The ballistic transport instability in Saturn’s rings – III. Numerical simulations

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ABSTRACT

Saturn’s inner B-ring and its C-ring support wavetrains of contrasting amplitudes but with similar length-scales, 100–1000 km. In addition, the inner B-ring is punctuated by two intriguing ‘flat’ regions between radii 93 000 and 98 000 km in which the waves die out, whereas the C-ring waves coexist with a forest of plateaus, narrow ringlets, and gaps. In both regions, the waves are probably generated by a large-scale linear instability whose origin lies in the meteoritic bombardment of the rings: the ballistic transport instability. In this paper, the third in a series, we numerically simulate the long-term non-linear evolution of this instability in a convenient local model. Our C-ring simulations confirm that the unstable system forms low-amplitude wavetrains possessing a preferred band of wavelengths. B-ring simulations, on the other hand, exhibit localized non-linear wave ‘packets’ separated by linearly stable flat zones. Wave packets travel slowly while spreading in time, a result that suggests the observed flat regions in Saturn’s B-ring are shrinking. Finally, we present exploratory runs of the inner B-ring edge which reproduce earlier numerical results: ballistic transport can maintain the sharpness of a spreading edge while building a ‘ramp’ structure at its base. Moreover, the ballistic transport instability can afflict the ramp region, but only in low-viscosity runs.

Key words: instabilities – waves – planets and satellites: rings.

1 INTRODUCTION

Planetary rings of low and intermediate optical depth are vulnerable to the ballistic transport instability (BTI) which arises from the continual bombardment of ring particles by hypervelocity micrometeoroids. Ejecta released via these impacts reaccrete on to the ring at various radii, and thus redistribute mass and angular momentum (Durisen 1984; Ip 1984; Lissauer 1984). A small positive perturbation in surface density will change the ring’s local transport properties, and if the overdense region releases less material than it absorbs relatively, then it will grow and an instability results (Durisen 1995; Latter et al. 2012, hereafter Paper 1). The BTI favours long length-scales $l_0 \sim 10$–$10^3$ km and long time-scales $t_\star \sim 10^6$–$10^7$ yr. Thus, the 100-km waves in the inner B-ring (between radii 93 000 and 98 000 km) and the 1000-km undulations in the C-ring (between 77 000 and 86 000 km) are possible manifestations of its non-linear development (figs 13.17 and 13.13 in Colwell et al. 2009; Charnoz et al. 2009).

This is the third paper in a series exploring the BTI’s generation of wave-like structure. The first paper presented a convenient local model with which to study the problem and worked through the BTI’s linear theory (Paper 1). The second paper established semi-analytically that the instability could sustain families of non-linear travelling wavetrains (Latter et al. 2014, hereafter Paper 2). For C-ring parameters, these waves saturate at low amplitude. For B-ring parameters, the ring exhibits bistability, with the system falling into one of two linearly stable states: the background homogeneous state (a ‘flat zone’) or a large-amplitude wave state (a ‘wave zone’). Both results are consistent with Cassini data and strengthen the connection between the observed wave features and the BTI.

In this paper, our earlier results are verified and extended with time-dependent simulations. Our numerical algorithm exploits the convolution form of the ballistic transport integrals and, as a result, can easily evolve the system on extremely long time-scales $\sim 1000 t_\star$ and length-scales $\sim 1000 l_0$. We calculate structure formation in three contexts: (a) low-optical depth models of the C-ring, (b) bistable models of the B-ring and (c) the spreading and structure of a sharp edge.

The low-optical depth simulations fulfil most of the expectations of Paper 2. After an initial period of wave competition, the system settles upon a linearly stable low-amplitude wavetrain. Our simulations, however, possess translational symmetry, a special constraint...
not shared by the C-ring. In order to eliminate its effects, we perform additional runs with ‘buffered’ boundaries that work similarly to outgoing wave conditions. Low-amplitude wavetrains dominate these simulations as well, but their dynamics is more complicated; in particular, wave activity can propagate out of the ring entirely leaving behind a state of very low amplitude and long wavelength. Both sets of simulations are consistent with Cassini observations of the very long C-ring undulations, and reinforce the attribution of these features to the BTI.

Our second group of simulations probes the dynamics of hysteresis in models of the B-ring. As the system supports both a stable homogeneous state and a wave state, we set up an initial condition in which these two states occupy different regions within the computational domain. We find that the ‘wave zones’ behave like wave packets, moving at the group velocity of their constituent waves. In addition, the wave packets spread, a non-linear effect due to the variation of the group velocity through the packet. This behaviour agrees qualitatively with the observations, though interesting discrepancies exist, which we discuss.

Finally, exploratory runs of a spreading ring edge are presented. Starting from a step function in optical depth between very thin and thick regions, we confirm the finding of Durisen et al. (1992, hereafter D92) that ballistic transport maintains the sharpness of an edge as it spreads, while building at its foot a ‘ramp’ (i.e. a region with a shallow optical depth gradient). Lower viscosity runs, however, indicate that this ramp is unstable to the BTI, with growing modes reaching large amplitudes. As such structures are absent in the Cassini observations, it may be possible to roughly constrain ring properties from this result.

The organization of the paper is as follows. In Section 2, we summarize the mathematical details of the physical model as well as our numerical method. Section 3 presents simulations that approximate the C-ring, Section 4 deals with the B-ring simulations, while Section 5 contains results on spreading ring edges. We draw our conclusions and point to future work in Section 6.

2 GOVERNING EQUATIONS AND NUMERICAL SET-UP

2.1 Mathematical formalism

Our simulations are undertaken in the shearing box, which is a local model of a planetary ring that ignores curvature effects and global gradients in ring properties. The box is centred at a fixed radius $r_0$, with $x$ denoting the local radial coordinate. Its radial size is $L$, and we must supply (potentially unrealistic) boundary conditions.

Following Paper 1, the time evolution of the dynamical optical depth $\tau$ is given by the following integro-differential equation:

$$\partial_t \tau = \mathcal{I} - \mathcal{J} + \frac{1}{2} \partial_x \left( \mathcal{K} + \mathcal{L} \right) + \partial_x (\mu \partial_x \tau),$$

(1)

where $\mu$ is a measure of the relative strength of viscous over ballistic transport. In contrast to Papers 1 and 2, we permit $\mu$ to vary with space in some runs. The integral operators $\mathcal{I}$ and $\mathcal{J}$ describe the direct transfer of mass by ballistic processes, while $\mathcal{K}$ and $\mathcal{L}$ describe the transfer of angular momentum. In equation (1), the units of time and space have been chosen so that the characteristic ballistic throw length $l_{\mu}$ and the characteristic ballistic erosion time $t_{\mu}$ are 1 (see Papers 1 or 2 for their exact definitions).

The integral operators in the governing equation may be compactly expressed using convolutions

$$\mathcal{I} = P \cdot (R \ast f), \quad \mathcal{J} = R \cdot (P \ast \tilde{f}),$$

(2)

$$\mathcal{K} = P \cdot (R \ast g), \quad \mathcal{L} = R \cdot (P \ast \tilde{g}).$$

(3)

Appearing in these expressions are the three key functions describing ballistic transport: the rate of ejecta emission per unit time and area $R(\tau)$, the probability of mass absorption from incoming ejecta $P(\tau)$, and the ejecta distribution function $f(x)$, defined so that $f(x) \, dx$ is the proportion of material thrown distances between $x$ and $x + dx$. The function $g$ is defined by $g(x) = x f(x)$, and the tilde denotes a reflection, so that $\tilde{f}(x) = f(-x)$ and $\tilde{g}(x) = g(-x)$. Note that more generally, $P$ is a function of both $\tau$ at the absorbing radius and $\tau$ at the emitting radius.

As in Papers 1 and 2, the functional forms for ejecta emission $R$ and absorption $P$ are

$$P(\tau) = 1 - \exp(-\tau/\tau_p),$$

(4)

$$R(\tau) = 0.933 \left[ 1 + \left( \frac{\tau}{\tau_s} - 1 \right) \exp(-\tau/\tau_s) \right],$$

(5)

the latter taken from Cuzzi and Durisen (1990). We fix the parameters so that the reference optical depths are $\tau_p = 0.5$ and $\tau_s = 0.28$. The throw distribution calculated by Cuzzi & Durisen (1990) is approximated by an off-centred Gaussian profile

$$f(\xi) = \frac{1}{\sqrt{2\pi d^2}} \exp \left[ - (\xi - \xi_0)^2 / (2d^2) \right],$$

(6)

in which we set the offset to $\xi_0 = 0.5$ and the standard deviation to $d = 0.6$.

2.2 Numerical approach

We employ a third-order Runge–Kutta time stepper to evolve equation (1) while computing the spatial derivatives and integrals with a Fourier pseudo-spectral method. Consequently, the radial domain is partitioned into $N$ nodes, each equally spaced by $\Delta x$. The $x$-derivatives of $\tau$ are calculated in Fourier space, using fast Fourier transforms, and the integrals are also evaluated in Fourier space using the convolution theorem. The non-linear terms, equations (2) and (3), are computed in real space. Note that by using the convolution theorem, we greatly speed up the algorithm because the ballistic transport integrals are only $O(N \log N)$ tasks, rather than $O(N^2)$. For typical simulations, this means each time-step is accomplished two orders of magnitude faster than if conventional quadrature were utilized.

Viscous diffusion imposes the primary limitation on the time-step $\Delta t$ in our problem. In order to avoid numerical instability, $\Delta t$ must satisfy a Courant condition,

$$\Delta t < C \left( \frac{\Delta x^2}{\max[\mu]} \right),$$

(7)

where $C$ is a constant, approximately 0.254 for a third-order Runge–Kutta scheme on a periodic domain (Canuto et al. 2006). We set $\Delta t$ in our simulations to the right-hand side multiplied by a small ‘safety factor’ $\approx 0.1$.

The grid spacing $\Delta x$ is limited by the (physical) viscous length $l_{\mu}$. The Gibbs phenomenon and worse ensues when the latter falls beneath $\Delta x$, because unsolvable gradients develop which can sharpen into discontinuities. In our units, the viscous length can be approximated by $l_{\mu} \approx \sqrt{\mu}$, and we take $\Delta x$ to be an order of
magnitude smaller. For additional safety, in some runs we de-alias the solution, using the 2/3 rule (Canuto et al. 2006).

2.3 Boundary conditions
We apply two different boundary conditions to equation (1), either (a) periodic boundaries or (b) ‘buffered’ periodic boundaries. The former forces the time-dependent solution $\tau(x, t)$ to satisfy

$$\tau(0, t) = \tau(L, t).$$

(8)

Buffered boundaries, on the other hand, permit information to freely leave the domain without reentering it from the opposite boundary. Periodic boundary conditions are retained, but we block wave transmission at the boundaries by increasing $\mu$ in two buffer zones encasing the two ends. Waves incident upon such zones decay rapidly to zero before reentering the domain on the other side. A convenient model profile for $\mu$ is

$$\mu = \frac{0.2 + \mu_0}{2} \left[ \tanh(x - L + l_B) - \tanh(x - l_B) + 2 \right] + \mu_0,$$

(9)

where $l_B$ is the radial size of each buffer and $\mu_0$ is the value of $\mu$ outside the buffers. This model lets $\mu$ rapidly increase to 0.2 in the buffers, about twice the value that can sustain BTI, as shown in fig. 6 of Paper 1.

2.4 Initial conditions and parameters
The initial condition is usually small-amplitude white noise atop the constant equilibrium state $\tau = \tau_0$, though in certain simulations, such as in numerical tests and with B-ring parameters, we employ the exact non-linear solutions computed in Paper 2. In the simulations of ring edges, we set $\tau$ to a boxcar profile

$$\tau = \begin{cases} 1.5, & 30 < x < 60 \\ 0.05, & \text{otherwise}, \end{cases}$$

(10)

and focus exclusively on the evolution of the inner edge near $x = 30$. The simulation is terminated once the inner and outer edge spread so far that they interact. When $L = 100$ this occurs only at extremely late times. The choice of upper and lower optical depths ($1.5$ and $0.05$, respectively) ensures that the BTI fails to appear in either location for $\mu = 0.02$ (see fig. 6 in Paper 1).

The two main physical parameters are $\tau_0$ and the parameter $\mu$. The former we set to either C-ring or inner B-ring values $\tau_0 \sim 0.1$ or $\tau_0 \sim 1$, and in almost all runs $\mu$ is fixed at 0.025, independent of $\tau$. The main numerical parameters are $L$ and $\Delta x$. Because $l_B \approx 0.16$, we set $\Delta x \approx 0.02$. The domain size $L$ we vary, but note that for the local model to be a good approximation $L \ll r_0$. Our model space unit is $l_{th}$, which falls between 50 and 500 km (Paper 2). At $r_0 \sim 10^3$ km, it follows that $L$ should take values below 200 and 20$l_{th}$, respectively.

2.5 Numerical tests
We present two tests that demonstrate the accuracy of our numerical tool. The first one checks that the code reproduces the linear growth rates of low-amplitude disturbances. We seed two wavelengths of an unstable mode of specified wavenumber $k$ and amplitude $\sim 10^{-6}$, and then evolve it forward until it grows by at least two orders of magnitude. We subsequently measure the growth rate and compare with the analytic dispersion relation of the linear modes (cf. equation 49 in Paper 1). Our results are plotted in Fig. 1. The agreement is good, with the relative errors below 1 per cent.

3 C-RING SIMULATIONS: LOW-AMPLITUDE WAVETRAINS
In this section, we explore the low-\(\tau\) regime relevant to the C-ring. As discussed in Paper 2, the BTI in this setting forms steady wave-trains of small amplitude, and we find that it is the emergence and competition between these structures that characterizes the non-linear evolution of the instability.
3.1 Periodic boundaries: free wavetrains

Our initial run employs periodic boundary conditions, and sets $\tau_0 = 0.175$ with a constant $\mu = 0.025$. The value for $\tau$ is slightly larger than observed but presents results that are easier to interpret. We restrict the size of the domain $L$ to 20 times $l_0$. As argued in Paper 2, structures in Saturn’s C-ring indicate that $l_0 \sim 500$ km, and so $20l_0$ approximately encompasses the extent of wave activity in the C-ring. It is also near the local model’s limit of applicability. The initial condition is small-amplitude white noise $\sim 10^{-5}$, and the simulation is run for 1000 erosion times.

Results are plotted in Figs 3 and 4. The first figure shows the evolution of the perturbation amplitude, which we measure by $\max[\tau - \tau_0]$. The second figure presents six snapshots of $\tau$ at different (unequally spaced) times. The early stages of the evolution witness the independent and exponential growth of competing linear BTI modes, with e-folding times $\sim 100$ (see fig. 1 in Paper 2). By $t = 150$, the system is dominated by the fastest growing mode, associated with the wavenumber $k = 2.827$ (thus nine wavelengths fit into the domain). This mode, low in amplitude and still roughly sinusoidal in shape, can be observed in the first panel of Fig. 4.

The exponential growth ceases after some 250 erosion times, as the system approaches the exact steady non-linear wavetrain solution associated with $k = 2.827$. However, as panels 2–5 in Fig. 4 indicate, the waveform exhibits significant modulational perturbations, conspicuous as early as $t = 220$. These correspond to the action of a secondary instability upon the $k = 2.827$ wavetrain; as shown in section 4 in Paper 2, the wavetrain is too short to be stable for these parameters. Not long after $t = 400$, the secondary instability destroys the $k = 2.827$ solution, which is superseded by the longer (and linearly stable) $k = 2.513$ solution (panel 6). Over the course of the next few hundred erosion times, the amplitude slowly relaxes to the value predicted by Paper 2 (roughly 0.19, see Fig. 3).

This general pattern of behaviour is reproduced by all parameter choices we have tried, and is not limited to the small $\tau_0$ regime. At first, the fastest growing mode dominates the evolution and the system approaches a wavetrain of the same wavenumber; but because this wavetrain is always unstable, the system eventually migrates away and seeks a linearly stable longer-wavelength solution. Depending on the size of the domain (and hence the number of available modes), this process can take one step (as above) or several steps. This is in contrast to analogous behaviour in the viscous overstability, where the wavelength selection procedure is more involved – mainly because the fastest growing wavelength and the first stable wavelength are much further apart (Latter & Ogilvie 2009, 2010). Similar behaviour also occurs when the initial condition is a long unstable non-linear wavetrain (with $k < 2.225$). These solutions...
break up relatively quickly and the system settles on a stable shorter wavelength wavetrain.

3.2 Buffered boundaries

The periodic box simulations of Section 3.1 indicate that the BTI, when present, always takes the system to a uniform and stable travelling wavetrain. This outcome, however, could be viewed as an artefact of the boundary conditions and the limited extent of the box. When periodic boundary conditions are imposed, the system ‘senses’ the translational symmetry of the domain after a sufficiently long time and is thus attracted to the steady wavetrain solutions admitted by this symmetry. But in the real rings, there is no radial periodicity or global translational symmetry. As a consequence, steady uniform wavetrains are not exact non-linear solutions globally – though they are approximate solutions locally – and hence cannot function as global attractors. The real rings will certainly exhibit non-linear wavetrains, and the secondary instabilities that assail them, yet the larger scale dynamics may be rather different to that shown in Section 3.1. They may instead exhibit a competition between different wavetrain solutions, propagating towards or away from each other, partly controlled by underlying gradients and inhomogeneities in the background state.

This subsection probes some of this behaviour by breaking the translational symmetry of the box. Buffer regions are imposed across both boundaries, as described in Section 2.3. These eliminate the attracting stable wavetrain solutions from the phase space, thus permitting a more realistic and interesting set of dynamics to develop. Our main simulation takes a mean optical depth \( \tau_0 = 0.175 \), a large domain \( L = 100 \) and buffers of size \( l_B = 10 \). White noise of moderate amplitude (\( \sim 0.01 \)) is seeded in order to speed up the evolution. The resulting evolution is summarized in Fig. 5, where we show six snapshots.

The fastest growing modes dominate the initial stages of the evolution, and after growing to appreciable amplitudes they undergo complicated interactions involving beating patterns and the formation of localized wave packets (panels 1 and 2). The group velocity of these waves \( c_p \) is negative and relatively large \( \sim -0.5 \) (see fig. 3b in Paper 2). As a result, disturbances propagate to the inner boundary on relatively short time-scales where, over the course of the simulation, they mostly disappear into the buffer zone. Ultimately, the system settles into a quasi-steady state characterized by low-amplitude long-wavelength waves with positive phase speeds (panel 6). These waves emerge from a ‘source’ at the inner buffer and decay to zero as their crests propagate outwards. Though their phase speed \( c_p \) is positive, the group velocity of these long waves remains negative (fig. 3b, Paper 2); thus disturbances at larger \( x \) cannot be replenished by activity in the inner regions. Activity persists in the inner regions, on the other hand, as ‘residue’ of the non-linear behaviour in the first 100 erosion times.

The key feature controlling these dynamics is the ‘convective’, as opposed to ‘absolute’, nature of the BTI: unstable waves do not grow in place, but travel as they grow. A disturbance will reach large amplitudes but it may leave the region of interest before it does so, especially if the growth time (the inverse of the linear growth rate \( s \)) is longer or similar to \( L/c_p \), i.e. the time it takes for information to traverse the domain. As a consequence, any given ring region may eventually return to its undisturbed state. (This scenario is in marked contrast to the unbuffered periodic simulation of Section 3.1, where travelling disturbances are never lost and indeed can positively reinforce themselves.) In the buffered simulation described by Fig. 5, \( 1/s \sim L/c_p \). As a result, the disturbances only just attain sufficiently large amplitudes for non-linear interactions to ensue, and hence for activity to be sustained, though at a fairly low level.

To further test this idea, we conducted additional simulations, one with \( L \) reduced by half and another with smaller amplitude initial conditions \( \sim 10^{-4} \). In both simulations, growing disturbances propagate out of the active zone before they achieve significant amplitudes, and eventually the ring returns to a near homogeneous state. We also ran a simulation illustrating the opposite extreme, in which the growth time \( 1/s \) is much smaller than \( L/c_p \); we set \( \tau \) to 0.5 and hence \( s \sim 0.4 \), an order of magnitude larger than in Fig. 5. The results of this last simulation are summarized in Fig. 6, which displays three snapshots at different times. Initially, the fastest-growing modes dominate the active region, leading to the emergence of wave packets travelling inwards. After an internal

![Figure 5](https://academic.oup.com/mnras/article-abstract/441/3/2773/1111510)

Figure 5. Six snapshots of a run with buffered domains. Here, \( L = 100 \) and \( l_B = 10 \). Main parameters are \( \tau = 0.175 \) and \( \mu_B = 0.025 \). Waves crests propagate inwards in panels 1 and 2, and mainly outwards in panels 3–6. The group velocity of the disturbances, however, always points inwards.
wavelength selection process (panel 2), the system settles on a long wavetrain with $k \approx 1.96$ (panel 3) and positive $c_p$. The key point is that, because of the large growth rates, disturbances attain large amplitudes before hitting the inner buffer, allowing the non-linear dynamics sufficient time to develop and sustain activity throughout the entire central region.

We expect some of this behaviour to characterize the low-amplitude undulations in the C-ring, which occur in a regime near marginal stability and in a relatively small domain ($\sim 20 l_0$). If left to evolve independently for sufficient time, it is possible that the present undulations may simply propagate out of the region and disappear at the inner edge of the C-ring. Alternatively, larger amplitude disturbances may have already left the system, leaving behind the low-amplitude waves we see today; these may then correspond to the undulations in panel 6 in Fig. 5. A third, more likely scenario, is that the C-ring suffers a continual supply of noise which reseeds the BTI and replenishes the undulations. A complication in all three hypotheses is that the waves travel through an environment punctuated by disruptive features, such as plateaus, ringlets, and gaps, as well as possible large-scale gradients in ring properties (figs 13.17 and 13.21 in Colwell et al. 2009; Fillachione et al. 2013; Hedman et al. 2013). Finally, we note the longer wavelengths produced by our simulations ($k \approx 1.5$), which suggest an estimate for $l_0$ closer to 300 km, shorter than estimates based on linear stability (see section 4 in Paper 2).

4 INNER B-RING SIMULATIONS: WAVETRAIN PULSES

This section presents simulations that approximate conditions in the inner B-ring. As discussed in Paper 2, this region is of special interest because it can exhibit bistability, whereby the system falls into a homogeneous ‘flat’ state, or a small set of large-amplitude ‘wave’ states. It is then possible that the ring radially breaks up into adjoining flat and wave zones, with the boundaries between the regions undergoing their own dynamics. We focus on this scenario here, and show that this is indeed possible.

Our fiducial B-ring run takes parameters $\tau_0 = 1.3$ and $\mu = 0.025$, which admit the linearly stable homogeneous state $\tau = \tau_0$. As explored in section 3.3 in Paper 2, these parameters also support a number of travelling non-linear wavetrains. Our initial condition comprises a portion of a wavetrain, possessing $k = 2.5$, inserted between $x = 10$ and 30. The remainder of the domain is set to $\tau = \tau_0$. The two ends of the wavetrain are abruptly smoothed to the homogeneous value over a few grid cells. The initial condition is hence a ‘wave packet’ or ‘wave zone’. The size of the box is $L = 50$, and we reinstate the periodic boundary conditions (without buffers).

Six snapshots of the evolution are displayed in Fig. 7 at equally spaced times. We find that within the packet the individual peaks of the wavetrain travel to the left with the phase speed $c_p \approx -0.014$ predicted by the calculations of Paper 2. Moreover, the wavepacket as a whole travels to the left with the faster group speed $c_g = \frac{d\omega}{dk} \approx -0.058$, associated with $k = 2.5$. As the wave packet propagates it also spreads, which is clear from the first and last snapshots in Fig. 7: after one traversal of the domain, the rear of the packet has lagged behind the packet’s leading front. This is because the packet’s component wavelengths at its front are shorter than at its rear. Consequently $c_g$ decreases throughout the packet. The rear moves slower, and the structure as a whole spreads.

The wave packets, understood as homoclinic orbits in phase space, share a number of features with localized oscillatory states in a variety of driven and/or bistable systems. Notable examples include the complex Ginzburg Landau equation (Aranson & Kramer 2002; Burke et al. 2008), plane Couette flow (Schneider et al. 2010), and magnetoh convection (Buckley & Bushby 2013). However, because $c_p$ varies throughout the packet and $\neq c_g$, our BTI structures resist the usual techniques of dynamical systems (e.g. Burke & Knobloch 2007). In particular, as the packet spreads in time, there are no distinct states comprising the classical ‘snakes and ladders’ pattern in the phase space. Instead, over time, the system progresses through a continuum of such states.

Qualitatively, these results agree relatively well with Cassini observations, which reveal that the inner B-ring radially splits into two ‘flat’ zones, where the photometric $\tau$ is constant, circumscribed by three ‘wave’ zones, exhibiting non-linear wavetrains of wavelength $\sim 100$ km and amplitude $\sim 1$ in $\tau$ (figs 13.11 and 13.13 in Colwell et al. 2009). We hence identify the flat zones with the inactive homogeneous state $\tau = \tau_0$ and the wave zones with travelling (and spreading) wave packets. Our results then suggest that the two flat zones are shrinking and will ‘evaporate’ entirely in $\sim 100 t_e$. Of course, the Janus/Epimetheus 2:1 inner Lindblad resonance situated between the two flat zones does complicate things, but the overall agreement encourages us to view the inner B-ring as controlled by the BTI’s bistable dynamics.

A number of intriguing discrepancies remain, however. As discussed in Paper 2, the troughs of the theoretical wave profiles (measured in dynamical $\tau$) are lower than those exhibited by the observed waves (measured in photometric $\tau$). The disagreement could disappear once the differences between photometric and dynamical optical depth are corrected for, but this is not yet certain. Another, potentially related, problem issues from the discrepancy between the observed mean photometric optical depth in the flat and wave zones (Colwell et al. 2009). If this indeed corresponds to significant differences in the surface density (hence dynamical $\tau$), then our picture of the dynamics will need to be modified. A final point is that it can be difficult to seed a travelling wave packet in our theoretical simulations; an arbitrary localized large-amplitude perturbation is usually insufficient. A wavelike or very large-amplitude disturbance does better. Sources for appropriate disturbances may be present in the Janus/Epimetheus resonance, the inner B-ring edge, or the...
The ballistic transport instability

Figure 7. Six snapshots showing the evolution of BTI wavepacket propagating through a bistable ring. The background is described by $\tau_0 = 1.3$, $\mu = 0.025$, and the numerical parameters are $L = 50l_0$ and $N = 1024$. The wave packet and its constituent waves travel to the left at different speeds. In addition, the front of the packet travels faster than the rear of the packet, leading to its spreading. An arrow is included indicating the direction of propagation.

transition to the high-$\tau$ regions in the mid-B-ring, but again this is uncertain.

5 SIMULATIONS OF THE INNER B-RING EDGE

We finish our numerical study with a handful of exploratory simulations of a sharp ring edge spreading under the action of viscosity and ballistic transport. So far we have reproduced the dynamics of the BTI mostly in isolation of strong inhomogeneities. In this section, we see how it fares in the presence of a dramatic variation in $\tau$.

We employ an initial condition representing an isolated wide ringlet possessing extremely sharp edges, as described by equation (10) in Section 2.4. In order to simplify the numerical results, we take the optical depth of the ringlet to be sufficiently high so that no wavetrains are possible, thus $\tau = 1.5$. Similarly, in the surrounding medium, we take $\tau$ to be sufficiently low so that the BTI is suppressed. We then focus exclusively on the dynamics of the inner edge of the ringlet, and treat it as an approximation to the inner edge of the B-ring. The viscosity, represented by the parameter $\mu$, is expected to vary between the low- and high-$\tau$ regions in our set-up (Daisaka et al. 2001). However, for our main runs, we let it remain a constant; additional simulations in which $\mu$ is a power law in $\tau$ do not exhibit qualitatively different results. In keeping with previous work, we associate $t_e$ with the erosion time of the high-$\tau$ region.

Our first runs take a relatively large $\mu$, equal to 0.65. Consequently, the BTI will not work for any $\tau$ (see fig. 2 in Paper 2). This set-up permits us to study the influence of ballistic transport on ring spreading in isolation of the instability itself. For comparison, we also perform a simulation with ballistic transport completely deactivated, leaving the edge to spread via viscous diffusion alone. Fig. 8 shows two snapshots taken at the same time from these two simulations. While the purely viscous simulation exhibits the self-similar tanh solution of the classical diffusion equation, the BT simulation reveals a more complex structure. A characteristic ‘ramp’ profile develops at the foot of the edge, exhibiting a shallower gradient than the tanh profile. Meanwhile the edge itself remains sharper in comparison to the pure viscous spreading case.

Figure 8. The $\tau$ profile of two ring edges at $t = 43.56$. The initial condition in both cases is a step function, but the left-hand panel shows an edge evolved under the action of viscous diffusion alone, while the right shows spreading under the action of both ballistic and viscous transport. In both simulations, $\mu = 0.065$ and is constant.
These results should be compared with the more viscous runs of D92; see their figs 4 and 5 (for which \( Y = 3 \times 10^5 \) and \( 1 \times 10^5 \), respectively). The most viscous D92 run (their fig. 4) resembles our Fig. 8(a), presumably because BT is sub-dominant. The corresponding \( \mu \) is difficult to calculate because of uncertainty in the value of \( l_0 \), and also because the D92 viscosity is not constant. Its mean value we estimate to be \( \gtrsim 0.1 \), consistent with our results. D92’s intermediate run (their fig. 5) yields an edge profile similar to our Fig. 8(b), and indeed possesses a comparable mean \( \mu \) (roughly 0.05). The main difference is the absence of a ‘hump’ on the high-\( \tau \) side of the edge. In fact, our \( \mu = 0.065 \) simulation does exhibit a hump at earlier times, but it has diffused away by \( t = 10 \). Perhaps this small discrepancy is due to our adoption of the simpler absorption probability function \( P \) (see discussion in Papers 1 and 2). Overall, however, we confirm the findings of D92 that ballistic transport can maintain the sharpness of a spreading edge, and concurrently develop a ‘ramp-like’ feature at its foot.

The ramp in Fig. 8(b) possesses values of \( \tau \) which place it in the favourable range for BTI (see fig. 2 in Paper 2). However, our choice of \( \mu \) prevents instability. To test whether the ramp region can indeed support instability, we reduce \( \mu \) to a value of 0.025 and redo the numerical calculation. In Fig. 9, we present six snapshots of the ensuing evolution. Indeed, as early as \( t = 18 \), a growing undulation appears at the foot of the edge (panel 2). As the ramp expands and generates greater \( \tau \), more and more of the region becomes unstable leading to the emergence of further growing modes. Throughout this phase, the edge itself remains relatively sharp and now exhibits the hump on the high-\( \tau \) side. At later times the ensuing wavetrain amplitudes reach levels \( \sim 1 \), and the waves propagate slowly inwards.

Similar growing modes are not observed in D92’s low viscosity run (see their fig. 6, which has \( Y = 3 \times 10^5 \)), even though it possesses a comparable mean \( \mu \) (\( \sim 0.01 \)). The most prominent feature in the D92 run, in fact, is the high-\( \tau \) wavetrain, which we have suppressed by our choice of \( \tau \) in the optically thick ringlet. Note, however, that the D92 simulation is only run till \( t = 41 \), and so it is possible that appreciable BTI waves would have emerged at later times. Moreover, more recent runs by the same research group witness growing ‘humps’ in the ramp for certain parameter choices (Durisen, private communication). We conclude that ramp stability is sensitive to input physics and its parameters.

Observations of the inner B-ring edge do not indicate the existence of BTI; the ramp, in fact, exhibits a remarkably unblemished linear profile (fig. 13.21, Colwell et al. 2009). It is also unlikely that the C-ring plateaus are related to the emergence of these BTI undulations. The morphologies of the two are dissimilar and there is no positive gradient in background \( \tau \) where plateaus exist, in contrast to our simulations; the observations indicate that ramps and plateaus appear separately. We hence conclude that the BTI is suppressed at the inner B-ring edge, though the responsible physical process is unclear, and obviously absent from our model. It is possible that our choice of distribution function \( f \), or the parameters that appear in it, may unrealistically boost instability in this region. But more work, with further refinements, is needed before this can be verified.

Finally, we speculate that, because the ramp increases in size as the edge spreads, the ramp width could be used as a diagnostic for the ring spreading time, possibly even allowing researchers to construct past morphologies and probe ring formation scenarios. For example, the observed ramp is \( \approx 1000 \) km, or possibly \( \approx 10 \) \( l_0 \); our simulations indicate that such a structure would take \( \approx 100 \) \( t_e \) to form, if the edge was initially very sharp.

6 CONCLUSION

In this paper, we have developed a reliable and efficient numerical tool with which to simulate the ballistic transport process in planetary rings. We have subsequently reproduced the non-linear evolution of the BTI in models of both the B-ring and C-ring, in addition to the spreading of a ring edge.

Both our B-ring and C-ring simulations validate the predictions of Paper 2’s semianalytic theory. C-ring simulations saturate by forming stable low-amplitude wavetrains within a narrow range of preferred wavelengths. But because of the translational symmetry of our local model this outcome is not a global solution to the real.
radially structured, C-ring. In order to eliminate some of the unrealistic effects of the translational symmetry, additional simulations were conducted with buffered boundaries. Near marginality, as in the C-ring, BTI modes possess low growth rates and yet retain relatively large group velocities. As a consequence, the BTI’s ‘convective’ character becomes important. Our buffered simulations show that unstable disturbances can propagate out of the region of interest before their non-linear dynamics develop and sustain appreciable amplitudes. Unless continually fed new perturbations, the BTI may saturate at a very low level of activity. It is possible that the C-ring has fallen into such a state. However, C-ring undulations compete with other features, such as plateaus, ringlets, and gaps that may interfere with their evolution, or alternatively help seed fresh BTI modes. Undoubtedly, the dynamics are complicated in this region and more irregular than predicted by the local shearing sheet with pure periodic boundary conditions.

Our simulations of the inner B-ring show that it is possible that the system splits into stable wave zones and stable homogeneous zones, in agreement with Cassini data. Moreover, the wave zones propagate through the homogeneous regions as independent wave packets while simultaneously spreading. Both the observed larger and smaller flat spots may evaporate in a time \( \sim 100 \tau_e \sim 10^7 \text{--} 10^9 \text{yr.} \) However, as discussed in Paper 2, the morphologies of the theoretical and observed B-ring waves exhibit troubling discrepancies. The troughs of the former are too deep, while the mean \( \tau \) in the former varies between flat and wavy zones. The disagreement could issue from the simple fact that the theoretical and observational profiles are measured in dynamical and photometric optical depths, respectively. But further work is needed to establish this directly.

Finally, we conducted exploratory simulations of the inner B-ring edge. We find that ballistic transport does not arrest the viscous spreading of the edge, but resculpts it as it spreads. In particular, ballistic transport forms a ramp-like feature at its base, while maintaining the sharpness of the edge – in agreement with previous simulations. We also find, in low-viscosity runs, that the ramp structure is susceptible to BTI. But as the observed ramp does not exhibit wave (or any other) features, we conclude that the BTI is suppressed in this region by physics not captured in our fiducial simulations.

In the future, we hope to refine our physical model and conduct more detailed simulations, especially of spreading edges. There are two obvious improvements: the inclusion of a \( \tau \)-dependent viscosity, and an absorption probability \( P \) that depends on \( \tau \) at both the emitting and absorbing radius. Preliminary simulations indicate little qualitative change when \( \mu \) is an increasing function of \( \tau \). On the other hand, a better \( P \) model may alter our results more significantly, not only for the C-ring dynamics but for conditions at the inner B-ring edge. Indeed, it may aid in the suppression of the BTI in the ramp region. Unfortunately the more realistic \( P \) precludes use of the convolution theorem and its many computational benefits. An intermediate model, however, need only incorporate the first few terms in an expansion of \( P \) (i.e. the first few ejecta ring-plane crossings), and the convolution theorem could then be applied to each term.

Simulations with the improved model may better reproduce, and help explain, the observed morphologies of spreading edges. They may also constrain the spreading time of the inner A-ring and B-ring by examining the widths of their ramp regions. Other targets for research include the C-ring plateaus. BTI does not generate these features and, being too narrow, nor does it emerge within them. But ballistic transport may sculp their structure, and in particular sharpen their front and rear edges. Future numerical work here would complement previous investigations which used a descendent of the D92 code (Estrada & Durisen 2010). It could also explore a possible connection between the C-ring plateaus and similarly sized narrow rings, such as the \( \epsilon \) ring in the Uranian system.

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