A covering theory of special relativity

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Abstract: Under the assumption of closed-path velocity of light invariant, we show both the general expression of velocity of light in an ordinary inertial reference frame and the generalized Lorentz transformation between the ordinary inertial reference frame and the absolute (privileged) reference frame. Although such assumption cannot determine theory ambiguously, some significant results can still be obtained by the assumption. Furthermore, the study shows that the relativity of simultaneity is not a universal concept.

Résumé: Dans l’hypothèse se de la vitesse fermé invariant chemin de la lumière, nous montrer à la fois l’expression générale de la vitesse de la lumière dans un cadre de référence inertiel ordinaire et le transformation de Lorentz généralisée entre le cadre de référence inertiel ordinaire et le cadre de référence absolu. Bien que cette hypothèse ne puisse pas déterminer uniquement la théorie, elle pourra nous conduire à certains résultats importants. En plus, cette étude montre que la relativité de la simultanéité n’est pas un concept universel.

Keywords: velocity of light, generalized Lorentz transformation.

1 Introduction

The special theory of relativity\textsuperscript{[1–6]}, one of the most significant achievements of physics, is the cornerstone of modern physics. It is the special theory of relativity that first took space and time as a whole and introduced abstract reasoning into the study of space-time. It has therefore improved our understanding of the nature of space-time. In particular, compared to the classical theory, there are many unique concepts in the special relativity, such as Lorentz contraction, the relativity of simultaneity and that there is no superluminal signal (Whether there is superluminal signal is a hotspot topic of theoretical and experimental studies\textsuperscript{[7–11]}). Essentially, all these unique concepts stem from the two postulates\textsuperscript{[1–6]}: relativity of inertial reference frames (relativity of IRFs), that is, the physical laws are independent of the state of motion of the reference frame, at least if the frame is not accelerated\textsuperscript{[12]}, and the invariant of one-way velocity of light (one-way VL), that is, the clocks can be synchronized in such a way that the propagation velocity of light ray in vacuum, measured by means of these clocks, becomes equal to a universal constant everywhere, provided that the coordinate system is not accelerated\textsuperscript{[13]}.

Bondi\textsuperscript{[5]}, Bergman\textsuperscript{[6]}, etc. claimed long time ago that there are mismatches between the postulate of relativity and cosmological observations. Cosmological observations tell us that there does exist a distinguished IRF, to which many cosmology phenomena respect\textsuperscript{[14]}. Furthermore, as well known, what we have measured is the two-way VL and the one-way VL depends on the synchronization of the clocks. Different synchronization scheme leads to different one-way VL and different form of theory even two-way VL is invariant.

It is, then, interesting to study a more general theory, nominated as covering theory\textsuperscript{[15, 16]} , than special relativity. The covering theory, a parametric extension of special theory that contradicts special relativity for all but one value of parameters, is very useful for exploring logical implications and deciding on the tradeoff between simplicity and predictive accuracy. There are studies on the covering theory\textsuperscript{[16, 19]} with two-way VL invariant\textsuperscript{[18, 19]} or without two-way VL invariant\textsuperscript{[19, 12]}.

These covering theories\textsuperscript{[15, 16]} are based on the
two points: 1) ”general Lorentz transformation” or Robertson transformation [20] between ordinary inertial reference frame and the absolute inertial reference; 2) the introduction of synchronization parameter. The physical meaning of synchronization parameter is obvious, of course. However, as shown in the refs. [16–18], such approach is always associated with a lengthy deduction.

Here we want to consider the covering theory in a different approach. In particular, we do not set the synchronization parameter. We first generalize the VL invariant of backwards-and-forwards way into that of any closed path. We assume that the closed-path VL is a universal constant, independent on IRF and closed path. Such observable assumption can be reexpressed as that the closed-path light-travel time is invariant under any shape, provided the length of the closed path is fixed. The assumption, as shown in context, puts a strong constraint on the covering theory. In fact, our study shows that parameters in the constrained theory are only one more than that in special relativity. We also investigate the validity of some unique concepts in special relativity.

The manuscript is organized as follows. In section 2, we study the velocity of light under the assumption. Then, the transformation between ordinary IRF and absolute reference of frame (ARF) is shown in section 3. The results are summarized in the last section.

2 Velocity of light

As is known to all, one can not measure one-way VL unambiguously. The logical situation becomes circular when one tries to measure one-way VL: On one hand to measure one-way VL one needs to synchronize clocks, while on the other hand to synchronize clocks one must know the one-way VL. To avoid the synchronization problem among clocks one may, for instance, use one clock.

Thus, all the laboratory experiments measured instead the two-way VL. Consider the following situation: In an inertial frame Σ, a flash of light leaves point A at time $t_1$, is reflected back in point B at times $t_2$, and returns point A at time $t_3$. Suppose VL along vector $\vec{AB}$ is $c(\vec{AB})$, VL along vector $\vec{BA}$ is $c(\vec{BA})$ respectively, we have

$$c(\vec{AB}) = \frac{|AB|}{t_2 - t_1}, \quad c(\vec{BA}) = \frac{|BA|}{t_3 - t_2}. \quad (1)$$

where $|\vec{AB}| = |\vec{BA}|$ and VLs, $c(\vec{AB})$ and $c(\vec{BA})$, depend on the propagation directions and IRF. In the above equation to determine $t_2 - t_1$ and $t_3 - t_2$ one needs to synchronize the clock at point A and clock at point B. However, if one considers only $t_3 - t_1$ the synchronization is not needed. Therefore, since $t_3 - t_1 = t_3 - t_2 + t_2 - t_1$ one can define two-way VL as

$$\bar{c} = \frac{|\vec{AB}| + |\vec{BA}|}{t_3 - t_1} = \frac{|\vec{AB}| + |\vec{BA}|}{c(\vec{AB})c(\vec{BA})} \quad (2)$$

where $|\vec{AB}| + |\vec{BA}|$ is the total length of the back-and-forward path, a special case of closed path, and $\frac{|\vec{AB}|}{c(\vec{AB})} + \frac{|\vec{BA}|}{c(\vec{BA})}$ is the total light-travel time along the back-and-forward path. Such measurement is just with one clock and irrespective of the synchronization. So far all the experiments point out that $\bar{c} = 3 \times 10^8$ m/s is a universal constant, that is, it is dependent neither on the direction and distance of $\vec{AB}$ nor on the IRF.

Such definition of two-way VL can be generalized to that of arbitrary closed-path VL. Suppose in a closed-path we choose a infinitesimal arc with the length $dl$, the total length of the closed path is $\int dl$. We then analogously define the close-path VL as

$$\bar{c} = \frac{\int dl}{\int t}. \quad (3)$$

where $t$ is the total light-travel time along the closed path. It is easy to see that Eq. (3) is a direct generalization of Eq. (2) and such definition of the closed-path VL $\bar{c}$ is irrespective of the synchronization. We think such generalization is very natural and two-way VL is a special case of closed-path VL. Suppose again that VL along the infinitesimal arc is $c(\phi)$, where $\phi$ is the propagation direction of the light, $t$ can be expressed as $t = \int \frac{dl}{c(\phi)}$. One finds that

$$\bar{c} = \frac{\int dl}{\int \frac{dl}{c(\phi)}}. \quad (4)$$

Since back-and-forward path is a special closed path and such two-way VL is a universal constant $3 \times 10^8$ m/s, we assume in this paper that closed-path velocity of light is also the same constant, which is independent on IRF and closed path. Thus from Eq. (4), with $\bar{c}$ universal constant, we have

$$\int \frac{dl}{c(\phi)} = \frac{1}{\bar{c}}\int dl \quad (4)$$

for arbitrary closed path in any IRF.
For simplification we first choose the closed path as a smooth plane convex curve, for instance, in x-y plane. Then, VL should be a periodic function of the angle, \( \varphi \in [0, 2\pi] \), between propagation direction and a fixed direction, in particular, x-axis. As shown in Fig. 1, we consider an infinitesimal arc, \( AB \), the length of which is \( dl = |\widehat{AB}| \), on the closed curve. Suppose the radius of the arc curvature is \( R \) and the arc angle expanded to the corresponding curvature (or polar axis) is \( \varphi \) and the angle between the tangent at point B and x-axis is \( \varphi + d\varphi \) respectively. It is easy to see that \( d\varphi = d\theta \). If the velocity of light along the arc is \( c(\varphi) \), the travel of light along the infinitesimal arc needs time interval

\[
  dt = \frac{Rd\theta}{c(\varphi)} = \frac{Rd\varphi}{c(\varphi)}.
\]  

(5)

Figure 1: Travel of light along a closed path. VL is a function of angle \( \varphi \).

Since the closed curve is convex, \( \varphi \) and point belonging to the closed curve are one-to-one. From Eq. (4) we get

\[
  \int R(\varphi)\frac{d\varphi}{c(\varphi)} = \frac{1}{c} \int R(\varphi)d\varphi,
\]

where observable quantity \( c \) is a universality constant. This equation is not an identical one but a direct deduction of equation (4).

In this paper we assume the universal constant \( c = 1 \) for simplification, we have then

\[
  \int R(\varphi)f(\varphi)d\varphi = \int_{0}^{2\pi} R(\varphi)f(\varphi)d\varphi \equiv 0,
\]

(6)

where \( f(\varphi) = \frac{1}{c(\varphi)} - 1 \) is a function of \( \varphi \) with period \( 2\pi \). For a closed convex curve in \( x-y \) plane, radius of curvature \( R(\varphi) \) is not an arbitrary positive function. It should and should only meet the requirements,

\[
  \int dx &= \int_{0}^{2\pi} R(\varphi)\cos\varphi d\varphi = 0,
  
  \int dy &= \int_{0}^{2\pi} R(\varphi)\sin\varphi d\varphi = 0.
\]

(7)

At the same time, periodic function \( f(\varphi) \) can be expanded by Fourier series,

\[
  f(\varphi) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi)
\]

(8)

We emphasize again that Eq. (3) puts a very strong constraint on the VL. Substituting Eqs. (7) and (5) into expression (5), one finds

\[
  \int_{0}^{2\pi} d\varphi R(\varphi)[a_0 + \sum_{n=2}^{\infty} (a_n \cos n\varphi + b_n \sin n\varphi)] = 0
\]

(9)

for arbitrary positive function \( R(\varphi) \) satisfying (1). One can prove that to meet the above requirement all \( a_0, a_n, b_n (n = 2, 3, \cdots \infty) \) should be zero. For instance, to prove \( a_0 = 0 \) one may choose \( R = 1 \). Therefore, \( f(\varphi) \) has only two free parameters, \( f(\varphi) = A_1 \cos \varphi + B_1 \sin \varphi = A_0 \cos(\varphi + \varphi_0) \), where \( A_1 \) and \( B_1 \) (or \( A_0 \) and \( \varphi_0 \)) are constant depending on the plane and IRF. It is easy to see that as long as \( A_0 \) is fixed, we can get all the VL provided the propagation direction is in the plane. In particular, in the case of \( A_0 = 0 \), VL is independent on the propagation direction, that is, it keeps invariant in the propagation plane. For \( A_0 \neq 0 \), we are able to choose a suitable polar axis to make \( \varphi_0 = 0 \). We have, then, three obvious conclusions: 1) VL is maximum(minimum) along polar axis if \( -1 < A_0 < 0 (0 < A_0 < 1) \). 2) The direction of the maximum VL and that of the minimum VL are anti-parallel. 3) The direction of the minimum (maximum) VL is unique in the plane.

For an inertial reference frame, VL is constant unity provided \( A_0 = 0 \) for all the plane. Such isoptropic inertial reference is nominated as absolute reference frame(ARF). Meanwhile, for an ordinary inertial reference frame in which VL varies with the propagation direction, we can always find a direction along which VL is minimum. This direction is also unique, otherwise one would obtain a contradictory result against conclusion 3) in the above paragraph.

If we choose such direction as our polar axis of the IRF, VL can be written as

\[
  c(\varphi) = \frac{1}{1 + A_0 \cos \varphi},
\]

(10)
AB

imal vector

infinitesimal arc can be approximated by an infinites-

the light is

x, y, z

are

where

A >

are

constant unity for arbitrary closed path, including

spatial closed path, provided VL is expressed by the

above formulae.

Here we show a simple proof in cartesian coordinate.
We set polar axis coincides with

z-axis, that is, 

φ

is the angle between the propagation direction and

z-axis. Without loss of generality, we assume that the

closed path is a smooth curve, and choose a infinites-

imal smooth arc along the propagation direction on

the curve, \( AB \), where the coordinates of \( A \) and \( B \)

are \((x, y, z)\), \((x + dx, y + dy, z + dz)\) respectively. The

infinitesimal arc can be approximated by an infinites-

imal vector \( \vec{AB} \). Thus the propagation direction

of the light is \( AB = (dx, dy, dz) \). Suppose the length of

\( AB \) is \( dl \), one finds that \( \cos \varphi = \frac{dx}{\sqrt{dx^2 + dy^2 + dz^2}} \). Therefore, the

travel of light along the infinitesimal arc needs time

interval

\[
    dt = \frac{dl}{c(\varphi)} = (1 + A \cos \varphi)dl = dl + Adz,
\]

where we have used the Eq. (10) and the expression

cos \( \varphi \). Thus, travel of light along the closed path

needs time

\[
    t = \int dt = \int (dl + Adz) = \int dl,
\]

provided the curve is closed, that is, \( \oint dz = 0 \).

3 Transformation between ARF and IRF

Under the assumption of closed-path VL invariant

we deduce in the section the constrained Robertson

transformation between ARF and an ordinary IRF, which

determines the covering theory.

We suppose there are two frames. One is isotropic

ARF, \( \Sigma_0 \), in which the one-way VL is constant, and

the other is an ordinary IRF, \( \Sigma \). In the case that the

observer is in \( \Sigma_0 \), he/she lets the frame \( \Sigma \) move

with velocity \( v_0 \) in the x direction with respect to

\( \Sigma_0 \). Meanwhile, in the case that the observer is in \( \Sigma \),

he/she finds that the frame \( \Sigma_0 \) moves with velocity

\(-v\) in the x direction with respect to \( \Sigma \). For the lack

of symmetry, the result \( v_0 = v \) is not held generally. However, we are entitled to assume that the x-axes of

the two frame are parallel to each other at all time.

To any system of values \((x_0, y_0, z_0, t_0)\), which completely defines space and time of an event in ARF, there is a system of values \((x, y, z, t)\) determining that event in IRF \( \Sigma \). Letting the axes of X in the two systems coincide, and their axes of Y and Z be parallel respectively, we now want to find the transformations connecting these two systems of values which depict the same event.

After suitable choices of origins of space and time, the transformation between the two frames has the following form

\[
    x_0 = a_{11}x + a_{14}t,
    y_0 = g_1y,
    z_0 = g_2z,
    t_0 = a_{41}x + a_{44}t.
\]

Since the velocity of \( \Sigma_0 \) with respect to \( \Sigma \) is \(-v\), we have \( a_{14} = a_{11}v \) immediately. It is apparent that the \( \Sigma \) has a rotation symmetry with respect to x-axis, therefore, \( g_1 = g_2 \). Furthermore, VL in \( \Sigma \) should have the form of \( c(\varphi) = \frac{1}{1 + A \cos \varphi} \) with \( \varphi \) the angle between propagation direction and x-axis direction.

At time \( t = t_0 = 0 \), when the origins of the two frames coincide, let a light flash be emitted therefrom, and be propagated with the velocity unit in \( \Sigma_0 \). If \((x_0, y_0, z_0, t_0)\) is an wavefront event in x-y plane, we have

\[
    t_0 = \sqrt{x_0^2 + y_0^2}.
\]

In frame \( \Sigma \) such event is depicted by

\[
    t = \frac{\sqrt{x^2 + u^2}}{1 + Au} = \sqrt{x^2 + y^2} = \frac{x}{u},
\]

with \( u = \cos \varphi = \frac{x}{\sqrt{x^2 + y^2}} \). In other words,

\[
    t = (A + \frac{1}{u})x. \tag{15}
\]

Substituting Eqs. (15) and (13) into Eq. (14) and noticing that Eq. (14) holds for arbitrary \( u \), one has

\[
    g_1(g_2) = a_{11}\sqrt{(1 + Av)^2 - v^2},
    a_{44} = a_{11}(1 + Av),
    a_{41} = a_{11}(v - A(1 + Av)). \tag{16}
\]

Since \( a_{11} \) is in fact a scale shift, one can simply set \( g_1 = g_2 = 1 \) and therefore \( a_{11} = \frac{1}{\sqrt{(1 + Av)^2 - v^2}} \) in the above equations.

Thus, the transformations between ordinary IRF and ARF are

\[
    x_0 = \gamma(x + vt),
    y_0 = y,
    z_0 = z,
    t_0 = \gamma[(v - A(1 + Av))x + (1 + Av)t], \tag{17}
\]
and
\[ x = \gamma((1 + Av)x_0 - vt_0), \]
\[ y = y_0, \]
\[ z = z_0, \]
\[ t = \gamma(-(v - A(1 + Av)x_0 + t_0), \]
with \( \gamma = \frac{1}{\sqrt{(1 + Av)^2 - v^2}} \). One can verify that in \( \Sigma_0 \), the observer finds that the velocity of \( \Sigma \) is \( v_0 = \frac{v}{1 + Av} \).

The fact that \( v_0 \neq v \) reflects the asymmetry of the two frames. In the above equations \( A \) is the function of \( v_0 \) (or \( v \)). Since \( v_0 = 0 \) or \( v = 0 \) implies that the two frames \( \Sigma_0 \) and \( \Sigma \) should be the same, we conclude that \( A(v) = 0 \) at \( v = 0 \) from Eqs. (17) and (18). We therefore rewrite the transformations utilizing \( v_0 \) and \( A = Bv_0 \) in the following
\[ x_0 = \gamma_0((1 - Bv_0^2)x + v_0t), \]
\[ y_0 = y, \]
\[ z_0 = z, \]
\[ t_0 = \gamma_0((1 - B)v_0x + t), \]
and
\[ x = \gamma_0(x_0 - v_0t_0), \]
\[ y = y_0, \]
\[ z = z_0, \]
\[ t = \gamma_0[-(1 - B)v_0x_0 + (1 - Bv_0^2)t_0], \]
with \( \gamma_0 = \frac{1}{\sqrt{1 - v_0^2}} \). Eqs. (17)-(18) can be regarded as the generalized Lorentz transformation. Therefore, the transformations between velocity \((u, y_0, z)\) in \( \Sigma_0 \) frame and velocity \((u, y, z)\) in \( \Sigma \) frame are
\[ u_x = \frac{(1 - Bv_0^2)u_x + v_0}{(1 - Bv_0^2)u_x + 1}, \]
\[ u_y = \frac{(1 - Bv_0^2)u_y + v_0}{(1 - Bv_0^2)u_y + 1}, \]
\[ u_z = \frac{(1 - Bv_0^2)u_z}{(1 - Bv_0^2)u_z + 1}, \]
and
\[ u_x = \frac{u_x - v_0}{u_x - (1 - Bv_0^2)}, \]
\[ u_y = \frac{u_y - v_0}{u_y - (1 - Bv_0^2)u_x}, \]
\[ u_z = \frac{u_z - v_0}{u_z - (1 - Bv_0^2)u_x}, \]
respectively.

We have thus shown the most generally possible form if we only take the assumption of close-path VL invariant. Parameter \( B \) depends on the velocity \( v \) (or \( v_0 \)) and the different dependency determines different theory.

From Eqs. (17) and (18), both the Lorentz contraction effect and the time dilation are valid if and only if the observer stays in ARF, \( \Sigma_0 \). These results, although from different starting-point, are in agreement with those in refs. [16][19]. It is obvious that our deductions are very concise. Furthermore, although theories are possibly very different, \( \gamma_0 = \frac{1}{\sqrt{1 - v_0^2}} \) means there is no superluminal object in ARF. Such statement is apparently valid in other IRF. Therefore, the closed-path VL invariant implies there is no superluminal signal. The communication with the past is then impossible, i.e. there is no Tolman’s paradox [21]. Noticing that here we make only the assumption of the closed-path VL invariant, we think the result is very significant.

For all the equivalent theories, we have two conventional choices:

1. One may choose \( B = 0 \). In this theory the one-way VL is constant in all the inertial reference frames. In other words, all the IRF are equivalent and we return to the special theory of relativity.

2. One may also choose \( B = 1 \). Now we get \( t_0 = \gamma_0t \). That is, there is a universal time in all the IRFs, including ARF. In this theory the simultaneity is absolute although there is time dilation. However, VL is anisotropic in an ordinary IRF except ARF. One may think that all the cosmological phenomena, such as, extragalactic galaxies redshift, microwave background radiation, are respect to ARF.

Therefore, the relativity postulate and the absoluteness of simultaneity cannot be valid in one theory. These two concepts can only be valid separately, that is, they can be valid in different theories. The only difference between these two theories is the different choice of the function \( B \). Due to the fact that such choice makes no physical observation, we conclude that these two concepts are not inconsistent [17][18].

4 Conclusions

Taking closed-path VL invariant as the starting-point, we discuss here the general form of VL. We find that such postulate puts a very strong constraint on VL and that there is only one free parameter in the expression of VL.

We also show the covering theory of special relativity under such postulate without relativity postulate. Since the sole parameter \( B \) determines the covering theory, there are many similarities among the covering theories. The two significant similarities are: 1) There is no superluminal signal; 2) the
Lorentz contraction effect and time dilation are valid partially except the special relativity, that is, these two concepts are valid in other theories only if the observer stays in ARF, $\Sigma_0$. However, the concept of relativity of simultaneity in the special theory of relativity is not universal. In other words, different covering theory has different opinion on the relativity of simultaneity. In particular, we have shown in the paper a special theory in which the simultaneity is absolute. Indeed, our study shows that the two concepts, relativity postulate (one-way VL invariant) and the absoluteness of simultaneity, can not be satisfied simultaneously in one theory. One can freely adopt conventional choice on different problem. But since such choice takes no physical effect, these two concepts are not inconsistent.

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