WEAK AND STRONG UNIQUENESS FOR SDES AND PARABOLIC EQUATIONS WITH SINGULAR COEFFICIENTS

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Abstract. The goal of this article is to present analytic conditions on the coefficients of Itô’s equations admitting certain singularities ensuring that there exists a strong solution and any solution is strong. The conditions are formulated in terms of Morrey spaces and the proofs are based on the solvability of the corresponding parabolic equations with singular coefficients in these spaces, which is a new result.

1. Introduction

Let \( \mathbb{R}^d \) be a \( d \)-dimensional Euclidean space of points \( x = (x^1, \ldots, x^d) \) with \( d \geq 2 \). Let \( (\Omega, \mathcal{F}, P) \) be a complete probability space, carrying a \( d_1 \)-dimensional Wiener process \( w_t \), where \( d_1 \geq d \).

Assume that on \( \mathbb{R}^{d+1} = \{(t,x) : t \in \mathbb{R}, x \in \mathbb{R}^d \} \) we are given \( \mathbb{R}^d \)-valued Borel functions \( b = (b^i), \sigma^k = (\sigma^{ik}), k = 1, \ldots, d_1 \). Set \( \sigma = (\sigma^k) = (\sigma^{ik}) \). We are going to investigate the equation

\[
x_t = \int_0^t \sigma^k(s, x_s) dw^k_s + \int_0^t b(s, x_s) ds,
\]

where and everywhere below the summation over repeated indices is understood.

There is an increasing interest in studying the properties of weak and strong solutions to (1.1) with singular coefficients \( b \), see e.g. [2], [12], [13], [16], [18], [19] and the references therein.

Many properties of solutions in a variety of settings are studied. For instance, the authors of [17] study the weak differentiability of a unique strong solution with respect to the starting point in the classical subcritical case. The authors of [11] consider the case in which \( b \) satisfies a subcritical mixed-norm condition with respect to \( x^1, \ldots, x^d \). By a subcritical case we usually mean that \( b \in L^{p,q} \) with \( p, q \) satisfying \( d/p + 2/q < 1 \).

In [10] (close to arxiv.org/abs/2207.03626) the author proposed rather general conditions on \( \sigma \) and \( b \) under which there exists a strong solution of (1.1) (these are Assumptions 2.7 and 2.8). However, uniqueness of these

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solutions was proved only under additional assumptions requiring the solutions to have some properties, which some strong solutions are proved to have. In this article we present explicit analytic conditions under which the above mentioned additional requirements are actually satisfied.

The closest to our results in the literature are those from L. Beck, F. Flandoli, M. Gubinelli, and M. Maurelli [1], Röckner and Zhao [14] and [15], and [10]. In these articles the strong uniqueness is proved only in restricted classes of strong solutions (the classes smaller than in [1] are found in [15] and [10]). Here we show that the strong uniqueness holds without any restriction on the solution. However, we cannot cover all the cases considered in [1], [15], and [10] (see the comparison of results in Remarks 2.6, 2.11, and 2.12).

We finish the introduction with some notation. Define

\[ B_\rho(x) = \{ y \in \mathbb{R}^d : |x - y| < \rho \}, \mathbb{R}^{d+1} := \{ z = (t, x) : t \in \mathbb{R}, x \in \mathbb{R}^d \}, \]

\[ C_\rho(t, x) = \{ (s, y) \in \times \mathbb{R}^{d+1} : |x - y| < \rho, t \leq s < t + \rho^2 \}, C_\rho = C_\rho(0) \]

and let \( \mathcal{B}_\rho \) be the collection of \( B_\rho(x), x \in \mathbb{R}^d \), and \( \mathcal{C}_\rho \) be the collection of \( C_\rho(z), z \in \mathbb{R}^{d+1}, \mathcal{C} = \{ C_\rho, \rho > 0 \} \).

For measurable \( \Gamma \subset \mathbb{R}^{d+1} \) set \( |\Gamma| \) to be its Lebesgue measure and when it makes sense set

\[ f_\Gamma = \int f \, dz = \frac{1}{|\Gamma|} \int f \, dz. \]

Similar notation is used for \( f = f(x) \).

For \( p, q \in [1, \infty) \) and domains \( Q \subset \mathbb{R}^{d+1} \) we introduce the space \( L_{p,q}(Q) \) as the space of Borel functions on \( Q \) such that

\[ \|f\|_{L_{p,q}(Q)}^q := \int \left( \int \mathcal{I}_Q(t, x)|f(t, x)|^p \, dx \right)^{q/p} \, dt < \infty \]

if \( p \geq q \) and

\[ \|f\|_{L_{p,q}(Q)}^p := \int \left( \int \mathcal{I}_Q(t, x)|f(t, x)|^q \, dt \right)^{p/q} \, dx < \infty \]

if \( q \geq p \). If \( Q = \mathbb{R}^{d+1} \), we drop \( Q \) in the above notation. Of course, if we write, say \( f \in L_{p,q,\text{loc}} \) we mean that \( f \in L_{p,q}(Q) \) for any bounded \( Q \). If \( C \in \mathcal{C} \) we set

\[ \|f\|_{L_{p,q}(C)} = \|IC\|_{L_{p,q}(C)}^{-1} \|f\|_{L_{p,q}(C)} \quad (\#1\|_{L_{p,q}(C)} = 1). \]

Observe that, if \( C \in \mathcal{C}_\rho \), \( \|IC\|_{L_{p,q}(C)} = N(d)\rho^{d/p+2/q} \). Finally, set

\[ \partial_t u = \frac{\partial u}{\partial t}, \quad D_t u = \frac{\partial u}{\partial t^1}, \quad D_u = (D_t u), \quad D_{ij} = D_i D_j, \quad D^2 u = (D_{ij} u) \]

2. Main results

Define \( a = (1/2)\sigma \sigma^* \).
Assumption 2.1. For a $\delta \in (0, 1]$, we have
\[ \delta^{-1}|\xi|^2 \geq a^{ij}(t, x)\xi^i\xi^j \geq \delta|\xi|^2 \]
for all $t, x, \xi$.

To state our assumption on $b$, fix some $p_0, q_0 \in [1, \infty)$ such that
\[ \frac{d}{p_0} + \frac{1}{q_0} = 1. \] (2.1)

Assumption 2.2. We have
\[ B := \|b\|_{L^{p_0, q_0}} < \infty. \]

As we know from [5] under Assumptions 2.1 and 2.2 there is a probability space, carrying a $d_1$-dimensional Wiener process $w_t$, such that equation (1.1) has a solution (weak existence).

For $\rho \in (0, \infty)$ set
\[ \hat{b}_\rho = \sup_{r \leq \rho} \sup_{C \in C_r} \|b\|_{L^{p_0, q_0}(C)}. \]

Assumption 2.3. There exists $\rho_b \in (0, 1]$ such that $\hat{b}_\rho < \bar{b}_0 \wedge \bar{b}_1$, where $\bar{b}_0 = \bar{b}_0(d, \delta, p_0)$ is introduced in Assumption 3.1 and $\bar{b}_1 = \bar{b}_1(d, \delta, p_0, \rho_0, \rho_\beta)$ is introduced in Theorem 3.14.

Denote
\[ \text{osc}_{x}(a, C_r(t, x)) = r^{-2}|B_r|^{-2} \int_t^{t+r^2} \int_{y,z \in B_r(x)} |a(s, y) - a(s, z)| \, dydzds, \]
\[ a^\sharp_R(x) = \sup_{r \leq R} C \in C_r \text{osc}(a, C). \]

Observe that if $a$ is independent of $x$, then $a^\sharp_R(x) = 0$. The value of $\theta > 0$ in the assumption below will be specified later.

Assumption 2.4. We have a $\rho_0 \in (0, \infty)$ such that $a^\sharp_{\rho_0} \leq \theta$.

Theorem 2.5. Let Assumption 2.4 be satisfied with $\theta = \theta_0(d, \delta, p_0, q_0) > 0$ from (3.17) and let Assumptions 2.1, 2.2 and 2.3 be also satisfied. Then all solutions of (1.1), perhaps on different probability spaces, have the same finite-dimensional distributions (weak uniqueness).

We prove this theorem at the end of this section by using the results of Section 3.

Remark 2.6. In the basic case in [14] the weak uniqueness is proved in the class of solutions admitting, as they call it, Krylov type estimate when $\sigma$ is constant and we have $p, q \in (1, \infty)$ such that
\[ \frac{d}{p} + \frac{2}{q} = 1, \quad \int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |b|^p \, dx \right)^{q/p} \, dt < \infty \] (2.2)
(Ladyzhenskaya-Prodi-Serrin condition).
Observe that if \( p \geq d + 1 \) and \( q' = q/2 \) \( (p \geq q') \), then for any \( \rho > 0 \), and \( C \in C_{\rho} \), by Hölder’s inequality we have

\[
\|b\|_{L_{p,q'}(C)} \leq \rho^{2/q} \|b\|_{L_{p,q}(C)}, \quad \|b\|_{L_{p,q'}(C)} \leq N(d)\rho^{-1} \|b\|_{L_{p,q}(C)}
\]  

(2.3)

and the last norm tends to zero as \( \rho \downarrow 0 \). This shows that Assumption 2.3 is satisfied for an appropriate \( \rho_b \) and our assumption is weaker than (2.2) but only if \( p \geq d + 1 \). We also do not impose any a priori conditions on solutions.

Actually, in case \( p < d + 1 \) the estimates (2.3) are still valid and show that Assumption 2.3 is satisfied for an appropriate \( \rho_b \), provided that the integral in (2.2) is replaced with

\[
\int_{\mathbb{R}^d} \left( \int_{\mathbb{R}} |b|^q \, dt \right)^{p/q} \, dx.
\]

This is because of the way the \( L_{p,q'} \)-norm is defined. Therefore, in case \( p < d + 1 \) and condition (2.2) one should rely on (preconditional) weak uniqueness proved in [14] (if \( \sigma \) is constant).

To have existence and uniqueness of strong solutions we need additional assumptions. Fix \( \rho_\sigma \in (0,1] \). The value of \( \theta > 0 \) in Assumption 2.7 below will be specified later.

**Assumption 2.7.** For any \( \rho \leq \rho_\sigma \) and any \( C \in C_{\rho} \)

\[
\int_C |\sigma(t,x) - \sigma(t)| \, dx \, dt \leq \theta,
\]

(2.4)

where

\[
\sigma(t) = \int_C \sigma(t,x) \, dx \, ds \quad \text{(note } t \text{ and } ds).\]

Observe that \( \sigma(t) \) is the average of \( \sigma(t,x) \) over a ball of radius \( \rho \) and, if \( \sigma \) is independent of \( x \), the left-hand side of (2.4) is zero.

**Assumption 2.8.** (i) We have \( d \geq 3 \);

(ii) For any \( t, \sigma(t,\cdot) \in W^1_{1,\text{loc}} \) and the tensor-valued \( D\sigma \) admits a representation \( D\sigma = D\sigma_M + D\sigma_B \) with Borel summands (“Morrey part” of \( D\sigma \) plus the “bounded part”) such that there exist a finite constant \( \hat{D}\sigma_M \) and \( p_{D\sigma} \in (2, d] \) for which

\[
\left( \int_B |D\sigma_M(t,x)|^{p_{D\sigma}} \, dx \right)^{1/p_{D\sigma}} \leq \rho^{-1} \hat{D}\sigma_M,
\]

whenever \( t \in \mathbb{R}, B \in B_{\rho}, \) and \( \rho \leq \rho_\sigma, \) and there exists a constant \( \hat{D}\sigma_B \in (0, \infty) \) such that

\[
\int_{\mathbb{R}} |\hat{D}\sigma_B(t)|^2 \, dt \leq (\hat{D}\sigma_B)^2, \quad \hat{D}\sigma_B(t) := \text{esssup}_{x \in \mathbb{R}^d} |D\sigma_B(t,x)|;
\]
(iii) The vector-valued $b = (b^i)$ admits a representation $b = b_M + b_B$ with Borel summands such that there exist $p_b \in (d/2 + 1, d]$ and a constant $\hat{b}_M < \infty$ for which
\[
\left( \int_B |b_M(t,x)|^{p_b} \, dx \right)^{1/p_b} \leq \hat{b}_M \rho^{-1},
\] (2.5)
whenever $t \in \mathbb{R}$, $B \in \mathcal{B}_\rho$, and $\rho \leq \rho_b$, and there exists a constant $\bar{b}_B \in (0, \infty)$ such that
\[
\int_{\mathbb{R}} \bar{b}_B^2(t) \, dt \leq \bar{b}_B^2(t) := \sup_{x \in \mathbb{R}^d} |b_B(t,x)|.
\]

**Theorem 2.9.** There exist constants $\alpha_\sigma, \alpha_b > 0$, depending only on $d, d_1, \delta, p_{D\sigma}, p_b$, with $\alpha_b$ also depending on $\rho_\sigma$, and there exists $\theta(d, d_1, \delta, p_b) > 0$ such that, if Assumptions 2.1, 2.7, and 2.8 are satisfied with this $\theta$ and
\[
\hat{D}\sigma_M \leq \alpha_\sigma, \quad \hat{b}_M \leq \alpha_b,
\] (2.6)
then there exists a strong solution of (1.1). If, additionally, the assumptions of Theorem 2.5 are satisfied then there exists only one solution of (1.1) and it is the strong solution.

Proof. The first assertion of the theorem follows from the first assertion of Theorem 2.3 of [10]. By using a deep observation by A. Cherny, the uniqueness is derived exactly as it is done in the proof of Theorem 2.3 of [10] on the basis of weak uniqueness from Theorem 2.5. The theorem is proved.

By using Remark 2.7 of [10] showing, basically, that condition (2.4) follows from Assumption 2.8 (ii) we can restate the last assertion of Theorem 2.9 in a shorter albeit somewhat vague way.

**Theorem 2.10.** Let Assumptions 2.1 and 2.8 be satisfied. Then there exists one and only one solution of (1.1) and it is the strong solution provided that $\hat{D}\sigma_M, \hat{b}_M,$ and $\hat{b}_0^+$ are sufficiently small.

**Remark 2.11.** Let $d \geq 3$, and let Ladyzhenskaya-Prodi-Serrin condition (2.2) be satisfied with $p > d$. Then, as it is shown in Remark 1.1 of [6], Assumption 2.8 (iii) is satisfied with $p_b = d$, any $p_b$, and any $b_M$ on the account of splitting $b$ appropriately. If, additionally $p \geq d + 1$ and $\sigma$ is constant, as we have seen in Remark 2.6, the assumptions of Theorem 2.5 are satisfied, and hence we have uniqueness of strong solutions, without requiring them to possess certain properties like in [1], [15], and [10].

In case $d < p < d + 1$, $\sigma$ is constant, and (2.2) holds, we only know from [10] that the strong uniqueness holds in a restricted class of solutions, somewhat similar to [1] and [15].

In comparison with [10] (general $\sigma$), we avoid restricting the set of strong solutions on the account of the assumptions in Theorem 2.5 (whose proof is, actually, based on showing that any solution of (1.1) has the property required in [10] for strong solutions).
It is also worth noting that if the coefficients $\sigma$ and $b$ are time-independent then the corresponding version of Theorem 2.9 can be found in [8] and it holds, basically, under only conditions in (2.6).

**Remark 2.12.** There examples showing that the assumption of Theorem 2.9 can be satisfied and (2.2) does not hold no matter what $p, q$ are. For instance, take $b(t, x)$ such that $|b| = cf$, where constant $c > 0$ and $f = (|x| + \sqrt{|t|})^{-1}I_{|x|<1, |t|<1}$. It turns out that, if $c$ is small enough, $b$ satisfies the assumptions of Theorem 2.9 for any $p_0, q_0$ satisfying (2.1).

Indeed, if $|x| + \sqrt{|t|} \leq 3\rho$ and $p_0 \geq q_0 \geq (p_0 + 25\rho^2, 25\rho^2) \times B_{3\rho}$ and

$$
\|f\|_{L_{p_0,q_0}(C_{\rho}(t,x))} \leq N\rho^{-q_0-1}\int_{-25\rho^2}^{25\rho^2} \left( \int_{\mathbb{R}^d} \frac{1}{(|y| + \sqrt{|s|})^{p_0}} dy \right)^{q_0/p_0} ds = N\rho^{-q_0}.
$$

Similar inequality holds if $p_0 < q_0$.

If $|x| + \sqrt{|t|} \geq 3\rho$, then $f \leq N/\rho$ on $C_{\rho}(t, x)$ and $\|f\|_{L_{p_0,q_0}(C_{\rho}(t,x))} \leq N/\rho$.

Also for any $p \in [1, d)$, if $|x| \leq 2\rho$

$$
\|f(t, \cdot)\|_{L_p(B_{\rho}(x))} \leq N\rho^{-d+1} \int_{B_{3\rho}} \frac{1}{|y|^d} dy = N\rho^{-d},
$$

and if $|x| > 2\rho$, then $f(t, \cdot) \leq \rho^{-1}$ on $B_{\rho}(x)$ and the inequality between the extreme terms of (2.7) holds again. Hence, if $c$ is sufficiently small the above $b$ satisfies the assumptions in Theorem 2.9 and (1.1) has a unique solution, say if $\sigma$ is constant, and this solution is strong.

However, there are no $p, q$ such that (2.2) holds. Indeed, for small $t$

$$
\int_{B_1} \frac{1}{(|x| + \sqrt{|t|})^p} dx
$$

is of order $|t|^{(d-p)/2}$, whose $(q/p)$th power is $|t|^{-1}$ which is not integrable near zero.

The proof of Theorem 2.5 is based on Itô’s formula, before coming to which we need some auxiliary facts. By $d_0 = d_0(d, d, p_0, \rho_0) \in (d/2, d)$ we denote the constant introduced in Remark 4.1 of [9]. Below we suppose that the assumptions of Theorem 2.5 are satisfied.

Here is a particular case of Theorem 4.10 of [9].

**Theorem 2.13.** Suppose

$$
p, q \in [1, \infty], \quad \frac{d_0}{p} + \frac{1}{q} \leq 1.
$$

Then for any $R \in (0, \infty)$ and Borel nonnegative $f$ given on $C_R$, we have

$$
E \int_0^{\tau_R} f(t, x_t) dt \leq N\|f\|_{L_{p,q}(C_R)},
$$

where $N$ is independent of $f$ and $\tau_R$ is the first exit time of $(t, x_t)$ from $C_R$. \hfill (2.9)
The Sobolev space $W^{1,2}_{p,q}(Q)$ is introduced as the space of functions $u \in L_{p,q}(Q)$ such that $\partial_t u, D^2 u, Du \in L_{p,q}(Q)$ provided with natural norm. If $Q = \mathbb{R}^{d+1}$ we drop $Q$ in the above notation.

Here is Itô’s formula we need.

**Theorem 2.14.** Assume that (2.8) holds and $p = p_0/\beta$, $q = q_0/\beta$, where $\beta \in (1, 2]$. Let $R \in (0, \infty)$, $u \in W^{1,2}_{p,q}(C_R)$. Assume that $Du \in L_{r,s}(C_R)$ where $(r, s) = (\beta - 1)^{-1}(p_0, q_0)$. Then, with probability one for all $t \geq 0$,

$$u(t, x) = u(0) + \int_0^t Du(s, x) \, dw^k + \int_0^t \left[ \partial_t u(s, x) + a^{ij} D_{ij} u(s, x) + b^i D_i u(s, x) \right] \, ds$$

and the stochastic integral above is a square-integrable martingale.

Proof. The last statement, of course, follows from Theorem 2.13 and the fact that $2p \leq r, 2q \leq s$. To prove the rest we approximate $u$ by smooth functions $u^{(\varepsilon)} = \zeta_\varepsilon u$, where $\zeta_\varepsilon (t, x) = \varepsilon^{-d-2} \zeta(t/\varepsilon^2, x/\varepsilon)$, and $\zeta \in C_0^\infty(\mathbb{R}^{d+1})$ has support in $(-1, 0) \times B_1$ and unit integral. Since $d/p + 2/q < 2$ ($d < 2d_0$), by embedding theorems $u \in C(\overline{C}_R)$ and, therefore, $u^{(\varepsilon)} \to u$ as $\varepsilon \downarrow 0$ uniformly in any $C_{R'}$ with $R' < R$.

Fix $R' < R$. Then for all sufficiently small $\varepsilon > 0$ by Itô’s formula

$$u^{(\varepsilon)}(t, x) = u^{(\varepsilon)}(0) + \int_0^t Du^{(\varepsilon)}(s, x) \, dw^k + \int_0^t \left[ \partial_t u^{(\varepsilon)}(s, x) + a^{ij} D_{ij} u^{(\varepsilon)}(s, x) + b^i D_i u^{(\varepsilon)}(s, x) \right] \, ds$$

We send $\varepsilon \downarrow 0$ and observe that $u^{(\varepsilon)} \to u$ in $W^{1,2}_{p,q}(C_{R'})$ and in $L_{r,s}(C_{R'})$.

Hence, Theorem 2.13 allows us easily to pass to the limit in (2.11), for instance, by using Hölder’s inequality we obtain

$$E \int_0^t |b| |Du - Du^{(\varepsilon)}|(t, x) \, dt \leq N \|b\|_{L_{p,q}(C_{R'})} \leq N \|b\|_{L_{p_0,q_0}(C_R)} \|Du - Du^{(\varepsilon)}\|_{L_{r,s}(C_{R'})} \to 0.$$ 

It follows that (2.11) holds with $u$ in place of $u^{(\varepsilon)}$. After that it only remains to send $R' \uparrow R$ and again use Theorem 2.13. The theorem is proved.

**Proof of Theorem 2.5.** Take a bounded Borel $c \geq 1, f$ on $\mathbb{R}^{d+1}$. By Remark 3.15 there is a bounded function $u$ defined uniquely by $a, b, c, f$, such that $u \in W^{1,2}_{p,q,loc}$, $Du \in L_{r,s,loc}$, and (2.8) is satisfied, where $p, q, r, s$ are taken from Theorem 2.14, and

$$\partial_t u + a^{ij} D_{ij} u + b^i D_i u - cu = -f.$$ 

By Theorem 2.14 and Itô’s formula applied to

$$u(t, x_t) \exp \left( - \int_0^t c(s, x_s) \, ds \right)$$
we obtain
\[ u(0) = E u(\tau_R, x_{\tau_R}) \exp \left( - \int_0^{\tau_R} c(s, x_s) \, ds \right) + E \int_0^{\tau_R} f(t, x_t) \exp \left( - \int_0^t c(s, x_s) \, ds \right) \, dt. \]

We send here \( R \to \infty \) taking into account that \( c \geq 1 \), \( u, f \) are bounded and \( \tau_R \to \infty \). Then we get that
\[ E \int_0^\infty f(t, x_t) \exp \left( - \int_0^t c(s, x_s) \, ds \right) \, dt \]
is uniquely defined by \( a, b, c, f \) (since it equals \( u(0) \)). For \( T > 0 \) and \( f = cI_{t<T} \) this shows that
\[ E \exp \left( - \int_0^T c(s, x_s) \, ds \right) \]
is uniquely defined by \( a, b, c, T \). The arbitrariness of \( c \) and \( T \) certainly proves the theorem.

3. **Parabolic equations with singular coefficients**

Here we are dealing with the operators
\[ L_0 u(t, x) = \partial_t u + a^{ij}(t, x) D_{ij} u(t, x), \]

\[ L u(t, x) = \partial_t u(t, x) + a^{ij}(t, x) D_{ij} u(t, x) + b^i(t, x) D_i u(t, x) \]

with symmetric \( (a^{ij}) \) satisfying Assumption 2.1.

We restate Assumption 2.2 of [9] in the following way.

**Assumption 3.1.** There exists \( \rho_0 \in (0, \infty) \) such that
\[ \bar{b}_{\rho_0} < \bar{b}_0, \]
where \( \bar{b}_0 = \bar{b}_0(d, \delta, \rho_0) = \bar{N}^{-1} \leq 1 \) and \( \bar{N} \) is taken from (2.15) of [9].

Throughout this section Assumptions 2.1 and 3.1 are supposed to be satisfied.

**Remark 3.2.** A simple covering argument shows that for \( \rho \geq \rho_0 \) and \( C \in C_\rho \)
\[ \|b\|_{L_{p_0,q_0}(C)} < N(d)(\rho/\rho_0)^{d-1/\rho_0 \bar{b}_{\rho_0} \rho^{-1}}. \]

For \( p, q \in [1, \infty) \) and domains \( Q \subset \mathbb{R}^{d+1} \) we introduce the space \( L_{p,q}(Q) \) as the space of Borel functions on \( Q \) such that
\[ \|f\|_{L_{p,q}(Q)}^q := \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} I_Q(t, x)|f(t, x)|^p \, dx \right)^{q/p} \, dt < \infty. \]

The Sobolev space \( W^{1/2}_{p,q}(Q) \) is introduced as the space of functions \( u \in L_{p,q}(Q) \) such that \( \partial_t u, D^2 u, Du \in L_{p,q}(Q) \) provided with natural norm. If \( Q = \mathbb{R}^{d+1} \) we drop \( Q \) in the above notation. Observe that \( L_{p,q} = L_{p,q} \) only if \( p \geq q \).

Here is a particular case of Theorem 5.1 of [9]. Fix \( \bar{R} \in (0, \infty) \).
Theorem 3.3. Let $0 < R \leq \bar{R}$, domain $Q \subset C_R$, and assume that

\[ p, q \in [1, \infty), \quad \frac{d_0}{p} + \frac{1}{q} = 1. \]

(3.1)

and that we are given a function $u \in W^{1,2}_{p,q,loc}(Q) \cap C(\bar{Q})$. Then on $Q$

\[ u \leq NR^{(2d_0-d)/p} \|I_{Q,u>0}(Lu)\|_{L_p,q} + \sup_{\partial' Q} u^+, \]

(3.2)

where $N = N(\delta, d, p_0, R, p, p_0)$ (and $\partial' Q$ is the parabolic boundary of $Q$). In particular (the maximum principle), if $\partial_t u + Lu \geq 0$ in $Q$ and $u \leq 0$ on $\partial' Q$, then $u \leq 0$ in $Q$.

Here is a particular case of Theorem 6.3 of [3].

Theorem 3.4. Let $p, q \in (1, \infty)$. There exists $\theta = \theta(d, \delta, p, q) > 0$ such that if Assumption 2.4 is satisfied with this $\theta$, then there exist $\lambda_0, N_0 \geq 0$, depending only on $d, \delta, p, q, \rho_0$, such that for any $\lambda \geq \lambda_0$ and $u \in W^{1,2}_{p,q}$ we have

\[ \|D^2 u\|_{L_p,q} + \sqrt{\lambda} \|Du\|_{L_p,q} + \lambda \|u\|_{L_p,q} \leq N_0 \|L_0 u - \lambda u\|_{L_p,q}. \]

(3.3)

Remark 3.5. For the purposes of this article it is important that, as it is not hard to derive from the arguments in [3] (see what is written in a few lines below (4.1) of [3]), one can interchange the integration order in the definition of the $L_{p,q}$-norm without affecting the validity of Theorem 3.4. Therefore, the assertion of Theorem 3.4 still holds if we replace $L_{p,q}$ and $W^{1,2}_{p,q}$ with $L_{p,q}$ and $W^{1,2}_{p,q}$, respectively.

Now to Assumptions 2.1 and 3.1 we add the following.

Assumption 3.6. Assumption 2.4 is satisfied with $\theta$ from Theorem 3.4.

From this moment on we suppose that Assumptions 2.1, 3.1, and 3.6 are satisfied.

Corollary 3.7. There exists $N$, depending only on $d, \delta, p, q, \rho_0$, such that for any $u \in W^{1,2}_{p,q}$ we have

\[ \|D^2 u\|_{L_p,q} + \|Du\|_{L_p,q} + \|u\|_{L_p,q} \leq N \|L_0 u\|_{L_p,q} + \|u\|_{L_p,q}. \]

(3.4)

By using a standard way of localizing such estimates given, for instance, in the proof of Lemma 2.4.4 of [4], one arrives at the following.

Lemma 3.8. There exists $N$, depending only on $d, \delta, p, q, \rho_0$, such that for any $0 < \rho_1 < \rho_2 \leq 1$ and $u \in W^{1,2}_{p,q,loc}(C_{\rho_2}) \cap C(C_{\rho_2})$ we have

\[ \|D^2 u\|_{L_p,q(C_{\rho_1})} \leq N \|L_0 u\|_{L_p,q(C_{\rho_2})} + N(\rho_2 - \rho_1)^{-2} \|u - l\|_{L_p,q(C_{\rho_2})}, \]

(3.5)

where $l$ is any affine function of $x$.

Here the appearance of $l$ is explained by the fact that $D^2 u$ and $L_0 u$ do not change if we take $u - l$ in place of $u$.

A new restriction on $\theta$ comes from the following.
Lemma 3.9. Let (3.1) be satisfied and take $\alpha > 0$. Then there is $\theta > 0$, $\theta \leq \theta (d, \delta, \rho, q)$, and $N, N_1, N_2$ depending only on $d, \delta, p, q$ with $N, N_1, N_2$ also depending on $\rho_0$ such that, if Assumption 2.4 is satisfied with this $\theta$, $0 < 4\rho_1 \leq \rho_2 \leq 1$, $u \in W^{1,2}_{p,q,\text{loc}}(C_{\rho_2}) \cap C(\overline{C}_{\rho_2})$, then

\[ \|D^2 u\|_{L_{p,q}(C_{\rho_2})} \leq N_1(\rho_2/\rho_1)^2 \|L_0 u\|_{L_{p,q}(C_{\rho_2})} + N(\rho_2/\rho_1)^{\alpha} \rho_2^{-2} \sup_{\partial C_{\rho_2}} |u - l|, \]

where $l$ is any affine function.

Proof. Observe that to prove (3.6) it suffices to concentrate on $\alpha < 1$. Having in mind obvious approximations we may assume that $u$ and $a$ are infinitely differentiable in $C_{\rho_2}$. Then by classical results there is smooth solution $v$ of the equation $Lv = 0$ in $C_{\rho_2}$ with boundary data $v = u$ on $\partial C_{\rho_2}$. This solution belongs to $W^{1,2}_{p,q}(C_{\rho_2})$ for any $r \geq 1$. We take and fix $r > (d+2)/\alpha$, $r \geq p, q$. Then by Lemma 3.8, for appropriately chosen $\theta$ we have

\[ \|D^2 v\|_{L_{r,r}(C_{\rho_2}/2)} \leq N(\rho_2 - \rho_2/2)^2 \sup_{C_{\rho_2}} |v - l| = N\rho_2^{-2} \sup_{\partial C_{\rho_2}} |u - l|, \]

where $l$ is any affine function and the equalities are due to the maximum principle. It follows by Hölder’s inequality that

\[ \|D^2 v\|_{L_{p,q}(C_{\rho_1})} \leq \|D^2 v\|_{L_{r,r}(C_{\rho_1})} \leq N(\rho_2/\rho_1)^{(d+2)/r} \|D^2 v\|_{L_{r,r}(C_{\rho_2}/2)} \leq N(\rho_2/\rho_1)^{(d+2)/r} \rho_2^{-2} \sup_{\partial C_{\rho_2}} |u - l|, \]

Since $(d+2)/r < \alpha$, we get

\[ \|D^2 v\|_{L_{p,q}(C_{\rho_1})} \leq N(\rho_2/\rho_1)^{\alpha} \rho_2^{-2} \sup_{\partial C_{\rho_2}} |u - l|. \]

Then a very particular case of (3.5) is that

\[ \|D^2 u\|_{L_{p,q}(C_{\rho_1})} \leq N \|L_0 u\|_{L_{p,q}(C_{\rho_1})} + N \rho_1^{-2} \sup_{C_{\rho_1}} |u - l'|, \]

where $l'$ is any affine function. Here $u - l' = w + (v - l')$, where $w = 0$ on $\partial C_{\rho_2}$ and $L_0 w = L_0 u$. It follows from Theorem 3.3 that $|w| \leq N\rho_2^2 \|L_0 u\|_{L_{p,q}(C_{\rho_2})}$. Hence the last term in (3.9) is dominated by

\[ N(\rho_2/\rho_1)^2 \|L_0 u\|_{L_{p,q}(C_{\rho_2})} + N \rho_1^{-2} \sup_{C_{\rho_1}} |v - l'|, \]

where the second term, for an appropriate choice of $l'$, is estimated by a constant times

\[ \|D^2 v\|_{L_{p,q}(C_{\rho_1})} + \|\partial v\|_{L_{p,q}(C_{\rho_1})} \leq N \|D^2 v\|_{L_{p,q}(C_{\rho_1})} \]
owing to the Poincaré inequality. After that, to get (3.6), it only remains
of these spaces proved in [7] when
\( L_{p,q}(C_\rho) \) and \( f := L_0 u \in L_{p,q}(C_\rho) \). Then there exists
\( \nu = \nu(\alpha, \beta, d, \rho, p, q) > 1 \) such that for any \( r \leq \rho/\nu \) we have
\[
\| \partial_t u, D^2 u \|_{L_{p,q}(C_r)} \leq N \sup_{r \leq s \leq \rho} s^\beta \| f \|_{L_{p,q}(C_s)} + N \rho^{\beta - 2} \sup_{C_\rho} |u - l|, \tag{3.10}
\]
where \( l \) is any affine function and the constants \( N \) depend only on \( \alpha, \beta, \rho_0, d, \delta, p \).

Proof. Since \( \partial_t u = f - a^{ij} D_{ij} u \), it suffices to concentrate on \( D^2 u \). Take
the smallest \( \nu \geq 4 \) such that \( N_2 \nu^{\alpha - \beta} \leq 1/2 \). Then, for \( r \leq \rho/\nu \), define
\( r_n = \nu^{n} r, m = \max\{n \geq 0 : r_{n+1} \leq \rho\} \),
\[ A_n = r_n^\beta \| D^2 u \|_{L_{p,q}(C_r)}, \quad B = \sup_{r \leq s \leq \rho} r^\beta \| f \|_{L_{p,q}(C_s)}. \]
For \( 0 \leq n \leq m, \rho_1 = r_n, \rho_2 = r_{n+1} \), estimate (3.7) yields
\[ A_n \leq N_1 \nu^{2-\nu} B + N_2 \nu^{\alpha - \mu} A_{n+1} \leq N_3 B + (1/2) A_{n+1}. \]
By iterating we obtain \( A_0 \leq 2 N_3 B + 2^{-m} A_m \) and arrive at
\[
r^\beta \| D^2 u \|_{L_{p,q}(C_r)} \leq N \sup_{r \leq s \leq \rho} s^\beta \| f \|_{L_{p,q}(C_s)} + 2^{-m} \rho^{\beta} \| D^2 u \|_{L_{p,q}(C_{rm})}. \tag{3.11}
\]
Here by (3.6)
\[ r_m^\beta \| D^2 u \|_{L_{p,q}(C_{rm})} \leq N r_m^\beta \| f \|_{L_{p,q}(C_{rm+1})} + N r_m^{\beta - 2} \sup_{C_{rm+1}} |u - l| \]
\[ \leq N \sup_{r \leq s \leq \rho} s^\beta \| f \|_{L_{p,q}(C_s)} + N \rho^{\beta - 2} \sup_{C_\rho} |u - l|, \]
where the inequality is due to the fact that \( r \leq r_{m+1} \leq \rho, r_{m+1}/\rho \geq \nu^{-1} \).
This proves the lemma.

For \( \beta \geq 0 \), introduce Morrey’s space \( E_{p,q,\beta} \) as the set of \( g \in L_{p,q,\text{loc}} \) such that
\[ \| g \|_{E_{p,q,\beta}} := \sup_{\rho \leq 1, C \in \mathcal{C}_\rho} \rho^\beta \| g \|_{L_{p,q}(C)} < \infty, \tag{3.12} \]
where \( \| g \|_{L_{p,q}(C)} \) is introduced in (1.2). Introduce
\[ E^2_{p,q,\beta} = \{ u : u, Du, D^2 u \in E_{p,q,\beta} \} \]
and provide \( E^2_{p,q,\beta} \) with an obvious norm. We are going to use some properties of these spaces proved in [7] when \( E^2_{p,q,\beta} \) is introduced in terms of \( \mathbb{L}_{p,q,S} \)-spaces. The proofs extend to our \( E^2_{p,q,\beta} \) without any trouble.

Lemma 3.10 (with \( \rho = 1 \)) and Remark 3.2 immediately lead to the fol-
lowing.
Corollary 3.11. Let (3.1) be satisfied and let $\beta > \alpha > 0$. Suppose that Assumption 2.4 is satisfied with $\theta = \theta(d, \delta, p, q, \alpha)$ from Lemma 3.9. Let bounded $u \in W^{1,2}_{p,q,\text{loc}}$ and $f := L_0 u \in E_{p,q,\beta}$. Then
\[ \|\partial_t u, D^2 u\|_{E_{p,q,\beta}} \leq N \|f\|_{E_{p,q,\beta}} + N \sup_{\mathbb{R}^{d+1}} |u|, \]  
(3.13)
where the constants $N$ depend only on $\alpha, \beta, \rho_0, d, \delta, p$.

To deal with the full operator $L$ we need the following.

Lemma 3.12. Take a number $\beta > 1$ such that $p = p_0/\beta > 1$ and $q = q_0/\beta > 1$. Then for any $u \in E^{1,2}_{p,q,\beta}$ we have
\[ \|b| Du| \|_{E_{p,q,\beta}} \leq N \hat{b}_{p_0} \|\partial_t u, D^2 u\|_{E_{p,q,\beta}} + N \hat{b}_{p_0} \rho^{-2} \|u\|_{E_{p,q,\beta}}, \]  
(3.14)
where $N = N(d, p_0, \beta)$.

Proof. Take $\zeta \in C_0^{\infty}(\mathbb{R}^{d+1})$ such that $1 \geq \zeta \geq 0$, $\zeta = 1$ on $C_{p_0}$ and $\zeta = 0$ outside $C_{2p_0}$. It easily follows from Remark 3.2 that for any $\rho > 0$ and $C \in C_\rho$
\[ \|\zeta \|_{L^{p_0,q_0}(C)} \leq N(d) \hat{b}_{p_0} \rho^{-1}. \]  
(3.15)
Therefore, by Remark 5.8 of [7], since $1 < \beta \leq d/p + 2/q$,
\[ \|\zeta b| D(\zeta u)| \|_{E_{p,q,\beta}} \leq N \hat{b}_{p_0} \sup_{\rho > 0} \sup_{C \in C_\rho} \rho^\beta \|\partial_t + \Delta)(\zeta u)\|_{L^{p,q}(C)}. \]  
(3.16)
Similarly to (3.15) the sup in the right-hand side is dominated by
\[ N\|\partial_t + \Delta)(\zeta u)\|_{E_{p,q,\beta}} \leq N\|\partial_t + \Delta)(\zeta u)\|_{E_{p,q,\beta}} + N \rho^{-2} \|u\|_{E_{p,q,\beta}} \]  
\[ + N \rho^{-1} \|Du\|_{E_{p,q,\beta}}. \]
By the interpolation Lemma 5.10 of [7] (owing to $0 < \beta \leq d/p + 2/q$), for any $\varepsilon \in (0, 1]$
\[ \|Du\|_{E_{p,q,\beta}} \leq N \varepsilon \|\partial_t u, D^2 u\|_{E_{p,q,\beta}} + N \varepsilon^{-1} \|u\|_{E_{p,q,\beta}}. \]
By gathering these estimates, for $\varepsilon = \rho_0$, we obtain (3.14) with the left-hand side replaced by the left-hand side of (3.16). Thus obtain inequality, obviously holds if we take any translation of $\zeta$. After that it only remains to observe that the supremum of the left-hand sides of (3.16) over translated $\zeta$ is definitely larger than the left-hand side of (3.14). The lemma is proved.

Remark 3.13. There exists $\beta_0 = \beta_0(d, \delta, p_0) \in (1, 2)$ such that $p = p_0/\beta_0 > 1$, $q = q_0/\beta_0 > 1$, and (3.1) holds. Indeed, we need
\[ 1 = \frac{d_0}{p} + \frac{1}{q} = \beta_0 \left( \frac{d_0}{p_0} + \frac{1}{q_0} \right) \]
and what is inside the parentheses is strictly less than one because $d_0 < d$ and bigger than $1/2$ because $d_0 > d/2, p_0 > d$. Therefore, $\beta_0 \in (1, 2)$. Then $p > d_0 > d/2 > 1$ and $q > 1$. 


Introduce
\[ \theta_0(d, \delta, p_0, q_0) = \theta(d, \delta, p_0/\beta_0, q_0/\beta_0, \beta_0/2), \]
where \( \theta(d, \delta, p, q, \alpha) \) is from Lemma 3.9 and \( \beta_0 = \beta_0(d, \delta, p_0) \) is introduced in Remark 3.13.

**Theorem 3.14.** Let Assumption 2.4 be satisfied with \( \theta = \theta_0(d, \delta, p_0, q_0) \) and set \((p, q) = (p_0, q_0) / \beta_0\). Then there is \( b_1 = b_1(d, \delta, p_0, \rho_0, \rho_\beta) \in (0, 1) \) such that, if \( \hat{b}_{p_0} \leq b_1 \), then for any \( \lambda \geq 0 \), Borel bounded \( c \geq 0 \) on \( \mathbb{R}^{d+1} \), and \( f \in E_{p,q,\beta_0} \) there exists a unique \( E_{1,2}^{p,q,\beta_0} \)-solution of \( Lu - (c + \lambda)u = f \). Furthermore, there exists a constant \( N \) depending only on \( \lambda, B, d, \delta, p_0, \rho_0, \rho_\beta \), and \( \sup c \), such that
\[
\| \partial_t u, D^2 u, Du, u \|_{E_{p,q,\beta_0}} \leq N \| f \|_{E_{p,q,\beta_0}}.
\]

Proof. As usual, it suffices to prove (3.18) as an a priori estimate assuming, additionally, that the coefficients of \( L \), \( u \), and \( f \) are smooth and bounded. In that case, by taking the solution of (1.1) with \( \sigma \) corresponding to \( a \) and applying Itô’s formula (as in the proof of Theorem 2.5) to
\[
u(t, x_t) \exp \left( -\lambda t - \int_0^t c(s, x_s) \, ds \right),
\]
by Theorem 4.8 of [9] we conclude that
\[
|u(0, 0)| \leq N \tilde{\lambda}^{(d-2d_0)/(2p)} \left( \int_0^\infty \left( \int_{\mathbb{R}^d} |\Psi_{\lambda} f|^p \, dx \right)^{q/p} \, dt \right)^{1/q},
\]
where
\[
\tilde{\lambda} = \lambda(1 \wedge \lambda) d/(2d_0 - d), \quad \lambda = \lambda \min(1, \lambda \rho_0^2),
\]
\[
\Psi_{\lambda}(t, x) = \exp(-\sqrt{\tilde{\lambda}}(|x| + \sqrt{t}) \tilde{\xi}/16), \quad \tilde{\xi} = \tilde{\xi}(d, \delta) > 0.
\]
Let \( Z = \{0, 1, \ldots\} \times \mathbb{Z}^d \) and for \( z = (z_1, z_2) \in Z \) let \( C^z = C_1(z) \). Observe that on \( C^z \) we have \( \Psi_{\lambda} \leq 1 \) and
\[
\Psi_{\lambda} \leq \exp(-\sqrt{\tilde{\lambda}}(|z_2| + \sqrt{z_1} - 1) \tilde{\xi}/16).
\]
Furthermore, for each \( z \in Z \)
\[
\| f \|_{L_p,q(C^z)} \leq N(d, p) \| f \|_{E_{p,q,\beta_0}}.
\]
Therefore, by noting that \( \Psi_{\lambda}|f| \leq \sum_Z \Psi_{\lambda}|f| |C^z| \) and using Minkowski’s inequality we get that the norm in (3.19) is dominated by
\[
N \| f \|_{E_{p,q,\beta_0}} \sum_Z \exp(-\sqrt{\tilde{\lambda}}(|z_2| + \sqrt{z_1} - 1) \tilde{\xi}/16).
\]
By majorating the last sum by an integral and denoting \( 2\mu = \sqrt{\lambda} \tilde{\xi}/16 \) we obtain that it is dominated by
\[
\int_0^\infty \int_{\mathbb{R}^d} e^{-\mu(|x|+\sqrt{t}-3)} \, dx \, dt \leq N + \int_0^\infty \int_{\mathbb{R}^d} e^{-2\mu(|x|+\sqrt{t}-3)} I_{|x|+\sqrt{t}>6} \, dx \, dt \leq N + \int_0^\infty \int_{\mathbb{R}^d} e^{-\mu(|x|+\sqrt{t})} I_{|x|+\sqrt{t}>6} \, dx \, dt.
\]
\[
\leq N + \int_0^\infty \int_{\mathbb{R}^d} e^{-\mu(|x|+\sqrt{t})} \, dx \, dt = N + N \mu^{-d-2}.
\]

Hence, at the origin we have
\[
|u| \leq N \hat{x}^{(d-2d_0)/(2p)} (1 + \hat{x}^{-(d+2)/2})\|f\|_{E_{p,q,\beta_0}}, \tag{3.20}
\]
Clearly, this also holds at any point.

Next, in light of (3.20), Corollary 3.11, and Lemma 3.12
\[
\|\partial_t u, D^2 u\|_{E_{p,q,\beta_0}} \leq N \|f + (\lambda + c) u - \tilde{b} D_i u\|_{E_{p,q,\beta_0}} + N \sup_{\mathbb{R}^{d+1}} |u|
\]
\[
\leq N \|f\|_{E_{p,q,\beta_0}} + N \hat{b}_{\rho_0} \|\partial_t u, D^2 u\|_{E_{p,q,\beta_0}}.
\]
We choose \(\tilde{b}_1\) so that \(N \hat{b}_1 \leq 1/2\) and then we obtain an estimate of \(\partial_t u, D^2 u\).
Now it only remains to observe that \(Du\) is estimated again thanks to Lemma 3.12 and (3.20). The theorem is proved.

Remark 3.15. The unique solution \(u\) from Theorem 3.14 possesses the following properties
a) obviously, \(u \in W^{1,2}_{p,q,\text{loc}}\); 
b) owing to Corollary 5.7 of [7], we have \(Du \in L_{r,s,\text{loc}}\), where \((r,s) = (\beta_0 - 1)^{-1}(p_0,q_0);\)
c) \(u\) is bounded.

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