Symmetric Unique Neighbor Expanders and Good LDPC Codes

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Abstract
An infinite family of bounded-degree 'unique-neighbor' expanders was constructed explicitly by Alon and Capalbo (2002). We present an infinite family $F$ of bounded-degree unique-neighbor expanders with the additional property that every graph in the family $F$ is a Cayley graph. This answers a question raised by Tali Kaufman. Using the same methods, we show that the symmetric LDPC codes constructed by Kaufman and Lubotzky (2012) are in fact symmetric under a simply transitive group action on coordinates.

Keywords: Unique neighbor expander, Error correcting code, Cayley graph

1. Introduction

An undirected graph $\Gamma = (V, E)$ is an $(\alpha, \epsilon)$-unique-neighbor expander if for every subset $X$ of $V$ such that $|X| \leq \alpha |V|$, there are at least $\epsilon |X|$ vertices in $V \setminus X$ that are adjacent to exactly one vertex in $X$. In [1], Alon and Capalbo construct families of bounded-degree $(\alpha, \epsilon)$-unique-neighbor expanders for some positive $\alpha, \epsilon$. One of the families constructed in [1] is an infinite family $F$ of 6-regular $(\alpha, \epsilon)$-unique-neighbor expanders. This construction is based on a notion of a product of graphs, forming a product graph $\Gamma$ from a $d$-regular graph $\Gamma'$ and a graph $\Delta$ on $d$ vertices. It is shown in [1] that if $\Gamma'$ is an 8-regular Ramanujan graph, and if $\Delta$ is a certain specific graph on 8 vertices, then the product graph $\Gamma$ is an $(\alpha, \epsilon)$-unique-neighbor expander for some positive absolute constants $\alpha, \epsilon$.

In response to a question of Tali Kaufman [2], who asked if there is an infinite family of $(\alpha, \epsilon)$-unique-neighbor expanders which are Cayley graphs, we develop a similar graph product in the context of Cayley graphs. The product of a graph $\Gamma'$ and a graph $\Delta$ is in general not symmetric, even if both graphs are Cayley graphs. However, we show that if $\Gamma'$ and $\Delta$ are Cayley graphs (with respect to the groups $G$ and $H$ respectively), and if further $\Gamma'$ is bipartite and simply-generator-symmetric with respect to the group $H$ (i.e. $H$ acts simply transitively on the generators of $G$ through group automorphisms of $G$), then the product graph $\Gamma$ can be formed 'symmetrically' and is then itself a Cayley graph. The graph $\Delta$ in [1] is indeed a Cayley graph on the cyclic group $C_8$. Fortunately, there are 8-regular bipartite Ramanujan graphs which are simply-generator-symmetric with respect to the group $C_8$. Such graphs were constructed explicitly by Lubotzky, Samuels and Vishne in [3], as a special case of the construction of Ramanujan complexes. Thus,

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we show that the method of [1], applied in a symmetric manner to this infinite family of Ramanujan graphs from [3], gives rise to an infinite family of unique-neighbor expanders which are Cayley graphs. This gives an affirmative answer to Kaufman’s question. We conclude:

**Theorem 1.1.** For some absolute positive constants $\alpha, \epsilon$, there is an explicit construction of an infinite family of $(\alpha, \epsilon)$-unique-neighbor expanders, such that every graph in the family is a Cayley graph.

Our method has another application: In [4], Kaufman and Lubotzky construct asymptotically good families of symmetric LDPC error correcting codes. A code $C \subset \mathbb{F}_2^X$ is symmetric with respect to a group $G$ if there is a transitive group action of $G$ on $X$, such that the corresponding coordinate-interchanging action on $\mathbb{F}_2^X$ preserves $C$. We say that a symmetric code is simply-symmetric if the action of $G$ on $X$ is simply transitive. The codes in [4] are based on a product of a large graph $\Gamma'$ and a small code $B$. This construction is due to Tanner ([5]) and Sipser-Spielman ([6]), and is referred to as an “expander code” in case the graph $\Gamma'$ is a good expander. In case the graph $\Gamma'$ is a Cayley graph, a more specific construction is defined in [7] by Kaufman and Wigderson, and is called a “Cayley code”. While Cayley codes are not necessarily symmetric, Kaufman and Wigderson construct a family symmetric Cayley codes with constant rate, but with normalized distance $\Omega\left(\frac{1}{(\log \log n)^2}\right)$, were $n$ is the code length (see Theorem 11 of [7]). In [4], using the Ramanujan graphs of [3], asymptotically good symmetric Cayley codes are constructed. The construction uses an infinite family of bipartite Ramanujan graphs, but the bipartiteness is not used in the proof (and similar constructions using non-bipartite generator-symmetric expanders are possible). We use the bipartiteness of these Ramanujan graphs, together with the fact that they are simply-generator-symmetric, to show that the codes of [4] are in fact simply symmetric. In addition to this extra feature, this construction has the benefit of exhibiting symmetric unique-neighbor expanders and symmetric error correcting codes in one framework. We conclude:

**Theorem 1.2.** There is an infinite family of asymptotically good simply-symmetric LDPC error correcting codes.

Finally, we define a variation of the codes of [4] which improves the bound on their density from 4094 to 20. This is achieved by a slight variation on the argument of Sipser-Spielman [6].

2. Edge Transitive Ramanujan Graphs

Let $\Gamma = \text{Cay} (G, S)$ be a Cayley graph where $S$ is a subset of the group $G$ satisfying $S = S^{-1}$. As a Cayley graph, $\Gamma$ is automatically vertex-transitive. We say that $\Gamma$ is generator-symmetric with respect to the group $H$ and the group homomorphism $\theta : H \to \text{Aut} (G)$ if $\theta (H)$ preserves $S$ and induces a transitive action of $H$ on $S$. In this case, the group $G \rtimes_\theta H$ acts transitively on the edges of $\Gamma$, and so $\Gamma$ is an edge-transitive graph. If the action of $H$ on $S$ is simply-transitive (in other words, if the action is both transitive and free), then we say that the Cayley graph $\Gamma$ is simply-generator-symmetric. This means, in particular, that $|H| = |S|$.

In this case, the action of $G \rtimes_\theta H$ on the directed edges of $\Gamma$ is simply transitive. However, some elements of $G \rtimes_\theta H$ act on $\Gamma$ by "reversing an edge". Therefore, if one regards $\Gamma$ as an undirected graph, the action
of $G \rtimes \theta H$ on edges, while still transitive, is no longer free. Our construction in section 3 is based on the observation that if $\Gamma$ is a bipartite simply-generator-symmetric Cayley graph, then there is a subgroup of $G \rtimes \theta H$ of index 2 which acts simply transitively on the undirected edges of $\Gamma$. For notational convenience, for every $h \in H$, we denote the automorphism $\theta (h) \in \text{Aut} (G)$ by $\theta_h$.

In [3], Lubotzky, Samuels and Vishne construct highly symmetric Ramanujan complexes, which we now specialize to the one dimensional case (i.e. graphs). We get two infinite families of $d$-regular Ramanujan graphs for each $d = q + 1$, where $q$ is an odd prime power. One is a family of bipartite graphs and the other is a family of non-bipartite graphs. All of these graphs are Cayley graphs and are simply-generator-symmetric with respect to the cyclic group $H = C_d$. For a brief overview of this special case see section 4 of [4].

Specializing further to the case of bipartite $8$-regular graphs, we get from [3] an infinite family of simply-generator-symmetric bipartite Ramanujan Cayley graphs $\{\Gamma' \}$ with $\Gamma' = \text{Cay} (\text{PGL}_2 (7^n), S(n))$ for some set of generators $S(n) \subset \text{PGL}_2 (7^n)$ defined in [3]. The group $\text{PSL}_2 (7^n)$ is of index 2 in $\text{PGL}_2 (7^n)$, and is disjoint from $S(n)$. As shown in [3], there is a group homomorphism $\theta_n : C_8 \to \text{Aut} (\text{PGL}_2 (7^n))$, inducing a simply-transitive action of $C_8$ on $S(n)$. For an explicit definition of the generators $S(n)$ and the action of $C_8$, see [3], Algorithm 9.2. In particular, Step 3 of this algorithm uses an embedding $C_8 \cong \mathbb{F}_2^* / \mathbb{F}_2^* \hookrightarrow \text{PGL}_2 (7^n)$ to define the generators $S(n)$ as the $C_8$-orbit of a certain element $b \in \text{PGL}_2 (7^n)$, where $C_8$ acts on $\text{PGL}_2 (7^n)$ by inner automorphisms through this embedding. Thus, the corresponding homomorphism $\theta_n : C_8 \to \text{Aut} (\text{PGL}_2 (7^n))$ is as required.

3. Line Graphs as Cayley Graphs

For a graph $\Gamma'=(V,E)$, the line graph $\Gamma$ of $\Gamma'$ is the graph whose vertex set is $E$, with vertices $e_1, e_2 \in E$ of $\Gamma$ adjacent in $\Gamma$ if and only if $e_1$ and $e_2$, as edges of $\Gamma'$, are incident to a common vertex of $\Gamma'$. In general, the line graph of a Cayley graph is not itself a Cayley graph. In this section, we show that the line graph of a simply-generator-symmetric bipartite Cayley graph is itself a Cayley graph. Let $G$ be a finite group and let $S \subset G$ be a symmetric subset of $G$ (i.e. $S = S^{-1}$) which generates $G$. Let $K$ be a subgroup of $G$ of index 2 such that $K$ and $S$ are disjoint. Fix a generator $s_0 \in S$. Then, the Cayley graph $\text{Cay} (G, S)$ is bipartite with the cosets $K$ and $Ks_0$ as its left and right sides respectively. The coset $K$ consists of the vertices connected to $1_G$ by a path of even length, and the coset $Ks_0$ consists of the vertices connected to $1_G$ by a path of odd length. Note that for a Cayley graph $\text{Cay} (G, S)$, the existence of an index 2 subgroup $K \leq G$ disjoint from $S$ is in fact equivalent to $\text{Cay} (G, S)$ being bipartite. Assume further that $\text{Cay} (G, S)$ is simply-generator-symmetric with respect to the group $H$ and the group homomorphism $\theta : H \to \text{Aut} (G)$. Then, for each element $h \in H$, we have $\theta_h (K) = K$. Therefore, we can define the semidirect product $K \rtimes_\theta H$. Finally, we let $T$ be a subset of $S$.

In this setting, we define two graphs $X$ and $\Gamma$, and show that they are isomorphic.
First, for every \( h \in H \), we define \( \sigma_1(h) \), \( \sigma_2(h) \in K \rtimes_\theta H \) by

\[
\sigma_1(h) = (1_K, h) \\
\sigma_2(h) = (s_0 \cdot \theta_h(s_0^{-1}), h)
\]

and we let \( X = \text{Cay}(K \rtimes_\theta H, \Sigma T_1 \cup \Sigma T_2) \), where

\[
\Sigma T_1 = \{ \sigma_1(t) \mid t \in T \} \\
\Sigma T_2 = \{ \sigma_2(t) \mid t \in T \}
\]

Second, we define a graph \( \Gamma = \Gamma(G, S, s_0, K, H, \theta, T) \). The vertices of \( \Gamma \) are the undirected edges of \( \text{Cay}(G, S) \), i.e. \( V(\Gamma) = E(\text{Cay}(G, S)) \). For each \( g \in G \), we let \( E_g \) be the set of undirected edges in \( \text{Cay}(G, S) \) which are incident to \( g \), and we define a bijection \( e_g : H \to E_g \) by

\[
e_g(h) = \begin{cases} 
(g, g \cdot \theta_h(s_0)) & g \in K \\
(g \cdot \theta_h(s_0^{-1}), g) & g \in Ks_0
\end{cases}
\]

The edges of \( \Gamma \) are \( E(\Gamma) = \{(e_g(h), e_g(ht)) \mid g \in G, h \in H, t \in T\} \).

Note that if \( T = T^{-1} \), then \( \Gamma \) is an undirected graph, and if \( T = S \), then \( \Gamma \) is the line graph of \( \text{Cay}(G, S) \).

**Proposition 3.1.** The bijection \( f : K \rtimes_\theta H \to E(\text{Cay}(G, S)) \) defined by

\[
f((k, h)) = (k, k \cdot \theta_h(s_0))
\]

is a graph isomorphism from \( X = \text{Cay}(K \rtimes_\theta H, \Sigma T_1 \cup \Sigma T_2) \) to \( \Gamma = \Gamma(G, S, s_0, K, H, \theta, T) \).

**Proof.** Let \( k \in K, h \in H \) and \( t \in T \). Direct computation shows that

\[
((k, h), (k, h) \cdot \sigma_1(t)) \xrightarrow{f \times f} (e_k(h), e_k(ht)) \\
((k, h), (k, h) \cdot \sigma_2(t)) \xrightarrow{f \times f} (e_{k \cdot \theta_h(s_0)}(h), e_{k \cdot \theta_h(s_0)}(ht))
\]

and

\[
(e_k(h), e_k(ht)) \xrightarrow{f^{-1} \times f^{-1}} ((k, h), (k, h) \cdot \sigma_1(t)) \\
(e_{k \cdot \theta_h(s_0)}(h), e_{k \cdot \theta_h(s_0)}(ht)) \xrightarrow{f^{-1} \times f^{-1}} ((k \cdot \theta_h(s_0^{-1}), h), (k \cdot \theta_h(s_0^{-1}), h) \cdot \sigma_2(t))
\]

Therefore, both \( f \) and \( f^{-1} \) are graph homomorphisms, and therefore \( f \) is a graph isomorphism. \( \square \)

In particular, Proposition 3.1 implies that the line graph of a generator symmetric bipartite Cayley graph is itself a Cayley graph. It may be interesting to characterize the Cayley graphs whose line graph is a Cayley graph.
4. Symmetric Unique Neighbor Expanders

We begin by describing the construction of an infinite family of 6-regular unique-neighbor expanders by Alon and Capalbo (see [1]). Let $\Gamma' = (V', E')$ be a $d$-regular graph and let $\Delta = (V(\Delta), E(\Delta))$ be a graph on $d$ vertices. For each vertex $v$ of $\Gamma'$, denote by $E'_v$ the set of $d$ edges of $\Gamma'$ which are incident to $v$. For each vertex $v$ of $\Gamma'$, fix a bijection between $E'_v$ and $V(\Delta)$. Note that these bijections do not need to be compatible with each other in any way. Form a new graph $\Gamma$ whose vertex set is $E'$, and where $\gamma_1, \gamma_2 \in E'$ are adjacent as vertices of $\Gamma$ if and only if $\gamma_1$ and $\gamma_2$ are both incident to a common vertex $v$ in $\Gamma'$, and are neighbors in $\Delta$ under the identification $E'_v \leftrightarrow V(\Delta)$. Note that the graph $\Gamma$ is a subgraph of the line graph of $\Gamma'$. By Theorem 2.1 of [1], if $\Gamma'$ is an $8$-regular Ramanujan graph, and if $\Delta$ is the $3$-regular graph on $8$ vertices $v_0, \ldots, v_7$ with $v_i$ adjacent to $v_{i-1}, v_{i+1}, v_{i+4}$ (indices taken modulo $8$), then $\Gamma$ is a $6$-regular $(\alpha, 1/10)$-unique-neighbor expander, for some positive constant $\alpha$.

In this way, an infinite family of (not necessarily bipartite) $8$-regular Ramanujan graphs gives rise to an infinite family of $6$-regular $(\alpha, 1/10)$-unique-neighbor expanders. In [1], the chosen family of Ramanujan graphs is the one constructed in [8] by Lubotzky, Phillips and Sarnak. These Ramanujan graphs are Cayley graphs and thus are vertex-transitive, but they are not known to be generator-symmetric.

Instead, we use the $8$-regular simply-generator-symmetric bipartite Ramanujan graphs of [3]. Keeping the notation of sections 2 and 3, we define for each $n \geq 1$,

$$\Gamma_n = \Gamma (\text{PGL}_2(7^n), S(n), b, \text{PSL}_2(7^n), C_8, \theta_n, \{1, 4, 7\})$$

On one hand, the graphs $\{\Gamma_n\}$ are a special case of the Alon-Capalbo construction. On the other hand, by Proposition 3.1, for each $n \geq 1$, the graph $\Gamma_n$ is isomorphic to $\text{Cay}(\text{PSL}_2(7^n) \rtimes_{\theta_n} C_8, \Sigma_1^{(1,4,7)} \cup \Sigma_2^{(1,4,7)})$.

We have thus proved the following more elaborate version of Theorem 1.1:

**Theorem 4.1.** For some constant $\alpha > 0$, there is an infinite family $\{\Gamma_n\}$ of $6$-regular $(\alpha, 1/10)$-unique-neighbor expanders, such that for every $n$, the graph $\Gamma_n$ is a Cayley graph on the group $\text{PSL}_2(7^n) \rtimes_{\theta_n} C_8$.

5. Symmetric Good LDPC Codes

We shall refer to linear error correcting codes simply as 'codes'. A code $C \subset \mathbb{F}_2^X$ is symmetric (resp. simply-symmetric) with respect to a group $G$ if there is a transitive (resp. simply-transitive) action of $G$ on $X$ such that the corresponding coordinate-interchanging action of $G$ on $\mathbb{F}_2^X$ preserves $C$. For a code $C \subset \mathbb{F}_2^X$, we denote its dual by $C^\perp \subset \mathbb{F}_2^X$ (this is the set of all vectors “orthogonal” to $C$), and think of it as the constraints defining the code $C$ (i.e. they define linear functionals whose common set of solutions is $C$). A spanning set for $C^\perp$ is called a set of defining constraints. By a standard abuse of terminology, we will refer to a code $C$ together with a specific set of defining constraints simply as a 'code'. Note that if $C$ is a symmetric code with respect to a group $G$, then the dual code $C^\perp$ is also symmetric with respect to
G. Finally, a family of codes is LDPC if the defining constraints of the codes in the family are of bounded Hamming weight.

One way to obtain symmetric codes is by 'codes defined on groups': For a group G, we say that a code \( C \subseteq \mathbb{F}_2^G \) is a code defined on G if C is invariant under the action of G. Equivalently, a code \( C \subseteq \mathbb{F}_2^G \) is a code defined on G if it is defined by a G-invariant set of constraints. Finally, a code C is defined on the group G if and only if it is simply-symmetric with respect to G.

Another way to obtain (usually not symmetric) codes is by 'codes defined on graphs': Let \( \Gamma = (V, E) \) be an l-regular graph and let \( B \subseteq \mathbb{F}_2^H \) be a 'small' code of length l (i.e. \( H \) is a set of cardinality l). For each vertex \( v \) of \( \Gamma \), fix a bijection \( h \mapsto e(v, h) \) from \( H \) to \( E_v \). Let \( C \subseteq \mathbb{F}_2^E \) be the code consisting of functions \( f : E \rightarrow \mathbb{F}_2 \) satisfying the following local constraint for each vertex \( v \) of \( \Gamma \):

\[
(f(e(v, h)))_{h \in H} \in B
\]

These local constraints are referred to as 'vertex consistency'. This idea is due to Tanner ([5]) and Sipser-Spielman ([6]), who refer to such codes (and similar codes) as expander codes in case the graph \( \Gamma \) is a good expander. If the graph \( \Gamma \) is a Cayley graph \( \Gamma = \text{Cay}(G, S) \) then, instead of arbitrarily fixing a bijection between \( H \) and \( E_v \) for every vertex \( v \) of \( \Gamma \), we can fix one bijection \( h \mapsto s_h \) from \( H \) to \( S \). Then, for each vertex \( v \) of \( \Gamma \), we define the bijection from \( H \) to \( E_v \) by \( e(v, h) = (v, v \cdot s_h) \) (where \( \cdot \) is multiplication in the group \( G \)), and proceed as before to define the code C. Codes defined on Cayley graphs in this manner are referred to as Cayley codes by Kaufman and Wigderson in [7] and are denoted by \( \text{Cay}(G, S, B) \).

As stated in [4], Cayley codes are in general not symmetric. Therefore, we assume further that \( H \) is a group (and not merely a set), that the code \( B \) is a code defined on \( H \), and that the Cayley graph \( \text{Cay}(G, S) \) is simply-generator-symmetric with respect to the group \( H \). Note that in this case we have \( |H| = |S| \). Let \( \theta : H \rightarrow \text{Aut}(G) \) be the group homomorphism inducing the transitive action of \( H \) on \( S \). Now, instead of arbitrarily fixing a bijection between \( H \) and \( S \), we only fix one generator \( s_0 \in S \). We define the bijection from \( H \) to \( S \) by \( h \mapsto \theta_h(s_0) \), and proceed as before to define the code C. This idea is due to Kaufman and Lubotzky and is presented in [4]. The resulting code C is symmetric with respect to the group \( G \rtimes \theta H \). However, the action of this group on the code C is not simply transitive (indeed, the cardinality of the group \( G \rtimes \theta H \) is not equal to the length of the code C; the size of the group is twice the length of the code). Therefore, C is not a code defined on the group \( G \rtimes \theta H \).

However, if the graph \( \text{Cay}(G, S) \) is bipartite, then there is an index 2 subgroup \( K \) of \( G \), disjoint from \( S \). Then, by Section [3] the code C is a code defined on \( K \rtimes \theta H \) (and not just a symmetric code with respect to \( G \rtimes \theta H \), as in [4]). If the small code \( B \) is defined by \( m H \)-orbits of constraints, then the code C is defined by \( 2m K \rtimes \theta H \)-orbits of constraints (or \( m G \rtimes \theta H \)-orbits of constraints, as in [4]). We summarize this result in the case where the small code \( B \) is defined by a single orbit of constraints (since this is sufficient for our application):

**Proposition 5.1.** For \( G, S, s_0, K, H, \theta \) as in section [3] and a 'small' code \( B \) defined on the group \( H \) by the
orbit of the single constraint \( \sum_{t \in T} f(t) = 0 \) for some \( T \subset H \), the Cayley code \( \text{Cay}(G, S, B) \) is the same as the code defined on the group \( K \rtimes H \) with defining constraints consisting of the \( K \rtimes H \)-orbits of the two constraints \( \sum_{t \in T} f((1, t)) = 0 \) and \( \sum_{t \in T} f((s_0 \cdot \theta_t(s_0^{-1}), t)) = 0 \).

Finally, note that in [4] the Cayley code construction is applied with a sequence \( \{\Gamma'_n\} \) of \( q + 1 \)-regular Ramanujan graphs from [3], where \( q = 4093 \), and a certain code \( B_0 \) defined on \( C_{q+1} \). Each graph \( \Gamma'_n \) in the sequence is a Cayley graph on \( \text{PGL}_2(q^n) \) and is simply-generator-symmetric with respect to \( C_{q+1} \). These graphs are in fact bipartite, although the proofs in [4] do not require bipartiteness. Therefore, by Proposition 5.1, we conclude that the codes constructed in [4] are in fact simply-symmetric. These codes are proved in [4] to have both rate and normalized distance bounded away from zero (i.e. they are asymptotically good codes). Also, they clearly are LDPC codes. We have thus proved the following more elaborate version of Theorem 1.2:

**Theorem 5.2.** There is an asymptotically good infinite family of LDPC codes \( \{K_n\} \), such that for every \( n \), the code \( K_n \) is simply-symmetric with respect to the group \( G_n = \text{PSL}_2(q^n) \rtimes \theta_n C_{q+1} \), where \( q = 4093 \). Furthermore, for every \( n \), the code \( K_n \) is defined by constraints consisting of two \( G_n \)-orbits of constraints.

### 6. Symmetric Good LDPC Codes with Improved Density

If \( C \) is a code (together with a set of defining constraints), then we define the density of \( C \) as the maximal Hamming weight of a defining constraint of \( C \). The density of the family of codes \( \{K_n\} \) is proved in [4] to have density bounded from above by 4094. We shall define a variation of this construction, with density 20. The density of an expander code on a graph \( \Gamma \) and a small code \( B \) equals the density of the small code \( B \).

Thus, our goal is to define an infinite family of simply-symmetric expander codes using a small code with density 20. We shall define later a code \( B' \) of length 158, rate \( \frac{40}{79} \), distance 15 and density 20. Let \( \{K'_n\} \) be the family of expander codes constructed symmetrically using the 158-regular simply-generator-symmetric Ramanujan graphs of [3] and the small code \( B' \). The family \( \{K'_n\} \) is a family of simply-symmetric codes of density 20. We now show that this is a family of asymptotically good codes.

By Lemma 15 of [4], an expander code on a \( k \)-regular graph \( \Gamma \) and a small code \( B \) has rate at least \( 2 \cdot \text{rate}(B) - 1 \) and normalized distance at least \( \left( \frac{\text{distance}(B) - \lambda}{k-\lambda} \right)^2 \) where \( \lambda \) is the second largest eigenvalue of the adjacency matrix of \( \Gamma \) (assuming distance \( (B) > \lambda \)). Thus, since rate \( (B') = \frac{40}{79} > \frac{1}{2} \), we have a guarantee that the codes \( \{K'_n\} \) have rate bounded away from zero. However, since distance \( (B') = 15 \), and since \( 2\sqrt{157} \approx 25.1 \), the method of [4] is not enough to show that the codes \( \{K'_n\} \) have normalized distance bounded away from zero. Nevertheless, we do have \( 15 > 1 + \sqrt{157} \approx 13.5 \), and thus the next lemma shows that the codes \( \{K'_n\} \) do have distance bounded away from zero:

**Lemma 6.1.** An expander code \( C \) defined on a \( k \)-regular Ramanujan graph \( \Gamma \) using a 'small' code \( B \) of length \( k \) and distance \( d \) larger than \( 1 + \sqrt{k-1} \) has normalized distance larger than some constant \( \alpha \) which depends only on \( k \) and \( d \).
result of interleaving two codewords of $B$ be the cyclic code of length 158 = $2 \cdot d$ Hamming weight smaller than $\frac{\epsilon}{C \sqrt{1 + \sqrt{k - 1}}}$ > 0, and let $\alpha = k^{-1/\delta - 1} > 0$. Let $f : E \to \mathbb{F}_2$ be a nonzero word of Hamming weight $w$ such that $w \leq \alpha |E|$. Let $S$ be the set of edges $e$ of $\Gamma$ such that $f(e) = 1$. Let $T$ be the subset of $V$ of all vertices in $\Gamma$ which are incident to at least one edge in $S$. Let $\Gamma[T]$ be the subgraph of $\Gamma$ induced on $T$. Since $0 < |T| \leq 2 \cdot |S| \leq 2 \alpha \cdot |E| = \alpha \cdot k \cdot |V| = k^{-1/\delta} \cdot |V|$, the average degree in $\Gamma[T]$ is at most $(1 + \sqrt{k - 1}) \cdot (1 + C \cdot \delta) = (1 + \sqrt{k - 1}) \cdot \left(1 + \frac{\epsilon}{C \sqrt{1 + \sqrt{k - 1}}}ight) = 1 + \sqrt{k - 1} + \epsilon < d$. Therefore, $\Gamma[T]$ has a vertex $v$ of degree less than $d$. The local subword of $f$ around the vertex $v$ is of positive Hamming weight smaller than $d$ and thus violates a constraint. Therefore, $f$ is not a codeword of $C$.

The improvement in this lemma over Lemma 15 of [4] boils down to the invocation of Kahale’s Theorem 4.2 in [9] instead of the Alon-Chung Lemma ([10], Lemma 2.3). This is the same tool that enables the proof of unique-neighbor expansion in [4].

Thus, we get the following improved version of Theorem 5.2.

Theorem 6.2. There is an asymptotically good infinite family of LDPC codes $\{K'_n\}$, such that for every $n$, the code $K'_n$ is simply-symmetric with respect to the group $G'_n = PSL_2(q^n) \rtimes C_{q+1}$, where $q = 157$. Each code $K'_n$ has density 20, and is defined by constraints consisting of two $G'_n$-orbits of constraints.

Note that the improvement in density over the symmetric good LDPC codes of [4] comes on the expense of a lower guaranteed normalized distance.

We now define a code $B'$ with the desired properties. See Chapter 6 of [11] for the theory of cyclic codes. Let $B''$ be the cyclic code of length 79 generated by

$$g(X) = X^{39} + X^{36} + X^{35} + X^{31} + X^{30} + X^{29} + X^{27} + X^{26} + X^{25} + X^{24} + X^{21} + X^{20} + X^{19} + X^{18} + X^{16} + X^{14} + X^{13} + X^{11} + X^{5} + X^{4} + X^{2} + X + 1$$

By [12], the code $B''$ has rate $\frac{40}{79}$ and distance 15. Direct computation shows that $h(X) = (X^{79} - 1)/g(X)$ has exactly 20 nonzero coefficients. Thus, the density of the code $B''$ is 20 (see Section 6.2 of [11]). Let $B'$ be the cyclic code of length $158 = 2 \cdot 79$ generated by $g(X)^2$. Note that the codewords of $B'$ are exactly the result of interleaving two codewords of $B''$. The code $B'$ has rate $\frac{40}{79}$, distance 15 and density 20.

7. Conclusion

We make use of of the simple-generator-symmetry of the Ramanujan graphs of [3], and of our observation that the line graph of a simply-generator-symmetric bipartite Cayley graph is itself a Cayley graph. This observation is crucial in the realization of some of the unique-neighbor expanders of [1] as Cayley graphs. It also allows us to see that the asymptotically good symmetric codes of [4] are in fact simply-symmetric. Finally, we improve the density of these codes.
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