Momentum and Position Representations for the
$q$-deformed Euclidean Quantum Space

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Abstract

We summarize some basics about mathematical tools of analysis for the $q$-deformed Euclidean space. We use the new tools to examine $q$-deformed eigenfunctions of the momentum or position operator within the framework of the star product formalism. We show that these two systems of functions are complete and orthonormal. With the $q$-deformed momentum or position eigenfunctions, we calculate matrix elements of the momentum or position operator. Considerations about expectation values and probability densities conclude the studies.

1 Introduction

The debate whether space and time are discrete or continuous began long ago. It was as early as in ancient times that the Greek philosopher Zenon of Elea confused his contemporaries. He baffled them by stating that Achilles, as the fastest runner of the Greeks, can never overtake a turtle with a one-meter lead if the distance between Achilles and the turtle is infinitely divisible.

Of course, the idea of a continuum can form a logical basis for the ordinary differential and integral calculus. Moreover, the physical theories formulated with the help of this differential and integral calculus are fruitful. Certain doubts, however, remain whether a continuum adequately describes space and time at small distances. To overcome the problem that it is not possible to calculate a finite self-energy of the electron with quantum field theories formulated on a space-time continuum Werner Heisenberg, for example, used a lattice-like space [7,8].

Even though such attempts have not yet been successful [5], some considerations within the framework of a future theory of quantum gravity suggest that space-time reveals a discrete structure at small distances [5]. For example, the attempt to increase the accuracy of a position measurement more and

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more, should disturb the background metric more and more \[24\]. Such a fundamental uncertainty in space could require a space-time algebra generated by non-commuting coordinates. Such a non-commutative algebra can be obtained, for example, by \(q\)-deformation. The general aim of my scientific work is to find out whether physical theories can be formulated on \(q\)-deformed space-time algebras free from contradictions. Furthermore, we could see if such an approach is suitable to solve existing physical problems.

In this article, I am going to give the reader some basics for dealing with free particles on the \(q\)-deformed Euclidean space. For this reason, let me summarize some well-known facts from quantum mechanics. If we want to describe the measurement of the position or momentum of a particle mathematically, we must assign a linear operator to each of these measurable quantities. The measurement corresponds to the action of the respective operator on a wave function that represents the state of the free particle. If the particle has a defined momentum or position after the measurement, an eigenfunction of the momentum operator or the position operator represents the particle state. In the case of the position measurement, these eigenfunctions are delta functions, and in the case of the momentum measurement, they are plane waves, i.e. exponential functions. A wave function that represents the state of a free particle can be expanded in terms of eigenfunctions of the position operator as well as in terms of eigenfunctions of the momentum operator. Moreover, each expansion coefficient determines the probability for obtaining the corresponding state of defined momentum or position in the position or momentum measurement.

In Chap. 4 of this article, I am going to discuss how the concepts described above apply to \(q\)-deformed quantum spaces within the framework of the so-called star product formalism. For this purpose, I am going to summarize some algebraic basics in Chap. 2.1 needed to provide mathematical tools of analysis for \(q\)-deformed quantum spaces. Among these tools, which we will briefly explain in the subsequent chapters, are star products, translations, actions of partial derivatives, integrals as well as Fourier transformations. In Chap. 4.1, we are going to apply these new tools to introduce eigenfunctions of the \(q\)-deformed position or momentum operator. In the subsequent chapters, we will show that the eigenfunctions of the \(q\)-deformed position or momentum operator are complete and orthonormal. In Chap. 4.5 and Chap. 4.6, we are going to use these results to write down transition matrix elements of the \(q\)-deformed position operator or the \(q\)-deformed momentum operator for both the \(q\)-deformed momentum eigenfunctions and the \(q\)-deformed position eigenfunctions. Finally, in Chap. 4.7, we are going to determine expectation values and probability densities for the position measurement or momentum measurement within the new formalism.

The reflections of this article show that we can formulate many concepts on \(q\)-deformed quantum spaces in far-reaching analogy to the undeformed case. The non-commutativity due to \(q\)-deformation, however, will lead to some complexity since we have several \(q\)-deformed variants for the considered mathematical objects.
2 Elements of multidimensional $q$-analysis

2.1 Euclidean quantum space

The $q$-deformed three-dimensional Euclidean quantum space $\mathbb{R}^3_q$ is a three-dimensional representation of the Hopf algebra $U_q(su_2)$ [15]. The latter is a deformation of the universal enveloping algebra of the Lie algebra $su_2$ [14], as the following relations of its generators show:

\[
\begin{align*}
q^{-1}T^+T^- - qT^-T^+ &= T^3, \\
q^2T^3T^+ - q^{-2}T^+T^3 &= (q + q^{-1})T^+, \\
q^2T^-T^3 - q^{-2}T^3T^- &= (q + q^{-1})T^-.
\end{align*}
\]  

(1)

Compared to an ordinary algebra, a Hopf algebra has not only a product and a unit but also a co-product, a co-unit and an antipode [12]. For example, the co-product of the $U_q(su_2)$-generators takes on the following form [15]:

\[
\begin{align*}
\Delta(T^+) &= T^+ \otimes 1 + \frac{1}{2}\tau \otimes T^+, \\
\Delta(T^-) &= T^- \otimes 1 + \frac{1}{2}\tau \otimes T^-, \\
\Delta(T^3) &= T^3 \otimes 1 + \tau \otimes T^3.
\end{align*}
\]  

(2)

Note that we have introduced the element $\tau = 1 - (q - q^{-1})T^3$ in the above expressions. The Hopf algebra $U_q(su_2)$ has the following spin representations [15]:

\[
\begin{align*}
T^3 |j,m\rangle &= q^{-1}[[2m]]_{q^{-2}} |j,m\rangle, \\
T^+ |j,m\rangle &= q^{-1}\sqrt{[[j + m + 1]]_{q^{-2}}[[j - m]]_{q^2}} |j,m + 1\rangle, \\
T^- |j,m\rangle &= q\sqrt{[[j + m]]_{q^{-2}}[[j - m + 1]]_{q^2}} |j,m - 1\rangle, \\
\tau |j,m\rangle &= q^{-4m} |j,m\rangle.
\end{align*}
\]  

(3)

Note also that the above representations depend on so-called anti-symmetric $q$-numbers:

\[
[[n]]_{q^2} = \frac{1 - q^{2n}}{1 - q^2}.
\]  

(4)

The fundamental states of the three-dimensional representation can be identified with the generators of the three-dimensional Euclidean quantum space:

\[
X^- = |1, -1\rangle, \quad X^3 = |1, 0\rangle, \quad X^+ = |1, 1\rangle.
\]  

(5)

The coordinates $X^3$, $X^+$, and $X^-$ form a light cone basis. Therefore they are related to the usual Cartesian coordinates $x^1$, $x^2$, and $x^3$ by the following classical limit [16]:

\[
\lim_{q \to 1} X^3 = x^3, \quad \lim_{q \to 1} X^\pm = i(x^1 \pm ix^2).
\]  

(6)

\footnote{If the deformation parameter $q$ tends to 1, we regain the commutation relations of the Lie algebra $su_2$.}
Tensor products of quantum spaces should have the same symmetry as the quantum spaces which they are composed of. In other words, tensor products of modules of a Hopf algebra must again be modules of the same Hopf algebra. The action of the Hopf algebra $U_q(su_2)$ on a tensor product of two Euclidean quantum spaces is calculated by using the co-product of $U_q(su_2)$. Specifically, it applies to an element $h$ of $U_q(su_2)$ and two elements $u$ and $v$ of the Euclidean quantum space $[12]$:

$$h ⊗ (u ⊗ v) = h_{(1)} ⊗ u ⊗ h_{(2)} ⊗ v.$$  \hfill (7)

Note that we have written the co-product of $h$ in the so-called Sweedler notation, i. e. $\Delta(h) = h_{(1)} ⊗ h_{(2)}$.

Due to the non-trivial co-product of $U_q(su_2)$, the multiplication on a tensor product of two quantum spaces can generally not be performed by using the usual twist, but with the help of a so-called braiding map. In the case of two coordinate generators this braiding map is given by the R-matrix $\hat{R}_{ABCD}$ for the three-dimensional $q$-deformed Euclidean space (together with a constant $k$):

$$(1 ⊗ Y^A) · (X^B ⊗ 1) = k \hat{R}_{AB}^{CD} X^C ⊗ Y^D.$$ \hfill (8)

Applying the action of the $U_q(su_2)$-generators to the above equations, a system of equations can be found to determine the entries of the R-matrix of the Euclidean quantum space $[15]$ The inverse matrix $\hat{R}^{-1}$ is obtained as a further solution of this system:

$$(1 ⊗ Y^A) · (X^B ⊗ 1) = k^{-1} \hat{R}^{-1}_{AB}^{CD} X^C ⊗ Y^D.$$ \hfill (9)

Eq. (8) and Eq. (9) are often referred to as braiding relations. These braiding relations can be generalized in a way that they apply to arbitrary elements of quantum spaces. To this end, we introduce the so-called universal R-matrix $R = R_{[1]} ⊗ R_{[2]} ∈ U_q(su_2) ⊗ U_q(su_2)$ and its inverse $[22]$, i. e.

$$(1 ⊗ u) · (v ⊗ 1) = R_{[2]} ⊗ v ⊗ R_{[1]} ⊗ u$$ \hfill (10)

or

$$(1 ⊗ u) · (v ⊗ 1) = R_{[1]}^{-1} ⊗ v ⊗ R_{[2]}^{-1} ⊗ u.$$ \hfill (11)

The R-matrix of the Euclidean quantum space has the following projector decomposition $[15]$:

$\hat{R} = P_S + q^{-6} P_T - q^{-4} P_A.$ \hfill (12)

The projector $P_A$ is a $q$-analogue of the antisymmetrizer which maps on the space of antisymmetric tensors of second rank. The projector $P_S$ is the $q$-deformed trace-free symmetrizer and $P_T$ is the $q$-deformed trace-projector.

The antisymmetric projector $P_A$ defines the commutation relations for the coordinates of the Euclidean quantum space ($A, B ∈ \{+, 3, -\}$):

$$(P_A)^{AB}_{CD} X^C X^D = 0.$$ \hfill (13)

\textsuperscript{2}The constant $k$ cannot be determined this way.
Explicitly, the commutation relations of the Euclidean quantum space coordinates read as [16]:

\[ X^3 X^+ = q^2 X^+ X^3, \]
\[ X^3 X^- = q^{-2} X^- X^3, \]
\[ X^- X^+ = X^+ X^- + (q - q^{-1}) X^3 X^3. \] (14)

The trace projector \( P_T \) leads us to a \( q \)-analog of the Euclidean metric. Concretely, it applies to the metric \( g^{AB} \) of the \( q \)-deformed Euclidean space and its inverse \( g_{CD} \) [16]:

\[ (P_T)^{AB}_{ \quad CD} = \frac{1}{g^{EF} g_{EF}} g^{AB} g_{CD}. \] (15)

This correspondence implies for the \( q \)-deformed Euclidean metric (row and column indices have the order \(+, 3, -\)):

\[ g_{AB} = g^{AB} = \begin{pmatrix} 0 & 0 & -q \\ 0 & 1 & 0 \\ -q^{-1} & 0 & 0 \end{pmatrix}. \] (16)

The Euclidean quantum space \( \mathbb{R}_q^3 \) is also a \( \ast \)-algebra, i.e. it has a semilinear, involutive and anti-multiplicative mapping, which we call quantum space conjugation. We indicate the conjugate elements of a quantum space by a bar. The properties of the conjugation on a quantum space can now be written down as follows (\( \alpha, \beta \in \mathbb{C} \) and \( u, v \in \mathbb{R}_q^3 \))[4]

\[ \bar{\alpha u} + \bar{\beta v} = \bar{\alpha} \bar{u} + \bar{\beta} \bar{v}, \quad \bar{u} = u, \quad \bar{uv} = \bar{v} \bar{u}. \] (17)

You can show that the conjugation on the quantum space \( \mathbb{R}_q^3 \) respects the commutation relations in Eq. (14) if the following applies [16]:

\[ \overline{X^A} = X_A = g_{AB} X^B. \] (18)

### 2.2 Star products

An \( N \)-dimensional quantum space can be seen as an algebra \( V_q \) which is spanned by non-commutative coordinates \( X^i \) with \( i = 1, \ldots, N \), i.e. the coordinates of the quantum space satisfy certain non-trivial commutation relations. The commutation relations of the quantum space coordinates generate a two-sided ideal \( \mathcal{I} \) which is invariant under actions of the Hopf algebra describing the symmetry of \( V_q \). From this point of view a quantum space is a quotient algebra which is formed by the free algebra \( \mathbb{C}[X^1, X^2, \ldots, X^N] \) and the ideal \( \mathcal{I} \):

\[ V_q = \frac{\mathbb{C}[X^1, X^2, \ldots, X^N]}{\mathcal{I}}. \] (19)

---

3A bar over a complex number indicates complex conjugation.
In general, a physical theory can only be verified if it predicts certain measurement results. The question, however, is: How can we associate the elements of a quantum space with real numbers? One solution to this problem is to interpret the quantum space coordinates \( X^i \) with \( i \in \{1, \ldots, N\} \) as operators acting on a ground state which is invariant under actions of the symmetry Hopf algebra. This way, the corresponding expectation values denoted as
\[
x^i = \langle X^i \rangle
\] (20)
can be seen as real-valued variables with their numbers depending on the underlying ground state. In the following, we will show how we can extend the above identity to normal ordered monomials of quantum space coordinates.

The normal ordered monomials of the quantum space coordinates \( X^i \) form a basis of the quantum space \( V_q \), i. e. each element \( F \in V_q \) can uniquely be written as a finite or infinite linear combination of monomials of a given normal ordering (Poincaré-Birkhoff-Witt property):
\[
F = \sum_{i_1, \ldots, i_N} a_{i_1 \ldots i_N} (X^1)^{i_1} \ldots (X^N)^{i_N} \quad \text{with} \quad a_{i_1 \ldots i_N} \in \mathbb{C}.
\] (21)

Since the set of monomials \( (x^1)^{i_1} \ldots (x^N)^{i_N} \) with \( i_1, \ldots, i_N \in \mathbb{N}_0 \) forms a basis of the commutative algebra \( V = \mathbb{C}[x^1, \ldots, x^N] \), we can define a vector space isomorphism between \( V \) and \( V_q \), i. e.
\[
\mathcal{W} : V \to V_q
\] (22)
with
\[
\mathcal{W}( (x^1)^{i_1} \ldots (x^N)^{i_N} ) = (X^1)^{i_1} \ldots (X^N)^{i_N}.
\] (23)

By linear extension follows
\[
V \ni f \mapsto F \in V_q,
\] (24)
where
\[
f = \sum_{i_1, \ldots, i_N} a_{i_1 \ldots i_N} (x^1)^{i_1} \ldots (x^N)^{i_N},
\]
\[
F = \sum_{i_1, \ldots, i_N} a_{i_1 \ldots i_N} (X^1)^{i_1} \ldots (X^N)^{i_N}.
\] (25)

The vector space isomorphism \( \mathcal{W} \) is nothing else but the so-called Moyal-Weyl mapping which gives an ‘operator’ \( F \in V_q \) to a complex valued function \( f \in V \) \[2\ [13\ [17\ [25\]. You can see that the inverse of the Moyal-Weyl mapping gives each quantum space coordinate its expectation value:
\[
\mathcal{W}^{-1}(X^i) = x^i = \langle X^i \rangle.
\] (26)
This relation can be used for normal ordered monomials as follows:
\[
\mathcal{W}^{-1}((X^1)^{i_1} \ldots (X^N)^{i_N}) = (x^1)^{i_1} \ldots (x^N)^{i_N} = \langle (X^1)^{i_1} \ldots (X^N)^{i_N} \rangle = \langle (X^1)^{i_1} \rangle \ldots \langle (X^N)^{i_N} \rangle.
\] (27)
We can go one step further. By linear extension, the vector space isomorphism \( W^{-1} \) can assign an expectation value \( f = \langle F \rangle \) to any element \( F \) of the quantum space \( V_q \):

\[
W^{-1}(F) = f = \langle F \rangle. \tag{28}
\]

As a look at Eq. (25) shows the expectation value \( f \) is a function of the commutative coordinates \( x^i \). In this way, the vector space isomorphism \( W^{-1} \) maps the non-commutative quantum space algebra \( V_q \) onto the commutative algebra \( V \) consisting of all power series with coordinates \( x^i \). We can even extend this vector space isomorphism to an algebra isomorphism if a new product is introduced on the commutative algebra \( V \). This so-called star product symbolized by \( \star \) satisfies the following homomorphism condition:

\[
W^{-1}(F \cdot G) = \langle F \rangle \star \langle G \rangle = W^{-1}(F) \star W^{-1}(G). \tag{29}
\]

With \( f \) and \( g \) being two formal power series of the commutative coordinates \( x^i \), the above condition can alternatively be written in the following form:

\[
W(f \star g) = W(f) \cdot W(g). \tag{30}
\]

Since the Moyal-Weyl mapping is invertible, we can write the star product as follows:

\[
f \star g = W^{-1}(W(f) \cdot W(g)). \tag{31}
\]

Thus, the star product realizes the non-commutative product of \( V_q \) on the commutative algebra \( V \).

To get explicit formulas for calculating the star product, we must define a suitable normal ordering for the non-commutative coordinate monomials. To derive these formulas, we have to expand the non-commutative product of two normal ordered monomials in terms of normal ordered monomials by using the commutation relations for the quantum space coordinates:

\[
(X^{1})^{i_1} \cdots (X^{N})^{i_N} \cdot (X^{1})^{j_1} \cdots (X^{N})^{j_N} = \sum_{k} B_{k} (X^{1})^{k_1} \cdots (X^{N})^{k_N}. \tag{32}
\]

In the case of the \( q \)-deformed Euclidean space, for example, we can obtain the following formula for calculating the star-product (\( \lambda = q - q^{-1} \))^4

\[
f(x) \star g(x) = \sum_{k=0}^{\infty} \lambda^k \frac{(x^3)^{2k}}{[k]!!q^k} D_{q^k,x} f(x) D_{q^k,x} g(x^3) \big|_{x^3 \rightarrow x}. \tag{33}
\]

Note that the above expression depends on the operators

\[
\hat{n}_A = x^A \frac{\partial}{\partial x^A} \quad \text{with} \quad q^{n_A} (x^A)^k = q^k (x^A)^k \tag{34}
\]

as well as the so-called Jackson derivatives \([10]\):

\[
D_{q^k,x} f = \frac{f(q^k x) - f(x)}{q^k x - x}. \tag{35}
\]

\(^4\)For the details see Ref. [20].
Eq. (33) shows that the formula for calculating the star product has the following structure:

\[
f \ast g = fg + \sum_{k>0} \lambda^k W_k(f, g). \tag{36}
\]

Normally, we have

\[
W_k(f, g) \neq W_k(g, f), \tag{37}
\]

i.e. the star product modifies the ordinary product of two commutative functions by correction terms that are responsible for the non-commutativity of the star product. In the undeformed limit \( q \to 1 \), these correction terms disappear because they depend on \( \lambda = q - q^{-1} \).

The algebra isomorphism \( \mathcal{W}^{-1} \) can also be used to carry over the conjugation properties of the non-commutative quantum space algebra \( V_q \) to the corresponding commutative coordinate algebra \( V \), i.e. the mapping \( \mathcal{W}^{-1} \) is also a \( * \)-algebra homomorphism. This way a conjugation is defined on the commutative coordinate algebra \( V \):

\[
\mathcal{W}(\bar{f}) = \mathcal{W}(\bar{f}) \Leftrightarrow \bar{f} = \mathcal{W}^{-1}(\mathcal{W}(\bar{f})). \tag{38}
\]

This relation implies the following conjugation property of the star product:

\[
\bar{f} \ast \bar{g} = g \ast f. \tag{39}
\]

Furthermore, a detailed analysis shows that a power series \( f \) in the commutative coordinates \( x^i \) becomes under conjugation \( \bar{f} \):

\[
\bar{f}(x^1, \ldots, x^N) = \sum_{i_1, \ldots, i_N} \bar{a}_{i_1, \ldots, i_N} (x^1)^{i_1} \cdots (x^N)^{i_N} = \sum_{i_1, \ldots, i_N} \bar{a}_{i_1, \ldots, i_N} (x_1)^{i_1} \cdots (x_N)^{i_N} = \sum_{i_1, \ldots, i_N} \bar{a}_{i_1, \ldots, i_N} \left( g_1 A_1 x^{A_1} \right)^{i_1} \cdots \left( g_N A_N x^{A_N} \right)^{i_N} = \bar{f}(x^1, \ldots, x^N). \tag{40}
\]

In this case \( \bar{a}_{i_1, \ldots, i_N} \) designates the complex conjugate of the coefficient \( a_{i_1, \ldots, i_N} \) and - according to Eq. (18) of the previous chapter - the covariant coordinates \( x_i \) are given by the following expressions:

\[
x_i = g_{ij} X^j. \tag{41}
\]

Now one recognizes that by conjugation of a power series \( f \) of the commutative coordinates \( x^i \) a new power series arises in which all expansion coefficients become complex conjugate and all contravariant coordinates are replaced by the corresponding covariant ones. In the following, we denote this new power series with \( \bar{f} \).
2.3 Translations

To perform translations on $q$-deformed quantum spaces, we replace every coordinate generator $X_i$ of a $q$-deformed quantum space $V_q$ by $X_i \otimes 1 + 1 \otimes Y_i$ \[19\]. Thus, we get a mapping from $V_q$ to the tensor product $V_q \otimes V_q$. Since each element of $V_q$ can be expanded in terms of normal ordered monomials, you only need to know how normal ordered monomials behave under translations. Accordingly, we apply the above substitutions to any normal ordered monomial of quantum space coordinates. The expression obtained in this way can again be expanded in terms of tensor products of two normal ordered monomials:

\[
(X^1 \otimes 1 + 1 \otimes Y^1)^{i_1} \cdots (X^N \otimes 1 + 1 \otimes Y^N)^{i_N} = \sum_{k,l} \alpha_{i_1 \cdots i_N; k_1 \cdots k_N} (X^1)^{k_1} \cdots (X^N)^{k_N} \otimes (Y^1)^{l_1} \cdots (Y^N)^{l_N}.
\] \[(42)\]

In order to get the expansion above, you need the braiding relations between coordinate generators of different quantum spaces [see Eq. (8) of Chap. 2.1] and the commutation relations for coordinate generators of the same quantum space. Since all non-commutative monomials are normal ordered in the above expressions, we can carry over the above formula to commutative coordinate monomials. This gives us a $q$-analog of the multidimensional binomial formula. From this $q$-deformed formula we can directly get an operator representation. With the help of this operator representation, we can calculate $q$-deformed translations for all those functions which we can write as a power series of the commutative coordinates $x^i$. In the case of the $q$-deformed Euclidean quantum space with three dimensions we can get the following formula for calculating $q$-translations \[26\]:

\[
f(x \oplus y) = \sum_{i=0}^{\infty} \sum_{i=0}^{\infty} \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-q^{-1} \lambda \lambda_i)^k (x^{-})^{i_1} (x^{+})^{i_2} (x^{3})^{i_3} (y^{-})^k}{[2k]_{q-2}!! [i_1-1]_{q-2}! [i_2-1]_{q-2}! [i_3-1]_{q-2}!} \times (D_{q^{-1}}^{i_1} y^{-} D_{q^{+1}}^{i_2} y^{+} D_{q^{3}}^{i_3} y^{3}) (q^{2(k-i_3)} y^{-}, q^{-2i_1+3}).
\] \[(43)\]

As mentioned above, the derivation of the $q$-translations is carried out with the help of the braiding relations for generators of different quantum spaces. However, there are two kinds of braiding relations [see also Eq. (8) and Eq. (9) of Chap. 2.1]. Accordingly, there are two versions of $q$-translations on each $q$-deformed quantum space. Whenever you want, the operator representations of the two $q$-translations can be transformed into each other by simple substitutions \[29\].

It should be mentioned that the $q$-deformed quantum spaces we have considered so far are so-called braided Hopf algebras \[22\]. From this point of view, the two versions of $q$-translations are nothing else but realizations of two braided co-products $\Delta$ and $\bar{\Delta}$ on the corresponding commutative coordinate algebras.
In the same way, you can implement the braided antipodes $\bar{S}$ and $\bar{S}$ on the corresponding commutative coordinate algebras:

\[
\begin{align*}
   f(\oplus x) &= (W^{-1} \circ \overline{S})(W(f)), \\
   f(\ominus x) &= (W^{-1} \circ \overline{S})(W(f)).
\end{align*}
\]  

(44)

The operations in Eq. (45) are referred to in the following as $q$-inversions. In the case of the $q$-deformed Euclidean quantum space, we can find the following operator representation for $q$-inversions [26]:

\[
\begin{align*}
   \hat{U}^{-1} f( \ominus x) = \sum_{i=0}^{\infty} (-q^{\lambda_{-}})^{i} \frac{(x_{-}^{+})^{i}}{[i]!} \frac{q^{-2\hat{n}_{+}(\hat{n}_{+} + \hat{n}_{-}) - 2\hat{n}_{-}(\hat{n}_{+} + \hat{n}_{-}) - \hat{n}_{+} \hat{n}_{-}}{q^{-2\hat{n}_{+} \hat{n}_{-} - 2\hat{n}_{+} \hat{n}_{-}} D_{q^{-2\hat{n}_{+}}} f}, \\
   \hat{U}^{-1} f = \sum_{k=0}^{\infty} \frac{(-\lambda)^{k}}{[k]!} \frac{(x_{-}^{+})^{2k}}{q^{-2\hat{n}_{+} \hat{n}_{-} - 2\hat{n}_{+} \hat{n}_{-} + k}} D_{q^{-2\hat{n}_{+}}} f,
\end{align*}
\]

(46)

It should be mentioned that the term $q$-inversion arises from the fact that the braided antipodes are subject to identities of the following form [18]:

\[
\sum_{i,j} \alpha_{i,j} (X^{1})^{i_{1}} \cdots (X^{N})^{i_{N}} \cdot \overline{S}((X^{1})^{j_{1}} \cdots (X^{N})^{j_{N}}) = 0.
\]

(47)

In this respect, the $q$-inversions are related in some way to an operation that replaces each coordinate $x^{i}$ with $-x^{i}$.

For the sake of completeness, we will show you how $q$-translations and $q$-inversions behave under quantum space conjugation. Since the quantum spaces we are looking at are so-called braided $\ast$-Hopf algebras, their braided co-products and antipodes behave under conjugation as follows [21]:

\[
\begin{align*}
   \tau \circ (\ast \otimes \ast) \circ \overline{\Delta} &= \overline{\Delta} \circ \ast, \\
   * \circ \overline{S} &= \overline{S} \circ \ast, \\
   \tau \circ (\ast \otimes \ast) \circ \overline{\Delta} &= \overline{\Delta} \circ \ast, \\
   * \circ \overline{S} &= \overline{S} \circ \ast.
\end{align*}
\]

(48)

5The quantum space conjugation is indicated by $\ast$. 

10
As we already know, $q$-translations and $q$-inversions are nothing else but realizations of co-products or antipodes. For this reason, we can immediately read off their behavior under conjugation from the identities above:

\[
\begin{align*}
\bar{f}(x \oplus y) &= \bar{f}(y \oplus x), \\
\bar{f}(\bar{x} \oplus y) &= \bar{f}(y \oplus \bar{x}), \\
\bar{f}(\bar{x} \bar{x}) &= \bar{f}(\bar{x} \bar{x}).
\end{align*}
\]  

(49)

2.4 Partial derivatives

On $q$-deformed quantum spaces partial derivatives with respect to the position coordinates can be introduced. These partial derivatives form a $q$-deformed quantum space, again. They commutate among each other in the same way as the position coordinates. Since there are two braiding mappings for two given quantum spaces, there are also two ways of commuting $q$-deformed partial derivatives with $q$-deformed coordinates. With the vector representation of the R-matrix, the following $q$-deformed Leibniz rules can be specified \[3, 31\]:

\[
\begin{align*}
\partial^i X^j &= g^{ij} + c (\hat{R}^{-1})^{ij}_{kl} X^k \partial^l, \\
\hat{\partial}^i X^j &= g^{ij} + c^{-1} \hat{R}^{ij}_{kl} X^k \hat{\partial}^l.
\end{align*}
\]  

(50)

For the three-dimensional $q$-deformed Euclidean space, the constant $c$ is 1.

By using the Leibniz rules above, we can calculate how the partial derivatives act on a normal ordered monomial of non-commutative coordinates. These actions can be carried over to commutative coordinate monomials with the help of the Moyal-Weyl mapping:

\[
\partial^i \triangleright (x^1)^{k_1} \ldots (x^N)^{k_N} = W^{-1} (\partial^i \triangleright (X^1)^{k_1} \ldots (X^N)^{k_N}).
\]  

(51)

Since the Moyal-Weyl mapping is linear, it is possible to extend the action above to functions that can be expanded as a power series:

\[
\partial^i \triangleright f = W^{-1} (\partial^i \triangleright W(f)).
\]  

(52)

This way we obtain operator representations for the $q$-deformed partial derivatives \[1\]. In the case of the covariant partial derivatives $\partial_A (= g_{AB} \partial^B)$ of the three-dimensional Euclidean quantum space, these representations are of the following form:\[\footnote{Note that the operator representations for the $q$-deformed partial derivatives depend on the choice of the normal ordering of the non-commutative coordinates.}]

\[
\begin{align*}
\partial_+ \triangleright f &= D_{q^1,x^+} f, \\
\partial_3 \triangleright f &= D_{q^2,x^3} f(q^2 x^+), \\
\partial_- \triangleright f &= D_{q^1,x^-} f(q^2 x^3) + \lambda x^+ D_{q^2,x^3} f.
\end{align*}
\]  

(53)

By applying the Leibniz rules of Eq. \[50\] repeatedly, we can commute $q$-deformed partial derivatives from the left side of a normal ordered monomial
of quantum space coordinates to the right side. This procedure leads us to left-representations of partial derivatives \( \partial^i \):

\[
\partial^i X^j = g^{ij} + c(\hat{R}^{-1})_{kl} X^k \partial^l \quad \Rightarrow \quad \partial^i \triangleright f,
\]

\[
\hat{\partial}^i X^j = g^{ij} + c^{-1} \hat{R}^{ij}_{kl} X^k \hat{\partial}^l \quad \Rightarrow \quad \hat{\partial}^i \triangleright f.
\]

(54)

We can also use the Leibniz rules to commute \( q \)-deformed partial derivatives from the right side of a normal ordered monomial to the left side. This way, we get right-representations of partial derivatives:

\[
X^i \partial^j = -g^{ij} + c(\hat{R}^{-1})_{kl} \partial^k X^l \quad \Rightarrow \quad \triangleright_{\partial^i} f,
\]

\[
X^i \hat{\partial}^j = -g^{ij} + c^{-1} \hat{R}^{ij}_{kl} \hat{\partial}^k X^l \quad \Rightarrow \quad \triangleright_{\hat{\partial}^i} f.
\]

(55)

The expressions for a given representation of \( q \)-deformed partial derivatives can be obtained from another representation by simple substitutions (see for example Ref. [1]). Moreover, the different actions of partial derivatives transform into each other by conjugation. In the case of the three-dimensional \( q \)-deformed Euclidean space, for example, the following applies [1]:

\[
\partial_A \triangleright f = \bar{f} \triangleright_{\partial_A}, \quad \bar{f} \triangleright_{\partial_A} = -\partial_A \triangleright f,
\]

\[
\hat{\partial}_A \triangleright f = \bar{f} \triangleright_{\hat{\partial}_A}, \quad f \triangleright_{\hat{\partial}_A} = -\hat{\partial}_A \triangleright \bar{f}.
\]

(56)

### 2.5 Integration

The operator representations of \( q \)-deformed partial derivatives consist of a term \( \partial_A^{\text{cla}} \) and a so-called correction term \( \partial_A^{\text{cor}} \):

\[
\partial^A \triangleright F = (\partial_A^{\text{cla}} + \partial_A^{\text{cor}}) \triangleright F.
\]

The term \( \partial_A^{\text{cla}} \) becomes an ordinary partial derivative in the undeformed limit \( q \to 1 \) and the term \( \partial_A^{\text{cor}} \) disappears in the undeformed limit. The difference equation \( \partial^A \triangleright F = f \) with given \( f \) is solved by the following expression:

\[
F = (\partial^A)^{-1} \triangleright f = (\partial_A^{\text{cla}} + \partial_A^{\text{cor}})^{-1} \triangleright f
\]

\[
= \sum_{k=0}^{\infty} [- (\partial_A^{\text{cla}})^{-1} \partial_A^{\text{cor}}]^{k} (\partial_A^{\text{cla}})^{-1} \triangleright f.
\]

(58)

In the case of the three-dimensional Euclidean quantum space, for example, the above formula leads to the expressions [28]

\[
(\partial_x) \triangleright f = D_{\frac{q^{-1}}{x}} f,
\]

\[
(\partial_y) \triangleright f = D_{\frac{q^{-1}}{x}} f(q^{-2} x^2),
\]

and

\[
\partial_+^{-1} \triangleright f = \sum_{k=0}^{\infty} q^{2k(k+1)} \left( -\lambda \frac{x^2}{D_{\frac{q^{-1}}{x}}} \right)^k D_{\frac{q^{-1}}{x}} f(q^{-2} x^3).
\]

(60)
Note that $D^{-1}_{q,x}$ stands for a Jackson integral with respect to the variable $x$. The explicit form of this Jackson integral depends on its integration limits and the value for the deformation parameter $q$. If $x > 0$ and $q > 1$, for example, the following applies:

\[
\int_0^x d_q z f(z) = (q - 1)x \sum_{j=1}^{\infty} q^{-j} f(q^{-j} x),
\]

\[
\int_x^{\infty} d_q z f(z) = (q - 1)x \sum_{j=0}^{\infty} q^j f(q^j x).
\]

(61)

By successively applying the $q$-integrals for the different coordinates, we can explain an integration over the entire coordinate space. With the exception of a normalization factor, this integration is not dependent on the sequence of the different integrations.[28][29]

\[
\int_{-\infty}^{+\infty} d_3 q x f(x^+ , x^3 , x^-) \sim (\partial_-)^{-1}\big|_{-\infty}^{+\infty} (\partial_3)^{-1}\big|_{-\infty}^{+\infty} (\partial_+)^{-1}\big|_{-\infty}^{+\infty} \vartriangleright f. \tag{62}
\]

We can also show that on the right side of the above relation the integrals to the different coordinates can be simplified to Jackson integrals.\[7\]

\[
\int_{-\infty}^{+\infty} d_3 q x f(x) \sim D^{-1}_{q^2,x^3,x^-} \big|_{-\infty}^{+\infty} D^{-1}_{q,x^3} \big|_{-\infty}^{+\infty} D^{-1}_{q^2,x^3} \big|_{-\infty}^{+\infty} f(x). \tag{63}
\]

Note that the Jackson integrals in the formula above refer to a smaller $q$-lattice. This is not a particular restriction since we can obtain each Jackson integral on the smaller $q$-lattice by adding two other Jackson integrals. These two Jackson integrals refer to two wider $q$-lattices shifted against each other. The scaling down of the $q$-lattice ensures that the $q$-integrals over the entire coordinate space form scalars with respect to the symmetry of the underlying quantum space.

The $q$-deformed integrals over the entire coordinate space show some important properties[29]. In this respect, $q$-deformed versions of Stokes’ theorem apply to the $q$-integration over the entire $q$-deformed Euclidean coordinate space:

\[
\int_{-\infty}^{+\infty} d_3 q x \partial^A \triangleright f = \int_{-\infty}^{+\infty} d_3 q x \tilde{\partial}^A = 0,
\]

\[
\int_{-\infty}^{+\infty} d_3 q x \tilde{\partial}^A \triangleright f = \int_{-\infty}^{+\infty} d_3 q x \partial^A = 0. \tag{64}
\]

This implies that $q$-integrals over the entire coordinate space are invariant with

\[7\]This simplification results from the fact that the integrated function must disappear at infinity.
respects to \( q \)-deformed translations:

\[
\int_{-\infty}^{+\infty} d^3_q x f(x) = \int_{-\infty}^{+\infty} d^3_q x f(y \oplus x) = \int_{-\infty}^{+\infty} d^3_q x f(y \otimes x)
\]

\[
= \int_{-\infty}^{+\infty} d^3_q x f(x \oplus y) = \int_{-\infty}^{+\infty} d^3_q x f(x \otimes y).
\]  

(65)

For later purposes, we provide a graphical representation\(^8\) of these identities in Fig. 1. The \( q \)-deformed Stokes’ theorem also implies the following rules for integration by parts:

\[
\int_{-\infty}^{+\infty} d^3_q x f \circ (\partial A \triangleleft g) = \int_{-\infty}^{+\infty} d^3_q x (f \circ \partial A) \triangleleft g,
\]

\[
\int_{-\infty}^{+\infty} d^3_q x f \circ (\hat{\partial} A \triangleleft g) = \int_{-\infty}^{+\infty} d^3_q x (f \circ \hat{\partial} A) \triangleleft g.
\]

(66)

Finally, it should be mentioned that the invariant integral for the \( q \)-deformed Euclidean quantum space behaves as follows under quantum space conjugation:

\[
\int_{-\infty}^{+\infty} d^3_q x f = \int_{-\infty}^{+\infty} d^3_q \bar{f}.
\]

(67)

2.6 Exponentials

The \( q \)-deformed exponentials are eigenfunctions of the \( q \)-deformed partial derivatives of a given quantum space \([20]\). In the following, we consider those \( q \)-deformed exponentials that are eigenfunctions of unconjugated left derivatives or conjugated right derivatives:

\[
i^{-1} \partial A \triangleright \exp_q (x | ip) = \exp_q (x | ip) \o p^A,
\]

\[
\exp_q (i^{-1} p | x) \circ \hat{\partial} A^{-1} = p^A \o \exp_q (i^{-1} p | x).
\]

(68)

---

*In App. A we have collected some information on the graphical calculus we use.*
For a better understanding, the above eigenvalue equations are shown graphically in Fig. 2. The $q$-exponentials are uniquely defined by their eigenvalue equations in connection with the following normalization conditions:

$$\exp_q(x|p)|_{x=0} = \exp_q(x|p)|_{p=0} = 1,$$

$$\exp_q(i^{-1}p|x)|_{x=0} = \exp_q(i^{-1}p|x)|_{p=0} = 1.$$  \hfill (69)

In order to get explicit formulas for the $q$-exponentials, we best consider the dual pairings between the coordinate algebra of the $q$-deformed position space and that of the corresponding $q$-deformed momentum space [20]:

$$\langle f(p), g(x) \rangle_{p, \bar{x}} = [f(i^{-1}\partial) \triangleright g(x)]_{x=0},$$

$$\langle f(x), g(p) \rangle_{x, \bar{p}} = [f(x) \triangleleft g(\partial i^{-1})]_{x=0}. \hfill (70)$$

Let $\{e^a\}$ be a basis of the $q$-deformed position space algebra and let $\{f_b\}$ be a basis of the corresponding $q$-deformed momentum space algebra. Furthermore, the elements of the two bases shall be dually paired in the following sense:

$$\langle \mathcal{W}_h(f_b), \mathcal{W}_x(e^a) \rangle_{p, \bar{x}} = \delta^a_b, \quad \langle \mathcal{W}_x(e^a), \mathcal{W}_p(f_b) \rangle_{x, \bar{p}} = \delta^a_b. \hfill (71)$$

Now, we are in a position to write the $q$-exponentials as canonical elements:

$$\exp_q(x|ip) \equiv \sum_a \mathcal{W}_x(e^a) \otimes \mathcal{W}_p(f_a),$$

$$\exp_q(ip|x) \equiv \sum_a \mathcal{W}_p(f_a) \otimes \mathcal{W}_x(e^a). \hfill (72)$$

The normal ordered monomials of non-commutative coordinates establish a basis of the quantum space under consideration. The elements of the dual basis can be obtained by the action of the partial derivatives on these normal ordered monomials. This way, we have found the following expressions for the $q$-exponentials:

\[^9\mathcal{W}_x\text{ and } \mathcal{W}_p\text{ denote the Moyal-Weyl mapping for the } q\text{-deformed position space algebra and that for the } q\text{-deformed momentum space algebra, respectively.}\]
In analogy to the undeformed case, the q-exponentials satisfy addition theorems of the following form [20, 29]:

\[
\exp_q(x|p) = \sum_{n=0}^{\infty} \frac{(q x^+)^n (x^3)^n (q^{-1} x^-)^n (i^{-1} p^+)^n (i p^3)^n (i^{-1} p^-)^n}{[[n_+]] q^n! [[n_3]] q^n! [[n_-]] q^n!},
\]

\[
\exp_q(i^{-1} p|x) = \sum_{n=0}^{\infty} \frac{(i p^+)^n (i^{-1} p^3)^n (i p^-)^n (q^{-1} x^+)^n (x^3)^n (q x^-)^n}{[[n_+]] q^n! [[n_3]] q^n! [[n_-]] q^n!}. \tag{73}
\]

In analogy to the undeformed case, the q-exponentials satisfy addition theorems of the following form [20, 29]:

\[
\exp_q(x \oplus y|ip) = \exp_q(x|\exp_q(y[ip] @ ip)),
\]

\[
\exp_q(ix|p \oplus \bar{p}) = \exp_q(x @ \exp_q(x|ip)|\bar{p}). \tag{74}
\]

Note that we can obtain further addition theorems from the above identities by substituting the position coordinates by momentum coordinates and vice versa. For a better understanding of the meaning of the two addition theorems in Eq. (74), we have depicted them graphically in Fig. 3.

We can also introduce inverse q-exponentials by using the q-inversions from Chap. 2.3:

\[
\exp_q(\bar{\otimes} x|ip) = \exp_q(ix|\bar{\otimes} p). \tag{75}
\]
Due to the addition theorems and the normalization conditions of the \( q \)-exponentials, the following applies:

\[
\exp_q \left( i x \otimes \exp_q (\bar{x} | i p) \otimes p \right) = \exp_q \left( x \bar{\otimes} (\bar{x} | i p) \right) = \exp_q (x | i p) \big|_{x=0} = 1. \quad (76)
\]

For a better understanding of these identities, we have given their graphic representation in Fig. 4.

Next, we will describe another way of obtaining \( q \)-exponentials. For this purpose, we exchange the two tensor factors of a \( q \)-exponential using the inverse universal R-matrix [see also Eq. (11) of Chap. 2.1 and the graphic representation in Fig. 5]:

\[
\begin{align*}
\exp_q^*(i p | x) &\equiv \tau \circ \left[(R_{[2]}^{-1} \otimes R_{[1]}^{-1}) \triangleright \exp_q(i x | \otimes p)\right] \\
&= \sum_a W_p \left(R_{[1]}^{-1} \triangleright f^a\right) \otimes W_x \left(R_{[2]}^{-1} \triangleright e_a\right), \quad (77)
\end{align*}
\]

\[
\begin{align*}
\exp_q^*(x | i p) &\equiv \tau \circ \left[(R_{[2]}^{-1} \otimes R_{[1]}^{-1}) \triangleright \exp_q(\otimes p | i x)\right] \\
&= \sum_a W_x \left(R_{[1]}^{-1} \triangleright e_a\right) \otimes W_p \left(R_{[2]}^{-1} \triangleright f^a\right). \quad (78)
\end{align*}
\]

Note that in the expressions above \( \tau \) denotes the ordinary twist operator. You can show that the new \( q \)-exponentials satisfy the following eigenvalue equations (also cf. Fig. 5):

\[
\begin{align*}
\exp_q^*(i p | x) &\triangleleft \partial^A = i p^A \otimes \exp_q^*(i p | x), \\
\partial^A \triangleright \exp_q^*(x | i^{-1} p) & = \exp_q^*(x | i^{-1} p) \otimes i p^A. \quad (79)
\end{align*}
\]

Finally, we write down how the \( q \)-exponentials of the three-dimensional quantum space behave under conjugation:

\[
\begin{align*}
\overline{\exp_q(x | i p)} &= \exp_q(i^{-1} p | x), \\
\overline{\exp_q^*(i p | x)} &= \exp_q^*(x | i^{-1} p). \quad (80)
\end{align*}
\]

The first identity can be derived directly from the expression in Eq. (73). The second identity then follows by taking the conjugation properties of the universal R-matrix into account.
3 Fourier transformations on quantum spaces

3.1 Definition

In Ref. [11], it was shown how to define Fourier transformations between two \(q\)-deformed quantum space algebras. Recall that the star product formalism allows us to identify \(q\)-deformed quantum space algebras with commutative coordinate algebras. Thus the different operations introduced for \(q\)-deformed quantum space algebras can be carried over to the corresponding commutative coordinate algebras. For this reason, we are able to implement the Fourier transformations for \(q\)-deformed quantum spaces on the corresponding commutative coordinate algebras. By using the star products from Chap. 2.2, the improper \(q\)-integrals from Chap. 2.5 and the \(q\)-exponentials from Chap. 2.6 we get:

\[
F_L(f)(p) = \int_{-\infty}^{+\infty} d_q^N x \cdot f(x) \otimes \exp_q(x|ip),
\]

\[
F_R(f)(p) = \int_{-\infty}^{+\infty} d_q^N x \exp_q(i^{-1}p|x) \otimes f(x).
\]

In analogy to the undeformed case, the \(q\)-deformed Fourier transformations of the function \(f(x) = 1\) result in \(q\)-deformed delta functions:

\[
\delta_N^L(p) = F_L(1)(p) = \int_{-\infty}^{+\infty} d_q^N x \exp_q(x|ip),
\]

\[
\delta_N^R(p) = F_R(1)(p) = \int_{-\infty}^{+\infty} d_q^N x \exp_q(i^{-1}p|x).
\]

Integration of the \(q\)-deformed delta functions, in turn, leads to the so-called \(q\)-deformed volume elements:

\[
\text{vol}_L = \int_{-\infty}^{+\infty} d_q^N p \delta_N^L(p) = \int_{-\infty}^{+\infty} d_q^N x \int_{-\infty}^{+\infty} d_q^N p \exp_q(x|ip),
\]

\[
\text{vol}_R = \int_{-\infty}^{+\infty} d_q^N p \delta_N^R(p) = \int_{-\infty}^{+\infty} d_q^N p \int_{-\infty}^{+\infty} d_q^N x \exp_q(i^{-1}p|x).
\]

As we can see from Eq. (73), the \(q\)-exponentials \(\exp_q(x|ip)\) and \(\exp_q(i^{-1}p|x)\) transform into each other by the substitutions \(x^A \rightarrow p^A\) and \(p^A \rightarrow -x^A\), i.e. the two \(q\)-exponentials differ from each other by an exchange of position coordinates and momentum coordinates. Therefore, the integration of both \(q\)-exponentials over the entire phase space must lead to the same result. For this reason, the two volume elements in Eq. (83) are identical:

\[
\text{vol} \equiv \text{vol}_L = \text{vol}_R.
\]

Up to now, we have considered Fourier transformations which map from the \(q\)-deformed position algebra into the \(q\)-deformed momentum algebra. If
we want to formulate Fourier transformations which map from the \(q\)-deformed momentum algebra into the \(q\)-deformed position algebra, we have to use the \(q\)-exponentials given in Eq. (77) and Eq. (78) of Chap. 2.6:

\[
\mathcal{F}_R^*(f)(x) = \int_{-\infty}^{+\infty} d_q^N p \exp_q^*(x|p^{-1}p) \otimes f(p), \\
\mathcal{F}_L^*(f)(x) = \int_{-\infty}^{+\infty} d_q^N f(p) \otimes \exp_q^*(ip|x).
\] (85)

From their defining expressions and the conjugation properties of \(q\)-integrals and \(q\)-exponentials, we can directly deduce that the \(q\)-deformed Fourier transformations of three-dimensional Euclidean quantum space behave as follows under conjugation:

\[
\overline{\mathcal{F}_L}(f) = \mathcal{F}_R(\bar{f}), \quad \overline{\mathcal{F}_L^*}(f) = \mathcal{F}_R^*(\bar{f}).
\] (86)

Correspondingly, we find for the volume element:

\[
\text{vol} = \text{vol}_L = \text{vol}_R = \text{vol}.
\] (87)

In addition to this, the \(q\)-deformed delta functions are subject to the following identities:

\[
\delta^N_L(p) = \mathcal{F}_L(1)(p) = \mathcal{F}_R(1)(p) = \delta^N_R(p).
\] (88)

### 3.2 Invertibility

The Fourier transformations on coordinate algebras of \(q\)-deformed quantum spaces can be inverted [11]. Specifically, we have the identities

\[
(\mathcal{F}_L \circ \mathcal{F}_L^*)(f(p)) = \text{vol}_L f(p), \\
(\mathcal{F}_R \circ \mathcal{F}_R^*)(f(p)) = \text{vol}_R f(p),
\] (89)

and

\[
(\mathcal{F}_L^* \circ \mathcal{F}_L)(f(x)) = \text{vol}_L f(x), \\
(\mathcal{F}_R^* \circ \mathcal{F}_R)(f(x)) = \text{vol}_R f(x).
\] (90)

In App. [11] we explain how to prove the above identities.

### 3.3 Transformation of actions and products

The Fourier transformations we have introduced in Eq. [81] show the characteristic property that they transform the action of a derivative operator into a product with a momentum coordinate [11]:

\[
\mathcal{F}_L(f(x) \triangleq \partial^k_x) = i \mathcal{F}_L(f(x)) \otimes p^k, \\
\mathcal{F}_R(\partial^k_x \triangleright f(x)) = ip^k \otimes \mathcal{F}_R(f(x)).
\] (91)
We can derive these identities in a direct way, as the following calculation helps to demonstrate:

\[
\mathcal{F}_L(f(x) \triangleleft \partial_x^k) = \int_{-\infty}^{+\infty} d_q^n x \ (f(x) \triangleleft \partial_x^k) \otimes \exp_q(x|ip) \\
= \int_{-\infty}^{+\infty} d_q^n x \ f(x) \otimes (\partial_x^k \triangleright \exp_q(x|ip)) \\
= i \int_{-\infty}^{+\infty} d_q^n x \ f(x) \otimes \exp_q(x|ip) \otimes p^k \\
= i \mathcal{F}_L(f(x)) \otimes p^k. \tag{92}
\]

The second identity follows from integration by parts [cf. Eq. (66) of Chap. 2.5] and the third identity is a consequence of the eigenvalue equations of the \(q\)-exponential [cf. Eq. (68) of Chap. 2.6].

In the other direction, the multiplication by coordinates is transformed into the action of partial derivatives:

\[
\mathcal{F}_L(f(x) \otimes x^k) = \mathcal{F}_L(f(x)) \bar{\triangleright} \partial_x^k i, \\
\mathcal{F}_R(x^k \otimes f(x)) = i \partial_x^k \triangleright \mathcal{F}_R(f(x)). \tag{93}
\]

These identities can be derived by calculations similar to those of Eq. (92):

\[
\mathcal{F}_L(f(x) \otimes x^k) = \int_{-\infty}^{+\infty} d_q^n x \ f(x) \otimes x^k \otimes \exp_q(x|ip) \\
= i \int_{-\infty}^{+\infty} d_q^n x \ f(x) \otimes \exp_q(x|ip) \bar{\triangleright} \partial_x^k \\
= i \mathcal{F}_L(f(x)) \bar{\triangleright} \partial_x^k. \tag{94}
\]

The \(q\)-deformed Fourier transformations given in Eq. (85) also transform actions of derivatives into products with coordinates and vice versa, i.e.

\[
\mathcal{F}_R^*(p^k \otimes f(p)) = i^{-1} \partial_p^k \triangleright \mathcal{F}_R^*(f(p)), \\
\mathcal{F}_L^*(f(p) \otimes p^k) = \mathcal{F}_L^*(f(p)) \bar{\triangleleft} \partial_p^{k-1}. \tag{95}
\]

and

\[
\mathcal{F}_R^*(i \partial_p^k \triangleright f(p)) = x^k \otimes \mathcal{F}_R^*(f(p)), \\
\mathcal{F}_L^*(f(p) \bar{\triangleleft} \partial_p^{k-1}) = \mathcal{F}_L^*(f(p)) \otimes x^k. \tag{96}
\]

We show that the identities of Eq. (95) can be derived from those of Eq. (91):

\[
\mathcal{F}_L(f(x) \triangleleft \partial_x^k) = i \mathcal{F}_L(f(x)) \otimes p^k \\
\Rightarrow \quad f(x) \triangleleft \partial_x^k = \text{vol}_L^{-1} \mathcal{F}_L(i \mathcal{F}_L(f(x)) \otimes p^k)(x) \\
\Rightarrow \quad \mathcal{F}_L^*(f(p)) \triangleleft \partial_p^k = \mathcal{F}_L^*(i f(p) \otimes p^k)(x). \tag{97}
\]
In the first step, the Fourier transformation $F^*_L$ was applied to the first identity of Eq. (91) and then we made use of Eq. (90). In the second step, $f$ was replaced by $F^*_L(f(p))$ and then Eq. (89) was used. Note that the identities in Eq. (96) can be derived in a similar way, as the following calculation shows:

$$F_L(f \oplus x^k) = iF_L(f) \bar{\otimes} \partial^k_p$$

$$\Rightarrow f \oplus x^k = \text{vol}^{-1}_L F_L(iF_L(f) \bar{\otimes} \partial^k_p)(x)$$

$$\Rightarrow F^*_L(f(p)) \oplus x^k = F^*_L(iF_L(f) \bar{\otimes} \partial^k_p)(x).$$ (98)

### 3.4 Fourier transforms of exponentials and delta functions

For some physical applications, the Fourier transforms of exponentials or delta functions are required. In the following we show that $q$-exponentials and $q$-delta functions are transformed into each other by $q$-deformed Fourier transforms.

We first calculate $q$-deformed Fourier transforms of ‘dual’ $q$-exponentials. With the help of graphical methods we get:

$$F_L(\exp^-_q(i|p|)x)(p) = \int^{+\infty}_{-\infty} d^N_q x \exp^*_q(i|p'|) \oplus \exp_q(x|i|p)$$

$$= \delta^N_L(\kappa^{-1}p') \oplus p),$$ (99)

$$F_R(\exp^*_q(x|i^{-1}p'))(p) = \int^{+\infty}_{-\infty} d^N_q x \exp_q(i^{-1}p|) \oplus \exp^*_q(x|i^{-1}p')$$

$$= \delta^N_R(p \oplus (\kappa^{-1}p')).$$ (100)

The graphical proof for Eq. (99) is shown in Fig. 6. We briefly explain the different steps of this graphical calculation. First, we pull the left strand past the integral as well as the $q$-exponential on the right side. By doing so, we take account of the identities shown in Fig. 10 of App. A as well as the trivial braiding properties of the $q$-exponentials. Now, we can apply the addition theorem for $q$-exponentials (cf. Fig. 3 of Chap. 2.6). We then identify the expression for the $q$-deformed delta function and rewrite the ‘conjugated’ co-product into its ‘unconjugated’ counterpart by taking into account the following identities [18, 23].

$$\Delta = \Psi^{-1} \circ \Delta, \quad \bar{\Delta} = \Psi \circ \Delta.$$ (101)

In our convention the mappings $\Psi$ and $\Psi^{-1}$ are exchanged in comparison with Ref. [18] or Ref. [23].
Similar considerations lead to the following identities:

\[ F_L^\ast (\exp_q(y|ip))(x) = \int_{-\infty}^{+\infty} d_q^N p \exp_q^\ast (ip|y) \exp_q(y|ip) \]
\[ = \delta^N_R (y \oplus (\ominus \kappa^{-1} x)), \quad (102) \]

\[ F_R^\ast (\exp_q(i^{-1}p|y))(x) = \int_{-\infty}^{+\infty} d_q^N p \exp_q^\ast (x|i^{-1}p) \exp_q(i^{-1}p|y) \]
\[ = \delta^N_L ((\ominus \kappa^{-1} x) \oplus y). \quad (103) \]

From the identities of Eq. (99) and Eq. (100) one can read off orthonormality relations for the \(q\)-exponentials:

\[ \delta^N_L ((\ominus \kappa^{-1} x) \oplus y) = \int_{-\infty}^{+\infty} d_q^N \exp_q^\ast (x|i^{-1}p) \exp_q(i^{-1}p|y) \]
\[ = \text{vol}_L \exp_q(y|ip), \quad (104) \]

The identities of Eq. (102) and Eq. (103) can be interpreted as completeness relations for the \(q\)-exponentials:

\[ \delta^N_R (y \oplus (\ominus \kappa^{-1} x)) = \int_{-\infty}^{+\infty} d_q^N p \exp_q(y|ip) \exp_q^\ast (ip|x) \]
\[ = \text{vol}_R \exp_q(i^{-1}p|y), \quad (105) \]

Next, we calculate Fourier transforms of \(q\)-delta functions. To this end, we apply the Fourier transformation \(F_L^\ast\) or \(F_R^\ast\) to the identity of Eq. (102) or Eq. (103) and take into account Eq. (89). Thus, we obtain:

\[ F_L^\ast (\delta^N_R (y \oplus (\ominus \kappa^{-1} x)))(p) = \int_{-\infty}^{+\infty} d_q^N x \exp_q(x|ip) \delta^N_R (y \oplus (\ominus \kappa^{-1} x)) \exp_q(y|ip) \]
\[ = \text{vol}_L \exp_q(y|ip), \quad (106) \]

\[ F_R^\ast (\delta^N_L ((\ominus \kappa^{-1} x) \oplus y))(p) = \int_{-\infty}^{+\infty} d_q^N x \exp_q(i^{-1}p|y) \delta^N_L ((\ominus \kappa^{-1} x) \oplus y) \exp_q(x|ip) \]
\[ = \text{vol}_R \exp_q(i^{-1}p|y). \quad (107) \]

Similarly, we can apply the Fourier transformation \(F_L^\ast\) or \(F_R^\ast\) to Eq. (99) or
Figure 6: Graphical proof of Eq. (99)

Eq. (100) while taking into account Eq. (90):

\[
F^* \langle \delta N_L ((\ominus \kappa^{-1} p') \oplus p) \rangle (x) = \int_{-\infty}^{+\infty} \mathrm{d}N_q p \exp_q^\ast (i p | x) \delta N_L ((\ominus \kappa^{-1} p') \oplus p) \oplus \exp_q^\ast (i p | x) = \text{vol}_L \exp_q^\ast (i p | x),
\]

(108)

\[
F^* \langle \delta N_R (p \oplus (\ominus \kappa^{-1} p')) \rangle (x) = \int_{-\infty}^{+\infty} \mathrm{d}N_q p \exp_q^\ast (x | i^{-1} p) \oplus \delta N_L (p \oplus (\ominus \kappa^{-1} p')) = \text{vol}_R \exp_q^\ast (x | i^{-1} p').
\]

(109)

3.5 Characteristic identities of delta functions

The \( q \)-delta functions satisfy some useful identities. In this respect, we have \( (G \in \{ L, R \}) \):

\[
f(x) = \text{vol}_G^{-1} \int_{-\infty}^{+\infty} \mathrm{d}N_q y \delta_G^N ((\ominus \kappa^{-1} x) \oplus y) \oplus f(y) = \text{vol}_G^{-1} \int_{-\infty}^{+\infty} \mathrm{d}N_q y f(y) \oplus \delta_G^N (y \oplus (\ominus \kappa^{-1} x)).
\]

(110)

Similarly, it holds:

\[
f(x) = \text{vol}_G^{-1} \int_{-\infty}^{+\infty} \mathrm{d}N_q y f(y) \oplus \delta_G^N ((\ominus \kappa^{-1} y) \oplus x) = \text{vol}_G^{-1} \int_{-\infty}^{+\infty} \mathrm{d}N_q y \delta_G^N (x \oplus (\ominus \kappa^{-1} y)) \oplus f(y).
\]

(111)

The following calculation serves as an example to show how the above identities can be proven. For this calculation, we use the completeness relations in
Eq. (105) and the invertibility of the $q$-deformed Fourier transformations:

\[
\text{vol}_R^{-1} \int_{-\infty}^{+\infty} d_q^N y \delta_G^N ((\ominus \kappa^{-1} x) \oplus y) \circ f(y) = \\
= \text{vol}_R^{-1} \int_{-\infty}^{+\infty} d_q^N y \int_{-\infty}^{+\infty} d_q^N p \exp_q^* (x|\bar{i}^{-1}p) \circ \exp_q (\bar{i}^{-1}p|y) \circ f(y) \\
= \text{vol}_R^{-1} \int_{-\infty}^{+\infty} d_q^N p \exp_q^* (x|\bar{i}^{-1}p) \circ \mathcal{F}_R(f)(p) \\
= \text{vol}_R^{-1} \mathcal{F}_R^*(\mathcal{F}_R(f))(x) = f. 
\]

(112)

We show now that some versions of the $q$-deformed delta function are identical. To this end, we replace the function $f$ in the identities of Eq. (110) and Eq. (111) by a delta function ($G \in \{L, R\}$):

\[
\int_{-\infty}^{+\infty} d_q^N z \delta_G^N ((\ominus \kappa^{-1} x) \oplus z) \circ \delta_G^N (z \oplus (\ominus \kappa^{-1} y)) = \\
= \text{vol}_G \delta_G^N (\ominus \kappa^{-1} x) \oplus y) = \text{vol}_G \delta_G^N (x \oplus (\ominus \kappa^{-1} y)). 
\]

(113)

This results in the following relation for $q$-deformed delta functions:

\[
\delta_G^N (\ominus \kappa^{-1} x) \oplus y) = \delta_G^N (x \oplus (\ominus \kappa^{-1} y)). 
\]

(114)

Similar reasonings using Eq. (84) lead to:

\[
\delta_G^N (\ominus \kappa^{-1} x) \oplus y) = \delta_L^N (x \oplus (\ominus \kappa^{-1} y)). 
\]

(115)

Using the above result in combination with Eq. (114) leads to:

\[
\delta_R^N (x \oplus y) = \delta_L^N (x \oplus y). 
\]

(116)

If the $y$-coordinates in the last two identities are set equal to zero, we can see the following identities:

\[
\delta_q^N (x) \equiv \delta_R^N (x) = \delta_L^N (x). 
\]

(117)

Let us mention that the identities in Eqns. (113)–(116) remain valid if we carry out the following substitutions:

\[
(\oplus, \ominus) \leftrightarrow (\bar{\oplus}, \bar{\ominus}), \quad \kappa \leftrightarrow \kappa^{-1}. 
\]

(118)

Finally, we give the conjugation properties of the $q$-deformed delta functions. Eq. (49) of Chap. 2.3 and Eq. (88) lead to the following result:

\[
\delta_L^N (\ominus \kappa^{-1} x) \oplus y) = \delta_R^N (y \ominus (\ominus \kappa^{-1} x)). 
\]

(119)
3.6 Fourier-Plancherel identities

Certain $q$-analogs of the so-called Fourier-Plancherel identities apply to the $q$-deformed Fourier transformations. These identities read as

\[
\int d_q^N x f(x) \circledast g(x) = \text{vol}^{-1} \int d_q^N p \mathcal{F}_L^*(f)(p) \circledast \mathcal{F}_R(g)(p) \tag{120}
\]

and

\[
\int d_q^N x f(x) \circledast g(x) = \text{vol}^{-1} \int d_q^N p \mathcal{F}_L(f)(p) \circledast \mathcal{F}_R^*(g)(p). \tag{121}
\]

In the following, we show how to prove the first identity (similar considerations hold for the second identity):

\[
\int d_q^N p \mathcal{F}_L^*(f)(p) \circledast \mathcal{F}_R(g)(p) = \\
= \int d_q^N x \int d_q^N y f(x) \circledast \int d_q^N p \exp_q^*(x|i^{-1}p) \circledast \exp_q(i^{-1}p|y) \circledast g(y) \\\n= \int d_q^N x f(x) \circledast \int d_q^N y \delta_q^N((i\kappa^{-1}x) \oplus y) \circledast g(y) \\\n= \text{vol} \int d_q^N x f(x) \circledast g(x). \tag{122}
\]

In the first step of the above calculation, we used the expressions for the Fourier transformations $\mathcal{F}_L^*$ and $\mathcal{F}_R$ [see Eq. (81) and Eq. (85)]. In the second step, we have written the momentum integral as a $q$-deformed delta function [see Eq. (105)]. Using the characteristic identities of the $q$-deformed delta functions [cf. Eq. (110)] is our final step.

4 Matrix representations on position and momentum space

4.1 Eigenfunctions of the momentum or position operator

The partial derivatives of a quantum space determine a momentum operator in analogy to the undeformed case. However, there are several ways for partial derivatives to act on a quantum space [cf. Eq. (54) and Eq. (55) of Chap. 2.4], and for this reason, we also have different $q$-deformed momentum operators:

\[
P^k \triangleright f(x) = i^{-1} \partial^k \triangleright f(x), \quad f(x) \triangleright P^k = f(x) \triangleright \partial^k i^{-1}, \\\nP^k \lhd f(x) = i^{-1} \hat{\partial}^k \lhd f(x), \quad f(x) \lhd P^k = f(x) \lhd \hat{\partial}^k i^{-1}. \tag{123}
\]

The momentum eigenfunctions describe states with defined momentum, i.e. plane waves. So they must be eigenfunctions of the partial derivatives of the $q$-deformed momentum operators.
quantum space in question. According to the different actions of partial derivatives, you have the following ways to formulate eigenvalue equations for momentum eigenfunctions:

\[ i^{-1} \partial^k \triangleright u_p(x) = u_p(x) \odot p^k, \quad u^p(x) \odot \partial^k i^{-1} = p^k \odot u^p(x), \]

\[ i^{-1} \partial^k \triangleleft \bar{u}_p(x) = \bar{u}_p(x) \odot p^k, \quad \bar{u}^p(x) \odot \partial^k i^{-1} = p^k \odot \bar{u}^p(x). \]  

(124)

We get the results for the eigenfunctions in the second line of Eq. (124) from those of the eigenfunctions in the first line by simple substitutions. For this reason, we do not consider the eigenfunctions of the second line in the following.

Remember that the quantum space exponentials of Chap. 2.6 are eigenfunctions of partial derivatives of a given quantum space. Consequently, the \( q \)-deformed momentum eigenfunctions take on the following form:

\[ u_p(x) = \text{vol}^{-1/2} \exp_q(x|ip), \quad u^p(x) = \text{vol}^{-1/2} \exp_q(i^{-1}p|x). \]  

(125)

We can also introduce ‘dual’ momentum eigenfunctions [cf. Eq. (124) and Eq. (125)] of Chap. 2.6:

\[ (u^*)_p(x) = \text{vol}^{-1/2} \exp^*_q(ip|x), \quad (u^*)^p(x) = \text{vol}^{-1/2} \exp^*_q(x|i^{-1}p). \]  

(126)

These ‘dual’ momentum eigenfunctions satisfy the following eigenvalue equations [cf. also Eq. (129) of Chap. 2.6]:

\[ (u^*)_p(x) \odot \partial^k i^{-1} = p^k \odot (u^*)_p(x), \]

\[ i^{-1} \partial^k \triangleleft (u^*)^p(x) = (u^*)^p(x) \odot p^k. \]  

(127)

The components of the position operator act on the functions of the position space as multiplication operators:

\[ X^k \triangleright f(x) = x^k \odot f(x), \quad f(x) \triangleleft X^k = f(x) \odot x^k. \]  

(128)

The corresponding position eigenfunctions are determined again by eigenvalue equations:

\[ X^k \triangleright u_y(x) = x^k \odot u_y(x) = u_y(x) \odot y^k, \]

\[ (u^*)_y(x) \triangleleft X^k = (u^*)_y(x) \odot x^k = y^k \odot (u^*)_y(x). \]  

(129)

The \( q \)-deformed delta functions satisfy the eigenvalue equations above:

\[ u_y(x) = \text{vol}^{-1} \delta^N_q(x \oplus (\ominus \kappa^{-1} y)) = \text{vol}^{-1} \delta^N_q((\ominus \kappa^{-1} x) \oplus y) = (u^*)_y(x), \]

\[ (u^*)_y(x) = \text{vol}^{-1} \delta^N_q(y \oplus (\ominus \kappa^{-1} x)) = \text{vol}^{-1} \delta^N_q((\ominus \kappa^{-1} y) \oplus x) = u^y(x). \]  

(130)

To prove this, we calculate e. g.

\[ \int d^N_q x (u^*)_y(x) \odot x^k \odot f(x) = \int d^N_q x \text{vol}^{-1} \delta^N_q((\oplus \kappa^{-1} y) \oplus x) \odot x^k \odot f(x) = y^k \odot f(y) \]  

(131)
and
\[ y^k \otimes f(y) = \text{vol}^{-1} y^k \otimes \int d_q^N x \delta_q^N ((\ominus \kappa^{-1} y) \oplus x) \otimes f(x) \]
\[ = \int d_q^N x y^k \otimes (u^*)(y)(x) \otimes f(x). \] (132)

Note that in both calculations, we have made use of the characteristic identities of the delta functions [cf. Eq. (110) and Eq. (111) of Chap. 3.5]. Comparing the results of Eq. (131) and Eq. (132) shows:
\[ \int d_q^N x ((u^*)(y)(x) \otimes x^k - y^k \otimes (u^*)(y)(x)) \otimes f(x) = 0. \] (133)

If we substitute \( f(x) \) by a \( q \)-deformed delta function and take the characteristic identities of the delta functions into account again, we finally get the eigenvalue equation for \((u^*)(y)\):
\[ 0 = \int d_q^N x' ((u^*)(y)(x') \otimes x'^k - y^k \otimes (u^*)(y)(x')) \otimes \delta_q^N (x' + (\ominus \kappa^{-1} x)) \text{vol}^{-1} = (u^*)(y)(x) \otimes x^k - y^k \otimes (u^*)(y)(x). \] (134)

For the sake of completeness, we must not forget that the position eigenfunctions satisfy the following identities:
\[ \partial^k_x \triangleright u_y(x) = u_y(x) \triangleleft \partial^k_y, \]
\[ (u^*)(y)(x) \triangleright \partial^k_x = \partial^k_y \triangleright (u^*)(y)(x). \] (135)

The following calculation serves as an example for proving the above identities:
\[ \partial^k_x \triangleright \delta_q^N (x \oplus (\ominus \kappa^{-1} y)) = \int_{-\infty}^{+\infty} d_q^N p \partial^k_x \triangleright \exp_q(x|ip) \otimes \exp_q^*(|ip|y) = \]
\[ = i \int_{-\infty}^{+\infty} d_q^N p \exp_q(x|ip) \otimes p^k \otimes \exp_q^*(|ip|y) = \]
\[ = \int_{-\infty}^{+\infty} d_q^N p \exp_q(x|ip) \otimes \exp_q^*(|ip|y) \triangleleft \partial^k_y = \]
\[ = \delta_q^N (x \oplus (\ominus \kappa^{-1} y)) \triangleleft \partial^k_y. \] (136)

The first step uses the completeness relations for \( q \)-deformed exponentials [cf. Eq. (105) of Chap. 3.4]. The second step follows from the eigenvalue equations of the \( q \)-exponentials [cf. Eq. (68) and Eq. (79) of Chap. 2.6]. The last step again is a consequence of the completeness relations for \( q \)-deformed exponentials.

The momentum eigenfunctions in Eq. (125) refer to a representation of the momentum operator in position space. Correspondingly, the position eigenfunctions in Eq. (130) refer to a representation of the position operator in position space. To a large extent, position space and momentum space can be treated
in the same way. For this reason, we can replace the momentum variables by position variables and vice versa in Eq. (125) and Eq. (130). This way, we obtain position or momentum eigenfunctions referring to a representation of the position or momentum operator in momentum space.

We know that \( q \)-exponential functions and \( q \)-delta functions transform into each other by \( q \)-deformed Fourier transformations. This correspondence is carried over to the momentum and position eigenfunctions. Thus, we have [cf. Eq. (100) and Eq. (102) of Chap. 3.4]

\[
\text{vol}^{-1/2} \mathcal{F}_R((u^*)_p(y))(x) = u_y(x)
\]

and

\[
\text{vol}^{-1/2} \mathcal{F}_L^*(u_p(y))(x) = (u^*)_y(x).
\]

Finally, we write down how momentum and position eigenfunctions behave under conjugation. Using Eq. (80) from Chap. 2.6 and Eq. (87) from Chap. 3.1 follows for the momentum eigenfunctions:

\[
\overline{u_p(x)} = u^p(x), \quad (u^*)_p(x) = (u^*)^p(x).
\]

Due to Eq. (119) of Chap. 3.5 the position eigenfunctions are subject to:

\[
\overline{u_y(x)} = u^y(x), \quad (u^*)_y(x) = (u^*)^y(x).
\]

### 4.2 Completeness of momentum eigenfunctions

In this section, we show that the momentum eigenfunctions constitute a complete system of functions. To this end, we remember that for a given wave function different expansions in terms of \( q \)-deformed momentum eigenfunctions exist. These expansions follow from the definitions of the \( q \)-deformed Fourier transformations [cf. Eq. (81) and Eq. (85) of Chap. 3.1] if we write the \( q \)-exponentials as momentum eigenfunctions, i.e.

\[
\psi_R(x) = \int \text{d}_q^N p u_p(x) \otimes c_p,
\]

\[
\psi_L(x) = \int \text{d}_q^N p c^p \otimes u^p(x)
\]

and

\[
\psi^*_R(x) = \int \text{d}_q^N p (u^*)^p(x) \otimes (c^*)^p,
\]

\[
\psi^*_L(x) = \int \text{d}_q^N p (c^*)^p \otimes (u^*)^p(x).
\]

It remains to determine the expansion coefficients. We start with the coefficients \( c_p \) for the wave function \( \psi_R \). Using the last identity in Eq. (80) of
Chap. 3.2 we can write $\psi_R(x)$ as follows:
\[ \psi_R(x) = \text{vol}^{-1} \mathcal{F}_R(\mathcal{F}_R^*(\psi_R))(x) \]
\[ = \text{vol}^{-1/2} \int d^N_q p u_p(x) \otimes \mathcal{F}_R^*(\psi_R)(p). \] (143)

If we compare the last expression in Eq. (143) with the corresponding expansion in Eq. (141), we read off:
\[ c_p = \text{vol}^{-1/2} \mathcal{F}_R^*(\psi_R)(p) = \int d^N_q x (u^*)_p(x) \otimes \psi_R(x). \] (144)

In the same way, we can show:
\[ c_p = \text{vol}^{-1/2} \mathcal{F}_R^*(\psi_R^*)(p) = \int d^N_q x (u^*)_p(x) \otimes \psi_R^*(x). \] (145)

Similar reasonings enable us to determine the coefficients $(c^*)_p$ for the wave function $\psi_L^*$. Using the first identity in Eq. (90) of Chap. 3.2 follows:
\[ \psi_L^*(x) = \text{vol}^{-1} \mathcal{F}_L^*(\mathcal{F}_L(\psi_L^*))(x) \]
\[ = \text{vol}^{-1/2} \int d^N_q p \mathcal{F}_L(\psi_L^*)(p) \otimes (u^*)_p(x). \] (146)

From this result, we read off:
\[ (c^*)_p = \text{vol}^{-1/2} \mathcal{F}_L(\psi_L^*)(p) = \int d^N_q x \psi_L^*(x) \otimes u_p(x). \] (147)

Similarly, we get:
\[ (c^*)_p = \text{vol}^{-1/2} \mathcal{F}_R(\psi_R^*)(p) = \int d^N_q x u_p^*(x) \otimes \psi_R^*(x). \] (148)

Now, we can write down completeness relations for $q$-deformed momentum eigenfunctions. To this end, we perform the following calculation [also see Eq. (143)]:
\[ \psi_R(x) = \text{vol}^{-1} \mathcal{F}_R(\mathcal{F}_R^*(\psi_R))(x) \]
\[ = \int d^N_q p u_p(x) \otimes \int d^N_q y (u^*)_p(y) \otimes \psi_R(y) \]
\[ = \int d^N_q y \left( \int d^N_q p u_p(x) \otimes (u^*)_p(y) \right) \otimes \psi_R(y). \] (149)

If you compare the result of the above calculation with the characteristic identities of the $q$-delta functions [cf. Eq. (111) of Chap. 3.5], you can see:
\[ \int d^N_q p u_p(x) \otimes (u^*)_p(y) = \text{vol}^{-1} \delta^N_q (x \oplus (\ominus \kappa^{-1} y)). \] (150)

Similar considerations lead to:
\[ \int d^N_q p (u^*)_p(y) \otimes u^p(x) = \text{vol}^{-1} \delta^N_q ((\ominus \kappa^{-1} y) \oplus x). \] (151)
4.3 Completeness of position eigenfunctions

We can develop each wave function in position space in terms of the position eigenfunctions. We have introduced them in Eq. (130) of Chap. 4.1. In this respect the following expansions apply:

\[
\psi_R(x) = \int \mathcal{d}^N_q y u_y(x) \odot c_y = \int \mathcal{d}^N_q y (u^*)_y(x) \odot (c^*)_y = \psi_R^*(x),
\]

\[
\psi_L(x) = \int \mathcal{d}^N_q y c_y \odot u_y(x) = \int \mathcal{d}^N_q y (c^*)_y \odot (u^*)_y(x) = \psi_L^*(x).
\] (152)

The above expansions are a direct consequence of the characteristic identities of the \(q\)-delta functions [cf. Eq. (110) and Eq. (111) of Chap. 3.5]. They also result from the fact that the position eigenfunctions are nothing else but \(q\)-delta functions.

The expansion coefficients in Eq. (152) are identical to the corresponding wave functions. To see this, let us take another look at the characteristic identities of the \(q\)-delta functions:

\[
\psi_R(x) = \text{vol}^{-1} \int \mathcal{d}^N_q y \delta_q^N((\ominus \kappa^{-1} x) \oplus y) \odot \psi_R(y),
\]

\[
\psi_L(x) = \text{vol}^{-1} \int \mathcal{d}^N_q y \psi_L(y) \odot \delta_q^N(y \oplus (\ominus \kappa^{-1} x)).
\] (153)

If you compare these identities with the expansions in Eq. (152) and take into account the expressions for the position eigenfunctions in Eq. (130) then you will recognize:

\[
c_y = \psi_R(y) = \psi_R^*(y) = (c^*)_y,
\]

\[
c^y = \psi_L(y) = \psi_L^*(y) = (c^*)_y.
\] (154)

Note that due to the above results, we can write the expansion coefficients as follows:

\[
c_y = \int \mathcal{d}^N_q x (u^*)_y(x) \odot \psi_R(x) = \int \mathcal{d}^N_q x u_y(x) \odot \psi_R^*(x) = (c^*)_y,
\]

\[
c^y = \int \mathcal{d}^N_q x \psi_L(x) \odot (u^*)_y(x) = \int \mathcal{d}^N_q x \psi_L^*(x) \odot u_y(x) = (c^*)_y.
\] (155)

The position eigenfunctions fulfill certain completeness relations, which we obtain by the following calculation:

\[
\psi_L(x) = \int \mathcal{d}^N_q y c^y \odot u_y(x) = \int \mathcal{d}^N_q y \int \mathcal{d}^N_q x' \psi_L(x') \odot (u^*)_y(x') \odot u_y(x) = \text{vol}^{-2} \int \mathcal{d}^N_q y \int \mathcal{d}^N_q x' \psi_L(x') \odot \delta_q^N(x') \odot (\ominus \kappa^{-1} y) \odot \delta_q^N(y \oplus (\ominus \kappa^{-1} x))
\]

\[
= \text{vol}^{-1} \int \mathcal{d}^N_q x' \psi_L(x') \odot \delta_q^N(x' \ominus (\ominus \kappa^{-1} x)).
\] (156)
If in the above calculation, we compare the third with the last expression, we find the wanted relations:

\[
\text{vol}^{-1} \delta^N_q (x' \oplus (\ominus \kappa^{-1} x)) = \int d^N_q y (u^*)_y (x') \oplus u^y (x) \\
= \int d^N_q y u_y (x') \oplus (u^*)_y (x).
\] (157)

4.4 Orthogonality of momentum or position eigenfunctions

We know from the last section that we can expand a wave function in terms of \(q\)-deformed momentum eigenfunctions. For this reason, the \(q\)-deformed momentum eigenfunctions fulfill certain completeness relations [cf. Eq. (150) and Eq. (151) of Chap. 4.2]. From an algebraic point of view, the \(q\)-deformed position and momentum coordinates behave in the same way. Therefore, we can replace the momentum coordinates in the completeness relations of the \(q\)-deformed momentum eigenfunctions by position coordinates and vice versa. In doing so, we can obtain orthonormality relations for the \(q\)-deformed momentum eigenfunctions from their completeness relations. We demonstrate this procedure with an example. We begin our considerations with the following completeness relation [cf. Eq. (150) of Chap. 4.2]:

\[
\int d^N_q y u_y (x) \oplus (u^*)_y (x) = \text{vol}^{-1} \int d^N_q y \exp_q (x|\i x) \oplus \exp^*_q (\i x|y) \\
= \text{vol}^{-1} \delta^N_q (x \oplus (\ominus \kappa^{-1} y)).
\] (158)

The exchange of position coordinates and momentum coordinates leads to the wanted orthonormality relation:

\[
\int d^N_q x u^p (x) \oplus (u^*)_p (x) = \text{vol}^{-1} \int d^N_q x \exp_q (\i^{-1} p|x) \oplus \exp^*_q (x|\i^{-1} p') \\
= \text{vol}^{-1} \delta^N_q (p \oplus (\ominus \kappa^{-1} p')).
\] (159)

In the same way, we obtain the following orthonormality relation:

\[
\int d^N_q x (u^*)_p (x) \oplus u_{p'} (x) = \text{vol}^{-1} \int d^N_q x \exp_q (ip|x) \oplus \exp^*_q (x|ip') \\
= \text{vol}^{-1} \delta^N_q ((\ominus \kappa^{-1} p) \oplus p').
\] (160)

Using a similar procedure, we find orthonormality relations for the \(q\)-deformed position eigenfunctions. For this purpose, we consider the completeness relations for the position eigenfunctions [cf. Eq. (157) from the last chapter]:

\[
\int d^N_q y u_y (x) \oplus (u^*)_y (x') = \int d^N_q y (u^*)_y (x) \oplus u^y (x') \\
= \text{vol}^{-2} \int d^N_q y \delta^N_q (x \oplus (\ominus \kappa^{-1} y)) \oplus \delta^N_q ((\ominus \kappa^{-1} y) \oplus x') \\
= \text{vol}^{-1} \delta^N_q (x \oplus (\ominus \kappa^{-1} x')).
\] (161)
If we exchange the $x$- and $y$-coordinates, we get the following result:

\[
\int d_q^N x' u^y(x) \otimes (u^*)' (x) = \int d_q^N x (u^*) y(x) \otimes u_{y'}(x) = \text{vol}^{-2} \int d_q^N x \delta_q^N (y \oplus (\ominus \kappa^{-1} x)) \otimes \delta_q^N ((\ominus \kappa^{-1} x) \oplus y') = \text{vol}^{-1} \delta_q^N (y \oplus (\ominus \kappa^{-1} y')).
\] (162)

### 4.5 Matrix representations on momentum space

The expansion coefficients in Eq. (144) or Eq. (145) of Chap. 4.2 form a vector of infinite dimension, with the momentum variable playing the role of an index to distinguish the components of the vector. From this point of view, the expansion coefficients under consideration can describe a physical state with respect to a basis of $q$-deformed momentum eigenfunctions. The same applies to the expansion coefficients in Eq. (147) or Eq. (148) of Chap. 4.2.

We now continue these considerations by determining matrix representations of the momentum or position operator with respect to a basis of $q$-deformed momentum eigenfunctions. There are different versions of these matrix representations since there are also different expansion coefficients for the same wave function. One way to represent the momentum operator in a momentum basis is [also see Eq. (96) of Chap. 3.3]:

\[
(P^A)_{p'p} = \text{vol}^{-1/2} \mathcal{F}_R^* (i^{-1} \partial_x^A \triangleright u_p)(p') = \int d^3 x (u^*)_{p'}(x) \otimes [i^{-1} \partial_x^A \triangleright u_p(x)] = \int d^3 x (u^*)_{p'}(x) \oplus u_p(x) \oplus p^A = \text{vol}^{-1} \delta_q^3 ((\ominus \kappa^{-1} p') \oplus p) \oplus p^A.
\] (163)

In the penultimate step of the above calculation, we have used the eigenvalue equation of the momentum operator. The last identity follows from the orthonormality relation in Eq. (160) of the previous chapter. A similar proceeding provides us with another way of calculating matrix elements of the $q$-deformed momentum operator:

\[
(P^A)^{pp'} = \text{vol}^{-1/2} \mathcal{F}_L^* (u^p \triangleright \partial_x^{A_1^{-1}})(p') = \int d^3 x [u^p(x) \triangleright \partial_x^{A_1^{-1}}] \oplus (u^*)^{p'}(x) = \int d^3 x p^A \oplus u^p(x) \oplus (u^*)^{p'}(x) = \text{vol}^{-1} p^A \oplus \delta_q^3 (p \oplus (\ominus \kappa^{-1} p')).
\] (164)

Next, we show that the action of the $q$-deformed momentum operator in momentum space is nothing else but multiplication by a momentum variable.
To this end, we replace the $q$-deformed plane wave $u_p(x)$ in the second expression of Eq. (163) by a general wave function. In doing so, we obtain in analogy to the result from Eq. (96) of Chap. 3.3:

\[
(P^A \triangleright \psi_R)_p = \text{vol}^{-1/2}\mathcal{F}_R^*(i^{-1}\partial_x^A \triangleright \psi_R)(p) = \text{vol}^{-1/2} p^A \otimes \mathcal{F}_R^*(\psi)(p) = p^A \otimes c_p.
\] (165)

In the last step of the above calculation, we could identify the expansion coefficient $c_p$ by its expression in Eq. (144) of Chap. 4.2. Due to the characteristic identities of the $q$-deformed delta functions, we can also write the above result as a kind of matrix multiplication [cf. Eq. (110) and Eq. (111) of Chap. 3.5]:

\[
p^A \otimes c_p = \text{vol}^{-1} \int d^3q' \delta_q^3(\ominus \kappa^{-1}p) \otimes p^A \otimes c_{p'}.
\] (166)

Note that in the last step, we were able to identify the expression for the momentum matrix element given in Eq. (163). Finally, we summarize the results in Eq. (165) and Eq. (166) as follows:

\[
(P^A \triangleright \psi_R)_p = \int d^3q' (P^A)_{pp'} \otimes c_{p'} = p^A \otimes c_p.
\] (167)

In the same way, one can show:

\[
(\psi_L \triangleleft P^A)_p = \text{vol}^{-1} \mathcal{F}_L^*(\psi_L \triangleleft \partial_x^A \ominus 1)(p) = \int d^3q' c_{p'} \otimes (P^A)_{p'} \otimes c_p = c_p \otimes p^A.
\] (168)

If two momentum operators act successively on a wave function, their representation matrices in the momentum space are multiplied with each other. We can read it off from repeated application of the first identity of Eq. (167):

\[
(P^A P^B \triangleright \psi_R)_p = (P^A \triangleright (P^B \triangleright \psi_R))_p = \int d^3q' (P^A)_{pp'} \otimes (P^B \triangleright \psi_R)_{p'} = \int d^3q' \int d^3q'' (P^A)_{pp'} \otimes (P^B)_{p'p''} \otimes c_{p''}.
\] (169)

Similarly, we have:

\[
(\psi_L \triangleleft P^A P^B)_p = \int d^3q' \int d^3q'' c_{p''} \otimes (P^A)_{p'} \otimes (P^B)_{p'p}.
\] (170)

Next, we calculate matrix representations of the position operator with respect to a basis of $q$-deformed momentum eigenfunctions. We do this in a similar
way as in Eq. (163):\[
\begin{align*}
(X^A)_p^{p'} &= \text{vol}^{-1/2} F_R^*(x^A \otimes u_p)(p') \\
&= \int d_3^q x (u^*) \otimes [x^A \otimes u_p(x)] \\
&= \int d_3^q x (u^*) \otimes u_p(x) \partial_i^{A^2} \\
&= \text{vol}^{-1} \delta_3^q((\otimes \kappa^{-1}p') \oplus p) \partial_i p^A. \quad (171)
\end{align*}
\]

Alternatively, we can calculate matrix elements of the position operator as follows:
\[
\begin{align*}
(X^A)^{p^{p'}} &= \text{vol}^{-1/2} F_L^*(u^p \otimes x^A)(p') \\
&= \int d_3^q x [u^p(x) \otimes x^A] \otimes (u^*)^{p'}(x) \\
&= \int d_3^q x i \partial_i^A \triangleright u^p(x) \otimes (u^*)^{p'}(x) \\
&= \text{vol}^{-1} i \partial_i^A \triangleright \delta_3^q(p \oplus (\otimes \kappa^{-1}p')). \quad (172)
\end{align*}
\]

The action of the \(q\)-deformed position operator on a wave function becomes the action of a derivative operator on the corresponding expansion coefficients in the momentum space since we have in analogy to Eq. (165):
\[
\begin{align*}
(X^A \triangleright \psi_R)_p &= \text{vol}^{-1/2} F_R^*(x^A \otimes \psi_R)(p) \\
&= \text{vol}^{-1/2} i \partial_i^A \triangleright F_R^*(\psi_R)(p) = i \partial_i^A \triangleright c_p. \quad (173)
\end{align*}
\]

In the second step of the above calculation, we have made use of the characteristic identities of the \(q\)-Fourier transformations [cf. Eq. (95) of Chap. 3.3]. In the last step, we have identified the expansion coefficients of \(\psi_R\) [cf. Eq. (144) of Chap. 4.2]. In addition to this, the following applies:
\[
\begin{align*}
i \partial_i^A \triangleright c_p &= \text{vol}^{-1} \int d_3^q p' \delta_3^q((\otimes \kappa^{-1}p') \oplus p') \otimes i \partial_i^A \triangleright c_{p'} \\
&= \text{vol}^{-1} \int d_3^q p' \delta_3^q((\otimes \kappa^{-1}p') \oplus p') \partial_i^A c_{p'} \\
&= \int d_3^q p' (X^A)_{pp'} \otimes c_{p'}. \quad (174)
\end{align*}
\]

Note that we have first used the characteristic identities of the \(q\)-delta functions and the rules for integration by parts. The last step then follows from Eq. (171). If we summarize the results of Eq. (173) and Eq. (174), we get:
\[
\begin{align*}
(X^A \triangleright \psi_R)_p &= \int d_3^q p' (X^A)_{pp'} \otimes c_{p'} = i \partial_i^A \triangleright c_p. \quad (175)
\end{align*}
\]
In the same way, we can show:

\[
\psi_L \triangleleft X^A p = \int d^3 p' \ c_{p'} \otimes (X^A)^{p'} \ = \ c_{p} \triangleleft \partial^A_i.
\] (176)

For the sake of completeness, we write down how the given matrix elements of the \(q\)-deformed momentum operator and that of the \(q\)-deformed position operator behave under conjugation. We can show that the different versions of matrix elements transform into each other by conjugation in the following way:

\[
(P_A)^{p'}_p = (P_A)_{p'p}, \quad (X_A)^{p'}_p = (X_A)_{p'p}.
\]

So far, we have only considered matrix representations referring to the expansion coefficients \(c_p\) or \(c^*_p\) [cf. Eq. (144) and Eq. (145) of Chap. 4.2]. However, we can also give matrix representations corresponding to the expansion coefficients \((c^*)_p\) or \((c^*)^p\) [cf. Eq. (147) and Eq. (148) of Chap. 4.2]:

\[
(P_A)^{p'}_p = \text{vol}^{-1/2} \mathcal{F}_L((u^*)_p \triangleleft \partial^A_i)^{-1}(p')
\]
\[
= \int d^3 x \ (u^*)_p(x) \triangleleft \partial^A_i^{-1} \otimes u_{p'}(x)
\]
\[
= \int d^3 x \ p^A \otimes (u^*)_p(x) \otimes u_{p'}(x)
\]
\[
= \text{vol}^{-1} p^A \otimes \delta_\eta((\otimes \kappa^{-1} p) \otimes p').
\] (177)

In the second step of the above calculation, we have written out the \(q\)-Fourier transformation. The penultimate step is a consequence of the eigenvalue equations for the \(q\)-deformed momentum eigenfunctions [cf. Eq. (127) of Chap. 4.1]. The last step follows from the orthonormality relations of the \(q\)-deformed momentum eigenfunctions [also see Eq. (160) of the previous chapter]. Similar considerations lead to:

\[
(P_A)^{p'}_p = \text{vol}^{-1/2} \mathcal{F}_R(i^{-1} \partial^A_i \triangleleft u^*)(p')
\]
\[
= \int d^3 x \ u^p(x) \triangleleft \partial^A_i^{-1} \otimes (u^*)^p(x)
\]
\[
= \text{vol}^{-1} \delta_\eta((p' \otimes (\otimes \kappa^{-1} p)) \otimes p^A).
\] (178)

Next, we consider the action of the \(q\)-deformed momentum operator on a wave function represented in the momentum space by the coefficients \((c^*)_p\) or \((c^*)^p\), i.e. [also see Eq. (91) of Chap. 3.3 and Eq. (147) of Chap. 4.2]

\[
(\psi_L \triangleleft P_A)_p = \text{vol}^{-1/2} \mathcal{F}_L(\psi^*_L \triangleleft \partial^A_i^{-1})(p)
\]
\[
= \text{vol}^{-1/2} \mathcal{F}_L(\psi^*_L)(p) \otimes p^A = (c^*)_p \otimes p^A
\] (179)

and

\[
(P_A \triangleleft \psi_R^*)^p = \text{vol}^{-1/2} \mathcal{F}_R(i^{-1} \partial^A_i \triangleleft \psi_R^*)(p)
\]
\[
= \text{vol}^{-1/2} p^A \otimes \mathcal{F}_R(\psi_R^*)(p) = p^A \otimes (c^*)^p.
\] (180)
On the other hand, the characteristic identities of the $q$-delta function together with Eq. (177) and Eq. (178) imply:

$$(c^*)_p \otimes p^A = \text{vol}^{-1} \int d_q^3 p' \ (c^*)_p' \otimes p'^A \otimes \delta_q^3((\ominus \kappa^{-1}p') \otimes p)$$

$$= \int d_q^3 p' \ (c^*)_p' \otimes (P^A)p'_p, \quad (181)$$

$$p^A \otimes (c^*)_p = \text{vol}^{-1} \int d_q^3 p' \ \delta_q^3(p \otimes (\ominus \kappa^{-1}p')) \otimes p'^A \otimes (c^*)_p'$$

$$= \int d_q^3 p' \ (P^A)p^A \otimes (c^*)_p'. \quad (182)$$

Finally, from the above considerations follows:

$$(\psi^*_L \triangleleft P^A)_p = (c^*)_p \otimes p^A = \int d_q^3 p' \ (c^*)_p' \otimes (P^A)p'_p,$$

$$(P^A \triangleright \psi^*_R)_p = p^A \otimes (c^*)_p = \int d_q^3 p' \ (P^A)p^A \otimes (c^*)_p'. \quad (183)$$

Accordingly, we can write the successive application of two $q$-deformed momentum operators as a kind of matrix multiplication:

$$(\psi^*_L \triangleleft P^A P^B)_p = \int d_q^3 p' \int d_q^3 p'' \ (c^*)_p' \otimes (P^A)p'_p \otimes (P^B)p''_p,$$

$$(P^A P^B \triangleright \psi^*_R)_p = \int d_q^3 p' \int d_q^3 p'' \ (P^A)p^A \otimes (P^B)p^B \otimes (c^*)_p''. \quad (184)$$

Again, we turn to the matrix representations of the $q$-deformed position operator. We can obtain these matrix representations by a calculation analogous to that of Eq. (177):

$$(X^A)_p p^p = \text{vol}^{-1/2} F_L((u^*)_p \otimes x^A)(p')$$

$$= \int d_q^3 x \ [(u^*)_p \otimes x^A] \otimes u_p^*(x)$$

$$= \int d_q^3 x \ [i\partial_{p} \triangleright (u^*)_p] \otimes u_p^*(x)$$

$$= \text{vol}^{-1} i\partial_{p} \triangleright \delta_q^3((\ominus \kappa^{-1}p) \otimes p'). \quad (185)$$

In the same way, we have:

$$(X^A)p^p = \text{vol}^{-1/2} F_R(x^A \otimes (u^*)^p)(p')$$

$$= \int d_q^3 x \ u^p(x) \otimes [x^A \otimes (u^*)^p(x)]$$

$$= \int d_q^3 x \ u^p(x) \otimes [(u^*)^p(x) \otimes \partial_{p} A^i]$$

$$= \text{vol}^{-1} \delta_q^3(p' \otimes (\ominus \kappa^{-1}p)) \triangleleft \partial_{p} A^i. \quad (186)$$
Next, we consider the action of the \(q\)-deformed position operator on a wave function. With the help of Eq. (93) from Chap. 3.3 we find:

\[
(\psi_L^* \bar{\partial} X^A)p = \frac{\text{vol}}{2} F_L(\psi_L^* \otimes X^A)(p) = \frac{\text{vol}}{2} F_L(\psi_L^*)(p) \bar{\partial} p^A_1 = (c^*)_p \bar{\partial} p^A_1.
\] (187)

Moreover, we have:

\[
(c^*)_p \bar{\partial} p^A_1 = \frac{1}{\text{vol}} \int d^3q \big[(c^*)_p \bar{\partial} p^A_1 \big] \otimes \delta^3_q((\ominus \kappa^{-1} p') \oplus p)
\]

\[
= \frac{1}{\text{vol}} \int d^3q' (c^*)_{p'} \otimes [i\partial^A_{p'} \otimes \delta^3_q((\ominus \kappa^{-1} p') \oplus p)]
\]

\[
= \int d^3q' (c^*)_{p'} \otimes (X^A)_{p'p}.
\] (188)

Finally, it follows:

\[
(\psi_L^* \bar{\partial} X^A)p = \int d^3q \big[(c^*)_p \bar{\partial} p^A_1 \big] \otimes (X^A)_{p'p} = (c^*)_p \bar{\partial} p^A_1.
\] (189)

In the same way, you can show:

\[
(X^A \triangleright \psi_R^*)p = \int d^3q \big[(X^A)_{p'p} \otimes (c^*)_p \big] = i\partial^A_{p'} \triangleright (c^*)^p.
\] (190)

### 4.6 Matrix representations on position space

In this chapter, we determine matrix representations of the position or momentum operator for a basis of \(q\)-deformed position eigenfunctions. We start with the matrix elements of the position operator. Using the characteristic identities for the \(q\)-deformed delta functions, we obtain the following expressions for the matrix elements of the position operator [cf. Eq. (110) and Eq. (111) of Chap. 3.5]:

\[
(X^A)_{y'y} = \int d^3x (u^*)_y(x) \otimes x^A \oplus u_y(x)
\]

\[
= \text{vol}^{-2} \int d^3x \delta^3_q((\ominus \kappa^{-1} y') \oplus x) \otimes x^A \oplus \delta^3_q((\ominus \kappa^{-1} x) \oplus y)
\]

\[
= \text{vol}^{-1} \delta^3_q((\ominus \kappa^{-1} y') \oplus y) \otimes y^A.
\] (191)

In the second step of the above calculation, we have written the position eigenfunctions in the form of \(q\)-delta functions [cf. Eq. (130) of Chap. 4.1]. In the same manner follows:

\[
(X^A)_{y'y} = \int d^3x u^v(y) \otimes x^A \oplus (u^*)_y(x)
\]

\[
= \text{vol}^{-1} y'^A \otimes \delta^3_q(y' \oplus (\ominus \kappa^{-1} y)).
\] (192)
Note that the matrix element of Eq. (191) is equal to that of Eq. (192) due to the following distribution equation:

$$\delta^3_q((\ominus \kappa^{-1} y') \oplus y) \otimes y^A = y'^A \otimes \delta^3_q(y' \oplus (\ominus \kappa^{-1} y)).$$

(193)

If we conjugate the above expressions for the matrix elements of the position operator and take into account the conjugation properties of volume elements, delta functions, and position coordinates, we will get the following result:

$$\overline{(X^A)_{y'y}} = \text{vol}^{-1} y_A \otimes \delta^3_q((y \oplus (\ominus \kappa^{-1} y')) = (X_A)^{y'y}.$$ 

(194)

Next, we consider the action of the $q$-deformed position operator on a wave function. If the $q$-deformed position operator acts on a wave function of position space from the left, the given wave function will be multiplied by position coordinates from the left:

$$(X^A \triangleright \psi_R)_y = \int d^3_y x (u^*)_y(x) \otimes x^A \otimes \psi_R(x)$$

$$= \text{vol}^{-1} \int d^3_y x \delta^3_q(y \oplus (\ominus \kappa^{-1} x)) \otimes x^A \otimes \psi_R(x)$$

$$= y^A \otimes \psi_R(y) = y^A \otimes c_y.$$ 

(195)

In the final step of the above calculation, we took into account that the wave function can be identified with its expansion coefficients referring to a basis of $q$-deformed position eigenfunctions [see Eq. (154) of Chap. 4.3]. On the other hand, we can also express $(X^A \triangleright \psi_R)_y$ by using the matrix elements from Eq. (191):

$$(X^A \triangleright \psi_R)_y = \text{vol}^{-1} \int d^3_y y' \delta^3_q(y \oplus (\ominus \kappa^{-1} y')) \otimes y'^A \otimes \psi_R(y')$$

$$= \int d^3_y y' (X^A)_{yy'} \otimes \psi_R(y') = \int d^3_y y' (X^A)_{yy'} \otimes c_{y'}.$$ 

(196)

We can summarize the results of Eq. (195) and Eq. (196) as follows:

$$(X^A \triangleright \psi_R)_y = \int d^3_y y' (X^A)_{yy'} \otimes c_{y'} = y^A \otimes c_y.$$ 

(197)

If the $q$-deformed position operator acts on the wave function from the right, the wave function must be multiplied by the position coordinates from the right:

$$(\psi^*_L \triangleleft X^A)_y = \int d^3_y x \psi^*_L(x) \otimes x^A \otimes u_y(x)$$

$$= \text{vol}^{-1} \int d^3_y x \psi^*_L(x) \otimes x^A \otimes \delta^3_q((\ominus \kappa^{-1} x) \oplus y)$$

$$= \psi^*_L(y) \otimes y^A = (c^*)_y \otimes y^A.$$ 

(198)
In analogy to Eq. (197), we can also write this result as follows:

\[
(\psi_L^* \ll X^A)_y = \int d^3 q^y (c^*)_y \otimes (X^A)_y^y
\]

\[
= (c^*)_y^y \otimes y^A.\tag{199}
\]

We now turn to the momentum operator and calculate its matrix elements for a basis of \(q\)-deformed position eigenfunctions. We proceed similarly to the calculation of the matrix elements of the position operator, i.e., we write the position eigenfunctions as \(q\)-deformed delta functions and apply the characteristic identities of the \(q\)-deformed delta functions. This way, we obtain the following expression for the matrix elements of the momentum operator:

\[
(P^A)_y^y = \int d^3 q^y (u^*)_y(x) \otimes [i^{-1} \partial^A \ll u^y(x)]
\]

\[
= \int d^3 q^y [(u^*)_y(x) \ll \partial^A y^{-1}] \otimes u^y(x)
\]

\[
= \mathrm{vol}^{-2} \int d^3 q^y [\delta^3_q((\ominus \kappa^{-1} y') \oplus x) \ll \partial^A y^{-1}] \otimes \delta^3_q((\ominus \kappa^{-1} x) \oplus y)
\]

\[
= \mathrm{vol}^{-1} \delta^3_q((\ominus \kappa^{-1} y') \oplus y) \ll \partial^A y^{-1}.\tag{200}
\]

Note that in the second step we made use of the \(q\)-deformed Stokes’ theorem [cf. Eq. (66) of Chap. 2.5]. It follows from similar arguments:

\[
(P^A)^y_y = \int d^3 q^y u^y(x) \ll \partial^A y^{-1} \otimes (u^*)_y^y(x)
\]

\[
= \mathrm{vol}^{-1} i^{-1} \partial^A \ll \delta^3_q(y \oplus (\ominus \kappa^{-1} y')).\tag{201}
\]

Taking into account the conjugation properties of volume elements, delta functions, and partial derivatives, the results of Eq. (200) and Eq. (201) imply that the matrix elements of the \(q\)-deformed momentum operator behave as follows under conjugation:

\[
(P^A)_y^y = (P^A)^y_y.
\]

Finally, we consider the action of the momentum operator on a general wave function. To this end, we proceed in analogy to the calculation of Eq. (196):

\[
(P^A \ll \psi_R)_y = \int d^3 q^y (u^*)_y(x) \otimes [i^{-1} \partial^A \ll \psi_R(x)]
\]

\[
= \mathrm{vol}^{-1} \int d^3 q^y \delta^3_q((\ominus \kappa^{-1} y) \oplus x) \otimes [i^{-1} \partial^A \ll \psi_R(x)]
\]

\[
= \mathrm{vol}^{-1} \int d^3 q^y \delta^3_q((\ominus \kappa^{-1} y) \oplus y') \ll \partial^A y^{-1}] \otimes \psi_R(y')
\]

\[
= \int d^3 q^y (P^A)_y^y \otimes c_y'.\tag{202}
\]
On the other hand, we have:

\[(P^A \tilde{\psi}_R)_y = \text{vol}^{-1} \int d^3x \, \delta^3(y \oplus (\ominus \kappa^{-1}x)) \otimes [i^{-1} \partial^A_x \tilde{\psi}_R(x)] = i^{-1} \partial^A_y \tilde{\psi}_R(y) = i^{-1} \partial^A_y \tilde{c}_y. \tag{203}\]

For this reason, it holds:

\[(P^A \tilde{\psi}_R)_y = \int d^3y' \, (P^A)_{yy'} \otimes c_{y'} = i^{-1} \partial^A_y \tilde{c}_y. \tag{204}\]

Similar arguments lead to:

\[(\psi_L \triangleleft P^A)_y = \int d^3y' \, c_{y'} \otimes (P^A)_{yy'} = c_y \triangleleft \partial^A y^{-1}. \tag{205}\]

Let us mention that in the Eqns. (200)-(205) we may carry out the following replacements:

\[\tilde{\triangleright} \leftrightarrow \triangleright, \, \triangleleft \leftrightarrow \triangleleft. \tag{206}\]

It should also be noted that all formulas in this chapter remain valid if we make the following replacements:

\[u_y \leftrightarrow (u^*)_y, \quad c_y \leftrightarrow (c^*)_y, \quad \psi_R \leftrightarrow \psi^*_R, \quad (u^*)_y \leftrightarrow u_y, \quad (c^*)_y \leftrightarrow c_y, \quad \psi^*_L \leftrightarrow \psi_L. \tag{207}\]

### 4.7 Expectation values and probability densities

To begin with, we describe a physical state on a $q$-deformed quantum space by a wave function $\psi_R(x)$ and a corresponding ‘dual’ wave function $\psi^*_L(x)$. Due to their probabilistic interpretation, these wave functions should fulfill a normalization condition of the following form:

\[\int d^3x \, \psi^*_L(x) \otimes \psi_R(x) = 1. \tag{208}\]

As the indices $L$ and $R$ indicate, the wave function $\psi^*_L$ is the left factor in the star product of the above expression and the wave function $\psi_R$ is the right factor.

In quantum mechanics, an operator $\hat{O}$ is assigned to a measurable quantity. If the $q$-deformed wave functions $\psi_R$ and $\psi^*_L$ describe the state of a physical system, the measured values of the physical quantity $\hat{O}$ scatter around an expectation value given by the following expression:

\[\langle \hat{O}_\psi \rangle = \int d^3x \, \psi^*_L(x) \otimes \hat{O} \triangleright \psi_R(x) = \int d^3x \, \psi^*_L(x) \triangleleft \hat{O} \otimes \psi_R(x). \tag{209}\]

We require that the expectation value behaves as follows under conjugation:

\[\langle \hat{O}_\psi \rangle = \langle \hat{O} \rangle_{\psi}. \tag{210}\]
Using the conjugation properties of star product and $q$-deformed integral, we can show that the above condition is satisfied if $\psi_R(x)$ and $\psi_L^*(x)$ are transformed into each other by quantum space conjugation, i.e.

$$\overline{\psi_R(x)} = \psi_L^*(x).$$  \hspace{1cm} (211)

If the operator $\hat{O}$ is self-adjoint, i.e.

$$\overline{\hat{O}} = \hat{O},$$

its expectation value is real in the following way:

$$\langle \hat{O} \rangle_\psi = \langle \hat{O} \rangle_\psi.$$  \hspace{1cm} (212)

We illustrate the above considerations using the example of the three-dimensional $q$-deformed position operator made up of the following components [also see Eq. (14) of Chap. 2.1]:

$$X^1 = \frac{i}{2}(-q^{-1/2}X^+ - q^{1/2}X^-),$$

$$X^2 = \frac{1}{2}(-q^{-1/2}X^+ + q^{1/2}X^-),$$

$$X^3 = X^3.$$  \hspace{1cm} (213)

Note that due to the conjugation properties of the quantum space generators $X^+$, $X^3$, and $X^-$ [cf. Eq. (18) of Chap. 2.1], the components in Eq. (213) are self-adjoint in the following way ($i \in \{1, 2, 3\}$):

$$\overline{X^i} = X^i.$$  \hspace{1cm} (214)

Next, we consider the expectation value of $X^i$. Using the conjugation properties of the invariant integral and those of the star product, we find:

$$\langle X^i \rangle_\psi = \int d^3x \psi_L^* \circ (X^i \triangleright \psi_R) = \int d^3x \psi_L^* \circ (x^i \bullet \psi_R)$$

$$= \int d^3x (x^i \bullet \psi_R) \circ \psi_L^* = \int d^3x \psi_R \bullet x^i \bullet \overline{\psi_L}$$

$$= \int d^3x \psi_L^* \circ (X^i \triangleright \psi_R) = \langle X^i \rangle_\psi.$$  \hspace{1cm} (215)

According to Eq. (152) of Chap. 4.3, the wave functions $\psi_R$ and $\psi_L^*$ can be expanded in terms of position eigenfunctions as follows:

$$\psi_R(x) = \int d^3y u_y(x) \circ \overline{c_y},$$

$$\psi_L^*(x) = \int d^3y (c^*)_y \circ (u^*)_y(x).$$  \hspace{1cm} (216)
Due to the formulas in Eq. (155) of Chap. 4.3 we also have the identities
\[
\psi_R(x) = \int d^3q \ y_u(x) \otimes c_y = \int d^3q \ y_u(x) \otimes \int d^3x' (u^*)_y(x') \otimes \psi_R(x')
\]
\[
= \int d^3q \ P_y \triangleright \psi_R(x)
\]
and
\[
\psi^*_L(x) = \int d^3q \ (c^*)_y \otimes (u^*)_y(x) = \int d^3q \int d^3x' \psi^*_L(x') \otimes u_y(x') \otimes (u^*)_y(x)
\]
\[
= \int d^3q \psi^*_L(x) \triangleleft P_y.
\]
(217)

Note that we have introduced the following operators:
\[
P_y \triangleleft \ldots = u_y(x) \otimes \int d^3x' (u^*)_y(x') \otimes \ldots,
\]
\[
\ldots \triangleleft P_y = \int d^3x' \ldots \otimes u_y(x') \otimes (u^*)_y(x).
\]
(218)

They are projectors corresponding to the different eigenvalues of the position operator:
\[
P_y \triangleright \psi_R(x) = u_y(x) \otimes c_y, \quad \psi^*_L(x) \triangleleft P_y = (c^*)_y \otimes (u^*)_y(x).
\]
(220)

Accordingly, these projectors are subject to following orthonormality relations:
\[
P_y P_{y'} \triangleright \ldots = (P_y \triangleright \ldots) \otimes \frac{1}{\text{vol}} \delta^3(y' \oplus (\ominus \kappa^{-1} y) \oplus y),
\]
\[
\ldots \triangleleft P_y P_{y'} = \frac{1}{\text{vol}} \delta^3(y' \oplus (\ominus \kappa^{-1} y)) \otimes (\ldots \triangleleft P_{y'}). 
\]
(221)

We can prove the above identities by calculations of the following type:
\[
P_{y'} P_y \triangleright \psi_R(x) = P_{y'} \triangleright u_y(x) \otimes c_y
\]
\[
= u_{y'}(x) \otimes \int d^3x' (u^*)_y(x') \otimes u_y(x') \otimes c_y
\]
\[
= u_{y'}(x) \otimes \frac{1}{\text{vol}} \delta^3((\ominus \kappa^{-1} y') \oplus y) \otimes c_y
\]
\[
= u_{y'}(x) \otimes c_{y'} \otimes \frac{1}{\text{vol}} \delta^3((\ominus \kappa^{-1} y') \oplus y)
\]
\[
= P_{y'} \triangleright \psi_R(x) \otimes \frac{1}{\text{vol}} \delta^3((\ominus \kappa^{-1} y') \oplus y).
\]
(222)

From Eq. (217) and Eq. (218), we finally see that the set of the projection operators is complete in some sense as the following relations hold:
\[
\int d^3y P_y \triangleright \ldots = \int d^3y \ldots \triangleleft P_y = \ldots
\]
(223)

\[\text{Note that the symbol "\ldots" stands for an expression the projectors are acting on.}\]
Identifying the wave functions \( \psi_R \) and \( \psi^*_L \) with their expansion coefficients \( c_y \) and \( (c^*)_y \) [cf. Eq. (154) of Chap. 4.3], we can write the expectation value of an operator \( \hat{O} \) as follows:

\[
\langle \hat{O} \rangle = \int d^3q \, \psi^*_L(x) \otimes \hat{O} \otimes \psi_R(x) = \int d^3q \, (c^*)_y \otimes \hat{O} \otimes c_y.
\]

(224)

Accordingly, we get for the expectation value of the position operator:

\[
\langle X^i \rangle = \int d^3q \, \psi^*_L(x) \otimes x^i \otimes \psi_R(x) = \int d^3q \, (c^*)_y \otimes y^i \otimes c_y.
\]

(225)

In this context, it should be mentioned that the normalization condition in Eq. (208) can also be written as an expectation value of the identity operator:

\[
\langle 1 \rangle = \int d^3q \, \psi^*_L(x) \otimes \psi_R(x) = \int d^3q \, (c^*)_y \otimes c_y = 1.
\]

(226)

If the measurement of the position coordinates of a particle yields the eigenvalues \( y^i \), the wave functions \( \psi_R \) and \( \psi^*_L \) of the particle are reduced to the expressions given in Eq. (220). The expectation value of the operator \( P_y \) is then equal to the probability density of finding the particle in a region represented by \( y = (y^i) \):

\[
\rho_\psi(y) = \int d^3q \, \psi^*_L(x) \otimes P_y \otimes \psi_R(x)
\]

\[
= \int d^3q \, (c^*)_y \otimes P_y \otimes \psi_R(x) = (c^*)_y \otimes c_y.
\]

(227)

Because of the normalization condition in Eq. (226), the following applies:

\[
\int d^3q \, \rho_\psi(y) = \int d^3q \, (c^*)_y \otimes c_y = 1.
\]

(228)

Considering the identifications in Eq. (154) of Chap. 4.3, we can state the above results in the following way. If the normalized wave functions \( \psi_R(x) \) and \( \psi^*_L(x) \) describe the state of a particle in position space, we can write down its probability density in position space as follows:

\[
\rho_\psi(x) = \psi^*_L(x) \otimes \psi_R(x).
\]

(229)

Recall now that we can expand a wave function in terms of momentum eigenfunctions [cf. Eq. (141) and Eq. (142) of Chap. 4.2]:

\[
\begin{align*}
\psi_R(x) &= \int d^3p \, u_p(x) \otimes c_p, \\
\psi^*_L(x) &= \int d^3p \, (c^*)_p \otimes (u^*)_p(x).
\end{align*}
\]

(230)
The corresponding expansion coefficients can be determined by $q$-deformed Fourier transformations [cf. Eq. (144) and Eq. (147) of Chap. 4.2]

$$c_p = \text{vol}^{-1/2} F_R^*(\psi_R)(p), \quad (c^*)_p = \text{vol}^{-1/2} F_L(\psi_R^*)(p). \quad (231)$$

The considerations in this chapter also apply to the expansions

$$\psi_R^*(x) = \int d^3 q (u^*)_p(x) \otimes (c^*)_p,$$
$$\psi_L(x) = \int d^3 q c_p \otimes u^p(x) \quad (232)$$

with the coefficients $c^p$ and $(c^*)_p$ given by the following expressions [cf. Eq. (145) and (148) of Chap. 4.2]

$$c^p = \text{vol}^{-1/2} F_L^*(\psi_L)(p), \quad (c^*)_p = \text{vol}^{-1/2} F_R(\psi_L^*)(p). \quad (233)$$

We can restrict ourselves, however, to the expansions in Eq. (230). We can do this since we obtain the results for the expansions in Eq. (232) by the following substitutions:

$$\psi^* \leftrightarrow \psi, \quad F \leftrightarrow F^*, \quad \triangleright \leftrightarrow \triangleleft, \quad \triangleleft \leftrightarrow \triangleright,$$
$$(u^*)_p \rightarrow u^p, \quad u^p \rightarrow (u^*)_p, \quad (c^*)_p \rightarrow c^p, \quad c_p \rightarrow (c^*)_p. \quad (234)$$

If we measure the values $p^i$ for the momentum coordinates of a particle, the wave function of the particle will be reduced to the component corresponding to $p = (p^i)$. We can introduce operators projecting onto these components, i. e.

$$P_p \triangleright \psi_R(x) = u_p(x) \otimes c_p, \quad \psi_R^*(x) \triangleleft P_p = (c^*)_p \otimes (u^*)_p(x) \quad (235)$$

with

$$P_p \triangleright \ldots = u_p(x) \otimes \int d^3 q x' (u^*)_p(x') \otimes \ldots, $$
$$\ldots \triangleleft P_p = \int d^3 q x' \ldots \otimes u_p(x') \otimes (u^*)_p(x). \quad (236)$$

Note that the set of these projection operators is complete as we have:

$$\int d^3 q P_p \triangleright \ldots = \int d^3 q \ldots \triangleleft P_p = \ldots \quad (237)$$

Furthermore, the projection operators to the momentum eigenvalues satisfy the following orthonormality conditions:

$$P_p P_p \triangleright \ldots = (P_p \triangleright \ldots) \otimes \frac{1}{\text{vol}} \delta_q^3 (\ominus \kappa^{-1} p') \otimes p), $$
$$\ldots \triangleleft P_p P_p' = \frac{1}{\text{vol}} \delta_q^3 (p' \ominus \ominus \kappa^{-1} p) \otimes (\ldots \triangleleft P_p'). \quad (238)$$
The expectation value of the projector $P_p$ provides the density for the probability that a momentum measurement yields $p = (p^i)$:

$$\rho_\psi(p) = \int d^3x \psi_L^*(x) \otimes P_p \triangleright \psi_R(x)$$

$$= \text{vol}^{-1} \mathcal{F}_L(\psi_L^*) \otimes \mathcal{F}_R^*(\psi_R) = (c^*)_p \otimes c_p. \quad (239)$$

Note that the second step is a consequence of the $q$-deformed Fourier-Plancherel identities [cf. Eq. (120) and Eq. (121) of Chap. 3.6]. Moreover, it follows:

$$\langle 1 \rangle_\psi = \int d^3x \psi_L^*(x) \otimes \psi_R(x) = \frac{1}{\text{vol}} \int d^3p \mathcal{F}_L(\psi_L) \otimes \mathcal{F}_R^*(\psi_R)(p)$$

$$= \int d^3p (c^*)_p \otimes c_p = \int d^3p \rho_\psi(p) = 1. \quad (240)$$

Next, we consider a self-adjoint version of the three-dimensional $q$-deformed momentum operator, whose components are defined in complete analogy to Eq. (213):

$$P^1 = \frac{i}{2}(-q^{-1/2}P^+ - q^{1/2}P^-),$$

$$P^2 = \frac{1}{2}(-q^{-1/2}P^+ + q^{1/2}P^-),$$

$$P^3 = P^3. \quad (241)$$

Using the $q$-deformed Fourier-Plancherel identities we obtain ($i \in \{1, 2, 3\}$):

$$\langle P^i \rangle_\psi = \int d^3x \psi_L^*(x) \otimes i^{-1} \partial^i \triangleright \psi_R(x) = \int d^3x \psi_L^*(x) \triangleright i^{-1} \partial^i \otimes \psi_R(x)$$

$$= \frac{1}{\text{vol}} \int d^3p \mathcal{F}_L(\psi_L^*)(p) \otimes \mathcal{F}_R^*(i^{-1} \partial^i \triangleright \psi_R)(p)$$

$$= \frac{1}{\text{vol}} \int d^3p \mathcal{F}_L(\psi_L^*) \triangleright i^{-1} \partial^i(p) \otimes \mathcal{F}_R^*(\psi_R)(p)$$

$$= \int d^3p (c^*)_p \otimes p^i \otimes c_p. \quad (242)$$

Finally, we examine how the expectation value of $P^i$ behaves under conjugation if we take into account the condition given in Eq. (211):

$$\langle \overline{P^i} \rangle_\psi = \int d^3x \overline{\psi_R} \triangleright \partial^i i^{-1} \otimes \overline{\psi}_L = \int d^3x \psi_L^* \triangleright i^{-1} \partial^i \triangleright \psi_R. \quad (243)$$

Since the actions $\partial^i \triangleright \psi$ and $\partial^i \triangleright \psi$ are generally not the same, the above result does not imply that the expectation value of $P^i$ is real in all cases. Looking at the last expression of Eq. (242) shows, however, that the condition $(c^*)_p = c_p$ leads to a real expectation value for $P^i$. 

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Figure 7: Graphical representation of the morphism $f : U \to V$

Figure 8: Braiding mappings

\[ \Psi_{U,V} = \begin{array}{c} U \\ \overline{V} \end{array} \hspace{1cm} \Psi_{U,V}^{-1} = \begin{array}{c} U \\ \overline{V} \end{array} \]

A Graphical representation of morphisms

The $q$-deformed quantum spaces we are dealing with are so-called braided Hopf algebras [18], i.e. objects of a braided tensor category. If $U$ and $V$ are objects of such a tensor category, a morphism $f : U \to V$ is symbolized by a downward pointing arrow. This arrow runs through a circle labeled by $f$ (see Fig. 7). In the following, we briefly discuss graphical representations of morphisms that are important for our considerations.

In Eq. (8) and Eq. (9) of Chap. 2.1, we have considered isomorphisms that can be used to exchange the factors of a tensor product. We represent these isomorphisms $\Psi_{U,V} : U \otimes V \to V \otimes U$ and $(\Psi_{U,V})^{-1} : V \otimes U \to U \otimes V$ known as braiding mappings by intersecting arrows (see Fig. 8).

We can draw the symbol for a morphism through a crossing that represents a braiding mapping (cf. Fig. 9). For our $q$-integrals represented graphically by a free node, this statement only applies with some restrictions. A detailed investigation shows the following. If we draw the free node representing a $q$-integral through an intersection, one of the two strands takes up a scaling (see Fig. 10). If the strand taking up the scaling symbolizes a function, all arguments of this function are multiplied by the same constant $\kappa$ or $\kappa^{-1}$. Note that in the case of the $q$-deformed Euclidean quantum space we have $\kappa = q^6$.

Being braided Hopf algebras, $q$-deformed quantum spaces have specific morphisms. In this respect, the algebraic structure of a quantum space $V$ is determined by a multiplication $m : V \otimes V \to V$ and a unit map $\eta : K \to V$. The
Figure 9: Commuting morphisms by $\Psi$ or $\Psi^{-1}$

Figure 10: Braiding properties of $q$-integrals
dual morphisms, i.e. the braided co-product \( \Delta : V \otimes V \to V \) and the (braided) co-unit \( \varepsilon : V \to \mathbb{K} \) together with the braided antipode \( \bar{S} : V \to V \) determine the (braided) Hopf structure of the quantum space algebra \( V \). In Chap. 2.2 and in Chap. 2.3 we have briefly explained how to realize these morphisms on a commutative coordinate algebra as star products, \( q \)-translations or \( q \)-inversions. For some calculations, it does not make sense to work with these commutative realizations. In this cases, it is much easier to perform calculations graphically. Therefore, we have summarized the graphical representations of the morphisms mentioned above in Fig. 11 (also see Ref. [22]).

**B Proof for invertibility of \( q \)-Fourier transformations**

Graphical methods provide the simplest way to prove the identities in Eq. (89) and Eq. (90) of Chap. 3.2. In this appendix, we briefly describe the most important steps of these graphical calculations taken from Ref. [11]. Note that we have adapted these graphical calculations to the needs of our formalism.

For the following considerations, we need a special graphical representation of the Fourier transformation \( F_q^\ast (f) \) [also see Eq. (85) of Chap. 3.1]. We show this representation in Fig. 12. Note that we can obtain the right diagram in Fig. 12 from the left one by dragging the downward strand past the node representing the \( q \)-integral and taking into account the braiding properties of the \( \bar{S} \)-integral (also see Fig. 10).

Fig. 13 shows the graphical prove for the first identity of Eq. (89). In the first step, we pull the antipode \( S \) through the crossing and apply Eq. (75) of Chap. 2.6. We have labeled all morphisms referring to the dual coordinate algebra, i.e. the momentum algebra, with a star. Furthermore, we keep in mind that we have made use of the scaling properties of the \( q \)-integral in the first step:

\[
\int d_q^N x f(\kappa^{-1}x) \otimes g(\kappa^{-1}x) = \kappa^N \int d_q^N x f(x) \otimes g(x).
\]  

\[\text{In contrast to Fig. 10 the scaling is given by } \kappa^{-1} \text{ since the } q \text{-integral refers to momentum space and the downward strand to position space.}\]
In the second step, we pull the upward strand over the lower integral and the lower exponential, taking into account the braiding properties of the $q$-integral. Now, we can make use of the addition theorem for quantum space exponentials [see Eq. (74) of Chap. 2.6]. The fourth step is a consequence of the following axioms of a braided Hopf algebra\footnote{Recall that in our convention the braiding mappings $\Psi$ and $\Psi^{-1}$ are interchanged in comparison to Ref. [18].}:

\[
m \circ (S^{-1} \otimes S^{-1}) \circ \Psi = m \circ \Psi \circ (S^{-1} \otimes S^{-1}) = S^{-1} \circ m,
\]

\[
m \circ (S \otimes S) \circ \Psi^{-1} = m \circ \Psi^{-1} \circ (S \otimes S) = S \circ m.
\] (245)

In the fifth step of Fig. 13 the strand symbolizing the function $f$ is drawn to the right over the $q$-integral and the $q$-exponential, taking into account the trivial braiding of the $q$-exponential as well as the braiding properties of the $q$-integral. For the sixth step, we use a graphical identity whose derivation we can find in Ref. [11]. In the seventh step, the $q$-exponential, together with the attached $q$-integral, is drawn over the left strand of the co-product. The right strand of the co-product is then deformed in such a way that we obtain the combination of a dual pairing (between a momentum space algebra and a position space algebra) with a canonical element (i.e. a $q$-exponential). Thus, we can apply the addition theorem for $q$-exponentials again. Then, we make use of the translation invariance of the $q$-integral (cf. Fig. 1 of Chap. 2.5). The penultimate step is nothing else but the graphical representation of the following property of a dual pairing:

\[
\langle f, 1 \rangle = \varepsilon(f).
\] (246)

Note that in the penultimate step, we have identified the right subdiagram with the $q$-deformed volume element (see Ref. [11] for a detailed explanation of this identification):

\[
\kappa^N \int_{-\infty}^{+\infty} d_q^N p \int_{-\infty}^{+\infty} d_q^N x \exp_q(q|x \oplus ip) = \kappa^N \int_{-\infty}^{+\infty} d_q^N p \delta^N_L (\ominus p) = \text{vol}_L.
\] (247)
Figure 13: Diagrammatic proof for the first identity of Eq. (89)
Finally, the last step results from the following axiom of a braided Hopf algebra:

\[(\text{id} \otimes \varepsilon) \circ \Delta = \text{id} = (\varepsilon \otimes \text{id}) \circ \Delta.\]  

(248)

Fig. 14 shows the graphical proof for the first identity of Eq. (90). Note that in the first diagram of this graphical proof the Fourier transformation $F^*_L$ is represented again in the form we have given in Fig. 12. We first drag the left integral over the downward strand and then apply the law of addition for the quantum space exponentials. The third step uses the same identity as the sixth step in Fig. 13. Now, we can identify the expression for a $q$-delta function [cf. Eq. (82) of Chap. 3.1]. The next step is nothing else but the graphical representation of the following characteristic identity for the $q$-delta function:

\[\int_{-\infty}^{+\infty} d^N_q x f(x) \otimes \delta^N_L(x) = \text{vol}_L f(0).\]  

(249)

You can find the proof for this identity in Ref. [11]. The last step is a consequence of the axiom in Eq. (248) again.

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