Ginsparg-Wilson Relation and ’t Hooft-Polyakov Monopole on Fuzzy 2-Sphere

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Abstract

We investigate several properties of Ginsparg-Wilson fermion on fuzzy 2-sphere. We first examine chiral anomaly up to the second order of the gauge field and show that it is indeed reduced to the correct form of the Chern character in the commutative limit. Next we study topologically non-trivial gauge configurations and their topological charges. We investigate ’t Hooft-Polyakov monopole type configuration on fuzzy 2-sphere and show that it has the correct commutative limit. We also consider more general configurations in our formulation.

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1 Introduction

Various matrix models have been proposed toward nonperturbative formulations of the superstring theory. In matrix models like the type IIB model \[1\], space-time itself is described by matrices and thus noncommutative (NC) geometries\[2\] naturally appear \[3, 4\]. Small fluctuations around the classical background give matter degrees of freedom and hence space-time and matter are unified in the same matrices. However it is not clear how space-time and matter are embedded in matrices, for example, how metric, topology etc. are described in matrices. A construction of configurations with non-trivial indices in finite NC geometries has been an important subject not only from the mathematical interest but also from the physical point of view such as the Kalza-Klein compactification of extra dimensions with non-trivial indices to realize four dimensional chiral gauge theories.

Topologically nontrivial configurations in finite NC geometries have been constructed based on algebraic K-theory and projective modules in refs.\[5, 6, 7, 8\] but the relations to local forms of chiral anomaly or indices of Dirac operators are not very clear because Dirac operators on the fuzzy sphere considered so far \[9, 10, 11\] are not suitable to discuss these problems in this kind of system with finite degrees of freedom. This is summarized later in this section. The most suitable framework will be to utilize the Ginsparg-Wilson (GW) relation\[12\] developed in lattice gauge theory (LGT), because the GW relation enables us to have the exact index theorem\[13, 14\] at a finite lattice spacing using the GW Dirac operator\[15\] and the modified chiral symmetry\[14, 16\].

The formulation of NC geometries in Connes’ prescription is based on the spectral triple \((\mathcal{A}, \mathcal{H}, \mathcal{D})\), where a chirality operator and a Dirac operator which anti-commute are introduced\[2\]. In ref.\[17\], we generalized the algebraic relation to the GW relation in general gauge field backgrounds and provided prescriptions to construct chirality operators and Dirac operators which satisfy the GW rela-
tion, so that we can define chiral structures in finite NC geometries. As a concrete example we considered fuzzy 2-sphere and constructed a set of chirality and Dirac operators. We showed that the Chern character is obtained by evaluating chiral anomaly up to the first order of the gauge field. The evaluation of the chiral anomaly is also considered in refs. Fuzzy 2-sphere is one of the simplest compact NC geometry and it can be regarded as the classical solution of the matrix model with a Chern-Simons term. Since it has finite degrees of freedom and UV/IR cutoffs, its stability can be studied analytically or numerically.

We here summarize various Dirac operators on fuzzy 2-sphere. There had been known two types of Dirac operators, $D_{WW}$ and $D_{GKP}$. Doubling problems of these operators are studied and based on this, these authors introduced two sets of chirality operators and a free Dirac operator satisfying GW relation. In [17], we generalized these free operators to an interacting case in general gauge field background configurations. Incorporation of gauge fields is essential in the GW formalism. Denoting this third Dirac operator as $D_{GW}$, various properties of these three types of the Dirac operators are summarized in Table 1. $D_{WW}$ has no chiral anomaly because of the doublers. Both of $D_{GKP}$ and $D_{GW}$ have no doublers and the chiral currents satisfy Ward identities with chiral anomaly, whose local forms were evaluated explicitly in ref. and ref. respectively. The global form of the chiral anomaly of $D_{GKP}$ was studied in refs. While in lattice gauge theories, the chiral symmetry and the no-doubler condition have been shown to be incompatible with some reasonable assumptions, table 1 implies the existence of analogous no-go theorem in finite NC geometries or matrix models.

In the present paper we proceed to study the chiral and topological properties of the GW fermion on fuzzy 2-sphere. After an introduction of fuzzy 2-sphere in subsection 2.1, we provide a set of GW Dirac operator and chirality operators on the fuzzy 2-sphere in subsection 2.2. We also define a topological charge and
Table 1: The properties of three types of Dirac operators on fuzzy 2-sphere are summarized. Each Dirac operator represents Watamura’s operator $D_{WW}$, Grosse et.al.’s operator $D_{GKP}$ and Ginsparg-Wilson Dirac operator $D_{GW}$.

| Dirac op. | chiral symmetry | no doublers | counterpart in LGT |
|-----------|-----------------|-------------|--------------------|
| $D_{WW}$  | $D_{WW}\Gamma + \Gamma D_{WW} = 0$ | $\bigcirc$ | $\times$ | naive fermion |
| $D_{GKP}$ | $D_{GKP}\Gamma + \Gamma D_{GKP} = O(\frac{1}{L})$ | $\times$ | $\bigcirc$ | Wilson fermion |
| $D_{GW}$  | $D_{GW}\hat{\Gamma} + \Gamma D_{GW} = 0$ | $\bigcirc$ | $\bigcirc$ | GW fermion |

provide an index theorem on fuzzy 2-sphere. We prove the index theorem in a more general formulation in appendix A which can be applied to any NC geometries. The topological charge we defined takes only integer values by definition, and we also show in appendix B that its value does not change under any small fluctuation of any parameters or any fields in the theory. We further need to show that it has the correct commutative limit, and also that it takes nonzero integer values for topologically-nontrivial configurations. The purpose of this paper is to study these two points.

In subsection 2.3, we evaluate chiral anomaly by calculating the non-trivial Jacobian under the local chiral transformation. In NC geometries, there is an ambiguity to define local transformations since the transformation parameter do not commute with the fields and the operators in the theory. We here consider two chiral transformations and corresponding chiral currents, which transform covariantly and invariantly under the gauge transformation respectively. We show that the correct form of the Chern character is obtained for the covariant current case by calculating the Jacobian up to the second order in gauge fields, and taking commutative limit. This confirms the previous result up to first order in the gauge field in [17]. Since the topological charge we defined on the fuzzy
2-sphere has the same form as the anomaly term, it has the correct commutative limit.

In section 3 we study topologically nontrivial gauge configurations and their topological charges. Even in the theories on the commutative sphere, after integrating the topological charge density over the sphere, the topological charge vanishes identically for any configurations. For the Dirac monopole, we need to introduce the notion of patches in the sphere and obtain nonzero values for the topological charge. For the 't Hooft-Polyakov (TP) monopole, we need to introduce the idea of spontaneous symmetry breaking, insert projector to pick up unbroken $U(1)$ component into the topological charge, and then obtain nonzero topological charge. These ideas correspond to introducing some projections into matrices in the NC theories since both space-time and gauge group space are embedded in matrices.

In subsection 3.1 we construct a TP monopole configuration on the fuzzy sphere and show that it becomes the correct form in the commutative limit. The normal component of the gauge field plays the role of the Higgs field. We also review the topological charge for the TP monopole in the commutative theory. In subsection 3.2 we define a topological charge for the TP monopole on the fuzzy 2-sphere by introducing a projector. In the commutative limit this projector becomes the one to pick up the unbroken $U(1)$ component, and thus this topological charge has the correct commutative limit. In subsection 3.3 we investigate TP monopole configurations with higher isospin. Since it is important that these configurations satisfy $SU(2)$ algebra, we study its meaning. We also define another topological charge with the Higgs field inserted, which has the correct commutative limit with the projector to pick up unbroken $U(1)$ component. We then see that the winding number of $\pi_2(SU(2)/U(1))$ for these configurations is 1. In subsection 3.4 we consider more general configurations.

Similar study was done from mathematical points of view in [5, 6, 7, 20, 8]. In this paper we give physical interpretation by studying its commutative limit.
Also, we believe that our derivation is simpler and more transparent.
Section 4 is devoted to conclusions and discussions.

2 Ginsparg-Wilson fermion on fuzzy 2-sphere

In the paper [17] we proposed a general prescription to construct chirality operators and Dirac operators satisfying the GW relation on general finite NC geometries and gave a simple example on the fuzzy 2-sphere. In this section we first review this construction and calculate the local form of anomaly up to the second order of the gauge field.

2.1 Brief review of fuzzy 2-sphere and $D_{GKP}$

We first briefly explain fuzzy 2-sphere and the Dirac operator $D_{GKP}$, which will be used when we construct the GW Dirac operator $D_{GW}$ in the next subsection.

NC coordinates of the fuzzy 2-sphere are given by

$$x_i = \alpha L_i$$ (2.1)

where $\alpha$ is a NC parameter, and $L_i$’s are $2L + 1$-dimensional irreducible representation matrices of $SU(2)$ algebra. Thus

$$[x_i, x_j] = i\alpha \epsilon_{ijk} x_k,$$ (2.2)

$$(x_i)^2 = \alpha^2 L(L + 1),$$ (2.3)

and

$$\rho = \alpha \sqrt{L(L + 1)}$$ (2.4)

gives the radius of the fuzzy sphere.

Wave functions on fuzzy 2-sphere are composed of $(2L+1) \times (2L+1)$ matrices, and can be expanded in terms of NC analogues of the spherical harmonics $\hat{Y}_{lm}$. They are traceless symmetric products of the NC coordinates. There is an upper
bound for the angular momentum $l$ in $\hat{Y}_{l,m}$: $l \leq 2L$. Derivatives along the Killing vectors on the sphere are given by the adjoint action of $L_i$ as

$$\mathcal{L}_i M = [L_i, M] = (L^L_i - L^R_i)M,$$

(2.5)

where $M$ is any wavefunction and the superscript L (R) in $L_i$ means that this operator acts from the left (right) on matrices. An integral over the 2-sphere is expressed by a trace over matrices:

$$\frac{1}{2L + 1} \text{tr} \leftrightarrow \int \frac{d\Omega}{4\pi}.$$  

(2.6)

The fermionic action with the Dirac operator $D_{GKP}$ is given by

$$S_{GKP} = \text{tr}(\bar{\psi} D_{GKP} \psi),$$

(2.7)

$$D_{GKP} = \sigma_i (L_i + \rho a_i) + 1.$$  

(2.8)

Here $i$ runs from 1 to 3, $\sigma_i$’s are Pauli matrices, and the fermion field $\psi$ and the gauge field $a_i$ are expressed as $(2L + 1) \times (2L + 1)$ Hermitian matrices. The free part of this theory does not have doubler modes.

$S_{GKP}$ is invariant under the following gauge transformation:

$$\psi \rightarrow U \psi, \quad \bar{\psi} \rightarrow \bar{\psi} U^\dagger,$$

(2.9)

$$a_i \rightarrow U a_i U^\dagger + \frac{1}{\rho} (U L_i U^\dagger - L_i),$$

(2.10)

since a combination

$$A_i \equiv L_i + \rho a_i$$

(2.11)

transforms covariantly under the gauge transformation as

$$A_i \rightarrow U A_i U^\dagger,$$

(2.12)

and the fermion $\psi$ transforms as the fundamental representation.

The normal component of $a_i$ to the sphere can be interpreted as the scalar field on the sphere. We define it covariantly as

$$\phi = \frac{A_i^2 - L(L + 1)}{2L + 1} \frac{1}{\rho} = \frac{\phi'}{\rho},$$

(2.13)

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Here we also defined a normalized scalar field $\phi'$ for later convenience.

The commutative limit can be taken by $\alpha \to 0, L \to \infty$ with $\rho$ fixed. Then, $D_{\text{GKP}}$ becomes a Dirac operator on the commutative 2-sphere,

$$D_{\text{com}} = \sigma_i (\tilde{\mathcal{L}}_i + \rho a_i) + 1,$$

where

$$\tilde{\mathcal{L}}_i = -i\epsilon_{ijk} x_j \partial_k.$$  

The scalar field (2.13) becomes the one on the commutative 2-sphere, $a_in_i$, where $n_i = x_i/\rho$.

More detailed explanations are given in refs. [11, 25, 17].

### 2.2 Ginsparg-Wilson fermion on fuzzy 2-sphere

In this subsection we review the construction of the GW Dirac operator [17].

We first introduce two hermitian chirality operators $\Gamma^R$ and $\hat{\Gamma}$ satisfying $\hat{\Gamma}^2 = (\Gamma^R)^2 = 1$ as follows:

$$\Gamma^R = a \left( \sigma_i L^R_i - \frac{1}{2} \right) = \frac{\sigma_i L^R_i - \frac{1}{2}}{\sqrt{(\sigma_i L^R_i - \frac{1}{2})^2}},$$

$$\hat{\Gamma} = \frac{H}{\sqrt{H^2}},$$

where

$$a = \frac{1}{L + \frac{1}{2}}$$

is introduced as a NC analogue of a lattice-spacing. The superscript $R$ of $L_i^R$ means that this operator acts from the right on matrices. We define the Hermitian operator $H$ as

$$H = a \left( \sigma_i A_i + \frac{1}{2} \right)$$

$$= \Gamma^R + a D_{\text{GKP}}$$

$$= \Gamma^L + a \rho \sigma_i a_i^L.$$
where

\[ \Gamma^L = a \left( \sigma_i L^L_i + \frac{1}{2} \right) = \frac{\sigma_i L^L_i + \frac{1}{2}}{\sqrt{(\sigma_i L^L_i + \frac{1}{2})^2}}, \] (2.22)

is the Hermitian chirality operator introduced in [24] and satisfies \((\Gamma^L)^2 = 1\). In eqs. (2.21), (2.22) the superscript \(L\) in \(a^L_i\) and \(L^L_i\) means that this operator acts from the left on matrices.

We thus see that \(\hat{\Gamma}\) in eq. (2.17) becomes \(\Gamma^L\) for vanishing gauge fields. We also note that both of the two chirality operators \(\Gamma^R\) and \(\Gamma^L\) become the chirality operator \(\gamma = \sigma_i n_i\) in the commutative limit. This was discussed in [24] for the free case without gauge field backgrounds.

We next define the GW Dirac operator as

\[ D_{GW} = -a^{-1}\Gamma^R (1 - \Gamma^R \hat{\Gamma}). \] (2.23)

The fermionic action

\[ S_{GW} = tr(\bar{\Psi} D_{GW} \Psi) \] (2.24)

is invariant under the gauge transformation (2.9), (2.12) because \(D_{GW}\) is composed of \(\Gamma^R\) and \(A_i\). The free part of the theory has no doublers. In the commutative limit \(D_{GW}\) becomes the commutative Dirac operator without coupling to the scalar field \(\phi\),

\[ D_{GW} \approx (D_{GKP} - \{\Gamma^R, D_{GKP}\}) \Gamma^R / 2 + O(1/L) \]

\[ \to \sigma_i (\tilde{\mathcal{L}}_i + \rho P_{ij} a_j) + 1, \] (2.25)

where \(P_{ij}\) is a projection operator on the sphere,

\[ P_{ij} = \delta_{ij} - n_i n_j. \] (2.26)

This satisfies \((P^2)_{ij} = P_{ij}\) and \(n_i P_{ij} = 0\).

We can see from the definition (2.23) that \(D_{GW}\) satisfies the GW relation:

\[ \Gamma^R D_{GW} + D_{GW} \hat{\Gamma} = 0. \] (2.27)
Then, as we show in appendix A, we can prove the following index theorem:

\[ \text{index} D_{GW} \equiv (n_+ - n_-) = \frac{1}{2} \text{Tr}(\Gamma_R + \hat{\Gamma}), \]

(2.28)

where \( n_\pm \) are the numbers of zero eigenstates of \( D_{GW} \) with a positive (or negative) chirality (for either \( \Gamma_R \) or \( \hat{\Gamma} \)) and \( \text{Tr} \) is a trace of operators acting on matrices. We also prove in appendix B that \( \text{Tr}(\hat{\Gamma}) \) is invariant under a small deformation of any parameter or any configuration such as gauge field in the operator \( H \). Furthermore, \( \frac{1}{2} \text{Tr}(\Gamma_R + \hat{\Gamma}) \) takes only integer values since both \( \Gamma_R \) and \( \hat{\Gamma} \) have a form of sign operator by the definitions (2.16) (2.17). Therefore, we may call this a topological charge.

In the next subsection we will investigate the commutative limit of this topological charge, and show that it becomes the Chern character in the commutative limit. In section 3 we will investigate topologically nontrivial configurations and their topological charges.

### 2.3 Chiral anomaly

In this subsection we calculate the local form of the chiral anomaly.

The fermionic action (2.24) is invariant under the global chiral transformation,

\[ \delta \Psi = i \hat{\Gamma} \Psi, \quad \delta \bar{\Psi} = i \bar{\Psi} \Gamma_R, \]

(2.29)

due to the GW relation (2.27). For a local transformation, however, we need to specify the ordering of the chiral transformation parameter \( \lambda \), the fermion field, and the chirality operator, since they are not commutable. This ambiguity is specific to the NC field theories and makes the analysis of the Ward-Takahashi(WT) identity complicated [29, 25, 17].

Here we consider two types of local chiral transformations. The first type of chiral transformation is defined as

\[ \delta \Psi = i \lambda \hat{\Gamma} \Psi, \quad \delta \bar{\Psi} = i \bar{\Psi} \lambda \Gamma_R, \]

(2.30)
where the chiral transformation parameter $\lambda$ should transform covariantly as $\lambda \rightarrow U\lambda U^\dagger$ under the gauge transformation (2.9), (2.10). The associated chiral current transforms covariantly. Another chiral transformation is defined as

$$\delta \Psi = i\hat{\Gamma}\Psi \lambda, \quad \delta \bar{\Psi} = i\lambda \bar{\Psi}\Gamma^R,$$

(2.31)

where the chiral transformation parameter $\lambda$ is assumed to be invariant under gauge transformations, so is the associated chiral current.

In the WT identity for the gauge-covariant current, the variation of the action under (2.30) gives the current-divergence term, and the variation of the integration measure gives the anomaly term,

$$2q^{\text{cov}}(\lambda) \equiv \mathcal{T}r(\lambda L\hat{\Gamma} + \lambda^L \Gamma^R)$$

$$= \text{tr}(1)\text{Tr}(\lambda \hat{\Gamma}) + \text{tr}(\lambda)\text{Tr}(\Gamma^R)$$

$$= \frac{2}{a}\text{Tr}(\lambda \hat{\Gamma}) - 2\text{tr}(\lambda),$$

(2.32)

where in the first line, $\mathcal{T}r$ is a trace of operators acting on matrices, and the superscript $L$ ($R$) means that this operator acts from the left (right) on matrices. In the second and third lines, $\text{Tr}$ is a trace over matrices and spinors, $\text{tr}$ is a trace over matrices, and $\Gamma^R$ and $\hat{\Gamma}$ are considered as mere matrices instead of operators acting on matrices. Similarly, in the WT identity for the gauge-invariant current, the variation of the measure under (2.31) gives the anomaly term,

$$2q^{\text{inv}}(\lambda) \equiv \mathcal{T}r(\lambda R\hat{\Gamma} + \lambda^R \Gamma^R)$$

$$= \text{tr}(\lambda)\text{Tr}(\hat{\Gamma}) + \text{tr}(1)\text{Tr}(\lambda \Gamma^R)$$

$$= \text{tr}(\lambda)\text{Tr}(\hat{\Gamma}) - 2\text{tr}(\lambda).$$

(2.33)

For a global chiral transformation, that is, $\lambda = 1$ case, both $q^{\text{cov}}(\lambda)$ and $q^{\text{inv}}(\lambda)$ become the topological charge defined in eq. (2.28). When background gauge fields vanish, $\hat{\Gamma} = \Gamma^L$, and $q^{\text{cov}}(\lambda)$ and $q^{\text{inv}}(\lambda)$ vanish.

For the covariant case (2.32), the chiral transformation parameter $\lambda$ and the gauge field $a_i$ in $\hat{\Gamma}$ are inserted in the same trace, while for the invariant case
\[\lambda\] and \(a_i\) are inserted in different traces. Since traces are replaced by the integrations in the commutative limit, this indicates that local WT identity can be written down for the covariant current, while some nonlocality must be introduced in the WT identity for the invariant current. This is consistent with the previous results \([29, 25]\). Gauge invariant operators can be defined only by taking traces over matrices, and thus this introduces nonlocality in NC spaces.

We now consider weak gauge field configurations, and see if the topological charge density \(q_{\text{cov}}(\lambda)\) reduces to the Chern character in the commutative limit. Expanding the topological charge density (2.32) with respect to the gauge field \(a_i\) up to the second order, and taking a trace over \(\sigma\) matrices, we obtain

\[
q_{\text{cov}}(\lambda) = \frac{a^2 \rho^2 \alpha}{i} \text{tr} \left( \lambda [L_i, a'_i] \right) \\
+ \text{tr} \left[ \left( \frac{3}{8} a^4 \rho^2 [L_i, a_i]^2 - 4 \left( \frac{\rho}{\alpha} \right)^2 (a'_i)^2 + 4 i \frac{\rho}{\alpha} L_i [L_j, a_j, a'_i] \right) \\
- 8 i \left( \frac{\rho}{\alpha} \right)^2 \epsilon_{ijk} L_i a'_j a'_k \right] \\
+ O(a^3),
\]

(2.34)

where

\[
a'_i = \frac{\alpha}{2 \rho} \epsilon_{ijk} (L_j a_k + a_k L_j)
\]

(2.35)

is the tangential component of the gauge field \(a_i\). The gauge field \(a_i\) are decomposed into the tangential component \(a'_i\) and the normal component, that is, the scalar field \(\phi\).

In the commutative limit, the second line of (2.34) vanishes, and we obtain

\[
q_{\text{cov}}(\lambda) \to \rho^2 \int \frac{d\Omega}{4\pi} \text{tr} \left[ \lambda \epsilon_{ijk} \frac{x_i}{\rho} F_{jk} \right],
\]

(2.36)

where the trace is taken over the non-Abelian gauge group, \(F_{jk}\) is the field strength defined as \(F_{jk} = \partial_j a'_k - \partial_k a'_j - i [a'_j, a'_k],\) and \(a'_i = \epsilon_{ijk} x_j a_k / \rho\). For the \(U(1)\) gauge theory, the trace in (2.36) is not necessary and the field strength is defined as such. All contributions from higher order terms in the gauge field \(a_i\) vanish in the commutative limit. It is now confirmed that the Chern character is reproduced
in all orders in $a_i$, which was previously checked up to the first order of $a_i$ in [17]. This topological charge density is nothing but the magnetic flux density penetrating the 2-sphere, and we obtain the correct form of anomalous WT identity in the commutative limit. Since the topological charge we defined in (2.28) has the form of $q_{\text{cov}}(\lambda)$ with $\lambda = 1$ in (2.32), our topological charge (2.28) becomes the Chern character in the commutative limit, at least locally.

However, for $\lambda = 1$, even in the commutative theories, (2.36) vanishes identically for any configurations after the integration over the sphere, if the gauge field $a_i$ is a single-valued function on the sphere for Abelian gauge theory, or if we naively take the trace for non-Abelian gauge theory. In order to describe topologically-nontrivial configurations and classify them by some topological charge, we need to introduce such ideas as patches in the Dirac monopole, or spontaneous symmetry breakings in the TP monopole. In NC theories, these ideas correspond to introducing some kind of projections into matrices, since both space-time and gauge group space are embedded in matrices in NC theory. We will study this subject in the next section.

3 't Hooft-Polyakov monopole on fuzzy 2-sphere

In this section we construct topologically nontrivial configurations which correspond to the 't Hooft-Polyakov (TP) monopole in the commutative theory. We then define a topological charge for nontrivial configurations by introducing projection operators. We show that the topological charge has the correct commutative limit. Similar study was done by using projective modules [5, 6, 7, 20, 8]. In the following we will give physical interpretation of the previous mathematical settings by studying its commutative limit.
3.1 TP monopole configuration on fuzzy 2-sphere

Let’s consider the following configuration in the $SU(2)$ gauge theory on fuzzy 2-sphere:

$$a_i = \frac{1}{\rho} 1_{2L+1} \otimes \frac{\tau_i}{2}, \quad (3.1)$$

where the first and second factors represent the NC space and the gauge group space respectively. Then, the combination (2.11) becomes

$$A_i = L_i^L \otimes 1_2 + 1_{2L+1} \otimes \frac{\tau_i}{2}. \quad (3.2)$$

We note here that $A_i$’s satisfy the $SU(2)$ algebra:

$$[A_i, A_j] = i\epsilon_{ijk} A_k. \quad (3.3)$$

We will use this property for constructing a nontrivial topological charge on the fuzzy sphere in the next subsection. Here we will first show that the configuration (3.1) corresponds to the section of the TP monopole configuration on $S^2$ in the commutative theory. We will also see that the scalar field $\phi$, the normal component of the gauge field $a_i$, has a role of the Higgs field in the $SU(2)$ adjoint representation.

In the commutative limit, (3.1) becomes

$$a_i(x) = \frac{1}{\rho} \frac{\tau_i}{2}, \quad (3.4)$$

or

$$a_i^a(x) = \frac{1}{\rho} \delta_{ia}, \quad (3.5)$$

if decomposed by $a_i = a_i^a \tau^a / 2$. We further decompose it into the tangential component on 2-sphere, $a'_i$ and normal component $\phi$ by

$$\begin{cases} 
    a'_i &= \epsilon_{ijk} n_j a_k, \\
    \phi &= n_i a_i, \\
    a_i &= -\epsilon_{ijk} n_j a'_k + n_i \phi, 
\end{cases} \quad (3.6)$$

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where \( n_i = x_i / \rho \). Then we obtain

\[
a_i^a = \frac{1}{\rho^2} \epsilon_i^{aj} x_j, \quad (3.8)
\]

\[
\phi^a = \frac{1}{\rho} n_a, \quad (3.9)
\]

for the configuration (3.5). These are nothing but the TP monopole configurations.

This configuration satisfies

\[
a_i^a = -\epsilon^{abc} \phi^b \partial_i \phi^c, \quad (3.10)
\]

where

\[
\phi^a = \frac{\phi^a}{\sqrt{(\phi^a)^2}} = n_a \quad (3.11)
\]

is the normalized scalar field which satisfy \( \sum_a (\phi^a)^2 = 1 \). For (3.10), the covariant derivative of \( \phi' \) and the field strength become

\[
(D_i \phi')^a \equiv \partial_i \phi'^a + \epsilon^{abc} a_i^b \phi'^c = 0, \quad (3.12)
\]

\[
F_{ij}^a \equiv \partial_i a_j^a - \partial_j a_i^a + \epsilon^{abc} a_i^b \phi'^c = -2\epsilon^{abc} (\partial_i \phi'^b)(\partial_j \phi'^c) + \epsilon^{bcd} \phi'^a \phi'^b (\partial_i \phi'^c)(\partial_j \phi'^d). \quad (3.15)
\]

If we extend this configuration to 3-dimensional space, by regarding \( \rho \) as the radial coordinate, we obtain the following asymptotic behavior:

\[
\begin{cases}
(\phi'_a)^2 \to 1, \\
D_i \phi' \to 0, \text{ for } \rho \to \infty, \\
F_{ij} \to O(1/\rho^2),
\end{cases} \quad (3.16)
\]

which assures the finiteness of the energy defined in 3-dimensions.

We next consider the winding number of \( \pi_2(SU(2)/U(1)) \). The magnetic flux of unbroken \( U(1) \) component penetrating the 2-sphere is written as

\[
Q = \frac{\rho^2}{4\pi} \int_{S^2} d\Omega tr(P_{\tau} \epsilon_{ijk} n_i F_{jk}) \quad (3.17)
\]

\[
= \frac{\rho^2}{8\pi} \int_{S^2} d\Omega \epsilon_{ijk} n_i \phi'^a F_{jk}^a \quad (3.18)
\]
where $P_r = \frac{1+i\eta r}{2}$ is the projector to pick up the unbroken $U(1)$ component. We note here that this is the Chern character (2.36) with the projection operator $P_r$ or $\phi'$ inserted. In the configuration (3.10), using (3.15), we obtain

$$Q = -\frac{\rho^2}{8\pi} \int_{S^2} d\Omega \epsilon_{ijk} n_i \epsilon^{abc} \phi'^a (\partial_j \phi'^b)(\partial_k \phi'^c).$$  \hspace{1cm} (3.19)

Thus $Q$ is the degree of mapping of $\phi'^a(x_i) : S^2_x \rightarrow S^2_{\phi'}$, $\pi_2(S^2) = \mathbb{Z}$. Inserting (3.11), this gives

$$Q = -\frac{1}{4\pi} \int_{S^2} d\Omega = -1.$$ \hspace{1cm} (3.20)

Thus the configuration (3.1) corresponds to the TP monopole configuration in the commutative theory.

3.2 Topological charge of TP monopole on fuzzy 2-sphere

In this subsection we study the topological charge for the configuration (3.1) by introducing a projection operator into the topological charge in (2.28). We see that this topological charge corresponds to that in commutative theory which was studied at the end of the previous subsection.

Since $A_i$’s in (3.2) satisfy $SU(2)$ algebra, we can decompose $A_i$ into irreducible representations using some unitary matrix $U$ as

$$A_i = U \begin{pmatrix} L^{(1)}_i \\ L^{(2)}_i \end{pmatrix} U^\dagger,$$ \hspace{1cm} (3.21)

where $L^{(1)}_i$ and $L^{(2)}_i$ are $L^{(1)} = L + \frac{1}{2}$ and $L^{(2)} = L - \frac{1}{2}$ representations respectively. We denote the Hilbert spaces on which the operator $L^{(1)}_i$ and $L^{(2)}_i$ act as $\mathcal{H}^{(1)}$ and $\mathcal{H}^{(2)}$ respectively. Each Hilbert space can be picked up by the following projection
operators:

\[
P^{(1)} = \frac{(A_i)^2 - L^{(2)}(L^{(2)} + 1)}{L^{(1)}(L^{(1)} + 1) - L^{(2)}(L^{(2)} + 1)} \quad (3.22)
\]

\[
= \frac{1 + \Gamma_\tau}{2} \quad (3.23)
\]

\[
= \rho \phi + \frac{L + \frac{1}{2}}{2L + 1} \quad (3.24)
\]

\[
P^{(2)} = \frac{L^{(1)}(L^{(1)} + 1) - (A_i)^2}{L^{(1)}(L^{(1)} + 1) - L^{(2)}(L^{(2)} + 1)} \quad (3.25)
\]

\[
= \frac{1 - \Gamma_\tau}{2} \quad (3.26)
\]

\[
= -\rho \phi + \frac{L + \frac{3}{4}}{2L + 1} \quad (3.27)
\]

where the Hermitian operator

\[
\Gamma_\tau = \frac{L_i \tau_i + \frac{1}{2}}{L + \frac{1}{2}} \quad (3.28)
\]

satisfies \((\Gamma_\tau)^2 = 1\) and becomes \(\tau_i n_i\) in the commutative limit. The scalar field \(\phi\) in (3.24) (3.27) is defined in (2.13).

We define the chirality operator \(\hat{\Gamma}\) as in (2.17). This has the following block diagonal form when \(A_i\) are decomposed into irreducible representations:

\[
\hat{\Gamma} = U \begin{pmatrix} \hat{\Gamma}^{(1)} \\ \hat{\Gamma}^{(2)} \end{pmatrix} U^\dagger, \quad (3.29)
\]

where

\[
\hat{\Gamma}^{(a)} = \frac{\sigma_i L^{(a)} + \frac{1}{2}}{L^{(a)} + \frac{1}{2}}, \quad (3.30)
\]

for \(a = 1, 2\).

We now consider the topological charge (2.28) with the projector (3.22)-(3.27)

\[
\frac{1}{2} \mathcal{T}_r P^{(a)}(\Gamma^R + \hat{\Gamma}) = \frac{1}{2} \mathcal{T}_r r_{(a)}(\Gamma^R + \hat{\Gamma}^{(a)}), \quad (3.31)
\]

where the trace \(\mathcal{T}_r\) is taken over the whole Hilbert space on which \(A_i, -L^R_i, \frac{\sigma_i}{2}\) operate. The trace \(\mathcal{T}_r_{(a)}\) is taken over the restricted Hilbert space \(\mathcal{H}^{(a)}\). We
note that, since \( A_i \)'s satisfy the \( SU(2) \) algebra, the projector \( P^{(a)} \) commutes with \( \Gamma^R \) and \( \hat{\Gamma} \). We can see that the topological charge (3.31) is invariant under any small perturbations and take only integer values since this has the form of a sign operator as we mentioned below (2.28) \(^1\). If we take the weak gauge limit and the commutative limit of (3.31) as we did in (2.36), we obtain

\[
\frac{1}{2} \text{Tr} P^{(a)} (\Gamma^R + \hat{\Gamma}) \rightarrow \rho^2 \int \frac{d\Omega}{4\pi} \text{tr} \left[ P^{(a)} \epsilon_{ijk} n_i F_{jk} \right].
\]  

(3.32)

By using (3.23) (3.24) for \( P^{(1)} \), this becomes (3.17) (3.18), while by using (3.27) for \( P^{(2)} \), this becomes a product of \(-1\) and (3.18). We thus see that the topological charge (3.31) becomes the desired form in the commutative limit.

We will now evaluate the topological charge (3.31). It was done in ref. [20], but we will give simpler evaluation below. For evaluating the first term, \( \frac{1}{2} \text{Tr} P^{(a)} \Gamma^R \), we introduce an operator \( J_i = \sigma_i^2 - L_i^R \). The Casimir operator of \( J_i \) is given by

\[
(J_i)^2 = J(J + 1) = L(L + 1) + \frac{3}{4} - \sigma_i L_i^R.
\]  

(3.33)

Denoting the degeneracy as \( m \), we obtain

- For \( J = L + \frac{1}{2} \), \( \Gamma^R = -1 \), \( m = (2L + 2)(2L^{(a)} + 1) \),
- For \( J = L - \frac{1}{2} \), \( \Gamma^R = 1 \), \( m = 2L(2L^{(a)} + 1) \),

and thus

\[
\frac{1}{2} \text{Tr} P^{(a)} \Gamma^R = -(2L^{(a)} + 1).
\]  

(3.34)

Similarly, for evaluating the second term, \( \frac{1}{2} \text{Tr} \hat{P}^{(a)} \hat{\Gamma}^{(a)} \), we introduce another operator \( J'_i = \frac{\sigma_i}{2} + L_i^{(a)} \). The Casimir operator of \( J'_i \) is calculated as

\[
(J'_i)^2 = J'(J' + 1) = L^{(a)}(L^{(a)} + 1) + \frac{3}{4} + \sigma_i L_i^{(a)}.
\]  

(3.35)

Then we obtain

\(^1\)We can also define a GW Dirac operator as (2.23) multiplied by the projection operator \( P^{(a)} \), and show that the index theorem is satisfied between the index for this Dirac operator and the topological charge (3.31). However, the meaning of this Dirac operator is not clear.
• For $J' = L^{(a)} + \frac{1}{2}$, $\hat{\Gamma}^{(a)} = 1$, $m = (2L^{(a)} + 2)(2L + 1)$,

• For $J' = L^{(a)} - \frac{1}{2}$, $\hat{\Gamma}^{(a)} = -1$, $m = 2L^{(a)}(2L + 1)$,

and

\[ \frac{1}{2} Tr_{(a)} \hat{\Gamma}^{(a)} = 2L + 1. \] (3.36)

From eqs. (3.34) (3.36), we obtain

\[ \frac{1}{2} Tr P^{(a)}(\Gamma^{R} + \hat{\Gamma}) = \frac{1}{2} Tr_{(a)}(\Gamma^{R} + \hat{\Gamma}^{(a)}) = 2(L - L^{(a)}). \] (3.37)

For $a = 1, 2$, $L^{(1)} = L + 1/2$, $L^{(2)} = L - 1/2$, and

\[ \frac{1}{2} Tr P^{(1)}(\Gamma^{R} + \hat{\Gamma}) = \frac{1}{2} Tr \left( \rho \phi (\Gamma^{R} + \hat{\Gamma}) \right) = -1, \] (3.38)

\[ \frac{1}{2} Tr P^{(2)}(\Gamma^{R} + \hat{\Gamma}) = -\frac{1}{2} Tr \left( \rho \phi (\Gamma^{R} + \hat{\Gamma}) \right) = 1. \] (3.39)

This result agrees with the monopole charge $Q$ in the commutative theory which was calculated in (3.20) in the previous subsection. We have thus shown that the non-trivial configuration (3.1) can be interpreted as the TP monopole configuration, and the topological charge (3.31) for this configuration gives the correct value.

### 3.3 Higher isospin

In this subsection we consider a configuration coupled to a fermion with higher isospin $T^{[5, 6, 20]}$

\[ a_i = \frac{1}{\rho} 1_{2L+1} \otimes T_i, \] (3.40)

where $T_i$'s are the $2T + 1$ dimensional representation of $SU(2)$ algebra. Since the combination

\[ A_i = L_i^L \otimes 1_{2T+1} + 1_{2L+1} \otimes T_i \] (3.41)
satisfies $SU(2)$ algebra, it can be decomposed into the irreducible representations of $SU(2)$ as

$$A_i \simeq \begin{pmatrix}
L_i^{(1)} \\
& L_i^{(2)} \\
& & \ddots \\
& & & L_i^{(2T+1)}
\end{pmatrix}$$  \hspace{1cm} (3.42)

where $L_i^{(a)}$'s ($a = 1, \cdots, 2T+1$) denote the $L^{(a)} = L + T + 1 - a$ representations respectively.

Projectors to pick up each Hilbert space on which $L_i^{(a)}$ acts can be defined as in (3.22)-(3.25),

$$P^{(a)} = \prod_{b \neq a} \frac{(A_i)^2 - L^{(b)}(L^{(b)} + 1)}{L^{(a)}(L^{(a)} + 1) - L^{(b)}(L^{(b)} + 1)}.$$  \hspace{1cm} (3.43)

Then we define the same topological charge as (3.31), and obtain the same result as (3.37).

Both of the configurations (3.2)-(3.41) satisfy the $SU(2)$ algebra (3.3). Since it is essential in the calculations of the topological charge (3.31), we here study its meaning. By inserting (2.11) into (3.3), taking the commutative limit, using (2.15), and decomposing $a_i$ into $a'_i$ and $\phi$ by (3.7), we obtain

$$\epsilon_{ilm}\epsilon_{jpq}n_l n_p F_{mq} + (n_i \epsilon_{jlm} - n_j \epsilon_{ilm}) n_l (D_m \phi) + \frac{1}{\rho} \epsilon_{ijk} n_k \phi = 0.$$  \hspace{1cm} (3.44)

Since (3.44) is a second rank antisymmetric tensor with indices $i,j$, we can decompose it into normal-tangential and tangential-tangential components on the sphere, and obtain

$$\epsilon_{ijk} n_j (D_k \phi) = 0,$$  \hspace{1cm} (3.45)

$$\epsilon_{ijk} n_i F_{jk} + \frac{2}{\rho} \phi = 0.$$  \hspace{1cm} (3.46)

Eq. (3.45) means that the tangential component of $D_i \phi$ vanishes, while (3.46) means that the normal component of the flux is equal to the scalar field. We see
that the finite energy condition \((3.16)\) is satisfied by \((3.45)\) since \(\phi = \phi'/\rho\) and \((\phi'_a)^2 = 1\). Also, by \((3.46)\) the topological charge \((3.32)\) becomes
\[
-2 \int \frac{d\Omega}{4\pi} \text{tr}[P^{(a)}\phi'].
\]
(3.47)
For \(T = 1/2\) case, by using \((3.24)\) \((3.27)\) for \(P^{(a)}\), we obtain the same result as \((3.38)\) \((3.39)\). For \(T \geq 1\), \(P^{(a)}\) is given by \((3.43)\). By using \((2.13)\), taking commutative limit of \((3.43)\), and inserting it into \((3.47)\), we can obtain the same result as \((3.37)\).

We now consider the commutative limit of the configuration \((3.40)\). We regard it as a configuration in the \(SU(2)\) gauge theory coupled with the fermion in \(2T + 1\) dimensional representation, although we could also regard it as a configuration in the \(SU(2T + 1)\) gauge theory coupled with the fermion in the fundamental representation. Then we obtain the same commutative limit as we did in subsection \(3.1\) Eqs. \((3.8)\) \((3.9)\) imply that the configuration \((3.40)\) has the winding number of 1.

We now define another topological charge:
\[
\frac{1}{2} \mathcal{T}_R[\phi'(\Gamma^R + \hat{\Gamma})],
\]
(3.48)
where we insert \(\phi'\) of \((2.13)\) instead of inserting the projection operator as in \((3.31)\). On the fuzzy 2-sphere it is evaluated as
\[
\sum_{a=1}^{2T+1} \frac{L^{(a)}(L^{(a)} + 1) - L(L + 1)}{2L + 1} 2(L - L^{(a)}) = \frac{2}{3} T(T + 1)(2T + 1)
\]
(3.49)
for the configuration \((3.40)\). On the other hand, by taking the weak gauge limit and the commutative limit, this becomes
\[
\frac{1}{2} \mathcal{T}_R[\phi'(\Gamma^R + \hat{\Gamma})] \rightarrow \rho^2 \int \frac{d\Omega}{4\pi} \text{tr}[\phi' \epsilon_{ijk} n_i F_{jk}],
\]
(3.50)
and then, by using \(SU(2)\) condition \((3.46)\), it is calculated as
\[
-2 \int \frac{d\Omega}{4\pi} \text{tr}[\phi'^2] = \frac{2}{3} T(T + 1)(2T + 1),
\]
(3.51)
which agrees with (3.49). Also, for the configurations (3.10), (3.50) becomes

\[-\frac{2}{3}T(T+1)(2T+1)\rho^2\int_{S^2} d\Omega \epsilon_{ijk} n_i \epsilon^{abc} \phi'^a (\partial_j \phi'^b)(\partial_k \phi'^c), \quad (3.52)\]

which is the winding number of $\pi_2(SU(2)/U(1))$. Comparing with the above values, we see that the winding number is 1 for the configuration (3.40), and for configurations which satisfy the $SU(2)$ algebra (3.3) in general.

We thus obtain a configuration with winding number 1, by suitably combining the coordinates of fuzzy 2-sphere and the gauge configuration, both of which satisfy the $SU(2)$ algebra. It will be an interesting attempt to construct configurations with general winding numbers by generalizing the procedure.

### 3.4 Other configurations

In this subsection we consider other configurations.

First we consider the case where the fermion is in $m$ dimensional representation of the gauge group, for example, $U(m)$ gauge theories coupled with the fermion in the fundamental representation. Then the Hilbert space on which $A_i$ acts is $m(2L+1)$ dimensional. We here assume that $A_i$ satisfy the $SU(2)$ algebra. Then $A_i$ is decomposed into irreducible representations as

\[A_i \simeq \begin{pmatrix}
L_i^{(1)} \\
L_i^{(2)} \\
\vdots \\
L_i^{(r)}
\end{pmatrix}, \quad (3.53)\]

where $L_i^{(a)}$'s are $L^{(a)}$ representations of $SU(2)$ for $a = 1, \cdots, r$. The projection operator to pick up the Hilbert space on which $L_i^{(a)}$ acts is defined to be (3.43). Then we have the same topological charge as (3.31) and the same result as (3.37).
Since \( \sum_{a=1}^{r} (2L^{(a)} + 1) = m(2L + 1) \), we have
\[
\frac{1}{2} Tr(\Gamma + \hat{\Gamma}) = \frac{1}{2} \sum_{a=1}^{r} Tr P^{(a)}(\Gamma + \hat{\Gamma}) \\
= \sum_{a=1}^{r} 2(L - L^{(a)}) \\
= (r - m)(2L + 1),
\]
which means that the topological charge without the projection operator gives a multiple number of \( 2L + 1 \).

Next we consider a NC analogue of \( U(1) \) Dirac monopoles. Let’s consider the configuration
\[
A_i \simeq \begin{pmatrix} L_i^{(1)} \\ c_i^{(1)} \\ \vdots \\ c_i^{(s)} \end{pmatrix},
\]
where \( L_i^{(1)} \) is \( L^{(1)} \) representation of \( SU(2) \) algebra, and \( c_i^{(a)} \)'s (\( a = 1, \cdots, s \)) are some numbers (\( 1 \times 1 \) matrices). We note here that the \( A_i \) does not satisfy the \( SU(2) \) algebra, but does satisfy the equation of motion in the reduced model of 3-dimensional Yang-Mills theory with the Chern-Simons term\[\text{II}]: [A_i, [A_i, A_j]] = -i\varepsilon_{jkl} [A_k, A_l]. \) The topological charge is calculated as
\[
\frac{1}{2} Tr_{(1)}(\Gamma + \hat{\Gamma}) = 2(L - L^{(1)}) = s.
\]
We may be able to regard this solution as “D2 with \( s \) D0’s”\[\text{30}], or “Dirac monopole with \( s \) Dirac strings”\[\text{31}]. But in the latter interpretation there are several problems which should be resolved.

4 Discussion

In this paper, after we briefly review the construction of GW Dirac operator and chirality operators on fuzzy 2-sphere \[\text{17}], we further investigate several topological properties. First we calculated the chiral anomaly from the non-trivial
Jacobian up to the second order in gauge fields and showed that the correct form of the Chern character is reproduced in the commutative limit. This result completed the first order calculation of the chiral anomaly in \[17\]. Thus we saw that the topological charge we defined on the fuzzy sphere has the correct commutative limit.

We then studied topologically nontrivial configurations. Even in the theories on the commutative sphere, if we naively integrate the topological charge density over the sphere, the topological charge vanishes identically for any configurations. In the \'t Hooft-Polyakov (TP) monopole, we introduce the idea of spontaneous symmetry breaking, insert the projector to pick up the unbroken $U(1)$ component into the topological charge, and obtain nonzero values for it. These idea and procedure correspond to introducing some projections into matrices in the noncommutative (NC) theories. We considered the TP monopole configurations and their topological charges on the fuzzy sphere, and showed that they have the correct commutative limit. We further studied other nontrivial configurations and their topological charges.

Projections into matrices may be necessary not only for describing the topologically nontrivial configurations, but for describing the smooth gauge configurations themselves, and the smooth space-time itself. Since the denominator of $\hat{\Gamma}$ is given by $\sqrt{H^2}$, we need to avoid zero eigenvalues of $H$, which corresponds to the admissibility conditions in the lattice gauge theories. Also, since space-time and field-theoretical degrees of freedom are considered to be embedded in the near-diagonal elements of the matrices, we need to project these elements from the full matrices to describe smooth field-configurations and space-time. Studies of embeddings of classical configurations in matrices \[32,33\] may be useful for this study.

It is also interesting to construct GW fermions on various NC geometries, such as NC torus \[34,35\] and NC $CP^n$, and consider the above mentioned problems in these models. Also, it may be interesting to study the Seiberg-Witten map \[36\] on
fuzzy 2-sphere\textsuperscript{37} to investigate nontrivial configurations and topological charges in commutative and NC theories.

We expect that our formalism can provide a clue for classifying the space-time topology as well as the topology of gauge field space, since space-time and matter are indivisible in matrix models or NC field theories. We hope that our formalism based on the GW relation will have important roles for considering topological structures of space-time and matter in these theories.

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\section{A Proof of Index Theorem}

In this appendix we prove the index theorem (2.28). We defined the chirality operator and the GW Dirac operator more generally in [17], so here we prove the index theorem in this general formulation. It is much easier to prove it in the formulation of this paper.

We defined in [17] the GW Dirac operator $D_{GW}$ as

$$f(a,\Gamma)D_{GW} = 1 - \Gamma \hat{\Gamma}, \quad (A.1)$$

where $\Gamma$ and $\hat{\Gamma}$ are the generalization of $\Gamma^R$ and $\hat{\Gamma}$ of this paper, which are Hermite, satisfy $\Gamma^2 = \hat{\Gamma}^2 = 1$, and become the commutative chirality operator in the commutative limit. The prefactor $f(a,\Gamma)$ can be any function of $a$ and $\Gamma$, but must be of the order of $a$, and must be invertible. Hermite conjugate of (A.1)
gives

\[ D_{GW}^\dagger f(a, \Gamma)^\dagger = 1 - \hat{\Gamma}. \]  
(A.2)

By multiplying (A.1) by \( \Gamma \) from the left, multiplying another (A.1) by \( \hat{\Gamma} \) from the right, and summing them, we obtain the following GW relation:

\[ \Gamma D_{GW} + D_{GW} \hat{\Gamma} = 0, \]
(A.3)

\[ D_{GW}^\dagger \Gamma + \hat{\Gamma} D_{GW}^\dagger = 0. \]
(A.4)

Also, since (A.1) multiplied by \( \Gamma \) both from the left and the right becomes (A.2),

\[ D_{GW}^\dagger = \Gamma f(a, \Gamma) D_{GW} \Gamma f(a, \Gamma)^\dagger^{-1}, \]
(A.5)

\[ D_{GW} = f(a, \Gamma)^{-1} \Gamma D_{GW}^\dagger f(a, \Gamma)^\dagger \Gamma. \]
(A.6)

Similarly, by multiplying (A.1) by \( \hat{\Gamma} \), we obtain

\[ D_{GW}^\dagger = \hat{\Gamma} f(a, \Gamma) D_{GW} \hat{\Gamma} f(a, \Gamma)^\dagger^{-1}, \]
(A.7)

\[ D_{GW} = f(a, \Gamma)^{-1} \hat{\Gamma} D_{GW}^\dagger f(a, \Gamma)^\dagger \hat{\Gamma}. \]
(A.8)

Now we introduce Fock space \( \mathcal{H} \) for the spinor matrices \( \psi \). In the case of 2-sphere, \( \mathcal{H} \) is an ensemble of \((2L + 1) \times (2L + 1)\) Hermitian matrices. The above GW Dirac operator \( D_{GW} \) and the chirality operators \( \Gamma, \hat{\Gamma} \) act on this space. We then decompose \( \mathcal{H} \) into the spaces of zero and nonzero eigenmodes for \( D_{GW} \), and also for \( D_{GW}^\dagger \):

\[ \mathcal{H} = \mathcal{H}_0 \oplus \bar{\mathcal{H}}_0, \]
(A.9)

\[ = \mathcal{H}_0' \oplus \bar{\mathcal{H}}_0', \]
(A.10)

where

\[ \mathcal{H}_0 = \{ \psi \in \mathcal{H} | D_{GW} \psi = 0 \}, \]
(A.11)

\[ \mathcal{H}_0' = \{ \psi \in \mathcal{H} | D_{GW}^\dagger \psi = 0 \}, \]
(A.12)

and \( \bar{\mathcal{H}}_0, \bar{\mathcal{H}}_0' \) are their complementary space.
First we show the following statement:

\[ H_0 = H'_0, \quad \bar{H}_0 = \bar{H}'_0, \]

\[ \psi \in H_0 = H'_0 \Rightarrow \Gamma \psi = \hat{\Gamma} \psi \in H_0 = H'_0. \tag{A.13} \]

Its proof is as follows: If \( \psi \in H_0 \), then due to (A.3), \( \hat{\Gamma} \psi \in H_0 \). (A.1) leads to \( \Gamma \psi = \hat{\Gamma} \psi \). Then \( \Gamma \psi \in H_0 \). Thus, due to (A.5), \( D_{GW}^\dagger \psi = 0 \), and then \( \psi \in H'_0 \). Hence, \( H_0 \subset H'_0 \). Similarly, if \( \psi \in H'_0 \), then due to (A.4), \( \Gamma \psi \in H'_0 \). Thus, due to (A.6), \( D_{GW} \psi = 0 \), and then \( \psi \in H_0 \). Hence, \( H'_0 \subset H_0 \). Therefore \( H'_0 = H_0 \). \( \bar{H}'_0 = \bar{H}_0 \) is its contraposition.

Next we show

\[ \psi \in \bar{H}_0 = \bar{H}'_0 \Rightarrow \Gamma \psi, \hat{\Gamma} \psi \in \bar{H}_0 = \bar{H}'_0. \tag{A.14} \]

We prove its contraposition as follows: If \( \Gamma \psi \in H'_0 \), \( D_{GW}^\dagger \Gamma \psi = 0 \). Then, due to (A.4), \( \hat{\Gamma} D_{GW}^\dagger \psi = 0 \). Multiplying it by \( \hat{\Gamma} \) from the left we have \( D_{GW}^\dagger \psi = 0 \). Thus \( \psi \in H'_0 \). Similarly, if \( \hat{\Gamma} \psi \in \bar{H}_0 \), \( D_{GW} \hat{\Gamma} \psi = 0 \). Then, due to (A.3), \( \Gamma D_{GW} \psi = 0 \). Multiplying it by \( \Gamma \) from the left we have \( D_{GW} \psi = 0 \). Thus \( \psi \in H_0 \).

Finally,

\[ \psi \in \bar{H}_0 = \bar{H}'_0 \Rightarrow \Gamma \psi, \hat{\Gamma} \psi \text{ anti-pairing} \tag{A.15} \]

since if \( \Gamma \psi = \pm \psi \), then due to (A.4), \( \hat{\Gamma}(D_{GW}^\dagger \psi) = -D_{GW}^\dagger \Gamma \psi = \mp(D_{GW}^\dagger \psi) \). Similarly, if \( \hat{\Gamma} \psi = \pm \psi \), then due to (A.3), \( \Gamma(D_{GW} \psi) = -D_{GW} \hat{\Gamma} \psi = \mp(D_{GW} \psi) \).

From (A.13), (A.14), and (A.15), we can prove the index theorem:

\[ Tr(\Gamma + \hat{\Gamma}) = Tr_{H_0}(\Gamma + \hat{\Gamma}) + Tr_{\bar{H}_0}(\Gamma + \hat{\Gamma}) \]

\[ = Tr_{H_0}(\Gamma + \hat{\Gamma}) \]

\[ = 2(n_+ - n_-) \]

\[ = 2 \text{ Index}(D_{GW}). \tag{A.16} \]
B Proof of $\delta \left( \mathcal{T} r \hat{\Gamma} \right) = 0$

In this appendix we show that if we define $\hat{\Gamma}$ as in (2.17), $\mathcal{T} r (\hat{\Gamma})$ is invariant under any small deformation of any parameter or any configuration such as gauge field in the operator $H$. Under the infinitesimal deformation of $H$, $H \rightarrow H + \delta H$, $\mathcal{T} r (\hat{\Gamma})$ varies as

$$
\delta \left( \mathcal{T} r \hat{\Gamma} \right) = \delta \left( \mathcal{T} r H \frac{1}{\sqrt{H^2}} \right) \\
= \mathcal{T} r \left( \delta H \frac{1}{\sqrt{H^2}} \right) + \mathcal{T} r \left( H \frac{1}{\sqrt{(H + \delta H)^2}} \right) - \mathcal{T} r \left( H \frac{1}{\sqrt{H^2}} \right).
$$

The second term can be evaluated as

$$
\mathcal{T} r \left( H \frac{1}{\sqrt{(H + \delta H)^2}} \right) \\
= \mathcal{T} r \left[ H \int_{-\infty}^{\infty} \frac{dt}{\pi \ t^2 + H^2} \frac{1}{\sqrt{H + \delta H} + \sqrt{H + (\delta H)^2}} \right] \\
= \mathcal{T} r \left[ H \int_{-\infty}^{\infty} \frac{dt}{\pi \ t^2 + H^2} \frac{1}{\sqrt{(H + \delta H)H + (\delta H)^2}} \right] + O((\delta H)^2).
$$

(B.1)

where we have utilized the cyclic property in $T \mathcal{r}$ and the following identities,

$$
\frac{1}{\sqrt{X^2}} = \int_{-\infty}^{\infty} \frac{dt}{\pi \ t^2 + X^2},
$$

(B.2)

$$
\frac{1}{2X^2 \sqrt{X^2}} = \int_{-\infty}^{\infty} \frac{dt}{\pi \ (t^2 + X^2)^2}.
$$

(B.3)

Therefore we obtain

$$
\delta \left( \mathcal{T} r \hat{\Gamma} \right) = 0.
$$

(B.4)

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