REGULAR CANONICAL COVERS

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Abstract. We construct three sequences of regular surfaces of general type with unbounded numerical invariants whose canonical map is 2-to-1 onto a canonically embedded surface. Only sporadic examples of surfaces with these properties were previously known.

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1. Introduction

Let $X$ be a smooth surface of general type and let $\varphi : X \to \Sigma \subseteq \mathbb{P}^{p_g(X)-1}$ be the canonical map of $X$, where $\Sigma$ is the image of $\varphi$. Suppose that $\Sigma$ is a surface and that $\varphi$ has degree $d \geq 2$. Let $\epsilon : S \to \Sigma$ be a desingularization of $\Sigma$. A classical result (cf. [Be], and also [Ba], [Ca1]) says that either $p_g(S) = 0$ or $S$ is of general type and $\epsilon : S \to \Sigma$ is the canonical map of $S$. In the latter case we have a dominant rational map $X \to S$ of degree $d$, a so-called good canonical cover of degree $d$ (see [CPT]).

A basic construction of sequences of good canonical covers with unbounded invariants is due to Beauville (see [Ca2], and [MP]). In [CPT] (see also §2) we have presented a more general construction which produces new infinite series of good canonical covers $X \to S$ of degree 2. All the examples obtained by this method have $q(X) \geq 2$ and, although some examples of canonical covers $X \to S$ with $X$ regular can be found in the literature (see [VdGZ], [Be] Proposition 3.6, [Ca1] Theorem 3.5, [Ci], [Pa2]), no infinite series of such examples has been presented so far.

In this note we fill up this gap. Our idea, roughly speaking, is to take a good canonical cover $X \to S$ and find a group $G$ acting on both $X$ and $S$ in such a way that $X/G \to S/G$ turns out to be again a good canonical cover and moreover $q(X/G) = 0$. This idea works nicely: for each infinite series of good canonical covers produced in [CPT] we are able to find a $\mathbb{Z}_3$—action which does the job (see §3). For one series of the above examples, we prove in addition that $X$ is simply connected (see §4).
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Notation: All varieties are projective varieties over the field of complex numbers. We write $\omega := e^{2\pi i / 3}$.

A map is a rational map and a morphism is a regular map. We do not distinguish between Cartier divisors and line bundles and use the additive and multiplicative notation interchangeably. If $X := X_1 \times X_2$ is a product of varieties with projections $p_i: X \to X_i$ and $L_i$ is a line bundle on $X_i$, $i = 1, 2$, then $L_1 \boxtimes L_2$ denotes the line bundle $p_1^*L_1 \otimes p_2^*L_2$.

The remaining notation is standard in algebraic geometry; we just recall here the notation for the invariants of a smooth surface $S$: $K_S$ is the canonical class, $p_g(S) = h^0(S, K_S)$ the geometric genus, $q(S) = h^1(S, \mathcal{O}_S)$ the irregularity, and $\chi(S) = 1 + p_g(S) - q(S)$ the Euler characteristic.

2. Good canonical covers and generating pairs

In this section we recall the definitions and the results that we need from [CPT] in the form most suitable for our purposes.

A canonical cover of degree $d \geq 2$ is a generically finite rational map $X \to S$ of degree $d$ between smooth surfaces of general type such that $p_g(X) = p_g(S)$. If in addition the canonical map of $S$ is birational onto its image, then we say that the canonical cover is good. By Proposition 4.1 of [Be], one has $d \leq 3$ for large values of the numerical invariants of $X$.

In [CPT], generalizing a construction of Beauville, it is shown how to construct an infinite series of good canonical covers of degree 2 starting with a so-called good generating pair. A good generating pair is a pair $(h: V \to W, L)$, where $h$ is a finite morphism of degree 2 between surfaces and $L$ is a line bundle on $W$ that satisfy a certain set of conditions (see [CPT], Definition 2.4 for the precise definition). In particular the surface $V$ is smooth, the involution $\iota$ acts freely in codimension 1 and $W$ is normal, hence $h: V \to W = V/\iota$ is the quotient map and the singularities of $W$ are $A_1$ points, which are the images of the fixed points of $\iota$. The general curve $C$ of $|L|$ is a smooth connected non hyperelliptic curve, whose genus $g$ is called the genus of the generating pair.

The main point of §2 of [CPT] is that a sequence of good canonical covers can be obtained from a good generating pair in the following
way. Denote by $\mathcal{L}(n)$ the line bundle $h^*L \otimes \mathcal{O}_{\mathbb{P}^1}(n)$ on $V \times \mathbb{P}^1$. For $n \geq 3$ we take $X \subset V \times \mathbb{P}^1$ to be a smooth element of $|\mathcal{L}(n)|$. By the properties of good generating pairs, $X$ is mapped to itself by the involution $\iota \times Id$ and the quotient surface $\Sigma \subset W \times \mathbb{P}^1$ is an element of the linear system $|L \otimes \mathcal{O}_{\mathbb{P}^1}(n)|$ with $A_1$ singularities. If we denote by $S$ the minimal resolution of $\Sigma$, then the induced rational map $X \to S$ is a good canonical cover of degree 2. The composite map $S \to W \times \mathbb{P}^1 \to \mathbb{P}^1$ is a fibration whose general fibre $F$ is a non hyperelliptic curve of genus $g$ (the map $S \to W \times \mathbb{P}^1 \to W$ identifies $F$ with a curve of $|L|$). The surface $S$ is regular, while the surface $X$ has irregularity $g - 1 \geq 2$. The invariants $\chi(X)$ and $K_X^2$ grow linearly with $n$, hence one obtains good canonical covers with arbitrarily large invariants.

We describe now briefly the three examples of generating pairs we know of. A more detailed description is contained in §3 of [CPT].

**Generating pair I:** Let $C$ be a smooth projective curve of genus 2. We set $V := \text{Pic}^1(C)$, we let $\iota$ be the involution of $V$ defined by $N \mapsto K_C \otimes N^{-1}$ and we let $h : V \to W := V/\iota$ be the quotient map. The involution $\iota$ has 16 fixed points. The map $C \hookrightarrow V$ defined by $Q \mapsto \mathcal{O}_C(Q)$ identifies $C$ with the set $\{N \in \text{Pic}^1(C)|h^0(N) > 0\}$, and the line bundle $L$ on $W$ is determined by the condition that $h^*L = \mathcal{O}_V(2C)$ (up to non canonical isomorphism, $V$ is an abelian surface with an irreducible principal polarization and $W$ is its Kummer surface).

This pair has genus $g = 3$, and the numerical invariants of the corresponding good canonical covers are the following:

\[
q(X) = 2; \quad p_g(X) = 4n - 3; \quad K_X^2 = 24n - 32
\]

\[
q(S) = 0; \quad p_g(S) = 4n - 3; \quad K_S^2 = 12n - 16.
\]

**Generating pair II:** Let $C$ be a smooth curve of genus 2 and, as in the description of the generating pair I, consider the natural embedding $C \hookrightarrow \text{Pic}^1(C)$. We set $M := \mathcal{O}_{\text{Pic}^1(C)}(C)$, we let $D$ be a smooth divisor of $|2M|$ and take $\pi : V \to \text{Pic}^1(C)$ to be the double cover of $\text{Pic}^1(C)$ branched on $D$ and such that $\pi_*\mathcal{O}_V = \mathcal{O}_{\text{Pic}^1(C)} \oplus M^{-1}$. The surface $V$ is smooth and the involution of $\text{Pic}^1(C)$ defined by $N \mapsto K_C \otimes N^{-1}$ can be lifted to an involution $\iota$ of $V$ with 20 fixed points. We take $h : V \to W := V/\iota$ to be the quotient map and we set $L := K_W$. This pair has genus $g = 3$, and the numerical invariants of the corresponding good canonical covers are the following:

\[
q(X) = 2; \quad p_g(X) = 5n - 3; \quad K_X^2 = 32n - 32
\]

\[
q(S) = 0; \quad p_g(S) = 5n - 3; \quad K_S^2 = 16n - 16.
\]
Generating pair III: Let $C$ be a smooth non hyperelliptic projective curve of genus 3. We set $V := S^2C$, we let $\iota$ be the involution of $V$ defined by $Q + R \mapsto K_C(-Q - R)$, we let $h: V \to W := V/\iota$ be the quotient map. The involution $\iota$ has 28 fixed points. We take $L := K_W$. This pair has genus $g = 4$, and the numerical invariants of the corresponding good canonical covers are the following:

$$q(X) = 3; \quad p_g(X) = 7n - 4; \quad K_X^2 = 48n - 48$$

$$q(S) = 0; \quad p_g(S) = 7n - 4; \quad K_S^2 = 24n - 24.$$ 

All the examples constructed in the next section involve taking quotients of surfaces by a $\mathbb{Z}_3$–action with isolated fixed points. Next we give the formulas that we need to compute the invariants of the minimal resolution of such a quotient. We recall that by Cartan’s lemma the representation of $\mathbb{Z}_3$ on the tangent space at an isolated fixed point $Q$ is the sum of two nontrivial characters. The image of $Q$ in the quotient surface is a canonical singularity of type $A_2$ if the two characters are different, and it is a $\frac{1}{3}(1, 1)$ singularity, i.e. a rational singularity solved by a smooth rational curve of self intersection $-3$, if the two characters are equal.

Proposition 2.1. Let $X$ be a projective surface with canonical singularities. Assume that $\mathbb{Z}_3$ acts on $X$ with finitely many fixed points, which are all smooth for $X$, and denote by $Y$ the minimal resolution of $X/\mathbb{Z}_3$. If $\alpha$ (resp. $\beta$) is the number of fixed points $Q$ of $\mathbb{Z}_3$ on $X$ such that the characters of the representation of $\mathbb{Z}_3$ on $T_QX$ are equal (resp. distinct), then:

$$K_X^2 = 3K_Y^2 + \alpha; \quad \chi(X) = 3\chi(Y) - \alpha/3 - 2\beta/3.$$ 

Proof. First of all it is easy to show that $\mathbb{Z}_3$ acts on the minimal desingularization of $X$ and that the quotient of the minimal desingularization is a partial resolution of $X/\mathbb{Z}_3$. In addition the invariants of $X$ are not affected by the resolution, since $X$ has canonical singularities. Hence we may assume that $X$ is smooth.

Let $Q \in X$ be a fixed point of $\mathbb{Z}_3$. Assume that $\mathbb{Z}_3$ acts on $T_QX$ with equal characters. If we blow up $Q$, then $\mathbb{Z}_3$ acts on the blown up surface fixing the exceptional curve pointwise. If instead $\mathbb{Z}_3$ acts on $T_QX$ with distinct characters, then the action on the blown up surface has two isolated fixed points $Q_1, Q_2$ on the exceptional curve, corresponding to the $\mathbb{Z}_3$–eigenspaces of $T_QX$. A local computation shows that $\mathbb{Z}_3$ acts with equal characters on the tangent space at $Q_1$ and $Q_2$. Summing up,
it is possible to blow up \( X \) in such a way that \( \mathbb{Z}_3 \) acts on the blow up \( \tilde{X} \) and the fixed locus of \( \mathbb{Z}_3 \) on \( \tilde{X} \) is a divisor. It follows that \( \tilde{Y} := \tilde{X}/\mathbb{Z}_3 \) is smooth and the quotient map \( \tilde{X} \to \tilde{Y} \) is a flat morphism. There is a commutative diagram:

\[
\begin{array}{ccc}
\tilde{X} & \longrightarrow & X \\
\downarrow & & \downarrow \\
\tilde{Y} & \longrightarrow & X/\mathbb{Z}_3
\end{array}
\]

where the map \( \tilde{X} \to X \) is the blow up at \( \alpha + 3\beta \) points and \( \tilde{Y} \to X/\mathbb{Z}_3 \) is the composition of the resolution \( Y \to X/\mathbb{Z}_3 \) with the map \( \tilde{Y} \to Y \) obtained by blowing up the \( \beta \) intersection points of the pairs of \(-2\) curves in the resolution of the \( A_2 \)-singularities of \( X/\mathbb{Z}_3 \). The statement now follows by using the standard formulas for cyclic covers of smooth surfaces to relate the invariants of \( \tilde{X} \) and \( \tilde{Y} \).

\[\square\]

3. Construction of the regular good canonical covers

In this section we construct infinite series of canonical covers \( Y \to T \) of degree 2 such that \( q(Y) = 0 \). We use the notation and the terminology introduced in \( \S 2 \).

Our strategy is the following. Let \( X \to \Sigma \) be a finite degree 2 map of surfaces, with \( X \) smooth and \( \Sigma \) normal, denote by \( S \) the minimal desingularization of \( \Sigma \) and assume that the induced map \( X \to S \) is a good canonical cover. Let \( G \) be a subgroup of \( \text{Aut}(X) \) that does not contain the canonical involution \( \sigma \) of \( X \). Since \( \sigma \) is in the center of \( \text{Aut}(X) \), it follows that \( G \) acts on \( \Sigma = X/\sigma \). Set \( \bar{Y} := X/G \) and \( \bar{T} = \Sigma/G \) and denote by \( \bar{\pi} : \bar{Y} \to \bar{T} \) the induced map, which is a finite morphism of degree 2. The surfaces \( \bar{Y} \) and \( \bar{T} \) have rational singularities. If \( Y \) is the minimal desingularization of \( \bar{Y} \) and \( T \) is the minimal desingularization of \( \bar{T} \), then there are natural identifications (cf. [St], Lemma 1.8 and 1.11): \( H^i(Y, \Omega^j_Y) \cong H^i(X, \Omega^j_X)^G \) and \( H^i(T, \Omega^j_T) \cong H^i(\Sigma, \Omega^j_\Sigma)^G \), for \( 0 \leq i, j \leq 2 \). It follows that \( p_g(Y) = p_g(T) \), namely the canonical map of \( Y \) factors through the induced map \( Y \to T \). If in addition the canonical map of \( T \) is birational and \( H^0(X, \Omega^1_X)^G = \{0\} \), then we have the required example.

We apply this idea to canonical covers obtained from the three generating pairs of \( \S 2 \), using \( \mathbb{Z}_3 \)-actions and giving an example for each one. One can obtain similar examples by using other groups. It is also possible to construct by the same method infinite series of good canonical covers \( Y \to T \) such that \( q(Y) = 1 \).
Example 1: Here we take $\mathbb{Z}_3$—quotients of good canonical covers obtained from the generating pair $I$ (cf. §2).

Assume that the curve $C$ of genus 2 has a $\mathbb{Z}_3$—action such that $C/\mathbb{Z}_3$ is rational (such a curve can be contructed using for instance Proposition 2.1 of [Pa1]). By the Hurwitz formula, $\mathbb{Z}_3$ has 4 fixed points on $C$. Denote by $\gamma$ the hyperelliptic involution of $C$ and by $\psi: C \to C/\gamma = \mathbb{P}^1$ the canonical map. Since $\gamma$ commutes with all the automorphisms of $C$, there is an induced action of $\mathbb{Z}_3$ on $\mathbb{P}^1$ that preserves the branch locus $B$ of $\psi$. Since $B$ consists of 6 points, it follows that it is the union of two orbits of the $\mathbb{Z}_3$—action on $\mathbb{P}^1$ and therefore the fixed points $P_0$, $P_1$ of $\mathbb{Z}_3$ on $\mathbb{P}^1$ do not belong to $B$. Let $x_0, x_1$ be homogeneous coordinates on $\mathbb{P}^1$ such that $P_0 = (1 : 0)$ and $P_1 = (0 : 1)$ and let $\tau_i = \psi^* x_i$, $i = 0, 1$. Then $\tau_0, \tau_1$ is a basis of $H^0(C, \omega_C)$ and we denote by $\xi$ the generator of $\mathbb{Z}_3$ such that $\xi^* \tau_0 = \omega \tau_0$ and $\xi^* \tau_1 = \omega^2 \tau_1$, where $\omega := e^{2\pi i/3}$. We write $Q_1 + Q_2$ for the divisor of zeros of $\tau_0$ on $C$ and $Q_3 + Q_4$ for the divisor of zeros of $\tau_1$. The points $Q_1 \ldots Q_4$ are distinct and they are the fixed points of $\mathbb{Z}_3$. Clearly, $\gamma$ maps $Q_1$ to $Q_2$ and $Q_3$ to $Q_4$.

The $\mathbb{Z}_3$—action on $C$ extends to a $\mathbb{Z}_3$—action on $\text{Pic}^1(C)$ compatible with the natural embedding $C \hookrightarrow \text{Pic}^1(C)$. Set $M := \mathcal{O}_{\text{Pic}^1(C)}(C)$. In the notation of §2, we have $M^{\otimes 2} = h^* L$. The line bundle $M$ can be identified with a subsheaf of the constant sheaf $\mathbb{C}(\text{Pic}^1(C))$, hence the action of $\mathbb{Z}_3$ on $\mathbb{C}(\text{Pic}^1(C))$ restricts to a linearization of $M$ such that $\mathbb{Z}_3$ acts trivially on the 1—dimensional vector space $H^0(\text{Pic}^1(C), M)$. We consider on $M^{\otimes 2}$ the $\mathbb{Z}_3$—linearization induced by that of $M$.

The above description of the $\mathbb{Z}_3$—action on $H^0(C, \omega_C)$ shows that $\mathbb{Z}_3$ acts trivially on $\omega_{\text{Pic}^1(C)} = \mathcal{O}_{\text{Pic}^1(C)}$. Consider the residue sequence:

$$0 \to M \to M^{\otimes 2} \to \omega_C^{\otimes 2} \to 0.$$ 

It is easy to check that the maps in the sequence are equivariant with respect to the chosen actions on $M$ and $M^{\otimes 2}$ and with respect to the natural action on $\omega_C^{\otimes 2}$. The space $H^0(C, \omega_C^{\otimes 2})$ is spanned by $\tau_0^2, \tau_1^2, \tau_0 \tau_1$ and one has: $\xi \tau_0 \tau_1 = \tau_0 \tau_1$, $\xi^* \tau_0^2 = \omega^2 \tau_0^2$ and $\xi^* \tau_1^2 = \omega \tau_1^2$. Since the restriction map on global sections $H^0(\text{Pic}^1(C), M^{\otimes 2}) \to H^0(C, \omega_C^{\otimes 2})$ is surjective by Kodaira vanishing, one can find eigenvectors $f_1, f_2, f_3 \in H^0(\text{Pic}^1(C), M^{\otimes 2})$ that map to $\tau_0 \tau_1, \tau_0^2, \tau_1^2$, respectively. It follows that a basis of eigenvectors for $H^0(\text{Pic}^1(C), M^{\otimes 2})$ is given by $f_0, f_1, f_2, f_3$, where $f_0$ is the square of a nonzero section of $M$, and thus it vanishes on $C$ of order 2. The base scheme of the pencil of $\text{Pic}^1(C)$ spanned by $f_0$ and $f_1$ is supported at the points $Q_1, Q_2, Q_3, Q_4$ of $C$ and it has length 2 at each of these points. It follows that the general curve of this pencil is smooth by Bertini’s theorem.
Consider the $\mathbb{Z}_3$–action on $\mathbb{P}^1$ defined by $\xi(x_0 : x_1) = (\omega x_0 : \omega^2 x_1)$ and let $\mathbb{Z}_3$ act diagonally on $\text{Pic}^1(C) \times \mathbb{P}^1$ by $\xi(a, b) = (\xi a, \xi b)$. We remark that this action has isolated fixed points. If we extend the $\mathbb{Z}_3$–action on $\text{Pic}^1(C) \times \mathbb{P}^1$ to $\mathcal{L}(3k) := M \otimes \mathcal{O}_\mathbb{P}^1(3k)$ by using the $\mathbb{Z}_3$–action that we have already defined on $M$ and the natural pull-back structure on $\mathcal{O}_\mathbb{P}^1(3k)$, then the following is a basis of the $\mathbb{Z}_3$–invariant subspace of $H^0(\text{Pic}^1(C) \times \mathbb{P}^1, \mathcal{L}(3k))$:

$$x_0^{3i} x_1^{3k-3i} f_0, \quad x_0^{3i} x_1^{3k-3i} f_1, \quad x_0^{2+3j} x_1^{3k-3j-2} f_2, \quad x_0^{1+3j} x_1^{3k-3j-1} f_3, \quad 0 \leq i \leq k, \quad 0 \leq j \leq k-1$$

Denote by $|X|$ the corresponding linear system on $\text{Pic}^1(C) \times \mathbb{P}^1$. The base set of $|X|$ consists of the 8 points $\{(Q_i, (1 : 0)), (Q_i, (0 : 1))\mid i = 1 \ldots 4\}$, which are all fixed points of $\mathbb{Z}_3$. The general surface of $|X|$ contains no other fixed point and one can check that it is smooth using Bertini’s theorem. We wish to determine the characters of the representation of $\mathbb{Z}_3$ on the tangent space to a general $X$ at the fixed points of $|X|$. We remark first of all that, since the tangent space to $\text{Pic}^1(C)$ is dual to $H^0(\omega_C)$, the $\mathbb{Z}_3$–action on it is the sum of the two nontrivial characters. Let $T'$ be the tangent space to $X$ at $(Q_1, (0 : 1))$. Since $f_3$ does not vanish at $Q_1$, for a general choice of $X \in |X|$ the projection $X \to \text{Pic}^1(C)$ induces an isomorphism of $T'$ with the tangent space to $\text{Pic}^1(C)$. Since $X \to \text{Pic}^1(C)$ is compatible with the $\mathbb{Z}_3$–actions on $X$ and $\text{Pic}^1(C)$, it follows that $\xi$ acts on $T'$ with eigenvalues $\omega$ and $\omega^2$. Now let $T''$ be the tangent space to $X$ at the point $(Q_1, (1 : 0))$. We have seen that the tangent space at $Q_1$ to the curves of the pencil of $\text{Pic}^1(C)$ spanned by $f_0$, $f_1$ is fixed and transversal to $C$. We denote by $T_1$ this space, which is clearly $\mathbb{Z}_3$–invariant. Since $f_2$ vanishes at $Q_1$, it follows that $T''$ is the direct sum of $T_1 \times \{0\}$ and of the tangent space to $Q_1 \times \mathbb{P}^1$ at $(Q_1, (1 : 0))$, on which $\xi$ acts with eigenvalue $\omega$. We observe that $T_1$ is a generator of $\omega_C$ in a neighbourhood of $Q_1$, hence $\xi$ acts by pull-back on the cotangent space to $C$ at $Q_1$ as the multiplication by $\omega^2$. Thus $\xi$ acts by differentiation on the tangent space $T_2$ to $C$ at $Q_1$ as the multiplication by $\omega^2$. Since the tangent space to $\text{Pic}^1(C)$ at $Q_1$ is the direct sum of $T_1$ and $T_2$, it follows that $\xi$ acts on $T_1$ as the multiplication by $\omega$. Summing up, we have shown that the action of $\xi$ on $T''$ is the multiplication by $\omega$. The representations of $\mathbb{Z}_3$ at the remaining six points can be studied in the same way, and therefore the characters of the representation of $\mathbb{Z}_3$ on the tangent space are distinct at the points $(Q_1, (0 : 1)), (Q_2, (0 : 1)), (Q_3, (1 : 0)), (Q_4, (1 : 0))$ and they are equal at the points $(Q_1, (1 : 0)), (Q_2, (1 : 0)), (Q_3, (0 : 1)), (Q_4, (0 : 1))$. Let $Y$ be the minimal desingularization of $X/\mathbb{Z}_3$. Then by Proposition
have:

\[ K^2_X - 4 = 3K^2_Y; \quad \chi(X) = 3\chi(Y) - \frac{4}{3} - \frac{8}{3}. \]

In addition, by the Lefschetz Theorem on hyperplane sections, the Albanese variety of \( X \) is \( \text{Pic}^0(C) \) and thus \( q(Y) = 0 \), since \( \mathbb{Z}_3 \) acts on \( \text{Pic}^0(C) \) with finitely many fixed points. Using the formulas for the invariants of \( X \) (cf. §2), we get:

\[ K^2_Y = 24k - 12; \quad q(Y) = 0; \quad p_g(Y) = 4k - 1. \]

As explained in §2, the canonical map of \( X \) has degree 2 and the corresponding involution \( \sigma \) of \( X \) is induced by the involution \( \iota \times \text{Id} \) of \( \text{Pic}^1(C) \times \mathbb{P}^1 \). The quotient surface \( \Sigma := X/\sigma \) is a normal regular surface with 48\( k \) singular points of type \( A_1 \) that are the images of the fixed points of \( \iota \times \text{Id} \) on \( X \). The \( \mathbb{Z}_3 \)-action on \( X \) commutes with \( \sigma \) and therefore there is an induced \( \mathbb{Z}_3 \)-action on \( \Sigma \). The involution \( \iota \) exchanges \( Q_1 \) with \( Q_2 \) and \( Q_3 \) with \( Q_4 \), and thus there are 4 fixed points of \( \mathbb{Z}_3 \) on \( \Sigma \), that are smooth points of \( \Sigma \). The eigenvalues of the action of \( \xi \) on the tangent space to \( \Sigma \) are equal at two of these points and they are different at the remaining two. If \( T \) is the minimal desingularization of \( \Sigma/\mathbb{Z}_3 \), then by Proposition 2.1 we have:

\[ K^2_\Sigma - 2 = 3K^2_T; \quad \chi(\Sigma) = 3\chi(T) - 2/3 - 4/3. \]

The invariants of \( \Sigma \) are computed in §2, and we get:

\[ K^2_T = 12k - 6; \quad q(T) = 0; \quad p_g(T) = 4k - 1. \]

Hence \( p_g(Y) = p_g(T) \). Finally, we are going to show that the canonical map of \( T \) is birational for \( k \geq 2 \). It is easy to check that the non hyperelliptic genus 3 fibration \( \Sigma \to \mathbb{P}^1 \) induced by the second projection \( W \times \mathbb{P}^1 \to \mathbb{P}^1 \) descends to a fibration \( \Sigma/\mathbb{Z}_3 \to \mathbb{P}^1/\mathbb{Z}_3 \) with the same general fibre \( F \), which in turn pulls back to a fibration \( p: T \to \mathbb{P}^1 \). For \( k \geq 2 \), we have \( p_g(T) \geq 7 \), hence the canonical map of \( T \) separates the fibres of \( p \). Thus, in order to show that the canonical map of \( T \) has degree 1 it is enough to show that the restriction of \( |K_T| \) to the general fibre of \( p \) is the complete canonical system of the fibre.

The surface \( \Sigma \subset W \times \mathbb{P}^1 \) is an element of the linear system \( |L \otimes \mathcal{O}_{\mathbb{P}^1}(3k)| \). The sheaf \( L \otimes \mathcal{O}_{\mathbb{P}^1}(3k) \) inherits from \( M^{\otimes 2} \otimes \mathcal{O}_{\mathbb{P}^1}(3k) \) a linearization such that \( \Sigma \) is defined by a \( \mathbb{Z}_3 \)-invariant section. The residue sequence:

\[ 0 \to K_{W \times \mathbb{P}^1} \to K_{W \times \mathbb{P}^1}(\Sigma) = L \otimes \mathcal{O}_{\mathbb{P}^1}(3k - 2) \to K_\Sigma \to 0 \]

is \( \mathbb{Z}_3 \)-equivariant with respect to the natural linearization of \( K_{W \times \mathbb{P}^1} \) and to the chosen linearization of \( L \otimes \mathcal{O}_{\mathbb{P}^1}(3k - 2) \).
For \( i = 0 \ldots 3 \), let \( g_i \in H^0(W, L) \) be such that \( h^*g_i = f_i \) (recall that we have \( H^0(V, M^{\otimes 2}) = h^*H^0(W, L) \)). A basis of the invariant subspace \( H \) of \( H^0(W \times \mathbb{P}^1, L \otimes \mathcal{O}_{\mathbb{P}^1}(3k - 2)) \) is the following:

\[
\begin{align*}
    x_0^{2+3i} x_1^{3k-4-3i} g_0, x_0^{2+3i} x_1^{3k-4-3i} g_1, x_0^{1+3m} x_1^{3k-3-3m} g_2, x_0^{3m} x_1^{3k-2-3m} g_3
\end{align*}
\]

\( 0 \leq i \leq k - 2, \ 0 \leq m \leq k - 1 \).

Consider a general fibre \( F \) with \( (x_0 : x_1) = (\lambda_0 : \lambda_1) \). Identify \( F \) with the corresponding curve \( C \) of \( |L| \) on \( W \). Then the above sections, restricted to \( F \), correspond to the restrictions of \( g_0 \ldots g_3 \) to \( C \). Hence, the restriction map \( H \to H^0(F, K_F) \) is surjective. Since the elements of \( H \) give sections of the canonical bundle of \( T \) and the fibrations \( \Sigma \to \mathbb{P}^1 \) and \( p: T \to \mathbb{P}^1 \) have the same general fibre, it follows that the restriction of \(|K_T|\) to the general fibre of \( p \) is also complete.

**Example 2:** Here we take \( \mathbb{Z}_3 \)-quotients of good canonical covers obtained from the generating pair \( \Pi \) (cf. \$2\).

We take a curve \( C \) of genus 2 with a \( \mathbb{Z}_3 \)-action with rational quotient as in Example 1 and we use the notation introduced there.

We take \( \pi: V \to \text{Pic}^1(C) \) to be a double cover such that \( \pi_*\mathcal{O}_V = \mathcal{O}_{\text{Pic}^1(C)} \oplus M^{-1} \), branched on a general curve \( D \) of \( |M^{\otimes 2}| \) invariant under the \( \mathbb{Z}_3 \)-action. We have seen in Example 1 that such curves are a 1-dimensional linear system with simple base points \( Q_i, i = 1 \ldots 4 \). Using the chosen \( \mathbb{Z}_3 \)-action on \( M \) it is possible to lift the automorphism \( \xi \) of \( \text{Pic}^1(C) \) to \( V \), hence there is an automorphism \( \zeta \) of \( V \) of order 6 such that \( \zeta^2 \) lifts \( \xi \) and \( \zeta^3 \) is the involution associated to the double cover \( \pi: V \to \text{Pic}^1(C) \). The fixed points of \( \zeta^2 \) are isolated, since the fixed points of \( \xi \) on \( \text{Pic}^1(C) \) are isolated, and they are of two types: the 10 inverse images \( P_1 \ldots P_{10} \) of the fixed points of \( \xi \) not lying on \( B \) and the points \( R_i := \pi^{-1}Q_i, i = 1 \ldots 4 \). Since the differential of \( \pi \) is nonsingular at \( P_1 \ldots P_{10}, \xi = \zeta^2 \) acts on the tangent space to \( V \) at \( P_1 \ldots P_{10} \) with eigenvalues \( \omega, \omega^2 \). The points \( R_1 \ldots R_4 \) are fixed points of \( h \). Now consider the minimal resolution \( Z \) of the quotient surface \( V/\xi \). Recall that \( V \) has invariants \( q(V) = p_g(V) = 2 \) and \( K_V^2 = 4 \). By Proposition 2.1, one has \( 4 = K_V^2 = \alpha + 3K_Z^2 \) and \( 1 = \chi(V) = 3\chi(Z) - (\alpha + 2\beta)/3 \), where \( \alpha \) (resp. \( \beta \)) is the number of isolated fixed points of \( \xi \) on \( V \) such that the eigenvalues of the action of \( \xi \) on the corresponding tangent space are equal (resp. distinct). By the above discussion, we have \( \beta \geq 10 \) and \( \alpha + \beta = 14 \), hence the only possibility is \( \alpha = 4, \beta = 10 \). Thus we have \( \chi(Z) = 3, p_g(Z) = 2, q(Z) = 0, K_Z^2 = 0 \).

In particular, \( p_g(Z) = p_g(V) \), namely the action of \( \mathbb{Z}_3 \) on \( H^0(V, K_V) \) is trivial. We let \( \xi \) act on \( \mathbb{P}^1 \) by \( \xi(x_0, x_1) = (wx_0, \omega x_1) \) and we consider the diagonal action on \( V \times \mathbb{P}^1 \). If \( f_0, f_1 \) is a basis of \( H^0(V, K_V) \) and
we consider the natural action of $\xi$ on $L(3k) = K_Y \boxtimes O_{\mathbb{P}^1}(3k)$, then the following is a basis of the subspace of invariant sections of $L(3k)$ with respect to the induced linearization:

\[
x_0^3 x_1^{3k-3i} f_0, \quad x_0^3 x_1^{3k-3i} f_1; \quad i = 0 \ldots k.
\]

It is easy to check that the base locus of the corresponding linear system $|X| \subset |L(3k)|$ is the union of the curves $\{R_i\} \times \mathbb{P}^1$, $i = 1 \ldots 4$, and that the general $X$ is smooth along the base locus. Hence the general $X$ is smooth by Bertini’s theorem. The fixed points of the action of $\xi$ on the general $X$ are the points $(R_i, (0 : 1))$ and $(R_i, (1 : 0))$, $i = 1 \ldots 4$. Arguing as in the previous case, it is not difficult to check that the irreducible characters of the representation of $\mathbb{Z}_3$ on the tangent space to $X$ are equal at four of these points and they are different at the remaining four. Denote by $Y$ the minimal desingularization of $X/\mathbb{Z}_3$.

By the formulas for the invariants of $X$ (cf. §2) and by Proposition 2.1, we have:

\[\chi(Y) = 5k; \quad K_Y^2 = 32k - 12.\]

In addition, we have $q(Y) = 0$, since the Albanese variety of $Y$ is $\text{Pic}^0(C)$ and $\mathbb{Z}_3$ acts on $\text{Pic}^0(C)$ with isolated fixed points. Thus we have $p_g(Y) = 5k - 1$. As explained in §2, the canonical map of $X$ has degree 2 and the canonical involution $\sigma$ of $X$ is the restriction of $\iota \times \text{Id}$. The action of $\xi$ on $X$ descends to $\Sigma = X/\sigma$ and $\xi$ has 4 fixed points occurring at smooth points $\Sigma$. The action of $\mathbb{Z}_3$ on the tangent space to $\Sigma$ has distinct characters at two of these points and equal characters at the remaining two. The invariants of the minimal resolution $T$ of $\Sigma/\mathbb{Z}_3$, which can be computed as before, are:

\[q(T) = 0; \quad p_g(T) = 5k - 1; \quad K_T^2 = 16k - 6.\]

Hence we have $p_g(Y) = p_g(T)$.

One can show that for $k \geq 1$ the canonical map of $T$ is birational, and thus $Y \to T$ is a canonical cover, by using an argument similar to that of Example 1. In order to do so, take a basis $g_0 \ldots g_4$ of $H^0(W, 2K_W)$ consisting of $\mathbb{Z}_3$–eigenvectors. The $\mathbb{Z}_3$–invariant subspace $H$ of the space $H^0(W \times \mathbb{P}^1, 2K_W \boxtimes O_{\mathbb{P}^1}(3k))$ is spanned by products of the $g_i$ with suitable monomials in $x_0, x_1$ and each of the $g_i$ occurs in at least one such product. Consider a general fibre $F$ of the map $\Sigma \subset W \times \mathbb{P}^1 \to \mathbb{P}^1$, with $(x_0 : x_1) = (\lambda_0 : \lambda_1)$. Identify $F$ with the corresponding curve $C$ of $|K_W|$ on $W$. Then the elements of the above mentioned basis of $H$, restricted to $F$, correspond to the restrictions of $g_0 \ldots g_4$ to $C$. Hence, the restriction map $H \to H^0(F, K_F)$ is surjective. Since the elements of $H$ give sections of the canonical bundle of $T$ and the fibrations $\Sigma \to \mathbb{P}^1$
and \( p: T \to \mathbb{P}^1/\mathbb{Z}_3 = \mathbb{P}^1 \) have the same general fibre, it follows that the restriction of \( |K_T| \) to the general fibre of \( p \) is also complete.

**Example 3:** Here we take \( \mathbb{Z}_3 \)-quotients of good canonical covers obtained from the generating pair III (cf. §2).

Consider \( \mathbb{P}^2 \) with homogeneous coordinates \( z_0, z_1, z_2 \) and let \( C \subset \mathbb{P}^2 \) be the curve defined by \( z_1z_2^3 = f(z_0, z_1) \) where \( f \) is a homogeneous polynomial of degree 4 with distinct roots and such that \( f(1, 0) \neq 0 \). 

\( C \) is a smooth non hyperelliptic curve of genus 3 and \((z_0 : z_1 : z_2) \mapsto (z_0 : z_1 : \omega z_2) \) defines an automorphism \( \xi \) of \( C \) of order 3. The fixed points of \( \xi \) are \( Q_0 := (0 : 0 : 1) \) and the points \( Q_1, \ldots, Q_4 \) defined by \( z_2 = f(z_0, z_1) = 0 \). We have \( H^0(C, \omega_C) = H^0(C, \mathcal{O}_C(1)) \) and \( \xi \) acts on \( 1 \)-forms as follows:

\[
\xi^* z_0 = \omega z_0, \quad \xi^* z_1 = \omega z_1, \quad \xi^* z_2 = \omega^2 z_2.
\]

The induced \( \mathbb{Z}_3 \)-action on \( V = S^2C \) clearly commutes with the involution \( \iota \) of \( V \). In order to write down the action of \( \mathbb{Z}_3 \) on \( H^0(V, K_V) \), we observe that there is a natural isomorphism \( \wedge^2 H^0(C, \omega_C) \cong H^0(V, K_V) \) (cf. §2).

If we denote by \( f_{ijk} \) the element of \( H^0(V, K_V) \) corresponding to \( z_j \wedge z_k \), with \((ijk)\) a permutation of \((012)\), then we have:

\[
\xi^* f_{0} = f_{0}; \quad \xi^* f_{1} = f_{1}; \quad \xi^* f_{2} = \omega^2 f_{2}.
\]

In addition, given a vector space \( U \) of dimension 3, the alternation map \( U \otimes \wedge^2 U \to \wedge^3 U \) gives a natural isomorphism \( \wedge^2 U \cong U^* \otimes (\wedge^3 U)^* \). For \( U = H^0(C, \omega_C) \) this gives a projective duality between \( \mathbb{P}^2 = \mathbb{P}(H^0(C, \omega_C)^*) \) and \( \mathbb{P}^2^* = \mathbb{P}(H^0(V, K_V)^*) \) such that \( z_0, z_1, z_2 \) and \( f_0, f_1, f_2 \) are dual homogeneous coordinates.

Hence the canonical map \( \varphi: V \to \mathbb{P}^2^* \) maps \( P + Q \in V \) to the line \( < P, Q > \). In particular, the zero locus of \( f_0 \) on \( V \) is the set of the \( P + Q \) such that the line determined by \( P \) and \( Q \) goes through \((1 : 0 : 0)\), the zero locus of \( f_1 \) is the set of \( P + Q \) such that the line determined by \( P \) and \( Q \) goes through \((0 : 1 : 0)\) and the zero locus of \( f_2 \) is the set of \( P + Q \) such that the line determined by \( P \) and \( Q \) goes through \((0 : 0 : 1)\).

As in the previous examples, we extend the \( \mathbb{Z}_3 \)-action to \( V \times \mathbb{P}^1 \) by setting \( \xi(P + Q, (x_0 : x_1)) = (\xi(P + Q), (\omega x_0 : \omega^2 x_1)) \). The fixed points of this action are the points \((Q_i + Q_j, (1 : 0)), (Q_i + Q_j, (0 : 1))\), \( 0 \leq i, j \leq 4 \).

For \( k \geq 1 \), we consider on \( \mathcal{L}(3k) = K_V \otimes \mathcal{O}_{\mathbb{P}^1}(3k) \) the \( \mathbb{Z}_3 \)-linearization induced by the natural one on \( K_V \) and by the linearization of \( \mathcal{O}_{\mathbb{P}^1}(3k) \) as a pull-back. The invariant subsystem \( |X| \) of \( |\mathcal{L}(3k)| \) is spanned by:

\[
x_0^3x_1^{3k-3i}f_0, \quad x_0^3x_1^{3k-3i}f_1, \quad x_0^{1+3m}x_1^{3k-3m-1}f_2, \quad 0 \leq i \leq k, \quad 0 \leq m \leq k-1
\]
The base scheme of $|X|$ consists of the 12 simple points $(Q_i + Q_j, (1 : 0))$, $(Q_i + Q_j, (0 : 1))$, $1 \leq i, j \leq 4$, $i \neq j$, hence the general surface of $|X|$ is smooth by Bertini’s theorem. The differential of $\xi$ on the tangent space to $V$ at $Q_i + Q_j$, $i \neq j$, $i, j \neq 0$, is the multiplication by $\omega$. For a general $X$ the projection $X \to V$ has non singular differential at the points $(Q_i + Q_j, (0 : 1))$, $i \neq j$, $i, j \neq 0$, hence the differential of $\xi$ at these points is also the multiplication by $\omega$. Arguing as in Example 1, one can also show that the differential of $\xi$ on the tangent space to $X$ at the points $(Q_i + Q_j, (1 : 0))$ has eigenvalues $\omega$ and $\omega^2$.

The invariants of the minimal desingularization $Y$ of $X/\mathbb{Z}_3$ can be computed as in the previous examples and they are the following:

$$K_Y^2 = 48k - 18; \quad \chi(Y) = 7k.$$

In addition, there are natural identifications $H^0(X, \Omega^1_X) \cong H^0(V, \Omega^1_V) \cong H^0(C, \omega_C)$, compatible with the various $\mathbb{Z}_3$–actions. It follows that $H^0(V, \Omega^1_V)^{\mathbb{Z}_3} = \{0\}$, and:

$$q(Y) = 0; \quad p_g(Y) = 7k - 1.$$

As explained in §2, the canonical map of $X$ has degree 2 and the corresponding involution $\sigma$ of $X$ is induced by the involution $\iota \times \text{Id}$ of $V \times \mathbb{P}^1$. The quotient surface $\Sigma := X/\sigma$ is a normal regular surface with 84 $k$ singular points of type $A_1$ that are the images of the fixed points of $\iota \times \text{Id}$ on $X$. The $\mathbb{Z}_3$–action on $X$ commutes with $\iota \times \text{Id}$ and therefore there is an induced $\mathbb{Z}_3$–action on $\Sigma$. The fixed points of $\mathbb{Z}_3$ on $\Sigma$ are the 6 image points of the fixed points of $\mathbb{Z}_3$ on $X$ and are smooth for $\Sigma$. The automorphism $\xi$ acts as the multiplication by $\omega$ on the tangent space at 3 of these points and it has eigenvalues $\omega$ and $\omega^2$ at the remaining 3. If $T$ is the minimal desingularization of $\Sigma/\mathbb{Z}_3$, then the usual computation gives:

$$K_T^2 = 24k - 9; \quad q(T) = 0; \quad p_g(T) = 7k - 1.$$

Hence $p_g(Y) = p_g(T)$.

One can use the same argument as in previous examples to show that for $k \geq 1$ the canonical map of $T$ is birational onto its image.

**Remark 3.1.** Notice that no good canonical cover $X \to S$ such that $q(S) > 0$ is known. It would be interesting to see whether such an example exists.
4. The fundamental group

In this section we prove that for the good canonical covers \( Y \to T \) of Example 1 of §3 the surface \( Y \) is simply connected. We use the notation of §2 and §3.

Our main tool is the following:

**Theorem 4.1.** Let \( Z \) be a connected simply connected manifold, let \( \Gamma \) be a group that acts properly discontinuously on \( Z \). Assume that the stabilizer of every point of \( Z \) is finite and let \( N < \Gamma \) be the subgroup generated by the elements that do not act freely. Then:

\[
\pi_1(Z/\Gamma) = \Gamma/N.
\]

**Proof.** The quotient map \( Z \to Z/\Gamma \) has the path lifting property up to homotopy by Proposition 3 of [Ar]. Hence the assumptions of Theorem 4 of [Ar] are satisfied and we have the result. \( \square \)

**Theorem 4.2.** Let \( Y \to T \) be one of the good canonical covers of Example 1 of §3. Then:

\[
\pi_1(Y) = 0.
\]

**Proof.** We write \( \tilde{Y} := X/Z_3 \), and we let \( \eta: Y \to \tilde{Y} \) be the minimal resolution of the singularities of \( \tilde{Y} \). The singularities of \( \tilde{Y} \) are points of type \( A_1, A_2 \) or \( \frac{1}{3}(1,1) \). A standard application of van Kampen’s theorem shows that \( \eta_\ast: \pi_1(Y) \to \pi_1(\tilde{Y}) \) is an isomorphism. Hence from now on we study \( \pi_1(\tilde{Y}) \).

By the Lefschetz Theorem on hyperplane sections, the inclusion \( X \to \Pic^1(C) \times \mathbb{P}^1 \) induces an isomorphism on the fundamental groups. Therefore, if we denote by \( p: \mathbb{C}^2 \times \mathbb{P}^1 \to \Pic^1(C) \times \mathbb{P}^1 \) the universal cover and we set \( \tilde{X} := p^{-1}X \), then the restricted map \( p: \tilde{X} \to X \) is the universal cover. We identify the curve \( C \) with \( C \times (1:0) \subset \Pic^1(C) \times \mathbb{P}^1 \) and we set \( \tilde{C} := p^{-1}C \). The restricted map \( p: \tilde{C} \to C \) is the covering of \( C \) associated to the commutator subgroup \( [\pi_1(C), \pi_1(C)] \). The automorphism \( \xi \) of \( \Pic^1(C) \times \mathbb{P}^1 \) lifts to the universal cover and we have an extension of groups:

\[
1 \to \pi_1(\Pic^1(C) \times \mathbb{P}^1) = \mathbb{Z}^4 \to \Gamma \to \mathbb{Z}_3 \to 1
\]

where \( \Gamma \) is a group that acts properly discontinuously on \( \mathbb{C}^2 \times \mathbb{P}^1 \). Clearly \( \Gamma \) acts also on \( \tilde{X} \) and \( \tilde{C} \) by construction, and \( \tilde{X}/\Gamma = \tilde{Y} \), \( \tilde{C}/\Gamma = C/\mathbb{Z}_3 = \mathbb{P}^1 \). If we denote by \( N \) the subgroup of \( \Gamma \) generated by the elements that do not act freely on \( \tilde{X} \) then we have \( \pi_1(\tilde{Y}) = \Gamma/N \) by Theorem 4.1. We denote by \( N_1 \) the subgroup of \( \Gamma \) generated by the elements that do not act freely on \( \tilde{C} \). The fixed points of an element
that does not act freely on $\tilde{C}$ necessarily map to fixed points of $\mathbb{Z}_3$ on $C$, namely to one of the points $Q_1 \times (1:0) \ldots Q_4 \times (1:0)$. Since these four points are also in $X$, it follows that $N_1 \subset N$. Consider now the universal cover $D \to C$ of $C$. The automorphism $\xi$ of $C$ lifts to $D$ and we have an extension of groups analogous to the one above:

$$1 \to \pi_1(C) \to \Gamma_0 \to \mathbb{Z}_3 \to 1.$$ 

We denote by $N_0 < \Gamma_0$ the subgroup generated by the elements that do not act freely on $D$. Since $D/\Gamma_0 = C/\mathbb{Z}_3 = \mathbb{P}^1$ is simply connected, we have $N_0 = \Gamma_0$ by Theorem 4.1. The commutator subgroup $[\pi_1(C), \pi_1(C)]$ is a characteristic subgroup of $\pi_1(C)$, hence it is normal in $\Gamma_0$. It follows that every element of $\Gamma_0$ induces an automorphism of $\tilde{C} = D/[\pi_1(C), \pi_1(C)]$. Hence there is a natural surjection $\Gamma_0 \to \Gamma$. Since the image of $N_0$ in $\Gamma$ is obviously contained in $N_1$, we have $N_1 = \Gamma$. Since $N_1 \subset N$, we also have $N = \Gamma$, hence $Y$ is simply connected by Theorem 4.1.  

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