Spectrum of quantized black hole, correspondence principle, and holographic bound

I.B. Khriplovich

Budker Institute of Nuclear Physics
630090 Novosibirsk, Russia,
and Novosibirsk University

Abstract

An equidistant spectrum of the horizon area of a quantized black hole does not follow from the correspondence principle or from general statistical arguments. On the other hand, such a spectrum obtained in loop quantum gravity (LQG) either does not comply with the holographic bound, or demands a special choice of the Barbero-Immirzi parameter for the horizon surface, distinct from its value for other quantized surfaces. The problem of distinguishability of edges in LQG is discussed, with the following conclusion. Only under the assumption of partial distinguishability of the edges, the microcanonical entropy of a black hole can be made both proportional to the horizon area and satisfying the holographic bound.

1khriplovich@inp.nsk.su
1. The idea of quantizing the horizon area of black holes was put forward many years ago by Bekenstein in the pioneering article [1]. It was based on the intriguing observation, made by Christodoulou and Ruffini [2, 3]: the horizon area of a nonextremal black hole behaves in a sense as an adiabatic invariant. Of course, the quantization of an adiabatic invariant is perfectly natural, in accordance with the correspondence principle.

One more conjecture made in [1] is that the spectrum of a quantized horizon area is equidistant. The argument therein was that a periodic system is quantized by equating its adiabatic invariant to $2\pi \hbar n$, $n = 0, 1, 2, ...$

Later it was pointed out by Bekenstein [4] that the classical adiabatic invariance does not guarantee by itself the equidistance of the spectrum, at least because any function of an adiabatic invariant is itself an adiabatic invariant. However, up to now articles on the subject abound in assertions that the form

$$A = \beta l_p^2 n, \quad n = 1, 2, ...$$  \hspace{1cm} (1)

for the horizon area spectrum\(^2\) is dictated by the respectable correspondence principle. The list of these references is too lengthy to be presented here.

Let us consider an instructive example of the situation when a nonequidistant spectrum arises in spite of the classical adiabatic invariance. We start with a classical spherical top of an angular momentum $J$. Of course, the $z$-projection $J_z$ of $J$ is an adiabatic invariant. If the $z$-axis is chosen along $J$, the value of $J_z$ is maximum, $J$, or $\hbar j$ in the quantum case. The classical angular momentum squared $J^2$ is also an adiabatic invariant, with eigenvalues $\hbar^2 j(j+1)$ when quantized. Let us try now to use the operator $\hat{J}^2$ for the area quantization in quite natural units of $l_p^2$. For the horizon area $A$ to be finite in the classical limit, the power of the quantum number $j$ in the result for $j \gg 1$ should be the same as that of $\hbar$ in $l_p^2$.[5]

With $l_p^2 \sim \hbar$, we arrive in this way at

$$A \sim l_p^2 \sqrt{j(j+1)}.$$

Since $\sqrt{j(j+1)} \rightarrow j + 1/2$ for $j \gg 1$, we have come back again to the equidistant spectrum in the classical limit. However, the equidistance can be avoided in the following way. Let us assume that the horizon area consists of sites with area on the order of $l_p^2$, and ascribe to each site $i$ its own quantum number $j_i$ and the contribution $\sqrt{j_i(j_i+1)}$ to the area. Then the above formula changes to

$$A \sim l_p^2 \sum_i \sqrt{j_i(j_i+1)}$$ \hspace{1cm} (2)

(in fact, this formula for a quantized area arises really as a special case in loop quantum gravity, see below). Of course, to retain a finite classical limit for $A$, we should require that $\sum_i \sqrt{j_i(j_i+1)} \gg 1$. However, any of $j_i$ can be well comparable with unity. So, in spite of the adiabatic invariance of $A$, its quantum spectrum [2] is not equidistant, though of course discrete.

\[^2\]Here and below $l_p^2 = \hbar k/c^3$ is the Planck length squared, $l_p = 1.6 \cdot 10^{-33}$ cm, $k$ is the Newton gravitational constant; $\beta$ is here some numerical factor.
One more quite popular argument in favour of the equidistant spectrum [1] is as follows [4, 6, 7]. On the one hand, the entropy $S$ of an horizon is related to its area $A$ through the Bekenstein-Hawking relation

$$A = 4l_p^2 S. \quad (3)$$

On the other hand, the entropy is nothing but $\ln g(n)$ where the statistical weight $g(n)$ of any quantum state $n$ is an integer. In [4, 6, 7] the requirement of integer $g(n)$ is taken literally, and results after simple reasoning not only in the equidistant spectrum [1], but also in the following allowed values for the numerical factor $\beta$ in this spectrum:

$$\beta = 4 \ln k, \quad k = 2, 3, \ldots$$

Let us imagine however that with some model for $S$ one obtains for $g(n)$, instead of an integer value $K$, a noninteger one $K + \delta, \quad 0 < \delta < 1$. Then, the entropy will be

$$S = \ln(K + \delta) = \ln K + \delta/K.$$ 

Now, the typical value of the black hole entropy $S = \ln K = A/4l_p^2$ is huge, something like $10^{76}$. So, the correction $\delta/K$ is absolutely negligible as compared to $S = \ln K$. Moreover, it is far below any conceivable accuracy of a description of entropy. Therefore, this correction can be safely omitted and forgotten. As usual for macroscopic objects, the fact that the statistical weight is an integer has no consequences for the entropy.

Thus, contrary to the popular belief, the equidistance of the spectrum for the horizon area does not follow from the correspondence principle and/or from general statistical arguments.

2. It does not mean however that any model leading to an equidistant spectrum for the quantized horizon area should be automatically rejected. Quite simple and elegant version of such a model, so called “it from bit”, for a Schwarzschild black hole was formulated by Wheeler [8]. The assumption is that the horizon surface consists of $\nu$ patches, each of them supplied with an “angular momentum” quantum number $j$ with two possible projections $\pm 1/2$. The total number $K$ of degenerate quantum states of this system is

$$K = 2^\nu. \quad (4)$$

Then the entropy of the black hole is

$$S_{1/2} = \ln K = \nu \ln 2. \quad (5)$$

And finally, with the Bekenstein-Hawking relation (3) one obtains for the area spectrum the following equidistant formula:

$$A_{1/2} = 4 \ln 2 l_p^2 \nu. \quad (6)$$

This model of a quantized Schwarzschild black hole looks by itself flawless.

Later this result was derived in Ref. [9] in the framework of loop quantum gravity (LQG) [10-14]. We discuss below whether the “it from bit” picture, if considered as a special case of the area quantization in LQG, can be reconciled with the holographic bound [15-17].
More generally, a quantized surface in LQG is described as follows. One ascribes to it a set of punctures. Each puncture is supplied with two integer or half-integer “angular momenta” $j^u$ and $j^d$:

$$j^u, j^d = 0, 1/2, 1, 3/2, \ldots$$

(7)

$j^u$ and $j^d$ are related to edges directed up and down the normal to the surface, respectively, and add up into an angular momentum $j^{ud}$:

$$j^{ud} = j^u + j^d; \quad |j^u - j^d| \leq j^{ud} \leq j^u + j^d.$$  

(8)

The area of a surface is

$$A = \beta l_p^2 \sum_i \sqrt{2j_i^u(j_i^u + 1) + 2j_i^d(j_i^d + 1) - j_i^{ud}(j_i^{ud} + 1)}.$$  

(9)

The overall numerical factor $\beta$ in (9) cannot be determined without an additional physical input. This ambiguity originates from a free (so-called Barbero-Immirzi) parameter \cite{18,19} which corresponds to a family of inequivalent quantum theories, all of them being viable without such an input.

The result (6) was obtained in \cite{9} under an additional condition that the gravitational field on the horizon is described by the $U(1)$ Chern-Simons theory. Formula (6) is a special case of general one (9) when all $j^d$ vanish and all $j^u$ equal $1/2$ (or vice versa). As to the overall factor $\beta$, its value here is $^3$

$$\beta = \frac{8 \ln 2}{\sqrt{3}}.$$  

(10)

Let us turn now to the holographic bound \cite{15-17}. According to it, the entropy $S$ of any spherically symmetric system confined inside a sphere of area $A$ is bounded as follows:

$$S \leq \frac{A}{4l_p^2},$$  

(11)

with the equality attained only for a system which is a black hole.

A simple intuitive argument confirming this bound is as follows \cite{17}. Let us allow the discussed system to collapse into a black hole. During the collapse the entropy increases from $S$ to $S_{bh}$, and the resulting horizon area $A_{bh}$ is certainly smaller than the initial confining one $A$. Now, with the account for the Bekenstein-Hawking relation (3) for a black hole we arrive, through the obvious chain of (in)equalities

$$S \leq S_{bh} = \frac{A_{bh}}{4l_p^2} \leq \frac{A}{4l_p^2},$$

at the discussed bound (11).

The result (11) can be formulated otherwise. Among the spherical surfaces of a given area, it is the surface of a black hole horizon that has the largest entropy.

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$^3$The common convention for the numerical factor in formula (9) is $8\pi \beta$; with it the parameter $\beta$ is smaller than ours by factor $8\pi$. 

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On the other hand, it is only natural that the entropy of an eternal black hole in equilibrium is maximum. This was used by Vaz and Witten [20] in a model of the quantum black hole as originating from a dust collapse. Then the idea was employed by us [21, 22] in the problem of quantizing the horizon of a black hole in LQG. In particular, the coefficient $\beta$ was calculated in Ref. [22] in the case when the area of a black hole horizon is given by the general formula (9) of LQG, as well as under some more special assumptions on the values of $j^u$, $j^d$, $j^{ud}$. Moreover, it was demonstrated in Ref. [22] for a rather general class of the horizon quantization schemes that it is the maximum entropy of a quantized surface which is proportional to its area.

Let us sketch the proof of this result (for more technical details see [22]). We consider here and below in the present paper the microcanonical entropy $S$ of a surface (though with fixed area instead of fixed energy). It is defined as the logarithm of the number of states of this surface with a fixed area $A$, i.e. with a fixed sum

$$N = \sum_i \sqrt{2j^u_i(j^u_i+1) + 2j^d_i(j^d_i+1) - j^{ud}_i(j^{ud}_i+1)}.$$  

(12)

Let $\nu_{im}$ be the number of punctures with a given set of momenta $j^u_i$, $j^d_i$, $j^{ud}_i$ and a given projection $m$ of $j^{ud}_i$. The total number of punctures is

$$\nu = \sum_{im} \nu_{im}.$$  

We will assume that the edges with the same set of the quantum numbers $im$ (i.e. with the same $j^u_i$, $j^d_i$, $j^{ud}_i$, and $m$) are indistinguishable, so that interchanging them does not result in new states. All other permutations, those among the edges with differing $im$, do create new states, so that such edges, with differing $im$, are distinguishable.

Then, the entropy is

$$S = \ln \left[ \nu! \prod_{im} \frac{1}{\nu_{im}!} \right].$$  

(13)

The structure of expressions (9) and (13) is so different that in a general case the entropy certainly cannot be proportional to the area. However, this is the case for the maximum entropy in the classical limit.

By combinatorial reasons, it is natural to expect that the absolute maximum of entropy is reached when all values of quantum numbers $j^{u,d,ud}_i$ are present. We assume also that in the

\[4\]

Let us note that the “it from bit” values (4) and (5) for the number of states and entropy, also follow from this assumption. Indeed, let $\nu$ be the total number of patches with $j = 1/2$, and let $\nu_+$ and $\nu_- = \nu - \nu_+$ patches have the projections $+1/2$ and $-1/2$, respectively. Then, the number of the corresponding states is obviously

$$\frac{\nu!}{\nu_+! (\nu - \nu_+)!},$$

and the total number of states is

$$K = \sum_{\nu_+=0}^{\nu} \frac{\nu!}{\nu_+! (\nu - \nu_+)!} = 2^\nu,$$

in agreement with (4).

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classical limit the typical values of puncture numbers $\nu_{im}$ are large. Then, with the Stirling formula for factorials, expression (13) transforms to

$$S = \sum_{im} \nu_{im} \times \ln \left( \sum_{i'm'} \nu_{i'm'} \right) - \sum_{im} \nu_{im} \ln \nu_{im}. \quad (14)$$

We are looking for the extremum of expression (14) under the condition

$$N = \sum_{i} \nu_{im} r_i = \text{const}, \quad (15)$$

where each partial contribution $r_i = \sqrt{2j_i^u(j_i^u + 1) + 2j_i^d(j_i^d + 1) - j_i^{ud}(j_i^{ud} + 1)}$ is independent of $m$. The problem reduces to the solution of the system of equations

$$\ln \left( \sum_{i'm'} \nu_{i'm'} \right) - \ln \nu_{im} = \mu r_i, \quad (16)$$

or

$$\nu_{im} = e^{-\mu r_i} \sum_{i'm'} \nu_{i'm'} = \nu e^{-\mu r_i}. \quad (17)$$

Here $\mu$ is the Lagrange multiplier for the constraining relation (15). Summing expressions (17) over $i, m$, we arrive at the equation for $\mu$:

$$\sum_{im} e^{-\mu r_i} = \sum_{i} g_i e^{-\mu r_i} = 1; \quad (18)$$

the statistical weight $g_i = 2j_i^{ud} + 1$ of a puncture arises here since $r_i$ are independent of $m$. On the other hand, when multiplying equation (16) by $\nu_{im}$ and summing over $i, m$, we arrive with the constraint (15) at the following result for the maximum entropy for a given value of the sum $N$, or the black hole area $A$:

$$S_{\text{max}} = \mu N = \frac{\mu}{\beta l_p^2} A. \quad (19)$$

One more curious feature of the obtained picture is worth noting: it gives a sort of the Boltzmann distribution for the occupation numbers (see (17)). In this distribution, the partial contributions $r_i$ to the area are analogues of energies, and the Lagrange multiplier $\mu$ corresponds (up to a factor) to the inverse temperature.

It should be emphasized that relation (19) is true not only in LQG, but applies to a more general class of approaches to the quantization of surfaces. The following assumption is really necessary here: the surface should consist of patches of different sorts, so that there are $\nu_{im}$ patches of each sort $i, m$, with a generalized effective quantum number $r_i$, and a statistical weight $g_i$. As necessary is the above assumption on the distinguishability of the patches.

Thus, it is the maximum entropy of a surface which is proportional in the classical limit to its area. This proportionality certainly takes place for a classical black hole. And this is one more strong argument in favour of the assumption that the black hole entropy is maximum.
Let us come back now to the result of Ref. [9]. If one assumes that the value \( \beta \) of the parameter \( \beta \) is the universal one (i.e., it is not special to black holes, but refers to any quantized spherical surface), then the value (5) is not the maximum one in LQG for a surface of the area (6). This looks quite natural: with the transition from the unique choice made in Ref. [9], \( j^d = 1/2, j^u = 0 \), to more extended and rich one, the number of the degenerate quantum states should, generally speaking, increase. And together with this number, its logarithm, which is the entropy of a quantized surface, increases as well.

We start the proof of the above statement with rewriting formula (5) as follows:

\[
S_{1/2} = \ln 2 \sqrt[4/3]{N} = 0.80 N; \quad N = \sqrt[3/4]{\nu}.
\]  

(20)

From now on, we consider this value of \( N \) as fixed one.

Let us start with a relatively simple example when \( j^u = 0 \), so that the general formula (9) for a surface area reduces to

\[
A = \beta l^2 p \sum_i \sqrt{j_i(j_i + 1)} = \beta l^2 p \sum_{j=1/2}^{\infty} \sqrt{j(j + 1)} \nu_j, \quad j = j^u
\]  

(21)

(and coincides with our naive model (2)). We will find the maximum entropy of such a surface for the fixed value of

\[
N = \sum_{j=1/2}^{\infty} \sqrt{j(j + 1)} \nu_j,
\]  

(22)

that should be equal to the “it from bit” one, \( \nu \sqrt{3/4} \). Here the statistical weight of a puncture with the quantum number \( j \) is \( g_j = 2j + 1 \), and equation (18) can be rewritten as

\[
\sum_{p=1}^{\infty} (p + 1) z^{p(p+2)} = 1, \quad p = 2j, \quad z = e^{-\mu/2}.
\]  

(23)

Its solution is \( \mu = -2 \ln z = 1.722 \) \[22\], and the maximum entropy in this case

\[
S_{max,1} = 1.72 N
\]  

(24)

exceeds the result (20).

As expected, in the general case, with \( N \) given by formula (12) with all values of \( j^u, j^d, j^u \) allowed and \( g_i = 2j^ud_i + 1 \), the maximum entropy is even larger \[22\]

\[
S_{max} = 3.12 N.
\]  

(25)

Thus, the conflict is obvious between the holographic bound and the result (20), as found within the LQG approach of [9].

One might try to avoid the conflict by assuming that the value (10) for the Barbero-Immirzi parameter \( \beta \) is special for black holes only, while for other quantized surfaces \( \beta \) is smaller. However, such a way out would be unattractive and unnatural.
3. We come back now to the essential assumption made in the previous section: the edges with the same set of the quantum numbers \( im \) are identical, the edges with differing \( im \) are distinguishable. In principle, one might try to modify this assumption of partial distinguishability of edges in two opposite ways.

One possibility, which might look quite appealing, is that of complete indistinguishability of edges. It means that no permutation of any edges results in new states. To simplify the discussion, let us confine here and below to expression (21) for the horizon area, instead of the most general one (9). Then, the total number of angular momentum states created by \( \nu_j = \sum_m \nu_{jm} \) indistinguishable edges of a given \( j \) with all \( 2j + 1 \) projections allowed, from \(-j\) to \( j\), is

\[
K_j = \frac{(\nu_j + 2j)!}{\nu_j! (2j)!}.
\] (26)

Those partial contributions \( s_j = \ln K_j \) to the black hole entropy \( S = \sum_j s_j \) that can potentially dominate the numerically large entropy, may correspond to the three cases: \( j \ll \nu_j \), \( j \gg \nu_j \), and \( j \sim \nu_j \gg 1 \). These contributions are as follows:

- \( j \ll \nu_j \): \( s_j \approx 2j \ln \nu_j \);
- \( j \gg \nu_j \): \( s_j \approx \nu_j \ln j \);
- \( j \sim \nu_j \gg 1 \): \( s_j \sim 4j \ln 2 \).

In all the three cases the partial contributions to the entropy \( S \) are much smaller parametrically than the corresponding contributions

\[ a_j \sim j \nu_j \]

to the area \( A = \sum_j a_j \). Thus, in all these cases \( S \ll A \), so that with indistinguishable edges of the same \( j \), one cannot make the entropy of a black hole proportional to its area. It was pointed out earlier in Refs. [23, 24].

Let us consider now the last conceivable option, that of completely distinguishable edges. In this case the total number of states is just \( K = \nu! \), instead of (13), with the microcanonical entropy \( S = \nu \ln \nu \). In principle, this entropy can be made proportional to the black hole area \( A \). The model (though not looking natural) could be as follows. We choose a large quantum number \( J \gg 1 \), and assume that the horizon area \( A \) is saturated by the edges with \( j \) in the interval \( J < j < 2J \), and with “occupation numbers” \( \nu_j \sim \ln J \). Then, the estimates both for \( S \) and \( A \) are \( \sim J \ln J \), and the proportionality between the entropy and the area can be attained.

However, though under the assumption of complete distinguishability the entropy can be proportional to the area, the maximum entropy for a given area is much larger than the area itself. Obviously, here the maximum entropy for fixed \( A \sim \sum_j \sqrt{j(j+1)} \nu_j \) is attained

\( 5^{\text{Perhaps, the simplest derivation of this formula is as follows. We are looking here effectively for the number of ways of distributing }\nu_j\text{ identical balls into }2j+1\text{ boxes. Then, the line of reasoning presented in }\cite{21}, \S 54, \text{ results in formula }\cite{20}\. I am grateful to V.F. Dmitriev for bringing to my attention that formula }\cite{20}\.}

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with all \( j \)'s being as small as possible, say, 1/2 or 1. Then, in the classical limit \( \nu \gg 1 \), the entropy of a black hole grows faster than its area, \( A \sim \nu \), while \( S = \nu \ln \nu \sim A \ln A \). Thus, the assumption of complete distinguishability is in conflict with the holographic bound, and therefore should be discarded.

There is no disagreement between this our conclusion and that of Refs. \[23, 25, 26\]: what is called complete distinguishability therein corresponds to our partial distinguishability.

I am grateful to V.F. Dmitriev, V.M. Khatsymovsky, and G.Yu. Ruban for discussions. The investigation was supported in part by the Russian Foundation for Basic Research through Grant No. 03-02-17612.

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