Comparison principles for the linear and semilinar time-fractional diffusion equations with the Robin boundary condition

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Abstract

The main objective of this paper is analysis of the initial-boundary value problems for the linear and semilinear time-fractional diffusion equations with a uniformly elliptic spatial differential operator of the second order and the Caputo type fractional derivative acting in the fractional Sobolev spaces. The boundary conditions are formulated in form of the homogeneous Neumann or Robin conditions. First we deal with the uniqueness and existence of the solutions to these initial-boundary value problems. Then we show a positivity property for the solutions and derive the corresponding comparison principles. In the case of the semilinear time-fractional diffusion equation, we also apply the monotonicity method by upper and lower solutions. As an application of our results, we present some a priori estimates for solutions to the semilinear time-fractional diffusion equations.

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1 Introduction

In this paper, we first deal with a linear time-fractional diffusion equation

\[
\partial_t^\alpha (u(x,t) - a(x)) = \sum_{i,j=1}^d \partial_i (a_{ij}(x) \partial_j u(x,t)) \\
+ \sum_{j=1}^d b_j(x,t) \partial_j u(x,t) + c(x,t) u(x,t) + F(x,t), \quad x \in \Omega, \ 0 < t < T,
\]  

(1.1)
where \( \partial_t^\alpha \) is the Caputo fractional derivative of order \( \alpha \in (0,1) \) defined in the fractional Sobolev spaces (see Section 2) and \( \Omega \subset \mathbb{R}^d \), \( d = 1, 2, 3 \) is a bounded domain with smooth boundary \( \partial \Omega \). All functions under consideration are assumed to be real-valued.

In what follows, we always assume that the following conditions are satisfied:

\[
\begin{align*}
    a_{ij} &= a_{ji} \in C^1(\Omega), \quad 1 \leq i, j \leq d, \\
    b_j, c &\in C^1([0, T]; C^1(\Omega)) \cap C([0, T]; C^2(\Omega)), \quad 1 \leq j \leq d, \\
    \text{and there exists a constant } \kappa > 0 \text{ such that } \\
    \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq \kappa \sum_{j=1}^d \xi_j^2, \quad x \in \Omega, \quad \xi_1, ..., \xi_d \in \mathbb{R}.
\end{align*}
\]

Using the notations \( \partial_j = \frac{\partial}{\partial x_j}, \) \( j = 1, 2, ..., d \) and \( \partial_t = \frac{\partial}{\partial t} \), we define the conormal derivative \( \partial_{\nu} A w \) with respect to \( \sum_{i,j=1}^d \partial_j(a_{ij}\partial_i) \) by

\[
\partial_{\nu} A w(x) = \sum_{i,j=1}^d a_{ij}(x) \partial_j w(x) \nu_i(x), \quad x \in \partial \Omega,
\]

where \( \nu = \nu(x) = (\nu_1(x), ..., \nu_d(x)) \) is the unit outward normal vector to \( \partial \Omega \) at the point \( x := (x_1, ..., x_d) \in \partial \Omega \).

For the equation (1.1), we consider the initial-boundary value problems with the Neumann boundary condition

\[
\partial_{\nu} A u = 0 \quad \text{on } \partial \Omega \times (0, T) \quad (1.4)
\]

or with the Robin boundary condition

\[
\partial_{\nu} A u + \sigma(x) u = 0 \quad \text{on } \partial \Omega \times (0, T), \quad (1.5)
\]

where \( \sigma \) is a smooth function on \( \partial \Omega \) that satisfies the condition \( \sigma \geq 0 \).

In the second part of the article, we treat a semilinear time-fractional diffusion equation

\[
\partial_t^\alpha(u(x,t) - a(x)) = \sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j u(x,t))
\]

\[
+ \sum_{j=1}^d b_j(x,t) \partial_j u(x,t) + c(x,t) u(x,t) + F(x,u(x,t), \nabla u(x,t)), \quad x \in \Omega, \quad 0 < t < T \quad (1.6)
\]

with the boundary condition (1.4) or (1.5).

For the parabolic partial differential equations which correspond to the case \( \alpha = 1 \) in the equations (1.1) and (1.6), several important qualitative properties of solutions to the initial-boundary value problems are known. In particular, we mention the maximum principle and the comparison principle ([29], [30]).

The main purpose of this article is in establishing the comparison principles and the monotonicity method for the linear and semilinear time-fractional diffusion equations (1.1) and (1.6) with the Neumann or Robin boundary conditions.

For the linear time-fractional diffusion equations of type (1.1) with the Dirichlet boundary condition, the maximum principle was studied and used in [19], [20], [22], [23], [24], [37]. For derivation of the maximum principle for the time-fractional transport equations we refer to [21].
Because the maximum principle involves the Dirichlet boundary values, it is difficult to formulate it in the case of the Neumann or Robin boundary conditions. However, for this kind of the boundary conditions, the comparison principle and the positivity of solutions can be proved that is the main objective of this article. One typical result of this sort says that the solution $u$ to the equation (1.1) with the boundary condition (1.5) and an appropriately formulated initial condition is non-negative in $\Omega \times (0, T)$ if the initial value $a$ and the non-homogeneous term $F$ are non-negative in $\Omega$ and in $\Omega \times (0, T)$, respectively. Such positivity properties and their applications were intensively derived and used for the parabolic partial differential equations ($\alpha = 1$ in the equations (1.1) and (1.6)), see e.g., [6], [7], [26] or [30]. Moreover, we refer to [2], [15], [26] for a discussion of the monotonicity method by upper and lower solutions for studying the semilinear parabolic partial differential equations. This method is based on the corresponding comparison principle.

However, to the best knowledge of the authors, there are no such works for the time-fractional diffusion equations in the case of the Neumann or Robin boundary conditions. The main purpose of this article is to fill this gap in the theory of the linear and semilinear time-fractional diffusion equations. The positivity property and the comparison principle for the linear equation (1.1) with the boundary condition (1.4) or (1.5) and an appropriately formulated initial condition are the subject of the first part of the article. In the second part, these result are extended to the semilinear time-fractional diffusion equation (1.6). The arguments employed in this article rely on an operator theoretical approach to the fractional integrals and derivatives in the fractional Sobolev spaces that is an extension of the theory developed for the case $\alpha = 1$ ([13], [27], [32]).

The rest of this article is organized as follows. In Section 2 some important results regarding the unique existence of solutions to the initial-boundary value problems for the linear time-fractional diffusion equations are presented. Section 3 is devoted to a proof of a key lemma that is a basis for the proofs of the comparison principles discussed in Sections 4 and 5. The lemma asserts that the solution mapping $\{a, F\} \rightarrow u_{a,F}$ preserves the ordering of $a$ and $F$, where $a$ and $F$ denote an initial condition and a source function of the problem under consideration, respectively, and $u_{a,F}$ denotes its solution in a certain class. In Section 4 the key lemma and the fixed point theorem are employed to prove the comparison principles and some of their corollaries for the initial-boundary value problems for the linear time-fractional diffusion equations. In Section 5 the unique existence of solutions to the initial-boundary value problems for the semilinear time-fractional diffusion equations are discussed. Section 6 presents the comparison principles and some of their corollaries for the initial-boundary value problems for the semilinear time-fractional diffusion equations. Finally, some conclusions and remarks are formulated in the last section.

2 Well-posedness for initial-boundary value problems for linear time-fractional diffusion equations

For $x \in \Omega$, $0 < t < T$, we define the operator

$$- Au(x, t) = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j v(x, t)) + \sum_{j=1}^{d} b_j(x, t)\partial_j u(x, t) + c(x, t)u(x, t) \quad (2.1)$$
and assume that the conditions (1.2) for the coefficients $a_{ij}(x), b_j(x, t), c(x, t)$ are satisfied.

In this section, we deal with the following initial-boundary value problem for the linear time-fractional diffusion equation with the fractional derivative of order $\alpha \in (0, 1)$

$$
\begin{align*}
\begin{cases}
\partial_t^\alpha (u(x, t) - a(x)) + Au(x, t) = F(x, t), & x \in \Omega, \ 0 < t < T, \\
\partial_{\nu^\alpha} u + \sigma(x) u(x, t) = 0, & x \in \partial \Omega, \ 0 < t < T,
\end{cases}
\end{align*}
$$

(2.2)

along with the initial condition (2.3) formulated below. To appropriately define the Caputo derivative $d_t^\alpha w(t), \ 0 < \alpha < 1$, we start with its definition on the space

$$
oC^1[0, T] := \{ u \in C^1[0, T]; \ u(0) = 0 \}
$$

that reads as follows:

$$
d_t^\alpha w(t) = \frac{1}{\Gamma(1 - \alpha)} \int_0^t (t - s)^{-\alpha} \frac{d w}{ds}(s)ds, \ w \in oC^1[0, T].
$$

Then we extend this operator from the domain $D(d_t^\alpha) := oC^1[0, T]$ into $L^2(0, T)$ taking into account the closability of the operator (36). As have been shown in [16], there exists a unique minimum closed extension of $d_t^\alpha$ with $D(d_t^\alpha) = oC^1[0, T]$. Moreover, the domain of this extension is the closure of $oC^1[0, T]$ in the Sobolev-Slobodecki space $H^\alpha(0, T)$. Let us recall that the norm $\| \cdot \|_{H^\alpha(0, T)}$ is defined as follows (11):

$$
\|v\|_{H^\alpha(0, T)} := \left( \|v\|_{L^2(0, T)}^2 + \int_0^T \int_0^T \frac{|v(t) - v(s)|^2}{|t - s|^{1 + 2\alpha}} dt ds \right)^{\frac{1}{2}}.
$$

By setting

$$
H_\alpha(0, T) := oC^1[0, T]_{H^\alpha(0, T)},
$$

we see that

$$
H_\alpha(0, T) = \begin{cases}
H^\alpha(0, T), & 0 < \alpha < \frac{1}{2}, \\
\{ v \in H^{\frac{1}{2}}(0, T); \int_0^T \frac{|v(t)|^2}{t} dt < \infty \}, & \alpha = \frac{1}{2}, \\
\{ v \in H^\alpha(0, T); v(0) = 0 \}, & \frac{1}{2} < \alpha < 1,
\end{cases}
$$

and

$$
\|v\|_{H_\alpha(0, T)} = \begin{cases}
\|v\|_{H^\alpha(0, T)}, & \alpha \neq \frac{1}{2}, \\
\left( \|v\|_{H^{\frac{1}{2}}(0, T)}^2 + \int_0^T \frac{|v(t)|^2}{t} dt \right)^{\frac{1}{2}}, & \alpha = \frac{1}{2}.
\end{cases}
$$

In what follows, we also use the Riemann-Liouville fractional integral operator $J^\beta, \ \beta > 0$ defined by

$$
J^\beta f(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} f(s) ds, \ \ 0 < t < T.
$$

Then, according to [11] and [16],

$$
H_\alpha(0, T) = J^\alpha L^2(0, T), \ \ 0 < \alpha < 1.
$$

We define

$$
\partial_t^\alpha = (J^\alpha)^{-1} \ \text{with} \ \mathcal{D}(\partial_t^\alpha) = H_\alpha(0, T).
$$
Then there exists a constant $C > 0$ depending only on $\alpha$ such that

$$C^{-1}\|v\|_{H^\alpha(0,T)} \leq \|\partial_t^\alpha v\|_{L^2(0,T)} \leq C\|v\|_{H^\alpha(0,T)} \quad \text{for all } v \in H^\alpha(0,T).$$

Now we can introduce a suitable form for an initial condition for the problem (2.2) as follows

$$u(x,\cdot) - a(x) \in H^\alpha(0,T) \quad \text{for almost all } x \in \Omega$$

and write down the complete formulation of the initial-boundary value problem for the linear time-fractional diffusion equation:

$$\begin{cases}
\partial_t^\alpha (u(x,t) - a(x)) + Au(x,t) = F(x,t), & x \in \Omega, \ 0 < t < T, \\
\partial_n u(x,t) + \sigma(x)u(x,t) = 0, & x \in \partial\Omega, \ 0 < t < T, \\
u(x,\cdot) - a(x) \in H^\alpha(0,T) & \text{for almost all } x \in \Omega.
\end{cases}$$

We note that the term $\partial_t^\alpha (u(x,t) - a(x))$ in (2.4) is well-defined via the third condition in (2.4). In particular, for $\frac{1}{2} < \alpha < 1$, the Sobolev embedding leads to the inclusions $H^\alpha(0,T) \subset H^2(0,T) \subset C[0,T]$. Thus, $u \in H^\alpha(0,T;L^2(\Omega))$ implies $u \in C([0,T];L^2(\Omega))$ and in this case, we can see that the initial condition is formulated as $u(\cdot,0) = a$ in $L^2$-sense.

In the following theorem, a fundamental result regarding the unique existence of the solution to the initial-boundary value problem (2.4) is formulated.

**Theorem 2.1.** For $a \in H^1(\Omega)$ and $F \in L^2(0,T;L^2(\Omega))$, there exists a unique solution $u(F,a) = u(F,a)(x,t) \in L^2(0,T;H^2(\Omega))$ to the initial-boundary value problem (2.4) such that $u(F,a) - a \in H^\alpha(0,T;L^2(\Omega))$. Moreover, there exists a constant $C > 0$ such that

$$\|u(F,a) - a\|_{H^\alpha(0,T;L^2(\Omega))} + \|u(F,a)\|_{L^2(0,T;H^2(\Omega))} \leq C(\|a\|_{H^1(\Omega)} + \|F\|_{L^2(0,T;L^2(\Omega))}).$$

**Proof.** For the proof, we introduce an elliptic operator $A_0$ in $L^2(\Omega)$ with an arbitrary $c_0 > 0$ as follows:

$$\begin{cases}
(-A_0v)(x) = \sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j v(x)) - c_0 v(x), & x \in \Omega, \\
D(A_0) = \{v \in H^2(\Omega); \partial_n v + \sigma v = 0 \text{ on } \partial\Omega\}.
\end{cases}$$

(2.5)

We recall that $\sigma$ is smooth and $\sigma \geq 0$ on $\partial\Omega$ and the coefficients $a_{ij}$, $b_j$ and $c$ satisfy the conditions (1.2).

Henceforth, by $\|\cdot\|$ and $(\cdot,\cdot)$ we denote the standard norm and the scalar product in $L^2(\Omega)$, respectively. It is well-known that the operator $A_0$ is self-adjoint and its resolvent is a compact operator. Moreover, for sufficiently large constant $c_0 > 0$, by Lemma 8.1 in Section 7 we can verify that $A_0$ is positive definite. Therefore, by choosing a large constant $c_0 > 0$, its spectrum consists entirely of discrete positive eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, which are numbered according to their multiplicities and $\lambda_n \to \infty$ as $n \to \infty$. Let $\varphi_n$ be an eigenvector corresponding to the eigenvalue $\lambda_n$ such that $(\varphi_n,\varphi_m) = 0$ if $n \neq m$ and $(\varphi_n,\varphi_n) = 1$. Note that $A\varphi_n = \lambda_n \varphi_n$ for $n \in \mathbb{N}$. Then the system $\{\varphi_n\}_{n \in \mathbb{N}}$ of the eigenvectors forms an orthonormal basis in $L^2(\Omega)$ and for any $\gamma \geq 0$ we can define the fractional powers $A_0^\gamma$ of the operator $A_0$ by the following relation (see, e.g., [27]):

$$A_0^\gamma v = \sum_{n=1}^\infty \lambda_n^\gamma (v,\varphi_n)\varphi_n.$$
where
\[ v \in \mathcal{D}(A^\gamma_0) := \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} (v, \varphi_n)^2 < \infty \right\} \]
and
\[ \|A^\gamma_0 v\| = \left( \sum_{n=1}^{\infty} \lambda_n^{2\gamma} (v, \varphi_n)^2 \right)^{\frac{1}{2}}. \]

We note that \( \mathcal{D}(A^\gamma_0) \subset H^{2\gamma}(\Omega) \).

The proof of Theorem 2.1 is similar to case of the homogeneous Dirichlet boundary condition ([11], [16]), and we define two operators \( S(t) \) and \( K(t) \) by
\[
S(t)a = \sum_{n=1}^{\infty} E_{\alpha,1}(-\lambda_n t^\alpha)(a, \varphi_n) \varphi_n, \quad a \in L^2(\Omega), \ t > 0 \tag{2.6}
\]
and
\[
K(t)a = -A_0^{-1} t S'(t)a = \sum_{n=1}^{\infty} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^\alpha)(a, \varphi_n) \varphi_n, \quad a \in L^2(\Omega), \ t > 0. \tag{2.7}
\]

Here \( E_{\alpha,\beta}(z) \) denotes the Mittag-Leffler function defined by a convergent series as follows:
\[
E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta \in \mathbb{C}, \ z \in \mathbb{C}.
\]

It follows directly from the definitions given above that \( A^\gamma_0 K(t)a = K(t)A^\gamma_0 a \) and \( A^\gamma_0 S(t)a = S(t)A^\gamma_0 a \) for \( a \in \mathcal{D}(A^\gamma_0) \). Moreover, since
\[
|E_{\alpha,1}(-\lambda_n t^\alpha)|, \ |E_{\alpha,\alpha}(-\lambda_n t^\alpha)| \leq \frac{C}{1 + \lambda_n t^\alpha} \quad \text{for all } t > 0
\]
(e.g., Theorem 1.6 (p.35) in [28]), we can prove
\[
\left\{ \begin{array}{l}
\|A^\gamma_0 S(t)a\| \leq C t^{-\alpha \gamma} \|a\|, \\
\|A^\gamma_0 K(t)a\| \leq C t^{\alpha(1-\gamma)-1} \|a\|, \quad a \in L^2(\Omega), \ t > 0, \ 0 \leq \gamma \leq 1 \tag{2.8}
\end{array} \right.
\]
([11]). In order to shorten the notations and focus on the dependence of the time variable \( t \), henceforth we sometimes omit the variable \( x \) in the functions of two variables \( x \) and \( t \), and write simply \( u(t) = u(\cdot, t) \), \( F(t) = F(\cdot, t) \), \( a = a(\cdot) \), etc.

Because of (2.8), the remaining estimation can be carried out analog to the one in the case of the fractional powers of generators of analytic semigroups ([13]).

To do this, we first formulate and prove the following lemma:

**Lemma 2.1.** Under the conditions formulated above, the following estimates hold true for \( F \in L^2(0,T; L^2(\Omega)) \) and \( a \in L^2(\Omega) \):
(i)
\[
\left\| \int_0^t A_0 K(t-s)F(s)ds \right\|_{L^2(0,T;L^2(\Omega))} \leq C \|F\|_{L^2(0,T;L^2(\Omega))},
\]
Then we employ the representation

\[ \int_0^t K(t-s)F(s)ds, \]

obtain the inequality

\[ \| S(t)a - a \|_{H^\alpha_n(0,T;L^2(\Omega))} + \| S(t)a \|_{L^2(0,T;H^2(\Omega))} \leq C\| a \|. \]

**Proof.** We start with the proof of the estimate (i). By (2.7), we have

\[ \left\| \int_0^t A_0 K(t-s)F(s)ds \right\|_{L^2(0,T;L^2(\Omega))} \]

\[ = \sum_{n=1}^{\infty} \left( \int_0^t \lambda_n(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})(F(s), \varphi_n)ds \right) \varphi_n \]

\[ = \sum_{n=1}^{\infty} \int_0^t \lambda_n(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)^{\alpha})(F(s), \varphi_n)ds \].

Therefore, using the Parseval equality and the Young inequality for the convolution, we obtain

\[ \left\| \int_0^t A_0 K(t-s)F(s)ds \right\|_{L^2(0,T;L^2(\Omega))} \]

\[ = \sum_{n=1}^{\infty} \int_0^T |(\lambda_n s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n s^{\alpha}) * (F(s), \varphi_n)|^2 ds \]

\[ = \sum_{n=1}^{\infty} \| \lambda_n s^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n s^{\alpha}) * (F(s), \varphi_n) \|_{L^2(0,T)}^2 \]

\[ \leq \sum_{n=1}^{\infty} \left( \lambda_n \int_0^t |t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha})|dt \right)^2 \| (F(t), \varphi_n) \|_{L^2(0,T)}^2. \]

Then we employ the representation

\[ \frac{d}{dt} E_{\alpha,1}(-\lambda_n t^{\alpha}) = -\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}), \quad (2.9) \]

and the complete monotonicity of the Mittag-Leffler function (10)

\[ E_{\alpha,1}(-\lambda_n t^{\alpha}) > 0, \quad \frac{d}{dt} E_{\alpha,1}(-\lambda_n t^{\alpha}) \leq 0, \quad t \geq 0, \quad 0 < \alpha \leq 1 \]

to reach the inequality

\[ \int_0^T |\lambda_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha})|dt = - \int_0^T \frac{d}{dt} E_{\alpha,1}(-\lambda_n t^{\alpha})dt \quad (2.10) \]

\[ = 1 - E_{\alpha,1}(-\lambda_n T^{\alpha}) \leq 1 \quad \text{for all } n \in \mathbb{N}. \]
By direct calculation, using (2.9), we obtain the formula

\[ \left\| \int_0^t A_0 K(t-s)F(s)ds \right\|_{L^2(0,T)}^2 \leq \sum_{n=1}^{\infty} \|(F(t), \varphi_n)\|_{L^2(0,T)}^2 \]

\[ = \int_0^T \sum_{n=1}^{\infty} |(F(t), \varphi_n)|^2 dt = \int_0^T \|F(\cdot, t)\|^2 dt = \|F\|_{L^2(0,T)}^2. \]

Hence,

\[ \int_0^t A_0 K(t-s)F(s)ds \]

Now we proceed with the proof of the estimate (ii). For \( 0 < t < T, n \in \mathbb{N} \) and \( f \in L^2(0,T) \), we set

\( (L_n f)(t) := \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(t-s)\alpha)f(s)ds. \)

Then

\[ \int_0^t K(t-s)F(s)ds = \sum_{n=1}^{\infty} (L_n f)(t) \varphi_n \]

in \( L^2(\Omega) \) for any fixed \( t \in [0,T] \).

First we prove that

\[ \begin{cases} L_n f \in H_{\alpha}(0,T), \\ \partial_t^\alpha (L_n f)(t) = -\lambda_n L_n f(t) + f(t), & 0 < t < T, \\ \|L_n f\|_{H_{\alpha}(0,T)} \leq C\|f\|_{L^2(0,T)}, & n \in \mathbb{N} \quad \text{for each } f \in L^2(0,T). \end{cases} \] (2.11)

In order to prove this, we apply to \( L_n f \) the Riemann-Liouville fractional integral operator \( J^\alpha \):

\[ J^\alpha(L_n f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}(L_n f)(s)ds \]

\[ = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left( \int_0^s (s-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(s-\xi)\alpha)f(\xi)d\xi \right) ds \]

\[ = \frac{1}{\Gamma(\alpha)} \int_\xi^t f(\xi) \left( \int_0^\xi (t-s)^{\alpha-1}(s-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(s-\xi)\alpha)ds \right) d\xi. \]

By direct calculation, using (2.9), we obtain the formula

\[ \frac{1}{\Gamma(\alpha)} \int_\xi^t (t-s)^{\alpha-1}(s-\xi)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n(s-\xi)\alpha)ds \]

\[ = -\frac{1}{\lambda_n} (t-\xi)^{\alpha-1} \left( E_{\alpha,\alpha}(-\lambda_n t\alpha) - \frac{1}{\Gamma(\alpha)} \right). \]

Therefore, we have

\[ J^\alpha(L_n f)(t) = -\frac{1}{\lambda_n} (L_n f)(t) + \frac{1}{\lambda_n} \int_0^t (t-\xi)^{\alpha-1} \frac{1}{\Gamma(\alpha)} f(\xi)d\xi \]

\[ = -\frac{1}{\lambda_n} (L_n f)(t) + \frac{1}{\lambda_n} (J^\alpha f)(t), \quad n \in \mathbb{N}, \]
that is,
\[(L_n f)(t) = -\lambda_n J^\alpha (L_n f)(t) + (J^\alpha f)(t), \quad 0 < t < T.\]

Hence, \(L_n f \in H_\alpha (0, T) = J^\alpha L^2(0, T).\) In view of \(\partial_t^\alpha = (J^\alpha)^{-1},\) we have
\[\partial_t^\alpha (L_n f) = -\lambda_n L_n f + f \quad \text{in} \ (0, T).\]

Using the inequality (2.10), we obtain
\[\lambda_n \|L_n f\|_{L^2(0, T)} \leq \lambda_n \|s^{\alpha-1} E_{0,\alpha}(-\lambda_n s^\alpha)\|_{L^1(0, T)} \|f\|_{L^2(0, T)} \leq \|f\|_{L^2(0, T)}.\]

Therefore,
\[\|L_n f\|_{H_\alpha(0, T)} \leq C \|\partial_t^\alpha (L_n f)\|_{L^2(0, T)} \leq C(\| -\lambda_n L_n f\|_{L^2(0, T)} + \|f\|_{L^2(0, T)}) \leq C \|f\|_{L^2(0, T)}, \quad n \in \mathbb{N}, \ f \in L^2(0, T).\]

Thus, the estimate (2.11) is proved.

Now we set \(f_n(s) := (F(s), \varphi_n)\) for \(0 < s < T\) and \(n \in \mathbb{N}.\) Since
\[\partial_t^\alpha \int_0^t K(t - s) F(s) ds = \sum_{n=1}^{\infty} \partial_t^\alpha (L_n f_n)(t) \varphi_n,\]
we have
\[\left\|\partial_t^\alpha \int_0^t K(t - s) F(s) ds\right\|_{L^2(\Omega)}^2 = \sum_{n=1}^{\infty} |\partial_t^\alpha (L_n f_n)(t)|^2.\]

Then by applying (2.11) we obtain
\[\left\|\partial_t^\alpha \int_0^t K(t - s) F(s) ds\right\|_{H_\alpha(0, T; L^2(\Omega))}^2 \leq C \left\|\partial_t^\alpha \int_0^t K(t - s) F(s) ds\right\|_{L^2(0, T; L^2(\Omega))}^2 \]
\[= C \sum_{n=1}^{\infty} \|\partial_t^\alpha (L_n f_n)\|_{L^2(0, T)}^2 \leq C \sum_{n=1}^{\infty} \|L_n f_n\|_{H_\alpha(0, T)}^2 \]
\[\leq C \sum_{n=1}^{\infty} \|f_n\|_{L^2(0, T)}^2 = C \int_0^T \sum_{n=1}^{\infty} |(F(s), \varphi_n)|^2 ds \]
\[= C \int_0^T \|F(s)\|_{L^2(\Omega)}^2 ds = C \|F\|_{L^2(0, T; L^2(\Omega))}^2.\]

Thus the proof of the estimate (ii) is completed.

The estimate (iii) in Lemma 2.1 follows from the standard estimates of the operator \(S(t)\) or can be derived by the same arguments as those that were employed in Section 6 of Chapter 4 in [16] for the case of the homogeneous Dirichlet boundary condition and we omit the technical details.

Now we proceed with the **Proof of Theorem 2.1**.
In the equation (2.12), we regard the terms $\sum_{j=1}^{d} b_j(x,t)\partial_j u$ and $(c_0 + c(x,t))u$ as non-homogeneous terms and rewrite it in terms of the operator $A_0$ as follows
\[
\begin{cases}
\partial_t^\alpha (u - a) + A_0u(x,t) = F(x,t) \\
\sum_{j=1}^{d} b_j(x,t)\partial_j u + (c_0 + c(x,t))u, \quad \forall x \in \Omega, 0 < t < T, \\
\partial_{\nu_A} u + \sigma(x)u = 0 \quad \text{on } \partial \Omega \times (0, T), \\
u(x, \cdot) - a(x) \in H_\alpha(0, T) \quad \text{for almost all } x \in \Omega.
\end{cases}
\] (2.12)

Then we represent the equation from (2.12) in the form ([11], [16])
\[
u(t) = S(t)a + \int_0^t K(t - s)F(s)ds + \int_0^t K(t - s)\left(\sum_{j=1}^{d} b_j(s)\partial_j u(s) + (c_0 + c(s))u(s)\right)ds, \quad 0 < t < T.
\] (2.13)

Moreover, it is known that if $u \in L^2(0, T; H^2(\Omega))$ satisfies the initial condition $u - a \in H_\alpha(0, T; L^2(\Omega))$ and the equation (2.13), then $u$ is a solution to the problem (2.12). With the notations
\[
\begin{align*}
G(t) &:= \int_0^t K(t - s)F(s)ds + S(t)a, \\
Qu(t) &:= Q(t)u(t) := \sum_{j=1}^{d} b_j(\cdot,t)\partial_j u(t) + (c_0 + c(\cdot,t))u(t), \\
Ru(t) &:= \int_0^t K(t - s)\left(\sum_{j=1}^{d} b_j(\cdot,s)\partial_j u(s) + (c_0 + c(\cdot,s))u(s)\right)ds,
\end{align*}
\] (2.14)

the equation (2.13) can be represented in form of a fixed point equation $u = Ru + G$ in $L^2(0, T; H^2(\Omega))$.

Lemma 2.1 yields that $G \in L^2(0, T; H^2(\Omega))$. Moreover, since $\|A_0^{\frac{1}{2}}a\| \leq C\|a\|_{H^1(\Omega)}$ and $\mathcal{D}(A_0^{\frac{1}{2}}) = H^1(\Omega)$ (e.g., [8]), the estimate (2.8) implies
\[
\|S(t)a\|_{H^2(\Omega)} \leq C\|A_0 S(t)a\| = C\|A_0^{\frac{1}{2}}S(t)A_0^{\frac{1}{2}}a\| \leq CT^{-\frac{\alpha}{2}}\|a\|_{H^1(\Omega)}
\]
and thus
\[
\|S(t)a\|_{L^2(0,T;H^2(\Omega))}^2 \leq C \left(\int_0^T t^{-\alpha}dt\right) \|a\|_{H^1(\Omega)}^2 \leq \frac{CT^{1-\alpha}}{1-\alpha} \|a\|_{H^1(\Omega)}^2.
\]

Consequently, the inclusion $S(t)a \in L^2(0, T; H^2(\Omega))$ holds valid.

For $0 < t < T$, we next estimate $\|Rv(\cdot,t)\|_{H^2(\Omega)}$ for $v(\cdot,t) \in \mathcal{D}(A_0)$ as follows:
\[
\|Rv(\cdot,t)\|_{H^2(\Omega)} \leq C\|A_0 Rv(\cdot,t)\|_{L^2(\Omega)}
\]
\[
\leq \int_0^t \|A_0^{\frac{1}{2}}K(t - s)A_0^{\frac{1}{2}}\left(\sum_{j=1}^{d} b_j(s)\partial_j u(s) + (c_0 + c(s))u(s)\right)\|ds
\]
\[
\leq C \int_0^t \|A_0^{\frac{1}{2}}K(t - s)\| \|A_0^{\frac{1}{2}}\left(\sum_{j=1}^{d} b_j(s)\partial_j u(s) + (c_0 + c(s))u(s)\right)\|ds
\]
\[
\leq C \int_0^t (t - s)^{\frac{\alpha}{2} - 1}\|v(s)\|_{H^2(\Omega)}ds = C \left(\Gamma\left(\frac{1}{2}\alpha\right) J^{\frac{1}{2}\alpha}\right) \|v\|_{H^2(\Omega)}(t).
\]
For derivation of this estimate, we employed the inequalities
\[
\|A_j b_j(s)\partial_j v(t)\| \leq C\|b_j(s)\partial_j v(s)\|_{H^1(\Omega)} \leq C\|v(s)\|_{H^2(\Omega)},
\]
\[
\|(c(s) + c_0)v(s)\|_{H^1(\Omega)} \leq C\|v(s)\|_{H^2(\Omega)}
\]
that hold true because \(b_j \in C^1(\Omega \times [0, T])\) and \(c + c_0 \in C([0, T]; C^1(\Omega))\).

Since \(J^\frac{\alpha}{2} w_1(t) \geq (J^\frac{\alpha}{2} w_2)(t)\) if \(w_1(t) \geq w_2(t)\) for \(0 \leq t \leq T\), and \(J^\frac{\alpha}{2} J^\frac{\alpha}{2} w = J^\alpha w\) for \(w_1, w_2, w \in L^2(0, T)\), we have
\[
\|R^2 v(t)\|_{H^2(\Omega)} = \|R(Rv)(t)\|_{H^2(\Omega)} \leq C \left( \Gamma \left( \frac{1}{2} \right) J^\frac{\alpha}{2} \left( CT \left( \frac{1}{2} \alpha \right) J^\frac{\alpha}{2} \|v\|_{H^2(\Omega)} \right) \right)(t)
\]
\[
= \left( CT \left( \frac{1}{2} \alpha \right) \right)^2 (J^\alpha \|v\|_{H^2(\Omega)})(t).
\]

Repeating this argumentation \(m\)-times, we obtain
\[
\|R^m v(t)\|_{H^2(\Omega)} \leq \left( CT \left( \frac{1}{2} \alpha \right) \right)^m \left( J^\frac{\alpha m}{2} \|v\|_{H^2(\Omega)} \right)(t)
\]
\[
\leq \frac{(CT)^m \left( \frac{1}{2} \alpha \right)^m}{\Gamma \left( \frac{1}{2} \alpha m + 1 \right)} \int_0^t (t - s)^{\frac{m}{2} - 1}\|v(s)\|_{H^2(\Omega)} ds, \quad 0 < t < T.
\]

Applying the Young inequality to the integral in the last estimate, we reach the inequality
\[
\|R^m v(t)\|^2_{L^2(0,T;H^2(\Omega))} \leq \left( \frac{(CT)^m \left( \frac{1}{2} \alpha \right)^m}{\Gamma \left( \frac{1}{2} \alpha m + 1 \right)} \right)^2 \|t^{\frac{m}{2} - 1}\|_{L^1(0,T)}^2 \|v\|^2_{L^2(0,T;H^2(\Omega))}
\]
\[
= \frac{(CT)^{2m} \left( \frac{1}{2} \alpha \right)^{2m}}{\Gamma \left( \frac{1}{2} \alpha m + 1 \right)^2} \|v\|^2_{L^2(0,T;H^2(\Omega))}.
\]

Employing the known asymptotic behavior of the gamma function, we obtain the relation
\[
\lim_{m \to \infty} \frac{(CT)^m \left( \frac{1}{2} \alpha \right)^m}{\Gamma \left( \frac{1}{2} \alpha m + 1 \right)} = 0
\]
that means that for sufficiently large \(m \in \mathbb{N}\), the mapping
\[
R^m : L^2(0,T;H^2(\Omega)) \to L^2(0,T;H^2(\Omega))
\]
is a contraction. Hence, by the Banach fixed point theorem, the equation (2.13) possesses a unique fixed point. Therefore, by the first equation in (2.3), we reach the inclusion \(\partial_t^\alpha (u - a) \in L^2(0,T;L^2(\Omega))\). Since \(\|\eta\|_{H_a(0,T)} \sim \|\partial_t^\alpha \eta\|_{L^2(0,T)}\) for \(\eta \in H_a(0,T)\) (16), we finally obtain the estimate
\[
\|u - a\|_{H_a(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^2(\Omega))} \leq C(\|a\|_{H^1(\Omega)} + \|F\|_{L^2(0,T;L^2(\Omega))}).
\]
The proof of Theorem 2.1 is completed.\[\square\]
3 Key lemma

For derivation of the comparison principles for solutions to the linear and semilinear time-fractional diffusion equations, we need some auxiliary results that are formulated and proved in this section.

In addition to the operator $-A_0$ defined by (2.5), we define an elliptic operator $-A_1$ with a positive zeroth-order coefficient:

$(-A_1(t)v)(x) := (-A_1v)(x)$

where $b_0(x,t) > 0$, $b_0(x,t) \in C^1([0,T]; C^1(\Omega)) \cap C([0,T]; C^2(\Omega))$, and $\min_{(x,t) \in \overline{\Omega} \times [0,T]} b_0(x,t)$ is sufficiently large. We recall that the pointwise Caputo derivative $d_\alpha^t$ is defined by

$$d_\alpha^t y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{dy}{ds}(s) ds \quad \text{for} \quad y \in W^{1,1}(0,T).$$

Henceforth we write $y'(t) = \frac{dy}{dt}(t)$ if there is no fear of confusion.

In what follows, we employ an extremum principle for the Caputo fractional derivative formulated below.

**Lemma 3.1 ([19]).** We assume that $y \in C[0,T]$ and $t^{1-\alpha}y' \in C[0,T]$. If $y = y(t)$ attains its minimum over the interval $[0,T]$ at the point $t_0 \in (0,T]$, then

$$d_\alpha^t y(t_0) \leq 0.$$

In the lemma, the assumption $t_0 > 0$ is essential.

In [19], Lemma 3.1 was formulated and proved under a weaker regularity condition posed on the function $y$, but for our arguments we can assume that $y \in C[0,T]$ and $t^{1-\alpha}y' \in C[0,T]$.

Employing Lemma 3.1 we now prove our key lemma that is a basis for further arguments in this article.

**Lemma 3.2 (Positivity of a smooth solution).** We assume that $\min_{(x,t) \in \overline{\Omega} \times [0,T]} b_0(x,t)$ is a sufficiently large positive constant. For $a \in H_0^1(\Omega)$ and $F \in L^2(0,T; L^2(\Omega))$, we assume that there exists a solution $u \in C([0,T]; C^2(\Omega))$ satisfying $t^{1-\alpha}\partial_t u \in C([0,T]; C(\Omega))$ to the initial-boundary value problem

$$\begin{cases}
\partial_\alpha^t (u - a) + A_1 u = F(x,t), & x \in \Omega, \ 0 < t < T, \\
\partial_{\nu_A} u + \sigma(x) u = 0 \quad \text{on} \ \partial \Omega \times (0,T), \\
u(x, \cdot) - a(x) \in H_\alpha(0,T) \quad \text{for almost all} \ x \in \Omega.
\end{cases}$$

If $F \geq 0$ in $\Omega \times (0,T)$ and $a \geq 0$ in $\Omega$, then

$$u \geq 0 \quad \text{in} \ \Omega \times (0,T).$$
We note that the regularity of the solution to the problem (3.3) at the point \( t = 0 \) is a more delicate question compared to the case \( \alpha = 1 \). In particular, we cannot expect that \( u(x, \cdot) \in C^1(0, T) \). This can be illustrated by a simple example of the equation \( \partial_t^\alpha y(t) = y(t) \) with \( y(t) - 1 \in H_\alpha(0, T) \) whose unique solution \( y(t) = E_{\alpha, 1}(t^\alpha) \) does not belong to the space \( C^1[0, T] \).

For the case of \( \alpha = 1 \), a similar positivity property for the time-fractional diffusion equation under the Robin boundary condition is well-known.

**Proof.** First we introduce an auxiliary function \( \psi \in C^1([0, T]; C^2(\Omega)) \) that satisfies the conditions

\[
\begin{cases}
A_1 \psi(x, t) = 1, & (x, t) \in \Omega \times [0, T], \\
\partial_{v, \alpha} \psi + \sigma \psi = 1 & \text{on } \partial \Omega \times [0, T].
\end{cases}
\]

(3.4)

The proof of the existence of such \( \psi \) is postponed to Appendix.

Now, choosing \( M > 0 \) sufficiently large and \( \varepsilon > 0 \) sufficiently small, we set

\[
w(x, t) := u(x, t) + \varepsilon(M + \psi(x, t) + t^\alpha), \quad x \in \Omega, \, 0 < t < T.
\]

(3.5)

For a fixed \( x \in \Omega \), by the assumption on the regularity of \( u \), we have

\[
t^{1-\alpha} \partial_t u(x, \cdot) \in C[0, T].
\]

(3.6)

Then, \( \partial_t u(x, \cdot) \in L^1(0, T) \), that is, \( u(x, \cdot) \in W^{1,1}(0, T) \). Moreover,

\[
u(x, 0) - a(x) = 0, \quad x \in \Omega.
\]

(3.7)

On the other hand, we have

\[
\partial_t^\alpha w = d_t^\alpha w = d_t^\alpha (w + c)
\]

with any constant \( c \), if \( w \in H_\alpha(0, T) \cap W^{1,1}(0, T) \) and \( w(0) = 0 \) (e.g., Theorem 2.4 of Chapter 2 in [16]).

Since \( u(x, \cdot) - a \in H_\alpha(0, T) \) and \( u(x, \cdot) \in W^{1,1}(0, T) \) for almost all \( x \in \Omega \), by (3.7) we see \( \partial_t^\alpha (u - a) = d_t^\alpha (u - a) = d_t^\alpha u \).

Furthermore, since \( \varepsilon(M + \psi(\cdot, t) + t^\alpha) \in W^{1,1}(0, T) \), we obtain

\[
d_t^\alpha w = d_t^\alpha (u + \varepsilon(M + \psi(x, t) + t^\alpha)) = d_t^\alpha u + \varepsilon d_t^\alpha (M + \psi(x, t) + t^\alpha)
\]

\[
= \partial_t^\alpha (u - a) + \varepsilon (d_t^\alpha (\psi + t^\alpha)) = \partial_t^\alpha (u - a) + \varepsilon (d_t^\alpha \psi + \Gamma(\alpha + 1))
\]

and

\[
A_1 w = A_1 u + \varepsilon A_1 \psi + \varepsilon A_1 t^\alpha + \varepsilon A_1 M
\]

\[
= A_1 u + \varepsilon + \varepsilon b_0(x, t) t^\alpha + b_0(x, t) \varepsilon M.
\]

Now we choose a constant \( M > 0 \) such that \( M + \psi(x, t) \geq 0 \) and \( d_t^\alpha \psi(x, t) + b_0(x, t) M > 0 \) for \( (x, t) \in \Omega \times [0, T] \), so that

\[
d_t^\alpha w + A_1 w = F + \varepsilon (\Gamma(\alpha + 1) + d_t^\alpha \psi + 1 + b_0(x, t) t^\alpha + b_0(x, t) M) > 0 \quad \text{in } \Omega \times (0, T).
\]

(3.8)
Moreover, because of the relation \( \partial_{v_A} w = \partial_{v_A} u + \varepsilon \partial_{v_A} \psi \), we obtain the following estimate:

\[
\partial_{v_A} w + \sigma w = \partial_{v_A} u + \sigma u + \varepsilon \varepsilon t^0 + \varepsilon M \geq \varepsilon + \varepsilon \varepsilon t^0 + \varepsilon M \geq \varepsilon \text{ on } \partial \Omega \times (0, T).
\]  

(3.9)

Evaluating the representation (3.5) at the point \( t = 0 \) immediately leads to the formula

\[
w(x, 0) = u(x, 0) + \varepsilon (\psi(x, 0) + M), \quad x \in \Omega.
\]

Let us assume that the inequality

\[
\min_{(x, t) \in \overline{\Omega} \times [0, T]} w(x, t) \geq 0
\]

does not hold, that is, there exists \((x_0, t_0) \in \overline{\Omega} \times [0, T]\) such that

\[
w(x_0, t_0) := \min_{(x, t) \in \overline{\Omega} \times [0, T]} w(x, t) < 0.
\]  

(3.10)

Since \( M > 0 \) is sufficiently large, by \( u(x, 0) \geq 0 \), we have

\[
w(x, 0) = u(x, 0) + \varepsilon (\psi(x, 0) + M) \geq u(x, 0) \geq 0, \quad x \in \overline{\Omega},
\]

and thus \( t_0 \neq 0 \) has to be valid.

Next we show that \( x_0 \notin \partial \Omega \). Indeed, let us assume that \( x_0 \in \partial \Omega \). Then the estimate (3.9) yields that \( \partial_{v_A} w(x_0, t_0) + \sigma(x_0)w(x_0, t_0) \geq \varepsilon \). By (3.10) and \( \sigma(x_0) \geq 0 \), we obtain

\[
\partial_{v_A} w(x_0, t_0) \geq -\sigma(x_0)w(x_0, t_0) + \varepsilon \geq \varepsilon > 0,
\]

which implies

\[
\partial_{v_A} w(x_0, t_0) = \sum_{i,j=1}^d a_{ij}(x_0) \nu_j(x_0) \partial_i w(x_0, t_0) = \nabla w(x_0, t_0) \cdot A(x_0) \nu(x_0)
\]

\[
= \sum_{i=1}^d (\partial_i w)(x_0, t_0)[A(x_0) \nu(x_0)]_i > 0.
\]  

(3.11)

Here \( A(x) = (a_{ij}(x))_{1 \leq i, j \leq d} \) and \([b]_i \) means the \( i \)-th element of a vector \( b \).

For sufficiently small \( \varepsilon_0 > 0 \) and \( x_0 \in \partial \Omega \), we now verify

\[
x_0 - \varepsilon_0 A(x_0) \nu(x_0) \in \Omega.
\]  

(3.12)

Indeed, since the matrix \( A(x_0) \) is positive-definite, the inequality

\[
(\nu(x_0) \cdot -\varepsilon_0 A(x_0) \nu(x_0)) = -\varepsilon_0 (A(x_0) \nu(x_0) \cdot \nu(x_0)) < 0
\]

holds true. In other words, we have

\[
\angle(\nu(x_0), (x_0 - \varepsilon_0 A(x_0) \nu(x_0)) - x_0) > \frac{\pi}{2}.
\]
Because the boundary $\partial \Omega$ is smooth, the domain $\Omega$ is locally located on one side of $\partial \Omega$. In a small neighborhood of the point $x_0 \in \partial \Omega$, the boundary $\partial \Omega$ can be described in the local coordinates composed of its tangential component in $\mathbb{R}^{d-1}$ and the normal component along $\nu(x_0)$. Consequently, if $y \in \mathbb{R}^d$ satisfies $\angle(\nu(x_0), y - x_0) > \frac{\pi}{2}$, then $y \in \Omega$. Therefore, for a sufficiently small $\varepsilon_0 > 0$, we have $x_0 - \varepsilon_0 A(x_0) \nu(x_0) \in \Omega$ and we proved the formula (3.12).

Moreover, for sufficiently small $\varepsilon_0 > 0$, we can prove that

$$w(x_0 - \varepsilon_0 A(x_0) \nu(x_0), t_0) < w(x_0, t_0).$$  

(3.13)

**Verification of (3.13).**

Indeed, the inequality (3.10) yields

$$\sum_{i=1}^d (\partial_i w)(x_0 - \eta A(x_0) \nu(x_0), t_0)[A(x_0) \nu(x_0)]_i > 0 \quad \text{if } |\eta| < \varepsilon_0.$$  

Then, by the mean value theorem, we obtain the inequality

$$w(x_0 - \xi A(x_0) \nu(x_0), t_0) - w(x_0, t_0) = \xi \sum_{i=1}^d \partial_i w(x_0 - \theta A(x_0) \nu(x_0), t_0)(- [A(x_0) \nu(x_0)]_i) < 0,$$

where $\theta$ is a number between 0 and $\xi \in (0, \varepsilon_0)$. Thus (3.13) is verified. $\square$

By combining (3.13) with (3.12), we conclude that there exists $\tilde{x}_0 \in \Omega$ such that $w(\tilde{x}_0, t_0) < w(x_0, t_0)$, which contradicts the assumption (3.10) and we have proved that $x_0 \notin \partial \Omega$.

By the definition, the function $w$ attains its minimum at the point $(x_0, t_0)$. Because $0 < t_0 \leq T$, Lemma 3.1 yields the inequality

$$d_t^2 w(x_0, t_0) \leq 0.$$  

(3.14)

Since $x_0 \in \Omega$, the necessary condition for an extremum point leads to the equality

$$\nabla w(x_0, t_0) = 0.$$  

(3.15)

Moreover, because $w$ attains its minimum at the point $x_0 \in \Omega$, in view of the sign of the Hessian, the inequality

$$\sum_{i,j=1}^d a_{ij}(x_0) \partial_i \partial_j w(x_0, t_0) \geq 0$$  

(3.16)

holds true (see e.g. the proof of Lemma 1 in Section 1 of Chapter 2 in [7]).

The inequalities $b(x_0, t_0) > 0$, $w(x_0, t_0) < 0$, and (3.14)-(3.16) let to derive the inequality

$$d_t^2 w(x_0, t_0) + A_1 w(x_0, t_0) = d_t^2 w(x_0, t_0) - \sum_{i,j=1}^d a_{ij}(x_0) \partial_i \partial_j w(x_0, t_0) - \sum_{i=1}^d (\partial_i a_{ij})(x_0) \partial_j w(x_0, t_0) - \sum_{i=1}^d b_i(x_0, t_0) \partial_i w(x_0, t_0) + b(x_0, t_0) w(x_0, t_0) < 0,$$

(3.17)
which contradicts the inequality \((3.8)\).

We thus have proved that

\[
  u(x, t) + \varepsilon(M + \psi(x, t) + t^\alpha) = w(x, t) \geq 0, \quad (x, t) \in \Omega \times (0, T).
\]

Since \(\varepsilon > 0\) is arbitrary, we let \(\varepsilon \downarrow 0\) to obtain the inequality \(u(x, t) \geq 0\) for \((x, t) \in \Omega \times (0, T)\) and the proof of Lemma 3.2 is completed.

Let us finally mention that the positivity of the function \(b_0\) in the operator \(-A_1\) is an essential condition for validity of our proof of Lemma 3.2. However, in the next section, we remove this condition while deriving the comparison principle for the solutions to the linear time-fractional diffusion equations.

4 Comparison principles for the linear time-fractional diffusion equations

According to the results formulated in Theorem 2.1, in this section, we consider the solutions to the initial-boundary value problem (2.4) within the class

\[
  \{u; u - a \in H_{\alpha}(0, T; L^2(\Omega)), u \in L^2(0, T; H^2(\Omega))\}.
\] (4.1)

By \(a \geq 0\), we always mean that the inequality \(a \geq 0\) holds almost everywhere in a set under consideration.

Our first main result concerning the comparison principle for the linear time-fractional diffusion equations is as follows:

**Theorem 4.1.** Let \(a \in H^1_0(\Omega)\) and \(F \in L^2(\Omega \times (0, T))\). If \(F(x, t) \geq 0\) in \(\Omega \times (0, T)\) and \(a(x) \geq 0\) in \(\Omega\), then \(u(F, a)(x, t) \geq 0\) in \(\Omega \times (0, T)\), where by \(u(F, a)\) we denote the solution \(u\) in the class (4.1) to the initial-boundary value problem (2.4) in the class (4.1) with the initial condition \(a\) and the right-hand side \(F\).

We emphasize that the non-negativity of solution \(u\) for (2.4) holds for general class (4.1), and \(u\) does not necessarily satisfy \(u \in C([0, T]; C^2(\overline{\Omega}))\) and \(t^{1-\alpha} \partial_t u \in C([0, T]; C(\Omega))\). Thus Theorem 4.1 is widely applicable.

Theorem 4.1 immediately yields the following comparison property:

**Corollary 4.1.** Let \(a_1, a_2 \in H^1_0(\Omega)\) and \(F_1, F_2 \in L^2(\Omega \times (0, T))\) satisfy \(a_1(x) \geq a_2(x)\) in \(\Omega\) and \(F_1(x, t) \geq F_2(x, t)\) in \(\Omega \times (0, T)\), respectively. Then \(u(F_1, a_1)(x, t) \geq u(F_2, a_2)(x, t)\) in \(\Omega \times (0, T)\).

**Proof.** We first prove Corollary 4.1. Setting \(a := a_1 - a_2\), \(F := F_1 - F_2\) and \(u := u(F_1, a_1) - u(F_2, a_2)\), we obtain that \(a \geq 0\) in \(\Omega\) and \(F \geq 0\) in \(\Omega \times (0, T)\) and

\[
  \begin{cases}
    \partial^\alpha_t (u - a) + Au = F \geq 0 & \text{in } \Omega \times (0, T), \\
    \partial_{\nu} a + \sigma u = 0 & \text{on } \partial \Omega.
  \end{cases}
\]

Therefore, Theorem 4.1 implies that \(u \geq 0\), that is, \(u(F_1, a_1) \geq u(F_2, a_2)\) in \(\Omega \times (0, T)\).  

\[\square\]
We apply Corollary 4.1 to derive lower and upper bounds for the solution $u$ by choosing initial values and non-homogeneous terms suitably. Here we are restricted to demonstrate only one example.

**Example 4.1.** Let

$$-Av(x) = \sum_{i,j=1}^{d} \partial_i(a_{ij}(x)\partial_j v(x)) + \sum_{j=1}^{d} b_j(x,t)\partial_j v(x),$$

where the coefficients $a_{ij}, b_j$, $1 \leq i,j \leq d$ satisfy (1.2). For constants $\beta \geq 0$ and $\delta > 0$, we assume that $a = 0$ in $\Omega$ and $F \in L^2(0,T;L^2(\Omega))$ satisfies

$$F(x,t) \geq \delta t^\beta, \quad x \in \Omega, 0 < t < T.$$  

Then

$$u(F,0)(x,t) \geq \frac{\delta \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} t^\alpha t^\beta, \quad x \in \Omega, 0 \leq t \leq T. \quad (4.2)$$

Indeed, setting

$$\underline{u}(x,t) := \frac{\delta \Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)} t^\alpha t^\beta, \quad x \in \Omega, t > 0,$$

we see that

$$\begin{cases} 
\partial_t^\alpha \underline{u} + A\underline{u} = \delta t^\beta & \text{in } \Omega \times (0,T), \\
\partial_{\nu} \underline{u} = 0 & \text{on } \partial \Omega \times (0,T), \\
\underline{u}(x,\cdot) \in H_\alpha(0,T). 
\end{cases}$$

By $F \geq \delta t^\beta$ in $\Omega \times (0,T)$, we apply Corollary 4.1 to $u$ and $\underline{u}$, and we can conclude (1.2).

In particular, noting that $u \in L^2(0,T;H^2(\Omega)) \subset L^2(0,T;C(\overline{\Omega}))$ by the Sobolev embedding and the spatial dimensions $d \leq 3$, we see that $u(F,0)(x,t) > 0$ for almost all $t > 0$ and all $x \in \overline{\Omega}$.

Now we proceed to the proof of Theorem 4.1.

**Proof.** We use the operators $Qu(t)$ and $G(t)$ defined by (2.14). In terms of these operators, the solution $u(t) := u(F,a)(t)$ satisfies the equation

$$u(F,a)(t) = G(t) + \int_{0}^{t} K(t-s)Qu(s)ds, \quad 0 < t < T. \quad (4.3)$$

We divide the proof of Theorem 4.1 into three parts.

(I) **First part of the proof of Theorem 4.1** existence of smoother solution.

In the formulation of Lemma 3.2 we assumed the existence of a solution $u$ to the initial-boundary value problem (3.3) that satisfies the inclusions $u \in C([0,T];C^2(\overline{\Omega}))$ and $t^{1-\alpha}\partial_t u \in C([0,T];C(\overline{\Omega}))$. On the other hand, Theorem 2.1 asserts the unique existence of solution $u$ to the initial-boundary value problem (2.1) in a class $u(F,a) \in L^2(0,T;H^2(\Omega))$ and $u(F,a) - a \in H_\alpha(0,T);L^2(\Omega))$.

In this part of the proof, we lift up the regularity of the solution $u \in L^2(0,T;H^2(\Omega)) \cap (H_\alpha(0,T);L^2(\Omega)) + \{a\}$ of the problem (3.3) with $a \in C_0^\infty(\Omega)$ and $F \in C_0^\infty(\Omega \times (0,T))$ to the inclusion $u \in C([0,T];C^2(\overline{\Omega}))$ satisfying the condition $t^{1-\alpha}\partial_t u \in C([0,T];C(\overline{\Omega}))$.

More precisely, we can state
Lemma 4.1. Let $a_{ij},$ $b_j,$ $c$ satisfy the condition \([1.2]\) and the inclusions $a \in C^0_0(\Omega),$ $F \in C^\infty_0(\Omega \times (0,T))$ hold true. Then the solution $u = u(F,a)$ to the problem \([2.3]\) satisfies

$$u \in C([0,T]; C^2(\Omega)), \quad t^{1-\alpha}\partial_t u \in C([0,T]; C(\overline{\Omega})).$$

**Proof.** We recall that $c_0 > 0$ is a positive fixed constant and

$$-A_0 v = \sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j v) - c_0 v, \quad \mathcal{D}(A_0) = \{v \in H^2(\Omega); \partial_{\nu_\alpha} v + \sigma v = 0 \text{ on } \partial\Omega\}.$$

Then $\mathcal{D}(A_0^\frac{1}{2}) = H^1(\Omega)$ and $\|A_0^\frac{1}{2} v\| \sim \|v\|_{H^1(\Omega)}$ \([8]\). Moreover, the estimates \([2.8]\) hold for the operators $S(t)$ and $K(t)$ defined by \([2.6]\) and \([2.7]\).

Here we write $u'(t) = \frac{d}{dt} u(t) = \frac{\partial u(t)}{\partial t}$ if there is no fear of confusion.

The relation \([4.3]\) shows that its solution $u(t)$ can be constructed as a fixed point of the equation

$$A_0 u(t) = A_0 G(t) + \int_0^t A_0^\frac{1}{2} K(t-s) A_0^\frac{1}{2} Q(s) u(s) ds, \quad 0 < t < T. \quad (4.4)$$

As already proved, the fixed point $u$ satisfies $u \in L^2(0,T; H^2(\Omega)) \cap (H_0(0,T; L^2(\Omega)) + \{a\}).$

We start with the derivation of some estimates for $\|A_0^\kappa u(t)\|$, $\kappa = 0, 1$ and $\|A_0 u'(t)\|$ for $0 < t < T$. We set

$$D := \sup_{0 < t < T} \left( \|A_0 F(t)\| + \|A_0 F'(t)\| + \|A_0^4 F(t)\| + \|a\|_{H^4(\Omega)} \right).$$

Since $F \in C^\infty_0(\Omega \times (0,T))$, we see $F \in L^\infty(0,T; \mathcal{D}(A_0^\frac{1}{2}))$ and $D < \infty$. Moreover, in view of \([2.8]\), for $\kappa = 1, 2$, we can estimate

$$\left\| A_0^\kappa \int_0^t K(t-s) F(s) ds \right\| \leq C \int_0^t \|K(t-s)\| A_0^\kappa F(s) ds \leq C \int_0^t \|K(t-s)\| A_0^\kappa F(s) ds \leq C (\int_0^t (t-s)^{\alpha-1} ds) \sup_{0 < s < T} \|A_0^\kappa F(s)\| \leq CD,$$

where $\alpha = \min\{\alpha, 1\}$.

$$\left\| A_0 \frac{d}{dt} \int_0^t K(t-s) F(s) ds \right\| = \left\| A_0 \int_0^t K(s) F(t-s) ds \right\|$$

$$\leq C \left\| A_0 \int_0^t K(s) F'(t-s) ds \right\| + C \left\| A_0 \int_0^t K(s) F(t-s) ds \right\|$$

$$\leq C \left\| A_0 \int_0^t K(s) F'(t-s) ds \right\| \leq C \int_0^t s^{\alpha-1} \|A_0 F'(t-s)\| ds < CD, \quad \text{etc.}$$

The regularity conditions in \([1.2]\) lead to the estimates

$$\|A_0^\frac{1}{2} Q(s) u(s)\| \leq C \|Q(s) u(s)\|_{H^1(\Omega)}$$

$$= C \left\| \sum_{j=1}^d b_j(s) \partial_j u(s) + (c_0 + c(s)) u(s) \right\|_{H^1(\Omega)}$$

$$\leq C \left\| A_0^\frac{1}{2} F(s) u(s) \right\|_{H^1(\Omega)}.$$
Moreover, \( \|A_0 S(t) a\| = \|S(t) A_0 a\| \leq C \|a\|_{H^2(\Omega)} \leq CD \) by (2.8). Then
\[
\|A_0 u(t)\| \leq CD + \int_0^t \|A_0^{\frac{3}{2}} K(t-s)\| \|A_0^{\frac{3}{2}} Q(s) u(s)\| ds \leq CD + C \int_0^t (t-s)^{\frac{3}{2}} \|A_0 u(s)\| ds, \quad 0 < s < T.
\]

The generalized Gronwall inequality yields
\[
\|A_0 u(t)\| \leq CD + C \int_0^t (t-s)^{\frac{3}{2}} D ds \leq CD, \quad 0 < t < T,
\]
which implies
\[
\|A_0 u\|_{L^\infty(0,T;H^2(\Omega))} \leq CD.
\]

Moreover, we can repeat the same arguments in the space \( C([0,T];L^2(\Omega)) \) as the ones employed for the iterations \( R^n \) of the operator in the proof of Theorem 2.1 and apply the fixed point theorem to the equation (4.3), so that \( A_0 u \in C([0,T];L^2(\Omega)) \). Therefore we reach
\[
u \in C([0,T];H^2(\Omega)), \quad \|u\|_{C([0,T];H^2(\Omega))} \leq CD. \tag{4.6}
\]

Choosing \( \varepsilon_0 > 0 \) sufficiently small, we have
\[
A_0^{\frac{3}{2}} u(t) = A_0^{\frac{3}{2}} G(t) + \int_0^t A_0^{\frac{4}{2}+\varepsilon_0} K(t-s) A_0^{\frac{4}{2}+\varepsilon_0} Q(s) u(s) ds, \quad 0 < t < T. \tag{4.7}
\]

Next, according to [8], the inclusion
\[
\mathcal{D}(A_0^{\frac{3}{2}-\varepsilon_0}) \subset H^{\frac{3}{2}-2\varepsilon_0}(\Omega)
\]
holds true. Now we proceed to the proof of the inclusion \( Q(s) u(s) \in \mathcal{D}(A_0^{\frac{3}{2}-\varepsilon_0}) \). By (2.8), we obtain the inequality
\[
\|A_0^{\frac{3}{2}} u(t)\| \leq CD + \int_0^t (t-s)^{\frac{3}{2}+\varepsilon_0 - \frac{1}{2}} \|A_0^{\frac{3}{2}-\varepsilon_0} Q(s) u(s)\| ds,
\]
which leads to the estimate
\[
\|u(t)\|_{H^3(\Omega)} \leq CD + \int_0^t (t-s)^{\frac{3}{2}+\varepsilon_0 - \frac{1}{2}} \|u(s)\|_{H^3(\Omega)} ds, \quad 0 < t < T.
\]

In the last estimate, we employed the inequality
\[
\|A_0^{\frac{3}{2}-\varepsilon_0} Q(s) u(s)\| \leq C \|Q(s) u(s)\|_{H^\frac{3}{2}(\Omega)} \leq C \|Q(s) u(s)\|_{H^2(\Omega)} \leq C \|u(s)\|_{H^3(\Omega)},
\]
which follows from by the regularity conditions (1.2) posed on the coefficients \( b_j, c \).
The generalized Gronwall inequality yields the estimate
\[ \|u(t)\|_{H^3(\Omega)} \leq C \left( 1 + t^{\alpha(\frac{1}{2} - \varepsilon_0)} \right) D \]
for \(0 < t < T\).

For the relation (4.7), we repeat the same arguments in \(C([0,T]; L^2(\Omega))\) as the ones employed in the proof of Theorem 2.1 to estimate \(A_{0}^{\frac{3}{2}} u(t)\) in the norm \(C([0,T]; L^2(\Omega))\) by the fixed point theorem and reach the inclusion \(A_{0}^{\frac{3}{2}} u \in C([0,T]; L^2(\Omega))\).

Collecting the last estimates, we obtain
\[
\begin{align*}
\left\{ \begin{array}{l}
\quad u \in C([0,T]; D(A_{0}^{\frac{3}{2}})) \subset C([0,T]; H^3(\Omega)), \\
\quad \|u(t)\|_{H^3(\Omega)} \leq C \left( 1 + t^{\alpha(\frac{1}{2} - \varepsilon_0)} \right) D, \quad 0 < t < T. 
\end{array} \right.
\end{align*}
\]
(4.8)

Next we estimate \(\|A_0 u'(t)\|\). First we represent \(u'(t)\) in the form
\[
\begin{align*}
u'(t) &= G'(t) + \frac{d}{dt} \int_{0}^{t} K(t-s)Q(s)u(s)ds \\
&= G'(t) + \frac{d}{dt} \int_{0}^{t} K(s)Q(t-s)u(t-s)ds \\
&= G'(t) + K(t)Q(0)u(0) \\
&\quad + \int_{0}^{t} K(s)(Q(t-s)u'(t-s) + Q'(t-s)u(t-s))ds, \quad 0 < t < T,
\end{align*}
\]
so that
\[
\begin{align*}
A_0 u'(t) &= A_0 G'(t) + A_0 K(t)Q(0)u(0) \\
&\quad + \int_{0}^{t} A_0^\frac{1}{2} K(s)A_0^\frac{1}{2} (Q(t-s)u'(t-s) + Q'(t-s)u(t-s))ds, \quad 0 < t < T.
\end{align*}
\]
(4.9)

Similarly to the arguments applied for derivation of (4.5), we obtain the inequality
\[ \|A_{0}^{\frac{3}{2}} (Q(t-s)u'(t-s) + Q'(t-s)u(t-s))\| \leq C \|A_{0} u'(t-s)\|, \quad 0 < t < T. \]

The inclusion \(Q(0)u(0) = Q(0) a \in C^2(\Omega) \subset D(A_0)\) follows from the regularity conditions (1.2) and the inclusion \(a \in C^\infty(\Omega)\). Furthermore, by (2.7) and (2.8), we see
\[ \|A_{0} S'(t)a\| = \|A_{0}^2 K(t)a\| = \|K(t)A_{0}^2 a\| \leq C t^{\alpha - 1}\|A_{0}^3 a\| \leq C t^{\alpha - 1}\|a\|_{H^4(\Omega)} \]
and
\[ \|K(t)A_0 (Q(0)a)\| \leq C t^{1-\alpha}\|A_0 (Q(0)a)\| \leq C t^{\alpha - 1}\|a\|_{H^3(\Omega)}. \]

Hence, the representation (4.9) leads to the estimate
\[ \|A_0 u'(t)\| \leq C t^{\alpha - 1}D + C \int_{0}^{t} s^{\frac{3}{2} - 1}\|A_0 u'(t-s)\|ds, \quad 0 < t < T. \]

We define a normed space
\[ \bar{X} := \{ v \in C([0,T]; L^2(\Omega)) \cap C^1((0,T]; L^2(\Omega)); t^{1-\alpha}\partial_t v \in C([0,T]; L^2(\Omega)) \} \]
and
\[ \|v\|_{\tilde{X}} := \max_{0 \leq t \leq T} \|t^{1-\alpha} \partial_t v(\cdot, t)\|_{L^2(\Omega)} + \max_{0 \leq t \leq T} \|v(\cdot, t)\|_{L^2(\Omega)}. \]

We readily verify that \( \tilde{X} \) is a Banach space.

Arguing similarly to the proof of Theorem 2.1 and applying the fixed point theorem in \( \tilde{X} \), we conclude that \( A_0 u \in \tilde{X} \), that is, \( t^{1-\alpha} A_0 u' \in C([0, T]; L^2(\Omega)) \). Using the inclusion \( D(A_0) \subset C(\overline{\Omega}) \) by the spatial dimensions \( d = 1, 2, 3 \), the Sobolev embedding theorem yields
\[ u' \in C(\overline{\Omega} \times (0, T]), \quad \|A_0 u'(t)\| \leq CD t^{\alpha-1}, \quad 0 \leq t \leq T. \quad (4.10) \]

Now we proceed to the estimation of \( A_0^2 u(t) \). Since \( \frac{d}{ds}(-A_0^{-1} S(s)) = K(s) \) for \( 0 < s < T \) by (2.7), the integration by parts yields
\[
\begin{align*}
\int_0^t K(t-s)Q(s)u(s)ds &= \int_0^t K(s)Q(t-s)u(t-s)ds \\
&= [-A_0^{-1} S(s)Q(t-s)u(t-s)]_{s=0}^{s=t} \\
&= -\int_0^t A_0^{-1} S(s)(Q'(t-s)u(t-s) + Q(t-s)u'(t-s))ds \\
&= A_0^{-1} Q(t)u(t) - A_0^{-1} S(t)Q(0)u(0) \\
&- \int_0^t A_0^{-1} S(s)(Q'(t-s)u(t-s) + Q(t-s)u'(t-s))ds, \quad 0 < t < T. \quad (4.11)
\end{align*}
\]

The Lebesgue convergence theorem and the estimate \(|E_{\alpha,1}(\eta)| \leq \frac{C}{t^{1-\eta}}\) for all \( \eta > 0 \) (Theorem 1.6 in [28]), we deduce
\[ \|(S(t) - 1)a\|^2 = \sum_{n=1}^{\infty} |(a, \varphi_n)|^2 (E_{\alpha,1}(-\lambda_n t^\alpha) - 1)^2 \to 0 \]
as \( t \to \infty \) for \( a \in L^2(\Omega) \).

Hence, \( u \in C([0, T]; L^2(\Omega)) \) and \( \lim_{t \to 0} \|(S(t) - 1)a\| = 0 \) and so
\[ \lim_{s \to t} S(s)Q(t-s)u(t-s) = S(0)Q(t)u(t) \quad \text{in} \quad L^2(\Omega) \]
and
\[ \lim_{s \to t} S(s)(t-s)u(t-s) = S(t)Q(0)u(0) \quad \text{in} \quad L^2(\Omega), \]
which justify the last equality in the formula (4.11).

Thus, in terms of (4.11), the representation (2.13) can be rewritten in the form
\[
A_0^2(u(t) - A_0^{-1} Q(t)u(t)) = A_0^2 G(t) - A_0 S(t)Q(0)u(0) \\
- \int_0^t A_0^\frac{1}{2} S(s)A_0^\frac{1}{2} (Q'(t-s)u(t-s) + Q(t-s)u'(t-s))ds, \quad 0 < t < T. \quad (4.12)
\]

Since \( u(0) = a \in C^\infty_0(\Omega) \) and \( F \in C^\infty_0(\Omega \times (0, T)) \), in view of (1.2) we have the inclusion
\[ A_0^2 G(\cdot) \in C([0, T]; L^2(\Omega)), \quad A_0 S(t)Q(0)u(0) = S(t)(A_0 Q(0)a) \in C([0, T]; L^2(\Omega)). \]
Now we use the conditions (1.2) and (2.8) and repeat the arguments employed for derivation of (4.5) by means of (4.6) and (4.10) to obtain the estimate

\[ \left\| \int_0^t A_0^{\frac{1}{2}} S(s) A_0^{\frac{1}{2}} (Q'(t-s)u(t-s) + Q(t-s)u'(t-s))ds \right\| \leq C \int_0^t s^{-\frac{1}{2}} \| Q'(t-s)u(t-s) + Q(t-s)u'(t-s) \|_{H^1(\Omega)} ds \]

\[ \leq C \int_0^t s^{-\frac{1}{2}} \| A_0 u'(t-s) \| + \| A_0 u(t-s) \| ds \leq C t^{\frac{1}{2}} D \]

and the inclusion

\[ - \int_0^t A_0^{\frac{1}{2}} S(s) A_0^{\frac{1}{2}} (Q'(t-s)u(t-s) + Q(t-s)u'(t-s))ds \in C([0,T]; L^2(\Omega)). \]

Therefore,

\[ A_0^2 (u(t) - A_0^{-1} Q(t) u(t)) = A_0 (A_0 u(t) - Q(t) u(t)) \in C([0,T]; L^2(\Omega)), \]

that is,

\[ A_0 u(t) - Q(t) u(t) \in C([0,T]; D(A_0)) \subset C([0,T]; H^2(\Omega)). \]

On the other hand, the estimate (4.8) implies

\[ Q(t) u(t) \in C([0,T]; H^2(\Omega)) \]

and we obtain

\[ A_0 u(t) \in C([0,T]; H^2(\Omega)). \]

(4.13)

For further arguments, we define the Schauder spaces \( C^\theta(\overline{\Omega}) \) and \( C^{2+\theta}(\overline{\Omega}) \) with \( 0 < \theta < 1 \) (see e.g., [9], [17]) as follows: A function \( w \) is defined to belong to the space \( C^\theta(\overline{\Omega}) \) if

\[ \sup_{x \neq x', x, x' \in \Omega} \frac{|w(x) - w(x')|}{|x - x'|^\theta} < \infty. \]

For \( w \in C^\theta(\overline{\Omega}) \), we define the norm

\[ \|w\|_{C^\theta(\overline{\Omega})} := \|w\|_{C(\overline{\Omega})} + \sup_{x \neq x', x, x' \in \Omega} \frac{|w(x) - w(x')|}{|x - x'|^\theta}. \]

and for \( w \in C^{2+\theta}(\overline{\Omega}) \), the norm is given by

\[ \|w\|_{C^{2+\theta}(\overline{\Omega})} := \|w\|_{C^2(\overline{\Omega})} + \sum_{|\tau| = 2} \sup_{x \neq x', x, x' \in \Omega} \frac{|\partial_\tau^\theta w(x) - \partial_\tau^\theta w(x')|}{|x - x'|^\theta} < \infty. \]

In the last formula, the notations \( \tau := (\tau_1, ..., \tau_d) \in (\mathbb{N} \cup \{0\})^d \), \( \partial_\tau^\theta := \partial_{\tau_1}^{\theta_1} \cdots \partial_{\tau_d}^{\theta_d} \), and \( |\tau| := \tau_1 + \cdots + \tau_d \) are employed.

For \( d = 1, 2, 3 \), the Sobolev embedding theorem says that \( H^2(\Omega) \subset C^\theta(\overline{\Omega}) \) with some \( \theta \in (0,1) \) (11).
Therefore, in view of (4.13), we see \( h := A_0u(\cdot, t) \in C^\theta(\overline{\Omega}) \) for each \( t \in [0, T] \). We apply the Schauder estimate (see e.g., [9] or [17]) for solutions to the elliptic boundary value problem

\[
A_0u(\cdot, t) = h \in C^\theta(\overline{\Omega}) \quad \text{in } \Omega
\]

with the boundary condition \( \partial_{\nu_A}u(\cdot, t) + \sigma(\cdot)u(\cdot, t) = 0 \) on \( \partial\Omega \) to arrive at the inclusion

\[
u \in C([0, T]; C^{2+\theta}(\overline{\Omega})).
\]

The inclusions (4.10) and (4.14) complete the proof of Lemma 4.1.

\[\square\]

(II) Second part of the proof of Theorem 4.1

In this part, we weaken the regularity conditions posed on \( u \) in Lemma 3.2 and prove the same results provided that \( u \in L^2(0, T; H^2(\Omega)) \) and \( u - a \in H_0(0, T; L^2(\Omega)) \).

Let \( F \in L^2(0, T; L^2(\Omega)) \) and \( a \in H_0^1(\Omega) \) be arbitrarily chosen such that \( F \geq 0 \) in \( \Omega \times (0, T) \) and \( a \geq 0 \) in \( \Omega \).

Now we apply the standard mollification procedure (see, e.g., [1]) and construct the sequences \( F_n \in C_0^\infty(\Omega \times (0, T)) \) and \( a_n \in C_0^\infty(\Omega) \), \( n \in \mathbb{N} \) such that \( F_n \geq 0 \) in \( \Omega \times (0, T) \) and \( a_n \geq 0 \) in \( \Omega \), \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \| F_n - F \|_{L^2(0, T; L^2(\Omega))} = 0 \) and \( \lim_{n \to \infty} \| a_n - a \|_{H_0^1(\Omega)} = 0 \). Then Lemma 4.1 yields the inclusion

\[
u(F_n, a_n) \in C([0, T]; C^2(\Omega)), \quad t^{1-\alpha} \partial_t u(F_n, a_n) \in C([0, T]; C(\Omega)), \quad n \in \mathbb{N}
\]

and thus Lemma 3.2 ensures the inequalities

\[
u(F_n, a_n) \geq 0 \quad \text{in } \Omega \times (0, T), \quad n \in \mathbb{N}.
\]

Since Theorem 2.1 hold true for the the initial-boundary value problem (3.3) with \( F \) and \( a \) replaced by \( F - F_n \) and \( a - a_n \), respectively, we have

\[
\| u(F, a) - u(F_n, a_n) \|_{L^2(0, T; H^2(\Omega))} \leq C(\| a - a_n \|_{H^1(\Omega)} + \| F - F_n \|_{L^2(0, T; L^2(\Omega))}) \to 0
\]

as \( n \to \infty \). Therefore, we can choose a subsequence \( m(n) \in \mathbb{N} \) such that \( u(F, a)(x, t) = \lim_{m(n) \to \infty} u(F_{m(n)}, a_{m(n)})(x, t) \) for almost all \( (x, t) \in \Omega \times (0, T) \). Then the inequality (4.15) leads to the desired result that \( u(F, a)(x, t) \geq 0 \) for almost all \( (x, t) \in \Omega \times (0, T) \).

(III) Third part: Completion of the proof of Theorem 4.1

Let \( a \geq 0, \in H_0^1(\Omega), \ F \geq 0, \in L^2(0, T; L^2(\Omega)), \) and let \( u = u(F, a) \in L^2(0, T; H^2(\Omega)) \) satisfy the initial condition \( u - a \in H_0(0, T; L^2(\Omega)) \) and the boundary value problem. Now we will prove Theorem 4.1 for \( u \) in this class without assumptions on the sign of the zeroth-order coefficient.

In (3.1), we choose a constant function \( b_0 > 0 \) as \( b_0(x, t) \), and assume that \( b_0 > 0 \) is sufficiently large. We define the operator \( A_1 \) by (3.1). Then we verify that the initial-boundary value problem (2.4) is equivalent to

\[
\begin{aligned}
\partial_t^\alpha (u - a) + A_1 u &= (b_0 + c(x, t))u + F, \quad (x, t) \in \Omega \times (0, T), \\
\partial_{\nu_A}u + \sigma u &= 0 \quad \text{on } \partial\Omega \times (0, T).
\end{aligned}
\]

In what follows, we choose sufficiently large \( b_0 > 0 \) such that \( b_0 \geq \| c \|_{C(\overline{\Omega} \times [0, T])}. \)
In the previous parts of the proof, we already interpreted the solution \( u \) as a unique fixed point for the equation \((4.3)\). Now let us construct an approximating sequence \( u_n \), \( n \in \mathbb{N} \) for \( u \) as follows. Setting

\[ u_1(x,t) = a(x) \geq 0, \quad (x,t) \in \Omega \times (0,T), \]

we inductively define a sequence \( \{u_n\}_{n \in \mathbb{N}} \) by solving \( u_{n+1} \) with given \( u_n \):

\[
\begin{aligned}
\partial_t^\alpha u_{n+1} - a + A_1 u_{n+1} &= (b_0 + c(x,t))u_n + F \quad \text{in } \Omega \times (0,T), \\
\partial_{\nu_A} u_{n+1} + \sigma u_{n+1} &= 0 \quad \text{on } \partial\Omega \times (0,T), \\
u_{n+1} - a &\in H_0(0,T;L^2(\Omega)), \quad n \in \mathbb{N}.
\end{aligned}
\]

(4.17)

We set \( u_0(x,t) := 0 \) for \( (x,t) \in \Omega \times (0,T) \). First we show that

\[ u_n(x,t) \geq 0, \quad (x,t) \in \Omega \times (0,T), \quad n \in \mathbb{N}. \]

(4.18)

Indeed, the inequality \((4.18)\) holds for \( n = 1 \). Now we assume that \( u_n \geq 0 \) in \( \Omega \times (0,T) \). Then \( (b_0 + c(x,t))u_n + F \geq 0 \) in \( \Omega \times (0,T) \), and thus by the results established in the second part of the proof of Theorem \((4.1)\) we obtain \( u_{n+1} \geq 0 \) in \( \Omega \times (0,T) \). Thus, by the principle of mathematical induction, the inequality \((4.18)\) holds true for all \( n \in \mathbb{N} \).

We rewrite \((4.17)\) as

\[
\partial_t^\alpha (u_{n+1}(t) - a) + A_0 u_{n+1}(t) = (Q(t)u_{n+1} - (c(t) + b_0)u_{n+1}) + (b_0 + c(t))u_n + F,
\]

where we recall that \( A_0 \) and \( Q(t) \) are defined by \((2.5)\) and \((2.14)\). Next we estimate \( w_{n+1} := u_{n+1} - u_n \). By the relation \((4.17)\), we obtain

\[
\begin{aligned}
\partial_t^\alpha w_{n+1} + A_0 w_{n+1} &= (Q(t)w_{n+1} - (c(t) + b_0)w_{n+1}) + (b_0 + c(t))w_n \\
in \Omega \times (0,T), \\
\partial_{\nu_A} w_{n+1} + \sigma w_{n+1} &= 0 \quad \text{on } \partial\Omega \times (0,T), \\
w_{n+1} &\in H_0(0,T;L^2(\Omega)), \quad n \in \mathbb{N}.
\end{aligned}
\]

(4.17)

In terms of the operator \( K(t) \) defined by \((2.7)\), acting similarly to our analysis of the fixed point equation \((4.3)\), we have

\[
\begin{aligned}
w_{n+1}(t) &= \int_0^t K(t-s)(Qw_{n+1}(s))ds - \int_0^t K(t-s)(c(s) + b_0)w_{n+1}(s)ds \\
&\quad + \int_0^t K(t-s)(b_0 + c(s))w_n(s)ds, \quad 0 < t < T,
\end{aligned}
\]

which leads to the estimation

\[
\begin{aligned}
\|A_0^\frac{1}{t} w_{n+1}(t)\| &\leq \int_0^t \|A_0^\frac{1}{t} K(t-s)\|\|Q(s)w_{n+1}(s)\|ds \\
&\quad + \int_0^t \|A_0^\frac{1}{t} K(t-s)\|\|(c(s) + b_0)w_{n+1}(s)\|ds + \int_0^t \|A_0^\frac{1}{t} K(t-s)\|\|(b_0 + c(s))w_n(s)\|ds \\
&\leq C \int_0^t (t-s)^{-\frac{1}{2} - \alpha} \|A_0^\frac{1}{t} w_{n+1}(s)\|ds + C \int_0^t (t-s)^{-\frac{1}{2} - \alpha} \|A_0^\frac{1}{t} w_n(s)\|ds \quad \text{for } 0 < t < T.
\end{aligned}
\]
Here, by (1.2) we used the norm estimates

\[ \|Q(s)w_{n+1}(s)\| \leq C\|w_{n+1}(s)\|_{H^1(\Omega)} \leq C\|A_0^{\frac{1}{2}}w_{n+1}(s)\| \]

and

\[ \|(c(s) + b_0)w_\ell(s)\| \leq C\|w_\ell(s)\|_{H^1(\Omega)} \leq C\|A_0^{\frac{1}{2}}w_\ell(s)\|, \quad \ell = n, n + 1. \]

Thus

\[ \|A_0^{\frac{1}{2}}w_{n+1}(t)\| \leq C\int_0^t (t-s)^{\frac{1}{2}a-1}\|A_0^{\frac{1}{2}}w_{n+1}(s)\|ds \]

\[ +C\int_0^t (t-s)^{\frac{1}{2}a-1}\|A_0^{\frac{1}{2}}w_n(s)\|ds, \quad 0 < t < T. \]

The generalized Gronwall inequality yields the inequality

\[ \|A_0^{\frac{1}{2}}w_{n+1}(t)\| \leq C\int_0^t (t-s)^{\frac{1}{2}a-1}\|A_0^{\frac{1}{2}}w_n(s)\|ds \]

\[ +C\int_0^t (t-s)^{\frac{1}{2}a-1}\left(\int_0^s (s-\xi)^{\frac{1}{2}a-1}\|A_0^{\frac{1}{2}}w_n(\xi)\|d\xi\right)ds. \]

For the second term of the right-hand side of the last inequality, the following calculations hold true:

\[ \int_0^t (t-s)^{\frac{1}{2}a-1}\left(\int_0^s (s-\xi)^{\frac{1}{2}a-1}\|A_0^{\frac{1}{2}}w_n(\xi)\|d\xi\right)ds \]

\[ = \int_0^t \|A_0^{\frac{1}{2}}w_n(\xi)\|\left(\int_0^t (t-s)^{\frac{1}{2}a-1}(s-\xi)^{\frac{1}{2}a-1}ds\right)d\xi \]

\[ = \frac{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}a\right)}{\Gamma(a)}\int_0^t (t-\xi)^{a-1}\|A_0^{\frac{1}{2}}w_n(\xi)\|d\xi \]

\[ = \frac{\Gamma\left(\frac{1}{2}a\right)^2}{\Gamma(a)}T^{\frac{1}{2}a}\int_0^t (t-s)^{\frac{1}{2}a-1}\|A_0^{\frac{1}{2}}w_n(s)\|ds. \]

Thus, we can choose a constant \( C > 0 \) dependent on \( \alpha \) and \( T \), such that

\[ \|A_0^{\frac{1}{2}}w_{n+1}(t)\| \leq C\int_0^t (t-\xi)^{\frac{1}{2}a-1}\|A_0^{\frac{1}{2}}w_n(s)\|ds, \quad 0 < t < T, n \in \mathbb{N}. \quad (4.19) \]

Recalling that

\[ \int_0^t (t-s)^{\frac{1}{2}a-1}\eta(s)ds = \Gamma\left(\frac{1}{2}a\right)(J^{\frac{1}{2}a}\eta)(t), \quad t > 0, \]

and setting \( \eta_n(t) := \|A_0^{\frac{1}{2}}w_n(t)\| \), we can rewrite (4.19) as follows:

\[ \eta_{n+1}(t) \leq CT \left(\frac{1}{2}a\right)(J^{\frac{1}{2}a}\eta_n)(t), \quad 0 < t < T, n \in \mathbb{N}. \quad (4.20) \]
Since $J^{\frac{1}{2}\alpha}$ preserves the signs of $\eta$ and $J^{\beta_1}(J^{\beta_2}\eta)(t) = J^{\beta_1+\beta_2}\eta(t)$ for each $\beta_1, \beta_2 > 0$, applying the inequality (4.20) repeatedly, we reach the inequality

$$\eta_n(t) \leq \left( C \Gamma \left( \frac{1}{2} \alpha \right) \right)^{n-1} \left( J^{(n-1) \frac{1}{2}\alpha} \eta_1 \right)(t)$$

$$= \frac{C}{\Gamma \left( \frac{2}{n} \right)(n-1)} \left( \int_0^t (t-s)^{(n-1) \frac{1}{2}\alpha-1} ds \right) \|A_0^\frac{1}{2} a\|$$

$$= \frac{C}{\Gamma \left( \frac{2}{n} \right)(n-1)} \frac{t^{(n-1) \frac{1}{2}\alpha}}{(n-1)^{\frac{1}{2}\alpha}} \|A_0^\frac{1}{2} a\| \leq C \left( C T^{\frac{1}{2}\alpha} \right)^{n-1} \frac{1}{\Gamma \left( \frac{2}{n} \right)(n-1)}.$$  

The known asymptotic behavior of the gamma function justifies the relation

$$\lim_{n \to \infty} \frac{\left( C T^{\frac{1}{2}\alpha} \right)^{n-1}}{\Gamma \left( \frac{2}{n} \right)(n-1)} = 0.$$  

Thus we have proved that the sequence $u_N = w_0 + \cdots + w_N$ converges to the solution $u$ in $L^\infty(0,T; H^1(\Omega))$ as $N \to \infty$. Therefore, we can choose a subsequence $m(n) \in \mathbb{N}$ for $n \in \mathbb{N}$ such that $\lim_{m(n) \to \infty} u_{m(n)}(x,t) = u(x,t)$ for almost all $(x,t) \in \Omega \times (0,T)$. This statement in combination with the inequality (4.18) means that $u(x,t) \geq 0$ for almost all $(x,t) \in \Omega \times (0,T)$. Thus the proof of Theorem 4.1 is completed. \hfill \Box

Now let us fix a source function $F = F(x,t) \geq 0$ and an initial value $a \in H_0^1(\Omega)$ in the initial-boundary value problem (2.3) and denote by $u(c,\sigma) = u(c,\sigma)(x,t)$ the solution to the problem (2.3) with the functions $c = c(x,t)$ and $\sigma = \sigma(x)$. Then the following comparison property regarding the coefficients $c$ and $\sigma$ is valid:

**Theorem 4.2.** Let $a \in H_0^1(\Omega)$ and $a \geq 0$ in $\Omega$ and $F \in L^2(\Omega \times (0,T))$ satisfy $F \geq 0$ in $\Omega \times (0,T)$.

(i) Let $c_1, c_2 \in C^1([0,T]; C^1(\Omega)) \cap C([0,T]; C^2(\Omega))$ satisfy $c_1(x,t) \geq c_2(x,t)$ in $\Omega$. Then $u(c_1, \sigma)(x,t) \geq u(c_2, \sigma)(x,t)$ in $\Omega \times (0,T)$.

(ii) Let $c(x,t) < 0$ for $(x,t) \in \Omega \times (0,T)$. We arbitrarily fix a constant $\sigma_0 > 0$. If the smooth functions $\sigma_1, \sigma_2$ on $\partial \Omega$ satisfy

$$\sigma_2(x) \geq \sigma_1(x) \geq \sigma_0 \quad \text{for } x \in \partial \Omega,$$

then $u(c, \sigma_1) \geq u(c, \sigma_2)$ in $\Omega \times (0,T)$.

**Proof.** (i) Because $a \geq 0$ in $\Omega$ and $F \geq 0$ in $\Omega \times (0,T)$, Theorem 4.1 yields the inequality $u(c_2, \sigma) \geq 0$ in $\Omega \times (0,T)$. Setting $u(x,t) := u(c_1, \sigma)(x,t) - u(c_2, \sigma)(x,t)$ for $(x,t) \in \Omega \times (0,T)$, we have

$$\begin{cases}
\partial_t u - \sum_{i,j=1}^d a_{ij} \partial_i \partial_j u - \sum_{j=1}^d b_j \partial_j u \\
- c_1(x,t)u = (c_1 - c_2)u(c_2, \sigma)(x,t) \quad \text{in } \Omega \times (0,T),
\end{cases}$$

$$\begin{align*}
\partial_{\nu_\alpha} u + \sigma u &= 0 \quad \text{on } \partial \Omega, \\
u &\in H_0^a(0,T; L^2(\Omega)).
\end{align*}$$
Let us now define an auxiliary function $H$. Hence, by (4.21) and $\sigma_c \geq 0$ for $(x,t) \in \Omega \times (0,T)$, which is equivalent to the inequality $u(c_1,\sigma)(x,t) \geq u(c_2,\sigma)(x,t)$ for $(x,t) \in \Omega \times (0,T)$.

(ii) Similarly to the procedure applied for the second part of the proof of Theorem 4.1, we choose the sequences $F_n \geq 0$, $C_0^\infty(\Omega \times (0,T))$ and $a_n \geq 0$, $C_0^\infty(\Omega)$, $n \in \mathbb{N}$ such that $F_n \rightarrow F$ in $L^2(\Omega \times (0,T))$ and $a_n \rightarrow a$ in $H_0^1(\Omega)$. Let $u_n$, $v_n$ be the solutions to initial-boundary value problem (2.4) with $F = F_n$, $a = a_n$ and with the coefficients $\sigma_1$ and $\sigma_2$ in the boundary condition, respectively. We note that $v_n$, $u_n \in C(\overline{\Omega} \times [0,T])$ and $F$, $\sigma_c \geq 0$, $t^{1-\alpha} \partial_t v_n$, $t^{1-\alpha} \partial_t u_n \in C([0,T] : C(\overline{\Omega}))$, $n \in \mathbb{N}$ by Lemma 4.2, and Theorem 4.1 yields that

$$v_n(x,t) \geq 0, \quad (x,t) \in \partial \Omega \times (0,T).$$

Then Theorem 2.1 yields the relation

$$\lim_{n \rightarrow \infty} \|u_n - u(c,\sigma_1)\|_{L^2(0,T;L^2(\Omega))} = \lim_{n \rightarrow \infty} \|v_n - u(c,\sigma_2)\|_{L^2(0,T;L^2(\Omega))} = 0. \quad (4.22)$$

Let us now define an auxiliary function $w_n := u_n - v_n$. Then,

$$t^{1-\alpha} \partial_t w_n \in C([0,T] : C(\overline{\Omega})), \quad w_n \in C([0,T] ; C^2(\overline{\Omega})), \quad n \in \mathbb{N} \quad (4.23)$$

and

$$\begin{cases} 
\partial_t^\alpha w_n + Aw_n = 0 & \text{in } \Omega \times (0,T), \\
\partial_\nu \sigma_1 w_n + \sigma_1 w_n - (\sigma_2 - \sigma_1) v_n & \text{on } \partial \Omega \times (0,T), \\
w_n(x,\cdot) \in H_\alpha(0,T) & \text{for almost all } x \in \Omega. 
\end{cases} \quad (4.24)$$

Hence, by (4.21) and $\sigma_2 \geq \sigma_1$ on $\partial \Omega$, we have

$$\partial_\nu \sigma_1 w_n + \sigma_1 w_n \geq 0 \quad \text{on } \partial \Omega \times (0,T). \quad (4.25)$$

We show a variant of Lemma 3.2.

**Lemma 4.2.** Let the elliptic operator $-A$ be defined by (2.1). We assume (1.2) and $c(x,t) < 0$ for $x \in \overline{\Omega}$ and $0 \leq t \leq T$ and there exists a constant $\sigma_0 > 0$ such that

$$\sigma(x) \geq \sigma_0 \quad \text{for all } x \in \partial \Omega. \quad \sigma_0 > 0$$

For $a \in H_0^1(\Omega)$ and $F \in L^2(\Omega \times (0,T))$, we further assume that there exists a solution $u \in C([0,T] ; C^2(\overline{\Omega}))$ satisfying $t^{1-\alpha} \partial_t u \in C([0,T] ; C(\overline{\Omega}))$ to

$$\begin{cases} 
\partial_t^\alpha(u - a) + Au = F & \text{in } \Omega \times (0,T), \\
\partial_\nu \sigma u + \sigma u \geq 0 & \text{on } \partial \Omega \times (0,T), \\
u(x,\cdot) - a \in H_\alpha(0,T) & \text{for almost all } x \in \Omega. 
\end{cases} \quad (4.22)$$

If $F \geq 0$ in $\Omega \times (0,T)$ and $a \geq 0$ in $\Omega$, then $u \geq 0$ in $\Omega \times (0,T)$.

In this lemma, at the expense of extra condition $\sigma > 0$ on $\partial \Omega$, we do not need assume that $\min_{(x,t) \in \overline{\Omega} \times [0,T]}(-c(x,t))$ is sufficiently large, which is the main difference from Lemma 3.2. The proof is much simpler than Lemma 3.2 and postponed to the end of this section.

Now we complete the proof of Theorem 4.1. Since $c(x,t) < 0$ for $(x,t) \in \Omega \times (0,T)$ and $\sigma_1 \geq \sigma_0 > 0$ on $\partial \Omega$, in terms of (4.23) and (4.24) we can apply Lemma 4.2 to (4.22) and deduce the inequality $w_n \geq 0$ in $\Omega \times (0,T)$, that is, $u_n \geq v_n$ in $\Omega \times (0,T)$ for $n \in \mathbb{N}$. In view of (4.22), we can choose a suitable subsequence of $u_n$, $n \in \mathbb{N}$ and pass to the limit, thus arriving at the inequality $u(c,\sigma_1) \geq u(c,\sigma_2)$ in $\Omega \times (0,T)$. The proof of Theorem 4.2 is completed. \qed
Finally, let us mention one direction for further research in connection with the results formulated and proved in this sections. In order to remove the negativity condition posed on the coefficient $c = c(x, t)$ in Theorem 4.2 (ii), one needs a unique existence result for solutions to the initial-boundary value problems of type (2.4) with non-zero Robin boundary condition similar to the one formulated in Theorem 2.1. There are several works that treat the case of the initial-boundary value problems with non-zero Dirichlet boundary conditions (see, e.g., [34] and the references therein). However, to the best of the authors’ knowledge, analogous results are not available for the case of the initial-boundary value problems with the Neumann and Robin boundary conditions. Thus, in Theorem 4.2 (ii), we assumed the condition $c < 0$ in $\Omega \times (0, T)$, although our conjecture is that this result is valid for an arbitrary coefficient $c = c(x, t)$.

We conclude this section with

**Proof of Lemma 4.2.** The proof is simple because we do not need the function $\psi$ defined by (3.4). We set

$$\tilde{w}(x, t) := u(x, t) + \varepsilon(1 + t^\alpha), \quad x \in \Omega, \ 0 < t < T.$$  

Using $c < 0$ on $\Omega \times [0, T]$ and $\sigma \geq \sigma_0 > 0$ on $\partial \Omega$ and calculating similarly to the proof of Lemma 3.2 we have

$$d_t^\alpha \tilde{w} + A\tilde{w} = F + \varepsilon \Gamma(\alpha + 1) - c(x, t)\varepsilon(1 + t^\alpha) > 0 \quad \text{in } \Omega \times (0, T),$$

$$\partial_\nu \tilde{w} + \sigma \tilde{w} = \partial_\nu u + \sigma u + \sigma\varepsilon(1 + t^\alpha) \geq \sigma_0 \varepsilon \quad \text{on } \partial \Omega \times (0, T)$$

and

$$\tilde{w}(x, 0) = a(x) + \varepsilon \geq \varepsilon \quad \text{in } \Omega.$$  

Therefore, we can follow the same arguments after (3.10) in the proof of Lemma 3.2. Thus the proof of Lemma 4.2 is complete. □

## 5 Well-posedness for an initial-boundary value problems for semilinear time-fractional diffusion equations

In order to extend the comparison principle to semilinear time-fractional diffusion equations, we have to establish the unique existence and a priori estimates for initial-boundary value problems, because the arguments rely on the construction of the solutions, as we see from the third part of the proof of Theorem 4.1 in Section 4.

Thus, in this section, we first formulate and prove some existence results for the following initial-boundary value problem for the semilinear time-fractional diffusion equation:

$$
\begin{cases}
\partial_t^\alpha (u(x, t) - a(x)) + Au(x, t) = f(u)(x, t), & x \in \Omega, \ 0 < t < T, \\
\partial_\nu A u + \sigma(x)u(x, t) = 0, & x \in \partial \Omega, \ 0 < t < T, \\
u(x, \cdot) - a(x) \in H_A(0, T) & \text{for almost all } x \in \Omega,
\end{cases}
$$

(5.1)

where the second order spatial differential operator $-A$ is defined by (2.1) and the source function $f(u)(\cdot, t)$ may depend not only on $u$ but also on $x$ and its spatial derivatives, provided that $f$ satisfies the conditions (5.2) stated below.
We recall that for a fixed but sufficiently large constant $c_0 > 0$, the operator $A_0$ in $L^2(\Omega)$ is defined by the relation
\[
\begin{align*}
(-A_0 v)(x) &= \sum_{i,j=1}^d \partial_i (a_{ij}(x) \partial_j v(x)) - c_0 v(x), \quad x \in \Omega, \\
\mathcal{D}(A_0) &= \{ v \in H^2(\Omega); \partial_{\nu A} v + \sigma v = 0 \text{ on } \partial \Omega \},
\end{align*}
\]
where $\sigma$ is smooth, $\sigma \geq 0$ on $\partial \Omega$, and the coefficients $a_{ij}, b_j$ and $c$ satisfy the conditions \([1.2]\).

Furthermore we recall that we number all the eigenvalues of $A$ as $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ with the multiplicities, and we can choose linearly independent $\varphi_n$, $n \in \mathbb{N}$ such that $A \varphi_n = \lambda_n \varphi_n$. Then $A_0^\gamma v = \sum_{n=1}^\infty \lambda_n^{2\gamma}(v, \varphi_n) \varphi_n$ and
\[
\mathcal{D}(A_0^\gamma) = \left\{ v \in L^2(\Omega) : \sum_{n=1}^\infty \lambda_n^{2\gamma}(v, \varphi_n)^2 < \infty \right\}
\]
with
\[
\|A_0^\gamma v\| = \left( \sum_{n=1}^\infty \lambda_n^{2\gamma}(v, \varphi_n)^2 \right)^{\frac{1}{2}}.
\]

Henceforth we choose a constant $\gamma$ such that
\[
\frac{3}{4} < \gamma \leq 1.
\]

For a semilinear term $f : \mathcal{D}(A_0^\gamma) \rightarrow L^2(\Omega)$, we assume that for some constant $m > 0$, there exists a constant $C_f = C_f(m) > 0$ such that
\[
\begin{align*}
\begin{cases}
(i) & \|f(v)\| \leq C_f, \quad \|f(v_1) - f(v_2)\| \leq C_f \|v_1 - v_2\|_{\mathcal{D}(A_0^\gamma)} \\
\text{if } \|v\|_{\mathcal{D}(A_0^\gamma)}, \|v_1\|_{\mathcal{D}(A_0^\gamma)}, \|v_2\|_{\mathcal{D}(A_0^\gamma)} \leq m
\end{cases}
\end{align*}
\]
and
\[
\begin{align*}
\begin{cases}
(ii) & \text{there exists a constant } \varepsilon \in \left(0, \frac{4}{3}\right) \text{ such that } \\
\|f(v)\|_{H^{2\varepsilon}(\Omega)} \leq C_f(m) \text{ if } \|v\|_{\mathcal{D}(A_0^\gamma)} \leq m.
\end{cases}
\end{align*}
\]

(5.2)

In what follows, $C > 0$, $C_0, C_1 > 0$, etc., denote generic constants which are independent of the functions $u, v$, etc. under consideration, and we write $C_f, C(m)$ when we need to specify the dependence on related quantities.

Before we state and prove the main results of this section, let us discuss some examples of the source functions which satisfy the condition \([5.2]\).

**Example 5.1.** For $f \in C^1(\mathbb{R})$, by setting $f(u) := f(u(x,t))$ for $(x,t) \in \Omega \times (0,T)$, we define $f : \mathcal{D}(A_0^\gamma) \rightarrow L^2(\Omega)$, $\frac{3}{4} < \gamma < 1$.

Let us verify that the function from Example 5.1 satisfies the condition \([5.2]\). By the Sobolev embedding and the spatial dimensions $d \leq 3$, we see that $\mathcal{D}(A_0^\gamma) \subset H^{2\varepsilon}(\Omega) \subset C(\overline{\Omega})$. Therefore $\|v\|_{\mathcal{D}(A_0^\gamma)} \leq m$ yields $|v(x)| \leq C_0 m$ for $a \in \Omega$. Hence, by using the mean value theorem, the first condition in \([5.2]\) is satisfied if we choose $C_f(m) = \|f\|_{C^1[-C_0 m, C_0 m]}$.

Next we have to verify the second condition in \([5.2]\). To this end, we prove the following lemma:
Lemma 5.1. Let \( f \in C^1[-C_0m, C_0m] \) and \( 0 < \varepsilon < \frac{3}{4} \). Then
\[
\|f(w)\|_{\mathcal{D}(A_0^s)} \leq C(1 + m)\|f\|_{C^1[-C_0m, C_0m] \cap \mathcal{D}(A_0^s)} \quad \text{if} \quad \|w\|_{\mathcal{D}(A_0^s)} \leq m.
\]

Proof. Because of the restrictions \( 0 < \varepsilon < \frac{3}{4} \), we have the equality \( \mathcal{D}(A_0^s) = H^{2\varepsilon}(\Omega) \) ([8], [12]). Therefore, since \( f \in C^1[-C_0m, C_0m] \), the following norm estimates in the Sobolev-Slobodecki space \( H^{2\varepsilon}(\Omega) \) are valid:
\[
\|f(w)\|^2_{H^{2\varepsilon}(\Omega)} = \|f(w)\|^2_{L^2(\Omega)} + \int_\Omega \int_\Omega \frac{|f(w(x)) - f(w(y))|^2}{|x - y|^{d+4\varepsilon}} \, dx \, dy
\]
\[
\leq \|f\|^2_{C^1[-C_0m, C_0m]} |\Omega| \|w\|^2_{H^{2\varepsilon}(\Omega)} + \int_\Omega \int_\Omega \frac{|f|^2_{C^1[-C_0m, C_0m]} |w(x) - w(y)|^2}{|x - y|^{d+4\varepsilon}} \, dx \, dy
\]
\[
\leq C(1 + \|w\|^2_{H^{2\varepsilon}(\Omega)}) \|f\|^2_{C^1[-C_0m, C_0m]} \leq C(1 + m^2) \|f\|^2_{C^1[-C_0m, C_0m]}.
\]

The result formulated in Lemma 5.1 ensures the validity of the second condition from (5.2) for the function defined in Example 5.1.

Example 5.2. Let
\[
f(v)(x) := \sum_{k=1}^d \mu_k(x)v(x)\partial_k v(x), \quad x \in \Omega,
\]
where \( \mu_k \in C^1(\overline{\Omega}) \). In particular, this semilinear term appears in the time-fractional Burgers equation \( \partial^\alpha_t u = \partial^2_x u - u\partial_x u \) among its particular cases.

Let us verify that the function defined in Example 5.2 satisfies the condition (5.2). Under the conditions \( \frac{3}{4} < \gamma < 1 \), the first condition from (5.2) is verified as follows. For \( v \in \mathcal{D}(A_0^s) \), it follows from \( \frac{3}{4} < \gamma < 1 \) that \( \|\nabla v\| \leq C\|v\|_{H^1(\Omega)} \leq C|A_0^s v| \). Moreover, \( d = 1, 2, 3 \) and the Sobolev embedding imply \( \|v\|_{C(\overline{\Omega})} \leq C\|v\|_{\mathcal{D}(A_0^s)} \).

Therefore, we obtain the desired inequalities:
\[
\|f(v)\| \leq C \sum_{k=1}^d \|v\| \partial_k v \| \leq C\|v\|_{C(\overline{\Omega})} \sum_{k=1}^d \|\partial_k v\|,
\]
\[
\leq C\|v\|_{C(\overline{\Omega})} \|v\|_{\mathcal{D}(A_0^s)} \leq C\|v\|^2_{\mathcal{D}(A_0^s)} \leq Cm^2 =: C_f(m)
\]
and
\[
\|f(v_1) - f(v_2)\| = \left\| \sum_{k=1}^d \mu_k(v_1 - v_2)\partial_k v_1 + \mu_kv_2\partial_k(v_1 - v_2) \right\|
\]
\[
\leq C(\|v_1 - v_2\|_{C(\overline{\Omega})}\|\nabla v_1\| + \|v_2\|_{C(\overline{\Omega})}\|\nabla(v_1 - v_2)\|)
\]
\[
\leq C \max\{\|v_1\|_{\mathcal{D}(A_0^s)}, \|v_2\|_{\mathcal{D}(A_0^s)}\}\|v_1 - v_2\|_{\mathcal{D}(A_0^s)}.
\]

Now we verify the second condition from (5.2). Since \( \mu_k \in C^1(\overline{\Omega}) \), it suffices to prove that we can choose some constant \( C(m) > 0 \) such that
\[
\|\partial_k v\|_{H^{2\varepsilon}(\Omega)} \leq C(m), \quad k = 1, \ldots, d \quad \text{if} \quad \|v\|_{\mathcal{D}(A_0^s)} \leq m.
\]
Since $d = 1, 2, 3$ and $\gamma > \frac{3}{4}$, the Sobolev embedding
\[
H^{2\gamma}(\Omega) \subset C^\theta(\overline{\Omega})
\] (5.4)
holds true, where $\theta < 2\gamma - \frac{3}{2}$ (e.g., Theorem 1.4.4.1 (p.27) in [12]). In (5.4), $C^\theta(\overline{\Omega})$ denotes the Schauder space of uniform Hölder continuous functions on $\overline{\Omega}$, which was defined at the end of the proof of Lemma 4.1.

Now we choose small $\varepsilon \in (0, 1)$ such that
\[
\varepsilon < \min\left\{\frac{1}{2}\theta, \frac{1}{4}\right\}. 
\] (5.5)

Now, using the inequalities
\[
\|v\|_{H^1(\Omega)} \leq C\|v\|_{\mathcal{D}(A_0^\gamma)} \leq Cm,
\]
we estimate
\[
\|v\partial_y v\|_{H^{2\gamma}(\Omega)}^2 = \|v\partial_y v\|_{L^2(\Omega)}^2 + \int_\Omega \int_\Omega \frac{|(v\partial_y v)(x) - (v\partial_y v)(y)|^2}{|x - y|^{d+4\varepsilon}} dxdy
\]
\[=: I_1 + I_2.\]
The inclusion (5.4) leads to the inequalities
\[
I_1 \leq \|v\|_{C(\overline{\Omega})}^2 \|\partial_y v\|_{L^2(\Omega)}^2 \leq C\|v\|_{\mathcal{D}(A_0^\gamma)}^2 \|v\|_{H^1(\Omega)}^2
\]
\[\leq C\|v\|_{\mathcal{D}(A_0^\gamma)}^4 \leq Cm^4.
\]
As for the term $I_2$, since
\[
|(v\partial_y v)(x) - (v\partial_y v)(y)|^2 \leq (|v(x)(\partial_y v(x) - \partial_y v(y))| + |v(x) - v(y)|\|\partial_y v(y)\|)^2
\]
\[\leq 2\|v\|_{C(\overline{\Omega})}^2 |\partial_y v(x) - \partial_y v(y)|^2 + 2|\partial_y v(y)|^2 |v(x) - v(y)|^2,
\]
we obtain the estimate
\[
I_2 \leq C\|v\|_{C(\overline{\Omega})}^2 \int_\Omega \int_\Omega \frac{|(\partial_y v)(x) - (\partial_y v)(y)|^2}{|x - y|^{d+4\varepsilon}} dxdy
\]
\[+ C\int_\Omega \int_\Omega |\partial_y v(y)|^2 \frac{|v(x) - v(y)|^2}{|x - y|^{d+4\varepsilon}} dxdy
\]
\[=: I_{21} + I_{22}.\]
The Sobolev embedding yields the inequality $\|v\|_{C(\overline{\Omega})} \leq C\|v\|_{\mathcal{D}(A_0^\gamma)}$. Moreover, since
\[
0 < \varepsilon < \frac{1}{4}
\]
by the condition (5.5), the inequality $\gamma > \frac{3}{4}$ leads to the estimates $2\varepsilon + 1 < 2\frac{1}{4} + 1 = \frac{3}{2} < 2\gamma$, so that we obtain the inequalities
\[
\|\partial_y v\|_{H^{2\gamma}(\Omega)} \leq C\|v\|_{H^{2\gamma+1}(\Omega)} \leq C\|v\|_{H^{2\gamma}(\Omega)} \leq C\|v\|_{\mathcal{D}(A_0^\gamma)}.
\]
Hence
\[
I_{21} \leq C\|v\|_{C(\overline{\Omega})}^2 \|\partial_y v\|_{H^{2\gamma}(\Omega)}^2 \leq C\|v\|_{C(\overline{\Omega})}^2 \|\partial_y v\|_{\mathcal{D}(A_0^\gamma)}^2 \leq Cm^4.
\]
Since \( \alpha \) is not globally Lipschitz continuous, and we need more cares for the arguments. However, in many semilinear equations, the constant \( \alpha \) independently of \( m > S_\alpha < \alpha < \) analytic semigroups. For the case \( 0 < \alpha \leq 22 \), there exists a constant

\[
\|u_a - u_b\|_{L^2(0,T;H^2(\Omega))} \leq C \|a - b\|_{L^2(0,T;H^2(\Omega))}
\]

provided that \( \|a\|_{L^2(0,T;H^2(\Omega))}, \|b\|_{L^2(0,T;H^2(\Omega))} \leq m \).

Theorem 5.1. Let a semilinear term \( f \) satisfy the condition (5.2) with \( m > 0 \) and \( \|a\|_{L^2(0,T;H^2(\Omega))} \leq m \).

Then there exists a constant \( T = T(m) > 0 \) such that the semilinear initial-boundary value problem (5.1) possesses a unique solution \( u = u_a \) satisfying the inclusions

\[
u_a \in L^2(0,T;H^2(\Omega)) \cap C([0,T]; \mathcal{D}(A_0^\gamma)), \quad u_a - a \in H_\alpha(0,T;L^2(\Omega)).
\]

Moreover there exists a constant \( C(m) > 0 \), such that

\[
\|u_a - u_b\|_{L^2(0,T;H^2(\Omega))} \leq C \|a - b\|_{L^2(0,T;H^2(\Omega))}
\]

In the theorem, the unique existence local in time of solution to an initial-boundary value problem for the semilinear time-fractional diffusion equation is established in the class (5.6). In the well-studied case of the conventional diffusion equation, i.e., \( \alpha = 1 \) in (5.1), one of the available methodologies for the proof of this result is the theory of the analytic semigroups. For the case \( 0 < \alpha < 1 \), our proof relies on a similar idea, that is, on employing the operators \( S(t) \) and \( K(t) \) defined by (2.6) and (2.7), even if the operator \( S(t) \) does not possess any semigroup properties. In several published works, a similar approach can be found for self-adjoint \( A \) in the case where the constant \( C_f \) can be chosen independently of \( m > 0 \). We can refer for example to Section 6.4.1 of Chapter 6 in [14]. However, in many semilinear equations, the constant \( C_f \) depends on \( m \), in other words, \( f \) is not globally Lipschitz continuous, and we need more cares for the arguments.

The results formulated in this theorem are similar to the ones well-known for the parabolic equations, that is, \( \alpha = 1 \) in (5.1), and see, e.g., [13] or [27].
Proof. First we recall the notation

\[ Q(t)u(t) := \sum_{j=1}^{d} b_j(\cdot, t) \partial_j u(\cdot, t) + (c(\cdot, t) + c_0) u(\cdot, t) \quad \text{in } \Omega, \ 0 < t < T. \]  

(5.8)

Moreover,

\[ \| A_0^{1/2} Q(t)u(t) \| \leq C \| Q(t)u(t) \|_{H^1(\Omega)} \leq C \| Q(t)u(t) \|_{H^2(\Omega)} \leq C \| A_0 u(t) \| \]  

(5.9)

for \( u \in D(A_0) \).

Similarly to (2.13), we can formally rewrite the problem (5.1) as follows:

\[ u(t) = S(t)a + \int_0^t K(t-s)Q(s)u(s)ds \]  

(5.10)

\[ + \int_0^t K(t-s)f(u(s))ds, \quad 0 < t < T. \]

For a fixed \( \gamma \in \left( \frac{3}{4}, 1 \right) \) in the condition (5.2) and a fixed initial value \( a \in D(A_0^\gamma) \), we define an operator \( L : L^2(0,T;L^2(\Omega)) \rightarrow L^2(0,T;L^2(\Omega)) \) by

\[ (Lu)(t) := S(t)a + \int_0^t K(t-s)Q(s)u(s)ds + \int_0^t K(t-s)f(u(s))ds, \quad 0 < t < T. \]

Choosing a constant \( m > 0 \) arbitrarily, we set

\[ V := \{ v \in C([0,T];D(A_0^\gamma)); \| u - S(\cdot)a \|_{C([0,T];D(A_0^\gamma))} \leq m \}. \]  

(5.11)

Then we prove the following lemma:

**Lemma 5.2.** Let \( H \in C([0,T];L^2(\Omega)) \). Then

\[ \int_0^t A_0^\gamma K(t-s)H(s)ds \in C([0,T];L^2(\Omega)), \]

that is,

\[ \int_0^t K(t-s)H(s)ds \in C([0,T];D(A_0^\gamma)). \]

**Proof.** Let \( 0 < \eta < t \leq T \). We have

\[
\begin{align*}
\int_0^t A_0^\gamma K(t-s)H(s)ds &= \int_0^\eta A_0^\gamma K(\eta-s)H(s)ds \\
&= \int_0^t A_0^\gamma K(s)H(t-s)ds - \int_0^\eta A_0^\gamma K(s)H(\eta-s)ds \\
&= \int_\eta^t A_0^\gamma K(s)H(t-s)ds + \int_0^\eta A_0^\gamma K(s)(H(t-s) - H(\eta-s))ds \\
&=: I_1 + I_2.
\end{align*}
\]

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For the first integral, by (2.8) and \( \gamma < 1 \), we have
\[
\|I_1\| \leq C \int_\eta^t s^{(1-\gamma)-1} \max_{0 \leq s \leq t} \|H(t-s)\| ds \\
\leq C \|H\|_{C([0,T];D(A_0^\gamma))} \frac{t^{(1-\gamma)} - \eta^{(1-\gamma)}}{(1-\gamma)} \to 0
\]
as \( \eta \uparrow t \).

Next we have the estimates
\[
\|I_2\| = \left\| \int_0^\eta A_0^\gamma K(s)(H(t-s) - H(\eta-s)) ds \right\| \\
\leq C \int_0^\eta s^{(1-\gamma)\alpha-1} \max_{0 \leq \eta \leq t \leq T} \|H(t-s) - H(\eta-s)\| ds.
\]
Hence, by \( H \in C([0,T];L^2(\Omega)) \), the function
\[
|s^{(1-\gamma)\alpha-1}| \max_{0 \leq \eta \leq t \leq T} \|H(t-s) - H(\eta-s)\|
\]
is an integrable function with respect to \( s \in (0,\eta) \) and
\[
\lim_{\eta \uparrow t} \int_0^\eta s^{(1-\gamma)\alpha-1} \max_{0 \leq \eta \leq t \leq T} \|H(t-s) - H(\eta-s)\| ds = 0
\]
for almost all \( s \in (0,\eta) \). Therefore, the Lebesgue convergence theorem implies \( \lim_{\eta \uparrow t} \|I_2\| = 0 \). Thus the proof of Lemma 5.2 is completed.

Now we proceed to the proof of Theorem 5.1. In view of (2.6), the inclusion \( a \in D(A_0^\gamma) \) implies
\[
S(t)a \in C([0,T];D(A_0^\gamma)). \quad (5.12)
\]
Indeed,
\[
\|A_0^\gamma(S(t)a - S(s)a)\|^2 = \|S(t)(A_0^\gamma a) - S(s)(A_0^\gamma a)\|^2 \\
= \sum_{n=1}^\infty |E_{\alpha,1}(-\lambda_n t^\alpha) - E_{\alpha,1}(-\lambda_n s^\alpha)|^2 |(A_0^\gamma a, \varphi_n)|^2.
\]
Applying the Lebesgue convergence theorem, in view of
\[
|E_{\alpha,1}(-\lambda_n t^\alpha)| \leq \frac{C}{1+\lambda_n t^\alpha} \quad \text{for all } n \in \mathbb{N} \text{ and } t > 0
\]
(e.g., Theorem 1.6 (p.35) in [28]), we can verify (5.12).

Because of the condition (5.2) and \( D(A_0^\gamma) \subset H^1(\Omega) \), for \( v \in C([0,T];D(A_0^\gamma)) \), we have \( f(v) \in C([0,T];L^2(\Omega)) \) and \( Qv \in C([0,T];L^2(\Omega)) \). Applying Lemma 5.2 in view of (5.12), we see
\[
Lv \in C([0,T];D(A_0^\gamma)) \quad \text{for } v \in C([0,T];D(A_0^\gamma)). \quad (5.13)
\]
For the further proof, we have to prove the following properties (i) and (ii) which are valid for sufficiently small \( T > 0 \):
The first condition in (5.2) implies that we obtain $u$ for $u \in V$. Let

Proof of (i).

Furthermore, (5.15) implies $u \in C([0,T];D(A_0^\gamma))$. Now we consider

$$A_0^\gamma (Lu(t) - S(t)a) = \int_0^t A_0^\gamma K(t-s)Q(s)u(s)ds + \int_0^t A_0^\gamma K(t-s)f(u(s))ds. \quad (5.14)$$

For any $u \in V$, using

$$\|a\|_{D(A_0^\gamma)} = \|A_0^\gamma a\| \leq m, \quad \|u - S(\cdot)a\|_{C([0,T];D(A_0^\gamma))} \leq m,$$

we obtain

$$\|u(t)\|_{D(A_0^\gamma)} \leq m + \|A_0^\gamma S(t)a\| = m + \|S(t)A_0^\gamma a\| \leq m + C_1 m : = C_2 m. \quad (5.15)$$

The first condition in (5.2) implies that

$$\|f(u(t))\| \leq C_f(C_2m) \quad \text{for all } u \in V \text{ and } 0 < t < T. \quad (5.16)$$

Furthermore, (5.15) implies

$$\|Q(s)u(s)\| \leq C\|u(s)\|_{H^1(\Omega)} \leq C_3 \|A_0^\gamma u(s)\| \leq C_4 \|u(t)\|_{D(A_0^\gamma)} \leq C_4 C_2 m \quad (5.17)$$

because of the inequalities $\gamma \geq 1/2$. Applying (5.16) and (5.17) in (5.14), by means of (2.8), we obtain the estimates

$$\|Lu(t) - S(t)a\|_{D(A_0^\gamma)}$$

$$= \left\| \int_0^t A_0^\gamma K(t-s)Q(s)u(s)ds + \int_0^t A_0^\gamma K(t-s)f(u(s))ds \right\|$$

$$\leq C \int_0^t (t-s)^{1-\gamma\alpha-1}(C_2 C_4 m + C_f(C_2 m))ds \leq C_5 t^{(1-\gamma\alpha)/(1-\gamma)} \leq C_5 T^{(1-\gamma\alpha)/(1-\gamma)}.$$

The constant $C_5 > 0$ depends on $m > 0$ but is independent on $T > 0$. Therefore, choosing $T > 0$ sufficiently small, we complete the proof of the property (i). □

Proof of (ii).

Estimate (5.15) yields that $\|u_1(t)\|_{D(A_0^\gamma)}$, $\|u_2(t)\|_{D(A_0^\gamma)} \leq C_2 m$ for each $u_1, u_2 \in V$. Therefore, the condition (5.2) yields

$$\|f(u_1(s)) - f(u_2(s))\| \leq C_f(C_2m)\|u_1(s) - u_2(s)\|_{D(A_0^\gamma)}, \quad 0 < s < T.$$
Hence, by (5.17) we have the following chain of estimates:

\[
\|Lu_1(t) - Lu_2(t)\|_{D(A_\gamma^0)} \\
= \left\| \int_0^t A_\gamma^0 K(t-s)Q(s)(u_1(s) - u_2(s))ds \\
+ \int_0^t A_\gamma^0 K(t-s)(f(u_1(s)) - f(u_2(s)))ds \right\| \\
\leq C \int_0^t (t-s)^{\alpha(1-\gamma)-1} \|(u_1 - u_2)(s)\|_{D(A_\gamma^0)} ds \\
+ C_f(C_{2m}) \int_0^t (t-s)^{\alpha(1-\gamma)-1} \|(u_1 - u_2)(s)\|_{D(A_\gamma^0)} ds \\
\leq C_6 T^{\alpha(1-\gamma)} \sup_{0<s<T} \|u_1(s) - u_2(s)\|_{D(A_\gamma^0)}.
\]

In the last inequality, the constant $C_6 > 0$ is independent of $T > 0$, and thus one can further choose a sufficiently small constant $T > 0$ satisfying the inequality $\rho := C_6 T^{\alpha(1-\gamma)} < 1$.

The proof of the property (ii) is completed. □

The properties (i) and (ii) allow to apply the contraction theorem to the equation $u = Lu$, which says that this equation has a unique solution $u \in V$ for $0 < t < T$. This solution $u \in C([0,T]; D(A_{\gamma}^0))$ satisfies the estimates (5.15) and the equation

\[
u(t) = S(t)a + \int_0^t K(t-s)Q(s)u(s)ds \\
+ \int_0^t K(t-s)f(u(s))ds, \quad 0 < t < T.
\] (5.18)

Next we have to prove the rest inclusions of (5.6). In the condition (5.2), we can choose $\varepsilon > 0$ such that $0 < \varepsilon < \frac{1}{2}$. By the equation (5.18), we obtain

\[
A_\gamma u(t) = A_\gamma^{1-\gamma} S(t)A_\gamma^0 a + \int_0^t A_\gamma^{1-\varepsilon} K(t-s)A_\gamma^\varepsilon Q(s)u(s)ds \\
+ \int_0^t A_\gamma^{1-\varepsilon} K(t-s)A_\gamma^\varepsilon f(u(s))ds, \quad 0 < t < T.
\]

In the last equation, $a \in D(A_\gamma^0)$ and the estimates

\[
\|A_\gamma^\varepsilon Q(s)u(s)\| \leq C \|A_\gamma^\varepsilon (Q(s)u(s))\| \leq C \|u(s)\|_{H^2(\Omega)} \leq C \|A_\gamma u(s)\|
\]

hold true because of (5.9). Furthermore, by the second condition in (5.2), the inequality (5.15) yields the estimate $\|A_\gamma^\varepsilon f(u(s))\| \leq C_f(C_{2m})$ by the second condition from (5.2).
Thus we reach the chain of the inequalities
\[
\|A_0 u(t)\| \leq C t^{-\alpha(1-\gamma)} \|A_0^\gamma a\| + C \int_0^t (t-s)^{\alpha\varepsilon-1} \|A_0 u(s)\| ds + C \int_0^t (t-s)^{\alpha\varepsilon-1} C_f(C_{2m}) ds \\
\leq C t^{-\alpha(1-\gamma)} \|A_0^\gamma a\| + C_f(C_{2m}) + C \int_0^t (t-s)^{\alpha\varepsilon-1} \|A_0 u(s)\| ds, \quad 0 < t < T.
\]

By $0 < \alpha < 1$, we have $-\alpha(1-\gamma) > -1$ and thus the inclusion $t^{-\alpha(1-\gamma)} \in L^1(0,T)$. Application of the generalized Gronwall inequality yields the estimates
\[
\|A_0 u(t)\| \leq (C t^{-\alpha(1-\gamma)} \|A_0^\gamma a\| + C_f(C_{2m})) \\
+ C \int_0^t (t-s)^{\alpha\varepsilon-1} (s^{-\alpha(1-\gamma)} \|A_0^\gamma a\| + C_f(C_{2m})) ds \\
\leq C t^{-\alpha(1-\gamma)} \|A_0^\gamma a\| + C_f(C_{2m}) + (\|A_0^\gamma a\| + C_f(C_{2m})) t^{\alpha(-\alpha(1-\gamma))}, \quad 0 < t < T.
\]

Therefore, noting that $-\alpha(1-\gamma) < \alpha(\varepsilon - (1-\gamma))$, we have
\[
\|A_0 u(t)\| \leq C_7 (1 + T^{\alpha\varepsilon})(t^{-\alpha(1-\gamma)} + 1), \quad 0 < t < T,
\]
where $C_7 > 0$ depends on $\|A_0^\gamma a\|$ and $C_f(C_{2m})$, $\alpha$, $\varepsilon$. For $\frac{1}{2} < \gamma \leq 1$, we can directly verify that $-2\alpha(1-\gamma) > -1$, so that $\int_0^T \|A_0 u(t)\|^2 dt < \infty$, that is, the inclusion
\[
u \in L^2(0,T;H^2(\Omega)) \tag{5.19}
\]
holds true.

It remains to prove $u - a \in H_\alpha(0,T;L^2(\Omega))$. The inequality (5.16) implies the inclusion $f(u) \in L^2(0,T;L^2(\Omega))$.

The estimate (5.11) and the inclusion (5.19) result in the inclusion $Qu \in L^2(0,T;L^2(\Omega))$.

Therefore, Lemma 2.1 (ii) yields the inclusion
\[
\int_0^t K(t-s)Q(s)u(s) ds + \int_0^t K(t-s)f(u(s)) ds \in H_\alpha(0,T;L^2(\Omega)).
\]

Applying Lemma 2.1 (iii) to the equation (5.10), we reach the inclusion $u - a \in H_\alpha(0,T;L^2(\Omega))$, which completes the proof of the relation (5.6) from the theorem.

Finally, we have to prove the estimate (5.7). By the construction of the solutions $u_a, u_b$ as the fixed points, we have the inequalities
\[
\|u_a\|_{L^2(0,T;H^2(\Omega))}, \|u_b\|_{L^2(0,T;H^2(\Omega))} \leq C(m). \tag{5.20}
\]

On the other hand,
\[
u_a(t) - u_b(t) = S(t)a - S(t)b + \int_0^t K(t-s)Q(s)(u_a - u_b)(s) ds \\
+ \int_0^t K(t-s)(f(u_a(s)) - f(u_b(s))) ds, \quad 0 < t < T.
\]

In view of (5.20), we can use the condition (5.2) and apply the generalized Gronwall inequality. Further details of the derivations are similar to the ones employed in the proof of Theorem 2.1 and we omit them here. Thus, the proof of Theorem 5.1 is completed.  \[\square\]
6 Comparison principles for the semilinear time-fractional diffusion equations

In this section, we derive comparison principles for semilinear time-fractional diffusion equations. To this end, we have to restrict semilinear terms \( f \) which depend only on \( x \) and \( u(x, t) \), but should not depend on its derivatives.

We introduce the class of semilinear terms in terms of \( f \in C^1(\Omega \times [-m, m]) \). For such a function, we can naturally define a mapping \( f : \{ v \in \mathcal{D}(A_0^\gamma) : \| v \|_{\mathcal{D}(A_0^\gamma)} \leq m \} \rightarrow L^2(\Omega) \) by

\[
f(v) := f(x, v(x)), \quad x \in \Omega, \ 0 < t < T.
\]

In what follows, we identify \( f \) with \( f \) through the relation (6.1). For a fixed constant \( M > 0 \), we set

\[
\mathcal{F}_M := \{ f \in C^1(\Omega \times [-m, m]) : \| f \|_{C^1(\Omega \times [-m, m])} \leq M \}.
\]

Now we are ready to formulate and to prove the first comparison principle for the initial-boundary value problems for semilinear time-fractional diffusion equation (5.1).

**Theorem 6.1.** For \( f_1, f_2 \in \mathcal{F}_M \) and \( a_1, a_2 \in H^1_0(\Omega) \), we assume that there exist solutions \( u(f_k, a_k), k = 1, 2 \) to the initial-boundary-value problem (5.1) with the semilinear terms \( f_k, k = 1, 2 \) and the initial values \( a_k, k = 1, 2 \), respectively, which satisfy the inclusions (5.6) and the estimates

\[
|u(f_k, a_k)(x, t)| \leq m, \quad x \in \Omega, \ 0 < t < T, \ k = 1, 2.
\]

If \( f_1 \geq f_2 \) on \( \Omega \times (-m, m) \) and \( a_1 \geq a_2 \) in \( \Omega \), then

\[
u(f_1, a_1) \geq u(f_2, a_2) \quad \text{in} \ \Omega \times (0, T).\]

**Proof.** Henceforth, for simplicity, we denote \( f_k(x, u_j(x, t)) \) by \( f_k(u_j) \) and \( u(f_k, a_k) \) by \( u_k \) for \( j, k = 1, 2 \).

For \( d = 1, 2, 3 \), application of the Sobolev embedding yields \( \mathcal{D}(A_0^\gamma) \subset H^{2\gamma}(\Omega) \subset C(\overline{\Omega}) \) by \( \gamma > \frac{3}{4} \). Hence, (5.6) implies

\[
u_1, u_2 \in C(\overline{\Omega} \times [0, T]).
\]

On the other hand, we have the representations

\[
f_1(u_1(x, t)) - f_2(u_2(x, t)) = f_1(u_1(x, t)) - f_1(u_2(x, t)) + (f_1 - f_2)(u_2(x, t))
\]

\[
\quad = g(x, t)(u_1(x, t) - u_2(x, t)) + H(x, t), \quad (x, t) \in \Omega \times (0, T),
\]

where we set

\[
g(x, t) := \begin{cases} \frac{f_1(u_1(x, t)) - f_1(u_2(x, t))}{u_1(x, t) - u_2(x, t)} & \text{if} \ u_1(x, t) \neq u_2(x, t), \\ f'_1(u_1(x, t)) & \text{if} \ u_1(x, t) = u_2(x, t), \end{cases}
\]

and

\[
H(x, t) := (f_1 - f_2)(u_2(x, t)), \quad (x, t) \in \Omega \times (0, T).
\]
By the inclusions \((6.5)\) and \(f \in C^1(\Omega \times [−m, m])\), we can verify that \(g, H \in C(\overline{\Omega} \times [0, T])\). Furthermore \(f_1 \geq f_2\) implies \(H \geq 0\) in \(\Omega \times (0, T)\).

Setting now \(y := u_1 - u_2\) and \(a := a_1 - a_2\), the function \(y\) is a solution to the following initial-boundary value problem:

\[
\begin{align*}
\partial_t^\alpha (y - a) + Ay - g(x, t)y &= H \geq 0 \quad \text{in } \Omega \times (0, T), \\
\partial_{\nu_A} y + \sigma y &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
y(x, \cdot) - a(x) &\in H_\alpha(0, T) \quad \text{for almost all } x \in \Omega.
\end{align*}
\]  
(6.6)

Since \(g \in C(\overline{\Omega} \times [0, T])\) does not in general satisfy the regularity condition \((1.2)\), we cannot directly apply Theorem \(4.1\). Thus, we first approximate \(g\) by \(g_n \in C^\infty(\Omega \times [0, T]), n \in \mathbb{N}\) such that \(g_n \rightarrow g\) in \(C(\Omega \times [0, T])\) as \(n \rightarrow \infty\) and consider a sequence of the following problems:

\[
\begin{align*}
\partial_t^\alpha (y_n - a) + Ay_n - g_n(x, t)y_n &= H \quad \text{in } \Omega \times (0, T), \\
\partial_{\nu_A} y_n + \sigma y_n &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
y_n(x, \cdot) - a(x) &\in H_\alpha(0, T) \quad \text{for almost all } x \in \Omega.
\end{align*}
\]  
(6.7)

Then Theorem \(2.1\) yields that for any \(n \in \mathbb{N}\) there exists a unique solution \(y_n \in L^2(0, T; H^2(\Omega))\) to the problem \((6.7)\) such that \(y_n - a \in H_\alpha(0, T; L^2(\Omega))\). Moreover, since \(g_n\) satisfies the regularity condition \((1.2)\), Theorem \(4.1\) yields that

\[
y_n \geq 0 \quad \text{in } \Omega \times (0, T) \quad \text{for each } n \in \mathbb{N}.
\]  
(6.8)

On the other hand, setting \(z_n := y_n - y\) in \(\Omega \times (0, T)\), the equations \((6.6)\) and \((6.7)\) allow to characterize \(z_n\) as solution to the problem

\[
\begin{align*}
\partial_t^\alpha z_n + A z_n &= (g_n - g)y + g_n z_n \quad \text{in } \Omega \times (0, T), \\
\partial_{\nu_A} z_n + \sigma z_n &= 0 \quad \text{on } \partial \Omega \times (0, T), \\
z_n(x, \cdot) &\in H_\alpha(0, T) \quad \text{for almost all } x \in \Omega.
\end{align*}
\]

Similarly to our treatment of the problem \((2.12)\), we rewrite the first equation in the form

\[
\partial_t^\alpha z_n + A_0 z_n \\
= (g_n - g)y + g_n z_n + \sum_{j=1}^d b_j(t) \partial_j z_n + (c_0 + c(t))z_n \quad \text{in } \Omega \times (0, T),
\]

and repeat the same arguments as the ones employed for \((2.13)\) to obtain the following
chain of the estimates:
\[
\|A_0^{1/2}z_n(t)\| = \left\| \int_0^t A_0^{1/2}K(t-s)(g_n-g)(s)y(s)ds \right. \\
+ \int_0^t A_0^{1/2}K(t-s) \left( (g_n(s) + c_0 + c(s))z_n(s) + \sum_{j=1}^d b_j(s)\partial_j z_n(s) \right) ds \right\| \\
\leq C \int_0^t (t-s)^{1/2}\alpha^{-1}\|g_n-g\|_{C(\overline{\Omega} \times [0,T])}\|y(s)\|ds \\
+ \int_0^t (t-s)^{1/2}\alpha^{-1}(\|g_n\|_{C(\overline{\Omega} \times [0,T])} + c_0 + \|c\|_{C(\overline{\Omega} \times [0,T])}) \\
+ \max_{1 \leq j \leq d} \left\| b_j \right\|_{C(\overline{\Omega} \times [0,T])}\|z_n(s)\|_{H^1(\Omega)}ds \\
\leq Ct^{1/2}\alpha\|g_n-g\|_{C(\overline{\Omega} \times [0,T])} + C \int_0^t (t-s)^{1/2}\alpha^{-1}\|A_0^{1/2}z_n(s)\|ds \\
\leq C_1\|g_n-g\|_{C(\overline{\Omega} \times [0,T])} + C_1 \int_0^t (t-s)^{1/2}\alpha^{-1}\|A_0^{1/2}z_n(s)\|ds, \quad 0 < t < T.
\]

Here the constant $C_1 > 0$ depends also on $\|y\|_{C([0,T];L^2(\Omega))}$. In the derivations, we used the relation $\sup_{n \in \mathbb{N}} \|g_n\|_{C(\overline{\Omega} \times [0,T])} < \infty$, which is justified by $g_n \rightarrow g$ in $C(\overline{\Omega} \times [0,T])$ and $y := u_1 - u_2 \in C(\overline{\Omega} \times [0,T])$ that follows from \((6.9)\).

Therefore, the generalized Gronwall inequality yields
\[
\|A_0^{1/2}z_n(t)\| \leq C_2\|g_n-g\|_{C(\overline{\Omega} \times [0,T])} + C_2 \int_0^t (t-s)^{1/2}\alpha^{-1}\|g_n-g\|_{C(\overline{\Omega} \times [0,T])}ds \\
\leq C_3\|g_n-g\|_{C(\overline{\Omega} \times [0,T])}, \quad 0 < t < T, \ n \in \mathbb{N}.
\]

Hence, $z_n \to 0$ in $L^\infty(0,T;\mathcal{D}(A_0^{1/2}))$ as $n \to \infty$, and thus $y_n \to y$ in $L^\infty(0,T;\mathcal{D}(A_0^{1/2}))$ as $n \to \infty$. Because of the relation \((6.8)\), we reach the inequality $y \geq 0$ in $\Omega \times (0,T)$. Thus the proof of Theorem \((6.1)\) is completed.

We note that in the formulation of Theorem \((6.1)\) the boundedness condition \((6.3)\) has been assumed. However, this condition is not automatically guaranteed for solutions to the initial-boundary value problem \((5.1)\). In the rest of this section, we present another comparison principle in terms of the upper and lower solutions, which does not require this boundedness condition.

**Definition 6.1.** Let $f \in \mathcal{F}_M$. The functions $\overline{u}$ and $\underline{u}$ satisfying \((5.6)\) are called an upper solution and a lower solution for the solution $u$ to the problem \((5.1)\), respectively, if
\[
\begin{aligned}
\partial_t \overline{u} + A\overline{u} &\geq f(\overline{u}) \quad \text{in} \ \Omega \times (0,T), \\
\overline{u}(x) &\geq u(x,0), \quad x \in \Omega, \\
\partial_{\nu} \overline{u} + \sigma \overline{u} &\equiv 0 \quad \text{on} \ \partial \Omega \times (0,T), \\
\overline{u} - \underline{u} &\in H_\Omega(0,T;L^2(\Omega)),
\end{aligned}
\]
and
\[
\begin{aligned}
\partial_t^\alpha (u - \bar{u}) + Au &\leq f(u) \quad \text{in } \Omega \times (0, T), \\
\bar{u}(x) &\leq u(x, 0), \quad x \in \Omega, \\
\partial_{\nu} A u + \sigma u &\equiv 0 \quad \text{on } \partial \Omega \times (0, T), \\
u - \bar{u} &\in H_\alpha(0, T; L^2(\Omega)),
\end{aligned}
\]
with $\bar{u}, \underline{u} \in L^2(\Omega)$.

We recall that $F_M$ is defined by (6.2).

Then the following result holds true:

**Theorem 6.2.** We arbitrarily choose $T > 0$. We assume that there exist an upper solution $\bar{u}$ and a lower solution $\underline{u}$ to the problem (5.1) and that $|\bar{u}(x, t)|, |\underline{u}(x, t)| \leq m$ for $x \in \Omega$ and $0 < t < T$.

Then there exists a unique solution $u_{\alpha}$ to the problem (5.1) in the class (5.6) and
\[
\underline{u}(x, t) \leq u_{\alpha}(x, t) \leq \bar{u}(x, t), \quad x \in \Omega, \ 0 < t < T.
\]

**Proof.** For the functions $\overline{y}, \underline{y} \in L^2(0, T; L^2(\Omega))$ satisfying the condition $\overline{y}(x, t) \leq \underline{y}(x, t)$ for $(x, t) \in \Omega \times (0, T)$, we set
\[
[y, \overline{y}] := \{ y \in L^2(0, T; L^2(\Omega)); y(x, t) \leq \overline{y}(x, t), \underline{y}(x, t) \}\,
\]
where the inequalities hold true for almost all $(x, t) \in \Omega \times (0, T)$.

The key element of our proof of Theorem 6.2 is the following fixed point theorem in an ordered Banach space:

**Lemma 6.1.** Let an operator $L : [y, \overline{y}] \rightarrow [y, \overline{y}] \subset L^2(0, T; L^2(\Omega))$ be compact and increasing, that is, $Lv \geq Lw$ in $\Omega \times (0, T)$ if $v, w \in [y, \overline{y}]$ and $v \geq w$ in $\Omega \times (0, T)$.

Then the operator $L$ possesses fixed points $y^*, y^- \in [y, \overline{y}]$ such that
\[
y^* = \lim_{k \rightarrow \infty} L^k y^- \quad \text{and} \quad y^- = \lim_{k \rightarrow \infty} L^k y^* \quad \text{in } L^2(0, T; L^2(\Omega)).
\]

The proof of Lemma 6.1 can be found in [2]. Lemma 6.1 implies the following result:

**Lemma 6.2.** In Lemma 6.1, we further assume that the operator $L$ possesses a unique fixed point. Then
\[
\underline{y} \leq y \leq \overline{y} \quad \text{in } \Omega \times (0, T).
\]

**Proof.** Proof of Lemma 6.2. Indeed, since $L[y, \overline{y}] \subset [y, \overline{y}]$, we see that $y \leq L\underline{y}$ and $L\overline{y} \leq \overline{y}$. Moreover, since the operator $L$ is increasing, the inequality $L\underline{y} \leq L\overline{y}$ holds true. Hence,
\[
\underline{y} \leq L\underline{y} \leq L\overline{y} \leq \overline{y} \quad \text{in } \Omega \times (0, T). \]

Applying the operator $L$ once again, we obtain
\[
\underline{y} \leq L\underline{y} \leq L^2\underline{y} \leq L^2\overline{y} \leq L\overline{y} \leq \overline{y}.
\]

Repeating the same arguments, we reach the inequalities
\[
\underline{y} \leq L^k \underline{y} \leq L^k \overline{y} \leq \overline{y} \quad \text{in } \Omega \times (0, T) \quad \text{for } k \in \mathbb{N}.
\]

By the uniqueness of the fixed point, letting $k \rightarrow \infty$, we see that $\underline{y} \leq y \leq \overline{y}$. \(\square\)
Thus is a compact embedding (see, e.g., [33]), we obtain that
\[ v - a \in H_{\alpha}(0,T; L^2(\Omega)), \]
\[ \partial_{\nu} u + \sigma v = 0 \quad \text{on } \partial \Omega \times (0,T), \]
where \( M > 0 \) is the constant in the definition (6.2) of the set \( \mathcal{F}_M \).

We verify that \( L: [u, \overline{u}] \subset L^2(0,T; L^2(\Omega)) \rightarrow L^2(0,T; L^2(\Omega)) \) is well-defined. Indeed let \( u \in [u, \overline{u}] \). By \( |u(x,t)|, |\overline{u}(x,t)| \leq m \) for all \( x \in \overline{\Omega} \) and \( 0 \leq t \leq T \), we see that \( |u(x,t)| \leq m \) for almost all \( x \in \Omega \) and \( t \in (0,T) \). Therefore \( f(u(x,t)) \) can be defined for almost all \( (x,t) \in \Omega \times (0,T) \), and \( f(u(x,t)) \in L^\infty(\Omega \times (0,T)) \). Hence, since \( (M + 1)u + f(u) \in L^\infty(\Omega \times (0,T)) \subset L^2(0,T; L^2(\Omega)) \), we apply Theorem 2.1 to (6.11), and we verify that there exists a unique solution \( v \in L^2(0,T; H^2(\Omega)) \) to (6.11) satisfying \( v - a \in H_{\alpha}(0,T; L^2(\Omega)) \), and
\[
\|v - a\|_{H_{\alpha}(0,T; L^2(\Omega))} + \|v\|_{L^2(0,T; H^2(\Omega))} \leq C((\alpha + 1)u + f(u))_{L^2(0,T; L^2(\Omega))}
\]
by using
\[
((M + 1)u + f(u))_{L^2(0,T; L^2(\Omega))} \leq (M + 1)\|u\|_{L^\infty(\Omega \times (0,T))} + \|f(u)\|_{L^\infty(\Omega \times (0,T))} \leq m(M + 1) + M.
\]
Thus \( L \) is well-defined and the estimate (6.12) holds. \( \square \)

For the operator \( L \), we will verify the following properties (i) - (iii):

(i) \( L \) is a compact operator from \([u, \overline{u}] \subset L^2(0,T; L^2(\Omega))\) into itself. In other words, the set \( L [u, \overline{u}] \) is relatively compact in \( L^2(0,T; L^2(\Omega)) \).

Verification of (i). Let \( u \in [u, \overline{u}] \). Then the estimate (6.12) means
\[
\|Lu\|_{L^2(0,T; H^2(\Omega))} + \|Lu - a\|_{H_{\alpha}(0,T; L^2(\Omega))} \leq C_4.
\]
For \( a \in H^1(\Omega) \), we deduce that
\[
\|Lu - a\|_{L^2(0,T; H^1(\Omega))} + \|Lu - a\|_{H_{\alpha}(0,T; L^2(\Omega))} \leq C_4 + \sqrt{T}\|a\|_{H^1(\Omega)}.
\]
Since
\[
L^2(0,T; H^1(\Omega)) \cap H_{\alpha}(0,T; L^2(\Omega)) \subset L^2(0,T; L^2(\Omega))
\]
is a compact embedding (see, e.g., [33]), we obtain that \( v \rightarrow Lv - a \) is compact from \([u, \overline{u}] \subset L^2(0,T; L^2(\Omega))\) to \( L^2(0,T; L^2(\Omega)) \). Thus the compactness of the operator \( L \) is proved. \( \square \)

(ii) \( Lv \geq Lw \) in \( \Omega \times (0,T) \) if \( v \geq w \) in \( \Omega \times (0,T) \) and \( v, w \in [u, \overline{u}] \).

Verification of (ii). Setting \( y := v - w \) and \( z := Lv - Lw \) and applying the mean value theorem for \( f(v) - f(w) \), we obtain the representation
\[
\partial_t^2(z - a) + Az + (M + 1)z = (M + 1)y + f'(\mu(x,t))y \quad \text{in } \Omega \times (0,T),
\]
where $\mu(x,t)$ is a number between $v(x,t)$ and $u(x,t)$.

The inclusion $f \in \mathcal{F}_M$ implies $|f'(\mu(x,t))| \leq M$. Therefore, in view of $y \geq 0$ in $\Omega \times (0,T)$, we estimate

$$(M + 1)y + f'(\mu(x,t))y \geq (M + 1 - M)y(x,t) \geq 0 \quad \text{in } \Omega \times (0,T).$$

Hence, the function $z$ satisfies

$$
\begin{cases}
\partial_t^\alpha (z - a) + Az + (M + 1)z \geq 0 & \text{in } \Omega \times (0,T), \\
\partial_{\nu_A} z + \sigma z = 0 & \text{on } \partial \Omega \times (0,T).
\end{cases}
$$

Theorem 2.1 implies that $z \in L^2(0,T;H^2(\Omega))$ and $z - a \in H_\alpha(0,T;L^2(\Omega))$ and Theorem 4.1 yields the inequality $z \geq 0$ in $\Omega \times (0,T)$. Therefore $Lv \geq Lw$ in $\Omega \times (0,T)$. □

 Verification of (iii). In order to prove the above inclusion, we show that $u \leq Lu$ and $L\overline{u} \leq \overline{u}$ in $\Omega \times (0,T)$. Indeed, let $\overline{u} \leq u \leq \overline{u}$ in $\Omega \times (0,T)$. Since $L$ is increasing, we have the inequalities $Lu \leq Lu \leq L\overline{u}$. Hence, $u \leq Lu \leq \overline{u}$ in $\Omega \times (0,T)$, which proves (iii). □

Proof of $u \leq Lu$.

Setting $v := Lu$, we have

$$
\begin{cases}
\partial_t^\alpha (v - a) + Av + (M + 1)v = (M + 1)u + f(u) & \text{in } \Omega \times (0,T), \\
v - a \in H_\alpha(0,T;L^2(\Omega)), \\
\partial_{\nu_A} v + \sigma v = 0 & \text{on } \partial \Omega \times (0,T).
\end{cases}
$$

On the other hand, since $u$ is a lower solution, we see that

$$
\begin{cases}
\partial_t^\alpha (u - a) + Au + (M + 1)u \leq (M + 1)u + f(u) & \text{in } \Omega \times (0,T), \\
u - a \in H_\alpha(0,T;L^2(\Omega)), \\
\partial_{\nu_A} u + \sigma u = 0 & \text{on } \partial \Omega \times (0,T).
\end{cases}
$$

Therefore, $z := v - u = Lu - u$ satisfies

$$
\begin{cases}
\partial_t^\alpha (z - (a - \overline{u})) + Az + (M + 1)z \geq 0 & \text{in } \Omega \times (0,T), \\
z - (a - \overline{u}) \in H_\alpha(0,T;L^2(\Omega)), \\
\partial_{\nu_A} z + \sigma z = 0 & \text{on } \partial \Omega \times (0,T), \\
a - \overline{u} \geq 0 & \text{in } \Omega.
\end{cases}
$$

According to Theorem 4.1, the inequality $z \geq 0$ holds true in $\Omega \times (0,T)$, that is, $Lu \geq u$ in $\Omega \times (0,T)$. By similar arguments, we prove that $L\overline{u} \leq \overline{u}$ in $\Omega \times (0,T)$. Thus the property (iii) is proved. □

The properties (i)-(iii) allow to apply Lemmas 6.1 and 6.2, which completes the proof of Theorem 6.2.

Theorem 6.2 ensures that, for a given $T > 0$, it is sufficient to determine an upper and a lower solutions to the problem 5.1 in order to guarantee the existence of the solution. This technique is called a monotonicity method which is also related to the Perron method or the concept of viscosity solutions. For applications of the monotonicity method to parabolic equations, that is, $\alpha = 1$ in 5.1, we refer, e.g., to [2], [15], [25], and [26]. Theorem 6.2 asserts a corresponding result for the case $0 < \alpha < 1$ and readily yields the following statement:

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Proposition 6.1. Let $u$ be a solution to the problem (5.1) satisfying (5.6) and $|u(x,t)| \leq m$ for $(x,t) \in \Omega \times (0,T)$ with some constant $m > 0$. Then:

(i) For any upper solution $\overline{u}$ to the problem (5.1) satisfying (5.6) and $|\overline{u}(x,t)| \leq m$ for $(x,t) \in \Omega \times (0,T)$, we have

$$u(x,t) \leq \overline{u}(x,t) \quad \text{for all } (x,t) \in \Omega \times (0,T).$$

(ii) For any lower solution $\underline{u}$ to the problem (5.1) satisfying (5.6) and $|\underline{u}(x,t)| \leq m$ for $(x,t) \in \Omega \times (0,T)$, we have

$$\underline{u}(x,t) \leq u(x,t) \quad \text{for all } (x,t) \in \Omega \times (0,T).$$

In the rest of this section, we focus on a special case of the operator $-A$ in the form

$$-Av := \text{div}(p(x)\nabla v(x)) + c(x)v, \quad x \in \Omega,$$

$$D(A) = \{v \in H^2(\Omega); \partial_{\nu_A}v + \sigma v = 0 \text{ on } \partial\Omega\},$$

where $p \in C^2(\overline{\Omega})$, $p > 0$ on $\overline{\Omega}$ and $c \in C^2(\overline{\Omega})$, $c \leq 0$, $c \not\equiv 0$ in $\Omega$.

For this operator, Theorem 6.2 leads to the following result:

Proposition 6.2. We assume that $f \in C^1(\mathbb{R})$ is a monotone decreasing function and that there exists a solution $u_\infty \in H^2(\Omega)$ to the boundary-value problem

$$\begin{cases}
    Au_\infty(x) = f(u_\infty(x)), & x \in \Omega, \\
    \partial_{\nu_A}u_\infty + \sigma u_\infty = 0 & \text{on } \partial\Omega, \\
    |u_\infty(x)| \leq m, & x \in \Omega,
\end{cases}$$

with the operator $-A$ defined by (6.13).

Then for any $T > 0$ and any $a \in D(A_0^\gamma)$, the initial-boundary value problem (5.1) possesses a unique solution $u_a$ in the class (5.6) such that the inequality

$$|u_a(x,t) - u_\infty(x)| \leq C e^{-\alpha t} \|\varphi_1(x)\|, \quad x \in \Omega, t \in (0,T)$$

holds true with a certain constant $C > 0$. In particular, we have

$$|u_a(x,t) - u_\infty(x)| \leq C t^{-\alpha} \|\varphi_1(x)\| \quad \text{for all } x \in \Omega \text{ as } t \to \infty. \quad (6.15)$$

Proof. The proof is similar to the one of Proposition 1.4 (pp.25-26) in [15], see also [26].

Since $c(x) \leq 0$, $\not\equiv 0$ on $\Omega$ and belongs to the space $C^2(\overline{\Omega})$, all the eigenvalues of the operator $A$ are known to be positive:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty,$$

where $\lambda_k$, $k \in \mathbb{N}$ are numbered according to their multiplicities. Now we choose an eigenfunction $\varphi_1$ for the minimum eigenvalue $\lambda_1$ with $\|\varphi_1\| = 1$. Then we apply the following well-known result.

Lemma 6.3. The multiplicity of the eigenvalue $\lambda_1$ is one and $\varphi_1(x) > 0$ for $x \in \overline{\Omega}$ or $\varphi_1(x) < 0$ for $x \in \overline{\Omega}$.
As for the proof, we can refer, for example, to Lemma 1.4 (p.96) and Theorem 1.2 (p.97) in [20].

Thus we choose such an eigenfunction \( \varphi_1 \) such that \( \varphi_1(x) > 0 \) for \( x \in \Omega \). Then we can find a sufficiently large constant \( M_1 > 0 \) such that

\[
    u_{\infty}(x) - M_1 \varphi_1(x) \leq a(x) \leq u_{\infty}(x) + M_1 \varphi_1(x), \quad x \in \overline{\Omega}.
\]

(6.17)

We set

\[
\begin{cases}
    \overline{u}(x) := \overline{u}(x, 0) = u_{\infty}(x) + M_1 \varphi_1(x), \\
    \underline{u}(x) := u_{\infty}(x) - M_1 \varphi_1(x), \quad x \in \Omega
\end{cases}
\]

and

\[
\begin{cases}
    \overline{u}(x, t) := u_{\infty}(x) - M_1 E_{\alpha, 1}(1 - \lambda_1 t^\alpha) \varphi_1(x), \\
    \underline{u}(x, t) := u_{\infty}(x) + M_1 E_{\alpha, 1}(1 - \lambda_1 t^\alpha) \varphi_1(x), \quad (x, t) \in \Omega \times (0, T),
\end{cases}
\]

where the Mittag-Leffler function \( E_{\alpha, 1} \) is defined by

\[
    E_{\alpha, 1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad z \in \mathbb{C}
\]

(see, e.g., [28]). Let \( T > 0 \) be arbitrary. Recalling that

\[
    -Au(x) = \text{div} (p(x) \nabla w) + c(x)w, \quad x \in \Omega
\]

with \( c(x) \leq 0, \neq 0 \) on \( \overline{\Omega} \) and \( \partial_t^\alpha (E_{\alpha, 1}(1 - \lambda_1 t^\alpha) - 1) = -\lambda_1 E_{\alpha, 1}(1 - \lambda_1 t^\alpha) \), we have \( \overline{u} - \underline{u} \in H_\alpha(0, T; L^2(\Omega)) \),

\[
    \partial_t^\alpha (\overline{u} - \underline{u}) = \lambda_1 M_1 E_{\alpha, 1}(1 - \lambda_1 t^\alpha) \varphi_1(x)
\]

and

\[
    A\underline{u} = Au_{\infty} - \lambda_1 M_1 E_{\alpha, 1}(1 - \lambda_1 t^\alpha) \varphi_1(x), \quad x \in \Omega, \quad 0 < t < T,
\]

so that

\[
    \partial_t^\alpha (\overline{u} - \underline{u}) + A\underline{u} = Au_{\infty} = f(u_{\infty}) \quad \text{in} \quad \Omega \times (0, T).
\]

It is known that \( E_{\alpha, 1}(1 - \lambda_1 t^\alpha) \geq 0 \) for \( t \geq 0 \) (see, e.g., [10]) and

\[
    \underline{u}(x, t) \leq u_{\infty}(x), \quad (x, t) \in \Omega \times (0, T).
\]

Since \( f \) is decreasing, we obtain \( f(u_{\infty}(x)) \leq f(\underline{u}(x, t)) \). Therefore,

\[
    \partial_t^\alpha (\overline{u} - \underline{u}) + A\underline{u} \leq f(\underline{u}) \quad \text{in} \quad \Omega \times (0, T).
\]

Since \( \partial_{\nu, A}u + \sigma u = 0 \) on \( \partial \Omega \times (0, T) \), the function \( \underline{u} \) is a lower solution to the initial-boundary value problem (5.1) with the operator \(-A\) defined by (6.13). Similarly we can verify that \( \overline{u} \) is an upper solution. Hence, Theorem 6.2 yields that there exists a unique solution \( u \in L^2(0, T; H^2(\Omega)) \) such that

\[
    u - a \in H_\alpha(0, T; L^2(\Omega)), \quad \underline{u}(x, t) \leq u(x, t) \leq \overline{u}(x, t), \quad (x, t) \in \Omega \times (0, T),
\]

that is,

\[
    |u(x, t) - u_{\infty}(x)| \leq M_1 E_{\alpha, 1}(1 - \lambda_1 t^\alpha) \varphi_1(x), \quad x \in \Omega, \quad 0 < t < T.
\]

Since \( T > 0 \) is arbitrary and \( |E_{\alpha, 1}(1 - \lambda_1 t^\alpha)| \leq \frac{C}{t^{\alpha}} \) as \( t \to \infty \) (Theorem 1.6 (p.35) in [28]), the proof of Proposition 6.2 is completed. \( \square \)
In general, the solution to the boundary-value problem (6.14) may be not unique. However, under our assumptions, the estimate (6.15) implies that \( u_\infty \) is uniquely determined as the limit of \( u_a \) as \( t \to \infty \). In the homogeneous case \( (f \equiv 0) \), the asymptotic behavior of the solution \( u_a \) is known (see, e.g., [16], [31], [35]): \( \| u_a(\cdot, t) \| = O(t^{-\alpha}) \) as \( t \to \infty \) that is the same as in the relation (6.16).

We close this section with two examples of the results presented in Proposition 6.1 in the case \( A = -\Delta \) with the homogeneous Neumann boundary condition, that is, \( \sigma = 0 \) on \( \partial \Omega \). In this case, the boundary condition can be represented as follows:

\[
\partial_{\nu} v = \partial_{\nu} v := \nabla v \cdot \nu \quad \text{on } \partial \Omega.
\]

We assume that there exists a solution \( u_a \) to the initial-boundary value problem (5.1) in the class (5.6). Then we present two examples, where we can estimate \( u(x, t) - a(x) \) for small \( t > 0 \).

**Example 6.1.** Let \( \sigma \equiv 0 \) on \( \partial \Omega \) and

\[
f(\eta) = -\frac{\eta}{1 + |\eta|}, \quad \eta \in \mathbb{R}.
\]

We assume that

\[
a \in C^2(\overline{\Omega}), \quad a \geq 0 \quad \text{in } \Omega, \quad \partial_{\nu} a = 0 \quad \text{on } \partial \Omega. \tag{6.18}
\]

Then we choose a constant \( \rho > 0 \) so large that

\[
\Delta a(x) \leq \Gamma(\alpha + 1) \rho, \quad x \in \overline{\Omega}. \tag{6.19}
\]

Then

\[
0 \leq u_a(x, t) - a(x) \leq \rho t^\alpha, \quad x \in \Omega, \quad 0 \leq t < T. \tag{6.20}
\]

**Proof of (6.20).** We set

\[
\overline{u}(x, t) = a(x) = 0, \quad \overline{u}(x, t) = \rho t^\alpha + a(x), \quad x \in \Omega, \quad 0 < t < T.
\]

Then the equation

\[
\partial_{t}^\alpha (\overline{u} - a) - \Delta \overline{u} - f(\overline{u}) = 0
\]

holds trivially, and so

\[
\left\{ \begin{array}{l}
\partial_{t}^\alpha (\overline{u} - a) - \Delta \overline{u} - f(\overline{u}) = 0 \quad \text{in } \Omega \times (0, T),
0 = a(x) \leq u(x, 0) = a(x), \quad x \in \Omega, \\
\partial_{\nu} \overline{u} = 0 \quad \text{on } \partial \Omega \times (0, T),
\end{array} \right.
\]

which means that \( \overline{u}(x, t) \equiv 0 \) is a lower solution. Moreover, the conditions (6.18) and (6.19) imply

\[
\left\{ \begin{array}{l}
\partial_{t}^\alpha (\overline{u}(x, t) - a(x)) - \Delta \overline{u} - f(\overline{u}) = \rho \Gamma(\alpha + 1) - \Delta a + \frac{a(x) + \rho t^\alpha}{1 + |a(x) + \rho t^\alpha|} \\
\geq \rho \Gamma(\alpha + 1) - \Delta a(x) \geq 0, \quad x \in \Omega, \quad 0 < t < T, \\
\overline{u}(x, 0) = a(x), \quad x \in \Omega, \\
\partial_{\nu} \overline{u} = 0 \quad \text{on } \partial \Omega \times (0, T).
\end{array} \right.
\]
Therefore, \( \Pi \) and \( \underline{u} \) defined above are an upper and a lower solutions, respectively, and Proposition \ref{prop:monotonicity} leads to the estimate \eqref{eq:estimate}. \( \Box \)

We note that for \( \alpha = 1 \) the equation considered in Example \ref{example:enzyme} is a semilinear parabolic equation

\[
\partial_t u - \Delta u = -\frac{u}{1 + |u|},
\]

which is a governing equation for an enzyme process. The monotonicity method is well-known for this type of semilinear parabolic equations (e.g., \cite{15} and \cite{25}).

**Example 6.2.** Let \( f \in C^1(0, \infty) \) be an increasing function in \((0, \infty)\). We assume that \( a \in C^2(\Omega) \), \( \partial_\nu a = 0 \) on \( \partial \Omega \).

Then for an arbitrary but fixed \( \varepsilon \in (0, \alpha) \), there exists \( T_1 = T_1(\varepsilon) > 0 \) such that

\[
\underline{u}(x, t) - a(x) \leq t^{\alpha - \varepsilon}, \quad 0 < t < T_1.
\]

In order to prove the inequality \eqref{eq:upper_solution}, we set \( \Pi(x, t) = t^{\alpha - \varepsilon} + a(x) \) for \((x, t) \in \Omega \times (0, T_1)\) with \( T_1 \) which will be selected later. Then

\[
\partial_t \Pi - \Delta \Pi = \frac{\Gamma(\alpha - \varepsilon + 1)}{\Gamma(-\varepsilon + 1)} t^{-\varepsilon} - \Delta a, \quad x \in \Omega, 0 < t < T_1.
\]

Moreover, setting \( M_1 := \max_{x \in \Omega} a(x) \), we deduce

\[
f(\Pi(x, t)) = f(t^{\alpha - \varepsilon} + a(x)) \leq f(T_1^{\alpha - \varepsilon} + M_1), \quad x \in \Omega, 0 < t < T_1.
\]

We note that \( T_1^{\alpha - \varepsilon} \) can be arbitrarily large if we choose \( T_1 > 0 \) sufficiently small. Therefore, in view of \( \alpha - \varepsilon > 0 \), we can choose \( T_1 > 0 \) sufficiently small such that

\[
\frac{\Gamma(\alpha - \varepsilon + 1)}{\Gamma(-\varepsilon + 1)} T_1^{-\varepsilon} \geq f(T_1^{\alpha - \varepsilon} + M_1) + \Delta a(x), \quad x \in \Omega.
\]

For this \( T_1 \), in terms of \eqref{eq:upper_solution} and \eqref{eq:lower_solution}, we can readily see

\[
\partial_t \Pi - \Delta \Pi \geq f(\Pi) \quad \text{in } \Omega \times (0, T_1).
\]

Since \( \Pi(x, 0) = a(x) = u_a(x, 0) \) and \( \partial_\nu \Pi = 0 \) on \( \partial \Omega \times (0, T_1) \), the function \( \Pi \) is an upper solution for \( 0 < t < T_1 \), which yields the inequality \eqref{eq:upper_solution}.

Next we look for a suitable lower solution. In addition to the conditions \eqref{eq:lower_solution}, we assume that there exists a constant \( \delta_1 > 0 \) such that

\[
a(x) \geq \delta_1 > 0 \quad \text{for all } x \in \Omega.
\]

Then we set

\[
M_2 := \max_{x \in \Omega} (-\Delta a(x)).
\]
The lower solution is introduced in the form

\[ u(x,t) = a(x) - \rho t^\alpha, \quad x \in \Omega, \; 0 < t < \left( \frac{\delta_1}{2\rho} \right)^{\frac{1}{\alpha}} =: T_2, \]

where \( \delta_1 > 0 \) is the constant in the condition (6.25) and the constant \( \rho > 0 \) will be selected later. Then, (6.25) implies

\[ u(x,t) \geq \frac{\delta_1}{2}, \quad x \in \Omega, \; 0 < t < T_2. \]

Moreover, the inequalities

\[ \partial_t^\alpha (u - a) - \Delta u = -\Gamma(\alpha + 1)\rho - \Delta a \leq -\Gamma(\alpha + 1)\rho + M_2 \]

and

\[ f(u(x,t)) = f(a(x) - \rho t^\alpha) \geq f(\delta_1 - \rho T_2^\alpha) = f\left( \delta_1 - \frac{\delta_1}{2} \right) = f\left( \frac{\delta_1}{2} \right) \]

hold true. Now we choose a constant \( \rho \) sufficiently large such that \( T_2 := \left( \frac{\delta_1}{2\rho} \right)^{\frac{1}{\alpha}} \) is sufficiently small and moreover

\[ \rho \geq \frac{1}{\Gamma(\alpha + 1)} \left( M_2 - f\left( \frac{\delta_1}{2} \right) \right). \]

Then

\[ \partial_t^\alpha (u - a) - \Delta u \leq f(u) \quad \text{in} \; \Omega \times (0,T_2). \]

Since \( \partial_n u = 0 \) on \( \partial \Omega \times (0,T_2) \) and \( u(x,0) = a(x) \) for \( x \in \Omega \), the function \( u \) is a lower solution, which leads to the estimate

\[ a(x) - \frac{1}{\Gamma(\alpha + 1)} \left( M_2 - f\left( \frac{\delta_1}{2} \right) \right) t^\alpha \leq u_a(x,t) \]

for \( 0 < t < T_2 \), where \( T_2 > 0 \) is sufficiently small.

Summarizing the results presented above, under the conditions (6.21) and (6.25), for any \( \varepsilon \in (0, \alpha) \), there exists a constant \( T_{\varepsilon,a} > 0 \) such that

\[ -\frac{1}{\Gamma(\alpha + 1)} \left( M_2 - f\left( \frac{\delta_1}{2} \right) \right) t^\alpha \leq u_a(x,t) - a(x) \]

\[ \leq t^{\alpha - \varepsilon}, \quad x \in \Omega, \; 0 < t < T_{\varepsilon,a}. \]

Finally let us emphasize that the construction of upper and lower solutions is not unique. For example, the right-hand side of the inequality (6.28) has the form \( t^{\alpha - \varepsilon} \).

However, under the conditions (6.21) we can obtain a different estimate as follows: For a sufficiently small \( T_3 > 0 \), we can choose a large constant \( M_3 = M_3(T_3) > 0 \) such that

\[ \| \Delta a \|_{C(\Omega)} \leq \frac{1}{2} M_3 \Gamma(\alpha + 1), \]

\[ f(M_3 T_3^\alpha + \| a \|_{C(\Omega)}) \leq \frac{1}{2} M_3 \Gamma(\alpha + 1). \]
We remark that sufficiently large $M_3(T) > 0$ can satisfy (6.29) for small $T > 0$, because
\[
\lim_{T \downarrow 0} f(M_3 T^\alpha + \|a\|_{C(\overline{\Omega})}) = f(\|a\|_{C(\overline{\Omega})}).
\]
Then the inequality
\[
u_a(x, t) - a(x) \leq M_3(T_3) t^\alpha, \quad x \in \Omega, \ 0 < t < T_3 \quad (6.30)
\]
holds true.

**Verification of (6.30).** We set $\overline{\pi}(x, t) := a(x) + M_3 t^\alpha$. Then the first condition in (6.29) yields
\[
da_t (\overline{\pi} - a) - \Delta \overline{\pi} = M_3 \Gamma(\alpha + 1) - \Delta a \geq \frac{1}{2} M_3 \Gamma(\alpha + 1).
\]
Moreover, since $f$ is monotone increasing, we have the estimate
\[
f(\overline{\pi}) = f(a(x) + M_3 t^\alpha) \leq f(\|a\|_{C(\overline{\Omega})} + M_3 T_3^\alpha)
\]
for $(x, t) \in \Omega \times (0, T_3)$. Therefore, by means of (6.29), we obtain
\[
da_t (\overline{\pi} - a) - \Delta \overline{\pi} \geq \frac{1}{2} M_3 \Gamma(\alpha + 1) \geq f(M_3 T_3^\alpha + \|a\|_{C(\overline{\Omega})}) \geq f(\overline{\pi}) \quad \text{in} \ \Omega \times (0, T_3),
\]
which means that $\overline{\pi}$ is another upper solution and thus the inequality (6.30) is verified. □

**7 Concluding remarks**

In this article, for derivation of the comparison principles and the monotonicity method, we considered a class of solutions $u \in L^2(0, T; H^2(\Omega))$ under the initial condition $u - a \in H_\alpha(0, T; L^2(\Omega))$. However, our main results remain valid for the solutions from a larger space of functions, for instance, for mild solutions to the semilinear equation (5.1) which are defined as solutions to the equation (5.10).

In order to prove the comparison principles for the mild solutions, one needs pointwise arguments in Lemma 3.2 and thus approximations of the solutions by smooth solutions $u \in C([0, T]; C^2(\Omega))$ satisfying $t^{1-\alpha} \partial_t u \in C([0, T]; C(\Omega))$ has to be considered. In other words, we first have to prove the comparison principles for smooth functions belonging to such a smoother class and then extend them to the desired class of solutions.

In our discussions, we restricted ourselves to the case of the homogeneous boundary conditions. However, in the definitions (6.9) and (6.10) of the upper and lower solutions, it is natural to replace the homogeneous boundary conditions by the inequalities $\partial_{\nu_A} \overline{\pi} + \sigma \overline{\pi} \geq 0$ and $\partial_{\nu_A} \underline{u} + \sigma \underline{u} \leq 0$ on $\partial \Omega \times (0, T)$. To this end, the unique existence results for the initial-boundary value problems with non-homogeneous boundary conditions are needed, which are not sufficiently studied. These problems will be considered elsewhere.

For the sake of simplicity of arguments, in this article we discussed only the case of the spatial dimensions $d = 1, 2, 3$. The case $d \geq 4$ can be treated similarly, but some stronger regularity conditions for the coefficients of the differential operators and more involved estimates are needed.
Appendix: Proof of existence of a function satisfying the conditions (3.4)

First Step.
In this step, we prove Lemma 8.1. We assume (1.2) and $M := \min_{(x,t) \in \overline{\Omega} \times [0,T]} b_0(x, t) > 0$ is sufficiently large. Then there exists a constant $\kappa_1 > 0$ such that

$$ (A_1(t)v, v) \geq \kappa_1 \|v\|_{H^1(\Omega)}^2 $$

\forall v \in H^2(\Omega) satisfying $\partial_{\nu_A} v + \sigma v = 0$ on $\partial \Omega$ and $\forall t \in [0, T]$.

In particular, Lemma 8.1 implies that all the eigenvalues of $A_0$ defined by (2.5) are positive if the constant $c_0 > 0$ is sufficiently large. Henceforth we write $b = (b_1, ..., b_d)$.

Proof. By using (1.2) and $\partial_{\nu_A} v + \sigma v = 0$ on $\partial \Omega$, integration by parts yields

$$ 0 = (A_1(t)v, v) = -\int_{\Omega} \sum_{i,j=1}^d \partial_i(a_{ij}(x)\partial_j v)vdx - \frac{1}{2} \int_{\Omega} \sum_{j=1}^d b_j(x, t)\partial_j(|v|^2)dx + \int_{\Omega} b_0(x, t)|v|^2dx $$

$$ = \int_{\Omega} \sum_{i,j=1}^d a_{ij}(x)(\partial_i v)(\partial_j v)dx - \int_{\partial \Omega} (\partial_{\nu_A} v)v dS $$

$$ + \frac{1}{2} \int_{\Omega} \text{div}(b)|v|^2dx - \frac{1}{2} \int_{\partial \Omega} (b \cdot \nu)|v|^2dS + \int_{\Omega} b_0(x, t)|v|^2dx $$

$$ \geq \kappa \int_{\Omega} \|
abla v\|^2dx + \int_{\Omega} \left( \min_{(x,t) \in \overline{\Omega} \times [0,T]} b_0(x, t) - \frac{1}{2}|\text{div} b| \right) |v|^2dx $$

$$ + \int_{\partial \Omega} (\sigma - \frac{1}{2}(b \cdot \nu)) |v|^2dS $$

$$ \geq \kappa \int_{\Omega} \|
abla v\|^2dx + \left( M - \frac{1}{2}||\text{div} b||_{C(\overline{\Omega} \times [0,T])} \right) \int_{\Omega} |v|^2dx - C \int_{\Omega} |v|^2dS. \hspace{1cm} (8.2) $$

Here and henceforth $C > 0, C_\varepsilon, C_\delta > 0$, etc. denote generic constants which are independent of $v$.

By the trace theorem (e.g., Theorem 9.4 (pp.41-42) in [18]), for $\delta \in (0, \frac{1}{2})$, there exists a constant $C_\delta > 0$ such that

$$ \|v\|_{L^2(\partial \Omega)} \leq C_\delta \|v\|_{H^{\delta+\frac{1}{2}}(\Omega)} \quad \text{for all } v \in H^1(\Omega). $$

We fix $\delta \in (0, \frac{1}{2})$. The interpolation inequality for the Sobolev spaces yields that for any $\varepsilon > 0$ we can find a constant $C_{\varepsilon, \delta} > 0$ such that

$$ \|v\|_{H^{\delta+\frac{1}{2}}(\Omega)} \leq \varepsilon \|
abla v\|_{L^2(\Omega)} + C_{\varepsilon, \delta} \|v\|_{L^2(\Omega)} \quad \text{for all } v \in H^1(\Omega) $$
for each \( t \) large such that \( t \). For each we can complete the proof of Lemma 8.1. for all \( v \) \( \| \cdot \| \) that \( w \). Now, we can derive: for each \( t \) such that for each \( t \), \( t \in \mathbb{N} \). Therefore, we have \( \theta \) \( \epsilon > 0 \) sufficiently small such that \( \kappa - 2C(\epsilon C_\delta)^2 > 0 \), since \( M > 0 \) is sufficiently large such that \( M > \frac{1}{2} \| \text{div} b \|_{C(\overline{\Omega} \times [0,T])} + 2(CC_\delta C_\varepsilon \delta)^2 \), we can complete the proof of Lemma 8.1.

Second Step. By (8.1), we can apply Theorem 3.2 (p.137) in [17] to see that there exists a constant \( \theta \in (0,1) \) such that for each \( t \in [0,T] \), a solution \( \psi(\cdot,t) \in C^{2+\theta}(\overline{\Omega}) \) to (8.4) exists uniquely. We set \( \eta(t) := \| \psi(\cdot,t) \|_{C^{2+\theta}(\overline{\Omega})} \), \( 0 \leq t \leq T \).

We note that \( \eta(t) < \infty \) for each \( t \in [0,T] \). The space \( C^{2+\theta}(\overline{\Omega}) \) is the Schauder space defined at the end of the proof of Lemma 4.1 in Section 4.

Furthermore, we obtain that for arbitrary \( G \in C^0(\overline{\Omega}) \), there exists a unique solution \( w = w(\cdot,t) \) to

\[
\left\{ \begin{array}{l}
A_1(t)w = G & \text{in } \Omega, \\
\partial_\nu w + \sigma w = 0 & \text{on } \partial\Omega
\end{array} \right.
\]

for each \( t \in [0,T] \).

Now, we can derive: for each \( t \in [0,T] \), there exists a constant \( C_t > 0 \) such that

\[
\|w(\cdot,t)\|_{C^{2+\theta}(\overline{\Omega})} \leq C_t \|G\|_{C^0(\overline{\Omega})}
\]

for all \( w \) satisfying (8.4). Here the constant \( C_t > 0 \) depends on \( \|a_{ij}\|_{C^1(\overline{\Omega})}, \|b_k\|_{C([0,T];C^2(\overline{\Omega}))} \) for \( 1 \leq i,j \leq d \) and \( 0 \leq k \leq d \), but not on respective choices of these coefficients.

Verification of (8.5). For each \( t \in [0,T] \), the inequality

\[
\|w(\cdot,t)\|_{C^{2+\theta}(\overline{\Omega})} \leq C_t(\|G\|_{C^0(\overline{\Omega})} + \|w(\cdot,t)\|_{C(\overline{\Omega})})
\]

holds true (see, e.g., the formula (3.7) on p. 137 in [17]). Then we have to eliminate the second term, namely the term \( \|w(\cdot,t)\|_{C(\overline{\Omega})} \), on the right-hand side of this inequality. This can be done by a classical compactness-uniqueness argument. More precisely, assume that (8.6) does not hold. Then there exist \( w_n \in C^{2+\theta}(\overline{\Omega}) \), \( G_n \in C^0(\overline{\Omega}) \) for \( n \in \mathbb{N} \) such that \( \|w_n\|_{C^{2+\theta}(\overline{\Omega})} = 1 \) and \( \lim_{n \to \infty} \|G_n\|_{C^0(\overline{\Omega})} = 0 \). By the Ascoli-Arzelà theorem, we can

(e.g., Chapter IV in [1] or Sections 2.5 and 11 of Chapter 1 in [18]). Therefore,

\[
\|v\|_{L^2(\partial\Omega)}^2 \leq (\varepsilon C_\delta \|\nabla v\|_{L^2(\Omega)} + C_\delta C_\varepsilon \|v\|_{L^2(\Omega)})^2
\]

\[
\leq 2(\varepsilon C_\delta)^2 \|\nabla v\|_{L^2(\Omega)}^2 + 2(C_\delta C_\varepsilon \delta)^2 \|v\|_{L^2(\Omega)}^2
\]

for all \( v \in H^1(\Omega) \). Substituting this inequality into (8.2), we obtain

\[
(\kappa - 2C(\varepsilon C_\delta)^2)\|\nabla v\|_{L^2(\Omega)}^2 + (M - \frac{1}{2} \|\text{div} b\|_{C(\overline{\Omega} \times [0,T])} - 2(CC_\delta C_\varepsilon \delta)^2) \|v\|_{L^2(\Omega)}^2 \leq 0.
\]

Choosing \( \varepsilon > 0 \) sufficiently small such that \( \kappa - 2C(\varepsilon C_\delta)^2 > 0 \), since \( M > 0 \) is sufficiently large such that

\[
M > \frac{1}{2} \|\text{div} b\|_{C(\overline{\Omega} \times [0,T])} + 2(CC_\delta C_\varepsilon \delta)^2,
\]

we can complete the proof of Lemma 8.1. □
extract a subsequence \( w_{k(n)} \) from \( w_n \) with \( n \in \mathbb{N} \) such that \( w_{k(n)} \rightarrow \tilde{w} \) in \( C(\bar{\Omega}) \) as \( n \rightarrow \infty \). Applying (8.4) to
\[
A_1(t)(w_{k(n)} - w_{k(m)}) = G_{k(n)} - G_{k(m)} \quad \text{in } \Omega
\]
with \( \partial_{\nu_A}(w_{k(n)} - w_{k(m)}) + \sigma(w_{k(n)} - w_{k(m)}) = 0 \) on \( \partial\Omega \), we obtain
\[
\|w_{k(n)} - w_{k(m)}\|_{C^{2+\theta}(\bar{\Omega})} \\
\leq C_1(\|G_{k(n)} - G_{k(m)}\|_{C^0(\bar{\Omega})} + \|w_{k(n)} - w_{k(m)}\|_{C(\bar{\Omega})}) \rightarrow 0
\]
as \( n, m \rightarrow \infty \). Hence, there exists \( w_0 \in C^{2+\theta}(\bar{\Omega}) \) such that \( w_{k(n)} \rightarrow w_0 \) in \( C^{2+\theta}(\bar{\Omega}) \), and so
\[
\|w_0\|_{C^{2+\theta}(\bar{\Omega})} = \lim_{n \rightarrow \infty} \|w_{k(n)}\|_{C^{2+\theta}(\bar{\Omega})} = 1
\]
and \( G_{k(n)} = A_1(t)w_{k(n)} \rightarrow A_1(t)w_0 \) in \( C^\theta(\bar{\Omega}) \).

Since \( \lim_{n \rightarrow \infty} \|G_{k(n)}\|_{C^0(\bar{\Omega})} = 0 \), we reach \( A_1(t)w_0 = 0 \) in \( \Omega \) with \( \partial_{\nu_A}w_0 + \sigma w_0 = 0 \) on \( \partial\Omega \). Then Lemma (8.4) yields \( w_0 = 0 \) in \( \Omega \), which contradicts \( \|w_0\|_{C^{2+\theta}(\bar{\Omega})} = 1 \). Thus the verification of (8.5) is complete. □

**Third Step.**

We have to prove that \( \psi(x,t) \) constructed in Second Step satisfy the inclusion \( \psi \in C^1([0,T];C^2(\bar{\Omega})) \).

Fixing \( t \in [0,T] \) arbitrarily, we verify that \( d(x,s) := \psi(x,t) - \psi(x,s) \) satisfies
\[
\begin{cases}
-A_1(t)d(\cdot,s) = (b_0(t) - b_0(s))\psi(\cdot,s) \\
- \sum_{j=1}^d (b_j(t) - b_j(s))\partial_j\psi(\cdot,s) & \text{in } \Omega \quad \text{for } 0 \leq s, t \leq T, \\
\partial_{\nu_A}d + |d| = 0 & \text{on } \partial\Omega, \quad 0 \leq s, t \leq T.
\end{cases}
\]
(8.7)

Let \( \delta > 0 \) be arbitrarily fixed. We set \( I_{\delta,t} := [0,T] \cap \{s; |t-s| \leq \delta\} \).

Again the application of (8.3) and (8.5) to (8.7) yields
\[
\|d(\cdot,s)\|_{C^{2+\theta}(\bar{\Omega})} \\
\leq C \left( \sum_{j=1}^d \|b_j(t) - b_j(s)\|_{C^0(\bar{\Omega})} \right) \leq C \max_{0 \leq j \leq d} \|b_j(t) - b_j(s)\|_{C^1(\bar{\Omega})} \sup_{s \in I_{\delta,t}} \eta(s).
\]
(8.8)

Setting
\[
h(\delta) := \max_{0 \leq j \leq d} \sup_{|s-t| \leq \delta} \|b_j(s) - b_j(t)\|_{C^1(\bar{\Omega})},
\]
by the condition \( b_j \in C([0,T];C^1(\bar{\Omega})) \) for \( 0 \leq j \leq d \), we deduce that \( \lim_{\delta \downarrow 0} h(\delta) = 0 \).

Moreover, in terms of \( \eta \) defined by (8.3), we can rewrite (8.7) as
\[
|\eta(s) - \eta(t)| \leq C h(\delta) \sup_{s \in I_{\delta,t}} \eta(s) \quad \text{for } s \in I_{\delta,t},
\]
(8.9)
and so

\[ \eta(s) \leq \eta(t) + Ch(\delta) \sup_{s \in I_{\delta,t}} \eta(s) \quad \text{for } s \in I_{\delta,t}. \]

Choosing \( \delta := \delta(t) > 0 \) sufficiently small for given \( t \in [0, T] \), we see that \( \sup_{s \in I_{\delta(t)}, t} \eta(s) \leq C_1 \eta(t) \). Varying \( t \in [0, T] \), we can choose a finite number of intervals \( I_{\delta(t)}, t \) covering \( [0, T] \), we obtain

\[ \|\psi\|_{L^\infty(0, T; C^{2+\theta}(\Omega))} \leq C_2 \]

with some constant \( C_2 > 0 \).

Substitution of (8.9) into (8.8) yields

\[ \|d(\cdot, s)\|_{C^{2+\theta}(\Omega)} = \|\psi(\cdot, t) - \psi(\cdot, s)\|_{C^{2+\theta}(\Omega)} \leq Ch(\delta) C_2 \]

for \( s \in I_{\delta(t), t} \). Consequently, \( \lim_{s \to t} \|\psi(\cdot, s) - \psi(\cdot, t)\|_{C^{2+\theta}(\Omega)} = 0 \), that is,

\[ \psi \in C([0, T]; C^{2+\theta}(\Omega)). \]

Finally we have to prove \( \psi \in C^1([0, T]; C^{2+\theta}(\Omega)) \). Since \(-A_1(\xi)\psi(x, \xi) = 1 \) in \( \Omega \) with \( \xi = t, s \), differentiating in \( t \) and \( s \), we can obtain

\[ \sum_{j=1}^d \partial_t (a_{ij}(x) \partial_j \partial_x \psi(x, \xi)) + \sum_{j=1}^d b_j(\xi) \partial_j \partial_x \psi(x, \xi) - b_0(\xi) \partial_x \psi(x, \xi) \]

\[ = -\sum_{j=1}^d \partial_x b_j(\xi) \partial_j \psi(x, \xi) + (\partial_x b_0(\xi) \psi(x, \xi) \quad \text{in } \Omega \]

with \( \xi = t, s \). Therefore, by subtracting the equation (8.11) with \( \xi = s \) from the one with \( \xi = t \), we deduce that \( d_1(x, s) := (\partial_t \psi)(x, t) - (\partial_t \psi)(x, s) \) satisfies

\[ -A_1(t)d_1(x, s) \]

\[ = \left[ -\sum_{i,j=1}^d (b_j(t) - b_j(s)) \partial_j \partial_s \psi(x, s) + (b_0(t) - b_0(s)) \partial_s \psi(x, s) \right] \]

\[ -\sum_{j=1}^d (\partial_t b_j(t) - \partial_t b_j(s)) \partial_j \psi(x, s) + (\partial_t b_0(t) - \partial_t b_0(s)) \psi(x, s) \right] \]

\[ + \left[ -\sum_{j=1}^d \partial_t b_j(t) (\partial_j \psi(x, t) - \partial_j \psi(x, s)) + \partial_t b_0(t) (\psi(x, t) - \psi(x, s)) \right] \]

\[ =: H_1(x, t, s) + H_2(x, t, s) \quad \text{in } \Omega \]

with \( \partial_{\nu,t} d_1 + \sigma d_1 = 0 \) on \( \partial \Omega \) for \( s, t \in [0, T] \). Thus, if we can verify the relation \( \lim_{s \to t} \|d_1(\cdot, s)\|_{C^{2+\theta}(\Omega)} = 0 \), then we can complete the proof of \( \psi \in C^1([0, T]; C^{2+\theta}(\Omega)) \).

To this end, by applying Theorem 3.2 (p.137) in [17] to (8.12), it suffices to prove that

\[ \lim_{s \to t} \|H_\ell(\cdot, t, s)\|_{C^0(\Omega)} = 0, \quad \ell = 1, 2. \]
Indeed the final limit implies that \( \lim_{s \to t} \| d_1(\cdot, t, s) \|_{C^{2+\theta}(\bar{\Omega})} \), that is, \( \partial_t \psi \in C^1([0, T]; C^{2+\theta}(\bar{\Omega})) \), so that the proof of the existence of \( \psi \) is complete.

**Verification of (8.13).**

Applying Theorem 3.2 in [17] to (8.11), in view of the regularity in (1.2) and (8.11), we see

\[
\| \partial_t \psi(\cdot, t) \|_{C^{2+\theta}(\bar{\Omega})}
\leq C \left( \sum_{j=1}^{d} \| (\partial_j \psi)(\cdot, t) \|_{C^{\theta}(\bar{\Omega})} + \| (\partial_t \psi)(\cdot, t) \|_{C^\theta(\bar{\Omega})} \right)
\leq C \sum_{j=0}^{d} \| b_j(\cdot, t) \|_{C^1([0, T]; C^1(\bar{\Omega}))} \| \psi(\cdot, t) \|_{C^{2+\theta}(\bar{\Omega})} \leq C_3 \quad \text{for } 0 \leq t \leq T.
\]

Hence (8.9) and (8.14) yield

\[
\| H_1(\cdot, t, s) \|_{C^\theta(\bar{\Omega})} \leq C_4 \sum_{k=0}^{1} \sum_{j=0}^{d} \| (\partial^k \psi)(\cdot, t) - (\partial^k \psi)(\cdot, s) \|_{C^1(\bar{\Omega})},
\]

Similarly we can prove

\[
\| H_2(\cdot, t, s) \|_{C^\theta(\bar{\Omega})} \leq C_4 \sum_{j=0}^{d} \| \partial_j b_j \|_{C([0, T]; C^1(\bar{\Omega}))} \sum_{k=0}^{1} \| \nabla^k \psi(\cdot, t) - \nabla^k \psi(\cdot, s) \|_{C^\theta(\bar{\Omega})}.
\]

Since \( \partial^k b_j \in C([0, T]; C^1(\bar{\Omega})) \) for \( k = 0, 1 \) and \( 0 \leq j \leq d \) by (1.2) and \( \nabla^k \psi \in C([0, T]; C^{1+\theta}(\bar{\Omega})) \) with \( k = 0, 1 \) by (8.10), from (8.15) and (8.16) it follows that \( \lim_{s \to t} \| H_\ell(\cdot, t, s) \|_{C^\theta(\bar{\Omega})} = 0 \)

\( \) for \( \ell = 1, 2 \). Thus the verification of (8.13) is complete. \( \square \)

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