Symmetric Triangle Quadrature Rules for Arbitrary Functions

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Abstract

Despite extensive research on symmetric polynomial quadrature rules for triangles, as well as approaches to their calculation, few studies have focused on non-polynomial functions, particularly on their integration using symmetric triangle rules. In this paper, we present two approaches to computing symmetric triangle rules for singular integrands by developing rules that can integrate arbitrary functions. The first approach is well suited for a moderate amount of points and retains much of the efficiency of polynomial quadrature rules. The second approach better addresses large amounts of points, though it is less efficient than the first approach. We demonstrate the effectiveness of both approaches on singular integrands, which can often yield relative errors two orders of magnitude less than those from polynomial quadrature rules.

Keywords: symmetric quadrature rules, quadrature rules for singularities, triangle quadrature rules, arbitrary functions

1. Introduction

Due to their efficiency, Gaussian quadrature rules are useful for numerical integration. For integrands that can be accurately approximated by polynomials, rules are typically employed that exactly integrate polynomials of increasing degree.

Because of their common use in two-dimensional discretizations, the development of quadrature rules for triangles is a popular research area. Several authors have developed methods for computing symmetric quadrature rules for polynomials [1, 2, 3, 4]. Symmetric rules are desirable because their mapping to the integration domain is straightforward and points are not heavily concentrated near some vertices. Asymmetric rules, on the other hand, require the determination of the vertex mapping, and point concentration at the vertices is inconsistent.

In Reference [1], the authors present quadrature rules for many polynomial degrees, up to degree 12. In Reference [2], the author provides quadrature rules for all degrees, up to degree 20. Reference [3] uses numerical optimization to compute even higher degrees of polynomials. In two dimensions, the optimal number of integration points is much less straightforward than for one dimension, and, for a given degree and number of points, there can be multiple solutions for the points and weights. Reference [4] presents an approach for determining all of the solutions for polynomials up to moderately high degrees, and the author presents these solutions as ancillary material in Reference [5]. Reference [6] presents an approach for computing quadrature rules for polynomials over arbitrary polygons.

However, for functions with singularities on the integration boundary, rules for polynomials do not converge monotonically or as rapidly as the number of integration points is increased. Provided they are integrable, these singular integrands may include unbounded derivatives at the boundary, where the integrand may be defined or undefined. In Reference [7], the authors present an approach for computing the quadrature rules associated with arbitrary one-dimensional functions and demonstrate the effectiveness of their technique for several sequences of functions, with and without singularities. To extend this approach to two dimensions, the authors of Reference [8] take an outer product of the one-dimensional rules and asymmetrically map the result to a triangle.

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Regardless of dimension and function sequence, the equations for computing quadrature rules are stiff and highly dependent on the initial guess \([2, 7, 3, 4]\). For one dimension, the authors of Reference \([7]\) use a continuation method, beginning with the polynomial quadrature rules and gradually transforming the sequence of polynomials to the desired sequence of functions, using the intermediate solutions as subsequent initial guesses. In higher dimensions, the problem is complicated by the potentially unknown number of optimal points \([9]\) and the potential existence of multiple solutions \([4, 5]\).

In this paper, we develop symmetric quadrature rules for triangles that integrate arbitrary functions, motivated by the need to address integrands with boundary singularities. This paper is organized as follows. In Section 2, we discuss the details of symmetric quadrature rules for triangles. In Section 3, we discuss singularities and the construction of one- and two-dimensional function sequences to be integrated exactly. In Section 4, we describe our first approach to computing symmetric quadrature rules, which is better suited for moderate amounts of functions and points (leading to about 6 or 7 digits in accuracy), and, in Section 5, we describe our second approach to computing symmetric quadrature rules, which is better suited for large amounts of functions and points (leading to machine accuracy). In Section 6, we demonstrate the two proposed approaches for a sample triangle and compare with polynomial rules. Finally, in Section 7, we provide an outlook for future work.

2. Quadrature Preliminaries

In this section, we describe the concepts we use to construct our approaches to computing symmetric quadrature rules for triangles that integrate singularities.

2.1. Quadrature Rules

An \(n\)-point quadrature rule exactly integrates a sequence of \(n_f\) functions \(f(x) = \{f_1(x), \ldots, f_{n_f}(x)\}\), such that

\[
\int_A f(x) dA = \sum_{i=1}^{n} w_i f(x_i).
\]

In Equation (1), the quadrature integration exactly computes the integrals by taking a linear combination of the function values at \(x_i\), which are weighted by weights \(w_i\), for \(i = 1, \ldots, n\). In one dimension, \(n_f = 2n\) and, for polynomials, \(f(x) = \{1, \ldots, x^{2n-1}\}\). For two dimensions, one could speculate \(n_f = 3n\) \([9]\), but the ability to achieve such efficiency is unproven, and, if the rules are required to be symmetric, the efficiency can be significantly lower \([9]\).

2.2. Symmetric Rules for Triangles

Symmetric rules for triangles are invariant to rotation and reflection about the medians for equilateral triangles, which can be isoparametrically transformed to arbitrary triangles. As several references \([1, 2, 3, 4]\) have described, the points are comprised of a combination of orbits. There are three types of orbits, which are shown in Figure 1.

The type-0 orbit consists of a point at the centroid, which is \(\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)\) in barycentric coordinates. The type-1 orbit consists of three points, each on a median, such that the coordinates are the three unique permutations of \((\alpha, \frac{1-\alpha}{2}, \frac{1-\alpha}{2})\). The type-2 orbit consists of six points, not on the medians, such that the coordinates are the six unique permutations of \((\alpha, \beta, 1 - \alpha - \beta)\).

In terms of orbits, the number of points \(n\) is

\[
n = n_0 + 3n_1 + 6n_2,
\]

where \(n_j\) is the number of type-\(j\) orbits. For type-0 orbits, \(n_0 = 0\) or \(n_0 = 1\). Type-1 and type-2 orbits can have arbitrary \(n_1\) and \(n_2\). We denote the orbit counts for a given \(n\) as the triplet \((n_0, n_1, n_2)\). For each orbit, the weights \(w_i\) for the points are the same. A type-0 orbit has an unknown weight, a type-1 orbit has an unknown weight and coordinate, and a type-2 orbit has an unknown weight and two unknown coordinates. Therefore, the total number of unknowns is \(n_0 + 2n_1 + 3n_2\).
2.3. Polynomial Integration

Polynomial rules capable of integrating polynomials up to degree \( d \) can exactly integrate linear combinations of the monomials \( x^p y^q \), where \( 0 \leq p \leq d \), \( 0 \leq q \leq d \), \( 0 \leq p + q \leq d \), totaling \( n_f = (d+1)(d+2)/2 \) monomials. This requirement can yield more equations than unknowns. For example, a \((0, 1, 0)\) rule can integrate polynomials up to \( d = 2 \), resulting in \( n_f = 6 \) monomials: \( f(x, y) = \{1, x, y, x^2, y^2, xy\} \). However, the number of unknowns is 2: \( \alpha \) and \( w \). This mismatch is reconcilable because the \( n_f = 6 \) equations of Equation (1) are not linearly independent.

To reduce the number of equations to an amount that is linearly independent, one can use de Moivre’s theorem to express the polynomials in polar coordinates [1, 2] or otherwise construct a sequence of invariant polynomials [3, 4].

On the other hand, instead of formulating the problem as the solution to a system of equations, one can formulate the problem as an unconstrained optimization problem:

\[
\arg \min_{\alpha, \beta, w} F(\alpha, \beta, w),
\]

where

\[
F(\alpha, \beta, w) = \sum_{j=1}^{n_f} \left( \frac{\sum_{i=1}^{n} w_i f_j(\alpha_i, \beta_i) - \int_{0}^{1} \int_{0}^{1-\beta} f_j(\alpha, \beta) \, d\alpha \, d\beta}{\int_{0}^{1} \int_{0}^{1-\beta} f_j(\alpha, \beta) \, d\alpha \, d\beta} \right)^2,
\]

with the expectation that the objective function \( F(\alpha, \beta, w) \) is reduced to zero. The scaling of the denominator in Equation (3) is optional and should be omitted if any of the integrals are zero. Additionally, the integrals in Equation (3) should be computed analytically when possible. In Equation (3), because of symmetry, the ability to integrate \( f(\alpha, \beta) \) indicates the ability to integrate \( f(\beta, \alpha) \), enabling the number of functions to be reduced to \( n_f = \lfloor (d+2)^2/4 \rfloor \), where \( \lfloor \cdot \rfloor \) denotes the floor function. For example, when \( d = 3 \), the \( n_f = 6 \) functions are \( f(\alpha, \beta) = \{1, \alpha, \alpha^2, \alpha \beta, \alpha^3, \alpha^2 \beta\} \).

3. Singularities and Function-Sequence Construction

Integrands with boundary singularities can have singularities located on edges and/or corners. At these locations, derivatives of the integrands are unbounded, and the integrand can be defined or undefined, provided the integrand is integrable. For example, in electromagnetic simulations, logarithmic singularities can appear at the integration boundaries [8, 10]. We present two approaches for constructing the sequence of functions to be exactly integrated by the quadrature rules.
3.1. One-Dimensional Functions

The integrands can be analyzed in terms of series expansions about the edges or corners. These expansions can take the form of alternating monomials and singularities; for example, \( f(x) = \{1, \ln x, x, x \ln x, \ldots\} \) [7] or \( f(x) = \{1, x, x \ln x, x^2, x^3, x^4, x^5, x^6, \ldots\} \) [10]. Note that, given the electromagnetic applications in mind, we concentrate on logarithmic functions here, but other singular functions can be considered. More generally, we denote these expansions as \( f(x) = \{m(x), s(x)\} \), where \( m(x) = \{1, x, \ldots, x^d\} \), and \( s(x) = \{s_1(x), s_2(x), \ldots, s_{n_s}(x)\} \) is the sequence of singular functions. The total number of functions is \( n_f = n_m + n_s \) and the number of monomials is \( n_m = d + 1 \).

For triangles, in barycentric coordinates, edge singularities (\( \alpha = 0 \)) can be modeled by \( s(\alpha) \) and corner singularities (\( \alpha = 1 \)) can be modeled by \( s(1-\alpha) \), assuming the singularities in \( s(x) \) occur at \( x = 0 \). For both corner and edge singularities, one can increase the number of entries in \( s(x) \); however, doing so may further reduce the maximum polynomial degree the rules are capable of integrating for a given number of points.

3.2. Two-Dimensional Functions

For this approach, \( m(x, y) \) is the sequence of \( n_m = \left\lceil \frac{(d+2)^2}{4} \right\rceil \) monomials described in Section 2.3. If two-dimensional characterizations of the singularities are known, they can be included in the singularity sequence \( s(x, y) \); otherwise, they can be obtained from series expansions, as is done in Section 3.1 for \( s(x) \). The total number of functions is \( n_f = n_m + n_s \).

We introduce the concept of function groups as an accounting mechanism. A function group contains either one singularity or the monomials with powers that sum to a particular degree. The function group associated with a polynomial of degree \( d \) contains \( 1 + \lfloor d/2 \rfloor \) monomials. In order to integrate a function group \( n_g \), we require that the quadrature rules be able to integrate the preceding groups.

4. Approach 1: Optimization for a Moderate Number of Functions

The goal of this work is to achieve the ability to integrate polynomials as efficiently as the symmetric polynomial rules, while being able to integrate singularities. Therefore, this approach uses the polynomial rules as a baseline.

For polynomial rules, Table 1 lists the maximum polynomial degree \( d \) per number of integration points \( n \), as well as the orbit counts [2, 4]. For each \( n \) in Table 1, the rules can integrate function group \( n_g = d \). Because these orbit counts have been shown to be the most efficient [2, 4], we use these choices for each \( n \) listed in Table 1.

| \( n \) | \( n_0 \) | \( n_1 \) | \( n_2 \) | \( n_g = d \) | \( n \) | \( n_0 \) | \( n_1 \) | \( n_2 \) | \( n_g = d \) |
|-------|---------|---------|---------|-------|-------|---------|---------|---------|-------|
| 1     | 1       | 0       | 0       | 1     | 27    | 0       | 5       | 2       | 11    |
| 3     | 0       | 1       | 0       | 2     | 33    | 0       | 5       | 3       | 12    |
| 4     | 1       | 1       | 0       | 3     | 37    | 1       | 6       | 3       | 13    |
| 6     | 0       | 2       | 0       | 4     | 42    | 0       | 6       | 4       | 14    |
| 7     | 1       | 2       | 0       | 5     | 48    | 0       | 6       | 5       | 15    |
| 12    | 0       | 2       | 1       | 6     | 52    | 1       | 7       | 5       | 16    |
| 13    | 1       | 2       | 1       | 7     | 61    | 1       | 8       | 6       | 17    |
| 16    | 1       | 3       | 1       | 8     | 70    | 1       | 9       | 7       | 18    |
| 19    | 1       | 4       | 1       | 9     | 73    | 1       | 8       | 8       | 19    |
| 25    | 1       | 2       | 3       | 10    | 79    | 1       | 10      | 8       | 20    |

Table 1: Maximum polynomial degree \( d \) per number of points \( n \).

For each \( n \), we construct a sequence of functions consisting of \( n_g + 1 \) function groups, using the same value for \( n_g \) as the polynomial rules. In doing so, we reduce the maximum degree of polynomials \( d \) that can be integrated exactly from Table 1 in exchange for the ability to integrate the singular functions.

When constructing the sequence of functions, one must weigh the amount of singular functions against the maximum polynomial degree that can be integrated. Additionally, whereas the ability to integrate polynomials includes the ability to integrate cross terms (e.g., the ability to integrate \( x^3 \) also indicates the
ability to integrate $x^2y$), the ability to integrate singular functions does not extend to cross terms. Therefore, there are three prioritized approaches to address this issue:

1. Use a two-dimensional characterization of the singularity (Section 3.2), if it is available.
2. Use a one-dimensional characterization of the singularity (Section 3.1), assuming the cross terms do not warrant an additional reduction in the maximum polynomial degree that can be integrated.
3. Include cross terms for the one-dimensional characterization (Section 3.1) at the expense of reducing the maximum polynomial degree that can be integrated.

Alternatively, one can use Approach 2, which is presented in Section 5.

With the function sequence constructed, we solve the optimization problem in Equation (2) with the expectation that the objective function in Equation (3) is zero. We use 64 digits of working precision and require that the objective function be less than $10^{-32}$. Upon doing so, we attempt to include subsequent groups of functions until the objective function exceeds $10^{-32}$. As an example, for a 13-point $(1, 2, 1)$ rule, Figure 2 depicts the eight unknowns: $\alpha_2, \alpha_3, \alpha_4, \beta_4, w_1, w_2, w_3,$ and $w_4$.

As mentioned in the introduction, the ability to compute quadrature rules is heavily dependent upon the initial guess for the iterative solver. Therefore, we use the points and weights presented in Reference [2] for polynomials as initial guesses. Points and weights for other polynomial rules can be used, such as those listed in Reference [5].

From barycentric coordinates, the points can be mapped to an arbitrary triangle with vertices $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ by

$$x = \alpha x_1 + \beta x_2 + (1 - \alpha - \beta) x_3, \quad y = \alpha y_1 + \beta y_2 + (1 - \alpha - \beta) y_3.$$ 

The weights are multiplied by twice the area of the arbitrary triangle: $w = 2A w'$.

5. **Approach 2: Quadrilateral Subdomains**

For polynomials, symmetric quadrature rules have been computed for high degrees. This is facilitated by the equation reduction employed by exploiting invariance, as mentioned in Section 2.3. Nonetheless, for higher degrees, the number of possible solutions increases [4, 5]. Additionally, for higher degrees, the optimal number of points is less straightforward.

For arbitrary function sequences, which are the focus of this paper, a systematic reduction in the number of equations is unavailable, the solution of Equation (2) is more susceptible to nonzero local minima, and
knowledge of the optimal number of points is unavailable. Therefore, for large amounts of functions, we employ \( n' \)-point one-dimensional rules that integrate the one-dimensional function sequences described in Section 3.1. Because the rules are one-dimensional, \( n_f = 2n' = n_m + n_s \).

To achieve symmetric quadrature rules, we do the following:

1. Compute one-dimensional rules \( \xi_i, w'_i \), for \( i = 1, \ldots, n' \) on the unit interval \( \xi \in [0, 1] \) using the continuation approach presented in Reference [7].
2. Take the outer product of the one-dimensional rules to obtain rules for the unit square \( (\xi, \eta) \in [0, 1] \times [0, 1] \), such that \( (\xi_i, \eta_j) = (\xi_i, \xi_j) \) and \( w'_{ij} = w'_i w'_j \).
3. Using the vertices, edge midpoints, and centroid, subdivide the triangle into 3 quadrilaterals: \((A, D, O, F)\), \((B, E, O, D)\), and \((C, F, O, E)\), where \( O \) denotes the centroid.
4. Bilinearly transform the unit square to each quadrilateral:
   \[
   x(\xi, \eta) = \sum_{k=1}^{n} x_k \psi_k(\xi, \eta), \quad y(\xi, \eta) = \sum_{k=1}^{n} y_k \psi_k(\xi, \eta),
   \]
   where
   \[
   \psi_1(\xi, \eta) = (1 - \xi)(1 - \eta), \quad \psi_2(\xi, \eta) = \xi (1 - \eta), \quad \psi_3(\xi, \eta) = \eta (1 - \xi), \quad \psi_4(\xi, \eta) = (1 - \xi) \eta,
   \]
   and \( \{x_1, x_2, x_3, x_4\} = \{x_A, x_D, x_O, x_F\}, \{x_B, x_E, x_O, x_D\}, \) or \( \{x_C, x_F, x_O, x_E\} \), and similarly for \( y_k \).
5. Compute the weights \( w_{ij} = J(\xi_i, \eta_j) w'_{ij} \), where \( J(\xi_i, \eta_j) \) is the determinant of the Jacobian:
   \[
   J(\xi_i, \eta_j) = \begin{vmatrix}
   \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\
   \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta}
   \end{vmatrix}.
   \]
   The transformation is shown in Figure 3. For an \( n' \)-point one-dimensional rule, the triangle has \( n = 3n'^2 \) integration points. Instead of performing the transformation for each triangular integration domain, the transformation can be performed on an arbitrary reference triangle, which is then linearly mapped to each triangular integration domain. The advantage of this approach is that it is also directly applicable to quadrilaterals exhibiting boundary singularities.

6. Numerical Example: Singular Functions Appearing in Electromagnetic Simulations

To demonstrate the approaches of the preceding sections, we consider the integrals
\[
I_c = \int_A \int_{A'} \frac{\cos(2\pi \|x - x'\|_2)}{\|x - x'\|_2} dA' dA, \quad I_s = \int_A \int_{A'} \frac{\sin(2\pi \|x - x'\|_2)}{\|x - x'\|_2} dA' dA,
\]
(4)
which commonly appear as the real and imaginary components in the integration of the free-space Green’s function, \( G(x, x') \propto e^{-i\kappa(x-x')/\|x-x\|_2} \), in electromagnetic simulations. The outer integration domain \( A \) is a triangle. For the inner integration domain \( A' \), we consider two domains: (1) \( A' = A \) and (2) \( A' \) is the co-planar reflection of \( A \) along an edge.

For both integrals and domains, the integral over \( A' \) can be integrated using a radial–angular transformation \([11, 10]\). While the integrand of \( I_s \) is smooth, the integrand of the outer integral of \( I_c \) exhibits logarithmic singularities along shared edges and corners. We have found that a suitable choice for a one-dimensional sequence \([11, 10]\) is

\[
\int_{s_i} \propto \frac{1}{\sqrt{x^2 + y^2}},
\]

\[
\text{for } i \in \mathbb{N}^+ [10],
\]

for both dimensions, the function sequences are constructed by placing each singularity after the polynomial with the degree that matches the power of \( x \) in the singularity. The one- and two-dimensional function sequences are reported in Tables 2 and 3, respectively.

| \( n \) | \( n' \) | Functions |
|------|------|----------|
| 3    | 1    | 1, \( x \) |
| 12   | 2    | \( x \ln x, x^2 \) |
| 27   | 3    | \( x^3, x^3 \ln x \) |
| 48   | 4    | \( x^4, x^5 \) |
| 75   | 5    | \( x^5 \ln x, x^6 \) |
| 108  | 6    | \( x^7, x^7 \ln x \) |

Table 2: Sequence of functions for one dimension.

Using Approach 1 from Section 4, for each \( n \), we attempt to increase \( n_g \). The final values of \( n_g \) are listed in Table 4, which provides a comparison with the initial \( n_g \), as well as a comparison with the maximum polynomial degree \( d \) from the polynomial rules. When, for a given \( n \), the final \( n_g \) is not greater than that from a lower \( n \), the higher \( n \) is eliminated. For example, for \( n = 13 \), \( n_g \) does not increase; however, for \( n = 12 \), \( n_g \) is increased by one, matching the final \( n_g \) for \( n = 13 \). Therefore, we eliminate \( n = 13 \). Diagrams of the points computed using Approach 1 are shown in Figures 4 and 5.

Though not shown in this paper, some missing final \( n_g \), including those for eliminated points, can be obtained through the following approaches (Reference [5] lists many of these options):

1. For a given \( n \) and orbit amount \((n_0, n_1, n_2)\), use another set of polynomial rules as the initial guess.
   As stated previously, the polynomial rules are often not unique and the optimization problem of Equation (2) is dependent upon initial guess.
2. For a given \( n \), use a different polynomial-rule orbit amount. For example, for \( n = 37 \), Reference [2] provides a \((1, 6, 3)\) rule, whereas Reference [4] provides a \((1, 4, 4)\) rule.
3. Use a slightly higher \( n \) that is less efficient as a polynomial rule. For example, a 13-point \((1, 2, 1)\) rule and a 15-point \((0, 1, 2)\) rule can both integrate polynomials of degree \( d = 7 \) \([4]\); however, using the 13-point rule from Reference [2] yields a final \( n_g = 7 \), whereas one of the 15-point rules from Reference [5] yields a final \( n_g = 8 \).

Using the functions in Table 2, we construct the one-dimensional rules used by Approach 2. Diagrams of the points computed using Approach 2 are shown in Figure 6.

To assess the performance of the rules arising from these approaches, we compute reference solutions for \( I_c \) and \( I_s \), as described in Reference [10], for \( A \) being defined by the vertices \((0, 0)\), \((1/20, 1/20)\), and \((-1/20, 1/20)\). For Domain 1, \( A' = A \) and, for Domain 2, \( A' \) is defined by the vertices \((0, 1/10)\), \((-1/20, 1/20)\), and \((1/20, 1/20)\).

Figures 7–8 show the relative errors \( \varepsilon = |\hat{I}_k - I_k|/I_k \), between the reference solution \( I_k \) and the quadrature solution \( \hat{I}_k \) for \( k \in \{c, s\} \) in Equation 4. These figures compare the quadrature rules presented in this
| \( n_g \) | Group Size | \( d \) | \( n_s \) | Functions |
|---|---|---|---|---|
| 0 | 1 | 0 | 0 | 1 |
| 1 | 1 | 1 | 0 | \( x \) |
| 2 | 1 | 1 | 1 | \( x \ln(y - 1 + \sqrt{x^2 + (y - 1)^2}) \) |
| 3 | 1 | 1 | 2 | \( x \ln(y + \sqrt{x^2 + y^2}) \) |
| 4 | 2 | 2 | 2 | \( x^2, xy \) |
| 5 | 2 | 3 | 2 | \( x^3, x^2y \) |
| 6 | 1 | 3 | 3 | \( x^3 \ln(y - 1 + \sqrt{x^2 + (y - 1)^2}) \) |
| 7 | 1 | 3 | 4 | \( x^3 \ln(y + \sqrt{x^2 + y^2}) \) |
| 8 | 3 | 4 | 4 | \( x^4, x^3y, x^2y^2 \) |
| 9 | 3 | 5 | 4 | \( x^5, x^4y, x^3y^2 \) |
| 10 | 1 | 5 | 5 | \( x^5 \ln(y - 1 + \sqrt{x^2 + (y - 1)^2}) \) |
| 11 | 1 | 5 | 6 | \( x^5 \ln(y + \sqrt{x^2 + y^2}) \) |
| 12 | 4 | 6 | 6 | \( x^6, x^5y, x^4y^2, x^3y^3 \) |
| 13 | 4 | 7 | 6 | \( x^7, x^6y, x^5y^2, x^4y^3 \) |
| 14 | 1 | 7 | 7 | \( x^7 \ln(y - 1 + \sqrt{x^2 + (y - 1)^2}) \) |
| 15 | 1 | 7 | 8 | \( x^7 \ln(y + \sqrt{x^2 + y^2}) \) |
| 16 | 5 | 8 | 8 | \( x^8, x^7y, x^6y^2, x^5y^3, x^4y^4 \) |
| 17 | 5 | 9 | 8 | \( x^9, x^8y, x^7y^2, x^6y^3, x^5y^4 \) |
| 18 | 1 | 9 | 9 | \( x^9 \ln(y - 1 + \sqrt{x^2 + (y - 1)^2}) \) |
| 19 | 1 | 9 | 10 | \( x^9 \ln(y + \sqrt{x^2 + y^2}) \) |
| 20 | 6 | 10 | 10 | \( x^{10}, x^9y, x^8y^2, x^7y^3, x^6y^4, x^5y^5 \) |

Table 3: Sequence of function groups for two dimensions.

| \( n \) | \( n_g \) | \( d \) | \( n_s \) | \( n_g \) | \( d \) | \( n_s \) | \( n_g = d \) |
|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 3 | 2 | 1 | 1 | 2 | 1 | 1 | 2 |
| 4 | 3 | 1 | 2 | 3 | 1 | 2 | 3 |
| 6 | 4 | 2 | 2 | 4 | 2 | 2 | 4 |
| 7 | 5 | 3 | 2 | 5 | 3 | 2 | 5 |
| 12 | 6 | 3 | 3 | 7 | 3 | 4 | 6 |
| 13 | 7 | 3 | 4 | 7 | 3 | 4 | 7 |
| 16 | 8 | 4 | 4 | 10 | 5 | 5 | 8 |
| 19 | 9 | 5 | 4 | 11 | 5 | 6 | 9 |
| 25 | 10 | 5 | 5 | 13 | 7 | 6 | 10 |
| 27 | 11 | 5 | 6 | 14 | 7 | 7 | 11 |
| 33 | 12 | 6 | 6 | 16 | 8 | 8 | 12 |
| 37 | 13 | 7 | 6 | 13 | 7 | 6 | 13 |
| 42 | 14 | 7 | 7 | 19 | 9 | 10 | 14 |
| 48 | 15 | 7 | 8 | 15 | 8 | 8 | 15 |
| 52 | 16 | 8 | 8 | 23 | 11 | 12 | 16 |

Table 4: Function groups integrated.
paper, as well as the polynomial quadrature rules for both domains and integrals. As shown in Figures 7a and 8a for $I_c$, both approaches generally outperform the polynomial quadrature rules, and Approach 1 often outperforms the polynomial quadrature rules by orders of magnitude. In Figure 7a, for example, Approach 1 outperforms the polynomial rules by four orders of magnitude for $n = 27$. Approach 1 is then appropriate when a moderate accuracy (e.g., 6 or 7 digits) is sufficient.

Because the integrand for $I_s$ is not singular, the polynomial rules perform the best; however, Approach 1 has similar efficiency, as shown in Figures 7b and 8b.

Though not as efficient as Approach 1 for $I_c$ or the polynomial rules for $I_s$, the relative error arising from Approach 2 decreases monotonically with respect to $n$, which is an important feature to guarantee improved accuracy when increasing $n$. Additionally, for large values of $n$, the points arising from Approach 2 take less time to compute than those from Approach 1 since they arise from one-dimensional rules.

![Figure 4: Points for different values of $n$ between 3 and 16 using Approach 1.](image-url)
Figure 5: Points for different values of $n$ between 19 and 52 using Approach 1.

Figure 6: Points for different values of $n$ between 3 and 108 using Approach 2.
7. Conclusions and Outlook for Future Work

In this paper, we presented two approaches for computing symmetric triangle quadrature rules for arbitrary functions. On an example problem, we demonstrated that, for several point amounts, our approaches can achieve relative errors two orders magnitude less than those achieved from polynomial rules.

The more novel Approach 1 of Section 4 yields efficient points, but, for larger amounts of points, their computation is costly and knowledge of their optimality and uniqueness is limited. Approaches to improve these shortcomings would be beneficial, though most likely highly dependent upon the function sequence.

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