Sigma Models with $\mathcal{N}=8$ Supersymmetries in 2+1 and 1+1 Dimensions

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Abstract

We introduce an $\mathcal{N}=8$ supersymmetric extension of the Bogomolny-type model for Yang-Mills-Higgs fields in 2+1 dimensions related with twistor string theory. It is shown that this model is equivalent to an $\mathcal{N}=8$ supersymmetric U($n$) chiral model in 2+1 dimensions with a Wess-Zumino-Witten-type term. Further reduction to 1+1 dimensions yields $\mathcal{N}=(8,8)$ supersymmetric extensions of the standard U($n$) chiral model and Grassmannian sigma models.
1 Introduction and Summary

Nonlinear sigma models in $k$ dimensions describe mappings of a $k$-dimensional manifold $X$ into a manifold $Y$ (target space). In particular, as target spaces one can consider Lie groups $G$ (chiral models) and homogeneous spaces $G/H$ for closed subgroups $H \subset G$. Sigma models and their $\mathcal{N}$-extended supersymmetric generalizations play an important role both in physics and mathematics (see e.g. [1, 2]). For instance, two-dimensional sigma models serve as a theoretical laboratory for the study of more complicated (quantum) super Yang-Mills theory since they share many of its features such as asymptotic freedom, nontrivial topological structure, the existence of instantons, ultraviolet finiteness for the $\mathcal{N}=4$ supersymmetric case etc. [3]. Moreover, supersymmetric two-dimensional sigma models are the building blocks for superstring theories [3, 4].

Recall that for two-dimensional nonlinear sigma models admitting a Lagrangian formulation the number of supersymmetries is intimately related to the geometry of the target space. Namely, it was argued that Lagrangian $\mathcal{N}=1$ models can be defined for any target space $Y$, for $\mathcal{N}=2$ the target space must be Kähler, for $\mathcal{N}=4$ it must be hyper-Kähler, and no Lagrangian models were introduced for $\mathcal{N}>4$ [5, 6]. Similar results hold for sigma models in three dimensions. In particular, this means that a target space $Y$ admits no more than $\mathcal{N}=1$ supersymmetry in the case of (non-Kähler) group manifolds $G$ and $\mathcal{N}\leq 2$ supersymmetries for homogeneous Kähler spaces $G/H$.

The field equations of the standard $G$ and $G/H$ sigma models in 1+1 and 2+0 dimensions can be obtained by dimensional reduction of the self-dual Yang-Mills (SDYM) equations in 2+2 dimensions, with a gauge group $G$ [7]. Concretely, the SDYM model reduced to two dimensions is equivalent to the sigma model with $G$-valued scalar fields, while the $G/H$ sigma model arises after imposing additional algebraic constraints. Similar reduction to 2+1 dimensions yields a modified integrable chiral model [8]. Recall that the SDYM model in 2+2 dimensions can be endowed with up to four supersymmetries [9, 10]. Reducing the $\mathcal{N}$-extended supersymmetric SDYM equations in 2+2 dimensions to 2+1 and 1+1 dimensions yields models which have twice as many supersymmetries (cf. [11] for reductions from 3+1 dimensions). We will show that for $G=U(n)$ and $\mathcal{N}=4$ these models are equivalent to $U(n)$ chiral models with $\mathcal{N}=8$ supersymmetries. These new supersymmetric sigma models in 2+1 and 1+1 dimensions are well defined on the level of equations of motion, but their Lagrangian formulation is not known yet.

In this note we concentrate on the reduction of the $\mathcal{N}=4$ SDYM equations (instead of arbitrary $\mathcal{N}\leq 4$) in 2+2 dimensions since for this case a Lagrangian can be written down at least in terms of the component fields of a reduced Yang-Mills-type supermultiplet. Moreover, it was shown by Witten [12] that the $\mathcal{N}=4$ SDYM model appears in twistor string theory, which is a B-type topological string with the supertwistor space $\mathbb{CP}^{3|4}$ as a target space\footnote{For other variants of twistor string models see [13].}. This fact gives additional arguments in favour of introducing $\mathcal{N}=8$ supersymmetric sigma models in 2+1 and 1+1 dimensions related with twistor string theory and of studying their properties.

2 $\mathcal{N}=4$ supersymmetric SDYM equations in 2+2 dimensions

Superspace $\mathbb{R}^{4|16}$. Let us consider the four-dimensional space $\mathbb{R}^{2,2} := (\mathbb{R}^4, g)$ with the metric
d$s^2 = g_{\mu\nu}dx^\mu dx^\nu = \det(dx^{\alpha\dot{\alpha}}) = dx^{1i}dx^{2\dot{i}} - dx^{2i}dx^{1\dot{i}}$ \hspace{1cm} (2.1)
with \((g_{\mu\nu}) = \text{diag}(-1, +1, +1, -1)\). Here \(\mu, \nu, \ldots = 1, \ldots, 4\) are vector indices and \(\alpha = 1, 2, \dot{\alpha} = \bar{1}, \bar{2}\) are spinor indices. We choose the real coordinates\(^2\) \((x^\mu) = (x^a, \bar{t}) = (t, x, y, \bar{t})\) with \(a, b, \ldots = 1, 2, 3\) such that
\[
x^{11} = \frac{1}{2}(t - y), \quad x^{12} = \frac{1}{2}(x + \bar{t}), \quad x^{21} = \frac{1}{2}(x - \bar{t}) \quad \text{and} \quad x^{22} = \frac{1}{2}(t + y). \tag{2.2}
\]

On the space \(\mathbb{R}^{2,2}\) one can introduce real Majorana-Weyl spinors and extend \(\mathbb{R}^{2,2}\) to a space with additional anticommuting (Grassmann) coordinates \(\theta^i\) and \(\eta_\dot{i}\) of helicity \(+\frac{1}{2}\) and \(-\frac{1}{2}\), respectively. Here index \(i = 1, \ldots, 4\) parametrizes fundamental and its conjugate representations of the R-symmetry group \(\text{SL}(4, \mathbb{R})\) with addition anticommuting (Grassmann) coordinates \(\theta^i\) and \(\eta_\dot{i}\) respectively. Here index \(i = 1, \ldots, 4\) parametrizes fundamental and its conjugate representations of the R-symmetry group \(\text{SL}(4, \mathbb{R})\) [9]. Thus, \((x^{\alpha\dot{\alpha}}, \eta_\dot{i}^\alpha, \theta^i\) are coordinates on superspace \(\mathbb{R}^{4|16}\).

**Supersymmetry algebra.** The \(\mathcal{N}=4\) supersymmetry algebra in 2+2 dimensions is generated by \(P_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} = \partial/\partial x^{\alpha\dot{\alpha}}\) and 16 real supercharges
\[
Q_{i\alpha} := \partial_{i\alpha} - \eta_\dot{i}^\alpha \partial_{\alpha\dot{\alpha}} \quad \text{and} \quad Q^i_{\alpha} := \partial^i_{\alpha} - \theta^i_{\dot{\alpha}} \partial_{\alpha\dot{\alpha}}, \tag{2.3}
\]
with \(\partial_{i\alpha} := \partial/\partial \theta^i\) and \(\partial^i_{\alpha} := \partial/\partial \eta_\dot{i}^\alpha\). The only nontrivial (anti)commutators in this superalgebra read
\[
\{Q_{i\alpha}, Q^j_{\beta}\} = -2\delta^j_i \partial_{\alpha\dot{\beta}}. \tag{2.4}
\]

In what follows we will also need superderivatives
\[
D_{i\alpha} := \partial_{i\alpha} + \eta_\dot{i}^\alpha \partial_{\alpha\dot{\alpha}} \quad \text{and} \quad D^i_{\alpha} := \partial^i_{\alpha} + \theta^i_{\dot{\alpha}} \partial_{\alpha\dot{\alpha}}, \tag{2.5}
\]
which anticommute with the operators (2.3) and satisfy
\[
\{D_{i\alpha}, D^j_{\beta}\} = 2\delta^j_i \partial_{\alpha\dot{\beta}}. \tag{2.6}
\]

**Antichiral superspace.** On the superspace \(\mathbb{R}^{4|16}\) we can introduce spin-tensor fields depending on both bosonic and fermionic coordinates (superfields) and impose on them various constraints. In particular, on any superfield \(\mathcal{A}\) one can impose the so-called antichirality conditions \(\mathcal{L}_Z \mathcal{A} = 0\), where \(\mathcal{L}_Z\) denotes the Lie derivative along a vector superfield \(Z\). One can easily solve these equations by using a coordinate transformation on superspace \(\mathbb{R}^{4|16}\),
\[
(x^{\alpha\dot{\alpha}}, \eta_\dot{i}^\alpha, \theta^i) \rightarrow (x^{\alpha\dot{\alpha}} = x^{\alpha\dot{\alpha}} - \theta^i \eta_\dot{i}^\alpha, \eta_\dot{i}^\alpha, \theta^i), \tag{2.7}
\]
under which \(\partial_{\alpha\dot{\alpha}}, D_{i\alpha}\) and \(D^i_{\alpha}\) transform to the operators
\[
\tilde{\partial}_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}, \quad \tilde{D}_{i\alpha} = \partial_{i\alpha}, \quad \tilde{D}^i_{\alpha} = \partial^i_{\alpha} + 2\theta^i \partial_{\alpha\dot{\alpha}}. \tag{2.8}
\]
The antichirality conditions then mean that a superfield \(\mathcal{A}\) satisfies the equations
\[
\tilde{D}_{i\alpha} \mathcal{A} = 0 \tag{2.9}
\]
meaning that \(\mathcal{A}\) is defined on superspace \(\mathbb{R}^{4|8} \subset \mathbb{R}^{4|16}\) called antichiral superspace with coordinates \((x^{\alpha\dot{\alpha}}, \eta_\dot{i}^\alpha)\). Note that for transformed supercharges we have
\[
\tilde{Q}_{i\alpha} = \partial_{i\alpha} - 2\eta_\dot{i}^\alpha \partial_{\alpha\dot{\alpha}} \quad \text{and} \quad \tilde{Q}^i_{\alpha} = \partial^i_{\alpha}. \tag{2.10}
\]
\(^2\)Our conventions are chosen to match those of [14] after reduction to the space \(\mathbb{R}^{2,1}\) with coordinates \((t, x, y)\).
N=4 SDYM in superfields. The field content of $\mathcal{N}=4$ supersymmetric SDYM is given by a supermultiplet $(A_{\a\a}, \chi^{\a}, \phi^{ij}, \tilde{\chi}^{\dot{\a}}, G_{\dot{\a}\dot{\b}})$ of fields on $\mathbb{R}^{4|8}$ of helicities $(+1, +\frac{1}{2}, 0, -\frac{1}{2}, -1)$. Here $A_{\a\a}$ are the components of a gauge potential with the field strength $F_{a\bar{a}, \b\bar{\b}} = \partial_{a\bar{a}} A_{\b\bar{\b}} - \partial_{\b\bar{\b}} A_{a\bar{a}} + [A_{a\bar{a}}, A_{\b\bar{\b}}]$. Note that the scalars $\phi^{ij}$ are antisymmetric in $ij$ and all the fields, including the fermionic ones $\chi^{\a}$ and $\tilde{\chi}^{\dot{\a}}$, live in the adjoint representation of the gauge group $U(n)$.

The $\mathcal{N}=4$ SDYM equations [15, 9] can be written in terms of superfields on antichiral superspace $\mathbb{R}^{4|8} [9, 16]$. Namely, all fields from the above $\mathcal{N}=4$ supermultiplet can be combined into superfields $A_{a\bar{a}}$ and $A_{\dot{a}}^i$ on $\mathbb{R}^{4|8}$ in terms of which the $\mathcal{N}=4$ SDYM equations read

$$[\nabla_{a\bar{d}}, \nabla_{b\bar{d}}] + [\nabla_{a\bar{b}}, \nabla_{b\bar{a}}] = 0 \, , \quad [\nabla_{a\bar{a}}, \nabla_{b\bar{b}}] + [\nabla_{a\bar{a}}, \nabla_{b\bar{b}}] = 0 \, , \quad [\nabla_{a\bar{a}}, \nabla_{b\bar{b}}] + [\nabla_{a\bar{a}}, \nabla_{b\bar{b}}] = 0 \, , \quad (2.11)$$

where we have introduced the covariant derivatives

$$\nabla_{a\bar{d}} := \partial_{a\bar{d}} + A_{a\bar{d}} \quad \text{and} \quad \nabla_{\dot{a}}^i := \partial_{\dot{a}}^i + A_{\dot{a}}^i \, . \quad (2.12)$$

Note that (2.11) can be combined into the manifestly supersymmetric equations

$$\{\nabla_{a\bar{d}}, \nabla_{b\bar{d}}\} + \{\nabla_{a\bar{d}}, \nabla_{b\bar{d}}\} = 0 \, \quad (2.13)$$

with

$$\tilde{\nabla}_{a\bar{d}}^{\dot{a}} := \nabla_{a\bar{d}} + 2\tilde{\eta}^i \nabla_{a\bar{d}} = \tilde{D}_{a\bar{d}}^\dot{a} + \tilde{A}_{a\bar{d}}^\dot{a} \quad \text{and} \quad \tilde{A}_{a\bar{d}}^\dot{a} := A_{a\bar{d}}^\dot{a} + 2\tilde{\eta}^i A_{a\bar{d}}^i \, , \quad (2.14)$$

where $A_{a\bar{d}}$ and $A_{a\bar{d}}^i$ depend only on $x^{a\bar{a}}$ and $\eta^{\dot{a}}$.

It is not difficult to see that equations (2.13) are the compatibility conditions for the linear system of differential equations

$$\lambda_{\pm}^\dot{a} (D_{a\bar{d}}^\dot{a} + \tilde{A}_{a\bar{d}}^\dot{a}) \psi_{\pm} = 0 \, , \quad (2.15)$$

where $\lambda_{\pm}^\dot{a} = \varepsilon^{\dot{a}\dot{b}} \lambda_{\dot{b}}^{\pm}$, $(\lambda^{\pm}_{\dot{b}}) = (1 \, \lambda^{+})^T$, $(\lambda^{-}_{\dot{b}}) = (\lambda^{-} \, 1)^T$ and the extra (local) coordinates $\lambda_{\pm}$ lie on patches $U_{\pm}$ covering the Riemann sphere $\mathbb{C}P^1 = U_+ \cup U_-$ (see e.g. [17]). Here $\psi_{\pm}$ are $n \times n$ matrices depending not only on $x^{a\bar{a}}$ and $\eta^{\dot{a}}$ but also (holomorphically) on $\lambda_{\pm} \in U_{\pm}$.

The field equations of the $\mathcal{N}=4$ SDYM model in the component fields read

$$F_{a\bar{d}} = 0 \, , \quad D_{a\bar{a}} \chi^{\a} = 0 \, , \quad D_{a\bar{a}} D^{\a\a} \phi^{ij} + \{\chi^{\a}, \chi^{\dot{\a}}\} = 0 \, , \quad (2.16a)$$

$$D_{a\bar{a}} \tilde{\chi}^{\dot{a}} + [\chi^{\a}, \phi_{ij}] = 0 \, , \quad \varepsilon^{\a\b} D_{a\bar{a}} G_{\b\bar{\b}} - \frac{1}{2} \{\chi^{\a}, \tilde{\chi}^{\dot{\a}}\} - \frac{1}{4} [\phi_{ij}, D_{a\bar{a}} \phi_{ij}] = 0 \, , \quad (2.16b)$$

where $F_{a\bar{d}, \b\bar{\b}} := -\frac{1}{2} \varepsilon^{\a\b} F_{a\bar{a}, \b\bar{\b}}$, $D_{a\bar{a}} := \partial_{a\bar{a}} + [A_{a\bar{a}}, \cdot]$ and $\phi_{ij} := \frac{1}{4!} \varepsilon_{ijkl} \phi^{kl}$. These equations can be extracted from (2.11) by using $\eta$-expansions and Bianchi identities (see e.g. [16]). We will not reproduce this derivation. Note only that (2.16) follows from the Lagrangian $[9, 12]$

$$\mathcal{L} = \text{tr} \left( G^{a\b} F_{a\b} + \tilde{\chi}^{\dot{a}} D_{a\bar{a}} \chi^{\a} + \phi_{ij} D_{a\bar{a}} D^{a\a} \phi^{ij} + \phi_{ij} \chi^{\a} \chi_{\dot{a}} \right) \, . \quad (2.17)$$
3 $\mathcal{N}=8$ supersymmetric sigma models in 2+1 dimensions

Reduction and spinors on $\mathbb{R}^{2,1}$. The $\mathcal{N}=8$ supersymmetric Bogomolny-type equations in 2+1 dimensions are obtained from the described $\mathcal{N}=4$ super SDYM equations by the dimensional reduction $\mathbb{R}^{2,2} \rightarrow \mathbb{R}^{2,1}$. Namely, we impose the $\partial_4$-invariance condition on all the fields $(A_{a\dot{a}}, \chi^i, \phi^j, \bar{\chi}^\dot{i}, \bar{G}_{\dot{a}\dot{b}})$ from the $\mathcal{N}=4$ supermultiplet. Also, the components $A_\mu$ of a gauge potential split into the components $A_\alpha$ in 2+1 dimensions and the Lie-algebra valued scalar field $\varphi := A_4$ (Higgs field). To see how this splitting looks in spinor notation, we briefly discuss spinors in 2+1 dimensions.

Recall that $\mathcal{N}=4$ SDYM theory on $\mathbb{R}^{2,2}$ has $\text{SL}(4,\mathbb{R}) \cong \text{Spin}(3,3)$ as an R-symmetry group \cite{9}. Analogously to the case of standard $\mathcal{N}=4$ super Yang-Mills (SYM) in Minkowski space with the $\text{Spin}(6)$ R-symmetry, the appearance of the group Spin(3,3) can be interpreted via a reduction of $\mathcal{N}=1$ SYM theory on space $\mathbb{R}^{5,5} \cong \mathbb{R}^{2,2} \times \mathbb{R}^{3,3}$ to $\mathbb{R}^{2,2}$ with internal space $\mathbb{R}^{3,3}$ \cite{10}. Furthermore, after reduction from $\mathbb{R}^{2,2}$ to $\mathbb{R}^{2,1}$ the R-symmetry group becomes Spin(4,4) and supersymmetry gets enlarged to $\mathcal{N} = 8$ with Spin(4,4) as the manifest R-symmetry group (cf. \cite{11} for Minkowski and \cite{18} for Euclidean signatures). Roughly speaking, this happens due to no distinction between dotted and undotted spinor indices in three dimensions. Recall that the rotation group SO(2,2) of $\mathbb{R}^{2,2}$ is locally isomorphic to SU(1,1) $\times$ SU(1,1) $\cong$ Spin(2,1) $\times$ Spin(2,1) $\cong$ Spin(4,4). Upon dimensional reduction to 2+1 dimensions, the rotation group of $\mathbb{R}^{2,1} = (\mathbb{R}^3, g)$ with $g = (a_{ab}) = \text{diag}(-1,1,1)$ is locally SU(1,1) $\cong$ Spin(2,1), which is the diagonal subgroup of Spin(2,1) $\times$ Spin(2,1) $\cong$ Spin(4,4). Therefore, the distinction between dotted and undotted indices disappear.

Coordinates and derivatives on $\mathbb{R}^{3|16}$. The $\partial_4$-invariance reduces superspace $\mathbb{R}^{4|16}$ with coordinates $x^\mu, \eta_i^a$ and $\theta^{i\alpha}$ to $\mathbb{R}^{3|16}$ with coordinates $x^\alpha, \eta_i^a$ and $\theta^{i\alpha}$. Furthermore, $x^\alpha$ and $\eta_i^a$ parametrize reduced antichiral superspace $\mathbb{R}^{3|8}$. For bosonic coordinates $x^{\alpha \beta} \rightarrow x^{\alpha \beta}$ in spinor notation we have

$$x^{\alpha \beta} = \frac{1}{2}(x^{\alpha \beta} + x^{\beta \alpha}) + \frac{1}{2}(x^{\alpha \beta} - x^{\beta \alpha}) = x^{(\alpha \beta)} + x^{[\alpha \beta]} .$$

(3.1)

Thus, we have coordinates

$$y^{\alpha \beta} := x^{(\alpha \beta)} \quad \text{with} \quad y^{11} = x^{11} = \frac{1}{2}(t - y), \quad y^{12} = \frac{1}{2}(x^{21} + x^{21}) = \frac{1}{2} x, \quad y^{22} = x^{22} = \frac{1}{2}(t + y) \quad \text{(3.2)}$$

on $\mathbb{R}^{2,1}$ and $x^{[\alpha \beta]} = -\varepsilon^{\alpha \beta} x^4 = -\varepsilon^{\alpha \beta} \tilde{t}$, where $\varepsilon^{12} = -\varepsilon^{21} = 1$.

For derivatives we obtain

$$\partial_{\alpha \beta} = \frac{1}{2} (\partial_{\alpha \beta} + \partial_{\beta \alpha}) + \frac{1}{2} (\partial_{\alpha \beta} - \partial_{\beta \alpha}) = \partial_{(\alpha \beta)} - \varepsilon_{\alpha \beta} \partial_4 = \partial_{(\alpha \beta)} - \varepsilon_{\alpha \beta} \partial_4 ,$$

(3.3)

where $\varepsilon_{12} = -\varepsilon_{21} = -1$ and

$$\partial_{(11)} = \frac{\partial}{\partial y^{11}} = \partial_t - \partial_y , \quad \partial_{(12)} = \partial_{(21)} = \frac{1}{2} \frac{\partial}{\partial y^{12}} = \partial_x , \quad \partial_{(22)} = \frac{\partial}{\partial y^{22}} = \partial_t + \partial_y \quad \text{(3.4)}$$

For the operators (2.8) acting on $\tilde{t}$-independent superfields we have

$$\check{D}_{i\alpha} = \partial_{i\alpha} \quad \text{and} \quad \check{D}_{\dot{a}} = \partial^i + 2\theta^{i\beta} \partial_{(\alpha \beta)} .$$

(3.5)

Similarly, supercharges (2.10) reduce to the operators

$$\hat{Q}_{i\alpha} = \partial_{i\alpha} - 2\eta^i_\beta \partial_{(\alpha \beta)} \quad \text{and} \quad \hat{Q}_\dot{a} = \partial^i .$$

(3.6)
anticommuting with (3.5).

\( N = 8 \) supersymmetric Bogomolny-type equations on \( \mathbb{R}^{2,1} \). After imposing the condition of \( \bar{t}\)-independence on all fields in the linear system (2.15), we obtain the equations

\[
\lambda_\pm^2 (\hat{D}_\alpha^i + \hat{A}_\alpha^i) \psi_\pm = 0
\]

with

\[
\hat{A}_\alpha^i = A_\alpha^i + 2 \theta^{i\beta} (A_{(\alpha\beta)} - \varepsilon_{\alpha\beta} \bar{\varphi}) ,
\]

and \( \hat{D}_\alpha^i \) given in (3.5). Here \( A_\alpha^i, A_{(\alpha\beta)} \) and \( \bar{\varphi} \) are superfields depending only on \( y^{\alpha\beta} \) and \( \eta_i^\beta \).

The compatibility conditions for the linear system (3.7) read

\[
\{ \hat{D}_\alpha^i + \hat{A}_\alpha^i, \hat{D}_\beta^j + \hat{A}_\beta^j \} + \{ \hat{D}_\beta^i + \hat{A}_\beta^i, \hat{D}_\alpha^i + \hat{A}_\alpha^i \} = 0 .
\]

As usual, these manifestly \( N = 8 \) supersymmetric equations are equivalent to equations in component fields,

\[
f_{\alpha\beta} + D_{\alpha\beta} \varphi = 0 , \quad D_{\alpha\beta} \chi^{ij} + \varepsilon_{\alpha\beta} [\varphi, \chi^{ij}] = 0 ,
\]

\[
D_{\alpha\beta} \phi^{ij} + 2 [\varphi, [\varphi, \phi^{ij}]] + \chi^{\alpha x} \chi_i^\alpha = 0 ,
\]

\[
D_{\alpha\beta} \tilde{\chi}_i^\alpha - \varepsilon_{\alpha\beta} [\varphi, \chi_i^\alpha] + [\chi_i^\alpha, \phi_{ij}] = 0 ,
\]

\[
\varepsilon^{\gamma\delta} D_{\alpha\gamma} G_{\delta\beta} + [\varphi, G_{\alpha\beta}] - \frac{1}{2} \{ \chi_i^\alpha, \tilde{\chi}_i^\alpha \} - \frac{1}{4} [\phi_{ij}, D_{\alpha\beta} \phi^{ij}] - \frac{1}{4} \varepsilon_{\alpha\beta} [\phi_{ij}, [\phi^{ij}, \varphi]] = 0 ,
\]

where \( D_{\alpha\beta} := \partial_{(\alpha\beta)} + [A_{(\alpha\beta)}, \cdot] \), \( f_{\alpha\beta} := -\frac{1}{2} \varepsilon^{\gamma\delta} [D_{\alpha\gamma}, D_{\beta\delta}] \) and \( \varphi := A_4 = A_{\bar{t}} \). Obviously, these equations are \( \partial_4 \)-reduction of (2.16).

Supersymmetric sigma models. Note that matrices \( \psi_\pm \) in (3.7) are defined up to a gauge transformation generated by a matrix which does not depend on \( \lambda_\pm \) and therefore one can choose a gauge such that

\[
\psi_+ = \Phi^{-1} + O(\lambda_+) \quad \text{and} \quad \psi_- = 1_n + \lambda_- Y + O(\lambda^2) ,
\]

where \( \Phi \) is a \( U(n) \)-valued superfield and \( Y \) is a \( u(n) \)-valued superfield both depending only on \( y^{\alpha\beta} \) and \( \eta_i^\alpha \). For this gauge, from (3.7) we obtain

\[
\hat{A}_1^i = 0 \quad \text{and} \quad \hat{A}_2^i = \Phi^{-1} \hat{D}_2^i \Phi ,
\]

and from (3.8) we have

\[
\hat{A}_1^i = 0 \quad \text{and} \quad \hat{A}_2^i = \Phi^{-1} \partial_2^i \Phi , \quad A_{(11)} = 0 \quad \text{and} \quad A_{(12)} - \bar{\varphi} = 0 ,
\]

\[
A_{(21)} = \Phi^{-1} \partial_{(2)} \Phi \quad \text{and} \quad A_{(22)} = \Phi^{-1} \partial_{(22)} \Phi .
\]

Substituting (3.12) into (3.9), we obtain equations

\[
\hat{D}_1^i (\Phi^{-1} \hat{D}_2^i \Phi) + \hat{D}_2^i (\Phi^{-1} \hat{D}_2^i \Phi) = 0
\]

which after using (3.5) and (3.13) read

\[
\partial_x (\Phi^{-1} \partial_x \Phi) + \partial_y (\Phi^{-1} \partial_y \Phi) - \partial_t (\Phi^{-1} \partial_t \Phi) + \partial_y (\Phi^{-1} \partial_t \Phi) - \partial_t (\Phi^{-1} \partial_y \Phi) = 0 ,
\]
\[ \partial_1^j (\Phi^{-1} \partial_x \Phi) - \partial_t (\Phi^{-1} \partial_2^j \Phi) + \partial_y (\Phi^{-1} \partial_3^j \Phi) = 0, \quad \partial_1^j (\Phi^{-1} \partial_t \Phi) + \partial_1^j (\Phi^{-1} \partial_y \Phi) - \partial_x (\Phi^{-1} \partial_2^j \Phi) = 0, \quad (3.16) \]

\[ \partial_1^j (\Phi^{-1} \partial_2^j \Phi) + \partial_1^j (\Phi^{-1} \partial_2^j \Phi) = 0. \quad (3.17) \]

Note that the last two terms in (3.15) are the Wess-Zumino-Witten terms which spoil the standard Lorentz invariance but yield an integrable U(n) chiral model in 2+1 dimensions. For reduction to 1+1 dimensions one should simply put \( \partial_y \Phi = 0 \) in (3.15)-(3.17) obtaining an \( \mathcal{N}=8 \) supersymmetric extensions of the standard U(n) chiral model in two dimensions with field equations

\[ \partial_t (\Phi^{-1} \partial_t \Phi) - \partial_x (\Phi^{-1} \partial_2^j \Phi) = 0, \quad \partial_1^j (\Phi^{-1} \partial_2^j \Phi) + \partial_1^j (\Phi^{-1} \partial_2^j \Phi) = 0, \quad (3.18a) \]

\[ \partial_1^j (\Phi^{-1} \partial_2^j \Phi) - \partial_t (\Phi^{-1} \partial_2^j \Phi) = 0, \quad \partial_1^j (\Phi^{-1} \partial_t \Phi) - \partial_x (\Phi^{-1} \partial_2^j \Phi) = 0. \quad (3.18b) \]

For \( \Phi \) taking values in the Grassmannian manifold \( \text{Gr}(k,n) \subset \text{U}(n) \), equations (3.15)-(3.17) and (3.18) describe correspondingly supersymmetric Grassmannian sigma models in 2+1 and 1+1 dimensions.

There is not yet a Lagrangian description of equations (3.15)-(3.17) or (3.18). However, using the equivalence of equations (3.10) to (3.14), one can write explicitly a Lagrangian in terms of the \( U(1) \) valued superfield \( \Phi \).

**Supersymmetry transformations.** For brevity, we consider only 2+1 dimensions, where the 16 supercharges have the form (3.6). Further reduction to 1+1 dimensions does not create any problem. From (3.6) we obtain

\[ \{ \tilde{Q}_{i\alpha}, \tilde{Q}^j_{\beta} \} = -2\delta_i^j \partial_{(\alpha\beta)} \quad (3.19) \]

On a (scalar) superfield \( \Sigma \) an infinitesimal supersymmetry transformation \( \hat{\delta} \) acts by

\[ \hat{\delta} \Sigma := \epsilon^i_\alpha \tilde{Q}_{i\alpha} \Sigma + \epsilon^j_\alpha \tilde{Q}^j_{\beta} \Sigma \quad (3.20) \]

where \( \epsilon^i_\alpha \) and \( \epsilon^j_\alpha \) are 16 Grassmann parameters. In particular, for coordinates \( y^{\alpha\beta} \) and \( \eta^\beta_i \) on the antichiral superspace \( \mathbb{R}^{3|8} \) we have \( \delta y^{\alpha\beta} = -2\epsilon^i (\alpha^\beta) \eta_i^\beta \) and \( \delta \eta_i^\alpha = \epsilon^i_\alpha \).

It is obvious that the sigma model field equations (3.14) are invariant under the supersymmetry transformations (3.20) because the operators \( \tilde{D}^i_{\alpha} \) as well as \( \tilde{D}_{\alpha i} \) anticommute with the supersymmetry generators \( \tilde{Q}_{i\alpha} \) and \( \tilde{Q}^j_{\beta} \). Note that these \( \mathcal{N}=8 \) supersymmetric extensions of the \( U(n) \) and \( \text{Gr}(k,n)=\text{U}(n)/\text{U}(k) \times \text{U}(n-k) \) sigma models in 2+1 and 1+1 dimensions are not the standard ones defined only for \( \mathcal{N} \leq 1 \) and \( \mathcal{N} \leq 2 \), respectively. It will be interesting to study this new kind of sigma models in more detail.

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