P-wave Superfluidity by Blockade Effects in a Rydberg-Dressed Fermi Gas*

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We systematically investigate the p-wave superfluidity of a Rydberg-dressed Fermi gas, where the soft-core effective interaction is of finite radius $R_c$ due to blockade effects. After solving the BCS gap equation and comparing the free energy, we obtain the quantum phase diagram, which is composed of three different phases: polar ($p_z$), axial ($p_x + i p_y$), and axi-planar ($p_x + i \beta p_y$) phases. The tri-critical point locates around $R_c k_F \sim 1$, where $k_F$ is the Fermi wave vector. We further derive the Ginzburg-Landau theory to explain the phase diagram, and estimate the transition temperature to be about $0.1E_F$ in the current experimental regime of $^6$Li. Our work paves the way for future studies on p-wave superfluids and related quantum phase transitions in ultracold atoms.

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Introduction: Remarkable progress in the studies of ultracold atomic Fermi gases [1, 2] has opened up prospects for creating novel phases not easily studied in traditional condensed matter systems. One of the most important examples is the p-wave superfluid phase in identical fermions [3–5], discussed in the contexts of superfluid $^3$He [6, 7], the fractional Quantum Hall effect [8], Sr$_2$RuO$_4$ [9], and fermionic cold atoms/molecules [10, 11]. Besides of the non-trivial gap symmetry, the topological p-wave pairing can be also degenerate in the presence of vortices, spanned by zero-energy Majorana modes at vortex cores [8, 12]. Such highly non-local character is expected to suppress the decoherence, and may be used for topologically protected quantum information processing [12].

Among these candidates of p-wave superfluid phases, systems of ultracold atoms are of special interest, because of the impurity-free environment and the large flexibility for parameter detuning. However, this fascinating p-wave superfluid phase usually has a very low critical temperature when away from the Feshbach resonance, but a strong three-body loss when near the resonance [13]. Proposals utilizing the long-range nature of the dipolar interaction between polar molecules are discussed recently [10, 11], but the experimental realization of quantum degenerate polar molecules is still challenging. On the other hand, recent successful experiments in cold Rydberg gas with the blockade effects [14–27] make it possible to realize a new strongly correlated system. Several theoretical proposals [28–32] and experimental works [33] have demonstrated the possibility of generating a supersolid phase in bosonic Rydberg atoms with a quiet long life time (≈0.5 sec). Along this line, therefore, fermionic Rydberg-dressed atoms (c.f. $^6$Li or $^{40}$K) should be a good candidate for investigating unique properties not fully realized in other condensed matter system, including the topological p-wave superfluid phase.

In this paper, we investigate the p-wave superfluid and its quantum phase diagram of fermionic atoms in three dimensional (3D) space, where the interatomic interaction is induced by weakly coupled to a Rydberg state through a two-photon transition (see Fig. 1(a)). By self-consistently solving the gap equation and comparing the total free energy, we can identify the following three distinct phases: polar ($p_z$), axial ($p_x + i p_y$), and axi-planar ($p_x + i \beta p_y$) states, where $\beta$ depends on the relative momentum of the Cooper pair. The first two phases are respectively equivalent to the $\beta$ and $A_1$ phases

![FIG. 1: (color online) (a) Schematic level plot for the off-resonant coupling of the ground state ($|g\rangle$) to a Rydberg state ($|c\rangle$) via an intermediate state ($|n1P\rangle$). The right panel is an effective single photon process with $\Omega_{ef} = \Omega_1 \Omega_2 / 2 \delta_1$ and $\delta_{eff} = \delta_1 + \delta_2$. (b) $nD - nD$ asymtote of Li atoms calculated by the quantum defect theory [34, 37] and normalized by their value at $n = 30$. (c) The obtained effective interaction between Rydberg-dressed atoms, where $V_0$ and $R_c$ are defined in the text. (d) Table of alkali atoms, which may have an attractive or repulsive interaction by selecting a specific quantum number of their Rydberg state [34, 37, 38].](image)
in $^3$He systems, while the last one is a completely new phase. The tri-critical point is found when the interaction range is about the inter-particle distance. The rich quantum phase diagram can be qualitatively understood from Ginzburg-Landau theory by considering the finite-range feature of the Rydberg-dressed interaction, which can also enhance the transition temperature ($T_c$) to the order of $0.1 E_F$ via nonlocal correlation effects. We then discuss the parameter regime for observing these p-wave superfluid phases in the current $^6$Li experiment.

**Interaction between Rydberg-dressed fermions:** We consider identical fermions in a 3D system (say $^6$Li or $^{40}$K). A far detuned two-photon transition (see Fig. 1(a)) is applied to couple the ground state ($|n_0S\rangle$) to a Rydberg excited state ($|n_2S\rangle$ or $|n_2D\rangle$) with an effective Rabi frequency ($\Omega_{\text{eff}}$) and detuning ($\delta_{\text{eff}}$). Similar to the repulsive potential considered in the bosonic case [29, 34–36], one may choose some specific Rydberg states so that all the van der Waals scattering channels are attractive ($C_6 < 0$), leading to the Cooper instability toward a superfluid phase. Using results of quantum defect theory [34, 37], we find that this condition can be fulfilled for $^6$Li when excited to $|n_2D\rangle$ state ($n_2 \geq 30$) (see Fig. 1(b)) or to $|n_2S\rangle$ state ($n_2 \geq 282$) [38]. Available regime for both pure attractive and repulsive interactions of other alkali atoms are also shown together in Fig. 1(d).

Since alkali atoms usually have a relatively small fine structure splitting (see Fig. 1(b)), the obtained effective interaction is almost isotropic in 3D space [29, 35, 36]. The effective interaction between Rydberg-dressed atoms can be calculated perturbatively within the adiabatic approximation [29, 34–36]:

$$V(r-r') = -\frac{V_0}{1 + |r-r'|^6}/R_c^6,$$

where $V_0 \equiv (\Omega_{\text{eff}}/2\delta_{\text{eff}})^4 |C_6|/R_c^6 > 0$. $R_c \equiv (|C_6|/2\delta_{\text{eff}})^{1/6}$ is the averaged soft-core radius ($h \equiv 1$), and $C_6$ is some averaged van der Waals coefficients after diagonalizing the whole scattering channels [35]. Eq. (1) is justified in the large detuning and weak field limit, i.e. the number of effectively excited atoms $N_{\text{Ryd}} = (\Omega_{\text{eff}}/2\delta_{\text{eff}})^2 N$ is smaller than one, where $N$ is the total atom number.

We note that the effective interaction above has some unique and important features for identical fermions we discussed here. (1) Such finite range interaction strongly enhances the p-wave scattering amplitude, making it possible to have a rapid evaporative cooling even without compensating cooling with another species. (2) The adiabatic approximation used for the effective potential, $V(r)$, should fail in the short-distance regime, because of the complicated level crossing between different the electronic states. However, such mechanism becomes less effective when considering identical fermions due to the Pauli exclusion principle. (3) Finally, the weak finite-range attractive interaction makes it possible to realize a stable BCS-BEC quantum phase transition [3, 5] without a strong loss rate in three-body collision, because the two-body bound state energy in the BEC side can be so small if $V(r)$ is weak enough [10, 11]. As a result, we believe that identical fermions of Rydberg-dressed interaction can be a better-controlled system for studying anisotropic the p-wave superfluidity.

**Order parameter of p-wave superfluid:** In this paper, we focus on the superfluid phase in the BCS side without two-body bound state (see Supplementary material for details). As a result, we can apply the first Born approximation for the weak soft-core interaction, i.e. using $V_{\mathbf{k}-\mathbf{k}'} = \int dV(r) e^{-i\mathbf{r}(\mathbf{k}-\mathbf{k}')/r}$ as the scattering amplitude, so that the BCS gap equation becomes ($\hbar = k_B \equiv 1$):

$$\Delta_{\mathbf{k}} = -\frac{1}{V} \sum_{\mathbf{k}''} \frac{V_{\mathbf{k}-\mathbf{k}'} V_{\mathbf{k}''-\mathbf{k}}}{2E_{\mathbf{k}''}} \tanh \left( \frac{E_{\mathbf{k}''}}{2T} \right) \Delta_{\mathbf{k}'},$$

where $E_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + |\Delta_{\mathbf{k}}|^2}$ is the quasi-particle excitation spectrum with $\xi_{\mathbf{k}} \equiv \mathbf{k}^2/2\mathcal{M} - \mu$ and $\Delta_{\mathbf{k}} = -\Delta_{-\mathbf{k}}$ being the antisymmetric gap function. $\mathcal{M}$ and $\mathcal{V}$ are the atom mass and system volume. The chemical potential $\mu$ must be determined by the density equation self-consistently:

$$n = \frac{1}{\mathcal{V}} \sum_{\mathbf{k}} \left[ 1 - \frac{\mathcal{V}}{\mathcal{N}} \tanh \left( \frac{\Delta_{\mathbf{k}}}{2T} \right) \right].$$

Before solving above equations, we first decompose the interaction matrix element and gap function into different angular momentum channels: $V_{\mathbf{k}-\mathbf{k}'} = -\sum_{lm} V_l (k, k') Y_{l,m}(k) Y_{l,m}(k')$, and $\Delta_{\mathbf{k}} = \sum_{lm} \Delta_{l,m}(k) Y_{l,m}(k)$, where $\sum_{lm}$ is the summation over all odd values of $l$ and $m = -l, \ldots, l$. In Fig. 2(a), we show the calculated $V_l (k, k')$ and find that the scattering amplitude is completely dominated by the p-wave channel. The density plot of $V_1 (k, k')$ shows that it is peaked around $k = k' \approx 2R_c^{-1}$.

As for the gap function, it is argued that it can be well described by its orientation alone [6, 39] when consider-
we show the obtained quantum phase diagram. Here, we need to consider a more general expression by separating the "amplitude part" and the "orientation part" of the relative momentum:

\[ \Delta_k = \Delta(k) \left( \sum_{m=0, \pm 1} \beta_m^m Y_{1,m}(\hat{k}) \right) \equiv \Delta(k) \Psi_k(\hat{k}), \]

where we keep the \( p \)-wave channel only and the complex coefficients, \( \{\beta_m^m\} \), is normalized as \( \sum_m |\beta_m^m|^2 = 1 \) (i.e. \( \int d\Omega |\Psi_k(\hat{k})|^2 = 1 \)) for all \( k \). Since \( p \)-wave pairing function generally break the gauge, time-reversal and rotational symmetries when below \( T_c \), we can classify them according to the residue symmetries as discussed in the inert state of spinor condensate \([40]\): (1) polar \( (p_z) \) state with \( U(1) \times S^2 \) symmetry (i.e. \( \beta_k^0 \neq 0 \) only), (2) axial \( (p_x \pm ip_y) \) state with \( SO(3) \) symmetry (i.e. \( \beta_k^\pm \neq 0 \) only), and (3) axi-planar \( (p_x \pm i\beta_p p_y) \) state with \( \beta_p \) depending on the momentum \( p \), which breaks rotational symmetry completely. Here \( U(1) \) stands for the rotation about the axis, and \( S^2 \) stands for the orientation about the unit sphere. We note that the axial phase has additional gauge-orbital symmetry \([7, 41]\), and therefore has coreless vortex structure as in \(^3\)He-A phase \([42]\).

**Quantum phase diagram:** Using the symmetry classification above, we may represent the ground state as the following general configuration:

\[ \Psi_k(\hat{k}) = \cos \gamma_k Y_{1,1}(\hat{k}) + \sin \gamma_k Y_{1,-1}(\hat{k}) \]

\[ \propto \sin(\gamma_k - \pi/4)k_x - \cos(\gamma_k - \pi/4)ik_y \]

(4) for \( 0 \leq \gamma_k \leq \pi/4 \). Note that all the phases mentioned above can be expressed by properly rotating the coordinate axis (for example, polar \( (p_0) \) can be obtained by using \( \gamma_k = \pi/4 \)). From our full numerical calculation (see Fig. 2(b) as an example), we find that the momentum dependence of \( \gamma_k \) is small, but finite (\( \sim 15\% \)). In the context of \(^3\)He system \([6, 39]\), it was generally assumed that \( \Psi_k(\hat{k}) \) is independent of the momentum magnitude, \( k \), because pairing occurs only near Fermi surface. However, as clearly shown below, this assumption fails to explain the full phase diagram here, due to the finite-range nature of the Rydberg-dressed interaction.

We calculate the ground state phase diagram via the following two steps: we first solve the gap equation (Eqs. (2)) for a given density self-consistently by iterating the initial configuration until convergence. The gap symmetry is specified in the initial configuration for polar and axial states. As for the axi-planar state, the relative strength of different components (i.e. the value of \( \gamma_k \)) evolves self-consistently until convergence. Next, we use the obtained gap function to calculate the corresponding ground state energy (see Supplementary Material) for determining the most favorite configuration. The finite temperature phases diagram can be also obtained similarly by minimizing the total free energy.

In Fig. 3 we show the obtained quantum phase diagram of the \( p \)-wave superfluid of Rydberg-dressed fermions. Three different pairing phases (see the text) are identified: polar \( (\boxtimes) \), axial \( (\bigcirc) \), and axi-planar \( (\bigtriangleup) \) states. Dashed lines are eye-guiding for the phase boundary and insets show the 3D spherical plot of the modulus of the orientation function, i.e. \( |\Psi_k(\hat{k})| \). For the axi-planar state, we draw the modulus for two different moments, \( k/k_F = 1.5 \) and 3, as an example. The table shows the corresponding total energies (in unit of Fermi energy) for five representative points indicated in quantum phase diagram. Here, "unstable" means such configuration is just a saddle point. The thick arrow indicates the possible position of tri-critical point.

**FIG. 3:** (Color online) Quantum phase diagram for \( p \)-wave superfluid of Rydberg-dressed fermions. Three different pairing phases (see the text) are identified: polar \( (\boxtimes) \), axial \( (\bigcirc) \), and axi-planar \( (\bigtriangleup) \) states. Dashed lines are eye-guiding for the phase boundary and insets show the 3D spherical plot of the modulus of the orientation function, i.e. \( |\Psi_k(\hat{k})| \). For the axi-planar state, we draw the modulus for two different moments, \( k/k_F = 1.5 \) and 3, as an example. The table shows the corresponding total energies (in unit of Fermi energy) for five representative points indicated in quantum phase diagram. Here, "unstable" means such configuration is just a saddle point. The thick arrow indicates the possible position of tri-critical point.

![Quantum phase diagram](image_url)

|          | A   | B    | C     | D     | E      |
|----------|-----|------|-------|-------|--------|
| Polar    | -3.34732 | 0.142 | unstable | unstable | 6.5168 |
| Axial    | -3.34273 | 0.788 | -2.1155 | 6.3203 | 7.3116 |
| Axi-Planar| -3.34799 | unstable | -2.1886 | 7.4806 | 9.3716 |
exclude the possibility of reentrant transition, as mentioned in Refs. [6].

To qualitatively understand the quantum phase diagram, it is better to start from the Ginzburg-Landau (GL) free energy ($F_{GL}$) with a small order parameter just below $T_c$. For simplicity, we just show the leading order expansion of the free energy for a constant $\gamma_k = \gamma$ (see Supplementary Material):

$$F_{GL} = \int \overline{\nu} \frac{\Delta(k)}{\overline{\nu}} = \int \frac{\Delta(k)}{\overline{\nu}} - \int \frac{\Delta^2(k)}{\overline{\nu}} + \frac{\Delta^4(k)}{\overline{\nu}}$$

where $\int_k \equiv \int_0^\infty \frac{\Delta^2(k)}{\overline{\nu}}$ and functions, $\tilde{\nu}_2$, $\tilde{\nu}_4$, $\tilde{\nu}_2$ and $\tilde{\nu}_4$ are defined explicitly in the Supplementary Material (we neglect the temperature dependence for simplicity). We first find that the $T_c$, which is determined solely by the sign of the quadratic order, is independent of the gap symmetry (i.e., $\gamma$). Secondly, when $T < T_c$ for a finite $\Delta(k)$, the quartic order is minimized either at $\gamma = 0$ (axial state) or $\gamma = \pi/4$ (polar state), depending on the sign of its coefficient. This explains why the axi-planar state with constant $\gamma_k = \gamma$ cannot be favoured in the ground state calculation. The polar state observed in strong interaction regime probably has to be understood by including higher order effects.

In the weak interaction regime, on the other hand, we find an axial state ($\gamma = 0$) can be energetically favoured, if only the momentum transfer ($\sim R_c^{-1}$) is smaller than Fermi momentum, $k_F$, so that pairing occurs near Fermi surface, similar to the 3He-A phase studied by Anderson and Morel [39]. This condition is satisfied in our system when $R_c k_F$ is large (i.e. the bottom right corner of Fig. 3). The phase transition between the polar and the axial state is expected to be first order due to breaking time reversal symmetry of the axial state.

Finally, in the weak interaction and smaller $R_c$ regime, the momentum transfer can be comparable to $k_F$ so that the possibility of momentum mixing cannot be neglected in the free energy (see Eq. (5)). This effect is expected to be important especially in the weak interaction limit, because energies of various configurations are very close to each other. This explains why axi-planar state has momentum dependence in $\gamma_k$ (i.e. gap function at different relative momentum are mixed due to scattering).

**Transition temperature, $T_c$:** We now estimate the critical temperature of the $p$-wave superfluid. We first solve the BCS gap equation (Eq. (2)) and the density equation self-consistently. The calculated gap function $\Delta(k)$ of a polar state are shown in Fig. 4(a) as a function of the momentum $k/k_F$ for various temperature $T/E_F$. In the inset, we show how the maximum value of the gap function decreases to zero when the temperature is increased from zero to above $T_c$.

In Fig. 4(b), we show the obtained $T_c$ as a function of the interaction strength, $\alpha$, for three different values of the soft-core radius. As expected in a standard BCS result, $T_c$ increases exponentially as a function of $\alpha$. In the large $\alpha$ or $R_c$ regime, $T_c$ can be comparable to Fermi energy. Although the BCS theory may not be justified in the strong interaction regime quantitatively, our results may still provide useful information because our self-consistent calculation has included the renormalization of the chemical potential, and the parameters here are still within the weak interaction limit for the first Born approximation without two-body bound states.

Besides of the numerical results, we can also estimate $T_c$ through the following analytic results by concentrating on the contribution of pairing near Fermi surface (see Supplementary Material):

$$\frac{T_c}{E_F} = \frac{\nu_0}{\pi} \exp \left( -\frac{1}{2\nu_0 V_1(k_F, k_F)} \right),$$

where $\nu_0$ is the density of state at Fermi surface. The energy scale, $\tilde{\nu}$, can be estimated within the Gor’kov and Melik-Barkhudarov (GM) approach [43, 44], which has taken into account the non-local correlation. Note that GM approach is important especially for finite-range interaction here, because it involves the pairing scattering away from the Fermi surface. The details of calculating $T_c$ within the GM approach can be found in the Supplementary Material. In Fig. 4(b), we show results of full numerical and GM methods together for comparison, and find that the transition temperature of the later is enhanced in the weak interaction limit, while it represents a strong suppression in the strong interaction regime. Such discrepancy has been observed in earlier works of 2D polar molecules [45]. Within the parameter regime we are interested in the quantum phase diagram...
(say, $\alpha < 0.04$ and $0.8 < R_c k_F < 1.2$), it seems very promising to have at least $T_c \sim 0.05 - 0.1 E_F$. We note that an even higher $T_c$ can be expected if one increases the interaction strength ($\alpha$) or interaction range ($R_c$) in a realistic experimental parameters (see below).

**Discussion:** Finally, we discuss some experimental conditions for preparing and observing of $p$-wave superfluid in Rydberg-dressed fermions. For the experimental preparation, one may consider $^6$Li by excited to a low-lying Rydberg state $n = 30$, which has an average van der Waal coefficient, $C_6 \sim 2 \text{ GHz}-\mu\text{m}^6$ [37]. For $\delta_{\text{eff}} = 2\pi \times 1 \text{ GHz}$, and $\Omega_{\text{eff}} = 2\pi \times 100 \text{ MHz}$, we have soft-core radius, $R_c \sim 0.74 \mu\text{m}$. If considering a low density regime, $n \sim 10^{12} \text{ cm}^{-3}$ (i.e. $E_F \sim 12.7 \text{ kHz}$ and $k_F \sim 3.9 \mu\text{m}^{-1}$), we may have $\alpha = 0.012$. Values of $\alpha$ and $R_c$ can be tuned by external fields in a wide range.

For a realistic Fermi gas in a harmonic trap, one may apply local density approximation so that the pairing wavefunction in the trap center (higher density) can be different from that near the edge. Near the boundary of a chiral $p$-wave (i.e. $p_x + ip_y$) phase, it is then possible to find a localized Majorana mode. More systematic calculation needs to be done by self-consistently solving the full Bogoliubov-de Gennes equation, and will be presented in other places.

To measure the $p$-wave superfluid we discuss here, one can first identify the superfluidity by observing vortices through a rotating experiment below $T_c$. The anisotropic $p$-wave pairing may not be easily measured from the gap anisotropy, since the corresponding frequency is in the radio-wave regime, where the angle resolution is very poor. But it may be observed from the anisotropic noise correlation [46], which should show different results along the axial direction and the planar direction if the atomic cloud is trapped in an anisotropic trap. The ground states of different pairing can be also distinguished by their vortices structure, since the symmetries of their order parameters are different. For example, axial phase is equivalent to $^3\text{He}-A_1$ phase [7] or the "ferromagnetic" state of spin-1 condensate [41]. As a result, it should have coreless vortices [7], which can be observed through interference pattern. Polar phase is equivalent to $^3\text{He}-\beta$ phase, and therefore shall have standard vortices, similar to $^3\text{He}-B$ phase. As for the vortices in the axi-planar phase, it has no counterpart in other $p$-wave systems and needs more careful classification. But its anisotropic feature should makes the vortex structure highly depending on the geometry of the trapping potential. We leave the detailed calculation of the vortex structure in the future.

**Summary:** In summary, we systematically investigate the novel BCS superfluid pairing of Rydberg-dressed identical fermions, identifying three distinct ground states according to their symmetries. Compared to other $p$-wave candidates in ultracold atoms/molecules, the finite-range interaction here leads not only rich quantum phase diagram as shown in Fig. 3, but also enhances the transition temperature in $p$-wave channel. Our studies therefore pave the way to study the exotic $p$-wave superfluid and related phenomena in the future.

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Supplementary material:
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A: Parameter regime without two-body bound state

In the system of Rydberg-dressed polarized Fermi gases, the effective two-body interaction is given by an isotropic soft-core potential $V(r)$ (See the Eq. (1) in the main text). As a result, the two-body bound state, if exists, can be described by the 3D Schrödinger equation with the separation of variables, i.e.

$$\psi(r) = \sum_{nlm} u_{nl}(r) Y_{lm}(\hat{r}),$$

where $r \equiv r_1 - r_2$ is the relative position between two particles, and $u_{nl}(r)$ has to satisfy the following equation ($\hbar \equiv 1$):

$$-\frac{1}{2M_r} \frac{d^2}{dr^2} [u_{nl}(r)] + \left[ \frac{-V_0}{1 + (r/R_c)^6} + \frac{l(l+1)}{2M_r r^2} \right] u_{nl}(r) = E_{n,l} u_{nl}(r).$$

(1)

Here $M_r = M/2$ is the reduced mass, and $E_{n,l}$ is the bound state eigen-energy of principle quantum number $n$ and angular momentum quantum number, $l$. For the purpose of this paper, we are looking at the situation when the attractive interaction strength ($V_0 > 0$) is so weak that at least a two-body bound state does not exist. Bound states with more than two identical fermions, although in principle possible, are very unlikely to be relevant due to Pauli exclusion principle. The criteria for such condition (no two-body bound states) can be easily obtained by taking the lowest energy, $E_{1,1} = 0$ (for identical fermionic atoms, the bound state must be antisymmetric, i.e. $l = 1, 3, \cdots$), such that the above equation determining the criteria of a bound state becomes ($\tilde{r} \equiv r/R_c$)

$$-\frac{d^2}{d\tilde{r}^2} [u_{1,1}(\tilde{r})] + \left[ \frac{-\tilde{V}_0}{1 + \tilde{r}^6} + \frac{2}{\tilde{r}^2} \right] u_{1,1}(\tilde{r}) = 0,$$

(2)

where

$$\tilde{V}_0 \equiv 2M_r R_c^2 V_0.$$  

(3)

is a dimensionless parameter. As a result, the functional relationship between the effective Rabi frequency, $\Omega_{eff}$, and effective detuning, $\delta_{eff}$, is:

$$\Omega_{eff} = \left( \frac{\tilde{V}_0}{M n^{1/3}} \right)^{1/4} (2\delta_{eff})^{5/6},$$

(4)

FIG. 1: (Color online) Parameter regions without a two-body bound state for a Rydberg-dressed interaction, as a function of the effective Rabi frequency $\Omega_{eff}$ and laser detuning $\delta_{eff}$. Results are based on the parameter of $^6$Li atoms excited to a Rydberg state of $n = 30$.

Below which we should have no two-body bound state between identical fermions. Numerical calculation gives $\tilde{V}_0 = 7.78$.

B: Ginzburg-Landau equation for $p$-wave superfluid

The total free energy of a $p$-wave superfluid with the BCS mean field approximation has been well-known in the literature (see for example, Ref. [1]), and is composed of kinetic energy, entropy of quasi-particles, and interaction energy as following:
\[ F = \sum_{\mathbf{k}} \left[ \xi_{\mathbf{k}} - E_{\mathbf{k}} + \frac{|\Delta_{\mathbf{k}}|^2}{E_{\mathbf{k}}} \frac{E_{\mathbf{k}}}{2T} - 2T \ln \left( 1 + e^{-E_{\mathbf{k}}/T} \right) \right] + \frac{1}{V} \sum_{\mathbf{k},\mathbf{k}'} \left[ \frac{\Delta_{\mathbf{k}}}{2E_{\mathbf{k}}} \frac{E_{\mathbf{k}}}{2T} + \frac{\Delta_{\mathbf{k}'}^*}{2E_{\mathbf{k}'}^*} \frac{E_{\mathbf{k}'}^*}{2T} \right], \quad (5) \]

where we keep the full expression of the gap function, \( \Delta_{\mathbf{k}} \) as well as the interaction matrix element, \( V_{\mathbf{k}-\mathbf{k}'} \). The Ginzburg-Landau free energy (\( F_{\text{GL}} \)) is then obtained by expanding the full free energy above to the leading orders of small \( |\Delta_{\mathbf{k}}| \). After some straightforward algebra, we can obtain \( F_{\text{GL}} \) to the quartic order:

\[
F_{\text{GL}} (\{\Delta_{\mathbf{k}}, \Delta_{\mathbf{k}}^*\}; T) / V = \frac{1}{\sqrt{2}} \sum_{\mathbf{k}} f_2(k; T)|\Delta_{\mathbf{k}}|^2 - \frac{1}{\sqrt{2}} \sum_{\mathbf{k}_1, \mathbf{k}_2} g_2(k_1, k_2; T)|\Delta_{\mathbf{k}_1}||\Delta_{\mathbf{k}_2}| - \frac{1}{\sqrt{2}} \sum_{\mathbf{k}} f_4(k; T)|\Delta_{\mathbf{k}}|^4 + \frac{1}{\sqrt{2}} \sum_{\mathbf{k}_1, \mathbf{k}_2} g_4(k_1, k_2; T) [\Delta_{\mathbf{k}_1}\Delta_{\mathbf{k}_2} + \Delta_{\mathbf{k}_1}^*\Delta_{\mathbf{k}_2}^*] |\Delta_{\mathbf{k}_1}|^2 + \mathcal{O}(\Delta^6), \quad (6)\]

where we have defined the following functions:

\[
f_2(k; T) \equiv \frac{\tanh(|\xi_{\mathbf{k}}|/2T)}{2|\xi_{\mathbf{k}}|}, \quad (7)
\]

\[
f_4(k; T) \equiv \frac{3}{8|\xi_{\mathbf{k}}|^2} \frac{\tanh(|\xi_{\mathbf{k}}|/2T)}{\sqrt{2}} - \frac{3}{16T|\xi_{\mathbf{k}}|^2} \text{sech}^2\left(\frac{|\xi_{\mathbf{k}}|}{2T}\right), \quad (8)
\]

\[
g_2(k_1, k_2; T) \equiv -V_{k_1-k_2} f_2(k_1; T)f_2(k_2; T), \quad (9)
\]

\[
g_4(k_1, k_2; T) \equiv -V_{k_1-k_2} f_2(k_1; T)f_4(k_2; T) + f_4(k_1; k_2; T), \quad (10)
\]

\[
f_4'(k; T) \equiv \frac{\text{sech}^2\left(\frac{|\xi_{\mathbf{k}}|}{2T}\right) \left[T \sinh\left(\frac{|\xi_{\mathbf{k}}|}{T}\right) - |\xi_{\mathbf{k}}|\right]}{8T|\xi_{\mathbf{k}}|^2}. \quad (11)
\]

Note that all of these function are regular and well-defined near the Fermi surface, \( \xi_{\mathbf{k}} = k^2/2M - \mu \to 0 \), and \( f_2, f_4 \) and \( f_4' \) are all dependent on the magnitude of \( \mathbf{k} \) only. We have selected the sign properly, so that all the functions, \( f_2, f_4, f_4', g_2(k_1, k_2; T) \), and \( g_4(k_1, k_2; T) \) are positive defined near Fermi surface (note that \( V_{k-k'} \) is negative for an attractive interaction here).

Since only \( p \)-wave component is relevant in our present system, we can use \( V_{k_1-k_2} \approx -V_{1m} \sum_{m=0,\pm 1} Y_{1,m}^* \hat{k} Y_{1,m}(\hat{k}') \) and consider the gap configuration ((See the Eq. (3) in the main text)). Integrating out the angular degree of freedom, we can rewrite Eq. (6) as following:

\[
F_{\text{GL}} (\Delta(k), \{\alpha_m\}; T) / V = \int_0^\infty \frac{k_1^2dk_1}{(2\pi)^3} \tilde{f}_2(k; T)\Delta(k_1)^2 - \int_0^\infty \frac{k_1^2dk_1}{(2\pi)^3} \int_0^\infty \frac{k_2^2dk_2}{(2\pi)^3} \tilde{g}_2(k_1, k_2; T)\Delta(k_1)\Delta(k_2)
\]

\[
- \int_0^\infty \frac{k_1^2dk_1}{(2\pi)^3} \int_0^\infty \frac{k_2^2dk_2}{(2\pi)^3} \tilde{f}_4(k; T)\Delta(k_1)^4 \sum_{m_1, \ldots, m_4} A_{m_1, m_2, m_3, m_4} \beta_{k_1}^{m_1} \beta_{k_2}^{m_2} \beta_{k_3}^{m_3} \beta_{k_4}^{m_4}
\]

\[
+ \int_0^\infty \frac{k_1^2dk_1}{(2\pi)^3} \int_0^\infty \frac{k_2^2dk_2}{(2\pi)^3} \tilde{g}_4(k_1, k_2; T)\Delta(k_1)\Delta(k_2)^3 \sum_{m_1, \ldots, m_4} A_{m_2, m_3, m_4, m_1} \beta_{k_2}^{m_2} \beta_{k_3}^{m_3} \beta_{k_4}^{m_4} + \mathcal{O}(\Delta^6), \quad (12)
\]

where we have used the renormalization condition: \( \sum_{m=0,\pm 1} |\beta_{\mathbf{k}}^m|^2 = 1 \), and

\[
\tilde{f}_2(k; T) \equiv f_2(k; T), \quad (13)
\]

\[
\tilde{f}_4(k_1, k_2; T) \equiv \frac{3}{20\pi^3} f_4(k; T), \quad (14)
\]

\[
\tilde{g}_2(k_1, k_2; T) \equiv V_{1}(k_1, k_2) f_2(k_1; T)f_2(k_2; T), \quad (15)
\]

\[
\tilde{g}_4(k_1, k_2; T) \equiv \frac{3}{20\pi} V_{1}(k_1, k_2) f_2(k_1; T)f_4' (k_2; T), \quad (16)
\]

\[
A_{m_1, m_2, m_3, m_4} \equiv \frac{20\pi}{3} \int d\Omega_{1m_1}(\hat{k}) Y_{1,m_2}^*(\hat{k}) Y_{1,m_3}(\hat{k}) Y_{1,m_4}(\hat{k}), \quad (17)
\]

\[
= 2 \text{ for } m_1 + m_2 = m_3 + m_4 = \pm 2
\]

\[
= 1 \text{ for } m_1 + m_2 = m_3 + m_4 = \pm 1
\]

\[
= 3 \text{ for } m_1 = m_2 = m_3 = m_4 = 0
\]

\[
= -1 \text{ for } m_1 + m_2 = m_3 + m_4 = 0, \text{ and } \sum_i |m_i| = 2
\]

\[
= 2 \text{ for } m_1 + m_2 = m_3 + m_4 = 0, \text{ and } \sum_i |m_i| = 4
\]

\[
= 0 \text{ otherwise}
\]
To understand the physical meaning of above GL free energy, we first note that the quadratic orders depends only on the “amplitude” of the gap function, i.e. $\Delta(k)$, and hence $T_c$ obtained by $\delta F_{GL}/\delta \Delta^*(k)|_{\Delta(k)\to 0} = 0$ is independent of the orientation. In order understand how the orientation of the order parameter ($\Delta(m)$) may affect the free energy, we take the orientation configuration ((See the Eq. (4) in the main text)), so that

$$
\sum_{m_1, \ldots, m_4} \frac{A_{m_1, m_2, m_3, m_4}}{\beta_k^{m_1} \beta_k^{m_2} \beta_k^{m_3} \beta_k^{m_4}} = 2(\cos^4 \gamma_k + \sin^4 \gamma_k) + 8 \cos^2 \gamma_k \sin^2 \gamma_k
$$

$$
= 2 + \sin^2(2\gamma_k)
$$

and

$$
\sum_{m_1, \ldots, m_4} A_{m_1, m_2, m_3, m_4} \beta_k^{m_1} \beta_k^{m_2} \beta_k^{m_3} \beta_k^{m_4}
$$

$$
= 2(\cos \gamma_k \cos^3 \gamma_k + \sin \gamma_k \sin^3 \gamma_k) 
+ 4 \sin(\gamma_k + \gamma_k) \sin \gamma_k \cos \gamma_k
$$

We note that, if we consider a simple case when the amplitude of the gap function is completely decoupled from it orientation (i.e. $\gamma_k = \gamma$ is independent of momentum magnitude, $k$, or equivalently, $\beta_k^m = \beta^m$ for all $k$), we can simplify above results and obtain

$$
\sum_{m_1, \ldots, m_4} A_{m_1, m_2, m_3, m_4} \beta_k^{m_1} \beta_k^{m_2} \beta_k^{m_3} \beta_k^{m_4}
$$

$$
= \sum_{m_1, \ldots, m_4} A_{m_2, m_3, m_4} \beta_k^{m_2} \beta_k^{m_3} \beta_k^{m_4}
$$

$$
= 2 + \sin^2(2\gamma).
$$

The Ginzburg-Landau free energy for such ansatz then becomes a simple function of $\gamma$:

$$
\frac{F_{GL}(\Delta(k), \gamma; T)}{2\mu} = \int_0^\infty \frac{k^2 dk}{2\pi} \int_0^\infty \frac{k'^2 dk'}{2\pi} \int_0^\infty \frac{k^2 dk_1}{(2\pi)^3} \int_0^\infty \frac{k'^2 dk'}{(2\pi)^3} \tilde{\gamma}(k, k; T) \tilde{\Delta}(k_1) \tilde{\Delta}(k_2)
$$

$$
+ \left[ -\int_0^\infty \frac{k'^2 dk'}{(2\pi)^3} \tilde{\bar{\bar{\gamma}}}(k, k; T) \tilde{\Delta}(k_1) \tilde{\Delta}(k_2)^3 \right](2 + \sin^2(2\gamma)) + \mathcal{O}(\Delta^6)
$$

C: Transition temperature of p-wave superfluid within the BCS theory

In this section, we derive semi-analytic results for the transition temperature ($T_c$) of p-wave pairing superfluid within the BCS theory. By decomposing the interaction matrix element and the gap function into different orbital angular momentum channels: $V_{k-k'} = -\sum_{l,m} V_l(k, k') Y_l^*(k') Y_{l,m}(k)$ and $\Delta_k = \sum_{l,m} \Delta_{l,m}(k) Y_{l,m}(k)$, where $\sum_{l,m}$ is summation over all odd values of $l$ and $m = -l, \ldots, l$, we find that the major contribution is from $p$-wave channel ($l = 1$) as shown in Fig 2(a). Since near $T_c$, pairing function of different channels are degenerate, below we will assume a polar state, $\Delta_k = \Delta_{1,0} Y_{1,0}(k)$, for the study of $T_c$. As a result, we may apply the theory developed in Ref. [2] in the context of fermionic dipolar gas to investigate $T_c$, where the gap function is also a $p_z$ type polar state.

We start from expanding the right hand side gap equation to the leading order, and rescale all quantities to dimensionless expression in unit of Fermi wavevector ($k_F$) and Fermi energy ($E_F$), i.e. $k \equiv k/k_F$, $k \equiv k/f_F$, $\tilde{T} \equiv T/E_F$, $\Delta_1 \equiv \Delta_{1,0}/E_F$, $\tilde{\nu} \equiv \mu/E_F$, and $\alpha \tilde{V}_1(k, k') \equiv V_1(k, k') k_F^2/2E_F/(2\pi)^3$, so that after integration of angular part, gap equation reduces to

$$
\tilde{\Delta}_1(\tilde{k}) = \alpha \int_0^\infty \tilde{k}^2 d\tilde{k}' \tilde{V}_1(\tilde{k}, \tilde{k}') \tanh \left( \frac{[\tilde{k}'^2 - \tilde{\mu}]}{2\tilde{T}} \right) \tilde{\Delta}_1(\tilde{k}')
$$

$$
(23)
$$

In order to single out the main contribution in the rhs of Eq. (23), we introduce a characteristic frequency or energy $\tilde{\omega}$ that obeys the constraint $T \ll \tilde{\omega}$ and is of the order of the Fermi energy. By introducing reduced kinetic energy, $k \equiv \tilde{k}^2$, and the density of state, $n(k) = k^2 dk/\tilde{\omega}$, we thus rewrite the r.h.s of Eq. (23) to be
\[ \Delta_1(\xi) = \alpha \left( \int_0^{\tilde{\mu} - \tilde{\omega}} + \int_{\tilde{\mu} - \tilde{\omega}}^{\tilde{\mu} + \tilde{\omega}} + \int_{\tilde{\mu} + \tilde{\omega}}^{\infty} \right) d\xi' \nu(\xi') \tilde{V}_1(\xi, \xi') \tanh \left( \frac{|\xi'| - \tilde{\mu}}{2T} \right) \Delta_1(\xi') \]

\[ = \alpha \left( \int_{-\tilde{\omega}}^{0} + \int_{0}^{+\tilde{\omega}} + \int_{-\tilde{\omega}}^{-\tilde{\omega}} \right) d\xi' \nu(\xi') \tilde{V}_1(\xi, \xi') \tanh \left( \frac{|\xi'|}{2T} \right) \Delta_1(\xi'), \quad (24) \]

where we have move the quasi-particle energy relative to the chemical potential (i.e. when \( \xi = 0 \)), redefine \( \tilde{\Delta}_1(\xi) = \Delta_1(\xi + \tilde{\mu}) \), \( \tilde{V}_1(\xi, \xi') \equiv \tilde{V}_1(\xi + \tilde{\mu}, \xi' + \tilde{\mu}) \), and let \( \tilde{\nu}(\xi) \equiv \nu(\xi + \tilde{\mu}) = \frac{1}{2} \sqrt{\xi + \tilde{\mu}} \).

For the three integral range above, it is easy to see that the major contribution comes from the quasi-particle excitation near the Fermi surface. Following the standard treatment, we first introduce the following function:

\[ G(\tilde{\xi}, \tilde{\xi}') = \tilde{\nu}(\tilde{\xi}') \tilde{V}_1(\tilde{\xi} + \tilde{\mu}, \tilde{\xi}' + \tilde{\mu}) \tilde{\Delta}_1(\tilde{\xi}'), \quad (25) \]

so that three integrals becomes:

\[ \alpha \int_{-\tilde{\omega}}^{+\tilde{\omega}} d\xi' \frac{\tanh(|\xi'|/2\tilde{T})}{|\xi'|} G(\xi, 0) \]

\[ + \alpha \left( \int_{-\tilde{\omega}}^{0} + \int_{0}^{+\tilde{\omega}} \right) d\xi' \frac{\tanh(|\xi'|/2\tilde{T})}{|\xi'|} G(\xi, \xi') \]

\[ + \alpha \int_{-\tilde{\omega}}^{+\tilde{\omega}} d\xi' \frac{\tanh(|\xi'|/2\tilde{T})}{|\xi'|} \left[ G(\xi, \xi') - G(\xi, 0) \right]. \quad (26) \]

The last two lines are considered as a minor contribution, because they involve the integration of particles away from Fermi surface. We will discuss this part in Appendix D later.

The first line is the major contribution of the right hand side, and therefore the transition temperature should be determined by the following equation:

\[ \tilde{\Delta}_1(\tilde{\xi}) = \alpha G(\tilde{\xi}, 0) \int_{-\tilde{\omega}}^{0} d\xi' \frac{\tanh(|\xi'|/2\tilde{T})}{|\xi'|} \]

\[ \approx 2\alpha\tilde{\nu}(0)\tilde{V}_1(\tilde{\xi}, 0)\tilde{\Delta}_1(0) \ln \left( \frac{2\tilde{\omega}e^\gamma}{\pi T} \right), \quad (27) \]

where we have evaluated the integral following the standard treatment (see Refs. [2–4]), where \( \gamma = 0.5772 \) is the Euler constant. Taking its result at Fermi surface, \( \tilde{\xi} = 0 \), we can rid of the gap function and obtain the following expression for the transition temperature in the standard BCS theory:

\[ \tilde{T}_c = \frac{2e^\gamma}{\pi} \ln \left( \frac{1}{2\alpha\tilde{\nu}(0)\tilde{V}_1(0, 0)} \right). \quad (28) \]

Note that the energy scale, \( \tilde{\omega} \), including the contribution of interaction away from the Fermi surface, has to be determined from the "minor" part of Eq. (26). When considering the momentum (or energy) dependence of the gap function, Eq. (27) can be applied to determine the behaviour of \( \tilde{\Delta}_1(\tilde{\xi}) \) from the interaction, i.e.

\[ \frac{\tilde{\Delta}_1(\tilde{\xi})}{\tilde{\Delta}_1(0)} = \frac{\tilde{V}_1(\tilde{\xi}, 0)}{\tilde{V}_1(0, 0)}. \quad (29) \]

In our full numerical calculation, we can see that the gap function is indeed proportional to the interaction matrix element as clearly shown in Fig. 2. The close relationship between the gap function and the interaction matrix element will help our further study in the determination of transition temperature.

D: Calculation of the prefactor \( \tilde{\omega} \) beyond the BCS theory

In order to estimate the transition temperature more precisely, here we have to determine the energy scale, \( \tilde{\omega} \), used in Eq. (26) for the separation of "major" contribution and "minor" contribution. Following the seminar work by Gor’kov-Melik-Barkhudarov [4] and later extended version for dipolar gas [2], we can estimate its contribution first by replacing \( \tanh(|\xi'|/2\tilde{T}) \) there via the step function and omitting the contribution from a narrow interval \( |\xi'| \lesssim 2\tilde{T} \ll \tilde{\omega} \). As a result, this minor contribution (the second and the third part of Eq. (26)) becomes
where the last form is obtained by the integration by parts for the three integrals independently and then combining their results. This is important to note that the integration of $\int_{-\bar{\omega}}^{\bar{\omega}}$ is composed by two divergent terms and therefore has to be treated differently.

Since the $T_c$ of Eq. (28) is based on the contribution of the "major" part (the first line of Eq. (26)), it is therefore meaningful to determine the unknown prefactor, $\tilde{\omega}$, by requiring the contribution from the "minor" part (the second and third lines of Eq. (26), or Eq. (30)) to be zero. Similar to the previous treatment, when considering interaction of finite range, we should multiply the density of state, $\tilde{\nu}(\tilde{\xi})$, and integrate these "minor" part over all relevant energy. As a result, the equation to determine $\tilde{\omega}$ is given by the following form:

$$ \int_{-\tilde{\mu}}^{\tilde{\mu}} \int_{-\tilde{\mu}}^{\tilde{\mu}} \frac{d}{d|\tilde{\xi}|} \tilde{G}(\tilde{\xi}, \tilde{\xi}') = 0, \tag{31} $$

where we define

$$ \tilde{G}(\tilde{\xi}, \tilde{\xi}') \equiv \tilde{\nu}(\tilde{\xi}) \tilde{\nu}(\tilde{\xi}') \tilde{V}_1(\tilde{\xi}, \tilde{\xi}') \tilde{\Delta}_1(\tilde{\xi}'). \tag{32} $$

As a result, we can separate the term of $\tilde{\omega}$ from others in Eq. (31) and obtain

$$ \ln \tilde{\omega} = \int_{-\tilde{\mu}}^{\tilde{\mu}} \int_{-\tilde{\mu}}^{\tilde{\mu}} \frac{d}{d|\tilde{\xi}|} \tilde{G}(\tilde{\xi}, \tilde{\xi}') \tilde{\nu}(\tilde{\xi}) \tilde{\nu}(\tilde{\xi}') \tilde{V}_1(\tilde{\xi}, \tilde{\xi}') \tilde{\Delta}_1(\tilde{\xi}') \tag{33} $$

where we have evaluated the following integral directly

$$ \left( \int_{-\tilde{\mu}}^{\tilde{\mu}} + \int_{0}^{\infty} \right) d\tilde{\xi} \frac{d}{d|\tilde{\xi}|} \tilde{G}(\tilde{\xi}, \tilde{\xi}) = \int_{0}^{\infty} d\tilde{\xi} \frac{d}{d|\tilde{\xi}|} \tilde{G}(\tilde{\xi}, \tilde{\xi}) = -2\tilde{G}(\tilde{\xi}, 0), \tag{34} $$

Replacing the definition of the function, $\tilde{G}$, and using Eq. (29), we can rewrite Eq. (33) to see how the energy scale, $\tilde{\omega}$, depends on the interaction matrix element and the gap function explicitly:

$$ \ln \tilde{\omega} = \frac{\int_{-\tilde{\mu}}^{\tilde{\mu}} \int_{-\tilde{\mu}}^{\tilde{\mu}} \frac{d}{d|\tilde{\xi}|} \tilde{\nu}(\tilde{\xi}) \tilde{\nu}(\tilde{\xi}') \tilde{V}_1(\tilde{\xi}, \tilde{\xi}') \tilde{\Delta}_1(\tilde{\xi}')}{-2\tilde{\nu}(0) \tilde{V}_1(0, 0)} \int_{-\tilde{\mu}}^{\tilde{\mu}} \int_{-\tilde{\mu}}^{\tilde{\mu}} \frac{d}{d|\tilde{\xi}|} \tilde{\nu}(\tilde{\xi}) \tilde{\nu}(\tilde{\xi}') \tilde{V}_1(\tilde{\xi}, \tilde{\xi}') \tilde{\Delta}_1(\tilde{\xi}')} \tag{35} $$

It is important to note that although the value of $\tilde{\omega}$ looks like depending on the second order of the interaction, but actually it is independent on the "magnitude" of the interaction, i.e. $\alpha$, as clearly seems from the original equation of such energy scale in Eq. (31). Therefore, calculating $\tilde{\omega}$ within the GM approach does not mean that we have to include the second order Born approximation for the BCS theory, which can still safely be assumed to be small. However, due to the integral over all energy regime, it is very clear that the shape or the core-core radius of the potential, $\tilde{V}_1$, about the Fermi energy can be very crucial. In our present system, we find that $\tilde{\omega} \sim 0.55$ when $R_c k_F = 1$.

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