Group Theory/Dynamical Systems

Explicit left orders on free groups extending the lexicographic order on free monoids

Ordres à gauche explicites sur les groupes libres étendant l'ordre lexicographique sur les monoïdes libres

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A B S T R A C T

For every finitely generated free group, we construct an explicit left order extending the lexicographic order on the free monoid generated by the positive letters. The order is defined by a left, free action on the orbit of 0 of a free group of piecewise linear homeomorphisms of the line. The membership in the positive cone is decidable in linear time in the length of the input word. The positive cone forms a context-free language closed under word reversal.

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Un groupe $G$ est ordonnable à gauche s'il existe un ordre total $\leq$ sur $G$ compatible avec la multiplication à gauche, i.e., pour tous les éléments $f$, $g$ et $h$ de $G$, l'inégalité $f \leq g$ entraîne $hf \leq hg$. Il est bien connu depuis les années 1940 que le groupe libre $F_k$ de rang $k$ est ordonnable à gauche (et, justement, bi-ordonnable). Pourtant, la plupart des preuves précédentes sont non constructives ou trop compliquées (souvent à cause d'une volonté de généraliser ultérieurement).

Peut-être la construction la plus explicite connue à l'heure actuelle d'un ordre sur les groupes libres est donnée par l’approche de Magnus–Bergman [1], basée sur le plongement de Magnus [2] du groupe libre $F_k = F(\Sigma_k)$ dans l’anneau des séries formelles à coefficients entiers en variables non commutatives dans $\Sigma_k = \{s_1, \ldots, s_k\}$. Les monômes sur $\Sigma_k$ sont ordonnés par longueur lexicographique et un élément $u$ de $F_k$ est déclaré positif si et seulement si le coefficient du monôme minimal (different de 1) dans la série formelle représentant $u$ dans le plongement de Magnus est positif.

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Pour tout groupe libre finiment engendré, nous construisons explicitement un ordre à gauche qui étend l’ordre lexicographique sur le monoïde libre engendré par les lettres positives. Cet ordre est défini par une action à gauche, libre, sur l’orbite de 0 d’un groupe libre d’homéomorphismes de la droite linéaires par morceaux. L’appartenance au cône positif est décidable en temps linéaire par rapport à la longueur du mot en comptant directement les sous-mots de longueur 2 d’un certain type (voir le critère de positivité dans le théorème 0.2). Le cône positif forme un langage non contextuel fermé par image miroir.

On divise le cercle $S^1 = \mathbb{R}/\mathbb{Z} = \{0, 1\}/\{0\} = 1$ en sept arcs de même longueur, notés $1' = \{0, 1/7, 2/7, 3/7, 4/7, 5/7, 6/7\}$. $A' = \{2/7, 3/7\}, A'' = \{1/7, 2/7, 3/7, 4/7, 5/7, 6/7\}$, et on définit trois homéomorphismes du cercle linéaires par morceaux (avec un nombre fini de singularités) conservant l’orientation $a, b$, et $c$ comme dans la Fig. 1 à gauche. Les inverses de $a, b$, et $c$, dénotés par $A, B$, et $C$, respectivement, sont représentés dans la moitié droite de la Fig. 1. Il est évident que les inclusions (1) et (2) sont satisfaites. Par conséquent, $F = F_3 = (a, b, c)$ est libre de rang 3. De plus, $F$ agit librement, par une action à gauche, sur le sous-ensemble $F'1$ du cercle (notons que $F'1$ a une mesure de Lebesgue égale à 1). En particulier, il agit librement sur l’orbite de 0.

On relève les sous-intervalle du cercle $S^1$ à des sous-ensembles de la droite $\mathbb{R}$ suivant la projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$, et on relève les applications $a, b$, et $c$ à des homéomorphismes de la droite linéaires par morceaux conservant l’orientation (avec un nombre fini de singularités dans chaque sous-ensemble compact) comme dans la Fig. 2. On ne change pas de notation pour des sous-ensembles ou des homéomorphismes relevés.

Notons que le groupe $F$ relevé est encore libre, agissant librement sur l’orbite de 0. Par conséquent, ceci définit un ordre à gauche sur $F$ en posant $g > h$ si et seulement si $g(0) > h(0)$ (de manière équivalente, si et seulement si $h^{-1}g(0) > 0$). Un critère simple pour la positivité est obtenu en « traçant » l’action sur 0, c’est-à-dire en calculant, pour tout mot réduit $u$, la distance signée, avec une erreur plus petite que 1/2, entre $u(0)$ et 0.

**Proposition 0.1.** Soit $w : F_3 \to \mathbb{R}$ la fonction poids définie par:

\[
w(u) = \#(\text{des sous-mots de } u \text{ de la forme } CB, \text{ ou } BA) - \#(\text{des sous-mots de } u \text{ de la forme } Ca, \text{ ou } Ba)
\]

\[
+ \frac{1}{2} \left\{ \begin{array}{ll}
1, & \text{si } u \text{ se termine par une lettre positive, i.e., par une des lettres } a, b, c, \\
-1, & \text{si } u \text{ se termine par une lettre negative, i.e., par une des lettres } A, B, C,
\end{array} \right.
\]

\[
0, \quad \text{si } u \text{ est le mot trivial,}
\]

où $u$ est un mot réduit sur $\{a, b, c\}$. Alors $w(0) > 0$ si et seulement si $w(u) > 0$.

En suivant une construction analogue (en divisant $S^1$ en $2k + 1$ morceaux, pour $k \geq 2$, et en définissant $h$ homéomorphismes de $S^1$ linéaires par morceaux $s_1, \ldots, s_k$, et ainsi de suite), on peut facilement établir le résultat suivant.

**Théorème 0.2.** Soit $\Sigma_k = \{s_1, \ldots, s_k\}$, avec $k \geq 2$. Pour $i = 1, \ldots, k$, posons $s_i = s_1^{-i}$. On peut définir un ordre à gauche sur le groupe libre $F_k = F(\Sigma_k)$ en étendant l’ordre lexicographique sur le monoïde libre $\Sigma_k$ basé sur l’ordre $s_1 < s_2 < \cdots < s_k$ sur l’alphabet des lettres positives, comme ci-dessous. Soit $w : F_k \to \mathbb{R}$ la fonction poids définie par:

\[
w(u) = \#(\text{des sous-mots de } u \text{ de la forme } s_j s_i, \text{ pour } j > i) - \#(\text{des sous-mots de } u \text{ de la forme } s_j s_i, \text{ pour } j > i)
\]

\[
+ \frac{1}{2} \left\{ \begin{array}{ll}
1, & \text{si } u \text{ se termine par une lettre positive } s_i, \ i = 1, \ldots, k,
\end{array} \right.
\]

\[
-1, \quad \text{si } u \text{ se termine par une lettre negative } s_i, \ i = 1, \ldots, k,
\]

\[
0, \quad \text{si } u \text{ est le mot trivial,}
\]

où $u$ est un mot réduit sur $\Sigma_k$. Alors, l’ensemble $P_k = \{u \in F_k : w(u) > 0\}$ est un cône positif de $F_k$ (i.e., $u > v$ dans $F_k \leftrightarrow w(v^{-1}u) > 0$).

1. **Left orders on free groups extending the lexicographic order**

A group $G$ is left orderable if there exists a linear order $\leq$ on $G$ that is compatible with the left multiplication, i.e., for all elements $f, g$ and $h$ in $G$, if $f \leq g$, then $hf \leq hg$. It has been known at least since the 1940s that the free group $F_k$ of rank $k$ is left orderable (and, in fact, bi-orderable, admitting an order that is compatible with both the left and the right multiplication simultaneously). In two of his papers [6,5] related to the subject, Neumann mentions that, in addition to himself, several other authors have stated this fact in published or unpublished works, including Tarski, G. Birkhoff, Shimbireva, and Iwasawa (despite his laudable effort to give credit to all, he was unaware of the simultaneous work of Vinogradov [8]). However, most of the early proofs are nonconstructive or too involved (often because of an attempt for greater generality).

Perhaps the most explicit, currently known, construction of an order on free groups is given by the Magnus–Bergman approach [1], based on the Magnus embedding [2] of the free group $F_k = F(\Sigma_k)$ into the ring of formal power series with integer coefficients in noncommuting variables from $\Sigma_k = \{s_1, \ldots, s_k\}$. The monomials over $\Sigma_k$ are ordered by short-lex and an element $u$ from $F_k$ is declared positive if and only if the coefficient in front of the smallest monomial (different from 1) in the power series representing $u$ under the Magnus embedding is positive.
For every finitely generated free group, we construct an explicit left order extending the lexicographic order on the free monoid generated by the positive letters. The order is defined by a left, free action on the orbit of 0 of a free group of piecewise linear homeomorphisms of the line. The membership in the positive cone is decidable in linear time in the free monoid generated by the positive letters. The order is defined by a left, free action on the orbit of 0 of a free group “tracing” the action on 0, i.e., by calculating, for reduced group words $u$ is a reduced group word over $a, b, c$.

\[
\begin{align*}
    a(S^1 \setminus A') &\subseteq a', & b(S^1 \setminus B') &\subseteq b', & c(S^1 \setminus C') &\subseteq c', \\
    A(S^1 \setminus a') &\subseteq A', & B(S^1 \setminus b') &\subseteq B', & C(S^1 \setminus c') &\subseteq C'.
\end{align*}
\]

Therefore, $F = F_3 = \langle a, b, c \rangle$ is free of rank 3. Moreover, $F$ acts freely, through a left action, on the subset $F' \setminus 0$ of the circle (note that $F'$ has Lebesgue measure 1). In particular, it acts freely on the orbit of 0.

We lift the subintervals from the circle $S^1$ to subsets of the line $\mathbb{R}$ along the projection $\mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1$, and we lift the maps $a, b$ and $c$ to piecewise linear, orientation preserving the homeomorphisms of the line (with finitely many breaks on every compact subset) as in Fig. 2. We do not change the notation for the lifted subsets or homeomorphisms.

Note that the lifted group $F$ is still free, acting freely on the orbit of 0. Therefore, a left order is defined on $F$ by declaring $g > h$ if and only if $g(0) > h(0)$ (equivalently, if and only if $h^{-1}(g(0)) > 0$). A simple criterion for positivity is obtained by “tracing” the action on 0, i.e., by calculating, for reduced group words $u$, the signed distance, within error smaller than $1/2$, between $u(0)$ and 0. Of course, since the action is given explicitly and the orbits of rational points are rational, $u(0)$ can be calculated exactly, but that takes longer, obscures the features of the order, and is not necessary.

**Proposition 1.1.** Define a weight function $w : \mathbb{F}_3 \to \mathbb{R}$ by

\[
w(u) = \#(\text{of subwords of } u \text{ of the form } cB, \ cA, \ \text{or } bA) - \#(\text{of subwords of } u \text{ of the form } Cb, \ Ca, \ \text{or } Ba)
\]

\[
+ \begin{cases} 1, & \text{if } u \text{ ends in a positive letter, i.e., one of the letters } a, b, c, \\ -1, & \text{if } u \text{ ends in a negative letter, i.e., one of the letters } A, B, C, \\ 0, & \text{if } u \text{ is the trivial word}, \end{cases}
\]

where $u$ is a reduced group word over $\{a, b, c\}$. Then

\[
u(0) > 0 \quad \text{if and only if} \quad w(u) > 0.
\]

Indeed, for any nontrivial word $u$, the point $u(0)$ is in exactly one of the intervals.
We claim that \( w(u) \) represents the midpoint of the interval to which \( u(0) \) belongs. By definition, the action of \( F \) is a left action, hence the last letter of \( u \) acts first on 0 and pushes it into the interval \((0, 1)\) if the letter is positive, or to \((-1, 0)\) if the letter is negative, and the part of the formula for \( w \) related to the last letter of \( u \) records this as \( 1/2 \) or \(-1/2\). From here on, we trace the action of \( u \) (from right to left) on the obtained point, but only record jumps into the next interval up (by adding 1 in our weight) and jumps into the next interval down (by subtracting 1 in our weight). Negative letters never produce jumps up (see the right half of Fig. 2) and positive letters never produce jumps down (see the left half of Fig. 2). By examining the action of the positive letters in the left half of Fig. 2, we see that jumps up occur exactly when the letter \( c \) is applied to a point in the regions \( B' \) and \( A' \), which is precisely when a subword of the form \( cb \) or \( ca \) occurs in \( u \), or when the letter \( b \) is applied to a point in the region \( A' \), which is precisely when a subword of the form \( bA \) occurs in \( u \). We exclude the possibility of applying the letter \( c \) to the region \( C' \), \( b \) to the region \( B' \), and \( a \) to the region \( A' \), because \( u \) is a reduced word. Similarly, by examining the action of the negative letters in the right half of Fig. 2, we see that jumps down occur exactly when the letter \( C \) is applied to a point in the regions \( b' \) and \( a' \), which is precisely when a subword of the form \( Cb \) or \( Ca \) occurs in \( u \), or when the letter \( B \) is applied to a point in the region \( a' \), which is precisely when a subword of the form \( Ba \) occurs in \( u \). We exclude the possibility of applying the letter \( C \) to the region \( c' \), \( b \) to the region \( b' \), and \( a \) to the region \( a' \), because \( u \) is a reduced word. We also exclude the possibility of applying any of the negative letters to the region \( 1' \), since the point 0 will never return to the region \( 1' \) under the action of a nontrivial word \( u \), after, in the very first step, the last letter of \( u \) moves it from there.

We claim that the given order extends the usual lexicographic order on the free monoid \( M_3 = [a, b, c]^\ast \) based on \( a < b < c \). All we need to verify is that, for all words \( v_1, v_2, v_3 \) in \( M_3 \), \( e < av_1 < bv_2 < cv_3 \), i.e., we need to verify that \( \langle w(u) \rangle, \langle w(v_1^{-1}Abv_2) \rangle, \langle w(v_2^{-1}Bcv_3) \rangle > 0 \). Since the weight of each of these words is \( 1/2 \), the claim is correct.

In fact, it is possible to see that the order, restricted to words in \( M_3 = [a, b, c]^\ast \) is lexicographic just by looking at the left half of Fig. 2. Namely, for \( u \) in \( M_3 \), \( u(0) \) is trapped in the interval \([0, 4/7)\), and, for all words \( v_1, v_2, v_3 \) in \( M_3 \), we have \( 0 < av_1(0) < bv_2(0) < cv_3(0) \), since, on the interval \([0, 4/7)\) the entire graph of the function \( a \) is above 0 and below the minimum of the function \( b \), and the entire graph of the function \( b \) is below the minimum of the function \( c \).

Following an analogous construction (subdividing \( S^1 \) into \( 2k + 1 \) pieces, for \( k \geq 2 \), defining \( k \) piecewise linear homeomorphisms \( s_1, \ldots, s_k \) of \( S^1 \), and so on), we may easily establish the following result.
Theorem 1.2. Let \( \Sigma_k = \{s_1, \ldots, s_k\} \), for some \( k \geq 2 \). For \( i = 1, \ldots, k \), denote \( S_i = s_i^{-1} \). A left order on the free group \( F_k = F(\Sigma_k) \) extending the lexicographic order on the free monoid \( \Sigma_k^* \) based on the order \( s_1 < s_2 < \cdots < s_k \) on the alphabet of positive letters may be defined as follows. Define a weight function \( w : F_k \to \mathbb{R} \) by:

\[
w(u) = \begin{cases} 1 & \text{if } u \text{ ends in any positive letter } s_i, \ i = 1, \ldots, k, \\ -1 & \text{if } u \text{ ends in any negative letter } s_i, \ i = 1, \ldots, k, \\ 0 & \text{if } u \text{ is the trivial word},
\end{cases}
\]

where \( u \) is a reduced group word over \( \Sigma_k \), and declare that the set:

\[
P_k = \{ u \in F_k \mid w(u) > 0 \}
\]

is the positive cone of \( F_k \) (i.e., \( u > v \) in \( F_k \) if and only if \( w(u^{-1}v) > 0 \)).

The membership problem for the positive cone \( P_k \) is rather easy and can be solved in linear time in the length of the input word. If we count the relevant subwords as we read, we can calculate the weight and tell if a word is in the positive cone by the time we finish reading the word.

It is known that the positive cone of a left order of a free group cannot be finitely generated as a monoid (this can be deduced from the work of McCleary [3], but was the first explicit proof is given by Navas [4]), which implies that it is closed under word reversal. Indeed, for a given word \( u \), with \( w(u) > 0 \), we have \( w(u^{-1}) = -w(u) < 0 \), where \( u^R \) denotes the word reversal of \( u \), since the transformation \( u^R \) just exchanges the positive and negative letters (while keeping them in the same order as in \( u \)), and the effect of this on the weight is to exactly exchange all positive and all negative contributions. Since \( w((u^R)^{-1}) < 0 \), we must have \( w(u^R) > 0 \).

The left order provided in Theorem 1.2 is not two-sided (unlike the Magnus–Bergman order, which is). In fact, it is clear that no order extending the lexicographic order can be two-sided (this is because \( s_1 < s_1s_1 \), but \( s_1s_2 > s_1s_1s_2 \)).

Variations of the construction presented above lead to other explicitly stated orders on free groups, not necessarily extending the lexicographic order (for instance, in case of \( F_2 \), add 1 to the weight for every subword of the form \( ab \) or \( ba \), subtract 1 for every subword of the form \( BA \) or \( Ba \), add plus or minus 1/2, depending on the last letter). In fact, Cantor sets of left orders on \( F_k \) may be constructed by stringing together variations of the above construction (subsequent work).

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