Connection between the Lieb–Thirring conjecture for Schrödinger operators and an isoperimetric problem for ovals on the plane

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Abstract. To determine the sharp constants for the one dimensional Lieb–Thirring inequalities with exponent $\gamma \in (1/2, 3/2)$ is still an open problem. According to a conjecture by Lieb and Thirring the sharp constants for these exponents should be attained by potentials having only one bound state. Here we exhibit a connection between the Lieb–Thirring conjecture for $\gamma = 1$ and an isoperimetric inequality for ovals in the plane.

1. Introduction

The Lieb–Thirring inequalities are one of the main tools in the proof of the stability of matter [16] (see also the review article [18] or [19]). Let $H = -\Delta + V$ be the Schrödinger operator acting on $L^2(\mathbb{R}^n)$, $n \geq 1$ and denote by $e_1 \leq e_2 \leq \cdots < 0$ the negative eigenvalues of $H$. The Lieb–Thirring inequalities are given by

$$\sum_{j \geq 1} |e_j| \leq L_{\gamma,n} \int_{\mathbb{R}^n} V_-(x)^{\gamma + n/2} \, dx,$$

(1.1)

where $V_-(x) \equiv \max(-V(x), 0)$ is the negative part of the potential. The above inequalities hold for $\gamma \geq 1/2$ when $n = 1$, for $\gamma > 0$ when $n = 2$, and for $\gamma \geq 0$ for $n \geq 3$. The case $\gamma = 1/2$, $n = 1$ was established by T. Weidl [21]. The case $\gamma = 0$, $n \geq 3$ was established independently by M. Cwikel, E.H. Lieb and G.V. Rosenbljum. One can show in general that $L_{\gamma,n} \geq L_{c,\gamma,n}$, where

$$L_{c,\gamma,n} = 2^{-n} \pi^{-n/2} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1 + n/2)}$$

(1.2)

are the semiclassical constants. Define $R_{\gamma,n} \equiv L_{\gamma,n}/L_{c,\gamma,n} \geq 1$. Aizenman and Lieb proved that $R_{\gamma,n}$ decreases as $\gamma$ increases [1]. In [17] it is proven that $L_{3/2,1} = L_{c,3/2,1}$ and thus, $L_{\gamma,1} = L_{c,\gamma,1}$, for all $\gamma \geq 3/2$. For $n > 1$, Laptev and Weidl [13] proved $L_{3/2,n} = L_{c,3/2,n}$ hence, $L_{\gamma,n} = L_{c,\gamma,n}$, for all $\gamma \geq 3/2$. The sharp constant for $\gamma = 1/2$ and $n = 1$, $L_{1/2,1} = 1/2$ was proved in [10]. For best constants up to date see [11].

For $n = 1$, the sharp constants $L_{\gamma,1}$ are not known for values of $\gamma$ in the interval $(1/2, 3/2)$. However, in 1976 Lieb and Thirring [17] conjectured that the
sharp constants are attained for potentials that have only one bound state, and therefore
\begin{equation}
L_{\gamma,1} = L_{\gamma,1}^1 = \frac{1}{\sqrt{\pi}} \frac{1}{\gamma - 1/2} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1/2)} \left( \frac{\gamma - 1/2}{\gamma + 1/2} \right)^{\gamma + 1/2}.
\end{equation}

In this manuscript we establish a connection between the Lieb–Thirring conjecture for $\gamma = 1$ and $n = 1$ and an isoperimetric inequality for closed curves in the plane which are smooth, have positive curvature and length $2\pi$. The rest of the article is organized as follows. In Section 2, we provide a new and direct method for maximizing the lowest eigenvalue of one dimensional Schrödinger operators. In Section 3 we establish the aforementioned connection with a problem for closed curves in the plane. We should emphasize that the isoperimetric problem that we allude to is also still open.

2. Maximizing the first eigenvalue

The problem of maximizing the lowest eigenvalue of the one-dimensional Schrödinger operator on the line subject to a constraint on integrals of powers of the potential was first considered by Joseph Keller in 1961 [12]. See also [17, 2, 4, 20].

Consider the Schrödinger operator,
\begin{equation}
H = -\frac{d^2}{dx^2} + V
\end{equation}
defined on $L^2(\mathbb{R})$, and let $-\lambda_1$ be the lowest eigenvalue. Then,
\begin{equation}
\lambda_1^\gamma \leq L_{\gamma,1}^1 \int_{-\infty}^{\infty} V_-(x)^{\gamma + 1/2} dx,
\end{equation}
for all $\gamma > 1/2$, where the sharp constants $L_{\gamma,1}^1$ are given by,
\begin{equation}
L_{\gamma,1}^1 = \frac{1}{\sqrt{\pi}} \frac{1}{\gamma - 1/2} \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + 1/2)} \left( \frac{\gamma - 1/2}{\gamma + 1/2} \right)^{\gamma + 1/2}.
\end{equation}
Keller’s proof uses the Direct Calculus of Variations. When the exponent $\gamma = 1$ there is a very simple argument to compute the best constant. We give the full argument in the sequel, because it is important in our later derivation of the connection between the Lieb–Thirring conjecture and an isoperimetric inequality for ovals in $\mathbb{R}^2$.

Let $u_1$ and $-\lambda_1$ be the normalized ground state and the lowest eigenvalue of the Schrödinger operator $H = -d^2/dx^2 - V$ on $L^2(\mathbb{R})$, where $V \geq 0$. Thus,
\begin{equation}
-u''_1 - V u_1 = -\lambda_1 u_1, \quad \text{in } \mathbb{R}.
\end{equation}
Multiplying (2.4) by $u_1$ and integrating in $\mathbb{R}$, we get
\begin{equation}
\lambda_1 = \int_{\mathbb{R}} V u_1^2 dx - \int_{\mathbb{R}} (u'_1)^2 dx.
\end{equation}
Since,
\begin{equation}
V u_1^2 \leq K V^{3/2} + \frac{4}{27 K^2} u_1^6,
\end{equation}
for all $K > 0$, from (2.5) we get,
\begin{equation}
\lambda_1 \leq K \int_{\mathbb{R}} V^{3/2} dx + \frac{4}{27 K^2} \int_{\mathbb{R}} u_1^6 dx - \int_{\mathbb{R}} (u'_1)^2 dx.
\end{equation}
However, if \( \int_R u_1^2 \, dx = 1 \),

\( \int_R (u_1')^2 \, dx \geq \frac{\pi^2}{4} \int_R u_1^6 \, dx, \)

so, choosing \( K = 4/(3\sqrt{3}\pi) \), we finally get

\( \lambda_1 \leq \frac{4}{3\sqrt{3}\pi} \int_R V^{3/2} \, dx = L_{1,1} \int_R V^{3/2} \, dx \)

which is Keller’s result for \( \gamma = 1 \). For completeness we give an elementary proof of (2.7). First make the change of variables \( x \to s \) given by

\( s = \int_{-\infty}^{x} u_1^2 \, dy. \)

Here, \( s : 0 \to 1 \), and \( ds/dx = u_1^2 \). With this change of variables we have,

\( \int_{-\infty}^{\infty} u_1^6 \, dx = \int_{0}^{1} u_1^4 \, ds, \)

and

\( \int_{-\infty}^{\infty} (u_1')^2 \, dx = \int_{0}^{1} (u_1')^2 u_1^2 \, ds, \)

where \( u_1 = du_1/ds \). Since \( u_1 \) goes to zero at \( x = \pm\infty \) we have \( u_1(s = 0) = u_1(s = 1) = 0 \). Finally, if we call \( w \equiv u_1^2 \), \( \int_{-\infty}^{\infty} u_1^2 \, ds \) and \( \int_{-\infty}^{\infty} (u_1')^2 \, dx = (1/4) \int_{0}^{1} \dot{w}^2 \, ds \). In terms of \( w(s) \), (2.7) is given by

\( \int_{0}^{1} \dot{w}^2 \, ds \geq \pi^2 \int_{0}^{1} w^2 \, ds, \)

which follows from the fact that the first Dirichlet eigenvalue of the interval \( (0,1) \) is \( \pi^2 \). One can obtain the cases with \( \gamma \neq 1 \) in (2.2) in a similar way (see the Appendix).

3. Maximizing the sum of the first two eigenvalues and the connection with a geometric problem in \( \mathbb{R}^2 \).

Consider the Schrödinger operator

\[ H = -\frac{d^2}{dx^2} - V, \]

on \( L^2(\mathbb{R}) \) with \( V \geq 0 \) such that \( \int V^{3/2} \, dx < \infty \). Assume \( H \) has at least two negative eigenvalues, and denote by \( -\lambda_1 \) and \( -\lambda_2 \) the lowest two eigenvalues and \( u_1, u_2 \) the corresponding normalized eigenfunctions. As before, we have

\( \lambda_1 = \int_R V u_1^2 \, dx - \int_R (u_1')^2 \, dx, \)

and

\( \lambda_2 = \int_R V u_2^2 \, dx - \int_R (u_2')^2 \, dx. \)

Adding these two equations and using the pointwise bound,

\[ V(u_1^2 + u_2^2) \leq K V^{3/2} + \frac{4}{27 K^2}(u_1^2 + u_2^2)^3, \]
we get
\begin{equation}
\lambda_1 + \lambda_2 \leq K \int_R V^{3/2} \, dx + \frac{4}{27 K^2} \int_R (u_1^2 + u_2^2)^3 \, dx - \int_R ((u'_1)^2 + (u'_2)^2) \, dx.
\end{equation}

In order to prove the Lieb–Thirring conjecture for $\gamma = 1$ in the special case of potentials having only two eigenvalues, it would be enough to prove
\begin{equation}
\int_R \left( (u'_1)^2 + (u'_2)^2 \right) \, dx \geq \frac{\pi^2}{4} \int_R (u_1^2 + u_2^2)^3 \, dx,
\end{equation}
for any pair of functions $u_1, u_2$ such that $\int_R u_1^2 \, dx = \int_R u_2^2 \, dx = 1$, and $\int_R u_1 u_2 \, dx = 0$ (i.e., for any pair of mutually orthogonal, normalized functions). For then, it would follow from (3.3) and (3.4) that
\begin{equation}
\lambda_1 + \lambda_2 \leq L_{1,1}^1 \int_R V^{3/2} \, dx.
\end{equation}

To prove (3.4) is still an open problem. Here we will show that (3.4) is equivalent to an (open) isoperimetric inequality for ovals on the plane. To establish this connection, we perform a change of variables similar to the one used in the previous section to prove Keller’s result on the lowest eigenvalue. First we change the independent variable
\begin{equation}
x \to s = \pi \int_{-\infty}^x \left( u_1^2 + u_2^2 \right) \, dy.
\end{equation}

Since $u_1$ and $u_2$ are both normalized, it follows that $s$ runs from 0 to $2\pi$. From (3.6) we have
\begin{equation}
\frac{ds}{dx} = \pi \left( u_1^2 + u_2^2 \right).
\end{equation}

Moreover, set
\begin{equation}
u_1 = \rho \cos \theta, \quad \text{and} \quad u_2 = \rho \sin \theta,
\end{equation}
so that
\begin{equation}
u_1^2 + u_2^2 = \rho^2, \quad \text{and} \quad (u_1^2 + u_2^2) = \rho^2 + \rho^2 \theta^2.
\end{equation}

With this change of variables we can write
\begin{equation}
\int_R \left( u_1^2 + u_2^2 \right) \, dx = \pi \int_0^{2\pi} \left( \rho^2 \dot{\rho}^2 + \rho^4 \dot{\theta}^2 \right) \, ds,
\end{equation}
and
\begin{equation}
\int_R \left( u_1^2 + u_2^2 \right)^3 \, dx = \frac{1}{\pi} \int_0^{2\pi} \rho^2 \, ds.
\end{equation}

Furthermore, set
\begin{equation}R = \rho^2,
\end{equation}
and
\begin{equation}\varphi = 2\theta.
\end{equation}

In these new variables, the desired inequality (3.4) is equivalent to
\begin{equation}
\frac{\int_0^{2\pi} \left( \dot{R}^2 + R^2 \varphi^2 \right) \, ds}{\int_0^{2\pi} R^2 \, ds} \geq 1,
\end{equation}
subject to the fact that $u_1$ and $u_2$ are orthonormal, fact that we ought to express in terms of the new variables. In the new variables,

$$0 = \int_R u_1 u_2 \, dx = \frac{1}{2\pi} \int_0^{2\pi} \sin \varphi(s) \, ds.$$  

Concerning the other side constraints (i.e., the fact that $u_1$ and $u_2$ are normalized), given the definition of $s$ and the fact that $s$ runs from 0 to $2\pi$, it is enough to consider the combination

$$0 = \int_R (u_1^2 - u_2^2) \, dx = \frac{1}{\pi} \int_0^{2\pi} \cos \varphi(s) \, ds.$$  

Thus, in the new variables, the fact that $u_1$ and $u_2$ are orthonormal imply

$$\int_0^{2\pi} \sin \varphi(s) \, ds = \int_0^{2\pi} \cos \varphi(s) \, ds = 0.$$  

These latter conditions can be given a simple geometrical interpretation. If one considers a closed curve in $\mathbb{R}^2$ and denote by $\cos \varphi(s)$ and $\sin \varphi(s)$ the components of the unit tangent, with respect to a fixed frame, as a function of arc-length, (3.12) just says that the curve in question is closed. Moreover, the curvature of the curve is given by

$$\kappa(s) = \frac{d\varphi}{ds}.$$  

Let’s denote by $C$ a closed curve in the plane, of length $2\pi$, with positive curvature, and let

$$H(C) \equiv -\frac{d^2}{ds^2} + \kappa^2$$  

acting on $L^2(C)$ with periodic boundary conditions. Then, (3.4), and for that matter (3.11), is equivalent to saying that the lowest eigenvalue of $H(C)$, $\lambda_1(C)$ say, is larger or equal to 1, for any closed curve on the plane of length $2\pi$. It is a simple fact to see that if $C$ is a circle of length $2\pi$, the lowest eigenvalue of $H(C)$ is precisely 1. Unfortunately we are far from proving the desired bound for general curves. It is relatively simple to show that the lowest eigenvalue of the Hamiltonian $H(C)$ is bounded below by $1/2$. To see this one first notes that the corresponding eigenfunction can be chosen to be positive. The quadratic form

$$(f, H(C)f) = \int_0^{2\pi} |f'(s)|^2 \, ds + \int_0^{2\pi} \kappa^2(s) f(s)^2 \, ds$$  

can be written as

$$\int_0^{2\pi} \left( \frac{d}{ds} (e^{i\varphi(s)} f(s)) \right)^2 \, ds,$$

which we have to minimize over non negative functions $f$ satisfying $\int_0^{2\pi} f(s)^2 \, ds = 1$. Expanding the function $e^{i\varphi(s)} f(s)$ into a Fourier series

$$e^{i\varphi(s)} f(s) = \sum_{n=-\infty}^{\infty} c_n e^{ins} \sqrt{2\pi},$$

we find that since $f(s) \geq 0$

$$|c_0|^2 \leq \frac{1}{2\pi} \left( \int_0^{2\pi} f(s) \, ds \right)^2.$$
Moreover, since the functions $1/\sqrt{2\pi}$ and $e^{i\varphi(s)}/\sqrt{2\pi}$ are orthogonal in the inner-product of $L^2([0,2\pi])$ we find that
\[
|c_0|^2 + \frac{1}{2\pi}(\int_0^{2\pi} f(s)ds)^2 \leq \int_0^{2\pi} f(s)^2ds = 1.
\]
Thus,
\[
|c_0|^2 \leq \min\left\{ \frac{1}{2\pi}(\int_0^{2\pi} f(s)ds)^2, 1 - \frac{1}{2\pi}(\int_0^{2\pi} f(s)ds)^2 \right\} \leq 1/2.
\]
Since $\sum_n |c_n|^2 = 1$ we learn that
\[
\sum_{n \neq 0} |c_n|^2 \geq \frac{1}{2}.
\]
Clearly,
\[
(f, H(C)f) = \sum_{n = -\infty}^{\infty} n^2 |c_n|^2 \geq \sum_{n \neq 0} |c_n|^2 \geq \frac{1}{2},
\]
hence $\lambda_1(C) \geq 1/2$.

Remarks:

i) A word of warning should be made at this point. In principle, the function $R$ defined from the eigenfunctions $u_1$ and $u_2$, via $\rho$ through equation (3.8) above, must vanish at $s = 0$ and $s = 2\pi$. For the curve problem, however, we drop this boundary condition. Thus, a priori the conjecture for the curve problem is stronger than the Lieb–Thirring conjecture for the two bound states, although we believe it amounts to the same.

ii) The best bound to date on $L_{1,1}$ is the bound of Eden and Foias [6] who proved,

(3.15)
\[
L_{1,1} \leq \frac{2}{9}\sqrt{3} \approx 0.3849\ldots
\]

Our bound $\lambda_1(C) \geq 1/2$ yields the bound
\[
L_{1,1} \leq \frac{4}{9\pi}\sqrt{6} \approx 0.3465\ldots,
\]
which although better than (3.15), only applies to Schrödinger operators with two bound states. Just for comparison, the conjectured sharp value for $L_{1,1}$ is $4\sqrt{3}/(9\pi) \approx 0.2450$.

iii) In recent years several authors have obtained isoperimetric inequalities for the lowest eigenvalues of a variant of $H(C)$, and we give a short summary of the main results in the sequel. Consider the Schrödinger operator

(3.16)
\[
H_g(C) \equiv -\frac{d^2}{ds^2} + g\kappa^2
\]
defined on $L^2(C)$ with periodic boundary conditions. As before, $C$ denotes a closed curve in $\mathbb{R}^2$ with positive curvature $\kappa$, and length $2\pi$. Here, $s$ denotes arclength. If $g < 0$, the lowest eigenvalue of $H_g(C)$, say $\lambda_1(g, C)$ is uniquely maximized when $C$ is a circle [5]. When $g = -1$, the second eigenvalue, $\lambda_2(-1, C)$ is uniquely maximized when $C$ is a circle [9]. If $0, g \leq 1/4$, $\lambda_1(g, C)$ is uniquely minimized when $C$ is a circle [7]. It is an open problem to determine the curve $C$ that minimizes $\lambda_1(g, C)$.
in the cases, $1/4 < g \leq 1$, and $g < 0, g \neq -1$. If $g > 1$ the circle is not a minimizer for $\lambda_1(g, C)$ (see, e.g., \[7, 8\] for more details on the subject).

To conclude this section we give an alternative interpretation of the minimization principle \((3.11)\) subject to the side constraints \((3.12)\). Interpret now $s$ as time (instead of arclength) and, given $R(s)$ and $\varphi(s)$ as before, define

$$x(s) = R(s) \cos \varphi(s),$$

and

$$y(s) = R(s) \sin \varphi(s).$$

Then, the minimization problem \((3.11), (3.12)\) is equivalent to the following,

\[(3.17)\]

$$\int_0^{2\pi} (\dot{x}^2 + \dot{y}^2) \, ds \geq 1,$$

where $x(s)$ and $y(s)$ are periodic, of period $2\pi$ and satisfy the side constraints,

\[(3.18)\]

$$\int_0^{2\pi} \frac{x(s)}{\sqrt{x(s)^2 + y(s)^2}} \, ds = \int_0^{2\pi} \frac{y(s)}{\sqrt{x(s)^2 + y(s)^2}} \, ds = 0.$$

Notice that \((3.17)\) certainly holds if one replaces the side constraints \((3.18)\) by $\int_0^{2\pi} x(s) \, ds = 0$ and $\int_0^{2\pi} y(s) \, ds = 0$, for then both functions $x(s)$ and $y(s)$ would be orthogonal to the constants and one would have $\int_0^{2\pi} \dot{x}^2 \, ds \geq \int_0^{2\pi} x(s)^2 \, ds$ and $\int_0^{2\pi} \dot{y}^2 \, ds \geq \int_0^{2\pi} y(s)^2 \, ds$, independently.

4. Appendix

To obtain inequality \((2.2)\) for $\gamma \neq 1$ we start from equation \((2.5)\) as before.

Using Hölder’s inequality we get

\[(4.1)\]

$$\lambda_1 \leq \left( \int_{-\infty}^{\infty} V^{\gamma + (1/2)} \, dx \right)^{2/2\gamma + 1} \left( \int_{-\infty}^{\infty} u_1^{2(2\gamma + 1)/(2\gamma - 1)} \, dx \right)^{(2\gamma - 1)/(2\gamma + 1)} - \int_{-\infty}^{\infty} (u_1')^2 \, dx.$$

We claim that if $\int_{-\infty}^{\infty} u_1^2 \, dx = 1$,

\[(4.2)\]

$$\int_{-\infty}^{\infty} u_1'^2 \, dx \geq c(\gamma) \left( \int_{-\infty}^{\infty} u_1^{2(2\gamma + 1)/(2\gamma - 1)} \, dx \right)^{2\gamma - 1},$$

where

$$c(\gamma) = \left[ \frac{\pi}{2} \Gamma(\gamma + 1/2) \right]^2 \frac{\gamma^\gamma \Gamma(\gamma + 1/2)}{\Gamma(\gamma + 1)(\gamma - 1/2)^{\gamma - 1/2}}.$$

Using the claim and denoting

$$A \equiv \left( \int_{-\infty}^{\infty} V^{\gamma + (1/2)} \, dx \right)^{2/2\gamma + 1},$$

and

$$Y \equiv \left( \int_{-\infty}^{\infty} u_1^{2(2\gamma + 1)/(2\gamma - 1)} \, dx \right)^{2\gamma - 1},$$

we get

\[(4.3)\]

$$\lambda_1 \leq AY^{1/(2\gamma + 1)} - c(\gamma)Y.$$
Maximizing the left side of (4.3) over \( Y \) (for \( \gamma > 1/2 \)), we get
\[
(4.4) \quad \lambda_1 \leq \tilde{c}(\gamma) \left( \int_{-\infty}^{\infty} V^{\gamma + (1/2)} \, dx \right)^{1/\gamma},
\]
where
\[
\tilde{c}(\gamma) = \frac{2\gamma}{c(\gamma)^{1/(2\gamma)}(2\gamma + 1)(2\gamma + 1)/(2\gamma)}.
\]
Hence,
\[
(4.5) \quad \lambda_1^\gamma \leq L_{1,1} \int_{-\infty}^{\infty} V^{\gamma + (1/2)} \, dx.
\]
To conclude we need only to prove the claim (4.2) whenever \( \int_{-\infty}^{\infty} u^2_1 \, dx = 1 \). Introducing the same change of variables as in Section 2, i.e.,
\[
x \to s = \int_{-\infty}^{x} u^2_1 \, dy,
\]
and
\[
w \equiv u^2_1,
\]
the claim reduces to proving
\[
(4.6) \quad \frac{1}{4} \int_0^1 \dot{w}^2 \, ds \geq c(\gamma) \left( \int_0^1 w^{2/2\gamma-1} \, ds \right)^{2\gamma-1},
\]
which follows from Sobolev’s inequality in one dimension.

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References

[1] M. Aizenman, and E. H. Lieb, On Semi–Classical Bounds for Eigenvalues of Schrödinger Operators. Physics Letters A 66 (1978), 427–429.

[2] M.S. Ashbaugh, and E.M. Harrell, Maximal and minimal eigenvalues and their associated nonlinear equations. J. Math. Phys. 28 (1987), 1770–1786.

[3] R. Benguria, and M. Loss, A simple proof of a theorem of Laptev and Weidl. Math. Res. Lett. 7 (2000), 195–203.

[4] C. Bennewitz, and E. J. M. Veling, Optimal bounds for the spectrum of a one-dimensional Schrödinger operator, in General inequalities, 6 (Oberwolfach, 1990), Internat. Ser. Numer. Math., Birkhäuser, Basel, 103 (1992), 257–268.

[5] P. Duclos, an P. Exner, Curvature–induced bound states in quantum waveguides in two and three dimensions. Rev. Math. Phys. 7 (1995), 73–102.

[6] Eden, A., Foias, C., A simple proof of the generalized Lieb–Thirring inequalities in one-space dimension. J. Math. Anal. Appl. 162 (1991), 250–254.

[7] P. Exner, E.M. Harrell, and M. Loss, Optimal eigenvalues for some Laplacians and Schrödinger operators depending on curvature. Mathematical results in quantum mechanics (Prague 1998), Oper. Theory Adv. Appl. 108 (1999), 47–58.

[8] E.M. Harrell, Gap estimates for Schrödinger operators depending on curvature, talk delivered at the 2002 UAB International Conference on Differential Equations and Mathematical Physics. Available electronically at http://www.math.gatech.edu/~harrell/

[9] E.M. Harrell, and M. Loss, On the Laplace operator penalized by mean curvature. Commun. Math. Phys., 195 (1998), 643–650.
[10] D. Hundertmark, E.H. Lieb, and L. Thomas, *A Sharp Bound for an Eigenvalue Moment of the One-Dimensional Schrödinger Operator*. Adv. Theor. Math. Phys. 2 (1998), 719–731.

[11] D. Hundertmark, A. Laptev, and T. Weidl, *New bounds on the Lieb–Thirring constants*. Invent. Math. 140 (2000), 693–704.

[12] Joseph B. Keller, *Lower bounds and isoperimetric inequalities for eigenvalues of the Schrödinger equation*. J. Mathematical Phys. 2 (1961), 262–266.

[13] A. Laptev, and T. Weidl, *Sharp Lieb–Thirring inequalities in high dimensions*. Acta Math. 184 (2000), 87–111.

[14] A. Laptev, and T. Weidl, *Recent results on Lieb–Thirring inequalities*. in “Journées Équations aux Dérivées Partielles” (La Chapelle sur Erdre, 2000), Exp. No. XX, 14 pp., Univ. Nantes, Nantes, 2000.

[15] E. H. Lieb, M. Loss, *Analysis. 2nd ed*. Graduate Studies in Mathematics. 14, Providence, RI, American Mathematical Society (AMS), 2001.

[16] E.H. Lieb, and W. Thirring, *Bounds for the kinetic energy of fermions which proves the stability of matter*. Phys. Rev. Lett. 35 (1975), 687–689. Errata: Phys. Rev. Lett. 35 (1975), 1116.

[17] E.H. Lieb, and W. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger hamiltonian and their relation to Sobolev inequalities*, in Studies in Mathematical Physics, Essays in Honor of Valentine Bargmann, edited by E.H. Lieb, B. Simon and A.S. Wightman, Princeton University Press, Princeton, NJ 1986, pp. 269–303.

[18] E.H. Lieb, *The Stability of Matter*. Reviews in Modern Physics 48 (1976), 553–569.

[19] E. H. Lieb, *Lieb–Thirring Inequalities* in Encyclopaedia of Mathematics, Suppl. II, Kluwer, Dordrecht 2000, pp. 311–312.

[20] E.J.M. Veling, *Lower bounds for the infimum of the spectrum of the Schrödinger operator in $\mathbb{R}^N$ and the Sobolev inequalities*. J. Inequal. Pure Appl. Math. 3 (2002), Article 63, 22 pp.

[21] T. Weidl, *On the Lieb–Thirring constants $L_{\gamma,1}$ for $\gamma \geq 1/2$*. Commun. Math. Phys., 178 (1996), 135–146.

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