WONDERFUL COMPACTIFICATION OF AN
ARRANGEMENT OF SUBVARIETIES

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1. Introduction

The purpose of this paper is to define the so-called wonderful compactification of an arrangement of subvarieties, to prove its expected properties, to give a construction by a sequence of blow-ups and to discuss the order in which the blow-ups can be carried out.

Fix a nonsingular algebraic variety $Y$ over an algebraically closed field (of arbitrary characteristic). An arrangement of subvarieties $S$ is a finite collection of nonsingular subvarieties such that all nonempty scheme-theoretic intersections of subvarieties in $S$ are again in $S$, or equivalently, such that any two subvarieties intersect cleanly and the intersection is either empty or a subvariety in this collection (see Definition 2.1).

Let $S$ be an arrangement of subvarieties of $Y$. A subset $G \subseteq S$ is called a building set of $S$ if $\forall S \in S \setminus G$, the minimal elements in $\{G \in G : G \supseteq S\}$ intersect transversally and the intersection is $S$. A set of subvarieties $G$ is called a building set if all the possible intersections of subvarieties in $G$ form an arrangement $S$ (called the induced arrangement of $G$) and $G$ is a building set of $S$ (see Definition 2.2).

For any building set $G$, the wonderful compactification of $G$ is defined as follows.

**Definition 1.1.** Let $G$ be a nonempty building set and $Y^\circ = Y \setminus \cup_{G \in G} G$. The closure of the image of the natural locally closed embedding

$$Y^\circ \hookrightarrow \prod_{G \in G} Bl_G Y,$$

is called the wonderful compactification of the arrangement $G$ and is denoted by $Y_G$.

The following description of $Y_G$ is the main theorem and is proved at the end of §2.3. A $G$-nest is a subset of the building set $G$ satisfying some inductive condition (see Definition 2.3).

**Theorem 1.2.** Let $Y$ be a nonsingular variety and $G$ be a nonempty building set of subvarieties of $Y$. Then the wonderful compactification $Y_G$ is a
nonsingular variety. Moreover, for each \( G \subset G \) there is a nonsingular divisor \( D_G \subset Y_G \), such that:

(i) The union of these divisors is \( Y_G \setminus Y^o \).
(ii) Any set of these divisors meet transversally. An intersection of divisors \( D_{T_1} \cap \cdots \cap D_{T_r} \) is nonempty exactly when \( \{T_1, \cdots, T_r\} \) form a \( G \)-nest.

This theorem is proved by a construction of \( Y_G \) through an explicit sequence of blow-ups of \( Y \) along nonsingular centers (Definition 2.12, Theorem 2.13).

Here are some examples of wonderful compactifications of an arrangement (see §4 for details).

(1) De Concini-Procesi’s wonderful model of subspace arrangements (4.11). In this case, \( Y \) is a vector space, \( S \) is a finite set of proper subspaces of \( Y \) and \( G \) is a building set with respect to \( S \).

(2) Suppose \( X \) is a nonsingular algebraic variety, \( n \) is a positive integer and \( Y \) is the Cartesian product \( X^n \). A diagonal of \( X^n \) is

\[
\Delta_I = \{(p_1, \ldots, p_n) \in X^n | p_i = p_j, \forall i, j \in I\}
\]

for \( I \subseteq [n], |I| \geq 2 \). A polydiagonal is an intersections of diagonals

\[
\Delta_{I_1} \cap \cdots \cap \Delta_{I_k}
\]

for \( I_i \subseteq [n], |I_i| \geq 2 \ (1 \leq i \leq k) \).

(a) Fulton-MacPherson configuration space \( X[n] \) (4.12). This is the wonderful compactification \( Y_G \) where \( G \) is the set of all diagonals in \( Y \) and the induced arrangement \( S \) is the set of all polydiagonals. It is a special example of Kuperberg-Thurston’s compactification \( X^\Gamma \) when \( \Gamma \) is the complete graph with \( n \) vertices.

(b) Ulyanov’s polydiagonal compactification \( X(n) \) (4.15). It is the wonderful compactification \( Y_G \) where \( S = G \) are the set of all polydiagonals.

(c) Kuperberg-Thurston’s compactification \( X^\Gamma \) where \( \Gamma \) is a connected graph with \( n \) labeled vertices (4.3). \( X^\Gamma \) is the wonderful compactification \( Y_G \) where \( G \) is the set of diagonals in \( Y \) corresponding to vertex-2-connected subgraphs of \( \Gamma \), and \( S \) is the set of polydiagonals generated by intersections of diagonals in \( G \).

(3) Moduli space of rational curves with \( n \) marked points \( \overline{M}_{0,n} \) (4.14). It is the wonderful compactification \( Y_G \) where \( Y = (\mathbb{P}^1)^{n-3} \) and \( G \) is set of all diagonals and augmented diagonals \( \Delta_{I,a} \) defined as

\[
\Delta_{I,a} := \{(p_1, \cdots, p_n) \in (\mathbb{P}^1)^{n-3} | p_i = a, \forall i \in I\}
\]

for \( I \subseteq \{4, \ldots, n\}, |I| \geq 2 \) and \( a \in \{0, 1, \infty\} \).
The moduli space $\overline{M}_{0,n}$ is also the wonderful compactification $Y_G$ where $Y = \mathbb{P}^{n-3}$ and $G$ is the set of all projective subspaces of $\mathbb{P}^{n-3}$ spanned by any subset of fixed $n - 1$ generic points [Ka93].

(4) Hu’s compactification of open varieties ([H3]). It is a wonderful compactification of $(Y, S, G)$ where $Y$ is a nonsingular algebraic variety, $S = G$ is an arrangement of subvarieties of $Y$ [Hu03].

During the study of the sequence of blow-ups, a natural question arises: in which order can we carry out the blow-ups to obtain the wonderful compactification? For example, the original construction of Fulton-MacPherson configuration space $X[n]$, Keel’s construction of $\overline{M}_{0,n}$, and Kapranov’s construction of $\overline{M}_{0,n}$, none of them are obtained by blowing up along the centers with increasing dimensions. If we change the order of blow-ups, do we still get the same variety?

We answer this question with the following theorem which is proved in §3. The notation $\tilde{G}$ stands for the so-called dominant transform of $G$ (see Definition 2.7) which is similar but slightly different to the strict transform: for a subvariety $G$ contained in the center of a blow-up, the strict transform $\text{Bl}_G Y$ is empty but the dominant transform $\tilde{G}$ is the preimage of $G$.

**Theorem 1.3.** Let $Y$ be a nonsingular variety and $G = \{G_1, \ldots, G_N\}$ be a nonempty building set of subvarieties of $Y$. Let $I_i$ be the ideal sheaf of $G_i \in G$.

(i) The wonderful compactification $Y_G$ is isomorphic to the blow-up of $Y$ along the ideal sheaf $I_1 I_2 \cdots I_N$.

(ii) If we arrange $G = \{G_1, \ldots, G_N\}$ in such an order that

(*) the first $i$ terms $G_1, \ldots, G_i$ form a building set for any $1 \leq i \leq N.$

then

$$Y_G = \text{Bl}_{G_N} \cdots \text{Bl}_{G_2} \text{Bl}_{G_1} Y,$$

where each blow-up is along a nonsingular subvariety.

**Example:** By Keel’s construction [Ke92] and the above theorem, $\overline{M}_{0,n}$ is isomorphic to the wonderful compactification $Y_G$ where $Y$ is $(\mathbb{P}^1)^{n-3}$ and $G$ is set of all diagonals and augmented diagonals. In other words, we can blow up along the centers in any order satisfying (*) (e.g., increasing dimension). As a consequence, we have

**Corollary 1.4.** Let $\psi : \mathbb{P}^1[n] \to (\mathbb{P}^1)^3$ be the composition of the natural morphism $\mathbb{P}^1[n] \to (\mathbb{P}^1)^n$ and let $\pi_{123} : (\mathbb{P}^1)^n \to (\mathbb{P}^1)^3$ be the projection to the first three components. Then $\overline{M}_{0,n}$ is isomorphic to the fiber of $\psi$ over the point $(0, 1, \infty) \in (\mathbb{P}^1)^3$. Equivalently, $\overline{M}_{0,n}$ is isomorphic to the fiber over any point $(p_1, p_2, p_3)$ where $p_1, p_2, p_3$ are three distinct points in $\mathbb{P}^1$.

Similarly, Kapranov’s construction does not delicately depend on the order of the blow-ups, for example we can blow up along the centers in any order of increasing dimension.
This article is built on the following previous works: Fulton-MacPherson [FM94], De Concini-Procesi [DP95], MacPherson-Procesi [MP98], Ulyanov [U02], Hu [Hu03].

The inspiring paper by De Concini and Procesi [DP95] gives a thorough discussion of an arrangement of linear subspaces of a vector space. Given a vector space \( Y \) and an arrangement of subspaces \( S \), De Concini and Procesi give a condition for a subset \( G \subseteq S \) such that there exists a so called wonderful model \( Y_G \) of the arrangement, in which the elements in \( G \) are replaced by simple normal crossing divisors. De Concini and Procesi call \( G \) a building set. Their paper also gives a criterion of whether the intersection of a collection of such divisors is nonempty by introducing the notion of a nest.

Later, this idea has been generalized to nonsingular varieties over \( \mathbb{C} \) with conical stratifications by MacPherson and Procesi. They consider conical stratifications in place of subspace arrangements in [DP95]. The notion of building set and nest is generalized in this setting. The idea of the construction of wonderful compactifications of arrangement of subvarieties in our paper is largely inspired by the beautiful paper [MP98].

In our paper, we give definitions of arrangements of subvarieties, building sets and nests. The wonderful compactifications are shown to have properties analogous to the ones in [DP95] or [MP98].

The paper is organized as follows. In section 2 we give the construction of the wonderful compactification \( Y_G \). In §2.1 we give the definition of arrangements, building sets and nests. §2.2 is the description of how the arrangements, building sets and nests vary under one blow-up. §2.3 gives the construction of \( Y_G \). In section 3 we discuss that in which order could the blow-ups be carried out to obtain \( Y_G \). Section 4 gives some examples of wonderful compactifications. In §5.1 we discuss clean intersections and transversal intersections. In §5.2 we give the proofs of previous statements. In §5.3 we discuss how different choices of blow-ups change the codimension of the centers. Finally in §5.4, we give the statements for a general (non-simple) arrangement (proofs omitted).

Acknowledgements. In many ways the author greatly indebted to Mark de Cataldo, his Ph.D. advisor. He is very grateful to William Fulton for valuable comments. He would also thank Blaine Lawson, Dror Varolin, Jun-Muk Hwang, especially Herwig Hauser, for their many useful comments and encouragement. He thanks Jonah Sinick for carefully proof-reading the paper. He thanks the referee for many constructive suggestions to improve the presentation.

2. ARRANGEMENTS OF SUBVARIETIES AND THE WONDERFUL COMPACTIFICATIONS

By a variety we shall mean a reduced and irreducible algebraic scheme defined over a fixed algebraically closed field (of arbitrary characteristic). A subvariety of a variety is a closed subscheme which is a variety. By a point of
a variety we shall mean a closed point of that variety. By the intersection of subvarieties \( Z_1, \ldots, Z_k \) we shall mean the set-theoretic intersection (denoted by \( Z_1 \cap \cdots \cap Z_k \)). We denote the ideal sheaf of a subvariety \( V \) of a variety \( Y \) by \( I_V \).

In this section, we will discuss arrangements, building sets and nests, based on which we define the wonderful compactifications of an arrangement. The idea is inspired by [DP95][MP98].

### 2.1. Arrangement, building set, nest

The following definition of arrangement is adapted from [Hu03]. For a brief review of the definitions of clean intersection and of transversal intersection, please see Appendix \textsection 5.1.

**Definition 2.1.** A simple arrangement of subvarieties of a nonsingular variety \( Y \) is a finite set \( S = \{S_i\} \) of nonsingular closed subvarieties \( S_i \) properly contained in \( Y \) satisfying the following conditions:

(i) \( S_i \) and \( S_j \) intersect cleanly (i.e., their intersection is nonsingular and the tangent bundles satisfy \( T(S_i \cap S_j) = T(S_i) \mid_{(S_i \cap S_j)} \cap T(S_j) \mid_{(S_i \cap S_j)} \)),

(ii) \( S_i \cap S_j \) either equals to some \( S_k \) or is empty.

The above definition is equivalent to say that \( S \) is an arrangement if and only if it is closed under scheme-theoretic intersections, cf. Lemma 5.1.

Although we will discuss only the simple arrangement for simplicity, most statements still hold, with minor revision, for general arrangements, i.e., instead of the above condition (2), we allow \( S_i \cap S_j \) to be a disjoint union of some \( S_k \)’s. (See Appendix \textsection 5.4.)

For a simple arrangement, the condition of transversality can be checked at one point (instead of at every point) of the intersection (Lemma 5.2).

**Definition 2.2.** Let \( S \) be an arrangement of subvarieties of \( Y \). A subset \( G \subseteq S \) is called a building set of \( S \) if \( \forall S \in S \), the minimal elements in \( \{G \in G : G \supseteq S\} \) intersect transversally and their intersection is \( S \) (by our definition of transversality \textsection 5.1, the condition is satisfied if \( S \in G \)). In this case, these minimal elements are called the \( G \)-factors of \( S \).

A finite set \( G \) of nonsingular subvarieties of \( Y \) is called a building set if the set of all possible intersections of collections of subvarieties from \( G \) forms an arrangement \( S \), and that \( G \) is a building set of \( S \). In this situation, \( S \) is called the arrangement induced by \( G \).

**Example:** Let \( X \) be a nonsingular variety of positive dimension and \( Y \) be the Cartesian product \( X^3 \).

1. The set \( G = \{\Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{123}\} \) is a building set whose induced arrangement is \( G \) itself.
2. The set \( G = \{\Delta_{12}, \Delta_{13}\} \) is a building set whose induced arrangement is \( \{\Delta_{12}, \Delta_{13}, \Delta_{123}\} \). On the other hand, \( G \) is not a building set of the arrangement \( \{\Delta_{12}, \Delta_{13}, \Delta_{23}, \Delta_{123}\} \).
Remark: The building set $G$ defined here is related to the one defined in [DP95] as follows. For any point $y \in Y$, define $S^*_y = \{T^\perp_{S,y}\}_{S \in S}$ and $G^*_y = \{T^\perp_{S,y}\}_{S \in G}$. We claim that the set $G$ is a building set if and only if $G^*_y$ is a building set for all $y \in Y$ in the sense of DeConcini and Procesi.

Indeed, $S$ being an arrangement is equivalent to the condition that for any $y \in Y$, $S^*_y$ is a finite set of nonzero linear subspaces of $T^*_y$ which is closed under sum, and such that each element of $S^*_y$ is equal to $T^\perp_{S,y}$ for a unique $S \in S$. The subset $G \subseteq S$ being a building set is equivalent to the condition that $\forall S \in S, \forall y \in S$, suppose $T^\perp_1, \ldots, T^\perp_k$ are all the maximal elements of $G^*_y$ contained in $T^\perp_{S,y}$, then they form a direct sum and

$$T^\perp_1 \oplus T^\perp_2 \oplus \cdots \oplus T^\perp_k = T^\perp_{S,y},$$

which is exactly the definition of building set in [DP95] §2.3 Theorem (2).

**Definition 2.3.** (cf. [MP98] §4) A subset $T \subseteq G$ is called $G$-nested (or a $G$-nest) if it satisfies one of the following equivalent conditions:

1. There is a flag of elements in $S$: $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_t$, such that $T = \bigcup_{i=1}^t \{A : A \text{ is a } G\text{-factor of } S_i\}$. (We say $T$ is induced by the flag $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_t$.)
2. Let $A_1, \ldots, A_k$ be the minimal elements of $T$, then they are all the $G$-factors of certain element in $S$. For any $1 \leq i \leq k$, the set $\{A \in T : A \supseteq A_i\}$ is also $G$-nested defined by induction.

Example: Let $X$ be a nonsingular variety of positive dimension and $Y$ be the Cartesian product $X^4$. Take the building set $G$ to be the set of all diagonals in $X^4$.

1. The set $T = \{\Delta_{12}, \Delta_{13}\}$ is a $G$-nest, since it can be induced by the flag $\Delta_{123} \subseteq \Delta_{12}$.
2. The set $T = \{\Delta_{12}, \Delta_{13}, \Delta_{123}\}$ is a $G$-nest, since it can be induced by the flag $\Delta_{1234} \subseteq (\Delta_{12} \cap \Delta_{34})$.
3. The set $T = \{\Delta_{12}, \Delta_{13}\}$ is not a $G$-nest. Indeed, the intersection of the minimal elements in $T$ is $\Delta_{123}$, which has only one $G$-factor: $\Delta_{123}$ itself. By condition (ii) of the definition, $T$ is not a $G$-nest.

Note that the intersection of elements in a $G$-nest $T$ is nonempty by (ii). Now we explain why the two conditions (i) and (ii) are equivalent. Given a set $T$ satisfying (ii), we can construct a flag as follows: define $S_1 = A_1 \cap \cdots \cap A_k$ which is the intersection of all subvarieties in $T$. Let $S_2$ be the intersection of the subvarieties in $T$ which are not minimal elements in $T$ that contain $S_1$. Then inductively let $S_{j+1}$ be the intersection of those
which are not minimal elements in \( \mathcal{T} \) that contains \( S_j \). It is easy to show that \( \mathcal{T} \) is induced by the flag \( S_1 \subseteq S_2 \subseteq \cdots \), therefore (ii) \( \Rightarrow \) (i). On the other hand, let \( S_{ij} = A_j \) \((1 \leq j \leq k)\) be the \( \mathcal{G} \)-factors of \( S_1 \). Note that for any \( 1 \leq i \leq \ell \), a \( \mathcal{G} \)-factor of \( S_i \) must contain exactly one element of \( A_1, \ldots, A_k \), otherwise the \( A_i \)'s will not intersect transversally. Let \( S_{ij} \) be the \( \mathcal{G} \)-factor of \( S_i \) that contains \( A_j \). (Define \( S_{ij} = Y \) if there is no such a \( \mathcal{G} \)-factor.) Then for each \( 1 \leq j \leq k \), there is a flag of elements \( S_{2j} \subseteq S_{3j} \subseteq \cdots \subseteq S_{\ell j} \), which induces the \( \mathcal{G} \)-nest \( \{ A \in \mathcal{T} : A \supseteq A_i \} \). This shows (i) \( \Rightarrow \) (ii).

We state some basic properties about arrangements and building sets.

**Lemma 2.4.** Let \( Y \) be a nonsingular variety and \( \mathcal{G} \) be a building set with the induced arrangement \( S \). Suppose \( S \in \mathcal{S} \) and \( G_1, \ldots, G_k \) are all the \( \mathcal{G} \)-factors of \( S \) (Definition 2.2). Then

(i) For any \( 1 \leq m \leq k \), the subvarieties \( G_1, \ldots, G_m \) are all the \( \mathcal{G} \)-factors of the subvariety \( G_1 \cap \cdots \cap G_m \).

(ii) Suppose \( F \in \mathcal{G} \) is minimal such that \( F \cap S \neq \emptyset \), \( F \subseteq G_1, \ldots, G_m \) and \( F \not\subseteq G_{m+1}, \ldots, G_k \). Then \( F, G_{m+1}, \ldots, G_k \) are all the \( \mathcal{G} \)-factors of the subvariety \( F \cap S \).

**Proof.** See Appendix §5.2. \( \square \)

Here is an immediate consequence of Lemma 2.4.

**Lemma 2.5.** If \( G_1, \ldots, G_k \in \mathcal{G} \) are all minimal and their intersection \( S \) is nonempty, then \( G_1, \ldots, G_k \) are all the \( \mathcal{G} \)-factors of \( S \).

Next, we introduce the notion of the \( F \)-factorization, which turns out to be a convenient terminology for the proof of the construction of wonderful compactifications.

**Definition-Lemma 2.6.** Suppose \( F \in \mathcal{G} \) is minimal. Then

(i) Any \( G \in \mathcal{G} \) either contains \( F \) or intersects transversally with \( F \).

(ii) Every \( S \in \mathcal{S} \) satisfying \( S \cap F \neq \emptyset \) can be uniquely expressed as \( A \cap B \) where \( A, B \in \mathcal{S} \cup \{ Y \} \) satisfy \( A \supseteq F \) and \( B \cap F \) (hence \( A \cap B \)). We call this expression \( S = A \cap B \) the \( F \)-factorization of \( S \).

(iii) Suppose the \( \mathcal{G} \)-factors of \( S \) are \( G_1, \ldots, G_k \) where \( G_1, \ldots, G_m \) contain \( F \) \((0 \leq m \leq k)\), the case \( m = 0 \) is understood to mean that no \( \mathcal{G} \)-factors of \( S \) contain \( F \) and let the \( F \)-factorization of \( S \) be \( A \cap B \).

Then \( G_1, \ldots, G_m \) are all the \( \mathcal{G} \)-factors of \( A \) and \( G_{m+1}, \ldots, G_k \) are all the \( \mathcal{G} \)-factors of \( B \), so \( A = \cap_{i=1}^m G_i \) and \( B = \cap_{i=m+1}^k G_i \). (Here we assume \( A = Y \) if \( m = 0 \), assume \( B = Y \) if \( m = k \).)

(iv) Suppose \( S' \in S \) such that \( S' \cap S \cap F \neq \emptyset \). Let \( S' = A' \cap B' \) be the \( F \)-factorization of \( S' \).

Then \( F \cap (B \cap B') \), therefore the \( F \)-factorization of \( S \cap S' \) is \((A \cap A') \cap (B \cap B')\).

**Proof.** See Appendix §5.2. \( \square \)
2.2. **Change of an arrangement after a blow-up.** Before considering a sequence of blow-ups, we first consider a single blow-up. Let $Y$ be a nonsingular variety and $G$ be a building set with the induced arrangement $S$. In Proposition 2.8 we show that, if $F \in G$ is minimal, then there exists a natural arrangement $\tilde{S}$ in $Bl_F Y$ induced from $S$ and a natural building set $\tilde{G}$ induced from $G$.

**Definition 2.7.** Let $Z$ be a nonsingular subvariety of a nonsingular variety $Y$ and $\pi : Bl_Z Y \to Y$ be the blow-up of $Y$ along $Z$.

For any irreducible subvariety $V$ of $Y$, we define the dominant transform of $V$, denoted by $\tilde{V}$ or $V^\sim$, to be the strict transform of $V$ if $V \not\subseteq G$ and to be the scheme-theoretic inverse $\pi^{-1}(V)$ if $V \subseteq G$.

For a sequence of blow-ups, we still denote the iterated dominant transform $\cdots ((\cdot \cdots (V^\sim)^\sim) \cdots)^\sim$ by $\tilde{V}$ or $V^\sim$.

**Remark:** The reason that we introduce the notion of dominant transform is because that the strict transform does not behave as expected: the strict transform of a subvariety contained in the center of a blow-up is empty, which is not what we need.

**Proposition 2.8.** Let $Y$ be a nonsingular variety and $G$ be a building set with the induced arrangement $S$. Let $F$ be a minimal element in $G$ and let $\pi : Bl_F Y \to Y$ be the blow-up of $Y$ along $F$. Denote the exceptional divisor by $E$.

(i) The collection $\tilde{S}$ of subvarieties in $Bl_G Y$ defined as

$$\tilde{S} := \{\tilde{S}\}_{S \in S} \cup \{\tilde{S} \cap E\}_{\emptyset \subseteq S \cap F \subseteq S}$$

is a (simple) arrangement of subvarieties in $Bl_G Y$.

(ii) $\tilde{G} := \{\tilde{G}\}_{G \in G}$ is a building set of $\tilde{S}$.

(iii) Given a subset $T$ of $G$, we define $\tilde{T} := \{\tilde{A}\}_{A \in T}$. Then $T$ is a $G$-nest if and only if $\tilde{T}$ is a $\tilde{G}$-nest.

The proof is in Appendix §5.2. The main ingredient is the following lemma.

**Lemma 2.9.** Assume the same notation as in Proposition 2.8. Assume $A$, $A_1$, $A_2$, $B$, $B_1$, $B_2$, $G$ are nonsingular subvarieties of $Y$.

(i) Suppose $A \supseteq F$. Then $\tilde{A} \cap E$ intersect transversally (hence cleanly).

(ii) Suppose $A_1 \not\subseteq A_2$, $A_2 \not\subseteq A_1$ and $A_1 \cap A_2 = F$ and the intersection is clean. Then $\tilde{A_1} \cap \tilde{A_2} = \emptyset$.

(iii) Suppose $A_1$ and $A_2$ intersect cleanly and $F \not\subseteq A_1 \cap A_2$. Then $\tilde{A_1} \cap \tilde{A_2} = (A_1 \cap A_2)^\sim$. Moreover, $\tilde{A_1}$ and $\tilde{A_2}$ intersect cleanly.

(iv) Suppose $B_1$ and $B_2$ intersect cleanly, and $G$ is transversal to $B_1$, $B_2$ and $B_1 \cap B_2$. Then $\tilde{B_1} \cap \tilde{B_2} = (B_1 \cap B_2)^\sim$. Moreover, $\tilde{B_1}$ and $\tilde{B_2}$ intersect cleanly.
(v) Suppose $A \ni B$, $F \subseteq A$ and $F \ni B$. Then $\hat{A} \cap \hat{B} = (A \cap B)^\sim$. Moreover, $\hat{A} \ni \hat{B}$ and $(E \cap \hat{A}) \ni \hat{B}$.

(vi) Assume $F \subseteq A$, $F \ni B_1 \ni B_2$, $G \subseteq F \cap B_1$ and $G \ni B_2$. Then $\hat{G} \cap \hat{A} \cap (B_1 \cap B_2)^\sim = \hat{G} \cap \hat{A} \cap \hat{B}_2$ where the latter is a transversal intersection.

Proof. See Appendix §5.2. □

2.3. A sequence of blow-ups and the construction of wonderful compactifications. Now we study a sequence of blow-ups, give different descriptions of a wonderful compactification, and study the relations of the arrangements occurred in the sequence of blow-ups.

Given $k$ morphisms between algebraic varieties with the same domain $f_i : X \to Y_i$, we adopt the notation $(f_1, f_2, \ldots, f_k) : X \to Y_1 \times \cdots \times Y_k$ to be the composition of the diagonal morphism $X \to X \times \cdots \times X$ with the morphism $f_1 \times \cdots \times f_k$.

Lemma 2.10. Let $V$ and $W$ be two nonsingular subvarieties of a nonsingular variety $Y$ such that either $V$ and $W$ intersect transversally or one of $V$ and $W$ contains the other. Let $f : Y_1 \to Y$ (resp. $g : Y_2 \to Y$) be the blow-up of $Y$ along $W$ (resp. $V$). Let $g' : Y_3 \to Y_1$ be the blow-up of $Y_1$ along the dominant transform $\hat{V}$. Then there exists a morphism $f' : Y_3 \to Y_2$ such that the following diagram commutes:

\[
\begin{array}{ccc}
Y_3 & \xrightarrow{f'} & Y_2 \\
\downarrow{g'} & & \downarrow{g} \\
Y_1 & \xrightarrow{f} & Y \\
\end{array}
\]

Moreover, $(g', f') : Y_3 \to Y_1 \times Y_2$ is a closed embedding.

Proof. Because of the universal property of blowing up ([Ha77] Proposition 7.14), to show the existence of $f'$, we need only to show that $(fg')^{-1}I_V \cdot O_{Y_3}$ is an invertible sheaf of ideals on $Y_3$. This is true because by our choice of $V$ and $W$, the sheaf $f^{-1}I_V \cdot O_{Y_1}$ is either $I_{\hat{V}}$ or $I_{\hat{V}}I_E$, where $E$ is the exceptional divisor of the blow-up $f : Y_1 \to Y$. Hence the ideal sheaf

\[(fg')^{-1}I_V \cdot O_{Y_3} = g'^{-1}(f^{-1}I_V \cdot O_{Y_1}) \cdot O_{Y_3}\]

is either $(g'^{-1}I_{\hat{V}}) \cdot O_{Y_3}$ or $g'^{-1}(I_{\hat{V}} \cdot I_E) \cdot O_{Y_3}$, both of which are invertible by the construction of $g'$, therefore the ideal sheaf $(fg')^{-1}I_V \cdot O_{Y_3}$ is invertible.

The fact that $(g', f')$ is a closed embedding can be checked using local parameters. □
Lemma 2.11. Suppose $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are nonsingular varieties such that the following diagram commutes,

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & X_3 \\
\downarrow{g_1} & & \downarrow{g_2} & & \downarrow{g_3} \\
Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & Y_3
\end{array}
$$

If $(g_1, f_1) : X_1 \to Y_1 \times X_2$ and $(g_2, f_2) : X_2 \to Y_2 \times X_3$ are closed embeddings, then $(g_1, f_2 f_1) : X_1 \to Y_1 \times X_3$ is also a closed embedding.

As a consequence, if we have the following commutative diagram

$$
\begin{array}{ccc}
X_1 & \xrightarrow{f_1} & X_2 & \xrightarrow{f_2} & \cdots & \xrightarrow{f_{k-1}} & X_k \\
\downarrow{g_1} & & \downarrow{g_2} & & \cdots & & \downarrow{g_k} \\
Y_1 & \xrightarrow{h_1} & Y_2 & \xrightarrow{h_2} & \cdots & \xrightarrow{h_{k-1}} & Y_k
\end{array}
$$

and $(g_i, f_i) : X_i \to Y_i \times X_{i+1}$ are closed embeddings for all $1 \leq i \leq k-1$, then $(g_1, f_{k-1} \cdots f_1) : X_1 \to Y_1 \times X_k$ is also a closed embedding.

Proof. The composition of two closed embeddings is still a closed embedding, so

$$
\phi := (g_1, g_2 f_1, f_2 f_1) : X_1 \to Y_1 \times Y_2 \times X_3
$$

is a closed embedding, whose image $\phi(X_1)$ is a closed subvariety of $Y_1 \times Y_2 \times X_3$ which is isomorphic to $X_1$. Consider the projection $\pi_{13} : Y_1 \times Y_2 \times X_3 \to Y_1 \times X_3$, and the morphism $\Gamma_{h_1 \times 1 X_3} : Y_1 \times X_3 \to Y_1 \times Y_2 \times X_3$.

Consider the projection $\pi_{13} : Y_1 \times Y_2 \times X_3 \to Y_1 \times X_3$, and the morphism $\Gamma_{h_1 \times 1 X_3} : Y_1 \times X_3 \to Y_1 \times Y_2 \times X_3$.

$$
\begin{array}{ccc}
X_1 & \xrightarrow{\phi} & Y_1 \times Y_2 \times X_3 \\
\downarrow{(g_1, f_2 f_1)} & & \downarrow{\Gamma_{h_1 \times 1 X_3}} \\
Y_1 \times X_3
\end{array}
$$

Notice that $\pi_{13} \circ (\Gamma_{h_1 \times 1 X_3})$ is the identity automorphism of $Y_1 \times X_3$, and $(\Gamma_{h_1 \times 1 X_3}) \circ \pi_{13} = \phi(X_1)$ is the identity automorphism of $\phi(X_1)$. It follows that $(g_1, f_2 f_1) : X_1 \to Y_1 \times X_3$ is a closed embedding. \qed

**Definition 2.12 (Inductive construction of $Y_G$).** Let $Y$ be a nonsingular variety, $\mathcal{S}$ be an arrangement of subvarieties and $\mathcal{G}$ be a building set of $\mathcal{S}$. Suppose $\mathcal{G} = \{G_1, \ldots, G_N\}$ is indexed in an order compatible with inclusion relations, i.e. $i \leq j$ if $G_i \subseteq G_j$. We define $(Y_k, \mathcal{S}^{(k)}, \mathcal{G}^{(k)})$ inductively with respect to $k$:

(i) For $k = 0$, define $Y_0 = Y$, $\mathcal{S}^{(0)} = \mathcal{S}$, $\mathcal{G}^{(0)} = \mathcal{G} = \{G_1, \ldots, G_N\}$, $G_i^{(0)} = G_i$ for $1 \leq i \leq N$.

(ii) Assume that $(Y_{k-1}, \mathcal{S}^{(k-1)}, \mathcal{G}^{(k-1)})$ is constructed.

* Define $Y_k$ to be the blow-up of $Y_{k-1}$ along the nonsingular subvariety $G_{k-1}^{(k-1)}$. 
Define $G^{(k)} := (G^{(k-1)})^\sim$ for $G \in \mathcal{G}$ and define
$$G^{(k)} := \{G^{(k)}\}_{G \in \mathcal{G}}.$$ 

- Define $S^{(k)}$ to be the induced arrangement of $G^{(k)}$.

(iii) Continue the inductive construction until $k = N$. We obtain
$$(Y_N, S^{(N)} , G^{(N)})$$
where all the subvarieties in the building set $G^{(N)}$ are divisors.

Remark: In step (ii) we need Proposition 2.8. Indeed, since $G^{(k-1)}_i$ for $i < k$ are all divisors hence are too large to be contained in $G^{(k-1)}_k$, $G^{(k-1)}_k$ is minimal in $G^{(k-1)}$. Proposition 2.8 then asserts the existence of a naturally induced arrangement $S^{(k)}$, and asserts that $G^{(k)} = \{G^{(k)}\}_{G \in \mathcal{G}}$ is a building set with respect to $S^{(k)}$.

**Proposition 2.13.** The variety $Y_N$ constructed in Definition 2.12 is isomorphic to the wonderful compactification $Y_G$ defined in Definition 1.1.

**Proof.** We prove by induction that $Y_k$ is the closure of the inclusion
$$Y^\circ \hookrightarrow \prod_{i=1}^k \text{Bl}_{G_i} Y.$$ 

The proposition is then the special case $k = N$.

Let $0 \leq i \leq k-1$. Since $G^{(i)}_{i+1}$ is minimal in $G^{(i)}$, Definition-Lemma 2.6 (i) asserts that there are only two possible relations between the nonsingular subvarieties $G^{(i)}_k$ and $G^{(i)}_{i+1}$ of $Y_i$: either $G^{(i)}_k \supseteq G^{(i)}_{i+1}$ or $G^{(i)}_k \cap G^{(i)}_{i+1}$. Therefore Lemma 2.10 applies. Since $G^{(i+1)}_k = (G^{(i)}_k)^\sim$, there exists a morphism $f'$ such that following diagram commutes.

The morphism $(g', f') : B_{G^{(i+1)}_k} Y_{i+1} \to Y_{i+1} \times B_{G^{(i)}_k} Y_i$ is a closed embedding. Using Lemma 2.11 on the following diagram

and the facts that $Y_k = B_{G^{(k-1)}_k} Y_{k-1}$, $G^{(0)}_k = G_k$ and $Y_0 = Y$, we conclude that the morphism
$$Y_k \to Y_{k-1} \times B_{G_k} Y$$
is a closed embedding. Because the composition of closed embeddings is still a closed embedding, the morphism

$$Y_k \rightarrow \prod_{i=1}^{k} \text{Bl}_{G_i} Y$$

is a closed embedding. Then since $Y^\circ$ is an open subset of $Y_k$ and since $Y_k$ is irreducible, from the following composition

$$Y^\circ \hookrightarrow Y_k \hookrightarrow Y \times \prod_{i=1}^{k} \text{Bl}_{G_i} Y$$

we see that the closure of $Y^\circ$ in $Y \times \prod_{i=1}^{k} \text{Bl}_{G_i} Y$ is $Y_k$.

Proof of Theorem 1.2. Since $Y^G_\ast \cong Y_N$, $Y^G_\ast$ is nonsingular, $D_G := G^{(N)}$ are codimension one nonsingular subvarieties of $Y^G_\ast$ and $Y^G_\ast \setminus Y^\circ = \cup D_G$. Therefore (i) is clear.

For any $T_1, \ldots, T_r$ in $G$ that form a $G$-nest, $D_{T_1}, \ldots, D_{T_r}$ form a $G^{(N)}$-nest, hence

$$D_{T_1} \cap \cdots \cap D_{T_r} \neq \emptyset$$

by the definition of nest. Conversely, given $T_1, \ldots, T_r$ in $G$ such that the above intersection is nonempty, Lemma 2.5 implies that $D_{T_1}, \ldots, D_{T_r}$ are all the $G^{(N)}$-factors of the intersection and therefore intersect transversally. Moreover, by the definition of nest, $D_{T_1}, \ldots, D_{T_r}$ form a $G^{(N)}$-nest. Proposition 2.8 then implies that $T_1, \ldots, T_r$ form a $G$-nest. So (ii) is clear. \hfill \Box

3. ORDER OF BLOW-UPs

In this section we shall prove Theorem 1.3. We shall use this theorem in §4.2, §4.3 and §4.4. For the proof, we need the following proposition which is stronger than Proposition 2.8 (2) in the sense that a building set still induces a building set after a blow-up even when the center of the blow-up is not assumed to be minimal.

Proposition 3.1. Suppose $G = \{G_1, \ldots, G_k\}$ is a building set of an arrangement $S$ in $Y$ and $F \in G$ is minimal. Let $\phi : \text{Bl}_F Y \rightarrow Y$ be the blow-up of $Y$ along $F$, let $\tilde{G}$ be the induced building set and let $\tilde{S}$ be the arrangement induced by $\tilde{G}$. Suppose $G_+ = \{G_0, \ldots, G_k\}$ is a building set and $S_+$ is the arrangement induced by $G_+$.

Then $\tilde{G}_+ := \tilde{G} \cup \{G_0\}$ is a building set of the induced arrangement

$$\tilde{S}_+ := \tilde{S} \cup \{\tilde{S} \cap \tilde{G}_0\}_{S \in S}.$$

Proof. As the proof of Proposition 2.8 we need to discuss different types of intersections of subvarieties. See Appendix §5.2. \hfill \Box
Lemma 3.2. Let $\mathcal{I}_1, \mathcal{I}_2$ be two ideal sheaves on a variety $Y$. Define $\text{Bl}_{\mathcal{I}_2} \text{Bl}_{\mathcal{I}_1} Y$ to be the blow-up of $Y' = \text{Bl}_{\mathcal{I}_1} Y$ along the ideal sheaf $\phi^{-1} \mathcal{I}_2 \cdot \mathcal{O}_Y$, where $\phi$ is the blow-up morphism $\phi : Y' \to Y$. Define $\text{Bl}_{\mathcal{I}_1} \text{Bl}_{\mathcal{I}_2} Y$ symmetrically. Then

$$\text{Bl}_{\mathcal{I}_1} \text{Bl}_{\mathcal{I}_2} Y \cong \text{Bl}_{\mathcal{I}_2} \text{Bl}_{\mathcal{I}_1} Y \cong \text{Bl}_{\mathcal{I}_1} \text{Bl}_{\mathcal{I}_2} Y.$$

Proof. We show the existence of two natural morphisms

$$f : \text{Bl}_{\mathcal{I}_2} \text{Bl}_{\mathcal{I}_1} Y \to \text{Bl}_{\mathcal{I}_1} \text{Bl}_{\mathcal{I}_2} Y,$$

$$g : \text{Bl}_{\mathcal{I}_1} Y \to \text{Bl}_{\mathcal{I}_2} \text{Bl}_{\mathcal{I}_1} Y,$$

from which we obtain the isomorphism $\text{Bl}_{\mathcal{I}_1} \mathcal{I}_2 Y \cong \text{Bl}_{\mathcal{I}_2} \text{Bl}_{\mathcal{I}_1} Y$. The other isomorphism $\text{Bl}_{\mathcal{I}_2} \mathcal{I}_1 Y \cong \text{Bl}_{\mathcal{I}_1} \text{Bl}_{\mathcal{I}_2} Y$ follows symmetrically.

For simplicity of notation, denote $Y_1 = \text{Bl}_{\mathcal{I}_1} Y$, $Y_2 = \text{Bl}_{\mathcal{I}_2} \text{Bl}_{\mathcal{I}_1} Y$ and $Y_3 = \text{Bl}_{\mathcal{I}_1} \mathcal{I}_2 Y$.

(i) We show the existence of $f$.

By the universal property of blowing up, it suffices to show that

$$(\phi_1 \phi_2)^{-1} (\mathcal{I}_1 \mathcal{I}_2) \cdot \mathcal{O}_{Y_2}$$

is an invertible sheaf. Indeed,

$$(\phi_1 \phi_2)^{-1} (\mathcal{I}_1 \mathcal{I}_2) \cdot \mathcal{O}_{Y_2} = \phi_2^{-1} ((\phi_1^{-1} \mathcal{I}_1 \cdot \mathcal{O}_{Y_1}) \cdot (\phi_1^{-1} \mathcal{I}_2 \cdot \mathcal{O}_{Y_1})) \cdot \mathcal{O}_{Y_2}$$

$$= (\phi_2^{-1} (\phi_1^{-1} \mathcal{I}_1 \cdot \mathcal{O}_{Y_1}) \cdot \mathcal{O}_{Y_2}) \cdot (\phi_2^{-1} (\phi_1^{-1} \mathcal{I}_2 \cdot \mathcal{O}_{Y_1}) \cdot \mathcal{O}_{Y_2}),$$

in the last expression both factors are invertible sheaves, therefore the product is also invertible.

(ii) We show the existence of $g$.

Since $(\phi^{-1} \mathcal{I}_1 \cdot \mathcal{O}_{Y_1}) \cdot (\phi^{-1} \mathcal{I}_2 \cdot \mathcal{O}_{Y_1}) = \phi^{-1} (\mathcal{I}_1 \mathcal{I}_2) \cdot \mathcal{O}_{Y_2}$ is invertible, both $(\phi^{-1} \mathcal{I}_1 \cdot \mathcal{O}_{Y_1})$ and $(\phi^{-1} \mathcal{I}_2 \cdot \mathcal{O}_{Y_1})$ are invertible. The invertibility of $(\phi^{-1} \mathcal{I}_1 \cdot \mathcal{O}_{Y_1})$ implies the existence of $h$ by the universal property of blowing up. Then, since $\phi_2$ is the blow-up of the ideal sheaf $(\phi_1^{-1} \mathcal{I}_2 \cdot \mathcal{O}_{Y_1})$ and

$$h^{-1} (\phi_1^{-1} \mathcal{I}_2 \cdot \mathcal{O}_{Y_1}) \cdot \mathcal{O}_{Y_3} = \phi_1^{-1} \mathcal{I}_2 \cdot \mathcal{O}_{Y_3}$$

is invertible, we can lift $h$ to $g$ by applying the universal property of blowing up again. This completes the proof. □
Now we are ready to prove Theorem 1.3 given in the Introduction.

**Proof of Theorem 1.3.** (i) We fix the indices of \{G_i\} in an order that is compatible with inclusion relations, i.e., \( i < j \) if \( G_i \subset G_j \). Consider the blow-up \( \phi : \tilde{Y} := Bl_{\mathcal{I}_1}Y \rightarrow Y \) where \( \mathcal{I}_1 \) is the ideal sheaf of \( G_1 \). Since \( G_i \) (\( i > 1 \)) either contains \( G_1 \) or is transversal to \( G_1 \) by Definition-Lemma 2.6, the ideal sheaf \( \phi^{-1}\mathcal{I}_{G_i} \cdot \mathcal{O}_{\tilde{Y}} \) is either \( \mathcal{I}_{G_i} \cdot \mathcal{I}_E \) or \( \mathcal{I}_{G_i} \). Since \( \mathcal{I}_E \) is invertible, the blow-up of \( \mathcal{I}_{G_i} \cdot \mathcal{I}_E \) is isomorphic to the blow-up of \( \mathcal{I}_{G_i} \), that is, the blow-up along the nonsingular subvariety \( \tilde{G}_i \). By the same argument, each blow-up \( Y_{k+1} \rightarrow Y_k \) is isomorphic to the blow-up of the ideal sheaf \( \psi^{-1}\mathcal{I}_{k+1} \cdot \mathcal{O}_{Y_k} \) where \( \psi : Y_k \rightarrow Y \) is the natural morphism. Therefore, by Lemma 3.2

\[
Y_G \cong Bl_{\mathcal{I}_N} \cdots Bl_{\mathcal{I}_2} Bl_{\mathcal{I}_1} Y \cong Bl_{\mathcal{I}_{G} \cdots \mathcal{I}_1} Y.
\]

(ii) Now assume the order of \{G_i\} is not necessarily compatible with inclusion relations, but satisfies (*).

The proof is by induction with respect to \( N \). The statement is obviously true for \( N = 1 \). Assume the statement (ii) is true for \( N \). Consider \( G_{+} = G \cup \{G_{N+1}\} \). We need to show that \( Y_{G_{+}} \) is isomorphic to the blow-up of \( Y_G \) along a nonsingular subvariety \( \tilde{G}_{N+1} \).

Suppose \( F \) is minimal in \( G \), \( \tilde{Y} = Bl_{\mathcal{I}_F} Y \) and \( \phi : \tilde{Y} \rightarrow Y \) is the natural morphism. Proposition 3.1 implies that \( \tilde{G}_{+} := \tilde{G} \cup \{\tilde{G}_{N+1}\} \) is a building set in \( \tilde{Y} \). There are two cases: if \( F \) is not minimal in \( G_{+} \), then \( G_{N+1} \) must be minimal and \( G_{N+1} \subsetneq F \), in this case \( \phi^{-1}\mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{\tilde{Y}} = \mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{\tilde{Y}} \). Now consider the case where \( F \) is minimal in \( G_{+} \). In this case \( G_{N+1} \) either contains \( F \) or is transversal to \( F \), thus \( \phi^{-1}\mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{\tilde{Y}} \) is either \( \mathcal{I}_{G_{N+1}} \cdot \mathcal{I}_E \) or \( \mathcal{I}_{G_{N+1}} \). In each situation, \( \phi^{-1}\mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{\tilde{Y}} \) is isomorphic to \( \mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{\tilde{Y}} \). Continue this procedure until all elements in \( G \) have been blown up. Let \( \psi : Y_G \rightarrow Y \) be the natural morphism. Then \( \psi^{-1}\mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{Y_G} \) is isomorphic to the ideal sheaf of the nonsingular subvariety \( \tilde{G}_{N+1} \subset Y_G \) up to an invertible sheaf, therefore the blow-up of \( Y_G \) along \( \psi^{-1}\mathcal{I}_{G_{N+1}} \cdot \mathcal{O}_{Y_G} \) is isomorphic to the blow-up of \( Y_G \) along \( \tilde{G}_{N+1} \). This completes the proof.

\[\square\]

4. Examples of wonderful compactifications

4.1. Wonderful model of subspace arrangements. If we let \( Y = V \) be a finite dimensional vector space and let \( S \) be any finite collection of subspaces of \( V \) and construct the wonderful compactification of any building set of subspaces of \( V \), one recovers the *wonderful model of subspace arrangements* by De Concini and Procesi.

It was discovered by De Concini and Procesi (DP95) that if a subset \( \mathcal{G} \subseteq S \) forms a so-called building set, the closure of the natural locally closed embedding

\[
i : V \setminus \bigcup_{W \in \mathcal{G}} W \hookrightarrow V \times \prod_{W \in \mathcal{G}} \mathbb{P}(V/W)
\]
is a nonsingular variety birational to $V$. Moreover, the subspaces in $S$ are replaced by a normal crossing divisor.

**Remark:** This idea motivated a generalized definition of the so-called wonderful conical compactifications for a complex manifold given by MacPherson and Procesi ([MP98]). Our definition of wonderful compactification is neither strictly general nor strictly less general than the wonderful compactification defined in [MP98]. On the one hand, our compactification does not include the conic case: all the subvarieties involved in our paper are assumed to be nonsingular. On the other hand, even over the complex field $\mathbb{C}$, many arrangements of nonsingular varieties are not conical.

4.2. **Fulton-MacPherson configuration spaces.** Let $X$ be a nonsingular variety, $Y = X^n$ and $G$ be the set of diagonals of $X^n$. Our wonderful compactification gives the Fulton-MacPherson configuration space $X[n]$. In [FM94], Fulton and MacPherson have constructed a compactification $X[n]$ of the configuration space $F(X,n)$ of $n$ distinct labeled points in a nonsingular algebraic variety $X$. This compactification is related to several areas of mathematics. In [FM94], Fulton and MacPherson use their compactification to construct a differential graded algebra which is a model for the configuration space $F(X,n)$ in the sense of Sullivan. Axelrod-Singer used an analogous construction in the setting of real smooth manifolds in Chern-Simons perturbation theory ([AS94]). Now we give a brief review of Fulton and MacPherson’s construction.

The configuration space $F(X,n)$ is an open subset of the Cartesian product $X^n$ defined as the complement of all diagonals:

$$F(X,n) := X^n \setminus \bigcup_{|I| \geq 2} \Delta_I = \{ (p_1, \ldots, p_n) \in X^n | p_i \neq p_j, \forall i \neq j \}.$$

The construction of $X[n]$ by Fulton and MacPherson is inductive. They define $X[1]$ to be $X$ and $X[n+1]$ is the variety that results from a sequence of blow-ups of $X[n] \times X$ along nonsingular subvarieties corresponding to all diagonals $\Delta_I$ where $I \subseteq [n+1]$, $|I| \geq 2$ and $I$ contains the number $n+1$.

For example, $X[2]$ is the blow-up of $X^2$ along the diagonal $\Delta_{12}$. The variety $X[3]$ is obtained from a sequence of blow-ups of $X[2] \times X$ along nonsingular subvarieties corresponding to $\{ \Delta_{123}; \Delta_{13}, \Delta_{23} \}$. More specifically, denote by $\pi$ the blow-up $X[2] \times X \to X^3$, we blow up first along $\pi^{-1}(\Delta_{123})$, then along the strict transforms of $\Delta_{13}$ and $\Delta_{23}$ (the two strict transforms are disjoint, so they can be blown up in any order). In general, the order of blow-ups in the construction of $X[n]$ can be expressed as $\Delta_{12}, \Delta_{123}, \Delta_{13}, \Delta_{23}, \Delta_{124}, \Delta_{134}, \Delta_{234}, \Delta_{14}, \Delta_{24}, \Delta_{34}, \Delta_{12345}, \Delta_{1235}, \ldots$.

It is easy to verify that the above sequence satisfies $(\ast)$ in Theorem 1.3, therefore the resulting variety $X[n]$ is indeed the wonderful compactification $Y_G$. Theorem 1.3 also implies that $X[n]$ can be obtained from a more
symmetric sequence of blow-ups in the order of ascending dimension:

$$\Delta_{12\cdot n}, \Delta_{12\cdot (n-1)}, \ldots, \Delta_{23\cdot n}, \ldots, \Delta_{12}, \ldots, \Delta_{(n-1)\cdot n}. $$

This more symmetric order of blow-ups is given by De Concini and Procesi [DP95], MacPherson and Procesi [MP98], and Thurston [T99].

In fact, graphs can be used to clarify the condition (\ast) by using the Kuperberg-Thurston’s compactification (cf. the discussion after Proposition 4.2 below).

4.3. Kuperberg-Thurston’s compactifications. In their paper [KT99], Kuperberg and Thurston have constructed an interesting compactification of the configuration space $F(X, n)$. Their construction is for real smooth manifolds and we adapt their compactification in this section to a nonsingular algebraic variety.

Let $\Gamma$ be a (not necessarily connected) graph with $n$ labeled vertices such that $\Gamma$ has no self-loops and multiple edges. Denote by $\Delta_{\Gamma}$ the polydiagonal in $X^n$ where $x_i = x_j$ if $i, j$ are connected in $\Gamma$. We call a graph $\Gamma$ vertex-2-connected if the graph is connected and will still be connected if we remove any vertex. In particular, a single edge is vertex-2-connected.

In [KT99] the authors state and sketch a proof that blowing up along $\Delta_{\Gamma'}$ for all vertex-2-connected subgraphs $\Gamma' \subseteq \Gamma$ gives a compactification $X^\Gamma$. When $\Gamma$ is the complete graph with $n$ vertices (i.e. any two vertices are joined with an edge), the compactification $X^\Gamma$ is exactly the Fulton-MacPherson compactification $X_{[n]}$.

Kuperburg-Thurston’s compactification $X^\Gamma$ is a special case of the wonderful compactification of an arrangement of subvarieties given in this paper. Indeed, let $Y = X^n$, let

$$\mathcal{G} := \{ \Delta_{\Gamma'} : \Gamma' \subseteq \Gamma \text{ is vertex-2-connected} \},$$

and let $\mathcal{S}$ be the set of polydiagonals of $X^n$ obtained by intersecting only the diagonals in $\mathcal{G}$.

Proposition 4.1. In the above notation, $\mathcal{G}$ is a building set with respect to $\mathcal{S}$. Therefore Kuperburg-Thurston’s compactification is the wonderful compactification $Y_\mathcal{G}$.

Proof. The proof is in two steps.

(i) We call $\Gamma' \subseteq \Gamma$ a full subgraph if the following are satisfied.

- $\Gamma'$ contains all vertices in $\Gamma$.
- For any edge $e \in \Gamma$, if its endpoints $p$ and $q$ are in the same connected component of $\Gamma'$, then $e \in \Gamma'$.

Then there is a one-one correspondence between the set of all full subgraphs of $\Gamma$ and the set $\mathcal{S}$. The correspondence is given by mapping a full subgraph $\Gamma'' \subseteq \Delta_{\Gamma'}$ to $\Delta_{\Gamma'}$.

(ii) Any full subgraph $\Gamma'$ has a unique decomposition into vertex-2-connected subgraphs $\Gamma_1, \ldots, \Gamma_k$. Notice that $\Delta_{\Gamma_1}, \ldots, \Delta_{\Gamma_k}$ are the minimal elements in
which contain $\Delta_{\Gamma'}$, and they intersect transversally with the intersection $\Delta_{\Gamma'}$. Therefore $\mathcal{G}$ is a building set by Definition 2.2.

**Remark:** It is also easy to describe a $\mathcal{G}$-nest: it corresponds to a set of vertex-2-connected subgraphs of $\Gamma$, where for any two subgraphs $\Gamma_1$ and $\Gamma_2$ one of the following holds.

(i) $\Gamma_1$ and $\Gamma_2$ are disjoint, or
(ii) $\Gamma_1$ and $\Gamma_2$ intersect at one vertex, or
(iii) $\Gamma_1 \subseteq \Gamma_2$ or $\Gamma_2 \subseteq \Gamma_1$.

The following proposition describes the relation between $X^{\Gamma_1}$ and $X^{\Gamma_2}$ for $\Gamma_1 \subseteq \Gamma_2$, which will help us to understand the construction of Fulton and MacPherson configuration spaces.

**Proposition 4.2.** Let $\Gamma_1 \subseteq \Gamma_2$ be two (not necessarily connected) graphs with $n$ labeled vertices without self-loops and multiple edges. Then $X^{\Gamma_2}$ can be obtained by a sequence of blow-ups of $X^{\Gamma_1}$ along nonsingular centers.

One such order is given as follows. Let $\{\Gamma''_j\}_{j=1}^{s}$ be the set of all vertex-2-connected subgraphs of $\Gamma_2$ that are not contained in $\Gamma_1$. Arrange the index such that $i < j$ if the number of vertices of $\Gamma''_i$ is greater than the number of vertices of $\Gamma''_j$. Then $X^{\Gamma_2}$ can be obtained by blowing up along the following nonsingular centers $\tilde{\Delta}_{\Gamma''_1}, \tilde{\Delta}_{\Gamma''_2}, \ldots, \tilde{\Delta}_{\Gamma''_s}$.

**Proof.** Let $\{\Gamma'_i\}_{i=1}^{s}$ be the set of all vertex-2-connected subgraphs of $\Gamma_1$, arrange the indices such that $i < j$ if the number of vertices of $\Gamma'_i$ is greater than the number of vertices of $\Gamma'_j$.

It is easy to verify that $\{\Delta_{\Gamma'_1}, \ldots, \Delta_{\Gamma'_s}, \Delta_{\Gamma''_1}, \ldots, \Delta_{\Gamma''_s}\}$ satisfies ($\ast$) in Theorem 1.3. Apply Theorem 1.3 we know that $X^{\Gamma_2}$ is the blowup of $X^n$ along nonsingular centers $\Delta_{\Gamma'_1}, \tilde{\Delta}_{\Gamma'_2}, \ldots, \tilde{\Delta}_{\Gamma''_1}, \ldots, \tilde{\Delta}_{\Gamma''_s}$.

On the other hand, after the first $s$ blow-ups, we get $X^{\Gamma_1}$. Therefore $X^{\Gamma_2}$ can be obtained by a sequence of blow-ups along nonsingular centers $\tilde{\Delta}_{\Gamma''_1}, \ldots, \tilde{\Delta}_{\Gamma''_s}$. This completes the proof. □

With the above proposition, Fulton and MacPherson’s original construction of $X[n]$ can be understood as specifying a chain of graphs: indeed, by

![Graphs](image)

**Figure 1.** Fulton and MacPherson’s construction of $X[4]$. 
Proposition 4.2, the first arrow corresponds to blowing up $X[4]$ along $\Delta_{12}$, the second corresponds to blowing up along $\Delta_{123}$, $\Delta_{13}$ and $\Delta_{23}$ (which correspond to all the vertex-2-connected subgraphs which are not in the previous graph), the last arrow corresponds to blowing up along $\Delta_{1234}$, $\Delta_{124}$, $\Delta_{134}$, $\Delta_{234}$, $\Delta_{14}$, $\Delta_{24}$, $\Delta_{34}$.

On the other hand, the symmetric construction of $X[4]$ corresponds to the chain containing only two graphs: the first graph and last graph in Figure 1.

To illustrate the idea a little more, we construct $X[3]$ corresponding to the chain of graphs in the figure below. The first step is to blow up along $\Delta_{12}$, the second step is to blow up along $\tilde{\Delta}_{23}$, the final step is to blow up along $\tilde{\Delta}_{123}$ and $\tilde{\Delta}_{13}$. Each blow-up is along a nonsingular subvariety.

\[ X[3] \rightarrow X[2] \times X \rightarrow B\Gamma_{\tilde{\Delta}_{23}}(X[2] \times X) \rightarrow X[3] \]

Figure 2. A new construction of $X[3]$.  

4.4. Moduli space $\overline{M}_{0,n}$ of rational curves with $n$ marked points. The moduli space $\overline{M}_{0,n}$ is the wonderful compactification of $((\mathbb{P}^1)^{n-3}, S, \mathcal{G})$ where $\mathcal{G}$ is set of all diagonals and augmented diagonals

$$\Delta_{I,a} := \{(p_4, \ldots, p_n) \in (\mathbb{P}^1)^{n-3} : p_i = a, \forall i \in I\}$$

for $I \subseteq \{4, \ldots, n\}$, $|I| \geq 2$ and $a \in \{0, 1, \infty\}$. $S$ is the set of all intersections of elements in $\mathcal{G}$.

This is a immediate consequence of Theorem 1.3 applied to Keel’s construction [Ke92]. Indeed, Keel gives the construction of $\overline{M}_{0,n}$ by a sequence of blow-ups in the following order:

$\Delta_{45,0}, \Delta_{45,1}, \Delta_{45,\infty}, \Delta_{456,0}, \Delta_{456,1}, \Delta_{456,\infty}, \ldots, \Delta_{46,0}, \ldots, \Delta_{456, \ldots}$

To be more precise: for $I$ such that max $I = 5$, blow up along $\Delta_{I,a}$ for those $I$ such that $|I| = 2$; for max $I = 6$, blow up $\Delta_{I,a}$ for $|I| = 3$, then $\Delta_I$ for $|I| = 3$; in general, for max $I = k$, blow up $\Delta_{I,a}$ for $|I| = k - 3$, then $\Delta_{I,a}$ for $|I| = k - 4$ and $\Delta_I$ for $|I| = n - 3$, then $\Delta_{I,a}$ for $|I| = k - 5$ and $\Delta_I$ for $|I| = n - 4$, ...

It is easy to check the order satisfies ($\ast$) in Theorem 1.3. So $\overline{M}_{0,n}$ is a wonderful compactification.

Notice that the above diagonals and augmented diagonals in $\mathbb{P}^{n-3}$ are just the restrictions of diagonals in $(\mathbb{P}^1)^n$ to the codimension three subvariety

$$Y = \{(p_1, p_2, \ldots, p_n) \in (\mathbb{P}^1)^n : p_1 = 0, p_2 = 1, p_3 = \infty\}.$$
Now blow up all the diagonals of \((\mathbb{P}^1)^n\) in order of increasing dimension and compare with the construction of Fulton-MacPherson configuration space, we get a relation between \(\overline{\mathcal{M}}_{0,n}\) and the Fulton-MacPherson space \(\mathbb{P}^1[n]\) in Corollary 1.4.

4.5. **Ulyanov’s compactifications.** Closely related to Fulton and MacPherson’s compactification, Ulyanov has discovered another compactification of the configuration space \(F(X,n)\), which he denoted by \(X(n)\) ([U02]). The construction consists of blowing up more subvarieties in \(X^n\) than Fulton-MacPherson’s construction does, i.e., it blows up not only diagonals but also polydiagonals. The order of the blow-ups in [U02] is the ascending order of the dimension. For example, \(X(4)\) is the blow-up of \(X^4\) along polydiagonals in the following order:

\[(1234), (123), (124), (134), (234), (12, 34), (13, 24), (14, 23), (12), \ldots, (34).\]

The polydiagonal compactification \(X(n)\) shares many similar properties with Fulton-MacPherson’s compactification. However, one difference is that in the case of characteristic 0, the isotropy group of any point in \(X(n)\) is abelian under the symmetric group action, while the isotropy group of a point in \(X[n]\) is not necessarily abelian (but always solvable) under the symmetric group action.

4.6. **Hu’s compactifications v.s. minimal compactifications.** We now consider the general situation where \(Y\) is nonsingular with an arrangement of subvarieties \(\mathcal{S}\). By blowing up all \(S \in \mathcal{S}\) in order of ascending dimension, we get a nonsingular variety \(Bl_{\mathcal{S}}Y\) ([Hu03]). Define \(Y^\circ := Y \setminus \cup_{S \in \mathcal{S}} D_S\), the open stratum of \(Y\). It is isomorphic to an open subset of \(Bl_{\mathcal{S}}Y\). Hu showed

(i) The boundary \(Bl_{\mathcal{S}}Y \setminus Y^\circ = \cup_{S \in \mathcal{S}} D_S\) is a simple normal crossing divisor.

(ii) For any \(S_1, \ldots, S_n \in \mathcal{S}\), the intersection of \(D_{S_1} \cdots D_{S_k}\) is nonempty if and only if \(\{S_i\}\) forms a chain, i.e., \(S_1 \subseteq \cdots \subseteq S_k\) with a rearrangement of indices if necessary.

Hu’s compactification generalized Ulyanov’s polydiagonal compactification and is a special case of the wonderful compactification of arrangement of subvarieties given in this paper where the building set \(\mathcal{G} = \mathcal{S}\). (In this special case, a \(\mathcal{G}\)-nest is simply a chain of subvarieties.)

Fixing an arrangement \(\mathcal{S}\), Hu’s compactification \(Y_{\mathcal{S}}\) is the maximal wonderful compactification. Indeed, it is not hard to show that for any building set \(\mathcal{G}\) of \(\mathcal{S}\), the natural birational map \(Y_{\mathcal{G}} \to Y_{\mathcal{S}}\) is a morphism. On the other extreme, there exists a minimal wonderful compactification for \(\mathcal{S}\), which can be defined by the set of so-called irreducible elements in \(\mathcal{S}\).

**Definition 4.3.** An element \(G\) in \(\mathcal{S}\) is called reducible if there are \(G_1, \ldots, G_k \in \mathcal{S}\) \((k \geq 2)\) with \(G = G_1 \cap \cdots \cap G_k\) and for every \(G' \supset G\) in \(\mathcal{S}\), there exist \(G'_i \in \mathcal{S}\) \((1 \leq i \leq k)\), for \(1 \leq i \leq k\), such that \(G' = G'_1 \cap \cdots \cap G'_k\).

\(G \in \mathcal{S}\) is called irreducible if it is not reducible.
By the same method as in [DP95], we can show that the irreducible elements in $S$ form a building set, denoted by $G_{\text{min}}$, and that every building set $G$ of the arrangement $S$ contains $G_{\text{min}}$. It is not hard to show that the natural birational map $Y_G \to Y_{G_{\text{min}}}$ is a morphism.

Out of the previous examples, Fulton-MacPherson configuration spaces, Kuperberg-Thurston’s compactifications and the moduli space $\mathcal{M}_{0,n}$ are minimal wonderful compactifications. Ulyanov’s polydiagonal compactifications are maximal.

5. Appendix

5.1. Clean intersection v.s. transversal intersection. Let $Y$ be a nonsingular variety. For a nonsingular subvariety $A$ (more generally, a subscheme whose connected components are nonsingular subvarieties) of $Y$, denote by $T_A$ the total space of the tangent bundle of $A$, denote by $T_{A,y}$ the tangent space of $A$ at the point $y \in A$. For a point $y \notin A$, define $T_{A,y}$ to be $T_y$, the tangent space of $Y$ at $y$. (This artificial definition will simplify the definition of transversal intersection.) In this paper, $T_{A,y}$ will be seen as a subspace of $T_y$ and $T_A$ will be seen as a subvariety of $T_Y$.

5.1.1. Clean intersection. The notion of cleanness can be traced back to Bott [B56] in the setting of differential geometry.

We say that the intersection of two nonsingular subvarieties $A$ and $B$ is clean if the set-theoretic intersection $A \cap B$ is a nonsingular subvariety (or, more generally, a scheme whose connected components are nonsingular subvarieties) and satisfies the condition

$$T_{A \cap B, y} = T_{A,y} \cap T_{B,y}, \quad \forall y \in A \cap B.$$ 

The following lemma gives a useful criterion for the cleanness of intersections.

**Lemma 5.1.** Suppose $A$ and $B$ are nonsingular closed subvarieties of $Y$ and the intersection $C = A \cap B$ is a disjoint union of nonsingular subvarieties. Let $\mathcal{I}_A$ (resp. $\mathcal{I}_B, \mathcal{I}_C$) denote the ideal sheaf of $A$ (resp. $B, C$). Then the following are equivalent:

(i) The subvarieties $A$ and $B$ intersect cleanly.
(ii) $\mathcal{I}_A + \mathcal{I}_B = \mathcal{I}_C$.

In other words, two subvarieties intersect cleanly if and only if their scheme-theoretic intersection is nonsingular.

**Proof.** Condition (i) is equivalent to

$$T_{A,y} \cap T_{B,y} = T_{C,y}, \quad \forall y \in A \cap B.$$ 

By definition of tangent space,

$$T_{A,y} = \{ v \in T_y | df(v) = 0, \forall f \in (\mathcal{I}_A)_y \}.$$
Define \( \phi : m_y / m_y^2 \) to be the natural quotient. Then \( T_{A,y} = \phi((I_A)_y) \), the \( \text{annihilator of } \phi((I_A)_y) \) in the dual space \((m_y / m_y^2)^* \cong T_y\). Therefore condition (1) is equivalent to
\[
\phi((I_A)_y) \perp \phi((I_B)_y) = \phi((I_C)_y) \perp, \quad \forall y \in A \cap B,
\]
which is equivalent to
\[
\phi((I_A)_y) + \phi((I_B)_y) = \phi((I_C)_y), \quad \forall y \in A \cap B.
\]
Since \( \phi((I_A)_y) = ((I_A)_y + m_y^2) / m_y^2 \) (similarly for \( B \) and \( C \)), the above condition is equivalent to
\[
(I_A)_y + (I_B)_y + m_y^2 = (I_C)_y + m_y^2, \quad \forall y \in A \cap B.
\]
On the other hand, two ideal sheaves on \( Y \) are the same if and only if their germs coincide at every closed point \( y \in Y \). So condition (ii) is equivalent to
\[
(I_A)_y + (I_B)_y = (I_C)_y, \quad \forall y \in Y
\]
where \( I_y \) denote the germ of a sheaf \( I \) at point \( y \). Therefore it suffices to show that \( (2) \Rightarrow (3) \).

Obviously \( (3) \Rightarrow (2) \). To see the implication \( (2) \Rightarrow (3) \), observe first that condition (3) holds for \( y \notin A \cap B \) and the inclusion \((I_A)_y + (I_B)_y \subseteq (I_C)_y\) holds for \( y \in A \cap B \). Thus it remains to show \((I_A)_y + (I_B)_y \supseteq (I_C)_y\) holds for \( y \in A \cap B \). Using local parameters it can be checked that \((I_C)_y \cap m_y^2 = (I_C)_y m_y\). Condition (2) then implies the surjection
\[
(I_A)_y + (I_B)_y \to ((I_C)_y + m_y^2) / m_y^2 \cong (I_C)_y / ((I_C)_y \cap m_y^2) \cong (I_C)_y / (I_C)_y m_y.
\]
Hence \((I_A)_y + (I_B)_y + (I_C)_y m_y = (I_C)_y\). Apply Nakayama's lemma, we have \((I_A)_y + (I_B)_y = (I_C)_y\). This completes the proof. \( \square \)

5.1.2. Transversal intersection. By definition, \( A \) and \( B \) intersect transversely (denoted by \( A \cap B \)) if \( T_{A,y} \perp T_{B,y} \) form a direct sum in the dual space \( T_y^* \cong m_y / m_y^2 \) of \( T_y \) for any point \( y \in Y \), or equivalently, if
\[
T_y = T_{A,y} + T_{B,y}, \quad \forall y \in Y.
\]

More generally, we define that a finite collection of \( k \) nonsingular subvarieties \( A_1, \ldots, A_k \) intersect transversally (denoted by \( A_1 \cap A_2 \cap \cdots \cap A_k \)) if either \( k = 1 \) or for any \( y \in Y \),
\[
T_{A_1,y} \perp T_{A_2,y} \perp \cdots \perp T_{A_k,y}
\]
form a direct sum in \( T_y^* \); or equivalently, if
\[
\text{codim}\left( \bigcap_{i=1}^k T_{A_i,y} \right) = \sum_{i=1}^k \text{codim}(A_i, Y);
\]
or equivalently, if for any \( y \in Y \), there exists a system of local parameters \( x_1, \ldots, x_n \) on \( Y \) at \( y \) which are regular on an affine neighborhood \( U \) of \( y \) such that \( y \) is defined by the maximal ideal \((x_1, \ldots, x_n)\), and there exist integers
0 = r_0 \leq r_1 \leq \cdots \leq r_k \leq n such that the subvariety \( A_i \) is defined by the ideal

\[(x_{r_{i-1}+1}, x_{r_{i-1}+2}, \ldots, x_{r_i}), \quad \forall 1 \leq i \leq k.\]

(In case that \( r_{i-1} = r_i \), the ideal is assumed to be the ideal containing units, which means geometrically that the restriction of \( A_i \) to \( U \) is empty.)

5.1.3. **Transversal intersection \( \Rightarrow \) clean intersection.** If \( A \) and \( B \) intersect transversally, we can choose local parameters around any point \( y \in A \cap B \) such that \( y \) is the origin and the restriction of \( A \) and \( B \) are defined by local parameters. Then it is obvious that \( T_{A \cap B,y} = T_{A,y} \cap T_{B,y}, \forall y \in A \cap B. \)

5.1.4. **Transversal intersection at one point + clean intersection \( \Rightarrow \) transversal intersection.**

**Lemma 5.2.** Let \( A_1 \) and \( A_2 \) be two nonsingular closed subvarieties of \( Y \) that intersect cleanly along a closed nonsingular subvariety \( A \). If \( A_1 \) and \( A_2 \) intersect transversally at a point \( y_0 \in A \), then they intersect transversally (at every point \( y \in A \)).

In general, let \( A_1, \ldots, A_k \) be subvarieties in a simple arrangement \( S \) (cf. Definition 2.1), let \( A = \cap_{i=1}^k A_i \). If \( A_1, \ldots, A_k \) intersect transversally at a point \( y_0 \in A \), then they intersect transversally (at every point).

**Proof.** We prove the general case. Without loss of generality, we need only to prove the transversality for points in \( A \). The irreducibility of \( A_i \) and \( A \) implies \( \dim T_{A_i,y}^\perp = \dim T_{A_i,y_0}^\perp \) and \( \dim T_{A,y}^\perp = \dim T_{A,y_0}^\perp \). By the definition of clean intersection, we have

\[T_{A_1,y}^\perp + \cdots + T_{A_k,y}^\perp = T_{A,y}^\perp.\]

On the other hand, by the transversality condition at point \( y_0 \),

\[T_{A_1,y_0}^\perp + \cdots + T_{A_k,y_0}^\perp = T_{A,y_0}^\perp.\]

Comparing the dimensions of the above two equalities, the left hand side of the first equality must form a direct sum, therefore \( A_1, \ldots, A_k \) intersect transversally at \( y \). \( \square \)

5.1.5. **Examples/nonexamples of clean and transversal intersections.**

- \( k(\leq n) \) hyperplanes \( H_i \) in \( \mathbb{A}^n \) defined by \( x_i = 0 \) intersect transversally, therefore any two of them intersect cleanly.
- Two (not necessarily distinct) lines in \( \mathbb{A}^3 \) passing through the origin intersect cleanly but not transversally.
- In \( \mathbb{A}^2 \), the intersection of the parabola \( y = x^2 \) and the line \( y = 0 \) is not clean, therefore not transversal.
5.2. Proofs of statements in previous sections.

Proof of Lemma \[2.4\] It is convenient to carry out the proof using the cotangent space \( T^*_y \). We use the same notation \( G^*_y, S, T^\perp_{S,y}, T^\perp \) as in the remark after Definition \[2.2\]. By \[DP95\] §2.3 Theorem (2), the definition of building set implies the following:

If \( S' \in S \) is such that \( S' \supseteq S \), then

\[
T^\perp_{S',y} = \bigoplus_{i=1}^k (T^\perp_{S',y} \cap T^\perp_i),
\]

moreover, if \( T^\perp_{S',y} = T^\perp_1 + \cdots + T^\perp_s \) where \( T^\perp_1, \ldots, T^\perp_s \) are the maximal elements in \( G^*_y \) contained in \( T^\perp_{S',y} \), then each term \((T^\perp_{S',y} \cap T^\perp_i)\) is a direct sum of some \( T^\perp_j \)'s.

Fix a point \( y \in S \). To show (i) it is enough to show the following:

Suppose that \( T^\perp_1, \ldots, T^\perp_k \) are all the maximal elements in \( G^*_y \) that are contained in \( T^\perp_{S,y} \). Suppose \( m \) is an integer such that \( 1 \leq m \leq k \), and define \( T^\perp := T^\perp_1 + \cdots + T^\perp_m \). Then \( T^\perp_1, \ldots, T^\perp_m \) are all the maximal elements in \( G^*_y \) which are contained in \( T^\perp \).

To show (ii) it is equivalent to show:

Suppose \( T^\perp_1, \ldots, T^\perp_k \) are all the maximal elements in \( G^*_y \) that are contained in \( T^\perp_{S,y} \). Suppose \( T^\perp \in G^*_y \) is maximal, \( T^\perp \supseteq T^\perp_1, \ldots, T^\perp_m \) and \( T^\perp \not\supseteq T^\perp_{m+1}, \ldots, T^\perp_k \). Then \( T^\perp, T^\perp_{m+1}, \ldots, T^\perp_k \) are all the maximal elements in \( G^*_y \) which are contained in \( T^\perp + T^\perp_{S,y} \).

Both statements can be shown by routine linear algebra. \( \square \)

Proof of Definition-Lemma \[2.6\] (i) This part follows directly from the definition of building set: if \( F \) is disjoint from \( G \) then of course \( F \cap G \); otherwise \( G \) contains some \( G \)-factor of \( F \cap G \). But a \( G \)-factor of \( F \cap G \) is either \( F \) or is transversal to \( F \) (which implies \( G \cap F \)).

(ii) Define \( A = \bigcap_{i=1}^m G_i \) and \( B = \bigcap_{i=m+1}^k G_i \). We claim that \( A \supseteq F \) and \( B \cap F \). \( A = \bigcap_{i=1}^m G_i \supseteq F \) because of the definition of \( m \). Lemma \[2.4\] (ii) asserts \( F \cap G_{m+1} \cap \cdots \cap G_k \), so \( F \) is transversal to \( B \). Then (iii) follows from Lemma \[2.4\] (i).

(ii) The above proof of (iii) shows the existence of an \( F \)-factorization. Now we show that such an factorization is unique. Given another factorization \( S = A' \cap B' \) such that \( A' \supseteq F \) and \( B' \cap F \). Since \( B' \supseteq F \cap B' = F \cap S \) and the \( G \)-factors of \( F \cap S \) are \( F, G_{m+1}, \ldots, G_k \) by Lemma \[2.4\] (ii), each \( G \)-factor \( G' \) of \( B' \) contains \( F \) or \( G_i \) for some \( m+1 \leq i \leq k \). But \( B' \cap F \) implies \( G' \cap F \), hence \( F \not\supseteq F \). So \( G' \supseteq G_i \) for some \( m+1 \leq i \leq k \). Intersecting all the \( G \)-factors \( G' \) of \( B' \), we have \( B' = \cap G' \supseteq \bigcap_{i=m+1}^k G_i = B \). Fixing a point \( y \in F \cap S \), we have

\[
T^\perp_{F,y} + T^\perp_{B',y} = T^\perp_{B,y} + T^\perp_{B',y}
\]

and \( T^\perp_{B,y} \supseteq T^\perp_{B',y} \) therefore \( T^\perp_{B,y} = T^\perp_{B',y} \) hence \( B = B' \). Similarly \( A = A' \).
(iv) Suppose the F-factorization of $S \cap S'$ is $A'' \cap B''$. Then $F \cap B''$ is the F-factorization of the intersection. Since $B \supseteq (F \cap S) = (F \cap B'')$ but $B \ni F$, $B \supseteq B''$. Similarly $B' \supseteq B''$. So $B \cap B' \supseteq B''$. By an analogous argument using the dual of the tangent space as in the proof of (ii), we can show that $B \cap B' = B''$. So $F \cap (B \cap B')$. Then it is easy to see that $A'' = A \cap A'$ and the F-factorization of $S \cap S'$ is indeed $(A \cap A') \cap (B \cap B')$.

**Proof of Lemma 2.9.** We give only the proof of (iii) since (ii) and (iv) can be proved similarly, while (i), (v) and (vi) can be easily checked using a system of local parameters.

In the complement of the exceptional divisor $E$, we have

$$(\bar{A}_1 \cap \bar{A}_2) \setminus E \cong (A_1 \setminus F) \cap (A_2 \setminus F) = (A_1 \cap A_2) \setminus G = (A_1 \cap A_2)^\sim \setminus E.$$ 

Inside $E$, we have

$$(\bar{A}_1 \cap \bar{A}_2) \cap E = \mathbb{P}(N_F A_1) \cap \mathbb{P}(N_F A_2) = \mathbb{P}(T_{A_1} / T_F) \cap \mathbb{P}(T_{A_2} / T_F) = \mathbb{P}(T_{A_1 \cap A_2} / T_F) = \mathbb{P}(N_F (A_1 \cap A_2)) = (A_1 \cap A_2)^\sim \cap E,$$

where $N_F (A_1 \cap A_2)$ stands for the normal bundle of $F$ in $A_1 \cap A_2$. Note that in the fourth equality we have used the condition that $A_1$ and $A_2$ intersect cleanly.

Hence $\bar{A}_1 \cap \bar{A}_2 = (A_1 \cap A_2)^\sim$.

According to Lemma 5.1, $A_1$ and $\bar{A}_2$ intersect cleanly if and only if

$$(4) \quad \mathcal{I}_{\bar{A}_1} + \mathcal{I}_{\bar{A}_2} = \mathcal{I}_{(A_1 \cap A_2)^\sim}.$$ 

But $\bar{A}_1 = \mathcal{R}(E, \pi^{-1}(A_1))$, where $\mathcal{R}(E, \pi^{-1}(A_1))$ is the residue scheme to $E$ in $\pi^{-1}(A_1)$ (see [Ke93] Theorem 1 or [F98] §9.2). By a property of residue schemes, we have

$$\mathcal{I}_{\mathcal{R}(E, \pi^{-1}(A_1))} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_1)},$$

which is the same as

$$\mathcal{I}_{\bar{A}_1} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_1)}.$$ 

Similarly, we have

$$\mathcal{I}_{\bar{A}_2} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_2)},$$

$$\mathcal{I}_{(A_1 \cap A_2)^\sim} \cdot \mathcal{I}_E = \mathcal{I}_{\pi^{-1}(A_1 \cap A_2)}.$$ 

Since $A_1$ and $A_2$ intersect cleanly, $\mathcal{I}_{A_1} + \mathcal{I}_{A_2} = \mathcal{I}_{A_1 \cap A_2}$, which implies

$$\mathcal{I}_{\pi^{-1}(A_1)} + \mathcal{I}_{\pi^{-1}(A_2)} = \mathcal{I}_{\pi^{-1}(A_1 \cap A_2)}.$$ 

Thus we get an equality

$$\mathcal{I}_{\bar{A}_1} \cdot \mathcal{I}_E + \mathcal{I}_{\bar{A}_2} \cdot \mathcal{I}_E = \mathcal{I}_{(A_1 \cap A_2)^\sim} \cdot \mathcal{I}_E.$$ 

Since $\mathcal{I}_E$ is an invertible sheaf, the above equality implies (4), hence (iii) is proved.

□
Proof of Proposition 2.8. (i) We need to check that any two elements in $\tilde{S}$ intersect cleanly, and the intersection is still in $\tilde{S}$. For this, we only need to check three cases: $\tilde{S} \cap \tilde{S}'$, $\tilde{S} \cap (\tilde{S}' \cap E)$ and $(\tilde{S} \cap E) \cap (\tilde{S}' \cap E)$. 

Suppose $S, S' \in S$. We can assume $S \cap S' \neq \emptyset$, otherwise $\tilde{S} \cap \tilde{S}'$ is obviously empty. Suppose the $F$-factorizations of $S$ and $S'$ are $S = A \cap B$ and $S' = A' \cap B'$, respectively. By Lemma 2.6 (iv), the $F$-factorization of $S \cap S'$ is $(A \cap A') \cap (B \cap B')$. Lemma 2.9 (v) asserts that $\tilde{S} = \tilde{A} \cap \tilde{B}$ and $\tilde{S}' = \tilde{A}' \cap \tilde{B}'$. To show that $\tilde{S}$ and $\tilde{S}'$ intersect cleanly along a subvariety in $\tilde{S}$, we consider three cases:

a) $F \subset A \cap A'$. In this case $(S \cap S')^\sim = (A \cap A')^\sim \cap (B \cap B')^\sim$ and $\tilde{S} \cap \tilde{S}' = (\tilde{A} \cap \tilde{A}') \cap (\tilde{B} \cap \tilde{B}') = (A \cap A')^\sim \cap (B \cap B')^\sim = (S \cap S')^\sim$.

Moreover, 

$$T_{\tilde{S}} \cap T_{\tilde{S}'} = T_{\tilde{A}} \cap T_{\tilde{B}} \cap T_{\tilde{A}'} \cap T_{\tilde{B}'} = T_{(A \cap A')^\sim} \cap T_{(B \cap B')^\sim} = T_{(S \cap S')^\sim},$$

where the first and third equalities hold because of Lemma 2.9 (v) and the second equality holds because of Lemma 2.9 (iii) and (iv). Thus $S$ intersects $S'$ cleanly along $(S \cap S')^\sim \in \tilde{S}$.

b) $F = A \cap A'$ but $F \neq A$ and $F \neq A'$. By Lemma 2.9 (ii), $\tilde{A} \cap \tilde{A}' = \emptyset$, hence $\tilde{S} \cap \tilde{S} = (\tilde{A} \cap \tilde{A}') \cap (\tilde{B} \cap \tilde{B}') = \emptyset$.

c) $F = A$ or $A'$. Without loss of generality, we assume $F = A$. Then $\tilde{S} \cap \tilde{S}' = (\tilde{A} \cap \tilde{A}') \cap (\tilde{B} \cap \tilde{B}') = (A \cap (A' \cap B \cap B'))^\sim$.

By Lemma 2.9 (i) and (v), 

$$T_{\tilde{S}} \cap T_{\tilde{S}'} = (T_E \cap T_{\tilde{B}}) \cap (T_{\tilde{A}} \cap T_{\tilde{B}'}) = (T_E \cap T_{\tilde{A}'}) \cap (T_{(B \cap B')^\sim}) = T_E \cap (A' \cap B \cap B')^\sim,$$

so again $\tilde{S}$ intersects $\tilde{S}'$ cleanly along $E \cap (A \cap B \cap B')^\sim \in \tilde{S}$. Therefore we have shown that $\tilde{S}$ and $\tilde{S}'$ intersect cleanly along a subvariety in $\tilde{S}$ in all possible cases.

(ii) To show that $\tilde{G} := \{\tilde{G}\}_{G \in G}$ forms a building set, we need to show that $\forall \tilde{S}$ (resp. $(\tilde{S} \cap E) \in \tilde{S}$, the $\tilde{G}$-factors of $\tilde{S}$ (resp. $(\tilde{S} \cap E)$) intersect transversally along $\tilde{S}$ (resp. along $(\tilde{S} \cap E)$).

By Definition-Lemma 2.6, we can assume $S = (G_1 \cap \cdots \cap G_m) \cap (G_{m+1} \cap \cdots \cap G_k), F \subseteq G_1, \ldots, G_m$, and $F \cap G_{m+1}, \ldots, G_k$. Define $A = G_1 \cap \cdots \cap G_m$ and $B = G_{m+1} \cap \cdots \cap G_k$. Then $\tilde{S} = \tilde{A} \cap \tilde{B}$ by Lemma 2.9 (v).

Two cases need to be considered: $F \subset A$ and $F = A$. We only give the proof for the first case, since the second case can be proved analogously. Assume $F \subset A$. 

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First, we show that $\forall \bar{S} \in \bar{S}$, the $\bar{G}$-factors of $\bar{S}$ (resp. of $(\bar{S} \cap E)$) intersect transversally along $\bar{S}$. Lemma 2.4 implies that
\[
\bar{S} = \bar{G}_1 \cap \cdots \cap \bar{G}_k
\]
and that $\bar{G}_1, \ldots, \bar{G}_k$ are all the $\bar{G}$-factors of $\bar{S}$. (Indeed, if some $\bar{G} \in \bar{G}$ contains $\bar{S}$, then $G = \pi(\bar{G})$ contains $S = \pi(\bar{S})$. Since $G_1, \ldots, G_k$ are all the minimal elements in $G$ that contain $S$, $G$ contains $G_r$ for some $1 \leq r \leq k$. The inclusion of their dominant transforms still holds: $\bar{G} \supseteq G_r$.) Therefore the $\bar{G}$-factors of $\bar{S}$ intersect transversally.

Next, we show that $\forall (\bar{S} \cap E) \in \bar{S}$, the $\bar{G}$-factors of $(\bar{S} \cap E)$ intersect transversally along $(\bar{S} \cap E)$. Noticing that
\[
\bar{S} \cap E = E \cap \bar{A} \cap B = E \cap \bar{G}_1 \cap \cdots \cap \bar{G}_k,
\]
we assert that $E, \bar{G}_1, \ldots, \bar{G}_k$ are all the $\bar{G}$-factors of $(\bar{S} \cap E)$ and the conclusion follows. Indeed, it is enough to show that given any $\bar{G} \subseteq \bar{G}$ containing $(\bar{S} \cap E)$, either $\bar{G} = E$ or $\bar{G} \supseteq \bar{G}_r$ for some $1 \leq r \leq k$. The inclusion $\bar{G} \supseteq (S \cap E)$ implies $G \supseteq (S \cap F)$ by taking the image of $\pi$. By Lemma 2.4 (ii), we know that $F, G_{m+1}, \ldots, G_k$ are all the $\bar{G}$-factors of $(S \cap F)$. Therefore $G$ contains either $F$ or one of $G_r$ for $m+1 \leq r \leq k$. In the latter case, we immediately get the conclusion. So we assume that $G$ contains $F$.

If $G = F$, then $\bar{G} = E$ from which the conclusion follows. Now we assume $G \supseteq F$. Since
\[
\bar{G} \cap E \cong \mathbb{P}(N_FG), \bar{S} \cap E \cong \mathbb{P}(N_FA|F \cap B),
\]
and that $\bar{G} \cap E$ contains $\bar{S} \cap E$, we have $(N_FG)_y \supseteq (N_FA)_y$ for any $y \in F \cap B$. But $(N_FG)_y \cong T_{G,y}/T_{F,y}$ and $(N_FA)_y \cong T_{A,y}/T_{F,y}$, therefore $T_{G,y} \supseteq T_{A,y}$ and $G$ contains $A$. Since $G_1, \ldots, G_m$ are the $\bar{G}$-factors of $A$ by Lemma 2.4 (i), $G$ contains $G_r$ for some $1 \leq r \leq m$.

(iii) “$T$ is a nest $\Rightarrow$ $\bar{T}$ is a nest”. Suppose $T$ is induced by the flag $S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k$. If $S_1 \not\subseteq F$ or $S_k \subseteq F$, then $\bar{T}$ is induced by the flag $\bar{S}_1 \subseteq \bar{S}_2 \subseteq \cdots \subseteq \bar{S}_k$; otherwise there exists an integer $m$, $1 \leq m \leq k-1$ where $\bar{S}_m \subseteq F$ but $S_{m+1} \not\subseteq F$. In this case it can be easily checked that $\bar{T}$ is generated by the flag
\[
(\bar{S}_1 \cap \bar{S}_{m+1}) \subseteq \cdots \subseteq (\bar{S}_m \cap \bar{S}_{m+1}) \subseteq (\bar{S}_{m+1} \cap E) \subseteq \cdots \subseteq (\bar{S}_k \cap E).
\]

“$T$ is a nest $\iff$ $\bar{T}$ is a nest”. Suppose $\bar{T}$ is induced by the flag $S_1^t \subseteq S_2^t \subseteq \cdots \subseteq S_{k'}^t$. If $S_1^t \not\subseteq E$, then $T$ is induced by the flag $\pi(S_1^t) \subseteq \pi(S_2^t) \subseteq \cdots \subseteq \pi(S_{k'}^t)$ and we are done. Now assume $S_1^t \subseteq E$, and denote by $m$ the maximal integer satisfying $S_m^t \subseteq E$. Since $E$ is both minimal and maximal in $\bar{G}$, the $E$-factorization of $S_m^t$ must be of the form $E \cap B_i^t$ for $1 \leq i \leq m$. Then it can be checked that $T$ is induced by the following flag
\[
(G \cap \pi(B_1^t)) \subseteq \pi(B_1^t) \subseteq \cdots \subseteq \pi(B_m^t) \subseteq \pi(S_{m+1}^t) \subseteq \cdots \subseteq \pi(S_{k'}^t).
\]

□
Proof of Proposition 2.8 The proof is similar to the proof of Proposition 2.8. The only new case is when $F$ is not minimal in $\mathcal{G}_+$. So throughout the proof we assume that $G_0$ is minimal and $G_0 \subsetneq F$.

(1) We show $\tilde{S}_+$ is an arrangement.

First we prove that the intersection $(\tilde{G}_0 \cap \tilde{S}) \cap \tilde{S}'$ is clean for $S, S' \in S$. Take the $F$-factorization $S = A \cap B$ and $S' = A' \cap B'$ in the arrangement $S$. Take the $G_0$-factorization $B = B_1 \cap B_2$ in the arrangement $\mathcal{S}_+$. Similar to the proof of Proposition 2.8 (i), we need to consider three cases:

a) $F \subsetneq A \cap A'$. Then

$$\quad (\tilde{G}_0 \cap \tilde{S}) \cap \tilde{S}' = \tilde{G}_0 \cap \tilde{A} \cap \tilde{B}_1 \cap \tilde{B}_2 \cap \tilde{A}' \cap \tilde{B}'$$

$$\quad = \tilde{G}_0 \cap \tilde{A} \cap \tilde{A}' \cap \tilde{B}_2 \cap \tilde{B}'$$

$$\quad = \tilde{G}_0 \cap (A \cap A')^c \cap (B_2 \cap B')^c$$

$$\quad = \tilde{G}_0 \cap (A \cap A' \cap B_2 \cap B')^c$$

The second equality holds because $T_{\tilde{B}_0} = \phi^{-1}I_{\tilde{B}_1} \subseteq \phi^{-1}I_{\tilde{G}_0} = I_{\tilde{G}_0}$, therefore $\tilde{B}_1 \supseteq \tilde{G}_0$. The third and fourth equalities are because of Lemma 2.9.

Moreover, we have

$$\quad T_{\tilde{G}_0 \cap \tilde{S} \cap \tilde{S}'} = T_{\tilde{G}_0} \cap T_{\tilde{A} \cap \tilde{A}'} \cap T_{\tilde{B}_2 \cap \tilde{B}'}$$

$$\quad = T_{\tilde{G}_0} \cap T_{\tilde{A} \cap \tilde{A}'} \cap T_{\tilde{B}_2} \cap T_{\tilde{B}'}$$

$$\quad = (T_{\tilde{G}_0} \cap T_{\tilde{A} \cap \tilde{B}_1} \cap T_{\tilde{B}_2}) \cap (T_{\tilde{A}'} \cap T_{\tilde{B}'})$$

$$\quad = T_{\tilde{G}_0 \cap \tilde{S}} \cap T_{\tilde{S}'}.$$  

b) $F = A \cap A'$ but $F \neq A$ and $F \neq A'$. It is easy to verify that $(\tilde{G}_0 \cap \tilde{S}) \cap \tilde{S}' = \emptyset$.

c) $F = A'$. The proof is similar to a) and we omit it.

Similarly, we can check that $(\tilde{G}_0 \cap \tilde{S}) \cap (\tilde{G}_0 \cap \tilde{S}')$ and $(\tilde{G}_0 \cap \tilde{S}) \cap (E \cap \tilde{S}')$ are clean intersections along some elements in $\tilde{S}_+$.

(2) We show $\tilde{G}_+$ is a building set. It is enough to show that the minimal elements in $\tilde{G}_+$ which contain $\tilde{G}_0 \cap \tilde{S}$ intersect transversally along $\tilde{G}_0 \cap \tilde{S}$. Assume the $F$-factorization of $S$ is $A \cap B$ where $G \subsetneq A$. (If $G = A$, then $\tilde{G}_0 \subseteq E$ hence we can replace $S$ by $B$ and keep $\tilde{G}_0 \cap \tilde{S}$ unchanged.) Assume the $G_0$-factorization of $B$ is $B_1 \cap B_2$.

We claim that the set of $\tilde{G}_+$-factors of $\tilde{G}_0 \cap \tilde{S}$ is:

$$\mathcal{P} := \{ \tilde{G}_0 \} \cup \{ \tilde{G}_+\text{-factors of } \tilde{A} \} \cup \{ \tilde{G}_+\text{-factors of } \tilde{B}_2 \}.$$  

It is easy to check that the subvarieties in $\mathcal{P}$ intersect transversally. To show that the subvarieties in $\mathcal{P}$ are all the $\tilde{G}_+$-factors, it suffices to show that any minimal element $\tilde{G} \in \tilde{G}_+$ that contains $\tilde{G}_0 \cap \tilde{S}$ ($= \tilde{G}_0 \cap \tilde{A} \cap \tilde{B}_2$) belongs to $\mathcal{P}$. 

Since $G = \phi(\tilde{G}) \supseteq \phi(\tilde{G}_0 \cap \tilde{A} \cap \tilde{B}_2) = G_0 \cap B_2$ and the $G_k$-factors of $G_0 \cap B_2$ are $G_0$ and $G$-factors of $B_2$, $G \supseteq G_0$ or $G \supseteq B_2$. If $G \supseteq B_2$, then the conclusion follows, so we can assume $G \supseteq G_0$. Then $G \ni F$ or $G \supseteq F$.

If $G \ni F$, then $G \supseteq G_0$ implies $G \supseteq \tilde{G}_0$. Since $\tilde{G}$ is chosen to be minimal, $\tilde{G} = \tilde{G}_0$ belongs to $\mathcal{P}$.

If $G \supseteq F$, then $\tilde{G} = \mathbb{P}(N_F G)$ and $\tilde{G}_0 \cap \tilde{S} = \mathbb{P}(N_{F,A} |_{G_0 \cap B})$. Therefore $G \supseteq A$ which implies that $G$ is a $G$-factor of $A$ and $\tilde{G}$ is a $G$-factor of $\tilde{A}$, hence $\tilde{G} \in \mathcal{P}$.

\[ \square \]

5.3. **Codimensions of the centers.** (Thanks the referee for suggesting this question.) In the original construction of the Fulton-MacPherson configuration space $X[n]$, each blow-up is along a nonsingular center of codimension $m$ or $m+1$, where $m$ is the dimension of $X$. But if we construct $X[n]$ by blowing up centers in ascending dimension, the codimensions of the centers are much larger: the first blow-up is along the smallest diagonal $\Delta_{[n]}$ which is of codimension $m(n-1)$.

In general, given a specified order of blow-ups, we can find the dimension, hence the codimension, of the centers easily.

**Proposition 5.3.** Let $Y$ be a nonsingular variety and $G = \{G_1, \ldots, G_N\}$ be a nonempty building set of subvarieties of $Y$ satisfying the condition (*) in Theorem 1.3. Let $j$ be an integer between 1 and $N$. Define $G' := \{G_1, \ldots, G_{j-1}\}$ and define $\mathcal{F} = \{G_{i_1}, G_{i_2}, \ldots, G_{i_k}\}$ to be the minimal elements of $\{G \in G' : G \supseteq G_j\}$.

Then in the construction of $Y_{G_j}$ by blowing up along $G_1, \ldots, G_N$ in order, the center of the $j$-th blow-up is of dimension

$$\dim G_j + \sum_{k=1}^{\ell} (d - 1 - \dim G_{i_k})$$

if $\mathcal{F} \neq \emptyset$, and of dimension $\dim G_j$ if $\mathcal{F} = \emptyset$.

**Proof.** The set $G' := \{G_1, \ldots, G_{j-1}\}$ is a building set by the condition (*) in Theorem 1.3. By the same theorem we can assume that $G_1, \ldots, G_{j-1}$ is in order of ascending dimension. Denote the variety obtained after the $i$-th blow-up by $Y_{G_{i-1}}$. We want to find the dimension of $G_j$ in $Y_{j-1}$.

Since (a) blowing up a center that does not contain $G_j$ will not change the dimension of $G_j$, and (b) $G$ does not contain $G_j$ if $G$ does not contain $G_j$, we can focus on the subset $G'' \subseteq G'$ of subvarieties that contain $G_j$. Let $\mathcal{F} := \{G_{i_1}, G_{i_2}, \ldots, G_{i_k}\}$ be the set of minimal elements in $G''$. Define $S := G_{i_1} \cap G_{i_2} \cap \cdots \cap G_{i_k}$. Then $\mathcal{F}$ is also the set of minimal elements in $G'$ that contain $S$. By the definition of building set, $G_{i_1} \ldots G_{i_k}$ intersect transversally. A subvariety $G \in G'' \setminus \mathcal{F}$ must contain a subvariety, say $G_i$, in $\mathcal{F}$. Then $G \supseteq G_i \supseteq G_j$, and in the variety $Y_{i-1}$ we have $G \supseteq G_j$. It can be easily checked that in the variety $Y_j$ (obtained by blowing up along $G_i$), $\tilde{G}$ will no longer contain $\tilde{G}_j$, hence the blow-up along $\tilde{G}$ will not change
the dimension of \( \tilde{G}_j \). In other words, we only need to find the change of 
\( \dim \tilde{G}_j \) for the blow-ups along the transversal subvarieties in \( \mathcal{F} \), which is simply 
\[
\sum_{k=1}^{\ell} (d - 1 - \dim G_{ik})
\]
\[\square\]

**Example:** Let \( m = \dim X \). In the original construction of the Fulton-MacPherson configuration space \( X[4] \) (cf. §4.2), the dimension of the first center is \( \dim \Delta_{12} = 3m \), the dimension of the second center \( \Delta_{123} \) is 
\[
\dim \Delta_{123} + (4m - 1 - \dim \Delta_{12}) = 2m + (m - 1) = 3m - 1,
\]

it can be easily checked that \( \dim \tilde{\Delta}_I \) is \( 3m \) if \( |I| = 2 \) and \( 3m - 1 \) otherwise, so the codimension is either \( m \) or \( m + 1 \). In general, using the order of blow-ups in the original construction of \( X[n] \) for \( n \geq 2 \), the codimension of each center \( \tilde{\Delta}_I \) is \( m \) if \( |I| = 2 \) and \( m + 1 \) otherwise.

**Example:** The codimension of each blow-up center is 2 in Keel’s construction of \( \overline{M}_{0,n} \). Since the blow-ups in Keel’s construction of \( \overline{M}_{0,n} \) can be obtained by restricting the blow-ups in the original Fulton-MacPherson’s construction of \( X[n] \) to a fiber of \( \pi_{123} \) (defined in Corollary 1.4), each blow-up center is of codimension \( m = 1 \) or \( m + 1 = 2 \). But blowing up along a center of codimension 1 does nothing. So we only need to carry out blow-ups along codimension-2 centers.

### 5.4. The statements for a general arrangement.

**Definition 5.4.** A *arrangement* of subvarieties of a nonsingular variety \( Y \) is a finite set \( S = \{S_i\} \) of nonsingular closed subvarieties \( S_i \) properly contained in \( Y \) satisfying the following conditions:

(i) \( S_i \) and \( S_j \) intersect cleanly,

(ii) \( S_i \cap S_j \) either equals to a disjoint union of some \( S_k \)'s or is empty.

**Definition 5.5.** Let \( S \) be an arrangement of subvarieties of \( Y \). A subset \( G \subseteq S \) is called a building set of \( S \) if there is an open cover \( \{U_i\} \) of \( Y \) such that the restriction of the arrangement \( S|_{U_i} \) is simple for each \( i \) and \( G|_{U_i} \) is a building set of \( S|_{U_i} \).

A finite set \( G \) of nonsingular subvarieties of \( Y \) is called a building set if the set of all possible intersections of collections of subvarieties from \( G \) forms an arrangement \( S \), and that \( G \) is a building set of \( S \) defined as above. In this situation, \( S \) is called the arrangement induced by \( G \).

**Definition 5.6.** A subset \( T \subseteq G \) is called \( G \)-nested (or a \( G \)-nest) if there is an open cover \( \{U_i\} \) of \( Y \), such that the restriction of the arrangement induced by \( G \) to each \( U_i \) is simple and \( T|_{U_i} \) is a \( G|_{U_i} \)-nest.

We define the wonderful compactification same as Definition 1.1. Then Theorem 1.2 and Theorem 1.3 still hold.
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