SOME RESULTS ASSOCIATED WITH BERNOULLI AND EULER NUMBERS WITH APPLICATIONS

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Abstract. In this paper, we present series representations of the remainders in the expansions for \(2/(e^t + 1)\), \(\text{sech } t\) and \(\text{coth } t\). For example, we prove that for \(t > 0\) and \(N \in \mathbb{N} := \{1, 2, \ldots\}\),

\[
\text{sech } t = \sum_{j=0}^{N-1} \frac{E_{2j} t^{2j}}{(2j)!} + R_N(t)
\]

with

\[
R_N(t) = \frac{(-1)^N 2t^{2N}}{N!} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + \frac{1}{2})^{2N-1}} \left( \frac{2^k + \pi^2(k + \frac{1}{2})^2}{2^k} \right).
\]

and

\[
\text{sech } t = \sum_{j=0}^{N-1} \frac{E_{2j} t^{2j}}{(2j)!} + \Theta(t, N) E_{2N} (2N)!
\]

with a suitable \(0 < \Theta(t, N) < 1\). Here \(E_n\) are the Euler numbers. By using the obtained results, we deduce some inequalities and completely monotonic functions associated with the ratio of gamma functions. Furthermore, we give a (presumably new) quadratic recurrence relation for the Bernoulli numbers.

1. Introduction

The Bernoulli polynomials \(B_n(x)\) and Euler polynomials \(E_n(x)\) are defined, respectively, by the generating functions:

\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} \quad (|t| < 2\pi) \quad \text{and} \quad \frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!} \quad (|t| < \pi).
\]

The numbers \(B_n = B_n(0)\) and \(E_n = 2^n E_n(\frac{1}{2})\), which are known to be rational numbers and integers, respectively, are called Bernoulli and Euler numbers.

It follows from [23, Chapter 4, Part I, Problem 154] that

\[
\sum_{j=1}^{2m} \frac{B_{2j} t^{2j}}{(2j)!} < \frac{t}{e^t - 1} - 1 + \frac{t}{2} < \sum_{j=1}^{2m+1} \frac{B_{2j} t^{2j}}{(2j)!} \tag{1.1}
\]

for \(t > 0\) and \(m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\), \(\mathbb{N} := \{1, 2, 3, \ldots\}\). The inequality [13] can be also found in [12,24]. It is also known [31, p. 64] that

\[
\frac{t}{e^t - 1} - 1 + \frac{t}{2} = \sum_{j=1}^{n} \frac{B_{2j} t^{2j}}{(2j)!} + (-1)^n t^{2n+2} \nu_n(t) \quad (n \in \mathbb{N}_0), \tag{1.2}
\]
where
\[
\nu_n(t) = \frac{2}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}(t^2 + 4\pi^2 k^2)}.
\] (1.3)

It is easily seen that (1.2) implies (1.1). Koumandos [12] gave the following integral representation of \(\nu_n(t)\):
\[
\nu_n(t) = \frac{(-1)^n}{(2n+1)!} \int_0^1 e^{xt} B_{2n+1}(x) \, dx.
\] (1.4)

**Remark 1.1.** From (1.4), it is possible to deduce (1.3) by making use of the expansion [20, p. 592, Eq. (24.8.2)]
\[
B_{2n+1}(x) = \frac{(-1)^{n+1}(2n+1)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin(2k\pi x)}{k^{2n+1}} \quad (n \in \mathbb{N}, \ 0 \leq x \leq 1).
\]

We then obtain from (1.4) that
\[
\nu_n(t) = -\frac{1}{e^t - 1} \frac{2}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \int_0^1 \frac{e^{xt} \sin(2k\pi x)}{k^{2n+1}} \, dx = \frac{2}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}(t^2 + 4\pi^2 k^2)}.
\]

An alternative derivation of (1.2) and another integral representation of the remainder function \(\nu_n(t)\) are given in the appendix.

Binet’s first formula [30, p. 16] for the logarithm of \(\Gamma(x)\) states that
\[
\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \int_0^\infty \left(\frac{t}{e^t - 1} - 1 + \frac{t^2}{2}\right) e^{-xt} \, dt \quad (x > 0).
\] (1.5)

Combining (1.2) with (1.5), Xu and Han [36] deduced in 2009 that for every \(m \in \mathbb{N}_0\), the function
\[
R_m(x) = (-1)^m \left[ \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x - x - \ln \sqrt{2\pi} - \sum_{j=1}^{m} \frac{B_{2j}}{2j(2j-1)x^{2j-1}} \right]
\] (1.6)
is completely monotonic on \((0, \infty)\). Recall that a function \(f(x)\) is said to be completely monotonic on an interval \(I\) if it has derivatives of all orders on \(I\) and satisfies the following inequality:
\[
(-1)^n f^{(n)}(x) \geq 0 \quad (x \in I, \ n \in \mathbb{N}_0).
\] (1.7)

For \(m = 0\), the complete monotonicity property of \(R_m(x)\) was proved by Muldoon [19]. Alzer [2] first proved in 1997 that \(R_n(x)\) is completely monotonic on \((0, \infty)\). In 2006, Koumandos [12] proved the double inequality (1.1), and then used (1.1) and (1.5) to give a simpler proof of the complete monotonicity property of \(R_m(x)\). In 2009, Koumandos and Pedersen [13, Theorem 2.1] strengthened this result.

Chen and Paris [9, Lemma 1] presented an analogous result to (1.1) given by
\[
\sum_{j=2}^{2m+1} \frac{(1 - 2^{2j})B_{2j}}{j} \frac{t^{2j-1}}{(2j-1)!} < \frac{2}{e^t + 1} - 1 + \frac{t}{2} < \sum_{j=2}^{2m} \frac{(1 - 2^{2j})B_{2j}}{j} \frac{t^{2j-1}}{(2j-1)!}
\] (1.8)
for \(t > 0\) and \(m \in \mathbb{N}\). The inequality (1.8) can also be written for \(t > 0\) and \(m \in \mathbb{N}_0\) as
\[
(-1)^{m+1} \left(\frac{2}{e^t + 1} - 1 - \sum_{j=1}^{m} \frac{(1 - 2^{2j})B_{2j}}{j} \frac{t^{2j-1}}{(2j-1)!}\right) > 0.
\] (1.9)
Based on the inequality (1.9), Chen and Paris [9, Theorem 1] proved that for every \(m \in \mathbb{N}_0\), the function
\[
F_m(x) = (-1)^m \left[ \ln \left( \frac{\Gamma(x + 1)}{\Gamma(x + \frac{1}{2})} \right) - \frac{1}{2} \ln x - \sum_{j=1}^{m} \left( 1 - \frac{1}{2^{2j}} \right) \frac{B_{2j}}{j(2j - 1)x^{2j - 1}} \right] (1.10)
\]
is completely monotonic on \((0, \infty)\). This result is similar to the complete monotonicity property of \(R_m(x)\) in (1.6). In analogy with (1.2), these authors also considered [9, Eq. (2.4)] the remainder \(r_m(t)\) in the expansion
\[
\frac{2}{e^t + 1} = 1 + \sum_{j=1}^{m} \frac{(1 - 2^{2j})B_{2j}}{j(2j - 1)!} t^{2j - 1} + r_m(t) 
\]
and gave an integral representation for \(r_m(t)\) when \(t > 0\).

Chen [6] proposed the following conjecture.

**Conjecture 1.1.** For \(t > 0\) and \(m \in \mathbb{N}_0\), let
\[
\mu_m(t) = \frac{e^{t/3} - e^{2t/3}}{e^t - 1} - \sum_{j=0}^{m} \frac{2B_{2j+1}(\frac{1}{3})}{(2j + 1)!} x^{2j}, \quad \nu_m(t) = \frac{e^{t/4} - e^{3t/4}}{e^t - 1} - \sum_{j=0}^{m} \frac{2B_{2j+1}(\frac{1}{4})}{(2j + 1)!} x^{2j},
\]
where \(B_n(x)\) denotes the Bernoulli polynomials. Then, for \(t > 0\) and \(m \in \mathbb{N}_0\),
\[
(-1)^m \mu_m(t) > 0 \quad (1.14)
\]
and
\[
(-1)^m \nu_m(t) > 0. \quad (1.15)
\]

Chen [6, Lemma 1] has proved the statements in Conjecture 1.1 for \(m = 0, 1, 2,\) and 3. He has also pointed out in [6] that, if Conjecture 1.1 is true, then it follows that the functions
\[
U_m(x) = (-1)^m \left[ \ln \left( \frac{\Gamma(x + 2/3)}{\Gamma(x + 1/3)} \right) - \sum_{j=1}^{m} \frac{B_{2j+1}(\frac{1}{3})}{j(2j + 1)x^{2j}} \right] (1.16)
\]
and
\[
V_m(x) = (-1)^m \left[ \ln \left( \frac{\Gamma(x + 2/4)}{\Gamma(x + 1/4)} \right) - \sum_{j=1}^{m} \frac{B_{2j+1}(\frac{1}{4})}{j(2j + 1)x^{2j}} \right] (1.17)
\]
for \(m \in \mathbb{N}_0\) are completely monotonic on \((0, \infty)\). The complete monotonicity properties of \(U_m(x)\) and \(V_m(x)\) are similar to the complete monotonicity property of \(F_m(x)\) in (1.10).

In this paper, we obtain the following results: (i) a series representation of the remainder \(r_m(t)\) in (1.11) (Theorem 2.1); (ii) a series representation of the remainder in the expansion of \(\text{sech} t\) involving the Euler numbers (Theorem 2.2), together with the double inequality for \(t > 0\) and \(m \in \mathbb{N}_0\),
\[
\sum_{j=0}^{2m+1} \frac{E_{2j}}{(2j)!} t^{2j} < \text{sech} t < \sum_{j=0}^{2m} \frac{E_{2j}}{(2j)!} t^{2j}, \quad (1.18)
\]
(iii) the proof of the inequality (1.15) for all \(m \in \mathbb{N}_0\), and a demonstration that the function \(V_m(x)\) in (1.17) is completely monotonic on \((0, \infty)\) (Remark 2.4); (iv) a series representation of the remainder in the expansion for \(\coth t\) (Theorem 2.3); and finally, (v) a quadratic recurrence relation for the Bernoulli numbers (Theorem 3.1).

2. Main results

**Theorem 2.1.** For \(t > 0\) and \(m \in \mathbb{N}\),

\[
\frac{2}{e^t + 1} = 1 + \sum_{j=1}^{m} \frac{(1 - 2^{2j})B_{2j}t^{2j-1} + (-1)^{m+1}t^{2m+1}s_m(t)}{j \cdot (2j - 1)!},
\]

where \(s_m(t)\) is given by

\[
s_m(t) = \frac{4}{\pi^2m} \sum_{k=0}^{\infty} \frac{1}{(2k + 1)^{2m}(t^2 + \pi^2(2k + 1)^2)}.
\]

**Proof.** Boole’s summation formula (see [31, p. 17, Theorem 1.4]) for a function \(f(t)\) defined on \([0, 1]\) with \(k\) continuous derivatives states that, for \(k \in \mathbb{N}\),

\[
f(1) = \frac{1}{2} \sum_{j=0}^{k-1} \frac{E_j(1)}{j!} \left( f^{(j)}(1) + f^{(j)}(0) \right) + \frac{1}{2(k-1)!} \int_0^1 f^{(k)}(x)E_{k-1}(x)dx.
\]

Noting [20, p. 590] that

\[E_n(1) = \frac{2(2^{n+1} - 1)}{n+1} B_{n+1} \quad (n \in \mathbb{N}),\]

we see that

\[E_{2j-1}(1) = \frac{(2^{2j} - 1)B_{2j}}{j} \quad \text{and} \quad E_{2j}(1) = 0 \quad (j \in \mathbb{N}).\]

The choice \(k = 2m + 1\) in (2.3) yields

\[
f(1) - f(0) = \sum_{j=1}^{m} \frac{(2^{2j} - 1)B_{2j}}{j \cdot (2j - 1)!} \left( f^{(2j-1)}(1) + f^{(2j-1)}(0) \right) + \frac{1}{(2m)!} \int_0^1 f^{(2m+1)}(x)E_{2m}(x)dx.
\]

Application of the above formula to \(f(x) = e^{xt}\) then produces

\[
\frac{2}{e^t + 1} = 1 + \sum_{j=1}^{m} \frac{(1 - 2^{2j})B_{2j}t^{2j-1} + r_m(t)}{j \cdot (2j - 1)!},
\]

where

\[r_m(t) = -\frac{1}{e^t + 1} \frac{t^{2m+1}}{(2m)!} \int_0^1 e^{xt}E_{2m}(x)dx.
\]

\(^1\)It is also possible to choose \(k = 2m\) in (2.3) and to use the Fourier expansion for \(E_{2m+1}(x)\) in [31, p. 16] to obtain the same result.
Using the following formula (see [31, p. 16]):

\[ E_{2m}(x) = (-1)^m \frac{4(2m)!}{\pi^{2m+1}} \sum_{k=0}^{\infty} \frac{\sin((2k+1)\pi x)}{(2k+1)^{2m+1}} \quad (m \in \mathbb{N}, \quad 0 \leq x \leq 1), \quad (2.8) \]

we obtain

\[
r_m(t) = \frac{(-1)^{m+1} 4t^{2m+1}}{e^t + 1} \sum_{k=0}^{\infty} \int_0^1 e^{xt} \sin((2k+1)\pi x) (2k+1)^{2m+1} \, dx
\]

\[
= (-1)^{m+1} \frac{4t^{2m+1}}{\pi^{2m+1}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2m} \left(t^2 + \pi^2(2k+1)^2\right)}.
\]

This completes the proof of Theorem 2.1. □

**Remark 2.1.** From (2.1) we retrieve (2.9).

**Remark 2.2.** From [20, p. 592, Eq. (24.7.9)] and [22, p. 43, Ex. 12(i)] we have

\[
E_{2n}(x) = (-1)^n \sin(\pi x) \int_0^\infty \frac{4t^{2n} \cosh(\pi t)}{\cos(2\pi t) - \cos(2\pi x)} \, dt \quad (0 < x < 1, \quad n \in \mathbb{N}_0),
\]

from which it follows that

\[
E_{4m}(x) > 0 \quad \text{and} \quad E_{4m+2}(x) < 0 \quad (0 < x < 1, \quad m \in \mathbb{N}_0).
\]

By combining these inequalities with (2.4) and (2.7) we immediately obtain (2.8).

**Corollary 2.1.** For \( t > 0 \) and \( m \in \mathbb{N}, \)

\[
(-1)^m \left( \frac{2e^t}{(e^t + 1)^2} - \sum_{j=1}^{m} \frac{(2j - 1)B_{2j}t^{2j-2}}{j \cdot (2j - 2)!} \right) > 0. \quad (2.9)
\]

**Proof.** Differentiating the expression in (2.1), we find

\[
- \frac{2}{(e^t + 1)^2} e^t = - \sum_{j=1}^{m} \frac{(2j - 1)B_{2j}t^{2j-2} + (-1)^{m+1}(t^{2m+1} s_m(t))'}{j \cdot (2j - 2)!}.
\]

(2.10)

It is easy to see that

\[ t^2 s_m(t) + s_{m-1}(t) = \frac{4}{\pi^{2m}} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2m}} = \frac{4}{\pi^{2m}} (1 - 2^{-2m}) \zeta(2m), \]

where \( \zeta(z) \) is the Riemann zeta function. This last expression can be written as

\[ t^2 s_m(t) = \frac{4}{\pi^{2m}} (1 - 2^{-2m}) \zeta(2m) - s_{m-1}(t). \quad (2.11) \]

Then, since \( s_m(t) \) is strictly decreasing for \( t > 0 \), we deduce from (2.11) that \( t^2 s_m(t) \) is strictly increasing for \( t > 0 \). Hence, \( t^{2m+1} s_m(t) \) is strictly increasing for \( t > 0 \), and we then obtain from (2.10) that

\[
(-1)^m \left( \frac{2e^t}{(e^t + 1)^2} - \sum_{j=1}^{m} \frac{(2j - 1)B_{2j}t^{2j-2}}{j \cdot (2j - 2)!} \right) = (t^{2m+1} s_m(t))' > 0
\]

for \( t > 0 \) and \( m \in \mathbb{N} \). The proof is complete. □
Theorem 2.2. For \( t > 0 \) and \( N \in \mathbb{N} \), we have

\[
\text{sech } t = \sum_{j=0}^{N-1} \frac{E_{2j}}{(2j)!} t^{2j} + R_N(t)
\]

with

\[
R_N(t) = \frac{(-1)^N 2^2N}{\pi^{2N-1}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k + \frac{1}{2})^{2N-1}} \left( t^2 + \pi^2 (k + \frac{1}{2})^2 \right),
\]

and

\[
\text{sech } t = \sum_{j=0}^{N-1} \frac{E_{2j}}{(2j)!} t^{2j} + \Theta(t, N) \frac{E_{2N}}{(2N)!} t^{2N}
\]

with a suitable \( 0 < \Theta(t, N) < 1 \).

Proof. It follows from \([34, p. 136]\) (see also \([5, p. 458, Eq. (27.3)]\)) that

\[
\pi \cosh \left( \frac{\pi x}{2} \right) = \sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)}{(2k+1)^2 + x^2},
\]

which can be written as

\[
\frac{\pi}{4 \cosh \left( \frac{\pi x}{2} \right)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2 + x^2}.
\]

Substitution of \( x = \frac{1}{2} \) in \((2.8)\) leads to

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2j+1}} = \frac{(-1)^j \pi^{2j+1}}{2^{2j+2} (2j)!} E_{2j}.
\]

Using the identity

\[
\frac{1}{1+q} = \sum_{j=0}^{\infty} (-1)^j q^j + (-1)^N \frac{q^N}{1+q} \quad (q \neq -1)
\]

and \((2.16)\), we obtain from \((2.15)\) that

\[
\text{sech } t = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left( 1 + \left( \frac{2t}{\pi (2k+1)} \right)^2 \right)
\]

Substitution of \( x = \frac{1}{2} \) in \((2.8)\) leads to

\[
\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2j+1}} = \frac{(-1)^j \pi^{2j+1}}{2^{2j+2} (2j)!} E_{2j}.
\]

Using the identity

\[
\frac{1}{1+q} = \sum_{j=0}^{\infty} (-1)^j q^j + (-1)^N \frac{q^N}{1+q} \quad (q \neq -1)
\]

and \((2.16)\), we obtain from \((2.15)\) that

\[
\text{sech } t = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \left( 1 + \left( \frac{2t}{\pi (2k+1)} \right)^2 \right)
\]

with

\[
R_N(t) = \frac{2}{\pi^{2N-1}} \sum_{k=0}^{\infty} \frac{(-1)^{N+k}}{(k + \frac{1}{2})^{2N-1}} \left( t^2 + \pi^2 (k + \frac{1}{2})^2 \right).
\]

Noting that \((2.16)\) holds, we find that \( R_N(t) \) can be written as

\[
R_N(t) = \Theta(t, N) \frac{E_{2N} t^{2N}}{(2N)!}, \quad \Theta(t, N) := \frac{F(t)}{F(0)},
\]
where

\[ F(t) := \sum_{k=0}^{\infty} (-1)^k \alpha_k, \quad \alpha_k := \frac{1}{(k + \frac{1}{2})^{2N-1}} \frac{1}{t^2 + \pi^2(k + \frac{1}{2})^2}. \]

Then it is easily seen that \( \alpha_{2k} > \alpha_{2k+1} \) for \( k \in \mathbb{N}_0, \ t > 0 \) and \( N \in \mathbb{N}; \) thus \( F(t) > 0 \) for \( t > 0. \) Differentiation yields

\[ F'(t) = -2t \sum_{k=0}^{\infty} \frac{(-1)^k \alpha_k}{t^2 + \pi^2(k + \frac{1}{2})^2} \]

and a similar reasoning shows that \( F'(t) < 0 \) for \( t > 0. \) Hence, for all \( t > 0 \) and \( N \in \mathbb{N}, \) we have \( 0 < F(t) < F(0) \) and thus \( 0 < \Theta(t, N) < 1. \) The proof of Theorem 2.2 is complete. \( \square \)

**Remark 2.3.** Recalling that

\[ E_{4m} > 0 \quad \text{and} \quad E_{4m+2} < 0 \quad (m \in \mathbb{N}_0), \]

we can deduce (1.18) from (2.11). Note that the inequality (1.18) can also be written as

\[ (-1)^{m+1} \left( \text{sech } t - \sum_{j=0}^{m} \frac{E_{2j}}{(2j)!} t^{2j} \right) > 0 \quad (t > 0, \ m \in \mathbb{N}_0). \]  \( (2.18) \)

**Remark 2.4.** It was shown in [6] that (1.13) can be written as

\[ \nu_m(t) = -\frac{1}{2 \cosh(\frac{t}{4})} + \sum_{j=0}^{m} \frac{E_{2j}}{2(2j)!} \left( \frac{t}{4} \right)^{2j} \]  \( (2.19) \)

and (1.13) is equivalent to (2.18). Hence, for \( t > 0 \) and \( m \in \mathbb{N}_0, \) (1.13) holds true.

It was also shown in [7] that

\[ V_m(x) = (-1)^m \left[ \int_0^{\infty} \left( \frac{e^{t/4} - e^{3t/4}}{e^t - 1} + \frac{1}{2} \right) e^{-xt} \frac{dt}{t} - \frac{2B_{2j+1}}{(2j+1)!} \int_0^{\infty} t^{2j-1} e^{-xt} \frac{dt}{t} \right] \]

\[ = \int_0^{\infty} (-1)^n \nu_m(t) e^{-xt} \frac{dt}{t}. \]  \( (2.20) \)

We obtain from (2.20) that for all \( m \in \mathbb{N}_0, \)

\[ (-1)^n V_m^{(n)}(x) = \int_0^{\infty} (-1)^n \nu_m(t) t^{n-1} e^{-xt} \frac{dt}{t} > 0 \]

for \( x > 0 \) and \( n \in \mathbb{N}_0. \) Hence, the function \( V_m(x), \) defined by (1.17), is completely monotonic on \((0, \infty).\)

Sondow and Hadjicostas [29] introduced and studied the generalized-Euler-constant function \( \gamma(z), \) defined by

\[ \gamma(z) = \sum_{n=1}^{\infty} z^{n-1} \left( \frac{1}{n} - \ln \frac{n+1}{n} \right), \]  \( (2.21) \)

where the series converges when \( |z| \leq 1. \) Pilehrood and Pilehrood [22] considered the function \( z \gamma(z) \ (|z| \leq 1). \) The function \( \gamma(z) \) generalizes both Euler’s constant \( \gamma(1) \) and the alternating Euler constant \( \ln \frac{4}{\pi} = \gamma(-1) \) [27][28]. An interesting comparison by Sondow [27] is the double integral and alternating series

\[ \ln \frac{4}{\pi} = \int_0^1 \int_0^1 \frac{x-1}{(1+xy) \ln(xy)} \, dx \, dy = \sum_{n=1}^{\infty} (-1)^{n-1} \left( \frac{1}{n} - \ln \frac{n+1}{n} \right). \]  \( (2.22) \)
The formula (2.20) can provide integral representations for the constant \( \pi \). For example, the choice \((x, m) = (1/4, 0)\) in (2.20) yields
\[
\int_0^\infty \left( \frac{e^{t/4} - e^{3t/4}}{e^t - 1} + \frac{1}{2} \right) \frac{2e^{-t/4}}{t} \, dt = \ln \frac{4}{\pi},
\]
which provides a new integral representation for the alternating Euler constant \( \ln \frac{4}{\pi} \). The choice \((x, m) = (3/4, 0)\) in (2.20) yields
\[
\int_0^\infty \left( \frac{e^{t/4} - e^{3t/4}}{e^t - 1} + \frac{1}{2} \right) \frac{2e^{-3t/4}}{t} \, dt = \ln \frac{\pi}{3}.
\]

Many formulas exist for the representation of \( \pi \), and a collection of these formulas is listed in [25, 26]. For more history of \( \pi \) see [3, 4, 10].

Noting [6, Eq. (3.26)] that \( B_{2n+1} (1/4) \) can be expressed in terms of the Euler numbers
\[
B_{2n+1} (1/4) = -\frac{(2n + 1) E_{2n}}{4^{2n+1}} \quad (n \in \mathbb{N}_0),
\]
we find that (1.17) can be written as
\[
V_m(x) = (-1)^m \left[ \ln \frac{\Gamma(x + \frac{3}{4})}{\sqrt{x + \frac{1}{2}}} + \sum_{j=1}^{m} \frac{E_{2j} j}{4^{2j+1} x^{2j}} \right].
\]

From the inequalities \( V_m(x) > 0 \) for \( x > 0 \), we obtain the following

**Corollary 2.2.** For \( x > 0 \),
\[
x^{1/2} \exp \left( -\sum_{j=1}^{2m} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{x^{2j}} \right) < \frac{\Gamma(x + \frac{3}{4})}{\Gamma(x + \frac{1}{2})} < x^{1/2} \exp \left( -\sum_{j=1}^{2m+1} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{x^{2j}} \right).
\]

The problem of finding new and sharp inequalities for the gamma function \( \Gamma \) and, in particular, for the Wallis ratio
\[
\frac{(2n - 1)!!}{(2n)!!} = \frac{1}{\sqrt{\pi}} \frac{\Gamma(n + \frac{1}{2})}{\Gamma(n + 1)}
\]
has attracted the attention of many researchers (see [8,10,14,16,18] and references therein). Here, we employ the special double factorial notation as follows:
\[
(2n)!! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n n!, \quad 0!! = 1, \quad (-1)!! = 1,
\]
\[
(2n - 1)!! = 1 \cdot 3 \cdot 5 \cdots (2n - 1) = \pi^{-1/2} 2^n \Gamma \left( n + \frac{1}{2} \right);
\]

see [1, p. 258]. For example, Chen and Qi [3] proved that for \( n \in \mathbb{N} \),
\[
\frac{1}{\sqrt{\pi(n + \frac{1}{4})}} \leq \frac{(2n - 1)!!}{(2n)!!} < \frac{1}{\sqrt{\pi(n + \frac{3}{4})}},
\]
where the constants \( \frac{1}{4} - 1 \) and \( \frac{1}{4} \) are the best possible. This inequality is a consequence of the complete monotonicity on \((0, \infty)\) of the function (see [4])
\[
V(x) = \frac{\Gamma(x + 1)}{\sqrt{x + \frac{1}{4}} \Gamma(x + \frac{1}{2})}.
\]
If we write (2.27) as
\[
\frac{1}{\sqrt{x}} \exp \left( \sum_{j=1}^{2m+1} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{x^{2j}} \right) < \frac{\Gamma(x + \frac{1}{4})}{\Gamma(x + \frac{3}{4})} < \frac{1}{\sqrt{x}} \exp \left( \sum_{j=1}^{2m} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{x^{2j}} \right)
\]
and replace \( x \) by \( x + \frac{1}{4} \), we find
\[
\frac{1}{\sqrt{x + \frac{1}{4}}} \exp \left( \sum_{j=1}^{2m+1} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{(x + \frac{1}{4})^{2j}} \right) < \frac{\Gamma(x + \frac{1}{4})}{\Gamma(x + 1)} < \frac{1}{\sqrt{x + \frac{1}{4}}} \exp \left( \sum_{j=1}^{2m} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{(x + \frac{1}{4})^{2j}} \right).
\]
(2.31)

Noting that (2.28) holds, we then deduce from (2.31) that
\[
\frac{1}{\sqrt{\pi(x + \frac{1}{4})}} \exp \left( \sum_{j=1}^{2m+1} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{(x + \frac{1}{4})^{2j}} \right) < \frac{(2n - 1)!!}{(2n)!!} < \frac{\pi(x + \frac{1}{4})}{\Gamma(x + 1)} \exp \left( \sum_{j=1}^{2m} \frac{E_{2j}}{j \cdot 4^{2j+1}} \frac{1}{(x + \frac{1}{4})^{2j}} \right),
\]
(2.32)

which generalizes a recently published result of Chen [6, Eq. (3.40)], who proved the inequality (2.32) for \( m = 1 \).

**Theorem 2.3.** For \( t > 0 \) and \( N \in \mathbb{N}_0 \), we have
\[
\coth t = \sum_{j=0}^{N} \frac{2^{2j} B_{2j}}{(2j)!} t^{2j-1} + \sigma_N(t),
\]
(2.33)
where
\[
\sigma_N(t) = \frac{(-1)^N 2^{N+1}}{\pi^{2N}} \sum_{k=1}^{\infty} \frac{1}{k^{2N}(t^2 + \pi^2 k^2)},
\]
(2.34)
and
\[
\coth t = \sum_{j=0}^{N} \frac{2^{2j} B_{2j}}{(2j)!} t^{2j-1} + \theta(t, N) \frac{2^{2N+2} B_{2N+2}}{(2N + 2)!} t^{2N+1}
\]
(2.35)
with a suitable \( 0 < \theta(t, N) < 1 \).

**Proof.** It follows from [20, p. 126, Eq. (4.36.3)] that
\[
\coth t = \frac{1}{t} + 2t \sum_{k=1}^{\infty} \frac{1}{\pi^2 k^2 + t^2} = \frac{1}{t} + 2t \sum_{k=1}^{\infty} \frac{1}{\pi^2 k^2 \left( 1 + \frac{t^2}{\pi^2 k^2} \right)}.
\]
(2.36)
It is well known that
\[
\sum_{k=1}^{\infty} \frac{1}{k^{2j}} = \frac{(-1)^{j-1} (2\pi)^{2j} B_{2j}}{2(2j)!}.
\]
(2.37)
Using (2.17) and (2.37), we obtain from (2.36) that
\[
\coth t = \frac{1}{t} + 2t \sum_{k=1}^{\infty} \frac{(-1)^j}{k^2\pi^2} \left( \sum_{j=0}^{N-1} (-1)^j \left( \frac{t}{k\pi} \right)^{2j} + (-1)^N \left( \frac{t}{k\pi} \right)^{2N} \right) + \sigma_N(t)
\]
\[
= \frac{1}{t} + \sum_{j=0}^{N-1} \frac{2^{2j+2} B_{2j+2} t^{2j+1}}{(2j+2)!} + \sigma_N(t)
\]
\[
= \sum_{j=0}^{N} \frac{2^{2j} B_{2j} t^{2j+1}}{(2j)!} + \sigma_N(t)
\]
with
\[
\sigma_N(t) = \frac{2(-1)^N}{\pi^{2N}} \sum_{k=1}^{\infty} \frac{t^{2N+1}}{k^{2N}(t^2 + \pi^2 k^2)}.
\]
Noting that (2.37) holds, we can rewrite \(\sigma_N(t)\) as
\[
\sigma_N(t) = \theta(t, N) \frac{2^{2N+2} B_{2N+2} t^{2N+1}}{(2N+2)!}
\]
where
\[
\theta(t, N) := \frac{f(t)}{f(0)}, \quad f(t) := \sum_{k=1}^{\infty} \frac{1}{k^{2N}(t^2 + \pi^2 k^2)}.
\]
Obviously, \(f(t) > 0\) and is strictly decreasing for \(t > 0\). Hence, for all \(t > 0\), \(0 < f(t) < f(0)\) and thus \(0 < \theta(t, N) < 1\). The proof of Theorem 2.3 is complete.

The following expansion for Barnes \(G\)-function was established by Ferreira and López [11, Theorem 1]. For \(|\arg(z)| < \pi\),
\[
\ln G(z+1) = \frac{1}{4} z^2 + z \ln \Gamma(z+1) - \left( \frac{1}{2} z^2 + \frac{1}{2} z + \frac{1}{12} \right) \ln z - \ln A
\]
\[
+ \sum_{k=1}^{N-1} \frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k+1}} + R_N(z) \quad (N \in \mathbb{N}),
\]
where \(B_{2k+2}\) are the Bernoulli numbers and \(A\) is the Glaisher–Kinkelin constant defined by
\[
\ln A = \lim_{n \to \infty} \left\{ \ln \left( \prod_{k=1}^{n} k^k \right) - \left( \frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n + \frac{n^2}{4} \right\}, \quad (2.38)
\]
the numerical value of \(A\) being 1.282427... For \(\Re(z) > 0\), the remainder \(R_N(z)\) is given by
\[
R_N(z) = \int_{0}^{\infty} \left( \frac{t}{e^t - 1} - \sum_{k=0}^{2N} \frac{B_k t^k}{k!} \right) e^{-zt} \frac{t^3}{3} dt. \quad (2.39)
\]
Estimates for \(|R_N(z)|\) were also obtained by Ferreira and López [11], showing that the expansion is indeed an asymptotic expansion of \(\ln G(z+1)\) in sectors of the complex plane cut along the negative axis. Pedersen [21, Theorem 1.1] proved that for any \(N \geq 1\), the function \(x \mapsto (-1)^N R_N(x)\) is completely monotonic on \((0, \infty)\).
Here, we present another proof of this complete monotonicity result. From (2.33), we obtain the following inequality:

\[
\sum_{j=0}^{2m} \frac{2^{2j} B_{2j} t^{2j-1}}{(2j)!} < \coth t < \sum_{j=0}^{2m+2} \frac{2^{2j} B_{2j} t^{2j-1}}{(2j)!} \quad (t > 0, \ m \in \mathbb{N}_0),
\]

which is equivalent to

\[
(-1)^N \left( \coth t - \sum_{j=0}^{N} \frac{2^{2j} B_{2j} t^{2j-1}}{(2j)!} \right) > 0 \quad (t > 0, \ N \in \mathbb{N}_0).
\]

Replacement of \( t \) by \( t/2 \) in (2.40) yields

\[
(-1)^N \left( \frac{t}{2} \coth \left( \frac{t}{2} \right) - \sum_{j=0}^{N} \frac{B_{2j} t^{2j}}{(2j)!} \right) > 0 \quad (t > 0, \ N \in \mathbb{N}_0).
\]

Accordingly, we obtain from (2.39) that the function

\[
(-1)^N R_N(x) = \int_{0}^{\infty} \left( -1 \right)^N \left( \frac{t}{2} \coth \left( \frac{t}{2} \right) - \sum_{k=0}^{N} \frac{B_{2k} t^{2k}}{(2k)!} \right) e^{-xt} t^{-3} dt
\]

is completely monotonic on \((0, \infty)\).

**Remark 2.5.** From (2.33), we can deduce (1.2). In fact, noting that

\[
\coth t = \frac{e^t + e^{-t}}{e^t - e^{-t}} = 1 + \frac{2}{e^{2t} - 1},
\]

we see that (2.33) can be written as

\[
x + \frac{2x}{e^{2x} - 1} = \sum_{j=0}^{N} \frac{B_{2j}}{(2j)!} (2x)^{2j} + \frac{(-1)^N x^{2N+2}}{\pi^{2N}} \sum_{k=1}^{\infty} \frac{2}{k^{2N}(x^2 + \pi^2 k^2)}. \tag{2.43}
\]

Replacement of \( x \) by \( t/2 \) in (2.43) yields (1.2).

### 3. A Quadratic Recurrence Relation for \( B_n \)

Euler (see [20, p. 595, Eq. (24.14.2)] and [35]) presented a quadratic recurrence relation for the Bernoulli numbers:

\[
\sum_{k=0}^{n} \binom{n}{k} B_k B_{n-k} = (1 - n)B_n - nB_{n-1} \quad (n \geq 1), \tag{3.1}
\]

which is equivalent to

\[
\sum_{j=1}^{n-1} \binom{2n}{2j} B_{2j} B_{2n-2j} = -(2n + 1)B_{2n} \quad (n \geq 2). \tag{3.2}
\]

The relation (3.2) can be used to show by induction that

\[
(-1)^{n-1} B_{2n} > 0 \quad \text{for all} \quad n \geq 1,
\]
i.e., the even-index Bernoulli numbers have alternating signs. Other quadratic recurrences for the Bernoulli numbers have been given by Gosper (see [33, Eq. (38)]) as

$$B_n = \frac{1}{1-n} \sum_{k=0}^{n} (1 - 2^{1-k})(1 - 2^{k-n+1}) \binom{n}{k} B_k B_{n-k}$$

(3.3)

and by Matiyasevitch [17] (see also [35]) as

$$B_n = \frac{1}{n(n+1)} \sum_{k=2}^{n-2} \left\{ n + 2 - 2 \binom{n+2}{k} \right\} B_k B_{n-k} \quad (n \geq 4).$$

(3.4)

Here, we present a (presumably new) quadratic recurrence relation for the Bernoulli numbers.

**Theorem 3.1.** The Bernoulli numbers satisfy the following quadratic recurrence relation:

$$B_n = \frac{1}{2^{n-1}} \sum_{k=2}^{n-2} (1 - 2^k) \binom{n}{k} B_k B_{n-k} \quad (n \geq 4).$$

(3.5)

**Proof.** If we replace \(t\) by \(t/2\) in (2.1), we find

$$\frac{2}{e^{t/2} + 1} = 1 + \sum_{j=2}^{\infty} b_j t^{j-1}, \quad b_j = \frac{2(1 - 2^j)}{2^{j-1} \cdot j!}.$$  

(3.6)

The Bernoulli numbers \(B_n\) are defined by the generating function

$$t e^t - 1 = \sum_{n=0}^{\infty} B_n \frac{t^n}{n!},$$

(3.7)

which yields

$$\frac{t/2}{e^{t/2} - 1} = \sum_{k=0}^{\infty} \frac{B_k t^k}{2^k k!}. $$

(3.8)

It then follows from (3.6) and (3.8) that

$$\frac{t}{e^t - 1} = \left(1 + \sum_{j=2}^{\infty} b_j t^{j-1}\right) \left(\sum_{k=0}^{\infty} \frac{B_k t^k}{2^k k!}\right)$$

$$= \sum_{k=0}^{\infty} \frac{B_k t^k}{2^k k!} + \sum_{j=2}^{\infty} b_j t^{j-1} \sum_{k=0}^{\infty} \frac{B_k t^k}{2^k k!}$$

$$= \sum_{j=0}^{\infty} \frac{B_j t^j}{2^j j!} + \sum_{j=1}^{\infty} \sum_{k=1}^{j-1} b_{k+2} \frac{B_{j-k-1} t^{j-k-1}}{2^{j-k-1} (j-k-1)!}.$$ 

that is

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} \left(\frac{B_j}{2^j \cdot j!} + \sum_{k=0}^{j-1} b_{k+2} \frac{B_{j-k-1}}{2^{j-k-1} \cdot (j-k-1)!}\right) t^j.$$ 

(3.9)

Equating coefficients of equal powers of \(t\) in (3.7) and (3.9), we see that

$$\frac{B_j}{j!} = \frac{B_j}{2^j \cdot j!} + \sum_{k=0}^{j-1} b_{k+2} \frac{B_{j-k-1}}{2^{j-k-1} \cdot (j-k-1)!} \quad (j \in \mathbb{N}_0).$$

(3.10)
Substitution of the coefficients \( b_j \) in (3.6) into (3.10) then yields

\[
B_j = \frac{j!}{2^j - 1} \sum_{k=0}^{j-1} \frac{2(1 - 2^{k+2})B_{k+2}B_{j-k-1}}{(k + 2)! \cdot (j - k - 1)!} \quad (j \in \mathbb{N}).
\] (3.11)

It is easy to see that

\[
B_j = \frac{j!}{2^j - 1} \left( \sum_{k=0}^{j-3} \frac{2(1 - 2^{k+2})B_{k+2}B_{j-k-1}}{(k + 2)! \cdot (j - k - 1)!} - \frac{(1 - 2^j)B_j}{j!} + \frac{2(1 - 2^{j+1})B_{j+1}}{(j + 1)!} \right)
\]

\[
= \frac{j!}{2^j - 1} \sum_{k=0}^{j-3} \frac{2(1 - 2^{k+2})B_{k+2}B_{j-k-1}}{(k + 2)! \cdot (j - k - 1)!} + B_j + \frac{2(1 - 2^{j+1})B_{j+1}}{(2^j - 1)(j + 1)}.
\]

We therefore obtain

\[
B_{j+1} = \frac{1}{2^{j+1} - 1} \sum_{k=0}^{j-3} \frac{2(1 - 2^{k+2})B_{k+2}B_{j-k-1}}{(k + 2)! \cdot (j - k - 1)!} \quad (n \in \mathbb{N} \setminus \{1, 2\}),
\]

which, upon replacing \( j \) by \( n - 1 \) and \( k \) by \( k - 2 \), yields (3.5). This completes the proof of Theorem 3.1. \( \square \)

**Appendix: An alternative proof of (1.2)**

Euler’s summation formula states that, for \( k \in \mathbb{N} \), (see [31, p. 9])

\[
f(1) = \int_0^1 f(x) \, dx + \sum_{j=1}^{k} (-1)^j \frac{B_j}{j!} \left( f^{(j-1)}(1) - f^{(j-1)}(0) \right) + \frac{(-1)^{k+1}}{k!} \int_0^1 f^{(k)}(x)B_k(x) \, dx.
\] (A.1)

The choice \( k = 2n \) in (A.1) yields

\[
\frac{f(1) + f(0)}{2} = \int_0^1 f(x) \, dx + \sum_{j=1}^{n} \frac{B_{2j}}{(2j)!} \left( f^{(2j-1)}(1) - f^{(2j-1)}(0) \right) - \frac{1}{(2n)!} \int_0^1 f^{(2n)}(x)B_{2n}(x) \, dx.
\]

Application of this formula to \( f(x) = e^x \) then yields

\[
\frac{t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{j=1}^{n} \frac{B_{2j}t^{2j}}{(2j)!} - \frac{t}{e^t - 1} \frac{t^{2n}}{(2n)!} \int_0^1 e^{xt}B_{2n}(x) \, dx.
\]

Using the following formula (see [31, p. 5])

\[
B_{2n}(x) = 2(-1)^{n+1}(2n)! \sum_{k=1}^{\infty} \frac{\cos(2k\pi x)}{(2\pi)^{2n}} \quad (n \in \mathbb{N}, \quad 0 \leq x \leq 1),
\]
we have
\[
\frac{t}{e^t - 1} - \left( 1 - \frac{t}{2} + \sum_{j=1}^{n} \frac{B_{2j} t^{2j}}{(2j)!} \right) = -\frac{t}{e^t - 1} \frac{t^{2n}}{(2n)!} \int_0^1 e^{xt} B_{2n}(x) \, dx
\]
\[
= \frac{(-1)^n 2^{2n+1}}{e^t - 1} \sum_{k=1}^{\infty} \frac{1}{e^{xt} \cos(2k\pi x)} \frac{e^{x(t-t_1)} (2k\pi)^{2n}}{(2k\pi)^{2n}} \, dx
\]
\[
= \frac{(-1)^n 2^{2n+2}}{(2\pi)^{2n}} \sum_{k=1}^{\infty} \frac{1}{k^{2n}(t^2 + 4\pi^2 k^2)}
\]
\[
= (-1)^n t^{2n+2} \nu_n(t) \quad \text{(A.2)}
\]

This gives another derivation of (1.2).

We obtain from (A.2) that
\[
\nu_n(t) = \frac{(-1)^{n-1}}{t(e^t - 1) \cdot (2n)!} \int_0^1 e^{xt} B_{2n}(x) \, dx,
\]
\[
\text{(A.3)}
\]

which provides an alternative integral representation of $\nu_n(t)$.

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