Minimal Quantities and Measurability. Gravity in Measurable Format and Natural Transition to High Energies

Alexander Shalyt-Margolin

Research Institute for Nuclear Problems, Belarusian State University, 11 Bobruiskaya str., Minsk 220040, Belarus

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Abstract

A new (more general) definition of the measurability concept not related to the principle of uncertainty is given. Then gravity is studied within the scope of this notion. The measurable format of General Relativity (GR) is constructed and it is shown that this format represents its deformation. Passage of the measurable analog of GR to the high energy (quantum gravitational) region is considered to show that this is quite natural in the physical sense. The results obtained are discussed; a further course of studies by the author is indicated.

1 Introduction

This paper presents the new results obtained within the scope of the approach to studies of a quantum theory and gravity in terms of the measurability notion, initiated in [1–8], with the aim to form the above-mentioned theories proceeding from the variations (increments) dependent on the existent energies.

These theories should not involve the infinitesimal variations \( dt, dx_i, dp_i, dE, i = 1, \ldots, 3 \) and, in general, any abstract small quantities \( \delta t, \delta x_i, \delta E, \delta p_i, \ldots \).

In this paper the results from [7,8] are significantly expanded.

Section 2 gives brief information that is necessary for further consideration. In Sections 3,4 the author logically develops the basics of the mathematical

\(^1\text{E-mail: a.shalyt@mail.ru; alexm@hep.by}\)
apparatus required to construct correctly the gravity in terms of measurable quantities at low energies \( E \ll E_p \).

Then the results obtained in Sections 3, 4, are directly applied to effect the above-mentioned construction in Section 5. Section 5 is subdivided into two parts: Subsection 5.1, where the author derives a measurable variant of Einstein Equations considered as deformation of the corresponding Einstein Equations in the canonical (classical) theory; and Subsection 5.2, where it is shown that at low energies \( E \ll E_p \) there is a measurable form of the Least Action Principle in Gravity generating the above equations. Section 6 demonstrates the ways to use the obtained results to solve the problems arising at the junction of the classical gravity and a quantum theory at low energies \( E \ll E_p \). Specifically, the question associated with involvement of the Closed Time-like Curves (CTC) and of the Hawking problems for black holes in the theory is touched upon.

In Section 7 it is demonstrated that the methods proposed for low energies \( E \ll E_p \) may be in a natural way extended to the region of high energies \( E \approx E_p \). In the process one can obtain a high-energy (quantum) variant of Einstein Equations in the measurable form. Finally, Section 8 outlines the problems of prime importance, the solutions of which are necessary for further studies in this direction, and the opportunities offered by the approach.

## 2 Measurability. Initial Information in Brief

Let us briefly consider the earlier results [1]–[6] laying the basis for this study.

It is assumed that there is a minimal (universal) unit for measurement of the length \( \ell \) corresponding to some maximal energy \( E_\ell = \hbar c \ell \) and a universal unit for measurement of time \( \tau = \ell/c \). Without loss of generality, we can consider \( \ell \) and \( \tau \) at Plank’s level, i.e. \( \ell = \kappa l_p, \tau = \kappa t_p \), where the numerical constant \( \kappa \) is on the order of 1. Consequently, we have \( E_\ell \propto E_p \) with the corresponding proportionality factor.

Then we consider a set of all nonzero momenta

\[
P = \{p_x\}, i = 1, \ldots, 3; |p_x| \neq 0.
\] (1)
and a subset of the **Primarily Measurable** momenta characterized by the property
\[ p_{x_i} \doteq p_{N_i} = \frac{\hbar}{N_i\ell}, \quad (2) \]

where \( N_i \) is an integer number and \( p_{x_i} \) is the momentum corresponding to the coordinate \( x_i \).

Formula (2) gives rise to the following definition:

**Definition 1.** Primary Measurability  
1.1. Any variation in \( \Delta x_i \) for the coordinates \( x_i \) and \( \Delta t \) of the time \( t \) is considered **primarily measurable** if
\[ \Delta x_i = N_{\Delta x_i} \ell, \quad \Delta t = N_{\Delta t} \tau, \quad (3) \]

where \( N_{\Delta x_i} \neq 0 \) and \( N_{\Delta t} \neq 0 \) are integer numbers.

1.2. Let us define any physical quantity as **primary or elementary measurable** when its value is consistent with point 1.1 of this Definition.

So, from **Definition 1.** it directly follows that all the momenta satisfying (2) are the **Primarily Measurable** momenta.

Then we consider formula (2) and **Definition 1.** with the addition of the momenta \( p_{x_0} = p_{N_0} = \frac{\hbar}{N_0\ell} \), where \( N_0 \) is an integer number corresponding to the time coordinate (\( N_{\Delta t} \) in formula (3)).

For convenience, we denote **Primarily Measurable Quantities** satisfying **Definition 1.** in the abbreviated form as **PMQ**.

It should be noted, that the space-time quantities
\[ \frac{\tau}{N_t} = p_{Ntc} \frac{\ell^2}{c\hbar}, \quad \frac{\ell}{N_i} = p_{Ni} \frac{\ell^2}{\hbar}, \quad 1 = 1, \ldots, 3, \quad (4) \]

where \( p_{Ni}, p_{Ntc} \) are **Primarily Measurable** momenta, up to the fundamental constants are coincident with \( p_{Ni}, p_{Ntc} \) and they may be involved at any stage of the calculations but, evidently, they are not **PMQ** in the general
Consequently PMQ is inadequate for studies of the physical processes. Therefore, it is reasonable to use Definition 2.

**Definition 2. Generalized Measurability**

We define any physical quantity at all energy scales as **generalized measurable** or, for simplicity, **measurable** if any of its values may be obtained in terms of PMQ specified by points 1.1, 1.2 of Definition 1.

It is important to make the following remark:

**Remark 2.1**

As long as \( \ell \) is a minimal measurable length and \( \tau \) is a minimal measurable time, values of all physical quantities should agree with this condition, i.e., their expressions should not involve the lengths \( l < \ell \) and the times \( t < \tau \) (and hence the momenta \( p > p_\ell \) and the energies \( E > E_\ell \)). Because of this, values of the length \( \ell/N_i \) and of the time \( \ell/N_t \) from formula (4) could not appear in expressions for physical quantities, being involved only in intermediate calculations, especially at the summation for replacement of the infinitesimal quantities \( dt, dx_i; i = 1, 2, 3 \) on passage from a continuous theory to its measurable variant.

The main target of the author is to form a quantum theory and gravity only in terms of measurable quantities (or of PMQ) in line with Remark 2.1.

Now we consider separately the two cases.

**A) Low Energies,** \( E \ll E_p \).

In \( P \) we consider the domain \( P_{LE} \subset P \) (LE is abbreviation of "Low Energies") defined by the conditions

\[
P_{LE} = \{p_{x_i}, i = 1, \ldots, 3; P_\ell \gg |p_{x_i}| \neq 0, \]

(5)
where $P_t = E_t/c$—maximal momentum.
In this case the formula of (2) takes the form

$$N_i = \frac{\hbar}{p_{x_i} \ell}, \text{or}$$

$$p_{x_i} = p_{N_i} = \frac{\hbar}{N_i \ell}$$

where the last row of the formula (6) is given by the requirement (5).

As shown in [7],[8], since the energies $E \ll E_\ell$ are low, i.e. $|N_i| \gg 1$, primarily measurable momenta are sufficient to specify the whole domain of the momenta to a high accuracy $P_{LE}$.

In the indicated domain a discrete set of primary measurable momenta $p_{N_i}, (i = 1, ..., 3)$ from formula (6) varies almost continuously, practically covering the whole domain.

That is why further $P_{LE}$ is associated with the domain of primary measurable momenta, satisfying the conditions of the formula (5) (or (6)).

Of course, all the calculations of point A) also comply with the primary measurable momenta $p_{N_{\ell}} \approx p_{N_0}$ in formula (4). Because of this, in what follows we understand $P_{LE}$ as a set of the primary measurable momenta $p_{x_{\mu}} = p_{N_{\mu}}, (\mu = 0, ..., 3)$ with $|N_{\mu}| \gg 1$.

It should be noted that, as all the experimentally involved energies $E$ are low, they meet the condition $E \ll E_\ell$, specifically for LHC the maximal energies are $\approx 10T \text{eV} = 10^4 \text{GeV}$, that is by 15 orders of magnitude lower than the Planck energy $\approx 10^{19} \text{GeV}$. But since the energy $E_\ell$ is on the order of the Planck energy $E_\ell \propto E_\mu$, in this case all the numbers $N_i$ for the corresponding momenta will meet the condition $\min |N_i| \approx 10^{15}$, i.e., the formula of (6). So, all the experimentally involved momenta are considered to be primary measurable momenta, i.e. $P_{LE}$ at low energies $E \ll E_\ell$.

**Note 2.1**

Further for the fixed point $x_{\mu}$ we use the notion $p_{x_{\mu}} = p_{N_{x_{\mu}}}$ or $p_{x_{\mu}} = p_{N_{\Delta x_{\mu}}}$.

Naturally, the small variation $\Delta p_{x_{\mu}}$ at the point $p_{x_{\mu}} = p_{N_{x_{\mu}}}$ of the momen-
tum space $P_{LE}$ is represented by the primary measurable momentum $p_{N'x}$ with the property $|N'x| \gg |N_{x}|$.

So, in the proposed paradigm at low energies $E \ll E_p$ a set of the primarily measurable $P_{LE}$ is discrete, and in every measurement of $\mu = 0, \ldots, 3$ there is the discrete subset $P_{x\mu} \subset P_{LE}$:

$$P_{x\mu} = \{\ldots, p_{N_{x\mu} - 1}, p_{N_{x\mu}}, p_{N_{x\mu} + 1}, \ldots\}. \quad (7)$$

In this case, as compared to the canonical quantum theory, in continuous space-time we have the following substitution:

$$\frac{\partial}{\partial p_\mu} \mapsto \Delta, \quad \frac{\partial F}{\partial p_\mu} \mapsto \frac{\Delta F(p_{N_{x\mu}})}{\Delta p_\mu} = \frac{\frac{F(p_{N_{x\mu}}) - F(p_{N_{x\mu} + 1})}{p_{N_{x\mu}} - p_{N_{x\mu} + 1}}}{p_{N_{x\mu}}(N_{x\mu} + 1)}. \quad (8)$$

It is clear that for sufficiently high integer values of $|N_{x\mu}|$, formula (8) reproduces a continuous paradigm in the momentum space to any preassigned accuracy.

Similarly for sufficiently high integer values of $|N_t|$ and $|N_i = N_{x_i}|$, the quantities $\tau/N_t, \ell/N_{x_i}$ from formula (4) may be arbitrary small.

Hence, for sufficiently high integer values of $|N_t|$ and $|N_i = N_{x_i}|$, the quantities $\tau/N_t, \ell/N_{x_i}$ are nothing but a measurable analog of the small quantities $\delta x_i, \delta t$ and the infinitesimal quantities $dx_i, dt$, i.e. $\delta x_\mu$, and $dx_\mu, \mu = 0, \ldots, 3$.

As follows from formula (4), for sufficiently high integer values of $|N_{x\mu}|, \mu = 0, \ldots, 3$, the primarily measurable momenta $P_{x\mu}$ (formula (7)) represent a measurable analog of small (and infinitesimal) space-time increments in the space-time variety $M \subset \mathbb{R}^4$.

Because of this, for sufficiently high integer values of $|N_{x\mu}|$, the space-time analog of formula (8) is as follows:

$$\frac{\partial}{\partial x_\mu} \mapsto \Delta, \quad \frac{\partial F}{\partial x_\mu} \mapsto \frac{\Delta F(x_\mu)}{\Delta x_\mu} = \frac{\frac{F(x_\mu + \ell/N_{x\mu}) - F(x_\mu)}{\ell/N_{x\mu}}}{\ell/N_{x\mu}}. \quad (9)$$
Note 2.2. In this way any point \(\{x_\mu\} \in \mathcal{M} \subset \mathbb{R}^4\) and any set of integer numbers high in absolute values \(\{N_{x_\mu}\}\) are correlated with a system of the neighborhoods for this point \((x_\mu \pm \ell/N_{x_\mu})\). It is clear that, with an increase in \(|N_{x_\mu}|\), the indicated system converges to the point \(\{x_\mu\}\). In this case all the ingredients of the initial (continuous) theory the partial derivatives including are replaced by the corresponding finite differences.

**Principle of Correspondence to Continuous Theory (PCCT).**

At low energies \(E \ll E_p\) (or same \(E \ll E_\ell\)) the infinitesimal space-time quantities \(dx_\mu; \mu = 0, ..., 3\) and also infinitesimal values of the momenta \(dp_i, i = 1, 2, 3\) and of the energies \(dE\) form the basic instruments ("construction materials") for any theory in continuous space-time. Because of this, to construct the measurable variant of such a theory, we should find the adequate substitutes for these quantities.

It is obvious that in the first case the substitute is represented by the quantities \(\ell/N_{x_\mu}\), where \(|N_{x_\mu}|\) - no arbitrary large (but finite!) integer, whereas in the second case by \(p_{N_{x_\mu}} = k_{N_{x_\mu}} \ell; i = 1, 2, 3\); \(E_{N_{x_0}} = \frac{\hbar}{N_{x_0} \ell}\), where \(N_{x_\mu}\) - integer with the above properties \(\mu = 0, ... 3\).

In this way in the proposed approach all the primary measurable momenta \(p_{N_{x_\mu}}, |N_{x_\mu}| \gg 1\) are small quantities at low energies \(E \ll E_\ell\) and primary measurable momenta \(p_{N_{x_\mu}}\) with sufficiently large \(|N_{x_\mu}| \gg 1\) being analogous to infinitesimal quantities of a continuous theory.

**B) High Energies, \(E \approx E_p\).**

In this case formula (2) takes the form

\[
N_i = \frac{\hbar}{p_{x_i} \ell}, \text{or} \quad p_{x_i} = p_{N_i} = \frac{\hbar}{N_i \ell}, \quad |N_i| \approx 1.
\]

where \(N_i\) is an integer number and \(p_{x_i}\) is the momentum corresponding to the coordinate \(x_i\). The discrete set \(p_{N_i} \doteq p_{N_{x_i}}\) is introduced as primarily measurable momenta.
The main difference of the case B) **High Energies** from the case A) **Low Energies** is in the fact that at **High Energies** the primary measurable momenta are *inadequate* for theoretical studies at the energy scales \( E \approx E_p \).

Indeed, as it has been shown in [6], the Generalized Uncertainty Principle (GUP), which is a generalization of the Heisenberg Uncertainty Principle (HUP) [9]–[18],

\[
\Delta x \geq \frac{\hbar}{\Delta p} + \alpha' l_p^2 \frac{\Delta p}{\hbar}, \tag{11}
\]

where \( \alpha' \) is a constant on the order of 1, leading to the minimal length \( l \) on the order of the Planck length \( l = \sqrt{\alpha'} \).

At high energies inevitably results in the momenta \( \Delta p(N_\Delta x, \text{GUP}) \) which are not primarily measurable:

\[
\Delta p \doteq \Delta p(N_\Delta x, \text{GUP}) = \frac{\hbar}{1/2(N_\Delta x + \sqrt{N_\Delta x^2 - 1})}. \tag{12}
\]

It is clear that for \( N_\Delta x \approx 1 \) the momentum \( \Delta p(N_\Delta x, \text{GUP}) \) is not a primary measurable momentum.

From Remark 2.1 in formula (12) it follows that the condition \( N_\Delta x \geq 2 \) should be fulfilled.

On the contrary, at low energies \( E \ll E_p, (E \ll E_{\ell}) \) the primary measurable space quantity \( \Delta x = N_\Delta x \ell \), where \( N_\Delta x \gg 1 \) is an integer number, due to the validity of the limit

\[
\lim_{N_\Delta x \to \infty} \sqrt{N_\Delta x^2 - 1} = N_\Delta x, \tag{13}
\]

leads to the momentum \( \Delta p(N_\Delta x, \text{HUP}) \):

\[
\Delta p \doteq \Delta p(N_\Delta x, \text{HUP}) = \frac{\hbar}{1/2(N_\Delta x + \sqrt{N_\Delta x^2 - 1})} \approx \frac{\hbar}{N_\Delta x} = \frac{\hbar}{\Delta x}. \tag{14}
\]

It is inferred that, for sufficiently high integer values of \( N_\Delta x \) the momentum \( \Delta p(N_\Delta x, \text{HUP}) \) within any high accuracy may be considered to be the primary measurable momentum.

Therefore, to study high (Planck’s) energies \( E \approx E_p \), we need not only primarily measurable momenta but also the generalized measurable

\[\]
momenta.

Remark 1. What is the main point of this Section?

1.a) At low energies $E \ll E_p$ we replace the abstract small and infinitesimal quantities $\delta x_\mu, dx_\mu, \delta p_\mu, dp_\mu$ incomparable with each other, by the specific small quantities $\ell/N_\mu, p_N x_\mu,$ which may be made however small at sufficiently high $|N_\mu|,$ still being ordered and comparable. It is very important that the quantities $\ell/N_\mu, p_N x_\mu$ are directly associated with the existing energies; for $|N'_\mu| > |N_\mu|$ the momentum $p_{|N'_\mu|} < p_{|N_\mu|}$ and $p_{|N'_\mu|}$ corresponds to lower energy than $p_{|N_\mu|}.$ The same is true for the space variations $\ell/N'_\mu, \ell/N_\mu.$

1.b) At low energies $E \ll E_p$ we should emphasize the difference between the primary measurable momenta $p_{N_\mu} \in P_{LE}$ and the space-time quantities $\ell/N_\mu$ corresponding to them in accordance with formula (7).

The first, that is $p_{N_\mu}$ represent the whole set of the momenta $P_{LE}$ at low energies $E \ll E_p$ in terms of measurable quantities, whereas the second, $\ell/N_\mu,$ represent only the measurable small variations of space-time quantities.

1.c) According to Definition 1., in the relativistic case the primary measurable energy is of the form

$$E = \frac{hc}{N_0 \ell}, N_0 = N_{x_0},$$

(15)

where $N_0$ is an integer number, and at low energies $E \ll E_p$ it is obvious that $N_0 \gg 1.$

Then at low energies $E \ll E_p$ from Remark 2.2. it follows naturally that primary measurable energies, to a high accuracy, cover the whole low-energy spectrum. Then, considering that the formula

$$E^2 = p^2 c^2 + m^2 c^4$$

low energies $E \ll E_p$ to a high accuracy is valid in terms of measurable quantities and all components of the vector $p$ are the primary measurable momenta, we can found the mass $m$ in terms of
the **measurability** notion as follows:

\[ m^2 = \frac{k^2}{c^2}\left(\frac{1}{N_0^2\ell^2} - \sum_{1 \leq i \leq 3} \frac{1}{N_i^2\ell^2}\right). \] (16)

1.d) Finally, it is important to note that actually the minimal quantities of length \( \ell \) and time \( \tau \), and also the maximal quantities of energy \( E_{\ell} \) and momentum \( p_{\ell} = E_{\ell}/c \), considered at the beginning of this section, are minimal and maximal **primarily measurable** quantities, respectively.

### 3 Measurable Metrics and Coordinate Transformations in Gravity

According to the results from the previous section, the **measurable** variant of gravity at low energies \( E \ll E_p \) should be formulated in terms of the **measurable** space-time quantities \( \ell/N_\Delta x_\mu \) or **primary measurable** momenta \( p_{N_\Delta x_\mu} \).

Let us consider the case of the random metric \( g_{\mu\nu} = g_{\mu\nu}(x) \) \[21],[22\], where \( x \in \mathbb{R}^4 \) is some point of the four-dimensional space \( \mathbb{R}^4 \) defined in **measurable** terms. Now, any such point \( x = \{x_\chi\} \in \mathbb{R}^4 \) and any set of integer numbers \( \{N_{x_\chi}\} \) dependent on the point \( \{x_\chi\} \) with the property \( |N_{x_\chi}| \gg 1 \) may be correlated to the "bundle" with the base \( \mathbb{R}^4 \) as follows:

\[ B_{N_{x_\chi}} \doteq \{x_\chi, \frac{\ell}{N_{x_\chi}}\} \mapsto \{x_\chi\}. \] (17)

It is clear that \( \lim_{|N_{x_\chi}| \to \infty} B_{N_{x_\chi}} = \mathbb{R}^4 \).

As distinct from the normal one, the "bundle" \( B_{N_{x_\chi}} \) is distinguished only by the fact that the mapping in formula (17) is not continuous (smooth) but discrete in fibers, being continuous in the limit \( |N_{x_\chi}| \to \infty \).

Then as a **canonically measurable prototype** of the infinitesimal space-time interval square \[21],[22\]

\[ ds^2(x) = g_{\mu\nu}(x)dx^\mu dx^\nu \] (18)
we take the expression

\[ \Delta s^2_{N_{x\chi}}(x) \equiv g_{\mu\nu}(x, N_{x\chi}) \frac{\ell^2}{N_{x\mu} N_{x\nu}}. \]  

(19)

Here \( g_{\mu\nu}(x, N_{x\chi}) \) – metric \( g_{\mu\nu}(x) \) from formula (18) with the property that minimal measurable variation of metric \( g_{\mu\nu}(x) \) in point \( x \) has form

\[ \Delta g_{\mu\nu}(x, N_{x\chi}) = g_{\mu\nu}(x + \ell/N_{x\chi}, N_{x\chi}) - g_{\mu\nu}(x, N_{x\chi}), \]  

(20)

Let us denote by \( \Delta g_{\mu\nu}(x, N_{x\chi}) \) quantity

\[ \Delta g_{\mu\nu}(x, N_{x\chi}) = \frac{\Delta g_{\mu\nu}(x, N_{x\chi})}{\ell/N_{x\chi}}. \]  

(21)

It is obvious that in the case under study the quantity \( \Delta g_{\mu\nu}(x, N_{x\chi}) \) is a measurable analog for the infinitesimal increment \( dg_{\mu\nu}(x) \) of the \( \chi \)-th component \( (dg_{\mu\nu}(x))_\chi \) in a continuous theory, whereas the quantity \( \Delta g_{\mu\nu}(x, N_{x\chi}) \) is a measurable analog of the partial derivative \( \partial_{\chi} g_{\mu\nu}(x) \).

In this manner we obtain the (17)-formula induced bundle over the metric manifold \( g_{\mu\nu}(x) \):

\[ B_{g,N_{x\chi}} \]  

(22)

Referring to formula (4), we can see that (19) may be written in terms of the primary measurable momenta \((p_{N_{i}} , p_{N_{t}})\) as follows:

\[ \Delta s^2_{N_{x\mu}}(x) = g_{\mu\nu}(x, N_{x\chi}) p_{N_{x\mu}} p_{N_{x\nu}}. \]  

(23)

Considering that \( \ell \propto l_P \) (i.e., \( \ell = \kappa l_P \)), where \( \kappa = const \) is on the order of 1, in the general case (23), to within the constant \( \ell^4/h^2 \), we have

\[ \Delta s^2_{N_{x\mu}}(x) = g_{\mu\nu}(x, N_{x\chi}) p_{N_{x\mu}} p_{N_{x\nu}}. \]  

(24)

As follows from the previous formulae, the measurable variant of General Relativity should be defined in the bundle \( B_{g,N_{x\chi}} \).
Let us consider any coordinate transformation \( x^\mu = x^\mu (\bar{x}^\nu) \) of the space–time coordinates in continuous space–time. Then we have

\[
dx^\mu = \frac{\partial x^\mu}{\partial \bar{x}^\nu} d\bar{x}^\nu. \tag{25}\]

As mentioned at the beginning of this section, in terms of measurable quantities we have the substitution

\[
dx^\mu \mapsto \frac{\ell}{N_{\Delta x^\mu}}; d\bar{x}^\nu \mapsto \frac{\ell}{\bar{N}_{\Delta \bar{x}^\nu}}, \tag{26}\]

where \( N_{\Delta x^\mu}, \bar{N}_{\Delta \bar{x}^\nu} \) – integers (\( |N_{\Delta x^\mu}| \gg 1, |ar{N}_{\Delta \bar{x}^\nu}| \gg 1 \)) sufficiently high in absolute value, and hence in the measurable case (25) is replaced by

\[
\frac{\ell}{N_{\Delta x^\mu}} = \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x^\mu}, 1/\bar{N}_{\Delta \bar{x}^\nu}) \frac{\ell}{\bar{N}_{\Delta \bar{x}^\nu}}. \tag{27}\]

Equivalently, in terms of the primary measurable momenta we have

\[
p_{N_{\Delta x^\mu}} = \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x^\mu}, 1/\bar{N}_{\Delta \bar{x}^\nu}) p_{\bar{N}_{\Delta \bar{x}^\nu}}, \tag{28}\]

where \( \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x^\mu}, 1/\bar{N}_{\Delta \bar{x}^\nu}) \) – corresponding matrix represented in terms of measurable quantities.

It is clear that, in accordance with formula (4), in passage to the limit we get

\[
\lim_{|N_{\Delta x^\mu}| \to \infty} \frac{\ell}{N_{\Delta x^\mu}} = \lim_{|\Delta_{x^\mu}| \to \infty} \frac{\ell}{\bar{N}_{\Delta \bar{x}^\nu}} = d\bar{x}^\nu = \frac{\partial x^\mu}{\partial \bar{x}^\nu} dx^\nu. \tag{29}\]

Equivalently, passage to the limit (29) may be written in terms of the primary measurable momenta \( p_{N_{\Delta x^\mu}}, p_{\bar{N}_{\Delta \bar{x}^\nu}} \) multiplied by the constant \( \ell^2/\hbar \).

How we understand formulae (26)–(29)?

The initial (continuous) coordinate transformations \( x^\mu = x^\mu (\bar{x}^\nu) \) gives the matrix \( \frac{\partial x^\mu}{\partial \bar{x}^\nu} \). Then, for the integers sufficiently high in absolute value \( \bar{N}_{\Delta \bar{x}^\nu}, |\bar{N}_{\Delta \bar{x}^\nu}| \gg 1 \), we can derive

\[
\frac{\ell}{\bar{N}_{\Delta \bar{x}^\nu}} = \frac{\partial x^\mu}{\partial \bar{x}^\nu} \frac{\ell}{N_{\Delta x^\mu}}, \tag{30}\]

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where \( |N_{\Delta x_\mu}| \gg 1 \) but the numbers for \( N_{\Delta x_\mu} \) are not necessarily integer. However it is easy to see the difference between \( \ell/N_{\Delta x_\mu} \) and \( \ell/[N_{\Delta x_\mu}] \) (and hence between \( p_{N_{\Delta x_\mu}} \) and \( p_{[N_{\Delta x_\mu}]} \)) is negligible.

Then substitution of \( [N_{\Delta x_\mu}] \) for \( N_{\Delta x_\mu} \) in the left-hand side of (30) leads to replacement of the initial matrix \( \partial x^\mu/\partial \bar{x}^\nu \) by the matrix \( \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) \) represented in terms of measurable quantities and, finally, to the formula (27). Clearly, for sufficiently high \( |N_{\Delta x_\mu}|, |\bar{N}_{\Delta \bar{x}_\nu}| \), the matrix \( \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) \) may be selected no matter how close to \( \partial x^\mu/\partial \bar{x}^\nu \).

Similarly, in the measurable format we can get the formula

\[
\frac{d\bar{x}^\mu}{\partial x^\nu} = \frac{\partial x^\mu}{\partial \bar{x}^\nu} \ dx^\nu. \tag{31}
\]

and correspondingly the matrix \( \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) \) instead of the matrix \( \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) \).

Thus, any coordinate transformations may be represented, to however high accuracy, by the measurable transformation (i.e., written in terms of measurable quantities), where the principal components are the measurable quantities \( \ell/N_{\Delta x_\mu} \) or the primary measurable momenta \( p_{N_{\Delta x_\mu}} \).

4 Measurability and Its Mathematical Instruments in Gravity

Using the results from the previous section, we can define the measurable analogs of the principal ingredients of General Relativity. In the canonical (continuous) theory the vector \( A^\mu \) is referred to as contravariant if it transforms, under the coordinate transformation, in the same way as coordinates do:

\[
A^\mu(x) = \frac{\partial x^\mu}{\partial \bar{x}^\nu} \bar{A}^\nu(\bar{x}). \tag{32}
\]

In the measurable case the formula for (32) is as follows:

\[
A^\mu(x, N_{\Delta x_\chi}) = \Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x_\mu}, 1/\bar{N}_{\Delta \bar{x}_\nu}) \bar{A}^\nu(\bar{x}, \bar{N}_{\Delta \bar{x}_\nu}). \tag{33}
\]
Here the contravariant vector $A^\mu(x, N_{\Delta x^\chi})$, by virtue of replacement of the matrix $\frac{\partial x^\mu}{\partial \bar{x}^\nu}$ by $\Delta_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x^\mu}, 1/N_{\bar{\Delta}x^\nu})$, is dependent not only on the point $x$ but also on the integer numbers $N_{\Delta x^\chi}$, $|N_{\Delta x^\chi}| \gg 1$ (i.e. on minimal variations in $\ell/N_{\Delta x^\chi}$ at the point $x$).

Similarly, in terms of $A_\mu(x, N_{\Delta x^\chi})$, we can define a measurable analog of the covariant vector $A_\mu(x)$ with the transformational properties

$$A_\mu(x, N_{\Delta x^\chi}) = \bar{\Delta}_{\mu\nu}(x^\mu, \bar{x}^\nu, 1/N_{\Delta x^\mu}, 1/\bar{N}_{\bar{\Delta}x^\nu}) \bar{A}^\nu(\bar{x}, \bar{N}_{\Delta x^\chi}).$$  \hspace{1cm} (34)

Let us consider in a continuous theory the equality of the two infinite similar intervals

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = \bar{g}_{\alpha\beta}(\bar{x}) d\bar{x}^\alpha d\bar{x}^\beta,$$

or

$$g_{\mu\nu}(x) = \frac{\partial \bar{x}^\alpha}{\partial x^\mu} \frac{\partial \bar{x}^\beta}{\partial x^\nu} \bar{g}_{\alpha\beta}(\bar{x}).$$  \hspace{1cm} (35)

Obviously, the measurable analog of the upper line of formula (35) should be the equality

$$\Delta s^2 \approx g_{\mu\nu}(x, N_{\Delta x^\chi}) \frac{\ell^2}{N_{\Delta x^\mu} N_{\Delta x^\nu}} \approx \bar{g}_{\alpha\beta}(\bar{x}, \bar{N}_{\bar{\Delta}x^\chi}) \frac{\ell^2}{\bar{N}_{\bar{\Delta}x^\alpha} \bar{N}_{\bar{\Delta}x^\beta}}.$$  \hspace{1cm} (36)

In formula (36) and in all subsequent formulae the approximation sign $\approx$ everywhere denotes however high accuracy for sufficiently great $|N_{\Delta x^\mu}|$, $|N_{\Delta x^\nu}|$, $|\bar{N}_{\bar{\Delta}x^\alpha}|$, $|\bar{N}_{\bar{\Delta}x^\beta}|$.

Neverthless, in the case of formula (35) we involve infinitesimal intervals, whereas in the case of formula (36) we consider the intervals having small but finite value. Because of this, it is assumed that formula (36) is valid when the second and the third terms in this formula have the same sign (or both are zero), so

$$g_{\mu\nu}(x, N_{\Delta x^\chi}) \frac{\ell^2}{N_{\Delta x^\mu} N_{\Delta x^\nu}} - \bar{g}_{\alpha\beta}(\bar{x}, \bar{N}_{\bar{\Delta}x^\chi}) \frac{\ell^2}{\bar{N}_{\bar{\Delta}x^\alpha} \bar{N}_{\bar{\Delta}x^\beta}} \approx 0.$$  \hspace{1cm} (37)

Then from the foregoing results it follows that a measurable analog of the lower line in formula (35) takes the form

$$g_{\mu\nu}(x, N_{\Delta x^\chi}) \approx \bar{\Delta}_{\mu\alpha}(x^\mu, \bar{x}^\alpha, 1/N_{\Delta x^\mu}, 1/\bar{N}_{\bar{\Delta}x^\alpha}) \bar{\Delta}_{\nu\beta}(x^\nu, \bar{x}^\beta, 1/N_{\Delta x^\nu}, 1/\bar{N}_{\bar{\Delta}x^\beta}) \bar{g}_{\alpha\beta}(\bar{x}, \bar{N}_{\Delta x^\chi}).$$  \hspace{1cm} (38)
Remark 4.1
It is clear that for sufficiently great \(|N_{\Delta x}|, |\bar{N}_{\Delta x}|\) all the above given formulae in a measurable variant may be no matter how close to the corresponding formulae associated with a continuous consideration. Without loss of generality, this may be assumed for lowering or growing indices of the contravariant or covariant vectors, tensors, etc. Specifically, in a measurable variant we have

\[
A_\mu(x, N_{\Delta x}) = g_{\mu\nu}(x, N_{\Delta x}) A_\nu(x, N_{\Delta x}) \approx g_{\mu\nu} A_\nu(x, N_{\Delta x})
\]

and \(g^{\mu\nu}(x, N_{\Delta x}) g_{\nu\alpha}(x, N_{\Delta x}) \approx g^{\mu\nu} = \delta^\mu_\alpha.\) (39)

Now we consider formula (35) in the assumption that the coordinates \(\bar{x}\) belong to a flat space, i.e. the metric \(\bar{g}_{\alpha\beta}(\bar{x})\) is the Lorentzian metric

\[
||\eta_{\alpha\beta}| | = ||\eta^{\alpha\beta}| | = \text{Diag} (-1, 1, 1, 1).
\] (40)

In this case formula (38) takes the form

\[
g_{\mu\nu}(x, N_{\Delta x}) = \bar{\Delta}_{\alpha} \bar{\Delta}_{\beta}(x^\alpha, 1/N_{\Delta x_{\alpha}}, 1/\bar{N}_{\Delta x_{\alpha}}) \eta_{\alpha\beta}(\bar{x}, \bar{N}_{\Delta x}).
\] (41)

Taking the determinant for both parts (41) and assuming that, by virtue of Remark 4.1, \(det ||\eta_{\alpha\beta}(\bar{x}, \bar{N}_{\Delta x})|| \approx det ||\eta_{\alpha\beta}|| = -1\), we obtain

\[
det ||g_{\mu\nu}(x, N_{\Delta x})|| \approx \frac{1}{J^2[\Delta_{\mu\alpha}(x^\mu, \bar{x}^\alpha, 1/N_{\Delta x_{\alpha}}, 1/\bar{N}_{\Delta x_{\alpha}})]},
\] (42)

where \(J[\Delta_{\mu\alpha}(x^\mu, \bar{x}^\alpha, 1/N_{\Delta x_{\alpha}}, 1/\bar{N}_{\Delta x_{\alpha}})]\) – measurable variant of the Jacobian \(J\) for the matrix \(\frac{\partial x^\mu}{\partial \bar{x}^\nu}\) when coordinates of the point \(\bar{x}\) belong to a flat space. In view of the lower line in formula (39), we have

\[
det ||g_{\mu\nu}(x, N_{\Delta x})|| = 1/det ||g^{\mu\nu}(x, N_{\Delta x})||.\] If, according to a continuous theory, we introduce the notation \(det ||g^{\mu\nu}(x, N_{\Delta x})|| \approx g(N_{\Delta x}),\) the for-
mula (42) should be rewritten as

\[
g(N\Delta x^\chi) \approx -J^2 [\tilde{\Delta}_{\mu\alpha}(x^\mu, \bar{x}^\alpha, 1/N\Delta x_\mu, 1/N\Delta \bar{x}_\alpha)],
\]

or

\[
J[\tilde{\Delta}_{\mu\alpha}(x^\mu, \bar{x}^\alpha, 1/N\Delta x_\mu, 1/N\Delta \bar{x}_\alpha)] \approx \frac{1}{\sqrt{-g(N\Delta x^\chi)}}, \tag{43}
\]

in a complete agreement with the well-known relation in a continuous theory

\[J = 1/\sqrt{-g} \quad \text{for} \quad g = \det |g_{\mu\nu}| \quad [20].\]

Let \(d\Omega\) be the element of integration with respect to the four-dimensional volume \(\Omega\) in the continuous case. We denote by \(\Delta_{(N\Delta x_\mu)}\Omega\) its measurable analog. \(\Delta_{(N\Delta x_\mu)}\Omega\) may be obtained by substitution of \(dx_\mu \mapsto \ell/N\Delta x_\mu\) in the formula for \(d\Omega\).

Obviously, we have

\[
\Delta_{(N\Delta x_\mu)}\Omega \equiv \frac{\ell^8}{\hbar^4} \Delta_{(gN\Delta x_\mu)}\Omega. \tag{44}
\]

The transition from the infinitesimal volume in a flat space \(d\bar{\Omega}\) to the infinitesimal volume in a curved space \(d\Omega\) with the metric \(g_{\mu\nu}\), that in the continuous case takes the form

\[d\bar{\Omega} \mapsto \sqrt{-g}d\Omega \quad \text{or equivalently} \quad d\bar{\Omega} \mapsto \sqrt{|g|}d\Omega, \tag{45}\]

in the measurable case is as follows:

\[
\Delta_{(\bar{N}\Delta x_\mu)}\bar{\Omega} \mapsto \sqrt{-g(N\Delta x^\chi)}\Delta_{(N\Delta x_\mu)}\Omega \quad \text{or equivalently} \quad \Delta_{(\bar{N}\Delta x_\mu)}\bar{\Omega} \mapsto \sqrt{|g(N\Delta x^\chi)|}\Delta_{(N\Delta x_\mu)}\Omega. \tag{46}
\]

In a similar way we can obtain other quantities in the four-dimensional "measurable" geometry as well. In particular, the infinitesimal element of the two-dimension surface in a continuous theory

\[df^{ik} = dx^i dx^j - dx^k dx^i \tag{47}\]

in a measurable variant is associated with the element

\[
\Delta f^{ik}(N\Delta x, N\Delta x') = \frac{\ell^2}{N\Delta x_\mu' N\Delta x'^k} - \frac{\ell^2}{N\Delta x_\mu N\Delta x'^i}. \tag{48}
\]
And the infinitesimal three-dimensional volume element
\[ dS^{ikl} = \det \begin{pmatrix} dx^i & dx^n & dx^{ni} \\ dx^k & dx^{nk} & dx^{nk} \\ dx^l & dx^{nl} & dx^{nl} \end{pmatrix} \]
is associated with the measurable element
\[ \Delta S^{ikl}(N_{\Delta x}, N_{\Delta x'}, N_{\Delta x''}) = \det \begin{pmatrix} \frac{\ell}{N_{\Delta x'}} & \frac{\ell}{N_{\Delta x}} & \frac{\ell}{N_{\Delta x''}} \\ \frac{\Delta x'}{N_{\Delta x}} & \frac{\Delta x}{N_{\Delta x'}} & \frac{\Delta x''}{N_{\Delta x''}} \\ \frac{\Delta x'}{N_{\Delta x}} & \frac{\Delta x}{N_{\Delta x'}} & \frac{\Delta x''}{N_{\Delta x''}} \end{pmatrix} \]
retaining, for sufficiently high \( |N_{\Delta x'}|, |N_{\Delta x''}|, |N_{\Delta x''}| \), all basic relations of a continuous geometry (paragraphs 6, 83 in [20]) to however high accuracy.

5 Gravity in Measurable Form at Low Energies

5.1 Einstein Equations in Measurable Format as Deformation of Canonical Theory

As directly follows from previous results, specifically from formulae (19)–(21), formulae of Sections 3, 4, the principal components involved in gravitational equations of General Relativity have measurable form. In particular, the Christoffel symbols [21], [22]
\[ \Gamma^\alpha_{\mu\nu}(x) = \frac{1}{2} g^{\alpha\beta}(x) \left( \partial_\nu g_{\beta\mu}(x) + \partial_\mu g_{\beta\nu}(x) - \partial_\beta g_{\mu\nu}(x) \right) \] (49)
have the measurable format [7]
\[ \Gamma^\alpha_{\mu\nu}(x, N_{\Delta x}) = \frac{1}{2} g^{\alpha\beta}(x, N_{\Delta x}) \left( \partial_\nu g_{\beta\mu}(x, N_{\Delta x}) + \Delta_\mu g_{\beta\nu}(x, N_{\Delta x}) + \Delta_\nu g_{\mu\beta}(x, N_{\Delta x}) - \Delta_\beta g_{\mu\nu}(x, N_{\Delta x}) \right) \] (50)
In is simple to derive measurable analogs of the well-known quantities in a continuous theory arising in the definition of the parallel translation.
Specifically, the formula

\[ \delta A_\mu = \Gamma^\nu_{\mu\alpha}(x) A_\nu(x) \, dx^\alpha, \]  

(51)
in a measurable variant is of the form

\[ \delta_{\text{meas}} A_\mu(x, N_{\Delta x}) = \Gamma^\nu_{\mu\alpha}(x, N_{\Delta x}) A_\nu(x, N_{\Delta x}) \frac{\ell}{N_{\Delta x_\alpha}}. \]  

(52)

Note that a formula for the covariant derivative

\[ A^\mu_{,\alpha} \equiv D_\alpha A^\mu = \partial_\alpha A^\mu + \Gamma^\nu_{\nu\alpha} A^\nu = (\partial_\alpha \delta^\nu_\mu + \Gamma^\nu_{\nu\alpha}) A^\nu, \]  

\[ A_{\mu;\alpha} \equiv D_\alpha A_\mu = \partial_\alpha A_\mu - \Gamma^\nu_{\mu\alpha} A_\nu = (\partial_\alpha \delta^\nu_\mu - \Gamma^\nu_{\mu\alpha}) A_\nu \]  

(53)
in a measurable format is given as

\[ \tilde{A}^\mu_{,\alpha} \equiv \tilde{D}_\alpha A^\mu(x, N_{\Delta x}) = \Delta \frac{\Delta}{\Delta N_{\Delta x_\alpha}} A^\mu(x, N_{\Delta x}) + \Gamma^\mu_{\nu\alpha}(x, N_{\Delta x}) A^\nu(x, N_{\Delta x}) = \]  

\[ = \left( \frac{\Delta}{\Delta N_{\Delta x_\alpha}} \delta^\mu_\nu + \Gamma^\mu_{\nu\alpha}(x, N_{\Delta x}) \right) A^\nu(x, N_{\Delta x}), \]  

\[ \tilde{A}_{\mu;\alpha} \equiv \tilde{D}_\alpha A_\mu(x, N_{\Delta x}) = \Delta \frac{\Delta}{\Delta N_{\Delta x_\alpha}} A_\mu(x, N_{\Delta x}) + \Gamma^\nu_{\mu\alpha}(x, N_{\Delta x}) A_\nu(x, N_{\Delta x}) = \]  

\[ = \left( \frac{\Delta}{\Delta N_{\Delta x_\alpha}} \delta^\nu_\mu + \Gamma^\nu_{\mu\alpha}(x, N_{\Delta x}) \right) A_\nu(x, N_{\Delta x}). \]  

(54)

where the operator \( \Delta/\Delta N_{\Delta x_\alpha} \) is taken from the formula (9) in which \( N_{x_\mu} = N_{\Delta x_\mu} \), in accordance with Note 2.1 from Section 2.

Similarly, for the Riemann tensor in a continuous theory we have [21,22]:

\[ R_{\nu\alpha\beta}(x) \equiv \partial_\alpha \Gamma^\mu_{\nu\beta}(x) - \partial_\beta \Gamma^\mu_{\nu\alpha}(x) + \Gamma^\mu_{\gamma\alpha}(x) \Gamma^\gamma_{\nu\beta}(x) - \Gamma^\mu_{\gamma\beta}(x) \Gamma^\gamma_{\nu\alpha}(x). \]  

(55)

With the use of formula (50), we can get the corresponding measurable analog, i.e. the quantity \( R_{\nu\alpha\beta}(x, N_{x_\nu}) \) [7].

In a similar way we can obtain the measurable variant of Ricci tensor, \( R_{\mu\nu}(x, N_{x_\nu}) \equiv R_{\mu\nu}(x, N_{x_\nu}) \), and the measurable variant of Ricci scalar.
\( R(x, N_{x\chi}) \equiv R_{\mu\nu}(x, N_{x\chi}) g^{\mu\nu}(x, N_{x\chi}) \) \[7\].

So, for the Einstein Equations (\(\mathcal{EE}\)) in a continuous theory \[21],[22\],

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \frac{1}{2} \Lambda g_{\mu\nu} = 8 \pi G T_{\mu\nu}
\] (56)

we can derive their measurable analog, for short denoted as (\(\mathcal{EE}_M\)) or Einstein Equations Measurable \[7\]:

\[
R_{\mu\nu}(x, N_{x\chi}) - \frac{1}{2} R(x, N_{x\chi}) g_{\mu\nu}(x, N_{x\chi}) - \frac{1}{2} \Lambda(x, N_{x\chi}) g_{\mu\nu}(x, N_{x\chi}) \approx \\
\approx 8 \pi G T_{\mu\nu}(x, N_{x\chi}),
\] (57)

where \(G\) – Newton’s gravitational constant.

For correspondence with a continuous theory, the following passage to the limit must take place for all the points \(x\):

\[
\lim_{|N_{x\chi}| \to \infty} \Lambda(x, N_{x\chi}) = \Lambda,
\] (58)

where the cosmological constant \(\Lambda\) is taken from formula (56).

Moreover, for high \(|N_{x\chi}|\), the quantity \(\Lambda(x, N_{x\chi})\) should be practically independent of the point \(x\), and we have

\[
\Lambda(x, N_{x\chi}) \approx \Lambda(x', N'_{x\chi}) \approx \Lambda,
\] (59)

where \(x \neq x'\) and \(|N_{x\chi}| \gg 1, |N'_{x\chi}| \gg 1\).

Actually, it is clear that formula (58) reflects the fact that (\(\mathcal{EE}_M\)) given by formula (17) represents deformation of the Einstein equations (\(\mathcal{EE}\)) (56) in the sense of the Definition given in [23] with the deformation parameter \(N_{x\chi}\) (or \(1/N_{x\chi}\)), and we have

\[
\lim_{|N_{x\chi}| \to \infty} \mathcal{EE}_M = \mathcal{EE}
\]

or same \[
\lim_{1/|N_{x\chi}| \to 0} \mathcal{EE}_M = \mathcal{EE}.
\] (60)

We denote this deformation as \(\mathcal{EE}_M[N_{x\chi}]\). Since at low energies \(E \ll E_P\) and to within the known constants we have \(\ell/N_{x\chi} = p_{N_{x\chi}}\), the following deformations of \(\mathcal{EE}\) are equivalent to

\[
\mathcal{EE}_M[N_{x\chi}] \equiv \mathcal{EE}_M[p_{N_{x\chi}}].
\] (61)
So, on passage from $\mathcal{E}$ to the measurable deformation of $\mathcal{E}\mathcal{M}[N_{x^i}]$ (or identically $\mathcal{E}\mathcal{M}[p_{N_{x^i}}]$) we can find solutions for the gravitational equations on the metric bundle $B_{g,N_{x^i}} \equiv g_{\mu \nu}(x, \{N_{x^i}\})$ (formula (22)).

### 5.2 Measurability and Least Action Principle in Gravity

#### 5.2.1. First, we consider the variational principle in a unidimensional case in the measurable form. Next, we use the terminology and notation of ([24], paragraph 31).

Let $S[\gamma] = \int_0^Q L(x, \dot{x}, t) dt; x = \{x^i\}, i = 1, ..., n$ be the action for the smooth curves $\gamma$ specified as $\gamma : x^i = x^i(t), a \leq t \leq b$, where the ends $a$ and $b$ are fixed and $\gamma$ connect the fixed pair of points $P = (x^i_1), Q = (x^i_2)$.

Then the transition from a continuous consideration to the measurable one is realized as

$$S[\gamma] = \int_0^Q L(x, \dot{x}, t) dt \to S_{1/N_t}[\gamma] = \sum_0^Q L(x(t), \frac{\Delta x(t)}{\Delta N_t}, t) \frac{1}{N_t}. \quad (62)$$

Here $x, \dot{x}$ are considered as "measurable" functions of $t$ (i.e. their infinitesimal variations at the point $t$ are of the form $1/\Delta N_t$), where $N_t$ is high (in its absolute value) integer, or $|N_t| \gg 1$.

It is clear that for high $|N_t|$ we have

$$S[\gamma] = \int_0^Q L(x, \dot{x}, t) dt \approx S_{1/N_t}[\gamma] = \sum_0^Q L(x(t), \frac{\Delta x(t)}{\Delta N_t}, t) \frac{1}{N_t}. \quad (63)$$

In the continuous case for great integer values of $N, |N| \gg 1$ we have

$$\lim_{\epsilon \to 0} \frac{S[\gamma + \epsilon \eta] - S[\gamma]}{\epsilon} \Rightarrow \lim_{|N| \gg 1} \frac{S[\gamma + \frac{1}{N} \eta] - S[\gamma]}{1/N} = \frac{\Delta}{\Delta N} S[\gamma + \frac{1}{N} \eta]|_{|N| \gg 1},$$

$$\lim_{\epsilon \to 0} \frac{S[\gamma + \epsilon \eta] - S[\gamma]}{\epsilon} = \frac{d}{d\epsilon} S[\gamma + \epsilon \eta]|_{\epsilon = 0} = \lim_{N \to \infty} \frac{S[\gamma + \frac{1}{N} \eta] - S[\gamma]}{1/N} = \frac{\Delta}{\Delta N} S[\gamma + \frac{1}{N} \eta]|_{N = \infty},$$

$$\frac{d}{d\epsilon} S[\gamma + \epsilon \eta]|_{\epsilon = 0} \approx \frac{\Delta}{\Delta N} S[\gamma + \frac{1}{N} \eta]|_{N \gg 1}. (64)$$
Consequently, the equation
\[
\frac{d}{d\epsilon} S[\gamma + \epsilon \eta]|_{\epsilon=0} \equiv 0
\] (65)
leads to
\[
\frac{\Delta}{\Delta N} S[\gamma + \frac{1}{N} \eta]|_{N \gg 1} \approx 0
\] (66)
and hence the variational derivative \( S[\gamma] \) in the continuous case
\[
\frac{\delta S}{\delta x^i} = \frac{\partial L}{\partial x^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i}
\] (67)
generates the \textit{measurable} variational derivative
\[
\frac{\delta_{\{\hat{N}\}} S}{\delta x^i} \doteq \frac{\Delta}{\Delta_{N^i}} L - \frac{\Delta}{\Delta_{N^t}} \left( \frac{\Delta}{\Delta_{\dot{x}^i}} L \right),
\] (68)
where \( \{\hat{N}\} \)–set of integer numbers sufficiently high in absolute value \( \{\hat{N}\} \doteq (N, N_t, N_{x^i}, N_{\dot{x}^i}) \).

Due to the fact that absolute values of all the integer numbers \( N, N_t, N_{x^i}, N_{\dot{x}^i} \) may be taken no matter how high, it holds true that \( \frac{\Delta}{\Delta_{\{\hat{N}\}}} \approx \frac{\Delta}{\Delta_{\{-\hat{N}\}}} \).

From the condition
\[
\frac{\delta S}{\delta x^i} = 0
\] (69)
it follows that
\[
\frac{\delta_{\{\hat{N}\}} S}{\delta x^i} \approx 0,
\] (70)
where, as demonstrated in Section 4, the sign \( \approx \) denotes however high accuracy for sufficiently high \( |\{\hat{N}\}| \).

It should be noted that the formulae (68), (70) for the \textit{measurable} variational derivative are easily derived from proof of the corresponding theorem.
in the continuous case (theorem 1 of paragraph 31 in [24]) with substitution of the sum for the integral and with replacement of all derivatives in the continuous consideration by their measurable analogs in the right-hand side of formulae (62) (63):

\[
\int_a^b \mapsto \sum_a^b; \quad \frac{dt}{N_t} \mapsto \frac{\Delta}{\Delta N_t}; \quad \frac{\partial}{\partial x^i} \mapsto \frac{\Delta}{\Delta N_{x^i}}; \ldots
\]  

(71)

Note that, as all the absolute values of the integers \(N, N_t, N_{x^i}, \ldots\) used in a measurable consideration are sufficiently high, it is obvious that a measurable analog of the "integration by parts" at a however high accuracy in this consideration is the case in the right side of formula (63) when using the substitution indicated in formula (71) in the process of proof. And in the final variant we have the formula (70).

5.2.2. In principle, consideration of the multidimensional case is no different from that of the unidimensional one. In this case in the continuous pattern formula for the variational derivative (67) is replaced by the formula ([24], paragraph 37):

\[
\frac{\delta I}{\delta f^i} = \frac{\partial L}{\partial f^i} - \sum_{\alpha=1}^n \frac{\partial}{\partial x^\alpha} (\frac{\partial L}{\partial f^i_{\alpha}}) , (1 \leq i \leq k),
\]  

(72)

where

\[
I[f] = \int_D L(x^\beta; f^i, f^i_{\alpha}) d^n x.
\]  

(73)

Here \(D\)-\(n\)-dimensional region, \(d^n x\)-\(n\)-dimensional form of the volume in \(\mathbb{R}^n\) and \(f^i_{\alpha}(x^\beta) = \frac{\partial}{\partial x^\alpha}(f^i(x^\beta)).\)

Accordingly, in the multidimensional measurable variant formula (68) is replaced by

\[
\frac{\delta_{\{N\}} I}{\delta f^i} = \frac{\Delta L}{\Delta N_{f^i}} - \sum_{\alpha=1}^n \frac{\Delta}{\Delta N_{x^\alpha}} (\frac{\Delta L}{\Delta N_{f^i_{\alpha}}}) , (1 \leq i \leq k),
\]  

(74)
where \( \{\hat{\mathbb{N}}\} \) — set of integer numbers sufficiently high in absolute value \( \{\hat{\mathbb{N}}\} \equiv (N, N_x, N_f, N_{\alpha}) \).

Then the condition \( \delta I/\delta f^i = 0 \) in the measurable format is equivalent to the condition

\[
\frac{\delta (\hat{\mathbb{N}})}{\delta f^i} \approx 0, \quad (1 \leq i \leq k).
\]

The functional \( I[f] \) (formula (73)) is associated with the sum

\[
I_{1/N_x}[f] = \sum_D L(x^\beta; f^i, f^i_{N_x})(\frac{\ell}{N_x})^n,
\]

where all the notation is taken from formula (73), with substitution of \( f^i_{N_x}(x^\beta) = \frac{\Delta f^i(x^\beta)}{\Delta N_x} \) for \( f^i_{x}(x^\beta) = \frac{\partial}{\partial x^\alpha}(f^i(x^\beta)) \) and of the measurable quantity \( (\ell/N_x)^n \) for the volume form \( d^n x \). (This means that in the standard expression for the form of the n-dimensional volume \( d^n x = dx^1 \wedge ... \wedge dx^n \) ([24], paragraph 37) the quantity \( dx^\alpha \) is replaced by the quantity \( \ell/N_x \) as at the end of Section 4 in formulae (46), (48), etc.).

Then in a multidimensional case at sufficiently high integer numbers \( |N_x| \) for formula (63) the following formula is its obvious analog:

\[
I[f] = \int_D L(x^\beta; f^i, f^i_{N_x}) d^n x \approx I_{1/N_x}[f] = \sum_D L(x^\beta; f^i, f^i_{N_x})(\frac{\ell}{N_x})^n.
\]

Similar to the unidimensional case, a measurable variant of the "integration by parts" is valid for the right-hand side of formula (77), with replacement of all derivatives by their measurable analogs as it has been performed in formula (74) compared with (72).

5.2.3. It is known that the gravitational action \( S_{EH} \) in a continuous consideration is of the form [20]–[22]

\[
S_{EH} = -\frac{1}{16\pi G} \int d^4 x \sqrt{|g|} \left( R + \Lambda \right) + S_M (g_{\mu\nu}, \text{ matt}) .
\]
Then, proceeding from the above-mentioned results, in a measurable variant it is associated with the quantity

\[ S_{EH}(N_{x}) = -\frac{1}{16\pi G} \sum \Delta(N_{x}) \Omega \sqrt{|g(N_{\Delta x})|} \left( R(x, N_{x}) + \Lambda(x, N_{x}) \right) + S_{M}(g_{\mu\nu}(x, N_{x}), \text{matt}) \] (79)

where the volume element in a measurable variant \( \Delta(N_{x}) \Omega \) is taken from formulae (44)–(46), the quantity \( g(N_{\Delta x}) \) – from the definition and formula (43) in Section 4; \( R(x, N_{x}) \) and \( \Lambda(x, N_{x}) \) are taken from formulae in Subsection 5.1. Finally, \( S_{M}(g_{\mu\nu}(x, N_{x}), \text{matt}) \) – term in the action corresponding to the measurable form of matter fields.

In canonical gravity the least action principle is associated with formulae [20]–[22]

\[
0 \approx \delta g S_{EH} = -\frac{1}{16\pi G} \delta g \int d^{4}x \sqrt{|g|} (g^{\mu\nu} R_{\mu\nu} + \Lambda) + \delta g S_{M} = \]

\[
= -\frac{1}{16\pi G} \int d^{4}x \left[ \left( \delta \sqrt{|g|} \right) \left( R + \Lambda \right) + \sqrt{|g|} \left( \delta g^{\mu\nu} \right) R_{\mu\nu} + \sqrt{|g|} \left( \delta R_{\mu\nu} \right) g^{\mu\nu} \right] + \delta g S_{M},(80)
\]

where \( \delta g \) \( \equiv [S(g + \delta g) - S(g)]_{\text{linear in } \delta g} \), and in the extremum of \( S \) we have that \( \delta g S = 0 \).

Due to the results from points 5.2.1. and 5.2.2. in a measurable consideration, formula (80) is associated with the expression

\[
0 \approx \delta_{g,N} S_{EH}(N_{x}) = -\frac{1}{16\pi G} \delta_{g,N} \sum \Delta(N_{x}) \Omega \sqrt{|g(N_{\Delta x})|} (g^{\mu\nu}(x, N_{x}) R_{\mu\nu}(x, N_{x}) + \Lambda(x, N_{x})) +
\]

\[
+ \delta_{g,N} S_{M} = \]

\[
= -\frac{1}{16\pi G} \sum \Delta(N_{x}) \Omega \left[ \left( \delta_{g,N} \sqrt{|g(N_{\Delta x})|} \right) (R(x, N_{x}) + \Lambda(x, N_{x})) + \right.
\]

\[
\left. + \sqrt{|g(N_{\Delta x})|} \left( \delta_{g,N} g^{\mu\nu}(x, N_{x}) \right) R_{\mu\nu}(x, N_{x}) + \right.
\]

\[
\left. + \sqrt{|g(N_{\Delta x})|} \left( \delta_{g,N} R_{\mu\nu}(x, N_{x}) g^{\mu\nu}(x, N_{x}) \right) \right] + \delta_{g,N} S_{M},(81)
\]

where, respectively, \( \delta_{g,N} \equiv [S(g + \frac{1}{N}\partial) - S(g)]_{\text{linear in } \frac{1}{N}\partial} \), the number \( N \) – integer, \( |N| \gg 1 \), and in the extremum of \( S \) we have that \( \delta_{g,N} S \approx 0 \) as in
formulae (64), (66). In this case, since the energies under consideration are low, according to formulae (58) and (59), $\delta_g N \Lambda(x, N_{x\chi}) \approx 0$ is satisfied.

The well-known formula for $\delta_g \sqrt{|g|}$ in the continuous pattern of a theory [20], [22]

$$\delta_g \sqrt{|g|} = -\frac{1}{2} \sqrt{|g|} g_{\mu \nu} \delta_g g^{\mu \nu}$$

(82)

in a measurable variant for integers sufficiently high in absolute value $N$ corresponds to the formula

$$\delta_{g,N} \sqrt{|g(N_{\Delta x\chi})|} \approx -\frac{1}{2} \sqrt{|g(N_{\Delta x\chi})|} g_{\mu \nu}(x, N_{x\chi}) \delta_{g,N} g^{\mu \nu}(x, N_{x\chi})$$

(83)

to no matter how high an accuracy. In [8] it has been shown that the strong principle of equivalence is valid for gravity in the measurable form. This can be demonstrated easily as within a random small neighborhood of any point there is even smaller neighborhood specified in terms of measurable quantities.

In the continuous pattern from this property it follows that in a sufficiently small neighborhood of the arbitrary point $x_0$ we have

$$g_{\mu \nu}(x_0) = \eta_{\mu \nu}, \quad \text{and} \quad \Gamma^\alpha_{\beta \gamma}(x_0) = 0.$$  

(84)

In the measurable form the relation of (84) is associated with the formula

$$g_{\mu \nu}(x_0, N_{(x_0)\chi}) = \eta_{\mu \nu}, \quad \text{and} \quad \Gamma^\alpha_{\beta \gamma}(x_0, N_{(x_0)\chi}) \approx 0.$$  

(85)

Initially, the first relation in formula (85) should have been written as $g_{\mu \nu}(x_0, N_{(x_0)\chi}) = \eta_{\mu \nu}(x_0, N_{(x_0)\chi})$, i.e. sets of the integers high in absolute value $N_{(x_0)\chi}$ and $N'_{(x_0)\chi}$ should be different. But we can demonstrate (see Section 3 in [8]) that, for sufficiently high absolute values of the integers from the sets $N_{(x_0)\chi}$, they may be considered as identical.

Without loss of generality, we can assume that the initial set of numbers $N_{(x_0)\chi}$ is exactly so.

In this way, using formula (85) and replacing the required formula in the continuous pattern by the corresponding analogs in the measurable form, we obtain the Least Action Principle for Gravity in a measurable variant.
Specifically, in the continuous pattern we have the equation (for example, formula (70) in \[22\])

\[
\int_{\mathcal{M}} d^4x \sqrt{|g|} g^{\mu\nu} \delta R_{\mu\nu} = \int_{\mathcal{M}} d^4x \sqrt{|g|} D_{\mu}\delta U^{\mu} = \oint_{\partial\mathcal{M}} d\Sigma^{\mu} \delta U^{\mu}, \quad (86)
\]

where \(\mathcal{M}\) is the space–time manifold under consideration and \(\partial\mathcal{M}\) is its boundary, \(d\Sigma^{\mu}\) is the four–vector normal to \(\partial\mathcal{M}\), whose modulus is the infinitesimal volume element of \(\partial\mathcal{M}\): \(d\Sigma^{\mu} = n_{\mu} \sqrt{|g^{(3)}|} d\xi\), where \(n_{\mu}\) is the normal vector to the boundary. And \(g^{(3)} = |\det g_{ij}|\) is the determinant of the induced three–dimensional metric, \(g_{ij}\), \(i = 1, 2, 3\), on the boundary \(\partial\mathcal{M}\) and \(\xi\) are the corresponding coordinates parametrizing the boundary, and \(\delta U^{\mu} = g^{\alpha\beta} \delta \Gamma_{\alpha\beta}^{\mu} - g^{\alpha\mu} \delta \Gamma_{\alpha\beta}^{\beta}\).

In a measurable consideration in formula (86) the substitution takes place

\[
\int_{\mathcal{M}} \mapsto \sum_{\mathcal{M}}, d^4x \mapsto \Delta_{(N_{x\chi})}\Omega, g^{\mu\nu} \mapsto g_{\mu\nu}(x, N_{x\chi}), g \mapsto g(N_{\Delta x\chi}),
\]

\[
\delta R_{\mu\nu} \mapsto \delta_{g, N} R_{\mu\nu}(x, N_{x\chi}), \delta U^{\mu} \mapsto \delta_{N} U^{\mu}(x, N_{x\chi}) = g^{\alpha\beta}(x, N_{x\chi})\delta_{g, N} \Gamma_{\alpha\beta}^{\mu}(x, N_{x\chi}) - g^{\alpha\mu}(x, N_{x\chi})\delta_{g, N} \Gamma_{\alpha\beta}^{\beta}(x, N_{x\chi}),
\]

\[
D_{\mu} \mapsto \tilde{D}_{\mu}, \oint_{\partial\mathcal{M}} \mapsto \sum_{\partial\mathcal{M}}, d\Sigma^{\mu} \mapsto \Delta\Sigma^{\mu} \simeq n_{\mu} \sqrt{|g^{(3)}(x, N_{x\chi})|} \prod_{i} N_{x_i}. \quad (87)
\]

In the last line of formula (87) \(\tilde{D}_{\mu}\) taken from (54), \(\sum_{\partial\mathcal{M}}\) is understood as a sum over the boundary \(\partial\mathcal{M}\), and, finally, \(\Delta\Sigma^{\mu}\) is derived from \(d\Sigma^{\mu}\) with the already known substitution (in this case for the three-dimensional case) \(g^{(3)} \mapsto g^{(3)}(x, N_{x\chi})\) and \(dx_i \mapsto \ell/N_{x_i}\).

In the continuous pattern, due to the Stokes formula,

\[
\oint_{\partial\mathcal{M}} d\Sigma^{\mu} \delta U^{\mu} = 0.
\]

But as \(\sum_{\partial\mathcal{M}} \Delta\Sigma^{\mu} \delta_{N} U^{\mu}(x, N_{x\chi}) \approx \sum_{\partial\mathcal{M}} d\Sigma^{\mu} \delta U^{\mu}\), in the measurable form we have \(\sum_{\partial\mathcal{M}} \Delta\Sigma^{\mu} \delta_{N} U^{\mu}(x, N_{x\chi}) \approx 0\).

From whence we obtain, in accordance with a continuous consideration, the formula for a measurable variant as follows:

\[
\delta_{g, N} S_{M} - \frac{1}{16 \pi G} \sum \Delta_{(N_{x\chi})}\Omega \sqrt{|g(N_{\Delta x\chi})|} [R_{\mu\nu}(x, N_{x\chi}) - \frac{1}{2} g_{\mu\nu}(x, N_{x\chi}) R(x, N_{x\chi}) - \frac{1}{2} g_{\mu\nu}(x, N_{x\chi}) \Lambda(x, N_{x\chi}) \delta_{g, N} g^{\mu\nu}(x, N_{x\chi})] \approx 0. \quad (88)
\]
In canonical gravity the variation \( \delta g S_M \) takes the form \[21\], \[22\]

\[
\delta g S_M = \delta \int d^4x \sqrt{|g|} \mathcal{L} = \int d^4x \left[ \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \delta g_{\mu\nu} \sqrt{|g|} + \mathcal{L} \delta \sqrt{|g|} \right] =
\]

\[
= \int d^4x \sqrt{|g|} \left[ \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \frac{1}{2} \mathcal{L} g_{\mu\nu} \right] \delta g_{\mu\nu} = \frac{1}{2} \int d^4x \sqrt{|g|} T_{\mu\nu} \delta g_{\mu\nu}, \quad (89)
\]

where \( \mathcal{L} \) – Lagrangian for matter fields and the corresponding energy-momentum tensor \( T_{\mu\nu} \) is defined as

\[
T_{\mu\nu} = 2 \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \mathcal{L} g_{\mu\nu}, \quad T_{\mu\nu} = T_{\nu\mu}. \quad (90)
\]

Using the foregoing results, it is easy to derive a measurable variant of (89), i.e. \( \delta g_{\mu\nu} S_M \), replacing all the required components in formula (89) by their measurable analogs from formula (87) in the assumption that \( \mathcal{L} = \mathcal{L}(g_{\mu\nu}(x, N_{x\chi}), \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \mapsto \frac{\Delta \mathcal{L}(g_{\mu\nu}(x, N_{x\chi}))}{\Delta N} \)

where

\[
\frac{\Delta \mathcal{L}(g_{\mu\nu}(x, N_{x\chi}))}{\Delta N} = \mathcal{L}(g_{\mu\nu}(x, N_{x\chi}) + \frac{1}{N} g_{\mu\nu}(x, N_{x\chi})) - \mathcal{L}(g_{\mu\nu}(x, N_{x\chi})), \quad (91)
\]

It is clear that in this case formula (90) is replaced by

\[
T_{\mu\nu, N} \approx 2 \frac{\Delta \mathcal{L}(g_{\mu\nu}(x, N_{x\chi}))}{\Delta N} - \mathcal{L} g_{\mu\nu}(x, N_{x\chi}), \quad T_{\mu\nu, N} \approx T_{\nu\mu, N}. \quad (92)
\]

In a continuous consideration we have the identity

\[
D^\mu T_{\mu\nu} = 0 \quad (93)
\]

that in a measurable variant takes the form

\[
\tilde{D}^\mu T_{\mu\nu, N} \approx 0, \quad (94)
\]

\( \tilde{D}^\mu \) – covariant derivative in a measurable form obtained from formula (54).
The identity (93) may be derived from the equation (for example, see formula (79) in [22])

\[ 0 \equiv \delta x S_M = \int_M d^4x \sqrt{|g|} T_{\mu\nu} \epsilon^\nu = \]

\[ = \int_M d^4x \sqrt{|g|} D^\mu (T_{\mu\nu} \epsilon^\nu) - \int_M d^4x \sqrt{|g|} \epsilon^\nu (D^\mu T_{\mu\nu}) = \]

\[ = \oint_{\partial M} d\Sigma T_{\mu\nu} \epsilon^\nu - \int_M d^4x \sqrt{|g|} \epsilon^\nu (D^\mu T_{\mu\nu}), \quad (95) \]

where \( T_{\mu\nu} = T_{\nu\mu} \), on going from the first to the second line, the integration by parts is performed, and then for the first term in the second line the Stokes theorem is used. It is assumed that \( \epsilon^\mu(x) \) is a small vector field such that \( \bar{x}^\mu = x^\mu + \epsilon^\mu(x) \), \( \epsilon^\mu(x) \big|_{\partial M} = 0 \) for all \( \mu \), and we have

\[ g^{\mu\nu}(\bar{x}) = g^{\mu\nu}(x + \epsilon) \approx g^{\mu\nu}(x) + \partial_\alpha g^{\mu\nu}(x) \epsilon^\alpha. \quad (96) \]

Then, in the assumption that \( \epsilon^\mu(x) \) is a small measurable vector field (i.e. a small vector field expressed in terms of measurable quantities), a measurable variant of formula (96) may be obtained as follows:

\[ g^{\mu\nu}(\bar{x}, N_{\bar{x}}) = g^{\mu\nu}(x + \epsilon, N_{(x+\epsilon)x}) \approx g^{\mu\nu}(x, N_{x}) + \Delta_\alpha g^{\mu\nu}(x, N_{x}) \epsilon^\alpha, \quad (97) \]

where \( \Delta_\alpha \) is taken from formula (21) in Section 3.

Next, by substitution of all the components in equation (95) according to formula (87) and with replacement of \( T_{\mu\nu} \mapsto T_{\mu\nu} \), of the sign = by \( \approx \) and assuming \( \epsilon^\alpha \) to be a small measurable vector field, we can obtain a measurable copy of the equation (95).

In this case the Stokes theorem and the ”integration by parts”, to a high accuracy, are fulfilled now not for the integrals but for the sums corresponding to these integrals and close in the values, as it has been already noted in points 5.2.1. and 5.2.2.
The "integration by parts" is of the form
\[ \sum_{\mathcal{M}} \Delta_{(N_\chi)} \Omega \sqrt{|g(N\Delta_\chi)|} T_{\mu\nu,N} \tilde{D}^\mu \epsilon^\nu \approx \]
\[ \approx \sum_{\mathcal{M}} \Delta_{(N_\chi)} \Omega \sqrt{|g(N\Delta_\chi)|} \tilde{D}^\mu (T_{\mu\nu,N} \epsilon^\nu) - \]
\[ - \sum_{\mathcal{M}} \Delta_{(N_\chi)} \Omega \sqrt{|g(N\Delta_\chi)|} \epsilon^\nu \left( \tilde{D}^\mu T_{\mu\nu,N} \right) \approx 0. \] (98)

Naturally assuming that at a small measurable vector field $\epsilon^\mu$ for all $\mu$ the condition $\epsilon^\mu(x)|_{\partial\mathcal{M}} = 0$ is fulfilled, we obtain the formula (94).

As $N$ satisfies the condition $|N| \gg 1$, from formula (94) it follows that $T_{\mu\nu,N}$ is practically independent of $N$, i.e. we have
\[ T_{\mu\nu,N} \approx T_{\mu\nu}, \quad |N| \gg 1, \] (99)

and we come to a measurable copy of the equation (56) given by formula (57).

So, the principal inference from this section is as follows:

at low energies $E \ll E_p$ Gravity (GR) in the continuous space-time may be redetermined in terms of the measurability notion using in GR, instead of the limiting quantities, the finite measurable quantities which are close to them and determined by the existing energies.

Let us call this theory redetermined in terms of measurable quantities the Measurable Gravity (or GRM). GRM is a discrete theory close to GR but not identical to it. Possible distinctions of GRM from GR are considered in the subsequent section.

However, in Section 7 it is shown that in the proposed approach the measurable form of Einstein Equations may be correctly defined at high energies $E \approx E_p$ as well. It should be noted that the indicated form completely satisfies the Correspondence Principle, i.e. in the limiting transition to low energies we involve $\mathcal{E}\mathcal{E}\mathcal{M}$ from formula (57).
6 Physical Meaning of Measurability and Some of Its Possible Inferences

6.1. The principal significance of introducing the **measurability** notion is in the fact that at low energies \( E \ll E_p \), instead of abstract, dependent on nothing small \( \delta x_{\mu} \) and infinitesimal \( dx_{\mu} \) space-time increments, we introduce the variations of \( \ell/N_{x_{\mu}}, |N_{x_{\mu}}| \gg 1 \), where \( N_{x_{\mu}} \) - integer.

These variations are determined by the **primarily measurable** momenta \( p_{N_{x_{\mu}}} \), i.e. by the particular energies. By virtue of their definition, the **primarily measurable** momenta \( p_{N_{x_{\mu}}} \), to a high accuracy, determine the whole totality of the momenta (and their variations) \( P_{LE} \) at low energies.

The "construction material" of a theory is changed considerably from the abstract \( \delta x_{\mu}, dx_{\mu}, dp_{x_{\mu}}, dE \) in a continuous consideration to \( \ell/N_{x_{\mu}}, p_{N_{x_{\mu}}} \) having specific values in the **measurable** pattern, as is directly indicated in formulae of Section 2 and in **Remark 1** at the very ending of this section.

The associated model is discrete but in this case, due to formulae (8), (9), it is very close to the initial continuous theory. Here the words "very close" do not mean that it is identical.

Let us return to formulae (57) and (60) of Section 5. As shown by these formulae, for high \( |N_{x_{\mu}}| \) **Einstein Equations Measurable** (EEM) and **Einstein Equations** (EE) are very close but in fact the solutions of these equations are differing.

Indeed, one metric \( g_{\mu\nu}(x) \) forming a solution of EE and given by formula (18) is associated with the whole class of metrics that is a layer over the "point" \( g_{\mu\nu}(x) \) in the bundle (22) and is given by formula (19). Then it is quite probable that some points based on the bundle (22) at the particular, quite natural conditions imposed on the numbers \( N_{x_{\mu}} \) have no prototype. Specifically, this primarily refers to the **Closed Time-like Curves** (CTC) \([25]–[28]\) from EE, which clearly are not the solutions having the physical meaning.

It should be recalled that the curve \( \gamma \) is referred to as the **time-like** curve if the norm of its tangent vector \( T^\mu = dx_\mu(t')/dt' \) is everywhere negative, i.e. \( g_{\alpha\beta} T^\alpha T^\beta < 0 \). [21]. Here \( t' \)–parameter along the curve.
It is obvious that in a measurable consideration we have the substitution
\[ g_{\alpha\beta}(x) \mapsto g_{\alpha\beta}(x, N_x) ; \]
\[ T^\mu = \frac{dx^\mu(t')}{dt'} \mapsto T^\mu_{N'} = \frac{\Delta x^\mu(t')}{\Delta N'} , \tag{100} \]
where \( N' \) — integer that is high in absolute value.

In this way the formulation of the CTC problem in the measurable case is varied and begins to be dependent on the parameters \( \{ N_x \} \) (and also on the additional small discrete parameter \( 1/N' \) along the curve \( \gamma \)).

Clearly, other quantities, e.g., lengths of the curves, in a measurable variant are varied too. In particular, for the space-like curve the transition from the quantity of length in a continuous variant \( L \) to that in a measurable variant \( L_{meas}(\{N_x\}, N') \) may be given by the formula
\[ L = \int [g_{\mu\nu}T^\mu T^\nu]^{1/2} dt \mapsto L_{meas}(N_x, N') = \]
\[ = \sum [g_{\mu\nu}(x, N_x)T^\mu_{N'_x'} T^\nu_{N'_x'}]^{1/2} \frac{1}{N'} . \tag{101} \]

The formula similar to (101) is written for the Time-like curves with changing in (101) of the sign before \( g_{\mu\nu} \) (and hence before \( g_{\mu\nu}(x, \{N_x\}) \) from ”+” to ”-” and with replacement of \( L \) by the intrinsic time \( T \) (and hence replacement of \( L_{meas} \) by \( T_{meas} \)). The absence of CTC in the measurable variant of gravity means that, under certain conditions for collection \( \{ N_x \} \) for metrics \( \tilde{g}_{\mu\nu}(x) \) in General Relativity generating the Closed Time-like Curves we have no prototype in the mapping (22).

So, the introduction of measurability allows for ”additional degrees of freedom ” in solution of the specific problems of General Relativity, CTC problem in particular.

At low energies \( E \ll E_p \) all measurable quantities are little different from the corresponding quantities in a continuous theory. Specifically, \( L \approx L_{meas} \).

Besides, it is obvious that all the above mentioned \( N_x \) should have the upper limit for their absolute values. As \( \ell \propto l_p \approx 10^{-33} cm \), all \( |N_x| ; i = 1, 2, 3 \) have the absolute upper limit of \( N \approx 10^{61} \):
The relation (102) is valid because all $|N_{x_i}|$ are determined by the real primarily measurable space quantities $N_{x_i}\ell$, but we understand that $|N_{x_i}|\ell$ never exceeds a radius of the visible part of the Universe $R_{Univ} \approx 10^{28}$ cm $\approx N\ell$. Similarly, the absolute upper limit $N_{t}$ may be introduced for $|N_{x_0}|$ as well.

In reality the upper limits for $|N_{x_i}|; i = 1, 2, 3$ and $|N_{x_0}|$ are considerably lower than $N$ and $N_{t}$, respectively, and are dependent on the problem at hand. For example, as the atomic radius has the characteristic scale $\approx 10^{-8}$ cm, we can demonstrate that in this case (102) is changed by (103):

$$1 \ll |N_{x_i}| \leq N_a \approx 10^{25}.$$ (103)

In the case of a nucleus, the radius of which is on the order of $\approx 10^{-12} \div \frac{10^{-12}}{cm}$ (103), is changed by (104)

$$1 \ll |N_{x_i}| \leq N_{nucl} \approx 10^{21}.$$ (104)

The formulae similar to (103) and (104) can be derived for $N_{x_0}$ too.

6.2. As in the well-known works by S.Hawking [40] – [42] all the results have been obtained within the scope of the semiclassical approximation, seeking for a solution of the above-mentioned problem is of primary importance. More precisely, we must find, how to describe thermodynamics and quantum mechanics using the “language” of the measurable variant of gravity and what is the difference (if any) from the continuous treatment in this case.

To have a deeper understanding of the problem, we should know about the transformations of the notion of quantum information for the measurable variant of gravity and quantum theory at low $E \ll E_P$ and at high $E \approx E_P$ energies. Possibly, a new approach to the solution of the Information Paradox Problem [40] will offer a better insight.

Because of this, it is very important to understand, which distinctive features has a semiclassical approximation in terms of measurable quantities. As the semiclassical approximation is consideration of the material quantum fields against the background of the classical space-time, it is important to study a dynamics of the system behavior in terms of measurability at low
energies $E \ll E_p$, i.e. for significantly different $N_{x_i}$. Specifically, of great importance is to get the answer to the following question: if we know system’s behavior for the energy $E \ll E_p$ that is associated with the numbers $N_{x_i}$, what is the behavior of the same system at higher energies $E' \ll E_p$ associated in the measurable form with the sets $N'_{x_i}$ so that we have

$$1 \ll |N'_{x_i}| \ll |N_{x_i}|.$$  \hspace{1cm} (105)

6.3. Finally, the proposed approach from the start is quantum in character due to the fact that the fundamental length $\ell$ is proportional to the Planck length $\ell \propto l_P$ and includes the whole three fundamental constants, the Planck constant $\hbar$ as well. Besides, it is naturally dependent on the energy scale: sets of the metrics $g_{\mu\nu}(x,\{N_{x_i}\})$ with the lowest value $|N_{x_i}|$ correspond to higher energies as they correspond to the momenta $\{p_{N_{x_i}}\}$ which are higher in absolute value. This is the case for all the energies $E$. However, minimal measurable increments for the energies $E \approx E_p$ are not of the form $\ell/N_{x_\mu}$ because the corresponding momenta $\{p_{N_{x_i}}\}$ are no longer primary measurable, as indicated by the results in Section 2.

So, in the proposed paradigm the problem of the ultraviolet generalization of the low-energy measurable gravity ($\mathcal{EM}[N_{x_i}]$ (formula (57))) is actually reduced to the problem: what becomes with the primary measurable momenta $\{p_{N_{x_i}}\}, |N_{x_i}| \gg 1$ at high Planck’s energies.

Just this problem is studied in the following section.

7 Transition to High Energies in the Framework of Measurable Gravity

As shown in this work, within the concept of measurability, at low energies $E \ll E_p$ all small space-time variations are determined by the primarily measurable momenta from formula (2). However, a simple example from Section 2 points to the fact that at high energies $E \approx E_p$ this is not the case. As we intend to construct a measurable variant of the theory at high energies $E \approx E_p$, in this case we should obtain the following:
A) some discrete model for the integers \( N_{x,\mu} \) so that \(|N_{x,\mu}| \approx 1\) (the condition "\(|N_{x,\mu}| \approx 1\)" should be included depending on the specific problem at hand);

B) the above-mentioned discrete model, according to the Principle of Correspondence, on going to low energies \( E \ll E_p \) should lead to a measurable variant of gravity from Sections 3-5 to a high accuracy.

Then it is natural to suppose that in such a discrete model all measurable analogs of the small space-time variations should be determined by some generalized measurable momenta \( p \equiv p(N_{x,\mu}); |N_{x,\mu}| \approx 1 \) so that on passage to low energies these momenta give primarily measurable momenta from formula (2). What do they look like these generalized measurable momenta \( p \equiv p(N_{x,\mu}); |N_{x,\mu}| \approx 1 \)?

In a relatively simple case of GUP in Section 2 we have the answer. And, using the fact that \((\mathcal{EEM})[N_{x,\chi}] \equiv (\mathcal{EEM})[p_{N_{x,\chi}}] \) (61), based on the results of Section 2, we can construct a correct high-energy passage to the Planck energies \( E \approx E_p \) [7].

\[(\mathcal{EEM})[p_{N_{x,\chi}}, |N_{x,\chi}| \gg 1] \mapsto (\mathcal{EEM})[p_{N_{x,\chi}}(GUP), |N_{x,\chi}| \approx 1], \quad (106)\]

where \( p_{N_{x,\chi}}(GUP) = \Delta p(\Delta x, GUP) \) according to formula (12) of Section 2. In this specific case, we can construct the natural ultraviolet generalization \((\mathcal{EEM})[p_{N_{x,\chi}}, |N_{x,\chi}| \gg 1] \mapsto (\mathcal{EEM})[p_{N_{x,\chi}}]\). The theoretical calculations \((\mathcal{EEM})[p_{N_{x,\chi}}(GUP), |N_{x,\chi}| \approx 1]\) derived at Planck’s energies are obviously discrete, measurable, and represent a high-energy deformation in the sense of the [23] measurable gravitational theory \((\mathcal{EEM})[p_{N_{x,\chi}}, |N_{x,\chi}| \gg 1]\).

GUP in Section 2 (formula (11)) is a partial case of a ”broader” GUP resultant in the minimal length \( \ell \) [17, 39]

\[\Delta x \Delta p \geq h(1 + \beta(\Delta p)^2 + \beta \langle p \rangle^2), \quad (107)\]

where \( \beta > 0 \).

Formula (107) gives rise to the absolutely smallest uncertainty in the positions

\[\Delta x_0 = 2h \sqrt{\beta} \doteq \ell. \quad (108)\]
To attain conformity of the formulae (11) and (107), in the right-hand side of (107) we use before $\hbar$ the factor 1, instead of $1/2$ in [17]. It is clear that (107) represents a more general form of GUP than (11) at least due to the fact that the minimal length $\ell$ involved with it not necessarily should be on the order of the Planck length $l_p$, i.e. proportional to it with the factor about 1. Nevertheless, we still assume in this work that the minimal length $\ell$ should be on the order of the Planck length $\ell \propto l_p$ because all the previous results point to the fact that for gravity the onset of a quantum regime begins at the Planck energies. [43]–[47]. But all the calculation are valid for the random minimal length $\ell$ too.

Assuming in (107) the equality $\langle p \rangle = 0$ and $\Delta x$ primarily measurable quantity, (i.e. $\Delta x = N \Delta x \ell$), we arrive at a formula for $\Delta p$ that is completely coincident with formula (12) for the partial case of GUP (11). In this way GUP, both in the partial and in the general cases, demonstrates that at high energies the primarily measurable momenta are inadequate for the construction of a theory and the generalized measurable momenta should be involved.

As noted in Section 2, the main target of the author is to construct a theory at all energy scales in terms of generalized measurable (or same measurable) quantities.

In this theory the values of the physical quantity $G$ may be represented by the numerical function $F$ in the following way:

$$G = F(N_i, N_t, \ell) = F(N_i, N_t, G, \hbar, c, \kappa),$$

(109)

where in the general case $N_i, N_t$–integer numbers from the formulae (2), (3) and $G, \hbar, c$ are the fundamental constants. The last equality in (109) is determined by the fact that $\ell = \kappa l_p$ and $l_p = \sqrt{G \hbar / c^3}$. As shown above, at low energies $E \ll E_p$ for the momenta we have $p_{N_{\mu}} = G$ because the latter are primarily measurable quantities, formula (109) is simplified and may be derived with the help of formula (6).

Let us assume that at high energies $E \approx E_p$ we have a certain formula (109) for the generalized measurable momenta $p_{N_{\mu}} = G, |N_{\mu}| \approx 1$. As demonstrated above, this is the case, for example, for GUP (11), (107). Then in a measurable variant of the theory, due to the Principle of Correspondence, these generalized measurable momenta at low energies
\( E \ll E_p \), to a high accuracy, should lead to the \textbf{primarily measurable} momenta

\[
p_{N_{x\chi}} ((|N_{x\chi}| \approx 1) \rightarrow |N_{x\chi}| \gg 1) \Rightarrow p_{N_{x\chi}} ((|N_{x\chi}| \gg 1), \tag{110}
\]

where momenta in the right-hand part of formula (110), i.e. \( p_{N_{x\chi}} ((|N_{x\chi}| \gg 1) \), are the \textbf{primarily measurable} momenta at low energies \( E \ll E_p \).

By formulae (12) and (14) of Section 2 it has been shown that (110) is valid for GUP (11)) and, proceeding from all the above, for a more general form of GUP (107).

Then, in accordance with formula (110), in the general case the transition from high \( E \approx E_p \) to low energies for a \textbf{measurable} variant of gravity is given as the low-energy deformation:

\[
(\mathcal{E}\mathcal{E}\mathcal{M})[p_{N_{x\chi}}, (|N_{x\chi}| \approx 1] \rightarrow (\mathcal{E}\mathcal{E}\mathcal{M})[p_{N_{x\chi}}, (|N_{x\chi}| \gg 1]. \tag{111}
\]

Next, in a quite natural way, we assume that in all the cases for a \textbf{measurable} variant of gravity the transition to the ultraviolet (i.e quantum) region may be realized by substitution of \( \frac{\ell^2}{\hbar} p_{N_{x\mu}}, (|N_{x\mu}| \approx 1 \) for the quantities \( \ell/N_{x\mu} = \frac{\ell^2}{\hbar} p_{N_{x\mu}}, (|N_{x\mu}| \gg 1 \); by the corresponding corrections of formulae (19)–(21) from Section 3, of all the components necessary for derivation of gravitational equations in a \textbf{measurable} variant \( \Gamma_{\mu\nu}(x, N_{x\chi}), R_{\mu\nu\alpha\beta}(x, N_{x\chi}), ... \)

Actually, this means that formula (19) for the \textit{canonically measurable prototype} of the infinitesimal space-time interval at low energies \( E \ll E_p \) is replaced by its quantum analog or the \textit{canonically measurable quantum prototype} for \( E \approx E_p \) taking the form

\[
\Delta s^2_{N_{x\chi}} (x, q) \equiv \frac{\ell^4}{\hbar^2} g_{\mu\nu}(x, N_{x\chi}, q)p_{N_{x\mu}}p_{N_{x\nu}}. \tag{112}
\]

Here there is no doubt that the numbers \( N_{x\mu}, N_{x\nu} \) belong to the set \( \{N_{x\chi}\} \), all the components of this set are integers with small absolute values, and \( p_{N_{x\chi}} \) are the \textbf{generalized measurable} momenta at high energies corresponding to formulae (110), (111).

As we have assumed that the values of \( p_{N_{x\chi}} \) are known, they are \textbf{measurable} analogs (or \textbf{measurable} variants) of \textit{small space-time variations} at
According to formula (110), we can easily obtain its variant in the space-time consideration as follows:

\[ l_H(p_{N_{x\chi}}) \approx \frac{\ell^2}{\hbar p_{N_{x\chi}}}; |N_{x\chi}| \approx 1. \]  

(113)

By the substitution \( \ell/N_{x\chi} \rightarrow l_H(p_{N_{x\chi}}) \) in formulae (20), (21) we can have quantum analogs of minimal measurable variations of the metric and of the partial derivative

\[ \Delta_q g_{\mu\nu}(x, N_{x\chi}, q) \approx g_{\mu\nu}(x + l_H(p_{N_{x\chi}}), N_{x\chi}, q) - g_{\mu\nu}(x, N_{x\chi}, q), \]

\[ \Delta_{x,q} g_{\mu\nu}(x, N_{x\chi}, q) \approx \frac{\Delta_q g_{\mu\nu}(x, N_{x\chi}, q)}{l_H(p_{N_{x\chi}})}. \]  

(115)

Then, using the substitution in formula (9)

\[ \frac{\ell}{N_{x\mu}} \rightarrow l_H(p_{N_{x\mu}}); \frac{\Delta}{\Delta_{N_{x\mu}}} \rightarrow \Delta_q, \]

\[ \frac{\Delta_q F(x_{\mu})}{\Delta_{N_{x\mu}}} = \frac{F(x_{\mu} + l_H(p_{N_{x\mu}})) - F(x_{\mu})}{l_H(p_{N_{x\mu}})} \]  

(116)

and applying this substitution to all the formulae in the measurable format of Subsection 5.1, we can derive at high energies \( E \approx E_p \) all the principal components of Einstein Equations in the measurable form \( \mathcal{EEM} \) (formula (57)). In what follows, we use for them \( q \) in parenthesis, as in formula (112). Substituting these components into Einstein Equations, we obtain a measurable variant of Einstein Equations at high energies. For short, we denote it as \( \mathcal{EEM}[q] \):

\[ \mathcal{EEM}[q] \doteq R_{\mu\nu}(x, N_{x\chi}, q) - \frac{1}{2} R(x, N_{x\chi}, q) g_{\mu\nu}(x, N_{x\chi}, q) - \frac{1}{2} A(x, N_{x\chi}, q) g_{\mu\nu}(x, N_{x\chi}, q) = 8 \pi G T_{\mu\nu}(x, N_{x\chi}, q). \]  

(117)
As a result, we have

$$\lim_{E \ll E_p} \mathcal{E}\mathcal{E}\mathcal{M}[q] = \mathcal{E}\mathcal{E}\mathcal{M} \quad \text{or} \quad \lim_{|N_{x\chi}| \gg 1} \mathcal{E}\mathcal{E}\mathcal{M}[q] = \mathcal{E}\mathcal{E}\mathcal{M}. \quad (118)$$

For $\mathcal{E}\mathcal{E}\mathcal{M}[q]$ the metrics $g^{\mu\nu}(x, N_{x\chi}, q)$ (formula (112)) represent the solutions.

**Comment 7.1.**

Thus, at high energies in a **measurable** variant we can have an analog of Einstein Equations (116) and of the metric (112). In the low-energy case $E \ll E_p$ we know the space-time manifold $\mathcal{M} \subseteq \mathcal{R}^4$ as well as its minimal variations $l/N_{x\mu}$, whereas at high energies $E \approx E_p$ in this consideration we, with certainty, present only "small" variations of the space-time positions $l_H(p_{N_{x\chi}})$ (formula (113)).

**Comment 7.2.**

When the metric $g_{\mu\nu}(x, N_{x\chi}); |N_{x\chi}| \gg 1$ in a **measurable** variant at low energies is varying practically continuously, the metric $g_{\mu\nu}(x, N_{x\chi}, q); |N_{x\chi}| \approx 1$ varies discretely and it should experience high fluctuations due to great fluctuations of the **generalized measurable** momenta $p_{N_{x\chi}}$, by virtue of $|N_{x\chi}| \approx 1$, in a good agreement with the results of J. A. Wheeler for the space-time foam at the Planck scale [29], [30], [31], [32]–[35].

**Comment 7.3.**

It is obvious that in the **measurable** pattern at high energies there is no direct analogy with the mathematical apparatus at low energies (second part of Section 3 and Sections 4,5). Clearly, the adequate mathematical apparatus in this case should meet the two following requirements:

a) it should be based on the discretely varying quantities expressed at high energies $E \approx E_p$ in terms of the **generalized measurable** momenta $p_{N_{x\chi}}; |N_{x\chi}| \approx 1$;

b) on going to low energies $E \ll E_p$, i.e. to the region $|N_{x\chi}| \gg 1$, in accordance with the **Principle of Correspondence**, the above-mentioned
mathematical apparatus should present the results given in Sections 3-5 to a high accuracy.

Comment 7.4.
It is important that Einstein equations for the spherically-symmetric horizon spaces [36], derived in the measurable form in [6],[5] and written at low energies in terms of the parameter $\alpha_a(HUP) = 1/N_a^2$ or at high energies in terms of the parameter $\alpha_a(GUP) = 1/[1/4(N_a + \sqrt{N_a^2 - 1})^2]$, where $a = N_a\ell$ is a primarily measurable quantity of the space radius, completely comply with their general form from Sections 5, 7.

Besides, in accordance with Remark 2.1, the condition $N_a \geq 2$ should be fulfilled as noted in Remark 3.4 of [5]. This fact was also noted in [37],[38], however, on the basis of another approach.

8 Final Comments and Further Prospects

In conclusion the author comments on the further course of his studies an the problems involved.

8.1. As at low energies $E \ll E_p$ all the spatial variables $\{x_i\}, i = 1,2,3$ are equitable, all the numbers $N_{x_i} \equiv N_{\Delta x_i}, |N_{x_i}| \gg 1$ should be sufficiently close.
At high energies $E \ll E_p$ the numbers $N_{x_i}$ should meet the condition $|N_{x_i}| \approx 1$ and it seems that they are close from the start. But in this case there are two objections:
a) at high energies the essence of the notions "close" and "far from" may be different;
b) in some currently used models it is assumed that at high energies the space becomes noncommutative (for example, [17],[18],[39])) and in this case the spatial positions could cease to be equitable.

8.2. According to Remark 1.d), $\ell$ – minimal primarily measurable length. At the beginning of this paper, we have assumed that it is close to $l_p$, i.e. $\ell \propto l_p$ or same $\ell = \kappa l_p, \kappa \approx 1$. But, actually, all the results of
In this paper, except of those associated with the simplest form of GUP \[^{(11)}\], are independent of the magnitude of the \textbf{primarily measurable} length $\ell$.

When $\ell \propto l_p \approx 10^{-33}\text{cm}$, it is clear that for $|N| \gg 1$ lengths of the form $\ell/N$ are very small.

Of course, for experimental physics at high energies far from the Planck energies, e.g. for those in LHC, $\ell$ may exceed the Planck length $\ell \gg l_p$ considerably and in this case all the fundamental energies available in LHC satisfy the condition $E \ll E_\ell \ll E_p$. Then $\ell$ (and hence $E_\ell$) determines the natural ultraviolet-cutoff bound in the corresponding quantum theory. Because of this, the following problem arises.

\textbf{8.3.} A correct (without ultraviolet and infra-red divergences) quantum theory with the parameter $\ell$ (same $E_\ell$) should be resolved in conformity to all experimental data of LHC. The behavior of such a theory in the high-energy (ultraviolet) region should be associated with the quantity $\ell$. The absence of infra-red divergences should be given by the natural upper bound for $N_x$ and determined from formulae \[^{(102)-(105)}\] in Section 6 that, due to the condition $\ell \gg l_p$, would be significantly lower and hence more realistic.

Moreover, for $\ell \approx l_p$, due to infinitesimal values of $\ell/N; |N| \gg 1$, the corresponding "\textbf{measurable}" theory, still being discrete, would be very close to (practically indistinguishable from) the initial continuous theory enabling us to solve the problems mentioned in points 6.1. and 6.2. But for $\ell \gg l_p$, considering formulae \[^{(102)-(105)}\], the situation may be different.

Indeed, as points 6.1. and 6.2. conform to the semiclassical approximation, i.e. to consideration of the quantized matter fields against the classical space-time pattern, the \textbf{primarily measurable} minimal length $\ell \gg l_p$ is more adequate to study these problems.

It seems that the absence of infra-red divergences directly follows from the lower limit for the momenta in a quantum theory with different generalizations of the Uncertainty Principle (for example, \[^{[39]}\]).

\textbf{8.4.} So, in this paper it is proposed in a quantum theory and in gravity to use, instead of abstract small and infinitesimal quantities $\delta x_\mu, dx_\mu, ...$, the small quantities derived from the \textbf{primarily measurable} minimal length $\ell$ with the help of the corresponding formulae in Sections 2-6 of the paper.
Note that $\ell$ may be both close to the Planck length $\ell \propto l_p$ and considerably higher than this length $\ell \gg l_p$.

In the suggested paradigm all variations of a physical system which may be regarded small, as distinct from the continuous space-time consideration, have particular values determined at low energies $E \ll E_\ell$ by the **primarily measurable** momenta and at high energies $E \approx E_\ell$ by the **generalized measurable** momenta from Section 2. In this way all small variations are dependent on the existing energies.

Since at the present time no direct or indirect experiments at the scales on the order of Planck’s scales (i.e. at the energies associated with the quantum gravity scales) are known, all theoretical studies in this field are to some or other extent speculative. Nevertheless, considering that gravity should be formulated with the use of the same terms at all the energy scales, it must be governed by the particular unified principles the formulation of which varies depending on the “available” energies. Because of this, the results from Section 7 seem to be important. Of course, these results are tentative and may be corrected during further studies of gravity in terms of the **measurability** notion. But they give the main idea and define the trend towards the derivation of a **measurable** variant of gravity: framing of a correct gravitational theory at all the energy scales, with the use of a set of discrete parameters $p(N_{\Delta x_\mu})$ for all nonzero integer values of $N_{\Delta x_\mu}$, that is close to the General Relativity at low energies $E \ll E_p$ and is a new (discrete) theory at high energies $E \approx E_p$.

**8.5.** In conclusion it may be stated that the principal result of this work is as follows.

**8.5.1.** At low energies far from the Planck energies $E \ll E_p$ we replace the space-time manifold $\mathcal{M} \subseteq \mathbb{R}^4$ by the lattice model (denoted by $Latt^{LE}_{\{N_{x_\mu}\}}\mathcal{M}$, where the upper index $^{LE}$ is the abbreviation for ”Low Energies”), with the nodes taken at the points $\{x_\mu\} \in \mathcal{M}$ so that all the edges belonging to $\{x_\mu\}$ have the size $\ell/N_{x_\mu}$, where $N_{x_\mu}$ - integers having the property $|N_{x_\mu}| \gg 1$.

As the edge lengths $\ell/N_{x_\mu}$, within a constant factor, are coincident with the **primarily measurable** momenta (formula (4)), the model $Latt^{LE}_{\{N_{x_\mu}\}}\mathcal{M}$ is dynamic and dependent on the existing energies. In this case all the main attributes of a gravitational theory in the manifold $\mathcal{M}$, including Einstein Equations, have their adequate analogs on the above-mentioned lat-
tice $Latt^{LE}_{\{N_{x\mu}\}}\mathcal{M}$, giving the low-energy deformation of General Relativity in terms of paper [23] (Section 4).

8.5.2. At high Planck’s energies $E \propto E_p$, the lattice model $Latt^{LE}_{\{N_{x\mu}\}}\mathcal{M}$ is replaced by $Latt^{HE}_{\{N_{x\mu}\}}\mathcal{M}$ (the upper index $HE$ is the abbreviation for ”High Energies”), the edges with the lengths $\ell/N_{x\mu}$ are replaced by those with the lengths $l_H(p_{N_{x\mu}})$ which, within a constant factor, are coincident with the generalized measurable momenta $p_{N_{x\mu}}$, where $N_{x\mu}$-integer number having the property $|N_{x\mu}| \approx 1$ (formula (113)). In this way $Latt^{HE}_{\{N_{x\mu}\}}\mathcal{M}$ also represents a dynamic model that is dependent on the existing energies and may be the basis for the construction of a correct variant of the high-energy deformation in General Relativity (Section 5).

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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