Affine Schubert calculus and double coinvariants

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Abstract

We define an action of the double coinvariant algebra $DR_n$ on the equivariant Borel-Moore homology of the affine flag variety $\tilde{F}_n$ in type $A$, which has an explicit form in terms of the left and right action of the (extended) affine Weyl group. Up to first order in the augmentation ideal, we show that it coincides with an action of the Cherednik algebra on the equivariant homology of the homogeneous affine Springer fiber $\tilde{S}_{n,m} \subset \tilde{F}_n$ due to Yun and the second author \cite{30}, and therefore preserves the non-equivariant homology group $\tilde{H}_\ast(\tilde{S}_{n,m}) \to \tilde{H}_\ast(\tilde{F}_n)$. We define a filtration by $\mathbb{Q}[x]$-submodules of $DR_n \cong H_\ast(\tilde{S}_{n,n+1})$ indexed by compositions of $n$, whose leading terms are the Garsia-Stanton "descent monomials" in the $y$-variables. We find an explicit presentation of the subquotients as submodules of the single-variable coinvariant algebra, using Schubert classes in $\tilde{S}$ due to Goresky, Kottwitz, and MacPherson. As a consequence, we obtain an explicit monomial basis of $DR_n$ which generalizes the Haglund-Loehr formula, independent of the results of \cite{5}.

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1 Introduction

The double coinvariant algebra is the quotient space of the polynomials algebra $\mathbb{Q}[x,y] = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ in $2n$ variables by the ideal generated by nonconstant diagonally symmetric polynomials

$$DR_n = \mathbb{Q}[x,y]/\mathfrak{m}_+(x,y), \quad \mathfrak{m}_+(x,y) = \left\langle \sum_k x_i^j y_k^j : (i, j) \neq (0, 0) \right\rangle.$$  

In [19], Haiman proved that this space has dimension $(n+1)^{n-1}$, and in [18], Haglund and Loehr conjectured the combinatorial formula for the bigraded Hilbert series in terms of certain parking function statistics

$$\sum_{\pi \in PF_n} q^{\text{dinv}(\pi)} t^{\text{area}(\pi)}.$$  

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This was first settled by the first author and Mellit in [5], as well as the more general “rational case” by Mellit [29].

Several articles due to Lusztig-Smelts, Gorsky-Mazin, Hikita, and Gorsky-Mazin-Vazirani have connected the combinatorics of the rational version of the Haglund-Loehr formula with a basis of the affine Springer fiber in type $A$, in which the $q$-grading corresponded with the homological grading, and the $t$-grading was a new statistic related to the “expected” dimension of a corresponding affine paving [11, 12, 13, 20, 28]. On the other hand, Oblomkov and Yun have shown that the cohomology of this affine Springer fiber $\tilde{S}_{n,m}$ is an irreducible module $L_{m/n}^{\text{triv}}$ over the rational Cherednik algebra $H_{\text{rat}}^{m/n}$ [30]. It was known from [9] that there is an isomorphism of graded spaces $DR_n$ and $L_{(n+1)/n}$, but this did not result in a proof of the Haglund-Loehr formula, because Hikita’s statistic could not be connected to the grading by the $y$-variables in $DR_n$.

The first main result of this paper defines an action of the double coinvariant algebra on the homology of the affine flag variety in type $A$, and shows that it preserves the homology of the $(n,m)$-affine Springer fibers:

**Theorem A.** There is an action of $DR_n$ on the homology of the affine flag variety that preserves $H_* (\tilde{S}_{n,m}, \mathbb{Q}) \subset H_* (\tilde{F}_{\ell}, \mathbb{Q})$, the Borel-Moore homology of the affine Springer fiber in type $A$. Here the $x$-variables act by dual Chern class operators which lower the homological degree, while the $y$-variables are homogeneous operators defined using the usual action of the (extended) affine Weyl group, which is compatible with the Springer action. In the case $m = n + 1$, this action induces an isomorphism $H_* (\tilde{S}_{n,n+1}) \cong DR_n$ by applying $f \in DR_n$ to the generator $[\tilde{S}_{n,n+1}] \in H_{2d} (\tilde{S}_{n,n+1})$, $d = \dim \tilde{S}_{n,n+1}$.

We then define a filtration $F_a DR_n \subset DR_n$ by $\mathbb{Q}[x]$-submodules, which is generated by monomials $y^b$ for in $b \leq_{\text{des}} a$ for a certain total order, whose leading terms are the “descent monomials” $g_\tau(y)$, which are well known to be a basis of the single coinvariant algebra $R_n(y)$ [8]. In our second main theorem, we describe this filtration under the isomorphism of Theorem A, and find an explicit presentation for the subquotients:

**Theorem B.** Given a composition $a \in \mathbb{Z}_{>0}^n$, the following three $\mathbb{Q}[x]$-modules are isomorphic:

a) The subquotients $F_a DR_n / F_{a-1} DR_n$, where $F_a DR_n = \langle y^b \rangle$, for $b \leq_{\text{des}} a$. 

b) The corresponding subquotients of a certain filtration on $H_\ast(S_{n,m})$, defined using a coarsening of the Bruhat order on the fixed point basis in the nil Hecke algebra, i.e. the equivariant Borel-Moore homology of the affine flag variety.

c) A certain submodule of the single coinvariant algebra $R_n(x)$ generated by an explicit polynomial depending on $a$.

Moreover, the third description actually has an explicit basis by monomials in the $x$-variables, leading to an explicit formula for the Hilbert series of all three modules, which in particular gives an independent proof the Haglund-Loehr formula.

This theorem resolves the problems of connecting the degree in the $y$-variables to Hikita’s statistics, produces a monomial basis of the double coinvariant algebra, and characterizes the subquotients as $\mathbb{Q}[x]$-modules.

Let us describe the monomial basis in the most elementary terms.

Given a permutation $\tau \in S_n$ the ordered set of descents $\text{Des}(\tau)$ consists of $i$ such that $\tau_i > \tau_{i+1}$. The descent monomial is defined by

$$g(\tau) = \prod_{i \in \text{Des}(\tau)} (y_{\tau_1} \cdots y_{\tau_i}).$$

If $|\text{Des}(\tau)| = k$ then we define a slight enlargement of the set of descents by $\text{Des}'(\tau)_0 = 0, \text{Des}'(\tau)_{k+1} = n$ and $\text{Des}'(\tau)_i = \text{Des}(\tau)_i$ for $1 \leq i \leq k$. Then for $i$ satisfying inequality $\text{Des}'(\tau)_{m-1} < i \leq \text{Des}'(\tau)_m$ we define

$$w_i(\tau) = (\text{Des}'(\tau)_m - i) + \{|j| \tau_j < \tau_i, \text{Des}'(\tau)_m < j \leq \text{Des}'(\tau)_{m+1}|\}.$$ 

**Corollary 1.** The double coinvariant algebra $DR_n$ has a monomial basis:

$$g(\tau)x_{\tau_1}^{k_1} \cdots x_{\tau_n}^{k_n}, \quad \tau \in S_n, \quad 0 \leq k_i \leq w_i(\tau) - 1.$$ 

We hope for several future directions:

1. The definition we present of the filtration on $H_\ast(S_{n,m})$ is defined algebraically using the equivariant homology groups. However, it seems clear that there should be a geometric description of this filtration, through topological subspaces $X_a \cap \bar{S}$, for $X_a \subset \mathcal{Fl}$. Our combinatorics turn out to be closely related to the description of the torus-fixed point set of the regular nilpotent Hessenberg varieties, but with Hessenberg
functions (i.e. Dyck paths) replaced with \((n, n+1)\) Dyck paths \([21, 32]\). It is likely that related varieties will be a part of any such geometric formulation.

2. We expect that Theorem B will be important for generalizing the Haglund-Loehr formula to other root systems, for which there are currently no available conjectures. We particularly hope that the desired geometric subspaces \(X_a\) will admit a natural generalization.

3. We have tested numerically that this filtration is compatible with taking invariants by Young subgroups, in the sense that the subquotients of

\[
F_a DR_n^{S_{\mu}} = F_a DR_n \cap DR_n^{S_{\mu}}, \quad S_{\mu} = S_{\mu_1} \times \cdots \times S_{\mu_l},
\]

produce the desired coefficients from the shuffle theorem \([5, 17]\). Extending our proof to this case should essentially follow from using affine parabolic flag varieties instead of the full flags.

4. While Theorem B applies to the case \(m = n + 1\), our filtration makes sense more generally. For coprime \((n, m)\), the replacement for \(DR_n\) should be an associated graded module of the finite-dimensional representations of the rational Cherednik algebra. One might also consider the intersection with affine Springer fibers which are not necessarily compact, such as the non-coprime case.

5. We expect the the description of the double coinvariants as a module will be important towards categorification, and hope for the expected applications to Khovanov-Rozansky knot homology, in which the \(x\)-variables are the variables that appear in Sörgel bimodules.

Let us comment on the main ideas behind the results in the paper. To prove Theorem A we explicitly construct the action of \(\mathbb{Q}[\epsilon][x, y]\) on the equivariant Borel-Moore homology \(H^*_C(\tilde{F}l)\) of the affine flag variety \(\tilde{F}l\), inducing an action of \(\mathbb{Q}[x, y]\) on nonequivariant homology. We then show that this action agrees up to first order in \(\epsilon\) with a noncommutative action due Oblomkov and Yun \([30]\) that preserves the subspace \(H^*_C(\tilde{S}_{m,n}) \hookrightarrow H^*_C(\tilde{F}l)\), implying \(\mathbb{Q}[x, y]\) preserves the subspace \(H_*(\tilde{S}_{m,n})\) in the nonequivariant setting. It follows from the explicit formulas for the action of \(x_i, y_i\) that the non-equivariant action of \(\mathbb{Q}[x, y]\) factors through an action of \(DR_n\). Finally, we check that the dimension of the \(DR_n\)-submodule generated by a
particular element $\Delta \in H_\bullet(\mathcal{S}_{n,n+1})$ has the correct dimension, identifying $DR_n \cong H_\bullet(\mathcal{S}_{n,n+1})$.

The proof of Theorem B is more involved and relies on a combination of several combinatorial descriptions of the set of parking functions of Haglund and others, as well as Gorsky-Mazin-Vazirani, and Hikita. It also uses a new bijection between parking functions, and certain subsets of $S_n$ which generalize the torus-fixed point sets of the regular nilpotent Hessenberg varieties, which we expect to be part of a larger geometric picture. A key insight in the proof is to lift $F_a DR_n$ to a $\mathbb{Q}[\epsilon]$-submodule $F_a H^C_\bullet(\mathcal{S}_{n,n+1})$, described using torus fixed points, which we prove has the desired Hilbert series.

The paper is divided into five sections. In section 2 we discuss the geometric results and definitions that we will need for the main construction, including the results of [30]. In the interest of making our paper readable to combinatorialists, we have compartmentalized the necessary algebraic facts from this section into Proposition 6 of section 4 so that it may be safely skipped. In section 3 we recall combinatorial facts about affine permutations and parking functions, and we give a new description of parking functions in terms of a bijection of Haglund [15], which turns out to be similar to the description of the fixed points of regular nilpotent Hessenberg varieties [21, 32]. Section 4 recalls the algebraic constructions of the affine Schubert polynomials and nil Hecke algebras [26]. Finally, in section 5 we state and prove the main results of the paper.

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2 Geometric preliminaries

We now recall some results about the affine Springer fiber and affine flag varieties that we will need for our main results in Chapter 5. The reader interested mainly in algebra can skip everything in this section, except for possibly the conventions for the root system in type $A$, provided they are willing to take Proposition 6 of section 4 on faith. In this paper, $(n, m)$ will always be coprime.
2.1 Root systems

In this section we fix our conventions on the root system for type $A$. Let $\mathfrak{g} = \mathfrak{sl}_n$, let $\mathfrak{g} = \mathfrak{sl}_n$ be the corresponding affine Lie algebra, and let $\mathfrak{t}$ denote the Lie algebra of the maximal torus $T \subset SL_n$. The dual $\mathfrak{t}^*$ of the maximal torus is spanned by the fundamental weights together with the imaginary root $\mathfrak{t}^* \in \langle \Lambda_0, ..., \Lambda_{n-1}, \delta \rangle$.

We also define weights for all integers $i$ satisfying $\lambda_1, ..., \lambda_n = \Lambda_1 - \Lambda_0$, $\Lambda_2 - \Lambda_1$, ..., $\Lambda_0 - \Lambda_{n-1} - \delta$, and $\lambda_{i+n} = \lambda_i - \delta$ for all $i$. The roots $\lambda_i$, $i \in \{1, \ldots, n\}$ form a basis of a subspace $\mathfrak{t}^* \oplus \langle \delta \rangle$. The simple roots are given by $\alpha_i = \lambda_i - \lambda_{i+1}$, $0 \leq i \leq n - 1$, and the action of the affine Weyl group is given by

$$s_i(\lambda_j) = \Lambda_j - \delta_{i,j} \alpha_i, \quad s_i(\delta) = \delta, \quad w(\lambda_j) = \lambda_{w^{-1}}$$

for $i, j \in \{0, \ldots, n-1\}$. The third equation follows from the first two, and in fact holds for any integer $j$, and defined below for extended affine permutations $w \in S_n$ as well.

2.2 The affine flag variety

Let $G$ be a complex algebraic group such that its Lie algebra $\mathfrak{g}$ is simple. We define $\mathcal{O} = \mathbb{C}[[t]]$ to be the ring of formal power series of $t$, and its quotient field is $\mathcal{F}$. Respectively, $G(\mathcal{F})$ is the group of formal loops and $K = G(\mathcal{F})$ is the subgroup of holomorphic loops. The quotient $Gr = G(\mathcal{F})/G(0)$ has the structure of the ind-scheme, as an inductive limit by smooth subschemes $Y$. For more details, see the survey [37].

The affine flag variety is the ind scheme $\tilde{Fl} = G(\mathcal{F})/I$ where $I \subset G(0)$ is the subgroup of elements $g(t) \in G(0)$ such that $g(0) \in B$. In this paper we assume that $G = SL(n)$ and $T \subset \mathcal{F} \subset G$ are the maximal torus and the Borel subgroup. Then the quotient $\tilde{Fl} = G(\mathcal{F})/I$ has $n$ connected components, and to simplify notations we use notation $\tilde{Fl}$ for the connected component that contains $I$. 

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The lattice inside of $\mathbb{C}^n \otimes \mathcal{F}$ is a subspace $\Lambda$ that is preserved by $O$ and the intersection $\Lambda \cap O^n$ is of finite codimension inside $\Lambda$ and $O^n$. The index $\text{ind}(\Lambda) = \text{codim}_1 \Lambda \cap O^n - \text{codim}_{O^n} \Lambda \cap O^n$ is well-defined for a lattice. The flag variety admits the following elementary description

$$\tilde{Fl} = \{ \mathbb{C}^n \otimes \mathcal{F} \supset \cdots \supset \Lambda_i \supset \Lambda_{i+1} \supset \cdots \supset 0 : i \in \mathbb{Z} \},$$

$$\Lambda_{i+1} \subset \Lambda_i \text{, } \Lambda_{i+n} = t\Lambda_i \text{, } \Lambda_i/\Lambda_{i+1} \cong \mathbb{C} \text{, } \text{ind}(\Lambda_0) = 0 \}.$$

In this description we have tautological line bundle $L_i$ over $\tilde{Fl}$ has fiber $\Lambda_i/\Lambda_{i+1}$ at the point $\Lambda_i \in \tilde{Fl}$.

The torus $T = T \times \mathbb{C}^*$ acts on $G(\mathcal{F})$: the torus $T$ acts by left multiplication and $\mathbb{C}^*$ acts by loop rotation $\lambda \cdot g(t) = g(\lambda^{-1} t) \text{ for } \lambda \in \mathbb{C}^*$. This action has isolated fixed points which are enumerated by the bijections $w : \mathbb{Z} \to \mathbb{Z}$. Indeed, if $e_0, \ldots, e_{n-1}$ be a basis of $\mathbb{C}^n$ that is fixed by $T$, then there is a unique flag of torus-invariant lattices $\Lambda^w \in \tilde{Fl}$ satisfying

$$\Lambda_i^w/\Lambda_{i+1}^w = \langle e_k t^{-m} \rangle, \text{ } w_i = mn + k, \text{ } 0 \leq k < n,$$

provided that $w$ satisfies:

$$w_i = w_{i-n} + n, \text{ } w_1 + \cdots + w_n = n(n+1)/2.$$

Thus there is a natural identification between $\tilde{Fl}_{T}$ and $\tilde{S}_n$.

There is a natural embedding $\iota : \tilde{S}_n \to G(\mathcal{F})$ such that $\iota(w) = \Lambda^w$. The Bruhat decomposition $G(\mathcal{F}) = \bigcup_{w \in \tilde{S}_n} IwI$ induces the decomposition of $\tilde{Fl}$ into affine cells $\tilde{F}_l = \bigcup_{w \in \tilde{S}_n} X_w^\circ$ where $X_w^\circ = IwI$ is the cell of dimension $\ell(w)$. The affine Schubert variety $X_w$ is the algebraic closure of $X_w^\circ$. The variety $X_w$ is the union of cells $X_w^\circ$ and the Bruhat order $\leq_{\text{bru}}$ is defined by condition $X_w = \bigcup_{x \in \tilde{S}_n} X_x^\circ$. The varieties $Y_k$ in the description of $\tilde{Fl}$ as an ind-variety can be taken to be the union of the cells with length at most $k$.

We remind the reader of the construction of the equivariant Borel-Moore homology from [14]. In this paper, all (equivariant) homology and cohomology groups will have coefficients in $\mathbb{Q}$. Let $Z$ be a scheme with a action of a linear algebraic group $G$. Let $V$ be a representation of $G$ and let $U \subset V$ be an open subset where $G$ acts freely. Then the equivariant cohomology and Borel-Moore homology are defined by:

$$H^i_G(Z) = H^i(U \times^G Z), \text{ } H^*_G(Z) = H_{j+2(\dim V - \dim G)}(U \times^G Z),$$
where \( U \times^G Z = (U \times Z)/G \), provided the complex codimension of \( V - U \) in \( V \) is greater than \( i/2 \) and \( \dim X - j/2 \).

Notice that in our definition the homological degree is bounded from above by \( 2 \dim Z \) and is not bounded from below. The main advantage of using equivariant Borel-Moore homology is we have a fundamental class \([Z] \in H^{2d}_G(Z)\), \( d = \dim Z \). In particular, fundamental class \([pt] \in H^G_0(pt)\) and cap product provide an identification \( H^G_*(pt) \) and \( H^*_G(pt) \). Let us also notice that \( H^G_*(pt) \) and \( H^*_G(pt) \) both have a ring structure and the above mentioned identification of both spaces respect the ring structure. Thus for any \( X \) with a \( \hat{T} \)-action, the spaces \( H^{\hat{T}}_*(X) \) and \( H^{\hat{T}}_*(X) \) are naturally \( S \)-modules and the natural pairing between these two spaces is \( S \)-linear.

The equivariant homology of the affine flag variety is defined as the direct limit
\[
H^{\hat{T}}_*(\hat{F}) = \lim_{\rightarrow} H^{\hat{T}}_*(Y_j).
\]
It has the structure of noncommutative ring with an explicit algebraic presentation, called the nil Hecke algebra, \( \mathcal{A}_{af} \) [23, 26]. The Schubert classes, \( A_w \in \mathcal{A}_{af} \) for \( w \in \hat{S}_n \) are defined as the fundamental classes \([X_w]\) of the closures of the Schubert cells \( \Omega_w \) again using Borel-Moore homology [25].

Since we define \( \hat{F} \) as inductive limit of finite-dimensional schemes \( Y_j \), it is natural to define the cohomology as inverse limit with respect to the pullback maps:
\[
H^*_T(\hat{F}) = \lim_{\leftarrow} H^*_T(Y_j),
\]
as graded modules, as described in the last paragraph of [14]. Then \( H^*_T(\hat{F}) \) is a module over the equivariant cohomology of the point \( S = \mathbb{Q}[\hat{t}^*] \), which may be identified as a submodule
\[
\Lambda \cong H^*_T(\hat{F}) \subset \text{Hom}_S \left( H^{\hat{T}}_*(\hat{F}), S \right). \tag{2}
\]
Then the affine Schubert polynomials may be defined as a dual basis to \( A_w \), see [24, 25, 26]. We will denote by \( x_i \) the first Chern class \( c_1(L_i) \in H_T(\hat{F}) \). These classes, together with the pullback of the equivariant cohomology of the affine Grassmannian, generate the equivariant cohomology as an \( S \)-module, with relations described in section 4.1.
2.3 The affine Springer fiber

Given an element $\gamma \in \mathfrak{g}[t]$ the authors of [22] attach a subset of $\tilde{F}$:

$$\tilde{S}_\gamma = \{gI| \text{Ad}_g^{-1}\gamma \in I\}.$$  

The lattice $L \subset T(F)$ consisting of elements commuting with $\gamma$ naturally acts on $\tilde{S}_\gamma$. It is shown in [22] that if $\gamma$ is a topologically nilpotent and regular semi-simple then the quotient $\tilde{S}_\gamma/L$ is a quasi-projective finite type scheme. The (ind)-scheme $\tilde{S}_\gamma$ is a variety called the affine Springer fiber.

The element $\gamma \in \mathfrak{g}[t]$ is called homogeneous if $\gamma(\lambda^{-1}t)$ is conjugate to $\gamma[t]$ for all $\lambda \in \mathbb{C}^*$. The topologically nilpotent regular semi-simple elements are classified in [30] and the corresponding affine Springer fibers have a natural $\mathbb{C}^*$-action. Their homologies provide a geometric model for the representations of the graded and rational Cherednik algebra of the corresponding type [30, 34, 35]. This paper deals only with the Springer theory in type $A$, and we now recall the relevant results.

Let us denote by $N \in \mathfrak{g}[t]$ an element such that

$$N(e_i) = e_{i+1}, \quad i = 0, \ldots, n - 2, \quad N(e_{n-1}) = te_0.$$  

This element is homogeneous and regular semi-simple, as is the element $\gamma_{n,m} = N^m$ for $m > 0$. If $(n, m)$ are coprime, then the affine Springer fiber $\tilde{S}_{n,m} = \tilde{S}_{\gamma_{n,m}}$ is a projective variety, that was first studied in [28]. Let $j : \tilde{S} \to \tilde{F}$ be the inclusion map.

The full torus $\tilde{T}$ does not preserve the Springer fiber, but the one-dimensional subtorus $U = \mathbb{C}^*$, $\phi : U \to \tilde{T}$

$$\phi(\lambda) = (\lambda^{c_1}, \ldots, \lambda^{c_n}, \lambda), \quad c_i = \frac{n+1-2i}{2n}, \quad (3)$$  

preserves $\tilde{S}_{n,m}$. We fix our conventions by setting $H^*_U(pt) = \mathbb{Q}[\epsilon]$. Strictly speaking, this map is not quite defined because the exponents may be fractions, but this has no effect, and the normalization $\delta = \epsilon$ is preferred for Cherednik algebras. Since $\tilde{F}U = \tilde{F}\tilde{T}$, the fixed point set $\tilde{S}_{n,m}^U$ is naturally a subset of $\tilde{S}_n$. This set is denoted $\text{Res}(n,m)$, and has explicit description given in section 3.1.

It was shown in [28] that $\tilde{S}_{n,m} \cap X_v^\circ$, $v \in \text{Res}(n,m)$ is an affine space of dimension $d_v(n,m) \geq 0$. Respectively, we denote by $X_v$ the closure of the intersection $\tilde{S}_{n,m} \cap X_v^\circ$. As in [14], there is a well-defined fundamental class $[X_w] \in H_T^*(\tilde{S}_{n,m})$. Then we have the following proposition:
Proposition 1. For $\tilde{S} = \tilde{S}_{n,m}$ with $(n, m)$ coprime, we have

a) The pushforward map $j_* : H^U(\tilde{S}) \to H^U(F)$ is injective.

b) The restriction map $j^* : H^*_U(F) \to H^*_U(\tilde{S})$ is surjective.

c) The localization map $i^*_{\text{Res}(n,m)} : H^*_U(F) \to H^*_U(\text{Res})$ to the fixed point set is injective.

d) The equivariant Borel-Moore homology is freely generated over $\mathbb{Q}[e]$ by the fundamental classes $[X_u] \in H^*_U(\tilde{S})$.

e) The equivariant Borel-Moore cohomology is freely generated by dual elements $[X^v] \in H^*_U(\tilde{S})$, such that the pairing of $[X_u]$ with $[X^v]$ is the delta function $\delta_{u,v}$.

Proof. Part (b) is due to Oblomkov and Yun [30, 31]. Parts (d) and (e) follow from the formality theorem for cohomology [10], and the formality of the homology [14], Proposition 2.1. Part (c) follows from parts (b) and (d), and part (a) follows from [7], Proposition 6. \hfill \square

2.4 Action of the Cherednik algebra

Let us recall the definition of the graded Cherednik algebra $\mathfrak{H}^{gr}$. As a $\mathbb{Q}$-vector space,

$\mathfrak{H}^{gr} = \mathbb{Q}[u, \delta] \otimes \text{Sym}(t^*) \otimes \mathbb{Q}[\tilde{S}_n],$

with grading given by

$\deg \tilde{w} = 0, \quad \tilde{w} \in \tilde{S}_n,$

$\deg(u) = \deg(\delta) = \deg(\xi) = 2, \quad \xi \in t^*.$

Let us fix notation $\hat{t}^* = t \oplus \langle \delta \rangle$. The algebra structure is defined by

1. $u$ is central.

2. $\mathbb{Q}[\tilde{S}_n]$ and $\text{Sym}(\hat{t}^*)$ are subalgebras

3. $s_i \xi - s_i(\xi)s_i = \langle \xi, \alpha_i^\vee \rangle u, \quad \xi \in \hat{t}^*, \quad i \in \{0, \ldots, n-1\}$
The element $\delta \in \mathbb{C}^*$ is also central, and thus for $\nu \in \mathbb{Q}$ we can define an algebra

$$\mathcal{H}_\nu^{gr} = \mathcal{H}^{gr}/(u + \nu \delta).$$

This is the *graded Cherednik algebra with the central charge* $\nu$. We set the image of $\delta = -u/\nu$ to be $\epsilon$. If we specialize $\epsilon$ to 1 we obtain the algebra $\mathcal{H}^{gr}_{\nu, \epsilon = 1}$ which is the trigonometric algebra in the literature.

The subalgebra $\mathbb{Q}[\epsilon] \otimes \mathbb{Q}[\hat{S}_n]$ has a trivial representation and the induced representation

$$\text{Ind}_{\mathbb{Q}[\epsilon] \otimes \mathbb{Q}[\hat{S}_n]}^{\mathcal{H}^{gr}_{\nu, \epsilon = 1}}(\mathbb{Q}[\epsilon]) = \mathbb{Q}[\epsilon] \otimes \text{Sym}(\mathbb{C}),$$

is called *polynomial representation* of $\mathcal{H}^{gr}_{\nu, \epsilon = 1}$. The subalgebra $\text{Sym}(\mathbb{C}^*)$ acts on the right by the left multiplication on this representation. On the other hand there is a standard action of $\hat{S}_n$ on $\mathbb{Q}[\epsilon] \otimes \text{Sym}(\mathbb{C})$ given by $[\mathbb{1}]$. The action of $\mathbb{Q}[\hat{S}_n] \subset \mathcal{H}^{gr}_{\nu, \epsilon = 1}$ is a deformation of the standard action, the generator $s_i$, $i \in \{0, \ldots, n\}$ acts by the (right) operator

$$s_i + \nu \epsilon \frac{1 - s_i}{x_i - x_{i+1}}. \quad (4)$$

The equivariant Chern classes $c_1(\mathcal{L}_i)$, $i = 1, \ldots, n - 1$ generate localized equivariant cohomology $H^i_U(\mathcal{F}\mathcal{L}) \otimes \mathbb{Q}(\epsilon)$. Hence there is a natural isomorphism $H^i_U(\mathcal{F}\mathcal{L}) = \text{Sym}(\mathbb{C}^*)$. Under this identification $H^i_{U, \epsilon = 1}(\mathcal{F}\mathcal{L})$ acquires structure of $\mathcal{H}^{gr}_{m/n, \epsilon = 1}$-module. Respectively, $H^i_{U, \epsilon = 1}(\mathcal{F}\mathcal{L})$ becomes an $\mathcal{H}^{gr}_{m/n}$-module. The embedding $j : \hat{S}_{n,m} \to \mathcal{F}\mathcal{L}$ induces the pullback map between the cohomology group. This map was studied in $[30, 31]$:

**Theorem 1.** (Oblomkov, Yun $[30, 31]$) For any coprime $(n, m)$ we have

a) The restriction map: $j^* : H^*_U(\mathcal{F}\mathcal{L}) \to H^*_U(\hat{S}_{n,m})$ is surjective.

b) The kernel of $j^*$ is preserved by $\mathcal{H}^{gr}_{m/n}$, i.e. $j^*$ is a homomorphism of $\mathcal{H}^{gr}_{m/n}$-modules.

c) The equivariant cohomology at $H^*_{U, \epsilon = 1}(\hat{S}_{n,m})$ is the unique irreducible finite dimensional $\mathcal{H}^{gr}_{m/n, \epsilon = 1}$-module $\mathcal{L}_{m/n}(\text{triv})$. 


3 Combinatorial results

We review some combinatorial preliminaries about parking functions and the restricted affine permutations, i.e. the torus fixed points of the affine Springer fiber. These were given a combinatorial description by Hikita, generalizing several previous bijections \cite{11, 12, 20}. This is described by the map called $\mathcal{PS}_m$ by Gorsky, Vazirani, Mazin in \cite{13}, who also discovered a second bijection, $\mathcal{A}_m$, that will play a major role in this paper. We then give a different description of parking functions in terms of normal permutations satisfying a condition that is similar to one that in the fixed points of Hessenberg varieties \cite{21, 32}.

3.1 Affine permutations

Let $\tilde{S}_n$ denote the affine permutations, i.e. those bijections $w : \mathbb{Z} \rightarrow \mathbb{Z}$ satisfying

$$w_i = w_{i-n} + n, \quad w_1 + \cdots + w_n = n(n + 1)/2.$$  

If the second condition is dropped, then $w$ is called an extended affine permutation. The set of extended affine permutations will be denoted $\tilde{S}'_n$. The window notation for any such permutation $w = [w_1, \ldots, w_n]$ may be used since $w$ is determined by its values on \{1, ..., n\}. We will single out two particularly important extended permutations by

$$\varphi_i = i + 1, \quad \varphi^{(j)}_i = i + n\delta_{i, \overline{j}},$$

where $\overline{i}$ is the class of $i$ modulo $n$.

The set of $m$-stable permutations is the subset

$$\text{Stab}(n, m) = \left\{ w \in \tilde{S}_n : w_{i+m} > w_i \text{ for all } i \right\}$$

The set of $m$-restricted permutations $\text{Res}(n, m)$ is the subset of affine permutations whose inverse is $m$-stable. This set is finite and was shown to have size $m^{n-1}$, and to parametrize the torus fixed points of the $(n, m)$-affine Springer fiber \cite{13, 20}.

3.2 Parking function statistics

An $(n, m)$-parking function is an $(n, m)$-Dyck path with rows labeled by numbers that are decreasing along each vertical wall, like the one shown in
They are uniquely determined by the composition of length \( n \) whose \( i \)th value is the number of boxes above the path in the row containing \( i \). In the example, for instance, the composition is \((2, 0, 4, 0)\).

In \cite{13}, Gorsky, Mazin, Vazirani defined two maps from \( \text{Stab}(n, m) \) to parking functions, described in terms of their associated compositions. Their first map,

\[
\mathcal{PS}_m : \text{Stab}(n, m) \to \mathbb{Z}_{\geq 0}^n
\]

defined for Pak-Stanley, is described by

\[
\mathcal{PS}_m(w)_i = \# \{ j : j > i, \ 0 < w_i^{-1} - w_j^{-1} < m \}.
\]

They proved that this map is in bijection with parking functions in the case \( m = kn + 1 \), and conjectured that it is a bijection for all \( m \). For the second, they defined a parking function \( \mathcal{A}_m(w) \) for each \( w \in \text{Stab}(n, m) \) as follows:

let \( M_w = \min \{ i : w_i > 0 \} \).

For each \( j \), there is a unique way to express \( w_j^{-1} - M_w \) as \( rm - kn \) for \( r \in \{ 0, ..., n - 1 \} \), which necessarily implies \( k \geq 0 \). Now define \( \mathcal{A}_m(w)_j = k \). They conjectured that

\[
\sum_{w \in \text{Stab}_{n,m}} q^{\text{area}(\mathcal{A}_m(w))} t^{\text{area}(\mathcal{PS}_m(w))}
\]

where

\[
\text{area}(a) = \frac{(m - 1)(n - 1)}{2} - |a|,
\]

is the symmetric \( q, t \) parking function generating function that appears in the rational shuffle conjecture.

For each \( w \in \text{Res}(n, m) \), let us define a composition by

\[
\mathcal{A}_m'(w)_i = b_{\sigma^{-1}i} - \mathcal{A}_m(w^{-1})_i, \quad \mathcal{PS}_m'(w)_i = b_{\sigma^{-1}i} - \mathcal{PS}_m(w)_i
\]

where \( \sigma \) is the permutation for which \( \sigma_i \) is the number in row \( i \) of the parking function diagram, and \( b_j = \lfloor m - jm/n \rfloor \) is the maximum number of boxes in row \( j \). Unlike the corresponding definition for \( \mathcal{PS}_m' \), we claim that this function does not depend on \( m \), and in fact can be described by

\[
\mathcal{A}'(w)_i = \frac{w'_i - (w'_i - 1 + 1)}{n} \tag{5}
\]

where \( w' = \varphi^{-\min(w_1, \ldots, w_n)}w \) is the unique left-shifted permutation whose minimum value in window notation is one.
Example 1. Let $w = (4, -2, 3, 5) \in \text{Res}(4, 7)$, so $w^{-1} = (0, 6, 3, 1)$, which is in $\text{Stab}(4, 7)$. Then it was shown in [13], Example 3.1 that $A_7(w^{-1})$ is the parking function $(2, 0, 4, 0)$, whose diagram is shown in Figure 1. On the other hand, we have $A'(w) = (1, 0, 1, 1)$, whose value in position $i$ is the number of remaining boxes in the row containing the number $i$.

3.3 A Hessenberg description of parking functions

We now give another description of the restricted permutations in the case of $(n, n + 1)$, using a bijection in Haglund’s book. In this subsection, we will assume $m = n + 1$.

Define the runs of a permutation $\tau \in S_n$ as the maximal consecutive increasing subsequences of $\tau$, so that the number of runs is the number of descents plus one. If there are $k$ runs, we decide that there is a $(k + 1)$st run consisting only of the number 0. Define the length of a row in an $(n, m)$-parking function to be the number of entire boxes between the path and the diagonal. Then for each parking function, there is a permutation $\tau$ such that the numbers (cars) in each row of length zero are the elements of the $k$th run of $\tau$, the numbers in rows of length one are the elements of the $k - 1$st run, etc. The set of parking functions associated to this permutation is denoted $\text{cars}(\tau)$. Then it is easy to check that

$$A_{n+1}(w) \in \text{cars}(\tau) \Leftrightarrow A'(w) = a(\tau),$$

where $a(\tau)$ is the composition such that $a(\tau)_i = j$ if $\tau_i$ is in the $k - j$th run, so that $\text{maj}(\tau) = |a(\tau)|$. We let $\text{Res}(a)$ denote those restricted permutations satisfying either side of equation (6).
Example 2. In Figure 4, Chapter 5 of Haglund’s book it is shown that
cars(τ) = \{(2, 1, 2, 0, 0), (1, 2, 2, 0, 0), (3, 1, 1, 0, 0), (1, 3, 1, 0, 0)\},
where τ = (3, 1, 2, 5, 4), and where we are writing parking functions as compositions. The corresponding restricted permutations are
{(4, 3, 10, −4, 2), (3, 4, 10, −4, 2), (5, 3, 9, −4, 2), (3, 5, 9, −4, 2)},
whose inverses map to the desired parking functions after applying \(A_6\). We can then check that these are precisely the restricted permutations satisfying
\(A'(w) = a(τ) = (1, 1, 2, 0, 1)\).

For each \(τ \in S_n\), consider the function \(α_τ : S_n \to \tilde{S}_n\) given by
\[
α_τ(σ) = ϕ^{-\text{maj}(τ)}(στ^{-1} + na(τ)) ∈ \tilde{S}_n
\]
where we identify affine permutations with \(n\)-tuples of integers by the window notation, and take the unique shift of the extended permutation in parentheses which is in \(\tilde{S}_n\). Then define
\[
C(τ) = \{σ ∈ S_n : α_τ(σ) ∈ \text{Res}(a(τ))\}.
\]

Example 3. Let \(τ = [1, 4, 2, 3]\), \(a(τ) = [1, 0, 0, 1]\). Then we would have
\[
C(τ) = α_τ^{-1}\{[5, −1, 2, 4], [5, 2, −1, 4], [6, −1, 0, 5], [6, 0, −1, 5]\} =
\{[3, 2, 1, 4], [3, 2, 4, 1], [4, 3, 1, 2], [4, 3, 2, 1]\}.
\]

We now give a parametrization of this set, which is essentially the same as a bijection in Chapter 5 of Haglund’s book as follows: if \(τ_i\) is in the \(j\)th run, we define \(r_i(σ)\) as the set of the indices of all those elements in the \(j\)th run that are greater than \(τ_i\), together with the set of elements in the \(j + 1\)st run that are less than \(τ_i\). For instance, writing \(r(σ)\) for the list of all the \(r_i(σ)\), we have
\[
r([3, 5, 1, 2, 7, 4, 6]) = ([2, 3, 4], [3, 4], [4, 5], [5], [6, 7], [7, 8], [8]).
\]

Let
\[
K(τ) = \{k ∈ \mathbb{Z}_{≥0} : k_i ≤ w_i(τ) − 1\},
\]
where \(w_i(τ) = |r_i(τ)|\).
**Definition 1.** We define a map $\beta_\tau : K_\tau \to S_n$ as follows: first start by setting $\sigma$ to be an arrangement starting with the number $n+1$, which we will think of as $\sigma_0$. Then for $i$ from $n$ to 1, insert the number $i$ to the right of the $k_i$th element of $r_i(\sigma)$, where the order is the opposite of the order in which they appear in $\sigma$, i.e. right to left. Finally, remove the leading $n+1$ and let $\beta_\tau(k) = \sigma^{-1}$. We define $H_\tau \subset S_n$ as the set of all $\beta_\tau(k)$ for $k \in K_\tau$.

**Example 4.** For $\tau = (3, 5, 1, 2, 7, 4, 6)$, and $k = (2, 1, 0, 0, 1, 0, 0) \in K_\tau$, the sequence would be

\[
8, 87, 876, 8756, 87546, 875436, 87541236,
\]

so $\beta_\tau(k)$ would be $(4, 5, 6, 3, 2, 7, 1)$.

**Lemma 2.** We have that $C_\tau = H_\tau$.

**Proof.** Recall from page 81 of Haglund’s book [15] that there is a bijection that we will call $\beta_1^\tau : K_\tau \to \text{cars}(\tau)$. The proof of the lemma is essentially the same as Haglund’s proof, but rewritten in terms of restricted permutations using $A_{n+1}$. To be precise, we have

$\alpha_\tau(\beta_\tau(k)) = A_{n+1}^{-1}(\beta_1^\tau(k))^{-1}$. \hspace{1cm} (7)

$\square$

**Example 5.** It is shown in Figure 4 of page 80 of Haglund’s book that the parking functions in Example 2 with $\tau = (3, 1, 2, 5, 4)$ are the elements $\beta_1^\tau(k)$ for $k$ in

$K_\tau = \{(0, 0, 0, 0, 0), (0, 1, 0, 0, 0), (1, 0, 0, 0, 0), (1, 1, 0, 0, 0)\}$.

On the other hand, the restricted permutations are also the images $\alpha_\tau(\sigma)$ for $\sigma$ in

$\mathcal{H}_\tau = \{(4, 3, 5, 2, 1), (4, 5, 3, 2, 1), (5, 3, 4, 2, 1), (5, 4, 3, 2, 1)\}$.

We also have

$|PS_{n+1}(\alpha_\tau(\beta_\tau(k)))| = \text{dinv}(\beta_1^\tau(k)) = |k|$, where the first equality comes from [13], and the second is built into Haglund’s construction. In particular, we have

$\sum_{w \in \text{Res}(\alpha(\tau))} q^{|PS_{n+1}(\alpha(\beta'(w)))|} = \sum_{\pi \in \text{cars}(\tau)} q^{\text{dinv}(\pi)} \text{area}(\pi) = \ldots \ldots = 17$
\[ t^{[\sigma(\tau)]} \sum_{k \in K(\tau)} q^{|k|} = t^{\text{maj}(\tau)} \prod_{i=1}^{n} [w_i(\tau)]_q, \] 

where \([k]_q\) is the \(q\)-number, using Corollary 5.3.1, page 82 of [15]. Then summing over all \(\tau\) in \(\mathcal{S}\) gives the character of the double coinvariant algebra \(DR_n\), assuming the Haglund-Loehr formula. For instance, the \(q,t\)-character of \(DR_3\) would be

\[(1 + q + q^2)(1 + q) + t^2 + t(1 + q) + t^2(1 + q) + t(1 + q)^2 + t^3.\] 

We now prove a third description of this set:

**Proposition 2.** We have that \(\mathcal{H}(\tau)\) is the set of all \(\sigma^{-1}\) such that

\[\sigma_{i-1} \leq \sigma_i + w_{\sigma_i}(\tau)\] 

for all \(i \in \{1, ..., n\}\), where as above, \(\sigma_0\) means \(n+1\).

**Proof.** Let \(\mathcal{H}(\tau)'\) denote the set described by the condition in the lemma. It is clear that (10) is satisfied at every step in the construction of Definition 1, because each number is added to the right of a number in \(r_i(\tau)\), and adding a smaller number to the left of any digit preserves the condition. This shows that \(\mathcal{H}(\tau) \subseteq \mathcal{H}(\tau)\).

To see the reverse, suppose that \(\sigma^{-1}\) satisfies the desired condition, and let \(\sigma'\) denote the result of adding \(\sigma_i\) immediately to the right of \(\sigma_j\) at every step in Definition 1 where \(j\) is the largest index satisfying \(j < i\), and \(\sigma_j > \sigma_i\), or \(j = 0\) if none exists. It is clear that \(\sigma' = \sigma\), and it remains to show that we necessarily have \(\sigma_j \in r_{\sigma_i}(\tau)\), so that \(\sigma' \in \mathcal{H}(\tau)\). To see this, we simply confirm the equation

\[\sigma_j \leq \sigma_{j+1} + w_{\sigma_{j+1}}(\tau) \leq \sigma_i + w_{\sigma_i}(\tau),\] 

establishing that \(\mathcal{H}(\tau)' \subseteq \mathcal{H}(\tau)\). \(\square\)

**Remark 1.** The reason we call (10) a “Hessenberg” condition is that it matches the description of the fixed point set of the normalized regular nilpotent Hessenberg variety with Hessenberg function given by \(h(i) = i + w_i(\tau)\) in type \(A\) [21, 32]. The only issue is that we do not have the condition \(h(i) \leq n\), only that \(h(i) \leq n+1\).
4 Affine Schubert calculus

We review some background on affine Schubert calculus, for which we refer to Goresky, Kottwitz, and MacPherson [10], as well as Lam [25], Kostant and Kumar [23], and the book of Lam, Lapointe, Morse, Schilling, Shimozono, and Zabrocki [26]. We follow the descriptions of the latter.

4.1 The nil Hecke and GKM rings

Let 
\[ S = \mathbb{Q}[\hat{t}^*], \quad F = \mathbb{Q}(\hat{t}^*), \]
and consider the noncommutative algebra
\[ F_{\hat{S}_n} = \bigoplus_{w \in \hat{S}_n} Fw, \]
with product given by
\[ (fu)(gv) = f(u(g))(uv), \]
where \( f, g \in F \), and the action of \( \hat{S}_n \) on \( F \) is determined by equation (1).

We define a subalgebra as follows: for any \( i \in \{0, ..., n-1\} \), let
\[ A_i = \frac{1 - s_i}{\alpha_i}. \]  
(11)
These operators satisfy the braid relations in type \( A \), and so we may define
\[ A_w = A_{i_1} \cdots A_{i_k} \]
whenever \( w = s_{i_1} \cdots s_{i_k} \) is a reduced word. The subring generated by these elements is called the affine nil Coxeter algebra. It is noncommutative and satisfies
\[ A_u A_v = \begin{cases} A_{uv} & l(uv) = l(u) + l(v), \\ 0 & \text{otherwise}. \end{cases} \]  
(12)
The subring generated by the \( A_i \) and \( S \subset F \) is called the affine nil Hecke algebra, and is denoted by \( \hat{A}_{af} \). It is graded by assigning the elements of \( \hat{t}^* \) and \( A_i \) degree 1, whence \( A_w \) has degree \( l(w) \). Now define
\[ \hat{\Lambda} = \left\{ f \in F_{\hat{S}_n}^*: f(w) \in S \right\}, \]
\[ \Lambda = \left\{ f \in \hat{A} : f(A_w) = 0 \text{ for all but finitely many } w \right\}. \]

Then \( \Lambda \) is a free \( S \)-module with a basis

\[ \xi^v(A_u) = \delta_{u,v}. \]

As explained in section 2, we have

**Proposition 3.** (Kostant, Kumar [23]) We have isomorphisms of graded \( S \)-modules

\[
H^\tilde{T}_\ast (\tilde{T}l) \cong \mathbb{A}_{af}, \quad H^\ast_\tilde{T}(\tilde{T}l) \cong \Lambda, \tag{13}
\]

in which the Schubert cycles \([X_w]\) map to \( A_w \), the dual classes in \([X^w]\) cohomology map to \( \xi^v \), and the pairing between homology and cohomology agrees with the pairing between \( \mathbb{A}_{af} \) and \( \Lambda \).

### 4.2 Affine Schubert polynomials

The left and right actions of \( \tilde{S}_n \) by the embedding \( \tilde{S}_n \subset \tilde{F}_n \) preserve \( \mathbb{A}_{af} \), and also induce dual actions on \( \Lambda \). If we view either as an \( S \)-module by left multiplication, then the action by right multiplication by \( \tilde{S}_n \) is linear over \( S \), whereas left multiplication has a nontrivial internal action on \( S \). There is also a compatible action of the extended affine permutations \( \tilde{S}_n' \) on \( \mathbb{A}_{af} \) by conjugation, in which

\[
\text{Ad}_\varphi \left( \sum_w a_w w \right) = \sum_w (\varphi(a_w)) (\varphi w \varphi^{-1}).
\]

We also have the dual actions on \( \Lambda \), and we will denote by the same letter \( \varphi = (\varphi^*)^{-1} \), which acts by \( \varphi^{-1} \) on scalars.

The cohomology ring is generated by two important sets of classes: first, we have the Chern classes of the natural line bundles \( x_i = c_1(\mathcal{L}_i) \), for all integers \( i \), determined as elements of \( \Lambda \) by

\[ x_i(v) = v(\lambda_i), \]

as well as the pullbacks of the generators of the cohomology of the affine Grassmannian

\[ h_k = \xi^v, \quad v = s_{k-1} \cdots s_0, \tag{14} \]

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where we identify the simple transpositions by \( s_k = s_{k-n} \), defining them for \( k > n \).

We will also denote the operator of multiplication by Chern classes by \( x_i \), as well as the dual operator on \( \mathbb{A}_{af} \), which acts diagonally on the classes of permutations,

\[
x_i \cdot w = w(\lambda_i)w.
\]  

Then we have

**Proposition 4.** The above assignments determine an surjective map of \( S \)-algebras

\[
S[h,x] = S[h_1, h_2, \ldots; x_1, \ldots, x_n] \to \Lambda
\]

whose kernel is is generated by elements of the form

\[
e_i(x_1, \ldots, x_n) - e_i(\lambda_1, \ldots, \lambda_n) - o_i(\delta), \quad m_\mu - o_\mu(\hat{t}^\mu),
\]

whenever \( \mu \) is a partition with \( \mu_1 > n-1 \), where \( m_\mu \) is the expression of the monomial symmetric function as a polynomial in the complete symmetric funtions \( h_k \), the \( o_i(\delta) \) are some multiples of \( \delta \) in \( S[h,x] \), and \( o_\mu(\hat{t}^\mu) \) are some elements in the ideal generated by \( \hat{t}^\mu \) in \( S[h] \).

We have the following formulas for the action of the (extended) affine Weyl group:

\[
x_i \cdot s_j = x_{s_j(i)}, \quad h_k \cdot s_j = \begin{cases} h_k + (x_1 - x_0)\rho_1(h_{k-1}) & j = 0, \\ h_k & \text{otherwise} \end{cases}
\]

for \( 0 \leq j \leq n-1 \), where \( \rho_i \) is the homomorphism over \( S \) determined by

\[
\rho_i^{-1}(h_k) = h_k + (\lambda_{k-1} - x_i)h_{k-1}, \quad \rho_i(x_j) = x_j.
\]

The action of left multiplication acts internally on \( S \), and on the generators by

\[
s_j \cdot x_i = x_i, \quad s_j \cdot h_k = \begin{cases} h_k + \lambda_{k-1}h_{k-1} & k = j + 1 \\ h_k & \text{otherwise} \end{cases}
\]

Finally, the action of conjugation by \( \varphi \) is given by

\[
\varphi x_i \varphi^{-1} = x_{i+1} + \delta/n, \quad \varphi h_k \varphi^{-1} = \rho_0(h_k).
\]
To prove the second statement in (17), for instance, we simply need to observe the corresponding matrix elements for dual operator of right multiplication by $s_i$ in terms of $A_w$. This is easy to do by solving for $s_i$ in (11), and applying (12).

Then we have

**Proposition 5.** (Kostant, Kumar [23]) The images $\xi^v$ in $\Lambda$ satisfy

$$\partial_i \xi^v = \begin{cases} \xi^{v s_i} & l(vs_i) < l(v) \\ 0 & \text{otherwise} \end{cases}, \quad \text{Ad}_\varphi(\xi^v) = \xi^{\varphi\varphi^{-1}},$$

(21)

where $\partial_i$ is the BGG operator

$$\partial_i(f) = \frac{f - f \cdot s_i}{x_i - x_{i+1}}.$$ 

In fact, they are determined uniquely by $\xi^1 = 1$, and either the first relation, or the second equation combined with the first for $i \neq 0$.

**Proof.** For the first statement, see [23 [24]. The second is a simple consequence of the first, and $\varphi s_i \varphi^{-1} = s_{i+1}$. 

\[\square\]

### 4.3 The nonequivariant limit

We can obtain the explicit relations in non-equivariant cohomology using the non-equivariant limit

$$H^*_Q(\tilde{F}l_n) \to H^*(\tilde{F}l_n).$$

If $M$ is an $S$-module, or some module with a clearly designated maximal ideal, define

$$\tilde{M} = M \otimes_S Q$$

where $Q$ is the quotient of $S$ by those terms of nonzero degree, i.e. setting the torus variables equal to zero. Then Proposition 4 becomes

$$H^*(\tilde{F}l_n) \cong \tilde{\Lambda} \cong \Lambda^{(n-1)} \otimes R_n(x),$$

$$H_*(\tilde{F}l_n) \cong \tilde{\Lambda}_{a f} \cong \Lambda_{(n-1)} \otimes R_n(x),$$

(22)

where

$$\Lambda_{(n-1)} = Q[h_1, \ldots, h_{n-1}], \quad \Lambda^{(n-1)} = Q[h_1, h_2, \ldots]/\langle m_\mu : \mu_1 \geq n - 1 \rangle$$
are the rings containing the $k$-Schur and affine Schur functions respectively, and

$$R_n(x) = \mathbb{Q}[x_1, ..., x_n]/\langle p_k(x) : k \geq 1 \rangle \cong H^*(\mathcal{F}_n)$$

is the ring of coinvariants in the $x$-variables. The first statement follows from taking the limit of the relations as the elements of $\hat{t}$ tend to zero. The second is through the identification of $\hat{A}_{af}$ as the dual space of $\Lambda$, where the pairing in equation (22) is the usual Hall pairing between $\Lambda_{(n-1)}$ and $\Lambda^{(n-1)}$, and the Poincaré duality pairing of $R_n(x) \cong H^*(\mathcal{F}_n)$ with itself:

$$(f, g) = fg(\Delta_n) \quad \Delta_n = \mathcal{S}_{w_0} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1} \in R_n(x). \quad (23)$$

Here $w_0 \in S_n$ is the maximal length permutation, $\mathcal{S}_w$ is the corresponding Schubert class in $H_*(\mathcal{F}_n)$, and $\Delta_n$ is the unique up to scalar element of maximal grading, which might also be represented by

$$\Delta_n = \frac{1}{n!} \prod_{i<j}(x_i - x_j),$$

to emphasize that $S_n$ acts on its span by the sign representation.

Under this limit, we have

$$\rho_i(f) = f[X + x_i], \quad (24)$$

which is the function that leaves the $x$-variables unchanged, and acts as plethystic substitution in the symmetric function part, i.e.

$$h_k \mapsto h_k + x_i h_{k-1} + \cdots + x_i^k.$$  

This determines the right action of $\hat{S}_n$ and conjugation by $\varphi$, which are easily seen to specialize to the rules given in [27]. The left action of course becomes trivial.

4.4 The affine Springer fiber

Fix coprime $(n, m)$, and consider the subtorus $U \cong \mathbb{C}^* \subset \hat{T}$ from [23]. The corresponding evaluation map $\hat{t}^* \rightarrow u^*$ is given by

$$\lambda_i = \left(\frac{n + 1 - 2i}{2n}\right) \epsilon, \quad \delta = \epsilon, \quad (25)$$
where $\epsilon \in u^*$ is the equivariant parameter. The affine Springer fiber $\tilde{S} = \tilde{S}_{n,m}$ is preserved by $U$, and as a subset of $\tilde{S}_n$, the fixed points are the $m$-restricted permutations $\text{Res}(n, m)$. Let $\mathbb{A}^U_{af}$, $\Lambda_U$ denote the corresponding spaces with coefficients in $\mathbb{Q}[\epsilon]$.

Now consider the kernel $I_{n,m} \subset \Lambda$ of the restriction map

$$i^*_{\text{Res}(n,m)} : \Lambda \rightarrow \bigoplus_{u \in \text{Res}(n,m)} \mathbb{Q}[\epsilon]u,$$

where the coefficient of $f$ is the evaluation of $f(u) \in \mathbb{Q}[\epsilon]$. We define

$$S_{af} = \{ c \in \mathbb{A}^U_{af} : f \in I_{n,m} \Rightarrow f(c) = 0 \}.$$

**Proposition 6.** There exist classes

$$C_v = \epsilon^{-d'_v} \sum_u c_{u,v} u \in S_{af}$$

for each $v \in \text{Res}(n, m)$ satisfying:

a) They freely generate $S_{af} \subset \mathbb{A}^U_{af}$ as a $\mathbb{Q}[\epsilon]$-module.

b) The coefficients are rational numbers satisfying

$$c_{u,v} \neq 0 \Rightarrow u \in \text{Res}(n, m), \hspace{1em} u \preceq_{br} v,$$

and $c_{v,v} \neq 0$.

c) The degrees are given by $d'_v = |\mathcal{PS}_m(v^{-1})|$.

**Remark 2.** The properties described in this proposition turn out to be sufficient for our purposes, even though they do not uniquely characterize the classes $C_v$, or their images in $H^*_s(\tilde{S}_{n,m})$ under the non-equivariant limit. An algebraic presentation of the coefficients $c_{u,v} \in \mathbb{Q}$ would be a fascinating result that is not part of this paper. However, see the appendix for some examples illustrating how such classes might be computed.

**Proof.** This follows from Proposition 4 in which $S_{af}$ is the image of $H^U_s(\tilde{S})$ under [13], and $C_u$ is the image of $[X_u]$. The degrees are the dimensions of the intersected Schubert cells, explained in section 3.2. \qed

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We choose a grading on $A_{af}$ so that $\epsilon$ has degree 1, which forces the degree of $A_w$ to decrease with the length of $w$. For each choice of $(n, m)$, we choose a the shifted grading

$$\deg(A_w) = \frac{(m-1)(n-1)}{2} - l(w), \quad \deg(\epsilon) = 1,$$

Then the operators from section 4.2 have degrees

$$\deg(x_i) = 1, \quad \deg(h_k) = -k,$$

and because of our choice of constant, we would have

$$\deg(C_w) = |PS'_m(w)|,$$

Since $d'_w = |PS_m(w^{-1})|$ from section 3.2. Notice that the gradings of $S_{af}$ are nonnegative, while the degree in $A_{af}$ may be negative.

The following operators were defined by Oblomkov and Yun [30] and are implicit in [35]:

**Proposition 7.** The modified operators

$$f *_m s_i = f \cdot s_i + \nu \epsilon \hat{\cdot} f, \quad \nu = m/n,$$

for $1 \leq 0 \leq n$ define a right action of $\tilde{S}_n$ on $A_U$. These operators, as well as conjugation by $\varphi$, preserve $I_{n,m}$, and hence the dual actions preserve $S_{af} \subset A_{af}$. In particular, the non-equivariant right action of $\tilde{S}_n$ and $\varphi$ preserve the subspace $\tilde{S}_{af} \cong H_*(\tilde{S}_{n,m})$.

**Proof.** First, note that the conjugation action of $\varphi$ preserves the kernel in equation (25), and so at least acts on $A_{af}$. It preserves the kernel simply because conjugation by $\varphi$ preserves the subset $\text{Res}(n, m) \subset \tilde{S}_n$.

The statement about the modified operators are due to Oblomkov and Yun [30, 31], but we give a simple algebraic proof in our case: as elements of $F_{\tilde{S}_n}$, we have

$$w *_{m} s_i = \left(\frac{m}{w_{i+1} - w_i}\right) w + \left(1 - \frac{m}{w_{i+1} - w_i}\right) w s_i.$$

Notice that this produces a $2 \times 2$ matrix that squares to the identity. From this, we see that the coefficient of $ws_i$ is zero if and only if $w_{i+1} - w_i = m$. It is straightforward to see that if $w \in \text{Res}(n, m)$, then

$$w_{i+1} - w_i = m \iff ws_i \notin \text{Res}(n, m).$$
Therefore the reflection operators preserve the span of Res(n, m) ⊂ FSn, and hence the dual reflection operators preserve In,m.

The statement that this defines an action of ˜Sn can also be proved algebraically. □

5 Double Coinvariants

In this section we will state and prove our main results.

5.1 Commuting variables

We have identified multiplication by xi as dual to the Chern class operators. We now construct a degree-preserving action of the yi variables, which together form an action of ˜Sn.

Definition 2. Define ˜Sn-linear maps xi, yj : MAf → MAf by

yi = z_i - 1,  \quad z_i(c) = \varphi \cdot c \cdot (\varphi(i))^{-1},

and xj is dual to multiplication by Chern classes [15]. Let ˜zi, and ˜yi denote the corresponding operators on ˜Aaf and ˜SAf, but using the modified action [26] in place of the usual one. Notice that while the zi act internally on ˜S, the induced operators on ˜Aaf are ˜Sn-linear.

We have our first theorem:

Theorem 2. The induced operators xi, yj on H*(˜Fl) commute, giving rise to an action of ˜Sn. Furthermore, this action satisfies the following properties:

a) The elements of m+(x, y) act by zero, giving us an action of DRn.

b) The subspaces H*(Sn, m) ⊂ H*(˜Fl) are preserved, i.e. are submodules.

c) The map DRn → H*(Sn, n+1) given by f → f ∙ Δn is an isomorphism.

d) There is an action of the extended Weyl group induced by the conjugation action, which is given by

wx_i = x_wi,  \quad wy_i = y_wi,  \quad σ(1) = (-1)σ,  \quad α(1) = 1 + y_n,

where w ∈ ˜Sn is any extended permutation, σ ∈ Sn, and we have identified the multiplication operators xi+n = xi, y_i+n = yi.
Remark 3. In fact, we could also have used the fundamental class $[\tilde{S}]$ as the generator, as we did in the introduction.

Proof. Let

$$\sigma^{(j)} = \varphi(\varphi^{(j)})^{-1} = s_{j+1}s_{j+2}\cdots s_{n-1}s_0s_1\cdots s_{j-1} \in \tilde{S}_n.$$ 

Using (20) and (24), we have

$$z_i(h_k) = \varphi \cdot h_k \cdot (\varphi^{(j)})^{-1} = \varphi \cdot h_k \cdot \varphi^{-1}\sigma^{(j)} =$$

$$h_k[X + x_1]s_{j+1}\cdots s_{n-1}s_0\cdots s_{j-1} =$$

$$h_k[X + x_1]s_0\cdots s_{j-1} = h_k[X + x_j]$$

on $H^*(\mathcal{F}l)$. We also see that $z_i(x_j) = x_j$, so in fact $z_i$ is precisely the homomorphism $\rho_i$ on $H^*(\mathcal{F}l) \cong \Lambda^{(n-1)} \otimes R_n(x)$. Since the pairing between $\Lambda_{(n-1)}$ and $\Lambda^{(n-1)}$ is the Hall inner product, the dual operator on $H^*(\mathcal{F}l) \cong \Lambda_{(n-1)} \otimes R_n(x)$ is the multiplication operator

$$z_i(f) = (1 + x_ih_1 + \cdots + x_i^{n-1}h_{n-1})f,$$  \hspace{1cm} (29)

proving that $x_i$ and $y_j$ commute.

Using (29) again, it follows easily that a nonconstant multisymmetric power sum in $x_i$ and $y_j$ is a multiplication operator by an element of $\Lambda_{(n-1)} \otimes R_n(x)$ whose coefficients in $h_\mu$ are elements of $m_+(x)$, and hence are zero, proving part $a)$. Next, notice that the modified actions in (26) preserve $\mathcal{S}_af$, and all limit to the usual right action modulo the relation $\epsilon = 0$, so part $b)$ follows from Proposition 7.

For part $c)$, notice that by Proposition 6 and degree reasons, we have that $C_{w_0} = A_{w_0}$ up to a nonzero multiple. The image of $A_{w_0}$ is actually the element $\Delta_n$, which corresponds to 1 in $\Lambda_{(n-1)} \otimes R_n(x)$. Since the dimensions match in part $c)$, it suffices to show that the map

$$DR_n \rightarrow \Lambda_{(n-1)} \otimes R_n(x)$$

is an injection. Interestingly, there is a proof of this in Haiman’s work, specifically [19], Proposition 4.5. In this paper, he deals with the map

$$DR_n \cong H^0_{Z_n}(P) \rightarrow H^0_U(i^*(P)) \cong \Lambda_{(n-1)} \otimes R_n(x),$$  \hspace{1cm} (30)

proving that $x_i$ and $y_j$ commute.
where $Z_n$ is the punctual Hilbert scheme, $P$ is the Procesi bundle, and $U$ is the open subset of $Z_n$ consisting of those ideals of the form

$$(x^n, y - (a_1x + \cdots + a_{n-1}x^{n-1})).$$

In the second isomorphism in (30), the variables $a_i$ are identified with $h_i$, while the $x$-variables have to do with the $x$-coordinates of distinct points in $\text{Hilb}_n$. We expect to generalize this relationship further in future papers.

Finally, the relations in part (d) hold equivariantly for the modified actions, and follow from definitions, as well as the twisting by the sign representation in $R_n(x) \otimes A_{(n-1)}$.

\(\square\)

### 5.2 A filtration by the descent order

We now describe a filtration on the homologies of the affine flag variety and Springer fiber by compositions, which we relate to the order on monomials in the $y$-variables that produce the “descent monomials” described below. For the rest of the paper, we will be concerned with the case $m = n + 1$, leaving applications to the rational case and other Springer fibers for future papers.

**Definition 3.** The descent order on compositions is defined by $a \leq_{\text{des}} b$ if

1. $\text{sort}(a) <_{\text{lex}} \text{sort}(b)$ or
2. $\text{sort}(a) = \text{sort}(b)$ and $a \leq_{\text{lex}} b$

where $\text{sort}(a)$ sorts a composition in decreasing order to produce a partition.

**Example 6.** For $n = 2$, we would have

$$(0, 0) < (0, 1) < (1, 0) < (1, 1) < (0, 2) < (2, 0) < \cdots$$

Notice that this is not a monomial order, and that, for example $(2, 2) < (0, 3)$, so it does not refine the total order by the norm.

**Definition 4.** For a composition $a$ of $n$, we have a filtration by compositions

$$F_aA_{af} = A_{af} \cap \bigoplus_{w:A'(w) \leq_{\text{des}} a} \mathbb{Q}(t) \cdot w,$$

(31)
in other words the set of all elements such that the coefficient of $w$ is zero unless $a \preceq_{\text{des}} a$. We also define

$$F_a S_{af} = F_a A_{af} \cap S_{af}$$  \hspace{1cm} (32)$$

where $F_a A_{af}$ is the image of $F_a A_{af}$ under the evaluation map. We define $F_a H_\ast(\hat{\cal I})$, $F_a H_\ast(\hat{\cal S})$ as the images of $F_a A_{af}$ and $F_a S_{af}$. Since the $x$-variables act diagonally in the fixed point basis, these are all $\mathbb{Q}[x]$-modules.

**Remark 4.** Definition 4 is of course unsatisfying. It would be highly preferable to find a filtration by topological subspaces $X_a \subset \hat{\cal I}$, and to define

$$F_a H_\ast(\hat{\cal I}) = H_\ast(X_a), \quad F_a H_\ast(\hat{\cal S}) = H_\ast(X_a \cap \hat{\cal S}).$$

It should then be possible to calculate the subquotients using the long exact sequence for relative homology. This is not available yet, though we hope to find such a presentation in future papers, as well as generalizations to other root systems.

**Remark 5.** Despite equation (32), we actually could not make a similar definition for usual homology, and in fact we may have

$$F_a H_\ast(\hat{\cal S}) \neq F_a H_\ast(\hat{\cal I}) \cap H_\ast(\hat{\cal S}).$$  \hspace{1cm} (33)$$

In other words, the non-equivariant limit does not respect the intersection $F_a A_{af} \cap S_{af} \subset A_{af}$. The first occurrence happens when $a = (0,1,2,0)$, and this is the only such example for $n \leq 4$, and $m = n + 1$. If this equation were true, we would not have needed to consider equivariant homology, and likely our results would have become much simpler. In other words, equivariant homology is a sufficiently refined middle ground between this “false” definition (33) and the even more refined potential definition in Remark 4.

The descent monomials defined in the introduction could also written as

$$g_\tau(y) = y^{a(\tau)},$$  \hspace{1cm} (34)$$

where $a(\tau)$ is the composition defined in section 3.3. A composition of the form $a(\tau)$ for some $\tau$ will be called a descent composition. These are known to be a basis of the coinvariant algebra $R_n(y)$, see, for instance [8]. The following proposition, due to E. E. Allen, shows that they are the leading terms in the descent order, and in fact the proof gives an algorithm for the reduction:
Proposition 8. (Allen [2]) For any composition \(a\), there exists a partition \(\mu\) and a composition \(c\) such that

\[y^c m_\mu(y) = y^a + \sum_{b \prec_{dc} a} c_b y^b,\]

where \(m_\mu(y)\) is the monomial symmetric function. Furthermore, \(\mu\) is the empty partition if and only if \(a\) is a descent composition.

5.3 Main result

We are now prepared to state our main result, which describes the subquotients of \(F_a DR_n\) as \(\mathbb{Q}[x]\)-modules. Note that whenever we have a filtration of a vector space \(V\) by a totally ordered set, order it makes sense to define \(F_a V/F_{a-1} V\) as the subquotient by the previous term, or by the zero space if there is no smaller term.

Theorem 3. The following \(\mathbb{Q}[x]\)-modules are isomorphic:

a) \(F_a DR_n/F_{a-1} DR_n\), where

\[F_a DR_n = \sum_{|b|=|a|, \ b \preceq_{ds} a} \mathbb{Q}[x] y^b.\]

b) \(F_a H_*(\tilde{S}_{n,n+1})/F_{a-1} H_*(\tilde{S}_{n,n+1})\).

c) The ideal \((f_\tau(x)) \subseteq R_\tau(x)\), where

\[f_\tau(x) = \prod_{i=1}^{n} \prod_{j=i+w_i(\tau)+1}^{n+1} (x_{\tau_i} - x_{\tau_j})\]

if \(a = a(\tau)\) for some \(\tau\), or the zero module otherwise. Here we are using Haglund’s convention that \(\tau_{n+1} = 0\), and we define \(x_0 = 0\).

Furthermore they all have a \(\mathbb{Q}\)-basis given by the monomials \(x_{\tau_1}^{k_1} \cdots x_{\tau_n}^{k_n}\) applied to the principal generator, ranging over all compositions \(k\) satisfying \(k_i \leq w_i(\tau) - 1\) for all \(i\).

In particular, we have an independent proof of the Haglund-Loehr formula [18], first proved in [5]:
Corollary 3. (Haglund-Loehr) The bigraded Hilbert series of the double coinvariants is given by

$$\sum_{\pi \in PF_n} q^{\operatorname{dinv}(\pi)} t^{\operatorname{area}(\pi)}.$$ 

Proof. The statement that $x^k$ form a basis of $F_a DR_n/F_{a-1} DR_n$ shows that the Hilbert series of the quotients is the same as the final expression in (8). Now sum over all $\tau$, and note that the $x$ degree satisfies $|a(\tau)| = \operatorname{maj}(\tau)$.

Example 7. If $\tau = (3, 1, 2)$, we would have $a = (0, 0, 1)$, and $f_\tau(x) = x_3$. Then the kernel of the map $\mathbb{Q}[x] \to R_n(x)$ of multiplication by $f_\tau(x)$ is the ideal

$$(x_1 + x_2 + x_3, x_3^2, x_2^2 + x_2x_3).$$

Since $a$ is already minimal in the descent order among compositions of degree 1, the theorem implies that each of these generators times $y^a = y_3$ are in the maximal ideal $m_+(x, y)$. Then $F_{a-1} DR_n$ is the zero module, and the basis of $F_a DR_n$ would be given by the monomials $\{y_3, y_3x_3, y_3x_2x_1, y_3x_3x_1\}$, contributing $t(1 + q)^2$ in (9).

We begin with some lemmas. Let $\tilde{z}^a = \tilde{z}_1^a \cdots \tilde{z}_n^a(A_{w_0})$, and recall that $A_{w_0} \in S_{af}$, and that it maps to the identity in $DR_n$.

Lemma 4. The set of all compositions $A'(\text{Res}(n, n+1))$ is the same as the set of all descent compositions $a(S_n)$, and there is a unique element $v \in \text{Res}(n, n+1)$ with $A'(v) = a(\tau)$ of minimal degree $|\mathcal{PS}'_{n+1}(v)| = 0$. Furthermore, we have that

$$\tilde{z}^a(\tau) = \sum_{u \in \text{Res}(n, n+1)} b_u(\epsilon) u \in S_{af},$$

(35)

where $b_u(\epsilon) = 0$ unless $u \leq br u v$, and $b_u(\epsilon) \neq 0$ for all $u \in \text{Res}(a(\tau))$.

Proof. The first statement follows from section 3.3, where the minimal permutation $v$ is the image of the zero composition $(0, ..., 0) \in \mathcal{K}(\tau)$ under the bijection $\alpha_\tau \beta_\tau$.

Now denote the minimal permutation associated to $a$ by $v(a)$. To check the relations, we first note that $v(a)$ and $\tilde{z}^a$ satisfy the parallel relations

1. If $a = (0, ..., 0)$, then $v(a) = w_0$, $\tilde{z}^a = A_{w_0}$, where $w_0$ is the maximal length permutation.
2. If $a_i > a_{i+1}$, then $v(a) = v(a')s_i$, $z^a = -z^{a'} s_{n+1} s_i$, where $a' = a \cdot s_i$.

3. If $a_n > 0$, then $v(a) = \varphi^{-1}v(a')\varphi$, $z^a = \varphi^{-1}z^{a'} \varphi$, where $a'$ is the composition $(a_n - 1, a_1, ... a_{n-1})$.

It is straightforward to check that the set of descent compositions $a(S_n)$ can be recursively described as the set of compositions satisfying one of these conditions, where the corresponding $a'$ is also a descent composition. In step 2, notice that $l(v(a)) = l(v(a')) + 1$.

By Proposition 7, there is some expression of the form (35), and it lives in $S_{af}$ since $A_{w_0} \in S_{af}$. To see the triangularity of the coefficients, first observe the following two properties of the Bruhat order:

1. If $u \leq_{bru} v$, and $v \leq_{bru} vs_i$, then $us_i \leq_{bru} vs_i$.

2. If $u \leq_{bru} v$, then $\varphi^k u \varphi^{-k} \leq \varphi^k v \varphi^{-k}$.

The triangularity follows from these properties, the recursive rules above for calculating $z^a$, and the fact that the modified action (26) sends $u$ to a linear combination of $u$ and $us_i$.

We prove the final nonvanishing statement inductively using the above recursive formula for the descent monomials $z^a$, by showing that we never have to worry about cancelation for the leading terms in $\text{Res}(a)$. In the base case where $a$ is the zero composition, we notice that $z^a = A_{w_0}$ from case 1 above, and that the coefficients of $u$ in $A_{w_0}$ are an overall multiple times $\pm 1$ for $u \in \text{Res}(a) = S_n$. Next, if $a, a'$ are a pair as in case 2, we can check that

$$u \in \text{Res}(a) \Rightarrow us_i \in \text{Res}(a').$$

From this, and the induction step, we know that the coefficient $b'_{us_i}(\epsilon)$ of $us_i$ in $z^{a'}$ is nonzero. Since $b'_u(\epsilon) = 0$, we have that

$$b_u(\epsilon) = \left(1 - \frac{m}{u_{i+1} - u_i}\right) b'_{us_i}(\epsilon),$$

because this is the only term that can possibly contribute to $z^a = z^{a'} \cdot s_i$ from (27). By (28), the coefficient is nonzero, showing that $b_u(\epsilon)$ is nonzero. A similar argument holds for case 3 since $\varphi$ simply permutes the restricted permutations.
An immediate consequence of the next lemma and the triangularity statement in Proposition 6 part b) is that
\[ F_a S_{af} \cong \bigoplus_{\mathcal{A}'(w) \leq a} \mathbb{Q}[\epsilon]C_w, \]  
\text{so in particular, it is free over } \mathbb{Q}[\epsilon].

**Lemma 5.** The descent order is compatible with the Bruhat order,
\[ u \leq_{\text{bru}} v \Rightarrow \mathcal{A}'(u) \leq_{\text{des}} \mathcal{A}'(v). \]

**Proof.** First, consider the case \(|a| = |b|\), where \(a, b = \mathcal{A}'(u), \mathcal{A}'(v)\). It follows fairly easily that
\[ |\mathcal{A}'(w)| = -\min(w) = -\min(w_1, ..., w_n), \]  
so we have that \(\min(u) = \min(v)\). Furthermore, by using \(\varphi\), we can see that it suffices to consider the case \(\min(u) \equiv 0 \pmod{n}\). In this case, \(\mathcal{A}'\) is the same as the composition corresponding to the left coset space in \(S_n \backslash \tilde{S}_n\). It is known that \(u \leq_{\text{bru}} v\) implies that \(a \leq_{\text{bru}} b\), where the Bruhat order on compositions is the same as the order on the coset spaces by taking minimal representatives in the affine Weyl group of \(GL_n\), see \([16]\). It follows immediately that \(a \leq_{\text{bru}} b\) implies that \(a \leq_{\text{des}} b\), proving this case.

Using (37) again, we see that \(u \leq_{\text{bru}} v\) implies that \(|a| \leq |b|\), so it remains to consider the case \(|a| < |b|\). Since \(a \neq b\), we only need to prove that \(\text{sort}(a) \leq_{\text{lex}} \text{sort}(b)\), as the tiebreaking case in Definition 3 will never come up. It is well known that
\[ u \leq_{\text{bru}} v \Rightarrow u' \leq_{\text{bru}} v' \]
where \(u', v'\) are the associated Grassmannian permutations, i.e. the permutations whose window notations have the same values as those of \(u, v\), but in increasing order. Since \(\mathcal{A}'(u') = \text{sort}(\mathcal{A}'(u))\), it suffices to assume that \(u, v\) are Grassmannian permutations.

In the case of Grassmannian permutations, there is an explicit description of the Bruhat order for the Bruhat order in terms of the “unit increasing monotone function” \(\mathbb{Z} \to \mathbb{Z}\) given by
\[ \varphi_w(j) = \sum_{i=1}^{n} \max \left( 0, \left\lfloor \frac{j - w_i}{n} \right\rfloor \right), \]  
33
see Theorem 6.3 of [3]. We make the following claim, which is straightforward to check using this description: given Grassmannian permutations with \( u \preceq_{bru} v \), if \( u_1 > v_1 \), then there exists \( v_i \leq j < i = u_1 \) such that \( w = t_{i,j}u \preceq_{bru} v \), where \( t_{i,j} \in \tilde{S}_n \) is the affine transposition that exchanges \( i \) and \( j \). It follows easily that \( \mathcal{A}'(w)_k \geq \mathcal{A}'(u)_k \) for all \( k \), so of course we have \( \mathcal{A}'(w) \leq_{\text{des}} \mathcal{A}'(v) \). But now inductively on \( |b| - |a| \), we may assume that \( \mathcal{A}'(w) \leq_{\text{des}} \mathcal{A}'(v) \), proving that \( \mathcal{A}'(u) \leq_{\text{des}} \mathcal{A}'(v) \).

\[ \square \]

Next, if \( A \) is a finite set, together with prescribed diagonal operators \( x_i \),

\[
x_i \in \text{End } (M'_A), \quad M'_A = \bigoplus_{u \in A} \mathbb{Q}[\epsilon],
\]

let \( M_A \) be the quotient module of \( \mathbb{Q}[x, \epsilon] \) by the relations satisfied by these operators. It is a simple observation that this module is isomorphic to the module generated by any element

\[
f = \sum_{u \in A} c_u(\epsilon)u \in M'_A, \quad (38)
\]

as long as all of the coefficients are nonzero. For any finite subset of \( S_n \) or \( \tilde{S}_n \), the endomorphisms will be the Chern class operators \( x_i \), with the restricted tori \( S \) or \( U \), respectively, where

\[
S = \text{diag}(1, z, \ldots, z^{n-1}), \quad \mathfrak{s} = \langle \epsilon \rangle, \quad z = \exp(\epsilon). \quad (39)
\]

For instance, we would have

\[
M_{\text{Res}(n,m)} \cong H^*_U(\tilde{S}_{n,m}),
\]

by the surjectivity in Theorem 6, and the injectivity of the localization map.

Recall that in section 3.3 we found a subset \( \mathcal{H}(\tau) \subset S_n \), which we showed is in bijection with \( \text{Res}(a(\tau)) \). The next lemma finds that the module \( M_{\mathcal{H}(\tau)} \) limits to the module in part (c) of the theorem, up to a change of variables.

**Lemma 6.** The submodule of \( M_{S_n} \) generated by

\[
g_\tau(x) = \prod_{i=1}^{n} \prod_{j=i+w_i(\tau)+1}^{n+1} (x_i - x_j - \epsilon), \quad (40)
\]

where \( x_{n+1} \) is defined to be zero, maps isomorphically to \( M_{\mathcal{H}(\tau)} \) under the obvious restriction map. Furthermore, the monomials \( x^k \) for \( k_i \leq w_i(\tau) - 1 \) are linearly independent in \( M_{\mathcal{H}(\tau)} \).
Proof. The restriction of the generator $g_{\tau}(x)$ to a fixed point $\sigma \in S_n$ is nonzero when none of the factors are zero. By comparing with the conditions in Proposition 2, this happens precisely when $\sigma \in \mathcal{H}(\tau)$. Thus the first statement follows from the observation surrounding equation (38). By Proposition 2, we have that

$$M_{\mathcal{H}(\tau)} \cong (g_{\tau}(x)) \subset M_{S_n},$$

with a grading shift.

Now notice that

$$\text{LT}(g_{\tau}(x)) = \prod_{i=1}^{n} x_i^{n-i+1-w_i(\tau)},$$

where LT is the leading term in the lexicographic order in the $x$-variables only, viewing $\epsilon$ as a scalar. Then we have that the polynomials $x^k g_{\tau}(x)$ for $k_i \leq w_i(\tau) - 1$ are triangular with respect to a subset of the standard basis $x^k \in M_{S_n} \cong H^*_S(\mathcal{F})$ for $k_i \leq n - i - 1$, showing that they are linearly independent.

\[ \square \]

Remark 6. Building on Remark 1, it can be similarly shown that

$$(g_{\tau}(x)) \cong x_1 \cdots x_{n-k} H^*(\mathcal{H}(f)),$$

where $k$ is the length of the final run of $\tau$ before the trailing run consisting only of zero, and the extra values with $j = n + 1$ then contribute to the monomial factors with $x_{n+1} = 0$. If $h$ is a Hessenberg function, and

$$g_h(x) = \prod_{i=1}^{n} \prod_{j=i+h(i)+1}^{n} (x_i - x_j - \epsilon),$$

one can show using a similar argument that the module generated by $(g_h(x))$ in $R_n(x)$ is isomorphic to the cohomology of the Hessenberg variety without the extra monomial, giving a different presentation from the explicit relations determined in [1]. We expect that similar varieties will appear in some way once a suitable geometric description of $F_\alpha$ can be found, as in Remark 4.

We may now complete the proof of Theorem 3.
Proof. First, we show that all three submodules are the zero module unless $a = a(\tau)$ for some $\tau$. Part a) follows immediately from Proposition 8. Part b) follows from Lemma 5. Part c) follows from definition.

Let $F'_aDR_n$ denote the filtration in part b), but taking the sum over all compositions $b \leq_{des} a$, not just those of equal degree. Notice that replacing $F_a$ by $F'_a$ has no effect on the subquotients, because the lower degree terms are simply canceled. We also have that $F'_aDR_n$ is unchanged if we replace $y^a$ by $z^a$, because the two monomials are triangular in the descent order. Now take the lifted filtration

$$F'_aS_{af} = \sum_{b \leq_{des} a} \mathbb{Q}[x]z^a,$$

so that $F'_aDR_n \cong F'_aS_{af} \otimes \mathbb{Q}$, using the isomorphism of Theorem 2. We will show that $F'_aS_{af} = F_aS_{af}$, proving the isomorphism between parts a) and b).

The inclusion $F'_aS_{af} \subset F_aS_{af}$ follows from Lemma 4 and equation (36) following Lemma 5. We clearly have $F'_aS_{af} = F_aS_{af}$ for $a = (0, \ldots, 0)$. To show equality for all $a$, we may suppose by induction on $a$ in the descent order that $F'_aS_{af} \cong F_aS_{af}$. For this, it suffices to show that the Hilbert series of $F'_aS_{af} / F'_aS_{af} = F_aS_{af} / F_aS_{af}$, which by (36) and (8) is

$$(1 - q)^{-1} \sum_{w \in \text{Res}(a)} q^{\text{Res}_n + 1(a)} = (1 - q)^{-1} \prod_{i=1}^{n} [w_i(\tau)]_q. \quad (42)$$

Because of the inclusion in the last paragraph, we already have one inequality between the coefficients of the two Hilbert series, and it suffices to prove the reverse. Now we have a restriction map $F'_aS_{af} \to M_{\text{Res}(a)}$, and by Definition 4, the kernel is just $F_aS_{af} / F_aS_{af}$, showing that the map $F_aS_{af} / F_aS_{af} \to M_{\text{Res}(a)}$ is injective. To see that it is surjective, recall that by the last statement from Lemma 4, the coefficients of the expansion of

$$z^a = \sum_{w \in \text{Res}(n,n+1)} b_w(\epsilon)w \in F_n,$$

are nonzero for $w \in \text{Res}(a)$. Surjectivity follows from this, and the observation before (38) again, applied to the image of $z^a$ in $M_{\text{Res}(a)}$.

Using the isomorphism from this last paragraph, it suffices to show that the Hilbert series of $M_{\text{Res}(a)}$ agrees with (42). We easily see that under the
bijection $\alpha_\tau : \mathcal{H}(\tau) \to \text{Res}(a)$, the action of the $x$-variables transform as
\[
x_i(\alpha_\tau(\sigma))\big|_U = -\frac{x_i(\sigma \tau^{-1})\big|_S}{n} + \frac{n + 1 + 2 \text{maj}(\tau)}{2n} \epsilon - a_i \epsilon,
\]
(43)
which implies the equality of the Hilbert series of $M_{\text{Res}(a)}$ and $M_{\mathcal{H}(\tau)}$. Here the restriction to $U$ refers to the evaluation map (25), whereas $S$ is the restricted torus acting on the usual flag variety from (39). The reverse inequality now follows from the independence of the monomials in Lemma 6.

Finally, by taking the nonequivariant limit of the relations in (43), we find that
\[
F'_a H_*(\tilde{S})/F'_a H_*(\tilde{S}) \cong \tau(M_{\mathcal{H}(a)})
\]
as quotients of $\mathbb{C}[x]$, where $\tau(M)$ of a quotient module $M$ is the quotient by the evaluation map $x_i = x_{\tau_i}$ of the corresponding relations.

\[\Box\]

\section{Examples: Luzstig-Schubert classes in the affine Springer fiber}

We now give examples of the classes guaranteed by Proposition 6 and how they can be computed.

\textbf{Example 8.} In the case $n = 2$ we have $\text{Res}(2, 3) = \{1, s_1, s_0\}$. In this case the intersected Schubert basis agrees with the Schubert basis, $C_w = A_w$, which is given using window notation by
\[
A_{1,2} = 1, \quad A_{2,1} = \epsilon^{-1}(s_1 - 1), \quad A_{0,3} = \epsilon^{-1}(s_0 - 1).
\]
Then the operators of right multiplication by $s_1, s_0$ are respectively given by
\[
A_{1,2} \mapsto A_{1,2} + 3\epsilon A_{2,1}, \quad A_{2,1} \mapsto -A_{2,1}, \quad A_{0,3} \mapsto A_{0,3} + 2A_{2,1} + 9\epsilon A_{3,0},
\]
\[
A_{1,2} \mapsto A_{1,2} + 3\epsilon A_{0,3}, \quad A_{2,1} \mapsto A_{2,1} + 2A_{0,3} + 9\epsilon A_{-1,4}, \quad A_{0,3} \mapsto -A_{0,3},
\]
while the duals of the BGG operators are
\[
\tilde{e}^*_1 : A_{1,2} \mapsto A_{2,1}, \quad A_{2,1} \mapsto 0, \quad A_{0,3} \mapsto A_{3,0},
\]
\[
\tilde{e}^*_0 : A_{1,2} \mapsto A_{0,3}, \quad A_{2,1} \mapsto A_{-1,4}, \quad A_{0,3} \mapsto 0.
\]

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Using equation (26), we find that the matrices of the dual of modified right multiplication are

\[ \begin{pmatrix} 1 & 0 & 0 \\ -6\epsilon & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6\epsilon & 2 & -1 \end{pmatrix}. \]

Setting \( \epsilon = 0 \), we recover the familiar matrices for the Springer action on \( H_\ast(\tilde{S}_2) \), which is two copies \( \mathbb{CP}^1 \) glued at a point. See [36], section 2.6.4, for instance.

**Example 9.** In the case \( (n, m) = (3, 4) \), there are 16 restricted affine permutations

\[
\text{Res}(3, 4) = \{(1, 2, 3), (0, 2, 4), (1, 3, 2), (2, 1, 3), (0, 4, 2), (2, 0, 4), (-1, 3, 4), (0, 1, 5), (3, 1, 2), (2, 3, 1), (-1, 4, 3), (1, 0, 5), (3, 2, 1), (4, -1, 3), (1, 5, 0), (-2, 2, 6)\}.
\]

The corresponding classes \( C_w \) are given by

\[
A_{1,2,3}, A_{0,2,4}, A_{1,3,2}, A_{2,1,3}, A_{0,4,2}, A_{2,0,4}, A_{-1,3,4}, A_{0,1,5}, A_{3,1,2}, A_{2,3,1}, A_{-1,4,3}, A_{1,0,5}, A_{3,2,1}, \epsilon A_{4,1,3} + A_{3,1,4} + A_{4,0,2}, \epsilon A_{1,5,0} + A_{0,5,1} + A_{2,4,0}, \epsilon A_{-2,2,6} + A_{-2,3,5} + A_{-1,1,6}.
\]

Before explaining how these are obtained, notice that

1. The coefficients have no negative powers of \( \epsilon \), so they are indeed elements of \( A_{af}^U \).
2. The Schubert classes \( A_w \) that appear do not include only the restricted permutations, but their expressions in the fixed-point basis must contain only these elements, as they are in \( S_{af} \). See equation (44) below, for instance.
3. The final three classes correspond to the three restricted permutations for which the Schubert cells and the Schubert-Springer cells have different dimensions. The degrees do indeed agree with the expected dimension count.
4. Somewhat unintuitively, the leading term of \( C_w \) is not \( A_w \) under the limit \( \epsilon \to 0 \), because this term will vanish if the dimension of the Schubert cell drops when intersected with the Springer fiber. Nevertheless, the map \( H_\ast(\hat{S}_{n,m}) \to H_\ast(\hat{F}_n) \) is injective, even though there is not an obvious triangularity statement.

By simply exhibiting these classes and checking the above statements, we have confirmed Proposition 6 in this case. However, this does not show that these are the Schubert-Springer classes, which could differ by a change of basis which is lower triangular in the Bruhat order.

In fact, we claim that these are the Schubert-Springer classes, and we now explain how they are calculated. It suffices to compute the classes \( C_w \) in the fixed-point basis, from which we can simply change basis to the \( A_w \) by inverting a matrix with coefficients in \( \mathbb{C}(\epsilon) \) that is triangular in the Bruhat order. For instance, let us explain how we would compute

\[
C_{4,-1,3} = -\frac{(-1, 4, 3)}{10 \epsilon^3} + \frac{(0, 2, 4)}{2 \epsilon^3} - \frac{3 (1, 2, 3)}{2 \epsilon^3} + \frac{(1, 3, 2)}{2 \epsilon^3} - \frac{(2, 0, 4)}{2 \epsilon^3} + \frac{3 (2, 1, 3)}{2 \epsilon^3} - \frac{(3, 1, 2)}{2 \epsilon^3} + \frac{(4, -1, 3)}{10 \epsilon^3}
\]

in window notation, noticing that now all terms are in \( \text{Res}(3, 4) \). Here we are using the normalization of \( \epsilon \) corresponding to the differential of the embedding \( U \to \hat{T} \), rather than the normalization of (3).

Even though the fixed points of Springer-Schubert varieties are not attractive for \( U \subset \hat{T} \), the coefficients may be determined from the (nonconvergent) Hilbert rational function of affine charts of the Schubert-Springer varieties by Brion [4], sections 4.2 and 4.4. For instance, the local chart of the Schubert-Springer cell about \((4, -1, 3)\) is given by

\[
\begin{pmatrix}
1 & a_{0,2} t^{-1} & 0 \\
0 & t^{-1} & 0 \\
t a_{4,3} & a_{4,2} t + a_{1,2} & t
\end{pmatrix}, \quad I_a = (a_{0,2} - a_{4,2}),
\]

where the coordinates are increasing moving leftward and upward, so that the matrix at \( a_{i,j} = 0 \) is the corresponding element of the Weyl group. The ideal \( I_a \) describes the relation that characterize the Springer fiber within the Bruhat cell.
The Schubert-Springer variety is the closure of this cell. It has an affine chart centered about $(1, 2, 3)$, for example, given by its intersection with the “big cell” in the Iwahori decomposition. It is given by

$$\begin{pmatrix}
1 & b_{0,2}t^{-1} & b_{0,1}t^{-1} \\
b_{2,3} & 1 + b_{-1,2}t^{-1} & b_{-1,1}t^{-1} \\
0 & b_{1,2} & 1
\end{pmatrix},$$

with relations in the ideal

$$I_b = (b_{0,1}b_{1,2} - b_{0,2}, b_{-1,2}b_{1,2}b_{2,3} - b_{0,2}^2b_{2,3} + b_{-1,2}b_{0,2},$$

$$-b_{0,1}b_{0,2}b_{2,3} + b_{-1,2}b_{0,1} + b_{-1,2}b_{2,3}, b_{-1,1}b_{1,2} - b_{-1,2},$$

$$-b_{0,1}b_{0,2}b_{2,3} + b_{-1,1}b_{0,2} + b_{-1,2}b_{2,3}, -b_{0,1}b_{2,3} + b_{-1,1}b_{0,1} + b_{-1,1}b_{2,3}).$$

These relations are determined by taking the rational map from the Bruhat cell $\text{Spec}(\mathbb{C}[a_{i,j}])$ to $\text{Spec}(\mathbb{C}[b_{i,j}])$, by computing a Cholesky decomposition assuming generic values of $a_{i,j}$, see the method presented in section 3.8.4 of [6]. This gives rise to a homogeneous ideal in $\mathbb{C}[a_{i,j}, b_{i,j}]$ by multiplying out by denominators, to which we then add the generators of $I_a$. Finally, we saturate the ideal by these denominators, and eliminate the $b$-variables, by taking just those elements in a Gr"obner basis that do not contain the $a$-variables, with respect to a monomial order in which the $a$-variables are given higher weight than all the $b$-variables. For a reference, see Stillman [33].

Ignoring the nonattracting nonissue (see Brion for how these equivariant weights are defined generally), the Hilbert series of the associated graded rings of these two cells with respect to the usual maximal ideals, with the grading given by the torus action are

$$\frac{1}{(1 - x^{-1})(1 - x^{-2})(1 - x^{-5})} = \frac{1}{10} \epsilon^{-3} + \frac{2}{5} \epsilon^{-2} + \cdots,$$

$$\frac{1 - (x^8 - x^6 - 2x^5 + 2x^3 + x^2)}{(1 - x^2)^2(1 - x^3)(1 - x)^3} = -\frac{3}{2} \epsilon^{-3} + \frac{3}{2} \epsilon^{-2} + \cdots$$

at $x = \exp(\epsilon)$. Essentially, the procedure described in [4] says that the rational coefficient in the expansion of $C_w$ is the coefficient of the lowest term $\epsilon^{-d}$, where $d$ is the Krull dimension of the local ring, and which agrees with the dimension of the corresponding cell. We can see that the leading terms are indeed the corresponding coefficients in (44).
References

[1] H. Abe, M. Harada, T. Horiguchi, and M. Masuda. The equivariant cohomology rings of regular nilpotent Hessenberg varieties in lie type A: research announcement. MORFISMOS, 2014. special volume in honor of Samuel Gitler, to appear.

[2] E. E. Allen. The descent monomials and a basis for the diagonally symmetric polynomials. Journal of Algebraic Combinatorics, (3):5–16, 1994.

[3] A. Björner and F. Brenti. Affine permutations of type A. Electr. J. Comb., 3, 1996.

[4] M. Brion. Equivariant chow groups for torus actions. Transformation Groups, 2(3):225–267, Sep 1997.

[5] E. Carlsson and A. Mellit. A proof of the shuffle conjecture. J. Amer. Math. Soc., 2015. to appear.

[6] M. de Cataldo, Th. Haines, and Li Li. Frobenius semisimplicity for convolution morphisms. Mathematische Zeitschrift, 289(1-2):119–169, Nov 2017.

[7] D. Edidin and W. Graham. Localization in equivariant intersection theory and the Bott residue formula. American Journal of Mathematics, 120(3):619–636, 1998.

[8] A. Garsia. Combinatorial methods in the theory of Cohen-Macaulay rings. Adv. Math., (38):229–266, 1980.

[9] I. Gordon. On the quotient ring by diagonal invariants. Inventiones Mathematicae, 153(3):503–518, Sep 2003.

[10] M.戈resky, R. Kottwitz, and R. MacPherson. Equivariant cohomology, Koszul duality, and the localization theorem. Inventiones Mathematicae, 131(1):25–83, Dec 1997.

[11] E. Gorsky and M. Mazin. Compactified Jacobians and $q, t$-Catalan numbers, I. Journal of Combinatorial Theory, Series A, 120(1):49 – 63, 2013.
[12] E. Gorsky and M. Mazin. Compactified Jacobians and \(q,t\)-Catalan numbers II. *Journal of Algebraic Combinatorics*, 39(1):153–186, 2014.

[13] E. Gorsky, M. Mazin, and M. Vazirani. Affine permutations and rational parking functions. *Trans. Amer. Math. Soc.* 368(12):8403–8445, 2016.

[14] W. Graham. Positivity in equivariant Schubert calculus. *Duke Mathematical Journal*, 109(3):599614, Sep 2001.

[15] J. Haglund. The \(q,t\)-Catalan numbers and the space of diagonal harmonics, volume 41 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2008. With an appendix on the combinatorics of Macdonald polynomials.

[16] J. Haglund, M. Haiman, and N. Loehr. A combinatorial formula for non-symmetric Macdonald polynomials. *American Journal of Mathematics*, 130(2):359–383, 2008.

[17] J. Haglund, M. Haiman, N. Loehr, J.B. Remmel, and A. Ulyanov. A combinatorial formula for the character of the diagonal coinvariants. *Duke J. Math.*, pages 195–232, 2005.

[18] J. Haglund and N Loehr. A conjectured combinatorial formula for the Hilbert series for diagonal harmonics. *Discrete Mathematics*, 298(1):189–204, 2005.

[19] M. Haiman. Vanishing theorems and character formulas for the Hilbert scheme of points in the plane. *Invent. Math.*, 2002.

[20] T. Hikita. Affine Springer fibers of type \(A\) and combinatorics of diagonal coinvariants. *Adv. Math.*, 263:88–122, 2014.

[21] E. Insko and J. Tymoczko. Affine pavings of regular nilpotent Hessenberg varieties and intersection theory of the Peterson variety. 2013. arXiv:1309.0484.

[22] D. Kazhdan and G. Lusztig. Fixed point varieties on affine flag manifolds. *Israel Journal of Mathematics*, 62(2):129–168, Jun 1988.

[23] B. Kostant and S. Kumar. The nil Hecke algebra and cohomology of \(G/P\) for a Kač-Moody group. *Advances in Mathematics*, 62:187–237, 1986.
[24] Sh. Kumar. *Kac-Moody Groups, their Flag Varieties and Representation Theory*. Progress in Mathematics 204. Birkhauser Basel, 1 edition, 2002.

[25] T. Lam. Schubert polynomials for the affine Grassmannian. *J. Amer. Math. Soc.*, 21(1), 2008.

[26] T. Lam, L. Lapointe, J. Morse, A. Schilling, M. Shimozono, and M. Zabrocki. *k-Schur functions and affine Schubert calculus*, volume 33 of *Fields Institute Monographs*. Springer-Verlag, New York, 2014.

[27] S. J. Lee. Combinatorial description of the cohomology of the affine flag variety. 2015. arXiv:1506.02390.

[28] G. Lusztig and J. M. Smelt. Fixed point varieties on the space of lattices. *Bulletin of the London Mathematical Society*, 23(3):213–218, May 1991.

[29] A. Mellit. Toric braids and \((m, n)\)-parking functions. arXiv:1604.07456, 2016.

[30] A. Oblomkov and Z. Yun. Geometric representations of graded and rational Cherednik algebras. *Advances in Mathematics*, 292:601–706, Apr 2016.

[31] A. Oblomkov and Z. Yun. The cohomology ring of certain compactified Jacobians. 2017.

[32] M. Precup. The Betti numbers of regular Hessenberg varieties are palindromic. *Transformation Groups*, 2017.

[33] M. Stillman. Methods for computing in algebraic geometry and commutative algebra rome, march 1990. *Topics in Computational Algebra*, pages 77–103, 1990.

[34] M. Varagnolo and E. Vasserot. Double affine Hecke algebras and affine flag manifolds, I. *Affine Flag Manifolds and Principal Bundles*, pages 233–289, 2010.

[35] E. Vasserot. Induced and simple modules of double affine Hecke algebras. *Duke Mathematical Journal*, 126(2):251–323, Feb 2005.

[36] Z. Yun. Lectures on Springer theories and orbital integrals. *PCMI procedings, to appear*, 2016. arXiv:1602.01451.
[37] X. Zhu. An introduction to affine Grassmannians and the geometric Satake equivalence. 2016.