RESTRICTED SIMPLE LIE ALGEBRAS AND THEIR
INFINITESIMAL DEFORMATIONS

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Abstract. In the first two sections, we review the Block-Wilson-Premet-
Strade classification of restricted simple Lie algebras. In the third section,
we compute their infinitesimal deformations. In the last section, we indicate
some possible generalizations by formulating some open problems.

1. Restricted Lie algebras

We fix a field \( F \) of characteristic \( p > 0 \) and we denote with \( F_p \) the prime field
with \( p \) elements. All the Lie algebras that we will consider are of finite dimen-
sion over \( F \). We are interested in particular class of Lie algebras, called restrict-
ed (or \( p \)-Lie algebras).

Definition 1.1 (Jacobson [JAC37]). A Lie algebra \( L \) over \( F \) is said to be restricted
(or a \( p \)-Lie algebra) if there exits a map (called \( p \)-map),
\[ [p] : L \rightarrow L, \quad x \mapsto x[p], \]
which verifies the following conditions:

1. \( \text{ad}(x[p]) = \text{ad}(x)^p \) for every \( x \in L \).
2. \( (\alpha x)[p] = \alpha^p x[p] \) for every \( x \in L \) and every \( \alpha \in F \).
3. \( (x_0 + x_1)[p] = x_0^p + x_1^p + \sum_{s=1}^{p-1} s_i(x_0, x_1) \) for every \( x, y \in L \), where the
   element \( s_i(x_0, x_1) \in L \) is defined by
   \[ s_i(x_0, x_1) = -\frac{1}{r} \sum_u \text{ad} x_u(1) \circ \text{ad} x_u(2) \circ \cdots \circ \text{ad} x_u(p-1)(x_1), \]
   the summation being over all the maps \( u : [1, \cdots, p-1] \rightarrow \{0,1\} \) taking
   \( r \)-times the value 0.

Example. (1) Let \( A \) an associative \( F \)-algebra. Then the Lie algebra \( \text{Der}_F A \)
of \( F \)-derivations of \( A \) is a restricted Lie algebra with respect to the \( p \)-map
\( D \rightarrow D^p := D \circ \cdots \circ D \).

(2) Let \( G \) a group scheme over \( F \). Then the Lie algebra \( \text{Lie}(G) \) associated to \( G \)
is a restricted Lie algebra with respect to the \( p \)-map given by the differential
of the homomorphism \( G \rightarrow G, \ x \mapsto x^p := x \circ \cdots \circ x \).

One can naturally ask when a \( F \)-Lie algebra can acquire the structure of a
restricted Lie algebra and how many such structures there can be. The following
criterion of Jacobson answers to that question.

Proposition 1.2 (Jacobson). Let \( L \) be a Lie algebra over \( F \). Then

(1) It is possible to define a \( p \)-map on \( L \) if and only if, for every element \( x \in L \),
   the \( p \)-th iterate of \( \text{ad}(x) \) is still an inner derivation.

(2) Two such \( p \)-maps differ by a semilinear map from \( L \) to the center \( Z(L) \) of
   \( L \), that is a map \( f : L \rightarrow Z(L) \) such that \( f(\alpha x) = \alpha^p f(x) \) for every \( x \in L \)
   and \( \alpha \in F \).

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A Dynkin diagrams, called associated Dynkin diagrams. It turns out that there are four infinite families of root systems and then the irreducible root systems are classified by means of their used to establish a correspondence between simple Lie algebras and irreducible classification proceeds as follows: first the non-degeneracy of the Killing form is.

Many of the modular Lie algebras that arise “in nature” are restricted. As an example of this principle, we would like to recall the following two results from the theory of finite group schemes and the theory of inseparable field extensions.

**Theorem 1.3.** There is a bijective correspondence

\[
\{\text{Restricted Lie algebras}/F\} \longleftrightarrow \{\text{Finite group schemes}/F \text{ of height } 1\},
\]

where a finite group scheme $G$ has height $1$ if the Frobenius $F : G \to G^{(p)}$ is zero. Explicitly to a finite group scheme $G$ of height $1$, one associates the restricted Lie algebra $\text{Lie}(G) := T_0 G$. Conversely, to a restricted Lie algebra $(L, [p])$, one associates the finite group scheme corresponding to the dual of the restricted enveloping Hopf algebra $U^{[p]}(L) := U(L)/(x^p - x^{[p]})$.

**Proof.** See [JAC62, Chapter V.7].

There is a bijective correspondence

\[
\{\text{Inseparable subextensions of exponent } 1\} \longleftrightarrow \{\text{Restricted subalgebras of } \text{Der}(F)\}
\]

where the inseparable subextensions of exponent $1$ are the subfields $E \subset F$ such that $F^p \subset E \subset F$ and $\text{Der}(F) := \text{Der}_{F^p}(F) = \text{Der}_{F^p}(F)$. Explicitly to any field $F^p \subset E \subset F$ one associates the restricted subalgebra $\text{Der}_{E}(F)$. Conversely, to any restricted subalgebra $L \subset \text{Der}(F)$, one associates the subfield $E_L := \{x \in F | D(x) = 0 \text{ for all } D \in L\}$.

**Proof.** See [DG70, Chapter 2.7].

2. Classification of restricted simple Lie algebras

Simple Lie algebras over an algebraically closed field of characteristic zero were classified at the beginning of the XIX century by Killing and Cartan. The classification proceeds as follows: first the non-degeneracy of the Killing form is used to establish a correspondence between simple Lie algebras and irreducible root systems and then the irreducible root systems are classified by mean of their associated Dynkin diagrams. It turns out that there are four infinite families of Dynkin diagrams, called $A_n$, $B_n$, $C_n$, $D_n$, and five exceptional Dynkin diagram, called $E_6$, $E_7$, $E_8$, $F_4$ and $G_2$. The four infinite families correspond, respectively, to the the special linear algebra $\mathfrak{sl}(n+1)$, the special orthogonal algebra of odd rank $\mathfrak{so}(2n+1)$, the symplectic algebra $\mathfrak{sp}(2n)$ and the special orthogonal algebra of even rank $\mathfrak{so}(2n)$. For the simple Lie algebras corresponding to the exceptional Dynkin diagrams, see the book [JAC71] or the nice account in [BAE02].

These simple Lie algebras admits a model over the integers via the (so-called) Chevalley bases. Therefore, via reduction modulo a prime $p$, one obtains a restricted Lie algebra over $\mathbb{F}_p$, which is simple up to a quotient by a small ideal. For example $\mathfrak{sl}(n)$ is not simple if $p$ divide $n$, but its quotient $\mathfrak{psl}(n) = \mathfrak{sl}(n)/(I_n)$ by the unit matrix $I_n$ becomes simple. There are similar phenomena occuring only for $p = 2, 3$ for the other Lie algebras (see [STRO04, Page 209] or [SEL67]). The restricted simple algebras obtained in this way are called algebras of classical type. Their Killing form is non-degenerate except at a finite number of primes. Moreover, they can be characterized as those restricted simple Lie algebras admitting a projective representation with nondegenerate trace form (see [BLO62], [KAP71]).

However, there are restricted simple Lie algebras which have no analogous in characteristic zero and therefore are called nonclassical. The first example of a nonclassical restricted simple Lie algebra is due to E. Witt, who in 1937 realized
that the derivation algebra $W(1) := \text{Der}_F(F[X]/(X^p))$ over a field $F$ of characteristic $p > 3$ is simple with a degenerate Killing form. In the succeeding three decades, many more nonclassical restricted simple Lie algebras have been found (see [JAC43], [FRA54], [AF54], [FRA64]). The first comprehensive conceptual approach to constructing these nonclassical restricted simple Lie algebras was proposed by Kostrikin and Shafarevich in 1966 (see [KS66]). They showed that all the known examples can be constructed as finite-dimensional analogues of the four classes of infinite-dimensional complex simple Lie algebras, which occurred in Cartan’s classification of Lie pseudogroups (see [CAR09]). These restricted simple Lie algebras, called of Cartan-type, are divided into four families, called Witt-Jacobson, Special, Hamiltonian and Contact algebras.

**Definition 2.1.** Let $A(n) := F[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$ the algebra of $p$-truncated polynomials in $n$ variables. Then the Witt-Jacobson Lie algebra $W(n)$ is the derivation algebra of $A(n)$:

$$W(n) = \text{Der}_F A(n).$$

For every $j \in \{1, \ldots, n\}$, we put $D_j := \frac{\partial}{\partial x_j}$. The Witt-Jacobson algebra $W(n)$ is a free $A(n)$-module with basis $\{D_1, \ldots, D_n\}$. Hence $\dim_F W(n) = np^n$ with a basis over $F$ given by $\{x^aD_j \mid 1 \leq j \leq n, x^a \in A(n)\}$.

The other three families are defined as $m$-th derived algebras of the subalgebras of derivations fixing a volume form, a Hamiltonian form and a contact form, respectively. More precisely, consider the natural action of $W(n)$ on the exterior algebra of differential forms in $dx_1, \ldots, dx_n$ over $A(n)$. Define the following three forms, called volume form, Hamiltonian form and contact form:

$$\begin{aligned}
\omega_S &= dx_1 \wedge \cdots \wedge dx_n, \\
\omega_H &= \sum_{i=1}^{m} dx_i \wedge dx_{i+m} \quad \text{if } n = 2m, \\
\omega_K &= dx_{2m+1} + \sum_{i=1}^{m} (x_{i+m} dx_i - x_i dx_{i+m}) \quad \text{if } n = 2m + 1.
\end{aligned}$$

**Definition 2.2.** Consider the following three subalgebras of $W(n)$:

$$\begin{aligned}
\tilde{S}(n) &= \{ D \in W(n) \mid D\omega_S = 0 \}, \\
\tilde{H}(n) &= \{ D \in W(n) \mid D\omega_H = 0 \}, \\
\tilde{K}(n) &= \{ D \in W(n) \mid D\omega_K \in A(n)\omega_K \}.
\end{aligned}$$

Then the Special algebra $S(n)$ ($n \geq 3$) is the derived algebra of $\tilde{S}(n)$, while the Hamiltonian algebra $H(n)$ ($n = 2m \geq 2$) and the Contact algebra $K(n)$ ($n = 2m + 1 \geq 3$) are the second derived algebras of $\tilde{H}(n)$ and $\tilde{K}(n)$, respectively.

We want to describe more explicitly the above algebras, starting from the Special algebra $S(n)$. For every $1 \leq i, j \leq n$ consider the following maps

$$D_{ij} = -D_{ji} : A(n) \rightarrow W(n) \quad f \mapsto D_j(f)D_i - D_i(f)D_j.$$
Suppose now that \( n = 2m \geq 2 \) and consider the map \( D_H : A(n) \to W(n) \) defined by
\[
D_H(f) = \sum_{i=1}^{m} [D_i(f)D_i + D_i(f)D_i],
\]
where, as before, \( D_i := \frac{\partial}{\partial x_i} \in W(n) \). Then the Hamiltonian algebra can be described as follows:

**Proposition 2.4.** The above map \( D_H \) induces an isomorphism
\[
D_H : A(n)_{\neq 1,x^e} \cong H(n),
\]
where \( A(n)_{\neq 1,x^e} = \{ x^a \in A(n) | x^a \neq 1, x^a \neq x^e := x_1^{p-1} \cdots x_n^{p-1} \} \). Therefore \( H(n) \) has dimension \( p^n - 2 \).

**Proof.** See [FSS88, Chapter 4.4].

Suppose finally that \( n = 2m + 1 \geq 3 \). Consider the map \( D_K : A(n) \to K(n) \) defined by
\[
D_K(f) = \sum_{i=1}^{m} [D_i(f)D_i + D_i(f)D_i] + \sum_{j=1}^{2m} x_j [D_n(f)D_j - D_j(f)D_n] + 2fD_n.
\]

Then the Contact algebra can be described as follows:

**Proposition 2.5.** The above map \( D_K \) induces an isomorphism
\[
K(n) \cong \begin{cases} A(n) & \text{if } p \nmid (m + 2), \\ A(n)_{\neq x^e} & \text{if } p \mid (m + 2), \end{cases}
\]
where \( A(n)_{\neq x^e} := \{ x^a \in A(n) | x^a \neq x^e := x_1^{p-1} \cdots x_n^{p-1} \} \). Therefore \( K(n) \) has dimension \( p^n - 1 \) if \( p \mid (m + 2) \) and \( p^n - 1 \) if \( p \nmid (m + 2) \).

**Proof.** See [FSS88, Chapter 4.5].

Kostrikin and Shafarevich (in the above mentioned paper [KS66]) conjectured that a restricted simple Lie algebras (that is a restricted algebras without proper ideals) over an algebraically closed field of characteristic \( p > 5 \) is either of classical or Cartan type. The Kostrikin-Shafarevich conjecture was proved by Block-Wilson (see [BW84] and [BW88]) for \( p > 7 \), building upon the work of Kostrikin-Shafarevich ([KS66] and [KS69]), Kac ([KAC70] and [KAC74]), Wilson ([WIL76]) and Weisfai ler ([WEI78]).

Recently, Premet and Strade (see [PS97], [PS99], [PS01], [PS04]) proved the Kostrikin-Shafarevich conjecture for \( p = 7 \). Moreover they showed that for \( p = 5 \) there is only one exception, the Melikian algebra ([MEL80]), whose definition is given below.

**Definition 2.6.** Let \( p = \text{char}(F) = 5 \). Let \( \widehat{W}(2) \) be a copy of \( W(2) \) and for an element \( D \in W(2) \) we indicate with \( \hat{D} \) the corresponding element inside \( \widehat{W}(2) \). The Melikian algebra \( M \) is defined as
\[
M = A(2) \oplus W(2) \oplus \widehat{W}(2),
\]

with Lie bracket defined by the following rules (for all \( D, E \in W(2) \) and \( f, g \in A(2) \)):

\[
\begin{align*}
[D, \tilde{E}] &:= [D, E] + 2 \text{div}(D)\tilde{E}, \\
[D, f] &:= D(f) - 2 \text{div}(D)f, \\
[f_1\tilde{D}_1 + f_2\tilde{D}_2, g_1\tilde{D}_1 + g_2\tilde{D}_2] &:= f_1g_2 - f_2g_1, \\
[f, \tilde{E}] &:= f\tilde{E}, \\
[f, g] &:= 2 (gD_2(f) - fD_2(g))\tilde{D}_1 + 2 (fD_1(g) - gD_1(f))\tilde{D}_2,
\end{align*}
\]

where \( \text{div}(f_1D_1 + f_2D_2) := D_1(f_1) + D_2(f_2) \in A(2) \).

In characteristic \( p = 2, 3 \), there are many exceptional restricted simple Lie algebras (see [STR04, page 209]) and the classification seems still far away.

3. Infinitesimal Deformations

An infinitesimal deformation of a Lie algebra \( L \) over a field \( F \) is a Lie algebra \( L' \) over \( F'[\epsilon]/(\epsilon^2) \) such that \( L' \times_{F'[\epsilon]/(\epsilon^2)} F \cong L \). Explicitly, \( L' = L + \epsilon L \) with Lie bracket \([-,-]'\) defined by (for any two elements \( X, Y \in L \subset L' \)):

\[
[X, Y]' = [X, Y] + \epsilon f(X, Y),
\]

where \([-,-]\) is the Lie bracket of \( L \) and \( f(\cdot, \cdot) \) is an 2-alternating function from \( L \) to \( L \), considered a module over itself via adjoint representation. The Jacobi identity for \([-,-]'\) forces \( f \) to be a cocycle and moreover one can check that two cocycles differing by a coboundary define isomorphic Lie algebras. Therefore the infinitesimal deformations of a Lie algebra \( L \) are parametrized by the second cohomology \( H^2(L, L) \) of the Lie algebra with values in the adjoint representation (see [GER64] for a rigorous treatment).

It is a classical result that simple Lie algebras in characteristic zero are rigid. We want to give a sketch of the proof of the following Theorem (see [HS97] for details).

**Theorem 3.1.** Let \( L \) be a simple Lie algebra over a field \( F \) of characteristic 0. Then, for every \( i \geq 0 \), we have that

\[
H^i(L, L) = 0.
\]

**Sketch of the Proof.** Since the Killing form \( \beta(x, y) = \text{tr}(\text{ad}(x)\text{ad}(y)) \) is non-degenerate (by Cartan’s criterion), we can choose two bases \( \{e_i\} \) and \( \{e'_i\} \) of \( L \) such that \( \beta(e_i, e'_j) = \delta_{ij} \). Consider the Casimir element \( C := \sum_i e_i \otimes e'_i \) inside the enveloping algebra \( U(L) \). One can check that:

1. \( C \) belongs to the center of the enveloping algebra and therefore it induces an \( L \)-homomorphism \( C : L \to L \), where \( L \) is a consider a module over itself via adjoint action. Moreover since \( \text{tr}_L(C) = \dim(L) \neq 0 \), \( C \) is non-zero and hence is an isomorphism by the simplicity of \( L \).

2. The map induced by \( C \) on the exact complex \( \{U(L) \otimes_k A^n L \}_{n} \to F \) is homotopic to 0.

Therefore the induced map on cohomology \( C_* : H^*(L, L) \to H^*(L, L) \) is an isomorphism by (1) and the zero map by (2), which implies that \( H^*(L, L) = 0 \).

The above proof uses the non-degeneracy of the Killing form and the non-vanishing of the trace of the Casimir element, which is equal to the dimension of the Lie algebra. Therefore the same proof works also for the restricted simple Lie algebras of classical type over a field of characteristic not dividing the determinant of the Killing form and the dimension of the Lie algebra. Actually Rudakov (see [RUD71]) showed that such Lie algebras are rigid if the characteristic of the
base field is greater or equal to 5 while in characteristic 2 and 3 there are non-rigid classical Lie algebras (see [CHE05, CK00, CKK00]).

It was already observed by Kostrikin and Đumadildaev ([DK78], [DZU80], [DZU81] and [DZU89]) that Witt-Jacobson Lie algebras admit infinitesimal deformations. More precisely: in [DK78] the authors compute the infinitesimal deformations of the Jacobson-Witt algebras of rank 1, while in [DZU80] Theorem 4], [DZU81] and [DZU89] the author describes the infinitesimal deformations of the Jacobson-Witt algebras of any rank but without a detailed proof.

In the papers [VIV1], [VIV2] and [VIV3], we computed the infinitesimal deformations of the restricted simple Lie algebras of Cartan type, Jacobson-Witt algebras of any rank but without a detailed proof.

Before stating the next theorem, we need some notations about \( n \)-tuples of natural numbers. We consider the order relation inside \( \mathbb{N}^n \) given by \( a = (a_1, \cdots, a_n) \leq b = (b_1, \cdots, b_n) \) if \( a_i \leq b_i \) for every \( i = 1, \cdots, n \). We define the degree of \( a \in \mathbb{N}^n \) as \( |a| = \sum_{i=1}^{n} a_i \) and the factorial as \( a! = \prod_{i=1}^{n} a_i! \). For two multindex \( a, b \in \mathbb{N}^n \) such that \( b \leq a \), we set \( \binom{a}{b} := \prod_{i=1}^{n} \binom{a_i}{b_i} = \frac{a!}{b!(a-b)!} \). For every integer \( j \in \{1, \cdots, n\} \) we call \( e_j \) the \( n \)-tuple having 1 at the \( j \)-th entry and 0 outside. We denote with \( \sigma \) the multindex \( (p-1, \cdots, p-1) \).

Assuming now that \( n = 2m \), we define the sign \( \sigma(j) \) and the conjugate \( j' \) of \( 1 \leq j \leq 2m \) as follows:

\[
\sigma(j) = \begin{cases} 
1 & \text{if } 1 \leq j \leq m, \\
-1 & \text{if } m < j \leq 2m,
\end{cases}
\]

and \( j' = \begin{cases} j + m & \text{if } 1 \leq j \leq m, \\
j - m & \text{if } m < j \leq 2m.
\end{cases}\)

Given a multindex \( a = (a_1, \cdots, a_{2m}) \in \mathbb{N}^{2m} \), we define the sign of \( a \) as \( \sigma(a) = \prod \sigma(i)^{a_i} \) and the conjugate of \( a \) as the multindex \( \check{a} \) such that \( a_i = a_{i'} \) for every \( 1 \leq i \leq 2m \).
Theorem 3.5. Let \( n = 2m \geq 2 \). Then if \( n \geq 4 \) we have that

\[
H^2(H(n), H(n)) = \bigoplus_{i=1}^{n} (\text{Sq}(D_H(x_i)))_{F} \bigoplus_{j \neq i}^{m} (\Pi_{ij})_{F} \bigoplus_{i=1}^{m} (\Pi_i)_{F} \bigoplus (\Phi)_{F},
\]

where the above cocycles are defined (and vanish outside) by

\[
\begin{align*}
\Pi_{ij}(D_H(x^a), D_H(x^b)) &= D_H(x^{e_{ij}^{-1}} x^{e_{ij}^{-1}} [D_i(x^a)D_j(x^b) - D_i(x^b)D_j(x^a)]), \\
\Pi_i(D_H(x_i x^a), D_H(x_i x^b)) &= D_H(x^{a+b+e_i}\epsilon_i\epsilon_i), \\
\Pi_i(D_H(x_i), D_H(x^{\sigma - e_i}\epsilon_i\epsilon_i)) &= -\sigma(k)D_H(x^{\sigma - e_i}) \text{ for } 1 \leq k \leq n, \\
\Phi(D_H(x^a), D_H(x^b)) &= \sum_{i \leq a, b} \binom{a}{\delta} \binom{b}{\delta} \sigma(\delta) \delta! \ D_H(x^{a+b-\delta}).
\end{align*}
\]

If \( n = 2 \) then we have that

\[
H^2(H(2), H(2)) = \bigoplus_{i=1}^{2} (\text{Sq}(D_H(x_i)))_{F} \bigoplus (\Phi)_{F}.
\]

Theorem 3.6. We have that

\[
H^2(M, M) = (\text{Sq}(1))_{F} \bigoplus (\text{Sq}(D_i))_{F} \bigoplus (\text{Sq}(D'_i))_{F}.
\]

4. Open Problems

Simple Lie algebras (not necessarily restricted) over an algebraically closed field \( F \) of characteristic \( p \neq 2, 3 \) have been classified by Strade and Wilson for \( p > 7 \) (see [SW91, STR90, STR92, STR91, STR93, STR94, STR98]) and by Premet-Strade for \( p = 5, 7 \) (see [PS97, PS99, PS01, PS04]). The classification says that for \( p \geq 7 \) a simple Lie algebra is of classical type (and hence restricted) or of generalized Cartan type. Those latter are generalizations of the Lie algebras of Cartan type, obtained by considering higher truncations of divided power algebras (not just \( p \)-truncated polynomial algebras) and by considering only the subalgebra of (the so called) special derivations (see [FS88] or [STR04] for the precise definitions). Again in characteristic \( p = 5 \), the only exception is represented by the generalized Melikian algebras. Therefore an interesting problem would be the following:

Problem 1. Compute the infinitesimal deformations of the simple Lie algebras.

Note that there is an important distinction between restricted simple Lie algebras and simple restricted Lie algebras. The former algebras are the restricted Lie algebras which do not have any nonzero proper ideal, while the second ones are the restricted Lie algebras which do not have any nonzero proper restricted ideal (or \( p \)-ideal), that is an ideal closed under the \( p \)-map. Clearly every restricted simple Lie algebra is a simple restricted Lie algebra, but a simple restricted Lie algebra need not be a simple Lie algebras. Indeed we have the following

Proposition 4.1. There is a bijection

\[
\{\text{Simple restricted Lie algebras}\} \leftrightarrow \{\text{Simple Lie algebras}\}.
\]

Explicitly to a simple restricted Lie algebra \( (L, [p]) \) we associates its derived algebra \( [L, L] \). Conversely to a simple Lie algebra \( M \) we associate the restricted subalgebra \( M^{[p]} \) of \( \text{Der}_F(M) \) generated by \( \text{ad}(M) \) (which is called the universal \( p \)-envelope of \( M \)).
Proof. We have to prove that the above maps are well-defined and are inverse one of the other.

- Consider a simple restricted Lie algebra $(L, [p])$. The derived subalgebra $[L, L] < L$ is a non-zero ideal (since $L$ cannot be abelian) and therefore $[L, L]_p = L$, where $[L, L]_p$ denotes the $p$-closure of $[L, L]$ inside $L$.

Take a non-zero ideal $0 \neq I \triangleleft [L, L]$. Since $[L, L]_p = L$, we deduce from [FS88, Chapter 2, Prop. 1.3] that $I$ is also an ideal of $L$ and therefore $I_p = L$ by restricted simplicity of $(L, [p])$. From loc. cit., it follows also that $[L, L] = [I_p, I_p] = [I, I] \triangleleft I$ from which we deduce that $I = L$. Therefore $[L, L]$ is simple.

Since $\text{ad} : L \to \text{Der}_F(L)$ is injective and $[L, L]_p = L$, it follows by loc. cit. that $\text{ad} : L \to \text{Der}_F([L, L])$ is injective. Therefore we have that $[L, L] \subset L \subset \text{Der}_F([L, L])$ and hence $[L, L]^{[p]} = [L, L]_p = L$.

- Conversely, start with a simple Lie algebra $M$ and consider its universal $p$-envelop $M < M^{[p]} < \text{Der}_F(M)$.

Take any restricted ideal $I <_p M^{[p]}$. By loc. cit., we deduce $[I, M^{[p]}] \subset I \cap [M^{[p]}, M^{[p]}] = I \cap [M, M] = I \cap M \triangleleft M$. Therefore, by the simplicity of $M$, either $I \cap M = M$ or $I \cap M = 0$. In the first case, we have that $M \subset I$ and therefore $M^{[p]} = I$. In the second case, we have that $[I, M^{[p]}] = 0$ and therefore $I = 0$ because $M^{[p]}$ has trivial center. We conclude that $M^{[p]}$ is simple restricted.

Moreover, by loc. cit., we have that $[M^{[p]}, M^{[p]}] = [M, M] = M$.

Therefore the preceding classifications of simple Lie algebras (for $p \neq 2, 3$) give a classification of simple restricted Lie algebras.

**Problem 2.** Compute the infinitesimal deformations of the simple restricted Lie algebras.

There is an important connection between simple restricted Lie algebras and simple finite group schemes.

**Proposition 4.2.** Over an algebraically closed field $F$ of characteristic $p > 0$, a simple finite group scheme is either the constant group scheme associated to a simple finite group or it is the finite group scheme of height 1 associated to a simple restricted Lie algebra.

**Proof.** Let $G$ be a simple finite group scheme. The kernel of the Frobenius map $F : G \to G^{(p)}$ is a normal subgroup and therefore, by the simplicity of $G$, we have that either $\ker(F) = 0$ or $\ker(F) = G$. In the first case, the group $G$ is constant (since $F = F$), and therefore it corresponds to an (abstract) simple finite group. In the second case, the group $G$ is of height 1 and therefore the result follows from Proposition 1.3.

The following problem seems very interesting.

**Problem 3.** Compute the infinitesimal deformations of the simple finite group schemes.

Since constant finite group schemes (or more generally étale group schemes) are rigid, one can restrict to the simple finite group schemes of height 1 associated to the simple restricted Lie algebras. Moreover, if $(L, [p])$ is the simple restricted Lie algebra corresponding to the simple finite group scheme $G$, then the infinitesimal deformations of $G$ correspond to restricted infinitesimal deformations of $(L, [p])$, that are infinitesimal deformations that admit a restricted structure. These are parametrized by the second restricted cohomology group $H^2_{\text{rest}}(L, L)$ (defined in [HOC54]). Therefore the above Problem 3 is equivalent to the following:
Problem 4. Compute the restricted infinitesimal deformations of the simple restricted Lie algebras.

The above Problem 4 is closely related to Problem 2 because of the following spectral sequence relating the restricted cohomology to the ordinary one (see [FAR91]):

\[ E^{p,q}_2 = \text{Hom}_{\text{Frob}} \left( \bigwedge^q L, H^p(L, L) \right) \Rightarrow H^{p+q}(L, L), \]

where \( \text{Hom}_{\text{Frob}} \) denote the homomorphisms that are semilinear with respect to the Frobenius.

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