Total angular momenta quantization of dielectric sphere modes

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Abstract. Spherical particles both dielectric and metallic are essential building blocks in nanophotonics. During the recent rapid development of Mie-ronic — nanophotonics devices heavily using various features of the Mie-resonances — the deep fundamental investigation of the eigenmodes of such particles by using the novel tools is still relevant and currently important. Moreover, eigenmodes of a sphere are closely related to the Vector Spherical Harmonics (VSH) which are widely used in the multipolar decomposition to analyze less symmetric structures. In this work, we study in detail the canonical spin and angular momenta (AM), helicity, and other properties of the eigenmodes of dielectric (nondispersive) and metallic (dispersive) spheres. We show that the canonical momentum density of the AM is quantized and has a close relation to the quantum picture of a single photon. Our work provides a solid platform for future studies and applications of the AM transfer from near fields of spherical particles to the matter in its vicinity.

1. Introduction
In recent years Mie-ronics started to gain popularity [1, 2, 3]. One of the key players in this branch of science are the Mie resonances. The first solution of the scattering of a plane by an arbitrary sphere was done by Gustav Mie in 1908 [13]. The scattered field was decomposed into the series of vector spherical harmonics (VSH), and weights in the decomposition contained all the information about all possible resonances in the particle. These ideas are heavily used today in a wide number of topics in modern optics such as metasurfaces [1], metalenses [14], second harmonic generation [5], and more.

In parallel, the Abraham-Minkowski controversy gained its momenta, especially in the terms of dispersive medium [7, 8]. Such approach was successfully implemented for the dielectric and metallic cylindrical waveguides [4]. In this work, we find new opportunities of this theory in the scope of spherical particles. Multipolar decomposition into the series of VSH shows its great power when it comes to the symmetry analyses of the studied structures [5, 6].
Spherical particles both dielectric and metallic are essential building blocks in nanophotonics. During the recent rapid development of "Mietronics" — nanophotonics devices heavily using various features of the Mie-resonances — the deep fundamental investigation of the eigenmodes of such particles by using the novel tools is still relevant and currently important. Moreover, eigenmodes of a sphere are closely related to the VSHs which are widely used in the multipolar decomposition to analyze less symmetric structures. In this work, we study in detail the canonical spin and angular momenta (AM) of the eigenmodes of dielectric (non-dispersive) spheres. We show that canonical momentum density of the AM is quantized and has a close relation to the quantum picture of a single photon. Our work provides a solid platform for future studies and applications of the AM transfer from near fields of spherical particles to the matter in its vicinity.

2. Eigenmodes of a sphere and canonical momenta of electromagnetic field

We consider a dielectric particle of radius $a$ placed at the origin which can be formally defined as

$$\varepsilon(r, \omega) = \begin{cases} 
\varepsilon_p(\omega), & \text{for } r < a \\
\varepsilon_{\text{host}}, & \text{for } r > a 
\end{cases} \quad \text{and} \quad \mu = 1. \quad (1)$$

The eigenmodes of a sphere are defined by a Helmholtz equation

$$\nabla \times \nabla \times \mathbf{E} - k^2 \varepsilon(r, \omega)\mathbf{E} = 0. \quad (2)$$

The solution is going to give the set of eigenmodes with corresponding complex eigen frequencies $\omega = \omega_{\text{Re}} - i\gamma$, where $\gamma$ is the dissipation rate.

We analyze the eigenmodes using canonical properties of the field introduced recently in [7] for the, generally, dispersive media, which is crucial for the case of metallic particles. The Brillouin energy $W$, linear momentum $\mathbf{P}$, spin $\mathbf{S}$, orbital $\mathbf{L}$, and total $\mathbf{J}$ angular momentum are given by [7, 8, 9]

$$W = \frac{1}{4} (\tilde{\varepsilon}\varepsilon_0 |\mathbf{E}|^2 + \tilde{\mu}\mu_0 |\mathbf{H}|^2),$$

$$\mathbf{P} = \frac{1}{4\omega} \text{Im} (\tilde{\varepsilon}\varepsilon_0 \mathbf{E}^\ast \cdot (\nabla)\mathbf{E} + \tilde{\mu}\mu_0 \mathbf{H}^\ast \cdot (\nabla)\mathbf{H}),$$

$$\mathbf{S} = \frac{1}{4\omega} \text{Im} (\tilde{\varepsilon}\varepsilon_0 \mathbf{E}^\ast \times \mathbf{E} + \tilde{\mu}\mu_0 \mathbf{H}^\ast \times \mathbf{H}),$$

$$\mathbf{L} = \mathbf{r} \times \mathbf{P}, \quad \mathbf{J} = \mathbf{L} + \mathbf{S}. \quad (6)$$
Here $(\tilde{\varepsilon}, \tilde{\mu}) = (\varepsilon, \mu) + \omega \partial_\omega (\varepsilon, \mu)$. Each of the components can be naturally decomposed into the electric and magnetic parts.

3. Angular momenta of eigenmodes of a dielectric sphere

Electromagnetic fields of the eigenmodes of the sphere are appeared to be connected with the electric field of a single photon. Akhiezer and Berestetsky showed in [10].

One of the key results of this work is that the total AM density normalized to the energy density per one photon is quantized separately for the electric and magnetic modes. We can define electric (TM) modes ($H_r = 0$) and magnetic (TE) modes ($E_r = 0$) through the complex VSH as

$E = N, \quad H = -\frac{i}{Z_0} M$ \hspace{1cm} TM modes (7)

$E = M, \quad H = -\frac{i}{Z_0} N$ \hspace{1cm} TE modes (8)

The vector structure of the spin, orbital, and total angular momenta is shown in Fig. 1. After some analytic calculations, we get

$$j_z = \frac{\omega J_z}{W} = m.$$ (9)

We illustrate this in Fig. 2. Otherwise it is non-integer. However, the absolute value does not follow the quantum limit

$$j^2(\theta = 0) = \frac{m^2}{\text{classical limit}} \neq \frac{n(n + 1)}{\text{quantum limit (1 photon)}},$$ (10)

Which is explicitly shown in [10] and the exact resolution is given in [11], where the problem of angular momenta of $N$ photons in free space was solved rigorously

$$j^2 = \frac{N^2 m^2 + Nn(n + 1) - m^2}{N^2}.$$ (11)

Figure 2. Normalized angular momentum density as a function of polar angle $\theta$ for $(n = 2, m = 1)$ mode (a) and for $(n = 2, m = 2)$ mode (b). Due to the azimuthal symmetry, there is no dependence on the polar angle $\varphi$. Here one can see that total angular momentum per one photon is quantized and equal $m$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2.png}
\caption{Normalized angular momentum density as a function of polar angle $\theta$ for $(n = 2, m = 1)$ mode (a) and for $(n = 2, m = 2)$ mode (b). Due to the azimuthal symmetry, there is no dependence on the polar angle $\varphi$. Here one can see that total angular momentum per one photon is quantized and equal $m$.}
\end{figure}
4. Conclusion
To summarize, in this work we have applied the canonical momenta analyses to the eigen modes of a dielectric sphere. We showed analytically that the \( z \)-component of the total angular momenta of light per one photon is quantized which structures a bridge between AM in classics electrodynamics and in quantum theory of photons. We also demonstrated the vector field structure of spin \( S \), orbital \( L \), and total \( J = S + L \) angular momenta of the eigen modes. Our work provides a solid platform for future studies and applications of the AM transfer from near fields of spherical particles to the matter in its vicinity.

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