Note on Dirac–Kähler massless fields

S. I. Kruglov

University of Toronto at Scarborough,
Physical and Environmental Sciences Department,
1265 Military Trail, Toronto, Ontario, Canada M1C 1A4

Abstract

We obtain the canonical and symmetrical Belinfante energy-momentum tensors of Dirac–Kähler’s fields. It is shown that the traces of the energy-momentum tensors are not equal to zero. We find the canonical and Belinfante dilatation currents which are not conserved, but a new conserved dilatation current is obtained. It is pointed out that the conformal symmetry is broken. The canonical quantization is performed and the propagator of the massless fields in the first-order formalism is found.

1 Introduction

The Dirac–Kähler (DK) fields [1], [2] are paid much attention to due to the development of quantum chromodynamics (QCD) on a lattice [3], [4], [5], [6], [7], [8], [9], [10]. Kähler postulated an equation in terms of inhomogeneous differential forms which is equivalent to a set of antisymmetric tensor fields [1], [11], [12]. In the matrix form the DK equation for massive fields can be represented as a direct sum of four Dirac equations. For massless fields, the DK equation comprises the additional projection operator and the DK equation is not a sum of Dirac equations.

In this paper, we investigate the massless DK boson fields. We imply that masses of boson fields can appear due to the Higgs mechanism. The relativistic wave equation describing massless DK fields is the Dirac-like $16 \times 16$ matrix equation with the additional projection operator [11], [12]. The Lagrangian for the DK massless boson fields possesses the internal symmetry group $SO(3, 1)$ [12], and generators of this group do not commute with the generators of the Lorentz group.

We study here the dilatation symmetry of the massless DK fields. The canonical, symmetrical Belinfante energy-momentum tensors, and the dilata-
tion current are obtained. The canonical quantization is performed for the massless DK fields in the first-order formalism.

The paper is organized as follows. In Sec. 2, the massless DK field equation is formulated in the matrix form. The canonical and the symmetrical Belinfante energy-momentum tensors are found in Sec. 3. We obtain the non-conserved canonical and Belinfante dilatation currents, and a new conserved dilatation current is found. In Sec. 4, we consider the canonical quantization of the DK massless fields and obtain the matrix propagator. We discuss results obtained and possible applications of the theory considered in Sec. 5.

The Heaviside units are chosen and the Euclidean metric is used, and $\hbar = c = 1$.

## 2 Dirac–Kähler equation for massless fields

The massless DK fields obey the following tensor equations [11], [12]:

\begin{align}
\partial_\nu \varphi_{\mu\nu} - \partial_\mu \varphi &= 0, \\
\partial_\nu \tilde{\varphi}_{\mu\nu} - \partial_\mu \tilde{\varphi} &= 0, \\
\partial_\mu \varphi_\mu &= \varphi, \\
\partial_\mu \tilde{\varphi}_\mu &= \tilde{\varphi}, \\
\varphi_{\mu\nu} &= \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu - \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \tilde{\varphi}_\beta,
\end{align}

where the dual tensor is defined as

\[ \tilde{\varphi}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \varphi_{\alpha\beta} \]

and $\varepsilon_{\mu\nu\alpha\beta}$ is an antisymmetric tensor with $\varepsilon_{1234} = -i$. We consider here the neutral fields when the fields $\varphi$, $\tilde{\varphi}$, $(\varphi_m, \varphi_0)$, $(\tilde{\varphi}_m, \tilde{\varphi}_0)$ are real values. The generalization for charged fields is straightforward: the fields become complex values. One may introduce the electric and magnetic sources in Eq. (1) [12]. Then, we have the Maxwell equations with electric and magnetic charges in the dual-symmetric form. The fields $\varphi$, $\tilde{\varphi}$ play the role of the general gauge [13]. To have the Lorentz gauge, one can put $\varphi = \tilde{\varphi} = 0$.

Equations (1)–(3) can be represented in the form of the Dirac-like equation with $16 \times 16$ dimensional Dirac matrices [11], [12]. It should be noted that the fields $\varphi$, $\tilde{\varphi}$, $\varphi_\mu$, $\tilde{\varphi}_\mu$ have different dimensions. Thus, in order to formulate the first-order equations, one needs to introduce the dimensional parameter. We use the notations $\psi_0 = -\varphi$, $\psi_\mu = \kappa \varphi_\mu$, $\psi_{[\mu\nu]} = \varphi_{\mu\nu}$, $\bar{\psi}_\mu = i\kappa \tilde{\varphi}_\mu$, $\bar{\psi}_0 = -i\tilde{\varphi}$,
\[ e_{\mu\alpha\beta} = i\varepsilon_{\mu\alpha\beta} \quad (\varepsilon_{1234} = 1), \] where the parameter \( \kappa \) has the dimension of the mass. With these notations, equations (1)–(3) may be rewritten as

\[
\begin{align*}
\partial_{\mu} \psi_{\mu} + \kappa \psi_{0} &= 0, \\
\partial_{\nu} \psi_{[\mu\nu]} + \partial_{\mu} \psi_{0} &= 0, \\
\partial_{\nu} \bar{\psi}_{\mu} - \partial_{\mu} \bar{\psi}_{\nu} - e_{\mu\alpha\beta} \partial_{\alpha} \bar{\psi}_{\beta} + \kappa \psi_{[\mu\nu]} &= 0,
\end{align*}
\] (5)

Introducing the 16-component wave function

\[
\Psi(x) = \{ \psi_A(x) \} = \begin{pmatrix} \psi_0(x) \\ \psi_{\mu}(x) \\ \psi_{[\mu\nu]}(x) \\ \bar{\psi}_{\mu}(x) \\ \bar{\psi}_0(x) \end{pmatrix},
\] (6)

where \( A = 0, \mu, [\mu\nu], \bar{\mu}, 0; \psi_{\mu} \equiv \bar{\psi}_{\mu}, \psi_0 \equiv \bar{\psi}_0, \) (5) can be cast in the form of the first-order wave equation [11], [12]:

\[
(\Gamma_{\nu} \partial_{\nu} + \kappa P) \Psi(x) = 0.
\] (7)

The 16 \times 16 matrices \( \Gamma_{\nu} \) are given by

\[
\Gamma_{\nu} = \beta_{\nu}^{(+)} + \beta_{\nu}^{(-)}, \quad \beta_{\nu}^{(+)} = \beta_{\nu}^{(1)} + \beta_{\nu}^{(0)}, \quad \beta_{\nu}^{(-)} = \beta_{\nu}^{(1)} + \beta_{\nu}^{(0)},
\]

\[
\beta_{\nu}^{(1)} = \varepsilon_{\nu}[\mu\nu] + \varepsilon_{[\mu\nu],\mu}, \quad \beta_{\nu}^{(1)} = \frac{1}{2} e_{\mu\rho\omega} \left( \varepsilon_{\mu[\rho\omega] + \varepsilon_{[\rho\omega],\mu}} \right),
\] (8)

\[
\beta_{\nu}^{(0)} = \varepsilon_{\nu,0} + \varepsilon_{0\nu}, \quad \beta_{\nu}^{(0)} = \varepsilon_{\nu,0} + \varepsilon_{0\nu}.
\]

Equation (7) describes the massless DK fields and it is not the direct sum of four Dirac equations because of the presence of the projection operator \( P \). In the case of massive fields, the parameter \( \kappa \) is replaced by the mass \( m \), and the projection operator \( P \) is replaced by the unit 16 \times 16 matrix. As a result the equation for massive DK fields represents the sum of four Dirac equations. But the Lorentz transformations for bosonic fields mix “flavors” [11], [12]. Matrices \( \beta_{\nu}^{(1)}, \beta_{\nu}^{(1)} \) and \( \beta_{\nu}^{(0)}, \beta_{\nu}^{(0)} \) obey the Petiau–Duffin–Kemmer algebra, and the \( \Gamma_{\nu} \) are the 16 \times 16 Dirac-like matrices:

\[
\Gamma_{\nu} \Gamma_{\mu} + \Gamma_{\mu} \Gamma_{\nu} = 2\delta_{\mu\nu}.
\] (9)

\[\text{In [11], [12], we have used instead of } \kappa \text{ the parameter } m_2; \text{ thus, } \kappa = m_2.\]
We have explored here the matrices $\varepsilon^{A,B}$ with the properties: $\varepsilon^{A,B}\varepsilon^{C,D} = \varepsilon^{A,D}\delta_{BC}$, $(\varepsilon^{A,B})_{CD} = \delta_{AC}\delta_{BD}$ and the indexes run $A, B, C, D = 1, 2, ... , 16$. The matrix $P$ is the projection matrix, $P^2 = P$, and is given by

$$P = \varepsilon^{0,0} + \frac{1}{2}\varepsilon^{[\mu\nu],[\mu\nu]} + \varepsilon^{\bar{0},\bar{0}}. \tag{10}$$

Now we consider the form-invariance of (7) under the Lorentz transformations. Coordinate transformations read as follows:

$$x'_\mu = L_{\mu\nu}x'_\nu, \tag{11}$$

where the Lorentz matrix $L = \{L_{\mu\nu}\}$ has the properties: $L_{\mu\alpha}L_{\nu\alpha} = \delta_{\mu\nu}$. The Lorentz transformations of coordinates (11) generate the wave function transformations

$$\Psi'(x') = T\Psi(x), \tag{12}$$

with the $16 \times 16$ matrix $T$. Then the first-order equation (7) becomes

$$\left(\Gamma_\mu \partial'_\mu + \kappa P\right)\Psi'(x') = (\Gamma_\mu L_{\mu\nu}\partial_\nu + \kappa P)T\Psi(x) = 0, \tag{13}$$

where $\partial'_\mu = L_{\mu\nu}\partial_\nu$. Equation (7) is form-invariant under the Lorentz transformations if the equations

$$\Gamma_\mu TL_{\mu\nu} = TT\Gamma_\nu, \quad PT = TP \tag{14}$$

hold. The matrix $T$ for the finite Lorentz transformations is given by

$$T = \exp\left(\frac{1}{2}\varepsilon_{\mu\nu}J_{\mu\nu}\right), \tag{15}$$

where $J_{\mu\nu}$ are the generators of the Lorentz group transformations. One can verify that the matrix $T$ with the generators for bosonic fields

$$J_{\mu\nu} = \frac{1}{4}\left(\Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu\right), \tag{16}$$

obeys (14). We have introduced the matrices $\Gamma_\nu$, which satisfy the Dirac algebra and are given by

$$\Gamma_\nu = \beta^{(+)}_\nu - \beta^{(-)}_\nu. \tag{17}$$

The matrices $\Gamma_\nu$ commute with $\Gamma_\mu$: $\Gamma_\mu \Gamma_\nu = \Gamma_\nu \Gamma_\mu$. At the infinitesimal Lorentz transformations (14) become

$$\Gamma_\mu J_{\alpha\nu} - J_{\alpha\nu} \Gamma_\mu = \delta_{\alpha\mu}\Gamma_\nu - \delta_{\nu\mu}\Gamma_\alpha, \quad PJ_{\mu\nu} = J_{\mu\nu}P. \tag{18}$$
The generators (16) obey (18).

The Lorentz-invariant is \( \Psi \Psi = \Psi^+ \eta \Psi \) (\( \Psi^+ \) is the Hermitian-conjugate wave function), where the Hermitian matrix, \( \eta \) is

\[
\eta = \Gamma_4 \tilde{\Gamma}_4.
\] (19)

From (8),(17), we obtain

\[
\eta = -\varepsilon^{0,0} + \varepsilon^{m,m} - \varepsilon^{4,4} + \varepsilon^{[m4],[m4]} - \frac{1}{2} \varepsilon^{[mn],[mn]} + \varepsilon^{0,0} + \varepsilon^{4,4} - \varepsilon^{m,\tilde{m}}. \] (20)

Taking into consideration that the fields \( \varphi, \tilde{\varphi}, (\varphi_m, \varphi_0), (\tilde{\varphi}_m, \tilde{\varphi}_0) \) are real values, we find from (20) the “conjugated” function:

\[
\Psi(x) = \Psi^+(x) \eta = (-\psi_0(x), \psi_\mu(x), -\psi_{[\mu\nu]}(x), \tilde{\psi}_\mu(x), -\tilde{\psi}_0(x))
\] (21)

which obeys the equation

\[
\Psi(x) \left( \Gamma_\mu \tilde{\partial}_\mu - \kappa P \right) = 0.
\] (22)

It follows from (7),(22) that the electric current

\[
J_\mu(x) = i \bar{\Psi}(x) \Gamma_\mu \Psi(x)
\] (23)

is conserved: \( \partial_\mu J_\mu(x) = 0 \). In addition, one may verify with the help of (6),(8),(21), that for the real fields, it vanishes, \( J_\mu(x) = 0 \), as the fields are neutral.

Solutions to (7) with definite energy and momentum are given by

\[
\Psi_s^{(\pm)}(x) = \sqrt{\frac{\kappa}{2k_0V}} u_s(\pm k) \exp(\pm ikx),
\] (24)

where \( V \) is the normalization volume, \( k^2 = k^2 - k_0^2 = 0 \), and \( s \) is the spin index which corresponds to scalar, vector, pseudovector and pseudoscalar states and runs eight values: \( s = 0, n, \tilde{n}, 0 \). The \( u_s(\pm k) \) obeys the equation for the field function in momentum space:

\[
(\pm i \hat{k} + \kappa P) u_s(\pm k) = 0,
\] (25)

where \( \hat{k} = \Gamma_\mu k_\mu \). One found in [11], [12] the minimal equation for the matrix of (25) \( B_\pm = \pm i \hat{k} + \kappa P \):

\[
B_\pm (B_\pm - \kappa) = 0,
\] (26)

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so that the projection operator \( (\alpha_{\pm} = \alpha_{\pm}^2) \) extracting solutions to (25) is given by

\[
\alpha_{\pm} = \frac{\kappa - B_{\pm}}{\kappa} = \mathcal{P} \pm \frac{i\hat{k}}{2}, \quad (27)
\]

and \( \mathcal{P} \) is the projection operator:

\[
\mathcal{P} = 1 - P = \varepsilon^{\mu,\mu} + \varepsilon^{\mu,\mu}, \quad (28)
\]

and \( \mathcal{P}P = P\mathcal{P} = 0 \). Every column of the matrix \( \alpha \) is the solution to (25). One can find the projection matrix-dyads, extracting solutions with different spins and spin projections in [11], [12]. From (23),(25), and the condition \( J_\mu(x) = 0 \) for the neutral fields, one obtains

\[
\pi_a(\pm k)\hat{k}u_a(\pm k) = 0, \quad \pi_a(\pm k)Pu_a(\pm k) = 0. \quad (29)
\]

Equation (27) will be used in the second quantization theory for obtaining the propagator of DK fields.

### 3 The energy-momentum tensor

The Lagrangian of massless DK fields in the first-order formalism can be written as

\[
\mathcal{L} = -\frac{1}{2}\overline{\Psi}(x)(\Gamma_\mu \partial_\mu + \kappa P)\Psi(x) + \frac{1}{2}\overline{\Psi}(x)(\Gamma_\mu \overset{\sim}{\partial}_\mu - \kappa P)\Psi(x). \quad (30)
\]

Equations (7),(22) follow from Lagrangian (30) by varying the corresponding action on the wave functions \( \overline{\Psi}(x), \Psi(x) \). For the neutral DK fields, the Lagrangian (30) reduces to

\[
\mathcal{L} = -\overline{\Psi}(x)(\Gamma_\mu \partial_\mu + \kappa P)\Psi(x). \quad (31)
\]

With the help of (6),(8),(21), Lagrangian (31) becomes

\[
\mathcal{L} = \psi_0\partial_\mu \psi_\mu - \psi_\mu \partial_\mu \psi_0 - \psi_\rho \partial_\mu \psi_{[\rho\mu]} + \psi_{[\rho\mu]}\partial_\mu \psi_\rho + *\psi_{[\rho\mu]}\partial_\mu \psi_\rho \\
-\bar{\psi}_\mu \partial_\mu *\psi_{[\rho\mu]} - \bar{\psi}_\mu \partial_\mu \bar{\psi}_0 + \bar{\psi}_0 \partial_\mu \bar{\psi}_\mu + \kappa \left( \psi_0^2 + \frac{1}{2}\psi_{[\rho\mu]}^2 + \bar{\psi}_0^2 \right), \quad (32)
\]

where

\[
*\psi_{[\mu\nu]} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\psi_{[\alpha\beta]} = i\bar{\psi}_{[\mu\nu]}, \quad (33)
\]
$\varepsilon_{\mu \nu \alpha \beta}$ is antisymmetric tensor ($\varepsilon_{1234} = 1$). It should be noted that $^{\ast}\psi_{[\mu \nu]}$ is not a dual tensor because $\varepsilon_{\mu \nu \alpha \beta} = i\varepsilon_{\mu \nu \alpha \beta}$, and the dual tensor is defined by Eq.(4). One can verify that the Euler–Lagrange equations

$$\frac{\partial L}{\partial \psi_A} - \partial_{\mu} \left( \frac{\partial L}{\partial (\partial_{\mu} \psi_A)} \right) = 0$$

(34)

with $L$ given by (32), where $A = 0, \mu, [\mu \nu], \tilde{\mu}, \tilde{0}$, lead to (5). For fields obeying the equations of motion (5) (or (7) and (22)) the Lagrangians (30),(31) and (32) vanish similarly to the Dirac Lagrangian.

The canonical energy-momentum tensor in the first-order formalism is given by

$$T_{\mu \nu}^{c} = \frac{\partial L}{\partial (\partial_{\mu} \Psi(x))} \partial_{\nu} \Psi(x) - \delta_{\mu \nu} L,$$

(35)

and using Eq.(31) it becomes

$$T_{\mu \nu}^{c} = \left( \partial_{\nu} \overline{\Psi}(x) \right) \Gamma_{\mu} \Psi(x).$$

(36)

We took into consideration here that for fields obeying equations of motion $L = 0$. One obtains from (6),(8),(21), and (36) the expression in the tensor form:

$$T_{\mu \nu}^{c} = \psi_0 \partial_{\nu} \psi_\mu - \psi_\mu \partial_{\nu} \psi_0 - \psi_\mu \partial_{\nu} \psi_{[\mu \nu]} + \psi_{[\mu \nu]} \partial_{\nu} \psi_\rho + {^{\ast}\psi}_{[\mu \nu]} \partial_{\nu} {^{\ast}\psi}_\rho$$

$$- {^{\ast}\psi}_\rho \partial_{\nu} {^{\ast}\psi}_{[\mu \nu]} - {^{\ast}\psi}_\mu \partial_{\nu} {^{\ast}\psi}_0 + {^{\ast}\psi}_0 \partial_{\nu} {^{\ast}\psi}_\mu.$$

(37)

It follows from the field equations that the energy-momentum tensor (37) (and (36)) is conserved tensor, $\partial_{\mu} T_{\mu \nu}^{c} = 0$. Contrary to classical electrodynamics [14] the energy-momentum tensor (37) is not the symmetric tensor, $T_{\mu \nu}^{c} \neq T_{\nu \mu}^{c}$.

Now, we investigate the dilatation symmetry [15]. The canonical dilatation current in the first-order formalism is given by

$$D_{\mu}^{c} = x_\nu T_{\mu \nu}^{c} + \Pi_{\mu} \Psi,$$

(38)

where

$$\Pi_{\mu} = \frac{\partial L}{\partial (\partial_{\mu} \Psi)} = -\overline{\Psi} \Gamma_{\mu}.$$

(39)

We note that for the bosonic fields, the matrix $d$ in [15] defining the field dimension, is the unit matrix. As the electric current $J_{\mu} = i\overline{\Psi} \Gamma_{\mu} \Psi$, for
neutral fields equals 0, the last term in Eq.(38) vanishes. As a result, we obtain non-zero divergence of the canonical dilatation current

$$\partial_\mu D^c_\mu = -\kappa \left( \psi^2_0(x) + \frac{1}{2} \psi^2_{[\mu\nu]}(x) + \bar{\psi}_0^2(x) \right).$$  (40)

The dilatation current $D^c_\mu$ is not conserved current. The similar expression was found in the first-order formulation of generalized electrodynamics with an additional scalar field [16]. Later, we will obtain new conserved current. Expression (40) also can be obtained from the relationship [15]

$$\partial_\mu D^c_\mu = 2\Pi_\mu \partial_\mu \Psi + \frac{\partial L}{\partial \Psi} \partial_\mu \Psi - 4L = \kappa \bar{\Psi}(x) P\Psi(x)$$  (41)

It should be noted that for the plane-wave solution (24), we find from Eq.(5)

$$\kappa \psi_{[\mu\nu]}(k) = k_\mu \psi_\nu(k) - k_\nu \psi_\mu(k) + e_{\mu\nu\alpha\beta} k_\alpha \bar{\psi}_\beta(k),$$  (42)

and

$$\frac{1}{2} \psi^2_{[\mu\nu]}(k) = -\bar{\psi}_0^2(k) - \bar{\psi}_0^2(k).$$  (43)

As a result, the right sides of (40),(41) vanish and the dilatation current is conserved, $\partial_\mu D^c_\mu = 0$. But, in the general configuration of fields, the dilatation current $D^c_\mu$ is not conserved.

Now, we find the symmetrical energy-momentum tensor. The general expression for the symmetrical Belinfante energy-momentum tensor is given by (see [15]):

$$T^B_{\mu\nu} = T^c_{\mu\nu} + \partial_\beta X_{\beta\mu\nu},$$  (44)

and

$$X_{\beta\mu\nu} = \frac{1}{2} \left[ \Pi_\beta J_{\mu\nu} \Psi - \Pi_\mu J_{\beta\nu} \Psi - \Pi_\nu J_{\beta\mu} \Psi \right].$$  (45)

The tensor (45) is antisymmetrical in indexes $\beta, \mu$, and therefore $\partial_\mu \partial_\beta X_{\beta\mu\nu} = 0$. As a result, $\partial_\mu T^B_{\mu\nu} = \partial_\mu T^c_{\mu\nu} = 0$. After some calculations, we obtain from (16) the expression for generators of the Lorentz group

$$J_{\mu\nu} = \varepsilon_{\mu,\nu} - \varepsilon^{[\lambda\nu],\lambda\mu} - \varepsilon^{[\mu\nu],\lambda\mu} + \varepsilon^{\lambda\mu,\nu} - \varepsilon^{\lambda\mu,\nu}.$$  (46)
Replacing expression (46) in Eq.(45), and taking into account (39), after calculations, we obtain

\[ X_{\alpha\mu\nu} = \delta_{\mu\nu}\psi_0\psi_\alpha - \delta_{\alpha\nu}\psi_0\psi_\mu + \delta_{\alpha\nu}\psi_\lambda\psi_{[\lambda\mu]} - \delta_{\mu\nu}\psi_\lambda\psi_{[\lambda\alpha]} + 2\psi_\nu\psi_{[\mu\alpha]} \]

\[ + \delta_{\mu\nu}\tilde{\psi}_0\tilde{\psi}_\alpha - \delta_{\alpha\nu}\tilde{\psi}_0\tilde{\psi}_\mu + \psi_\nu^*\psi_{[\mu\alpha]} + \epsilon_{\beta\alpha\mu\lambda}\tilde{\psi}_\beta\psi_{[\lambda\nu]} \]  

From (37),(44),(47), we obtain the Belinfante energy-momentum tensor

\[ T_{\mu\nu}^B = -2\psi_\mu\partial_\nu\psi_0 - 2\tilde{\psi}_\mu\partial_\nu\tilde{\psi}_0 + 2\psi_{[\lambda\mu]}\partial_\nu\psi_\lambda - \tilde{\psi}_\beta\partial_\nu^*\psi_{[\beta\mu]} + \psi_{[\beta\mu]}\partial_\nu\tilde{\psi}_\beta \]

\[ + \partial_\alpha \left[ 2\psi_\nu\psi_{[\mu\alpha]} + \psi_\nu^*\psi_{[\mu\alpha]} + \epsilon_{\beta\alpha\mu\lambda}\tilde{\psi}_\beta\psi_{[\lambda\nu]} + \delta_{\mu\nu} \left( \psi_0\psi_\alpha + \tilde{\psi}_0\tilde{\psi}_\alpha - \psi_\lambda\psi_{[\lambda\alpha]} \right) \right] \]

Expression (48) with the help of equations of motion (5) can be represented in the symmetrical form

\[ T_{\mu\nu}^B = -2\psi_\mu\partial_\nu\psi_0 - 2\tilde{\psi}_\mu\partial_\nu\tilde{\psi}_0 - 2\tilde{\psi}_\nu\partial_\nu\tilde{\psi}_0 - 2\tilde{\psi}_\nu\partial_\mu\tilde{\psi}_0 + \kappa \left( 2F_{\mu\alpha}F_{\alpha\nu} + G_{\mu\alpha}G_{\alpha\nu} - *G_{\mu\alpha}G_{\alpha\nu} + *F_{\mu\alpha}G_{\alpha\nu} + *G_{\mu\alpha}F_{\alpha\nu} \right) \]

\[ + \delta_{\mu\nu} \left[ \partial_\alpha \left( \psi_0\psi_\alpha + \tilde{\psi}_0\tilde{\psi}_\alpha - \psi_\lambda\psi_{[\lambda\alpha]} \right) + \tilde{\psi}_\alpha\partial_\alpha\tilde{\psi}_0 \right] \]

where we use the notation

\[ \kappa F_{\mu\nu} = \partial_\mu\psi_\nu - \partial_\nu\psi_\mu, \quad \kappa G_{\mu\nu} = \partial_\mu\tilde{\psi}_\nu - \partial_\nu\tilde{\psi}_\mu. \]

It is easy to find, with the help of field equations (5), the trace of the Belinfante energy-momentum tensor (49):

\[ T_{\mu\mu}^B = 4\partial_{\mu} \left( \psi_0\psi_\mu + \tilde{\psi}_0\tilde{\psi}_\mu \right). \]

The modified dilatation current is given by [15]

\[ D_{\mu}^B = x_\alpha T_{\mu\alpha}^B + V_\mu, \]

where the field-virial \( V_\mu \) is defined as

\[ V_\mu = \Pi_\mu\Psi - \Pi_\alpha J_{\alpha\mu} = \overline{\Psi}\Gamma_\alpha J_{\alpha\mu}\Psi \]

\[ = \psi_\lambda\psi_{[\lambda\mu]} + \tilde{\psi}_\lambda^*\psi_{[\lambda\mu]} - 3 \left( \psi_0\psi_\mu + \tilde{\psi}_0\tilde{\psi}_\mu \right). \]
It is easy to verify that $X_{a\mu} = -V_\mu$. As a result, the divergence of the Belinfante dilatation current becomes

$$\partial_\mu D_\mu^B = T_{\mu\alpha}^B + \partial_\mu V_\alpha = \partial_\mu D_\mu^c = T_{\mu\nu}^c.$$  \hfill (54)

The divergences of the Belinfante and canonical dilatation currents are the same. Thus, the currents $D_\mu^c$, $D_\mu^B$ are not conserved, but because the trace of the Belinfante energy-momentum tensor (51) is a total divergence, we can introduce a new conserved current\footnote{I am grateful to Yu Nakayama for his remarks.}

$$D_\mu = x_\alpha T_{\mu\alpha}^B - 4 \left( \psi_0 \psi_\mu + \bar{\psi}_0 \bar{\psi}_\mu \right),$$  \hfill (55)

so that $\partial_\mu D_\mu = 0$. Thus, massless DK fields possess the dilatation symmetry with new dilatation current (55). The similar conserved current can be introduced also for generalized electrodynamics \cite{16}, \cite{17}. It should be noted that the conformal invariance is broken because the field-virial $V_\mu$ is not a total derivative of some local quantity \cite{15}.

4 Canonical quantization

To perform the canonical quantization, we define the momenta in the matrix form from Eq.(31):

$$\pi(x) = \frac{\partial L}{\partial (\partial_0 \Psi(x))} = i\overline{\Psi}\Gamma_4.$$  \hfill (56)

Then, using the quantum commutator

$$[\Psi_M(x,t), \pi_N(y,t)] = i\delta_{MN}\delta(x - y),$$

we obtain from (56) the commutation relation

$$\left[ \Psi_M(x,t), \left( \overline{\Psi}(y,t) \Gamma_4 \right)_N \right] = \delta_{MN}\delta(x - y).$$  \hfill (57)

Equation (57), with the help of (6),(8),(21), leads to simultaneous field commutators for the components $\psi_A(x)$:

$$[\psi_0(x,t), \psi_4(y,t)] = \delta(x - y), \quad \left[ \bar{\psi}_0(x,t), \bar{\psi}_4(y,t) \right] = \delta(x - y).$$  \hfill (58)
\[ \psi_{[ml]}(x, t), \psi_{n}(y, t) = \delta_{mn} \delta(x - y), \quad \psi_{[mn]}(x, t), \bar{\psi}_{k}(y, t) = \epsilon_{mnl} \delta(x - y), \]

where \( \epsilon_{mnl} \) is antisymmetric tensor \((\epsilon_{123} = 1)\).

The operators for the neutral field, in the second quantized theory, can be written as follows:

\[
\Psi(x) = \sum_{k,s} \left[ a_{k,s} \Psi_{s}^{(+)}(x) + a_{k,s}^{+} \Psi_{s}^{(-)}(x) \right],
\]

\[
\bar{\Psi}(x) = \sum_{k,s} \left[ a_{k,s}^{+} \Psi_{s}^{(+)}(x) + a_{k,s} \Psi_{s}^{(-)}(x) \right],
\]

where the positive and negative parts of the wave function are given by (24). The creation and annihilation operators of particles, \( a_{k,s}^{+}, a_{k,s} \), obey the commutation relations \([11], [12]\):

\[
[a_{k,s}, a_{k',s'}^{+}] = \varepsilon_{s} \delta_{ss'} \delta_{kk'}, \quad [a_{k,s}, a_{k',s'}] = [a_{k,s}^{+}, a_{k',s'}^{+}] = 0,
\]

where \( \varepsilon_{s} = 1 \) at \( s = 0, m, \) and \( \varepsilon_{s} = -1 \) at \( s = 0, \bar{m} \). There is no summation in the index \( s \) in (60). The operators \( a_{k,s}, a_{k,s}^{+} \) at \( s = 0, \bar{m} \), corresponding to scalar and pseudovector states, satisfy the commutation relation with the “wrong” sign \((-\)) and one should introduce the indefinite metric \([12]\). The energy density follows from (36), and is

\[
\mathcal{E} = -T_{44} = \pi(x) \partial_{0} \Psi(x) - \mathcal{L} = i \bar{\Psi}(x) \Gamma_{4} \partial_{0} \Psi(x).
\]

With the help of (59)-(61), and the normalization conditions, we obtain the Hamiltonian

\[
H = \int \mathcal{E} d^{3}x = \sum_{k,s} \kappa_{0} \varepsilon_{s} \left( a_{k,s}^{+} a_{k,s} + a_{k,s} a_{k,s}^{+} \right).
\]

After the introduction of the indefinite metric the eigenvalues of the Hamiltonian (62) are positive values but the classical Hamiltonian is not positive-definite. As a result, scalar and pseudovector states are ghost states \([12]\).

From (59),(60), one finds commutation relations for different times:

\[
[\Psi_{M}(x), \Psi_{N}(x')] = [\bar{\Psi}_{M}(x), \bar{\Psi}_{N}(x')] = 0,
\]

\[
[\Psi_{M}(x), \bar{\Psi}_{N}(x')] = S_{MN}(x, x'),
\]

\[
S_{MN}(x, x') = S_{MN}^{+}(x, x') - S_{MN}^{-}(x, x'),
\]

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\[
S_{MN}^+(x, x') = \sum_{k,s} \varepsilon_s (\Psi^+_s(x)) M (\overline{\Psi}^+_s(x'))_N, \tag{66}
\]
\[
S_{MN}^-(x, x') = \sum_{k,s} \varepsilon_s (\Psi^-_s(x)) M (\overline{\Psi}^-_s(x'))_N. \tag{67}
\]

From (24),(64)-(67), we obtain:
\[
S_{MN}^\pm(x, x') = \sum_{k,s} \frac{\kappa}{2k_0V} \varepsilon_s (u_s(\pm k)) M (\overline{u}_s(\pm k))_N \exp[\pm ik(x - x')]. \tag{68}
\]

From (27), and taking into consideration that the sum of all spin projection operators is unity (see [12]), one finds
\[
\sum_s \varepsilon_s (u_s(\pm k)) M (\overline{u}_s(\pm k))_N = (\kappa \overline{P} \mp i \hat{k})_MN. \tag{69}
\]

With the help of (69), we obtain from (68):
\[
S_{MN}^\pm(x, x') = \sum_k \frac{1}{2k_0V} (\kappa \overline{P} \mp i \hat{k})_MN \exp[\pm ik(x - x')]
= \left(\kappa \overline{P} - \Gamma_\mu \frac{\partial}{\partial x_\mu}\right)_MN D_\pm(x - y), \tag{70}
\]

where we exploit the singular functions [18]
\[
D_+(x) = \sum_k \frac{1}{2k_0V} \exp(ikx), \quad D_-(x) = \sum_k \frac{1}{2k_0V} \exp(-ikx). \tag{71}
\]

Introducing the function [18]
\[
D_0(x) = i (D_+(x) - D_-(x)), \tag{72}
\]

from (65)-(67),(70), we arrive at
\[
S_{MN}(x, x') = -i \left(\kappa \overline{P} - \Gamma_\mu \frac{\partial}{\partial x_\mu}\right)_MN D_0(x - x'). \tag{73}
\]

One can prove, with the help of (8),(10),(28), that the relations \(\Gamma_\mu \overline{P} = P \Gamma_\mu, \overline{P} P = 0\) are valid. Thus, we find the equation
\[
(\Gamma_\mu \partial_\mu + \kappa P) \left(\kappa \overline{P} - \Gamma_\mu \partial_\mu\right) = -\partial^2_\alpha. \tag{74}
\]
With the aid of the singular function properties \[18\], one can verify the relation
\[
\left( \Gamma_\mu \frac{\partial}{\partial x_\mu} + \kappa P \right) S^\pm (x, x') = 0. \tag{75}
\]
The propagator (the vacuum expectation of the chronological pairing of operators) is defined by the equation
\[
\langle T \Psi_M(x) \overline{\Psi}_N(y) \rangle_0 = S^c_{MN}(x - y) \tag{76}
\]
\[
= \theta(x_0 - y_0) S^+_M(x - y) + \theta(y_0 - x_0) S^-_{MN}(x - y),
\]
where the theta-function is \(\theta(x)\). Using the function \(D_c(x - y)\) \[18\]:
\[
D_c(x - y) = \theta(x_0 - y_0) D_+(x - y) + \theta(y_0 - x_0) D_-(x - y), \tag{77}
\]
we obtain the propagator
\[
\langle T \Psi_M(x) \overline{\Psi}_N(y) \rangle_0 = \left( \kappa \overline{\mathcal{P}} - \Gamma_\mu \frac{\partial}{\partial x_\mu} \right)_{MN} D_c(x - y). \tag{78}
\]
Taking into account the equation \[18\]
\[
\partial^2_\mu D_c(x) = i \delta(x),
\]
we arrive at
\[
\left( \Gamma_\mu \frac{\partial}{\partial x_\mu} + \kappa P \right) (T \Psi(x) \cdot \overline{\Psi}(y))_0 = -i \delta(x - y). \tag{79}
\]
The propagator (79) corresponds to Dirac–Kähler’s massless fields including scalar, vector, pseudovector and pseudoscalar fields.

5 Discussion

We have considered the first-order formulation of the theory of Dirac–Kähler massless fields. The form-invariance of the first-order relativistic wave equation was proven. This is a new type of formulation of Dirac–Kähler antisymmetric tensor fields. This allows us to obtain in the simple manner the canonical and symmetrical Belinfante energy-momentum tensors. The traces
of the energy-momentum tensors do not equal zero and the divergences of the canonical and Belinfante dilatation currents do not vanish. Nevertheless, we obtain a new conserved dilatation current but the conformal invariance is broken. The canonical quantization requires the introduction of indefinite metrics. This is connected with the presence of ghosts (the pseudovector and scalar states). The propagator of the massless fields in the matrix form has been found and can be used for some calculations in the quantum theory.

If we impose the conditions $\psi_0(x) = 0$, $\bar{\psi}_0(x) = 0$, we arrive at the two-potential formulation of electrodynamics which is convenient for considering magnetic monopoles [12].

Let us discuss the possible important application of the theory considered with the nonperturbative QCD on a lattice. First, the Dirac–Kähler formulation of fermions on the lattice is an interesting problem [19]. We notice that all tensor fields considered are described by the 16-component wave function, i.e. they form the same multiplet. To have a connection with flavor degrees, one needs to make the transformation of the matrices and the wave function:

$$\Gamma'_\mu = S \Gamma_\mu S^{-1} = I \otimes \gamma_\mu, \quad \Psi'(x) = S \Psi(x) = \{ \psi_i \} \quad (i = 1, 2, 3, 4),$$

where $I$ represents the $4 \times 4$ unit matrix, and $\gamma_\mu$ are the Dirac matrices. In this case the matrices $\Gamma'_\mu$ represent the direct sum of four Dirac matrices. Then, we formally can express four Dirac spinors $\psi_i$ (flavor degrees) through tensor fields. It should be noted, however, that transformations of tensor fields and Dirac spinors under the Lorentz group are different [11], [12]. We also mention that the Dirac–Kähler formulation on a lattice and the staggered fermion formulation (which is widely used for lattice QCD) are equivalent [20]. There is another possible application of Dirac–Kähler formulation considered. It has been pointed out that the Dirac–Kähler fermion formalism is essentially equivalent to the twisting of topological field theory generating SUSY [21]. The Dirac–Kähler formulation has also a fundamental connection with the regularization of fermions, and is related to the twisting of supersymmetry and leads to the corresponding lattice SUSY formulation. Applicability of the current formulation of Dirac–Kähler fields to the above mentioned investigations will be the subject of further work.

Acknowledgement

I am grateful to a referee of EPJC for his valuable remarks.
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