Growth of nonlinear structures on the interface between dielectric liquids in a strong vertical electric field

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Abstract. Nonlinear dynamics of the interface between dielectric liquids exposed to a strong vertical electric field is studied. Two types of exact solutions for quadratically nonlinear equations of motion (periodic solutions involving a finite number of Fourier harmonics and spatially localized rational solutions) are analyzed. Description of the interfacial evolution reduces to solving a finite number of ordinary differential equations either on amplitudes of harmonics, or, through the analytical continuation into the complex plane from the interface, for the poles motion. The common property of the solutions is a tendency for the growth of interface perturbations in the direction of the liquid with a lower permittivity.

1. Introduction
It is well known [1–4] that free surfaces or interfaces of liquids are unstable in a strong vertical electric field. The behavior of dielectric liquids subjected to an electric field is actively investigated both experimentally [5–8] and theoretically [9–15]. External electric field directed tangentially to the unperturbed boundary has a stabilizing effect [16]; its influence on the nonlinear dynamics of free surfaces or interfaces of dielectric liquids has been studied in recent works [17–23]. The possibility of controlling the behavior of liquids by an applied electric field makes such studies important for applications.

As demonstrated in [9, 24], the nonlinear evolution of the free surface of liquids can be effectively studied by analytical methods in the strong-field limit, where the surface motion is completely determined by electrostatic forces. In [25–27], these results were generalized for the interface between two immiscible dielectric liquids. In the present work, we will compare different types of exact analytical solutions describing the development of weakly nonlinear spatially localized and periodic structures at the interface of liquids in a strong vertical electric field. For boundary perturbations of different geometries, it will be shown that their growth rate is significantly higher in the direction of the fluid with a lower dielectric constant than in the opposite direction.

2. Governing equations
Let us introduce a Cartesian coordinate system such that the $z$ axis is directed along the normal to the flat unperturbed interface (coinciding with the $z = 0$ plane) of two liquids.
Deviation of the interface from the unperturbed state is described by the $\eta(x, y, t)$ function. The external electric field is assumed to be oriented along the $z$ axis, i.e., vertically. Then, the field strengths $E_{1,2}$ in the lower and upper liquids are related as $\varepsilon_1 E_1 = \varepsilon_2 E_2$, where $\varepsilon_1$, $\varepsilon_2$ are the corresponding permittivities (here and below, subscripts 1 and 2 refer to the lower and upper liquids, respectively). The electric field potentials $\varphi_{1,2}$ obey the Laplace equations, $\nabla^2 \varphi_{1,2} = 0$, with the following boundary conditions on the interface:

$$\varphi_1 = \varphi_2, \quad \varepsilon_1 \partial_n \varphi_1 = \varepsilon_2 \partial_n \varphi_2, \quad z = \eta,$$

where $\partial_n$ is the derivative in the direction normal to the boundary.

We assume that both liquids are incompressible and their flows are potential. Then the velocity potentials $\Phi_{1,2}$, as well as the electric field potentials, satisfy the Laplace equations, $\nabla^2 \Phi_{1,2} = 0$, and the interface motion is defined by the so-called dynamic boundary condition (the nonstationary Bernoulli equation),

$$\rho_1 \left( \frac{\partial \Phi_1}{\partial t} + \frac{(\nabla \Phi_1)^2}{2} \right) - \rho_2 \left( \frac{\partial \Phi_2}{\partial t} + \frac{(\nabla \Phi_2)^2}{2} \right) = -\frac{\varepsilon_0 (\varepsilon_1 - \varepsilon_2)}{2} (E_1 E_2 - \nabla \varphi_1 \nabla \varphi_2), \quad z = \eta,$$

where $\varepsilon_0$ is the electric constant and $\rho_{1,2}$ are the densities of liquids. The right-hand side of (1) contains terms that are responsible for the electrostatic and hydrostatic pressures.

The evolution of the interface must also obey the kinematic boundary condition, according to which the velocity of the interface coincides with the velocities of both liquids,

$$\eta \left( 1 + (\nabla \eta)^2 \right)^{-1/2} = \partial_n \Phi_1 = \partial_n \Phi_2, \quad z = \eta,$$

where $\nabla \eta$ is the two-dimensional gradient in the $\{x, y\}$ plane. The system of the equations of motion is closed by the conditions at infinity, where the liquids are at rest and the electric field is homogeneous,

$$\Phi_{1,2} \to 0, \quad \varphi_{1,2} - E_{1,2} z, \quad z \to \mp \infty.$$

For convenience of the further analysis, we switch to dimensionless notations as follows:

$$\Phi_{1,2} \to \Phi_{1,2} \frac{E_1}{k_0} \sqrt{\frac{\varepsilon_0 \varepsilon_1}{\rho_1}}, \quad \varphi_{1,2} \to \varphi_{1,2} \frac{E_1}{k_0}, \quad r \to \frac{r}{k_0}, \quad t \to \frac{t}{E_1 k_0 \sqrt{\frac{\rho_1}{\varepsilon_0 \varepsilon_1}}},$$

where $k_0$ is the characteristic wave number.

The above system can be sufficiently simplified by considering only quadratically nonlinear terms (see [12] for details):

$$\psi_t - \frac{2A_E^2}{1 - A_E} \hat{k} \eta = \frac{A(1 + A)}{4} \left( (\hat{k} \psi)^2 - (\nabla \perp \psi)^2 \right) + \frac{A_E^2}{1 - A_E} \left( (\hat{k} \eta)^2 - (\nabla \perp \eta)^2 \right)$$

$$+ \frac{2A_E^2}{1 - A_E} \left( \hat{k} (\eta \hat{k} \eta) + \nabla \perp (\eta \nabla \perp \eta) \right), \quad (3)$$

$$\eta_t - \frac{1 + A}{2} \hat{k} \psi = -\frac{A(1 + A)}{2} \left( \hat{k} (\eta \hat{k} \psi) + \nabla \perp (\eta \nabla \perp \psi) \right), \quad (4)$$

where $A = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$ is the Atwood number, $A_E = (\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2)$ is its electric analogue, $\hat{k}$ is the two-dimensional integral operator with a difference kernel whose Fourier transform is equal to the absolute value of the wave vector, and $\psi = [\Phi_1 - (\rho_2/\rho_1) \Phi_2]_{z=\eta}$ is an auxiliary function responsible for the velocity distribution. In the next section, we will consider two different types of exact solutions of the system (3), (4).
3. Analysis of exact solutions

Consider the behavior of the interface between two dielectric fluids in a vertical electric field for the case where the ratio of the permittivities of the fluids is equal to the inverse ratio of their densities, i.e., \( A_E = -A \) (examples of such liquids can be found in [12]). In such a situation, as was shown in [10], a special regime of fluids motion can be realized where the functions \( \eta \) and \( \psi \) are related as

\[
2A\eta = (1 + A)\psi.
\]

This relation allows us to obtain a single equation describing weakly nonlinear dynamics of the interface,

\[
\eta_t = A\hat{k}\eta - A^2 \left( \hat{k}\eta \hat{k}\eta + \nabla_\perp(\eta\nabla_\perp\eta) \right).
\]

We will study the development of the electrohydrodynamic instability of the interface in the framework of this integro-differential equation. It is important that this equation admits exact analytical solutions in the form of nonlinear spatially localized and periodic structures. Numerical and analytical study of these solutions to equation (5) is the main goal of the work.

Let us consider plane symmetric perturbations of the boundary depending only on \( x \). In this case, the operator \( \hat{k} \) can be expressed in the form \( \hat{k} = -\partial_x \hat{H} \), where \( \hat{H} \) is the Hilbert transform (its action in Fourier space is defined as \( \hat{H}k = i\text{sgn}(k) \)). It is convenient to rewrite the equation (5) in complex form,

\[
\eta^\pm_t = -iA\eta_x - A^2 \hat{P}(\eta^\pm\eta_x^\mp)_x,
\]

where \( \eta^\pm \) are analytical functions (in the upper and lower half-planes of the complex variable \( x \), respectively) and \( \hat{P} = (1 - i\hat{H})/2 \) is the projection operator.

At first, consider the evolution of spatially localized perturbation of the interface given by the expression

\[
\eta(x, t) = \frac{Sa(t)}{x^2 + a^2(t)}, \quad \text{or} \quad \eta^+(x, t) = \frac{iS/2}{x + ia(t)},
\]

where \( S \) is a real constant whose sign determines the direction of the perturbation, and \( a(t) \) is a positive function of time that sets the location of the pole on the imaginary axis. By substituting expression (7) into (6), one can get the following nonlinear ordinary differential equation for the function \( a(t) \):

\[
\frac{da}{dt} = -A + \frac{SA^2}{4a^2}.
\]

As can be seen, if the pole is at a considerable distance from the real axis, then the second (nonlinear) term on the right-hand side of the equation is small compared to the first one and, hence, the pole moves to the origin with a constant velocity along the imaginary axis. The nonlinear term begins to play a significant role when the pole approaches the real axis.

In the case of negative \( S \), the solution (7), (8) describes the collapse of the boundary perturbation. The perturbation amplitude becomes infinite in a finite time, and its shape tends to the Dirac delta function, \( \eta \rightarrow -\pi S\delta(x) \). In the case of \( S > 0 \), the system behavior is non-singular. At \( t \rightarrow +\infty \), the boundary shape asymptotically tends to an equilibrium configuration with \( a = \sqrt{SA}/2 \). In figure 1, (a), the time dependence of the interfacial amplitude (in absolute value) is shown at the point \( x = 0 \) for perturbations of different signs of the parameter \( S \) under the initial condition \( a(0) = 10 \). Hereinafter, we put \( A = -A_E = 0.1 \) that corresponds to the situation where \( \rho_1 > \rho_2 \) and \( \varepsilon_1 < \varepsilon_2 \). As it is seen, equation (8) describes the explosive growth of the boundary perturbation amplitudes for \( S < 0 \) and the relaxation to steady state for \( S > 0 \). In general, the behavior of the boundary is similar to the Laplacian growth process, observed in various physical situations [28–30].
Another class of exact solutions of the equation (6) is superposition of a finite number of Fourier harmonic,

\[ \eta^+(x, t) = \sum_{n=1}^{N} q_n(t) \exp(inx), \]  

where \( q_n(t) \) are the amplitudes of the corresponding modes. Substitution of this expression into (6) yields a system of ordinary differential equations for the complex amplitudes,

\[ \frac{dq_n}{dt} = Aq_n - 2A^2 n \sum_{m=1}^{N-n} m\overline{q}_m q_{m+n}, \quad n = 1, 2, \ldots, N, \]  

where the bar denotes the complex conjugation. The fact that the exact solution of the equation (6) can be represented as a finite Fourier series makes it possible to investigate the dynamics of an interface of almost arbitrary shape solving the system (10) numerically.

Using the representation (9), we consider the evolution of a periodic perturbation of the boundary of the following shape (it is close to (7)):

\[ \eta(x, 0) = s \exp(-x^2), \quad \eta(-h/2, t) = \eta(h/2, t), \]  

where \( s \) is the parameter characterizing the initial amplitude of the boundary perturbation and its direction; \( h \) is the spatial period (i.e., the function \( \eta \) is expanded into Fourier series on the
Figure 2. Evolution of the interface for the periodic solution (11) for the parameters $s = -0.1$ (a) and $s = 0.1$ (b).

interval $-h/2 \leq x \leq h/2$). We will approximate the interface shape by a finite Fourier series with $N = 25$ and with the spatial period $h$ chosen to be 10. Numerical solution of the system (10) will be constructed on the basis of the explicit fourth order Runge–Kutta method.

In figure 1, (b), time dependencies $|\eta(0, t)|$ for the solution (9)–(11) are shown for different signs of the parameter $s$ (compare with figure 1, (a)). The main difference in the dynamics of periodic perturbations from the above considered case of spatially localized perturbations is that the interfacial amplitude grows without limit for any sign of $s$. A common feature for two types of solutions is that the growth rate is significantly larger for perturbations initially directed downwards.

An important property of the rational solution (7) is that the corresponding boundary perturbation remains structurally stable during its evolution. The nonlinearity leads only to either acceleration or deceleration of growth of spatially localized perturbations of the boundary. This process is not accompanied by the emission of wave packets.

Figure 2 shows the evolution of the periodic perturbation of the boundary (9)–(11) for $s$ with different signs. It can be seen that, for negative $s$, the interface behavior is qualitatively similar to that demonstrated by solution (7) at $S < 0$: the initial perturbations directed towards the liquid with a lower dielectric constant grow rapidly (figure 2, (a)). The situation changes qualitatively for $s > 0$, when the perturbation is initially directed upwards. The waves with explosively growing amplitudes are generated at the periphery of the original perturbation. Then, as for $s < 0$, the boundary perturbation directed downwards begins to develop rapidly (figure 2, (b)).
Such a behavior is radically different from that described by (7) and (8) for $S > 0$.

4. Conclusion

In the present work, the behavior of the interface between two liquid dielectrics in a strong vertical electric field has been studied. In the case where the ratio of the permittivities of the liquids equals the inverse ratio of their densities ($\varepsilon_1/\varepsilon_2 = \rho_2/\rho_1$ or, which is the same, $A_E = -A$), two classes of exact solutions of the electrohydrodynamics equations were analyzed. The first type of exact solutions describes the evolution of spatially localized perturbations of the interface between the liquids. Depending on the perturbation sign, the obtained solutions describe either the wave collapse or the formation of a stationary configuration of the boundary. The second type of solutions for the boundary evolution represents the superposition of a finite number of interacting Fourier harmonics. For this case, the energy does not transfer to higher harmonics and, as a consequence, discontinuities cannot be formed in the solutions. The common property of two types of solutions is that the perturbations directed towards the liquid with a lower permittivity grow significantly faster than those directed in the opposite direction.

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