SOME LARGE DEVIATION RESULTS FOR NEAR INTERMEDIATE RANDOM GEOMETRIC GRAPHS

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Abstract. We find large deviation principles for the degree distribution and the proportion of isolated vertices for the near intermediate random geometric graph models on \( n \) vertices placed uniformly in \([0,1]^d\), for \( d \in \mathbb{N} \). In the course of the proof of these large deviation results we find joint large deviation principle for the empirical locality measure of the coloured random geometric graphs, (Canning & Penman, 2003).

1. Introduction

In this article we study random graph model, the random geometric graph RGG, where \( n \) vertices or nodes or points are placed uniformly at random in \([0,1]^d\), and any two points distance at most \( r_n \) apart are connected. See (Penrose, 2003). The connectivity radius \( r_n \) plays similar role as the connection probability \( p_n \) in the Erdős-Rényi graph model. Several large deviation results about the Erdős-Rényi graph have been established recently. See (O’Connell, 1998), (Biggins and Penman, 2009), (Doku-Amponsah and Moerters, 2010), (Doku-Amponsah, 2006), (Bordenave and Caputo, 2013), (Mukherjee, 2013) and (Doku-Amponsah, 2014[a]).

Until recently few or no large deviation result about the degree distribution of the RGG have been found. Doku-Amponsah (2014[b]) proved some large deviation principle for the degree distribution of the classical Erdős-Rényi graph, where \( n \) points are uniformly chosen in \([0,1]^d\) and \( \lambda_n \) edges are randomly inserted among the points.

This article presents a full large deviation principle (LDP) for the empirical degree measure and the proportion of isolated vertices of near intermediate RGG presented. Specifically, we prove an LDP for the degree measure of the coloured RGG. We Refer to (Doku-Amponsah and Moerters) for similar result for the Erdő-Rényi graphs. From the LDP for the empirical degree measure; we derive an LDP for the proportion of isolated vertices. See, O’Connell [OC98] for similar result for the Erdős-Rényi graphs.

In the course of the proofs of this LDP we obtain joint the empirical pair measure and the empirical locality measures for coloured RGGs. Refer to (Doku-Amponsah & Moerters, 2010) or (Doku-Amponsah, 2006) for similar results for the coloured random graphs.

We note that physical quantities such as the degree distribution, number of edges per vertex and the proportion of isolated vertices of RGGs are crucial for understanding many biological systems.

In the remainder of the paper we state and prove our LDP results. In Section 2 we state our LDPs, Theorem 2.1, Corollary 2.2 and Theorem 2.3. In Section 3 we combine (Doku-Amponsah, Theorem 2.1, AMS Subject Classification: 60F10, 05C80

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2014[b]) and (Doku-Amponsah,Theorem 2.1, 2014[c]) to obtain the Theorem 2.3 using the setup and result of (Biggins, 2004) to ‘mix’ the LDPs. The paper concludes with the proofs of our main results Theorem 2.1 and Corollary 2.2 which are given in Section 4.

2. Statement of the results

2.1 Large deviations results for the random geometric graphs. The RGG is obtained when we sample points \( W_1, \ldots, W_n \) at independently according to the uniform probability distribution on \([0, 1]^d\), for \( d \geq 2 \) and given a fixed \( r_n > 0 \) we connect \( W_i, W_j \) (\( i \neq j \)) if 
\[
\|W_i - W_j\| \leq r_n.
\]

See [Pen03]. Various cases of the graph can be describe in terms of the quantity \( nr^d \), which is a measure of the average degree of the graph. See, MCDiarmid and Müller [MM05]. Our main aim in this article is to present and prove LDPs for the empirical degree measure and the proportion of isolated nodes to the number of vertices of the RGG when the connectivity radius satisfies \( nr^d \to \alpha \), for \( \alpha > 0 \). Thus, we consider the near intermediate case.

The first theorem in this subsection is the LDP for the degree distribution of the RGGs. We assume \( d \geq 2 \) is finite and write
\[
\rho(d) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d+2}{2}\right)},
\]
where Gamma is the gamma function.

**Theorem 2.1.** Suppose \( D \) is the degree distribution of the random graph \( G(n, r_n) \), where the connectivity radius \( r_n \in (0, 1] \) satisfies \( nr_n^d \to c \in (0, \infty) \). Then, as \( n \to \infty \), \( D \) satisfies an LDP in the space \( \mathcal{P}(\mathbb{N} \cup \{0\}) \) with good rate function
\[
\eta_1(\delta) = \begin{cases} \frac{1}{2} \langle \delta \rangle \log \left(\frac{\langle \delta \rangle}{\rho(d)c}\right) - \frac{1}{2} \langle \delta \rangle + \frac{\rho(d)c}{2} + H(d \| q_{\delta}^*), & \text{if } \langle \delta \rangle < \infty, \\ \infty, & \text{if } \langle \delta \rangle = \infty, \end{cases}
\]
where \( q_x \) is a Poisson distribution with parameter \( x \) and \( \langle \delta \rangle := \sum_{m=0}^{\infty} m \delta(m) \).

Next we give a similar result as in O’Connell [OC98], the LDP for the proportion of isolated vertices of the RGG.

**Corollary 2.2.** Suppose \( D \) is the degree distribution of the random graph \( G(n, r_n) \), where the connectivity radius \( r_n \in (0, 1] \) satisfies \( nr_n^d \to c \in (0, \infty) \). Then, as \( n \to \infty \), the proportion of isolated vertices, \( D(0) \) satisfies an LDP in \([0, 1]\) with good rate function
\[
\xi_1(y) = y \log y + \rho(d)c y(1 - y/2) - (1 - y) [\log (\frac{\rho(d)c}{a}) - \frac{(a - \rho(d)c(1-y)^2)}{2 \rho(d)c(1-y)}],
\]
where \( a = a(y) \) is the unique positive solution of \( 1 - e^{-a} = \frac{\rho(d)c}{a} (1 - y) \).

From Lemma 2.2 we deduce that on a typical random geometric graphs the number of isolated vertices will grow like \( ne^{-\rho(d)c} \). Thus, as \( n \to \infty \), the number of isolated vertices in the R.G graphs converges to \( ne^{-\rho(d)c} \) in probability. In our last theorem in this subsection we give the LDP for the proportion of edges to the number of vertices of the R.G.

2.2 Large deviation principles for empirical measures of the coloured random geometric graphs. In this subsection we shall look at a more general model of random geometric graphs, the coloured RGGs in which the connectivity radius depends on the type or colour or symbol or spin...
of the nodes. The empirical pair measure and the empirical locality measure are our main object of study.

Given a probability measure $\nu$ on $\mathcal{W}$ and a function $r_n: \mathcal{W} \times \mathcal{W} \to (0, 1]$ we may define the randomly coloured random geometric graph or simply coloured random geometric graph $X$ with $n$ vertices as follows: Pick vertices $W_1, ..., W_n$ at random independently according to the uniform distribution on $[0, 1]^2$. Assign to each vertex $W_j$ colour $X(W_j)$ independently according to the colour law $\mu$. Given the colours, we join any two vertices $W_i, W_j (i \neq j)$ by an edge independently of everything else, if

$$\|W_i - W_j\| \leq r_n[X(W_i), X(W_j)].$$

In this article we shall refer to $r_n(a, b)$, for $a, b \in \mathcal{W}$ as a connection radius, and always consider

$$X = ((X(W_i), X(W_j)) : i, j = 1, 2, 3, ..., n), E)$$

under the joint law of graph and colour. We interpret $X$ as coloured RGG with vertices $Y_1, ..., Y_n$ chosen at random uniformly and independently from the vertices space $[0, 1]^2$. For the purposes of this study we restrict ourselves to the near intermediate cases i.e. the connection radius $r_n$ satisfies the condition $nr_n^2(a, b) \to C(a, b)$ for all $a, b \in \mathcal{W}$, where $C: \mathcal{W}^2 \to [0, \infty)$ is a symmetric function, which is not identically equal to zero.

For any finite or countable set $\mathcal{W}$ we denote by $\mathcal{P}(\mathcal{W})$ the space of probability measures, and by $\mathcal{P}_c(\mathcal{W})$ the space of finite measures on $\mathcal{W}$, both endowed with the weak topology. By convention we write $\mathbb{N} = \{0, 1, 2, \ldots\}$.

We associate with any coloured graph $X$ a probability measure, the empirical colour measure $L_1^1 \in \mathcal{P}(\mathcal{W})$, by

$$L^1_X(a) := \frac{1}{n} \sum_{j=1}^{n} \delta_{X(W_j)}(a), \quad \text{for } a \in \mathcal{W},$$

and a symmetric finite measure, the empirical pair measure $L^2_X \in \mathcal{P}_c(\mathcal{W}^2)$, by

$$L^2_X(a, b) := \frac{1}{n} \sum_{(i, j) \in E} \left[ \delta_{(X(W_i), X(W_j))} + \delta_{(X(W_j), X(W_i))}\right](a, b), \quad \text{for } (a, b) \in \mathcal{W}^2.$$

The total mass $\|L^2_X\|$ of the empirical pair measure is $2|E|/n$. Finally we define a further probability measure, the empirical locality measure $M_X \in \mathcal{P}(\mathcal{W} \times \mathbb{N})$, by

$$M_X(a, \ell) := \frac{1}{n} \sum_{j=1}^{n} \delta_{(X(W_j), L(W_j))}(a, \ell), \quad \text{for } (a, \ell) \in \mathcal{W} \times \mathbb{N},$$

where $L(v) = (l^v(b), b \in \mathcal{W})$ and $l^v(b)$ is the number of vertices of colour $b$ connected to vertex $v$.

For any $\mu \in \mathcal{P}(\mathcal{W} \times \mathbb{N})$ we denote by $\mu_1$ the $\mathcal{W}$- marginal of $\mu$ and for every $(b, a) \in \mathcal{W} \times \mathcal{W}$, let $\mu_2$ be the law of the pair $(a, l(b))$ under the measure $\mu$. Define the measure (finite), $<\mu(\cdot, \ell), l(\cdot)> \in \mathcal{P}(\mathcal{W} \times \mathcal{W})$ by

$$\mathcal{H}_1(\mu)(b, a) := \sum_{l(b) \in \mathbb{N}} \mu_2(a, l(b))l(b), \quad \text{for } a, b \in \mathcal{W}$$

and write $\mathcal{H}_1(\mu) = \mu_1$. We define the function $\mathcal{H}: \mathcal{P}(\mathcal{W} \times \mathbb{N}) \to \mathcal{P}(\mathcal{W}) \times \hat{\mathcal{P}}(\mathcal{W} \times \mathcal{W})$ by $\mathcal{H}(\mu) = (\mathcal{H}_1(\mu), \mathcal{H}_2(\mu))$ and note that $\mathcal{H}(M_X) = (L^2_X, L^1_X)$. Observe that $\mathcal{H}_1$ is a continuous function but $\mathcal{H}_2$ is discontinuous in the weak topology. In particular, in the summation $\sum_{l(b) \in \mathbb{N}} \mu_2(a, l(b))l(b)$ the
function \( l(b) \) may be unbounded and so the functional \( \mu \to \mathcal{H}_2(\mu) \) would not be continuous in the weak topology. We call a pair of measures \((\varpi, \mu) \in \tilde{\mathcal{P}}(W \times W) \times \mathcal{P}(W \times \mathbb{N}^W)\) sub-consistent if
\[
\mathcal{H}_2(\mu)(b, a) \leq \varpi(b, a), \quad \text{for all } a, b \in W,
\]
and consistent if equality holds in (2.2). For a measure \( \varpi \in \tilde{\mathcal{P}}_*(W^2) \) and a measure \( \omega \in \mathcal{P}(W) \), define
\[
\mathcal{J}^d_C(\varpi \| \omega) := H(\varpi \| \rho(d)C\omega \otimes \omega) + \rho(d)\|C\omega \otimes \omega\| - \|\varpi\|,
\]
where the measure \( C\omega \otimes \omega \in \tilde{\mathcal{P}}(W \times W) \) is defined by \( C\omega \otimes \omega(a, b) = C(a, b)\omega(a)\omega(b) \) for \( a, b \in W \).

It is not hard to see that \( \mathcal{J}^d_C(\varpi \| \omega) \geq 0 \) and equality holds if and only if \( \varpi = \rho(d)C\omega \otimes \omega \). For every \((\varpi, \mu) \in \tilde{\mathcal{P}}_*(W \times W) \times \mathcal{P}(W \times \mathbb{N})\) define a probability measure \( Q = Q[\varpi, \mu] \) on \( W \times \mathbb{N} \) by
\[
Q(a, \ell) := \mu_1(a) \prod_{b \in W} e^{-\varpi(a, b)_{\mu_1(a)}} \frac{1}{\ell(b)!} \left( \frac{\varpi(a, b)}{\mu_1(a)} \right)^{\ell(b)}, \quad \text{for } a \in W, \ell \in \mathbb{N}.
\]

We now state the principal theorem in this section the LDP for the empirical pair measure and the empirical neighbourhood measure.

**Theorem 2.3.** Suppose that \( X \) is a coloured RGG graph with colour law \( \mu \) and connection radii \( r_n : W \times W \to [0, 1] \) satisfying \( nr_n^3(a, b) \to C(a, b) \) for some symmetric function \( C : W \times W \to [0, \infty) \) not identical to zero. Then, as \( n \to \infty \), the pair \((\mathcal{L}_X^1, \mathcal{M}_X)\) satisfies an LDP in \( \tilde{\mathcal{P}}_*(W \times W) \times \mathcal{P}(W \times \mathbb{N}) \) with good rate function
\[
J(\varpi, \mu) = \begin{cases} 
H(\mu \| Q) + H(\mu_1 \| \nu) + \frac{1}{2} \mathcal{J}^d_C(\varpi \| \mu_1) & \text{if } (\varpi, \mu) \text{ consistent and } \mu_1 = \varpi, \\
\infty & \text{otherwise.}
\end{cases}
\]

**Remark 1** Note that on typical coloured RGG graph we have, \( \omega = \mu_1, \varpi = \rho(d)C \mu \otimes \mu \) and
\[
\mu(a, \ell) = \nu(a) \prod_{b \in W} e^{-\rho(d)C(a, b)\nu(b)} \frac{(\rho(d)C(a, b)\nu(b))^{\ell(b)}}{\ell(b)!}, \quad \text{for all } (a, \ell) \in W \times \mathbb{N}.
\]
This is the law of a pair \((a, \ell)\) where \( a \) is distributed according to \( \mu \) and, given the value of \( a \), the random variables \( \ell(b) \) are independently Poisson distributed with parameter \( \rho(d)C(a, b)\nu(b) \). Hence, as \( n \to \infty \), the empirical neighbourhood measure \( \mathcal{M}_X(a, \ell) \) converges to \( \mu(a, \ell) \) in probability.

### 3. Proof of Theorem 2.3

For any \( n \in \mathbb{N} \) we define
\[
\mathcal{P}_n(W) := \{ \omega \in \mathcal{P}(W) : n\omega(a) \in \mathbb{N} \text{ for all } a \in W \},
\]
\[
\tilde{\mathcal{P}}_n(W \times W) := \{ \varpi \in \tilde{\mathcal{P}}_*(W \times W) : \frac{n}{1+\mathbb{I}_{a=b}} \varpi(a, b) \in \mathbb{N} \text{ for all } a, b \in W \}.
\]

We denote by \( \Theta_n := \mathcal{P}_n(W) \times \tilde{\mathcal{P}}_n(W \times W) \) and \( \Theta := \mathcal{P}(W) \times \tilde{\mathcal{P}}_*(W \times W) \). With
\[
P^{(n)}(\omega_n, \varpi_n) := \mathbb{P}\{ \mathcal{M}_X = \mu_n \mid \mathcal{H}(\mathcal{M}_X) = (\omega_n, \varpi_n) \},
\]
\[
P^{(n)}(\omega_n, \varpi_n) := \mathbb{P}\{ (\mathcal{L}_X^1, \mathcal{L}_X^2) = (\omega_n, \varpi_n) \}
\]
the joint distribution of \( \mathcal{L}_X^1, \mathcal{L}_X^2 \) and \( \mathcal{M}_X \) is the mixture of \( P^{(n)}(\omega_n, \varpi_n) \) with \( P^{(n)}(\omega_n, \varpi_n) \) defined as
\[
d\tilde{\mathcal{P}}^{(n)}(\omega_n, \varpi_n, \mu_n) := dP^{(n)}(\omega_n, \varpi_n)(\mu_n) dP^{(n)}(\omega_n, \varpi_n).
\]

(Biggins, Theorem 5(b), 2004) gives criteria for the validity of large deviation principles for the mixtures and for the goodness of the rate function if individual large deviation principles are known. The following three lemmas ensure validity of these conditions.
Lemma 3.1 (Doku-Amponsah, 2014b). The family of measures \( (P^n : n \in \mathbb{N}) \) is exponentially tight on \( \Theta \).

Lemma 3.2 (Doku-Amponsah & Moerters, 2010). The family of measures \( (\tilde{P}^n : n \in \mathbb{N}) \) is exponentially tight on \( \Theta \times \mathcal{P}(\mathcal{W} \times \mathbb{N}) \).

Define the function
\[
\tilde{J} : \Theta \times \mathcal{P}(\mathcal{W} \times \mathbb{N}) \to [0, \infty],
\]
where
\[
\tilde{J}((\omega, \varpi) \mu) = \begin{cases} 
H(\mu \| Q_{\text{poi}}) & \text{if } (\omega, \mu) \text{ is consistent and } \mu_1 = \omega_2 \\
\infty & \text{otherwise.}
\end{cases}
\]

Lemma 3.3 (Doku-Amponsah & Moerters, 2010). \( \tilde{J} \) is lower semi-continuous.

By (Biggins, Theorem 5(b), 2004) the two previous lemmas and the large deviation principles we have established in (Doku-Amponsah, Theorem 2.1, 2014b) and (Doku-Amponsah, Theorem 2.1, 2014c) ensure that under \( (\tilde{P}^n) \) the random variables \( (\omega_n, \varpi_n, \mu_n) \) satisfy a large deviation principle on \( \mathcal{P}(\mathcal{W}) \times \mathcal{P}(\mathcal{W} \times \mathcal{N}) \times \mathcal{P}(\mathcal{W} \times \mathbb{N}) \) with good rate function
\[
\tilde{J}(\omega, \varpi, \mu) = \begin{cases} 
H(\omega \| \nu) + \frac{1}{2} \eta_C(\varpi \| \omega) + H(\mu \| Q_{\text{poi}}) & \text{if } (\omega, \mu) \text{ is consistent and } \mu_1 = \omega_2 \\
\infty & \text{otherwise.}
\end{cases}
\]

By projection onto the last two components we obtain the large deviation principle as stated in Theorem 2.3 from the contraction principle, see e.g. (Dembo et al., 1998, Theorem 4.2.1).

4. Proof of Theorem 2.1 and Corollary 2.2

We derive the theorems from Theorem 2.3 by applying the contraction principle, see e.g. (Dembo & Zeitouni, Theorem 4.2.1, 1998). In fact Theorem 2.3 and the contraction principle imply a large deviation principle for \( D \) if it just remains to simplify the rate functions.

4.1 Proof of Theorem 2.1
Note that, in the case of an uncoloured RGG graphs, the function \( C \) degenerates to a constant \( c, L^2 = |E|/n \in [0, \infty) \) and \( M = D \in \mathcal{P}(\mathbb{N} \cup \{0\}) \). Theorem 2.3 and the contraction principle imply a large deviation principle for \( D \) with good rate function
\[
\eta_1(\delta) = \inf \{ J(x, \delta) : x \geq 0 \} = \inf \{ H(\delta \| q_x) \frac{1}{2} x \log x - \frac{1}{2} x \log \rho(d) c + \frac{1}{2} \rho(d) c - \frac{1}{2} x : \langle \delta \rangle \leq x \},
\]
which is to be understood as infinity if \( \langle d \rangle \) is infinite. We denote by \( \eta^\alpha(\delta) \) the expression inside the infimum. For any \( \alpha > 0 \), we have
\[
\eta^\alpha(\delta) = \frac{e}{2} + \frac{\delta - e}{2} \log \frac{\delta}{\rho(d)c} \geq \frac{e}{2} + \frac{\delta - e}{2} \left( \frac{\rho(d)c}{\rho(d)c} \right) + \frac{e}{2} \log \frac{\delta}{\rho(d)c} > 0,
\]
so that the minimum is attained at \( x = \langle \delta \rangle \).

4.2 Proof of Corollary 2.2
Corollary 2.2 follows from Theorem 2.1 and the contraction principle applied to the continuous linear map \( G : \mathcal{P}(\mathbb{N} \cup \{0\}) \to [0, 1] \) defined by \( G(\delta) = \delta(0) \). Thus, Theorem 2.1 implies the large deviation principle for \( G(D) = W \) with the good rate function
\[
\xi_1(y) = \inf \{ \eta_1(\delta) : \delta(0) = y, \langle \delta \rangle < \infty \}.
\]
We recall the definition of \( \eta^\alpha \) and observe that \( \xi_2(y) \) can be expressed as
\[
\xi_2(y) = \inf_{b \geq 0} \inf_{d \in \mathcal{P}(\mathbb{N} \cup \{0\})} \left\{ \frac{1}{2} c + y \log y + \frac{b^2}{2 \rho(d)c} + \sum_{k=1}^{\infty} \delta(k) \log \frac{\delta(k)}{\rho(d)c(k)} - b(1 - y) \right\}.
\]
Now, using Jensen’s inequality, we have that
\[ \sum_{k=1}^{\infty} \delta(k) \log \frac{\delta(k)}{q_0(k)} \geq (1 - y) \log \frac{(1-y)}{(1-e^{-y})}, \]
with equality if \( \delta(k) = \frac{(1-y)}{(1-e^{-y})} q_0(k) \), for all \( k \in \mathbb{N} \). Therefore, we have the inequality
\[ \inf \{ \eta(\delta) : \delta(0) = y, \langle \delta \rangle < \infty \} \geq \{ \frac{1}{2} c + y \log y + \frac{b^2}{2 \rho(d)c} + (1 - y) \log \frac{(1-y)}{(1-e^{-y})} - b(1-y) : b \geq 0 \}. \]

Let \( y \in [0, 1] \). Then, the equation \( a(1-e^{-a}) = \rho(d)c(1-y) \) has a unique positive solution. Elementary calculus shows that the global minimum of \( b \mapsto \frac{1}{2} \rho(d)c + y \log y + \frac{b^2}{2 \rho(d)c} + (1 - y) \log \frac{(1-y)}{(1-e^{-y})} - b(1-y) \) on \( (0, \infty) \) is attained at the value \( b = a \), where \( a \) is the positive solution of our equation. We obtain the form of \( \xi \) in Corollary \( 2.2 \) by observing that
\[ \frac{a(y)^2 + (\rho(d)c)^2 - 2 \rho(d)c a(y)(1-y)}{2 \rho(d)c} = \frac{\rho(d)c}{2} (2 - y) + \frac{1}{2 \rho(d)c} (a(y) - \rho(d)c(1-y))^2. \]

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