Exponential integration for efficient and accurate multibody simulation with stiff viscoelastic contacts

Bilal Hammoud1 · Luca Olivieri2 · Ludovic Righetti1 · Justin Carpentier3 · Andrea Del Prete2

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Abstract
The simulation of multibody systems with frictional contacts is a fundamental tool for many fields, such as robotics, computer graphics, and mechanics. Hard frictional contacts are particularly troublesome to simulate because they make differential equations stiff, calling for computationally demanding implicit integration schemes. We suggest to tackle this issue by using exponential integrators, a long-standing class of integration schemes (first introduced in the 1960s) that in recent years has enjoyed a resurgence of interest. This scheme can be applied to multibody systems subject to stiff viscoelastic contacts, leading to integration errors similar to implicit Euler, but at much lower computational costs (between 2 to 100 times faster). In our tests with quadruped and biped robots, our method demonstrated a stable behavior with large time steps (10 ms) and stiff contacts (10^5 N/m). Its excellent properties, especially for fast and coarse simulations, make it a valuable candidate for many applications in robotics, such as simulation, model predictive control, reinforcement learning, and controller design.

Keywords Exponential integrators · Spring-damper · Robotics · Contact simulation

✉ B. Hammoud
bah436@nyu.edu

L. Olivieri
luca.olivieri-1@unitn.it

L. Righetti
lr114@nyu.edu

J. Carpentier
justin.carpentier@inria.fr

A. Del Prete
andrea.delprete@unitn.it

1 Tandon School of Engineering, New York University, New York, NY, USA
2 Department of Industrial Engineering, University of Trento, Via Sommarive 9, Trento, 38123, Italy
3 INRIA, Paris, France
1 Introduction

The interest of the robotics community for fast and reliable methods to simulate multibody systems subject to frictional contacts has been constantly growing in the last two decades [1–5]. This is reasonable considering that simulation is at the core of many robotics applications, such as the development and testing of novel controllers before deployment on hardware. Moreover, many advanced control and planning techniques, such as model predictive control [6] (MPC) and optimal control [7], rely on the ability to predict the future behavior of the system. Finally, the current bottleneck of many learning algorithms [8, 9] is their need for huge amounts of data, which therefore can greatly benefit from fast and accurate simulation methods.

The simulation of articulated rigid multibody systems without contacts is a solved problem [10]. The same is not the case for systems with stiff contacts, which can be treated in two ways, each leading to a hard (but different) numerical problem. The first approach, which more closely follows the physical phenomenon of contacts, consists in expressing contact forces as a function of the penetration between bodies. Often, linear spring dampers have been used in this context [5, 11]. This leads to stiff differential equations, which are simple to evaluate but difficult to integrate because of their numerical stiffness [12]. The second approach tries to circumvent these numerical challenges by assuming contacts to be infinitely rigid. This approach effectively gets rid of the numerical stiffness, but in exchange for nonsmoothness. One method, which has been particularly successful for dealing with the resulting nonsmooth equations, is velocity-impulse time-stepping [1, 4, 13, 14]. This has become the standard for robot simulation [15], demonstrating stable behaviors with large time steps (several ms)—even though it must solve a numerically hard linear complementarity problem (LCP). Several authors have tried to improve this approach by getting rid of the strict complementarity constraints [2, 3, 16], which are the source of the numerical challenge. However, none of these approaches is currently widely accepted in the robotics community, mainly because of the unclear effects of the introduced numerical regularization/relaxations on the physics (i.e., relaxations can be interpreted as implicit spring/dampers but are not an explicit part of the modeling).

The approach we advocate for in this paper is based on a well-known soft contact model, the linear spring damper. Instead of using explicit integration schemes, which require small time steps, or implicit schemes, which require solving nonlinear systems of equations and introduce artificial viscosity leading to nonphysical behaviors, we use exponential integrators (EIs) [17, 18]. EIs are a long-standing class of integration schemes [19] particularly suited for stiff differential equations. EIs were initially considered unpractical because of the computational challenges related to the matrix exponential [20]. However, novel numerical methods to compute the matrix exponential [21–23] have recently unlocked the potential of EIs. This has already been used in computer graphics for simulating deformable objects, modeled as systems of particles [24–26]. This model is particularly suited for EIs because the stiff part of the dynamics is linear, which however is not the case for articulated systems in contact with the environment.

Our main contribution is a simulation algorithm that exploits EIs to simulate articulated robots in contact with a stiff viscoelastic environment. Particularly, this paper addresses the following questions. (1) Does the proposed simulation scheme provide an improvement in terms of speed vs accuracy when compared to classic explicit and implicit methods? (2) Is it possible to develop a simulation scheme that is less sensitive to the choice of contact stiffness and damping? (3) Is it possible to get stable simulations with increased time step (integration interval)? The last question is of particular importance for using MPC and reinforcement learning.
To address all the above questions in an efficient way, we apply EIs only to the contact dynamics (which is stiff) while using an explicit Euler scheme for the remaining terms (which are not stiff). Our simulation results on quadruped and biped robots (see Fig. 1) show the superior performance of our method compared to standard integration schemes in terms of accuracy, speed, and stability. To our knowledge, this class of integrators has never been used before by the robotics community. The paper is organized as follows. Section 2 introduces the problem of multibody simulation and the basic theory of EI. Section 3 explains how EI can be used for multibody simulation with bilateral contacts. This method is then extended to frictional contacts in Sect. 4. Section 5 discusses the implementation details of the algorithm. Section 6 presents the results, and Sect. 7 concludes the paper.

2 Background

2.1 Multibody dynamics and soft contact model

We want to simulate a multibody mechanical system with the following dynamics [10]:

\[
M(q)\ddot{v} = u(q, v) + J(q)\lambda^\top \lambda,
\]

where \(q\) is the robot joint configuration, \(v\) is the robot joint velocity, \(M\) is the joint space mass matrix, \(u\) contains gravity, nonlinear effects, and actuator forces, \(J\) is the contact point Jacobian, and \(\lambda\) contains the stacked 3D contact forces. We assume a linear spring-damper contact model, which means that the contact forces \(\lambda\) are proportional to the interpenetration of contacting bodies:

\[
\lambda = -K \underbrace{(p(q) - p_0)}_{\Delta p} - B \underbrace{(\dot{p}(q, v) - \dot{p}_0)}_{\Delta \dot{p}},
\]

(2)
where \( p \) and \( \dot{p} \) contain the stacked 3D contact point positions, and velocities, \( p_0 \) and \( \dot{p}_0 \) contain the stacked 3D anchor point positions and velocities, and \( K \) and \( B \) are the diagonal stiffness and damping matrices, respectively. The anchor point \( p_0 \) is a virtual point to which the virtual spring and damper are attached. It is typically set to the contact point location when contact is first detected, and as long as contacts are sticking, \( \dot{p}_0 = 0 \). However, when slipping occurs, \( \dot{p}_0 \neq 0 \). A limitation of the “anchor point” model is that to generate tangential forces, some lateral motion of the contact point is always necessary. Consequently, pure static friction cannot be modeled with this approach, but it can be well approximated by using large lateral stiffnesses. Dependencies on \( q \) and \( v \) are dropped in the following to ease notation.

### 2.2 Explicit integration schemes

The classic approach to integrate this dynamical system starts by writing it in standard form. Defining the state as \( x_q \triangleq (q, v) \), its dynamics is

\[
\frac{d}{dt} \begin{bmatrix} q \\ v \end{bmatrix} = \begin{bmatrix} v \\ f(x_q, u) \end{bmatrix}. \tag{3}
\]

We can integrate (3) with any numerical integration scheme, such as a high-order Runge–Kutta scheme or even a simple explicit Euler [12] (very common in robotics)

\[
x_q^{+} = x_q + \delta t f(x_q, u), \tag{4}
\]

where \( x_q^{+} \) represents the next value of the state, and \( \delta t \) is the integration time step. The problem with this approach is that for large values of \( K \) and \( B \), the differential equations (3) are stiff [12]. This means that they require very small integration steps for numerical stability. This is the main reason why soft contact models have been mostly abandoned in the last decade, in favor of complementarity-based models (and their relaxations) and time-stepping integration [2, 4, 27].

### 2.3 Exponential integrators (EIs)

EIs [24–26] are integration schemes particularly suited for stiff dynamical systems for which the stiffness comes from a linear part of their dynamics:

\[
\dot{x} = f(x) + Ax. \tag{5}
\]

In this case, using an explicit integration scheme would result in the problems mentioned above. Instead, EIs exploit the linearity of the stiff part of the dynamics, which can be solved analytically using the matrix exponential, thanks to the well-known solution of linear dynamical systems:

\[
\dot{x}(t) = Ax(t) + b, \\
x(t) = e^{tA}x(0) + \int_0^t e^{\tau A} d\tau b. \tag{6}
\]
First-order EIs apply solution (6) to the nonlinear system (5) by interpreting $f$ as $b$ and assuming that it remains constant during the integration step:

$$x(t) = e^{tA}x(0) + \int_0^t e^{tA} \frac{d}{dt} f(x(t)).$$  \(7\)

Since the stiff part of the equations is integrated via the matrix exponential, large integration steps can be taken.

### 3 Bilateral contacts

Our approach consists in using EIs to simulate system (3). The standard approach to apply EIs to arbitrary dynamics is to use a first-order Taylor expansion

$$\dot{x}_q(t_0 + t) \approx \dot{x}_q(t_0) + \frac{\partial f}{\partial x_q}(x_q(t_0 + t) - x_q(t_0)).$$  \(8\)

However, this would require two demanding computations, the dynamics Jacobian and a matrix exponential with the size of $x_q$. In the following, we present instead an approach that i) does not require the dynamics Jacobian and ii) only computes a matrix exponential with twice the size of $\lambda$, which is typically smaller than $x_q$ in legged locomotion.

To get a differential equation of the form (5), we start by projecting (1) in contact space premultiplying both sides by $J M^{-1}$:

$$J \dot{v} - J M^{-1} J^\top \lambda = J M^{-1} u.$$  \(9\)

Then we use the relationship $\ddot{p} = J \dot{v} + J \dot{v}$ to express the contact point accelerations as functions of the robot accelerations:

$$\ddot{p} - \Upsilon \lambda = J \dot{v} + J \dot{v},$$

$$\ddot{p} + \Upsilon K \Delta p + \Upsilon B \Delta \dot{p} = J \dot{v} + J \dot{v}.$$  \(10\)

Since for bilateral contacts, $\dot{p}_0$ is always null, we have that $\ddot{p}_0 = 0$, and thus we can write the contact point dynamics as

$$\frac{d}{dt} \begin{bmatrix} \Delta p \\ \Delta \dot{p} \end{bmatrix} = \begin{bmatrix} 0 & I \\ -\Upsilon K & -\Upsilon B \end{bmatrix} \begin{bmatrix} \Delta p \\ \Delta \dot{p} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}. $$  \(11\)

This dynamical system does not have the same form as (5) because $\Upsilon$ (and thus $A$) depends on $q$. However, $\Upsilon$ is typically a well-conditioned function, meaning that it changes little for small variations of $q$. The same holds for $\ddot{p}$ (and thus $b$), which is why multibody systems without contacts can typically be integrated with large time steps ($\approx 5$ ms). This means that we can approximate $A$ and $b$ as constants during the integration step and therefore treat (11) as linear. We can now express the contact forces as

$$\lambda(t) = \begin{bmatrix} -K \\ -B \end{bmatrix} x(t) = D e^{tA} x(0) + D \int_0^t e^{tA} \frac{d}{dt} f(x(t)).$$  \(12\)
Substituting (12) into (3), we can compute the robot accelerations:

\[
\dot{v}(t) = M^{-1}(u + J^\top \dot{\lambda}(t)) = \ddot{v} + M^{-1} J^\top D \dot{x}(t),
\]
where we consider all terms constant during the integration step, except for \(x(t)\). Now we can integrate to get the new velocities \(v^+\):

\[
v^+ = v + \int_0^{\delta t} \dot{v}(\tau) d\tau = v + \delta t \ddot{v} + M^{-1} J^\top D \int_0^{\delta t} x(\tau) d\tau.
\]

Finally, we integrate twice to get the new configuration \(q^+\):

\[
q^+ = q + \int_0^{\delta t} v(\tau) d\tau = q + \delta t v + \frac{\delta t^2}{2} \ddot{v} + M^{-1} J^\top D \int_0^{\delta t} \int_0^\tau x(\tau_1) d\tau_1 d\tau.
\]

### 3.1 Integration of matrix exponentials

Equations (14) and (15) are straightforward to compute, except for their last terms, which are

\[
x_{int}(t) \triangleq \int_0^t x(\tau) d\tau,
\]

\[
x_{int2}(t) \triangleq \int_0^t \int_0^\tau x(\tau_1) d\tau_1 d\tau,
\]

where

\[
x(t) = e^{tA}x(0) + \int_0^t e^{\tau A} d\tau b.
\]

When \(A\) is invertible, we can express the integral of \(e^{tA}\) as an algebraic function of \(e^{tA}\):

\[
\int_0^t e^{\tau A} d\tau = A^{-1}(e^{tA} - I).
\]

However, \(A\) is not invertible if the contact Jacobian \(J\) is not full-row rank. Luckily, the computation of integrals involving matrix exponentials has been thoroughly investigated \([28, 29]\). In Sect. 5, we show how to compute these integrals indirectly by simply computing the matrix exponentials of an augmented system.

### 3.2 Extension to non-Euclidian spaces

When \(q\) does not belong to an Euclidian space (as in the case of legged robots, where \(q\) includes the orientation of the base link), the integration of \(q\) is slightly more complicated (whereas the integration of \(v\) remains unchanged). Given the definition

\[
v_{\text{mean}} \triangleq v + \frac{\delta t}{2} \ddot{v} + \frac{1}{\delta t} M^{-1} J^\top D x_{\text{int2}}(\delta t),
\]
the integration step of $q$ is computed as

$$q^+ = \text{integrate}(q, \delta t \nu_{\text{mean}}),$$

(20)

where the function $\text{integrate}(\cdot)$ performs integration in the non-Euclidian space of $q$.

### 4 Frictional contacts

So far we have assumed that contact forces were bilateral. However, we typically want to simulate unilateral contacts, where forces oppose penetration but do not oppose detachment of bodies. Assuming that the contact forces are expressed in a local reference frame with the $z$ direction aligned with the contact normal, the unilateral forces must satisfy

$$f^z_i \geq 0 \quad \forall i.$$  

(21)

Moreover, tangential forces are typically limited as well. Assuming the Coulomb friction model, we have

$$\sqrt{(f^x_i)^2 + (f^y_i)^2} \leq \mu f^z_i \quad \forall i,$$  

(22)

where $\mu \in \mathbb{R}^+$ is the coefficient of friction.\(^1\) We can represent constraints (21) and (22) as $\lambda \in \mathcal{K}_{\mu}$ with $\mathcal{K}_{\mu}$ being a second-order cone.

#### 4.1 Force projection

To account for these constraints, when the value of $\lambda(t)$ computed by (12) is outside $\mathcal{K}_{\mu}$, we should project it on the boundaries of $\mathcal{K}_{\mu}$. However, we do not know how to check this constraint in continuous time. In the same spirit of time-stepping simulators [1], we suggest to check friction constraints on the average value of $\lambda(t)$ during the integration step, which is

$$\tilde{\lambda} \triangleq \frac{1}{\delta t} \int_0^{\delta t} \lambda(\tau) \, d\tau = \frac{1}{\delta t} D x_{\text{int}}(\delta t).$$

(23)

If $\tilde{\lambda} \notin \mathcal{K}_{\mu}$, then we compute its projection on the boundaries of the friction cone $\lambda_{\text{pr}} = \text{proj}_{\mathcal{K}_{\mu}}(\tilde{\lambda})$ and use it to compute the next state:

$$\dot{v}_{\text{pr}} \triangleq M^{-1}(u + J^\top \lambda_{\text{pr}}),$$

$$v^+ = v + \delta t \dot{v}_{\text{pr}}, \quad q^+ = q + \delta t v + \frac{\delta t^2}{2} \dot{v}_{\text{pr}}.$$  

(24)

Note that in case $\tilde{\lambda} \in \mathcal{K}_{\mu}$, we have $\lambda_{\text{pr}} = \tilde{\lambda}$, and the velocity update in (24) is equivalent to (14). However, the position update in (24) approximates the double integral of $\lambda(t)$ assuming a constant force ($\lambda_{\text{pr}}$), and so it is not equivalent to (15) in general. To exploit also

\(^1\)We assume that static and dynamic coefficients of friction are equal.
the double integral of $x(t)$, we can check the friction cone constraints on the average of the average $\bar{\lambda}(t)$, computed as

$$\bar{\lambda} = \frac{2}{\delta t^2} \int_{0}^{\delta t} \int_{0}^{t} \lambda(\tau_1) \, d\tau_1 \, d\tau = \frac{2}{\delta t^2} D x_{\text{int}}(\delta t).$$

If $\bar{\lambda} \notin K_\mu$, then we project it on the boundaries of the friction cone $\lambda_{pr2} = \text{proj}_{K_\mu}(\bar{\lambda})$ and use it to compute the next position:

$$v_{pr2} = M^{-1}(u + J^T \lambda_{pr2}),$$

$$q^+ = q + \delta t v + \frac{\delta t^2}{2} \dot{v}_{pr2}.$$  \hfill (26)

Using (26) for the position update and (24) for the velocity update, both updates are equivalent to the original ones in case of no slippage.

### 4.2 Anchor point update

When slippage occurs, the tangent anchor point state $(p_0^t, \dot{p}_0^t)$ (where the index $t$ indicates the tangent directions) changes, which has two main implications. First, the assumption $\dot{\lambda}^n = 0$ that we took to write the contact point dynamics as (11) is no longer valid. This means that during slippage, (11) is an approximation of the contact point dynamics, based on a “business as usual” assumption (i.e., that the anchor point $p_0$ continues slipping at constant velocity). Second, the anchor point state should be updated so that the contact forces at the end of the time step are inside the friction cones. When a contact is slipping, the tangent anchor point velocity converges to $\dot{p}_t^t$. We show it now for the case of a 2D contact, but a similar reasoning can be applied to the 3D case. While a contact is slipping, the tangential force $\lambda^t$ remains on the boundary of the friction cone, so we have

$$\dot{\lambda}^t = \mu \dot{\lambda}^n,$$

$$K^t(\dot{p}_0^t - \dot{p}^t) + B^t(\ddot{p}_0^t - \ddot{p}^t) = \mu \dot{\lambda}^n,$$

$$\ddot{p}_0^t - \ddot{p}^t = -(B^t)^{-1}K^t(\dot{p}_0^t - \dot{p}^t) + \mu (B^t)^{-1}\dot{\lambda}^n.$$  \hfill (27)

The last equation shows that if $\dot{\lambda}^n = 0$, then we have an exponential convergence to zero of $(\dot{p}_0^t - \dot{p}^t)$ with rate $(B^t)^{-1}K^t$. Since typically $(B^t)^{-1}K^t$ is large, whereas $\mu (B^t)^{-1}\dot{\lambda}^n$ is small, we can expect this convergence to be fast. For instance, if $(B^t)^{-1}K^t = 10^5$ and $\dot{\lambda}^n = 0$, then after 3 ms, $(\dot{p}_0^t - \dot{p}^t)$ will be 5% of its initial value. Given this fast convergence, we neglect the transient, and as soon as slippage starts, we set $p_0^t := \dot{p}^t$. Then we compute $p_0^t$ so that the contact force is on the boundary of the friction cone:

$$\lambda := \text{proj}_{K_\mu}(\lambda),$$

$$p_0^t := p^t + (K^t)^{-1}\lambda^t.$$  \hfill (28)

### 5 Computational aspects

The computational bottleneck of the presented approach is the computation of $x_{\text{int}}$ and $x_{\text{int2}}$ defined in (16). This section shows how to compute these quantities with a matrix exponential and how this computation can be sped up.
5.1 Computing $x_{\text{int}}$ and $x_{\text{int}2}$

Using the results presented in [29], we can compute $x_{\text{int}}$ and $x_{\text{int}2}$ as

$$
\begin{bmatrix}
  x_{\text{int}}(t) \\
  x_{\text{int}2}(t)
\end{bmatrix} =
\begin{bmatrix}
  I_n & 0_{n \times 3}
\end{bmatrix}
\begin{bmatrix}
  0_{(n+1) \times 2} \\
  I_2
\end{bmatrix},
$$

(29)

where $n$ is the size of $A$, and $\bar{A} \in \mathbb{R}^{(n+3) \times (n+3)}$ is an augmented matrix,

$$
\bar{A} \triangleq \begin{bmatrix}
  A & b & x(0) \\
  0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0
\end{bmatrix}.
$$

(30)

5.2 Computing the matrix exponential

Using (29), we have transformed the problem of computing (16) into a matrix exponential evaluation. Computing the matrix exponential is a challenging but well-understood numerical problem [21–23, 30]. We have used as starting point the scaling&squaring method, as revisited by Higham [21], widely used for computing the exponential of small–medium size dense matrices. The method scales the matrix by a power of 2 to reduce the norm to order 1, computes a Padé approximant to the matrix exponential, and then repeatedly squares to undo the effect of the scaling. A Padé approximant of a function is its “best” approximation achievable by a ratio of two polynomials $D_j$ and $N_j$ of order $j$:

$$
e^A \approx D_j(A)^{-1} N_j(A).
$$

(31)

These approximants are only accurate around zero, so they cannot be used directly if $\|A\|$ is large. When that is the case, the scaling&squaring method is used to reduce $\|A\|$ by exploiting this property of the exponential:

$$
e^A = (e^{A/(2^s)})^{2^s}.
$$

(32)

The integer scaling parameter $s$ is chosen so that $\|e^{A/(2^s)}\|$ is sufficiently small.

5.3 Boosting the matrix exponential computation

Our problem has two features, which we can exploit to speed up computation:

1. We do not need double machine precision, i.e., $\approx 10^{-16}$ (which is the target of the algorithm of [21]) because we are typically fine with much larger numerical integration errors, e.g., $\approx 10^{-4}$.
2. We do not need the whole matrix exponential, but only its product with a two-column matrix, as shown in (29).

The first point is easily exploitable. The choice of the scaling parameter $s$ and the polynomial order $j$ is usually optimized to achieve double machine precision with the minimum amount of matrix–matrix multiplications. We have empirically found that for our tests, we can set $s = 0$ and use a relatively low order $j \in [1, 2, 3, 5, 7]$ corresponding to $[0, 1, 2, 3, 4]$ matrix–matrix multiplications, respectively. Which polynomial order is optimal depends on the specific test and is discussed in the next section.
Regarding the second point, given a matrix \( V \), we can directly compute the product \( e^AV \) by performing operations in the following order:

\[
V_1 := N_j(A) V, \\
e^AV := D_j(A)^{-1} V_1.
\] (33)

This is faster than computing \( e^A \) and then multiplying it times \( V \) because we have to solve the linear system with a much smaller right-hand-side (\( V_1 \) rather than \( N_j \)).

Finally, we have also observed that the preprocessing step suggested in [21], which uses matrix balancing, is extremely effective at reducing \( \|A\| \) in our tests. This is crucial to achieve accurate results with low polynomial orders, therefore speeding up computations. Further details can be found in our open-source online repository.2

6 Results

We assess the performance of our simulation algorithm (\textit{Expo}) comparing it to Implicit Euler (\textit{Eul-imp}), Runge-Kutta 4 (\textit{RK4}), and explicit Euler (\textit{Eul-exp}). Our implementation of \textit{Eul-imp} is described in the Appendix. Our implementation of \textit{RK4} is standard, whereas \textit{Eul-exp} was implemented as follows:

\[
v^+ = v + \delta t \dot{v}, \quad q^+ = \text{integrate}(q, \delta t v + \frac{\delta t^2}{2} \ddot{v}).
\] (34)

Our results try to answer to the following questions:

1. Can our approach (compared to the others) achieve higher accuracy for equal computation time or equal accuracy for smaller computation time? (Sect. 6.3)
2. How sensitive is the simulator accuracy to contact stiffness and damping? (Sect. 6.4)
3. What is the maximum integration time step that results in a stable\(^3\) motion? (Sect. 6.5)
4. How accurately can (11) predict future contact forces when assuming constant \( A \) and \( b \)? (Sect. 6.6)
5. How much computation time is spent in the different operations of our simulator? Is there room for improvement? (Sect. 6.7)

6.1 Accuracy metric

Following an approach similar to [15], we measure accuracy with \textit{local integration error}. We compute the ground truth trajectory \( x_q(t) \) using the simulator under analysis with an extremely small time step \( \delta t = 1/64 \) ms. Let us define \( \tilde{x}_q(t; t - \delta t_c, x_q(t - \delta t_c)) \) as the state at time \( t \) obtained by numerical integration starting from the ground-truth state \( x_q(t - \delta t_c) \), where \( \delta t_c (\geq \delta t) \) is the time step of the controller. We define the \textit{local} integration error as the error accumulated over one control time step:

\[
e(t) \triangleq \|x_q(t) \ominus \tilde{x}_q(t; t - \delta t_c, x_q(t - \delta t_c))\|_\infty.
\]

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2https://github.com/andreadelprete/consim.

3We say that a simulation is “stable” if the robot state remains bounded.
where $\ominus$ is a difference operator on the space of $x_q$. In the numerical integration literature [12] the local integration (or truncation) error is typically defined using the integration step $\delta t$ rather than the controller step $\delta t_c$. We chose to use the controller step to make errors comparable across tests with different integration steps (as in [15]).

### 6.2 Test description

To evaluate the trade-off between accuracy and computation time, we tested each simulator with different time steps. For Expo, RK-4, and Eul-imp, we have started from $\delta t = 1/8$ ms up to the controller time step $\delta t_c$ with a logarithmic step of 2 (i.e., $1/8$, $1/4$, $\ldots$, $\delta t_c/2$, $\delta t_c$). For Eul-exp, we have used the same approach, but starting from a value of $\delta t$ resulting in roughly the same computation time of Expo. For every test, we have set $\delta t_c$ to the largest value that still ensured control stability (see Table 1). Since our main interest lies in legged robots, our tests focused on quadrupeds and bipeds:

- **Solo-squat**: Quadruped robot Solo [31] performing a squatting motion.
- **Solo-jump**: Quadruped robot Solo jumping in place.
- **Solo-trot**: Quadruped robot Solo trotting forward.
- **Romeo-walk**: Humanoid robot Romeo [32] taking two walking steps.

It is important to note that the quadruped Solo has a total of 13 links, 18 degrees of freedom, 12 of which are actuated revolute joints, and 4 contact points (one on each foot). As for the humanoid Romeo, it consists of 32 links, 37 degrees of freedom, 31 actuated revolute joints, and a total of 8 contact points (four on each foot). In all tests, the control torques have been computed with a feedback controller, either a linear controller, or a Task-Space Inverse Dynamics controller. If not specified otherwise, we have used a contact stiffness $K = 10^5$ N/m, and a contact damping $B = 300$ Ns/m, which are reasonable values for contacts with a hard floor. For homogeneity, we have used the same value of friction coefficient $\mu = 1$ across all our tests, even though this large friction was only needed for control stability of the quadruped jumping motion. Besides testing the default Expo simulator, we also tested five other versions of the same scheme, where we used a reduced polynomial order in the Padé approximant of the matrix exponential. This leads to a reduced number of matrix–matrix multiplications (mmm), between 0 and 4 (see Sect. 5.3). This results in a faster but potentially less accurate computation of the matrix exponential.

All the code has been implemented in C++ and binded with Python. For all dynamics computation, we have used the Pinocchio library [33].

### 6.3 Accuracy-speed results

Figures 2 and 3 summarize the results for the four tests. Figure 2 plots local errors vs real-time factor, which measures how many times the simulation was faster than real time. Figure 3 instead plots local errors vs integration time step. Even though our main interest is in the trade-off between computation time and accuracy, which is depicted in Fig. 2, we decided to report also the accuracy as a function of integration time in Fig. 3 to provide more information about the behavior of the different methods.
Fig. 2  Local integration errors vs real-time factors. The label \textit{mmm-1} in the legend corresponds to using the default number of matrix–matrix multiplications (mmm) in the computation of the matrix exponential.

Overall, \textit{Expo} outperformed the other methods in all tests, showing faster computation for equal accuracy or greater accuracy for equal computation time. Surprisingly, the second best method overall was the simple \textit{Eul-exp}, even though it was partially beaten by \textit{Eul-imp}. 
6.4 Stiffness and damping

In this subsection, we investigate the sensitivity to contact stiffness and damping ratio of Expo, Eul-exp, and Eul-imp. The damping ratio is defined as $\frac{B}{2\sqrt{K}}$. A damping ratio of 1 corresponds to a critically damped contact. These results are based on the “solo-trot” scenario. For Eul-exp, we have used $\delta t = 1/2$ ms (real-time factor $\approx 50$). Then we have set $\delta t = 2$ ms for Expo, so that it had roughly the same computation time, and $\delta t = 2$ ms for Eul-imp, so that it performed similarly to Expo (even though with much larger computation times).

Figure 4 shows the local integration error as we vary the contact stiffness (with fixed damping ratio) and the damping ratio (with fixed contact stiffness). Expo performs consistently as damping ratio and stiffness increase up to $K = 10^8$, which roughly corresponds to a ground penetration of 0.01 mm for 100 kg of weight on a single contact point. The error of Eul-exp instead is highly affected by both stiffness and damping. Eul-imp performed slightly better than Expo in most cases (but at the cost of being 50 times slower), except for very stiff contacts ($K \geq 10^6$), where it led to larger errors.
To test the stability of the simulators, we have repeated the previous tests but without resetting the state to the ground truth after every control loop. Table 2 reports, for Expo, Eul-exp, and Eul-imp, the largest integration time step for which the system remained stable. Expo and Eul-imp showed similar stability, both remaining stable for large time steps, mostly between 5 and 40 ms. In the tests, “solo-squat” and “solo-jump” Expo was stable even with a larger time step than Eul-imp, whereas the opposite happened in “romeo-walk”. Eul-exp instead showed poor stability, needing a time step between 4 and 512 times smaller than Expo to remain stable.

6.6 Force prediction

To gain some insights into the internal computations of Expo, we show in Fig. 5 the normal contact forces predicted with (11) assuming constants $A$ and $b$—which is the key assumption of our method. Since $A$ and $b$ depend on $q$ and $v$, which vary during the time step, we could expect that neglecting their variations would result in significant force prediction errors. However, Fig. 5 shows that the force prediction can be accurate over a rather long time horizon (20 ms). These forces were generated at the beginning of the “solo-squat” test using different initial velocities. Since a linear spring-damper model is used, a sudden discontinuity in the contact force is expected when a point reaches contact with nonzero velocity. To demonstrate this, the normal contact force at a single contact point is plotted for the cases of the trotting quadruped and the walking biped in Fig. 6. Depending on the velocity at contact time, a finite jump in the contact force is observed. In terms of numerical issues, no major problems are observed in the simulations. This is due to the fact that these events are rather rare during the simulation. However, it is also observed that these moments are those where the integration error typically increases the most, as it was expected.
Exponential integration for efficient multibody simulation

Fig. 6 Normal contact force during quadruped trotting and biped walking

Table 3 Computation times of Expo for “solo-trot”, using zero mmm for the matrix exponential (in parentheses the values using the standard matrix exponential routine)

| Operation               | Mean time [µs] | Percentage of total time [%] |
|-------------------------|----------------|-----------------------------|
| step                    | 39 (94)        | 100                         |
| computeIntegrals        | 13 (67)        | 33 (72)                     |
| prepareExpLDS           | 13             | 33 (14)                     |
| computeContactForces    | 8              | 20 (8)                      |
| Eul-exp step            | 9              | -                           |

6.7 Computation times

We report here a breakdown of the computation time of our method. The times shown in Table 3 are for the “solo-trot” test, which means that \( v \in \mathbb{R}^{18} \) and most of the times \( \lambda \in \mathbb{R}^6 \). Most computation time (86%) is spent in three operations: computeIntegrals, prepareExpLDS, and computeContactForces. computeIntegrals boils down to computing a matrix exponential. This takes 72% of the total time when using a standard expm routine (without balancing and reduced matrix–matrix multiplications), but it goes down to 33% with our optimized version using zero matrix–matrix multiplications—we have seen in Fig. 2 that often this results in only a small loss of integration accuracy. The preparation of the linear dynamical system (11) (prepareExpLDS, which includes the computations of \( h(q, v) \) with RNEA, \( M(q) \) with CRBA, and \( \Upsilon \) with a custom sparse Cholesky decomposition) takes an equal amount of time: 13 µs on average, namely 33% of the total time. The third operation (computeContactForces) takes 20% of the total time and includes the computation of all kinematic quantities (contact point positions, velocities, accelerations, Jacobian) and the contact detection.

We believe that computation times could be improved, especially for the first two operations. In computeIntegrals, we could test novel techniques [23] to compute the matrix exponential, exploit the sparse structure of the matrix \( A \), and warm-start the computation using quantities computed at the previous cycle. In prepareExpLDS, the inverse contact-space inertia matrix \( \Upsilon \) could be computed faster using a customized algorithm, rather than
with products between \( J, M^{-1}, \) and \( J^\top \) [34]. Overall, it seems impossible to reach the same efficiency of a simple \( \text{Eul-exp} \) step (9 \( \mu s \)), but we think we could reach computation times in the range [20, 30] \( \mu s \).

7 Conclusions

In this paper, we presented a new approach to simulate articulated systems subject to stiff viscoelastic frictional contacts. The novelty of the approach lies in the numerical integration, which applies a first-order exponential integrator scheme to the contact point dynamics to obtain a time-varying expression of the contact forces. Then these contact forces are integrated analytically, exploiting theoretical results on the integrals of the matrix exponential [29] and advanced numerical algorithms for its fast computation [21]. Comparison with standard integration schemes, both implicit and explicit, highlighted the benefits of the proposed approach in terms of speed-accuracy trade off and stability. Overall, the proposed approach performed similarly to an implicit scheme in terms of stability and accuracy, but without the excessive computational burden.

Given its good behavior in the high-speed/low-accuracy regime, we believe that this simulation technique could be an excellent candidate for MPC. To do that, we will need to differentiate the integration scheme, which should be feasible. We also plan to investigate the improvement, in terms of computational efficiency, of the needed dynamics quantities and the matrix exponential.

Appendix: Implicit Euler

This Appendix reports some details about our implementation of implicit Euler. The state of our robots lies in a Lie group, so we have taken into account the derivatives of the integrate and difference functions, which need to be used in place of the simple + and − operators:

\[
\text{difference}(x^+, \text{integrate}(x, \delta tf (x^+, u))) = 0.
\] (35)

We have computed the dynamics Jacobian using the appropriate functions of the Pinocchio library. In particular, we have used the analytical derivatives of the \( \text{ABA} \) function (28 \( \mu s \) for the Solo quadruped robot), and we have added to it the derivatives of our contact model (which are fast to compute because they depend on already computed quantities). For the line search, we have used the \( \text{ABA} \) function (8 \( \mu s \) for the Solo quadruped robot). This resulted in a rather efficient implementation of implicit Euler.

For solving the nonlinear equations, we have implemented Newton’s method with line search and regularization. For solving the linear system of equations, we have used the LU decomposition with partial pivoting provided by the Eigen library. We initialized the search with the solution of an explicit Euler integration (which proved to be better than initializing with the current state).

We have measured where computation time is spent in implicit Euler to make sure that our implementation was efficient and to reason about potential improvements. For the “solo-squat” test, with an integration step \( \delta t = 10 \) ms, the average computation time was 0.92 ms. \( \text{Expo} \) was roughly 10 times faster for the same test and gave roughly the same integration
error. The Newton method took on average 8 iterations to converge to an error norm below $10^{-6}$. Computation time was distributed in the following way:\footnote{Percentages do not add up to 100% because of rounding errors.}

- Prepare linear system: 51%
  - Dynamics Jacobian: 29%
    \* ABA derivatives: 24%
    \* Contact model derivatives: 4%
  - Lie-group derivatives: 21%
- Line search: 29%
- Solve linear system: 12%
- Other: 8%

Most of the time was spent preparing the linear system (51%) and performing the line search (29%). Given the high efficiency of the Pinocchio library, we believe that there is not much room for improvement in the steps “ABA derivatives” and “line search” (which mainly consists of calls to ABA, integrate, and difference). The only significant room for improvement is in the “Lie-group derivatives” step, which includes two matrix–matrix multiplications of the dynamics Jacobian and the sparse Jacobians of the integrate and difference functions. Since these Jacobians are sparse, these multiplications could be computed more efficiently than what we currently do. However, this could lead to a computational gain of at most 15%. Therefore there seems to be little hope to make implicit Euler 10 times faster (i.e., as efficient as Expo).

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