Characterization of Exponential Distribution and
Sukhatme-Renyi Decomposition of Exponential
Maxima

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Abstract

The exponential distribution is characterized by a distributional equation between a pair of maxima of independent and identically distributed random variables. In particular, it is proven that, under some regularity assumptions, the well-known Sukhatme-Renyi necessary condition for the maximum of exponential variables is sufficient to guarantee that the underlying distribution is exponential. An argument due to Arnold and Villasenor (2013) based on the Maclaurin series expansion of the probability density plays a crucial role in the proof of the main result.

1 Introduction and Main Results

Various characterizations of the exponential distribution are available as is apparent from the surveys [1], [3], and [8]. A number of known results are based on the distributional equation \( X + T \overset{d}{=} Y \) involving a pair of random variables \((X, Y)\) and a random translator (shift) variable \( T \), independent of \( X \). Characterizations based on this equation when \( X, Y, \) and \( T \) are either order statistics or record values were obtained in [13] and [5], among others. In [6], previous results were generalized by exploring uniqueness results for non-linear Volterra integral equations.

Suppose \( X_1, X_2, \ldots, X_n \) is a sample of size \( n \geq 2 \) from a distribution with cdf \( F \), assumed to be absolutely continuous with \( F(0) = 0 \) and with density function \( f \). Denote by \( X_{k,k} \) for \( k \geq 2 \) the maximum in a sample of size \( k \).

Recently, Arnold and Villasenor [4] obtained a series of characterizations of the exponential distribution based on a random sample of size two. In particular, they proved that for a sample \( X_1, X_2 \) from a continuous distribution with cdf \( F \) such that \( F(0) = 0 \), under some additional regularity assumptions,

\[ X_1 + \frac{1}{2}X_2 \overset{d}{=} X_{2,2} \]
characterizes the exponential distribution with some positive parameter. In [4], a list of conjectures for possible extensions to larger samples was also given. In [14] and [7] some of the results in [4] were extended to the case of maxima for samples of size \( n \geq 3 \). For instance, it was proven in [7] that, under the abovementioned assumptions on \( F \), if for a fixed \( n \)

\[
X_{n-1:n-1} + \frac{1}{n} X_n \overset{d}{=} X_{n:n},
\]

then the underlying distribution is exponential with some positive parameter. Milošević and Obradović [10] obtained that, under the abovementioned assumptions on \( F(x) \), for a fixed \( n \) and fixed \( 1 \leq k \leq n-1 \),

\[
X_{k-1:n-1} + \frac{1}{n} X_n \overset{d}{=} X_{k:n} \quad \text{for } 1 < k \leq n,
\]

characterizes the exponential distribution. In the present paper we prove a generalization of [10] based on an equation in distribution between maximum of \( n \) and \( n-s \geq 2 \) i.i.d. random variables. A crucial role in the proof is played by an argument from [4] based on the Maclaurin series expansion for the probability density \( F' \). The following is our main result.

**Theorem** Let the pdf \( f(x) \) be analytic at 0 for \( x > 0 \). If for fixed integer \( n \) and \( s \) such that \( 1 \leq s \leq n-1 \)

\[
X_{n-s:n-s} + \sum_{j=n-s+1}^{n} \frac{X_j}{j} \overset{d}{=} X_{n:n},
\]

then \( X \) is exponential with some positive parameter.

Let us point out that some particular cases of the Theorem have already been applied in constructing goodness-of-fit tests for exponential distribution in [9] and [12].

In comparison with [5], [6], and [13], the above Theorem has stronger assumptions on the density of the underlying random variable. However, the condition in [5], [6], and [13] that the translator random variable \( T \) has certain known distribution is omitted here.

The following direct corollary of the Theorem is of independent interest.

**Corollary** Under the assumptions of the theorem, if for fixed \( n \)

\[
\sum_{j=1}^{n} \frac{X_j}{j} \overset{d}{=} X_{n:n},
\]

then \( X \) is exponential with some positive parameter.

Note that (3) is the particular case for \( k = n \) of the well-known Sukhatme-Rényi decomposition (e.g., [2], p.73) of the kth order statistic in a sample.
$X_1, X_2, \ldots, X_n$ from exponential distribution

$$
\sum_{j=n-k+1}^{n} \frac{X_j}{j} = X_{k:n}, \quad 1 \leq k \leq n.
$$

(4)

Let us point out that the Corollary was conjectured in [4] and proved in [10]. As it was remarked in [4], if we assume for i.i.d. variables $X_1, X_2, \ldots$ with $E|X_1| < \infty$, that (3) holds for every $n$, then necessarily the $X_i$’s have a common exponential distribution. In fact, it is enough to assume that the equation between the first moments $E[X_{n:n}] = E \left[ \sum_{j=1}^{n} \frac{X_i}{i} \right]$ holds for every $n$, since the sequence of expected maxima determines the distribution. It turns out that, under the assumptions of the Corollary, for $X_i$’s to be exponential it is sufficient that (3) holds for one $n$ only.

In the next section we state several lemmas, which will be used in Section 3 where the Theorem will be proven. Section 4 contains the proofs of the lemmas. In the last section we give some concluding remarks.

2 Preliminaries

In this section we formulate five lemmas to be used in the proof of the Theorem. Let $F_n(x)$ be the cdf of $X_{n:n}$ and

$$
r_j(t) := 2t - \sum_{k=1}^{j-1} i_k, \quad 1 \leq j \leq s,
$$

where $i_k$ are some integers and by convention $\sum_{k=1}^{0} (\cdot) = 0$. Further on, for simplicity, we will drop the letter $t$ and will write $r_j$ instead of $r_j(t)$.

The first lemma calculates the derivatives of the probability density of the left-hand side of (2).

**Lemma 1** Let $f_{LHS}(z)$ denote the density function of the left-hand side of (2) and $t = n - s - 1$. The $(2t+s)$th derivative of $f_{LHS}(x)$ at $x = 0$ equals

$$
(t + s + 1) \ldots (t + 1) \sum_{i_1=0}^{r_1} \cdots \sum_{i_s=0}^{r_s} a_{i_1:i_s} Q_{i_1:i_s}(0) F_{t+1}^{(i_1+1)}(0)
$$

(5)

where

$$
a_{i_1:i_s} := (t + 2)^{r_s-i_s} \prod_{j=1}^{s-1} (t + s - j + 2)^{i_j}, \quad Q_{i_1:i_s}(0) := f^{(r_s-i_s)}(0) \prod_{j=1}^{s-1} f^{(i_j)}(0).
$$

We turn to the derivatives of the density of the right-hand side of (2).

**Lemma 2** Let $f_{RHS}(z)$ denote the density function of the right-hand side of (2) and $t = n - s - 1$. The $(2t+s)$th derivative of $f_{RHS}(x)$ at $x = 0$ equals

$$
(t + s + 1) \ldots (t + 1) \sum_{i_1=0}^{r_1} \cdots \sum_{i_s=0}^{r_s} b_{i_1:i_s} Q_{i_1:i_s}(0) F_{t+1}^{(i_1+1)}(0)
$$

(6)
where
\[
b_{i;1:s} := \binom{r_s + 1}{i_1 + 1} \prod_{j=1}^{s-1} \binom{r_j + s - j + 1}{i_j}, \quad Q_{i;1:s}(0) := f^{(r_s-i_s)}(0) \prod_{j=1}^{s-1} f^{(i_j)}(0).
\]

Lemma 3 below establishes two combinatorial identities, which might be of independent interest. Define for all non-negative integers \(n, i, \) and any real number \(x\)
\[
H_{n,i}(x) := \sum_{j=0}^{n} (-1)^j \binom{n}{j} (x - j)^i.
\]  
(7)
It is known (cf. [11]) that for any non-negative integer \(n\) and real \(x\)
\[
H_{n,i}(x) = \begin{cases} 
  n! & \text{if } i = n; \\
  0 & \text{if } 0 \leq i \leq n - 1.
\end{cases}
\]  
(8)

**Lemma 3** Let \(s\) and \(r\) be positive integers. Then
\[
(i) \quad \sum_{j=0}^{r-1} \binom{r}{j+1} H_{s,j}(s+1) = \frac{1}{s+1} H_{s+1,r}(s+2).
\]  
(9)
\[
(ii) \quad \sum_{j=0}^{r-1} (s+2)^{r-1-j} H_{s,j}(s+1) = \frac{1}{s+1} H_{s+1,r}(s+2).
\]  
(10)
Making the change of indexes \(i = r - 1 - j\) one can see that (9) and (10) are equivalent to
\[
\sum_{i=0}^{r-1} \binom{r}{i} H_{s,r-1-i}(s+1) = \frac{1}{s+1} H_{s+1,r}(s+2) = \sum_{i=0}^{r-1} (s+2)^i H_{s,r-1-i}(s+1).
\]  
(11)

Next lemma, proven in [11], is needed for the induction step in the proof of the Theorem.

**Lemma 4** If for \(0 \leq r \leq t-1\)
\[
f^{(r)}(0) = \left[ \frac{f'(0)}{f(0)} \right]^{r-1} f'(0),
\]  
(12)
then for \(t \leq i \leq 2t\)
\[
\frac{d^{i+1}}{dx^{i+1}}F_{t+1}(x)|_{x=0} = (t+1)[f'(0)]^{t-i} [f(0)]^{2t+1-i} H_{t,i}(t+1).
\]  
(13)
The last lemma, taken from [4], provides a crucial argument in the proof of the Theorem.
Lemma 5 If $F(0) = 0$, the pdf $f(x)$ of $X$ is analytic at 0 for $x > 0$ and

$$f^{(k)}(0) = \left[ \frac{f'(0)}{f(0)} \right]^{k-1} f'(0), \quad k = 1, 2, \ldots,$$  \hspace{1cm} (14)

then $X$ is exponential with some positive parameter.

3 Proof of the Theorem

We begin by differentiating the probability densities of both sides of (2). Let $t = n - s - 1$. For the $(2t+s)$th derivatives evaluated at 0, according to Lemma 1 and Lemma 2, we have

$$\sum_{i_1=0}^{r_1} \cdots \sum_{i_s=0}^{r_s} a_{i_1; i_s} Q_{i_1; i_s; t}(0) F_{t+1}^{(i_s+1)}(0) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_s=0}^{r_s} b_{i_1; i_s} Q_{i_1; i_s; t}(0) F_{t+1}^{(i_s+1)}(0), \quad (15)$$

where

$$Q_{i_1; i_s}(0) = f^{(r_s - i_s)}(0) \prod_{j=1}^{s-1} f^{(i_j)}(0).$$

(For more details about the transition from (2) to (15) we refer to the proofs of Lemma 1 and Lemma 2 in Section 4.) In view of Lemma 5, to complete the proof it suffices to show that

$$f^{(k)}(0) = \left[ \frac{f'(0)}{f(0)} \right]^{k-1} f'(0), \quad k \geq 1.$$  \hspace{1cm} (16)

We shall prove (16) by (strong) induction on $k$. The base case $k = 1$ is immediate. Assume that (16) holds for $1 \leq k \leq t - 1$. Using (15) we shall prove (16) for $k = t$.

Denote $c_{i_1; i_s} := a_{i_1; i_s} - b_{i_1; i_s}$. Since $F_{t+1}^{(i+1)}(0) = 0$ for $1 \leq i \leq t - 1$, the first $t - 1$ terms in the most inner sum of (15) are zeros. Hence, (15) is equivalent to

$$\sum_{i_1=0}^{r_1} \cdots \sum_{i_s-1=0}^{r_s-1} \sum_{i_s=t}^{r_s} c_{i_1; i_s} Q_{i_1; i_s; t}(0) F_{t+1}^{(i_s+1)}(0) = 0.$$

Splitting the above sum by extracting the terms with factor $f^{(t)}(0)$, we obtain

$$\sum_{\mathcal{I}_0} c_{i_1; i_s} Q_{i_1; i_s; t}(0) F_{t+1}^{(i_s+1)}(0) + \sum_{\mathcal{I}_0} c_{i_1; i_s} [f(0)]^{s-1} f^{(t)}(0) F_{t+1}^{(t+1)}(0) = 0, \quad (17)$$

where $\mathcal{I}_0$ is the set of vectors $(i_1, \ldots, i_s)$ such that $i_s = t$ and among the other $i - 1$ components: (i) all are zeros or (ii) exactly one is $t$ and the others are zeros. That is $\mathcal{I}_0 = \{(0, 0, \ldots, 0, t)\} \cup \{(t, 0, \ldots, 0, t), \ldots, (0, 0, \ldots, 0, t)\}$. Therefore, $r_s - i_s$
\[ Q_{i_1:i_s:t}(0) = f^{(r_s-i_s)}(0) \prod_{j=1}^{s-1} f^{(i_j)}(0) \]

are of order less than or equal to \( t - 1 \). Applying Lemma 4 and the induction hypothesis to (17), we find

\[ [f'(0)]^t \sum_{\mathcal{I} \setminus \mathcal{J}_0} c_{i_1:i_s} H_{t,i_s}(t + 1) + f(t)(0) \prod_{j=1}^{t-1} c_{i_1:i_s} H_{t,i_s}(t + 1) = 0, \]

Thus, to prove (16) for \( k = t \), it is sufficient to prove

\[ \sum_{\mathcal{I} \setminus \mathcal{J}_0} c_{i_1:i_s} H_{t,i_s}(t + 1) + \sum_{\mathcal{I}_{0}} c_{i_1:i_s} H_{t,i_s}(t + 1) = \sum_{\mathcal{I}} c_{i_1:i_s} H_{t,i_s}(t + 1) = 0 \]

or, equivalently,

\[ \sum_{i_1=0}^{r_1} \cdots \sum_{i_s=t}^{r_s} a_{i_1:i_s} H_{t,i_s}(t + 1) = \sum_{i_1=0}^{r_1} \cdots \sum_{i_s=t}^{r_s} b_{i_1:i_s} H_{t,i_s}(t + 1). \quad (18) \]

Consider the right-hand side of (18). Since \( H_{t,i}() = 0 \) for \( 0 \leq i \leq t - 1 \), the most inner sum can start at \( i_s = 0 \). Recalling the definition of \( b_{i_1:i_s} \), we have

\[ \sum_{i_1=0}^{r_1} \sum_{i_{s-1}=i_s}^{r_{s-1}} \sum_{i_s=t}^{r_s} b_{i_1:i_s} H_{t,i_s}(t + 1) \]

\[ = \sum_{i_1=0}^{r_1} \left( \begin{array}{c} r_1 + s \\ i_1 \end{array} \right) \sum_{i_{s-1}=i_s}^{r_{s-1}} \left( \begin{array}{c} r_{s-1} + 2 \\ i_{s-1} \end{array} \right) \sum_{i_s=0}^{r_s} \left( \begin{array}{c} r_s + 1 \\ i_s + 1 \end{array} \right) H_{t,i_s}(t + 1). \quad (19) \]

We shall simplify (19), working on the sums from inside out. Applying (9) and (11), we obtain

\[ \sum_{i_{s-1}=0}^{r_{s-1}} \left( \begin{array}{c} r_s + 2 \\ i_{s-1} \end{array} \right) \sum_{i_s=0}^{r_s} \left( \begin{array}{c} r_s + 1 \\ i_s + 1 \end{array} \right) H_{t,i_s}(t + 1) \]

\[ = \frac{1}{t+1} \sum_{i_{s-1}=0}^{r_{s-1}} \left( \begin{array}{c} r_s + 2 \\ i_{s-1} \end{array} \right) H_{t+1,r_{s-1}+1}(t + 2) \]

\[ = \frac{1}{t+1} \sum_{i_{s-1}=0}^{r_{s-1}+1} \left( \begin{array}{c} r_s + 2 \\ i_{s-1} \end{array} \right) H_{t+1,r_{s-1}+1-i_{s-1}+1}(t + 2) \]

\[ = \frac{1}{(t+1)(t+2)} H_{t+2,r_{s-1}+2}(t + 3). \]
Repeating this argument, we find for the right-hand side of (19)

\[
\sum_{i_1=0}^{r_1} \left( r_1 + s \right) \cdots \sum_{i_{s-1}=0}^{r_{s-1}+1} \left( r_{s-1} + 2 \right) \sum_{i_s=0}^{r_s} \left( r_s + 1 \right) H_{t, i_s} (t + 1)
\]

\[
= H_{t+s, 2t+s} \left( t + s + 1 \right) \frac{(t+1) \ldots (t+s)}{(t+1) \ldots (t+s)} .
\]

Let us turn to the left-hand side of (18). Recalling that \( H_{t, i_s} (t) = 0 \) for \( 0 \leq i_s \leq t - 1 \) and the definition of \( a_{i_1 \ldots i_s} \), we have

\[
\sum_{i_1=0}^{r_1} \cdots \sum_{i_{s-1}=0}^{r_{s-1}} \sum_{i_s=0}^{r_s} a_{i_1 \ldots i_s} H_{t, i_s} (t + 1)
\]

\[
= \sum_{i_1=0}^{r_1} (t + s + 1)^{i_1} \cdots \sum_{i_{s-1}=0}^{r_{s-1}} (t + 3)^{i_{s-1}} \sum_{i_s=0}^{r_s} (t + 2)^{r_s - i_s} H_{t, i_s} (t + 1).
\]

Applying (10) and (11) to the two most inner sums in (21) we find

\[
\sum_{i_{s-1}=0}^{r_{s-1}} (t + 3)^{i_{s-1}} \sum_{i_s=0}^{r_s} (t + 2)^{r_s - i_s} H_{t, i_s} (t + 1)
\]

\[
= \frac{1}{t+1} \sum_{i_{s-1}=0}^{r_{s-1}+1} (t + 3)^{i_{s-1}} H_{t+1, r_{s-1}+1} (t + 2)
\]

\[
= \frac{1}{t+1} \sum_{i_{s-1}=0}^{r_{s-1}+1} (t + 3)^{i_{s-1}} H_{t+1, r_{s-1}+1+i_{s-1}} (t + 2)
\]

\[
= \frac{1}{(t+1)(t+2)} H_{t+2, r_{s-1}+2} (t + 3).
\]

Repeating this argument, for the right-hand side of (21) we obtain

\[
\sum_{i_1=0}^{r_1} (t + s + 1)^{i_1} \cdots \sum_{i_{s-1}=0}^{r_{s-1}} (t + 3)^{i_{s-1}} \sum_{i_s=0}^{r_s} (t + 2)^{r_s - i_s} H_{t, i_s} (t + 1)
\]

\[
= H_{t+s, 2t+s} \left( t + s + 1 \right) \frac{(t+1) \ldots (t+s)}{(t+1) \ldots (t+s)} .
\]

Equations (19)-(22) imply (18), thus completing the induction. The proof of the Theorem is also complete.

4 Proofs of Lemmas 1-3

Recall that \( F_n (x) \) and \( f_n (x) \) denote the cdf and pdf of the maximum \( X_{n:n} \), respectively. That is, for any \( n = 1, 2, \ldots \)

\[
F_n (x) = \int_0^x f_n (t) \, dt = F^n (x).
\]
4.1 Proof of Lemma 1

Let \( f_{n-1,n}(x) \) denote the density of \( X_{n-1}/(n-1)+X_n/n \) and \( G_k(x) := F^k(x) f(x) \) for \( k \geq 1 \). Setting \( s = 2 \) in (2), for the density \( f_{LHS}(z \mid s = 2) \) of the left-hand side we find

\[
f_{LHS}(z \mid s = 2) = \int_0^z f_{n-2}(x) f_{n-1,n}(z - x) \, dx \]

\[
= \int_0^z (n-2)G_{n-3}(x) n(n-1) \int_0^{z-x} f(nx_1) f((n-1)(z - x - x_1)) \, dx_1 \, dx \]

\[
= (n)_3 \int_0^z G_{n-3}(x) K_{n,1}(z - x) \, dx,
\]

where \( (n)_3 := n(n-1)(n-2) \) and

\[
K_{n,1}(y) := \int_0^y f(nx_1) f((n-1)(y - x_1)) \, dx_1.
\]  

Similarly, for \( s = 3 \), a straightforward manipulation gives

\[
f_{LHS}(z \mid s = 3) = (n)_4 \int_0^z G_{n-4}(x) K_{n,2}(z - x) \, dx,
\]

where

\[
K_{n,2}(y) = \int_0^y \int_0^{y-x_1} f(nx_1) f((n-1)x_2) f((n-2)(y - x_1 - x_2)) \, dx_2 \, dx_1
\]

\[
= \int_0^y f(nx_1) K_{n-1,1}(y - x_1) \, dx_1.
\]

Repeating this argument, for the density \( f_{LHS}(z) \) of the left-hand side of (2) for any \( s \) such that \( 2 \leq s \leq n-1 \), we obtain

\[
f_{LHS}(z) = (n)_{s+1} \int_0^z G_{n-s-1}(x) K_{n,s-1}(z - x) \, dx \tag{25}
\]

where the following recursive relation holds for \( 3 \leq s \leq n-1 \)

\[
K_{n,s-1}(y) = \int_0^y f(nx_1) K_{n-1,s-2}(y - x_1) \, dx_1
\]

\[
= \int_0^y \int_0^{y_1} \cdots \int_0^{y_{s-2}} \prod_{j=1}^{s-1} f((n - j + 1)x_j) f((n - s + 1)y_{s-1}) \, dx_{s-1} \cdots dx_1,
\]

with \( y_j := y - \sum_{i=1}^j x_i \) for \( 1 \leq j \leq s-1 \).

First, we show that the \( m \)th derivative of \( K_{n,j}(y) \) at 0 for any \( m \geq j \geq 1 \) and any \( n \geq 2 \) is given by

\[
K^{(m)}_{n,j}(0) = \sum_{i_1=0}^{d_1} \cdots \sum_{i_j=0}^{d_j} \left( \prod_{k=1}^{j} c_k f^{(i_k)}(0) \right) c_{j+1} f^{(d_{j+1})}(0), \tag{26}
\]
where \( d_k = m - j - \sum_{i=1}^{k-1} t_i \) and \( c_k = n - k + 1 \) for \( 1 \leq k \leq j + 1 \), \( (\sum_{i=0}^{0} i) = 0 \).

We prove (26) by induction on \( j \). Differentiating (24) \( m \) times we see that the base of the induction is true, i.e., for any \( m \geq 1 \) and any \( n \geq 2 

\[
K_{n,1}^{(m)}(0) = \sum_{i=0}^{m} n^i (n-1)^{m-i} f^{(i)}(0) f^{(m-i)}(0). \tag{27}
\]

Assuming (26) with \( K_{n,j}^{(m)}(0) \), we shall prove it for \( K_{n,j+1}^{(m)}(0) \). Taking into account that \( K_{n-1,j}^{(i)}(0) = 0 \) for \( 0 \leq i \leq j - 1 \), we obtain

\[
K_{n,j+1}^{(m)}(0) = \frac{d^m}{dz^m} \left[ \int_0^z f(ny)K_{n-1,j}(z-y) \, dy \right]_{z=0} \tag{28}
\]

\[
= \frac{d^{m-j}}{dz^{m-j}} \left[ \int_0^z f(ny)K_{n-1,j}^{(j)}(z-y) \, dy \right]_{z=0}
\]

\[
= \frac{d^{m-j-1}}{dz^{m-j-1}} \left[ f(nz)K_{n-1,j}^{(j)}(0) + \int_0^z f(ny)K_{n-1,j}^{(j+1)}(z-y) \, dy \right]_{z=0}
\]

\[
= \sum_{i=0}^{m-j} n^i f^{(i)}(0)K_{n-1,j}^{(m-1-i)}(0).
\]

We complete the induction step by applying the hypothesis to \( K_{n-1,j}^{(m-1-i)}(0) \) in the last equation.

Next, we differentiate (26) using (20). Let \( t = n - s - 1 \). Then (26) can be rewritten as

\[
f_{LHS}(z) = (t + s + 1)_{s+1} \int_0^z G_t(x)K_{n,s-1}(z-x) \, dx
\]

Hence, for the \((2t+s)\)th derivative of \( f_{LHS}(z)/(t+s+1)_{s+1} \) at 0, recalling that \( K_{n-1,j}^{(i)}(0) = 0 \) for \( 0 \leq i \leq j - 1 \), we have

\[
\frac{d^{2t+s}}{dz^{2t+s}} \left[ \int_0^z G_t(x)K_{n,s-1}(z-x) \, dx \right]_{z=0}
\]

\[
= \frac{d^{2t+s-1}}{dz^{2t+s-1}} \left[ G_t(z)K_{n,s-1}(0) + \int_0^z G_t(x)K_{n,s-1}^{(1)}(z-x) \, dx \right]_{z=0}
\]

\[
= \frac{d^{2t}}{dz^{2t}} \left[ G_t(z)K_{n,s-1}^{(s)}(0) + \int_0^z G_t(x)K_{n,s-1}^{(s)}(z-x) \, dx \right]_{z=0}
\]

\[
= \sum_{i=0}^{2t} K_{n,s-1}^{(2t+s-i-1)}(0)G_t^{(i)}(0).
\]

Finally, substituting \( K_{n,s-1}^{(2t+s-i-1)}(0) \) from (26), it is not difficult to see that the last result equals

\[
\sum_{i_1=0}^{r_1} \cdots \sum_{i_s=0}^{r_s} \left( \prod_{j=1}^{s-1} (t + s + 2 - j)^{i_j} f^{(i_j)}(0) \right) (t + 2)^{r_s-i_s} f^{(r_s-i_s)}(0) F_{t+1}^{(i_s+1)}(0),
\]

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where, as before, \( r_k = 2t - \sum_{k=0}^{i-1} i_k \) for \( 1 \leq k \leq s \) and \( G_t(x) = F_t(x) f(x) \). This, in turn, is equivalent to (23), which completes the proof.

### 4.2 Proof of Lemma 2

Referring to (23), for the density \( f_{\text{RHS}}(z) \) of the right-hand side of (2), we have

\[
f_{\text{RHS}}(z) = n f(z) F_{n-1}(z)
\]

\[
= n f(z) \int_0^z f_{n-1}(x_1) \, dx_1
\]

\[
= n(n-1) f(z) \int_0^z f(x_1) F_{n-2}(x_1) \, dx_1
\]

\[
= n(n-1)(n-2) f(z) \int_0^z \int_0^{x_1} f(x_1) f(x_2) F_{n-3}(x_2) \, dx_2 \, dx_1.
\]

Repeating this argument \((s-2)\) times we obtain

\[
\frac{f_{\text{RHS}}(z)}{(n)_s} = f(z) \int_0^z \int_{D_1} \left( \prod_{k=1}^s f(x_k) \right) F_{n-s-1}(x_s) \, dA_s \, dx_s,
\]

where for convenience we denote

\[
\int_{D_j} (.) \, dA_{j+1} := \int_0^{x_{j+1}} \cdots \int_0^{x_{s-1}} (.) \, dx_s \cdots dx_{j+1}, \quad 1 \leq j \leq s-1.
\]

Setting

\[
g_i(x_i) := F_t(x_i) \prod_{j=1}^{s} f(x_j) \quad 1 \leq i \leq s-1,
\]

for the \((2t+s)\)th derivative of \( f_{\text{RHS}}(z)/(n)_s \) at 0, we find

\[
\frac{d^{2t+s}}{dz^{2t+s}} \left[ f(z) \int_0^z \int_{D_1} g_1(x_1) \, dA_2 \, dx_1 \right] \bigg|_{z=0}
\]

\[
= \sum_{i_1=0}^{2t+s} \binom{2t+s}{i_1} f^{(i_1)}(0) \frac{d^{2t+s-i_1}}{dz^{2t+s-i_1}} \left[ \int_0^z f(x_1) \int_{D_1} g_2(x_2) \, dA_2 \, dx_1 \right] \bigg|_{z=0}
\]

\[
= \sum_{i_2=0}^{2t+s-1} \binom{2t+s}{i_1} f^{(i_1)}(0) \frac{d^{2t+s-1-i_2}}{dz^{2t+s-1-i_2}} \left[ f(z) \int_0^z \int_{D_2} g_2(x_2) \, dA_3 \, dx_2 \right] \bigg|_{z=0}
\]

\[
= \sum_{i_2=0}^{2t+s-2} \binom{2t+s-2}{i_1} f^{(i_1)}(0) \sum_{i_2=0}^{2t+s-3} \binom{2t+s-3-i_2}{i_2} (2t+s-3-i_2) \bigg[ f(z) \int_0^z \int_{D_3} g_3(x_3) \, dA_4 \, dx_3 \bigg] \bigg|_{z=0}.
\]
Recall that \( r_j := 2t - \sum_{k=1}^{j-1} i_k \) for \( j = 1, \ldots, s \). Repeating the last argument \((s-2)\) times, it is not difficult to obtain

\[
\sum_{i_1=0}^{r_1} \cdots \sum_{i_{s-1}=0}^{r_{s-1}} \left( \prod_{j=1}^{s-1} \left( r_j + s - j + 1 \right) \right) f^{(i_s)}(0) \frac{d^{r_s+1}}{dx^{r_s+1}} [f(x)F_{t+1}(x)]_{x=0}
\]

\[
= \sum_{i_1=0}^{r_1} \cdots \sum_{i_{s-1}=0}^{r_{s-1}} \left( \prod_{j=1}^{s-1} \left( h_j \right) f^{(i_j)}(0) \right) \sum_{i_1=0}^{r_s+1} \left( \frac{r_s + 1}{i} \right) f^{(r_s+1-i)}(0)F_{t+1}(0)
\]

\[
= \sum_{i_1=0}^{r_1} \cdots \sum_{i_{s-1}=0}^{r_{s-1}} \left( \prod_{j=1}^{s-1} \left( h_j \right) f^{(i_j)}(0) \right) \sum_{i_1=0}^{r_s+1} \left( \frac{r_s + 1}{i} + 1 \right) f^{(r_s+i)}(0)F_{t+1}(0),
\]

where \( h_j := r_j + s - j + 1 \) for \( 2 \leq j \leq s - 1 \). This is equivalent to \((8)\), which needed to be proven.

### 4.3 Proof of Lemma 3

(i) Indeed, using the definition of \( H_{s,j}(x) \) in \((7)\), we have

\[
\sum_{j=0}^{r-1} \binom{r}{j+1} H_{s,j}(s+1) = \sum_{j=0}^{r-1} \binom{r}{j+1} \sum_{i=0}^{s} (-1)^i \binom{s}{i} (s+1-i)^j
\]

\[
= \sum_{i=0}^{s} (-1)^i \binom{s}{i} \sum_{k=1}^{r} \binom{r}{k} (s+1-i)^{k-1}
\]

\[
= \sum_{i=0}^{s} (-1)^i \binom{s}{i} \frac{1}{s+1-i} \left[ \sum_{k=0}^{r} \binom{r}{k} (s+1-i)^k - 1 \right]
\]

\[
= \frac{1}{s+1} \sum_{i=0}^{s} (-1)^i \binom{s+1}{i} [(s+2-i)^r - 1]
\]

\[
= \frac{1}{s+1} \sum_{i=0}^{s+1} (-1)^i \binom{s+1}{i} (s+2-i)^r - \frac{1}{s+1} \sum_{i=0}^{s+1} (-1)^i \binom{s+1}{i}
\]

\[
= \frac{1}{s+1} \sum_{i=0}^{s+1} (-1)^i \binom{s+1}{i} (s+2-i)^r
\]

\[
= \frac{1}{s+1} H_{s+1,r}(s+2).
\]

(ii) We have

\[
\sum_{j=0}^{r-1} (s+2)^{r-1-j} H_{s,j}(s+1) = \sum_{j=0}^{r-1} (s+2)^{r-1-j} \sum_{i=0}^{s} (-1)^i \binom{s}{i} (s+1-i)^j
\]

\[
= \sum_{i=0}^{s} (-1)^i \binom{s}{i} (s+2)^{r-1-j} \sum_{j=0}^{r-1} \binom{s+1-i}{j} (s+1-i)^j
\]

\[
= \sum_{i=0}^{s} (-1)^i \binom{s}{i} (s+2)^{r-1} \sum_{j=0}^{r-1} \binom{s+1-i}{j} (s+1-i)^j.
\]
\[
\begin{align*}
    &= \sum_{i=0}^{s} (-1)^i \binom{s}{i} \frac{1}{i+1} [(s+2)^r - (s+1-i)^r] \\
    &= \frac{1}{s+1} \sum_{i=0}^{s} (-1)^i \binom{s+1}{i+1} [(s+2)^r - (s+1-i)^r] \\
    &= \frac{1}{s+1} \sum_{j=0}^{s+1} (-1)^j \binom{s+1}{j} (s+2-j)^r - \frac{(s+2)^r}{s+1} \sum_{j=0}^{s+1} (-1)^j \binom{s+1}{j} \\
    &= \frac{1}{s+1} H_{s+1,r}(s+2).
\end{align*}
\]

5 Concluding Remarks

The main result in this paper is a characterization of the exponential distribution via a distributional relationship involving a pair of maxima of i.i.d. continuous random variables. It may be considered as part of the group of characterization results based on the distributional equation with independent shift \(X + T \overset{d}{=} Y\). In this paper the shift (translator) \(T\) is a sum of i.i.d. variables without a specified distribution. In particular, as a corollary, it is obtained that the Sukhatme-Rényi decomposition of the maximum of i.i.d. variables is a characterization property for the exponential distribution.

A key tool in the proof of the main theorem is a lemma due to Arnold and Villaseñor [4], which requires analyticity of the density function at 0. It is an open question if this assumption can be weakened by using a different method of proof.

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