Brane Webs and Random Processes

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We study \((p,q)\) 5-brane webs dual to certain \(N\) M5-brane configurations and show that the partition function of these brane webs gives rise to cylindric Schur process with period \(N\). This generalizes the previously studied case of period 1. We also show that open string amplitudes corresponding to these brane webs are captured by the generating function of cylindric plane partitions with profile determined by the boundary conditions imposed on the open string amplitudes.

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I. INTRODUCTION

Topological strings on toric Calabi-Yau threefolds provide an interesting set of examples which are quite well understood. The topological vertex \cite{1} and refined topological vertex \cite{2} formalism provide an exact solution of the topological string partition functions in the unrefined and the refined case respectively. The topological vertex formalism reduces the calculation of the topological string partition function to sums over functions of Young diagrams. In case the toric Calabi-Yau threefold gives rise to gauge theory, via geometric engineering \cite{4}, the Young diagrams appearing in the partition functions of topological string can be directly be related to Young diagrams in the Nekrasov’s instanton calculus \cite{5} which label the fixed points on the instanton moduli spaces. It has been shown that in certain cases these gauge theory partition functions can be thought of as sums over probability measures on the set of Young diagrams \cite{6}.

In \cite{7} it was shown that the partition function of 5D \(\mathcal{N} = 1\) \(U(1)\) gauge theory with an adjoint hypermultiplet compactified on a circle gives a probability measure of a random process studied by Borodin in \cite{8} called the periodic Schur process with period one. This gauge theory also arises from mass deformation of an M5-brane with a transverse direction compactified to a circle \cite{9}.

In this short note we show that the probability measure associated with periodic Schur process with period \(N\) is given by the gauge theory partition function arises from N-M5-branes threading a circle. The gauge theory this configuration gives rise to is the \(U(1)^N\) gauge theory with bifundamental matter.

The paper is organized as follows. In section 2 we discuss the various generalizations of the Plancherel measure which arise from the \(N = 1\) case. In section 3 we discuss the M5-brane configuration and the corresponding dual \((p,q)\) brane web configuration which corresponds to the period \(N\) periodic Schur process. In this section we also calculate the topological string partition function, using the topological vertex formalism, of the Calabi-Yau threefold dual to the brane web and show that it is gives the periodic Schur process of period \(N\). In section 4 we show that the periodic Schur process with non-trivial profile corresponds to certain open topological string amplitudes. In section 5 we present our conclusions and future directions.

II. PROBABILITY MEASURES AND GAUGE THEORIES

In \cite{3} it was shown that the partition function of the four dimensional \(U(1)\) gauge theory with adjoint hypermultiplet can be written as sum over Young diagram of a function which can be thought as a probability measure on the set of Young diagrams. In this section we study some generalizations of the Plancherel measure which follow from 5D \(U(1)\) and 6D \(U(1)\) gauge theory with adjoint hypermultiplet.
A. Plancherel Measure

The case of the Plancherel measure was shown to follow from the four dimensional \( \mathcal{N} = 2 \) \( U(1) \) gauge theory. This four dimensional gauge theory can be obtained from the five dimensional theory by circle compactification, the 5D theory can be geometrically engineered by M-theory compactification on the resolved conifold. The 5D theory can also be realized on a \((p,q)\) 5-brane web, shown in Fig. 1, in type IIB string theory.

The above is precisely the poissonized Plancherel measure \[ Z_{\mathcal{P}} = \sum_{\lambda} Q^{[\lambda]} s^\rho{}_{\lambda}(1-t^\rho) = \sum_{\lambda} Q^{[\lambda]} \prod_{(i,j)\in\lambda} h_{\lambda}(i,j)^2 \] (1)

defines a probability measure on the set of partitions for \( q, t, Q < 1 \) which is a generalization of the poissonized Plancherel measure. To see this take \( q = t = e^\epsilon, Q = \epsilon^2 \Lambda \) and consider the limit \( \epsilon \to 0 \),

\[ \lim_{\epsilon \to 0} \mathcal{P}_\lambda(e^\epsilon, e^\epsilon, \epsilon^2) = e^{-\Lambda} \Lambda^{[\lambda]} \prod_{(i,j)\in\lambda} h_{\lambda}(i,j)^2 = e^{-\Lambda} \Lambda^{[\lambda]} \left( \frac{\dim \lambda}{|\lambda|} \right)^2. \] (3)

The above is precisely the poissonized Plancherel measure. In the above we have used

\[ \lim_{\epsilon \to 0} Z(e^\epsilon, e^\epsilon, \epsilon^2) = \lim_{\epsilon \to 0} \prod_{i,j} \left( 1 - \epsilon^2 \Lambda e^{(i+1,j-1)} \right) = e^\Lambda \]

and the hook length formula,

\[ \dim \lambda = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} h_{\lambda}(i,j)}, \] (4)

where \( \dim \lambda \) is the number of standard Young tableaus of shape \( \lambda \).

In this case if we consider \( t = e^{\beta^\epsilon} \), and then take \( \epsilon \to 0 \) we do not get any \( \beta \) deformed measure instead just get the same measure with \( \Lambda \to \Lambda/\beta \).

B. Nekrasov-Okounkov Measure and its \((q, t)\) Deformation

The 5D \( U(1) \) gauge theory can be obtained from 5D \( U(1) \) gauge theory with an adjoint hypermultiplet in the limit that the mass of the adjoint goes to infinity. The mass deformed 5D theory is realized by the brane web shown in Fig. 2.

\[ Z_{5D} = Z_{4D} \sum_{\lambda} \mathcal{M}_\lambda \]

(5)

\[ \mathcal{M}_\lambda = Q^{[\lambda]} \prod_{s\in\lambda} \left( 1 - Q q^{f(s)+1} + \bar{q}^{\mu(s)+1} \right) \]

(6)

defines a probability measure on the set of partitions which is a \((q, t)\) deformation of the Nekrasov-Okounkov measure.

\[ \lim_{\epsilon \to 0} \mathcal{P}_\lambda(e^{\epsilon r}, e^{\epsilon r}, e^{\epsilon^2}) = \frac{Q^{[\lambda]} \prod_{s\in\lambda} \left( h(s)^2 - \mu^2 \right)}{\prod_{k=1}^{\infty} (1 - k^2 \mu^2 - 1)} \]

(7)

which gives the \( \beta \) deformed Nekrasov-Okounkov measure.
C. Generalization of Nekrasov-Okounkov Measure Involving Theta Function

The further compactification of the web shown in Fig. 2 by changing the space on which the web lives from a cylinder to a torus gives a generalization of the Nekrasov-Okounkov measure involving theta functions.

\[
\begin{align*}
\text{The partition function in this case is given by} & \quad Z_{6D} = Z_{5D} \sum_{\lambda} M_\lambda \\
M_\lambda &= Q^{|\lambda|}_\rho \prod_{s \in \lambda} \frac{\theta_1(m + z(s))\theta_1(m + z(s) - \epsilon_+)}{\theta_1(z(s))\theta_1(z(s) - \epsilon_-)}
\end{align*}
\]

where \(z(s) = (\ell(s) + 1)\epsilon_1 - a(s)\epsilon_2\) and \(\epsilon_+ = \epsilon_1 + \epsilon_2\).

\[
\mathbb{P}_\lambda = \mathbb{W}^{-1} M_\lambda
\]

where \(\mathbb{W} = \frac{Z_{6D}}{Z_{5D}}\).

- \(m \rightarrow 0\) gives uniform measure \(Q^{|\lambda|}_\rho\).
- \(\tau \rightarrow i\infty\) gives the \((p,q)\) deformed Nekrasov-Okounkov measure.

III. PERIODIC SCHUR PROCESS AND BRANE WEBS

A. Periodic Schur process of period \(N\)

In this section we review, following [8], the periodic Schur process and the probability measure associated with it.

We denote by \(Y\) the set of Young diagrams. Then [8] defines the periodic Schur process of period \(N\) to be a random process defined on \(\mathbb{Y}^{2N}\) which assigns to the set of partitions \(\{\nu^0, \mu^1, \nu^1, \mu^2, \ldots, \nu^{N-1}, \mu^N\} \subset \mathbb{Y}^{2N}\) the weight,

\[
Q_{\rho}^{(\mu)} \left[ \prod_{a=0}^{N-1} \frac{s_{\nu^a/\mu^{a+1}}(x_{a+1})s_{\nu^{a+1}/\mu^a}(y_{a+1})}{Z_N} \right]
\]

where \(\nu^N = \nu^0\) and \(Z_N\) is the normalization which is also the partition function of this random process,

\[
Z_N = \sum_{\nu^a, \mu^a} Q_{\rho}^{(\mu)} \prod_{a=0}^{N-1} s_{\nu^a/\mu^{a+1}}(x_{a+1})s_{\nu^{a+1}/\mu^a}(y_{a+1})
\]

We will see in the next section that for a particular specialization \(x_a\) and \(y_a\) the above partition function of the periodic Schur process will be exactly the partition function of a configuration of \(N\) M5-branes.

B. \((p,q)\) 5-brane webs and M5-branes

The case of periodic Schur process of period 1 was discussed in detail in [7]. The brane web for this case is shown in Fig. 2. An M5-brane realization of this theory was studied in [9] where it was shown that the \((p,q)\) 5-brane web of Fig. 2 is dual to a configuration in which we have a single M5-brane compactified on a circle. The space transverse to the M5-brane is \(\mathbb{R}^3\) and there is \(U(1)\) action on the \(\mathbb{C}^2 \subset \mathbb{R}^5\) one goes around the transverse circle,

\[
U(1) : (w_1, w_2) \rightarrow (e^{2\pi i m} w_1, e^{-2\pi i m} w_2).
\]

This M5-brane configuration is, however, dual to another configuration in which the M5-brane is not wrapped but there is a circle transverse to the M5-brane. When the M5-brane is wrapped on a circle the massive modes come from Kaluza-Klein reduction of the 6D free tensor multiplet on a circle as shown in [9]. In the case when the M5-brane is not wrapped but there is a circle transverse to it the massive modes come from the M2-brane starting and ending of the M5-brane and wrapping the transverse circle.

Now consider the brane configuration in which we have multiple coincident M5-branes and a circle transverse to them. We can separate the M5-branes on that circle and mass deform the configuration as given in Eq. (11). This brane configuration is dual to a \((p,q)\) 5-brane web given in Fig. 4(a).

\[
\begin{align*}
Z_N &= \sum_{\lambda} \prod_{a=0}^{N-1} \left[ (-Q_a)^{|\lambda(a)|} W_{\lambda(a)\lambda(a+1)} \right]
\end{align*}
\]
where

\[ W_{\lambda^\prime(\mu+1)} = \sum_{\mu} (-Q_m)^{|\mu|} C_{\lambda^\prime(\mu+1)}(t, q) C_{\lambda(\mu+1)} \phi(q, t) \]

and

\[ C_{\lambda \phi}(t, q) = \left( \frac{q}{t} \right)^{\frac{|\lambda|-|\phi|}{2}} t^{\frac{|\lambda|}{2}} \widetilde{Z}_\varphi(t, q) \]

\[ \times \sum_{\eta} \left( \frac{q}{t} \right)^{|\eta|} s_{\lambda / \eta}(t^{-\rho}) s_{\mu / \eta}(q^{-\rho}) \]  \hfill (13)

is the refined topological vertex. For notation see Appendix A. The length of the slanted lines in Fig. 4 are all equal to \( m \) and we have defined \( Q_m = e^{-m} \), similarly the length of the horizontal lines is \( T_a \) and we have defined \( Q_a = e^{-T_a} \) such that \( -\log(Q_a Q_m) \) is the distance between the two vertical lines. In Eq. (12) \( W_{\lambda \nu} \) is the open string amplitude corresponding the brane configuration shown in Fig. 5 below and is given by

\[ W_{\lambda \nu} = \left( \frac{q}{t} \right)^{\frac{|\lambda|-|\nu|}{2}} \sum_{\eta_1, \eta_2} \left( \frac{q}{t} \right)^{\frac{|\eta_1|-|\eta_2|}{2}} s_{\lambda / \eta_1}(t^{-\rho}) s_{\mu / \eta_2}(q^{-\rho}) \]

\[ \times \sum_{\lambda'} (-Q_m)^{|\lambda'|} s_{\lambda'/\eta_1}(q^{-\rho}) s_{\mu'/\eta_2}(t^{-\rho}) \]  \hfill (14)

FIG. 5. The open string amplitude which is building block of the topological string partition function. After summing over \( \mu \) an auxiliary partition \( \tau \) appears which is same as the one appearing in the definition of the periodic Schur process.

Using the identity

\[ \sum_{\lambda} s_{\lambda'/\eta}(x) s_{\lambda/\sigma}(y) = \prod_{i,j} (1 + x_i y_j) \]

\[ \times \prod_{\tau} s_{\lambda'/\tau}(x) s_{\mu'/\tau}(y) \]

the sum over \( \mu \) in Eq. (14) can be carried out and we get,

\[ W_{\lambda \nu} = \Pi(Q_m) \left( \frac{q}{t} \right)^{\frac{|\lambda|-|\nu|}{2}} \sum_{\eta_1, \eta_2} \left( \frac{q}{t} \right)^{\frac{|\eta_1|-|\eta_2|}{2}} s_{\lambda / \eta_1}(t^{-\rho}) s_{\mu / \eta_2}(t^{-\rho}) \]  \hfill (15)

\[ s_{\mu'/\eta_2}(q^{-\rho}) \sum_{\tau} (-Q_m)^{|\tau|} s_{\eta_1'/\tau}(-Q_m t^{-\rho}) s_{\eta_2'/\tau}(-Q_m q^{-\rho}) \]

where

\[ \Pi(x) = \prod_{i,j=1}^{\infty} \left( 1 - x q^{-\rho_i} t^{-\rho_j} \right) \]  \hfill (16)

Using the following properties of the skew-schur functions,

\[ s_{\lambda'/\sigma}(q^{-\rho}) = s_{\lambda/\sigma}(-q^{-\rho}) \]

\[ \sum_{\eta} s_{\lambda/\eta}(x) s_{\eta/\sigma}(y) = s_{\lambda/\sigma}(x, y), \]

we get,

\[ W_{\lambda \nu} = \left( \frac{q}{t} \right)^{\frac{|\lambda|-|\nu|}{2}} (-1)^{|\nu|} \Pi(Q_m) \widetilde{W}_{\lambda \nu} \]

\[ \widetilde{W}_{\lambda \nu} = \sum_{\tau} Q_{m \lambda} s_{\lambda/\tau}(a) s_{\nu/\tau}(b). \]

In the above equation,

\[ a = \{ Q_m \sqrt{\frac{q}{t}} t^{\rho}, t^{-\rho} \} \]

\[ b = \{ Q_m \sqrt{\frac{t}{q}} q^{-\rho}, q^{\rho} \} \]  \hfill (17)

The partition function in Eq. (12) can now be written as,

\[ Z_N := \Pi(Q_m)^N \sum_{\lambda} \prod_{a=0}^{N-1} \left[ Q_a^{(\lambda_{a|})} \widetilde{W}_{\lambda_{a|}\lambda_{a+1|}} \right] \]  \hfill (18)

The partition function \( \widetilde{Z}_N := Z_N / \Pi(Q_m)^N \) is thus given by,

\[ \widetilde{Z}_N := \sum_{\lambda} \prod_{a=0}^{N-1} \left[ Q_a^{(\lambda_{a|})} \widetilde{W}_{\lambda_{a|}\lambda_{a+1|}} \right] \]

\[ = \sum_{\lambda_\tau} \prod_{\lambda_\tau} \left[ Q_m^{(\lambda_{\tau|})} \widetilde{W}_{\lambda_{\tau|}\lambda_{\tau+1|}} \right] \left( x_{\tau+1} \right) \left( y_{\tau+1} \right) \]  \hfill (19)

where we have defined new variables:

\[ Q_m = \prod_{a=0}^{N-1} (Q_a Q_m), \quad Q_{1, a+1} = (Q_1 Q_{2} \cdots Q_a) Q_m^a \]

and

\[ x_{a+1} = Q_m^{-1/2} Q_{1, a+1} a, \]

\[ y_{a+1} = Q_m^{-1/2} Q_{1, a+1} b. \]  \hfill (20)

\(-\log(Q_m)\) is precisely the circumference of the circle. In writing last line of Eq. (22) we have used the following identity,

\[ \sum_{a=0}^{N-1} \lambda_{a\tau} \prod_{a=0}^{N-1} Q_a^{(\lambda_{a\tau})} = \prod_{a=0}^{N-1} Q_a^{(\lambda_{a\tau})} \prod_{a=0}^{N-1} Q_{1, a+1}^{(\lambda_{a\tau})} \]  \hfill (21)

We can describe the partition function graphically by associating the partition \( \tau_a \), with each M5-brane and partitions \( \lambda_a \) with the interval between the M5-branes as shown in Fig. 6 below.
FIG. 6. After summing over the partitions associated with the slanted lines auxiliary partitions $\tau_a$ appear sandwiched between $\lambda_{a-1}$ and $\lambda_a$. The periodic Schur process is defined in terms of the set $\{\lambda_a, \tau_a\}$.

From last line of Eq.(22) and Eq.(10) we see that the partition function of this configuration of $N$ M5-branes is precisely the partition function of the periodic schur process with period $N$. In the limit $Q_\rho \to \infty$ we get $\lambda_{(N)} = \lambda_{(0)} = 0$ and we get usual Schur process. Thus the usual Schur process is associated with linear configuration of M5-branes.

IV. OPEN STRING AMPLITUDES AND CYLINDRIC PARTITIONS

In this section we show that open string amplitudes corresponding to the brane configuration shown in Fig. 7 are also given by cylindric plane partitions \[8, 11, 12\] with non-trivial profile which captures the partitions $\nu_{(a)}$.

Let us begin by considering the cylindric partition with trivial profile as shown below in Fig. 8(a).

 FIG. 8. (a) Cylindric partition with empty partition as a profile. (b) Cylindric partition with profile given by a $R \times M$ partition. The vertical dotted lines indicate gluing so that the plane partitions live on a cylinder.

non-trivial partition at each of the corners as shown in Fig. 9. It was shown in \[10\] that the generating function associated with Fig. 9(a) is precisely the open string amplitude corresponding to the brane configuration in Fig. 7(a).

 FIG. 9. (a) region in the shape of $\nu$ is excised and plane partitions are put in the remaining region. (b) Regions in the shape of $\nu_1$ and $\nu_2$ are excised and plane partitions put in the remaining region.

If we non-trivial partition $\nu_1$ and $\nu_2$ in the two corners of Fig. 9(b) then the generating function of cylindric plane partitions is given by,

$$G_{\sigma}^{t,n} (q) = \prod_{s \in \sigma} (1 - q^{h(s)})^{-1} \prod_{k = 1}^{\infty} (1 - Q^k_{\rho})^{-1} \prod_{i,j=1}^{\ell,n} (1 - Q^k_{\rho} q^{h(s)})$$

where $Q_\rho = q^{n+\ell}$ and $h(i, j) = \sigma_i + \sigma_j - i - j + 1$. We take the partition $\sigma = (\nu^{(1)}_1 + R, \cdots, \nu^{(1)}_{M-(\nu^{(1)}_1)} + R, R, R, \cdots, R, \nu^{(2)}_1, \cdots, \nu^{(2)}_{M-(\nu^{(2)}_1)})$.

The above is precisely the open string amplitude associated with brane configuration shown in Fig. 7(b) for,

$$Q_{m_1} = q^{R}, \quad Q_{m_2} = q^{\ell - R}, \quad Q_1 := q^{n-M}, \quad (24)$$

where $Q_{m_1, 2}$ are the parameters associated with the slanted lines and $Q_1$ is the parameter associated with middle horizontal line.
V. CONCLUSIONS

In this short note we have shown that generalizations of Nekrasov-Okounkov measure (which itself generalizes the Plancherel measure) follows from considering gauge theories in four, five and six dimensions. These gauge theories arise from certain 5-brane configurations when the plane in which the brane lives is compactified to a cylinder and then to a torus. We also saw that $U(1)^N$ quiver gauge theories arising on a stack of $N$ M5-branes separated on a circle have a partition function which is exactly the partition function of periodic Schur process of period $N$. The correlation function of chiral operators in these gauge theories are also given by sum over young diagrams [13] and it would be interesting to relate these to the expectation value of random variables in the periodic Schur process.

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