COHOMOLOGY OF THE LIE ALGEBRAS OF DIFFERENTIAL OPERATORS: LIFTING FORMULAS

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INTRODUCTION

The lifting formulas are formulas for the cocycles of the Lie algebra $\mathfrak{A}$, constructed from an associative algebra $\mathfrak{A}$ with the bracket $[a, b] = a \cdot b - b \cdot a$. These cocycles are constructed with the following data:

1) a trace $\text{Tr}: \mathfrak{A} \rightarrow \mathbb{C}$ on the associative algebra $\mathfrak{A}$
2) a set $\mathcal{D} = \{D_1, D_2, \ldots \}$ of the (exterior) derivations of the associative algebra $\mathfrak{A}$ satisfying the following conditions:

(i) $\text{Tr}(D_i A) = 0$ for any $A \in \mathfrak{A}$ and any $D_i \in \mathcal{D}$
(ii) $[D_i, D_j] = \text{ad}(Q_{ij})$ — inner derivation ($Q_{ij} \in \mathfrak{A}$) for any $D_i, D_j \in \mathcal{D}$
(iii) $\text{Alt}_{i,j,k}(D_k(Q_{ij})) = 0$ for all $i, j, k$.

The main example of such a situation is the Lie algebra $\Psi_{\text{Diff}}(S^1)$ of the formal pseudodifferential operators on $(S^1)^n$ (see [A]). The trace $\text{Tr}$ in this example is the “noncommutative residue”, $\text{Tr}(D) = \text{the coefficient of the term } x_i^{-1} \cdot x_j^{-1} \cdot \ldots \cdot x_n^{-1} \cdot \partial_1^{-1} \cdot \ldots \cdot \partial_n^{-1}$ of $D \in \Psi_{\text{Diff}}(S^1)$ (in any coordinate system). It is easy to verify that $\text{Tr}[D_1, D_2] = 0$ for any $D_1, D_2 \in \Psi_{\text{Diff}}(S^1)$. Furthermore, $\ln x_i (i = 1, \ldots, n)$ are (exterior) derivations of $\Psi_{\text{Diff}}(S^1)$ with respect to the adjoint action; the symmetry between the operators $x$ and $\frac{d}{dx}$ allows us to define exterior derivations $\ln \partial_i (i = 1, \ldots, n)$. (Actually, one can define $\ln D (D \in \Psi_{\text{Diff}}(S^1))$ in much greater generality — see Sect. 2.4 and 2.8).

We prove in §1 that the noncommutative residue $\text{Tr}$ on the associative algebra $\Psi_{\text{Diff}}(S^1)$ and the set of $2n$ derivations $\{\ln x_i, \ln \partial_i; i = 1, \ldots, n\}$ satisfy conditions (i)–(iii) above, and this is our main (and in some sense the unique) example.

In the case of the one derivation $D$ such a construction appeared in [KR], where two 2-cocycles on the Lie algebra $\Psi_{\text{Diff}}(S^1) = \Psi_{\text{Diff}}(S^1)$ were constructed:

$$\Psi^{(1)}(A_1, A_2) = \text{Tr}([\ln x, A_1] \cdot A_2)$$
$$\Psi^{(2)}(A_1, A_2) = \text{Tr}([\ln \partial, A_1] \cdot A_2).$$

Both these cocycles are cohomologous to zero when restricted to the Lie algebra $\text{Diff}_1$ of the (polynomial) differential operators on $\mathbb{C}^1$; on the other hand, our aim is to construct cocycles on this Lie algebra. We are able to accomplish this by the simultaneous application of both $\ln x_i$ and $\ln \partial_i$.

Now the problem is solved only for $n = 1$ (§1) and $n = 2$ (§3). But the Second Version of the Main Conjecture (§4) gives us an explicit formula for arbitrary $n$ (when conditions (i)–(iii) above hold). We obtain this formulas using a some simple trick, and all that

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remains to prove is the fact that this formula gives in fact *cocycles*. We will see in §3 ($n = 2$) that this is very difficult problem.

In this way we obtain elements in $H^{2n+1}(\text{Dif}_n; \mathbb{C})$. We assume that $H^{2n+1}(\text{Dif}_n; \mathbb{C}) = \mathbb{C}$ (but we don’t use this fact elsewhere). There is a deformation of the Lie algebra $\text{Poiss}_{2n}$ of the functions on $\mathbb{R}^{2n}$ with the Poisson bracket to the Lie algebra $\text{Dif}_n$, and the well-known result is the vanishing theorem: $H^i(\text{Poiss}_{2n}; \mathbb{C}) = 0$ when $0 < i \leq 2n$. Hence, we study the first nontrivial case.

It is easy to construct a $(2n + 1)$-cocycle on $\text{Poiss}_{2n}$ (or on the Lie algebra $\text{Poiss}_{2n}(S^1)$ of the functions on the torus $(T^2)_n$ with the Poisson bracket), and if the lifting problem can be solved, the corresponding $(2n + 1)$-cocycle on $\Psi\text{Dif}_n(S^1)$ is a deformation of this $(2n + 1)$-cocycle on $\text{Poiss}_{2n}(S^1)$. The lifting problem is equivalent to the deformation problem (§2).

B. Feigin explained me that such formulas should exist; numerous talks with him and with M. Kontsevich made me interested in this subject. I express my deep gratitude to them. I am also grateful to Jeremy Bem for helping with my English and to Seva Kordonsky for carefully typing this manuscript.

1. COHOMOLOGY OF THE LIE ALGEBRA $\text{Dif}_1$

1.1. Let $\text{Tr} = \text{pr}_{x^{-1}\partial^{-1}}$ be a trace on the associative algebra $\Psi\text{Dif}(S^1)$, $D_1 = \text{ad}(\ln \partial)$ and $D_2 = \text{ad}(\ln x)$ be two its exterior derivations.

**Lemma.**

(i) $[D_1, D_2] = \text{ad } Q (Q \in \Psi\text{Dif}(S^1))$ is an inner derivation of the associative algebra $\Psi\text{Dif}(S^1)$.

(ii) $Q = x^{-1}\partial^{-1} + \frac{1}{2}x^{-2}\partial^{-2} + \frac{2}{3}x^{-3}\partial^{-3} + \ldots + \frac{(n-1)!}{n}x^{-n}\partial^{-n} + \ldots$

**Proof.** It is sufficient to show that $[D_1, D_2](x) = [Q, x]$ and $[D_1, D_2](\partial) = [Q, \partial]$. This is a straightforward calculation.

**Theorem.**

$$\Psi_3(A_1, A_2, A_3) = \text{Alt}_{A,D} \text{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot A_3) + \text{Alt}_{A} \text{Tr}(Q \cdot A_1 \cdot A_2 \cdot A_3)$$

is a 3-cocycle on the Lie algebra $\Psi\text{Dif}(S^1)$. This cocycle is not cohomologous to zero, and it remains noncohomologous to zero after restriction to the Lie algebra $\text{Dif}_1$.

We will prove this theorem in Sect. 1.2–1.4.

1.2.

**Lemma.** $\text{Tr}(D_i A) = 0$ for any $A \in \Psi\text{Dif}(S^1)$ and $i = 1, 2$.

**Proof.** It is clear that $[\Psi\text{Dif}(S^1), \Psi\text{Dif}(S^1)] \subset \text{Ker} \text{Tr } D_i A$, but $\text{codim}(\Psi\text{Dif}(S^1), \Psi\text{Dif}(S^1)) = 1$. Hence it is sufficient to verify the Lemma for $A = x^{-1}\partial^{-1}$.

**Corollary.**

(1) $\text{Alt}_{A,D} \text{Tr}(D_1(D_2 A_1 \cdot A_2 \cdot A_3 \cdot A_4)) = 0$

for any $A_1, A_2, A_3, A_4 \in \Psi\text{Dif}(S^1)$.

The alternation here is just decoration.
1.3. Let us calculate the l.h.s. of (1), using the Leibniz rule; we have:

(2) \[ \text{l.h.s. } (1) = \text{Alt}_A \text{Tr}(D_2 A_1 \cdot D_1 A_2 \cdot A_3 \cdot A_4 + D_2 A_1 \cdot A_2 \cdot A_3 \cdot D_1 A_4) + \]
\[ + \text{Alt}_A \text{Tr}([Q, A_1] \cdot A_2 \cdot A_3 \cdot A_4). \]

Note that \( \text{Alt}_A \text{Tr}(D_2 A_1 \cdot A_2 \cdot D_1 A_3 \cdot A_4) \equiv 0 \) in view of the main property of \( \text{Tr} \) and the symmetry conditions.

(3) \[ \text{r.h.s. } (2) = -2 \text{Alt}_A \text{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot A_3 \cdot A_4) + \text{Alt}_A \text{Tr}([Q, A_1] \cdot A_2 \cdot A_3 \cdot A_4) \]

Let us denote the first summand in (3) by \( \alpha \) and the second one by \( \beta \)

**Lemma.** (i) \( \alpha = d(-2 \text{Alt}_A \text{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot A_3))(A_1, A_2, A_3); \)

(ii) \( \beta = d(-2 \text{Alt}_A \text{Tr}(Q \cdot A_1 \cdot A_2 \cdot A_3))(A_4, A_1, A_2, A_3). \)

**Proof.** We will use the following expression for the differential in the cochain complex of the Lie algebra:

(4) \[ (d\Psi_n)(A_1, \ldots, A_{n+1}) = \frac{1}{2} \sum_{i=1}^{n+1} (-1)^{i-1} \cdot \]
\[ \cdot (\Psi_n([A_i, A_1], A_2, \ldots, \hat{A}_i, A_{n+1}) + \Psi_n(A_1, [A_i, A_2], \ldots, \hat{A}_i, A_{n+1}) + \ldots) \]

Let us prove (i):

(5) \[ d(\text{Alt}_A \text{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot A_3))(A_4, A_1, A_2, A_3) = \]
\[ = \frac{1}{2} \text{Alt}_{A_1, A_2} \text{Tr}(D_1[A_4, A_1] \cdot D_2 A_2 \cdot A_3 + \]
\[ + D_1 A_1 \cdot D_2[A_4, A_2] \cdot A_3 + D_1 A_1 \cdot D_2 A_2 \cdot [A_4, A_3]). \]

Then we subtract from r.h.s. of (5)

\[ \gamma = \frac{1}{2} \text{Alt}_A \text{Tr}([A_4, D_1 A_1] \cdot D_2 A_2 \cdot A_3 + \]
\[ + D_1 A_1 \cdot [A_4, D_2 A_2] \cdot A_3 + D_1 A_1 \cdot D_2 A_2 \cdot [A_4, A_3]), \]

(it is clear that \( \gamma = 0 \)).

We have:

r.h.s. (5) \[ = \frac{1}{2} \text{Alt}_{A_1, A_2} \text{Tr}([D_1 A_4, A_1] \cdot D_2 A_2 \cdot A_3 + D_1 A_1 \cdot [D_2 A_4, A_2] \cdot A_3), \]

\[ \text{Alt}_A \text{Tr}(D_1 A_4 \cdot A_1 \cdot D_2 A_2 \cdot A_3 - D_1 A_1 \cdot A_2 \cdot D_2 A_4 \cdot A_3) = 0 \]

in view of the symmetry conditions, and

r.h.s (5) \[ = \frac{1}{2} \cdot 2 \cdot \text{Alt}_A \text{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot A_3 \cdot A_4). \]

The proof of (ii) is similar.
1.4. It follows from Corollary 1.2 and Lemma 1.3 that
\[ d_{A,D}(\text{Alt } \text{Tr}(D_1A_1 \cdot D_2A_2 \cdot A_3) + \text{Alt } \text{Tr}(Q \cdot A_1 \cdot A_2 \cdot A_3)) = 0, \]
and therefore
\[ \Psi_3(A_1, A_2, A_3) = \text{Alt } \text{Tr}(D_1A_1 \cdot D_2A_2 \cdot A_3) + \text{Alt } \text{Tr}(Q \cdot A_1 \cdot A_2 \cdot A_3) \]
is a 3-cocycle on the Lie algebra \( \Psi \text{Dif}(S^1) \). To complete the proof of Theorem 1.1, it is sufficient to verify that the restriction of \( \Psi_3 \) on \( \text{Dif}_1 \) is nonhomologous to zero cocycle. The check is straightforward: \( 1 \wedge x \wedge \partial \) is a cycle on \( \text{Dif}_1 \) and \( \Psi_3(1 \wedge x \wedge \partial) \neq 0. \)

1.5. In [F] B. Feigin formulated a conjecture about \( H^*(\text{Dif}_1; \mathbb{C}) \). According to this conjecture,
\[ H^*(\text{Dif}_1, \mathbb{C}) = \wedge^*(\Psi_3, \Psi_5, \Psi_7, \ldots) \otimes S^*(c_4, c_6, c_8, \ldots) \]
(the lower index denotes the grading). This conjecture has not been proved yet. However, our methods allow to construct some cocycles \( \tilde{\Psi}_5, \tilde{\Psi}_7, \tilde{\Psi}_9, \ldots \); unfortunately, even the proof of their nontriviality runs into trouble because there are not explicit formulas for the cycles. Also, it seems that one can’t obtain a proof by passing to the Hamiltonian limit. Note that no representatives for \( c_4, c_6, c_8, \ldots \) have been found.

**Theorem.** For \( i = 2, 3, 4, \ldots \)
\[ \tilde{\Psi}_{2i+1}(A_1, \ldots, A_{2i+1}) = \text{Alt } \text{Tr}(Q \cdot A_1 \cdots A_{2i+1}) + \]
\[ + \text{Alt } \text{Tr}(D_1A_1 \cdot D_2A_2 \cdot A_3 \cdots A_{2i+1} + D_1A_1 \cdot A_2 \cdot A_3 \cdot D_2A_4 \cdot A_5 \cdots A_{2i+1} + \]
\[ + D_1A_1 \cdot A_2 \cdots A_s \cdot D_2A_{s+1} \cdot A_{s+2} \cdots A_{2i+1}) \]
where \( \begin{cases} s = i + 1, & a_s = \frac{1}{2}, \text{ if } i \text{ is even} \\ s = i, & a_s = 1, \text{ if } i \text{ is odd} \end{cases} \)
is a \((2i+1)\)-cocycle on the Lie algebra \( \Psi \text{Dif}(S^1) \) (or \( \text{Dif}_1 \)).

**Proof.** It is a direct calculation, similar to the Proof of Theorem 1.1. \( \square \)

**Conjecture.** \( \Psi_3, \tilde{\Psi}_5, \tilde{\Psi}_7, \tilde{\Psi}_9 \ldots \) generate the exterior algebra in \( H^*(\text{Dif}_1, \mathbb{C}) \).

Note, that from this Conjecture follows the statement about spectral sequence from [F]. Therefore, if this Conjecture is true, all we need to prove Feigin’s conjecture about \( H^*(\text{Dif}_1, \mathbb{C}) \) is the “main theorem of the invariant theory” for the Lie algebra \( \mathfrak{gl}(\lambda) \) — see [F].

2. THE LIFTING PROBLEM AND THE DEFORMATION PROBLEM

2.1. Let \( \mathfrak{A} \) be an associative algebra, let us denote by the same symbol the corresponding Lie algebra with the bracket \([a, b] = a \cdot b - b \cdot a\). Suppose that there is an invariant form \( \langle , \rangle \) on the Lie algebra \( \mathfrak{A} \) (not necessary nondegenerate).

For any Lie algebra \( \mathfrak{g} \) there is a canonical map \( \theta_{i+1} : H^{i+1}(\mathfrak{g}; \mathbb{C}) \to H^i(\mathfrak{g}; \mathfrak{g}^*). \) Inner derivations \( \text{Inn}(\mathfrak{A}) \) of the associative algebra \( \mathfrak{A} \) form an ideal in the Lie algebra \( \text{Der}_{\text{Ass}}(\mathfrak{A}) \) of all derivations of the associative algebra \( \mathfrak{A} \). \( \text{Inn}(\mathfrak{A}) \) is also an ideal in the Lie algebra
DerLie(\mathfrak{A}) of the derivations of the Lie algebra \mathfrak{A}. There is a map DerAss(\mathfrak{A}) \rightarrow DerLie(\mathfrak{A})
and a map DerAss(\mathfrak{A})/\text{Inn}(\mathfrak{A}) \rightarrow DerLie(\mathfrak{A})/\text{Inn}(\mathfrak{A}). The last space is equal to \( H^1(\mathfrak{A}, \mathfrak{A}) \).

Let \( D_1, \ldots, D_i \in \text{DerAss}(\mathfrak{A}) \); we will use the same symbols for their images in \( H^1(\mathfrak{A}, \mathfrak{A}) \).

There is a map \( H^m(\mathfrak{A}, \mathfrak{A}) \otimes H^n(\mathfrak{A}, \mathfrak{A}) \rightarrow H^{m+n}(\mathfrak{A}, \mathfrak{A}) \) because \( \mathfrak{A} \) is an associative algebra.

Let \( \alpha_{D_1, \ldots, D_i} = \text{Alt}(D_1 \cdot D_2 \cdot \ldots \cdot D_i) \in H^i(\mathfrak{A}; \mathfrak{A}) \) or, in the explicit form,

\[
\alpha_{D_1, \ldots, D_i}(A_1, \ldots, A_i) = \text{Alt}(D_1(A_1) \cdot D_2(A_2) \cdot \ldots \cdot D_i(A_i)).
\]

The form \( (, ) \) determines the invariant map \( \mathfrak{A} \rightarrow \mathfrak{A}^* \), and we have the element \( \alpha^*_{D_1, \ldots, D_i} \in H^i(\mathfrak{A}; \mathfrak{A}^*) \).

**Question.** When does there exist an element \( \Psi_{i+1} \in H^{i+1}(\mathfrak{A}, \mathbb{C}) \) such that \( \theta_{i+1}(\Psi_{i+1}) = \alpha^*_{D_1, \ldots, D_i} \). If such an element exists, it is called lifting of \( \alpha^*_{D_1, \ldots, D_n} \). (Note, that \( \Psi_{i+1} \), if it exists, is not uniquely determined.)

Furthermore, if \( D_j \in H^1(\mathfrak{A}; \mathfrak{A}) \), \( a_j \in \mathbb{C} \), we have an element \( \sum k \alpha^*_{D_1^{(k)}, \ldots, D_i^{(k)}} \), and there is the same lifting problem for this element.

2.2. We will be concerned only with the case when there is a trace \( \text{Tr}: \mathfrak{A} \rightarrow \mathbb{C} \) on the associative algebra \( \mathfrak{A} \) (i.e., \( \text{Tr}[A_1, A_2] = 0 \) for any \( A_1, A_2 \in \mathfrak{A} \)) and \( \langle A_1, A_2 \rangle = \text{Tr}(A_1 \cdot A_2) \). Suppose also, that the condition (i) from the Introduction holds: \( \text{Tr}(D) = 0 \) for all \( A \in \mathfrak{A} \) and \( D \in D = \{D_1, D_2, \ldots, D_i\} \). Such derivations form an Lie subalgebra \( \text{Der}^1_{\text{Ass}}(\mathfrak{A}) \subset \text{DerAss}(\mathfrak{A}) \); we will denote by \( H^1(\mathfrak{A}; \mathfrak{A}) \) the image of this subalgebra in \( H^1(\mathfrak{A}; \mathfrak{A}) \).

**Lemma.** Suppose that \( \sum k \cdot D_1^{(k)} \cdot \cdots \cdot D_i^{(k)} \) is a cycle in Lie algebra \( \text{Der}^1_{\text{Ass}}(\mathfrak{A}) \) (but not only in \( H^1(\mathfrak{A}; \mathfrak{A}) \)). Then \( \Psi_{i+1}(A_1, \ldots, A_{i+1}) = \text{Alt} \text{Tr}(\sum k \cdot D_1^{(k)} \cdot \cdots \cdot D_i^{(k)} A_i \cdot A_{i+1}) \) is a cocycle, and the corresponding element in \( H^{i+1}(\mathfrak{A}; \mathbb{C}) \) is a lifting of \( \sum k \alpha^*_{D_1^{(k)}, \ldots, D_i^{(k)}} \).

**Proof.** It is sufficient to prove that \( \Psi_{i+1} \) is a cocycle. As in the proof of Theorem 1.1 we have to write an expression on \( A_1, \ldots, A_{i+2} \) which is a coboundary and which a priori is equal to zero.

For simplicity we suppose, that \( D_1 \wedge \cdots \wedge D_i \) is a cycle in \( \text{Der}^1_{\text{Ass}}(\mathfrak{A}) \).

Consider the following expression:

\[
(6) \quad \text{Alt} \text{Tr}_{A,D}(D_i(\text{Cycle}(D_i A_1 \cdot \cdots \cdot D_i A_i \cdot A_i \cdot A_{i+2}) \cdot A_{i+2})).
\]

(Here \text{Cycle} is the sum on all the cyclic permutations; it contains \( i + 1 \) summands.)

The part of (6) which contains \( [D_a, D_b] \) is equal to zero because \( D_1 \wedge \cdots \wedge D_i \) is a cycle.

The remaining summands are of 3 types:

\[
\begin{align*}
\text{Alt} \text{Tr}(D \cdot D \cdot \cdots \cdot A \cdot D \cdot D \cdot A), \\
\text{Alt} \text{Tr}(D \cdot D \cdot \cdots \cdot D \cdot A \cdot D \cdot A), \\
\text{and} \ \text{Alt} \text{Tr}(D \cdot D \cdot \cdots \cdot D \cdot D \cdot A \cdot A).
\end{align*}
\]

(Here \( A \) denotes \( A_a \) and \( D \) denotes \( D_b A_a \).) The summands of the first two types are *not* coboundaries, but the summands of the third type are. Therefore we have to eliminate the summands of the first two types.
We have:

\[
(7) \quad (6) = \text{Alt}_{A,D} \text{Tr}(\text{Cycle}(D_1 A_1 \cdots D_{i-1} A_{i-1} \cdot D_i A_i \cdot A_{i+1}) \cdot A_{i+2}) + \\
+ \text{Alt}_{A,D} \text{Tr}(\text{Cycle}(D_1 A_1 \cdots D_{i-1} A_{i-1} \cdot A_i \cdot D_{i+1} A_{i+1}) \cdot A_{i+2}) + \\
+ \text{Alt}_{A,D} \text{Tr}(\text{Cycle}(D_1 A_1 \cdots D_{i-1} A_{i-1} \cdot A_i \cdot A_{i+1}) \cdot D_i A_{i+2})
\]

The first summand in (7) cancels with the second one because of the symmetry and the main property of Tr; the third summand is

\[
(8) \quad i \cdot \text{Alt}_{A,D} \text{Tr}(D_1 A_1 \cdots D_i A_i \cdot A_{i+1} \cdot A_{i+2}) - \\
- \text{Alt}_{A,D} \text{Tr}(D_1 A_1 \cdots D_{i-1} A_{i-1} \cdot A_i \cdot D_i A_{i+1} \cdot A_{i+2})
\]

Furthermore let us consider an expression

\[
(9) \quad \text{Alt}_{A,D} \text{Tr}(D_1(D_1 A_1 \cdots D_{i-1} A_{i-1} \cdot A_i \cdot A_{i+1} \cdot A_{i+2})).
\]

It is clear that

\[
(8) + (9) = (i + 2) \text{Alt}_{A,D} \text{Tr}(D_1 A_1 \cdots D_i A_i \cdot A_{i+1} \cdot A_{i+2}) = \\
= (i + 2)d(\text{Alt}_{A,D} \text{Tr}(D_1 A_1 \cdots D_i A_i \cdot A_{i+1}))(A_{i+2}, A_1, \ldots, A_{i+1}).
\]

The terms which contain \([D_a, D_b]\) again cancel because \(D_1 \land \cdots \land D_i\) is a cycle. 

2.3. We give here two examples of Lemma 2.2.

2.3.1.

Example. Let \(\mathfrak{g} = \mathfrak{gl}_n \otimes \mathbb{C}[t_1, \ldots, t_n, t_1^{-1}, \ldots, t_n^{-1}]\). Define \(\text{Tr}: \mathfrak{g} \to \mathbb{C}\) to be the composition \(\text{Tr} = \text{Res}_{t_1, \ldots, t_n} \circ \text{Tr}_{\mathfrak{gl}_n}\). The associative algebra \(\mathfrak{g}\) has \(n\) exterior derivations: \(D_1 = \frac{d}{dt_1}, \ldots, D_n = \frac{d}{dt_n}\). Therefore, Lemma 2.2 gives the \((n + 1)\)-cocycle on the Lie algebra \(\mathfrak{g}\). In particular, in the case \(n = 1\) we obtain the standard Kac–Moody 2-cocycle on the current algebra.

2.3.2.

Example. (i) Let \(\mathfrak{A}\) be an associative algebra, \(\mathfrak{h}\) an two-sided ideal in \(\mathfrak{A}\). Then \(\mathfrak{A}\) acts on the associate algebra \(\mathfrak{h}\) via the adjoint action. Suppose, that the image of \(\mathfrak{A}\) in \(\text{Der}_{\text{Ass}}(\mathfrak{h})\) lies in \(\text{Der}_{\text{Ass}}^{\text{Tr}}(\mathfrak{h})\). Then Lemma 2.2 gives a map \(q: H_i(\mathfrak{A}; \mathbb{C}) \to H^{i+1}(\mathfrak{h}; \mathbb{C})\).

(ii) In particular, let \(\mathfrak{gl}_\infty^{\text{Jac}}\) be an associative algebra of infinite matrices with a finite number (depending on the matrix) of nonzero diagonals, and let \(\mathfrak{gl}_\infty^{\text{fin}}\) be the ideal in \(\mathfrak{gl}_\infty^{\text{Jac}}\) consisting of the matrices with finite support. Then we have a map

\[
q: H_i(\mathfrak{gl}_\infty^{\text{Jac}}; \mathbb{C}) \to H^{i+1}(\mathfrak{gl}_\infty^{\text{fin}}; \mathbb{C}).
\]

Recall (\([F1], [Fu]\)) that

\[
H^*(\mathfrak{gl}_\infty^{\text{Jac}}; \mathbb{C}) = S^*(c_2, c_4, c_6, \ldots)
\]

\[
H^*(\mathfrak{gl}_\infty^{\text{fin}}; \mathbb{C}) = \Lambda^*(\xi_1, \xi_3, \xi_5, \ldots)
\]
This construction has an evident generalization: let $A$ be an associative algebra; then we have a map

$$q: H_*(\mathfrak{gl}^{\text{lac}}_{\infty}(A); \mathbb{C}) \to H^{*+1}_{\text{fin}}(\mathfrak{gl}^{\text{fin}}_{\infty}(A); \mathbb{C}).$$

2.4. The case we have discussed in Sect. 2.2 is the simplest case of the lifting problem. The next step is to extend our construction to cycles in $H^1_{\text{Tr}}(A; A)$. But we can do this only in the case $i = 2$ (see §1). In the cases $i = 4, 6, 8, \ldots$ the most we can hope is to solve the problem with some additional conditions. We thus assume that the following conditions (i)–(iii) hold:

(i) $D_j \in \text{Der}_{\text{Ass}}^{\text{fin}}(A)$ for all $j$;
(ii) we solve problem only for one monomial $D_1 \wedge \cdots \wedge D_i$ ($k = 1$), and

$$[D_a, D_b] = \text{ad} Q_{ab} — \text{inner derivations (}Q_{ab} \in A);$$
(iii) $\text{Alt}_{a,b,c} D_c(Q_{ab}) = 0$ for all $a, b, c$.

**Main Conjecture** (First Version). *Under the conditions (i)–(iii) the lifting problem can be solved for $\alpha^*_1, \ldots, \alpha^*_i$.*

We expect that

$$\Psi_{i+1} (A_1, \ldots, A_{i+1}) = \text{Alt}_{A, D} \text{Tr}(D_1 A_1 \cdots D_i A_i \cdot A_{i+1}) +$$

$$+ (\text{terms, linear in } Q_{ab}) + (\text{terms, quadratic in } Q_{ab}) + \ldots$$

(by analogy with Theorem 1.1).

**Remark.** It follows easily from the Jacobi identity that $\text{Alt}_{a,b,c} D_c(Q_{ab})$ lies in the center $Z$ of Lie algebra $\mathfrak{A}$. Hence, codition (iii) is not very strong; in particular, it holds for the Lie factor algebra $\mathfrak{A}/Z$.

Here the Conjecture will be proved only for $n = 1, 2$; but we will conject in Sect. 4.6 an explicit formula for all $n$.

The lifting formula for $n = 2$ contains terms quadratic in $Q_{ab}$.

2.5. The main example when conditions (i)–(iii) hold is an associative algebra $\Psi_{\text{Dif}}(S^1)$ of pseudo-differential operators on $(S^1)^n$ and $2 \cdot n$ exterior derivations of $\Psi_{\text{Dif}}(S^1)$: $\ln x_1, \ldots, \ln x_n, \ln \partial_1, \ldots, \ln \partial_n$. In fact, we can define derivation $\ln D$ (for $D \in \Psi_{\text{Dif}}(S^1)$) in much more generality. It seems to be true that to do this we need only this condition:

There exists $D^* \in \Psi_{\text{Dif}}(S^1)$ such that $[D, D^*] = 1$.

But, first, we have to remember about conditions (i)–(iii), and, second, we will see in Sect. 2.8 that in fact the choice $\{\ln x_1, \ldots, \ln x_n; \ln \partial_1, \ldots, \ln \partial_n\}$ is, in the same sense, most general.

2.6. There is a deformation of the Lie algebra Poiss$2n(S^1)$ to the Lie algebra $\Psi_{\text{Dif}}(S^1)$. If we suppose that that the lifting problem can be solved, it is interesting to find the Hamiltonian limit of the lifting formulas. It is easy to see that in (10), this limit depends only on the leading term, but not on the terms containing $Q_{ab}$. Let us describe this cocycles.

Let $p_1, \ldots, p_n; q_1, \ldots, q_n$ be standard Poisson coordinates, $\{p_i, q_j\} = \delta_{ij}$. 
Let $\text{Tr}: \text{Poiss}_{2n}(S^1) \to \mathbb{C}$ be the linear functional $\text{Tr} = \text{Res}_{p_1} \circ \ldots \circ \text{Res}_{p_n} \circ \text{Res}_{q_1} \circ \ldots \circ \text{Res}_{q_n}$.

**Lemma.** $\text{Tr}\{f, g\} = 0$ for any $f, g \in \text{Poiss}_{2n}(S^1)$.

**Proof.**

$$\{f, g\} = \sum_k \left( \frac{\partial f}{\partial q_k} \frac{\partial q}{\partial p_k} - \frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} \right) = \sum_k \left( \frac{\partial}{\partial p_k} \left( f \cdot \frac{\partial q}{\partial q_k} \right) - \frac{\partial}{\partial q_k} \left( f \cdot \frac{\partial q}{\partial p_k} \right) \right).$$

\[\Box\]

2.7. Any $\mathcal{D} \in \text{Poiss}_{2n}(S^1)$, $\mathcal{D} \neq 0$, determines an exterior derivation

$\ln \mathcal{D}: \text{Poiss}_{2n}(S^1) \to \text{Poiss}_{2n}(S^1)$ by the formula

$$\ln \mathcal{D}, f) = \left\{ \frac{\mathcal{D}, f}{\mathcal{D}} \right\}.$$

**Lemma.** For any $\mathcal{D}_1, \ldots, \mathcal{D}_{2n} \in \text{Poiss}_{2n}(S^1)$ the formula

$$\Psi_{2n+1}^{\mathcal{D}_1, \ldots, \mathcal{D}_{2n}}(f_1, \ldots, f_{2n+1}) = \text{Alt Tr} \left( \ln \mathcal{D}_1, f_1 \right) \circ \ldots \circ \ln \mathcal{D}_{2n}, f_{2n} \circ f_{2n+1}$$

defines a $(2n + 1)$-cocycle on $\text{Poiss}_{2n}(S^1)$.

**Proof.** We will prove, that

$$\Psi_{2n+1}^{\mathcal{D}_1, \ldots, \mathcal{D}_{2n}}(f_1, \ldots, f_{2n+1}) = \text{Alt Tr} \left( F \cdot \frac{\partial f_1}{\partial p_1} \cdot \ldots \cdot \frac{\partial f_n}{\partial p_n} \cdot \frac{\partial f_{n+1}}{\partial q_1} \cdot \ldots \cdot \frac{\partial f_{2n+1}}{\partial q_{2n}} \cdot f_{2n+1} \right)$$

defines a $(2n + 1)$-cocycle on $\text{Poiss}_{2n}(S^1)$. It is easy to see, that

$$\Psi_{2n+1}^{\mathcal{D}_1, \ldots, \mathcal{D}_{2n}} = \Psi_{2n+1}^F$$

for $F = \frac{\det \left( \frac{\partial \mathcal{D}_j}{\partial \xi_j} \right)}{\mathcal{D}_1 \cdot \ldots \cdot \mathcal{D}_{2n}}$ (here $\xi_1 = p_1, \ldots, \xi_n = p_n, \xi_{n+1} = q_1, \ldots, \xi_{2n} = q_n$).

We have:

$$\frac{1}{2} \text{Alt Tr} \left[ F \cdot \frac{\partial f_{2n+2}}{\partial p_1} \cdot \ldots \cdot \frac{\partial f_{2n+2}}{\partial p_2} \cdot \frac{\partial f_{2n+2}}{\partial p_3} \cdot \ldots \cdot \frac{\partial f_{2n+2}}{\partial q_{2n}} \cdot f_{2n+2} \cdot f_{2n+1} \right].$$

We subtract from the r.h.s. of (12) the expression

$$\frac{1}{2} \text{Alt Tr} \left( \left\{ f_{2n+2}, F \cdot \frac{\partial f_{2n+2}}{\partial p_1} \cdot \ldots \cdot \frac{\partial f_{2n+2}}{\partial q_{2n}} \cdot f_{2n+1} \right\} \right),$$

which is equal to zero by Lemma 2.6.

We have:

$$\text{r.h.s. (12)} = -\frac{1}{2} \text{Alt Tr} \left( \left\{ f_{2n+2}, F \cdot \frac{\partial f_{2n+2}}{\partial p_1} \cdot \ldots \cdot \frac{\partial f_{2n+2}}{\partial q_{2n}} \cdot f_{2n+1} \right\} \right)$$

$$+ \frac{1}{2} \text{Alt Tr} \left[ F \cdot \left\{ \frac{\partial}{\partial p_1} f_{2n+2}, f_1 \right\} \cdot \frac{\partial f_{2n+2}}{\partial p_2} \cdot \ldots \cdot \frac{\partial f_{2n+2}}{\partial q_{2n}} \cdot f_{2n+2} \cdot f_{2n+1} \right].$$
The second summand is equal to the first one by Lemma 2.5 and the symmetry, and we have:

\[(d\Psi_{2n+1}^{F})(f_{2n+2}, f_1, \ldots, f_{2n+1}) = \text{Alt} \text{Tr} \left( \{ F, f_{2n+2} \} \cdot \frac{\partial}{\partial p_1} f_1 \cdots \frac{\partial}{\partial q_n} f_{2n} \cdot f_{2n+1} \right). \]

Furthermore, \( \{ F, f_{2n+2} \} = \sum_k \left( \frac{\partial F}{\partial p_k} \frac{\partial f_{2n+2}}{\partial q_k} - \frac{\partial F}{\partial q_k} \frac{\partial f_{2n+2}}{\partial p_k} \right) \) and every summand in (14) is equal to zero via the alternation.

2.8. Although Lemma 2.7 gives us a lot of cocycles, it is easy to see that \( \Psi_{2n+1}^{F_1} \sim \Psi_{2n+1}^{F_2} \) when \( \text{Res} F_1 = \text{Res} F_2 \); therefore, we have only one interesting cocycle, which is \( \Psi_{2n+1}^{2n, p_1, \ldots, p_n, q_1, \ldots, q_n} \). We will denote it by \( \Psi_{2n+1}^0 \).

**Lemma.** (i) If the lifting problem for \( \Psi_{\text{Dif}_n}(S^1) \) can be solved, the Hamiltonian limit of any lifting is \( \Psi_{2n+1}^0 \);

(ii) \( \Psi_{2n+1}^0 (1 \wedge p_1 \wedge \cdots \wedge p_n \wedge q_1 \wedge \cdots \wedge q_n) \neq 0. \)

This Lemma implies that the lifting problem is the deformation problem.

On the other hand, we see that we don’t need try to generalize the construction for \( \Psi_{\text{Dif}_n}(S^1) \) to other derivations of the form \( \text{ln} D \); the case \( D_1 = \text{ln} x_1, \ldots, D_{2n} = \text{ln} \partial_n \) is most general.

**Corollary.** Any \((2n + 1)\)-cocycle on the Lie algebra \( \Psi_{\text{Dif}_n}(S^1) \) of the form (10) \( (D_1 = \text{ln} x_1, \ldots, D_{2n} = \text{ln} \partial_n) \) is not cohomologous to zero.

Another proof follows from the fact that every cocycle of the form (10) is a lifting of \( \alpha^*_{\text{ln} x_1, \ldots, \text{ln} \partial_n} \), which is a nonzero element in \( H^{2n}(\Psi_{\text{Dif}_n}(S^1); \Psi_{\text{Dif}_n}(S^1)^*) \).

### 3. Computation for \( n = 2 \)

#### 3.1. The main result of this Section is the following

**Theorem.** Let \( \mathfrak{A} \) be an associative algebra, \( \text{Tr}: \mathfrak{A} \rightarrow \mathbb{C} \) be a trace on \( \mathfrak{A} \) and \( D_1, D_2, D_3, D_4 \) — four (exterior) derivations on \( \mathfrak{A} \), which satisfy conditions (i)–(iii) from Sect. 2.4. Then

\[\psi_5(A_1, A_2, A_3, A_4, A_5) = \text{Alt} \left\{ D_1 A_1 \cdot D_2 A_2 \cdot D_3 A_3 \cdot D_4 A_4 \cdot A_5 \right\} \text{ terms, linear in } Q_{ij} \]

\[+ A_1 Q_{12} A_2 \cdot D_3 A_3 \cdot D_4 A_4 \cdot A_5 \]

\[+ D_1 A_1 \cdot A_2 \cdot Q_{23} A_3 \cdot D_4 A_4 \cdot A_5 \]

\[+ D_1 A_1 \cdot D_2 A_2 \cdot A_3 \cdot Q_{34} A_4 \cdot A_5 \]

\[+ A_1 Q_{12} A_2 A_3 Q_{34} A_4 A_5 \} \text{ term, quadric in } Q_{ij} \]

is a 5-cocycle on the Lie algebra \( \mathfrak{A} \), which is the lifting of \( \alpha^*_{D_1, D_2, D_3, D_4} \) (see Sect. 2.1). In the case \( \mathfrak{A} = \Psi_{\text{Dif}_2}(S^1) \), \( D_1 = \text{ln} \partial_1, D_2 = \text{ln} x_1, D_3 = \text{ln} \partial_2, D_4 = \text{ln} x_2, \) \( \psi_5 \) is not cohomologous to zero: \( \psi_5 (1 \wedge x_1 \wedge x_2 \wedge \partial_1 \wedge \partial_2) \neq 0. \)

**Remark.** \( \text{ad} Q_{ij} = [D_i, D_j] \); however, in formula (15), we don’t alternate symbols \( i \) and \( j \) in \( Q_{ij} \).
We will give a sketch of the proof of this Theorem in Sect. 3.2–3.5.

3.2.

Lemma.

(i) \[ d(\operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_1 \cdot A_2 \cdot A_3 \cdot D_2 A_4 \cdot Q \cdot A_5))(A_6, A_1, \ldots, A_5) = \]
\[ = \operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_1 \cdot A_2 \cdot A_3 \cdot D_2 A_6 \cdot A_4 \cdot Q \cdot A_5) \]

(ii) \[ d(\operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot A_3 \cdot A_4 \cdot Q \cdot A_5))(A_6, A_1, \ldots, A_5) = \]
\[ = \operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_1 \cdot D_2 A_6 \cdot A_2 \cdot A_3 \cdot A_4 \cdot Q \cdot A_5) \]

(iii) \[ d(\operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot A_3 \cdot A_4 \cdot Q \cdot A_5))(A_6, A_1, \ldots, A_5) = \]
\[ = \operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot A_3 \cdot Q \cdot A_6 \cdot A_4 \cdot A_5) \]

Proof. Straightforward (see Sect. 1.3).

3.3.

Lemma.

(i) \[ d(\operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot D_2 A_5 \cdot Q))(A_6, A_1, \ldots, A_5) = \]
\[ = \operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_6 \cdot A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot D_2 A_5 \cdot Q - D_1 A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot D_2 A_5 \cdot [A_6, Q]) \]

(ii) \[ d(\operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_1 \cdot A_2 \cdot D_2 A_3 \cdot A_4 \cdot A_5 \cdot Q))(A_6, A_1, \ldots, A_5) = \]
\[ = \operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_6 \cdot A_1 \cdot A_2 \cdot D_2 A_3 \cdot A_4 \cdot A_5 \cdot Q - D_1 A_1 \cdot A_2 \cdot D_2 A_3 \cdot A_4 \cdot A_5 \cdot [A_6, Q]) \]

(iii) \[ d(\operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_1 \cdot A_2 \cdot D_2 A_3 \cdot Q \cdot A_4 \cdot A_5))(A_6, A_1, \ldots, A_5) = \]
\[ = \operatorname{Alt}_{A,D} \operatorname{Tr}(D_1 A_6 \cdot A_1 \cdot A_2 \cdot D_2 A_3 \cdot Q \cdot A_4 \cdot A_5 - D_1 A_1 \cdot A_2 \cdot D_2 A_3 \cdot [A_6, Q] \cdot A_4 \cdot A_5) \]
3.4. Lemma.

(i) \( d(\text{Alt} \, \text{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot D_3 A_3 \cdot D_4 A_4 \cdot A_5))(A_6, A_1, \ldots, A_5) = \) \\
\( = \text{Alt} \, \text{Tr}(D_1 A_1 \cdot D_2 A_2 \cdot D_3 A_3 \cdot D_4 A_4 \cdot A_5 \cdot A_6) \)

(ii) \( d(\text{Alt} \, \text{Tr}(Q_1 \cdot Q_2 \cdot A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5))(A_6, A_1, \ldots, A_5) = \) \\
\( = - \text{Alt} \, \text{Tr}(Q_1 \cdot Q_2 \cdot A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot A_6) \)

(iii) \( d(\text{Alt} \, \text{Tr}(Q_1 \cdot A_1 \cdot Q_2 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5))(A_6, A_1, \ldots, A_5) = \) \\
\( = \text{Alt} \, \text{Tr}(Q_1 \cdot A_6 \cdot A_1 \cdot Q_2 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5) \)

3.5. To prove Theorem 3.1, we consider an expression in \( A_1, \ldots, A_6 \) which a priori is equal to zero (we use \( \text{Tr}(DA) = 0 \) — condition (i) from Sect. 2.4), and then prove that this expression is a coboundary of \( \Psi_5(A_1, \ldots, A_5) \). Then \( \Psi_5 \) is a 5-cocycle. Our main tools are Lemmas 3.2–3.4.

Suppose that

\[
E_1 = \text{Alt} \, \text{Tr}(2D_2(Q_{43} \cdot D_1 A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot A_6) + 2D_2(D_1 A_1 \cdot Q_{43} \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot A_6))
\]

\[
E_2 = \text{Alt} \, \text{Tr}(D_2(D_1 A_1 \cdot A_2 \cdot A_3 \cdot Q_{43} \cdot A_4 \cdot A_5 \cdot A_6) + D_2(D_1 A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot Q_{43} \cdot A_5 \cdot A_6))
\]

\[
E_3 = \text{Alt} \, \text{Tr}(D_4(D_1 A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot Q_{43} \cdot A_5 \cdot A_6))
\]

Note that \( E_1 = E_2 = E_3 = 0 \).

Furthermore suppose that

\[
I = \text{Alt} \, \text{Tr}(D_2((D_1 A_1 \cdot A_2 \cdot A_3 \cdot A_4) \cdot D_4(A_6 \cdot D_3 A_5)))
\]

\[
II = \text{Alt} \, \text{Tr}(D_2((A_3 \cdot A_4 \cdot D_1 A_1 \cdot A_2) \cdot D_4(A_6 \cdot D_3 A_5)))
\]

\[
III = \text{Alt} \, \text{Tr}(D_2((A_2 \cdot A_3 \cdot A_4 \cdot D_1 A_1) \cdot D_4(D_3 A_5 \cdot A_6)))
\]

\[
IV = \text{Alt} \, \text{Tr}(D_2((A_4 \cdot D_1 A_1 \cdot A_2 \cdot A_3) \cdot D_4(D_3 A_5 \cdot A_6)))
\]

Note also that \( I = II = III = IV = 0 \).
We state that
\[
E_1 - E_2 + \frac{4}{3}E_3 - I + II - III + IV = d\tilde{\Psi}_5
\]
where
\[
\tilde{\Psi}_5 = \operatorname{Alt}_{A,D} \operatorname{Tr}(-2D_1A_1 \cdot D_2A_2 \cdot D_3A_3 \cdot D_4A_4 \cdot A_5) \text{ term without } Q_{ij}
\]
\[
+ D_1A_1 \cdot D_2A_2 \cdot Q_{43} \cdot A_3 \cdot A_4 \cdot A_5
\]
\[
+ D_1A_1 \cdot D_2A_2 \cdot A_3 \cdot A_4 \cdot A_5 \cdot Q_{43}
\]
\[
+ 2 \cdot D_1A_1 \cdot Q_{43} \cdot D_2A_2 \cdot A_3 \cdot A_4 \cdot A_5
\]
\[
- D_1A_1 \cdot A_2 \cdot D_2A_3 \cdot Q_{43} \cdot A_4 \cdot A_5
\]
\[
- D_1A_1 \cdot A_2 \cdot D_2A_3 \cdot A_4 \cdot A_5 \cdot Q_{43}
\]
\[
- 4Q_{21} \cdot Q_{43} \cdot A_1 \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5
\]
\[
+ 2Q_{21} \cdot A_1 \cdot A_2 \cdot Q_{43} \cdot A_3 \cdot A_4 \cdot A_5
\]
\[
- 2\{Q_{21}, A_1\} \cdot \{Q_{43}, A_2\} \cdot A_3 \cdot A_4 \cdot A_5
\]
terms, linear in \(Q_{ij}\)

To prove this, we need condition (iii) from Sect. 2.4:
\[
\operatorname{Alt}_{i,j,k} D_k(Q_{ij}) = 0 \quad \text{for all } i, j, k.
\]

The proof of (16) is a very long straightforward calculation, using Lemmas 3.2–3.4 and other observations.

Furthermore, after simple manipulations
\[
\tilde{\Psi}_5 - \operatorname{Alt}_{A,D} \operatorname{Tr}(D_2(D_1A_1 \cdot Q_{43} \cdot A_2 \cdot A_3 \cdot A_4 \cdot A_5)) - \operatorname{Alt}_{A,D} \operatorname{Tr}(D_1(A_1 \cdot Q_{43} \cdot D_2A_2 \cdot A_3 \cdot A_4 \cdot A_5))
\]
will have a form as in Th. 3.1.

\[\square\]

4. Lifting formulas: the general case

4.1. In this Section the lifting formulas for any number of derivations \(D_1, \ldots, D_l\) (with conditions (i)–(iii) from Sect. 2.4) appear. The fact that polylinear skew-symmetric functions on \(A\), defined by these formulas, are cocycles, is the Main Conjecture of this Section. As we have seen in §3, in our situation it is highly nontrivial to check that a given formula in fact defines a cocycle.

The main idea is the following. We consider the expression
\[
\operatorname{Alt}_{A,D} \operatorname{Tr}(D_1A_1 \cdot D_2A_2 \cdots D_{2n}A_{2n} \cdot A_{2n+1})
\]
where \(D_1, \ldots, D_{2n}\) are inner derivations, \(D_iA = D_i \cdot A - A \cdot D_i\). We want to add to (17) some terms containing \(Q_{ij} = [D_i, D_j]\) and in this way obtain a cocycle. It is meaningless from the cohomological viewpoint, because this cocycle will be cohomologous to 0 (see Remark 4.2 and Remark 4.5), but using this trick we obtain explicit formulas.

We have:
\[
\operatorname{Alt}_{A,D} \operatorname{Tr}((D_1 \cdot A_1 - A_1 \cdot D_1) \cdot (A_2 \cdot D_2 - D_2 \cdot A_2) \cdots (D_{2n} \cdot A_{2n} - A_{2n} \cdot D_{2n}) \cdot A_{2n+1})
\]

Note also that our formulas are true for any number of derivations, not only for an even number.
4.2.

**Lemma.** Let $\mathfrak{A}$ be an associative algebra; $D_1, \ldots, D_{2n} \in \mathfrak{A}$. Then

$$\alpha(A_1, \ldots, A_{2n+1}) = Alt_{A,D} Tr(D_1 \cdot A_1 \cdot D_2 \cdot A_2 \cdots D_{2n} \cdot A_{2n} \cdot A_{2n+1})$$

is a $(2n + 1)$-cocycle on the Lie algebra $\mathfrak{A}$.

**Proof.**

(18) $d\alpha(A_{2n+2}, A_1, \ldots, A_{2n+1}) =$

$$= \frac{1}{4} Alt_{A,D} Tr\{D_1 \cdot [A_{2n+2}, A_1] \cdot D_2 \cdot A_2 \cdots D_{2n} \cdot [A_{2n}, A_{2n+1}] +$$

$$+ D_1 \cdot A_1 \cdot D_2 \cdot [A_{2n+2}, A_2] \cdot D_3 \cdot A_3 \cdots D_{2n} \cdot [A_{2n}, A_{2n+1}] +$$

$$+ \ldots + D_1 \cdot A_1 \cdot D_2 \cdot A_2 \cdots D_{2n-1} \cdot [A_{2n+2}, A_{2n-1}] \cdot D_{2n} \cdot [A_{2n}, A_{2n+1}] +$$

$$+ D_1 \cdot A_1 \cdots D_{2n} \cdot [[A_{2n+2}, [A_{2n}, A_{2n+1}]]].$$

The last summand in (18) is equal to 0 by the Jacobi identity. The remaining summands cancel by the trace property and symmetry conditions. For example,

$$Alt_{A,D} Tr\{D_1 \cdot [A_{2n+2}, A_1] \cdot D_2 \cdot A_2 \cdots D_{2n} \cdot [A_{2n}, A_{2n+1}] +$$

$$+ D_1 \cdot A_1 \cdot D_2 \cdots D_{2n-1} \cdot [A_{2n+2}, A_{2n-1}] \cdot D_{2n} \cdot [A_{2n}, A_{2n+1}]) = 0.$$

Note also that the proof remains the same in the case when there are an odd number of $D_i$.

**Remark.** Indeed, $\alpha$ is a coboundary, $\alpha = a \cdot d(D_1 \cdot A_1 \cdot \ldots \cdot D_{2n} \cdot A_{2n})$ ($a \in \mathbb{Z}$).

4.3. Let $D_i \in \mathfrak{A}$ for all $i$. Then

(19) $Alt_{A,D} Tr([D_1, A_1] \cdots [D_{2n}, A_{2n}] \cdot A_{2n+1}) = k\alpha + S$ ($k \in \mathbb{Z}$)

where

(20) $S = \text{(sum of the terms which contains } D_i \cdot D_{i+1} \text{ for some } i)$

We replace in the all summands of $S \cdot D_i D_{i+1}$ by $[D_i, D_{i+1}]$ because of the symmetry condition. Hence, these terms have the same form as the terms in the lifting formulas, and our problem is to represent $S$ as a sum of the terms of the form

$$Alt_{A,D} Tr([D_1, A_1] \cdots A_{i_1} \cdot Q_{i_1,i_1+1} A_{i_1+1} \cdot [D_{i_1+2}, A_{i_1+2}]$$

$$\cdots A_{i_2} \cdot Q_{i_2,i_2+1} A_{i_2+1} \cdot [D_{i_2+2}, A_{i_2+2}] \cdots).$$

Then, when we subtract $S$ from (19) we obtain $k\alpha$, which is a cocycle by Lemma 2.1. Indeed, in this case $(D_i \in A)$, $k\alpha$ is a coboundary. But it turns out that when the $D_i$ are exterior derivations satisfying conditions (i)–(iii) from Sect. 2.4 and $n = 1, 2$, the lifting formulas from §1.3 have the same form. This fact allows us to formulate the Main Conjecture in Sect. 4.6.
4.4. We will denote summands from $S$ by closed intervals with marked points. If $2n$ is the total number of $D_i$, the length of the interval is $2n - 2$, and some integral points on it are marked. Let us denote by $1, \ldots, 2n - 1$ the integral points of the interval; the point $i$ is marked, if in the corresponding summand in $S$, $D_i$ and $D_{i+1}$ are neighbors (i.e. not separated by $A_i$ or $A_{i+1}$, or $A_i \cdot A_{i+1}$). For example, the interval

\[ \begin{array}{ccccccc}
1 & \cdot & i_1 & \cdot & i_2 & \cdot & i_3 & \cdot & 2n - 1
\end{array} \]

corresponds to the expression

\[ \text{Alt Tr}_A(D_1 \cdot D_2 \cdot D_3 \cdot [D_4, A_3] \cdot [D_5, A_4] \cdot A_5). \]

It is clear that the distance between the marked points is $\geq 2$. In the general case, the interval

\[ \begin{array}{ccccccc}
1 & \cdot & i_1 & \cdot & i_2 & \cdot & i_3 & \cdot & 2n - 1
\end{array} \]

corresponds to the expression

\[ \text{Alt Tr}_A([D_1, A_1] \cdot [D_2, A_2] \cdot \ldots \cdot A_{i_1} \cdot D_{i_1} \cdot D_{i_1+1} \cdot A_{i_1+1} \cdot [D_{i_1+2}, A_{i_1+2}] \ldots \cdot A_{i_2} \cdot D_{i_2} \cdot D_{i_2+1} \cdot A_{i_2+1} \cdot \ldots) \]

Let $N$ be the total number of the marked points, $1 \leq N \leq n$.

The first summand in $S$ is a sum of all intervals with $N = 1$, with sign “$-$” ($2n - 1$ intervals):<ref>

\[ S_1 = - \begin{array}{ccccccc}
1 & \cdot & i_1 & \cdot & i_2 & \cdot & i_3 & \cdot & 2n - 1
\end{array} \]

\text{Figure 1. $n = 3$} \]

Some summands of $S$ are counted in $S_1$ with multiplicities $> 1$. We subtract from $S_1$ all summands with $N = 2$ pairs of $D_i$, and then we add every interval with $N = 2$ marked points with its sign in $S$. We have:

\[ S_2 = -2 \cdot \begin{array}{ccccccc}
1 & \cdot & i_1 & \cdot & i_2 & \cdot & i_3 & \cdot & 2n - 1
\end{array} + 1 \cdot \begin{array}{ccccccc}
1 & \cdot & i_1 & \cdot & i_2 & \cdot & i_3 & \cdot & 2n - 1
\end{array} \]

\text{Figure 2. $S_2$ for $n = 3$} \]

We wrote the coefficient $-2$ because every interval with $N = 2$ marked points is “contained” in two intervals with $N = 1$ marked points with the sign “$-$”; for example, the interval \[\begin{array}{ccccccc}
1 & \cdot & i_1 & \cdot & i_2 & \cdot & i_3 & \cdot & 2n - 1
\end{array}\] “contains” in \[\begin{array}{ccccccc}
1 & \cdot & i_1 & \cdot & i_2 & \cdot & i_3 & \cdot & 2n - 1
\end{array}\] and in \[\begin{array}{ccccccc}
1 & \cdot & i_1 & \cdot & i_2 & \cdot & i_3 & \cdot & 2n - 1
\end{array}\]. In the sum $S_1 + S_2$ only the terms with $N \geq 3$ pairs of $D_i$ are counted incorrectly. In our example ($n = 3$) the maximal value for $N$ is $N = 3$, and there exists just one unique interval with $N = 3$: \[\begin{array}{ccccccc}
1 & \cdot & i_1 & \cdot & i_2 & \cdot & i_3 & \cdot & 2n - 1
\end{array}\].
It contains with the sign “+” in 3 intervals with \( N = 1 \) marked points, namely, in Figure 3.

and with the sign “−” in 3 intervals with \( N = 2 \) marked points:

Hence,
\[
S_3 = 3 \cdot \text{---} - 3 \cdot \text{---} - 1 \cdot \text{---} = -1 \cdot \text{---}.
\]

Note that all coefficients in \( S_1, S_2, S_3 \) are equal to \(-1\).

We have: \( S = S_1 + S_2 + S_3 \).

4.5. Let us denote by \( \Sigma_{N,2n-1} \) the sum of all intervals of length \( 2n - 2 \) with \( N \) marked points (\( 1 \leq N \leq n \), the distance between marked points is \( \geq 2 \)).

**Lemma.**
\[
S = -\Sigma_{1,2n-1} - \Sigma_{2,2n-1} - \ldots - \Sigma_{n,2n-1}.
\]

**Proof.** Suppose that the summands in \( S \) with \( N \leq k \) pairs of \( D_i \) are counted correctly in \( S^{(k)} = -\Sigma_{1,2n-1} - \ldots - \Sigma_{k,2n-1} \). Then every summand with \( N = k + 1 \) pairs of \( D_i \) is “contained” in \( \Sigma_i \) with the multiplicity \( C_i^{k+1} \) and with the sign \((-1)^{i-1} \) \( i \leq k \).

Hence we add to \( S^{(k)} \)
\[
(22) \quad (-1) \cdot \left( \sum_{i=1}^{k} (-1)^i C_i^{k+1} \right) \Sigma_{k+1,2n-1} + (-1)^{k+1} \Sigma_{k+1,2n-1}
\]

It is clear that in the sum \( S^{(k)} + (22) \) all terms with \( N \leq k + 1 \) pairs of \( D_i \) are correctly calculated; on the other hand, \( (22) = -1\Sigma_{k+1,2n-1} \) by the binomial formula for \((1 - 1)^{k+1} = 0 \).

**Corollary.** For \( D_1, \ldots, D_{2n} \in \mathfrak{A} \),
\[
\text{Alt}_A \text{Tr}([D_1, A_1] \cdot \ldots \cdot [D_{2n}, A_{2n}] \cdot A_{2n+1}) + \Sigma_{1,2n-1} + \ldots + \Sigma_{n,2n-1}
\]
is a cocycle on the Lie algebra \( \mathfrak{A} \).

**Proof.** Follows from Lemma 4.2.

**Remark.** It follows from Remark 4.2 that this cocycle is cohomologous to 0.
4.6.

**Main Conjecture (Second Version).** Let \( \mathfrak{A} \) be an associative algebra with \( \text{Tr}, \ D_1, \ldots, \ D_{2n} \) be its derivations satisfying conditions (i)–(iii) from Sect. 2.4. Then (in the notations of Sect. 4.4, 4.5)

\[
\Psi_{2n+1}(A_1, \ldots, A_{2n+1}) = \text{Alt}_{A,D} \text{Tr}(D_1A_1 \cdot \ldots \cdot D_{2n}A_{2n} \cdot A_{2n+1}) + \Sigma_{1,2n-1} + \Sigma_{2,2n-1} + \ldots + \Sigma_{n,2n-1}
\]

is a cocycle on the Lie algebra \( \mathfrak{A} \) (and, hence, it solves the lifting problem).

**Remark.** Let \( \mathfrak{A} \) be an associative algebra, \( \mathfrak{h} \) be a two-sided ideal in \( \mathfrak{A} \), \( \text{Tr} \) — a trace on \( \mathfrak{h} \), \( \mathfrak{A} \) acts on \( \mathfrak{h} \) by the adjoint action (see Ex. 2.3.2). Then the above formula is true without conditions (ii)–(iii) from Sect. 2.4, but any cocycle obtained in this way is cohomologous to 0.

4.7.

**Example (\( n = 3 \)).** In §1 we obtained the lifting formula for \( \text{Diff}_1 \), in §3 we found it for \( \text{Diff}_2 \). Main Conjecture 4.6 states that the following formula is a lifting formula for \( \text{Diff}_3 \):

\[
\Psi_3(A_1, \ldots, A_7) = \text{Alt}_{A,D} \text{Tr}\{D_1A_1 \cdot \ldots \cdot D_6A_6 \cdot A_7
\]

\[
+ A_1 \cdot Q_{12} \cdot A_2 \cdot D_3A_3 \cdot D_4A_4 \cdot D_5A_5 \cdot D_6A_6 \cdot A_7
\]

\[
+ D_1A_1 \cdot A_2 \cdot Q_{23} \cdot A_3 \cdot D_4A_4 \cdot D_5A_5 \cdot D_6A_6 \cdot A_7
\]

\[
+ D_1A_1 \cdot D_2A_2 \cdot A_3 \cdot Q_{34} \cdot A_4 \cdot D_5A_5 \cdot D_6A_6 \cdot A_7
\]

\[
+ D_1A_1 \cdot A_2 \cdot D_3A_3 \cdot A_4 \cdot Q_{45} \cdot A_5 \cdot D_6A_6 \cdot A_7
\]

\[
+ D_1A_1 \cdot A_2 \cdot D_3A_3 \cdot A_4 \cdot A_5 \cdot Q_{56} \cdot A_6 \cdot A_7
\]

\[
+ A_1 \cdot Q_{12} \cdot A_2 \cdot A_3 \cdot Q_{34} \cdot A_4 \cdot D_5A_5 \cdot D_6A_6 \cdot A_7
\]

\[
+ A_1 \cdot Q_{12} \cdot A_2 \cdot A_3 \cdot Q_{34} \cdot A_4 \cdot Q_{45} \cdot A_5 \cdot D_6A_6 \cdot A_7
\]

\[
+ A_1 \cdot Q_{12} \cdot A_2 \cdot D_3A_3 \cdot A_4 \cdot Q_{45} \cdot A_5 \cdot D_6A_6 \cdot A_7
\]

\[
+ D_1A_1 \cdot A_2 \cdot Q_{23} \cdot A_3 \cdot D_4A_4 \cdot A_5 \cdot Q_{56} \cdot A_6 \cdot A_7
\]

\[
+ D_1A_1 \cdot A_2 \cdot D_3A_3 \cdot A_4 \cdot Q_{45} \cdot A_5 \cdot Q_{56} \cdot A_6 \cdot A_7
\]

\[
+ A_1 \cdot Q_{12} \cdot A_2 \cdot A_3 \cdot Q_{34} \cdot A_4 \cdot A_5 \cdot Q_{56} \cdot A_6 \cdot A_7
\}

Recall that we don’t alternate symbols \( i, j \) in \( Q_{ij} \).

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