The Binary Invariant Differential Operators on Weighted Densities on the superspace $\mathbb{R}^{1|n}$ and Cohomology

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Abstract

Over the $(1,n)$-dimensional real superspace, $n > 1$, we classify $\mathcal{K}(n)$-invariant binary differential operators acting on the superspaces of weighted densities, where $\mathcal{K}(n)$ is the Lie superalgebra of contact vector fields. This result allows us to compute the first differential cohomology of $\mathcal{K}(n)$ with coefficients in the superspace of linear differential operators acting on the superspaces of weighted densities—a superisation of a result by Feigin and Fuchs. We explicitly give 1-cocycles spanning these cohomology spaces.

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1 Introduction

This work is a direct continuation of [9, 10] and [2, 3] listed among other things, binary differential operators invariant with respect to a supergroup of diffeomorphisms and computed cohomology of polynomial versions of various infinite dimensional Lie superalgebras.

Let $\text{vect}(1)$ be the Lie algebra of polynomial vector fields on $\mathbb{R}$. Consider the 1-parameter deformation of the $\text{vect}(1)$-action on $\mathbb{R}[x]$: $L^\lambda_{\frac{d}{dx}}(f) = Xf' + \lambda Xf,$

where $X, f \in \mathbb{R}[x]$ and $X' := \frac{dX}{dx}$. Denote by $\mathcal{F}_\lambda$ the $\text{vect}(1)$-module structure on $\mathbb{R}[x]$ defined by $L^\lambda$ for a fixed $\lambda$. Geometrically, $\mathcal{F}_\lambda = \{fdx^\lambda \mid f \in \mathbb{R}[x]\}$ is the space of polynomial weighted densities of weight $\lambda \in \mathbb{R}$. The space $\mathcal{F}_\lambda$ coincides with the space of vector fields, functions and differential 1-forms for $\lambda = -1, 0$ and 1, respectively.

Denote by $D_{\lambda,\mu} := \text{Hom}_{\text{diff}}(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ the $\text{vect}(1)$-module of linear differential operators with the natural $\text{vect}(1)$-action. Feigin and Fuchs [5] computed $H^1_{\text{diff}}(\text{vect}(1); D_{\lambda,\mu})$, where $H^1_{\text{diff}}$ denotes the differential cohomology; that is, only cochains given by differential operators are considered. They showed that non-zero cohomology $H^1_{\text{diff}}(\text{vect}(1); D_{\lambda,\mu})$ only appear for particular values of weights that we call resonant which satisfy $\mu - \lambda \in \mathbb{N}$. These spaces arise

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in the classification of infinitesimal deformations of the \( \text{vect}(1) \)-module \( S_{\mu-\lambda} = \bigoplus_{k=0}^{\infty} \mathcal{F}_{\mu-\lambda-k} \), the space of symbols of \( D_{\lambda,\mu} \).

On the other hand, Grozman \[10\] classified all \( \text{vect}(1) \)-invariant binary differential operators on \( \mathbb{R} \) acting in the spaces \( \mathcal{F}_\lambda \). He showed that all invariant operators are of order \( \leq 3 \) and can be expressed as a composition of the Rham differential and the Poisson bracket, except for one called Grozman operator.

It is natural to study the simplest super analog of the problems solved respectively in \([5]\) and \([10]\), namely, we consider the superspace \( \mathbb{R}^{1|n} \) endowed with its standard contact structure defined by the 1-form \( \alpha_n \), and the Lie superalgebra \( \mathcal{K}(n) \) of contact polynomial vector fields on \( \mathbb{R}^{1|n} \). We introduce the \( \mathcal{K}(n) \)-module \( \mathbb{F}_\lambda^n \) of \( \lambda \)-densities on \( \mathbb{R}^{1|n} \) and the \( \mathcal{K}(n) \)-module of linear differential operators, \( \mathcal{D}^{n}_{\lambda,\mu} := \text{Hom}_{\text{diff}}(\mathbb{F}_\lambda^n, \mathbb{F}_\mu^n) \), which are super analogues of the spaces \( \mathcal{F}_\lambda \) and \( D_{\lambda,\mu} \), respectively. The classification of the \( \mathcal{K}(1) \)-invariant binary differential operators on \( \mathbb{R}^{1|1} \) acting in the spaces \( \mathbb{F}_1^n \) is due to Leites et al. \[9\], while the space \( H^1_{\text{diff}}(\mathcal{K}(1); \mathcal{D}^1_{\lambda,\mu}) \) has been computed by Basdouri et al. \[11\] (see also \[3\]) and the space \( H^1_{\text{diff}}(\mathcal{K}(2); \mathcal{D}^2_{\lambda,\mu}) \) has been computed by the second author \[2\]. We also mention that Duval and Michel studied a similar problem for the group of contactomorphisms of the supercircle \( S^{1|n} \) instead of \( \mathcal{K}(n) \) related to the link between discrete projective invariants of the supercircle, and the cohomology of the group of its contactomorphisms \[4\].

In this paper we classify all \( \mathcal{K}(n) \)-invariant binary differential operators on \( \mathbb{R}^{1|n} \) acting in the spaces \( \mathbb{F}_\lambda^n \) for \( n > 1 \). We use the result to compute \( H^1_{\text{diff}}(\mathcal{K}(n); \mathcal{D}^n_{\lambda,\mu}) \) for \( n > 2 \). We show that, as in the classical setting, non-zero cohomology \( H^1_{\text{diff}}(\mathcal{K}(n); \mathcal{D}^n_{\lambda,\mu}) \) only appear for resonant values of weights which satisfy \( \mu - \lambda \in \frac{1}{2} \mathbb{N} \). Moreover, we give explicit basis of these cohomology spaces. These spaces arise in the classification of infinitesimal deformations of the \( \mathcal{K}(n) \)-module \( S^n_{\mu-\lambda} = \bigoplus_{k=0}^{\infty} \mathbb{F}^n_{\mu-\lambda-k} \), a super analogue of \( S_{\mu-\lambda} \), see \[11\].

2 Definitions and Notation

2.1 The Lie superalgebra of contact vector fields on \( \mathbb{R}^{1|n} \)

Let \( \mathbb{R}^{1|n} \) be the superspace with coordinates \( (x, \theta_1, \ldots, \theta_n) \), where \( \theta_1, \ldots, \theta_n \) are the odd variables, equipped with the standard contact structure given by the following 1-form:

\[
\alpha_n = dx + \sum_{i=1}^{n} \theta_i d\theta_i. \tag{2.1}
\]

On the space \( \mathbb{R}[x, \theta] := \mathbb{R}[x, \theta_1, \ldots, \theta_n] \), we consider the contact bracket

\[
\{F, G\} = FG' - F'G - \frac{1}{2} (-1)^{|F|} \sum_{i=1}^{n} \eta_i(F) \cdot \eta_i(G), \tag{2.2}
\]

where \( \eta_i = \frac{\partial}{\partial \theta_i}, - \theta_i \frac{\partial}{\partial x} \) and \( |F| \) is the parity of \( F \). Note that the derivations \( \eta_i \) are the generators of \( n \)-extended supersymmetry and generate the kernel of the form \( (2.1) \) as a module over the ring of polynomial functions. Let \( \text{Vect}_{\text{pol}}(\mathbb{R}^{1|n}) \) be the superspace of polynomial vector fields on \( \mathbb{R}^{1|n} \):

\[
\text{Vect}_{\text{pol}}(\mathbb{R}^{1|n}) = \left\{ F_0 \partial_x + \sum_{i=1}^{n} F_i \partial_{\theta_i} \mid F_i \in \mathbb{R}[x, \theta] \text{ for all } i \right\}.
\]
where $\partial_i = \frac{\partial}{\partial \theta_i}$ and $\partial_x = \frac{\partial}{\partial x}$, and consider the superspace $\mathcal{K}(n)$ of contact polynomial vector fields on $\mathbb{R}^{1|n}$. That is, $\mathcal{K}(n)$ is the superspace of vector fields on $\mathbb{R}^{1|n}$ preserving the distribution singled out by the 1-form $\alpha_n$:

$$\mathcal{K}(n) = \{ X \in \text{Vect}_{\text{Pol}}(\mathbb{R}^{1|n}) \mid \text{there exists } F \in \mathbb{R}[x, \theta] \text{ such that } L_X(\alpha_n) = F \alpha_n \}.$$ 

The Lie superalgebra $\mathcal{K}(n)$ is spanned by the fields of the form:

$$X_F = F \partial_x - \frac{1}{2} \sum_{i=1}^{n} (-1)^{|F|} \eta_i(F) \partial_i, \quad \text{where } F \in \mathbb{R}[x, \theta].$$

In particular, we have $\mathcal{K}(0) = \text{vect}(1)$. Observe that $L_{X_F}(\alpha_n) = X_1(F)\alpha_n$. The bracket in $\mathcal{K}(n)$ can be written as:

$$[X_F, X_G] = X_{\{F, G\}}.$$  

### 2.2 Modules of weighted densities

We introduce a one-parameter family of modules over the Lie superalgebra $\mathcal{K}(n)$. As vector spaces all these modules are isomorphic to $\mathbb{R}[x, \theta]$, but not as $\mathcal{K}(n)$-modules.

For every contact polynomial vector field $X_F$, define a one-parameter family of first-order differential operators on $\mathbb{R}[x, \theta]$:

$$L^\lambda_{X_F} = X_F + \lambda F', \quad \lambda \in \mathbb{R}. \quad (2.3)$$

We easily check that

$$[L^\lambda_{X_F}, L^\mu_{X_G}] = L^\lambda_{\{X_F, X_G\}}. \quad (2.4)$$

We thus obtain a one-parameter family of $\mathcal{K}(n)$-modules on $\mathbb{R}[x, \theta]$ that we denote $\mathbb{F}^n_\lambda$, the space of all polynomial weighted densities on $\mathbb{R}^{1|n}$ of weight $\lambda$ with respect to $\alpha_n$:

$$\mathbb{F}^n_\lambda = \left\{ F \alpha_\lambda^n \mid F \in \mathbb{R}[x, \theta] \right\}. \quad (2.5)$$

In particular, we have $\mathbb{F}^0_\lambda = \mathcal{F}_\lambda$. Obviously the adjoint $\mathcal{K}(n)$-module is isomorphic to the space of weighted densities on $\mathbb{R}^{1|n}$ of weight $-1$.

### 2.3 Differential operators on weighted densities

A differential operator on $\mathbb{R}^{1|n}$ is an operator on $\mathbb{R}[x, \theta]$ of the form:

$$A = \sum_{k=0}^{M} \sum_{\varepsilon=(\varepsilon_1, \ldots, \varepsilon_n)} a_{k,\varepsilon}(x, \theta) \partial_x^{\varepsilon_1} \partial_1^{\varepsilon_1} \cdots \partial_n^{\varepsilon_n}; \quad \varepsilon_i = 0, 1; \quad M \in \mathbb{N}. \quad (2.6)$$

Of course any differential operator defines a linear mapping $F \alpha^n_\lambda \mapsto (AF)\alpha^n_\mu$ from $\mathbb{F}^n_\lambda$ to $\mathbb{F}^n_\mu$ for any $\lambda, \mu \in \mathbb{R}$, thus the space of differential operators becomes a family of $\mathcal{K}(n)$-modules $\mathbb{D}^n_{\lambda, \mu}$ for the natural action:

$$X_F \cdot A = L^\mu_{X_F} \circ A - (-1)^{|A||F|} A \circ L^\lambda_{X_F}. \quad (2.7)$$
Similarly, we consider a multi-parameter family of $\mathcal{K}(n)$-modules on the space $\mathbb{D}^n_{\lambda_1,\ldots,\lambda_m;\mu}$ of multi-linear differential operators: $A : \mathbb{F}_\lambda^\mu \otimes \cdots \otimes \mathbb{F}_\lambda^\mu \to \mathbb{F}_\mu^\mu$ with the natural $\mathcal{K}(n)$-action:

$$X_F \cdot A = \mathbb{L}_{X_F}^\mu \circ A - (-1)^{|A||F|} A \circ \mathbb{L}_{X_F}^\lambda \cdots \lambda_m,$$

where $\mathbb{L}_{X_F}^{\lambda_1,\ldots,\lambda_m}$ is defined by the Leibnitz rule. We also consider the $\mathcal{K}(n)$-module $\Pi \left( \mathbb{D}^n_{\lambda_1,\ldots,\lambda_m;\mu} \right)$ with the $\mathcal{K}(n)$-action ($\Pi$ is the change of parity operator):

$$X_F \cdot \Pi(A) = \Pi \left( \mathbb{L}_{X_F}^\mu \circ A - (-1)^{|(1|+1)|F|} A \circ \mathbb{L}_{X_F}^\lambda \cdots \lambda_m \right).$$

Since $-\eta_i^2 = \partial_x$, and $\partial_i = \eta_i - \theta_i \eta_i^2$, every differential operator $A \in \mathbb{D}^n_{\lambda,\mu}$ can be expressed in the form

$$A(F)^{\lambda_1}_{\alpha_n} = \sum_{\ell=(\ell_1,\ldots,\ell_n)} a_{\ell}(x,\theta) \eta_1^{\ell_1} \cdots \eta_n^{\ell_n}(F)^{\mu_1}_{\alpha_n}, \quad (2.8)$$

where the coefficients $a_{\ell}(x,\theta)$ are arbitrary polynomial functions.

The Lie superalgebra $\mathcal{K}(n-1)$ can be realized as a subalgebra of $\mathcal{K}(n)$:

$$\mathcal{K}(n-1) = \left\{ X_F \in \mathcal{K}(n) \mid \partial_n F = 0 \right\}.$$ 

Therefore, $\mathbb{D}^n_{\lambda_1,\ldots,\lambda_m;\mu}$ and $\mathbb{F}_\lambda^\mu$ are $\mathcal{K}(n-1)$-modules. Note also that, for any $i \in \{1, 2, \ldots, n-1\}$, $\mathcal{K}(n-1)$ is isomorphic to

$$\mathcal{K}(n-1)^i = \left\{ X_F \in \mathcal{K}(n) \mid \partial_i F = 0 \right\}.$$ 

**Proposition 2.1.** As a $\mathcal{K}(n-1)$-module, we have

$$\mathbb{D}^n_{\lambda,\mu;\nu} \simeq \mathbb{D}^n_{\lambda,\mu;\nu} := \mathbb{D}^{n-1}_{\lambda,\mu;\nu} \oplus \mathbb{D}^{n-1}_{\lambda,\mu+\frac{1}{2};\mu+\frac{1}{2}+\frac{1}{2}} \oplus \mathbb{D}^{n-1}_{\lambda,\mu+\frac{1}{2};\mu+\frac{1}{2}+\frac{1}{2}} \oplus \mathbb{D}^{\mu}_{\lambda,\mu;\nu+\frac{1}{2}} \oplus \mathbb{D}^{\mu}_{\lambda,\mu+\frac{1}{2};\mu+\frac{1}{2}+\frac{1}{2}} \oplus \Pi \left( \mathbb{D}^{n-1}_{\lambda,\mu;\nu+\frac{1}{2}} \oplus \mathbb{D}^{n-1}_{\lambda,\mu+\frac{1}{2};\mu+\frac{1}{2}+\frac{1}{2}} \oplus \mathbb{D}^{n-1}_{\lambda,\mu+\frac{1}{2};\mu+\frac{1}{2}+\frac{1}{2}} \oplus \mathbb{D}^{n-1}_{\lambda,\mu;\nu+\frac{1}{2}} \oplus \mathbb{D}^{n-1}_{\lambda,\mu+\frac{1}{2};\mu+\frac{1}{2}+\frac{1}{2}} \right). \quad (2.9)$$

**Proof.** For any $F \in \mathbb{R}[x,\theta]$, we write

$$F = F_1 + F_2 \theta_n \quad \text{where} \quad \partial_n F_1 = \partial_n F_2 = 0$$

and we prove that

$$\mathbb{L}_{X_F}^\lambda F = \mathbb{L}_{X_F}^\lambda F + (\mathbb{L}_{X_F}^{\lambda+\frac{1}{2}} F_2) \theta_n.$$ 

Thus, it is clear that the map

$$\varphi_{\lambda} : \mathbb{F}_\lambda^\mu \to \mathbb{F}_\lambda^\mu \oplus \Pi(\mathbb{F}_\mu^{\lambda+\frac{1}{2}})$$

$$F^\lambda_{\alpha_n} \mapsto (F_1^{\lambda}_{\alpha_n-1}, \Pi(F_2^{\lambda}_{\alpha_n-1})) \quad (2.10)$$

is $\mathcal{K}(n-1)$-isomorphism. So, we get the natural $\mathcal{K}(n-1)$-isomorphism from $\mathbb{F}_\lambda^\mu \otimes \mathbb{F}_\mu^\mu$ to

$$\mathbb{F}_\lambda^\mu \otimes \mathbb{F}_\mu^\mu \oplus \Pi(\mathbb{F}_\mu^{\lambda+\frac{1}{2}}) \oplus \Pi(\mathbb{F}_\mu^{\lambda+\frac{1}{2}}) \oplus \mathbb{F}_\mu^\mu \oplus \Pi(\mathbb{F}_\mu^{\lambda+\frac{1}{2}}) \oplus \Pi(\mathbb{F}_\mu^{\lambda+\frac{1}{2}}) \oplus \Pi(\mathbb{F}_\mu^{\lambda+\frac{1}{2}}) \oplus \Pi(\mathbb{F}_\mu^{\lambda+\frac{1}{2}}).$$
denoted \( \psi_{\lambda,\mu} \). Therefore, we deduce a \( \mathcal{K}(n-1) \)-isomorphism:

\[
\Psi_{\lambda,\mu,\nu} : \mathcal{D}_{\lambda,\mu,\nu}^{n-1} \rightarrow \mathcal{D}_{\lambda,\mu,\nu}^{n-1} \quad A \mapsto \varphi_{\nu}^{-1} \circ A \circ \psi_{\lambda,\mu}.
\] (2.11)

Here, we identify the \( \mathcal{K}(n-1) \)-modules via the following isomorphisms:

\[
\begin{align*}
\Pi \left( \mathcal{D}_{\lambda,\mu;\nu}^{n-1} \right) & \rightarrow \text{Hom}_{\text{diff}} \left( F_{\lambda}^{n-1} \otimes F_{\mu}^{n-1}, \Pi(F_{\nu}^{-1}) \right), \quad \Pi(A) \mapsto \Pi \circ A, \\
\Pi \left( \mathcal{D}_{\lambda',\mu';\nu'}^{n-1} \right) & \rightarrow \text{Hom}_{\text{diff}} \left( F_{\lambda'}^{n-1} \otimes \Pi(F_{\mu'}^{n-1}), F_{\nu'}^{-1} \right), \quad \Pi(A) \mapsto A \circ (1 \otimes \Pi), \\
\Pi \left( \mathcal{D}_{\lambda',\mu';\nu'}^{n-1} \right) & \rightarrow \text{Hom}_{\text{diff}} \left( \Pi(F_{\lambda'}^{n-1}) \otimes F_{\mu'}^{n-1}, F_{\nu'}^{-1} \right), \quad \Pi(A) \mapsto A \circ (\Pi \otimes \sigma), \\
\mathcal{D}_{\lambda,\mu;\nu}^{n-1} & \rightarrow \text{Hom}_{\text{diff}} \left( F_{\lambda}^{n-1} \otimes \Pi(F_{\mu}^{n-1}), F_{\nu}^{-1} \right), \quad A \mapsto \Pi \circ A \circ (1 \otimes \Pi), \\
\mathcal{D}_{\lambda',\mu';\nu'}^{n-1} & \rightarrow \text{Hom}_{\text{diff}} \left( \Pi(F_{\lambda'}^{n-1}) \otimes F_{\mu'}^{n-1}, F_{\nu'}^{-1} \right), \quad A \mapsto A \circ (\Pi \otimes \sigma), \\
\mathcal{D}_{\lambda',\mu;\nu'}^{n-1} & \rightarrow \text{Hom}_{\text{diff}} \left( \Pi(F_{\lambda'}^{n-1}) \otimes F_{\mu}^{n-1}, F_{\nu'}^{-1} \right), \quad A \mapsto \Pi \circ A \circ (\Pi \otimes \sigma),
\end{align*}
\]

where \( \lambda' = \lambda + \frac{1}{2}, \mu' = \mu + \frac{1}{2}, \nu' = \nu + \frac{1}{2} \) and \( \sigma(F) = (-1)^{|F|} F \).

\section{\( \mathcal{K}(n) \)-Invariant Binary Differential Operators}

In this section, we will classify all \( \mathcal{K}(n) \)-invariant binary differential operators acting on the spaces of weighted densities on \( \mathbb{R}^{1|n} \) for \( n \geq 2 \). As a first step towards these classifications, we shall need the list of binary \( \mathcal{K}(1) \)-invariant differential operators acting on the spaces of weighted densities on \( \mathbb{R}^{1|1} \).

\subsection{\( \mathcal{K}(1) \)-invariant binary differential operators}

In [3], Leites et al. classified all binary \( \mathcal{K}(1) \)-invariant differential operators

\[
F_{\lambda}^{1} \otimes F_{\mu}^{1} \rightarrow F_{\nu}^{1}, \quad F\alpha_{1}^{\lambda} \otimes G\alpha_{1}^{\mu} \mapsto T_{\lambda,\mu,\nu}(F, G)\alpha_{1}^{\nu}.
\]
Recall that their list consists of (here $\nu_k = \lambda + \mu + \frac{k}{2}$ for $k = 0, 1, 2, 3$)

\[
\begin{align*}
T_{\lambda,\mu,\nu_0}(F, G) &= FG, \\
T_{0,0,\frac{1}{2}}^{0,b}(F, G) &= a(-1)^{|F|}F\eta(G) + b\eta(F)G, \quad a, b \in \mathbb{R}, \\
T_{\lambda,\mu,\nu_1}(F, G) &= \mu F\eta(G) - \lambda(-1)^{|F|}F\eta(G), \\
T_{\lambda,\mu,\nu_2}(F, G) &= \mu F'G - \frac{1}{2}(-1)^{|F|}\eta(F)\eta_1(G) - \lambda FG', \\
T_{0,\mu,\nu_3}(F, G) &= S(F, G) - 2\mu\eta(F')G, \\
T_{\lambda,0,\nu_3}(F, G) &= S(F, G) - 2\lambda(-1)^{|F|}F\eta_1(G'), \\
T_{0,0,2}(F, G) &= F''G' + (-1)^{|F|}(\eta_1(F')\eta_1(G) - \eta(F)\eta_1(G')), \\
T_{-\frac{1}{2},0,\frac{1}{2}}(F, G) &= 3FG'' - (-1)^{|F|}M(F, G) + 2F'G', \\
T_{0,-\frac{1}{2},\frac{1}{2}}(F, G) &= 3F''G + (-1)^{|F|}M(G, F) + 2F'G', \\
T_{\lambda,-\lambda-1,\frac{1}{2}}(F, G) &= \lambda(-1)^{|F|}F\eta_1(G') + (\lambda + 1)\eta_1(F')G + (\lambda + \frac{1}{2})S(F, G),
\end{align*}
\]

(3.12)

where

\[
M(F, G) = 2\eta_1(F)\eta_1(G') + \eta_1(F')\eta_1(G) \quad \text{and} \quad S(F, G) = \eta_1(F)G' + (-1)^{|F|}F'\eta_1(G).
\]

Observe that the operation $T_{\lambda,\mu,\nu_2}$ is nothing but the well-known Poisson bracket on $\mathbb{R}^1$ and the operation $T_{\lambda,\mu,\nu_1}$ is just the Buttin bracket in coordinates $\theta$ and $p := \Pi(\alpha_1)$ with $x$ serving as parameter (see, e.g. [6] [7] [8] [9]).

### 3.2 $K(n)$-invariant binary differential operators for $n \geq 2$

Now, we describe the spaces of $K(n)$-invariant binary differential operators $\mathbb{F}_n^\lambda \otimes \mathbb{F}_n^\mu \rightarrow \mathbb{F}_n^\nu$ for $n \geq 2$. We prove that these spaces are nontrivial only if $\nu = \lambda + \mu$ or $\nu = \lambda + \mu + 1$ and they are, in some sense, spanned by the following even operators defined on $\mathbb{R}[x, \theta] \otimes \mathbb{R}[x, \theta]$:

\[
\begin{align*}
a(F, G) &= FG, \\
b(F, G) &= \mu F'G - \lambda FG' - \frac{1}{2}(-1)^{|F|}\sum_{i=1}^n \eta_1(F)\eta_i(G), \\
c(F, G) &= (-1)^{|F|}(\eta_1(F)\eta_2(G) - \eta_2(F)\eta_1(G)) + 2\eta_2(\eta_1(F))G, \\
d(F, G) &= (-1)^{|F|}(\eta_1(F)\eta_2(G) - \eta_2(F)\eta_1(G)) + 2\lambda F\eta_2(G), \\
e(F, G) &= (-1)^{|F|}(\lambda + \frac{1}{2}) (\eta_1(F)\eta_2(G) - \eta_2(F)\eta_1(G)) + \lambda F\eta_2(G) + (\lambda + 1)\eta_1(\eta_2(F))G.
\end{align*}
\]

(3.13)

More precisely, we have

**Theorem 3.1.** Let $n \geq 2$ and

\[
\mathbb{F}_n^\lambda \otimes \mathbb{F}_n^\mu \rightarrow \mathbb{F}_n^\nu, \quad Fa_1^\lambda \otimes G\alpha_n^\mu \mapsto T_{\lambda,\mu,\nu}(F, G)\alpha_n^\nu
\]

be a nontrivial $K(n)$-invariant binary differential operator. Then

$\nu = \lambda + \mu$ or $\nu = \lambda + \mu + 1$.

Moreover,

(a) If $\nu = \lambda + \mu$ then $T_{\lambda,\mu,\nu} = \alpha a$.

(b) If $\nu = \lambda + \mu + 1$ then, for $n = 2$ or $n = 2$ but $\lambda\mu\nu \neq 0$ we have $T_{\lambda,\mu,\nu} = \alpha b$ and if $n = 2$ and $\lambda\mu\nu = 0$ then $T_{\lambda,\mu,\nu}$ has the form $\alpha b + \beta c$, $\alpha b + \beta d$ or $\alpha b + \beta e$ in accordance with $\lambda = 0$, $\mu = 0$ or $\nu = 0$. Here, $\alpha, \beta \in \mathbb{R}$ and $a, b, c, d, e$ are defined by (3.13).
Proof (i) First assume that \( n = 2 \). The \( \mathcal{K}(2) \)-invariance of any element of \( \mathbb{D}^2_{\lambda,\mu,\nu} \) is equivalent to invariance with respect just to the vector fields \( X_F \in \mathcal{K}(2) \) such that \( \partial_1 \partial_2 F = 0 \) that generate \( \mathcal{K}(2) \). That is, an element of \( \mathbb{D}^2_{\lambda,\mu,\nu} \) is \( \mathcal{K}(2) \)-invariant if and only if it is invariant with respect just to the two subalgebras \( \mathcal{K}(1) \) and \( \mathcal{K}(1)^1 \). Obviously, the \( \mathcal{K}(1) \)-invariant elements of \( \Pi(\mathbb{D}^1_{\lambda,\mu,\nu}) \) can be deduced from those given in (3.12) by using the following \( \mathcal{K}(1) \)-isomorphism

\[
\mathbb{D}^1_{\lambda,\mu,\nu} \rightarrow \Pi(\mathbb{D}^1_{\lambda,\mu,\nu}), \quad A \mapsto \Pi(A \circ (\sigma \otimes \sigma)) \tag{3.14}
\]

Now, by isomorphism (2.11) we exhibit the \( \mathcal{K}(1) \)-invariant elements of \( \mathbb{D}^2_{\lambda,\mu,\nu} \). Of course, these elements are identically zero if \( 2(\nu - \mu - \lambda) \neq -1, 0, 1, 2, 3, 4, 5, 6 \). More precisely, any \( \mathcal{K}(1) \)-invariant element \( \mathfrak{T} \) of \( \mathbb{D}^2_{\lambda,\mu,\nu} \) can be expressed as follows

\[
\mathfrak{T} = \sum_{j,\ell,k=0,1} \Omega_{\lambda,\mu,\nu}^{j,\ell,k} \Psi_{\lambda,\mu,\nu} \left( \Pi^{j+\ell+k} \left( T_{\lambda+\frac{j}{2},\mu+\frac{j}{2},\nu+\frac{k}{2}} \circ (\sigma^{j+\ell+k} \otimes \sigma^{j+\ell+k}) \right) \right) + \sum_{j,\ell,k=0,1} \Omega_{\lambda,\mu,\nu}^{j,\ell,k,a} \Psi_{\lambda,\mu,\nu} \left( \Pi^{j+\ell+k} \left( T_{\lambda+\frac{j}{2},\mu+\frac{j}{2},\nu+\frac{k}{2}} \circ (\sigma^{j+\ell+k} \otimes \sigma^{j+\ell+k}) \right) \right)
\]

where \( T_{\lambda+\frac{j}{2},\mu+\frac{j}{2},\nu+\frac{k}{2}} \) are defined by (3.12). The coefficients \( \Omega_{\lambda,\mu,\nu}^{j,\ell,k} \) and \( \Omega_{\lambda,\mu,\nu}^{j,\ell,k,a} \) are, a priori, arbitrary constants, but the invariance of \( \mathfrak{T} \) with respect \( \mathcal{K}(1) \) imposes some supplementary conditions over these coefficients and determines thus completely the space of \( \mathcal{K}(2) \)-invariant elements of \( \mathbb{D}^2_{\lambda,\mu,\nu} \). By a direct computation, we get:

\[
\Omega_{\lambda,\mu,\nu}^{0,0,0} = \Omega_{\lambda,\mu,\nu}^{0,1,1} = \Omega_{\lambda,\mu,\nu}^{1,0,1}, \quad \Omega_{\lambda,\mu,\nu}^{0,0,1} = 2\Omega_{\lambda,\mu,\nu}^{1,0,1} = \Omega_{\lambda,\mu,\nu}^{0,1,0}.
\]

All other coefficients vanish except for \( \nu = \lambda + \mu + 1 \) with \( \lambda \mu \nu = 0 \), in which case we have also the following non-zero coefficients:

\[
\begin{align*}
\Omega_{\lambda,\mu,\nu}^{0,0,0} &= -\frac{1}{2} \Omega_{\lambda,\mu,\nu}^{0,1,0} = \frac{2\lambda + 1}{2} \Omega_{\lambda,\mu,\nu}^{1,0,0} = \frac{1}{2} \Omega_{\lambda,\mu,\nu}^{1,1,1} \quad \text{for} \quad \lambda \neq -\frac{1}{2}, \\
\Omega_{\lambda,\mu,\nu}^{0,0,1} &= -\frac{1}{2} \Omega_{\lambda,\mu,\nu}^{0,1,0} = -\Omega_{\lambda,\mu,\nu}^{0,1,0} = \frac{1}{2} \Omega_{\lambda,\mu,\nu}^{1,1,1} \quad \text{for} \quad \lambda = -\frac{1}{2}, \\
\Omega_{\lambda,\mu,\nu}^{0,1,0} &= -\frac{1}{2} \Omega_{\lambda,\mu,\nu}^{0,0,1} = \frac{1}{2} \Omega_{\lambda,\mu,\nu}^{1,0,0} = \frac{1}{2} \Omega_{\lambda,\mu,\nu}^{1,1,1} \quad \text{for} \quad \mu \neq -\frac{1}{2}, \\
\Omega_{\lambda,\mu,\nu}^{1,0,0} &= -\Omega_{\lambda,\mu,\nu}^{0,1,0} = \frac{1}{2} \Omega_{\lambda,\mu,\nu}^{0,0,1} = \Omega_{\lambda,\mu,\nu}^{1,1,1} \quad \text{for} \quad \mu = -\frac{1}{2}, \\
\Omega_{\lambda,\mu,0}^{0,1,0} &= \Omega_{\lambda,\mu,0}^{1,0,0} = \Omega_{\lambda,\mu,0}^{0,0,1} = (2\lambda + 1) \Omega_{\lambda,\mu,0}^{1,1,1} \quad \text{for} \quad \lambda \neq -\frac{1}{2}, \\
\Omega_{\lambda,\mu,0}^{1,0,1} &= \Omega_{\lambda,\mu,0}^{1,1,0} = -\Omega_{\lambda,\mu,0}^{1,0,0} = -2\Omega_{\lambda,\mu,0}^{1,1,1} \quad \text{for} \quad \lambda = -\frac{1}{2}.
\end{align*}
\]

Thus, we easily check that Theorem 3.1 is proved for \( n = 2 \).

(ii) Now, we assume that \( n \geq 3 \) and then we proceed by recurrence over \( n \). First note that the \( \mathcal{K}(n) \)-invariance of any element of \( \mathbb{D}^n_{\lambda,\mu,\nu} \) is equivalent to invariance with respect just to the fields \( X_F \in \mathcal{K}(n) \) such that \( \partial_1 \cdots \partial_n F = 0 \) that generate \( \mathcal{K}(n) \). That is, an element of \( \mathbb{D}^n_{\lambda,\mu,\nu} \) is \( \mathcal{K}(n) \)-invariant if and only if it is invariant with respect to the subalgebras \( \mathcal{K}(n - 1) \) and \( \mathcal{K}(n - 1)^1 \), \( i = 1, \ldots, n - 1 \). Thus, as before, we prove that our result holds for \( n = 3 \). Assume that it holds for \( n \geq 3 \). Then, by recurrence assumption and isomorphism (2.11), we deduce that any nontrivial \( \mathcal{K}(n) \)-invariant element \( \mathfrak{T} \) of \( \mathbb{D}^{n+1}_{\lambda,\mu,\nu} \) only can appear if \( 2(\nu - \mu - \lambda) = -1, 0, 1, 2, 3, 4, \) and it has the general following form:

\[
\mathfrak{T} = \sum_{j,\ell,k=0,1} \Delta_{\lambda,\mu,\nu}^{j,\ell,k} \Psi_{\lambda,\mu,\nu} \left( \Pi^{j+\ell+k} \left( T_{\lambda+\frac{j}{2},\mu+\frac{j}{2},\nu+\frac{k}{2}} \circ (\sigma^{j+\ell+k} \otimes \sigma^{j+\ell+k}) \right) \right).
\]
As before, the coefficients \( \Delta^{j,\ell,k}_{\lambda,\mu,\nu} \) are, a priori, arbitrary constants, but the invariance of \( \mathfrak{F} \) with respect \( \mathcal{K}(n)^i, \ i = 1, \ldots, n \), shows that

\[
\begin{align*}
\Delta^{0,0,0}_{\lambda,\mu,\lambda+\mu} &= \Delta^{0,1,1}_{\lambda,\mu,\lambda+\mu} &= \Delta^{1,0,1}_{\lambda,\mu,\lambda+\mu}, \\
\Delta^{0,0,0}_{\lambda,\mu,\lambda+\mu+1} &= 2\Delta^{1,0,0}_{\lambda,\mu,\lambda+\mu+1} &= \Delta^{0,1,1}_{\lambda,\mu,\lambda+\mu+1} &= \Delta^{1,0,1}_{\lambda,\mu,\lambda+\mu+1}
\end{align*}
\]

and all other coefficients are identically zero. Therefore, we easily check that \( \mathfrak{F} \) is expressed as in Theorem 3.1.

3.3 Poisson superalgebra of weighted densities

For \( n \geq 2 \), the even operation

\[
\mathfrak{F}^n_{\lambda,\mu,\lambda+\mu+1}(F, G) = \mu F' G - \lambda F G' - \frac{1}{2} (-1)^{|F|} \sum_{i=1}^{n} \eta_i(F) \eta_i(G)
\]  

(3.15)

defines a structure of Poisson Lie superalgebra on \( \mathbb{R}^{1|n} \). Indeed, consider the continuous sum (direct integral) of all spaces \( \mathbb{F}^n \):

\[
\mathbb{F}^n = \bigcup_{\lambda \in \mathbb{R}} \mathbb{F}^n_{\lambda,\mu,\lambda+\mu+1}.
\]

The collection of the operations (3.15) defines a bilinear map \( \mathfrak{F}^1: \mathbb{F}^n \otimes \mathbb{F}^n \to \mathbb{F}^n \). The following statement can be checked directly.

Proposition 3.1. The operation \( \mathfrak{F}^1 \) satisfies the Jacobi and Leibniz identities, then it equips the space \( \mathbb{F}^n \) with a Poisson superalgebra structure.

Note that this Proposition is a simplest generalization of a result by Gargoubi and Ovsienko for \( n = 1 \) (see [7]).

4 Cohomology

Let us first recall some fundamental concepts from cohomology theory (see, e.g., [3]). Let \( \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) be a Lie superalgebra acting on a superspace \( V = V_0 \oplus V_1 \) and let \( \mathfrak{h} \) be a subalgebra of \( \mathfrak{g} \). (If \( \mathfrak{h} \) is omitted it assumed to be \( \{0\} \).) The space of \( \mathfrak{h} \)-relative \( n \)-cochains of \( \mathfrak{g} \) with values in \( V \) is the \( \mathfrak{g} \)-module

\[
\mathcal{C}^n(\mathfrak{g}, \mathfrak{h}; V) := \text{Hom}_h(A^n(\mathfrak{g}/\mathfrak{h}); V).
\]

The coboundary operator \( \delta_n : \mathcal{C}^n(\mathfrak{g}, \mathfrak{h}; V) \to \mathcal{C}^{n+1}(\mathfrak{g}, \mathfrak{h}; V) \) is a \( \mathfrak{g} \)-map satisfying \( \delta_n \circ \delta_{n-1} = 0 \). The kernel of \( \delta_n \), denoted \( Z^n(\mathfrak{g}, \mathfrak{h}; V) \), is the space of \( \mathfrak{h} \)-relative \( n \)-cocycles, among them, the elements in the range of \( \delta_{n-1} \) are called \( \mathfrak{h} \)-relative \( n \)-coboundaries. We denote \( B^n(\mathfrak{g}, \mathfrak{h}; V) \) the space of \( n \)-coboundaries.

By definition, the \( n \)th \( \mathfrak{h} \)-relative cohomolgy space is the quotient space

\[
\mathcal{H}^n(\mathfrak{g}, \mathfrak{h}; V) = Z^n(\mathfrak{g}, \mathfrak{h}; V)/B^n(\mathfrak{g}, \mathfrak{h}; V).
\]

We will only need the formula of \( \delta_n \) (which will be simply denoted \( \delta \) in degrees 0 and 1: for \( v \in \mathcal{C}^0(\mathfrak{g}, \mathfrak{h}; V) = V^\mathfrak{h}, \delta v(g) := (-1)^{p(g)p(v)} g \cdot v \), where

\[
V^\mathfrak{h} = \{ v \in V \mid h \cdot v = 0 \ \text{for all} \ h \in \mathfrak{h} \},
\]

and for \( \Upsilon \in \mathcal{C}^1(\mathfrak{g}, \mathfrak{h}; V), \delta(\Upsilon)(g, h) := (-1)^{|g||\Upsilon|} g \cdot \Upsilon(h) - (-1)^{|h|(|g|+|\Upsilon|)} h \cdot \Upsilon(g) - \Upsilon([g, h]) \) for any \( g, h \in \mathfrak{g} \).
4.1 The space $\text{H}^1_{\text{diff}}(\mathcal{K}(n); \mathbb{D}^n_{\lambda, \mu})$

In this subsection, we will compute the first differential cohomology spaces $\text{H}^1_{\text{diff}}(\mathcal{K}(n); \mathbb{D}^n_{\lambda, \mu})$ for $n \geq 3$. Our main result is the following:

**Theorem 4.1.** The space $\text{H}^1_{\text{diff}}(\mathcal{K}(n); \mathbb{D}^n_{\lambda, \mu})$ has the following structure:

$$\text{H}^1_{\text{diff}}(\mathcal{K}(n); \mathbb{D}^n_{\lambda, \mu}) \simeq \begin{cases} \mathbb{R} & \text{if } \begin{cases} n = 3 \text{ and } \mu - \lambda = 0, \frac{1}{2}, \frac{3}{2}, \\ n = 4 \text{ and } \mu - \lambda = 0, 1, \\ n \geq 5 \text{ and } \mu - \lambda = 0, \end{cases}, \\ 0 & \text{otherwise.} \end{cases} \quad (4.16)$$

A base for the nontrivial $\text{H}^1_{\text{diff}}(\mathcal{K}(n); \mathbb{D}^n_{\lambda, \mu})$ is given by the cohomology classes of the 1-cocycles:

$$\Upsilon^3_{\lambda, \lambda}(X_G) = G'$$

$$\Upsilon^3_{\lambda, \lambda + \frac{1}{2}}(X_G) = \begin{cases} \eta_3 \eta_2 \eta_1(G) & \text{if } \lambda \neq \frac{1}{2}, \\ \partial_3(G) \eta_1 \eta_2 - \eta_1 \eta_2 (\partial_3(G)) \zeta_4 - (-1)^{|G|} \partial_3 M_{\eta_3(G)} \eta_3 & \text{if } \lambda = \frac{1}{2} \end{cases}$$

$$\Upsilon^3_{\lambda, \lambda + 1}(X_G) = \begin{cases} \Xi_G + 2 \lambda \eta_3 \eta_2 \eta_1(G') + \eta_3 \eta_2 \eta_1(G) \eta_1^2 & \text{if } \lambda \neq -1, \\ \Xi_G' + \sum_{1 \leq i < j \leq 3} (-1)^{i+j} \eta_0 - i - j(G') \eta_j \eta_i & \text{if } \lambda = -1. \end{cases}$$

where

$$M_G = (-1)^{|G|} \sum_{i=1}^2 (-1)^{r} \eta_{3-i}(G) \eta_i, \quad \Xi_G = (-1)^{|G|} \sum_{1 \leq i < j \leq 3} (-1)^{i+j} \eta_j \eta_i(G) \eta_{6-i-j},$$

$$Q_G = (-1)^{|G|} \sum_{1 \leq i < j < k \leq 4} (-1)^{i+j+k} \eta_j \eta_k \eta_i(G) \eta_{10-i-j-k},$$

$$A_G = (-1)^{|G|} \sum_{i=3}^4 (-1)^{i} \left( \eta_1 \eta_2 (\partial_3 \zeta_i(G)) \zeta_{7-i} - \partial_3 \zeta_i(G) \eta_1 \eta_2 \right) \partial_{7-i} \text{ with } \zeta_i = 1 - \theta_{7-i} \eta_{7-i}. \quad (4.17)$$

The proof of Theorem 4.1 will be the subject of subsection 4.3. In fact, we need first the description of $\text{H}^1_{\text{diff}}(\mathcal{K}(n-1); \mathbb{D}^n_{\lambda, \mu})$ and $\text{H}^1_{\text{diff}}(\mathcal{K}(n), \mathcal{K}(n-1)^2; \mathbb{D}^n_{\lambda, \mu})$.

4.2 The space $\text{H}^1_{\text{diff}}(\mathcal{K}(n-1); \mathbb{D}^n_{\lambda, \mu})$

The space $\text{H}^1_{\text{diff}}(\mathcal{K}(n-1); \mathbb{D}^n_{\lambda, \mu})$ is closely related to $\text{H}^1_{\text{diff}}(\mathcal{K}(n-1); \mathbb{D}^{n-1}_{\lambda, \mu})$. Therefore, for comparison and to build upon, we first recall the description of $\text{H}^1_{\text{diff}}(\mathcal{K}(2); \mathbb{D}^2_{\lambda, \mu})$. This space was calculated in [2]. The result is as follows:

$$\text{H}^1_{\text{diff}}(\mathcal{K}(2); \mathbb{D}^2_{\lambda, \mu}) \simeq \begin{cases} \mathbb{R}^2 & \text{if } \mu - \lambda = 0, 2, \\ \mathbb{R} & \text{if } \mu - \lambda = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (4.18)$$
The following 1-cocycles span the corresponding cohomology spaces:

\[ Y^2_{\lambda,\lambda}(X_G) = G' \]
\[ \tilde{Y}^2_{\lambda,\lambda}(X_G) = \begin{cases} \eta_1 \eta_2(G) & \text{if } \lambda = 0 \\ 2 \lambda \eta_1 \eta_2 \left( \theta_2 \partial_2(G) \right) - (-1)^{|G|} \sum_{i=1}^2 \eta_i \left( \theta_2 \partial_2(G) \right) \eta_{3-i} & \text{if } \lambda \neq 0 \end{cases} \]
\[ Y^2_{\lambda,\lambda+1}(X_G) = \begin{cases} \eta_1 \eta_2(G') & \text{if } \lambda \neq -\frac{1}{2} \\ \eta_1 \eta_2(G') + M_{G'} & \text{if } \lambda = -\frac{1}{2} \end{cases} \]
\[ Y^2_{\lambda,\lambda+2}(X_G) = (2\lambda + 1) \left( \frac{2\lambda}{3} G'' - H_{G''} \right) - 2\eta_2 \eta_1(G') \eta_2 \eta_1 \]
\[ \tilde{Y}^2_{\lambda,\lambda+2}(X_G) = \begin{cases} M_{G''} + 2\lambda \eta_2 \eta_1(G'') - 2\eta_2 \eta_1(G') \partial_x & \text{if } \lambda \neq -1 \\ (M_{G''} - \eta_2 \eta_1(G')) \partial_x + M_{G''} - G'' \eta_2 \eta_1 & \text{if } \lambda = -1, \end{cases} \]

where, for \( G \in \mathbb{R}[x, \theta] \), \( M_G \) is as \( (2.17) \) and \( H_G = (-1)^{|G|} \sum_{i=1}^2 \eta_i(G) \eta_i \)

**Proposition 4.1.** As a \( K(n - 1) \)-module, we have

\[ D^n_{\lambda,\mu} \simeq D_{\lambda,\mu}^{n-1} \oplus D_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^{n-1} \oplus \Pi \left( D_{\lambda,\mu+1}^{n-1} \oplus D_{\lambda+\frac{1}{2},\mu}^{n-1} \right). \]

Proof. By isomorphism \( (2.10) \), we deduce a \( K(n - 1) \)-isomorphism:

\[ \Phi_{\lambda,\mu} : D_{\lambda,\mu}^{n-1} \oplus D_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^{n-1} \oplus \Pi \left( D_{\lambda,\mu+1}^{n-1} \oplus D_{\lambda+\frac{1}{2},\mu}^{n-1} \right) \to D^n_{\lambda,\mu} \]

\[ A \mapsto \varphi_{\mu}^{-1} \circ A \circ \varphi_{\lambda}. \]

Here, we identify the \( K(n - 1) \)-modules via the following isomorphisms:

\[ \Pi \left( D_{\lambda,\mu+\frac{1}{2}}^{n-1} \right) \to \text{Hom}_{\text{diff}} \left( F_{\lambda}^{n-1}, \Pi(F_{\mu+\frac{1}{2}}^{n-1}) \right) \quad \Pi(A) \mapsto \Pi \circ A, \]
\[ \Pi \left( D_{\lambda+\frac{1}{2},\mu}^{n-1} \right) \to \text{Hom}_{\text{diff}} \left( \Pi(F_{\lambda+\frac{1}{2}}^{n-1}), F_{\mu}^{n-1} \right) \quad \Pi(A) \mapsto A \circ \Pi, \]
\[ D_{\lambda+\frac{1}{2},\mu+\frac{1}{2}}^{n-1} \to \text{Hom}_{\text{diff}} \left( \Pi(F_{\lambda+\frac{1}{2}}^{n-1}), \Pi(F_{\mu+\frac{1}{2}}^{n-1}) \right) \quad A \mapsto \Pi \circ A \circ \Pi. \]

**Corollary 4.2.** The space \( H^1_{\text{diff}}(K(2); D^3_{\lambda,\mu}) \) has the following structure:

\[ H^1_{\text{diff}}(K(2); D^3_{\lambda,\mu}) \simeq \begin{cases} \mathbb{R}^4 & \text{if } \mu - \lambda = 0, 2, \\ \mathbb{R}^3 & \text{if } \mu - \lambda = \frac{1}{2}, \frac{3}{2}, \\ \mathbb{R}^2 & \text{if } \mu - \lambda = \frac{1}{2}, 1, \frac{5}{2}, \\ 0 & \text{otherwise}. \end{cases} \]

The corresponding spaces \( H^1_{\text{diff}}(K(2); D^3_{\lambda,\lambda+\frac{k}{2}}) \) are spanned by the cohomology classes of the 1-cocycles \( \Theta_{\lambda,\lambda+\frac{k}{2}}^{3,j,\ell} \) and \( \tilde{\Theta}_{\lambda,\lambda+\frac{k}{2}}^{3,j,\ell} \), defined by

\[ \Theta_{\lambda,\lambda+\frac{k}{2}}^{3,j,\ell} (X_G) = \Phi_{\lambda,\lambda+\frac{k}{2}} \left( \Pi^{j+\ell} \left( \sigma^{j+\ell} \circ \tilde{Y}^2_{\lambda,\lambda+\frac{k+2j+2\ell}{2}}(X_G) \right) \right) \]
and
\[
\Theta^{3,i,\ell}_{\lambda,\mu} (X_G) = \Phi_{\lambda,\lambda+\frac{1}{2}} \left( \Pi^{i+\ell} \left( \sigma^{i+\ell} \circ \tilde{Y}_{\lambda,\lambda+\frac{1}{2}}^2 (X_G) \right) \right),
\]
where \( j, \ell = 0, 1, k \in \{ -1, \ldots, 5 \} \), \( \mathcal{T}_{\lambda,\mu}^2, \tilde{Y}_{\lambda,\mu}^2 \) are as in \([4.19]\), \( \Phi_{\lambda,\mu} \) is as in \([4.21]\). Furthermore, the space \( H^1_{\text{diff}} (\mathcal{K}(2); \mathbb{D}^3_{\lambda,\mu} \oplus \mathbb{D}^3_{\lambda,\mu}) \) has the same parity as the integer \( k \).

Proof. First, it is easy to see that the map \( \chi : \mathbb{D}^n_{\lambda,\mu} \to \Pi \left( \mathbb{D}^n_{\lambda,\mu} \right) \) defined by \( \chi (A) = \Pi (\sigma \circ A) \) satisfies
\[
\mathbb{L}_{X_G}^{\lambda,\mu} \circ \chi = (-1)^{[G]} \chi \circ \mathbb{L}_{X_G}^{\lambda,\mu} \quad \text{for all } X_G \in \mathcal{K}(n).
\]
Thus, we deduce the structure of \( H^1_{\text{diff}} (\mathcal{K}(n); \Pi (\mathbb{D}^n_{\lambda,\mu})) \) from \( H^1_{\text{diff}} (\mathcal{K}(n); \mathbb{D}^n_{\lambda,\mu}) \). Indeed, to any \( \Upsilon \in Z^1_{\text{diff}} (\mathcal{K}(n); \mathbb{D}^n_{\lambda,\mu}) \) corresponds \( \chi \circ \Upsilon \in Z^1_{\text{diff}} (\mathcal{K}(n); \Pi (\mathbb{D}^n_{\lambda,\mu})) \). Obviously, \( \Upsilon \) is a coboundary if and only if \( \chi \circ \Upsilon \) is a coboundary.

Second, according to Proposition \([4.1]\) we obtain the following isomorphism between cohomology spaces:
\[
H^1_{\text{diff}} (\mathcal{K}(n-1); \mathbb{D}^n_{\lambda,\mu}) \cong H^1_{\text{diff}} (\mathcal{K}(n-1); \mathbb{D}^{n-1}_{\lambda,\mu}) \oplus H^1_{\text{diff}} (\mathcal{K}(n-1); \mathbb{D}^{n-1}_{\lambda,\mu+\frac{1}{2}}) \oplus H^1_{\text{diff}} (\mathcal{K}(n-1); \Pi (\mathbb{D}^{n-1}_{\lambda+\frac{1}{2},\mu})) \oplus H^1_{\text{diff}} (\mathcal{K}(n-1); \Pi (\mathbb{D}^{n-1}_{\lambda+\frac{1}{2},\mu+\frac{1}{2}})).
\]
Thus, we deduce the structure of \( H^1_{\text{diff}} (\mathcal{K}(2); \mathbb{D}^3_{\lambda,\mu}) \).

\[ \square \]

4.3 The spaces \( H^1_{\text{diff}} (\mathcal{K}(n), \mathcal{K}(n-1)^i; \mathbb{D}^n_{\lambda,\mu}) \)

As a first step towards the proof of Theorem \([4.1]\) we shall need to study the \( \mathcal{K}(n-1)^i \)-relative cohomology \( H^1_{\text{diff}} (\mathcal{K}(n), \mathcal{K}(n-1)^i; \mathbb{D}^n_{\lambda,\mu}) \). Hereafter all \( \epsilon \)'s are constants and we will use the superscript \( i \) when we consider the superalgebra \( \mathcal{K}(n)^i \) instead of \( \mathcal{K}(n) \).

Let \( \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p} \) be a Lie superalgebra, where \( \mathfrak{h} \) is a subalgebra and \( \mathfrak{p} \) is a \( \mathfrak{h} \)-module such that \([\mathfrak{p}, \mathfrak{p}] = \mathfrak{h}\). Consider a 1-cocycle \( \Upsilon \in Z^1 (\mathfrak{g}; V) \), where \( V \) is a \( \mathfrak{g} \)-module. The cocycle relation reads
\[
(-1)^{|\mathfrak{g}| |\Upsilon|} g \cdot \Upsilon (h) = (-1)^{|\mathfrak{h}| (|\mathfrak{g}| + |\Upsilon|)} h \cdot \Upsilon (g) - \Upsilon ([g, h]) = 0 \quad \text{for any } g, h \in \mathfrak{g}.
\]
Denote \( \Upsilon_{\mathfrak{h}} = \Upsilon_{\mid \mathfrak{h}} \) and \( \Upsilon_{\mathfrak{p}} = \Upsilon_{\mid \mathfrak{p}} \). Obviously, if \( \Upsilon_{\mathfrak{h}} = 0 \) then \( \Upsilon \) is \( \mathfrak{h} \)-invariant, therefore, the \( \mathfrak{h} \)-relative cohomology space \( H^1 (\mathfrak{g}, \mathfrak{h}; V) \) is nothing but the space of cohomology classes of 1-cocycles vanishing on \( \mathfrak{h} \). In our situation, \( \mathfrak{g} = \mathcal{K}(n), \mathfrak{h} = \mathcal{K}(n-1)^i, \mathfrak{p} = \Pi (\mathbb{F}^{n-1, i}) \) and \( V = \mathbb{D}^n_{\lambda,\mu} \). Furthermore, in this case, the 1-cocycle relation yields the following equations:
\[
(-1)^{|\mathfrak{g}| |\Upsilon|} X_g \cdot \Upsilon_{\mathfrak{p}} (X_{h_{\theta_i}}) - (-1)^{|\mathfrak{h}| (|\mathfrak{g}| + |\Upsilon|)} X_{h_{\theta_i}} \cdot \Upsilon_{\mathfrak{h}} (X_g) - \Upsilon_{\mathfrak{p}} ([X_g, X_{h_{\theta_i}}]) = 0, \quad (4.25)
\]
\[
(-1)^{|\mathfrak{g}| |\Upsilon|} X_{g_{\theta_i}} \cdot \Upsilon_{\mathfrak{p}} (X_{h_{\theta_i}}) - (-1)^{|\mathfrak{h}| (|\mathfrak{g}| + |\Upsilon|)} X_{h_{\theta_i}} \cdot \Upsilon_{\mathfrak{h}} (X_{g_{\theta_i}}) - \Upsilon_{\mathfrak{p}} ([X_{g_{\theta_i}}, X_{h_{\theta_i}}]) = 0, \quad (4.26)
\]
where \( g, h \in \mathbb{R} [x, \theta_1, \ldots, \theta_i, \ldots, \theta_n] \) and \( |\bar{h}| = |h| + 1 \).

**Theorem 4.3.** For all \( n \geq 1 \) and for all \( i = 1, \ldots, n \), we have
\[
H^1_{\text{diff}} (\mathcal{K}(n), \mathcal{K}(n-1)^i; \mathbb{D}^n_{\lambda,\mu}) \cong \begin{cases} \mathbb{R} & n = 2 \text{ and } \lambda = \mu \neq 0, \\ 0 & n = 3 \text{ and } (\lambda, \mu) = (-\frac{1}{2}, 0), \end{cases} \quad (4.27)
\]
Moreover, for fixed \( n = 2 \) or \( 3 \), non-zero relative cohomology \( H^1_{\text{diff}}(K(n), K(n-1)^i; \mathbb{D}_{\lambda,\mu}^n) \) are spanned by classes of some \( K(n-1)^i \)-relative cocycles which are cohomologous.

Proof. For \( n = 1 \), the result holds from [1], (Lemma 4.1). For \( n = 2 \), we deduce the result from [2] (Proposition 4.2). Moreover, the space \( H^1_{\text{diff}}(K(2), K(1); \mathbb{D}_{\lambda,\mu}^2) \), for \( \lambda \neq 0 \), is spanned by the cohomology class of the \( K(1) \)-relative 1-cocycle \( \tilde{\Upsilon}_{\lambda,\lambda} \) defined by (4.19). Note that \( \tilde{\Upsilon}_{\lambda,\lambda}|_{K(1)^1} \) is a coboundary, namely, \( 2\delta(\theta_2\partial_1 + \theta_1\partial_2) \). Therefore, \( \tilde{\Upsilon}_{\lambda,\lambda} \) is coboundaries to the \( K(1)^1 \)-relative 1-cocycle \( \tilde{\Upsilon}_{\lambda,\lambda} = 2\delta(\theta_2\partial_1 + \theta_1\partial_2) \) generating the space \( H^1_{\text{diff}}(K(2), K(1)^1; \mathbb{D}_{\lambda,\mu}^2) \). Now, we deduce the result for \( n \geq 3 \) from the following Proposition.

**Proposition 4.2.**

1) For \((\lambda, \mu) \neq (-\frac{1}{2}, 0)\), any element of \( Z^1_{\text{diff}}(K(3); \mathbb{D}_{\lambda,\mu}^3) \) is a coboundary over \( K(3) \) if and only if at least one of its restrictions to the subalgebras \( K(2)^i \) is a coboundary.

2) For \((\lambda, \mu) = (-\frac{1}{2}, 0)\), there exists a unique, up to a scalar factor and a coboundary, nontrivial 1-cocycle \( \Upsilon^3_{-\frac{1}{2}, 0} \in Z^1_{\text{diff}}(K(3); \mathbb{D}_{\lambda,\mu}^3) \) such that its restrictions to \( K(2), K(2)^1 \), and to \( K(2)^2 \) are coboundaries. This 1-cocycle is odd and it is given by:

\[
\Upsilon^3_{-\frac{1}{2}, 0}(X_G) = \partial_5(G)\eta_1\eta_2 - \eta_1\eta_2 (\partial_5(G)) (1 - \theta_3\partial_3) - (-1)^{G}\theta_3 M_{\theta_3(G)}\eta_3,
\]

where, for \( G \in \mathbb{R}[x, \theta] \), \( M_G \) is as (4.17).

Proof. Let \( \Upsilon \in Z^1_{\text{diff}}(K(n); \mathbb{D}_{\lambda,\mu}^n) \) and assume that the restriction of \( \Upsilon \) to some \( K(n-1)^i \) is a coboundary, that is, there exists \( b \in \mathbb{D}_{\lambda,\mu}^n \) such that

\[
\Upsilon(X_F) = \delta(b)(X_F) = (-1)^{|F||b|}X_F \cdot b \quad \text{for all} \quad X_F \in K(n-1)^i.
\]

By replacing \( \Upsilon \) by \( \Upsilon - \delta b \), we can suppose that \( \Upsilon|_{K(n-1)^i} = 0 \). Thus, the map \( \Upsilon \) is \( K(n-1)^i \)-invariant and therefore the equation (4.29) becomes:

\[
(-1)^{|\tilde{\theta}|}\Upsilon(X_{g\theta_i} \cdot \Upsilon(X_{h\theta_i} - (-1)^{|\tilde{\theta}|(\|\tilde{\theta}\|+|\Upsilon|)}X_{h\theta_i} \cdot \Upsilon(X_{g\theta_i}) = 0.
\]

According to the isomorphism (2.10), the map \( \Upsilon \) is decomposed into four components

\[
\begin{align*}
\Pi(F_{-\frac{1}{2}}^{n-1,i}) \otimes F_{\lambda,i}^{n-1,i} & \rightarrow F_{\mu}^{n-1,i}, \\
\Pi(F_{-\frac{1}{2}}^{n-1,i}) \otimes F_{\lambda,i}^{n-1,i} & \rightarrow \Pi(F_{\mu+\frac{1}{2}}^{n+1,i}), \\
\Pi(F_{-\frac{1}{2}}^{n-1,i}) \otimes F_{\lambda,i}^{n-1,i} & \rightarrow \Pi(F_{\mu+\frac{1}{2}}^{n-1,i}).
\end{align*}
\]

So, each of these bilinear maps is \( K(n-1)^i \)-invariant. Therefore, their expressions are given by Theorem 3.1 with the help of isomorphisms (2.11) and (3.14). More precisely, using equation (4.29), we get up to a scalar factor:

- For \( n \geq 3 \) with \( (\lambda, \mu) \neq (-\frac{1}{2}, 0) \) if \( n = 3 \),

\[
\Upsilon = \begin{cases} 
\delta(\theta_i) & \text{if } \mu = \lambda - \frac{1}{2}, \\
\delta(1 - 2\theta_i\partial_i) & \text{if } \mu = \lambda, \\
\delta(\partial_i) & \text{if } \mu = \lambda + \frac{1}{2}, \\
0 & \text{otherwise}.
\end{cases}
\]

(4.31)
• For \( n = 3 \) and \( \mu = \lambda + \frac{1}{2} = 0 \),
\[
\Upsilon = \epsilon_1 \Upsilon^{3,i}_{-\frac{1}{2},0} + \epsilon_2 \delta (\partial_i),
\] (4.32)
where \( \Upsilon^{3,i}_{-\frac{1}{2},0} \) is the 1-cocycle on \( \mathcal{K}(3) \) with coefficients in \( \mathbb{D}^{3}_{-\frac{1}{2},0} \) defined by
\[
\Upsilon^{3,i}_{-\frac{1}{2},0}(X_G) = \partial_i(G)\eta_\ell \eta_k - \eta_\ell \eta_k (\partial_i(G)) (1 - \theta_i \partial_i) + \theta_i (\eta_\ell \eta_k(G)\eta_\ell - \eta_\ell \eta_k(G)\eta_k) \eta_i
\] (4.33)
with \( \ell, k \neq i \) and \( \ell < k \). Obviously, \( \Upsilon^{3,i}_{-\frac{1}{2},0} = \Upsilon^{3,i}_{-\frac{1}{2},0} \) with \( \Upsilon^{3,i}_{-\frac{1}{2},0} \) is as in (4.23), and a direct computation shows that for \( j = 1, 2 \):
\[
\Upsilon^{3,i}_{-\frac{1}{2},0} + (-1)^j \Upsilon^{3,j}_{-\frac{1}{2},0} = 2(-1)^j \delta ((\theta_3 \eta_j + \theta_j \eta_3)\eta_{3-j}).
\] (4.34)
Thus, up to a scalar factor and a coboundary, \( \Upsilon = \Upsilon^{3,i}_{-\frac{1}{2},0} \). Therefore, in order to complete the proof of Proposition [4.2], we have to study the cohomology class of the 1-cocycle \( \Upsilon^{3,i}_{-\frac{1}{2},0} \) in \( H^1_{\text{diff}}(\mathcal{K}(3), \mathbb{D}^{3}_{-\frac{1}{2},0}) \).

**Lemma 4.3.** The 1-cocycle \( \Upsilon^{3,i}_{-\frac{1}{2},0} \) defines a nontrivial cohomology class over \( \mathcal{K}(3) \). Its restrictions to \( \mathcal{K}(2), \mathcal{K}(2)^1 \) and to \( \mathcal{K}(2)^2 \) are coboundaries.

Proof. It follows from equation (4.31) that the restriction of \( \Upsilon^{3,i}_{-\frac{1}{2},0} \) to \( \mathcal{K}(2) \) vanishes and to \( \mathcal{K}(2)^1 \) and to \( \mathcal{K}(2)^2 \) are coboundaries. Now, assume that there exists an odd operator \( A \in \mathbb{D}^{3}_{-\frac{1}{2},0} \) such that \( \Upsilon^{3,i}_{-\frac{1}{2},0} \) is equal to \( \delta A \). By isomorphism (4.21), the operator \( A \) can be expressed as \( A = \Phi_{-\frac{1}{2},0}(A_1, A_2, \Pi(A_3), \Pi(A_4)) \), where \( A_1 \in \mathbb{D}^{2}_{-\frac{1}{2},0}, A_2 \in \mathbb{D}^{2}_{0,\frac{1}{2}} \), \( A_3 \in \mathbb{D}^{2}_{-\frac{1}{2},\frac{1}{2}} \) and \( A_4 \in \mathbb{D}^{2}_{0,0} \). Thus, since the map
\[
\mathbb{D}^{2}_{\lambda,\mu} \rightarrow \Pi(\mathbb{D}^{2}_{\lambda,\mu}), \quad B \mapsto \Pi(B \circ \sigma)
\]
is a \( \mathcal{K}(2) \)-isomorphism, the condition \( \Upsilon^{3,i}_{-\frac{1}{2},0|\mathcal{K}(2)} = 0 \) tell us that \( A_1, A_2, A_3 \circ \sigma \) and \( A_4 \circ \sigma \) are \( \mathcal{K}(2) \)-invariant linear maps. Therefore, up to a scalar factor, each of \( A_1, A_2, A_3 \circ \sigma \) and \( A_4 \circ \sigma \) is the identity map \( \mathbb{S} \): \( \mathbb{F}^2_{\lambda} \rightarrow \mathbb{F}^2_{\lambda}, \quad F \alpha^2_{\lambda} \mapsto F \alpha^2_{\lambda} \). Thus, we obtain
\[
A(F \alpha^2_{\frac{1}{2}}) = \epsilon \partial_3(F).
\]
Finally, it is easy to check that the equation \( \Upsilon^{3,i}_{-\frac{1}{2},0} = \delta(A) \) has no solutions contradicting our assumption. Lemma 4.3 is proved. Thus we have completed the proof of Proposition 4.2 \( \square \)

**Corollary 4.4.** Up to a coboundary, any 1-cocycle \( \Upsilon \in Z^1_{\text{diff}}(\mathcal{K}(3); \mathbb{D}^{3}_{\lambda,\mu}) \) has the following general form:
\[
\Upsilon(X_F) = \sum a_{\ell_1 \ell_2 \ell_3} m_1 m_2 m_3 \eta^1_{\ell_1} \eta^2_{\ell_2} \eta^3_{\ell_3} (F) \eta_1 \eta_2 \eta_3, \quad (4.35)
\]
where the coefficients \( a_{\ell_1 \ell_2 \ell_3} \) are functions of \( \theta_i \), not depending on \( x \).
Proof. By (2.8), we can see that the operator $\Upsilon$ has the form (4.35) where, a priori, the coefficients $a_{\ell_1 \ell_2 \ell_3 k_1 k_2 k_3}$ are some functions of $x$ and $\theta_i$, but we shall now prove that $\partial_x a_{\ell_1 \ell_2 \ell_3 k_1 k_2 k_3} = 0$. To do this, we shall simply show that $X_1 \cdot \Upsilon = 0$.

We have

$$(X_1 \cdot \Upsilon)(X_F) := X_1 \cdot \Upsilon(X_F) - \Upsilon([X_1, X_F]) \quad \text{for all } F \in \mathbb{R}[x, \theta].$$

But, from Proposition 4.2 and Corollary 4.2, it follows that, up to a coboundary, $\Upsilon(X_1) = 0$, and therefore the equation (4.36) becomes

$$(X_1 \cdot \Upsilon)(X_F) = X_1 \cdot \Upsilon(X_F) - (-1)^{|F|}|\tau| X_F \cdot \Upsilon(X_1) - \Upsilon([X_1, X_F]).$$

The right-hand side of (4.37) vanishes because $\Upsilon$ is a 1-cocycle. Thus, $X_1 \cdot \Upsilon = 0$. □

The following lemma gives a description of all coboundaries over $K(2)^i$, vanishing on the subalgebra $K(1)^m_i$, where $m_i \in \{1, 2, 3\} \setminus \{i\}$. This description will be useful in the proof of Theorem 4.1.

**Lemma 4.5.** (see [2]) Any coboundary $B_{\lambda, \mu}^{i, m_i} \in B^1_{\text{diff}}(K(2)^i; \mathbb{D}^3_{\lambda, \mu})$ vanishing on $K(1)^{m_i}$ is, up to a scalar factor, given by

$$B_{\lambda, \mu}^{i, m_i} = \begin{cases} 
\delta \left( \epsilon_1 \partial_{m_i} + \epsilon_2 \eta_{6-i-m_i} \left( \theta_{m_i} \eta_{m_i} - 1 \right) \right) & \text{if } (\lambda, \mu) = (0, \frac{1}{2}) \\
\delta \left( \eta_{m_i} - \epsilon_2 \theta_{m_i} \eta_{6-i-m_i} \right) & \text{if } (\lambda, \mu) = (-\frac{3}{2}, 0) \\
\delta \left( \theta_{m_i} \eta_{6-i-m_i} \right) & \text{if } (\lambda, \mu) = (-\frac{1}{2}, \frac{1}{2}) \\
\delta \left( \partial_{m_i} \right) & \text{if } \mu = \lambda + \frac{1}{2} \text{ and } \lambda \neq 0, -\frac{1}{2} \\
0 & \text{otherwise.}
\end{cases}$$

**4.4 Proof of Theorem 4.1**

(i) According to Proposition 4.2, the restriction of any nontrivial differential 1-cocycle $\Upsilon$ of $K(3)$ with coefficients in $\mathbb{D}^3_{\lambda, \mu}$, to $K(2)^i$, for $i = 1, 2, 3$, is a nontrivial 1-cocycle except for

$$\Upsilon = \epsilon \Upsilon_{\frac{3}{2}, 0} + \delta A,$$

where $\Upsilon_{\frac{3}{2}, 0}$ is as (4.28), $\epsilon \neq 0$ and $A \in \mathbb{D}^3_{\frac{3}{2}, 0}$. So, if $2(\mu - \lambda) \neq -1, 0, 1, 2, 3, 4, 5$, then, by Corollary 4.2, the corresponding cohomology spaces $H^1_{\text{diff}}(K(3); \mathbb{D}^3_{\lambda, \mu})$ vanish.

For $2(\mu - \lambda) = -1, 0, 1, 2, 3, 4, 5$, let $\Upsilon$ be a 1-cocycle from $K(3)$ to $\mathbb{D}^3_{\lambda, \mu}$. The map $\Upsilon_{|K(2)^i}$ is a 1-cocycle of $K(2)^i$. Therefore, using Corollary 4.2 together with Lemma 4.5 and Theorem 4.3 with the help of isomorphism (4.21), we deduce that, up to a coboundary, the non-zero restrictions of the cocycle $\Upsilon$ on $K(2)^i$ can be expressed as (here $\tau = \mu - \lambda$):

For $\tau = \frac{3}{2}, 2$,

$$\Upsilon_{|K(2)^i} = \begin{cases} 
(\lambda - \frac{3}{2} - 1 \lambda_{0, 0}) + \frac{4(3-\lambda)}{2} b \left( \lambda_{0, 0} \right) + \lambda_{0, 0} \left( \lambda_{1, 1} + \lambda_{1, 1} \right) \quad & \text{if } \tau = \frac{1}{2}, \lambda = 0 \\
(\lambda_{1, 1} + \lambda_{0, 0}) + b \left( \lambda_{2, 3} - \lambda_{1, 3} \right) \left( \lambda_{0, 0} + \lambda_{0, 0} \right) \quad & \text{if } \tau = -\frac{1}{2}, \lambda \neq 0 \\
(\lambda_{0, 0} - \lambda_{0, 0}) \quad & \text{if } \tau = \frac{1}{2} \\
2(\lambda + 2) \lambda_{0, 0} + (2\lambda + 3) \lambda_{1, 1} \quad & \text{if } \tau = 2
\end{cases}$$

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For \( \tau = 0 \),
\[
\mathcal{Y}_{|_{\mathcal{K}(2)^{0}}} = \begin{cases} 
\gamma \left( \Theta_{\lambda, \mu}^{1,0,0} + \Theta_{\lambda, \mu}^{1,1,1} \right) + b_{1} \left( \Theta_{\lambda, \mu}^{1,0,0} + \Theta_{\lambda, \mu}^{1,1,1} \right) + \\
\Lambda_{1,3} \left( 2b_{1} \left( \Gamma_{\lambda, \mu}^{2,3,0,1} - \Gamma_{\lambda, \mu}^{2,3,0,1} \right) \right) - t \left( \Gamma_{\lambda, \mu}^{2,3,0,0} + \Gamma_{\lambda, \mu}^{2,3,1,1} \right) \right) + \\
\Lambda_{2,3} \left( b_{3} \delta(A) + b_{2} \left( \Gamma_{\lambda, \mu}^{1,3,0,1} - \Gamma_{\lambda, \mu}^{1,3,1,0} \right) + \frac{t}{2} \left( \Gamma_{\lambda, \mu}^{1,3,0,0} + \Gamma_{\lambda, \mu}^{1,3,1,1} \right) \right) \right) & \text{if } \mu \neq 0, -\frac{1}{2} \\
a \left( \Theta_{\lambda, \mu}^{1,0,0} + \Theta_{\lambda, \mu}^{1,1,1} \right) + b_{1} \left( \Theta_{\lambda, \mu}^{1,0,0} + \Theta_{\lambda, \mu}^{1,1,1} \right) - 2\Lambda_{1,3} \left( b_{2} \Gamma_{\lambda, \mu}^{2,3,0,1} + \\
b_{3} \left( \Gamma_{\lambda, \mu}^{2,3,0,1} + b_{1} \left( \Gamma_{\lambda, \mu}^{2,3,0,1} - \Gamma_{\lambda, \mu}^{2,3,0,1} \right) + \frac{t}{2} \left( \Gamma_{\lambda, \mu}^{2,3,0,0} + \Gamma_{\lambda, \mu}^{2,3,1,1} \right) \right) + \\
\Lambda_{2,3} \left( b_{3} \delta(A) - \Gamma_{\lambda, \mu}^{1,3,0,1} + b_{2} \left( \Gamma_{\lambda, \mu}^{1,3,0,1} - \Gamma_{\lambda, \mu}^{1,3,1,0} \right) + \\
b_{1} \left( \Gamma_{\lambda, \mu}^{1,3,0,0} + \frac{t}{2} \left( \Gamma_{\lambda, \mu}^{1,3,1,1} + \Gamma_{\lambda, \mu}^{1,3,1,1} \right) \right) \right) \right) & \text{if } \mu = 0 \\
\end{cases}
\]

For \( \tau = \frac{1}{2} \),
\[
\mathcal{Y}_{|_{\mathcal{K}(2)}} = \begin{cases} 
\gamma \left( \Theta_{\lambda, \mu}^{1,0,1} - \Theta_{\lambda, \mu}^{1,1,0} \right) + \left( \frac{1}{2} \Lambda_{2,3} - \Lambda_{1,3} \right) \times \\
\left( \frac{1}{2} \Lambda_{2,3} - \Lambda_{1,3} \right) \right) \times \left( \Gamma_{\lambda, \mu}^{1,3,0,0} + \Gamma_{\lambda, \mu}^{1,3,1,1} \right) - b\Lambda_{i,2,3} \delta(A_{10}) \right) \right) & \text{if } \mu = 0 \\
\gamma \left( \Theta_{\lambda, \mu}^{1,0,1} + b\Theta_{\lambda, \mu}^{1,1,0} \right) + t \left( \frac{1}{2} \Lambda_{2,3} - \Lambda_{1,3} \right) \times \\
\left( \frac{1}{2} \Lambda_{2,3} - \Lambda_{1,3} \right) \right) \times \left( \Gamma_{\lambda, \mu}^{1,3,0,0} + \Gamma_{\lambda, \mu}^{1,3,1,1} \right) - b\Lambda_{i,2,3} \delta(A_{10}) \right) \right) & \text{if } \mu = 0 \\
\gamma \left( \Theta_{\lambda, \mu}^{1,0,1} + b\Theta_{\lambda, \mu}^{1,1,0} \right) + t \left( \frac{1}{2} \Lambda_{2,3} - \Lambda_{1,3} \right) \times \\
\left( \frac{1}{2} \Lambda_{2,3} - \Lambda_{1,3} \right) \right) \times \left( \Gamma_{\lambda, \mu}^{1,3,0,0} + \Gamma_{\lambda, \mu}^{1,3,1,1} \right) - b\Lambda_{i,2,3} \delta(A_{10}) \right) \right) & \text{if } \mu = 0 \\
\end{cases}
\]

where (recall that \( \mathcal{B}_{\lambda, \mu}^{i, m_{j}} \) depend on \( \epsilon_{1}, \epsilon_{2} \))
\[
\Gamma_{\lambda, \mu, \epsilon_{1}, \epsilon_{2}} (X_{G}) = \Phi_{\lambda, \mu} \left( \Pi_{i, t_{\epsilon}} \left( \sigma_{i, t_{\epsilon}} \circ \mathcal{B}_{\lambda, \mu}^{i, m_{j}} \left( X_{G} \right) \right) \right), \\
\Lambda_{11} = \Lambda_{10} = \Lambda_{2,3} = (i-r)(i-s),
\]
\( \Theta_{\lambda, \mu}^{i, j, \ell} \) and \( \tilde{\Theta}_{\lambda, \mu}^{i, j, \ell} \) are defined by (4.23)-(4.24) and the coefficients \( a, b, b_{i} \) and \( t \) are constants.
So, by Proposition 4.2 \( H_{1, \text{diff}}(\mathcal{K}(3); \mathcal{D}_{\lambda, \mu}) = 0 \) for \( \mu - \lambda = 1, \frac{5}{2} \).
Now, by Corollary 4.4, we can write
\[
\mathcal{Y}(X_{ht_{1, \cdots, t_{3}}}) = \sum_{m, k, \epsilon = (\epsilon_{1}, \epsilon_{2}, \epsilon_{3})} \left( a_{m, k, \epsilon} + \sum_{j=1}^{3} \sum_{1 \leq i_{1} < \cdots < i_{j} \leq 3} \sum_{a_{i_{1} \cdots i_{j}, m, k, \epsilon} \theta_{i_{1}} \cdots \theta_{i_{j}} \gamma(h) \delta_{m}^{i_{1}} \delta_{i_{1}}^{x} \delta_{i_{2}}^{y} \delta_{i_{3}}^{z} \right)
\]
with \( \varepsilon_i = 0, 1 \). For each case, we solve the equations (4.25) and (4.26) for \( a, b, b_i, t, a_{0,m,k,\varepsilon}, a_{1-i,j,m,k,\varepsilon} \). We obtain

1) For \( 2(\mu - \lambda) = -1, 4 \), the coefficient \( a \) vanishes; so, by Proposition 4.2, \( \Upsilon \) is a coboundary. Hence \( H^1_{\text{diff}}(K(3); D^3_{\lambda,\mu}) = 0 \).

2) For \( \mu = \lambda \), the coefficients \( b_i \) vanish and, up to a coboundary, \( \Upsilon \) is a multiple of \( \Upsilon^3_{\lambda,\lambda} \), see Theorem 4.1. Hence \( \dim H^1_{\text{diff}}(K(3); D^3_{\lambda,\lambda}) = 1 \).

3) For \( 2(\mu - \lambda) = 1 \), the coefficient \( b \) vanishes and, up to a coboundary, \( \Upsilon \) is a multiple of \( \Upsilon^3_{\lambda,\lambda + \frac{1}{2}} \), see Theorem 4.1. Hence \( \dim H^1_{\text{diff}}(K(3); D^3_{\lambda,\lambda + \frac{1}{2}}) = 1 \).

4) For \( 2(\mu - \lambda) = 3 \), \( \Upsilon \) is a multiple of \( \Upsilon^3_{\lambda,\lambda + \frac{3}{2}} \). Hence \( \dim H^1_{\text{diff}}(K(3); D^3_{\lambda,\lambda + \frac{3}{2}}) = 1 \).

(ii) Note that, by Proposition 4.2, the restriction of any nontrivial differential 1-cocycle \( \Upsilon \) of \( K(4) \) with coefficients in \( D^4_{\lambda,\mu} \) to \( K(3)^i \), for \( i = 1, \ldots, 4 \), is a nontrivial 1-cocycle. Furthermore, using arguments similar to those of the proof of Corollary 4.2 together with the above result, we deduce that \( H^1_{\text{diff}}(K(3)^i; D^4_{\lambda,\mu}) = 0 \) if \( 2(\mu - \lambda) \neq -1, 0, 1, 2, 3, 4 \). Then, we consider only the cases where \( 2(\mu - \lambda) = -1, 0, 1, 2, 3, 4 \) and, as before, we get the result for \( n = 4 \).

(iii) We proceed by recurrence over \( n \). In a similar way as in (ii), we get the result for \( n = 5 \). Now, we assume that it holds for some \( n \geq 5 \). Again, the same arguments as in the proof of Corollary 4.2 together with recurrence assumption show that \( H^1_{\text{diff}}(K(n)^i; D^4_{\lambda,\mu}) = 0 \) if \( 2(\mu - \lambda) \neq -1, 0, 1 \). So, we consider only the cases where \( 2(\mu - \lambda) = -1, 0, 1 \), we proceed as in (i) and we get the result for \( n + 1 \). \( \square \)

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