JET MODULES FOR THE CENTERLESS VIRASORO-LIKE ALGEBRA

XIANGQIAN GUO AND GENQIANG LIU

Abstract. In this paper, we studied the jet modules for the centerless Virasoro-like algebra which is the Lie algebra of the Lie group of the area-preserving diffeomorphisms of a 2-torus. The jet modules are certain natural modules over the Lie algebra of semi-direct product of the centerless Virasoro-like algebra and the Laurent polynomial algebra in two variables. We reduce the irreducible jet modules to the finite-dimensional irreducible modules over some infinite-dimensional Lie algebra and then characterize the irreducible jet modules with irreducible finite dimensional modules over $\mathfrak{sl}_2$. To determine the indecomposable jet modules, we use the technique of polynomial modules in the sense of [BB, BZ]. Consequently, indecomposable jet modules are described using modules over the algebra $\mathcal{E}_+$, which is the “positive part” of a Block type algebra studied first by [DZ] and recently by [IM, I].

Keywords: Virasoro algebra, Virasoro-like algebra, Witt algebras, jet modules, irreducible modules, indecomposable modules, $\mathfrak{sl}_2$-modules, Block type algebras.

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1. Introduction

In the study of the representation theory of infinite-dimensional Lie algebras, one of the most important objects is to classify all irreducible Harish-Chandra modules, that is, weight modules with finite-dimensional weight spaces, for some important algebras. There are some known such classification results for several infinite-dimensional Lie algebras such as (generalized) Virasoro algebras ([M, S1, LuZ1, BF1, Ma, GLuZ1]), (generalized) Heisenberg-Virasoro algebra ([LuZ2, GLiZ, LG]), loop-Virasoro algebra ([GLuZ2]), Witt algebra ([BF2]), a class of Lie algebras of Weyl type ([S2]) and some infinite-dimensional Lie algebras of Block type ([S3, S4, CGZ, SXX]).

In particular, to obtain such classification for the Witt algebras Billig and Futorny ([BF2]) established a completely new method, where they used the notions of polynomial modules and jet modules for the Witt algebras. The motivation of this paper is to generalize their method to the centerless Virasoro-like algebra. As one of the steps, we consider the jet modules for this algebra. The centerless Virasoro-like algebra is also a kind of $W_\infty$ algebras in Physics([Sc]). It is the Lie algebra of $\text{SDiff}(T^2)$, the Lie group of the area-preserving diffeomorphisms of a 2-torus. It turns out that the generators of the Virasoro algebra (with or without central charge) can be constructed as a linear combination of infinitely many $\text{SDiff}(T^2)$ generators, see [ADFL].

1Corresponding author: Genqiang Liu; email: liugenqiang@amss.ac.cn
Let $\mathbb{Z}$ be the group of all integers and $\mathbb{C}$ the field of all complexes. Let $A = \mathbb{C}[t_1^{\pm1}, t_2^{\pm1}]$ be the Laurent polynomial algebra in two variables $t_1$ and $t_2$. The centerless Virasoro-like algebra $\mathcal{G}$ is the Lie subalgebra of the rank-2 Witt algebra (the derivation algebra of $A$) spanned by the operators with divergence zero as follows:

$$d_m = t_1^{m_1} t_2^{m_2} \left( m(2) t_1 \frac{\partial}{\partial t_1} - m(1) t_2 \frac{\partial}{\partial t_2} \right), \quad m \in \mathbb{Z}^2 \setminus \{0\}.$$ 

It is straight to check that

$$(1.1) \quad [d_m, d_n] = \det \left( \begin{array}{c} n \\ m \end{array} \right) d_{m+n}, \quad \forall \ m, n \in \mathbb{Z}^2 \setminus \{0\}, i = 1, 2,$$

where $m = (m(1), m(2)), n = (n(1), n(2))$ and $\det \left( \begin{array}{c} n \\ m \end{array} \right) = n(1)m(2) - n(2)m(1)$.

Note that $\mathcal{G}$ has no weight zero part, i.e., no Cartan subalgebra. So it does not even have an intrinsic way to define weight modules. To get around this, several authors look at $\mathbb{Z}^2$-graded and $\mathbb{Z}$-graded modules over $\mathcal{G}$. Many irreducible $\mathbb{Z}^2$-graded and $\mathbb{Z}$-graded modules over $\mathcal{G}$ have been constructed. For example, a large class of uniformly bounded $\mathbb{Z}^2$-graded $\mathcal{G}$-module were constructed in [LT1], a large class of $\mathbb{Z}^2$-graded irreducible generalized highest weight $\mathcal{G}$-module were constructed in [BZ], and a class of $\mathbb{Z}$-graded irreducible highest weight $\mathcal{G}$-modules were constructed in [BGLZ] (these modules can be regarded as certain kind of Whittaker modules in [GL]). In [WZ], the $\mathbb{Z}^2$-graded Verma modules over $\mathcal{G}$ relative to some total order were investigated. In [LT2], the structure of irreducible $\mathbb{Z}^2$-graded modules over the universal central extension $\hat{\mathcal{G}}$ (which is called the Virasoro-like algebra) of $\mathcal{G}$ with nonzero central actions and all homogeneous spaces finite dimensional were determined. Recently, the results of Whittaker modules for $\hat{\mathcal{G}}$ in [GL] were generalized to the algebra of semi-direct product of $\mathcal{G}$ and the rank-2 Laurent polynomials in [TWX].

In [LS], the irreducible $\mathbb{Z}^2$-graded and $\mathbb{Z}$-graded $\mathcal{G}$-modules were divided into two classes: generalized highest weight modules and uniformly bounded modules. However, they did not give a description of the structure of irreducible uniformly bounded modules. We want to settle this problem for the $\mathbb{Z}^2$-graded case using the technique developed by Billig and Futorny in [BF2]. To this aim, we first studied the jet modules in this paper. Another reason to research jet modules over $\mathcal{G}$ is that the natural modules coming from the 2-torus, and the tensor field modules are jet modules. And it is reasonable to guess that any irreducible uniformly bounded $\mathbb{Z}^2$-graded $\mathcal{G}$-module is isomorphic some subquotient of tensor field modules.

Our paper is organized as follows. In Section 2, we first introduce the notation of jet modules for the Virasoro-like algebra $\mathcal{G}$ and reduce the study of the jet modules to the study of finite dimensional modules over a Lie algebra $\mathcal{L}$. In Section 3, we provided an (equivalent) weaker condition for irreducible $\mathcal{G}$-modules to be jet modules. Then irreducible $\mathcal{L}$-modules are classified in Section 4 using $\mathfrak{sl}_2$-modules. This give rise to a completely description of irreducible jet modules via finite dimensional irreducible $\mathfrak{sl}_2$-modules. The last section is devoted to the indecomposable jet modules, which are shown to be polynomial modules. More precisely, we show that the indecomposable jet modules are in one-to-one
correspondence with the finite dimensional indecomposable module over a Lie algebra $\mathcal{B}_{+}$, which is the “positive part” of a special Block type Lie algebra, studied first by Djokovic and Zhao ([DZ]) and recently by Iohara and Mathieu ([IM], [I]). We mention that using the results of Section 5, one can also describe the irreducible jet modules as in Section 4, however, our methods in Section 4 are more straightforward and does not involve the concept of polynomial modules as in [B].

When we were submitting this paper, we learned that there are also several papers which deal with similar problems. For example, in [JL], the authors classified the irreducible $G$-modules with the condition that the action of $A$ being associative. In [BT], the authors considered the same kind of modules for the algebras of divergence zero vector fields on $n$-dimensional tori, which is just the Virasoro-like algebra considered in this paper when $n = 2$. However, they also require that the action of $A_{n} = \mathbb{C}[t_{1}^{\pm 1}, \cdots, t_{n}^{\pm 1}]$ are associative. The main differences of the present paper with the above mentioned papers is that we only assume that the action of $A$ is nonzero; and this is just the main difficulty of our methods inducing associativity from being nonzero. In fact, such kind of modules are of particular importance when we try to classify all irreducible weight Virasoro-like modules with finite-dimensional weight spaces, as similar situation occurs in the classification of such modules for the Witt algebras (see [BF2]).

2. Jet modules for the Virasoro-like algebra $G$

Let $\mathbb{Z}_{+}$ and $\mathbb{N}$ be the sets of all nonnegative integers and all positive integers respectively. All vector spaces and algebras are over $\mathbb{C}$. For a Lie algebra $\mathcal{L}$, we denote its enveloping algebra by $U(\mathcal{L})$. For any $m \in \mathbb{Z}^2$, we denote by $m(1), m(2)$ its entries, i.e., $m = (m(1), m(2))$; let $|m| = m(1) + m(2)$. For any $r \in \mathbb{Z}^2_+$ and $m \in \mathbb{Z}^2$ we denote $m^{r} = (m(1))^{r(1)} (m(2))^{r(2)}$ and $r! = (r(1))! (r(2))!$. We also have the generalized binomial coefficients $\binom{n}{m} = \binom{n(1)}{m(1)} \binom{n(2)}{m(2)}$ for $m, n \in \mathbb{Z}^2_+$ with $m(1) \leq n(1)$ and $m(2) \leq n(2)$.

In [B], Y. Billig introduced the concept of jet modules for the Witt algebra $W_{n}$, that is, the Lie algebra of vector fields on an $n$-dimensional torus. The jet modules originate from the modules of the jet bundles of tensor fields on the torus. Similarly, we can define the jet modules for the Virasoro-like algebra $G$.

For convenience denote $t^{n} = t_{1}^{n(1)} t_{2}^{n(2)}$ for any $n \in \mathbb{Z}^2$. Then $A$ has a natural $G$-module structure:

\begin{equation}
    d_{m} t^{n} = \det \left( \frac{n}{m} \right) t^{m+n}, \quad \forall \; m, n \in \mathbb{Z}^2 \setminus \{0\}, \; i = 1, 2.
\end{equation}

We can form the semi-direct Lie algebra $G \ltimes A$ whose Lie bracket is

\begin{equation}
    [x + f, y + g] = [x, y] + xg - yf, \quad \forall \; x, y \in G, f, g \in A.
\end{equation}

It is clear that $A$ is an abelian Lie subalgebra of $G \ltimes A$.

**Definition 1.** For any $\lambda \in \mathbb{C} \setminus \{0\}$, a $G$-module $V$ is called a *jet module* of parameter $\lambda$ if
(J1) $V$ is a $\mathbb{Z}^2$-graded $G \times A$-module with bounded dimensions of homogenous spaces, i.e.,

\begin{equation}
V = \bigoplus_{m \in \mathbb{Z}^2} V_m, \quad \dim V_m \leq N \text{ for some } N \in \mathbb{N},
\end{equation}

and the $G$-module structure is just the restriction;

(J2) The action of all $t^m(m \in \mathbb{Z}^2 \setminus \{0\})$ are bijective and quasi-associative, i.e.,

\begin{equation}
t^m t^n v = \lambda t^{m+n} v, \quad \forall \ m, n \in \mathbb{Z}^2 \setminus \{0\} \text{ with } m + n \neq 0, v \in V.
\end{equation}

Denote the category consisting of all jet $G$-modules by $\mathcal{J}$. A jet module is called irreducible (resp. indecomposable) if it is an irreducible (resp. indecomposable) $(G \times A)$-module.

A homomorphism $f$ between two jet modules $V = \bigoplus_{m \in \mathbb{Z}^2} V_m, W = \bigoplus_{m \in \mathbb{Z}^2} W_m$ is an isomorphism if there is a bijection $\varphi$ of $\mathbb{Z}^2$ such that $f(V_m) = W_{\varphi(m)}$ for any $m \in \mathbb{Z}^2$.

**Remark 2.1.** Note that an irreducible jet $G$-module and an irreducible $G$-module are not the same thing in general. Indeed, there exist jet modules which are irreducible as $(G \times A)$-modules but decomposable as $G$-modules, for example, $A$ is an irreducible $(G \times A)$-module via the natural way but $A = A' \oplus \mathbb{C}t^0$ as a $G$-module, where $A' = \text{span}\{t^m | m \in \mathbb{Z}^2 \setminus \{0\}\}$.

**Remark 2.2.** Our concept of jet modules are slightly different from that of Billig in [B]. Billig’s jet modules for $W_n$ require that all $t^m$ acts associatively, i.e., $\lambda = 1$ in our case. However, all properties within our interest can be deduced if we replace “associativity” with “quasi-associativity” (See [CLZ]). Indeed, if we take $(t^m)' = t^m/\lambda$, then we have

\[(t^m)'(t^n)' v = (t^{m+n})' v, \quad \forall \ m, n \in \mathbb{Z}^2, v \in V,
\]

which justify the terminology quasi-associative. We also note that the bijectivity of the actions of $t^m, 0 \neq m \in \mathbb{Z}^2$ on a jet module $V$ indicates $\dim V_n = \dim V_{m+n}$ for all $m, n \in \mathbb{C}^2$.

**Remark 2.3.** Note that $G \times A = (G \times A') \oplus \mathbb{C}t^0$ is a direct sum of ideals, so the action of $t^0$ is a scalar and does not affect the module structure for an irreducible $(G \times A)$-module. As we will see in Section 3 when the module $V$ is irreducible over $G \times A$, the second condition can be weakened as

(J2') The action of $A'$ is nonzero.

Let $V = \bigoplus_{m \in \mathbb{Z}^2} V_m$ be a jet $G$-module. Thanks to Remark 2.2, we can choose linear transformations $L(m) : V_0 \to V_0$ such that

\begin{equation}
d_m v = t^m L(m) v, \quad \forall \ m \in \mathbb{Z}^2 \setminus \{0\}, v \in V_0.
\end{equation}

Using the Lie bracket (1.1), (2.1) and (2.2), we can deduce that

\begin{equation}
d_m (t^m v) = t^{m+n} L(m) v + \det(n_m) t^{m+n} v, \quad \forall \ m, n \in \mathbb{Z}^2 \setminus \{0\}, v \in V_0
\end{equation}

and the commutator

\begin{equation}
[L(m), L(n)] = \det(n_m) (L(m + n) - L(m) - L(n)), \quad \forall \ m, n \in \mathbb{Z}^2 \setminus \{0\}.
\end{equation}

Denote by $\mathcal{L}$ the Lie algebra defined by the Lie bracket in (2.7) and by abuse of language we may assume that $L(m), m \in \mathbb{Z}^2$ forms a basis $\mathcal{L}$, that is, $\mathcal{L} = \bigoplus_{m \in \mathbb{Z}^2 \setminus \{0\}} \mathbb{C}L(m)$. Thus
V₀ can be viewed as a finite dimensional module over ℒ. In Section 4 we will describe the irreducible finite dimensional modules over ℒ. This allow us to characterize the irreducible jet modules using ℱ₂-modules. In Section 5 we will show that L(\( m \)), as a linear transformation on V₀, is a polynomial in \( m \in \mathbb{Z}^2 \). Hence V₀ is a polynomial ℳ-module in the sense of [BB, BZ]. Recall that a finite dimensional ℳ-module W is called a polynomial module if there is a basis \{v₁, \ldots, vₖ\} of W such that

\[
L(\( m \))vᵢ = \sum_{j=1}^{k} f_j(\( m \))v_j, \quad \( m = (m(1), m(2)) \in \mathbb{Z}^2 \), 1 ≤ i ≤ k,
\]

where all \( f_j(\( m \)) \) are polynomials in \( m(1), m(2) \).

3. Irreducible Jet Modules

In this section and the next, we will deduce the structure of irreducible jet modules. Before this, we first weaken the second condition for an irreducible jet module, which is of interest in its own right. First note that \( \mathcal{G} \ltimes A = (\mathcal{G} \ltimes A') \otimes \mathbb{C}t^0 \), where \( A' = \sum_{m \in \mathbb{Z}^2 \setminus \{0\}} \mathbb{C}t^m \). Hence, for any irreducible \( (\mathcal{G} \ltimes A)-\)module, the action of \( t^0 \) is a scalar and its value does not affect the module structure. Thus we can change the value of \( t^0 \) appropriately due to our convenience.

**Theorem 3.1.** Let V be an irreducible \( (\mathcal{G} \ltimes A)-\)module satisfying the conditions (J1) and (J2'), then V is a jet module of parameter λ, for some nonzero scalar \( \lambda \in \mathbb{C} \).

To prove Theorem 3.1 we need the following technique lemmas. Their proofs are easy and we leave them to the reader. For a proof of the first lemma please see Lemma 3.1 of [GLZ]. The proof of the second one follows easily from the fact that [\( [U(A), [U(A), U(\mathcal{G} \ltimes A)] ] = 0 \), similar to the proof of Lemma 3.2 of [GLZ].

**Lemma 3.2.** Let \( V = \bigoplus_{n \in \mathbb{Z}^2} V_n \) be an irreducible \( \mathbb{Z}^2 \)-graded module with bounded dimensions of homogeneous spaces over the associative algebra A, then \( \dim V_n \leq 1 \) for all \( n \in \mathbb{Z}^2 \).

**Lemma 3.3.** Suppose that V is an irreducible \( (\mathcal{G} \ltimes A)-\)module and \( x \in U(A) \). If \( xv = 0 \) for some nonzero \( v \in V \), then \( x \) is locally nilpotent on V.

**Proposition 3.4.** Let V be an irreducible \( (\mathcal{G} \ltimes A)-\)module satisfying condition (J1). Then either all \( t^m, m \neq 0 \) act injectively on V or all \( t^m, m \neq 0 \) act locally nilpotently on V.

**Proof.** Suppose that \( V = \bigoplus_{n \in \mathbb{Z}^2} V_n \) and \( t^m \) acts injectively on V for some \( m \in \mathbb{Z}^2 \setminus \{0\} \).

**Claim 1.** For any \( n \in \mathbb{Z}^2 \) with \( \det (\begin{pmatrix} n \\ m \end{pmatrix}) \neq 0 \), \( t^n \) acts nilpotently on V if it acts locally nilpotently on V.

Take any \( n \in \mathbb{Z}^2 \) such that \( \det (\begin{pmatrix} n \\ m \end{pmatrix}) \neq 0 \) and \( t^n \) acts locally nilpotently on V. Then \( t^{-n}t^n \) is locally nilpotent on V and hence nilpotent on each homogeneous space of V. Since the dimensions of all homogeneous spaces have an upper bound, there exists \( N \in \mathbb{N} \) such that \( (t^{-n}t^n)^N V = 0 \), i.e., \( (t^{-n})^N (t^n)^N V = 0 \). Here we note that \( (t^n)^k \) denotes \( t^n \cdots t^n \) rather than \( t^{kn} \) for all \( k \in \mathbb{N} \) and one should be careful with this notion throughout this section.
Now suppose that there exist positive integers $k$ and $N_k$ with $1 \leq k \leq N$ such that \((t^{-n})^k(t^n)^{N_k}V = 0\). Then we have
\[
0 = d_{m+n}(t^{-n})^k(t^n)^{N_k}V \\
= N_k \det \left( \frac{n}{m} \right) t^{m+2n}(t^{-n})^k(t^n)^{N_k-1}V - k \det \left( \frac{n}{m} \right) t^m(t^{-n})^{k-1}(t^n)^{N_k}V.
\]

Applying $t^n$ to the above equation, we deduce that $t^m(t^{-n})^{k-1}(t^n)^{N_k+1}V = 0$ and hence $(t-n)^{k-1}(t^n)^{N_k+1}V = 0$ since $t^m$ is injective. By downward induction on $k$, we obtain that $(t^n)^{N_0}V = 0$ for some $N_0 \in \mathbb{N}$. Claim 1 is proved.

**Claim 2.** \(t^{qm/p}\) acts injectively on $V$ for any $p, q \in \mathbb{Z}$ whenever \(qm/p \in \mathbb{Z}^2 \setminus \{0\}\).

Assume \(q/p < 0\) without loss of generality. Suppose on the contrary that the action of \(t^{qm/p}\) on $V$ is not injective, then \(t^{qm/p}\) acts on $V$ locally nilpotently by Lemma 3.3. Consequently, $(t^{n})^{-q(t^{qm/p})} \text{ acts nilpotently on each homogeneous space and hence also on the whole space } V$. There exists an $N' \in \mathbb{N}$ such that $(m,q(t^{qm/p})) N' V = 0$. Since $t^m$ is injective, we have $(t^{qm/p})^{N'}V = 0$. Choose a minimal $N'' \in \mathbb{N}$ such that $(t^{qm/p})^{N''}V = 0$.

Take any $n \in \mathbb{Z}^2$ such that $\det (\frac{n}{m}) \neq 0$, then we have
\[
0 = d_{n-qm/p}(t^{qm/p})^{N''}V = N'' \det \left( \frac{qm/p}{n} \right) t^n(t^{qm/p})^{N''-1}V = 0.
\]

Since $(t^{qm/p})^{N''-1}V \neq 0$ by the choice of $N''$, we see that $t^n$ is locally nilpotent on $V$ by Lemma 3.3. By Claim 1, $t^n$ is nilpotent on $V$ and we can choose a minimal $N_0 \in \mathbb{N}$ such that $(t^n)^{N_0}V = 0$. Then
\[
0 = d_{m-n}(t^n)^{N_0}V = N_0 \det \left( \frac{n}{m} \right) t^m(t^n)^{N_0-1}V = 0,
\]
which implies that $t^n$ is not injective, contradiction. Claim 2 follows.

**Claim 3.** $t^n$ is injective on $V$ for all $n \in \mathbb{Z}^2 \setminus \{0\}$.

By Claim 2, it suffices to prove this claim for $n \in \mathbb{Z}^2$ with $\det (\frac{n}{m}) \neq 0$. Take one such $n$ and suppose that $t^n$ is locally nilpotent on $V$. Then Claim 1 implies that $t^n$ is nilpotent on $V$. Find $k \in \mathbb{N}$ such that $(t^n)^k V = 0$.

Without loss of generality, we also assume that $\dim V_0 \geq \dim V_r$ for all $r \in \mathbb{Z}^2$. Then we have $t^m V_{im} = V_{(i+1)m}$ and $\dim V_{im} = \dim V_0$ for all $i \in \mathbb{Z}$. Then for any $v \in V_{-km}$ we have
\[
0 = (d_{m-n})^k(t^n)^k V = k! \det \left( \frac{n}{m} \right) (t^m)^k V + t^n u,
\]
for some $u \in V_{-n}$. This means that the map
\[
t^n : V_{-n} \to V_0
\]
is surjective. The fact that $\dim V_0 \geq \dim V_{-n}$ indicates that $t^n$ is bijective on $V_{-n}$. Replacing $V_0$ with $V_{-m}$ in the above argument, we may deduce that $t^n$ is bijective on all $V_{-m}, i \in \mathbb{N}$. Thus $t^n$ can not be nilpotent, contradiction. Claim 3 follows.
Lemma 3.5. Let $V$ be an irreducible $(\mathcal{G} \times A)$-module satisfying condition (J1). If $t^m$ acts injectively on $V$ for any $m \in \mathbb{Z}^2 \setminus \{0\}$, then there exists $\lambda \in \mathbb{C} \setminus \{0\}$ such that $(t^m t^n - \lambda t^{m+n})V = 0$ for all $m, n, m + n \in \mathbb{Z}^2 \setminus \{0\}$.

Proof. Suppose that $V = \bigoplus_{n \in \mathbb{Z}^2} V_n$. Since each $t^m$ is injective, we have $\dim V_m = \dim V_n$ for all $m, n \in \mathbb{Z}^2$. By Lemma 3.2 we see that $V$ contains an irreducible, $A$-submodule $W = \bigoplus_{m \in \mathbb{Z}^2} W_m$ with $W_m \subset V_m$. From Remark 2.3 we may assume that $t^0$ acts as a nonzero scalar $\lambda \in \mathbb{C}$. Then there exist $\lambda_{m,n} \in \mathbb{C} \setminus \{0\}$ such that

$$t^m t^n v = \lambda_{m,n} t^{m+n} v, \ \forall \ m, n \in \mathbb{Z}^2, v \in W_0.$$ 

Set

$$T_{m,n} = t^m t^n - \lambda_{m,n} t^{m+n}, \ \forall \ m, n \in \mathbb{Z}^2.$$

We can adjust the value of $\lambda$ so that $\lambda_{(1,0),(-1,0)} = \lambda_{(1,0),(1,0)}$. Note that $\lambda_{m,n} = \lambda_{n,m}$ and $\lambda_{m,0} = \lambda$. By Lemma 3.3 we have that $T_{m,n}$ is locally nilpotent on $V$ for all $m, n \in \mathbb{Z}^2$.

Choose a basis $\{v_1, \ldots, v_k\}$ of $V_0$, then $\{t^m v_1, \ldots, t^m v_k\}$ is a basis of $V_m$ for all $m \in \mathbb{Z}^2$. There exist $k \times k$ matrices $A_{m,n}$ such that

$$t^m t^n (v_1, \ldots, v_k) = t^{m+n} (v_1, \ldots, v_k) A_{m,n}.$$

Since $A$ is commutative, we see that

$$A_{m_1,n_1} A_{m_2,n_2} = A_{m_2,n_2} A_{m_1,n_1}, \ \forall \ m_1, m_2, n_1, n_2 \in \mathbb{Z}^2.$$

So there exists a complex matrix $S$ such that $B_{m,n} = S^{-1} A_{m,n} S$ are upper triangular matrices for all $m, n \in \mathbb{Z}^2$. Set

$$(w_1, \ldots, w_k) = (v_1, \ldots, v_k) S.$$ 

We have

$$t^m t^n (w_1, \ldots, w_k) = t^{m+n} (w_1, \ldots, w_k) B_{m,n}$$

and

$$T_{m,n} (w_1, \ldots, w_k) = t^{m+n} (w_1, \ldots, w_k) (B_{m,n} - \lambda_{m,n} I_k),$$

where $I_k$ is the identity matrix of rank $k$. Since $T_{m,n}$ is locally nilpotent on $V$, we see that all the diagonal entries of $B_{m,n}$ are $\lambda_{m,n}$. In particular,

$$T_{m,n} w_l = \sum_{i=0}^{l-1} C^{m+n} w_i, \ \forall \ m, n \in \mathbb{Z}^2, l = 1, \ldots, k$$

where $w_0 = 0$, and hence $(T_{m,n})^k w_k = 0$. 

Finally, this lemma follows from Claim 3 and Lemma 3.3.
For any \( m, n \in \mathbb{Z}^2 \) and \( u \in V_{-m-n} = \sum_{i=1}^{k} C t^{-m-n} w_i \), we have by (3.3) that

\[
T_{m,n} u \in \sum_{i=0}^{k} C T_{m,n} t^{-m-n} w_i = \sum_{i=0}^{k} C t^{-m-n} T_{m,n} w_i
\]

(3.4)

\[
\subseteq \sum_{i=0}^{k-1} C t^{-m-n} t^{m+n} w_i \subseteq \sum_{i=0}^{k-1} C (T_{m+n,-m-n} + \lambda_{m+n,-m-n} t^0 \circ) w_i
\]

Set \( V'_0 = \sum_{i=0}^{k-1} C w_i \), then we have \( T_{m,n} V_{-m-n} \subseteq V'_0 \) for all \( m, n \in \mathbb{Z}^2 \).

Given any \( m, n, r \in \mathbb{Z}^2 \), set \( r' = m + n + r \) and we have

\[
0 = t^{-r'} d_r T_{m,n} w_k \equiv t^{-m-n-r} [d_r, T_{m,n}] w_k \mod V'_0
\]

\[
\equiv t^{-r'} \left( \det \begin{pmatrix} m & n \end{pmatrix} t^{m+r} t^n + \det \begin{pmatrix} n & r \end{pmatrix} t^{m+n+r} - \det \begin{pmatrix} m+n & r \end{pmatrix} \lambda_{m,n} t^{m+n+r} \right) w_k \mod V'_0
\]

(3.5)

\[
\equiv t^{-r'} \left( \det \begin{pmatrix} m & n \end{pmatrix} \lambda_{m+r,n} + \det \begin{pmatrix} n & r \end{pmatrix} \lambda_{m,n+r} - \det \begin{pmatrix} m+n & r \end{pmatrix} \lambda_{m,n} \right) t^{m+n+r} w_k \mod V'_0.
\]

By the injectivity of \( t^m, t^n \) and \( t^r \), we obtain that

(3.6)

\[
\det \begin{pmatrix} m & n \end{pmatrix} \lambda_{m+r,n} + \det \begin{pmatrix} n & r \end{pmatrix} \lambda_{m,n+r} - \det \begin{pmatrix} m+n & r \end{pmatrix} \lambda_{m,n} = 0, \forall m, n, r \in \mathbb{Z}^2.
\]

Take \( m = j n \) for some \( j \in \mathbb{Z} \) and \( r \in \mathbb{Z}^2 \) such that \( \det \begin{pmatrix} n & r \end{pmatrix} \neq 0 \) in (3.5), we have

\[
j \lambda_{j n+r,n} + \lambda_{j n,n+r} - (j+1) \lambda_{j n,n} = 0, \forall \det \begin{pmatrix} n & r \end{pmatrix} \neq 0.
\]

Take \( j = 1 \) we get \( \lambda_{n+r,n} = \lambda_{n,n} \) for all \( \det \begin{pmatrix} n & r \end{pmatrix} \neq 0 \). Then we have \( \lambda_{j n+r,n} = \lambda_{n,n} \) and

\[
\lambda_{j n,n+r} = \lambda_{n+r,n+r} = \lambda_{n+r,n} = \lambda_{n,n}.
\]

Substitute these into (3.6), we can deduce that \( \lambda_{j n,n} = \lambda_{n,n} \) for \( j \neq -1 \). Consequently, \( \lambda_{m,n} = \lambda_{n,n} \) for all \( m, n, m+n \neq 0 \).

Taking \( r = -m - n \) in (3.5), we see that \( \lambda_{-m,n} = \lambda_{-m,m} \) for all \( \det \begin{pmatrix} m & n \end{pmatrix} \neq 0 \) and hence for all \( m, n \neq 0 \). Combining these results with the assumption that \( \lambda_{(1,0),(1,0)} = \lambda_{(1,0),-(1,0)} \), we obtain that \( \lambda_{m,n} = \lambda_{n,n} \) for all \( m, n \neq 0 \). Note that

\[
(t^{m} t^{-m})(t^{n} t^{-n}) w_1 = \lambda_{-m,-n} t^0 (t^{m} t^{-m}) w_1 = \lambda_{m,-m} \lambda_{n,-n} (t^0)^2 w_1
\]

\[
= (t^{m} t^{n})(t^{-m} t^{-n}) w_1 = \lambda_{m,n} \lambda_{-m,-n} t^{m+n} t^{-m-n} w_1 = \lambda_{m,n} \lambda_{-m,-n} \lambda_{m+n,-m-n} t^0 w_1.
\]

Recalling that \( t^0 \) acts as the scalar \( \lambda \), we have \( \lambda_{m,n} = \lambda \) for all \( m, n \in \mathbb{Z}^2 \), as desired.

Finally, we have

\[
w_1 \in V' = \{ v \in V \mid (t^{m} t^{n} - \lambda t^{m+n}) v = 0, \forall m, n \in \mathbb{Z}^2 \},
\]
which is nonzero. Since the space spanned by $t^m t^n - \lambda t^{m+n}, m, n \in \mathbb{Z}^2$, is stable under the action of $\mathcal{G} \ltimes A$, so $V'$ is an $(\mathcal{G} \ltimes A)$-submodule of $V$. By the irreducibility of $V$, we see that $V' = V$. Our result follows.

**Lemma 3.6.** Let $V$ be an irreducible $(\mathcal{G} \ltimes A)$-module satisfying condition (J1). If $t^m$ acts locally nilpotently on $V$ for any $m \in \mathbb{Z}^2 \setminus \{0\}$, then $t^m V = 0$ for all $m \in \mathbb{Z}^2 \setminus \{0\}$.

**Proof.** Without loss of generality, we may assume that $t^0 V = 0$. Set \( \deg(d_m) = \deg(t^m) = m \), then we have a $\mathbb{Z}^2$-gradation on $\mathcal{G} \ltimes A$ and $U(A)$. Denote $U = U(A)$ for short and let $U_m$ be the homogeneous component of degree $m$. It is clear that any element in $U_0$ is nilpotent on each $V_m$. Since $\dim V_m, m \in \mathbb{Z}^2$ have a uniform upper bound, we have $U_0^N V = 0$ for some $N \in \mathbb{N}$. In particular, $(t^m t^{-n}) V = 0$ for any $n \neq 0$.

Choose any $m, n \in \mathbb{Z}^2$ such that $\det \begin{pmatrix} m \\ n \end{pmatrix} \neq 0$, then for any $v \in V$ we have

$$0 = d_m (t^m t^{-n})^N v = \det \begin{pmatrix} n \\ m \end{pmatrix} N t^{m+n} (t^n)^{N-1} (t^{-n})^N v - \det \begin{pmatrix} n \\ m \end{pmatrix} N t^{m-n} (t^{-n})^{N-1} (t^n)^N v.$$  

Applying $t^n$ to the above equation, we deduce

$$t^{m-n} (t^{-n})^{N-1} (t^n)^{N+1} V = 0, \forall \det \begin{pmatrix} m \\ n \end{pmatrix} \neq 0.$$  

First we want to prove the following equation for all $1 \leq j \leq N$ by induction on $j$:  

$$t^{m_1-n} \cdots t^{m_j-n} (t^{-n})^{N-j} (t^n)^{N+j} V = 0,$$

for any $m_1, \ldots, m_j \in \mathbb{Z}^2$ with $\sum_{i=1}^j \epsilon_i m_i \notin \mathbb{Q} n, \epsilon_i \in \{0, 1\}$. When $j = 1$, (3.8) is just (3.7). Now suppose that (3.8) holds for some $j$ with $1 \leq j < N$. Applying $d_m$ to (3.8) for any $m$ with $m + \sum_{i=1}^j \epsilon_i m_i \notin \mathbb{Q} n$ for all $\epsilon_i \in \{0, 1\}$, we have

$$0 = d_m t^{m_1-n} \cdots t^{m_j-n} (t^{-n})^{N-j} (t^n)^{N+j} V$$

$$= (N+j) \det \begin{pmatrix} n \\ m \end{pmatrix} t^{m+n} t^{m_1-n} \cdots t^{m_j-n} (t^{-n})^{N-j} (t^n)^{N+j-1} V$$

$$- (N-j) \det \begin{pmatrix} n \\ m \end{pmatrix} t^{m-n} t^{m_1-n} \cdots t^{m_j-n} (t^{-n})^{N-j} (t^n)^{N+j} V.$$  

Applying $t^n$ to this equation, we get

$$t^{m_1-n} \cdots t^{m_{N-j}-n} (t^{-n})^{N-j-1} (t^n)^{N+j+1} V = 0,$$

that is, (3.8) holds for all $j$ with $1 \leq j \leq N$. Take $j = N$ in (3.8), we get $t^{m_1-n} \cdots t^{m_{N-n}} (t^n)^{2N} V = 0$.

Next we prove for all $0 \leq j \leq 2N$ that

$$t^{m_1-n} \cdots t^{m_{N+j-n}} (t^n)^{2N-j} V = 0, \forall \sum_{i=1}^j \epsilon_i m_i \notin \mathbb{Q} n, \epsilon_i \in \{0, 1\}.$$  

The equation (3.9) for $j = 0$ has been proved. Now suppose that (3.9) holds for some $j$ with $0 \leq j < 2N$. Applying $d_{m-2n}$ to (3.9) for any $m$ with $m + \sum_{i=1}^j \epsilon_i m_i \notin \mathbb{Q} n$ for all $\epsilon_i \in \{0, 1\}$, we have

$$0 = d_{m-2n} t^{m_1-n} \cdots t^{m_{N+j-n}} (t^n)^{2N-j} V$$

$$= (2N-j) \det \begin{pmatrix} n \\ m \end{pmatrix} t^{m-n} t^{m_1-n} \cdots t^{m_{N+j-n}} (t^n)^{2N-j-1} V,$$
which is just (3.9) for $j + 1$. So (3.9) holds for all $0 \leq j \leq 2N$. Take $j = 2N$ there we obtain that

$$t^{m_1-n} \cdots t^{m_{3N}-n}V = 0, \quad \forall \sum_{i=1}^{3N} \epsilon_i m_i \not\in \mathbb{Q}n, \epsilon_i \in \{0, 1\}. \tag{3.10}$$

Now for any $n_1, \cdots, n_{3N}$, take any $n \in \mathbb{Z}^2 \setminus \sum_{i \in \{0, 1\}} \mathbb{C} \sum_{i=1}^{3N} \epsilon_i n_i$, and let $m_i = n_i + n$ for all $i = 1, \cdots, 3N$. By (3.10), we have that $t^{m_1} \cdots t^{m_{3N}}V = 0$, that is $A^{3N}V = 0$. Then it is clear that the subspace

$$V' = \{v \in V \mid Av = 0\}$$

is a nonzero $(G \ltimes A)$-submodule of $V$. Therefore $V' = V$ and $AV = 0$, as desired. \qed

Now we can give the proof of the Theorem 3.1. Indeed, let $V$ be an irreducible $(G \ltimes A)$-module satisfying the conditions $(J1)$ and $(J2)$, then by Proposition 3.4, Lemma 3.5 and Lemma 3.6 we see that condition $(J2)$ holds. Thus $V$ is a jet module.

4. Finite dimensional modules over $\mathcal{L}$

In this section, we will give a description of the irreducible jet modules over $G$. From the argument in the last paragraph of Section 2, we see that the homogeneous space of a jet module admits a natural $\mathcal{L}$-module structure, where $\mathcal{L}$ is the Lie algebra defined by (2.7). To determine this module structure, we first make some preparations on the ideals of this Lie algebra $\mathcal{L}$. For any $k \geq 2$, define the subspace

$$\mathcal{I}_k = \left\{ \sum_{i=1}^{n} a_i L(m_i) \mid \sum_{i=1}^{n} a_i m_i^r = 0, \quad \forall r \in \mathbb{Z}^2_+, 2 \leq |r| \leq k \right\},$$

where $|r| = r(1) + r(2), m^r = m(1)^{r(1)} m(2)^{r(2)}$ and we have make the convention that $0^0 = 1$. Then we have the following properties:

**Proposition 4.1.** Assume $k, l \in \mathbb{N}$ with $k, l \geq 2$, then we have

1. $\mathcal{I}_k$ are co-finite dimensional ideals of $\mathcal{L}$ for all $k \geq 2$;
2. $\mathcal{I}_{k+1} \subseteq \mathcal{I}_k$;
3. $\bigcap_{k=2}^{\infty} \mathcal{I}_k = 0$;
4. $[\mathcal{I}_k, \mathcal{I}_l] \subseteq \mathcal{I}_{k+l-1}$;
5. $\mathcal{L}/\mathcal{I}_2 \cong \mathfrak{sl}_2$;
6. $[\mathcal{L}, \mathcal{L}] = \sum_{i=1}^{n} a_i L(m_i) \mid \sum_{i=1}^{n} a_i m_i = 0, a_1, \ldots, a_n \in \mathbb{C}$;
7. $\mathcal{L}/(\mathcal{I}_2 \cap [\mathcal{L}, \mathcal{L}]) \cong \mathfrak{sl}_2 \oplus \mathbb{C}z_1 \oplus \mathbb{C}z_2$, where $z_1$ and $z_2$ are central.

**Proof.** (1) Take any $\sum_{i=1}^{n} a_i L(m_i) \in \mathcal{I}_k$. Then for any $n \in \mathbb{Z}^2$, we have

$$[L(n), \sum_{i=1}^{n} a_i L(m_i)] = \sum_{i=1}^{n} a_i \det \left( \frac{m_i}{n} \right) (L(m_i + n) - L(m_i) - L(n)).$$
Then for any $r \in \mathbb{Z}_+^2$, $2 \leq |r| \leq k$, we have

$$
\sum_{i=1}^{n} a_i \det \left( \frac{m_i}{n} \right) \left((m_i + n)^r - (m_i)^r - (n)^r \right)
$$

$$
= \sum_{i=1}^{n} a_i (m_i(2)n(1) - m_i(1)n(2)) \sum_{s, r - s \in \mathbb{Z}_+^2 \setminus \{0\}} \binom{r}{s} m_i^s n^{r-s}
$$

$$
= \sum_{s, r - s \in \mathbb{Z}_+^2 \setminus \{0\}} \binom{r}{s} \left(n^{r-s+(1,0)} \sum_{i=1}^{n} a_i m_i^{s+(0,1)} - n^{r-s+(0,1)} \sum_{i=1}^{n} a_i m_i^{s+(1,0)} \right) = 0.
$$

Thus $[L(n), \sum_{i=1}^{n} a_i L(m_i)] \in \mathcal{I}_k$ and $\mathcal{I}_k$ is an ideal of $\mathcal{L}$.

For any element $x = \sum_{i=1}^{n} a_i L(m_i)$ we can assign a vector $v_x = (\sum_{i=1}^{n} a_i m_i^r)_{r \in \mathbb{Z}_+^2, 2 \leq |r| \leq k}$. Note that there exist $x_1, \ldots, x_l$ such that $\text{span}\{v_{x_1}, \ldots, v_{x_l}\} = \text{span}\{v_x \mid x \in \mathcal{L}\}$. Then it is easy to see that $\mathcal{L} = \text{span}\{x_1, \ldots, x_l\} + \mathcal{I}_k$, that is, $\mathcal{I}_k$ has finite codimension in $\mathcal{L}$.

(2) is clear. To prove (3) we take any $\sum_{i=1}^{n} a_i L(m_i) \in \cap_{k=2}^{+\infty} \mathcal{I}_k$, then $\sum_{i=1}^{n} a_i m_i^r = 0$ for all $r \in \mathbb{Z}_+, |r| \geq 2$. Since all $m_i$ are distinct, we may choose $r \in \mathbb{N}^2$ such that all $m_i^r$ are distinct. Then we have $\sum_{i=1}^{n} a_i (m_i^r)^l = 0$ for all $l \in \mathbb{N}$, forcing all $a_i = 0$. Hence $\cap_{k=2}^{+\infty} \mathcal{I}_k = 0$.

(4) Take any $x = \sum_{i=1}^{p} a_i L(m_i) \in \mathcal{I}_k$ and $y = \sum_{j=1}^{q} b_j L(n_j) \in \mathcal{I}_l$, then we have

$$
[\sum_{i=1}^{p} a_i L(m_i), \sum_{j=1}^{q} b_j L(n_j)] = \sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j \det \left( \frac{n_j}{m_i} \right) (L(m_i + n_j) - L(m_i) - L(n_j)).
$$

For any $r \in \mathbb{Z}_+^2$ with $2 \leq |r| \leq k + l - 1$, we can compute

$$
\sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j \det \left( \frac{n_j}{m_i} \right) \left((m_i + n_j)^r - (m_i)^r - (n_j)^r \right)
$$

$$
= \sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j \det \left( \frac{n_j}{m_i} \right) \sum_{s, r - s \in \mathbb{Z}_+^2 \setminus \{0\}} \binom{r}{s} (m_i)^s (n_j)^{r-s}
$$

$$
= \sum_{s, r - s \in \mathbb{Z}_+^2 \setminus \{0\}} \binom{r}{s} \sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j (m_i(2)n_j(1) - m_i(1)n_j(2))(m_i)^s (n_j)^{r-s}.
$$

Note that

$$
\sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j m_i(2)n_j(1)(m_i)^s (n_j)^{r-s} = \left( \sum_{i=1}^{p} a_i m_i(2)(m_i)^s \right) \left( \sum_{j=1}^{q} b_j n_j(1)(n_j)^{r-s} \right) = 0,
$$

where the last equation holds because either $2 \leq |s| + 1 \leq k$ or $2 \leq |r - s| + 1 \leq l$. Similarly

$$
\sum_{i=1}^{p} \sum_{j=1}^{q} a_i b_j m_i(1)n_j(2)(m_i)^s (n_j)^{r-s} = 0.
$$

This indicates the element in (4.1) equals to 0 and hence $[x, y] \in \mathcal{I}_{k+j-1}$ as desired.
(5) Denote by $e, f, h$ the images of $-L(1,0), L(0,1)$ and $L(1,1) - L(1,0) - L(0,1)$ in $\mathcal{L}/\mathcal{I}_2$. It is easy to check that $L(i,j) - ij(L(1,1) - L(1,0) - L(0,1)) - i^2 L(0,1) - j^2 L(0,1) \in \mathcal{I}_2$ for all $i, j \in \mathbb{Z}_+$, hence $\mathcal{L}/\mathcal{I}_2 = \mathbb{C}e \oplus \mathbb{C}h \oplus \mathbb{C}f$. From $(2.7)$ and the definition of $\mathcal{I}_2$, we get
\[ [-L(1,0), L(0,1)] = L(1,1) - L(1,0) - L(0,1), \]
\[ [L(1,1) - L(1,0) - L(0,1), -(L(1,0))] = -[(L(2,1) - L(1,1) - L(1,0)) + (L(1,1) - L(0,1) - L(1,0)) - 2L(1,0) - (L(2,1) - 2L(1,1) - 2L(0,1) + L(0,1)) \in -2L(1,0) + \mathcal{I}_2, \]
\[ [L(1,1) - L(1,0) - L(0,1), L(0,1)] = -[(L(1,2) - L(1,1) - L(0,1)) + (L(1,1) - L(0,1) - L(1,0)) - 2L(0,1) - (L(1,2) - 2L(1,1) - 2L(0,1) + L(0,1)) \in -2L(0,1) + \mathcal{I}_2 \]
Then $[e, f] = h, [h, e] = 2e, [h, f] = -2f$. So $\mathcal{L}/\mathcal{I}_2 \cong \mathfrak{sl}_2$ as Lie algebras.

(6) By $(2.7), L(m) - m(1)L(1,0) - m(2)L(0,1) \in [\mathcal{L}, \mathcal{L}]$ for any nonzero $m \in \mathbb{Z}^2$. Thus (6) is true.

(7) We note that $[\mathcal{L}, \mathcal{L}] + \mathcal{I}_2 = \mathcal{L}$ and hence $[\mathcal{L}, \mathcal{L}]/(\mathcal{I}_2 \cap [\mathcal{L}, \mathcal{L}]) = \mathcal{L}/\mathcal{I}_2 \cong \mathfrak{sl}_2$ and $\mathcal{I}_2/(\mathcal{I}_2 \cap [\mathcal{L}, \mathcal{L}]) = \mathcal{L}/[\mathcal{L}, \mathcal{L}]$ is 2-dimensional. More precisely, we have $\mathcal{L}/[\mathcal{L}, \mathcal{L}] = \mathbb{C}z_1 \oplus \mathbb{C}z_2$, where $z_1$ and $z_2$ are the images of $L(1,0)$ and $L(0,1)$ in $\mathcal{L}/[\mathcal{L}, \mathcal{L}]$ respectively. As in $(5)$ we still denote by $e, f, h$ the images of $-L(1,0), L(0,1)$ and $L(1,1) - L(1,0) - L(0,1)$ in $\mathcal{L}/\mathcal{I}_2$ respectively. Thus
\[
\frac{\mathcal{L}}{\mathcal{I}_2 \cap [\mathcal{L}, \mathcal{L}]} = \frac{[\mathcal{L}, \mathcal{L}]}{\mathcal{I}_2 \cap [\mathcal{L}, \mathcal{L}]} \oplus \frac{\mathcal{I}_2}{\mathcal{I}_2 \cap [\mathcal{L}, \mathcal{L}]} \cong \frac{\mathcal{L}}{\mathcal{I}_2} \oplus \frac{\mathcal{L}}{[\mathcal{L}, \mathcal{L}]} \cong \mathfrak{sl}_2 \oplus \mathbb{C}z_1 \oplus \mathbb{C}z_2,
\]
which is a reductive Lie algebra with center $\mathbb{C}z_1 \oplus \mathbb{C}z_2$. The corresponding projection is given by
\[
\pi : \mathcal{L} \to \mathfrak{sl}_2 \oplus \mathbb{C}z_1 \oplus \mathbb{C}z_2, \quad \pi(L(m)) = \begin{pmatrix}
    m(1)m(2) \\
m(2)^2 \\
-m(1)m(2)
\end{pmatrix} + m(1)z_1 + m(2)z_2.
\]
\[\square\]

For a Lie algebra $\mathcal{I}$, denote $\mathcal{I}^2 = [\mathcal{I}, \mathcal{I}], \mathcal{I}^3 = [\mathcal{I}, \mathcal{I}^2], \ldots, \mathcal{I}^{k+1} = [\mathcal{I}, \mathcal{I}^k]$ and $\mathcal{I}^w = \bigcap_{k=2}^{\infty} \mathcal{I}^k$.

**Lemma 4.2.** If $\mathcal{I}$ is a finite dimensional Lie algebra such that $\mathcal{I}^w = 0$, then $\mathcal{I}$ is a nilpotent Lie algebra.

**Proof.** Since $\mathcal{I}$ is finite dimensional, there is a positive integer $n$ such that $\mathcal{I}^n = \mathcal{I}^w = 0$. So $\mathcal{I}$ is nilpotent. \[\square\]

Now we can give the description of finite dimensional irreducible modules over $\mathcal{L}$.

**Theorem 4.3.** Let $V$ be a finite dimensional irreducible $\mathcal{L}$-modules. Then $\mathcal{I}_2 \cap [\mathcal{L}, \mathcal{L}]$ acts trivially on $V$, i.e., $V$ is an irreducible module over $\mathfrak{sl}_2 \oplus \mathbb{C}z_1 \oplus \mathbb{C}z_2$. \[\square\]
Proof. Let \( \rho : \mathcal{L} \to \mathfrak{gl}(V) \) be the representation of \( \mathcal{L} \) in \( V \). By Proposition \( 4.1 \) (2)(3)(4), \((\mathcal{L}_2/\ker(\rho))^w = 0 \). Then from Lemma \( 1.2 \), \( \mathcal{L}_2/\ker(\rho) \) is a nilpotent ideal of \( \mathcal{L}/\ker(\rho) \). Thus \( \mathcal{L}_2/\ker(\rho) \) is in the radical of \( \mathcal{L}/\ker(\rho) \). By a well known result on finite dimensional Lie algebra (see Proposition 19.1 of \([1]\)), we know that \( \mathcal{L}/\ker(\rho) \) is a reductive Lie algebra with trivial or 1-dimensional center. So \( \dim(\mathcal{L}_2/\ker(\rho)) \leq 1 \) and all elements in \( \mathcal{L}_2 \) acts as scalars on \( V \). Note that any element in \( \mathcal{L}_2 \cap [\mathcal{L}, \mathcal{L}] \) has trace 0 on \( V \), so \( \mathcal{L}_2 \cap [\mathcal{L}, \mathcal{L}] \subseteq \ker(\rho) \). \( \square \)

Using Theorem \( 4.3 \) and formula \( (2.6) \), we can determine the structure of irreducible jet modules explicitly. Before doing this, we first define some irreducible jet modules. For any irreducible finite dimensional \( \mathfrak{sl}_2 \)-module \( U \), \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2 \) and \( \lambda \in \mathbb{C} \), we can define a \((\mathcal{G} \ltimes \mathcal{A})\)-module structure on \( \mathbb{C}[t_1^\pm, t_2^\pm] \otimes U \) by

\[
 t^m(t^n \otimes v) = \lambda t^{m+n} \otimes v
\]

and

\[
d_m(t^n \otimes v) = t^{m+n} \otimes \left( \begin{pmatrix} m(1)m(2) & -m(1)^2 \\ m(2)^2 & -m(1)m(2) \end{pmatrix} + \det \begin{pmatrix} \alpha + n \\ m \end{pmatrix} \right) v
\]

for any \( m, n \in \mathbb{Z}^2 \) and \( v \in U \). We denote this module by \( F(\alpha, \lambda, U) \). One check that \( F(\alpha, \lambda, U) \cong F(\alpha', \lambda', U') \) as jet modules if and only if \( \alpha - \alpha' \in \mathbb{Z}^2, \lambda = \lambda' \) and \( U \cong U' \) as \( \mathfrak{sl}_2 \)-modules.

**Theorem 4.4.** If \( V \) is an irreducible jet \( \mathcal{G} \)-module, then \( V \cong F(\alpha, U, \lambda) \) for some irreducible finite dimensional \( \mathfrak{sl}_2 \)-module \( U \) and \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2, \lambda \in \mathbb{C} \setminus \{0\} \).

**Proof.** Let \( V = \bigoplus_{m \in \mathbb{Z}^2} V_m \) be an irreducible jet \( \mathcal{G} \)-module. We denote \( t^m v \) by \( t^m \otimes v \) for all \( v \in V_0, m \in \mathbb{Z}^2 \setminus \{0\} \). Then there exists nonzero \( \lambda \in \mathbb{C} \) such that \( t^m(t^n \otimes v) = \lambda t^{m+n} \otimes v \) for all \( m, n \in \mathbb{Z}^2 \) and \( v \in V_0 \). By the argument at the end of Section 2, we see that \( V_0 \) is a finite dimensional irreducible module over the infinite dimensional Lie algebra \( \mathcal{L} \) defined by \( (2.7) \). By Theorem \( 4.3 \), \( V_0 \) can be viewed as a finite dimensional irreducible module over \( \mathfrak{sl}_2 \oplus \mathbb{C}z_1 \oplus \mathbb{C}z_2 \), as given in Proposition \( 4.1 \) \( (7) \).

We can give the action of \( \mathcal{L} \) on \( V_0 \) explicitly. There exist \( \alpha_1, \alpha_2 \in \mathbb{C} \) such that \( z_1 \) and \( z_2 \) act as scalars \(-\alpha_2\) and \( \alpha_1 \) respectively. Using the projection given by \( (4.3) \), we can write out the \( \mathcal{L} \)-module action explicitly as

\[
 L(m)v = \begin{pmatrix} m(1)m(2) & -m(1)^2 \\ m(2)^2 & -m(1)m(2) \end{pmatrix} v + (m(2)\alpha_1 - m(1)\alpha_2)v.
\]

Since \( t^m \) acts bijectively on \( V \), we can identify \( V_n = t^nV_0 \). Then by \((2.5)\) and \((2.6)\) we can deduce that

\[
d_m v = t^m \otimes L(m)v \\
= \begin{pmatrix} m(1)m(2) & -m(1)^2 \\ m(2)^2 & -m(1)m(2) \end{pmatrix} t^m \otimes v + \begin{pmatrix} \alpha \\ m \end{pmatrix} t^m \otimes v,
\]
where \( \alpha = (\alpha_1, \alpha_2) \) and
\[
d_m(t^n \otimes v) = d_m t^n v = t^n \otimes d_m v + \left| \frac{n}{m} \right| t^{m+n} \otimes v
\]
\[
= \left( \begin{array}{c} m(1)m(2) \\ m(2) \end{array} \right) t^{m+n} \otimes v + \left| \frac{\alpha + n}{m} \right| t^{m+n} \otimes v,
\]
for all \( m, n \in \mathbb{Z}^2 \) and \( v \in V_0 \). So \( A \otimes V_0 \) is submodule of \( V \). By the irreducibility of \( V \), we have \( V = A \otimes V_0 \). Then \( V \cong F(\alpha, U, \lambda) \).

5. Indecomposable Jet Modules

In this section we will give a description of indecomposable jet \( \mathcal{G} \)-modules, which is an analogue of the main result of [B]. The arguments in Section 3 and Section 4 depend on the irreducibility of the modules, so the methods there do not generally apply to indecomposable modules. We will follow the idea of [B], using the technique of polynomial modules.

From now on, we fix an indecomposable jet \( \mathcal{G} \)-modules \( V \) with gradation as in (2.3). We first show that \( V_0 \) is a polynomial module over \( \mathcal{L} \). Indeed, we have

**Theorem 5.1.** Any finite dimensional \( \mathcal{L} \)-module \( W \) is a polynomial module.

**Proof.** Fix any \( s \in \mathbb{Z} \setminus \{0\} \), and we define the difference derivative inductively by \( \partial^0 L(s, 1) = L(s, 1) \) and \( \partial^{l+1} L(s, 1) = \partial^l L(s + 1, 1) - \partial^l L(s, 1) \) for all \( l \in \mathbb{Z}_+ \). One can check the following explicit formula by induction on \( l \):
\[
\partial^l L(s, 1) = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} L(s + i, 1).
\]

**Claim 1.** \( \partial^l L(s, 1) \in \mathcal{I}_k \) for all \( s \in \mathbb{Z} \) and \( l, k \in \mathbb{Z}_+ \) with \( l > k \geq 2 \).

For convenience, we define the linear operators on \( \mathcal{L} \):
\[
\phi_r \left( \sum_{i=1}^{n} a_i L(m_i) \right) = \sum_{i=1}^{n} a_i m_i^r, \quad \forall \ m_i \in \mathbb{Z}^2, \ r \in \mathbb{Z}^2_+, \ n \in \mathbb{N}.
\]
Then we have \( \mathcal{I}_k = \bigcap_{2 \leq |r| \leq k} \ker \phi_r \) for any \( k \geq 2 \). Note that
\[
\phi_{(j,j')} (\partial^l L(s, 1)) = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} (s + i)^j, \quad \forall \ j, j' \in \mathbb{Z}_+.
\]

Given any \( k \geq 2 \), we need only to show
\begin{align}
\sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} (s + i)^j = 0, \quad \forall \ j, l \in \mathbb{Z}_+ \text{ with } l > k \geq j.
\end{align}
The result for $j = 0$ is clear. Now suppose that (5.11) holds for any $j \in \mathbb{Z}_+$ with $j \leq k - 1$, then we have

$$
\sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} (s+i)^{i+1} = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} (s+i)^j
$$

$$
= l \sum_{i=1}^{l} (-1)^{l-i} \binom{l-1}{i-1} (s+i)^j = -l \sum_{i=0}^{l-1} (-1)^{l-i} \binom{l-1}{i} (s+i+1)^j = 0,
$$

where the first and last equalities follow from induction hypothesis. The claim is proved.

By a similar argument, we can deduce that

$$
\partial^3 (L(r, 0)) = L(r + 3, 0) - 3L(r + 2, 0) + 3L(r + 1, 0) - L(r, 0) \in \mathcal{I}_2, \ \forall \ r \in \mathbb{Z}.
$$

A direct calculation shows that

$$
[\partial^L(s, 1), L(r, 0)] = \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} [L(s+i, 1), L(r, 0)]
$$

$$
= r \sum_{i=0}^{l} (-1)^{l-i} \binom{l}{i} (L(s+r+i, 1) - L(s+i, 1) - L(r, 0))
$$

$$
= r \left( \partial^L(s+r, 1) - \partial^L(s, 1) \right).
$$

Then

$$
[\partial^L(s - r, 1), \partial^3 L(r, 0)]
$$

$$
= [\partial^L(s - r, 1), L(r + 3, 0) - 3L(r + 2, 0) + 3L(r + 1, 0) - L(r, 0)]
$$

$$
= (r + 3) \left( \partial^L(s + 3, 1) - \partial^L(s - r, 1) \right) - 3(r + 2) \left( \partial^L(s + 2, 1) - \partial^L(s - r, 1) \right)
$$

$$
= 3(r + 1) \left( \partial^L(s + 1, 1) - \partial^L(s - r, 1) \right) - r \left( \partial^L(s, 1) - \partial^L(s - r, 1) \right)
$$

$$
= r \partial^3 L(s, 1) + 3\partial^3 L(s + 3, 1) - 6\partial^3 L(s + 2, 1) + 3\partial^L(s + 1, 1).
$$

Replacing $r$ with $r + 1$ and making difference, we obtain

$$
[\partial^L(s - r - 1, 1), \partial^3 L(r + 1, 0)] - [\partial^L(s - r, 1), \partial^3 L(r, 0)] = \partial^{3k} L(s, 1).
$$

Noticing that $\partial^L(s, 1), \partial(r, 0) \in \mathcal{I}_2 \cap [\mathcal{L}, \mathcal{L}]$ for all $l \geq 3$, we may deduce by induction that

$$
\partial^{3k} L(s, 1) \subseteq (\mathcal{I}_2 \cap [\mathcal{L}, \mathcal{L}])^k, \ \forall \ k, l \in \mathbb{N}.
$$

Now $W$ is a finite dimensional $\mathcal{L}$-module and hence has a composition series of submodules. By Theorem 4.3, we see that $(\mathcal{I}_2 \cap [\mathcal{L}, \mathcal{L}])^k$ acts trivially on $W$ for sufficiently large $k \in \mathbb{N}$. In particular, there exists $N \geq 2$ such that $\partial L(s, 1) \in \ker \rho$ for $l > N$, where $\rho$ is the corresponding representation of $\mathcal{L}$ in $W$. Then Claim 1 indicates that $\rho(\partial L(s, 1)) = 0$ for all $l \in \mathbb{Z}_+$ with $l > N$. By Lemma 4.2 in [B], there exists a polynomial $f_s$ in one variable with coefficients in $\text{End}(W)$ such that $\deg(f_s) \leq N$ and $\rho(L(s + i, 1)) = f_s(s + i)$ for all $i \in \mathbb{Z}_+$. Replacing $s$ with any other $s' \in \mathbb{Z}$, we get the polynomial $f_{s'}$ such that $\rho(L(s' + i, 1)) = f_{s'}(s' + i)$ for all $i \in \mathbb{Z}_+$. Since $f_s$ and $f_{s'}$ have equal value for infinitely
many elements in \( \mathbb{Z}_+ \), we have \( f_s = f_{s'} \) for all \( s, s' \in \mathbb{Z} \). Denote this polynomial by \( f \). Then we have \( \rho(L(i, 1)) = f(i) \) in \( \text{End}(W) \) for all \( i \in \mathbb{Z} \).

Similarly, we can find polynomial \( g \) with coefficients in \( \text{End}(W) \) such that \( \rho(L(1, i)) = g(i) \) for all \( i \in \mathbb{Z} \). Then the commutator (2.7) gives that

\[
(ij - 1)\rho(L(i + 1, j + 1)) = g(j)f(i) - f(i)g(j) + (ij - 1)(g(j) + f(i)),
\]

for all \((i, j) \in \mathbb{Z}^2 \setminus \{(-1, -1)\}\) or equivalently,

\[
(ij - i - j)\rho(L(i, j)) = F(i, j), \quad \forall (i, j) \in \mathbb{Z} \setminus \{0\},
\]

where \( F(i, j) = g(j - 1)f(i - 1) - f(i - 1)g(j - 1) + (ij - i - j)(g(j - 1) + f(i - 1)) \). Note that \( ij - i - j = 0 \) implies \( F(i, j) = 0 \), so \( F(i, j) \) can be divided by \( ij - i - j \) as polynomials in \( i, j \). Denote \( G(i, j) = F(i, j)/(ij - i - j) \), we have

\[
(5.2) \quad \rho(L(i, j)) = G(i, j), \quad \forall ij - i - j \neq 0.
\]

Replacing \( L(s, 1) \) and \( L(1, s) \) with \( L(s, 0) \) and \( L(0, s) \) respectively in the previous argument, we may get another polynomial \( H(i, j) \) such that

\[
(5.3) \quad \rho(L(i, j)) = H(i, j), \quad \forall ij \neq 0.
\]

Noticing that \( G(i, j) = H(j, i) \) for \( ij \neq 0 \) and \( ij - i - j \neq 0 \), we have \( G(i, j) = H(i, j) \) for all \( i, j \in \mathbb{Z}^2 \). Combing with the equations (5.2) and (5.3), we conclude that \( \rho(L(i, j)) = G(i, j) \), for all \((i, j) \in \mathbb{Z}^2 \setminus \{0\}\), as desired. \( \square \)

Regard \( L(m), m \in \mathbb{Z}^2 \setminus \{0\} \) as linear transformations on \( V_0 \), and applying Theorem 5.1 to the \( \mathcal{L} \)-module \( V_0 \) we can find \( D_1 \in \text{End} V_0, i \in \mathbb{Z}_+^2 \), independent of \( m \), such that

\[
(5.4) \quad L(m) = \sum_{i=\mathbb{Z}_+^2} \frac{m^i}{i!} D_1, \quad \forall m \in \mathbb{Z}^2 \setminus \{0\},
\]
where only finitely many $D_i \neq 0$. Substitute the above equation into the commutator of $L(m)$ in $[\mathbb{Z}, \mathbb{Z}]$, we have
\[
\sum_{i \in \mathbb{Z}_+^2} \sum_{j \in \mathbb{Z}_+^2} \frac{m^i n^j}{i! j!} [D_i, D_j] = \det \left( \begin{pmatrix} n \\ m \end{pmatrix} \right) \left( \sum_{i \in \mathbb{Z}_+^2} \frac{(m + n)^i}{i!} D_i - \frac{m^i}{i!} D_i - \frac{n^i}{i!} D_i \right)
= \det \left( \begin{pmatrix} n \\ m \end{pmatrix} \right) \sum_{i \in \mathbb{Z}_+^2 \setminus \{0\}} \frac{m^i n^j}{i! j!} D_{i+j} - \det \left( \begin{pmatrix} n \\ m \end{pmatrix} \right) D_{(0,0)}
= \sum_{i \in \mathbb{Z}_+^2 \setminus \{0\}} \frac{m^i}{i!} D_{i} - \sum_{i \in \mathbb{Z}_+^2 \setminus \{0\}} \frac{n^j}{j!} D_{i+j}
= \det \left( \begin{pmatrix} n \\ m \end{pmatrix} \right) D_{(0,0)}
= \sum_{i, j \in \mathbb{Z}_+^2, |i| \geq 2, |j| \geq 2} \left( \frac{m^i n^j}{i! j!} j(1) i(2) D_{i+j-(1,1)} - \frac{m^i n^j}{i! j!} i(1) j(2) D_{i+j-(1,1)} \right)
\]
Simplifying it and comparing the coefficients of $\frac{m^i n^j}{i! j!}$, we can obtain
\[
[D_i, D_j] = \begin{cases} 
\det \left( \begin{pmatrix} j \\ i \end{pmatrix} \right) D_{i+j-(1,1)}, & \forall \ i, j \in \mathbb{Z}_+^2 \text{ with } |i| \geq 2 \text{ and } |j| \geq 2 \\
D_{(0,0)} \text{ or } -D_{(0,0)}, & \text{if } i = (1, 0), j = (0, 1) \text{ or } i = (0, 1), j = (1, 0) \\
0, & \forall \text{ other } i, j \in \mathbb{Z}_+^2 
\end{cases}
\] (5.5)

Denote the Lie algebra defined by the above commutator by $\mathcal{B}_+$ and by abusing language we still use $\{D_i \mid i \in \mathbb{Z}_+^2\}$ as a basis of $\mathcal{B}_+$. We have
\[
\mathcal{B}_+ = \mathcal{B}_+^\prime \oplus \mathcal{Z},
\]
where $\mathcal{B}_+^\prime = \text{span}\{D_j \mid j \in \mathbb{Z}_+^2, |j| \geq 2\}$ and $\mathcal{Z} = \text{span}\{D_{(0,0)}, D_{(0,1)}, D_{(1,0)}\}$. Consequently, $V_0$ becomes a finite dimensional $\mathcal{B}_+$-module.

**Remark 5.2.** Note that the Lie algebra $\mathcal{B}_+^\prime$ is part of the Lie algebra $\mathcal{B}_n = \text{span}\{D_i \mid i \in \mathbb{Z}^n\}$ ($n = 2$ case) subject to the following Lie bracket
\[
[D_i, D_j] = \det \left( \begin{pmatrix} \pi(j) \\ \pi(i) \end{pmatrix} \right) D_{i+j-(1,1)}, \forall \ i, j \in \mathbb{Z}^n,
\]
where $\pi : \mathbb{Z}^n \to \mathbb{C}^2$ is an additive map. This algebra is a special case of the Lie algebras of Block type, first introduced and studied by D. Djokovic and K. Zhao [DZ]. In a recent paper [IK], K. Iohara and O. Mathieu have classified the $\mathbb{Z}^n$-graded simple Lie algebras $\mathfrak{g} = \oplus_{i \in \mathbb{Z}^n} \mathfrak{g}_i$ with $\text{dim } \mathfrak{g}_i = 1$ for any $i \in \mathbb{Z}^n$, which turns out to be isomorphic to the algebra $\mathcal{B}_n$ or the
Kac-Moody algebra $A_1^{(2)}$ or $A_2^{(2)}$. Moreover, K. Iohara \[1\] studied the graded $\mathcal{B}_n$-modules with 1-dimensional homogeneous spaces.

The finite dimensional $\mathcal{B}_+$-module can be characterized by the following

**Lemma 5.3.** Any finite codimensional ideal of $\mathcal{B}_+^+$ is of the form $\mathcal{J}_k = \text{span}\{D(j) \mid |j| \geq k\}$. Moreover $[\mathcal{J}_k, \mathcal{J}_l] \subseteq \mathcal{J}_{k+l-2}$ for all $k, l \geq 2$ and $\mathcal{B}_+^+/\mathcal{J}_3 \cong \mathfrak{sl}_2$.

**Proof.** Let $\mathcal{J}$ be an ideal of $\mathcal{B}_+^+$ with $\dim(\mathcal{B}_+^+ / \mathcal{J}) < +\infty$. We first show that $D_1 \in \mathcal{J}$ implies $\mathcal{J}_k \subseteq \mathcal{J}$, where $k = |i| \geq 2$. Indeed, if $D_1 \in \mathcal{J}$, then

$$[D_{(1,2)}, D_i] = (2(i(1) - i(2)))D_{i+(0,1)} \in \mathcal{J}$$

and

$$[D_{(2,1)}, D_i] = (i(1) - 2i(2))D_{i+(1,0)} \in \mathcal{J}$$

imply that $D_{1+(0,1)} \in \mathcal{J}$ or $D_{1+(1,0)} \in \mathcal{J}$. Inductively we can find $j \in \mathbb{Z}_+^2$ such that $D_j \in \mathcal{J}$ and $|j| = k'$ for any $k' \geq k$. The following equations

$$[D_{(2,0)}, D_{(i,j)}] = \det \begin{pmatrix} i & j \\ 2 & 0 \end{pmatrix} D_{(i+1,j-1)}, \quad \forall i \geq 0, j \geq 1,$$

$$[D_{(0,2)}, D_{(i,j)}] = \det \begin{pmatrix} i & j \\ 0 & 2 \end{pmatrix} D_{(i-1,j+1)}, \quad \forall i \geq 1, j \geq 0,$$

implies that $D_j \in \mathcal{J}$ for all $|j| \geq k$. Hence $\mathcal{J}_k \subseteq \mathcal{J}$.

Then by the commutator \[5.5\], we have

$$[D_{(1,1)}, D_j] = (j(1) - j(2))D_j, \quad \forall j \in \mathbb{Z}_+^2 \text{ with } |j| \geq 2.$$  

We see that $\mathcal{J}$ is the direct sum of eigenspaces with respect to the action of $D_{(1,1)}$. Take any nonzero eigenvector of $D_{(1,1)}$, say $x \in \mathcal{J}$ with eigenvalue $l \in \mathbb{Z}$. Without loss of generality, we may assume that $l \geq 0$ and $x = \sum_{i=0}^n a_i D_{(i+l,i)}$ for some $a_i \in \mathbb{C}, i_0, n \in \mathbb{Z}_+$ and $a_n \neq 0$. If $n > i_0$, then a direct calculation shows that

$$(\text{ad } D_{(2,0)})^n(x) = \det \begin{pmatrix} n + l & n \\ 2 & 0 \end{pmatrix} \cdots \det \begin{pmatrix} 2n + l - 1 & 1 \\ 2 & 0 \end{pmatrix} a_n D_{(2n+l,0)} \in \mathcal{J}.$$  

By the arguments in the last paragraph, we have $\mathcal{J}_{2n+l} \subseteq \mathcal{J}$ and in particular $D_{(n+l,n)} \in \mathcal{J}$. Inductively, we get $D_{(i+l,i)} \in \mathcal{J}$ for all $i = i_0, \ldots, n$. Thus let $k \in \mathbb{N}$ be the smallest integer such that $\mathcal{J}_k \cap \mathcal{J} \neq 0$, then $\mathcal{J} = \mathcal{J}_k$.

Finally from the commutator

$$[D_{(1,1)}, D_{(2,0)}] = 2D_{(2,0)}, \quad [D_{(1,1)}, D_{(0,2)}] = -2D_{(0,2)} \quad \text{and} \quad [D_{(0,2)}, D_{(2,0)}] = 4D_{(1,1)}$$

we see that $\mathcal{B}_+^+/\mathcal{J}_3 \cong \mathfrak{sl}_2$. \[
\]

Now we can give the description of the indecomposable and irreducible jet $\mathcal{G}$-modules using finite dimensional $\mathcal{B}_+$-modules.

**Theorem 5.4.** Let $V$ be an indecomposable jet $\mathcal{G}$-module with gradation as in \[23\]. Then $V_0$ admits a $\mathcal{B}_+$-module structure. Moreover, we have
(1) There exists a one-to-one correspondence between indecomposable jet $G$-modules and indecomposable finite dimensional $B_+$-modules given by
\[ V \mapsto V_0 \quad \text{and} \quad U \mapsto V = \mathbb{C}[t_1^+, t_2^+] \otimes U, \]
where the $(G \times A)$-action on $\mathbb{C}[t_1^+, t_2^+] \otimes U$ is defined as $t^m t^n \otimes v = \lambda t^{m+n} \otimes v$ and
\[
d_m t^n \otimes v = t^{m+n} \otimes \left( \det \begin{pmatrix} n \\ m \end{pmatrix} + \sum_{i \in \mathbb{Z}_+^2} \frac{m!}{i!} D_i \right) v, \quad \forall \, m, n \in \mathbb{Z}^2, v \in U.
\]

(2) There exists a one-to-one correspondence between irreducible jet $G$-modules and irreducible finite dimensional $\mathfrak{sl}_2 \oplus \mathbb{C} z_1 \oplus \mathbb{C} z_2$-modules given by
\[ V \mapsto V_0 \quad \text{and} \quad U \mapsto V = \mathbb{C}[t_1^+, t_2^+] \otimes U, \]
where the $(G \times A)$-action on $\mathbb{C}[t_1^+, t_2^+] \otimes U$ is defined as $t^m t^n \otimes v = \lambda t^{m+n} \otimes v$ and
\[
d_m t^n \otimes v = t^{m+n} \otimes \left( \begin{pmatrix} m(1)m(2) \\ m(2)^2 \end{pmatrix} - \begin{pmatrix} 1 \end{pmatrix} m(1)^2 \right) + \det \begin{pmatrix} \alpha + n \\ m \end{pmatrix} \right) v
\]
for any $m, n \in \mathbb{Z}^2$ and $v \in U$, where $\alpha = (\alpha_1, \alpha_2) \in \mathbb{C}^2$, and $z_1$ and $z_2$ act on $U$ as scalars $-\alpha_2$ and $\alpha_1$ respectively.

**Proof.** By the arguments previous Lemma 5.3, we see that any jet $G$-module yields a finite dimensional $B_+$-module. On the other hand, given any finite dimensional $B_+$-module $U$, only finitely many $D_i, i \in \mathbb{Z}_+^2$, have nonzero actions on $U$ by Lemma 5.3, hence the formula (5.4) defines a representation of $\mathcal{L}$ on $U$. Setting $V = \bigoplus_{n \in \mathbb{Z}^2} V_n, V_n = t^n \otimes U$, then the formula (2.6) makes $V$ into a $(G \times A)$-module. It is easy to see that $V$ is an indecomposable (resp. irreducible) $(G \times A)$-module if and only if $U = V_0$ is an indecomposable (resp. irreducible) $B_+$-module.

Now suppose that $U$ is an irreducible $B_+$-module, and let $J$ be the kernel of the corresponding representation. Then $B_+/J$ is reductive. Since $\mathcal{J}_3 \subseteq \mathcal{J}_4$ and $B'_+/\mathcal{J}_3 \cong \mathfrak{sl}_2$, we see that $\mathcal{J}_4 \subseteq J$ and $\mathcal{J}_3/\mathcal{J}$ is contained in the radical of $B_+/J$. So all elements in $\mathcal{J}_3$ act as scalars. On the other hand, we have $\mathcal{J}_3 \subseteq [B'_+, \mathcal{J}_3]$, hence $\mathcal{J}_3 \subseteq J$. We conclude that $U$ is a $B'_+/\mathcal{J}_3 \cong \mathfrak{sl}_2$-module with $D_{(0,0)}, D_{(0,1)}$ and $D_{(1,0)}$ acting as scalars. In particular, $D_{(0,0)} = [D_{(1,0)}, D_{(0,1)}]$ acts as 0.

Choose a standard basis of $B'_+/\mathcal{J}_3 \cong \mathfrak{sl}_2$, say, $e = -\frac{D_{(2,0)}}{2}, f = \frac{D_{(0,2)}}{2}, h = D_{(1,1)}$. Then for any $\mathfrak{sl}_2$-module $U$, we can regard $U$ as a $B_+$-module via the above isomorphism and actions of $D_{(0,0)}, D_{(0,1)}, D_{(1,0)}$ as scalars $0, \alpha(1), -\alpha(2)$ respectively. Forming the $(G \times A)$-module $V = A \otimes U$, the module action can be written as $t^m(t^n \otimes v) = \lambda t^{m+n} \otimes v$ for some $\lambda \neq 0$ and
\[
d_m t^n \otimes v = t^{m+n} \otimes \left\{ \det \begin{pmatrix} n \\ m \end{pmatrix} - \alpha(2)m(1) + \alpha(1)m(2) \right\} v
\]
\[
= t^{m+n} \otimes \left( \begin{pmatrix} m(1)m(2) \\ m(2)^2 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} m(1)^2 \right) + \det \begin{pmatrix} \alpha + n \\ m \end{pmatrix} \right) v
\]
for all $m, n \in \mathbb{Z}^2$ and $v \in U$. \qed
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X.G.: Department of Mathematics, Zhengzhou university, Zhengzhou 450001, Henan, P. R. China. Email: guoxq@amss.ac.cn

G.L.: College of Mathematics and Information Science, Henan University, Kaifeng 475004, China. Email: liugenqiang@amss.ac.cn