PROBABILISTIC CONTINUITY OF A PULLBACK RANDOM ATTRACTOR IN TIME-SAMPLE

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Abstract. Given a time-sample dependent attractor of a random dynamical system, we study its lower semi-continuity in probability along the time axis, and the criteria are established by using the local-sample asymptotically compactness for a triple-continuous system. The abstract results are applied to the non-autonomous stochastic p-Laplace equation on an unbounded domain with weakly dissipative nonlinearity. Without any additional hypotheses, we prove that the pullback random attractor is probabilistically continuous in both time and sample parameters.

1. Introduction. A core concept in the theory of non-autonomous random dynamical systems is the so-called pullback random attractor (PRA), which seems to be first involved in Crauel et al. [8] and systemically studied by Wang [39].

A PRA is formulated by a bi-parametric set \( \mathcal{A} = \{ \mathcal{A}(\tau, \omega) \} \), where \( \tau \in \mathbb{R} \) is the current time and \( \omega \) is the sample in a measurable dynamical system \( (\Omega, \mathcal{F}, P, \theta_t) \). Its properties for a fixed time-fiber had been researched, see [10, 11, 12, 29, 42, 45].

Recently, Cui, Kloeden and Wu [14] have studied the upper semi-continuity of a PRA in time-parameter:

\[
\lim_{\tau \to \tau_0} \text{dist}(\mathcal{A}(\tau, \omega), \mathcal{A}(\tau_0, \omega)) = 0, \ \forall \tau_0 \in \mathbb{R}, \ \omega \in \Omega.
\] (1)

In this paper, we consider a more challenging problem on the lower semi-continuity:

\[
\lim_{\tau \to \tau_0} P\{ \omega \in \Omega : \text{dist}(\mathcal{A}(\tau_0, \omega), \mathcal{A}(\tau, \omega)) \geq \delta \} = 0, \ \forall \delta > 0, \tau_0 \in \mathbb{R}.
\] (2)

In general, it is impossible to obtain the lower semi-continuity even if the upper semi-continuity holds true, many examples can be found in [19, 32, 33]. Only in the special case, such as, the number of equilibria is finite, the lower semi-continuity was
established in [2, 5, 15, 16, 21, 27], in these references, the target of investigation is a family of attractors \( \{ A_\alpha \} \) rather than a single attractor \( A \) as in (2).

It is hard to directly deal with the lower semi-continuity (2). We will convert the lower semi-continuity into an upper semi-continuity. In fact, we will prove in Theorem 2.4 that the lower semi-continuity (2) can be deduced from the upper semi-continuity in sample:

\[
\lim_{s \to s_0} \text{dist}(A(\tau, \theta, s, \omega), A(\tau, \theta, s_0, \omega)) = 0, \quad \forall s_0, \tau \in \mathbb{R}, \omega \in \Omega.
\] (3)

For an (autonomous) random attractor \( A = \{ A(\omega) \} \) ([4, 6, 7, 9, 36, 40]), both (1) and (2) are trivially true, however, (3) is given by \( \text{dist}(A(\theta, s, \omega), A(\theta, s_0, \omega)) \to 0 \), which is nontrivial. Maybe (3) is more meaningful than (2).

Next, our main task is to establish criteria for (3) (and thus for (2)). An essential criterion for (3) is local-sample compactness (Theorem 2.11), which means that \( \bigcup_{|s| \leq 1} A(\tau, \theta, s, \omega) \) is pre-compact for each \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \).

In this case, we need that the system (cocycle) is triple-continuous (Definition 2.10) in three variables: time, sample and space, where the continuity in sample seems to be a new idea in the literature. We also require that the attracted universe is local-sample closed (Definition 2.6).

Furthermore, we establish the criteria in terms of the cocycle. A main criterion is that the cocycle is local-sample limit-set compact, which means the usual limit-set compactness is uniform in local samples. Also, we show that it is equivalent to the local-sample asymptotic compactness, see Theorem 2.9. All abstract results are summarized in Theorem 2.12.

In the application part, we consider the \( p \)-Laplace equation perturbed by additive noise and generally dissipative nonlinearity on \( \mathbb{R}^m \):

\[
du + (\lambda u - \text{div}(|\nabla u|^{p-2} \nabla u) + f(x, u) - g(t, x))dt = h(x)dW,
\] (4)

where \( p > 2, \lambda > 0 \). The equation is said to be strongly dissipative if \( p \leq q \) and weakly dissipative if \( p > q \), where \( q > 2 \) such that \( q - 1 \) is the growth order of \( f \).

It seems to be open even for the existence of a random attractor in the weakly dissipative case, although the strongly dissipative case had been considered in [23, 24, 25, 26, 30, 37, 43, 44]. In a bounded domain, it is not necessary to distinguish the strong or weak dissipation, see [17, 18].

In this paper, the nonlinearity can be both strongly and weakly dissipative.

Now, let us describe technique and novelty how to prove both existence and probabilistic lower semi-continuity of a random \( \mathcal{D} \)-pullback attractor \( A \) in \( L^2(\mathbb{R}^m) \), where \( \mathcal{D} \) is the usual universe forming from all tempered sets.

In section 3, we mainly prove the solution operator is triple-continuous, especially in sample. So, the equation (4) generates a triple-continuous cocycle \( \Phi \).

In section 4, we take a local-sample closed sub-universe \( \mathcal{D}_0 \) of \( \mathcal{D} \). By an inductive method, we prove the existence of a \( \mathcal{D}_0 \)-pullback (local-sample) absorbing set in \( X_k = L^2 \cap L^{q_k} \), where \( q_k := k(q - 2) + 2 \). Let \( N = \{(p - 2)/(q - 2)\} \), then \( q_N \geq p \) and so the absorption can reach the \( L^p \)-level.

By the same inductive method, we can prove a triple-compactness result of the cocycle on the localization of space, time and sample, where the usual localization method [25, 26] can be generalized from one space-variable to three variables.

In section 5, we take a sequence of local-sample closed universes \( \mathcal{D}_k \) in \( X_k \). We then present the local-uniform tail-estimate when the initial data belong to \( \mathcal{D}_{N-1} \).
In section 6, we show our main application results. On one hand, we can prove that the cocycle is local-sample (and local-time) asymptotically compact on $D_{N-1}$ and then on $D_0$. So, we obtain a $D_0$-pullback attractor $A_0$, which is local-sample compact and local-time compact.

On the other hand, the usual method can show the existence of a $D$-pullback random attractor $A$. We then prove $A_0 = A$ and so $A_0$ is still measurable. Therefore, we obtain a $D$-pullback random attractor $A \in D_0$ such that $A$ is local-sample (and local-time) compact, and thus it satisfies (2), (3) and conveniently (1).

2. Abstract results on probabilistic lower semi-continuity.

2.1. A bi-parametric attractor for a cocycle. Let $X$ be a separable Banach space with the Borel algebra $\mathcal{B}(X)$. Let $(\Omega, \mathfrak{F}, P)$ be a probability space with a group $\{\theta_s\}_{s \in \mathbb{R}}$ such that each $\theta_s : \Omega \rightarrow \Omega$ is measure-preserving.

A measurable mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X \mapsto X$ is said to be a **cocycle** if

$$\Phi(0, \tau, \omega) = I_X, \quad \Phi(t + s, \tau, \omega) = \Phi(t, \tau + \theta_s \omega) \Phi(s, \tau, \omega)$$

for all $t, s \geq 0, \tau \in \mathbb{R}, \omega \in \Omega$. We always assume that $\Phi(t, \tau, \omega)x$ is continuous in $t \geq 0, \tau \in \mathbb{R}$ and $x \in X$ respectively.

We use the terminology of a **bi-parametric set** $B = \{B(\tau, \omega)\}$ to denote a set-valued map $B : \mathbb{R} \times \Omega \mapsto 2^X \setminus \{\emptyset\}$. A bi-parametric set $B(\tau, \omega)$ is called compact if each component $B(\tau, \omega)$ is compact.

Let $D$ be a universe of some bi-parametric sets in $X$.

**Definition 2.1.** $A \in D$ is called a **$D$-pullback bi-parametric attractor** for a cocycle $\Phi$ if it is compact, invariant, that is,

$$\Phi(t, \tau, \omega)A(\tau, \omega) = A(t + \tau, \theta_t \omega), \quad \forall t \geq 0,$$

and $D$-pullback **attracting**, that is, for each $D \in D$,

$$\lim_{t \to +\infty} \text{dist}(\Phi(t, \tau - t, \theta_{-t} \omega)D(\tau - t, \theta_{-t} \omega), A(\tau, \omega)) = 0.$$

A bi-parametric attractor $A$ is called a **$D$-pullback random attractor** (PRA) if it is measurable, which means that the mapping $\omega \rightarrow d(x, A(\tau, \omega))$ is $(\mathfrak{F}, \mathcal{B}(\mathbb{R}^+))$ measurable for each $x \in X$ and $\tau \in \mathbb{R}$.

For the details of a PRA, one can refer to Wang [39].

2.2. Probabilistic lower semi-continuity of a PRA. We denote the Hausdorff distance by

$$\text{dist}_h(A, B) = \max\{\text{dist}(A, B), \text{dist}(B, A)\}, \quad \forall A, B \subset X.$$

**Lemma 2.2.** Suppose a random attractor $A$ is upper semi-continuous in sample:

$$\lim_{s \to 0} \text{dist}(A(\tau, \theta_s \omega), A(\tau, \omega)) = 0, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega,$$

which is equivalent to (3). Then, we have the probabilistic continuity in sample, that is, for each $\delta > 0$,

$$\lim_{s_n \to s_0} P\{\omega \in \Omega : \text{dist}_h(A(\tau, \theta_{s_0} \omega), A(\tau, \theta_{s_n} \omega)) \geq \delta\} = 0, \quad \forall \tau, s_0 \in \mathbb{R}. \quad (7)$$
Lemma 2.3. \[41, \text{Theorem 2.3}\]

Proof. Since an almost everywhere convergent sequence is convergent in probability, (6) implies the probabilistic upper semi-continuity in sample:

\[\lim_{s_n \to s_0} P\{\omega \in \Omega : \text{dist}(A(\tau, \theta_{s_n}\omega), A(\tau, \theta_{s_n}\omega)) \geq \delta\} = 0, \ \forall \delta > 0, \tau \in \mathbb{R}. \quad (8)\]

Notice that each \(\theta_{s_n} : \Omega \to \Omega\) is measure-preserving, we have

\[P\{\omega \in \Omega : \text{dist}(A(\tau, \theta_{s_n}\omega), A(\tau, \theta_{s_n}\omega)) \geq \delta\} = P\{\theta_{s_n}\omega \in \Omega : \text{dist}(A(\tau, \theta_{s_n}\omega), A(\tau, \theta_{s_n}\omega)) \geq \delta\} = P\{\omega \in \Omega : \text{dist}(A(\tau, \theta_{s_0-s_n}\omega), A(\tau, \omega)) \geq \delta\} \quad (9)\]

By (8), the last term in (9) tends to zero as \(s_0 - s_n \to 0\). Therefore, both (8) and (9) imply the probabilistic continuity (7). \(\square\)

We also need a type of (automatic) continuity for a PRA.

Lemma 2.3. \[41, \text{Theorem 2.3}\]. Suppose a cocycle has a PRA \(A\), then

\[\lim_{\tau \to \tau_0} \text{dist}_h(A(\tau_0, \theta_{\tau_0}\omega), A(\tau, \theta_{\tau_0}\omega)) = 0, \ \forall \omega \in \Omega. \quad (10)\]

Now, we convert the lower semi-continuity (2) into the upper semi-continuity (6).

Theorem 2.4. Let a cocycle \(\Phi\) have a pullback random attractor \(A\).

(i) If \(A\) is upper semi-continuous in sample (i.e. (6) holds true), then \(A\) is probabilistic lower semi-continuous in time:

\[\lim_{\tau \to \tau_0} P\{\omega \in \Omega : \text{dist}(A(\tau_0, \omega), A(\tau, \omega)) \geq \delta\} = 0, \ \forall \delta > 0, \tau_0 \in \mathbb{R}. \quad (11)\]

(ii) Conversely, the lower semi-continuity (11) implies the probabilistic continuity in sample (i.e. (7) holds true).

Proof. (i) It suffices to prove that (11) holds true for any sequence \(\{\tau_n\}\) such that \(\tau_n \to \tau_0\). By the triangle inequality of Hausdorff semi-distance, for \(P\)-a.s. \(\omega \in \Omega\),

\[\text{dist}(A(\tau_0, \omega), A(\tau_n, \omega)) \leq \text{dist}(A(\tau_0, \omega), A(\tau_0, \theta_{\tau_0-\tau_n}\omega)) + \text{dist}(A(\tau_0, \theta_{\tau_0-\tau_n}\omega), A(\tau_n, \omega)).\]

By the sub-additivity of the probability, we have, for each \(\delta > 0\),

\[P\{\omega \in \Omega : \text{dist}(A(\tau_0, \omega), A(\tau_n, \omega)) \geq \delta\} \leq P\{\omega \in \Omega : \text{dist}(A(\tau_0, \omega), A(\tau_0, \theta_{\tau_0-\tau_n}\omega)) \geq \delta/2\} \]

\[+ P\{\omega \in \Omega : \text{dist}(A(\tau_0, \theta_{\tau_0-\tau_n}\omega), A(\tau_n, \omega)) \geq \delta/2\}. \quad (12)\]

Since \(A\) is upper semi-continuous in sample and \(\tau_0 - \tau_n \to 0\), it follows from Lemma 2.2 that

\[\lim_{n \to \infty} P\{\omega \in \Omega : \text{dist}(A(\tau_0, \omega), A(\tau_0, \theta_{\tau_0-\tau_n}\omega)) \geq \delta/2\} = 0. \quad (13)\]

On the other hand, by (10) in Lemma 2.3, we have

\[\text{dist}(A(\tau_0, \theta_{\tau_0}\omega), A(\tau_n, \theta_{\tau_n}\omega)) \to 0, \ \text{for a.e. } \omega \in \Omega.\]
Hence, the sequence \( \{ \text{dist}(A(\tau_0, \theta_{\tau_0}), A(\tau_n, \theta_{\tau_n})) \}_n \) of random variables is convergent in probability. Furthermore, by using the measure-preserving property of each \( \theta_{\tau_n} : \Omega \to \Omega \), we have

\[
P\{ \omega \in \Omega : \text{dist}(A(\tau_0, \theta_{\tau_0} \omega), A(\tau_n, \omega)) \geq \frac{\delta}{2} \} \leq P\{ \theta_{\tau_n} \omega \in \Omega : \text{dist}(A(\tau_0, \theta_{\tau_0} \omega), A(\tau_n, \omega)) \geq \frac{\delta}{2} \} \]

\[
= P\{ \omega \in \Omega : \text{dist}(A(\tau_0, \theta_{\tau_n} \omega), A(\tau_n, \theta_{\tau_n} \omega)) \geq \frac{\delta}{2} \} \to 0
\]

as \( n \to \infty \). Now, we substitute (13) and (14) into (12) to obtain

\[
P\{ \omega \in \Omega : \text{dist}(A(\tau_0, \omega), A(\tau_n, \omega)) \geq \frac{\delta}{2} \} \to 0
\]

as \( n \to \infty \), which proves (11) for any sequence \( \{ \tau_n \} \) with \( \tau_n \to \tau_0 \).

(ii) Suppose (11) is true. Then, for \( s_n \to 0 \) and \( \tau \in \mathbb{R} \),

\[
P\{ \omega \in \Omega : \text{dist}(A(\tau, \omega), A(\tau - s_n, \omega)) \geq \frac{\delta}{2} \} \to 0.
\]

Since \( \tau - s_n \to \tau \), it follows from (10) in Lemma 2.3 that

\[
\text{dist}(A(\tau - s_n, \theta_{\tau - s_n} \omega), A(\tau, \theta_\tau \omega)) \to 0, 
\]

which holds true in probability. Noting that \( \theta_{s_n - \tau} \) is measure-preserving, we have

\[
P\{ \omega \in \Omega : \text{dist}(A(\tau - s_n, \omega), A(\tau, \theta_{s_n} \omega)) \geq \frac{\delta}{2} \} \leq P\{ \omega \in \Omega : \text{dist}(A(\tau, \omega), A(\tau - s_n, \omega)) \geq \frac{\delta}{2} \}
\]

\[
+ P\{ \omega \in \Omega : \text{dist}(A(\tau - s_n, \omega), A(\tau, \theta_{s_n} \omega)) \geq \frac{\delta}{2} \} \to 0
\]

as \( n \to \infty \). Since \( \theta_{s_n} \) is measure-preserving, it further implies that

\[
P\{ \omega \in \Omega : \text{dist}(A(\tau, \theta_{s_n} \omega), A(\tau, \omega)) \geq \delta \} \to 0
\]

as \( n \to \infty \). We obtain the probabilistic continuity in sample, i.e. (7) is true. \( \square \)

2.3. **Local-sample compactness of a bi-parametric attractor**. In this subsection, we do not require the measurability of the attractor.

**Definition 2.5.** A bi-parametric attractor \( A \) is called **local-sample compact** if the local-sample union \( A_{LS} \) is compact in \( X \), where

\[
A_{LS}(\tau, \omega) := \bigcup_{|s| \leq 1} A(\tau, \theta_s \omega), \quad \forall \tau \in \mathbb{R}, \omega \in \Omega.
\]

A universe \( D \) is **inclusion-closed** [34] if \( \bar{D} \in D \) whenever \( D \in D \) and \( \bar{D} \subset D \). This condition ensures \( A \in D \). We also need a special property of \( D \).

**Definition 2.6.** A universe \( D \) is called **local-sample closed** if \( D_{LS} \in D \) whenever \( D \in D \), where \( D_{LS} \) is the local-sample union of \( D \), as defined by (15).

We then generalize the concepts of limit-set compactness [22] and asymptotic compactness [3] to the local-sample version.
Definition 2.8. A cocycle $\Phi$ is called local-sample limit-set compact on $\mathcal{D}$ if
\[
\lim_{T \to +\infty} \kappa_X \left( \bigcup_{t \geq T} \bigcup_{|s| \leq 1} \Phi(t, \tau - t, \theta_{s-t}\omega) D(\tau - t, \theta_{-t}\omega) \right) = 0
\]
for each $D \in \mathcal{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, where, the Kuratowski measure $\kappa_X(\cdot)$ is defined by the minimal one among diameters of sets forming a finite cover.

Definition 2.8. A cocycle $\Phi$ is called local-sample asymptotically compact on $\mathcal{D}$ if, for each $(\tau, \omega, D) \in \mathbb{R} \times \Omega \times \mathcal{D}$, the sequence
\[
\{ \Phi(t_n, \tau - t_n, \theta_{s_n-t_n}\omega)x_n \}_{n}
\]
is pre-compact whenever $|s_n| \leq 1$, $t_n \to +\infty$ and $x_n \in D(\tau - t_n, \theta_{-t_n}\omega)$.

We recall that a bi-parametric set $\mathcal{B}$ is called a $\mathcal{D}$-pullback absorbing set for a cocycle $\Phi$ if, for each $D \in \mathcal{D}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$, there is a $T := T(D, \tau, \omega) > 0$ such that
\[
\Phi(t, \tau - t, \theta_{-t}\omega) D(\tau - t, \theta_{-t}\omega) \subset \mathcal{B}(\tau, \omega), \quad \forall t \geq T.
\]

Theorem 2.9. Let $\mathcal{D}$ be a local-sample closed and inclusion-closed universe. Then, the cocycle $\Phi$ possesses a unique $\mathcal{D}$-pullback bi-parametric attractor $\mathcal{A}$ with local-sample compactness if
(i) $\Phi$ has a closed $\mathcal{D}$-pullback absorbing set $\mathcal{B} \in \mathcal{D}$, and
(ii) $\Phi$ is local-sample limit-set compact on $\mathcal{D}$, or equivalently,
(ii*) $\Phi$ is local-sample asymptotically compact on $\mathcal{D}$.

Proof. Suppose (i) and (ii) hold true. By taking $s = 0$ in Definition 2.7, we know $\Phi$ is (usual) limit-set compact. By [35, theorem 2.2], there is a unique $\mathcal{D}$-pullback attractor $\mathcal{A}$ given by
\[
\mathcal{A}(\tau, \omega) = \alpha_{\mathcal{B}}(\tau, \omega) := \bigcap_{T > 0} \bigcup_{t \geq T} \Phi(t, \tau - t, \theta_{t}\omega) \mathcal{B}(\tau - t, \theta_{t}\omega).
\]

We then introduce the notion of local-sample limit-set by
\[
\Gamma_D(\tau, \omega) := \bigcap_{T > 0} \bigcup_{t \geq T} \bigcup_{|s| \leq 1} \Phi(t, \tau - t, \theta_{s-t}\omega) D(\tau - t, \theta_{-t}\omega). \quad (16)
\]

We claim that $\Gamma_D$ is nonempty compact for $D \in \mathcal{D}$. Indeed, let
\[
K_T(\tau, \omega) := \bigcup_{t \geq T} \bigcup_{|s| \leq 1} \Phi(t, \tau - t, \theta_{s-t}\omega) D(\tau - t, \theta_{-t}\omega). \quad (17)
\]

By the condition (ii), $\{K_T(\tau, \omega)\}_{T > 0}$ is a decreasing family of closed sets in $X$ such that
\[
\kappa_X(\overline{K_T(\tau, \omega)}) = \kappa_X(\overline{K_T(\tau, \omega)}) \to 0 \text{ as } T \to +\infty.
\]

By [31, lemma 2.5], the intersection $\cap_{T > 0} \overline{K_T(\tau, \omega)}$ (which is just $\Gamma_D(\tau, \omega)$) is nonempty compact.

Since $\mathcal{D}$ is local-sample closed and $\mathcal{B} \in \mathcal{D}$, we know $\mathcal{B}_{LS} \in \mathcal{D}$ and thus $\Gamma_{\mathcal{B}_{LS}}$ is compact as proved above. On the other hand,
\[
\bigcup_{|s| \leq 1} \mathcal{A}(\tau, \theta_s\omega) = \bigcup_{|s| \leq 1} \alpha_{\mathcal{B}}(\tau, \theta_s\omega) = \bigcup_{|s| \leq 1} \bigcap_{T > 0} \bigcup_{t \geq T} \Phi(t, \tau - t, \theta_{s-t}\omega) \mathcal{B}(\tau - t, \theta_{-t}\omega)
\]
\[
\subset \bigcap_{T > 0} \bigcup_{t \geq T} \bigcup_{|s| \leq 1} \Phi(t, \tau - t, \theta_{s-t}\omega) \mathcal{B}_{LS}(\tau - t, \theta_{-t}\omega).
\]
Indeed, if \( (18) \) is not true, then there are \( \eta > |\cdot| \) as \( n \to \infty \) and \( x_n \in D(\tau - t_n, \theta - t_n, \omega) \) with \( D \in \mathcal{D} \). By (ii), we have
\[
\kappa_X \left( \left\{ \Phi(t_n, \tau - t_n, \theta_{s_n - t_n} \omega)x_n \right\}_{n=k}^{\infty} \right)
\leq \kappa_X \left( \bigcup_{t \geq t_k} \bigcup_{|s| \leq 1} \Phi(t, t - t, \theta_{s - t} \omega)D(\tau - t, \theta_{- t} \omega) \right) \to 0
\]
as \( k \to \infty \). By [30, lemma 2.3], \( \left\{ \Phi(t_n, \tau - t_n, \theta_{s_n - t_n} \omega)x_n \right\} \) is pre-compact.

In order to show (ii*) \( \Rightarrow \) (ii), we first notice that \( y \in \Gamma_D(\tau, \omega) \) if and only if there are \( |s_n| \leq 1 \), \( t_n \uparrow +\infty \) and \( z_n \in D(\tau - t_n, \theta - t_n, \omega) \) such that
\[
\Phi(t_n, \tau - t_n, \theta_{s_n - t_n} \omega)z_n \to y \quad \text{in} \quad X.
\]

We then show that \( \Gamma_D(\tau, \omega) \) is still compact under the condition (ii*). Indeed, let \( \{y_n\}_{n=1}^{\infty} \) be a sequence taken from \( \Gamma_D(\tau, \omega) \). We can sequentially choose \( |s_n| \leq 1 \), \( t_n \geq \max\{n, t_{n-1}\} \) ensuring \( t_n \uparrow +\infty \) and \( z_n \in D(\tau - t_n, \theta - t_n, \omega) \) such that
\[
\|y_n - \Phi(t_n, \tau - t_n, \theta_{s_n - t_n} \omega)z_n\| \leq \frac{1}{n}, \quad \forall n \in \mathbb{N}.
\]
By the condition (ii*), passing to a subsequence, \( \Phi(t_{n}, \tau - t_{n}, \theta_{s_{n} - t_{n}} \omega)z_{n} \to y_{0} \) in \( X \) and \( y_{0} \in \Gamma_D(\tau, \omega) \). Moreover,
\[
\|y_{n_{*}} - y_{0}\| \leq \|y_{n_{*}} - \Phi(t_{n_{*}}, \tau - t_{n_{*}}, \theta_{s_{n_{*}} - t_{n_{*}}} \omega)z_{n_{*}}\| + \|\Phi(t_{n_{*}}, \tau - t_{n_{*}}, \theta_{s_{n_{*}} - t_{n_{*}}} \omega)z_{n_{*}} - y_{0}\| \to 0
\]
as \( n_{*} \to \infty \). So, \( \Gamma_D(\tau, \omega) \) is compact as desired.

We prove that \( \Gamma_D(\tau, \omega) \) is locally sample uniformly attracts \( D \in \mathcal{D} \) in the following sense:
\[
\lim_{t \to +\infty} \sup_{|s| \leq 1} \text{dist}_X(\Phi(t, \tau - t, \theta_{- t} \omega)D(\tau - t, \theta_{- t} \omega), \Gamma_D(\tau, \omega)) = 0. \tag{18}
\]
Indeed, if (18) is not true, then there are \( \eta > 0 \), \( |s_n| \leq 1 \), \( t_n \uparrow +\infty \) and \( z_n \in D(\tau - t_n, \theta - t_n, \omega) \) such that
\[
\text{dist}_X(\Phi(t_n, \tau - t_n, \theta_{s_n - t_n} \omega)z_n, \Gamma_D(\tau, \omega)) \geq \eta, \quad \forall n \in \mathbb{N}.
\]
By the condition (ii*), passing to a subsequence, \( \Phi(t_n, \tau - t_n, \theta_{s_n - t_n} \omega)z_n \to y \in \Gamma_D(\tau, \omega) \), which gives a contradiction. So (18) holds true.

Given now \( \varepsilon > 0 \), by (18), we can choose a \( T_1 > 0 \) such that for all \( T \geq T_1 \),
\[
\text{dist}_X(K_T(\tau, \omega), \Gamma_D(\tau, \omega)) < \varepsilon,
\]
where \( K_T \) is defined by (17). Hence, \( K_T(\tau, \omega) \subset N_\varepsilon(\Gamma_D(\tau, \omega)) \) for all \( T \geq T_1 \), where \( N_\varepsilon(\cdot) \) denotes the \( \varepsilon \)-neighborhood.

On the other hand, by the compactness of \( \Gamma_D \) as proved above, we can find \( y_1, y_2, \cdots, y_m \in \Gamma_D(\tau, \omega) \) such that \( \Gamma_D(\tau, \omega) \subset \bigcup_{k=1}^m N_\varepsilon(y_k) \). Hence,
\[
K_T(\tau, \omega) \subset N_\varepsilon(\Gamma_D(\tau, \omega)) \subset \bigcup_{k=1}^m N_{2\varepsilon}(y_k),
\]
which implies that \( \kappa_X(K_T(\tau, \omega)) \leq 4\varepsilon \) for all \( T \geq T_1 \). Therefore, \( \kappa_X(K_T(\tau, \omega)) \to 0 \)
as \( T \to +\infty \), which means the condition (ii). \( \square \)
2.4. Upper semi-continuity in sample. Now, we establish the criterion for
the upper semi-continuity (3), which leads to the lower semi-continuity (2) in view of
Theorem 2.4. In this case, we need the triple-continuity of the cocycle.

Definition 2.10. A cocycle $\Phi$ is called \textit{triple-continuous} if $\Phi$ is
continuous in time, sample and space, more precisely,

$$
\|\Phi(t_n, \tau_{s_n}, \omega)x_n - \Phi(t_0, \tau_{s_0}, \omega)x_0\| \to 0, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega
$$

whenever $t_n \to t_0$, $s_n \to s_0$ and $\|x_n - x_0\| \to 0$.

Since $\{\theta_s\}_{s \in \mathbb{R}}$ is a group, it is easy to show that each $\theta_s : \Omega \to \Omega$ is bijective. Thereby, we only require that (19) holds true at $s_0 = 0$.

Theorem 2.11. Let a triple-continuous cocycle $\Phi$ have a $\mathcal{D}$-pullback attractor $\mathcal{A}$,
where $\mathcal{D}$ is local-sample closed. Then, $\mathcal{A}$ is upper semi-continuous in sample if and
only if $\mathcal{A}$ is local-sample compact.

Proof. Since each $\theta_s : \Omega \to \Omega$ is bijective, it is easy to show that (3) is equivalent to

$$
\lim_{s \to 0} \text{dist}(\mathcal{A}(\tau, \theta_s \omega), \mathcal{A}(\tau, \omega)) = 0, \quad \forall \tau \in \mathbb{R}, \omega \in \Omega.
$$

Now, suppose $\mathcal{A}$ is local-sample compact, we argue (20) by contradiction. If (20) is
not true, then, there are $\eta > 0$, $s_n \to 0$, $y_n \in \mathcal{A}(\tau, \theta_{s_n} \omega)$ with $\tau \in \mathbb{R}$ and $\omega \in \Omega$
such that

$$
\text{dist}(y_n, \mathcal{A}(\tau, \omega)) \geq \eta, \quad \forall n \in \mathbb{N}.
$$

Since $\mathcal{D}$ is local-sample closed, the local-sample union $\mathcal{A}_{LS} \in \mathcal{D}$. So, the attractor
$\mathcal{A}$ can attract $\mathcal{A}_{LS}$, and thus there is a $T > 0$ such that

$$
\text{dist}(\Phi(T, \tau - T, \theta_{-T} \omega)\mathcal{A}_{LS}(\tau - T, \theta_{-T} \omega), \mathcal{A}(\tau, \omega)) < \eta.
$$

By the invariance of $\mathcal{A}$ (at $\theta_{s_n} \omega$), there is $z_n \in \mathcal{A}(\tau - T, \theta_{s_n - T} \omega)$ such that

$$
y_n = \Phi(T, \tau - T, \theta_{s_n - T} \omega)z_n, \quad \forall n \in \mathbb{N}.
$$

We assume without loss of generality that $|s_n| \leq 1$, then,

$$
z_n \in \mathcal{A}(\tau - T, \theta_{s_n - T} \omega) \subset \mathcal{A}_{LS}(\tau - T, \theta_{-T} \omega), \quad \forall n \in \mathbb{N}.
$$

By the assumption, $\mathcal{A}_{LS}(\tau - T, \theta_{-T} \omega)$ is compact, and so, passing to a subsequence,

$$
z_n \to z \in \mathcal{A}_{LS}(\tau - T, \theta_{-T} \omega).
$$

By the triple-continuity of $\Phi$, we have

$$
\Phi(T, \tau - T, \theta_{s_n - T} \omega)z_n \to \Phi(T, \tau - T, \theta_{-T} \omega)z \in X.
$$

Therefore, if $n$ is large enough,

$$
\text{dist}(y_n, \mathcal{A}(\tau, \omega)) \leq \|\Phi(T, \tau - T, \theta_{s_n - T} \omega)z_n - \Phi(T, \tau - T, \theta_{-T} \omega)z\|
$$

+ $\text{dist}(\Phi(T, \tau - T, \theta_{-T} \omega)\mathcal{A}_{LS}(\tau - T, \theta_{-T} \omega), \mathcal{A}(\tau, \omega)) < \eta,$

which contradicts with (21). Therefore, $\mathcal{A}$ is upper semi-continuous in sample.

On the contrary, suppose $\mathcal{A}$ is upper semi-continuous in sample. We take a sequence $y_n \in \mathcal{A}(\tau, \theta_{s_n} \omega)$ with $|s_n| \leq 1$. Passing to a subsequence, $s_n \to s_0$ and so

$$
\text{dist}(y_{n^*}, \mathcal{A}(\tau, \theta_{s_0} \omega)) \leq \text{dist}(\mathcal{A}(\tau, \theta_{s_0} \omega), \mathcal{A}(\tau, \theta_{s_0} \omega)) \to 0
$$
as $n^* \to \infty$. On the other hand, for each $n^*$, there is a $z_{n^*} \in \mathcal{A}(\tau, \theta_{s_0} \omega)$ such that

$$
d(y_{n^*}, z_{n^*}) \leq \text{dist}(y_{n^*}, \mathcal{A}(\tau, \theta_{s_0} \omega)) + \frac{1}{n^*} \to 0.
$$
Since $A_3(\tau, \theta_{s_0}, \omega)$ is compact, the sequence $\{z_n\}$ is pre-compact and so is $\{y_n\}$. Therefore, $A_3$ is local-sample compact.

By combining Theorems 2.4, 2.9 and 2.11, we summarize our abstract results.

**Theorem 2.12.** Let $\mathcal{D}$ be a local-sample closed and inclusion-closed universe. Assume a cocycle $\Phi$ satisfies the following three conditions:

(i) it is triple-continuous;
(ii) it has a closed $\mathcal{D}$-pullback absorbing set $B \in \mathcal{D}$;
(iii) it is local-sample limit-set (or asymptotically) compact on $\mathcal{D}$.

Then, there is a unique $\mathcal{D}$-pullback bi-parametric attractor $A$ with the following properties:

(a) $A$ is local-sample compact.
(b) $A$ is upper semi-continuous in sample (i.e. (3) holds true).
Moreover, $A$ is random if $B$ is random. In this case, we have
(c) $A$ is probabilistically continuous in sample (i.e. (7) holds true).
(d) $A$ is probabilistically lower semi-continuous in time (i.e. (2) holds true).

3. **Stochastic non-autonomous $p$-Laplace equations.** The equation can be read as

$$\begin{cases}
\frac{du}{dt} + \lambda u - \text{div}(|\nabla u|^{p-2}\nabla u) + f(x, u) - g(t, x))dt = h(x)dW, \\ u(\tau, x) = u_\tau(x), \quad x \in \mathbb{R}^m,
\end{cases}$$

where $p > 2$ and $\lambda > 0$.

3.1. **Hypotheses and transformation for the equation.** We use $\|\cdot\|_r$ to denote the norm in $L^r(\mathbb{R}^m)$ and delete the subscript when $r = 2$.

**Hypothesis F.** $f : \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$ is continuously differential, and for all $x \in \mathbb{R}^m$, $s \in \mathbb{R}$,

$$f(x, s) \geq \alpha_1|s|^q + \psi_1(x), \quad |f(x, s)| \leq \alpha_2|s|^{q-1} + \psi_2(x),$$

$$\frac{\partial f}{\partial s}(x, s) \geq -\alpha_3, \quad \left| \frac{\partial f}{\partial s}(x, s) \right| \leq \alpha_4|s|^{q-2} + \psi_3(x),$$

where $q > 2$, $\alpha_1 \geq 0$, $\psi_1 \in L^1 \cap L^N$, $\psi_2 \in L^2 \cap L^N$, $\psi_3 \in L^2$, and

$$N = \left\{ \frac{p-2}{q-2} \right\} = \min \left\{ n \in \mathbb{Z} : n \geq \frac{p-2}{q-2} \right\}.$$  

**Hypothesis H.** The density function $h \in W^{1,p} \cap W^{1,Np} \cap L^1 \cap L^\infty$.

**Hypothesis G.** The force $g \in L^2_{\text{loc}}(\mathbb{R} ; L^2(\mathbb{R}^m)) \cap L^N_{\text{loc}}(\mathbb{R} ; L^N)$ satisfies:

$$G(\tau) := \int_{-\infty}^{0} e^{\lambda s} (|g(s+\tau)|^2 + |g(s+\tau)|_N^N)ds < +\infty, \quad \forall \tau \in \mathbb{R}. \quad (25)$$

We will frequently use a local estimate of $G(\cdot)$:

$$\sup_{|s| \leq 1} G(\tau + s) \leq e^{2\lambda} G(\tau + 1) < +\infty, \quad \forall \tau \in \mathbb{R}. \quad (26)$$

We consider a measurable dynamical system $(\Omega, \mathcal{F}, P, \theta_s)$, where $\Omega = \{ \omega \in C(\mathbb{R}, \mathbb{R}) : \omega(0) = 0 \}$, $\mathcal{F} = \mathcal{B}(\Omega)$ is the Borel algebra on $\Omega$ equipping with the Fréchet metric, $P$ is the two-sided Wiener measure and $\theta_s \omega(\cdot) = \omega(s + \cdot) - \omega(s)$. Note that $\{\theta_s : s \in \mathbb{R}\}$ is a group of measure-preserving transformations on $\Omega$.

As usual, we consider a special Wiener process $W(s, \omega) = \omega(s)$. The pathwise continuous solution of $dz + \lambda z = dW$ is denoted by $z(\theta_s \omega)$. It is easy to prove that
Proposition 4.3.3 was imposed), we have
\[ Z(\theta s, \omega) := |z(\theta s, \omega)| + |z(\theta s, \omega)|^M \leq e^{1/s}|\theta(\omega)| \text{ for all } s \in \mathbb{R}, \ \omega \in \Omega, \quad (27) \]
where \( M = \max\{Nq, Np\} \) and \( \rho(\cdot) \) is a tempered random variable. By setting
\[ v(s, \tau, \omega) = u(s, \tau, \omega) - hz(\theta s, \omega), \]
we can translate the equation (22) into a random equation:
\[ \frac{dv}{ds} + \lambda v + A(v + hz(\theta s, \omega)) + f(x, v + hz(\theta s, \omega)) = g(s), \quad s \geq \tau, \quad (28) \]
where the Laplace operator \( A : W^{1, p} \rightarrow W^{-1, p} \) is defined by
\[ \langle Au, u \rangle = \int_{\mathbb{R}^m} |\nabla u|^p \nabla u \cdot \nabla u dx, \quad u_1, u_2 \in W^{1, p}. \]

3.2. Basic estimate of the solution. We will frequently use a priori estimate.

Lemma 3.1. For each \( s \geq \tau, \ \omega \in \Omega \) and \( r \in L^2 \), we have
\[ \|v(s, \tau, \omega, r)\|^2 \leq c \Upsilon(s, \tau, \omega, v_r), \quad (29) \]
\[ \int_{\tau}^{s} e^{\lambda(s-s)}(\|\nabla u(s, \tau, \omega, r)\|_p^p + \|v(s)\|_q^q) d\xi \leq c \Upsilon(s, \tau, \omega, v_r), \quad (30) \]
where
\[ \Upsilon(s, \tau, \omega, v_r) := e^{\lambda(s-s)}\|v_r\|^2 + \int_{\tau}^{s} e^{\lambda(s-s)}(1 + \|g(s)\|^2 + Z(\theta s, \omega))d\xi. \]

Proof. By the inner product of (28) with \( v = v(s, \tau, \omega, r) \), we obtain
\[ \frac{1}{2} \frac{d}{ds} \|v\|^2 + \lambda \|v\|^2 + \langle Au, v \rangle + (f(x, u), v) = (g(s), v). \quad (31) \]
By (23) and the inequality \( |a|^q \geq 2^{-q}|b|^q - |a - b|^q \), we have
\[ f(x, u) v = f(x, u) u - f(x, u) hz(\theta s, \omega) \geq \alpha_1 |v|^q - |\psi_1| - c hz(\theta s, \omega)||u|^q - 1 - |\psi_2 hz(\theta s, \omega)| \geq \alpha_1 |v|^q - |\psi_1| - c hz(\theta s, \omega)|^q - |\psi_2 hz(\theta s, \omega)|. \quad (32) \]
We integrate (32) over \( \mathbb{R}^m \) to obtain
\[ (f(x, u), v) \geq \alpha_1 \|v\|^q - c(1 + Z(\theta s, \omega)), \quad (33) \]
where \( Z \) is given in (27). By the Young inequality,
\[ |(g(s), v)| \leq \frac{\alpha_1}{2q^2} \|v\|^q_c + c \|g(s)\|^2. \]
Since \( \nabla u = \nabla v + z(\theta s, \omega) \nabla h \), we have
\[ \langle Au, v \rangle = \int_{\mathbb{R}^m} |\nabla u|^p \nabla u \cdot \nabla v dx \]
\[ = \int_{\mathbb{R}^m} |\nabla u|^p dx - \int_{\mathbb{R}^m} |\nabla u|^p \nabla u \cdot \nabla h (\theta s, \omega) dx \]
\[ \geq \frac{1}{2} \int_{\mathbb{R}^m} |\nabla u|^p dx - c Z(\theta s, \omega). \quad (34) \]
We substitute (33)-(34) into (31) to obtain (for $c_1 = \alpha_1/2^p$)

$$\frac{d}{ds} \|v\|^2 + \lambda \|v\|^2 + \|\nabla u\|^2_{L^p} + c_1 \|v\|^2_{L^q} \leq c(1 + \|g(s)\|^2 + Z(\theta_s \omega)).$$

Therefore, both (29) and (30) follow from the Gronwall inequality over $[\tau, s]$. \qed

By a standard method, Lemma 3.1 implies the well-posedness (cf. [25, 26]).

**Lemma 3.2.** For each $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $v_\tau \in X_0 := L^2(\mathbb{R}^m)$, Eq.(28) has a unique weak solution $v(t, \tau, \omega; v_\tau)$ such that $v(\tau, \tau, \omega; v_\tau) = v_\tau$ and

$$v \in C([\tau, +\infty); X_0) \cap L^p_{loc}(\tau, +\infty; W^{1,p}) \cap L^q_{loc}(\tau, +\infty; L^q).$$

Furthermore, for each fixed $\omega \in \Omega$, the solution $v(s, \tau, \omega; v_\tau)$ is continuous in $s$, $\tau$, $v_\tau \in X_0$, where $s \geq \tau$.

By Lemma 3.2, we obtain a cocycle $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times X_0 \rightarrow X_0$ defined by

$$\Phi(t, \tau, \omega)v_\tau = v(t + \tau, \tau, \theta_{-\tau} \omega; v_\tau), \ \forall t \geq 0, \tau \in \mathbb{R},$$

(35)

where the $\mathfrak{F}$-measurability of $\Phi$ can be proved by the same method as given in [13].

**3.3. Triple-continuity of the cocycle.** We need to show $\Phi$ is triple-continuous.

**Proposition 1.** Let $s_n \rightarrow s_0$, $\tau_n \rightarrow 0$ and $v_{\tau_n} \rightarrow v_\tau$ as $n \rightarrow \infty$. Then, for any $\sigma < s_0$ and $\omega \in \Omega$, we have

$$v(s_n, \sigma, \theta_{\tau_n} \omega, v_{\tau_n}) \rightarrow v(s_0, \sigma, \omega, v_\tau) \text{ in } L^2.$$  (36)

**Proof.** By the continuity as given in Lemma 3.2, it suffices to prove that

$$\|v(s_n, \sigma, \theta_{\tau_n} \omega, v_{\tau_n}) - v(s_0, \sigma, \omega, v_{\tau_n})\| \rightarrow 0.$$  (36)

We assume that $\sigma < s_n \leq s_0 + 1$, and define $V_n(s) := v^1_n(s) - v^0_n(s)$ for $s \in [\sigma, s_0 + 1]$, where $v^1_n(s) := v(s, \sigma, \theta_{\tau_n} \omega, v_{\tau_n})$ and $v^0_n(s) := v(s, \sigma, \omega, v_{\tau_n}).$

By the difference of Eq.(28), we have

$$\frac{dV_n}{ds} + \lambda V_n + A(v^1_n + hz(\theta_{s+\tau_n} \omega)) - A(v^0_n + hz(\theta_s \omega))$$

$$+ f(x, v^1_n + hz(\theta_{s+\tau_n} \omega)) - f(x, v^0_n + hz(\theta_s \omega)) = 0.$$  (37)

By multiplying (37) with $V_n$, we obtain

$$\frac{1}{2} \frac{d}{ds} \|V_n\|^2 + \lambda \|V_n\|^2 + I_A + I_f = 0,$$

(38)

where

$$I_A := \langle A(v^1_n + hz(\theta_{s+\tau_n} \omega)) - A(v^0_n + hz(\theta_s \omega)), V_n \rangle,$$

$$I_f := \langle f(x, v^1_n + hz(\theta_{s+\tau_n} \omega)) - f(x, v^0_n + hz(\theta_s \omega)), V_n \rangle.$$

Since $s \rightarrow z(\theta_s \omega)$ is continuous, we have the uniform continuity:

$$Z_n := \sup_{s \in [\sigma, s_0 + 1]} |z(\theta_{s+\tau_n} \omega) - z(\theta_s \omega)| \rightarrow 0.$$
By (24) and the boundedness of $Z_n$, it follows from the mean value theorem that
\[
I_f = \frac{\partial f}{\partial r}(x, r)(V_n + (z(\theta_s + r\omega) - z(\theta_s\omega))h, V_n)
\geq -\alpha_3||V_n||^2 - Z_n||\frac{\partial f}{\partial r}(x, r), V_n||
\geq -\alpha_3||V_n||^2 - cZ_n \int_{\mathbb{R}^n} |h||v_n^1|^q - 2 + |v_n^0|^q - 2 + |h|^q - 2 + |\psi_3||V_n|dx
\geq -\alpha_3||V_n||^2 - cZ_n \int_{\mathbb{R}^n} |\psi_3||V_n|dx
- cZ_n \int_{\mathbb{R}^n} |h||v_n^1|^q - 2 + |v_n^0|^q - 2 + |h|^q - 2 + |\psi_3||V_n|dx
\geq - 2\alpha_3||V_n||^2 - cZ_n^2||\psi_3||^2 - cZ_n(||h||^q + ||v_n^1||^q + ||v_n^0||^q)
\geq - 2\alpha_3||V_n||^2 - cZ_n(||v_n^1||^q + ||v_n^0||^q + 1),
\] (39)
where we have used $h \in L^\infty \cap L^1$ (so $h \in L^q$) and the triple Young inequality with exponents $(q, \frac{q}{q-2}, q)$.

We split $I_A$ as $I_A = I_{A1} - I_{A2}$, then the monotonicity of $A$ implies
\[
I_{A1} := \langle A(v_n^1 + hz(\theta_s + r\omega)) - A(v_n^0 + hz(\theta_s + r\omega)), V_n \rangle \geq \gamma ||\nabla V_n||_p^p.
\]
By the mean value theorem again,
\[
I_{A2} := \langle A(v_n^0 + hz(\theta_s + r\omega)) - A(v_n^0 + hz(\theta_s + r\omega)), V_n \rangle
= \int_{\mathbb{R}^n} \left(|\nabla(v_n^0 + hz(\theta_s + r\omega))|^p - 2 |\nabla(v_n^0 + hz(\theta_s + r\omega))|\right) \cdot \nabla V_n dx
\leq cZ_n \int (||v_n^0||^{p-2} |\nabla h||\nabla V_n| + |\nabla h|^{p-1} |\nabla V_n|).
\]
Since $\frac{p-2}{p} + \frac{1}{p} + \frac{1}{p} = 1$, the triple Young inequality gives
\[
cZ_n \int (||v_n^0||^{p-2} |\nabla h||\nabla V_n| = c \int |\nabla V_n| (||v_n^0||^{p-2} Z_n^{\frac{p-2}{p}}) (|\nabla h| Z_n^\frac{2}{p})
\leq \frac{\gamma}{2} ||\nabla V_n||_p^p + cZ_n^2||\nabla V_n||_p^p + cZ_n^2||\nabla h||_p^p \leq \frac{\gamma}{2} ||\nabla V_n||_p^p + cZ_n (||v_n^0||_p^p + 1),
\]
where we use $h \in W^{1,p}$ in Hypothesis H.

By the Young inequality and the boundedness of $Z_n$,
\[
cZ_n \int |\nabla h|^{p-1} |\nabla V_n| \leq \frac{\gamma}{2} ||\nabla V_n||_p^p + cZ_n^2 \frac{p}{p-1} \leq \frac{\gamma}{2} ||\nabla V_n||_p^p + cZ_n.
\]
By all estimates mentioned above, we have
\[
I_A = I_{A1} - I_{A2} \geq -cZ_n(||\nabla v_n^0||_p^p + 1).
\] (40)

We substitute (39)-(40) into (38) to obtain
\[
d ds||V_n||^2 \leq c ||V_n||^2 + cZ_n(||v_n^1||_q^q + ||v_n^0||_q^q + ||\nabla v_n^0||_p^p + 1).
\]
Note that $V_n(\sigma) = 0$. The Gronwall inequality over $[\sigma, s_n]$ gives
\[
||V_n(s_n)||^2 \leq cZ_n \int_{\sigma}^{s_n+1} (||v_n^1(s)||_q^q + ||v_n^0(s)||_q^q + ||\nabla v_n^0(s)||_p^p + 1)ds.
\]
By the basic estimate (30) at the sample \( \theta_{\tau_n, \omega} \), we have
\[
\int_{\sigma}^{\sigma_0 + 1} \| v_{n}^{t}(s, \sigma; \theta_{\tau_n, \omega}, v_{\sigma, n}) \|^2 ds \leq c \| v_{\sigma, n} \|^2 + c \int_{\sigma}^{\sigma_0 + 1} Z(\theta_{s + \sigma, \omega}) ds + c,
\]
which is bounded, since \( \| v_{\sigma, n} \| \) and \( Z(\theta_{s + \sigma, \omega}) \) are bounded. By (30) again,
\[
\int_{\sigma}^{\sigma_0 + 1} (\| v_{n}^{t}(s, \sigma; \theta_{\tau_n, \omega}, v_{\sigma, n}) \|^2 + \| \nabla v_{n}^{t}(s) \|^2 ) ds \leq c \| v_{\sigma, n} \|^2 + c \int_{\sigma}^{\sigma_0 + 1} Z(\theta_{s, \omega}) ds + c,
\]
which is still bounded in \( n \). Therefore, \( \| V_n(s_n) \| \to 0 \) and thus (36) holds true. \( \square \)

3.4. Attracted universes. We use \( \mathcal{D} \) to denote the usual universe of all tempered sets in \( X_0 = L^2(\mathbb{R}^m) \), where \( D = \{ D(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega \} \) is called tempered if
\[
\lim_{t \to +\infty} e^{-\lambda t} \| D(\tau - t, \theta_{-t}\omega) \|^2 = 0, \quad \forall \tau \in \mathbb{R}, \ \omega \in \Omega. \tag{41}
\]
Unfortunately, the universe \( \mathcal{D} \) is not local-sample closed (Def.2.6). So, we introduce a sub-universe \( \mathcal{D}_0 \subset \mathcal{D} \) such that \( D_0 \in \mathcal{D}_0 \) if and only if
\[
\lim_{t \to +\infty} e^{-\lambda t} \sup_{|s| \leq 1} \| D_0(\tau + s - t, \theta_{-t}\omega) \|^2 = 0, \quad \forall \tau \in \mathbb{R}, \ \omega \in \Omega. \tag{42}
\]
It is easy to show that \( \mathcal{D}_0 \) is local-time closed (41), which means \( D_{LT, 0} \in \mathcal{D}_0 \) as long as \( D_0 \in \mathcal{D}_0 \), where \( D_{LT} \) is the local-time union given by
\[
D_{LT}(\tau, \omega) := \bigcup_{|s| \leq 1} D_{0}(\tau + s, \omega).
\]
Moreover, \( \mathcal{D}_0 \) is local-sample closed as proved below.

Lemma 3.3. The universe \( \mathcal{D}_0 \) is local-sample closed.

Proof. Let \( D_0 \in \mathcal{D}_0 \). Then its local-time union \( D_{LT, 0} \in \mathcal{D}_0 \). Let \( D_{LT} \) be the local-time union as given in (15). Let \( \hat{t} = t - \sigma \) with \( |\sigma| \leq 1 \). If \( t \to +\infty \), then \( \hat{t} \to +\infty \) uniformly in \( |\sigma| \leq 1 \). Hence, as \( t \to +\infty \),
\[
e^{-\lambda \hat{t}} \sup_{|s| \leq 1} \| D_{LT, 0}(\tau + s - t, \theta_{-t}\omega) \|^2 = e^{-\lambda \hat{t}} \sup_{|s| \leq 1} \| D_0(\tau + s - \sigma - \hat{t}, \theta_{-t}\omega) \|^2 \\
\leq e^{\lambda} e^{-\lambda \hat{t}} \sup_{|s| \leq 1} \| D_0(\tau + s - \sigma - \hat{t}, \theta_{-t}\omega) \|^2 = e^{\lambda} e^{-\lambda \hat{t}} \sup_{|s| \leq 1} \| D_{LT, 0}(\tau + s - \hat{t}, \theta_{-t}\omega) \|^2 \to 0,
\]
in view of \( D_{LT, 0} \in \mathcal{D}_0 \). Therefore, \( \mathcal{D}_0 \) is local-sample closed. \( \square \)

In order to discuss the weakly dissipative case \( (q < p) \), we also introduce a sequence of universes \( \mathcal{D}_k \) in \( X_k = L^2 \cap L^{q_k} \) such that \( D_k \in \mathcal{D}_k \) if and only if
\[
\lim_{t \to +\infty} e^{-\lambda t} \sup_{|s| \leq 1} (\| D_k(\tau + s - t, \theta_{-t}\omega) \|^2 + \| D_k(\tau + s - t, \theta_{-t}\omega) \|^2_{q_k}) = 0 \tag{43}
\]
for all \( \tau \in \mathbb{R} \) and \( \omega \in \Omega \), where \( q_k = k(q - 2) + 2 \). We have \( \mathcal{D}_{N-1} \subset \cdots \subset \mathcal{D}_0 \subset \mathcal{D} \), which follows from the following interpolation:
\[
a^{r_2} \leq ca^{r_1} + c(\epsilon)a^{r_3}, \quad \text{for } a > 0, \quad 0 < r_1 < r_2 \leq r_3. \tag{44}
\]
4. Inductive estimates and triple-compactness for localization.

**Proposition 2.** For \( D_0 \in \mathcal{D}_0, \tau \in \mathbb{R} \) and \( \omega \in \Omega \), there is \( T = T(D_0, \tau, \omega) \geq 1 \) such that for all \( t \geq T \) and \( v_{\tau-t} \in D_0(\tau-t, \theta_{-\omega}) \),

\[
\sup_{s \in [\tau-1, \tau]} \sup_{|\sigma| \leq 1} \|v(s, \tau - t, \theta_{\sigma - \omega}; v_{\tau-t})\|^2 \leq c_0 R(\tau, \omega), \tag{45}
\]

\[
\sup_{|\sigma| \leq 1} \int_{\tau-t}^\tau (\|\nabla u_s(\tau - t, \theta_{\sigma - \omega}; v_{\tau-t})\|_p^p + \|v_s(\tau - t, \theta_{\sigma - \omega}; v_{\tau-t})\|_p^p) d\zeta \leq c_0 R(\tau, \omega), \tag{46}
\]

where \( R(\tau, \omega) = 1 + G(\tau) + \rho(\omega) \) with \( G(\tau) \), \( \rho(\omega) \) as given by (25), (27) respectively.

**Proof.** We take the sample \( \theta_{\sigma - \omega} \) in (29) of Lemma 3.1 to obtain

\[
\sup_{s \in [\tau-1, \tau]} \sup_{|\sigma| \leq 1} \|v(s, \tau - t, \theta_{\sigma - \omega}; v_{\tau-t})\|^2 \leq c \sup_{s \in [\tau-1, \tau]} \int_{\tau-t}^s e^{\lambda(s-s)(1 + \|g(s)\|^2)} d\zeta
\]

\[
+ c \sup_{s \in [\tau-1, \tau]} \sup_{|\sigma| \leq 1} \int_{\tau-t}^s e^{\lambda(s-s)} Z(\theta_{\sigma + \sigma - \omega}) d\zeta + \sup_{s \in [\tau-1, \tau]} e^{\lambda(t-s)} \|v_{\tau-t}\|^2
\]

\[
= I_1 + I_2 + I_3.
\]

By Hypothesis G, \( I_1 \leq c(1 + G(\tau)) < \infty \). By (27),

\[
I_2 \leq c \sup_{|\sigma| \leq 1} \int_{\tau-t}^0 e^{\lambda s} Z(\theta_{\sigma + \sigma - \omega}) d\zeta \leq c \rho(\omega) \sup_{|\sigma| \leq 1} \int_{\tau-t}^0 e^{\lambda s} e^{\frac{\lambda}{2}(c + |\sigma|)} d\zeta \leq c \rho(\omega).
\]

While, \( I_3 \leq e^{-\lambda t} \|D_0(\tau - t, \theta_{-\omega})\|^2 \to 0 \) as \( t \to +\infty \). Hence, (45) holds true. Similarly, (46) follows from (30) at the sample \( \theta_{\sigma - \omega} \).

**Corollary 1.** The cocycle \( \Phi \) has a \( \mathcal{D}_0 \)-pullback, local-sample absorbing set \( \mathcal{K}_0 \in \mathcal{D}_0 \), given by

\[
\mathcal{K}_0(\tau, \omega) := \{ u \in X_0 : \|u\|^2 \leq c_0 R(\tau, \omega) \}. \tag{47}
\]

**Proof.** By taking \( s = \tau \) in (45), we obtain the local-sample absorption:

\[
\sup_{|\sigma| \leq 1} \|v(\tau, \tau - t, \theta_{\sigma - \omega}; v_{\tau-t})\|^2 \leq c_0 R(\tau, \omega), \forall t \geq T. \tag{48}
\]

Next, we prove \( \mathcal{K}_0 \in \mathcal{D}_0 \). Indeed, by (26),

\[
e^{-\lambda t} \sup_{|\sigma| \leq 1} R(\tau + s - t, \theta_{-\omega}) = e^{-\lambda t} \sup_{|\sigma| \leq 1} G(\tau + s - t) + e^{-\lambda t} \rho(\theta_{-\omega})
\]

\[
\leq e^{-\lambda t} + e^{-\lambda t} e^{2\lambda t} G(\tau + 1 - t) + e^{-\lambda t} \rho(\theta_{-\omega}) \to 0 \tag{49}
\]

as \( t \to +\infty \), in view of the fact that \( G(\tau + 1) \) and \( \rho(\omega) \) are tempered.

We need to show the local-sample absorption in \( L^{q_k} \), where \( q_k = k(q - 2) + 2, k = 0, \ldots, N - 1 \). This is unnecessary in the strongly dissipative case \( (N = 1) \).

**Proposition 3.** For each \( k \in \{0, 1, \cdots, N - 1\} \), \( D_0 \in \mathcal{D}_0 \), \( \tau \in \mathbb{R}, \omega \in \Omega \), there exists \( T_k = T_k(D_0, \tau, \omega) \) such that for all \( t \geq T_k \) and \( v_{\tau-t} \in D_0(\tau-t, \theta_{-\omega}) \),

\[
\sup_{|\sigma| \leq 1} \|v(\tau, \tau - t, \theta_{\sigma - \omega}; v_{\tau-t})\|_{q_k}^{q_k} \leq c_k R(\tau, \omega). \tag{50}
\]
Proof. By using an induction method, we prove the following stronger conclusions:

\[
\sup_{s \in [\tau - \frac{1}{k+1}, \tau]} \sup_{|\sigma| \leq 1} \|v(s, \tau - t, \theta_{\sigma - \tau} \omega; v_{\tau - t})\|_{q_k}^{q_k} \leq c_k R(\tau, \omega), \tag{51}
\]

\[
\sup_{|\sigma| \leq 1} \int_{\tau - \frac{1}{k+1}}^{\tau} \|v(s, \tau - t, \theta_{\sigma - \tau} \omega; v_{\tau - t})\|_{q_{k+1}}^{q_{k+1}} ds \leq c_k R(\tau, \omega) \tag{52}
\]

for all \( t \geq T_k \) and \( v_{\tau - t} \in D_{q}(\tau - t, \theta_{\tau} \omega) \). By Proposition 2, the case of \( k = 0 \) is true. We assume that (51) and (52) hold true for \( k - 1 \) with \( 1 \leq k \leq N - 1 \).

By taking the inner product of Eq. (28) with \( |v|^{q_k - 2}v \), where \( v(s) = v(s, \tau - t, \theta_{\sigma - \tau} \omega; v_{\tau - t}) \), we have

\[
\frac{1}{q_k} \frac{d}{ds} \|v\|_{q_k}^{q_k} + \lambda \|v\|_{q_k}^{q_k} + \langle Au, |v|^{q_k - 2}v \rangle + (f(x, u), |v|^{q_k - 2}v) = (g(s), |v|^{q_k - 2}v). \tag{53}
\]

By the production of (32) with \( |v|^{q_k - 2} \) and by \( q + q_k - 2 = q_{k+1} \), we have

\[
f(x, u)v|v|^{q_k - 2} \geq \frac{\alpha_1}{2q} |v|^{q_{k+1}} - c(|\psi_1| + |h z(\theta_{s+\sigma-\tau} \omega)|^q + |\psi_2 h z(\theta_{s+\sigma-\tau} \omega)|)|v|^{q_k - 2}.
\]

Let \( \mu_k = 1/(1 - \frac{q_k - 2}{q_{k+1}}) = k + 1 - \frac{2k}{q} \), then, the Young inequality gives

\[
f(x, u)v|v|^{q_k - 2} \geq \frac{\alpha_1}{2q^{k+1}} |v|^{q_{k+1}} - c(|\psi_1|^{\mu_k} + |h z(\theta_{s+\sigma-\tau} \omega)|^{\sigma_k} + |\psi_2 h z(\theta_{s+\sigma-\tau} \omega)|^{\mu_k}).
\]

Obviously, \( 1 < \mu_k \leq N \) for \( 1 \leq k \leq N - 1 \). By (44), \( |\psi_1|^{\mu_k} \leq c(|\psi_1| + |\psi_1|^{N}) \). Since \( 2 < q_{k+1} \leq qN \), the interpolation (44) implies

\[
|h z(\theta_{s+\sigma-\tau} \omega)|^{\mu_k} \leq c(|h z(\theta_{s+\sigma-\tau} \omega)|^{N} + |h z(\theta_{s+\sigma-\tau} \omega)|^N).
\]

It is easy to verify \( \frac{\mu_k}{N - \mu_k} \in [1, qN] \). The interpolation (44) gives

\[
|h z(\theta_{s+\sigma-\tau} \omega)|^{\mu_k} \leq c(|h z(\theta_{s+\sigma-\tau} \omega)|^N + c|h z(\theta_{s+\sigma-\tau} \omega)|^{N} - |\psi_2|^{N} + c|h z(\theta_{s+\sigma-\tau} \omega)|^{N}.
\]

By integrating over \( \mathbb{R}^m \) and using (27), we have

\[
(f(x, u), v|v|^{q_k - 2}) \geq \frac{\alpha_1}{2q^{k+1}} \|v\|_{q_{k+1}}^{q_{k+1}} - c(1 + Z(\theta_{s+\sigma-\tau} \omega)). \tag{54}
\]

Since \( \nabla(|v|^{q_k - 2}v) = (q_k - 1)|v|^{q_k - 2}\nabla v \), we can estimate the Laplace term:

\[
\langle Au, |v|^{q_k - 2}v \rangle = (q_k - 1) \int_{\mathbb{R}^m} |\nabla u|^{q_k - 2} |\nabla u| v^{q_k - 2} \nabla v dx
\]

\[
= (q_k - 1) \int_{\mathbb{R}^m} |v|^{q_k - 2} \left( |\nabla u|^{p} - |\nabla u|^{q_k - 2} |\nabla u| \nabla h z(\theta_{s+\sigma-\tau} \omega) \right) dx
\]

\[
\geq - c \int_{\mathbb{R}^m} |v|^{q_k - 2} |\nabla h z(\theta_{s+\sigma-\tau} \omega)|^p dx.
\]

Since \( \frac{q_{k+1}}{q_k - 2} \) and \( \mu_k \) are conjugate exponents, and \( p < p \mu_k \leq pN \), we have

\[
\langle Au, |v|^{q_k - 2}v \rangle \geq - \frac{\alpha_1}{2q^{k+1}} \|v\|_{q_{k+1}}^{q_{k+1}} - c \int_{\mathbb{R}^m} |\nabla h z(\theta_{s+\sigma-\tau} \omega)|^{\mu_k} dx
\]

\[
\geq - \frac{\alpha_1}{2q^{k+1}} \|v\|_{q_{k+1}}^{q_{k+1}} - c(\|\nabla h\|_p^{pN} + \|\nabla h\|_{pN} Z(\theta_{s+\sigma-\tau} \omega)
\]

\[
\geq - \frac{\alpha_1}{2q^{k+1}} \|v\|_{q_{k+1}}^{q_{k+1}} - c Z(\theta_{s+\sigma-\tau} \omega). \tag{55}
\]
Let \( \nu_k := 1/(1 - \frac{q_k - 1}{q_k + 1}) = k + 1 - \frac{k-1}{q-1} \) and so \( 2 \leq \nu_k \leq N \). The interpolation (44) gives

\[
|g(s)|v^{q_k-2}v| \leq \alpha_1 \frac{1}{2q^2+3} \|v\|_{q_k+1}^{q_k+1} + c\|g(s)\|_{\nu_k}^{\nu_k} \\
\leq \alpha_1 \frac{1}{2q^2+3} \|v\|_{q_k+1}^{q_k+1} + c(\|g(s)\|^2 + \|g(s)\|_N^N).
\]

(56)

Substituting (54)-(56) into (53), we obtain, for \( \hat{c} = \frac{\alpha_1 q_k}{2q^2+2} \),

\[
\frac{d}{ds} \|v\|_{q_k}^q + \lambda \|v\|_{q_k}^q + \hat{c} \|v\|_{q_k+1}^{q_k+1} \leq c(1 + \|g(s)\|^2 + \|g(s)\|_N^N + Z(\theta_{k+\sigma-\omega})).
\]

Integrating (57) from \( \hat{s} \in [\tau - \frac{1}{k}, \tau - \frac{1}{k+1}] \) to \( s \in [\tau - \frac{1}{k+1}, \tau] \), we obtain that for \( |\sigma| \leq 1 \),

\[
\|v(s)\|_{q_k}^q \leq \|v(\hat{s})\|_{q_k}^q + c \int_{\tau-1}^{\tau} \left( 1 + \|g(s)\|^2 + \|g(s)\|_N^N + Z(\theta_{k+\sigma-\omega}) \right) d\zeta.
\]

Integrating it \( \hat{s} \) over \( [\tau - \frac{1}{k}, \tau - \frac{1}{k+1}] \), we obtain that for all \( t \geq 1 \),

\[
\sup_{s \in [\tau - \frac{1}{k+1}, \tau]} \sup_{|\sigma| \leq 1} \int_{\tau-1}^{\tau} \|v(s, \tau - t, \theta_{\sigma-\omega}; v_{\tau-t})\|_{q_k}^q d\zeta \\
\leq \sup_{|\sigma| \leq 1} \int_{\tau-1}^{\tau} \|v(\zeta, \tau - t, \theta_{\sigma-\omega}; v_{\tau-t})\|_{q_k}^q d\zeta \\
+c(1 + G(\tau)) + c \sup_{|\sigma| \leq 1} \int_{\tau-1}^{\tau} Z(\theta_{k+\sigma-\omega}) d\zeta.
\]

By (27), the last term is bounded by

\[
cp(\omega) \int_{\tau-1}^{\tau} e^{\frac{1}{2}(|\zeta|)} d\zeta \sup_{|\sigma| \leq 1} e^{\frac{1}{2}|\sigma|} \leq cp(\omega).
\]

Therefore, by the inductive hypothesis (52) for \( k - 1 \), we have proved (51) for \( k \).

On the other hand, we integrate (57) over \( [\tau - \frac{1}{k+1}, \tau] \) to obtain, for all \( t \geq T_k \) and \( v_{\tau-t} \in D_k(\tau - t, \theta_{-\omega}) \),

\[
\sup_{|\sigma| \leq 1} \int_{\tau-1}^{\tau} \|v(s, \tau - t, \theta_{\sigma-\omega}; v_{\tau-t})\|_{q_k+1}^{q_k+1} d\zeta \\
\leq c \sup_{|\sigma| \leq 1} \|v(\tau - \frac{1}{k+1}, \tau - t, \theta_{\sigma-\omega})\|_{q_k}^q + c(1 + G(\tau)) \\
+c \sup_{|\sigma| \leq 1} \int_{\tau-1}^{\tau} Z(\theta_{k+\sigma-\omega}) d\zeta \leq cR(\tau, \omega),
\]

which completes the inductive proof.
Proposition 4. (Triple-compactness) Let

By Lemma 3.1 and using boundedness of

Since

By the same induction method as given in (52) for

Noting that

By the Gronwall lemma and by

Note that

(59)

Note that

Next, we prove a triple-compactness result on the time-space-sample localization. We denote by \( O_r = \{ x \in \mathbb{R}^m : |x| < r \}, r \in \mathbb{N} \) and recall that

\[ N = \left\{ \frac{r}{2} \right\} \geq 1. \]

Proposition 4. (Triple-compactness) Let \( \{ v_{0,n} \}_n \) be a bounded sequence in \( X_0 \) and let \( \sigma_n \to \sigma_0, \tau \in \mathbb{R}, \omega \in \Omega \). Then, there are \( s_0 \in (\tau - \frac{1}{N}, \tau) \), a subsequence \( \{ n^* \} \) of \( \{ n \} \) and \( v_0 \in X_0 \) such that, as \( n^* \to \infty \),

\[ v(s_0, \tau - 1, \theta_{\sigma_n} \omega; v_{0,n^*}) \to v_0 \text{ strongly in } L^2(O_r) \text{ for each } r \in \mathbb{N}. \]

Proof. We assume without lose of generality that \( |\sigma_n - \sigma_0| < 1 \) and let

\[ v_n := v_n(s) = v(s, \tau - 1, \theta_{\sigma_n} \omega; v_{0,n}), \forall s \in [\tau - 1, \tau]. \]

By Lemma 3.1 and using boundedness of \( \{ v_{0,n} \} \), we have

\[ \sup_{s \in [\tau - 1, \tau]} \| v_n(s) \|^2 + \int_{\tau - 1}^{\tau} \| \nabla v_n(s) \|_p^p ds \leq c \| v \|^2 + c \| v \|_{q_N}^q \leq c. \] (60)

By the same induction method as given in (52) for \( k = N - 1 \), we have

\[ \int_{\tau - \frac{1}{N}}^{\tau} \| v_n(s) \|_{q_N}^q ds \leq c + c \int_{\tau - 1}^{\tau} Z(\theta_{\sigma_n} \omega) ds \leq c. \] (61)

Since \( q_N \geq p > 2 \), by the interpolation (44), it follows from (60) and (61) that

\[ \int_{\tau - \frac{1}{N}}^{\tau} \| v_n(s) \|_p^p ds \leq \int_{\tau - \frac{1}{N}}^{\tau} (\| v_n(s) \|^2 + \| v_n(s) \|_{q_N}^q) ds \leq c. \] (62)

Noting that \( \| v \|_{W^{1,p}} = \| v \|_p + \| \nabla v \|_p \), we see from (60) and (62) that

\[ \{ v_n \} \text{ bounded in } L^\infty(\tau - \frac{1}{N}, \tau; X_0) \cap L^p(\tau - \frac{1}{N}, \tau; W^{1,p}(\mathbb{R}^m)). \] (63)

On the other hand, it is easy from (63) to prove that for each \( r \in \mathbb{N} \),

\[ \{ A(v_n + hz(\theta_{\sigma_n} \omega)) \} \text{ bounded in } L^\hat{p}(\tau - \frac{1}{N}, \tau; W^{-1,\hat{p}}(O_r)) \]

\[ \{ f(x, v_n + hz(\theta_{\sigma_n} \omega)) \} \text{ bounded in } L^{\hat{q}}(\tau - \frac{1}{N}, \tau; L^{\hat{q}}(O_r)), \]
where $\hat{p}$, $\hat{q}$ are conjugate exponents of $p, q$. Given $\mu = \min\{\hat{p}, \hat{q}\}$, by Eq.(28), we obtain that for each $r \in \mathbb{N}$,
\[
\{\frac{dv_n}{ds}\} \text{ bounded in } L^\mu(\tau - \frac{1}{N}, \tau; W^{-1, \hat{p}}(\mathcal{O}_r) + L^{\hat{q}}(\mathcal{O}_r)). \tag{64}
\]
By (63), there is $\tilde{v} \in L^2(\tau - \frac{1}{N}, \tau; X_0)$ such that, passing to a subsequence,
\[
v_n \to \tilde{v} \text{ weakly in } L^2(\tau - \frac{1}{N}, \tau; L^2(\mathbb{R}^m)).
\]

Let $r \in \mathbb{N}$ be fixed. By applying the Aubin’s compactness theorem [38] on three spaces $Y_0 = W^{1,p}(\mathcal{O}_r)$, $Y = L^2(\mathcal{O}_r)$ and $Y_1 = W^{-1, p}(\mathcal{O}_r) + L^{q}(\mathcal{O}_r)$, we see from (63) and (64) that there is a subsequence $\{v_n^r\}$ of $\{v_n^{r-1}\}$ such that
\[
v_n^r \to \tilde{v} \text{ strongly in } L^2(\tau - \frac{1}{N}, \tau; L^2(\mathcal{O}_r)).
\]
We still use $\{v_n\}$ to denote the diagonal subsequence $\{v_n^0\}$, then
\[
v_n \to \tilde{v} \text{ strongly in } L^2(\tau - \frac{1}{N}, \tau; L^2(\mathcal{O}_r)), \forall r \in \mathbb{N}. \tag{65}
\]
For each $r \in \mathbb{N}$, by (65), we can subsequently choose a set $I_r \subset [\tau - \frac{1}{N}, \tau]$ of measure $\frac{1}{N}$ and a subsequence $\{v_{n_r}^r\}$ of $\{v_{n_r}^{r-1}\}$ such that
\[
v_{n_r}^r(s) = v(s, \tau - 1, \theta_{\sigma_n, \omega}; v_{0,n}) \to \tilde{v}(s) \text{ strongly in } L^2(\mathcal{O}_r) \text{ for all } s \in I_r.
\]
Therefore, the diagonal subsequence $\{v_{n_r}\} = \{v_{n_r}^r\}$ satisfies: as $n^* \to \infty$,
\[
v_{n^*}(s) = v(s, \tau - 1, \theta_{\sigma_n, \omega}; v_{0,n^*}) \to \tilde{v}(s) \text{ strongly in } L^2(\mathcal{O}_r)
\]
for each $r \in \mathbb{N}$ and $s \in \cap_{n=1}^{\infty} I_r$. Since $\cap_{n=1}^{\infty} I_r$ has a positive measure, there is at least one $s_0 \in \cap_{n=1}^{\infty} I_r$, which finishes the proof. \hfill \qed

5. Local-sample uniform estimate of the tail. We show that the tail of the solution is uniformly small in local samples when the initial data belong to $X_{N-1}$.

**Proposition 5.** If $D_{N-1} \in \mathcal{D}_{N-1}$, $\tau \in \mathbb{R}$ and $\omega \in \Omega$ are fixed, then, for each $\varepsilon > 0$, there exist $T = T(\varepsilon)$ and $r_0 = r_0(\varepsilon)$ such that
\[
\sup_{s \in [\tau - 1, \tau]} \sup_{\{\sigma \leq 1, |\sigma| \geq \tau \}} \int_{|x| \geq r_0} |v(s, \tau - t, \theta_{\sigma, -t \omega}v; v_{\tau - t})|^2 dx \leq \varepsilon, \tag{66}
\]
whenever $t \geq T$ and $v_{\tau-t} \in D_{N-1}(\tau - t, \theta_{-t \omega})$.

**Proof.** We consider a cut-off function defined by
\[
\xi_r(x) = \xi\left(\frac{|x|^2}{r^2}\right), \quad x \in \mathbb{R}^m, \quad r > 0,
\]
where $\xi : \mathbb{R}^+ \to [0, 1]$ is a smooth function such that $\xi \equiv 0$ on $[0, 1]$ and $\xi \equiv 1$ on $[4, +\infty)$. It is easy to verify that $\|\xi_r\|_{\infty} \leq 1$ and $\|\nabla \xi_r\|_{\infty} \leq C/r$ for all $r > 0$.

We then multiply Eq.(28) with $\xi_r v$ to find
\[
\frac{1}{2} \frac{d}{ds} \int_{\mathbb{R}^m} \xi_r |v|^2 dx + \lambda \int_{\mathbb{R}^m} \xi_r |v|^2 dx + \langle Au, \xi_r v \rangle + \langle f(x, u), \xi_r v \rangle = \langle g(s), \xi_r v \rangle. \tag{67}
\]
By (27), we have
\[
\langle Au, \xi_r v \rangle = \int_{\mathbb{R}^m} v|\nabla u|^p - 2\nabla u \cdot \nabla \xi_r dx + \int_{\mathbb{R}^m} \xi_r |\nabla u|^p - 2\nabla u \cdot \nabla v dx
\]
\[
\geq - \frac{c}{r} (||u||_p^p + ||\nabla u||_p^p) + \int_{\mathbb{R}^m} \xi_r |\nabla u|^p dx - \int_{\mathbb{R}^m} \xi_r |\nabla u|^p - 2\nabla u \cdot \nabla h z dx
\]
\[
\geq - \frac{c}{r} (||u||_p^p + ||\nabla u||_p^p) - cZ(\theta_{s + \sigma - \tau} \omega) \int_{|x| \geq r} |\nabla h|^p dx. \tag{68}
\]
For the nonlinearity, since \( \xi_r \geq 0 \), by (32), we have
\[
(f(x, u), \xi_r v) \geq - c \int_{\mathbb{R}^m} \xi_r (|\psi_1| + |h z|^q + |\psi_2 h z|) dx
\]
\[
\geq - c \int_{|x| \geq r} (|\psi_1| + \psi_2^q + Z(\theta_{s + \sigma - \tau}) (h^2 + |h|^q)) dx.
\]
The Young inequality implies
\[
|\langle g(s), \xi_r v \rangle| \leq \frac{\lambda}{4} \int_{\mathbb{R}^m} \xi_r |v|^2 dx + c \int_{|x| \geq r} g^2(s, x) dx. \tag{69}
\]
We substitute (68)-(69) into (67) to obtain
\[
\frac{d}{ds} \int_{\mathbb{R}^m} \xi_r |v|^2 dx + \frac{3\lambda}{2} \int_{\mathbb{R}^m} \xi_r |v|^2 dx
\]
\[
\leq \frac{c}{r} (||u||_p^p + ||\nabla u||_p^p) + \int_{|x| \geq r} (|\psi_1| + \psi_2^q) + c \int_{|x| \geq r} g^2(s) + cZ(\theta_{s + \sigma - \tau} \omega) \int_{|x| \geq r} (h^2 + |h|^q + |\nabla h|^p).
\]
Given \( \varepsilon > 0 \), there is a \( r_1 > 0 \) such that for all \( r \geq r_1 \),
\[
\frac{d}{ds} \int_{\mathbb{R}^m} \xi_r |v|^2 dx + \lambda \int_{\mathbb{R}^m} \xi_r |v|^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^m} \xi_r |v|^2 dx
\]
\[
\leq \varepsilon ||u||_p^p + \varepsilon ||\nabla u||_p^p + \varepsilon (1 + Z(\theta_{s + \sigma - \tau} \omega)) + c \int_{|x| \geq r} g^2(s, x) dx. \tag{70}
\]
Applying the Gronwall lemma on (70) over \([\tau - t, \tau] \), we have, for all \( r \geq r_1 \),
\[
\sup_{|\sigma| \leq 1} \int_{\tau - t}^{\tau} e^{\lambda(s - \tau)} \int_{\mathbb{R}^m} \xi_r |v(s, \tau - t, \theta_{\sigma - \tau} \omega; v_{\tau - t})|^2 dx ds
\]
\[
\leq c e^{-\lambda t} \int_{\mathbb{R}^m} \xi_r |v_{\tau - t}|^2 dx + \varepsilon (J_1 + J_2 + J_3) + J_4(r), \tag{71}
\]
where \( J_1, J_2, J_3, J_4(r) \) are defined and estimated as follows.

Since \( D_{N-1} \in \mathbb{D}_{N-1} \), by Corollary 2, there is a \( T > 0 \) such that for all \( t \geq T \) and \( v_{\tau - t} \in D_{N-1}(\tau - t, \theta_{\sigma - \tau} \omega) \),
\[
J_1 := c \sup_{|\sigma| \leq 1} \int_{\tau - t}^{\tau} e^{\lambda(s - \tau)} ||v(s, \tau - t, \theta_{\sigma - \tau} \omega; v_{\tau - t})||_p^p ds \leq c R(\tau, \omega). \tag{72}
\]
By (46) in Proposition 2,
\[
J_2 := c \sup_{|\sigma| \leq 1} \int_{\tau - t}^{\tau} e^{\lambda(s - \tau)} ||\nabla u(s, \tau - t, \theta_{\sigma - \tau} \omega; v_{\tau - t})||_p^p ds \leq c R(\tau, \omega). \tag{73}
\]
By (27), we have, for all $t > 0$,
\[ J_3 := c \sup_{|\sigma| \leq 1} \int_{\tau-t}^\tau e^{\lambda(s-\tau)}(1 + Z(\theta_{s+\sigma-\tau}\omega))ds \leq c(1 + \rho(\omega)). \] (74)

The Lebesgue controlled convergence theorem implies that as $r \to +\infty$,
\[ J_4(r) := c \int_{\tau-t}^\tau e^{\lambda(s-\tau)} \int_{|x| \geq r} g^2(s, x)dxds \to 0. \] (75)

Finally, since $D_{N-1} \in \mathcal{D}_{N-1} \subset \mathcal{D}_0$ and $\|\xi_r\|_{\infty} = 1$, we have
\[ e^{-\lambda t} \int_{\mathbb{R}^m} |\xi_r| |v_{\tau-t}|^2 dx \to 0 \text{ as } t \to +\infty. \] (76)

We substitute (72)-(76) into (71) to find
\[ \sup_{|\sigma| \leq 1} \int_{\tau-t}^\tau e^{\lambda(s-\tau)} \int_{\mathbb{R}^m} \xi_r |v(s, \tau - t, \theta_{\sigma-\tau}\omega; v_{\tau-t})|^2 dxds < c\varepsilon, \] (77)
uniformly in $t \geq T$, $r \geq r_2(\geq r_1)$ and $v_{\tau-t} \in D_{N-1}(\tau - t, \theta_{-\tau}\omega)$.

On the other hand, we integrate (70) from $\hat{s} \in [\tau - 2, \tau - 1]$ to $s \in [\tau - 1, \tau]$, the result is
\[ \int_{\mathbb{R}^m} \xi_r |v(s, \tau - t, \theta_{\sigma-\tau}\omega; v_{\tau-t})|^2 dx \]
\[ \leq \int_{\mathbb{R}^m} \xi_r |v(\hat{s})|^2 dx + \varepsilon \int_{\tau-2}^s (\|\nabla u(\varsigma)\|_p^p + \|v(\varsigma)\|_p^p) d\varsigma \]
\[ + \varepsilon \int_{\tau-2}^s (1 + Z(\theta_{\varsigma+\sigma-\tau}\omega))d\varsigma + c \int_{\tau-2}^s \int_{|x| \geq r} g^2(\varsigma, x)d\varsigma ds. \] (78)

Then, by integrating (78) about $\hat{s} \in [\tau - 2, \tau - 1]$, we see from (77) and (72)-(75) that
\[ \sup_{s \in [\tau - 1, \tau]} \sup_{|\sigma| \leq 1} \int_{\mathbb{R}^m} \xi_r |v(s, \tau - t, \theta_{\sigma-\tau}\omega; v_{\tau-t})|^2 dx \]
\[ \leq \sup_{|\sigma| \leq 1} \int_{\tau-2}^\tau \int_{\mathbb{R}^m} \xi_r |v(\hat{s}, \tau - t, \theta_{\sigma-\tau}\omega; v_{\tau-t})|^2 dxds \]
\[ + c\varepsilon(J_1 + J_2 + J_3) + cJ_4(r) \leq c\varepsilon, \]
for all $t \geq T$, $r \geq r_2$ and $v_{\tau-t} \in D_{N-1}(\tau - t, \theta_{-\tau}\omega)$. Therefore, by $\int_{|x| \geq 2r} |v|^2 \leq \int_{\mathbb{R}^m} \xi_r |v|^2$, we obtain (66) as desired. \qed

6. Existence and probabilistic continuity of a PRA. In the last section, we summarize and prove the application results.

**Theorem 6.1.** Let $\Phi$ be the cocycle generated from the $p$-Laplace equation, $\mathcal{D}_0$ is the local-uniform tempered universe and $\mathcal{D}$ is the tempered universe.

(i) $\Phi$ has a unique $\mathcal{D}_0$-pullback attractor $\mathcal{A}_0 \in \mathcal{D}_0$ such that $\mathcal{A}_0$ is local-sample compact in $X_0 = L^2(\mathbb{R}^m)$.

(ii) $\Phi$ has a unique $\mathcal{D}$-pullback random attractor $\mathcal{A} \in \mathcal{D}$ such that $\mathcal{A} = \mathcal{A}_0$.

(iii) $\mathcal{A}$ is upper semi-continuous in sample:
\[ \lim_{s \to \infty} \text{dist}(\mathcal{A}(\tau, \theta_\omega), \mathcal{A}(\tau, \theta_{s_0}\omega)) = 0, \forall \tau, \omega \in \Omega. \] (79)

(iv) $\mathcal{A}$ is probabilistically continuous in sample, i.e. (7) holds true.
(v) $A$ is probabilistically lower semi-continuous in time:
$$\lim_{\tau \to \tau_0} P\{\omega \in \Omega : \text{dist}(A(\tau_0, \omega), A(\tau, \omega)) \geq \delta\} = 0, \forall \delta > 0, \tau_0 \in \mathbb{R}.$$ \hfill (80)

Proof. We mainly prove (i) in three steps.

**Step 1.** The cocycle $\Phi$ is local-sample asymptotically compact on $\mathcal{D}_{N-1}$. More precisely, the sequence
$$\{\Phi(t_n, \tau - t_n, \theta_{\sigma_n - t_n}\omega)_{\nu \rightarrow t_n}\}_{n} = \{\nu(\tau, \tau - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n})\}_{n} \quad \text{(81)}$$
is pre-compact in $X_0$, whenever
$$|\sigma_n| \leq 1, \ t_n \to +\infty, \ \nu_{\tau - t_n} \in D_{N-1}(\tau - t_n, \theta_{t_n}\omega) \text{ with } D_{N-1} \in \mathcal{D}_{N-1}.$$  

Indeed, passing to a subsequence, we can assume that $\sigma_n \to \sigma_0$. Since $D_{N-1} \in \mathcal{D}_{N-1} \subset \mathcal{D}_0$, it follows from Proposition 2 that
$$\{\nu_{0n}\}_{n} := \{\nu(\tau - 1, \tau - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n})\}_{n} \quad \text{bounded in } X_0.$$  

By the triple-compactness in Proposition 4, there are $s_0 \in (\tau - 1, \tau)$ and $v_0 \in X_0$ such that, passing to a subsequence,
$$\nu(s_0, \tau - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n}) = \nu(s_0, \tau - 1, \theta_{\sigma_n - t_n}\omega; v_{0n})$$
$$\to v_0 \quad \text{strongly in } L^2(O_r) \text{ for each } r \in \mathbb{N}. \hfill (82)$$

Given $\varepsilon > 0$, by Proposition 5, there are $r_1 \in \mathbb{N}$ and $n_1 \in \mathbb{N}$ such that
$$\int_{|x| \geq r_1} |\nu(s_0, \tau - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n})|^2 \nu_{0n} \, dx < \varepsilon, \ \forall n \geq n_1. \hfill (83)$$

Since $v_0 \in X_0$, one can choose $r_2 \in \mathbb{N}$ with $r_2 \geq r_1$ such that
$$\int_{|x| \geq r_2} |v_0(x)|^2 \nu_{0n} \, dx < \varepsilon. \hfill (84)$$

By (82), $\nu(s_0, \tau - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n}) \to v_0 \in L^2(O_{r_2})$. We choose $n_2 \geq n_1$ such that
$$\int_{|x| < r_2} |\nu(s_0, \tau - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n}) - v_0|^2 \nu_{0n} \, dx < \varepsilon, \ \forall n \geq n_2. \hfill (85)$$

Hence, by (83)-(85), for all $n \geq n_2$,
$$\|\nu(s_0, \tau - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n}) - v_0\|^2_{L^2(\mathbb{R}^m)}$$
$$\leq \int_{|x| < r_2} |\nu(s_0, \tau - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n}) - v_0|^2 \nu_{0n} \, dx$$
$$+ 2 \int_{|x| \geq r_2} (|\nu(s_0, \tau - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n})|^2 + |v_0|^2) \nu_{0n} \, dx < 5\varepsilon.$$  

In a conclusion, $\nu(s_0, \tau - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n}) \to v_0$ strongly in $X_0$.

By the triple-continuity as proved in Proposition 1, we obtain

$$\nu(\tau, s_0, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n})$$
$$= \nu(\tau, s_0, \theta_{\sigma_n - t_n}\omega; \nu(s_0, \tau - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n}))$$
$$\to v(\tau, s_0, \theta_{\sigma_n - t_n}\omega; v_0), \ \text{strongly in } X_0.$$  

**Step 2.** The cocycle $\Phi$ is local-sample asymptotically compact on $\mathcal{D}_0$. It suffices to prove $\{\nu(\tau, t - t_n, \theta_{\sigma_n - t_n}\omega; \nu_{\tau - t_n})\}_{n}$ is pre-compact, whenever $|\sigma_n| \leq 1, \ t_n \to +\infty$ and $\nu_{\tau - t_n} \in D_0(\tau - t_n, \theta_{t_n}\omega)$ with $D_0 \in \mathcal{D}_0$. 

For this end, we define a bi-parametric set $\mathcal{K}_{N-1}$ in $X_{N-1} = X_0 \cap L^q_{N-1}$ by

$$\mathcal{K}_{N-1}(\tau, \omega) = \mathcal{K}_0(\tau, \omega) \cap \{ u \in L^q_{N-1} : \| u \|_{L^q_{N-1}} \leq c_{N-1} R(\tau, \omega) \},$$

where $\mathcal{K}_0$ is the absorbing set given in (47). By (49), we know $\mathcal{K}_{N-1} \in \mathcal{D}_{N-1}$.

By Proposition 3, $\mathcal{K}_{N-1}$ locally-uniformly absorbs $D_0$ in the sense of (50). In particular, for each fixed $i \in \mathbb{N}$, there exists a large time $T_i = T_i(D_0, \tau, \omega)$ such that

$$| v(\tau - i, \tau - i - t, \theta_{\sigma_i - (\tau - i)} \theta_i \omega ; v_{\tau - i - t}, v_{\tau - i}) \rangle \in \mathcal{K}_{N-1}(\tau - i, \theta_i \omega)$$

whenever $|\sigma| \leq 1$, $t \geq T_i$ and $v_{\tau - i - t} \in D_0(\tau - i - t, \theta_i \omega)$.

Since $t_n \to +\infty$, we can choose a subsequence $\{n_i\}$ of $\{n\}$ such that $t_{n_i} - i \geq T_i$ for each $i \in \mathbb{N}$. Since $v_{\tau - t_{n_i}} \in D_0(\tau - t_{n_i}, \theta_i \omega)$, we can rewrite

$$v_{\tau - t_{n_i}} \in D_0((\tau - i - (t_{n_i} - i), \theta_{-(t_{n_i} - i)} \theta_i \omega).$$

Note that $|\sigma_{n_i}| \leq 1$, it follows from (86) that

$$y_i = v(\tau - i, \tau - i - (t_{n_i} - i), \theta_{\sigma_{n_i} - (\tau - i)} \theta_i \omega; v_{\tau - t_{n_i}}) \in \mathcal{K}_{N-1}(\tau - i, \theta_i \omega).$$

By Step 1, $\Phi$ is local-sample asymptotically compact on $\mathcal{K}_{N-1} \in \mathcal{D}_{N-1}$. Hence,

$$\{ v(\tau, \tau - t_{n_i}, \theta_{\sigma_{n_i} - \tau} \omega; v_{\tau - t_{n_i}}) \}_{i=1}^\infty = \{ v(\tau, \tau - i, \theta_{\sigma_{n_i} - \tau} \omega; y_i) \}_{i=1}^\infty$$

has a convergent subsequence.

**Step 3.** $\Phi$ has a unique $\mathcal{D}_0$-pullback attractor $\mathcal{A}_0 \in \mathcal{D}_0$ such that $\mathcal{A}_0$ is local-sample compact. Indeed, by Corollary 1, $\Phi$ has a $\mathcal{D}_0$-pullback absorbing set $\mathcal{K}_0 \in \mathcal{D}_0$. By Step 2, $\Phi$ is $\mathcal{D}_0$-pullback asymptotically compact. By [39, Theorem 2.23], $\Phi$ has a unique $\mathcal{D}_0$-pullback attractor $\mathcal{A}_0 = \alpha_{\mathcal{K}_0} \in \mathcal{D}_0$.

On the other hand, by Lemma 3.3, the universe $\mathcal{D}_0$ is local-sample closed. By Step 2, $\Phi$ is local-sample asymptotically compact on $\mathcal{D}_0$. So, it follows from Theorem 2.9 that $\mathcal{A}_0$ is local-sample compact.

(ii) By the usual method, one can show that $\mathcal{K}_0$ is a $\mathcal{D}$-pullback random absorbing set and $\Phi$ is $\mathcal{D}$-pullback asymptotically compact. So, $\Phi$ has a unique random $\mathcal{D}$-pullback attractor $\mathcal{A} \in \mathcal{D}$. In addition, both $\mathcal{A}$ and $\mathcal{A}_0$ are the $\alpha$-limit set of the absorbing set $\mathcal{K}_0$, that is,

$$\mathcal{A}(\tau, \omega) = \alpha_{\mathcal{K}_0}(\tau, \omega) = \mathcal{A}_0(\tau, \omega).$$

(iii)-(v) By Proposition 1, the cocycle is triple-continuous. By Lemma 3.3, the universe $\mathcal{D}_0$ is local-sample closed. By Step 2, $\Phi$ is $\mathcal{D}_0$-pullback local-sample asymptotically compact. By (ii), $\mathcal{A} = \mathcal{A}_0 \in \mathcal{D}_0$. Therefore, the assertions (iii)-(v) follow from the abstract Theorem 2.12 immediately.

Notice that $\mathcal{D}_0$ is also local-time closed. Hence, by using the same method as given in [14, 41], one can show that $\mathcal{A}_0$ is local-time compact, that is, $\cup_{|s| \leq 1} \mathcal{A}_0(\tau + s, \omega)$ is pre-compact. By combining Theorem 6.1 and [41, theorem 2.2], we obtain the probabilistic continuity of the attractor $\mathcal{A}$.

**Corollary 3.** The attractor $\mathcal{A} = \mathcal{A}_0 \in \mathcal{D}_0$ in Theorem 6.1 has the following properties.

(I) $\mathcal{A}$ is both local-time compact and local-sample compact.

(II) $\mathcal{A}$ is upper semi-continuous in time-sample:

$$\lim_{n \to \infty} \text{dist}(\mathcal{A}(\tau_n, \theta_{s_n} \omega), \mathcal{A}(\tau_0, \theta_{s_0} \omega)) = 0, \text{ whenever } \tau_n \to \tau_0, s_n \to s_0.$$
(III) $A$ is probabilistically continuous in time-sample: for $\delta > 0$, $\tau_n \to \tau_0$ and $s_n \to s_0$,

$$\lim_{n \to \infty} P\{\omega \in \Omega : \text{dist}\,h(A(\tau_0, \theta_{s_0} \omega), A(\tau_n, \theta_{s_n} \omega)) \geq \delta\} = 0. \tag{88}$$

Remark 1. It remains open even for the existence of an attractor in the most weakly dissipative case that $q = 2 < p$. In this case, $q_k \equiv 2$, the induction method in Proposition 3 loses effectiveness and thus the tail-estimate in Proposition 5 cannot be established.

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