On the Riemann Hypothesis

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Abstract

In this working paper I try to prove the Riemann hypothesis

let $\zeta$ the zeta function and $\eta$ the dirklet function $\forall s \in \mathbb{C}$ with $Re(s) > 0$ $\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^s}$

We know that $\forall s \in \mathbb{C}$ with $Re(s) > 0$ $(1 - 2^{1-s})\zeta(s) = \eta(s)$

Let $s = a + ib$ a complex number with $a, b \in \mathbb{R}$; $0 < a < 1, b \neq 0$ such that $\zeta(s) = 0$

We have also $\zeta(1 - s) = 0$

So $\eta(s) = 0$ (because $s \neq 1 + \frac{2k\pi i}{\ln 2}, k \in \mathbb{Z}$) and also $\eta(1 - s) = 0, \eta(s) = 0$ and $\eta(1 - s) = 0$

Since $\eta(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{x^{s-1}}{e^x + 1} = 0$ we have $\int_{0}^{\infty} \frac{x^{s-1}}{e^x + 1} = 0$ and also $\int_{0}^{\infty} \frac{x^{s-1}}{e^x + 1} = 0$

an integration by substitution ($x = e^t$) gives $\int_{-\infty}^{\infty} \frac{e^{xt}}{e^{t+1}} = 0$ and also $\int_{-\infty}^{\infty} \frac{e^{(1-s)t}}{e^{t+1}} = 0$

Let the complex function $f$ $\forall z \in \mathbb{C}$ $f(z) = \frac{e^{\pi z}}{e^{z+1}}$ $f$ is meromorphic and poles of $f$ are:

$z_{k,k'} = \ln((2k + 1)\pi) + sgn(2k + 1)i\frac{\pi}{2} + i2k'\pi$ $k, k' \in \mathbb{Z}$ where $sgn(2k + 1)$ is the sign of $(2k + 1)$

$z_{k,k'} = \ln((2k + 1)\pi) + i\frac{\pi}{2} + i2k'\pi$ $k \in \mathbb{N}, k' \in \mathbb{Z}$

See that $Re(z_{k,k'})$ is strictly positive

Let $n, m \in \mathbb{N}'$ and $\varepsilon \in \mathbb{R}$ with $0 < \varepsilon < \frac{1}{2}$ and $A \in \mathbb{R}, A = A_n = \ln((2n + \varepsilon)\pi)$

Let $K_{(n,m)}$ the compact set in $\mathbb{C}$ (the rectangle)

$K_{(n,m)} = \{x + iy, x, y \in \mathbb{R} - m \leq x \leq A_n and 0 \leq y \leq 2\pi\}$

Poles of $f$ in $K_{(n,m)}$ are

$z_k = \ln((2k + 1)\pi) + i\frac{\pi}{2}$ and $z'_{k} = \ln((2k + 1)\pi) + i\frac{3\pi}{2}$ $0 \leq k \leq (n - 1)$

(see the graph below)

The residu formula gives

$\int_{K_{(n,m)}} f(z)dz = 2\pi i(\sum_{k=0}^{n-1} \text{Res}(f, z_k) + \sum_{k=0}^{n-1} \text{Res}(f, z'_{k}))$
\[
\begin{align*}
\gamma_1 & \quad 0 \\
\gamma_2 & \quad \frac{i\pi}{2} \\
\gamma_3 & \quad i2\pi \\
\gamma_4 & \quad -i\pi \\
m & \quad \gamma_1 \\
A_n & \quad \gamma_3 \\
\end{align*}
\]
\[
\begin{align*}
\hat{f}_1 f(z) \, dz + \hat{f}_2 f(z) \, dz + \hat{f}_3 f(z) \, dz + \hat{f}_4 f(z) \, dz &= 2\pi i \left( \sum_{k=0}^{(n-1)} \text{Res}(f, z_k) + \sum_{k=0}^{(n-1)} \text{Res}(f, z'_k) \right) \\
\int_{-m}^{A} \frac{e^{zt}}{e^{(1+t)+1}} \, dt + i \int_{0}^{2\pi} \frac{e^{i(t+A)}}{e^{(1+t)+1}} \, dt - \int_{-m}^{A} \frac{e^{i(t+2\pi t)}}{e^{(1+t)+1}} \, dt - i \int_{0}^{2\pi} \frac{e^{i(t-m)}}{e^{(1-t)+1}} \, dt \\
&= 2\pi i \left( \sum_{k=0}^{(n-1)} \text{Res}(f, z_k) + \sum_{k=0}^{(n-1)} \text{Res}(f, z'_k) \right) \\
(1 - e^{2\pi i}) \int_{-m}^{A} \frac{e^{zt}}{e^{(1+t)+1}} \, dt + i \int_{0}^{2\pi} \frac{e^{i(t+A)}}{e^{(1+t)+1}} \, dt - ie^{s_{m}} \int_{0}^{2\pi} \frac{e^{is_{t}}}{e^{(1-t)+1}} \, dt \\
&= 2\pi i \left( \sum_{k=0}^{(n-1)} \text{Res}(f, z_k) + \sum_{k=0}^{(n-1)} \text{Res}(f, z'_k) \right)
\end{align*}
\]

Let's calculate \( \lim_{m \to +\infty} e^{-sm} \int_{0}^{2\pi} \frac{e^{is_{t}}}{e^{(1-t)+1}} \, dt \) 

\( \forall z \in \mathbb{C} \) with \(|z| \leq 1 \) \(|e^{z} + 1| \neq 0 \) so the functionz \( |e^{z} + 1| \) has amnima \( p > 0 \) On the cmopact 

\( \{z \in \mathbb{C} \) with \(|z| \leq 1 \}\)

So \( \forall z \in \mathbb{C} \) with \(|z| \leq 1 \) \(|e^{z} + 1| \geq p \)

\( \forall m \in \mathbb{N} \) \( \forall t \in [0, 2\pi] \) \(|e^{it-m}| = e^{(-m)} \leq 1 \) so \(|e^{e^{it-m}} + 1| \geq p \)

\( \forall m \in \mathbb{N} \) \( \forall t \in [0, 2\pi] \) \(|e^{it} - e^{-it} \mid \leq e^{-bt} \)

Since \( \int_{0}^{2\pi} e^{-bt} \, dt < \infty \) So \( \lim_{m \to +\infty} e^{-sm} \int_{0}^{2\pi} \frac{e^{is_{t}}}{e^{(1-t)+1}} \, dt = 0 \)

When \( m \) tends to \(+\infty\) the equation (1) becomes

\( (1 - e^{2\pi i}) \int_{-\infty}^{A} \frac{e^{zt}}{e^{(1+t)+1}} \, dt + i \int_{0}^{2\pi} \frac{e^{i(t+A)}}{e^{(1+t)+1}} \, dt = 2\pi i \left( \sum_{k=0}^{(n-1)} \text{Res}(f, z_k) + \sum_{k=0}^{(n-1)} \text{Res}(f, z'_k) \right) \)

Since \( \int_{-\infty}^{A} \frac{e^{zt}}{e^{(1+t)+1}} \, dt = 0 \) we have \( \int_{A}^{+\infty} \frac{e^{zt}}{e^{(1+t)+1}} \, dt = -\int_{A}^{+\infty} \frac{e^{zt}}{e^{(1+t)+1}} \, dt \)

\( -(1 - e^{2\pi i}) \int_{A}^{+\infty} \frac{e^{zt}}{e^{(1+t)+1}} \, dt + i \int_{0}^{2\pi} \frac{e^{i(t+A)}}{e^{(1+t)+1}} \, dt = 2\pi i \left( \sum_{k=0}^{(n-1)} \text{Res}(f, z_k) + \sum_{k=0}^{(n-1)} \text{Res}(f, z'_k) \right) \)

Let's calculate \( \sum_{k=0}^{(n-1)} \text{Res}(f, z_k) \)

\( \text{Res}(f, z_k) = \frac{e^{zt}}{e^{z} + e^{zt}} = \frac{e^{zt}}{e^{z} - 1} = -e^{(z-1)z} = -e^{(z-1)\left( \ln((2k+1)+i\frac{\pi}{2}) \right)} = -\pi^{(z-1)}e^{(z-1)(i\frac{\pi}{2}) \times \frac{1}{(2k+1)(1-s)}} \)

\( \sum_{k=0}^{(n-1)} \text{Res}(f, z_k) = -\pi^{(z-1)}e^{(z-1)(i\frac{\pi}{2}) \times \frac{1}{(2k+1)(1-s)}} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)(1-s)} \)

By the same we have

\( \sum_{k=0}^{(n-1)} \text{Res}(f, z'_k) = -\pi^{(z-1)}e^{(z-1)(i\frac{\pi}{2}) \times \frac{1}{(2k+1)(1-s)}} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)(1-s)} \)

So

\( -(1 - e^{2\pi i}) \int_{A}^{+\infty} \frac{e^{zt}}{e^{(1+t)+1}} \, dt + i \int_{0}^{2\pi} \frac{e^{i(t+A)}}{e^{(1+t)+1}} \, dt = -2\pi^{(z-1)}e^{(z-1)(i\frac{\pi}{2}) \times \frac{1}{(2k+1)(1-s)}} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)(1-s)} \)

\( -(1 - e^{2\pi i}) \int_{A}^{+\infty} \frac{e^{zt}}{e^{(1+t)+1}} \, dt + i \int_{0}^{2\pi} \frac{e^{i(t+A)}}{e^{(1+t)+1}} \, dt = -2\pi^{(z-1)}e^{(z-1)(i\frac{\pi}{2}) \times \frac{1}{(2k+1)(1-s)}} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)(1-s)} \)

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\[-(1 - e^{-2\pi i}) \int_A^{+\infty} \frac{e^{st}}{e^{sA} + 1} dt + i \int_0^{+\infty} \frac{e^{s(\frac{\pi}{2} + i + A)}}{e^{s(\frac{\pi}{2} + i + A)} + 1} dt = -2\pi^s (e^{\frac{\pi s}{2}} - e^{\frac{3\pi s}{2}}) \sum_{k=0}^{n-1} \frac{1}{(2k+1)(-s)} \]  

So equality (2) becomes

\[-(1 - e^{-2\pi i}) \int_A^{+\infty} \frac{e^{st}}{e^{sA} + 1} dt + i \int_0^{+\infty} \frac{e^{s(\frac{\pi}{2} + i + A)}}{e^{s(\frac{\pi}{2} + i + A)} + 1} dt = -2\pi^s (e^{\frac{\pi s}{2}} - e^{\frac{3\pi s}{2}}) \sum_{k=0}^{n-1} \frac{1}{(2k+1)(-s)} \]  

So equality (3) becomes

\[-(1 + e^{2\pi i}) \int_A^{+\infty} \frac{e^{st}}{e^{sA} + 1} dt + i e^{sA} \int_0^{+\infty} \frac{e^{s(-\sin t + i \cos t)}}{e^{s(-\sin t + i \cos t)} + 1} dt - i e^{sA} \int_0^{+\infty} \frac{e^{s(-\sin t + i \cos t)}}{e^{s(-\sin t + i \cos t)} + 1} dt + i e^{sA} \int_0^{+\infty} \frac{e^{s(-\sin t + i \cos t)}}{e^{s(-\sin t + i \cos t)} + 1} dt = -2\pi^s \sum_{k=0}^{n-1} \frac{1}{(2k+1)(-s)} \]  

\[-(1 + e^{2\pi i}) \int_A^{+\infty} \frac{e^{st}}{e^{sA} + 1} dt + i e^{sA} \int_0^{+\infty} \frac{e^{s(-\sin t + i \cos t)}}{e^{s(-\sin t + i \cos t)} + 1} dt - i e^{sA} \int_0^{+\infty} \frac{e^{s(-\sin t + i \cos t)}}{e^{s(-\sin t + i \cos t)} + 1} dt + i e^{sA} \int_0^{+\infty} \frac{e^{s(-\sin t + i \cos t)}}{e^{s(-\sin t + i \cos t)} + 1} dt = -2\pi^s \sum_{k=0}^{n-1} \frac{1}{(2k+1)(-s)} \]
We multiply by $e^{(3-s)A}$ we get

$$-(1 + e^{s\pi i})e^{-\frac{s\pi i}{2}}e^{(3-s)A}\int_{A}^{+\infty} \frac{e^{st}}{e^{t}+1} dt + ie^{3A}\left(\int_{0}^{\frac{\pi}{2}} e^{-sit} e^{A(t+cost)} dt - \int_{0}^{\frac{\pi}{2}} e^{sit} e^{A(t-cost)} dt\right) - \frac{1}{s} e^{3A}$$

$$= -2\pi^S e^{(3-s)A}\sum_{k=0}^{(n-1)} \frac{1}{(2k+1)(1-s)}$$

Let's calculate $\lim_{n\to+\infty} e^{(3-s)A} \int_{A}^{+\infty} \frac{e^{st}}{e^{t}+1} dt$

$$\left|\int_{A}^{+\infty} \frac{e^{st}}{e^{t}+1} dt\right| \leq \int_{A}^{+\infty} \frac{|e^{st}|}{|e^{t}+1|} dt = \int_{A}^{+\infty} \frac{e^{at}}{e^{t}} dt \leq \frac{1}{e^{(\frac{1}{2}A)}} f_{A}^{+\infty} \frac{e^{at}}{e^{t}} dt$$

$$\Rightarrow \lim_{n\to+\infty} e^{(3-s)A} \int_{A}^{+\infty} \frac{e^{st}}{e^{t}+1} dt = 0$$

So

$$-2\pi^S e^{(3-s)A}\sum_{k=0}^{(n-1)} \frac{1}{(2k+1)(1-s)}$$

$$= e^{3A}\left(\int_{0}^{\frac{\pi}{2}} e^{-sit} e^{A(t+cost)} dt - i \int_{0}^{\frac{\pi}{2}} e^{sit} e^{A(t-cost)} dt\right) - \frac{1}{s} e^{3A} + o(1)$$

$$e^{3A}\left(\int_{0}^{\frac{\pi}{2}} e^{-sit} e^{A(t+cost)} dt - i \int_{0}^{\frac{\pi}{2}} e^{sit} e^{A(t-cost)} dt\right) = e^{3A}\left(\int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt - i \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt\right)$$

$$+ e^{3A}\left(\int_{0}^{\frac{\pi}{2}} (s-1)(e^{-2it-e^{-it}}) \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt - i \int_{0}^{\frac{\pi}{2}} (s-1)(e^{2it-e^{it}}) \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt\right)$$

$$+ e^{3A}\left(\int_{0}^{\frac{\pi}{2}} e^{-sit} e^{it} e^{-it} \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt - i \int_{0}^{\frac{\pi}{2}} e^{sit} e^{it} e^{it} \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt\right)$$

**Lemma:**

$$\lim_{n\to+\infty} e^{3A} \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt = 0$$

**Proof:**

$$e^{3A} \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt = e^{3A} \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt + e^{3A} \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt$$

$$\leq e^{3A} \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt \leq \frac{\pi}{2} e^{A(t-cost)} dt$$

$$\leq \frac{1}{e^{(\frac{1}{2}A)}} \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt$$

$$e^{3A} \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt \leq \frac{\pi}{2} e^{A(t-cost)} dt$$

So

$$\lim_{n\to+\infty} e^{3A} \int_{0}^{\frac{\pi}{2}} e^{A(t-cost)} dt = 0$$
\[
\int_0^\pi \frac{t^3}{e^{A(t+i\cos t)}} dt \leq \int_0^\pi \frac{t^3}{e^{A(t+i\cos t)}} dt \leq \int_0^\pi \frac{\sqrt{2}}{e^{A(t)}} dt = \int_0^{\frac{\sqrt{2}}{2}} (\arcsin(u))^3 \times \frac{1}{\sqrt{1-u^2}} du
\]

By substitution \( t = \arcsin(u) \)

We have \( \forall u \in \left[ 0, \frac{\sqrt{2}}{2} \right] \frac{1}{\sqrt{1-u^2}} \leq \sqrt{2} \) so

\[
\int_0^\pi \frac{t^3}{e^{A(t+i\cos t)}} dt \leq \sqrt{2} \int_0^{\frac{\sqrt{2}}{2}} (\arcsin(u))^3 du
\]

The taylor formula with gives

\( \forall u \in \left[ 0, \frac{\sqrt{2}}{2} \right] \arcsin(u) = \arcsin(0) + u \times \frac{1}{\sqrt{1-(\xi_u)^2}} \) where \( \xi_u \in ]0,u[ \)

So \( \forall u \in \left[ 0, \frac{\sqrt{2}}{2} \right] \arcsin(u) \leq u\sqrt{2} \) (because \( -\frac{1}{\sqrt{1-(\xi_u)^2}} \leq \sqrt{2} \))

So \( \forall u \in \left[ 0, \frac{\sqrt{2}}{2} \right] (\arcsin(u))^3 \leq 2\sqrt{2}u^3 \)

So \( \int_0^\pi \frac{t^3}{e^{A(t+i\cos t)}} dt \leq 4 \int_0^{\frac{\sqrt{2}}{2}} \frac{u^3}{e^{A(t)}} du = 4e^{-4A} \int_0^{\frac{\sqrt{2}}{2}} \frac{v^3}{e^{v}} dv \leq 4e^{-4A} \int_0^{\infty} \frac{v^3}{e^{v}} dv \)

(By substitution \( e^A u = v \))

So \( e^{3A} \int_0^\pi \frac{t^3}{e^{A(t+i\cos t)}} dt \leq 4e^{3A} \int_0^{\infty} \frac{v^3}{e^{v}} dv \)

Since \( \int_0^{\infty} \frac{v^3}{e^{v}} dv < \infty \) we get \( \lim_{n \to +\infty} e^{3A} \int_0^\pi \frac{t^3}{e^{A(t+i\cos t)}} dt = 0 \)

So \( \lim_{n \to +\infty} e^{3A} \int_0^\pi \frac{t^3}{e^{A(t+i\cos t)}} dt = 0 \)

We deduce that \( \lim_{n \to +\infty} e^{3A} \int_0^\pi \frac{t^3}{e^{A(t+i\cos t)}} dt = 0 \)

Let’s prove that \( \lim_{n \to +\infty} e^{3A} \int_0^\pi \frac{\pi e^{i\pi t}}{e^{A(t+i\cos t)}} dt = 0 \) and

\( \lim_{n \to +\infty} e^{3A} \int_0^\pi \frac{\pi e^{-i\pi t}}{e^{A(t+i\cos t)}} dt = 0 \)

The taylor formula with integral gives

\( \exists M \in \mathbb{R}^+ \forall t \in \left[ 0, \frac{\pi}{2} \right] |e^{-i\pi t} - e^{-i\pi t - (s-1)}(e^{-2i\pi t - e^{-i\pi t}} - \frac{1}{2}(-s^2 + 3s - 2)t^2| \leq Mt^3 \)

So \( \left| \int_0^\pi \frac{\pi e^{i\pi t}}{e^{A(t+i\cos t)}} dt \right| \leq M \int_0^\pi \frac{t^3}{e^{A(t+i\cos t)}} dt \)

\( \left| \int_0^\pi \frac{\pi e^{-i\pi t}}{e^{A(t+i\cos t)}} dt \right| \leq Me^{3A} \int_0^\pi \frac{t^3}{e^{A(t+i\cos t)}} dt \)

Using the lemma we get \( \lim_{n \to +\infty} e^{3A} \int_0^\pi \frac{\pi e^{i\pi t}}{e^{A(t+i\cos t)}} dt = 0 \)

So \( \lim_{n \to +\infty} e^{3A} \int_0^\pi \frac{\pi e^{-i\pi t}}{e^{A(t+i\cos t)}} dt = 0 \)

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By the same we get: 
\[
\lim_{n \to +\infty} e^{3A} \int_0^{\pi} \frac{e^{i(t-\pi)}(e^{2it-e^{il}}-\frac{1}{2}(s^2+3s-2)t^2)}{e^{At-i\cos t} + 1} \, dt = 0
\]

We deduce that
\[
e^{3A} \left( i \int_0^{\pi} \frac{e^{-it}}{e^{At-i\cos t} + 1} \, dt - i \int_0^{\pi} \frac{e^{it}}{e^{At-i\cos t} + 1} \, dt \right) = e^{3A} \left( i \int_0^{\pi} \frac{e^{-it}}{e^{At-i\cos t} + 1} \, dt - i \int_0^{\pi} \frac{e^{it}}{e^{At-i\cos t} + 1} \, dt \right) + o(1)
\]

We deduce from equality (4)
\[
-2\pi e^{(3-s)A} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)(1-s)} = e^{3A} \left( i \int_0^{\pi} \frac{e^{-it}}{e^{At-i\cos t} + 1} \, dt - i \int_0^{\pi} \frac{e^{it}}{e^{At-i\cos t} + 1} \, dt \right) + o(1)
\]

Let the complex function \( h \) \( \forall z \in \mathbb{C} \) \( h(z) = \frac{e^{qt}}{e^{z^2+1}} \) \( q \in \mathbb{N}^* \)

The residu formula on \( K_{(n,m)} \) gives
\[
(1 - e^{q2\pi i}) \int_{-m}^{m} \frac{e^{qt}}{e^{z^2+1}} \, dt + i \int_{-m}^{m} \frac{e^{q(it+A)}}{e^{z^2+1}} \, dt - ie^{st} \int_{-m}^{m} \frac{e^{qst}}{e^{z^2+1}} \, dt = 2\pi i \left( \sum_{k=0}^{(n-1)} \text{Res}(h,z_k) + \sum_{k=0}^{(n-1)} \text{Res}(h,z'_k) \right)
\]

When \( m \) tends to +\( \infty \) we get
\[
i \int_{0}^{2\pi} \frac{e^{q(it+A)}}{e^{z^2+1}} \, dt = -2\pi e^{q2\pi} \left( e^{q(n-1)} - e^{q2\pi} \right) \frac{1}{(2k+1)(1-s)}
\]

By the same as above we have
\[
i \int_{0}^{\pi} \frac{e^{q(it+A)}}{e^{z^2+1}} \, dt - \frac{1}{q} e^{qA} e^{\frac{q\pi}{2}} = -2\pi q \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)(1-s)}
\]

\[
i \int_{0}^{\pi} \frac{e^{q(it+A)}}{e^{z^2+1}} \, dt - \frac{1}{q} e^{qA} e^{\frac{q\pi}{2}} = -2\pi q \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)(1-s)} + \frac{1}{q} e^{qA} e^{\frac{q\pi}{2}}
\]

\[
ie^{3A} \int_0^{\pi} \frac{e^{-qit}}{e^{At-i\cos t} + 1} \, dt - ie^{3A} \int_0^{\pi} \frac{e^{qit}}{e^{At-i\cos t} + 1} \, dt = e^{3A} \left( i \int_0^{\pi} \frac{e^{-qit}}{e^{At-i\cos t} + 1} \, dt - i \int_0^{\pi} \frac{e^{qit}}{e^{At-i\cos t} + 1} \, dt \right)
\]

\[
e^{3A} \left( i \int_0^{\pi} \frac{e^{-qit}}{e^{At-i\cos t} + 1} \, dt - i \int_0^{\pi} \frac{e^{qit}}{e^{At-i\cos t} + 1} \, dt \right) = -2\pi e^{(3-q)A} \sum_{k=0}^{(n-1)} \frac{1}{(2k+1)(1-s)} + \frac{1}{q} e^{3A}
\]
Let’s calculate $e^{3A}\left(i \int_0^{\pi/2} \frac{e^{-it}}{e^{A(sin t + cost)_{+1}}} dt - i \int_0^{\pi/2} \frac{e^{it}}{e^{A(sin t - cost)}_{+1}} dt\right)$

Using equality (6) for $q = 1$ we get

$$e^{3A}\left(i \int_0^{\pi/2} \frac{e^{-it}}{e^{A(sin t + cost)_{+1}}} dt - i \int_0^{\pi/2} \frac{e^{it}}{e^{A(sin t - cost)}_{+1}} dt\right) = -2\pi e^{2A}n + e^{3A} = -\pi e^{2A}(2n + \varepsilon - \varepsilon) + e^{3A}$$

$$= -\pi e^{2A}(2n + \varepsilon) + \pi\varepsilon e^{2A} + e^{3A} = -e^{3A} + \pi\varepsilon e^{2A} + e^{3A} = \pi\varepsilon e^{2A}$$

So $e^{3A}\left(i \int_0^{\pi/2} \frac{e^{-it}}{e^{A(sin t + cost)_{+1}}} dt - i \int_0^{\pi/2} \frac{e^{it}}{e^{A(sin t - cost)}_{+1}} dt\right) = \pi\varepsilon e^{2A}$

Let’s calculate $e^{3A}\left(i \int_0^{\pi/2} \frac{e^{-2it} - e^{-it}}{e^{A(sin t + cost)_{+1}}} dt - i \int_0^{\pi/2} \frac{e^{2it} - e^{it}}{e^{A(sin t - cost)}_{+1}} dt\right)$

Using equality (6) for $q = 2$ we get

$$e^{3A}\left(i \int_0^{\pi/2} \frac{e^{-2it} - e^{-it}}{e^{A(sin t + cost)_{+1}}} dt - i \int_0^{\pi/2} \frac{e^{2it} - e^{it}}{e^{A(sin t - cost)}_{+1}} dt\right) = -2\pi^2 e^A n^2 + \frac{1}{2} e^{3A}$$

$$= -\pi^2 e^A (2n^2 + 2\pi\varepsilon e^A) = -\frac{1}{2} e^A \left(-4\pi^2 n^2 + (2n + \varepsilon)^2 - 2\pi\varepsilon(2n + \varepsilon)\right)$$

$$= \frac{1}{2} e^A n^2 - 4\pi^2 n^2 + (2n + \varepsilon)^2 - 2\pi\varepsilon(2n + \varepsilon) = \frac{1}{2} e^A n^2 - 4\pi^2 n^2 + 4\pi n + \varepsilon e^2 - 4\pi n - 2\pi\varepsilon - 2\pi\varepsilon$$

$$= \frac{1}{2} \pi^2 e^2 e^A$$

By the same we can calculate $e^{3A}\left(i \int_0^{\pi/2} \frac{e^{t^2} - e^{-it}}{e^{A(sin t + cost)_{+1}}} dt - i \int_0^{\pi/2} \frac{e^{-t^2} - e^{it}}{e^{A(sin t - cost)}_{+1}} dt\right)$ we find

$$e^{3A}\left(i \int_0^{\pi/2} \frac{e^{t^2} - e^{-it}}{e^{A(sin t + cost)_{+1}}} dt - i \int_0^{\pi/2} \frac{e^{-t^2} - e^{it}}{e^{A(sin t - cost)}_{+1}} dt\right) = C + o(1) \text{ where } C \text{ is constante depending only on } \varepsilon$$

(By using the equation (5) which is also true for $q \in \mathbb{N}$’ we can take for example $q = 3$ there is a lot of calculus).

So equality (5) becomes

$$e^{3A}\left(i \int_0^{\pi/2} \frac{e^{-st}}{e^{A(sin t + cost)_{+1}}} dt - i \int_0^{\pi/2} \frac{e^{st}}{e^{A(sin t - cost)}_{+1}} dt\right) = \pi\varepsilon e^{2A} - \frac{1}{2} (s - 1) \pi^2 s^2 e^A + \frac{1}{2} (-s^2 + 3s - 2) C + o(1)$$

Thus equality (4) gives

$$-2\pi s e^{(3-s)A} \sum_{k=0}^{n-1} \frac{1}{(2k+1)^{(1-s)}} = \pi\varepsilon e^{2A} - \frac{1}{2} (s - 1) \pi^2 s^2 e^A + \frac{1}{2} (-s^2 + 3s - 2) C + o(1)$$

$$= \pi\varepsilon e^{2A} - \frac{1}{2} (s - 1) \pi^2 s^2 e^A + \frac{1}{2} (-s^2 + 3s - 2) C + o(1)$$

$$2\pi s e^{(3-s)A} \sum_{k=0}^{n-1} \frac{1}{(2k+1)^{(1-s)}} = e^{3A} - \pi\varepsilon e^{2A} + \frac{1}{2} (s - 1) \pi^2 e^A + \frac{1}{2} s (s^2 - 3s + 2) C + o(1)$$
Let $C'(s) = \frac{1}{2} s(s^2 - 3s + 2)C$ so

$$2\pi^s se^{(3-s)A} \sum_{k=0}^{n-1} \frac{1}{(2k+1)^{(1-s)}} = e^{3A} - \pi \varepsilon s e^{2A} + \frac{1}{2} s(s-1)\pi^2 \varepsilon^2 e^A + C'(s) + o(1)$$

$$2\pi^s s((2n + \varepsilon)\pi)^{(3-s)} \sum_{k=0}^{n-1} \frac{1}{(2k+1)^{(1-s)}}$$

$$= ((2n + \varepsilon)\pi)^3 - \pi \varepsilon s((2n + \varepsilon)\pi)^2 + \frac{1}{2} s(s-1)\pi^2 \varepsilon^2 (2n + \varepsilon)\pi + C'(s) + o(1)$$

$$2s(2n + \varepsilon)^{(3-s)} \sum_{k=0}^{n-1} \frac{1}{(2k+1)^{(1-s)}} = (2n + \varepsilon)^3 - \varepsilon s(2n + \varepsilon)^2 + \frac{1}{2} s(s-1)\varepsilon^2 (2n + \varepsilon) + C'(s) + o(1) \quad (7)$$

We have also

$$2s \sum_{k=0}^{n-1} \frac{1}{(2k+1)^{(1-s)}}$$

$$= (2n + \varepsilon)^s - \varepsilon s(2n + \varepsilon)^{(s-1)} + \frac{1}{2} s(s-1)\varepsilon^2 (2n + \varepsilon)^{(s-2)} + C'(s)(2n + \varepsilon)^{(s-3)} + o((2n + \varepsilon)^{(s-3)}) \quad (8)$$