THE RICCATI SYSTEM AND A DIFFUSION-TYPE EQUATION

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Abstract. We discuss a method of constructing solution of the initial value problem for diffusion-type equations in terms of solutions of certain Riccati and Ermakov-type systems. A nonautonomous Burgers-type equation is also considered.

1. Introduction

A goal of this note, complementary to our recent paper [37], is to elaborate on the Cauchy initial value problem for a class of nonautonomous and inhomogeneous diffusion-type equations on $\mathbb{R}$. A corresponding nonautonomous Burgers-type equation is also analyzed as a by-product. Here, we use explicit transformations to the standard forms and emphasize natural relations with certain Riccati and Ermakov-type systems, which seem are missing in the available literature. Similar methods are applied to the corresponding Schrödinger equation (see, for example, [6], [7], [8], [9], [11], [24], [25], [26], [27], [36], [38], [39] and references therein). A group theoretical approach to a similar class of partial differential equations is discussed in Refs. [15], [28] and [34].

For an introduction to fundamental solutions for parabolic equations, see chapter one of the book by Friedman [14]. Among numerous applications, we only elaborate here on an important role of fundamental solutions in probability theory [10], [21]. Consider an Itô diffusion $X = \{X_t : t \geq 0\}$ which satisfies the stochastic differential equation

$$dX_t = b(X_t, t) \ dt + \sigma(X_t, t) \ dW_t, \quad X_0 = x, \quad (1.1)$$

in which $W = \{W_t : t \geq 0\}$ is a standard Wiener process. The existence and uniqueness of solutions of (1.1) depends on the coefficients $b$ and $\sigma$. (See Ref. [21] for conditions of unique strong solution to (1.1).) If the equation (1.1) has a unique solution, then the expectations

$$u(x, t) = E_x[\phi(X_t)] = E[\phi(X_t) | X_0 = x] \quad (1.2)$$

are solutions of the Cauchy problem

$$u_t = \frac{1}{2} \sigma^2(x, t) \ u_{xx} + b(x, t) \ u_x, \quad u(x, 0) = \phi(x). \quad (1.3)$$

This PDE is known as Kolmogorov forward equation [10], [21]. Thus if $p(x, y, t)$ is the appropriate fundamental solution of (1.3), then one can compute the given expectations according to

$$E_x[\phi(X_t)] = \int_{\Omega} p(x, y, t) \phi(y) \ dy. \quad (1.4)$$

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In this context, the fundamental solution is known as the probability transition density for the process and
\[ \int_{\Omega} p(x, y, t) \, dy = 1. \] (1.5)

See also Refs. [1] and [20] for applications to stochastic differential equations related to Fokker–Planck and Burgers equations.

2. Transformation to the Standard Form

We present the following result.

Lemma 1. The nonautonomous and inhomogeneous diffusion-type equation
\[ \frac{\partial u}{\partial t} = a(t) \frac{\partial^2 u}{\partial x^2} - (g(t) - c(t) x) \frac{\partial u}{\partial x} + (d(t) + f(t) x - b(t) x^2) u, \] (2.1)
where \( a, b, c, d, f, g \) are suitable functions of time \( t \) only, can be reduced to the standard autonomous form
\[ \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2}, \] (2.2)
with the help of the following substitution:
\[ u(x, t) = \frac{1}{\sqrt{\mu(t)}} e^{\alpha(t)x^2 + \delta(t)x + \kappa(t)} v(\xi, \tau), \] (2.3)
\[ \xi = \beta(t) x + \epsilon(t), \quad \tau = \gamma(t). \]

Here, \( \mu, \alpha, \beta, \gamma, \delta, \epsilon, \kappa \) are functions of \( t \) that satisfy
\[ \frac{\mu''}{2\mu} + 2\alpha \alpha + d = 0 \] (2.4)
and
\[ \begin{align*}
\frac{d\alpha}{dt} + b - 2c\alpha - 4a\alpha^2 &= 0, \\
\frac{d\beta}{dt} - (c + 4a\alpha) \beta &= 0, \\
\frac{d\gamma}{dt} - a\beta^2 &= 0, \\
\frac{d\delta}{dt} - (c + 4a\alpha) \delta &= f - 2\alpha g, \\
\frac{d\epsilon}{dt} + (g - 2a\delta) \beta &= 0, \\
\frac{d\kappa}{dt} + g\delta - a\delta^2 &= 0. 
\end{align*} \] (2.5)

Equation (2.5) is called the Riccati nonlinear differential equation [32], [40], [42] and we shall refer to the system (2.5)–(2.10) as a Riccati-type system.

The substitution (2.4) reduces the nonlinear Riccati equation (2.5) to the second order linear equation
\[ \mu'' - \tau(t) \mu' - 4\sigma(t) \mu = 0, \] (2.11)
where
\[ \tau (t) = \frac{a'}{a} + 2c - 4d, \quad \sigma (t) = ab + cd - d^2 + \frac{d}{2} \left( \frac{a'}{a} - \frac{d'}{d} \right), \] (2.12)
which shall be referred to as a characteristic equation [37].

It is also known [37] that the diffusion-type equation (2.1) has a particular solution of the form
\[ u = \frac{1}{\sqrt{\mu(t)}} e^{\alpha(t)x^2 + \beta(t)xy + \gamma(t)y^2 + \delta(t)x + \varepsilon(t)y + \kappa(t)}, \] (2.13)
provided that the time dependent functions \( \mu, \alpha, \beta, \gamma, \delta, \varepsilon, \kappa \) satisfy the Riccati-type system (2.4)–(2.10).

A group theoretical approach to a similar class of partial differential equations is discussed in Refs. [15], [28] and [34].

3. Fundamental Solution

By the superposition principle one can solve (formally) the Cauchy initial value problem for the diffusion-type equation (2.1) subject to initial data \( u(x,0) = \varphi(x) \) on the entire real line \( -\infty < x < \infty \) in an integral form
\[ u(x,t) = \int_{-\infty}^{\infty} K_0(x,y,t) \varphi(y) \, dy \] (3.1)
with the fundamental solution (heat kernel) [37]:
\[ K_0(x,y,t) = \frac{1}{\sqrt{2\pi \mu_0(t)}} e^{\alpha_0(t)x^2 + \beta_0(t)xy + \gamma_0(t)y^2 + \delta_0(t)x + \varepsilon_0(t)y + \kappa_0(t)}, \] (3.2)
where a particular solution of the Riccati-type system (2.9)–(2.10) is given by
\[ \alpha_0(t) = -\frac{1}{4a(t)} \frac{\mu_0'(t)}{\mu_0(t)} - \frac{d(t)}{2a(t)}, \] (3.3)
\[ \beta_0(t) = \frac{h(t)}{\mu_0(t)}, \quad h(t) = \exp \left( \int_0^t (c(s) - 2d(s)) \, ds \right), \] (3.4)
\[ \gamma_0(t) = \frac{d(0)}{2a(0)} - \frac{a(t) h^2(t)}{\mu_0(t) \mu_0'(t)} - 4 \int_0^t \frac{a(s) \sigma(s) h(s)}{(\mu_0'(s))^2} \, ds \] (3.5)
\[ = \frac{d(0)}{2a(0)} - \frac{1}{2} \frac{\mu_1(t)}{\mu_0(t)}, \] (3.6)
\[ \delta_0(t) = \frac{h(t)}{\mu_0(t)} \int_0^t \left[ \left( f(s) + \frac{d(s)}{a(s)} g(s) \right) \mu_0(s) + \frac{g(s)}{2a(s)} \mu_0'(s) \right] \frac{ds}{h(s)}, \] (3.7)
\[ \varepsilon_0(t) = -2a(t) \frac{h(t)}{\mu_0'(t)} \delta_0(t) - 8 \int_0^t \frac{a(s) \sigma(s) h(s)}{(\mu_0'(s))^2} (\mu_0(s) \delta_0(s)) \, ds \] (3.8)
\[ + 2 \int_0^t \frac{a(s) h(s)}{\mu_0'(s)} \left( f(s) + \frac{d(s)}{a(s)} g(s) \right) \, ds, \]
\[
\kappa_0(t) = -\frac{a(t)\mu_0(t)}{\mu'_0(t)}\delta_0(t) - 4\int_0^t a(s)\sigma(s)\left(\frac{\mu_0(s)}{\mu'_0(s)}\right)^2(\mu_0(s)\delta_0(s))^2\,ds + 2\int_0^t a(s)\mu_0(s)\delta_0(s) \left[ f(s) + \frac{d(s)}{a(s)}g(s) \right] \,ds 
\] (3.9)

with \(\delta(0) = g(0)/(2a(0))\), \(\varepsilon(0) = -\delta(0)\), \(\kappa(0) = 0\). Here, \(\mu_0\) and \(\mu_1\) are the so-called standard solutions of the characteristic equation (2.11) subject to the following initial data:

\[
\mu_0(0) = 0, \quad \mu_0'(0) = 2a(0) \neq 0, \quad \mu_1(0) \neq 0, \quad \mu_1'(0) = 0. \quad (3.10)
\]

Solution (3.3)–(3.9) shall be referred to as a fundamental solution of the Riccati-type system (2.5)–(2.10); see (3.27)–(3.31) and (3.32) for the corresponding asymptotics.

Lemma 2. The Riccati-type system (2.4)–(2.10) has the following (general) solution:

\[
\mu(t) = -2\mu(0)\mu_0(t)(\alpha(0) + \gamma_0(t)), \quad (3.11)
\]

\[
\alpha(t) = \alpha_0(t) - \frac{\beta_0^2(t)}{4(\alpha(0) + \gamma_0(t))}, \quad (3.12)
\]

\[
\beta(t) = -\frac{\beta(0)\beta_0(t)}{2(\alpha(0) + \gamma_0(t))}, \quad (3.13)
\]

\[
\gamma(t) = \gamma(0) - \frac{\beta_0^2(0)}{4(\alpha(0) + \gamma_0(t))}, \quad (3.14)
\]

and

\[
\delta(t) = \delta_0(t) - \frac{\beta_0(t)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \quad (3.15)
\]

\[
\varepsilon(t) = \varepsilon(0) - \frac{\beta(0)(\delta(0) + \varepsilon_0(t))}{2(\alpha(0) + \gamma_0(t))}, \quad (3.16)
\]

\[
\kappa(t) = \kappa(0) + \kappa_0(t) - \frac{(\delta(0) + \varepsilon_0(t))^2}{4(\alpha(0) + \gamma_0(t))}, \quad (3.17)
\]

in terms of the fundamental solution (3.3)–(3.9) subject to arbitrary initial data \(\mu(0), \alpha(0), \beta(0), \gamma(0), \delta(0), \varepsilon(0), \kappa(0)\).

Proof. Use (2.13)–(3.2), uniqueness of the solution and the elementary integral:

\[
\int_{-\infty}^{\infty} e^{-ay^2 + 2by} \, dy = \sqrt{\frac{\pi}{a}} e^{b^2/a}, \quad a > 0. \quad (3.18)
\]

Computational details are left to the reader. \(\Box\)

Remark 1. It is worth noting that our transformation (2.3), combined with the standard heat kernel (20),

\[
K_0(\xi, \eta, \tau) = \frac{1}{\sqrt{4\pi(\tau - \tau_0)}} \exp\left[ -\frac{(\xi - \eta)^2}{4(\tau - \tau_0)} \right] \quad (3.19)
\]

for the diffusion equation (2.2) and (3.11)–(3.17), allows one to derive the fundamental solution (3.2) of the diffusion-type equation (2.1) from a new perspective.
Lemma 3. Solution (3.11)–(3.17) implies:

\[ \mu_0 = \frac{2\mu}{\mu(0)\beta^2(0)}(\gamma - \gamma(0)), \]  
\[ \alpha_0 = \alpha_0(t) - \frac{\beta}{4(\gamma - \gamma(0))}, \]  
\[ \beta_0 = \frac{\beta(0)\beta}{2(\gamma - \gamma(0))}, \]  
\[ \gamma_0 = -\alpha(0) - \frac{\beta^2(0)}{4(\gamma - \gamma(0))}, \]  

and

\[ \delta_0 = \delta - \frac{\beta(\varepsilon - \varepsilon(0))}{2(\gamma - \gamma(0))}, \]  
\[ \varepsilon_0 = -\delta(0) + \frac{\beta(0)(\varepsilon - \varepsilon(0))}{2(\gamma - \gamma(0))}, \]  
\[ \kappa_0 = \kappa - \kappa(0) - \frac{(\varepsilon - \varepsilon(0))^2}{4(\gamma - \gamma(0))}, \]  

which gives the following asymptotics

\[ \alpha_0(t) = -\frac{1}{4a(0)t} - \frac{c(0)}{4a(0)} + \frac{a'(0)}{8a^2(0)} + O(t), \]  
\[ \beta_0(t) = \frac{1}{2a(0)t} - \frac{a'(0)}{4a^2(0)} + O(t), \]  
\[ \gamma_0(t) = -\frac{1}{4a(0)t} + \frac{c(0)}{4a(0)} + \frac{a'(0)}{8a^2(0)} + O(t), \]  
\[ \delta_0(t) = \frac{g(0)}{2a(0)} + O(t), \]  
\[ \varepsilon_0(t) = -\frac{g(0)}{2a(0)} + O(t), \]  
\[ \kappa_0(t) = O(t) \]  

as \( t \to 0^+ \).

(The proof is left to the reader.)

These formulas allows to establish a required asymptotic of the fundamental solution (3.2):

\[ K_0(x, y, t) \sim \frac{1}{\sqrt{4\pi a(0)t}} \exp \left[ -\frac{(x - y)^2}{4a(0)t} \right] \]  
\[ \times \exp \left[ \frac{a'(0)}{8a^2(0)}(x - y)^2 - \frac{c(0)}{4a(0)}(x^2 - y^2) \right] \exp \left[ \frac{g(0)}{2a(0)}(x - y) \right]. \]  

(Here, \( f \sim g \) as \( t \to 0^+ \), if \( \lim_{t \to 0^+} (f/g) = 1 \). The proof is left to the reader.)

By a direct substitution one can verify that the right hand sides of (3.11)–(3.17) satisfy the Riccati-type system (2.4)–(2.10) and that the asymptotics (3.27)–(3.31) result in the continuity
with respect to initial data:
\[
\lim_{t \to 0^+} \mu (t) = \mu (0), \quad \lim_{t \to 0^+} \alpha (t) = \alpha (0), \quad \text{etc.}
\] (3.33)

The transformation property (3.11)–(3.17) allows one to find solution of the initial value problem in terms of the fundamental solution (3.3)–(3.9) and may be referred to as a nonlinear superposition principle for the Riccati-type system.

4. Eigenfunction Expansion and Ermakov-type System

With the help of transformation (2.3) one can reduce the diffusion equation (2.1) to another convenient form
\[
\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2} + \xi^2 v, \quad (4.1)
\]
which allows to find solution of the Cauchy initial value problem in terms of an eigenfunction expansion similar to the case of the corresponding Schrödinger in Refs. [24] and [38]. This method requires an extension the Riccati-type system (2.5)–(2.10) to a more general Ermakov-type system [24], which is integrable in quadratures once again in terms of solutions of the characteristic equation (2.11). Further details are left to the reader.

5. Nonautonomous Burgers Equation

The nonlinear equation
\[
\frac{\partial v}{\partial t} + a (t) \left( v \frac{\partial v}{\partial x} - \frac{\partial^2 v}{\partial x^2} \right) - c (t) \left( x \frac{\partial v}{\partial x} + v \right) + g (t) \frac{\partial v}{\partial x} = 2 \left( 2 b (t) x - f (t) \right),
\] (5.1)
when \( a = 1 \) and \( b = c = f = g = 0 \), is known as Burgers’ equation [2], [3], [5], [17], [19], [35], [41] and we shall refer to (5.1) as a nonautonomous Burgers-type equation.

**Lemma 4.** The following identity holds
\[
v_t + a (v v_x - v_{xx}) + (g - c x) v_x - c v + 2 (f - 2 b x) = -2 \left( \frac{u_t - Qu}{u} \right)_x, \quad (5.2)
\]
if
\[
v = -2 \frac{u_x}{u} \quad \text{(The Cole–Hopf transformation)} \quad (5.3)
\]
and
\[
Qu = a u_{xx} - (g - c x) u_x + (d + f x - b x^2) u \quad (5.4)
\]
\( (a, b, c, d, f, g \text{ are functions of } t \text{ only}).

(This can be verified by a direct substitution.)

The substitution (5.3) turns the nonlinear Burgers-type equation (5.1) into the diffusion-type equation (2.1). Then solution of the corresponding Cauchy initial value problem can be represented as
\[
v (x, t) = -2 \frac{\partial}{\partial x} \ln \left[ \int_{-\infty}^{\infty} K_0 (x, y, t) \exp \left( -\frac{1}{2} \int_0^y v (z, 0) \, dz \right) \, dy \right], \quad (5.5)
\]
where the heat kernel is given by (3.2), for suitable initial data \( v(z,0) \) on \( \mathbb{R} \).

6. Traveling Wave Solutions of Burgers-type Equation

Looking for solutions of our equation (5.1) in the form
\[
v = \beta(t) F(\beta(t)x + \gamma(t)) = \beta F(z), \quad z = \beta x + \gamma
\]
(\( \beta \) and \( \gamma \) are functions of \( t \) only), one gets
\[
F'' = (c_0 + c_1) F' + FF' + 2c_2 z + c_3
\]
provided that
\[
\beta' = c\beta, \quad \gamma' = c_0 a\beta^2, \quad g = c_1 a\beta, \quad b = -\frac{1}{2} c_2 a\beta^4,
\]
\[
f = \frac{1}{2} a\beta^3 (2c_2 \gamma + c_3)
\]
(\( c_0, c_1, c_2, c_3 \) are constants). From (6.2):
\[
F' = (c_0 + c_1) F + \frac{1}{2} F^2 + c_2 z^2 + c_3 z + c_4,
\]
where \( c_4 \) is a constant of integration. The substitution
\[
F = -2 \frac{\mu'}{\mu}
\]
transforms the Riccati equation (6.6) into a special case of generalized equation of hypergeometric type:
\[
\mu'' - (c_0 + c_1) \mu' + \frac{1}{2} (c_2 z^2 + c_3 z + c_4) \mu = 0,
\]
which can be solved in general by methods of Ref. [30]. Elementary solutions are discussed, for example, in [22] and [23].

7. Some Examples

Now we consider from a united viewpoint several elementary diffusion and Burgers equations that are important in applications.

Example 1 For the standard diffusion equation on \( \mathbb{R} \):
\[
\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2}, \quad a = \text{constant} > 0
\]
the heat kernel is given by
\[
K(x, y, t) = \frac{1}{\sqrt{4\pi at}} \exp \left[ -\frac{(x - y)^2}{4at} \right], \quad t > 0.
\]
(See [4], [29] and references therein for a detailed investigation of the classical one-dimensional heat equation.)
Example 2  In mathematical description of the nerve cell a dendritic branch is typically modeled by using cylindrical cable equation [18]:

\[
\tau \frac{\partial u}{\partial t} = \lambda \frac{\partial^2 u}{\partial x^2} + u, \quad \tau = \text{constant} > 0.
\]  

(7.3)

The fundamental solution on \( \mathbb{R} \) is given by

\[
K_0 (x, y, t) = \frac{\sqrt{\tau} e^{t/\tau}}{\sqrt{4\pi \lambda^2 t}} \exp \left[-\frac{\tau (x - y)^2}{4 \lambda^2 t}\right], \quad t > 0.
\]  

(7.4)

(See also [16] and references therein.)

Example 3  The fundamental solution of the Fokker-Planck equation [33], [43]:

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} + u
\]  

(7.5)

on \( \mathbb{R} \) is given by [37]:

\[
K_0 (x, y, t) = \frac{1}{\sqrt{2\pi} (1 - e^{-2t})} \exp \left[-\frac{(x - e^{-t}y)^2}{2 (1 - e^{-2t})}\right], \quad t > 0.
\]  

(7.6)

Here,

\[
\lim_{t \to \infty} K_0 (x, y, t) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad y = \text{constant}.
\]  

(7.7)

Example 4  Equation

\[
\frac{\partial u}{\partial t} = a \frac{\partial^2 u}{\partial x^2} + (g - kx) \frac{\partial u}{\partial x}, \quad a, k > 0, \quad g \geq 0
\]  

(7.8)

corresponds to the heat equation with linear drift when \( g = 0 \) [28]. In stochastic differential equations this equation corresponds the Kolmogorov forward equation for the regular Ornstein–Uhlenbech process [10]. The fundamental solution is given by

\[
K_0 (x, y, t) = \frac{\sqrt{k e^{kt/2}}}{\sqrt{4\pi a \sinh (kt)}} \exp \left[-\frac{\left(k \left(xe^{-kt/2} - ye^{kt/2}\right) + 2g \sinh (kt/2)\right)^2}{4ak \sinh (kt)}\right], \quad t > 0.
\]  

(7.9)

(See Refs. [10] and [37] for more details.)

Example 5  The viscous Burgers equation [2], [3], [19], [23], [41]:

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = a \frac{\partial^2 v}{\partial x^2}, \quad a = \text{constant} > 0
\]  

(7.10)

can be linearized by the Cole–Hopf substitution [5], [17]:

\[
v = -\frac{2a}{u} \frac{\partial u}{\partial x}.
\]  

(7.11)
which turns it into the diffusion equation (7.1). Solution of the initial value problem has the form:

\[ v(x, t) = -\frac{a}{\sqrt{\pi at}} \frac{\partial}{\partial x} \ln \left[ \int_{-\infty}^{\infty} \exp \left( -\frac{(x - y)^2}{4at} - \frac{1}{2a} \int_0^y v(z, 0) \, dz \right) \, dy \right] \]  

(7.12)

for \( t > 0 \) and suitable initial data on \( \mathbb{R} \).

**Example 6** Equation (7.10) possesses a solution of the form:

\[ v = F(x + V t), \quad V = \text{constant} \]  

(7.13)

(we follow the original Bateman paper [2] with slightly different notations), if

\[ VF' + FF'' = aF'' \]  

(7.14)

or

\[ (F + V)^2 \pm A^2 = 2aF' \]  

(7.15)

where \( A \) is a positive constant. The solution is thus either

\[ v + V = A \tan \left( \frac{A(x + V t - c)}{2a} \right) \]  

(7.16)

or

\[ \frac{A - v - V}{A + v + V} = \exp \left[ \frac{A}{a} (x + V t - c) \right], \]  

(7.17)

according as the + or − sign is taken. In the first case there is no definite value of \( v \) when \( a \) tends to zero, while in the second case the limiting value of \( v \) is either \( A - V \) or \( A + V \) according as \( x + V t \) is less or greater than \( c \). The limiting form of the solution is thus discontinuous [2].

Further examples can be found in Refs. [10], [23], [26], [28] and [37].

8. **Conclusion**

In this note, we have discussed connections of certain nonautonomous and inhomogeneous diffusion-type equation and Burgers equation with solutions of the Riccati and Ermakov-type systems that seem are missing in the available literature. Traveling wave solutions of the Burgers-type equations are also discussed.

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