The Iterative Simplicity of Basic Topological Operations

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Abstract

Semigroups generated by topological operations such as closure, interior or boundary are considered. It is noted that some of these semigroups are in general finite and noncommutative. The problem is formulated whether they are always finite.

1. Iterating Closure, Interior and other Topological Operations

Let $X$ be a topological space with the set $T \subseteq P(X)$ of open subsets. For convenience, we shall denote by $c(A)$ and $i(A)$ the closure, respectively, interior of a subset $A \subseteq X$.

As is well known, some of the iterates of $c$ and $i$ play an important role in topology. For instance, a subset $A \subseteq X$ is called nowhere dense, iff $i(c(A)) = \phi$. Further, a countable union of nowhere dense subsets is called of first Baire category, the essential fact in this regard being that no complete metric space is of first Baire category.

Motivated by the above, here various iterates of $c$ and $i$, as well as of other related basic operations will be considered. In this regard we can note that all such operations are mappings of $P(X)$ into itself, thus
their compositions are associative. Consequently, we can consider the free semigroup $S(T)$ of mappings of $\mathcal{P}(X)$ into itself generated by these topological operations which will be listed below. This semigroup is in general obviously noncommutative.

The nontrivial aspect involved is that, in view of well known relations, such as

\begin{equation}
(1.1) \quad c^2 = c, \quad i^2 = i
\end{equation}

a number of elements in this noncommutative semigroup $S(T)$ correspond to the same mappings of $\mathcal{P}(X)$ into itself. Therefore, their identification, that is, the identification of the different elements in $S(T)$ is of interest. Here we shall be interested in such identification which hold for all topological spaces $(X, T)$. Obviously, in the case of particular topological spaces, one can find more such identifications.

Let us list now the other mappings of $\mathcal{P}(X)$ which we shall consider. One of them is the operation of taking the complementary, namely, $\mathcal{C}(A) = X \setminus A$, with $A \subseteq X$. Here we have further reductions in the different elements of $S(T)$, since

\begin{equation}
(1.2) \quad c(\mathcal{C}(A)) = \mathcal{C}(i(A)), \quad i(\mathcal{C}(A)) = \mathcal{C}(c(A)), \quad A \subseteq X
\end{equation}

hence

\begin{equation}
(1.3) \quad e \mathcal{C} = \mathcal{C} i, \quad i \mathcal{C} = \mathcal{C} c
\end{equation}

One also defines the exterior $e(A)$ of a subset $A \subseteq X$, given by

\begin{equation}
(1.4) \quad e = i \mathcal{C}
\end{equation}

Another important topological operation is that of boundary of a subset $A \subseteq X$, which we shall denote by $b(A)$. Here we have the obvious relation
\[(1.5) \quad b(A) = b(\mathcal{C}(A)), \quad A \subseteq X\]

or simply
\[(1.6) \quad b \mathcal{C} = b\]

Also
\[(1.7) \quad b(A) = \mathcal{C}(i(A) \cup i(\mathcal{C}(A))), \quad A \subseteq X\]

thus
\[(1.8) \quad b = \mathcal{C}(i \cup i \mathcal{C})\]

or in view of (1.4)
\[(1.7) \quad b = \mathcal{C}(i \cup e)\]

Let us recall that a subset \( A \subseteq X \) is called a \textit{boundary set}, iff
\[(1.8) \quad i(A) = \phi\]

thus \( A \) is nowhere dense, iff
\[(1.9) \quad i(c(A)) = \phi\]

Let us now consider the related operations
\[(1.10) \quad b_i(A) = b(A) \cap A, \quad b_e(A) = b(A) \cap \mathcal{C}(A), \quad A \subseteq X\]

called respectively the \textit{internal} and \textit{external} boundary of \( A \). Clearly, we have
\[(1.11) \quad b = b_i \cup b_e\]

There are several further frequently used topological operations. One of them is the \textit{derived set} \( d(A) \) of a subset \( A \subseteq X \), defined by
\[(1.12) \quad d(A) = \{ x \in X \mid \forall \ G \in T : (A \cap G) \setminus \{ x \} \neq \emptyset \}\]

And now we define

\[(1.13) \quad S(T)\]

as the free semigroup generated by the set of mappings \( \{ c, i, \mathcal{C}, e, b, b_i, b_e, d \} \), and as customary with semigroups, we assume that it contains the neutral element \( id_X \) which maps each \( A \subseteq X \) into itself.

2. A Simpler Problem

Let us start with the simpler problem of studying the subsemigroup of \( S(T) \) which is generated by the two operations \( c \) and \( i \) alone. This subsemigroup, in view of (1.1), is obviously given as follows

\[(2.1) \quad S_{c, i}(T) = \{ id_X, c, i, ci, ic, cic, cicic, icic, \ldots \}\]

Let us consider the partial order relation \( \alpha \to \beta \) between mappings of \( \mathcal{P}(X) \) into itself, defined by

\[(2.2) \quad \alpha(A) \subseteq \beta(A), \quad A \subseteq X\]

Then obviously

\[(2.3) \quad i \to id_X \to c\]

On the other hand, we have

\[(2.4) \quad i(A) \subseteq i(B), \quad c(A) \subseteq c(B), \quad A \subseteq B \subseteq X\]

Lemma 2.1.

\( S_{c, i}(T) \) is a partially ordered semigroup with respect to \( \to \), and in general it is noncommutative.
Proof.

Let $\alpha, \beta, \gamma \in S_{c,i}(T)$, with $\alpha \rightarrow \beta$.
We show that $\gamma \alpha \rightarrow \gamma \beta$. Let $A \subseteq X$. Then (2.2) gives $\alpha(A) \subseteq \beta(A)$, hence (2.1), (2.4) result in $\gamma(\alpha(A)) \subseteq \gamma(\beta(A))$.
Similarly, $\alpha \gamma \rightarrow \beta \gamma$. Indeed, for $A \subseteq X$, we have $\gamma(A) \subseteq X$, hence (2.2) gives $\alpha(\gamma(A)) \subseteq \beta(\gamma(A))$.

We show that, in general, none of the relations holds

(2.5) \hspace{1cm} ic = ci, \hspace{1cm} ic \rightarrow ci, \hspace{1cm} ci \rightarrow ic

Let $(X, T) = \mathbb{R}$. If $A = [0, 1] \cap \mathbb{Q}$, then $c(i(A)) = \phi \subseteq (0, 1) = i(c(A))$.
Thus the first and the second of the above relations do not hold. Let now $A = [0, 1]$, then $i(c(A)) = (0, 1) \not\subseteq [0, 1] = c(i(A))$, hence the third relation above cannot hold.

Lemma 2.2

The relations hold

(2.6) \hspace{1cm} i \rightarrow ci \rightarrow c \\
\hspace{1cm} i \rightarrow ic \rightarrow c \\
\hspace{1cm} i \rightarrow ici \rightarrow ic \rightarrow c \\
\hspace{1cm} i \rightarrow ci \rightarrow cic \rightarrow c \\
\hspace{1cm} i \rightarrow ci \rightarrow cic \rightarrow cic \rightarrow c \\
\hspace{1cm} i \rightarrow ici \rightarrow ici \rightarrow ic \rightarrow c

Consequently we have

(2.7) \hspace{1cm} cic = ci, \hspace{1cm} icic = ic

Proof.

For the first relation in (2.6) we compose (2.3) on the left with $c$ and obtain $ci \rightarrow c \rightarrow c^2$, thus $ci \rightarrow c$. Composing now (2.3) on the right with $i$, the result is $i^2 \rightarrow i \rightarrow ci$, or $i \rightarrow ci$.

The second relation in (2.6) follows by composing (2.3) on the right
with \(c\), and thus obtaining \(ic \rightarrow c \rightarrow c^2\) or \(ic \rightarrow c\). While composing (2.3) on the left with \(i\), it follows that \(i^2 \rightarrow i \rightarrow ic\), or \(i \rightarrow ic\).

For the third relation in (2.6) we compose the first relation in it with \(i\) on the left and obtain \(i^2 \rightarrow ici \rightarrow ic\), and then recall the second relation in (2.6).

The fourth relation in (2.6) is obtained by composing the second relation in it with \(c\) on the left, with the result \(ci \rightarrow cic \rightarrow c^2\), and then use the first relation in (2.6).

The fifth relation in (2.6) results from the composition on the left with \(c\) of the third relation in (2.6), which gives \(ci \rightarrow cic \rightarrow cic \rightarrow c^2\), after which we recall the first relation in (2.6).

The sixth relation in (2.6) comes from composing the fourth relation in (2.6) with \(i\) on the left, thus having \(i^2 \rightarrow ici \rightarrow icic \rightarrow ic\), and then recalling the second relation in (2.6).

For (2.7) we proceed as follows.

We have \(ci \rightarrow cic\) from the fifth relation in (2.6). On the other hand, the fourth relation in (2.6) gives \(cic \rightarrow c\), which composed on the right with \(i\), yields \(cici \rightarrow ci\).

As for \(icic \rightarrow ic\), it follows from the sixth relation in (2.6). Now the third relation in (2.6) gives \(i \rightarrow ici\), which composed with \(c\) on the right results in \(ic \rightarrow icic\).

The above lead now to

**Theorem 2.1.**

The typically noncommutative semigroup generated by the closure and interior operations \(c\), respectively \(i\), is given by

\[
S_{c,i}(T) = \{ id_X, c, i, ci, ic, ici, cic \}
\]
thus it has at most seven elements. With respect to the partial order relation \( \rightarrow \), these elements are in general arranged as follows

\[
\begin{array}{c}
i \quad i \quad i \quad i \quad c \quad c \quad c \quad i
\end{array}
\]

\[(2.9)\]

\[
\begin{array}{c}
i \quad id_X \quad c
\end{array}
\]

\[
\begin{array}{c}
i \quad ic \quad ici \quad ic \quad cic
\end{array}
\]

Remark 2.1.

We typically have the relations

\[(2.10)\]

\[ cic \neq ci, \quad ici \neq ic \]

as the following simple example shows it. Let \((X, T) = \mathbb{R} \). If \( A = [0, 1] \cap \mathbb{Q} \), then \( c(i(A)) = \phi \), while \( c(i(c(A))) = [0, 1] \). Also \( i(c(A)) = (0, 1) \), while \( i(c(i(A))) = \phi \).
3. A Larger Semigroup

Let us consider now the subsemigroup $S_{c, i, \mathcal{C}}(T)$ of $S(T)$ which is generated by the three operations $c$, $i$ and $\mathcal{C}$. This subsemigroup, in view of (1.3), (2.8) and of the relation $\mathcal{C}\mathcal{C} = id_X$, is obviously given by

$$(3.1) \quad S_{c, i, \mathcal{C}}(T) = \{ id_X, c, i, ci, cic, \mathcal{C}, \mathcal{C}c, \mathcal{C}i, cic, cici, \mathcal{C}cic \}$$

thus in general, it has at most fourteen elements.

4. An Open Problem

A next issue is the structure of the generally noncommutative semigroup

$$(4.1) \quad S_{c, i, \mathcal{C}, b}(T)$$

generated by the five operations $c$, $i$, $\mathcal{C}$, $b$. In view of relations such as (1.6) and (1.8), one can expect a certain simplification in the structure of that semigroup. This leads, among others, to the following

Open Problem

1) Is the noncommutative semigroup $S_{c, i, \mathcal{C}, b}(T)$ finite, in the case of general topological spaces $(X, T)$?

2) The same question for the noncommutative semigroup $S(T)$ in (1.13).