Large-amplitude ferrofluid surface waves and jets

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Abstract. A non-mechanical technique for investigating fluid surface waves and jets is presented. The method uses a ferrofluid driven by electromagnets, eliminating complicated mechanical wave generating systems. The waves are imaged by a fast digital camera, and associated diagnostics. Experimental results, with accompanying modelling, are presented for cylindrical standing waves, with the maximum amplitude wave being investigated. Exceeding the maximum amplitude results in a dramatic jet, with a maximum acceleration exceeding 70g.

1. Introduction

In this paper we present a novel method of generating and analysing purely hydrodynamic surface waves in a magnetic liquid. A vessel containing the liquid is surrounded by an electromagnetic coil, and a time-dependent current is applied. The resulting magnetic field causes the magnetic liquid, in this case a ferrofluid (FF), to respond via the consequent magnetization force, explained in detail in the next section. This technique affords considerable flexibility in both the driving geometry and the time profile of the excitation force. It is also suitable for perturbing small liquid volumes.

Conventional methods for exciting surface waves on a fluid use mechanical systems to drive the disturbance, either by a moving flap (Rayleigh [1]), or by parametric excitation, in which the table on which the fluid vessel sits is driven to oscillate vertically (Zeff et al [2], Sawada et al [3], Müller [4]), or by some variant of these methods. Whilst undoubtedly effective for large volumes of fluid, these techniques can be limited, either when only small amounts of liquid are to be excited, or when the forcing geometry is complex.

In this paper, we report on a fresh approach to this problem, in which conventional hydrodynamical surface waves in a small volume of ferrofluid are excited by exploiting the body force caused by a magnetic field gradient (see section 2). This method allows for localized
excitation of the fluid surface with a high degree of control of the stimulation region. As the fluid body force is a direct function of the current flowing through the driving electromagnet this gives total flexibility of the time variation, allowing arbitrary driving waveforms, over a wide range of frequencies. Since the fluid is directly driven this method avoids any mechanical stimulation of the vessel or supporting structures, and as such, is less prone to stray harmonics than other methods.

Electromagnets have been used in the past to excite surface waves in FF (e.g. Müller [4], Bacri et al [5]). In such experiments the objective was to study the interaction of the surface waves with the onset of the Rosensweig or peaks instability. In [4] a near-critical permanent magnetic field was present, normal to the free surface in equilibrium. The fluid was then stimulated either by mechanical means, or by the application of ac currents to a surrounding coil, in order to investigate the evolution of the peaks instability, and the consequent surface wave behaviour as a function of the applied field.

In the experiments reported here, there is no Rosensweig instability because the applied magnetic field is far below the requisite threshold for such behaviour, and is being used only as a driver for the study of the evolution of unsteady hydrodynamical surface waves.

The concept of a maximum amplitude standing wave has long been appreciated experimentally (Taylor [6]), and theoretical analysis of the shape of the one-dimensional maximum wave (Penney and Price [7], Longuet-Higgins [8]) agrees with the experiment in that they each recover a maximum angle at the crest of a standing wave to be 90°, with the theoretical assumption that the maximum vertical acceleration associated with this critical wave is equal to \(-g\), the acceleration due to gravity. This problem was revisited by Okamura [9], who extended the Penney and Price calculation and derived a maximum wave steepness, defined to be \(\pi a/\lambda\), where \(a\) is the amplitude of the wave of wavelength \(\lambda\), of 0.61, but essentially confirmed the 90° critical angle.

Exceeding the maximum steepness of a standing wave eventually results in fluid jetting or splashing. An comprehensive review of the topic is given by Peregrine [10] where mathematical models are developed. Numerical simulations of splashing (Anderson et al [11]) agree closely, and very recent experimental studies (Zeff et al [2]) offer further insight into the role of singularities and bubbles in the evolution of jets from over-forced standing surface waves.

Our experimental studies of maximum amplitude waves and fluid jetting are based on driving FF magnetically, without any mechanical intervention. The next section discusses the nature of FF, and its response to non-uniform magnetic fields. The apparatus and experimental method are detailed in section 3, with a detailed presentation of the experimental results in wave dynamics and jetting threshold in section 4. A concluding discussion completes the paper.

## 2. Ferrofluid

A FF is a stable, colloidal suspension of sub-micrometre-sized single domain magnetic particles in a liquid carrier, usually a light hydrocarbon solvent, an ester or simply water (see, for example, Rosensweig [12]). Each particle behaves like a spherical magnet and, in the absence of the applied magnetic field, the fluid has zero net magnetization. However, in the presence of an external magnetic field, the dipoles experience a torque which aligns them with the field, resulting in a magnetization of the fluid. If the applied field is inhomogenous, then the FF responds by moving to the region where the field gradient is greatest, analogous to the behaviour of an electric dipole in the presence of a non-uniform electric field (Wade et al [13]).
For a magnetized FF, the force per unit volume is given by

$$f = \mu_0 M \cdot \nabla H_0$$

where $M$ is the magnetization of the fluid, and $H_0$ is the applied magnetic field. For applied fields that vary with time much more slowly than the relaxation time for the magnetization ($\sim 10^{-7}$ s), the magnetization will be in the same direction as the applied field, and closely follows the Langevin law for paramagnets. In this case, assuming that the fluid is non-conducting ($\nabla \times H = 0$) the force density equation may be reduced to

$$f = \mu_0 M \nabla H_0.$$  \hspace{1cm} (2)

In this way, the FF is forced to move to the region of strongest magnetic field, offering a straightforward means of manipulating the fluid in order to excite surface waves.

The FF used in the series of experiments described here was Ferrosound APGJ12 [14], with viscosity 40 cp at 24 °C and low-field permeability of 1.8. The depth of FF used was 25 mm. The cylindrical Pyrex vessel had an interior diameter of 86 mm, and the driving coil had 42 turns, with a maximum current of 12 A.

### 3. The experimental set-up

The fluid is driven by a coil wrapped around a cylindrical vessel, as shown in figure 1. Since the coil is very short, it is dominated by end effects, and therefore produces a magnetic field which is small at the centre of the vessel compared with the perimeter; figure 2 shows calculated contours of the magnetic field strength in the absence of the FF. Hence there is a strong magnetic field gradient concentrated at the edges of the vessel, which by equation (2) pulls the FF away from the centre and up the vessel walls. The force at the vessel centre is very small, a factor of 100 less than that near the walls. Switching the current off releases the fluid, and by cycling the current in time the FF can therefore be stimulated into harmonic motion.

For all experiments, the temperature of the fluid was maintained at 32 ± 2 °C in order to keep the viscosity consistently low. The fluid was then excited over a range of frequencies with a 50% duty cycle square wave. Note that although in principle any time profile of the driving...
Once the fluid was in motion, the driving system needed only to compensate for viscous losses, and so the additional energy injected into the resonant motion by the driver was small in comparison with the existing energy of the fluid motion, as can be seen from the relatively narrow peaks in figure 5 corresponding to the resonances, showing a $Q = f / \Delta f \approx 10$ for the first harmonic. The next harmonic present in the square-wave driving signal is at three times this frequency, but only a third of the amplitude. This mode is not resonant, and is also heavily damped by viscous losses. Hence the square-wave driver applied to the resonant system did not excite significant high harmonics.

Imaging the fluid surface presented considerable technical problems, given that FF is black and very opaque, somewhat like used engine oil. The diffuse reflections are therefore very weak, and the specular reflections relatively bright. To get a profile of the axially symmetric surface produced by the fluid dynamics, the whole arrangement was lit with a vertical sheet of laser light, aligned perpendicular to the camera.

Adequate surface illumination was provided by a 5 mW diode laser line generator, which gave a very bright 635 nm line, 1 mm wide over an 85° fan. As the surface profiles could be very steep sided, it was necessary to illuminate the surface from more than one direction. This was achieved using the mirror arrangement shown in figure 1.

The fluid motion was captured using a fast progressive scan CCD camera (Pulnix TM6701 [15]), and the images were streamed into computer memory by a MATRIX PCimage-SG frame grabber [16]. Frames could be gathered at a rate of 60 Hz, at a resolution of $640 \times 482$ pixels with 256 levels of grey scale, or at twice the frame rate with half the vertical resolution. The camera was angled at 35° to the horizontal to allow it to see into the bottom of the surface-wave troughs. As a result, the images obtained were vertically compressed, and distorted by perspective. Imaging a square grid in the plane of the laser light allowed the precise nature and extent of this distortion to be measured, allowing the images to be corrected.

This experimental set-up produced images consisting of a dim line from the diffuse surface reflections, and various bright specular reflections from spurious sources. A typical example of such an image, rendered in false colour, together with the calibration grid, is shown in figure 3.

**Figure 2.** Detail of wave vessel showing contours of magnetic field strength and FF position.
Figure 3. False colour image showing a typical captured image. The dim green line is the diffuse reflection from the surface, and the bright spots are unwanted specular reflections. Superimposed is the calibration grid, used for perspective correction.

The surface data was extracted automatically from the images using an algorithm written specifically for this task. The software searched the raw image for the characteristic intensity profile of the diffuse reflections, by estimating the brightness, width, background noise offset and vertical position based on the same data set for the image at the previous timestep. This iterative method proved very stable and accurate, and was able to extract very clean and precise surface wave profiles unpolluted by specular reflection. These computer-extracted profiles were further processed to extract information on the surface speed and acceleration at each time step. Examples of surface wave profiles recovered by this technique are shown in figure 4.

4. Results

The experimental observations are presented here, together with analysis of the data, and some simple modelling. Two aspects of the driven fluid were explored: resonant standing waves, and wave breaking and jetting.

4.1. Surface-wave response as a function of driving frequency

Here the FF was subjected to a driving field of moderate amplitude (driving current 7 A), for various different frequencies. The resonant response at 4.41 Hz is clearly seen in figure 4, together with a further resonance near 6.2 Hz. A further weak resonance around \( (7.9 \pm 0.5) \) Hz was also observed, and has been included for completeness. Note that, at resonance, the waves have finite amplitude, and are well below the threshold for jetting. Because of the camera angle
Figure 4. Experimentally recovered wave profiles for low amplitude oscillations at different frequencies. True vessel diameter is indicated by the extent of the horizontal axis.

Table 1. Comparison of small amplitude dispersion relation with experimental observations.

| $\nu_n$ | $\beta_n$ | $\nu_n/\beta_n^{1/2}$ | $\nu_n^*$ | $\nu_n^*/\beta_n^{1/2}$ |
|---------|-----------|------------------------|-----------|-------------------------|
| 4.41    | 3.83      | 2.25                   | 4.6       | 2.36                    |
| 6.17    | 7.01      | 2.33                   | 6.25      | 2.36                    |
| 7.94    | 10.2      | 2.47                   | 7.94      | 2.47                    |

in the imaging system as arranged for this particular experiment, the edges of the vessel at the wave height cannot be viewed, and so the profiles of the fluid in figure 4 do not show the full width of the disturbance; instead the axes have been extended to indicate the true width of the vessel. However, since the prime goal here was to show wave resonances as a function of driving frequency, this is not a significant drawback, particularly since the jetting experiments concentrate on the fundamental mode. Clearer data for this 4.4 Hz fundamental mode are shown in figure 6, where the full width of the disturbance is presented.
Figure 5. Plot of amplitude response as a function of frequency, showing the first two resonances. The driving current was 7 A. Note that a weak resonance was also observed at 7.9 Hz.

A graph of the wave amplitude in response to the driving frequency is shown in figure 5, demonstrating clearly the resonances at 4.4 Hz and 6.2 Hz. Note that the resonances are relatively wide and asymmetric, reflecting the fact that the fluid is viscous, allowing off-resonance coupling, and the waves being excited at resonance have finite amplitude. Unfortunately, the resonance at 7.94 Hz has poor signal-to-noise ratio when displayed in this graph, but it is observed experimentally, and therefore worth quoting.

For comparison, the linear solution for the wave profile for cylindrically symmetric water waves is (for example, see Stoker [17]) \( \sin(\omega t)J_0(kr) \), where \( J_0 \) is the Bessel function of order zero, \( r \) is the radial coordinate, \( \omega \) is the wave frequency, \( k \) is the wavenumber and \( \omega^2 = kg \). Applying the appropriate boundary conditions for the edge-driven surface wave we have \( k_n = \beta_n/R \), where \( \beta_n \) is the \( n \)th zero of \( J'_0 \) and \( R \) the radius of the vessel. This form of solution ensures that there is always an anti-node at \( r = R \).

Using this small-amplitude theory, the observed resonant frequencies \( \nu_n = \omega_n/(2\pi) \) ought to be proportional to \( \beta_n^{1/2} \). The data and theory are compared in table 1. The first column in table 1 details the resonant frequencies corresponding to the largest amplitude waves. Column 3 of the table shows the quantity \( \nu_n/\beta_n^{1/2} \), which should be a constant, according to the linear theory. The results show relatively poor agreement. However, as is clear from figure 4, these resonant waves have significant amplitude, leading to a clear asymmetry in the resonances in the amplitude–frequency plot of figure 5. If we allow for the downward drift of wave frequency as the amplitude becomes significantly nonlinear, then it is clear that the \( \nu_n \) in the first column of table 1 are lower than the appropriate linear frequencies. Correcting for this frequency drift by taking the small-amplitude wave frequency to be at the centre of each resonance in figure 5 yields \( \nu^*_n \), detailed in the fourth column of table 1. The agreement with the dispersion relation (fifth column) is now more satisfactory, at least for the first two modes. The weak resonance at around 7.9 Hz is a poorer fit, but lies within experimental error.
Figure 6. Experimentally recovered wave profiles for low (top) and high (bottom) amplitude oscillations for the lowest frequency mode (around 4.4 Hz). The driving currents were 5 A and 8 A respectively. Dashed curves indicate the profile when the peak is rising; full curves indicate when the peak is falling. Animated versions can be found for the low amplitude case, and for the high amplitude case.

4.2. Resonant surface-wave response as function of driving amplitude

In this set of experiments, the amplitude of the driving field was varied at the frequency of the lowest resonant mode (at around 4.4 Hz, allowing for the fact that the resonant frequency drops as the wave amplitude increases). Typical responses for low and high amplitude forcing are presented in figure 6.

In the theoretical descriptions (see Penney and Price [7]) the contention is that the maximum downward acceleration at the crest of the surface wave must not exceed that due to gravity, otherwise the fluid surface will become unstable via Rayleigh–Taylor instabilities. This is also consistent with the concept that the fluid cannot withstand tension, so that the pressure just inside the liquid must be positive or zero. This can be quantified in the equation for the vertical motion of the fluid at the crest, where it is assumed that at the instant the crest reaches its maximum
height it is stationary:
\[
\frac{\partial v}{\partial t} = -g - \frac{1}{\rho} \frac{\partial p}{\partial y}
\]  
(3)
where \( y \) is the vertical coordinate, \( v \) is the vertical component of fluid velocity, \( g \) is the acceleration due to gravity, and \( p \) is the fluid pressure. Consequently, assuming \( \frac{\partial p}{\partial y} \leq 0 \) at the crest means that
\[
-\frac{\partial v}{\partial t} - g \leq 0.
\]  
(4)

In figure 7, the measured acceleration of the crest as a function of applied amplitude, driven resonantly, is plotted. Experiment shows that jetting begins if the driving current exceeds \( I_{max} = 9.7 \) A. For low-amplitude waves, that is waves driven by currents well below \( I_{max} \), the downward acceleration of the crest is less than the upward, but for high-amplitude waves this trend is reversed as the slope of the downward acceleration curve decreases markedly, and tends to a value of just under \( 12 \) m s\(^{-2} \) at threshold. Clearly, this maximum downward acceleration exceeds the theoretical maximum by 20%, and so there must be another downward force in addition to gravity. To check that this effect was purely hydrodynamic, and not affected by the magnetic field, experiments were done where a single cycle was removed from the driving signal. These showed the same acceleration for the subsequent wave. The extra downwards force must therefore be provided by the surface tension, which provides an additional significant pressure contribution when the curvature of the surface wave becomes important.

**Figure 7.** Acceleration at the crest as a function of driver amplitude.
The pressure excess associated with a fluid region of convex curvature can be described by the Young–Laplace equation (Middleman [18])

$$p_{\text{int}} - p_{\text{ext}} = \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right)$$

(5)

where $p_{\text{int}}$ and $p_{\text{ext}}$ are the internal and external fluid pressures, $\sigma$ is the surface tension, and $R_1$, $R_2$ are the radii of curvature in two perpendicular directions. For the simple case of a hemisphere of fluid of radius $R$, the extra acceleration $\delta a$ due to the surface tension is given by the total force divided by the cross-sectional area and the density:

$$\delta a = \frac{3\sigma}{R^2 \rho}.$$ 

(6)

Applying this formula to the large-amplitude cylindrical waves in our experiment is an approximation, since the crest surface is not precisely hemispherical: in fact, for the peak of a stable wave just below the jetting limit, the radius of curvature varies from 0.48 cm to 1.0 cm. However, using these limiting values of the radius $R$, in combination with the physical fluid parameters ($\sigma = 0.043$ N m$^{-1}$, $\rho = 1.01$ kg m$^{-3}$), the possible values of $\delta a$ lie in the range 0.96 m s$^{-2}$ to 4.2 m s$^{-2}$, comfortably bracketing the observed excess acceleration over $g$ of 2.2 m s$^{-2}$. Hence we can attribute the enhanced downward acceleration to the effect of surface tension.

Figure 8. Sequence of frames showing a rapidly moving jet evolving from an overforced standing wave; ● denotes non-zero coil current. An animated version is available.
4.3. Surface jets

Choosing the lowest harmonic at around 4.4 Hz, the amplitude of the standing wave was increased to beyond the maximum amplitude, so that the current in the coil exceeded $I_{\text{max}}$. The fluid response is shown in figure 8 as a subset of 15 frames, selected from a complete sequence covering a 633 ms period. The full set of images shows the fluid evolution and coil current at 17 ms intervals over this whole time period. The early frames show the extent of the nonlinearity in the fluid motion, with the profile flattening very clear at 233 ms. At 417 ms an extraordinarily fine jet of 0.2 mm diameter is seen to erupt from an otherwise relatively flat surface, and is a precursor to a much more substantial jet evident some 50 ms later. Taking the surface speed to be approximately zero at the centre at 400 ms, the precursor jet has travelled a vertical distance of approximately 105 mm in the 17 ms prior to the next frame, which equates to an average acceleration of 726 ms$^{-2}$, with a corresponding maximum speed of 12.3 m s$^{-1}$. In common with Zeff et al [2], we take the initiation of the jet to be a singular event, and neglect gravity.

Maintaining this acceleration for the next 50 ms would yield a jet height of approximately 1.6 m, which is consistent with the experimental observations of a maximum jet height of around 2 m. The corresponding maximum jet speed over this interval would be 48.6 m s$^{-1}$, similar to those observed in the experiments of Zeff et al [2]. However, note that this acceleration is not uniformly maintained over even short intervals, and simple dynamical estimates should be treated with the utmost caution. Our data suggest that the initial jet erupts extraordinarily rapidly, but is retarded very quickly, and our speed estimates are derived from image frame analysis, rather than the direct measurement technique of Zeff et al. In compensation, however, we can image the jet over a greater proportion of its lifetime.

The rapid retardation of the jet it is clear in the next frame, at 550 ms, since the discontinuity in the jet diameter has not risen as rapidly as during the initial growth stage of the instability.
Figure 10. The left-hand image shows the actual surface profile. The right-hand image shows the digitized half profile and the theoretical fit.

The collapse of the jet and the onset of a further instability is evident from the frames corresponding to the time period 583 ms to 633 ms. At 667 ms the new upward moving jet collides with the drops in free-fall, creating a plateau of horizontally-moving FF.

Jetting occurs because the surface tension is broken by the appearance of a sharp feature, associated with the flattening of the surface wave profile in the cycle immediately before the onset of instability.

Figure 9 shows this effect very clearly: not only has the wave top flattened, but a dimple has appeared in the centre. When the main oscillation moves into its downward stroke, the edges of the dimple meet in a sharp feature that destroys the surface tension, leading to the eruption of a jet from this singular event.

Accounting for this singularity in the fluid dynamics, Zeff and co-authors [2] use a similarity method and power-law scaling in order to infer important characteristics about the singularity of the fluid equations at the point of jet creation. This surface discontinuity can be seen in our experimental results. Recall that small-amplitude solutions have a wave profile \( J_0(kr) \). Immediately after the jet forms, the surface profile is closely fitted by the functional form

\[
a J_0(k_1r) - b Y_0(k_1r) + c J_0(k_2r) + d,
\]

where \( Y_0 \) denotes the Bessel function of the second kind. An optimized superposition of this kind is presented in figure 10. The left-hand image shows a colour enhanced surface profile frame just after the jet formation, captured from an experiment in which the driving amplitude was slowly and continuously increased to critical amplitude and beyond. The profile was digitized and is presented in the right-hand image, along with the theoretical fit, for which the parameters are \( a = -0.0045, b = 0.5007, c = -0.1026 \) and \( d = 0.1952 \).

It is interesting to note that this simple analysis models the surface profile rather well, through the inclusion of the logarithmically singular \( Y_0 \) form, and also shows the underlying presence of at least the first harmonic, consistent with the generation of higher harmonics in this nonlinear behaviour. It is also satisfying that such a solution is the generalization of the continuous, small-amplitude Bessel solution quoted in section 4.2. It suggests that the breaking of the surface tension admits the singular component of the general solution, which before was excluded on physical grounds.

Note that the magnetic field is not present at the time of the initial jet, and so the field does not alter the fluid dynamics at instability. That this is so can be demonstrated by the fact that the FF is not magnetically saturated at any time. The maximum magnetic field generated...
by the coil carrying 10 A is 0.05 T in the absence of FF, equating to less than 0.09 T in the FF. This field is an order of magnitude less than that required for 90% saturation of the FF used in this experiment. The field is also far below the threshold for the peaks (or Rosensweig) instability.

Moreover, figure 11 presents experimental results showing the vertical position of the centre point (coincident with the crest), together with the coil current, as a function of time. It is clear that the magnetic field is not present as the surface wave breaks and jetting begins.

5. Conclusions

The experimental results presented here show how surface waves for even small volumes of a viscous liquid (in this case, 145 cm$^{-3}$) can be excited by non-mechanical means. The magnetic forcing allowed off-resonance stimulation of surface disturbances, and higher harmonics were readily investigated. The wave evolution as a function of forcing amplitude was also investigated, resulting in the creation of rapidly accelerating jets which were imaged by a fast, high-resolution CCD camera. Careful study of the magnetic field and its possible influence on the jet production and evolution led to the conclusion that since only a small field was required to induce the wave breaking, and that the field was not present during the jet eruption, the results presented here are truly hydrodynamic in nature, free of any magnetization effects. A simple analytical model of the jetting surface suggests that the singular solution for stable cylindrical waves, which is inadmissible on physical grounds, provides a close fit to the jetting surface profile after the breaking of the surface tension. However, exciting possibilities exist for fresh investigations of jet behaviour much closer to saturation magnetization, which will extend the significance of these results into the magnetohydrodynamical area, with particular significance to astrophysical jets. We hope to report on these experiments in the near future.

Figure 11. Graphs of coil current and crest height as function of time.
Acknowledgments

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