Large-distance asymptotic behaviour of multi-point correlation functions in massless quantum models

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Abstract. We provide a microscopic model setting that allows us to readily access the large-distance asymptotic behaviour of multi-point correlation functions in massless, one-dimensional, quantum models. The method of analysis we propose is based on the form factor expansion of the correlation functions and does not build on any field theory reasoning. It constitutes an extension of the restricted sum techniques leading to the large-distance asymptotic behaviour of two-point correlation functions obtained previously.

Keywords: correlation functions, form factors, correlation functions (theory), critical exponents and amplitudes (theory)

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1. Introduction

Form factor expansions constitute a natural means for studying correlation functions in the case of massive integrable quantum field theories in the infinite volume limit [17, 32]. Indeed, the presence of a gap in the spectrum leads both to regularity and well-ordering of the form factor series as an effective asymptotic series in the large-distance regime. However, these two crucial properties are no longer satisfied for massless models; their lack constituted an obstruction to an efficient formulation or even handling of form factor series in the infinite volume limit of a model in the massless regime. The main problem in the case of massless models is that the form factors of local operators vanish as a non-integer power law in the model’s volume $L$. This originates from the operator’s non-integer conformal dimension. When computing a form factor expansion, this non-integer power law vanishing of individual form factors in $L$ should be compensated by certain multi-dimensional sums (which contain a number of summands diverging in $L$) over appropriate sub-classes of low-lying excited states, hence making this setting rather intricate to tackle. Recently, however, the authors, in collaboration with Slavnov, have set forth a method of summing up, in the large-distance regime and for large but finite
volume, the form factor expansion of two-point correlation functions in massless quantum
integrable models [21]. The issue is that, as we have observed, in the large-distance regime,
the sums over form factors associated with scanning the whole spectrum of the theory
localize to so-called restricted sums. The latter correspond, in physical terms, to summing
up over all low-lying energy (of the order $1/L$, with $L$ being the volume assumed to
be large but finite) excitations on the Fermi surface associated, in the thermodynamic
limit, with different Umklapp excitation momenta. Although the number of terms in these
multiple restricted sums tends to infinity with the volume, it has been shown in [21] that
they can be evaluated exactly using purely combinatorial identities. Certain instances of
these identities have already appeared in the context of harmonic analysis on the infinite-
dimensional symmetric group [18] (see [28] for a first practical application to $z$-measures
on Young diagrams).

In this paper we continue developing the technique of summation, in the large-distance
regime, of form factor expansions, now for general $n$-point correlation functions. More
precisely, we start from a model in large but finite volume $L$ and make some technical
assumptions on its spectrum and form factors. These properties are verified for quantum
integrable models like the XXZ spin-1/2 Heisenberg model and the 1D Bose gas at finite
positive coupling; however we do trust that they should also hold for a much wider class
of massless models. Building on these settings we develop summation techniques for form
factor expansions that allow us to access the large-distance regime of general multi-point
correlation functions. The corresponding restricted sums generalize those obtained for the
two-point case [21]. In fact, in the case of general $n$-point correlation functions (with $n \geq 3$)
the corresponding sums can be interpreted as multiple restricted sums of the previous
type (obtained for two-point functions), however highly coupled between themselves,
hence rendering the needed summation identity quite non-intuitive. The derivation of this
identity for the restricted sums corresponding to general $n$-point correlation functions
follows from the two possible representations for the large-size asymptotic behaviour of a
Toeplitz determinant generated by Fisher–Hartwig symbols: the one corresponding to the
Fisher–Hartwig asymptotic behaviour formula [8] and the one issuing from a form factor
like expansion of this same Toeplitz determinant.

The large-distance asymptotic summation of form factor method we develop
does reproduce, within a microscopic approach to the model, the conformal field
theory/Luttinger liquid predictions for the asymptotic behaviour of correlation functions
in massless models with central charge equal to one. As an example, the large-distance
behaviour of four-point functions in the XXZ spin-1/2 chain is obtained as

$$
\langle \sigma_{m_1}^x \cdot \sigma_{m_2}^x \cdot \sigma_{m_3}^x \cdot \sigma_{m_4}^x \rangle \simeq 2|F^+|^4 \left\{ \left( \frac{(m_2 - m_1) \cdot (m_4 - m_3)}{(m_3 - m_1) \cdot (m_4 - m_1) \cdot (m_3 - m_2) \cdot (m_4 - m_2)} \right)^{\theta} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) \right\} + \cdots,
$$

matching the predictions given in [11, 26]. Above, $\theta$ is an explicit exponent that is
expressed in terms of quantities parametrizing the low-lying excitations of the XXZ spin-
1/2 chain and the dots refer to higher order (sub-leading) terms. The amplitude $F^+$
corresponds to a form factor of the $\sigma^+$ operator that is properly normalized in the length
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L of the chain taken between the model’s overall ground state \(|\Psi_g\rangle\) and the local ground state in the sector with one spin being flipped down \(|\Psi_{\langle -1 \rangle}^g\rangle\),

\[
\mathcal{F}^+=\lim_{L\to+\infty}\left\{\left(\frac{L}{2\pi}\right)^\theta \langle \Psi_g | \sigma^+ | \Psi_{\langle -1 \rangle}^g \rangle\right\}.
\]

The critical exponent \(\theta\) was already present in the field theory predictions [11, 26]. We do stress however that our method goes beyond, in that it gives a direct access and interpretation of the amplitudes arising in front of the algebraically decaying terms (along with the numerical pre-factor of 2). It allows one readily to extract the large-distance asymptotic behaviour of any multi-point correlation function. Moreover, our method is solely based on a genuine microscopic formulation of the model and at no stage of our analysis do we invoke a conjectural correspondence of the discrete model with some continuous field theory. Furthermore, with minor modifications, it should open the possibility to treat the case of time-dependent correlation functions as was done for two-point functions in [23].

We would like to stress that the general dependence in the distance we obtain is similar to the one for correlation functions of exponents of free fields, cf [6, 7]. In fact, such a form already appeared in the late 1960s paper of Kadanoff [16]. Recently, the same structure of the large-distance asymptotic behaviour has been proven, for the Ising correlation functions at the critical point, by Palmer [29], essentially based on the Sato, Miwa, Jimbo formalism [13]–[15]. It is in fact in these pioneering works [13]–[15] that the first rigorous approach to characterizing and computing multi-point correlation functions in a free fermion model were given. The analysis developed there allowed these authors to provide, in particular, a characterization of multi-point correlation functions in a massless model for the case of the one-dimensional impenetrable Bose gas [12] and to extract explicitly the large-distance asymptotic behaviour of the so-called one-particle reduced density matrix in that model. Multi-point correlation functions in free fermion models have also been investigated through Pfaffian or integrable integral operator methods. In [1], Abraham, Barouch and McCoy derived a Pfaffian representation for four-point functions that allowed them to access the time-dependent spin–spin correlation functions. In [2, 3] Bariev derived the large-distance asymptotic behaviour of three and then multi-point correlation functions in the off-critical Ising model. In [31] Slavnov obtained a system of partial differential equations characterizing the \(n\)-point off-diagonal correlation functions in the one-dimensional impenetrable Bose gas.

This paper is organized as follows. In section 2 we provide a definition of the microscopic model and of all the quantities of interest in the problem. These definitions provide the general setting that allows for an application of our form factor summation method. It surely does hold for quantum integrable models such as the XXZ spin-1/2 chain or the quantum non-linear Schrödinger model, but should also be valid for numerous non-integrable models as well and in particular for a wide class of one-dimensional quantum liquids. Then, in section 3, we outline the main steps of our method and describe the overall features behind its philosophy. In particular we present the restricted sums of interest which are at the root of our approach to the large-distance asymptotic behaviour of the multi-point correlation function. Finally, we apply our general result to the case of...
2. Overall setting for the analysis of multi-point correlation functions

This paper aims at introducing a method of analysis of the large-distance asymptotic behaviour of \( r \)-point ground state correlation functions of the type

\[
C(\mathbf{x}_r; \mathbf{o}_r) = \langle \Psi_g | \mathcal{O}_1(x_1) \cdots \mathcal{O}_r(x_r) | \Psi_g \rangle,
\]

where we agree to represent \( r \)-dimensional vectors as

\[
\mathbf{x}_r = (x_1, \ldots, x_r) \quad \text{and} \quad \mathbf{o}_r = (o_1, \ldots, o_r).
\]

On the level of (2.1), the correlation function \( C(\mathbf{x}_r; \mathbf{o}_r) \) is defined for a model in finite volume \( L \). In this formula, \( \langle \Psi_g \rangle \) represents the ground state of the model in finite volume whereas \( \mathcal{O}_a(x) \) are some elementary (in the sense defined below) local operators associated with the model and located at position \( x \). Depending on the nature of the model (continuous or discrete), the position variables can be continuous or discrete. Throughout the paper, we shall build on the assumption that the states of the model are constructed out of a fixed number of pseudo-particles, this number possibly changing from one state to another. Within such a setting, we assume that one can associate a fixed integer \( o_a \) to each operator \( \mathcal{O}_a(x) \). These integers refer to the jump in the pseudo-particle number of the states connected by the operator. More precisely, we assume that the operator \( \mathcal{O}_a(x) \) is an elementary local operator that connects only states having \( N + o_a \) and \( N \) pseudo-particles, \( N \) being arbitrary but \( o_a \) fixed. In other words, the only non-zero form factors of the operator \( \mathcal{O}_a(x) \) are given by the matrix elements

\[
\langle \Phi_N | \mathcal{O}_a(x) | \Psi_{N+o_a} \rangle
\]

in which \( |\Psi_{N+o_a}\rangle \) is some \( N + o_a \) pseudo-particle eigenstate of the model’s Hamiltonian whereas \( |\Phi_N\rangle \) corresponds to an eigenstate built out of \( N \) pseudo-particles. Any local operator can be decomposed in terms of a finite number of such elementary operators \( \mathcal{O}_a(x) \). Finally, we focus solely on the case of translation invariant models (periodic boundary conditions) meaning that one can explicitly factor out the \( x \)-dependence of (2.3) in terms of the relative momentum of the states \( |\Psi_{N+o_a}\rangle \) and \( |\Phi_N\rangle \).

Our aim is to extract the large-distance asymptotic behaviour of the correlation function, in the infinite volume limit, \( L \to +\infty \), by using its form factor expansion obtained by inserting the closure relation in between the operators \( \mathcal{O}_a(x) \). The key matter is that the latter can be re-summed in the appropriate regimes of interest. In fact, we shall focus on the following regime of parameters

\[
1 \ll |x_l - x_k| \cdot p_F \quad \text{where} \ l, k = 1, \ldots, r - 1, \ l \neq k, \quad L \gg x_k \quad k = 1, \ldots, r,
\]

and \( p_F \) denotes the Fermi momentum associated with the model.
In the following, we describe a general framework such that, if the spectrum, eigenstates and form factors of a model fit within its grasp, then the present analysis holds. As we shall argue, this setting is verified for a number of quantum integrable models such as the non-linear Schrödinger model or the XXZ spin-1/2 chain in their massless phase. However, we are deeply convinced that the setting is, in fact, much more general and, in particular, encompasses also numerous instances of so-called one-dimensional quantum liquids, see e.g. [10]. Indeed, our setting provides one with a microscopic alternative to the handling of such models. In particular, the saddle-point arguments we develop in section 3 correspond, on the field theory side, to the heuristic arguments raised within the non-linear Luttinger liquid approach which lead to the projection of the model’s Hamiltonian onto sub-band effective Hamiltonians which, ultimately, lead to an effective description of one-dimensional quantum liquids through the mobile quantum impurity model. In fact, our framework goes even beyond the field theory approach in that it allows one to treat effectively the effects of multi-particle excitations (and not solely one) located far from the model’s Fermi endpoints. Such features become relevant in the study of the time- and distance-dependent correlation functions, see [23] for more details.

2.1. The spectrum

We assume that, in the large-$L$ limit, the spectrum of the model is of particle–hole type\(^3\) and that the ground state $|\Psi_g\rangle$ is constructed within an $N$ pseudo-particle sector. We shall focus on states located in nearby sectors containing $N_s$ pseudo-particles,

$$N_s = N + \bar{\sigma}_s, \quad \bar{\sigma}_s = \sum_{a=1}^s o_a. \quad (2.5)$$

Within our setting, the states of the system in an $N_s$ pseudo-particle sector are labelled by two sets of $n$ integers associated with so-called particle and hole excitations, $n = 0, 1, \ldots N_s$;

$$p_1^{(s)} < \cdots < p_n^{(s)} \quad \text{and} \quad h_1^{(s)} < \cdots < h_n^{(s)}$$

with

$$\begin{align*}
   p_a^{(s)} &\in [-M_L^{(1)}; M_L^{(2)}] \setminus [1; N_s], \\
   h_a^{(s)} &\in [1; N_s].
\end{align*} \quad (2.6)$$

The values taken by the integers $M_L^{(a)}$, $a = 1, 2$, strongly depend on the model. Typically, for models having no upper bound on their energy, one has $M_L^{(a)} = +\infty$, while for models having an upper bound, $M_L^{(1)}$, $M_L^{(2)}$ are both finite but such that $M_L^{(a)} - N$, $a = 1, 2$, both go to $+\infty$ sufficiently fast with $L$. Also note that throughout this paper we shall adopt, for definiteness, the convention that the integers arising in the parametrization of a particle–hole excited state in the $N_s$ pseudo-particle sector always bear a superscript $(s)$.

\(^3\) Such an assumption is not very restrictive as there are indications that the bound state part of the spectrum does not contribute to the algebraic part of the large-distance asymptotic behaviour of correlation functions. For instance, this is supported by exact calculations for the large-distance asymptotic behaviour of two-point functions in the XXZ spin-1/2 chain [19]. This is however most probably not the case for large-distance and large-time asymptotic behaviour.
A given choice of integers
\[ I_n^{(s)} = \{ (p_{an}^{(s)})^n_1; (h_{an}^{(s)})^n_1 \} \]
for an excited state in the \( N_s \) pseudo-particle sector gives rise to a set of rapidities \( \hat{\mu}_{pa}^{(s)} \) for the particle and \( \hat{\mu}_{ha}^{(s)} \) for the holes associated with this state. The former and latter are computed from the integers \( I_n^{(s)} \) by the use of a so-called counting function \( \hat{\xi}_{I_n^{(s)}} \) as the unique solutions to
\[ \hat{\xi}_{I_n^{(s)}}(\hat{\mu}_{pa}^{(s)}) = p_{an}^{(s)} \quad \text{and} \quad \hat{\xi}_{I_n^{(s)}}(\hat{\mu}_{ha}^{(s)}) = h_{an}^{(s)} \] (2.7)
We do stress that the counting function depends on the choice of the excited state, i.e. on the choice of integers \( I_n^{(s)} \). Hence, (2.8) is, in fact, a quite complicated set of equations. We assume that all counting functions admit the large volume \( L \) asymptotic expansion
\[ \hat{\xi}_{I_n^{(s)}}(\omega) = \xi(\omega) + \frac{1}{L} \xi_{-1}(\omega) - \frac{1}{L} F_{R_n^{(s)}}(\omega) + O\left(\frac{1}{L^2}\right) \] (2.9)
The above asymptotic expansion involves three functions \( \xi, \xi_{-1} \) and \( F_{R_n^{(s)}} \).

- \( \xi \) is the asymptotic counting function. It is the same for all excited states and defines a set of rapidities \( \{ \mu_a \}_{a \in \mathbb{Z}} \) as the unique solutions to the equation
  \[ \xi(\mu_a) = \frac{a}{L} \quad \text{so that} \quad \hat{\mu}_{pa}^{(s)} \simeq \mu_{pa}^{(s)} \quad \text{and} \quad \hat{\mu}_{ha}^{(s)} \simeq \mu_{ha}^{(s)} \] (2.10)
at leading order in \( L \).

- \( F_{R_n^{(s)}} \) is called the shift function (of the given excited state with respect to the model’s ground state). It is a function of the macroscopic rapidities
  \[ R_n^{(s)} = \{ (\mu_{pa}^{(s)})^n_1; (\mu_{ha}^{(s)})^n_1 \} \] (2.11)
and also depends on the deviation \( N_s - N = \overline{a}_s \) of the number \( N_s \) of pseudo-particles in the excited state with respect to the one for the ground state \( N \).

Recall that within Landau’s phenomenological approach [27] to interacting Fermi systems, an \( n \) particle–hole excited state is realized by first considering the free, i.e. non-interacting, version of the model. The \( n \)-fold particle–hole excitations for the free model are described in terms of quantum numbers \( n_a \). One then turns the interaction on and evolves it up to the desired value. Then, the pseudo-particles forming such an excited state in the interacting model acquire a phase shift which can be expressed in terms of the shift function \( F_{R_n^{(s)}}(\lambda) \) [10]. Independently, the shift function measures the local deformation, at rapidity \( \lambda \), of the model’s Fermi zone as an effect of the presence of interactions in the system. More precisely, a quasi-particle constituting an \( n \) particle–hole excited state associated with a quantum number \( n_a \) will be characterized by the rapidity \( \hat{\mu}_{na}^{(s)} \). The quasi-particle corresponding to the same quantum number taken in the model’s ground state will be characterized by
the rapidity \( \hat{\lambda}_{na} \). The shift function \( F_{R_n^{(s)}} \) measures the deviation of \( \hat{\mu}_{na}^{(s)} \) with respect to its ground state position \( \hat{\lambda}_{na} \), in units of the local density of rapidities,

\[
F_{R_n^{(s)}}(\mu_{na}^{(s)}) = \frac{\hat{\mu}_{na}^{(s)} - \hat{\lambda}_{na}}{L \xi' (\hat{\mu}_{na}^{(s)})} \left( 1 + O\left( \frac{1}{L} \right) \right).
\]

It is interesting to stress that the critical exponents driving the correlation function asymptotic behaviour are just determined by the values of the shift function evaluated at the Fermi zone boundaries for the various Umklapp excitations.

- \( \xi_{-1} \) represents the \( 1/L \) corrections to the ground state’s counting functions. It is this quantity that drives the non-trivial part of the first sub-leading corrections to the ground state’s energy. It appears for normalization purposes so that the shift function for the ground state vanishes, i.e. \( F_{R_0^{(s)}} = 0 \).

In the large-\( L \) limit and within such a setting, the rapidities for the ground state form a dense distribution on \([-q,q]\)—the so-called Fermi zone of the model—with density \( \xi' \). The endpoints \( \pm q \) are called the Fermi boundaries. In this setting, the shift function allows one to measure how the interaction shifts the local position of the individual constituents of the Fermi zone upon removing some of the rapidities that densely fill the Fermi zone and adding as well additional rapidities outside of this zone. It is the shift functions that, per se, are responsible for the dressing of the bare quantities that characterize the excitations of the quasi-particle.

For instance, the observables of the model such as the relative momentum and energy are parametrized by the particle–hole rapidities

\[
\Delta \mathcal{E}(\mathcal{I}_n^{(s)}) \equiv \mathcal{E}(\mathcal{I}_n^{(s)}) - \mathcal{E}(\mathcal{I}_0^{(0)}) = \sum_{a=1}^{n} (\varepsilon(\mu_{pa}^{(s)}) - \varepsilon(\mu_{ha}^{(s)})) + O\left( \frac{1}{L} \right), \tag{2.12}
\]

\[
\Delta \mathcal{P}(\mathcal{I}_n^{(s)}) \equiv \mathcal{P}(\mathcal{I}_n^{(s)}) - \mathcal{P}(\mathcal{I}_0^{(0)}) = \sum_{a=1}^{n} (p(\mu_{pa}^{(s)}) - p(\mu_{ha}^{(s)})) + O\left( \frac{1}{L} \right). \tag{2.13}
\]

Above, \( \mathcal{I}_0^{(0)} = \{0;0\} \) refers to the set of integers which parametrizes the ground state of the model and \( \mathcal{P}(\mathcal{I}_n^{(s)}) \) and \( \mathcal{E}(\mathcal{I}_n^{(s)}) \) are respectively the momentum and the energy of the state parametrized by the set of integers \( \mathcal{I}_n^{(s)} \). Furthermore, \( p \) is the so-called dressed momentum and similarly \( \varepsilon \) is the dressed energy. Typically, for quantum integrable models, these functions are given as solutions to linear integral equations [9]. Their construction, for more complex, non-integrable models, is a priori much more complicated but carries direct physical meaning. Note that the particle–hole interpretation of the spectrum is manifest at the level of (2.12)–(2.13): the excitations consist of adding ‘particles’ with rapidities \( \mu_{p_1}^{(s)}, \ldots, \mu_{p_n}^{(s)} \) and associated energies \( \varepsilon(\mu_{p_1}^{(s)}), \ldots, \varepsilon(\mu_{p_n}^{(s)}) \) along with creating ‘holes’ in the Fermi zone, with rapidities \( \mu_{h_1}^{(s)}, \ldots, \mu_{h_n}^{(s)} \) and associated energies \( -\varepsilon(\mu_{h_1}^{(s)}), \ldots, -\varepsilon(\mu_{h_n}^{(s)}) \). The superscript \( (s) \) refers to the fact that the excitation labelled by \( \mathcal{I}_n^{(s)} \) takes place above the lowest lying energy level in the \( N_s = N + \sigma \) quasi-particle sector. We henceforth shall assume that the only roots on \( \mathbb{R} \) of the dressed energy \( \varepsilon \) are at \( \pm q \) and that the dressed momentum \( p \) is a strictly monotonic function.
There is however a huge degeneracy in what concerns the O(1) part of the excitation energy $\Delta E(I(s)_n)$: numerous choices of the set $I(s)_n$ will lead to the same O(1) part of $\Delta E(I(s)_n)$. For instance, it follows from (2.9) to (2.10) that, given any integer $\ell$,

$$
\mu_k - \mu_{k+\ell} = O \left( \frac{\ell}{L} \right),
$$

so that $I(s)_n = \{ \{ p_a \}_1^n; \{ h_a \}_1^n \}$ and

$$
\tilde{I}(s)_n = \{ \{ p_a + k_a \}_1^n; \{ h_a + t_a \}_1^n \},
$$

(2.14)

where $k_a, t_a$ are some $L$-independent integers, will lead to the same value for the O(1) part of $\Delta E$, i.e.

$$
\Delta E(I(s)_n) - \Delta E(\tilde{I}(s)_n) = O \left( \frac{1}{L} \right).
$$

(2.15)

At this stage of the description of our formalism, one may raise the question of the computability—for a given model—of quantities such as the dressed energy $\varepsilon$, and the dressed momentum $p$ or the shift function $F_{R(s)}$. Clearly, for a given general model, the task is hopeless because it would amount to diagonalizing the Hamiltonian, and then extracting an explicit description thereof in the $L \to +\infty$ limit. Still, such calculations are within the grasp of quantum integrable models. Furthermore, these quantities can be computed, perturbatively, around a free theory [30]. Likewise, they can be accessed through experiments. For instance, the dispersion relation $p \mapsto \varepsilon(p)$ for holes or particles can be read off from the density structure factors or the spectral functions. Finally, specific values of the shift function which, in fact, are those relevant to the descriptions of the large-distance behaviour of the model’s correlation functions, can be read off from the $1/L$-corrections to the energy of excited states located in the immediate vicinity of an endpoint of the Fermi zone.

2.2. The form factors

In the analysis of the multi-point correlation function $C(x_r; o_r)$ (2.1) through its form factor expansion, one needs to access the form factors of the local operators $O_s, s = 1, \ldots, r$, between two excited states of the model. According to our assumptions on the structure of the model’s spectrum, these will be labelled by two sets of multi-indices $I(s-1)_m$ and $I(s)_n$ corresponding to the outgoing (bra) and ingoing (ket) states. In fact, since we imposed periodic boundary conditions on the model and required translation invariance, the form factors will satisfy

$$
\langle \Psi(I(s-1)_m) | O_s(x) | \Psi(I(s)_n) \rangle = e^{ix(\Delta P)_{s-1}} \langle \Psi(I(s-1)_m) | O_s(0) | \Psi(I(s)_n) \rangle,
$$

(2.16)

$$(\Delta P)_{s-1}^s = \mathcal{P}(I(s-1)_m) - \mathcal{P}(I(s)_n).$$

Scalar observables introduced in the last subsection were parametrized, in the large-$L$ limit, solely by the macroscopic set of rapidities $R(s)_n$ subordinate to the set of multi-indices $I(s)_n$. This is no longer the case for form factors as we demonstrated in our previous work [22]. Within our setting, the latter are parametrized, in the large-$L$ limit, not only by the sets of macroscopic rapidities $R(s)_n, R(s-1)_m$, but also by the sets of discrete integers...
If the form factors take the form
\[
\mathcal{F}_{\mathcal{O}_s}(I^{(s-1)}_m | I^{(s)}_n) = \langle \Psi(I^{(s)}_n) | \mathcal{O}_s(0) | \Psi(I^{(s-1)}_m) \rangle 
\]
the following apply in this decomposition.

- \( S^{(O_s)} \) is called the smooth part since it only depends on the macroscopic rapidities \( R^{(s)}_m, R^{(s-1)}_n \), and this in a smooth manner. As a consequence, a small (of the order \( O(1) \)) change in the value of the integers parametrizing the state, say \( p_a \leftrightarrow p_a + \kappa \), will not change the value of the smooth part (up to \( 1/L \) corrections). Furthermore, being a set function, it is invariant under permutation of the particle or hole rapidities.

- The smooth part enjoys hypergeometric-like reduction properties. Namely, if within a given set of macroscopic rapidities a particle’s rapidity coincides with a hole rapidity, then this dependence simply disappears. More precisely

\[
S^{(O_s)}(R^{(s-1)}_m; R^{(s)}_n)_{\mu^{(s)}_{k_1}=\mu^{(s)}_{k_2}} = S^{(O_s)}(R^{(s-1)}_m; \hat{R}^{(s)}_n) 
\]

with \( \hat{R}^{(s)}_n = \{\{\mu^{(s)}_{k_1}\}_{1\neq k}; \{\mu^{(s)}_{\ell}\}_{1\neq \ell}\} \).

- \( D^{(s)} \) is called the discrete part since it depends not only on the macroscopic rapidities \( R^{(s)}_m, R^{(s-1)}_n \) but also explicitly (i.e. not through the parametrization by the macroscopic rapidities) on the sets of integers labelling the excited states \( I^{(s)}_n \), \( I^{(s-1)}_m \). The main effect of such a dependence is that a small change in the value of the integers parametrizing the state does imply a significant change (of the order of \( O(1) \)) in the value of \( D^{(s)} \). The discrete part thus keeps track of the microscopic details of the different excited states.

The smooth part represents, in fact, a non-universal part of the model’s form factors. Its explicit expression not only depends on the operator \( O_s \) but also varies strongly from one model to another, see \([22, 25]\) for examples issuing from quantum integrable models. However, the part \( D^{(s)} \) is entirely universal within the present setting of the description of the model’s spectrum. It solely depends on the values of the pseudo-particle sectors that the operator connects, namely the integer \( o_s = N_s - N_{s-1} \). Its general explicit expression plays no role in our analysis, in the sense that we shall only need the expression for specific excited states, namely the one belonging to the so-called \( \ell_s \) critical classes that we define below.

We refer the interested reader to \([21, 22]\) for a more thorough discussion relative to the origin of the discrete \( D^{(s)} \) and smooth \( S^{(O_s)} \) parts in the framework of Bethe ansatz solvable models.
2.3. The critical $\ell_s$ class

As we shall argue in the following, only a very specific class of excited states will play an effective role in our analysis—the so-called critical states. These excited states are characterized by the fact that, in the $L \to +\infty$ limit, all macroscopic rapidities describing the particle and hole excitations ‘collapse’ on the model’s Fermi boundary,

$$\mu_{p_a}^{(s)} \simeq \pm q \quad \text{and} \quad \mu_{h_a}^{(s)} \simeq \pm q. \quad (2.18)$$

There, the ± sign depends on whether the particle or hole rapidity collapses on the right or left Fermi boundary.

A set of integers $I^{(s)}_n$ is said to parametrize a critical excited state if the associated particle–hole integers $\{p_a^{(s)}\}_1^n$ and $\{h_a^{(s)}\}_1^n$ can be represented as

$$
\{p_a^{(s)}\}_1^n = \{N_s + p_{a;+}\}_1^{n_{p;+}} \cup \{1 - p_{a;+}\}_1^{n_{p;-}} \quad \text{and} \quad
\{h_a^{(s)}\}_1^n = \{1 + N_s - h_{a;+}\}_1^{n_{h;+}} \cup \{h_{a;-}\}_1^{n_{h;-}},
$$

where the integers $p_{a;\pm}, h_{a;\pm} \in \mathbb{N}$ are ‘small’ compared to $L$, i.e.

$$
\lim_{L \to +\infty} \frac{p_{a;\pm}}{L} = \lim_{L \to +\infty} \frac{h_{a;\pm}}{L} = 0, \quad (2.20)
$$

and the integers $n_{p;\pm}$ and $n_{h;\pm}$ satisfy to the constraint

$$
n_{p;+} + n_{p;-} = n_{h;+} + n_{h;-} = n. \quad (2.21)
$$

Within this setting, one can readily check that the critical excited state described above will have $n_{p;+}$ or $n_{p;-}$ particles, resp. $n_{h;+}$ or $n_{h;-}$ holes, on the right or the left end of the Fermi zone $[-q; q]$ associated with the $N_s$ pseudo-particle sector.

In fact, one can distinguish between various critical states by organizing them into so-called $\ell_s$ critical classes. This classification takes its origin from the fact that all such states have a vanishing excitation energy (up to $O(1/L)$ corrections) but can be gathered into classes depending on the value of their macroscopic momenta $2\ell_s p_F$, where $p_F = p(q)$ is the so-called Fermi momentum and

$$
\ell_s = n_{p;+} - n_{h;+} = n_{h;-} - n_{p;-}. \quad (2.22)
$$

The index $s$ in $\ell_s$ is there so as to keep track of the $N_s$ pseudo-particle sector with which the critical class is associated. However, for the sake of simplifying the notations in formulae, we chose not to emphasize the $s$ dependence in the left or right particle/hole numbers $n_{p/h;\pm}$.

In fact, one has the following large-$L$ expansion for the relative excitation momentum (2.13) associated with the $\ell_s$ critical class excited state described above:

$$
\Delta P(I^{(s)}_n) = 2\ell_s p_F + \frac{2\pi}{L} \left\{ \sum_{a=1}^{n_{p;+}} p_{a;+} + \sum_{a=1}^{n_{h;+}} (h_{a;+} - 1) \right\} \\
- \frac{2\pi}{L} \left\{ \sum_{a=1}^{n_{p;-}} (p_{a;-} - 1) + \sum_{a=1}^{n_{h;-}} h_{a;-} \right\} + \cdots \quad (2.23)
$$

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where we do stress that all the terms included in the dots are either of the order of $O(1/L)$ but do not depend on the integers $p_{a;\pm}$ and $h_{a;\pm}$ or they depend on these integers but are of the order of $O(1/L^2)$.

It is convenient, for such $\ell_s$ critical excited states, to use a reparametrization of the sets of integers labelling the excitations. It is readily seen that the excited states of an $\ell_s$ critical class are described by two sets of integers $J^{(s)}_{n;\pm} \setminus m_{h;\pm}$ each containing all the information on the local integers on the right (+) or left (−) Fermi boundaries. Here, we agree upon

$$\mathcal{J}^{(s)}_{n;m} = \{ \{p_a^{(s)}\}_1^n; \{h_a^{(s)}\}_1^m \}. \quad (2.24)$$

Note that the value of $\ell_s$ is encoded in the very notation $\mathcal{J}^{(s)}_{n;\pm} \setminus m_{h;\pm}$ by means of the identification (2.22). Since there is a one-to-one correspondence between the set $I^{(s)}_a$ and $\mathcal{J}^{(s)}_{n;\pm} \setminus m_{h;\pm}$, we shall identify the two sets. We do stress that the parameter $s$ does play a role in this correspondence, cf (2.19).

2.4. Large-$L$ expansion of form factors connecting critical states

As we have already mentioned, solely critical states enter in the analysis of the form factor expansion at large spacial separation between the operators. Thus, we now present the explicit expression for the form factor taken between such states. Given two excited states

$$I^{(s-1)}_m \equiv \mathcal{J}^{(s-1)}_{m_{p;+}+m_{h;+}} \setminus m_{h;+} \quad \text{and} \quad I^{(s)}_n \equiv \mathcal{J}^{(s)}_{n_{p;+}+n_{h;+}} \setminus m_{h;+} \quad (2.25)$$

belonging respectively to the

$$\ell_{s-1} = m_{p;+} - m_{h;+} = m_{h;+} - m_{p;+} \quad \text{and} \quad \ell_s = n_{p;+} - n_{h;+} = n_{h;+} - n_{p;+} \quad (2.26)$$

critical classes, we shall assume that the form factors of local operators take the form\(^4\)

$$F_{O_s}(I^{(s-1)}_m | I^{(s)}_n) = F_{O_s}(\ell_{s-1}, \ell_s) \cdot C^{(\ell_{s-1}; \ell_s) \cdot \nu_s^+ \nu_s^-} \times \mathcal{F}^{(+)}[\mathcal{J}^{(s-1)}_{m_{p;+}+m_{h;+}} \setminus m_{h;+}; \mathcal{J}^{(s)}_{n_{p;+}+n_{h;+}} \setminus m_{h;+} | \nu_s^+] \cdot \mathcal{F}^{(-)}[\mathcal{J}^{(s-1)}_{m_{p;+}+m_{h;+}}; \mathcal{J}^{(s)}_{n_{p;+}+n_{h;+}} | \nu_s^-]. \quad (2.27)$$

The constituents of the above formula are parametrized by the values

$$\nu_s^+ = \nu_s(q) - \phi_s \quad \text{and} \quad \nu_s^- = \nu_s(-q) \quad (2.28)$$

that the relative shift function between the $\ell_s, \ell_{s-1}$ critical states,

$$\nu_s(\lambda) = F_{s-1}(\lambda) - F_s(\lambda), \quad (2.29)$$

\(^4\) A priori, this formula could be modified by the presence of a sign factor $(-1)^{\sigma_{s-1}+\sigma_{s-1}}$ originating from the determination of the square root one chooses for the norms. The integer $\sigma_s$ depends, a priori, on $I^{(s)}_s$. However, the very structure of such a sign factor makes its contribution to the form factor expansion irrelevant as it always appears twice. We have therefore disregarded its presence from the very beginning. One can also allow for the multiplication of (2.27) by an $L$-dependent phase factor of the form $e^{i\phi_s(h_{s-1} - h_{s-1})}$, with $\phi_s$ depending on $I^{(s)}_s$. Once again, the appearance of such a phase factor has no influence on the form factor expansion (A.33), hence we simply omit it here (note however that the presence of such a phase factor should be taken into account in the definition (2.30) of $F_{O_s}(\ell_{s-1}, \ell_s)$).
takes on the right/left endpoints of the Fermi zone, up to subtracting the level \( \omega_s \) of the operator \( \mathcal{O}_s \) in the case of the right endpoint. In formula (2.27), all the pre-factors but \( \mathcal{F}_{\mathcal{O}_s}(\ell_{s-1}, \ell_s) \) issue from the ‘relevant’ part of the discrete term. Their explicit expressions are given below. Even though these may appear as slightly complicated, the physical interpretation of each factor in (2.27) is crystal clear. Indeed, the quantity \( \mathcal{F}_{\mathcal{O}_s}(\ell_{s-1}, \ell_s) \) represents the properly normalized finite and non-universal (i.e. model and operator dependent) part of the large-\( L \) behaviour of the form factor of the operator \( \mathcal{O}_s \) taken between fundamental representatives of the \( \ell_s \) and \( \ell_{s-1} \) critical classes. More precisely, it is defined as:

\[
\mathcal{F}_{\mathcal{O}_s}(\ell_{s-1}, \ell_s) = \lim_{L \to +\infty} \left\{ \left( \frac{L}{2\pi} \right)^{\rho_s(\nu_s^+) + \rho_s(\nu_s^-)} \langle \Psi(\mathcal{L}^{(s-1)}_{\ell_{s-1}}) | \mathcal{O}_s(0) | \Psi(\mathcal{L}^{(s)}_{\ell_s}) \rangle \right\}
\]

(2.30)

where \( \ell_{s-1} \) and \( \ell_s \) are the right and left Fermi boundary critical form factors.

The relative shift function \( r_{\mathcal{O}_s}(\ell_{s-1}, \ell_s) \) is defined as

\[
r_{\mathcal{O}_s}(\ell_{s-1}, \ell_s) = \lim_{L \to +\infty} \left\{ \left( \frac{L}{2\pi} \right)^{\rho_s(\nu_s^+) + \rho_s(\nu_s^-)} \langle \Psi(\mathcal{L}^{(s-1)}_{\ell_{s-1}}) | \mathcal{O}_s(0) | \Psi(\mathcal{L}^{(s)}_{\ell_s}) \rangle \right\}
\]

(2.31)

in which the sets of integers \( \mathcal{L}^{(s-1)}_{\ell_{s-1}} \) and \( \mathcal{L}^{(s)}_{\ell_s} \) parametrizing the excited states correspond to the fundamental representatives of the \( \ell_{s-1} \) and \( \ell_s \) critical classes. Namely, \( \mathcal{L}^{(s)}_{\ell_s} \) is the set of particle–hole integers living on the Fermi boundary in the \( N_s \) pseudo-particle sector such that

\[
\mathcal{L}^{(s)}_{\ell_s} = \left\{ \{ \{ a_{1+:} = a \}^*_{\ell_s}; \{ 0 \} \} \cup \{ \{ a_{1:-} = a \}^*_{\ell_s}; \{ 0 \} \} \right\} \quad \text{if } \ell_s \geq 0
\]

\[
\mathcal{L}^{(s)}_{\ell_s} = \left\{ \{ \{ 0 \} \} \cup \{ \{ a_{1:-} = a \}^*_{\ell_s}; \{ 0 \} \} \right\} \quad \text{if } \ell_s \leq 0.
\]

The power of the volume \( L \) arising in (2.30) involves the right \( \rho_s(\nu_s^+) \) and left \( \rho_s(\nu_s^-) \) scaling dimensions whose generic expression reads

\[
\rho_s(\nu) = \frac{1}{2}(\ell_s - \ell_{s-1})^2 + \frac{1}{2} \nu^2 - (\ell_s - \ell_{s-1}) \nu.
\]

The factor \( \mathcal{F}^{(+)} \) corresponds to the contributions of the excitations on the right Fermi boundary of the model. It is a function of the value taken on the right Fermi boundary by the relative shift function \( \nu_s \) associated with the \( \ell_s \) class of interest. Also, \( \mathcal{F}^{(+)} \) depends on the sets of integers \( \mathcal{J}_{\ell_{s-1}}^{(s-1)} \) and \( \mathcal{J}_{\ell_s}^{(s)} \) parametrizing the excitations on the right boundary for the \( s-1 \) and \( s \) excited states. Given two sets of integers

\[
\mathcal{J}_{\ell_{s-1}}^{(s-1)} = \{ \{ a \}^n_1; \{ a \}^n_1 \} \quad \text{and} \quad \mathcal{J}_{\ell_s}^{(s)} = \{ \{ a \}^n_1; \{ a \}^n_1 \},
\]

the right Fermi boundary critical form factor reads

\[
\mathcal{F}^{(+)}(\mathcal{J}_{\ell_{s-1}}^{(s-1)}; \mathcal{J}_{\ell_s}^{(s)} | \nu) = \left( \frac{2\pi}{L} \right)^{\rho_s(\nu)} \cdot (-1)^{\nu} \cdot \left( \frac{\sin[\pi \nu]}{\pi} \right)^{n+1} \cdot \mathcal{F}^{(+)}(\mathcal{J}_{\ell_{s-1}}^{(s-1)}; \mathcal{J}_{\ell_s}^{(s)} | \nu)
\]

(2.32)

\[
\times \prod_{a>b}^{\mathcal{J}_{\ell_{s-1}}^{(s-1)}} \prod_{a=1}^{\mathcal{J}_{\ell_s}^{(s)}} \prod_{b=1}^{\mathcal{J}_{\ell_s}^{(s)}} (p_a - p_b) \cdot \prod_{b=1}^{\mathcal{J}_{\ell_s}^{(s)}} (a_{a:-} - a_{b:-}) \cdot \prod_{a=1}^{\mathcal{J}_{\ell_s}^{(s)}} \prod_{b=1}^{\mathcal{J}_{\ell_s}^{(s)}} (k_a - k_b) \cdot \prod_{a=1}^{\mathcal{J}_{\ell_s}^{(s)}} (t_a - t_b)
\]

(2.33)

\[
\times \Gamma \left( \{ p_a + \nu \}^{\mathcal{J}_{\ell_{s-1}}^{(s-1)}} \cdot \{ a_{a:-} - \nu \}^{\mathcal{J}_{\ell_s}^{(s)}} \cdot \{ k_a - \nu \}^{\mathcal{J}_{\ell_s}^{(s)}} \cdot \{ t_a + \nu \}^{\mathcal{J}_{\ell_s}^{(s)}} \right).
\]

\(5\) We do stress here that within our setting the volume-renormalized form factors are supposed to be given data within our approach. Given a specific model, these should be computed by some alternative means, for instance by the use of a perturbative expansion around a free fermion point or some explicit result for a quantum integrable model.
Note that $\mathcal{F}^{(+)}$ decays as an algebraic power of the volume $L$ with a scaling dimension $\rho_s(\nu)$. Likewise, the factor $\mathcal{F}^{(-)}$ corresponds to the contributions of the excitations on the left Fermi boundary of the model. It is a function of the value $\nu^-_s$ taken on the left Fermi boundary by the relative shift function $\nu_s$ and of the sets $\mathcal{J}^{(s-1)}_{m_p \cdots m_h \cdots}$ and $\mathcal{J}^{(s)}_{m_p \cdots m_h \cdots}$ of integers parametrizing the excitations on the left Fermi boundary for the $s - 1$ and $s$ excited states. Given sets as in (2.33) for the parametrization of the excitations on the left Fermi boundary, one has

$$
\mathcal{F}^{(-)}[\mathcal{J}_{m_p; m}; \mathcal{J}_{n_k; n_t} \mid \nu] = \left(\frac{2\pi}{L}\right)^{\rho_s(\nu)} \cdot (-1)^{n_k} \cdot \left(\frac{\sin[\pi \nu]}{\pi}\right)^{n_k + n_t} \cdot \varpi[\mathcal{J}_{m_p; m}; \mathcal{J}_{n_k; n_t} \mid -\nu] \times \frac{\prod_{a > b (p_a - p_b)} \cdot \prod_{a \leq b (h_a - h_b)} \cdot \prod_{a > b (k_a - k_b)} \cdot \prod_{a \leq b (t_a - t_b)}}{\prod_{a=1}^{n_p} \prod_{b=1}^{n_h} (p_a + h_b - 1) \prod_{a=1}^{n_k} \prod_{b=1}^{n_t} (k_a + t_b - 1)} \times \Gamma \left( \left\{ \{p_a - \nu\}, \{h_a + \nu\}, \{k_a + \nu\}, \{t_a - \nu\} \right\} \right).
$$

Just as for the right Fermi boundary critical form factor, the left one decays algebraically with the volume of the model with a left scaling dimension $\rho_s(\nu)$.

The expressions for the right and left Fermi boundary factors involve products of $\Gamma$-functions written as

$$
\Gamma \left( \left\{ \left\{ v_{a_1}^{n_1} \right\} \right\} \right) = \Gamma \left( \left\{ v_1, \ldots, v_n \right\} \right) = \prod_{a=1}^{n_p} \Gamma(v_a) \prod_{a=1}^{n_h} \Gamma(w_a).
$$

The $\Gamma$ functions encode the structure of excitations within a given $(s)$-sector of excitations on the Fermi zone. We do stress that, in order to lighten slightly the notations, we have dropped the indices of the arguments of the $\Gamma$-functions arising in the definitions of the left and right local factors $\mathcal{F}^{(+)}$.

The expressions for the left and right local factors $\mathcal{F}^{(+)}$ also involve the function $\varpi$ whose presence translates the interactions between integers parametrizing the neighbouring sectors $(s - 1)$ and $(s)$ that are connected by the operator $\mathcal{O}_s$. Taking two sets of integers as in (2.33), one has

$$
\varpi[\mathcal{J}_{m_p; m}; \mathcal{J}_{n_k; n_t} \mid \nu] = \prod_{a=1}^{n_p} \left\{ \prod_{b=1}^{n_h} \left(1 - k_b - h_a + \nu\right) \right\} \cdot \prod_{a=1}^{n_h} \left\{ \prod_{b=1}^{n_t} \left(t_a - h_b + \nu\right) \right\}.
$$

Finally, the function $C^{(\ell_s - 1 : \ell_s)}(\nu_s^+, \nu_s^-)$ is a normalization constant. It provides the appropriate normalization of the expression for the general critical form factor. It is such that it precisely cancels out the $L$-independent contributions of the right and left critical form factors when focusing on the fundamental representative of the $\ell_s, \ell_{s-1}$ critical classes. Its explicit form can be computed in terms of the Barnes $G$-function [4] and reads

$$
C^{(\ell_s - 1 : \ell_s)}(\nu_s^+, \nu_s^-) = G \left( \left\{ 1 + \nu_s^-, 1 - \nu_s^+, 1 + \ell_s - \nu_s^-, 1 - \ell_s + \nu_s^+ \right\} \right).
$$

Note that above we have adopted similar product conventions as for $\Gamma$-functions, cf (2.34).

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3. Asymptotic behaviour of multi-point correlation functions

3.1. Large-distance asymptotic expansion of multi-point functions

In this section, building on our previous assumptions on the structure of the spectrum and form factors of local operators, we derive the large-distance asymptotic behaviour of the correlator \( C(\mathbf{x}_r; \mathbf{o}_r) \), cf (2.1). As already discussed, our method builds on a microscopic analysis of the form factor expansion for this correlator. Hence, in between the operators \( s \) and \( s + 1 \) we insert a decomposition of the identity. In principle, we should thus sum up over all the states of the model, namely the states generated by the particle–hole excitations along with the more complex states related to bound states. However, building up over all the states of the model, namely the states generated by the particle–hole states, as described in section 2.1. Thus, for each \( s (s = 1, \ldots, r - 1) \) we sum up over all possible choices of the sets \( \{ \mathcal{I}^{(s)} \} \) as given in (2.7), with \( n^{(s)} = 0, 1, \ldots, \).

In other words the multi-point correlator is recast as

\[
C(\mathbf{x}_r; \mathbf{o}_r) = \prod_{s=1}^{r-1} \left\{ \sum_{\{ \mathcal{I}^{(s)} \}_{\nu_1^{(s)}}^{\nu_{\nu_1^{(s)}}^{(s)}}} \right\} \cdot \prod_{s=1}^{r-1} \left\{ \exp[i(x_{s+1} - x_s) \cdot \Delta \mathcal{P}(\mathcal{I}^{(s)}_{\nu_1^{(s)}})] \right\} \cdot \prod_{s=1}^{r} \mathcal{F}_s(\mathcal{I}^{(s)}_{\nu_1^{(s)}} | \mathcal{I}^{(s)}_{\nu_1^{(s)}}),
\]

(3.1)

in which \( \Delta \mathcal{P}(\mathcal{I}^{(s)}_{\nu_1^{(s)}}) \) stands for the relative excitation momentum defined in (2.13).

We do stress that the equality (3.1) is to be understood in the sense of having neglected all the contributions from the bound states to the form factor expansion.

We now follow exactly the reasoning outlined in [21]. The multi-dimensional sum in (3.1) contains quickly oscillating terms with speed measured by the magnitude of \( |x_{k+1} - x_k| \cdot p_F, k = 1, \ldots, r - 1 \). Thus, by analogy with a multiple integral of oscillatory type, the leading asymptotic behaviour will issue from localizing the sums at the endpoints of the summation domain or at the saddle points of the oscillating exponent. However, due to monotonicity of the dressed momentum function \( p \), the relative excitation momentum \( \Delta \mathcal{P}(\mathcal{I}^{(s)}_{\nu_1^{(s)}}) \) does not have saddle points. Hence, the leading \( |x_k - x_l| \cdot p_F \gg 1, k \neq l \) asymptotic behaviour will be given by localizing the summand on the states belonging to \( \ell_1 \) critical classes, \( s = 1, \ldots, r - 1 \). Upon inserting the local expression for the model’s form factors, a straightforward calculation leads to the conclusion that

\[
C(\mathbf{x}_r; \mathbf{o}_r) \approx \sum_{\ell_1} \left( \frac{2\pi}{L} \right)^{\nu_1^{(s)}(\ell_1, \nu_1^{(s)})} \cdot \prod_{s=1}^{r-1} \left\{ \exp \left[ i \left( x_{s+1} - x_s \right) \cdot \frac{\Delta \mathcal{P}(\mathcal{I}^{(s)}_{\nu_1^{(s)}})}{(2\pi)^{\nu_1^{(s)}}(\ell_1, \nu_1^{(s)})} \right] \right\} \cdot \prod_{s=1}^{r} \mathcal{F}_s(\mathcal{I}^{(s)}_{\nu_1^{(s)}} | \mathcal{I}^{(s)}_{\nu_1^{(s)}}),
\]

(3.2)

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The summand at fixed \( \ell_{r-1} \in \mathbb{Z}^{-1} \) is weighted by an algebraic factor in the volume \( L \). The associated exponent \( \vartheta(\ell_{r-1}, o_r) \) reads

\[
\vartheta(\ell_{r-1}, o_r) = \frac{1}{2} \sum_{s=1}^{r} \left\{ (\nu_s^+) + (\nu_s^-) - 2 \ell_s \right\} - 2 \sum_{s=2}^{r-1} \ell_s \ell_{s-1}.
\] (3.3)

We do insist that each relative shift function \( \nu_s \) does depend \textit{a priori} on the integers \( \ell_{s-1}, \ell_s \) labelling the critical class of the excited states being connected by the \( s \)th operator.

The form factor expansion has been recast with the help of multi-point restricted sums \( \mathcal{S}^{\pm}_{\ell_{r-1}} \). These refer to multi-dimensional sums that correspond to summing up the whole set of contributions coming from excitations on Fermi surfaces, for the intermediate states labelled by \( s = 1, \ldots, r - 1 \), and belonging to given critical classes labelled by the vector \( \ell_{r-1} \). The + and the − sums correspond respectively to the contributions stemming from the right and the left Fermi boundaries.

The multi-point restricted sums of interest are given as the multi-dimensional sums below,

\[
\mathcal{S}^{\pm}_{\ell_{r-1}}(\{\ell_s\}_{1}^{r-1}, \{\nu_s\}_{1}) = \prod_{s=1}^{r-1} \sum_{n_{\ell_s}^{(s)}(s) \in \mathbb{N}^*} \mathcal{S}_{\ell_{r-1}}^{\pm}(\{\nu_s\}_{1}^{r-1}; \nu_s, \nu_{s+1}; t_s)
\]

\[
\times \prod_{s=2}^{r-1} \mathcal{S}^{\pm}_{\ell_{r-1}}(\{\ell_s\}_{1}^{r-1}, \{\nu_s\}_{1}) | \pm \nu_s \). \] (3.4)

The summation in the above formula runs through all the possible choices of the sets of integers

\[
\mathcal{S}_{\ell_{r-1}}^{(s)} \equiv \{ \{ p_a^{(s)} \}_{1}^{n_p} \}; \{ h_a^{(s)} \}_{1}^{n_h} \}
\] (3.5)

parametrizing the excitations on the relevant boundary (left for − and right for +), provided that these lead to an \( \ell_s \) critical class. The summations are repeated for \( s = 1, \ldots, r - 1 \). In more explicit terms, the formula corresponds to a multiple sum over particle-like \( p_a^{(s)} \) and hole-like \( h_a^{(s)} \) integers with respectively the number \( n_p^{(s)} \) of particle-like variables and \( n_h^{(s)} \) of hole-like variables, taking all admissible values in \( \mathbb{N} \) that are compatible with the condition \( n_p^{(s)} - n_h^{(s)} = \pm \ell_s \). These particle and hole integers satisfy the constraints

\[
p_1^{(s)} < \cdots < p_{n_p^{(s)}} \quad \text{with} \quad p_{a}^{(s)} \in \mathbb{N}^* \quad \text{and} \quad h_1^{(s)} < \cdots < h_{n_h^{(s)}} \quad \text{with} \quad h_{a}^{(s)} \in \mathbb{N}^*.
\] (3.6)
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In what concerns the functions composing the summand, \( \varpi \) has been defined in (2.35) whereas the functions \( \mathcal{R}^\pm \) read

\[
\mathcal{R}^-(\mathcal{J}_{n_p:n_h} | \nu, \eta; t) = \left( -\frac{\sin[\pi \nu]}{\pi} \cdot \frac{\sin[\pi \eta]}{\pi} \right)^{n_h} \cdot \frac{\prod_{a=b}^{n_p} (p_a - p_b)^2 \cdot \prod_{a<b}^{n_h} (h_a - h_b)^2}{\prod_{a=1}^{n_p} \prod_{b=1}^{n_h} (p_a + h_b - 1)^2} \times \prod_{a=1}^{n_p} \left\{ e^{it(1-p_a)} \Gamma \left( \frac{p_a + \nu, p_a - \eta}{p_a, p_a} \right) \right\} \cdot \prod_{a=1}^{n_h} \left\{ e^{-it h_a} \Gamma \left( \frac{h_a - \nu, h_a + \eta}{h_a, h_a} \right) \right\}
\]

and

\[
\mathcal{R}^+(\mathcal{J}_{n_p:n_h} | \nu, \eta; t) = \left( -\frac{\sin[\pi \nu]}{\pi} \cdot \frac{\sin[\pi \eta]}{\pi} \right)^{n_h} \cdot \frac{\prod_{a=b}^{n_p} (p_a - p_b)^2 \cdot \prod_{a<b}^{n_h} (h_a - h_b)^2}{\prod_{a=1}^{n_p} \prod_{b=1}^{n_h} (p_a + h_b - 1)^2} \times \prod_{a=1}^{n_p} \left\{ e^{it p_a} \Gamma \left( p_a - \nu, p_a + \eta \right) \right\} \cdot \prod_{a=1}^{n_h} \left\{ e^{it (h_a-1)} \Gamma \left( h_a + \nu, h_a - \eta \right) \right\}
\]

Above, the set \( \mathcal{J}_{n_p:n_h} \) is as given in (2.33). Quite remarkably, the multi-particle restricted sums can be re-summed in a quite compact way, generalizing the results of [21, 28]

\[
\mathcal{S}_{\ell, r}^\pm (\{t_b\}_{b=1}^{r-1}; \{\nu_s\}_{s=1}^{s}) = \prod_{s=1}^{r-1} \left\{ e^{\pm i t_s (\ell_s+1)/2} G \left( \pm (\ell_s - \nu_s), \pm (\ell_s + \nu_s+1) \right) \right\} \cdot \prod_{s=2}^{r} G \left( \pm (\ell_s - \nu_s), \pm (\ell_s + \nu_s+1) \right) \cdot \prod_{b>a} \left( 1 - e^{\pm i \sum_{s=a}^{b-1} t_s} (\nu_a + \kappa_a)(\nu_b + \kappa_b) \right).
\] (3.7)

The product appearing at the bottom right of (3.7) involves integers \( \kappa_a, a = 1, \ldots, s \) which are defined in terms of the integers \( \ell_a, a = 1, \ldots, s - 1 \), as

\[
\kappa_s = \ell_{s-1} - \ell_s \quad \text{for } s = 1, \ldots, r \quad \text{so that } \sum_{a=1}^{r} \kappa_a = 0 \quad (3.8)
\]

where, as above, we agree upon \( \ell_0 = \ell_r = 0 \).

In the present paper we do not pretend to prove the above summation formulae. We however give a formal derivation. These arguments do not lead, however, to a proof in that several justifications relative to the exchangeability of symbols and limits are omitted. We leave the rigorous proof of the summation formulae to some subsequent publication. Our present formal justification is based on the use of an auxiliary, ‘model’, form factor series. The latter can be recast in two ways,

- as a Toeplitz determinant generated by a symbol with Fisher–Hartwig singularities, this by generalizing the techniques of form factor summations developed in [24];

- as an asymptotic series whose individual summands directly involve multi-particle restricted sums.

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Comparisons of the large-$N$ limit of both expressions yield the sought identities. We refer the reader to the appendix for our formal proof.

Building on the above results, allowing one to sum up the multi-point restricted sums and taking into account their most natural parametrization in terms of the $\kappa_a$ integers, we change the summation variables to the more convenient $\kappa$-like reparametrization,

$$
\ell_s(\kappa_r) = \sum_{a=s+1}^{r} \kappa_a \quad \text{for } s = 1, \ldots, r - 1.
$$

It is then readily checked that, for any set of parameters $\{t_a\}$ and $\{\nu_a\},$

$$
\sum_{s=1}^{r-1} \ell_s(\kappa_r) \cdot t_s = \sum_{s=2}^{r} \ell_{s-1} \kappa_s,
$$

and

$$
2 \sum_{s=1}^{r} [\ell_s(\kappa_r)]^2 - 2 \sum_{s=2}^{r-1} \ell_s(\kappa_r) \ell_{s-1}(\kappa_r) + 2 \sum_{s=1}^{r-1} ((\nu_{s+1} - \nu_s) \ell_s(\kappa_r))
\quad + \sum_{s=1}^{r} \nu_s^2 = \sum_{s=1}^{r} (\nu_s + \kappa_s)^2,
$$

where in (3.10) we have used a similar notation as in (2.5). Upon using the explicit expressions for the multi-particle restricted sums, we are thus led to

$$
C(x_r; o_r) = \sum_{\kappa_r \in \mathbb{Z}^r} \prod_{s=1}^{r} \left\{ e^{2i\pi \nu_s x_s} \right\} \cdot \mathcal{F}(\{\kappa_a\}_1^r; \{o_a\}_1^r)
\quad \times \prod_{s=1}^{r} \left\{ \frac{(1/2)[\theta^+_{s}(\kappa_s)]^2 + (1/2)[\theta^-_{s}(\kappa_s)]^2}{L} \right\} \cdot \prod_{b>a}^{r} \left\{ 1 + e^{(2\pi i/L)(x_b-x_a)}[\theta^+_{b}(\kappa_b)\theta^-_{a}(\kappa_a) - \theta^-_{b}(\kappa_b)\theta^+_{a}(\kappa_a)] \right\}
\quad \times \left[ 1 - e^{-(2\pi i/L)(x_b-x_a)}\theta^-_{b}(\kappa_b)\theta^+_{a}(\kappa_a) \right].
$$

There, we have set

$$
\theta^\pm_{b}(\kappa_b) = \nu^\pm_{b} + \kappa_b.
$$

We do stress that $\theta^\pm_{b}$ has an explicit dependence on $\kappa_b$ but also an implicit one through the relative shift function $\nu^\pm_{b}.$ Furthermore, we have made use of the fact that the relative shift functions $\nu^\pm_{a}$ only depend on the parameter $\kappa_a.$ Finally, the asymptotic formula involves the factor

$$
\mathcal{F}(\{\kappa_a\}_1^r; \{o_a\}_1^r) = \prod_{s=1}^{r} \mathcal{F}_{\mathcal{O}_s}(\ell_{s-1}(\kappa_r), \ell_s(\kappa_r)).
$$

The above amplitude has a crystal clear interpretation: the pre-factor in front of the power-law decay of the multi-point correlation function associated with an excitation belonging to the $\ell_{r-1}(\kappa_r)$ critical classes is precisely given by the product of form factors.
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of local operators taken between the typical representatives of the $\ell_{r-1}(\kappa_r)$ critical classes of interest. Note that, for a given $\kappa_r$, the form of the $L$-dependence in (3.12) is already reminiscent of the expression for multi-point correlation functions in a conformal invariant theory of a strip of width $L$ [5].

It is straightforward to take the $L \to +\infty$ limit in (3.12). This leads to

$$C(\mathbf{x}_r; \mathbf{O}_r) = \sum_{\kappa_r \in \mathbb{Z}} \prod_{s=1}^{r} \left\{ e^{2i p_F \kappa_s \mathbf{x}_s} \right\} \cdot \mathcal{F}(\{\kappa_s\}_1^r; \{\mathbf{o}_a\}_1^r) \cdot \prod_{b>a} \left\{ [i(\mathbf{x}_b - \mathbf{x}_a)]^{\theta_+^e(\kappa_b) \theta_-^e(\kappa_a)} \right\} \times \bigg[ -i(\mathbf{x}_b - \mathbf{x}_a) \bigg]^{\theta_+^e(\kappa_b) \theta_-^e(\kappa_a)} \bigg) \right\}, \tag{3.15}$$

where we have used that

$$\sum_{s=1}^{r} \theta_+^e(\kappa_s) = 0. \tag{3.16}$$

Note that the above asymptotic expansion provides one with an expression that is symmetric under a simultaneous permutation

$$(\mathbf{x}_r, \mathbf{O}_r) \mapsto (\mathbf{x}^{\sigma}_r, \mathbf{O}^{\sigma}_r) \quad \text{with} \quad \mathbf{x}^{\sigma}_r = (x^{\sigma}(1), \ldots, x^{\sigma}(r)) \quad \sigma \in \mathfrak{S}_r. \tag{3.17}$$

This translates the fact that the local operators $\mathcal{O}_r(\mathbf{x}_r)$ commute at different distances and, in particular, in the large-distance regime.

4. Applications

4.1. The conformal regime

The asymptotic formula (3.15) is quite general in that it does not assume any specific order of magnitude for the spacing between the various ‘space’ parameters $x_a$. The sole constraint is that all have to be pairwise large, namely

$$|x_k - x_\ell| \cdot p_F \gg 1 \quad \text{for} \quad k \neq \ell. \tag{4.1}$$

Thus, in such a general setting, determining the leading term of the asymptotics (3.15) leads to a very complex minimization problem. However, the situation gets much simpler in the so-called conformal regime of the correlation functions. In the latter case, all distances scale with the same magnitude $R$,

$$x_k = R \cdot z_k \quad 0 < \epsilon < |z_k - z_\ell| < \epsilon^{-1} \quad \text{for} \quad k \neq \ell \quad \text{and} \quad \text{some} \quad \epsilon > 0. \tag{4.2}$$

One then has

$$C(R \cdot \mathbf{z}_r; \mathbf{O}_r) \simeq \sum_{\kappa_r \in \mathbb{Z}} \prod_{s=1}^{r} \left\{ \left( \frac{1}{R} \right)^{[\theta_+^e(\kappa_s)]^2/2 + [\theta_-^e(\kappa_s)]^2/2} \right\} \prod_{s=1}^{r} \left\{ e^{2i p_F R \kappa_s \mathbf{z}_s} \right\} \cdot \mathcal{F}(\{\kappa_s\}_1^r; \{\mathbf{o}_a\}_1^r) \times \prod_{b>a} \left\{ [i(z_b - z_a)]^{\theta_+^e(\kappa_b) \theta_-^e(\kappa_a)} \cdot \bigg[ -i(z_b - z_a) \bigg]^{\theta_+^e(\kappa_b) \theta_-^e(\kappa_a)} \right\} \tag{4.3}$$

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using again

$$\sum_{s=1}^{r} \theta^{\pm}_{s}(\kappa_{s}) = 0. \quad (4.4)$$

The leading asymptotics are then obtained by choosing an integer vector $\kappa_{r} \in \mathbb{Z}^{r}$ realizing the minimum of

$$\kappa_{r} \mapsto \sum_{s=1}^{r} \left[ \theta^{+}_{s}(\kappa_{s}) \right]^{2} + \left[ \theta^{-}_{s}(\kappa_{s}) \right]^{2}. \quad (4.5)$$

In practical situations (such as Bethe ansatz solvable models, cf later on) the existence and uniqueness of the minimum, for generic values of the coupling constants, follows by mimicking the reasonings presented in [8]. Let us also mention that such an asymptotic behaviour is characteristic of conformal field theories with central charge equal to one.

4.2. Bethe ansatz solvable models

In this subsection we specialize our results for the cases of two Bethe ansatz solvable models, the XXZ spin-1/2 chain and the non-linear Schrödinger model. The key matter is that one has quite explicit expressions for the shift functions arising in the description of these models in terms of solutions to linear integral equations. The latter thus give access to the critical exponents.

The XXZ spin-1/2 chain corresponds to the Hamiltonian

$$H_{XXZ} = \sum_{k=1}^{L} \left( \sigma^{x}_{k} \sigma^{x}_{k+1} + \sigma^{y}_{k} \sigma^{y}_{k+1} + \Delta (\sigma^{z}_{k} \sigma^{z}_{k+1} - 1) \right) - \frac{h}{2} \sum_{k=1}^{L} \sigma^{z}_{k}. \quad (4.6)$$

Here $\sigma^{x,y,z}_{k}$ are the spin operators (Pauli matrices) acting on the $k$th site of the chain, $h$ is an external magnetic field and the model is subject to periodic boundary conditions. The XXZ spin chain above exhibits different phases depending on the value of the anisotropy parameter $\Delta$. We only focus on the critical regime $-1 < \Delta < 1$ where we set $\Delta = \cos \zeta$. It is known that, in its massless phase, the excitations in the XXZ chain can be either of a bound state nature (so-called string solutions) or a particle–hole one. As argued previously, we solely focus on the particle–hole part of the spectrum.

The non-linear Schrödinger model corresponds to the Hamiltonian

$$H_{NLS} = \int_{0}^{L} \left\{ \partial_{y} \Phi^{\dagger}(y) \partial_{y} \Phi(y) + c \Phi^{\dagger}(y) \Phi^{\dagger}(y) \Phi(y) \Phi(y) - \mu \Phi^{\dagger}(y) \Phi(y) \right\} dy. \quad (4.7)$$

The model is defined on a circle of length $L$, so that the canonical Bose fields $\Phi, \Phi^\dagger$ are subject to $L$-periodic boundary conditions. We solely focus on the repulsive regime $c > 0$ in the presence of a positive chemical potential $\mu > 0$. In this model, one can show that the excitations are only given by particles and holes.

One can show using Bethe ansatz methods that, for both models, the general shift functions—in the sense of (2.9)—take the form

$$F_{\mathcal{R}_{n};s}(\lambda) = -\sigma_{s} \left( \frac{Z(\lambda)}{2} + v \phi(\lambda, q) \right) - v \sum_{a=1}^{n} [\phi(\lambda, \mu_{pa}) - \phi(\lambda, \mu_{ha})], \quad (4.8)$$

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in which the rapidities are the unique solutions to

$$\xi(\mu_a) = \frac{a}{L} \quad \text{with} \quad \xi(\omega) = \frac{p(\omega)}{2\pi} + \frac{D}{2},$$

with \( p \) representing the dressed momentum associated with the model, and \( \sigma_s \) has been defined in (2.5). Although the form for \( \xi \) and \( F_{R,s} \) is similar for both models, the definitions of the functions arising in their expressions differ. Also, one should set

- \( \nu = 1 \) for the non-linear Schrödinger model
- \( \nu = -1 \) for the XXZ chain.

The function \( \phi \) is the dressed phase and \( Z \) the dressed charge. Setting

$$\theta(\lambda) = i \ln \left( \frac{ic + \lambda}{ic - \lambda} \right) \quad \text{for NLSM} \quad \text{and}$$

$$\theta(\lambda) = i \ln \left( \frac{\sinh(i\zeta + \lambda)}{\sinh(i\zeta - \lambda)} \right) \quad \text{for XXZ},$$

one has that \( p \) solves the integro-differential equation under the requirement that \( p(\lambda) = -p(-\lambda) \):

$$p(\lambda) - \nu \int_{-\pi}^{\pi} \theta(\lambda - \mu) \cdot p'(\mu) \cdot \frac{d\mu}{2\pi} = p_0(\lambda)$$

with \( p_0(\lambda) = \begin{cases} \lambda & \text{for NLSM} \\ i \ln \left( \frac{\sinh(i\zeta/2 + \lambda)}{\sinh(i\zeta/2 - \lambda)} \right) & \text{for XXZ}. \end{cases} \)

The functions \( Z \) and \( \phi \) solve the Lieb integral equations

$$Z(\lambda) - \nu \int_{-\pi}^{\pi} \theta'(\lambda - \mu)Z(\mu) \cdot \frac{d\mu}{2\pi} = 1$$

$$\phi(\lambda, \nu) - \nu \int_{-\pi}^{\pi} \theta'(\lambda - \mu)\phi(\mu, \nu) \cdot \frac{d\mu}{2\pi} = \frac{\theta(\lambda - \nu)}{2\pi}.$$

As a consequence, the relative shift function \( \nu_s \) between critical excited states belonging to the \( \ell_s \) and \( \ell_{s-1} \) classes and having a change in the pseudo-particle number of \( \sigma_s \) takes the form

$$\nu_s(\lambda) = \sigma_s \left( \frac{Z(\lambda)}{2} + \nu \phi(\lambda, q) \right) + (\ell_{s-1} - \ell_s)(Z(\lambda) - 1).$$

In particular, then, the critical exponents take the explicit expression

$$\theta_s^\pm(\kappa) = \kappa Z(q) + \frac{\sigma_s}{2} Z^{-1}(q).$$
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These data are already sufficient to extract, in the large-distance conformal limit, the asymptotic behaviour of multi-point correlation functions of the XXZ spin-1/2 chain. We shall discuss the example of the correlator

\[ C_{xxxx} = \langle \Psi_g | \sigma^x_{x_1} \cdot \sigma^x_{x_2} \cdot \sigma^x_{x_3} \cdot \sigma^x_{x_4} | \Psi_g \rangle. \] (4.15)

In order to apply the method developed in this paper, one ought to decompose the operators \( \sigma^x_{x_a} \) onto operators which change the particle number in a definite way, namely the \( \sigma^\pm_{x_a} \) operators. Due to the conservation of the longitudinal total spin, such a decomposition leads to

\[ C_{xxxx} = \sum_{\kappa_1, \kappa_2, \kappa_3, \kappa_4} \left\langle \Psi_g | \sigma^{\kappa_1}_{x_1} \cdot \sigma^{\kappa_2}_{x_2} \cdot \sigma^{\kappa_3}_{x_3} \cdot \sigma^{\kappa_4}_{x_4} | \Psi_g \right\rangle. \] (4.16)

We shall consider the large-\( x \) asymptotic behaviour of \( C_{xxxx} \) in the conformal scaling regime where \( x_a = R \cdot z_a, \) \( R \) is the large parameter and the \( z_a \) are fixed and pairwise distinct. Then, it follows from the previous calculations that

\[
\left\langle \Psi_g | \sigma^x_{x_1} \cdot \sigma^x_{x_2} \cdot \sigma^x_{x_3} \cdot \sigma^x_{x_4} | \Psi_g \right\rangle = \sum_{\kappa_1, \kappa_2, \kappa_3, \kappa_4} \prod_{s=1}^{4} \left\{ \left( \frac{1}{R} \right)^{[\vartheta_{z_a}(\kappa_a)]^2/2 + [\vartheta_{-z_a}(\kappa_a)]^2/2} \right\}
\times \prod_{s=1}^{4} \left\{ e^{i\pi RF_{\kappa_1}Z_{\kappa_2}} \right\} \cdot \mathcal{F} (\{ \kappa_a \}_{s=1}^{r}; \{ \epsilon_a \}_{s=1}^{r})
\times \prod_{b > a}^{r} \left\{ i(z_b - z_a) \vartheta_{-z_b}(\kappa_b) \vartheta_{-z_a}(\kappa_a) \right\}. \] (4.17)

Above \( \mathcal{F} (\{ \kappa_a \}_{s=1}^{r}; \{ \epsilon_a \}_{s=1}^{r}) \) represents the properly normalized, in the volume \( L \), product of form factors of operators \( \sigma^{\kappa_a} \). Also, we agree upon

\[ \vartheta_a(\kappa) = \kappa Z(q) - \frac{\epsilon}{2Z(q)}. \] (4.18)

In order to access the leading asymptotics in the distance one should choose the combination of integers \( \kappa_4 = (\kappa_1, \ldots, \kappa_4) \) that minimizes the exponent of \( R \). In fact, independently of the operator (i.e. the sign of \( \epsilon_a \)) one has

\[ [\vartheta_{z_a}(\kappa_a)]^2 + [\vartheta_{-z_a}(\kappa_a)]^2 = 2(\kappa_a Z(q))^2 + \frac{1}{2Z^2(q)}. \] (4.19)

As a consequence, independently of the choice of the \( \epsilon_a \)'s, the leading asymptotics are given by the choice \( \kappa_4 = 0 \). It is then straightforward to see that

\[
C_{xxxx} = 2\mathcal{F}_{\sigma^x}^2 (0, 0) \mathcal{F}_{\sigma^x}^2 (0, 0) \cdot \left\{ \left( \frac{x_2 - x_1}{x_3 - x_1} \cdot \frac{x_4 - x_3}{x_4 - x_1} \cdot \frac{x_4 - x_2}{x_4 - x_1} \cdot \frac{x_4 - x_3}{x_4 - x_1} \right)^{1/2Z^2(q)} \right\}^{1/2Z^2(q)} + (2 \leftrightarrow 3) + (2 \leftrightarrow 4) + \cdots. \] (4.20)
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Above, \( F_{\sigma^{\pm}}(0,0) \) are the afore-discussed form factors of the \( \sigma^{\pm} \) operators between appropriate states of the 0 critical class.

Although we have focused in the present example on ultra-local elementary operators acting on a single site, our approach would work just as well for higher composite operators such as finite products of local operators on an adjacent site, such as

\[
O(x) = \prod_{a=1}^{p} \sigma_{x+a}^{\epsilon_a}.
\]

5. Conclusion

In this paper, we have generalized to the multi-point case the restricted sum formalism developed in [21] for the study of the form factor expansion of two-point correlation functions in massless models. In this framework, the computation of the form factor series, for large but finite volume and in the large-distance regime, reduces to the computation of multi-dimensional sums over particular classes of low-energy excited states. Our formalism naturally applies to quantum integrable models such as the XXZ spin-1/2 chain in the massless regime or the quantum non-linear Schrödinger model in the repulsive regime. It also applies, with quite reasonable assumptions regarding the way of parametrizing the model’s spectrum and the structure of the model’s form factors, to more general massless one-dimensional quantum models corresponding in the thermodynamic limit to conformal field theories with central charge equal to one. We believe that more general cases (with different values of the central charge) could also be considered, but require further studies. Let us finally mention that this approach to multi-point correlation functions, developed here in the static case, can be extended, as in [23], to the time-dependent case, i.e. to the study of the large-distance and long-time asymptotic behaviour of multi-point dynamical correlation functions.

Appendix. The summation identity

In order to establish the summation identities of interest, we shall compute the large-\( N \) behaviour of the sum

\[
S_N^{(r)} = \prod_{k=1}^{r-1} \left\{ \sum_{\ell_k^{(a)} \in \mathbb{Z}, \ell_k^{(a)} < \cdots < \ell_N^{(a)}} \right\} \left( \frac{\sin[\pi \nu_r]}{\pi} \right)^N \prod_{s=1}^{r-1} \prod_{a=1}^{N} \frac{\sin[\pi \nu_s]}{\pi} \exp\{i\epsilon_a^{(s)} \} \right\} \right\} \prod_{k=1}^{r} \left\{ \det_{N} \left[ \frac{1}{\lambda_{(k-1)}^{(a-1)} - \lambda_{(k)}^{(b)}} \right] \right\}
\]

in two ways.

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The definition of $S_N^{(r)}$ involves the numbers $\lambda_a^{(k)}$ defined as

$$
\lambda_a^{(0)} = a - \frac{N + 1}{2} \quad \text{and} \quad \lambda_a^{(k)} = a - \frac{N + 1}{2} - \bar{\nu}_k \quad \text{for } k = 1, \ldots, r.
$$

(A.2)

Above, we have adopted the convenient shorthand notation: for a vector $\eta_k$ we denote

$$
\bar{\eta}_k = \sum_{s=1}^k \eta_s.
$$

(A.3)

Finally, in (A.1), $\nu_s$ and $t_s$ are some auxiliary complex numbers such that

$$
\text{Re}(\nu_s) \not\in \frac{1}{2} + Z \quad \text{and} \quad \bar{t}_s \in [0; 2\pi], \quad s = 1, \ldots, r - 1.
$$

(A.4)

Furthermore, the two boundary sequences of integers read

$$
\ell_a^{(0)} = \ell_a^{(r)} = a.
$$

(A.5)

The $(r-1) \times N$-fold sum (A.1) defining $S_N^{(r)}$ can be recast in two different ways. On the one hand, one can relate it to a Toeplitz matrix. On the other hand, one can represent it as a sum over particle–hole like excitations associated with each intermediate state arising in the expansion. The latter generates the multi-dimensional restricted sums (3.4) that arise in the context of form factor expansions of multi-point correlation functions.

**A.1. Toeplitz determinant representation**

Let $S_{N; \ell_N^{(n)}}$ satisfy the induction

$$
S_{N; \ell_N^{(n+1)}}^{(n+1)}(a) = \sum_{\ell_1^{(n)} < \cdots < \ell_N^{(n)} \in \mathbb{Z}} \prod_{a=1}^N \left\{ \left( \frac{\sin[\pi \nu_{a+1}]}{\pi} \right) \exp \left( i t_a (\ell_a^{(n)}) - (N+1)/2 - \bar{\nu}_a \right) \right\} \cdot \frac{1}{\prod_{b=1}^N \left( \ell_b^{(n+1)} - \ell_a^{(n+1)} + \nu_{a+1} \right)}
$$

$$
\times S_{N; \ell_N^{(n)}}^{(n)}(a),
$$

(A.6)

and be subject to the initialization condition

$$
S_{N; \ell_N^{(1)}}^{(1)}(a) = \left( \frac{\sin[\pi \nu_1]}{\pi} \right)^N \cdot \frac{1}{\prod_{b=1}^N \left( \ell_b^{(1)} - \ell_a^{(1)} + \nu_1 \right)}
$$

(A.7)

It is then readily seen that with $\ell_N^{(r)} = (1, \ldots, N)$

$$
S_{N; \ell_N^{(r)}}^{(r)} = S_N^{(r)}.
$$

(A.8)

Furthermore, we shall establish by induction on $n$ that

$$
S_{N; \ell_N^{(n+1)}}^{(n+1)} = \prod_{s=1}^n \left( \exp \{-iN \bar{\nu}_s \} \right) \cdot \prod_{a=1}^N \left\{ \exp \left( i t_a (\ell_a^{(n+1)}) - (N+1)/2 \right) \right\} \cdot \frac{1}{\prod_{b=1}^N \left( \ell_b^{(n+1)} - \ell_a^{(n+1)} + \nu_{a+1} \right)}.
$$

(A.9)
where we have set
\[ c_k[f] = \int_0^{2\pi} e^{-ik\theta} f(\theta) \cdot \frac{d\theta}{2\pi}. \] (A.10)

The function \( \chi_n \) is expressed in terms of elementary jump like Fisher–Hartwig symbols
\[
\chi_n = \chi_{n,0} \cdot \prod_{s=1}^{n-1} \chi_{n+1,s}
\] (A.11)
where
\[
\chi_{\delta,\varphi}(\theta) = e^{i(\varphi-\varphi+\delta)} \{ 1_{[0,\varphi]} + e^{-2i\pi \delta} 1_{[\varphi,2\pi]} \}. \] (A.12)

One can recast \( S^{(1)}_{N,\ell_N} \) in terms of a Toeplitz determinant. Indeed, one has
\[
c_j[\chi_{n,0}] = e^{-i\pi \nu_1} \int_0^{2\pi} e^{i(\nu_1-j)\theta} \cdot \frac{d\theta}{2\pi} = e^{i\pi \nu_1} - e^{-i\pi \nu_1} = \sin[\pi \nu_1] \] (A.13)

Thus, all in all,
\[
S^{(1)}_{N,\ell_N} = \det_N \left[ e_{\ell^{(1)}_{a \rightarrow b}}[\chi_{n,0}] \right], \] (A.14)
so that the induction hypothesis does indeed hold for \( n = 1 \). Then, assume it holds for some \( n \). The antisymmetry of the determinants allows one to replace the summation over a fundamental simplex by one over the whole of \( \mathbb{Z}^N \) normalized by \( 1/N! \). Then, using the antisymmetry of the determinants, one can replace one of the determinants by \( N! \) times the product of its diagonal entries. This ultimately yields
\[
S^{(n+1)}_{N,\ell_N} = \prod_{s=1}^n \left\{ e^{-iN\pi \ell_s} \right\} \cdot \prod_{a=1}^N \left\{ e^{i\pi \nu_{a+1}} \right\} \cdot \det_N \left[ e_{\ell^{(n+1)}_{a \rightarrow b}}[\chi_{n,0}] \right]. \] (A.15)

Entering the sums into the lines of the determinant, one gets the representation
\[
S^{(n+1)}_{N,\ell_N} = \prod_{s=1}^n \left\{ e^{-iN\pi \ell_s} \right\} \cdot \prod_{a=1}^N \left\{ e^{i\pi \nu_{a+1}} \right\} \cdot \det_N \left[ M^{(n)}_{\ell^{(n+1)}_a b} \right] \] (A.16)
where we have set
\[
M^{(n)}_{ab} = \sum_{\ell \in \mathbb{Z}} \frac{\sin[\pi \nu_{a+1}]}{\pi} \cdot e^{i\pi \nu_{a+1}} \cdot \frac{c_{\ell-b}[\chi_n]}{\ell - a + \nu_{a+1}} = \frac{\sin[\pi \nu_{a+1}]}{\pi} \cdot e^{-i\pi \nu_{a+1}}
\times \int_0^{2\pi} e^{i\ell \theta} \sin[\pi \nu_1] \cdot \sum_{\ell \in \mathbb{Z}} \frac{e^{i\pi \nu_{a+1}}}{\ell - a + \nu_{a+1}} \cdot \frac{d\theta}{2\pi}. \]

\[ \text{doi:10.1088/1742-5468/2014/05/P05011} \]
Using that for $0 < t < 2\pi$
\[
\sum_{\ell \in \mathbb{Z}} \frac{e^{i\ell t}}{\ell + a} = \frac{2i\pi e^{-iat}}{1 - e^{-2i\pi a}},
\]
(A.17)
one gets
\[
\sum_{\ell \in \mathbb{Z}} \frac{e^{i(t_\ell - \theta)}}{\ell - a + \nu_{n+1}} = \frac{\pi e^{i(\nu_{n+1} - a)}}{\sin[\pi(\nu_{n+1} - a)]} \cdot e^{-i(\nu_{n+1} - a)\theta} \times \left\{1_{[0,\pi]}(\theta) + e^{-2i\pi\nu_{n+1}}1_{[\pi,2\pi]}(\theta)\right\}
\]
leading to
\[
M_{ab}^{(n)} = c_{a-b}[\chi_{n+1}],
\]
(A.19)
hence establishing the induction hypothesis at $n + 1$. Then, the large-$N$ asymptotic behaviour of $S_N^{(r)}$ can be obtained, say, from the results established in [8],
\[
S_N^{(r)} = \prod_{s=1}^{r-1} \left\{e^{-iNt_s p_s} \cdot e^{iN\kappa_{s+1} t_s}\right\} \cdot \prod_{s=1}^{r} \left\{\frac{G(1 + \nu_s + \kappa_s, 1 - \nu_s - \kappa_s)}{N^{(\nu_s + \kappa_s)^2}}\right\} \times \prod_{a \neq b} (1 - e^{i(t_{a-1} - t_{b-1})})^{(\nu_a + \kappa_a)(\nu_b + \kappa_b)} \cdot (1 + o(1)),
\]
(A.20)
where $G$ is the Barnes function. Furthermore, $\kappa_r \in \mathbb{Z}^r$ is an integer valued vector such that $\kappa_r$ maximizes
\[
\sum_{s=1}^{r} (\nu_s + \kappa_s)^2,
\]
under the constraint $\kappa_r = 0$. Due to the hypothesis on $\nu_a$, $a = 1, \ldots, r$, the maximizer exists and is unique.

**A.2. The form factor expansion representation**

In order to get the form factor like expansion, we relabel the integers $\ell_a^{(s)}$ in terms of the particle–hole like integers $\{p_a^{(s)}\}_{1}^{n(s)}$ and $\{h_a^{(s)}\}_{1}^{n(s)}$, for some $n^{(s)} = 0, \ldots, N$. Namely, for any sequence $\ell_1^{(s)} < \cdots < \ell_N^{(s)}$ we define the integer $n^{(s)}$ and the integers
\[
p_1^{(s)} < \cdots < p_{n^{(s)}}^{(s)} \quad \text{with} \quad p_a^{(s)} \in \mathbb{Z} \setminus [1; N] \quad \text{and} \quad h_1^{(s)} < \cdots < h_{n^{(s)}}^{(s)} \quad \text{with} \quad h_a^{(s)} \in [1; N],
\]
(A.22)
as
\[
\ell_a^{(s)} = a \quad \text{for} \quad a \in [1; N] \setminus \{h_1^{(s)}, \ldots, h_{n^{(s)}}^{(s)}\}
\]
and
\[
\ell_a^{(s)} = p_a^{(s)} \quad \text{for} \quad a \in [1; n^{(s)}].
\]
(A.23)
Then, after some algebra, one gets that

\[
\det \frac{1}{N} \left[ \lambda^{(s-1)}_{a} - \lambda^{(s)}_{b} \right] \cdot \frac{1}{n} \det \frac{1}{n^{(s-1)}} \left[ \lambda^{(s-1)}_{a} - \lambda^{(s-1)}_{b} \right] \cdot \det \frac{1}{n^{(s)}} \left[ \lambda^{(s)}_{a} - \lambda^{(s)}_{b} \right] \\
\times \prod_{b=1}^{n^{(s-1)}-1} \prod_{a=1}^{n^{(s)}} \left\{ (p^{(s-1)}_{b} - h^{(s)}_{a} + \nu) (h^{(s-1)}_{b} - p^{(s)}_{a} + \nu) \right\} \\
\times \prod_{a=1}^{n^{(s-1)}-1} \sin \left[ \pi \nu \right] \cdot \frac{1}{\pi} \left( \frac{N + 1 - h^{(s)}_{a} + \nu, h^{(s-1)}_{a} - \nu}{N + 1 - h^{(s)}_{a}, h^{(s-1)}_{a}, 1 - p^{(s)}_{a} + i0^{+}, N + 1 - p^{(s)}_{a} + \nu} \right) \\
\times \prod_{a=1}^{n^{(s-1)}-1} \sin \left[ \pi \nu \right] \cdot \frac{1}{\pi} \left( \frac{N + 1 - h^{(s)}_{a} - \nu, h^{(s-1)}_{a} + \nu, 1 - p^{(s-1)}_{a} - \nu}{N + 1 - h^{(s)}_{a}, h^{(s-1)}_{a}, 1 - p^{(s-1)} + i0^{+}, N + 1 - p^{(s-1)} - \nu} \right) \cdot \Gamma \left( \frac{N + 1 - h^{(s)}_{a} - \nu, h^{(s-1)}_{a} + \nu, 1 - p^{(s-1)} - \nu}{N + 1 - h^{(s)}_{a}, h^{(s-1)}_{a}, 1 - p^{(s-1)} + i0^{+}, N + 1 - p^{(s-1)} - \nu} \right) \right) .
\]

(A.24)

Thus, the series expansion for \( S_{N} \) takes the form

\[
S_{N}^{(r)} = G_{N} \left\{ \{ \nu_{s} \}_{1}^{r} \right\} \cdot \prod_{a=1}^{r-1} \left\{ \sum_{n^{(s)}=0}^{N} \sum_{h_{a}^{(s)}<\cdots<h_{a}^{(s)}_{n^{(s)}}} \sum_{p_{a}^{(s)}<\cdots<p_{a}^{(s)}_{n^{(s)}}} \prod_{a=1}^{n^{(s)}-1} \prod_{b=1}^{n^{(s)-1}} \prod_{a=1}^{n^{(s)-1}} \right\} \\
\times \prod_{s=1}^{r-1} \mathcal{H}_{N} \left( \{ p_{a}^{(s)} \}_{1}^{n^{(s)}} \left| h_{a}^{(s)} \right; \nu_{s+1}^{(s)}, t_{s}^{(s)} \right) \\
\times \prod_{s=1}^{r} \prod_{b=1}^{n^{(s)-1}} \prod_{a=1}^{n^{(s)-1}} \left\{ \frac{(h^{(s)}_{b} - h^{(s)}_{a} + \nu)}{(h^{(s)}_{b} - h^{(s)}_{a} + \nu)} \right\} \\
\times \prod_{a=1}^{n} \Gamma \left( \frac{N + 1 - h - \nu, h_{a} + \nu, 1 - p_{a} + i0^{+}}{N + 1 - h_{a}, h_{a}, 1 - p_{a} + i0^{+}, N + 1 - p_{a} + \nu} \right) \\
\times \prod_{a=1}^{n} \Gamma \left( \frac{N + 1 - h - \eta, h_{a} + \eta, 1 - p_{a} - \eta}{N + 1 - h_{a}, h_{a}, 1 - p_{a} + i0^{+}, N + 1 - p_{a} - \eta} \right) .
\]

(A.25)

where

\[
\mathcal{H}_{N} \left( \{ p_{a} \}_{1}^{n} \left| h_{a} \right; \nu, \eta, t \right) = (-1)^{n} \cdot \left( \frac{\sin \left[ \pi \nu \right]}{\pi} \cdot \frac{\sin \left[ \pi \eta \right]}{\pi} \right)^{n} \cdot \frac{1}{n} \left[ \prod_{a=1}^{n} \left\{ e^{i(t p_{a} - h_{a})} \right\} \right] \\
\times \prod_{a=1}^{n} \Gamma \left( \frac{N + 1 - h_{a} + \nu, h_{a} - \nu, 1 - p_{a} + \nu}{N + 1 - h_{a}, h_{a}, 1 - p_{a} + i0^{+}, N + 1 - p_{a} + \nu} \right) \\
\times \prod_{a=1}^{n} \Gamma \left( \frac{N + 1 - h_{a} - \eta, h_{a} + \eta, 1 - p_{a} - \eta}{N + 1 - h_{a}, h_{a}, 1 - p_{a} + i0^{+}, N + 1 - p_{a} - \eta} \right) .
\]

(A.26)

In (A.25), \( G_{N} \left( \{ \nu_{s} \}_{1}^{r} \right) \) is built out of products of so-called background form factors. These have been considered in [20]. In this free-fermionic setting, each such form factor can be computed explicitly in terms of Barnes functions. More explicitly,
one has
\[ G_N(S_1, \ldots, S_n) \equiv \prod_{s=1}^{r-1} \left\{ e^{-iN\nu_s} \right\} \prod_{s=1}^{r} \left\{ \left( \frac{\sin(\pi \nu_s)}{\pi} \right)^N \det_{N} \left[ \frac{1}{a - b + \nu_s} \right] \right\} \]
\[ = \prod_{s=1}^{r-1} \left\{ e^{-iN\nu_s} \right\} \cdot \prod_{s=1}^{r} \left\{ \left( 1 + \nu_s, 1 - \nu_s \right) \right\} \times \prod_{s=1}^{r} \left\{ \left( N + 1, N + 1 \right) \right\} \]
\[ \times \prod_{s=1}^{r-1} \left\{ e^{-iN\nu_s} \right\} \cdot \prod_{s=1}^{r-1} \left\{ \left( 1 + \nu_s, 1 - \nu_s \right) \right\} \cdot (1 + o(1)). \] (A.27)

Now we focus in more detail on the sum over the integers \{p^{(s)}_a\}^{n^{(s)}}_1 and \{h^{(s)}_a\}^{n^{(s)}}_1. We will reorganize the sum into one over excitations of Umklapp type. This will allow us to identify clearly the leading oscillating power, hence making a comparison with the Fisher–Hartwig asymptotics. In the following, we agree upon
\[ H_L = \left\{ 1, \ldots, \left\lfloor \frac{N}{2} \right\rfloor \right\}, \quad H_R = \left\{ \left\lfloor \frac{N}{2} \right\rfloor + 1, \ldots, N \right\}, \quad P_L = -N^* \quad \text{and} \quad P_R = N + N. \] (A.28)

Above \(\lfloor \cdot \rfloor\) stands for the floor function. It then follows that one can recast the various sums defining \(S_N^{(r)}\) as
\[ S_N^{(r)} = G_N(S_1, \ldots, S_n) \prod_{s=1}^{r-1} \left\{ \sum_{n^{(s)}=0}^{N} \sum_{m^{(s)}=0}^{N} \sum_{r^{(s)}=0}^{n^{(s)}} f((m^{(s)}), (r^{(s)}), (n^{(s)})) \right\} \]
where
\[ f((m^{(s)}), (r^{(s)}), (n^{(s)})) \]
\[ = \prod_{s=1}^{r-1} \left\{ \sum_{h^{(s)}_a \in H_L} h^{(s)}_a \sum_{h^{(s)}_a \in H_R} h^{(s)}_a \sum_{p^{(s)}_a \in P_L} p^{(s)}_a \sum_{p^{(s)}_a \in P_R} p^{(s)}_a \right\} \times \prod_{s=1}^{r-1} \left\{ \mathcal{H}_N((p^{(s)}_a), (h^{(s)}_a), (n^{(s)})) \right\} \]
\[ \times \prod_{s=1}^{r} \prod_{b=1}^{n^{(s)-1}} \prod_{a=1}^{n^{(s)}} \left\{ (p^{(s)-1}_b - h^{(s)}_a + \nu_s)(p^{(s)}_b - p^{(s)-1}_a - p^{(s)}_a + \nu_s) \right\}. \] (A.30)

It then remains to make several simplifications in each of these summands by using the fact that some terms, due to the largeness of \(N\), can be simplified. For this purpose,
for each set of right or left type integers, we reparametrize the variables in terms of ‘small’ ones by

\[ p^{(s)}_a = 1 - p^{(s)}_{a-} \quad \text{for } a = 1, \ldots, r^{(s)}, \quad p^{(s)}_a = N + p^{(s)}_{a-r^{(s)}+} \]
\[ \text{for } a = r^{(s)} + 1, \ldots, n^{(s)}, \]

(A.31)

and

\[ h^{(s)}_a = h^{(s)}_{a-} \quad \text{for } a = 1, \ldots, m^{(s)}, \quad h^{(s)}_a = N - h^{(s)}_{a-m^{(s)}+} + 1 \]
\[ \text{for } a = 1 + m^{(s)}, \ldots, n^{(s)}. \]

(A.32)

Since we are solely interested in the leading \( N \to +\infty \) asymptotics of each term subordinate to a choice of \( n^{(s)}, r^{(s)} \) and \( m^{(s)} \), we shall make several simplifying assumptions. First, we shall assume that we can extend the summation up to \( N^* \) for all of the variables, this without altering the leading \( N \to +\infty \) asymptotics. Then, we shall work as if the series were convergent in a strong sense. This will allow us to treat the variables \( \{p^{(s)}_{a-}\} \) and \( \{h^{(s)}_{a-}\} \) as ‘small’ in respect of \( N \). Finally, expanding the summand into powers of \( N \), we shall only keep the leading contribution.

A long but straightforward calculation then shows that \( S^{(r)}_N \) admits the following large-\( N \) behaviour

\[ S^{(r)}_N \simeq \prod_{s=1}^r \left\{ G(1 + \nu_s, 1 - \nu_s) \cdot N^{-\nu_s^2} \right\} \sum_{\ell_{s-1}}^{r-1} \prod_{s=1}^{\nu_s+1} \left\{ (e^{iN\ell_s} - \nu_s) \cdot N^{-\nu_s^2} N^{2(\nu_s-\nu_{s+1})} \ell_s \right\} \]
\[ \times \prod_{s=2}^{r-1} \left\{ N^2 \ell_{s-1} \right\} \]
\[ \times \mathcal{J}^{-}_{\ell_{s-1}} \left( \{t_s\}_{1}^{r-1}, \{\nu_s\}_{1}^{r} \right) \cdot \mathcal{J}^{+}_{\ell_{s-1}} \left( \{t_s\}_{1}^{r-1}, \{\nu_s\}_{1}^{r} \right). \]

(A.33)

Above, the functions \( \mathcal{J}^{\pm}_{\ell_{s-1}} \) are as defined in (3.4).

### A.3. Identification of the asymptotics and the summation formula

In order to identify the asymptotics one shall change the variables of summation in (A.33) according to (3.8)–(3.9). Upon using the identities (3.10) and (3.11), we are, all in all, led to

\[ S^{(r)}_N = \prod_{s=1}^{r-1} \left\{ e^{-iN\ell_s} \right\} \sum_{\sum_{s=1}^{r} \nu_s = \nu_s} \prod_{s=1}^{r} G(1 + \nu_s, 1 - \nu_s) N^{(\nu_s+\kappa_s)^2} \prod_{s=1}^{r-1} \left\{ e^{iN\kappa_{s+1}} \ell_s \right\} \]
\[ \times \mathcal{J}^{-}_{\ell_{s-1}(\kappa_s)} \left( \{t_s\}_{1}^{r-1}, \{\nu_s\}_{1}^{r} \right) \cdot \mathcal{J}^{+}_{\ell_{s-1}(\kappa_s)} \left( \{t_s\}_{1}^{r-1}, \{\nu_s\}_{1}^{r} \right). \]

(A.34)
The identification of the leading asymptotics of the Toeplitz determinant with the leading $N$ term of (A.33) yields

\[
\mathcal{J}^{-}_{\ell_{r-1}(\kappa_s)}\left(\{t_s\}_1^{r-1}, \{\nu_s\}_1^r\right) \cdot \mathcal{J}^{+}_{\ell_{r-1}(\kappa_s)}\left(\{t_s\}_1^{r-1}, \{\nu_s\}_1^r\right) = \prod_{s=1}^{r} \left\{ \frac{G(1 - \nu_s - \kappa_s)G(1 + \nu_s + \kappa_s)}{G(1 - \nu_s)G(1 + \nu_s)} \right\} \\
\times \prod_{s \neq b} (1 - e^{i(\bar{t}_a - t_{b-1})}) (\nu_a + \kappa_a)(\nu_b + \kappa_b).
\]

The parts relative to $\mathcal{J}^{-}_{\ell_{r-1}(\kappa_s)}\left(\{t_s\}_1^{r-1}, \{\nu_s\}_1^r\right)$ (resp. $\mathcal{J}^{+}_{\ell_{r-1}(\kappa_s)}\left(\{t_s\}_1^{r-1}, \{\nu_s\}_1^r\right)$) can be readily inferred by applying a Wiener–Hopf factorization on $\mathbb{C} \setminus \mathbb{R}$. In order to ensure uniqueness of such a factorization, one should fix the asymptotic behaviour of $\mathcal{J}^{\pm}_{\ell_{r-1}(\kappa_s)}\left(\{t_s\}_1^{r-1}, \{\nu_s\}_1^r\right)$ at $t \to \infty$, this non-tangentially to $\mathbb{R}$ and under the constraint $\pm \text{Im}(t) > 0$. The latter are readily read off from the explicit expressions (3.4) for $\mathcal{J}^{\pm}_{\ell_{r-1}(\kappa_s)}\left(\{t_s\}_1^{r-1}, \{\nu_s\}_1^r\right)$. Namely

\[
\mathcal{J}^{+}_{\ell_{r-1}(\kappa_s)}\left(\{t_s\}_1^{r-1}, \{\nu_s\}_1^r\right) \sim_{t \to +\infty} \prod_{s=1}^{r-1} \left\{ e^{2\nu_s(t_s(\ell_s+1)/2)} G\left(1 + \ell_s - \nu_s, 1 + \ell_s + \nu_s+1\right) \right\} \\
\times \prod_{s=2}^{r} G\left(1 + \nu_s, 1 + \ell_{s-1} - \ell_s + \nu_s, 1 - \ell_s + \nu_s\right),
\]

and

\[
\mathcal{J}^{-}_{\ell_{r-1}(\kappa_s)}\left(\{t_s\}_1^{r-1}, \{\nu_s\}_1^r\right) \sim_{t \to +\infty} \prod_{s=1}^{r-1} \left\{ e^{-2\nu_s(t_s(\ell_s+1)/2)} G\left(1 - \ell_s + \nu_s, 1 - \ell_s - \nu_s+1\right) \right\} \\
\times \prod_{s=2}^{r} G\left(1 - \nu_s, 1 - \ell_{s-1} + \ell_s - \nu_s, 1 - \ell_s - \nu_s\right).
\]

It is then readily checked that one has the decomposition

\[
\prod_{s=1}^{r} \frac{G(1 - \nu_s - \kappa_s)G(1 + \nu_s + \kappa_s)}{G(1 - \nu_s)G(1 + \nu_s)} = \prod_{s=1}^{r-1} \left\{ \frac{1 + \ell_s - \nu_s, 1 + \ell_s + \nu_{s+1}, 1 - \ell_s + \nu_s, 1 - \ell_s - \nu_{s+1}}{1 - \nu_s, 1 + \nu_{s+1}, 1 + \nu_s, 1 - \nu_{s+1}} \right\} \\
\times \prod_{s=2}^{r-1} \left\{ \frac{1 + \mu_s, 1 + \ell_{s-1} - \ell_s + \nu_s, 1 - \nu_s, 1 - \ell_{s-1} + \ell_s - \nu_s}{1 - \ell_s + \nu_s, 1 + \ell_{s-1} + \nu_s, 1 - \ell_{s-1} - \nu_s, 1 + \ell_s - \nu_s} \right\}.
\] (A.35)

The latter was the last missing piece so as to obtain the formulae (3.7).

\[ \square \]

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