OPTIMAL PROPORTIONAL REINSURANCE AND PAIRS TRADING UNDER EXPONENTIAL UTILITY CRITERION FOR THE INSURER

PENGXU XIE\textsuperscript{1}, LIHUA BAI\textsuperscript{1,2} AND HUAYUE ZHANG\textsuperscript{1,2,*}

\textsuperscript{1}School of Mathematical Sciences, Nankai University, Tianjin 300071, China
\textsuperscript{2}School of Finance, Nankai University, Tianjin 300071, China

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ABSTRACT. This paper studies the optimal proportional reinsurance and investment strategy for an insurer who invests one paired assets, where their price spread is described by Ornstein-Uhlenbeck (O-U) processes. The insurer’s objective is to maximize the expected exponential utility of the terminal wealth in a finite time horizon under two risk models: a classical risk model and a diffusion model. Using the classical stochastic control approach based on the Hamilton-Jacobi-Bellman equation, we characterize the optimal strategies and provide a verification result for the value function via the exponential integrability of the square of an O-U process. Finally, numerical examples are performed to obtain sensitivity analysis.

1. Introduction. Investment and reinsurance have become a very effective tool for insurance companies to increase income and manage risk. In recent years, a large number of literatures have discussed the optimal reinsurance and optimal investment of insurance companies in financial market. In these works, aiming at minimizing the ruin probability or maximizing the utility of terminal wealth, the optimal strategy and value function of insurance companies are obtained by using stochastic control theory and related methods. Browne [6] used a Brownian motion with drift to describe the surplus of the insurance company and found the optimal investment strategy. The diffusion approximation was used by Promislow and Young [16] to solve the problem of minimizing the probability of ruin where the investment and reinsurance strategy are both considered. Yang and Zhang [20] considered the optimal investment problem, for a risk process modeled by a jump-diffusion process. In addition, Bai and Guo [1] considered two optimization problems: the problem of maximizing the expected exponential utility of terminal wealth and the problem of minimizing the probability of ruin under the no-shorting constraint. Brachetta and Schmidli [5] considered a diffusion approximation to an insurance risk model where an external driver models a stochastic environment. They found the optimal investment strategy, proportional and excess-of-loss reinsurance strategies. For related studies, readers can be easily found more in the references mentioned above.

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*Corresponding author: Huayue Zhang.
These works focused on risky assets following the Black-Scholes model. In finance, pair trading is a risk neutral investment strategy that considers two highly correlated stocks. Assume that the spread between the two stock prices fluctuates randomly and that the spread has a long-run mean. Sometimes, the spread process diverges from the long-run mean, and sometimes it converges. If the spread widens, the expensive stock is sold and the cheap stock is purchased. As the spread narrows again, profit is achieved by unwinding the position of the pair. Various quantitative methods have been developed and applied to pairs trading in the literature. Three commonly used techniques are: distance method, co-integration and stochastic spread. We also refer the readers Gatev et al. [8], Vidyamurthy [18] and Lin et al. [13] for more on this. The stochastic spread method models the mean reverting process of pairs trading as an Ornstein-Uhlenbeck (O-U) process. Elliott et al. [7] provided an analytic framework of pairs trading, which laid the ground for prediction and decision-making based on the hidden O-U process. We also refer the readers to Vladislav [19], Boguslavsky and Boguslansky [4] and Bertram [3] for a comprehensive survey on pairs trading strategy based on stochastic spread.

To the best of our knowledge, the concept of pairs trading is not yet to be incorporated into the insurer’s investment problem with utility maximization. In this paper, we study optimal reinsurance-investment problem in the framework with the paired asset. On the one hand, the insurer transfers risks and reduces losses by purchasing proportional reinsurance from reinsurance companies and pay reinsurance premiums at the same time. On the other hand, insurance company manages his wealth by investing their surplus in a paired assets. Compared with previous work, the innovation of this paper is to use the pairs trading strategy to study the wealth management and risk control of insurance companies. The surplus process is more complex after introducing the pairs trading and reinsurance strategy, we use the approach of Hamilton-Jacobi-Bellman (HJB) equation to obtain the optimal pairs trading strategy and the optimal proportional reinsurance.

There are three main contributions: Firstly, because of the presence of O-U process, the classical verification theorem does not hold. To tackle the difficulty, we prove the exponential integrability of the square of an O-U process and finally give some sufficient conditions for the verification theorem. Secondly, we find an interesting phenomenon that optimal pairs trading strategy are identical under two risk models. Thirdly, we analyze numerically how the spread and the key parameters of the model affect optimal pairs trading strategy.

Here is a brief outline of this paper. In section 2, we consider the classical risk model and formulate the assumptions and the problem of maximizing the exponential utility. The optimal proportional reinsurance and pairs trading strategies are obtained by solving the HJB equation. In section 3, the same problem is solved by the analogous method for the diffusion approximation model. Section 4 presents numerical examples. Section 5 concludes the paper.

2. The classical risk model.

2.1. The market. Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a filtered probability space which satisfies the usual conditions. We consider a financial market that contains a risk-free asset and two risky assets that are traded continuously within the time horizon \(T > 0\). The risk-free asset \(R_t\) satisfies the dynamics:

\[
dR_t = r R_t \, dt,
\]
geometric Brownian motion:

\[
dS_t^2 = \delta S_t^2 dt + \xi S_t^2 dW_t^2,
\]

where \( \delta \in \mathbb{R} \) is the drift, \( \xi \in \mathbb{R}^+ \) is the volatility and \( W_t^2 \) is a standard Brownian motion defined on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). Let \( Z_t \) denote the spread of the two stocks at time \( t \), defined as

\[
Z_t = \ln S_t^1 - \ln S_t^2.
\]

According to Mudchanatongsuk et al. [15], \( Z_t \) can be assumed to follow an Ornstein-Uhlenbeck process, that is,

\[
dZ_t = a(b-Z_t)dt + \sigma dW_t^Z,
\]

where \( a \in \mathbb{R}^+ \) is the rate of reversion, \( b \in \mathbb{R} \) is the equilibrium level, \( \sigma \in \mathbb{R}^+ \) is the volatility and \( W^Z \) is a standard Brownian motion defined on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\), which is correlated with \( W_t^2 \) with the correlation coefficient \( \rho \). By using (1), (2) and the Itô formula, we can obtain the dynamics of \( S_t^1 \) by

\[
dS_t^1 = [\delta + a(b-Z_t) + \frac{1}{2}\sigma^2 + \rho \xi \sigma]S_t^1 dt + \xi S_t^1 dW_t + \sigma S_t^1 dW_t^Z.
\]

### 2.2. Maximizing the exponential utility of terminal wealth.

We consider a classical risk model

\[
dx_t = cd\tau - d \sum_{i=1}^{N_t} Y_t,
\]

where \( \{N_t\} \) is a poisson process with intensity \( \lambda > 0 \) and \( \{Y_t\} \), independent of \( \{N_t\} \), is a sequence of positive i.i.d variables with distribution function \( F(y) \) having finite first and second moments \( \mu_j \) for \( j = 1, 2 \). We denote the claim times by \( \{\sigma_i, i = 1, 2, \ldots\} \). Let \( Y \) be a generic random variable which has the same distribution as \( \{Y_t\} \). For any \( l < +\infty \), the moment generating function of \( Y \), \( M_Y(l) = Ee^{lY} \), is well defined and \( \lim_{l \to +\infty} M_Y(l) = +\infty \). The premium rate \( c \) is assumed to be calculated via the expected value principle, i.e. \( (1+\eta)\lambda \mu_1 \) with safety loading \( \eta > 0 \).

We assume that \( \sum_{i=1}^{N_t} Y_t \) is independent of \( \{W_t^2\} \) and \( \{W_t^Z\} \).

Suppose that the insurer is allowed to invest its wealth in a financial market. Let \( u_t \) and \( \tilde{u}_t \) denote the amount invested in the stocks \( S^1 \) and \( S^2 \) at time \( t \), respectively. We are only allowed to go short one of them and long the other in equal amount, that is to say,

\[
u_t = -\tilde{u}_t.
\]

In addition to investment, we assume that the insurer can purchase proportional reinsurance to reduce the underlying insurance risk. Let \( p_t \in [0, 1] \) represents the retention level of reinsurance acquired at time \( t \). It means that the insurer pays \( p_t Y \) of a claim occurring at time \( t \) and the reinsurer pays \((1-p_t)Y \). For this reinsurance, the premium has to be paid at rate \((1 + \theta)(1 - p_t)\lambda \mu_1 \), where \( \theta > \eta \) represents the safety loading of the reinsurer.
A strategy $\pi$ is described by a two-dimensional stochastic process $(u_t, p_t)$, we model the controlled reserve process by

$$
\begin{align*}
   dX_t^\pi &= \left\{ rX_t^\pi + u_t a(b-Z_t) + \lambda \mu_1[p_t(1+\theta) - (\theta - \eta)] \right\} dt + u_t \sigma dW_t^Z \\
   -d \sum_{i=1}^{N_t} p_\sigma Y_i,
\end{align*}
$$

(3)

where $\bar{b} = b + \sigma^2/2a + \rho \xi \sigma/a$.

**Definition 2.1.** A strategy $\pi = \{(u_t, p_t), t \in [0, T]\}$ is said to be admissible if

(i) $(u_t, p_t)$ is $F_t$ progressively measurable;

(ii) $u_t$ satisfy

$$
   \int_0^T u_t^2 ds < +\infty, \quad a.s.;
$$

(iii) $p_t \in [0, 1]$;

(iv) the equation (3) has a strong solution.

Denote $\Pi$ is the set of all admissible strategies.

Suppose now that the insurer is interested in maximizing the expected utility function of his terminal wealth, say at time $T$. We focus on the standard approach which assumes that the utility function $U(x)$ is strictly increasing and concave ($U''(x) < 0$). For a strategy $\pi$, we define the utility attained by the insurer from state $x$ at time $t$ as

$$
V_x(t, x) = E[U(X_T^\pi)|X_t^\pi = x].
$$

Our objective is to find the optimal value function

$$
V(t, x) = \sup_{\pi \in \Pi} V_x(t, x)
$$

(4)

and the optimal strategy $\pi^*$ such that $V_{\pi^*}(t, x) = V(t, x)$.

Assume that the insurer has an exponential utility function

$$
U(x) = -e^{-\gamma x},
$$

(5)

where $\gamma > 0$. The utility function (5) plays a prominent role in insurance mathematics and actuarial practice, as it is the only utility function under which the principle of “zero utility” gives a fair premium that is independent of the level of reserve of an insurance company (see Gerber [9]).

2.3. **The solution of HJB equation.** In this section, we will solve the problem (4). Define the second-order partial differential operator:

$$
\mathcal{L}^{u,p}[v](t, x, z) = v_t + \left\{ \frac{1}{2} v_{xx} \sigma^2 u^2 + [v_x a(\bar{b} - z) + \sigma^2 v_{xz}] u \right\}
$$

$$
+ \left\{ v_x \lambda \mu_1 (1 + \theta) p + \lambda E[v(t, x - pY, z) - v(t, x, z)] \right\}
$$

$$
+ r x v_x + \lambda \mu_1 (\eta - \theta) v_x + v_x a(\bar{b} - z) + \frac{1}{2} \sigma^2 v_{zz}.
$$

From standard arguments, we know that if $V \in C^{1,2,2}$, then $V$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$
\sup_{(u,p) \in R \times [0,1]} \mathcal{L}^{u,p}[v](t, x, z) = 0,
$$

(6)
where

$$v(T, x, z) = -e^{-\gamma x}.$$  

(7)

For convenience, before giving the solution of HJB equation (6), we define the following functions:

$$A(t) := -\frac{a^2}{2\sigma^2} (T - t),$$  

(8)

$$B(t) := \frac{a^3}{2\sigma^2} (b - b)(T - t)^2 + \frac{a^2b}{\sigma^2} (T - t),$$  

(9)

$$C_1(t) := -\frac{a^4}{6\sigma^2} (b - b)^2 (T - t)^3 - \left[ \frac{a^2b}{2\sigma^2} (b - b) + \frac{a^2}{4} \right] (T - t)^2$$  

$$- \left( \frac{a^2b}{2\sigma^2} + \lambda \right) (T - t) + \frac{\lambda \mu_1}{r} (1 + \eta) \gamma \left[ 1 - e^{r(T - t)} \right]$$  

$$+ \lambda \int_0^{T-t} M_Y(\gamma e^s) ds,$$  

(10)

$$C_2(t) := -\frac{a^4}{6\sigma^2} (b - b)^2 (T - t)^3 - \left[ \frac{a^2b}{2\sigma^2} (b - b) + \frac{a^2}{4} \right] (T - t)^2$$  

$$- \left\{ \frac{a^2b}{2\sigma^2} + \lambda \mu_1 (1 + \theta) \bar{p} - \lambda [M_Y(\bar{p}) - 1] \right\} (T - t)$$  

$$+ \frac{\lambda \mu_1 (\eta - \theta) \gamma}{r} \left[ 1 - e^{r(T - t)} \right],$$  

(11)

$$C_3(t) := \begin{cases} C_2(t) + k, & t \in [0, T - \frac{1}{r} \ln \bar{p}_\gamma], \\ C_1(t), & t \in (T - \frac{1}{r} \ln \bar{p}_\gamma, T], \end{cases}$$  

(12)

where

$$k := \frac{\lambda \mu_1 (1 + \theta) \bar{p}}{r} \ln \frac{\bar{p}}{\gamma} + \frac{\lambda \mu_1 (1 + \theta) \gamma}{r} \left[ 1 - \frac{\bar{p}}{\gamma} \right] + \lambda M_Y(\bar{p})$$  

$$+ \lambda \int_{\frac{1}{r} \ln \bar{p}_\gamma}^{T-t} M_Y(\gamma e^s) ds,$$

and $\bar{p}$ is an unique positive solution of $M'_Y(p) = (1 + \theta) \mu_1$.

**Theorem 2.2.** Assume that $T - t < \frac{1}{2\sigma}$. Let $v$ be defined as follows:

(i) if $\bar{p} > e^{rT}$,

$$v(t, x, z) = -\exp\{-\gamma x e^{r(T - t)} + A(t)z^2 + B(t)z + C_1(t)\};$$

(ii) if $\bar{p} \leq \gamma$,

$$v(t, x, z) = -\exp\{-\gamma x e^{r(T - t)} + A(t)z^2 + B(t)z + C_2(t)\};$$

(iii) if $\gamma < \bar{p} \leq e^{rT}$,

$$v(t, x, z) = -\exp\{-\gamma x e^{r(T - t)} + A(t)z^2 + B(t)z + C_3(t)\},$$

(13)

where $A(t), B(t), C_1(t), C_2(t)$ and $C_3(t)$ are defined in (8)-(12). Then $v$ satisfies (6)-(7) and the maximal function $(u^*(t, x, z), p^*(t, x, z))$ is given by

$$(u^*(t, x, z), p^*(t, x, z))$$

$$= \left( \frac{a(b - z)[1 + a(T - t)] + \frac{a^2}{4}(b - b)(t - T)^2}{\sigma^2 \gamma}, \bar{p} e^{-r(T - t)} \right),$$

(14)
Proof. We only verify Case (iii). Case (i) and Case (ii) can be similarly proved. It is easy to show that \( v \) defined by (13) is twice continuously differentiable. Note that \( v \) satisfies that \( v_x > 0 \) and \( v_{xx} < 0 \). From lemma 5.1 in Appendix, the maximal function of \( L^u_p[v](t, x, z) \) without restriction is

\[
\begin{align*}
&u^0(t, x, z) = \frac{a(b - z)[1 + a(T - t)] + \frac{a^3}{2}(b - b)(t - T)^2}{\sigma^2 \gamma} e^{-r(T-t)}, \\
p^0(t, x, z) = \frac{b}{\gamma} e^{-r(T-t)}. 
\end{align*}
\]

When \( 0 \leq t < T - \frac{1}{r} \ln \frac{\bar{p}}{\gamma} \), we have that \( 0 < p^0(t, x, z) < 1 \). In this case, \((u^0(t, x, z), p^0(t, x, z))\) is in the interior of the control region \( R \times [0, 1] \). This means that the maximal function \((u^*(t, x, z), p^*(t, x, z)) = (u^0(t, x, z), p^0(t, x, z))\). Substituting it into \( L_u^p[v](t, x, z) \) yields

\[
\sup_{(u, p) \in R \times [0, 1]} L_u^p[v](t, x, z) = L_u^0(t, x, z), p^0(t, x, z)[v](t, x, z)
\]

\[
\begin{align*}
&= \left[ A' - \frac{1}{2} \frac{(2\sigma^2 A - a)^2}{\sigma^2} - 2aA + 2\sigma^2 A^2 \right] z^2 \\
&+ \left[ B' - \frac{1}{2} \frac{2(2\sigma^2 A - a)(a\bar{b} + \sigma^2 B)}{\sigma^2} + 2abA - aB + 2\sigma^2 AB \right] z \\
&+ \left[ C'_3 - \frac{1}{2} \frac{(a\bar{b} + \sigma^2 B)^2}{\sigma^2} - \lambda \mu_1 (1 + \theta) \bar{p} + \lambda [M_y(\bar{p}) - 1] \\
&- \lambda \mu_1 (\eta - \theta) \gamma e^{r(T-t)} + abB + \frac{1}{2} \sigma^2 B^2 + \sigma^2 A \right].
\end{align*}
\]

It is easy to verify that \( A(t), B(t), C_3(t) \) are the solutions of the following systems of ordinary differential equations (ODEs)

\[
\begin{align*}
&A' - \frac{1}{2} \frac{(2\sigma^2 A - a)^2}{\sigma^2} - 2aA + 2\sigma^2 A^2 = 0, \quad A(T) = 0, \\
&B' - \frac{1}{2} \frac{2(2\sigma^2 A - a)(a\bar{b} + \sigma^2 B)}{\sigma^2} + 2abA - aB + 2\sigma^2 AB = 0, \quad B(T) = 0, \\
&C'_3 - \frac{1}{2} \frac{(a\bar{b} + \sigma^2 B)^2}{\sigma^2} - \lambda \mu_1 (1 + \theta) \bar{p} + \lambda [M_y(\bar{p}) - 1] \\
&- \lambda \mu_1 (\eta - \theta) \gamma e^{r(T-t)} + abB + \frac{1}{2} \sigma^2 B^2 + \sigma^2 A = 0, \quad C_3(T) = 0.
\end{align*}
\]

Therefore, \( v \) of (13) satisfies that

\[
\sup_{(u, p) \in R \times [0, 1]} L_u^p[v](t, x, z) = 0, \quad 0 \leq t < T - \frac{1}{r} \ln \frac{\bar{p}}{\gamma}.
\]

When \( T - \frac{1}{r} \ln \frac{\bar{p}}{\gamma} \leq t \leq T \), we have that \( p^0(t, x, z) \geq 1 \). Since \( f(p) \) is a concave function, we have that \((u^*(t, x, z), p^*(t, x, z)) = (u^0(t, x, z), 1)\). Substituting it into
\[ \mathcal{L}^{u,p}[v](t, x, z) \text{ yields} \]
\[
\sup_{(u,p)\in R\times[0,1]} \mathcal{L}^{u,p}[v](t, x, z) \\
= \mathcal{L}^{u}(t,x,z,1)[v](t, x, z) \\
= \left[ A' - \frac{1}{2} \cdot \frac{(2\sigma^2 A - a)^2}{\sigma^2} - 2aA + 2\sigma^2 A^2 \right] z^2 \\
+ \left[ B' - \frac{1}{2} \cdot \frac{2(2\sigma^2 A - a)(\bar{a}b + \sigma^2 B)}{\sigma^2} + 2abA - aB + 2\sigma^2 AB \right] z \\
+ \left[ C'_1 - \frac{1}{2} \cdot \frac{(\bar{a}b + \sigma^2 B)^2}{\sigma^2} - \lambda\mu_1(1 + \eta)\gamma e^{\gamma(T-t)} + \lambda[M_Y(\gamma e^{\gamma(T-t)}) - 1] \\
+ abB + \frac{1}{2} \sigma^2 B^2 + \sigma^2 A \right].
\]

It is easy to verify that \( A(t), B(t), C_1(t) \) are the solutions of the following systems of ordinary differential equations (ODEs)

\[
\begin{cases}
A' - \frac{1}{2} \cdot \frac{(2\sigma^2 A - a)^2}{\sigma^2} - 2aA + 2\sigma^2 A^2 = 0, & A(T) = 0, \\
B' - \frac{1}{2} \cdot \frac{2(2\sigma^2 A - a)(\bar{a}b + \sigma^2 B)}{\sigma^2} + 2abA - aB + 2\sigma^2 AB = 0, & B(T) = 0, \\
C'_1 - \frac{1}{2} \cdot \frac{(\bar{a}b + \sigma^2 B)^2}{\sigma^2} - \lambda\mu_1(1 + \eta)\gamma e^{\gamma(T-t)} \\
+ \lambda[M_Y(\gamma e^{\gamma(T-t)}) - 1] + abB + \frac{1}{2} \sigma^2 B^2 + \sigma^2 A = 0, & C_1(T) = 0.
\end{cases}
\]

Therefore, \( v \) of (13) satisfies that

\[
\sup_{(u,p)\in R\times[0,1]} \mathcal{L}^{u,p}[v](t, x, z) = 0, \quad T - \frac{1}{r} \ln \frac{\bar{p}}{\gamma} < t \leq T
\]

and

\[
v(T, x, z) = -e^{-\gamma x}.
\]

In conclusion, \( v \) of (13) satisfies HJB equation and the maximal function is given by (14).

\( \square \)

2.4. Verification theorem. Denote

\[
\Pi_c := \left\{ \pi \in \Pi : \pi \text{ satisfies that } \{v(\tau_1 \wedge T, X_{\tau_1 \wedge T}, X_{\tau_1 \wedge T}, i = 1, 2, 3...) \text{ is uniformly integrable} \right\},
\]

where

\[
\tau_i^* = \inf \{ s \geq t, (|X_s^x|, |Z_s|) \notin [0, i) \times [0, i) \}, \quad i = 1, 2, 3...
\]

is a sequence of stopping time. In the following, we give the verification theorem.

**Theorem 2.3.** (Verification Theorem)
Let \((u^*(t, x, z), p^*(t, x, z))\) be defined by (14). Assume that \(T - t < 1/2a\), then the strategy
\[
\pi^* = (u^*_s, p^*_s) := (u^*(s, X^s_*, Z_s), p^*(s, X^s_*, Z_s)), \quad t \leq s \leq T
\]
is optimal on \(\Pi_e\), where \(X^s_*\) is the reserve process with the strategy \(\pi^*\).

Proof. For convenience, let \(T^*_e = \tau^*_e \wedge T\) and \(T^*_i := T^*_{e}^i\). Firstly, we need to verify \(\pi^* \in \Pi_e\). It is sufficient to prove that, for some \(q > 1\) (its domain will be given by (16)),
\[
\sup \mathbb{E}^{t, x, z} \left| \pi(T^*_e, X^*_e, Z_{T^*_e}) \right|^q < \infty.
\]

By Itô formula and HJB equation, we can conclude that
\[
v(T^*_e, X^*_e, Z_{T^*_e}) = v(t, x, z) - \frac{a}{\sigma} \int_t^{T^*_e} v(s, X^s_*, Z_s)(\bar{b} - Z_s) dW_s^Z
\]
\[
- \lambda \int_t^{T^*_e} v(s, X^s_*, Z_s) \left[ M_Y(\gamma e^{r(T-s)} p^*) - 1 \right] ds
\]
\[
+ \sum_{t \leq s \leq T^*_i} \left[ v(s, X^s_*, Z_s) - v(s-, X^s_*, Z_{s-}) \right].
\]

Then solving the stochastic differential equation SDE with jump (15) yields
\[
v(T^*_e, X^*_e, Z_{T^*_e}) = v(t, x, z) \cdot \exp \left\{ - \frac{a}{\sigma} \int_t^{T^*_e} (\bar{b} - Z_s) dW_s^Z - \frac{1}{2} \cdot \frac{a^2}{\sigma^2} \int_t^{T^*_e} (\bar{b} - Z_s)^2 ds \right\}
\cdot \exp \left\{ - \lambda \int_t^{T^*_e} \left[ M_Y(\gamma e^{r(T-s)} p^*) - 1 \right] ds \right\} \cdot \prod_{j=1}^{N_{T^*_e}} \exp \left[ \gamma e^{r(T-\sigma_j)} p^*_{\sigma_j Y_j} \right].
\]

Then by Hölder inequality
\[
\sup \mathbb{E}^{t, x, z} \left\{ \exp \left\{ - \frac{qa}{\sigma} \int_t^{T^*_e} (\bar{b} - Z_s) dW_s^Z - \frac{1}{2} \cdot \frac{qa^2}{\sigma^2} \int_t^{T^*_e} (\bar{b} - Z_s)^2 ds \right\}
\cdot \exp \left\{ - \lambda q \int_t^{T^*_e} \left[ M_Y(\gamma e^{r(T-s)} p^*) - 1 \right] ds \right\} \cdot \prod_{j=1}^{N_{T^*_e}} \exp \left[ \gamma q e^{r(T-\sigma_j)} p^*_{\sigma_j Y_j} \right] \right\}
\leq \sup \mathbb{E}^{t, x, z} \left\{ \exp \left[ - \frac{2qa}{\sigma} \int_t^{T^*_e} (\bar{b} - Z_s) dW_s^Z - 2q^2 \frac{a^2}{\sigma^2} \int_t^{T^*_e} (\bar{b} - Z_s)^2 ds \right] \right\}^{1/2}
\leq \sup \mathbb{E}^{t, x, z} \left\{ \exp \left[ \left( \frac{2q^2 - q)a^2}{\sigma^2} \int_t^{T^*_e} (\bar{b} - Z_s)^2 ds \right] \cdot \prod_{j=1}^{N_{T^*_e}} \exp [2q p Y_j] \right\}^{1/2}
\leq \sup \mathbb{E}^{t, x, z} \left\{ \exp \left[ \left( \frac{2q^2 - q)a^2}{\sigma^2} \int_t^{T} (\bar{b} - Z_s)^2 ds \right] \cdot \prod_{j=1}^{N_T} \exp [2q p Y_j] \right\}^{1/2},
\]
\[
= M \left\{ \mathbb{E}^{t, x, z} \left[ \exp \left[ \left( \frac{2q^2 - q)a^2}{\sigma^2} \int_t^{T} (\bar{b} - Z_s)^2 ds \right] \right] \right\}^{1/2},
\]
where
\[
M = \left\{ E_t^{t,x,z} \prod_{j=1}^{N_T} \exp[2qP Y_j] \right\}^{\frac{1}{2}} = \left\{ \exp[\lambda T(M_Y(2qP) - 1)] \right\}^{\frac{1}{2}} < \infty,
\]
the first and second inequality follow from the fact \(\gamma e^{r(T-s)}p^*_s \leq \bar{p}\) for any \(s \in [t,T]\) and Doob’s Optional Stopping Theorem respectively. Take \(q\) from the set defined by
\[
Q = \left\{ q > 1 : a(T-t) < \frac{1}{4q^2 - 2q} < \frac{1}{2} \right\}.
\]
According to Lemma 4.3 of Benth and Karlsen [2] and Young’s inequality, then we have that
\[
E_t^{t,x,z} \exp \left[ \frac{(2q^2 - q)a^2}{\sigma^2} \int_t^T (\bar{b} - Z_s)^2 ds \right] < \infty.
\]
Thus, we prove that
\[
\sup_i E_t^{t,x,z} |v(T_i^*, X_{T_i^*}^*, Z_{T_i^*})|^q < \infty, \quad q \in Q.
\]
which implies \(\pi^* \in \Pi_c\).

By HJB equation and Itô formula, for any admissible strategy \(\pi \in \Pi_c\), we get
\[
E_t^{t,x,z}[U(X_T^\pi)] = \lim_{i \to \infty} E_t^{t,x,z}v(T_i^\pi, X_{T_i^\pi}^\pi, Z_{T_i^\pi}^\pi)
\]
\[
= v(t, x, z) + \lim_{i \to \infty} E_t^{t,x,z} \int_t^{T_i^\pi} \mathcal{L}^\pi[v](s, X_s^\pi, Z_s) ds
\]
\[
\leq v(t, x, z).
\]
and for the strategy \(\pi^*\), we get
\[
E_t^{t,x,z}[U(X_T^\pi^*)] = v(t, x, z).
\]
Thus, we obtain that
\[
E_t^{t,x,z}[U(X_T^\pi^*)] = \sup_{\pi \in \Pi_c} E_t^{t,x,z}[U(X_T^\pi)].
\]

**Proposition 2.4.** Assume that \(T-t < 1/2a\). For \(r = 0\), the value function and the optimal strategy are given as follows:

(i) if \(\bar{p} > \gamma\), then the value function \(V(t, x, z)\) and the optimal strategy \(\pi^*\) are given by

\[
V(t, x, z) = -\exp\{-\gamma x + A(t)z^2 + B(t)z + \hat{C}_1(t)\},
\]

and

\[
(w^*(t, X_t^\pi^*, Z_t), p^*(t, X_t^\pi^*, Z_t))
= \left( \frac{a(\bar{b} - z)[1 + a(T-t)] + \frac{a^2}{2}(\bar{b} - \bar{b})(t - T)^2}{\sigma^2 \gamma}, 1 \right);
\]

(ii) if \(\bar{p} \leq \gamma\), then the value function \(V(t, x, z)\) and the optimal strategy \(\pi^*\) are given by

\[
V(t, x, z) = -\exp\{-\gamma x + A(t)z^2 + B(t)z + \hat{C}_2(t)\},
\]
(u^*(t, X_t, Z_t, W_t, Y_t), p^*(t, X_t, Z_t, W_t, Y_t)) = \left( \frac{a(b - z)}{\sigma^2 \gamma} + \frac{a^2 b(t - T)}{\gamma}, \frac{p}{\gamma} \right),

where A(t) and B(t) are defined in (8) and (9) respectively, and

\begin{align*}
\hat{C}_1(t) &= -\frac{a^4}{6\sigma^2} (b - \bar{b})^2 (T - t)^3 - \left[ \frac{a^3 \bar{b}}{2\sigma^2} (b - \bar{b}) + \frac{a^2}{4} \right] (T - t)^2 \\
&\quad - \left\{ \frac{a^2 \bar{b}^2}{2\sigma^2} + \lambda \mu_1 (1 + \eta) \gamma - \lambda [M_Y(\gamma) - 1] \right\} (T - t), \\
\hat{C}_2(t) &= -\frac{a^4}{6\sigma^2} (b - \bar{b})^2 (T - t)^3 - \left[ \frac{a^3 \bar{b}}{2\sigma^2} (b - \bar{b}) + \frac{a^2}{4} \right] (T - t)^2 \\
&\quad - \left\{ \frac{a^2 \bar{b}^2}{2\sigma^2} + \lambda \mu_1 (1 + \theta) \bar{p} + \lambda \mu_1 (\eta - \theta) \gamma - \lambda [M_Y(\bar{p}) - 1] \right\} (T - t).
\end{align*}

Remark 2.5. Note that

u^*(t, X_t, Z_t) = \frac{a(b - Z_t)}{\sigma^2 \gamma} + \frac{2A(t)Z_t + B(t)}{\gamma}.

The optimal strategy u^*(t, X_t, Z_t) has two components, the first term is proportional to the risk premium of the paired assets, which is similar to the results of Merton [14]. The second component is related to the correlation structure between two stocks, and is time-dependent and vanishes at time t = T. Indeed, we partially unwind the positions progressively over time as we approach the time horizon T.

Remark 2.6. The optimal reinsurance strategy in our model is the same as the results under the model without paired assets, see Liang et al. [12]. Indeed, the insurer purchases reinsurance to avoid the risk, while it invests in financial market to make profit. This is accord with the logic of asset liability management.

3. The diffusion model. Let us now turn to a diffusion model. According to Grandell [10] and similarly Schmidli [17] or Højgaard and Taksar [11], we model the controlled reserve process by

\begin{align*}
dX_t^r &= [rX_t^r + u(t,Z_t) + \lambda \mu_1 (\theta \rho_t - \theta + \eta)]dt + \sigma \bar{d}W_t^Z + \sqrt{\lambda \mu_2}dW_t^1,
\end{align*}

where \( \bar{b} = b + \sigma^2/2\alpha + \rho \xi \sigma / \alpha \) and \{W_t^1\} is a standard Brownian motion with respect to \( \mathcal{F}_t \). Here we assume that \{W_t^1\} is independent of \{W_t^Z\}.

The formulation and assumption of the exponential utility problem are the same as section 2. Now we define the second-order partial differential operator:

\begin{align*}
\mathcal{L}^{u,p}[V](t,x,z) &= V_t + \left\{ \frac{1}{2} \sigma^2 u_x^2 + [\sigma^2 \alpha(b - z) + \alpha^2 V_x]u \right\} \\
&\quad + \left\{ \frac{1}{2} \lambda \mu_2 \bar{v}_x^2 + \lambda \mu_1 \theta \bar{v}_x \right\} \\
&\quad + \lambda \mu_1 \eta \bar{v}_x + \lambda \mu_1 (\eta - \theta) V_x + \lambda \mu_1 (\eta - \theta) V_z + \frac{1}{2} \sigma^2 V_{zz}.
\end{align*}

From standard arguments, we know that if \( V \in C^{1,2,2} \), then \( V \) satisfies the following Hamilton-Jacobi-Bellman (HJB) equation.
\[
\sup_{(u,p) \in \mathbb{R} \times [0,1]} \mathcal{L}^{u,p}[v](t,x,z) = 0,
\]
with boundary condition
\[
v(T,x,z) = -e^{-\gamma x}.
\]
Define
\[
C_4(t) := -\frac{a^4}{6\sigma^2} (\tilde{b} - b)^2 (T - t)^3 - \left[ \frac{a^3 \tilde{b}}{2\sigma^2} (\tilde{b} - b) + \frac{a^2}{4} \right] (T - t)^2 \frac{2\sigma^2}{2a^2} (T - t) + \frac{\lambda \mu_2 \gamma^2}{4r} \left[ e^{2r(T-t)} - 1 \right] + \frac{\lambda \mu_1 \eta \gamma}{r} [1 - e^{r(T-t)}],
\]
\[
C_5(t) := -\frac{a^4}{6\sigma^2} (\tilde{b} - b)^2 (T - t)^3 - \left[ \frac{a^3 \tilde{b}}{2\sigma^2} (\tilde{b} - b) + \frac{a^2}{4} \right] (T - t)^2 
- \left( \frac{a^2 \tilde{b}^2}{2\sigma^2} + \frac{\lambda \mu_2 \theta^2}{4\mu_2} \right) (T - t) + \frac{\lambda \mu_1 (\eta - \theta) \gamma}{r} [1 - e^{r(T-t)}],
\]
\[
C_6(t) := \begin{cases} 
C_5(t) + \bar{k}, & t \in [0, T - (\ln \mu_1 \theta - \ln \mu_2 \gamma)/r]; \\
C_4(t), & t \in (T - (\ln \mu_1 \theta - \ln \mu_2 \gamma)/r, T],
\end{cases}
\]
where
\[
\bar{k} := \frac{1}{2} \cdot \frac{\lambda \mu_2 \theta^2}{\mu_2} \cdot \frac{\ln \mu_1 \theta - \ln \mu_2 \gamma}{r} - \frac{3}{4} \cdot \frac{\lambda \mu_2 \theta^2}{r\mu_2} + \frac{\lambda \mu_1 \theta \gamma}{r} - \frac{\lambda \mu_2 \gamma^2}{4r}.
\]
These functions will appear in the following theorem.

**Theorem 3.1.** Assume that \( T - t < \frac{1}{2a} \). Let \( v \) be defined as follows:

(i) if \( \mu_1 \theta > \mu_2 \gamma e^{rT} \),
\[
v(t,x,z) = -\exp\{-\gamma xe^{r(T-t)} + A(t)z^2 + B(t)z + C_4(t)\};
\]

(ii) if \( \mu_1 \theta \leq \mu_2 \gamma \),
\[
v(t,p,z) = -\exp\{-\gamma xe^{r(T-t)} + A(t)z^2 + B(t)z + C_5(t)\};
\]

(iii) if \( \mu_2 \gamma < \mu_1 \theta \leq \mu_2 \gamma e^{rT} \),
\[
v(t,x,z) = -\exp\{-\gamma xe^{r(T-t)} + A(t)z^2 + B(t)z + C_6(t)\},
\]
where \( A(t), B(t), C_4(t), C_5(t) \) and \( C_6(t) \) is defined in (8), (9), (19), (20) and (21), respectively. Then, \( v \) satisfies (17)-(18) and the maximal function \( (u^*(t,x,z), p^*(t,x,z)) \) is given by
\[
(u^*(t,x,z), p^*(t,x,z)) = \left( \frac{a(b - z)[1 + a(T - t)] + \frac{3}{2} (\tilde{b} - b)(t - T)^2}{\sigma^2 \gamma} e^{-r(T-t)}, \frac{\mu_1 \theta}{\mu_2 \gamma} e^{-r(T-t)} \land 1 \right).
\]

**Proof.** We only verify Case (iii). Case (i) and Case (ii) can be similarly proved. Direct verification shows that \( v \) defined by (22) is twice continuously differentiable.
Note that \( v \) of (22) satisfies that \( v_x > 0 \) and \( v_{xx} < 0 \). Therefore, the maximal function of \( \bar{L}^{u,p}[v](t, x, z) \) without restriction is

\[
\begin{align*}
    u^0(t, x, z) &= \frac{a(b - z)(1 + a(T - t)) + \frac{a^2}{2}(b - b)(t - T)^2}{\sigma^2} e^{-r(T - t)}, \\
    p^0(t, x, z) &= \frac{\mu_1 \theta}{\mu_2} e^{-r(T - t)}.
\end{align*}
\]

When \( 0 \leq t < T - (\ln \mu_1 \theta - \ln \mu_2 \gamma)/r \), we have that \( 0 < p^0(t, x, z) < 1 \). In this case, \( (u^0(t, x, z), p^0(t, x, z)) \) is in the interior of the control region \( R \times [0, 1] \). This means that the maximal function \( (u^*(t, x, z), p^*(t, x, z)) = (u^0(t, x, z), p^0(t, x, z)) \). Substituting it into \( \bar{L}^{u,p}[v](t, x, z) \) yields

\[
\sup_{(u,p) \in R \times [0,1]} \bar{L}^{u,p}[v](t, x, z) = \bar{L}^{u^0(t,x,z),p^0(t,x,z)}[v](t, x, z)
= \left[ A' - \frac{1}{2} \frac{(2\sigma^2 A - a)^2}{\sigma^2} - 2aA + 2\sigma^2 A^2 \right] z^2
+ \left[ B' - \frac{1}{2} \frac{2(2\sigma^2 A - a)(ab + \sigma^2 B)}{\sigma^2} + 2abA - aB + 2\sigma^2 AB \right] z
+ \left[ C' - \frac{1}{2} \frac{(ab + \sigma^2 B)^2}{\sigma^2} - \frac{\lambda \mu_1 \eta}{\lambda \mu_1 (\eta - \theta)} - \lambda \mu_1 (\eta - \theta) \gamma e^{r(T - t)} \right]
+ abB + \frac{1}{2} \sigma^2 B^2 + \sigma^2 A.
\]

It is easy to verify that \( A(t), B(t), C(t) \) are the solutions of the following systems of ordinary differential equations (ODEs)

\[
\begin{align*}
    A' - \frac{1}{2} \frac{(2\sigma^2 A - a)^2}{\sigma^2} - 2aA + 2\sigma^2 A^2 &= 0, \\
    B' - \frac{1}{2} \frac{2(2\sigma^2 A - a)(ab + \sigma^2 B)}{\sigma^2} + 2abA - aB + 2\sigma^2 AB &= 0, \\
    C_0' - \frac{1}{2} \frac{(ab + \sigma^2 B)^2}{\sigma^2} - \frac{\lambda \mu_1 \eta}{\lambda \mu_1 (\eta - \theta)} - \lambda \mu_1 (\eta - \theta) \gamma e^{r(T - t)}
+ abB + \frac{1}{2} \sigma^2 B^2 + \sigma^2 A &= 0.
\end{align*}
\]

Therefore, \( v \) of (22) satisfies that

\[
\sup_{(u,p) \in R \times [0,1]} \bar{L}^{u,p}[v](t, x, z) = 0, \quad 0 \leq t < T - (\ln \mu_1 \theta - \ln \mu_2 \gamma)/r.
\]

When \( T - (\ln \mu_1 \theta - \ln \mu_2 \gamma)/r \leq t \leq T \), we have that \( p^0(t, x, z) \geq 1 \). In this case, by \( v_{xx} < 0 \), we have that \( (u^*(t, x, z), p^*(t, x, z)) = (u^0(t, x, z), 1) \). Substituting it
into \( \mathcal{L}^{u,p}[v](t, x, z) \) yields
\[
\sup_{(u,p) \in R \times [0,1]} \mathcal{L}^{u,p}[v](t, x, z) = \mathcal{L}^{u}(t, x, z) \]
\[
= \left[ A' - \frac{1}{2} \frac{(2\sigma^2 A - a)^2}{\sigma^2} - 2aA + 2\sigma^2 A^2 \right] z^2 + \left[ B' - \frac{1}{2} \frac{2(2\sigma^2 A - a)(a\bar{b} + \sigma^2 B)}{\sigma^2} + 2abA - aB + 2\sigma^2 AB \right] z + \left[ C_4' - \frac{1}{2} \frac{(a\bar{b} + \sigma^2 B)^2}{\sigma^2} + \frac{1}{2} \lambda \mu_2 \gamma^2 \alpha^2 - \lambda \mu_1 \eta \gamma e^{r(T-t)} \right] + \bar{a}B + \frac{1}{2} \sigma^2 B^2 + \sigma^2 A. \]

It is easy to verify that \( A(t), B(t), C_4(t) \) are the solutions of the following systems of ordinary differential equations (ODEs)
\[
\begin{aligned}
A' &- \frac{1}{2} \frac{(2\sigma^2 A - a)^2}{\sigma^2} - 2aA + 2\sigma^2 A^2 = 0, \quad A(T) = 0, \\
B' &- \frac{1}{2} \frac{2(2\sigma^2 A - a)(a\bar{b} + \sigma^2 B)}{\sigma^2} + 2abA - aB + 2\sigma^2 AB = 0, \quad B(T) = 0, \\
C_4' &- \frac{1}{2} \frac{(a\bar{b} + \sigma^2 B)^2}{\sigma^2} + \frac{1}{2} \lambda \mu_2 \gamma^2 \alpha^2 - \lambda \mu_1 \eta \gamma e^{r(T-t)} + \bar{a}B + \frac{1}{2} \sigma^2 B^2 + \sigma^2 A = 0, \quad C_4(T) = 0.
\end{aligned}
\]

Therefore, \( v \) of (22) satisfies that
\[
\sup_{(u,p) \in R \times [0,1]} \mathcal{L}^{u,p}[v](t, x, z) = 0, \quad T - (\ln \mu_1 \theta - \ln \mu_2 \gamma)/r < t \leq T,
\]
and
\[
v(T, x, z) = -e^{-\gamma x}.
\]

In conclusion, \( v \) of (22) satisfies (17)-(18) and the maximal function is given by (23).

Denote
\[
\Pi_d := \left\{ \pi \in \Pi : \pi \text{ satisfies that } \{ v(t \wedge \tau_i, X_{t \wedge \tau_i}, Z_{t \wedge \tau_i}) \}, i = 1, 2, 3... \right\}
\]
is uniformly integrable,

where
\[
\tau_i = \inf\{ s \geq t, (|X_s|^i, |Z_s|) \notin [0, i) \times [0, i) \}, \quad i = 1, 2, 3...
\]

In the following, we give the verification theorem.

**Theorem 3.2. (Verification Theorem)**
Assume that \( T - t < 1/2a \). Then the strategy
\[
\pi^* = (u_i^*, p_i^*) := (u^*(t, X_i^{\pi^*}, Z_i), p^*(t, X_i^{\pi^*}, Z_i))
\]
is optimal on \( \Pi_d \), where the function \( u^*(t, x, z) \) and \( p^*(t, x, z) \) are defined by (23) and \( X_t^\pi^* \) is the reserve process with the strategy \( \pi^* \).

**Proof.** For convenience, let \( T_t^\pi = \tau_t^\pi \land T \) and \( T_t^{\pi^*} = T_t^\pi^* \). Firstly, we need to verify \( \pi^* \in \Pi_d \). It is sufficient to prove that, for some \( q > 1 \) (Its domain will be given by (16)),

\[
\sup_t E^{t, x, z}[v(T_t^*, X_t^{\pi^*}, Z_{T_t^*})] < \infty.
\]

By Itô formula and HJB equation, we can conclude that

\[
v(T_t^*, X_t^{\pi^*}, Z_{T_t^*}) = v(t, x) - \frac{a}{\sigma} \int_t^{T_t^*} v(s, X_s^{\pi^*}, Z_s)(\dd s - Z_s) + \gamma \sqrt{\lambda \mu_2} \int_t^{T_t^*} e^{r(T-s)}p_s^*dW_s^1.
\]

Then solving the stochastic differential equation SDE (24) yields

\[
v(T_t^*, X_t^{\pi^*}, Z_{T_t^*}) = v(t, x, z) \cdot \exp \left[ -\frac{a}{\sigma} \int_t^{T_t^*} (\dd b - Z_s)dW_s^Z - \gamma \sqrt{\lambda \mu_2} \int_t^{T_t^*} e^{r(T-s)}p_s^*dW_s^1 \right] \\
\cdot \exp \left[ -\frac{1}{2} \cdot \frac{a^2}{\sigma^2} \int_t^{T_t^*} (\dd b - Z_s)^2 ds - \frac{1}{2} \cdot \gamma^2 \lambda \mu_2 \int_t^{T_t^*} e^{2r(T-s)}(p_s^*)^2 ds \right].
\]

Noting \( p^* \in [0, 1] \) is a determined function and by Hölder inequality

\[
\sup_t E^{t, x, z}\left\{ \exp \left[ -\frac{q a}{\sigma} \int_t^{T_t^*} (\dd b - Z_s)dW_s^Z - q \gamma \sqrt{\lambda \mu_2} \int_t^{T_t^*} e^{r(T-s)}p_s^*dW_s^1 \right] \\
\cdot \exp \left[ -\frac{1}{2} \cdot \frac{q a^2}{\sigma^2} \int_t^{T_t^*} (\dd b - Z_s)^2 ds - \frac{1}{2} \cdot q \gamma^2 \lambda \mu_2 \int_t^{T_t^*} e^{2r(T-s)}(p_s^*)^2 ds \right] \right\}
\leq \sup_t \left\{ E^{t, x, z}\left[ -\frac{q a}{\sigma} \int_t^{T_t^*} (\dd b - Z_s)dW_s^Z - 2q \gamma \sqrt{\lambda \mu_2} \int_t^{T_t^*} e^{r(T-s)}p_s^*dW_s^1 \\
-2q^2 \frac{a^2}{\sigma^2} \int_t^{T_t^*} (\dd b - Z_s)^2 ds - 2q \gamma^2 \lambda \mu_2 q^2 \int_t^{T_t^*} e^{2r(T-s)}(p_s^*)^2 ds \right] \right\} \frac{1}{2}
\cdot \left\{ E^{t, x, z}\left[ (2q^2 - q)a^2 \int_t^{T_t^*} (\dd b - Z_s)^2 ds \\
+(q^2 - q)\gamma^2 \lambda \mu_2 \int_t^{T_t^*} e^{2r(T-s)}(p_s^*)^2 ds \right] \right\} \frac{1}{2}
\]

\leq \sup_t \left\{ E^{t, x, z}\left[ (2q^2 - q)a^2 \int_t^{T_t^*} (\dd b - Z_s)^2 ds \\
+(q^2 - q)\gamma^2 \lambda \mu_2 \int_t^{T_t^*} e^{2r(T-s)}(p_s^*)^2 ds \right] \right\} ^\frac{1}{2},
\]

\[
\leq M \left\{ E^{t, x, z}\left[ (2q^2 - q)a^2 \int_t^{T_t^*} (\dd b - Z_s)^2 ds \right] \right\} ^\frac{1}{2},
\]
where
\[ M = \left\{ E^{t,x,z} \exp \left[ (2q^2 - q)\gamma^2 \lambda \mu Z t e^{2r(T-s)}(p^*_s)^2 ds \right] \right\}^{\frac{1}{2}}, \]
and the second equality follows from the Doob’s Optional Stopping Theorem. Take \( q \) from the set \( Q \) defined by (16). According to Lemma 4.3 of Benth and Karlsen [2] and Young’s inequality, then we have that
\[ E^{t,x,z} \exp \left[ (2q^2 - q)\frac{a^2}{\sigma^2} \int_t^T (\bar{b} - Z_s)^2 ds \right] < \infty. \]
Furthermore, we show that
\[ \sup_i E^{t,x,z} |v(T^*_i, X^*_T, Z^*_T)|^q < \infty. \]

Remark 3.3. From Theorem 2.3 and 3.2, we can find that the optimal pairs trading strategies are identical under two models, while the optimal reinsurance strategies are not the same. It is what we can anticipate.

4. Numerical results. This section provides numerical examples that demonstrate how the optimal solution works and gives some advice for practice. For this set of parameters, we set \( a = 0.2, b = 1, \sigma = 0.3, r = 0.01, T = 1 \) (one year), \( \delta = 0.3, \xi = 1 \) and \( \rho = 0.3 \). Figure 1 demonstrates the spread process \( Z \) and gives a sample path of the optimal strategy \( u^* \). We can see that \( u^* \) switches between long and short, and it depends on the spread \( z \).

Let constant parameters \( b = 1, \sigma = 0.3, r = 0.01, T = 1 \) (one year), \( \rho = 0.3, \xi = 0.1 \). With this parameters setting, we will examine the impact of mean-reversion rate \( a \), and spread \( z \) on \( u^* \) with different risk aversion \( \gamma \).

We can observe from Figure 2 that the mean reversion rate \( a \) has a great impact on \( u^* \), the larger \( a \) is, the more insurer prefer active investment. It is not surprising that an increasing \( a \) leads directly to a shorter investment period. Concluding, the results are in line with the literature – mean-reversion is driver for successful and fast termination of our pairs trading strategy.

Figure 3 shows that the optimal pairs trading \( u^* \) of the insurer depends on spread \( z \) for different \( \gamma \), we consider four scenarios: (1) \( z \) is below than \( \bar{b} \); (2) \( z \) is close to
b; (3) $z$ is higher than $\bar{b}$; and (4) $u^*$ is against $z$ for different $\gamma$. Top-left in Figure 3 shows that the insurer purchases the underpriced spread. When $z$ is close to $\bar{b}$, the insurer sells risky asset. The amount of short selling decreases with the larger $\gamma$. In this case, the insurer’s investment strategy switches from long to short. When $z$ is higher than $\bar{b}$, the bottom-left shows that the insurer will sell the overpriced spread. Moreover, the insurer with smaller $\gamma$ will sell short more quality with the increase of $z$. Figure 3 demonstrates that the insurer is not willing to invest in financial market when $t$ is close to $T$. At the same time, the insurer with the lower $\gamma$ is eager to participate in pairs trading to increasing its profit, while the insurer with the larger $\gamma$ have no particular interest in searching pairs asset in the financial market, but to manage insurable risks.

**Figure 1.** The sample path of $u^*$

**Figure 2.** The influence of $a$ on $u^*$
5. Conclusion. This paper investigates the optimal reinsurance and pairs trading strategy to maximize the expected exponential utility for an insurer under two risk models. We assume that the price spread of paired assets follows an O-U process. The explicit optimal solutions are derived by using the dynamic programming approach. The numerical examples indicate that although a financial market with paired assets implies the existence of statistical arbitrage opportunities, insurers with a high level of risk aversion may not be interested in pair trading. For future research, we identify three possible directions. First, the model may be extended by considering different weight ratio in the formation of pair, that is, \( u_t = -k\bar{u}_t (k \neq 1) \); Second, we should consider the different market state by applying markov regime-switching to investigate how the market state impact on the optimal trading; Third, we consider multi-stocks paring for maximizing the exponential utility which is worthy of our further exploitation direction.

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Appendix.

Lemma 5.1. For fixed \((t, x, z)\), define the function \(f\) as

\[
 f(p) = v_x(t, x, z)\lambda \mu_1 (1 + \theta)p + \lambda E [v(t, x - pY, z) - v(t, x, z)] .
\]

Then \(f(p)\) is a concave function and takes the maximum value at \(\tilde{p}e^{-r(T-t)}/\gamma\).

Proof. Note that \(v < 0\) and

\[
 f(p) = v_x(t, x, z)\lambda \mu_1 (1 + \theta)p + \lambda E [v(t, x - pY, z) - v(t, x, z)]
 = -\lambda \gamma \mu_1 (1 + \theta)e^{r(T-t)} v(t, x, z)p + \lambda v \left[ M_Y(\gamma e^{r(T-t)}) - 1 \right],
\]

then we get

\[
 f'(p) = \lambda \gamma e^{r(T-t)} v(t, x, z)[M_Y'(\gamma e^{r(T-t)}) - \mu_1 (1 + \theta)],
 f''(p) = \lambda^2 \gamma^2 e^{2r(T-t)} v(t, x, z)M_Y''(\gamma e^{r(T-t)}) < 0.
\]

It is easy to verify that the function \(M_Y'(l)\) is a strictly increasing convex function and \(M_Y'(0) = \mu_1 < (1 + \theta)\mu_1\). Then, by \(\lim_{l \to +\infty} M_Y'(l) = +\infty\), we have that \(\tilde{p}e^{-r(T-t)}/\gamma\) is an unique positive solution of \(f'(p) = 0\). Hence, the results follow. \(\Box\)

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E-mail address: 1120180026@mail.nankai.edu.cn
E-mail address: lhbai@nankai.edu.cn
E-mail address: byzhang69@nankai.edu.cn