Polygons in hyperbolic geometry 2:
Maximality of area
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1. Introduction

The topic of this part is the maximum question for the area of polygons in the hyperbolic plane with fixed sidelengths. As a main result it will be shown: Among all polygons in the hyperbolic plane with fixed positive sidelengths there exist polygons of maximal area. Each such maximal polygon is either oriented-convex and cocyclic or else collinear. In the first case the maximal area is positive, in the second case it is zero. A more detailed version will be given in Theorem 4.9. As a corollary one obtains that among the non-collinear polygons the only copies for which a rigidity can be hoped for are the oriented-convex cocyclic ones. The cocyclicity means that the vertices are situated on a distance circle, a distance line or a horocycle, the three types of cycles in the hyperbolic plane. The phenomenon of different circle types stands in salient contrast to the Euclidean case and pays for various difficulties in the hyperbolic discussion.

The corresponding result in the Euclidean plane has been discussed several times in the literature: In Yaglom/Boltjanski [1951] a proof of the cocyclicity for maximal polygons in \( \mathbb{R}^2 \) is given within the class of simply connected polygons, using the general isoperimetric inequality. It is stated there that a proof without this tool would be extremely difficult. Other treatments for the Euclidean case are in Blaschke [1956], Kryzhanovskij [1959], and Knebelman [1941]. Of course, there are some ideas from the Euclidean situation which are also worthwhile in the non-Euclidean case, but new phenomena and difficulties occur. In particular, this is true for the notion of area itself. Also familiar constructions from Euclidean geometry are no more available. For instance there is no circumferential angle theorem in the hyperbolic circle theory and no similarities exist for figures in the hyperbolic space.

In classical expositions of hyperbolic geometry, there prevails the relation of area to the angle sum. However, this relation is directly applicable only for polygons which are bounding because it rests on the Gauss/Bonnet integral theorem. The polygons to be considered here are more general: no a-priori assumptions on their form (convexity, simple closedness, etc.) have to be made. Therefore, the angle sum is hard to handle for our purpose. Since the sidelengths are strongly involved it is more advisable to view the whole problem in the context of distance geometry in the sense of Menger [1928] and Blumenthal [1970]. So an effort is necessary to express the area by different means, in particular by distances instead of angles. This is achieved by a more analytical definition of the area functional in combination with explicit expressions to be found in a paper of Bilinski [1969].

For basics on hyperbolic geometry we refer to part 1. In particular the circle model of Cayley/Klein will be considered throughout. It has the advantage that geodesics are Euclidean lines such that hyperbolic convexity properties are very near to their Euclidean relatives. This is not true for the angles but they are almost not entering anyway.
The polygon area

The Cayley/Klein model of the hyperbolic plane consists of the open unit ball \( \mathbb{B} \) in \( \mathbb{R}^2 \) where the hyperbolic lines are just the chords of the horizon \( \mathbb{S} := \partial \mathbb{B} \) (which itself doesn’t belong the hyperbolic plane). In this model, the riemannian metric on \( \mathbb{B} \) is given by

\[
\begin{align*}
    ds^2 &= \frac{1}{(1 - \xi^2 - \eta^2)^2} \left( (1 - \eta^2)d\xi^2 + 2\xi\eta d\xi d\eta + (1 - \xi^2)d\eta^2 \right),
\end{align*}
\]

where \((\xi, \eta)\) are cartesian coordinates in \( \mathbb{B} \). The area element of this metric, viewed as a 2-form, sounds

\[
\mu = \frac{d\xi \wedge d\eta}{\sqrt{1 - \xi^2 - \eta^2^3}}.
\]

It is the exterior derivative of a certain 1-form, namely

\[\mu = d\omega, \quad \omega := \frac{-\eta \, d\xi + \xi \, d\eta}{1 - \xi^2 - \eta^2 + \sqrt{1 - \xi^2 - \eta^2}}.\]

This is the decisive access to the area here. The form \( \omega \) plays the same role in the hyperbolic plane as the form \(-\eta \, d\xi + \xi \, d\eta\) does in the Euclidean plane. It admits the calculation of the area \( F(A) \) of any compact subset \( A \subset \mathbb{B} \) with ‘good’ oriented boundary \( \partial A = K \) by a curve integral:

\[
(2.1b) \quad F(A) = \int_K \omega,
\]

in complete analogy to the Leibniz formula in the Euclidean case. A ‘good’ boundary is e.g. a closed continuous \( C^1 \)-chain without selfintersections, in particular a polygon chain without selfintersections (see Cartan [1967], Sects. 4.2–4.4).

However, the integral in (2.1) is much more general. It yields, for any oriented chain \( K \) composed of compact \( C^1 \)-arcs with real weights, a real value which depends linearly under the addition and scalar multiplication of such chains. This value is well defined insofar as it is independent of the representation of the chain. In this generality, the value can also be zero or negative, depending on the orientation of the chain. So we are dealing with signed areas.

By Bilinski [1969], Eqn. (6.2), the signed area of a 3-gon (i.e. a triangle) in \( \mathbb{B} \) with vertices \( A, B, C \) is explicitly given by

\[
(2.2) \quad F(ABC) = 2 \arctan \left[ \frac{[a, b, c]}{\langle a, b \rangle + \langle b, c \rangle + \langle c, a \rangle + 1} \right],
\]

where \( a, b, c \in \mathbb{R}^3 \) are the normalized point vectors of the vertices. For the pseudo-Euclidean scalar product involved here and its rules see part 1, Sect. 2.

Now, combining the chain integral from (2.1b) with the expression (2.2), one obtains the following explicit representation for the signed area of any \( n \)-gon \( P := Z_1 \ldots Z_n \):

\[
(2.3) \quad F(P) = 2 \sum_{k=1}^{n} \arctan \left[ \frac{[z, z_k, z_{k+1}]}{\langle z, z_k \rangle + \langle z, z_{k+1} \rangle + \langle z_k, z_{k+1} \rangle + 1} \right].
\]
Here, $Z$ can be any point in $\mathbb{B}$, and all point vectors occurring must be normalized. The properties of the chain integral (2.1b) ensure that this expression is indeed independent of the choice of the ‘origin’ $Z$. One may call Eqn. (2.3) the parachute formula since all connecting triangles with $Z$ are summed together with the right account of signs. The corresponding Euclidean formula is much simpler and is sometimes named after Gauß.

This independence immediately yields the following facts:

### 2.1. Lemma.

Each oriented-convex polygon has positive area. Each collinear polygon has area zero.

**Proof.** The special choice $Z := Z_1$ in (2.3) leads to

$$F(P) = 2 \sum_{k=2}^{n-1} \arctan \frac{[z_1, z_k, z_{k+1}]}{(z_1, z_k) + (z_1, z_{k+1}) + (z_k, z_{k+1}) + 1}.$$  

In the first case, every numerator in the sum is positive, so is $F(P)$. In the second case every numerator is 0, so is $F(P)$. □

Also, the existence of polygons with maximal area can be deduced from (2.3), using the general maximum principle:

### 2.2. Lemma.

For each $n$-gon there exists an $n$-gon of maximal area with same sidelengths.

**Proof.** A $n$-gon $Z_1 \ldots Z_n$ can be represented by a point in the cartesian product $\mathbb{B}^n := \mathbb{B} \times \ldots \times \mathbb{B}$ ($n$ factors), say equipped with the maximum metric $d_n$. Without loss of generality it is possible to fix the point $Z_1$ for all polygons to be considered. As is obvious from Eqn. (2.3), the area $F(Z_1, \ldots, Z_n)$ depends continuously on $(Z_2, \ldots, Z_n)$. The definition set consists of all points $(Z_2, \ldots, Z_n) \in \mathbb{B}^{n-1}$ with $d(Z_k, Z_{k+1}) = L_k = \text{const.}, k = 1, \ldots, n$. It is compact, namely bounded and closed: The boundedness follows from the estimate

$$d_{n-1}((Z_2, \ldots, Z_n), (Z_1, \ldots, Z_1)) = \max\{d(Z_2, Z_1), d(Z_3, Z_1), \ldots, d(Z_n, Z_1)\} \leq (n - 1) \max\{L_1, \ldots, L_n\},$$

and the closedness is deduced from the continuity of the functions $(Z_2, \ldots, Z_n) \mapsto d(Z_k, Z_{k+1})$, $k = 1, \ldots, n$. Thus $F(Z_1, \ldots, Z_n)$ has on this definition set (which is not vacuous) a finite maximum. □

The main question is of course: How do the maximal $n$-gons look like? As in the Euclidean case the final answer will be: In general they are cocyclic. But on the way to this goal one needs an analytical characterization for the cocyclicity, at least for low $n$. Non-collinear triangles always have a circum-circle. So the next interesting case are the quadrangles. For three and four points in the hyperbolic plane there are several identities and characterizations which will prepare the answer (see Sect. 3).
2.3. Remarks.

(i) In Eqn. (2.2), the absolute value of the determinant $|a, b, c|$ is expressible by Gram’s identity in terms of the pairwise scalar products of the point vectors $a, b, c$, so finally in terms of the sidelengths. With some fancy conversions for hyperbolic functions this is converted to a formula of classical L’Huilier type:

\[(2.4a) \quad |F(ABC)| = 4 \arctan \sqrt{\tanh \frac{S}{4} \tanh \frac{D_1}{4} \tanh \frac{D_2}{4} \tanh \frac{D_3}{4},}\]

where

\[S := L_1 + L_2 + L_3\]
\[D_1 := -L_1 + L_2 + L_3\]
\[D_2 := L_1 - L_2 + L_3\]
\[D_3 := L_1 + L_2 - L_3,\]

thus expressing the absolute value of the area of a triangle solely by its sidelengths $L_1, L_2, L_3$.

The quantities $D_1, D_2, D_3$ are just the differences from the triangle inequalities, so $D_1 \geq 0, D_2 \geq 0, D_3 \geq 0$. The appropriate definition set of the right hand side of (2.4a) is

\[\Lambda := \{(L_1, L_2, L_3) \mid L_1 > 0, L_2 > 0, L_3 > 0, D_1 \geq 0, D_2 \geq 0, D_3 \geq 0\}.\]

Denote by $H : \Lambda \to \mathbb{R}$ the L’Huilier-function, given by the right hand side of (2.4a). For $D_1 > 0, D_2 > 0, D_3 > 0$, $H$ depends real-holomorphically on $(L_1, L_2, L_3)$. At the boundary of $\Lambda$ ($D_1 = 0$ or $D_2 = 0$ or $D_3 = 0$) $H$ is still continuous but no more differentiable. But just this singularity will be helpful later on for the growth of the area. The growth of $H$ is controlled by the following limit relations:

\[(2.5) \quad \lim_{X \uparrow L_2 + L_3} \frac{\partial H}{\partial X}(X, L_2, L_3) = -\infty \quad \text{if} \quad L_2 > 0, L_3 > 0\]
\[(2.6) \quad \lim_{X \downarrow L_1 - L_2} \frac{\partial H}{\partial X}(L_1, L_2, X) = \infty \quad \text{if} \quad L_1 > L_2 > 0.\]

They immediately follow from the corresponding partial derivatives in the interior of $\Lambda$.

(ii) Besides the L’Huilier expression, there are some other formulas for the absolute value of the triangle area which will be needed; see Bilinski [1969], Eqns. (11.3) and (11.5):

\[(2.7) \quad |F(ABC)| = 2 \arctan \frac{\sinh L_1 \sinh L_2 \sin \gamma}{\cosh L_1 + \cosh L_2 + \cosh L_3 + 1},\]
\[(2.8) \quad |F(ABC)| = 2 \arccos \frac{\cosh L_1 + \cosh L_2 + \cosh L_3 + 1}{4 \cosh \frac{L_1}{2} \cosh \frac{L_2}{2} \cosh \frac{L_3}{2}},\]

where $\gamma \in [0, \pi]$ is the angle opposite to the side of length $L_3$. Instead of the ‘over-determined’ formula (2.7) one may use a variant which arises by substituting $\cosh L_3$ according to the
cosine law:

\[(2.9) \quad |F(ABC)| = 2 \arctan \frac{\sinh L_1 \sinh L_2 \sin \gamma}{(1 + \cosh L_1)(1 + \cosh L_2) - \sinh L_1 \sinh L_2 \cos \gamma}.\]

Eqn. (2.9) is the sole instance where angles enter the game. It will be needed for ‘parallelogram-like’ quadrangles which occur as exceptional cases in the area maximizing problem.

3. Identities for triples und quadruples

Here, certain properties of triples and quadruples of points will be expressed solely by distances. In particular this applies to the property of cocyclicity. The following abbreviations will be used:

\[(3.1) \quad K_{PQ} := \cosh d(P, Q), \quad S_{PQ} := \sinh \frac{d(P, Q)}{2}, \quad P, Q \in \mathbb{H}.\]

In most cases, the hyperbolic distances \(d(P, Q)\) enter in form of the sinh-quantities \(S_{PQ}\), but in a few cases the expressions become easier with the cosh-quantities \(K_{PQ}\).

**Point triples**

The vertices of a triangle are collinear if their images on the quadric shell \(\mathbb{H} \subset \mathbb{R}^3\) lie in a vector plane. If the vertices are not collinear then they are always cocyclic because their images lie in a plane of \(\mathbb{R}^3\) which doesn’t contain \(0\). The image points in \(\mathbb{R}^3\) are also not collinear since \(\mathbb{H}\) cuts any straight line of \(\mathbb{R}^3\) at most twice. So the plane and hence the circum-circle is uniquely determined. Certainly, this circle needs not to have a center, i.e. it can also be a distance line or a horocycle. Which case occurs can be read off solely from the sidelengths.

**3.1. Lemma.** Let \(A, B, C\) be non-collinear points in \(\mathbb{H}\) and \(a, b, c\) their image points on \(\mathbb{H}\). Then the plane in \(\mathbb{R}^3\), affinely spanned by \(a, b, c\) has the equation

\[(3.2) \quad \langle u, x \rangle = p \quad \text{with} \quad u := a \times b + b \times c + c \times a, \quad p := [a, b, c].\]

With the corresponding distance quantities \(3.1\), the following representations hold true:

\[(3.3) \quad \langle u, u \rangle = 8(S_{AB}^2S_{BC}^2 + S_{BC}^2S_{CA}^2 + S_{CA}^2S_{AB}^2) - 4(S_{AB}^4 + S_{BC}^4 + S_{CA}^4)
\] 

\[= 4(S_{AB} + S_{BC} + S_{CA})(S_{AB} + S_{BC} - S_{CA})(S_{BC} + S_{CA} - S_{AB})(S_{CA} + S_{AB} - S_{BC})
\][\(a, b, c\)]

\[= 16S_{AB}^2S_{BC}^2S_{CA}^2 + \langle u, u \rangle .\]

**Proof.** The vector \(u\) from (3.2) doesn’t vanish because

\[a \times b + b \times c + c \times a = (b - a) \times (c - a).\]
The plane affinely spanned by \(a, b, c\) has indeed the equation \(\langle u, x \rangle = p\) with the values of \(u\) and \(p\) as in (3.2) because each of \(a, b, c\) satisfies it, e.g.

\[
\langle a \times b + b \times c + c \times a, a \rangle = \langle b \times c, a \rangle = [a, b, c].
\]

This proves the first part.

From \(u\) in (3.2) follows for \(\langle u, u \rangle\) by the Lagrange identity, since \(\|a\| = \|b\| = \|c\| = 1:\)

\[
\langle u, u \rangle = 1 - \langle a, b \rangle^2 + 1 - \langle b, c \rangle^2 + 1 - \langle c, a \rangle^2 \\
+ 2(\langle a, b \rangle \langle b, c \rangle - \langle c, a \rangle) + 2(\langle b, c \rangle \langle c, a \rangle - \langle a, b \rangle) + 2(\langle c, a \rangle \langle a, b \rangle - \langle b, c \rangle).
\]

Substituting the scalar products according to the formula

\[
(3.4) \quad \langle a, b \rangle = \cosh d(A, B) = 1 + 2 \sinh^2 \frac{d(A, B)}{2} = 1 + 2S_{AB}^2
\]

and also its cyclic extensions then yields the first representation in (3.3) and by factorizing the second one.

Finally, \([a, b, c]^2\) is by Gram’s identity expressible in terms of scalar products and then in the same manner by the corresponding sinh-quantities. The third line follows from this by comparison with the second line.

In part 1, Theorem 4.3 it has been stated that the sidelengths already determine on which circle type the vertices of an oriented-convex cocyclic polygon are situated. Now, for triangles, this can be made completely explicit:

### 3.2. Corollary

Define for non-collinear points \(A, B, C\) in \(\mathbb{B}\) the invariant

\[
\Delta := (S_{AB} + S_{BC} - S_{CA})(S_{BC} + S_{CA} - S_{AB})(S_{CA} + S_{AB} - S_{BC}).
\]

Then the quantity \(\Delta\) which solely depends on the pairwise distances of the points determines the type of the circum-circle of \(A, B, C\), namely:

\[
\Delta > 0 \iff \text{the circum-circle is a distance circle} \\
\Delta < 0 \iff \text{the circum-circle is a distance line} \\
\Delta = 0 \iff \text{the circum-circle is a horocycle}.
\]

**Proof.** Comparison of the middle equation in (3.3) with the classification of the circle types in part 1, Sect. 2. \(\square\)

### Point quadruples

Let \(a, b, c, e\) points in the pseudo-Euclidean space \(\mathbb{R}^3\). The decision whether these points are coplanar, i.e. are contained in a plane, depends on the quadruple quantity

\[
(3.5) \quad [a, b, c, e] := [a, b, c] - [b, c, e] + [c, e, a] - [e, a, b].
\]
3. Identities for triples and quadruples

Namely, the vectors \( b - a, c - a, e - a \) are linearly dependent iff the determinant \( [b - a, c - a, a - e] \) vanishes. Expanding this determinant by the multilinear and alternating rules just results in the quadruple quantity. Thus:

\[
(3.6) \quad a, b, c, e \in \mathbb{R}^3 \text{ coplanar } \iff [a, b, c, e] = 0.
\]

On the other hand, the vectors \( a, b, c, e \) are always linearly dependent. This can be formulated by means of the hyperbolic analogue of the so-called Cayley/Menger determinant (see Blumenthal [1970], Ch. IV, § 40 and Ch. XII, § 106). Since, here, the relation to the quadruple quantity is needed, an equivalent expression will be deduced directly as follows: A ‘universal’ dependency relation formulated without any scalar product is \( V(a, b, c, e) = 0 \), where

\[
(3.7) \quad V(a, b, c, e) := [a, b, c]e - [b, c, e]a + [c, e, a]b - [e, a, b]c.
\]

3.3. Lemma (17-identity).

For any points \( A, B, C, E \) in \( \mathbb{B} \) one always has:

\[
0 = K_{AB}K_{CE}^2 + K_{AC}^2K_{BE}^2 + K_{AE}^2K_{BC}^2
- 2K_{AB}K_{AC}K_{BE}K_{CE} - 2K_{AB}K_{AE}K_{BC}K_{CE} - 2K_{AC}K_{AE}K_{BC}K_{BE}
+ 2K_{AB}K_{AC}K_{BC} + 2K_{AB}K_{AE}K_{BE} + 2K_{AC}K_{AE}K_{CE} + 2K_{BC}K_{BE}K_{CE}
- K_{AB}^2 - K_{AC}^2 - K_{AE}^2 - K_{BC}^2 - K_{BE}^2 - K_{CE}^2 + 1.
\]

Proof. This is just the relation \( \langle V(a, b, c, e), V(a, b, c, e) \rangle = 0 \), followed by expansion with means of Gram’s identity, and expressing the occurring scalar products according to the first part of (3.4). \( \square \)

The announced relation then sounds:

3.4. Lemma. Let \( A, B, C, E \) be points in \( \mathbb{B} \) and \( a, b, c, e \) their images in \( \mathbb{H} \). Then between the quantities (3.5) and (3.7) the following identity holds true:

\[
\langle V(a, b, c, e), V(a, b, c, e) \rangle =
4(S_{AB}S_{CE} + S_{AC}S_{BE} + S_{AE}S_{BC})[a, b, c, e]^2 +
64(S_{AC}S_{BE} - S_{AE}S_{BC} - S_{AB}S_{CE})(S_{AB}S_{CE} - S_{AC}S_{BE} - S_{AE}S_{BC})
(S_{AE}S_{BC} - S_{AC}S_{BE} - S_{AB}S_{CE}).
\]

Proof. If the computation of Lemma 3.3 is continued by replacing the cosh-quantities by the
Then there hold the inequalities
\begin{align*}
\Delta_1 := S_{AC}S_{BE} - S_{AE}S_{BC} - S_{AB}S_{CE} \\
\Delta_2 := S_{AB}S_{CE} - S_{AC}S_{BE} - S_{AE}S_{BC} \\
\Delta_3 := S_{AE}S_{BC} - S_{AC}S_{BE} - S_{AB}S_{CE}.
\end{align*}
(3.11)
Then there hold the inequalities
\begin{align*}
\Delta_1 &\leq 0, \\
\Delta_2 &\leq 0, \\
\Delta_3 &\leq 0.
\end{align*}
(3.12)
If \(A, B, C, E\) are not collinear then \(A, B, C, E\) are cocyclic if and only if, in (3.12), the equals sign occurs at least once, i.e.
\begin{align*}
\Delta_1 = 0 &\text{ or } \Delta_2 = 0 &\text{ or } \Delta_3 = 0.
\end{align*}
(3.13)
Proof.

For (3.12): Eqn. (3.8) says indeed

\[(3.14) \quad -(S_{AB}S_{CE} + S_{AC}S_{BE} + S_{AE}S_{BC})[a, b, c, e]^2 = 16\Delta_1\Delta_2\Delta_3,
\]

hence always

\[(3.15) \quad \Delta_1\Delta_2\Delta_3 \leq 0.
\]

Moreover

\[(3.16) \quad \Delta_1 + \Delta_2 = -2S_{AE}S_{BC} \leq 0
\]

\[(3.16) \quad \Delta_2 + \Delta_3 = -2S_{AC}S_{BE} \leq 0
\]

\[(3.16) \quad \Delta_1 + \Delta_3 = -2S_{AB}S_{CE} \leq 0.
\]

Case \(\Delta_1\Delta_2\Delta_3 < 0\): If one of these factors were positive, say \(\Delta_1 > 0\) then, from (3.15), the other two must be of different sign, so another factor has to be positive, say \(\Delta_2 > 0\). Then, from (3.16), a contradiction can be read off. Hence the assertion in this case, and indeed \(\Delta_1 < 0, \Delta_2 < 0, \Delta_3 < 0\).

Case \(\Delta_1\Delta_2\Delta_3 = 0\): At least one of these factors vanishes, say \(\Delta_1 = 0\). Then from (3.16):

\(\Delta_2 \leq 0, \Delta_3 \leq 0.
\)

For the remaining assertion:

If the non-collinear points \(A, B, C, E\) are cocyclic then the points \(a, b, c, e\) are coplanar in \(\mathbb{R}^3\), thus from (3.6) and (3.14):

\(\Delta_1\Delta_2\Delta_3 = 0.
\)

For the converse, assume \(\Delta_1\Delta_2\Delta_3 = 0\) and consider the two cases for the affine hull \(A\) of \(a, b, c, e\) in \(\mathbb{R}^3\): (a) \(\dim A = 1\); (b) \(\dim A \geq 2\).

In case (a), there exist two different points among \(a, b, c, e\), say \(a \neq b\) in \(\mathbb{H}\), and then \(c, d \in a \vee b\). Since any straight line in \(\mathbb{R}^3\) cuts \(\mathbb{H}\) at most twice: \(c, e \in \{a, b\}\). So \(A, B, C, E\) are collinear: Case (a) cannot happen.

In case (b), there exist three points among \(a, b, c, e\) in general position, say \(a, b, c\), a fortiori pairwise different. Then \(S_{AB}S_{CE} + S_{AC}S_{BE} + S_{AE}S_{BC} > 0\) because not both of \(S_{CE}, S_{BE}\) can vanish. Thus, by (3.14): \([a, b, c, e] = 0\). The points \(a, b, c, e\) affinely span a plane in \(\mathbb{R}^3\) which doesn’t contain \(0\) since \(A, B, C, E\) are not collinear. So \(A, B, C, E\) are cocyclic.

The Ptolemy equations (3.13) have the disadvantage that each of them contains all six pairwise distances while a quadrangle is generally determined by five distances. The following results work against this disadvantage.

3.6. Corollary. If the non-collinear points \(A, B, C, E \in \mathbb{B}\) are cocyclic then at least one of the following equations is valid:

\[(3.17) \quad (S_{AB}S_{BC} + S_{AE}S_{CE})S_{AC}^2 = (S_{AB}S_{CE} + S_{AE}S_{BC})(S_{AB}S_{AE} + S_{BC}S_{CE})
\]

\[(3.18) \quad -(S_{AB}S_{BC} - S_{AE}S_{CE})S_{AC}^2 = (S_{AB}S_{CE} - S_{AE}S_{BC})(S_{AB}S_{AE} - S_{BC}S_{CE}).
\]
Also, at least one of the two equations which arise from (3.17), (3.18) by permuting the points A, B, C, D holds true, in particular at least one of the equations

\[(3.19) \quad (S_{AB}S_a + S_{BC}S_C)v^2 = (S_{AB}S_C + S_{BC}S_a)(S_{AB}S_B + S_{AE}S_C)\]

\[(3.20) \quad -(S_{AB}S_a - S_{BC}S_C)v^2 = (S_{AB}S_C - S_{BC}S_a)(S_{AB}S_B - S_{AE}S_C).\]

is valid.

Proof. Clearly, it suffices to proof that (3.17) or (3.18) holds.

These equations arise by eliminating \(S_{BE}\) from \([a,b,c,e] = 0 \) and \(\langle V(a,b,c,e), V(a,b,c,e) \rangle = 0\), with the left hand sides expressed by (3.10) and (3.9). Both equations are biquadratic w.r.t. variable \(S_{AC}\), i.e. they only contain \(S_{AC}^4\) and \(S_{AC}^2\). The leading coefficients are \(S_{AC}^2\) resp. \((1+S_{AC}^2)S_{AC}^2\). In case \(S_{AC} \neq 0\), the elimination can be done via the resultant of two quadratic polynomials (cf. e.g. van der Waerden [2003], § 30). The resultant comes out as

\[R := S_{AC}^4 \cdot ((S_{AB}S_B + S_{AE}S_C)S_{AC}^2 - (S_{AB}S_C + S_{AE}S_BC)(S_{AB}S_A + S_{BC}S_C))^2\]

\[\quad \cdot ((S_{AB}S_B - S_{AE}S_C)S_{AC}^2 + (S_{AB}S_C - S_{AE}S_BC)(S_{AB}S_A - S_{BC}S_C))^2.\]

This implies the assertion if \(S_{AC} \neq 0\).

In case \(S_{AC} = 0\), i.e. \(C = A\), the points \(A, B, E\) are always situated on a circle (if not collinear), and also Eqn. (3.18) is always satisfied.

Assuming convexity, one can say more:

3.7. Lemma (Perron [1964]).

For any oriented-convex cocyclic 4-gon \(ABCE\) in \(B\) there hold Eqns. (3.17) and (3.19).

Proof. Using the means of part 1, in particular Lemma 3.1, this can be done be straightforward calculation. A cocyclic 4-gon \(ABCE\) is oriented-convex iff, in a suitable representation of the circum-circle, the group parameters of the vertices are in monotonic order. E.g. for a distance circle of hyperbolic radius \(R\) in standard position, the quantities \(S_{AB}\), etc. sound by Eqn. (2.32) of part 1:

\[S_{AB} = \varrho \cdot \sin \frac{\varphi_B - \varphi_A}{2}, \ldots, \quad \varrho := \sinh R,\]

where \(\varphi_A, \ldots, \varphi_E\) are the parameter values of the points \(A, \ldots, E\) in the representation (2.22) of part 1. By \(\varphi_A < \varphi_B < \varphi_C < \varphi_E\) and \(0 < \varphi_E - \varphi_A < 2\pi\), all these sine-values are positive and, by due trigonometric conversions, both sides of (3.17) resp. (3.19) turn out to be equal, namely

\[\varrho^2 \cdot \sin^2 \frac{\varphi_C - \varphi_A}{2} \quad \text{resp.} \quad \varrho^2 \cdot \sin^2 \frac{\varphi_E - \varphi_B}{2}.\]

Similar calculations confirm Eqns. (3.17), (3.19) for distance lines and horocycles as circum-circles.

Even more important is the converse since the equations of Corollary 3.6 contain one variable less then the Ptolemy equations of Corollary 3.5.
3.8. Theorem. Let $ABCE$ be a 4-gon in $\mathbb{B}$ with $A \neq C$ such that $B$ and $E$ lie on different sides of the diagonal line $A \lor C$. Then, the relation

$$S_{AC}^2 = \frac{(S_{AB}S_{CE} + S_{AE}S_{BC})(S_{AB}S_{AE} + S_{BC}S_{CE})}{S_{AB}S_{BC} + S_{AE}S_{CE}},$$

implies that the 4-gon $ABCE$ is cocyclic.

Proof. By Theorem 5.3 of part 1, there exists an oriented-convex cocyclic 4-gon $A'B'C'E'$ with same sidelengths as $ABCE$. Its diagonal length $d(A', C')$ is, by Lemma 3.7, computed from the sidelengths $d(A', B')$, $d(B', C')$, $d(C', E')$, $d(A', E')$ by the same formula as, by assumption (3.21), $d(A, C)$ is calculated from the sidelengths $d(A, B)$, $d(B, C)$, $d(C, E)$, $d(A, E)$. This implies $d(A', C') = d(A, C)$. So the triangles $ABC$ und $A'B'C'$ are congruent, and the same is true for the triangles $CEA$ und $C'E'A'$. Moreover, the points $B'$, $C'$ lie on different sides of the line $A' \lor C'$. For the vertex $E'$, there are left two possible positions which only differ by reflection on $A' \lor C'$. Exactly one of this positions has the property that $E$, $B$ are on different sides of the line $A' \lor C$. For this position the 4-gon $ABCE$ is congruent to the 4-gon $A'B'C'E'$. So, the 4-gon $ABCE$ must be cocyclic. □

4. Polygons with maximal area

We already know from Lemma 2.2 that the maximum problem of the polygon area for fixed sidelengths is solvable. Here, the maximal copies will be determined as expressed in detail in the main result 4.9. When we speak of maximal polygons or of the enlargement of polygons this always refers to the area functional for fixed sidelengths.

In general, the vertices of a $n$-gon will be denoted by $Z_1, \ldots, Z_n$. In order to keep compliance with the special annotations of Sect. 3, we follow the identifications $A = Z_1, B = Z_2, C = Z_3, E = Z_4$ without further mention.

The following fact is helpful in order to exclude eventual degenerate cases:

4.1. Lemma. A $n$-gon for which two non-adjacent vertices coincide can always be enlarged.

Proof. Without loss of generality, assume $Z_1 = Z_m$ for an index $m \in [3, n-1]$. The ‘residual’ polygon $Z_m, \ldots, Z_n$ can be rotated about the vertex $Z_m = Z_1$ such that the ‘arriving’ edgeline $Z_{m-1} \lor Z_m$ and the ‘leaving’ edgeline are different and also that the triangle $Z_{m-1}, Z_m, Z_{m+1}$ is negatively oriented. (The new vertices will not be denoted anew.) This process doesn’t change the sidelengths nor the areas of the partial polygons $Z_1, \ldots, Z_{m-1}$ and $Z_m, \ldots, Z_n$ and of the whole polygon. Now, the vertex $Z_m$ can be replaced by its mirror point $Z'_m$ w.r.t. the line $Z_{m-1} \lor Z_{m+1}$. This produces a positively oriented kite quadrangle $Z_{m-1}, Z'_m, Z_{m+1}, Z_m$ as can be seen from the standard position:
The kite quadrangle has positive area
\[ F' = \int_{[Z_{m-1},Z_{m}']} \omega + \int_{[Z_m,Z_{m+1}]} \omega + \int_{[Z_{m+1},Z_m]} \omega + \int_{[Z_m,Z_{m-1}]} \omega, \]
and this implies that the new \( n \)-gon is bigger than the old one since
\[ \int_{[Z_{m-1},Z_{m}']} \omega + \int_{[Z_m,Z_{m+1}]} \omega = F' - \int_{[Z_{m+1},Z_m]} \omega - \int_{[Z_m,Z_{m-1}]} \omega \]
\[ = F' + \int_{[Z_m,Z_{m+1}]} \omega + \int_{[Z_{m-1},Z_m]} \omega \]
\[ > \int_{[Z_{m-1},Z_{m}']} \omega + \int_{[Z_m,Z_{m+1}]} \omega. \]

So, we can replace the polygon \( Z_1, \ldots, Z_{m-1}, Z_m, Z_{m+1}, \ldots, Z_n \) (in the new form modified by the above rotation) by the bigger polygon \( Z_1, \ldots, Z_{m-1}, Z'_m, Z_{m+1}, \ldots, Z_n \) with same sidelengths. □

As a result, a maximal \( n \)-gon always has pairwise distinct vertices.

The next arguments concern the low cases \( n = 3, 4 \).

4.2. Lemma. If an oriented-convex triangle \( Z_1Z_2Z_3 \) with fixed sidelengths \( L_1, L_2 \) has maximal area then its parallelogram completion is cocyclic.

Of course, the third sidelength \( L_3 \) is not fixed here. The parallelogram completion is the 4-gon \( Z_1Z_2Z_3Z_4 \) such that \( Z_4 \) is the point on the opposite side of \( Z_1 \lor Z_3 \) to \( Z_2 \) with \( d(Z_3, Z_4) = L_1 \) and \( d(Z_1, Z_4) = L_2 \).

Proof of 4.2. With fixed sidelengths \( L_1, L_2 \) and variable enclosed angle \( \gamma \), the area of the triangle \( Z_1Z_2Z_3 \) is given by the function \( f : ]0, \pi[ \to \mathbb{R}^+ \):
\[ f(\gamma) := 2 \arctan \frac{\sinh L_1 \sinh L_2 \sin \gamma}{(1 + \cosh L_1)(1 + \cosh L_2) - \sinh L_1 \sinh L_2 \cos \gamma}; \]
see Eqn. (2.9). The function $f$ has the continuous extension $f(0) := f(\pi) := 0$. The derivative can easily be calculated, and the condition $f'(\gamma) = 0$ turns out to be equivalent to

\begin{equation}
\cos \gamma = \frac{\sinh L_1 \sinh L_2}{(1 + \cosh L_1)(1 + \cosh L_2)}.
\end{equation}

So, there is exactly one maximal point of $f$, namely contained in the open interval $]0, \pi[$. Combined with the cosine law, it is seen that condition (4.1) is equivalent to

\begin{equation}
\cosh d(Z_1, Z_3) = \cosh L_1 + \cosh L_2 - 1.
\end{equation}

By the definition of $Z_4$ and switching to the notation of Sect. 3 we have $K_{AE} = K_{BC}$, $K_{CE} = K_{AB}$ and, from (4.2), $K_{AC} = K_{AB} + K_{BC} - 1$. Substituting these values into the 17-identity (Lemma 3.3) yields:

\begin{align*}
(K_{AB} + K_{BC}) \cdot (K_{BE} + 1 - K_{AB} - K_{BC}) \cdot \\
(-K_{BC}^2 + K_{BC}K_{BE} + 2K_{AB}K_{BC} - K_{BC} - 2K_{BE} - K_{AB}^2 + 2 - K_{AB} + K_{BE}K_{AB}) &= 0.
\end{align*}

The vanishing of the second and third parenthesis each time leads to a conditional equation for $K_{BE}$ with the unique solution

\begin{align*}
K_{BE} &= K_{AB} + K_{BC} - 1 \quad \text{resp.} \quad K_{BE} = \frac{(K_{AB} - K_{BC})^2}{K_{AB} + K_{BC} - 2} + 1.
\end{align*}

This means, for the corresponding $S$-values

\begin{align*}
S_{BE} &= \sqrt{S_{AB}^2 + S_{BC}^2} \quad \text{resp.} \quad S_{BE} = \pm \frac{S_{AB}^2 - S_{BC}^2}{\sqrt{S_{AB}^2 + S_{BC}^2}}.
\end{align*}

Transcribing Eqn. (4.2) to the $S$-values gives

\begin{equation}
S_{AC} = \sqrt{S_{AB}^2 + S_{BC}^2}.
\end{equation}

Still, by assumption: $S_{AE} = S_{BC}$ and $S_{CE} = S_{AB}$. If this, together with (4.4) and the alternatives (4.3) is inserted into the Ptolemy conditions (3.13) it turns out that always one of these conditions is satisfied, namely the first one in case of the first alternative (4.3), the second one in case of the plus sign, and the third one in case of the minus sign of the second alternative in (4.3).

\[\square\]

4.3. Remark. The assumption that $Z_4$ should lie on the other side of $Z_2$ w.r.t. $Z_1 \lor Z_3$ has not been used in this proof. So, under the same hypothesis, also the points $Z_1, Z_2, Z_3, Z_4^*$ are cocyclic where $Z_4^*$ is the point on the same side of $Z_2$ w.r.t. $Z_1 \lor Z_3$, satisfying $d(Z_3, Z_4^*) = L_1$ and $d(Z_1, Z_4^*) = L_2$. However, this will not be needed in the sequel.

The existence of polygons with prescribed sidelengths (part 1, Theorem 5.3) can be completed to a continuous variant in case of triangles:
4.4. Lemma. Given positive real numbers $L_1, L_2, L_3$ satisfying the three triangle inequalities in a strict manner, there exist triangles in $B$ of a given orientation with these sidelengths in continuous dependency on $L_1, L_2, L_3$.

Proof. For the vertices the following ansatz can be made:

$$z_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad z_3 = \begin{pmatrix} \cosh L_3 \\ \sinh L_3 \\ 0 \end{pmatrix}, \quad z_3 = \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix}, \quad \xi > 0, \quad \xi^2 - \eta^2 - \zeta^2 = 1.$$

Then $d(Z_1, Z_3) = L_3$, and the demands $d(Z_1, Z_2) = L_1$, $d(Z_2, Z_3) = L_2$ translate to conditional equations for $\xi, \eta, \zeta$ with the following solutions:

$$\begin{align*}
\xi &= \cosh L_1 \\
\eta &= \frac{\cosh L_1 \cosh L_3 - \cosh L_2}{\sinh L_3} \\
\zeta &= \frac{\pm 2}{\sinh L_3} \sqrt{\sinh \frac{S}{2} \sinh \frac{D_1}{2} \sinh \frac{D_2}{2} \sinh \frac{D_3}{2}}
\end{align*}$$

where $S := L_1 + L_2 + L_3$, $D_1 := -L_1 + L_2 + L_3$, $D_2 := L_1 - L_2 + L_3$, $D_3 := L_1 + L_2 - L_3$.

Here, the quantities $D_1, D_2, D_3$ are just the differences from the triangle quantities as in Eqn. (2.4b), so $D_1 \geq 0$, $D_2 \geq 0$, $D_3 \geq 0$. In front of the $\zeta$ there occurs the minus resp. plus sign according as the triangle $Z_1, Z_2, Z_3$ is oriented positively or negatively. In particular, for positive $D_k$, these formulas show the continuous (even real holomorphic) dependency as asserted.

4.5. Lemma. If a 4-gon $Z_1, Z_2, Z_3, Z_4$ in $B$ is maximal then it is cocyclic or non-strict (hence collinear).

Proof. Consider in the strict case the following possibilities:

Case I.a: Neither the points $Z_1, Z_2, Z_3$ nor the points $Z_3, Z_4, Z_1$ are collinear.

Both triangles are then strict. Look at further 4-gons $Z_1', Z_2', Z_3', Z_4'$ with same sidelengths under analogous assumptions. These triangles are determined by the distance $L := d(Z_1', Z_3')$ if their orientations are kept fixed. In order to apply differential calculus on the function $L \mapsto F(Z_1', Z_2', Z_3') + F(Z_3', Z_1', Z_4')$ it must be clarified that, for any $L$ near $L_0 := d(Z_1, Z_3) > 0$, there are such further 4-gons. But this follows from Lemma 4.4, observing that the assumptions are characterized by strict inequalities between continuous functions. As a parameter for these neighbouring quadrangles one may use the distance $L$. (In the notation for these quadrangles, the prime will be skipped from now on.)

In addition, one may assume that both triangles $Z_1, Z_2, Z_3$ and $Z_3, Z_4, Z_1$ are positively oriented since otherwise a genuine enlargement of the whole area would be possible by reflections on the line $Z_1 \lor Z_3$. Then the points $Z_2$ and $Z_4$ lie on different sides of this line,
and by Eqn. (2.8) the whole area is given by
\[
F(Z_1, Z_2, Z_3, Z_4) = 2 \arccos \frac{\cosh L_1 + \cosh L_2 + \cosh L + 1}{4 \cosh \frac{L_1}{2} \cosh \frac{L_2}{2} \cosh \frac{L}{2}} + 2 \arccos \frac{\cosh L_3 + \cosh L_4 + \cosh L + 1}{4 \cosh \frac{L_3}{2} \cosh \frac{L_4}{2} \cosh \frac{L}{2}}.
\]

With the abbreviations
\[
a_k := \cosh \frac{L_k}{2}, \quad k = 1, \ldots, 4 \quad x := \cosh \frac{L}{2}
\]
and by means of the doubling formulas for the cosh-function this amounts to the discussion of the function
\[
f(x) := \arccos \frac{a_1^2 + a_2^2 + x^2 - 1}{2a_1a_2x} + \arccos \frac{a_3^2 + a_4^2 + x^2 - 1}{2a_3a_4x}
\]
or, by setting
\[
b_{12} := \frac{a_1^2 + a_2^2 - 1}{2a_1a_2}, \quad c_{12} := \frac{1}{2a_1a_2}, \quad b_{34} := \frac{a_3^2 + a_4^2 - 1}{2a_3a_4}, \quad c_{34} := \frac{1}{2a_3a_4},
\]
to
\[
f(x) = \arccos \left( \frac{b_{12}}{x} + c_{12}x \right) + \arccos \left( \frac{b_{34}}{x} + c_{34}x \right).
\]
The derivative is calculated to be
\[
f' = A_{12} + A_{34},
\]
where
\[
A_{ij} := \frac{N_{ij}}{W_{ij}}, \quad N_{ij} := \frac{b_{ij}}{x^2} - c_{ij}, \quad W_{ij} := \sqrt{1 - \left( \frac{b_{ij}}{x} + c_{ij}x \right)^2}, \quad ij \in \{12, 34\}.
\]
The requirement \(f'(x) = 0\) implies \(A_{12} - A_{12}^2 = 0\). With regard to \(W_{ij} \neq 0\) and \(x \neq 0\) this can be written as an algebraic equation \(P(x) = 0\) where the polynomial \(P(x)\) is biquadratic (only containing \(x^4\) and \(x^2\)). The search for its zeros is very much simplified when one replaces the cosh-quantities (4.6) by the corresponding sinh-quantities:

\[
(4.7) \quad S_{AB} := \sinh \frac{L_1}{2}, \quad S_{BC} := \sinh \frac{L_2}{2}, \quad S_{CE} := \sinh \frac{L_3}{2}, \quad S_{AE} := \sinh \frac{L_4}{2},
\]
\[
S_{AC} := \sinh \frac{L}{2}.
\]
The equation \(P(x) = 0\) for \(x = \cosh \frac{L}{2}\) is then equivalent to an algebraic equation \(Q(y) = 0\) for \(y = \sinh \frac{L}{2}\). The decisive point is the polynomial factorization \(Q(y) = Q_+(y) \cdot Q_-(y)\) where
\[
Q_+(y) := (S_{AB}S_{BC} + S_{CE}S_{AE})y^2 - (S_{AB}S_{AE} + S_{BC}S_{CE})(S_{AB}S_{CE} + S_{BC}S_{AE})
\]
\[
Q_-(y) := (S_{AB}S_{BC} - S_{CE}S_{AE})y^2 + (S_{AB}S_{AE} - S_{BC}S_{CE})(S_{AB}S_{CE} - S_{BC}S_{AE}).
\]
If we had $Q_+(S_{AC}) = 0$ then we could deduce the cocyclicity of the 4-gon $ABCE$ from Theorem 3.8. If $Q_+(S_{AC}) \neq 0$ and thus $Q_-(S_{AC}) = 0$ nothing can be derived in the first instance, nevertheless in connection with the next case:

**Case I.b:** Neither the points $Z_2, Z_3, Z_4$ nor the points $Z_4, Z_1, Z_2$ are collinear.

Here, completely analogous to case I.a, the maximality of the area implies $R_+(S_{BE}) = 0$ or $R_-(S_{BE}) = 0$ for the two polynomials

\[
R_+(z) := (S_{BC}S_{CE} + S_{AE}S_{AB})z^2 - (S_{BC}S_{AB} + S_{CE}S_{AE})(S_{BC}S_{AE} + S_{CE}S_{AB})
\]
\[
R_-(z) := (S_{BC}S_{CE} - S_{AE}S_{AB})z^2 + (S_{BC}S_{AB} - S_{CE}S_{AE})(S_{BC}S_{AE} - S_{CE}S_{AB}).
\]

This case arises from case I.a by cyclically proceeding in the list of vertices.

**Case I:** Each three consecutive vertices of the 4-gon are not collinear.

Then both assumptions of the cases I.a and I.b are satisfied, so $Q_+(S_{AC})Q_-(S_{AC}) = 0$ and $R_+(S_{BE})R_-(S_{BE}) = 0$. If $Q_+(S_{AC}) = 0$ or $R_+(S_{BE}) = 0$ the cocyclicity is ensured by Theorem 3.8. It remains the discussion if $Q_-(S_{AC}) = 0$ and $R_-(S_{BE}) = 0$.

If both leading coefficients of the polynomials $Q_-, R_-$ don’t vanish then the equations $Q_-(S_{AC}) = R_-(S_{BE}) = 0$ can be solved for $S_{AC}^2$ resp. $S_{BE}^2$, and the solutions imply by multiplication

\[
S_{AC}^2 \cdot S_{BE}^2 = (S_{AB}S_{CE} - S_{BC}S_{AE})^2.
\]

Thus one of the Ptolemy equations is satisfied, namely the second or the third one in (3.13).

If the leading coefficient of $Q_-$ is zero then so is the leading coefficient of $R_-$, by the specific design of $R_-$. This yields $S_{AB} = S_{AB}$ and $S_{AE} = S_{BC}$. Then Lemma 4.2 implies the cocyclicity of $ABCE$ (by applying this lemma to the triangle $ABC$).

If the leading coefficient of $R_-$ vanishes, so does the leading coefficient of $Q_-$ with the same result as before.

**Case II:** The points $Z_1, Z_2, Z_3$ are collinear and $Z_4$ is not on the line of $Z_1, Z_2, Z_3$.

Set $D_{13} := d(Z_1, Z_3) > 0$. Without loss, we may assume $F(Z_3, Z_4, Z_1) > 0$.

Three collinear and pairwise distinct points always form a non-strict 3-gon. For a non-strict 3-gon, exactly one of the triangle inequalities is converted to an equality. So there exist three subcases

(II.a) $\quad D_{13} = L_1 + L_2, \quad L_1 < L_2 + D_{13}, \quad L_2 < L_1 + D_{13}$

(II.b) $\quad D_{13} < L_1 + L_2, \quad L_1 = L_2 + D_{13}, \quad L_2 < L_1 + D_{13}$

(II.c) $\quad D_{13} < L_1 + L_2, \quad L_1 < L_2 + D_{13}, \quad L_2 = L_1 + D_{13}$

In all these cases, a suitable variation of $D_{13}$ produces a 4-gon with larger area and like sidelengths $L_1, L_2, L_3, L_4$. This is possible since the area of $Z_3Z_4Z_1$ depends differentiably on $D_{13}$ while for the area of $Z_1Z_2Z_4$ one of the relations [2.5], [2.6] is appropriate. For example, in case (II.a) one has to diminish $D_{13}$ in such a way that in all relations of this row the
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genuine smaller sign remains. Then, for the limit $D_{13} \uparrow L_1 + L_2$, Eqn. (2.5) is effective such that the area development of $Z_1, Z_2, Z_3$ exceeds that of $Z_3, Z_4, Z_1$ if $D_{13}$ is sufficiently close to $L_1 + L_2$. Similar arguments apply in the cases (II.b) and (II.c), where here $D_{13}$ must be enlarged suitably, and Eqn. (2.6) must be used instead.

So in case II the given quadrangle cannot be maximal.

**Case III:** The vertices $Z_1, Z_2, Z_3$ are collinear as well as the vertices $Z_3, Z_4, Z_1$.

Then all vertices are collinear (by $Z_1 \neq Z_3$) and the whole area is zero. On the other hand there exists a cocyclic and oriented-convex 4-gon with same sidelengths by part 1, Theorem 5.3. This 4-gon has positive area (Lemma 2.1) such that the original area was not maximal.

**Remark.** The special case of Lemma 4.5 for three *equal* sidelengths has been discussed in Leichtweiß [2005], Lemma 5.9 with a different method. Generally, in this paper, Leichtweiß solved the maximum area problem for curves in $\mathbb{B}$ of fixed constant width.

**4.6. Corollary.** If a $n$-gon $Z_1 \ldots Z_n$ in $\mathbb{B}$ has maximal area compared to all $n$-gons with same sidelengths then it is either cocyclic or else non-strict (hence collinear).

**Proof.** Consider the following cases I and II:

**Case I:** There exists an index $k \in [3, n]$ such that the polygon $Z_1 \ldots Z_k$ is not strict. Then choose $k$ maximal with this property.

If $k = n$ we are done.

Now assume $k \leq n - 1$. Then $Z_1, \ldots, Z_k$ are collinear, due to part 1, Theorem 5.3 (ii). By the maximal choice of $k$, the polygon $Z_1 \ldots Z_{k+1}$ is strict. The point $Z_{k+1}$ is then outside the line of $Z_1, \ldots, Z_k$ because otherwise the area of the polygon $Z_1, \ldots, Z_{k+1}$ could be increased (from 0 to a positive value), according to part 1, Theorem 5.3 (i). Also the polygon $Z_1, \ldots, Z_n$ could be enlarged. By the same reason, the 4-gon $Z_{k-2}Z_{k-1}Z_kZ_{k+1}$ cannot be enlarged (it cannot be collinear). But a hyperbolic circle cannot contain three pairwise distinct collinear points. Thus this situation is not existent.

**Case II:** For any index $k \in [3, n]$ the polygon $Z_1 \ldots Z_k$ is strict.

The triangle $Z_1, Z_2, Z_3$ cannot be enlarged because otherwise the whole $n$-gon could be made larger. This triangle is not collinear since otherwise a non-collinear triangle with positive area and same sidelengths would exist (part 1, Theorem 5.3 (i)). Thus, the 3-gon $Z_1Z_2Z_3$ is cocyclic, as any non-collinear 3-gon.

Now, an induction can be started:

The 4-gon $Z_1Z_2Z_3Z_4$ is strict and cannot be enlarged, hence it is cocyclic (Lemma 4.5).

The 4-gon $Z_2Z_3Z_4Z_5$ is strict. Otherwise, these points were collinear what is not possible with regard to the points $Z_3, Z_4, Z_5$; these points are cocyclic by the above argument. Moreover, the two circum-circles of $Z_1, Z_2, Z_3, Z_4$ and of $Z_2, Z_3, Z_4, Z_5$ are equal since they have the points $Z_2, Z_3, Z_4$ in common.
Obviously, one can proceed in the same way and realize successively that the 4-gons
\[ Z_3Z_4Z_5Z_6, \quad Z_4Z_5Z_6Z_7, \quad \ldots \]
are all cocyclic with a fixed circum-circle. Thus, the whole \( n \)-gon is cocyclic. \( \square \)

In order to establish the oriented convexity of the maximal polygons we use a reduction lemma which is well known in Euclidean convexity (see e.g. Moret/Shapiro [1990] or Pinelis [2006]). In the present context it reads as:

4.7. Lemma (reduction lemma). For \( n \geq 3 \) let points \( Z_1, \ldots, Z_{n+1} \in \mathbb{B} \) be given. If, for any \( k \in \{1, \ldots, n+1\} \), the points \( Z_1, \ldots, \hat{Z}_k, \ldots, Z_{n+1} \) form an oriented-convex \( n \)-gon then the points \( Z_1, \ldots, Z_{n+1} \) form an oriented-convex \( (n+1) \)-gon.

The roof over an element in a list means omission of the element.

Proof of 4.7. The argument is not very different from the Euclidean situation since the convexity notions are rather near in both geometries.

Let the given \( n + 1 \) points be denoted somewhat differently, namely as \( Z_1, \ldots, Z_n, Z^* \). By hypothesis, the polygon \( Z_1, \ldots, Z_n \) is oriented-convex, and for this polygon the cyclic index convention from part 1 will be maintained, so \( Z_{n+1} := Z_1 \), etc.

In the polygon \( Z_1 \ldots Z_nZ^* \) there is one additional point compared to the polygon \( Z_1 \ldots Z_n \) and there are two edgelines more, namely \( Z_n \lor Z^* \) and \( Z^* \lor Z_1 \). (In return, the edgeline \( Z_n \lor Z_1 \) is omitted.) In order to gain the defining conditions of oriented convexity (part 1, Sect. 3) for \( Z_1, \ldots, Z_n, Z^* \) we must show:

- the relations of the ‘new’ point \( Z^* \) with the ‘old’ edgelines, i.e.
  
  (a) \[ [Z_k, Z_{k+1}, Z^*] > 0, \quad k = 1, \ldots, n-1; \]

- the relations of the two ‘new’ edgelines with the ‘old’ points, i.e.

  (b) \[ [Z_n, Z^*, Z_k] > 0, \quad k = 1, \ldots, n-1 \]
  
  (c) \[ [Z^*, Z_1, Z_k] > 0, \quad k = 2, \ldots, n. \]

For (a): Among the old vertices one can omit one, e.g. one with an index \( j \in \{1, \ldots, k-1\} \) (only possible for \( k \geq 2 \)) or one with an index \( j \in \{k+2, \ldots, n\} \) (only possible for \( k \leq n-2 \)). In the first case, the relation \( [Z_k, Z_{k+1}, Z^*] > 0 \) is deduced from the oriented convexity of

\[ Z_1 \ldots \hat{Z}_j \ldots Z_kZ_{k+1} \ldots Z_nZ^*, \]

and in the second case from the oriented convexity of

\[ Z_1 \ldots Z_k \hat{Z}_{k+1} \ldots \hat{Z}_j \ldots Z_nZ^*. \]
4. Polygons with maximal area

For (b): Here, one succeeds by omitting an index \( j \in \{1, \ldots, n\} \setminus \{k, n\} \), meaning that one considers the oriented-convex polygon \( Z_1 \ldots \hat{Z}_j \ldots Z_n Z^* \) and deducing from this \([Z_n, Z^*, Z_k] > 0\).

For (c): Again, by omitting an index \( j \in \{1, \ldots, n\} \setminus \{1, k\} \), one deduces from the oriented convexity of \( Z_1 \ldots \hat{Z}_j \ldots Z_n Z^* \) that \([Z^*, Z_1, Z_k] > 0\).

4.8. Lemma. If a cocyclic \( n \)-gon \( Z_1 \ldots Z_n \) in \( \mathbb{B} \) has maximal area compared to all \( n \)-gons with same sidelengths then it is oriented-convex.

Proof. From the maximality follows at any rate that the vertices are pairwise distinct and from the cocyclicity that the \( n \)-gon is not collinear. Now the proof proceeds by induction on \( n \), using the foregoing Lemma 4.7.

Initial step \( n = 3 \): For three non-collinear points \( Z_1, Z_2, Z_3 \) there are only the two possibilities \([Z_1, Z_2, Z_3] < 0 \) and \([Z_1, Z_2, Z_3] > 0\). Due to the maximality, only the second possibility is left over.

Induction step from \( n \) to \( n + 1 \) for \( n = 3 \): Let the \((n + 1)\)-gon \( Z_1 \ldots Z_{n+1} \) be cocyclic and of maximal area. Consider, for any \( k \in \{1, \ldots, n + 1\} \), the points \( Z_1, \ldots, \hat{Z}_k, \ldots, Z_{n+1} \). They form a cocyclic \( n \)-gon (by \( Z_{k-1} \neq Z_{k+1} \)). It has maximal area because otherwise the polygon \( Z_1 \ldots Z_{n+1} \) could be increased. By the induction hypothesis, the \( n \)-gon \( Z_1 \ldots \hat{Z}_k \ldots Z_{n+1} \) is oriented-convex. This being true for any \( k \), the preceding reduction Lemma 4.7 shows that also the polygon \( Z_1, \ldots, Z_{n+1} \) must be oriented-convex.

□

The main theorem now arises by combining the above results with those of part 1:

4.9. Theorem. For any \( n \)-gon in the hyperbolic plane with sidelengths \( L_1, \ldots, L_n \) there exist a \( n \)-gon of maximal area with same sidelengths. This maximal copy is either cocyclic and oriented-convex or else collinear and monotonically arranged.

In both cases the maximal copy is uniquely determined up to hyperbolic motion. In the first case, its area is positive, in the second case the area vanishes.

Which case occurs is solely determined by the behaviour of the sidelengths as real numbers: If, for all \( k = 1, \ldots, n \), there holds

\[
L_k < L_1 + \cdots + \hat{L}_k + \cdots + L_n
\]

then the first case is present, otherwise the second.

Proof. The existence follows from Lemma 2.2, the alternative from Corollary 4.6, the oriented convexity from Lemma 4.8, the sign of the maximal area from Lemma 2.1 and the uniqueness from part 1, Theorem 5.3.

□
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