On certain representations of pricing functionals

Carlo Marinelli*

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Abstract

We revisit two classical problems: the determination of the law of the underlying with respect to a risk-neutral measure on the basis of option prices, and the pricing of options with convex payoffs in terms of prices of call options with the same maturity (all options are European). The formulation of both problems is expressed in a language loosely inspired by the theory of inverse problems, and several proofs of the corresponding solutions are provided that do not rely on any special assumptions on the law of the underlying and that may, in some cases, extend results currently available in the literature.

1 Introduction

Let $S, \beta: \Omega \times [0, T] \to \mathbb{R}_+$ denote the price processes of an asset and of a numéraire (that we shall assume to be the money-market account, for simplicity), respectively, in an arbitrage-free market, modeled on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$, where $T > 0$ is a fixed time horizon and $\mathcal{F}_0$ is the trivial $\sigma$-algebra. Assuming that pricing takes place with respect to a risk-neutral probability measure $Q$, the price at time zero of a European option with maturity $T$ and payoff profile $g: \mathbb{R}_+ \to \mathbb{R}$ on the asset with price process $S$ is given by

$$
\pi(g) = \mathbb{E}_Q \beta_T^{-1} g(S_T).
$$

We shall call the map $g \mapsto \pi(g)$, defined on the set of all measurable functions $g$ such that the right-hand side is finite, the pricing functional.

A rather general and natural question, of clear relevance also for practical purposes, is the following: suppose that the action of $\pi$ is known on a set of functions $G$, i.e. that $\pi(g)$ is known for every $g \in G$. Is it possible to enlarge the set of functions $G$ where $\pi$ is determined, i.e. to compute $\pi(f)$ for some functions $f$ that do not belong to $G$? We are going to discuss some questions of this type (although not in this generality), through the representation of the pricing functional as a (Stieltjes) measure, that is

$$
\pi(g) = \int_{\mathbb{R}_+} g(x) dF(x),
$$

where $F$ is the (right-continuous version of the) distribution function of $S_T$ with respect to the measure $(dQ/d\mathbb{P})\beta_T^{-1} \cdot \mathbb{P}$, i.e. the measure with density with respect to $\mathbb{P}$ equal to the stochastic discount factor.

If a collection $G$ of payoff profiles $g$ is such that the prices $\pi(g)$ of the corresponding options are known, the set $M := \{(g, \pi(g)) \in G \}$ will be called a measurement set, and a measurement set that determines $F$ will be called a representation. That is to say, if knowing $M$ allows one to reconstruct

*Department of Mathematics, University College London, Gower Street, London WC1E 6BT, United Kingdom. URL: goo.gl/4GKJP
A kind of representation where a sequence of measures converging towards $\pi$ intervenes. Let $(\mathscr{F})$, we shall use some elementary facts from measure theory and convexity, that we recall for convenience. Let $(\mathscr{F})$, motivated by an empirical problem on non-parametric pricing of options treated in [8].

We conclude in this method is that it provides a particularly handy way to make computations, also in cases that do not directly follow from the setups of the previous two sections. We conclude in §4-5, that is, using the theory of distributions. An interesting aspect of this method is that it provides a particularly handy way to make computations, also in cases that do not directly follow from the setups of the previous two sections.

The reconstruction of $F$ from a set of option prices is interesting in its own right, but sometimes less information is enough for the problem at hand (roughly speaking, this is just the idea behind static hedging). Using the above terminology, if one needs a measure $M$, it may be possible to determine a measurement set $M'$ that contains $M$, without necessarily recovering $F$ first. The simplest example is the pricing of options with continuous piecewise linear payoff profile. Another one is the pricing of options with payoff function equal to the difference of convex functions in terms of call options. Even though, in the latter case, the measurement set of all call options is already a representation, there is an alternative pricing formula that avoids the differentiation of $C$, which might be preferable for numerical purposes. Such pricing formula for options with convex payoff profiles is not new, but we give nonetheless several proofs: a very concise one, a longer one that (hopefully) highlights the role of convexity, and a third one that is extremely simple if sufficient regularity is present. We also show that similar ideas can be used to “localize” the pricing formula, i.e. to price options with payoff profile that is piecewise the difference of convex functions.

The main content is organized as follows: we collect in §2 some useful (elementary) facts from measure theory, convexity, and the theory of distribution. Definitions, motivations, basic properties, and examples pertaining to pricing functionals, measurement sets, and representations are given in §3. Qualitative properties of the functions $P$ and $C$, as defined above, are discussed in §4 without any assumption on $F$. Moreover, we show that $F$ is the right derivative of $P$ by two methods, that is, using the integration by parts formula for càdlàg functions of finite variation and by a denseness argument, respectively. In §5 we revisit the fact that prices of options with convex profile are determined by prices of call options for all positive strikes. This is proved in two ways: by an integration argument, that uses essentially only the Fubini theorem, and via the above-mentioned integration by parts formula. The results of the previous two sections are derived by yet another approach in §4.5 that is, using the theory of distributions. An interesting aspect of this method is that it provides a particularly handy way to make computations, also in cases that do not directly follow from the setups of the previous two sections.

We conclude in §7 considering a kind of representation where a sequence of measures converging towards $F$ intervenes. This is motivated by an empirical problem on non-parametric pricing of (vanilla) options treated in [8].

## 2 Preliminaries

We shall use some elementary facts from measure theory and convexity, that we recall for convenience. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces, and $\mu$ a measure on the former. If $\phi: X \to Y$
is a measurable function, then the image measure or push-forward of $\mu$ through $\phi$ is the measure on $(Y, \mathcal{B})$ defined by $\phi_*\mu : B \mapsto \mu(\phi^{-1}(B))$. If $g : Y \to \mathbb{R}$ is a measurable function, then

$$\int_Y g \, d\phi_*\mu = \int_X g \circ \phi \, d\mu,$$

(2.1)

in the sense that $g$ is $\phi_*\mu$-integrable if and only if $g \circ \phi$ is $\mu$-integrable, and in this case the integrals coincide (see, e.g., [3 §2.6.8]). Interpreting precomposition as pull-back, hence writing $\phi^* g := g \circ \phi$, and using the notation $m(f) := \langle m, f \rangle := \int f \, dm$ for any function $f$ integrable with respect to a measure $m$, the identity (2.1) can be written in the simple and suggestive form

$$\langle \phi^* g, \mu \rangle = \langle g, \phi_*\mu \rangle.$$

We shall extensively use an integration-by-parts formula for Lebesgue-Stieltjes integrals. Let the functions $F, G : \mathbb{R} \to \mathbb{R}$ be càdlàg (i.e. right-continuous with left limits) and with finite variation (i.e. having bounded variation on every bounded interval). Then, for any two real numbers $a < b$,

$$F(b)G(b) - F(a)G(a) = \int_{[a,b]} G(x-) \, dF(x) + \int_{[a,b]} F(x) \, dG(x).$$

(2.2)

If $G$ is continuous, one can obviously replace $G(x-)$ by $G(x)$. Whenever $dG$ is an atomless measure we shall just write $\int_{[a,b]} F \, dG$ instead of denoting the interval of integration as a subscript. Moreover, we set $\mathbb{R}_+ := [0, +\infty[.$

Let $I \subset \mathbb{R}$ be an open interval and $f : I \to \mathbb{R}$ be a convex function. Then $f$ is everywhere left- and right-differentiable, that is, for any $x \in I$ the left and right derivatives

$$D^- f(x) := \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}, \quad D^+ f(x) := \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

exist and are finite, and $D^- f(x) \leq D^+ f(x)$. Moreover, both $D^- f$ and $D^+ f$ are increasing functions, $D^- f$ is left-continuous, and $D^+ f$ is right-continuous. It follows that $f$ is differentiable except at the countable set of points where $D^- f$ and $D^+ f$ do not coincide. The subdifferential of $f$ at $x$ is defined as

$$\partial f(x) = \{ z \in \mathbb{R} : f(y) - f(x) \geq z(y-x) \quad \forall y \in I \}.$$

It can be shown that $\partial f(x) = [D^- f(x), D^+ f(x)]$ and that, for any $x_1, x_2 \in I$, $x_1 < x_2$, it holds $D^+ f(x_1) \leq D^- f(x_2)$, hence $\partial f(x_1) \cap \partial f(x_2)$ is either empty, if the last inequality is strict, or equal to $\{D^- f(x_2)\}$, if the last inequality is an equality. The right derivative $D^+ f$, being increasing, hence of bounded variation, and right-continuous, defines a (Lebesgue-Stieltjes) measure $m$ via the prescription

$$D^+ f(b) - D^+ f(a) = m([a, b]), \quad a, b \in I, \ a \leq b.$$

In this sense, the positive measure $m$ can be interpreted as the second derivative of $f$. For a proof of these results on convex functions see, e.g., [10] Chapter 1.

We shall also use elementary properties of distributions, for which we refer to, e.g., [9]. Assume that $f : \mathbb{R} \to \mathbb{R}$ is piecewise of class $C^1$, with discontinuity points $(x_n)$. Using the standard notation $\Delta f(x) := f(x^+) - f(x^-)$, one has (see [9 p. 37])

$$f' = \sum_n \Delta f(x_n) \delta_{x_n} + [f'],$$

where $f'$ stands for the derivative of $f$ in the sense of distributions, and $[f']$ for the derivative in the classical sense over the open intervals $]x_n, x_{n+1}[$ where $f$ is continuously differentiable (a
corresponding result for higher-order derivatives can be obtained by induction). If \( f \) is a function with finite variation, then \( f' \) coincides, in the sense of distributions, with the Lebesgue-Stieltjes measure \( df \). We shall need to consider functions \( f \) that are piecewise differences of convex functions, i.e. of the form

\[
f = \sum_{n} f_n, \quad f_n: [a_n, a_{n+1}] \to \mathbb{R},
\]

where the sum is (at most) countable, and for every \( n \) there exist convex functions \( h_n^1, h_n^2 \) on \( \mathbb{R} \) such that \( f_n = h_n^1 - h_n^2 \) on \([a_n, a_{n+1}]\). Then \( f \) is càdlàg and has finite variation. We are going to compute the first and second distributional derivatives of \( f \). It is clear that it is enough to consider, without loss of generality, \( f \) with support equal to \([a, b] \) and \( f = h^1 - h^2 \) on \([a, b] \), for \( h^1 \) and \( h^2 \) convex functions on \( \mathbb{R} \).

**Lemma 2.1.** Let \( f: \mathbb{R} \to \mathbb{R} \) be a càdlàg function with finite variation, and \([a, b] \subset \mathbb{R} \) a compact interval. The distributional derivative of \( f_{[a, b]} \) is

\[
f' = df|_{[a, b]} + f(a)\delta_a - f(b-)\delta_b.
\]

**Proof.** Let us denote \( f_{[a, b]} \), for the purposes of this proof only, simply by \( f \). The distributional derivative \( f' \) is defined by the identity \( \langle f', \phi \rangle = -\langle f, \phi' \rangle \) for every \( \phi \in D(\mathbb{R}) \), where

\[
\langle f, \phi' \rangle = \int_{\mathbb{R}} f \phi' = \int_{a}^{b} f \, d\phi = \int_{[a, b]} f \, d\phi
\]

and, thanks to the integration-by-parts formula,

\[
f(b)\phi(a) - f(a)\phi(b) = \int_{[a, b]} f \, d\phi + \int_{[a, b]} \phi \, df,
\]

hence

\[
\int_{[a, b]} f \, d\phi = -\int_{[a, b]} \phi \, df + f(b)\phi(b) - f(a)\phi(a),
\]

i.e.

\[
\langle f', \phi \rangle = \int_{[a, b]} \phi \, df + f(a)\phi(a) - f(b)\phi(b)
\]

\[
= \int_{[a, b]} \phi \, df + \phi(b)(f(b) - f(b-)) + f(a)\phi(a) - f(b)\phi(b)
\]

\[
= \int_{[a, b]} \phi \, df + f(a)\phi(a) - f(b-)\phi(b)
\]

\( \square \)

**Proposition 2.2.** Let \( f: \mathbb{R} \to \mathbb{R} \) be a difference of convex functions and let \([a, b] \subset \mathbb{R} \) be a compact interval. Then the first and second distributional derivatives of \( f_{[a, b]} \), denote by \( f' \) and \( f'' \), respectively, are

\[
f' = df|_{[a, b]} + f(a)\delta_a - f(b-)\delta_b,
\]

\[
f'' = DD^+f|_{[a, b]} + f(a)\delta'(a) - f(b)\delta'_b + DD^+f(a)\delta_a - DD^+f(b-)\delta_b.
\]

**Proof.** The function \( f \) is continuous on \( \mathbb{R} \) and, being the difference of convex functions, has finite variation. The previous lemma then yields the expression for \( f' \). To compute \( f'' \), let us recall that \( f \) is absolutely continuous on with (classical) derivative equal to \( DD^+f \) a.e., so that

\[
\langle f'', \phi \rangle = -\langle f', \phi'' \rangle = -\int_{[a, b]} \phi'' \, df = f(a)\phi'(a) + f(b-)\phi'(b)
\]

\[
= -\int_{[a, b]} DD^+f \, d\phi = f(a)\phi'(a) + f(b-)\phi'(b).
\]
Since \(D^+ f\) is càdlàg and of finite variation, the integration-by-parts formula yields
\[
D^+ f(b)\phi(b) - D^+ f(a)\phi(a) = \int_{[a,b]} D^+ f \, d\phi + \int_{[a,b]} \phi \, dD^+ f,
\]
hence
\[
-\int_{[a,b]} D^+ f \, d\phi = -\int_{[a,b]} D^+ f \, d\phi
= \int_{[a,b]} \phi \, dD^+ f + D^+ f(a)\phi(a) - D^+ f(b)\phi(b)
= \int_{[a,b]} \phi \, dD^+ f + D^+ f(a)\phi(a) - D^+ f(b-)\phi(b).
\]
Collecting terms concludes the proof:
\[
(f'', \phi) = \int_{[a,b]} \phi \, dD^+ f - f(a)\phi'(a) + f(b-)\phi'(b) + D^+ f(a)\phi(a) - D^+ f(b-)\phi(b).
\]

**Remark 2.3.** Let \(a < b\) be real numbers and \(I\) be any interval with endpoints \(a\) and \(b\). Note that \(fI_{[a,b]}\) coincides in \(\mathcal{D}'\) with \(fI_{t}\), for any choice of \(I\). Therefore their distributional derivatives are also the same.

## 3 Pricing functionals, measurements and representations

### 3.1 Pricing functionals

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space endowed with a filtration \((\mathcal{F}_t)_{t \in [0,T]}\), with \(T > 0\) a fixed time horizon, and let \(S: \Omega \times [0,T] \to \mathbb{R}_+\) be the price process of an asset. We assume also that \(\beta: \Omega \times [0,T] \to [0,\infty]\) is the price process of a further asset used as numéraire, normalized with \(\beta_0 = 1\) and uniformly bounded from below, and that the market where both assets are traded is free of arbitrage, so that the set \(\mathbb{Q}\) of probability measures \(\mathbb{Q}\) equivalent to \(\mathbb{P}\) such that the discounted price process \(\beta^{-1}S\) is a \(\mathbb{Q}\)-local martingale is not empty. For any \(\mathcal{F}_T\)-measurable claim \(X\) such that \(\beta_T^{-1}X\) is bounded, the value \(\mathbb{E}_\mathbb{Q} \beta_T^{-1}X\) is an arbitrage-free price at time zero of \(X\) for every \(\mathbb{Q} \in \mathbb{Q}\). From now on we shall fix a measure \(\mathbb{Q} \in \mathbb{Q}\). For any measurable bounded function \(g: \mathbb{R}_+ \to \mathbb{R}\), the (bounded) \(\mathcal{F}_T\)-measurable random variable \(g(S_T)\) is the payoff of a European option on \(S\) with payoff profile \(g\), the price of which at time zero is \(\mathbb{E}_\mathbb{Q} \beta_T^{-1} g(S_T)\).

We shall call *pricing functional* the map
\[
\pi: g \mapsto \mathbb{E}_\mathbb{Q} \beta_T^{-1} g(S_T) = \mathbb{E} \frac{d\mathbb{Q}}{d\mathbb{P}} \beta_T^{-1} g(S_T),
\]
defined first on the set of measurable bounded functions \(g: \mathbb{R}_+ \to \mathbb{R}\). Let \(\mu\) be the measure on \(\mathcal{F}_T\) defined by
\[
\mu(A) := \mathbb{E}_\mathbb{Q} \beta_T^{-1} 1_A,
\]
that is, \(\mu\) is the measure on \(\mathcal{F}_T\) the Radon-Nikodym derivative of which with respect to \(\mathbb{P}\) is
\[
\frac{d\mu}{d\mathbb{P}} = \frac{d\mathbb{Q}}{d\mathbb{P}} \beta_T^{-1}.
\]
Note that \(\mu\) is (in general) not a probability measure: in fact, \(\mu(\Omega) = \mathbb{E}_\mathbb{Q} \beta_T^{-1}\) need not be one, and could be interpreted as the price at time zero of a zero-coupon bond maturing at time \(T\) with
face value equal to one. In this case, $\mu$ is a sub-probability measure, i.e. $\mu(\Omega) \leq 1$. The pricing functional can then be written as

$$\pi: g \mapsto \int_{\Omega} g(S_T) \, d\mu.$$  

Denoting the push-forward of $\mu$ through $S_T$ by $S_*\mu$, i.e. the measure on the Borel $\sigma$-algebra of $\mathbb{R}$ defined by

$$S_*\mu: B \mapsto \mu(S_T^{-1}(B)),$$

one has

$$\mathbb{E}_{\mathbb{Q}} \beta_T^{-1} g(S_T) = \int_{\Omega} g(S_T) \, d\mu = \int_{\mathbb{R}} g \, d(S_*\mu).$$

Therefore, denoting the distribution function of the measure $S_*\mu$ by $F$, i.e.

$$F(x) := \mu(S_T \leq x) = \mathbb{E}_{\mathbb{Q}} \beta_T^{-1} 1_{\{S_T \leq x\}},$$

the pricing functional can be written as

$$\pi: g \mapsto \int_{\mathbb{R}_+} g \, dF.$$

In other words, the pricing functional can be identified with $F$, or with $S_*\mu$. Note also that the pricing functional can naturally be extended to every $g \in L^1(dF)$. If $g \geq 0$, as is mostly the case for payoff functions, then $\pi(g)$ is simply the norm of $g$ in $L^1(dF)$.

**Remark 3.1.** If one just assumes that there exists a pricing functional, defined as a positive linear functional on a certain set of functions, then an integral representation of $\pi$ holds in many cases. This is essentially the content of various forms of the Riesz representation theorem. For instance, if $\pi$ is continuous on $C_0(\mathbb{R})$, the Banach space of continuous functions that are zero at infinity, endowed with the supremum norm, then there exists a unique finite Radon measure $m$ on $\mathbb{R}$ such that

$$\pi(g) = \int g \, dm$$

for every $g \in C_0(\mathbb{R})$. If $\pi$ is continuous on $C_c(\mathbb{R})$, the space of continuous functions with compact support with the topology of uniform convergence on compact sets, then there exists a unique Radon measure $m$ on $\mathbb{R}$ (not necessarily finite) such that $\pi(g) = \int g \, dm$ for every $g \in C_0(\mathbb{R})$. On the other hand, if $\pi$ is just assumed to be continuous on $L^\infty(\mathcal{B})$, the Banach space of bounded functions with the supremum norm, then an integral representation of $\pi$ is only possible with respect to a bounded additive set function, not a measure. A completely analogous situation arises if continuity of $\pi$ is assumed on $L^\infty(\mathbb{R})$. On the other hand, if $\pi$ is assumed to be weak* continuous on $L^\infty(\mathbb{R})$, then there exists $\phi \in L^1_+(\mathbb{R})$ such that $\pi(g) = \int \phi g$ for every $g \in L^\infty(\mathbb{R})$. However, the weak* convergence of a sequence $(g_n)$ in $L^\infty$, i.e. the existence of $g \in L^\infty$ such that

$$\lim_{n \to \infty} \int f g_n = \int f g \quad \forall f \in L^1(\mathbb{R}),$$

does not seem to have a clear economic interpretation.

### 3.2 Measurements and representations

Depending on the problem at hand, the pricing functional $\pi: g \mapsto dF(g)$ may or may not be known. If $dF$ is assumed a priori to be known, for instance in the Black-Scholes model with given volatility, then $\pi$ is trivially known. Analogously, one may assume that $dF$ belongs to a certain family of finite measures $(dF_\theta)_{\theta \in \Theta}$ parametrized by a finite-dimensional parameter $\theta$, and by statistical procedures an estimate $\hat{\theta}$ is obtained, so that $dF_{\hat{\theta}}$ is then used in the definition of $\pi$, thus falling back to the previous (quite tautological) case. Strictly speaking, this procedure produces an estimator of the pricing functional, but we are not going to discuss any issues pertaining to statistics. On the other
hand, in many other situations, for instance when no parametric assumptions on \( dF \) are made, the pricing functional \( \pi \) is only known through its action on a set of “test functions” \((g_j)_{j \in J}\), e.g. with \( g_j \) the payoff profile of a call or put option with a strike price indexed by \( j \in J \). The next definition is hence quite natural.

**Definition 3.2.** A measurement (of \( dF \)) is a pair \((g, \pi(g))\), where \( g : \mathbb{R}_+ \to \mathbb{R} \) is a measurable function integrable with respect to \( dF \). A measurement set (of \( dF \)) is a collection of measurements.

A typical situation of practical relevance is given by \((g_j)\) being a collection of payoff profiles of (European) options. For instance, for any \( j \geq 0 \), let \( g_j \) be the payoff function of a put option with strike price \( j \), that is, \( g_j : x \mapsto (j - x)^+ \). If the price of the put option with strike \( j \) is known for every \( j > 0 \), then we have a measurement \( M = (g_j, \pi_j)_{j \in J} \) setting \( J = \mathbb{R}_+ \), \( g_j : x \mapsto (j - x)^+ \), and \( \pi_j = dF(g_j) \).

**Remarks 3.3.** a) Let \( M = (g_j, \pi_j)_{j \in J} \) be a measurement set \((J \text{ is just an index set})\). The set of numbers \((\pi_j)\) is included in the definition of \( M \) just for convenience, as it is clearly redundant being uniquely determined by \((g_j)\) and \( dF \), the latter of which is assumed to be fixed, even though treated as unknown. b) A measurement set is just a subset of the graph of \( \pi \). c) The term “measurement”, by no means standard, somewhat mimics an analogous one used in the theory of inverse problems, where, in a (usually) more rigid functional setting, the expression “measurement operator” is sometimes used. d) In view of the linearity of integration, if \( \pi \) is known on a set \( G \subseteq L^1(dF) \), then it is known also on the vector space generated by \( G \). Similarly, it would seem natural to augment \( M \) with its accumulation points, i.e. to take its closure, in \( L^1(dF) \times \mathbb{R} \). However, since we treat \( dF \) as unknown, this operation would not be plausible. Some accumulation points can be added nonetheless, as we shall see below, so long as they are constructed without using \( dF \) or, more precisely, assuming that all is known about \( dF \) is (the vector space generated by) \( M \).

Measurement sets can be ordered by inclusion, hence they can be compared. If \( M \) is a measurement set, the vector space generated by \( M \), itself a measurement set, will be denoted by \( M \).

**Definition 3.4.** Let \( M_1 \) and \( M_2 \) be two measurement sets. One says that \( M_1 \) is finer than \( M_2 \) if \( M_1 \) contains \( M_2 \), and that \( M_1 \) and \( M_2 \) are equivalent if \( M_1 \) is finer than \( M_2 \) and \( M_2 \) is finer than \( M_1 \), i.e. if \( M_1 = M_2 \). A representation is a measurement set finer than \((1_A, dF(A))_{A \in \mathcal{A}}, \) where \( \mathcal{A} \) is any set of subsets of \( \mathbb{R}_+\) generating the Borel \( \sigma \)-algebra \( \mathcal{B}(\mathbb{R}_+) \).

Apart from the natural inverse problem of recovering the measure \( dF \) (or equivalently the function \( F \)) from a sufficiently rich collection of option prices, possibly providing an algorithm to do so, it is interesting also to describe relations between measurement sets. For instance, if one needs \( F \) only to price a certain set of options, instead of reconstructing \( F \) it could suffice to identify a measurement set that already allows to accomplish the task. In the simplest case, if \( g \) is the payoff profile of the option to price and \( g \) belongs to the vector space generated by an available measurement set \( M \), there is clearly no need to recover \( F \). In spite of its simplicity, this is precisely how one can proceed to price options with continuous piecewise linear payoff profile. In fact, as is well known, these options can be priced in terms of linear combinations (independent of \( F \)) of prices of put options with strikes at the “juncture” points of the piecewise linear profile. A more sophisticated fact is that call option prices for every positive strike price allow to price option with arbitrary convex payoff. In this case, however, if \( g \) is an arbitrary convex function and \( \pi(M) \), the projection on \( L^1(dF) \) of the measurement set \( M \), is the vector space generated by \((x \mapsto (x - k)^+)_{k \in \mathbb{R}_+} \), it is not true in general that \( g \in \pi(M) \). It is true, however, that \( g \) is an accumulation point of \( \pi(M) \), as discussed in \( \S 5 \) below.

It was mentioned above that it would not be meaningful to extend a measurement set \( M \) taking its closure in \( L^1(dF) \times \mathbb{R} \), as \( F \) is considered unknown. However, one can indeed add some
cluster points, if they are defined by procedures that do not involve \( F \). In particular, at least two possibilities exist:

(a) let \( (g_n) \subseteq \text{pr}_1 M \) be a sequence that converges pointwise to \( g \) and for which there exists \( h \in L^1(dF) \) such that \( |g_n(x)| \leq h(x) \) for all \( x \in \mathbb{R}^+ \). The dominated convergence theorem then implies that \( g \in L^1(dF) \) and that \( \pi(g) = \lim_{n \to \infty} \pi(g_n) \);

(b) let \( (g_n) \subseteq \text{pr}_1 M \) be such that \( g_n \uparrow g \), i.e. \( (g_n) \) is an increasing sequence that convergence pointwise to \( g \), and such that \( (\pi(g_n)) \) is bounded from above, i.e. \( \sup_n \pi(g_n) < \infty \). Then \( 0 \leq g_n - g_0 \uparrow g - g_0 \) and, by the monotone convergence theorem,

\[
\pi(g - g_0) = \lim_n \pi(g_n - g_0) = \sup_n \pi(g_n) - \pi(g_0) < \infty,
\]

hence \( g - g_0 \in L^1(dF) \), i.e. \( g \in L^1(dF) \) with \( \pi(g) = \sup_n \pi(g_n) \).

The cluster points constructed in (a) and (b) do not depend on knowing \( F \), hence they could reasonably be added to the measurement set \( M \). The measurement sets obtained by adding to \( M \) the cluster points described in (a) and (b) will be denoted by \( M^d \) and \( M^{m} \), respectively. We shall see that if \( M_1 \) is measurement set of all call options, and \( M_2 \) the measurement set of all convex options, then \( M_1^d \) is finer than \( M_2 \). Since \( M_2 \) is finer than \( M_1^m \) (the pointwise supremum of a family of convex functions is convex), \( M_1^m \) and \( M_2 \) are equivalent measurement sets. In other words, one cannot replicate a convex payoff with just call payoffs, but one can approximate a convex payoff by a combination of call payoffs with any pricing accuracy.

Taking suitable limits of sequences of measurements is not the only possible way to enrich a measurement set. In fact, one can also perform several operations on \( (\pi_J)_{J \subseteq \mathbb{R}} \), using the structure of \( \mathbb{R} \): they can for instance be added, multiplied, and functions \( \phi: \mathbb{R}^n \to X \) can be applied to \( n \) of them, with \( X \) suitable sets, and so on. Note that \( M \) could also be seen as a linear map from the space of finite measures \( \mathcal{M}(\mathbb{R}^+) \) to \( \mathcal{R}^J \), mapping \( dF \) to \( (dF(g_j))_{J \subseteq \mathbb{R}} \). Viewing elements of \( \mathcal{R}^J \) as functions from \( J \) to \( \mathbb{R} \), the problem at hand may imply that these functions in the codomain have additional properties, for instance they may be monotone, or convex, or continuous, or differentiable, and outputs on the \( (g_j) \). Depending on the range of \( M \) in the codomain \( \mathcal{R}^J \), different operations may be applied. For instance, taking derivatives on \( \mathcal{R}^J \) or on \( C(J) \) would not make sense, but it would make sense on \( C^1(J) \), or in the a.e. sense if we knew that the range is made of Lipschitz continuous functions. We shall see that this point of view is also fruitful, showing that the right derivative of put prices, seen as a function \( P \) of the strike price, is equal to \( F \). We shall also see that the price of an option with arbitrary convex payoff can be written in terms of an integral of \( C \), where \( C(k) \) is the price of the call option with strike \( k \).

In some cases one does not observe a measurement directly, but a function of a measurement. This is the case, for instance, of implied volatility. If \( g_k: x \mapsto (k - x)^+ \) is the payoff function of a put option with strike \( k \), there is a one-to-one correspondence between \( \pi_k := \pi(g_k) \) and the (Black-Scholes) implied volatility, given by a function \( \nu: \mathbb{R}^+ \to \mathbb{R}^+ \) such that \( \pi_k = \text{BS}(S_0, k, T, \nu(\pi_k)) \). Here \( \text{BS}(S_0, k, T, \sigma) \) denotes the Black-Scholes price (at time zero) of a put option on an underlying with price at time zero equal to \( S_0 \), strike \( k \), time to maturity \( T \), volatility \( \sigma \), and interest rate as well as dividend rate equal to zero (or to any other constants). In particular, if the implied volatility is known for every strike \( k > 0 \), inverting the function \( \nu \) we obtain the measurement set of put prices \( M = (g_k)_{k \geq 0} \), which is a representation. In other words, implied volatility for all strikes uniquely determines the pricing functional or, equivalently, the measure \( dF \). We may then say, with a slight abuse of terminology, that implied volatility is a representation.

Let \( X \) be a locally compact space and \( \phi: \mathbb{R} \to X \) be a measurable isomorphism, i.e. a bijection such that both \( \phi \) and \( \phi^{-1} \) are measurable. This is the case, for instance, if \( \phi \) is a homeomorphism. Then, for any \( g \in L^1(dF) \), one has

\[
dF(g) = \langle g, dF \rangle = \langle \phi^*(\phi^{-1})^* g, F \rangle = \langle (\phi^{-1})^*g, \phi_* F \rangle.
\]
This change of parametrization can also be interpreted in terms of measurement sets, saying that the measurement set \( M = (g_j, dF(g_j)) \) of \( \phi_*dF \). Even though the two measurements are isomorphic (as sets), they may have quite different character. Let us consider, for instance, the reparametrization from price to logarithmic return: setting \( S_T = S_0 \exp(\sigma X_T + m) \), where \( \sigma > 0 \) and \( m \) are constants, the pricing functional can be written as

\[
\pi: g \mapsto \int g(S_0e^{x+m}) dF_X(x),
\]

where \( F_X \) is the (right-continuous) distribution function of the measure \( (X_T)_\mu \), the support of which is \( \mathbb{R} \). If \( g \) is the payoff function of a put option with strike \( k \), then \( x \mapsto g(S_0e^{x+m}) = (k - S_0e^{x+m})^+ \) does not have compact support. This is clearly in stark contrast to the expression of \( \pi(g) \) in terms of \( dF \), where the intersection of the supports of \( g \) and \( dF \) is compact. As will be seen, several analytic arguments strongly depend on this property, that hence cannot be used with the new parametrization, even though the values of the corresponding integrals are the same.

Finally, we remark that it is sometimes useful to extend the definition of measurement set adding \( (dF_j) \), a collection of (possibly signed) measures for which a relation to \( dF \) is known. For instance, let \( g_k \), for any \( k \geq 0 \), be the payoff function of a put option with strike price \( k \), that is, \( g_k: x \mapsto (k - x)^+ \). Moreover, let \( (dF_n)_{n \in \mathbb{N}} \) be a sequence of Radon measures converging weakly to \( dF \) as \( n \to \infty \). Each measure \( dF_n \) can be thought of as an approximation to the law \( dF \), and \( dF_n(g_k) \) as the price of a put option with strike \( k \) under the approximating law \( dF_n \). If all such prices can be observed, then we have an “extended” measurement set \( M' = (g_j, \pi_j, dF_j)_{j \in J} \), where \( J = \mathbb{R}_+ \times \mathbb{N}, F_j = F_{kn}, F_{kn} = F_n \) for every \( k, j = g_{kn} = g_k \) for every \( n \), and \( \pi_j = \pi_{kn} = F_n(g_k) \). Note that \( g_k \in C_0(\mathbb{R}) \) for every \( k \), hence \( F_n(g_k) \to F(g_k) \) as \( n \to \infty \). In particular, if we define the (standard) measurement set \( M = (g_j, \pi_j)_{j \in \mathbb{R}_+} \) as \( g_j: x \mapsto (j-x)^+ \) and \( \pi_j = dF(g_j) \), then we could say that \( M \) “implies” \( M' \). That is, for every \( g \in \Pr_1 M' \) there exists a sequence \( (dF_n) \subset \Pr_1 M \) such that \( \pi(g) = dF(g) \) is the limit of \( dF_n(g) \subset \Pr_1 M \). An analogous example motivated by (empirical) non-parametric option pricing is discussed in §7 below.

4 Put and call option prices and the pricing functional

Let us define the numerical functions \( P, C: \mathbb{R}_+ \to [0, +\infty] \) by

\[
P(k) := \int \mathbb{R}_+ (k - x)^+ dF(x), \quad C(k) := \int \mathbb{R}_+ (x - k)^+ dF(x).
\]

Note that \( P(k) \) is finite for all \( k \) as \( \mathbb{R}_+ \ni x \mapsto (k-x)^+ \) is bounded (has even compact support), but \( C(k) \) is finite if and only

\[
\int k x dF(x) < \infty,
\]

hence \( C \) is everywhere finite if and only \( dF \) has a finite mean

\[
\overline{dF} := \int \mathbb{R}_+ x dF(x).
\]

The assumption \( \overline{dF} < \infty \) also implies that \( dF \) is a finite measure, hence \( F(\infty) := \lim_{x \to \infty} F(x) \) is finite. In fact, rather obviously, \( F(\infty) = F(\infty) - F(1) + F(1) \) and

\[
F(\infty) - F(1) = \int_{|1, \infty]} dF \leq \int_{|1, \infty]} x dF(x) \leq \overline{dF} < \infty.
\]
The financial interpretation of $dF < \infty$ is that $E_Q \beta_T^{-1} S_T$ must be finite. This is clearly not a limitation. However, we shall mention the hypothesis anyway because some of the considerations to follow may be interesting for general $F$, irrespective of the underlying financial interpretation.

The functions $P$ and $C$ will play a central role, so we discuss some of their properties. They are all rather basic, but they are given here in full detail because it is simpler to prove them than to look for a suitable reference.

**Proposition 4.1.** The function $P$ is increasing, locally Lipschitz continuous, Lipschitz continuous if $dF$ is finite, convex, and satisfies the inequality $P(k) \leq kF(k)$ for every $k \geq 0$. Moreover, $P(0) = 0$ and

$$\lim_{k \to \infty} \frac{P(k)}{k} = F(\infty),$$

hence, in particular, $\lim_{k \to \infty} P(k) = \infty$.

**Proof.** Since $k \mapsto (k-x)^+$ is increasing for every $x \in \mathbb{R}_+$ and integration (with respect to a positive measure) is positivity preserving, $P$ is increasing. Similarly, as $k \mapsto k^+$ is 1-Lipschitz continuous, $k \mapsto (k-x)^+$ is 1-Lipschitz continuous uniformly with respect to $x$, hence $P$ is locally Lipschitz continuous as well and globally Lipschitz continuous if $F(\infty) < \infty$. To prove convexity, note that, for any $x \in \mathbb{R}_+$, $k \mapsto k - x$ is affine, in particular convex, and $y \mapsto y^+$ is convex increasing, hence the composite function $k \mapsto (k - x)^+$ is convex. Finally, integration with respect to a positive measure preserves convexity, hence $P$ is convex. The identity $P(0) = 0$ follows immediately by the definition of $P$, as does the estimate $P(k) \leq kF(k)$, where $F(k) \leq F(\infty)$. Finally,

$$\frac{P(k)}{k} = \frac{1}{k} \int_{[0,k]} (k-x) dF(x) = \int_{\mathbb{R}_+} \left(1 - \frac{x}{k}\right) 1_{[0,k]} dF(x),$$

where $(1 - x/k) 1_{[0,k]} \to 1$ for all $x \geq 0$ as $k \to \infty$ and $(1 - x/k) 1_{[0,k]} \in [0,1]$ for all $x,k \geq 0$, hence the dominated convergence theorem implies

$$\lim_{k \to \infty} \frac{P(k)}{k} = \int_{\mathbb{R}_+} dF = F(\infty).$$

**Proposition 4.2.** Assume that $dF < \infty$. The function $C$ is decreasing, Lipschitz continuous, and convex. Moreover, $C(0) = dF$ and $\lim_{k \to \infty} C(k) = 0$.

**Proof.** The proof of monotonicity, Lipschitz continuity, and convexity are entirely similar to the corresponding proof for put options, noting that $k \mapsto (x-k)^+$ is decreasing. The definition of $C$ immediately implies that $C(0) = \int_{\mathbb{R}_+} x dF(x)$, and also that

$$C(k) = \int_k^\infty (x-k) dF(x) \leq \int_{[k,\infty]} x dF(x),$$

where the right-hand side converges to zero as $k \to \infty$ because $\int_{\mathbb{R}_+} x dF(x)$ is finite by assumption.

By a direct computation one can obtain estimates for local and global Lipschitz constants. In fact, the 1-Lipschitz continuity of $x \mapsto x^+$, hence also of $k \mapsto (k-x)^+$, yields, for any $k_1, k_2 \geq 0$,

$$|P(k_2) - P(k_1)| \leq \int_{[0,k_1 \vee k_2]} |(k_2 - x)^+ - (k_1 - x)^+| dF(x)$$

$$= \int_{[0,k_1 \vee k_2]} |(k_2 - x)^+ - (k_1 - x)^+| dF(x)$$

$$\leq \int_{[0,k_1 \vee k_2]} |k_2 - k_1| dF(x) = |k_2 - k_1| F_-(k_1 \vee k_2),$$

where $F_-$ denotes the tail distribution function.
where \( F_\cdot \) stands for the left-continuous version of \( F \), defined by \( F_\cdot(x) := \lim_{h \downarrow 0} F(x - h) \). The same estimate holds for \( P \) replaced by \( C \). One can actually show, using subdifferentials, that the Lipschitz continuity estimates thus obtained are sharp. In fact, for any \( k_1, k_2 \geq 0 \), convexity implies

\[
P(k_2) \geq P(k_1) + \partial P(k_1)(k_2 - k_1)
\]

where \( \partial \) stands for the subdifferential in the sense of convex analysis. Hence, if \( k_1 \geq k_2, P(k_1) \geq P(k_2) \) because \( P \) is increasing, which also implies that \( \partial P(x) \subset \mathbb{R}_+ \) for every \( x > 0 \), hence

\[
|P(k_1) - P(k_2)| = P(k_1) - P(k_2) \leq \partial P(k_1)(k_1 - k_2) = \partial P(k_1)|k_1 - k_2|.
\]

Similarly, if \( k_1 \leq k_2 \),

\[
|P(k_1) - P(k_2)| = P(k_2) - P(k_1) \leq \partial P(k_2)(k_2 - k_1) = \partial P(k_2)|k_1 - k_2|.
\]

Recalling that \( \partial P(k) = [D^- P(k), D^+ P(k)] \) for every \( k > 0 \), it easily follows that

\[
|P(k_1) - P(k_2)| \leq D^- P(k_1 \vee k_2)|k_1 - k_2|.
\]

As \( D^+ P = F \) and the left-continuous version of \( D^+ P \) is \( D^- P \), it follows that \( D^- P \equiv F_\cdot \).

We are going to show that \( F \) is the right derivative of \( P \), and that a similar relation holds between the call price function \( C \) and \( F \). We give two proofs, one that relies on the integration-by-parts formula for cadlag functions, and one a bit indirect based on a denseness result: we show that the set of put payoffs are total in \( L^1(dF) \), i.e. that for any \( g \in L^1(dF) \) there exist a sequence of finite linear combinations of pay payoffs that converges to \( g \) in \( L^1(dF) \). This connects with another formulation of representation that we have discussed, i.e. by a kind of closure operation. Then we show that the two approaches are in fact equivalent. A third approach, using distributions, will be given in (3) below.

### 4.1 Reconstruction of \( F \) via integration by parts

We shall apply the integration-by-parts formula to establish formulas relating the distribution function \( F \) and the price functions for put and call options \( P \) and \( C \).

**Theorem 4.3.** One has \( P^\prime = F \) a.e. in \( \mathbb{R}_+ \) and \( D^+ P(x) = F(x) \) for every \( x \in \mathbb{R}_+ \). Moreover, if the measure \( dF \) has finite mean, then \( C^\prime = F - F(\infty) \) a.e. in \( \mathbb{R}_+ \) and \( D^+ C(x) = F(x) - F(\infty) \) for every \( x \in \mathbb{R}_+ \).

**Proof.** Let \( k \geq 0 \) and \( G: x \mapsto k - x \). The integration-by-parts formula

\[
G(k)F(k) - G(0)F(0) = \int_{[0,k]} G(x) dF(x) + \int_{[0,k]} F(x) dG(x),
\]

yields

\[
\int_0^k F(x) \ dx = kF(0) + \int_{[0,k]} (k - x) dF(x)
= \int_{[0,k]} (k - x) dF(x)
= \int_{\mathbb{R}_+} (k - x)^+ dF(x) = P(k).
\]

\(^4\)Since \( \partial P(k_1) \) is in general a set, one should write \( P(k_2) \geq P(k_1) + g(k_2 - k_1) \) for every \( g \in \partial P(k_1) \). This slight abuse of notation shall not create any harm though.
The Lebesgue differentiation theorem then implies that $P' = F$ a.e. in $\mathbb{R}_+$. Moreover, since $F$ is right-continuous by definition, and $P$ is convex, hence right-differentiable, we also have $D^+P(x) = F(x)$ for every $x \in \mathbb{R}_+$.

Obtaining a relation between $C$ and $F$ along the same lines is a bit more involved: if $k > 0$ and $G: x \mapsto x - k$, one has, for any $a > k$,
\[
G(a)F(a) - G(k)F(k) = \int_{[k,a]} G(x) \, dF(x) + \int_{[k,a]} F(x) \, dG(x),
\]
i.e.
\[
(a-k)F(a) = \int_{[k,a]} (x-k) \, dF(x) + \int_k^a F(x) \, dx,
\]
which is equivalent to
\[
\int_{[k,a]} (x-k) \, dF(x) = \int_k^a (F(a) - F(x)) \, dx.
\]
Therefore, by the monotone convergence theorem,
\[
\lim_{a \to \infty} \int_{[k,a]} (x-k) \, dF(x) = \lim_{a \to \infty} \int_{\mathbb{R}_+} 1_{[k,a]}(x-k) \, dF(x) = \int_{[k,\infty]} (x-k) \, dF(x) = \int_{\mathbb{R}_+} (x-k)^+ \, dF(x) = C(k),
\]
as well as
\[
\lim_{a \to \infty} \int_k^a (F(a) - F(x)) \, dx = \lim_{a \to \infty} \int_{\mathbb{R}_+} 1_{[k,a]}(F(a) - F(x)) \, dx = \int_{k}^{\infty} (F(\infty) - F(x)) \, dx,
\]
hence
\[
C(k) = \int_k^{\infty} (F(\infty) - F(x)) \, dx. \tag{4.1}
\]
This implies $C' = F - F(\infty)$ a.e. as well as, by right continuity of $F$ and convexity of $C$, $D^+C(x) = F(x) - F(\infty)$ for every $x \in \mathbb{R}_+$. \hfill $\square$

The finiteness of the integral on the right-hand side of (4.1) is implied by the finiteness of $C(k)$, which in turn follows by the assumption that $dF$ has finite mean. One may also easily see directly that the last assumption implies that the integral is finite. In fact, this produces another proof of the identity (4.1): by Tonelli’s theorem,
\[
\int_k^{\infty} (F(\infty) - F(x)) \, dx = \int_k^{\infty} \int_{[x,\infty]} \, dF(y) \, dx = \int_{[k,\infty]} \int_k^{\infty} \, dx \, dF(y) = \int_{[k,\infty]} (k-y) \, dF(y) = \int_{\mathbb{R}_+} (k-y)^+ \, dF(y) = C(k),
\]
The relation between $C$ and $F$ can of course be obtained also from put-call parity, once the relation between $P$ and $F$ has been obtained: if follows from the identity $x - k = (x-k)^+ - (k-x)^+$, upon integrating with respect to $dF$, that
\[
\int_{\mathbb{R}_+} x \, dF(x) - k \int_{\mathbb{R}_+} dF = C(k) - P(k),
\]
hence, by Lebesgue’s differentiation theorem, $-F(\infty) = C'(k) - P'(k) = C'(k) - F(k)$ for a.a. $k \in \mathbb{R}_+$, as well as $D^+C(k) = F(k) - F(\infty)$ for every $k \in \mathbb{R}_+$ by the same argument used above.
4.2 Reconstruction of $F$ by approximation in $L^1(dF)$

Let $V$ be the vector space generated by put payoff profiles, i.e. by the family of functions $\mathbb{R}_+ \ni x \mapsto (k-x)^+$, $k \geq 0$. We are going to show the following approximation result.

**Lemma 4.4.** Let $a > 0$. For any $\varepsilon > 0$ there exists $\phi \in V$ such that

$$\|\phi - 1_{[0,a]}\|_{L^1(dF)} < \varepsilon.$$  

**Proof.** Since $F$ is right-continuous, there exists $b > a$ such that $F(b) - F(a) < \varepsilon$. Set $\phi_a(x) := (a-x)^+$, $\phi_b(x) := (b-x)^+$, $\alpha = 1/(b-a)$, and $\phi = \alpha \phi_b - \alpha \phi_a$. Then easy computations show that $\phi: \mathbb{R}_+ \to [0,1]$ is a continuous function with support $[0,b]$, equal to one on $[0,a]$. More precisely,

$$\phi(x) = \begin{cases} 
\alpha(b-a) = 1, & 0 \leq x \leq a, \\
\alpha b - \alpha x, & a \leq x \leq b, \\
0, & x \geq b.
\end{cases}$$

Since $\phi = 1_{[0,a]} + \phi_{1_{[a,b]}}$, we have

$$|\phi - 1_{[0,a]}| = \phi_{1_{[a,b]}} \leq 1_{[a,b]},$$

hence

$$\|\phi - 1_{[0,a]}\|_{L^1(dF)} \leq \int_{[0,b]} 1_{[a,b]} dF = F(b) - F(a) < \varepsilon. \quad \Box$$

This shows that we can explicitly approximate $F$ by $P$. The (proof of the) lemma also shows that $D^+ P = F$: for any $a > 0$, take a sequence $(b_n)$ converging to $a$ from the right, and call $\phi_n$ the corresponding approximating sequence converging to $1_{[0,a]}$ in $L^1(dF)$, for which

$$\int \phi_n dF = \int \frac{1}{b_n-a} ((b_n-x)^+ - (a-x)^+) dF(x) = \frac{P(b_n) - P(a)}{b_n-a},$$

hence

$$F(a) = \lim_{n \to \infty} \int \phi_n dF = \lim_{n \to \infty} \frac{P(b_n) - P(a)}{b_n-a} = D^+ P(a).$$

This approach to proving that $D^+ P = F$ (that, by the way, does not require any further condition on $F$) is probably the most elementary. Note that the approach via integration by parts of the previous subsection also implies

$$F(a) = D^+ P(a) = \lim_{n \to \infty} \frac{P(b_n) - P(a)}{b_n-a} = \lim_{n \to \infty} \int \phi_n dF,$$

while here we prove the seemingly more precise limiting relation $\phi_n \to 1_{[0,a]}$ in $L^1(dF)$. This, however, can be deduced from $F$. Riesz’s lemma\footnote{This result is often called Scheffé’s lemma: in a general measure space with measure $\mu$, if $f_n \to f$ $\mu$-a.e. and $\int |f_n| d\mu \to \int |f| d\mu$, then $f_n \to f$ in $L^1(\mu)$.} since both $1_{[0,a]}$ and $\phi_n$ are positive, $\phi_n \to 1_{[0,a]}$ a.e. and $\int \phi_n dF \to \int 1_{[0,a]} dF$, it follows that $\phi_n \to 1_{[0,a]}$ in $L^1(dF)$. Therefore also the integration-by-parts proof of $D^+ P = F$, together with F. Riesz’s lemma, implies that indicator functions of intervals can be obtained as limits in the $L^1(dF)$ norm of linear combinations of put payoff profiles, that are explicitly determined.

Even though the previous lemma is enough to obtain $F$ from $P$, a more general denseness result holds.

**Proposition 4.5.** The vector space $V$ generated by put payoff profiles is dense in $L^1(dF)$. 


Proof. Let \( g \in L^1(dF) \) and \( \varepsilon > 0 \). Then there exists \( n \in \mathbb{N} \) and \( A_i := [a_i, b_i], 0 \leq a_i \leq b_i \), and \( c_i \in \mathbb{R}, i = 1, \ldots, n \), such that

\[
\|g - \sum_{i=1}^n c_i 1_{A_i}\|_{L^1(dF)} \leq \varepsilon/2.
\]

By the previous lemma, the indicator function of any interval of \( \mathbb{R}_+ \) open to the left and closed to the right can be approximated by an element of \( V \). Therefore, for every \( i = 1, \ldots, n \) there exists \( \phi_i \in V \) such that (all norms until the end of the proof are meant to be in \( L^1(dF) \))

\[
\|\phi_i - 1_{A_i}\| \leq \frac{1}{n|c_i|} \frac{\varepsilon}{2},
\]

hence, setting \( \phi := \sum c_i \phi_i \),

\[
\|g - \phi\| \leq \|g - \sum_{i=1}^n c_i 1_{A_i}\| + \sum_{i=1}^n |c_i| \|1_{A_i} - \phi_i\|
\leq \varepsilon/2 + \sum_{i=1}^n |c_i| \frac{1}{n|c_i|} \frac{\varepsilon}{2} = \varepsilon.
\]

Since \( \phi \) clearly belongs to \( V \), the proof is completed. \( \square \)

5 Convex payoffs

We are going to show that prices of call options for all strikes determine the price of any option with arbitrary convex payoff function (the result is not new — see, e.g., [7] pp. 51-52, with a different proof), thus also for options with payoff function that can be written as the difference of two convex functions.

Using the language of \( \S 3 \), let \( M_1 = (g, dF(g))_{g \in G} \) be the measurement set with \( G \) the set of convex functions on \( \mathbb{R}_+ \) (satisfying the assumption below), and \( M_2 = (g_k, dF(g_k))_{k \in \mathbb{R}_+}, g_k : x \mapsto (x - k)^+, \) the measurement set of call options (for all strikes). It is evident that \( M_1 \) is finer than \( M_2 \). We shall show that \( M_2^n \) is finer than \( M_1 \), hence that \( M_1 \) and \( M_2^n \) are equivalent (in particular, \( M_1 \) is a representation). The proof will actually establish that, for any \( g \in G, \pi(g) \) can be written in terms of an integral of the function \( C : k \mapsto \pi(g_k) \). This will then be shown to belong to \( M_2^n \).

Throughout this section we assume that \( g : \mathbb{R}_+ \to \mathbb{R} \) is the restriction to \( \mathbb{R}_+ \) of a convex function \( h \) on an open set \( I \supset \mathbb{R}_+ \). In particular, \( D^+g(0) > -\infty \). In order to avoid trivialities, we also assume that \( g \in L^1(dF) \). We recall that \( g \) is continuous, differentiable almost everywhere, right-differentiable on \( [0, \infty] \), and that \( D^+g \) is increasing and càdlàg. In particular, \( D^+g \) has finite variation, thus generates a Lebesgue-Stieltjes measure that we shall denote by \( m \), or also by \( dg' \). The positive measure \( m \) can also be identified with the second derivative of \( g \) in the sense of distributions.

Proposition 5.1. Assume that \( \overline{dF} < \infty \) and let \( C : \mathbb{R}_+ \to \mathbb{R}_+ \) be the call option price function. Then

\[
\int_{\mathbb{R}_+} g \, dF = g(0) F(\infty) + D^+ g(0) \overline{dF} + \int_{[0, \infty]} C \, dm.
\]

(5.1)

Proof. We have

\[
g(x) = g(0) + \int_0^x D^+ g(y) \, dy,
\]
where \( D^+ g(y) - D^+ g(0) = m([0,y]) \) for every \( y > 0 \), hence, by Tonelli’s theorem,

\[
g(x) = g(0) + D^+ g(0)x + \int_0^x \int_{[0,y]} dm(k) \, dy
\]

\[
= g(0) + D^+ g(0)x + \int_{[0,\infty]} \int_{[k,x]} dy \, dm(k)
\]

\[
= g(0) + D^+ g(0)x + \int_{[0,\infty]} (x-k)^+ \, dm(k).
\]

Integrating both sides with respect to \( dF \) and appealing again to Tonelli’s theorem completes the proof.

Note that

\[
\int_{[0,\infty]} C \, dm = C(0)m(\{0\}) + \int_{[0,\infty]} C \, dm = D^+ g(0)dF + \int_{[0,\infty]} C \, dm,
\]

i.e. (5.1) could be written in the more symmetric form

\[
\int g \, dF = g(0)F(\infty) + \int_{\mathbb{R}_+} C \, dm.
\]

Analogously, since

\[
\int g \, dF = g(0)F(0) + \int_{[0,\infty]} g \, dF,
\]

(5.1) could also be written as

\[
\int_{[0,\infty]} g \, dF = g(0)(F(\infty) - F(0)) + D^+ g(0)dF + \int_{[0,\infty]} C \, dm.
\]

**Corollary 5.2.** Let \( I \subseteq \mathbb{R} \) be an open set containing \( \mathbb{R}_+ \) and \( h_1, h_2 : I \to \mathbb{R}_+ \) convex functions belonging to \( L^1(dF) \). If \( g = h_1 - h_2 \) and \( \nu \) is the Lebesgue-Stieltjes (signed) measure induced by \( D^+ h_1 - D^+ h_2 \), i.e. \( \nu([0,x]) := D^+ h_1(x) - D^+ h_2(x) \), then

\[
\int_{\mathbb{R}_+} g \, dF = g(0)F(\infty) + \int_{[0,\infty]} C \, dm.
\]

Slightly more generally, one can also write

\[
\beta_T^{-1} g(S_T) = g(0)\beta_T^{-1} + D^+ g(0)\beta_T^{-1} S_T + \int_{[0,\infty]} \beta_T^{-1}(S_T - k)^+ \, d\nu(k),
\]

hence, taking conditional expectation with respect to \( \mathcal{F}_t \), for any \( t \in [0,T] \), and multiplying by \( \beta_t \),

\[
\beta_t E_Q[\beta_T^{-1} g(S_T)|\mathcal{F}_t] = g(0)\beta_t E_Q[\beta_T^{-1}|\mathcal{F}_t]
\]

\[
+ D^+ g(0)\beta_t E_Q[\beta_T^{-1} S_T|\mathcal{F}_t]
\]

\[
+ \int_{[0,\infty]} \beta_t E_Q[\beta_T^{-1}(S_T - k)^+|\mathcal{F}_t] \, d\nu(k)
\]

\[
= g(0)B(t,T) + D^+ g(0)S_t + \int_{[0,\infty]} C_t(k) \, d\nu(k),
\]

where \( C_t(k) := \beta_t E_Q[\beta_T^{-1}(S_T - k)^+|\mathcal{F}_t] \) is the price at time \( t \) of the call option with strike \( k \).

It is actually possible to prove Proposition 5.1 using only the integration by parts formula (2.2). Even though the proof is longer than the previous one, some of its ingredients may be interesting in their own right. We begin with a useful reduction step.
Lemma 5.3. Assume that $\bar{d}F < \infty$. The claim of Proposition 5.1 holds if and only if it does under the additional assumptions that $g(0) = D^+ g(0) = 0$ and $m$ has compact support.

Proof. Clearly only sufficiency needs a proof. The extra assumption $g(0) = D^+ g(0) = 0$ comes at no loss of generality as one can reduce to this situation simply replacing the function $g$ by the function $x \mapsto g(x) - g(0) - D^+ g(0) x$, which is still convex, being the sum of a convex function and an affine function, as well as in $L^1(dF)$, because $\bar{d}F$ is finite by assumption. Let us then assume that $g(0) = D^+ g(0) = 0$. Let $(\chi_n)$ be a sequence of smooth cutoff functions such that $\chi_n : \mathbb{R}_+ \to [0,1]$ has support equal to $[0,n+1]$ and is equal to one on $[0,n]$. Setting, for every $n \in \mathbb{N}$, $m_n := \chi_n m$ and

$$g^{(1)}_n(x) := m_n([0,x]) = \int_{[0,x]} \chi_n \, dm, \quad g_n(x) := \int_0^x g^{(1)}_n(y) \, dy,$$

it is immediately seen that $g^{(1)}_n$ is positive, $g'_n = g^{(1)}_n$ a.e. and $D^+ g_n = g^{(1)}_n$, and $g_n$ is convex. Therefore, by hypothesis,

$$\int_{\mathbb{R}_+} g_n \, dF = \int_{[0,\infty]} C \, dm_n = \int_{[0,\infty]} C \chi_n \, dm.$$

Several applications of the monotone convergence theorem imply that $(g_n)$ converges pointwise from below to $g$, hence, finally, that

$$\int_{\mathbb{R}_+} g \, dF = \int_{[0,\infty]} C \, dm. \quad \square$$

Note that the “normalizing” assumptions $g(0) = 0$ and $D^+ g(0) = 0$ imply that

$$\int_{\mathbb{R}_+} g \, dF = \int_{[0,\infty]} g \, dF$$

and that

$$\int_{[0,\infty]} C \, dm = \int_{\mathbb{R}_+} C \, dm,$$

respectively. The former is evident, and the latter follows from $m(\{0\}) = D^+ g(0) = 0$. Therefore

$$\int_0^\infty g \, dF = \int_{[0,\infty]} g \, dF = \int_{[0,\infty]} C \, dm = \int_{\mathbb{R}_+} C \, dm.$$

An alternative proof of Proposition 5.1. We shall assume, as the previous lemma allows to do, that $g(0) = D^+ g(0) = 0$ and that $m$ has compact support, which implies that, for all $x$ sufficiently large, $g$ is differentiable at $x$ and $g'(x)$ is constant. For the rest of the proof, we shall write, with a harmless abuse of notation, $g'$ to denote $D^+ g$. Since $g$ is continuous and $F$ is càdlàg, the integration by parts formula (2.2) yields, for any $a \in \mathbb{R}_+$,

$$g(a) F'(a) - g(0) F(0) = \int_{[0,a]} g(x) \, dF(x) + \int_{[0,a]} F(x) \, dg(x).$$

Therefore, as $g(0) = 0$ and the Lebesgue-Stieltjes measure $dg$ is absolutely continuous with respect to Lebesgue measure with density $g'$,

$$\int_{[0,a]} g(x) \, dF(x) = g(a) F(a) - \int_0^a g'(x) F(x) \, dx,$$
hence
\[ \int_0^\infty g(x) \, dF(x) = \lim_{a \to +\infty} \left( g(a) F(a) - \int_0^a g'(x) F(x) \, dx \right). \]

Since \( g' \) is increasing and càdlàg, and \( C \) is continuous, another application of the integration by parts formula yields, for any \( a \in \mathbb{R}_+ \),
\[ g'(a) C(a) - g'(0) C(0) = \int_0^a g'(x) \, dC(x) + \int_{[0,a]} C(x) \, dg'(x), \]

hence, recalling that \( g'(0) = 0 \),
\[ \int_0^a g'(x) \, dC(x) = g'(a) C(a) - \int_{[0,a]} C \, dm. \]

Moreover, the identity \( C' = F - F(\infty) \) a.e. implies
\[ \int_0^a g'(x) \, dC(x) = -F(\infty) g(a) + \int_0^a g'(x) F(x) \, dx, \]

hence
\[ -\int_0^a g'(x) F(x) \, dx = -F(\infty) g(a) - \int_0^a g'(x) \, dC(x) = -F(\infty) g(a) - g'(a) C(a) + \int_{[0,a]} C \, dm, \]

thus also
\[ \int_{\mathbb{R}_+} g(x) \, dF(x) = \lim_{a \to +\infty} \left( g(a) (F(a) - F(\infty)) - g'(a) C(a) + \int_{[0,a]} C \, dm \right). \]

Note that \( g' \) is increasing by convexity of \( g \) and \( g'(0) = 0 \), hence \( g' \) is positive, therefore \( g \) is increasing and positive because \( g(0) = 0 \). Therefore
\[ |g(a) (F(a) - F(\infty))| = g(a) (F(\infty) - F(a)) = \int_{[a,+,\infty]} g(a) \, dF \leq \int_{[a,+,\infty]} g(x) \, dF(x), \]

where the last term converges to zero as \( a \to +\infty \) because \( g \in L^1(dF) \) by assumption. In particular,
\[ \lim_{a \to +\infty} g(a) (F(a) - F(\infty)) = 0. \]

Moreover, as \( g' \) is constant at infinity and \( C \) tends to zero at infinity, we also have
\[ \lim_{a \to +\infty} g'(a) C(a) = 0, \]

which allows to conclude that
\[ \int_{\mathbb{R}_+} g \, dF = \int_{[0,\infty]} C \, dm. \]

Let us show that \( \int_{[0,\infty]} C \, dm \in M^n_m \). By Tonelli’s theorem,
\[ \int_{[0,\infty]} C \, dm = \int_{[0,\infty]} \int_{\mathbb{R}_+} (x-k)^+ \, dF(x) \, dm(k) \]
\[ = \int_{\mathbb{R}_+} \int_{[0,\infty]} (x-k)^+ \, dm(k) \, dF(x). \]
Let \((k_i)_{i=0,\ldots,2^n}\) be a dyadic partition of \([0,n]\). Then
\[
\sum_{i=1}^{2^n} (x - k_{i+1})^{+} 1_{[k_i,k_{i+1}]}(k) \uparrow (x - k)^{+} \quad \forall x, k \in \mathbb{R}_+
\]
as \(n \to \infty\), hence, again by Tonelli’s theorem,
\[
\int_{[0,\infty]} \sum_{i=1}^{2^n} (x - k_{i+1})^{+} 1_{[k_i,k_{i+1}]}(k) \ dm(k)
\]
\[
= \sum_{i=1}^{2^n} m([k_i,k_{i+1}]) (x - k_{i+1})^{+} \uparrow \int_{[0,\infty]} (x - k)^{+} \ dm(k) \quad \forall x \in \mathbb{R}_+.
\]
Then
\[
g_n := \sum_{i=1}^{2^n} m([k_i,k_{i+1}]) (x - k_{i+1})^{+}
\]
defines a sequence of elements in the vector space generated by \(M_2\) that monotonically converges pointwise to the function \(x \mapsto \int_{[0,\infty]} (x - k)^{+} \ dm(k)\), which belongs to \(L^1(dF)\) by assumption, therefore also to \(M_2^n\).

**Remark 5.4.** It is more convenient to work with the call price function \(C\), rather than with the put price function \(P\), because \(C\) vanishes at infinity, while \(P\) grows linearly at infinity (see Propositions 4.1 and 4.2). However, the identity
\[
x - k = (x - k)^{+} - (x - k)^{-} = (x - k)^{+} - (k - x)^{+}
\]
yields, upon integrating both sides with respect to \(dF\),
\[
\int_{\mathbb{R}_+} x \ dF(x) - k \int_{\mathbb{R}_+} dF(x) = \overline{dF} - kF(\infty) = C(k) - P(k),
\]
i.e.
\[
C(k) = P(k) - kF(\infty) + \overline{dF}, \tag{5.2}
\]

hence \(k \mapsto P(k) - kF(\infty) + \overline{dF} \in L^1(m)\), even though, in general, \(P\) need not belong to \(L^1(dF)\). A formula relating the integral of \(g\) with respect to \(dF\) with the integral of \(P\) with respect to \(m\) for a special class of functions \(g\) will be discussed in the next section.

**Remark 5.5.** A small variation of the argument used in the proof of Lemma 5.3 shows that every \(C^2\) function \(g\) is the difference of two convex functions \(h_1 \) and \(h_2\) (taking the positive and negative part of \(g^{''}\)). A simple sufficient condition ensuring that the functions \(h_1 \) and \(h_2\) can be chosen in \(L^1(dF)\) is that there exists a function \(h \in L^1(dF)\) with \(h^{''} = |g^{''}|\).

### 6 A distributional approach

We are going to show that most properties of the functions \(F\), \(P\) and \(C\) discussed in the previous sections can also be obtained using Schwartz’s distributions. The main advantage of this approach is that several results reduce, in the formal aspect, to simple calculus for distributions. Some work is needed, however, to remove the regularity assumptions on test functions typical of this approach.

Throughout this section, the functions \(F\), \(P\), and \(C\) (the last one if \(\overline{dF}\) is finite) are assumed to be extended to \(\mathbb{R}\) setting them equal to zero on \([-\infty,0]\). All of them are obviously locally in \(L^1(\mathbb{R})\), hence they can be considered as distributions in \(\mathcal{D}'(\mathbb{R})\). For instance,
\[
\langle F, \phi \rangle := \int_{\mathbb{R}} F(x) \phi(x) \ dx, \quad \phi \in \mathcal{D}(\mathbb{R})
\]

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(as is customary, we shall use the same symbols to define both a functions and the corresponding distribution). Moreover, the measure \( dF \) can be identified with the distributional derivative \( F' \) of \( F \). In fact, for any \( \phi \in \mathscr{D}(\mathbb{R}) \),

\[
(F', \phi) = -\langle F, \phi' \rangle = -\int_{\mathbb{R}} F(x)\phi'(x) \, dx,
\]

where, thanks to the integration-by-parts formula,

\[
-\int_{\mathbb{R}} F(x)\phi'(x) \, dx = \int_{[0,\infty]} \phi(x) \, dF(x).
\]

The price function for put options \( P \) can be written in terms of convolutions of distributions. In fact, denoting the function \( x \mapsto x^+ \) by \((\cdot)^+\), the function \( P \) is the convolution of \((\cdot)^+\) with the measure \( dF \), therefore, since \( dF = F' \) in \( \mathscr{D}' \) and both \( F' \) and \((\cdot)^+\), interpreted as elements of \( \mathscr{D}' \), are supported on \( \mathbb{R}_+ \), the convolution of \((\cdot)^+\) and \( F' \) is well-defined in the sense of distributions, and \( P = (\cdot)^+ * F' \) in \( \mathscr{D}' \). Let \( H = 1_{\mathbb{R}_+} \) denote the (right-continuous) Heaviside function. Standard calculus in \( \mathscr{D}' \) then yields

\[
P = (\cdot)^+ * F' = ((\cdot)^+)' * F = H * F,
\]

thus also, denoting the Dirac measure at the origin by \( \delta \),

\[
P' = H' * F = \delta * F = F,
\]

and \( P'' = F' \), all as identities in \( \mathscr{D}' \). As already observed, \( F' \) coincides with the Lebesgue-Stieltjes measure \( dF \), hence it is a positive distribution. As is well known, a distribution with positive second derivative is a convex function, hence we recover the convexity of \( P \). This and the identity \( P' = F \) in \( \mathscr{D}' \) also imply that \( P'' = F \) holds also in the a.e. sense in \( \mathbb{R} \), and that one can choose a right-continuous version of \( P' \), so that \( D^+ P = F \). The properties of \( P \) have thus been obtained starting from the properties of its second distributional derivative, that is reversing the path followed in the previous section, where convexity of \( P \) was proved first, which implied first-order differentiability outside a countable set of points first, hence second-order differentiability in the sense of measures.

On the other hand, it seems not possible to treat the call option price function \( C \) by similar arguments, because one would formally have \( C = (\cdot)^- * F' \), where, however, the convolution is not well-defined in the sense of distributions. In fact, \((\cdot)^- \) and \( F' \) do not have their support “on the same side” of \( \mathbb{R} \), and none of them has compact support. Nonetheless, properties of \( C \) can be deduced from those of \( P \) taking \((5.2)\) into account. Since we are considering \( C \) and \( P \) as distributions on \( \mathbb{R} \), it is convenient to rewrite \((5.2)\) as

\[
C = P - F(\infty)(\cdot)^+ + dFH,
\]

which can be interpreted both as an identity of càdlàg functions on \( \mathbb{R} \), as well as an identity in \( \mathscr{D}'(\mathbb{R}) \). Differentiating in \( \mathscr{D}'(\mathbb{R}) \) yields

\[
C' = P' - F(\infty)H + dF' \delta,
\]

\[
C'' = P'' - F(\infty)\delta + dF'' \delta'.
\]

Since \( \delta' \) is not a measure, the function \( C \) is not convex on \( \mathbb{R} \) (this also clearly follows from \( C(0) = dFF \) and \( C(k) = 0 \) for all \( k < 0 \)). On the other hand, one also infers that

\[
C' = P' - F(\infty), \quad C'' = P'' \quad \text{in } \mathscr{D}'(\mathbb{R}_+),
\]

hence \( C \) is convex on \( [0,\infty[ \), then also on \( \mathbb{R}_+ \), and one can choose a right-continuous version of \( C' \) on \( \mathbb{R}_+ \) such that \( C'(x) = F(x) - F(\infty) \) for a.a. \( x \in \mathbb{R}_+ \), with \( D^+ C(x) = F(x) - F(\infty) \) for every \( x \in \mathbb{R}_+ \).
We are now going to show how to prove (5.1) using distribution arguments. Note that, assuming that \( g(0) = Dg^+(0) = 0 \), (5.1) could heuristically be written as \( (C'', g) = \langle C, g'' \rangle \), which seems very natural indeed. It is clear, however, that it makes no sense if \( g \) is just a convex function in \( L^2(dF) \). However, note that the identity has a meaning if \( \langle \cdot, \cdot \rangle \) is interpreted as the duality between measures and continuous functions, rather than between distributions and test functions.

Let us start from the identity \( P'' = F' = dF \) in \( \mathcal{D}'(\mathbb{R}) \) that was proved above. Then we immediately obtain \( \langle P'', g \rangle = \langle P, g'' \rangle \) for every \( g \in \mathcal{D}(\mathbb{R}) \), hence also, since \( P'' \) is a distribution of order at most two,

\[
\langle P'', g \rangle = \int_{\mathbb{R}} g \, dF = \langle P, g'' \rangle \quad \forall g \in C^2_0(\mathbb{R}).
\]

Therefore, using identity (6.1),

\[
\langle P'', g \rangle = \int_{\mathbb{R}} g \, dF = (C'', g) + F(\infty)(\delta, g) - \overline{\Delta} F(\delta', g) = (C'', g) + F(\infty)g(0) + \overline{\Delta} g'(0).
\]

We have thus obtained (5.1) under the assumption \( g \in C^2(\mathbb{R}) \), or, equivalently, \( g \in C^2(\mathbb{R}_+) \) such that \( g(x) = 0 \) for \( x \) sufficiently large.

Let us now assume that \( g \in C^2(\mathbb{R}) \). As discussed above, we can and shall assume, without loss of generality, that \( g(0) = g'(0) = 0 \). Let \( (\chi_n) \) be a sequence of smooth cutoff function taking values in \([0, 1]\), equal to one on \([-a, a]\), and equal to zero on \([a + 1/n, \infty] \). Then \( g\chi_n \in C^2(\mathbb{R}) \) and

\[
(g\chi_n)'' = g''\chi_n + 2g'\chi'_n + g\chi''_n,
\]

hence

\[
\int_{\mathbb{R}_+} g\chi_n \, dF = \langle C'', g\chi_n \rangle = \langle C, (g\chi_n)'' \rangle = \langle C, g''\chi_n \rangle + 2\langle C, g'\chi'_n \rangle + \langle C, g\chi''_n \rangle.
\]

We are going to pass to the limit as \( n \to \infty \). One has

\[
\int_{\mathbb{R}_+} g\chi_n \, dF = \int_{[0,a]} g \, dF + \int g\chi_n I_{[a,a+1/n]} \, dF,
\]

where \( g\chi_n I_{[a,a+1/n]} \to 0 \) pointwise, hence, by the dominated convergence theorem,

\[
\lim_{n \to \infty} \int_{\mathbb{R}_+} g\chi_n \, dF = \int_{[0,a]} g \, dF.
\]

An entirely similar, slightly simpler reasoning shows that

\[
\lim_{n \to \infty} \langle C, g''\chi_n \rangle = \lim_{n \to \infty} \int_{\mathbb{R}_+} Cg'' \chi_n = \int_0^a Cg''.
\]

Moreover,

\[
\langle C, g'\chi'_n \rangle = \int R C(x)g'(x)\chi'_n(x) \, dx = \int_{a+1/n}^{a+1/n} C(x)g'(x)\chi'_n(x) \, dx,
\]

where \( -\chi'_n \) converges to \( \delta_a + R \) in the sense of distributions, where \( \delta_a \) is the Dirac measure at \( a \) and \( R \) a distribution with support contained in \([a, -a] \), hence

\[
\lim_{n \to \infty} \langle C, g'\chi'_n \rangle = -C(a)g'(a).
\]
The term \((C, g\chi''_n)\) is more difficult to treat because \(\chi''_n\) converges to \(-\delta'\) in the sense of distributions (modulo terms with support in the strictly negative reals, that we are going to ignore), but \(C\) is just right-differentiable, not of class \(C^1\). We can nonetheless argue as follows: let \((\rho_m)\) be a sequence of mollifiers with support contained in \([-1/m, 0]\) and set \(C_m := C \ast \rho_m\). Then \(C_m \in C^\infty(\mathbb{R})\) and

\[
(C_m, g\chi''_n) = (C_m, g) = -((C_m g)'', \chi''_n),
\]

hence

\[
lm{\eta}{\infty} (C_m, g\chi''_n) = C'_m(a)g(a) + C_m(a)g'(a).
\]

Thus one has

\[
\int_0^a C'_m g = \int_0^a C_m g'' + C'_m(a)g(a) - C_m(a)g'(a).
\]

We can now pass to the limit as \(m \to \infty\): the continuity of \(C\) implies that \(C_m\) converges to \(C\) uniformly on \([0, a]\), hence

\[
lm{\eta}{\infty} \int_0^a C_m g'' = \int_0^a C g'', \quad \text{lim}_{m \to \infty} C_m(a) = C(a).
\]

Setting \(dF_m := C'_m = dF \ast \rho_m\) and \(\tilde{\rho}_m : x \mapsto \rho_m(-x)\), so that the support of \(\tilde{\rho}_m\) is contained in \([0, 1/m]\), one has

\[
\int_0^a gC_m'' = \int g1_{[0, a]} dF_m = \int g1_{[0, a]} * \tilde{\rho}_m dF,
\]

where

\[
\lim_{n \to \infty} g1_{[0, a]} * \tilde{\rho}_n(x) = g(x) \quad \forall x \in [0, a],
\]

\[
\lim_{n \to \infty} g1_{[0, a]} * \tilde{\rho}_n(0) = 0,
\]

\[
\lim_{n \to \infty} g1_{[0, a]} * \tilde{\rho}_n(a) = g(a),
\]

i.e.

\[
\lim_{n \to \infty} g1_{[0, a]} * \tilde{\rho}_n(x) = g1_{[0, a]}(x) \quad \forall x \in \mathbb{R},
\]

or, in other words, \(g1_{[0, a]} * \tilde{\rho}_n\) converges to the càglàd version of \(g1_{[0, a]}\). Therefore, by the dominated convergence theorem,

\[
\lm{\eta}{\infty} \int_0^a gC_m'' = \lim_{m \to \infty} \int g1_{[0, a]} * \tilde{\rho}_m dF = \int g1_{[0, a]} dF = \int g dF,
\]

where the last equality follows from \(g(0) = 0\).

Since \(C_m \in C^\infty(\mathbb{R})\) and \(C\) is right-differentiable with increasing incremental quotients (because it is convex), the dominated convergence theorem yields

\[
C'_m(a) = D^+ C_m(a) = \lim_{h \to 0^+} \frac{C_m(a + h) - C_m(a)}{h} = \lim_{h \to 0^+} \int_{\mathbb{R}} \frac{C(a - y + h) - C(a - y)}{h} \rho_m(y) dy
\]

\[
= \int_{\mathbb{R}} D^+ C(a - y) \rho_m(y) dy,
\]

hence also, recalling that the support of \(\rho_m\) is contained in \(\mathbb{R}_-\) and that \(D^+ C\) is right-continuous,

\[
\lim_{m \to \infty} C'_m(a) - D^+ C(a) = \lim_{m \to \infty} \int_{\mathbb{R}} (D^+ C(a - y) - D^+ C(a)) \rho_m(y) dy = 0.
\]
We have thus shown that
\[
\int_{[0,a]} g\,dF = \int_0^a Cg'' - C(a)g'(a) + D^+C(a)g(a)
\]
for every \( g \in C^2(\mathbb{R}) \). To remove the assumption that \( g \in C^2 \), assuming instead that it is convex, we can apply the same regularization by convolution: let \( g \) be convex and set \( g_n := g \ast \rho_n \), with the sequence of mollifiers \((\rho_n)\) chosen as before. Then \( g_n \in C^\infty \) and
\[
\int_{[0,a]} g_n\,dF = \int_0^a Cg''_n - C(a)g'_n(a) + D^+C(a)g_n(a),
\]
where \( g_n \to g \) uniformly on \([0,a]\) and \( \lim_{n \to \infty} g'_n(a) = D^+g(a) \). Moreover, using the same argument as before,
\[
\lim_{n \to \infty} \int_0^a Cg''_n = \lim_{n \to \infty} \int C1_{[0,a]} \ast \tilde{\rho}_n \, dm = \int_{[0,a]} C \, dm.
\]
We conclude that
\[
\int_{[0,a]} g\,dF = \int_{[0,a]} g\,dF = \int_{[0,a]} C\,dm - C(a)D^+g(a) + D^+C(a)g(a). \tag{6.2}
\]
Note that until here we have not used the assumption that \( g \in L^1(dF) \). To complete the proof of (5.1), we let \( a \) tend to infinity using two lemmas proved next, according to which the last two terms on the right-hand side of (6.2) tend to zero. It is precisely at this point that we use the assumption that \( g \in L^1(dF) \).

**Lemma 6.1.** Assume that \( dF \) is a finite measure and let \( g \in L^1(dF) \) be increasing. Then
\[
\lim_{a \to \infty} g(a)(F(\infty) - F(a)) = 0.
\]
In particular, if \( \overline{df} < \infty \) then \( \lim_{a \to \infty} D^+C(a)g(a) = 0 \).

**Proof.** Assume first that \( g(0) = 0 \), so that \( g \) is positive. Then, as \( g \) is increasing,
\[
g(a)(F(\infty) - F(a)) = \int_{[a,\infty]} g(a)\,dF(x) \leq \int_{[a,\infty]} g(x)\,dF(x),
\]
and
\[
\lim_{a \to \infty} \int_{[a,\infty]} g(x)\,dF(x) = 0
\]
because \( g \in L^1(dF) \). If \( g(0) < 0 \), then consider the function \( \tilde{g} := |g(0)| + g \), which is increasing and belongs to \( L^1(dF) \). The identity
\[
g(a)(F(\infty) - F(a)) = \tilde{g}(a)(F(\infty) - F(a)) - |g(0)|(F(\infty) - F(a)) = 0
\]
immediately implies the claim. \( \square \)

**Lemma 6.2.** Assume that \( \overline{df} < \infty \). Let \( g \in L^1(dF) \) be absolutely continuous and such that \( g' \) is increasing (possibly after a suitable modification on a set of Lebesgue measure zero). Then
\[
\lim_{a \to \infty} g'(a) \int_a^\infty (F(\infty) - F(y)) \, dy = 0,
\]
or, equivalently, \( \lim_{a \to \infty} C(a)g'(a) = 0 \).
Proof. The assumption \( \frac{dF}{x} < \infty \) guarantees that the function \( C \) is well-defined and
\[
C(a) = \int_a^\infty (F(\infty) - F(y)) \, dy \quad \forall a \in \mathbb{R}_+.
\]
Then we can write
\[
g'(a)C(a) = \int_a^\infty g'(a)(F(\infty) - F(y)) \, dy \\
\leq \int_a^\infty g'(y)(F(\infty) - F(y)) \, dy \\
= \int_a^\infty g'(y) \int_{[y, \infty]} dF(x) \, dy \\
= \int_{[a, \infty]} \int_a^\infty g'(y) \, dy \, dF(x) = \int_{[a, \infty]} g(x) \, dF(x) - \int_{[a, \infty]} g(a) \, dF(x),
\]
where
\[
\lim_{a \to \infty} \int_{[a, \infty]} g(x) \, dF(x) = 0
\]
because \( g \in L^1(dF) \). Moreover,
\[
\left| \int_{[a, \infty]} g(a) \, dF(x) \right| \leq \int_{[a, \infty]} |g(a)| \, dF(x).
\]
Let us first consider the case that \( g'(0) \geq 0 \), so that \( g' \) is positive and \( g \) is increasing. If there exists \( a_0 \in \mathbb{R}_+ \) such that \( g(a_0) \geq 0 \), then
\[
\lim_{a \to \infty} \int_{[a, \infty]} g(a) \, dF(x) \leq \lim_{a \to \infty} \int_{[a, \infty]} g(x) \, dF(x) = 0.
\]
Otherwise, if \( g(x) \leq 0 \) for all \( x \in \mathbb{R}_+ \), then \( |g| = -g \) is decreasing, therefore
\[
\lim_{a \to \infty} \int_{[a, \infty]} |g(a)| \, dF(x) \leq \lim_{a \to \infty} |g(1)| (F(\infty) - F(a)) = 0.
\]
Let us now consider the case that \( g'(0) < 0 \): introduce the function \( \tilde{g}(x) := g(x) + |g'(0)|x \), for which \( \tilde{g}'(0) = g'(0) + |g'(0)| > 0 \), and note that \( \tilde{g}' \) is increasing. The assumption \( \frac{dF}{x} < \infty \) implies that \( \tilde{g} \in L^1(dF) \), hence the previous part of the proof shows that \( \lim_{a \to \infty} C(a) \tilde{g}'(a) = 0 \). Writing \( C(a)g'(a) = C(a)\tilde{g}'(a) - C(a)|g'(0)| \) and recalling that \( \lim_{a \to \infty} C(a) = 0 \), the proof is completed.

Even though the set of functions that can be written as the difference of two convex functions is quite rich (see, e.g., [2]), it does not contain any discontinuous function. So, for instance, for digital options we cannot produce a pricing formula such as (5.1). However, the distributional approach allows to obtain in a quite efficient way pricing formulas for options the payoff of which can be written piecewise as the difference of convex functions. The formulas involve, apart from integrals of \( C \), also pointwise evaluations of \( C \) and \( F \). Let \( g_0 : \mathbb{R} \to \mathbb{R} \) be a convex function, \( [a, b] \subset \mathbb{R}_+ \) a compact interval, and \( g := g_0 \mid_{[a, b]} \) a cadlag restriction of \( g_0 \) that will serve as payoff function of an option, of which we are going to compute the price.

One has
\[
\int_{\mathbb{R}_+} g \, dF = \int_{[a, b]} g \, dF = \int_{[a, b]} g \, dF + g(a) \Delta F(a),
\]

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where \( \Delta F(a) = D^+ C(a) - D^+ C(a^-) \) and, by dominated convergence,
\[
\int_{[a,b]} g \, dF = \lim_{x \to b^-} \int_{[a,x]} g \, dF.
\]
Since
\[
\int_{[a,x]} g \, dF = \int_{[0,x]} g \, dF - \int_{[0,a]} g \, dF,
\]
it follows by (6.2) that
\[
\int_{[a,x]} g \, dF = \int_{[a,b]} C \, dm + C(a) D^+ g(a) - D^+ C(a) g(a) - C(x) D^+ g(x) + D^+ C(x) g(x).
\]
Taking the limit for \( x \) going to \( b \) from the left, one obtains, recalling that \( C \) is continuous and both \( g \) and \( D^+ C \) are càdlàg,
\[
\int_{[a,b]} g \, dF = \int_{[a,b]} C \, dm + C(a) D^+ g(a) - D^+ C(a) g(a) - C(b) D^+ g(b^-) + D^+ C(b^-) g(b^-).
\]
There is an alternative way to obtain the same formula, using Proposition 2.2, that is slightly longer but that starts from very basic principles and shows how the distributional approach allows to compute very quickly the price of an option under the very mild assumption that \( F \) admits a continuous density. Let us consider \( g := g_1 1_{[a,b]} \) as a distribution, and assume first that \( F \) is of class \( C^1 \), which implies that \( C \) is of class \( C^2 \). Then
\[
\int_{\mathbb{R}^+} g \, dF = \int_a^b g \, dF = \langle g, C'' \rangle = \langle g'', C \rangle,
\]
where, thanks to Proposition 2.2,
\[
\langle g'', C \rangle = \int_{[a,b]} C \, dm + C(a) D^+ g(a) - C''(a) g(a) - C(b) D^+ g(b^-) + C''(b) g(b^-).
\]
If \( C \) is not twice continuously differentiable, setting \( C_n := C * \rho_n \), with \( \rho_n \) a sequence of mollifiers chosen as before, then \( C_n \) and \( dF_n := dF * \rho_n \) are both in \( C^\infty \) and
\[
\int_a^b g \, dF_n = \langle g'', C_n \rangle = \int_{[a,b]} C_n \, dm + C_n(a) D^+ g(a) - C_n''(a) g(a) - C_n(b) D^+ g(b^-) + C_n''(b) g(b^-).
\]
We are now going to pass to the limit as \( n \to \infty \): \( C_n \) converges to \( C \) uniformly on compact sets, hence \( C_n(a) \) and \( C_n(b) \) converge to \( C(a) \) and \( C(b) \), respectively, and
\[
\lim_{n \to \infty} \int_{[a,b]} C_n \, dm = \int_{[a,b]} C \, dm.
\]
As before, the choice of \( \rho_n \) and the right continuity of \( D^+ C \) imply that \( C_n'(a) \) and \( C_n'(b) \) converge to \( D^+ C(a) \) and \( D^+ C(b) \), respectively, hence
\[
\lim_{n \to \infty} \int_a^b g \, dF_n = \int_{[a,b]} C \, dm + C(a) D^+ g(a) - D^+ C(a) g(a) - C(b) D^+ g(b^-) + D^+ C(b) g(b^-).
\]
Writing
\[ \int_a^b g \, dF_n = \int g \cdot 1_{[a,b]} \ast \tilde{\rho}_n \, dF \]
we can use again an argument already met before, which shows that
\[ \lim_{n \to \infty} g \cdot 1_{[a,b]} \ast \tilde{\rho}_n (x) = g_- (x) \quad \forall x \in \mathbb{R}, \]
where \( g_- \) denotes the càglàd version of \( g \). Therefore, by dominated convergence,
\[ \lim_{n \to \infty} \int_a^b g \, dF_n = \int_{[a,b]} g_- \, dF = \int_{[a,b]} g_- \, dF + g(b-)\Delta F(b) \]
\[ = \int_{[a,b]} g \, dF + g(b-)(D^+ C(b) - D^+ C(b-)). \]
Rearranging terms we are left with
\[ \int_{[a,b]} g \, dF = \int_{[a,b]} C \, dm + C(a)D^+ g(a) - D^+ C(a)g(a) - C(b)D^+ g(b-) + D^+ C(b-)g(b-), \]
as before.

7 A representation through approximated laws of logarithmic returns

In the standard Black-Scholes (BS) model one assumes that \( S_T = \exp(\varsigma \sqrt{T} Z - \varsigma^2 T/2) \) in law, where the volatility \( \varsigma \) is constant and \( Z \) is a standard Gaussian random variable. This family of random variables (indexed by \( \varsigma \), with time to maturity \( T \) fixed as before) can be embedded in the larger class defined by \( S_T = \exp(\sigma X + m) \), where \( \sigma \) and \( m \) are constants, and \( X \) is a random variable with density \( f \in L^2 := L^2(\mathbb{R}) \). This rather general family of laws can be used as setup for empirical non-parametric option pricing, essentially by projecting the density \( f \) on radial basis functions (see [8]). More precisely, we consider expansions of \( f \) in terms of Hermite functions, so that the lognormal distribution of returns corresponds exactly to the zeroth order expansion of \( f \). The approach can thus be thought of as a perturbation of the BS model at fixed time. The following problem then arises: let \((f_n) \subset L^1 \cap L^2 \) be a sequence of functions converging to \( f \) in \( L^2 \), and let \( p_n(k) \) be the “fictitious” price of a put option with strike \( k \), obtained replacing the density \( f \) with its approximation \( f_n \). Suppose that the \( p_n(k) \) are known for all \( k \in \mathbb{R}_+ \) and \( n \in \mathbb{N} \). Is this information enough to determine the function \( P \), i.e. the put option prices in the “true” model?

Assuming, for simplicity, that \( S_0 = 1 \), \( \sigma = 1 \) and \( m = 0 \), and denoting the distribution function of \( X = \log S_T \) with respect to the measure \( \mu \) by \( F_X \), it is immediately seen that \( F_X(x) = F(e^x) \) for every \( x \in \mathbb{R} \) and that
\[ P(k) = \int_{\mathbb{R}} (k - e^x)^+ \, dF_X(x) = \int_{\mathbb{R}} (k - e^x)^+ f(x) \, dx, \]
hence,
\[ p_n(k) = \int_{\mathbb{R}} (k - e^x)^+ f_n(x) \, dx. \]
Note that \( F_X \) and \( f \) are supported on the whole real line and that \( F \) and \( F_X \) are in bijective correspondence, hence \( F_X \) is in bijective correspondence also with \( P \).

The sequence \( p_n(k) \) does not necessarily converge to \( P(k) \) as \( n \to \infty \), because the function \( x \mapsto (k - e^x)^+ \) belongs to \( L^\infty \) but not to \( L^2 \), hence it induces a continuous linear form on \( L^1 \), but not on \( L^2 \). Moreover, convergence in \( L^2(\mathbb{R}) \) does not imply convergence in \( L^1(\mathbb{R}) \).
We are going to show that the function \( P \) can be reconstructed from approximation to option prices with payoff of the type

\[
\theta_{k_1, k_2}(x) := (k_2 - e^x)^+ - \frac{k_2}{k_1}(k_1 - e^x)^+, \quad k_1, k_2 > 0.
\]

More precisely, to identify the pricing functional \( P \), it suffices to know, for any sequence \((f_n)\) converging to \( f \) in \( L^2 \), the values \( \langle \theta_{k_1, k_2}, f_n \rangle \) for all \( k_1, k_2 > 0 \) and all \( n \geq 0 \), where we recall that \( \langle \cdot, \cdot \rangle \) stands for the scalar product of \( L^2 \).

In fact, for any \( k_1, k_2 > 0 \), the function \( \theta_{k_1, k_2} \) is in \( L^2 \), hence, for any sequence \( (f_n) \subset L^1 \cap L^2 \) converging to \( f \) in \( L^2 \) (weak convergence in \( L^2 \) would also suffice), one has

\[
P_n(k_2) - \frac{k_2}{k_1} P_n(k_1) = \langle \theta_{k_1, k_2}, f_n \rangle \longrightarrow \langle \theta_{k_1, k_2}, f \rangle = P(k_2) - \frac{k_2}{k_1} P(k_1).
\]

Moreover,

\[
\frac{k_2}{k_1} P(k_1) = \int_\mathbb{R} \frac{k_2}{k_1} (k_1 - e^x)^+ f(x) \, dx,
\]

where \((k_1 - e^x)^+ \in [0, k_1]\) for all \( x \in \mathbb{R} \), hence \( \frac{k_2}{k_1} (k_1 - e^x)^+ \in [0, k_2] \) for all \( x \in \mathbb{R} \), and

\[
\frac{k_2}{k_1} (k_1 - e^x)^+ = \begin{cases} k_2 - \frac{k_2}{k_1} e^x, & \text{if } x \leq \log k_1, \\ 0, & \text{if } x \geq \log k_1,
\end{cases}
\]

hence

\[
\lim_{k_1 \to 0} \frac{k_2}{k_1} (k_1 - e^x)^+ = 0 \quad \forall x \in \mathbb{R}.
\]

Therefore the function \( x \mapsto \frac{k_2}{k_1} (k_1 - e^x)^+ \) converges to zero as \( k_1 \to 0 \) in \( L^p \) for every \( p \in [1, \infty[ \) by the dominated convergence theorem. In particular, since \( f \in L^2 \),

\[
\lim_{k_1 \to 0} \frac{k_2}{k_1} P(k_1) = \lim_{k_1 \to 0} \int_\mathbb{R} \frac{k_2}{k_1} (k_1 - e^x)^+ f(x) \, dx = 0. \tag{7.1}
\]

We have thus shown that

\[
\lim_{k_1 \to 0} \lim_{n \to \infty} \langle \theta_{k_1, k_2}, f_n \rangle = P(k_2) \quad \forall k_2 > 0,
\]

thus also the following statement.

**Proposition 7.1.** Let \((f_n) \subset L^1 \cap L^2 \) be a sequence converging to \( f \) in \( L^2 \). There is a bijection between

\[
\left( \langle \theta_{k_1, k_2}, f_n \rangle \right)_{k_1, k_2 > 0, n \geq 0}
\]

and \( P \).

Completely analogously, if \( P(k_1) \) is known, then

\[
P(k_2) = \frac{k_2}{k_1} P(k_1) + \lim_{n \to \infty} \langle \theta_{k_1, k_2}, f_n \rangle = \frac{k_2}{k_1} P(k_1) + \lim_{n \to \infty} \left( P_n(k_2) - \frac{k_2}{k_1} P_n(k_1) \right) .
\]

**Remark 7.2.** The function \( x \mapsto \frac{k_2}{k_1} (k_1 - e^x)^+ \) does not converge to zero in \( L^\infty \) as \( k_1 \to 0 \), as

\[
\sup_{x \in \mathbb{R}} \frac{k_2}{k_1} (k_1 - e^x)^+ = k_2.
\]

However, the convergence in (7.1) also holds with \( f \in L^1 \), i.e. without any extra integrability assumption on \( f \), because \( \frac{k_2}{k_1} (k_1 - e^x)^+ f(x) \leq k_2 f(x) \) for every \( x \in \mathbb{R} \), hence the result follows by dominated convergence.
Note that Proposition 7.1 can be interpreted as a representation of \( dF \), but, as discussed at the end of §3, it cannot be formulated in the language introduced there. Even the extended measurements of the type \((g_j, \pi_j, dF_j)\), with \( dF_j \) a family of measures weakly converging to \( dF \), is not enough. In fact, it is not difficult to check that the push-forward of \( f_n \, dx \) through \( x \mapsto e^x \), denoted by \( dF_n \), does not converge weakly to \( dF \), in general. However, setting 
\[ M_1 = (g_k, dF_n(g_k), dF_n)_{k>0, n \in \mathbb{N}}, \]
where \( g_k \colon x \mapsto (k - x)^+ \), we have shown that \( M_1 \) “implies” 
\[ M_2 = (\theta_{k_1, k_2}, \pi_{k_1, k_2}, k_1, k_2 > 0), \]
where implication is meant as in the last paragraph of §3 and that \( M_2^m \) is finer than \( M \), the measurement set composed of put prices, which is a representation.

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