Hawking Effect in 2-D String Theory

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Abstract

We use the matrix model to study the final state resulting from a coherent high energy pulse in 2-d string theory at large string coupling. We show that the outgoing signal produced via reflection off the potential has a thermal spectrum, with the correct temperature and profile to be identified with Hawking radiation. We confirm its origin as geometrical radiation produced by the gravitational background. However, for a total incoming energy $M$, the amount of energy carried by the thermal radiation scales only as $\log M$. Most of the incoming energy is returned via the transmitted wave, which does not have a thermal spectrum, indicating the absence of macroscopic black hole formation.
1. Introduction

The correspondence between 2-d string theory and the \( c = 1 \) matrix model is the oldest example of a holographic duality [1]. Recently, it has been further clarified via the identification of the matrix model with the world-line degrees of freedom of unstable D-particles [2, 3, 4, 5, 6, 7]. Given that the classical low energy target space theory supports a 2-d black hole space-time [8, 9], it is natural to ask whether the \( c = 1 \) S-matrix reveals characteristic features that would underscore its possible interpretation in terms of black hole formation and evaporation.

The cleanest formulation of the matrix model is in terms of free fermions moving in an upside-down harmonic oscillator potential. In this language, the matrix model is exactly soluble and admits an explicit S-matrix description [10]. The collective perturbations of the Fermi surface are described by a scalar field \( \varphi \), which in turn are related to the space-time profile of the closed string fields via a non-local leg-pole transformation. In the case of type 0B non-critical string theory, the closed string fields are the tachyon and axion. The presence of two space-time scalar fields connects with the fact that the Fermi sea of the matrix model splits in two halves, one on each side of the inverted harmonic potential [11, 12]. For the case of the axion, the leg-pole transform reads

\[
C(u) = \frac{\Gamma\left(\frac{1}{2}(1 - \partial_u)\right)}{\Gamma\left(\frac{1}{2}(1 + \partial_u)\right)} \varphi(u)
\]

where \( \varphi(u) \) denotes the anti-symmetric combination of \( \varphi \) [11, 12]. This leg pole factor is an essential ingredient in the holographic dictionary of the matrix model; in particular, it ensures a causal description of gravitational bulk scattering [13, 14].

In this paper, we will study the final state produced by a large incoming matter pulse. For the incoming state, we take a general coherent state of the \( \varphi \)-field

\[
| \text{in} \rangle = V_f | 0 \rangle, \quad V_f = \exp\left(\int du f(u) \partial_u \varphi(u)\right)
\]

Since the leg-pole transformation is linear, this also represents a coherent state in terms of the space-time fields. However, a sharply localized incoming matter pulse in the target space amounts to a smeared distribution in terms of the \( \varphi \)-field.

The calculation of the final state is quite straightforward [10]. First one re-expresses the above state in terms of the fermion field via the replacement \( \partial \varphi = \psi^\dagger \psi \). Next one applies the single particle S-matrix to each fermionic mode to obtain the corresponding
Fig 1: Pictorial description of the scattering process. Matter comes in from one side of the well. The portion reflecting back has the form of Hawking radiation. Since both asymptotic regions are identified in the target space, both the Hawking radiation and the transmitted signal return to the same physical spacetime location.

fermionic out-state. The bosonic target space out-state is then obtained by rebosonization and applying the leg-pole transform.

Our main result will be that the final state contains a clearly identifiable thermal component, with the correct temperature to support its interpretation as Hawking radiation [15]. As indicated in Fig 1, we will find that the thermal component consists of the matrix model fermion modes that have reflected back off the potential. This reflected subsector is correlated with the transmitted subsector, and thus forms a mixed state. The last two steps of the scattering recipe will not alter the thermal nature of this part of the final state: bosonization preserves thermality, and since the leg-pole factor is just an energy dependent phase, it drops out of a thermal density matrix.

This paper is organized as follows. After reviewing the mechanics of matrix model scattering, we give a short description of gravitational backreaction and the Hawking effect in 2-d dilaton gravity. We then show that the out-state resulting from a localized incoming matter pulse contains Hawking radiation. We will find, however, that the Hawking radiation carries only a small fraction of the total energy. Most of the energy is returned via the transmitted signal, which does not have a thermal spectrum. We give a possible space-time interpretation of the total scattering process in the final section.

Some recent related work on high energy scattering and particle creation in 2-d string theory includes [16, 17, 18, 19, 20].
2. Scattering as Fourier Transform

We begin with a description of the exact scattering matrix of the matrix model given in [10], and highlight its interpretation as a simple Fourier transformation.

After reduction to the eigenvalues, the $c = 1$ matrix model becomes a system of non-interacting fermions, each described by a single particle Hamiltonian

$$H(p,x) = \frac{1}{2} p^2 - \frac{1}{2} x^2,$$

where $x$ denotes the matrix eigenvalue, and $p$ its canonical momentum. The number of fermions equals the rank $N$ of the hermitian matrix. To characterize the large $N$ dynamics, it is convenient to introduce a second quantized fermion language. The fermionic field operators are expanded as

$$\psi_\alpha(x) = \int \frac{d\omega}{2\pi} b_\alpha(\omega) \ |x|^{-i\omega - \frac{1}{2}},$$
$$\psi_\alpha^\dagger(x) = \int \frac{d\omega}{2\pi} b_\alpha^\dagger(\omega) \ |x|^{i\omega - \frac{1}{2}},$$

where the subscript $\alpha = \pm$ denotes the restriction of $x$ to the positive or negative half-line. The modes $b_\alpha^\dagger(\omega)$ and $b_\alpha(\omega)$ respectively create and annihilate a quantum mechanical eigenvalue, with a wave function $|x|^{i\omega}$, supported on the corresponding half-line. They satisfy the canonical anti-commutation relation

$$\{b_\alpha(\omega), b_\beta^\dagger(\xi)\} = 2\pi \delta_{\alpha\beta} \delta(\omega - \xi).$$

In the ground state of the matrix model, the fermions fill a Fermi sea of states with energy smaller than the Fermi energy $\mu$:

$$b_\alpha(\omega)|\mu\rangle = 0 \quad \omega > -\mu,$$
$$b_\alpha^\dagger(\omega)|\mu\rangle = 0 \quad \omega < -\mu.$$

Note that the fermion Hilbert space is similar to that of a field theory in Rindler space. This correspondence will play an important role in what follows.

The fermions evolve individually via the Hamiltonian (3). This Hamiltonian evolution maps asymptotic single particle $in$-states to asymptotic single particle $out$ states. The resulting single particle $S$-matrix preserves energy, and for given $\omega$, acts as a two-by-two matrix [10]

$$\psi_\alpha^{out}(\omega) = S_\alpha^\beta(\omega) \psi_\beta^{in}(\omega)$$

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where
\[ S_\pm^\pm(\omega) = \frac{1}{\sqrt{2\pi}} \mu^{-i\omega} e^{\mp \omega \Gamma(\frac{1}{2} - i\omega)}, \]
\[ S_\pm^\pm(\omega) = \frac{i}{\sqrt{2\pi}} \mu^{-i\omega} e^{-\frac{\omega}{2} \pi / 2 \Gamma(\frac{1}{2} - i\omega)}. \] (8)
The relation
\[ |S^\pm(\omega)|^2 + |S^\pm(\omega)|^2 = 1 \] (9)
shows that the transition from \textit{in} to \textit{out} states preserves probability.

This result for the \( c = 1 \) S-matrix has a simple interpretation as a Fourier transformation. Let us denote the \textit{in} coordinate and momentum by \((x,p)\) and the \textit{out} coordinate and momentum by \((y,q)\). We claim that the above \( S \)-matrix in effect amounts to the identification \( y = p \) and \( x = q \), so that the \textit{in} and \textit{out} Kruskal coordinates are each others canonical momenta: \([x,y] = i\). Explicitly, let us introduce two independent \textit{in} and \textit{out} one particle wave functions of frequency \( \omega \) via
\[ \langle x|\omega,\text{in}\rangle_\pm = |x|^{-\frac{1}{2} - i\omega} \theta(\pm x), \]
\[ \langle y|\omega,\text{out}\rangle_\pm = |y|^{-\frac{1}{2} + i\omega} \theta(\pm y). \] (10)
These \textit{in} and \textit{out} states are linearly dependent. The transition from \( x \) to \( y \) is just a Fourier transformation, and it is easily checked that this leads to the following relation
\[ \beta \langle \omega',\text{out}|\omega,\text{in}\rangle_\alpha = S^\beta_\alpha(\omega) \delta(\omega - \omega') \] (11)
where \( \alpha, \beta = \pm \). This representation of the \( S \)-matrix as a Fourier transform can be directly understood from its definition as the evolution of a particle in an (inverted) harmonic potential: for a right-side up harmonic oscillator, the wave function evolves into its Fourier transform every quarter of a period. The hyperbolic phase space trajectory that connects the \textit{in} and \textit{out} state for the upside-down harmonic oscillator can be thought of as obtained via analytic continuation from \( 1/4 \) of an elliptic orbit of the right-side up oscillator.

The time-evolution and \( S \)-matrix preserve an infinite set of conserved quantities
\[ W_{mn} = \int_{-\infty}^{\infty} dx :\psi^\dagger(x)\{p^m, x^n\}\psi(x): \] (12)
that span a $\mathcal{W}_\infty$-algebra. The S-matrix acts on these charges by exchanging the two labels $n$ and $m$

$$\mathcal{W}_{nm}^{out} = \mathcal{W}_{mn}^{in} \quad (13)$$

A special case of this equation is conservation of total energy $H = \mathcal{W}_{11}$. Two charges that will play a special role in the following are

$$P \equiv \mathcal{W}_{10} = \int dx : \psi^\dagger(x) p \psi(x) :$$

$$Q \equiv \mathcal{W}_{01} = \int dx : \psi^\dagger(x) x \psi(x) : \quad (14)$$

The operator $P$ is the total incoming Kruskal momentum, and $Q$ is identified with the total outgoing Kruskal momentum via the S-matrix. As part of the $\mathcal{W}$-algebra, they satisfy the “canonical” commutation relation

$$[P, Q] = \Lambda \quad (15)$$

with

$$\Lambda \equiv \mathcal{W}_{00} = \int dx : \psi^\dagger(x) \psi(x) : \quad (16)$$

the central element of the $\mathcal{W}_\infty$-algebra. This relation will be of use later on.

3. BACKREACTION AND HAWKING EFFECT IN 2-D DILATON GRAVITY

Our goal is to compare the $c = 1$ matrix quantum mechanics with the expected gravitational physics of the target space string theory. By now, there is a relatively complete understanding of how the weak coupling dynamics of 1+1-d string theory is encoded in the matrix model [7, 21]. In the strong coupling regime, on the other hand, the dictionary is much less clear. In this regime, it is expected that gravitational interactions become important.

The gravitational part of the 2-d low energy effective field theory is described by the well-known dilaton gravity action

$$S_{grav} = \int d^2 x \sqrt{-g} e^{-2\phi} \left( R + 4(\nabla \phi)^2 + \lambda \right) \quad (17)$$
Here $\lambda = 8/\alpha'$ and $\phi$ is the dilaton. This gravitational sector does not support any local dynamics; its equations of motion are solved by

$$ds^2 = e^{2\phi}dx^+dx_- \quad (18)$$

$$e^{-2\phi} = -\lambda x^+x^- + a x^+ + b x^- + c, \quad (19)$$

with $a$, $b$ and $c$ integration constants. The vacuum solution is the linear dilaton background

$$e^{-2\phi} = -\lambda x^+x^- \quad \pm x^\pm > 0. \quad (20)$$

The coordinates $(x^+, x^-)$ are similar to Kruskal coordinates, related to the Minkowski light-cone coordinates $(u, v)$ via

$$(x^+, x^-) = (e^u, -e^{-v}). \quad (21)$$

The string theory dynamics in the linear dilaton background are believed to be accurately described by the matrix model. For general values of the integration constants in (18), however, the geometry is that of a two-dimensional black hole [8].

The central question we wish to address is whether, starting from the matrix model, we can recognize this black hole background as the intermediate state produced by a high energy incoming matter pulse. In particular, we wish to identify the component of the final state that describes the Hawking radiation generated by the black hole geometry. This Hawking effect is expected to occur regardless of whether the system actually develops a true event horizon or not, except that in case it does not, the thermal radiation is produced only for a rather short amount of time. Conversely, intermediate black hole formation is signalled by a final state that looks thermal for a sufficiently long time interval.

Let us recall how the Hawking effect arises from the perspective of the low energy effective field theory. Besides the dilaton gravity fields, 2-d type 0B string theory contains matter degrees of freedom in the form of the NS-NS tachyon field $T$ and the RR axion $C$ [11]. To leading order in $\alpha'$, the action for the axion is simply that of a massless scalar field

$$S_C = \int d^2x \sqrt{-g} (\nabla C)^2. \quad (22)$$

Small excitations of the $C$-field are therefore expected to travel along light-like trajectories, at least in the region of weak curvature and string coupling.
Now consider a large incoming matter pulse traveling along a light-like trajectory \( x^+ = q^+ \), with Kruskal momentum \( p_+ \) and energy \( M \). The energy and Kruskal momentum are related via

\[
M = q^+ p_+ \tag{23}
\]

The corresponding space-time geometry is obtained by gluing together, along the matter trajectory, the linear dilaton vacuum solution \( x^+ < q^+ \), to a static black hole solution for \( x^+ > q^+ \)

\[
e^{-2\phi} = -\lambda x^+ (x^- + \frac{1}{\lambda} p_+) + M \tag{24}
\]

Note that for this solution, the dilaton and metric are continuous along the matter pulse. Assuming that the solution is sufficiently accurate, it describes the formation of a black hole with a horizon located at \( x^- = -\frac{1}{\lambda} p_+ \). It is useful, however, to redefine coordinates such that the relationship (21) between \( x^- \) and the Minkowski light-cone coordinate \( v \) is restored, in other words, to shift the definition of \( x^- \) such that the observable part of space-time is still given by the region \( x^- < 0 \). In this new coordinate system, right-moving geodesics undergo a shift

\[
x^- \rightarrow x^- + \delta x^- \quad \delta x^- = \frac{1}{\lambda} p_+ \tag{25}
\]

upon crossing the matter trajectory. As emphasized in particular by 't Hooft and in [22, 23], this shift leads to a characteristic commutation relation between the \( \text{in} \) and \( \text{out} \)-modes.

Let us introduce the \( \text{in} \) quantum operator \( V_{p_+} \) that creates the incoming matter pulse. We imagine that \( V_{p_+} \, |0\rangle \) is an optimal approximation of an eigenstate of the Kruskal momentum operator \( \hat{P}_+ \). Now let \( \varphi_{\text{out}}(x^-) \) denote a local \( \text{out} \) operator. If we assume that the shift (25) is the only direct interaction between the two modes, they satisfy an exchange algebra of the form

\[
\varphi_{\text{out}}(x^-) \, \hat{V}_{p_+} = \hat{V}_{p_+} \, \varphi_{\text{out}}(x^- - \frac{1}{\lambda} p_+.) \tag{26}
\]

This relation directly gives rise to the Hawking effect [22, 24]. In terms of wave modes with given frequency \( e^{i \omega v} \), the exchange algebra reads

\[
a_{\omega} \hat{V}_{p_+} = \hat{V}_{p_+} \int_0^\infty d\xi \frac{1}{2\pi} \left( \alpha_{\omega \xi} a_{\xi} + \beta_{\omega \xi} a_{\xi}^\dagger \right) \tag{27}
\]
with
\[ \beta_{\omega \xi}(p_+) = \left( \frac{p_+}{\lambda} \right) ^{i(\omega + \xi)} \sqrt{\frac{\xi}{\omega}} \frac{\Gamma(1 - i\omega)\Gamma(i(\omega + \xi))}{\Gamma(1 + i\xi)} \] (28)
and \( \alpha_{\omega \xi} = \beta_{\omega, -\xi} \). Now let us imagine that \( \varphi \) represents a massless scalar field. The linear combination of creation and annihilation modes on the right-hand-side then represents a Bogolubov transformation. The resulting expectation value of the particle number operator is
\[ \langle N(\omega) \rangle = \langle 0 | \hat{V}^\dagger p_+ a_\omega^\dagger a_\omega \hat{V} p_+ | 0 \rangle = \int_0^\infty d\xi \frac{|\beta_{\omega \xi}|^2}{2\pi} \] (29)
A simple calculation gives the expected thermal spectrum
\[ \langle N(\omega) \rangle = \frac{A}{e^{2\pi \omega} - 1} - B \] (30)
with
\[ A = \int_\omega^\infty \frac{d\xi}{\xi}, \quad B = \int_\omega^\infty \frac{d\xi}{\xi} \frac{1}{e^{2\pi \xi} - 1}. \] (31)
The integral \( A \) in (29) as it stands diverges; this divergence is a consequence of our idealized representation of the incoming matter pulse. In the following section we will repeat the corresponding calculation in the matrix model, while taking into account that the incoming matter pulse contains a finite total energy \( M \).

4. **Hawking Effect and Backreaction in the Matrix Model**

In this section, we will consider an incoming high energy pulse, represented by a coherent state of the form
\[ |in\rangle = V_h |0\rangle, \quad V_h = \exp\left( \int du h(u) \partial_u C(u) \right) \] (32)
with \( h(u) \) some sharply localized function. We will find that the resulting outgoing state will contain a thermal component, with the correct temperature to be identified with Hawking radiation. The total energy contained in the radiation is rather small, however: it scales as \( \log M \) for a total incoming energy \( M \).
We will be focusing on the high energy regime, in which, from the matrix model perspective, the incoming signal consists of eigenvalues with energy $\omega \gg \mu$. For simplicity, we will therefore set $\mu = 0$ in this section.

We start with a simple observation. Given the form (32) of the incoming state, it immediately follows that in terms of the fermionic matrix model language, the final state will take the form $\exp \mathcal{B} |0\rangle$ with $\mathcal{B}$ some bilinear operator in terms of the fermions. The final state is therefore uniquely specified once we know the two-point function of the outgoing fermions. A special two-point function is the fermion number density as a function of frequency. Let us define the $2 \times 2$ matrix of number densities

$$ N_{\alpha\beta}(\omega) = b_{\alpha}^\dagger(\omega) b_{\beta}(\omega) $$

where $\alpha$ and $\beta$ refer to the left or right half, see Eqn (4). The in and out number operators are related via the single fermion S-matrix via

$$ N_{\alpha\beta}^{\text{out}}(\omega) = S_{\alpha\gamma}^\dagger N_{\gamma\delta}^{\text{in}}(\omega) S_{\delta\beta} $$

Using the explicit expression (7) for the S-matrix, we find

$$ N_{++}^{\text{out}}(\omega) = \frac{1}{1 + e^{2\pi \omega}} N_{++}^{\text{in}}(\omega) + \frac{1}{1 + e^{-2\pi \omega}} N_{--}^{\text{in}}(\omega) - i \frac{N_{++}^{\text{in}}(\omega) - N_{--}^{\text{in}}(\omega)}{e^{\pi \omega} + e^{-\pi \omega}} $$

$$ N_{+-}^{\text{out}}(\omega) = \frac{1}{1 + e^{2\pi \omega}} N_{+-}^{\text{in}}(\omega) + \frac{1}{1 + e^{-2\pi \omega}} N_{-+}^{\text{in}}(\omega) - i \frac{N_{+-}^{\text{in}}(\omega) - N_{-+}^{\text{in}}(\omega)}{e^{\pi \omega} + e^{-\pi \omega}} $$

This equation is already quite instructive: it shows that, if we would consider an incoming excitation localized on only one side of the well, so that only $N_{++}^{\text{in}}(\omega)$ is non-vanishing, the reflected part of the out-state has a number density given by

$$ N_{++}^{\text{out}}(\omega) = \frac{1}{1 + e^{2\pi \omega}} N_{++}^{\text{in}}(\omega). $$

Therefore if the incoming fermion number density $N_{++}^{\text{in}}(\omega)$ would be constant over some sufficiently large frequency range, the reflected part of the outgoing state would be thermal. In a moment we will show that, for a generic incoming localized matter pulse, $N_{++}^{\text{in}}(\omega)$ will indeed be constant over a large range of frequencies.

General incoming signals will have support on both sides, so that other number densities besides $N_{++}^{\text{in}}$ will be non-vanishing. This spoils the thermal form of the outgoing fermion number density. Our interpretation of the above formulas, however, is that the total scattering matrix naturally splits up into different S-matrix channels, each of which
describes a different geometrical contribution to the total scattering process. Our main observation will be that the S-matrix channel associated with reflection off the matrix model potential describes the part of the outgoing state produced via the Hawking effect. Indeed, the thermal prefactor in (36) has the correct temperature to support this interpretation.

Since we have set $\mu = 0$, reflection off the potential is in fact a subdominant effect for large energies. In the regime of interest, the deformations of the Fermi sea extend far above the top of the potential, and as a result, most of the incoming wave will be transmitted to the other side. Via the matrix model/target space dictionary, this transmitted wave describes matter that is reflected back from the strongly curved target space region. We will further comment on the interpretation of the transmitted signal in the next subsection.

To proceed, let us compute the incoming fermion number density associated with a given coherent state of the form as given in (2). In fermionized form, it is given by

$$ V_f|0\rangle = \exp \left[ i \int_{-\infty}^{\infty} du f(u)\psi^\dagger(u)\psi(u) \right]|0\rangle. $$

(37)

$V_f$ can be thought of as an operator that implements a specific Bogolubov transformation on the fermions. It satisfies the following exchange algebra with the fermion field

$$ \psi(u) V_f = V_f \psi(u) e^{-i f(u)}. $$

(38)

Let us write $\Psi(u) = e^{-i f(u)} \psi(u)$, and let us denote the frequency modes of $\Psi(u)$ by $B_\omega$ and $B_\omega^\dagger$. We thus find for the incoming fermion number density

$$ N^{in}(\omega) = \langle 0 | V_f b_\omega^\dagger b_\omega V_f | 0 \rangle = \langle 0 | B_\omega^\dagger B_\omega | 0 \rangle = \int_0^{\infty} \frac{d\xi}{2\pi} |\beta_{\omega\xi}|^2 $$

(39)

where and $\beta_{\omega\xi}$ is the Bogolubov coefficient in the relation $B_\omega = \alpha_{\omega\xi} b_\xi + \beta_{\omega\xi} b_\xi^\dagger$ between the $B$ and $b$ modes. It given by the integral

$$ \beta_{\omega\xi} = \int_{-\infty}^{\infty} du e^{i(\omega+\xi)u} e^{-i f(u)} $$

(40)

Computing this integral yields the incoming number distribution for any profile $f(u)$. For our purpose, it is sufficient to evaluate it using the stationary phase approximation. For
The spectral fermion number density \( N(\omega)d\omega \) is given by the phase space volume of the strip between \( \omega \) and \( \omega + d\omega \) that is filled by the perturbed Fermi sea. Each stationary point \( \bar{u} \), satisfying \( f'(\bar{u}) = \xi \), one obtains a contribution \( \sqrt{2\pi} \frac{d\xi}{f''(\bar{u})} \). Plugging this back into (39) we find

\[
N^{in}(\omega) \simeq \int \frac{d\xi}{f''(\bar{u})} \simeq \int d\xi \frac{d\bar{u}}{d\xi} \simeq \bar{u}_{\text{max}}(\omega) - \bar{u}_{\text{min}}(\omega)
\]

where \( \bar{u}_{\text{max}}(\omega) \) and \( \bar{u}_{\text{min}}(\omega) \) denote the maximum and minimum of the two solutions to \( f'(\bar{u}) = \omega \) (see Fig 2). As seen from the figure, this result has a clear semi-classical interpretation: it tells us that the spectral fermion number density \( N(\omega)d\omega \) is given by the phase space volume of the strip between \( \omega \) and \( \omega + d\omega \) that is filled by the perturbed Fermi sea.

Next we need to determine the approximate shape of the Fermi sea corresponding to a localized matter pulse in the target space. The profile \( f(u) \) of the coherent state (2) of the matrix model scalar field \( \varphi \) is related to the target space profile \( h(u) \) in (32) via the inverse leg pole transform

\[
f(u) = \frac{\Gamma(\frac{1}{2}(1 - \partial_u))}{\Gamma(\frac{1}{2}(1 + \partial_u))} h(u).
\]

Since \( h(u) \) is assumed to be sharply localized, we may approximate it by

\[
h(u) \simeq H_0 \delta(u - u_0),
\]

where the amplitude \( H_0 \) is assumed to be large. The properties of the leg pole transform (42) are discussed in some detail in [13]. For our purpose, it will be sufficient to know the
leading order behavior of \( f(u) \) at early times relative to \( u_0 \). It turns out that in this regime, the non-local leg-pole transform is dominated by the pole in the function \( \Gamma(\frac{1}{2}(1 - \partial_u)) \) at \( \partial_u = 1 \). So at time \( u \) sufficiently early compared to \( u_0 \), we can approximate

\[
f(u) \simeq H_0 e^{u - u_0} \quad u_0 - u \gg 1.
\] (44)

The contribution from the higher order poles lead to higher powers of \( e^{u - u_0} \), and are therefore subdominant for \( u_0 - u \gg 1 \). As a related point, we remark that if, instead of the axion field \( C(u) \), we would consider an incoming tachyon matter wave, the corresponding leg pole transform would give \( f(u) \) decaying as \( e^{2(u - u_0)} \) — tachyon matter waves in this respect are subdominant to axion matter waves.

Given the result (44), we can now estimate the incoming fermion number density \( N^{in}(\omega) \) for frequencies \( \omega \) much smaller than \( H_0 \), via (41). From \( f'(\bar{u}_{\text{min}}) = \omega \) and (44) we find \( \bar{u}_{\text{min}} \simeq u_0 - \log(H_0/\omega) \). Setting \( \bar{u}_{\text{max}} \simeq u_0 \) we obtain

\[
N^{in}(\omega) \simeq \log(H_0/\omega).
\] (45)

Combined with (36), this tells us that the reflected out-signal indeed has an (approximately) thermal spectrum of the form*

\[
N^{out}_{\text{refl}}(\omega) \simeq \frac{\log H_0}{e^{2\pi \omega} + 1}.
\] (46)

It is interesting to compare this result with the divergent answer (30)-(31) obtained earlier, via an idealized calculation in the low energy effective field theory. After bosonizing (46), we can directly compare the two. We note that the matrix model result essentially amounts to a UV regulation of the frequency integral \( A \) in (31), cutting it off at a frequency \( \xi_{\text{max}} = H_0 \) equal to the maximal energy carried by any of the matrix model eigenvalues. Conversely, within the matrix model, one can formally recover the exact result (28) for the Bogolubov coefficients by taking the limit \( H_0 \to \infty, u_0 \to \infty \), while keeping \( H_0 e^{-u_0} \) fixed.

Based on the correspondence with the semi-classical target space physics, we would like to make the following identification between the amplitude \( H_0 \), the instant \( u_0 \) and the total incoming Kruskal momentum \( p_+ \) of the incoming matter pulse

\[
H_0 e^{-u_0} \simeq p_+ / \lambda
\] (47)

*Since the amplitude \( H_0 \) is assumed to be very large compared to the typical Hawking frequency \( \omega \), we dropped the subleading \( \log \omega \)-term.
where $\lambda$ is the coefficient of $e^{-2\phi} \sqrt{-g}$ in the dilaton gravity Lagrangian (17). Combined with (44), this identification tells us that relative to operators defined at early times, the operator $V_f$ behaves as

$$V_f \sim e^{\frac{\lambda}{2} p + Q^+}$$

with

$$Q^+ = \int_{-\infty}^{\infty} du e^{u} \partial_u \varphi_{in} = \int_{0}^{\infty} dx :\psi_{in}^\dagger(x) x \psi_{in}(x):$$

In terms of the out-modes,

$$Q^+ = \int_{0}^{\infty} dq :\psi_{out}^\dagger(q) q \psi_{out}(q):$$

with $\psi_{out}(q)$ the outgoing fermionic mode with given Kruskal momentum $q$. We see that $Q^+$ represents the total outgoing Kruskal momentum, restricted to the subsector $q > 0$. Consequently, the operator $V_f$ that creates the incoming matter pulse has the following effect on the out-modes (with $q > 0$)

$$\psi_{out}(q) V_f = V_f \psi_{out}(q) e^{\frac{\lambda}{2} p + q}$$

This relation can be recognized as the momentum space version of the target space exchange algebra (26): it tells us that, also in the matrix model, an incoming matter pulse with Kruskal momentum $p_+$ results in a coordinate shift identical to (25) on (part of) the outgoing modes. We believe this correspondence provides an important hint for how the gravitational target space dynamics is encoded in the matrix model, as well as concrete support for the interpretation of the reflected wave as Hawking radiation.

The proposed linear relation (47) between the amplitude $H_0$ and the Kruskal momentum is directly motivated by the correspondence between (51) and (26). It has actually a slightly unexpected form, since energy-momentum is typically proportional to the square of the amplitude. Suppose, however, that the vacuum state satisfies

$$\int_{0}^{\infty} dx :\psi^\dagger(x) \psi(x): \left| 0 \right\rangle = \lambda \left| 0 \right\rangle$$

From Eqns (15), (16) and (48), we can then deduce that the in state $V_f \left| 0 \right\rangle$ is indeed an (approximate) eigenstate of the total Kruskal momentum with eigenvalue $p_+$. In bosonized
language, (52) translates to a finite, non-zero vacuum value for the zero-mode $\int \partial \varphi$ of the matrix model scalar field.

The relation (52) looks rather subtle, since typically one would define the normal ordered expression such that the right-hand side vanishes. Our proposed interpretation, however, is as follows. In order to establish a correspondence with target space physics, one must find the proper identification of the target space quantities such as the total energy and total incoming and outgoing Kruskal momenta. These quantities are defined as normal ordered expressions as in (12) and (14). The meaning of Eqn (52) is that it specifies the normal ordering prescription one needs to use in defining the space-time charges.

We end with a comment concerning the tachyon field. Given that, relative to the axion field, a localized tachyon matter wave gives rise to a subdominant Fermi sea profile for early times $u$, we must conclude that a collapsing shell of tachyon matter leads to a subdominant, or perhaps even absent, gravitational Hawking effect. This is somewhat mysterious, since it implies that the gravitational backreaction due to a tachyon matter pulse is much less pronounced than that of an axion matter pulse. Presumably, this difference arises due to strong self-interactions of the tachyon.

5. FATE OF THE MATTER PULSE

We have identified the Hawking radiation in the matrix model as the signal that reflects off the inverted harmonic potential. In the high energy regime, where the deformations of the Fermi sea extend far above the top of the potential, most of the signal simply propagates through to the other side. From the target space perspective, this transmitted wave describes matter that is reflected back via some non-linear dynamics that takes place in the strongly curved region. Unlike most higher dimensional black holes, the surface gravity of the black hole in 2-d string theory is independent of its mass or size, and always of order the string scale. Stringy effects can therefore kick in, and lead to drastic modifications of the low energy effective dynamics before the infalling matter has a chance to retreat behind the horizon. Our present calculation indicates that such a non-linear modification indeed takes place, and results in a complete reflection of all infalling matter.

Given our simple geometric characterization of the S-matrix as Fourier transformation in Kruskal coordinates, it is easy to determine the rough location where the reflection takes place. Suppose we consider a incoming matrix model eigenvalue with a wave function of
Fig 3. The space-time interpretation of the high energy scattering in the matrix model. The straight arrows represent infalling matter that gets reflected back via a non-linear process in the strongly curved region. The wavy arrows represent geometric Hawking radiation that appears a short time (of order \( \log M \)) before most of the matter gets returned.

The form

\[
\psi_{q,p}(x) = e^{-\frac{1}{2}(x-q)^2} e^{ipx}
\]  

(53)

This eigenvalue has an approximate Kruskal position equal to \( x \simeq q \) and incoming Kruskal momentum equal to \( p \). Its energy is approximately \( \omega \simeq pq \). Now, the S-matrix simply exchanges \( p \) and \( q \):

\[
\psi_{q,p} = S \psi_{p,q}.
\]  

(54)

The effective location of reflection is therefore given by \( x^+x^- \simeq \omega \). As seen from (44), the most energetic matrix model eigenvalue has an energy proportional to \( H_0 \), which scales as \( \sqrt{M} \). A good estimate for the outgoing time \( v_{refl} \), when most of the matter wave gets returned, is therefore \( v_{refl} \simeq u_0 - \frac{1}{2} \log M \). This should be compared with the moment when the Hawking radiation turns on, which is roughly at \( v_{rad} \simeq u_0 - \log M \). According to these estimates, the Hawking radiation precedes the return of the large matter pulse by a short time interval of order \( \frac{1}{2} \log M \) (see Fig 2).
6. Conclusion

We have proposed a dictionary between the matrix model and two-dimensional string theory in the high energy regime where the semi-classical gravitational backreaction of the incoming matter wave is expected to give rise to black hole radiance. We have found that the Hawking effect indeed takes place, but that the Hawking radiation carries off only a relatively small portion of the total energy $M$, proportional to $\log H_0$ where $H_0$ is the amplitude of the incoming matter wave. Given that $M \sim H_0^2$ and that the Hawking flux is of order one in string units, this tells us that the radiation is produced for a short amount of time of order $\log M$.

Although our finding that high energy incoming matter fails to produce a long-lived black hole counts as somewhat of a disappointment, our paper still supports some positive conclusions. We have found, within the matrix model, a clear signal of semi-classical target space physics. As such, our results represent one of the first time-dependent tests of the correspondence principle between a holographic theory and its gravitational dual.

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References

[1] J. Polchinski, “What is String Theory?,” hep-th/9411028.

[2] J. McGreevy and H. Verlinde, “Strings from Tachyons: The c = 1 Matrix Reloaded,” JHEP 12 (2003) 054, hep-th/0304224.

[3] E. J. Martinec, “The annular report on non-critical string theory,” hep-th/0305148.

[4] I. R. Klebanov, J. Maldacena, and N. Seiberg, “D-brane decay in two-dimensional string theory,” JHEP 07 (2003) 045, hep-th/0305159.

[5] J. McGreevy, J. Teschner, and H. Verlinde, “Classical and quantum D-branes in 2D string theory,” JHEP 01 (2004) 039, hep-th/0305194.

[6] Y. Nakayama, “Liouville field theory: A decade after the revolution,” Int. J. Mod. Phys. A19 (2004) 2771–2930, hep-th/0402009.

[7] E. J. Martinec, “Matrix models and 2D string theory,” hep-th/0410136.

[8] E. Witten, “On string theory and black holes,” Phys. Rev. D44 (1991) 314–324.

[9] G. Mandal, A. M. Sengupta, and S. R. Wadia, “Classical solutions of two-dimensional string theory,” Mod. Phys. Lett. A6 (1991) 1685–1692.

[10] G. W. Moore, M. R. Plesser, and S. Ramgoolam, “Exact S-Matrix for 2-D String Theory,” Nucl. Phys. B377 (1992) 143–190, hep-th/9111035.

[11] M. R. Douglas et al., “A New Hat for the c = 1 Matrix Model,” hep-th/0307195.

[12] T. Takayanagi and N. Toumbas, “A matrix model dual of type 0B string theory in two dimensions,” JHEP 07 (2003) 064, hep-th/0307083.

[13] M. Natsume and J. Polchinski, “Gravitational Scattering in the c = 1 Matrix Model,” Nucl. Phys. B424 (1994) 137–154, hep-th/9402156.

[14] J. Polchinski, “On the Nonperturbative Consistency of d = 2 String Theory,” Phys. Rev. Lett. 74 (1995) 638–641, hep-th/9409168.

[15] S. W. Hawking, “Particle creation by black holes,” Commun. Math. Phys. 43 (1975) 199–220.
[16] J. L. Karczmarek, A. Maloney, and A. Strominger, “Hartle-Hawking vacuum for c = 1 tachyon condensation,” hep-th/0405092.

[17] S. R. Das, J. L. Davis, F. Larsen, and P. Mukhopadhyay, “Particle production in matrix cosmology,” Phys. Rev. D70 (2004) 044017, hep-th/0403275.

[18] P. Mukhopadhyay, “On the problem of particle production in c = 1 matrix model,” JHEP 08 (2004) 032, hep-th/0406029.

[19] E. Martinec and K. Okuyama, “Scattered results in 2D string theory,” JHEP 10 (2004) 065, hep-th/0407136.

[20] J. Karczmarek, J. Maldacena, and A. Strominger, To appear.

[21] P. H. Ginsparg and G. W. Moore, “Lectures on 2-D gravity and 2-D string theory,” hep-th/9304011.

[22] K. Schoutens, H. Verlinde, and E. Verlinde, “Quantum Black Hole Evaporation,” Phys. Rev. D48 (1993) 2670–2685, hep-th/9304128.

[23] Y. Kiem, H. Verlinde, and E. Verlinde, “Black Hole Horizons and Complementarity,” Phys. Rev. D52 (1995) 7053–7065, hep-th/9502074.

[24] S. B. Giddings and W. M. Nelson, “Quantum emission from two-dimensional black holes,” Phys. Rev. D46 (1992) 2486–2496, hep-th/9204072.