Cooperative Resource Sharing with Adamant Player

Shiksha Singhal and Veeraruna Kavitha,
IEOR, Indian Institute of Technology Bombay, India

Abstract— Cooperative game theory deals with systems where players want to cooperate to improve their payoffs. But players may choose coalitions in a non-cooperative manner, leading to a coalition-formation game. We consider such a game with several players (willing to cooperate) and an adamant player (unwilling to cooperate) involved in resource-sharing. Here, the strategy of a player is the set of players with whom it wants to form a coalition. Given a strategy profile, an appropriate partition of coalitions is formed; players in each coalition maximize their collective utilities leading to a non-cooperative resource-sharing game among the coalitions, the utilities at the resulting equilibrium are shared via Shapley-value; these shares define the utilities of players for the given strategy profile in coalition-formation game. We also consider the utilitarian solution to derive the price of anarchy (PoA). We considered a case with symmetric players and an adamant player; wherein we observed that players prefer to stay alone at Nash equilibrium when the number of players \(n\) is more than 4. In contrast, in majority of the cases, the utilitarian partition is grand coalition. Interestingly the PoA is smaller with an adamant player of intermediate strength. Further, PoA grows like \(O(n)\).

I. INTRODUCTION

Resource sharing problem is a well-known problem that aims to find an optimal allocation of shared resources. We consider a scenario in which a common resource is to be shared amongst several users as in Kelly’s mechanism \([1]\); the utility of any player is proportional to its bid and inversely proportional to the weighted sum of bids of all players, with the weights representing the influence factors. This proportional allocation problem is considered in a variety of other contexts; e.g., \([2]\) considers real-time performance in time-shared operating systems, \([3]\) considers rate allocation for communication networks, \([4]\) considers resource allocation in wireless network slicing, \([5]\) considers online auctions, etc. We consider a similar game theoretic formulation with some important differentiating features: i) possibility of cooperation among the willing players; and, ii) the presence of an adamant player, not interested in cooperation. Usually, the market giants (e.g., Amazon) tend to strive alone while smaller business entities (e.g., Flipkart, Walmart) try to look for collaboration opportunities.

Any transferable utility cooperative game is defined by a set of players \(N\) and the worth of each possible coalition \(\nu(S); S \subset N\) (e.g., \([4]\), \([5]\)). Majority of the analysis related to cooperative games discuss the emergence of grand coalition (includes all players) as a successful partition and then consider the division of worth among the players; Shapley value, Core etc., are some such solution concepts (e.g., \([4]\)). But one can find many example scenarios, in which a partition of strict coalitions (subsets) of \(N\) might emerge out at some appropriate equilibrium (\([15]\), \([14]\)). In this context, one of the key challenges is to generate a partition, i.e., an exhaustive and disjoint division of the set of agents, such that the performance of the system is optimized (see for example, \([6]\) and references therein). This leads to a utilitarian solution. In contrast we consider a non-cooperative approach to generate partitions (e.g. as in \([1]\), \([15]\)); basically the solution/partition would be stable against unilateral deviation. These are in general called as coalition formation games (CFGs) (\([6]\)).

Another important aspect of cooperative games is characteristic form games and partition form games \([6]\). Majority of the literature focuses on the former type of games and a very little attention has been given to a more general class of partition form games. In the former type of cooperative games, the worth of a coalition depends only on the members of the coalition, while in the latter type, the worth is also influenced by the partition of the players outside the given coalition (\([6]\)). These inter-coalitional dependencies, play a crucial role in many real-world applications where agents have either conflicting or overlapping goals (e.g., \([13]\), \([3]\)). Our problem falls into the latter category.

We consider a CFG in the presence of an adamant player (not willing to cooperate) and seek for a non-cooperative solution. In our game, the strategy of a player is the set of players with whom it wants to form coalition as in \([1]\). Given the strategies of all players, basically the preferences of all the players, an appropriate partition of coalitions is formed; and players in each coalition maximize their collective utilities. This leads to a non-cooperative resource sharing game (RSG) among the coalitions. The utilities at the resulting equilibrium are shared via Shapley value (confined to each coalition); these shares define the utilities of individual players for the given coalition suggestive preferences of all players in CFG.

We derived the solution of this non-cooperative CFG for the special case of symmetric players (players with same influence factor). For smaller number of players \((n \leq 4)\), the partitions at NE depend upon the relative strength of the adamant player and that of the others (referred as \(\eta\)). The partitions at NE are not monotone with \(\eta\); coarser partitions result at lower and higher values of \(\eta\), while we have finer partitions for intermediate values. This non-monotone behaviour is absent for \(n \geq 4\); the players prefer to remain alone at equilibrium (irrespective of \(\eta\)), i.e., the finest partition emerges out.

We also consider the utilitarian solution (maximizes the sum of utilities of all players) to derive the price of anarchy (PoA), which captures the loss of players resulting due to their rational behaviour. The utilitarian solutions are also non-monotone with \(\eta\), however majority of times grand coalition
Observe that a partition \( P_{\alpha} \) of two coalitions is the solution. For few cases a partition with two coalitions is the solution. Interestingly, the \( P_{\alpha} \) in higher in the absence of an adamant player even when the players do not share resources with an extra player. The \( P_{\alpha} \) increases with \( n \) and \( \eta \). It also increases with decrease in \( \eta \) to zero. Interestingly, the limit in all the cases equals that in the system without adamant player.

II. PROBLEM DESCRIPTION AND BACKGROUND

Consider a system with \((n+1)\) players involved in a resource sharing game (RSG) and let \( N = \{0,1,2,\cdots,n\} \) denote the set of players along with an adamant player represented by index 0. The \( n \) players (other than the adamant player) are interested in forming coalitions, and these are referred to as C-players. These players are willing to cooperate with each other if they can obtain higher individual share while the adamant player is not interested in cooperation.

The utility of a player is proportional to their actions which also includes a proportional cost. Thus, when players choose respective actions \( (a_0,a_1,\cdots,a_n) \), the utility of player \( i \) equals

\[
\varphi_i = \frac{\lambda_i a_i}{\sum_{j=0}^{n} \lambda_j a_j} - \gamma a_i \quad \text{for} \quad i \in N, \quad \text{where}, \quad (1)
\]

\( \gamma \) represents the cost factor, \( \lambda_i \) represents the influence factor of \( i^{th} \) player and, 
\( a_i \in (0,\hat{a}) \) for some \( n/\gamma < \hat{a} < \infty \) which ensures the existence of a unique Nash Equilibrium (NE).

The first component of equation (1) is the fraction of resource allocated to player \( i \) and the other component represents the cost. This resembles the utility of players in well known Kelly-mechanism for resource sharing and is relevant in various applications including communication networks (e.g., [8], [9] etc.), online-auctions ([10], etc).

In this paper, we consider the case with symmetric C-players, i.e., \( \lambda_i = \lambda \) for all \( i \in N_c \) where \( N_c := \{1,\cdots,n\} \) is the set of C-players and \( \lambda_0 = \eta \lambda \) with \( \eta \in [0,\infty) \). Here, \( \eta = 0 \) implies the absence of adamant player (considered in Section [VII]).

When the players choose their actions in a fully non-cooperative manner, i.e., when none of the C-players are interested in forming coalitions, it results in a strategic form game with utilities as in (1): basically the rational and intelligent players choose their respective actions to improve their own utility and the utility derived by any player equals that at Nash Equilibrium (NE) [4].

When the C-players are looking for opportunities to form coalitions and work together, a set/collection of coalitions emerge at an equivalent equilibrium (details in later sections); say \( P = \{S_0, S_1, \cdots, S_k\} \) represents the partition of \( N \) into different coalitions where \( S_0 = \{0\} \) denotes the adamant player. Observe that a partition \( P \) is a set of coalitions such that

\[
\bigcup_{i=0}^{k} S_i = N \quad \text{and} \quad S_i \cap S_j = \emptyset, \quad \text{null set,} \quad \forall i \neq j. \quad (2)
\]

The players in a coalition \( S_i \) choose their strategies together with an aim to optimize their social objective function (of their own coalition) and hence the utility of a coalition is given by:

\[
\varphi_{S_m}(a_m, a_{-m}) = \frac{\lambda \sum_{l \in S_m} a_l}{\lambda_0 a_0 + \lambda \sum_{l=1}^{n} a_l} - \gamma \sum_{l \in S_m} a_l; \quad m \geq 1 \quad (3)
\]

\[
\varphi_S(a_m, a_{-m}) = \frac{\lambda_0 a_0}{\lambda_0 a_0 + \lambda \sum_{l=1}^{n} a_l} - \gamma a_0, \quad \text{where}, \quad (4)
\]

\( a_m = \{a_i, i \in S_m\}, a_{-m} = \{a_i, i \notin S_m\}, \forall S_m \in P \),

which is the sum of their individual utilities. The players will now try to derive maximum utility for their own coalition and hence there would again be a non-cooperative game, but now among coalitions. Thus we have a reduced RSG (one for every \( P \)) with each coalition representing one (aggregate) player and the utilities given by (3) and (4); utility of any coalition equals that at the corresponding NE. This utility is divided among the members of the coalition using the well-known Shapley values (computed within the coalition) as described in Section [III-C].

This is the problem setting and our aim is to study the coalitions/partitions that emerge out successfully (at an appropriate equilibrium), when the C-players (henceforth referred as players) seek opportunities to come together in a non-cooperative manner. There is a brief initial study of this problem in [5], for the special case when players only form grand coalition, i.e., when \( P = \{\{0\},\{1,\cdots,n\}\} \). For this case, it has been shown that:

i) The utility of grand coalition at CNE (Cooperative NE) is higher than the sum of individual utilities of players at the unique NCNE (Non-Cooperative NE) for majority of the scenarios. The paper also provides example scenarios for the case where the sum of utilities at NCNE is larger.

ii) Moreover, Shapley value does not always share this utility in a fair manner; the sum of utilities might be larger, but the shares derived via Shapley value by some players is smaller (especially ones with higher influence factors). This aspect is not further investigated in the current paper.

The above study lead to new questions: a) can the players derive even better utilities if they form strict sub-coalitions instead of grand coalition; b) when is it beneficial for the players to cooperate; c) how stable are these resultant coalitions (e.g., against unilateral deviation). One can have more questions with asymmetric C-players and we wish to investigate this in future (we already have some initial result [3].

We build an appropriate non-cooperative framework to study these aspects. We also consider solutions that optimize social objective function and derive the Price of Anarchy.

III. ADAMANT COALITION FORMATION GAMES

We use non-cooperative framework to study this coalition formation game (CFG) as in [11]. For each C-player, i.e., for \( i \in N_c \), strategy \( x_i \) is defined as the set of players with whom player \( i \) wants to form coalitions, i.e., \( x_i \subseteq N_c \) and the corresponding strategy set \( X_i \) is defined as:

\[ X_i \]

With one asymmetric player and no adamant player, we found that grand coalition can also emerge at NE, even for large \( n \).
$X_i = \{x_i : i \in x_i \text{ and } x_i \subseteq N_C\}$. 
To find a strategic form game we need to define the utility of all players for any given strategy profile, i.e., for any $\bar{x} = (x_1, x_2, \cdots, x_n)$ with $x_i \in X_i$ for each $i \in N_C$. 
As a first step, one needs to define appropriate partition(s) of coalitions (referred as $P(\bar{x})$, and made up of subsets of $N$) that can result for any given strategy profile $\bar{x}$. 

A. Partition for a given strategy profile $\bar{x}$ 
We say a partition $P'$ is (strictly) better than partition $P$, represented by the symbol $P' \prec P$, if every coalition of the latter is a subset of a coalition of the former (with at least one of them being a strict subset), i.e., if $P' \neq P$, and, for all $S \in P \ni S' \in P'$ such that $S \subset S'$. (5) 
Note that the size (number of coalitions) of the better partition is strictly smaller than that of the other; in other words, there exists at least two coalitions $S_1, S_2 \in P$ such that $S_1 \cup S_2 \subset S$ for some $S \in P'$.
Partition $P(\bar{x})$ formed by $\bar{x}$: We say $\bar{x} \rightarrow P(\bar{x})$, if it satisfies the following two conditions as in (6): 

i) respects the preferences, a coalition $S$ is an element of partition $P(\bar{x})$, i.e., $S \in P(\bar{x})$, if it satisfies: 

\[
i \in x_i \text{ and } j \in x_j \text{ for all } i, j \in S; \text{ and,}
\]

ii) minimal partition, there exists no other (see (5)) (better) partition $P'$ formed by $\bar{x}$, such that $P' \prec P$. (7) 

Hence, a partition formed by $\bar{x}$ is a (minimal) subset of $2^N := \{S : S \subset N\}$ such that (8) and (7) are satisfied and all its coalitions satisfy (6) with $\bar{x}$. Using these rules, we may obtain multiple partitions for some strategy profiles (examples in Tables III and VII).
To summarize, if $\bar{x} = (x_1, \cdots, x_n)$ is the strategy profile, let $n(\bar{x})$ represent the number of possible partitions corresponding to $\bar{x}$ and let the partitions formed be represented by the following: $P_1(\bar{x}), P_2(\bar{x}), \ldots, P^{n(\bar{x})}(\bar{x})$. 
We now define the utilities derived by (all) the coalitions and then the individual players. We begin with $n(\bar{x}) = 1$. 

B. Utilities of coalitions in a given partition 
Let $P(\bar{x}) = \{S_0, S_1, \cdots, S_k\}$ be a partition of $N$ with $k$ coalitions of C-players, corresponding to $\bar{x}$. We now aim to find the utility of coalitions in $P(\bar{x})$, represented by $\varphi^e_{S_m}(P)$ for all $m \in \{0, 1, 2, \cdots, k\}$. We will see that these utilities depend upon the strength of the adamant player, via $\eta := \lambda_0 / \lambda$, the relative ratio of the influence factors (recall $S_0 = \{0\}$ is the coalition with only adamant player).
As already mentioned, the resource sharing game (RSG) is now reduced to a $(k+1)$-(aggregated) player non-cooperative strategic form game which is given by the tuple, 

\[
\left(\{0, 1, \cdots, k\}, \{[0, \hat{a}]{\mid S_0} \times \cdots \times [0, \hat{a}]{\mid S_k}\}, \varphi\right), \tag{8}
\]
where $|S_m|$ represents the cardinality of coalition $S_m$ and $\varphi = \{\varphi_{S_0}, \varphi_{S_1}, \cdots, \varphi_{S_k}\}$, the vector of utilities is given by (3) and (4). This kind of a game is analysed in [5, Lemma 2] for the special case with grand coalition (GC) of C-players. Since, we consider all possible exhaustive and disjoint collection of players, i.e., all possible partitions (corresponding to various coalition suggestive strategy profiles), we extend the above result to a general partition in the following:

Theorem 1: [Utilities of coalitions] The game (8) can have multiple NE, but the utilities at NE are unique and are given by (for any $1 \leq m \leq k$), 

\[
\varphi^e_{S_m}(P) = \frac{\lambda^2}{(\lambda + k\lambda_0)^2} \frac{1}{1 + k^2(1 - \|B\|)} = \frac{1}{(1 + k\eta)^2} 1 + \frac{1}{1 - \|B\|}, \quad (9)
\]

with indicator $\|B\| := \|_B > \frac{k}{k}\|, k = |P| - 1, \eta := \lambda_0 / \lambda$ and,

\[
\varphi^e_{S_0}(P) = \left(\frac{1 - k}{\lambda + k\lambda_0}\right)^2 \frac{1}{1 + k^2}, \quad (10)
\]

Further the optimal actions at any NE satisfy: 

\[
\bar{a}^*_m := \sum_{j \in S_m} a^*_j = \frac{k\lambda_0}{\gamma(\lambda + k\lambda_0)^2} \text{ and, } \bar{a}^*_0 = \frac{k\lambda^2((1 - k)\lambda + k\lambda_0)}{\gamma(\lambda + k\lambda_0)^2}. \tag{11}
\]

Proof: The proof is almost similar to the one in [5] and is available in Appendix A. 

From (11), some or all of the players in a coalition can choose actions such that the sum of these actions equal corresponding $\bar{a}^*_m$; all such actions constitute NE; hence multiple NE exist. However, the utilities of coalitions are uniquely defined by (9).

Significant Adaman Player: The adamant player gets non-zero utility at NE when $\|B\| = 1$, i.e., when $\eta > 1 - 1/k$, we then say the adamant player is significant otherwise, it is insignificant. However, it is always significant when grand coalition is formed, i.e., when $k = 1$ (see equation (10)). This condition will play an important role in our CFG.
To summarize the utilities of any coalition of any given partition $P$ are given by (9) and (10), which are the utilities at NE of the reduced RSG with coalitions as the players.

C. Division of worth within a coalition 
The next step is to divide the worth of a coalition among its members using Shapley value confined to each coalition as in (9). For symmetric players, the utility of a coalition gets divided equally among its members because of equal influence factors. Hence from (9), the utility of player $i$ under partition $P$ is given by (if $i \in S_m$): 

\[
\varphi^e_{i}(P) = \frac{\|B\|}{|S_m| (1 + k\eta) + \frac{1 - \|B\|}{k^2 |S_m|}} \text{ with } k = |P| - 1. \tag{12}
\]

D. Utility of a player 
We define the utility of a player, say $i$ as the minimum utility among all the possible partitions, i.e., (see (12)) 

\[
U_i(\bar{x}) = \min_{P(\bar{x})} \varphi^e_{i}(P(\bar{x})). \tag{13}
\]

This definition ensures minimum guaranteed utility to each player for the given strategy profile $\bar{x}$ and is similar to the
security value used in game theory (1)). Basically, when a strategy profile (recall it represents the coalition formation interests of all the players) can lead to multiple partitions, the eventual partition formed may depend on some further negotiations. Hence it is best to define the utility of each player as the worst possible utility.

E. Coalition Formation Game: Ingredients

We now have a non-cooperative CFG with, i) $N_C$ as the set of players; ii) $X_i$ is the strategy set of player $i$; and iii) Utilities of players, $\{U_i(x)\}_{i}^N$ given by (12). Recall these utilities are defined via their Shapley value corresponding to the coalition that they belonged (based on their and others strategies), the worth of which is computed using NE of the reduced RSG.

We study this game and consider two types of solution concepts: NE and Social Optima and also discuss the price of anarchy in the coming sections.

IV. INITIAL ANALYSIS AND SOLUTIONS

In this section we consider some partition-wise analysis which later leads to the analysis of Nash equilibrium. We also define the solution concepts used in this paper.

A. Partition resulting from a unilateral deviation

Recall a strategy profile $x$ leads to partition $P$, represented by $x \rightarrow P$, if $P$ results from $x$ as explained in section III-A i.e., if it satisfies (6) with $x$, (2) and (7). We say, $x$ leads to unique partition $P$, represented by $x \rightarrow_! P$, further, if $P$ is unique such partition, i.e., if $n(x) = 1$.

Consider any partition $P = \{S_0, \cdots, S_k\}$ and say $x \rightarrow_! P$. Now consider a unilateral deviation of player $i$, from $x_i$ to $\{i\}$ (strategy of being alone) in $x$ and say $i \in S_l$. Then, the following lemma shows that the new strategy profile $(x')$ also leads to a unique partition with $S_l$ (coalition getting split into two; $\{i\}$ and $S_l/\{i\}$ (the rest as one sub-coalition):

Lemma 1: Consider a strategy profile $x \rightarrow_! P$, where $i \in S_l$. Let $x' := \{\{i\}, x_{-i}\}$ be the strategy obtained by the above unilateral deviation, then $x' \rightarrow_! P$, where:

$\mathcal{P}_{-i} := \{S_0, S_1, \cdots, S_{l-1}, \{i\}, S_l(\{i\}, S_{l+1}, \cdots, S_k)\}$.  

Proof: The proof is in Appendix B.

We call partition $\mathcal{P}_{-i}$ of the above Lemma as the $i$-unilateral deviation partition, $i$-u.d.p., of the pair $(x, P)$.

B. Weak Partition

A partition is defined to be weak if for all $x \rightarrow_! P$, there exists a player $i$ which gets strictly better utility at its $i$-u.d.p., i.e., if $\mathcal{P}_{-i}$ defined in Lemma 1,

$$U_i(\mathcal{P}_{-i}) > U_i(\mathcal{P})$$

With the above definitions in place, we have the following result for characterizing the weak partitions:

Lemma 2: Consider a partition $P$ with $|P| = (k + 1)$. Let $m^* := \max_{S_i \in P} |S_i|$, be the size of the largest coalition. If $m^* > (k + 1)^2/k^2$, then $P$ is weak.

Proof: The proof is in Appendix B.

Remark: Say for all the strategy profiles $x$ leading to $P$ it is the unique such one (i.e., $x \rightarrow_! P$). Further, if it satisfies the above conditions, it cannot be a partition at NE. However, if there is a strategy profile leading to multiple partitions with one of them being $P$, then $P$ can still emerge at a NE. We will investigate these aspects in the immediate following.

C. Nash Equilibrium

To study the CFG (see section III-E), we again consider the solution concept Nash Equilibrium (NE) (recall this solution ensures that no player can get better on unilateral deviation). The NE is now in terms of coalition suggestive strategy profile, but one might be more interested in NE-partitions. Lemma 2 characterizes weak partitions, and one may think weak partitions cannot result from a NE. However, as discussed before, if a weak partition is one amongst the multiple partitions emerging from a NE, then a weak partition can also be a NE-partition. Thus we have:

Lemma 3: [NE $\not\rightarrow$ Weak Partition] Assume that the game does not have multiple partitions at NE. Then, if a partition $\mathcal{P}$ is weak, it cannot be a NE-partition.

Proof: The proof is straightforward.

If for a given set of parameters, it is known a priori that none of the NE lead to multiple partitions, then by the above Lemma, a weak partition can’t emerge from a NE. We will then concentrate on partitions that are not weak. We will use these intermediate results to derive the NE. Before we proceed with this we discuss the relevant social objective function.

D. Social Optima

In this paper we are primarily studying the CFGs, in which the players choose their partners in a non-cooperative manner; basically the players are interested in coalition formation, so as to improve their own objective function (selfishly) and one requires a solution which is stable against unilateral deviations. But if instead the players attempt to optimize a social/utilitarian objective (sum of utilities of all the players), they would have achieved much better utilities; this aspect is well understood in literature (2) and references therein) and we study the same in our context. A utilitarian solution, referred to as SO (social optimizer), is any strategy profile $x^s$ that maximizes:

$$\sum_{i \in N_C} U_i(x^s) = \max_{\tilde{x}} \sum_{i \in N_C} U_i(\tilde{x}).$$

In [5] authors illustrated that the sum utility of the C-players improve significantly, when all players come together to form a grand coalition (as $n$ increases). However we will see in this paper that for $n > 4$, the only NE-partition is ALC (all alone). Because of the selfish nature of the players, the efficiency of a system degrades and the utility received by players at NE is lower than that at SO. We study this loss using the well known concept, Price of Anarchy.

One might be interested in the NE or SO, basically the strategies that represent the solutions. However in our context, the more interesting entities are the partitions at various equilibrium/optimal solutions; we are interested in NE-partitions
and the SO-partitions. When one directly optimize using partitions; it is easy to see that the SO-partition, \( P^* \), satisfies the following:

\[
U_{SO} := \sum_{S_i \in P^*_i; i \neq 0} U_{S_i}(P^*_S) = \max_{P} \sum_{S_i \in P; i \neq 0} U_{S_i}(P'),
\]

where \( P' \) includes all possible partitions (see (9)-(12)).

**Some more notations:** Let \( P_k \) represent any partition with \( k \) coalitions of C-players, i.e., \(|P_k| = 1 + k\). Let (All Alone Coalitions) \( \text{ALC} := \{\{0\}\}, \{\{1\}\}, \{\{2\}\}, \cdots , \{\{n\}\}\) (all players are alone). The strategy \( x_i = \{i\} \) is the ALC strategy for any \( i \), and GC (Grand Coalition) partition implies partition \( \{\{0\}, N_C\} \), while GC strategy implies \( x_i = N_C \).

**Two groups of partitions:** As seen in (10), at some equilibrium the adamant player becomes insignificant, i.e., gets 0 utility. We distinguish these equilibrium partitions from the others using superscript \(^6\). Thus, for example, ALC is the NE-partition if adamant player is significant at that NE, otherwise, ALC\(^o\) is the NE-partition.

With the above notations in place, we have the following result completely characterizing the SO-partitions:

**Lemma 4:** [SO-partitions] i) When \( \eta > 0.707 \) or when \( \eta \leq 0.414 \), then GC is the SO-Partition.
ii) When \( 0.414 \leq \eta \leq 0.5 \), any \( P^*_j \) is the SO-partition.
iii) Any \( P^*_2 \) is a SO-partition for rest, \( (0.5 < \eta \leq 0.707) \).

**Proof:** The proof is in Appendix B.

**E. Price of Anarchy and SO-partition**

Price of Anarchy (\( P_oA \)) is defined as the ratio between the sum utilities at ‘social optima’ and the sum utilities at the social optima and the SO-partitions. When one directly optimize using partitions; it is easy to see that the SO-partition, \( P^*_S \), satisfies the following:

\[
P_oA = \frac{\max_{P} \sum_{S_i \in P; i \neq 0} U_{S_i}}{\min_{P^*} \sum_{S_i \in P^*; i \neq 0} U_{S_i}} = \frac{U_{SO}}{\min_{P^*} \sum_{S_i \in P^*; i \neq 0} U_{S_i}},
\]

where \( P^* \) is any NE-partition.

**ALC/ALC\(^o\)** is always an NE-partition

When all others choose to be alone, i.e., if \( x_i = \{i\} \) for all \( i \neq j \), then it is clear that the best response of \( j \) includes \( x_j = \{j\} \). This is true for any \( j \). This leads to an NE. From (10), the adamant player becomes insignificant at ALC when \( \eta \leq 1 \) \( 1/n \), then the NE-partition is ALC\(^o\), otherwise ALC is the NE-partition.

**V. LARGE NUMBER OF PLAYERS, \( n > 4 \)**

For the case with \( n > 4 \), we have the following two results using Lemma 3 (proofs in Appendix A):

**Corollary 1:** [Weak Partitions] All partitions other than ALC/ALC\(^o\) are weak.

**Theorem 2:** [No Multiple Partitions at NE] Any strategy profile leading to multiple partitions cannot be an NE.

In view of the above two results and Lemma 3 only ALC/ALC\(^o\) is the NE-partition. Further using (10), we have:

**Corollary 2:** There is a unique NE-partition. ALC is the NE-partition if \( \eta > (n-1)/n \), else ALC\(^o\) is the NE-partition.

**Remarks:** In [5], authors defined BoC (benefit of cooperation) as the normalized improvement in sum of utilities that the players achieve at GC in comparison with that achieved when they compete alone. They showed that BoC increases significantly as \( n \) increases ([5, Lemma 3]). Despite the fact that BoC is large for large \( n \), by the above Corollary we have that players prefer to remain alone at NE. Thus the price paid for anarchy (\( P_oA \)) can be significantly high.

**A. Price of Anarchy**

From Corollary 2, we have ALC/ALC\(^o\) as the only NE-partition and from Lemma 4 GC is the SO-Partition when \( \eta \geq 0.707 \). Hence \( P_oA \) equals (see (9)):

\[
P_oA = \frac{1}{n(1+\eta^2)} = \frac{(1+n\eta)^2}{n(1+\eta)^2} \quad \text{when } \eta \geq 0.707.
\]

We compute \( P_oA \) for the remaining cases in a similar way and the results are in Table I. Clearly as \( n \to \infty \), \( P_oA \) grows like \( n \), i.e., \( P_oA = O(n) \); this is another instance of strategic behaviour where the players pay high price for being strategic.

| \( \eta \) | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| \( \eta \geq \frac{2}{\sqrt{3}} \) | ALC | GC | \( \left( \frac{1+\eta}{1+\eta^2} \right)^2 \) | \( \frac{n}{(1+\eta)^2} \) | \( \frac{n}{(1+\eta)^2} \) |
| \( 0.707 \leq \eta \leq \frac{\sqrt{7}}{2} \) | ALC\(^o\) | GC | ALC\(^o\) | P\(^o\) | ALC\(^o\) |
| \( 0.5 < \eta \leq 0.707 \) | ALC\(^o\) | P\(^o\) | ALC\(^o\) | P\(^o\) | ALC\(^o\) |
| \( 0.414 \leq \eta \leq 0.5 \) | ALC\(^o\) | P\(^o\) | ALC\(^o\) | P\(^o\) | ALC\(^o\) |
| \( 0 < \eta \leq 0.414 \) | ALC\(^o\) | GC | ALC\(^o\) | GC | ALC\(^o\) |

**TABLE I**

**NE-partitions, SO-partitions and \( P_oA \) for \( n > 4 \)**

**VI. SMALL NUMBER OF PLAYERS, \( n \leq 4 \)**

In this section, we identify the NE-partitions and derive the \( P_oA \) for \( n \leq 4 \), by direct computations.

When \( n = 2 \): Here, GC and ALC (or ALC\(^o\)) are the only possible partitions. Some strategy profiles and the corresponding partitions can be seen from Table II.

**TABLE II**

**PARTITIONS AT \( n = 2 \)**

| \( x_1 \) | \( x_2 \) | \( P \) |
|---|---|---|
| GC | GC | GC |
| GC | GC | \{GC,GC\} |
| GC | ALC | \{ALC,GC\} |
| ALC | GC | \{ALC,GC\} |

**TABLE III**

**PARTITIONS AT \( n = 3 \)**

We begin with deriving the best responses (BR). Consider the case with \( \eta \geq 0.707 \). Then from (12), BR of player 2 against player 1’s strategy, \( x_1 = \{1, 2\} \) is GC, because:

\[
\frac{1}{2} \left( \frac{\lambda}{\lambda + 2\lambda_0} \right)^2 \geq \left( \frac{\lambda}{\lambda + 2\lambda_0} \right)^2.
\]

Thus both GC and ALC are NE-partitions when \( \eta \geq 0.707 \). In a similar way one can verify that the only NE-partition is ALC for \( 0.5 < \eta \leq 0.707 \) (see Table IV).

When \( 0.414 \leq \eta \leq 0.5 \), the adamant player is insignificant (gets 0 at NE) and ALC\(^o\) is the unique NE-partition. Interestingly, below \( \eta \leq 0.414 \), the C-players find it beneficial (again) to cooperate, note GC is also a NE. Thus we observe
interesting non-monotone phenomenon with ratio of influence factors, $\eta$.

When $n = 3$: In this case, we can have three types of partitions: GC, ALC, and $P_2$ type partitions. In any $P_2$ type partition, two of the C-players are together in one coalition, while the remaining one is alone. Some strategy profiles and their partitions are in Table III.

We derive the analysis by directly computing the BRs as in the previous case. The results are summarized in Table IV.

Important observations are: a) If GC is a NE-partition, all others are also NE-partitions; b) recall ALC/ALC$^0$ is always a NE-partition; c) the utilities of all the players at GC are bigger than those at ALC, when TTC is a NE-partition, thus TTC is the preferred NE. The non-monotone phenomena observed for the case with $n = 2$ can also be seen for $n = 3$ and $n = 4$.

$P_{oA}$ and Observations

The $P_{oA}$ for smaller $n$ is computed in the Tables V and VI and the overall observations related to $P_{oA}$ are: i) From Table I as the number of players increases the $P_{oA}$ also increases, and, $P_{oA} = O(n)$ when $n \to \infty$; ii) For any $n$ as the adamant player grows strong (as $\eta \to \infty$), the $P_{oA} \uparrow n$ (see Tables II [IV, V, VII]; and iii) Similarly when the adamant player becomes weak ($\eta \to 0$), the $P_{oA}$ again increases to $n$.

VII. Without Adamant Player

We now consider the same model as in previous sections, but without adamant player. Majority of the analysis goes through as in previous cases, we will only mention the differences. The utility of any partition $P^o = \{S_1, \ldots, S_k\}$ and that of the individual players, using Theorem I and Shapley value simplify to:

$$U_{S_m} = \frac{1}{|P^o|^2} \forall m, \text{ and } U_i = \frac{1}{|P^o|^2|S_m|} \text{ if } i \in S_m. \quad (14)$$

These utilities are exactly the same as those in the previous model with insignificant adamant player, except for GC.

The results for the case with $n > 4$ are exactly the same because of the following: i) Lemmas II and III are independent of adamant player; ii) Theorem II is also applicable, since only $G_1 := \{N_C, \ldots, N_C\}$ leads to GC, and that toc $G_1 \to! GC$; and iii) The proof of Lemma II can easily be adapted.

Smaller $n$: One can compute NE for all these cases as before, and the results are in Table VII. In similar way the SO-partition is GC$^0$ (proof in Appendix B):

**Lemma 5:** GC$^0$ is the SO-partition in the absence of adamant player, for all $n$. 

| $P$ at NE | Range of parameters | $P$ at SO | $P_{oA}$ |
|-----------|---------------------|-----------|-----------|
| GC        | $\eta \geq 0.707$  | GC        | $\eta$   |
| ALC       | $0.5 < \eta \leq 0.707$ | P$^2$  | 1         |
| ALC$^0$   | $0.414 \leq \eta \leq 0.5$ | P$^2$  | 2         |
| GC$^0$    | $0 \leq \eta \leq 0.414$ | GC$^0$ | $\eta$   |

**Table IV**

NE-partitions, SO-partitions and $P_{oA}$ for $n = 2$
With adamant player absent, the players naturally derive larger shares. However, we observe (from all the tables) that $P_{oA}$ is larger without adamant player. In all the cases, the $P_{oA}$ with adamant player increases to that without adamant player either when $\eta \to \infty$ or when $\eta \to 0$.

### VIII. Conclusions

We consider a coalition formation game with players exploring cooperation opportunities in a non-cooperative manner, where the utilities of players/coalitions are resultant of a resource sharing game. We developed a framework to study the partitions (non-overlapping and exhaustive set of coalitions) that emerge at equilibrium. The strategy of a player is the set of players with whom it wants to form coalition, while the utilities of players are defined via (Shapley values of) the utilities of their coalitions and these coalitions/partition is formed based on the choice of all players; the resulting coalitions involve in a non-cooperative game along with an adamant player (not willing to cooperate) and the utilities at the equilibrium define the utilities of the coalitions.

Our primary aim is to identify the NE-partitions, we also derive the partitions that result at utilitarian solution (maximizes the sum of utilities). We observe that the agents derive much lower utilities at NE than that at utilitarian solution, and this loss is because of their strategic behaviour. We considered $P_{oA}$ (price of anarchy) to estimate the loss.

We considered a case study with symmetric players (with and without adamant player), and have the following important results: i) when $n > 4$, the players prefer to stay alone at NE, while their preferences are coarser partitions (either grand coalition or a partition with two coalitions) at utilitarian solution; ii) players derive larger utilities at coarser (with smaller number of coalitions) partitions, if the latter emerges at equilibrium; iii) thus the price of anarchy is significantly high, in fact increases as $O(n)$. iii) We see a non-monotone behaviour in the NE-partitions when $n \leq 4$, with the strength of the adamant player measured via the ratio of influence factors; and this behaviour is primarily because the adamant player gets insignificant at more number of partitions, as its strength reduces; iv) the $P_{oA}$ is smaller when the adamant player is of intermediate strength; it increases either as the strength of adamant player increases or as the strength reduces to zero; the limit in both the cases equals that in the system without adamant player.

### APPENDIX A

**Proof of Theorem 1** The utility of a coalition $S_m$, given by equation (3) and (4) can be re-written as:

$$\varphi_{S_m} = \frac{\lambda \bar{\alpha}_m}{\lambda_0 \alpha_0 + \lambda \sum_{i=1}^k \alpha_i} - \gamma \bar{\alpha}_m,$$

where aggregate actions, $\bar{\alpha}_m := \sum_{j \in S_m} a_j$ for each $1 \leq m \leq k$ and, $\varphi_{S_0} = \frac{\lambda_0 \alpha_0}{\lambda_0 \alpha_0 + \lambda \sum_{i=1}^k \alpha_i} - \gamma \alpha_0$.

Now, our game is reduced to a similar game as studied in [5] (with action of each coalition given by the aggregate action) and the result follows from [5] Theorem 1; the aggregate actions at NE are given by (11) and the utilities are given by equations (9) and (10).

Every action profile, in which the aggregate actions of each coalition equals (11) forms a NE for RSG. Thus one can have multiple NE, but the aggregate actions and utility of each coalition are the same at all NE.

**Proof of Theorem 2** Consider a strategy profile $\mathbf{x}$ which leads to multiple partitions. Then as in (13) we define utility of a player to be the minimum utility among all the possible partitions emerging from $\mathbf{x}$.

Let $(k_m + 1)$ be the size of the biggest partition emerging from $\mathbf{x}$ (call it $P^*$), i.e., $k_m + 1 = \max_{P(\mathbf{x})} |P(\mathbf{x})|$. $P^* = \{S_0, S_1, \ldots, S_t, S_{t+1} \ldots\}$, with $|P^*| = k + 1$.

Now, if suppose player $i$ in coalition $S_t$ of size $m > 1$ (we can always find such a player since otherwise all players are alone in this partition and we cannot have multiple partitions because of (6) and (7)) deviates unilaterally to the strategy of being alone, i.e., to $\{i\}$ (changing strategy profile to $\mathbf{x}'$), then we can have a partition with size at maximum $k_m + 2$, call it $P^*_{i-}$ (after splitting as in Lemma 1 since remaining players in $S_t/i$ may merge with some other coalition keeping the partition size intact).

$P^*_{i-} = \{S_0, S_1, \ldots, \{i\}, S_t/i, S_{t+1} \ldots\}$ with $|P^*_{i-}| = k + 2$.

We have three cases based on adamant player:

**Case 1:** When $\eta > 1 - 1/(k_m + 1)$: In this case the adamant player gets non-zero utility in both the partitions, i.e., partition with $k_m + 1$ as well as $k_m + 2$ coalitions (see (10)). Then, utility of player $i$ with strategy profile $\mathbf{x}'$ using (5),

$$U_i(\mathbf{x}') \geq \frac{\lambda^2}{(\lambda + (k_m + 1)\lambda_0)^2} = \frac{1}{(1 + (k_m + 1)\eta)^2}. \quad (15)$$

The inequality above follows because the utility of a player decreases with increasing number of coalitions.

The utility of same player $i$ under strategy profile $\mathbf{x}$ equals,

$$U_i(\mathbf{x}) \leq \frac{\lambda^2}{m_m(\lambda + k_m \lambda_0)^2} = \frac{1}{m_m(1 + k_m \eta)^2}.$$ 

It can be seen from equation (12) that the utility of players decreases when the partition size, i.e., $k$ increases.
since the utility of a player is defined to be the minimum utility among all possible partitions and \( \min(z_1, z_2, \cdots, z_n) \leq z_i \forall i \). From (13) and Lemma 2, we have (as \( m_m > 1 \) and because one can’t have \( m_m = 2 \) and \( k_m = 2 \) simultaneously for \( n > 4 \)):

\[
U_i(x) \leq \frac{1}{m_m(1 + k_m \eta)^2} < \frac{1}{(1 + (k_m + 1) \eta)^2} \leq U_i(x').
\]

**Case 2:** When \( 1 - 1/k_m \eta \leq \frac{n}{k_m + 1} \). From (10), the adamant player gets non-zero utility in partition with \( k_m + 1 \) coalitions but zero utility with \( k_m + 2 \) coalitions. Once again, from (9) the utility of player \( i \) with strategy profile \( x' \) (adversary insignificant),

\[
U_i(x') \geq \left( \frac{1}{1 + k_m \eta} \right)^2.
\]

Thus the utility of player \( i \) with strategy profile \( x \) equals (inequality as explained in Case 1),

\[
U_i(x) \leq \frac{1}{m_m(1 + k_m \eta)^2} \leq \left( \frac{1}{1 + k_m \eta} \right)^2 \leq U_i(x').
\]

**Case 3:** When adamant player gets zero utility in both the partitions: Once again, the utility of player \( i \) with strategy profile \( x' \),

\[
U_i(x') \geq \left( \frac{1}{1 + k_m \eta} \right)^2.
\]

As before:

\[
U_i(x) \leq \frac{1}{m_m k_m^2}.
\]

As in Lemma 2 \( \sqrt{m_m k_m} > (k_m + 1) \) and hence

\[
U_i(x) \leq \frac{1}{m_m k_m^2} < \left( \frac{1}{1 + k_m \eta} \right)^2 \leq U_i(x').
\]

Thus, player \( i \)'s player finds it strictly better to deviate and hence the result. 

---

**APPENDIX B**

**Proof of Lemma 1** We prove it in two steps: i) \( x' \rightarrow P_{-i} \) and ii) \( x' \rightarrow P_{-i} \).

To prove \( x' \rightarrow P_{-i} \): it is clear by definition that every coalition of \( P_{-i} \) satisfies the requirement (6) (with \( x' \)). Hence, it suffices to prove that it is minimal as in (7).

If possible consider a (better) partition \( P' \) which satisfies (6) and such that \( P' \prec P_{-i} \). This means, from (7), there exist at least a pair of coalitions \( S_1, S_2 \in P_{-i} \) and \( S \in P' \) such that \( S \cup S_1 \cup S_2 \subset S \). Observe that \( \{i\} \in P' \cap P_{-i} \), as player \( i \) deviates unilaterally to \( \{i\} \).

If all such merging coalitions in \( P_{-i} \) are not equal to \( S_{\{i\}} \), the merging coalitions will also belong to \( P \) (i.e., for example if \( S_1 \neq S_2 \neq S_{\{i\}} \), then \( S_1, S_2 \) also belong to \( P \), then one can construct a better partition \( P'' \prec P \) and \( x \rightarrow P'' \), which contradicts \( x \rightarrow P \).

On the other hand, if one of the merging coalitions equal \( S_{\{i\}} \), then \( P' \) is not comparable with \( P \) as in (5) (i.e., neither is better than the other), as \( \{i\} \in P' \). Further \( P' \) satisfies (6) with \( x \) and hence \( x \rightarrow P' \). That means \( x \) leads to multiple partitions and this contradicts the hypothesis that \( x \rightarrow P \). This proves (i).

Next we prove uniqueness in (ii). If possible \( x' \) leads to multiple partitions, say \( P_{-i} \) (defined in hypothesis) and \( P' \). This implies \( P_{-i} \) is not comparable to \( P' \). Further observe \( \{i\} \in P' \) and hence \( P' \) is not even comparable to \( P \). Further more, it is easy to verify that any coalition that satisfies (6) with \( x' \) also satisfies (6) with \( x \). In all we have that \( x \rightarrow P' \), which again contradicts the uniqueness of \( x \rightarrow P \). 

---

**Proof of Lemma 2** Wlog we can assume that the \( n \) C-players (i.e., with influence factor \( \lambda \)) form \( k \) coalitions where \( k \leq n \), i.e.,

\[
P = \{ \{0\}, \{1, \cdots, m_1\}, \{m_1 + 1, \cdots, m_2\}, \cdots, \{m_{k-1} + 1, \cdots, m_k\} \}.
\]

Consider the best response of (say \( m_1 = m^* \)) player 1 against any strategy profile \( x \rightarrow P \); player 1 could either choose to remain alone (i.e., \( x_1 = \{1\} \)) or could form coalition with all or a subset of \( \{m_1 - 1\} \) players (i.e., \( x_1 \subset \{1, \cdots, m_1\} \)) resulting into a new strategy profile \( x' \). In particular, we would show that forming coalition with all players (as given by \( x \rightarrow P \)) is strictly inferior to remaining alone, i.e., player 1 could get higher utility by unilaterally deviating to \( \{1\} \).

**Case 1:** When \( \eta > 1 - 1/\{k+1\} \): In this case the adamant player gets non-zero utility in both the partitions, i.e., partition with \( k+1 \) as well as \( k+2 \) coalitions.

Then, from (9) utility of player 1 when it chooses to remain alone (with strategies of the others remaining the same),

\[
U_1(x') = \left( \frac{\lambda}{\lambda + (k+1)\eta} \right)^2 = \left( \frac{1}{1 + (k+1)\eta} \right)^2.
\]

Similarly, utility of player 1 when it proposes to form coalition with all \( \{m_1 - 1\} \) players,

\[
U_1(x) = \frac{1}{m_1} \left( \frac{\lambda}{\lambda + k\eta} \right)^2 = \frac{1}{m_1} \left( \frac{1}{1 + k\eta} \right)^2.
\]

\( ^{6} \)Partition \( P'' \) contains all coalitions of \( P' \), except that \( \{i\} \) and \( S_{\{i\}} \) are merged in \( P'' \).
Since $m_1 > 1$, from (18), player $i$ finds it better to deviate:

$$U_i(x') = \left(\frac{1}{1 + (k + 1)\eta}\right)^2 > \frac{1}{m_1} \left(\frac{1}{1 + k\eta}\right)^2 = U_i(x).$$

if $\sqrt{m_1}\eta > k + 1$.

**Case 2:** When $1 - 1/k \leq \eta < 1 - 1/(k + 1)$: From (10), the adamant player gets non-zero utility in partition with $k + 1$ coalitions but zero utility with $k + 2$ coalitions.

Now, utility of player 1 when it proposes to form coalitions with $(m_1 - 1)$ players,

$$U_i(x) = \frac{1}{m_1} \left(\frac{1}{1 + k\eta}\right)^2. \quad (19)$$

When player 1 chooses to remain alone (with strategies of other players remaining the same) then, utility of player 1 is given by

$$U_i(x') = \left(\frac{1}{k + 1}\right)^2. \quad (20)$$

By the conditions of Case 2 we have $k\eta \geq (k - 1)$

$$\sqrt{m_1}(1 + k\eta) \geq \sqrt{m_1(1 + k - 1)} = \sqrt{m_1}k.$$ 

Hence from (20):

$$U_i(x') = \left(\frac{1}{k + 1}\right)^2 > \frac{1}{m_1} \left(\frac{1}{1 + k\eta}\right)^2 = U_i(x),$$

if $\sqrt{m_1}\eta > k + 1$.

**Case 3:** When adamant player gets zero utility in both the partitions

Once again the utility of player 1 when it proposes to form coalitions with $(m_1 - 1)$ players,

$$U_i(x) = \frac{1}{m_1} \left(\frac{1}{k}\right)^2. \quad (21)$$

When player 1 chooses to remain alone (with strategies of other players remaining the same) then, utility of player 1 is given by

$$U_i(x') = \left(\frac{1}{k + 1}\right)^2. \quad (22)$$

Hence from (22):

$$U_i(x') = \left(\frac{1}{k + 1}\right)^2 > \frac{1}{m_1} \left(\frac{1}{k}\right)^2 = U_i(x).$$

if $\sqrt{m_1}k > k + 1$.

**Proof of Lemma 4** Let $\eta \geq 1$ (adamant player gets non-zero utility in all such partitions). From (9), one can verify that $\max\{(m_1)\}$-size of coalition, $k + 1$-size of partition:

$$U_{SO} := \max_{\rho \in \mathcal{P}^o} \sum_{S_i \in \rho, i \neq 0} U_{S_i} = \max_{\{m_1\} \leq k} \sum_{j=1}^{k} m_1 (\lambda + k\lambda_0)^2 = \frac{k^2\lambda^2}{(\lambda + k\lambda_0)^2}. \quad (23)$$

One can equivalently minimize:

$$\min_{1 \leq k \leq n} \frac{(\lambda + k\lambda_0)^2}{k^2} = \frac{\lambda^2}{\min_{1 \leq k \leq n} \left(\lambda^2 + 2k\lambda_0\lambda + k^2\lambda_0^2\right)}.$$

or equivalently consider:

$$\min_{1 \leq k \leq n} \frac{\lambda^2}{k} + k\lambda_0^2. \quad (24)$$

By relaxing $k$ to real numbers, and equating the derivative to zero (verify the second derivative is positive) we obtain:

$$-\frac{\lambda^2}{k^2} + \lambda_0^2 = 0$$

This implies (by convexity) that the optimizer among integers is $k^* = 1$ when $\eta \geq 1$, i.e., GC is the SO-partition. On the other hand, when $\eta \leq 1 - 1/2 = 0.5$, from (10) the adversary gets insignificant in all partitions other than GC, and one needs to maximize

$$U_{SO} = \max \left\{\frac{\lambda^2}{(\lambda + \lambda_0)^2}, \frac{\lambda^2}{(\lambda + \lambda_0)^2}, \frac{1}{k}\right\},$$

When $\eta \leq \sqrt{2} - 1 = 0.414$, GC is the SO-partition and for $0.414 \leq \eta \leq 0.5$ any $P^o_2$ is an SO-partition, which completes the proof of part (ii). Similarly when $\eta \leq 1 - 1/3$ we have

$$U_{SO} = \max \left\{\frac{\lambda^2}{(\lambda + \lambda_0)^2}, \frac{2\lambda^2}{(\lambda + 2\lambda_0)^2}, \frac{1}{k}\right\}.$$ 

Progressing this way, for any $\eta \leq 1 - 1/k$ (as in (23)):

$$U_{SO} = \max \left\{\max_{k' < k} \frac{k'^2\lambda^2}{(\lambda + k\lambda_0)^2}, \frac{1}{k}\right\} \text{ for any } k \leq n,$$

But with $\eta > 1/2$, the relaxed $k^* = 1/\eta < 2$. Thus by convexity of (24) the maximizer of the first term among integers is either at 1 or 2, i.e., when $\eta \leq 1 - 1/k$:

$$U_{SO} = \max \left\{\frac{\lambda^2}{(\lambda + \lambda_0)^2}, \frac{2\lambda^2}{(\lambda + 2\lambda_0)^2}, \frac{1}{k}\right\} \text{ if } k \leq n,$$

We have GC is best among the first two if

$$1 + 2\eta \geq \sqrt{2}(1 + \eta) \text{ or if } \eta \geq 1/\sqrt{2} = 0.707.$$ 

further in this range for all $n$, GC is better than the third possibility also (if it is feasible). In a similar way one can prove that $P_2$ is optimal for all other values of $\eta$. Thus we proved the Lemma.

**Proof of Lemma 5** Consider $\{m_1\}$ to be the size of coalition $S_i$, i.e., $|S_i|$ and $k$ be the size of partition $P^o$, i.e., $|P^o|$.

$$U_{SO} := \max_{P^o} \sum_{S_i \in P^o} U_{S_i} = \max_{\{m_1\} \leq k} \sum_{i=1}^{k} \sum_{j=1}^{m_1} \frac{1}{k^2}.$$ 

Since the minimum possible value of $k$ is 1, we have GC $^o$ is the only SO-partition in this case.

**REFERENCES**

[1] S. Nevrekar. A theory of coalition formation in constant sum games. 2015.

[2] R. Johari and J. N. Tsitsiklis. Efficiency loss in a network resource allocation game. Mathematics of Operations Research, 29(3):407–435, 2004.

[3] I. E. Hafalir. Efficiency in coalition games with externalities. Games and Economic Behavior, 61(2):242–258, 2007.

[4] Y. Narahari. Game theory and mechanism design, volume 4. World Scientific, 2014.

[5] R. Dhounchak, V. Kavitha, and Y. Hayel. To participate or not in a coalition in adversarial games. In Network Games, Control, and Optimization, pages 125–144. Springer, 2019.
[6] W. Saad, Z. Han, M. Debbah, A. Hjørungnes, and T. Basar. Coalitional game theory for communication networks: A tutorial. arXiv preprint arXiv:0905.4057, 2009.

[7] R. J. Aumann and J. H. Dreze. Cooperative games with coalition structures. International Journal of game theory, 3(4):217–237, 1974.

[8] F. P. Kelly, A. K. Maulloo, and D. K. Tan. Rate control for communication networks: shadow prices, proportional fairness and stability. Journal of the Operational Research society, 49(3):237–252, 1998.

[9] Y. K. Tun, N. H. Tran, D. T. Ngo, S. R. Pandey, Z. Han, and C. S. Hong. Wireless network slicing: Generalized kelly mechanism-based resource allocation. IEEE Journal on Selected Areas in Communications, 37(8):1794–1807, 2019.

[10] I. Koutsopoulos and G. Iosifidis. Auction mechanisms for network resource allocation. In 8th International Symposium on Modeling and Optimization in Mobile, Ad Hoc, and Wireless Networks, pages 554–563. IEEE, 2010.

[11] F. Kelly. Charging and rate control for elastic traffic. European transactions on Telecommunications, 8(1):33–37, 1997.

[12] I. Stoica, H. Abdel-Wahab, K. Jeffay, S. K. Baruah, J. E. Gehrke, and C. G. Plaxton. A proportional share resource allocation algorithm for real-time, time-shared systems. In 17th IEEE Real-Time Systems Symposium, pages 288–299. IEEE, 1996.

[13] S.-S. Yi. Endogenous formation of economic coalitions: a survey of the partition function approach. Endogenous Formation of Economic Coalitions, Edward Elgar, Cheltenham, UK, pages 80–127, 2003.

[14] W. Saad, Z. Han, M. Debbah, and A. Hjørungnes. A distributed merge and split algorithm for fair cooperation in wireless networks. In ICC Workshops-2008 IEEE International Conference on Communications Workshops, pages 311–315. IEEE, 2008.

[15] W. Saad, Z. Han, M. Debbah, A. Hjørungnes, and T. Basar. Coalitional game theory for communication networks: A tutorial. arXiv preprint arXiv:0905.4057, 2009.