Galois descent for completed Algebraic K-theory

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Abstract

In this paper we consider the problem of Galois descent for suitably completed algebraic K-theory of fields. One of the main results is a suitable form of rigidity for Borel-style generalized equivariant cohomology with respect to certain spectra. In order to apply this to the problem at hand, we need to invoke a derived Atiyah-Segal completion theorem for pro-groups. In the present paper, the authors apply such a derived completion theorem proven by the first author elsewhere. These two results provide a proof of the Galois descent problem for equivariant algebraic K-theory as formulated by the first author, at least when restricted to the case where the absolute Galois groups are pro-$l$ groups for some prime $l$ different from the characteristic of the base field and the K-theory spectrum is completed at the same prime $l$. Work in progress hopes to remove these restrictions.

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1 Introduction

The main goal of this paper is to provide a proof of a conjecture, at least in several important cases, due to the second author relating the equivariant K-theory of the algebraic closure of geometric fields $F$ (i.e. fields containing an algebraically closed subfield $k$) with the equivariant K-theory of $k$, where the group action in both cases are with respect to the absolute Galois group of $F$.

Our approach to a solution of this conjecture has two main steps: one part is a rigidity theorem for Borel-style generalized equivariant cohomology with respect to spectra which are $l$-primary torsion for a fixed prime $l$ different from the characteristic of the base field. This rigidity theorem may be viewed as an extension of well-known rigidity theorems in the non-equivariant setting. The strategy of the proof is by doing a reduction to the non-equivariant case by means of a sheaf-theoretic argument somewhat similar to using a Leray-spectral sequence. The main result here is Theorem 3.0.7 which is also restated as Theorem 1.1.2.

The remaining part of the proof is to show that suitable derived completions of the $l$-primary equivariant K-theory spectrum of the algebraic closure of a geometric field $F$, (i.e. a field containing an algebraically closed subfield $k$), equivariant with respect to the action of the absolute Galois group of $F$ is weakly-equivalent to its Borel-style equivariant $l$-primary K-theory spectrum and that a similar weak-equivalence holds for the corresponding $l$-primary equivariant K-theory spectrum for the field $k$ with respect to the trivial action of the above absolute Galois group. Combining this result with the rigidity statements proven in Theorem 3.0.7 therefore, completes the proof of Theorem 1.1.4.

In [C13], the first author has already established a form of this derived Atiyah-Segal completion theorem, under certain restrictions. The Borel-style generalized equivariant cohomology defined in [C13] makes use of a different model for the classifying spaces of profinite groups, i.e. different from the geometric classifying spaces commonly used in the motivic contexts: see, for example, [MV] or [Tot]. Observe that the rigidity theorem above is proven making use of these geometric classifying spaces. Therefore, a second main result in this paper is to relate these two forms of Borel-style generalized equivariant cohomology for actions of profinite groups. This appears in Theorem 4.0.10. With this result in place, the rigidity theorem in Theorem 1.1.2 provides a rigidity theorem for the Borel-style generalized equivariant cohomology considered in [C13]. Combining these two steps, provides a proof of the conjecture under the hypotheses of [C13].

While the hypotheses of [C13] are fairly general, they do not address the conjecture in its most general form for geometric fields. Work in progress by the authors is expected to address this.

Here is a brief outline of the paper. We discuss the basic framework and the main results in the rest of this section. The second section is devoted to a discussion of Borel-style generalized equivariant cohomology theories defined with respect to spectra. Since this is carried out using geometric models for the classifying spaces of algebraic groups, we discuss these in the framework of admissible gadgets as in [MV] and also [K]. We first define such generalized equivariant cohomology theories for the action of a single algebraic group. Then we discuss actions of pro-groups. We prove that the generalized equivariant cohomology theories so defined are independent of the choice of an inverse system of geometric classifying spaces: this proof makes use of the motivic Postnikov towers as in [Voev2] and [Lev].

The third section is devoted to a discussion of rigidity, where we establish rigidity for Borel-style generalized equivariant cohomology theories, the main result being Theorem 1.1.2. One may view this theorem as an equivariant version of well-known rigidity theorems: see [Sus]. In fact the proof is by reducing rigidity for such equivariant cohomology theories to rigidity for the corresponding non-
equivariant cohomology theories. The fourth section is devoted to a comparison with the generalized equivariant cohomology theories defined using a tower construction as in [C13]. With such a comparison theorem, we obtain rigidity for the generalized equivariant cohomology theories defined using the tower construction.

The fifth section is devoted to a proof of Theorem 1.1.4 by combining Theorem 4.0.10 along with a derived Atiyah-Segal completion theorem proved in [C13] using the generalized equivariant cohomology theories defined using the tower construction.

1.1 Statement of results

We adopt standard conventions on motivic homotopy theory. We fix a base scheme, which for the most part will be a perfect infinite field $k$, and restrict to the category of smooth schemes over this base scheme.

For the most part, we will restrict to the case where the field $k$ is in fact algebraically closed. But some of our constructions and intermediate results hold in the above more general framework.

The latter category will be denoted $\text{Sm}/k$. For the most part the site we put on $\text{Sm}/k$ will be the Nisnevich site, which will be denoted $\text{Sm}/k_{\text{Nis}}$. The unstable model structure on pointed simplicial presheaves on this site, will be the local projective model structure of $\mathcal{B}$ where the generating cofibrations are injective maps of simplicial presheaves of the form $\Lambda[n]+ \cup U \to \Delta[n]+ \cup U$ and $\delta\Delta[n]+ \cup U \to \Delta[n]+ \cup U$, where $\Delta[n]$, $\Lambda[n]$ and $\delta\Delta[n]$ are the obvious simplicial sets and $U$ is an object of the chosen site. In the local projective model structure, the weak-equivalences will be stalk-wise weak-equivalences and fibrations will be characterized by the right-lifting property with respect to trivial cofibrations. The $\mathbb{A}^1$-model structure may be obtained by localizing the local projective model structure with respect to $\mathbb{A}^1$-equivalences: see [Hov-2] (or [Dund2] or [CJ1]) for more details.

One starts with the $\mathbb{A}^1$-localized local projective model category of pointed simplicial presheaves on the Nisnevich site of $\text{Sm}/k$. Then an $\mathbb{S}^1$-spectrum in this category is a system of pointed simplicial presheaves $\{E_n| n \geq 0\}$ equipped with maps $\mathbb{S}^1 \wedge E_n \to E_{n+1}$ which are compatible as $n$ varies. We will let $\text{Spt}_{\mathbb{S}^1}(k)$ denote the category of such $\mathbb{S}^1$-spectra. This is a symmetric monoidal category with respect to the operation of smash product, denoted $\wedge: \Sigma$, the sphere spectrum, will denote the unit of this symmetric monoidal category. We will provide $\text{Spt}_{\mathbb{S}^1}(k)$ with the stable model structure as discussed in [Hov-2] section 5]. The presheaves of spectra $E$ that we consider in the paper will always be assumed to satisfy the following hypotheses (see [Lev] section 2.1 and also [Yag] Definition 1.2):

1.1.1

(i) $E$ is $N$-connected for some integer $N$. (The $N$ usually will be $-1$.)

(ii) $E$ is homotopy invariant, i.e. for each smooth scheme $X$ of finite type over $k$, and a closed subscheme $Y$ (not necessarily smooth), the map $\Gamma_Y(X, E) \to \Gamma_{Y \times \mathbb{A}^1}(X \times \mathbb{A}^1, E)$ is a weak-equivalence.

(iii) $E$ satisfies Zariski excision. Recall Zariski excision means the following: for each open immersion $U \to X$ of schemes in $\text{Sm}/k$ with $W \subseteq X$ closed (not necessarily smooth) and contained in $U$, the induced map $\Gamma_W(X, E) \to \Gamma_W(U, E)$ is a weak-equivalence.

(iv) $E$ satisfies Nisnevich excision. Recall that Nisnevich excision means the following: for each $X' \to X$ an étale map in $\text{Sm}/k$, and $W \subseteq X$ closed (not necessarily smooth), $W' = W \times_X X'$ so that the induced map $W' \to W$ is an isomorphism, the induced map $\Gamma_W(X, E) \to \Gamma_{W'}(X', E)$ is a weak-equivalence.
(v) $E$ satisfies Zariski localization, i.e. if $X$ is in $\text{Sm}/k$ and $Y \subseteq X$ is a closed subscheme (not necessarily smooth), one obtains the stable cofiber sequence: $\Gamma_Y(X, E) \to \Gamma(X, E) \to \Gamma(X - Y, E)$.

(vi) $E$ has the property that the $\mathbb{P}^1$-suspension induces a weak-equivalence. i.e. If $X$ is in $\text{Sm}/k$ and $Y \subseteq X$ is a closed subscheme (not necessarily smooth), then the map $\Gamma_Y(X, E) \to \Gamma_{Y \times 0}(X \times \mathbb{A}^1, E)$ induced by the $\mathbb{P}^1$-suspension is a weak-equivalence.

(vii) $E$ has the homotopy purity property. i.e. Let $Z \subseteq Y \subseteq X$ be closed immersions of schemes in $\text{Sm}/k$. Let $N$ denote the normal bundle associated to the closed immersion $Y \subseteq X$. Let $B(X, Y)$ denote the deformation space obtained by blowing up $Y \times 0$ in $X \times \mathbb{A}^1$. Then $N \to B(X, Y)$ and $X \to B(X, Y)$ are closed immersions and induce weak-equivalences:

$$\Gamma_Z(N, E) \leftarrow \Gamma_{Z \times h^1}(B(X, Y), E) \to \Gamma_Z(X, E).$$

(viii) (Extension to inverse systems of smooth schemes with affine structure maps.) If $\{X_\alpha\}_{\alpha}$ denotes an inverse system of smooth schemes in $\text{Sm}/k$ with affine structure maps, we let $X = \lim_{\alpha} X_\alpha$. Then we define $\Gamma(X, E) = \colim_{\alpha} \Gamma(X_\alpha, E)$. We require that, so defined, the left-hand-side is independent (up to weak-equivalence) of the choice of the inverse system $\{X_\alpha\}_{\alpha}$ whose inverse limit is $X$.

(ix) (Invariance under purely inseparable maps.) If $f : W' \to W$ is a purely inseparable map of smooth schemes, $Y \subseteq W$ is a closed subscheme with $Y' = Y \times W'$, then the induced map $f_* : \Gamma_Y(W, E) \to \Gamma_Y(W', E)$ is a weak-equivalence provided the presheaf of homotopy groups of $E$ are $l$-primary torsion, where $l \neq \text{char}(k)$.

Observe that the Zariski localization and Nisnevich excision properties imply that the spectrum has what is often called the Brown-Gersten property and therefore cohomological descent on the Nisnevich site. Therefore, one obtains the weak-equivalence of spectra for any $E \in \mathbf{Spt}_{S^1}(k)$:

$$\mathbb{H}(X, E) \simeq \mathcal{M}ap(\Sigma_{S^1}X, E)$$

where the right-hand-side is the usual mapping spectrum, and where $\mathbb{H}(X, E)$ denotes the hypercohomology spectrum computed on the Nisnevich site. One may also verify that spectrum representing algebraic K-theory (viewed as an $S^1$-spectrum), satisfies all of the above properties. We show below in Lemma 2.5.1 that if $E$ is a spectrum satisfying the above properties, then the spectrum

$$\text{E}_{\Sigma} \bigwedge_{\Sigma}^{\text{L}} H(Z/\ell) \cdots \bigwedge_{\Sigma}^{\text{L}} H(Z/\ell)$$

also satisfies the same properties, where $l$ is a fixed prime different from the characteristic of $k$ and the right-hand-side involves the derived smash product of the Eilenberg-Maclane spectrum over the sphere spectrum as in [CT1].

Remarks 1.1.1. (i) Observe that the properties above, except for Nisnevich excision and the last property, together with the hypothesis that the homotopy groups of the spectrum $E$ are $l$-primary torsion are the hypotheses needed in [Yag] to ensure rigidity for the cohomology theory represented by the spectrum $E$.

(ii) In [Lev, section 2.1, (A3)], there is an additional property that is required of the spectra. However, that applies only to the case where the base field $k$ would be finite. Since we always assume the base field $k$ is infinite, we do not need to consider this last property.
(iii) There are different, but related notions of spectra in the motivic context. The $S^1$-spectra here mean spectra in the usual sense. On replacing $S^1$ by $\mathbb{P}^1$ one obtains $\mathbb{P}^1$-spectra. These two categories are related by considering the intermediate category of bi-spectra where there are two suspension functors: one, the simplicial suspension with respect to $S^1$ and the other, the suspension with respect to $G_m$. See [Lev] section 8 or [Voev3] for a discussion of these different categories. A key observation for us is that the spectrum representing algebraic $K$-theory is a $\mathbb{P}^1$-spectrum and therefore one may consider the associated bi-spectra to apply results on $S^1$-spectra to it.

A main result in this paper is the following rigidity statement for Borel style generalized equivariant cohomology. Let $k$ denote a fixed algebraically closed field, $F$ a field containing $k$ and let $F^{sep}(F)$ denote a separable closure of $F$ (an algebraic closure of $F$, respectively). We make the following observations that will enable us to simplify our constructions and arguments.

(i) First, if $F$ is of infinite transcendence degree over $k$, one may write $F$ as a direct limit of subfields $F_\alpha$ of finite transcendence degree over $k$. Then the algebraic closure of $F$ will be the direct limit of the algebraic closures of the subfields $F_\alpha$. This observation enables us to effectively reduce to considering fields $F$ of finite transcendence degree over $k$, i.e. to generic points of schemes in $\text{Sm}/k$.

(ii) Second, the algebraic closure of a field $K$ identifies with the splitting field of all polynomials in one variable over $K$. Therefore, it may be written as a direct limit of a sequence of finite separable and a finite purely inseparable extension. Moreover the automorphism group of each such extension will identify with the Galois group of the corresponding finite separable extension. Finally making use of the property: \textit{Invariance under purely inseparable maps} considered above, together with the above observations, make it unnecessary to distinguish between the separable closure and the algebraic closure of any field under consideration.

For example, $F$ could be the field of rational functions on a $k$-variety and $\overline{F}$ an algebraic closure of $F$. Let $G_F = \{G_\alpha | \alpha \in A\}$ denote the absolute Galois group of the field $F$. We then view $\text{Spec } k$ with the trivial action of $G_F$ as a pro-scheme. $\text{Spec } F = \{(\text{Spec } F_\alpha) | \alpha \in A\}$ is clearly a pro-scheme together with an action of $G_F$ on $\text{Spec } F$, i.e. a family of compatible actions by $G_\alpha = \text{Gal}_F(F_\alpha)$ on $F_\alpha$ which is a finite Galois extension of $F$ with Galois group $G_\alpha$. Let $Y = \{Y_\alpha | \alpha \in A\}$ denote a pro-object of smooth schemes of finite type over $k$ provided with compatible actions by $G_F = \{G_\alpha | \alpha \in A\}$. The $G_F$-equivariant hypercohomology $\mathbb{H}_{G_F}$ is defined in 2.4.6. Observe that this is always computed on the Nisnevich site.

**Theorem 1.1.2.** Assume that the base field $k$ is algebraically closed. Let $E$ denote any ring spectrum in $\text{Spt}_{S^1}(k)$ satisfying the hypotheses in 1.1.1. Then the Galois-equivariant map $\text{Spec } \overline{F} \to \text{Spec } k$ (where the Galois group acts trivially on $\text{Spec } k$) induces a weak-equivalence in the following cases:

$$\mathbb{H}_{G_F}(\text{Spec } k \times Y, E/\ell) \to \mathbb{H}_{G_F}(\text{Spec } \overline{F} \times Y, E/\ell),$$

$$\mathbb{H}_{G_F}(\text{Spec } k \times Y, \underbrace{E \wedge H(\mathbb{Z}/\ell)}_m) \to \mathbb{H}_{G_F}(\text{Spec } \overline{F} \times Y, \underbrace{E \wedge H(\mathbb{Z}/\ell)}_m)$$

and

$$\mathbb{H}_{G_F}(\text{Spec } k \times Y, \underbrace{E \wedge H(\mathbb{Z}/\ell) \cdots \wedge H(\mathbb{Z}/\ell)}_m) \to \mathbb{H}_{G_F}(\text{Spec } \overline{F} \times Y, \underbrace{E \wedge H(\mathbb{Z}/\ell) \cdots \wedge H(\mathbb{Z}/\ell)}_m)$$

for any positive integer $m$.

Here (as elsewhere in the paper), $\ell$ is a prime different from the characteristic of $k$. $E/\ell = E \wedge M(\ell)$ denotes the smash product of $E$ with a mod-$\ell$ Moore-spectrum, $\mathbb{H}(\mathbb{Z}/\ell)$ denotes the Eilenberg-Maclane spectrum with the non-trivial homotopy group in degree 0 where it is $\mathbb{Z}/\ell$. Given a commutative ring
spectrum \( R \) and module spectra \( M, N \) over \( R \), \( M_R^L \wedge N \) denotes the derived smash-product defined as \( M_R^L \wedge \tilde{N} \), where \( \tilde{N} \to N \) is a suitable cofibrant replacement of \( N \). In particular, the above weak-equivalences hold with the spectrum \( E = K \) which denotes the spectrum representing algebraic K-theory.

With this theorem in place, we are able to continue on the work of the first author as in \([C13]\) where a derived completion theorem is proven for the actions pro-finite Galois group, resulting in the following theorems which prove the above mentioned conjecture in many important cases.

For the rest of the paper, we will assume that \( G_F \) is a pro-l group. Let \( E_G^C_F \) denote the pro-scheme defined in \([C13]\): see 4.0.8.

**Theorem 1.1.3.** (i) Assume \( E \) denotes a ring spectrum and the base field \( k \) is algebraically closed as in the last theorem. Then one obtains the weak-equivalence:

\[
\mathbb{H}_{G_F}(\chi \times E_G^C_F, E) \simeq \text{Map}(\chi \times G_F \times E_G^C_F, E)
\]

where \( \chi = \{X_a|a\} \) is a pro-scheme with an action by \( G_F = \{G_a|a\} \) and with each \( X_a \in \text{Sm}/k \) as in Definition 4.0.9. The left-hand-side is the Borel-style generalized equivariant cohomology defined in 2.4.6 with respect to the spectrum \( E \), \( \text{Map} \) denotes the obvious mapping spectrum and the term on the right-hand-side is defined as in Definition 4.0.9.

(ii) Assume \( E \) is a ring spectrum as in (i). Then we obtain the weak-equivalence:

\[
\text{Map}(\text{Spec } k \times G_F \times E_G^C_F, E_\ell) \simeq \text{Map}(\text{Spec } F \times G_F \times E_G^C_F, E_\ell)
\]

where \( E_\ell \) denotes one of the following spectra: \( E/\ell = E \wedge M(\ell), E_{\Sigma}^{L} \wedge H(\mathbb{Z}/\ell) \) or \( E_{\Sigma}^{L} \wedge H(\mathbb{Z}/\ell) \cdots \wedge H(\mathbb{Z}/\ell) \) for some \( m \geq 1 \).

Making use of the above weak-equivalences, we obtain the main result of the paper which is the following corollary.

**Theorem 1.1.4.** Assume the above situation. Then we obtain a weak-equivalence

\[
K(\text{Spec } k, G_F)^{I_G,\ell} \simeq K(\text{Spec } F)^{\rho_\ell}
\]

where \( I_G,\ell \) denotes the derived completion along \( I_G,\ell : K(\text{Spec } k, G) \to K(\text{Spec } k) \wedge H(\mathbb{Z}/\ell) \) and \( \rho_\ell \) denotes completion along the map \( \rho_\ell : \Sigma \to H(\mathbb{Z}/\ell) \) (i.e. the completion at the prime \( \ell \).)

2 Borel style equivariant K-theory and generalized equivariant cohomology

The first step in defining Borel style generalized equivariant cohomology, is to define the geometric classifying space of a linear algebraic group. Since different choices are possible for such geometric classifying spaces, we proceed to consider this in the more general framework of admissible gadgets as defined in \([MV]\) section 4.2. The following definition is a variation of the above definition in \([MV]\) and appears in \([K]\) Definition 2.2.
2.1 Admissible gadgets associated to a given $G$-scheme

Let $G$ denote a linear algebraic group over $k$. We shall say that a pair $(W, U)$ of smooth schemes over $k$ is a good pair for $G$ if $W$ is a $k$-rational representation of $G$ and $U \subseteq W$ is a $G$-invariant open subset which is a smooth scheme with a free action by $G$. It is known (cf. [Tot, Remark 1.4]) that a good pair for $G$ always exists.

**Definition 2.1.1.** A sequence of pairs $\{(W_m, U_m)\}_{m \geq 1}$ of smooth schemes over $k$ is called an admissible gadget for $G$, if there exists a good pair $(W, U)$ for $G$ such that $W_m = W^{\oplus m}$ and $U_m \subseteq U$ is a $G$-invariant open subset such that the following hold for each $m \geq 1$.

1. $(U_m \oplus W) \cup (W \oplus U_m) \subseteq U_{m+1}$ as $G$-invariant open subsets.
2. $\text{codim}_{U_{m+2}} (U_{m+2} \setminus (U_{m+1} \oplus W)) > \text{codim}_{U_{m+1}} (U_{m+1} \setminus (U_m \oplus W))$.
3. $\text{codim}_{W_{m+1}} (W_{m+1} \setminus U_{m+1}) > \text{codim}_{W_m} (W_m \setminus U_m)$.
4. $U_m$ is a smooth scheme over $k$ with a free $G$-action.

An example of an admissible gadget for $G$ can be constructed as follows. The first step in constructing an admissible gadget is to start with a good pair $(W, U)$ for $G$. The choice of such a good pair will vary depending on $G$.

The following choice of a good pair is often convenient, but see Proposition 2.1.2 for a different choice when $G$ is a finite $l$-group where $l$ is different from the characteristic of $k$. Choose a faithful $k$-rational representation $R$ of $G$ of dimension $n$. i.e. $G$ admits a closed immersion into $\text{GL}(R)$. Then $G$ acts freely on an open subset $U$ of $W = R^{\oplus m} = \text{End}(W)$. (i.e. Take $U = \text{GL}(R)$.) Let $Z = W \setminus U$.

Given a good pair $(W, U)$, we now let

$$W_m = W^{\oplus m}, U_1 = U \text{ and } U_{m+1} = (U_m \oplus W) \cup (W \oplus U_m) \text{ for } m \geq 1. \quad (2.1.1)$$

Setting $Z_1 = Z$ and $Z_{m+1} = U_{m+1} \setminus (U_m \oplus W)$ for $m \geq 1$, one checks that $W_m \setminus U_m = Z_m$ and $Z_{m+1} = Z^{m+1} \cup U$. In particular, $\text{codim}_{W_m} (W_m \setminus U_m) = m(\text{codim}_W(Z))$ and $\text{codim}_{U_{m+1}} (Z_{m+1}) = (m+1)d - m(\text{dim}(Z)) - d = m(\text{codim}_W(Z))$, where $d = \text{dim}(W)$. Moreover, $U_m \rightarrow U_m/G$ is a principal $G$-bundle and the quotient $V_m = U_m/G$ exists at least as a smooth algebraic space since the $G$-action on $U_m$ is free. We will often use $\text{EG}^{\text{gm}, m}$ to denote the $m$-th term of an admissible gadget $\{U_m\}_m$.

The following proposition provides an alternative construction of a good pair when the group $G$ is a finite $l$-group, with $l$ a prime different from the characteristic of $k$.

**Proposition 2.1.2.** Let $G$ be a finite $l$-group with $l$ a prime different from the characteristic of $k$. Then there exist $k$-rational representations $\rho_i, i = 1, \ldots, s$ so that the action of $G$ on $\Pi_{i=1}^s S(\rho_i)$ is free where $S(\rho_i)$ denotes the complement of the origin in the representation $\rho_i$.

**Proof.** We proceed by induction on $n = \log_l(|G|)$. For $n = 1$, this result is clear since any cyclic $l$-group acts freely on a single $S(\rho)$. Suppose the result is known for $n < i$, and we are given an $l$-group of order $l^n$. The group $G$ is solvable, and therefore has a quotient $Q$ of order $l^{i-1}$. Therefore, there exist representations $\rho_1, \ldots, \rho_n$ of $Q$ (and therefore of $G$) so that the action of $Q$ on

$$\Pi_{j=1}^i S(\rho_j)$$

is free. The kernel $K$ of the projection $G \rightarrow Q$ is a cyclic group of order $l$, and is in the center of $G$. Let $\sigma$ denote a complex representation of $K$ so that the $K$-action is free on $S(\sigma)$. Construct the
induced representation $\tau = i^G_K(\sigma)$ of $G$. It is clear from the usual induction/restriction formulae that the restriction of $\tau$ to $K$ also has the property that the $K$ action on $S(\tau)$ is free. It now follows directly that the action of $G$ on

$$S(\tau) \times \prod_{j=1}^r S(\rho_j)$$

is free.

**Proposition 2.1.3.** (i) If $\{(W_m, U_m)|m\}$ is an admissible gadget for a linear algebraic group $G$, it is also an admissible gadget for any closed subgroup $H$ of $G$.

(ii) For $i = 1, 2$, let $G(i)$ denote linear algebraic groups and let $(W(i), U(i))$ denote a good pair for $G(i)$. For $i = 1, 2$, let $\{(W(i)_m, U(i)_m)|m \geq 1\}$ denote the corresponding admissible gadgets. Then $(W(1) \times W(2), U(1) \times U(2))$ is a good pair for $G(1) \times G(2)$. Under the identification

$$(W(1) \times W(2))^m = W(1)^m \times W(2)^m,$$

$$(W(1)_m \times W(2)_m, U(1)_m \times U(2)_m)|m \geq 1\}$$

is an admissible gadget for $G(1) \times G(2)$.

(iii) Assume $G(i) = G$, $i = 1, 2$. Then $(W(1) \times W(2), U(1) \times W(2) \cup W(1) \times U(2))$ is a good pair for the diagonal action of $G$. Moreover if $U_m = U(1)_m \times W(2)^m \cup W(1)^m \times U(2)_m$, then $(W_m = W(1)^m \times W(2)^m, U_m)$ is an admissible gadget for the diagonal action of $G$.

**Proof.** The first assertion, being obvious, we will only consider the second assertion. Observe that, under the identification in $\text{(2.1.2)}$ $(U(1)_m \times U(2)_m) \times (W(1) \times W(2) \cup W(1) \times W(2) \cup (U(1)_m \times U(2)_m)$ is identified with $U(1)_m \times W(1) \times U(2)_m \times W(2) \cup W(1) \times (U(1)_m \times W(2) \times U(2)_m \times W(2))$. Since

$$U(1)_{m+1} \times U(2)_{m+1} \supseteq (U(1)_m \times W(1) \cup W(1) \times U(1)_m) \times (U(2)_m \times W(2) \cup W(2) \times U(2)_m)$$

it is clear that

$$U(1)_{m+1} \times U(2)_{m+1} \supseteq (U(1)_m \times U(2)_m) \times W(1) \times W(2) \cup W(1) \times (U(1)_m \times U(2)_m)$$

thereby proving that the first hypothesis in Definition $\text{(2.1.1)}$ is satisfied.

Now observe that

$$U(1)_{m+1} \times U(2)_{m+1} \cup U_m(1) \times W(1) \times U(2)_m \times W(2) = (U(1)_{m+1} \cup U_m(1) \times W(1)) \times U(2)_{m+1} \cup U(1)_{m+1} \times U(2)_{m+1} \times W(2)$$

Therefore,

$$\text{codim}_{U(1)_{m+1} \times U(2)_{m+1}} (U(1)_{m+1} \times U(2)_{m+1} \cup U_m(1) \times W(1) \times U(2)_m \times W(2)) = \min(\text{codim}_{U(1)_{m+1}} (U(1)_{m+1} \times U(1)_m \times W(1)), \text{codim}_{U(2)_{m+1}} (U(2)_{m+1} \times U(2)_m \times W(2)))$$

Similarly,

$$\text{codim}_{U(1)_{m+2} \times U(2)_{m+2}} (U(1)_{m+2} \times U(2)_{m+2} \cup U_m(1) \times W(1) \times U(2)_{m+1} \times W(2)) = \min(\text{codim}_{U(1)_{m+2}} (U(1)_{m+2} \times U(1)_{m+1} \times W(1)), \text{codim}_{U(2)_{m+2}} (U(2)_{m+2} \times U(2)_{m+1} \times W(2)))$$

Therefore, one may observe that the second hypothesis in Definition $\text{(2.1.1)}$ is satisfied. One verifies similarly that the third hypothesis in Definition $\text{(2.1.1)}$ is satisfied. Since $G(i)$ acts freely on $U(i)_m$ for
each $i = 1, 2$, the last hypothesis in Definition 2.1.1 is also satisfied thereby completing the proof of the proposition.

Next we will consider the third statement briefly. Now one readily observes that

$$U_m \oplus W(1) \oplus W(2) = (U_m(1) \oplus W) \times W(2)^{m+1} \cup W(1)^{m+1} \times (U_m(2) \oplus W(2))$$

and

$$W(1) \oplus W(2) \oplus U_m = (W \oplus U_m(1)) \times W(2)^{m+1} \cup W(1)^{m+1} \times (W(2) \oplus U_m(2))$$

Since both $U_{m+1}(1)$ and $U_{m+2}(2)$ have the first property in Definition 2.1.1 it follows that this is contained in $U_{m+1}(1) \times W(2)^{m+1} \cup W(1)^{m+1} \times U_{m+1}(2) = U_{m+1}$. This verifies the first property in Definition 2.1.1 for $\{U_m| m\}$. To verify the second property, one observes that

$$U_{m+1} - (U_m \oplus W(1) \oplus W(2)) = (U_{m+1}(1) - (U_m(1) \oplus W(1))) \times W(2)^{m+1} \cup W(1)^{m+1} \times (U_{m+1}(2) - (U_m(2) \oplus W(2)))$$

Now one may readily compute

$$\text{codim}_{U_{m+1}}(U_{m+1}(1) - (U_m \oplus W(1) \oplus W(2)))$$

$$= \min(\text{codim}_{U_m(1)}(U_{m+1}(1) - (U_m(1) \times W(1))), \text{codim}_{U_m(2)}(U_{m+1}(2) - (U_m(2) \times W(2))))$$

Now one may argue as in the proof of (ii) to complete the proof. \qed

### 2.2 Borel-style equivariant K-theory and generalized equivariant cohomology for actions of algebraic groups

Let $G$ denote a linear algebraic group over $S = \text{Spec } k$. Let $\{(W_m, U_m)| m \geq 1\}$ denote an admissible gadget for $G$. For $m \geq 1$, we let

$$EG^{gm,m} = U_m, BG^{gm,m} = V_m = U_m/G.$$

We let

$$EG^g = \lim_{m \to \infty} U_m$$

and

$$BG^g = \lim_{m \to \infty} V_m$$

where the colimit is taken over the closed embeddings $U_m \to U_{m+1}$ and $V_m \to V_{m+1}$ corresponding to the embeddings $Id \times \{0\} : U_m \to U_m \times W \subseteq U_{m+1}$. These are viewed as sheaves on $(\text{Sm}/k)_{\text{Nis}}$ or on $(\text{Sm}/k)_{\text{et}}$.

Given a smooth scheme $X$ of finite type over $S$ with a $G$-action, we let $U_m \times X$ denote the obvious balanced product, where $(u, x)$ and $(ug^{-1}, gx)$ are identified for all $(u, x) \in U_m \times X$ and $g \in G$. Since the $G$-action on $U_m$ is free, a geometric quotient again exists at least as an algebraic space in this setting.

**Definition 2.2.1.** (Borel style equivariant K-theory and generalized cohomology) Assume first that $G$ is an algebraic group or a finite group viewed as an algebraic group by imbedding it in some $GL_n$. We define the Borel style equivariant K-theory of $X$ to be $K(EG^g \times X)$. More generally, given a spectrum $E$, we define the corresponding Borel style generalized equivariant cohomology of $X$ to be

$$H^G_G(X, E) = \lim_{m \to \infty} \text{Map}(\Sigma^{gm,m}_G X, E) \in \text{holim}_{m \to \infty} \text{Map}(\Sigma^{gm,m}_G X, E)$$

$$= \text{holim}_{m \to \infty} \text{Map}(\Sigma^{gm,m}_G X, E) = \text{holim}_{m \to \infty} \text{Map}(\Sigma^{gm,m}_G X, E).$$
Here the hypercohomology denotes hypercohomology computed on the Nisnevich site of the scheme $U^m \times X$. \textit{Map} denotes the simplicial mapping functor computed again on the same site. It is often convenient to identify this with

$$
\mathbb{H}_{GL_n}(GL_n \times X, E) = \mathbb{H}(EGL_n^{gm} \times GL_n \times X, E) = \lim_{\longrightarrow \leftarrow m} \mathbb{H}(EGL_n^{gm,m} \times GL_n \times X, E)
$$

$$
= \lim_{\longrightarrow \leftarrow m} \text{Map}(\Sigma S_1 EGL_n^{gm,m} \times GL_n \times X, E)
$$

(2.2.1)

2.3 Essential uniqueness of the geometric Borel construction

Here we follow the discussion in [MV] and [K].

The following are shown in [K, section 2]:

- Let $\{EG^{gm,m}|m\}$ denote the ind-scheme obtained from an admissible gadget associated to the algebraic group G. Then if $X$ is any scheme or algebraic space over $k$, then $\lim_{m \to \infty} EG^{gm,m} \times X \cong \lim_{m \to \infty} EG^{gm} \times X$. (This follows readily from the observation that the $G$ action on $EG^{gm,m}$ is free and that filtered colimits commute with the balanced product construction above.)

- If $\{\widetilde{EG}^{gm,m}|m\}$ denotes the ind-scheme obtained from another choice of an admissible gadget for the algebraic group $G$, one obtains an isomorphism $EG^{gm} \times X \cong \widetilde{EG}^{gm} \times X$ in the motivic homotopy category.

It follows therefore, that if $E$ is any $\mathbb{A}^1$-local spectrum, then one obtains a weak-equivalence:

$$
\text{Map}(EG^{gm} \times X, E) = \lim_{\longrightarrow \leftarrow m} \{\text{Map}(EG^{gm,m} \times X, E)|m\} \cong \lim_{\longrightarrow \leftarrow m} \{\text{Map}(\widetilde{EG}^{gm,m} \times X, E)|m\}
$$

$$
= \text{Map}(\widetilde{EG}^{gm} \times X, E).
$$

The following is also proven in [MV, Lemma 2.9, p.139]: if $Y$ is a smooth scheme of finite type over $k$, with a free $G$-action and $X$ is any smooth scheme with a $G$-action, then the obvious map $EG^{gm} \times (X \times Y) \to G \times X \times G Y$ is an $\mathbb{A}^1$-equivalence. (Here $G$ is assumed to act diagonally on $X \times Y$ so that this action is also free.)

2.4 Borel style equivariant K-theory and generalized equivariant cohomology for pro-group actions

In this section we will extend the constructions of the last section to actions of pro-groups on pro-schemes.

2.4.1 Construction of the inverse system

Let $\mathcal{A}$ denote a directed set and let $\{\phi(\alpha) : G_a \to G_b|\alpha : b \to a \in \mathcal{A}\}$ denote an inverse system of algebraic groups indexed by $\mathcal{A}^{op}$. We will denote the order on $\mathcal{A}$ by $\leq$ and assume that if $a \leq b$ and $b \leq a$, then $a = b$. We will also assume that given any $a \in \mathcal{A}$, the full subcategory $\mathcal{A}/a$ has only a finite number of objects and morphisms, i.e. the set $\mathcal{A}^{op}$ viewed as a cofiltered category in the obvious way is cofinite. For the most part we will be interested in the case where this forms the inverse system of finite quotient groups of a profinite group $G$. As shown in [RZ, Corollary 1.1.18], we may assume that
each of the structure maps $\phi(\alpha) : G_a \to G_b$ is surjective, so that there are only finitely many non-trivial quotient groups for any $G_a$. Therefore, there is no loss of generality in assuming the hypothesis that each of the subcategories $A/a$ has only a finite number of objects and morphisms, i.e. the underlying category associated to $A^{op}$ is cofiltered and cofinite.

Next suppose $\mathcal{X} = \{X_a | a \in \mathcal{A}\}$ is an inverse system of schemes, so that each $X_a$ is provided with an action by $G_a$ and the these actions are compatible. Suppose further that structure maps of the inverse system $\{X_a | a \in \mathcal{A}\}$ are flat. Then for each map $\alpha : b \to a$ in $\mathcal{A}$, one obtains the following induced maps:

$$G(X_b, G_b) \longrightarrow G(X_b, G_a) \longrightarrow G(X_a, G_a) \quad (2.4.2)$$

where the map

$$G(X_b, G_b) \to G(X_b, G_a)$$

is induced by restriction of the group action and the map

$$G(X_b, G_a) \to G(X_a, G_a)$$

is induced by pull-back along the flat maps $X_a \to X_b$. One may readily see that this provides a direct system $\{G(X_a, G_a) | a \in \mathcal{A}\}$. A corresponding result holds for $K$-theory in the place of $G$-theory: since $K$-theory is always contravariantly functorial, one need not assume that the structure maps of the inverse system $\{X_a | a \in \mathcal{A}\}$ are flat.

In view of the above discussion, we will assume that for each $a \in \mathcal{A}$, one is provided with a linear algebraic group $\mathfrak{G}_a$ (not necessarily part of any inverse system of groups indexed by $\mathcal{A}$) and that the inverse system of linear algebraic groups $\{G_a | a \in \mathcal{A}\}$ is defined with

$$\mathfrak{G}_a = \Pi_{b \leq a} \mathfrak{G}_b$$

with the structure maps of the inverse system $\{G_a | a \in \mathcal{A}\}$ defined by the obvious projection $\mathfrak{G}_a = \Pi_{b \leq a} \mathfrak{G}_b \to \mathfrak{G}_x = \Pi_{y \leq x} \mathfrak{G}_y$, for $x \leq a$.

Next assume that one is given an inverse system of algebraic groups $\{G_a | a \in \mathcal{A}\}$, (which for the most part will be finite groups, viewed as algebraic groups) $\{G_a | a \in \mathcal{A}\}$. Let $\phi_{a,b} : G_a \to G_b$ denote the structure map of the inverse system $\{G_a | a \in \mathcal{A}\}$. Then letting $\mathfrak{G}_a = G_a$ for each $a \in \mathcal{A}$ shows that one obtains a new inverse system of groups $\{G_a | a \in \mathcal{A}\}$. Moreover mapping $G_a \to G_a$ by sending $G_a$ by the identity map to $G_a$ and by the structure map $\phi_{a,b}$ to $G_b$, for $b \leq a$, provides a strict map $\{G_a | a \in \mathcal{A}\} \to \{G_a | a \in \mathcal{A}\}$ of inverse systems which in each degree is a closed immersion and a group homomorphism.

Next we consider non-negative integral valued functions (or sequences) $s$ defined on $\mathcal{A}$. Given two sequences $s$ and $t$, we let $s \leq t$ if $s(a) \leq t(a)$ for all $a \in \mathcal{A}$. Clearly given two such sequences $s$ and $t$, one has a third sequences $u$ dominating both $s$ and $t$ so that the collection of such sequences is a directed set. We say a sequence $s$ is non-decreasing if $s(b) \leq s(a)$ for all $b \leq a$ in $\mathcal{A}$. Since each object $a \in \mathcal{A}$ has only finitely many objects $b \leq a$, we see that the subset of non-decreasing sequences is cofinal in the directed set of all sequences. We will denote this directed set of non-decreasing sequences on $\mathcal{A}$ by $K$.

**Definition 2.4.1.** In this situation, we assume that for each $a \in \mathcal{A}$ one has chosen a good pair for the group $\mathfrak{G}_a$ and that $\{E_{\mathfrak{G}_a}^{gm,m} | m\}$ is the corresponding admissible gadget, fixed throughout the following discussion. Associated to the sequence $s$, we now let

$$U_a^s = E_{\mathfrak{G}_a}^{gm,s} = \Pi_{b \leq a} E_{\mathfrak{G}_b}^{gm,s(b)}. \quad (2.4.4)$$
We also let
\[ EG^{gm,s} = \{ EG^s_a | a \}. \]

One observes that for each fixed sequence \( s \) as above, \( \{ U^s_a | a \in A \} \) defines an inverse system of schemes (with the structure map \( U^s_a \to U^s_b \) induced by the obvious projection for each \( b \leq a \)) and so that each \( U^s_a \) is provided with an action by \( G_a \) which are compatible as \( a \) varies.

**Remark 2.4.2.** For many situations, it suffices to consider non-negative integer valued sequences \( s \) defined on \( A \) that are constant. In this case we will denote such constant sequences by their common integral value. However, this will *not suffice* in general. Anticipating this, we are setting up a framework here that will apply to more general situations readily.

**Definition 2.4.3.** (Borel style generalized equivariant cohomology for pro-objects.) Assume in addition to the above situation that \( X = \{ X_a | a \in A \} \) is an inverse system of schemes with compatible actions by \( \{ G_a | a \in A \} \). Then we let
\[
H(EG^{gm,s} \times_G X, E) = \operatorname{colim}_{a \in A} \{ \mathbb{H}(EG^{gm,s}_a \times_{G_a} X_a, E) | a \in A \} \quad \tag{2.4.5}
\]
\[
\mathbb{H}_G(X, E) = \operatorname{holim}_s \mathbb{H}(EG^{gm,s} \times_G X, E)
\]

In particular, when the spectrum \( E \) denotes \( K \)-theory, one obtains the weak-equivalences (since the \( G_a \)-action on \( EG^{gm,s}_a(a) \) is free):
\[
K(EG^{gm,s} \times X, G) = \operatorname{colim}_{a \in A} \{ K(EG^{gm,s}_a(a) \times X_a, G_a) | a \in A \} \simeq \operatorname{colim}_{a \in A} \{ K(EG^{gm,s}_a(a) \times X_a, G_a) | a \in A \}
\]
\[
K(EG^{gm} \times X, G) \simeq \operatorname{holimcolim}_{s \in K \atop a \in A} \{ K(EG^{gm}_a(a) \times X_a, G_a) | a \in A \}
\]

A corresponding result holds for \( G \)-theory, when the structure maps of the inverse system \( \{ X_a | a \in A \} \) are flat.

Next let \( G = \{ G_a | a \in A \} \) denote an inverse system of finite groups, acting on the inverse system of schemes \( X = \{ X_a | a \in A \} \), with \( G_a \) acting on \( X_a \). Then we let \( \Phi_b \) denote a general linear group into which \( G_a \) admits a closed immersion. We then form the inverse system \( \{ G_a = \bigoplus_{b \leq a} \Phi_b | a \in A \} \) where the structure maps are the obvious projections. Now each \( G_a \) admits a closed immersion into \( G_a \) by mapping into each \( \Phi_b \) through the homomorphism into \( G_b \). We then let \( X_a = G_a \times X_a \) and observe the weak-equivalence
\[
\mathbb{H}_G(X, E) = \operatorname{holimcolim}_{s \in K \atop a \in A} \{ EG^{gm}_a(a) \times_{G_a} G_a \times X_a, E \} \simeq \operatorname{holimcolim}_{s \in K \atop a \in A} \{ EG^{gm}_a(a) \times_{G_a} X_a, E \} \quad \tag{2.4.6}
\]

Therefore, we denote \( \operatorname{holimcolim}_{s \in K \atop a \in A} \{ EG^{gm}_a(a) \times_{G_a} X_a, E \} \) by \( \mathbb{H}_G(X, E) \).

### 2.5 Essential uniqueness of the geometric classifying spaces for pro-group actions

In this section, we proceed to prove that the Borel style generalized equivariant cohomology considered above is independent of the choice of the geometric classifying spaces for the groups involved. The proof of this statement follows in outline the proof of the corresponding statements for actions of inverse systems.
of topological groups on inverse systems of topological spaces. Recall this makes use of Postnikov towers for the corresponding spectrum: see [Fausk, Proposition 11.17]. Our proof, therefore, will make use of the corresponding motivic Postnikov towers as defined in [Voev2] and [Lev]. Next, we digress to recall the relevant properties of the motivic Postnikov towers.

Recall that $\text{Spt}_{S^1}(k)$ denotes the category of motivic $S^1$ spectra. We will begin with the following rather technical (but useful) result.

**Lemma 2.5.1.** Assume $E$ is a ring spectrum in $\text{Spt}_{S^1}(k)$ satisfying the properties in 1.1.1. Then, for any prime $l \neq \text{char}(k)$, the spectrum $\underbrace{E_L^{\Sigma}H(Z/\ell)^{\cdots}}_{m} \cdots \underbrace{E_L^{\Sigma}H(Z/\ell)^{\cdots}}_{0}$ in $\text{Spt}_{S^1}(k)$ also satisfies the same properties.

**Proof.** Clearly it suffices to consider the case $m = 1$. The hypothesis that a spectrum $E$ has Nisnevich excision, implies it has what is called the Brown-Gersten property (see [MV, section 3, Definition 1.13]), so that it has Nisnevich descent. Now one may observe readily that if $E$ has Nisnevich excision and Zariski localization, then so does $E_L^{\Sigma}H(Z/\ell)$. Therefore, for every scheme $X$ in $\text{Sm}/k$ and a closed subscheme $Y \subseteq X$, the obvious map $\Gamma_Y(X, E_L^{\Sigma}H(Z/\ell)) \rightarrow R\Gamma_Y(X, E_L^{\Sigma}H(Z/\ell))$ is a weak-equivalence. This readily proves that all the properties in 1.1.1(i) through (vii) hold for $E_L^{\Sigma}H(Z/\ell)$ if they hold for $E$. $\square$

The work of [Lev] and [Pel] show that now one can define a sequence of functors $f_n : \text{Spt}_{S^1}(k) \rightarrow \text{Spt}_{S^1}(k)$ so that the following properties are true:

### 2.5.1

(i) One obtains a map $f_n E \rightarrow E$ that is natural in the spectrum $E$.

(ii) One also obtains a tower of maps $\cdots \rightarrow f_{n+1} E \rightarrow f_n E \rightarrow \cdots \rightarrow f_0 E = E$. Let $s_p E$ denote the canonical homotopy cofiber of the map $f_{p+1} E \rightarrow f_p E$ and let $P_{\leq q-1} E$ be defined as the canonical homotopy cofiber of the map $f_q E \rightarrow E$. Then one may observe readily (see, for example, [Pel, Proposition 3.1.19]) that the canonical homotopy fiber of the induced map $P_{\leq q} E \rightarrow P_{\leq q-1} E$ identifies also with $s_q(E)$. One then also obtains a tower of fibrations $\cdots \rightarrow P_{\leq q} E \rightarrow P_{\leq q-1} E \rightarrow \cdots$.

(iii) Let $Y$ be a smooth scheme of finite type over $k$ and let $W \subseteq Y$ denote a closed not necessarily smooth subscheme so that $\text{codim}_Y(W) \geq q$ for some $q \geq 0$. Then the map $f_q E \rightarrow E$ induces a weak-equivalence (see [Lev, Lemma 7.3.2] and also [L-K, Lemma 2.3.2]):

$$\text{Map}(\Sigma_{S^1}(Y/Y - W)_+, f_q E) \rightarrow \text{Map}(\Sigma_{S^1}(Y/Y - W)_+, E), E \in \text{Spt}_{S^1}(k).$$

(iv) It follows that, then,

$$\text{Map}(\Sigma_{S^1}(Y/Y - W)_+, P_{\leq q-1} E) \simeq *$$

under the same hypotheses on $Y$ and $W$. (This makes use of the property that if $E$ is homotopy invariant and has Nisnevich excision and Zariski localization, then the terms $f_p E$ and hence $s_p E$ have the same properties. These follow readily by identifying the terms $f_p E$ with the terms in the homotopy coniveau tower as in [Lev].)
(v) Let $X$ be a smooth scheme of finite type over $k$ and of dimension $d$, and let $n$ be a non-negative integer. Then, since $E$ assumed to be $-1$-connected, $\pi_n(\text{Map}(\Sigma^1 X_+, s_pE)) = 0,$ if $p > \text{dim}(X) + n$: see [Lev] proof of Proposition 2.1.3.

(vi) Recall that $s_{p+m}E$ identifies with the homotopy fiber of the induced map $P_{\leq p+m}E \to P_{\leq p+m-1}E$. It follows by making use of the Milnor exact sequence for $\pi_n(\text{holim}\text{Map}(\Sigma^1 X_+, P_{\leq p+m}E))$, that for $p > \text{dim}(X) + n + 1$,

$$\pi_n(\text{holim}\text{Map}(\Sigma^1 X_+, P_{\leq p+m}E)) \cong \pi_n(\text{Map}(\Sigma^1 X_+, P_{\leq p}E)).$$

To see this, consider the long-exact-sequence for $i = 0, 1$, with $m \geq 1$:

$$\pi_{n+i}\text{Map}(\Sigma^1 X_+, s_{p+m}E) \to \pi_{n+i}\text{Map}(\Sigma^1 X_+, P_{\leq p+m}E) \to \pi_{n+i}\text{Map}(\Sigma^1 X_+, P_{\leq p+m-1}E)$$

$$\to \pi_{n+i-1}(\text{Map}(\Sigma^1 X_+, s_{p+m}E))$$

If $p > \text{dim}(X) + n + 1$, then $p + m > \text{dim}(X) + n + 1 + m \geq \text{dim}(X) + n + 1 > \text{dim}(X) + n - 1$, so that

$$\pi_{n+j}(\text{Map}(\Sigma^1 X_+, s_{p+m}E)) = 0, \text{ for } j = -1, 0, 1$$

so that the map $P_{\leq p+m}E \to P_{\leq p+m-1}E$ induces an isomorphism

$$\pi_{n+i}(\text{Map}(\Sigma^1 X_+, P_{\leq p+m}E)) \cong \pi_{n+i}(\text{Map}(\Sigma^1 X_+, P_{\leq p+m-1}E)), i = 0, 1.$$

(vii) Now one may make use of [Lev] Proposition 2.1.3 to conclude that, if $E$ is also homotopy invariant, then $\pi_n(\text{holim}\text{Map}(\Sigma^1 X_+, P_{\leq p+m}E)) \cong \pi_n(\text{Map}(\Sigma^1 X_+, E))$ so that if $p > \text{dim}(X) + n + 1$, then

$$\pi_n(\text{Map}(\Sigma^1 X_+, P_{\leq p}E)) \cong \pi_n(\text{Map}(\Sigma^1 X_+, E)).$$



**Remarks 2.5.2.** We use the notation $P_{\leq q}E$, for obvious reasons, to denote the motivic Postnikov truncation that kills all the slices $s_pE$ for $p > q$. It is shown in [Pel] that the functors $f_q$, $s_q$ and $P_{\leq q}$ lift to the level of model categories, though this fact is not important for us.

Next we proceed to adapt the motivic Postnikov truncation functors in the context of an inverse system of Borel constructions as in Definition 2.4.3. We will consider associated to each $t \in K$,

$$H(EG^{gm,s} \times_G X, P_{\leq t}E) = \text{colim}\{\text{Ind}(EG^{gm,s}_a \times_G X_a, P_{\leq t(a)}E) | a \in A\} \quad (25.2)$$

$$\text{Ind}_G(X, P_{\leq t}E) = \text{holim}_{t \in K}\text{Ind}_G(EG^{gm,s} \times_G X, P_{\leq t}E)$$

$$\text{holim}_{t \in K}\text{Ind}_G(X, P_{\leq t}E) = \text{holim}_{t \in K}\text{holim}_{s \in K}\text{Ind}_G(EG^{gm,s} \times_G X, P_{\leq t}E)$$

**Proposition 2.5.3.** Assume the above situation. Then the two spectra

$$\text{holim}_{s \in K}\text{Ind}_G(EG^{gm,s} \times_G X, E) \text{ and } \text{holim}_{t \in K}\text{holim}_{s \in K}\text{Ind}_G(EG^{gm,s} \times_G X, P_{\leq t}E)$$

are weakly-equivalent.
Proof. Observe that $$\text{holim}_{s \in \mathcal{K}}(E^{\text{gm},s})_{G}^{X,E}$$ identifies with
$$\text{holim}_{s \in \mathcal{K}}(E^{\text{gm},s})_{G,a}^{X_{a},E} |a \in A$$ while $$\text{holim}_{s \in \mathcal{K}}(E^{\text{gm},s})_{G}^{X,P_{<\ell}(E)}$$ identifies with
$$\text{holim}_{s \in \mathcal{K}}(E^{\text{gm},s})_{G,a}^{X_{a},P_{<\ell}(a)} |a \in A$$

$$= \text{holim}_{s \in \mathcal{K}}(E^{\text{gm},s})_{G,a}^{X_{a},P_{<\ell}(a)} |a \in A$$.}

Assume that the geometric classifying spaces $$\{E^{\text{gm},b}\}$$ are defined using the choice of a good-pair $$(W_{b},U_{b})$$. Observe then that the dimension of $$E_{G}^{\text{gm},s} \times \Pi_{b \leq a} A_{b} X_{a}$$ is $$\sum_{b \leq a} s(b) \dim_{k}(W_{b}) + \dim_{k}(X_{a}) - (\sum_{b \leq a} \dim_{k}(A_{b}))$$. For any fixed non-negative integer $$n$$, and given any sequence $$s \in \mathcal{K}$$, one may clearly choose a sequence $$t \in \mathcal{K}$$, so that for each $$a \in A$$, $$t(a) \geq \dim_{k}(E_{G}^{\text{gm},s} \times \Pi_{b \leq a} A_{b} X_{a}) + n + 2$$. It follows from the last property in (2.5.3) that this implies the map
$$E_{G}^{\text{gm},s} \times \Pi_{b \leq a} A_{b} X_{a}, P_{<\ell}(a) \rightarrow E_{G}^{\text{gm},s} \times \Pi_{b \leq a} A_{b} X_{a}, E$$
induces an isomorphism on taking the $$n$$-th and $$n+1$$-st homotopy groups and for all $$a \in A$$. It follows now that the induced map
$$\text{holim}_{s \in \mathcal{K}}(E_{G}^{\text{gm},s})_{G,a}^{X_{a},P_{<\ell}(a)} E \rightarrow \text{holim}_{s \in \mathcal{K}}(E_{G}^{\text{gm},s})_{G,a}^{X_{a},P_{<\ell}(a)} E$$
is a weak-equivalence for each fixed $$s \in \mathcal{K}$$ and therefore again is a weak-equivalence on taking $$\text{holim}_{s \in \mathcal{K}}$$.

Next assume that $$G$$ is a linear algebraic group and $$(W,U)$$, $$(\bar{W},\bar{U})$$ are both good pairs for $$G$$. Let $$\{W_{m}/U_{m}|m \geq 1\}$$ and $$\{W_{m}/\bar{U}_{m}|m \geq 1\}$$ denote the associated admissible gadgets. Then, since $$G$$ acts freely on both $$U_{m}$$ and $$\bar{U}_{m}$$, it is easy to see that $$(W \times \bar{W}, U \times \bar{W} \cup W \times \bar{U})$$ is also a good pair for $$G$$ with respect to the diagonal action on $$W \times W$$. Moreover, under the same hypotheses, Proposition (2.3.3)(iii) shows that $$\{W_{m} \times \bar{W}_{m}, U_{m} \times \bar{U}_{m} \cup W_{m} \times \bar{U}_{m}|m \geq 1\}$$ is also an admissible gadget for $$G$$ for the diagonal action on $$W \times \bar{W}$$. Let $$X$$ denote a smooth scheme of finite type over $$k$$ on which $$G$$ acts.

Since $$G$$ acts freely on both $$U_{m}$$ and $$\bar{U}_{m}$$, it follows that $$G$$ has a free action on $$U_{m} \times \bar{U}_{m}$$ and also on $$W_{m} \times \bar{U}_{m}$$. We will let $$\bar{U}_{m} = U_{m} \times \bar{W}_{m} \cup W_{m} \times \bar{U}_{m}$$ for the following discussion. One may now compute the codimensions
$$\text{codim}_{\bar{U}_{m} \times X_{G}}(\bar{U}_{m} \times X - U_{m} \times \bar{W}_{m} \times X) = \text{codim}_{W_{m}}(W_{m} - U_{m}) \text{, and (2.5.3)}$$
$$\text{codim}_{\bar{U}_{m} \times X_{G}}(\bar{U}_{m} \times X - W_{m} \times \bar{U}_{m} \times X) = \text{codim}_{W_{m}}(\bar{W}_{m} - \bar{U}_{m}) \text{ (2.5.4)}$$

In view of (2.5.4), it follows that the induced maps
$$\text{Map}(\Sigma_{S_{1}} U_{m} \times_{G} X_{+}, P_{<q-1} E) \simeq \text{Map}(\Sigma_{S_{1}} (U_{m} \times \bar{W}_{m}) \times_{G} X_{+}, P_{<q-1} E)$$
$$\rightarrow \text{Map}(\Sigma_{S_{1}} \bar{U}_{m} \times_{G} X_{+}, P_{<q-1} E) \text{ and (2.5.5)}$$
$$\text{Map}(\Sigma_{S_{1}} \bar{U}_{m} \times_{G} X_{+}, P_{<q-1} E) \simeq \text{Map}(\Sigma_{S_{1}} (W_{m} \times U_{m}) \times_{G} X_{+}, P_{<q-1} E)$$
$$\rightarrow \text{Map}(\Sigma_{S_{1}} \bar{U}_{m} \times_{G} X_{+}, P_{<q-1} E)$$
are both weak-equivalences if \( \text{codim}_{W_m}(W_m - U_m) \) and \( \text{codim}_{W_m}(\bar{W}_m - \bar{U}_m) \) are both greater than or equal to \( q \). The first weak-equivalences in \((2.5.5)\) are provided by the homotopy property for the spectrum \( E \), which is inherited by the Postnikov-truncations (as shown in \([\text{Lev}] (2.2)(2)\)).

Next assume that \( Y \) is a smooth scheme of finite type over \( k \) provided with a free action by \( G \). Now \( G \) acts freely on both \( W_m \times Y \times X \) as well as on \( U_m \times Y \times X \). One may compute

\[
\text{codim}_{W_m \times G}(X \times Y)(W_m \times G (X \times Y) - U_m \times G (X \times Y)) = \text{codim}_{W_m}(W_m - U_m)
\]

Again, in view of \((2.5.1)\)[iv], it follows that the induced map

\[
\text{Map}(\Sigma_S t; U_m \times G (X \times Y)_+, P_{\leq q-1} E) \rightarrow \text{Map}(\Sigma_S t; W_m \times G (X \times Y)_+, P_{\leq q-1} E) \\
\simeq \text{Map}(\Sigma_S t; G \times Y)_+, P_{\leq q-1} E)
\]

is a weak-equivalence if \( \text{codim}_{W_m}(W_m - U_m) \geq q \). The last weak-equivalence is again provided by the homotopy property for the spectrum \( E \), which is inherited by the Postnikov-truncations (as shown in \([\text{Lev}] (2.2)(2)\)). One may recall that if the admissible gadgets are chosen as in \((2.1.1)\), then \( \text{codim}_{W_m}(W_m - U_m) = m.\text{codim}_{W}(W - U) \) and \( \text{codim}_{W_m}(W_m - \bar{U}_m) = m.\text{codim}_{W}(\bar{W} - \bar{U}) \), so that both \( \text{codim}_{W_m}(W_m - U_m) \geq q \) and \( \text{codim}_{W_m}(W_m - \bar{U}_m) \geq q \) if \( m \) is chosen large enough.

Next assume the situation considered in Definition \(2.4.3\). Assume in addition that for each \( b \in A \), \((\bar{W}_b, \bar{U}_b)\) and \((\bar{W}_b, \bar{U}_b)\) are two choices of good pairs and that \( \{(\bar{W}_{b,m}, \bar{U}_{b,m})| m \geq 1\} \) and \( \{(\bar{W}_{b,m}, \bar{U}_{b,m})| m \geq 1\} \) are the corresponding associated admissible gadgets chosen as in \((2.1.1)\). We will denote the geometric classifying spaces obtained from \( \{\bar{W}_{b,m}, \bar{U}_{b,m}| m \geq 1\} \) \((\{\bar{W}_{b,m}, \bar{U}_{b,m}| m \geq 1\} \) \( \{EG^m_b | b, m\} \), respectively). Assume further that one is provided with compatible actions by \( G_a = \Pi_{b \leq a} G_b \) on \( X_a \) and \( Y_a \) which are smooth schemes of finite type over \( k \) and that the action of \( G_a \) on \( Y_a \) is free.

**Theorem 2.5.4.** Assume the above situation. (i) Then one obtains weak-equivalences:

\[
\text{holim}_{t \in K} \left( \text{colim}_{s \in K} \right) (EG^{gm,s}_a \times G_a X_a, P_{\leq t} E) \simeq \text{holim}_{t \in K} \left( \text{colim}_{s \in K} \right) (EG^{gm,s}_a \times G_a X_a, P_{\leq t} E). \quad (2.5.8)
\]

Moreover \( \text{holim}_{t \in K} \left( \text{colim}_{s \in K} \right) (EG^{gm,s}_a \times G_a X_a, P_{\leq t} E) \simeq \text{holim}_{t \in K} \left( \text{colim}_{s \in K} \right) (EG^{gm,s}_a \times G_a X_a, E) \) and

\[
\text{holim}_{t \in K} \left( \text{colim}_{s \in K} \right) (EG^{gm,s}_a \times G_a X_a, P_{\leq t} E) \simeq \text{holim}_{t \in K} \left( \text{colim}_{s \in K} \right) (EG^{gm,s}_a \times G_a X_a, E).
\]

(ii) \( \text{holim}_{t \in K} \left( \text{colim}_{s \in K} \right) (EG^{gm,s}_a \times G_a (X_a \times Y_a), E) \simeq \text{holim}_{t \in K} \left( \text{colim}_{s \in K} \right) (EG^{gm,s}_a \times G_a (X_a \times Y_a), P_{\leq t} E) \)

\[
\simeq \text{holim}_{t \in K} \left( \text{colim}_{s \in K} \right) (X_a \times G_a Y_a, P_{\leq t} E) \simeq \text{colim}_{t \in K} (X_a \times G_a Y_a, E).
\]

**Proof.** For each fixed sequence \( t \in K \), one may clearly choose a sequence \( s \in K \) so that for each \( b \in A \), \( s(b) \text{codim}_{W_b}(W - U) \geq t(b) + 1 \) and \( s(b) \text{codim}_{W_b}(\bar{W} - \bar{U}) \geq t(b) + 1 \). It now follows from \((2.5.5)\) that with this choice of \( s \), one obtains a weak-equivalence:

\[
\text{colim}_{a \in A} (EG^{gm,s}_a \times G_a X_a, P_{\leq t} E) \simeq \text{colim}_{a \in A} (EG^{gm,s}_a \times G_a X_a, P_{\leq t} E)
\]

It follows that for each fixed \( t \in K \), one obtains a weak-equivalence:

\[
\text{holim}_{s \in K} \left( \text{colim}_{a \in A} \right) (EG^{gm,s}_a \times G_a X_a, P_{\leq t} E) \simeq \text{holim}_{s \in K} \left( \text{colim}_{a \in A} \right) (EG^{gm,s}_a \times G_a X_a, P_{\leq t} E)
\]
Taking holim now, we obtain the first weak-equivalence in (i). Observe that the two homotopy inverse limits over \( s \) and \( t \in \mathcal{K} \) commute. Therefore, one may now take the outer homotopy inverse limit to be holim. For each fixed non-negative integer \( n \), and \( s \in \mathcal{K} \), one may choose a \( t \in \mathcal{K} \), so that \( t(a) \geq \Sigma_{b \leq a} s(b) \dim_k(W_b) + \dim_k(X_a) - \Sigma_{b \leq a} \dim_k \mathfrak{g}_b + n + 2 \). Then, Proposition 2.5.3 completes the proof of the remaining assertions in (i).

Next we consider (ii). Recall \( E \mathfrak{g}_a^{gm,s} = \Pi_{b \leq a} E \mathfrak{g}_b^{gm,s(b)} = \Pi_{b \leq a} U_b^{s(b)} \) and that \( \mathfrak{g}_a = \Pi_{b \leq a} \mathfrak{g}_b \). Therefore, the weak-equivalence in (ii) is provided by (2.5.7) which shows for each fixed \( t \in \mathcal{K} \), one may choose an \( s \in \mathcal{K} \) so that the map

\[
\colim_{t \in \mathcal{K}} \Pi_{b \leq a} U_b^{s(b)} \times \mathfrak{g}_a(X_a \times Y_a), P_{\leq t} E \rightarrow \colim_{t \in \mathcal{K}} \Pi_{b \leq a} W_b^{s(b)} \times \mathfrak{g}_a(X_a \times Y_a), P_{\leq t} E
\]

is a weak-equivalence. It follows therefore that the map

\[
\holim_{t \in \mathcal{K}} \colim_{a \in \mathcal{A}} \Pi_{b \leq a} U_b^{s(b)} \times \mathfrak{g}_a(X_a \times Y_a), P_{\leq t} E \rightarrow \holim_{t \in \mathcal{K}} \colim_{a \in \mathcal{A}} \Pi_{b \leq a} W_b^{s(b)} \times \mathfrak{g}_a(X_a \times Y_a), P_{\leq t} E
\]

is also a weak-equivalence. Now the statement in (i) shows that one may omit the first homotopy inverse limit holim. As in (2.5.7), the homotopy property for the spectrum \( E \), which is inherited by the Postnikov-truncations then provides the weak-equivalence:

\[
\holim_{t \in \mathcal{K}} \colim_{a \in \mathcal{A}} \Pi_{b \leq a} W_b^{s(b)} \times \mathfrak{g}_a(X_a \times Y_a), P_{\leq t} E \simeq \holim_{t \in \mathcal{K}} \colim_{a \in \mathcal{A}} (X_a \times \mathfrak{g}_a Y_a), P_{\leq t} E.
\]

Since \( X_a \) and \( Y_a \) are schemes of finite type over \( k \), for each non-negative integer \( n \), one may choose a sequence \( t \in \mathcal{K} \), so that \( t(a) \geq \dim_k(X_a \times \mathfrak{g}_a Y_a) + n + 2 \). Then, 2.5.1(vi) shows that the induced map

\[
\colim_{a \in \mathcal{A}} \colim_{t \in \mathcal{K}} \Pi_{b \leq a} W_b^{s(b)} \times \mathfrak{g}_a(X_a \times Y_a), P_{\leq t} E \rightarrow \colim_{a \in \mathcal{A}} \colim_{t \in \mathcal{K}} (X_a \times \mathfrak{g}_a Y_a), E
\]

is an isomorphism on the \( n \)-th homotopy groups. Therefore, the last weak-equivalence in (ii) follows.

\[\square\]

**Corollary 2.5.5.** Assume \( E \) denotes any one of the following spectra (a) the \( \mathbb{P}^1 \)-ring spectrum \( K \) representing algebraic \( K \)-theory on \( S \mathbb{P}^1/k \) or (b) the \( \mathbb{P}^1 \)-ring spectrum \( K \supset \mathbb{H}(Z/\ell) \cdots K \supset \mathbb{H}(Z/\ell) \) for some \( m \geq 1 \). Assume also that the remaining hypotheses of the last theorem hold. Then we obtain the weak-equivalences:

\[
\holim_{t \in \mathcal{K}} \colim_{a \in \mathcal{A}} \Pi_{b \leq a} \mathfrak{g}_a^{gm,s} \times \mathfrak{g}_a X_a, E \simeq \holim_{t \in \mathcal{K}} \colim_{a \in \mathcal{A}} \Pi_{b \leq a} \mathfrak{g}_a^{gm,s} \times \mathfrak{g}_a X_a, E \\
Theorem 2.5.1 \text{vii)} \text{ shows that the induced map}
\]

\[
\colim_{a \in \mathcal{A}} \colim_{t \in \mathcal{K}} \Pi_{b \leq a} \mathfrak{g}_a^{gm,s} \times \mathfrak{g}_a (X_a \times Y_a), E \rightarrow \colim_{a \in \mathcal{A}} \colim_{t \in \mathcal{K}} \mathfrak{g}_a (X_a \times Y_a), E
\]

is an isomorphism on the \( n \)-th homotopy groups. Therefore, the last weak-equivalence in (ii) follows.

\[\square\]

**Proof.** The \( \mathbb{P}^1 \)-spectrum representing algebraic \( K \)-theory corresponding to the \( \mathbb{P}^1 \)-spectrum satisfies all the required properties listed in Theorem 2.5.1. Moreover, it satisfies the defining property to be an \( \Omega \mathbb{S}_1 \)-spectrum except for the 0-th term. Therefore, the results of the last theorem apply. \[\square\]
3 Rigidity for mod-\(\ell\) Borel style generalized cohomology and equivariant K-theory

This section is devoted to a complete proof of Theorem 1.1.2. The main result is Theorem 3.0.7 whose proof depends on certain technical results that are discussed later on in this section, in Propositions 3.1.1 and 3.2.8. The strategy of the proof of Theorem 1.1.2 is as follows. We will first assume \(G\) is a finite group acting on smooth schemes \(X\) and \(Y\) of finite type over \(k\). We will then show that the following identification, in view of Lemma 3.0.6 below,

\[
\text{Spec}(O^{gm}_{V_m, x_n}) \times U_m \to \text{Spec}(O_{V_m, x_n})
\]

is a finite étale map since \(G\) is a finite group and the action of \(G\) on \(U_m\) is free. Therefore, \(\text{Spec}(O_{V_m, x_n}) \times U_m \to \text{Spec}(O_{V_m, x_n})\) breaks up as the

\[
\begin{align*}
U^m &= EG^{gm, m_1}_G \times \cdots \times EG^{gm, m_k}_G \times Y, \\
V^m &= EG^{gm, m_1}_G \times \cdots \times EG^{gm, m_k}_G \times Y, \\
U^m(i) &= EG^{gm, m}_G \times Y \text{ and } V^m(i) = EG^{gm, m}_G \times Y
\end{align*}
\]

where the geometric classifying spaces are defined making use of some choice of admissible gadgets as in Definition 2.1.1. Let \(\pi = \pi^m : U^m \times X = EG^{gm, m_1}_G \times \cdots \times EG^{gm, m_k}_G \times (Y \times X) \to V^m\) denote the corresponding projection. We begin with the weak-equivalences (where the hypercohomology is computed on the Nisnevich site):

\[
\mathbb{H}(U^m \times X, E) \simeq \mathbb{H}(V^m, R\pi^m_*E).
\]

This follows readily from the definition of \(R\pi^m_*E\) as \(\pi^m_*(GE)\) where \(E \to GE\) is a fibrant replacement or from the definition of \(R\pi^m_*\) if \(E\) is already assumed to be fibrant in the injective model structure. One may also observe that

\[
\begin{align*}
\mathbb{H}(EG^{gm}_G \times \cdots \times EG^{gm}_G \times (Y \times X), E) &= \operatorname{holim}_{- (m_1, \ldots, m_k)} \mathbb{H}(U^m \times X, E) \text{ and} \\
\mathbb{H}(EG^{gm}_G \times \cdots \times EG^{gm}_G \times Y, R\pi^m_*E) &= \operatorname{holim}_{- (m_1, \ldots, m_k)} \mathbb{H}(V^m, R\pi^m_*E).
\end{align*}
\]

Next we proceed to analyze the stalks of the presheaf \(R\pi^m_*E\). Afterwards we will extend the framework to include in the place of the scheme \(X\), also pro-objects in the category of smooth schemes of finite type over \(k\) provided with actions by inverse systems of groups. We will then show that the required rigidity statements follow by taking \(X\) to be either Spec \(k\) or Spec \(\overline{F}\).

Observe first that the points of \(\lim_{(m_1, \ldots, m_k) \to \infty} V^m\) are the residue fields at points of \(\bigcup_{n \geq 0}(S^{n_1, \ldots, n_k})\) where \(S^{n_1, \ldots, n_k} = (U_{m_1}(1) - U_{m_1-1}(1)) \times \cdots \times (U_{m_k}(k) - U_{m_k-1}(k))\). Let \(x_n\) denote such a fixed point belonging to the stratum \(S^n\), for some \(n = (n_1, \ldots, n_k) \geq 1\). Observe that if \(O^{gm, h}_{m'}\) denotes the Henselization of \(O_{V_m}\) at the point \(x_n\), then there are compatible maps \(O^{m, h}_{m'} \to O^{x_n, h}_{m'}\) for \(m' \geq m\).

Next we proceed to compute the stalks of the Nisnevich sheaf \(R\pi^m_*E\) at a point \(x_n \in S^n\). One obtains the following identification, in view of Lemma 3.0.6 below,

\[
(R\pi^m_*E)_{x_n} = \lim_{\overline{V}} \mathbb{H}(V \times U^m \times X, E) \simeq \mathbb{H}(\text{Spec}(O^{h}_{V_m, x_n}) \times U^m \times X, E).
\]

The induced map \(\text{Spec}(O^{h}_{V_m, x_n}) \times U^m \to \text{Spec}(O^{h}_{V_m, x_n})\) is a finite étale map since \(G\) is a finite group and the action of \(G\) on \(U_m\) is free. Therefore, \(\text{Spec}(O^{h}_{V_m, x_n}) \times U^m \to \text{Spec}(O^{h}_{V_m, x_n})\) breaks up as the
finite disjoint union of Hensel local rings \( B_i \): see [Mi, Theorem 4.2 and Corollary 4.3Chapter I]. Since \( G \) acts freely on \( U^n \), one can see that each fiber over the point \( x_n \) of \( \pi^m \) consists of the closed points of these local rings which are permuted by \( G \). Therefore, since each \( \text{Spec } B_i \) must be connected, one may in fact observe that elements of \( G \) act transitively on \( \sqcup_i \text{Spec } B_i \) and that all the Hensel rings \( B_i \) are isomorphic. (To see \( \text{Spec } B_i \) must be connected, one may argue as follows. Suppose that it is not connected. Then there exist nonzero idempotents \( e_i, i = 1, 2, \) so that \( e_1 + e_2 = 1 \) and with \( e_1.e_2 = 0 \). But since \( B_i \) is a local ring, it has no idempotents other than 0 and 1.) It follows one obtains the isomorphism

\[
\text{Spec } (O_{V^m,x_n}^b) \times U^m \cong \sqcup_{g \in G} \text{Spec } B \tag{3.0.10}
\]

where \( B \) denotes any one of the Hensel rings \( B_i \). Therefore, we obtain the identification

\[
V \times U^m \times X \cong (\text{Spec } B) \times \text{Spec } k \tag{3.0.11}
\]

Moreover, the definition of \( \text{Spec } B \) shows that it is independent of \( X \).

Next suppose \( \{X_a|a \in A\}, \{Y_a|a \in A\} \) are inverse systems of schemes and \( \{G_a|a \in A\} \) is an inverse system of finite groups, so that each \( X_a \) and \( Y_a \) is provided with an action by the group \( G_a \) and these actions are compatible. Suppose further that the structure maps of the inverse systems \( \{X_a|a \in A\} \) and \( \{Y_a|a \in A\} \) are flat as maps of schemes. In this situation, we may start with a closed immersion \( G_a \to \mathfrak{G}_a \) for each \( a \in A \), with \( \mathfrak{G}_a \) being a linear algebraic group. Then we may form an inverse system of algebraic groups by replacing each \( \mathfrak{G}_a \) with the finite product \( G_a = \prod_{b \leq a} \mathfrak{G}_b \) where the structure maps are induced by the obvious projection maps: see the construction in [2.1.1] for more details. Let \( \{s(a)|a \in A\} \) denote a sequence with each \( s(a) \) a non-negative integer. Now we let

\[
U_a^{s(a)} = \prod_{b \leq a} E \mathfrak{G}_b^{s(b)} \times Y_a, \tag{3.0.12}
\]

\[
V_a^{s(a)} = (\prod_{b \leq a} E \mathfrak{G}_b^{s(b)}) \times Y_a.
\]

We let \( \pi^{s(a)}: U_a^{s(a)} \times X_a \to V_a^{s(a)} \) denote the corresponding projections.

A point \( x \) of \( \{V_a^{s(a)}|a \in A\} \) corresponds to a compatible collection of points \( \{x_a|a \in A\} \) with each \( x_a \) being a point of \( V_a^{s(a)} \). Then the analysis above (see \( 3.0.10 \)) shows that one obtains a decomposition:

\[
\text{Spec } (O_{V^{s(a)}_a,x_a}^b) \times U_a^{s(a)} \cong \sqcup_{g \in G_a} \text{Spec } B_a \tag{3.0.13}
\]

Let \( E_a \) denote the restriction of the given presheaf of spectra \( E \) to the Nisnevich site of \( U_a^{s(a)} \times X_a \). In view of Proposition \( 3.2.8 \) we obtain the identification:

\[
R\pi^{s}_a(\Phi(\{E_a|a \in A\})) = \Phi(\{R\pi^{s}_aE_a|a \in A\})
\]

where \( n^{s(a)}: \text{Top}(U_a^{s(a)} \times X_a) \to \text{Top}(V^{s(a)}) \) and \( \pi^{s}: \text{Top}(\{U_a^{s(a)} \times X_a|a \in A\}) \to \text{Top}(\{V_a^{s(a)}|a \in A\}) \) are the induced maps of sites. Therefore, (see Corollary \( 3.2.3 \)) one obtains the following identification of the
stalk at $x = \{x_a|a \in \mathcal{A}\}$:

$$R\pi^* \Phi(\{E_a|a \in \mathcal{A}\}) \simeq \text{colim}_{a \in \mathcal{A}} \mathbb{H}(\text{Spec} B_a \times \mathcal{X}_a, E_a)$$  \hspace{1cm} (3.0.14)

where $\text{Spec} B_a$ depends only on $U^s_a(a)$ and not on $\mathcal{X}_a$. Next observe that the diagonal imbedding of $\mathcal{A}$ in $\mathcal{A} \times \mathcal{A}$ is cofinal, so that denoting by $B$ another copy of $\mathcal{A}$, one may identify the stalks above with

$$\text{colim}_{a \in \mathcal{A}} \text{colim}_{b \in B} \mathbb{H}(\text{Spec} B_a \times \mathcal{X}_b, E) \simeq \text{colim}_{a \in \mathcal{A}} \text{Map}_{s_1}(\Sigma S^1 \mathcal{X}_b+, (\text{Spec} B_a)+, E)$$

$$\simeq \text{colim}_{a \in \mathcal{A}} \text{Map}_{s_1}(\Sigma S^1 \mathcal{X}_b+, \text{Map}_{s_1}(\Sigma S^1 (\text{Spec} B_a)+, E))$$  \hspace{1cm} (3.0.15)

where $\text{Map}_{s_1}$ denotes the internal hom in the category of $\mathbb{P}^1$-spectra on the Nisnevich site of $\text{Sm}/k$.

Next assume that the field $k$ is algebraically closed and that $F$ is a field containing $k$. In view of our remarks in [1,1,1] we may assume without loss of generality that $F$ has finite transcendence degree over $k$. Let $G = G_F$ denote the absolute Galois group of $F$. Let $X = \{X_a|a \in \mathcal{A}\}$ denote the constant inverse system given by $X_a = \text{Spec} k$, where $k$ is given algebraically closed field $k$ and provided with the trivial action by the absolute Galois group $G_F = \{G_a|a \in \mathcal{A}\}$ of the field $F$. Let $Z = \{Z_a|a \in \mathcal{A}\}$ denote the inverse system of finite normal extensions $\{\text{Spec} F_a|a \in \mathcal{A}\}$ of the field $F$ together with the obvious action of $G_F$ on $\text{Spec} F$, i.e. a family of compatible actions by $G_a = \text{Gal}_F(F_a)$ on $F_a$. (Observe that a finite normal extension is the composition of a finite separable extension and finite purely inseparable extension. The automorphism group of such a finite normal extension identifies with the automorphism group of the corresponding separable extension, i.e. with the Galois group $G_a$.) In this situation, the obvious map of pro-objects $Z = \{Z_a|a \in \mathcal{A}\} \to X = \{X_a|a \in \mathcal{A}\}$ induced by the imbeddings $\{k \to F \to F_a|a \in \mathcal{A}\}$ is $G_F$-equivariant. Therefore, one obtains an inverse system of commutative diagrams for each fixed sequence $s$:

$$U^s_a \times Z_a \longrightarrow U^s_a \times X_a$$

$$\downarrow \hspace{1cm} \downarrow$$

$$V^s_a \longrightarrow V^s_a$$

$$\hspace{1cm} \text{id}$$

$$\downarrow$$

$$V^s_a \times X_a \longrightarrow V^s_a$$

$$\hspace{1cm} \downarrow$$

$$\downarrow$$

Lemma 3.0.6. Let $G_a$ denote a finite group acting on a smooth scheme $X_a$ of finite type over $k$. Let $\pi : U^s_a \to V^s_a$ denote the principal $G_a$-bundle as before. Then for any $V \to V^s_a$ the square

$$\begin{array}{ccc}
(V \times U^s_a)_{G_a} & \longrightarrow & V \\
\downarrow & & \downarrow \\
U^s_a \times X_{G_a} & \longrightarrow & V^s_a
\end{array}$$

is a cartesian square.

Proof. Let $Z$ denote a scheme with compatible maps $\alpha : Z \to V$, $\beta : Z \to U^s_a \times X_{G_a}$ and $\gamma : Z \to V^s_a$. Since the $G_a$-action on $U^s_a$ is free one may view $U^s_a \times X_{G_a}$ as the quotient stack $[(U^s_a \times X_{G_a})/G_a]$
one observes that the given map $\beta : Z \to [(U^{s(a)}_a \times \mathcal{X}_a)/G_a]$ corresponds to giving a principal $G_a$-bundle $z : \tilde{Z} \to Z$ together with a $G_a$-equivariant map $\phi : \tilde{Z} \to U^{s(a)}_a \times \mathcal{X}_a$. Now the map $z$ together with the given maps of $Z$ provide compatible $G_a$-equivariant maps $\tilde{\alpha} = \alpha \circ z : \tilde{Z} \to V$, $\tilde{\gamma} = \gamma \circ z : \tilde{Z} \to V^{s(a)}_a$ and $\tilde{\beta} = \phi : \tilde{Z} \to V^{s(a)}_a \times \mathcal{X}_a$. Since the square

$$
\begin{array}{ccc}
V \times U^{s(a)}_a \times \mathcal{X}_a & \longrightarrow & V \\
\downarrow & & \downarrow \\
U^{s(a)}_a \times \mathcal{X}_a & \longrightarrow & V^{s(a)}_a
\end{array}
$$

is evidently a cartesian square of $G_a$-schemes, one obtains an induced $G_a$-equivariant map $\tilde{\phi} : \tilde{Z} \to V \times U^{s(a)}_a \times \mathcal{X}_a$. Since $V \times U^{s(a)}_a \times \mathcal{X}_a \to V \times U^{s(a)}_a \times \mathcal{X}_a$ is a principal $G_a$-bundle, the map $\tilde{\phi}$ descends to define a map $Z \to V \times U^{s(a)}_a \times \mathcal{X}_a$, thereby proving the lemma.

**Theorem 3.0.7.** Assume in addition to the above situation that the spectrum $E$ is one of the spectra as in Theorem 1.1.2 i.e. $E = E'/\ell$, $E' \wedge H(Z/\ell) \cdots E' \wedge H(Z/\ell)$ for some ring spectrum $E \in \text{Spt}_S(k)$. Then the corresponding induced maps of the stalks

$$R\pi^a_*(\mathcal{F}(\{E_a|a \in \mathcal{A}\})) \to R\pi^a_*\mathcal{F}(\{E_a|a \in \mathcal{A}\})$$

is a weak-equivalence. It follows therefore, that the induced map

$$\text{colim}_{a \in \mathcal{A}} \mathbb{H}_{G_a}(U^{s(a)}_a \times (\text{Spec } k)/G_a, E) \to \text{colim}_{a \in \mathcal{A}} \mathbb{H}_{G_a}(U^{s(a)}_a \times (\text{Spec } F_a)/G_a, E)$$

is a weak-equivalence for each fixed sequence $s$. Therefore, one also obtains a weak-equivalence on taking the homotopy inverse limit over all $s$ of the above colimits. i.e. The induced map

$$\mathbb{H}_{G_p}(\text{Spec } k \times Y, E) = \text{holim}_{s \in K} \text{colim}_{a \in \mathcal{A}} \mathbb{H}_{G_a}(U^{s(a)}_a \times (\text{Spec } k)/G_a, E)$$

$$\to \text{holim}_{s \in K} \text{colim}_{a \in \mathcal{A}} \mathbb{H}_{G_a}(U^{s(a)}_a \times (\text{Spec } F_a)/G_a, E) = \mathbb{H}_{G_p}(\text{Spec } \tilde{F} \times Y, E) \quad (3.0.17)$$

is a weak-equivalence, where $Y = \{\mathcal{Y}_a|a \in \mathcal{A}\}$. In particular, taking $Y = \text{Spec } k$, one obtains the weak-equivalence:

$$\mathbb{H}_{G_p}(\text{Spec } k, E) \to \mathbb{H}_{G_p}(\text{Spec } \tilde{F}, E).$$

**Proof.** The first statement of the theorem, providing the weak-equivalence of the stalks follows readily from the identification in (3.0.15), the observation that the choice of Spec $B_a$ depends only on the $U^{s}_a$ and Proposition 3.1.1 applied with the scheme $W$ denoting Spec $B_a$ for a fixed $a$. This will first show that the induced maps of the term in (3.0.15) will be a weak-equivalence on taking the colimit over $B$ for each fixed $a \in \mathcal{B}$. Then one may also take the colimit over $\mathcal{A}$ to obtain a weak-equivalence of the corresponding stalks.

The second statement then follows readily from Proposition 3.2.8. The third statement then follows from the second by taking the homotopy inverse limit holim. Finally the last statement follows from this
by observing from Definition 2.4.3 that now $U_a^{s(a)} = \Pi_{b \leq a} E \mathcal{O}_b^{s(b)}$ so that
\[
\mathbb{H}_{G_p}(\text{Spec } k, E) = \operatorname{holim}_{s \in K} \mathbb{H}_{G_a}(U_a^{s(a)} \times (\text{Spec } k), E) \quad \text{and that}
\]
\[
\mathbb{H}_{G_p}(\text{Spec } \bar{F}, E) = \operatorname{holim}_{s \in K} \mathbb{H}_{G_a}(U_a^{s(a)} \times (\text{Spec } F_a), E).
\]

\[
\square
\]

Observe that the last theorem proves Theorem 1.1.2 when the spectrum $E = E/l = E \wedge M(l)$. We proceed to consider the other situations considered in the theorem. When the spectrum $E$ has been replaced by $E \wedge_{\Sigma} \mathbb{H}(\mathbb{Z}/\ell)$ the conclusion follows as in the last case since the homotopy groups of this spectrum are all $l$-primary torsion. It follows that the same conclusion holds when the spectrum $E$ has been replaced by the iterated derived smash product: $E \wedge_{\Sigma} \mathbb{H}(\mathbb{Z}/\ell) \wedge_{\Sigma} \mathbb{H}(\mathbb{Z}/\ell) \cdots \wedge_{\Sigma} \mathbb{H}(\mathbb{Z}/\ell)$. This completes the proof of Theorem 1.1.2.

3.1 Rigidity for spectra (after Yagunov): see [Yag]

Proposition 3.1.1. (i) Let $E$ denote a spectrum and let $X \mapsto h(X, E) = \operatorname{Map}(\Sigma S^1 X_+, E)$ denote the generalized cohomology theory defined with respect to the spectrum $E$. If $W$ is a fixed smooth scheme of finite type over $k$, then the functor $X \mapsto h(X \times W, E) = \operatorname{Map}(\Sigma S^1 X_+ \wedge W_+, E)$ is represented by the $S^1$-spectrum $\operatorname{Map}(\Sigma S^1 W_+, E)$. This spectrum also satisfies all the properties in 1.1.1 except possibly for Nisnevich excision and the last two properties. In addition, if $l.h^*(X, E) = 0$ for all smooth schemes $X$ of finite type over $k$, then $l.h^*(X \times W, E) = 0$ also. (Here $h^*$ denotes the homotopy groups of the spectrum $h(\quad, E)$ re-indexed in the obvious manner.)

(ii) Suppose $E$ is a spectrum so that $l.h^*(X, E) = 0$ for all smooth schemes $X$ of finite type over $k$. Then if $W$ is any smooth scheme of finite type over $k$, the obvious map $\text{id} \times i : W \times \text{Spec } \bar{F} \to \text{Spec } k$

\[
h^*(W \times \text{Spec } k, E) \to h^*(W \times \text{Spec } \bar{F}, E)
\]

Proof. (i) is more or less obvious. However, we will provide some details for the convenience of the reader. Since $X \mapsto h(X \times W, E)$ is also a cohomology theory, it suffices to verify that it extends to define a cohomology theory for pairs of smooth schemes $(X, Y)$ with $Y$ locally closed in $X$ satisfying the properties discussed in [Yag] Definition 1.2]. One may extend such a generalized cohomology theory to such pairs $(X, Y)$ by defining $h_{Y \times W}(X \times W, E)$ to be the homotopy fiber of $h(X \times W, E) \to h(Y \times W, E)$. If $X$ and $Y$ are both smooth schemes of finite type over $k$ and $Y$ is locally closed in $X$, we call $(X, Y)$ a smooth pair. Then one needs to check this extension to smooth pairs satisfies the following:

- Suspension: if $(X, Y)$ is a smooth pair, then $h_{Y \times W \times 0}(X \times W \times \mathbb{A}^1, E) \simeq h_{Y \times W}(X \times W, E)$
- Zariski excision: if $Z \subseteq X_0 \subseteq X$ are smooth schemes with $Z$ closed in $X_0$ and $X_0$ open in $X$, then, one obtains a weak-equivalence: $h_{Z \times W}(X_0 \times W, E) \simeq h_{Z \times W}(X \times W, E)$
- Homotopy invariance: If $(X, Y)$ is a smooth pair, then $h_{Y \times W \times \mathbb{A}^1}(X \times W \times \mathbb{A}^1) \simeq h_{Y \times W}(X \times W)$.
- Homotopy Purity: If $Z \subseteq Y \subseteq X$ are closed immersions of smooth schemes, $N$ is the normal bundle to the immersion of $Y$ in $X$ and $B(X, Y)$ denotes the deformation space obtained by blowing
up $X \times A^1$ along $Y \times 0$ and removing the divisor $P(N)$. Then one obtains weak-equivalences:

$$h_{Z \times W}(N \times W) \simeq h_{Z \times W \times A^1}(B(X, Y) \times W) \simeq h_{Z \times W}(X \times W).$$

Of these, the first three properties follow readily from the fact they hold for the generalized cohomology theory represented by the spectrum $E$. The last property may be verified using the observation that $B(X, Y) \times W \cong B(X \times W, Y \times W)$ and the homotopy purity theorem proven in [MV] Theorem 2.23, p. 115.

Now the statement (ii) follows from [Yag] Theorem 1.10 since the statement in (i) verifies that all the hypotheses in [Yag] Theorem 1.10 are satisfied.

3.2 Sites for pro-schemes

Let $\text{Sch}(S)$ denote the category of schemes of finite type over the fixed Noetherian base scheme $S$. Observe that each scheme $X \in \text{Sch}(S)$ is automatically quasi-compact. We will let $\mathcal{A}$ denote a partially ordered directed set which is cofinite as in [2.4.1] i.e. for each fixed $a \in \mathcal{A}$, the set of elements $b \in \mathcal{A}$ so that $b \leq a$ is finite and the relation $\leq$ is anti-symmetric. In particular, this means that there is at most one map between any two objects in $\mathcal{A}$. Then a pro-object in $\text{Sch}(S)$ will mean an inverse system of schemes $\{X_a|a \in \mathcal{A}\}$. Observe that, there is at most one structure map between any two $X_a$ and $X_b$ that are part of the inverse system.

A level representation of a map $f : \mathcal{X} \rightarrow \mathcal{Y}$ of pro-schemes will mean a choice of isomorphisms $\mathcal{X} \cong \{X_a|a \in \mathcal{A}\}$ and $\mathcal{Y} \cong \{Y_a|a \in \mathcal{A}\}$ of pro-schemes together with a compatible collection of maps $\{f_a : X_a \rightarrow Y_a|a \in \mathcal{A}\}$ which represents $f$. Recall (see [AM] Appendix 3.2]) any finite diagram of maps of pro-schemes with no loops may be replaced up to isomorphism by a diagram of level-maps. Therefore, when considering such finite diagrams of pro-objects, we will always consider level maps.

One begins with a small site on $\text{Sch}(S)$, which we will denote by $\text{Top}$. The objects of the corresponding small site for a pro-object $\mathcal{X} = \{X_a|a \in \mathcal{A}\}$ will be pro-objects $U = \{U_a|a \in \mathcal{A}\}$ provided with a level-map $U \rightarrow \mathcal{X}$ of pro-objects, so that for each $a \in \mathcal{A}$, the map $U_a \rightarrow X_a$ belongs to the given site $\text{Top}(X_a)$. Morphisms between two objects $U = \{U_a|a \in \mathcal{A}\} \rightarrow V = \{V_a|a \in \mathcal{A}\}$ will be obvious commutative triangles of level maps of pro-objects over $\mathcal{X}$. A covering of a pro-object $\mathcal{X}$ will be a collection $\{U^\alpha|\alpha\}$ of objects in the site over $\mathcal{X}$ so that for each fixed $a \in \mathcal{A}$, $\{U^\alpha_a|\alpha\}$ will be a cover of $X_a$. One may contrast this with $\{\text{Top}(X_a)|a \in \mathcal{A}\}$ which is a fibered site, fibered over $\mathcal{A}$.

It is clear that if $f : \mathcal{X} = \{X_a|a \in \mathcal{A}\} \rightarrow \mathcal{Y} = \{Y_a|a \in \mathcal{A}\}$ is a level-map, it induces a morphism of sites: $f_* : \text{Top}(\mathcal{X}) \rightarrow \text{Top}(\mathcal{Y})$ where $f_*(U) = \{U_a \times X_a|a \in \mathcal{A}\}$, where $U = \{U_a|a \in \mathcal{A}\}$.

Let $a_0 \in I$ denote fixed index and let $(\cdot)_{a_0}$ denote the functor sending an object $\mathcal{X} = \{X_a|a \in \mathcal{A}\}$ to $X_{a_0}$. Moreover, the restriction functor $(\cdot)_{a_0} : \text{Top}(\mathcal{X}) \rightarrow \text{Top}(X_{a_0})$ defines a map of sites $\text{Top}(X_{a_0}) \rightarrow \text{Top}(\mathcal{X})$ sending $U = \{U_a|a \in \mathcal{A}\} \rightarrow U_{a_0}$. Henceforth a presheaf (sheaf) will generically denote a presheaf (sheaf) of pointed sets (pointed simplicial sets, abelian groups or rings etc.) Next observe that a presheaf (sheaf) on the fibered site $\{\text{Top}(X_a)|a \in \mathcal{A}\}$ is given by a collection of presheaves (sheaves) $\{P_a|\text{Top}(X_a)|a \in \mathcal{A}\}$ together with compatibility data for the structure maps of the inverse system. Let $\text{Psh}(\mathcal{X})$ (Sh(\mathcal{X})) denote the corresponding category of presheaves (sheaves, respectively) of pointed sets on $\text{Top}(\mathcal{X})$. We let $\{\text{Psh}(X_a)|a \in \mathcal{A}\}$ denote the category of presheaves of pointed sets on the fibered site $\{\text{Top}(X_a)|a \in \mathcal{A}\}$. Then we define a functor

$$\Phi : \{\text{Psh}(X_a)|a \in \mathcal{A}\} \rightarrow \text{Psh}(\mathcal{X})$$
as follows. Let \( F = \{ F_a | a \in \mathcal{A} \} \in \{ \text{Psh}(X_a) | a \in \mathcal{A} \} \). Then
\[
\Gamma(\mathcal{U}, \Phi(F)) = \lim_{a \in \mathcal{A}} \Gamma(U_a, F_a) = \lim_{a \in \mathcal{A}} \Gamma(\mathcal{U}, (\lambda_a)(F_a)) \tag{3.2.1}
\]

One may now observe that \( \Phi(F) = \lim_{i \to \infty} \Phi(F_a) \), where \( F = \{ F_a | a \in \mathcal{A} \} \).

**Definition 3.2.1.** A point for the site \( \text{Top}(\mathcal{X}) \) will denote a compatible collection of points \( p = \{ p_a | a \in \mathcal{A} \} \) with \( p_a \) being a point of the site \( \text{Top}(X_a) \). (Recall this means if \( \lambda_{a,b} : X_a \to X_b \) is the structure map of \( \mathcal{X} \), then \( \lambda_{a,b} \circ p_a = p_b \), for every \( a \) and \( b \).)

**Proposition 3.2.2.** Let \( p = \{ p_a | a \in \mathcal{A} \} \) denote a point of the site \( \text{Top}(\mathcal{X}) \) and let \( U_{a_0} \) denote a neighborhood of \( p_{a_0} \) in \( \text{Top}(X_{a_0}) \), for a fixed \( a_0 \in \mathcal{A} \). Then there exists a neighborhood \( \mathcal{U} \) of \( p \) in \( \text{Top}(\mathcal{X}) \) so that the map \( U_{a_0} \to X_{a_0} \) factors through the given map \( U_{a_0} \to X_{a_0} \).

**Proof.** We construct \( \mathcal{U} \) in steps.

**Step 1.** Recall that \( a_0 \) has only finitely many descendants. A string \( b_n \leq b_{n-1} \leq \cdots \leq b_1 \leq b_0 = a_0 \) of descendants of \( a_0 \) will be called a path of length \( n \) starting at \( a_0 \) and ending at \( b_n \). For each such path, one starts with a neighborhood of \( p_{b_n} \) and takes iterated pull-backs to define neighborhoods of \( p_{b_i} \), \( i = n, n-1, \cdots, 1, 0 \). For each vertex \( b \) on such a path, take the fibered product over \( X_b \) of all such neighborhoods of \( p_b \) obtained from varying paths starting at \( a_0 \) and passing through descendants of \( a_0 \). Finally take the fibered product over \( X_{a_0} \) of such a neighborhoods of \( p_{a_0} \) obtained this way with the given neighborhood \( U_{a_0} \). At this point, we will replace the original neighborhood \( U_{a_0} \) by this neighborhood: clearly we have now a compatible system of neighborhoods for \( p_{a_0} \) and all the \( \{ p_b | b < a_0 \} \).

**Step 2.** Let \( a' \in \mathcal{A}, a' \neq a_0 \). If \( a' \) is a descendant of \( a_0 \), we have already constructed a neighborhood of \( p_{a'} \) compatible with the neighborhood \( U_{a_0} \). Therefore, without loss of generality, we may assume \( a' \) is not a descendant of \( a_0 \). Since \( a' \) has only finitely many descendants, there are only finitely many paths that start at \( a' \) and passing through its descendants.

For each path that starts at \( a' \) and ends at a descendant \( d_{a_0} \) of \( a_0 \), one may take iterated pull-back of the already chosen neighborhood \( U_{d_{a_0}} \) to define neighborhoods of every \( p_b \), for every vertex \( b \) on such a path. For each path that starts at \( a' \) and ends at a vertex \( v \) that is not a descendant of \( a_0 \), one chooses \( X_v \) as the neighborhood of each \( p_v \), for a vertex \( b \) on this path.

For each of the descendants \( b \) of \( a' \) (including \( a' \)) there will be only finitely many such paths that start at \( a' \) and pass through \( b \); now take the iterated fibered product over \( X_b \) of all the neighborhoods of \( p_b \), \( b \) which are descendants of \( a_0 \), chosen in step 1. Moreover if \( a' \) is a descendant of \( a'' \), then the system of neighborhoods constructed this way for paths starting at \( a'' \) leaves intact the neighborhoods constructed for any \( p_c \), where \( c \) is a descendant of \( a' \). Therefore, the system of neighborhoods constructed this way for each \( p_a \), defines a compatible system of neighborhoods \( \mathcal{U} \) for the point \( p \). Since the structure map \( U_{a_0} \to X_{a_0} \) factors through the given map \( U_{a_0} \to X_{a_0} \) one started out with, the last statement in the proposition is clear. \( \square \)

**Corollary 3.2.3.** Let \( F = \{ F_a | a \in \mathcal{A} \} \in \{ \text{Psh}(X_a) | a \in \mathcal{A} \} \) and let \( p = \{ p_a | a \in \mathcal{A} \} \) denote a compatible collection of points of \( \mathcal{X} = \{ X_a | a \in \mathcal{A} \} \). Then
\[
\Phi(F)_p = \lim_{a \in \mathcal{A}} (F_a)_{p_a} \tag{3.2.2}
\]
Proof. This is clear in view of the last statement in the last Proposition.

Remark 3.2.4. In several cases of interest the directed set $\mathcal{A}$ would be a product of $\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. In this case therefore the inverse system of schemes $\{X_\alpha | \alpha \in \mathcal{A}\}$ will reduce to a product of towers of schemes, where each tower is indexed by $\mathbb{N}$. In this case the construction in Proposition 3.2.2 is particularly simple, as there is a single path that starts at a given vertex and passes through its descendants.

### 3.2.3 Model structures on simplicial presheaves

The local projective model structure (as in [Bl]) or the local flasque model structure (as in [Isak]) will be the ones we put on the topos of simplicial presheaves on the above site. We proceed to recall their definitions in a general context.

Let $\mathcal{C}$ denote any essentially small site and let $\text{SPrsh}(\mathcal{C})$ denote the category of all pointed simplicial presheaves on $\mathcal{C}$. Then the local projective model structure on $\text{SPrsh}(\mathcal{C})$ has the following structure: the generating trivial cofibrations will be those maps of the form $\Lambda[n]_+ \land U_+ \to \Delta[n]_+ \land U_+$, where $\Lambda[n]$ denotes a horn of $\Delta[n]$. If $K$ is a simplicial set, $K_+ \land U_+$ is the simplicial presheaf defined by $(K_+ \land U_+)_\alpha = (\sqcup_{\epsilon \in K_\alpha} U_\epsilon)_+$ with the structure maps induced from those of $K$. The generating cofibrations are those maps of the form: $\delta \Delta[n]_+ \land U_+ \to \Delta[n]_+ \land U_+$, with $U \in \mathcal{C}$, $n \geq 1$. The weak-equivalences will be those maps $f : F' \to F$ of simplicial presheaves that induce weak-equivalence stalk-wise and the fibrations are defined by the right-lifting-property with respect to trivial cofibrations. Then this model category is a cofibrantly generated simplicial model category which is also proper and cellular.

Next we discuss the flasque model structure. Let $X \in \mathcal{C}$ and let $\mathcal{U} = \{U_\alpha \to X | \alpha = 1, \cdots, n\}$ denote a finite collection of monomorphisms in $\mathcal{C}$. Then one defines $\mathcal{U} \mathcal{U} = U_\alpha \sqcup \cdots \sqcup U_\beta$ to denote the co-equalizer of the diagram: $\sqcup_{\alpha, \beta} U_\alpha \times U_\beta \to \sqcup_{\alpha} U_\alpha$, where the top row (bottom row) is the projection to the first (second, respectively) factor. Given a map $f : F \to G$ in $\text{SPrsh}(\mathcal{C})$ and a map $g : K \to L$ of pointed simplicial sets, the pushout-product $f \Box g$ is the map $G \land K \lor_{F \land K} F \land L \to G \land L$. Then we define the flasque model structure on $\text{SPrsh}(\mathcal{C})$ to have the following structure: the generating trivial cofibrations are of the form $f \Box g$, where $f : \mathcal{U} \mathcal{U} \to U$ in $\text{SPrsh}(\mathcal{C})$ and $g : \Lambda[n] \to \Delta[n]$, $n \geq 1$ and the generating cofibrations are of the form $f \Box g$, where $f$ is as above and $g : \delta \Delta[n] \to \Delta[n]$, $n \geq 1$. The weak-equivalences are those maps $f : F' \to F$ of simplicial presheaves that induce weak-equivalence stalk-wise and the fibrations are defined by the right-lifting-property with respect to trivial cofibrations. Then this model category is a a cofibrantly generated simplicial model category which is also left-proper and cellular.

The weak-equivalences in both of the above model structures will often be referred to as local weak-equivalences. In [Bl] and [Isak], what is considered is the unpointed case; but their results readily extend to the pointed case as shown in [Hov-1] Proposition 1.1.8 and Lemma 2.1.21. The main use of the above model structures is to be able to prove the following result, which is used in the body of the paper.

**Proposition 3.2.5.** Suppose $f : E' \to E$ is a local weak-equivalence between fibrant objects in $\text{SPrsh}(\mathcal{C})$ provided with any of the above model structures. Then $\Gamma(U, f)$ is also a weak-equivalence for any $U \in \mathcal{C}$. (Maps $f$ as above, for which $\Gamma(U, f)$ is a weak-equivalence for every $U \in \mathcal{C}$ will be called section-wise weak-equivalences.)

**Proof.** We will first prove this result under the assumption that $f$ is also a fibration. Let $* = \Delta[0]$, let $n \geq 1$ be any integer and let $U \in \mathcal{C}$. In this case we observe that it suffices to show that both the diagrams...
have a lifting from the left-bottom corner to the right-top corner. To see this, observe that a class in \( \pi_{n-1}(\Gamma(U, K)) \) corresponds to a map \( \delta \Delta[n] \to \Gamma(U, K) \) for any fibrant object in \( K \) in \( \text{SPrsh}(C) \). Therefore, the lifting in the first square shows that the induced map \( \pi_{n-1}(\Gamma(U, f)) \) is surjective. A lifting in the second square then shows that the induced map \( \pi_{n-1}(\Gamma(U, f)) \) is injective as well.

Next assume that \( f \) is a fibration as well. Then the two diagrams above correspond to the two diagrams:

\[
\begin{array}{ccc}
* \times U & \to & E' \\
\downarrow & & \downarrow \\
\delta \Delta[n] \times U & \to & \Gamma(U, E')
\end{array}
\quad \quad \begin{array}{ccc}
\delta \Delta[n] \times U & \to & \Gamma(U, E') \\
\downarrow & & \downarrow \\
\Delta[n] \times U & \to & \Gamma(U, E)
\end{array}
\]

The required liftings then correspond to liftings in the above squares, i.e. maps from the left-bottom corner to the right-top corner. But these exist since \( f \) is a trivial fibration and both the maps \( * \times U + \to \delta \Delta[n] \times U + \) and \( \Delta[n] \times U \to \Delta[n] \times U + \) are cofibrations both in the local projective model structure and the local flasque model structure.

Finally we consider the general case where \( f \) is required to be only a map that is a weak-equivalence and both \( E' \) and \( E \) are required to be fibrant in either of the above model structures. This then follows by invoking what is called Ken Brown’s lemma: see [Hov-1 Lemma 1.1.12]. We will provide some details on this for the convenience of the reader. Let \( f : E' \to E \) be the given map. Form the product \( E' \times E \) with the projection to the factor \( E' \) (denoted \( p_1 \) (\( p_2 \), respectively)). The maps \( f : E' \to E \) and the identity \( \text{id} : E' \to E' \) induce a map \( E' \to E' \times E \). Factor this map as the composition of a trivial cofibration \( i \) followed by a fibration \( q \). Then \( p_1 \circ q \circ i = \text{id}_{E'} \) and \( p_2 \circ q \circ i = f \): so \( p_1 \circ q \) and \( p_2 \circ q \) are both weak-equivalences as well as fibrations. Therefore, by what is proven above, \( \Gamma(U, p_1 \circ q) \) and \( \Gamma(U, p_2 \circ q) \) are both weak-equivalences for any \( U \in C \). But \( \Gamma(U, \text{id}) = \Gamma(U, p_1 \circ q \circ i) = \Gamma(U, p_1 \circ q) \circ \Gamma(U, i) \) so that \( \Gamma(U, i) \) is a weak-equivalence. Therefore, \( \Gamma(U, f) = \Gamma(U, p_2 \circ q \circ i) = \Gamma(U, p_2 \circ q) \circ \Gamma(U, i) \) is also a weak-equivalence. This completes the proof of the proposition.

**Proposition 3.2.6.** Assume the site \( C \) has enough points. For an \( F \in \text{SPrsh}(C) \), let \( \mathcal{G} F = \text{holim}_{\Delta} \{ G^n F | n \} \) denote the simplicial presheaf defined by the Godement resolution \( \{ G^n F | n \} \) which is a cosimplicial object of \( \text{SPrsh}(C) \) augmented by \( F \). Then the obvious maps \( \Gamma(U, \mathcal{G} F) \to * \) is a fibration for any \( U \in C \) and \( \mathcal{G} F \) is fibrant in the local projective model structure if \( F \) is stalk-wise fibrant. Therefore, if \( f : F' \to F \) is a stalk-wise weak-equivalence in \( \text{SPrsh}(C) \) between pointed simplicial presheaves that are stalk-wise fibrant, then \( \Gamma(U, \mathcal{G} f) \) is a weak-equivalence for every \( U \in C \).

**Proof.** In order to prove the first statement, it suffices to show that one has liftings in the diagrams (from the left bottom corner to the right top corner):

\[
\begin{array}{ccc}
\Delta[n] & \to & \Gamma(U, \mathcal{G} F) \\
\downarrow & & \downarrow \\
* & \to & \mathcal{G} F
\end{array}
\quad \quad \begin{array}{ccc}
\Delta[n] \times U & \to & \Gamma(U, \mathcal{G} F) \\
\downarrow & & \downarrow \\
* \times U & \to & \mathcal{G} F
\end{array}
\]

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Clearly a lifting in the second diagram implies a lifting in the first diagram, so that it suffices to establish a lifting in the second diagram. Next consider the model structure on cosimplicial objects of \(SPrsh(C)\) defined by viewing cosimplicial objects as diagrams of type \(\Delta\) and invoking [Hirsch, Theorem 11.6.1]; recall the fibrations (weak-equivalences) in this model structure are those maps \(f : X^\bullet \to Y^\bullet\) which are fibrations (weak-equivalences) for each fixed cosimplicial index \(n\) and cofibrations are generated by the free diagram construction applied to the generating cofibrations in \(SPrsh(C)\). Making use of the definition of the homotopy inverse limit, this amounts to obtaining lifting in the following diagram:

\[
\begin{array}{ccc}
\Lambda[n]_+ \land U_+ \land B(\Delta \downarrow -)^{op} & \xrightarrow{\{G^n F|n\}} & \{G^n F|n\} \\
\downarrow & & \downarrow \\
\Delta[n]_+ \land U_+ \land B(\Delta \downarrow -)^{op} & \xrightarrow{*} & *
\end{array}
\]

Now \(B(\Delta \downarrow -)^{op}\) is a cofibrant object in this model structure on cosimplicial objects of \(SPrsh(C)\) so that the left-vertical map in the above diagram is a trivial cofibration. Therefore, the required lifting exists, thereby proving the first statement. The second statement then follows from Proposition 3.2.5.

**Proposition 3.2.7.** Let \(\phi : C \to D\) denote a morphism of sites. Assume \(SPrsh(C)\) and \(SPrsh(D)\) are both provided with the local projective model structures. Then \(\phi_*\) preserves fibrations and trivial fibrations. It also preserves weak-equivalences between fibrant objects. In particular, if \(f : F' \to F\) is a stalk-wise weak-equivalence between objects in \(SPrsh(C)\) that are fibrant stalk-wise, then \(\Gamma(V, \phi_*(G f))\) is a weak-equivalence for every \(V \in D\).

**Proof.** Now consider the squares:

\[
\begin{array}{ccc}
\Lambda[n]_+ \land V_+ & \xrightarrow{\phi_*(F')} & \phi_*(F') \\
\downarrow & \phi_*(f) & \downarrow \phi_*(f) \\
\Delta[n]_+ \land V_+ & \xrightarrow{\phi_*(F)} & \phi_*(F)
\end{array}
\]

\[
\begin{array}{ccc}
\delta\Lambda[n]_+ \land V_+ & \xrightarrow{f_*(F')} & f_*(F') \\
\downarrow & \phi_*(f) & \downarrow \phi_*(f) \\
\delta\Delta[n]_+ \land V_+ & \xrightarrow{\phi_*(F)} & \phi_*(F)
\end{array}
\]

In order to prove the first statement, it suffices to show that if \(f\) is a fibration (trivial fibration) then there is a lifting in the first (second, respectively) square. By adjunction, the above squares correspond to:

\[
\begin{array}{ccc}
\Lambda[n]_+ \land \phi^{-1}(V)_+ & \xrightarrow{F'} & F' \\
\downarrow & f & \downarrow f \\
\Delta[n]_+ \land \phi^{-1}(V)_+ & \xrightarrow{F} & F
\end{array}
\]

\[
\begin{array}{ccc}
\delta\Lambda[n]_+ \land \phi^{-1}(V)_+ & \xrightarrow{F'} & F' \\
\downarrow & f & \downarrow f \\
\delta\Delta[n]_+ \land \phi^{-1}(V)_+ & \xrightarrow{F} & F
\end{array}
\]

A lifting exists in the first square (second square) when \(f\) is a fibration (trivial fibration, respectively). Now use the adjunction one more to carry over these liftings to the liftings in the original squares. This proves the first statement. The second statement then follows by invoking Ken Brown’s lemma: see [Hov-1, Lemma 1.1.12]. The proof of the last statement is now clear.

Next let \(\mathcal{X} = \{X_a|a \in \mathcal{A}\}\) denote a pro-object of \(Sch(S)\). We will provide the site \(Top(\mathcal{X})\) as well as the fibered site \(\{Top(X_a)|a \in \mathcal{A}\}\) with the local projective model structures. Accordingly the generating cofibrations (generating trivial cofibrations) in \(Top(\mathcal{X})\) are given by \(\{\delta\Lambda[n]_+ \land U_+ \to \Delta[n]_+ U_+ | n \geq 1, U \in Top(\mathcal{X})\}\) (\(\{\Lambda[n]_+ \land U_+ \to \Delta[n]_+ U_+ | n \geq 1, U \in Top(\mathcal{X})\}\), respectively). The generating cofibrations (generating trivial cofibrations) in the fibered site \(\{Top(X_a)|a \in \mathcal{A}\}\) are given by \(\{\delta\Lambda[n]_+ \land U_+ \to \Delta[n]_+ \land U_+ | n \geq 1, U \in Top(\mathcal{X})\}\), respectively.)
$U_+|n \geq 1, U \in \text{Top}(X_a)$, for some $a$ \{(\Lambda[n]_+ \wedge U_+ \to \Delta[n]_+ \wedge U_+|n \geq 1, U \in \text{Top}(X_a)$, for some $a$, respectively).

**Proposition 3.2.8.** Assume the above situation. Then the functor $\Phi : \{\text{SPrsh}(\text{Top}(X_a))|a \in A \} \to \text{SPrsh}(\text{Top}(\mathcal{X}))$ preserves fibrations and trivial fibrations. It sends local weak-equivalences between fibrant objects to section-wise weak-equivalences. In particular, if $f : \text{F} = \{\text{F}_a|a \in A \} \to \text{F} = \{\text{F}_a|a \in A \}$ is a stalk-wise weak-equivalence, then the induced map $\Gamma(U, \mathcal{G})$ is also a weak-equivalence for every $U \in \text{Top}(\mathcal{X})$. If $f : \mathcal{X} \to \mathcal{Y}$ is a map of pro-objects in Sch(S), the functor $\Phi$ commutes with $f$.

**Proof.** Let $f : \text{F} \to \text{F}$ denote a map in $\{\text{SPrsh}(\text{Top}(X_a))|a \in A \}$ which is a fibration, i.e. each $f_a : \text{F}_a \to \text{F}_a$ is a fibration in $\text{SPrsh}(\text{Top}(X_a))$. To prove the first statement, it suffices to show that one has a lifting in the squares:

$$
\begin{align*}
\Lambda[n]_+ \wedge U_+ & \xrightarrow{\Phi(\text{F}')_{\text{a}}} \Phi(\text{F}') \\
\Delta[n]_+ \wedge U_+ & \xrightarrow{\Phi(\text{F})} \Phi(\text{F}) \\
\delta \Delta[n]_+ \wedge U_+ & \xrightarrow{\Phi(\text{F}')_{\text{a}}} \Phi(\text{F}') \\
\Delta[n]_+ \wedge U_+ & \xrightarrow{\Phi(\text{F})} \Phi(\text{F})
\end{align*}
$$

These liftings correspond to liftings in the squares:

$$
\begin{align*}
\Lambda[n]_+ & \xrightarrow{\text{colim}(U_a, F'_a)} \text{colim}(U_a, F'_a) \\
\Delta[n]_+ & \xrightarrow{\text{colim}(U_a, F_a)} \text{colim}(U_a, F_a)
\end{align*}
$$

Now the smallness of finite simplicial sets shows that the above squares correspond to squares:

$$
\begin{align*}
\Lambda[n]_+ & \xrightarrow{\Gamma(U_{a_0}, F'_{a_0})} \Gamma(U_{a_0}, F'_a) \\
\Delta[n]_+ & \xrightarrow{\Gamma(U_{a_0}, F_a)} \Gamma(U_{a_0}, F_f)
\end{align*}
$$

for some $a_0 \in A$. Clearly these squares correspond to:

$$
\begin{align*}
\Lambda[n]_+ \wedge U_{a_0} & \xrightarrow{F'_{a_0}} \text{F}'_{a_0} \\
\Delta[n]_+ \wedge U_{a_0} & \xrightarrow{F_{a_0}} \text{F}_{a_0}
\end{align*}
$$

The left vertical map in the first square is a trivial cofibration in $\text{SPrsh}(\text{Top}(X_{a_0}))$ and the left vertical map in the second square is a cofibration in $\text{SPrsh}(\text{Top}(X_{a_0}))$ when they are provided with the local projective model structures. Therefore, the required lifting exists in the first square when $f$ is a fibration and the required lifting exists in the second square when $f$ is a trivial fibration. Tracing back, and using the fact that $A$ is cofiltered, one may see that these show the required liftings exist in (3.2.4).

Next suppose $f : \text{F}' \to \text{F}$ is a stalk-wise weak-equivalence between fibrant objects. Recall each $F'_{a_0}$ and $F_a$ is fibrant in the local projective model structure on $\text{Top}(X_a)$. Therefore, Proposition 3.2.6 shows that for each $U_a \in \text{Top}(X_a)$, $\Gamma(U_a, f_a)$ is a weak-equivalence. If $\mathcal{U} = \{U_a|a \in A \} \in \text{Top}(\mathcal{X})$, then $\Gamma(\mathcal{U}, f) = \text{colim}(U_a, f_a)$ so that $\Gamma(\mathcal{U}, f)$ is also a weak-equivalence. The last but one statement is now clear in view of Proposition 3.2.6.
Recall that $\Phi(F) = \operatorname{colim}(a(F_a))$ if $F = \{F_a | a \in A\}$. Therefore,

$$
\Gamma(\mathcal{U}, f_*(\Phi(F))) = \Gamma((f^{-1}(\mathcal{U}), \Phi(F)) = \operatorname{colim}_{a \in A} \Gamma((f^{-1}(\mathcal{U}), (a(F_a))) = \operatorname{colim}_{a \in A} \Gamma(f^{-1}(U_a), F_a)
$$

$$
= \operatorname{colim}_{a \in A} \Gamma(U_a, f_*((a(F_a)))) = \operatorname{colim}_{a \in A} \Gamma(\mathcal{U}, (a(f_*(a(F_a)))) = \Gamma(\mathcal{U}, \Phi(\{f_*(a(F_a)) | a \in A\})).
$$

This proves the last statement and completes the proof. 

\[\square\]

### 3.2.5 Hypercohomology spectra and the right derived functor of the direct image functor

Let $\mathcal{X}$ denote a pro-scheme. In view of the above observation, one may define both of these making use of the Godement resolutions. Alternatively one may choose a model structure where the fibrations and weak-equivalences are defined to be those which are object-wise, i.e. $E \to B$ is a fibration (weak-equivalence) if and only if $\Gamma(\mathcal{U}, E) \to \Gamma(\mathcal{U}, B)$ is a fibration (weak-equivalence, respectively) for every $\mathcal{U} \in \mathcal{D}(\mathcal{X})$ and cofibrations defined by the lifting property. Given a presheaf of spectra $E$ on $\mathcal{D}(\mathcal{X})$ or on $\{\mathcal{D}(X_a) | a \in A\}$, we let $\mathcal{G}E$ denote such a fibrant replacement. Therefore, one defines

$$
\mathbb{H}(\mathcal{X}, E) = \Gamma(\mathcal{X}, \mathcal{G}E).
$$

If we start with an $E' = \{E'_a = \text{a presheaf of spectra on } \mathcal{D}(X_a) | a \in A\}$, then $R\Phi(E') = \Phi(\mathcal{G}E')$. Then one readily observes that $\mathbb{H}(\mathcal{X}, R\Phi(E')) = \operatorname{colim}_{a \in A} \mathbb{H}(X_a, E'_a)$. In fact one may see that the functor $\Phi$ preserves object-wise fibrations and weak-equivalences. So $R\Gamma(\mathcal{U}, R\Phi(E'))$ identifies up to weak-equivalence with $\Gamma(\mathcal{U}, R\Phi(E'))$, for any $\mathcal{U} \in \mathcal{D}(\mathcal{X})$.

Next let $f : \mathcal{X} = \{X_a | a \in A\} \to \mathcal{Y} = \{Y_a | a \in A\}$ denote a level map of pro-schemes. Then we let

$$
Rf_*(E) = f_*(\mathcal{G}E).
$$

In case $E = \Phi(E')$ as above, then $Rf_*(\Phi(E'))$ identifies up to weak-equivalence with $f_*(R\Phi(E')) = \operatorname{colim}_{a \in A} \{Rf_*(a(E'_a)) | a \in A\}$.

### 4 Comparison with equivariant cohomology defined by the tower construction

In this section we proceed to obtain a comparison of the generalized Borel-style equivariant cohomology theories defined in this paper for pro-group actions with a corresponding theory defined in [C13]. Let $F$ denote a field containing the algebraically closed field $k$ and let $G = G_F$ denote the absolute Galois group of $F$. A basic assumption here is that

the the profinite Galois group $G_F$ is is a pro-$l$-group. (4.0.6)

Then it is proven in [C13], Proposition 3.2] that it is totally torsion free, i.e. for every closed subgroup $K \subseteq G_F$, the abelianization $K^{ab}$ is torsion free. A monomial representation (also called an affine $l$-adic representation in [C13], sections 4, 5] $\rho$ of $G$ is a continuous homomorphism $\rho : G \to \Sigma_n \ltimes \mathbb{Z}_l^n$. $n$ is the dimension of the representation. For each pair $(\rho, m)$ where $\rho$ is a monomial representation as above and $m \in \mathbb{N}$, we let $EG^C_{\rho,m}$ denote the pro-scheme constructed in [C13]. Recall $EG^C_{\rho,m}$ is a tower indexed by $i \in \mathbb{N}$ so that

$$
(EG^C_{\rho,m})_i = ((\mathbb{G}_m)^n)^m
$$

and where the structure map $EG^C_{\rho,m} \to EG^C_{\rho,m-1}$ is the $l$-th power map. The action of the Galois group $G$ on this tower is induced from the diagonal action of $\Sigma_n \ltimes \mathbb{Z}_l^n$ through representations of $G$ in $\Sigma_n \ltimes \mathbb{Z}_l^n$. 29
The group \( \Sigma_n \) acts by permuting the \( n \)-factors of \( \mathbb{G}_m^n \) while \( \mathbb{Z}_l \) acts on a factor of the tower formed by the \( \mathbb{G}_m \) as follows. One first fixes a system of primitive \( l^i \)-th roots of unity in the field \( k \), for all \( i \geq 1 \). If \( \mathbb{G}_m(i) \) denotes the \( i \)-th term of the tower of \( \mathbb{G}_m \), then one lets \( \mathbb{Z}_l \) act on it through its quotient \( \mathbb{Z}/l^i \) by multiplication by the \( l^i \)-th primitive roots of unity in \( k \). We let

\[
EG^C = \Pi_{\rho,m}EG^C_{\rho,m}
\]

(4.0.8)

Then a basic result we need is the following result proved in [C13, Proposition 5.9]. Let \( G = \{ G_a | a \in \mathcal{A} \} \) with \( G_a \) running over finite quotient groups of \( G \). Then there exists a pro-scheme \( \{ \mathcal{Y}_a | a \in \mathcal{A} \} \) provided with an action on \( \mathcal{Y}_a \) by \( G_a \), and with all the actions compatible as \( a \) varies over \( \mathcal{A} \). Moreover each \( \mathcal{Y}_a \) is smooth and the action of \( G_a \) on \( \mathcal{Y}_a \) is free and one obtains the weak-equivalences:

\[
\operatorname{Map}((\text{Spec } \bar{F}) \times_{G_F} EG_F^C, K_\ell) \simeq \operatorname{colim}_{a \in \mathcal{A}} \operatorname{Map}((\text{Spec } F_a)_{a} \times_{G_a} \mathcal{Y}_a, K_\ell)
\]

(4.0.9)

\[
\operatorname{Map}((\text{Spec } k) \times_{G_F} EG_F^C, K_\ell) \simeq \operatorname{colim}_{a \in \mathcal{A}} \operatorname{Map}((\text{Spec } k)_{a} \times_{G_a} \mathcal{Y}_a, K_\ell)
\]

where the left-hand-side of the first equation (second equation) is defined as \( \operatorname{colim}_{b \in \mathcal{A}} \operatorname{Map}((\text{Spec } F_b)_{b} \times_{G_b} \mathcal{Y}_b, K_\ell) \) (\( \operatorname{colim}_{b \in \mathcal{A}} \operatorname{Map}((\text{Spec } k)_{b} \times_{G_b} \mathcal{Y}_b, K_\ell) \), respectively). \( G_b \) is a finite quotient of \( G_F \) that acts on the \( b \)-th stage of the tower \( EG_F^C \) as well as on the finite normal extension \( F_b \) and \( K_\ell \) denotes any of the spectra considered in Theorem 1.1.2 with \( E = K \) being the spectrum representing algebraic K-theory on \( \text{Sm}/k \).

In this situation, we may start with a closed immersion \( G_a \rightarrow \mathcal{G}_a \) for each \( a \in \mathcal{A} \), with \( \mathcal{G}_a \) being a linear algebraic group. Then we may form an inverse system of algebraic groups by replacing each \( \mathcal{G}_a \) with the finite product \( G_a = \prod_{b \leq a} \mathcal{G}_b \) where the structure maps are induced by the obvious projection maps: see the construction in 2.4.1 for more details. Clearly one may imbed \( G_a \) into \( G_a \). Let \( \{ s(a) | a \in \mathcal{A} \} \) denote a non-decreasing sequence with each \( s(a) \) a non-negative integer. (Recall \( K \) denotes the directed set of all such sequences.)

In the above situation, we let \( Y_a = G_a \times_{G_a} \mathcal{Y}_a \), for each \( a \in \mathcal{A} \). Next suppose \( \{ \mathcal{X}_a | a \in \mathcal{A} \} \) is an inverse systems of smooth schemes of finite type over \( k \) so that each \( \mathcal{X}_a \) is provided with an action by the group \( G_a \) and the these actions are compatible. Suppose further that structure maps of the inverse system \( \{ \mathcal{X}_a | a \in \mathcal{A} \} \) are flat as maps of schemes. We let \( X_a = G_a \times_{\mathcal{G}_a} \mathcal{X}_a \).

**Definition 4.0.9.** Assume the above situation. In view of the weak-equivalence 4.0.9, for any spectrum \( E \in \text{Spt}_{S_1}(k) \), we let:

\[
\operatorname{Map}(X \times_{G_F} EG_F^C, E) = \operatorname{colim}_{a \in \mathcal{A}} \operatorname{Map}(X_a \times_{G_a} \mathcal{Y}_a, E).
\]

Now Theorem 2.5.4 readily provides the following comparison result.

**Theorem 4.0.10.** Assume that \( F \) is a field containing the algebraically closed field \( k \), let \( G = G_F \) denote the absolute Galois group of \( F \). Assume that \( G \) is a free pro-\( l \) group with \( l \) prime to the characteristic of \( k \).

Then we obtain the string of weak-equivalences:

\[
\operatorname{holimcolim}_{a \in \mathcal{A}} H(EG_a^{gm,s} \times_{G_a} (X_a \times \mathcal{Y}_a), K_\ell) \simeq \operatorname{holimcolim}_{a \in \mathcal{A}} H(EG_a^{gm,s} \times_{G_a} (X_a \times \mathcal{Y}_a), K_\ell)
\]

30
\[ \simeq \lim_{\to \ell} \lim_{a, \in A} \mathbb{H}(E_{g, a}^{g_m}, X_a \times Y_a), P_{\leq l}(K_{\ell}) \simeq \lim_{\to \ell} \lim_{a, \in A} \mathbb{H}(X_a \times G_a, Y_a, P_{\leq l}(K_{\ell})) \]
\[ \simeq \lim_{a, \in A} \mathbb{H}(X_a \times G_a, Y_a, K_{\ell}) \simeq \lim_{a, \in A} \mathbb{H}(X_a \times G_a, Y_a, K_{\ell}) = \text{Map}(X \times G_{F}, EG_{F}, K_{\ell}). \]

The last comparison result along with the rigidity theorem, Theorem \[3.0.7\] now provides rigidity for mod-\(l\) or \(l\)-primary Borel-style equivariant K-theory defined in [C13].

**Theorem 4.0.11.** Assume the above situation. Then we obtain the weak-equivalence:
\[
\text{Map}(\text{Spec } k \times G_{F}, EG_{F}, K_{\ell}) \to \text{Map}(\text{Spec } \bar{F} \times G_{F}, EG_{F}, K_{\ell}) \quad (4.0.10)
\]

**Proof.** Let \(Y = \{Y_a | a \in A\}\) denote the inverse system of smooth schemes obtained from the tower \(EG_{F}^{g_m}\). Then the last theorem identifies the left-hand-side (right-hand-side) up to weak-equivalence with \(\lim_{\to \ell} \lim_{a, \in A} \mathbb{H}(E_{g, a}^{g_m}, X_a \times Y_a, K_{\ell}) = \lim_{\to \ell} \lim_{a, \in A} \mathbb{H}(E_{g, a}^{g_m}, X_a \times G_a, Y_a, K_{\ell}),\) respectively. Then the assertion that the map represented by the arrow is a weak-equivalence is Theorem \[1.1.2\]. \(\square\)

5 **Proof of Theorem [1.1.4]**

**Proof.** Let \(G_{F}\) be denoted \(G\) throughout the following discussion. Then, the main result of [C13, Proposition 9.1] provides the weak-equivalences
\[
(K(\text{Spec } k, G), G)_{l, \ell} \simeq \mathbb{H}(\text{Spec } k, K_{\ell}) \simeq \text{Map}(\text{Spec } k \times G, EG_{F}, K_{\ell}) \quad (5.0.11)
\]
\[
(K(\text{Spec } \bar{F}, G), G)_{l, \ell} \simeq \mathbb{H}(\text{Spec } \bar{F}, K_{\ell}) \simeq \text{Map}(\text{Spec } \bar{F} \times G, EG_{F}, K_{\ell})
\]
in both rows. Here the subscript \(l\) denotes any of the spectra considered in Theorem \[1.1.2\] with \(E\) denoting the spectrum representing algebraic K-theory, which is denoted \(K\). Finally Theorem \[1.1.2\] provides a weak-equivalence between the right-hand-sides of the above weak-equivalences and one uses Galois descent to identify \(K(\text{Spec } F, G)\) with \(K(\text{Spec } F)\). This proves all but the last weak-equivalence: \(K(\text{Spec } F, G) \simeq K(\text{Spec } F)\). This follows from Proposition \[5.0.13\] below. These prove a mod-\(l\) version of the required result.

Since the subscript \(l\) refers to the mod-\(l\) spectra which are defined as the partial derived completion with respect to \(\rho_l : \Sigma \to H(\mathbb{Z}/\ell)\), one may now take the homotopy inverse limit of the partial completions to obtain the derived completion with respect to \(\rho_l\). This identifies the term \(\lim\text{holim}_{\to \ell} K(\text{Spec } F, G)_{l, \ell} \simeq K(\text{Spec } F)_{\rho_l, m}\) with \(K(\text{Spec } F)_{\rho_l, m}\). The term
\[
\lim\text{holim}_{\to \ell} K(\text{Spec } k, G)_{l, m} = \lim\text{holim}_{\to \ell} K(\text{Spec } k, G)_{l, m} \simeq \lim\text{holim}_{\to \ell} K(\text{Spec } k, G)_{l, n} \simeq \lim\text{holim}_{\to \ell} K(\text{Spec } k, G)_{l, m}.
\]

That this identifies with the derived completion \(K(\text{Spec } k, G)_{l, \ell} \simeq K(\text{Spec } k, G)_{\rho_l, m}\) follows from Corollary \[5.0.15\]. \(\square\)
Remark 5.0.12. Observe that
\[ \text{Map}(\text{Spec } k \times_G E_G, \mathbb{K}_G) = \colim_{a \in A} \text{Map}(\text{Spec } k \times_G \mathcal{Y}_a, \mathbb{K}_G) \] and
\[ \text{Map}(\text{Spec } F \times_G E_G, \mathbb{K}_G) = \colim_{a \in A} \text{Map}(\text{Spec } F_a \times_G \mathcal{Y}_a, \mathbb{K}_G) \]
Assume that \( \mathbb{K}_G = \mathbb{K} \wedge H(Z/\ell) \cdots \wedge H(Z/\ell) \) for some \( m \). Since the spectrum \( \mathbb{K}_G \) has Nisnevich excision, it has Nisnevich descent and therefore
\[ \colim_{a \in A} \text{Map}(\text{Spec } k \times_G \mathcal{Y}_a, \mathbb{K}_G) \simeq \colim_{a \in A} \text{Map}(\text{Spec } k \times_G \mathcal{Y}_a, \mathbb{K} \wedge H(Z/\ell) \cdots \wedge H(Z/\ell) \text{ and} \]
\[ \colim_{a \in A} \text{Map}(\text{Spec } F_a \times_G \mathcal{Y}_a, \mathbb{K}_G) \simeq \colim_{a \in A} \text{Map}(\text{Spec } F_a \times_G \mathcal{Y}_a, \mathbb{K} \wedge H(Z/\ell) \cdots \wedge H(Z/\ell) \text{.} \]

Proposition 5.0.13. Let \( M \in \text{Mod}(K(\text{Spec } k, G)), \) where \( G \) is either a profinite group or an algebraic group. Assume that the augmentation ideal \( I_G = ker(R(G) \to \mathbb{Z}) \) acts trivially on \( \pi_*(M) \). Then the obvious map \( M \to M \wedge_{I_G} \) is a weak-equivalence.

Proof. The proof is essentially in [C13 Corollary 5.3]. Making use of the spectral sequence in [C08 Theorem 7.1], one immediately reduces to the case where \( M \in \text{Mod}(R(G)) \). In this situation, one first observes the commutative diagram:

\[
\begin{array}{ccc}
\mathbb{Z} & \xrightarrow{id} & R(G) \\
\downarrow{id} & & \downarrow{id} \\
\mathbb{Z} & \xrightarrow{I_G} & \mathbb{Z} \\
\end{array}
\]

where the maps in the top row are the obvious ones. By working in the category of diagrams of dg-algebras one may find a dg-algebra \( \mathbb{Z}^\bullet_{R(G)} \) which is a cofibrant replacement of \( \mathbb{Z} \) by dg-algebras over \( R(G) \) provided with maps \( \mathbb{Z} \to \mathbb{Z}^\bullet_{R(G)} \) and \( \mathbb{Z}^\bullet_{R(G)} \to \mathbb{Z} \) whose composition is the identity. Now let \( S(T) \) denote the triple defined on \( \text{Mod}(\mathbb{Z}) \) sending \( N \) to \( N \) (sending \( M \) to \( M \otimes \mathbb{Z}/\ell_{R(G)} \), respectively). Then the above commutative diagram provides maps of triples \( S \to T \to S \). Observe that the cosimplicial objects defined by the triple \( S(T) \) defines the derived completion with respect to the identity map \( \mathbb{Z} \to \mathbb{Z} \) (\( I_G \), respectively). It follows from [C08 Theorem 2.15] that these cosimplicial objects define weakly equivalent homotopy inverse limits. \( \square \)

Let \( l \) denote a fixed prime different from the characteristic of the base field \( k \). Let \( \wedge_{\mathbb{Z}/\ell}(H) \) denote a cofibrant replacement of \( H(\mathbb{Z}/\ell) \) in the category of commutative algebra spectra over the \( S^1 \)-sphere spectrum \( \Sigma \). Let \( S = \text{Spec } k \) and let \( \mathbb{K}^\wedge(S) \) denote a cofibrant replacement of \( K(S) \) in the category of commutative algebra spectra over \( K(S, G) \) where \( G \) denotes either an algebraic group or an inverse system of such groups. Let \( \text{Mod}(K(S, G)) \) (\( \text{Mod}(K(S, G) \wedge_{\mathbb{Z}/\ell}(H)) \)), \( \text{Mod}(K(S)) \) and \( \text{Mod}(K(S) \wedge_{\mathbb{Z}/\ell}(H)) \))
denote the category of module-spectra over the commutative ring spectrum \( K(S, G) \) \((K(S, G) \wedge \tilde{H}(\mathbb{Z}/\ell)), \widetilde{K}(S) \) and \( \widetilde{K}(S) \wedge \tilde{H}(\mathbb{Z}/\ell) \), respectively).

Let \( \rho_l : K(S, G) \to K(S, G) \wedge \tilde{H}(\mathbb{Z}/\ell) \) denote the map induced by the mod-\( l \) reduction \( \Sigma \to \tilde{\mathbb{H}}(\mathbb{Z}/\ell) \). Let \( I_G : K(S, G) \to \widetilde{K}(S) \) denote the obvious map induced by the restriction \( K(S, G) \to \widetilde{K}(S) \). Clearly these are maps of commutative ring spectra. We will also let \( \rho_l \) denote the map \( K(S) \to K(S) \wedge \tilde{H}(\mathbb{Z}/\ell) \) induced by \( \rho_l \) while \( I_G \) will also denote the map \( \widetilde{K}(S) \to \widetilde{K}(S) \wedge \tilde{H}(\mathbb{Z}/\ell) \) induced by the mod-\( l \) reduction map \( \Sigma \to \tilde{\mathbb{H}}(\mathbb{Z}/\ell) \).

**Proposition 5.0.14.** Then pull-back and push-forward by \( \rho_l \) defines triples: \( \rho_{l*} \circ \rho_l^* : \text{Mod}(K(S, G)) \to \text{Mod}(K(S, G)) \) and \( \rho_{l*} \circ \rho_l^* : \text{Mod}(\widetilde{K}(S)) \to \text{Mod}(\widetilde{K}(S)) \). Similarly pull-back and push-forward by \( I_G \) defines triples: \( I_{G*} \circ I_G^* : \text{Mod}(K(S, G)) \to \text{Mod}(K(S, G)) \) and \( I_{G*} \circ I_G^* : \text{Mod}(K(S, G) \wedge \tilde{H}(\mathbb{Z}/\ell)) \to \text{Mod}(K(S, G) \wedge \tilde{H}(\mathbb{Z}/\ell)) \).

Then \( \rho_{l*} \) commutes with \( I_{G*} \) and \( I_G^* \) while \( \rho_l^* \) commutes with \( I_G^* \) and \( I_{G*} \).

**Proof.** Given an \( M \in \text{Mod}(K(S, G)) \),

\[
I_G^*(\rho_l^*(M)) = (M \wedge_{K(S, G)} K(S, G) \wedge \tilde{H}(\mathbb{Z}/\ell)) \wedge_{K(S, G) \wedge \tilde{H}(\mathbb{Z}/\ell)} \widetilde{K}(S) \wedge \tilde{H}(\mathbb{Z}/\ell) \\
\cong M \wedge_{K(S, G)} \widetilde{K}(S) \wedge \tilde{H}(\mathbb{Z}/\ell) = \rho_l^*(I_G^*(M)).
\]

This proves that \( \rho_l^* \) commutes with \( I_G^* \). Observe that \( I_{G*} \) sends an \( N \in \text{Mod}(\widetilde{K}(S)) \) to \( N \) viewed as a module-spectrum over \( K(S, G) \) using the map \( K(S, G) \to \widetilde{K}(S) \). Now \( \rho_l^* \) sends

\[
I_{G*}(N) \text{ to } N \wedge_{K(S, G)} K(S, G) \wedge \tilde{H}(\mathbb{Z}/\ell) = N \wedge \tilde{H}(\mathbb{Z}/\ell).
\]

On the other hand \( \rho_l^* \) sends \( N \) to \( N \wedge_{K(S)} K(S, G) \wedge \tilde{H}(\mathbb{Z}/\ell) = N \wedge \tilde{H}(\mathbb{Z}/\ell) \). Therefore, \( I_{G*}(\rho_l^*(N)) \) is the above spectrum viewed as a module spectrum over \( K(S, G) \wedge \tilde{H}(\mathbb{Z}/\ell) \) using the map \( K(S, G) \to K(S) \). This proves \( \rho_l^* \) commutes with \( I_{G*} \). Since \( \rho_l^* \) commutes with \( I_G^* \), their right-adjoints, \( \rho_{l*} \) and \( I_{G*} \) commute. If \( M \in \text{Mod}(K(S, G) \wedge \tilde{H}(\mathbb{Z}/\ell)) \), \( I_G^* \rho_{l*}(M) \) is \( M \wedge_{K(S, G)} \widetilde{K}(S) \) where one views \( M \) first as a module over \( K(S, G) \). One may see that \( \rho_{l*}I_G^*(M) \) is also the same object, which proves that \( \rho_{l*} \) commutes with \( I_G^* \).

**Corollary 5.0.15.** Let \( M \in \text{Mod}(K(S, G)) \). Then one obtains a weak-equivalence: \( \holim_{\infty \leftarrow m} (M \wedge_{\rho_{l,m}} I_{G,m}) \cong M \wedge_{I_{G,l}} \).

**Proof.** Observe that
\[
M^\leq_{\rho_i,n} \widehat{I}_{G,m} = \text{holim}_{\Delta^\leq_m} \left\{ (I_{G*} \circ I_G^* \circ \cdots \circ (I_{G*} \circ I_G^*)|i \leq m \} \right\} \text{holim}_{\Delta^\leq_n} \left\{ (\rho_{I*} \circ \rho_I^* \circ \cdots \circ (\rho_{I*} \circ \rho_I^*)(M)|j \leq n \} \right\}.
\]

By fixing an \(i \leq m\) and taking the homotopy inverse limit on varying the \(j \leq n\), one sees by an application of [C08, Proposition 2.9], that the above term identifies up to weak-equivalence with

\[
\text{holim} \left\{ (I_{G*} \circ I_G^* \circ \cdots \circ (I_{G*} \circ I_G^*)|i \leq m \} \right\} \left\{ (\rho_{I*} \circ \rho_I^* \circ \cdots \circ (\rho_{I*} \circ \rho_I^*)(M)|j \leq n \} \right\}.
\]

This is the iterated homotopy inverse limit of the double cosimplicial object

\[
\left\{ (I_{G*} \circ I_G^* \circ \cdots \circ (I_{G*} \circ I_G^*)|i \leq m \} \right\} \left\{ (\rho_{I*} \circ \rho_I^* \circ \cdots \circ (\rho_{I*} \circ \rho_I^*)(M)|j \leq n \} \right\}
\]

truncated to degrees \(\leq m, n\). The corresponding diagonal cosimplicial object is given by

\[
\left\{ (I_{G*} \circ I_G^* \circ \cdots \circ (I_{G*} \circ I_G^*)|i \leq m \} \right\} \left\{ (\rho_{I*} \circ \rho_I^* \circ \cdots \circ (\rho_{I*} \circ \rho_I^*)(M)|i \geq 0 \} \right\}.
\]

Clearly one may replace \(\text{holim} \circ \text{holim}\) applied to

\[
\text{holim}_{\Delta^\leq_m} \left\{ (I_{G*} \circ I_G^* \circ \cdots \circ (I_{G*} \circ I_G^*)|i \leq m \} \right\} \text{holim}_{\Delta^\leq_n} \left\{ (\rho_{I*} \circ \rho_I^* \circ \cdots \circ (\rho_{I*} \circ \rho_I^*)(M)|j \leq n \} \right\}
\]

by \(\text{holim}\) applied to

\[
\text{holim}_{\Delta^\leq_m} \left\{ (I_{G*} \circ I_G^* \circ \cdots \circ (I_{G*} \circ I_G^*)|i \leq m \} \right\} \text{holim}_{\Delta^\leq_n} \left\{ (\rho_{I*} \circ \rho_I^* \circ \cdots \circ (\rho_{I*} \circ \rho_I^*)(M)|i \leq m \} \right\}.
\]

Making use of the last proposition, the truncated cosimplicial object

\[
\left\{ (I_{G*} \circ I_G^* \circ \cdots \circ (I_{G*} \circ I_G^*)|i \leq m \} \right\}
\]

identifies with

\[
\left\{ (\rho_{I*} I_{G*} I_G^* \rho_I^*) \circ \cdots \circ (\rho_{I*} I_{G*} I_G^* \rho_I^*)(M)|i \leq m \} \right\}
\]

Observe that the triple \((\rho_{I*} I_{G*} I_G^* \rho_I^*)\) defines the derived completion with respect to the map \(K(S,G) \to \widehat{K(S)^\wedge \mathbb{H}(Z/@\ell)}\), i.e. the derived completion with respect to \(I_{G,1}\). This completes the proof of the corollary.

\[\square\]

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