On alphabetic presentations of Clifford algebras and their possible applications

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In this paper, we address the problem of constructing a class of representations of Clifford algebras that can be named "alphabetic (re)presentations." The Clifford algebra generators are expressed as \(m\)-letter words written with a three-character or a four-character alphabet. We formulate the problem of the alphabetic presentations, deriving the main properties and some general results. At the end, we briefly discuss the motivations of this work and outline some possible applications.

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I. INTRODUCTION

The irreducible representations of Clifford algebras have been classified in Ref. 1. Convenient reformulations of this result can be found, e.g., in Refs. 2 and 3, where some topics, such as the connection with division algebras, are also discussed.

The \(\text{Cl}(p,q)\) Clifford algebra over the real is the enveloping algebra generated by the \(\gamma_i\) real matrices \((i=1,\ldots,p+q)\) and quotiented by the relation

\[
\gamma_i \gamma_j + \gamma_j \gamma_i = 2 \eta_{ij} \mathbf{1},
\]

where \(\eta_{ij}\) is a diagonal matrix with \(p\) positive entries \(+1\) and \(q\) negative entries \(-1\). In the following, a basis of \(p+q\) gamma matrices \(\gamma_i\) satisfying (1) will be called a gamma basis.

The real irreducible representations are, up to similarity transformations, unique for \(p-q\neq 1,5\) mod 8, while for \(p-q=1,5\) mod 8, there are two inequivalent irreducible representations, which can be recovered by flipping the sign \((\gamma_i \rightarrow -\gamma_i)\) of all gamma basis generators. The size \(n\) of an \(n \times n\) real matrix irreducible representation is specified in terms of \(p\) and \(q\).

Both in Refs. 2 and 4, the given gamma basis representatives of a \(\text{Cl}(p,q)\) real irreducible representation were explicitly constructed (for any \(p,q\) pair) up to an overall sign flipping in terms of tensor products of four basic \(2 \times 2\) real matrices. In Ref. 4, the four matrices were named \(\sigma_1, \sigma_2, \sigma_A, \mathbf{1}\), and are defined as follows:

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Without loss of generality, e.g., the three irreducible gamma generators of, let us say, \(\text{Cl}(3,0)\), can be explicitly given by

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\( \gamma_1 = 1 \otimes \sigma_1, \quad \gamma_2 = 1_2 \otimes \sigma_2, \quad \gamma_3 = \sigma_A \otimes \sigma_A \) (3)

[any different presentation for the Cl(3,0) gamma basis is equivalent by similarity].

Extending this result, the \( p+q \) generators of a given real irreducible Cl\((p,q)\) Clifford algebra can be expressed as strings of tensor products of the above four matrices, taken \( m \) times (if \( n \) is the size of the irreducible representation, therefore \( n = 2^m \); in the previous \( p=3, q=0 \) example, \( n = 4 \) and \( m = 2 \)).

In the above type of gamma basis presentations, a few points should be noticed. At first, the introduction of the tensor product symbol “\( \otimes \)” is redundant. Once we understood that we are dealing with tensor products, we do not need to write it explicitly. For the same reason, the four matrices given in (2) can be expressed with four characters of some given alphabet. For our purposes here, we choose the four characters being given by \( I, X, Z, A \); we associate them with the above gamma matrices according to

\[ 1_2 = I, \quad \sigma_1 = X, \quad \sigma_2 = Z, \quad \sigma_A = A \] (4)

\( A \) stands for antisymmetric since \( \sigma_A \) is the only antisymmetric matrix in the above set.

In the above example, the three gamma matrices \( \gamma_i \) can be more compactly expressed through the positions

\[ \gamma_1 = IX, \quad \gamma_2 = IZ, \quad \gamma_3 = AA. \] (5)

With the above identifications, for any \( (p,q) \) pair (with the exception of the trivial \( p = 1, q = 0 \) case) and up to an overall sign factor, we can always write down the \( p+q \) generators of a gamma basis as \( m \)-letter words (the value \( m \) is common to all words of the basis), written with the four \( I, X, Z, A \) characters. For obvious reasons, we call this type of gamma matrix presentations “alphabetic presentations” or “alphabetic representations,” according to the context.

Not all representations are alphabetic according to the previous definition. The Cl\((2,0)\) Clifford algebra admits \( X \) and \( Z \) as gamma bases. An equivalent gamma basis can be expressed, e.g., through the “entangled” matrices \( \bar{X} = 1/\sqrt{2}(X+Z), \bar{Z} = 1/\sqrt{2}(X-Z) \).

In any case, due to the results in Refs. 2 and 4, it is always possible to produce a four-character alphabetic presentation of an irreducible gamma basis with words of a given length \( m \). In the Euclidean case \( (q=0) \), for instance, \( m \) is explicitly given by the formula

\[ m = \log_2 G(k+1) + 4r + 1, \] (6)

where \( p \geq 2 \) is parametrized according to

\[ p = 8r + k + 2, \] (7)

with \( r = 1, 2, \ldots \) and \( k = 0, 1, 2, 3, 4, 5, 6, 7 \), while \( G(k+1) \) is given by the Radon–Hurwitz function

\[ G(n) = \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
1 & 2 & 4 & 4 & 8 & 8 & 8 & 8
\end{array} \] (8)

The mod 8 property is a consequence of the famous Bott’s periodicity.

We can therefore concentrate on the subclass of the alphabetic presentations, as previously defined. Several questions can now be addressed. How many inequivalent alphabetic presentations can be defined? The notion of the equivalence group should not be based, of course, on the class of similarity transformations connecting real-valued Clifford algebras, instead the notion of a finite equivalence group of suitably defined moves transforming characters and words of an alphabetically presented gamma basis into a new, equivalent, alphabetically presented gamma basis should be given. Further questions can be addressed. Given the fact that \( A \) is the only character whose square is negative \( [A^2 = -I] \), with the position (4)), any alphabetically presented Euclidean gamma basis (for \( q = 0 \)) admits words with even numbers of \( A \)'s only (in case (5) above \( \gamma_3 \) contains two \( A \)'s, while \( \gamma_1, \gamma_2 \) contain no \( A \)'s). Is it possible to define, for any \( p \), Euclidean alphabetic presen-
tations containing only the three characters \( I, X, Z \) (namely, limiting ourselves to a three-character alphabet)? Furthermore, which is the minimal length \( m \) of the three-character Euclidean words? Under which conditions \( m \) coincides with \( m \) given in (6)? This is just a partial list of the questions that we are addressing (and partially solve) in this paper. To our knowledge, this type of program has never been investigated in literature. Due to the recognized importance of Clifford algebras in several areas of mathematics and physics (for our purposes here, it is sufficient to mention the applications to higher-dimensional unification theories such as supergravities or superstrings\(^6\), or the applications to robotics\(^7\)), we feel that it deserves being duly investigated. At the end of this paper, we provide a very rough and preliminary list of possible topics that could benefit from it.

The main core of this paper is devoted to the formulation of the problem and the presentation of general results and partial answers. The scheme of this paper is as follows. In Sec. II we prove the Euclidean completeness of the three-character alphabetic presentations, introduce the equivalence group and a set of invariant functions. In Sec. III we furnish a few algorithmic constructions to induce inequivalent three-character alphabetic presentations, compute the admissible invariants, and present the results of an extensive computer search (for three-, four-, five-, and six-letter words). A table with the minimal lengths for three-character alphabetic presentations of \( \text{Cl}(p,0) \) is also given. Further issues and an outline of four-character alphabetic presentations will be discussed in Sec. IV. We will also mention there some topics that could benefit from the present investigation program.

II. ALPHABETIC PRESENTATIONS

In Sec. I we defined the alphabetic presentations of the gamma basis generators of a \( \text{Cl}(p,q) \) Clifford algebra as given by \( p+q \) words of \( m \)-letters constructed with the four alphabetic characters \( I, A, X, Z \) [the alphabetic characters are in one-to-one correspondence (4) with the four \( 2 \times 2 \) matrices (2)]. We also pointed out that for Euclidean Clifford algebras (\( q=0 \)) with \( p \geq 2 \), three-character alphabetic presentations of the \( p \) gamma basis generators could exist. Their words are constructed with the \( I, X, Z \) characters alone. It is indeed easily proved that a three-character alphabetic presentation is Euclidean complete. This means the following: for any \( p \), it is always possible to find \( p \) words satisfying (1) and written with \( I, X, Z \) alone. The completeness of the four-character alphabetic presentations is guaranteed by the results given, e.g., in Refs. 2 and 4. If the given \( \text{Cl}(p,0) \) alphabetic presentation contains no \( A \)'s, a three-character presentation immediately follows. If at least one word contains an \( A \) in the \( j \)th position, we can replace all \( j \)th letter characters by two characters (in \( j \)th and \( j+1 \)th positions) according to, for instance, \( I \rightarrow II, X \rightarrow XX, Z \rightarrow ZX, A \rightarrow IZ \). [This position leaves the anticommutation relation (1) between two different characters unchanged. The square of \( A \) changes its sign. This, however, has no overall effect since each word of the Euclidean gamma basis contains an even number of \( A \)'s.] If the original words possess \( m \) letters, the transformed words possess \( m+1 \) letters. We can repeat the procedure every time we need to get rid of all \( A \)'s. The replacement leaves the relation (1) unchanged. Applying the transformations to the \( \text{Cl}(3,0) \) gamma basis (5), we obtain, for instance, the three-character presentation

\[
\gamma_1 = II X, \quad \gamma_2 = I I Z, \quad \gamma_3 = I Z I Z.
\]

(9)

It follows that a three-character presentation (not necessarily with minimal length words) can always be found for any \( p \). Translated back into the matrix language (tensor products of \( 2 \times 2 \) matrices), it produces representations of (1) \( q=0 \) generating relations in terms of matrices that are not necessarily irreducible. “Alphabetic” irreducibility should not be confused with matrix irreducibility.

A. The alphabetic group of equivalence

We are now in the position to introduce the finite group of equivalence acting on alphabetic presentations. It is easier to discuss at first the three-character alphabetic presentations. It is convenient to arrange the \( p \) words of \( m \) letters each of a given alphabetic \( \text{Cl}(p,0) \) gamma basis
into a $p \times m$ rectangular matrix, whose entries are the three alphabetic characters. The equivalence group $G$ acting on the $p \times m$ rectangular matrices is obtained by combining the following three types of moves:

(i) permutations of the rows (they correspond to irrelevant reordering of the $p$ words);
(ii) permutations of the columns [the anticommutative property (1) between two distinct given words is unaffected by this operation]; and
(iii) transmutation of the characters in a given column: $X, Z$ are exchanged ($X \leftrightarrow Z$), while $I$ is unchanged [as before, the anticommutative property (1) between two distinct given words is unaffected by this operation].

It should be noticed that the rectangular matrices can be simplified, without affecting the relation (1), by erasing the columns possessing either a single character or the two characters $I$ and $X$ or $I$ and $Z$ (the columns possessing both $X$ and $Z$ as entries cannot be erased). The process of erasing columns will be referred to as “simplification of the rectangular matrix.” A simple rectangular matrix is a rectangular matrix that cannot be further simplified. It produces a simple alphabetic presentation of a gamma basis. To be explicit, the three-character presentation of the Cl(3,0) gamma basis (9) is associated with a $3 \times 4$ rectangular matrix, which can be simplified, erasing the first and the second columns, to produce a $3 \times 2$ rectangular matrix according to

\[
\begin{align*}
I & \ I \ X \ X \ X \ X \\
I & \ I \ Z \ X \rightarrow Z \ X \\
I & \ Z \ I \ Z \ I \ Z
\end{align*}
\]

The simple rectangular matrix on the right hand side corresponds to three two-letter (length 2) words. This is the minimal length for an alphabetic presentation of Cl(3,0). It coincides with the minimal length of the presentation (5), which, on the other hand, requires four characters instead of just three.

Two problems will be addressed in Sec. III.

(1) Which is the minimal length $\bar{m}$ of the words for a three-character alphabetic presentation of Cl$(p,0)$?

(2) How many inequivalent simple presentations of length $m$ can be found for a three-character alphabetic presentation of Cl$(p,0)$?

The second problem can be investigated with the help of invariants that detect the inequivalent classes under the finite group of transformations defined above. We introduce a few invariants, a “horizontal invariant” and the “vertical invariants.”

### B. Alphabetic invariants

The horizontal invariant is defined as follows: at first, the number $m_I$ of $I$ entries in any one of the $p$ rows is computed. Let us suppose we obtain $i$ different results $k_1, \ldots, k_i$. We order them according to $k_1 > k_2 > \cdots > k_i \geq 0$. Let $h_r$ be the number of rows producing the $k_r$ result ($r = 1, 2, \ldots, i$). Obviously, $h_1 + h_2 + \cdots + h_i = p$. The horizontal invariant $\text{hor}$ is expressed as an ordered set of the $h_r$ values with $k_r$ as the suffix. We write it as $\text{hor}(h_{1k}, h_{2k}, \ldots, h_{ik})$. It is easily checked that $\text{hor}$ is invariant under the group transformations (permutations and transmutations). As an example, the $\text{hor}$ invariant of the simple rectangular matrix in the right hand side of (10) is $\text{hor}(1, 2_0)$.

The first vertical invariant $\text{ver}$ is analogously defined; the difference is that the number $n_I$ of $I$ entries is computed in terms of the columns. Let us suppose we get $j$ different results $l_1, \ldots, l_j$, ordered according to $l_1 > l_2 > \cdots > l_j \geq 0$. Let $v_l$ be the number of columns producing the $l_j$ result ($v_1 + v_2 + \cdots + v_j = m$). The vertical invariant $\text{ver}$ is expressed as $\text{ver}(v_{l_1}, v_{l_2}, \ldots, v_{l_j})$. The $\text{ver}$ invariant of the simple rectangular matrix in the right hand side of (10) is explicitly given by $\text{ver}(1_1, 1_0)$. 


The second vertical invariant \( \widetilde{\text{ver}} \) is defined as \( \text{ver} \), but instead of counting the number \( n_I \) of \( I \)'s in a given column, we compute the absolute difference \( n_{XZ} = |n_X - n_Z| \) between the number of \( X \) and the number of \( Z \) entries in any given column. Applied to (10), we obtain \( \text{ver}(1, 1, 0) \). A less refined invariant under the group generated by permutations and transmutations is the total number \( N_I \) of \( I \) entries in a simple rectangular matrix.

A more refined invariant is \( \text{ver} \), counting the number \( v_{(k, l)} \) of columns presenting the given pair \( (n_k = k_1, n_{XZ} = l) \). The result is presented as \( \text{ver}(v_{(k_1, l_1)}, v_{(k_2, l_2)}, \ldots, v_{(k_p, l_p)}) \) (the pairs are conveniently ordered). Applied to (10), we obtain \( \text{ver}(1(1,0), 1(0,1)) \).

In Sec. III it is sufficient to use the invariants \( \text{hor} \) and \( \text{ver} \) (based on the counting of \( I \)'s) to detect the inequivalent three-letter and four-letter alphabetic presentations.

For \( m=1 \) (single-letter word), we have a unique \( \text{Cl}(2,0) \) gamma basis given by \( \{X, Z\} \).

For \( m=2 \), we have four equivalent (under permutations and transmutations) presentations of \( \text{Cl}(3,0) \), given by \( \{XX, ZX, IZ\}, \{XX, XZ, ZX\}, \{ZZ, XZ, IX\}, \{ZZ, ZX, XI\} \).

In Sec. III we discuss the construction of three-character alphabetic presentations with \( m \)-letter words for higher values of \( m \).

III. INEQUIVALENT THREE-CHARACTER ALPHABETIC PRESENTATIONS

In Sec. II we furnished the \( m \)-letter three-character alphabetic presentations for \( m=1,2 \). We discuss now the situation for \( m \geq 3 \). In order to do that, besides the already introduced notion of “simple alphabetic presentation,” we also need to define the notion of “maximally extended alphabetic presentation.” It corresponds to an \( m \)-letter gamma basis \( B \) such that no further word, anticommuting with all the words in \( B \), can be added (in the following, explicit examples of nonmaximally extended gamma basis will be given; they are obtained by erasing at least one word from a maximally extended gamma basis). It turns out that, at any given \( m \), the classification of the inequivalent gamma basis is recovered from the classification of the simple, maximally extended, gamma basis.

In Ref. 4 an algorithmic presentation was given to induce new gamma basis from the previously known ones. In a very simple form (which is applied to the Euclidean case), it corresponds to produce an \((m+1)\)-letter gamma basis for the \( \text{Cl}(p+1,0) \) Clifford algebra in terms of an \( m \)-letter gamma basis for \( \text{Cl}(p,0) \). If we denote with \( \gamma_i \) the words in the \( \text{Cl}(p,0) \) gamma basis, it is sufficient to express the \( \tilde{\gamma}_j \) words \( (j=1,2,\ldots,p+1) \) in the \( \text{Cl}(p+1,0) \) gamma basis as

\[
\tilde{\gamma}_j = \gamma_i X, \\
\tilde{\gamma}_{p+1} = f^{(m)}Z, \quad f^{(m)} = I_1, \ldots, I \quad \text{(taken \( m \) times).} \quad (11)
\]

It is easily shown that the above position, in general, does not exhaust the class of inequivalent (in the sense specified in Sec. II) \((m+1)\)-letter alphabetic presentations of \( \text{Cl}(p+1,0) \). A general algorithm can be presented through the following construction. Let \( A, B_1, B_2 \) be the three sets of \( m \)-letter words (whose respective cardinalities are \( n_A, n_{B_1}, n_{B_2} \)) satisfying the following properties: both \( C_1 = A \cup B_1 \) and \( C_2 = A \cup B_2 \) are a gamma basis and, furthermore, the words in \( B_1 \) commute with all the words in \( B_2 \). Under these conditions, an \((m+1)\)-letter presentation \( \tilde{B} \) of a \( \text{Cl}(n_A + n_{B_1} + n_{B_2},0) \) gamma basis can be produced by setting, symbolically,

\[
\tilde{B} = \{AI, B_1X, B_2Z\} \quad (12)
\]

One should notice that \( A \) could be the empty set, while both \( B_1, B_2 \) must necessarily be nonempty in order for \( \tilde{B} \) to be a simple gamma basis.

We applied this algorithm to induce, for \( m=3,4 \), the whole set of inequivalent, simple, maximally extended, gamma basis. In parallel, we produce a systematic computer search of the inequivalent, simple, maximally extended gamma basis for \( m=3,4,5,6 \). The results are reported below.
A. Three-letter alphabetic presentations

For \( m = 3 \), there are only three inequivalent, simple, maximally extended gamma bases (two for \( p = 4 \), one for \( p = 5 \)). The representatives in each given class and their associated invariants are explicitly given by

\[
\begin{align*}
\alpha(p = 4): & \quad X \ X \ X \\
& \quad \rightarrow [\text{hor}(1_2,1_1,2_0); \ \text{ver}(1_2,1_1,1_1); \ \mathcal{N}_I = 3], \\
\alpha(p = 4): & \quad X \ X \ Z \\
& \quad \rightarrow [\text{hor}(4_1); \ \text{ver}(2_3,1_0); \ \mathcal{N}_I = 4], \\
\beta(p = 4): & \quad Z \ I \ I \\
\beta(p = 4): & \quad Z \ I \ X \\
& \quad \rightarrow [\text{hor}(3_1,2_0); \ \text{ver}(3_1); \ \mathcal{N}_I = 3].
\end{align*}
\]

One should notice that two inequivalent \( p = 4 \) nonmaximally extended gamma bases are obtained by erasing one word from \( \beta \); if the word to be erased is \( \text{XXX} \), we obtain a gamma basis with horizontal invariant \( \text{hor}(3_1,1_1) \), while if the word to be erased is \( \text{XIZ} \), we obtain a gamma basis \( \{\text{XXX}, \text{IZX}, \text{XZI}, \text{ZZZ}\} \) with horizontal invariant \( \text{hor}(2_1,2_0) \). The \( \mathcal{N}_I \) invariant of the first case (the \( \{\text{XIZ}, \text{IZX}, \text{XZI}, \text{ZZZ}\} \) gamma basis) is \( \mathcal{N}_I = 3 \), which means that it is not sufficiently refined to detect a difference between this nonmaximally extended representation and the maximally extended \( \alpha \) gamma basis. Erasing from both cases above an extra, conveniently chosen, word, we produce two inequivalent \( p = 3 \) simple nonmaximally extended gamma bases. They are given by \( \{\text{XIZ}, \text{XZI}, \text{IZX}\} \) with horizontal invariant \( \text{hor}(3_1) \) and \( \{\text{XXX}, \text{XIZ}, \text{ZZZ}\} \) with \( \text{hor}(1_1,2_0) \).

On the other hand, erasing a word from either \( \alpha \) or \( \beta \) produces, in both cases, a \( p = 3 \) nonsimple gamma basis.

It is quite illustrative to show how \( \alpha, \beta \), and \( \delta \) in (13) can be algorithmically computed in terms of (12). We get

\[
\begin{align*}
\mathcal{A} = \{\text{IZ}\} & \rightarrow \{\text{IZ}, \text{ZX}, \text{XX}, \text{IXZ}\} \in \alpha, \\
\mathcal{B}_1 = \{\text{ZX}, \text{XX}\} & \Rightarrow \{\text{IZ}, \text{ZX}, \text{XX}, \text{IXZ}\} \in \alpha, \\
\mathcal{B}_2 = \{\text{IX}\} & \\
\mathcal{A} = \{\text{XX}, \text{XZ}\} & \Rightarrow \{\text{XX}, \text{XZ}, \text{IZX}, \text{IZZ}\} \in \beta, \\
\mathcal{B}_1 = \{\text{XX}, \text{IXZ}\} & \\
\mathcal{B}_2 = \{\text{IX}, \text{IZ}\} & \\
\mathcal{A} = \{\text{XX}, \text{ZZX}\} & \Rightarrow \{\text{XX}, \text{ZZX}, \text{IZX}, \text{IZZ}\} \in \delta.
\end{align*}
\]
B. Four-letter alphabetic presentations

Starting from $m=4$, a new feature arises. Simple, maximally extended gamma basis with nonminimal length words are produced. Indeed, four inequivalent such representations for $p=5$ can be found. On the other hand, as we have seen, a $p=5$ gamma basis is already encountered for $m=3$. Translated back into matrix representations, the four $p=5$, $m=4$ gamma bases produce reducible (in matrix, not alphabetic, sense) $16 \times 16$ gamma matrices, whose size is twice the $8 \times 8$ irreducible representation obtained from $\mathfrak{g}$ in (13). The representatives of the four inequivalent $p=5$, $m=4$ gamma bases and their associated invariants are explicitly given by

$$X \ X \ X \ X$$
$$X \ X \ I \ Z$$

$5_a(p=5): I \ Z \ X \ I \ [\text{hor}(1,2,1,2)\); \ ver(4); \ N_t=4]$, $Z \ I \ Z \ Z$

$$Z \ Z \ Z \ X$$

$$X \ X \ X \ X$$
$$X \ X \ X \ Z$$

$5_b(p=5): X \ X \ Z \ I \ [\text{hor}(1,3,1,2,1)\); \ ver(1,3,1,2,1,1)\); \ N_t=6]$, $X \ Z \ I \ I$

$$Z \ I \ I \ I$$

$$X \ X \ X \ I$$
$$X \ X \ Z \ I$$

$5_c(p=5): X \ Z \ I \ X \ [\text{hor}(1,3,4)\); \ ver(2,1,1,1)\); \ N_t=7]$, $X \ Z \ I \ Z$

$$Z \ I \ I \ I$$

$$X \ X \ X \ I$$
$$X \ X \ Z \ I$$

$5_d(p=5): X \ Z \ I \ I \ [\text{hor}(3,2,2)\); \ ver(2,3,1,2,1)\); \ N_t=8]$. $Z \ I \ I \ I \ X$

Here, $m=4$ is the minimal length for an alphabetic presentation of the Euclidean Clifford algebra with $p=6, 7, 8$. The complete list (representatives and their associated invariants) of inequivalent, simple, maximally extended gamma basis for $p=6,7,8$, and $m=4$ is explicitly given by

$$X \ X \ X \ X$$
$$Z \ I \ X \ X$$

$6_a(p=6): X \ Z \ I \ X \ I \ X \ Z \ X \ [\text{hor}(1,3,3,2)\); \ ver(3,2,1,0)\); \ N_t=6]$, $Z \ Z \ Z \ X$

$I \ I \ I \ Z$
All the gamma bases entering (15) and (16) can be algorithmically produced with the construction (12). For simplicity, we limit ourselves to present the algorithmic construction of the largest of such representations, the gamma basis 8 in (16) which generates Cl(8,0). The sets $\mathcal{A}$, $\mathcal{B}_1$, $\mathcal{B}_2$ are given by

$$\mathcal{A} = \{XXX, ZZZ\}, \quad \mathcal{B}_1 = \{ZXI, XI, IZX\} \quad \mathcal{B}_2 = \{XZI, ZIX, IXZ\}$$

$$\Rightarrow \{XXX, ZZZ, ZXI, ZIX, XIZ, XIX, IXI, IXZ, IXX, IXZ, XIZ, IXZ, IXZ\} \in \mathcal{S}. \quad (17)$$

This is the first example of the subclass of “cyclic” algorithmic constructions that will be discussed later.

We made an exhaustive computer search and listed all inequivalent, simple, maximally extended, three-character alphabetic presentations for $m=5$ and $m=6$. To save space, we just limit ourselves to mention that five-letter words can produce a Euclidean gamma basis for at most $p=9$, while six-letter words can produce a gamma basis for at most $p=10$. 
C. The minimal lengths

We are now in the position to present a table with the minimal length \( \bar{m} \) required to produce a three-character alphabetic presentation of Cl\((p, 0)\) at a given \( p \). We compare \( \bar{m} \) with \( m \), the minimal length for four-character alphabetic presentations, given by (6). We get

\[
\begin{array}{cccccccccccccccc}
  p & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & \ldots \\
  m & 1 & 2 & 3 & 4 & 4 & 4 & 4 & 5 & 6 & 7 & 7 & 8 & 8 & 8 & 9 & \ldots \\
  \bar{m} & 1 & 2 & 3 & 4 & 4 & 4 & 4 & 5 & 6 & 7 & 7 & 8 & 8 & 8 & 9 &? & 10? & \ldots
\end{array}
\]

(18)

The \( \bar{m} \) values for \( p=13, 17, 18 \) are conjectured since a formal proof is lacking.

The above table is the result of an explicit computer search for \( \bar{m} \leq 6 \), combined with algorithmic constructions for \( \bar{m} > 6 \).

D. The cyclic prescription and another algorithm

There is a class of gamma basis (let us call them cyclic), obtained by a specific choice of \( A \), \( B_1 \), \( B_2 \) entering (12).

For integral values \( n=1, 2, 3, \ldots \), we construct \( B_1 \) as a set of \( 2n+1 \) words of \((2n+1)\)-length obtained by cyclically permuting \( IZXZX\cdots ZX=I(ZX)^{(n)} \), while \( B_2 \) is the set of \( 2n+1 \) words of \((2n+1)\)-length obtained by cyclically permuting \( IZXZX\cdots XZ=I(XZ)^{(n)} \),

\[ B_1 = \{ I(ZZ)^{(n)} \text{ and its cyclic permutations} \} \]

\[ B_2 = \{ I(ZX)^{(n)} \text{ and its cyclic permutations} \} \]  

Clearly, the words in \( B_1 \) commute with the words in \( B_2 \).

Two subcases are now considered.

Subcase (i): For odd values \( n \), \( A \) is given by the two word sets

\[ A = \{ Z(ZZ)^{(n)}, X(XX)^{(n)} \} \]  

Subcase (ii): For even values \( n \), \( A \) is the empty set

\[ A = \emptyset \]  

The prescription (12) gives us, in both cases, a \((2n+2)\)-letter gamma basis such that subcase (i) for odd values \( n \), \( p=2(2n+1)+2=4(n+1) \) and subcase (ii) for even values \( n \), \( p=2(2n+1)=4n+2 \).

For \( n=1 \) we recover the construction of the 8 gamma basis given in (17).

As a result, we obtain a relation, for cyclic three-character representations, between \( p \) and the length \( \bar{m} \) of their words, given by

\[
\begin{array}{cccccccccccccccc}
  p & 8 & 10 & 16 & 18 & \ldots \\
  \bar{m} & 4 & 6 & 8 & 10 & \ldots
\end{array}
\]

We know that in subcase (i), for \( p=8k \ (k=1, 2, \ldots) \), \( \bar{m}=4k \) is a minimal length because it coincides with the known minimal length for four-character presentations. On the other hand, we explicitly checked that in subcase (ii), for \( p=10 \), \( \bar{m}=6 \) corresponds to a minimal length, while subcase (ii) provides an upper bound for the minimal length for \( p=18 \).

Another algorithmic construction, different from the cyclic prescription and generalizing the algorithm (11), allows us to prove that \( \bar{m}=7 \) in (18) is indeed the three-character minimal length for \( p=12 \). Let \( C_1, C_2 \) be two gamma basis for, respectively, Cl\((p_1, 0)\), Cl\((p_2, 0)\) with \( m_1, m_2 \) length of their words. Let \( \gamma \) be a word of \( C_1 \) and \( \bar{C}_1 \) the complement set of \( \{ \gamma \} \) in \( C_1 \). A new gamma basis \( C \) for Cl\((p_1+p_2-1, 0)\), with words of length \( m=m_1+m_2 \), is symbolically given by
By taking, e.g., 5 in (13) as \( C_1 \) and 8 in (16) as \( C_2 \), we obtain a three-character gamma basis with \( p = 5 + 8 - 1 = 12 \) and \( m = 3 + 4 = 7 \).

### IV. CONCLUSIONS AND OUTLOOK

In this work, we investigated the alphabetic representations of the Cl\((p, q)\) Clifford algebra gamma basis. The gamma basis generators are expressed as words written in up to four character alphabets. The four characters, \( I, X, Z, A \), are associated with four \( 2 \times 2 \) matrices according to (4) (\( I \) corresponds to the identity matrix, \( A \) to the antisymmetric matrix, etc.) and satisfy the anticommutation relation (1). The words of an alphabetic representation are in correspondence with the matrix tensor products (in the correspondence, the tensor product symbol is omitted).

The interesting alphabets to consider are the whole four-character alphabet or a three-character alphabet. A two-character alphabet given by, e.g., \( X \) and \( Z \), is too poor; indeed, it can only produce a Euclidean gamma basis for \( p = 1, 2 \). On the other hand, the three-character alphabet given by \( I, X, Z \) is Euclidean complete. It produces Cl\((p, 0)\) Euclidean gamma basis for any value of \( p \). For this alphabet, we introduced the notion of the alphabetic group of equivalence, constructed invariants, and derived general and partial results (concerning, e.g., the minimal length of the words which produce a gamma basis for a given \( p \)). The alphabetic group of equivalence \( G \) can be extended to the whole four-character alphabet or to a three-character alphabet containing \( A \) (namely, the character associated with the antisymmetric matrix). \( G \) is based on three types of moves, the permutations (of rows and columns) and the transmutations of characters. In the extended case, the transmutations have to be suitably restricted since an \( A \leftrightarrow X \) (or an \( A \leftrightarrow Z \)) transmutation maps a Cl\((p, q)\) gamma basis into a Cl\((p', q')\) gamma basis (the constraint \( p' + q' = p + q \) is satisfied; in the general case, on the other hand, \( p' \) differs from \( p \)). A viable restriction in the definition of the alphabetic group of equivalence consists in disregarding the transmutations involving the \( A \) character. Besides the invariants discussed in Sec. II B, extra horizontal and vertical invariants, counting the number of the \( A \)'s character, have to be introduced. The analysis of the four-character case (invariants, inequivalent alphabetic presentations, etc.) is left for the forthcoming publications. It is worth pointing out that the introduction of a fourth character greatly increases the time needed for computer search of the inequivalent alphabetic presentations.

To our knowledge, this investigation program has not been addressed in literature. We have proven here that it is based on a well-posed mathematical problem admitting interesting and quite nontrivial solutions.

We have postponed so far discussing its possible applications. In the light of this, we should mention that the whole idea of constructing and analyzing the alphabetic presentations was deeply rooted in the investigations in our respective fields. Clifford algebras (in their alphabetic presentations) are the basis to construct representations of the \( N \)-extended supersymmetric quantum mechanics. These representations are nicely encoded in a graphical interpretation (see Refs. 9 and 10) in terms of colored, oriented, graphs. The equivalence group of transformations acting on graphs is related to the alphabetic group of transformations of the associated Clifford algebra.

The applications of Clifford algebras to robotics have been detailed, e.g., in Ref. 7. An interesting possibility is offered by the construction of cellular automata, which manipulate words in an alphabetic presentation of Clifford algebras. The three-character alphabet, here investigated in detail, is the simplest of such settings which allows the necessary complexity (Euclidean completeness, inequivalent alphabetic presentations, etc.).

At the end, let us just mention a seemingly far-fetched possibility, which, nevertheless, we believe deserves being duly investigated. The DNA codon problem concerns the yet to be explained degeneracies found in associating amino acids with the triplets of the DNA nucleotides, cytosine (C), adenine (A), thymine (T), guanine (G) for DNA or their respective G, U (for uracil), A, C complements for mRNA. In the vertebral mitochondrial code, for instance, the \( 4^3 = 64 \) nucleotides triples are associated with 20 amino acids and a stop signal according to a decomposition assigning 2, 4, or 6 different words to each amino acid and the stop signal: \( 64 = 2 \times 6 + 7 \).
One can consult Ref. 11 for an updated discussion of the codon problem and the attempted solutions (based on \(p\)-adic distance, deformed superalgebras, etc.). It is quite tempting to reformulate this problem in terms of alphabetic presentations of Clifford algebras (identifying each nucleotide with one of the four characters \(I\), \(X\), \(Z\), and \(A\)) and check whether the alphabetic invariants could play a role in the association with the amino acids.

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