RATIONAL DYCK TILINGS

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Abstract. We introduce rational Dyck tilings, or \((a, b)\)-Dyck tilings, and study them by the decomposition into \((1, 1)\)-Dyck tilings. This decomposition allows us to make use of combinatorial models for \((1, 1)\)-Dyck tilings such as the Hermite history and the Dyck tiling strip bijection. Together with \(b\)-Stirling permutations associated to the rational Dyck tilings, we obtain a correspondence between an \((a, b)\)-Dyck tiling and a tuple of \(ab\) \((1, 1)\)-Dyck tilings.

1. Introduction

A rational Dyck tiling, also called an \((a, b)\)-Dyck tiling, is a tiling by rational Dyck tiles in the region above \(\lambda\) and below \(\mu\), where \(\lambda\) and \(\mu\) are rational Dyck paths satisfying \(\lambda \leq \mu\). There are two types of rational Dyck tilings: one is a cover-inclusive tiling, and the other is a cover-exclusive tiling. In this paper, we study the rational Dyck tilings by decomposing them into a tuple of Dyck tilings.

Dyck tilings naturally appear in relation with the computation of the parabolic Kazhdan–Lusztig polynomials \(P_{\lambda, \mu}^{\pm}\) for the maximal parabolic subgroups of type \(A\) [1, 21]. The computation of \(P_{\lambda, \mu}^{-}\) requires cover-exclusive tilings as shown in [1]. Similarly, one make use of cover-inclusive tilings to compute \(P_{\lambda, \mu}^{+}\) as in [21]. Cover-inclusive Dyck tilings also appear in research areas in mathematical physics [3, 7, 8, 9, 10, 14, 15, 16]. There are several generalizations of Dyck tilings. First generalization is to consider the Kazhdan–Lusztig polynomials of other types, and this leads to ballot tilings for type \(B\) [17], and Dyck tilings of type \(D\) in [20]. Second is to impose a symmetry on Dyck tilings, and this leads to symmetric Dyck tilings studied in [5, 19]. Symmetric Dyck tilings have common properties of both type \(A\) and type \(B\). Third is to change the structure of Dyck path. In other words, we consider \(b\)-Dyck paths or more generally \((a, b)\)-Dyck paths. This generalization gives \(b\)-Dyck tilings studied in [5], and \((a, b)\)-Dyck tilings which are the main object in this paper.

Since we decompose \((a, b)\)-Dyck tilings into \(ab\) \((1, 1)\)-Dyck tilings, main tools to study \((a, b)\)-Dyck tilings can be reduced to the tools used for \((1, 1)\)-Dyck tilings. We mainly make use of two approaches for \((1, 1)\)-Dyck tilings: the Hermite history, and the DTS bijection [11, 18]. Both approaches behave nicely with the inversion number of a Dyck tiling compared to other approaches. One can use a rooted tree to describe a \((1, 1)\)-Dyck tiling (see for example [11, 18, 21]). Then, the labels on the edges of the tree are given by the DTS bijection for a \((1, 1)\)-Dyck tiling [11, 18, 19]. The labels are strictly increasing from the root to leaves of the tree. Then, the post-order word of the labels gives a permutation which characterizes the \((1, 1)\)-Dyck tiling together with the shape of the tree. On the other hand, to characterize an \((a, b)\)-Dyck tiling, we make use of a multi-permutation instead of a permutation. In fact, we consider a special class of multi-permutations, \(b\)-Stirling permutations [4, 12, 13]. This restriction of multi-permutations corresponds to capture the shape of the tree in case of \((1, 1)\)-Dyck tilings. This is because that a \(b\)-Stirling permutation is one-to-one to the \((b + 1)\)-ary tree with labels (see for example [2]).

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To relate cover-inclusive \((a,b)\)-Dyck tilings with cover-exclusive \((a,b)\)-Dyck tilings, we introduce an incidence matrix which expresses the cover-exclusive tilings. The parabolic Kazhdan–Lusztig polynomials \(P_{\lambda,\mu}^{\pm}\) are dual to each other, i.e., \(P_{\lambda,\mu}^{\pm}\) is obtained from \(P_{\lambda,\mu}^{\pm}\) by taking the transpose of the inverse of it. Therefore, the cover-inclusive Dyck tilings are expressed in terms of the incidence matrix by taking the inverse.

We introduce two types of decompositions of \((a,b)\)-Dyck paths: the horizontal and the vertical decompositions. The horizontal decomposition was defined for \((1,b)\)-Dyck paths in [6]. A decomposition of an \((a,b)\)-Dyck tiling into \(ab\) \((1,1)\)-Dyck tilings imposes some constraints on them. These constraints give relations among \((1,1)\)-Dyck tilings, which insure that we have a \(b\)-Stirling permutation for the \((a,b)\)-Dyck tiling. Besides those, we can distinguish a \((a,b)\)-Dyck tiling with or without non-trivial \((a,b)\)-Dyck tiles by looking at \((1,1)\)-Dyck tilings.

By the horizontal decomposition of an \((1,b)\)-Dyck tiling, we obtain \((1,1)\)-Dyck tilings. By combining the DTS bijection with the constraints on \((1,1)\)-Dyck tilings, we can compute the weight of the \((1,b)\)-Dyck tiling through \((1,1)\)-Dyck tilings. Further, one can show that the Hermite history of the \((1,b)\)-Dyck tiling is compatible with the Hermite histories of the \((1,1)\)-Dyck tilings. We have similar results for the vertical decomposition.

The paper is organized as follows. In Section 2, we introduce the notions of \((a,b)\)-Dyck paths and \((a,b)\)-Dyck tilings. Then, we briefly review the Hermite history of a \((1,1)\)-Dyck tiling and the Dyck tiling strip bijection. We establish relations between cover-inclusive and cover-exclusive rational Dyck tilings in Section 3. In Section 4, we decompose an \((a,b)\)-Dyck tilings into \(ab\) Dyck tilings by use of the horizontal and vertical decompositions and \(b\)-Stirling permutations. In Section 5, we give a description of the weight of an \((a,b)\)-Dyck tiling in terms of words of \((1,1)\)-Dyck tilings obtained by the DTS bijections.

## 2. Rational Dyck Tilings

### 2.1. Rational Dyck paths

Let \((a,b) \in \mathbb{N}^2\) be relatively prime positive integers, and \(n \in \mathbb{N}\). A rational Dyck path of size \(n\) is a lattice path from \((0,0)\) to \((bn, an)\), which does not go below the line \(y = ax/b\). We call a rational path an \((a,b)\)-Dyck path when we emphasize \((a,b)\). Since an \((a,b)\)-Dyck path \(p\) is a lattice path, \(p\) consists of north steps and east steps. Here, north (resp. east) step means the vector \((0,1)\) (resp. \((1,0)\)). By assigning a north step \(N\) and an east step \(E\), we simply write the path \(p\) as a word of \(N\)’s and \(E\)’s.

**Definition 2.2.** We denote by \(\mathcal{D}_n^{(a,b)}\) the set of \((a,b)\)-Dyck paths of size \(n\).

Figure 2.1 shows an example of a rational Dyck path with \((a,b) = (3,5)\) and \(n = 1\). The path is written as \(NENENE^3\).
Since the region, which is surrounded by a rational Dyck path $p$, the line $x = 0$, and the line $y = an$, can be regarded a Young diagram $\pi$, we identify $p$ with $\pi$ by abuse of notation. In Figure 2.1, the path is identified with the Young diagram $(2, 1)$.

Let $\lambda$ and $\mu$ be two rational Dyck paths. Then, we say that $\mu$ is above $\lambda$ (or equivalently $\lambda$ is below $\mu$) if and only if the Young diagrams $\lambda$ and $\mu$ satisfy $\lambda \supseteq \mu$. We denote by $\lambda \leq \mu$ if $\mu$ is above $\lambda$. Note that when $\mu$ is above $\lambda$, one can consider a skew shape $\lambda/\mu$.

Example 2.3. we have two $(2, 3)$-Dyck paths for $n = 1$. They are $N^2 E^3$ and $N E E^2$. The former is a $(2, 3)$-enlarged Dyck path, but the latter is not.

2.2. Rational Dyck tilings. Let $\lambda \leq \mu$ be two rational Dyck paths. A ribbon is a skew shape $\lambda/\mu$ which does not contain a two-by-two box. Then, an $(a, b)$-Dyck tile $d$ is defined as a ribbon such that the centers of boxes form an $(a, b)$-enlarged Dyck path $p^{(a,b)}$. We say that the tile $d$ is characterized by the path $p^{(a,b)}$ of size $n$.

We introduce a special class of $(a, b)$-Dyck paths which we call enlarged Dyck paths. Let $p$ be a $(1, 1)$-Dyck path expressed as a word of $N$’s and $E$’s. We replace each $N$ and $E$ in $p$ by $N^a$ and $E^b$ and denote the new path by $p^{(a,b)}$. Obviously, the path $p^{(a,b)}$ is a $(a, b)$-Dyck path. We call the path $p^{(a,b)}$ an $(a, b)$-enlarged Dyck path. If the size of a $(1, 1)$-Dyck path $p$ is $n$, we define the size of $p^{(a,b)}$ is also $n$.

Definition 2.4. Let $\lambda \leq \mu$ be two rational $(a, b)$-Dyck paths. A rational Dyck tiling (or a $(a, b)$-Dyck tiling) in the region $\lambda/\mu$ is a tiling by $(a, b)$-enlarged Dyck tiles.

A box $(x, y)$ means a box whose center is $(x, y)$. Let $b$ be a box $(x, y)$. Then, a box $(x - 1, y + 1)$ is said to be NW (north-west) of $b$, a box $(x, y + 1)$ is N of $b$, and a box $(x - 1, y)$ is W of $b$.

In Definition 2.4, we have no constraints on rational Dyck tiles. Below, we consider the two special classes of rational Dyck tilings in the region $\lambda/\mu$.

We consider the following conditions on rational Dyck tiles. Let $d_1$ and $d_2$ be rational Dyck tiles.

(I) Then, if we move $d_1$ by $(1, -1)$, then it is contained by another Dyck tile $d_2$ or below the path $\lambda$.

(II) If there exists a box of $d_1$ N, W, or NW of a box $d_2$, then all boxes N, W, or NW of a box of $d_2$ belong to $d_1$ or $d_2$.

Roughly speaking, the condition (I) means the sizes of rational Dyck tiles are weakly decreasing from south-east to north-west direction. On the other hand, the condition (II) means that the sizes of tiles are strictly increasing from south-east to north-west direction.

Definition 2.5. A rational Dyck tiling is said to be cover-inclusive (resp. cover-exclusive) if and only if all rational Dyck tiles satisfy the condition (I) (resp. (II)).

Figure 2.6 shows two examples of cover-inclusive $(2, 3)$-Dyck tilings. The tiling in the left picture consists of four single boxes. The tiling in the right picture contains a $(2, 3)$-Dyck tile of size one.

We define three statistics tiles, area and art on a rational Dyck tile $d$. 
Figure 2.6. Two cover-inclusive (2, 3)-Dyck tilings

Suppose that a rational Dyck tile $d$ is characterized by an $(a, b)$-enlarged Dyck path of size $n$. Then, we define

\[
\begin{align*}
tiles(d) & := 1, \\
area(d) & := an + bn + 1, \\
art(d) & := (a \cdot tiles(d) + b \cdot area(d))/(a + b), \\
        & = bn + 1.
\end{align*}
\]

**Remark 2.7.** The statistics $tiles$ counts the number of tiles forming a rational Dyck tile $d$, which is one, area counts the number of boxes forming $d$. One may define the statistics $art$ by

\[
art(d) := (b \cdot tiles(d) + a \cdot area(d))/(a + b), \\
        = an + 1.
\]

Since we have a natural bijection between $(a, b)$-Dyck tilings and $(b, a)$-Dyck tilings by reflecting the picture along the line $y = -x$, one can choose one of the definitions of $art$ without loss of generality.

Let $D$ be a $(a, b)$-Dyck tiling.

**Definition 2.8.** We define the weight of $D$ as

\[
wt(D) = \sum_{d \in D} art(d),
\]

where $d$ is a Dyck tile in $D$.

**Example 2.9.** The weights of two $(2, 3)$-Dyck tilings in Figure 2.6 are both four.

Let $D^{(a,b)}(\lambda)$ be the set of rational cover-inclusive Dyck tiling above $\lambda$.

**Definition 2.10.** The generating function $Z^{(a,b)}(\lambda)$ of rational Dyck tilings above $\lambda$ is defined as

\[
Z^{(a,b)}(\lambda) = \sum_{D \in D^{(a,b)}(\lambda)} q^{wt(D)},
\]

where $q$ is an indeterminate.

**Remark 2.11.** The generating function $Z^{(a,b)}(\lambda)$ can not be expressed in terms of $q$-integers. For example, we have

\[
Z^{(2,3)}(NEN^2E^3NE^2) = 1 + 2q + 3q^2 + 3q^3 + 3q^4 + q^5 + q^6.
\]

However, $Z^{(1,1)}(\lambda)$ can be expressed in terms of $q$-integers in a simple form. See [8, 10, 11, 18] for details.
2.3. Hermite histories and Cover-inclusive Dyck tilings. In this subsection, we summarize some properties of cover-inclusive Dyck tiling above a Dyck path $D_{n}^{(1,1)}$ following [11, 18].

Let $p$ be a $(1, 1)$-Dyck path and $D(p)$ be a Dyck tiling above $p$. We denote by $q$ the top path of $D(p)$. Let $d$ be a Dyck tile in $D(p)$. We call the right-most edge of $d$ entry and the vertical edge at the bottom of $d$ exit. We connect the entry and the exit of $d$ by a line. We call this line a trajectory. We concatenate trajectories of $D(p)$ if and only if it the entry of a Dyck tile is attached to the exit of another Dyck tile.

Note that a concatenated trajectory may start from a $N$-step $s_{N}'$ is a Dyck path $p'$ which is above $p$. In this case, we say this trajectory is attached to the $N$ step $s_{N}$ in $p$. We have an obvious bijection between a trajectory and a $N$ step in $p$. We say such a trajectory is attached to the up step $N$ in $p$. We have an obvious bijection between a trajectory and a $N$ step in $p$. The set of trajectories is called an Hermite history.

Let $S_{N}$ be a $N$ step in $p$. We denote by $l(S_{N})$ the sum of the size of Dyck tiles and the number of Dyck tiles on the trajectory attached to the step $S_{N}$. Here, the size of a Dyck tile is the size of $D_{n}$ which characterizing this Dyck tile. Thus, a single box is a Dyck tile of size 0.

We introduce a chord of a Dyck path $p$. Since $p$ consists of $N$'s and $E$'s, and they are balanced, we make a pair of $N$ and $E$ next to each other in this order. Then, by ignoring such pairs, we continue to make pairs. A pair of $N$ and $E$ obtained in this way is called a chord of $p$.

We assign an integer in $[1, n]$ to a chord of $p$ as follows. Let $S_{N}$ be a $N$ step $p$. We assign $l(S_{N})$ to the chord containing $S_{N}$.

We will define a permutation $\omega'(p)$ from the labeled chords in $p$. We read the labels on chords of $p$ in the pre-order. Here, pre-order means that we read the labels from the left-most and bottom-most chord, then left chords, and right chords. We continue this process until we read all the labels on the chords. We denote by $\omega(p)$ the inverse of the permutation obtained as above. We say that $\omega(p)$ is obtained by an Hermite history.

Remark 2.12. The labels on a chord are increasing from upper-left to bottom-right for a Dyck tiling obtained by an Hermite history.

Example 2.13. We consider the two Dyck tilings associated to Dyck paths in $D_{4}^{(1,1)}$ as below.

$D_1 = \begin{array}{c}
\framebox{1} \\
\framebox{2} \\
\framebox{3} \\
\multicolumn{1}{c}{A}
\end{array}$, $D_2 = \begin{array}{c}
\framebox{1} \\
\framebox{2} \\
\framebox{3} \\
\multicolumn{1}{c}{A}
\end{array}$

The pre-order words for these two Dyck tilings are 3214 and 4213. Thus, $\omega(D_1) = 3214$ and $\omega(D_2) = 3241$.

Let $w$ be a permutation on the alphabets $[1, n]$. We construct a non-negative integer sequence $h(w) := (h_1, \ldots, h_n)$ as follows. Let $w_j$ be a permutation consisting of integers $[1, i]$ in $w$. We define $h_j$ by the position of $j$ in $w_j$ from left minus one. For example, we have $h(w) = (0, 1, 0, 2)$ if $w = 3142$.

Definition 2.14. We call $h(w)$ the insertion history of a permutation $w$.

Let $u(p)$ be a step sequence of the Dyck path $p$, and $D$ be a Dyck tiling above $p$ without non-trivial Dyck tiles. We denote by $q$ the top Dyck path in $D$.

Proposition 2.15. The integer sequence $u(p) - h(\omega(D))$ is the step sequence of $q$. 
Proof. We prove the statement by induction. We first consider a unique Dyck tiling $D_0$ such that the top path and the bottom path are both $p$. By construction of an Hermite history, $\omega_0 := \omega(D_0)$ is the permutation with maximal inversions, i.e., $\omega_0 = (n, n - 1, \ldots, 1)$. The insertion history for $\omega_0$ is $(0, \ldots, 0)$, which implies $u(p) - \mathfrak{h}(\omega_0) = u(p)$. Since $q = p$, $u(p)$ gives the step sequence of $q$.

We assume that the statement is true up to all path $q'$ below $q$. The Dyck tiling $D$ above $p$ and below $q$ is obtained from some Dyck tiling $D'$ above $p$ and below $q'$ by adding a single box. This addition comes from the fact that $D$ and $D'$ have no non-trivial Dyck tiles. The addition of boxes to $D'$ implies the following operation on the labels on chords of $p$. The addition of a single box results in an extension of a trajectory by a single box. Without loss of generality, we assume that the trajectory associated to a chord $c_1$ is extended. Let $l$ be the label on $c_1$ for $D'$. Then, addition of a single box is equivalent to changing the label $l$ on $c_1$ and a label $l'$ on some $c_2$ such that the chord $c_2$ is left to $c_1$ and $l'$ is minimal label larger than $l$. Since we have no non-trivial Dyck tiles, the chord $c_2$ should be left next to $c_1$. Thus, the permutation $\omega(D)$ is obtained from $\omega(D')$ by exchanging some integer $m$ and $m + 1$. The condition that labels on chords are increasing from a left-top chord to a right-bottom chord insures that there is no integer $m' < m$ between $m$ and $m + 1$ in $\omega(D)$. Then, the exchange of $m$ and $m + 1$ in $\omega(D')$ means that the $m + 1$-th entry of $\mathfrak{h}(\omega(D'))$ is increased by one. From these observations, the step sequence of $q$ is also expressed as $u(p) - \mathfrak{h}(\omega(D))$. This completes the proof.

2.4. Dyck tiling strip and cover-inclusive Dyck tilings. In Section 2.3, we introduce an Hermite history, and construct a permutation $\omega(p)$ from a Dyck tiling $p$. In this subsection, we introduce another construction of a permutation, called Dyck tiling strip (DTS for short) following [11, 18].

Let $p$ be a Dyck path of size $n - 1$ and $D$ be a Dyck tiling above $p$. First, we define an operation called spread of $D$ at $y = -x + m$ for $0 \leq m \leq 2(n - 1)$. We divide $D$ into two pieces by the line $y = -x + m$. Then, we move the up-right piece by $(1, 1)$-direction. By reconnecting two pieces by $NE$ steps, we obtain a new Dyck tiling of size $n$.

In a spread, if the line $y = -x + m$ passes through a Dyck tile, the size of the Dyck tile is increased by one. Figure 2.16 is an example of a spread of a Dyck tiling at $y = -x + 3$.

![Figure 2.16. A spread at $y = -x + 3$ of a Dyck tiling](image)

Let $D'$ be a new Dyck tiling of size $n$ after a spread of $D$. We perform an addition of boxes as follows. We attach a single box to each $N$ step in $D$ such that the $N$ step is right to the line $y = -x + m + 1$. We denote by $D_{\text{new}}$ the new Dyck tiling obtained from $D'$. This process is called right strip-growth. Similarly, if we attach boxes to each $E$ step in $D$ such that the $E$ step is left to the line $y = -x + m + 1$. This process is called left strip-growth.

Since a spread increases the number of chords for the path $p$ by one, we have a new added chord. Then, we assign a label $n$ to the newly added chord. In this way, we obtain labels on chords for a Dyck tiling $D_{\text{new}}$. We call these processes to obtain $D_{\text{new}}$ from $D$ Dyck tiling strip.
**Definition 2.17.** We say Dyck tiling strip is right (resp. left) Dyck tiling strip if we perform right (resp. left) strip-growths. If we do not specify right or left, a Dyck tiling strip means right Dyck tiling strip.

Given a Dyck tiling $D$, we have a label on chords of $p$. We denote by $\nu(D)$ the word read by the post-order. Here, post-order means that we read the labels of chords by the following order: 1) the left chords, 2) the right chords, then 3) the bottom chords.

**Example 2.18.** We consider the same Dyck tilings as Example 2.13.

$$D_1 = \begin{array}{c}
A \quad 1 \\
2 
\end{array}, \quad D_2 = \begin{array}{c}
1 \\
3 \quad 2
\end{array}$$

The post-order words for $D_1$ and $D_2$ are $\nu(D_1) = 2431$ and $\nu(D_2) = 4321$ respectively.

**Remark 2.19.** The labels on chords are decreasing from left-top to bottom-right for a Dyck tiling obtained by the DTS bijection.

Labels on chords in an Hermite history and those in the DTS bijection are related as below.

**Proposition 2.20.** Let $D$ be a Dyck tiling above $p$, and $h_1(c)$ (resp. $h_2(c)$) are a label of a chord $c$ in $p$ obtained from $D$ for an Hermite history (resp. DTS bijection). Then, we have

\begin{equation}
(2.1) \quad h_1(c) + h_2(c) = n + 1.
\end{equation}

**Proof.** We prove the proposition by induction. For a Dyck tiling $D$, we denote by $q$ the top path. The statement is obvious when $p = q$. We assume that Eqn. (2.1) holds for all $q'$ below $q$.

As in the proof of Proposition 2.15, we extend a trajectory by length one in the Hermite history. On the other hand, the strip-growth in the DTS bijection corresponds to extend a trajectory by one. Further, the labels on chords in the Hermite history are increasing from left-top chords to right-bottom chords, while the labels in the DTS bijection are increasing from left-top to right-bottom. Thus, the extension of a trajectory increases a label in the Hermite history and decrease the corresponding label in the DTS bijection. From these observations, we have Eqn. (2.1). $\square$

**3. Incidence matrix**

Let $\pi := \pi_1 \ldots \pi_r$ be a path of length $r$ consisting of $N$’s and $E$’s. Suppose that $\pi_i = N$ and $\pi_j = E$ for $1 \leq i < j \leq r$. We say that $(i, j)$ is $(a, b)$-admissible if and only if the partial path $\pi' := \pi_{i+1} \ldots \pi_{j-1}$ is an $(a, b)$-enlarged Dyck path, and $\pi'$ cannot be written as a concatenation of two $(a, b)$-enlarged Dyck paths.

Suppose that $\pi_i = N$ in $\pi$. Then, by definition of $(a, b)$-admissibility, we have at most one $j$ such that the pair $(i, j)$ is $(a, b)$-admissible. Note that if $\pi_i = N$ and $\pi_{i+1} = E$, the pair $(i, i+1)$ is $(a, b)$-admissible for any pair of $a$ and $b$.

We define the operation, which we call $NE$-flipping, on a path $\pi$. Suppose that the pair $(i, j)$ is $(a, b)$-admissible in $\pi$. The $NE$-flip is defined to exchange $\pi_i$ and $\pi_j$, namely, we have a new path $\pi'$ such that $\pi'_i = E, \pi'_j = N$ and $\pi'_k = \pi_k$ for $k \neq i, j$. We denote this relation by $\pi' \leftrightarrow \pi$, or equivalently $\pi \rightarrow \pi'$. 
Let $\mathcal{F}(\pi)$ be the set of $(a,b)$-admissible pairs in $\pi$, and $f_i \in \mathcal{F}(\pi)$ for $1 \leq i \leq |\mathcal{F}(\pi)|$. We consider the sequence

\[(3.1) \quad \pi \xrightarrow{i_1} \pi_1 \xrightarrow{i_2} \pi_2 \ldots \xrightarrow{i_p} \pi_p,\]

with $1 \leq i_1 < i_2 < \ldots < i_p \leq |\mathcal{F}(\pi)|$. We denote by $\pi_j \xrightarrow{i_j+1} \pi_{j+1}$ the NE-flip by the pair $f_{i_{j+1}}$.

As already mentioned above, we have no chance to have two $(a,b)$-admissible pairs $(i, j)$ and $(i', j')$ with $j \neq j'$ or $(i, j)$ and $(i', j)$ with $i \neq i'$. Thus, we can choose the order of $i_j$ with $1 \leq j \leq p$ as above.

Given two paths $\pi$ and $\pi'$, we denote by $\pi \rightarrow \pi'$ if there exists a sequence (3.1) with $\pi_p = \pi'$.

We introduce two types of weight which is given to a NE-flip. We define the weight of type $I$ of a NE-flip as $wt^I(\pi' \leftarrow \pi) = -q$ where $\pi'$ is obtained by a single NE-flip from $\pi$. Then, in general, we define $wt^I(\pi' \leftarrow \pi) = (-q)^p$ when $\pi' \leftarrow \pi$ as a sequence (3.1).

Let $(i, j) \in \mathcal{F}(\pi)$. Then, we denote by $n_{E}$ the number of $E$'s in the partial path $\pi_{i+1} \ldots \pi_{j-1}$. The weight of type $II$ of a NE-flip is given by $wt^{II}(\pi' \leftarrow \pi) = -q^{p+E}$. For general two paths $\pi$ and $\pi'$, the weight $wt^{II}(\pi' \leftarrow \pi)$ is given by the product of the weights of NE-flips.

For both types $I$ and $II$, the weight $wt^X(\pi' \leftarrow \pi)$ with $X = I$ or $II$ is given by the monomial of $q$. When there exists no sequence (3.1) starting from $\pi$ and ending with $\pi'$, we define $wt^X(\pi' \leftarrow \pi) = 0$.

We define two types of incidence matrices as follows.

**Definition 3.1.** The matrices $M$ and $N$ are defined by

\[
M_{\pi', \pi} = wt^I(\pi' \leftarrow \pi),
\]

\[
N_{\pi', \pi} = wt^{II}(\pi' \leftarrow \pi).
\]

**Proposition 3.2.** The entry $M_{\pi', \pi}$ (resp. $N_{\pi', \pi}$) gives the cover-exclusive $(a,b)$-Dyck tiling in the region above $\pi'$ and below $\pi$. The weight of a tile is given by the statistics tiles (resp. art).

**Proof.** By definition, the entry $\pi'$ is obtained from $\pi$ by exchanging an $N$ and an $E$ in $\pi$. If we connect a pair of these $N$ and $E$ by a arc in $\pi$, arcs never intersect. In terms of a Dyck tiling, it is clear that $\pi'$ is obtained from $\pi$ by a cover-exclusive tiling. Since the weight of NE-flip is a monomial of $q$, the weight of a tiling above $\pi'$ and below $\pi$ is given by the statistic tile.

We have a similar proof for $N_{\pi', \pi}$. Since the weight of NE-flip is equal to the statistic art, the weight of a tiling above $\pi'$ and below $\pi$ is given by art. This completes the proof. □

The inverse matrices of $M$ and $N$ are characterized by $(a,b)$-Dyck tilings as follows.

**Theorem 3.3.** The entry $M_{\pi', \pi}^{-1}$ (resp. $N_{\pi', \pi}^{-1}$) gives the cover-inclusive $(a,b)$-Dyck tiling in the region above $\pi'$ and below $\pi$. The weight of a tile is given by the statistics tiles (resp. art).

**Proof.** The theorem follows from the Principle of Inclusion-Exclusion in [22]. See also the proof of Theorem in [21]. □
Example 3.4. We consider rational $(1, 2)$-Dyck tilings. The order of bases is $\text{NNEE}, \text{NENE}, \text{NEEN}, \text{NENE}, \text{ENNE}, \text{ENEN}, \text{ENEEN}, \text{EENEN}$, and $\text{EEENN}$.

Let $D$ be an $(a, b)$-Dyck tiling. The main purpose of this section is to introduce a decomposition of $D$ into $ab$ $(1, 1)$-Dyck tilings. In [6], a strip decomposition of $(1, b)$-Dyck paths is introduced and studied. The strip decomposition assigns a $(1, b)$-Dyck path to $b$ $(1, 1)$-Dyck paths. We call this strip decomposition the horizontal decomposition. When $a \neq 1$, one can also consider a strip decomposition which is in the vertical direction. Similarly, a vertical decomposition gives a $(a, 1)$-Dyck path to $a$ $(1, 1)$-Dyck paths. Thus, for $(a, b)$-Dyck paths, we can introduce the decomposition which is both horizontal and vertical. This decomposition assigns a $(a, b)$-Dyck path to $ab$ $(1, 1)$-Dyck paths. In the language of Dyck tilings, $(a, b)$-Dyck paths correspond to a $(a, b)$-Dyck tiling consisting of single boxes, which means that the tiling do not contain non-trivial $(a, b)$-Dyck tiles. In this section, we generalize this decomposition from $(a, b)$-Dyck paths to $(a, b)$-Dyck tilings.

4. Decomposition

Let $D$ be a $(a, b)$-Dyck tiling. We introduce the step and height sequences. Let $\pi$ be an $(a, b)$-Dyck path of size $n$. We introduce the step sequence and the height sequence to the path $\pi$ following [6].

The step sequence $\mathbf{u}(\pi) = (u_1, \ldots, u_{an})$ is a sequence of non-negative integers defined for $\pi$ as

$u_1 \leq u_2 \leq \ldots \leq u_{an},$

$u_k \leq b(k - 1)/a, \quad \forall k \in [1, an].$

The entry $u_k$ in the step sequence indicates that the path $\pi$ passes through the edge connecting $(u_k, k - 1)$ and $(u_k, k)$. 

Note that when $\pi' = \text{ENEEN}$ and $\pi = \text{NNEE}$, we have a non-trivial Dyck tiling and this tiling contributes as $q$ in $M^{-1}$. This tiling also contributes as $q^3$ in $N^{-1}$. 

\[
M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-q & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -q & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -q & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -q & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & q^2 & -q & 0 & -q & 1 & 0 & 0 & 0 \\
-q & 0 & q^2 & -q & 0 & -q & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & q & 0 & 1 & 0 \\
q^2 & 0 & 0 & 0 & 0 & q & q & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & q & 1 \\
\end{pmatrix},
\]

\[
M^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^2 & q & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
q^3 & q^2 & q & 1 & 0 & 0 & 0 & 0 & 0 \\
q^2 & q & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
q^3 & q^2 & q & 0 & q & 1 & 0 & 0 & 0 \\
q + q^4 & q^3 & q^2 & q & q^2 & q & 1 & 0 & 0 \\
q^4 & q^3 & q^2 & 0 & q^2 & q & 0 & 1 & 0 \\
q^5 & q^4 & q^3 & q^2 & q^3 & q^2 & q & 0 & 1 \\
q^6 & q^5 & q^4 & q^3 & q^4 & q^3 & q^2 & q^2 & q & 1 \\
\end{pmatrix}.
\]
Similarly, the height sequence $h(\pi)$ is a sequence of positive integers satisfying
\[
\begin{align*}
    h_1 &\leq h_2 \leq \ldots \leq h_n, \\
    h_k &\geq \lfloor ka/b \rfloor, \quad \forall k \in [1, bn].
\end{align*}
\]
The entry $h_k$ in the height sequence indicates that the path $P$ passes through the edge connecting $(k-1, h_k)$ and $(k, h_k)$.

**Example 4.1.** Let $(a, b) = (2, 3)$, $n = 2$ and $\pi = N\text{NE}\text{NE}\text{NE}\text{NE}\text{E}$. The step sequence is $u(\pi) = (0, 1, 2, 4)$ and the height sequence is $h(\pi) = (1, 2, 3, 3, 4, 4)$.

4.2. **Horizontal and vertical strip decomposition.** Let $\pi$ be an $(a, b)$-Dyck path of size $n$ and $h(\pi) := (h_1, h_2, \ldots, h_{bn})$ be its height sequence. We construct $b$ integer sequences $H_i := (H_1^i, \ldots, H_n^i)$, $1 \leq i \leq b$, from $h(\pi)$ by
\[
    H_j^i := h_{(j-1)b+i}, \quad j \in [1, n].
\]
The integer sequence $H_i$ defines a lattice path $p_i$ from $(0, 0)$ to $(n, an)$. Note that a path $p_i$ may not be an $(a, 1)$-Dyck path of size $n$.

**Definition 4.2.** We call the map from $\pi$ to $b$ paths $p_i$, $1 \leq i \leq b$ the horizontal strip decomposition of $\pi$. We denote this map by $\theta_h : \pi \mapsto (p_1, \ldots, p_b)$.

Let $u(\pi) := (u_1, \ldots, u_{an})$ be the step sequence of $\pi$. We construct $a$ integer sequences $U_i := (U_1^i, \ldots, U_n^i)$, $1 \leq i \leq a$, from $u(\pi)$ by
\[
    U_j^i := u_{(j-1)a+i}, \quad j \in [1, n]
\]
The integer sequence $U_i$ defines a lattice path $q_i$ from $(0, 0)$ to $(bn, n)$.

**Definition 4.3.** We call the map from $\pi$ to $a$ paths $q_i$, $1 \leq i \leq a$ the vertical strip decomposition of $\pi$. We denote this map by $\theta_v : \pi \mapsto (q_1, \ldots, q_a)$.

We give another description of the horizontal and vertical decompositions of $\pi$. We first consider the horizontal decomposition. We define $b$ non-negative integer sequences $v_i$, $1 \leq i \leq b$, recursively as follows.

**Algorithm A:**

1. Set $i := b$, $u' := u(\pi)$ and $v_0 = (0, \ldots, 0)$.
2. Define
\[
    v_{b+1-i} := \left\lfloor \frac{u'}{i} \right\rfloor.
\]
3. Decrease $i$ by one. Replace $u'$ by $u' - v_{b+1-i}$. Then, go to (2). The algorithm stops when $i = 1$.

**Proposition 4.4.** Let $v_i$, $1 \leq i \leq b$, be integer sequences defined as above. Then, $v_i$ is the step sequence of a Dyck path $p_i$.

**Proof.** By a horizontal decomposition of $\pi$, we have $b$ lattice paths $p_i$, $1 \leq i \leq b$. The $i$-th Dyck path $p_i$ consists of $(j-1)b+i$-th columns for $1 \leq j \leq n$. Thus, it is straightforward that $v_i$ in Eqn. (4.1) is the step sequence of $p_i$. □

Given a partition $\lambda := (\lambda_1, \ldots, \lambda_l)$ where $l$ is the length of $\lambda$, we define the transposition of $\lambda$, denoted by $\lambda_i^t := (\lambda_1^t, \ldots, \lambda_l^t)$, as
\[
\lambda_i^t := \# \{ j | \lambda_j \geq i \}.
\]
We append several 0’s to $\lambda^t_i$ if necessary. For example, we consider a rational Dyck path in $\mathcal{D}_2^{(2,3)}$. If we have a step sequence $(0, 1, 2, 4)$, then its transposition is given by $(3, 2, 1, 0, 0)$.

We consider the vertical decomposition of $P$. We replace $u(\pi)$ by its transposition $u(\pi)^t$, $v_i$ by $w_i$ in Algorithm A.

**Proposition 4.5.** Let $w_i$, $1 \leq i \leq b$, be integer sequences defined as above. Then, $w_i^t$ is the step sequence of a Dyck path $q_i$.

**Proof.** Let $P = P_1P_2 \ldots P_{(a+b)n} \in \mathcal{D}_n^{(a,b)}$ be a rational Dyck path. We define $N^t = E$ and $E^2 = N$. The vertical decomposition of $P$ is equivalent to the horizontal decomposition of the $P^2 \in \mathcal{D}_n^{(b,a)}$ where $P^2$ is the Dyck path $P^2 = p_1^{(a+b)n} \ldots p_1^2$.

Note that taking the transposition means we apply the operation $\sharp$. Then, the statement in the Proposition is a direct consequence of Proposition 4.4 by the transposition. $\square$

The following proposition is clear from the definitions of $\theta_h$ and $\theta_v$.

**Proposition 4.6.** Let $\pi$ be an $(a,b)$-Dyck path. Then, we have

$$\theta_h \circ \theta_v = \theta_v \circ \theta_h.$$ 

**Definition 4.7.** We define $\vartheta := \theta_v \circ \theta_h$. We call $\vartheta$ Dyck path decomposition.

The Dyck path decomposition $\vartheta$ sends an $(a,b)$-Dyck path of size $n$ to a set of $ab$ paths of size $n$.

**Example 4.8.** We consider a Dyck path $P = NENENE^2NE^2 \in \mathcal{D}_2^{(2,3)}$. The actions of $\theta_h$ and $\theta_v$ on $P$ are given by

$$\theta_h(P) = (NEN^2EN, N^2EN^2E, N^3ENE),$$

$$\theta_v(P) = \left( \begin{array}{c} NE^2NE^4 \\ ENENENE^2 \end{array} \right)$$

Then, the map $\vartheta$ gives 6 lattice paths

$$\vartheta(P) = \left( \begin{array}{c} NENE NENE NNEE \\ ENEN NENE NENE \end{array} \right)$$

Note that paths in each column (resp. row) of $\vartheta(P)$ are obtained by the vertical (resp. horizontal) decomposition of $\theta_h(P)$ (resp. $\theta_v(P)$).

4.3. $b$-Stirling permutations. A $b$-Stirling permutation of size $n$ is a permutation of the multiset $\{1^b, 2^b, \ldots, n^b\}$ such that if an integer $j$ appears between two $i$’s, we have $j > i$.

**Definition 4.9.** We denote by $\mathcal{S}_n^{(b)}$ the set of $b$-Stirling permutations of size $n$.

We construct a $b$-Stirling permutation from a non-negative integer sequence $u := (u_1, \ldots, u_n)$ such that $u_i \leq b(i-1)$ for all $1 \leq i \leq n$. Since $u_1 = 0$ for any integer sequence $u$, we put $b$ 1’s in line. By definition, $p_1 := 1^b$ is a $b$-Stirling permutation. Let $p_i$ be a $b$-Stirling permutation consisting of integers in $[1, i]$. The multi-permutation $p_i$ can be recursively obtained from $p_{i-1}$ by inserting $i^b$ into the $u_i$-th position in $p_{i-1}$.

**Definition 4.10.** Let $u$ and $p_n$ be sequences as above. We denote by $\mu : u \mapsto p_n$ the map from a non-negative integer sequence to a $b$-Stirling permutation. We call $p_n$ the insertion history of $u$. 

Remark 4.11. In Definition 2.14, we introduce an insertion history for a permutation. The map \( \mu \) is a generalization of an insertion history for a \( b \)-Stirling permutation.

It is obvious that \( \mu \) has an inverse from \( \mathfrak{p}_n \) from \( u \). We denote by \( \mu^{-1} \) the inverse of \( \mu \).

Example 4.12. Let \( u = (0, 1, 2, 4) \) and \( b = 3 \). We have a sequence of \( b \)-Stirling permutations:

\[
111 \rightarrow 1222111 \rightarrow 123322111 \rightarrow 1233444322111.
\]

We give a description of the Dyck path decomposition in terms of \( b \)-Stirling permutations. We first consider the horizontal decomposition of a rational Dyck path, then consider the vertical decomposition. Note that the order of horizontal and vertical decompositions is irrelevant to the result by Proposition 4.6.

Let \( P \) be a rational Dyck path in \( \mathfrak{S}^{(a, b)}_n \) and \( u(P) \) be its step sequence. Let \( \mu := \mu(u(P)) = (\mu_1, \mu_2, \ldots, \mu_m) \) be the \( b \)-Stirling permutation obtained from \( u(P) \).

We define \( b \) permutations of size \( n \) denoted by \( \nu^i := (\nu^i_1, \ldots, \nu^i_n) \), \( 1 \leq i \leq b \), from \( \mu \) by

\[
\nu^i_j := \mu(j-1)b+i,
\]

for \( 1 \leq j \leq n \).

Lemma 4.13. Eqn. (4.2) is well-defined, i.e., \( \nu^i \) is a permutation.

Proof. In a \( b \)-Stirling permutation \( \mu \), each integer in \( [1, n] \) appears exactly \( b \) times. By definition, we may select integers larger than \( i \) between two integer \( i \)'s. By construction, the number of such integers is zero modulo \( b \). We take integers separated by \( b \) steps in \( \mu \) to obtain \( \nu^i \) in Eqn. (4.2), which insures that \( \nu^i \) is a permutation. \( \square \)

We construct \( b \) integer sequences \( \xi^i, 1 \leq i \leq b \), from permutations \( \nu^i \) by the map \( \mu^{-1} \), i.e., we define

\[
\xi^i = \mu^{-1}(\nu^i), \quad \forall i \in [1, b].
\]

Lemma 4.14. The integer sequences \( \xi^i, 1 \leq i \leq b \), are weakly increasing.

Proof. Let \( \mu^{(i)} \) be the \( b \)-Stirling permutation consisting of integers in \( [1, i] \). Since \( u(P) \) is the step sequence of \( P \), it is a non-decreasing integer sequence. This implies that integers \( i-1 \) and \( i \) are next to each other in \( \mu^{(i)} \) if \( i \) is left to \( i-1 \), or \( i \) is right to \( i-1 \). From Eqn. (4.2), it is obvious that the relative positions of integers \( i \) and \( i-1 \) in \( \mu \) are preserved. More precisely, the integer \( i \) is right next to \( i-1 \), or \( i \) is right to \( i-1 \) if we ignore the integers larger than \( i \). Since \( \xi^k, 1 \leq k \leq b \), is obtained by Eqn. (4.3), the constraint on \( i \) and \( i-1 \) in \( \nu^i \) insures that \( \xi^k \) is weakly increasing. \( \square \)

Let \( p_i \) be Dyck paths in \( \mathfrak{D}^{(a, 1)}_n \) obtained from \( P \) by the horizontal strip-decomposition as in Definition 4.2.

Proposition 4.15. The weakly increasing sequences \( \xi^i, 1 \leq i \leq b \), are the step sequences of \( p_i \).

Proof. From Proposition 4.4, it is enough to show that \( \xi^i \) coincides with \( v_i \), for \( 1 \leq i \leq b \). However, this is clear form the constructions of \( \mu \) and \( \nu^i \). \( \square \)

We apply a vertical strip-decomposition to \( \xi^i := (\xi^i_1, \ldots, \xi^i_n) \), \( 1 \leq i \leq b \). We define \( \rho^{i,j} := (\rho^{i,j}_1, \ldots, \rho^{i,j}_n) \) with \( 1 \leq i \leq b \) and \( 1 \leq j \leq a \) from \( \xi^i \) by

\[
\rho^{i,j}_k := s_{a(k-1)+j},
\]
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for \( k \in [1,n] \).

Let \( \vartheta(P) := (\vartheta_{i,j}) \) for \( 1 \leq i \leq b \) and \( 1 \leq j \leq a \) be a Dyck path decomposition of \( P \).

**Proposition 4.16.** The ab weakly increasing sequences \( \rho^{i,j} \) for \( 1 \leq i \leq b \) and \( 1 \leq j \leq a \) are the step sequences of \( \vartheta_{i,j} \).

**Proof.** As in the proof of Proposition 4.5, the horizontal and vertical strip decompositions are dual to each other. By applying Proposition 4.15 to the transposed vertical strip decomposition, we obtain weakly increasing sequences \( \rho^{i,j} \) which are the step sequences of \( \vartheta_{i,j} \). \( \square \)

**Example 4.17.** Let \( P = NENENE^2NE \in \mathcal{D}^{(2,3)}_2 \). Since \( \mu = 12334432211 \), we have three permutations

\[ \nu^1 = 1342, \quad \nu^2 = 2431, \quad \nu^3 = 3421. \]

By applying \( \mu^{-1} \) to \( \nu^i \)'s, we have

\[ \xi^1 = 0112, \quad \xi^2 = 0011, \quad \xi^3 = 0001. \]

Finally, \( \rho^{i,j} \) are given by

\[ \rho^{1,1} = 01, \quad \rho^{2,1} = 01, \quad \rho^{3,1} = 00, \]
\[ \rho^{1,2} = 12, \quad \rho^{2,2} = 01, \quad \rho^{3,2} = 01. \]

The integer sequences \( \rho^{i,j} \) are the step sequence of \( \vartheta(P) \) in Example 4.8.

4.4. \((a,b)\)-Dyck paths and Dyck path decomposition. Let \( \pi \) be an \((a,b)\)-Dyck path. Let \( (p_1, \ldots, p_b) \) be a horizontal strip decomposition of \( \pi \). We consider a vertical strip decomposition of \( p_i \), and denote it by \( (r_{i,1}, \ldots, r_{i,a}) \) for \( 1 \leq i \leq b \). We place \( ab \) paths \( r_{i,j} \) with \( 1 \leq i \leq b \) and \( 1 \leq j \leq a \) as

\[
\begin{array}{cccc}
  r_{1,1} & r_{2,1} & \ldots & r_{b,1} \\
  r_{1,2} & r_{2,2} & \ldots & r_{b,2} \\
  \vdots & \vdots & \ddots & \vdots \\
  r_{1,a} & r_{2,a} & \ldots & r_{b,a}
\end{array}
\]

(4.4)

Then, the Dyck path decomposition is given by

\[ \vartheta : \pi \mapsto \{ r_{i,j} : 1 \leq i \leq b, 1 \leq j \leq a \}. \]

**Remark 4.18.** The paths \( r_{i,j} \) for \( i \in [1,b] \) and \( j \in [1,a] \) are all lattice paths from \((0,0)\) to \((n,n)\). By embedding these paths in the set paths of larger size by adding \( N^m \) from left and \( E^m \) from right for some \( m > 0 \), one can regard these paths as Dyck paths of size \( m + n \). This is the reason why we call \( \vartheta \) Dyck path decomposition.

**Proposition 4.19.** Let \( r_{i,j} \) be a path obtained by Dyck path decomposition of \( \pi \). Then, we have \( r_{i,j} \leq r_{i+1,j} \) and \( r_{i,j+1} \leq r_{i,j} \).

**Proof.** The paths \( p_i, 1 \leq i \leq b \), obtained by the horizontal strip decomposition of \( \pi \), satisfy \( p_i \leq p_{i+1} \) since the height sequence of \( \pi \) is weakly increasing. The paths \( r_{i,j}, 1 \leq j \leq a \), are obtained by the vertical strip decomposition of \( p_i \). The step sequence of \( p_i \) is also weakly increasing, which implies that \( r_{i,j} \leq r_{i+1,j} \) and \( r_{i,j+1} \leq r_{i,j} \). \( \square \)
We introduce two relations $\preceq_h$ and $\preceq_v$ between two paths as follows. Let $r$ and $s$ be two paths from $(0,0)$ to $(n,n)$ and $s$ is above $r$.

Let $\mathbf{u}(r) := (u_1, \ldots, u_n)$ be the step sequence of $r$. Then, we define a path $r'$ whose step sequence $\mathbf{u}(r') := (u'_1, \ldots, u'_n)$ is given by

$$u'_i := \begin{cases} 0, & \text{if } u_i = 0, \\ u_i - 1, & \text{if } u_i \geq 1 \text{ and } u_{i-1} < u_i. \end{cases}$$

for some $i \in [2,n]$. Then, we define

$$r \preceq_h s \Rightarrow r \leq s \leq r'.$$

Similarly, let $\mathbf{h}(r)$ be the height sequence of $r$. Then, we define a path $r''$ whose height sequence $\mathbf{h}(r'') := (h'_1, \ldots, h''_n)$ is given by

$$h''_i := \begin{cases} n, & \text{if } h_i = n, \\ h_i + 1, & \text{if } h_i \leq n \text{ and } h_i < h_{i+1}. \end{cases}$$

for some $i \in [1,n-1]$. Then, we define

$$r \preceq_v s \Rightarrow r \leq s \leq r''.$$

We impose the two relations $\preceq_h$ and $\preceq_v$ between adjacent elements in (4.4). More precisely, we impose $r_{i,j} \preceq_h r_{i+1,j}$ for $i \in [1, b-1]$ and $j \in [1, a]$ and $r_{i,j+1} \preceq_v r_{i,j}$ for $i \in [1, b]$ and $j \in [1, a-1]$. Furthermore, we also impose $r_{1,j} \preceq_h r_{b,j}$ for $j \in [1, a]$ and $r_{i,a} \preceq_v r_{i,1}$ for $i \in [1, b]$.

**Definition 4.20.** We denote by $R(\pi)$ the set of $\{r_{i,j} : 1 \leq i \leq b, 1 \leq j \leq a\}$ with relations $\preceq_h$ and $\preceq_v$ as above.

**Remark 4.21.** The relations $\preceq_h$ and $\preceq_v$ come from the strip decomposition of a Dyck path. In the strip decompositions, we take every three entries in the step or the height sequences. This means that the number of the entries with the same value in $r_{i,j}$ differs at most one. Therefore, we have the relations for $r_{i,j}$.

Let $\pi'$ be an $(a,b)$-Dyck path satisfying $\pi \leq \pi'$. Suppose the Dyck path decomposition of $\pi'$ gives the set $R(\pi') = \{r'_{i,j} : 1 \leq i \leq b, 1 \leq j \leq a\}$ with relations.

If $r_{i,j} \leq r'_{i,j}$ for all $i \in [1,b]$ and $j \in [1,a]$, we denote by $R(\pi) \preceq R(\pi')$.

**Proposition 4.22.** Let $\pi$ be an $(a,b)$-Dyck path. Then, the cardinality of the set

$$\{\pi' : R(\pi) \preceq R(\pi')\},$$

(4.7)

gives the number of $(a,b)$-Dyck paths above $\pi$.

**Proof.** Two conditions Eqn. (4.5) and Eqn. (4.6) correspond to deleting a single box from the Young diagram characterized by $r$. The new diagram is also a Young diagram characterized by $r'$ or $r''$. Since we have a natural bijection between Young diagrams and rational Dyck paths, an element of the set defined in Eqn. (4.7) corresponds to a rational Dyck paths $\pi'$ above $\pi$.

Conversely, if we have a rational Dyck path $\pi'$ above $\pi$, there is a unique Young diagram corresponding to $\pi'$. This Young diagram satisfy Eqn. (4.5) or Eqn. (4.6).
4.5. Dyck tilings and Dyck path decomposition. In Section 4.4, we consider \((a, b)\)-Dyck paths above the path \(\pi\). In this subsection, we consider a non-trivial Dyck tiling above \(\pi\). Here, "non-trivial Dyck tiling" means that the tiling contains at least one rational Dyck tile of size \(n \geq 1\). Recall that given a path \(\pi\), we have the set of \(ab\) paths, denoted by \(R(\pi)\).

Let \(\pi\) be a path and \(Y(\pi)\) be the Young diagram characterized by the path \(\pi\). A path \(r_{i,j} \in R(\pi)\) can be identified with the Young diagram \(Y(r_{i,j})\) which is above \(r_{i,j}\), right to the line \(x = 0\), and below the line \(y = n\). Then, by definition of the Dyck path decomposition of \(\pi\), there is an obvious bijection between the boxes in \(Y(\pi)\) and the boxes in \(Y(r_{i,j})\) for \((i, j) \in [1, b] \times [1, a]\).

Let \(r\) be a lattice path from \((0, 0)\) to \((n, n)\). Suppose that \(r\) contains a partial path \(NE\), namely, a path consisting of three points \((x, y), (x, y + 1)\) and \((x + 1, y + 1)\). Then, the lattice point \((x, y + 1)\) is said to be a peak of \(r\).

Let \(v\) be a peak of \(r_{i,j} \in R(\pi)\), and \((x, y)\) is its coordinate. Let \(b(v)\) be the box in \(Y(r_{i,j})\) whose center is \((x - 1/2, y + 1/2)\) if it exists. By the bijection between the boxes in \(Y(\pi)\) and \(Y(r_{i,j})\), we have a box corresponding to \(b(v)\), denoted by \(\overline{b(v)}\). Note that the box \(\overline{b(v)}\) is in \(Y(\pi)\). We denote by \((\overline{x} - 1/2, \overline{y} + 1/2)\) the coordinate of the center of the box \(\overline{b(v)}\). We consider the set of lattice points \((\overline{x} + m, \overline{y} - m)\) with \(m \geq 0\). There exists a unique lattice points \((\overline{x} + m, \overline{y} - m)\) with some \(m\), which is on the \((a, b)\)-Dyck path \(\pi\). If such a lattice point is a peak of \(\pi\), we say that the lattice point \((x, y)\) (or equivalently the peak \(v\)) in \(r_{i,j}\) is admissible. Peaks which is not admissible are called non-admissible peak.

Let \(\pi'\) be a path satisfying \(\pi \leq \pi'\). The two paths \(r_{i,j} \in R(\pi)\) and \(r'_{i,j} \in R(\pi')\) determines a skew shape \(S_{i,j}\) since \(r_{i,j} \leq r'_{i,j}\) from Proposition 4.22. We consider a \((1, 1)\)-Dyck tiling in the skew shape \(S_{i,j}\). Let \(v\) be a non-admissible peak in \(r_{i,j}\) and \((v_x, v_y)\) be its coordinate. Then, we consider the condition on Dyck tilings in \(S_{i,j}\) as follows:

\((\triangledown)\) The box \(b'\) whose center is \((v_x - 1/2, v_y + 1/2)\) is contained in a single box in \(S_{i,j}\). This is equivalent to that the box \(b'\) cannot be contained in a Dyck tile of size \(n \geq 1\).

Let \(D_{i,j}\) be a \((1, 1)\)-Dyck tilings in \(S_{i,j}\) satisfying the condition \((\triangledown)\). Then, we have the following theorem.

**Theorem 4.23.** The number of \((a, b)\)-Dyck tilings above \(\pi\) and below \(\pi'\) with at least one non-trivial Dyck tiles is equal to the cardinality of the set

\[ D := \{(D_{1,1}, \ldots, D_{a,b}) : \text{ at least one } D_{i,j} \text{ contain a non-trivial Dyck tile}\}, \]

where \(D_{i,j}\) is a \((1, 1)\)-Dyck tiling in \(S_{i,j}\). Further, there exists a bijection between non-trivial \((a, b)\)-Dyck tilings above \(\pi\) and the elements in \(D\).

**Proof.** Recall that we have a natural bijection between a box in \(Y(\pi)\) and a box in \(r_{i,j}\) for some \(i \in [1, b]\) and \(j \in [1, a]\).

Suppose that we have no non-trivial Dyck tiles in \(Y(\pi)\). Since all Dyck tiles above \(\pi\) and below \(\pi'\) are single boxes, all Dyck tiles in \(r_{i,j}\) are single boxes as well.

Suppose that we have at least one non-trivial Dyck tiles in \(Y(\pi)\). Recall that the shape of a non-trivial Dyck tile \(d\) is an \((a, b)\)-enlarged path. There exists at least one pair \((i, j)\) such that \(r_{i,j}\) contains at least three boxes corresponding to boxes in \(d\). These three boxes are a left-top box \(b_0\) in \(d\), a box \(b\) step right to \(b_0\), and a box \(a\) step below \(b_0\) in \(d\). In \(r_{i,j}\), these three boxes are next to each other, and form a Dyck tile of size one. If \(r_{i,j}\) contains more than three boxes corresponding to boxes in \(d\), it has a Dyck tile of size larger than one. From these observations, we have at least one non-trivial Dyck tile in \(r_{i,j}\) for some \(i\) and \(j\) if a Dyck tiling in \(Y(\pi)\) contains a non-trivial
(a, b)-Dyck tile. By construction, if this map from a Dyck tiling in $Y(\pi)$ to a Dyck tiling in $r_{i,j}$ is injective.

Conversely, if $r_{i,j}$ contains a non-trivial Dyck tile for some $i$ and $j$, we have at least one $(a, b)$-Dyck tiles in $Y(\pi)$ by the correspondence between boxes in $r_{i,j}$ and in $Y(\pi)$. Again, this map from a Dyck tiling in $r_{i,j}$ to a Dyck tiling in $Y(\pi)$ is injective.

We have a bijection between a Dyck tiling in $Y(\pi)$ and $ab$ Dyck tilings $(D_{1,1}, \ldots, D_{a,b})$. This completes the proof. \hfill $\Box$

5. Weight of a cover-inclusive rational Dyck tiling

In this section, we study the relation between a $(a, b)$-Dyck tilings and strip decompositions of them in view of the weights of Dyck tilings. First, we consider the horizontal decomposition of $(1, b)$-Dyck tilings, then deal with the vertical decomposition of $(a, 1)$-Dyck tilings. Finally, we apply these two cases to the general $(a, b)$-Dyck tilings.

In Sections 4.4 and 5.5, we have an description of an $(a, b)$-Dyck tiling $D$ in terms of the Dyck path decomposition $(r_{i,j})$ with $1 \leq i \leq b$ and $1 \leq j \leq a$. There, we have used the bijection between boxes in $D$ and boxes in $r_{i,j}$. In this section, we give a different description of $D$ in terms of $r_{i,j}$. We make use of an Hermite history of $D$ to establish the correspondence between $D$ and $r_{i,j}$.

5.1. $(1, b)$-Dyck tilings and weight of a Dyck tiling. In this subsection, we study rational Dyck tilings for $(a, b) = (1, b)$, and its relation to $b$-Stirling permutations. We consider the horizontal strip decomposition of Dyck tilings. Let $P$ be a rational Dyck path $D_n^{(1,b)}$ and $p := (p_1, \ldots, p_b)$ be lattice paths obtained from the horizontal strip decomposition. We denote by $D$ a $(1, b)$-Dyck tiling above $P$. From Section 4.4, the lattice paths $p_i, 1 \leq i \leq b$, satisfy $p_i \preceq_{h} p_{i+1}$ for $1 \leq i \leq b-1$ and $p_1 \preceq_{h} p_b$.

Let $D_i$ be a Dyck tiling above $p_i$, and $\nu(D_i)$ be the post-order word obtained from $D_i$ by the DTS bijection (see Section 2.4). We denote by $t_i$ a top path of the Dyck tiling $D_i$. Since the top paths $t_i, 1 \leq i \leq b$, also form a top path of $D$, we have $t_i \preceq_{h} t_{i+1}$. We denote by $\preceq$ the lexicographic order of words. The order for alphabets is $1 < 2 < 3 \ldots < n$.

**Lemma 5.1.** Suppose $p_i \preceq_{h} p_{i+1}$ and $t_i = t_{i+1}$. We have $\nu(D_{i+1}) \preceq \nu(D_i)$ if both Dyck tilings consist of single Dyck tiles.

**Proof.** Let $h := (h_1, \ldots, h_n)$ be an integer sequences such that $h_j$ is the label on the $j$-th chord in an Hermite history. We denote by $h$ and $h'$ the integer sequences for $D_i$ and $D_{i+1}$ respectively.

The condition that $p_i \preceq_{h} p_{i+1}$ means that we have some $j$ such that $h'_j = h_j - 1$. This is because that the top paths of $D_i$ and $D_{i+1}$ are the same. Since we read the word by post-order, we have the same word, or we exchange $k$ and $k'$ in the Hermite history such that $k'$ is the maximal integer smaller than $k$ and left to $k$. From Proposition 2.20, a chord has a label $n + 1 - j$ in the DTS bijection if the same chord has a label $j$ in the Hermite history. Thus, if we read the post-order word in the DTS bijection, $\nu(D_i) \succeq \nu(D_{i+1})$. \hfill $\Box$

**Example 5.2.** In Figure 5.3, we show some examples of Dyck tiling with the top path $N^3E^3E^3$ and without non-trivial Dyck tiles.

**Remark 5.4.** We remark that if a Dyck tiling contains Dyck tiles of size larger than zero, we have similar properties to Lemma 5.1 (see Figure 5.7). However, the comparison between a Dyck tiling without non-trivial Dyck tiles and a Dyck tiling with non-trivial Dyck tiles is not obvious.
Example 5.5. We consider the \((1,3)\)-Dyck path \(\lambda\) with the step sequence \((0,2,2,8)\). We have eight non-trivial Dyck tilings. For example, three of them are depicted in Figure 5.6.

Let \(p\) be a \((1,1)\)-Dyck path. We consider \((1,1)\)-Dyck tilings with non-trivial Dyck tiles. Suppose that a Dyck tiling \(D_0\) contains a Dyck tile of size \(m\), and a Dyck tiling \(D_1\) contains a Dyck tile of size \(m'\) with \(m' < m\). Then, the post-order words \(\nu(D_0)\) and \(\nu(D_1)\) satisfy \(\nu(D_0) < \nu(D_1)\) in the lexicographic order. Figure 5.7 shows an example. The red lines indicate the order of two elements.
We construct a (1, b)-Dyck tiling \( D \) from \( b \) Dyck tilings \( D_i, 1 \leq i \leq b \). Recall that an Hermite history consists of trajectories starting from the \( N \) step in the lowest Dyck path (see Section 2.3). We define a non-negative integer sequences \( g^i := (g^i_1, \ldots, g^i_n) \), where \( g^i_j \) is the \( l(S_N) - 1 \) for the \( j \)-th \( N \) step \( S_N \) in \( D_i \). We define
\[
G^i_j(D_1, \ldots, D_b) := \sum_{1 \leq i \leq b} g^i_j,
\]
where \( 1 \leq j \leq n \).

We define a (1, b)-Dyck tiling \( D \) such that the Hermite history of \( D \) is \((G_1, \ldots, G_n)\).

We abbreviate \( \nu(D_i) \) as \( \nu^i \). We denote by \( N(\nu^1, \ldots, \nu^n) \) the sum of entries in \( \mu(\nu^i) \) for \( 1 \leq i \leq b \). Let \( Y(P) \) be the sum of entries of step sequence of the lowest path \( P \).

**Proposition 5.8.** We have
\[
\text{wt}(D) = Y(P) - N(\nu^1, \ldots, \nu^n).
\]

**Proof.** We prove Eqn. (5.1) by induction. We first show that Eqn. (5.1) holds when the Dyck tiling \( D \) has no Dyck tiles. If we read the words consisting of the labels on chords of the lowest path \( P \) in \( D \) in the pre-order, we have the identity permutation. We use the post-order words to obtain \( \nu^i \). We denote by \( u(p_i) := (u_1, \ldots, u_n) \) the step sequence of \( p_i \). Suppose \( u_{i-1} = u_i \). Then, the integer \( i \) appears just before \( i - 1 \) in the post-order words. Thus in \( \nu^i \), the insertion history \( \mu(\nu^i) := (\mu_1, \ldots, \mu_n) \) satisfies \( \mu_{i-1} = \mu_i \). Suppose \( u_{i-1} < u_i \). By a similar argument, we have \( \mu_{i-1} < \mu_i \). From these observations, \( h_i \) is the sum of the numbers of boxes left to the chord with label \( i \). Therefore, the insertion history is nothing but the step sequence of \( P \). We obtain \( \text{wt}(D) = 0 \) since \( N(\lambda) = Y(P) \).

We assume that Eqn. (5.1) is true for all Dyck tiling \( D' \) with the top path \( Q' \). We show Eqn. (5.1) is also true for \( Q \) such that the number of boxes in \( Y(Q) \) is one plus the number of boxes in \( Y(Q') \). We denote by \( \tilde{D} \) the Dyck tiling in \( Y(Q) \). In \( D_i \) for some \( i \), the number of boxes is increased by one, or the size of a non-trivial Dyck tile is increased by one. Then, it is obvious that \( N(\lambda(D')) \) is increased by one from the construction of the post-order words. We have \( N(\lambda(D)) = N(\lambda(D')) + 1 \).

We obtain \( \text{wt}(D) = Y(P) - N(\lambda(D)) = \text{wt}(D) + 1 \). This completes the proof.

**Example 5.9.** We consider the third example in Figure 5.6. The corresponding (1, 3)-Dyck tiling is depicted as

Note that \( g^1 = (0, 0, 0, 3) \), \( g^2 = (0, 1, 0, 3) \) and \( g^3 = (0, 0, 0, 2) \) give \( G = (G_1, G_2, G_3, G_4) = (0, 1, 0, 8) \). The step sequence of the lowest path is \( (0, 2, 2, 8) \), which implies \( Y(p) = 12 \). Since \( \mu(2431) = (0, 0, 1, 1), \mu(3421) = (0, 0, 0, 1) \) and \( \mu(3421) = (0, 0, 0, 0) \), we obtain \( N(\nu^1, \nu^2, \nu^3) = 3 \). Thus, the weight of the (1, 3)-Dyck tiling is \( 12 - 3 = 9 \).

5.2. (a, 1)-Dyck tilings and weight of a Dyck tiling. In this subsection, we study the relation between (a, 1)-Dyck tilings and \( a \)-Stirling permutations. The results are parallel to those of (1, b)-Dyck tilings by changing \( a \) and \( b \). However, the statistics \( \text{art}(D) \) for a Dyck tiling is not preserved by the exchange of \( a \) and \( b \).
Let $P$ be a rational Dyck path in $D_n^{(a,1)}$, and $q := (q_1, \ldots, q_a)$ be lattice paths obtained by the vertical strip decomposition. From Section 4.4, lattice paths $q_i$, $1 \leq i \leq a$, satisfy $q_{i+1} \preceq q_i$ for $1 \leq i \leq b - 1$ and $q_b \succeq q_1$.

Let $D_i$ be a Dyck tiling above $q_i$, and $\nu_i := \nu(D_i)$ be the post-order word reading from right to left obtained from $D_i$ by the left DTS bijection. The top paths $t_i$ of $D_i$ satisfies $t_{i+1} \preceq \nu t_i$.

By a similar argument to the proof of Lemma 5.1, we obtain the following lemma.

**Lemma 5.10.** Suppose $q_{i+1} \preceq q_i$, and $D_i$ and $D_{i+1}$ have the same top path. We have $\nu(D_i) \leq \nu(D_{i+1})$ if neither $D_i$ and $D_{i+1}$ have non-trivial Dyck tiles.

**Remark 5.11.** Note that the order is reversed compared to the horizontal strip decomposition. This is because we make use of left DTS. We also read the labels of chords from right to left in the post-order.

Let $N(\nu_1, \ldots, \nu_n)$ be the sum of entries of the insertion history $\mu(\nu_i)$, $1 \leq i \leq a$.

**Proposition 5.12.** We have
\begin{equation}
\text{wt}(D) = Y(P) - N(\nu_1, \ldots, \nu_n) - M(a - 1),
\end{equation}
where $M$ is the sum of the sizes of a non-trivial Dyck tiles in $D$.

**Proof.** The proof is essentially the same as the proof of Proposition 5.8. The difference is that the statistic $\text{art}(D)$ for a non-trivial Dyck tile is not equal to the size of a non-trivial Dyck tile in $D$. In Eqn. (5.2), this difference is computed by $M(a - 1)$. This completes the proof. \qed

To obtain the $(a, 1)$-Dyck tiling from $a$ $(1, 1)$-Dyck tilings, we make use of the Hermite history as in case of a $(1, b)$-Dyck tiling. Here, we have to use the Hermite history attached to the $E$ steps in the lowest path.

**Example 5.13.** We consider a $(3, 1)$-Dyck tiling as in Figure 5.14. Since $\nu_1 = 4231$, $\nu_2 = 2341$ and $\nu_3 = 4312$, we have $\mu(\nu_1) = (0, 0, 1, 0)$, $\mu(\nu_2) = (0, 0, 1, 2)$ and $\mu(\nu_3) = (0, 1, 0, 0)$. We have $N(\nu_1, \nu_2, \nu_3) = 4$ We have $Y(P) = 15$, $M = 2$ and $a = 3$ for the $(3, 1)$-Dyck tiling $D$. We obtain $\text{wt}(D) = 15 - 5 - 4 = 6$.

5.3. $(a, b)$-Dyck tilings. Let $D$ be a $(a, b)$-Dyck tiling, and $(r_{i,j})$ for $1 \leq i \leq b$ and $1 \leq j \leq a$ be its Dyck path decomposition. Note that each row gives a horizontal strip decomposition of some $(1, b)$-Dyck tiling, and each column gives a vertical strip decomposition of some $(a, 1)$-Dyck tiling. Therefore, one can apply Propositions 5.8 or 5.12 to rows or columns of $(r_{i,j})$. 

![Figure 5.14. Decomposition of a (3,1)-Dyck tiling](image-url)
References

[1] F. Brenti, *Kazhdan–Lusztig and R-polynomials, YounG’s lattice, and Dyck partitions*, Pacific. J. Math. 207 (2002), no. 2, 257–286.

[2] C. Ceballos and R. S. González D’León, *Signature Catalan Combinatorics*, J. Comb. 10.4 (2019), 725–773, arXiv:1805.03863.

[3] I. Fischer and P. Nadeau, *Fully packed loops in a triangle: matchings, paths and puzzles*, J. Combin. Theory Ser. A 130 (2015), 64–118, arXiv:1209.1262.

[4] I. Gessel and R. P. Stanley, *Stirling polynomials*, J. Combinatorial Theory Ser. A 24 (1978), no. 1, 24–33.

[5] J. S. Kim, *Proofs of two conjectures of Kenyon and Wilson on Dyck tilings*, J. Combin. Theory Ser. A 119 (2012), no. 8, 1692–1710, arXiv:1108.5558.

[6] A. Karrila, K. Kytölä, and E. Peltola, *Conformal blocks, q-combinatorics, and quantum group symmetry*, Ann. Inst. Henri Poincaré D 6 (2019), 449–487, arXiv:1709.00249, doi.

[7] R. W. Kenyon and D. B. Wilson, *Double-dimer pairings and skew Young diagrams*, Electron. J. Combin. 18 (2011), no. 1, P130, arXiv:1007.2006.

[8] L. Patimo, *Bases of the Intersection Cohomology of Grassmannian Schubert Varieties*, preprint (2019), arXiv:1908.11606.

[9] E. Peltola and H. Wu, *Global and Local Multiple SLEs for κ ≤ 4 and Connection Probabilities for Level Lines of GFF*, Comm. Math. Phys. 366 (2019), 469–536, arXiv:1703.00898.

[10] K. Shigechi, *Ballot tilings and increasing trees*, preprint (2017), arXiv:1705.06434.

[11] K. Shigechi, *Bijections on Dyck tilings: DTS/DTR bijections, Dyck tableaux and tree-like tableaux*, preprint (2019), arXiv:1910.08913.

[12] K. Shigechi, *Symmetric Dyck tilings, ballot tableaux and tree-like tableaux of shifted shapes*, preprint (2020), arXiv:2011.07296.

[13] K. Shigechi, *Dyck tilings of type D*, preprint (2021), arXiv:2104.01391.

[14] R. P. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge University Press, 1997.

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