When one of the space-time dimension is compactified on $S^1$, the QCD exhibits the chiral phase transition at some critical radius. When we further turn on a background $\theta$ term which depends on the $S^1$ compactified coordinate, a topological ordered phase appears at low energy via the winding of $\theta$. We discuss what kind of theories can describe the physics near the critical point by requiring the matching of topological field theories in the infrared. As one of the possibilities, we propose a scenario where the $\rho$ and $\omega$ mesons form a $U(N_f)$ gauge theory near the critical point. In the phase where the chiral symmetry is restored, they become the dual gauge boson of the gluon related by the level-rank duality between the three dimensional gauge theories, $SU(N)_{N_f}$ and $U(N_f)-N$. 
1 Introduction

The low energy limit of QCD is described by pions whose properties and interactions have information of the global symmetry of QCD. The lowest dimensional interaction terms can be determined once we know the coset space of which the pions are the coordinate. Also, the inconsistency in gauging a part of the global symmetry, i.e., the 't Hooft anomaly, is encoded in the Wess-Zumino (WZ) term in the low energy effective Lagrangian [1,2].

The long distance behavior is also important near the critical point of a phase transition. If the phase transition is smooth enough, one can consider an effective theory of an order parameter which obtains a vacuum expectation value (VEV) in the broken phase. The global structure of the theory such as anomaly, should also be kept in the effective theory for consistency. Moreover, the 't Hooft anomaly results in a matching conditions which constrain the realization of the vacuum structure and the infrared degrees of freedom [3–5]. In the case of the finite temperature QCD above the QCD scale, such an anomaly matching is usually trivially satisfied. The finite temperature system can be regarded as an $S^1$ compactified QCD, and the effective theory is, therefore, a three dimensional theory that has no chiral anomaly.

The study of phase transitions in the three dimensional gauge theory has a long history [6–16] with some important recent developments [17–30]. Even though there is no chiral anomaly, there are topological orders in the low energy effective theories. For example, $SU(N)$ gauge theory can have the Chern-Simons (CS) term with an integer level, $k$. The CS term dominates the infrared physics, and it reduces to the CS theory, denoted as $SU(N)_k$, which has a gap, but the Wilson lines have non-trivial values depending on their topology. This non-trivial topological behavior should be matched when we discuss the effective description near the phase transition. Based on this discussion, the dualities between CS theories coupled to fermions and those coupled to bosons have been proposed and checked. The precise forms of the dualities are listed in Ref. [18]. Based on these dualities, the phase diagrams of the three dimensional QCD (QCD$_3$) have been discussed [24]. In particular, it has been conjectured that the $SU(N)_k$ theory with $N_f (> 2|k|)$ fermions undergoes the symmetry breaking $U(N_f) \rightarrow U(N_f/2 + k) \times U(N_f/2 - k)$ when the fermion masses are smaller than some critical value. There is a phase transition between the symmetry broken and unbroken phases as the fermion masses are varied. If the transition is of the second order, near the critical point, there is a dual description of the theory by the $U(N_f/2 + k)_N$ or $U(N_f/2 - k)_N$ theory coupled to $N_f$ scalar fields whose Higgs phenomenon describes the phase transition.
In the unbroken phase in QCD$_3$ the CS theory describes the low energy physics. The fermionic theory flows to the $SU(N)_{\pm N_f/2+k}$ theory while it is the $U(N_f/2 \pm k)_{-N}$ theory in the bosonic theory. These two theories are related by the level-rank duality and are known to give the same physics. It is quite not trivial that the matching of the low energy limit is realized in this way.

In this paper, motivated by the symmetry breaking and its dual description in QCD$_3$, we discuss the low energy limits of the $S^1$ compactified QCD in four space-time dimensions $[24, 31, 64]$ with the hope that the three dimensional duality may give us some hints of the four dimensional physics. Indeed for abelian gauge theories it has been pointed out that the three dimensional duality is lifted up to the S-duality in the four dimensional theory $[65]$. We consider a background $\theta$ term which depends on the coordinate of the $S^1$ direction. In particular, the function $\theta$ can have windings along the $S^1$ direction $[66, 67]$, which determines the CS level in the three dimensional effective theory. As the most interesting example, one can take the background with the winding number, $N_f$, which is the number of quark flavors in four dimensions. For a small radius, one finds that the low energy theory is $SU(N)_{N_f}$ where the vacuum is gapped. At a large radius, the theory is better described by the chiral Lagrangian for pions with the WZ terms. Since the low energy limits are different, there must be a phase transition at some critical radius. The background $\theta$ induces a winding of the $\eta$ meson ($\eta'$ in the three-flavor language), which gives rise to the non-trivial WZ term in the three dimensional theory. The two limits of the theory can be interpolated by the Higgs mechanism of the $U(N_f)_{-N}$ theory coupled to $2N_f$ flavors of scalar fields. Here, again, in the unbroken phase, the theory is related by the level-rank duality. If this picture is correct, the natural candidate for this dual $U(N_f)$ gauge bosons are the $\rho$ and the $\omega$ mesons (in the spirit of Ref. $[68]$), which we know, phenomenologically, to be successfully described in terms of gauge fields $[69, 75]$. See also $[76, 78]$ for the interpretation of the Seiberg duality as the gauge theory of vector mesons. The “gauge bosons” are massive anyway by the CS term even in the unbroken phase in the three dimensional effective theory. Therefore, they are not quite effective degrees of freedom, but their existence is important for having a non-trivial topological order required from the matching of the low energy physics.

We discuss the similarities between the QCD$_3$ for small $k$ and QCD$_4$, and propose an exotic possibility that the $\rho$ and $\omega$ mesons continuously becoming a dual gauge boson of gluons related by the level-rank duality. This conjectured picture is at least consistent when the winding of $\theta$ is less than $N_f$, especially, when it is zero at which case the system can be regarded as the finite temperature QCD.
The paper is organized as follows: In section 2 we discuss the pattern of the flavor symmetry breaking in QCD$_3$ with $2N_f$ fermions and its breaking to the chiral symmetry group associated with QCD$_4$. We identify the order parameter which leaves the correct Nambu-Goldstone bosons massless to match the one of QCD$_4$. The corresponding effective low energy theory is written down with an emphasis on the CS terms and the related WZ terms, also in the presence of external gauge fields. In particular, we identify the term associated with baryon number. These terms are related to the flavor anomalies in QCD$_4$. They were recently discussed also by Komargodski in [28]. In section 3 we put QCD$_4$ on $M^3 \times S^1$ including a $\theta$ angle which winds along $S^1$ and background gauge fields which depend on the $S^1$ coordinate. We explore the theory for small radius ($\Lambda_4 R \ll 1$) and large radius ($\Lambda_4 R \gg 1$) and argue for the existence of a phase transition at some critical $R_*$. In section 4 we speculate/conjecture on the possible behavior of the hadronic vector mesons ($\rho$, $\omega$, ...) near the critical point and the nature of the theory at the critical point. In particular, we put forward a scenario in which the hadronic vector mesons become gauge bosons and give rise to an $U(N_f)$ gauge theory at the critical point. In section 5 we present holographic QCD-like models represented by quiver diagrams which capture this conjectured scenario of vector mesons as gauge bosons. In section 6 we comment briefly on the implications of our study on finite temperature QCD and possible scenarios for the nature of the phase transition at $T_* = 1/R_*$. Lattice simulations may decide between these various scenarios. Section 7 is devoted to discussion. There are three appendices. In Appendix A we write down the Lagrangian associated with the quiver diagram in section 5. Appendix B addresses the issue of the integration over $S^1$ when a winding $\theta$ term is present. In Appendix C we discuss the issue of the Baryon number and the (winding) configuration of $\eta$ ($\eta'$ for $N_f > 2$) by considering the WZ term and the associated anomaly of chiral $U(1)_A$ current, $U(1)_{EM}$ electromagnetic current and the $U(1)$ baryon number. In particular we can consider the situation that $\eta$ winding happens on a finite sheet and localized on $S^1$. The sheet configuration is just the Hall droplet studied in [28]. Following [28] we can identify the Baryonic configuration which resides on the boundary of the finite region. It is “amusing” to note that one can also identify a non-local configuration which corresponds to the quark with its Wilson line going into the bulk. In this sense the quark appears as a “soliton” in the hadronic effective theory.
2 QCD$_3$ and chiral symmetry breaking

Based on the studies of the three dimensional dualities, it has been conjectured that the low energy theory of three dimensional $SU(N)_0$ QCD with $2N_f$ fermions is described by a non-linear sigma model with the target space,

$$ \frac{U(2N_f)}{U(N_f) \times U(N_f)} $$

(1)

for small enough fermion masses [24,79]. There is an upper bound on $N_f$ although the precise location is unknown [80]. It is noted that an appropriate WZ term should be added in the Lagrangian. The same low energy theory is obtained by a linear sigma model with $U(N_f)$ or $U(N_f) - N$ gauge theory coupled to $2N_f$ scalar fields. Beyond the critical value of the fermion mass, both the fermionic and bosonic theories flow to topological field theories related by the level-rank duality, $SU(N)_{\pm N_f} \leftrightarrow U(N_f)_{\pm N}.$

We would like to discuss the relation between this symmetry breaking phenomena and the chiral symmetry breaking in QCD$_4$. In QCD$_4$ with $N_f$ Dirac fermions (corresponding to $2N_f$ fermions in QCD$_3$), the low energy theory is a non-linear sigma model with

$$ \frac{SU(N_f)_L \times SU(N_f)_R}{SU(N_f)_{L+R}}. $$

(2)

This coset space is a subspace of (1). One can reduce the space (1) to (2) by adding an explicit breaking terms in the Lagrangian.

By denoting $\psi$ and $\bar{\psi}$ as $N_f + N_f$ flavors of the QCD$_3$, one can, for example, introduce the explicit breaking term by coupling a massive adjoint scalar field $a_3$ to QCD$_3$,

$$ L_{a_3} = -\bar{\psi}a_3\psi + \bar{\tilde{\psi}}a_3\tilde{\psi}, $$

(3)

this interaction breaks the $U(2N_f)$ symmetry to $U(N_f) \times U(N_f)$. In QCD$_4$, the role of $a_3$ is played by the extra component of the gauge boson.

The symmetry breaking pattern in Eq. (1) suggests that the order parameter is

$$ \langle \bar{\psi}\psi - \bar{\tilde{\psi}}\tilde{\psi} \rangle $$

(4)

and/or

$$ \langle \bar{\tilde{\psi}}\tilde{\psi} \rangle $$

(5)

which are not equivalent once the $U(2N_f)$ breaking terms are introduced. If the latter VEV is vanishing, the $U(N_f) \times U(N_f)$ symmetry is left unbroken since the former VEV is invariant
under the symmetry. In that case, all the Nambu-Goldstone modes obtain masses by the explicit breaking term in Eq. (3).

By embedding this model to a four dimensional theory, one can show that the former VEV is vanishing along the line with the Bloch theorem which states that there is no spontaneous electric current in the ground state [81, 82]. As we will see later, this three dimensional model can be obtained as the low energy effective theory of the $S^1$ compactified QCD$_4$ with $N_f$ Dirac fermions, $\Psi$. The model with massless fermions can be realized at a meta-stable vacuum when $N_f < N$. The periodic boundary condition is taken for $\Psi$. In this language, the former and the latter VEVs correspond to $\langle \bar{\Psi}\gamma^3\Psi \rangle$ and $\langle \bar{\Psi}\Psi \rangle$, respectively, where $x_3$ is the $S^1$ direction. When we consider modifying the model by adding $\partial_3 \epsilon \bar{\Psi} \gamma^3 \Psi$ to the Lagrangian with a real function $\epsilon$, the vacuum energy density at the meta-stable vacuum is given by

$$E = E_0 - \partial_3 \epsilon \langle \bar{\Psi}\gamma^3\Psi \rangle + O(\epsilon^2),$$

where $E_0$ is the vacuum energy density for $\epsilon = 0$, and the VEV is the one at $\epsilon = 0$. If $\langle \bar{\Psi}\gamma^3\Psi \rangle$ is non-vanishing, the vacuum energy density is locally shifted by $O(\epsilon)$. On the other hand, one can eliminate the $\partial_3 \epsilon \bar{\Psi}\gamma^3\Psi$ term in the Lagrangian by an appropriate redefinition of the field,

$$\Psi \rightarrow e^{i\epsilon} \Psi.$$  

The boundary condition of $\Psi$ is maintained by taking $\epsilon$ a periodic function up to an $2\pi$ shift. In this basis, there is no shift of the vacuum energy density. Therefore, $\langle \bar{\Psi}\gamma^3\Psi \rangle \neq 0$ is inconsistent, and thus the latter VEV, $\langle \bar{\psi}\tilde{\psi} \rangle$, should be chosen.

For $\langle \bar{\psi}\tilde{\psi} \rangle \neq 0$, the $U(N_f) \times U(N_f)$ chiral symmetry is broken to $U(N_f)$, leaving a part of the Nambu-Goldstone modes massless and matches to the QCD$_4$ up to an anomalous axial $U(1)_A$ that can also be explicitly broken. Therefore, it is possible that the chiral symmetry breaking in QCD$_4$ has something to do with the phase transition in QCD$_3$ where there is a dual description by the Higgs mechanism of $U(N_f)_{\pm N}$ gauge theory. In QCD$_4$, it is well-known that the physics of the vector mesons, $\rho$ and $\omega$, is nicely described by the color-flavor locked phase of $U(N_f)$ gauge theory. The structure of the dual bosonic theory in three dimension is indeed of this type as we see below.

Let us look at the low energy effective theory. We first discuss the theory without the explicit breaking terms in Eq. (3). In this case, the non-zero VEV $\langle \bar{\psi}\tilde{\psi} \rangle$ breaks $U(2N_f)$ to $U(N_f) \times U(N_f)$. In the dual $U(N_f)_{\pm N}$ bosonic theory, this symmetry breaking corresponds to a color-flavor locked phase: the $U(N_f)$ gauge symmetry is completely Higgsed, and the
\(U(2N_f)\) flavor symmetry is broken, while a part of \(U(2N_f)\) flavor rotation together with a \(U(N_f)\) gauge rotation is unbroken. The unbroken flavor symmetry is \(U(N_f) \times U(N_f)\). The pions in this symmetry breaking have the WZ term as discussed in Ref. \[24\]. The WZ term can be obtained from the CS term in the bosonic \(U(N_f)_{\pm N}\) theory. The uneaten \(2N_f^2\) Nambu-Goldstone fields in Eq. \[1\] are introduced as a \(2N_f \times 2N_f\) matrix:

\[
\xi = \exp \left[ i \frac{\sqrt{2}}{2} \pi^a \begin{pmatrix} 0 & T^a \\ T^a & 0 \end{pmatrix} + i \frac{\sqrt{2}}{2} \tilde{\pi}^a \begin{pmatrix} 0 & iT^a \\ -iT^a & 0 \end{pmatrix} \right], \tag{8}
\]

where \(T^a\) are the generators of \(U(N_f)\) group. The Higgsed \(U(N_f)\) gauge field, \(b_\mu\) couples to \(\xi\) as

\[
\mathcal{L}_{\text{NLSM}} = \frac{f^2}{4} \left| \partial_\mu \xi + i \xi \begin{pmatrix} b_\mu & 0 \\ 0 & 0 \end{pmatrix} \right|^2. \tag{9}
\]

In the low-energy limit, the gauge field \(b_\mu\) can be integrated out, since this gauge field is massive by the Higgs mechanism. The equation of motion for \(b_\mu\) gives

\[
b_{ij} = (\xi^{-1} d\xi)_{ij}, \tag{10}
\]

where \(i, j = 1, \cdots, N_f\) run the first half of the \(2N_f\) indices. Substituting this into the CS term,

\[
S_{\text{CS}} = \pm \frac{N}{4\pi} \int_{M^3} \text{Tr} \left( bdb + \frac{2}{3} b^3 \right), \tag{11}
\]

one obtains the WZ terms.

We now introduce the explicit breaking term in Eq. \[3\]. A half of pions obtain masses when we include the interaction to break the \(U(2N_f)\) symmetry to the chiral symmetry. For example, one can introduce a spurion field,

\[
X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{12}
\]

and write down a \(U(2N_f)\) breaking term,

\[
\text{Tr} \left( \xi^{-1} X\xi X \right). \tag{13}
\]

This term gives a mass to \(\tilde{\pi}\) while leaving \(\pi\) massless.

In the low-energy limit where \(\tilde{\pi}^a\) are decoupled, one can set \(\tilde{\pi}^a = 0\), and the WZ term among pions vanishes. However, one can trace the existence of the WZ term by turning on
external gauge fields. Let us introduce the background gauge fields for the unbroken \( U(N_f) \) global symmetry \( A^\mu \), as the one which couples to the \( U(N_f) \) vector current, \( \bar{\psi} \gamma^\mu T^a \psi + \tilde{\bar{\psi}} \gamma^\mu T^a \tilde{\psi} \). The \( U(1) \) part is the baryon number. The equation of motion for \( b \) now gives, \( b = A + \cdots \), and thus we have

\[
S_{\text{WZ}} = \pm \frac{N}{4\pi} \int_{M^3} \text{Tr} \left( A dA + \frac{2}{3} A^3 \right) + \cdots .
\]  

In particular, we have a term

\[
\pm \frac{1}{4\pi} \int_{M^3} B \text{Tr}(dA),
\]

where \( B \) is the baryon number normalized such that the quarks have the charge \( 1/N \). The above term gives the baryon number \( (B = 1) \) for a monopole that has the unit magnetic charge of the \( U(1) \) subgroup of \( U(N_f) \). For example, the monopole made of the \((11)\) component of \( b \) has the baryon number \( B = 1 \).

### 3 QCD\(_4\) on a circle

We discuss the four-dimensional QCD compactified on \( S^1 \) and look for a relation to the phase transition in QCD\(_3\). We start with the action of \( SU(N) \) gauge theory coupled with massless \( N_f \) Dirac fermions, \( \Psi_i \), \((i = 1, \cdots, N_f)\), on \( M^3 \times S^1 \),

\[
S = \int_{M^3 \times S^1} d^4x \left[ -\frac{1}{2g^2} \text{Tr} (f_{MN} f^{MN}) + \frac{\theta(x_3)}{32\pi^2} \epsilon_{MNPQ} \text{Tr} (f^{MN} f^{PQ}) 
\right.
\]

\[
+ i \bar{\Psi} \gamma^M (\partial_M - ia_M) \Psi_i 
\]

\[
- \partial_M \alpha^i_L(x_3) \bar{\Psi} \gamma^M \Psi_i - \partial_M \alpha^i_R(x_3) \bar{\Psi} \gamma^M P_R \Psi_i \right].
\]  

The periodic boundary conditions are imposed on gauge fields. The Lorentz indices, \( M, N, P, Q \), run from 0 to 3, where \( x_3 \) is the \( S^1 \) direction. \( P_{L,R} \) are projection operators of chirality, \( P_{L,R} = (1 \mp \gamma_5)/2 \). We introduced \( x_3 \) dependent background fields, \( \theta(x_3) \) and \( \alpha^i_{L,R}(x_3) \). The boundary condition of the quarks are

\[
\Psi_i(x_3 + 2\pi R) = e^{i\nu} \Psi_i(x_3),
\]

where \( 0 \leq \nu < 2\pi \).

The \( S^1 \) compactification requires that \( \theta \) and \( \alpha \) are also valued on \( S^1 \) that allows

\[
\int_{S^1} d\theta = 2\pi k, \quad \int_{S^1} d\alpha^i_{L,R} = 2\pi m^i_{L,R},
\]  

8
where \( k \) and \( m_{iL,R} \) are integers. The integral on \( S^1 \) should be properly defined as in Ref. [66] so that the partition function does not depend on the coordinate system on \( S^1 \). (See Appendix B for the definition.) There is a redundancy due to the anomalous chiral symmetry, \( \Psi_{L(R)i} \rightarrow e^{i\beta_{L(R)i}}\Psi_{L(R)i} \):

\[
\theta(x_3) \rightarrow \theta(x_3) - \sum_i (\beta_R^i(x_3) - \beta_L^i(x_3)), \quad \alpha_{L,R}^i(x_3) \rightarrow \alpha_{L,R}^i(x_3) + \beta_{L,R}^i(x_3),
\]

where

\[
\int_{S^1} d\beta_{L,R}^i = 2p\pi,
\]

with \( p \) integers to maintain the boundary conditions.

In the following discussion, we are particularly interested in the theory with

\[
k = N_f, \quad m_{iL,R} = 0.
\]

since the vacuum structure looks the same as the three dimensional case discussed in the previous section. By the chiral rotations, this theory is equivalent to, for example,

\[
k = 0, \quad m_R^i = 1, \quad m_L^i = 0.
\]

The physics should depend on the combination:

\[
\tilde{\theta}(x_3) = \theta(x_3) + \sum_i (\alpha_R^i(x_3) - \alpha_L^i(x_3)), \quad \tilde{k} = k + \sum_i (m_R^i - m_L^i).
\]

We discuss the phase structure of the theory as a function of the radius \( R \). The dynamical scale \( \Lambda_4 \) is defined as

\[
\Lambda_4^b = \Lambda^b e^{-8\pi^2/\rho_4^2(\Lambda)}, \quad b = \frac{11}{3} N - \frac{2}{3} N_f,
\]

where \( \Lambda \) is an arbitrarily high scale. For \( \Lambda_4 R \gg 1 \), the low energy dynamics is described by hadrons on an \( S^1 \) compactified background. In the other limit, \( \Lambda_4 R \ll 1 \), the low energy description is a three dimensional gauge theory on \( M^3 \) via the Kaluza-Klein decomposition. The gauge coupling constant in the three dimensional effective theory is given by

\[
\frac{1}{g_3^2} = \frac{2\pi R}{g_4^2(1/R)},
\]

and the dynamical scale in the three dimensional theory is defined as

\[
\Lambda_3 = \frac{g_3^2 N}{8\pi} = \frac{g_3^2(1/R)N}{16\pi^2 R} = \frac{1}{R} \left( -\frac{2b}{N} \log(\Lambda_4 R) \right)^{-1}.
\]

One can see that for a small enough \( \Lambda_4 R \), there is an energy gap between the dynamical scale \( \Lambda_3 \) and the mass of the first Kaluza-Klein mode, \( 1/R \).
3.1 Small radius, $\Lambda_4 R \ll 1$

Let us use the basis with $k = 0$. The Kaluza-Klein expansion of the fermions can be done as

$$\Psi(x, x_3) = \sum_{n=-\infty}^{\infty} \left( \psi_n(x) \phi_n(x_3) \right),$$

(27)

with

$$\phi_n(x_3) = \frac{1}{\sqrt{2\pi R}} \exp \left[ i\alpha_L(x_3) + i \left( \frac{n}{R} + \frac{m_L}{R} + \frac{\nu}{2\pi R} \right) x_3 \right],$$

(28)

and

$$\tilde{\psi}_n(x_3) = \frac{1}{\sqrt{2\pi R}} \exp \left[ i\alpha_R(x_3) + i \left( \frac{n}{R} + \frac{m_R}{R} + \frac{\nu}{2\pi R} \right) x_3 \right].$$

(29)

The three dimensional effective action is given by

$$S_{\text{eff}} = \int d^3x \left[ -\frac{1}{2g_3^2} \text{Tr}(f_{\mu\nu}f^{\mu\nu}) + \sum_n i\bar{\psi}_n \gamma^\mu (\partial_\mu - ia_\mu) \psi_n + \sum_n i\bar{\tilde{\psi}}_n \gamma^\mu (\partial_\mu - ia_\mu) \tilde{\psi}_n - \sum_n \left(\frac{m_L}{R} - \frac{n}{R} - \frac{\nu}{2\pi R}\right) \bar{\psi}_n \psi_n - \sum_n \left(\frac{m_R}{R} + \frac{n}{R} + \frac{\nu}{2\pi R}\right) \bar{\tilde{\psi}}_n \tilde{\psi}_n + \frac{1}{g_3^2} \text{Tr}(D_\mu a_3 D^\mu a_3) - (\bar{\psi}_3 a_3 \psi - \bar{\tilde{\psi}}_3 \tilde{\psi}_3) - V(a_3) \right].$$

(30)

We have dropped the massive Kaluza-Klein modes of gauge fields. At this stage, one can see that the effects of $m_L$ and $m_R$ are to shift the Kaluza-Klein spectrum of the fermions by $1/R$, and thus can be absorbed by the redefinition of $n$. However, in three dimensions, the signs of the fermion masses are important, and thus one cannot simply ignore $m_L, R$.

The potential $V(a_3)$ is generated at the one-loop level \[47, 83\] and it depends on the boundary condition, $\nu$. For example, by putting an ansatz,

$$a_3 = \frac{1}{2\pi R} \text{diag.}(\xi, \xi, \cdots, \xi, -(N-1)\xi),$$

(31)

it is given by

$$V(a_3) = \frac{1}{8\pi^5 R^3} \left[ -4(N-1) \sum_{n=1}^{\infty} \frac{\cos n N \xi}{n^4} ight.$$

$$+ 2N_f (N-1) \sum_{n=1}^{\infty} \left( \frac{\cos n(\xi + \nu + 2\pi m_L)}{n^4} + (L \rightarrow R) \right)$$

$$+ 2N_f \sum_{n=1}^{\infty} \left( \frac{\cos n(-(N-1)\xi + \nu + 2\pi m_L)}{n^4} + (L \rightarrow R) \right) \right].$$

(32)
The shapes of the potential of $\xi$ at $\nu = 0$ ($\nu = \pi$) is shown in the left (right) panel of Fig. 1 for $N = 3$ and $N_f = 2$. In general, for $N_f < N$, there are $N$ minima at $\xi = \pm 2p\pi/N$, ($p = 0, \cdots, [N/2]$). (For even $N$, $\xi = \pi$ and $\xi = -\pi$ are equivalent.) At $\nu = 0$, $\xi = 0$ is a local minimum, and the true minimum is at $\xi = \pm (N-1)\pi/N$ for odd $N$ and $\xi = \pi$ for even $N$. For $\nu = \pi$, the $\xi = 0$ point is the true vacuum.

The potential has a symmetry $\nu \rightarrow \nu + 2\pi n/N$, $n \in \mathbb{Z}$, together with $\xi \rightarrow \xi - 2\pi n/N$ that is the reflection of the fact that the action of $\mathbb{Z}_N$ elements in $U(1)_B$ is the same as that of the gauge group $SU(N)$ \[84\]. For even $N$, $\nu = 0$ and $\xi = \pi$ is equivalent to $\nu = \pi$ and $\xi = 0$. For odd $N$, they are not equivalent. The $\nu = \pi$ point is equivalent to $\nu = \pi/N$ by an appropriate shift of $\xi$. There is a first order transition in between $\nu = 0$ and $\nu = \pi/N$.

In all the $N$ minima, the $SU(N)$ gauge symmetry is unbroken as the Wilson loop along the $x_3$ direction is a phase times the unit matrix. The fermion masses for $\psi_n$ and $\tilde{\psi}_n$ are, respectively,

$$m_n^{(\psi)} = \frac{n}{R} - \frac{\nu}{2\pi R} - \frac{m_L}{2\pi R} + \frac{\xi}{2\pi R}, \quad m_n^{(\tilde{\psi})} = \frac{n}{R} + \frac{\nu}{2\pi R} + \frac{m_R}{2\pi R} - \frac{\xi}{2\pi R}. \quad (33)$$

By following the global minimum of the potential, in the entire region of $\nu$, the fermion masses are non-vanishing. Therefore, the low energy limit of the 4d QCD on $M^3 \times S^1$ is $SU(N)$ pure gauge theories on $M^3$ for a small radius. There is a mass gap, but the low energy limit can be a topological field theory. For $m_R = 1$ and $m_L = 0$, the fermion masses for $\tilde{\psi}_n$ are shifted by $1/R$. The shift changes the sign of $N_f$ fermion masses from negative to positive. Therefore, the low energy theory obtains the CS level $N_f$. ($N_f/2$ to integrate in the negative ones, and another $N_f/2$ to integrate out the positive ones.)

The CS level is consistent with the result in the basis of $k = N_f$. The $\theta$ term can be expressed as

$$\frac{1}{8\pi^2} \int_{M^3 \times S^1} \theta \text{Tr}(ff) = \frac{1}{8\pi^2} \int_{M^3 \times S^1} \text{Tr} \left( ada + \frac{2}{3} a^3 \right) d\theta, \mod 2\pi. \quad (34)$$

Again, it is important that the integral on $S^1$ is properly defined \[66\]. Since $d\theta$ is single valued, one can naively use the right-hand side of the integral over $S^1$, which reduces to the CS term with the level $k$ for the lowest KK mode of the gauge field.

In summary, the low energy limit of QCD$_4$ on $M^3 \times S^1$ with Eq. (21) for a small radius, $\Lambda_4 R \ll 1$, is the topological field theory, $SU(N)_{N_f}$, that has the dual description by $U(N_f)_{-N}$. 

11
3.2 Large radius, $\Lambda_4R \gg 1$

One can also analyze the low energy limit of QCD$_4$ on $M^3 \times S^1$ for a large radius as we know that the low energy theory is described by pions. We discuss the effect of $\bar{\theta}(x_3)$ in the low energy effective theory. In order to see the $\bar{\theta}(x_3)$ dependence of the theory, one needs to introduce $\eta$ meson ($\eta'$ in the three-flavor language) in addition to the massless pions.

The effective theory is given in terms of the $N_f \times N_f$ unitary matrix $U = e^{i\pi^a T^a + \bar{\eta} \eta}$. While we implicitly assume here that $\eta$ behaves as the Nambu-Goldstone boson, we do not assume that we are in the large $N$ limit. We treat $\eta$ as a heavy meson as we see below. The field $U$ transforms as

$$U \rightarrow g_L^{-1} U g_R,$$

under the $g_L \in SU(N_f)_L$ and $g_R \in SU(N_f)_R$ chiral transformations. Under the axial $U(1)_A$, it transforms as $U \rightarrow e^{2i\beta}U$. The effective action is given by

$$S_{\text{eff}} = \int_{M^3 \times S^1} d^4x$$
$$\times \left[ f_\pi^2 \text{Tr} |\partial_\mu U|^2 - \frac{m_\eta^2 f_\pi^2}{N_f} \left| \log(e^{-i\bar{\theta}} \det U) \right|^2 + \cdots \right].$$

The effect of the boundary condition, $\nu$, can be taken into account by introducing a background gauge field for the baryon number, $(\nu/2\pi R)\bar{\Psi} \gamma^3 \Psi$. The effect of this term appears in the WZ terms. The parameter $\nu$ couples to the topological current, i.e., the Skyrmions.

The equation of motion for $\eta$ is

$$\frac{\partial^2}{\partial x_3^2} \eta = m_\eta^2 \left( \eta - \frac{\bar{\theta}}{N_f} \right).$$

12
For the periodicity of $\bar{\theta}$, the potential for $\eta$ has $N_f$ domains where the $\eta - \bar{\theta}/N_f$ is minimized at $2n\pi/N_f$ with $n$ integers. Transition between two other domains requires a treatment beyond this effective theory. Once we restrict ourselves that the shape of the function $\bar{\theta}$ is not very rapid so that the effective theory can be used, $\eta$ should stay in one of the domains to minimize the energy, which means $\eta$ develops a winding under Eq. (21):

$$\eta(x_3 + 2\pi R) = \eta(x_3) + 2\pi.$$

(38)

This is consistent with the $S^1$ compactification. One should not be confused with the $\eta$ winding as the jump of domains. We are treating $\eta$ as a heavy field and we are working within the effective theory. Also, the argument of the winding of $\eta$ depends on the basis. We can eliminate $\bar{\theta}$ by the field redefinition of $\eta$. In that basis, $\eta$ does not get winding while we obtain the same effects at low energy as we see below.

The three dimensional low energy effective theory is the non-linear sigma model with the coset in Eq. (2), but there are effects from the $\eta$ winding. By turning on the external gauge field, $A$, which couple to the vector current as we discussed in the previous section, a part of the WZ term,

$$S_{WZ} = -\frac{N}{8\pi^2} \int_{M^3 \times S^1} \text{Tr} \left( A dA + \frac{2}{3} A^3 \right) d\eta,$$

(39)

reduces to Eq. (14) where the minus sign is chosen, i.e., it is the same as the one obtained from $U(N_f)_{-N}$ theory. It is interesting that the external magnetic field carries baryon number in this $\eta$ winding background. See Appendix C for the relation between the profile of $\eta$ and the baryons. The parameter set in Eq. (22) corresponds to the basis where $\eta - \bar{\theta}/N_f$ is redefined to be $\eta$. In that bases, there is no winding of $\eta$, but the same WZ term appears by shifting $\eta$ in Eq. (39). One can confirm the consistency of the appearance of the WZ term by comparing the $\bar{\theta} \rightarrow AA$ amplitude to that in QCD$_4$.

In summary, we find that the low energy limit of QCD$_4$ with Eq. (21) is a topological field theory, $SU(N)_{N_f}$, for small $\Lambda_4 R$ and a non-linear sigma model with the WZ term for a large radius, $\Lambda_4 R \gg 1$. There must be a phase transition between these two extreme regions. It is interesting to find that the two limits are the same as the conjectured limit of the three dimensional $SU(N)_0$ theory with $2N_f$ fermions with large and small fermion masses. It is therefore possible to anticipate that the phase transition between a large and a small radius is described by the critical point of $SU(N)_0$ QCD$_3$ with $2N_f$ fermions with the explicit $U(2N_f)$ symmetry breaking terms. The phase transition can also be consistent with the description by the three dimensional $U(N_f)_{-N}$ theory with $2N_f$ scalar fields.
For $|\bar{k}| < N_f$ instead of Eq. (21), the $\eta$ winding should be accompanied with the winding of $\pi$’s. In order to satisfy the boundary condition, $U(x_3 + 2\pi R) = U(x_3)$, the winding $\eta(x_3 + 2\pi R) = \eta(x_3) + 2\pi \bar{k}/N_f$ should be accompanied by

$$e^{i\pi T^a(x_3+2\pi R)} = e^{-2\pi i\bar{k}/N_f} e^{i\pi T^a(x_3)},$$

that can be realized as a configuration of $\pi^a$ as the phase factor is an element of $SU(N_f)$. For example, for $\bar{k} = 1$, one of the diagonal components of $U$ acquires a winding by $2\pi$, and the non-trivial WZ term appears only for that component of the external gauge fields. In general for $|\bar{k}| \leq N_f$, the same effective three dimensional theory can be obtained by the theories of pions interacting with a Higgsed $U(|\bar{k}|)_{\pm N}$ gauge group. This part matches the conjectured dualities in three dimensions between $SU(N)_{\bar{k} - N_f}$ with $2N_f$ fermions and $U(\bar{k})_{-N}$ with $2N_f$ scalars.

4 Hadrons near the critical point

Here we proceed to speculations on the possible behavior of the vector mesons based on the discussion in the previous section. As we discussed, QCD$_4$ in the background of Eq. (21) provides us with the same low energy theories of QCD$_3$ both in the broken and unbroken phases of the global symmetry once an explicit breaking term of the $U(2N_f)$ symmetry is added. The phase transition in three dimensions has a dual picture by the $U(N_f)_{-N}$ gauge theory. Below, we consider the possibility that the dual picture also describes the QCD$_4$ near the phase transition.

Indeed, it is interesting that the extension of the chiral Lagrangian to a $U(N_f)$ gauge theory is known to give a great success to describe the phenomenology of the vector mesons, $\rho$ and $\omega$. Therefore, the vector mesons are the natural candidates for the gauge bosons of the $U(N_f)$ dual theory. If such an interpretation is true, in the phase where chiral symmetry is restored, the $\rho$ and $\omega$ mesons are in the topological phase rather than in the Higgs phase under the background in Eq. (21).

The possible behavior of the hadrons as the function of the radius is as follows. Starting from a large radius where the hadrons describe physics effectively, as the radius approaches to the critical point, $R_* \sim 1/\Lambda_4$, the lowest mode of hadrons start to form a $U(N_f)_{-N}$ gauge theory in the Higgs phase. The members are pions, $\rho$, $\omega$ and other scalar mesons. At the critical point, the chiral symmetry is restored and the $\rho$ and $\omega$ mesons get into the topological phase described by the $U(N_f)_{-N}$ theory, that is dual to $SU(N)_{N_f}$. As further decreasing the
radius, the picture of weakly interacting gluons and quarks makes sense at the energy scale between $\Lambda_3$ and $1/R$. All the fermions as well as KK modes of gluons decouple below $1/R$. Although the description in terms of hadrons gets ineffective in this energy region, the low energy limit of the theory stays the same.

Since $U(N_f)$ gauge group is spontaneously broken, there are vortex configurations in three dimensions made of $\rho$ and $\omega$, which carry magnetic and electric charges of $U(1)^{N_f} (\subset U(N_f))$ [85]. The electric charge is a consequence of the CS term. In the color-flavor locked phase, the electric charge is identified as the baryon number. The vortex with the unit magnetic charge has $B = 1$. We will discuss how this vortex configuration extends to the $S^1$ direction in the next section.

5 Holographic model

The vector mesons as gauge bosons are nicely described by the holographic QCD where the gauge bosons are propagating into an extra dimension [73][75]. Based on the holographic model, we here try to find a dual model of QCD$_4$ which reproduces the story in the previous section. For the duality to work, we need the same low energy limits both in the broken and unbroken phase of the chiral symmetry. In particular, one needs to arrange the theory such that there is an unbroken gauge group, $U(N_f)_+-N$, in the symmetric phase. The chiral symmetry breaking is described by the VEVs of scalar fields which simultaneously make the gauge group Higgsed.

Of course, we are not aware if the phase transition is smooth enough that such an effective description exists. We here assume that is the case and look for a dual theory. In this sense, this is a construction of the Nambu-Jona-Lasinio model or the Ginzburg-Landau model while taking into account the consistency with topology. For $\tilde{k} = 0$ it was a trivial task since the low energy limit of symmetric phase is trivially gapped. But for $\tilde{k} \neq 0$, we need some gauge theory to survive to match the topological field theory.

The holographic QCD describes the vector mesons and pions as gauge fields propagating into the fifth dimension, $b_{L,R}$. The gauge group is $U(N_f)_L \times U(N_f)_R$ which is broken down to $U(N_f)_{L+R}$ somewhere in the extra dimension. The five-dimensional space has a boundary. The boundary conditions are taken to be

$$b_{L,R}^\mu \big|_{\text{boundary}} = A_{L,R}^\mu, \quad (\mu = 0, 1, 2, 3),$$

(41)

where the right-hand side is external gauge fields which couple to chiral currents. The pions
appear as the extra dimensional component of 4\(b_L^4 - b_R^4 \sim \partial^4 \pi\). The boundary condition makes the gauge bosons massive, and the lightest modes are identified as the \(\rho\) and \(\omega\) mesons. The WZ terms can be reproduced by

\[
S_{CS} = -\frac{N}{24\pi^2} \int_{X_5} (\omega_5(b_L) - \omega_5(b_R)). \tag{42}
\]

See Ref. [86] for details. The local chiral transformation shifts the external gauge fields and modifies the boundary conditions, that can be absorbed by the gauge transformation of the bulk gauge fields, and that in turn provides a boundary term from the gauge transformation of the CS term. This procedure results in the WZ terms on the boundary written in terms of the pions and the external gauge fields. They are necessary to match the ’t Hooft anomaly in QCD

The deconstructed version of the model can be built as in the left panel of Fig. 2. The explicit form of the Lagrangian is given in Appendix A. The leftmost sites with open circles are the boundary. No gauge fields are living there. At other sites with double circles, there are \(U(N_f)\) gauge fields which are all Higgsed by eating the link fields. By the most right link, denoted \(\Phi\), the \(U(N_f)_L \times U(N_f)_R\) symmetry is broken down to \(U(N_f)_{L+R}\). Since the number of links are larger by one than the number of double circles, there are \(N_f^2\) Nambu-Goldstone bosons left uneaten. They are identified as the pions and \(\eta\). The thick lines in each site represent \(N\) chiral fermions. The dashed line between fermions represents the mass terms. They are all massive, but necessary to reproduce the CS term in the five dimensional theory. Since the fermions are always massive even in the chiral symmetric phase as we see later, we can think of fermions as auxiliary degrees of freedom. We also gauge the \(Z_N\) subgroup of \(U(1)_B\) which transforms fermions. The gauging is necessary to match the global symmetry
$U(1)_B/Z_N$ in the original QCD. The gauging requires that one should sum up all the sectors with boundary conditions of fermions twisted by elements of $Z_N$ in the $x_3$ direction.

Now let us turn on the external gauge fields. The external gauge fields couple to fermions as

$$A_L^\mu (\bar{q}_L \gamma_\mu q_L + \cdots) + (L \leftrightarrow R).$$

(43)

The fermions in the upper and lower wings couple to $A_L$ and $A_R$, respectively. The mass terms of fermions except for the one with the link $\Phi$ do not break the global symmetry, $U(N_f)_L \times U(N_f)_R$. The combination of $A_L + A_R$ couples to the conserved vector current. By integrating out the massive fermions, one obtains the correct WZ terms. At this stage, we have not included the mass term of $\eta$. One can introduce it by writing a mass term to break the $U(1)_{L-R}$ gauge symmetry for $\Phi$, such as

$$|\log \det \Phi|^2.$$ 

(44)

The axial $U(1)$ is now explicitly broken, and the $\eta$ obtains the mass. In the background of Eq. (16), we have a term

$$|\log e^{-i\theta} \det(\Phi)|^2.$$ 

(45)

This term cause the winding of the trace part of $\Phi$. The winding gives important effects through the WZ term as we discussed before. In addition to the WZ terms among external fields and Nambu-Goldstone modes, we also have

$$S_{WZ} = -\frac{N}{8\pi^2} \int_{M^3 \times S^1} \text{Tr} \left( bdb + \frac{2}{3}b^3 \right) d(-i \log \det \Phi)/N_f,$$

(46)

where $b$ is a gauge field of $U(N_f)_{L+R}$ part of the right most gauge sites. They are massive modes which correspond to the $\rho$ and $\omega$ mesons.

One can modify the model by introducing Higgs fields, $H_L$ and $H_R$, as in the right panel of Fig. 2. Two of the links are replaced by the Higgs fields. When the VEV of $\langle H_{L,R} \rangle$, proportional to the unit matrix, is large, one can identify the sites on the both sides of $H_{L,R}$, and it comes back to the model in the left panel. The location of the links to be replaced with $H_{L,R}$ can be anywhere in the wings.

For small $\langle H_{L,R} \rangle$, the vector part of the gauge bosons in the most right sites, i.e., $\rho$ and $\omega$, gets light, and for $\langle H_{L,R} \rangle = 0$, the gauge bosons as well as the fermions (marked as blue lines) become “massless.” They obtain masses when the $x_3$ direction is compactified on $S^1$. 

17
All the link fields are eaten by the gauge fields, and thus massless pions disappear. Therefore, the chiral symmetry is now recovered in this phase. In this way, the model interpolates the chiral Lagrangian and the linear sigma model by changing the sizes of $\langle H_{L,R} \rangle$.

In the phase of $\langle H_{L,R} \rangle = 0$, we have the term in Eq. (46) but now $b$ represents the massless gauge boson. For the background with $k = N_f$, the winding of $\Phi$ gives the winding $\theta$ term for vector mesons.

Let us consider the vacuum of the theory after the compactification of the $x_3$ direction by the one-loop effective potential near the phase transition point, $\langle H_{L,R} \rangle = 0$. In this region, the massless degrees of freedom are the $U(N_f)$ gauge field and $4N$ chiral fermions. The scalar fields $H_{L,R}$ can be light, but the contribution of the scalars has the same shape as the fermions with the opposite sign and thus for $N_f < N$, the effects can be ignored. The half of $4N$ massless fermions are charged under $U(N_f)$ while the rest are neutral. We first discuss the contribution to the effective potential of $b_3$ from the $2N$ chiral fermions. We take an ansatz of the VEV of $b_3$ to be

$$b_3 = \frac{1}{2\pi R} \text{diag.(}\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_{N_f})$$

where $\sum_{i=1}^{N_f} \tilde{\xi}_i = 0$ (mod $2\pi$). The boundary condition of the fermions is

$$q(x_3 + 2\pi R) = e^{i\nu}q(x_3),$$

where $0 \leq \nu < 2\pi$. The KK spectra of the gauge fields and the fermions are, respectively,

$$M_{ij,n}^2 = \frac{1}{R^2} \left(n - \frac{\tilde{\xi}_i - \tilde{\xi}_j}{2\pi}\right)^2,$$

$$m_{i,n}^2 = \frac{1}{R^2} \left(n + \frac{\nu - \tilde{\xi}_i}{2\pi}\right)^2.$$

By using these, the one-loop effective potential is given by

$$V(b_3) = \frac{1}{4\pi^5 R^3} \left(- \sum_{i,j=1}^{N_f} \sum_{n=1}^{\infty} \cos(n(\tilde{\xi}_i - \tilde{\xi}_j)) \frac{1}{n^4} + 2N \sum_{i=1}^{N_f} \sum_{n=1}^{\infty} \cos(n(\nu - \tilde{\xi}_i)) \frac{1}{n^4}\right).$$

The first term in the right hand side is minimized when $\tilde{\xi}_1 = \tilde{\xi}_2 = \cdots = \tilde{\xi}_{N_f} = \tilde{\xi}$. Furthermore, the second term is minimized at $\tilde{\xi} = \nu - \pi$. Therefore, for $N_f < N$, the lowest minimum is at $\int_{S^1} b = (\pi - \nu) \cdot 1$, which gives the anti-periodic boundary conditions for the fermions. The $U(N_f)$ group is unbroken at the minimum, and the $2N$ fermions get massive.
Next, we consider the one-loop effects of the gauged $\mathbb{Z}_N$ which is the subgroup of $U(1)_B$ symmetry. The effective potential is given by

$$V(\xi') = \frac{N}{2\pi^3 R^3} \sum_{n=1}^{\infty} \frac{\cos(n(\nu - \xi'))}{n^4},$$

where $\xi' = 2\pi m/N$ ($m \in \mathbb{Z}$) is the VEV of the gauged $\mathbb{Z}_N$ field. The different VEV corresponds to the sector of different boundary conditions twisted by the $\mathbb{Z}_N$ elements. Since $\mathbb{Z}_N$ is gauged, we are summing up all the values of $\xi'$ in the path integral. The free energy is minimized at $m = N(\nu - \pi)/(2\pi)$, which means the path integral is dominated by this sector. The anti-periodic boundary condition is chosen for even $N$, and the sector which is the closest to the anti-periodic one is chosen for odd $N$. Therefore, all the fermions decouple and the effective three dimensional theory is the $U(N_f)_{-N}$ CS theory where the CS level stems from the winding of $\Phi$. This theory is dual to $SU(N)_{N_f}$ that is the low energy limit of QCD$_4$ for a small $S^1$ radius.

It is important that the axial $U(1)$ is kept broken in the phase of $\langle H_{L,R} \rangle = 0$ where chiral symmetry $SU(N_f)_L \times SU(N_f)_R$ is unbroken. The explicit breaking is important to obtain the correct CS level via the winding of $\Phi$. Therefore, for the scenario to work the axial $U(1)$ should be broken during the chiral phase transition.

The baryon number as the topological charge in the $U(N_f)_{-N}$ theory can be seen in this model. The baryon number in this model is identified as the trace part of the external $U(N_f)_L + U(N_f)_R$. The unbroken baryon number is rearranged to be a sum of all the $U(1)$ part of each site, and thus the fermions which are integrated out to obtain Eq. (46) are then charged under the baryon number. Therefore, by turning on the background gauge field, $B$, for the baryon number we obtain a term in the three dimensional effective theory:

$$S_{\text{baryon}} = -\frac{1}{4\pi} \int_{M^3} B \text{Tr}(db),$$

under the winding of $\Phi$.

As in the three dimensional effective picture, there are vortex configurations made of $\rho$ and $\omega$. Under the winding of $\Phi$, there is a winding $\theta$ term for the $U(N_f)$ gauge group from Eq. (46). In the presence of this $\theta$ term (which we call it $\tilde{\theta}$), the Abrikosov-Nielsen-Olesen (ANO) vortex string [87, 88] which goes around $S^1$ cannot be connected since the magnetic flux obtains the electric charge as it goes around the $S^1$ direction by the Witten effect [89]. In order to have a stable string loop which goes around $S^1$, one needs to have some non-trivial configuration which carries the electric charge, i.e., the baryon number.
In the background with a constant $d\tilde{\theta}$, one can look for a static field configuration which is $x_3$ independent. The field equations are then the same as the CS case, and thus one finds the solution with a finite energy. The baryon number, $B = 1$, is indeed carried by the string through the electric charge of this solution. For a general background, this configuration will be relaxed to a solution of the field equations with a finite energy. We call it the $B = 1$ string.

Another possibility is to unwind $\Phi$, i.e., $\eta$, by forming a Hall droplet described in Ref. [28]. The droplet is a configuration of the $\eta$ field. It is a sheet with a boundary, and the value of $\eta$ changes by $\pm 2\pi$ when we go across the sheet. The ANO $\rho - \omega$ string can be connected when it goes across the sheet as the Witten effect is cancelled by the change of $\tilde{\theta}$. The net baryon number of this configuration is $B = 0$, and thus we call it the $B = 0$ string. This string is penetrating the Hall droplet. The droplet cannot shrink to nothing, as there is no $B = 0$ string in the background without the droplet. The $B = 1$ string discussed above cannot smoothly deform into this string due to the different baryon number. Interestingly, it is proposed in Ref. [28] that the droplet has an excitation of the edge mode with $B = 1$ and that object is identified as the baryon such as $\Delta^{++} \sim uuu$. Therefore, the $B = 1$ string can deform into a $B = 0$ string together with the excitation of the edge mode of the droplet with $B = 1$. Since there is no such stable string configuration in full QCD, we expect that there are monopoles to cut the $B = 0$ string. A pair of a monopole and an anti-monopole can cut and eliminate the string. Now the $B = 1$ string can decay into a baryon via the deformation into a $B = 0$ string and a $B = 1$ droplet and then the $B = 0$ string part is eliminated. This is a good candidate of the fate of the $B = 1$ string.

The phase transition by the VEV of $H_{L,R}$ can be extended to the case of $0 \leq \bar{k} < N_f$. In that case, the unbroken gauge group is $U(\bar{k})_\ast \times U(N_f - \bar{k})_0$. In order for the theory to have the same low energy limit as QCD, the $U(N_f - \bar{k})_0$ factor should decouple. The $SU(N_f - \bar{k})_0$ part exhibits a mass gap by the confinement. The $U(1)$ part also confines by the instantons. As we discussed in the previous section, we need monopoles (in four dimensions) to cut the stable $\rho$ and $\omega$ strings. The monopole configurations in (012) directions are instantons in three dimensions, and the path integral including such instantons causes the confinement of the $U(1)$ factor [90,91]. The presence of the monopole does not make the $U(\bar{k})_\ast$ part to confine as the gauge bosons have a mass term from the CS term [92]. Therefore, we obtain the correct low energy limit, $U(\bar{k})_\ast$ theory.

There is another possibility that the $U(N_f - \bar{k})$ part of the VEV is kept non-vanishing for $H_{L,R}$ while the chiral symmetry is restored by cutting the $U(N_f - \bar{k})$ part the link $\Phi$. The
$U(N_f - \bar{k})$ gauge group is kept in the Higgs phase, and the only $U(\bar{k})_N$ part remains at low energy.

What happens for $\rho$ and $\omega$ mesons is qualitatively different in the above two cases. When the $S^1$ radius is large, they are vector mesons which we are familiar with. As the radius approaches to the critical point, they behave as the gauge bosons in the Higgs phase. In particular, the meta-stable vortex strings made of $\rho$ and $\omega$ appear. Beyond the critical radius, the chiral symmetry is restored, and the $U(\bar{k})$ part of them goes into the topological phase, whereas the $U(N_f - \bar{k})$ part goes into either the confining phase or remains in the Higgs phase. The rest of them stays in the Higgs phase.

The model described here can be viewed as the Nambu-Jona-Lasino model for chiral symmetry breaking under the background of the imaginary chiral chemical potential (once we take the compactification direction to be the time direction in the Euclidean space.) The $Z_N$ twisting boundary condition can be naturally identified as the VEV of the Polyakov loop. For a small radius (high temperature), $\langle H_{L,R} \rangle$ is vanishing, and thus the fermions choose a particular boundary condition by minimizing the free energy. This corresponds to the non-vanishing VEV of the Polyakov loop, and thus describing the deconfined phase. On the other hand, for a large radius where $\langle H_{L,R} \rangle$ is large, the fermions decouple, and all the boundary conditions equally contribute to the path integral. This corresponds to vanishing VEV for the Polyakov loop, and thus the quarks are confined.

### 6 Finite temperature QCD

We discussed a somewhat exotic scenario for the chiral phase transition which happens at some critical radius, $R_*$. For $\bar{k} = 0$ with the anti-periodic boundary condition for quarks, $\nu = \pi$, one can think of this system as the finite temperature QCD by taking the Euclidean metric. At some critical temperature, $T_* = 1/R_*$, the chiral phase transition happens.

In the model we discussed, there is a consistent scenario where $U(\bar{k})_N \times U(N_f - \bar{k})_0$ theory remains in the infrared and $U(N_f - \bar{k})_0$ factor confines due to instantons. For $\bar{k} = 0$, the confining $U(N_f)_0$ gauge field is the $\rho$ and $\omega$ mesons.

There are other possibilities as we discussed already. There is also a possibility that $U(N_f)$ gauge theory is not a good picture at all. For example, the gauged Nambu-Jona-Lasino model in Ref. [93] can give $SU(N)_k$ factor in the chiral symmetric phase. The model is simply adding to QCD a scalar field, $X$, which transforms as $(N_f, \bar{N}_f)$ under the $U(N_f)_L \times U(N_f)_R$ chiral
symmetry, and coupling it to quarks as $\bar{q}Xq$. This model gives the correct non-linear sigma model in the broken phase where the scalar field has a VEV, while it reduces to $SU(N)_k$ theory in the symmetric phase. The $\rho$ and $\omega$ mesons do not appear as the field to describe the phase transition phenomena.

The question of which picture is the most appropriate near the chiral phase transition should be able to be tested by the lattice simulations. By looking at the behavior of the two point functions of the vector currents, one may check if the $\rho$ and the $\omega$ mesons get “massless.” Although they have no mass term in the four dimensional Lagrangian, they have thermal masses and also masses from instantons (monopoles in four dimensions) in the actual spectrum. We will leave the study of these effects as well as that of actual methods in the lattice QCD to distinguish the scenarios.

7 Discussion

The three dimensional CS matter systems exhibit various non-trivial topological phases at low energy, and it has been conjectured that gauge theories with fermions and bosons describe the same physics near the critical point of the parameter spaces. Although this duality is tightly related to the peculiar anyon statistics in three space-time dimensions, the symmetry breaking phenomena and dualities conjectured in QCD$_3$ with small number of the CS level look quite similar to our QCD vacuum in four dimensions.

The winding $\theta$ background on an $S^1$ compactified space can directly relate the three and four dimensional theories by comparing the low energy limits. We find that in QCD$_4$ the chiral phase transition should happen at a critical radius, and there can be a description of the phase transition as the Higgs mechanism of the $U(N_f)$ gauge theory where the gauge bosons are the vector mesons.

Under the winding $\theta$, the WZ terms in the chiral Lagrangian leave non-trivial WZ terms in the three dimensional effective theories, where the baryon number can be identified as the magnetic flux of QED. The origin of this unusual relation between baryons and monopoles can be understood as the ’t Hooft anomaly in QCD$_4$.

We left the discussion of how to test the possibility of the vector mesons becoming gauge bosons. One of the natural frameworks to discuss this point is the holographic QCD where the vector mesons are already gauge bosons in the Higgs phase. The specific holographic model such as the Sakai-Sugimoto model [73] may be able to be used to study the dynamics
of the phase transition in the winding $\theta$ background. Also, if the feature of gauge bosons getting “massless” in the four dimensional language remains in the trivial $\theta = 0$ background, the lattice QCD may be able to directly test the scenario.

Another non-trivial prediction of the model is the existence of the monopoles which cut the string made of $\rho$ and $\omega$ \cite{94}. We are not sure what this objects to be identified in the hadron spectrum. Due to the color-flavor locking, the monopoles really carries the magnetic charge of QED while they are confined by the string. It is certainly interesting to look for the candidates of hadrons which are made of the monopole-string system.

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A Lagrangian of quiver diagram

Here, we show the Lagrangian that is described by the quiver diagram in the left panel of Fig. 2.

\[ \mathcal{L} = i\bar{q}^{(L)0} \gamma^M (\partial_M - iA_M^{(L)}) P_L q^{(L)0} + i\bar{q}^{(R)0} \gamma^M (\partial_M - iA_M^{(R)}) P_R q^{(R)0} \]

\[ + \sum_{i=1}^{n_L} i\bar{q}^{(L)i} \gamma^M (\partial_M - i\theta_M^{(L)i}) q^{(L)i} + \sum_{i=1}^{n_R} i\bar{q}^{(R)i} \gamma^M (\partial_M - i\theta_M^{(R)i}) q^{(R)i} \]

\[- \sum_{i=1}^{n_L} \frac{1}{2g_i} \text{Tr} \left( f^{(L)i}_M f^{(L)i}_M \right) - \sum_{i=1}^{n_R} \frac{1}{2g_i} \text{Tr} \left( f^{(R)i}_M f^{(R)i}_M \right) \]

\[ + \text{Tr} \left( \partial_M U_{01}^{(L)} - iA_M^{(L)} U_{01}^{(L)} + iU_{01}^{(L)i} U_{1i}^{(L)i} \right)^2 + \sum_{i=1}^{n_L-1} \text{Tr} \left( \partial_M U_{i,i+1}^{(L)} - i\theta_M^{(L)i} U_{i,i+1}^{(L)} \right)^2 + \sum_{i=1}^{n_R-1} \text{Tr} \left( \partial_M U_{i+1,i}^{(R)} - i\theta_M^{(R)i} U_{i+1,i}^{(R)} \right)^2 \]

\[ + \text{Tr} \left( \partial_M \Phi - i\theta_M^{(L)n_L} \Phi + i\Phi \theta_M^{(R)n_R} \right)^2 \]

\[- \sum_{i=0}^{n_L-1} m_{i,i+1}^{(L)i} U_{i,i+1}^{(L)i} P_L q^{(L)i} - \sum_{i=0}^{n_R-1} m_{i+1,i}^{(R)i} \bar{q}^{(R)i} U_{i+1,i}^{(R)i} P_L q^{(R)i} + \text{h.c.} \]

\[ - m_{\Phi} \bar{q}^{(R)n_R} \Phi^{*} P_L q^{(L)n_L} + \text{h.c.}, \quad (54) \]

where \( n_L \) and \( n_R \) are the number of double circle nodes in the upper and lower lines, respectively.

B Winding \( \theta \) term

Here, we review how to treat the \( \theta \) term with a winding number. When \( \theta \) has winding number along a compact direction, the \( \theta \) term is not well defined on one patch.

Let us consider an integral

\[ \frac{1}{2\pi} \int_{S^1} \theta dq, \]

with

\[ \int_{S^1} d\theta = 2\pi k, \quad \int_{S^1} dq = 2\pi n, \quad (k, n \in \mathbb{Z}). \]

We would like to define the integral up to \( 2\pi m \) (\( m \in \mathbb{Z} \)) since the integral will be exponentiated in the path integral. In Ref. [66], a general prescription to define such an integral is
discussed. By taking \( t \), \((0 \leq t < 2\pi)\) as the coordinate on \( S^1 \), the prescription gives

\[
\frac{1}{2\pi} \int dt \theta(t) \dot{q}(t) := \frac{1}{2\pi} \int_0^{2\pi} dt \theta(t) \dot{q}(t) - kq(2\pi).
\] (57)

Similarly, the integration of \( \dot{\theta}(t)q(t) \) can be defined by

\[
\frac{1}{2\pi} \int dt \dot{\theta}(t)q(t) := \frac{1}{2\pi} \int_0^{2\pi} dt \dot{\theta}(t)q(t) - n\theta(2\pi).
\] (58)

The definitions of the integral in Eqs. (57) and (58) have the following desired features. The integral is invariant modulo \( 2\pi \) under the shifts, \( \theta \rightarrow \theta + 2\pi \) and \( q \rightarrow q + 2\pi \). Also, the integral does not depend (modulo \( 2\pi \)) on the choice of the \( t = 0 \) point on \( S^1 \).

These definitions are consistent with the integration by parts (modulo \( 2\pi \)), i.e.,

\[
\frac{1}{2\pi} \int dt \theta(t) \dot{q}(t) = \frac{1}{2\pi} \int_0^{2\pi} dt \theta(t) \dot{q}(t) + kq(2\pi) = -\frac{1}{2\pi} \int_0^{2\pi} dt \dot{\theta}(t)q(t) + n\theta(0)
\]

\[
= -\frac{1}{2\pi} \int dt \dot{\theta}(t)q(t), \mod 2\pi.
\] (59)

C Baryon number and the configuration of \( \eta \)

We discuss a configuration to give the baryon number, \( B \neq 0 \), in QCD under a non-trivial background of \( \eta \). We now take the space-time as the Minkowski space, \( M^4 \). The WZ term in QCD contains the following term:

\[
S_{WZ} = -\frac{N}{8\pi^2} \int \text{Tr} \left( AdA + \frac{2}{3} A^3 \right) d\eta,
\] (60)

where the \( N_f \times N_f \) matrix \( A \) is the external gauge field which couples to the \( U(N_f) \) vector current, and \( \eta \) is the \( U(1) \) part of the Nambu-Goldstone mode, \( U = e^{i\pi a T^a + i\eta} \). The trace part of the gauge field normalized as, \( B = (N/N_f)\text{Tr}A \), is the source for the baryon number. The baryon charge density can be read off by differentiating with respect to \( B \) as

\[
\rho_B = \frac{1}{4\pi^2}\epsilon_{ijk}\text{Tr}(\partial_i A_j)\partial_k \eta + \cdots.
\] (61)

Let us consider a configuration with \( \eta = 0 \) at \( x_3 = -\infty \) and \( \eta = 2\pi \) at \( x_3 = +\infty \). We also apply an external magnetic field of the (11) component of \( A \) in the \( x_3 \) direction, with

\[
\int dA^{(11)} = 2\pi,
\] (62)

i.e., putting a monopole and an anti-monopole at \( x_3 = \mp\infty \).
This configuration provides $B = 1$ as one can see from the baryon density in Eq. (61). One can also understand this as the Witten effect. For the $(11)$ component of $A$, there is an effective $\theta$ term from Eq. (60),

$$S_{\theta} = -\frac{N}{8\pi^2} \int A^{(11)} dA^{(11)} d\eta, \quad (63)$$

which varies as a function of $x_3$. Therefore, as we move a monopole from $x_3 = -\infty$ to $x_3 = +\infty$, the monopole obtains the electric charge, $N$, to couple to $A^{(11)}$ by the Witten effect. Since $A^{(11)} = B/N + \cdots$, the dyon carries the baryon number $B = 1$. Although the magnetic field is eliminated by this move, the baryon number remains.

Now we consider the situation that the change of the value of $\eta$ happens in a finite region on the $(x_1, x_2)$-plane and at a localized location in the $x_3$ coordinate. This sheet-like configuration is called the Hall droplet in Ref. [28]. If a monopole goes through the Hall droplet, it becomes a dyon with $B = 1$ by the Witten effect. This means that if we put magnets on the both sides of the droplet, the magnetic lines cannot just go through the droplet. In order to let the monopole line to go through, one needs to throw in a baryon. Conversely, starting from a configuration where the magnetic lines are penetrating the droplet, when the magnets are turned off or taken away, the system should relax to a state with a finite baryon number. It has been discussed in Ref. [28] that the excitation of the edge mode of the droplet corresponds to the baryon state with spin $N/2$. This state is a good candidate of the remnant of the system.

The flavor quantum numbers of systems can be read off as in the same way as the baryon number. Let us take the cases with $N = 3$ and $N_f = 2$ as in real QCD, where $A^{(11)}$ couples to the current of the up quark. The configuration of the unit magnetic line of $A^{(11)}$ going through the droplet now has the quantum number of the operator $uau$, i.e., it has the electric charge $Q = 2$. This is indeed the same as the baryon discussed in Ref. [28].

Let’s consider another example where only one of the diagonal components of the Nambu-Goldstone field has the non-trivial configurations:

$$(\pi^a T^a + \eta) \bigg|_{x_3 = -\infty} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (\pi^a T^a + \eta) \bigg|_{x_3 = +\infty} = \begin{pmatrix} 2\pi & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \quad (64)$$

When the $U(1)$ baryon (or QED as in the real world) is gauged, the minimal magnetic charge

26
\[ \int \frac{dB}{N} = \int dA^{(11)} = \ldots = \int dA^{(N_fN_f)} = \frac{2\pi}{N}, \quad (65) \]

as in the well-known magnetic monopole in grand unified theories \[ 95,96 \]. The Dirac quantization conditions for quarks are satisfied by taking into account the \( Z_N \) magnetic flux of \( SU(N) \) carried by the monopole \[ 97,98 \]. The configuration that the magnetic line of this monopole, \textit{i.e.}, the ’t Hooft line, penetrates the Hall droplet has now the baryon number \( B = 1/N \). Therefore, once the magnets are removed, the system should relax to a quark! Again by the Witten effects, the ’t Hooft line accompanies a Wilson line when it goes across the droplet. When we turn off the magnetic part, the Wilson line which ends on the sheet remains. There should be a quark at the end point of the Wilson line. Indeed, there is an anyon excitation of the Hall droplet with the baryon number \( B = 1/N \). (See \[ 99 \] for a review.) It is interesting that the quark is described as a soliton made of hadrons!

The discussion here is closely related to the chiral soliton lattice studied in Ref. \[ 100 \], where the pions get winding under strong magnetic fields and a chemical potential of baryons. Microscopically, one baryon can be converted into a configuration of a Hall droplet with one unit of the magnetic flux penetrating through it.

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