Spectral Theory of the Riemann Zeta-Function

Chapter 6: Appendix

Leaving the hyperbolic plane \( \mathcal{H} \) behind, we now stay on the group \( G = \text{PSL}(2, \mathbb{R}) \). The aim of the present chapter is to review what has been related above by changing our vantage point to the theory of \( \Gamma \)-automorphic representations of \( G \), with \( \Gamma = \text{PSL}(2, \mathbb{Z}) \). We shall gain, in particular, a geometrical understanding of the sum formulas involving Kloosterman sums as well as the explicit formula for the fourth moment of the Riemann zeta-function. We shall obtain also a unified approach to the mean values of individual automorphic \( L \)-functions, a subject which is naturally an extension of the fourth moment of the zeta-function but does not admit any analogous treatment, thus requiring a genuinely new method.

We shall develop a relatively self-contained treatment of the spectral theory of the space \( L^2(\Gamma \backslash G) \); in fact, this chapter could be read as a particular episode in the theory of unitary representations of Lie groups. Our reasoning is mostly explicit.

6.1 The group

To begin with, we introduce a coordinate system into \( G \) by means of the Iwasawa decomposition

\[
G = \text{NAK},
\]

where

\[
N = \left\{ n[x] = \begin{bmatrix} 1 & x \\ 1 & 1 \end{bmatrix} : x \in \mathbb{R} \right\},
\]

\[
A = \left\{ a[y] = \begin{bmatrix} \sqrt{y} \\ 1/\sqrt{y} \end{bmatrix} : y > 0 \right\},
\]

\[
K = \left\{ k[\theta] = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} : \theta \in \mathbb{R}/\pi \mathbb{Z} \right\}.\]

In fact we have \( G \ni \begin{bmatrix} a & b \\ c & d \end{bmatrix} = n[x] a[y] k[\theta], \) with

\[
x = \frac{ac + bd}{c^2 + d^2}, \quad y = (c^2 + d^2)^{-1}, \quad \exp(i\theta) = \frac{d - i c}{|d - i c|}.
\]
The first two give also that if \( n[x]a[y]k[\theta] = n[x_1]a[y_1]k[\theta_1] \), then \( x = x_1, y = y_1 \), and thus \( k[\theta] = k[\theta_1] \) or \( \theta \equiv \theta_1 \mod \pi \mathbb{Z} \). Hence (6.1.1) is indeed a coordinate system on \( G \). We shall always read the notation \( G \ni g = nak \) or \( n[x]a[y]k[\theta] \) in this context.

Let \( x + iy \in H \) correspond to the coset \( n[x]a[y]K \in G/K \). For \( g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \), we have, by (6.1.3), \( gn[x]a[y] = n[x_1]a[y_1]k[\theta] \), with

\[
x_1 = \frac{acy^2 + (ax + b)(cx + d)}{(cy)^2 + (cx + d)^2}, \quad y_1 = \frac{y}{(cy)^2 + (cx + d)^2},
\]

\[
\exp(i\theta) = \left( \frac{j(g, x + iy)}{\|j(g, x + iy)\|} \right)^{-1}.
\]

(6.1.4)

The notation \( j(g, x + iy) \) is introduced in Section 2.2; here \( g \) is regarded as an element in \( \mathbb{T}(H) \), that is, \( g(x + iy) = x_1 + iy_1 \). Namely, \( gn[x]a[y]K = n[x_1]a[y_1]K \), and we have an exact correspondence between the elements of \( H \) and \( G/K \) and the one between the actions of elements of \( G \) upon them. In this way we identify the pair \( (H, \mathbb{T}(H)) \) with the pair \( (G/K, G) \). Note that (6.1.4) is the result of the left multiplication or translation by \( g \).

We then turn to the differentiable structure on \( G \). The coordinate system (6.1.1) suggests that we should work with operators \( \partial_x, \partial_y, \) and \( \partial_\theta \). However, since the harmonic analysis on \( G \) should contain that on \( H \) which is based on the invariance of the hyperbolic Laplacian \( \Delta \), we need a differentiable structure on \( G \) which commutes with the left translations by elements of \( G \), because the identification of \( (H, \mathbb{T}(H)) \) and \( (G/K, G) \) is built upon those translations. A simple and natural way to realize such a construction is to define the procedure of differentiation on \( G \) in a manner independent of left translations; on the other hand the procedure needs to be the result of group actions on \( G \), that is, those done without leaving \( G \) as the analogy to \( \mathbb{R} \) dictates. To satisfy both, it is logical to utilize right translations.

To be precise, let us put

\[
X_1 = \begin{pmatrix} 1 \\ \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & -1 \\ \end{pmatrix}, \quad X_3 = \begin{pmatrix} -1 & 1 \\ \end{pmatrix},
\]

(6.1.5)

and observe that

\[
N = \{ \exp(X_1t) : t \in \mathbb{R} \}, \quad A = \{ \exp(X_2t) : t \in \mathbb{R} \}, \quad K = \{ \exp(X_3t) : t \in \mathbb{R}/\pi \mathbb{Z} \}.
\]

(6.1.6)

Namely (6.1.1) means that these three one-parameter subgroups generate \( G \) and at each of its element the three curves pass through. Hence we utilize the
differentiation to the directions $X_j$, and define, for any smooth function $f$ on $G$,
\[
(x_j f)(g) = \left[ \frac{d}{dt} \right]_{t=0} f(g \exp(X_j t)).
\] (6.1.7)

It is obvious that the differential operators $x_j$ are left invariant or commute with any left translation, for they are defined via right translations. One may use (6.1.3) to see that at $g = n[x[a[y]\theta]$\]
\[
x_1 = y(\cos 2\theta)\partial_x + y(\sin 2\theta)\partial_y + (\sin \theta)^2 \partial_\theta,
\]
\[
x_2 = -2y(\sin 2\theta)\partial_x + 2y(\cos 2\theta)\partial_y + (\sin 2\theta)\partial_\theta,
\]
\[
x_3 = \partial_\theta.
\] (6.1.8)

The set $\{x_1, x_2, x_3\}$ generates a non-commutative algebra $\mathcal{U}$ over $\mathbb{C}$ with respect to the operator multiplication, which is the set of all left invariant differential operators on $G$. Its linear subspace $\mathfrak{g}$ generated by $\{x_1, x_2, x_3\}$ is a Lie algebra, for we have the commutator relations

\[
[x_1, x_2] = -2x_1, \quad [x_1, x_3] = -x_2, \quad [x_2, x_3] = 4x_1 - 2x_3, \quad (6.1.9)
\]

with $[x_i, x_j] = x_i x_j - x_j x_i$, which can be confirmed via (6.1.8); and the Jacobi identity holds naturally. The space $\mathfrak{g}$ is isomorphic to the Lie algebra generated by the elements (6.1.5), for the relations in (6.1.9) hold with $X_j$ under an obvious correspondence, as can be verified easily.

What is important for our purpose is to fix the center of $\mathcal{U}$, and there is a general method to compute an element in the center. Thus, let the map $\text{ad} x$ acting on $\mathfrak{g}$ be defined by $(\text{ad} x)(y) = [x, y]$. Then

\[
\text{trace of } (\text{ad} x) \cdot (\text{ad} y)
\]
is called the Killing form on $\mathfrak{g} \times \mathfrak{g}$. Let $(k_{ij})$ be the inverse matrix of the one attached to the form, with respect to the basis $\{x_1, x_2, x_3\}$. Then the element $\sum k_{ij} x_i x_j$ is in the center of $\mathcal{U}$. The relations (6.1.9) give the matrix for the form, and we find after some rudimentary computation that

\[
\Omega = -x_1^2 - \frac{1}{4} x_2^2 + \frac{1}{2} x_1 x_3 + \frac{1}{2} x_3 x_1
\] (6.1.10)

should be in the center, which one may, however, verify directly by using (6.1.9). This is the Casimir operator on $G$.

For the sake of a later purpose we put

\[
\begin{align*}
\mathbf{w} &= x_3 = \partial_\theta, \\
\mathbf{e}^+ &= 2ix_1 + x_2 - ix_3 = e^{2i\theta}(2iy\partial_x + 2y\partial_y - i\partial_\theta), \\
\mathbf{e}^- &= -2ix_1 + x_2 + ix_3 = e^{-2i\theta}(-2iy\partial_x + 2y\partial_y + i\partial_\theta).
\end{align*}
\] (6.1.11)
The $e^\pm$ are termed the Maass operators, under our normalization. We have the relations

$$[w, e^+] = 2i e^+, \quad [w, e^-] = -2i e^-, \quad [e^+, e^-] = -4i w, \quad (6.1.12)$$

$$\Omega = -\frac{1}{4} e^+ e^- + \frac{1}{4} w^2 - \frac{1}{2} i w. \quad (6.1.13)$$

From (6.1.11) and (6.1.13) we have also

$$\Omega = -y^2((\partial_x)^2 + (\partial_y)^2) + y \partial_x \partial_\theta. \quad (6.1.14)$$

If a smooth function $f$ on $G$ is $K$-trivial, that is, $f(nak) = f(na)$, then we have $\Omega f = \Delta f$. Since $f$ can be regarded as a function on $\mathbb{H}$, this relation characterizes the non-Euclidean Laplacian as a restriction of the Casimir operator.

In passing, we remark that $\Omega$ commutes not only with left translations but also with right translations. In fact, we have, for any smooth $f$,

$$\int_0^a (x_j \Omega f)(g \exp(X_j t)) dt = \Omega \int_0^a (x_j f)(g \exp(X_j t)) dt. \quad (6.1.15)$$

By the definition (6.1.7), the left side is equal to $(\Omega f)(g \exp(X_j a)) - (\Omega f)(g)$, while the right side to $\Omega(f(g \exp(X_j a)) - f(g))$. That is, $(\Omega f)(g \exp(X_j a)) = \Omega(f(g \exp(X_j a)))$, which gives the assertion.

### 6.2 Spectral resolution of the Casimir operator

We now turn to a spectral resolution of the Casimir operator. First of all we need to fix a measure on $G$ which generalizes the non-Euclidean area element $d\mu$, and is invariant against the left translation. The latter is naturally required, because of the aforementioned relation between $(\mathbb{H}, T(\mathbb{H}))$ and $(G/K, G)$. We put, in an a priori manner,

$$dn = dx, \quad da = dy/y, \quad dk = d\theta/\pi,$$

$$dg = d\nu dk = dxdyd\theta/y^2, \quad (6.2.1)$$

where $g = n[x]a[y]k[\theta]$, and $dx, dy, d\theta$ are ordinary Lebesgue measures. It is immediate that $dn, da, dk$ are invariant measures on the groups $N, A, K$. As to the invariance of $dg$, we observe that what is essential, in the present context, about the change of variable $g \mapsto hg$ with a fixed $h$ is the nature of the coset map $gK \mapsto hgK$, for $K$ is abelian and the measure $dk$ is not affected by the map. Thus by the invariance of $d\mu$ we get that of $dg$, via (6.1.4).
The measure $dg$ is unimodular, i.e., invariant against the right translation as well. To verify this, we write $k[\theta]h = n[\xi(\theta)]a[u(\theta)]k[\vartheta(\theta)]$. Then we have $gh = n[x + \xi(\theta)y]a[u(\theta)y]k[\vartheta(\theta)]$. Thus the Jacobian of the right translation by $h$ is equal to $u(\theta)\vartheta'(\theta)$. An explicit computation of $\exp(\vartheta(\theta)i)$ via (6.1.3) gives $u(\theta)\vartheta'(\theta) \equiv 1$, which proves the assertion. In passing, we note further the invariance of $dg$ against $g \mapsto g^{-1}$, which is, however, irrelevant to our subsequent discussion.

With this, we consider (left) $\Gamma$-automorphic functions $f$ on $G$; that is, $f(\gamma g) = f(g)$ for any pair $(\gamma, g) \in \Gamma \times G$. If $f$ is smooth, then $uf$ is also $\Gamma$-automorphic for any $u \in U$, since $\Gamma$ acts from the left. The Hilbert space where we work is

$$L^2(\Gamma \setminus G) = \left\{ f : \text{left } \Gamma\text{-automorphic and } \int_{\Gamma \setminus G} |f(g)|^2 dg < +\infty \right\},$$  \hspace{1cm} (6.2.2)

with the natural inner product

$$\langle f_1, f_2 \rangle = \int_{\Gamma \setminus G} f_1(g)\overline{f_2(g)} dg.$$

One may choose $\mathcal{F} \times K$ as $\Gamma \setminus G$, with a minor abuse of notation, where $\mathcal{F}$ is specified in (1.1.3). According to Lemma 1.1 and (6.1.4), any coset $gK$ can be mapped by a $\gamma \in \Gamma$ so that $\gamma g \in \mathcal{F} \times K$, and if two inner points $x + iy, x_1 + iy_1$ of $\mathcal{F}$ satisfy $\gamma n[x]a[y]k[\theta] = n[x_1]a[y_1]k[\theta_1]$ with a $\gamma \in \Gamma$, then we must have $\gamma = 1$, whence $x = x_1, y = y_1, \theta = \theta_1 \mod \pi \mathbb{Z}$. In fact, one may choose any measurable domain $D$ on $G$ such that $\gamma D$ ($\gamma \in \Gamma$) cover $G$ without overlapping except for sets of null measure. If $D_1$ is another domain of this property, then for any integrable $\Gamma$-automorphic function $f$

$$\int_{D_1} f(g) dg = \sum_{\gamma \in \Gamma} \int_{\gamma D \cap D_1} f(g) dg = \sum_{\gamma \in \Gamma} \int_{D \cap \gamma^{-1} D_1} f(\gamma g) dg = \int_{D} f(g) dg, \hspace{1cm} (6.2.3)$$

where the second line is due to the left invariance of $dg$ and the automorphy of $f$. Hence one may put $\Gamma \setminus G$ without specifying which domain is under consideration. In addition, we note that for $f$ and $D$ as above it holds that

$$\int_{D} f(gh) dg = \int_{D} f(g) dg, \hspace{1cm} (6.2.4)$$

with any $h \in G$. For $f(gh)$ is left $\Gamma$ automorphic, $dg$ is unimodular, and $Dh$ can obviously stand for $D_1$ in (6.2.3).
The last observation has an important consequence: With smooth vectors $f_1, f_2 \in L^2(\Gamma \setminus G)$, we have, for any $u \in \mathcal{U}$,

$$\langle uf_1, f_2 \rangle = \langle f_1, u^* f_2 \rangle,$$

(6.2.5)

where

$$u = \sum c_{j_1j_2...j_k} x_{j_1} x_{j_2} \cdots x_{j_k},$$

$$u^* = \sum (-1)^k c_{j_1j_2...j_k} x_{j_k} x_{j_{k-1}} \cdots x_{j_1},$$

with $x_{j_i} \in \{x_1, x_2, x_3\}$. In fact we have

$$\langle x_j f_1, f_2 \rangle = \int_{\Gamma \setminus G} \frac{d}{dt} \bigg|_{t=0} f(g \exp(X_j t)) f_2(g) dg$$

$$= \frac{d}{dt} \bigg|_{t=0} \int_{\Gamma \setminus G} f(g \exp(X_j t)) f_2(g) dg$$

$$= \frac{d}{dt} \bigg|_{t=0} \int_{\Gamma \setminus G} f(g) f_2(g \exp(-X_j t)) dg = \langle f_1, -x_j f_2 \rangle,$$

where the third line is due to (6.2.4). For instance, we have

$$\Omega^* = \Omega.$$ 

(6.2.6)

Namely, $\Omega$ is self-adjoint.

Now, let $f \in L^2(\Gamma \setminus G)$ be smooth and bounded. Since $f(g)$ is of period $\pi$ in $\theta$, we have the Fourier expansion

$$f(g) = \sum_{p=-\infty}^{\infty} f_p(g) \exp(2pi\theta),$$

(6.2.7)

with

$$f_p(g) = \frac{1}{\pi} \int_0^\pi f(gk[\xi]) \exp(-2pi\xi) d\xi.$$ 

We may regard the latter as an integral over $K$, and have, by the ordinary Parseval identity,

$$\| f(g) \|^2_K = \sum_{p=-\infty}^{\infty} \| f_p(g) \|^2_K,$$

(6.2.8)

with the natural norm, which implies the orthonormal decomposition

$$L^2(\Gamma \setminus G) = \bigoplus_{p=-\infty}^{\infty} L^2_p(\Gamma \setminus G).$$

(6.2.9)
Here

\[ L_p^2(\Gamma \backslash G) = \{ f \in L^2(\Gamma \backslash G) : f(gk|\xi) = f(g) \exp(2\pi i \xi) \} \]. \tag{6.2.10} \]

We then put \( g(x + iy) = f(n[x]a[y]) \) for any \( f \) in \( L_p^2(\Gamma \backslash G) \), and let \( \gamma \in \Gamma \) be such that \( \gamma(x + iy) = x_1 + iy_1 \). We have, by (6.1.4),

\[
\begin{align*}
g(\gamma(x + iy)) &= f(n[x_1]a[y_1]) = f((\gamma n[x]a[y]) \cdot k[-\vartheta]) \\
&= f(\gamma n[x]a[y]) \exp(-2\pi i \vartheta) = f(n[x]a[y]) \exp(-2\pi i \vartheta),
\end{align*}
\]

and thus, with \( z = x + iy \),

\[
g(\gamma(z)) = g(z) \left( \frac{j(\gamma, z)}{|j(\gamma, z)|} \right)^{2p}; \tag{6.2.11}
\]

that is, \( g \) is of weight \( 2p \). Namely, we have

\[
L^2(\Gamma \backslash G) = \bigoplus_{p=-\infty}^{\infty} L_p^2(\Gamma \backslash G/K), \tag{6.2.12}
\]

with

\[
L_p^2(\Gamma \backslash G/K) = L_p^2(\Gamma \backslash \mathcal{H})
\]

\[
= \{ g \text{ satisfying (6.2.11) and } \int_{\mathcal{H}} |g(z)|^2 \mu(z) < +\infty \}. \tag{6.2.13}
\]

On the other hand, applying \( \Omega \) to both sides of (6.2.7), we have, by (6.1.14),

\[
(\Omega f)(g) = \sum_{p=-\infty}^{\infty} (\Delta_p f_p)(n[x]a[y]) \exp(2\pi i \vartheta), \tag{6.2.14}
\]

with \( \Delta_p \) as in (3.2.30), where the smoothness of \( f_p \) comes from that of \( f \).

This means that the problem of the spectral resolution of \( \Omega \) over \( L^2(\Gamma \backslash G) \) is replaced by that of \( \Delta_p \) over \( L_p^2(\Gamma \backslash \mathcal{H}) \). We should, however, take a caution.

For, given \( \Delta_p f_p = \lambda f_p \) with a certain constant \( \lambda \) and with a \( p \), we are unable to assert immediately that \( f \) itself is an eigenfunction of \( \Omega \) with the eigenvalue \( \lambda \), although the converse is trivial. We shall show in the sequel that this is in fact the case. Namely, we shall see that the eigenvalues and eigenvectors of \( \Delta_p \) over \( L_p^2(\Gamma \backslash G/K) \) with varying \( p \) are closely related to each other. Behind this mechanism is the existence of the Maass operators introduced at (6.1.11).

Now, let \( f \in L_p^2(\Gamma \backslash G) \) be a \( C^2 \)-class function such that \( \Omega f = (\kappa^2 + \frac{1}{4}) f \); note that \( \kappa^2 + \frac{1}{4} \in \mathbb{R} \), because of (6.2.6); here \( f \) is such that \( f_p \equiv 0 \) for
$p \neq p'$ in the above notation. We are going to apply differential operators to $f$, and it is expedient to have an extension of Lemma 1.4. Thus, let us put $g(x + iy) = f([x,a[y])]$ as above; we have $\Delta_p g = (\kappa^2 + \frac{1}{4}) g$. Since $g$ is of $C^2$-class and of period 1 with respect to $x$, we have the Fourier expansion

$$g(x + iy) = \sum_{n=-\infty}^{\infty} b(n,y)e(nx).$$

Applying $\Delta_p$ to both sides, we see that $b(n,y)$ satisfies the differential equation

$$-y^2 b''(n,y) + ((2\pi ny)^2 - 4\pi npy - \kappa^2 - \frac{1}{4})b(n,y) = 0. \quad (6.2.15)$$

If $n \neq 0$, then we have, by Lemma 3.8,

$$b(n,y) = \rho(n)W_{sp,ik}(4\pi|n|y), \quad \delta = \text{sgn}(n), \quad (6.2.16)$$

with a certain constant $\rho(n)$. We have used that $\|g\| = \|f\| < \infty$, which gives also $\rho(n) \ll e^{c|n|}$ with any small $\varepsilon > 0$, in view of (3.2.33).

We shall show that $b(0,y) \equiv 0$. The argument is similar to that in the corresponding part of the proof of Lemma 1.4, though it is a little bit more involved. We may restrict ourselves to the situation with $i\kappa < 0$, and $b(0,y) = \rho(0)y^{\frac{1}{2}} + iy$. We are going to show that $\rho(0) = 0$. To this end, we consider the identity

$$0 = \int_{\gamma} \left\{ \Delta_p g(z)\overline{E_p(z,s)} - g(z)\overline{\Delta_p E_p(z,s)} \right\} d\mu(z),$$

where $\gamma$ is as in (1.1.30), $E_p$ defined by (3.2.24), and $s = \frac{1}{2} - i\kappa$. Note that since we may assume that $p \neq 0$, the expansion (3.2.27) implies that this value of $E_p$ is finite. Integration by parts gives

$$0 = \int_{\gamma} \left( \frac{\partial g}{\partial \hat{n}}(z)\overline{E_p(z,s)} - g(z)\frac{\partial}{\partial \hat{n}}\overline{E_p(z,s)} + 2ipg(z)\overline{E_p(z,s)} \right) \frac{|dz|}{y},$$

where $\partial/\partial \hat{n}$ is the non-Euclidean outer-normal differentiation introduced in Section 1.1. Let $\gamma \in \Gamma$. Since $\gamma$ commutes with $\partial/\partial \hat{n}$, we have, by (3.2.26) and (6.2.11),

$$\frac{\partial g}{\partial \hat{n}}(\gamma(z))\overline{E_p(\gamma(z),s)} = \frac{\partial \gamma g}{\partial \hat{n}}(\gamma(z))\overline{E_p(\gamma(z),s)}$$

$$= \frac{\partial g}{\partial \hat{n}}(z)\overline{E_p(z,s)} + g(z)\overline{E_p(z,s)} \left( \frac{\gamma(z)}{|\gamma(z)|} \right)^{-2p} \frac{\partial}{\partial \hat{n}} \left( \frac{\gamma(z)}{|\gamma(z)|} \right)^{2p}.$$
Putting \( j(\gamma, z) = cz + d \), one may compute the last derivative explicitly. We have
\[
\left( \frac{j(\gamma, z)}{|j(\gamma, z)|} \right)^{-2p} \frac{\partial}{\partial \bar{z}} \left( \frac{j(\gamma, z)}{|j(\gamma, z)|} \right)^{2p} = -2 \pi y \frac{d}{dz} \log |j(\gamma, z)|
= \pi y \frac{d}{dz} \log \left( \frac{\text{Im} \gamma(z)}{\text{Im} z} \right).
\]
Collecting these, we find that
\[
0 = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left( \frac{\partial g}{\partial y}(z) E_p(z, s) - g(z) \frac{\partial}{\partial y} E_p(z, s) + 2i \pi y^{-1} g(z) E_p(z, s) \right) \bigg|_{y=Y} \, dx,
\]
which gives \( \rho(0) = 0 \).

That is, we have, for any \( f \in L^2_p(\Gamma \setminus G) \) such that \( \Omega f = (\kappa^2 + \frac{1}{4})f \),
\[
f(g) = e^{2\pi i \theta} \sum_{n=-\infty \atop n \neq 0}^{\infty} \rho(n) W_{\delta_p, i\kappa}(4\pi |n| y) e(nx)
\]
With this, let us consider \( f^{-1} = e^{-f}. \) We have, by (6.1.11), \( w f^{-1} = e^{-w} f - 2i e^{-f} f = 2i(p-1)f^{-1} \). Moreover, by (6.1.13) and (6.2.5)
\[
\|f^{-1}\|^2 = (e^{-f}, e^{-f}) = (f, -e^+ e^{-f})
= (f, (4\Omega - w^2 + 2i w) f) = 4(\kappa^2 + (p-\frac{1}{2})^2)\|f\|^2; \quad (6.2.17)
\]
whence \( f^{-1} \in L^2_{p-1}(\Gamma \setminus G); \) moreover, \( \Omega f^{-1} = (\kappa^2 + \frac{1}{4})f^{-1} \), for \( \Omega \) is in the center of \( U \). Analogously, with \( f^{(+1)} = e^{+f} \), we have \( f^{(+1)} \in L^2_{p+1}(\Gamma \setminus G) \), \( \Omega f^{(+1)} = (\kappa^2 + \frac{1}{4})f^{(+1)} \) as well as
\[
\|f^{(+1)}\|^2 = 4(\kappa^2 + (p+\frac{1}{2})^2)\|f\|^2. \quad (6.2.18)
\]
We may repeat the procedure like ascending and descending aerial strata. There are three possible cases:

1. \( \kappa \geq 0 \).
2. \( \text{Im} \kappa > 0 \) but \( \kappa \neq i(q - \frac{1}{2}) \) for any integer \( q \),
3. \( \kappa = i(q - \frac{1}{2}) \), with an integer \( q > 0 \).

In both the cases (1) and (2) with \( \pm p > 0 \) we have that \( f^{(\mp p)} \neq 0 \) is K-trivial, i.e., in \( L^2(\Gamma \setminus \Delta) \). Namely, there exists a real analytic cusp form \( \psi \) such that \( \Delta \psi = (\kappa^2 + \frac{1}{4})\psi \) and \( f^{(\mp p)} = c \psi \) with a constant \( c \). Then Lemma 1.4 implies that the case (2) is impossible under our assumption that \( \Gamma = \text{PSL}(2, \mathbb{Z}) \). Also, with (1) we have in fact \( \kappa > 3.815 \); and the procedure can be reversed. We
may express this fact that \( f \) can be reached by either ascending or descending from a real analytic cusp form \( \psi \), i.e., \((e^\pm_\psi)\psi = f\).

On the other hand, in the case (3) with \( p > 0 \) the descent terminates, for we have \( f^{(-p+q-1)} \equiv 0 \) as (6.2.17) implies. The Fourier coefficients of \( f^{(-p+q)} \) satisfy (6.2.15) with \( p = q \) and \( \kappa^2 + \frac{1}{4} = -q(q - 1) \). We may use the second identity in (3.2.35). On noting that \( W_{-q,q-\frac{1}{4}}(y) \) satisfies the same differential equation as that for \( W_{q,q-\frac{1}{2}}(-y) \), we have

\[
f^{(-p+q)}(g) = y^q e^{2iq\theta} \sum_{n=-\infty}^{\infty} \rho(n) \exp(-2\pi|n|y + 2\pi inx).
\]

The last sum, denoted by \( h(z) \), \( z = x + iy \), should converge absolutely for any \( z \in \mathcal{F} \). The equation \( \Omega f^{(-p+q)} = -q(q - 1)f^{(-p+q)} \) implies that

\[
\Delta h(z) = 8\pi^2 qy \sum_{n<0} |n|\rho(n) \exp(-2\pi|n|y + 2\pi inx).
\]

This gives \( \rho(n) = 0 \) for all \( n < 0 \), since \( \Delta h \equiv 0 \). Hence we have found that

\[
f^{(-p+q)}(g) = y^q e^{2iq\theta} \sum_{n=1}^{\infty} \rho(n)e(nz).
\]

By the same way as the derivation of (6.2.11) we have \( h(\gamma(z)) = (j(\gamma,z))^2 h(z) \) for any \( \gamma \in \Gamma \); that is, \( h(z) \) is a holomorphic cusp form of weight \( 2q \) with respect to \( \Gamma \). The case (2) with \( p < 0 \) is analogous, and the counterpart of \( h \) turns out to be anti-holomorphic, i.e.,

\[
f^{(-p-q)}(g) = y^q e^{-2iq\theta} \sum_{n=1}^{\infty} \rho(n)e(-nz),
\]

in which the complex conjugate of the sum is a holomorphic cusp form of weight \( 2q \).

What remains then is to make precise the contribution of the continuous spectrum. This can also be dealt with in a fashion similar to the above; that is, the action of the Maass operators is again the key. Thus, from what we have seen in the above, we expect that except for those vectors originating from holomorphic cusp forms the space \( L^p_0(\Gamma\backslash G) \), \( p \geq 0 \), should be spanned by the \((e^\pm_\nu)^p\)-images of smooth vectors of \( L^2_0(\Gamma\backslash G) = L^2(\Gamma\backslash \mathcal{H}) \).

In order to verify this proposition, we first re-define the Eisenstein series \( E_p \) introduced at (3.2.24). We put, for any \( p \in \mathbb{Z} \),

\[
\phi_p(g,\nu) = y^{\nu+\frac{1}{2}} e^{2pi\theta}, \quad (6.2.19)
\]
and
\[ E_p(g, \nu) = \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \phi_p(\gamma g, \nu), \quad \text{Re} \nu > \frac{1}{2}. \] (6.2.20)

Via (6.1.4), we have that
\[ E_p(g, \nu) = E_p(z, \nu + \frac{1}{2}) e^{2i\theta}, \quad z = x + iy, \] (6.2.21)
where the \( E_p \) on the right side stands for (3.2.24); there should not be any notational confusion. On noting that
\[ e^{\pm \phi_p(g)} = (2\nu + 1 \pm 2p) \phi_{p \pm 1}(g), \] (6.2.22)
we have, for \( p \geq 0 \),
\[ (e^{\pm})^p E_0(g, \nu) = \prod_{\ell=0}^{p-1} (2\nu + 1 \pm 2\ell) \cdot E_{p \pm 1}(g, \nu). \] (6.2.23)

Then, let us consider an \( f \) in \( L^2_p(\Gamma \setminus G) \) with \( p > 0 \), each partial derivative of which is of fast decay; c.f., (1.1.29). Since \( (e^{-})^p f \in L^2(\Gamma \setminus \mathcal{H}) \), we have, by Theorem 1.1 with a minor rearrangement,
\[ (e^{-})^p f(g) = \sum_{j=0}^{\infty} \langle (e^{-})^p f, \psi_j \rangle \psi_j(g) \]
\[ + \frac{1}{2\pi} \int_0^\infty E_0(g, it) \mathcal{E}(t, (e^{-})^p f) dt, \] (6.2.24)

By (6.2.23) we rewrite this as
\[ (e^{-})^p f(g) = (-1)^p \sum_{j=0}^{\infty} \langle f, (e^{+})^p \psi_j \rangle \psi_j(g) \]
\[ + \frac{(-1)^p}{2\pi} \int_0^\infty E_0(g, it) \mathcal{E}_p(t, f) dt, \] (6.2.25)
with
\[ \mathcal{E}_p(t, f) = 2^p \frac{\Gamma(\frac{1}{2} - it + p)}{\Gamma(\frac{1}{2} - it)} \int_{\Gamma \setminus G} f(g) \overline{E_p(g, it)} dg. \] (6.2.26)

We then observe that (6.1.12), (6.1.13), and (6.2.23) give
\[ (e^{-})^p (e^{+})^p \psi_j(g) = (-4)^p \frac{\Gamma(\frac{1}{2} + i\kappa_j + p)}{\Gamma(\frac{1}{2} + i\kappa_j)^2} \cdot \psi_j(g), \]
\[ (e^{-})^p (e^{+})^p E_0(g, it) = (-2)^p \frac{\Gamma(\frac{1}{2} + it + p)}{\Gamma(\frac{1}{2} + it)} \cdot (e^{-})^p E_p(g, it). \] (6.2.27)
A combination of (6.2.25)–(6.2.27) yields that $(e^-)^p f^*(g) = 0$, with

$$f^*(g) = f(g) - 2^{-2p} \sum_{j=0}^{\infty} (f, (e^+)^p \psi_j) \frac{\Gamma(\frac{1}{2} + ik_j)^2}{\Gamma(\frac{1}{2} + ik_j + p)^2} \cdot (e^+)^p \psi_j(g)$$

$$- \frac{1}{2\pi} \int_0^\infty E_p(g, it) E_p(t, f) dt.$$  \hspace{1cm} (6.2.28)

We assert that

$$f^* = \sum_{\ell=0}^p (e^+)^{p-\ell} \varphi_\ell,$$  \hspace{1cm} (6.2.29)

where $y^{-\ell} \varphi_\ell(n[x]a[y])$ is a holomorphic cusp form of weight $2\ell$; note that we have actually $\ell \geq 6$, since there exist no holomorphic cusp forms of weight less than 12 over $\Gamma$. We prove (6.2.29) by induction with respect to $p$. Thus, let $f_1 \in L^2_{p+1}(\Gamma \sm G)$ be such that $(e^-)^{p+1} f_1 \equiv 0$. By the inductive assumption, we have $e^- f_1 = \sum_{\ell=0}^p (e^+)^{p-\ell} \varphi_{1,\ell}$, where the specification of the right side is as that of (6.2.29). Applying $e^+$ to both sides, we have, by (6.1.13),

$$(\Omega + p(p + 1)) f_1 = -\frac{1}{4} \sum_{\ell=0}^p (e^+)^{p+1-\ell} \varphi_{1,\ell}$$

$$= -\frac{1}{4} \sum_{\ell=0}^p (e^+)^{p+1-\ell} \frac{(\Omega + p(p + 1))}{\ell(1-\ell) + p(p + 1)} \varphi_{1,\ell}.$$  

That is, $(\Omega + p(p + 1)) f_2 = 0$ with

$$f_2 = f_1 + \frac{1}{4} \sum_{\ell=0}^p (e^+)^{p+1-\ell} \varphi_{1,\ell}.$$  

Thus $y^{-p-1} f_2(n[x]a[y])$ is a holomorphic cusp form of weight $2(p + 1)$, which ends the proof of (6.2.29).

The formula (6.2.28) with (6.2.29) reveals the spectral structure of the space $L^2_G(\Gamma \sm G)$, $p > 0$. It is generated by the $(e^+)^p$-images of real analytic cusp forms and integrals of Eisenstein series and by the vectors of the type (6.2.29). The case $p < 0$ is analogous.

### 6.3 Automorphic representations

The above discussion essentially completes the spectral resolution of $\Omega$ over $L^2(\Gamma \sm G)$. We see horizontal strata in the space, all of which whirl with individual rates by the action of $K$ from the right; thus, no mixing takes place.
There are, however, lifts to climb up and down the strata. Their vertical ways naturally never cross each other, that is, they are invariant against the action of $G$ from the right, as is implicitly asserted at (6.1.15). Some penetrate the hyperbolic plane, and some start or terminate without touching it. We are now about to render this structure in terms of the $\Gamma$-automorphic representation of the Lie group $G$.

To begin with, we rearrange the somewhat complicated formula (6.2.32) by using a certain integral transform due to H. Jacquet. As an orientation, we observe that in view of the Fourier expansion (3.2.27) of $E_p$ on the right side of (6.2.24) it is natural to expect that there should exist an integral transform connecting $\phi_{p}(g, \nu)$ with the Whittaker function $W_{p, \nu}(y)$, which defines the latter in terms of elements and actions of $G$, and makes it possible to understand the basic differential equation (3.2.32) in terms of the pair $(G, g)$.

The integral transform we are concerned is defined by

$$A^\delta f(g) = \int_{-\infty}^{\infty} e(-\delta \xi) f(w_n[\xi]g) d\xi,$$  \hspace{1cm} (6.3.1)

whenever the integral converges absolutely; here $\delta = \pm 1$, and $w = \begin{bmatrix} -1 & 1 \end{bmatrix}$ is the Weyl element of $G$. A basic property of $A^\delta$ is that it commutes with right translations, and is an inter-twining operator; that is, we have, for any $u \in \mathcal{U}$,

$$u(A^\delta f)(g) = A^\delta(u f)(g),$$  \hspace{1cm} (6.3.2)

provided $f$ is smooth. In view of (6.1.7), this is immediate with $u = x_j$, and the general case as well. As we shall see in Section 6.5 below, $A^\delta$ is closely related to the Fourier expansion of Poincaré series on $G$, with respect to the left action of $N$.

By (6.1.3) the map $g \mapsto w_n[\xi]g$ is equivalent to

$$x \mapsto \frac{-x - \xi}{\sqrt{y^2 + (x + \xi)^2}}, \quad y \mapsto \frac{y}{y^2 + (x + \xi)^2},
\quad e^{2p\theta} \mapsto e^{2p\theta} \left(\frac{x + \xi - iy}{x + \xi + iy}\right)^p.$$  \hspace{1cm} (6.3.3)

Hence

$$A^\delta \phi_{p}(g, \nu) = \exp(2p\theta e(\delta x)y^{-\nu+\frac{1}{2}}) \int_{-\infty}^{\infty} \frac{e(y\xi)}{(\xi^2 + 1)^{\nu+\frac{1}{2}}(\xi - i)^{\delta p}} d\xi
\quad = (-1)^p e^p \exp(2p\theta e(\delta x)) \frac{W_{\delta p, \nu}(4\pi y)}{\Gamma(\delta p + \nu + \frac{1}{2})}.$$  \hspace{1cm} (6.3.4)
The first line is valid for $\text{Re} \nu > 0$, and the second line for all $\nu \in \mathbb{C}$ because of (3.2.31) and Lemma 3.8. On noting the first identity in (3.2.35), this implies that we may rewrite (1.1.43) as

$$\psi_j(g) = \frac{\Gamma\left(\frac{1}{2} + i\kappa_j\right)}{2\pi^{\frac{1}{2} + i\kappa_j}} \sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{\rho_j(n)}{\sqrt{|n|}} A_{\text{sgn}(n)} \phi_0(a[n]|g, i\kappa_j)$$

$$= \sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{\varrho_j(n)}{\sqrt{|n|}} A_{\text{sgn}(n)} \phi_0(a[n]|g, i\kappa_j), \quad (6.3.5)$$

with the new normalization of the Fourier coefficients:

$$\varrho_j(n) = \frac{\Gamma\left(\frac{1}{2} + i\kappa_j\right)}{2\pi^{\frac{1}{2} + i\kappa_j}} \rho_j(n). \quad (6.3.6)$$

By virtue of (6.3.2), we have

$$(e^+)^p \psi_j(g) = \sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{\varrho_j(n)}{\sqrt{|n|}} A_{\text{sgn}(n)} (e^+)^p \phi_0(a[n]|g, i\kappa_j). \quad (6.3.7)$$

Then we put

$$\lambda_j^{(p)}(g) = \frac{\Gamma\left(\frac{1}{2} + i\kappa_j\right)}{2^{p} \Gamma\left(\frac{1}{2} + i\kappa_j + p\right)} (e^+)^p \psi_j(g). \quad (6.3.8)$$

We see readily that

$$\lambda_j^{(p)}(g) = \sum_{n=-\infty \atop n \neq 0}^{\infty} \frac{\varrho_j(n)}{\sqrt{|n|}} A_{\text{sgn}(n)} \phi_p(a[n]|g, i\kappa_j). \quad (6.3.9)$$

We have

$$\langle \lambda_j^{(p)}, \lambda_l^{(p)} \rangle = \delta_{jl}. \quad (6.3.10)$$

In fact, this is the same as the first formula in (6.2.27). Hence, we may rewrite (6.2.28) as

$$f(g) = f^*(g) + \sum_{j=0}^{\infty} \langle f, \lambda_j^{(p)} \lambda_j^{(p)}(g) \rangle + \frac{1}{2\pi} \int_0^\infty E_p(g, it) E_p(t, f) dt, \quad (6.3.11)$$

for any smooth $f \in L^2_p(\Gamma \backslash G)$. 

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We extend (6.3.9) to those vectors of $L^2_p (\Gamma \setminus G)$ which come from holomorphic cusp forms. Thus, let $C_k(\Gamma)$ with $k \leq p$ be the vector space of holomorphic cusp forms of weight $2k$ that is introduced in Section 2.2. Let \{\psi_{j,k} : 1 \leq j \leq \vartheta(k)\} be its orthonormal base defined in (2.2.2). We then put

$$\lambda^{(k)}_{j,k}(g) = y^k \psi_{j,k}(x + iy) \exp(2ki\theta).$$

(6.3.12)

Also, corresponding to (6.3.6), we introduce the renormalization of the Fourier coefficients of $\psi_{j,k}$:

$$\varrho_{j,k}(n) = (-1)^k \frac{\Gamma(2k)\frac{1}{2}}{2^{2k} \pi^{k+p} \varrho_{j,k}(n)}.$$

(6.3.13)

By the first line of (6.3.4), we may rewrite (2.2.3) as

$$\lambda^{(k)}_{j,k}(g) = \pi^k \Gamma(2k) \varrho_{j,k}(n) \sum_{n=1}^{\infty} \frac{\varrho_{j,k}(n)}{\sqrt{n}} A^+ \phi_k(a[n]g, k - \frac{1}{2}),$$

(6.3.14)

which is a counterpart of (6.3.5). Further, we put

$$\lambda^{(p)}_{j,k}(g) = 2^{k-p} \left( \frac{\Gamma(p)}{\Gamma(p - k + 1)\Gamma(p + k)} \right) \frac{1}{2} \exp(p-k \lambda^{(k)}_{j,k}(g)).$$

(6.3.15)

We have

$$\lambda^{(p)}_{j,k}(g) = \pi^k \left( \frac{\Gamma(p + k)}{\Gamma(p - k + 1)} \right) \frac{1}{2} \sum_{n=1}^{\infty} \frac{\varrho_{j,k}(n)}{\sqrt{n}} A^+ \phi_p(a[n]g, k - \frac{1}{2}).$$

(6.3.16)

As an analogue of (6.3.10), we have, for any $p \geq k$,

$$\langle \lambda^{(p)}_{j,k}, \lambda^{(p)}_{l,k} \rangle = \delta_{j,l}.$$  

(6.3.17)

Hence (6.2.33) can be expressed as

$$f^*(g) = \sum_{k=6}^{p} \sum_{j=1}^{\vartheta(k)} \langle f, \lambda^{(p)}_{j,k} \rangle \lambda^{(p)}_{j,k}(g).$$

(6.3.18)

In fact, $f^*$ is obviously in the space spanned by \{\lambda^{(p)}_{j,k} : 1 \leq j \leq \vartheta(k), k \leq p\}, which is orthogonal to the space spanned by \{\lambda^{(p)}_{j} : j = 1, 2, \ldots, \infty\} and integrals of the Eisenstein series $E_p$. For instance, $\langle \lambda^{(p)}_{j}, \lambda^{(p)}_{l,k} \rangle = 0$ follows
from the identity \( \langle \Omega \lambda^{(p)}_{j}, \lambda^{(p)}_{l,k} \rangle = \langle \lambda^{(p)}_{j}, \Omega \lambda^{(p)}_{l,k} \rangle \); the Eisenstein series is treated analogously.

The space \( L^2_p(\Gamma \backslash G) \) with \( p < 0 \) has essentially the same spectral structure, except that anti-holomorphic cusp forms fill the rôle of holomorphic cusp forms. Their orthonormal base vectors are given by

\[
\lambda^{(p)}_{j,k}(g) = \pi^{1-k} \left( \frac{\Gamma(|p| + k)}{\Gamma(|p| - k + 1)} \right)^{\frac{1}{2}} \sum_{n=1}^{\infty} \frac{\varrho_{j,k}(n)}{\sqrt{n}} A^{-\phi_p(a[n]g, k - \frac{1}{2})},
\]

(6.3.19)

with \( \varrho_{j,k}(n) \) as in (6.3.16). The involution \( g = n a k \mapsto n^{-1} a k^{-1} \) sends anti-holomorphic cusp forms to holomorphic cusp forms, and vice versa.

Collecting the above discussion, we obtain

**Theorem 6.1** (The spectral resolution of the Casimir operator) Let \( 0L^2(\Gamma \backslash G) \) be the cuspidal subspace spanned by those vectors in \( L^2(\Gamma \backslash G) \) whose Fourier expansion with respect to the left action of \( N \) have vanishing constant terms, and \( eL^2(\Gamma \backslash G) \) the subspace spanned by integrals of Eisenstein series \( E_p \) of all even integral weights which are defined by (6.2.20). Then we have the orthogonal decomposition

\[
L^2(\Gamma \backslash G) = C \cdot 1 \oplus 0L^2(\Gamma \backslash G) \oplus eL^2(\Gamma \backslash G).
\]

(6.3.20)

More precisely, we have the orthonormal decomposition

\[
0L^2(\Gamma \backslash G) = \bigoplus V,
\]

(6.3.21)

where \( V \) runs over all

\[
V_j = \bigoplus_{p=-\infty}^{\infty} \mathbb{C} \lambda^{(p)}_j, \quad V_{l,k}^{\pm} = \bigoplus_{p=k}^{\infty} \mathbb{C} \lambda^{(p)}_{l,k},
\]

(6.3.22)

with the base elements being defined by (6.3.9), (6.3.16), and (6.3.19), respectively; thus \( j \) varies from 1 to infinity, \( k \) from 6 to infinity, and \( 1 \leq l \leq \vartheta(k) \), with \( \vartheta(k) \) as in (2.2.2). Also we have the orthonormal decomposition

\[
eL^2(\Gamma \backslash G) = \bigoplus_{p=-\infty}^{\infty} E^{(p)},
\]

(6.3.23)

where \( E^{(p)} \) consists of

\[
\frac{1}{4\pi i} \int_{(0)} E_p(g, \nu) h(\nu) d\nu,
\]

(6.3.24)
with $h$ being ordinary square integrable functions over the imaginary axis $(0)$.

A few points are missing in our discussion so far developed. We have started with (6.2.7); and each component $f_p$ has been spectrally expanded as in (6.3.11) together with (6.3.18), although there is a slight notational confusion. We have not mentioned the detail about several convergence issues, for their verifications are immediate as far as $f$ on $G$ is left $\Gamma$-automorphic and such that $uf$ with $u \in U$ of sufficiently high order are all of fast decay; and the set of those $f$ is dense in $L^2(\Gamma \setminus G)$. It would be expedient to remark here that the uniform fast decay of projections of $f$ to any of the subspaces listed in (6.3.22) and (6.3.23) could be derived readily from that of $uf$; for instance,

$$\langle f, \lambda_j^{(p)} \rangle = \frac{1}{(\kappa_j^2 + \frac{1}{4} + 2\pi i)^a} \langle f, (\Omega - w)^a \lambda_j^{(p)} \rangle$$

$$= \frac{1}{(\kappa_j^2 + \frac{1}{4} + 2\pi i)^a} \langle (\Omega + w)^a f, \lambda_j^{(p)} \rangle$$

$$\ll (\kappa_j^2 + |p|^{-a})||\Omega + w|^a f||,$$  \hspace{1cm} (6.3.25)

with any fixed integer $a \geq 0$, gives what is needed, and the same device works for all other subspaces. We have neither mentioned explicitly the Parseval formula which generalizes (1.1.49) to the whole $L^2(\Gamma \setminus G)$; however, this could readily be inferred from a combination of (6.2.7), (6.2.8), and (6.3.11), since the last is in fact a rearrangement of (6.2.24), which is in turn a direct consequence of (1.1.47).

We are now at the stage to express the assertion of the last theorem in the language of automorphic representations. Thus, let us consider the right translation

$$\omega(h) : f(g) \mapsto f(gh),$$ \hspace{1cm} (6.3.26)

with any $f \in L^2(\Gamma \setminus G)$. For each $h \in G$, $\omega(h)$ is a unitary map of $L^2(\Gamma \setminus G)$ into itself, because of the unimodularity of $dg$; and $\omega$ is a homomorphism. This configuration is expressed that $\omega$ is a $\Gamma$-automorphic unitary representation of the Lie group $G$.

If $W$ is a closed subspace of $L^2(\Gamma \setminus G)$ and $\omega(h)W \subset W$ for all $h \in G$, then $W$ is called an invariant subspace. We shall prove that those $V$ in (6.3.21) are all invariant subspaces. Thus, let $V$ be the closed subspace generated by $\bigcup_h \omega(h)V$ with $h$ varying throughout $G$. Since $\Omega$ commutes with any right translation as is shown at (6.1.15), all smooth elements in $V$ are eigenfunctions of $\Omega$ with the same eigenvalue. The Fourier coefficients, with respect to the right action of $K$, of a particular eigenfunction come from cusp forms either real analytic or holomorphic or anti-holomorphic over $\mathcal{H}$; hence $V$ splits into a
finite number of subspaces among those listed in (6.3.22) which share the same eigenvalue with $V$. This means that $G$ splits into the same number of cosets $Hh$ with $H$ being the closed subgroup of $G$ composed of all elements that send $V$ to itself. Let us assume that this number is larger than 1, and consider the cosets $H \exp(tX_j), t \in \mathbb{R}$. With either $j = 1$ or 2, there should be at least two different cosets. This is, however, a contradiction, for $\exp(tX_j)$ is of course a continuous curve on $G$.

Those $V$’s are in fact irreducible representations; that is, any invariant subspace contained in $V$ is either $V$ itself or $\{0\}$. The proof of this fact requires some preparation which appears to be an excess for our present purpose. In fact, what we need genuinely is not the irreducibility but a realization of the structure of each $V$ in terms of an ordinary functional space that we shall develop in the next section. Nevertheless, we shall see that the latter gives a somewhat unconventional proof of the former as well. Thus in the closing paragraph of the next section we shall prove

**Theorem 6.2** The identity (6.3.21) gives the decomposition of the cuspidal subspace into irreducible subspaces with respect to the unitary representation (6.3.26) of $G$.

Representations and invariant subspaces are obviously inter-changeable concepts. With this convention, we may call $V$’s in (6.3.21) as irreducible representations of $G$ occurring in the Hilbert space $L^2(\Gamma \setminus G)$. Those $V_j$ arising from real analytic cusp forms belong to the unitary principal series of irreducible representations, and $V_{\pm l,k}$ to the holomorphic and the anti-holomorphic discrete series, respectively. In general there can be additional series of representations coming from exceptional eigen vectors of the Casimir operator (the complementary series); in our situation with $\Gamma$, such does not occur.

Here we introduce a major simplification of notation. We write

$$
\lambda_V^{(p)}(g) = \sum_{n=-\infty}^{\infty} \frac{\vartheta_V(n)}{\sqrt{|n|}} A^{\vartheta_V(n)} \phi_p(a|n|g, \nu_V),
$$

$$
\lambda_V^{(p)}(g) = \pi^{-\nu_V} \left( \frac{\Gamma(|p| + \nu_V + \frac{1}{2})}{\Gamma(|p| - \nu_V + \frac{1}{2})} \right) \frac{1}{2} \sum_{n=1}^{\infty} \frac{\vartheta_V(n)}{\sqrt{n}} A^{\pm} \phi_p(a|n|g, \nu_V),
$$

$$
E_p(g, \nu) = y^{2\nu} \exp(2pi\theta) + (-1)^p \frac{\Gamma(\frac{1}{2} + \nu)^2 \varphi_{\nu}(\frac{1}{2} + \nu)}{\Gamma(\frac{1}{2} + \nu + p) \Gamma(\frac{1}{2} + \nu - p)} \exp(2pi\theta)
$$

$$
+ \sum_{n=-\infty}^{\infty} \frac{|n|^{-\nu} \sigma_{2\nu}(|n|)}{\zeta(1 + 2\nu) \sqrt{|n|}} A^{\vartheta_V(n)} \phi_p(a|n|g, \nu). \tag{6.3.29}
$$
The first is equivalent to (6.3.9), the second to either (6.3.16) or (6.3.19), and the third to (3.2.27). This is possible, for the Fourier coefficients \( g_j(n) \) and \( g_{j,k}(n) \) do not depend on the weight \( 2p \) but only on the representation \( V \); and the eigenvalues of the Casimir operator have the same property. As a matter of fact, our normalization (6.3.6) and (6.3.13) as well as the use of the operator \( A^\delta \) have been done with (6.3.27)–(6.3.29) in mind. More precisely we have now

\[ \nu_V = i\kappa_j \quad \text{or} \quad k - \frac{1}{2}, \]

\[ g_V(n) = \frac{\Gamma(\frac{1}{2} + i\kappa_j)}{2\pi^{\frac{1}{2}+i\kappa_j}} \rho_j(n) \quad \text{or} \quad (-1)^k \frac{\Gamma(2k)^{\frac{1}{2}}}{2^{2k-1} \pi^k} \rho_{j,k}(n), \]

according as \( V \) belongs to either the unitary principal or holomorphic/anti-holomorphic discrete series. We have, in place of (6.3.22),

\[ V = \bigoplus_{p=-\infty}^{\infty} \mathbb{C} \Lambda_V^{(p)}, \]

for any \( V \), where an obvious convention is in force when \( V \) is not in the unitary principal series.

As to the action of Hecke operators, the definitions (3.1.3) and (3.1.19) can be translated into

\[ T(n)f(g) = n^{-\frac{1}{2}} \sum_{\tau \in \Gamma \setminus \mathbb{M}(n)} f(n^{-\frac{1}{2}} \tau g) \]

\[ = n^{-\frac{1}{2}} \sum_{d|n} \sum_{b=1}^{d} f(n|b/d|[a[n/d^2]|g]), \]

with \( \mathbb{M}(n) \) as in Section 3.1. We have, in place of (3.1.13) and (3.1.20),

\[ T(n)\Lambda_V^{(p)} = t_V(n)\Lambda_V^{(p)}, \]

with \( t_V(n) = t_j(n) \) or \( t_{j,k}(n) \). This is due to (6.3.5) and (6.3.12) as well as to that \( T(n) \) is defined with left translations in (6.3.33); that is, \( T(n) \) commutes with the action of \( g \), and each \( V \) is Hecke invariant. We have, for any integer \( n \),

\[ g_{tV}(n) = \begin{cases} \epsilon_V^{(1-\text{sgn}(n))} g_{tV}(1) t_V(|n|) & \text{unitary principal series}, \\ \frac{i}{2}(1 + \text{sgn}(n)) g_V(1) t_V(n) & \text{discrete series}, \end{cases} \]

where \( \epsilon_V = \epsilon_j \) is as in (3.1.15). The bound (4.4.4) implies that uniformly for all \( V \)

\[ t_V(n) \ll n^{\frac{1}{4} + \delta}, \]
Finally, the automorphic $L$-functions $L_j$, $L_{j,k}$, and their corresponding Hecke series $H_j$, $H_{j,k}$ which are introduced in Section 3.2 are replaced by

\[ L_V(s) = \sum_{n=1}^{\infty} \varrho_V(n)n^{-s}, \quad H_V(s) = \sum_{n=1}^{\infty} t_V(n)n^{-s}, \]

(6.36)

together with the normalization (6.30). Also the definition (3.2.3) of the Rankin $L$-function is now extended by

\[ L_{V \otimes V'}(s) = \zeta(2s) \sum_{n=1}^{\infty} \varrho_V(n)\varrho_{V'}(n)n^{-s}, \]

(6.37)

for any pair $V, V'$ of irreducible representations.

### 6.4 Realization of representations

We shall try to investigate the structure of individual subspaces $V$ listed in (6.3.22) by means of a realization of representations. This will lead us, in particular, to a geometrical understanding of those exotic Bessel transforms (2.3.17), (2.4.8), (2.5.7), and (2.5.15) which are involved in the sum formulas of Kloosterman sums. More precisely, we shall find that those integral transforms are closely related to the action of the Weyl element $w = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ in each subspace $V$. One should note here that the Bruhat decomposition

\[ G = NA \cup \text{NwNA} \]

(6.4.1)

holds, as we have

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{cases} n[a/c]a[a^2] & \text{if } c = 0, \\ n[a/c]w[n[c]a[a^2] & \text{if } c \neq 0. \end{cases} \]

(6.4.2)

Thus the realization of the actions of $w, n[x], a[y]$ in each $V$ is a fundamental issue, and an answer to it is given in Lemma 6.1 below.

In the formulas (6.3.9), (6.3.16), and (6.3.19) we see a correspondence between the base elements of $V$ and the simple function $\phi_p(g, n_V)$ in terms of the operator $A^\pm$ and Fourier expansions of cusp forms over $\Gamma \backslash \mathbb{H}$. The metrical properties (6.3.10) and (6.3.17) are in fact solely due to (6.2.25), (6.3.2) and the original orthonormality of the base system of those cusp forms. As each subspace $V$ is generated from a particular cusp form and the actions of the
Maass operators upon it, it is naturally surmised that the structure of $V$ could be described only with the behaviour of $\mathcal{A}^\pm$ in the space spanned by $\phi_p$, without recoursing to the nature of the cusp form generating $V$. In this way, we reach the notion of models. What we are about to develop is, in fact, such a model.

Thus, returning to (6.3.1), we consider first the validity of the equation

$$A^k\phi(g, \nu) = \sum_{p=-\infty}^{\infty} c_p A^k \phi_p(g, \nu), \quad (6.4.3)$$

with

$$\phi(g, \nu) = \sum_{p=-\infty}^{\infty} c_p \phi_p(g, \nu). \quad (6.4.4)$$

Here $\phi$ is assumed to be smooth, i.e., $c_p$ decays faster than any negative power of $|p|$ as $p$ tends to infinity. Integration by parts gives

$$A^k \phi_p(a[y], \nu) = \frac{y^{-\frac{1}{2} - \nu}}{2\pi i} \int_{-\infty}^{\infty} \frac{(1 + 2\nu)\xi + 2\delta p)i e(y\xi)}{(\xi^2 + 1)^{\frac{1}{2} + \nu}} (\frac{\xi + i}{\xi - i})^p d\xi, \quad (6.4.5)$$

which implies that (6.4.3) holds for $\text{Re} \nu > -\frac{1}{2}$ via analytic continuation with respect to $\nu$. Repeating the same procedure, we find that (6.4.3) holds for any $\nu \in \mathbb{C}$.

Then, we define the Kirillov operator $\mathcal{K}$ by

$$\mathcal{K}\phi(u) = \mathcal{A}^{\text{sgn}(u)} \phi(a[|u|]), \quad u \in \mathbb{R}^\times, \quad (6.4.6)$$

where $\mathbb{R}^\times = \mathbb{R}\setminus\{0\}$; hereafter we shall often omit the parameter $\nu$ to avoid otherwise heavy notations. This concept will play a crucial rôle in the sequel, for it gives a realization of each $V$ in terms of a fairy ordinary function space, that is, $L^2(\mathbb{R}^\times, \pi^{-1}d^\times)$, $d^\times u = du/|u|$, as is made explicit in the following three lemmas.

**Lemma 6.1** Let $\phi$ be smooth as in (6.4.4). We have, with the right translation $\omega$,

$$\mathcal{K}\omega(n[x])\phi(u) = e(nx)\mathcal{K}\phi(u), \quad \mathcal{K}\omega(a[y])\phi(u) = \mathcal{K}\phi(uy). \quad (6.4.7)$$

Also, if $|\text{Re} \nu| < \frac{1}{2}$, then

$$\mathcal{K}\omega(w)\phi(u) = \int_{\mathbb{R}^\times} j_\nu(u\lambda)\mathcal{K}\phi(\lambda)d^\times \lambda. \quad (6.4.8)$$
Here

\begin{equation}
\nu(u) = \pi \sqrt{\frac{|u|}{\sin \pi \nu}} \left( J_{-2\nu}^{\text{sgn}(u)}(4\pi \sqrt{|u|}) - J_{2\nu}^{\text{sgn}(u)}(4\pi \sqrt{|u|}) \right), \tag{6.4.9}
\end{equation}

where \( J_{\nu}^+ = J_{\nu} \) and \( J_{-\nu}^- = I_{\nu} \) with the ordinary notation for Bessel functions.

**Proof.** Since (6.4.7) is immediate, we deal with (6.4.8) only. For this sake we consider the Mellin transform

\begin{equation}
\Gamma_p(s) = \Gamma_p(s, \nu) = \int_0^\infty A^p(a[y])y^{s-\frac{3}{2}}dy. \tag{6.4.10}
\end{equation}

We shall show that \( \Gamma_p(s) \) continues meromorphically to \( \mathbb{C} \), and satisfies the local functional equation

\begin{equation}
(-1)^p \Gamma_p(s) = 2^{1-2s} \pi^{-2} \Gamma(s+\nu) \Gamma(s-\nu) \times (\cos \pi s \Gamma_p(1-s) + \cos \pi \nu \Gamma_{-p}(1-s)), \tag{6.4.11}
\end{equation}

provided \( \text{Re} \nu > -\frac{1}{2} \). In fact, by the first line of (6.3.4), we have, for \( \text{Re} s > \text{Re} \nu > 0 \),

\begin{equation}
\Gamma_p(s) = \int_0^\infty y^{s-\nu-1} \int_{\text{Im} \xi = \frac{1}{2}} \frac{e(y\xi)}{\left(\xi + i\right)^p} \left(\frac{\xi + i}{\xi - i}\right)^p d\xi dy.
\end{equation}

Assuming temporarily that \( 0 < \text{Re} \nu < \frac{1}{4} < \text{Re} s < \frac{1}{2} \), we exchange the order of integration, and compute the resulting inner integral. We find that

\begin{equation}
\Gamma_p(s) = (2\pi)^{\nu-s} \Gamma(s-\nu) \times \left[ \exp(-\frac{1}{4} \pi i(s-\nu))L_p(s) + \exp(\frac{1}{4} \pi i(s-\nu))L_{-p}(s) \right], \tag{6.4.12}
\end{equation}

with

\begin{equation}
L_p(s) = \int_0^\infty \frac{\xi^{-s+\nu}}{(\xi^2 + 1)^{\nu+\frac{3}{2}}} \left(\frac{\xi + i}{\xi - i}\right)^p d\xi. \tag{6.4.13}
\end{equation}

By analytic continuation, the expression (6.4.12) holds if \(-\text{Re} \nu < \text{Re} s < 1 + \text{Re} \nu \). Under this condition, we observe that the change of variable \( \xi \rightarrow \xi^{-1} \) gives \( L_p(s) = (-1)^p L_{-p}(1-s) \). Then a rearrangement gives (6.4.11) and the meromorphic continuation of \( \Gamma_p(s) \) follows via analytic continuation.

We are now going to show that (6.4.8) with \( \phi = \phi_p \) and (6.4.11) are in fact a Mellin pair; that is, the Mellin inversion of (6.4.11) yields a special case of (6.4.8). To this end, we note first that if \( |\text{Re} \nu| < \text{Re} s < \frac{1}{4} \), then

\begin{equation}
\int_0^\infty j\nu(u)u^{s-\frac{3}{2}}du = 2^{1-2s} \pi^{-2s} \cos(\pi s) \Gamma(s+\nu) \Gamma(s-\nu), \tag{6.4.14}
\end{equation}

and that if $|\text{Re}\nu| < \text{Re}s$, then
\[
\int_{-\infty}^{0} j_\nu(u)|u|^{s-\frac{3}{2}}du = 2^{1-2s}\pi^{-2s}\cos(\pi\nu)\Gamma(s+\nu)\Gamma(s-\nu). \tag{6.4.15}
\]

The former is a consequence of (4.4.11) and the latter of (3.2.38) with (1.1.27); both integrals are absolutely convergent. One could then appeal to the Parseval formula for $L^2$-pairs of Mellin transforms. Here we develop instead a direct reasoning. Thus, by the last two formulas we transform (6.4.11) into
\[
(-1)^p\Gamma_p(s) = \int_{R^+} j_\nu(u)|u|^{s-\frac{3}{2}}\Gamma_{\text{sgn}(u)p}(1-s)d^su.
\]

We replace $p$ by $\text{sgn}(v)p$, multiply both sides by the factor $|v|^{\frac{1}{2}-s}/2\pi i$, and integrate over the vertical line $\text{Re}s = \beta$, with $|\text{Re}\nu| < \beta < \frac{1}{4}$. We have
\[
\frac{(-1)^p}{2\pi i} \int_{(\beta)} \Gamma_{\text{sgn}(v)p}(s)|v|^{\frac{1}{2}-s}ds
= \int_{R^+} j_\nu(u)\left\{ \frac{1}{2\pi i} \int_{(\beta)} \Gamma_{\text{sgn}(uv)p}(1-s)|u/v|^{s-\frac{3}{2}}ds \right\}d^su. \tag{6.4.16}
\]

The absolute convergence that is needed to verify the exchange of the order integration is due to the exponential decay of $\Gamma_p(s)$ which can be confirmed by turning the contour in (6.4.13) through a small angle appropriately. The left side of (6.4.16) is equal to
\[
(-1)^p\mathcal{A}^\phi_{\text{sgn}(v)p}(a[|v|]) = \mathcal{A}^{\text{sgn}(v)\phi_p}(a[|v|]w) = \mathcal{K}_\omega(w)\phi_p(v)
\]
in view of the first line of (6.3.4); also, the inner-integral to
\[
\mathcal{A}^\phi_{\text{sgn}(uv)p}(a[|u/v|]) = \mathcal{K}\phi_p(u/v),
\]
which yields (6.4.8) in the case of $\phi = \phi_p$ with $|\text{Re}\nu| < \frac{1}{4}$. To widen this range of $\nu$, we remark that we have, for $\text{Re}\nu > -\frac{1}{2}$, $0 < y \leq 1$,
\[
\mathcal{A}^\phi\phi_p(a[y]) \ll (|p| + |\nu| + 1)y^{\frac{1}{2}-|\text{Re}\nu|}\log y, \tag{6.4.17}
\]
and, for $\text{Re}\nu > -\frac{1}{2}$, $y \geq 1$,
\[
\mathcal{A}^\phi\phi_p(a[y]) \ll (|p| + |\nu| + 1)y^{\frac{1}{2}-\text{Re}\nu}\exp\left(-\frac{y}{|\nu| + |p| + 1}\right). \tag{6.4.18}
\]
The implied constants in both bounds are absolute. In fact, the first line of (6.3.4) gives

\[ A^0 \phi_p(a[y]) = A^0 \phi_0(a[y]) + y^{\frac{1}{2} - \nu} \int_{-\infty}^{\infty} \frac{e(y\xi)}{(\xi^2 + 1)^{\nu}} \left( \frac{\xi + i}{\xi - i} \right)^{\delta_p} - 1 \, d\xi \]

\[ = \frac{2\pi^{\frac{1}{2} + \nu}}{\Gamma(\frac{1}{2} + \nu)} y^{\frac{1}{2}} K_\nu(2\pi y) + O \left( y^{\frac{1}{2} - \text{Re} \nu} (|p| + 1) \right), \quad (6.4.19) \]

provided \( \text{Re} \nu > -\frac{1}{2} \), and (6.4.17) follows. As to (6.4.18), it suffices to shift the contour in (6.4.5) to \( \text{Im} \xi = (|\nu| + |p| + 1)^{-1} \). Via these bounds we get the desired analytic continuation to \( |\text{Re} \nu| < \frac{1}{2} \). The assertion (6.4.8) with a general smooth \( \phi \) is now immediate. This ends the proof of the lemma.

Next, we shall show that the Kirillov operator is in fact a unitary map of a simple nature:

**Lemma 6.2.** We assume that \( \nu \in i\mathbb{R} \), and introduce the Hilbert space

\[ U_\nu = \bigoplus_{p=-\infty}^{\infty} \mathbb{C} \phi_p, \quad \phi_p(g) = \phi_p(g; \nu), \quad (6.4.20) \]

equipped with the ordinary norm

\[ \|\phi\|_{U_\nu}^2 = \sum_{p=-\infty}^{\infty} |c_p|^2, \quad \phi = \sum_{p=-\infty}^{\infty} c_p \phi_p. \quad (6.4.21) \]

Then the operator \( \mathcal{K} \) is a unitary map from \( U_\nu \) onto \( L^2(\mathbb{R}^\times, \pi^{-1}d^x) \). In particular, \( \omega \) and \( \mathcal{K}\omega \mathcal{K}^{-1} \) are equivalent unitary representations of \( G \) in \( U_\nu \) and \( L^2(\mathbb{R}^\times, \pi^{-1}d^x) \), respectively.

**Proof.** We shall first treat the second assertion, while assuming the validity of the first. The unitarity of \( \mathcal{K}\omega(n[x])\mathcal{K}^{-1} \) and \( \mathcal{K}\omega(n[y])\mathcal{K}^{-1} \) with respect to \( L^2(\mathbb{R}^\times, \pi^{-1}d^x) \) is obvious from (6.4.7). Also, the unitarity of \( \omega(k[\theta]) \) on \( U_\nu \) is fairly obvious. Hence the assertion follows.

Let us prove the unitarity of \( \mathcal{K} \). We shall employ an explicit reasoning. Thus, by (6.3.4) and (6.4.6), we have, for \( \nu \in i\mathbb{R} \) and \( p, q \in \mathbb{Z} \),

\[ \frac{1}{\pi} \int_{\mathbb{R}^\times} \mathcal{K} \phi_p(u) \overline{\mathcal{K} \phi_q(u)} d^x u \]

\[ = \frac{(-1)^{p+q}}{\Gamma(p + \nu + \frac{1}{2})\Gamma(q - \nu + \frac{1}{2})} \int_0^\infty W_{p,\nu}(y) W_{q,\nu}(y) \frac{dy}{y} \]

\[ + \frac{(-1)^{p+q}}{\Gamma(-p + \nu + \frac{1}{2})\Gamma(-q - \nu + \frac{1}{2})} \int_0^\infty W_{-p,\nu}(y) W_{-q,\nu}(y) \frac{dy}{y}. \quad (6.4.22) \]
where we have used the fact that \(W_{p,\nu}(y)\) is real because of (3.2.34). To evaluate these integrals, we appeal to the following formula: For any \(\alpha, \beta \in \mathbb{C}\) and \(|\text{Re}\, \mu| < \frac{1}{2}\), it holds that

\[
\int_0^\infty W_{\alpha,\mu}(y)W_{\beta,\mu}(y) \frac{dy}{y} = \frac{\pi}{(\alpha - \beta) \sin(2\pi \mu)} \times \left[ \frac{1}{\Gamma\left(\frac{1}{2} - \alpha + \mu\right)\Gamma(\frac{1}{2} - \beta - \mu)} - \frac{1}{\Gamma\left(\frac{1}{2} - \alpha - \mu\right)\Gamma(\frac{1}{2} - \beta + \mu)} \right], \quad (6.4.23)
\]

together with

\[
\int_0^\infty (W_{\alpha,\mu}(y))^2 \frac{dy}{y} = \frac{\pi}{\sin(2\pi \mu)} \times \left[ \frac{1}{\Gamma\left(\frac{1}{2} - \alpha + \mu\right)\Gamma\left(\frac{1}{2} - \alpha - \mu\right)}\left[ \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - \alpha + \mu\right) - \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} - \alpha - \mu\right) \right] \right], \quad (6.4.24)
\]

By (6.4.23) we see readily that the right side of (6.4.22) vanishes whenever \(p \neq q\); and by (6.4.24) it is equal to 1 if \(p = q\). Hence

\[
\frac{1}{\pi} \int_{\mathbb{R}^+} \mathcal{K}\phi_p(u)\overline{\mathcal{K}\phi_q(u)} d^\times u = \delta_{p,q}, \quad (6.4.25)
\]

which is equivalent to the unitarity of \(\mathcal{K}\).

To show (6.4.23), we use the Whittaker differential equation (3.2.32). We have

\[
- \alpha \int_0^\infty W_{\alpha,\mu}(y)W_{\beta,\mu}(y) \frac{dy}{y} = \lim_{\varepsilon \to 0^+} \int_\varepsilon^\infty \left[ \left( \frac{d}{dy} \right)^2 - \frac{1}{4} + \frac{4 - \mu^2}{y^2} \right] W_{\alpha,\mu}(y)W_{\beta,\mu}(y) \frac{dy}{y} \\
= \lim_{\varepsilon \to 0^+} \left[ -W'_{\alpha,\mu}(\varepsilon)W_{\beta,\mu}(\varepsilon) + W_{\alpha,\mu}(\varepsilon)W'_{\beta,\mu}(\varepsilon) \right] - \beta \int_0^\infty W_{\alpha,\mu}(y)W_{\beta,\mu}(y) \frac{dy}{y} , \quad (6.4.26)
\]

To compute the last limit we invoke that near the origin

\[
W_{\alpha,\mu}(y) = \frac{\Gamma(-2\mu)}{\Gamma\left(\frac{1}{2} - \alpha - \mu\right)} y^{\mu + \frac{1}{2}} + \frac{\Gamma(2\mu)}{\Gamma\left(\frac{1}{2} + \alpha + \mu\right)} y^{-\mu + \frac{1}{2}} \left(1 + O(y)\right), \\
W'_{\alpha,\mu}(y) = \left(\frac{\mu + \frac{1}{2}}{\Gamma\left(\frac{1}{2} - \alpha - \mu\right)} y^{\mu - \frac{1}{2}} - \frac{\mu - \frac{1}{2}}{\Gamma\left(\frac{1}{2} + \alpha + \mu\right)} y^{-\mu + \frac{1}{2}} \right) \left(1 + O(y)\right). \quad (6.4.27)
\]
where the implied constant is bounded as far as $|\text{Re} \mu| < \frac{1}{2}$. After a rearrangement we get (6.4.23).

It remains for us to show the surjectivity of $\mathcal{K}$. Thus, let $\nu \in i\mathbb{R}$ and assume that a smooth function $g$, compactly supported on $\mathbb{R} \setminus \mathcal{K}\phi_p$, is orthogonal to all $\mathcal{K}\phi_p$. Multiply (6.4.5) by $g$ and integrate, change the order of integration, and undo the integration by parts with respect to the outer integral. We have

$$0 = \int_{\mathbb{R} \times} g(u)\mathcal{K}\phi_p(u) d^\times u$$

$$= \int_{-\infty}^{\infty} \frac{1}{(\xi^2 + 1)^{\frac{1}{2} + \nu}} \left( \frac{\xi + i}{\xi - i} \right)^p \left( \int_{-\infty}^{\infty} g(u)|u|^{\frac{1}{2} + \nu} e(-u\xi) du \right) d\xi.$$ 

Then we note that the system $\{(\xi + i)/(\xi - i))^p : p \in \mathbb{Z}\}$ is complete orthonormal in the space $L^2(\mathbb{R}, (\pi(\xi^2 + 1))^{-1} d\xi)$. Hence the Fourier transform of $g(u)|u|^{-\frac{1}{2} + \nu}$ vanishes identically, whence the assertion. This ends the proof of the lemma.

In passing, we make a remark on the complementary series, i.e., the situation with $-\frac{1}{2} < \nu < \frac{1}{2}$, although such a representation of $G$ does not occur in $L^2(\Gamma \setminus G)$. It is easy to see that Lemma 6.1 remains valid. The definition (6.4.20) is the same, but (6.4.21) has to be replaced by the norm

$$\pi^\nu \left( \sum_{p = -\infty}^{\infty} \frac{\Gamma(p + \frac{1}{2} - \nu)}{\Gamma(p + \frac{1}{2} + \nu)} |c_p|^2 \right)^{\frac{1}{2}}.$$ 

With this, the above proof extends readily, and Lemma 6.2 holds for these $\nu$ as well.

On the other hand, in dealing with the holomorphic discrete series, (6.4.20) has to be replaced by the Hilbert space

$$D_k = \bigoplus_{p = k}^{\infty} \mathbb{C}\phi_p, \quad \phi_p(g) = \phi_p(g; k - \frac{1}{2}),$$ 

with an integer $k \geq 1$, which is equipped with the norm

$$\|\phi\|_{D_k} = \pi^{k-\frac{1}{2}} \left( \sum_{p = k}^{\infty} \frac{\Gamma(p - k + 1)}{\Gamma(p + k)} |c_p|^2 \right)^{\frac{1}{2}}, \quad \phi = \sum_{p = k}^{\infty} c_p\phi_p.$$ 

Since $\mathcal{A}^-$ annihilates $D_k$, we are concerned with $\mathcal{A}^+$ only. The expression (6.3.4), $\delta = +$, holds without changes. With this, the operator $\mathcal{K}$ is defined by (6.4.6) again; it should be noted that

$$j_{k-\frac{1}{2}}(u) = \begin{cases} 0, & \text{if } u < 0, \\ 2\pi(-1)^k \sqrt{u} J_{2k-1}(4\pi\sqrt{u}), & \text{if } u > 0. \end{cases}$$
Lemma 6.3. The operator $\mathcal{K}$ is a unitary map of $D_k$ onto $L^2((0, \infty), \pi^{-1}d^x)$. Also, for any smooth $\phi \in D_k$, we have (6.4.7) and (6.4.8) with $\nu = k - \frac{1}{2}$. With these changes the second assertion of the previous lemma holds. The analogue for the anti-holomorphic discrete series is obtained by applying the involution $g = nak \mapsto n^{-1}ak^{-1}$.

Proof. The third assertion is immediate. As to the unitarity of $\mathcal{K}$, it is proved with a minor change of the above argument. In fact, the Whittaker function $W_{p,k-\frac{1}{2}}(u)$ is a product of $u^k \exp(-\frac{1}{2}u)$ and a polynomial on $u$ of degree $p - k$, as (6.3.4) implies. Thus the proof of (6.4.23) can be carried out also for the product $W_{p,k-\frac{1}{2}}(u)W_{p,k-\frac{1}{2}}(u)$ with integers $p, q$, although the condition on $\Re \mu$ there is violated. The result is equal to the limit of (6.4.23) as $(\alpha, \beta, \nu)$ tends to $(p, q, k - \frac{1}{2})$. About the surjectivity, we argue as follows: Let $g$ be smooth and compactly supported on $(0, \infty)$. If $g$ is orthogonal to all $\mathcal{K}\phi_p$, $p \geq k$, then we have, by the remark just made on $W_{p,k-\frac{1}{2}}(u)$,

$$
\int_0^\infty g(u) \exp(-2\pi u) u^{p-1} du = 0, \quad p \geq k.
$$

(6.4.32)

This implies that the Fourier transform of $g(u) \exp(-2\pi u) u^{k-1}$ vanishes identically; in fact it suffices to expand the additive character into a power series and integrate termwise. Hence $g \equiv 0$. On noting (6.4.31), the counterpart of (6.4.8), with $\phi = \phi_p, \nu = k - \frac{1}{2}$, can be proved in much the same way as before. The extension to any smooth $\phi$ is immediate via (4.24) and

$$
\mathcal{K}\phi_p(u) = A^+\phi_p(a|u|) \ll \min(u, |p| + 1)u^{-k}, \quad u > 0,
$$

(6.4.33)

which comes from (6.3.4) and (6.4.5). This ends the proof of the lemma.

Here we summarize our discussion in the present section: Let $V$ be a subspace listed in (6.3.22); we assume for instance that $\nu_V \in i\mathbb{R}$. We put $\mathcal{L}(\lambda_V^{(p)}) = \phi_p(\cdot, \nu_V)$, and extend $\mathcal{L}$ to the whole of $V$ in an obvious manner. Then $\mathcal{L}$ is an isometry mapping $V$ onto $U_{\nu_V}$, where the metric of the latter is defined in Lemma 6.2. We put

$$
\omega_V = (\mathcal{K}\mathcal{L})_\omega (\mathcal{K}\mathcal{L})^{-1},
$$

(6.4.34)

with the right translation $\omega$ acting in $V$. Then $\omega_V$ is a representation of $G$ in the Hilbert space $L^2(\mathbb{R}^\times, \pi^{-1}d^x)$. In this way, the representation $V$ is realized in terms of $L^2(\mathbb{R}^\times, \pi^{-1}d^x)$. On the other hand, $\omega_\nu = \mathcal{K}\omega\mathcal{K}^{-1}$ with $\omega$ acting in $U_\nu$ is also a representation of $G$ in $L^2(\mathbb{R}^\times, \pi^{-1}d^x)$, for any $\nu \in i\mathbb{R}$. The explicit description of the actions of $G$ under $\omega_\nu$ is given in Lemma 6.1. The extension to the discrete series of representations is given in Lemma 6.3.
Now, as we have promised, we shall give a proof of Theorem 6.2. We may assume that $V$ be such that $\nu \in i\mathbb{R}$, for other cases are in fact easier. Naturally, it suffices to prove that $\omega_\nu$ is an irreducible representation. Let $f_j \in Y_j$ be arbitrary; hereafter, equalities are in the $L^2$-sense. Since $\omega_\nu(u[\lambda])f_1 \in Y_1$ for any real $x$, the orthogonality of $Y_1$ and $Y_2$ implies that the Fourier transform of $f_1(u)f_2(u)/|u|$ vanishes identically, because of the first identity in (6.4.7). That is, $f_1f_2 = 0$. Then, by the second identity in (6.4.7) we see that $f_1(uy)f_2(u) = 0$ for any $y > 0$. This means that $f_1(u)f_2(v) = 0$ for $uv > 0$. Consequently we may assume without loss of generality that any $f \in Y_1$ is such that $f(u) = 0$ for $u < 0$. By (6.4.8) we have, for $u < 0$,  

$$0 = \omega_\nu(w)f(u) = \int_0^\infty j_\nu(u\lambda)f(\lambda)\frac{d\lambda}{\lambda}. \quad (6.4.35)$$

Let $\tilde{f}(s)$ be the $L^2$-Mellin transform of $f$, which should exist for $s \in i\mathbb{R}$. We observe that by (6.4.15)

$$\int_0^\infty j_\nu(u\lambda)\lambda^{s-2}d\lambda = 4\pi \cos(\pi\nu)(2\pi)^{-2s}\Gamma(s-\frac{1}{2}+\nu)\Gamma(s-\frac{1}{2}-\nu)|u|^{1-s},$$

for Re $s > \frac{1}{2}$ and $u < 0$. Hence, by the Mellin-Parseval identity, (6.4.35) is equivalent to

$$\int_{(1)} (2\pi)^{-2s}\Gamma(s-\frac{1}{2}+\nu)\Gamma(s-\frac{1}{2}-\nu)\tilde{f}(1-s)|u|^{-s}ds = 0,$$

for any $u < 0$. The integrand has to vanish, and $\tilde{f}(s) = 0$ for $s \in i\mathbb{R}$. Namely, we have $\|f\| = 0$ with the norm in $L^2(\mathbb{R}^\times, \pi^{-1}d^\times)$. We end the proof of Theorem 6.2.

### 6.5 Revisits

The aim of this section is to review the sum formulas of Kloosterman sums and the explicit formula for the fourth moment of the zeta-function in the light of automorphic representations. We shall, however, restrict ourselves to the structural aspect of the new argument leading to those formulas, as a fuller account would not be a help, rather a hindrance to see the essentials. Thus, for instance, convergence issues will be ignored. We shall first discuss the sum formulas and then turn to the zeta-function.
Thus, let us reformulate the sum formulas: For any non-zero integers \( m, n \) and for appropriate weight functions \( f, \varphi \), we have

\[
\sum_{V} \frac{g_{V}(m)g_{V}(n)f(\nu_{V})}{4\pi i} + \int_{(0)} \frac{\sigma_{r}(m)\sigma_{r}(n)}{(mn)^{r}\zeta(1+2r)\zeta(1-2r)}f(r)dr
\]

\[
= \delta_{mn} \frac{i}{4\pi^{2}} \int_{(0)} r \tan(\pi r) f(r) dr + \sum_{\ell=1}^{\infty} \frac{1}{\ell} S(m, n; \ell) A^{\delta} f\left(\frac{4\pi}{\ell} \sqrt{|mn|}\right),
\]

(6.5.1)

as well as

\[
\sum_{\ell=1}^{\infty} \frac{1}{\ell} S(m, n; \ell) \varphi\left(\frac{4\pi}{\ell} \sqrt{|mn|}\right) = \sum_{V} \frac{g_{V}(m)g_{V}(n)B^{\delta} \varphi(\nu_{V})}{\sqrt{|mn|}}
\]

\[
+ \frac{1}{4\pi i} \int_{(0)} \frac{\sigma_{r}(m)\sigma_{r}(n)}{(mn)^{r}\zeta(1+2r)\zeta(1-2r)}B^{\delta} \varphi(r) dr.
\]

(6.5.2)

Here \( \delta = \text{sgn}(mn) \) and \( \sum_{V}^{a} \) indicates that the sum is restricted to all irreducible representations in the unitary principal series; in (6.5.2) \( V \) runs over all irreducible representations listed in (6.3.22). Also,

\[
A^{\delta} f(x) = \frac{i}{4\pi} \int_{(0)} \frac{J_{-2\nu}(x) - J_{\nu}(x)}{\sin(\pi \nu)} \nu \tan(\pi \nu) f(\nu) d\nu,
\]

(6.5.3)

\[
B^{\delta} \varphi(\nu) = 2\pi \int_{0}^{\infty} \frac{J_{-2\nu}(x) - J_{\nu}(x)}{\sin(\pi \nu)} \varphi(x) \frac{dx}{x},
\]

(6.5.4)

where \( J_{\pm \nu} \) are as in (6.4.9). The formula (6.5.1) is equivalent to Theorems 2.2 and 2.4; (6.5.2) to Theorems 2.3 and 2.5; also, (6.5.3) to (2.3.17) and (2.5.7); and (6.5.4) to (2.4.8) and (2.5.15). Note that the present \( f, \varphi \) are not the same as in those theorems, which is due partly to the renormalization (6.3.30).

Although it does not matter for practical purposes, our proofs of Theorems 2.2–2.5 or (6.5.1)–(6.5.2) that are developed in Chapter 2 are admittedly highly technical. Also, the emergence of holomorphic cusp forms in the statement of Theorem 2.3 or the same in (6.5.2) remains mysterious and unexplained. With what we have developed in the present chapter, one may infer the latter phenomenon should be related to the assertion (6.4.8) as a minor modification of the kernel \( j_{\nu} \), the Bessel function of representations under our specification with \( G = \text{PSL}(2, \mathbb{R}) \), appears in the Bessel transforms \( A^{\delta}, B^{\delta} \). Also the latter phenomenon should be related to the spectral decomposition (6.3.20) with (6.3.21)–(6.3.24), as there cusp forms of all types play their rôles, without any notable discrimination among them, through irreducible representations generated by them. That is, the sum formulas should better be captured as a
consequence of Theorem 6.1 augmented by Lemmas 6.1–6.3, not as that of Theorem 1.1. Namely, we need to devise an argument to prove the sum formulas in the framework of the space $L^2(\Gamma \setminus G)$.

To this end, we consider a Poincaré series on $G$. Let the seed function $q(g)$ be defined on the big Bruhat cell, i.e., the set $NwNA$ in (6.3.1) in such a way that $q(n|x_1|\text{wn}[x_2]\text{a}[u]) = \exp(2\pi imx_1)\eta(x_2)g(u)$ with an integer $m \neq 0$ and smooth functions $\eta, g$ of fast decay; and $q(g) = 0$ on the small cell, i.e., NA. In particular we have $q(n|\xi|g) = \exp(2\pi im\xi)q(g)$ for any real $\xi$. We then put

$$Q(g) = \sum_{\gamma \in \Gamma \setminus \Gamma} q(\gamma g), \quad (6.5.5)$$

with $\gamma$ as in (1.1.4). In much the same way as the derivation of (1.1.6), we have

$$Q(g) = q(g) + \sum_{n = -\infty}^{\infty} \sum_{\ell = 1}^{\infty} S(m, n; \ell) \int_{-\infty}^{\infty} e(-n\xi)q(n|x_1|\text{a}[y_1]\text{k}[\theta_1])d\xi, \quad (6.5.6)$$

where

$$x_1 = -\frac{\xi + x}{\ell^2((\xi + x)^2 + y^2)}, \quad y_1 = \frac{y}{\ell^2((\xi + x)^2 + y^2)},$$

$$\exp(i\theta_1) = \exp(i\theta)\frac{\xi - iy}{(\ell^2 + y^2)^{1/2}}.$$

We observe that

$$n|x_1|\text{a}[y_1]\text{k}[\theta_1] = a|\ell^{-2}|\text{wn}[\xi]g, \quad (6.5.7)$$

and that the last integral at $g = 1$ is equal to

$$\int_{-\infty}^{\infty} e(-n\xi)q(\text{wn}[\ell^2\xi|a[\ell^2]])d\xi = \ell^{-2}\hat{\eta}(2\pi n/\ell^2)g(\ell^2), \quad (6.5.8)$$

with the Fourier transform $\hat{\eta}$.

On the other hand, the projection of $Q(g)$ to an irreducible subspace $V$ in the unitary principal series is equal to

$$\sum_{p = -\infty}^{\infty} \langle Q, \lambda_V^{(p)} \rangle \lambda_V^{(p)}(g) = \sum_{p = -\infty}^{\infty} \lambda_V^{(p)}(g) \int_{\Gamma \setminus \text{NwNA}} q(h)\lambda_V^{(p)}(h)dh.$$  

The value at $g = 1$ of the $n$-th Fourier coefficient of this expression is equal to

$$\frac{\varphi_V(m)\varphi_V(n)}{\sqrt{|mn|}} \sum_{p = -\infty}^{\infty} A_{\text{sgn}(n)}\phi_p(a[|n|])$$

$$\times \int_{0}^{\infty} \int_{-\infty}^{\infty} \eta(\xi)g(u)A_{\text{sgn}(m)}\phi_p(a[|m|]wn[\xi]|a[u])d\xi du \frac{\pi u}{\pi u}, \quad (6.5.9)$$
where \( \phi_p = \phi_p(\cdot, \nu V) \), and we have used the fact that the Jacobian of the change of variables

\[
\begin{align*}
n[x]a[y]k[\theta] &\mapsto n[x_1]wn[\xi]a[u],
\end{align*}
\]

is equal to \( y^2/u \), that is, \( dg = dx_1d\xi du/(\pi u) \). We have

\[
A^{\text{sgn}(m)}\phi_p(a||m||)wn[\xi]a[u] = \mathcal{K}\omega(wn[\xi]a[u])\phi_p(m) = \int_{\mathbb{R}^\times} j_{\nu}(m\lambda)\omega(n[\xi]a[u])\mathcal{K}\phi_p(\lambda)d^\times\lambda = \int_{\mathbb{R}^\times} j_{\nu}(m\lambda)e(\xi\lambda)\mathcal{K}\phi_p(u\lambda)d^\times\lambda,
\]

(6.5.10)
in which the second line is due to (6.4.7), and the third to (6.4.8). Thus the double integral in (6.5.9) is equal to

\[
\int_{\mathbb{R}^\times} \left( \int_0^\infty \hat{\eta}(2\pi\lambda/u)g(u)j_{\nu}(m\lambda/u)\frac{du}{u} \right)\mathcal{K}\phi_p(\lambda)d^\times\lambda,
\]

where we have used that \( j_{\nu} \) is real valued. Then, the sum in (6.5.9) is equal to

\[
\pi \int_0^\infty \hat{\eta}(2\pi n/u)g(u)j_{\nu}(mn/u)\frac{du}{u},
\]

(6.5.11)

for \( A^{\text{sgn}(n)}\phi_p(a||n||) = \mathcal{K}\phi_p(n) \), and \( \{\mathcal{K}\phi_p : p \in \mathbb{Z}\} \) is a complete orthonormal system of the space \( L^2(\mathbb{R}^\times, \pi^{-1}d^\times) \), according to Lemma 6.2.

Collecting these, we see that the contribution of the irreducible representation \( V \) to the spectral expansion of the sum

\[
\sum_{\ell=1}^\infty \frac{1}{\ell^2}S(m,n;\ell)\hat{\eta}(2\pi n/\ell^2)g(\ell^2)
\]
is equal to

\[
\frac{\vartheta_{\nu}(m)\vartheta_{\nu}(n)}{\sqrt{|mn|}} \int_0^\infty \hat{\eta}(2\pi n/u)g(u)j_{\nu}(mn/u)\frac{du}{u}.
\]

Assuming that \( m, n > 0 \), we put \( \hat{\eta}(x) = \hat{\eta}(mx) \), \( g(u) = \hat{g}(mn/u) \), and further \( \hat{\eta}(2\pi u)\hat{g}(u) = \varphi(4\pi\sqrt{u})/(4\pi\sqrt{u}) \). Then, we recover the integral transform \( B^+ \) defined by (6.5.4) and the relevant part of (6.5.2). The contribution of the representations in the discrete series and that of the continuous spectrum are treated fairly analogously. In this way we have obtained a proof of Theorem 2.3, i.e., (6.5.2) with \( mn > 0 \) via the theory of automorphic representations, as far as we restrict ourselves to the present choice of the weight function \( \varphi \). Also, the case with \( mn < 0 \) can be treated similarly. The transforms \( B^\pm \) have
turned out indeed to be equivalent to (6.4.8) and its relevant statement given in Lemma 6.3.

The mechanism can be summarized as this: The Fourier expansion of Poincaré series with respect to the left action of \( N \) takes us to the notion of the operator \( A^\delta \) as (6.5.7) dictates, and to the big Bruhat cell. The former demands a theory of representations expressed in terms of \( A^\delta \), and this leads us to the theory of the Kirillov model as we have given in Lemmas 6.1–6.3. The big Bruhat cell is of course characterized by the presence of the Weyl element \( w \), and its action has to be realized if any practical application of the harmonic analysis has to be performed. There naturally emerges the Bessel function \( j_\nu \) of representations. On the other hand it now becomes expedient for us to work with functions defined in the big Bruhat cell as (6.5.8) shows clearly. With this, the rest of the procedure is quite plain as (6.5.9)–(6.5.11) is simply a logical rearrangement, although it is true that the inversion argument at (6.5.11) of a Fourier type is of some interest.

The above discussion is, however, highly formal. There are a few missing points. One is the treatment of the convergence issue; and the other is the expansion of the space of weight functions so that the full statement of Theorems 2.3 and 2.5 be recovered. These are, however, technical issues, and could be regarded as being outside our present aim. A more essential problem than them is the derivation of Theorems 2.2 and 2.4 from 2.3 and 2.5, respectively; that is, the Spectral–Kloosterman sum formula (6.5.1) is to be derived from the Kloosterman–Spectral sum formula (6.5.2), the direction of which is exactly opposite to the reasoning in Sections 2.4 and 2.5. Its solution is naturally a logical necessity as far as we proceed as in the present chapter. Our answer to this is that for both \( \delta = \pm 1 \)

The sum formulas (6.5.1) and (6.5.2) are equivalent to each other.  (6.5.12)

We skip the proof, for it would be a digression too long.

We turn to a review of Theorem 4.2. Thus, we should consider rather (4.2.5) than the theorem itself. The exploitation of the view point provided there has been the main motivation for us to develop an account on automorphic representations. Since we are now working with matrices in projective sense, (4.2.5) should be reformulated as

\[
\sum_{n=1}^{\infty} n^{-z-\frac{d}{2}} \sum_{d|n} \sum_{b=1}^{d} \mathcal{P}_F(n|b/d|a[n/d^2]|g),
\]

\[
\mathcal{P}_F(g) = \sum_{\gamma \in \Gamma} F(\gamma g),
\]
where the definition (6.3.32) of Hecke operators is taken into account, and the convergent factor \( n^{-z} \) is inserted, with \( \Re z \) being sufficiently large. We are now to carry out the computation of the spectral decomposition of the Poincaré series \( \mathcal{P}_F(g) \) via Theorem 6.1 and Lemmas 6.1–6.3; note that our interest is in fact in the special value at \( g = 1 \). Our argument is again formal. This time we take into account the action of Hecke operators, i.e., the assumption (6.3.32)–(6.3.34).

Thus, the value at \( g = 1 \) of the projection of (6.5.13) to an irreducible subspace \( V \) in the unitary principal series is equal to

\[
H_V(z) \sum_{p = -\infty}^{\infty} \langle \mathcal{P}_F, \lambda_V^{(p)} \rangle \lambda_V^{(p)}(1),
\]

(6.5.15)

where \( H_V \) is as in (6.3.35). This sum is

\[
\sum_{p = -\infty}^{\infty} \lambda_V^{(p)}(1) \int_G F(g) \overline{\lambda_V^{(p)}(g)} dg
\]

\[
= \varrho_V(1) \sum_{p = -\infty}^{\infty} \lambda_V^{(p)}(1) \sum_{m=1}^{\infty} \frac{t_V(m)}{\sqrt{m}} (\Phi_p^+ + \epsilon_V \Phi_p^-) F(m, \nu_V)
\]

\[
= |\varrho_V(1)|^2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{t_V(m)t_V(n)}{\sqrt{mn}}
\]

\[
\times \left( \mathcal{B}^{(+,+)} + \mathcal{B}^{(-,-)} + \epsilon_V \mathcal{B}^{(+,-)} + \epsilon_V \mathcal{B}^{(-,+)} \right) F(a[n]; m, \nu_V),
\]

(6.5.16)

where

\[
\mathcal{B}^{(\delta_1, \delta_2)} F(g; m, \nu_V) = \sum_{p = -\infty}^{\infty} \Phi_p^{\delta_1} F(m, \nu_V) A^{\delta_2} \phi_p(g),
\]

(6.5.17)

with

\[
\Phi_p^{\delta} F(m, \nu) = \int_G F(g) A^{\delta} \phi_p(a[n]g) dg, \quad \phi_p(g) = \phi_p(g, \nu).
\]

(6.5.18)

We have, in terms of the Kirillov operator,

\[
\mathcal{B}^{(\delta_1, \delta_2)} F(a[n]; m, \nu) = \sum_{p = -\infty}^{\infty} \Phi_p^{\delta_1} F(m, \nu) X \phi_p(\delta_2 n).
\]

(6.5.19)

We then proceed just as in (6.5.9)–(6.5.11). Since the integral in (6.5.18) can be restricted to the big Bruhat cell, we perform the change of variables accordingly.

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We have, with \( h = n[x_1]w[nx_2]a[u] \),

\[
\Phi_\delta F(m, \nu) = \int_{\text{NwNA}} F(h) \omega(h) \overline{X \phi_p(\delta m)} dx_1 dx_2 \frac{du}{\pi u} \\
= \int_{\mathbb{R}^2} \left( \int_{\text{NwNA}} F(h) j_\nu(\delta \lambda / u) \right) \times e(-\delta mx_1 u - \lambda x_2 / u) dx_1 dx_2 \overline{X \phi_p(\lambda) d^\lambda \lambda}, \quad (6.20)
\]

where we have applied Lemma 6.1. Inserting this into (6.19), we get, via Lemma 6.2, that

\[
\mathcal{B}(\delta_1, \delta_2) F(a[n]; m, \nu) = \int_0^\infty j_\nu(\delta_2 m n / u) \\
\times \left( \int_{\mathbb{R}^2} F(n[x_1]w[nx_2]a[u]) e(-\delta_1 m x_1 - \delta_2 n x_2 / u) dx_1 dx_2 \right) \frac{du}{u}.
\]

Thus,

\[
\mathcal{B}(\delta_1, \delta_2) F(a[n]; m, \nu) = \int_0^\infty j_\nu(\delta_1 \delta_2 / u) \\
\times \left( \int_{\mathbb{R}^2} F(n[x_1/m]w[nx_2]a[mn u]) e(-\delta_1 x_1 - \delta_2 x_2) dx_1 dx_2 \right) du. \quad (6.21)
\]

One may desire to compute the double sum (6.16) and the last double integral into closed forms. In the applications to \( Z_2(g) \), we are in a fortuitous situation that the double sum is transformed into a product of two values of \( H_\nu \). As to the double integral, it is a Fourier transform over the Euclidean plane, and thus, in principle, can be expressed in terms of a Bessel transform as can be seen in (6.24) below. With \( Z_2(g) \), the situation turns out in fact to be as such. Hence the matter seems to depend much on the specific nature of the seed \( F \). Nevertheless, with any smooth \( F \), one might appeal to Mellin transform of several variables, and (6.21) could be pushed to a more closed form.

Before finishing this section, we render the spectral decomposition of \( Z_2(g) \) in terms of notions from the theory of \( I \)-automorphic representations: Thus, let us put

\[
\Theta(\nu; g) = \int_0^\infty \left( \frac{u}{u + 1} \right)^{1/2} g_c \left( \log (1 + 1/u) \right) \Xi(u; \nu) d^\times u, \quad (6.23)
\]

\[
\Xi(u; \nu) = \int_{\mathbb{R}^2} j_0(-v) j_\nu \left( \frac{u}{v} \right) \frac{d^\times \nu}{v}.
\]

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Then (4.7.1) is expressed as

\[ Z_2(g) = \left\{ Z_2^{(r)} + Z_2^{(c)} + Z_2^{(e)} \right\}(g), \]  

(6.5.25)

where

\[ Z_2^{(c)}(g) = \sum_V \alpha_V H_V \left( \frac{1}{2} \right)^3 \Theta(\nu_V; g), \]  

(6.5.26)

\[ Z_2^{(e)}(g) = \frac{1}{2\pi i} \int_{(0)} \frac{1}{|\zeta(1 + 2\nu)|^2} \Theta(\nu; g) d\nu, \]  

(6.5.27)

with \( \alpha_V = |\nu_V(1)|^2 + |\nu_V(-1)|^2 \). The \( V \) runs over a complete system of Hecke-invariant cuspidal irreducible \( \Gamma \)-automorphic representations of \( G \). The \( Z_2^{(r)}(g) \) is the same as \( Z_2^{(c)}(g) \) in (4.7.1). The equivalence between (4.7.2) and (6.5.24) may independently be verified by using (6.4.14) and (6.4.15). The factor \( j_\nu \) in (6.5.24) has come from the same involved in (6.5.21).

### 6.6 Mean values of automorphic \( L \)-functions

The aim of the present section is to develop a unified treatment of mean values of individual automorphic \( L \)-functions associated with the spectral decomposition of \( L^2(\Gamma \backslash G) \). We shall establish a complete spectral expansion for

\[ \mathcal{M}(U, g) = \int_{-\infty}^{\infty} |L_U(\frac{1}{2} + it)|^2 g(t) dt, \]  

(6.6.1)

where \( U \) is any irreducible representation listed in (6.3.22), and

- the weight \( g \) is even, entire, real valued on \( \mathbb{R} \), and
- of rapid decay in any fixed horizontal strip,  

(6.6.2)

which is assumed for the sake of simplicity and could be replaced by the less stringent assumption given in the introduction of Chapter 4. Our argument is unified in the sense that it is equally applicable to any \( U \), whereas hitherto known arguments are applicable only either to the zeta-function, which corresponds to the continuous spectrum, or to those \( L_U \) with \( U \) in the discrete series. It will be seen that the theory of automorphic representations is genuinely needed in our solution of the problem. This is in contrast to the situation in the previous section where the theory has been utilized to gain a geometric understanding of the sum formulas and the explicit formula for \( Z_2(g) \), and could be dispensed with otherwise. In other words, the theory of automorphic

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representations leads us to a genuinely new assertion in the theory of mean values of the zeta and \(L\)-functions as well. We shall discuss mainly the case with \(U\) in the unitary principal series because of an obvious reason.

We begin, nevertheless, with a brief discussion on irreducible representations in the discrete series, in order to illustrate the main problematics that we have to resolve when we treat \(\mathcal{M}(U, g)\) with general \(U\). Thus, let us assume temporarily that \(U\) be in the holomorphic discrete series and \(\nu_U = k - \frac{1}{2}\) with an integer \(k \geq 6\). Corresponding to (4.3.1), we consider

\[
\mathcal{J}_U(u, v; g) = \int_{-\infty}^{\infty} L_U(\bar{u} + it)L_U(v + it)g(t)dt,
\]

(6.6.3)

which is an entire function over \(\mathbb{C}^2\). We have \(\mathcal{M}(U, g) = \mathcal{J}_U(\frac{1}{2}, \frac{1}{2}; g)\). In the region of absolute convergence it holds that

\[
\mathcal{J}_U(u, v; g) = \frac{L_U \otimes U(2(u + v))}{\zeta(2(u + v))}g^*(0) + \mathcal{J}_U^{(1)}(u, v; g) + \mathcal{J}_U^{(1)}(\bar{v}, \bar{u}; g),
\]

(6.6.4)

where

\[
\mathcal{J}_U^{(1)}(u, v; g) = \sum_{f, n=1}^{\infty} \frac{\theta_U(n)\theta_U(n + m)}{n^u(n + m)^v}g^*(\log \frac{n + m}{n}),
\]

(6.6.5)

with \(g^*\) as in (4.1.6). By the Mellin inversion,

\[
\mathcal{J}_U^{(1)}(u, v; g) = \frac{1}{2\pi i} \int_{(\eta)} \left\{ \sum_{m=1}^{\infty} m^{-s}D_U(u + v - s, m) \right\} \times \tilde{g}(s, s - u - k + \frac{3}{2})ds,
\]

(6.6.6)

where \(\tilde{g}\) is defined by (4.1.7), and

\[
D_U(s, m) = \sum_{n=1}^{\infty} \frac{\theta_U(n)\theta_U(n + m)}{(n + m)^s} \left(\frac{n}{n + m}\right)^{k-\frac{1}{2}}.
\]

(6.6.7)

If \(u + v > \eta + \frac{3}{2} > \frac{5}{2}\), then (6.6.6) converges absolutely, for we have (6.3.34)–(6.3.35). On noting that

\[
|\lambda_U^{(k)}(g)|^2 = \frac{2^{2k-1}}{\Gamma(2k)}g^{2k}\left|\sum_{n=1}^{\infty} \theta_U(n)n^{k-\frac{1}{2}}e(nz)\right|^2, \quad z = x + iy,
\]

(6.6.8)

is in \(L^2(\Gamma\backslash \mathbb{H})\) and that for \(\text{Re} s > 1\)

\[
D_U(s, m) = 4\frac{(4\pi)^{s-2}\Gamma(2k)}{\Gamma(s + 2k - 1)}(|\lambda_U^{(k)}|^2, P_m(\cdot, \bar{s})),
\]

(6.6.9)
with $P_m$ as in (1.1.4), we may compute a decomposition of $D_U(s, m)$ over
the spectrum of the Casimir operator; here the inner product is over $\Gamma \setminus \mathcal{H}$, i.e., $\Gamma \setminus G/K$. Thus, by Theorem 6.1 or rather by Theorem 1.1 together with
(2.1.19)–(2.1.20), we find that

$$D_U(s, m) = \frac{m^{\frac{1}{2}} \Gamma(2k)}{\Gamma(s) \Gamma(s + 2k - 1)} \times \left\{ \sum_{\nu} \frac{\varphi_{\nu}(m)}{\pi^{\nu} \Gamma(\frac{1}{2} + \nu)} \Gamma(s - \frac{1}{2} + \nu) \Gamma(s - \frac{1}{2} - \nu) \langle \lambda^{(k)}_{U}, \lambda^{(0)}_{V} \rangle \right\}$$

$$+ \frac{1}{4\pi i} \int_{(0)} \sigma_{-2\nu}(m) L_{\nu}^{*} \Gamma(\frac{1}{2} + \nu) \Gamma(s - \frac{1}{2} + \nu) \Gamma(s - \frac{1}{2} - \nu) d\nu \right\}, \quad (6.6.10)$$

where $\sum_{\nu}$ is as in (6.5.1), and

$$L_{U \otimes U}^{*}(s) = (2\pi)^{2(1-s)} \Gamma(2k)^{-1} \Gamma(s) \Gamma(s + 2k - 1) L_{U \otimes U}(s) \quad (6.6.11)$$

is the normalized Rankin $L$-function attached to $U$.

The next step is to insert the decomposition (6.6.10) into (6.6.6), and try
to exchange the order of the sum and the integral. Here we face a problem
about the uniform growth rate of individual terms on the right side of (6.6.10),
anything similar to which we have not experienced in dealing with $\mathcal{Z}_2(g)$. Thus,
in general we may expect at most that the factor $\hat{g}(s, s - m + k + \frac{1}{2})$ in (6.6.6)
decays faster than any negative power of $|s|$ while Re $s$ is bounded; consequently
the polynomial growth of the right side of (6.6.10) is essential for the success
of the argument. Note that the same about the contribution of the continuous
spectrum is immediate, via the functional equation $L_{U \otimes U}^{*}(s) = L_{U \otimes U}(1 - s)$.
Hence we need in turn the polynomial growth of

$$\langle \lambda^{(k)}_{U}, \lambda^{(0)}_{V} \rangle \exp(\frac{1}{2} \pi |\nu|)$$ \quad (6.6.12)

with respect to the parameter $\nu$, in view of the estimation (2.3.2) and Stirling’s formula. As a matter of fact, an assertion exists that guarantees such a bound for (6.6.12). However, the reasoning employed there is highly specific to
that $U$ is in the discrete series, and it does not extend to the general situation
where we have an arbitrary irreducible representation in place of $U$. Because
of this, it is useless for us to proceed further along the above argument. Nevertheless,
it might be worth stating the following analogue of (6.5.23)–(6.5.24): The contribution of the irreducible cuspidal representation $V$ to $M(U, g)$, $\nu = k - \frac{1}{2}$
with an integer $k \geq 6$, is equal to

$$(-1)^k \frac{(2\pi)^{2k-1} \Gamma(2k) H_{\nu}(\frac{1}{2}) \langle \lambda^{(k)}_{U}, \lambda^{(0)}_{V} \rangle}{\cos(\pi \nu) \Gamma(2k - \frac{1}{2} + \nu) \Gamma(2k - \frac{1}{2} - \nu)} \frac{\Theta_k(\nu; g)}{\pi^{\frac{1}{2} - \nu} \Gamma(\frac{1}{2} + \nu)}, \quad (6.6.13)$$

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where

\[
\Theta_k(\nu; g) = \int_0^{\infty} \left( \frac{u}{u + 1} \right)^{1-k} g_c(\log (1 + 1/u)) \Xi_k(u, \nu) d^\times u, \tag{6.6.14}
\]

\[
\Xi_k(u, \nu) = \int_{\mathbb{R}^\times} j_{k-\frac{1}{2}}(-v)j_\nu \left( \frac{u}{v} \right) |v|^{k-1} d^\times v. \tag{6.6.15}
\]

With this, we now turn to the unitary principal series, so that hereafter \( U \) is an arbitrary irreducible representation with \( \nu_U \in i\mathbb{R} \). One may follow the above argument, with necessary changes, up to (6.6.7). We face, however, a serious obstacle already at (6.6.8); that is, this time we have \( \lambda^{(0)}_{\nu_U} \), and the factor \( K_{\nu_U}(2\pi|y|e(n\nu)) \) arises in place of the additive character \( e(nz) \). Hence we are unable to readily attain an expression analogous to (6.6.9). On the other hand, despite this difficulty there exists an argument that extends the polynomial growth of (6.6.12) to \(|\langle |\lambda^{(0)}_{\nu_U}|^2, \lambda^{(0)}_{\nu_U} \rangle| \exp(\frac{c}{2} \pi |\nu_U|)\); but we skip the details, since we are about to exhibit an alternative argument that resolves as well the above difficulty pertaining to the \( K \)-Bessel factors.

Our discussion depends much on uniform bounds for \( A\phi_p(a[y], \nu), A = A^+ \), such as (6.4.17) and (6.4.18). In order to make our argument applicable to any irreducible cuspidal representation, we derive from (6.4.18) a bound that is somewhat weaker than (6.4.17) but still sufficient for our purpose; in fact, the proof of (6.4.18) works for all cases. Let us assume that \( \nu \in i\mathbb{R} \). We divide the integral (6.4.10) at \( y = |p| + |\nu| + 1 \). To the part with smaller argument we apply the fact that \( A^{\text{reg}(p)}\phi_p(a[u]; \nu) \) is a unit vector in \( L^2(\mathbb{R}^\times, d^\times \nu) \). Hence this part is \( \ll (|p| + |\nu| + 1)^{Re s - \frac{1}{2}} \). On the other hand, by (6.4.18) the remaining part is \( \ll (|p| + |\nu| + 1)^{Re s} \). We then invoke the identity

\[
\Gamma_p(s, \nu) = 4\pi \cdot \pi \Gamma_p(s + 2, \nu) - p \Gamma_p(s + 1, \nu) \tag{6.6.16}
\]

which can be proved via integration by parts, on noting that

\[
\mathcal{D}_\nu A\phi_p(a[y], \nu) = -4\pi p A\phi_p(a[y], \nu),
\]

\[
\mathcal{D}_\nu = (d/dy)^2 - (2\pi)^2 - (\nu^2 - \frac{1}{4})y^{-2}, \tag{6.6.17}
\]

as \( A\phi_p(a[y], \nu) \) is a constant multiple of the Whittaker function \( W_{p, \nu}(4\pi y) \); see (3.2.32). Then, by the Mellin inversion, we have

\[
A\phi_p(y, \nu) = -2i \int_{(c)} (\pi \Gamma_p(s + 2, \nu) - p \Gamma_p(s + 1, \nu)) \frac{y^{\frac{1}{2} - s}}{s^2 - \nu^2} ds, \tag{6.6.18}
\]

with any small constant \( \varepsilon > 0 \). Inserting the above bound for \( \Gamma_p(s, \nu) \), we conclude that

\[
A\phi_p(a[y], \nu) \ll y^{\frac{1}{2} - \varepsilon}(|p| + |\nu| + 1)^{2+\varepsilon}, \tag{6.6.19}
\]

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where the implied constant depends only on \( \varepsilon \).

The discussion on the discrete series is analogous. Actually, any combination of \( p, \nu \) such that either \( -\frac{1}{2} < \nu < \frac{1}{2} \) or \( \nu = \ell - \frac{1}{2} \) with \( 1 \leq \ell \in \mathbb{Z} \), \( \ell \leq |p| \), could also be dealt with, as an explicit evaluation of the norm of \( \mathcal{A}_{\text{sign}}(a|u|; \nu) \) in \( L^2(\mathbb{R}^\nu, d^\nu / \pi) \) can be performed by using (6.4.23). In passing, we remark that the operator \( \mathcal{D}_\nu \) is connected with \( \partial_\theta \) via the Kirillov map.

We return to (6.6.3) but with the present choice of \( U \); we shall mostly omit the symbol \( U \) to avoid otherwise heavy notation, so that hereafter we have, for instance, \( \varrho(n) = \varrho_U(n) \). We then replace (6.6.4) by

\[
J(u, v; g) = \frac{L_{U \otimes V}(u + v)}{\zeta(2(u + v))} g^\ast(0) + J(u, v; g) + J(\bar{v}, \bar{u}; g), \tag{6.6.20}
\]

where

\[
J(u, v; g) = \sum_{f=1}^{\infty} \sum_{n=1}^{\infty} \frac{g(n)g(n + m)}{(2n + m)^{u+v}} \left( \frac{\sqrt{n(n + m)}}{2n + m} \right)^{2\alpha} g_*(m/(2n + m); u, v), \tag{6.6.21}
\]

with

\[
g_*(x; u, v) = 2^{u+v+2\alpha} \frac{g^\ast(\log((1 + x)/(1 - x)))}{(1 - x)^{\alpha/2 + \alpha} (1 + x)^{v+\alpha}}, \quad 0 \leq x \leq 1. \tag{6.6.22}
\]

Here \( \alpha \) is a sufficiently large positive integer, which is implicit throughout the sequel. Let \( \tilde{g} \) be the Mellin transform of \( g_* \); note that the definition of \( \tilde{g} \) has been changed from (4.1.7). It is immediate to see that \( \tilde{g}(s; u, v) \) is of rapid decay with respect to \( s \), provided \( \text{Re} \ s \) and \( u, v \) are bounded; moreover, \( \tilde{g}(s; u, v) / \Gamma(s) \) is entire over \( \mathbb{C}^3 \). Thus, by Mellin’s inversion,

\[
J(u, v; g) = \frac{1}{2\pi i} \int_{(\eta)} \left\{ \sum_{m=1}^{\infty} m^{-s} D(u + v - s, m) \right\} \tilde{g}(s; u, v) ds, \tag{6.6.23}
\]

where

\[
D(s, m) = \sum_{n=1}^{\infty} \frac{\varrho(n)g(n + m)}{(2n + m)^{s}} \left( \frac{\sqrt{n(n + m)}}{2n + m} \right)^{2\alpha}. \tag{6.6.24}
\]

It is assumed temporarily that \( \text{Re} (u + v) > \max\{2, 1 + \eta\} \) is sufficiently large.

We now try to imitate (6.6.8) with a vector in \( U \) that is generated by \( \lambda^0_U(g) \). What is essential for our purpose is the fact that the Fourier coefficients \( \varrho(n) \) are stable in this generating process, and the subspace \( U \) thus obtained
is unitarily equivalent to the space $L^2(\mathbb{R}^\times, \pi^{-1}d^\times)$ as is stated in Lemma 6.2. With this in mind, we apply the inverse Kirillov map $\mathcal{K}^{-1}$ to the function

$$w(y, \tau) = \begin{cases} 0 & \text{if } y \leq 0, \\ y^{\alpha + \frac{1}{2}} \exp(-\tau y) & \text{if } y > 0, \end{cases}$$

with $\Re \tau > 0$, which is in $L^2(\mathbb{R}^\times, \pi^{-1}d^\times)$; all implicit constants in the sequel may depend on $U$, $\alpha$, and $\Re \tau$ at most. Namely, according to the mechanism explained around (6.4.34), we have that

$$\Phi(g, \tau) = \sum_{n=-\infty}^{\infty} \frac{\phi(n)}{\sqrt{|n|}} A^{\nu n}(n) \mathcal{K}^{-1}w(a[n]g, \tau)$$

is a vector in $U$ such that

$$\Phi(n[x]a[y], \tau) = \sum_{n=1}^{\infty} \frac{\phi(n)}{\sqrt{n}} w(ny, \tau) \exp(2\pi inx).$$

More precisely, we have, by Lemma 6.2,

$$\Phi(g, \tau) = \sum_{p=-\infty}^{\infty} a_p(\tau) \lambda^{(p)}_g,$$

with

$$a_p(\tau) = \frac{1}{\pi} \int_0^{\infty} w(y, \tau) \overline{A\phi_p(a[y]; \nu_U)} \frac{dy}{y}.$$

The function $\Phi(g, \tau)$ is regular for $\Re \tau > 0$, provided $\alpha > 2$, since we have

$$\lambda^{(p)}_g \ll (|p| + 1)^2,$$

$$a_p(\tau) \ll (|\tau| + 1)^{2\alpha}(|p| + 1)^{-\alpha}.$$  

The former can be shown by (6.4.18). To prove the latter, we use the operator $\mathcal{D}_\nu$ defined in (6.6.17): We may assume that $p \neq 0$; then,

$$a_p(\tau) = -\frac{1}{4\pi^2} \int_0^{\infty} w(y, \tau) \overline{\mathcal{D}_\nu A\phi_p(a[y]; \nu_U)} dy.$$  

We integrate by parts, repeat the procedure $\alpha$ times, and use the fact that $\|A\phi_p(a[y]; \nu_U)\| \leq 1$ in $L^2(\mathbb{R}^\times, \pi^{-1}d^\times)$.  

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Next, we put $\Psi(g, \tau) = \Phi(g, \tau)\overline{\Phi(g, \tau)}$. The Parseval formula in $L^2(\Gamma \backslash G)$ gives that

$$
\Psi(g, \tau) = \frac{3}{\pi} \langle \Psi, 1 \rangle + \sum_{V} \sum_{p=-\infty}^{\infty} \langle \Psi, \lambda_V^{(p)} \rangle \lambda_V^{(p)}(g) 
+ \sum_{p=-\infty}^{\infty} \int_{(0)} \langle \Psi, E_p(\cdot, \nu) \rangle E_p(g, \nu) \frac{d\nu}{4\pi i},
$$

with $V$ running over all irreducible cuspidal representations; the sums over the discrete series need to be modified appropriately. The convergence is absolute and fast, provided $\alpha$ is sufficiently large. In fact, we have

$$
\langle \Psi, \lambda_V^{(p)} \rangle \lesssim (|\tau| + 1)^{4\alpha} (|\nu_V| + |p|)^{-\frac{1}{4}\alpha},
$$

with which and (6.6.30) the assertion follows. The proof of this bound and the discussion on the continuous spectrum are to be given later.

Picking up the $m$-th Fourier coefficient on both sides of (6.6.33) with $g = n[x]a[y]$, while invoking (6.6.37), we get, for any $m > 0$,

$$
g^{2n+1} \sum_{n=1}^{\infty} \frac{\vartheta(n) \varphi(n+m)(n(n+m))^\alpha}{\sqrt{m}} \exp(-2n+m)\tau y 
= \sum_{V} \frac{\vartheta_V(m)}{\sqrt{m}} X_V(my; \tau) + \int_{(0)} \frac{m^{-\nu} \sigma_{2\nu}(m)}{\sqrt{m}\zeta(1+2\nu)} Y_{\nu}(my; \tau) \frac{d\nu}{4\pi i},
$$

with

$$
X_V(y; \tau) = \sum_{p=-\infty}^{\infty} \langle \Psi, \lambda_V^{(p)} \rangle A\phi_p(a[y]; \nu_V),
$$

$$
Y_{\nu}(y; \tau) = \sum_{p=-\infty}^{\infty} \langle \Psi, E_p(\cdot, \nu) \rangle A\phi_p(a[y]; \nu).
$$

A combination of (6.4.18), (6.6.19), and (6.6.34) yields the uniform bound

$$
X_V(y; \tau) \ll (|\tau| + 1)^{4\alpha} (|\nu_V| + 1)^{-\frac{1}{4}\alpha} y^{\frac{1}{4}-\varepsilon}(y+1)^{-\frac{1}{4}\alpha}.
$$

It should be noted that this bound holds for any $V$, since (6.4.18) and (6.6.19) holds for all relevant $\nu$. The function $Y_{\nu}$ will be treated later.

We are about to verify (6.6.34). We have, by (6.3.25),

$$
|\langle \nu_V^2 - \frac{1}{4} + i(2\nu)^2 \rangle | |\langle \Psi, \lambda_V^{(p)} \rangle | = |\langle \Omega - i\partial_{\nu}^2 \rangle \lambda_V^{(p)} | 
= \langle (\Omega + i\partial_{\nu}^2)^{2} \Psi, \lambda_V^{(p)} | \leq \| (\Omega + i\partial_{\nu}^2)^{2} \Psi \|,
$$

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for any integer \( q \geq 0 \). By (6.1.11) and (6.1.13), \( \Omega \lambda_U^{(k)} \lambda_U^{(l)} \) is a linear combination of \( \lambda_U^{(k+j)} \lambda_U^{(l+j)} \), \( j = -1, 0, 1 \), the coefficients of which are polynomials of the second degree on \( k, l \). Thus, by (6.6.28),

\[
(\Omega + i\partial^2_\theta)^q \Psi(g) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} b_{k,l}^{(q)}(\tau) \lambda_U^{(k)}(g) \lambda_U^{(l)}(g),
\]

(6.6.40)

where

\[
b_{k,l}^{(q)}(\tau) = \sum_{j=-q}^{q} d_j(k,l) a_{k+j}(\tau) a_{l+j}(\overline{\tau}),
\]

(6.6.41)

with a polynomial \( d_j(k,l) \) of degree 2 on \( k, l \); and by (6.6.31), we have

\[
b_{k,l}^{(q)}(\tau) \ll (|\tau| + 1)^{4\alpha} (|k| + |l| + 1)^{2\nu} (|k| + 1)(|l| + 1)^{\alpha},
\]

(6.6.42)

uniformly for \( k, l \), and \( \tau \) with \( \text{Re} \tau > 0 \). We put \( q = \lfloor \frac{1}{2} \alpha \rfloor \), and get the uniform bound \( (\Omega + i\partial^2_\theta)^q \Psi(g, \tau) \ll (|\tau| + 1)^{4\alpha} \), which and (6.6.39) give (6.6.34).

We turn to \( \tilde{Y}_\nu \) defined by (6.6.37). We first invoke the functional equation for \( E_p \) that follows from (3.2.28) via (6.2.21), and have, for \( \text{Re} \nu = 0 \),

\[
\langle (\Omega + i\partial^2_\theta)^q \Psi(\cdot, \tau), E_p(\cdot, \nu) \rangle = \pi^{-2\nu} \zeta(1+2\nu) \Gamma\left(\frac{1}{2} + \nu + p\right) \Gamma\left(\frac{1}{2} - \nu + p\right) \times \int_{\Gamma \backslash G} (\Omega + i\partial^2_\theta)^q \Psi(g, \tau) E_{-p}(g, \nu) dg.
\]

(6.6.43)

Assuming temporarily that \( \text{Re} \nu > \frac{1}{2} \), we unfold the last integral, and see via (6.6.40) that it is equal to

\[
\frac{L_{U \otimes U}(\nu + \frac{1}{2})}{\zeta(2\nu + 1)} W(\nu, \tau; p, q),
\]

(6.6.44)

where

\[
W(\nu, \tau; p, q) = \sum_{l=-\infty}^{\infty} b_{l+p, l}^{(q)}(\tau) \times \sum_{\delta = \pm} \int_{0}^{\infty} A^{\delta} \phi_{l+p}(a[y], \nu_U) A^{\delta} \phi_{l}(a[y], \nu_U) y^{p - \frac{1}{2}} dy.
\]

(6.6.45)

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Again by (6.4.18), (6.6.19), and (6.6.43), we see that \( W(\nu, \tau; p, q) \) is regular and \( \ll (|\tau| + 1)^{4\alpha} \) for \( \Re \tau > 0 \) and \( \Re \nu > -\frac{1}{2} \). Hence, in the same domain,

\[
Y_\nu(y, \tau) = \frac{L_{U \otimes U}(\nu + \frac{1}{2})}{\zeta(1 - 2\nu)} Y_\nu^*(y, \tau), \quad (6.6.46)
\]

with

\[
Y_\nu^*(y, \tau) = \pi^{-2\nu} \sum_{p=-\infty}^{\infty} \frac{W(\nu, \tau; p, q)}{(\nu^2 - \frac{1}{4} - i(2p)^2)^{\frac{1}{2}}} \frac{\Gamma\left(\frac{1}{2} + \nu + p\right)}{\Gamma\left(\frac{1}{2} - \nu + p\right)} A_{\phi_p}(a[y]; \nu). \quad (6.6.47)
\]

One may conclude, via (6.4.17)–(6.4.18), that

\[
Y_\nu^*(y, \tau) \ll (|\tau| + 1)^{4\alpha} (|\nu| + 1)^{-\frac{1}{2} - \left|\Re \nu\right| - \epsilon} (y + 1)^{-\frac{1}{2} - \epsilon}, \quad (6.6.48)
\]

for \( \Re \tau > 0 \) and \( \Re \nu > -\frac{1}{2} \).

Now we set \( \tau = s \). We multiply both sides of (6.6.35) by \( y^{s-2} \) and integrate. We have

\[
D(s, m) = \sum_V m^{\frac{1}{2} - s} g_V(m) \Xi_V(s) + \int_{(0)} m^{\frac{1}{2} - \nu - s} \sigma_V(m) \frac{L_{U \otimes U}(\nu + \frac{1}{2})}{\zeta(1 - 2\nu)} \Upsilon_V(s) \frac{d\nu}{4\pi i}, \quad (6.6.49)
\]

where

\[
\Xi_V(s) = \frac{s^{s+2\alpha}}{\Gamma(s + 2\alpha)} \int_0^\infty y^{s-2} X_V(y, s) dy,
\]

\[
\Upsilon_V(s) = \frac{s^{s+2\alpha}}{\Gamma(s + 2\alpha)} \int_0^\infty y^{s-2} Y_V^*(y, s) dy. \quad (6.6.50)
\]

The bound (6.6.38) implies that \( \Xi_V(s) \) is regular and \( \ll |s|^{4\alpha + \frac{1}{2}} (|\nu| + 1)^{-\frac{1}{2} - \alpha} \) for \( \Re s > \frac{1}{2} \). Also, (6.6.48) implies that \( \Upsilon_V(s) \) is regular and \( \ll |s|^{4\alpha + \frac{1}{2}} (|\nu| + 1)^{-\frac{1}{2} - \alpha} \) for \( \Re s > \frac{1}{2} + |\Re \nu| \) and \( \Re \nu > -\frac{1}{2} \).

Therefore we have proved

**Lemma 6.4.** The function \( D(s, m) \) admits the spectral decomposition (6.6.49) which converges absolutely and uniformly for \( \Re s > \frac{1}{2} \). In particular, \( D(s, m) \) is regular and of polynomial growth for \( \Re s > \frac{1}{2} \).
With this, we return to the expression (6.6.23) of the function \( J(u, v; g) \); thus we impose \( \text{Re} \, (u + v) > \max\{2, 1 + \eta\} \) initially. In view of the fast decay of \( \tilde{g}(s; u, v) \), the last lemma yields immediately that

\[
J(u, v; g) = \sum_V L_V(u + v - \frac{1}{2}) \Theta_V(u, v; g)
\]

\[
+ \frac{1}{4\pi i} \int_{(0)} \frac{\zeta(u + v - \frac{1}{2} + \nu)\zeta(u + v - \frac{1}{2} - \nu)}{\zeta(1 + 2\nu)\zeta(1 - 2\nu)}
\times L_{U \otimes U} \left( \frac{1}{2} + \nu \right) \Lambda_\nu(u, v; g)dv,
\]

(6.6.51)

where

\[
\Theta_V(u, v; g) = \frac{1}{2\pi i} \int_{(\eta)} \Xi_V(u + v - \xi)\tilde{g}(\xi; u, v)d\xi,
\]

(6.6.52)

\[
\Lambda_\nu(u, v; g) = \frac{1}{2\pi i} \int_{(\eta)} \Upsilon_\nu(u + v - \xi)\tilde{g}(\xi; u, v)d\xi.
\]

(6.6.53)

We have

\[
\Theta_V(u, v; g) \ll (|\nu\nu| + 1)^{-\frac{4}{3}e}, \quad \Lambda_\nu(u, v; g) \ll (|\nu| + 1)^{-\frac{4}{3}e},
\]

(6.6.54)

where \( u, v \) are bounded; the first holds uniformly in the domain \( \text{Re} \, (u + v) > \frac{1}{2} \), and the second in \( \text{Re} \, (u + v) > \frac{1}{2} + |\text{Re} \, \nu|, \text{Re} \, \nu > -\frac{1}{2} \).

We shall discuss the analytic continuation of the expansion (6.6.51). Let \( c > 0 \) be a sufficiently small constant. We may move the contour in (6.6.52) to \((c)\), provided \( \text{Re} \, (u + v) > \frac{c}{2} \), say. Hence, the expansion (6.6.51) holds under \( \text{Re} \, (u + v) > \frac{c}{2} \). This condition is required to get the factors \( L_V(u + v - \frac{1}{2}) \) and \( \zeta(u + v - \frac{1}{2} \pm \nu) \). However, the former is entire and of a polynomial order in \( \nu \nu \) if \( u, v \) are bounded. Thus the cuspidal part of \( J(u, v; g) \) is regular in a neighbourhood of the point \((\frac{1}{2}, \frac{1}{2})\) at which it takes the value

\[
\sum_V L_V \left( \frac{1}{2} \right) \Theta_V(g),
\]

(6.6.55)

with \( \Theta_V(g) = \Theta_V \left( \frac{1}{2}, \frac{1}{2}; g \right) \).

As we are about to deal with the continuous spectrum, we should remark that \( \Lambda_\nu(u, v; g) \) remains regular in the three complex variables and of fast decay in \( \nu \), throughout the procedure below, because of the property of \( \Upsilon_\nu(s) \) mentioned above. Thus, we temporarily restrict \((u, v)\) so that \( 2 > \text{Re} \, (u + v) > \frac{c}{2} \).

Then, in (6.6.51) one may shift the contour to \((\frac{1}{2} + c)\), with \( c \) as above, encountering the pole at \( u + v - \frac{3}{2} \) with the residue

\[
-\frac{L_{U \otimes U}(u + v - 1)}{\zeta(2(2 - u - v))} \Lambda_{u + v - \frac{3}{2}}(u, v; g)
\]

(6.6.56)
as well as those of the factor \( L_U \otimes U(\frac{1}{2} + \nu) / \zeta(1 - 2\nu) \); we may assume, without any loss of generality, that \( u, v \) are such that all the residues in question are finite. This yields a meromorphic continuation of the continuous spectrum part, so that one may move \((u, v)\) close to \((\frac{1}{2}, \frac{1}{2})\) as far as \( \text{Re}(u + v) > 1 \) is satisfied, which is needed to have the last \( \Lambda \) factor defined well. Then, shift the \( \nu \)-contour back to the original. All the residual contribution coming from \( L_U \otimes U(\frac{1}{2} + \nu) / \zeta(1 - 2\nu) \) cancel out those arising from the previous shift of the contour. Only the pole at \( \frac{3}{2} - u - v \) contributes newly. The resulting integral is regular at \((\frac{1}{2}, \frac{1}{2})\); we get the factor \( \Lambda_\nu(g) = \Lambda_\nu(\frac{1}{2}, \frac{1}{2}; g), \text{ Re } \nu = 0. \)

Collecting all the above, we obtain

**Theorem 6.3** We have the spectral decomposition

\[
M(U, g) = m(U, g) + 2\text{Re}\left\{ \sum_V L_V(\frac{1}{2}) \Theta_V(g) + \int (0) \frac{\zeta\left(\frac{1}{2} + \nu\right) \zeta\left(\frac{1}{2} - \nu\right)}{\zeta\left(1 + 2\nu\right) \zeta\left(1 - 2\nu\right)} L_U \otimes U(\frac{1}{2} + \nu) \Lambda_\nu(g) \frac{dv}{4\pi i} \right\}, \tag{6.67}
\]

where \( m(U, g) \) is the value at \((\frac{1}{2}, \frac{1}{2})\) of the function

\[
\begin{align*}
\frac{L_U \otimes U(2(u + v))}{\zeta(2(u + v))} g^*(0) \\
+ \left\{ L_U \otimes U(u + v - 1) \Lambda_{u+v-\frac{3}{2}}(u, v; g) \\
+ L_U \otimes U(1 - u - v) \Lambda_{\frac{3}{2}-u-v}(u, v; g) \\
+ L_U \otimes U(u + v - 1) \Lambda_{u+v-\frac{3}{2}}(v, u; g) \\
+ L_U \otimes U(1 - u - v) \Lambda_{\frac{3}{2}-u-v}(v, u; g) \right\} / \zeta(2(2 - u - v)). \tag{6.68}
\end{align*}
\]

Albeit the \( \Lambda \) factors in the last expression is defined so far only under the condition \( 2 > \text{Re}(u + v) > 1 \), the expression can in fact be continued to a neighbourhood of \((\frac{1}{2}, \frac{1}{2})\), for \( J(u, v; g) \) and all other parts in the spectral expansion of \( J(u, v; g) \) and \( J(\bar{v}, \bar{u}; g) \) are regular there.