ABSTRACT. In this paper we obtain a classification of rigid isotopy classes of totally reducible trigonal curves lying on a Hirzebruch surface \( \Sigma_n \), and having a maximal number of non-degenerated double points. Such curves correspond to morsifications of a totally real semiquasihomogeneous singularity of weight \((3, 3n)\) (the union of three smooth real branches intersecting each other with multiplicity \(n\)). We obtain this classification by studying combinatorial properties of dessins.

In the appendix, we prove that any morsification of a totally real semiquasihomogeneous singularity of weight \((3, 3n)\) can be realized (up to isotopy) by the restriction of the equation to the Newton diagram and adding monomials under the Newton diagram.

CONTENTS

1. Trigonal curves and dessins
   1.1. Ruled surfaces and trigonal curves
   1.2. Dessins
   1.3. Trigonal curves and their dessins
   1.4. Toiles
2. Trigonal morsifications on Hirzebruch surfaces
   2.1. Wiring diagrams
   2.2. Decomposition of dessins and wiring diagrams
Appendix A. E. Shustin. Polynomialsity of morsifications of trigonal singularities
   A.1. Main result
   A.2. Proof of Theorem A.1
Acknowledgements
References

A morsification of a real plane singularity is a real deformation with the maximal possible number of hyperbolic nodes. Morsifications were introduced by N. A’Campo \[17, 18\] and S. Gusein-Zade \[20, 21\] as an important tool for the study of Dynkin diagrams, monodromy, topology of the singularity link, and other characteristics of singularities. Many interesting questions related to the geometry of morsifications still remain open, e.g., the problem of the existence of morsifications for arbitrary singularities (see \[23\]), or the relation between different morsifications of singularities of the same topological type and mutational equivalence of quivers (see \[22\]). The present paper addresses the problem of isotopy classification of morsifications of singularities and completely solves the problem for singularities combined of three smooth real branches. Note that for simple ADE singularities the classification of morsifications is well known (see, for example, \[19\]).
In Section 1 we introduce dessins, a combinatorial tool encoding the geometry of trigonal curves lying on ruled surfaces. We restrain ourselves to the case of real trigonal curves lying on the Hirzebruch surfaces $\Sigma_n$ having 3 reducible real components and $3n$ non-degenerated double points. We refer to such curves as trigonal morsifications. They represent morsifications of quasihomogeneous singularities of weight $(3, 3n)$ (the union of three smooth real branches intersecting each other with multiplicity $n$) in the real plane. We state combinatorial properties of the dessins associated to trigonal morsifications.

In Section 2 we introduce wiring diagrams, a sequential representation of the projective equivalence class of a trigonal morsification. We use wiring diagrams to state Theorem 2.6, which allow us to describe, in an inductive (on $n$) manner, all projective equivalence classes of trigonal morsifications. This description leads to the isotopy classification of trigonal morsifications.

Additionally, we proved unicity for the rigid isotopy class of trigonal morsifications on $\Sigma_n$ whose wiring diagram has a maximal number of consecutive equal entries. We introduce Reidemeister moves as a topological operation on the real part of a morsification and we prove that such operation can be realized algebraically, and that for every $n$, there exists a unique isotopy class of trigonal morsification up to Reidemeister moves.

The Appendix A, by E. Shustin, is devoted to the semiquasihomogeneous case. We prove that any morsification of a totally real semiquasihomogeneous singularity of weight $(3, 3n)$ (the union of three smooth real branches intersecting each other with multiplicity $n$) can be realized (up to isotopy) by a polynomial consisting of the restriction of the equation to the Newton diagram and adding monomials under the Newton diagram.

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1. Trigonal curves and dessins

In this section we introduce trigonal curves and dessins, which are the principal tool we use in order to study trigonal morsifications on the Hirzebruch surfaces $\Sigma_n$. The content of this chapter is based on the book [2] and the article [3].

1.1. Ruled surfaces and trigonal curves.

1.1.1. Basic definitions. A compact complex surface $\Sigma$ is a (geometrically) ruled surface over a curve $B$ if $\Sigma$ is endowed with a projection $\pi : \Sigma \rightarrow B$ of fiber $\mathbb{CP}^1$ as well as a special section $E$.

Definition 1.1. A reducible trigonal curve is a curve $C$, lying in a ruled surface $\Sigma$ such that $C$ contains neither the exceptional section $E$ nor a fiber as component, and the restriction $\pi|_C : C \rightarrow B$ is a degree 3 map.

A trigonal curve $C \subset \Sigma$ is proper if it does not intersect the exceptional section $E$. A singular fiber of a trigonal curve $C \subset \Sigma$ is a fiber $F$ of $\Sigma$ intersecting $C \cup E$ geometrically in less than 4 points.

1.1.2. Deformations. We are interested in the study of trigonal curves up to deformation. In the real case, we consider the curves up to equivariant deformation (with respect to the action of the complex conjugation, cf. 1.1.6).

In the Kodaira–Spencer sense, a deformation of the quintuple $(\pi : \Sigma \rightarrow B, E, C)$ refers to an analytic space $X \rightarrow S$ fibered over an marked open disk $S \ni o$ endowed
with analytic subspaces $B,E,C \subset X$ such that for every $s \in S$, the fiber $X_s$ is diffeomorphic to $\Sigma$ and the intersections $B_s := X_s \cap B, E_s := X_s \cap E$ and $C_s := X_s \cap C$ are diffeomorphic to $B, E$ and $C$, respectively, and there exists a map $\pi_s : X_s \rightarrow B_s$ making $X_s$ a geometrically ruled surface over $B_s$ with exceptional section $E_s$, such that the diagram in Figure 1 commutes and $(\pi_o : X_o \rightarrow B_o, E_o, C_o) = (\pi : \Sigma \rightarrow B, E, C)$.

\[ 
\begin{array}{ccc}
E_s & \iff & E \\
\downarrow & & \downarrow \\
C_s & \iff & C \\
\pi_s|_{C_s} & \downarrow & \pi \\
B_s & \iff & B \\
\end{array}
\]

**Figure 1.** Commuting diagram of morphisms and diffeomorphisms of the fibers of a deformation.

**Definition 1.2.** An elementary deformation of a trigonal curve $C \subset \Sigma \rightarrow B$ is a deformation of the quintuple $(\pi : \Sigma \rightarrow B, E, C)$ in the Kodaira-Spencer sense.

An elementary deformation $X \rightarrow S$ is equisingular if for every $s \in S$ there exists a neighborhood $U_s \subset S$ of $s$ such that for every singular fiber $F$ of $C$, there exists a neighborhood $V_{\pi(F)} \subset B$ of $\pi(F)$, where $\pi(F)$ is the only point with a singular fiber for every $t \in U_s$. An elementary deformation over $D^2$ is a degeneration or perturbation if the restriction to $D^2 \setminus \{0\}$ is equisingular and for a set of singular fibers $F_t$ there exists a neighborhood $V_{\pi(F_t)} \subset B$ where there are no points with a singular fiber for every $t \neq 0$. In this case we say that $C_t$ degenerates to $C_0$ or $C_0$ is perturbed to $C_t$, for $t \neq 0$.

1.1.3. **Weierstraß equations.** For a trigonal curve, the Weierstraß equations are an algebraic tool which allows us to study the behavior of the trigonal curve with respect to the zero section and the exceptional one. They give rise to an auxiliary morphism of $j$-invariant type, which plays an intermediary role between trigonal curves and dessins. Let $C \subset \Sigma \rightarrow B$ be a proper trigonal curve. Mapping a point $b \in B$ of the base to the barycenter of the points in $C \cap F_B$ (weighted according to their multiplicity) defines a section $B \rightarrow Z \subset \Sigma$ called the zero section; it is disjoint from the exceptional section $E$.

The surface $\Sigma$ can be seen as the projectivization of a rank 2 vector bundle, which splits as a direct sum of two line bundles such that the zero section $Z$ corresponds to the projectivization of $\mathcal{Y}$, one of the terms of this decomposition. In this context, the trigonal curve $C$ can be described by a Weierstraß equation, which in suitable affine charts has the form

\[ x^3 + g_2x + g_3 = 0, \]

where $g_2, g_3$ are sections of $\mathcal{Y}^2, \mathcal{Y}^3$ respectively, and $x$ is an affine coordinate such that $Z = \{x = 0\}$ and $E = \{x = \infty\}$. For this construction, we can identify $\Sigma \setminus B$ with the total space of $\mathcal{Y}$ and take $x$ as a local trivialization of this bundle. Nonetheless, the sections $g_2, g_3$ are globally defined. The line bundle $\mathcal{Y}$ is determined by $C$. The sections $g_2, g_3$ are determined up to change of variable defined by $$(g_2, g_3) \rightarrow (s^2g_2, s^3g_3), \ s \in H^0(B, \mathcal{O}_B).$$
Hence, the singular fibers of the trinodal curve $C$ correspond to the points where the equation (1) has multiple roots, i.e., the zeros of the discriminant section
\[ \Delta := -4g_2 \lambda - 27g_3 \in H^0(B, \mathcal{O}_B(\lambda^2)). \]
A Nagata transformation over a point $b \in B$ changes the line bundle $\mathcal{O}$ to $\mathcal{O} \otimes \mathcal{O}_B(b)$ and the sections $g_2$ and $g_3$ to $s^2g_2$ and $s^3g_3$, where $s \in H^0(B, \mathcal{O}_B)$ is any holomorphic function having a zero at $b$.

**Definition 1.3.** Let $C$ be a non-singular trinodal curve with Weierstraß model determined by the sections $g_2$ and $g_3$ as in (1). The trinodal curve $C$ is **almost generic** if every singular fiber corresponds to a simple root of the determinant section $\Delta = -4g_2 \lambda - 27g_3$ which is not a root of $g_2$ nor of $g_3$. The trinodal curve $C$ is **generic** if it is almost generic and the sections $g_2$ and $g_3$ have only simple roots.

**1.1.4. The $j$-invariant.** The $j$-invariant describes the relative position of four points in the complex projective line $\mathbb{CP}^1$. We describe some properties of the $j$-invariant in order to use them in the description of the dessins.

**Definition 1.4.** Let $z_1, z_2, z_3, z_4 \in \mathbb{CP}^1$. The **$j$-invariant** of a set $\{z_1, z_2, z_3, z_4\}$ is given by
\[ j(z_1, z_2, z_3, z_4) = \frac{4(\lambda^2 - \lambda + 1)^3}{27\lambda^2(\lambda - 1)^2}, \]
where $\lambda$ is the cross-ratio of the quadruple $(z_1, z_2, z_3, z_4)$ defined as
\[ \lambda(z_1, z_2, z_3, z_4) = \frac{z_1 - z_3}{z_2 - z_3} : \frac{z_1 - z_4}{z_2 - z_4}. \]

The cross-ratio depends on the order of the points while the $j$-invariant does not. Since the cross-ratio $\lambda$ is invariant under Möbius transformations, so is the $j$-invariant. When two points $z_1, z_2$ coincide, the cross-ratio $\lambda$ equals either 0, 1 or $\infty$, and the $j$-invariant equals $\infty$.

Let us consider a polynomial $z^3 + g_2z + g_3$. We define the $j$-invariant $j(z_1, z_2, z_3)$ of its roots $z_1, z_2, z_3$ as $j(z_1, z_2, z_3, \infty)$. If $\Delta = -4g_2 \lambda - 27g_3$ is the discriminant of the polynomial, then
\[ j(z_1, z_2, z_3, \infty) = \frac{-4g_2^3}{\Delta}. \]

A subset $A$ of $\mathbb{CP}^1$ is real if $A$ is invariant under the complex conjugation. We say that $A$ has a nontrivial symmetry if there is a nontrivial permutation of its elements which extends to a linear map $z \mapsto az + b$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$.

**Lemma 1.5** (2). The set $\{z_1, z_2, z_3\}$ of roots of the polynomial $z^3 + g_2z + g_3$ has a nontrivial symmetry if and only if its $j$-invariant equals 0 (for an order 3 symmetry) or 1 (for an order 2 symmetry).

**Proposition 1.6** (2). Assume that $j(z_1, z_2, z_3) \in \mathbb{R}$. Then, the following holds
- The $j$-invariant $j(z_1, z_2, z_3) < 1$ if and only if the points $z_1, z_2, z_3$ form an isosceles triangle. The special angle seen as a function of the $j$-invariant is a increasing monotone function. This angle tends to 0 when $j$ tends to $-\infty$, equals $\pi$ at $j = 0$ and tends to $\pi$ when $j$ approaches 1.
- The $j$-invariant $j(z_1, z_2, z_3) \geq 1$ if and only if the points $z_1, z_2, z_3$ are collinear. The ratio between the lengths of the smallest segment and the longest segment $\frac{z_1z_3}{z_2}$ seen as a function of the $j$-invariant is a decreasing monotone function. This ratio equals 1 when $j$ equals 1, and 0 when $j$ approaches $\infty$. 

1.1.5. The \( j \)-invariant of a trigonal curve. Let \( C \) be a proper trigonal curve. We use the \( j \)-invariant defined for triples of complex numbers in order to define a meromorphic map \( j_C \) on the base curve \( B \). The map \( j_C \) encodes the topology of the trigonal curve \( C \). The map \( j_C \) is called the \( j \)-invariant of the curve \( C \) and provides a correspondence between trigonal curves and dessins.

**Definition 1.7.** For a proper trigonal curve \( C \), we define its \( j \)-invariant \( j_C \) as the analytic continuation of the map

\[
\begin{align*}
B^\# & \longrightarrow \mathbb{C} \\
b & \longmapsto j\text{-invariant of } C \cap F_b^0 \subset F_b^0 \cong \mathbb{C}.
\end{align*}
\]

We call the trigonal curve \( C \) isotrivial if its \( j \)-invariant is constant.

If a proper trigonal curve \( C \) is given by a Weierstraß equation of the form \([1]\), then

\[
(4) \quad j_C = -\frac{4g_3^3}{\Delta}, \quad \text{where } \Delta = -4g_2^3 - 27g_3^2.
\]

**Theorem 1.8** ([2]). Let \( B \) be a compact curve and \( j: B \longrightarrow \mathbb{C}P^1 \) a non-constant meromorphic map. Up to Nagata equivalence, there exists a unique trigonal curve \( C \subset \Sigma \longrightarrow B \) such that \( j_C = j \).

Following the proof of the theorem, \( j_B \longrightarrow \mathbb{C}P^1 \) leads to a unique minimal proper trigonal curve \( C_j \), in the sense that any other trigonal curve with the same \( j \)-invariant can be obtained by positive Nagata transformations from \( C_j \).

An equisingular deformation \( C_s, s \in S, \) of \( C \) leads to an analytic deformation of the couple \((B, j_C)\).

**Corollary 1.9** ([2]). Let \((B, j)\) be a couple, where \( B \) is a compact curve and \( j: B \longrightarrow \mathbb{C}P^1 \) is a non-constant meromorphic map. Then, any deformation of \((B, j)\) results in a deformation of the minimal curve \( C_j \subset \Sigma \longrightarrow B \) associated to \( j \).

The \( j \)-invariant of a generic trigonal curve \( C \subset \Sigma \longrightarrow B \) has degree \( \deg(j_C) = 6d \), where \( d = -E^2 \). A positive Nagata transformation increases \( d \) by one while leaving \( j_C \) invariant. The \( j \)-invariant of a generic trigonal curve \( C \) has a ramification index equal to 3, 2 or 1 at every point \( b \in B \) such that \( j_C(b) \) equals 0, 1 or \( \infty \), respectively. We can assume, up to perturbation, that every critical value of \( j_C \) is simple. In this case we say that \( j_C \) has a generic branching behavior.

1.1.6. Real trigonal curves. We are mostly interested in real trigonal curves. A real structure on a complex variety \( X \) is an anti-holomorphic involution \( c: X \longrightarrow X \).

We define a real variety as a couple \((X, c)\), where \( c \) is a real structure on a complex variety \( X \). We denote by \( X_\mathbb{R} \) the fixed point set of the involution \( c \) and we call \( X_\mathbb{R} \) the set of real points of \( c \).

A geometrically ruled surface \( \pi: \Sigma \longrightarrow B \) is real if there exist real structures \( c_\Sigma: \Sigma \longrightarrow \Sigma \) and \( c_B: B \longrightarrow B \) compatible with the projection \( \pi \), i.e., such that \( \pi \circ c_\Sigma = c_B \circ \pi \). We assume the exceptional section is real in the sense that it is invariant by conjugation, i.e., \( c_\Sigma(E) = E \). Put \( \pi_\mathbb{R} := \pi|_{\Sigma_\mathbb{R}}: \Sigma_\mathbb{R} \longrightarrow B_\mathbb{R} \). Since the exceptional section is real, the fixed point set of every fiber is not empty, implying that the real structure on the fiber is isomorphic to the standard complex conjugation on \( \mathbb{C}P^1 \). Hence all the fibers of \( \pi_\mathbb{R} \) are isomorphic to \( \mathbb{R}P^1 \). Thus, the map \( \pi_\mathbb{R} \) establishes a bijection between the connected components of the real part \( \Sigma_\mathbb{R} \) of the surface \( \Sigma \) and the connected components of the real part \( B_\mathbb{R} \) of the curve \( B \). Every connected component of \( \Sigma_\mathbb{R} \) is homeomorphic either to a torus or to a Klein bottle.
If $\Sigma = \mathbb{P}(\mathcal{Y})$, with $\mathcal{Y} \in \text{Pic}(B)$, we put $\mathcal{Y}_i := \mathcal{Y}_i|_{B_i}$ for every connected component $B_i$ of $B_R$. Hence $\Sigma_i := \Sigma_i|_{B_i}$ is orientable if and only if $\mathcal{Y}_i$ is topologically trivial, i.e., its first Stiefel-Whitney class $w_1(\mathcal{Y}_i)$ is zero.

**Definition 1.10.** A real trigonal curve $C$ is a trigonal curve contained in a real ruled surface $(\Sigma, c_\Sigma) \rightarrow (B, c_B)$ such that $C$ is $c_\Sigma$-invariant, i.e., $c_\Sigma(C) = C$.

If a real trigonal curve is proper, then $\mathcal{Y}$ is real as well as its $j$-invariant, seen as a morphism $j_C: (B, c_B) \rightarrow (\mathbb{CP}^1, z \mapsto \bar{z})$, where $z \mapsto \bar{z}$ denotes the standard complex conjugation on $\mathbb{CP}^1$. In addition, the sections $g_2$ and $g_3$ can be chosen real.

Let us consider the restriction $\pi|_{C_B}: C_R \rightarrow B_R$. We put $C_i := \pi|_{C_B}^{-1}(B_i)$ for every connected component $B_i$ of $B_R$. We say that $B_i$ is hyperbolic if $\pi|_{C_i}: C_i \rightarrow B_i$ has generically a fiber with three elements. The trigonal curve $C$ is hyperbolic if its real part is non-empty and all the connected components of $B_R$ are hyperbolic.

1.2. **Dessins.** The dessins d’enfants were introduced by A. Grothendieck (cf. [7]) in order to study the action of the absolute Galois group. We use a modified version of dessins d’enfants which was proposed by S. Orevkov [8].

1.2.1. **Trichotomic graphs.** Let $S$ be a compact connected topological surface. A graph $D$ on the surface $S$ is a graph embedded into the surface and considered as a subset $D \subset S$. We denote by $\text{Cut}(D)$ the cut of $S$ along $D$, i.e., the disjoint union of the closure of connected components of $S \setminus D$.

**Definition 1.11.** A trichotomic graph on a compact surface $S$ is an embedded finite directed graph $D \subset S$ decorated with the following additional structures (referred to as colorings of the edges and vertices of $D$, respectively):

- every edge of $D$ is color solid, bold or dotted,
- every vertex of $D$ is black ($\bullet$), white ($\circ$), cross ($\times$) or monochrome (the vertices of the first three types are called essential),

and satisfying the following conditions:

1. $\partial S \subset D$,
2. every essential vertex is incident to at least 2 edges,
3. every monochrome vertex is incident to at least 3 edges,
4. the orientations of the edges of $D$ form an orientation of the boundary $\partial \text{Cut}(D)$ which is compatible with an orientation on $\text{Cut}(D)$,
5. all edges incident to a monochrome vertex are of the same color,
6. $\times$-vertices are incident to incoming dotted edges and outgoing solid edges,
7. $\bullet$-vertices are incident to incoming solid edges and outgoing bold edges,
8. $\circ$-vertices are incident to incoming bold edges and outgoing dotted edges.

Let $D \subset S$ be a trichotomic graph. A region $R$ is an element of of $\text{Cut}(D)$. The boundary $\partial R$ of $R$ contains $n = 3k$ essential vertices. A region with $n$ essential vertices on its boundary is called an $n$-gonal region. We denote by $D_{\text{solid}}, D_{\text{bold}}, D_{\text{dotted}}$ the monochrome parts of $D$, i.e., the sets of vertices and edges of the specific color. On the set of vertices of a specific color, we define the relation $u \preceq v$ if there is a monochrome path from $u$ to $v$, i.e., a path formed entirely of edges of the same color. We call the graph $D$ admissible if the relation $\preceq$ is a partial order, equivalently, if there are no directed monochrome cycles.

**Definition 1.12.** A trichotomic graph $D$ is a dessin if

1. $D$ is admissible;
2. every trigonal region of $D$ is homeomorphic to a disk.

The orientation of the graph $D$ is determined by the pattern of colors of the vertices on the boundary of every region.
1.2.2. Complex and real dessins. Let $S$ be an orientable surface. Every orientation of $S$ induces a chessboard coloring of $\text{Cut}(D)$, i.e., a function on $\text{Cut}(D)$ determining if a region $R$ endowed with the orientation set by $D$ coincides with the orientation of $S$.

**Definition 1.13.** A real trichotomic graph on a real compact surface $(S, c)$ is a trichotomic graph $D$ on $S$ which is invariant under the action of $c$. Explicitly, every vertex $v$ of $D$ has as image $c(v)$ a vertex of the same color; every edge $e$ of $D$ has as image $c(e)$ an edge of the same color.

Let $D$ be a real trichotomic graph on $(S, c)$. Put $\mathcal{S} := S/c$ as the quotient surface and put $\mathcal{D} \subset \mathcal{S}$ as the image of $D$ by the quotient map $S \rightarrow S/c$. The graph $\mathcal{D}$ is a well defined graph on the surface $S/c$.

In the inverse sense, let $S$ be a compact surface, which can be non-orientable or can have non-empty boundary. Let $D \subset S$ be a trichotomic graph on $S$. Consider its complex double covering $\hat{S} \rightarrow S$ (cf. [1] for details), which has a real structure given by the deck transformation, and put $\hat{D} \subset \hat{S}$ the inverse image of $D$. The graph $\hat{D}$ is a graph on $\hat{S}$ invariant by the deck transformation. We use these constructions in order to identify real trichotomic graphs on real surfaces with their images on the quotient surface.

**Proposition 1.14 ([2]).** Let $S$ be a compact surface. Given a trichotomic graph $D \subset S$, then its oriented double covering $\hat{D} \subset \hat{S}$ is a real trichotomic graph. Moreover, $\hat{D} \subset \hat{S}$ is a dessin if and only if so is $D \subset S$. Conversely, if $(S, c)$ is a real compact surface and $D \subset S$ is a real trichotomic graph, then its image $\mathcal{D}$ in the quotient $\mathcal{S} := S/c$ is a trichotomic graph. Moreover, $\mathcal{D} \subset \mathcal{S}$ is a dessin if and only if so is $D \subset S$.

**Definition 1.15.** Let $D$ be a dessin on a compact surface $S$. Let us denote by $\text{Ver}(D)$ the set of vertices of $D$. For a vertex $v \in \text{Ver}(D)$, we define the index $\text{Ind}(v)$ of $v$ as half of the number of incident edges of $\bar{v}$, where $\bar{v}$ is a preimage of $v$ by the double complex cover of $S$ as in Proposition 1.14.

A vertex $v \in \text{Ver}(D)$ is **singular** if

- $v$ is black and $\text{Ind}(v) \not\equiv 0 \mod 3$,
- or $v$ is white and $\text{Ind}(v) \not\equiv 0 \mod 2$,
- or $v$ has color $\times$ and $\text{Ind}(v) \geq 2$.

We denote by $\text{Sing}(D)$ the set of singular vertices of $D$. A dessin is **non-singular** if none of its vertices is singular.

**Definition 1.16.** Let $B$ be a complex curve and let $j : B \rightarrow \mathbb{CP}^1$ a non-constant meromorphic function, in other words, a ramified covering of the complex projective line. The dessin $D := \text{Dssn}(j)$ associated to $j$ is the graph given by the following construction:

- as a set, the dessin $D$ coincides with $j^{-1}(\mathbb{RP}^1)$, where $\mathbb{RP}^1$ is the fixed point set of the standard complex conjugation in $\mathbb{CP}^1$;
- black vertices (●) are the inverse images of $0$;
- white vertices (○) are the inverse images of $1$;
- vertices of color $\times$ are the inverse images of $\infty$;
- monochrome vertices are the critical points of $j$ with critical value in $\mathbb{CP}^1 \setminus \{0, 1, \infty\}$;
- solid edges are the inverse images of the interval $[\infty, 0]$;
- bold edges are the inverse images of the interval $[0, 1]$;
- dotted edges are the inverse images of the interval $[1, \infty]$;
- orientation on edges is induced from an orientation of $\mathbb{RP}^1$. 

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**TRIGONAL MORSIFICATIONS ON HIRZEBRUCH SURFACES APPENDIX BY E. SHUSTIN**

7
Lemma 1.17 [2]. Let \( S \) be an oriented connected closed surface. Let \( j : S \to \mathbb{C}\mathbb{P}^1 \) a ramified covering map. The trichotomic graph \( D = \text{Dssn}(j) \subset S \) is a dessin. Moreover, if \( j \) is real with respect to an orientation-reversing involution \( c : S \to S \), then \( D \) is \( c \)-invariant.

Let \((S,c)\) be a compact real surface. If \( j : (S,c) \to (\mathbb{C}\mathbb{P}^1,z \to \bar{z}) \) is a real map, we define \( \text{Dssn}(j) := \text{Dssn}(j)/c \subset S/c \).

Theorem 1.18 [2]. Let \( S \) be an oriented connected closed surface (and let \( c : S \to S \) an orientation-reversing involution). A (real) trichotomic graph \( D \subset S \) is a (real) dessin if and only if \( D = \text{Dssn}(j) \) for a (real) ramified covering \( j : S \to \mathbb{C}\mathbb{P}^1 \).

Moreover, \( j \) is unique up to homotopy in the class of (real) ramified coverings with dessin \( D \).

The last theorem together with the Riemann existence theorem provides the next corollaries, for the complex and real settings.

Corollary 1.19 [2]. Let \( D \subset S \) be a dessin on a compact closed orientable surface \( S \). Then there exists a complex structure on \( S \) and a holomorphic map \( j : S \to \mathbb{C}\mathbb{P}^1 \) such that \( \text{Dssn}(j) = D \). Moreover, this structure is unique up to deformation of the complex structure on \( S \) and the map \( j \) in the Kodaira-Spencer sense.

Corollary 1.20 [2]. Let \( D \subset S \) be a dessin on a compact closed surface \( S \). Then there exists a complex structure on its double cover \( \tilde{S} \) and a holomorphic map \( \tilde{j} : \tilde{S} \to \mathbb{C}\mathbb{P}^1 \) such that \( \tilde{j} \) is real with respect to the real structure \( c \) of \( \tilde{S} \) and \( \text{Dssn}(\tilde{j}) = D \). Moreover, this structure is unique up to equivariant deformation of the complex structure on \( S \) and the map \( j \) in the Kodaira-Spencer sense.

1.2.3. Deformations of dessins. In this section we describe the notions of deformations which allow us to associate classes of non-isotrivial trigonal curves and classes of dessins, up to deformations and equivalences that we explicit.

Definition 1.21. A deformation of coverings is a homotopy \( S \times [0,1] \to \mathbb{C}\mathbb{P}^1 \) within the class of (equivariant) ramified coverings. The deformation is simple if it preserves the multiplicity of the inverse images of 0, 1, \( \infty \) and of the other real critical values.

Any deformation is locally simple except for a finite number of values \( t \in [0,1] \).

Proposition 1.22 [2]. Let \( j_0, j_1 : S \to \mathbb{C}\mathbb{P}^1 \) be (c-equivariant) ramified coverings. They can be connected by a simple (equivariant) deformation if and only their dessins \( D(j_0) \) and \( D(j_1) \) are isotopic (respectively, \( D_c(j_0) \) and \( D_c(j_1) \)).

Definition 1.23. A deformation \( j_t : S \to \mathbb{C}\mathbb{P}^1 \) of ramified coverings is equisingular if the union of the supports

\[
\bigcup_{t \in [0,1]} \text{supp}\{ (j^*_t(0) \mod 3) + (j^*_t(1) \mod 2) + j^*_t(\infty) \}
\]

considered as a subset of \( S \times [0,1] \) is an isotopy. Here \( * \) denotes the divisorial pullback of a map \( \phi : S \to S' \) at a point \( s' \in S' \):

\[
\phi^*(s') = \sum_{s \in \phi^{-1}(s')} r_s s,
\]

where \( r_s \) if the ramification index of \( \phi \) at \( s \in S \).
A dessin $D_1 \subset S$ is called a perturbation of a dessin $D_0 \subset S$, and $D_0$ is called a degeneration of $D_1$, if for every vertex $v \in \text{Ver}(D_0)$ there exists a small neighboring disk $U_v \subset S$ such that $D_0 \cap U_v$ only has edges incident to $v$ and $D_1 \cap U_v$ contains essential vertices of at most one color.

**Theorem 1.24 (2).** Let $D_0 \subset S$ be a dessin, and let $D_1$ be an admissible perturbation. Then there exists a map $j_1 : S \to \mathbb{CP}^1$ such that

1. $D_0 = \text{Dssn}(j_0)$ and $D_1 = \text{Dssn}(j_1)$;
2. $j_1|_{S \setminus \cup_v v_c} = j_0|_{S \setminus \cup_v v_c}$ for every $t, t'$;
3. the deformation restricted to $S \times [0, 1]$ is simple.

**Corollary 1.25 (2).** Let $S$ be a complex compact curve, $j : S \to \mathbb{CP}^1$ a non-constant holomorphic map, and let $\text{Dssn}_z(j) = D_0, D_1, \ldots, D_n$ be a chain of dessins in $S$ such that for $i = 1, \ldots, n$ either $D_i$ is a perturbation of $D_{i-1}$, or $D_i$ is a degeneration of $D_{i-1}$, or $D_i$ is isomorphic to $D_{i-1}$. Then there exists a piecewise-analytic deformation $j_t : S_t \to \mathbb{CP}^1$, $t \in [0, 1]$, of $j_0 = j$ such that $\text{Dssn}_z(j_t) = D_n$.

**Corollary 1.26 (2).** Let $(S, c)$ be a real compact curve, $j : (S, c) \to (\mathbb{CP}^1, \bar{z} \mapsto z)$ a real non-constant holomorphic map, and let $\text{Dssn}_z(j) = D_0, D_1, \ldots, D_n$ be a chain of real dessins in $(S, c)$ such that for $i = 1, \ldots, n$ either $D_i$ is a equivariant perturbation of $D_{i-1}$, or $D_i$ is a equivariant degeneration of $D_{i-1}$, or $D_i$ is equivariantly isomorphic to $D_{i-1}$. Then there is a piecewise-analytic real deformation $j_t : (S_t, c_t) \to (\mathbb{CP}^1, \bar{z})$, $t \in [0, 1]$, of $j_0 = j$ such that $\text{Dssn}_z(j_t) = D_n$.

Due to Theorem 1.24 the deformation $j_t$ given by Corollaries 1.25 and 1.26 is equisingular in the sense of Definition 1.23 if and only if all perturbations and degenerations of the dessins on the chain $D_0, D_1, \ldots, D_n$ are equisingular.

### 1.3. Trigonal curves and their dessins.

In this section we describe an equivalence between dessins.

#### 1.3.1. Correspondence theorems.

Let $C \subset \Sigma \to B$ be a non-isotrivial proper trigonal curve. We associate to $C$ the dessin corresponding to its $j$-invariant $\text{Dssn}(C) := \text{Dssn}(j_C) \subset B$. In the case when $C$ is a real trigonal curve we associate to $C$ the real dessin corresponding to its $j$-invariant, $\text{Dssn}_r(C) := \text{Dssn}(j_C) \subset B/e_B$, where $e_B$ is the real structure of the base curve $B$.

So far, we have focused on one direction of the correspondences: we start with a trigonal curve $C$, consider its $j$-invariant and construct the dessin associated to it. Now, we study the opposite direction. Let us consider a dessin $D$ on a topological orientable closed surface $S$. By Corollary 1.19 there exist a complex structure $B$ on $S$ and a holomorphic map $j_D : B \to \mathbb{CP}^1$ such that $\text{Dssn}(j_D) = D$. By Theorem 1.8 and Corollary 1.9 there exists a trigonal curve $C$ having $j_D$ as $j$-invariant; such a curve is unique up to deformation in the class of trigonal curves with fixed dessin. Moreover, due to Corollary 1.26 any sequence of isotopies, perturbations and degenerations of dessins gives rise to a piecewise-analytic deformation of trigonal curves, which is singular if and only if all perturbations and degenerations are.

In the real framework, let $(S, c)$ a compact close oriented topological surface endowed with an orientation-reversing involution. Let $D$ be a real dessin on $(S, c)$. By Corollary 1.20 there exists a real structure $(B, e_B)$ on $(S, c)$ and a real map $j_D : (B, e_B) \to (\mathbb{CP}^1, \bar{z} \mapsto z)$ such that $\text{Dssn}_r(j_D) = D$. By Theorem 1.8 Corollary 1.9 and the remarks made in Section 1.1.6 there exists a real trigonal curve $C$ having $j_D$ as $j$-invariant; such a curve is unique up to equivariant deformation in the class of real trigonal curves with fixed dessin. Furthermore, due to Corollary 1.26 any sequence of isotopies, perturbations and degenerations of dessins
(a) Monochrome modification

(b) Creating/destroying a bridge

(c) •-in/•-out

(d) •-in/•-out

(e) ○-in/○-out

(f) ○-in/○-out

Figure 2. Elementary moves.

gives rise to a piecewise-analytic equivariant deformation of real trigonal curves, which is equisingular if and only if all perturbations and degenerations are.

Definition 1.27. A dessin is reduced if
- for every $v$ •-vertex one has $\text{Ind} v \leq 3$,
- for every $v$ ○-vertex one has $\text{Ind} v \leq 2$,
- every monochrome vertex is real and has index 2.

A reduced dessin is generic if all its •-vertices and ○-vertices are non-singular and all its x-vertices have index 1.

Any dessin admits an equisingular perturbation to a reduced dessin. The vertices with excessive index (i.e., index greater than 3 for •-vertices or than 2 for ○-vertices) can be reduced by introducing new vertices of the same color.

In order to define an equivalence relation of dessins, we introduce elementary moves. Consider two reduced dessins $D, D' \subset S$ such that they coincide outside a closed disk $V \subset S$. If $V$ does not intersect $\partial S$ and the graphs $D \cap V$ and $D' \cap V$ are as shown in Figure 2(a), then we say that performing a monochrome modification on the edges intersecting $V$ produces $D'$ from $D$, or vice versa. This is the first type of elementary moves. Otherwise, the boundary component inside $V$ is shown in light gray. In this setting, if the graphs $D \cap V$ and $D' \cap V$ are as shown in one of the subfigures in Figure 2, we say that performing an elementary move of the corresponding type on $D \cap V$ produces $D'$ from $D$, or vice versa.

Definition 1.28. Two reduced dessins $D, D' \subset S$ are elementary equivalent if, after a (preserving orientation, in the complex case) homeomorphism of the underlying surface $S$ they can be connected by a sequence of isotopies and elementary moves between dessins, as described in Figure 2.

This definition is meant so that two reduced dessins are elementary equivalent if and only if they can be connected up to homeomorphism by a sequence of isotopies, equisingular perturbations and degenerations.

The following theorems establish the equivalences between the deformation classes of trigonal curves we are interested in and elementary equivalence classes of certain dessins. We use these links to obtain different classifications of curves via the combinatorial study of dessins.
Theorem 1.29 ([2]). There is a one-to-one correspondence between the set of
equivariant equisingular deformation classes of non-isotrivial proper real trigonal
curves $C \subset \Sigma \rightarrow (B, c)$ with $\bar{A}$ type singular fibers only and the set of elementary
equivalence classes of reduced real dessins $D \subset B/c$.

This correspondence can be extended to trigonal curves with more general sin-
gular fibers (see [2]).

Definition 1.30. Let $C \subset \Sigma \rightarrow B$ be a proper trigonal curve. We define the
degree of the curve $C$ as $\deg(C) := -3E^2$ where $E$ is the exceptional section of $\Sigma$.
For a dessin $D$, we define its degree as $\deg(D) = \deg(C)$ where $C$ is a minimal
proper trigonal curve such that $\text{Dssn}(C) = D$.

1.3.2. Real generic curves. Let $C$ be a generic real trigonal curve and let $D := 
\text{Dssn}_{\text{re}}(C)$ be a generic dessin. The real part of $D \subset S$ is the intersection $D \cap \partial S$.
For a specific color $\ast \in \{\text{solid, bold, dotted}\}$, $D_\ast$ is the subgraph of the corresponding
color and its adjacent vertices. The components of $D_\ast \cap \partial S$ are either components
of $\partial S$, called monochrome components of $D$, or segments, called maximal mono-
chrome segments of $D$. We call these monochrome components or segments even
or odd according to the parity of the number of $\ast$-vertices they contain.

Moreover, we refer to the dotted monochrome components as hyperbolic compo-
nents.

1.4. Toiles. Within the moduli space of trigonal curves of fixed degree, generic
trigonal curves are smooth and the discriminant (i.e., the set of singular trigonal
curves, cf. [3]) has a stratification which we can describe by means of dessins.
Singular proper trigonal curves have singular dessins and the singular points are
represented by singular vertices. A generic singular trigonal curve $C$ has exactly
one singular point, which is a non-degenerate double point (node). Moreover, if $C$
is a proper trigonal curve, then the double point on it is represented by a $\times$-vertex
of index 2 on its dessin. In addition, if $C$ has a real structure, the double point
is real and so is its corresponding vertex, leading to the cases where the $\times$-vertex of
index 2 has dotted real edges (representing the intersection of two real branches) or
has solid real edges (representing one isolated real point, which is the intersection
of two complex conjugated branches).

Definition 1.31. Let $D \subset S$ be a dessin on a compact surface $S$. A nodal vertex
(node) of $D$ is a $\times$-vertex of index 2. The dessin $D$ is called nodal if all its singular
vertices are nodal vertices. We call a toile a non-hyperbolic real nodal dessin on
$(\mathbb{C}P^1, z \mapsto \bar{z})$.

Since a real dessin on $(\mathbb{C}P^1, z \mapsto \bar{z})$ descends to the quotient, we represent toiles
on the disk.

In a real dessin, there are two types of real nodal vertices, namely, vertices
having either real solid edges and interior dotted edges, or dotted real edges and
interior solid edges. We call isolated nodes of a dessin $D$ those $\times$-vertices of index 2
corresponding to the former case and non-isolated nodes those corresponding to the
latter.

Definition 1.32. Given a dessin $D$, a subgraph $\Gamma \subset D$ is a cut if it consists of a
single interior edge connecting two real monochrome vertices. An axe is an interior
edge of a dessin connecting a $\times$-vertex of index 2 and a real monochrome vertex.

Let us consider a dessin $D$ lying on a surface $S$ having a cut or an axe $T$. Assume
that $T$ divides $S$ and consider the connected components $S_1$ and $S_2$ of $S \setminus T$. Then,
we can define two dessins $D_1$, $D_2$, each lying on the compact surface $\overline{S_i} \subset S$,
respectively for $i = 1, 2$, and determined by $D_i := (D \cap S_i) \cup \{T\}$. If $S \setminus T$ is
connected, we define the surface \( S' = (S \setminus T) \cup T_1 \cup T_2 / \varphi_1, \varphi_2 \), where \( \varphi : T' \to S \) is the inclusion of one copy \( T' \) of \( T \) into \( S \), and the dessin \( D' = (D \setminus T) \cup T_1 \cup T_2 / \varphi_1, \varphi_2 \).

By these means, a dessin having a cut or an axe determines either two other dessins of smaller degree or a dessin lying on a surface with a smaller fundamental group. Moreover, in the case of an axe, the resulting dessins have one singular vertex less. Considering the inverse process, we call a dessin \( D \) with itself along \( T \) or the gluing of \( D_1 \) and \( D_2 \) along \( T \) or the gluing of \( D' \) with itself along \( T_1 \) and \( T_2 \).

**Definition 1.33.** Let \( C \subseteq \Sigma \) be a nodal proper trigonal curve with nodes \( n_1, n_2, \ldots, n_l \). Consider a Weierstraß model of \( C \) determined by sections \( g_2 \) and \( g_3 \). We say that the trigonal curve \( C \) is *almost generic* if every singular fiber different from the fiber at \( \pi(n_i) \) corresponds to a simple root of the determinant section \( \Delta = -4g_2^3 - 27g_3^2 \) which is not a root of \( g_2 \) nor of \( g_3 \). The nodal trigonal curve \( C \) is *generic* if it is almost generic and the sections \( g_2 \) and \( g_3 \) only have simple roots.

**Definition 1.34.** Given a real dessin \( D \) and a vertex \( v \in \text{Ver}(D) \), we call the *depth* of \( v \) the minimal number \( n \) such that there exists an undirected inner chain \( v_0, \ldots, v_n \) in \( D \) from \( v_0 = v \) to a real vertex \( v_n \) and we denote the depth of \( v \) by \( \text{dp}(v) \). The depth of a dessin \( D \) is defined as the maximum of the depth of the black and white vertices of \( D \) and it is denoted by \( \text{dp}(D) \).

**Definition 1.35.** A *generalized cut* of a dessin \( D \) is an inner undirected chain formed entirely of inner edges of the same color, either dotted or solid, connecting two distinct real nodal or monochrome vertices.

Analogously to a cut, cutting a dessin \( D \subseteq S \) by a generalized cut produces two dessins of lower degree or a dessin of the same degree in a surface with a simpler topology, depending on whether the inner chain divides or not the surface \( S \).

**Proposition 1.36 ([5]).** Let \( D \subseteq \mathbb{D}^2 \) be a toile of degree greater than 3. Then, there exists a toile \( D' \) weakly equivalent to \( D \) such that either \( D' \) has depth 1 or \( D' \) has a generalized cut.

**Corollary 1.37.** Let \( D \) be as in Proposition 1.36. If there exists a toile \( D' \) weakly equivalent to \( D \) with depth 1, then \( D' \) can be chosen bridge-free.

**Proposition 1.38 ([5]).** Let \( D \) be a toile of degree at least 6 and depth at most 1. Then, there exists a toile \( D' \) weakly equivalent to \( D \) such that \( D' \) has a generalized cut. Moreover, if \( D \) has isolated real nodal \( \times \)-vertices, the generalized cut is dotted or a solid axe.

2. Trigonal morsifications on Hirzebruch surfaces

**Definition 2.1.** A *trigonal morsification* on \( \Sigma_n \) is a hyperbolic trigonal curve on \( \Sigma_n \) having as singular points exactly \( 3n \) real nodal points.

Henceforth, we only consider trigonal morsifications on the Hirzebruch surfaces \( \Sigma_n \).

2.1. Wiring diagrams. Given a morsification \( C \) and a real white vertex \( v \in \text{Dsn}(C) \), we associate to \( v \) a sequence \( (a_i)_{i=1}^{3n} \) in the following way. First, let us fix an orientation of the real part of the dessin. Then, enumerate accordingly the set of nodal \( \times \)-vertices as \( \{u_1, u_2, \ldots, u_{3n}\} \). Lastly, put the number \( a_i = 1 \) if the arc \((v, u_i)\) has an even number of white vertices, and \( a_i = 2 \) otherwise.

Since the elementary moves of dessins do not modify the parity of segment between nodal \( \times \)-vertices, this sequence is well defined in the equivalence class of dessins.
If \( n \) is odd, the sequence \((a_{3n-i})_{i=1}^{3n}\) corresponds to the opposite orientation. On the other hand, if \( n \) is even, the sequence \((a'_{i})_{i=1}^{3n}\), with \( a'_{i} = 3 - a_{3n-i} \), corresponds to the opposite orientation. We say that two sequences \((a_i)_{i=1}^{3n}\) and \((b_i)_{i=1}^{3n}\) are equivalent if they are equal, if \( b_i = 3 - a_i, i = 1, 2, \ldots, 3n \), if \( b_i = a_{3n-i}, i = 1, 2, \ldots, 3n \) or if \( b_i = 3 - a_{3n-i}, i = 1, 2, \ldots, 3n \).

We call the wiring diagram of \( D \) with respect to \( v \) the sequence \((a_i)_{i=1}^{3n}\) associated to \( v \) up to equivalence.

We put \( \sigma_m \in S_m \) as the permutation

\[
\sigma_m = \begin{pmatrix}
1 & 2 & \cdots & m-1 & m \\
2 & 3 & \cdots & m & 1
\end{pmatrix},
\]

and we define \( \sigma_{3n} \cdot (a_i)_{i=1}^{3n} \) as the sequence given by \((a_{\sigma_{3n}(i)})_{i=1}^{3n}\) for \( n \) even, or by the sequence

\[
(\sigma_{3n} \cdot (a_i)_{i=1}^{3n})_j = \begin{cases}
    a_{\sigma_{3n}(j)} & \text{if } j \neq 3n, \\
    3 - a_1 & \text{if } j = 3n,
\end{cases}
\]

for \( n \) odd.

We say that two sequences \((a_i)_{i=1}^{3n}, (b_i)_{i=1}^{3n}\) are projective equivalent if there exists \( k \in \mathbb{N} \) such that \((b_i)_{i=1}^{3n} = \sigma_k^{3n} \cdot (a_i)_{i=1}^{3n}\).

Lemma 2.2. Let \( C \) be a trigonal morsification and let \( D = D_{ssn}(C) \) be its dessin. If \( v, v' \) are real white vertices of \( D \), then, the wiring diagrams of \( D \) with respect to \( v \) and \( v' \) are projective equivalent.

Proof. Fix an orientation of \( \mathbb{R}^2 \), the boundary of \( D \). Let \( k \) and \( l \) be the number of nodal \( \times \)-vertices and white vertices, respectively, in the arc \((v, v')\). Let \((a_i)_{i=1}^{3n}, (a'_i)_{i=1}^{3n}\) the wiring diagrams of \( D \) with respect to \( v \) and \( v' \), respectively. If \( n \) is even, then \( a'_i = a_j \) if \( l \) is even or \( a'_i = 3 - a_j \) if \( l \) is odd, for \( 1 \leq j \leq 3n \). If \( n \) is odd, then \( a'_i = a_j \), \( 1 \leq j \leq 3n - k \) and \( a'_i = 3 - a_j \), \( 3n - k < j \) if \( l \) is even or \( a'_i = 3 - a_j \), \( 1 \leq j \leq 3n - k \) and \( a'_i = a_j \), \( 3n - k < j \) if \( l \) is odd. Therefore, we have that \((a_i)_{i=1}^{3n} = \sigma_k^{3n} (a_i)_{i=1}^{3n}\) or \((a_i)_{i=1}^{3n} = \sigma_k^{3n} (3 - a_i)_{i=1}^{3n}\), depending whether \( l \) is even or odd, respectively. \( \square \)

Definition 2.3. Given a morsification \( C \) on \( \Sigma_n \), we define as the wiring diagram of \( C \) the projective equivalence class of the wiring diagram of \( D_{ssn}(D) \) with respect to any real white vertex.

Given a sequence \((a_i)_{i=1}^{3n}\), we define its length vector as the vector \( o \) whose \( i \)-th entry \( o_i \) correspond to the length of the \( i \)-th maximal set of indexes \( I_i = \{i_0, i_0 + 1, \ldots, i_0 + o_i\} \) such that \( a_j = a_i, \forall j, l \in I_i \). The vector \( o \) is an ordered partition of \( 3n \). We denote by \( o(a_i) \) the entry in \( o \) corresponding to the maximal set of indexes containing \( i \).

Given a projective equivalence class of a sequence \((a_i)_{i=1}^{3n}\), we define its cyclic length vector as the length vector \( o \) of a projective equivalent sequence \((a'_i)_{i=1}^{3n}\) such that \( a_1 \neq a_{3n} \) if \( n \) is even, or such that \( a_1 = a_{3n} \) if \( n \) is odd. (If such a sequence does not exist, we put its cyclic length vector as \((3n)\)). We consider this vector \( o \) as an element of \( \mathbb{Z}^m / \langle \sigma_m, \tau_m \rangle \), where \( \tau_m \in S_m \) is the permutation

\[
\tau_m = \begin{pmatrix}
1 & 2 & \cdots & m-1 & m \\
m & m-1 & \cdots & 2 & 1
\end{pmatrix},
\]

so in that manner, it does not depend on the representative of the projective equivalent class.

Definition 2.4. Given a morsification \( C \) on \( \Sigma_n \), we define as the cyclic length vector of \( C \) the cyclic length vector of its wiring diagram.
Then, either \( \{ 1 \} \) ANDRÉS JARAMILLO PUENTES

Every projective equivalence class of sequences is determined by its cyclic length vector.

Proof. Let \( \mu \) be two sequences such that they have the same cyclic length vector \( \mu \). After a shifting \( \sigma_{3n} \) and \( \tau_{3n}^l \), the sequences \( (a_i)_{i=1}^{3n} \) and \( (b_i)_{i=1}^{3n} \) have the same length vector. Then, either \( a_i = 3 - \sigma_{3n} \circ \tau_{3n}^l (b_i), i = 1, 2, \ldots, 3n \) or \( a_i = 3 - \sigma_{3n} \circ \tau_{3n}^l (b_i), i = 1, 2, \ldots, 3n \).

Thus, the sequences \( (a_i)_{i=1}^{3n} \) and \( (b_i)_{i=1}^{3n} \) are projective equivalent.

2.2. Decomposition of dessins and wiring diagrams. Due to Proposition 1.38, the dessin associated to a morsification \( C \subset \Sigma_n \) is elementary equivalent to the gluing of \( n \) cubic dessins. There are two kinds of gluings: through a dotted cut (see Figure 3a) or through a solid generalized cut (see Figure 3b).

In the case the gluing is though a dotted cut, the dessin \( D \) of a morsification is elementary equivalent to the gluing \( D' - H^{***} \), where \( H^{***} \) is the hyperbolic cubic dessin corresponding to three real lines in general position in \( \mathbb{R}^2 \), after a strict transform with respect to the blow up of a point in \( \mathbb{R}^2 \) which does not lie on any line. Then the wiring diagram of \( D \) would have either 1121 or 2212 corresponding to one of the nodal \( \ast - \)vertices of \( D' \) and the nodal \( \ast - \)vertices of \( H^{***} \).

In the case the gluing is though a solid generalized cut, the dessin \( D \) of a morsification is elementary equivalent to the gluing \( D' - I^{**}_2 \), where \( I^{**}_2 \) is the non-hyperbolic cubic corresponding to a line and a non-singular conic in \( \mathbb{R}^2 \) intersecting in two different real points, after a strict transform with respect to the blow up of a point in \( \mathbb{R}^2 \) which does not lie on any component. Then the wiring diagram of \( D \) would have either 1212 or 2121 corresponding to one of the nodal \( \ast - \)vertices of \( D' \) and the nodal \( \ast - \)vertices of the gluing with \( I^{**}_2 \).

With this property in mind, we define the following operation on sequences. Given a sequence \( (a_i)_{i=1}^{3n} \), we put

\[
\eta_p((a_i)_{i=1}^{3n}) = \begin{cases} 
(a_1, a_2, \ldots, a_n, 1, 2, 1) & \text{if } a_n = 1, \\
(a_1, a_2, \ldots, a_n, 2, 1, 2) & \text{if } a_n = 2,
\end{cases}
\]

\[
\eta_\times((a_i)_{i=1}^{3n}) = \begin{cases} 
(a_1, a_2, \ldots, a_n, 2, 1, 2) & \text{if } a_n = 1, \\
(a_1, a_2, \ldots, a_n, 1, 2, 1) & \text{if } a_n = 2.
\end{cases}
\]

These function are not invariant under the action of \( \sigma_{3n} \).
Theorem 2.6. Every wiring diagram $\omega$ of a morsification can be obtained as the projective equivalent class of a sequence of operations $\eta_0, \eta_\times$ and shifts of the wiring diagram $(1,2,1)$, i.e., there exist $\eta_i \in \{\eta_0, \eta_\times\}$, $k_i \in \mathbb{N}$, $i = 1,2, \ldots, n-1$ such that $\omega$ is the projective equivalence class of the sequence

$$\eta_{n-1} \circ \sigma_{3n-3}^{k_{n-1}} \circ \cdots \circ \sigma_2^k \circ \eta_1 \circ \sigma_3^1(1,2,1).$$

Proof. Let $C$ be a trigonal morsification such that $\omega$ is its wiring diagram. Let $D = \text{Dsu}(C)$ be its dessin. By Proposition 1.38 the dessin $D$ is elementary equivalent to a dessin having a decomposition $D_{3n-3} - D_3$, where $D_{3n-3}$ and $D_3$ are dessins of degree $3n - 3$ and $3$, respectively. Fix a white vertex $v \in \text{Ver}(D)$ and an orientation on the real part of $D$ such that $\omega$ is written as the wiring diagram of $D$ with respect to $v$.

Let $k_{n-1}$ the number of nodal $\times$-vertices in the arc $(v,u)$, where $u$ is the vertex on the generalized cut passing from $D_3$ to $D_{3n-3}$ with respect to the aforementioned order of the real part of the dessin $D$.

Let $\eta_{n-1}$ be the operation $\eta_0$ if $D_3$ is the cubic dessin $H^{***}$ or the operation $\eta_\times$ if $D_3$ is the cubic dessin $I^{**}_3$. In the latter case, we contract the solid segment of $D_{3n-3}$ to obtain the dessin $D'_{3n-3}$ of a morsification of degree $3n - 3$. We iterate this decomposition $n - 1$ times. In the eventual case where the last cubic dessin has a resulting wiring diagram $(2,1,2)$, we add one to $k_1$ in order to obtain the desire decomposition.

Corollary 2.7. If $C$ is a morsification of degree $3n$, then, all the entries on its cyclic length vector are less or equal to $n$.

Moreover, if its cyclic length vector has $n$ as an entry, then, the wiring diagram of $C$ is the only projective equivalence class with cyclic length vector

$$(1,n,1,2,2,\ldots,2)$$

$n-1$ times.

Proof. We proceed by induction on $n$. For $n = 1$, there is a unique deformation class of morsifications in $\Sigma_1$, corresponding to the strict transform of three lines in $\mathbb{R}^2$ in general position, after the blow-up of a point which does not lie in any line.

Let us assume that $n > 1$ and the statement holds for $n - 1$. Let $\omega$ be the wiring diagram of $C$. By Theorem 2.6 we have that $\omega$ can be obtained as the projective equivalence class of $\eta_{n-1}(a_1)_{i=1}^{3n-3} \circ \cdots \circ \eta_1 \circ \sigma_3^1(1,2,1)$. Denote by $o$ the cyclic length vector of $\omega$ and by $o_{n-1}$ the cyclic length vector of the projective equivalence class of $(a_1)_{i=1}^{3n-3}$.

In the case when $\eta_{n-1} = \eta_0$, we have that $o(a_{3n-3})$, the entry in $o$ corresponding to $a_{3n-3}$, increases at most by $1$ since $\eta_0$ concatenates to the sequence exactly one more element that equals $a_{3n-3}$. Also, the entry $o(a_1) \leq o_{n-1}(a_1) + 1$. Equality holds when $a_1 = a_{3n-3}$ and $n - 1$ is odd or when $a_1 \neq a_{3n-3}$ and $n - 1$ is even. Every other entry in $o$ either comes from an entry in $o_{n-1}$ unaffected by $\eta_{n-1}$ or is the entry $1$ corresponding to the middle number in the added elements. Therefore, since every entry in $o_{n-1}$ is at most $n - 1$, then every entry in $o$ is at most $n$.

Moreover, if $o(a_{3n-3})$ or $o(a_1)$ equals $n$, then $o_{n-1}(a_{3n-3})$ or $o_{n-1}(a_1)$ equals $n - 1$. By the induction hypothesis, we have that $o_{3n-3}, a_1$ are distinct for $n - 1$ even, or equal for $n - 1$ odd and that either $o_{n-1}(a_{3n-3}) = n - 1, o_{n-1}(a_1) = 1,$ or $o_{n-1}(a_{3n-3}) = 1, o_{n-1}(a_1) = n - 1,$ respectively. Therefore, in both cases $o(a_{3n-3}) = o_{n-1}(a_{3n-3}) + 1$ and $o(a_1) = o_{n-1}(a_1) + 1$. Thus, the cyclic length vector $o$ equals $(1,n,1,2,2,\ldots,2)$

In the case when $\eta_{n-1} = \eta_\times$, we have that $o(a_{3n-3})$ cannot increase since $\eta_\times$ concatenates to the sequence an element that differs from $a_{3n-3}$. The entry $o(a_1) \leq o_{n-1}(a_1) + 1$. Equality holds when $a_1 = a_{3n-3}$ and $n - 1$ is even or when
$a_1 \neq a_{3n-3}$ and $n - 1$ is odd. Every other entry in $o$ either comes from an entry in $o_{n-1}$ unaffected by $\eta_{n-1}$ or is the entry 1 corresponding to the middle number in the added elements. Therefore, since every entry in $o_{n-1}$ is at most $n - 1$, then every entry in $o$ is at most $n$.

We cannot have that $o(a_1) = n$. Indeed, if $o(a_1) = n$, then $o_{n-1}(a_1) = n - 1$ and $a_1 = a_{3n-3}$ if $n - 1$ is even or $a_1 \neq a_{3n-3}$ if $n - 1$ is odd. On the other hand, since $o(a_1) = n$ and only the last element in $\eta_n(a_1)_{i=1}^{3n-3}$ add to it, then $a_1 = a_2 = \cdots = a_{n-1}$. Then, the fact that $a_1 = a_{3n-3}$ if $n - 1$ is even or $a_1 \neq a_{3n-3}$ if $n - 1$ is odd, implies that the entry $o_{n-1}(a_1) \geq n$, which is a contradiction. □

It was remarked that in degree 3 the wiring diagram $(1, 1, 1, 2, 2, 2, 1, 1, 1)$ does not exist. Indeed, its cyclic length vector being $(3, 3, 3)$ does not correspond to a morsification.

**Definition 2.8.** Let $M \subset \Sigma_n$ be a trigonal morsification and let $X \subset \mathbb{R} \Sigma_n \to \mathbb{R}^1$ be a topological subspace. If there exists an arc $\alpha \subset \mathbb{R}^1$ such that $(\mathbb{R} \Sigma_n|_{\mathbb{R}^1\backslash\alpha}, X)$ is homeomorphic to $(\mathbb{R} \Sigma_n|_{\mathbb{R}^1\backslash\alpha}, M)$, and $(\mathbb{R} \Sigma_n|_{\alpha}, X), (\mathbb{R} \Sigma_n|_{\alpha}, M)$ are homeomorphic to the curves in Figure 4, we say that $X$ is obtained from $M$ by a Reidemeister move.

A trigonal morsification $M' \subset \Sigma_n$ is said to be obtained from $M$ by a Reidemeister move if $RM \subset \mathbb{R} \Sigma_n$ is.

We say that two trigonal morsifications $M, M' \subset \Sigma_n$ are Reidemeister equivalent if there exists a chain $M_0 = M, M_1, \ldots, M_k = M'$ of trigonal morsifications such that $M_i$ is obtained from $M_{i-1}$ by a Reidemeister move, for $i = 1, 2, \ldots, k$.

**Corollary 2.9.** Let $M \subset \Sigma_n$ be a trigonal morsification. If $X \subset \mathbb{R} \Sigma_n$ is obtained from $M$ by a Reidemeister move, then, there exists a trigonal morsification $M' \subset \Sigma_n$ such that $(\mathbb{R} \Sigma_n, X) \cong (\mathbb{R} \Sigma_n, \mathbb{R} M')$.

**Proof.** Let us consider $n \geq 2$ since the statement holds trivially for $n = 1$. Let $D \subset \mathbb{C} \mathbb{P}^1/\{z \mapsto \bar{z}\}$ be the dessins associated to $M$. Let us consider the segment $S_\alpha \subset D$ corresponding to the arc $\alpha \subset \mathbb{R}^1$ such that $(\mathbb{R} \Sigma_n|_{\alpha}, X)$ and $(\mathbb{R} \Sigma_n|_{\alpha}, M)$ are homeomorphic to the curves in Figure 4.

We can assume $D$ is bridge-free (see [5] Lemma 2.3]). Let us denote by $n_1, n_2, n_3$ the $x$-vertices in $S_\alpha$ corresponding to the nodal vertices. Up to elementary moves of type $o$-in, we can assume there is exactly one real white vertex $w_{12}$ neighbouring $n_1$ and $n_2$ and exactly one real white vertex $w_{23}$ neighbouring $n_2$ and $n_3$ (see Figure 5).
Figure 5. Decorations of the segment of the dessin corresponding to the arc of a Reidemeister move.

Up to monochrome modification, the vertices $n_1, w_{12}, n_2, w_{23}$ and $n_3$ are adjacent to an inner black vertex $b$.

If $n_1$ and $n_3$ have neighboring white vertices $w_1 \neq w_{12}$ and $w_3 \neq w_{23}$, the creation of an inner bold monochrome vertex adjacent to $w_1$, $b$ and $w_3$ bring us to the configuration shown in Figure 6a.

Otherwise, if there is a monochrome vertex $m$ adjacent to $n_1$ or $n_3$, up to an elementary move of type ◦-in, the vertex $m$ is connected to an inner white vertex $w$. Up to a monochrome modification, the vertex $w$ is adjacent to $b$. Then, up to the creation of a dotted bridge or a monochrome modification, the dessin $D$ has a subgraph shown in Figure 6b.

In each case, replacing this subgraph with the alternative one in Figure 6 produces a dessin $D'$ whose corresponding trigonal curve $M' \subset \Sigma_n$ is a morsification such that $(\mathbb{R}\Sigma_n, X) \cong (\mathbb{R}\Sigma_n, \mathbb{R}M')$ since outside the segment $S_\alpha$, there are no changes and within the segment $S_\alpha$ the parity of the number of white vertices in the dotted segments neighboring $n_1$ and $n_3$ changed.

Corollary 2.10. If $M, M' \subset \Sigma_n$ are trigonal morsatifications, then, they are Reidemeister equivalent.

Proof. We prove that every trigonal morsification $M \subset \Sigma_n$ is Reidemeister equivalent to the trigonal morsification $M^*$ whose wiring diagram is

$$\omega^*_{i} := \eta_{i}^{-1}(1, 2, 1).$$

Let us remark that $\omega^*_{i}$ is invariant under the action of $\sigma_{3n}$, for all $n$, and the fact that a Reidemeister move on a morsification change a block of entries of its wiring diagram of the form $(1, 2, 1)$ for a block of the form $(2, 1, 2)$ and vice versa. Therefore, for every $i$, we have that the morsifications corresponding to $\eta_i \circ \sigma_i^k \omega_i^*$ and $\omega^*_i$ are Reidemeister equivalent.
The statement follows by applying these facts to the decomposition \( \eta_{n-1} \circ \sigma_{a_{2n-3}}^k \circ \cdots \circ \eta_2 \circ \sigma_a^2 \circ \eta_1 \circ \sigma_b^1 (1, 2, 1) \) of the wiring diagram \( \omega \) of \( M \) given by Theorem 2.6. \( \square \)

**Appendix A. E. Shustin. Polymomiality of morsifications of trigonal singularities**

**A.1. Main result.** By a singularity we always mean a germ \((C, z) \subset \mathbb{C}^2\) of a plane reduced analytic curve at its singular point \(z\). Irreducible components of the germ \((C, z)\) are called branches of \((C, z)\). Let \(f(x, y) = 0\) be an (analytic) equation of \((C, z)\), where \(f\) is defined in the closed ball \(B(z, \varepsilon) \subset \mathbb{C}^2\) of radius \(\varepsilon > 0\) centered at \(z\). The ball \(B(z, \varepsilon)\) is called the **Milnor ball** of \((C, z)\) (and is denoted in the sequel \(B_{C,z}\)) if \(z\) is the only singular point of \(C\) in \(B(z, \varepsilon)\), and \(\partial B(z, \eta)\) intersects \(C\) transversally for all \(0 < \eta \leq \varepsilon\). A **nodal deformation** of a singularity \((C, z)\) is a family of analytic curves \(\{f_t(x, y) = 0\}\), where \(f_t(x, y)\) is analytic in \(x, y, t\) for \((x, y) \in B(C, z)\) and \(t\) varying in an open disc \(D_\zeta \subset \mathbb{C}\) of some radius \(\zeta > 0\) centered at zero, and where \(C_0 = C\), \(C_t\) is smooth along \(\partial B_{C,z}\), intersects \(\partial B_{C,z}\) transversally for all \(t \in D_\zeta\), the curve \(C_t\) has only ordinary nodes in \(B_{C,z}\) for any \(t \neq 0\), and the number of nodes does not depend on \(t\). The maximal number of nodes in a nodal deformation of \((C, z)\) in \(B\) equals \(\delta(C, z)\), the \(\delta\)-invariant (see, for instance, [11 §10]).

Let \((C, z)\) be a totally real singularity, i.e., invariant with respect to the complex conjugation, with \(z \in C\) a real singular point and all branches real. Let \(C_t = \{f_t(x, y) = 0\}, t \in D_\zeta\), be an equivariant nodal deformation of a real singularity \((C, z)\). Its restriction to \(t \in [0, \zeta)\) is called a **real nodal deformation**. A real nodal deformation is called a **real morsification** of \((C, z)\) if any \(C_t, t > 0\), has \(\delta(C, z)\) hyperbolic real nodes (i.e., intersection points of two smooth transverse real branches).

Denote by \(T_k(3, 3n)\) the class of real plane curve singularities having three real smooth branches intersecting each other with multiplicity \(n\).

**Theorem A.1.** Let \((C, 0) \subset B(0, \varepsilon) \subset \mathbb{C}^2\) be a germ of a real plane curve singularity of type \(T_k(3, 3n)\), given by an analytic equation

\[
F(x, y) = f_{3n} + \sum_{3i+nj > 3n} a_{ij}x^iy^j = 0, \quad f_{3n} = y^3 + a_{n,2}x^{2n}y^2 + a_{2n,1}x^{2n}y + a_{3n,0}x^{3n},
\]

1Here and further on, **equivariant** means commuting with the complex conjugation.

\[\begin{array}{ccc}
\text{(a)} & \text{(b)}
\end{array}\]
where $f_{3n}$ is squarefree, and let $F_t(x,y), |t| < \eta$, be a morsification of the given singularity $(C,0)$ (respectively having $3n$ hyperbolic nodes in $B^{\varepsilon}(0,\varepsilon)$ for each $0 < t < \eta$). Then there exists a polynomial
\begin{equation}
G(x,y) = f_{3n} + \sum_{3i+nj < 3n} a_{ij}x^iy^j
\end{equation}
such that $\{G = 0\} \cap B(0,\varepsilon)$ is equivariantly isotopic to any curve $\{F_t = 0\} \cap B(0,\varepsilon)$, $|t| < \eta$, relative to $\partial B(0,\varepsilon)$.

In the proof, we explore the idea originated in the proof of [13, Corollary], later used in [10] Proof of Theorem 3, and elaborated further in [12] Proof of Theorem 5. It, however, appears to be rather more involved technically than [12] Proof of Theorem 5, and we present it in several parts.

**Remark A.2.** The statement of Theorem A.1 yields the following fact: if $(C,z)$ is a real quasihomogeneous singularity of type $T_2(3,3n)$, and $(C',z)$ is a real quasihomogeneous singularity of type $T_2(3,3n)$ obtained from $(C,z)$ by a sufficiently small equisingular deformation, then the set of isotopy types of morsifications of $(C,z)$ is naturally included into the set of isotopy types of morsifications of $(C',z)$. Indeed, if, in the notation of Theorem A.1, a morsification of $(C,z) = \{f_{3n} = 0\}$ is isotopic to $\{G = 0\} \cap B(z,\varepsilon)$, where
\begin{equation}
G(x,y) = \prod_{i=1}^{3} \left( y + \sum_{j=0}^{n} \alpha_{ij}x^j \right),
\end{equation}
then a small variation of the factors of $G(x,y)$ preserves all real nodes of the curve $\{G = 0\}$, and hence its equivariant isotopy type.

A.2. Proof of Theorem A.1

A.2.1. Part I. We start with reducing the problem to the independence of simultaneous morsifications of singularities of a real curve given by a polynomial with the the Newton polygon under the line $\{i + nj = 3n\}$.

(1) Observe that there exists $\alpha$ such that after the coordinate change $(x,y) \mapsto (x,y + \alpha x^n)$, the coefficients of $f_{3n}$ satisfy the relation
\begin{equation}
9a_{3n,0} - a_{n,2}a_{2n,1} \neq 0.
\end{equation}
Indeed, the above coordinate change turns the left-hand side of (6) into
\begin{equation}
2\alpha(3a_{2n,1} - a_{n,2}^2) + 9a_{3n,0} - a_{2n,1}a_{n,2},
\end{equation}
which vanishes identically only if $f_{3n}$ is the cube of a binomial against our assumptions. Since the coordinate change we used does not affect the statement of proposition, we can assume that (6) holds.

(2) For a given
\begin{equation}
F_t(x,y) = \sum_{i,j \geq 0} b_{ij}(t), \quad b_{ij}(t) = \begin{cases} a_{ij}, & i + nj \geq 3n, \\ 0, & i + nj < 3n, \end{cases}
\end{equation}
there exist analytic functions $\alpha(t), \beta_0(t), ..., \beta_{n-1}(t)$ vanishing at zero such that $F_t(x + \alpha(t),y + \beta_0(t),x + \beta_1(t)x + ... + \beta_{n-1}(t)x^{n-1})$ does not contain the monomials $y^2, x^{n-1}y^2$, and $x^{3n-1}$. Indeed, the required property reduces to a system of equations on $\alpha, \beta_1, ..., \beta_{n-1}$ with the Jacobian at $t = 0$ equal to $3^{n-1}(9a_{3n,0} - a_{n,2}a_{2n,1})$, non-vanishing according to (6). Thus, without loss of generality, we can assume that $b_{0,2}(t) = ... = b_{n-1,2}(t) = b_{3n-1,0}(t) \equiv 0$. 
Since $F_t = 0$ has no singularity of type $T_3(3, 3n)$ as $t \neq 0$, we get $\sum_{i+nj<3n} |b_{ij}(t)| > 0$ for $t \neq 0$. Hence, for a fixed $c_0 > 0$, there exists a positive function $\tau(t)$ on the interval $(0, \eta)$ such that

\begin{equation}
\sum_{i+nj<3n} |b_{ij}(t)|\tau(t)^{3n-i-nj} = c_0, \quad t > 0.
\end{equation}

In view of (7), $\lim_{t \to 0} \tau(t) = \infty$. Thus, in the family

\begin{equation}
\tau(t)^{3n}F_t(x\tau(t)^{-1}, y\tau(t)^{-n}) = \sum_{i+nj>3n} b_{ij}(t)\tau^{3n-i-nj}x^iy^j
\end{equation}

the first sum converges to zero, while the second one converges to $f_{3n}(x, y)$ as $t \to 0$. Moreover, by (5), there exists a sequence $t_m \to 0$, $m = 1, 2, \ldots$, such that the third sum in (9) converges as well so that

\begin{equation}
\lim_{m \to \infty} \tau(t_m)^{3n}F_t(x\tau(t_m)^{-1}, y\tau(t_m)^{-n}) = f_{3n}(x, y) + \sum_{i+nj<3n} a_{ij}x^iy^j =: R(x, y),
\end{equation}

where

\begin{equation}
\sum_{i+nj<3n} |a_{ij}| = c_0 > 0 \quad \text{and} \quad a_{0,2} = \ldots = a_{n-1,2} = a_{3n-1,0} = 0.
\end{equation}

Suppose that $c_0 > 0$ is chosen so that the curve $\{R(x, y) = 0\} \subset \mathbb{C}^2$ is smooth outside $B(0, \varepsilon)$ and intersects with all spheres $\partial B(0, \zeta)$, $\zeta \geq \varepsilon$, transversally. Thus, by construction we have:

(R1) $\{R(x, y) = 0\}$ has no singularity of type $T_3(3, 3n)$ in $B(0, \varepsilon)$, and each its singular point is a center of two or three smooth real branches; furthermore, since the singularities of $\{R = 0\}$ admit a deformation into $3n$ nodes in total, $R(x, y)$ splits into the product of $(y - Q_1(x))(y - Q_2(x))(y - Q_3(x))$, where $\deg Q_i = n$, $i = 1, 2, 3$;

(R2) there exists an equivariant analytic deformation of $\{R(x, y) = 0\}$, simultaneously realizing morsifications of all singular points of $\{R = 0\}$ so that the resulting curve in $B(0, \varepsilon)$ is equivariantly isotopic to $\{F_t(x, y) = 0\} \cap B(0, \varepsilon)$, $t > 0$, relative to $\partial B(0, \varepsilon)$.

Denote by $P(k) \subset \mathbb{C}[x, y]$ the linear subspace spanned by the monomials $x^iy^j$ with $i + nj = k$, and put

\begin{equation}
P(k) = \bigoplus_{i \leq k} P(i), \quad P^{(k)} = \bigoplus_{i \geq k} P(i).
\end{equation}

So, we will complete the proof of Theorem [A.1] when showing that the germ at $R$ of the affine space $R + P_{3n-1}$ induces one-parameter deformations simultaneously realizing morsifications of prescribed isotopy types for all singular points of the curve $\{R = 0\} \cap B(0, \varepsilon)$. We shall prove this fact by induction on $n$. For $n = 1$, we have a $D_4$ singularity, for which the statement is evident. The induction step splits into several cases treated below.

Before proceeding further on, we make one more coordinate change $(x, y) \mapsto (x, y + \alpha x^2)$, which annihilates the coefficient of $x^ny^2$, i.e., leads to

\begin{equation}
\sum_{i+nj<3n} |a_{ij}| = c'_0 > 0 \quad \text{and} \quad a_{0,2} = \ldots = a_{n-1,2} = a_{n,2} = 0,
\end{equation}

while, of course, preserving the properties (R1), (R2).
A.2.2. Part II. Suppose that $R(x, y) = yS(x, y)$. In view of [10], $S(x, y) = y^2 - Q(x)^2$, where a polynomial $Q(x)$ of degree $n$ has at least two roots and all the roots are real. That is, the curve $C(R) := \{ R = 0 \}$ has singularities of types $T_{x_i}(3, 3n_i)$, $1 \leq i \leq k$ ($k \geq 2$), where $n_1 + \ldots + n_k = n$. For each singular point $z_i$ of $C(R)$, $i = 1, \ldots, k$, denote by $I_i \subset \mathcal{O}_{\mathbb{C}^2, z}$ the ideal defined by vanishing of the coefficients of the monomials lying strictly below the line $i + n_ij = 3n_i$. Then the statement of the induction step can be reformulated as the surjectivity of the natural map $f_{3n_1} + \mathcal{P}_{3n_1-1} \to \mathbb{C}[z]/I_i$.

To prove this surjectivity, we consider $C(R)$ as a curve in the linear system $|3D|$ on the Hirzebruch surface $\mathcal{F}_n$, where $D$ is the divisor class of the section disjoint from the exceptional curve. Introduce the zero-dimensional subscheme $Z \subset \mathcal{F}_n$, $Z = Z_0 \cup Z_1 \cup \ldots \cup Z_k$, where $Z_0$ consists of the three reduced points of intersection of $C(R)$ with the fiber $x = \infty$, and $Z_i$ is defined at the point $z_i$ by the ideal $I_i$, $i = 1, \ldots, k$. Then the required statement admits the following cohomological reformulation:

$$H^i(\mathcal{F}_n, \mathcal{J}_Z|_{\mathcal{F}_n}(3D)) = 0,$$

where $\mathcal{J}_Z$ is the ideal sheaf of the subscheme $Z$. The curve $C(R)$ splits into three components $C_1, C_2, C_3 \in \{ D \}$. We apply “le metodo d’Horace” (see, [3]). Consider the three exact sequences

$$0 \to \mathcal{J}_{Z,C_1}/\mathcal{F}_n(2D) \xrightarrow{c_2} \mathcal{J}_{Z}/\mathcal{F}_n(3D) \to \mathcal{J}_{Z,C_1}/C_1(3D) \to 0,$$
$$0 \to \mathcal{J}_{Z,C_1}/\mathcal{F}_n(2D) \xrightarrow{c_3} \mathcal{J}_{Z,C_1}/\mathcal{F}_n(3D) \to \mathcal{J}_{Z}(C_1,C_2)/\mathcal{F}_n(2D) \to 0,$$
$$0 \to \mathcal{O}_{\mathcal{F}_n} \xrightarrow{c_3} \mathcal{J}_{Z,C_1}/\mathcal{F}_n(3D) \to \mathcal{J}_{Z}(C_1,C_2)/\mathcal{F}_n(2D) \to 0,$$

which yield the exact cohomology sequences

$$H^1(\mathcal{F}_n, \mathcal{J}_{Z,C_1}/\mathcal{F}_n(2D)) \to H^1(\mathcal{F}_n, \mathcal{J}_{Z}/\mathcal{F}_n(3D)) \to H^1(C_1, \mathcal{J}_{Z,C_1}/C_1(3D)),$$
$$H^1(\mathcal{F}_n, \mathcal{J}_{Z,C_1}/\mathcal{F}_n(2D)) \to H^1(\mathcal{F}_n, \mathcal{J}_{Z,C_1}/\mathcal{F}_n(3D)) \to H^1(C_2, \mathcal{J}_{Z}(C_1,C_2)/C_2(2D)),$$
$$0 = H^1(\mathcal{O}_{\mathcal{F}_n} \to H^1(\mathcal{F}_n, \mathcal{J}_{Z,C_1}/\mathcal{F}_n(3D)) \to H^1(C_3, \mathcal{J}_{Z}(C_1,C_2)/C_3(3D)),$$

(here $Z : (C_1C_2) \subset C_3$) and hence (11) can be derived from

$$H^1(C_1, \mathcal{J}_{Z,C_1}/C_1(3D)) = H^1(C_2, \mathcal{J}_{Z}(C_1,C_2)/C_2(2D)) = H^1(C_3, \mathcal{J}_{Z}(C_1,C_2))/C_3(3D) = 0.$$

To establish these $h^1$-vanishing relations, we use the Riemann-Roch criterion, which for the smooth rational curves $C_1, C_2, C_3$ respectively reads

$$3D^2 - \deg(Z \cap C_1) > D^2 + DK_{\mathcal{F}_n} = -2, \quad 2D^2 - \deg((Z : C_1) \cap C_2) > -2,$$
$$D^2 - \deg(Z : (C_1C_2)) > -2,$$

and all these conditions are fulfilled in view of $D^2 = n$ and

$$\deg(Z \cap C_1) = 3n_1 + 1, \quad \deg((Z : C_1) \cap C_2) = 2n_1 + 1, \quad \deg(Z : (C_1C_2)) = n + 1.$$

A.2.3. Part III. Suppose that $R(x, y)$ is not divisible by $y$, and the curve $C(R)$ contains a singular point of multiplicity 3. It follows from the properties (R1), (R2), and (10) that this point $z = (x_0, y_0)$ must be of type $T_{x_i}(3, 3n_i)$ with $1 \leq i < n$, and that $R(x_0, y) = y^3$. So, after the shift $(x, y) \mapsto (x, y + \alpha_1x + \ldots + \alpha_mx^m)$, we obtain a polynomial $\tilde{R}(x, y)$ with the Newton triangle $\text{Conv}\{3m, 0, (0, 0), (0, 3)\}$, whose truncations to the edges $[3m, 0, (0, 0)]$ and $[3m, 0, (0, 3)]$ are squarefree. The polynomial $\tilde{R}(x, y) := x^{-3m}\tilde{R}(x, x^m)\tilde{y}$ has Newton triangle $\text{Conv}\{(0, 0), (3m-3m, 0), (0, 3)\}$ and defines a curve with the same singularities in $C^2 \setminus \{ x = 0 \}$ as the polynomial $R(x, y)$. Thus, by the induction assumption, there exists a deformation of the coefficients of the monomials under the segment $[(3m-3m, 0), (0, 3)]$, $\tilde{R}_t(x, y)$, $0 \leq t < \eta$, simultaneously realizing morsifications of prescribed isotopy types for all the singularities.
of \( \{ \tilde{R} = 0 \} \). Hence the deformation of \( \tilde{R}(x,y) \) given by \( \tilde{R}_t(x,y) := x^{2m} \tilde{R}_t(x,x^{-m}y) \), \( 0 \leq t < \eta \), simultaneously realizes morsifications of prescribed isotopy types for all the singularities of \( \{ \tilde{R} = 0 \} \), while keeping the singularity of type \( T_{3k}(3,3m) \) at the origin. By the induction assumption, Theorem \( A.1 \) and Remark \( A.2 \) apply to the singularities of \( \{ \tilde{R}_0 = 0 \} \) (with some fixed \( 0 < t_0 < \eta \)), which means that for any chosen morsification of the singularity of \( \{ \tilde{R} = 0 \} \) at the origin, there exists a real polynomial \( P(x,y) \) with Newton triangle \( \text{Conv}\{ (0,0),(3m,0),(0,3) \} \), the same truncation of the edge \( [(3m,0),(0,3)] \) as \( \tilde{R}_0(x,y) \), and such that the curve \( \{ P = 0 \} \cap B(0,\varepsilon) \) is nodal and equivariantly isotopic to the chosen morsification in \( B(0,\varepsilon) \) (relative to \( \partial B(0,\varepsilon) \)). By \( [14] \) Theorems 3.1 and 4.1(1), we can “patchwork the polynomials \( \tilde{R}_n(x,y) \) and \( P(x,y) \) and obtain a family of real polynomials with Newton triangle \( \text{Conv}\{ (0,0),(3m,0),(0,3) \} \) simultaneously realizing morsifications of prescribed isotopy types for all the singularities of the original curve \( \{ R = 0 \} \).

A.2.4. **Part IV.** So, we are left with the case of the curve \( C(R) \) having only double singular points, i.e. points of intersection of two local smooth real branches. We shall show that, in this case, the linear system \( |3D| \) on the Hirzebruch surface \( F_n \) induces a joint versal deformation of all singular points of the curve \( C(R) \subset F_n \).

(1) It is well-known that a versal deformation of each point \( (x_0,y_0) \in \text{Sing}(C(R)) \) is generated by any basis of the linear space \( \mathbb{C}\{ x-x_0, y-y_0 \}/\langle R, R_x, R_y \rangle \), where \( \mathbb{C}\{ *, * \} \) denotes the ring of locally convergent power series, and \( \langle R, R_x, R_y \rangle \) is the Tjurina ideal (generated by \( R, R_x, R_y \)). Hence, it is enough to prove the surjectivity of the following two projections:

\[
\text{pr}_1 : \mathbb{C}\{ x, y \}/\langle R, R_x, R_y \rangle \rightarrow \bigoplus_{(x_0,y_0) \in \text{Sing}(C(R))} \mathbb{C}\{ x-x_0, y-y_0 \}/\langle R, R_x, R_y \rangle ,
\]

\[
\text{pr}_2 : \mathcal{P}_{3n} \rightarrow \mathbb{C}\{ x, y \}/\langle R, R_x, R_y \rangle .
\]

To prove the surjectivity of \( \text{pr}_1 \), introduce the zero-dimensional subscheme \( Z \) of the Hirzebruch surface \( F_n \), concentrated at \( \text{Sing}(C(R)) \) and defined by the local Tjurina ideal at each point. Then we have an exact sequence of sheaves

\[
0 \rightarrow J_{Z/F_n}(mD) \rightarrow \mathcal{O}_{F_n}(mD) \rightarrow \mathcal{O}_Z \rightarrow 0 ,
\]

\( J_{Z/F_n} \) being the ideal sheaf of \( Z \). Since for \( m \gg 0 \), \( H^1(J_{Z/F_n}(mD)) = 0 \), we get

\[
H^0(\mathcal{O}_{F_n}(mD))/H^0(J_{Z/F_n}(mD)) \cong H^0(\mathcal{O}_Z) = \bigoplus_{(x_0,y_0) \in \text{Sing}(\{ R=0 \})} \mathbb{C}\{ x-x_0, y-y_0 \}/\langle R, R_x, R_y \rangle ,
\]

which, in fact, yields that \( \text{pr}_1 \) is an isomorphism.

(2) Now we prove the surjectivity of \( \text{pr}_2 \).

Since \( R(x,0) \neq 0 \) and \( C(R) \) is the union of three smooth real sections, there exist \( x_0, x_\infty \in \mathbb{R} \) such that \( R(x_0,0) = 0 \) and \( R(x_\infty,0) \neq 0 \). Performing an equivariant automorphism of the base of \( F_n \), we can set \( x_0 = 0 \), \( x_\infty = \infty \), while keeping the property \( [10] \). Note, that then \( f_{3n}(x,y) = y^3 + a_{2n,1}x^{2n}y + a_{3n,0}x^{3n} \) with \( a_{2n,1}a_{3n,0} \neq 0 \), since \( F_{3n} \) splits into three real factors. Write

\[
R(x,y) = f_{3n} + \sum_{i=1}^{s} f_{k_i}, \quad f_{k_i} \in \mathcal{P}(k_i) \setminus \{0\}, \quad 3n > k_1 > \ldots > k_s \geq 0, \quad s \geq 1 .
\]

Observe that \( k_s \geq 0 \) and \( f_{n,y} \neq 0 \), that is,

\[
a_{0,0} = R(0,0) = 0 \quad \text{and} \quad a_{0,1} = R_y(0,0) \neq 0 .
\]

The former relation - by construction, the latter relation - due to \( [10] \) and the absence of triple singular points.
Since $f_{3n}$ is square-free, the derivatives $f_{3n,x} \in \mathcal{P}(3n-1)$ and $f_{3n,y} \in \mathcal{P}(2n)$ are coprime. Hence, any polynomial in $\mathcal{P}(4n-1)$ is congruent to a polynomial in $\mathcal{P}(4n-2)$ modulo $(R_x, R_y)$.

Notice that

$$f_{3n,x} = 2a_{2n,1}x^{2n-1}y + 3a_{3n,0}x^{3n-1} = nx^{2n-1}g, \quad g = 2a_{2n,1}y + 3a_{3n,0}x^n.$$ 

To show that

$$\mathcal{P}(m) \equiv \mathcal{P}(m-1) \mod (R_x, R_y) \text{ for all } 3n \leq m \leq 4n-2,$$

which is equivalent to the surjectivity of $pr_2$, we construct a finite sequence of polynomials $R_i \in (R_x, R_y)$, $i \geq 1$, such that the final polynomial $R_p(x, y)$ has the leading $(1, n)$-homogeneous form

$$g_k = ax^{k-n}y + bx^k \notin \langle g \rangle, \quad k < 3n - 1.$$ 

Since (see (10))

$$f_{3n,y} = 3y^2 + a_{2n,1}x^{2n}, \quad f_{3n,x} = 2a_{2n,1}x^{2n-1}y + 3a_{3n,0}x^{3n-1},$$

$$\mathcal{P}(m) = \text{Span}\{x^{m-3n}y^3, x^{m-2n}y^2, x^{m-n}x^m-n, x^k\}, \quad 3n \leq m \leq 4n-2,$$

relation (15) will imply that

$$\mathcal{P}(m) = \mathcal{P}(m-2n) \cdot f_{3n,y} + \mathcal{P}(m-3n+1) \cdot f_{3n,x} \cdot \mathcal{P}(m-k) \cdot g_k, \quad 3n \leq m \leq 4n-2,$$

and hence (14).

We start with the required sequence $\{R_i(x, y)\}_{i \geq 1}$. If $k_1 = 3n - 1$ and $f_{k_1} \in \langle g \rangle$, then we can perform a coordinate change $(x, y) \mapsto (x + \alpha, y)$, which annihilates the form $f_{3n-1}$, but does not affect the properties (R1), (R2), and (10). Thus, we can assume that

(16) \quad either $f_{k_1} \notin \langle g \rangle$, or $k_1 = 3n - d, d \geq 2, f_{k_1} \in \langle g \rangle$.

Euler’s identities $3nf_{3n} = xf_{3n,x} + nyf_{3n,y}$ and $k_1f_{k_1} = xf_{k_1,x} + nyf_{k_1,y}$ yield

(17) \quad $R_1(x, y) := 3nf_{3n,x} - xR_x(x, y) - nR_y(x, y) = (3n-k_1)f_{k_1} + \sum_{i \geq 2} (3n-k_i)f_{k_i}.$

The former option in (16) leads to the situation of (15) with $g_{k_1} = f_{k_1}$, and hence to (14) as required.

So, assume now that $f_{k_1} \in \langle g \rangle, k_1 = 3n - d, d \geq 2$. Starting with the polynomials $R_0 := R_x$ and $R_1$, we perform the following operation. Suppose that, for some $i \geq 0$, we have a couple of polynomials $R_i(x, y), R_{i+1}(x, y)$, containing only monomials of type $x^k, x^iy$, having the highest $(1, n)$-quasihomogeneous forms $\lambda_i x^{m_i}g$ and $\lambda_{i+1} x^{m_{i+1}}g$, respectively, where $\lambda_i > \lambda_{i+1} \neq 0$. If, say, $m_i \geq m_{i+1}$, we define $R_{i+2} = \lambda_{i+1} R_i - \lambda_i x^{m_i - m_{i+1}} R_i$. If the leading form of $R_{i+2}$ does not belong to $\langle g \rangle$, we are done (see the above argument). If $R_{i+2} \notin \langle g \rangle$, and the leading form of $R_{i+2}$ belongs to $\langle g \rangle$, then we arrive to the pair $R_{i+1}, R_{i+2}$ with the total degree $m_{i+1} + m_{i+2} < m_i + m_{i+1}$.

In the worst case, we end up with some $R_m(x, y) \equiv 0$, which means that

$$R_x(x, y) = P_1(x)Q(x, y), \quad R_1 = P_2(x)Q(x, y),$$

where

$$Q(x, y) = x^m g(x, y) + \sum_{0 \leq j \leq 1, n\geq 1, n+j \leq m+n} b_{ij} x^iy^j, \quad \deg P_1 = 2n-1-m, \deg P_2 = 2n-d.$$

It follows from (13) that $P_2(0) \neq 0$, and hence $R_1(x, 0)$ has the root $x = 0$ of some multiplicity $r > 0$, while $R_2(x, 0)$ has the root $x = 0$ with multiplicity at least $r$. However, this contradicts the fact that

$$3nR(x, 0) = (xR_x(x, y) + nyR_y(x, y) + R_1(x, y))y = xR_x(x, 0) + R_1(x, 0)$$
has the root $x = 0$ of multiplicity $r$.

The proof of Theorem A.1 is completed.

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