MATRICIAL $R$-CIRCULAR SYSTEMS AND RANDOM MATRICES

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ABSTRACT. We introduce and study matricial $R$-circular systems of operators which play the role of matricial analogs of circular operators. They are obtained from canonical decompositions of ‘matricial circular systems’ studied recently in the context of the Hilbert space realization of the asymptotic joint $*$-distributions of symmetric blocks of independent block-identically distributed Gaussian random matrices with respect to partial traces. As compared with those systems, matricial $R$-circular systems describe the asymptotic joint $*$-distributions of blocks rather than those of symmetric blocks. We prove that they are $*$-symmetrically matricially free with respect to the corresponding array of states. We also study their cyclic cumulants defined in terms of cyclic non-crossing partitions as well as the corresponding cyclic $R$-transform.

1. INTRODUCTION

Circular and semicircular systems of operators were introduced by Voiculescu [14] in the context of random matrices and their asymptotics and applied to free group factors. This result followed his fundamental asymptotic freeness result for independent Hermitian and non-Hermitian Gaussian random matrices with complex i.i.d. entries, respectively [13]. Decomposition of these matrices into blocks corresponds to decompositions of circular and semicircular operators into circular and semicircular systems, respectively.

Recently, we obtained related results in the case when the considered independent Gaussian random matrices have independent block-identically distributed complex entries, which we abbreviate $i.b.i.d.$ [5,7]. We showed that in order to give an operatorial model for the asymptotic joint ($*-$) distributions of blocks of these matrices under partial traces rather than under the ‘complete’ trace, one needs to replace ($*-$) free summands in the decompositions of (circular) semicircular operators by their matricial counterparts, related to each other by a matricial generalization of ($*-$) freeness with respect to an array of scalar-valued states.

The underlying concept of independence is that of matricial freeness [4] which can be described by means of the intuitive equation

$\text{matricial freeness} = \text{freeness} \& \text{matriciality}$

and its symmetrized version called symmetric matricial freeness [5] which also plays an important role in our developments. A connection between these concepts and blocks of large random matrices was established in [5] for one matrix and in [7] for an ensemble of independent matrices.

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For instance, if we are given an ensemble of independent non-Hermitian $n \times n$ random matrices \( \{Y(u, n) : u \in \mathcal{U}\} \) whose entries are suitably normalized i.b.i.d. complex Gaussian random variables with zero mean for each natural $n$, then the mixed *-moments of their symmetric blocks \( \{T_{p,q}(u, n) : 1 \leq p \leq q \leq r, u \in \mathcal{U}\} \) converge under partial traces to arrays of certain bounded non-self-adjoint operators, which we write informally

\[
\lim_{n \to \infty} T_{p,q}(u, n) = \eta_{p,q}(u),
\]

and the operators \( \eta_{p,q}(u) \) are called matricial circular operators. The whole family

\[
\{\eta_{p,q}(u) : 1 \leq p \leq q \leq r, u \in \mathcal{U}\}
\]

will be called a matricial circular system. Let us remark that our approach allows us to treat all rectangular blocks, including those which are unbalanced or evanescent [7]. We would also like to mention that the first operatorial treatment of the asymptotic distributions under the ‘complete’ trace of Gaussian random matrices with non-identically distributed entries is due to Shlyakhtenko [11], who used free probability with operator-valued states for that purpose.

The array of $C^*$-algebras $\mathcal{A}_{p,q}$, each generated by the matricial circular operators indexed by $p, q$ and by the corresponding unit, are related to each other by symmetric matricial freeness. In a similar result for i.b.i.d. Hermitian Gaussian random matrices (in the Hermitian case, we understand that variables are block-identically distributed if the covariances of complex variables are equal within blocks), one obtains $C^*$-algebras generated by matricially free Gaussian operators or their symmetrizations, symmetrized Gaussian operators [5,7]. An application of this scheme to products of independent random matrices leads to multivariate Fuss-Narayana polynomials and free multiplicative convolutions of Marchenko-Pastur laws [8].

All our operators, including matricially free Gaussian operators, symmetrized Gaussian operators as well as matricial circular operators, live in the matricially free Fock space of tracial type (Definition 2.1), where each summand is built by the ‘matricial’ action of free creation operators onto a unit vector $\Omega_q$, where $1 \leq q \leq r$. This Fock space gives a canonical model for a matricial circular system. We associate the state

\[
\Psi_q : B(\mathcal{M}) \to \mathbb{C} \quad \text{by} \quad \Psi_q(a) = \langle a\Omega_q, \Omega_q \rangle
\]

with each $\Omega_q$ as it is always done for vector states. The presence of a family of vacuum vectors allows us then to compute distributions with respect to the state

\[
\Psi = \sum_{q=1}^{r} d_q \Psi_q,
\]

where $d_1 + \ldots + d_r = 1$, which corresponds to the canonical normalized trace of large random matrices. In general, we are interested in (\( * \)) distributions with respect to any $\Psi$ of the above form, for which it suffices to determine those with respect to all vector states $\Psi_q$. 
We also use them to construct the array of states on $B(\mathcal{M})$ with respect to which matricial freeness or symmetric matricial freeness is proved, namely

$$(\Psi_{p,q}) = \begin{pmatrix}
\Psi_1 & \Psi_2 & \cdots & \Psi_r \\
\Psi_1 & \Psi_2 & \cdots & \Psi_r \\
\vdots & \vdots & \ddots & \vdots \\
\Psi_1 & \Psi_2 & \cdots & \Psi_r
\end{pmatrix}.$$

In the present paper, we can identify $\Psi_{q}$ with $\Phi_q = \varphi \otimes \psi_q$, where $\varphi$ is the vacuum expectation on the $C^*$-algebra $\mathcal{A}$ generated by a system of free circular operators and $\psi_q$ is the state associated with the basis vector $e(q)$ of $\mathbb{C}^r$. Under the symmetry assumption on the covariance matrices, the considered matricial circular operators $\eta_{p,q}(u)$ have circular *-distributions with respect to the two associated states $\Psi_p$ and $\Psi_q$, which explains why we call them ‘circular’. Another reason is that the arrays $(\eta_{p,q}(u))$ are matricial analogs of circular operators even in the case when the covariance matrices are not symmetric (see Proposition 4.3).

One can use circular systems of Voiculescu (more generally, generalized circular systems) to find their matrix realization. Let $\mathcal{A}$ be the $C^*$-algebra generated by the system of free (generalized) circular operators

$$\{g(p, q, u) : 1 \leq p, q \leq r, u \in \mathcal{U}\}$$

and define operators

$$\zeta_{p,q}(u) = g(p, q, u) \otimes e(p, q) \in M_r(\mathcal{A})$$

for any $p, q, u$. Then, the matricial circular operators can be identified with

$$\eta_{p,q}(u) = \begin{cases} 
\zeta_{p,q}(u) + \zeta_{q,p}(u) & \text{if } p < q \\
\zeta_{q,q}(u) & \text{if } p = q
\end{cases},$$

which also explains why we call them ‘matricial’. The arrays $(\zeta_{p,q}(u))$ which arise in these decompositions are the main objects of our study. Note that they are the generators of free group factors used by Voiculescu in his proof of free group factors isomorphisms [14, Theorem 3.3].

In particular, we prove that both arrays, $(\eta_{p,q}(u))$ and $(\zeta_{p,q}(u))$, are *-symmetrically matricially free with respect to $(\Psi_{p,q})$ for any $u \in \mathcal{U}$. The proof reminds that for symmetrized Gaussian operators in [7, Proposition 3.5]. However, in the present paper, we use the matricial realization given above, which allows us to phrase it in the language of free probability. In our previous papers on matricial systems of operators [5,7], we relied on the Hilbert space approach and its implications for random matrix theory.

The diagonal operators $\zeta_{q,q}(u)$ are clearly circular under the corresponding states $\Psi_q$. The off-diagonal ones $\zeta_{p,q}(u)$ correspond to canonical decompositions of circular operators $\eta_{p,q}(u)$. The family

$$\{\zeta_{p,q}(u) : 1 \leq p, q \leq r, u \in \mathcal{U}\}$$

will be called a matricial $R$-circular system and its elements will be called matricial $R$-circular operators since their cyclic $R$-transforms defined by cyclic cumulants give canonical decompositions of the $R$-transform for circular operators. In the simplest case of an off-diagonal pair $\zeta_1 = \zeta_{p,q}(u)$ and $\zeta_2 = \zeta_{q,p}(u)$, the cyclic $R$-transform of $\{\zeta_1, \zeta_1^*\}$ and $\{\zeta_2, \zeta_2^*\}$ in the state $\Psi_q$ assume the form

$$R_{\zeta_1, \zeta_1^*}(z_1, z_2; q) = z_2 z_1$$

and

$$R_{\zeta_2, \zeta_2^*}(z_1, z_2; q) = z_1 z_2.$$
if the covariances are equal to one, respectively. Thus, by multilinearity of cumulants, the cyclic \( R \)-transform of \( \{\eta, \eta^*\} \) in the state \( \Psi_q \), where \( \eta = \eta_{p,q}(u) \), takes the form
\[
R_{\eta,\eta^*}(z_1, z_2; q) = z_1 z_2 + z_2 z_1
\]
and thus it agrees with its \( R \)-transform in that state. Similar formulas are obtained for the state \( \Psi_p \), whereas the remaining cyclic \( R \)-transforms of the above operators, all of which are labelled by \( p, q \), are identically equal to zero. This formalism allows us to show that the cyclic \( R \)-transform of matricial \( R \)-circular systems become ‘hermitian forms’ in the corresponding arrays of non-commutative indeterminates.

Moreover, we show in this paper that the arrays of \( R \)-circular operators are of importance in random matrix theory. To catch a glimpse of that importance, let us go back to non-Hermitian Gaussian random matrices with i.b.i.d. entries. Write each matrix \( Y = Y(u,n) \) in the block form
\[
Y = \begin{pmatrix}
S_{1,1} & S_{1,2} & \ldots & S_{1,r} \\
S_{2,1} & S_{2,2} & \ldots & S_{2,r} \\
& \ddots & \ddots & \\
S_{r,1} & S_{r,2} & \ldots & S_{r,r}
\end{pmatrix}
\]
and consider the asymptotic joint *-distributions of blocks \( S_{p,q} = S_{p,q}(u,n) \) under partial traces and their realizations on \( \mathcal{M} \). We assume that, for any given \( n \), the dimensions of blocks do not depend on \( u \) and we identify each block with its image under the embedding in the algebra of \( n \times n \) matrices.

The operators \( \zeta_{p,q}(u) \) describe the asymptotic joint *-distributions of blocks \( S_{p,q}(u,n) \) as \( n \to \infty \) with sizes of blocks growing proportionately to \( n \) or slower. Namely, we show that their limit joint *-distributions under partial traces coincide with the corresponding *-distributions of \( \zeta_{p,q}(u) \), which we write informally
\[
\lim_{n \to \infty} S_{p,q}(u,n) = \zeta_{p,q}(u),
\]
for any \( u \in \mathcal{U} \) and \( 1 \leq p, q \leq r \). Since blocks \( S_{p,q}(u) \) refer to the most general model of i.b.i.d. Gaussian random matrices, the systems of \( R \)-circular operators comprise the basic constituents of the asymptotic operatorial models. Clearly, we can also use them to reproduce the results on the asymptotics of symmetric random blocks of i.b.i.d. Gaussian random matrices [7]. In the special case when \( Y(u,n) \) has i.i.d. entries and we divide it into blocks, we obtain in the large \( n \) limit a matricial decomposition of the standard circular operator in terms of \( R \)-circular ones and its \( R \)-transform decomposes as the sum of cyclic \( R \)-transforms.

Voiculescu’s asymptotic freeness result allowed him to show that a decomposition of square Gaussian random matrices with i.i.d. complex entries into blocks leads to a natural decomposition of circular operators into a circular system, and that a similar connection holds for Hermitian matrices and semicircular systems [14] (see also [15]). Triangular random matrix models and the corresponding decompositions of circular operators and more general operators called DT-operators were found by Dykema and Haagerup [2,3]. Our approach in [5,7] can be treated as a (symmetric) block refinement of Voiculescu’s result and the present paper goes one step further by giving the mapping between arbitrary blocks and \( R \)-circular operators [14]. Some implications concerning the triangular operator will be given in a forthcoming paper. In general, one of our main goals in [5,7] and in the present paper is to construct a universal operator system.
describing the asymptotic *-distributions of a large class of random matrices, including
their sums and products, which would not make it necessary to return to classical
probability for different random matrix models.

The paper is organized as follows. In Section 2, we recall the definitions of certain
matricial systems of operators from [7] and we introduce matricial $R$-circular op-
erators. We also establish a connection between $C^*$-algebras generated by matricially
free creation operators and Cuntz-Krieger algebras [1]. In Section 3, we find matrix
realizations of our matricial systems of operators, using systems of operators in free
probability. Section 4 is devoted to computations of *-distributions of circular and $R$-
circular operators in the corresponding states. In Section 5, we show that their arrays
are *-symmetrically matricially free with respect to the array $(\Psi_{p,q})$. In Section 6, we
study the joint *-distributions of $R$-circular operators. Their cyclic cumulants and the
cyclic $R$-transforms are determined in Section 7. The moment series and their relation
to the cyclic $R$-transforms is studied in Section 8. Finally, in Section 9, we show
that *-distributions of blocks of i.i.d. Gaussian random matrices under partial traces
converge to the *-distributions of the $R$-circular operators.

2. Operators constructed from partial isometries

Our study of the asymptotic distributions of random matrices in [5] and [7] was based
on the construction of the appropriate Hilbert space of Fock type, in which ‘matriciality’
is added to ‘freeness’ and in which certain systems of operators called matricially free
Gaussian operators replace semicircular systems. This Hilbert space is the matricially
free Fock space of tracial type and is derived from the slightly simpler matricially free
Fock space and the matricially free product of Hilbert spaces introduced in [4]. In this
section, after recalling some definitions, we show a connection between the system of
matricially free Gaussian operators and Cuntz-Krieger algebras [1].

In the general case, we assume that $J \subseteq [r] \times [r]$, where $[r] := \{1, 2, \ldots, r\}$, although
in this paper we will mainly deal with the situation when these sets are equal. However,
passing from $[r] \times [r]$ to any proper subset presents no difficulty and can be achieved
by setting certain operators to be zero. Moreover, we will consider another finite set of
indices $\mathcal{U}$, setting for convenience $\mathcal{U} = [t]$ for some integer $t$. To each $(p, q) \in J$ and
$u \in \mathcal{U}$ we then associate a Hilbert space $\mathcal{H}_{p,q}(u)$. Using this family of Hilbert spaces,
we can construct our Fock space, a matricial version of the free Fock space.

Definition 2.1. By the matricially free Fock space of tracial type we understand the
direct sum of Hilbert spaces

$$
\mathcal{M} = \bigoplus_{q=1}^{r} \mathcal{M}_q,
$$

where each summand is of the form

$$
\mathcal{M}_q = \bigoplus_{m=1}^{\infty} \mathcal{H}_{p_1, p_2}(u_1) \otimes \mathcal{H}_{p_2, p_3}(u_2) \otimes \cdots \otimes \mathcal{H}_{p_m, q}(u_m),
$$

endowed with the canonical inner products.

Note that this definition is equivalent to the one in [7], but it is considerably simpler
since it does not use boolean and free Fock spaces associated with the off-diagonal and
diagonal pairs of indices, respectively. Observe also that the neighboring Hilbert spaces
can coincide if and only if they have diagonal indices. This shows that there is an essential difference between the diagonal Hilbert spaces and the off-diagonal ones in this structure.

**Proposition 2.1.** If $\mathcal{H}_{p,q}(u) = C e_{p,q}(u)$ for any $p, q, u$, where $e_{p,q}(u)$ is a unit vector, the canonical orthonormal basis $B$ of the matricially free Fock space $\mathcal{M}$ consists of

$$e_{p_1,p_2}(u_1) \otimes e_{p_2,p_3}(u_2) \otimes \ldots \otimes e_{p_m,q}(u_m)$$

where $p_1, \ldots, p_m, q \in [r]$, $u_1, \ldots, u_m \in \mathcal{U}$ and $m \in \mathbb{N}$, and of vacuum vectors $\Omega_1, \ldots, \Omega_r$.

**Proof.** This fact is obvious. ■

Let us recall the definition of the matricially free creation operators. In this work, we prefer to give a concrete definition by showing their action onto the basis vectors. An equivalent, more abstract definition was given in [7].

**Definition 2.2.** Let $B(u) = (b_{p,q}(u))$ be an array of positive real numbers for any $u \in \mathcal{U}$. We associate with each such matrix the matricially free creation operators whose non-trivial action onto the basis vectors is

$$\varphi_{p,q}(u)\Omega_q = \sqrt{b_{p,q}(u)} e_{p,q}(u)$$

$$\varphi_{p,q}(u)(e_{q,t}(s)) = \sqrt{b_{p,q}(u)}(e_{p,q}(u) \otimes e_{q,t}(s))$$

$$\varphi_{p,q}(u)(e_{q,t}(s) \otimes w) = \sqrt{b_{p,q}(u)}(e_{p,q}(u) \otimes e_{q,t}(s) \otimes w)$$

for any $p, q, t \in [r]$ and $u, s \in \mathcal{U}$, where $e_{q,t}(s) \otimes w$ is a basis vector. Their actions onto the remaining basis vectors give zero. The corresponding matricially free annihilation operators are their adjoints denoted $\varphi_{p,q}^*(u)$. If $b_{p,q}(u) = 1$ we will call the associated operators standard.

**Definition 2.3.** A symmetrization procedure leads to symmetrized creation operators of the form

$$\hat{\varphi}_{p,q}(u) = \begin{cases} \varphi_{p,q}(u) + \varphi_{q,p}(u) & \text{if } p \neq q \\ \varphi_{q,p}(u) & \text{if } p = q \end{cases}$$

and their adjoints are called symmetrized annihilation operators (in general, we do not assume that the matrices $B(u)$ are symmetric).

**Definition 2.4.** Certain linear combinations of the matricially free creation and annihilation operators are of special interest:

1. matricial $R$-semicircular operators
   $$\omega_{p,q}(u) = \varphi_{p,q}(u) + \varphi_{p,q}^*(u),$$

2. matricial semicircular operators
   $$\hat{\omega}_{p,q}(u) = \hat{\varphi}_{p,q}(u) + \hat{\varphi}_{p,q}^*(u),$$

3. matricial $R$-circular operators
   $$\zeta_{p,q}(u) = \varphi_{p,q}(2u - 1) + \varphi_{q,p}^*(2u),$$

4. matricial circular operators
   $$\eta_{p,q}(u) = \hat{\varphi}_{p,q}(2u - 1) + \hat{\varphi}_{p,q}^*(2u),$$
where \( u \in \mathcal{U} = [t] \) and \( p, q \in [r] \). The corresponding families of arrays of operators will be called matricial \( R \)-semicircular, semicircular, \( R \)-circular and circular systems, respectively.

**Remark 2.1.** Let us make a few comments on the above definition.

1. The operators \( \omega_{p,q}(u) \) were called matricially free Gaussian operators in \([5,7]\) since they play the role of the Gaussian operators in matricially free probability and they are matricially free with respect to the array of states \( (\Psi_{p,q}) \). Here, we prefer the name ‘matricial \( R \)-circular operators’ since they have a special form of the cyclic \( R \)-transform (the corresponding terms in free probability are for instance: ‘free semicircular operators’ and ‘free Gaussian operators’ for the same objects).

2. Similarly, the operators \( \tilde{\omega}_{p,q}(u) \) were called symmetrized Gaussian operators in \([5,7]\). Moreover, the arrays of both matricial semicircular and circular operators are symmetric and it would suffice to use upper-triangular arrays as far as the operators are concerned. However, with any off-diagonal operator indexed by \((p,q)\) we will associate two states, \( \Psi_p \) and \( \Psi_q \), and in some cases it will be convenient to keep square arrays in order to use two operators, one indexed by \((q,p)\) and the other indexed by \((p,q)\).

3. As in the case of circular operators studied by Voiculescu \([14]\), who used \( 2k \) creation and annihilation operators to find a nice realization of \( k \) circular operators

\[
\ell_{2j-1} + \ell^*_{2j},
\]

where \( j = 1, \ldots, k \), we need to double the number of creation and annihilation operators to produce arrays of matricial circular operators in a similar fashion. Thus, we need to take a pair of symmetrized creation operators for any \( u \in \mathcal{U} \). For convenience, understanding that \( \mathcal{U} = [t] \), we label them by \( 2u - 1 \) and \( 2u \), respectively. Recall from \([7]\) that this procedure requires that

\[
B(2u - 1) = B(2u)
\]

for any \( u \in [t] \), even if these matrices are not symmetric.

4. In the case when we want to consider proper subsystems of the above systems, it is convenient to set \( b_{p,q}(u) = 0 \), although we then avoid applying the terminology of Definition 2.4 to single operators if they are trivial.

We close this section with showing that the \( C^* \)-algebras generated by the matricially free creation and annihilation operators are closely related to Cuntz-Krieger algebras. In fact, they are \( C^* \)-algebras of Toeplitz-Cuntz-Krieger type, each with a projection onto the \( r \)-dimensional vacuum space. We obtain a Cuntz-Krieger algebra if we take the quotient by the two-sided ideal generated by this projection.

**Theorem 2.1.** Denote \( \mu = (p,q,u) \) and let us suppose that \( b_{p,q}(u) = 1 \) for any \( (p,q) \in \mathcal{J} \subseteq [r] \) and \( u \in \mathcal{U} \), where \( \mathcal{U} \) is finite. Let

\[
S_{\mu} = \varphi_{p,q}(u) \quad \text{for any} \quad \mu = (p,q,u) \in \mathcal{J} \times \mathcal{U}.
\]

Then

1. each \( S_{\mu} \) is a partial isometry and the \( C^* \)-algebra \( \mathcal{T} \) generated by the family \( \{S_{\mu} : \mu \in \mathcal{J} \times \mathcal{U}\} \) is a Toeplitz-Cuntz-Krieger algebra with a rank \( r \) projection \( p_{\Omega} \) onto the vacuum space,
(2) If $\mathcal{I}$ is the two-sided ideal generated by $p_\Omega$, then the quotient $C^*$-algebra $\mathcal{C} = \mathcal{T}/\mathcal{I}$ is the Cuntz-Krieger algebra associated with the matrix

$$A(\nu, \mu) = \begin{cases} 1 & \text{if } \nu \sim \mu \\ 0 & \text{otherwise} \end{cases}$$

where $\nu = (i, j, u) \sim (k, l, s) = \mu$ if and only if $j = k$.

**Proof.** It follows from the definition of $S_\mu = \varphi_{p,q}(u)$ that it is a partial isometry, where $\mu = (p, q, u)$. Its range projection $s_\mu = S_\mu S_\mu^*$ is the orthogonal projection onto

$$N_\mu = \bigoplus_{\mu} \bigoplus_{\mu \sim \mu_1 \cdots \sim \mu_m} H_\mu \otimes H_{\mu_1} \otimes \cdots \otimes H_{\mu_m},$$

where $H_\mu = H_{p,q}(u)$ and similar notations are used for other Hilbert spaces. The corresponding source projection $r_\mu = S_\mu^* S_\mu$ is the orthogonal projection onto

$$K_\mu = \mathbb{C} \Omega_q \oplus \bigoplus_{\mu \sim \nu} N_\nu,$$

for $\mu = (p, q, u)$, where the direct sum $\bigoplus_{\mu \sim \nu}$ is understood as the sum over those $\nu$ which are matricially related to $\mu$ in the given order, i.e. $\mu \sim \nu$. It can be observed that $r_\mu$ depends only on $q$. Now, the above formulas lead to the relations between range and source projections of the form

$$\sum_{\nu \sim \mu} s_\mu = r_\nu - p_\Omega_q$$

$$\sum_{\mu} s_\mu = 1 - p_\Omega$$

for any $\nu = (p, q, u)$, where $p_\Omega_q$ and $p_\Omega$ are the orthogonal projections onto $\mathbb{C} \Omega_q$ and onto the vacuum space $\Omega = \bigoplus_q \mathbb{C} \Omega_q$, respectively. We can write the first relation in the form

$$\sum_{\nu} A(\nu, \mu) s_\mu = r_\nu - p_\Omega_q,$$

using the matrix $A$ defined above. This proves that the $C^*$-algebra $\mathcal{T}$ generated by partial isometries $S_\mu$ is the Toeplitz-Cuntz-Krieger algebra with the projection $p_\Omega$ of rank $r$. This algebra has a two-sided ideal $\mathcal{I}$ generated by $p_\Omega$. The quotient algebra $\mathcal{T}/\mathcal{I}$ is clearly the Cuntz-Krieger algebra associated with the matrix $A$, which completes the proof. \hfill \blacksquare

**Example 2.1.** One of the simplest examples of Cuntz-Krieger algebras obtained in the framework of matricial freeness is that associated with $\mathcal{U}$ consisting of one element and

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

where we take the basis vectors $\mu = (p, q)$ in the following order: $(1, 1), (1, 2), (2, 1), (2, 2)$.

For bigger $r$, we obtain matrices with similar patterns of ones and zeros. Of course, if we consider $\mathcal{J}$ to be a proper subset of $[r] \times [r]$ or consider larger sets $\mathcal{U}$, then we can obtain a much larger class of Cuntz-Krieger algebras.
3. Matricial realizations

Instead of the Hilbert space approach used in [5,7], we will find matricial realizations of our systems of operators by using circular and semicircular systems [14]. In this realization, all operators are obtained by taking linear combinations of tensor products of free creation and annihilation operators and matrix units.

In our study of distributions of the matricial systems of Definition 2.4, we use the family of vector states $\Psi_q$ on $B(M)$ of the form

$$\Psi_q(a) = \langle a\Omega_q, \Omega_q\rangle,$$

where $q \in [r]$. The states $\Psi_q$ are then used to construct the array $(\Psi_{p,q})$ defined in the Introduction, needed when considering the definition of matricial freeness or symmetric matricial freeness.

We have shown in [5] that one array $(\omega_{p,q})$ is matricially free with respect to $(\Psi_{p,q})$ and that the corresponding array $(\hat{\omega}_{p,q})$ is ‘symmetrically matricially free’. In [7], this result was generalized to the corresponding arrays of collective operators obtained by summing over $u \in U$. Another proof which uses free probability rather than Hilbert space methods will be given in Section 5. In order to be able to use this new language, we will find new realizations of the considered matricial systems of operators.

Let $A$ be a unital *-algebra and let $\varphi$ be a state on $A$. We then obtain a *-probability space $(A, \varphi)$. Consider then the algebra of matrices $M_r(A) \cong A \otimes M_r(\mathbb{C})$ with the natural involution

$$(a \otimes e(p,q))^* = a^* \otimes e(q,p)$$

for any $a \in A$ and $p, q \in [r]$. Consider the states $\Phi_1, \ldots, \Phi_r$ on $M_r(A)$ of the form

$$\Phi_q = \varphi \otimes \psi_q$$

for any $q \in [r]$, where $\psi_q(b) = \langle be(q), e(q)\rangle$ and $e(1), \ldots, e(r)$ is the canonical orthonormal basis in $\mathbb{C}^r$. If $A$ is a $C^*$-algebra, we can work in the category of $C^*$-probability spaces.

We would like to compare the (*-) distributions of the operators defined in Definition 2.4 in the states $\Psi_q$ with the corresponding (*-) distributions of certain elements of $M_r(A)$ in the states $\Phi_q$, respectively. When speaking of (*-) distributions of a family of elements of a *-algebra equipped with a family of states we shall understand the family of all mixed (*-) moments of these elements in all states from this family.

For that purpose, let us suppose that in the given $C^*$-probability space $(A, \varphi)$ we have a family of free creation operators,

$$\{\ell(p, q, u) : p, q \in [r], u \in U\}$$

which is *-free with respect to $\varphi$, with $\ell(p, q, u)^*$ being the free annihilation operator corresponding to $\ell(p, q, u)$, for which

$$\ell(p, q, u)^*\ell(p', q', u') = \delta_{p,p'}\delta_{q,q'}\delta_{u,u'}b_{p,q}(u),$$

for any $p, q, u, p', q', u'$, where the corresponding matrices $B(u) = (b_{p,q}(u))$ consist of non-negative numbers. Thus, in general, we do not assume that our free creation and annihilation operators are standard since the entries of $B(u)$ do not have to be equal to one. The number $b_{p,q}(u)$ will be called the covariance of $\ell(p, q, u)$. A family of arrays of the above form will be called a system of free creation operators. If it contains only standard free creation operators, we will say that this system is standard.
By a generalised circular element of a *-probability space \((\mathcal{A}, \varphi)\) we understand an element whose *-distribution agrees with the *-distribution of the sum of the form
\[
c = \ell_1 + \ell_2^*,
\]
where \(\ell_1\) and \(\ell_2\) are free creation operators with covariances \(\alpha > 0\) and \(\beta > 0\), respectively, which are *-free with respect to \(\varphi\). If we need to be more specific, we will call the above element an \((\alpha, \beta)\)-circular element. In particular, when \(\alpha = \beta\), it is called the circular element with covariance \(\alpha\). If \(\alpha = 1\), the circular element will be called standard. Let us point out that if we have a family of states on a given *-algebra, the state with respect to which a given element has (generalized) circular *-distribution has to be specified. If \(\mathcal{A}\) is a C*-algebra, it is customary to speak of operators instead of elements.

We shall use the above system of free creation operators as well as semicircular and (generalized) circular systems [14]. By a (generalized) circular system in \((\mathcal{A}, \varphi)\) we shall understand the family
\[
\{g(p, q, u) : p, q \in [r], u \in \mathcal{U}\}
\]
where each \(g(p, q, u)\) is a (generalized) circular element and the whole family is *-free with respect to \(\varphi\). We will also use a semicircular system in \((\mathcal{A}, \varphi)\) of the form
\[
\{f(p, u) : p \in [r], u \in \mathcal{U}\}
\]
where each \(f(p, u)\) is a semicircular element and the whole family is free with respect to \(\varphi\). We will understand, as in [14, Proposition 2.8], that this semicircular system is free from the (generalized) circular system defined above. In fact, one can construct both systems from one semicircular family as in [14]. The same terminology will be used for subfamilies of these families.

We are ready to give matricial representations of the matricial systems of Definition 2.4. In particular, this will enable us to observe that the \(R\)-circular operators are generators of free group factors used by Voiculescu [14, Theorem 3.3].

**Lemma 3.1.** With the above notations, suppose that the covariance of each \(\ell(p, q, u)\) is \(b_{p,q}(u)\) and that each \(g(p, q, u)\) is \((\alpha, \beta)\)-circular, where \(\alpha = b_{p,q}(u)\) and \(\beta = b_{q,p}(u)\).

1. The joint distributions of the operators
\[
(\ell(p, q, u) \otimes e(p, q)) + (\ell(p, q, u)^* \otimes e(q, p)),
\]
where \(p, q \in [r]\) and \(u \in \mathcal{U}\), agree with the joint distributions of the corresponding matricial \(R\)-semicircular operators \(\omega_{p,q}(u)\).

2. The joint distributions of the operators
\[
\begin{cases}
g(p, q, u) \otimes e(p, q) + (g(p, q, u)^* \otimes e(q, p)) & \text{if } p < q \\
f(p, u) \otimes e(p, p) & \text{if } p = q
\end{cases}
\]
where \(p, q \in [r]\) with \(p \leq q\) and \(u \in \mathcal{U}\), agree with the joint distributions of the corresponding matricial semicircular operators \(\widehat{\omega}_{p,q}(u)\).

3. The joint *-distributions of operators
\[
\begin{cases}
g(p, q, u) \otimes e(p, q) + g(q, p, u) \otimes e(q, p) & \text{if } p < q \\
g(p, p, u) \otimes e(p, p) & \text{if } p = q
\end{cases}
\]
where \(p, q \in [r]\) with \(p \leq q\) and \(u \in \mathcal{U}\), agree with the joint *-distributions of the corresponding matricial circular operators \(\eta_{p,q}(u)\).
(4) The joint *-distributions of the operators
\[ g(p, q, u) \otimes e(p, q) \]
where \( p, q \in [r] \) and \( u \in \mathcal{U} \), agree with the joint *-distributions of the corresponding matricial \( R \)-circular operators \( \zeta_{q,p}(u) \).

Here, (*-) distributions of the given operators in the states \( \Phi_q \) are compared against those of Definition 2.4 in the states \( \Psi_q \).

Proof. Let us first show that the *-distributions of the arrays
\[ \{ \ell(p, q, u) \otimes e(p, q) : p, q \in [r], u \in \mathcal{U} \} \]
with respect to the states \( \Phi_q \) agree with the corresponding *-distributions of the arrays of matricially free creation operators
\[ \{ \phi_{p,q}(u) : p, q \in [r], u \in \mathcal{U} \} \]
with respect to the states \( \Psi_q \), respectively. For that purpose, we shall use the isometric embedding of the second underlying Hilbert space into the first one as described below. To obtain shorter formulas, we restrict ourselves to the case when all elements of \( B(\mathcal{U}) \) are equal to one for all \( u \in \mathcal{U} \). Let \( \mathcal{F}(\mathcal{H}) \) be the free Fock space over
\[ \mathcal{H} = \bigoplus_{1 \leq p,q \leq r} \mathcal{H}(p, q, u), \]
where \( \mathcal{H}(p, q, u) = \mathbb{C} e(p, q, u) \) for any \( p, q, u \) with each \( e(p, q, u) \) being a unit vector and denote by \( \Omega \) the vacuum vector in this Fock space. The embedding
\[ \tau : \mathcal{M} \rightarrow \mathcal{F}(\mathcal{H}) \otimes \mathbb{C}^r \]
is given by the formulas
\[ \tau(\Omega_q) = \Omega \otimes e(q) \]
\[ \tau(e_{p_1,p_2}(u_1) \otimes \ldots \otimes e_{p_n,q}(u_n)) = e(p_1, p_2, u_1) \otimes \ldots \otimes e(p_n, q, u_n) \otimes e(p_1), \]
for any \( q, p_1, \ldots, p_n \) and \( u_1, \ldots, u_n \), where \( \{ e(1), \ldots, e(r) \} \) is the canonical basis in \( \mathbb{C}^r \).

We then have
\[ \tau \phi_{p,q}(u) = (\ell(p, q, u) \otimes e(p, q)) \tau \]
since
\[ \tau(\phi_{p,q}(u)\Omega_q) = \tau(e_{p,q}(u)) = e(p, q, u) \otimes e(p) \]
\[ = (\ell(p, q, u) \otimes e(p, q)) (\Omega \otimes e(q)) = (\ell(p, q, u) \otimes e(p, q)) \tau(\Omega_q) \]
and
\[ \tau(\phi_{p,q}(u) e_{q,p_2}(u_1) \otimes \ldots \otimes e_{p_n,t}(u_n)) \]
\[ = \tau(\phi_{p,q}(u) e_{q,p_2}(u_1) \otimes \ldots \otimes e_{p_n,t}(u_n)) \]
\[ = e(p, q, u) \otimes e(q, p_2, u_1) \otimes \ldots \otimes e(p_n, t, u_n) \otimes e(p) \]
\[ = (\ell(p, q, u) \otimes e(p, q)) (e(q, p_2, u_1) \otimes \ldots \otimes e(p_n, t, u_n) \otimes e(q)) \]
\[ = (\ell(p, q, u) \otimes e(p, q)) \tau(e_{q,p_2}(u_1) \otimes \ldots \otimes e_{p_n,t}(u_n)) \]
for any values of indices and arguments, whereas the actions onto the remaining basis vectors gives zero. This proves that \( \phi_{p,q}(u) \) intertwines with \( \ell(p, q, u) \otimes e(p, q) \). Therefore, the *-distributions of \( \phi_{p,q}(u) \) under the states \( \Psi_k \) agree with the corresponding *-distributions of \( \ell(p, q, u) \otimes e(p, q) \) under the states \( \Phi_k \), respectively, which finishes the
proof of (1). Now, the remaining operators are linear combinations of the matricially free creation and annihilation operators. If we identify these with \( \ell(p, q, u) \otimes e(p, q) \) and \( \ell(p, q, u)^* \otimes e(q, p) \), respectively, we obtain

\[
\hat{\omega}_{p,q}(u) = (\ell(p, q, u) + \ell(q, p, u)^*) \otimes e(p, q) \\
+ (\ell(q, p, u) + \ell(p, q, u)^*) \otimes e(q, p) \\
= g(p, q, u) \otimes e(p, q) + g(p, q, u)^* \otimes e(q, p),
\]

for any \( p < q \) and any \( u \), where

\[
g(p, q, u) = \ell(p, q, u) + \ell(q, p, u)^*
\]

is a generalized circular operator. The realization for diagonal operators is

\[
f(p, u) = \ell(p, p, u) + \ell(p, p, u)^*
\]

for any \( p, u \). Clearly, \(*\)-freeness with respect to \( \varphi \) of the system consisting of circular operators \( g(p, q, u) \) for all \( p < q \) and \( u \) and semicircular ones \( f(p, u) \) for all \( p, u \) follows from \(*\)-freeness of the system of creation operators \( \ell(p, q, u) \) for all \( p, q, u \), which gives (2). Similarly,

\[
\eta_{p,q}(u) = (\ell(p, q, 2u - 1) + \ell(q, p, 2u)^*) \otimes e(p, q) \\
+ (\ell(q, p, 2u - 1) + \ell(p, q, 2u)^*) \otimes e(q, p) \\
= g(p, q, u) \otimes e(p, q) + g(q, p, u) \otimes e(q, p),
\]

where

\[
g(p, q, u) = \ell(p, q, 2u - 1) + \ell(q, p, 2u)^*
\]

are generalized circular operators for any \( p < q \) and any \( u \in [t] \). We need to remember here that in the procedure of doubling the number of creation and annihilation operators we assume that \( B(2u - 1) = B(2u) \). This leads to \( \ell(p, q, 2u - 1) \) and \( \ell(p, q, 2u) \) with the same covariances, say \( b_{p,q}(u) \), for any \( p, q \in [r] \), but possibly different from the covariances of \( \ell(q, p, 2u - 1) \) and \( \ell(q, p, 2u) \), equal to, say \( b_{q,p}(u) \). Again, the realization for diagonal operators and \(*\)-freeness with respect to \( \varphi \) of the whole family of circular operators are obvious, which proves (3). Finally, it easily follows from this that

\[
\zeta_{p,q}(u) = g(p, q, u) \otimes e(p, q)
\]

with exactly the same \( g(p, q, u) \) as above, where the family of circular operators is \(*\)-free with respect to \( \varphi \), which proves (4).

\[\square\]

4. Distributions and \(*\)-distributions

Let us collect certain facts about the \((\ast^{-})\) distributions of the matricial operators. Some of them were proved in [5] and [7], but most facts are new.

In order to use the language of free probability, we will use the realizations of Lemma 3.1 and thus we will speak of the \((\ast^{-})\) distributions in the states \( \Phi_q \).

**Proposition 4.1.** Let \( p, q, u \) be arbitrary.

1. If \( p \neq q \), the distribution of \( \omega_{p,q}(u) \) in the state \( \Phi_q \) is the Bernoulli law with mean zero and variance \( b_{p,q}(u) \).
2. If \( p = q \), the distribution of \( \omega_{p,q}(u) \) in the state \( \Phi_q \) is the semicircle law with mean zero and variance \( b_{q,q}(u) \).
(3) If $b_{p,q}(u) = b_{q,p}(u)$, then the distributions of $\hat{\omega}_{p,q}(u)$ in the states $\Phi_p$ and $\Phi_q$ are semicircle laws with mean zero and variance $b_{p,q}(u)$.

(4) If $b_{p,q}(u) = b_{q,p}(u)$, then the *-distributions of $\eta_{p,q}(u)$ in the states $\Phi_p$ and $\Phi_q$ are circular laws with mean zero and covariance $b_{p,q}(u)$.

Proof. For (1)-(3), see [7, Proposition 2.3 and Proposition 3.5]. For instructive reasons, let us show how to prove these, using the language of free probability and matrix calculus instead of the canonical Hilbert space representation. If we identify an off-diagonal $\omega_{p,q}(u)$ with

$$\ell(p, q, u) \otimes e(p, q) + \ell(p, q, u)^* \otimes e(q, p),$$

we can observe that its only non-trivial *-moments are equal to

$$\Phi_q((\ell(p, q, u)^* \otimes e(q, p))(\ell(p, q, u) \otimes e(p, q)))^k = (b_{p,q}(u))^k$$

and therefore only these contribute to the moments of even orders of $\omega_{p,q}(u)$, which proves (1). In turn, the action of a diagonal $\omega_{q,q}(u)$ on $\mathcal{M}_q$ can be identified with the action of a free Gaussian operator of mean zero and variance $b_{q,q}(u)$ on the free Fock space, which proves (2). In a similar manner we prove (3). To prove (4) for the off-diagonal case (the diagonal one is obvious), fix $p \neq q$ and identify $\eta_{p,q}(u)$ with

$$\eta_1 + \eta_2^* + \eta_3 + \eta_4^*,$$

where

$$\eta_1 = \ell_1 \otimes e(p, q), \quad \eta_2 = \ell_2 \otimes e(q, p), \quad \eta_3 = \ell_3 \otimes e(q, p), \quad \eta_4 = \ell_4 \otimes e(p, q),$$

where $\ell_1, \ell_2, \ell_3, \ell_4$ are free creation operators with respect to $\varphi$ and their covariances are all equal. Let us show that there is a bijection between the *-moments of $\eta_{p,q}(u)$ and the *-moments of a circular operator of the form $\ell_1 + \ell_2^*$. When we compute the *-moments of $\eta_{p,q}(u)$ in the state $\Phi_q$, we obtain mixed *-moments of the form

$$\Phi_q(\eta_{i_1}^{\epsilon_1} \eta_{i_2}^{\epsilon_2} \cdots \eta_{i_m}^{\epsilon_m}),$$

where $i_1, \ldots, i_m \in \{1, 2, 3, 4\}$ and $\epsilon_1, \ldots, \epsilon_m$ are either * or no symbol. The non-vanishing mixed *-moments of this form are in bijection with the non-vanishing mixed *-moments

$$\varphi(\ell_{k_1}^{\epsilon_1} \ell_{k_2}^{\epsilon_2} \cdots \ell_{k_m}^{\epsilon_m}),$$

where the bijection is implemented by the mapping

$$\ell_1 \rightarrow \begin{cases} \eta_1 & \text{at even positions} \\ \eta_3 & \text{at odd positions} \end{cases} \quad \ell_2 \rightarrow \begin{cases} \eta_2 & \text{at odd positions} \\ \eta_4 & \text{at even positions} \end{cases}$$

and positions in words are counted from the left. The starred operators $\ell_1^*, \ell_2^*$ are mapped onto the corresponding $\eta_1^*, \eta_2^*, \eta_3^*, \eta_4^*$, respectively, with the odd positions interchanged with the even ones. Note that the definition of this bijection follows from the fact that operators with odd and even positions, respectively, must have matrix units $e(q, p)$ and $e(p, q)$, respectively. An analogous proof holds for $\Psi_p$ and thus the proof of (4) is completed.

\[\blacksquare\]

Remark 4.1. If $b_{p,q}(u) = \gamma_1$ and $b_{q,p}(u) = \gamma_2$ are not the same, then the *-distributions of $\eta_{p,q}(u)$ in the states $\Psi_p$ and $\Psi_q$ are neither circular nor generalized circular. In the
proof of Proposition 4.1 we have to replace standard free creation operators by those with covariances
\[ \varphi(\ell_4^*\ell_1) = \varphi(\ell_4^*\ell_4) = \gamma_1 \quad \text{and} \quad \varphi(\ell_2^*\ell_2) = \varphi(\ell_3^*\ell_3) = \gamma_2. \]

Then the *-distribution of an off-diagonal \( \eta_{p,q}(u) \) in the state \( \Phi_q \) does not agree with the *-distribution of an \((\alpha, \beta)\)-circular operator. In order to see this, it suffices to use the bijection described in the proof of Proposition 4.1, where \( \ell_1 \) is replaced either by \( \eta_1 \) or \( \eta_3 \) and \( \ell_2 \) by \( \eta_2 \) or \( \eta_4 \) depending on the positions of these operators in the given *-moments and both these pairs have different covariances. In the computation of *-moments of an \((\alpha, \beta)\)-circular operator \( c = \ell_1 + \ell_2^* \) it is not the case and the pairings \( \ell_1, \ell_1^* \) and \( \ell_2, \ell_2^* \) contribute \( \alpha \) and \( \beta \), respectively, irrespective of their positions in the *-moments. Instead of a formal proof, we shall present a simple example below.

**Example 4.1.** Consider one off-diagonal operator \( \eta = \eta_{p,q} \) of the form

\[ \eta = (\ell_1 + \ell_2^*) \otimes e(p,q) + (\ell_3 + \ell_4^*) \otimes e(q,p). \]

where \( \varphi(\ell_1^*\ell_1) = \varphi(\ell_2^*\ell_2) = \varphi(\ell_3^*\ell_3) = \varphi(\ell_4^*\ell_4) = \gamma_1 \) and \( \varphi(\ell_2^*\ell_2) = \varphi(\ell_3^*\ell_3) = \gamma_2 \). We obtain

\[
\begin{align*}
\Phi_q(\eta^*\eta^*\eta) & = \varphi(\ell_1^*\ell_1^*\ell_1^*\ell_1) + \varphi(\ell_2^*\ell_2^*\ell_2^*\ell_2) = \gamma_1^2 + \gamma_1 \gamma_2 \\
\Phi_q(\eta^*\eta^*\eta^*) & = \varphi(\ell_1^*\ell_1^*\ell_2^*\ell_2^*) + \varphi(\ell_2^*\ell_2^*\ell_3^*\ell_3^*) = \gamma_2^2 + \gamma_1 \gamma_2 \\
\Phi_q(\eta^*\eta^*\eta) & = \varphi(\ell_1^*\ell_3^*\ell_3^*\ell_1) = \gamma_1 \gamma_2 \\
\Phi_q(\eta^*\eta^*\eta^*) & = \varphi(\ell_2^*\ell_2^*\ell_2^*\ell_4^*) = \gamma_1 \gamma_2
\end{align*}
\]

and it can be seen that these *-moments do not agree with the corresponding *-moments of any generalized circular operator if \( \gamma_1 \neq \gamma_2 \). A similar result is obtained for \( \Phi_p \).

**Remark 4.2.** Under the same assumptions as in Remark 4.1, we also obtain generalizations of semicircle distributions. As we showed in [7, Example 5.2] (see also [7, Corollary 3.2]), the distribution of \( \tilde{\omega}_{p,q}(u) \) in the state \( \Psi_q \) is associated with the 2-periodic sequence of Jacobi coefficients \( (\gamma_1, \gamma_2, \gamma_1, \gamma_2, \ldots) \), where \( \gamma_1 = b_{q,p}(u) \) and \( \gamma_2 = b_{p,q}(u) \), and will be called the \((\gamma_1, \gamma_2)\)-semicircle law, or simply a generalized semicircle law.

Let us finally study the *-distributions of the \( R \)-circular operators.

**Proposition 4.2.** Let \( \zeta := \zeta_{p,q}(u) \), \( \zeta^* := \zeta_{p,q}^*(u) \) and suppose that \( b := b_{p,q}(u) = b_{q,p}(u) \) for fixed \( p,q,u \).

1. If \( p = q \), then the *-distribution of \( \zeta \) with respect to \( \Phi_q \) is circular with mean zero and covariance \( b \).
2. If \( p \neq q \), then the *-distributions of \( \zeta \) with respect to \( \Phi_q \) and \( \Phi_p \) are given by *-moments of the form

\[ \Phi_q((\zeta^*\zeta)^k) = \Phi_p((\zeta^*\zeta)^k) = C_k b^k \]

where \( C_0 = 1 \) and \( C_k \) is the \( k \)-th Catalan number for \( k \in \mathbb{N} \), with the remaining *-moments equal to zero.

**Proof.** By Lemma 3.1, we can use the realization

\[ \zeta = c \otimes e(p,q) \]
where $c$ is a circular operator with covariance $b$ with respect to $\varphi$. Thus, the computation of the $*$-distribution of $\zeta$ reduces to the computation of the $*$-distribution of $c$. In the diagonal case, when $\zeta = \zeta_{p,q}(u)$, we obtain

$$\Phi_q(\zeta^{e_1} \ldots \zeta^{e_k}) = \varphi(c^{e_1} \ldots c^{e_k})$$

for any $e_1, \ldots, e_k \in \{1, *\}$ and $k \in \mathbb{N}$, which proves (1). In the off-diagonal case, when $\zeta = \zeta_{p,q}(u)$, the non-trivial $*$-moments of $\zeta$ in the state $\Phi_q$ are alternating since $\zeta^2 = 0$ and must end with $\zeta$ since $\zeta^* \Omega_q = 0$. Thus, they are of the form

$$\Phi_q((\zeta^* \zeta)^k) = \varphi((c^* c)^k) = C_k b^k$$

where $k \in \mathbb{N}$. The remaining $*$-moments vanish; in particular, $\Phi_q((\zeta^* \zeta)^k) = 0$ for any $k \in \mathbb{N}$. A similar computation gives the $*$-distribution of $\zeta$ with respect to $\Phi_p$, except that the non-trivial $*$-moments must end with $\zeta^*$, which completes the proof of (2). ■

It is well known that a circular operator can be represented as a square matrix of $*$-free circular operators, which follows from a generalization of [13, Proposition 2.8] to circular families. In our approach, it is a consequence of [7, Theorem 9.1], which leads to the representation of a circular operator as a sum of $R$-circular operators. An analogous result for a semicircular operator is a consequence of [7, Theorem 5.1].

The random matrix context on which we concentrated in [7] involves the so-called dimension matrix

$$D = \text{diag}(d_1, \ldots, d_r)$$

consisting of the asymptotic dimensions of blocks of a given sequence of random matrices (see Section 9). The limit ($*$)-distributions of blocks are described in terms of various systems of operators with (co)variances $b_{p,q}(u)$, where the corresponding matrices are of the form

$$B(u) = DV(u),$$

where $V(u)$ is a matrix of block (co)variances. If we assume that the considered random matrices have i.i.d. instead of i.b.i.d. variables, we can set all these (co)variances to be equal to one. Then $b_{p,q}(u) = d_p$ for any $p, q, u$, which explains why we use asymptotic dimensions when describing decompositions of the semicircular and circular operators of the form given below (a slight generalization of the well known Voiculescu’s decomposition of semicircular and circular operators [14]).

**Proposition 4.3.** Let $\{g(p,q) : p, q \in [r]\}$ be a generalized circular system, where $g(p,q)$ is $(d_p, d_q)$-circular, $*$-free from a semicircular system $\{f(q,q) : q \in [r]\}$, where $f(q,q)$ has variance $d_q$, in a $C^*$-probability space $(A, \varphi)$. Then the matrix

$$\begin{pmatrix}
  f(1,1) & g(1,2) & \cdots & g(1,r) \\
  g(1,2) & f(2,2) & \cdots & g(2,r) \\
  \vdots & \vdots & \ddots & \vdots \\
  g(1,r) & g(2,r) & \cdots & f(r,r)
\end{pmatrix}$$

has the standard semicircular distribution and the matrix

$$\begin{pmatrix}
  g(1,1) & g(1,2) & \cdots & g(1,r) \\
  g(2,1) & g(2,2) & \cdots & g(2,r) \\
  \vdots & \vdots & \ddots & \vdots \\
  g(r,1) & g(r,2) & \cdots & g(r,r)
\end{pmatrix}$$
has the standard circular distribution with respect to the state \( \Phi = \sum_{q=1}^{r} d_q \Phi_q \) on \( M_r(A) \cong A \otimes M_r \).

Proof. It is well known that a Hermitian Gaussian random matrix \( Y(n) = Y(u,n) \) with i.i.d. complex entries with covariances one, where \( u \) is fixed (and thus omitted) converges under the trace to the standard semicircular operator. On the other hand, if we divide \( Y(n) \) into blocks and use [7, Theorem 5.1], we obtain

\[
Y(n) \to \sum_{p,q=1}^{r} \omega_{p,q}
\]
as \( k \to \infty \) under the trace, where convergence is in the sense of *-moments under any partial trace and thus under the trace to the *-moments of the operator on the right-hand side with respect to \( \Phi \), where \( \omega_{p,q} \) has the variance \( b_{p,q} = d_p \). By Lemma 3.1(2), we obtain the desired matrix representation of the limit operator.

We obtain the standard circular distribution in the non-Hermitian case in a similar way. Dividing \( Y(n) = Y(u,n) \) into blocks and using [7, Theorem 9.1] gives

\[
Y(n) \to \sum_{p,q=1}^{r} \zeta_{p,q}
\]
as \( k \to \infty \) under the trace. By Lemma 3.1(4), we obtain the desired matrix representation of \( \zeta \). This completes the proof. In particular, if we set \( d_q = 1/r \) for each \( q \), we obtain the representation in Voiculescu’s paper [14]. \( \blacksquare \)

Let us also describe the decomposition of an off-diagonal circular operator as the sum of two \( R \)-circular ones in a free probability framework. For that purpose, we shall use the free Fock space

\[
\mathcal{F}(Ce_1 \oplus Ce_2) = \mathcal{F}_1 \oplus \mathcal{F}_2,
\]
where

\[
\mathcal{F}_1 = \bigoplus_{k=1}^{\infty} (Ce_1 \oplus Ce_2)^{\otimes (2k-1)}
\]
\[
\mathcal{F}_2 = \mathbb{C} \Omega \oplus \bigoplus_{k=1}^{\infty} (Ce_1 \oplus Ce_2)^{\otimes (2k)}
\]
are its odd and even subspaces, respectively, with \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) denoting the associated orthogonal projections. We denote by \( \varphi \) the vacuum expectation.

**Proposition 4.4.** Under the assumptions of Proposition 4.2, the joint *-distribution of the off-diagonal pair \( \{\zeta_{p,q}, \zeta_{q,p}\} \) with respect to \( \Phi_q \) agrees with that of the pair \( \{c\mathcal{P}_2, c\mathcal{P}_1\} \) with respect to \( \varphi \), where \( c \) is a circular operator with mean zero and covariance \( b \).

Proof. In the proof of Proposition 4.1, we used the decomposition

\[
\eta_{p,q} = \zeta_{p,q} + \zeta_{q,p} = (\ell_1 + \ell_2^*) \otimes e(p,q) + (\ell_3 + \ell_4^*) \otimes e(q,p).
\]
Invoking the bijection described in that proof, we can observe that when the covariances are symmetric and we compute the joint *-distribution of the pair \( \{\zeta_{p,q}, \zeta_{q,p}\} \) in the state \( \Phi_q \), we can replace \( \ell_3 + \ell_4^* \) by \( \ell_1 + \ell_2^* \) and observe that \( \zeta_{q,p} \) corresponds to the
action of $c = \ell_1 + \ell_2^*$ onto the odd subspace $\mathcal{F}_1$, whereas $\zeta_{p,q}$ corresponds to the action of $c$ onto the even subspace $\mathcal{F}_2$, which completes the proof.

One can also derive a canonical product realization of each $R$-circular operator. For that purpose, we can use the fact that any so-called $R$-diagonal element $y$ (its definition is given in Section 7) in a $C^*$-probability space can be realized in the form of a product $y = vw$ of two elements of some $C^*$-probability space $(\mathcal{A}, \varphi)$, where $v$ is a Haar unitary and $w$ is positive with distribution $\sqrt{y^*y}$ and $v, w$ are *-free. In particular, the circular elements $c$ are of this type and if $c$ has covariance one, $w$ is the so-called quarter-circular distribution given by density $\frac{1}{\pi} \sqrt{4 - t^2} dt$ on $[0, 2]$. Recall that by a Haar unitary we understand a unitary $v$ such that $\varphi(v^n) = \delta_{n,0}$ for any $n \in \mathbb{Z}$. Of course, all *-moments of a Haar unitary can be easily determined since $v$ commutes with $v^*$ and thus, by unitarity, $\varphi(v^k(v^*)^l) = \delta_{k,l}$. The framework of $W^*$-probability spaces offers the advantage of $v, w$ being the elements of the same space. This decomposition for circular elements (in the $W^*$-probability case called polar decomposition) is due to Voiculescu [14]. For a more general setting of $R$-diagonal elements, see [10].

**Proposition 4.5.** Let $c$ be a standard circular element in a $W^*$-probability space with a faithful trace $\varphi$ and let $\Phi_q = \varphi \otimes \psi_q$. Then the *-distribution of $\zeta = c \otimes e(p,q)$ with respect to $\Phi_q$ can be realized in the form $\theta \chi$, where $\theta$ is a partial isometry and $\chi$ is positive with the quarter-circular distribution. Moreover, if $p = q$, then the pair $(\theta, \chi)$ is *-free, and if $p \neq q$, then the pair $(\theta, \chi)$ is *-monotone independent.

**Proof.** Let $c = vw$, where $v$ is a Haar unitary and $w$ is positive with a quarter circular distribution with respect to $\varphi$. Therefore, we obtain the decomposition

$$\zeta = \theta \chi$$

where

$$\theta = v \otimes e(p,q) \quad \text{and} \quad \chi = w \otimes e(q,q).$$

In fact, this type of decomposition was used by Voiculescu in the construction of generators of free group factors [14, Theorem 3.3]. It is easy to see that $\theta$ is a partial isometry with the range and source projections

$$\theta \theta^* = 1 \otimes e(p,p) \quad \text{and} \quad \theta^* \theta = 1 \otimes e(q,q),$$

respectively. Moreover, $\chi$ is positive and has the quarter-circular distribution with respect to $\Phi_q$ since $w$ has these properties with respect to $\varphi$. Let us now prove the properties related to independence of $\theta$ and $\chi$. The diagonal case easily follows from *-freeness of $v$ and $w$. Let us consider the off-diagonal case. A *-polynomial in $\theta$ is of the form

$$a = \alpha(v \otimes e(p,q)) + \beta(v^* \otimes e(q,p)) + \gamma(1 \otimes e(q,q)) + \delta(1 \otimes e(p,p))$$

where we used the properties of the matrix units and unitarity of $v$. In turn, a polynomial in $\chi$ is of the form

$$b = p(w) \otimes e(q,q)$$
where \( p(w) \) is a polynomial in \( w \). If \( a_1, \ldots, a_n \) are \(*\)-polynomials in \( \theta \) and \( b_1, \ldots, b_{n-1} \) are polynomials in \( \chi \), we obtain

\[
\Phi_q(a_1b_1a_2 \ldots b_{n-1}a_n) = \gamma_1\gamma_2 \ldots \gamma_n\Phi_q(p_1(w) \ldots p_{n-1}(w) \otimes e(q, q)) = \Phi_q(a_1)\Phi_q(a_2) \ldots \Phi_q(a_n)\Phi_q(b_1 \ldots b_{n-1})
\]

which gives \(*\)-monotone independence of \( \theta \) and \( \chi \).  

**Remark 4.3.** An \( R \)-diagonal partial isometry is called an \((\alpha, \beta)\)-Haar partial isometry with respect to \( \varphi \) if \( \varphi(v^*v) = \alpha \) and \( \varphi(vv^*) = \beta \) [10]. Using this terminology, the \( R \)-diagonal isometry \( \theta \) can be called a \((1,0)\)-Haar partial isometry with respect to \( \Phi_q \). At the same time, \( \theta \) is a \((0,1)\)-Haar partial isometry with respect to \( \Phi_p \).  

We already know from Proposition 4.1 that if \( b_{p,q}(u) = b_{q,p}(u) \), each operator \( \eta_{p,q}(u) \) is circular and thus its polar decomposition is well known. However, one can also use Proposition 4.4 to derive it as a consequence of the polar decompositions of \( R \)-circular operators.  

**Corollary 4.1.** Under the assumptions of Proposition 4.4, the \(*\)-distribution of \( \eta_{p,q}(u) \) with respect to \( \Phi_q \) can be realized in the form \( \theta \chi \), where \( \theta \) is a Haar unitary and \( \chi \) is positive with the quarter-circular distribution. Moreover, \( \theta \) and \( \chi \) are \(*\)-free with respect to \( \Phi_q \).  

**Proof.** Since \( \zeta_{q,q}(u) = \eta_{q,q}(u) \), the polar decomposition of \( \zeta_{q,q}(u) \) coincides with that for the circular operator \( \eta_{q,q}(u) \). In the off-diagonal case, we have

\[
\eta_{p,q}(u) = \zeta_{p,q}(u) + \zeta_{q,p}(u)
\]

and we can use Proposition 4.4 for both summands and realize the \(*\)-distributions of \( \zeta_{p,q}(u) \) and \( \zeta_{q,p}(u) \) in the form

\[
\theta_{p,q}\chi_{p,q} = (u_1 \otimes e(p, q))(w_1 \otimes e(q, q))
\]

\[
\theta_{q,p}\chi_{q,p} = (u_2 \otimes e(q, p))(w_2 \otimes e(p, p)),
\]

respectively, where \( u_1, u_2 \) are \(*\)-free and \( w_1, w_2 \) are free with respect to \( \varphi \). The \(*\)-distribution of \( \eta_{p,q}(u) \) agrees with that of \( \theta \chi \), where

\[
\theta = u_1 \otimes e(p, q) + u_2 \otimes e(q, p)
\]

\[
\chi = w_1 \otimes e(q, q) + w_2 \otimes e(p, p)
\]

since cross multiplications give zero. Now, it is not hard to show that \( \theta \) is a Haar unitary with respect to \( \Phi_q \). Namely,

\[
\theta^*\theta = u_1^*u_1 \otimes e(q, q) + u_2^*u_2 \otimes e(p, p) = 1 \otimes e(q, q) + 1 \otimes e(p, p)
\]

which can be identified with the unit in the unital \(*\)-algebra generated by \( \eta_{p,q}(u) \). A similar equation holds for \( \theta\theta^* \). Moreover,

\[
\Phi_q(\theta^n) = \varphi(u_{i_1} \ldots u_{i_n}) = 0
\]

where \( i_1 \neq \ldots \neq i_n \) and a similar equation gives \( \Phi_q((\theta^*)^n) = 0 \). Next, it is easy to see that \( \chi \) has the quarter circular distribution with respect to \( \Phi_q \). Finally, the proof that \( \theta \) and \( \chi \) are \(*\)-free with respect to \( \Phi_q \) follows from \(*\)-freeness of \( u_1, u_2, w_1, w_2 \) (the details are omitted). This completes the proof of our assertions.
Remark 4.4. Proposition 4.5 can be generalized to \( \zeta = c \otimes e(p, q) \), where \( c \) is a generalized circular element. A polar decomposition of \((1, \beta)\)-circular elements has been found by Shlyakhtenko [12] for \( 0 < \beta < 1 \). Namely, again \( c = vw \), where \( v, w \) are *-free, except that \( v \) is now a \((1, \beta)\)-Haar partial isometry and \( w \) is positive with distribution of \( \sqrt{c^*c} \), where

\[
\frac{\sqrt{4\lambda - (t - (1 + \lambda))^2}}{2\pi \lambda t} dt
\]

is the distribution of \( c^*c \) (all moments except the zeroth moment agree with \( 1/\lambda \) times the moments of the Marchenko-Pastur law with shape parameter \( \lambda \)). Using this decomposition, one can derive a decomposition of \( \zeta \) in the general case (to treat the case when \( c \) is \((\alpha, \beta)\)-circular, a suitable rescaling is needed).

5. Matricial freeness and symmetric matricial freeness

We would like to recall that the elements of each matricial system of operators are related to each other through the concept of matricial freeness, the definition of which is given below.

Let \( \mathcal{A} \) be a unital algebra, equipped with \( r \) normalized linear functionals \( \varphi_1, \ldots, \varphi_r \), from which we construct the array \( \varphi_{p,q} = \varphi_q \) for any \( p, q \). Let \((\mathcal{A}_{p,q})\) be an array of (in general, non-unital) subalgebras of \( \mathcal{A} \). When considering products of elements from different algebras, we shall need the sets of ‘matricially free tuples’ of various lengths,

\[
\mathcal{K}_m = \{(p_1, p_2), (p_2, p_3), \ldots, (p_m, q_m) : (p_1, p_2) \neq (p_2, p_3) \neq \ldots \neq (p_m, q_m)\},
\]

where the neighboring pairs of indices are not only different (as in free products), but are related to each other as in matrix multiplication. Finally, each \( \mathcal{A}_{p,q} \) is equipped with an internal unit \( 1_{p,q} \), for which \( 1_{p,q} a = a1_{p,q} = a \) for any \( a \in \mathcal{A}_{p,q} \) and by \( \mathcal{I} \) we denote the subalgebra of \( \mathcal{A} \) generated by all internal units, assumed to be commutative. If \( \mathcal{A} \) is a *-algebra or a \( C^* \)-algebra, we make additional natural assumptions that the functionals are positive and the units are projections.

Definition 5.1. The array \((1_{p,q})\) is a matricially free array of units if for any \( q \) it holds that

(a) \( \varphi_q(1_{k,l}) = 1_{q,l} \) for any \( q, k, l \),

(b) \( \varphi_q(b_1 a b_2) = \varphi_q(b_1) \varphi_q(a) \varphi_q(b_2) \) for any \( a \in \mathcal{A} \) and \( b_1, b_2 \in \mathcal{I} \),

(c) if \( a_r \in \mathcal{A}_{p,q} \cap \text{Ker}\varphi_{p,q} \), where \( 1 < r \leq m \), then

\[
\varphi_q(a_1, a_{p_1, q_1}, a_2, \ldots, a_m) = \begin{cases} 
\varphi_q(a a_2 \ldots a_m) & \text{if } ((p_1, q_1), \ldots, (p_m, q_m)) \in \mathcal{K}_m \\
0 & \text{otherwise}
\end{cases}
\]

where \( a \in \mathcal{A} \) is arbitrary and \((p_1, q_1) \neq \ldots \neq (p_m, q_m)\).

Definition 5.2. We say that the array \((\mathcal{A}_{p,q})\) is matricially free with respect to \((\varphi_{p,q})\) if the array of internal units \((1_{p,q})\) is the associated matricially free array of units and

\[
\varphi_q(a_1 a_2 \ldots a_n) = 0 \text{ whenever } a_k \in \mathcal{A}_{p_k,q_k} \cap \text{Ker}\varphi_{p_k,q_k}
\]

for any state \( \varphi_q \), where \((p_1, q_1) \neq \ldots \neq (p_m, q_m)\). The variables \((a_{p,q})\) in a unital (*) algebra \( \mathcal{A} \) are said to be (*) matricially free with respect to the given array if the corresponding array of (*) subalgebras \((\mathcal{A}_{p,q})\), where each \( \mathcal{A}_{p,q} \) is (*) generated by \( a_{p,q} \) and \( 1_{p,q} \), is matricially free.
One can prove matricial freeness of \((\omega_{p,q}(u))\), using the language of free probability and Lemma 3.1 instead of applying the Hilbert space methods as in [7].

**Theorem 5.1.** Let \((A, \varphi)\) be a noncommutative \(*\)-probability space and let
\[
\{\ell(p, q, u) : 1 \leq p, q \leq r, u \in \mathcal{U}\}
\]
be a standard system of free creation operators in \((A, \varphi)\), where \(\mathcal{U}\) is finite. Let \(M_{p,q}\) be the \(*\)-subalgebra of \(M_r(A)\) generated by
\[
\{\ell(p, q, u) \otimes e(p, q) : u \in \mathcal{U}\} \cup \{1_{p,q}\},
\]
where the units are
\[
1_{p,q} = t(p, q) \otimes e(p, p) + 1 \otimes e(q, q)
\]
and
\[
t(p, q) = \begin{cases} \sum_{u \in \mathcal{U}} \ell(p, q, u)\ell(p, q, u)^* & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}
\]
for any \(p, q\). Then the array \((M_{p,q})\) is matricially free with respect to the array \((\Phi_{p,q})\), where \(\Phi_{p,q} = \varphi \otimes \psi_q\) for any \(p, q\), where \(\psi_q\) is the vector state on \(M_r(\mathbb{C})\) associated with the basis unit vector \(e(q) \in \mathbb{C}^r\).

**Proof.** First, let us show that the array \((1_{p,q})\) is a matricially free array of units in the sense of Definition 5.1. Each \(1_{p,q}\) is indeed an internal unit in \(M_{p,q}\) since any product of the generators and their adjoints is in one of the following forms:
\[
w_1 \otimes e(p, q), \ w_2 \otimes e(p, p), \ w_3 \otimes e(q, p), \ w_4 \otimes e(q, q)
\]
where the last two are preserved by \(1 \otimes e(q, q)\) and killed by the \(t(p, q) \otimes e(p, p)\) due to multiplication of matrix units, whereas the first two are preserved by \(t(p, q) \otimes e(p, p)\) since words \(w_1\) and \(w_2\) begin with a creation operator of type \(\ell(p, q, u)\) and are killed by \(1 \otimes e(q, q)\). A similar reasoning holds when we multiply such terms by the unit from the right. Moreover,
\[
\varphi_q(1_{k,l}) = \varphi_q(t(k, l) \otimes e(k, k) + 1 \otimes e(l, l)) = \delta_{q,l}
\]
thus either
\[
\varphi(\ell(k, l, u)\ell(k, l, u)^*) = 0
\]
for any \(k \neq l\) and any \(u\) and thus condition (a) of Definition 5.1 is satisfied, or \(k = l\) and then \(t(k, l) = 0\). For the same reason both \(t(k, l)\) and \(t(l, s)\) are eliminated from the equation
\[
\varphi_q(1_{k,l}a1_{r,s}) = \varphi_q((1 \otimes e(l, l))a(1 \otimes e(s, s)))
\]
\[
= \delta_{q,l}\delta_{q,s}\varphi_q(a)
\]
\[
= \varphi_q(1_{k,l})\varphi_q(a)\varphi_q(1_{r,s})
\]
for any \(q, k, l, r, s\) and any \(a \in M_r(A)\), which gives condition (b) of Definition 5.1. In order to show (c), we need to compute the moments
\[
\varphi_q(a_1p_1q_1a_2 \ldots a_m),
\]
where the product \(a_2 \ldots a_m\) is in the matricially free kernel form. In order to act with \(1_{p_1q_1}\) onto \(a_2 \ldots a_m\) from the left, note that the latter is a linear combinations of elements of the form
\[
a = w \otimes e(p_2, q_m)
\]
where $w$ is a word which begins with $\ell(p_2, q_2, u)$ for some $u \in \mathcal{U}$. In fact, one can always use the relations between creation and annihilation operators to reduce $w$ to such a form. This means that

$$(1 \otimes e(q, q))a = a \quad \text{and} \quad (t(p, q) \otimes e(p, p))a = 0$$

whenever $q = p_2$ since $1$ preserves $w$ and $q_{p,q}$ kills it since $t(p, q)$ ends with operators of the form $\ell(p, q, s)^*$ which are free from $\ell(q, q_2, u)$ for any $s, u \in \mathcal{U}$. This proves that the array $(1_{p,q})$ is a matricially free array of units. It remains to show that the condition of Definition 5.2 is satisfied. This is actually quite straightforward since products $a_1 \ldots a_m$ for which

1. if $(p_1, q_1) \neq \ldots \neq (p_m, q_m)$ but $(p_1, q_1), \ldots, (p_m, q_m) \notin \mathcal{K}_m$ we get zero moments since the associated product of matrix units vanishes,
2. if $((p_1, q_1), \ldots, (p_m, q_m)) \in \mathcal{K}_m$, we also get zero moments since in this case the moment $\varphi_q(a_1 \ldots a_m)$ is a sum of moments of type $\varphi(a) \otimes \psi_q(e(p_1, q_m))$, where $a \in \ker \varphi$, so even if $q = p_1 = q_m$, we get zero.

Therefore, our proof has been completed. ■

Remark 5.1. The sums of matricially free Gaussian operators called Gaussian pseudomatrices in [5,7] decompose as

$$\omega(u) = \sum_{p,q} \omega_{p,q}(u) = L(u) + L(u)^*$$

for any $u$, where $L(u), L(u)^*$ are generalizations of matrices studied by Shlyakhtenko [12, Theorems 5.1-5.2] since

$$L(u) = \sum_{p,q} \ell(p, q, u) \otimes e(p, q)$$

by Lemma 3.1. Thus, if $b_{p,q}(u) = d_p$ for any $p, q, u$ and $\sum_q d_q = 1$, the operators $L(u)$ are of the same form as those in [12] since $d_p$ is the covariance of $\ell(p, q, u)$ (Definition 2.2). In particular, the fact that these operators have the same joint $*$-distributions with respect to $\Phi = \sum_q d_q \Phi_q$ as free creation and annihilation operators is a consequence of the decomposition of the Gaussian random matrix with i.i.d. complex entries into blocks of asymptotic dimensions $d_1, \ldots, d_r$.

It follows from the definition of matricial freeness that if an array $(a_{p,q})$ is $(*-)$ matricially free with respect to $(\varphi_{p,q})$, then the joint $(*-)$ distribution of this array with respect to any state $\varphi_{p,q}$ is uniquely determined by the array of $(*-)$ distributions of the variables $a_{p,q}$ with respect to the corresponding states $\varphi_{p,q}$, respectively. Thus, the situation is similar to that in free probability, where the joint $(*-)$ distribution of a $(*-)$ free family with respect to a distinguished state $\varphi$ is uniquely determined by the family of $(*-)$ distributions of all members of this family.

The same property holds for $(*-)$ symmetrically matricially free arrays, whose definition is recalled below. This notion is a kind of symmetrization of matricial freeness and it was introduced in [5] (with a correction of the definition of the array of symmetrized units in [7]). In this setting, we need to distinguish odd and even elements.
Definition 5.3. In the setting used for matricial freeness, assume in addition that the arrays \( (A_{p,q}) \) and \((1_{p,q})\) are symmetric. If, for some \( a \in A^0_{p,q}, \) where \( p \neq q \), it holds that 
\[
\varphi_q(b_{1,p,p}a) = \varphi_q(ba) \text{ and } \varphi_q(b_{1,q,q}a) = 0
\]
or
\[
\varphi_q(b_{1,q,q}a) = \varphi_q(ba) \text{ and } \varphi_q(b_{1,p,p}a) = 0
\]
for any \( b \in A^0_{p,q}, \) then we will say that \( a \) is odd or even, respectively. The subspaces of \( A^0_{p,q} \) spanned by even and odd elements will be called even and odd, respectively. Since two states, \( \varphi_p \) and \( \varphi_q \), are associated with \( A_{p,q} \), we understand that the above properties hold for both states. If each off-diagonal \( A^0_{p,q} \) is a vector space direct sum of an odd subspace and an even subspace, the array \( (A_{p,q}) \) will be called decomposable.

There was a slight imprecision concerning indices in this definition in [7] and that is why we wrote the conditions using two equations instead of one. In the framework of matrices with operatorial entries, odd and even elements have a quite natural interpretation shown in the proposition given below.

Proposition 5.1. Let \((A, \varphi)\) be a noncommutative probability space. If \( a \) is a product of an odd (even) number of elements taken from the set
\[
\{c \otimes e(p,q), c \otimes e(q,p) : c \in A\}
\]
where \( p \neq q \) are fixed, then \( a \) is odd (even), where
\[
1_{p,q} = \begin{cases} 
1 \otimes e(p,p) + 1 \otimes e(q,q) & \text{if } p \neq q \\
1 \otimes e(q,q) & \text{if } p = q
\end{cases}
\]
defines the array of units and the states are of the form \( \Phi_q = \varphi \otimes \psi_q \), where \( q \in [r] \).

Proof. If we compute moments in the state \( \Phi_q \), it suffices to consider the following alternating products of matrix units:
\[
\begin{align*}
& e(p,q)e(q,p) \ldots e(p,q) = e(p,q) \\
& e(q,p)e(p,q) \ldots e(p,q) = e(q,q).
\end{align*}
\]
Now, \( 1_{p,p} \) preserves products of the form \( a := c_1 \ldots c_{2k-1} \otimes e(p,q) \) acting on \( \Omega \otimes e(q) \) and \( 1_{q,q} \) kills it, which proves that 
\[
\Phi_q(b_{1,p,p}a) = \Phi_q(ba) \text{ and } \Phi_q(b_{1,q,q}a) = 0
\]
which proves our claim for \( a \) odd. The proof for \( a \) even is similar. In fact, analogous equations are obtained for \( \Phi_p \), \( \square \)

The main difference between matricial freeness and symmetric matricial freeness is that ordered pairs of indices of type \( (p,q) \) are replaced by sets \( \{p,q\} \). In the definition, we need to consider the tuples
\[
\hat{K}_m = \{(\{p_1,q_1\}, \ldots, \{p_m,q_m\}) : p_k, q_k \} \cap \{p_{k+1}, q_{k+1}\} \neq \emptyset \text{ for } 1 \leq k \leq m - 1
\]
where \( m \in \mathbb{N} \). We will also use the abbreviated notation \( w_k = \{p_k, q_k\} \). Another important difference is that we have to use decomposable arrays in the sense of Definition 5.3. Using the above notations and definitions, one can write the conditions on the symmetrically matricially free array of units.
Definition 5.4. Assume that the array \((A_{p,q})\) is symmetric and decomposable. We say that a symmetric array \((1_{p,q})\) is a symmetrically matricially free array of units if for any state \(\varphi_q\) it holds that

1. \(\varphi_q(u_1 a u_2) = \varphi_q(u_1) \varphi_q(a) \varphi_q(u_2)\) for any \(a \in A\) and \(u_1, u_2 \in I\),
2. \(\varphi_q(1_{k,l}) = 2^{-1}(\delta_{q,k} + \delta_{q,l})\) for any \(q, k, l\),
3. if \(a_r \in A_{p,q_r} \cap \text{Ker} \varphi_{p,r},\) where \(1 < r < m\) and \((w_1, \ldots, w_m) \in \tilde{K}_m\), then
   \[\varphi_q(a_1 p_1 q_1 a_2 \ldots a_m) = \varphi_q(a a_2 \ldots a_m)\]

for any \(a \in A\), whenever one of the following cases holds:
(a) \(w_1 \cap w_2 \neq w_2 \cap w_3\) and \(a_2\) is odd,
(b) \(w_1 \cap w_2 = w_2 \cap w_3\) and \(a_2\) is diagonal or even,
where we set \(w_3 = \{q, q\}\) for \(m = 2\), and for any other \(w_1 \neq \ldots \neq w_m\) the moment vanishes.

The above definition seems to be somewhat technical due to the abstract formulation, but in the framework of matrices with operatorial entries studied in this paper it reflects the action of the natural units of Proposition 5.1 onto their entries. More explicitly, this can be seen in the proof of Theorem 5.2.

Definition 5.5. We say that a symmetric decomposable array \((A_{p,q})\) is symmetrically matricially free with respect to \((\varphi_{p,q})\) if

1. for any \(a_k \in \text{Ker} \varphi_{p,q_k} \cap A_{p,q_k}\), where \(k \in \{m\}\) and \(w_1 \neq \ldots \neq w_m\) and for any state \(\varphi_q\) it holds that
   \[\varphi_q(a_1 a_2 \ldots a_m) = 0\]
2. \((1_{p,q})\) is the associated symmetrically matricially free array of units. The variables \((a_{p,q})\) in a unital \((*)\) algebra \(A\) are said to be \((*)\) symmetrically matricially free with respect to the given array if the corresponding array of \((*)\) symmetric subalgebras \((A_{p,q})\), where each \(A_{p,q}\) is \((*)\) generated by \(a_{p,q}\) and \(1_{p,q}\), is symmetrically matricially free.

We have shown in [7, Proposition 3.4] that matricial symmetrized Gaussian operators are symmetrically matricially free. Here, we prove a more general theorem, which allows us to extend this result to three different matricial systems of operators studied in this paper.

Theorem 5.2. Let \((A, \varphi)\) be a noncommutative \((*)\)-probability space and let
\[\{\ell(p, q, u) : p, q \in [r], u \in \mathcal{U}\}\]
be a standard system of free creation operators in \((A, \varphi)\), where \(\mathcal{U}\) is finite. Let \(M_{p,q}\) be the \((*)\)-subalgebra of \(M_r(A)\) generated by
\[\{\ell(p, q, u) \otimes e(p, q), \ell(q, p, u) \otimes e(q, p) : u \in \mathcal{U}\}\]
for any fixed \(p, q \in [r]\). Then the array \((M_{p,q})\) is symmetrically matricially free with respect to \((\Phi_{p,q})\), where \(\Phi_{p,q} = \varphi \otimes \psi_q\) for any \(p, q\), with the array of symmetrized units of Proposition 5.1.

Proof. It is immediate that the units defined in Proposition 5.1 belong to the \((*)\)-subalgebras \(M_{p,q}\) and are indeed their internal units (the algebra generated by them is clearly commutative). The proof that they satisfy the conditions of Definition 5.4 is similar to that of Theorem 5.1. In particular, checking (1) and (2) is easy. In order to
prove (3), notice that the array $\mathcal{M}_{p,q}$ is decomposable and even as well as odd elements can be easily determined by Proposition 5.1. Suppose now that $\{p_1, q_1\} \cap \{p_2, q_2\} \neq \{p_2, q_2\} \cap \{p_3, q_3\}$ and that $a_2$ is odd, for instance $p := q_1 = p_2$ and $q := q_2 = p_3$ with $p \neq q$. Then $a_2$ is of the form

$$\alpha c_1 \otimes e(p, q) + \beta c_2 \otimes e(q, p)$$

but the second expression can be dropped since it kills $a_3$, whereas the first one is preserved by $1_{p_1,q_1}$. In turn, if $a_2$ is even, it is of the form

$$\alpha c_1 \otimes e(p, p) + \beta c_2 \otimes e(q, q)$$

and again the first expression can be dropped since it kills $a_3$, whereas the second one is killed by $1_{p_1,q_1}$. Suppose now that $\{p_1, q_1\} \cap \{p_2, q_2\} = \{p_2, q_2\} \cap \{p_3, q_3\}$ and $a_2$ is even, for instance $p := q_1 = p_2 = p_3$ and $q := q_2$. Then $a_2$ is of the form

$$\alpha c_1 \otimes e(p, p) + \beta c_2 \otimes e(q, q)$$

and the second expression can be dropped since it kills $a_3$, whereas the first one is preserved by $1_{p_1,q_1}$. In turn, if $a_2$ is odd, it is of the form

$$\alpha c_1 \otimes e(p, q) + \beta c_2 \otimes e(q, p)$$

and the first expression can be dropped since it kills $a_3$, whereas the second one is killed by $1_{p_1,q_1}$. The case when $a_2$ is diagonal is obvious. This proves that the array of units satisfies the conditions of Definition 5.4. The proof of the first condition of Definition 5.5 is straightforward and similar to that in the case of matricial freeness.

Let us remark that the essential difference between matricial freeness of arrays in Theorem 5.1 and symmetric matricial freeness of those in Theorem 5.2 is that in the first case we have free creation operators associated only with off-diagonal matrix units of one type, namely $e(p, q)$, whereas in the second case the free creation operators are associated with two off-diagonal matrix units, namely $e(p, q)$ and $e(q, p)$.

**Corollary 5.1.** Let $(\mathcal{M}_{p,q})$ be the array of *-subalgebras of $M_r(\mathcal{A})$ such that one of the following cases holds:

1. each $\mathcal{M}_{p,q}$ is generated by $\{\tilde{\omega}_{p,q}(u) : u \in \mathcal{U}\}$,
2. each $\mathcal{M}_{p,q}$ is generated by $\{\eta_{p,q}(u) : u \in \mathcal{U}\}$,
3. each $\mathcal{M}_{p,q}$ is generated by $\{\zeta_{p,q}(u) : u \in \mathcal{U}\}$,

and let the array of units be given by Proposition 4.1. Then $(\mathcal{M}_{p,q})$ is symmetrically matricially free with respect to $(\Phi_{p,q})$, where $\Phi_{p,q} = \varphi \otimes \psi_q$ for any $p, q$.

**Proof.** It is obvious that all three matricial systems of operators give arrays of *-algebras of the form required by Theorem 5.2 or arrays of *-subalgebras of such *-algebras and that implies symmetric matricial freeness.

6. **Mixed *-moments of $R$-circular operators**

The combinatorics of mixed *-moments of $R$-circular operators is similar to that for matricial circular operators [7]. However, there are some differences resulting from the fact that the latter are sums of two $R$-circular operators. Mixed moments of matricial semicircular and $R$-semicircular operators were computed in [5,7].
If $\pi$ is a non-crossing pair-partition of the set $[m]$, where $m$ is an even positive integer, which is denoted $\pi \in \mathcal{NC}_m^2$, the set
\[ B(\pi) = \{\pi_1, \pi_2, \ldots, \pi_s\} \]
is the set of its blocks, where $m = 2s$. If $\pi_i = \{l(i), r(i)\}$ and $\pi_j = \{l(j), r(j)\}$ are two blocks of $\pi$ with left legs $l(i)$ and $l(j)$ and right legs $r(i)$ and $r(j)$, respectively, then $\pi_i$ is inner with respect to $\pi_j$ if $l(j) < l(i) < r(i) < r(j)$. In that case $\pi_j$ is outer with respect to $\pi_i$. It is the nearest outer block of $\pi$, if there is no block $\pi_k = \{l(k), r(k)\}$ such that $l(j) < l(k) < l(i) < r(i) < r(k) < r(j)$. It is easy to see that the nearest outer block, if it exists, is unique, and we write in this case $\pi_j = o(\pi_i)$. If $\pi_i$ does not have an outer block, it is called a covering block. In that case we set $o(\pi_i) = \pi_0$, where $\pi_0 = \{0, m + 1\}$ is the additional block called imaginary. The partition of the set $\{0, 1, \ldots, m + 1\}$ consisting of the blocks of $\pi$ and of the imaginary block will be denoted by $\hat{\pi}$.

Before we go into more details, let us compute a simple mixed *-moment of $R$-circular operators, in which non-crossing partitions start to play a role. This computation also shows the difference between matricial $R$-circular operators and matricial circular operators whose mixed *-moments were computed in [7]

**Example 6.1.** Consider one off-diagonal $R$-circular operator $\zeta_{p,q}$ of the form
\[ \zeta_{p,q} = (\ell_1 + \ell_2^*) \otimes e(p,q) \]
where $p \neq q$ and $\ell_1, \ell_2$ are free creation operators with covariances $\gamma_1$ and $\gamma_2$, respectively. We have
\[ \Phi_q(\zeta_{p,q}^* \zeta_{p,q} \zeta_{p,q}^* \zeta_{p,q}) = \varphi(\ell_1^* \ell_1 \ell_2^* \ell_2) + \varphi(\ell_1^* \ell_1 \ell_2^* \ell_2) = \gamma_1^2 + \gamma_1 \gamma_2 \]
whereas all remaining *-moments of $\zeta_{p,q}$ in the state $\Phi_q$ vanish. Of course, the *-moments of $\zeta_{p,q} + \zeta_{q,p}$ add up to give those of $\eta_{p,q}$ in Example 4.1.

**Definition 6.1.** The tuple $(e(p_1, q_1), \ldots, e(p_m, q_m))$ of matrix units in $M_r(\mathbb{C})$ will be called cyclic if
\[ q_1 = p_2, q_2 = p_3, \ldots, q_m = p_1. \]
We will say that $\pi \in \mathcal{NC}_m$ is adapted to $(e(p_1, q_1), \ldots, e(p_m, q_m))$ if this tuple and all tuples associated with the blocks of $\pi$ are cyclic. The set of all non-crossing pair partitions adapted to the tuple $(e(p_1, q_1), \ldots, e(p_m, q_m))$, where $p_1 = q_m$, will be denoted by $\mathcal{NC}_m(e(p_1, q_1), \ldots, e(p_m, q_m))$.

**Example 6.2.** For a tuple of off-diagonal matrix units $(e(p,q), e(q,p), e(p,q), e(q,p))$, there are three non-crossing partitions which are adapted to it, namely
\[ \{\{1, 2, 3, 4\}\}, \{\{1, 4\}, \{2, 3\}\} \text{ and } \{\{1, 2\}, \{3, 4\}\} \]
since only the tuples of matrix units associated to these partitions are cyclic. In turn, the partitions
\[ \{\{1, 2, 5\}, \{3, 4\}\}, \{\{1, 2, 3, 4, 5\}\} \text{ and } \{\{1, 2, 3\}, \{4, 5\}\} \]
are the only non-crossing partitions adapted to the tuple of off-diagonal matrix units of the form $(e(p,q), e(q,t), e(t,p), e(p,t), e(t,p))$. 
Definition 6.2. If we are given $a_j = c_j \otimes e(p_j, q_j) \in M_r(\mathcal{A})$ for $j = 1, \ldots, m$, where $c_j \in \mathcal{A}$, we will denote by
\[
\mathcal{NC}_m(a_1, \ldots, a_m)
\]
the set of non-crossing partitions of $[m]$ which are adapted to $(e(p_1, q_1), \ldots, e(p_m, q_m))$. These partitions will be called adapted to the tuple $(a_1, \ldots, a_m)$. Its subset consisting of pair partitions will be denoted $\mathcal{NC}^2_m(a_1, \ldots, a_m)$.

In order to find combinatorial formulas for the mixed moments of the operators $\zeta_{p,q}(u)$, we need to use colored non-crossing pair partitions. It will suffice to color each $\pi \in \mathcal{NC}_m^2$, where $m$ is even, by numbers from the set $[r]$. We will denote by $F_r(\pi)$ the set of all mappings $f: \mathcal{B}(\pi) \rightarrow [r]$ called colorings. By a colored non-crossing pair partition we then understand a pair $(\pi, f)$, where $\pi \in \mathcal{NC}_m^2$ and $f \in F_r(\pi)$. The set of pairs
\[
\mathcal{B}(\pi, f) = \{(\pi_1, f), (\pi_2, f), \ldots, (\pi_k, f)\}
\]
will play the role of the set of its blocks. We will always assume that also the imaginary block is colored by a number from the set $[r]$ and thus we can speak of a coloring of $\hat{\pi}$. Examples of colored non-crossing pair partitions are given in Fig.1.

For any given $r \times r$ covariance matrix $B(u)$, there is a natural way to assign its entries to the blocks of non-crossing pair partitions colored by the set $[r]$. By multiplicativity over the blocks, we can then define the associated functions on non-crossing pair partitions.

Definition 6.3. Let a covariance matrix $B(u) = (b_{p,q}(u)) \in M_r(\mathbb{R})$ be given for any $u \in \mathfrak{u}$. For any $\pi \in \mathcal{NC}_m^2$ and $f \in F_r(\pi)$, let
\[
b_q(\pi, f) = \prod_{k=1}^{s} b_q(\pi_k, f)
\]
where
\[
b_q(\pi_k, f) = b_{s,t}(u),
\]
whenever $\pi_k = \{i, j\}$ is colored by $s$, its nearest outer block $o(\pi_k)$ is colored by $t$ and $u_i = u_j = u$ and we assume that the imaginary block is colored by $q \in [r]$, and otherwise we set $b_q(\pi_k, f) = 0$.

It remains to determine which colorings are natural for $R$-circular operators. It is convenient to introduce the following definition.
Definition 6.4. Let \( \pi \in \mathcal{NC}_m^2(a_1, \ldots, a_m) \), where \( a_j = (c_j(u_j) \otimes e(p_j, q_j))^\epsilon_j \) and \( \epsilon_j \in \{1, *\} \) for \( j \in [m] \) and \( m \) is even. A coloring \( f : \mathcal{B}(\pi) \rightarrow [r] \) will be called adapted to \((a_1, \ldots, a_m)\) if
\[
f(\pi_k) = \begin{cases} p & \text{if } (\epsilon_i, \epsilon_j) = (*, 1) \\ q & \text{if } (\epsilon_i, \epsilon_j) = (1, *) \end{cases}
\]
whenever \( \pi_k = \{i, j\} \) is a block and \((p_i, q_i) = (p_j, q_j) = (p, q)\).

Lemma 6.1. With the above notations, let \( a_j = \zeta_{p_j,q_j}^\epsilon(u_j) \), where \( p_j, q_j \in [r], u_j \in \mathcal{U} \), and \( \epsilon_j \in \{1, *\} \) for \( j \in [m] \) and \( m \in \mathbb{N} \). Then
\[
\Phi_q(a_1 \ldots a_m) = \sum_{\pi \in \mathcal{NC}_m^2(a_1, \ldots, a_m)} b_q(\pi, f)
\]
where \( q \) is equal to the second index of the matrix unit associated with \( a_m \) and \( f \) is the unique coloring of \( \pi \) adapted to \((a_1, \ldots, a_m)\). In the remaining cases, the moment vanishes.

Proof. For notational simplicity, we assume that \( \mathcal{U} \) consists of one element, which allows to skip \( u \), with no essential loss to the generality of our proof. We have
\[
\zeta_{p,q} = (\ell(p, q, 1) + \ell(q, p, 2)^*) \otimes e(p, q)
\]
\[
\zeta_{p,q}^* = (\ell(p, q, 1)^* + \ell(q, p, 2)) \otimes e(q, p)
\]
for any \( p, q \). We would like to compute the mixed *-moments
\[
\Phi_q(\zeta_{p_1,q_1}^{\epsilon_1} \ldots \zeta_{p_m,q_m}^{\epsilon_m})
\]
where \( \epsilon_1, \ldots, \epsilon_m \in \{*, 1\} \) and \( m \) is even (we adopt the convention that if \( m \) is odd, then all corresponding sets of pair partitions are empty). It is clear that only non-crossing partitions which are adapted to the tuple of associated matrix units can give a non-zero contribution and only if the second index of the last matrix unit (this could be \( p_m \) or \( q_m \) depending on whether \( \epsilon_m = * \) or \( \epsilon_m = 1 \), respectively) is equal to \( q \). On the other hand, since we have free creation and annihilation operators at the first positions, these must be pair partitions. More precisely, non-zero contributions come from all non-crossing pairings which match \( \ell(p, q, 1)^*, \ell(p, q, 1) \) and \( \ell(q, p, 2)^*, \ell(q, p, 2) \) and this corresponds to the pairings of \( \zeta_{p,q}^*, \zeta_{p,q} \) and \( \zeta_{p,q}^*, \zeta_{p,q}^* \), respectively. Note that these operators are *-free unless they have the same triples of indices \((p, q, u)\), where \( u \in \{1, 2\} \). This implies that the pair partitions must be adapted in the sense of Definition 6.2. Thus, if \( \pi = \{\pi_1, \ldots, \pi_s\} \) is such a partition, we must have
\[
\epsilon_i \neq \epsilon_j \quad \text{and} \quad (p_i, q_i) = (p_j, q_j)
\]
whenever \( \{i, j\} \) is a block, which shows that \( \pi \in \mathcal{NC}_m^2(a_1, \ldots, a_m) \). It remain to check how the colors can be assigned to blocks of such \( \pi \) in order to reproduce the appropriate product of covariances. Let \( \pi_k = \{i, j\} \) be a block and let \( o(\pi_k) = \{o(i), o(j)\} \) be its nearest outer block. This notation also applies to the blocks whose nearest outer block is the imaginary block to which we assign the pair \((q, q)\). We always assign to a given block the color equal to this index of the pair of matrix units which is inner since the outer ones have to match those associated with other blocks. For instance, suppose that we have the pairing of type \( \zeta_{p,q}^*, \zeta_{p,q} \), i.e. \( \zeta_{p,q}^* \) is first and \( \zeta_{p,q} \) is second, where \((p_i, q_i) = (p_j, q_j) = (p, q)\). The situation is like in the first block below:
The corresponding second matrix unit is \( e(p_j, q_j) = e(p, q) \) and the color assigned to the block is \( p \). Here, \( q_j \) is the second index associated with the right leg of the considered block. It must agree with the first matricial index assigned to the right leg of the nearest outer block (not shown in the above picture). This is obvious if the nearest outer block is immediately after the given block. If there is a sequence \( \pi_{k_1}, \ldots, \pi_{k_s} \) of blocks of the same depth in between the given block \( \pi_k \) and its nearest outer block \( o(\pi_k) \) when we move from left to right, then the matricial agreement of the corresponding matrix units must ‘transfer’ \( q_j \) as the index to be matched to the right leg of the nearest outer block. A similar argument applies to pairings of type \( \zeta_{p,q}, \zeta'_{p,q} \) to which the second block in the above picture refers. Therefore, these two kinds of pairings contribute

\[
\Phi_q(\zeta_{p,q}, \zeta'_{p,q}) = b_{p,q} \text{ and } \Phi_p(\zeta_{p,q}, \zeta'_{p,q}) = b_{q,p},
\]

respectively, where we use the fact that the covariances assigned to 1 and 2 are equal (of course, in general we would say that of \( 2u - 1 \) and \( 2u \)). It can be seen that these contributions correspond to the way covariances are assigned to blocks in the sense of Definitions 6.3 and 6.4 for one value of \( u \). As we remarked, treating an arbitrary \( \mathcal{U} \) is done in a similar fashion. Therefore, our proof is completed.

\section*{7. Cyclic \( R \)-transform for \( R \)-circular systems}

We would like to examine \( R \)-circular operators from the point of view of cumulants. First, let us recall some definitions from free probability referring to free cumulants and \( R \)-diagonal elements [10].

If \( a \) is a random variables in a *-probability space \( (\mathcal{A}, \varphi) \), then a free cumulant \( \kappa_{2n}(a_1, \ldots, a_{2n}) \) with arguments from the set \( \{a, a^*\} \) is called alternating if there does not exist any \( a_i \) \((1 \leq i \leq 2n - 1)\) with \( a_{i+1} = a_i \). Cumulants with an odd number of arguments are considered as not alternating. A random variable \( a \in \mathcal{A} \) is called \( R \)-diagonal if for all \( n \in \mathbb{N} \)

\[
\kappa_n(a_1, \ldots, a_n) = 0
\]

whenever its arguments are not alternating.

The \( R \)-transform of \( r \) elements \( a_1, \ldots, a_r \) of the *-probability space \( (\mathcal{A}, \varphi) \) is a formal power series in noncommutative indeterminates \( z_1, \ldots, z_r \) of the form

\[
R_{a_1, \ldots, a_r}(z_1, \ldots, z_r) = \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1}^{r} \kappa_n(a_{i_1}, \ldots, a_{i_n}) z_{i_1} \cdots z_{i_n}.
\]

In particular, the \( R \)-transform of the pair \( \{a, a^*\} \) in the case when \( a \) is \( R \)-diagonal is a formal series in noncommuting indeterminates \( z_1, z_2 \) of the form

\[
R_{a, a^*}(z_1, z_2) = f(z_1 z_2) + g(z_2 z_1),
\]
where
\[ f(z) = \sum_{n=0}^{\infty} \alpha_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} \beta_n z^n \]
and \( \alpha_n, \beta_n \) are alternating *-cumulants of \( a \) and \( a^* \).

Circular elements are important examples of R-diagonal elements. If \( c \) is a standard circular element (i.e., has covariance one), then the R-transform of \( c, c^* \) takes the simple form
\[ R_{c,c^*}(z_1, z_2) = z_1 z_2 + z_2 z_1 \]
since the only non-vanishing cumulants for \( c \) and \( c^* \) are \( \kappa_2(c, c^*) = \kappa_2(c^*, c) = 1 \).

We will examine matricial R-circular operators from a similar point of view. In particular, we will study the distributions of the R-circular operators \( \zeta_{p,q} \) with respect to the array \( (\Phi_{p,q}) \), where \( \Phi_{p,q} = \Phi_q \) and
\[ \Phi_q = \varphi \otimes \psi_q \]
for any \( p, q \in [r] \), where \( \psi_q \) is associated with the basis unit vector \( e(q) \in \mathbb{C}^r \). However, we will need a new concept of cumulants. Of course, the situation is rather clear with the cumulants of diagonal operators with respect to the corresponding diagonal states since they are circular with respect to these states. However, the off-diagonal ones exhibit new features.

Naturally, this situation should lead to a family of cumulants indexed by \( q \) and for that reason our cumulants will have the additional parameter \( q \).

**Definition 7.1.** A family of multilinear functions \( \kappa_{s}[\cdot; q] \), where \( q \in [r] \), of matricial variables
\[ a_j = c_j \otimes e(p_j, q_j) \in M_r(\mathcal{A}), \]
where \( m \in [r] \), will be called cyclically multiplicative over the blocks \( \pi_1, \ldots, \pi_s \) of \( \pi \in \mathcal{NC}_m(a_1, \ldots, a_m) \) if
\[ \kappa_{s}[a_1, \ldots, a_m; q] = \prod_{j=1}^{s} \kappa(\pi_j)[a_1, \ldots, a_m; q(j)], \]
for any \( a_1, \ldots, a_m \), where
\[ \kappa(\pi_j)[a_1, \ldots, a_m; q(j)] = \kappa_s(a_{i_1}, \ldots, a_{i_l}; q(\ell)) \]
for the block \( \pi_1 = (i(1) < \ldots < i(l)) \), i.e., \( q(j) \) is equal to the last matrix index in the block, where \( \{\kappa_s(\cdot; q) : s \geq 1, q \in [r]\} \) is a family of multilinear functions. If \( \pi \notin \mathcal{NC}_m(a_1, \ldots, a_m) \), then we set \( \kappa_{s}[a_1, \ldots, a_m; q] = 0 \) for any \( q \).

**Definition 7.2.** With the above notations, by the cyclic cumulants we shall understand the family of multilinear cyclically multiplicative functionals over the blocks of non-crossing partitions
\[ \pi \rightarrow \kappa_{s}[\cdot; q], \]
defined by the moment-cumulant formulas
\[ \Phi_q(a_1 \ldots a_m) = \sum_{\pi \in \mathcal{NC}_m(a_1, \ldots, a_m)} \kappa_{s}[a_1, \ldots, a_m; q], \]
where \( q \in [r] \) and \( \Phi_q = \varphi \otimes \psi_q \) are states on \( M_r(\mathcal{A}) \) as defined before.
The cyclic cumulants remind similar objects defined for \( R \)-cyclic matrices by Nica, Shlyakhtenko and Speicher [9,10]. We have a similar restriction on the non-crossing partitions (cyclicity), but we consider a family of states rather than one state. It is convenient to think of the cyclic cumulants labelled by \( q \) as of cumulants ‘under condition \( q \)’. In fact, their values depend on the index \( q \) which labels the corresponding state. When speaking of cumulants, we will always mean ‘cyclic cumulants’ and the usual ones will be called ‘free cumulants’.

Example 7.1. We would like to compute simple cumulants for one off-diagonal \( R \)-circular operator \( \zeta = \zeta_{p,q} \) and its adjoint,

\[
\zeta_{p,q} = (\ell_1 + \ell_2^*) \otimes e(p,q) \quad \text{and} \quad \zeta_{p,q}^* = (\ell_1^* + \ell_2) \otimes e(q,p),
\]

considered before in Example 6.1. Using the properties of the free creation and annihilation operators, we can easily see that all odd cumulants vanish. Now, the non-trivial second order cumulants are

\[
\kappa_2(\zeta_{p,q}^*, \zeta_{p,q}; q) = \Phi_q(\zeta_{p,q}^* \zeta_{p,q}^* q) = \varphi(\ell_1^* \ell_1) = \gamma_1
\]

\[
\kappa_2(\zeta_{p,q}, \zeta_{p,q}^*; p) = \Phi_p(\zeta_{p,q} \zeta_{p,q}^*) = \varphi(\ell_2^* \ell_2) = \gamma_2
\]

whereas all fourth order cumulants vanish. For instance,

\[
\kappa_2(\zeta_{p,q}^*, \zeta_{p,q}, \zeta_{p,q}^*, \zeta_{p,q}; q) = \Phi_q(\zeta_{p,q}^* \zeta_{p,q}^* \zeta_{p,q}^* \zeta_{p,q}^* q) - \kappa_2(\zeta_{p,q}^*, \zeta_{p,q}; q) \kappa_2(\zeta_{p,q}, \zeta_{p,q}^*; p)
\]

\[
= \gamma_1^2 + \gamma_1 \gamma_2 - \gamma_1^2 - \gamma_1 \gamma_2 = 0,
\]

where we used the moment computed in Example 6.1. It is not hard to observe that this cumulant vanishes due to the fact that we used Definition 7.2 and not the definition of free cumulants with one state. If we used the latter (computed with respect to \( \Phi_q \)), we would not get zero since \( \kappa_2(\zeta_{p,q}, \zeta_{p,q}^*; p) = \gamma_2 \) would have to be replaced by \( \kappa_2(\zeta_{p,q}, \zeta_{p,q}^*; q) = 0 \).

In fact, all cumulants of matricial \( R \)-circular operators of orders higher than two vanish. This fact is proved in the theorem given below, where we derive a formula for the family of cyclic \( R \)-transforms of the array \( \zeta := \zeta_{p,q} \) of matricial \( R \)-circular operators and the array of their adjoints \( \zeta^* := (\zeta_{p,q}^*) \). These are formal power series in \( 2r^2 \) noncommuting indeterminates arranged in arrays \( z := (z_{p,q}) \) and \( z^* := (z_{p,q}^*) \) of the form

\[
R_{\zeta, \zeta^*}(z, z^*; q) = \sum_{n=1}^{\infty} \sum_{p_1, \ldots, p_n, q_1, \ldots, q_n} \sum_{\varepsilon_1, \ldots, \varepsilon_n} \kappa_n(\zeta_{p_1, q_1}, \ldots, \zeta_{p_n, q_n}; q) z_{p_1, q_1}^{\varepsilon_1} \cdots z_{p_n, q_n}^{\varepsilon_n},
\]

where it is understood that \( p_1, \ldots, p_n, q_1, \ldots, q_n \in [r] \) and \( \varepsilon_1, \ldots, \varepsilon_n \in \{1, *\} \) with 1 playing the role of ‘no symbol’. The theorem proved below shows that the cyclic \( R \)-transforms are the canonical transforms of \( R \)-transform type to describe \( R \)-circular systems (the first indication of that fact was Example 7.1).

Theorem 7.1. If \( \zeta := \zeta_{p,q} \) is the square array of matricial \( R \)-circular operators and \( \zeta^* := (\zeta_{p,q}^*) \), then its cyclic \( R \)-transforms are of the form

\[
R_{\zeta, \zeta^*}(z, z^*; q) = \sum_{p=1}^{r} (b_{p,q} z_{p,q}^* z_{p,q} + b_{p,q}^* z_{p,q}^* z_{p,q})
\]
for any $q \in [r]$, where $b_{p,q} = \kappa_2(\xi^*,\zeta_{p,q}; q) = \kappa_2(\xi_{q,p},\xi^*_{p,q}; q)$ for any $p, q$.

Proof. It suffices to consider cumulants of even orders since the odd ones vanish. The non-vanishing cumulants of second order of the off-diagonal operators $\zeta_{p,q}, \zeta^*_{p,q}$ were computed in Example 7.1. The case of diagonal operators is easy since this case is reduced to the circular one. We have

$$
\Phi_q(\zeta^1_{p_1,q_1} \cdots \zeta^m_{p_m,q_m}) = \sum_{\pi \in NC_m(\zeta^1_{p_1,q_1}, \ldots, \zeta^m_{p_m,q_m})} \kappa_\pi[\zeta^1_{p_1,q_1}, \ldots, \zeta^m_{p_m,q_m}; q]
$$

where

$$
\kappa_\pi[\zeta^1_{p_1,q_1}, \ldots, \zeta^m_{p_m,q_m}; q] = \prod_{k=1}^s \kappa(\pi_k)[\zeta^1_{p_1,q_1}, \ldots, \zeta^m_{p_m,q_m}; q(k)].
$$

The contribution from an adapted pair-partition is a product of cumulants of order two and it takes the form

$$
\kappa(\pi_k)[\zeta^1_{p_1,q_1}, \ldots, \zeta^m_{p_m,q_m}; q(k)] = \begin{cases} 
\kappa_2(\zeta^*_{p_i,q_i}, \zeta_{p_i,q_i}; q_i) & \text{if } (p_i, q_i) = (p_j, q_j) \\
\kappa_2(\zeta_{p_i,q_i}, \zeta^*_{p_i,q_i}; p_i) & \text{if } (p_i, q_i) = (p_j, q_j) \\
0 & \text{otherwise}
\end{cases}
$$

for any block $\pi_k = \{i, j\}$ of $\pi$, where $i < j$. In order to obtain the result in terms of covariances $b_{p,q}$, we substitute

$$
\kappa_2(\zeta^*_{p,q}, \zeta_{p,q}; q) = \phi(\ell(p, q, 1)^*\ell(p, q, 1)) = b_{p,q}
$$

and

$$
\kappa_2(\zeta_{p,q}, \zeta^*_{p,q}; p) = \phi(\ell(q, p, 2)^*\ell(q, p, 2)) = b_{q,p}
$$

for any $p, q$. Now, comparing the expression that we obtained here with the formula for mixed moments of Lemma 6.1, we can see that all non-crossing pair partitions which are adapted to the given tuple $(\zeta^1_{p_1,q_1}, \ldots, \zeta^m_{p_m,q_m})$ appear in the expression for their mixed moment as well as in the moment-cumulant formula. By induction, this suffices to claim that all mixed cumulants of orders higher than two vanish since in the moment-cumulant formula for any mixed moment of even order $m$ there is one cumulant of order $m$ and products of cumulants of even orders smaller than $m$. To begin with, all cumulants of order four vanish since the products of cumulants of order two which correspond to non-crossing pair partitions adapted to $(\zeta^1_{p_1,q_1}, \ldots, \zeta^m_{p_m,q_m})$ and under condition $q$ appear in the formula for the moment and in the sum of cumulants associated with pair partitions. Continuing this process, we obtain our claim. That proves that the only non-vanishing cumulants are of order two. We have already computed them above and it suffices to exchange $p$ with $q$ in the second cumulant since we want to compute the cyclic $R$-transform associated with the state $\Phi_q$. If we fix $q$ and multiply all of these by the noncommutative variables $z_{p,q}$ and $z^*_{p,q}$ corresponding to $\xi_{p,q}$ and $\xi^*_{p,q}$, we obtain the desired expression for the cyclic $R$-transforms.

**Corollary 7.1.** If $\eta = (\eta_{p,q})$ is the upper triangular array of matricial circular operators and $\eta^* = (\eta^*_{p,q})$, then their cyclic $R$-transforms are of the form

$$
R_{\eta,\eta^*}(z, z^*; q) = \sum_{p=1}^q (b_{p,q}z^*_{p,q}z_{p,q} + b_{p,q}z_{p,q}z^*_{p,q})
$$
where \( q \in [r] \) and \( b_{p,q} = \kappa_2(\eta_{p,q}^*, \eta_{p,q}; q) = \kappa_2(\eta_{p,q}^*, \eta_{p,q}; q) \) for any \( p \leq q \).

Proof. We can use Theorem 7.1 and the multilinearity of the cumulants since

\[
\eta_{p,q} = \zeta_{p,q} + \zeta_{q,p} \quad \text{and} \quad \eta_{q,q} = \zeta_{q,q}
\]

for any \( p < q \), which gives

\[
\kappa_2(\eta_{p,q}^*, \eta_{p,q}; q) = \kappa_2(\zeta_{p,q}^*, \zeta_{p,q}; q) = b_{p,q}
\]

\[
\kappa_2(\eta_{p,q}^*, \eta_{p,q}; q) = \kappa_2(\zeta_{q,p}^*, \zeta_{q,p}; q) = b_{p,q}
\]

for \( p \leq q \) and similar computations give \( b_{q,p} \) for the cumulants of \( \eta_{q,p}, \eta_{p,q}^* \) under condition \( q \). Thus, it suffices to put \( z_{p,q} = z_{q,p} \) for \( p > q \) in Theorem 7.1 to obtain \( R_{\eta,\eta^*}(z, z^*; q) \) (for upper-triangular arrays) from \( R_{\zeta,\zeta^*}(z, z^*; q) \) (for square arrays). This gives the desired form of the cyclic \( R \)-transforms.

In the results given above, we dealt with cyclic \( R \)-transforms of square \( r \times r \) arrays of \( R \)-circular operators. However, one can also consider pairs consisting of one fixed operator and its adjoint. In the off-diagonal case, we then obtain two non-trivial cyclic \( R \)-transforms of this pair and they take a very simple form.

**Corollary 7.2.** If \( p, q \in [r] \) are fixed and \( p \neq q \), then

1. the non-trivial cyclic \( R \)-transforms of \( h := \zeta_{p,q}, h^* := \zeta_{p,q}^* \) are

\[
R_{h,h^*}(z_1, z_2; q) = b_{p,q}z_2z_1 \quad \text{and} \quad R_{h^*,h}(z_1, z_2; p) = b_{q,p}z_1z_2.
\]

2. the non-trivial cyclic \( R \)-transform of \( c := \zeta_{q,q}, c^* := \zeta_{q,q}^* \) is

\[
R_{c,c^*}(z_1, z_2; q) = b_{q,q}(z_1z_2 + z_2z_1),
\]

3. the non-trivial cyclic \( R \)-transforms of \( \eta := \eta_{p,q}, \eta^* := \eta_{p,q}^* \) are

\[
R_{\eta,\eta^*}(z_1, z_2; q) = b_{p,q}(z_1z_2 + z_2z_1)
\]

\[
R_{\eta^*,\eta}(z_1, z_2; p) = b_{q,p}(z_1z_2 + z_2z_1).
\]

Proof. The formulas for these cyclic \( R \)-transforms are obtained from Theorem 7.1 or Corollary 7.2 by taking suitable subarrays.

**Remark 7.1.** It seems natural that the cyclic \( R \)-transforms of \( \eta_{p,q}, \eta_{p,q}^* \) in the states \( \Phi_q \) and \( \Phi_p \) coincide with the \( R \)-transforms of circular operators since the *-distributions of \( \eta_{p,q} \) in these states are circular by Proposition 4.1. However, cyclic \( R \)-transforms are defined differently than the usual \( R \)-transforms and thus we prefer to treat this property as a decomposition of the \( R \)-transform of circular operators in terms of cyclic \( R \)-transforms of \( R \)-circular operators rather than an obvious fact. A more general decomposition property of this type is formulated below.

**Corollary 7.3.** If \( b_{p,q} = d_p \) for any \( p, q \), then \( c := \sum_{p,q} \zeta_{p,q} \) is circular with respect to \( \Phi = \sum_{q=1}^{r} d_q \Phi_q \) and the corresponding \( R \)-transform takes the form

\[
R_{c,c^*}(z_1, z_2) = \sum_{q=1}^{r} d_q R_{\zeta,\zeta^*}(z, z^*; q),
\]
where $R_{\xi,\xi^*}(z, z^*; q)$ is the cyclic $R$-transform of Theorem 7.1 in which all entries of arrays $z$ and $z^*$ are identified with $z_1$ and $z_2$, respectively.

Proof. If we assume that all block variances in [7, Theorem 9.1] are equal to one, the moments of the matrix $Y(n)$ converge under the normalized trace to those of the standard circular operator $c : = \sum_{p,q} \zeta_{p,q}$ under $\Phi$ by Proposition 4.3. We have

$$\kappa_2(c^*, c) = \Phi(c^* c) \quad \text{and} \quad \kappa_2(\zeta_{p,q}^*, \zeta_{p,q}; q) = \Phi_q(\zeta_{p,q}^* \zeta_{p,q}),$$

where we denote by $\kappa_n$ the free cumulants with respect to $\Phi$. Since

$$\Phi(\zeta_{p,q}^* \zeta_{p,q}; q) = \frac{1}{d_q} \kappa_2(\zeta_{p,q}^*, \zeta_{p,q}; q),$$

we obtain

$$\kappa_2(c^*, c) = \sum_{q=1}^r d_q \kappa_2(c^*, c; q).$$

A similar decomposition holds if we interchange $c$ and $c^*$. Free cumulants of $c, c^*$ of higher orders vanish since $c$ is circular. The cyclic cumulants of higher orders of $c, c^*$ also vanish by multilinearity in view of Theorem 7.1. Therefore, the $R$-transform for $c, c^*$ has the desired decomposition in terms of their cyclic $R$-transforms which can be identified with $R_{\xi,\xi^*}(z, z^*; q)$ for $q \in [r]$, where $z_{p,q} = z_1$ and $z_{p,q}^* = z_2$ for any $p, q$. Of course, both sides of this equation are equal to $z_1 z_2 + z_2 z_1$. $lacksquare$

We proceed in a similar way in the case of matricial $R$-semicircular and semicircular operators. The corresponding cyclic $R$-transforms become ‘quadratic forms’ similar to those for the matricial $R$-circular and circular operators.

**Theorem 7.2.** If $\omega : = (\omega_{p,q})$ is the square array of matricial $R$-semicircular operators, then its cyclic $R$-transforms are of the form

$$R_{\omega}(z; q) = \sum_{p=1}^r b_{p,q} z_{p,q}^2$$

for any $q \in [r]$, where $z = (z_{p,q})$ and $b_{p,q} = \kappa_2(\omega_{p,q}, \omega_{p,q}; q)$ for any $p, q$.

Proof. The proof is similar to that of Theorem 7.1. The only non-vanishing cyclic cumulants under condition $q$ are of the form $\kappa_2(\omega_{p,q}, \omega_{p,q}; q) = b_{p,q}$, where $p, q$ are arbitrary, which gives the desired equation. $lacksquare$

**Corollary 7.4.** If $\hat{\omega} : = (\hat{\omega}_{p,q})$ is the upper triangular array of matricial semicircular operators, then its cyclic $R$-transforms are of the form

$$R_{\hat{\omega}}(z; q) = \sum_{p=1}^q b_{p,q} z_{p,q}^2 + \sum_{p=q+1}^r b_{p,q} z_{p,q}^2$$

where we use the notations of Theorem 7.2.

Proof. The only non-vanishing cyclic cumulants under condition $q$ are of the form $\kappa_2(\hat{\omega}_{p,q}, \hat{\omega}_{p,q}; q) = b_{p,q}$ for $p \leq q$ and $\kappa_2(\hat{\omega}_{q,p}, \hat{\omega}_{q,p}; q) = b_{q,p}$ for $p > q$, which gives the desired equation. $lacksquare$
Corollary 7.5. If \( b_{p,q} = d_p \) for any \( p, q \), then \( s := \sum_{p,q} \omega_{p,q} \) is semicircular with respect to \( \Phi = \sum_{q=1}^{r} d_q \Phi_q \) and the corresponding R-transform takes the form

\[
R_s(w) = \sum_{q=1}^{r} d_q R_\omega(z; q),
\]

where the \( R_\omega(z; q) \) are the cyclic R-transform of Theorem 7.2 in which all entries of the array \( z \) are identified and denoted by \( w \).

Proof. The proof is similar to that of Corollary 7.3 except that we use [7, Theorem 5.1]. The moments of the matrix \( Y(n) \) converge then under the normalized trace to those of the standard semicircular operator \( s := \sum_{p,q} \omega_{p,q} \) under \( \Phi \) by Proposition 4.3. \( \blacksquare \)

Remark 7.2. For simplicity, we considered so far the case of one array and its adjoint. However, one can easily extend the results of this section to a finite number of arrays \( (\zeta_{p,q}(u)) \) and \( (\zeta^*_{p,q}(u)) \), where \( u \in \mathcal{U} \). In that case, in order to avoid lengthy notations, we replace the triples \( (p, q, u) \) by one symbol \( \mu \) as in Section 2. We then have a family of arrays of \( R \)-circular operators

\[
\zeta := \{ \zeta_\mu : \mu \in [r] \times [r] \times \mathcal{U} \}
\]

and, denoting the corresponding family of their adjoints by \( \zeta^* \), we can define the corresponding family of cyclic R-transforms

\[
R_{\zeta, \zeta^*}(z, z^*; q) = \sum_{n=1}^{\infty} \sum_{\mu_1, \ldots, \mu_n} \sum_{\epsilon_1, \ldots, \epsilon_n} \kappa_n(\zeta_{\mu_1, \ldots, \mu_n}, q) z_{\mu_1}^{\epsilon_1} \cdots z_{\mu_n}^{\epsilon_n}
\]

where \( q \in [r] \) and \( z := (z_\mu); z^* := (z^*_\mu) \). It can be proved that these cyclic R-transforms take the form

\[
R_{\zeta, \zeta^*}(z, z^*; q) = \sum_{\mu=(p,q,u)} b_\mu z_\mu^* z_{\mu} + \sum_{\mu=(p,q,u)} b_\mu z_\mu^* z_{\mu^*},
\]

where the summation runs over all triples of the form \( \mu = (p, q, u) \), with \( \mu_1 = (q, p, u) \) and \( b_\mu = b_{p,q}(u) \) for any \( p, q, u \). In fact, the proof is identical to that of Theorem 7.1 since operators labelled by \( u \) and \( u' \) cannot be paired if \( u \neq u' \) and the proof given here for one array does not change if we take more arrays. Thus, we get additivity of the cyclic R-transforms corresponding to different pairs of arrays labelled by \( u \) and their adjoints.

One can also observe that all systems of operators considered in this paper (see Definition 2.4) are matricial analogs of \( R \)-diagonal operators, which requires us to replace the \( R \)-transform by the cyclic \( R \)-transforms. Moreover, matricial \( R \)-circular operators can also be viewed as non-tracial analogs of circular operators.

A general treatment of cyclic cumulants and cyclic \( R \)-transforms in the framework of matricial freeness and symmetric matricial freeness goes beyond the scope of this article. We expect a close relation here, similar to that between free cumulants and \( R \)-transform and the notion of freeness. The concept of the matricial \( R \)-transform introduced in [6] seems to be different since the latter refers to a slightly different concept of independence called strong matricial freeness. Actually, this is the main reason why we used the term ‘cyclic \( R \)-transforms’ instead of ‘matricial \( R \)-transform’.

Nevertheless, a unified approach to both concepts seems to be possible.
8. Moment series for $R$-circular systems

The matricial moment series for the arrays $\zeta = (\zeta_{p,q})$ and $\zeta^* = (\zeta^*_{p,q})$ is the family of formal series of the form

$$M_q(z, z^*) = \sum_{m=1}^{\infty} \sum_{p_1, q_1, \ldots, p_m, q_m} \Phi_q(\zeta_{p_1, q_1}^{e_1} \cdots \zeta_{p_m, q_m}^{e_m}) z_{p_1, q_1}^{e_1} \cdots z_{p_m, q_m}^{e_m}$$

where $q \in [r]$ and $z = (z_{p,q})$ and $z^* = (z^*_{p,q})$ are the arrays of $2r^2$ noncommuting indeterminates.

Proposition 8.1. The moment series for the arrays $\zeta, \zeta^*$ satisfy the relations

$$M_q = 1 + \sum_{p=1}^{r} b_{p,q}(z_{p,q}^* M_p z_{p,q}^* M_q + z_{q,p}^* M_p z_{p,q}^* M_q)$$

for any $q \in [r]$, where $M_s = M_s(z, z^*)$ for any $s \in [r]$.

Proof. We can express each moment $\Phi_q(\zeta_{p_1, q_1}^{e_1} \cdots \zeta_{p_m, q_m}^{e_m})$ in terms of cumulants according to the formula

$$\Phi_q(\zeta_{p_1, q_1}^{e_1} \cdots \zeta_{p_m, q_m}^{e_m}) = \sum_{\pi \in \mathcal{N}(\zeta_{p_1, q_1}^{e_1}, \ldots, \zeta_{p_m, q_m}^{e_m})} \kappa_{\pi}(\zeta_{p_1, q_1}, \ldots, \zeta_{p_m, q_m}; q),$$

where each partitioned cumulant $\kappa_{\pi}(:, q)$ is a product of cumulants over the 2-blocks of $\pi$. In order to find a typical relation between moment series, it now suffices to single out in each $\pi$ the 2-block containing 1, say $\{1, k\}$. The cumulant corresponding to this block is $b_{p,q}$ for some $p$ and the corresponding matrix units must be $e(q, p)$ and $e(p, q)$, respectively. All cumulants corresponding to the inner blocks of this block are collected in the series $M_p(z, z^*)$ since they correspond to the mixed moments of variables lying in between $\zeta_{p_1, q_1}^{e_1}$ and $\zeta_{p_k, q_k}^{e_k}$ computed ‘under condition $p$’ since the corresponding product of matrix units must be $e(p, p)$. In turn, all cumulants corresponding to the blocks involving numbers greater than $q$ are collected in the series $M_q(z, z^*)$ since they correspond to the mixed moments of $\zeta_{p_{k+1}, q_{k+1}}^{e_{k+1}}, \ldots, \zeta_{p_m, q_m}^{e_m}$ and the corresponding product of matrix units must be $e(q, q)$. Of course, $z_{p,q}$ is assigned to $\zeta_{q,p}^*$ and $z_{p,q}$ is assigned to its adjoint $\zeta_{p,q}^*$ and, since our indeterminates do not commute, we have to place the series $M_p(z, z^*)$ in between them if $\{1, k\}$ is associated with the pairing $(\zeta_{q,p}^*, \zeta_{p,q})$. In turn, the series $M_p(z, z^*)$ is placed in between $z_{q,p}^*$ and $z_{p,q}^*$ if $\{1, k\}$ is associated with the pairing $(\zeta_{q,p}, \zeta_{q,p}^*)$. This gives the formula

$$M_q = 1 + \sum_{p=1}^{r} b_{p,q}(z_{p,q}^* M_p z_{p,q}^* M_q + z_{q,p}^* M_p z_{p,q}^* M_q)$$

where $M_s = M_s(z, z^*)$ for any $s$.

One can express the functional formulas of Proposition 8.1 in terms of the cyclic $R$-transforms of $\zeta, \zeta^*$. For that purpose, it is convenient to replace the array $z^*$ by its transpose $w = (w_{p,q})$, where $w_{p,q} = z_{q,p}^*$ for any $p, q \in [r]$. Moreover, let

$$M(z, w) : = \text{diag}(M_1, \ldots, M_r)$$

$$R(z, w) : = \text{diag}(R_1, \ldots, R_r),$$

$$M(z, w) : = \text{diag}(M_1, \ldots, M_r)$$

$$R(z, w) : = \text{diag}(R_1, \ldots, R_r),$$
where $M_q = M_q(z, w)$ and $R_q = R_{\zeta'}(z, w; q)$ for any $q \in [r]$ and we abuse the notation in the sense that we replace each $z^*_{p,q}$ by $w_{q,p}$ and we still write $w$ as the second argument of each $M_q$ and $R_q$.

**Corollary 8.1.** With the above notations, let
\[
R_q(z, w) = \sum_{p=1}^r b_{p,q}(w_{q,p}z_{p,q} + z_{q,p}w_{p,q})
\]
for any $q \in [r]$. Then, the functional equation holds
\[
M(z, w) = I + R(zM(z, w), wM(z, w)),
\]
where $zM(z, w)$ and $wM(z, w)$ are products of arrays.

**Proof.** We can substitute $z_{s,t} \rightarrow z_{s,t}M_t = (zM)_{s,t}$ and $w_{s,t} \rightarrow w_{s,t}M_t = (wM)_{s,t}$ for $s,t \in \{p,q\}$, which gives the formula of Proposition 8.1. This completes the proof. ■

From the formulas of Proposition 8.1 we can obtain a concise formula for the diagonal matrix of moment series for $\zeta = \sum_{q} \zeta_q$ in the states $\Psi_q$. Let $D : M_r(A) \rightarrow M_r(A)$ be the linear mapping given by
\[
D(A) = \text{diag}(A_1, \ldots, A_r)
\]
for any $A \in M_r(A)$, where
\[
A_q = \sum_{p=1}^r a_{p,q}
\]
for any $q$. This formula will be similar to that derived in [7] for the moment series for the sums of matricial $R$-semicircular (or, semicircular) operators $\omega = \sum_{p,q} \omega_{p,q}$.

**Corollary 8.2.** Let $M = \text{diag}(M_1, \ldots, M_r)$, where each $M_q$ is given by
\[
M_q(z, z^*) = \sum_{m=1}^\infty \sum_{\epsilon_1, \ldots, \epsilon_m} \Phi_q(\zeta^{\epsilon_1} \ldots \zeta^{\epsilon_m})z^{\epsilon_1} \ldots z^{\epsilon_m}
\]
for the sum $\zeta = \sum_{p,q} \zeta_{p,q}$. Then $M$ satisfies the functional equation
\[
M = I + D(z^*MBz + zMBz^*M),
\]
where $B = (b_{p,q})$ and $z,z^*$ are two noncommuting variables.

**Proof.** We set $z_{s,t} = z$ and $z^*_{s,t} = z^*$ for any $s,t$ in the formulas of Proposition 8.1, which gives
\[
M_q = 1 + \sum_{p=1}^r (z^*M_pb_{p,q}zM_q + zM_pb_{p,q}z^*M_q)
\]
for any $q$, where we put $b_{p,q}$ between $M_p$ and $M_q$ to anticipate the product of matrices $M_p, B, M_q$. Now, it is easy to see that $z^*M_pb_{p,q}zM_q$ is the $(p,q)$-th entry of the matrix $z^*MBzM$, where $z$ and $z^*$ are to be understood as diagonal matrices with $z$ and $z^*$ on the diagonal, respectively. Similarly, $zM_pb_{p,q}z^*M_q$ is the $(p,q)$-th entry of $zMBz^*M$. In turn, summation over $p$ for fixed $q$ corresponds to the mapping $D$, which completes the proof. ■
The formulas of Proposition 8.1 for the moment series for arrays of \( R \)-circular operators as well as those of Corollary 8.2 for the moment series for their sums are equivalent to the definition of their cumulants (which are recursions for moments) and thus it is not surprising that their solutions can be found in the form of recursions analogous to those for Catalan numbers. For simplicity, we restrict our attention to the case discussed in Corollary 8.1.

**Proposition 8.2.** The matrix-valued moment series \( M \) of Corollary 8.2 takes the form

\[
M(z, z^*) = \sum_{m=0}^{\infty} \sum_{\epsilon_1, \ldots, \epsilon_m} C_m(\epsilon_1, \ldots, \epsilon_2m) z^{\epsilon_1} \ldots z^{\epsilon_2m},
\]

where the coefficients are diagonal \( r \times r \) matrices satisfying the recurrences

\[
C_m(\epsilon_1, \ldots, \epsilon_m) = \sum_{j+k=m-1} D(C_j(\epsilon_2, \ldots, \epsilon_{2j-1})BC_k(\epsilon_{2j+1}, \ldots, \epsilon_{2m})),
\]

for any \( m \in \mathbb{N} \), with \( C_0 = I \), where we set \( C_m(\epsilon_1, \ldots, \epsilon_m) = 0 \) whenever \( (\epsilon_1, \ldots, \epsilon_m) \) is not associated with a non-crossing pairing consisting of \((1, *)\) and \((*, 1)\). These recurrences have a unique solution.

**Proof.** It suffices to substitute \( M(z, z^*) \) in the above form to the equation for \( M \) of Corollary 8.2 and observe that if \( (\epsilon_1, \epsilon_2j) = (*, 1) \), then \( C_j(\epsilon_2, \ldots, \epsilon_{2j-1}) \) comes from the first \( M \) in the product \( z^*MBzM \) and \( C_k(\epsilon_{2j-1}, \ldots, \epsilon_{2m}) \) comes from the second \( M \) in that product. All terms for which \( (\epsilon_1, \epsilon_2j) = (1, *) \) correspond to the product \( zMBz^*M \) in a similar way. This gives the desired recurrence for the matrix-valued coefficients. It is obvious that these recurrences have a unique solution.

**Corollary 8.3.** If \( b_{p,q} = d_p \) for any \( p, q \in [r] \), then the moment series \( M_0 = M_0(z, z^*) \) for the circular operator \( \zeta = \sum_{p,q} \zeta_{p,q} \) associated with \( \Phi = \sum_{q=1}^{r} d_q \Phi_q \) satisfies the functional equation

\[
M_0 = 1 + z^*M_0zM_0 + zM_0z^*M_0
\]

**Proof.** It holds that

\[
M_0(z, z^*) = \sum_{m=1}^{\infty} \sum_{\epsilon_1, \ldots, \epsilon_m} \Phi(\epsilon_1 \ldots \epsilon_m) z^{\epsilon_1} \ldots z^{\epsilon_m} = \sum_{q=1}^{r} d_q M_q(z, z^*)
\]

and it suffices to take the sum over \( q \) of the equation in the proof of Corollary 8.2 and set \( b_{p,q} = d_q \) to obtain the desired equation.

**Remark 8.1.** The moment series for a family of arrays labelled by \( u \) is defined in a similar way. Accordingly, all results of this section hold for such families, but we do not formulate them in the whole generality for the sake of simplicity. In order to obtain the general formulation it suffices to label each pair \( \zeta_{p,q}, \zeta_{p,q}^* \) by an additional argument \( u \) and use the fact that two operators corresponding to any 2-block \( \{i, j\} \) of any non-crossing pair partition give a non-trivial contribution to the moment series and to the cyclic \( R \)-transforms only if they are labelled by the same \( u \). The effect is that the definition of the series \( M_q \) acquires additional summations over \( u_1, \ldots, u_m \in \mathcal{U} \) and appropriate additional summations appear in the formulas of Proposition 8.1 (over
u), Corollary 8.1 (over u), Corollary 8.2 (over \(u_1, \ldots, u_m\)) and Proposition 8.2 (over \(u_1, \ldots, u_{2m}\)).

**Remark 8.2.** The recurrence of Proposition 8.2 reminds that for Catalan matrices in [7, Theorem 8.3]. Note, however, that the latter involved the matrix of \(K\)-transforms

\[
K(z) = \text{diag}(K_{1,1}(z), \ldots, K_{r,r}(z))
\]

and not the corresponding matrix of moment generating functions \(M_{j,j}\). Nevertheless, the sequence of Catalan matrices \((C_m)_{m \geq 0}\) can be viewed as the matricial analog of the sequence of Catalan numbers since that recurrence involves a nice bracketing rule and, after all, the \(K\)-transform of the standard semicircle law agrees with its Cauchy transform. Clearly, the matrix of moment series can be computed from \(K(z)\) as in the scalar-valued case, namely

\[
M(z) = \text{diag}(M_{1,1}(z), \ldots, M_{r,r}(z)) = \sum_{m=0}^{\infty} (zK(1/z))^m.
\]

Its matrix-valued coefficients satisfy a different recurrence than that for Catalan matrices. Of course, these coefficients give another sequence of matricial analogs of Catalan numbers which is directly related to moments.

Instead of using convolutions and \(K\)-transforms [7], let us now briefly state the results for the moment series for matricial \(R\)-semicircular operators in the same spirit as we did for the matricial \(R\)-circular operators in this section. It is not hard to see that we obtain functional equations similar to those in Proposition 8.1 and Corollary 8.1. We state these results without proofs since they are very similar to those given above.

**Proposition 8.3.** The moment series for the array \((\omega_{p,q})\) of the form

\[
M_q(z) = \sum_{m=1}^{\infty} \sum_{p_1,q_1, \ldots, p_m,q_m} \Phi_q(\omega_{p_1,q_1} \cdots \omega_{p_m,q_m}) z_{p_1,q_1} \cdots z_{p_m,q_m}
\]

where \(z = (z_{p,q})\) is the array of \(r^2\) noncommuting indeterminates, satisfy the relations

\[
M_q(z) = 1 + \sum_{p=1}^{r} b_{p,q} z_{p,q} M_p(z) z_{p,q} M_q(z)
\]

for any \(q \in [r]\).

**Corollary 8.4.** The matrix-valued moment series \(M = \text{diag}(M_1, \ldots, M_r)\) for the sum

\[
\omega = \sum_{p,q} \omega_{p,q},\text{ where each } M_q = M_q(z) \text{ is given by}
\]

\[
M_q(z) = \sum_{m=1}^{\infty} \Phi_q(\omega^m) z^m,
\]

satisfies the functional equation

\[
M(z) = I + z^2 D(M(z)BM(z)),
\]

where \(B = (b_{p,q})\) and \(z\) is a variable. Its unique solution is of the form

\[
M(z) = \sum_{m=0}^{\infty} C_m z^{2m},
\]
where the coefficients are diagonal $r \times r$ matrices satisfying the recurrence

$$C_m = \sum_{j+k=m-1} \mathcal{D}(C_j B C_k)$$

for any $m \in \mathbb{N}$, with $C_0 = I$.

**Remark 8.3.** One can also consider the moment series for the arrays of matricial creation and annihilation operators $\phi = (\phi_{p,q})$ and $\phi^* = (\phi^*_{p,q})$ of the form

$$M_q(z, z^*) = \sum_{m=1}^{\infty} \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq \left\lfloor \frac{r}{2} \right\rfloor} \Phi_q(\phi_{i_1,i_1}^{i_1} \cdots \phi_{i_m,i_m}^{i_m}) z_{i_1,i_1}^{i_1} \cdots z_{i_m,i_m}^{i_m}$$

where $q \in [r]$ and $z = (z_{p,q})$ and $z^* = (z^*_{p,q})$ are the arrays of $2r^2$ noncommuting indeterminates. Then the corresponding cyclic $R$-transforms are of the form

$$R_{\phi,\phi^*}(z, z^*; q) = \sum_{p=1}^{r} b_{p,q}^* z_{p,q} z_{p,q}^*$$

and the functional equation for the matrix-valued moment series takes the same form as for the matricial $R$-circular operators in Corollary 8.1, namely

$$M(z, w) = I + R(zM(z, w), wM(z, w)),$$

where $w = (w_{p,q})$ is the transpose of $z$, i.e. $w_{p,q} = z_{q,p}^*$. If we set $z_{p,q}^* = z_{p,q}$, we can write in the same form the functional equation for the $R$-semicircular operators involving the cyclic $R$-transforms of Theorem 7.2.

**9. Asymptotics of random blocks**

Matricial semicircular (circular) operators give operatorial realizations of the limit joint distributions ($\ast$-distributions) under partial traces of symmetric random blocks of Hermitian (non-Hermitian) random matrices with i.i.d. complex entries, respectively [5,7].

By partial traces we understand normalized traces composed with classical expectations of the form

$$\tau_q(n) = \frac{1}{|N_q|} (\text{tr}_q \circ \mathbb{E})$$

where $q \in [r] := \{1, 2, \ldots, r\}$ and $\text{tr}_q$ is the trace over the sets of basis vectors of $\mathbb{C}^n$ indexed by $N_q$, where

$$[n] := N_1 \cup \ldots \cup N_r$$

is a partition into disjoint non-empty intervals (their dependence on $n$ is suppressed). Further, it is assumed that

$$\lim_{n \to \infty} \frac{|N_q|}{n} = d_q \geq 0 \quad \text{for any } q,$$

where $D$ is the dimension matrix.

The following theorem has been proved in [7] (see also [5] for the case of one matrix).

**Theorem 9.1.** Let $\{Y(u, n) : u \in \mathcal{U}\}$ be a family of Gaussian random matrices with i.i.d. complex entries, each with symmetric blocks $(T_{p,q}(u, n))$ and symmetric block variance matrix $V(u)$ and associated matrix $B(u) = D V(u)$. Then
(1) if the matrices are Hermitian,
\[
\lim_{n \to \infty} T_{p,q}(u,n) = \hat{\omega}_{p,q}(u),
\]
for any \(1 \leq p < q \leq r\) and \(u \in \mathcal{U}\), where convergence is in the sense of mixed moments and \(^*\)-moments under partial traces, respectively.

(2) if the matrices are non-Hermitian, then
\[
\lim_{n \to \infty} T_{p,q}(u,n) = \eta_{p,q}(u),
\]
for any \(1 \leq p < q \leq r\) and \(u \in \mathcal{U}\), where convergence is in the sense of mixed \(^*\)-moments and \(^*\)-moments under partial traces, respectively.

We would like to complete this picture with the asymptotics of blocks
\[
S_{p,q}(u,n) = \sum_{i,j \in N_p \times N_q} Y_{i,j}(u,n) \otimes e_{i,j}(n)
\]
where \(p,q \in [r]\), from which the symmetric blocks are constructed:
\[
T_{p,q}(u,n) = \begin{cases} 
S_{p,q}(u,n) + S_{q,p}(u,n) & \text{if } p \neq q \\
S_{q,q}(u,n) & \text{if } p = q 
\end{cases}
\]

Of course, the right framework is non-Hermitian since we do not want to assume any dependence among the entries. It is natural to expect that \(R\)-circular operator will give the asymptotics of blocks \(S_{p,q}(u,n)\).

**Theorem 9.2.** Let \(\{Y(u,n) : u \in \mathcal{U}\}\) be a family of Gaussian random matrices with i.b.i.d. complex entries, each with blocks \((S_{p,q}(u,n))\) and symmetric block covariance matrix \(V(u)\) and associated matrix \(B(u) = DV(u)\), respectively. Then
\[
\lim_{n \to \infty} S_{p,q}(u,n) = \zeta_{p,q}(u)
\]
for any \(1 \leq p, q \leq r\) and \(u \in \mathcal{U}\), where convergence is in the sense of mixed \(^*\)-moments under partial traces.

**Proof.** We need to show that
\[
\lim_{n \to \infty} \tau_q(n)(S_{p_1,q_1}(u_1,n) \ldots S_{p_m,q_m}(u_m,n)) = \Psi_q(\zeta_{p_1,q_1}(u_1) \ldots \zeta_{p_m,q_m}(u_m))
\]
for any \(\epsilon_1, \ldots, \epsilon_m \in \{1,*\}\), \(u_1, \ldots, u_m \in \mathcal{U}\) and \(q, p_1, q_1, \ldots, p_m, q_m \in [r]\). Here, the symbol \(S_{p,q}(u,n)\) denotes the block consisting of variables indexed by \((i,j) \in N_p \times N_q\) and \(S_{p,q}(u,n)\) is its adjoint (we identify blocks with the subblocks of \(n \times n\) matrices).

By [7, Theorem 9.1], we have
\[
\lim_{n \to \infty} \tau_q(n)(T_{p_1,q_1}(u_1,n) \ldots T_{p_m,q_m}(u_m,n)) = \Psi_q(\eta_{p_1,q_1}^\epsilon(u_1) \ldots \eta_{p_m,q_m}^\epsilon(u_m))
\]
for any \(\epsilon_1, \ldots, \epsilon_m \in \{1,*\}\), \(u_1, \ldots, u_m \in \mathcal{U}\) and \(q, p_1, q_1, \ldots, p_m, q_m \in [r]\). By definition,
\[
\eta_{p,q}^\epsilon = \begin{cases} 
\zeta_{p,q}^\epsilon(u) + \zeta_{q,p}^\epsilon(u) & \text{if } p \neq q \\
\zeta_{q,q}^\epsilon(u) & \text{if } p = q 
\end{cases}
\]
for any \(p, q, u, \epsilon\). It suffices to analyze the contributions from various mixed \(^*\)-moments of \(R\)-circular operators to the combinatorial formula for the mixed \(^*\)-moments of operators \(\eta_{p,q}(u)\) given in [7, Proposition 9.1]. Namely, let \(\mathcal{NC}^2_m(\pi_1(u_1, \epsilon_1), \ldots, (w_m, u_m, \epsilon_m))\) be the set of those \(\pi \in \mathcal{NC}^2_m\) which are adapted to \((w_1, u_1, \epsilon_1), \ldots, (w_m, u_m, \epsilon_m))\) and to the number \(q\), where \(w_k = \{p_k, q_k\}\) for any \(k\). This means that if \(\{i,j\}\) is
a block of $\pi$, where $i < j$, then there exists a tuple $((v_1, u_1), \ldots, (v_m, u_m))$, where $v_k \in \{(p_k, q_k), (q_k, p_k)\}$ for any $k$, such that

$$v_i = v_j, \ u_i = u_j, \ \varepsilon_i \neq \varepsilon_j \text{ and } q_j = p_{o(j)}$$

where $\{o(i), o(j)\}$ is the nearest outer block of $\{i, j\}$, where $o(i) < o(j)$ and it is understood that we assign $\{q, q\}$ to the imaginary block. Then

$$\Psi_q(\eta_{p_1,q_1}^{\pi}(u_1) \ldots \eta_{p_m,q_m}^{\pi}(u_m)) = \sum_{\pi \in \mathcal{N}C_{m,q}^2((w_1, u_1, \varepsilon_1), \ldots, (w_m, u_m, \varepsilon_m))} b_q(\pi, f),$$

where $f$ is the coloring of $\pi$ defined by $((w_1, u_1, \varepsilon_1), \ldots, (w_m, u_m, \varepsilon_m))$ and by the number $q$. Therefore, is suffices to identify the limit contributions from various mixed *-moments of the blocks $S_{p,q}(n, n)$ with the corresponding mixed *-moments of the operators $\zeta_{p,q}(u)$. The diagonal blocks and the diagonal operators are the same as in [7, Theorem 9.1] and thus it suffices to analyze the off-diagonal ones. By definition, we have

$$\eta_{p,q}(u) = \zeta_{p,q}(u) + \zeta_{q,p}(u),$$

where

$$\zeta_{p,q}(u) = \varphi_{p,q}(2u - 1) + \varphi_{q,p}^*(2u)$$

$$\zeta_{q,p}(u) = \varphi_{q,p}(2u - 1) + \varphi_{p,q}^*(2u)$$

for any $p \neq q$ and any $u$. If $\{i, j\}$ is a block of $\pi \in \mathcal{N}C_{m,q}^2((w_1, u_1, \varepsilon_1), \ldots, (w_m, u_m, \varepsilon_m))$ and $(\varepsilon_i, \varepsilon_j) = (*, 1)$, then $\eta_{p,q}^*(u_1), \eta_{p,q}^*(u_j)$ forms a pairing and thus there are two possible pairings of $R$-circular operators which may contribute:

$$\zeta_{p,q}^*(u_i), \zeta_{p,q}^*(u_j) \text{ or } \zeta_{q,p}^*(u_i), \zeta_{q,p}^*(u_j).$$

The choice of the ‘right’ pairing depends on the vector of $\mathbb{C}^n$ onto which $\eta_{p,q}(u_j)$ is supposed to act. Depending on whether it is $e(q_j)$ or $e(p_j)$, the first or the second pairing contributes, respectively. Observe that these pairings must correspond to the pairings of blocks

$$S_{p,q}^*(u_i, n), S_{p,q}^*(u_m, n) \text{ or } S_{q,p}^*(u_i, n), S_{q,p}^*(u_m, n),$$

respectively, since in a given pair of symmetric blocks $T_{p,q}^*(u_i, n), T_{p,q}^*(u_j, n)$ we get a non-trivial action of the second member of these pairs of subblocks onto vectors of $\mathbb{C}^n$ indexed either by $k \in N_{q_j}$ or $k \in N_{p_j}$, respectively. A similar correspondence holds if $(\varepsilon_i, \varepsilon_j) = (1, *)$ and thus our formula for the limit mixed *-moments of $R$-circular operators is proved. 

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