Research Article
Cycle Intersection for $\text{SO}(p, q)$-Flag Domains

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A real form $G_0$ of a complex semisimple Lie group $G$ has only finitely many orbits in any given compact $G$-homogeneous projective algebraic manifold $Z = G/Q$. A maximal compact subgroup $K_0$ of $G_0$ has special orbits $C$ which are complex submanifolds in the open orbits of $G_0$. These special orbits $C$ are characterized as the closed orbits in $Z$ of the complexification $K$ of $K_0$. These are referred to as cycles. The cycles intersect Schubert varieties $S$ transversely at finitely many points. Describing these points and their multiplicities was carried out for all real forms of $\text{SL}(n, C)$ by Brecan (Brecan, 2014) and (Brecan, 2017) and for the other real forms by Abu-Shoga (Abu-Shoga, 2017) and Huckleberry (Abu-Shoga and Huckleberry). In the present paper, we deal with the real form $\text{SO}(p, q)$ acting on the $\text{SO}(2n, C)$-manifold of maximal isotropic full flags. We give a precise description of the relevant Schubert varieties in terms of certain subsets of the Weyl group and compute their total number. Furthermore, we give an explicit description of the points of intersection in terms of flags and their number. The results in the case of $G/Q$ for all real forms will be given by Abu-Shoga and Huckleberry.

1. Background

Let $G$ be a complex semisimple Lie group and $B$ a Borel subgroup. The compact algebraic homogeneous space $Z = G/B$ is called a complex flag manifold. Let $G_0$ be a real form of $G$ in the sense that there is an antiholomorphic involution $\tau : G \rightarrow G$ such that $G_0$ is the fixed point set of $\tau$. It is well known [1] that there are only finitely many $G_0$-orbits on $Z$. Therefore, at least one of them must be open. Such open orbits are called flag domains.

Fix a Cartan involution $\theta : G_0 \rightarrow G_0$. The fixed point set of $\theta$ in $G_0$ is a maximal compact subgroup of $G_0$ denoted by $K_{0,\theta}$ and $K$ is its complexification.

If $D$ is a flag domain on $Z$, then $K_0$ has exactly one open orbit in $D$ that is a complex submanifold and denoted by $C_0$. The complex submanifold $C_0$ is both the unique $K$-orbit in $D$ that is compact and the unique $K_0$-orbit in $D$ that is complex. This is the origin of what is known as “Matsuki duality.”

Consider an Iwasawa decomposition $G_0 = K_0A_0N_0$ of the real form $G_0$. We refer to a Borel subgroup $B \subset G$ as an Iwasawa–Borel subgroup of $G$ if it contains an Iwasawa factor $A_0N_0$. Of course, these are just the Borel subgroups which occur as the isotropy groups at points of the closed $G_0$-orbit in $G/B$.

In general, given any Borel subgroup $B$ in $G$ and a point $z \in G/B$, the closure $S = \text{cl}(\mathcal{O})$ of the orbit $\mathcal{O} = B.z$ is called the Schubert variety or Schubert cycle. An Iwasawa–Schubert variety $S = \text{cl}(\mathcal{O})$ is transversal (relative to an open $G_0$-orbit $D$) if $S \cap D$ is nonempty and contained in the open $B$-orbit $\mathcal{O}$, and $\text{codim} S = q - r$ where $r = \text{dim} C_0$ and the intersection $S \cap C_0$ is transversal at each of its points.

Thus, although the determination of the $r$-codimensional Iwasawa–Schubert varieties which have nonempty intersection with $D$, along with their points of intersection with $C_0$, is apparently a problem of a combinatorial nature, such information has complex analytic significance.

Since $S \cap D \neq \emptyset$ implies that $S \cap C_0 \neq \emptyset$, the first goal of this paper is to determine which Schubert varieties $S$ have nonempty intersection with $C_0$. After doing so, we then describe this intersection in the case where $S$ is of complementary dimension to $C_0$. The Schubert varieties are determined by the elements of the Weyl group $W_I$ of a distinguished maximal torus $T_I$ in the Borel subgroup $B_I$ which fixes a certain base point in $\gamma_{cI}$. These Schubert varieties are denoted by $S_w$ where $w \in W_I$. Consequently,
our results are formulated in terms of conditions on elements \( w \) in the Weyl group \( W_f \). In this paper, we discussed both even and odd cases for \( p \) and \( q \). (In some steps, we discuss only the even case because the odd case is included in the even case.)

2. Flag Intersection for \( \text{SO}(p, q) \)

2.1. Preliminaries. Consider the semisimple Lie group \( G = \text{SO}(m, C) \) with the real form \( \text{SO}(p, q) \). In this case, it is convenient to choose the bilinear form \( b \) on \( C^{2m} \) depending on \( p \) and \( q \). If \( p \) or \( q \) is even, then we choose it in the usual way:

\[
b(v, w) = -\sum_{i=1}^{q} v_i w_i + \sum_{i=q+1}^{m} v_i w_i.
\]

Let \( \sigma \) be the complex conjugation \( \sigma(v) = \bar{v} \), then the Hermitian form \( h(v, w) \) of signature \( (p, q) \) which defines the real form is defined by

\[
h(v, w) = b(v, \sigma(w)) = -\sum_{i=1}^{q} v_i \sigma(w_i) + \sum_{i=q+1}^{m} v_i \sigma(w_i).
\]

If both \( p \) and \( q \) are odd, then the complex bilinear form is

\[
b(v, w) = -\sum_{i=1}^{q-1} v_i w_i + \sum_{i=q+1}^{m} v_i w_i,
\]

and the Hermitian form \( h(v, w) \) is defined by

\[
h(v, w) = -\sum_{i=1}^{q} v_i \bar{w}_i + \sum_{i=q+1}^{m} v_i \bar{w}_i.
\]

The real form is \( G_0 = \text{SO}(p, q) = \text{SO}(2n, C) \cap \text{SU}(p, q) \). A Cartan involution \( \theta = \text{SO}(p, q) \rightarrow \text{SO}(p, q) \) in the Lie algebra level is given by \( \theta(g) = -g^\theta \), and the maximal compact subgroup associated with it is \( K_0 = \text{SO}(p) \times \text{SU}(q) \subset \text{SU}(p) \times \text{U}(q) \).

2.2. Introduction to the Flags in \( Z \). Recall that the (indefinite metric) orthogonal group \( G = \text{SO}(p, q) \) acts on the space of full flags by two orbits: closed orbit and open orbit. The closed orbit is the set of all maximally \( b \)-isotropic full flags \( F \) with respect to \( b \), where the maximally \( b \)-isotropic full flag \( F \) is defined as a sequence of \( (2n+1) \)-vector spaces:

\[
F = \{0\} \subset V_0 \subset V_1 \subset \ldots \subset V_{2n} = C^{2n},
\]

such that \( \dim V_i = i \), for all \( 0 \leq i \leq 2n \), and \( V_{2n-i} = V_i^\perp \), for \( 1 \leq i \leq n \). In particular, \( V_i \) is isotropic for \( 1 \leq i \leq n \). The set of all maximal isotropic flags is denoted by \( Z \). The manifold structure of \( Z \) arises from the fact that the Lie group \( G \) acts transitively on it with \( Z = G/B \) where \( B \) is the stabilizer of any particular maximally \( b \)-isotropic flag in \( Z \).

The signature of a flag \( F = \{0 \subset V_1 \subset \ldots \subset V_{2n}\} \subset Z \) with respect to \( h \) consists of three sequences:

\[
a : 0 \leq a_1 \leq a_2 \leq \ldots \leq a_{2n},
b : 0 \leq b_1 \leq b_2 \leq \ldots \leq b_{2n},
d : 0 \leq d_1 \leq d_2 \leq \ldots \leq d_{2n},
\]

with sign \( (V_i) = (a_i, b_i, d_i) \) where \( a_i \) (resp., \( b_i \)) denotes the dimension of a maximal negative (positive) subspace and \( d_i \) the degeneracy of the restriction of \( h \) to \( V_i \) and \( a_i + b_i + d_i = i \) \( \forall i \leq 1 \leq 2n \). The 3-tuple \( (a, b, d) \) is called the signature of the flag \( F \). If \( d = 0 \), we refer to \( F \) as being nondegenerate and write sign \( (F) = (a, b) \).

2.3. Base Points. To parametrize the Schubert cells, it is important to focus on a maximal Torus \( T_I \) in an Iwasawa–Borel subgroup \( B_I \) of \( G \) which fixes the associated flag \( F_I \). On the contrary, we need another maximal torus \( T_S \) which defines a basis to determine the base cycles called the standard basis. These two tori are conjugate.

In the following, we define bases depending on \( p \) and \( q \) as well as the positive and negative spaces which will be essential for understanding the base cycle. Although the goal of the paper is to handle the case where \( m \) is even, since in this paragraph the discussion for the odd case is essentially the same, we include it. Also, the fundamental maximal torus \( T_S \) is defined in each case by requiring that each basis vector is a \( T_S \)-eigenvector.

(i) If \( m = 2n \) and \( q \) are even, the ordered \( b \)-isotropic basis is

\[
e_1 + ie_2, e_3 + ie_4, \ldots, e_{2n-1} + ie_{2n}, e_{2n-1} - ie_{2n}, e_{2n-1} - ie_{2n}, \ldots, e_1 - ie_2,
\]

where

\[
E^- = \langle e_1 + ie_2, e_3 + ie_4, \ldots, e_{q-1} + ie_q, e_1 - ie_2, e_3 - ie_4, \ldots, e_{q-1} - ie_q \rangle,
\]

\[
E^+ = \langle e_{q+1} + ie_{q+2}, \ldots, e_{2n-1} + ie_{2n}, e_{q+1} - ie_{q+2}, \ldots, e_{2n-1} - ie_{2n} \rangle.
\]

(ii) If \( m = 2n \) and \( q \) are odd, the ordered \( b \)-isotropic basis is

\[
e_1 + ie_2, e_3 + ie_4, \ldots, e_{2n-1} + ie_{2n}, e_q - ie_{q-1}, e_{2n-1} - ie_{2n}, \ldots, e_1 - ie_2,
\]

where

\[
E^- = \langle e_1 + ie_2, e_3 + ie_4, \ldots, e_{q-1} + ie_q, e_1 - ie_2, e_3 - ie_4, \ldots, e_{q-1} - ie_q \rangle,
\]

\[
E^+ = \langle e_{q+1} + ie_{q+2}, \ldots, e_{2n-1} + ie_{2n}, e_{q+1} - ie_{q+2}, \ldots, e_{2n-1} - ie_{2n} \rangle.
\]

Note that the vectors \( e_q \) and \( e_{q+1} \) are isotropic and \( b(e_q, e_{q+1}) = -1 \). Moreover, \( h(e_3, e_5) = -1 \) and
\[ h(e_q, e_q) = 1 \] which means that \( e_q \in E^- \) and \( e_{q+1} \in E^+ \).

(iii) If \( m = 2n + 1 \) and \( q \) are even, the ordered \( b \)-isotropic basis is
\[
e_1 + i e_2, e_3 + i e_4, \ldots, e_{2n} + i e_{2n+1}, e_{q+1}, e_{2n} - i e_{2n+1}, \ldots, e_2 - i e_3,
\]
where
\[
E^- = \langle e_1 + i e_2, e_3 + i e_4, \ldots, e_{q-1} + i e_q, e_1 - i e_2, e_3 - i e_4, \ldots, e_{q-2} - i e_{q-1} \rangle,
\]
\[
E^+ = \langle e_{q+1}, e_{q+2} + i e_{q+3}, \ldots, e_{2n+1} + i e_{2n+2}, e_{q}, \ldots, e_{q+3}, \ldots, e_{2n+1} - i e_{2n+2} \rangle.
\]

(iv) If \( m = 2n + 1 \) and \( q \) are odd, the ordered \( b \)-isotropic basis is
\[
e_1 + i e_2, e_3 + i e_4, \ldots, e_{2n} + i e_{2n+1}, e_{q}, e_{2n} - i e_{2n+1}, \ldots, e_2 - i e_3,
\]
where
\[
E^- = \langle e_1 + i e_2, e_3 + i e_4, \ldots, e_{q-1} + i e_q, e_1 - i e_2, e_3 - i e_4, \ldots, e_{q-2} - i e_{q-1} \rangle,
\]
\[
E^+ = \langle e_{q+1}, e_{q+2} + i e_{q+3}, \ldots, e_{2n+1} + i e_{2n+2}, e_{q}, \ldots, e_{q+3}, \ldots, e_{2n+1} - i e_{2n+2} \rangle.
\]

Remark 1. If \( m = 2n + 1 \), we have the vector \( e_q \) or \( e_{q+1} \) in the ordered \( b \)-isotropic basis. In this case, the vector sits in a fixed position in the middle of the basis and the terms at the beginning skip over \( e_q \) or \( e_{q+1} \).

The above bases define the standard maximal torus \( T_S \) in each case as the subgroup of diagonal matrices, i.e.,
\[
T_S = \{ g \in G_0 : g = \text{diag}(t_1, t_2, \ldots, t_m, -t_m, -t_{2m}, -t_1), t_i \in \mathbb{C} \} \quad \text{if } m = 2n,
\]
or \( T_S = \{ g \in G_0 : g = \text{diag}(t_1, t_2, \ldots, t_m, 0, -t_m, \ldots, -t_{2m}, t_1), t_i \in \mathbb{C} \} \) if \( m = 2n + 1 \).

For any of the ordered bases above, we denote by \( F_S \) the associated flag in \( Z \), where the reordering on these bases will determine the flag domains, the base cycles, and the intersection points.

Just as in the case of \( T_S \), the maximal torus \( T_I \) is defined to have a certain basis of eigenvectors which depends on \( m \) being odd or even.

(i) If \( m \) is even, then the basis is
\[
e_1 + e_2, e_3 + e_4, \ldots, e_{2n} + e_{2n+1}, e_{q+1}, e_{2n} - e_{2n+1}, \ldots, e_2 - e_3,
\]
\[
\text{if } m = 2n \quad (16)
\]

(ii) If \( m \) is odd, then the basis is
\[
e_1 + e_2, e_3 + e_4, \ldots, e_{2n} + e_{2n+1}, e_{q}, e_{2n} - e_{2n+1}, \ldots, e_2 - e_3,
\]
\[
\text{if } m = 2n + 1 \quad (17)
\]

2.4. Weyl Groups, Flag Domains, and Base Cycles. Let us use shorthand notation for the bases used above. By \( (r, s) \) – form, we mean a basis \( (r_1, \ldots, r_m, s_n, \ldots, s_1) \) where \( b(r_i, r_j) = b(s_i, s_j) = 0 \) and \( b(r_i, s_j) = \delta_{ij} \) for all \( i \) and \( j \). This occurs in the even-dimensional case. By \( (r, t, s) \)-form, which occurs in the odd-dimensional case, \( r_i \) and \( s_i \) satisfy the same conditions and \( t \) is a single vector with \( b(t, t) = \pm 1 \) and \( b(t, r_i) = b(t, s_i) = 0 \) for all \( i \).

Here, we discuss the Weyl groups \( W(T_S) \) and \( W(T_I) \) by their actions on these bases. If \( T \) is either of these tori, then it stabilizes the spaces spanned by \( r_i \) and \( s_i \) for \( i = 1, \ldots, n \) for both kinds of bases. In this, we regard \( T \) as a product \( T = T_{i_1} \cdot \ldots \cdot T_{i_n} \) in the second case acting trivially on the space spanned by \( t \). In both cases, an arbitrary permutation in \( S_n \) acting diagonally on these spaces by \( (r_1, s_1) \mapsto (r_i, s_i) \) normalizes the \( T \)-action and is in the given orthogonal group \( G \).

A simple reflection \( (r_i, s_i) \mapsto (s_i, r_i) \) for one such \( i \), and being the identity on the other 2-dimensional spaces, is denoted by a sign change. This normalizes \( T \) but has negative determinant. In the first case, this means that only an even number of sign changes are allowed. In the second case, we may couple the sign change with the map \( t \mapsto -t \) so that this slightly modified sign change has positive determinant. Thus, in both cases, we have the action of \( S_n \times \mathbb{Z}_2^3 \) normalizing the \( T \)-representation on the basis at hand. This is in fact the action of the full Weyl group.

As in the cases of \( \text{Sp}(2n, \mathbb{R}) \) and \( \text{SO}^+(2n) \), to describe the flag domains and their \( K_0 \)-base cycles, we use the Weyl group \( W(T_S) \) of the standard torus \( T_S \). The base point for \( \alpha \) is the flag \( F_S \) defined by the standard basis. Then, all base points are \( F_{\alpha \omega} = w(F_S) \) where \( \omega \in \mathbb{Z}_2^{m-1} \) if \( m = 2n \) and \( \omega \in \mathbb{Z}_2^n \) if \( m = 2n + 1 \) is associated with the sign-change vector \( \alpha \). As before, the base cycles and flag domains are the orbits \( K_0 F_{\alpha} \) and \( G_0 F_{\alpha} \), respectively, and \( \alpha \) defines a non-degenerate sign with \( \text{sign}(F_{\alpha}) = (a, b) \) and then \( D_\alpha = D_{ab} \).

In our particular case, there are two maximal tori, the maximal torus \( T_I \) and the maximal torus \( T_S \), which were
defined earlier. Recall that these two maximal tori $T_f$ and $T_s$ are conjugates, and a conjugation induces an isomorphism $\psi$ of the associated Weyl groups. In this case, the bijective map $\psi$ is described, depending on the case, as follows:

(i) If $m = 2n$ and $q$ are even, define $W_{T_s}$ to be the Weyl group with respect to the basis (7) and let $W_f$ be the Weyl group with respect to the basis (16). Then, the bijective map in this case is $\psi : W_f \rightarrow W_{T_s}$ defined by $\psi(\mp (2i - 1)) = \mp i, \psi(\mp 2i) = \pm (q - i + 1)$ if $1 \leq i \leq q/2$ and $\psi(\mp i) = \mp i$ if $i > q$.

(ii) If $m = 2n$ and $q$ are odd, define $W_{T_s}$ to be the Weyl group with respect to the basis (9) and let $W_f$ be the Weyl group with respect to the basis (16); then, the bijective map in this case is $\psi : W_f \rightarrow W_{T_s}$ defined by $\psi(\mp (2i - 1)) = \pm i, \psi(\mp 2i) = \pm (q - i) - 1$ if $1 \leq i \leq (q - 1)/2$ and $\psi(\mp i) = \pm (i - 1)$ if $i > q$ and $\psi(-q) = -n$.

(iii) If $m = 2n + 1$ and $q$ are even, define $W_{T_s}$ to be the Weyl group with respect to the basis (11) and let $W_f$ be the Weyl group with respect to the basis (17); then, we can define the bijective map between them to be $\psi(\mp (2i - 1)) = \pm i, \psi(\mp 2i) = \pm (q - i + 1)$ if $1 \leq i \leq q/2$ and $\psi(\mp i) = \pm i$ if $i > q$.

(iv) If $m = 2n + 1$ and $q$ are odd, define $W_{T_s}$ to be the Weyl group with respect to the basis (13). Let $W_f$ be the Weyl group with respect to the basis (17). The bijective map $\psi$ between $W_f$ and $W_{T_s}$ can be defined as $\psi(\mp (2i - 1)) = \pm i, \psi(\mp 2i) = \pm (q - i)$ if $1 \leq i \leq (q - 1)/2$ and $\psi(\mp i) = \pm (i - 1)$ if $i > q$.

Moreover, through an argument quite similar to Proposition 1, the base cycle $C_\alpha$ is the set

$$C_\alpha = \{ F \in Z : V_1 = (V_1 \cap E^-) \oplus (V_1 \cap E^+), 1 \leq i \leq m \},$$

where $E^-$ and $E^+$ defined in Section 2.3 and the intersection dimensions are determined by $\alpha$ (resp., $(a, b)$). The flag domains are parametrized by the signature of their base cycles.

2.5. The Length of the Elements of $S_n \ltimes \mathbb{Z}_2^2$. Recall that every B-Schubert cell $S$ contains exactly one $T$-fixed point. Hence, at least in $Z = G/B$ where the Weyl group acts freely and transitively on $\text{Fix}(T)$, we should be able to compute $\ell_m c_S$ in terms of a corresponding Weyl element. In fact, if $z_0$ is the $T$-fixed point in $Z = G/B$ and $w \in W$, then the complex dimension of the Schubert cell $S_w = B.w(z_0)$ is the length $\ell(w)$.

As indicated above, we used a special way of writing the Weyl elements which is defined in the paper [2]. For dimension computations, let us state how we can compute the length of elements $w \in S_n \ltimes \mathbb{Z}_2^2$ relative to the notation for the Weyl group elements used in [2].

**Lemma 1** (see [2]). Fix $w \in S_n \ltimes \mathbb{Z}_2^2$ and construct $\bar{w} \in S_n \ltimes \mathbb{Z}_2^2$ by the following algorithm:

1. Starting from left to right in $w$, using simple reflections, place all positive numbers in $w$ step by step in a sequence of $n$-empty boxes beginning from the first one in $\bar{w}$, in the same order as they appeared in $w$.

2. From left to right in $w$, replace a negative number with its absolute value in the $n$-empty boxes starting from right to left.

If $\bar{w} = (\bar{w}(1), \bar{w}(2), \ldots, \bar{w}(n))$, then define $L(\bar{w}) = (n^2 - n)/2 - \text{number of } [\bar{w}(i) : i < k \text{ and } \bar{w}(i) < \bar{w}(k)]$, and if we have $m$ negative signs in $w$ in the following positions $j_1, j_2, \ldots, j_m$, then define $f(w) = \sum_{i=1}^m [(n - j_i)]$. It follows that the length of $w$ is

$$\ell(w) = L(\bar{w}) + f(w) + m. \quad (19)$$

**Example 1.** Let $w = 5(\bar{3}4)(\bar{1}2)$, then by following the above remark, we have

$$5(\bar{3}4)(\bar{1}2) \rightarrow 542(\bar{3}\bar{1}2) \rightarrow 542(\bar{1}2), \quad (20)$$

so $w = 5(\bar{3}4)(\bar{1}2)$ and $\bar{w} = 542(\bar{1}2)$, then $L(\bar{w}) = 8$ and $f(w) = 4$.

Hence, $\ell(w) = 8 + 4 + 2 = 14$.

2.6. Description of the $B_f$-Orbits. In the previous section, we have chosen a flag $F_f$ defined by the basis

$$e_1 - e_{2n}, e_2 - e_{2n-1}, \ldots, e_n - e_{n+1}, e_n + e_{n+1}, \ldots, e_2 + e_{2n-1}, e_1 + e_{2n}, \quad (21)$$

which belongs to the $G_0$-closed orbit $y^f$. Given a Weyl element $w \in W$, we consider the Schubert cell $B_f w$, and an element $F = b(F_{y^f})$ in it which is defined by an ordered basis $(b_{v_f}, v_{f+1}, \ldots, v_{f+m})$, $w \in W_f$, where $v_f = e_f - e_{2n-f+1}, 1 \leq i \leq n$, and $v_i = e_{2n-i+1} + e_i, n + 1 \leq i \leq 2n$. If $F \in C_{y^f}$, then it can be defined by an ordered basis $(e_1, \ldots, e_{2n})$ where for all $k$ the vector $e_k$ is either in $v_f$ or in $v_{f+1}$ and is a linear combination

$$e_k = \sum_{f \in k} c_e j_{f} b_{v_f}. \quad (22)$$

In this case, we call the basis a split basis. According to the above discussion for $\text{Sp}(2n, C)$, we can write the formulas for $b_{v_{f+1}}$ for any element $b \in B_f$, as follows:

$$b_{v_{f+1}} = e_{w_{f+1}} e_{w_{f+2}} + e_{w_{f+3}}, \quad 1 \leq w_{f+1} \leq n, \quad (23)$$

where $B_f$ does not involve the basis vectors $e_{w_{f+1}}$ and $e_{w_{f+2}}$.

**Remark 2.** When discussing the $B_f$-orbit of the base point $F_f$, it is important to explicitly understand the orbits $B_f (e_i - e_{2n-i+1})$ and $B_f (e_i + e_{2n-i+1})$. As we explained above, we have the following facts.

If $F_f$ and $B_f$ are as above, then the orbits of interest are $B_f (e_i - e_{2n-i+1})$, $i = 1, \ldots, n$. In this case, the orbits $B_f (e_i + e_{2n-i+1})$ and $B_f (e_i - e_{2n-i+1})$ have points of the forms
\[ b(e_i + e_{2n-i-1}) = \lambda (e_i + e_{2n-i-1}) + \cdots + b_n(e_n + e_{n+1}) + a_n(e_n - e_{n+1}) + \cdots + a_i(e_i - e_{2n}), \]

\[ b(e_i - e_{2n-i-1}) = \lambda (e_i - e_{2n-i-1}) + a_n(e_{n+1} - e_{n+2}) + \cdots + a_i(e_i - e_{2n}), \]

\[ (24) \]

respectively, with \( \lambda \neq 0 \). Note that, in the above orbits, if \( b \) is chosen appropriately, then we can arrange \( b(e_i + e_{2n-i-1}) = e_i \) or \( b(e_i + e_{2n-i-1}) = e_{2n-i-1} \) for all \( i \). Does this play a role in the proof of Theorem 1.

2.7. Flag Domains and Base Cycles. In order to parametrize the flag domains, it is enough to parametrize the closed \( K \)-orbits (see [3], Sections 4.2 and 4.3). For this, we choose a base point \( F \in \text{Fix}(T) \). It follows that the set \( \mathcal{C} \) of closed \( K \)-orbits can be identified with the Weyl group orbit \( W_G(T)F \) (see [3], Corollary 4.2.4). As we have observed above, \( W_G(T) \cong S_\alpha \times Z_2 \) where the \( S_\alpha \) factor can be identified with \( W_K(T) \). In fact, it is exactly the stabilizer of \( KF \) in \( \mathcal{C} \). Consequently, \( \mathcal{C} \) can be identified with \( Z_2 \times F \). We regard an element of \( Z_2 \times F \) as a vector \( \alpha \) of length \( n \) consisting of plus and minus signs, and thus, \( \mathcal{C} \) can be identified with the set of such vectors. In concrete terms, the \( K \)-orbits (resp., \( G_0 \)-orbits) of base points \(( \pm e_1, \ldots, \pm e_n)\) are exactly the base cycles (resp., flag domains) in \( Z \). The respective flag domains are denoted by \( D_\alpha \).

Observe that \( \alpha \) defines a nondegenerate signature \((a, b)\) and that \( D_\alpha \subset D_{a,b} := \{ F : \text{sign}(F) = (a, b) \} \). Conversely, given \( F \) with \( \text{sign}(F) = (a, b) \), we may choose a basis \((v, w) := (v_1, \ldots, v_n, w_0, \ldots, w_1) \) which defines \( F \) and which has the following properties:

1. \( b \) is in canonical form
2. \( (v, w) \) is \( h \)-orthogonal
3. \( \|v_j\|^2_h = \pm 1 \) with \( \text{sgn}(\|v_j\|^2_h) = \alpha_j \)
4. \( \|w_j\|^2_h = -\|v_j\|^2_h \)

If \( \text{sign}(F) = \text{sign}(\bar{F}) \), then we choose bases \((v, w) \) for \( F \) and \((\bar{v}, \bar{w}) \) for \( \bar{F} \), respectively, and note that the transformation which takes one basis to the other is in \( G_0 \).

This argument shows that \( D_{a,b} = D_\alpha \).

Denote \( E_+ = \langle e_1, \ldots, e_n \rangle \) and \( E_- = \langle e_{n+1}, \ldots, e_{2n} \rangle \), where \( C^{2n} = E_+ \oplus E_- \).

**Proposition 1.** For a fixed open orbit \( D_\alpha \), the base cycle \( C_\alpha \) is the set

\[ C_\alpha = \{ F \in Z : V_i = (V_i \cap E) \oplus (V_i \cap E^\perp) \}, 1 \leq i \leq 2n \].

**Proof.** Firstly, it was shown above that there is a base point \( F_\beta \) with the splitting condition. Since \( K_0 \) is the product \( K_0^+ \times K_0^- \), it acts transitively on all such flags. Let \( F \in C_\alpha \) where \( V_i^+ = V_i \cap E \) and \( V_i^- = V_i \cap E^\perp \). Define \( F^+ \) and \( F^- \) to be the sets of maximally isotropic flags in \( E^+ \) and \( E^- \), respectively. Recall that \( K_0 = K_0^+ \times K_0^- \) where \( K_0^+ \) and \( K_0^- \) act transitively on the sets \( F^+ \) and \( F^- \), respectively, which implies that \( K_0 \) acts transitively on the set of maximal isotropic flags obtained by putting flags from \( F^+ \) and \( F^- \) in a way such that the new flag has signature \( \alpha \). Hence, \( C_\alpha \) is a complex manifold. Therefore, \( C_\alpha \) is the base cycle.

\[ \square \]

3. Conditions for \( S_w \cap C_\alpha \neq \emptyset \)

Here, we deal with the homogeneous space \( Z \) of maximally isotropic full flags of the complex orthogonal symmetric group \( G = \text{SO}(2n, \mathbb{C}) \) equipped with the action of the real form \( G_0 = \text{SO}(p, q) \). The general results here are stated in terms of algorithms (See Definitions 1 and 2); in fact, it seems impossible to avoid this. In Corollary 1, we give concrete formulas for the intersection points in \( S_w \cap C_\alpha \) if the intersection is nonempty and \( C_\alpha \) of complementary dimension. Also, the number of intersection points with \( C_\alpha \) is explicitly computed in Theorem 2.

Since the Schubert variety \( S_w \) is determined by the Weyl element \( w \), we will describe in this section the conditions for an element \( w \) of the Weyl group which parametrize the Schubert variety \( S_w \) that satisfies \( S_w \cap D_\alpha \neq \emptyset \) for some flag domain \( D_\alpha \). As would be expected, a special type of permutation plays a fundamental role.

**Definition 1.** An element \( w \in W \) is called a harmonic permutation if it satisfies the following conditions:

If \( q \) is even, the number \( -(2i - 1), 1 \leq i \leq q/2 \) sits in any place to the left of the number \( (2i) \) or \( -(2i), 1 \leq i \leq q/2, \) in the one line notation of the permutation and the order of the numbers or \( -(q + i) \), where \( 1 \leq i \leq p - q \) is arbitrary.

If \( q \) is odd, the number \( -(2i - 1), 1 \leq i \leq (q - 1)/2, \) sits in any place to the left of the number \( (2i), 1 \leq i \leq (q - 1)/2, \) and the number \( -q \) sits in the last position in the one line notation of the permutation and the order of the numbers \( q + i \) or \( -(q + i) \), where \( 1 \leq i \leq (p - q)/2 \) is arbitrary.

**Example 2.** In \( \text{SO}(4, 2) \), the relevant pairs are \((-12)\) and \((-1 - 2)\). As a result, we have 6 harmonic permutations. They are \((-12 - 3), (-1 - 32), (-3 - 12), (-1 - 23), (-13 - 2), \) and \((3 - 1 - 2)\).

Recall that the fixed point in the closed orbit \( v^i \) is the flag associated with the following ordered basis:

\[ e_1 + e_2 + \cdots + e_q + e_{q+1} + e_{2q+1} + i e_{2q+2} + i e_{2q+3} + i e_{2q+4} + \cdots + i e_{2n-1} = i e_n \]

\[ e_{2n-1} - i e_{2n} + e_{2n+1} - i e_{2n+2} + i e_{2n+3} - i e_{2n+4} + e_{2n+5} - i e_{2n+6} + \cdots + i e_{3q-1} + e_{3q} \]

\[ (26) \]

(ii) If \( m \) is odd, then the basis is
Theorem 3. For \(v_i\), any such basis vector, and \(b \in B_1\), the form of \(b v_i\) is given as follows:

\[
\begin{align*}
(1) & \quad b \cdot (e_i - e_{2q+1}) = \lambda (e_i - e_{2q+1}) + \cdots + b_n (e_q - e_{q+1}) + a_n (e_q + e_{q+1}) + \cdots + a_1 (e_1 + e_2) \\
(2) & \quad b \cdot (e_{2q+2j-1} - e_{i-1}) = \lambda (e_{2q+2j-1} - e_{i-1}) + a_{q+j} (e_{2q+2j-1} + e_{i-1}) + \cdots + a_1 (e_1 + e_2) \\
(3) & \quad b \cdot (e_{2q+2j} - e_{i+1}) = \lambda (e_{2q+2j} - e_{i+1}) + a_{q+j} (e_{2q+2j} + e_{i+1}) + \cdots + a_1 (e_1 + e_2) \\
(4) & \quad b \cdot (e_{2q+2j+1} - e_{i}) = \lambda (e_{2q+2j+1} - e_{i}) + a_{q+j} (e_{2q+2j+1} + e_{i}) + \cdots + a_1 (e_1 + e_2) \\
\end{align*}
\]

where \(\lambda \neq 0\) in all cases above. Note that, in the above orbits, if \(b\) is chosen appropriately, we can then arrange all linear combination for every one of the above vectors to be in the standard basis of \(T_S\)-eigenvectors. See Theorem 1.

In the following result, \(\psi : W_l \rightarrow W_s\) denotes the bijective map between Weyl groups which was introduced in Section 2.

Proposition 2. If \(w\) is a harmonic permutation, then the flag \(F_{\psi(w)}\) belongs to the orbit \(B_1 (F_w)\).

Proof. We handle the case where \(m = 2n\). The proof for \(m = 2n + 1\) goes analogously. Let us first prove the theorem for the case that \(q\) is even. For this, let \(w \in W_l\) be a harmonic permutation and define \(F_w = w (F_i)\) to be the isotropic full flag associated with \(w\). Denote by

\[
\{0\} \subset \langle e_{\ell} \rangle \subset \langle u_{\ell}, u_{\ell-1} \rangle \subset \cdots \subset \langle u_{\ell}, \ldots, u_{1} \rangle,
\]

the first \(n\) subspaces of \(F_w = w (F_i)\), where \(u_{\ell} \subset \ldots \subset u_{1}\) is a basis from the basis above. Let \(\bar{w} \in W_s\) be the image of \(w\) under the bijective map \(\psi\), and let \(Y_{\bar{w}}\) be the isotropic flag associated with \(\bar{w}\) such that the first half of \(Y_{\bar{w}}\) is

\[
\{0\} \subset \langle e_{\ell} \rangle \subset \langle e_{\ell}, e_\ell \rangle \subset \cdots \subset \langle e_{\ell}, \ldots, e_\ell \rangle.
\]

Our claim here is that this flag is an intersection point in \(Y_{\bar{w}} \cap C_{\bar{w}}\). To prove this, we will construct \(b\) with \(b (F_w) = Y_{\bar{w}}\). From the definition of harmonic permutation, there are two possibilities for \(w_i\): \(|w_i| = 2i - 1 \leq q\) or \(|w_i| > q\).

Case 1: if \(|w_i| > q\), then \(w_i = e_{\ell} \), so the orbit \(B_1 \cdot \langle u_{\ell} \rangle\) contains the point \(\langle e_{\ell} \rangle\).

Case 2: if \(|w_i| = 2i - 1 \leq q\), then we must consider the orbit \(B_{\ell} \cdot \langle e_{\ell+1} - e_{2q+1} \rangle\). By Remark 3 above, the orbit \(B_{\ell} \cdot \langle e_{\ell+1} - e_{2q+1} \rangle\) contains points of the form

\[
y = \left( a_1 (e_{\ell+1} - e_{2q+1}) + a_2 (e_{2q+1} + e_{2q+1}) + a_3 (e_{2q+1} - e_{2q+1}) \right),
\]

where \(a_1 = \pm a_4\) and \(a_2 = \pm a_3\). By taking \(a_1 = a_4 = 1/2\) and \(a_2 = a_3 = (1/2)i\), it follows that \(y = \langle e_{\ell+1} + e_{2q+1} \rangle\). To construct the \(j\)-vector of \(b (v_{\ell+1})\) to obtain the subspace \(V_{\ell+1}\), we must consider three cases:

Case 1: if \(|w_j| > q\), then \(w_j = \bar{w}_j\) and \(e_{\ell+1} = u_{\ell+1}\), so the orbit \(B_{\ell+1} \cdot \langle u_{\ell+1} \rangle\) contains the point \(e_{\ell+1}\).

Case 2: if \(|w_i| = 2i - 1 \leq q\), then our job goes through the orbit \(B_{\ell+1} \cdot \langle e_{\ell+1} - e_{2q+1} \rangle\). By using Remark 3, we see that the orbit \(B_{\ell+1} \cdot \langle e_{\ell+1} - e_{2q+1} \rangle\) contains points of the form

\[
y = \left( a_1 (e_{\ell+1} + e_{2q+1}) + a_2 (e_{2q+1} + e_{2q+1}) + a_3 (e_{2q+1} - e_{2q+1}) \right),
\]

where \(a_1 = \pm a_4\) and \(a_2 = \pm a_3\). By taking \(a_1 = a_4 = 1/2\) and \(a_2 = a_3 = (1/2)i\), it follows that \(y = \langle e_{\ell+1} + e_{2q+1} \rangle\). Therefore, \(b\) is constructed to obtain the flag

\[
\{0\} \subset \langle e_{\ell+1} \rangle \subset \langle e_{\ell+1}, e_{\ell+1} \rangle \subset \cdots \subset \langle e_{\ell+1}, \ldots, e_{\ell+1} \rangle.
\]

Thus, by induction, we observe that \(b \in B_1\) can be constructed with \(b (F_w) = Y_{\bar{w}}\). We complete the proof by handling the case where \(q\) is odd. For this point, we can repeat the steps of the proof above for \(w_j > q\) and \(|w_j| = 2i - 1\). For the case where \(|w_j| = 2i\), we apply the same method as above, only changing \(2|w_j|\) by \((2|w_j| + 1)\). If \(w_j = -q\), then \(w_j = -n\). In this case, the orbit of relevance is \(B_{\ell+1} \cdot \langle e_{\ell+1} - e_{2q+1} \rangle\). As we see from Remark 3, \(y = \langle e_{\ell+1} + e_{2q+1} \rangle \) belongs to this orbit. The desired result is then obtained by taking \(a_1 = 1, a_2 = -1, a_3 = a_4\), and the point \(y = e_{2q+1}\) belongs to the orbit \(B_{\ell+1} \cdot \langle e_{\ell+1} - e_{2q+1} \rangle\). Therefore, the element of \(B_{\ell+1}\) is also constructed in the case where \(q\) is odd.
Theorem 1 (harmonic permutation theorem). The following are equivalent:

(i) \( w \) is harmonic

(ii) \( B_j(F_w) \cap D_\alpha \neq \emptyset \) for some \( \alpha \)

Under either of these conditions for every \( \alpha \), the nonempty intersection \( B_j(F_w) \cap C_\alpha \) contains a \( T_\alpha \)-fixed point.

Proof. (i) \( \implies \) (ii) is exactly the statement of Proposition 2.

(ii) \( \implies \) (i) Assuming that \( w \) is not harmonic permutation, we will show that \( S_\alpha \cap D_\alpha = \emptyset \), for all \( \alpha \), i.e., \( S_\alpha \) has no \( T_\alpha \)-fixed points. Assume on the contrary that there exists \( b \in B_j \) such that \( b.F_w \in S_\alpha \cap C_\alpha \). For \( b.F_w \) to be fixed, \( b \) has to have a certain shape, and at each stage where the condition of harmonicity is violated, we should prove that there is no such \( b \). For this, recall that the complex bilinear form \( b \) has been defined to satisfy the following orthogonality condition:

\[
\begin{align*}
&b(e_j - e_{2g-j+1}, e_j + e_{2g-j+1}) = 1, \quad 1 \leq j \leq q, \\
&b(e_j \pm e_{2g-j+1}, e_k \pm e_{2g-k+1}) = 0, \quad \text{for } k \neq j.
\end{align*}
\]

(34)

Since \( w \) is not a harmonic permutation, there exists a pair of the form \((-2i-1, 2i) \) or the form \((-2i-1, -2i)\), where \( 1 \leq i \leq q/2 \), such that \( -2i \) sits to the left of \(-2i-1\). Assume that \( w_i = 2i \) is the first even number which appears in \( w \) such that \( 2i \) sits to the left of \(-2i-1\). Then, \( b \) does in fact yield a \( T_\alpha \)-fixed partial flag, i.e.,

\[
[0] \subset \langle e_{w_1} \rangle \subset \langle e_{w_1}, e_{w_2} \rangle \subset \ldots \subset \langle e_{w_1}, \ldots, v_{w_i} \rangle.
\]

(35)

Now, we check that there is no \( b \in B_j \) so that \( b.(v_{w_i}) \) defines an extended partial flag which is \( T_\alpha \)-fixed. For this, we consider the orbit \( B_j.(e_{w_j} + e_{2w_j-1}) \) which contains points of the form

\[
\alpha_1(e_1 + e_{2g}) + \alpha_2(e_2 + e_{2g-1}) + \cdots + \alpha_j(e_{w_j} + e_{2w_j-1}).
\]

(36)

This is a linear combination of \( h \)-isotropic vectors for all \( \alpha_j \). Recall that the flag of intersection has non-\( h \)-isotropic vectors, and to have one of these vectors in this step which is linearly independent of all vectors in the flag (35) above, we should add the point \( \langle e_{w_j-1} + i e_{w_j} \rangle \) from the flag (35) to the linear combination in (36). But the flag (35) does not contain any of these points. So again, for all \( b \in B_j \), the flag \( b.F_w \) is not \( T_\alpha \)-fixed, and consequently, \( S_\alpha \cap D_\alpha = \emptyset \). \( \square \)

4. Introduction to the Combinatorics

For the remainder of this paper, we only discuss the intersection properties of the Iwasawa–Schubert cells which are of complementary dimension to \( C_\alpha \). Recall that the maximal compact subgroup of \( SO(p, q) \) is \( K_0 := SO(p) \times O(q) \subset SU(p) \times U(q) \). For \( E^* \) and \( E^* \) as in Section 2.4, the base cycle in the flag domain \( D_{ab} \) is given by

\[
C_0 = \left\{ F \in Z : \dim V_j \cap E^* = \sum_{j=1}^i a_j \text{ and } \dim V_j \cap E^* = \sum_{j=1}^i b_j, 1 \leq i \leq m \right\}.
\]

(38)

If \( m = 2n \), we have two cases: If \( q \) is even, the dimension of \( C_0 \) is \( p(p-2)/4 + q(q-2)/4 \). If \( q \) is odd, then \( C_0 \) has the dimension \( (p-1)/2 + (q-1)/2 \). If \( m = 2n + 1 \), then the dimension of the base cycle is \( (p-1)/4 + (q-2)/4 \) if \( q \) is even and is \( (p-2)/4 + (q-2)/4 \) if \( q \) is odd. Since we have restricted to the case where \( Z = G(B) \) is the manifold of complete flags, it follows that \( \dim S_w = pq/2 \) if \( q \) and \( n \) are even and \( \dim S_w = (pq-1)/2 \) if \( q \) or \( n \) is odd.

Definition 2. A harmonic permutation \( w \in S_n \times \mathbb{Z}^{n-1} \) is called a perfect harmonic permutation if it is constructed by the following algorithm:

(A) Start with a sequence of \( n \) empty boxes which are to be filled in order to construct \( w \).

(B) Consider the pairs \((-2j-1, 2j)\) and \((-2j-1, -2j)\) for all \( 1 \leq j \leq q/2 \).

(C) If \( q \) is even,

(1) The pairs in step B are \((-1, 2), (-1, 2), (-3, 4), \ldots, (-q, 1), q, (-q, 1), -q\).

(2) Step by step, starting from \((-1, 2) \) until \((-q, q)\), for each \( 1 \leq j \leq q/2 \), we have two pairs of the forms \((-2j-1, 2j) \) and \((-2j-1, -2j)\), so choose only one pair of them for each step and omit the other from the above list.

(3) If we choose the pair of the form \((-2j-1, 2j)\), place this pair in any box in \( w \) such that the components \(-2j-1 \) and \( 2j \) of this pair sit as close as possible to each other.

(4) If we choose the pair of the form \((-2j-1, -2j)\), place this pair in any box in \( w \) such that the components \(-2j-1 \) and \( -2j \) of this pair sit as close as possible to each other and all pairs \((-2i-1, 2i)\) or \((-2i-1, -2i)\) with \( i > j \) are sitting to the left of this pair and the pairs of the form \((-2i-1, -2i)\) with \( i < j \) are sitting in a decreasing order with respect to \( i \) to the right of \((-2j-1, -2j)\).
Remark 4. Once a pair is placed, its position can be ignored so that the places at the immediate left and right of this pair become adjacent.

(6) After all pairs are placed, the remaining numbers \( \pm (q + i) \) for \( 1 \leq i \leq (p - q)/2 \) are placed in the remaining spots in the strictly increasing order with respect to \( |w(i)| \) such that all numbers \( \pm (q + i) \) for \( 1 \leq i \leq (p - q)/2 \) are sitting to the left of the pairs \((-2i - 1), -2i \) \(1 \leq i \leq q/2\). If the number of negative signs in all pairs from steps 2 and 3 is even, then all the remaining numbers are positive, and if the number of negative signs is odd, then all the remaining numbers are positive except the number \( n \) is negative.

(D) If \( q \) is odd,

(1) The pairs in step B are \((-1, 2), (-3, 4), (-5, 6), \ldots, (-q, -2), q - 1, -q\).

(2) In the last box of \( w \), put the number \(-q\).

(3) Step by step, starting from \((-1, 2)\) until \((-q - 2), q - 1\), choose a pair and place this pair in any box of the first \( n - 1 \) boxes in \( w \) such that the components \((-2i - 1) \) and \(2i \) of this pair sit as close as possible to each other. This means that once a pair is placed, it cannot be ignored so that the places at the immediate left and right of this pair become adjacent.

(4) After all pairs are placed, the numbers \( \pm (q + i) \) for \( 1 \leq i \leq p - q/2 \) are placed in the remaining spots in the strictly increasing order with respect to \( |w(i)| \). If the number of negative signs in all pairs from steps 1 and 2 is even, then all the remaining numbers are positive, and if the number of negative signs is odd, then all the remaining numbers are positive except the number \( n \) is negative.

**Remark 4.** If \( q \) is odd, then the signature of the flag domain \( D_a \) is + in the last position.

**Remark 5.** If \( W = S_n \times Z_{n}^m \), then a perfect harmonic permutation consists only of pairs of the form \((-2i - 1), 2i\) for all \( 1 \leq j \leq q/2 \). It is constructed as above, in particular such that the sign of \( n \) is +.

**Example 3.** If \( p = 6, q = 4 \), then the perfect harmonic permutations are

\[
\begin{align*}
&(-12) \quad (-34)5, \quad (-12)5 \quad (-34), \quad 5 \quad (-12) \quad (-34), \quad (-34) \quad (-12)5, \\
&\cdot \quad (-34)5 \quad (-12), \quad 5 \quad (-34) \quad (-12), \\
&(-12) \quad -5 \quad (-3-4), \quad -5 \quad (-12) \quad (-3-4), \quad -5 \quad (-3-4) \quad (-12), \\
&\cdot \quad (-34) \quad -5 \quad (-1-2), \quad -5 \quad (-34) \quad (-1-2), \\
&5 \quad (-3-4) \quad (-1-2), \quad (-3-124)5, \quad 5 \quad (-3-124), \\
&\cdot \quad -5 \quad (-3-12-4).
\end{align*}
\]

**Example 4.** If \( p = 10, q = 6 \), then the element \( 78 \quad (-3-4) \quad (-12) \) is a perfect harmonic permutation, while the element \( 78 \quad (-3-4) \quad (-12) \quad (-56) \) is not a perfect harmonic permutation.

Recall that the dimension of the cells corresponds to the length of the word \( w \) in the Weyl group, i.e., if \( F_w = wF_1 \), then \( \dim(BF_w) = l(w) \). So if we want to discuss the dimension of the Schubert cell, it is enough to discuss the length of Weyl elements.

**Proposition 3.** Every perfect harmonic permutation \( w \in S_n \times Z_{n}^{m-1} \) has dimension \( pq/2 \) if \( q \) is even and \( (pq/2) \) if \( q \) is odd.

**Proof.** Given a perfect harmonic permutation \( w \), we consider three cases. These depend on which of the pairs \((-1, 2), (-3, 4), (-1 - 2)\), or \((-3 - 4)\) is contained in \( w \). In each of these cases, our proof goes by induction on the dimension of the flag manifold. Without loss of generality, let \( p > q \geq 6 \) because if \( q \leq 6 \), then the permutation has only one or two pairs and the proof becomes trivial.

**Case 1.** The permutation \( w \) contains the pairs \((-1, 2) \) and \((-3, 4)\).

First, we remove the pairs \((-1, 2) \) and \((-3, 4) \) from \( w \) to have a new permutation \( v \) consisting of the numbers \( \{5, 6, \ldots, n\} \). We define a function \( f : \{5, 6, \ldots, n\} \rightarrow \{1, 2, \ldots, n - 1\} \) by \( f(i) = i - 4 \) for all \( 1 \leq i \leq n - 4 \). This is a bijective map which sends \( v \to \tilde{w} \in S_{n-4} \times Z_{n-1}^{m-5} \). Note that \( \tilde{w} \) is a perfect harmonic permutation in \( S_n \times Z_{n-1}^{m-5} \). Thus by the induction assumption, \( l(\tilde{w}) = (p - 4) \quad (q - 4)/2 \). Since \( v = f^{-1}(\tilde{w}) \), it follows that \( v \) has the same length as \( \tilde{w} \). So we put the numbers 1234 to the left of \( v \) to have an element \( \bar{w} \in S_{n-1} \times Z_{n-1}^{m-1} \) with length \( (p - 4)(q - 4)/2 \). To split the sign of \( 1 \) and \( 3 \) (i.e., to change the positive sign to the negative sign), we add \( n - 4 + 1 \) to \((p - 4)(q - 4)/2 \) to send 3 to the last position and add \( n - 4 + 2 \) to send 1 to the position before the last position. Consequently, we have the element \( (24)v(13) \) with length \((p - 4)(q - 4)/2 + n - 4 + 1 + n - 4 + 2 = (p - 4)(q - 4)/2 + 2n - 5 \), and it follows that the element \( (24)v(-3 - 1) \) has length \((p - 4)(q - 4)/2 + 2n - 4 \). We then return to the original \( w \) and remove only the pair \((-1, 2) \). Then, we define \( g \) to be the number of positions to the left of \((-34) \). As a result, we have \( n - 4 - g \) positions to the left of \((-34) \), and \(-3 \) should cross \( n - 4 - g + 1 \) positions to end up in the last position and 4 should cross \( g \) boxes to end up in the first position.

Finally, we return to the original \( w \) and define \( h \) to be the number of positions to the left of \((-12) \) and \( n - 2 - f \) to the right. In this situation, \(-1 \) must cross \( n - 2 - h + 1 \) positions to end up in the last position and 2 must cross \( g \) positions to end up in the first position. It follows that the length of \( w \) is \((p - 4)(q - 4)/2 + 2n - 4 + n - 4 - g + 1 + g + n - 2 = h - 1 + h = pq/2 \).

If \( q \) is odd, then the length of \( \bar{w} \in S_{n-1} \times Z_{n-1}^{m-1} \) is \( l(\bar{w}) = (p - 4)(q - 4) - 1/2 \). After applying the same steps as above, it follows that \( l(w) = (pq - 1)/2 \).

**Case 2.** The permutation \( w \) contains the pair \((-1 - 2) \).

Note that this case only occurs if \( q \) is even. Since \( w \) is a perfect harmonic permutation, the pair \((-1 - 2) \) sits in the
last 2 positions of $w$. We remove the pair $(-1 - 2)$ from $w$ to obtain $\tilde{v}$. Since $q$ is even, a similar argument to that shown above shows that $I(\tilde{v}) = ((p - 2)(q - 2))/2$ because $q$ is even. We put the pair $(1 2)$ to the left of $\tilde{v}$. It follows that $\tilde{v} = (1 2)\tilde{v}$ has the same length as $\tilde{v}$. To split the position of $1$ and $2$, each of them must cross $n - 2$ positions. Having made this move, we then apply 2 reflections to obtain the pair $(-3 - 4)$ in the last two positions. Hence, the length of $w$ becomes $l(w) = ((p - 2)(q - 2))/2 + 2(n - 2) + 2 = pq/2$.

Case 3. The permutation $w$ contains the pairs $(-1 2)$ and $(-3 - 4)$.

Note that this case appears only if $q$ is even. In this case, these two pairs appear in $w$ in the following form: $(-3 - 4 - 12)$ or $(-3 - 12 - 4)$, or the pair $(-3 - 4)$ sits in the last two positions of $w$. If the pair $(-3 - 4)$ sits in the last two positions of $w$, then the argument goes as in Case 2 above.

If we have the form $(-3 - 12 - 4)$, then we must add $n - 2$ to $(p - 2)(q - 2)/2$ to put $3$ in the last position and add $n - 2$ to $(p - 2)(q - 2)/2$ to put $4$ in the last position. Then, we must add $1$ to split the signs and $3$ to send $-3$ to its position in the original; then, the length of $w$ becomes $((p - 2)(q - 2))/2 + (n - 4) + (n - 2) + 1 + 3 = pq/2$.

If we have the form $(-3 - 4 - 12)$, then we must add $n - 2$ to $(p - 2)(q - 2)/2$ to put $3$ in the last position and add $n - 2$ to $(p - 2)(q - 2)/2$ to put $4$ in the position before the last position. Then, we must add $1$ to split the signs, $3$ to send $-3$ to its position in the original $w$, and $1$ to send $-4$ to its position in the original $w$. It follows that the length of $w$ is $((p - 2)(q - 2))/2 + (n - 4) + (n - 3) + 1 + 3 + 1 = pq/2$.

Proposition 4. Every perfect harmonic permutation $w \in S_n \times \mathbb{Z}_2$ has dimension $pq/2$.

Proof. The argument goes exactly along the lines as that of the above proposition. Here, it is in fact simpler because only the pairs $(-1 2)$ and $(-3 - 4)$ appear.

5. Intersection Points of Schubert Duality

Let $w \in W_f$ be a perfect harmonic permutation, in particular so that $S_w$ is of complementary dimension to the cycles. Recall that, in this case, either $S_w \cap C_\alpha = \emptyset$ or is pointwise $T_\gamma$-fixed (see Section 2.3). The main goal in this section is to compute all such intersection points. The argument in the case of SO($p, q$) is carried out by means of algorithms. Nevertheless, we are able to provide formulas for the cardinality of $S_w \cap C_\alpha$ when it is nonzero and the total number of cycles $C_\alpha$ for which this intersection is nonempty (see Theorem 2).

Proposition 5. If $w$ is a perfect harmonic permutation so that $B_1 F_w$ intersects a cycle $C_\alpha$ at a point given by the $\epsilon$-basis $(\epsilon_1, \ldots, \epsilon_m)$ of $T_\gamma$-eigenvectors, then for any such eigenvector $\epsilon_k$, it follows that the $\epsilon$-basis is given by $\epsilon_k = \epsilon_{2j-1} + i \epsilon_{2j}$ or $\epsilon_{2j-1} - i \epsilon_{2j}$ if $q$ and $p$ are even and $\epsilon_k = \epsilon_{2j-1} + i \epsilon_{2j}$, $\epsilon_{2j-1} - i \epsilon_{2j}$, $\epsilon_k = \epsilon_{2j} + i \epsilon_{2j+1}$, or $\epsilon_{2j+1}$ if $q$ or $p$ is odd, depending on the dimension $m$ and the signature $\alpha$.

Proof. This is a consequence of the following:

(1) $w$ is a perfect harmonic permutation and the flag basis is that of $w(F_1)$.

(2) We have the following cases:

(i) $b.\eta = \eta_1 (e_r + e_{2q-r+1}) + \eta_1 (e_r - e_{2q-r+1}) + \eta_1 (e_{2q-r+1} + e_{2q-r-1}) + \eta_1 (e_{2q-r+1} - e_{2q-r-1}) + B_j$

$= K_j + B_j$

(ii) $b.\eta = \eta_1 (e_{2q-r+1} + i e_{2q-r-1}) + B_j = K_j + B_j$

Here, $\eta_1 = \pm \eta_1^* \pm \eta_1^*$ and $B_j$ does not involve the basis vectors $e_r, e_{r+1}, e_{2q-r+1}$, and $e_{2q-r} - r$.

(iii) $b.\eta = \eta_1 (e_{2q+r+1}) + B_j = K_j + B_j$, if $m = 2n + 1$

(3) $\eta_1 \neq 0$ and $\eta_1^* \neq 0$.

(4) The intersection $S_w \cap C_\alpha$ is a flag defined by $T_\gamma$-eigenvectors.

From the expression for $\eta_1$, it is obvious that all of the possibilities in the statement occur. Furthermore, since $\eta_1 \neq 0$ and $\eta_1^* \neq 0$, for every $j$, a nonzero contribution from $K_j$ occurs in the sum

$$v_j = \sum_{k \leq j} c_{kj} b.\eta_{w_k}.$$ (40)

Since $e_{2j-1} + i e_{2j}$ and $e_{2j-1} - i e_{2j}$ and $e_r$ or $e_{q+1}$ do not occur in $b.\eta_{w_k}$ for $k < j$, it follows that $\epsilon_j = K_j + E_j$ in the standard basis. Finally, since $\epsilon_j$ is a $T_\gamma$-eigenvector, it follows that $s_j = K_j$ where $K_j$ of the $\epsilon_j$ forms indicated in the proposition.

As a result of the above proposition, the following corollary gives us all intersection points of $S_w \cap C_\alpha$.

Corollary 1. Let $D_a$ be a flag domain parametrized by a sequence $\alpha$ and $w \in W_f$ be a perfect harmonic permutation such that $S_w \cap D_a \neq \emptyset$. Then, the following algorithm produces us all intersection points of $S_w \cap C_\alpha$.

If $q$ is even,

(i) Consider a copy of $\alpha$ denoted by $\beta$.

(ii) For any pair $(-2j - 1, 2j)$, $1 \leq j \leq q/2$, in $w$, if the corresponding signature of it in $\beta$ is $+-$, then replace this $+-$ in $\beta$ by

$$\langle e_{2q-2j+1} + i e_{2q-2j+2}, e_{2j-1} - i e_{2j} \rangle,$$

or $$\langle e_{2q-2j+1} - i e_{2q-2j+2}, e_{2j-1} + i e_{2j} \rangle,$$ (41)

and if the corresponding signature of it in $\beta$ is $-+$, then replace this $-+$ in $\beta$ by

$$\langle e_{2j-1} + i e_{2j}, e_{2q-2j+1} - i e_{2q-2j+2} \rangle,$$

or $$\langle e_{2j-1} - i e_{2j}, e_{2q-2j+1} + i e_{2q-2j+2} \rangle.$$ (42)

(iii) For any pair $(-2j - 1, -2j)$, $1 \leq j \leq q/2$, in $w$, if the corresponding signature of it in $\beta$ is $--$, then replace this $--$ in $\beta$ by
\begin{align}
(e_{2q-2j+1}^\ast + ie_{2q-2j+2}, e_{2j-1}^\ast + ie_{2j}) \\
\text{or } (e_{2q-2j+1}^\ast - ie_{2q-2j+2}, e_{2j-1}^\ast - ie_{2j}).
\end{align}

(43)

and if the corresponding signature of it in \( \beta \) is +, replace this \(-\) in \( \beta \) by
\begin{align}
(e_{2j-1}^\ast + ie_{2j}, e_{2q-2j+1}^\ast + ie_{2q-2j+2}) \\
\text{or } (e_{2j-1}^\ast - ie_{2j}, e_{2q-2j+1}^\ast - ie_{2q-2j+2}).
\end{align}

(44)

(iv) For the remaining numbers, for each \( q + 1 \leq j \leq n - 1 \), replace the corresponding + in \( \beta \) by
\begin{align}
(e_{2j-1}^\ast + ie_{2j}, e_{2q-2j+1}^\ast - ie_{2q-2j+2}) \\
\text{or } (e_{2j-1}^\ast - ie_{2j}, e_{2q-2j+1}^\ast + ie_{2q-2j+2}).
\end{align}

(45)

and if the corresponding signature of it in \( \beta \) is -, replace this \(+\) in \( \beta \) by
\begin{align}
(e_{2j-1}^\ast + ie_{2j}, e_{2q-2j+1}^\ast - ie_{2q-2j+2}) \\
\text{or } (e_{2j-1}^\ast - ie_{2j}, e_{2q-2j+1}^\ast + ie_{2q-2j+2}).
\end{align}

(46)

Each point obtained from this algorithm is a point of the intersection of \( S_w \cap D_n \).

\textbf{Theorem 2.} A Schubert variety \( S_w \) which is parametrized by a perfect harmonic permutation \( w \) intersects \( 2^{q^2} \) flag domains if \( q \) is even and intersects the base cycles of these flag domains in \( 2^q \) points. If \( q \) is odd, it intersects \( 2^{(q+1)^2}/2 \) flag domains and intersects the base cycles of these flag domains in \( 2^{q+1} \) points.

\textbf{Proof.} Let \( w \in W_q \) be a perfect harmonic permutation. We first show that if \( q \) is even, then \( S_w \cap C_a \) consists of \( 2^q \) points and \( 2^{q-1} \) points if \( q \) is odd. Since \( w \) is a perfect harmonic permutation, we have two cases. In the first case, if \( w \) contains the pair \((-1, 2)\), then the pair \((-1, 2)\) sits inside consecutive boxes in \( w \). The goal here is to show that there are exactly 4 possibilities for this pair which can be completed to the maximal isotropic flag. We begin by considering the \( B_1 \)-orbit of \( (e_1, e_2^\ast) \). By Remark 3, the elements in this orbit have the form
\[ \beta_1(e_1 + e_2) + \cdots + \beta_{2n}(e_1 - e_2). \]

(47)

The question is how many 1-dimensional subspaces (spanned by vectors in Proposition 5) do we have such that these subspaces can be completed to the maximal isotropic flag. To compute this number, we denote by \( v_1 \) the vector we have from the first step which spans the 1-dimensional subspace; in the second step, we consider the orbit \( B_1(e_1 + e_2^\ast) \), which has points of the form
\[ \beta_1(e_1 + e_2) + \beta_2(e_1 + e_2^\ast) \]

(48)

The 2-dimensional subspace in the flag is spanned by linear combinations of the form
\[ v_2 = \beta_1(e_1 + e_2) + \beta_2(e_1 + e_2^\ast) + \gamma v_1. \]

Note that \( v_2 \) should be in \( E^\ast \) or in \( E^\ast \) and of the form stated in Proposition 5, and therefore, \( v_1 \) should contain the terms \( e_1 \) and \( e_2 \) or the terms \( e_{2q-1} \) and \( e_{2q} \). Thus, we have the following 4 possibilities of \( v_1 \) as follows: \( e_1 - ie_2, e_1 + ie_2, e_{2q-1} + ie_{2q}, \text{ and } e_{2q-1} - ie_{2q} \).

If \( v_1 = e_1^\ast + ie_2 \), then for \( \beta_1 = \beta_2 = 1, \gamma = i, \) the vector \( v_2 \) is \( v_2 = e_{2q-1} + ie_{2q}, \) so the 2-dimensional subspace corresponding to the pair \((-1, 2)\) is spanned by
\[ (e_1 + ie_2, e_{2q-1} + ie_{2q}). \]

(50)

If \( v_1 = e_{2q-1} + ie_{2q} \), then for \( \beta_1 = 1, \beta_2 = -1, \gamma = i, \) it follows that \( v_2 \) is \( v_2 = e_{2q-1} + ie_{2q}, \) so the 2-dimensional subspace corresponding to the pair \((-1, 2)\) is spanned by
\[ (e_{2q-1} + ie_{2q}, e_1 + ie_2). \]

(51)

Having constructed the 2-dimensional space, we ignore the pair \((-1, 2)\) in \( w \) and repeat the same steps for the next pairs of step by step. So if \( w \) contains the pair \((-2j-1, 2j), 1 \leq j \leq q/2 \), then in the same way, the only possible 2-dimensional subspaces which can be completed to maximal isotropic flags are the subspaces spanned by
\[ (e_{2q-2j+1} + ie_{2q-2j+2}, e_{2q-2j+1} + ie_{2q-2j+2}). \]

(52)

The second case is where \( w \) contains the pair \((-1, 2)\) which sits inside consecutive boxes in \( w \). Look at the orbit \( B_1(e_1 - e_{2q}) \), then this orbit has points of the form
\[ \beta_1(e_1 + e_{2q}) + \cdots + \beta_{2n-1}(e_2 - e_{2q}) + \beta_{2n}(e_1 - e_{2q}). \]

(53)

We also consider the orbit \( B_1(e_2 - e_{2q-1}) \) which has points of the form
\[ \beta_1(e_1 + e_{2q}) + \cdots + \beta_{2n-1}(e_2 - e_{2q-1}). \]

(54)

Then, if we have 1-dimensional subspace spanned by \( v_1 \) from the first step, the 2-dimensional subspace in our flag spanned by \( v_1 \) and a vector \( v_2 \) is of the form
\[ v_2 = \beta_1(e_1 + e_{2q}) + \beta_2(e_2 + e_{2q-1}) + \beta_3(e_2 - e_{2q-1}) + \gamma v_1. \]

So we have 4 possibilities of \( v_1 \) which can be extended to the maximal isotropic flag. They are \( e_1 - ie_2, e_1 + ie_2, e_{2q-1} + ie_{2q}, \text{ and } e_{2q-1} - ie_{2q} \).
If \( v_1 = e_1 + ie_2 \), then for \( \beta_1 = \mp i, \beta_2 = 0, \beta_3 = -1 \), and \( \gamma = \pm i \), the vector \( v_2 \) is \( v_2 = e_{2q-1} + ie_{2q} \). Thus, the 2-dimensional subspace corresponding to the pair \((1 - 2)\) is spanned by

\[
\langle e_1 + ie_2, e_{2q-1} + ie_{2q} \rangle. \tag{55}
\]

If \( v_1 = e_{2q-1} + ie_{2q} \), then for \( \beta_1 = 1, \beta_2 = 0, \beta_3 = \mp i \), and \( \gamma = \mp i \), the vector \( v_2 \) is \( v_2 = e_1 + ie_2 \). As a result, the 2-dimensional subspace corresponding to the pair \((1 - 2)\) is spanned by

\[
\langle e_{2q-1} + ie_{2q}, e_1 + ie_2 \rangle. \tag{56}
\]

Having determined the 2-dimensional subspace, we ignore the pair \((1 - 2)\) from \( w \) and repeat the same steps for the next pairs step by step. More generally, if \( w \) contains the pair \((2j - 1, -2j)\), \( 1 \leq j \leq q/2 \), then by the same method, the only possible 2-dimensional subspaces which can be completed to the maximal isotropic flag are

\[
\langle e_{2j-1} + ie_{2j}, e_{2q-2j+1} + ie_{2q-2j+2} \rangle \quad \text{or} \quad \langle e_{2q-2j+1} + ie_{2q-2j+2}, e_{2j-1} + ie_{2j} \rangle. \tag{57}
\]

Therefore, for each pair of \( w \), we have 4 possibilities. For the remaining numbers, recall that \( w \) is a perfect harmonic permutation. In particular, all numbers in the remaining boxes sit in an increasing order. Thus, the only possibility for these which can be completed to the maximal isotropic flag is the following point: if \( n \) is positive, the point is the flag associated with the ordered basis

\[
e_{2q+1} + ie_{2q+2}, \ldots, e_{2n-1} + ie_{2n}, \tag{58}
\]

and if \( n \) is negative, then the point is the flag associated with the ordered basis

\[
e_{2q+1} + ie_{2q+2}, \ldots, e_{2n-3} + ie_{2n-2}, e_{2n-1} - ie_{2n}. \tag{59}
\]

Hence, if \( q \) is even, then \( w \) contains \( q/2 \) pairs and each pair has 4 possibilities. Therefore, in this case, the number of possible intersection points is \( 4^{q/2} \cdot 1 = 2^q \). If \( q \) is odd, then \( w \) contains \((q - 1)/2\) pairs and each pair has 4 possibilities; then, the number of possible intersection points is \( 4^{(q-1)/2} \cdot 1 = 2^{q-1} \).

Finally, we show that the points described above belong to exactly \( q/2 \) flag domains if \( q \) is even and to \((q - 1)/2\) flag domains if \( q \) is odd. For this, recall that, for each pair in \( w \), we have 4 possibilities of 2-dimensional subspaces and note that the signature of these subspaces is ++ and ++, and the signature of the remainder is ++. Thus, the number of flag domains which have nonempty intersection with \( S_w \) for the given \( w \) is \( 2^{q/2} \) if \( q \) is even and is \( 2^{(q-1)/2} \) if \( q \) is odd.

Also, since each two of these 4 possibilities of 2-dimensional subspaces have the same signature, then for any fixed signature for a flag domain, there are \( 2^{q/2} \) points belonging to the base cycle of that flag domain if \( q \) is even and \( 2^{(q-1)/2} \) points belonging to the same base cycle of that flag domain if \( q \) is odd.

\[\square\]

Remark 6. In the case of the group \( SL_{2n}^* \) with the real form \( SU(p,q) \), Brecan [4, 5] shows that the number of Iwasawa–Schubert varieties which intersect at least one base cycle and have the minimal dimension \( pq \) is \((2n - 1) . (2n - 3) . . . (2n - 2q + 1) \). But for the group \( SO(2n,C) \) with the real form \( SO(p,q) \), if \( q \) is even, the number of Schubert varieties \( S_w \) which have the minimal dimension \( pq/2 \) and intersect at least one base cycle is \( n(2n-1)(n-3)\ldots(n-q+2) \), and if \( q \) is odd, the number of Schubert varieties \( S_w \) which have the minimal dimension \( pq/2 \) and intersect at least one base cycle is \((n-2)(n-4)\ldots(n-q+1) \).

In the case of \( SO(2n-1,C) \) with the real form \( SO(p,q) \), if \( q \) is even, the number of Schubert varieties \( S_w \) which have the minimal dimension \( pq/2 \) and intersect at least one base cycle is \((n-1)(n-3)\ldots(n-q+1) \), and if \( q \) is odd, the number of Schubert varieties \( S_w \) which have the minimal dimension \( pq/2 \) and intersect at least one base cycle is \((n-2)(n-4)\ldots(n-q+1) \).

The following remark, which is a consequence of the proof of Theorem 2, describes all intersection points of \( S_w \) with the base cycles \( C_n \).

Remark 7. To determine all intersection points between the base cycles and the Iwasawa–Schubert variety \( S_w \) of complementary dimension, we will define a set for each case of \( q \):

If \( q \) is even, let \( w \in W_1 \) be a perfect harmonic permutation and define the following:

\[\text{Swite}_w := \{ \psi(w_p) : w_p \text{ is obtained from } w \text{ by switching none, or some, or all pairs } (-2i - 1, 2i) \text{ or } (2i, -2i - 1) \text{ or } ((2i, -2i) \text{ and switching none, some, or all pairs } (-2i - 1, -2i) \text{ by } (-2i, 2i - 1) \text{ or } (2i, 2i - 1) \text{ or switch point } (2i - 1, 2i), 1 \leq i \leq q/2) \text{ Define } F_p(\text{Fix} T_q) \text{ to be the set of all maximally } b \text{-isotropic flags associated with the basis}\]

\[
e_1 + ie_2, e_3 + ie_4, \ldots, e_{2n-1} + ie_{2n}, e_{2n-1} - ie_{2n-1}, \]

\[e_{2n}, \ldots, e_1 - ie_2. \tag{60}\]

Let \( M_w \subset F_p(\text{Fix} T_q) \) be the set of all maximally \( b \)-isotropic flags associated with all elements in \( \text{Swite}_w \). Note that we have \( q/2 \) of the pairs \((-2i - 1, 2i)\) and \((-2i - 1, -2i)\), \(1 \leq i \leq q/2\), in any \( w \in W_1 \), and for each pair, we have 4 possibilities to switch it, so the cardinality of \( \text{Swite}_w \) is \( 4^{q/2} = 2^q \). The set \( \text{Swite}_w \) gives us all intersection points of \( S_w \) and each \( 2^{q/2} \) of these points belongs to only one flag domain where these points of intersection sit in the base cycle of that flag domain.

If \( q \) is odd, let \( w \in W_1 \) be a perfect harmonic permutation and define the following:

\[\text{Swite}_w := \{ \psi(w_p) : w_p \text{ is obtained from } w \text{ by switching none, some, or all pairs } (-2i - 1, 2i) \text{ by } (-2i, -2i - 1) \text{ or } (-2i, 2i - 1) \text{ or } ((2i - 1, 2i), 1 \leq i \leq q/2) \text{ Define } F_p(\text{Fix} T_q) \text{ to be the set of all maximally } b \text{-isotropic flags associated with the basis}\]

\[
e_1 + ie_2, e_3 + ie_4, \ldots, e_{2n-1} + ie_{2n} e_{2n-1} - ie_{2n-1} \]

\[e_{2n}, \ldots, e_1 - ie_2. \tag{60}\]
Example 6. In $G_0 = SO(5, 3)$, fix $w = (−124 − 3)$ as a perfect harmonic permutation, then

$$Swito_w = \{(251 − 3 − 4), (25 − 1 − 34), (−2513 − 4), (−25 − 134), (−3512 − 4),$$

$$(351 − 2 − 4), (−35 − 124), (35 − 1 − 24), (25 − 4 − 31), (−25 − 431), (254 − 3 − 1), (−2543 − 1),$$

$$(−3542 − 1), (354 − 2 − 1)\}. \quad (62)$$

Recall that, in the cases of $SP(2n, \mathbb{R})$ and $SO^+(2n)$, every flag domain intersects all Schubert varieties of complementary dimension. But in the case of $SO(p, q)$, we do not have this property except in a very special case. We explain this case in the following example.

Example 7. If $n = q + 1$, then the flag domain parametrized by the sequence

$$\alpha = +−−−+−−−+−−−++ \quad (64)$$

intersects all Schubert varieties of dimension $pq/2$ if $q$ is even. And the flag domain parametrized by the sequence

$$\beta = +−−−+−−−++ \quad (65)$$

intersects all Schubert varieties of dimension $(pq−1)/2$ if $q$ is odd.

Data Availability

The research data used to support this study are included within the article.

Disclosure

Parts of this work are from the author’s PhD thesis at Ruhr-Universität, Bochum, Germany [6].

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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