On a Subclass of p-Valent Functions with Negative Coefficients Defined by Using Rafid Operator

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Abstract

By using Rafid operator we define the subclass \(R_{\mu,p}^\delta(\alpha; A, B)\) and \(P_{\mu,p}^\delta(\alpha; A, B)\) of analytic and p-valent functions with negative coefficients we investigate some sharp results including coefficients estimates, distortion theorem, radii of starlikeness, convexity, close-to-convexity, and modified-Hadamard product. Finally, we give an application of fractional calculus and Bernadi-Libora-Livingston operator.

Keywords and phrases: analytic, p-valent functions, Hadamard product, differential subordination, fractional calculus.

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1. Introduction

Let \(T(p)\) denotes the class of normalized p-valent functions \(f\) which are analytic in \(U = \{z \in \mathbb{C} : |z| < 1\}\), and given by

\[
f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \quad (a_k \geq 0, p \in \mathbb{N} = \{1, 2, 3\ldots\}),
\]

(1.1)

A function \(f \in T(p)\) is called p-valent starlike of order \(\alpha (0 \leq \alpha < p)\), if and only if

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in U),
\]

(1.2)

we denote by \(T^*(p, \alpha)\) the class of all p-valent starlike functions of order \(\alpha\). Also a function \(f \in T(p)\) is called p-valent convex of order \(\alpha (0 \leq \alpha < p)\), if and only if

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in U),
\]

(1.3)

we denote by \(C(p, \alpha)\) the class of all p-valent convex functions of order \(\alpha\). For more informations about the subclasses \(T^*(p, \alpha)\) and \(C(p, \alpha)\), see [6]. Motivated by Atshan and Rafid see [1], we introduce the following p-valent analogue \(R_{\mu,p}^\delta : T(p) \rightarrow\)
For $0 \leq \mu < 1$ and $0 \leq \delta \leq 1$,

\[
R_{\mu,p}^\delta f(z) = \frac{1}{\Gamma(p+\delta)(1-\mu)p+\delta} \int_0^\infty t^{p-1}e^{-(\frac{t}{1-\mu})}f(zt)dt
\]  \hspace{1cm} (1.4)

where $\Gamma$ stands for Euler’s Gamma function (which is valid for all complex numbers except the non-positive integers).

Let $f$ and $g$ be analytic in $U$. Then we say that the function $g$ is subordinate to $f$ if there exists an analytic function in $U$ such that $|w(z)| < 1$ ($z \in U$) and $g(z) = f(w(z))$. For this subordination, the symbol $g(z) \prec f(z)$ is used. In case $f(z)$ is univalent in $U$, the subordination $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(U) \subset f(U)$, see [7].

For the function $f$ given by (1.1) and $g(z) = z^p - \sum_{k=p+1}^\infty a_k b_k z^k$, the modified Hadamard product (or convolution) of $f$ and $g$ is denoted by $f \ast g$ and is given by

\[
(f \ast g)(z) = z^p - \sum_{k=p+1}^\infty a_k b_k z^k = (g \ast f)(z).
\]  \hspace{1cm} (1.5)

**Definition 1.** For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < p$, let $R_{\mu,p}^\delta(\alpha; A, B)$ be the subclass of functions $f \in T(p)$ for which:

\[
\frac{z(R_{\mu,p}^\delta f(z))'}{R_{\mu,p}^\delta f(z)} \prec (p - \alpha)\frac{1 + A z}{1 + B z} + \alpha,
\]  \hspace{1cm} (1.6)

that is, that

\[
R_{\mu,p}^\delta(p, \alpha; A, B) = \left\{ f \in T(p) : \left| \frac{z(R_{\mu,p}^\delta f(z))'}{R_{\mu,p}^\delta f(z)} - p}{\frac{z(R_{\mu,p}^\delta f(z))'}{R_{\mu,p}^\delta f(z)} - [Bp(A-B)(p-\alpha)]} < 1, z \in U \right\}. \hspace{1cm} (1.7)
\]

Note that $\text{Re}\{ (p - \alpha)\frac{1 + A z}{1 + B z} + \alpha \} > \frac{1 - A + \alpha(A-B)}{1 - B}$.

Also, for $-1 \leq B < A \leq 1$ and $0 \leq \alpha < p$, let $P_{\mu,p}^\delta(\alpha; A, B)$ be subclass of functions $f \in T(p)$ for which:

\[
1 + \frac{z(R_{\mu,p}^\delta f(z))''}{(R_{\mu,p}^\delta f(z))'} \prec (p - \alpha)\frac{1 + A z}{1 + B z} + \alpha,
\]  \hspace{1cm} (1.8)

For (1.6) and (1.8) it is clear that

\[
f(z) \in P_{\mu,p}^\delta \iff \frac{zf'(z)}{p} \in R_{\mu,p}^\delta
\]  \hspace{1cm} (1.9)

**2. Main Results**

Unless otherwise mentioned, we assume in the remainder of this paper that, $0 \leq \alpha < p$, $0 \leq \mu < 1$, $0 \leq \delta \leq 1$,

$-1 \leq B < A \leq 1$, $p \in \mathbb{N}$ and $z \in U$. 


2.1. Coefficients Estimate

**Theorem 1.** Let the function \( f(z) \) be given by (1.1). Then \( f(z) \in R^\delta_{\mu,p}(\alpha; A, B) \), if and only if
\[
\sum_{k=p+1}^{\infty} [(1 - B)(k - p) + (A - B)(p - \alpha)] (1 - \mu)^{k-p} \frac{\Gamma(k + \delta)}{\Gamma(p + \delta)} a_k \leq (A - B)(p - \alpha).
\] (2.1)

**Proof.** Assume that the inequality (2.1) holds true. We find from (1.1) and (2.1) that thus we have
\[
\left| z(R^\delta_{\mu,p}f(z))' - p(R^\delta_{\mu,p}f(z)) \right| - \left| B \left[ z(R^\delta_{\mu,p}f(z))' \right] - [Bp + (A - B)(p - \alpha)] [R^\delta_{\mu,p}f(z)] \right|
\]
\[
\leq \sum_{k=p+1}^{\infty} [(1 - B)(k - p) + (A - B)(p - \alpha)] \frac{\Gamma(k + \delta)}{\Gamma(p + \delta)} (1 - \mu)^{k-p} a_k - (A - B)(p - \alpha) \leq 0.
\]
Hence, by the maximum modulus theorem, we have
\[
\left| \frac{z(R^\delta_{\mu,p}f(z))'}{R^\delta_{\mu,p}f(z)} - p \right| < 1.
\]
Thus \( f \in R^\delta_{\mu,p}(\alpha; A, B) \).

Conversely, let \( f \in R^\delta_{\mu,p}(\alpha; A, B) \) be given by (1.1), then from (1.1) and (1.7), we have
\[
\left| \frac{z(R^\delta_{\mu,p}f(z))'}{R^\delta_{\mu,p}f(z)} - p \right| = \left| B \left[ z(R^\delta_{\mu,p}f(z))' \right] - [Bp + (A - B)(p - \alpha)] [R^\delta_{\mu,p}f(z)] \right|
\]
\[
= \left| \sum_{k=p+1}^{\infty} (k - p)(1 - \mu)^{k-p} \frac{\Gamma(k + \delta)}{\Gamma(p + \delta)} a_k z^k \right| < 1.
\]
Since \( \text{Re}(z) \leq |z| \) for all \( z \), we have
\[
\text{Re} \left\{ \frac{\sum_{k=p+1}^{\infty} (k - p)(1 - \mu)^{k-p} \frac{\Gamma(k + \delta)}{\Gamma(p + \delta)} a_k z^k}{(A - B)(p - \alpha)z^p + \sum_{k=p+1}^{\infty} [-B(k - p) + (A + B)(p - \alpha)](1 - \mu)^{k-p} \frac{\Gamma(k + \delta)}{\Gamma(p + \delta)} a_k z^k} \right\} < 1,
\] (2.2)
choose values of \( z \) on the real axis so that \( \frac{z(R^\delta_{\mu,p}f(z))'}{R^\delta_{\mu,p}f(z)} \) is real. It is clearing the denominator in (2.2) and letting \( z \to 1^- \) through real values, we have
Theorem 3. If the function \( f(z) \) defined by (1.1) is in the class \( R_{\mu,p}^\delta(\alpha; A, B) \). Then

\[
\sum_{k=p+1}^{\infty} (k-p)(1-\mu)k-p\frac{\Gamma(k+\delta)}{\Gamma(p+\delta)}a_k \leq (A-B)(p-\alpha) - \sum_{k=p+1}^{\infty} [-B(k-p)+(A-B)(p-\alpha)](1-\mu)k-p\frac{\Gamma(k+\delta)}{\Gamma(p+\delta)}a_k.
\]

(2.3)

This gives the required condition.

**Corollary 1.** Let the function \( f(z) \) defined by (1.1) be in the class \( R_{\mu,p}^\delta(\alpha; A, B) \). Then

\[
a_k \leq \frac{(A-B)(p-\alpha)}{[(1-B)(k-p)+(A-B)(p-\alpha)](1-\mu)k-p\frac{\Gamma(k+\delta)}{\Gamma(p+\delta)}} (k \geq p+1),
\]

(2.4)

the result is sharp for the function \( f_0 \) given by

\[
f_0(z) = z^p - \frac{(A-B)(p-\alpha)}{[(1-B)(k-p)+(A-B)(p-\alpha)](1-\mu)k-p\frac{\Gamma(k+\delta)}{\Gamma(p+\delta)}} z^k \quad (k \geq p+1).
\]

(2.5)

By using (1.9) and Theorem 1, it is easily to obtain the following result.

**Theorem 2.** Let the function \( f(z) \) be given by (1.1). Then \( f \in P_{\mu,p}^\delta(\alpha; A, B) \) if and only if

\[
\sum_{k=p+1}^{\infty} \frac{k}{p}(1-B)(k-p)k-p\frac{\Gamma(k+\delta)}{\Gamma(p+\delta)}a_k \leq (A-B)(1-\alpha)\frac{\Gamma(k+\delta)}{\Gamma(p+\delta)}
\]

(2.6)

**Corollary 2.** Let the function \( f(z) \) defined by (1.1) be in the class \( P_{\mu,p}^\delta(\alpha; A, B) \). Then

\[
a_k \leq \frac{(A-B)(p-\alpha)}{\frac{k}{p}(1-B)(k-p)k-p\frac{\Gamma(k+\delta)}{\Gamma(p+\delta)}} (k \geq p+1),
\]

(2.7)

the result is sharp for the function \( f_1(z) \) give by

\[
f_1(z) = z^p - \frac{(A-B)(p-\alpha)}{\frac{k}{p}(1-B)(k-p)k-p\frac{\Gamma(k+\delta)}{\Gamma(p+\delta)}} z^k \quad (k \geq p+1),
\]

(2.8)

### 2.2. Distortion Theorem

**Theorem 3.** If the function \( f(z) \) defined by (1.1) is in the class \( R_{\mu,p}^\delta(\alpha; A, B) \). Then

\[
\left| \delta(p,m) - \frac{(A-B)(p-\alpha)(p+1)!}{(p+1-m)!(1-B)+(A-B)(p-\alpha)(1-\mu)\frac{\Gamma(p+\delta+1)}{\Gamma(p+\delta)}} |z| \right| \leq |f^{(m)}(z)| \leq \left| \delta(p,m) + \frac{(A-B)(p-\alpha)(p+1)!}{(p+1-m)!(1-B)+(A-B)(p-\alpha)(1-\mu)\frac{\Gamma(p+\delta+1)}{\Gamma(p+\delta)}} |z| \right| |z|^{p-m}.
\]

(2.6)

\((m \in \mathbb{N}_0, p > \{m\})\)
The result is sharp for the function $f_0$ given by

$$f_0(z) = z^p - \frac{(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)\frac{\Gamma(p + \mu + 1)}{\Gamma(p + \delta)}(1 - \mu)}z^{p+1}$$

**Proof.** In view of Theorem 1, we have

$$\frac{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)\frac{\Gamma(p + \mu + 1)}{\Gamma(p + \delta)}}{(A - B)(p - \alpha)(p + 1)!}\sum_{k=p+1}^{\infty} k!a_k \leq \frac{\sum_{k=p+1}^{\infty} [(1 - B)(k - p) + (A - B)(p - \alpha)](1 - \mu)^{k-p}\frac{\Gamma(k + \delta)}{\Gamma(p + \delta)}a_k}{((1 - B) + (A - B)(p - \alpha))(1 - \mu)\frac{\Gamma(p + \mu + 1)}{\Gamma(p + \delta)}},$$

which readily yields

$$\sum_{k=p+1}^{\infty} k!a_k \leq \frac{(A - B)(p - \alpha)(p + 1)!}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)\frac{\Gamma(p + \mu + 1)}{\Gamma(p + \delta)}}.$$

Now by differentiating both sides of (1.1) $m$-times we have

$$f^{(m)}(z) = \delta(p, m)z^{p-m} - \sum_{k=p+1}^{\infty} \delta(k, m)a_kz^{k-m}.$$  \hspace{1cm} (2.8)

and Theorem 3 would follow from (2.7) and (2.8).

### 2.3. Radii of Starlikeness, Convexity and Close-to-Convexity

**Theorem 4.** Let the function $f(z)$ defined by (1.1) be in the class $R^\delta_{\mu, p}(\alpha; A, B)$, then

(i) $f(z)$ is $p$-valently starlike of order $\zeta(0 \leq \zeta < p)$ in $|z| < r_1$, where

$$r_1 = \inf_k \left[ \frac{[(1 - B)(k - p) + (A - B)(p - \alpha)](1 - \mu)^{k-p}\frac{\Gamma(k + \delta)}{\Gamma(p + \delta)}(p - \zeta)}{(A - B)(p - \alpha)k(k - \zeta)} \right]^{\frac{1}{k-p}} (k \geq p + 1),$$  \hspace{1cm} (2.9)

(ii) $f(z)$ is $p$-valently convex of order $\zeta(0 \leq \zeta < p)$ in $|z| < r_2$, where

$$r_2 = \inf_k \left[ \frac{[(1 - B)(k - p) + (A - B)(p - \alpha)](1 - \mu)^{k-p}\frac{\Gamma(k + \delta)}{\Gamma(p + \delta)}(p(p - \zeta)}{(A - B)(p - \alpha)k(k - \zeta)} \right]^{\frac{1}{k-p}} (k \geq p + 1),$$  \hspace{1cm} (2.10)

(iii) $f(z)$ is $p$-valently close-to-convex of order $\zeta(0 \leq \zeta < p)$ in $|z| < r_3$ where

$$r_3 = \inf_k \left[ \frac{[(1 - B)(k - p) + (A - B)(p - \alpha)](1 - \mu)^{k-p}\frac{\Gamma(k + \delta)}{\Gamma(p + \delta)}(p - \zeta)}{(A - B)(p - \alpha)k} \right]^{\frac{1}{k-p}} (k \geq p + 1),$$  \hspace{1cm} (2.11)

Each of these results are sharp for the function $f(z)$ given by (2.5)

**Proof.** It is sufficient to show that
\[
\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \zeta \quad (|z| < r_1; 0 \leq \zeta < p),
\]
(2.12)

or

\[
\left| \frac{zf'(z)}{f(z)} - p \right| = \left| \sum_{k=p+1}^{\infty} (k-p)a_k z^{k-p} \right| \\
\leq \frac{\sum_{k=p+1}^{\infty} (k-p)a_k |z|^{k-p}}{1 - \sum_{k=p+1}^{\infty} a_k |z|^{k-p}}.
\]
(2.13)

Inequality (2.12) holds true, when

\[
\sum_{k=p+1}^{\infty} (k-p)a_k |z|^{k-p} \\
\leq p - \zeta,
\]
or, when

\[
\sum_{k=p+1}^{\infty} \left( \frac{k-\zeta}{p-\zeta} \right) a_k |z|^{k-p} \leq 1,
\]
(2.14)

using inequality (2.1), then (2.14) holds true if

\[
\left( \frac{k-\zeta}{p-\zeta} \right) a_k |z|^{k-p} \leq \frac{[(1-B)(k-p) + (A-B)(p-\alpha)](1-\mu)^{-p} \Gamma(k+\delta)}{(A-B)(p-\alpha)} a_k, \quad (k \geq p+1),
\]
(2.15)

or

\[
|z| \leq \left\{ \frac{[(1-B)(k-p) + (A-B)(p-\alpha)](1-\mu)^{-p} \Gamma(k+\delta)}{(A-B)(p-\alpha)} \left( \frac{p-\zeta}{k-\zeta} \right) \right\}^{\frac{1}{p-k}} \quad (k \geq p+1),
\]
(2.16)

or

\[
\quad r_1 = \inf_k \left\{ \frac{[(1-B)(k-p) + (A-B)(p-\alpha)](1-\mu)^{-p} \Gamma(k+\delta)}{(A-B)(p-\alpha)} \left( \frac{p-\zeta}{k-\zeta} \right) \right\}^{\frac{1}{p-k}} \quad (k \geq p+1).
\]
(2.17)

This completes the proof (2.9).

To prove (ii) and (iii) it is sufficient to note that

\[
\left| 1 + \frac{zf''(z)}{f'(z)} - p \right| \leq p - \zeta \quad (|z| < r_2; 0 \leq \zeta < p)
\]
(2.18)

and

\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \zeta \quad (|z| < r_3; 0 \leq \zeta < p)
\]
(2.19)

respectively.
2.4 Modified-Hadamard Product

In this subsection, we obtain some results of the modified Hadamard product of functions $f_1$ and $f_2$, which are defined by

$$f_v(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,v} z^k \quad (a_{k,v} \geq 0, v = 1, 2), \quad (2.20)$$

**Theorem 5.** If $f_v \in R^{\delta}_{\mu,p}(\alpha; A, B)$ ($v = 1, 2$) defined by (2.20), then $(f_1 \ast f_2)(z) \in R^{\delta}_{\mu,p}(\lambda; A, B)$, where

$$\lambda = p - \frac{(1 - B)(A - B)(p - \alpha)^2}{[(1 - B) + (A - B)(p - \alpha)]^2 (1 - \mu) \frac{\Gamma(p + \delta + 1)}{\Gamma(p + \delta)}} - \frac{[(A - B)(p - \alpha)]}{(1 - \mu) \frac{\Gamma(p + \delta + 1)}{\Gamma(p + \delta)}}. \quad (2.21)$$

The result is sharp for that function $f_v(z)$ ($v = 1, 2$) given by

$$f_v(z) = z^p - \frac{(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu) \frac{\Gamma(p + \delta + 1)}{\Gamma(p + \delta)}} z^{p+1}. \quad (2.22)$$

**Proof.** Employing the technique used earlier by Schild and Silverman [5], we need to find the largest $\lambda$ such that

$$\sum_{k=p+1}^{\infty} \frac{[(1 - B)(k - p) + (A - B)(p - \lambda)](1 - \mu)^{k-p} \frac{\Gamma(k + \delta)}{\Gamma(p + \delta)}}{(A - B)(p - \lambda)} a_{k,1} a_{k,2} \leq 1. \quad (2.23)$$

Since $f_v \in R^{\delta}_{\mu,p}(p, \alpha; A, B)$ ($v = 1, 2$), we readily see that

$$\sum_{k=p+1}^{\infty} \frac{[(1 - B)(k - p) + (A - B)(p - \alpha)](1 - \mu)^{k-p} \frac{\Gamma(k + \delta)}{\Gamma(p + \delta)}}{(A - B)(p - \alpha)} a_{k,v} \leq 1 \quad (v = 1, 2).$$

Therefore, by the Cauchy-Schwarz inequality, we obtain

$$\sum_{k=p+1}^{\infty} \frac{[(1 - B)(k - p) + (A - B)(p - \alpha)](1 - \mu)^{k-p} \frac{\Gamma(k + \delta)}{\Gamma(p + \delta)}}{(A - B)(p - \alpha)} \sqrt{a_{k,1} a_{k,2}} \leq 1. \quad (2.24)$$

From (2.23) and (2.24), we need only to show that

$$\frac{[(1 - B)(k - p) + (A - B)(p - \lambda)]}{(p - \lambda)} \sqrt{a_{k,1} a_{k,2}} \leq \frac{[(1 - B)(k - p) + (A - B)(p - \alpha)]}{(p - \alpha)} a_{k,1} a_{k,2} \quad (k \geq p + 1),$$

or, equivalently, that

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{(p - \lambda)[(1 - B)(k - p) + (A - B)(p - \alpha)]}{(p - \alpha)(1 - B)(k - p) + (A - B)(p - \lambda)} \quad (k \geq p + 1). \quad (2.25)$$

Hence, in the light of inequality (2.24). It is sufficient to prove that

$$\frac{(A - B)(p - \alpha)}{[(1 - B)(k - p) + (A - B)(p - \alpha)](1 - \mu)^{k-p} \frac{\Gamma(k + \delta)}{\Gamma(p + \delta)}}$$
In our present investigation, we shall make use of the familiar integral operator

\[ J_{c,p}(f)(z) = \frac{c + p}{z^{c}} \int_{0}^{z} t^{c-1} f(t) \, dt \quad (f \in T(p); c > -p), \] (3.1)

also, the fractional integral of order \( \eta \) is defined, for a function \( f \), by

\[ D_{z}^{-\eta}f(z) = \frac{1}{\Gamma(\eta)} \int_{0}^{z} f(\xi) \left( \frac{z}{z - \xi} \right)^{1-\eta} d\xi \quad (\eta > 0), \] (3.2)

3. Applications of Fractional Calculus

In our present investigation, we shall make use of the familiar integral operator \( J_{c,p} \) defined by (see [2], [3] and [4])

\[ \lambda \leq \frac{(p - \lambda)[(1 - B)(k - p) + (A - B)(p - \alpha)]}{(p - \alpha)[(1 - B)(k - p) + (A - B)(p - \lambda)]} \quad (k \geq p + 1). \] (2.26)

It follows from (2.26) that,

\[ \lambda \leq p - \frac{(1 - B)(k - p)(A - B)(p - \alpha)^2}{[(1 - B)(k - p) + (A - B)(p - \alpha)]^2(1 - \mu)k^{p+\eta} - [(A - B)(p - \alpha)]^2} \quad (k \geq p + 1). \]

Now, defining the function \( \Phi(k) \) by

\[ \Phi(k) = p - \frac{(1 - B)(k - p)(A - B)(p - \alpha)^2}{[(1 - B)(k - p) + (A - B)(p - \alpha)]^2(1 - \mu)^{k+\eta} - [(A - B)(p - \alpha)]^2} \quad (k \geq p + 1). \]

We see that \( \Phi(k) \) is an increasing function of \( k \) \((k \geq p + 1)\). Therefore, we conclude that

\[ \lambda = \Phi(p + 1) = p - \frac{(1 - B)(A - B)(1 - \alpha)^2}{[(1 - B) + (A - B)(p - \alpha)]^2(1 - \mu)^{p+\eta} - [(A - B)(p - \alpha)]^2}, \] (3.2)

this completes the proof.

**Theorem 6.** If \( f_1 \in R^{\delta}_{\mu,p}(\alpha; A, B) \) and \( f_2 \in R^{\delta}_{\mu,p}(\beta; A, B) \), which are defined by (2.22), then \((f_1 * f_2)(z) \in R^{\delta}_{\mu,p}(\xi; A, B)\), where

\[ \xi = p - \frac{(1 - B)(A - B)(p - \alpha)(p - \beta)}{[(1 - B) + (A - B)(p - \alpha)][(1 - B) + (A - B)(p - \beta)][(1 - \mu)^{k+\eta} - (A - B)^2(p - \alpha)(p - \beta)]} \] (3.28)

the result is the best possible for the functions

\[ f_1(z) = z^{p} - \frac{(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)]^2(1 - \mu)^{p+\eta} - [(A - B)(p - \alpha)]^2} z^{p+1}, \] (3.29)

and

\[ f_2(z) = z^{p} - \frac{(A - B)(p - \beta)}{[(1 - B) + (A - B)(p - \beta)]^2(1 - \mu)^{p+\eta} - [(A - B)(p - \beta)]^2} z^{p+1}. \] (3.30)
where the function $f$ is analytic in a simply-connected domain of the complex plane containing the origin and the multiplicity of $(z - \xi)^{-\eta}$ is removed by requiring $\log(z - \xi)$ to be real when $z - \xi > 0$.

The fractional derivative of order $\eta$ is defined, for a function $f$, by

$$D_z^\eta f(z) = \frac{1}{\Gamma(1 - \eta)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z - \xi)^{\eta}} d\xi \quad (0 \leq \eta < 1),$$

where the function $f(z)$ is constrained, and the multiplicity of $(z - \xi)^{-\eta}$ is removed, as above. In this section, we investigate the distortion properties of functions in the class $R^\delta_{\mu,p}(\alpha; A, B)$ involving the operator $J_{c,p}$ and the fractional calculus $D_z^{-\eta}$ and $D_z^\eta$. By using (3.1), (3.2), (3.3) and (1.1) it is easily deduced that:

$$D_z^\eta (J_{c,p}f(z)) = \frac{\Gamma(p + 1)}{\Gamma(2 - \eta)} z^{p-\eta} - \sum_{k=p+1}^{\infty} \frac{(c + p)\Gamma(k + 1)}{(c + k)\Gamma(k - \eta + 1)} a_k z^{-k - \eta},$$

and

$$D_z^{-\mu} (J_{c,p}f(z)) = \frac{\Gamma(p + 1)}{\Gamma(2 + \eta)} z^{p+\eta} - \sum_{k=p+1}^{\infty} \frac{(c + p)\Gamma(k + 1)}{(c + k)\Gamma(k + \eta + 1)} a_k z^{k + \eta}.$$

**Theorem 7.** Let the function $f$ defined by (1.1) be in the class $R^\delta_{\mu,p}(\alpha; A, B)$. Then

$$|D_z^{-\eta} (J_{c,p}f(z))| \geq \left\{ \frac{\Gamma(p + 1)}{\Gamma(p + 1 + \eta)} - \frac{(c + p)\Gamma(p + 2)(B - A)(p - \alpha)}{(c + p + 1)\Gamma(p + 2)(1 - B)(A - B)(p - \alpha)\Gamma(2 - \eta)} |z| \right\} |z|^{p + \eta},$$

and

$$|D_z^{-\eta} (J_{c,p}f(z))| \leq \left\{ \frac{\Gamma(p + 1)}{\Gamma(p + 1 + \eta)} + \frac{(c + p)\Gamma(p + 2)(A - B)(p - \alpha)}{(c + p + 1)\Gamma(p + 2)(1 - B)(A - B)(p - \alpha)\Gamma(2 + \eta)} |z| \right\} |z|^{p + \eta},$$

these results are sharp.

**Proof.** In view of Theorem 1 we have

$$\sum_{k=p+1}^{\infty} \frac{[(1 - B) + (A - B)(p - \alpha)](1 - \mu)\Gamma(p + \delta + 1)}{(A - B)(p - \alpha) \Gamma(p + \delta) \Gamma(p + \eta)} a_k \leq \frac{(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu) \Gamma(p + \delta + 1) \Gamma(p + \eta)}.$$

which readily yields

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{(A - B)(p - \alpha)}{[(1 - B) + (A - B)(p - \alpha)](1 - \mu) \Gamma(p + \delta + 1) \Gamma(p + \eta)}.$$

Suppose that function $F(z)$ defined in $U$ by

$$F(z) = \frac{\Gamma(p + \eta + 1)}{\Gamma(p + 1)} z^{-\eta} [D_z^{-\eta} (J_{c,p}f(z))] = z^p - \sum_{k=p+1}^{\infty} \frac{(c + p)\Gamma(k + 1)\Gamma(p + \eta + 1)}{(c + k)\Gamma(p + 1)\Gamma(k + \eta + 1)} a_k z^k$$

$$= z^p - \sum_{k=p+1}^{\infty} \Upsilon(k) a_k z^k,$$

(3.9)
where

$$\Upsilon(k) = \frac{(c+p)\Gamma(k+1)\Gamma(p+\eta+1)}{(c+k)\Gamma(p+1)\Gamma(k+\eta+1)}.$$  \hspace{1cm} (3.10)

Since \(\Upsilon(k)\) is a decreasing function of \(k\),

\[
0 < \Upsilon(k) \leq \Upsilon(p+1) = \frac{(c+p)(p+1)\Gamma(p+\eta+1)}{(c+p+1)(p+\eta+1)}.
\]  \hspace{1cm} (3.11)

By using (3.9) and (3.11) we have

\[
|F(z)| = \left| z^p - \sum_{k=p+1}^{\infty} \Upsilon(k) a_k z^k \right| \geq |z|^p - |z|^p \ Upsilon(p+1) \sum_{k=p+1}^{\infty} a_k z^k
\]  \hspace{1cm} (3.12)

and

\[
|F(z)| = \left| z^p + \sum_{k=p+1}^{\infty} \Upsilon(k) a_k z^k \right| \leq |z|^p - |z|^p \ Upsilon(p+1) \sum_{k=p+1}^{\infty} a_k z^k
\]  \hspace{1cm} (3.13)

which yield the inequalities (3.6) and (3.7) of Theorem 8.

This equalities in (3.6) and (3.7) are attained for the function \(f\) of which

\[
D_z^{-\eta} (J_{c,p} f(z)) = \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1+\eta)} - \frac{(c+p)\Gamma(p+2)(A-B)(p-\alpha)}{(c+p+1)\Gamma(p+1)\Gamma(p+\eta+2)(1-B)+(A-B)(p-\alpha)(1-\mu)\Gamma(p+1+\eta)} z \right\} z^{p+1},
\]  \hspace{1cm} (3.14)

or, equivalently

\[
J_{c,p} f(z) = z^p - \frac{(c+p)(A-B)(p-\alpha)}{(c+p+1)(1-B)+(A-B)(p-\alpha)(1-\mu)\Gamma(p+1+\eta)} z^{p+1},
\]

Thus the proof of Theorem 7 is completed.

Another inequalities can be given and the proof is omitted.

**Theorem 8.** Let the function \(f\) defined by (1.1) be in the class \(R^\alpha_{\mu,p}(\alpha; A, B)\). Then

\[
|D_z^n (J_{c,p} f(z))| \geq \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1+\eta)} - \frac{(c+p)\Gamma(p+2)(A-B)(p-\alpha)}{(c+p+1)\Gamma(p+1)\Gamma(p+\eta+2)(1-B)+(A-B)(p-\alpha)(1-\mu)\Gamma(p+1+\eta)} z \right\} |z|^{p-\eta},
\]  \hspace{1cm} (3.15)

and

\[
|D_z^n (J_{c,p} f(z))| \leq \left\{ \frac{\Gamma(p+1)}{\Gamma(p+1+\eta)} + \frac{(c+p)\Gamma(p+2)(A-B)(p-\alpha)}{(c+p+1)\Gamma(p+1)\Gamma(p+\eta+2)(1-B)+(A-B)(p-\alpha)(1-\mu)\Gamma(p+1+\eta)} z \right\} |z|^{p-\eta}.
\]  \hspace{1cm} (3.16)

Each of the assertions (3.15) and (3.16) is sharp.

Then, we can easily obtain the following two theorems and the proofs are omitted.

**Theorem 9.** Let the function \(f\) defined by (1.1) be in the class \(R^\alpha_{\mu,p}(\alpha; A, B)\). Then

\[
|J_c (D_z^n f(z))| \geq \left\{ \frac{(c+p)}{\Gamma(p+1-\eta)} - \frac{(c+p)\Gamma(p+2)(A-B)(p-\alpha)}{(c+p+1)\Gamma(p+1)\Gamma(p+\eta+2)(1-B)+(A-B)(p-\alpha)(1-\mu)\Gamma(p+1+\eta)} z \right\} |z|^{p-\eta},
\]  \hspace{1cm} (3.17)
and

\[ |J_c \, (D^\eta_z \, f(z))| \leq \left\{ \frac{(c+p)}{(c+\eta+1)\Gamma(p+1-\eta)} - \frac{(c+p)\Gamma(p+2)(A-B)(p-\alpha)}{(c+p+1)\Gamma(p+\eta+2)(1-B)+(A-B)(p-\alpha)[(1-\mu)\Gamma(p+\eta+1)]} \right\} |z|^{p-\eta}. \]  

(3.18)

**Theorem 10.** If the function \( f \) given by (1.1) be in the class \( R^\delta_{\mu,p}(\alpha; A, B) \). Then

\[ |J_c \, (D^{-\eta}_z \, f(z))| \geq \left\{ \frac{(c+p)}{(c+\eta+1)\Gamma(p+1+\eta)} - \frac{(c+p)\Gamma(p+2)(A-B)(p-\alpha)}{(c+p+1)\Gamma(p+\eta+2)(1-B)+(A-B)(p-\alpha)[(1-\mu)\Gamma(p+\eta+1)]} \right\} |z|^{p+\eta}, \]  

(3.19)

and

\[ |J_c \, (D^{-\eta}_z \, f(z))| \leq \left\{ \frac{(c+p)}{(c+\eta+1)\Gamma(p+1+\eta)} + \frac{(c+p)\Gamma(p+2)(A-B)(p-\alpha)}{(c+p+1)\Gamma(p+\eta+2)(1-B)+(A-B)(p-\alpha)[(1-\mu)\Gamma(p+\eta+1)]} \right\} |z|^{p+\eta}. \]  

(3.20)

**Remark**

By using the coefficients estimates (given by Theorem 2) of functions belonging to the subclass \( P^\delta_{\mu,p}(\alpha; A, B) \) and performing the same techniques of proofs given during Sections 2 and 3, then we can obtain the corresponding results of \( P^\delta_{\mu,p}(\alpha; A, B) \).

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