Calculation of the triangular panels’ orthotropic covers for the retained domes

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Abstract. The experience of design and construction of single-mesh spherical dome coverings based on meshes with triangular cells was studied. Constructive solutions of wooden domes made of triangular panels with orthotropic plywood covers are considered. The method analysis for constructing a bar model of an orthotropic plate in the form of an oblique triangle has been carried out. It is proposed to reduce the calculation of the triangular dome panel’s rod-sheet structure to the calculation of the hinge-bar system. The stiffness matrices of the contour and intra-contour rods of the super cell are obtained.

Introduction
Bar spatial structures, forming a mesh system on the surface are widely used in modern construction practice. Single-layer structures have a curved mesh surface and are called single mesh. Two-layer structures have two parallel mesh surfaces connected by rigid bar ties. Such structures are called two-mesh structures. One of the effective forms of spatial structures are single-mesh spherical dome coverings made of metal and wood. World experience in design and construction shows that currently single-mesh domes are designed primarily on the basis of meshes with triangular cells. [1, 2, 3, 4, 5, 7, 11, 13]. Domestic and foreign experience of dome construction confirms the rationality of using single-mesh spherical dome coverings made of wood and materials based on it.

When calculating single-mesh domes, many authors use an approximate technique based on the use of a continuous computational model in the form of an isotropic spherical shell [8, 9, 10, 12, 14]. In this case, the assumption that the structural network of the dome is rather dense and therefore each elementary cell of it is an equilateral triangle, is introduced. In addition, in these works, the elementary cell of the dome’s structural network, located in such a way on the surface of the sphere that its height turns out to be lying in the meridional plane, is considered. The disadvantage of this approach is quite obvious, since the junctions between the panels in general do not coincide with the parallels and meridians, and the cells themselves have the shape of an arbitrary oblique triangle. It should also be noted that within the dome panel, the cladding can be made of an orthotropic material, and in this case a situation when the main directions of elasticity do not coincide with the directions of the plate sides is possible. To eliminate the noted drawbacks, below is a solution to the problem of constructing a bar model of an orthotropic plate in the form of an oblique triangle, reinforced along the contour by ribs.

Materials and methods
Based on the method proposed in [6] for constructing a bar model of an orthotropic plate in the form of an oblique triangle, it seems possible to reduce the calculation of a bar-and-sheet structure of a
polyhedral dome to the calculation of a hinge-bar system consisting of a finite number of super cells interconnected at nodes. We will assume that the super cell is a triangular dome panel, which includes an orthotropic plate and the contour edges of the panel frame that reinforce it. As a design model of a triangular orthotropic plate, an element in the form of an oblique triangle, referred to the rectangular coordinate system $\mathcal{XOY}$, is taken. One of its vertices 1 coincides with the origin, and the side 1-2 is aligned with the direction of the axis 0Y (Figure 1). Contour bars of the model 1-2, 2-3 and 3-1 are pivotally connected at the vertices of the triangle. The elastic modulus of these bars will be considered equal to $E$, and the cross-sectional area is assumed to be the same and equal to $A$. Let the interior angles at vertices 1, 2 and 3 be respectively equal to $\theta_1$, $\theta_2$ and $\theta_3$, and the lengths of the contour bars - $L_1$, $L_2$ and $L_3$, moreover $L_1 + L_2 + L_3 = 2S$. Point $D$ is connected to the vertices of the triangle using the elastic internal contour bars $AD$, $BD$ and $CD$, which length is respectively equal to $l_1$, $l_2$ and $l_3$. The cross-sectional areas of these bars are assigned so that

$$A_{11}: A_{12}: A_{13} = \cos(\theta_1/2): \cos(\theta_2/2): \cos(\theta_3/2).$$

Let us introduce the designation of the bars’ stiffness that are a part of the super cell. Let the stiffness of the contour bars of the design model of the orthotropic plate be $E_A$, and the stiffness of the internal contour bars - $E_iA_i$. To determine the values $E_A$ and $E_iA_i$, it is recommended to use the appropriate formulas given in the above-mentioned work:

$$E_A = \frac{E_x}{2}h_2/B,$$

$$E_iA_i = EA(4sL_3/L_1L_2)(B - C)/D\cos(\theta_i/2),$$

where $B = l_{31}^2(l_{31}^2 - l_{23}^2) + \left(\frac{E_x}{G_{xy}}\right)l_{31}m_{31}(l_{31}m_{31} - l_{23}m_{23}) + \nu_{xy}(l_{23}m_{31}^2 + l_{31}m_{23}^2 - 2l_{31}m_{31}) + (E_x/E_y)m_{31}^2m_{31} - m_{23}^2),$

$$C = (h_2/h_2)\left[l_{12}^4 + \left(\frac{E_x}{G_{xy}} - 2\nu_{xy}\right)l_{23}m_{23} + \left(E_x/E_y\right)m_{31}^4\right],$$

$$D = l_{12}^2[l_{12}^2 + (h_3/h_1)l_{23}^2 + (h_3/h_1)l_{31}^2] + \left(E_x/G_{xy}\right)l_{12}m_{12}l_{12}m_{12} + (h_3/h_1)l_{23}m_{23} + (h_3/h_1)l_{31}m_{31},$$

$$\nu_{xy}\left[2l_{12}m_{12}^2 + \left(h_1\right)^2l_{23}^2m_{23}^2 + l_{23}^2m_{12}^2 + (h_3/h_2)(l_{12}m_{31}^2 + l_{23}m_{12}) + (h_3/h_2) \times (l_{12}m_{31}^2 + l_{23}m_{12}) + (E_x/E_y)m_{31}^2 \times (m_{12}^2 + (h_3/h_1)m_{23}^2 + (h_3/h_2)m_{31}^2).$$

**Figure 1.** Design model of a triangular orthotropic plate.

Here, the following notation for the parameters of an orthotropic triangular plate is used:
$E_x, E_y$ – are the elasticity moduli in the main directions $X$ and $Y$;

$G_{xy}$ – is the shear modulus in planes parallel to the median surface of the plate;

$h_y$ – is the height dropped from the top $i$ to the opposite side;

t – defines plate thickness;

$l_{ij}$ and $m_{ij}$ – are the direction cosines of the rib $ij$.

The stiffness of the contour ribs supporting the triangular covers of the dome panels is denoted by $E_s A_s$.

**Results**

When using the super cell approach for calculating domes, stiffness matrices of individual super cells are compiled, from which the stiffness matrix for the entire structure is formed. The expression for the stiffness matrix of the super cell contour bars, taking into account the stiffness of the contour ribs reinforcing the covering, is obtained in the form:

$$\alpha_{ks} = (1 + k_1) \frac{E_s A_s}{L_1} \begin{bmatrix}
 a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\
 a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
 a_{33} & a_{34} & a_{35} & a_{36} \\
 a_{44} & a_{45} & a_{46} \\
 a_{55} & a_{56} \\
 a_{66}
\end{bmatrix}$$

(4)

$k_1 = EA/(E_s A_s)$. Matrix elements $\alpha_{ks}$ are calculated in the following way:

$$a_{11} = -a_{15} = \frac{l_1}{L_3} \sin^2 \theta_1,$$

$$a_{12} = -a_{16} = a_{25} = \frac{l_1}{L_3} \sin \theta_1 \cos \theta_1,$$

$$a_{13} = a_{14} = a_{23} = 0,$$

$$a_{22} = 1 + \frac{l_1}{L_3} \cos^2 \theta_1,$$

$$a_{24} = -1,$$

$$a_{26} = -\frac{l_1}{L_3} \cos^2 \theta_1,$$

$$a_{33} = -a_{35} = \frac{l_1}{L_2} \sin^2 \theta_2,$$

$$a_{34} = -a_{36} = -a_{45} = -\frac{l_1}{L_2} \sin \theta_2 \cos \theta_2,$$

$$a_{44} = 1 + \frac{l_1}{L_2} \cos^2 \theta_2,$$

$$a_{46} = -\frac{l_1}{L_2} \cos \theta_2,$$

$$a_{55} = a_{11} + a_{33},$$

$$a_{56} = a_{12} + a_{34},$$

$$a_{66} = -(a_{26} + a_{46}).$$

Let us now proceed to the stiffness matrices construction of the model’s internal contour bars (Figure 1). Let us express the displacement $u_7$ and $u_8$ points 4 through the displacements of the nodal points of the super cell 1, 2 and 3.

$$u_7(8) = au_1(2) + bu_3(4) + cu_5(6),$$

(5)

where

$$a = 1 - \frac{l_1}{L_1} \frac{\sin \theta_1/2}{\sin \theta_1} \left( 1 + \frac{l_1}{L_2} \right);$$

(6)

$$b = \frac{l_1}{L_1} \frac{\sin \theta_1/2}{\sin \theta_1};$$

(7)

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The expression for the stiffness matrix of the internal contour bars has the form:

\[
\mathbf{\alpha}_i = k_2 \frac{E_s A_s}{L_1} \begin{bmatrix}
    b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & b_{16} \\
    b_{22} & b_{23} & b_{24} & b_{25} & b_{26} \\
    b_{33} & b_{34} & b_{35} & b_{36} & b_{37} \\
    b_{44} & b_{45} & b_{46} & b_{47} & b_{48} \\
    b_{55} & b_{56} & b_{57} & b_{58} & b_{59} \\
    b_{66} & b_{67} & b_{68} & b_{69} & b_{70}
\end{bmatrix}
\]

symmetrically

\[c = \frac{L_1 \sin \theta_1 / 2}{L_3 \sin \theta_1}, \quad (8)\]

where \(k_2 = E_1 A_1 L_1 / (E_s A_s l_1)\).

The elements of the matrix expression included in (9) are as follows:

\[
b_{11} = (a - 1)^2 \sin^2 \frac{\theta_1}{2} + a^2 \left( \sin^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta_2 - \theta_1}{2} \right),
\]

\[
b_{12} = \frac{1}{2} \left( (a - 1)^2 \sin \theta_1 + a^2 \left[ \sin \theta_2 - \sin \theta_1 \right] - \cos \theta_2 \cos \theta_1 \right),
\]

\[
b_{13} = ab \left( \sin^2 \frac{\theta_1}{2} + \sin \frac{\theta_2}{2} + \cos \frac{\theta_2 - \theta_1}{2} \right) - \left( b \sin^2 \frac{\theta_1}{2} + a \cos^2 \frac{\theta_2 - \theta_1}{2} \right),
\]

\[
b_{14} = \frac{1}{2} \left( ab \left[ \sin \theta_1 - \sin \theta_2 + \sin \left( \theta_2 - \theta_1 \right) \right] + \left( \cos \theta_2 - \cos \theta_1 \right) \right),
\]

\[
b_{15} = ac \left( \sin^2 \frac{\theta_1}{2} + \sin \frac{\theta_2}{2} + \cos \frac{\theta_2 - \theta_1}{2} \right) - \left( c \cos^2 \frac{\theta_1}{2} + a \cos^2 \frac{\theta_2 - \theta_1}{2} \right),
\]

\[
b_{16} = b_{25} = \frac{1}{2} \left( ab \left[ \sin \theta_1 - \sin \theta_2 + \sin \left( \theta_2 - \theta_1 \right) \right] - \left[ \cos \theta_1 + \cos \left( \theta_2 - \theta_1 \right) \right] \right),
\]

\[
b_{22} = (a - 1)^2 \cos^2 \frac{\theta_1}{2} + a^2 \left( \cos^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_2 - \theta_1}{2} \right),
\]

\[
b_{23} = \frac{1}{2} \left( ab \left[ \sin \theta_1 - \sin \theta_2 + \sin \left( \theta_2 - \theta_1 \right) \right] + \left( \cos \theta_2 - \cos \theta_1 \right) \right),
\]

\[
b_{24} = ab \left( \cos^2 \frac{\theta_1}{2} + \cos \frac{\theta_2}{2} + \sin \frac{\theta_2 - \theta_1}{2} \right) - \left( b \cos^2 \frac{\theta_1}{2} + a \sin^2 \frac{\theta_2 - \theta_1}{2} \right),
\]

\[
b_{25} = ac \left( \cos^2 \frac{\theta_1}{2} + \cos \frac{\theta_2}{2} + \sin \frac{\theta_2 - \theta_1}{2} \right) - \left( c \cos^2 \frac{\theta_1}{2} + a \cos^2 \frac{\theta_2 - \theta_1}{2} \right),
\]

\[
b_{26} = d \left( 1 - b \right)^2 \sin^2 \frac{\theta_1}{2} + b^2 \left( \sin^2 \frac{\theta_2}{2} + \cos^2 \frac{\theta_2 - \theta_1}{2} \right),
\]

\[
b_{33} = \frac{1}{2} \left( b \left[ \sin \theta_1 - \sin \theta_2 + \sin \left( \theta_2 - \theta_1 \right) \right] - d \left( 1 - b \right)^2 \sin \theta_2 \right),
\]

\[
b_{35} = bc \left( \sin^2 \frac{\theta_1}{2} + \sin \frac{\theta_2}{2} + \cos \frac{\theta_2 - \theta_1}{2} \right) - \left( c \sin^2 \frac{\theta_2}{2} + b \cos^2 \frac{\theta_2 - \theta_1}{2} \right),
\]

\[
b_{36} = b_{45} = \frac{1}{2} \left( bc \left[ \sin \theta_1 - \sin \theta_2 + \sin \left( \theta_2 - \theta_1 \right) \right] + \left[ \cos \theta_2 - \cos \left( \theta_2 - \theta_1 \right) \right] \right),
\]

\[
b_{44} = d \left( 1 - b \right)^2 \cos^2 \frac{\theta_1}{2} + b^2 \left( \cos^2 \frac{\theta_2}{2} + \sin^2 \frac{\theta_2 - \theta_1}{2} \right),
\]

\[
b_{46} = bc \left( \cos^2 \frac{\theta_1}{2} + \cos \frac{\theta_2}{2} + \sin \frac{\theta_2 - \theta_1}{2} \right) - \left( c \cos^2 \frac{\theta_2}{2} + b \sin^2 \frac{\theta_2 - \theta_1}{2} \right),
\]

\[
b_{55} = f \left( 1 - c \right)^2 \cos^2 \frac{\theta_2 - \theta_1}{2} + c^2 \left( \sin^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta_2}{2} \right),
\]

\[
b_{56} = \frac{1}{2} \left( f \left( 1 - c \right)^2 \sin \theta_2 + c^2 \left( \sin \theta_2 - \sin \theta_1 \right) \right),
\]

\[
b_{66} = f \left( 1 - c \right)^2 \sin^2 \frac{\theta_2 - \theta_1}{2} + c^2 \left( \cos^2 \frac{\theta_1}{2} + \cos^2 \frac{\theta_2}{2} \right).
\]

The following notation is introduced here: \(d = \frac{\sin \theta_2}{\sin \theta_1}, f = \frac{\sin \theta_1}{\sin \theta_1}\).

Summary

The stiffness matrix of the super cell is written in its final form as the sum of two matrices (4) and (9):

\[
\mathbf{\alpha}_e = \mathbf{\alpha}_k + \mathbf{\alpha}_l.
\]

(10)
Having the stiffness matrices for triangular super cells in the local coordinate system XDY and using the congruent transformations, we further form the stiffness matrices of these elements in a common coordinate system. After constructing the stiffness matrix of the entire spatial configuration, the unknown displacements of nodes and stresses in the bar and plate elements of the multiple retained dome are determined. The validity of the proposed super cell approach is confirmed by the results obtained when solving the problem of calculating a dome for the vertical load action applied at its top [6].

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