A SHORT PROOF OF THE UNOBSTRUCTEDNESS OF PAIRS
(VARIETY WITH TRIVIAL CANONICAL BUNDLE, LINE BUNDLE)

DONATELLA IACONO AND MARCO MANETTI

Abstract. We investigate the deformations of pairs \((X, L)\), where \(L\) is a line bundle on a smooth projective variety \(X\), defined over an algebraically closed field \(K\) of characteristic 0. In particular, we prove that the DG-Lie algebra controlling the deformations of the pair \((X, L)\) is homotopy abelian whenever \(X\) has trivial canonical bundle, and so these deformations are unobstructed.

1. Introduction

Let \(X\) be a smooth projective variety with trivial canonical bundle defined over an algebraically closed field \(K\) of characteristic 0. It is well known that the deformations of \(X\) are unobstructed by the Bogomolov-Tian-Todorov Theorem.

The aim of this paper is to give a short and purely algebraic proof that, for every line bundle \(L\) on \(X\) as above, the deformations of pairs \((X, L)\) are unobstructed.

Theorem 1.1. Let \(L\) be a line bundle on a smooth projective variety \(X\) defined over an algebraically closed field \(K\) of characteristic 0. If \(X\) has trivial canonical bundle, then the pair \((X, L)\) has unobstructed deformations.

This result was inspired by the ideas contained in [IM17] and, mainly, by the very recent preprint [LP19]. In the fifth version of [LP19], the authors proved the above theorem in a completely different way over the field of complex numbers, while in the previous versions of the preprint the same result was proved under some additional assumptions on the pair \((X, L)\). Their proof is also valid for varieties with torsion canonical bundle and relies on Beauville-Bogomolov decomposition Theorem and on some explicit computations on the Hodge loci in the period domain.

Here, we show Theorem 1.1 by investigating the differential graded Lie algebra (DG-Lie algebra) associated with the deformation problem. A DG-Lie algebra is homotopy abelian if it is quasi-isomorphic to an abelian DG-Lie algebra. This implies that the associated deformation functor is isomorphic to the deformation functor associated with an abelian DG-Lie algebra that is smooth. Therefore, if a geometric deformation problem is controlled by an homotopy abelian DG-Lie algebra, then it is smooth, i.e., the geometric deformation problem is unobstructed.

For example, Bogomolov-Tian-Todorov Theorem is a consequence of the stronger fact that the DG-Lie algebra associated with the infinitesimal deformations of \(X\) is homotopy abelian [GM90, Ma04, IM10, Ia17]. In [KKP08, Ia15], the authors extended this result to the deformations of pairs \((X, \Delta)\), showing in particular the homotopy abelianity of the DG-Lie algebra associated with the infinitesimal deformations of a log Calabi-Yau pair \((X, \Delta)\), i.e., \(\Delta\) is a smooth anticanonical divisor in a smooth projective variety \(X\). Following these ideas, we are able to prove the following stronger result that implies Theorem 1.1.

Date: February 28, 2019.

2010 Mathematics Subject Classification. 14D15, 17B70, 13D10, 32G08.

Key words and phrases. Deformations of manifold and line bundle, differential graded Lie algebras.
Theorem 1.2. Let $L$ be a line bundle on a smooth projective variety $X$ defined over an algebraically closed field $\mathbb{K}$ of characteristic 0. If $X$ has trivial canonical bundle, then the DG-Lie algebra controlling the deformations of the pair $(X, L)$ is homotopy abelian.

It is well known (see e.g. [Hu95, IM17]) that the DG-Lie algebra controlling the deformations of the pair $(X, L)$ is the algebra $R\Gamma(X, D^1(L))$ of the derived sections of the sheaf of first-order differential operators on $L$: this is an object in the homotopy category of DG-Lie algebras and then it is represented by a DG-Lie algebra up to quasi-isomorphism. Over the complex numbers, a possible representative of $R\Gamma(X, D^1(L))$ is given by the Dolbeault resolution of $D^1(L)$ [Mar12, Example 2.12].

However, in this paper we work over any algebraically closed field $\mathbb{K}$ of characteristic 0 and so we adopt the purely algebraic construction, described in [IM17, Sections 6 and 7], of the Thom-Whitney-Sullivan totalization with respect to any affine open cover.

The main idea behind the proof of Theorem 1.2 is the following. Given a pair $(X, L)$, we construct a new pair $(Y, \Delta)$, where $Y$ is a $\mathbb{P}^1$-bundle on $X$ and $\Delta$ is a smooth divisor in $Y$. Whenever $X$ has trivial canonical bundle, the pair $(Y, \Delta)$ is a log Calabi-Yau pair.

Then, we conclude the proof showing that there exists a quasi isomorphism between the DG-Lie algebra controlling the deformations of the pair $(X, L)$ and the homotopy abelian DG-Lie algebra controlling the deformations of the pair $(Y, \Delta)$ (Lemma 2.2). Hence, we deduce Theorem 1.1.

2. The proof

Let $L$ be a line bundle on a smooth algebraic variety $X$ of dimension $n$ over an algebraically closed field $\mathbb{K}$ of characteristic 0 and denote by $\mathcal{L} = \mathcal{O}_X(L)$ the invertible sheaf of its sections. According to [IM17, Section 5], we denote by $\mathcal{D}(X, \mathcal{L})$ the sheaf of derivations of pairs which is the subsheaf of $\mathcal{D}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \times \mathcal{H}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L})$ consisting of pairs $(h, u)$ such that $u(ax) = h(a)x + au(x)$ for every $a \in \mathcal{O}_X$ and $x \in \mathcal{L}$, i.e.,

$$\mathcal{D}(X, \mathcal{L}) = \{(h, u) \in \mathcal{D}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X) \times \mathcal{H}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) | u(\alpha x) - au(e) = h(a)e, \forall a \in \mathcal{O}_X, e \in \mathcal{L}\}.$$ 

It is almost immediate to see that $\mathcal{D}(X, \mathcal{L})$ is a sheaf of Lie algebras over $\mathbb{K}$ and that the projection on the second factor $(h, u) \mapsto u$ induces an isomorphism with the sheaf $D^1(\mathcal{L})$ of first-order differential operators [IM17, Example 5.2]. In particular $\mathcal{D}(X, \mathcal{L})$ is locally free of rank $n + 1$ and there exists the following exact sequence

$$0 \to \mathcal{H}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \to \mathcal{D}(X, \mathcal{L}) \to \Theta_X \to 0,$$

where $\Theta_X = \mathcal{D}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$ denotes the tangent sheaf of $X$. Consider the $\mathbb{P}^1$-bundle

$$p: Y = \mathbb{P}(\mathcal{O}_X \oplus \mathcal{L}) = \mathbb{P}(\mathcal{L}^{-1} \oplus \mathcal{O}_X) \to X$$

together with the two distinguished sections $\Delta_0$ and $\Delta_\infty$ corresponding to the two direct summands, namely:

$$Y = \Delta_\infty \cup \text{Spec}_X(\oplus_{n \geq 0} \mathcal{L}^n) = \Delta_0 \cup \text{Spec}_X(\oplus_{n \leq 0} \mathcal{L}^n).$$

If $\Delta = \Delta_0 + \Delta_\infty$, then we have the adjunction formula $p^*K_X = K_Y + \Delta$: this follows from the relative Euler exact sequence [Ha77, Exercise III.8.4]. It can be also proved by noticing that if $\omega$ is a rational $n$-form on $X$ then $p^*\omega \wedge \frac{dt}{t}$, where $t$ is a local coordinate frame on the fibres of $L$, is a well defined rational $n + 1$-form on $Y$. Note that if $X$ has trivial canonical bundle, then $\Delta$ is an anticanonical divisor in $Y$, i.e., $(Y, \Delta)$ is a log Calabi-Yau pair.

We denote by $\Theta_Y$ the tangent sheaf of $Y$ and by $\Theta_Y(- \log \Delta)$ the subsheaf of vector fields that are tangent to the smooth divisor $\Delta$. Note that $\Theta_Y(- \log \Delta)$ is the subsheaf of the derivations of the sheaf $\mathcal{O}_Y$ preserving the ideal sheaf of $\Delta$. Moreover, since $\Delta \subset Y$ is smooth, there exists the following exact sequence

$$0 \to \Theta_Y(- \log \Delta) \to \Theta_Y \to N_{\Delta/Y} \to 0.$$
Lemma 2.1. In the above notation $R^i p_* \Theta_Y(- \log \Delta) = 0$ for every $i > 0$ and there exists a natural $\mathcal{O}_X$-linear isomorphism of sheaves of Lie algebras

$$\Psi : D(X, L) \xrightarrow{\cong} p_* \Theta_Y(- \log \Delta).$$

Proof. In the sequel, we shall denote by $U = Y - \Delta_{\infty}$ the total space of the dual bundle of $L$. Assume first that $X = \text{Spec} \, A$ is an affine scheme and that $L$ is the trivial line bundle. Thus $Y = \mathbb{P}^1 \times X$, $\Delta = \{0, \infty\} \times X$ and then

$$\Theta_Y(- \log \Delta) = p^* \Theta_X \oplus q^* \Theta_{\mathbb{P}^1}(-0 - \infty),$$

where $q$ is the projection onto $\mathbb{P}^1$. Since $\Theta_{\mathbb{P}^1}(-0 - \infty)$ is trivial, we have that $\Theta_Y(- \log \Delta) = p^* \Theta_X \oplus \mathcal{O}_Y$. Since $p_* \mathcal{O}_Y = \mathcal{O}_X$ and $R^i p_* \mathcal{O}_Y = 0$ for $i > 0$ [Ha77, Exercise III.8.4], by the projection formula [Ha77, Exercise III.8.3] we have

$$p_* \Theta_Y(- \log \Delta) = \Theta_X \oplus \mathcal{O}_X, \quad R^i p_* \Theta_Y(- \log \Delta) = 0, \quad i > 0.$$

We point out that $p_* \Theta_Y(- \log \Delta)$ is a locally free sheaf of rank $n + 1$, whose sections are of type $\chi + a \frac{d}{dt}$, where $\chi \in \Theta_X$, $a \in \mathcal{O}_X$ and $t$ is a linear coordinate on the fibres of $L$.

We have $U = \text{Spec} \, R$, $R = \Gamma(X, \oplus_{n \geq 0} \mathcal{L}_n)$; the choice of an isomorphism $z : \mathcal{O}_X \rightarrow \mathcal{L}$ provides an isomorphism of $A$-algebras $R = A[z]$. In this setting, there exists a unique $A$-linear morphism of Lie algebras

$$\Psi : \Gamma(X, D(X, L)) \rightarrow \Gamma(X, p_* \mathcal{O}_U) = \Gamma(U, \mathcal{O}_U) = \text{Der}_R(R, R),$$

such that $\Psi(h, u)(a) = h(a)$ and $\Psi(h, u)(m) = u(m)$ for every $a \in A \subset R$ and every $m \in \mathcal{L}$. The unicity is clear by Leibniz formula: for the existence, using the isomorphism $R = A[z]$ it is sufficient to define

$$\Psi(h, u) = h + u(z) \frac{d}{dz}.$$ 

We have $\left(h + u(z) \frac{d}{dz}\right)(a) = h(a)$ for every $a \in A$. Every section of $\mathcal{L}$ is of type $az$ for some $a \in A$ and then

$$\left(h + u(z) \frac{d}{dz}\right)(az) = h(a)z + u(z)a = u(az).$$

Notice that, since $u(z) = zk$ for some $k \in A$, the vector field $\Psi(h, u)$ is tangent to $\Delta$ and then belongs to $\Gamma(X, p_* \Theta_Y(- \log \Delta))$.

The local unicity allows to glue the morphisms $\Psi$ on affine subsets to a morphism of quasi-coherent sheaves $D(X, L) \rightarrow p_* \Theta_U$ whose image is contained in $p_* \Theta_Y(- \log \Delta)$. Moreover, the explicit local description of $\Psi$ implies that $\Psi : D(X, L) \rightarrow p_* \Theta_Y(- \log \Delta)$ is an isomorphism of locally free sheaves of rank $n + 1$. \hfill \Box

Given a coherent sheaf of Lie algebra $\mathcal{F}$ over $X$, we denote by $R \Gamma(X, \mathcal{F})$ the DG-Lie algebra of derived sections. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open affine cover of $X$, we denote by $C^*(\mathcal{U}, \mathcal{F})$ the Čech complex of $\mathcal{F}$, i.e., the cochain complex associated with the semicosimplicial Lie algebra:

$$\mathcal{F}(\mathcal{U}) : \prod_i \mathcal{F}(U_i) \xrightarrow{\partial_h} \prod_{i,j} \mathcal{F}(U_{ij}) \xrightarrow{\partial_h} \prod_{i,j,k} \mathcal{F}(U_{ijk}) \xrightarrow{\partial_h} \cdots,$$

where the face operators $\partial_h : \prod_{i_0, \ldots, i_{k-1}} \mathcal{F}(U_{i_0 \ldots i_{k-1}}) \rightarrow \prod_{i_0, \ldots, i_{k}} \mathcal{F}(U_{i_0 \ldots i_k})$ are given by

$$\partial_h(x)_{i_0 \ldots i_k} = x_{i_0 \ldots i_{k-1}}|_{U_{i_0 \ldots i_k}}, \quad \text{for } h = 0, \ldots, k.$$

An explicit model of $R \Gamma(X, \mathcal{F})$ is given by the Thom-Whitney totalization $\text{Tot}(\mathcal{U}, \mathcal{F})$ associated with the semicosimplicial Lie algebra $\mathcal{F}(\mathcal{U})$, see e.g. [FMM12, IM17]. Note that the homotopy class of the DG-Lie algebra $\text{Tot}(\mathcal{U}, \mathcal{F})$ does not depend on the choice of the
open affine cover and, by Whitney’s theorem (see e.g. [IM10, Sec. 2]), there exists a canonical quasi-isomorphism of complexes

\[ I: \text{Tot}(\mathcal{U}, \mathcal{F}) \rightarrow C^*(\mathcal{U}, \mathcal{F}). \]

As we already point out, the sheaf of Lie algebras \( \mathcal{D}(X, \mathcal{L}) \) is isomorphic to the sheaf \( \mathcal{D}^1(L) \), and so the DG-Lie algebra \( R\Gamma(X, \mathcal{D}(X, \mathcal{L})) \) controls the deformations of the pair \( (X, L) \) [IM17, Theorem 7.5]. As regard the deformations of the pair \( (Y, \Delta) \), these are controlled by the DG-Lie algebra \( R\Gamma(Y, \Theta_Y(-\log \Delta)) \) [KKP08, Section 4.3.3 (i)] or [Ia15, Theorem 4.3].

**Lemma 2.2.** The morphism \( \Psi: \mathcal{D}(X, \mathcal{L}) \rightarrow p_*\Theta_Y(-\log \Delta) \) induces a quasi-isomorphism of DG-Lie algebras

\[ \Psi: R\Gamma(X, \mathcal{D}(X, \mathcal{L})) \rightarrow R\Gamma(Y, \Theta_Y(-\log \Delta)). \]

Therefore the DG-Lie algebra controlling the deformations of the pair \( (X, L) \) is quasi-isomorphic to the DG-Lie algebra controlling the deformations of the pair \( (Y, \Delta) \).

**Proof.** Let \( \mathcal{U} = \{ U_i \}_{i \in I} \) be an open affine cover of \( X \) and take an open affine cover \( \mathcal{V} = \{ V_j \}_{j \in J} \) of \( Y \) together with a refining map \( r: J \rightarrow I \) such that \( p(V_j) \subset U_{r(j)} \) for every \( j \).

The above data give a morphism of Čech complexes

\[ C^*(\mathcal{U}, p_*\Theta_Y(-\log \Delta)) \rightarrow C^*(\mathcal{V}, \Theta_Y(-\log \Delta)) \]

which is a quasi-isomorphism by Leray spectral sequence (see e.g., [Vo12, Theorem 16.11]). Therefore, the morphism \( \Psi \) of Lemma 2.1 gives a quasi-isomorphism of Čech complexes

\[ \Psi: C^*(\mathcal{U}, \mathcal{D}(X, \mathcal{L})) \rightarrow C^*(\mathcal{V}, \Theta_Y(-\log \Delta)). \]

Similarly, \( \Psi \) and the refining map induce a morphism of semicosimplicial Lie algebra

\[ \mathcal{D}(X, \mathcal{L})(\mathcal{U}) \rightarrow \Theta_Y(-\log \Delta)(\mathcal{V}) \]

and so a DG-Lie algebras morphism of the Thom-Whitney totalizations

\[ \Psi: \text{Tot}(\mathcal{U}, \mathcal{D}(X, \mathcal{L})) \rightarrow \text{Tot}(\mathcal{V}, \Theta_Y(-\log \Delta)), \]

which is a quasi-isomorphism by Whitney’s Theorem. \( \square \)

If \( X \) has trivial canonical bundle then \( (Y, \Delta) \) is a log Calabi-Yau pair, thus Theorem 1.2 is an immediate consequence of Lemma 2.2 and of the following theorem [Ia15, Corollary 5.4] or [KKP08, Lemma 4.19], cf. [Ia17, Sec. 4.2].

**Theorem 2.3.** Let \( Y \) be a smooth projective variety defined over an algebraically closed field of characteristic 0 and \( \Delta \subset Y \) a smooth divisor. If \( (Y, \Delta) \) is a log Calabi-Yau pair, then the DG-Lie algebra \( R\Gamma(Y, \Theta_Y(-\log \Delta)) \) is homotopy abelian.

Theorem 1.2 implies that DG-Lie algebra \( R\Gamma(X, \mathcal{D}(X, \mathcal{L})) \) controlling the deformations of the pair \( (X, L) \) is homotopy abelian, whenever \( X \) has trivial canonical bundle. This immediately implies Theorem 1.1, i.e., if \( X \) has trivial canonical bundle, then the pair \( (X, L) \) has unobstructed deformations.

**References**

[FMM12] D. Fiorenza, M. Manetti and E. Martinengo: *Cosimplicial DGLAs in deformation theory*, Communications in Algebra 40 (2012), 2243-2260; arXiv:0803.0399. 3

[GM90] W.M. Goldman and J.J. Millson: *The homotopy invariance of the Kuranishi space*, Illinois J. Math., 34, (1990), 337-367. 1

[Ha77] R. Hartshorne: Algebraic geometry. Graduate texts in mathematics 52, Springer-Verlag, New York/Berlin, (1977). 2, 3

[Hu95] L. Huang: *On joint moduli spaces*, Math. Ann. 302 (1995), 61-79. 2

[Ia15] D. Iacono: *Deformations and obstructions of pairs \((X,D)\)*, Internat. Math. Res. Notices (IMRN), Volume 2015, Issue 19, 9660-9695; arXiv:1302.1149 1, 4

[Ia17] D. Iacono: *On the abstract Bogomolov-Tian-Todorov theorem*, Rend. Mat. Appl. 38, (2017), 175-198; www1.mat.uniroma1.it/ricerca/rendiconti/ 1, 4
A SHORT PROOF OF THE UNOBSTRUCTEDNESS OF PAIRS $(X, L)$

[IM10] D. Iacono and M. Manetti: An algebraic proof of Bogomolov-Tian-Todorov theorem, Deformation Spaces, 39, (2010), 113-133; arXiv:0902.0732 [math.AG].

[IM17] D. Iacono and M. Manetti: On deformations of pairs (manifold, coherent sheaf), to appear in Canad. J. Math., arXiv:1707.06612.

[KKP08] L. Katzarkov, M. Kontsevich and T. Pantev: Hodge theoretic aspects of mirror symmetry, From Hodge theory to integrability and TQFT tt*-geometry, Proc. Sympos. Pure Math., 78, Amer. Math. Soc., Providence, (2008), 87-174; arXiv:0806.0107.

[Ma04] M. Manetti: Lectures on deformations of complex manifolds, Rend. Mat. Appl., (7) 24, (2004), 1-183; arXiv:math.AG/0507286.

[Mar12] E. Martinengo: Infinitesimal deformations of Hitchin pairs and Hitchin map, Internat. J. Math., 23 no. 7, (2012), 30pp.; arXiv:1003.5531.

(LP19) L. Shizhang and P. Xuanyu: Unobstructedness of deformations of Calabi-Yau varieties with a line bundle; preprint arXiv:1310.7162v5 (2019).

[Vo12] C. Voisin: Théorie de Hodge et géométrie algébrique complexe. Société Mathématique de France, Paris (2002).

Università degli Studi di Bari,
Dipartimento di Matematica,
Via E. Orabona 4, I-70125 Bari, Italy.
E-mail address: donatella.iacono@uniba.it
URL: www.dm.uniba.it/~iacono/

Università degli studi di Roma “La Sapienza”,
Dipartimento di Matematica “Guido Castelnuovo”,
P.le Aldo Moro 5, I-00185 Roma, Italy.
E-mail address: manetti@mat.uniroma1.it
URL: www.mat.uniroma1.it/people/manetti/