Phase transition for extremes of a family of stationary multiple-stable processes

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Abstract. We investigate a family of stationary processes that may exhibit either long-range or short-range dependence, depending on the parameters. The processes can be represented as multiple stable integrals, and there are two parameters for the processes, the memory parameter $\beta \in (0,1)$ and the multiplicity parameter $p \in \mathbb{N}$. We investigate the macroscopic limit of extremes of the process, in terms of convergence of random sup-measures, for the full range of parameters. Our results show that (i) the extremes of the process exhibit long-range dependence when $\beta p := p\beta - p + 1 \in (0,1)$, with a new family of random sup-measures arising in the limit, (ii) the extremes are of short-range dependence when $\beta p < 0$, with independently scattered random sup-measures arising in the limit, and (iii) there is a delicate phase transition at the critical regime $\beta p = 0$.

MSC2020 subject classifications: Primary 60F17, 60G70; secondary 60G52, 60K05
Keywords: stable regenerative set, random sup-measure, long-range dependence, infinite ergodic theory, phase transition, regular variation, multiple integral, renewal process

1. Introduction and main results

1.1. Background

Extensive investigations on stationary stable processes started in 1980s. These processes extend naturally Gaussian processes, have regularly-varying tails, and they exhibit very rich dependence structures. The monograph by Samorodnitsky and Taqqu \cite{55} summarizes foundational earlier developments of stable processes, and remains a classic on the topic.

In early developments of stable processes, one of the main driving questions was to understand their ergodic properties. A modern approach in this direction was initiated by Rosiński \cite{48}, who first related every stationary stable process to a dynamical system, more precisely an underlying flow on a certain space, and then revealed that the ergodic properties of the stationary stable process of interest can be derived from the one of the underlying flow. Then, instead of working on stable processes directly, one benefits from known results on dynamical systems, which shed light in turn on the studies of stable processes. Thanks to the seminal work of Rosiński and another fundamental contribution later from Samorodnitsky \cite{52}, it is now well understood that all stationary stable processes can be divided into three categories according to the ergodic-theoretic classification of the underlying flows, and each category of stable processes is with distinct ergodic properties. More precisely, the underlying flow can be positively recurrent, null recurrent (a.k.a. null-conservative) and dissipative: when the flow is dissipative, the corresponding stable process has a mixed moving average representation and is known to be mixing; when the flow is positively recurrent, the stable process is known to be non-ergodic; when the flow is null recurrent, the stable process is ergodic and weakly mixing, and yet it does not necessarily have the mixing property.

From a different aspect, stable processes driven by positively- or null-recurrent flows are often associated with the notion of long-range dependence (a.k.a. long memory), while those driven by dissipative flows are often associated
with the notion of short-range dependence. The investigation of stochastic processes with long-range dependence has a long history [11, 43], and in these studies stable processes provide important examples with representative properties. The recent monograph by Samorodnitsky [53] provided a first systematic presentation on stable processes (and more generally infinitely-divisible processes) incorporating the points of view of both ergodic-theoretic classifications and long-/short-range dependence.

Some later advances focus on further refinements of the aforementioned classifications of stable processes. When the underlying flow is dissipative, it is known that the stable process is a mixed moving average process, and further classifications can be found in Pipiras and Taqqu [44], summarizing mostly a series of extensive investigations by the same authors since 2000s. Within this class of stochastic processes with relatively simple structures, the situation is already very delicate. At the same time, it was argued in Samorodnitsky [53] that the most challenging case is when the underlying flow is null-recurrent. In words, for processes within this category, while they are ergodic, they already exhibit abnormal asymptotic behaviors in terms of limit fluctuations.

Many recent advances on stable processes concern their limit theorems: for the partial-sum processes, for the extremes, and for total variations, just to mention a few. Sometimes, qualitatively different asymptotic behaviors are shown in complete correspondence to the aforementioned classifications based on ergodic properties, which is not surprising; but often much more elaborated information on the asymptotic behaviors in terms of limit theorems for fluctuations have been established, and new stochastic processes arise in such investigations. Our object of interest is a class of stable processes driven by null-recurrent flows and its extensions to be explained below. We shall focus on limit theorems regarding their extremes, following the recent developments [36, 40, 56]. For recent limit theorems of other types for stable and infinitely-divisible processes, we mention [8–10, 21, 39, 53].

1.2. Random sup-measures

The asymptotic extremes at macroscopic level are commonly characterized by convergence of random sup-measures. The foundation of random sup-measures and the corresponding weak convergence theory have been established in the 1980s and 1990s, and standard references are [38, 58]. We shall provide a brief introduction below where all the claims can be found in [58].

In this paper, we shall focus on sup-measures over the unit interval $[0, 1]$ taking value in $\mathbb{R} = [-\infty, \infty]$. In particular, suppose $\mathcal{G}$ is the collection of open subsets of $[0, 1]$ (w.r.t. subspace topology). A sup-measure is a mapping $m : \mathcal{G} \to \mathbb{R}$ that satisfies

$$m \left( \bigcup_{\lambda} G_{\lambda} \right) = \sup_{\lambda} m(G_{\lambda}),$$

for any family $\{G_{\lambda}\}_{\lambda}$, $G_{\lambda} \in \mathcal{G}$, with the convention $m(\emptyset) = -\infty$. A sup-measure $m$ defined on open subsets admits a canonical extension to an arbitrary subset $A \subset [0, 1]$ via $m(A) := \inf_{G \supseteq A, G \in \mathcal{G}} m(G)$. The space of such sup-measures, denoted as $\text{SM} = \text{SM}([0, 1], \mathbb{R})$, can be equipped with the so-called sup vague topology, which is generated by the subbase consisting of subsets of SM in the forms $\{m \in \text{SM} : m(G) > x\}$ and $\{m \in \text{SM} : m(F) < x\}$ with $G \in \mathcal{G}$, $F \in \mathcal{F}$ and $x \in \mathbb{R}$, where $\mathcal{F}$ denotes the collection of closed subsets of $[0, 1]$. Equipped with the sup vague topology, the space SM is compact and second-countable, and hence metrizable.

A random sup-measure $M$ is a random element taking value in the measurable space $(\text{SM}, \mathcal{B}(\text{SM}))$, where $\mathcal{B}(\text{SM})$ denotes the Borel $\sigma$-field on SM with respect to the sup vague topology. In addition, for each $A \in \mathcal{G} \cup \mathcal{F}$, the mapping $\text{SM} \to \mathbb{R}, m \mapsto m(A)$ is measurable and hence each $M(A)$ is a ($\mathbb{R}$-valued) random variable. Suppose $\mathcal{G}_0$ is the collection of all nonempty open subintervals of $[0, 1]$. Then a mapping $M$ from the underlying probability space to SM is a random sup-measure if and only if every $M(I)$ is a random variable for any $I \in \mathcal{G}_0$. Furthermore, the law of $M$ in $(\text{SM}, \mathcal{B}(\text{SM}))$ is uniquely determined by the finite-dimensional distributions of the set-indexed process $\{M(I)\}_{I \in \mathcal{G}_0}$. At last, the weak convergence of a sequence of random sup-measures $\{M_n\}_{n \in \mathbb{N}}$ to a limit random sup-measure $M$ in $(\text{SM}, \mathcal{B}(\text{SM}))$, denoted by

$$M_n \Rightarrow M,$$

in $(\text{SM}, \mathcal{B}(\text{SM}))$ as $n \to \infty$, is equivalent to

$$\{M_n(I)\}_{I \in \mathcal{G}_0} \xrightarrow{f.d.d.} \{M(I)\}_{I \in \mathcal{G}_0}$$

as $n \to \infty$, which convergence above means convergence of finite-dimensional distributions of set-indexed stochastic processes, and $\mathcal{G}_0 = \{I \in \mathcal{G}_0 : P(M(I) = M(\text{cl}(I)) = 1\}$, with $\text{cl}(I)$ denoting the closure of $I$. In fact, for all the limit random sup-measures we shall encounter in this paper, we have $\mathcal{G}_0 = \mathcal{G}_0$. 


As an example of random sup-measure, consider first

\[ M_{\alpha}(G) := \sup_{x \in G} \frac{1}{1/\alpha} \mathbb{1}_{\{u_i \in G\}}, G \in \mathcal{G}, \]

where \( \alpha > 0 \), \( \{\Gamma_i\}_{i \in \mathbb{N}} \) are consecutive arrival times of a standard Poisson process, and \( \{U_i\}_{i \in \mathbb{N}} \) are i.i.d. uniform random variables over \((0, 1)\) independent from \( \{\Gamma_i\}_{i \in \mathbb{N}} \). The random sup-measure \( M_{\alpha} \) is known as an independently scattered \( \alpha \)-Fréchet random sup-measure with Lebesgue control measure on \([0, 1]\). Here, it is \( \alpha \)-Fréchet in the sense that \( \mathbb{P}(M_{\alpha} \leq x) = e^{-Leb(G)x^{-\alpha}} \) for all \( x > 0, G \in \mathcal{G} \), and it is independently scattered in the sense that for disjoint \( G_1, \ldots, G_m \in \mathcal{G} \), \( M_{\alpha}(G_1), \ldots, M_{\alpha}(G_m) \) are independent. It is a classical result that, for i.i.d. random variables \( \{X_k\}_{k \in \mathbb{N}} \) with \( \mathbb{P}(X_k > x) \sim x^{-\alpha} \) as \( x \to \infty \),

\[ \frac{1}{c_n} M_n \Rightarrow M_{\alpha} \text{ with } M_n(G) := \max_{k/n \in G} X_k, \quad G \in \mathcal{G}, \]

with \( c_n = n^{1/\alpha} \); here and below, we understand a maximum or a supremum over an empty set as \(-\infty\).

For a general stationary sequence of random variables, the above (1.3) may continue to hold with possibly different normalizations \( c_n \), and also different random sup-measures in place of \( M_{\alpha} \). Heuristically, the dependence of extremes of \( \{X_k\}_{k \in \mathbb{N}} \) is considered weak, or of local nature, if the same \( M_{\alpha} \) arises in the limit with possibly a different normalization \( c_n \). O’Brien, Torfs and Vervaat [38] advocated using such a framework as (1.3) to characterize macroscopic limit of extremes of a stationary sequence when the limit is not independently scattered. Therein, a thorough characterization of all possible limit random sup-measures (from a general sequence not necessarily with regularly varying tails) was carried out, and in particular, the shift-invariance and self-similarity properties of the limit are established. For the model of our interest later, we shall see that independently scattered Fréchet random sup-measures as well as another family, denoted by \( M_{\alpha, \beta, p} \) below, may arise in the limit, depending on the range of the parameters of the model.

For readers familiar with functional central limit theorems (Donsker’s theorem) for the partial-sum process but not as much with extremal limit theorems, it is worth noting that the convergence of random sup-measures reveals strictly more information than the convergence of the partial-maximum process

\[ \frac{1}{c_n} \left\{ \max_{k=1, \ldots, \lfloor nt \rfloor} X_k \right\} \quad \xrightarrow{f.d.d.} \quad \{M_t\}_{t \in [0, 1]}, \quad n \to \infty. \]

This is because different limits of random sup-measures may correspond to the same time-indexed process \( \{M_t\}_{t \in [0, 1]} \) [22, Proposition A.1].

### 1.3. Stable-regenerative models

The model of our interest stems from a family of stable processes introduced first by Rosiński and Samorodnitsky [49]. The original model can be viewed as a prototype of stable processes driven by a null-recurrent flow. The investigations of its limit theorems dated back to the early 2000s [46, 47, 51]. Key advances appeared recently in [30, 40, 41, 56], and it is understood since these works that the large-scale behavior of this family is closely related to stable-regenerative sets [12], and hence we name the original discrete-time stable processes stable-regenerative stable processes or stable-regenerative models. (We prefer the latter as the two ‘stable’ in the former name may cause confusion.)

Let \( \{X_k\}_{k \in \mathbb{N}} \) denote such a stable-regenerative model. This is a symmetric \( \alpha \)-stable process a parameter \( \alpha \in (0, 2) \) and a memory parameter \( \beta \in (0, 1) \). The representation is intrinsically related to renewal processes, and for which we introduce some notations. Consider a discrete-time renewal process starting at the origin with the consecutive renewal times denoted by \( \tau := \{\tau_0, \tau_1, \tau_2, \ldots\} \), where \( \tau_0 = 0 < \tau_1 < \tau_2 < \ldots \) We let \( F \) denote the renewal distribution function, that is \( F(x) = \mathbb{P}(\tau_{i+1} - \tau_i \leq x), i \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}, x \in \mathbb{N} \). Throughout, we assume

\[ \bar{F}(x) = 1 - F(x) \sim C_F x^{-\beta} \quad \text{as } x \to \infty \quad \text{with} \quad \beta \in (0, 1), \]

and the following technical assumption, with \( f(n) := F(n) - F(n - 1) \) denoting the probability mass function of the renewal distribution,

\[ \sup_{n \in \mathbb{N}} \frac{nf(n)}{F(n)} < \infty. \]

In this paper a renewal process is often shifted so that the starting point may not be the origin. An important notion is the stationary shift distribution of the renewal process, denoted by \( \pi \). Since the renewal distribution has infinite mean,
the stationary shift distribution $\pi$ is a $\sigma$-finite and infinite measure on $\mathbb{N}_0$ unique up to a multiplicative constant. We shall work with

$$\pi(\{k\}) := F(k), \quad k \in \mathbb{N}_0.$$  \hfill (1.6)

Then it is well-known that the law of the shifted renewal process $d + \tau := \{d, d + \tau_1, d + \tau_2, \ldots\}$ is shift-invariant (see, e.g., [6]). Here by the law of

$$\tau^*: = d + \tau,$$

we mean the pushforward of the product measure between $\pi$ and the law of $\tau$. We shall also need

$$w_n := \sum_{k=1}^n \pi(\{k\}) = \sum_{k=1}^n F(k) \sim \frac{C_F}{1 - \beta} n^{1 - \beta} \text{ as } n \to \infty. \quad (1.7)$$

We are now ready to define the stable-regenerative model through a series representation. Let $\pi$ be as in (1.6), and consider

$$\sum_{i=1}^\infty \delta_{(\eta_i, d_i)} \overset{d}{\rightarrow} \text{PPP} \left( (0, \infty] \times \mathbb{N}, B_{\pi} \right).$$

Suppose $\{\tau^{(i)}\}_{i \in \mathbb{N}}$ are i.i.d. copies of the non-shifted renewal process $\tau$ which are independent of the point process above. Set $\tau^{(d, i)} := d_i + \tau^{(i)}, i \in \mathbb{N}$. Then, the stable-regenerative model is defined as

$$\{X_k\}_{k \in \mathbb{N}} = \left\{ \sum_{i=1}^\infty \epsilon_i \eta_i \mathbf{1}_{\{k \in \tau^{(d, i)}\}} \right\}_{k \in \mathbb{N}}, \quad (1.8)$$

where $\{\epsilon_i\}_{i \in \mathbb{N}}$ are i.i.d. Rademacher random variables independent from all other random elements previously introduced.

The model (1.8) exhibits long-range dependence in terms of limit theorems as revealed in [40, 41, 56]. In particular, it was shown in [56] that random sup-measures with long-range clustering, denoted by $\mathcal{M}_{\alpha, \beta}$, arise as macroscopic limits of extremes of (1.8). Thanks to translation invariance and self-similarity, it suffices to restrict to random sup-measures on $[0, 1]$, and we do so throughout.

The random sup-measure $\mathcal{M}_{\alpha, \beta}$ is built upon independent stable regenerative sets that we now recall some basics. A $\beta$-stable-regenerative set, say $\mathcal{R}_\beta$, can be defined as the scaling limit of $\tau/n$, viewed as a random closed set, as $n \to \infty$, with $\tau$ as the set of renewal times of non-shifted renewal processes with renewal distribution corresponding to (1.4); or, $\mathcal{R}_\beta$ can be defined as the closure of the image of a $\beta$-stable subordinator which satisfies $0 \in \mathcal{R}_\beta$ almost surely. Then, by a randomly shifted $\beta$-stable regenerative set we refer to

$$\mathcal{R}_\beta := \mathcal{R}_\beta + B_{1 - \beta,1},$$

where $B_{1 - \beta,1}$ is a Beta$(1 - \beta, 1)$ distributed shift ($\mathbb{P}(B_{1 - \beta,1} \leq x) = x^{1 - \beta}, x \in [0, 1])$ independent from $\mathcal{R}_\beta$. The stable-regenerative sets have very rich structures and many fundamental developments took place in 1980s and 1990s [12, 25–27]. They are important examples of random closed sets, of which our reference is [37]. The appearance of $B_{1 - \beta,1}$ is necessary to make the randomly shifted stable-regenerative sets stationary (a.k.a. shift-invariant) in an appropriate sense, and this has been well understood (see e.g. [56]).

Let $\{\mathcal{R}_{\beta,i}\}_{i \in \mathbb{N}}$ denote i.i.d. copies of $\mathcal{R}_\beta$. We also need the following (see [56]) regarding their intersections:

$$\bigcap_{i=1}^q \mathcal{R}_{\beta,i} \overset{d}{=} \begin{cases} \mathcal{R}_{\beta, q}, & \text{if } q = 1, \ldots, p', \\ \emptyset, & \text{otherwise}, \end{cases} \quad \text{with } p' := \max\{q \in \mathbb{N} : \beta q > 0\}. \quad (1.9)$$

Now we introduce $\mathcal{M}_{\alpha, \beta}$. Note that our representation below is not the one used in [56] as the definition, but it was already used in the proofs therein. Recall $\{\epsilon_i\}_{i \in \mathbb{N}}$ are i.i.d. Rademacher random variables. Let $\{\tau_i\}_{i \in \mathbb{N}}$ be consecutive arrival times of a standard Poisson process on $[0, \infty)$. Suppose the three sequences $\{\mathcal{R}_{\beta,i}\}_{i \in \mathbb{N}}, \{\epsilon_i\}_{i \in \mathbb{N}}$ and $\{\tau_i\}_{i \in \mathbb{N}}$ are independent of each other. We set

$$\mathcal{M}_{\alpha, \beta}(G) := \sup_{J \subset \mathbb{N},|J| \leq p'} \mathcal{M}_{\alpha, \beta,J}(G) \quad \text{with } \mathcal{M}_{\alpha, \beta,J}(G) := \begin{cases} \sum_{i \in J} \epsilon_i \tau_i / n, & \text{if } \mathcal{R}_{\beta,J} \cap G \neq \emptyset, \\ -\infty, & \text{otherwise}. \end{cases} \quad (1.10)$$
We interpret that each $\mathcal{M}_{\alpha,\beta,J}$ representing an aggregated cluster of individual clusters, with each individual cluster indexed by $i \in J$ having magnitude $\varepsilon_i G_i^{1/\alpha}$ and locations $R_{\beta,i}$. The aggregated cluster then has magnitude and locations represented by
\[
\sum_{i \in J} \varepsilon_i i^{1/\alpha - 1/\beta} \quad \text{and} \quad R_{\beta,J} = \bigcap_{i \in J} R_{\beta,i}.
\]
We interpret the clustering as long-range clustering, as the locations are represented by unbounded random sets.

**Remark 1.1.** Note that for our convention, the aggregated clusters and individual clusters may have negative values, and hence they do not represent extremal clusters in the usual sense. But our interpretation is convenient when extending the results later.

When $\beta \leq 1/2$, $\mathcal{M}_{\alpha,\beta}$ is an $\alpha$-Fréchet random sup-measure, in the sense that for all $G_1, \ldots, G_d \in \mathcal{G}$, $(\mathcal{M}_{\alpha,\beta}(G_i))_{i=1,\ldots,d}$ has multivariate $\alpha$-Fréchet distribution, that is, for all $a_1, \ldots, a_d > 0, \max_{i=1,\ldots,d} a_i \mathcal{M}_{\alpha,\beta}(G_i)$ has an $\alpha$-Fréchet distribution [57], and yet it is no longer independently scattered due to the presence of long-range clustering. Note that $p' = 1$ when $\beta \leq 1/2$ so there is no aggregation in this regime. However, with $\beta > 1/2$, due to the appearance of aggregation even the marginal distribution $\mathcal{M}_{\alpha,\beta}(G)$ is not $\alpha$-Fréchet.

The random sup-measure $\mathcal{M}_{\alpha,\beta}$ first appeared in [36] with $\beta \in (0, 1/2)$ and [56] with $\beta \in (0, 1)$. It was proved therein that, with the empirical random sup-measure $M_n$ as in (1.3), as $n \to \infty$,
\[
\frac{1}{n^{1/\alpha}} M_n \Rightarrow \mathcal{M}_{\alpha,\beta}.
\]

### 1.4. A multiple-stable version of stable-regenerative models

The model of interest in this paper is the following extension of (1.8). With the same ingredients defining (1.8) above, consider
\[
\{X_k\}_{k \in \mathbb{N}} = \left\{ \left[ \sum_{0 \leq i_1 < \cdots < i_p} \varepsilon_{i_1} \cdots \varepsilon_{i_p} \right] \eta_k \right\} \chi_{\{k \in \mathbb{N}^p \cap \tau^{(\varepsilon,\eta)}\}} \quad \forall k \in \mathbb{N},
\]
where $[\varepsilon_{i_1} \times \cdots \times \varepsilon_{i_p}]$, and similar notation for $[\eta_{i_1}]$. It is known that such multiple series converges almost surely and unconditionally (e.g., [54, Theorem 1.3 and Remark 1.5]). Note that when $p = 1$ it recovers the model (1.8).

From now on, we shall use the term stable-regenerative model to refer to the more general class of models (1.12). The stable-regenerative model and its variants have been investigated in literature recently. In particular, when $p \geq 2$ and $\beta_p = p\beta_p - p + 1 \in (0, 1)$, a functional central limit theorem has been established for (1.12) in [4], where the limits are a new family of a self-similar multiple-stable process with stationary increments. See also [3, 5] for variations of the model that scale to multiple-Gaussian processes known as Hermite processes. Our main motivation is to investigate the case $\beta_p \leq 0$, about which little has been known in the literature.

**Remark 1.2.** It might be helpful to interpret the stable-regenerative model as follows. Consider the case $p = 1$ in (1.8) for the sake of simplicity. Then, one may view each renewal process $\tau^{(\varepsilon,\eta)}$ recording the visit times of a certain state of a null-recurrent Markov chain, and each value $\varepsilon_{i_1} \cdots \varepsilon_{i_p}$ is the reward collected by the chain when the Markov chain visits the state. This interpretation has a flavor of aggregated random walks in random scenery, as explained in [59].

**Remark 1.3.** It is well-known that a series representation as in (1.12) can be alternatively expressed as a multiple stable integral, whence the name multiple-stable processes. We sketch the representation, starting with $p = 1$. In this case, (1.8) has the equivalent stochastic-integral representation
\[
\{X_k\}_{k \in \mathbb{N}} \overset{d}{=} \left\{ \int_{\Omega \times \mathbb{N}} 1_{\{k \in s + \tau(\xi)\}} M_{\alpha}(d\tau d\xi) \right\} \quad \forall k \in \mathbb{N},
\]
for some explicit constant $C_{\alpha}$, where the $(\Omega, \bar{F}, \bar{P})$ is a measure space equipped with a probability measure $\bar{P}$ (different from the underlying probability space), and moreover on $(\Omega, \bar{F}, \bar{P})$, $\tau = \{\tau_0, \tau_1, \tau_2, \ldots\}$, $\tau_0 = 0 < \tau_1 < \tau_2 < \ldots$, is a renewal process with renewal distribution $F$ as before, and $M_{\alpha}$ is a symmetric $\alpha$-stable random measure on $\Omega \times \mathbb{N}$ with control measure $\bar{P} \times \pi$, with $\pi$ as in (1.6). The above equivalence holds only for $\alpha \in (0, 2)$ [55], while the stochastic integral representation is valid for $\alpha = 2$, and in this case the stochastic integral is with respect to a Gaussian random measure.
The multiple-stable integral representations would take more effort to explain. In particular, the integral excludes the diagonals. Since the development of our proofs relies exclusively on the series representation, we choose not to further elaborate the multiple-stable-integral representation, but refer the interested readers to [4, Section 3.1 and Example 4.2] for more details.

Remark 1.4. For processes represented by multiple-integrals with respect to Gaussian random measures, most commonly known as Gaussian chaos, the literature is extensive. However, much less has been devoted to multiple-stable (non-Gaussian) processes since early investigations in the 1980s and 1990s [33, 35, 50, 54]. In most of the aforementioned papers, the focus is rather on the properties of multiple-stable integrals, instead of the stochastic processes represented by such integrals.

We also mention that the multiple-stable process (1.12) is of its own interest as it has led to a new representation of Hermite processes in terms of stable regenerative sets, their joint intersections, and the corresponding joint local times [2].

1.5. Main results: phase transition for extremes

Our main results reveal a phase transition on the asymptotic behaviors of extremes for the multiple-stable version of stable-regenerative models. Before stating the limit theorems, we first introduce the limit random sup-measures at the super-critical regime, denoted by $M_{\alpha, \beta, p}$ below. This extends the limit random sup-measures in [56] corresponding to $p = 1$ here.

For $p \geq 2$, we extend the definition (1.10) as follows. Throughout, denote

$$\mathcal{D}_p = \{i = (i_1, \ldots, i_p) \in \mathbb{N}^p : i_1 < \cdots < i_p\}.$$ 

Let $\{\mathcal{R}_{\beta, i}\}_{i \in \mathbb{N}}$, $\{\varepsilon_i\}_{i \in \mathbb{N}}$ and $\{\Gamma_i\}_{i \in \mathbb{N}}$ be as introduced before (1.10). The point process

$$\sum_{i \in \mathcal{D}_p} \delta_{\{i_p/[\Gamma_i]^{1/\alpha}, \mathcal{R}_{\beta, i}\}}$$

encodes the magnitude and locations of each individual cluster now indexed by $i \in \mathcal{D}_p$. To describe the index set of each aggregated cluster is a little more involved than in the case $p = 1$. This time, each aggregated cluster is indexed by some $c = (c_1, \ldots, c_q) \in \mathcal{D}_q$ for some $q \geq p$ as follows. Introduce the multi-index set

$$J(c) := \{i = (i_1, \ldots, i_p) \in \mathcal{D}_p : i \subset c\},$$

where by $i \subset c$ we mean $\{i_1, \ldots, i_p\} \subset \{c_1, \ldots, c_q\}$. In particular, $|J(c)| = (q)^p$. For example, if $p = 3, q = 4$, then $J((1, 3, 5, 6)) = \{(1, 3, 5), (1, 3, 6), (1, 5, 6), (3, 5, 6)\}$. Next, define the following countable collection of index sets

$$\mathcal{J}_{p, p'} := \{J(c) : c \in \mathcal{D}_q, q = p, \ldots, p'\},$$

where $p'$ is as in (1.9). Now we say each $J \in \mathcal{J}_{p, p'}$ corresponds to an aggregated cluster, of which the magnitude and locations are represented by

$$\sum_{i \in J} \frac{[\varepsilon_i]}{[\Gamma_i]^{1/\alpha}} \quad \text{and} \quad \mathcal{R}_{\beta, J} = \bigcap_{i \in J} \mathcal{R}_{\beta, i},$$

respectively. This time, the contributions to the aggregated cluster are from individual clusters indexed by $i \in J$. Note that by our choice of $p'$, $\mathcal{R}_{\beta, J} \neq \emptyset$ almost surely for all $J \in \mathcal{J}_{p, p'}$ (see (1.9)). The expressions in (1.14) when $p = 1$ can be identified with those in (1.11) (in particular $\mathcal{J}_{1, p'} = \{J \subset \mathbb{N} : 1 \leq |J| \leq p'\}$).

We now introduce

$$M_{\alpha, \beta, p}(G) := \sup_{J \in \mathcal{J}_{p, p'}} M_{\alpha, \beta, J}(G) \quad \text{with} \quad M_{\alpha, \beta, J}(G) := \begin{cases} \sum_{i \in J} \frac{[\varepsilon_i]}{[\Gamma_i]^{1/\alpha}}, & \text{if } \mathcal{R}_{\beta, J} \cap G \neq \emptyset, \\ -\infty, & \text{otherwise}, \end{cases}$$

Note that we use the notation $M_{\alpha, \beta, J}$ both here and previously in (1.10). The two places are consistent in the sense that from now on we view $M_{\alpha, \beta, J}$ in (1.10) as a special case of the above with $p = 1$. 
Remark 1.5. The definition of $M_{\alpha,\beta,p}$ in (1.15) readily yields a random sup-measure. Indeed, it is easily verified that each $M_{\alpha,\beta,J}$ in (1.15) is a random sup-measure which takes 2 values, and hence so is the countable sup of $M_{\alpha,\beta,J}$ in (1.15). It follows from that fact $P(x \in R_{\beta,i}) = 0$ for any $x \in [0,1]$ (e.g., [12, Proposition 1.9]) that the property $G_0 = G_0$ mentioned below (1.1) holds for $M_{\alpha,\beta,p}$. An alternative representation of $M_{\alpha,\beta,p}(G)$ is provided in Section 3.1 which may shed light on the size of $M_{\alpha,\beta,p}$ as the limit in the super-critical regime.

Remark 1.6. In view of (1.15), the number $\nu'$ defined in (1.9) may be called the largest 'size' of an aggregated cluster given $\beta$ and $p$. From (1.9), we see that there is another layer of phase transitions within the supercritical regime $\beta > 1 - 1/p$, in the sense that as $\beta$ increases towards 1, the maximum 'size' $\nu'$ of the aggregated cluster will also increase to infinity.

Remark 1.7. In the case $p = 1$, we have a further simplified representation as follows. First, as an alternative representation of (1.10) is

$$M_{\alpha,\beta,1}(G) = \sup_{J \subset \mathbb{N}, |J| \leq p'} M_{\alpha,\beta,J}^+(G) \quad \text{with} \quad M_{\alpha,\beta,J}^+(G) := \begin{cases} \sum_{i \in J} (\varepsilon_i)^+, & \text{if } R_{\beta,J} \cap G \neq \emptyset, \\ -\infty, & \text{otherwise.} \end{cases}$$

Once one realizes that for those $i \in \mathbb{N}$ such that $\varepsilon_i = -1$, $\varepsilon_i/\Gamma_i^1/\alpha$ has no contribution to the limit. Then, one may further drop all these terms by a standard thinning argument of Poisson point process, and arrive at

$$\{M_{\alpha,\beta,1}(G)\}_{G \in \mathcal{G}} \overset{d}{=} \left\{ 2^{-1/\alpha} \sup_{J \subset \mathbb{N}, |J| \leq p'} \sum_{i \in J} \frac{1}{\Gamma_i} 1\{R_{\beta,i} \cap G \neq \emptyset\} \right\}_{G \in \mathcal{G}}$$

For $p \geq 2$, such a simplified representation without any Rademacher random variables seems no longer available. Instead, since $M_{\alpha,\beta,p}(G) > 0$ for all non-empty $G \in \mathcal{G}$ (see Proposition 3.1), we have the following simplification:

$$M_{\alpha,\beta,p}(G) = \sup_{J \in \mathcal{J}^+_{p,p'}} M_{\alpha,\beta,J}^+(G) \quad \text{with} \quad \mathcal{J}^+_{p,p'} := \left\{ J \in \mathcal{J}_{p,p'} : \sum_{i \in J} \frac{|\varepsilon_i|}{[1/\Gamma_i]^{1/\alpha}} > 0 \right\}.$$ 

In words, those individual clusters with negative magnitudes do not have any impact on the law of $M_{\alpha,\beta,p}$, and those indexed by $J \in \mathcal{J}^+_{p,p'}$ are the (long-range) extremal clusters in the common sense.

The main result of this paper is the following.

**Theorem 1.8.** We have the follow weak convergence in the space of sup-measures $SM([0,1], \mathbb{R})$: as $n \to \infty$,

$$\frac{1}{c_n} M_n \Rightarrow \begin{cases} \mathcal{C}_{F,p}' M_{\alpha,\beta,p}, & \text{if } \beta_p > 0, \\ \mathcal{C}_{F,p}' M_{\alpha,\beta,s}, & \text{if } \beta_p \leq 0, \end{cases}$$

with

$$c_n = \begin{cases} n^{(1-\beta_p)/\alpha}, & \text{if } \beta_p > 0, \\ \left( n \frac{\log \log n}{\log n} \right)^{p-1} \frac{1}{\alpha}, & \text{if } \beta_p = 0, \\ (n \log^{p-1} n)^{1/\alpha}, & \text{if } \beta_p < 0, \end{cases}$$

the random sup-measures $M_{\alpha,\beta,p}$ in (1.15) and $M_{\alpha,s}$ in (1.2), and the constant $\mathcal{C}_{F,p}$, depending on $F$ (and hence $\beta$) and $p$ only, to be specified in (3.4), (4.1) and (5.1) in later sections (where we shall restate the limit theorem before the proof in each regime again for convenience).
As a corollary, we summarize the marginal limit theorem for the partial maxima. Note that \( M_{\alpha,\beta,p}([0,1]) \) is almost surely equal to the random variable

\[
Z_{\alpha,\beta,p} := \sup_{J \in \mathcal{J}_{p,p'}} \frac{\sum_{i \in J} |\varepsilon_i|}{|\Gamma_i|^{1/\alpha}},
\]

(1.16)

because each \( \mathcal{R}_{\beta,J} \cap [0,1] \neq \emptyset \) for all \( J \in \mathcal{J}_{p,p'} \) (see [56, Corollary B.3]). Some further simplification may be obtained. If \( p' = p \), then \( Z_{\alpha,\beta,p} = \sup_{i \in \mathbb{N}} |\varepsilon_i|/|\Gamma_i|^{1/\alpha} \), and if \( p' = p = 1 \) (which is exactly the case \( p = 1, \beta \in (0,1/2) \)), then \( Z_{\alpha,\beta,1} = 2^{-1/\alpha} \Gamma^{-1/\alpha} \) (see Remark 1.7).

**Corollary 1.9.** With \( c_n \) and \( \mathcal{C}_{F,p} \) as above, we have

\[
\lim_{n \to \infty} \mathbb{P} \left( \frac{1}{c_n} \max_{k=1}^{\infty} X_k \leq x \right) = \begin{cases} \mathbb{P}(\mathcal{C}_{F,p}^{1/\alpha} Z_{\alpha,\beta,p} \leq x), & \text{if } \beta_p > 0, \\ \exp(-\mathcal{C}_{F,p} x^{-\alpha}), & \text{if } \beta_p \leq 0, \end{cases}
\]

with \( Z_{\alpha,\beta,p} \) defined in (1.16). In particular, \( Z_{\alpha,\beta,p} \) is not an \( \alpha \)-Fréchet random variable unless \( p = 1, \beta \in (0,1/2) \).

There are three different regimes as illustrated by Theorem 1.8.

1. **Super-critical regime:** \( \beta_p > 0 \). The limit theorem of this regime is a generalization of [56], which corresponds to \( p = 1, \beta_p = \beta \) here, and in this case \( M_{\alpha,\beta} = M_{\alpha,\beta,1} \). It is worth emphasizing that not only \( M_{\alpha,\beta,p} \) is not independently scattered with \( \beta_p > 0 \), but also that except the case \( p = 1, \beta \in (0,1/2) \), \( M_{\alpha,\beta,p} \) is not even \( \alpha \)-Fréchet, indicating very strong dependence of the original model. The dependence is of an aggregation nature, and a more refined phase transition within this regime regarding the size of the aggregated cluster of extremes is explained in Remark 1.6.

2. **Sub-critical regime:** \( \beta_p < 0 \). In this regime, the limit random sup-measure is independently scattered, and hence illustrates that the extremes of the original model are of short-range dependence. Note that it is a macroscopic property for the limit random sup-measure to be independently scattered, and in this case we obtain an immediate consequence that the well-known extremal index (e.g. [34]) of the process is \( \theta = 2p(1-p)\mathcal{C}_{F,p} = q_{F,p} D_{\beta,p} \in (0,1) \), where \( q_{F,p} \) is given in (2.2) and \( D_{\beta,p} \) is given in (4.3). However, limit theorems for random sup-measures are unable to characterize precisely the local-clustering of extremes. As a more precise elaboration we investigate the tail processes [7, 34] in the accompanying paper [6].

3. **Critical regime:** \( \beta_p = 0 \). In this regime, the limit random sup-measure is again independently scattered, and yet in the limit theorem the normalization sequence \( c_n \) is of strictly smaller order than \((n \log n)^1/\alpha \) as in the sub-critical regime. It follows immediately that the extremal index is zero, suggesting that the local clustering of extremes is unbounded (with infinite size). This again can be made precise by characterizing the tail processes as addressed in [6].

The critical regime is of particular interest in the sense that it provides a rare example where at the microscopic level the extremes of the process do not exhibit typical behavior of short-range dependence (which normally has local clustering with finite size), and yet at the macroscopic level they do. See Remark 1.10 for more details.

**Remark 1.10.** Combined with the results in the accompanying paper [6] on local clustering of extremes, we have a complete picture regarding limit extremes at both macroscopic and microscopic levels, as summarized in Table 1 below. We first describe the tail processes and the limit theorem in the accompanying paper [6]. Let \( \Theta^* = \{\Theta^*_k\}_{k \in \mathbb{N}_0} \) be a \( \{0,1\}\)-valued sequence defined as follows: let \( \{\tau^{(r)}\}_{r=1}^{p} \) denote i.i.d. copies of a standard (non-shifted) renewal process, with the renewal distribution function \( F \) as in (1.4), and consider

\[
\Theta^*_k := \begin{cases} 1, & \text{if } k \in \eta, \\ 0, & \text{otherwise}, \end{cases} \quad k = 0, 1, \ldots \quad \text{with} \quad \eta := \bigcap_{r=1}^{p} \tau^{(r)}.
\]

In particular, \( \Theta^*_0 = 1 \) since \( 0 \in \tau^{(r)}, r = 1, \ldots, p \) by definition. Note that \( \eta \) is a non-shifted renewal process (\( 0 \in \eta \)), and it is possibly terminating (i.e., \( \eta_l = \infty \) with strictly positive probability; this is the case when the renewal distribution has a mass at infinity). The renewal process \( \eta \) is terminating if and only if \( \beta_p < 0 \), and in this case with probability one, \( \Theta^*_k = 0 \) for all \( k \) large enough. (More precisely, the terminating rate is shown to be the so-called candidate extremal index of the tail process; see [6] for details.) It is proved in [6] that for all \( m \in \mathbb{N} \) (extending the definition of the stationary process to \( \{X_n\}_{n \in \mathbb{N}_0} \) to follow the convention)

\[
\mathcal{L} \left( \frac{X_0}{|X_0|}, \ldots, \frac{X_m}{|X_0|}, \frac{|X_0|}{|X_0|} > x \right) \to \mathcal{L}(\varepsilon \Theta_0, \ldots, \varepsilon \Theta^*_m),
\]

where \( \varepsilon > 0 \).
as \( x \to \infty \), where \( \varepsilon \) is a Rademacher random variable independent of \( \{\Theta_k^*\} \). The left-hand side above is understood as the conditional law of the finite-dimensional distribution given \( |X_0| > x \), and the right hand-side is the law of finite-dimensional distribution the spectral tail process, \( \{\varepsilon \Theta_k^*\}_{k \in \mathbb{N}_0} \), of the model \( \{X_n\}_{n \in \mathbb{N}_0} \). We say the (spectral) tail process is terminating, if \( \Theta_k^* = 0 \) for \( k \) large enough with probability one. Now we have the following summary of the phase transition at both levels.

| regime              | tail process (microscopic) | limit random sup-measure (macroscopic) |
|---------------------|-----------------------------|--------------------------------------|
| super-critical, \( \beta_p > 0 \) | non-terminating             | \( \mathcal{M}_{\alpha,\beta,p} \) |
| critical, \( \beta_p = 0 \)          | non-terminating             | \( \mathcal{M}_{\alpha,\beta,0} \) |
| sub-critical, \( \beta_p < 0 \)      | terminating                 | \( \mathcal{M}_{\alpha,\beta,0} \) |

**Table 1**
Summary of phase transition.

**Remark 1.11.** We are unaware of any other examples of phase transition of extremes for a stationary regularly-varying stochastic processes, except the recent result in [23] for the so-called heavy-tailed Karlin model with multiplicative noise. It is worth noticing that therein the random sup-measures arising at the critical regime are different from the ones in both super- and sub-critical regimes therein, and also that the random sup-measures in both super-critical and critical regimes therein are different from \( \mathcal{M}_{\alpha,\beta,p} \) here.

### 1.6. Comments

We conclude the introduction with several remarks regarding our main results, proofs, and follow-up questions.

**Remark 1.12.** We assume \( \overline{F} \) to be regularly varying without a slowly varying function that goes to either zero or infinity. This constraint is for convenience only, as the proof is quite involved already. We do not expect the slowly varying function will change qualitatively the limiting objects in either super or sub-critical regime. In the critical regime, however, the limit tail process studied in the accompanying paper [6] may have qualitatively different behaviors when the limit of slowly varying function is not a finite non-zero constant. We shall leave the investigation of the effects of slowly-varying functions at the critical regime in a future work.

**Remark 1.13.** One could also choose to work with multiple-infinitely-divisible process with other types of tails. If the tail decays like \( C|x|^{-\alpha} \) at \( |x| \to \infty \) for \( \alpha \in (0, \infty) \) (including our case here with \( \alpha \in (0, 2) \)), the same phase transition shall occur (note that the limits \( \mathcal{M}_{\alpha,\beta,p} \) and \( \mathcal{M}_{\alpha,0} \) are both valid for all \( \alpha > 0 \)). If the tail decays like \( L(x) x^{-\alpha} \) for some slowly varying function, then this shall introduce some delicacy, but we do not expect an essential difference from our results here. We hence do not pursue such a generalization. A more challenging problem is to consider other types of heavy tails that are not regularly varying. For \( p = 1 \), Chen and Samorodnitsky [15, 16] recently revealed a very delicate phase transition from this respect: restricted to stable-regenerative model with heavy-tailed distributions (but not necessarily with regularly-varying tails), they proved that with different heaviness of the tails, different random sup-measures may arise in the limit.

**Remark 1.14.** We provide here some heuristic explanations for each case of \( \beta_p > 0 \) and \( \beta_p \leq 0 \) to illustrate the different mechanisms of the formulation of extremes in the limit.

In the super-critical regime \( \beta_p > 0 \), working with (1.8), the key observation is that for every \( i_1 < \cdots < i_p \), the intersection \( \bigcap_{r=1}^p \tau^{(r,i)} \) scaled properly has a limit law as that of the shifted \( \beta_p \)-stable-regenerative set \( \mathcal{R}_{\beta_p} \). In words, every collection of \( p \) renewals indexed by \( i_1 < \cdots < i_p \) has a contribution to the limit random sup-measure.

For the sub-critical and critical regime \( \beta_p \leq 0 \), \( \prod_{r=1}^p \tau^{(r,i)} \) is eventually an empty set, and the picture is completely changed. In words, for each fixed \( i_1 < \cdots < i_p \), the event that \( \bigcap_{r=1}^p \tau^{(r,i)} \cap \{1, \ldots, n\} \neq \emptyset \) becomes a rare one for \( n \) large, and essentially we shall identify an appropriately chosen region \( D^* \subset \{(i_1, \ldots, i_p) \in \mathbb{N}^p : i_1 < \cdots < i_p \} \), so that on this region a Poisson limit theorem holds for the total number of occurrence of a large collection of rare events in the (exact rare events to be considered, additionally the magnitude of \( |\varepsilon_i|/\eta_i \) shall exceed an properly chosen growing threshold, which we omit in this remark). In words, the formulation of extreme values in this regime is due to the occurrence of rare events, and any single collection of \( p \) renewals indexed by \( i_1 < \cdots < i_p \) does not have contribution to the limit.

**Remark 1.15.** Our proof for the critical and sub-critical regimes is based on the two-moment method for Poisson approximation Arratia, Goldstein and Gordon [1]. This step is the most involved argument in our paper, due to the underlying delicate mechanism of the formulation (more precisely, the region \( D^* \) in the previous remark) of rare events. First, there is another phase transition within the sub-critical regime: there is another sub-regime so that one can apply a straightforward application of the two-moment method, and yet for the rest one has to truncate the process first carefully to avoid
certain local dependence that may block the method (see Remark 4.10 for a more detailed explanation; at a high level, the delicacy is similar to extremes for random energy model as explained in [32], a classical example for extremes of Gaussian random variables with strong dependence). Second, the critical regime also differs slightly from the sub-critical regime (see Remark 5.2).

Remark 1.16. Our results, actually, suggest a full point-process convergence of

$$\sum_{k=1}^{n} \delta_{(X_k / c_n, k/n)}$$

as in [7]. Such a full point-process convergence has many other consequences, including the convergence of the random sup-measures as in Theorem 1.8. Another notable consequence would be a functional central limit theorem for the partial-sum process.

However, the widely applied classical method for proving such a full point-process convergence [7, 17, 18, 34] does not apply here. This is closely related to the fact that for our model at sub-critical regime, the candidate extremal index \( \vartheta = \vartheta_{F,p} \) and extremal index \( \theta = \vartheta_{F,p} \beta \) do not equal, and yet the classical approach requires necessarily \( \vartheta = \theta \). The fact and related background on why/how the two indices are not the same is explained in full details in the accompanying paper [6, Section 1.4].

We expect the two-moment method can be adapted to prove the full point-process convergence, and this will be addressed in another paper.

The paper is organized as follows. Section 2 provides related backgrounds, notably on multiple-stable processes and renewal processes. For Theorem 1.8, the super-critical, sub-critical and critical regimes are proved in Sections 3, 4 and 5, respectively.

2. Preliminary results

2.1. Intersections of renewal processes with infinite mean

Throughout, our references on discrete-time renewal processes are Giacomin [28, Appendix A.5] and Bingham, Goldie and Teugels [14, Section 8.7.1]. Besides the notations and properties of renewal processes in Section 1, we shall also use the renewal mass function of a standard renewal process (no shift) with renewal distribution \( F \):

$$u(k) := P(k \in \tau), \; k \in \mathbb{N}_0.$$  

Note that \( u(0) = 1 \). It is well-known (e.g., [14, Theorem 8.7.3]) that the relation

$$u(n) \sim \frac{n^{\beta-1}}{C_F \Gamma(\beta) \Gamma(1-\beta)}, \quad \text{as } n \to \infty,$$

implies the assumption (1.4), and furthermore under the assumption (1.5) the two are equivalent (cf. [20]).

Next, we recall some properties of the intersected process of \( p \) i.i.d. copies of \( \tau \). \( \eta := \bigcap_{r=1}^{p} \tau^{(r)} = \{ \eta_0 = 0, \eta_1, \eta_2, \ldots \} \). This is again a (non-shifted) renewal process, although when \( \beta_p < 0 \), it is terminating, namely, \( \eta_1 = \infty \) with strictly positive probability

$$q_{F,p} = P(\eta_1 = \infty) = \lim_{n \to \infty} F_{\eta}(n) \in (0,1).$$  

(2.2)

Here and below \( F_{\eta}(x) = 1 - F_{\eta}(x) \) with \( F_{\eta} \) denoting the cumulative distribution function of \( \eta_1 \). Recall the renewal process \( \eta \) is said to be null-recurrent, if \( P(\eta_1 < \infty) = 1 \) and \( \mathbb{E} \eta_1 = \infty \).

Lemma 2.1. If \( \beta_p \geq 0 \), then the renewal process \( \eta \) is null-recurrent and

$$F_{\eta}(n) \sim \begin{cases} \frac{n^{-\beta} (C_F \Gamma(\beta) \Gamma(1-\beta))^p}{\Gamma(\beta)(\Gamma(1-\beta_p))}, & \text{if } \beta_p > 0, \\ \frac{(C_F \Gamma(\beta) \Gamma(1-\beta))^p}{\log n}, & \text{if } \beta_p = 0. \end{cases}$$

(2.1)

If \( \beta_p < 0 \), then the renewal process \( \eta \) is terminating with \( P(\eta_1 = \infty) = q_{F,p} \) in (2.2).
Proof. Let \( u_p(n) \) denote the renewal mass function of \( \eta \). Clearly by independence,

\[
 u_p(n) = u(n)^{\beta_p} \sim \frac{n^{\beta_p - 1}}{(C_F \Gamma(\beta)^{1-\beta})^p}, \quad \text{as } n \to \infty, \tag{2.3}
\]

regardless of the value of \( \beta_p \) (and \( u_p(0) = 1 \) always). In the case \( \beta_p > 0 \), and necessarily \( \beta_p < 1 \), (2.3) implies

\[
 F_p(n) \sim n^{-\beta_p} \frac{C_F \Gamma(\beta)^{1-\beta}}{\Gamma(1-\beta_p)}.
\]

This follows again from [14, Theorem 8.7.3] and was already used in [56, Lemma A.1]. The critical regime \( \beta_p = 0 \) is similar. In this case,

\[
 \sum_{k=1}^{n} u_p(k) \sim \frac{1}{(C_F \Gamma(\beta))^{1-\beta}} \log n,
\]

and hence by [14, Theorem 8.7.3], it follows that

\[
 F_p(n) \sim \left( \frac{C_F \Gamma(\beta)}{\Gamma(1-\beta_p)} \right)^{1-\beta} \log n. \tag{2.4}
\]

The null-recurrence in these two cases readily follows.

In the case \( \beta_p < 0 \), consider

\[
 G = \sum_{n=0}^{\infty} 1_{\{n \in \eta\}}.
\]

The conclusion follows since

\[
 \mathbb{E}G = \sum_{n=0}^{\infty} u_p(n) = \sum_{n=0}^{\infty} u(n)^p < \infty.
\]

(In fact, by the renewal property, \( G \) is a geometric random variable satisfying \( \mathbb{P}(G = k) = q_{F,p}(1 - q_{F,p})^{k-1}, k \in \mathbb{N} \), so that \( \mathbb{E}G = 1/q_{F,p} \).) \[ \square \]

2.2. A representation of multiple-stable processes

We shall work with the following equivalent representation of finite-dimensional distributions of \( \{X_k\}_{k=1,\ldots,n} \) in (1.12):

\[
 \{X_k\}_{k=1,\ldots,n} \overset{d}{=} \{X_{n,k}\}_{k=1,\ldots,n} := \left\{ \left\lbrack w_{\pi}^{\alpha} \sum_{0 < i_1 < \cdots < i_p} \frac{[\xi]^{i_1}}{[\Gamma]^1} \mathbb{1}_{\{k \in \cap_{i=1}^{n} R_{n,i}\}} \right\rbrack_{k=1,\ldots,n} \right\}, \tag{2.5}
\]

where \( w_n \) is as in (1.7), \( \{\xi_i\}_{i \in \mathbb{N}} \) are i.i.d. Rademacher random variables, \( \{\Gamma_i\}_{i \in \mathbb{N}} \) are consecutive ordered points from a standard Poisson process on \( [0, \infty) \), and \( \{R_{n,i}\}_{i \in \mathbb{N}} \) are i.i.d. copies of certain random closed set \( R_n \), all three families are independent. Here \( R_n \) is described as follows. Suppose \( \tau^* \) is a shifted renewal process with the stationary shift distribution \( \pi \) and renewal distribution \( F \) defined on a certain measurable space with respect to certain infinite measure \( \mu^* \) (since \( \pi \) is an infinite measure). Then, one can introduce a probability measure \( \mu_n \) on the same measurable space via

\[
 \frac{d\mu_n}{d\mu^*} = \frac{1_{\{\tau^* \cap \{1,\ldots,n\} \neq \emptyset\}}}{\mu^*\{\tau^* \cap \{1,\ldots,n\} \neq \emptyset\}} = \frac{1_{\{\tau^* \cap \{1,\ldots,n\} \neq \emptyset\}}}{w_n}.
\]

Then, the law of \( R_n \) is the one induced by \( \tau^* \) with respect to the probability measure \( \mu_n \). Moreover, it is immediately verified that

(i) \( \mathbb{P}(k \in R_n) = 1/w_n, k = 1, \ldots, n \) (shift invariance).

(ii) \( \mathbb{P}(\min(R_n \cap \{k+1, k+2, \ldots\}) \leq k+j \mid k \in R_n) = F(j) \) (Markov/renewal property).

We refer to [41, 56] for details. Note that we write \( \{X_{n,k}\}_{k=1,\ldots,n} \) to emphasize that the last expression of (2.5) depends on \( n \in \mathbb{N} \), although they correspond consistently to the same process \( \{X_k\}_{k \in \mathbb{N}} \).
3. Convergence for random sup-measures, $\beta_p > 0$

Throughout $G$ denotes a generic positive constant whose value may change from line to line.

3.1. Alternative representations of random sup-measures

In this subsection, we first provide an alternative representation of the limit random sup-measure $M_{\alpha,\beta,p}$. This alternative representation will not actually be used in the proof of the limit theorem. It is, however, instructive for understanding the arise of $M_{\alpha,\beta,p}$ in the limit theorem from discrete model $\{X_k\}_{k\in\mathbb{N}}$, which is also related to the original definition of the limit random sup-measure in [56] when $p = 1$. Introduce the following set-indexed process: for $G \in \mathcal{G}$,

$$M'_{\alpha,\beta,p}(G) := \sup_{t \in G} \zeta_{\alpha,\beta,p}(t) \quad \text{with} \quad \zeta_{\alpha,\beta,p}(t) := \sum_{0 < i_1 < \cdots < i_p} \frac{[\varepsilon_i]}{[1]}^{1/\alpha} \mathbf{1}_{\{t \in \cap_{j=1}^p \mathcal{R}_{\beta,j}\}}, \quad t \in [0,1],$$

where the consecutive Poisson points $\{\Gamma_i\}_{i\in\mathbb{N}}$, i.i.d. Rademacher sequences $\{\varepsilon_i\}_{i\in\mathbb{N}}$, and i.i.d. shifted stable regenerative sets $\{\mathcal{R}_{\beta,i}\}_{i\in\mathbb{N}}$ are as described in Section 1.3. The function $\zeta_{\alpha,\beta,p}(t)$ may be viewed as a continuous-time analog of the representation (2.5) (or (1.8)) for $\{X_k\}_{k\in\mathbb{N}}$. Note that the measurability of $M'_{\alpha,\beta,p}(G)$ is not immediately clear by the definition.

**Proposition 3.1.**

(i) For $M_{\alpha,\beta,p}$ defined in (1.15), we have for every $G \in \mathcal{G}$, almost surely

$$M_{\alpha,\beta,p}(G) > 0.$$

(ii) Assume in addition that the underlying probability space is complete. Then for $M'_{\alpha,\beta,p}$ defined in (3.1), we have for every $G \in \mathcal{G}$, almost surely

$$M_{\alpha,\beta,p}(G) = M'_{\alpha,\beta,p}(G).$$

**Proof of Proposition 3.1.** We first prove the first part. The case of $G = \emptyset$ is trivial since both sides of (1.15) are $-\infty$. From now on, fix an nonempty $G \in \mathcal{G}$. First, we show that $M_{\alpha,\beta,p}(G) > 0$ almost surely. To see this, note that

$$\mathbb{P}\left(\sup_{J \in J_{p',p}} M_{\alpha,\beta,J}(G) \leq 0\right) \leq \mathbb{P}(M_{\alpha,\beta,J}(G) \leq 0, J = \{(1,\ldots,p)\}, \{(p+1,\ldots,2p)\}, \ldots)$$

$$= \lim_{n \to \infty} \left[\mathbb{P}(\mathcal{R}_{\beta,(1,\ldots,p)} \cap G \neq \emptyset) \mathbb{P}(\{\varepsilon_{(1,\ldots,p)} = -1\} \mathbb{P}(\mathcal{R}_{\beta,(1,\ldots,p)} \cap G = \emptyset)\right]^n,$$

where we used independence in the last equality. The limit above is zero since $\mathbb{P}(\{\varepsilon_{(1,\ldots,p)} = -1\} = 1/2$ and $\mathbb{P}(\mathcal{R}_{\beta,(1,\ldots,p)} \cap G \neq \emptyset) \geq \mathbb{P}(B_{1-\beta,p,1} \in G) > 0$, recalling $\mathcal{R}_{\beta,(1,\ldots,p)} \overset{d}{=} \mathcal{R}_{\beta,p} + B_{1-\beta,p,1}$ with $\mathcal{R}_{\beta,p}$ and $B_{1-\beta,p,1}$ as described in Section 1.3.

Now we prove the second part. We start by showing that on an event with probability one, $M'_{\alpha,\beta,p}(G) \leq M_{\alpha,\beta,p}(G)$. (Note that a priori the measurability of $M'_{\alpha,\beta,p}$ is not clear yet.) Introduce the random index set $J_t := \{i \in D_p : t \in \mathcal{R}_{\beta,i}\}$. Then for $t \in G$, with the notation in (1.11), we have

$$\zeta_{\alpha,\beta,p}(t) = \sum_{i \in J_t} \frac{[\varepsilon_i]}{[1]}^{1/\alpha} = \sum_{i \in J_t} \frac{[\varepsilon_i]}{[1]}^{1/\alpha} \mathbf{1}_{\{\mathcal{R}_{\beta,i} \cap G \neq \emptyset\}} = M_{\alpha,\beta,J_t}(G),$$

where we understand a summation over $J_t$ as 0 if $J_t = \emptyset$. Then

$$\mathbb{P}(\Omega_0) = 1 \quad \text{with} \quad \Omega_0 := \{\mathcal{R}_{\beta,J} = \emptyset \text{ for all } J \subset \mathbb{N}, |J| > p'\},$$

by the choice of $p'$ in (1.9), and it follows that for all $\omega \in \Omega_0$, $J_t(\omega) \in J_{p',p'}$ for all $t \in G$ such that $J_t \neq \emptyset$. Also, let $\Omega_1 = \{M_{\alpha,\beta,p}(G) > 0\}$. So we have

$$M'_{\alpha,\beta,p}(G) = \sup_{t \in G} \zeta_{\alpha,\beta,p}(t) = \left(\sup_{t \in G : J_t \neq \emptyset} M_{\alpha,\beta,J_t}(G)\right) \leq M_{\alpha,\beta,p}(G) \quad \text{on } \Omega_0 \cap \Omega_1.$$
Next, we show that \( M'_{\alpha, \beta, p}(G) \geq M_{\alpha, \beta, p}(G) \) on the event of probability one. First, since \( |\Gamma_4|^{-1/\alpha} \to 0 \) when \( \max(i) \to \infty \) almost surely, it follows that the supremum over \( J \in J_{p, p'} \) in \( M_{\alpha, \beta, p}(G) \) is attainable almost surely. Let \( \Omega_2 \) denote this event. So on \( \Omega_0 \cap \Omega_1 \cap \Omega_2 \),

\[
M_{\alpha, \beta, p}(G) = \mathcal{M}_{\alpha, \beta, J(\bar{c})}(G) > 0
\]

for some random \( \bar{c} \in D_q \) with random \( q \in \{p, p+1, \ldots, p'\} \), and in particular \( R_{\beta, J(\bar{c})} \cap G \neq \emptyset \). It suffices to show that

\[
J(\bar{c}) = J_t \text{ for some } t \in R_{\beta, J(\bar{c})} \cap G \text{ on } \Omega_0 \cap \Omega_1 \cap \Omega_2.
\]

(In words, there is a location \( t \in R_{\beta, J(\bar{c})} \cap G \) exactly covered by the aggregated cluster corresponding to \( J(\bar{c}) \).) Note that on the event \( \Omega_0 \cap \Omega_1 \cap \Omega_2 \), for all \( t \in R_{\beta, J(\bar{c})} \cap G \), necessarily \( J(\bar{c}) \subset J_t \). It remains to show the inclusion is in fact an equality on the event mentioned above. We now prove by contradiction: suppose on a sub-event say \( \max(J_{p, p'}) \) by some aggregated cluster indexed by \( J(\bar{c}) \) such that \( \bar{c} \supseteq \hat{c} \). Recall that in view of (3.2), \( q \leq p' - 1 \) on the event \( E \). It then follows that on the event \( E \),

\[
\emptyset \neq R_{\beta, J(\bar{c})} \cap G \subset \left( \bigcup_{t \in R_{\beta, J(\bar{c})} \cap G} R_{\beta, J_t} \right) \cap G \subset \left( \bigcup_{c' \in D_{q+1}, \bar{c} \subset c'} R_{\beta, J(c')} \right) \cap G.
\]

But almost everywhere on the sub-event of \( E \), \( R_{\beta, J(\bar{c})} \cap G \) has Hausdorff dimension \( \beta_q \), while each \( R_{\beta, J(c')} \) in the last expression above with \( c' \in D_{q+1} \) has Hausdorff dimension \( \beta_{q+1} < \beta_q \) and hence the countable union in the last expression has Hausdorff dimension \( \beta_{q+1} < \beta_q \) in view of the countable stability property (cf. [24, Section 3.2]). So the event \( E \) cannot have strictly positive probability.

In summary, we have proved that on an event of probability one \( \mathcal{M}_{\alpha, \beta, p}(G) = M'_{\alpha, \beta, p}(G) \). By completeness of probability space, \( \mathcal{M}'_{\alpha, \beta, p}(G) \) is a random variable. This completes the proof. \( \square \)

3.2. Proof of Theorem 1.8, \( \beta_p > 0 \)

Recall the model in (1.12) and consider the associated empirical random sup-measure \( M_n \) as in (1.3). The goal of this section is to prove the following case of Theorem 1.8 (recall the characterization of the weak convergence in the space of sup-measures discussed around (1.1)), \( \beta_p > 0 \):

\[
\frac{1}{w_n^{1/\alpha}} \{ M_n(I) \}_{I \in \Theta_0} \overset{f.d.d.}{\to} \{ M_{\alpha, \beta, p}(I) \}_{I \in \Theta_0} \overset{d}{=} \left\{ \sup_{J \subset J_{p, p'}} \mathcal{M}_{\alpha, \beta, J}(I) \right\}_{I \in \Theta_0}.
\]

(3.3)

In view of (1.7), in Theorem 1.8,

\[
\mathfrak{C}_{F, p} = \left( \frac{C_F}{1 - \beta} \right)^p.
\]

(3.4)

We start with a truncation argument. Define first

\[
R_{n, \widehat{i}} := \bigcap_{k=1}^p R_{n, i_k}, \quad \widehat{i} = (i_1, \ldots, i_p) \in D_p.
\]

Recall that \( M_n(G) = \max_{i/n \in G} X_i \) with \( X_i \) in (2.5). For each \( \ell \in \mathbb{N} \) and \( G \in \mathcal{G} \), consider

\[
M_{n, \ell}(G) := \max_{k/n \in G} \frac{1}{w_n^{1/\alpha}} \sum_{i \in D_p, i_p \leq \ell} \frac{[\varepsilon_i]}{|\Gamma_4|^{1/\alpha}} \mathbf{1}_{\{k \in R_{n, i}\}}.
\]

(3.5)

**Lemma 3.2.** For any \( \varepsilon > 0 \), we have

\[
\lim_{\ell \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \max_{k=1, \ldots, n} \left| \sum_{i \in D_p, i_p > \ell} \frac{[\varepsilon_i]}{|\Gamma_4|^{1/\alpha}} \mathbf{1}_{\{k \in R_{n, i}\}} \right| > \varepsilon \right) = 0.
\]
Proof. The argument is similar to those in [4, 56]. Fix finite \( M \geq 2p/\alpha \). It suffices to show for any \( 0 \leq q \leq p - 1 \) and fixed \( i = (i_1, \ldots, i_q) \in D_q \) with \( i_q \leq M \) that

\[
\lim_{\ell \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \max_{k=1, \ldots, n} \left| \frac{\varepsilon_{i_q}}{\Gamma_{i_q}} \right|^{1/\alpha} \sum_{M<i_q+1} \frac{\varepsilon_{i_q+1}}{\Gamma_{i_q+1}} 1\{k \in R_{n,i}\} \right| > \epsilon \right) = 0.
\]

Here and below we write \( \varepsilon_{i_q} = (\varepsilon_{i_q,1}, \ldots, \varepsilon_{i_q}) \) and \( \varepsilon_{i_q+1} = \prod_{r=0}^{b} \varepsilon_{i_r} \). By Markov inequality, union bound and stationarity in \( k \), the probability above is bounded by

\[
\frac{n}{\epsilon^r} \mathbb{E} \left[ \left| \frac{\varepsilon_{i_q}^{1/\alpha}}{\Gamma_{i_q}} \right|^{1/\alpha} \sum_{M<i_q+1} \frac{\varepsilon_{i_q+1}}{\Gamma_{i_q+1}} 1\{k \in R_{n,i}\} \right]^\gamma
\]

for some \( r > 0 \) satisfying \( 1/r = 1/r' + 1/\gamma \), where \( \gamma \) is chosen large enough so that

\[
\gamma > \frac{2}{1 - \beta},
\]

and \( r' \) is chosen so that \( (p - 1)2r'/\alpha < 1 \) which ensures \( \mathbb{E}(\Gamma_{i_q}^{-r'/\alpha}) < \infty \) for all \( q \leq p - 1 \) (cf. [54, eq. (3.2)]). By Hölder’s inequality, it remains to show

\[
\lim_{\ell \to \infty} \limsup_{n \to \infty} \frac{n}{\epsilon^r} \mathbb{E} \left[ \left| \frac{\varepsilon_{i_q}^{1/\alpha}}{\Gamma_{i_q}} \right|^{1/\alpha} \sum_{M<i_q+1} \frac{\varepsilon_{i_q+1}}{\Gamma_{i_q+1}} 1\{k \in R_{n,i}\} \right]^\gamma = 0.
\]

By a generalized Khinchine inequality [54, Theorem 1.3(ii)], exploring the orthogonality induced by \( \varepsilon_{i_q+1} \), and the fact \( \mathbb{P}(1 \in R_{n,i}) = \mathbb{P}(1 \in R_{n,1})^{p-q} = w_n^{-p+q} \), the expectation in the displayed line above is bounded from above by, up to a multiplicative constant,

\[
\left( \mathbb{E} \left( \left| \sum_{M<i_q+1} \frac{\varepsilon_{i_q+1}}{\Gamma_{i_q+1}} 1\{k \in R_{n,i}\} \right|^{2/\alpha} \right) \right)^{\gamma/2} \leq w_n^{-\gamma(p-q)/2} \left( \mathbb{E} \left| \sum_{M<i_q+1} \frac{\varepsilon_{i_q+1}}{\Gamma_{i_q+1}} 1\{k \in R_{n,i}\} \right|^{-2/\alpha} \right)^{\gamma/2}.
\]

By the choice of \( \gamma \), we have \( nw_n^{-\gamma(p-q)/2} \leq nw_n^{-\gamma/2} \to 0 \). To conclude the proof, we apply the bound

\[
\mathbb{E} \left| \Gamma_{i_q+1} \right|^{-2/\alpha} \leq C \left| \Gamma_{i_q+1} \right|^{-2/\alpha} \tag{3.6}
\]

which holds since \( i_q+1 > M \geq 2p/\alpha \geq 2(p - q)/\alpha \) [54, eq. (3.2)], so that the last multiple sum above is bounded. \( \square \)

Next, we define for any \( J = \{i_1, \ldots, i_r\}, i_s = (i_{s,1}, \ldots, i_{s,p}), s = 1, \ldots, r, \) that

\[
\cup J := \{i_{s,h} : s = 1, \ldots, r, \ h = 1, \ldots, p \} \subset \mathbb{N}
\]

and introduce of a finite truncation of \( J_{p,p'} \) in (1.13) as

\[
J_{p,p'}(\ell) := \{J \in J_{p,p'} : \max(\cup J) \leq \ell \}.
\]

Next, as a discrete analog of Proposition 3.1, we approximate \( M_{n,\ell}(G) \) in (3.5) by

\[
\tilde{M}_{n,\ell}(G) := \max_{J \subset J_{p,p'}(\ell)} \tilde{M}_{n,J}(G),
\]

where

\[
\tilde{M}_{n,J}(G) := \left\{ \begin{array}{ll}
\frac{w_n^{p/\alpha}}{n} \sum_{i \in J} 1_{\{i \}} \frac{1}{\Gamma_{i}} 1/\alpha, & \text{if } \frac{1}{n} R_{n,J} \cap G \neq \emptyset, \\
-\infty, & \text{otherwise},
\end{array} \right.
\]

with \( R_{n,J} := \bigcap_{i \in J} R_{n,i} \).
Lemma 3.3. For every $G \in \mathcal{G}$ fixed,
\[
\lim_{n \to \infty} \mathbb{P} \left( M_{n, \ell}(G) = \tilde{M}_{n, \ell}(G) \right) = 1.
\]

Proof. Suppose $G$ is nonempty; otherwise the result is trivial. Consider the following event:
\[
\Omega_{n,1}(\ell) := \left\{ \max_{k=1, \ldots, n} \sum_{i=1}^{\ell} 1_{\{k \in R_{n,i}\}} \leq p' \right\}.
\]
Restricted to $\Omega_{n,1}(\ell)$, one readily checks that $M_{n, \ell}(G) \leq \tilde{M}_{n, \ell}(G)$. For the other direction, introduce further
\[
\Omega_{n,2}(J) := \left\{ R_{n,J} \cap G = \emptyset \right\} \cup \left\{ \exists k \in \{1, \ldots, n\} \text{ s.t. } k \in R_{n,J}, k \notin \bigcup_{i \in \{1, \ldots, \ell\} \setminus \{J\}} R_{n,i} \right\},
\]
and
\[
\Omega_{n,2}(\ell) := \bigcap_{J \in \mathcal{J}_{n,p',(\ell)}} \Omega_{n,2}(J).
\]
Restricted to $\Omega_{n,2}(\ell)$, one has $\tilde{M}_{n, \ell}(G) \leq M_{n, \ell}(G)$ because the $J$ set which maximizes $(3.7)$ can always be realized by $\{i = (i_1, \ldots, i_p) \in \mathcal{D}_p, \ i_p \leq \ell : k \in R_{n,1}\}$ for some $k$. Therefore, it remains to prove that $\lim_{n \to \infty} \mathbb{P}(\Omega_{n,1}(\ell)) = 1$ for $\ell = 1, 2$. For $\Omega_{n,1}(\ell)$, this follows from the fact that $R_{n,1} \Rightarrow \emptyset$ (e.g. [56, Theorem 3.1]) for all $i \in \mathcal{D}_{p'+1}$. The fact that $\lim_{n \to \infty} \mathbb{P}(\Omega_{n,2}(\ell)) = 1$ has played a crucial role in [56, Lemma 5.5] (see also [59, Lemma 3.7]). □

Proof of Theorem 1.8, $\beta_p > 0$. The goal is to show $(3.3)$. Recall $M_{n, \ell}$ in $(3.5)$ and $\tilde{M}_{n, \ell}$ in $(3.7)$, which serve as approximations for $M_n$. Introduce the truncated version of $\mathcal{M}_{\alpha, \beta, p}$ as
\[
\mathcal{M}_{\alpha, \beta, p}^{(\ell)}(G) := \max_{J \in \mathcal{J}_{p',(\ell)}} \mathcal{M}_{\alpha, \beta, p}(G) \quad G \in \mathcal{G}.
\]
We first show
\[
\left\{ \frac{1}{w_n^{p/\alpha}} \mathcal{M}_{n, \ell}(I) \right\}_{I \in \mathcal{G}_0} \overset{\text{f.d.d.}}{\longrightarrow} \left\{ \mathcal{M}_{\alpha, \beta, p}^{(\ell)}(I) \right\}_{I \in \tilde{\mathcal{G}}_0},
\]
Indeed, the above follows if for all $I_s \in \mathcal{G}_0$, $s = 1, \ldots, d, \ d \in \mathbb{N},$
\[
\left\{ \frac{1}{w_n^{p/\alpha}} \mathcal{M}_{n, \ell}(I_s) \right\}_{J \in \mathcal{J}_{p',(\ell)}, s = 1, \ldots, d} \Rightarrow \left\{ \mathcal{M}_{\alpha, \beta, p}(I_s) \right\}_{J \in \mathcal{J}_{p',(\ell)}, s = 1, \ldots, d},
\]
Comparing the two sides (see $(3.8)$ and $(1.15)$), it suffices to show
\[
\left\{ \frac{1}{n} R_{n,i} \right\}_{i = (i_1, \ldots, i_p) \in \mathcal{D}_p} \Rightarrow \left\{ R_{\beta, i} \right\}_{i = (i_1, \ldots, i_p) \in \mathcal{D}_q},
\]
a joint convergence in distribution of finitely many random closed sets. This joint convergence, a non-trivial result since $R_{n,i}$ and $R_{n,i'}$ are dependent when $i \cap i' \neq \emptyset$, was established in [56, Theorem 5.4]. The desired $(3.10)$ and hence $(3.9)$ now follow. Next, for any $I \in \mathcal{G}_0$, we have
\[
\lim_{\ell \to \infty, n \to \infty} \limsup \mathbb{P} \left( \frac{1}{w_n^{p/\alpha}} \left| M_{n}(I) - \tilde{M}_{n, \ell}(I) \right| \geq \epsilon \right) = 0.
\]
Above $\tilde{M}_{n, \ell}(I)$ may be $-\infty$ while $M_{n}(I)$ (and also $M_{n, \ell}(I)$ appearing below) is almost surely finite for all large enough $n$; we understand $\left| M_{n}(I) - \tilde{M}_{n, \ell}(I) \right|$ as $\infty$ when $M_{n}(I)$ is finite and $\tilde{M}_{n, \ell}(I) = -\infty$. Indeed, the above follows from Lemmas 3.2 and 3.3 applied to the two probabilities respectively on right-hand side of following inequality
\[
\mathbb{P} \left( \frac{1}{w_n^{p/\alpha}} \left| M_{n}(I) - \tilde{M}_{n, \ell}(I) \right| \geq \epsilon \right) \leq \mathbb{P} \left( \frac{1}{w_n^{p/\alpha}} \left| M_{n}(I) - M_{n, \ell}(I) \right| > \epsilon \right) + \mathbb{P} \left( M_{n, \ell}(I) \neq \tilde{M}_{n, \ell}(I) \right).
\]
In addition, by monotonicity, one has 
\[ \lim_{\ell \to \infty} \mathcal{M}^{(\ell)}_{\alpha,\beta,p}(I) = \mathcal{M}_{\alpha,\beta,p}(I). \]
The desired result now follows from a routine triangular approximation argument [13, Theorem 3.2].

4. Convergence for random sup-measures, \( \beta_p < 0 \)

The goal of this section is to prove the following case of Theorem 1.8 (recall the characterization of the weak convergence in the space of sup-measures discussed around (1.1)), with \( \beta_p < 0 \):
\[ \frac{1}{c_n} \{ M_n(I) \}_{I \in \mathcal{G}_0} \xrightarrow{f.d.d.} \left\{ c_F^{1/\alpha, \mathcal{M}_{\alpha}^{(\infty)}(I)} \right\}_{I \in \mathcal{G}_0}, \tag{4.1} \]
where we have
\[ c_n = (n \log p^{-1} n)^{1/\alpha} \quad \text{and} \quad c_F^{1/\alpha, \mathcal{M}_{\alpha}^{(\infty)}} = \frac{1}{2p!(p-1)!} q_{\beta,p} D_{\beta,p}, \tag{4.2} \]
with \( q_{\beta,p} \) as in (2.2), and
\[ D_{\beta,p} := \sum_{s=q_{\beta,p}}^{p} (-1)^{p-s} \binom{p}{s} (-\beta_s)^{p-1} \quad \text{with} \quad q_{\beta,p} := \min \{ q \in \mathbb{N} : \beta_q < 0 \}. \tag{4.3} \]
It can be shown that the parameter \( D_{\beta,p} \) takes value in \( (0,1) \) when \( \beta \in (0, 1 - 1/p) \) (see [6]). Recall from (1.7) that \( w_n \sim C_F (1 - \beta)^{-1} n^{1-\beta} \). Throughout this section we set
\[ r_n = \frac{w_n^p}{c_n^p} \sim \left( \frac{C_F}{1 - \beta} \right)^p n^{-\beta_p} \log^{p-1} n. \tag{4.4} \]
To prove (4.1), the strategy is to first approximate the series representation (2.5) by a truncated version, and then to establish the point-process convergence of block maxima for the truncated process. For each \( n \in \mathbb{N} \), we recall the series representation of finite-dimensional distribution \( \{ X_{n,k} \}_{k=1,\ldots,n} \overset{d}{=} \{ X_k \}_{k=1,\ldots,n} \) in (2.5): Introduce, for a sequence \( d_n \to \infty \) such that \( k_n := [n/d_n] \to \infty \),
\[ \mathcal{I}_{n,j} := \{(j-1)d_n + 1, \ldots, jd_n\}, \quad j = 1, \ldots, k_n, \]
which are \( k_n \) non-overlapping blocks each of length \( d_n \). Introduce a sub-domain of \( \mathcal{D}_p \) with a product constraint as
\[ \mathcal{D}_p(x) := \{ i = (i_1, \ldots, i_p) \in \mathbb{N}^p : i_1 < \cdots < i_p, [i] = i_1 \ldots i_p \leq x \}, \quad x > 0, \]
as well as
\[ \mathcal{H}(n,K) := \{ i \in \mathcal{D}_p(Kr_n) : i_p \leq w_n \}, \tag{4.5} \]
where \( r_n \) is as in (4.4). Note that \( \mathcal{H}(n,K) = \mathcal{D}_p^*(Kr_n) \) if \( r_n \leq Kw_n \), which holds for large \( n \) if \( \beta_{p-1} \geq 0 \). It is crucial to work with \( \mathcal{H}(n,K) \) instead of \( \mathcal{D}_p^*(Kr_n) \) when \( \beta_{p-1} < 0 \). See Remark 4.10 for explanations.

Now we introduce the truncated process, for \( K > 0 \),
\[ X_{n,k}^{(K)} := \sum_{i \in \mathcal{H}(n,K)} \frac{w_n^p/\alpha}{|i|^{1/\alpha}} 1_{\{k \in R_{n,i}\}}, \quad k = 1, \ldots, n, \]
and a point process involving block maxima:
\[ \xi_{n,K} := \sum_{j=1}^{k_n} \delta_{\left( \frac{X_{n,k}^{(K)}}{c_n/k_n} \right)} 1_{\left\{ \frac{X_{n,k}^{(K)}}{c_n/k_n} > 0 \right\}} \quad \text{with} \quad \hat{\xi}_{n,k} := \max_{k \in \mathcal{I}_{n,j}} X_{n,k}^{(K)} , \quad j = 1, \ldots, k_n. \]
It is worth mentioning that since we consider limit theorem for maximum, the negative values of \( \tilde{m}_{n,j}^{(K)} \) will play no role and hence are excluded in the point process \( \xi_{n,K} \); see the relation (4.38) in Section 4.3 below. Establishing weak convergence of the point process \( \xi_{n,K} \) as \( n \to \infty \) is a key step in the proof of (4.1). Throughout, we let \( \mathcal{M}_p((0, \infty] \times [0, 1]) \) denote the space of Radon point measures on \( (0, \infty] \times [0, 1] \) with the vague topology. The standard reference is Resnick [45].

**Proposition 4.1.** With \( \beta_p < 0, d_n \to \infty \) and \( k_n \to \infty \) as \( n \to \infty \), we have the following weak convergence

\[
\xi_{n,K} \Rightarrow \xi_K := \sum_{i=1}^{\infty} \delta \left( e_{\beta_p r_{\xi}}^{-1/\alpha} \right) 1 \left\{ e_{\beta_p r_{\xi}}^{-1/\alpha} > K^{-1/\alpha} \right\},
\]

in \( \mathcal{M}_p((0, \infty] \times [0, 1]) \), where \( \mathcal{C}_{\beta_p} \) is as in (4.2).

Note that the limit \( \xi_K \) in (4.6) is a Poisson point process.

In order to prove Proposition 4.1, we shall work with the following point processes

\[
\eta_{n,K} := \sum_{j=1}^{k_n} \sum_{i \in H(n,K)} \delta \left( \frac{1}{r_{\xi_i}} - 1/\alpha \right) \{ R_{n,i} \cap I_{n,j} \neq \emptyset, [\epsilon_j] = 1 \},
\]

\[
\tilde{\eta}_{n,K} := \sum_{j=1}^{k_n} \sum_{i \in H(n,K)} \delta \left( \frac{1}{r_{\xi_i}} - 1/\alpha \right) \{ R_{n,i} \cap I_{n,j} \neq \emptyset, [\epsilon_j] = 1 \}.
\]

Note that there is only one difference between the two above: the random product \([\Gamma_i]\) in \( \eta_{n,K} \) replaced by the non-random product \([i]\) in \( \tilde{\eta}_{n,K} \). An overview of the approximations behind Proposition 4.1 is as follows. For some test set \( E \),

\[
\xi_{n,K}(E) \approx \eta_{n,K}(E) \approx \tilde{\eta}_{n,K}(E) \Rightarrow \xi_K(E).
\]

The approximations above are understood with letting \( n \to \infty \), with \( E \) fixed and \( K \) picked large enough depending on \( E \). The meaning of the approximations ‘\( \approx \)’ will be made precise in the lemmas indicated. In particular, we shall consider sets of the form

\[
E := (y_0, y_1] \times \bar{I} \subset [0, \infty] \times [0, 1] \quad \text{with} \quad \bar{I} = (a, b) \text{ or } [0, b] \subset [0, 1],
\]

for some \( 0 < y_0 < y_1 \leq \infty \), and each set satisfying

\[
0 < K^{-1/\alpha} < y_0.
\]

We mention that for proving Proposition 4.1 we shall fix \( K \) and consider all \( E \) described above, whereas for proving (4.1), we shall work with a fixed collection of \( E_1, \ldots, E_d \), and let \( K \to \infty \) so that (4.9) is eventually satisfied.

At last, say for the marginal convergence in (4.1) with \( I \in \mathcal{G}_0 \), we have for \( y > K^{-1/\alpha} \),

\[
\mathbb{P} (M_n(I) \leq y) \approx \mathbb{P} (M_{n,K}(I) \leq y) \approx \mathbb{P} (\xi_{n,K}((y, \infty] \times I) = 0),
\]

where \( M_{n,K}(I) \) is defined as \( M_n(I) \) with \( X_k \) (or \( X_{n,k} \)) replaced by the truncated version \( X_{n,k}^{(K)} \).

The most involved step is a Poisson approximation between \( \tilde{\eta}_{n,K} \) and \( \xi \) over certain test sets, which will be proved first in Section 4.1. We then prove Proposition 4.1 in Section 4.2 and finally (4.1) in Section 4.3.

### 4.1. Two-moment method for Poisson approximation

**Lemma 4.2.** For \( K > 0 \) fixed and for all disjoint sets \( E_s = (y_{s,0}, y_{s,1}] \times \bar{I}_s, s = 1, \ldots, d \), of the form (4.8), suppose

\[
0 < K^{-1/\alpha} < \min_{s=1,\ldots,d} y_{s,0}.
\]

We have as \( n \to \infty \),

\[
(\tilde{\eta}_{n,K}(E_1), \ldots, \tilde{\eta}_{n,K}(E_d)) \Rightarrow (\xi_K(E_1), \ldots, \xi_K(E_d)).
\]
Moreover, $\pi_{n,k} \Rightarrow \xi_K$ in $\mathbb{M}_p([0, \infty] \times [0, 1])$.

Lemma 4.2 is a multivariate Poisson limit theorem for sums of dependent Bernoulli variables

$$\chi_{i,j} := 1_{(R_{n,i} \cap I_{n,j}) \neq \emptyset, |\varepsilon_i|=1},$$

with summation index sets

$$\mathcal{I}_{n,s} := \left\{(i,j) \in \mathcal{H}(n,K) \times \{1, \ldots, k_n\} : ((i)_{r_n} r_n^{-1/\alpha}, j/k_n) \in E_s\right\}, \ s = 1, \ldots, d.$$  

In this way, we write

$$\pi_{n,K}(E_s) = \sum_{(i,j) \in \mathcal{I}_{n,s}} \chi_{i,j}, \ s = 1, \ldots, d.$$  

We apply the well-known two-moment method of Arratia, Goldstein and Gordon [1, Theorem 2]. The dependence structure of the Bernoulli random variables $\chi_{i,j}$ can be described by a dependency graph ([42, Section 2.1]), where disconnection of subsets of nodes implies independence. Obviously for $\chi_{i,j}$, we can define $(i, j) \sim (i', j')$, namely, these two nodes to be connected, if $i \cap i' \neq \emptyset$ (regardless of $j$ and $j'$). Here and below $i \cap i'$ is understood in the obvious way by regarding $i$ and $i'$ as subsets of $\mathbb{N}$ and the elements of $i \cap i'$ listed in increasing order if it is nonempty.

Before stating the conditions for the two-moment method, we introduce a few notations. First, introduce the following counting numbers (recall $\mathcal{H}(n,K)$ in (4.5)):

$$C_{n,1}(K) := |\mathcal{H}(n,K)|, \quad C_{n,2}(r,K) := |\{(i, i') : i, i' \in \mathcal{H}(n,K), |i \cap i'| = r\}|, \ r = 1, \ldots, p,$$

and the following probabilities

$$\rho_n := \mathbb{E}(\chi_{i,j} = 1) \cdot \mathbb{P}(R_{n,i} \cap I_{n,1} \neq \emptyset) = \frac{1}{2} \cdot \mathbb{P}(R_{n,i} \cap I_{n,1} \neq \emptyset),$$

$$\rho_{n,i,i',j,j'} := \mathbb{E}(\chi_{i,j} \chi_{i',j'}) \leq \mathbb{P}(R_{n,i} \cap I_{n,j} \neq \emptyset, R_{n,i'} \cap I_{n,j'} \neq \emptyset).$$

Note that $C_{n,2}(p, K) = C_{n,1}(K)$, and for all $j, j'$ fixed, $\rho_{n,i,i',j,j'}$ has an identical value for all $i, i'$ such that $|i \cap i'| = r$. Note also that in (4.15) we bound $\mathbb{P}(|\varepsilon_i| = 1, |\varepsilon_{i'}| = 1) \leq 1$. Introduce

$$\rho_n(r) := \begin{cases} \sum_{j=1}^{k_n} \sum_{j'=1}^{k_n} \rho_n(1, \ldots, p-1, r, j, j'), & \text{if } r = 1, \ldots, p-1, \\ \sum_{j,j' \in \{1, \ldots, k_n\} \setminus j \neq j'} \rho_n(1, \ldots, p, r, j, j'), & \text{if } r = p. \end{cases}$$

In this way, we have

$$\mathbb{E}\pi_{n,K}(E_s) = \sum_{(i,j) \in \mathcal{I}_{n,s}} \mathbb{E}(\chi_{i,j}) = |\mathcal{I}_{n,s}| \cdot \rho_n, \ s = 1, \ldots, d.$$  

Next, we introduce, with $\mathcal{I}_n := \mathcal{H}(n,K) \times \{1, \ldots, k_n\},$

$$b_{n,1}(K) := \sum_{(i,j) \in \mathcal{I}_n} \mathbb{E}(\chi_{i,j}) = k_n^2 \sum_{i,i' \in \mathcal{H}(n,K)} \mathbb{E}(\chi_{i,j} \chi_{i',j'}) \mathbb{P}(|i' \cap i'| > 0) \sum_{r=1}^{p} \rho_n^2 = k_n^2 p \sum_{r=1}^{p} C_{n,2}(r, K) \rho_n(r),$$

$$b_{n,2}(K) := \sum_{(i,j) \in \mathcal{I}_n} \mathbb{E}(\chi_{i,j}^2) = \sum_{i,i' \in \mathcal{H}(n,K)} \mathbb{E}(\chi_{i,j} \chi_{i',j'}) \mathbb{P}(|i' \cap i'| = r) \sum_{r=1}^{p} \rho_n(r) = \sum_{r=1}^{p} C_{n,2}(r, K) \rho_n(r).$$
Then, by [1, Theorem 2], the Poisson convergence (4.11) follows once we show
\[
\lim_{n \to \infty} \mathbb{E} \mathbf{1}_{n,K}(E_s) = \mathbb{E} \xi_K(E_s), \quad s = 1, \ldots, d, \quad \text{and} \quad \lim_{n \to \infty} (b_{n,1}(K) + b_{n,2}(K)) = 0. \tag{4.20}
\]

It remains to provide estimates for \( C_{n,1}(K), C_{n,2}(r,K), \rho_n \) and \( \rho_n(r) \), and we shall proceed one by one. Note that we need exact asymptotics of \( C_{n,1}(K) \) and \( \rho_n \), and only the orders of \( C_{n,2}(r,K) \) and \( \rho_n(r) \).

**Lemma 4.3.** As \( r \to \infty \), we have
\[
|D_p^r(r)| \sim \frac{r \log^{p-1}(r)}{p!} \tag{4.21}
\]

**Proof.** The key is to argue the integral approximation \( |D_p^r(r)| \sim \int_{[1,r]^p} 1_{\{x_1 \cdots x_p \leq r\}} \, dx \) as \( r \to \infty \). Here we omit its proof, which is similar to a more sophisticated case below in the proof of Lemma 4.4. Then by change of variables,
\[
\int_{[1,r]^p} 1_{\{x_1 \cdots x_p \leq r\}} \, dx = \int_{[0,\log(r)]^p} e^{y_1 + \cdots + y_p} 1_{\{y_1 + \cdots + y_p \leq \log(r)\}} \, dy
\]
\[
= \frac{1}{(p-1)!} \int_0^{\log(r)} z^{p-1} e^z \, dz \sim r \log^{p-1}(r) \tag{4.22}
\]
as \( r \to \infty \).

**Lemma 4.4.** Assume \( \beta_p < 0 \). We have, for all \( K > 0 \),
\[
C_{n,1}(K) = |\mathcal{H}(n,K)| \sim \frac{K}{p!(p-1)!} \mathbb{D}_{\beta,p} \frac{w_p^n}{n},
\]
where
\[
\mathbb{D}_{\beta,p} := \sum_{s=q_{\beta,p}}^p (-1)^{s-p} \binom{p}{s} (-\beta_s)^{p-1} \quad \text{with} \quad q_{\beta,p} := \min \{ q \in \mathbb{N} : \beta_q < 0 \}.
\]

**Proof.** We shall approximate summations by the corresponding integrals as follows. For a general summable function \( f : \mathbb{N}^p \to \mathbb{R} \),
\[
\sum_{i=1}^n f(i) = \int_{\mathbb{N}^p} f([x_1], \ldots, [x_p]) \, dx = \int_{\{0,1\}^p} f([x_1] + 1, \ldots, [x_p] + 1) \, dx.
\]
Here and below \([x] ([x] \text{ resp.})\) denotes the greatest (smallest, resp.) integer less than or equal to (greater than or equal to, resp.) \( x \), which should be distinguished with \([x] \) which denotes the product of all components of a vector \( x \). We first derive a crude upper bound. Let \( U_1, \ldots, U_p \) be i.i.d. uniform random variables in \([0,1]\). Then
\[
C_{n,1}(K) \leq \frac{1}{p!} \int_{[0,\infty)^p} 1_{\{x_1 + \cdots + x_p \leq K r_n, x_i \leq w_n, i=1, \ldots, p\}} \, dx
\]
\[
\leq \frac{1}{p!} \int_{0 \leq x_1 + \cdots + x_p \leq K r_n} dx \leq \frac{w_p^n}{p!} \mathbb{P}(U_1 \cdots U_p \leq K r_n / w_p^n)
\]
\[
\sim \frac{w_p^n}{p!} \frac{K(r_n / w_p^n)((- \log(r_n / w_p^n))^{p-1}}{(p-1)!} \sim \frac{K \alpha^{p-1}}{p!(p-1)!} r_n \log^{p-1}(n), \tag{4.23}
\]
where we used the fact that \( r_n / w_p^n = c_n^{-\alpha} \to 0 \) as \( n \to \infty \), and recalled in the third step that for \( s \in (0,1) \),
\[
\mathbb{P}(U_1 \cdots U_p \leq s) = s \sum_{k=0}^{p-1} (-\log s)^k \frac{k!}{k!} \sim \frac{s(- \log s)^{p-1}}{(p-1)!}
\]
as \( s \downarrow 0 \) [54, Lemma 3.1]. (This upper bound (4.23) is of the correct order \( O(w_p^n / n) \), but not sharp in the multiplicative constant as shown below.)

Next we aim at precise upper and lower bounds. Write \( C_{n,1}(K) = A_n + B_n \) with
\[
A_n := |\{ \mathbb{i} \in \mathbb{N}^p : [\mathbb{i}] \leq K r_n, 2 \leq i_1 < \ldots < i_p \leq w_n \}|,
\]
\[
B_n := |\{ \mathbb{i} \in \mathbb{N}^p : [\mathbb{i}] \leq K r_n, i_1 = 1, i_2 < \ldots < i_p \leq w_n \}|.
\]
The crude upper bound (4.23) implies that $B_n = O(r_n(\log(n))^{p-2})$, which will be eventually negligible. On the other hand,

$$p! A_n = \{i \in \mathbb{N}^p : [i] \leq Kr_n, 2 \leq k_s \leq w_n, s = 1, \ldots, p, i_s \neq i_t \text{ if } s \neq t\}$$

$$= G_n - O(r_n(\log(n))^{p-2}),$$

where

$$G_n = \{i \in \mathbb{N}^p : [i] \leq Kr_n, 2 \leq k_s \leq w_n, s = 1, \ldots, p\},$$

and the negligible $O(r_n(\log(n))^{p-2})$ term follows from the upper bound (4.23) with $p$ replaced by $p-1, p-2, \ldots, 1$, which correspond to the number of distinct $k_s$'s.

Now we provide precise upper and lower bounds for $G_n$:

$$G_n = \int_{[1,\infty)^p} \mathbf{1}_{\{|x_1| \ldots |x_p| \leq Kr_n, [x_i] \leq w_n, i=1,\ldots,p\}}(x) \, dx \leq \int_{[1,w_n]^p} \mathbf{1}_{\{|x| \leq Kr_n\}}(x) \, dx$$

and

$$G_n = \int_{[1,\infty)^p} \mathbf{1}_{\{|(x_1+1) \ldots (x_p+1)| \leq Kr_n, [x_i] \leq w_n, i=1,\ldots,p\}}(x) \, dx \geq \int_{[2,w_n]^p} \mathbf{1}_{\{|x| \leq Kr_n\}}(x) \, dx.$$

So for a fixed constant $a > 0$, we are left to compute the asymptotics of

$$G_n(a) := \int_{[a,w_n]^p} \mathbf{1}_{\{|x| \leq Kr_n\}}(x) \, dx = (w_n - a)^p \mathbb{P}(U_{n,1} \times \cdots \times U_{n,p} \leq Kr_n),$$

where $U_{n,1}, \ldots, U_{n,p}$ are i.i.d. uniform random variables over the interval $(a, w_n)$. The density of products of i.i.d. uniform random variables can be derived by an induction method [29] (or, see [19] for a simple complex-analysis argument), and one can derive the corresponding cumulative distribution function. In particular, from [29, (3.9)] we have, for all $a > 0$ fixed,

$$G_n(a) := (w_n - a)^p \mathbb{P}(U_{n,1} \cdots U_{n,p} \leq Kr_n)$$

$$= \sum_{s=0}^p \binom{p}{s} (-1)^s \left( Kr_n \sum_{j=1}^{p-1} \frac{(-1)^{j+1}}{(p-j)!} \left( \log \frac{Kr_n}{a^{p-s} w_n^s} \right)_+^{p-j} + (-1)^{p-1} \left( Kr_n - a^{p-s} w_n^s \right)_+ \right)$$

$$\sim \frac{Kr_n}{(p-1)!} \sum_{s=0}^p (-1)^s \binom{p}{s} \left( \log \frac{r_n}{w_n^s} \right)_+^{p-1} \sim \frac{K}{(p-1)!} \sum_{s=0}^p (-1)^{p-s} \binom{p}{s} (-\beta_s)^{p-1} \frac{w_n^p}{n},$$

where for the last asymptotic equivalence we have used (4.4). Note that the last expression does not depend on $a$. In summary, we have proved $C_{n,K} \sim A_n \sim G_n/p! \sim G_n(1)/p!$ and the desired result follows.

\[\square\]

**Lemma 4.5.** For $r = 1, \ldots, p-1$, we have

$$C_{n,2}(r, K) \leq \begin{cases} 
\frac{C w_n^{2p}}{n^2} \log^{-2r} n, & \text{if } \beta_r > 0, \\
\frac{C w_n^{2p}}{n^2} \log^{-2r} n, & \text{if } \beta_r = 0, \\
\frac{C w_n^{2p-r}}{n} \log^{-r} n, & \text{if } \beta_r < 0.
\end{cases} \quad (4.24)$$

**Proof.** Write

$$C_{n,2}(r, K) = \sum_{i, i' \in \mathcal{H}(n, K)} \mathbf{1} \sum_{i' \in \mathcal{H}(n, K)} \mathbf{1} \sum_{i, i' \in \mathcal{H}(n, K)} \mathbf{1}$$
\[
\leq C\left(\frac{\beta}{r}\right) \sum_{i \in \mathcal{H}(n, K)} [\bar{t}_{r+1:p}] \leq C \sum_{i_{r+1} \leq \cdots \leq t_p \leq w_n} [\bar{t}_{r+1:p}] \left| D^r \left( \frac{r_n}{[\bar{t}_{r+1:p}]} \right) \right|
\]
\[
\leq Cr_n \log^{r-1} r_n \sum_{i_{r+1} \leq \cdots \leq t_p \leq w_n} 1 \leq Cr_n u_n^{p-r} \log^{r-1} r_n \sim C \frac{u_n^{2p-r}}{n} \log^{r-1} r_n,
\]
(4.25)

where we have used (4.21) in the first inequality of the last line.

On the other hand, using an integral re-expression of sum as in the proof of Lemma 4.3, and the fact \( u/2 \leq |v| \) for \( u \geq 1 \), we have (recall \( |v| \) denotes product of all components in \( v \))
\[
C_{n,2}(r, K) \leq \int_{x, y \in [1, \infty]^r} 1(|x||z| \leq 2^p K_{r_n}, |y||z| \leq 2^p K_{r_n}) dx dy dz
\]
\[
= \int_{x \in [1, \infty]^r} \left( \int_{y \in [1, \infty]^r} 1(|z| \leq 2^p K_{r_n}) dz \right)^2 dx
\]
\[
\leq C r_n^2 \log(r_n)^{(2p-r-1)} \int_{x \in [1, \infty]^r} 1(|z| \leq 2^p K_{r_n}) \frac{1}{|z|^2} dz \leq C r_n^2 \log(r_n)^{(2p-r-1)},
\]
(4.26)

where in the third step we applied (4.22).

With our choice of \( r_n, \log r_n \sim C \log n \). Then, the bound (4.25) (4.26) resp.) is sharper if \( \beta_r < 0 \) (\( \beta_r > 0 \) resp.). If \( r < p, \beta_r = 0 \), then (4.25) yields an upper bound \( C u_n^{2p} \log^{r-1} n/n^2 \), and (4.26) \( C u_n^{2p} \log^{2p-r} n/n^2 \). We simply use the latter upper bound for our purpose later, even it might not always be better than the former. The desired (4.24) now follows.

**Lemma 4.6.** For \( \rho_n \) in (4.14), and with any \( d_n \to \infty \) as \( n \to \infty \), we have
\[
\rho_n \sim \frac{1}{2} q_{F,p} \frac{n}{K_r u_n^p},
\]
(4.27)

where \( q_{F,p} \) is as in (2.2).

**Proof.** Introduce
\[
q_{n,1}^{(p)} := \mathbb{P} \left( R_n(1,\ldots,p) \cap \{1,\ldots,d_n\} \neq \emptyset, \max R_n(1,\ldots,p) \leq d_n \right),
\]
\[
q_{n,2}^{(p)} := \mathbb{P} \left( R_n(1,\ldots,p) \cap \{1,\ldots,d_n\} \neq \emptyset, \max R_n(1,\ldots,p) > d_n \right).
\]
Then, \( q_{n,1}^{(p)} \leq 2 \rho_n \leq q_{n,1}^{(p)} + q_{n,2}^{(p)} \). For \( q_{n,1}^{(p)} \) we can apply the last-renewal decomposition and Markov property:
\[
q_{n,1}^{(p)} = \sum_{i=1}^{d_n} \mathbb{P} \left( \max R_n(1,\ldots,p) = i \right)
\]
\[
= \sum_{i=1}^{d_n} \mathbb{P} \left( \max R_n(1,\ldots,p) = i \right) \mathbb{P} \left( i \in R_n(1,\ldots,p) \right) = \sum_{i=1}^{d_n} q_{F,p} \frac{1}{u_n} = \frac{q_{F,p} d_n}{u_n^p}.
\]

For \( q_{n,2}^{(p)} \), write first by a similar decomposition based on the last renewal before time \( d_n \),
\[
q_{n,2}^{(p)} = \sum_{i=1}^{d_n} \mathbb{P} \left( \max(R_n(1,\ldots,p) \cap \{1,\ldots,d_n\}) = i, \max R_n(1,\ldots,p) > d_n \right)
\]
\[
\leq \sum_{i=1}^{d_n} \mathbb{P} \left( i \in R_n(1,\ldots,p), \max R_n(1,\ldots,p) > d_n \right)
\]
\[
= \sum_{i=1}^{d_n} \mathbb{P} \left( i \in R_n(1,\ldots,p) \right) \mathbb{P} \left( \max R_n(1,\ldots,p) > d_n \mid i \in R_n(1,\ldots,p) \right) \leq \sum_{i=1}^{d_n} \frac{1}{u_n^p} \sum_{j=d_n-i+1}^{\infty} u(j)^p.
\]
The last step above follows from the renewal property and then the union bound. With \( v(i) := \sum_{j=1}^{\infty} u(j)^p \downarrow 0 \) as \( i \to \infty \), the last expression becomes \( w_n^{1-p} \sum_{i=1}^{d_n} v(i) = d_n w_n^{1-p} \sum_{i=1}^{d_n} (v(i)/d_n) = d_n w_n^{1-p} o(1) = o(q_{n,1}^{(p)}) \). We have thus proved (4.27).

**Lemma 4.7.** With \( \beta_p < 0 \), if \( d_n \to \infty, n/d_n \to \infty \) as \( n \to \infty \), then \( \rho_n(r) \) in (4.16) satisfies

\[
\rho_n(r) \leq \begin{cases} 
\frac{Cn^{1+\beta_p}}{w_n^{2p-r}}, & \text{if } \beta_r > 0, \\
\frac{Cn \log n}{w_n^{2p-r}}, & \text{if } \beta_r = 0, \\
\frac{Cn}{w_n^{2p-r}}, & \text{if } \beta_r < 0, \quad r < p, \\
\frac{Cn d_n^\beta}{w_n^p} \log d_n, & \text{if } r = p.
\end{cases}
\]

(4.28)

In fact, the \( \log d_n \) term in the case \( r = p \) can be dropped if \( \beta_p \neq -1 \). We keep it for all values of \( \beta_p \) for the sake of simplicity.

**Proof.** We shall often use the following fact: \( u(n) \leq Cn^{\beta-1} \) for some constant \( C > 0 \) for all \( n \in \mathbb{N} \) (recall (2.1)). We first show that for all \( i, i' \),

\[
\sum_{j,j'=1,\ldots,k_n \atop |j-j'|>1} \rho_n,i,i',j,j' \leq \frac{Cn^{1+\beta_p}}{w_n^{2p-r}}, \quad \text{if } \beta_r > 0, \\
\frac{Cn \log k_n}{w_n^{2p-r}}, \quad \text{if } \beta_r = 0, \quad \text{with } r = |i \cap i'|, \\
\frac{Cn \log k_n}{w_n^{2p-r}}, \quad \text{if } \beta_r < 0,
\]

(4.29)

The relation above can be obtained in the following a unified argument for all 3 cases. Write

\[
\rho_n,i,i',j,j' \leq \mathbb{P}(R_{n,i} \cap \mathcal{I}_{n,j} \neq \emptyset, R_{n,i'} \cap \mathcal{I}_{n,j'} \neq \emptyset) \leq \sum_{k \in \mathcal{I}_{n,j}} \sum_{k' \in \mathcal{I}_{n,j'}} \mathbb{P}(k \in R_{n,i}, k' \in R_{n,i'})
\]

\[
= \frac{1}{w_n^{2p-2r}} \sum_{k \in \mathcal{I}_{n,j}} \sum_{k' \in \mathcal{I}_{n,j'}} ^{\mathcal{I}_{n,j}} \mathbb{P}(k, k' \in R_{n,i',\gamma'}) = \frac{1}{w_n^{2p-2r}} \sum_{k \in \mathcal{I}_{n,j}} \sum_{k' \in \mathcal{I}_{n,j'}} \left( \frac{u(|k-k'|)}{w_n} \right)^r
\]

\[
\leq \frac{C_n \beta}{w_n^{2p-r}} \left( \frac{(d_n(|j-j'|-1))^\beta}{w_n} \right)^r \leq \frac{C_n 1+\beta_p}{w_n^{2p-r}} |j-j'|^{\beta_p-1}, \quad |j-j'| > 1,
\]

and hence

\[
\sum_{j,j'=1,\ldots,k_n \atop |j-j'|>1} \rho_n,i,i',j,j' \leq \frac{C_n 1+\beta_p}{w_n^{2p-r}} \sum_{j=1}^{k_n} (k_n-j) j^{\beta_p-1}.
\]

Note \( k_n \sim n/d_n \to \infty \). Now, (4.29) follows by applying

\[
\sum_{j=1}^{n} (n-j)^{\gamma-1} \leq \begin{cases} 
Cn^{\gamma+1}, & \text{if } \gamma > 0, \\
Cn \log n, & \text{if } \gamma = 0, \\
Cn, & \text{if } \gamma < 0.
\end{cases}
\]
For $|j - j'| \leq 1$, consider first $j' = j + 1$. Then, with $r = |i \cap i'|$,

$$
\rho_{n,i,i',j,j+1} \leq \sum_{\ell=1}^{2d_n} \sum_{k \in \mathbb{Z}_{n,j}, k' \in \mathbb{Z}_{n,j+1}, |k-k'|=\ell} \frac{u(\ell)^r}{w_n^{2p-r}} \leq \frac{C}{w_n^{2p-r}} \sum_{\ell=1}^{2d_n} \ell \cdot |\rho^{j,j+1}|^{-1} \leq \begin{cases} \frac{Cd_n^{\beta_r+1}}{w_n^{2p-r}}, & \text{if } \beta_r > -1, \\ \frac{C\log d_n}{w_n^{2p-r}}, & \text{if } \beta_r = -1, \\ \frac{C}{w_n^{2p-r}}, & \text{if } \beta_r < -1. 
\end{cases}
$$

Similarly,

$$
\rho_{n,i,i',j,j} \leq \sum_{\ell=1}^{d_n} \sum_{k \in \mathbb{Z}_{n,j}, |k-k'|=\ell} \frac{u(\ell)^r}{w_n^{2p-r}} \leq \frac{C}{w_n^{2p-r}} \sum_{\ell=1}^{d_n} (d_n - \ell) |\rho^{j,j'}|^{-1} \leq \begin{cases} \frac{Cd_n^{\beta_r+1}}{w_n^{2p-r}}, & \text{if } \beta_r > 0, \\ \frac{Cn \log d_n}{w_n^{2p-r}}, & \text{if } \beta_r = 0, \\ \frac{Cn}{w_n^{2p-r}}, & \text{if } \beta_r < 0, 
\end{cases}
$$

One readily checks that the bounds for $\rho_{n,i,i',j,j}$ are of equal or larger order than those for $\rho_{n,i,i',j,j+1}$, regardless of the values of $\beta_r$. Therefore, we arrive at

$$
\sum_{j,j'=1,\ldots,\kappa_n \atop |j-j'|\leq 1} \rho_{n,i,i',j,j'} \leq Cn \rho_{n,i,i',1,1} \leq \begin{cases} \frac{Cd_n^{\beta_r}}{w_n^{2p-r}}, & \text{if } \beta_r > 0, \\ \frac{Cn \log d_n}{w_n^{2p-r}}, & \text{if } \beta_r = 0, \\ \frac{Cn}{w_n^{2p-r}}, & \text{if } \beta_r < 0, r < p. 
\end{cases}
$$

(4.30)

For the case $r = p$, recall that the summation now excludes $j = j'$ and hence in addition to (4.29) we only consider $|j-j'| = 1$. Then, by the bound on $\rho_{n,i,i',j,j+1}$ above, we have

$$
\sum_{j,j'=1,\ldots,\kappa_n \atop |j-j'|=1} \rho_{n,i,i',j,j'} \leq \frac{Cn d_n^{(|\beta_r|+1)}}{w_n^{p}} \log d_n, 
$$

(4.31)

and the $\log d_n$ factor can be dropped if $\beta_p \neq -1$. Combining (4.29), (4.30) and (4.31), we obtain (4.28). □

Proof of Lemma 4.2. Now we complete the proof of (4.20). It is straightforward to show

$$
\lim_{n \to \infty} \mathbb{E} \xi_{n,K}(E_s) = \mathbb{E} \xi_{K}(E_s), \ s = 1, \ldots, d,
$$

Indeed, consider $E = (y_0, y_1) \times (a, b)$ or $E = (y_0, y_1) \times [0, b]$ (with $a = 0$ in the latter case). Then from (4.17), by Lemma 4.4 (with $y_i^{-1/\alpha}, i = 0, 1$, playing the role of $K$) and the restriction $K^{-1/\alpha} < y_0 < y_1$, Lemma 4.6 and the restriction $K^{-1/\alpha} < y_0 < y_1$, we have

$$
\mathbb{E} \xi_{n,K}(E) \sim (C_{n,1}(y_0^{-\alpha}) - C_{n,1}(y_1^{-\alpha})) \cdot (b - a) k_n \cdot \rho_n
$$

$$
\sim (y_0^{-\alpha} - y_1^{-\alpha}) \frac{D_{\beta,p}}{p!(p-1)!} \frac{w_n^p}{n} \cdot (b - a) k_n \cdot \frac{1}{2} q_{F,p} \aleph_n \frac{n}{k_n w_n^p}
$$

$$
\to \frac{1}{2p!(p-1)!} (y_0^{-\alpha} - y_1^{-\alpha}) D_{\beta,p} q_{F,p} (b - a) = \mathbb{E} \xi_{K}(E).
$$
Next, among the bounds for $C_{n,2}(r, K)$ in Lemma 4.5, the $r = 1$ (note that $\beta_1 = \beta > 0$) case is of the dominating order for $r = 1, \ldots, p - 1$. Recall also that $C_{n,2}(p, K) = C_{n,1}(K)$, for which we have the precise estimate in Lemma 4.4. Combining these into (4.18) we have

$$
\begin{align*}
b_{n,1}(K) &\leq k_n^2 (CC_{n,2}(1, K) + C_{n,1}(K)) \rho_n^2 \\
&\leq CK_n^2 \left( \frac{w_n^p}{n^q} \log^{-2} n + r_n \log^{p-1} n \right) \left( \frac{n}{\kappa_n w_n} \right)^2 \leq C \left( \frac{1}{\log^2 n} + n^\beta \right) \to 0
\end{align*}
$$

as $n \to \infty$, where for the last inequality we have used (4.4). At last by Lemmas 4.4, 4.5, 4.7, and the assumption $w_n \sim \frac{C}{1-\beta} n^{1-\beta}$, and in view of (4.19), we have $\lim_{n \to \infty} b_{n,2}(K) = 0$. We have thereby completed the proof of (4.20) and hence (4.11).

The fact that (4.11) implies the weak convergence of the point processes $\tilde{\eta}_{n,K} \Rightarrow \xi_K$ in $\mathfrak{M}_p((0, \infty] \times [0, 1])$ as $n \to \infty$ is well-known (e.g., by combining Kallenberg [31, Theorem 4.11(ii)] with Cramér–Wold Theorem.) Note that we shall need to apply (4.11) with a relaxed assumption on the test sets $E = (y_0, y_1] \times \tilde{I}$ so that $y_0 > 0$ (instead of $y_0 > K^{-1/\alpha}$ above). The modification does not effect the proof above as for the point processes of interest, there are no points of which the first coordinate has a value less than $K^{-1/\alpha}$ by our construction.

\section*{4.2. Proof of Proposition 4.1}

Recall the overview of the approximations we shall work with in (4.7).

Next, we approximate $\xi_{n,K}$ by $\tilde{\eta}_{n,K}$, with an intermediate approximation by $\eta_{n,K}$ as seen in the proof. Recall the set $E$ in (4.8). Introduce accordingly

$$
E^{\pm,\epsilon} := (y_0(1 + \epsilon)^{1/\alpha}, y_1(1 + \epsilon)^{1/\alpha}] \times \tilde{I},
$$

where $\tilde{I} = (a, b]$ or $[0, b]$ (with $a = 0$ in the latter case), $0 \leq a < b \leq 1$. We shall assume $0 < y_0 < y_1$ and eventually $\epsilon > 0$ small enough so that $y_0(1 + \epsilon)^{1/\alpha} < y_1(1 - \epsilon)^{1/\alpha}$. Otherwise, by convention $E^{+,\epsilon} = \emptyset$.

**Lemma 4.8.** With the notations above, for all $\epsilon > 0$, with probability going to 1 as $n \to \infty$,

$$
\tilde{\eta}_{n,K}(E^{+,\epsilon}) \leq \xi_{n,K}(E) \leq \tilde{\eta}_{n,K}(E^{-,\epsilon}).
$$

**Proof.** Consider the following events

$$
\begin{align*}
\Omega_{n,1}(K) &:= \bigcap_{j=1}^{k_n} \left\{ \exists \text{ at most one } i \in \mathcal{H}(n, K) \text{ s.t. } R_{n,i} \cap I_{n,j} \neq \emptyset \right\}, \\
\Omega_{n,2}(K, \epsilon) &:= \left\{ \frac{|I_i|}{|I_j|} \in (1 - \epsilon, 1 + \epsilon), \text{ for all } i \in \mathcal{H}(n, K), R_{n,i} \neq \emptyset \right\}.
\end{align*}
$$

Recall $\tilde{m}_{n,j}^{(K)}(K) = \max_{k \in \mathbb{Z}} \sum_{i \in \mathcal{H}(n, K)} \frac{|I_i|}{|I_j|} 1_{k \in R_{n,i}}$. Fix $K > 0$. Then, we claim that under $\Omega_{n,1}(K)$, for each $j = 1, \ldots, k_n$, the following two subsets of $(0, \infty)$ coincide:

$$
\{ \tilde{m}_{n,j}^{(K)} : \tilde{m}_{n,j}^{(K)} > 0 \} = \left\{ \left( |I_i|/r_n \right)^{-1/\alpha} : i \in \mathcal{H}(n, K), R_{n,i} \cap I_{n,j} \neq \emptyset, |\varepsilon_i| = 1 \right\}.
$$

In particular, both are empty if for any $i \in \mathcal{H}(n, K)$, we have either $R_{n,i} \cap I_{n,j} = \emptyset$ or $|\varepsilon_i| = -1$; otherwise, both are the same singleton set $\{ \left( |I_i|/r_n \right)^{-1/\alpha} \}$ where $i$ is the unique index under $\Omega_{n,1}(K)$ for which $R_{n,i} \cap I_{n,j} \neq \emptyset$. It follows that $\xi_{n,K} = \eta_{n,K}$ under $\Omega_{n,1}(K)$.
Next, note that the only difference between $\overline{\eta}_{n,K}$ and $\eta_{n,K}$ is that for each $i$, the random $|\Gamma_i|$ in $\eta_{n,K}$ is replaced by the nonrandom $[i]$ in $\overline{\eta}_{n,K}$. If restricted to $\Omega_{n,1}(K, \epsilon) \cap \Omega_{n,2}(K, \epsilon)$, it easily follows from monotonicity that $\overline{\eta}_{n,K}(E^{\epsilon, \gamma}) \leq \eta_{n,K}(E^{\epsilon, \gamma})$.

Combining the above, to prove the desired result, it remains to show that the events $\Omega_{n,1}(K)$ and $\Omega_{n,2}(K)$ each occur with probability tending to one as $n \to \infty$.

We first prove $\lim_{n \to \infty} P(\Omega_{n,1}(K)) = 1$. This follows from the fact that the limit of $\overline{\eta}_{n,K}$ is Poisson. A more detailed argument can be given by using the estimates needed in the Poisson approximation. Indeed, setting $N_n(K) = \sum_{j=1}^{k_n} \sum_{i \neq i' \in \mathcal{H}(n, K)} 1\{R_{n,i} \cup \mathcal{I}_{n,j} \neq \emptyset, R_{n,i'} \cup \mathcal{I}_{n,j} \neq \emptyset\}$, then

$$
\mathbb{P}(\Omega_{n,1}(K)) = \mathbb{P}(N_n(K) \geq 1) \leq \mathbb{E}N_n(K) = \sum_{j=1}^{k_n} \sum_{i \neq i' \in \mathcal{H}(n, K)} \rho_n(i, i', j, j) \leq C_n(K)^2 k_n \rho_n^2 + \sum_{r=1}^{p-1} C_n(K) \rho_n(r)
$$

where the last bound is obtained by dividing the double sum over $(i, i')$ into two cases: $i \cap i' = \emptyset$ and $i \cap i' \neq \emptyset$. The first term in the bound, in view of Lemmas 4.4 and 4.6, is bounded up to a constant by $k_n^{-1} \to 0$ as $n \to \infty$. The second term in the bound is $b_{n,2}(K)$ in (4.19) which tends to zero as $n \to \infty$ as already shown.

At last, we show that $\lim_{n \to \infty} \mathbb{P}(\Omega_{n,2}(K, \epsilon)) = 1$. As an intermediate step, we claim that for all $m \in \mathbb{N}$ fixed,

$$
\lim_{n \to \infty} \mathbb{P}(\exists i = (i_1, \ldots, i_p) \in \mathcal{H}(n, K), \text{ s.t. } R_{n,i} \neq \emptyset, i_1 \leq m) = 0.
$$

Indeed, by a union bound, Lemma 4.6, (4.4) and (4.23) ($p$ there replaced by $p - 1$), the probability displayed above is bounded by

$$
nw_n^{-p} \left| \left\{ i \in \mathcal{H}(n, K) : i_1 \leq m \right\} \right| \leq nw_n^{-p} m \left| \left\{ (i_2, \ldots, i_p) \in \mathcal{D}_p-1(Kr_n) : i_p \leq w_n \right\} \right| \leq \frac{C}{\log(n)} \to 0
$$

as $n \to \infty$. Then, the desired result follows from the strong law of large numbers: $\lim_{m \to \infty} \sup_{i > m} |\Gamma_i/i - 1| = 0$ almost surely.

**Proof of Proposition 4.1.** Consider any collection of $E_1, \ldots, E_d$ as in Lemma 4.2 above. Then, based on Lemmas 4.8 and 4.2, the almost sure convergence $\xi_K(E_\epsilon^{\epsilon, \gamma}) \to \xi_K(E_\epsilon)$ as $\epsilon \downarrow 0$, we conclude that as $n \to \infty$,

$$
(\xi_{n,K}(E_1), \ldots, \xi_{n,K}(E_d)) \Rightarrow (\xi_K(E_1), \ldots, \xi_K(E_d)).
$$

The desired result now follows.

**4.3. Proof of (4.1)**

We first provide a uniform control on the remainder of the truncation.

**Lemma 4.9.** We have

$$
\lim_{K \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \frac{1}{\epsilon_n} \max_{k=1, \ldots, n} \left| X_{n,k} - X_{n,k}^{(K)} \right| > \epsilon \right) = 0 \text{ for all } \epsilon > 0.
$$

**Proof.** Recall $X_{n,k}^{(K)} = \sum_{i \in \mathcal{H}(n, K)} \frac{|\epsilon|}{|\Gamma_i|} 1\{k \in R_{n,i}\}$. With probability tending to one as $n \to \infty$, we claim that

$$
X_{n,k}^{(K)} = \overline{X}_{n,k}^{(K)} = \sum_{i \in \mathcal{D}_p^*(Kr_n)} \frac{|\epsilon|}{|\Gamma_i|} 1\{k \in R_{n,i}\}, \ k = 1, \ldots, n.
$$

Indeed, Note that if $\epsilon_n \leq C w_n$ for some constant $C > 0$ (equivalently $\beta_{p-1} \geq 0$), then for $n$ large enough, $\mathcal{H}(n, K) = \mathcal{D}_p^*(Kr_n)$, and there is nothing to prove. So assume $\beta_{p-1} < 0$. The key observation is that for $X_{n,k}^{(K)}$ and $\overline{X}_{n,k}^{(K)}$ to differ
for some \( k \), then necessarily there exists \( \mathbf{i} = (i_1, \ldots, i_p) \in D^*_p(Kr_n) \) such that \( i_p > w_n \) and \( k \in R_n,i \). However, \( i_p > w_n \) implies necessarily that \( (i_1, \ldots, i_{p-1}) \in D^*_{p-1}(Kr_n/w_n) \). But,

\[
\mathcal{D}^*_p(Kr_n/w_n) := \{ \mathbf{i} \in D^*_{p-1}(Kr_n/w_n) : R_n,i \neq \emptyset \}
\]

is an empty set with probability tending to one as \( n \to \infty \), in the case \( \beta_{p-1} < 0 \). Indeed, using Lemma 4.6 (with \( d_n = n \), (4.21) and (4.4), we have

\[
\mathbb{E} \left| \mathcal{D}^*_p(Kr_n/w_n) \right| \leq C \left| D^*_{p-1}(Kr_n/w_n) \right| \frac{n}{w_n^{p-1}} \leq C \frac{r_n}{w_n^{p-2}} r_n \leq \frac{C}{\log n} \to 0
\]

as \( n \to \infty \).

Therefore, it suffices to show (4.33) with \( X_{n,k}^{(r)} \) replaced by \( \bar{X}_{n,k} \). To do so, we introduce fixed \( m \in \mathbb{N} \) to be chosen later. By (4.4), a triangular inequality, and the inequality \( \max_k (\sum_\ell a_{k\ell}) \leq \sum_\ell \max_k (a_{k\ell}) \), \( a_{k\ell} \in \mathbb{R} \), we have

\[
\frac{1}{c_n} \max_{k = 1, \ldots, n} X_{n,k} - \bar{X}_{n,k} \leq \max_{k = 1, \ldots, n} \sum_{\mathbf{i} \in D_p} D^*_E(Kr_n) \left( \frac{r_n^{1/\alpha}}{\Gamma_k^{1/\alpha}} \right) 1_{\{k \in R_n,i\}} \left( \mathbf{i} \right) 
\]

\[
\leq \sum_{q = 0}^{p-1} \sum_{i_1 < \cdots < i_q \leq m} \left( \frac{1}{\Gamma_{i_1}^{1/\alpha}} \right) Z_{n,i_1,\ldots,i_q}(K) 
\]

with

\[
Z_{n,i_1,\ldots,i_q}(K) := \max_{k = 1, \ldots, n} \left| \sum_{m \leq i_{q+1} < \cdots < i_p > K_r} \frac{[\xi_{i_{q+1}+p}]}{\Gamma_{i_{q+1}+p}^{1/\alpha}} 1_{\{k \in R_n,i\}} \right| , \quad q = 0, \ldots, p - 1. 
\]

When \( q = 0 \), we understand \( Z_{n,i_1,\ldots,i_q}(K) \) above as

\[
Z_{n,0}(K) := \max_{k = 1, \ldots, n} \left| \sum_{m < i_1 < \cdots < i_p > K_r} \frac{[\xi_1]}{\Gamma_1^{1/\alpha}} 1_{\{k \in R_n,i\}} \right| . 
\]

Therefore, it suffices to show that for each \( q = 0, \ldots, p - 1 \) and \( i_1 < \cdots < i_q \leq m \) fixed,

\[
\lim_{K \to \infty} \lim_{n \to \infty} \mathbb{P} \left( Z_{n,i_1,\ldots,i_q}(K) > \epsilon \right) = 0 \quad \text{for all } \epsilon > 0. 
\]

Bounding the max by a sum, using the orthogonality induced by \( [\xi_{i_{q+1}+p}] \), the fact \( P(k \in R_n,i) = w_n^{-p} \) and (3.6) (for which it suffices to choose \( m \geq 2p/\alpha \)), we have

\[
\mathbb{E} Z_{n,i_1,\ldots,i_q}(K)^2 \leq \mathbb{E} \left( \max_{k = 1, \ldots, n} \left| \sum_{m < i_{q+1} < \cdots < i_p > K_r} \frac{[\xi_{i_{q+1}+p}]}{\Gamma_{i_{q+1}+p}^{1/\alpha}} 1_{\{k \in R_n,i\}} \right| ^2 \right) 
\]

\[
\leq C \frac{n^{2/\alpha}}{w_n^{p/\alpha}} \sum_{m < i_{q+1} < \cdots < i_p > K_r/m^q} (i_{q+1} \cdots i_p)^{-2/\alpha}. 
\]

Now we estimate the multiple sum above by an integral approximation and suitable change of variables:

\[
\sum_{m < i_{q+1} < \cdots < i_p > K_r/m^q} (i_{q+1} \cdots i_p)^{-2/\alpha} \leq \int_{[m,\infty)^{p-q}} (x_{q+1} \cdots x_p)^{-2/\alpha} 1_{\{(x_{q+1}+1) \cdots (x_p+1) > K_r/m^q\}} dx
\]

\[
= m^{(p-q)(1-2/\alpha)} \int_{[0,\infty)^{p-q}} e^{(1-2/\alpha)(y_1+\cdots+y_{p-q})} 1_{\{(m+1) \cdots (m+p-q+1) > K_r/m^q\}} dy
\]
\[
\leq m^{(p-q)(1-2/\alpha)} \int_{[0,\infty)^{p-q}} e^{(1-2/\alpha)(y_1+\cdots+y_{p-q})} 1_{\{y_1+\cdots+y_{p-q}>\log(Kr_n)-p\log(m+1)\}} dy
\]
\[
\leq C \int_{\log(Kr_n)}^\infty e^{(1-2/\alpha)z} z^{-p-q-1} dz \leq C(Kr_n)^{1-2/\alpha} \log^{p-q-1}(Kr_n),
\]
where in the second inequality above we used \(me^{y_j} + 1 \leq (m+1)e^{y_j}, j = 1, \ldots, p-q\). Hence with (4.4) one concludes
\[
\mathbb{E}Z_{n,i_1,\ldots,i_q}(K)^2 \leq Cn^{-p/\alpha}(Kr_n)^{1-2/\alpha} \log^{p-q-1}(Kr_n)
\]
\[
\leq \frac{CK^{1-2/\alpha} \log^{p-q-1}(K)}{\log^{p-q-1} n}, \quad q = 0, \ldots, p-1. \quad (4.36)
\]
So (4.35) follows from Markov inequality, and hence the whole proof is finished.

**Proof of (4.1).** Since the limit \(\mathcal{M}_n^\alpha(I) > 0\) almost surely for any \(I \in \mathcal{G}_0\), it suffices to show for any \(x_i > 0\) and any \(I_i \in \mathcal{G}_0, i = 1, \ldots, d\), that
\[
\lim_{n \to \infty} P\left(\frac{1}{c_n} M_n(I_i) \leq x_i, i = 1, \ldots, d\right) = P\left(\mathcal{M}_n^\alpha(I_i) \leq x_i, i = 1, \ldots, d\right)
\]
\[
= P\left(\xi \left(\bigcup_{i=1}^d ((x_i, \infty] \times I_i)\right) = 0\right). \quad (4.37)
\]
We have established Proposition 4.1, which implies that
\[
\lim_{n \to \infty} P\left(\xi_{n,K} \left(\bigcup_{i=1}^d ((x_i, \infty] \times I_i)\right) = 0\right) = \lim_{n \to \infty} P\left(\frac{1}{c_n} \max_{j/k_n \in G_i} \tilde{m}_{n,j}^{(K)} \leq x_i, i = 1, \ldots, d\right)
\]
\[
= \lim_{n \to \infty} P\left(\frac{1}{c_n} \max_{j/k_n \in I_i} \tilde{m}_{n,j}^{(K)} \leq x_i, i = 1, \ldots, d\right) = P\left(\xi_K \left(\bigcup_{i=1}^d ((x_i, \infty] \times I_i)\right) = 0\right). \quad (4.38)
\]
where in the first equality in the line of (4.38), we have used the assumption \(x_i > 0, i = 1, \ldots, d\) and the convention \(\max_0 = -\infty\). This will be shown to imply that
\[
\lim_{n \to \infty} P\left(\frac{1}{c_n} \max_{k/n \in G_i} \tilde{X}_{n,k}^{(K)} \leq x_i, i = 1, \ldots, d\right) = P\left(\xi_K \left(\bigcup_{i=1}^d ((x_i, \infty] \times I_i)\right) = 0\right). \quad (4.39)
\]
Indeed, for any open sub-interval \(I \subset [0,1]\) and large enough \(n\), the two terms \(c_n^{-1} \max_{k/n \in I} X_{n,k}^{(K)}\) and \(c_n^{-1} \max_{j/k_n \in G_i} \tilde{m}_{n,j}^{(K)}\) could differ due to two possibilities: first, the sets \(\{k \in \mathbb{N} : k/n \in I\}\) and \(\{k \in \mathcal{T}_n,j : j/k_n \in I\}\) possibly differ near the boundaries of \(I\) with at most \(O(n)\) different indices; second, if \(\sup I = 1\), the first term involves also the last partial block \(\mathcal{T}_n,n+1 := \{d/n, n+1, \ldots, n\}\). Suppose first \(I = (a,b)\) with \(0 < a < b < 1\). It is clear that for \(\epsilon > 0\) small enough so that \(0 < a - \epsilon < a < b - \epsilon < b + \epsilon < 1\), and \(n\) large enough, we have
\[
\max_{k/n \in (a+\epsilon, b-\epsilon)} \tilde{m}_{n,j}^{(K)} \leq \max_{k/n \in (a,b)} \tilde{m}_{n,j}^{(K)}.
\]
Therefore, (4.39) follows from a sandwich approximation argument based on (4.38). Similar arguments applies if \(0 = \inf I < \sup I < 1\). When \(\sup I = 1\), it suffices to additionally apply the following fact due to stationarity: \(c_n^{-1} \max_{k/n \in I_{n,k_n+1}} X_{n,k}^{(K)} = c_n^{-1} \max_{k/n \in I_{n,k_n+1}} X_{n,k}^{(K)} = o_p(1)\), where the \(o_p(1)\) claim again follows from a similar approximation argument using (4.38).

Now (4.39) has been verified. The desired convergence in (4.37) follows from a standard triangular-array approximation (e.g. [13, Theorem 3.2]) based on Lemma 4.9. To do so, notice that when \(n\) is large enough so that each of the \(\max_{k/n \in I}\) below is over an nonempty set, we have
\[
\left|\frac{1}{c_n} \max_{k/n \in I} X_{n,k} - \frac{1}{c_n} \max_{k/n \in I} X_{n,k}^{(K)}\right| \leq \frac{1}{c_n} \max_{k=1,\ldots,n} \left|X_{n,k} - X_{n,k}^{(K)}\right|, \quad i = 1, \ldots, d.
\]
In addition, one readily checks \(\xi_K \Rightarrow \xi\) so that the last expression of (4.38) tends to the last expression of (4.37) as \(K \to \infty\). Hence the triangular-array approximation is justified.
4.2 We emphasize that we use the same symbols for random variables $\xi_{n,K}, \eta_{n,K}, \overline{\eta}_{n,K}$ defined as in the previous section, but now with the rates chosen above.

The key behind (5.1) is again the following (in place of Proposition 4.1), but now with a different normalizing constant in the limit $\xi$ and hence $\xi_K$.

**Proposition 5.1.** With $\beta \neq 0$, $d_n \to \infty$, 

\[
\xi_{n,K} := \sum_{j=1}^{k_n} \delta\left(\frac{\hat{\mathcal{M}}_{n,j}}{c_n}, j/k_n\right) 1\{\hat{\mathcal{M}}_{n,j} > 0\} \Rightarrow \xi_K := \sum_{\ell=1}^{\infty} \delta\left(\mathcal{C}_{F,p}^\alpha \frac{1}{\ell} \mathcal{F}_{\ell} \frac{\log n}{\log \log n} \mathcal{M}_{\ell} \right) 1\{\mathcal{C}_{F,p}^\alpha \frac{1}{\ell} \mathcal{F}_{\ell} \frac{\log n}{\log \log n} \mathcal{M}_{\ell} > K\},
\]
in $\mathcal{M}_{p}((0, \infty] \times [0, 1])$, where $\mathcal{C}_{F, p}$ is as in (5.2).

The proof is similar as the one with $\beta_{p} < 0$ in Section 4, and proceeds by a series of approximations as summarized below in place of (4.7) and (4.10). For the proof of Proposition 5.1, for some test set $E$ as in (4.8), we have

$$
\xi_{n, K}(E) \approx \eta_{n, K}(E) \approx \xi_{n, K}(E) \Rightarrow \xi_{K}(E).
$$

Then, for the proof of, say the marginal convergence in (5.1), this time we have

$$
P(M_{n}(G) \leq y) \approx P(M_{n, K}(G) \leq y) \approx P(\xi_{n, K}((y, \infty] \times G) = 0).
$$

Lemmas 5.10, 5.4 and 5.11 respectively correspond to Lemmas 4.8, 4.2 and 4.9 for the subcritical case $\beta_{p} < 0$ in Section 4.

**Remark 5.2.** Despite the fact that in both regimes the limit extremes are independently scattered, geometries underlying the Poisson approximation for local extremes are different. Therefore so are the proofs for Lemmas 5.4 and 5.11 and Lemmas 4.2 and 4.9 mentioned above. The different geometries can be viewed from the following two aspects.

(a) The shape of the region that contributes in each regime is different. In the sub-critical regime, it is possible that $r_{n} \gg w_{n}$ (this is the case if $\beta_{p} = 0$, and the shape has a delicate influence in the limit through the constant $D_{\beta, p}$), while at critical regime $r_{n} \ll w_{n}$ by our assumption. In particular, for $n$ large enough $\mathcal{H}(n, K) = D_{p}^{*}(K r_{n})$ and there is no need to work with $\mathcal{H}(n, K)$.

(b) Second, in terms of the size $d_{n}$, reflecting the size of each local extremal cluster, we see that the order is drastically different from the sub-critical regime. Recall that in that regime only $d_{n} \to \infty$ and $n/d_{n} \to \infty$ are assumed. Here, however, we have more restrictive constraints on both upper and lower bounds of the rate. Indeed, in order to apply the two-moment method we actually need (see Lemmas 5.5 and 5.6 below)

$$
k_{n}^{\beta_{p} - 1} = n^{1/p} \geq C \log n \quad \text{and} \quad \log k_{n} = o(\log d_{n}). \tag{5.5}
$$

For a lower bound on the rate of $d_{n} \to \infty$, the first part of Lemma 5.6 enforces the constraint

$$
C_{n, 1}(K) k_{n} \mathbb{P}(R_{n, i} \cap I_{n, 1}) \sim C_{n} \frac{\log n \cdot k_{n} d_{n}}{n \log d_{n}} = O(1),
$$

which implies already a lower bound $\log d_{n} \geq C \log n$ on the rate $d_{n} \to \infty$. The estimate above is sharp and hence $d_{n} \to \infty$ has to grow at least at a polynomial rate, which is different from the sub-critical regime.

**Remark 5.3.** The more restrictive second constraint in (5.5) on the lower bound, coming from (5.8), requires that $d_{n} \sim n L(n)$ for some slowly varying function $L$ as $n \to \infty$. We do not know whether (5.8) is optimal. For the upper bound on $d_{n} \to \infty$, the first constraint in (5.5), coming from Lemma 5.9 below, is more restrictive than $n/d_{n} \to \infty$. We do not know if our estimate on this part is optimal either.

### 5.1. Two-moment method

**Lemma 5.4.** With $\beta_{p} = 0$, we have

$$
\mathcal{P}_{n, K} = \sum_{j=1}^{k_{n}} \sum_{i \in D_{\beta, p}^{*}(n, K)} \delta_{\left(\left(\frac{i}{r_{n}}\right)^{-1} = j/k_{n}\right)} \mathbb{1}_{\{R_{n, i} \cap I_{n, j} \neq \emptyset, |i| = 1\}} \Rightarrow \xi_{K}
$$

as $n \to \infty$ in $\mathcal{M}_{p}((0, \infty] \times [0, 1])$.

Again we check the conditions of two-moments method as in (4.20). Recall that $C_{n, 1}(K)$, $C_{n, 2}(r, K)$, $\rho_{n}$ and $\rho_{p}(r)$ are defined the same way as before (see (4.12), (4.13), (4.14), (4.15)), but with different rates $r_{n}$, $c_{n}$, $d_{n}$, $k_{n}$. The following estimates collect results that are similar as in Section 4.

**Lemma 5.5.** We have

$$
C_{n, 1}(K) \sim \frac{K}{p! (p - 1)!} \left(\frac{C_{F}}{1 - \beta}\right)^{p} \log n,
$$
and
\[ C_{n,2}(r, K) \leq C \frac{\log^2 n}{(\log \log n)^{2r}}, \quad r = 1, \ldots, p - 1. \] (5.6)

Furthermore, as \( n \) and \( d_n \to \infty \),
\[ \rho_n = \frac{1}{2} \cdot \mathbb{P}\left( R_{n,i} \cap \mathcal{I}_{n,j} \neq \emptyset \right) \sim \frac{1}{2} \frac{d_n}{w_n} \left( C \frac{\Gamma(\beta) \Gamma(1 - \beta)}{\log d_n} \right)^p. \] (5.7)

**Proof.** Recall in this case \( \mathcal{H}(n, K) = \mathcal{D}_p(Kr_n) \) when \( n \) is large enough and \( r_n \) is now as in (5.4). The first relation follows from Lemma 4.3. The second relation (5.6) follows from (4.26). For the third relation, we first recall the theorem in the \( p \)-intersection of renewal processes, and \( \mathcal{F}_p \) in (2.4). Thus, using a last-renewal decomposition, we have
\[ \text{\textbf{Proof.}} \]

The estimate of \( \rho_n(r) \) in (4.16) is summarized below.

**Lemma 5.6.** Furthermore,
\[ \rho_n(r) \leq \begin{cases} \frac{C}{d_n^{r-1} \log d_n} + \frac{C}{\log^2 d_n}, & \text{if } r = 1, \ldots, p - 1, \\ \frac{C}{\log^{3/2} d_n} + \frac{C \log k_n}{\log^2 d_n}, & \text{if } r = p. \end{cases} \]

The proof is divided into the following three lemmas. We first estimate the summation “off the diagonal” in (4.16).

**Lemma 5.7.** We have for \( r := |i \cap i'| = 1, \ldots, p - 1, \)
\[ \sum_{j,j' = 1, k_n \atop |j-j'| \geq 1} \mathbb{P}\left( R_{n,i} \cap \mathcal{I}_{n,j} \neq \emptyset, R_{n,i'} \cap \mathcal{I}_{n,j'} \neq \emptyset \right) \leq \frac{C}{\log^2 d_n}, \]

and for \( r = p \) (hence \( i = i' \))
\[ \sum_{j,j' = 1, k_n \atop |j-j'| \geq 2} \mathbb{P}\left( R_{n,i} \cap \mathcal{I}_{n,j} \neq \emptyset, R_{n,i} \cap \mathcal{I}_{n,j'} \neq \emptyset \right) \leq \frac{C \log k_n}{\log^2 d_n}. \] (5.8)

**Proof.** Introduce
\[ \hat{f}_{n,i,j} := \min(R_{n,i} \cap \mathcal{I}_{n,j}) \quad \text{and} \quad \hat{\ell}_{n,i,j} := \max(R_{n,i} \cap \mathcal{I}_{n,j}). \] (5.9)

Recall again that one can find a constant \( C > 0 \) such that \( u(n) \leq Cn^{\beta-1} \) for all \( n \in \mathbb{N} \). Write, by the first-renewal decomposition,
\[ \mathbb{P}\left( R_{n,i} \cap \mathcal{I}_{n,1} \neq \emptyset, R_{n,i'} \cap \mathcal{I}_{n,j+1} \neq \emptyset \right) = \sum_{k \in \mathcal{I}_{n,1}} \sum_{k' \in \mathcal{I}_{n,j+1}} \mathbb{P}\left( \hat{f}_{n,i,1} = k, \hat{f}_{n,i',j+1} = k' \right) \]
\[ = \sum_{k \in \mathcal{I}_{n,1}} \sum_{k' \in \mathcal{I}_{n,j+1}} u(k' - k)^p \frac{1}{w_n^{p-1}} \mathcal{F}_p((j+1)d_n - k') \]
\[ \leq C \sum_{k \in \mathcal{I}_{n,1}} \sum_{k' \in \mathcal{I}_{n,j+1}} (k' - d_n)^{(\beta-1)r} \frac{1}{w_n^{p-1}} \mathcal{F}_p((j+1)d_n - k') \]
\[ = C \mathbb{P}\left( R_{n,i} \cap \mathcal{I}_{n,1} \neq \emptyset \right) \frac{1}{w_n^{p-1}} \sum_{k = (j-1)d_n+1}^{jd_n} k^{\beta-1} \mathcal{F}_p(jd_n - k). \] (5.10)
If \( j \geq 2, r = 1, \ldots, p \), the summation above can be bounded as (recall \( \beta_r < 1 \) always)
\[
\sum_{k=(j-1)d_n+1}^{jd_n} k^{\beta_r-1} F_p(jd_n - k) \leq ((j-1)d_n + 1)^{\beta_r-1} \sum_{k=0}^{d_n-1} F_p(k) \leq C j^{\beta_r-1} d_n^{\beta_r} \log d_n.
\]

If \( j = 1, r < p \), and \( d_n \) is sufficiently large, exploiting monotonicity we have
\[
\sum_{k=1}^{d_n} k^{\beta_r-1} F_p(d_n - k) \leq F_p(d_n/2) \sum_{k=1}^{\lfloor d_n/2 \rfloor} k^{\beta_r-1} + (d_n/2)^{\beta_r-1} \sum_{k=\lfloor d_n/2 \rfloor + 1}^{d_n} F_p(d_n - k) \leq C \frac{d_n^{\beta_r}}{\log d_n}, \tag{5.11}
\]
(Notice that we used \( \beta_r < 0 \) above and hence \( r = p \) is excluded.) Therefore combining the above to (5.10) with also (5.7), we arrive at
\[
\mathbb{P} \left( R_{n,i} \cap \mathcal{I}_{n,1} \neq \emptyset, R_{n,i'} \cap \mathcal{I}_{n, j+1} \neq \emptyset \right) \leq C \frac{\rho_n}{\log d_n} \frac{d_n^{\beta_r} j^{\beta_r-1}}{\log d_n} \leq C \frac{d_n^{\beta_r+1}}{n^{\beta_r+1} \log^2 d_n j^{\beta_r-1}},
\]
for all (i) \( r = 1, \ldots, p-1, j, n \in \mathbb{N} \) or (ii) \( r = p, j \geq 2, n \in \mathbb{N} \), for some constant \( C > 0 \). Therefore, for \( |i \cap i'| = r \leq p-1 \),
\[
\sum_{j,j' = 1, \ldots, k_n \atop |j-j'| \geq 1} \mathbb{P} \left( R_{n,i} \cap \mathcal{I}_{n,j} \neq \emptyset, R_{n,i'} \cap \mathcal{I}_{n,j'} \neq \emptyset \right) \leq C \frac{d_n^{\beta_r+1} k_n^{\beta_r+1}}{n^{\beta_r+1} \log^2 d_n} \leq C \frac{\log k_n}{\log^2 d_n},
\]
and
\[
\sum_{j,j' = 1, \ldots, k_n \atop |j-j'| \geq 2} \mathbb{P} \left( R_{n,i} \cap \mathcal{I}_{n,j} \neq \emptyset, R_{n,i'} \cap \mathcal{I}_{n,j'} \neq \emptyset \right) \leq C \frac{d_n k_n \log k_n}{n \log^2 d_n} \leq C \frac{\log k_n}{\log^2 d_n},
\]
as desired. \( \square \)

With \( r = p \) for adjoint blocks (\( |j-j'| = 1 \)), a little more care is needed, as the previous method leads to an inferior control with \( r = p \). Below is an improved estimate in this case.

**Lemma 5.8.** We have
\[
\sum_{j,j' = 1, \ldots, k_n \atop |j-j'| = 1} \mathbb{P} \left( R_{n,i} \cap \mathcal{I}_{n,j} \neq \emptyset, R_{n,i'} \cap \mathcal{I}_{n,j'} \neq \emptyset \right) \leq C \frac{\log^{3/2} d_n}{d_n}.
\]

**Proof.** Introduce another sequence of integers \( \{s_n\}_{n \in \mathbb{N}} \) such that \( s_n \to \infty \) and \( s_n/d_n \to 0 \). For comparison purpose we write the proof for general \( i, i' \) (so not necessarily \( r = p \)), and we shall see that at the end the method in this proof is superior than in the previous lemma only when \( r = p \). Then recalling (5.9), we proceed as
\[
\mathbb{P} \left( R_{n,i} \cap \mathcal{I}_{n,1} \neq \emptyset, R_{n,i'} \cap \mathcal{I}_{n,2} \neq \emptyset \right) \leq \sum_{k=1}^{d_n-s_n} \mathbb{P} \left( \hat{f}_{n,i,1} = k \right) \sum_{k'=d_n+1}^{2d_n} \mathbb{P} \left( \hat{f}_{n,i',2} = k' \mid \hat{f}_{n,i,1} = k \right) + \mathbb{P} \left( \hat{f}_{n,i,1} \in \{d_n-s_n+1, \ldots, d_n\} \right).
\]
We focus on the first double summation, which becomes
\[
\sum_{k=1}^{d_n-s_n} \mathbb{P} \left( \hat{f}_{n,i,1} = k \right) \sum_{k'=d_n+1}^{2d_n} u(k' - k)^{r(\beta - 1)} \frac{1}{w_n^p r} F_p(2d_n - k') \leq \frac{C d_n^2}{n \log d_n} \sum_{k'=d_n+1}^{2d_n} s_n^{\beta_r - 1} \frac{1}{w_n^p r} F_p(2d_n - k') \leq \frac{C d_n^2 s_n^{\beta_r - 1}}{n w_n^p r \log^2 d_n},
\]
where for the previous line we have applied (5.7) and (2.4). At the same time,
\[
\mathbb{P} \left( \hat{f}_{n,i,1} \in \{ d_n - s_n + 1, \ldots, d_n \} \right) \leq \sum_{k=d_n-s_n+1}^{d_n} \mathbb{P} \left( \hat{f}_{n,i,1} = k \right) = \sum_{k=d_n-s_n+1}^{d_n} \frac{1}{w_n^p r} F_p(d_n - k) \leq \frac{C s_n}{n \log s_n}.
\]

Assume that \( s_n \) grows at least at a polynomial rate. Then, the optimal rate is achieved if \( d_n^2 / (w_n^p r \log d_n) \sim s_n^{2 - \beta_r} \) (balancing the two bounds above), or equivalently
\[
s_n \sim \left( \frac{d_n^2}{n \beta_r \log d_n} \right)^{1/(2 - \beta_r)}, \quad r = 1, \ldots, p.
\]

(This sequence is indeed with polynomial growth.) Therefore, with \( s_n \) as above
\[
\sum_{j,j'=1,\ldots,k_n \atop |j-j'|=1} \mathbb{P} \left( R_{n,i} \cap I_{n,j} \neq \emptyset, R_{n,i'} \cap I_{n,j'} \neq \emptyset \right) \leq \frac{C k_n s_n}{n \log d_n} \sim \frac{C k_n d_n^{2/(2 - \beta_r)}}{n^{2/(2 - \beta_r)} \log^{(3 - \beta_r)/(2 - \beta_r)} d_n}
\]

\[
\sim C \left( k_n^{\beta_r} n^{\beta_r - 1} \log d_n \right)^{-1}.
\]

The desired result follows with \( r = p \). Notice that with \( r < p \), the above is inferior from what is obtained in Lemma 5.8.

With Lemmas 5.7 and 5.8 above, we have control over the summation "off the diagonal" for \( \rho_n \) in (4.14). We are left with controlling the summation "on the diagonal".

**Lemma 5.9.** For \( r = |\mathbf{i} \cap \mathbf{i}'| = 1, \ldots, p - 1 \),
\[
\mathbb{P} \left( R_{n,i} \cap I_{n,1} \neq \emptyset, R_{n,i'} \cap I_{n,1} \neq \emptyset \right) \leq \frac{C}{k_n^{\beta_r+1} \log d_n}.
\]

**Proof.** Recall that \( \beta_r > \beta_p = 0 \). It is clear that \( \mathbb{P}(R_{n,i} \cap R_{n,i'} \cap I_{n,1} \neq \emptyset) \leq \sum_{k \in I_{n,1}} \mathbb{P}(k \in R_{n,i} \cap R_{n,i'}) = d_n / w_n^{2p-r} \) which compared with the desired bound is of smaller order. So we focus on
\[
\mathbb{P} \left( R_{n,i} \cap I_{n,1} \neq \emptyset, R_{n,i'} \cap I_{n,1} \neq \emptyset, R_{n,i} \cap R_{n,i'} \cap I_{n,1} = \emptyset \right),
\]
which can be bounded from above by
\[
2 \sum_{1 \leq k < k' \leq d_n} \mathbb{P}(k \in R_{n,i}) \mathbb{P}(\max R_{n,i'} = k' \mid k \in R_{n,i})
\]
\[
= \frac{2}{w_n^p r} \sum_{1 \leq k < d_n} (d_n - k) \mathbb{P} \left( 1 \in R_{n,i}, \max R_{n,i'} = 1 + k \right)
\]
\[
\leq \frac{C}{w_n^{2p-r}} \sum_{k=1}^{d_n} (d_n - k) k^{\beta_r - 1} F_p(d_n - k) \leq C \frac{d_n^{\beta_r+1}}{w_n^{2p-r} \log d_n} \sim C \frac{1}{k_n^{\beta_r+1} \log d_n},
\]
where for the last inequality above we have applied the bound (5.11).

Now we have obtained Lemma 5.6 by combining Lemmas 5.7 to 5.9.

**Proof of Lemma 5.4.** The proof is similar to the proof of Lemma 4.2 by applying the two-moments method, and in particular, by verifying (4.20) based on the estimates in Lemmas 5.5 and 5.6 with the rate of $d_n$ specified in (5.3). We omit the details, but only point out that the normalization constant $CE_{F,p}$ is determined as (see the computation of $\lim n \rightarrow \infty E_{n,K}(E_n)$ in the proof of Lemma 4.2)

$$
\lim_{n \rightarrow \infty} C_{n,1}(K)k_n\rho_n = \frac{1}{2}K\left(\frac{CE_{F}(\beta)\Gamma(1-\beta)}{p!(p-1)!}\right) = KCE_{F,p}.
$$

5.2. **Proof of Proposition 5.1**

**Lemma 5.10.** *With the setup for $\beta_p = 0$, the conclusion of Lemma 4.8 continues to hold.*

**Proof.** The proof is almost identical to the proof of Lemma 4.8. We only mention that (4.32) is replaced by

$$
P \left( \exists i = (i_1, \ldots, i_p) \in \left| D_{p}^n(Kn_1) \right|, \text{ s.t. } R_{n,i} \neq 0, i_1 \leq m \right) \leq \frac{C}{\log(n)}\left| D_{p-1}^n(Kn_1) \right| \leq \frac{Cr_p\log^{p-2}(rn_1)}{\log(n)} \leq \frac{C}{\log\log n}.
$$

**Proof of Proposition 5.1.** The proof is similar to that of Proposition 4.1 in Section 4.2, which follows from Lemmas 5.10 and 5.4.

5.3. **Proof of (5.1)**

**Lemma 5.11.** *With the setup for $\beta_p = 0$, the uniform control (4.33) continues to hold.*

**Proof.** The first part of the proof is as in the proof of Lemma 4.9. With $\beta_p = 0$, we are actually in the simpler case $X_{n,k}^{(k)} = \tilde{X}_{n,k}^{(k)}$ therein, and we arrive at (4.35) as before. That is, with

$$Z_{n,i_1,\ldots,i_q}(K) := \max_{k=1,\ldots,n} r_n^{1/\alpha} \left| \sum_{m \leq i_{q+1} \ldots < i_{p}, i|K_n} \frac{[\epsilon_{k+1:p}]}{\Gamma(k+1:p)} 1_{\{k \in R_{n,i} \}} \right|, \quad q = 0, \ldots, p - 1,
$$

it suffices to show that for each $q = 0, \ldots, p - 1, i_1, \ldots, i_q \leq m$ fixed,

$$\lim_{K \rightarrow \infty} \limsup_{n \rightarrow \infty} P \left( Z_{n,i_1,\ldots,i_q}(K) > \epsilon \right) = 0 \text{ for all } \epsilon > 0.
$$

(5.12)

Recall in the proof of Lemma 4.9 for the case $\beta_p < 0$, the relation above was established by an $L^2$ estimation directly. In the case $\beta_p = 0$ here, we need a refinement (the previous estimate (4.36) no longer works: now $nw_n^{-p}r_n\log^{q-1}(rn_1) \sim C_{\log\log n} \rightarrow \infty$). Instead, this time we first control

$$\tilde{Z}_{n,i_1,\ldots,i_q}(K) := \max_{k=1,\ldots,n} r_n^{1/\alpha} \left| \sum_{m \leq i_{q+1} \ldots < i_{p}, i|K_n} \frac{[\epsilon_{k+1:p}]}{\Gamma(k+1:p)} 1_{\{k \in R_{n,i} \}} \right|, \quad q = 0, \ldots, p - 1,
$$

with

$$\tilde{r}_n := r_n(\log n)^{\frac{1}{\alpha}} \gg r_n.
$$
The previous $L^2$-estimate (4.36) now gives
\[
\mathbb{E} \tilde{Z}_{n,i_1,\ldots,i_q}(K)^2 \leq C n w_n^{-p} r_n^{2/\alpha} (K r_n)^{1-2/\alpha} \log^{p-q-1}(r_n) \\
\leq C K^{1-2/\alpha} \frac{r_n}{\log n} \log^{p-q-1}(r_n) \sim C K^{1-2/\alpha} (\log \log n)^q.
\]
So (5.12) holds with $Z_{n,i_1,\ldots,i_q}(K)$ replaced by $\tilde{Z}_{n,i_1,\ldots,i_q}(K)$.

Next, we examine the difference
\[
\left| \tilde{Z}_{n,i_1,\ldots,i_q}(K) - Z_{n,i_1,\ldots,i_q}(K) \right| \leq r_n^{1/\alpha} \max_{k=1,\ldots,n} \left| \sum_{m<i_{q+1}<\cdots<i_p \atop K r_n \leq |k| \leq K r_n} \varepsilon_{i_{q+1} \ldots i_p} \frac{1}{\Gamma_{i_{q+1} \ldots i_p}^1/\alpha} 1_{\{k \in R_{n,k} \}} \right|,
\]
and consider the following event
\[
\Omega_n = \Omega_n(i_1,\ldots,i_q) := \left\{ \exists k = 1,\ldots,n: \sum_{m<i_{q+1}<\cdots<i_p \atop K r_n \leq |k| \leq K r_n} 1_{\{k \in R_{n,k} \}} \geq 2 \right\}.
\]
The key observation is that in the event $\Omega_n^c$, for every $k = 1,\ldots,n$, the summation on the right-hand side of (5.13) has at most one non-zero term, and therefore, in the event $\Omega_n^c$, (5.13) becomes
\[
\left| \tilde{Z}_{n,i_1,\ldots,i_q}(K) - Z_{n,i_1,\ldots,i_q}(K) \right| \leq r_n^{1/\alpha} \max_{m<i_{q+1}<\cdots<i_p \atop K r_n \leq |k| \leq K r_n} (K r_n)^{-1/\alpha} \left| \sum_{m<i_{q+1}<\cdots<i_p \atop K r_n \leq |k| \leq K r_n} \varepsilon_{i_{q+1} \ldots i_p} \frac{1}{\Gamma_{i_{q+1} \ldots i_p}^1} \leq K^{-1/\alpha} \sup_{i \in \mathbb{N}} \left( \frac{\Gamma_i}{i} \right)^{-p/\alpha}.
\]
However, $\sup_{i \in \mathbb{N}} (\Gamma_i/i)^{-p/\alpha}$ is a finite random variable, and hence
\[
\lim_{K \to \infty} \lim_{n \to \infty} \mathbb{P} \left( \left| \tilde{Z}_{n,i_1,\ldots,i_q}(K) - Z_{n,i_1,\ldots,i_q}(K) \right| > \epsilon \right) \cap \Omega_n^c = 0, \text{ for all } \epsilon > 0.
\]
It remains to examine $\Omega_n$. We have (recall that $i_1,\ldots,i_q \leq m$ are fixed),
\[
\mathbb{P}(\Omega_n) \leq \sum_{m<i_{q+1}<\cdots<i_{p+1} \leq K r_n} \mathbb{P} \left( R_{n,i_1,\ldots,i_{p+1}} \neq \emptyset \right) \leq \left( \frac{K r_n}{p-q+1} \right) \frac{n}{w_n^{p+1}} \leq C \frac{n^{p-q+1}}{w_n} \to 0.
\]
So combining the two relations above completes the proof.

\textbf{Proof of (5.1).} The proof is the same as for (4.1) in Section 4.3, with Proposition 4.1 and Lemma 4.9 replaced by Proposition 5.1 and Lemma 5.11, respectively.

\textbf{Acknowledgments}

The authors would like to thank Larry Goldstein, Takashi Owada and Gennady Samorodnitsky for very helpful discussions. The authors would like to thank an anonymous referee for the critical and yet constructive report that has helped us significantly during the revision.

\textbf{Funding}

The second author was partially supported by Army Research Office, USA (W911NF-20-1-0139).
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