\lambda t) dt, \) and \( H(t) \in L^2(0,1) \) if and only if

\[
w(\lambda) = \prod_{j=1}^{\infty} \frac{\lambda_j^2 - \lambda^2}{(\lambda_j^0)^2},
\]

where \( \lambda_j = \lambda_j^0 - \frac{B}{j} + \beta_j \), \( \beta_j \) are some numbers satisfying the condition: \( \sum_{j=1}^{\infty} |\beta_j|^2 < \infty \), \( \lambda_j \) are the roots of the even function \( w(\lambda) \) and \( \lambda_j^0 = (j - \frac{1}{2})\pi, j = 1, 2, ...., \) are the positive roots of \( \cos(\lambda) \).

The equality

\[
\prod_{j=1}^{\infty} \frac{\lambda_j^2 - \lambda^2}{(\lambda_j^0)^2} = \gamma \prod_{j=1}^{\infty} \left(1 - \frac{\lambda^2}{\lambda_j^0}\right), \tag{2.3''}
\]

where \( \gamma \) is defined in (2.3'), is easy to prove: if \( w \) is the left-hand side and \( v \) the right-hand side of the above equality, then \( w \) and \( v \) are entire functions of \( \lambda \), the infinite products converge absolutely, \( \frac{\lambda_j^2 - \lambda^2}{(\lambda_j^0)^2} = \frac{\lambda_j^2}{(\lambda_j^0)^2} \left(1 - \frac{\lambda^2}{\lambda_j^0}\right) \), and taking the infinite product and using (2.3'), one concludes that \( \frac{w}{v} = 1 \), as claimed.

In fact, one can establish formula (2.3'') and prove that \( \gamma \) in (2.3'') is defined by (2.3') without assuming a priori that (2.3') holds and without using Claim 1. The following assumption suffices for the proof of (2.3''):

i) \( \lambda_j^2 = (\lambda_j^0)^2 + O(1), \ (\lambda_j^0)^2 = \pi^2 (j - \frac{1}{2})^2. \)

Indeed, if i) holds then both sides of (2.3'') are entire functions with the same set of zeros and their ratio is a constant. This constant equals to 1 if there is a sequence of points at which this ratio converges to 1. Using the known formula: \( \cos(\lambda) = \prod_{j=1}^{\infty} \frac{(\lambda_j^0)^2 - \lambda^2}{(\lambda_j^0)^2} \), and the assumption i) one checks easily that the ratio of the left- and right-hand sides of (2.3'') tends to 1 along the positive imaginary semiaxis. Thus, we have proved formulas (2.3)-(2.3') without reference to Claim 1.
The above claim is used with $w(s) = \Psi'(1, s)$ in our paper. The fact that $\Psi'(1, s)$ admits the representation required in the claim is checked by means of the formula for $\Psi'(1, s)$ in terms of the transformation operator: $\Psi(z, s) = \sin(sz) + \int_0^z K(z, t) \frac{\sin(st)}{s} dt$, and the properties of the kernel $K(z, t)$ are studied in [3]. Thus, $\Psi'(1, s) = \cos(s) + \frac{K(1, s)}{s} + \int_0^1 K_z(1, t) \frac{\sin(st)}{s} dt$. This is the representation of $\Psi'(1, s) := w(s)$ used in Claim 1.

Let us derive a formula for $\alpha_j := \parallel \Psi_j \parallel^2$. Denote $\dot{\Psi} := \frac{d\Psi}{dv}$, differentiate (2.1), with $s^2$ replaced by $\nu$, with respect to $\nu$ and get:

$$-\dot{\Psi}'' - \nu \dot{\Psi} - q \dot{\Psi} = \Psi.$$ (2.4)

Since $q(z)$ is assumed real-valued, one may assume $\psi$ real-valued. Multiply (2.4) by $\Psi$ and (2.1) by $\dot{\Psi}$, subtract and integrate over $(0, 1)$ to get

$$0 < \alpha_j := \int_0^1 \Psi_j^2 dz = \left( \Psi_j \dot{\Psi}_j - \Psi_j \dot{\Psi}_j' \right) \bigg|_0^1 = -\Psi_j(1) \dot{\Psi}_j(1),$$ (2.5)

where the boundary conditions $\Psi_j(0) = \Psi_j'(1) = \dot{\Psi}_j(0) = 0$ were used.

From (2.3) with $s^2 = \nu$ one finds the numbers $b_j := \dot{\Psi}_j'(1)$:

$$b_j = \gamma \frac{d}{dv} \prod_{j' = 1}^{\infty} \left( 1 - \frac{\nu}{\lambda_j^2} \right) \bigg|_{v = \lambda_j^2} = -\gamma \prod_{j' = 1}^{\infty} \prod_{j' \neq j} \left( 1 - \frac{\lambda_j^2}{\lambda_{j'}^2} \right).$$ (2.6)

Claim 2: The data $\psi_j^2(1) = \frac{\Psi_j^2(1)}{\alpha_j} := t_j$, where $\alpha_j := \parallel \Psi_j(z) \parallel^2$, and equation (2.5) determine uniquely $\alpha_j$.

Indeed, the numbers $b_j$ are the known numbers from formula (2.6). Denote by $t_j := \psi_j^2(1)$ the quantities known from the data (1.10). Then it follows from (2.5) that $\alpha_j^2 = t_j \alpha_j b_j$, so that

$$\alpha_j = t_j b_j.$$ (2.7)

Claim 2 is proved.

Thus, the data (1.10) determine $\alpha_j := \parallel \Psi_j \parallel^2$ uniquely and analytically by the above formula, and consequently $q(z)$ is uniquely determined by the following known theorem (see for example, [3]):

*The spectral function of the operator $L$ determines $q(z)$ uniquely.*

The spectral function $\rho(\lambda)$ of the operator $L$ is defined by the formula (see [3, formula (10.5)]):

$$\rho(\lambda) = \sum_{\lambda_j^2 < \lambda} \frac{1}{\alpha_j}.$$ (2.8)

The Gelfand-Levitan algorithm [3] allows one to reconstruct analytically $q(z)$ from the spectral function $\rho(\lambda)$ and therefore from the data (1.10), since, as we have proved already, these data determine the spectral function $\rho(\lambda)$ uniquely.

Theorem 1.1 is proved. □
Let us describe an algorithm for calculation of \( q(z) \) from the data \( g(x^1) \):

**Step 1:** Calculate \( G(\lambda) \), the Fourier transform of \( g(x^1) \). Given \( G(\lambda) \), find its poles \( \pm i\lambda_j \), and consequently the numbers \( \lambda_j \); then find its residues, and consequently the numbers \( \psi_j(1)f_j \).

**Step 2:** Calculate the function (2.3), and the constant \( \gamma \) by formulas (2.3) and (2.3'). Calculate the numbers \( b_j \) by formula (2.6) and \( \alpha_j \) by formula (2.7). Calculate the spectral function \( \rho(\lambda) \) by formula (2.8).

**Step 3:** Use the known Gel’fand-Levitan algorithm (see [3]-[5]) to calculate \( q(z) \) from \( \rho(\lambda) \).

This completes the description of the inversion algorithm for IP.

**Remark 2.1.** There are inaccuracies in [1]. We point out two of these, of which the first invalidates the approach in [1].

In [1, p.128, line 2] the \( \alpha_n \) are not the same as \( \alpha_n \) in formula [1, (3.3)]. If one uses \( \alpha_n \) from formula [1, (3.3)], then one has to use in [1, p.128, line 2] the coefficients \( \alpha_n\phi_n(h) \), according to formula [1, (1.5)]. In [1] \( h \) is the width of the layer, which we took to be \( h = 1 \) in our paper without loss of generality. However, the numbers \( \phi_n(h) \) are not known in the inverse problem, since the coefficient \( n(z) \) is not known. Therefore formula [1, (3.9)] is incorrect. This invalidates the approach in [1].

In [1, p.128] a negative decreasing sequence of real numbers \( a_n \) is defined by equation (3.1), which we give for \( h = 1 \):

\[
k\sqrt{1 - a_n^2} = (n + \frac{1}{2})\pi + O\left(\frac{1}{n}\right) \quad (*) .
\]

Such a sequence does not exist: if \( a_n < 0 \) and \( a_n \) has a finite limit then the right-hand side of (\( \ast \)) cannot grow to infinity, and if \( a_n \to -\infty \), then the left-hand side of (\( \ast \)) cannot stay positive for large \( n \), and therefore cannot be equal to the right-hand side of (\( \ast \)).
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