I. INTRODUCTION

Contrary to intuition, two fully noisy quantum channels can still transmit classical information when they are combined in a superposition of spatial or temporal trajectories, e.g., in a superposition of causal orders where the order of application of the channels is indeterminate \[^1\]. This counter-intuitive effect is impossible to achieve using either one channel alone or a cascade of such fully noisy channels in a definite causal order. Causal activation of communication was invoked to explain this result and also demonstrated to enable transmission of quantum information \[^2\], even with a zero capacity channel \[^3\]. Experimental realizations have recently tested this new activation of communication \[^4\] \[^5\]. In addition, the transmission of classical information has also been evidenced controlling the path followed by the target system through one of two fully noisy channels with a quantum superposition \[^5\]. The physical origin of each kind of communication enhancement through quantum coherent control is thus currently the matter of stimulating discussions \[^7\] \[^8\]. Interestingly, a generalization of quantum Shannon theory was proposed in \[^7\] \[^8\] encompassing both origins for enhancement. The well-established quantification of the internal degree of freedom of the information and/or channels is presented as a first level. The quantification of external degree of freedom, i.e. connections between channels, either through superposition of causal orders or superposition of paths, is considered as a second quantization level of the quantum Shannon theory of information.

As an example of superposition of causal orders, the operation known as quantum switch has been initially designed by Chiribella et al \[^9\]. This primitive has subsequently been theoretically proposed as a novel resource for applications to quantum information theory \[^10\] \[^11\], quantum communication complexity \[^12\], quantum communication \[^2\] \[^13\], non-local games \[^14\] and quantum metrology \[^15\] \[^16\]. Moreover, the quantum switch has been implemented experimentally \[^13\] \[^17\] \[^20\]. In the quantum switch, a target system \(\rho\) undergoes a superposition of two different causal orders of application of two quantum channels. A control system \(\rho_c\) is used to route target system; the state \(\rho_c = |1\rangle \langle 1|\) encodes for order where channel one is applied before channel two while the state \(\rho_c = |2\rangle \langle 2|\) encodes for channel one after channel two. By placing \(\rho_c\) in superposition, i.e. \(\rho_c = |+\rangle \langle +|\), where \(|+\rangle_c = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle)\), \(\rho\) shall experience both orders simultaneously.

By increasing the number \(N\) of quantum channels, the number of possible causal orders increases as \(N!\). This, in turn, provides a large number of indeterminations, i.e. superposition of causal orders. The partial or total use of indeterminations in a quantum \(N\)-switch becomes a rich quantum resource, as we demonstrate in this paper for \(N = 3\). Indeed, this wealth of combinations of causal orders has been not yet exploited as it does not appear when only two channels are used in a quantum 2-switch. In a recent work \[^13\], we presented a general procedure to study the transmission of classical information through \(N\) quantum noisy channels using the quantum \(N\)-switch. We use it here the transmission of information beyond the two-channel paradigm resulting from fine quantum control of three noisy channels. We uncover new features of the transmission of information. Furthermore, the \(N = 3\) case is attainable with nowadays technologies and we sketch an optical implementation.

The paper is organized as follows. Section \[^11\] introduces...
the quantum 3-switch. It gives the necessary formalism and results to investigate the transmission of classical information by three fully noisy quantum channels. We follow the formalism described in Ref. [13] and obtain the quantum 3-switch matrix to calculate the Holevo information, which quantifies transmission of information [21]. In Section IV, we discuss the possible combinations of causal order that one can use to transmit information with three channels, and we analyze the effects on the transmission of information by superimposing $m$ causal orders, where $m = 1, 2, \ldots, 3!$. We further get and compare the Holevo information as a function of the number of causal orders involved in the superposition. In Section V, we suggest an optical implementation for the quantum 2-switch and quantum 3-switch to test our predictions. Finally, in Section V, we give conclusions and perspectives of our work.

II. TRIPARTITE QUANTUM SWITCH

Three parties A, B and C can be in $\sum_{m=2}^{N!} \binom{N!}{m}$ superpositions of $m$ causal orders with $N = 3$, where $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ is the binomial coefficient. Figure 1 describes possible superpositions of causal orders. In the context of quantum communications, A, B and C are three quantum noisy channels $N_1$, $N_2$ and $N_3$ respectively. We model the action of a noisy channel $N$ on a qudit (d-dimensional) system $\rho$ as a depolarising quantum channel [22]

$$N(\rho) = q\rho + (1-q)\text{Tr}[\rho] \frac{1}{d},$$

where $\frac{1}{d}$ is the identity operator, which represents a maximally mixed state for the target system when the channel is completely depolarizing, i.e., $q = 0$. In this work, we will focus on the transmission of information through completely depolarizing channels. We use the Kraus decomposition $N(\rho) = \sum K_i \rho K_i^\dagger$ to mathematically represent the action of a channel $N$ on the quantum state $\rho$ such that $\sum K_i K_i^\dagger = \frac{1}{d}$ [22].

For three noisily channels $N_1$, $N_2$ and $N_3$, the control system $\rho_c = \left\ket{\psi_c} \bra{\psi_c}\right|$ coherently controls the target system $\rho$ via $3!$ states, see Fig. IV top. The quantum state of the control system writes

$$\left|\psi_c\right> = \sum_{k=1}^{d} \sqrt{P_k} \left|k\right>$$

where $P_k$ is the probability associated to the definite causal order $\left|k\right> \bra{k}$ such that $\sum_{k=1}^{d} P_k = 1$. If $\rho_c$ is at the state $\left|1\right>$, then the order to apply the channels will be $N_1 \circ N_2 \circ N_3$. Likewise, if $\rho_c$ is on the states $\left|2\right>$, $\left|3\right>$, $\left|4\right>$, $\left|5\right>$ or $\left|6\right>$, the orders will be $N_1 \circ N_3 \circ N_2$, $N_2 \circ N_1 \circ N_3$, $N_2 \circ N_3 \circ N_1$, $N_3 \circ N_1 \circ N_2$ and $N_3 \circ N_2 \circ N_1$ respectively. Setting the control state as in equation (2), yields a superposition of several causal orders with their respective weights $P_k$. We refer to this type of superposition as the quantum 3-switch (Q3S) which is an extension of the quantum 2-switch (Q2S).

If the Kraus operators of the channels $N_1$, $N_2$ and $N_3$ are $\{K_i^{(1)}\}$, $\{K_i^{(2)}\}$ and $\{K_i^{(3)}\}$ respectively, then the Kraus operators $W_{ijk}$ of the full quantum 3-switch channel is

$$W_{ijk} = K_i^{(1)} K_j^{(2)} K_k^{(3)} \left|1\right> \bra{1} + K_i^{(1)} K_j^{(3)} K_k^{(2)} \left|2\right> \bra{2} + K_i^{(2)} K_j^{(1)} K_k^{(3)} \left|3\right> \bra{3} + K_i^{(2)} K_j^{(3)} K_k^{(1)} \left|4\right> \bra{4} + K_i^{(3)} K_j^{(1)} K_k^{(2)} \left|5\right> \bra{5} + K_i^{(3)} K_j^{(2)} K_k^{(1)} \left|6\right> \bra{6}. \quad (3)$$

The action of the quantum 3-switch $S(N_1, N_2, N_3)$ over an input $\rho \otimes \rho_c$ can be expressed through the Kraus operators $W_{ijk}$ as

$$S(N_1, N_2, N_3)(\rho \otimes \rho_c) = \sum_{ijk} W_{ijk}(\rho \otimes \rho_c) W_{ijk}^\dagger. \quad (4)$$

Following the procedure described in Ref. [13], we found the quantum 3-switch matrix is a $6 \times 6$ block-symmetry matrix

$$S = \begin{pmatrix} A_1 & B & C & D & E & F \\ B & A_2 & G & H & I & J \\ C & G & A_3 & K & L & M \\ D & H & K & A_4 & N & P \\ E & I & L & N & A_5 & Q \\ F & J & M & P & Q & A_6 \end{pmatrix}, \quad (5)$$

where $S \equiv S(N_1, N_2, N_3)(\rho \otimes \rho_c)$, diagonal and off-diagonal elements are $d \times d$ matrices, linear combinations of the identity matrix $\frac{1}{d}$ and the target system is the density matrix $\rho$, see Appendix A of Ref. [13]. This quantum 3-switch matrix [5] provides the output state of the quantum switch as a function of all the parameters involved: the dimension of the target system $d$, the depolarization strengths $q_i$‘s, the probabilities $P_k$ and the density matrix $\rho$.

We compute the Holevo information $\chi(S)$ of the full quantum 3-switch channel $S(N_1, N_2, N_3)$ through a generalization of the mutual information, see Refs [1, 23]. $\chi(S)$ is found by maximizing the mutual information, and it can be shown that maximization over pure states $\rho$ for the target system is sufficient [23]. The Holevo information is then given by

$$\chi(S) = \log d + H(\hat{\rho}_c^{(3)}) - H^{\text{min}}(S) \quad (6)$$

where $d$ is the dimension of the target system, $H^{\text{min}}(S)$ is the minimum of the entropy over pure states $\rho$ at the output of the channel $S$. $H(\hat{\rho}_c^{(3)})$ is the von-Neumann entropy of the output state of the control system $\hat{\rho}_c^{(3)}$ for three channels which was found in Ref. [13] Eq. (44). The diagonalization and minimization of $H^{\text{min}}(S)$ are performed numerically.
III. FINE QUANTUM CONTROL OF $N = 3$ CHANNELS

A. Information transmission for full and partial superpositions

As the number of channels increases, the number of possible causal orders increases as well: 2 for $N = 2$, 6 for $N = 3$, following a $N!$ law. This in turn increases the number of possible superpositions of combinations. We analyze here in details the Holevo information with respect to these superpositions in the case of three channels. Note that this constitutes a new issue, since for $N = 2$ there is no room to play with superposition of different causal orders.

Each definite causal order is associated to the control state $|k\rangle$ with probability $P_k$. We consider all possible superposition of different causal orders with equally weighted probabilities $P_k$, i.e., for each superposition of $m = \{1;N = 6\}$ causal orders, we fix $m$ probabilities $P_k$ to $\frac{1}{m}$ and the rest of $P_k$'s to zero. We restrict our analysis to the case in which the three channels are completely depolarizing, i.e., $q_1 = q_2 = q_3 = 0$.

There are $\frac{3!}{2}$ possible superpositions of $m$ different causal orders for three channels. For $m = 1$ causal order, there are six possible configurations, each in a specific definite causal order. We label each causal order, depending on $\rho_c$, we have $3^3$ possibilities to combine the channels in a definite causal order. Figures from (b) to (e) illustrate a single partial superposition among all possible combinations of causal orders of Table III, see main text.

FIG. 1: Tripartite quantum switch. In the quantum switch with three parties, A, B and C, there are $\binom{3}{m}$ (with $m = 1, 2, \ldots, 6$) available superpositions of causal orders. Here we represent A, B and C as three different quantum channels $N_1$, $N_2$ and $N_3$ respectively. $\rho_c = |1\rangle \langle 1|$ encodes a causal order $N_1 \circ N_2 \circ N_3$, i.e., $N_3$ is applied first to the target system $\rho$; $\rho_c = |2\rangle \langle 2|$ encodes $N_1 \circ N_3 \circ N_2$; $\rho_c = |3\rangle \langle 3|$ encodes $N_2 \circ N_1 \circ N_3$; $\rho_c = |4\rangle \langle 4|$ encodes $N_2 \circ N_3 \circ N_1$; $\rho_c = |5\rangle \langle 5|$ encodes $N_3 \circ N_1 \circ N_2$; $\rho_c = |6\rangle \langle 6|$ encodes $N_3 \circ N_2 \circ N_1$; finally, if $\rho = |+\rangle_m \langle +_m|$, where $|+\rangle_m = \frac{1}{\sqrt{m}} \sum_{k=1}^{m} |k\rangle$ is a superposition of $m$ different causal orders. (a) For $m = 6$ causal orders, there is only 1 combination to superimpose six definite causal orders. (b) For $m = 5$ causal orders, there are $\binom{5}{m} = 6$ combinations to superimpose five definite causal orders. (c) For $m = 4$ causal orders, there are $\binom{4}{m} = 15$ combinations to superimpose four definite causal orders. (d) For $m = 3$ causal orders, there are $\binom{3}{m} = 20$ combinations to superimpose three definite causal orders. (e) For $m = 2$ causal orders, there are $\binom{2}{m} = 15$ combinations to superimpose two definite causal orders. (f) For $m = 1$ causal order, depending on $\rho_c$, we have $3^3$ possibilities to combine the channels in a definite causal order. Figures from (b) to (e) illustrate a single partial superposition among all possible combinations of causal orders of Table III, see main text.
FIG. 2: **Superposition of m causal orders.** Holevo information $\chi_{Q3S}$ as the number of causal orders $m$ involved in $\rho_c$ is varied, for dimension $d = 2$ (blue) and $d = 3$ (red) of $\rho$. The Holevo information can take 2 different values $\chi_{\max}$ and $\chi_{\min}$ when there is a superposition of $m = 2, 3$ and $4$ causal orders, depending on the chosen causal orders in the superposition. $\chi_{\max}$ is indicated by the top end of each color bar (for each $d = 2, 3$) and $\chi_{\min}$ is given by the position of the clear horizontal line bar with a colored centered dot. The numerical values of $\chi_{\max}$ and $\chi_{\min}$ are given in the Table II. For a fixed number of causal order $m$, the Holevo information decreases as $d$ increases. Note that for superposition of two causal orders, $\chi_{\max}$ is equal to the value of Holevo information of two channels $\chi_{Q2S}$. See main text for explanation. The two triangles correspond to the values obtained in Ref. [13].

$d = 3$ (red). Table II gives the possible combinations of $m$ causal orders related to the values of Fig. 2.

The main findings in Fig. 2 are:

- For a fixed dimension $d$, the transmission of information mostly increases as the number $m$ of involved causal orders increases.

- This behavior is nonetheless not strictly monotonous. It is therefore unnecessary to waste resource to go from the $m = 3$ to the $m = 4$ case as the 3-order-combination achieves better information transmission.

- The analysis of limited number of combinations is a novelty that could not be studied in the previous $N = 2$ works. For $m = 2$, $m = 3$ and $m = 4$, the possible combinations fall into 2 categories whether they transmit $\chi_{\min}$ or $\chi_{\max}$, $\chi_{\min} < \chi_{\max}$, see Table II that exhibits the values of Holevo information for $m = 1, 2, \ldots, 6$ causal orders and distinct values of $d$, the dimension of the target state.

In more details and for increasing $m$,

*Case m = 1.* In the case of $m = 1$ the Holevo information $\chi_{Q3S}$ reduces to that of a definite causal order scheme, i.e. no information can be extracted.

*Case m = 2.* For $m = 2$ the 15 possible combinations are evaluated to be one of two values $\chi_{\min} = 0$ and $\chi_{\max}$, the maximal one $\chi_{\max}$ is endorsed by 6 superpositions of different combinations (see Table II) and it coincides with the Holevo information obtained exploiting fully the two channel configuration i.e., $\chi_{\max} = \chi_{Q2S}$ ($q_i = 0$).[II]

*Case m = 3.* By increasing the causal order resource exploitation for the three channels case to $m > 2$, the Holevo information is enhanced. Note that for $m = 3$ (20 possible selections for the superpositions involved in the control states), superposition activates information transmission for all combinations of causal orders, and transmission is maximal for two specific combinations for which the transmitted information is 1.67 times larger than the transmitted information using two channels. It is interesting to notice that it is possible to surpass the bound in the transmission of information for the quantum 2-switch, combining three causal orders instead of involving all causal orders in the quantum 3-switch. From the experimental point of view, this can help reducing the complexity of implementations.

*Case m = 4.* For $m = 4$, the Holevo information is smaller than those combinations of $m = 3$ where the transmitted information is maximum.

*Case m = 5.* The two values collapse into a single one for the 6 possible combinations associated to $m = 5$.

*Case m = 6.* Remarkably, when the 3-channel resources are fully exploited, for the single equally weighted combination of the $m = 6$ case, the Holevo information is approximately two times that of the two channel configuration up to $d = 10$ as it was found in Ref. [13].

**B. Realistic quantum switch and efficiency of transmission of local and global combinations**

The case $m = 2$ can be understood as follows. For those combinations of causal orders where superposition activation is on, i.e. $\chi_{Q3S} = \chi_{\max}$, the quantum 3-switch is switching globally all channels, i.e. all channels are combined in such a way that they all have changed positions in the ordering. In the particular case when there is information transmission $\chi_{Q3S} = \chi_{\max} \neq 0$, the quantum 3-switch is in fact switching only two channels: one channel $\mathcal{N}_i$ and another composite channel $\mathcal{N}_{jk} = \mathcal{N}_j \circ \mathcal{N}_k$. The quantum 3-switch thus indeed behaves as an effective quantum 2-switch. Hence $\chi_{Q3S}(m = 2)$ is equal to $\chi_{Q2S}$.

| $m$ | $\chi_{\max}$ | $\chi_{\min}$ |
|-----|---------------|---------------|
|     | $d = 2$      | $d = 3$      | $d = 2$      | $d = 3$      |
| 1   | 0             | 0             | 0             | 0             |
| 2   | 0.0487        | 0.0183        | 0             | 0             |
| 3   | 0.0817        | 0.0325        | 0.0333        | 0.0122        |
| 4   | 0.0640        | 0.0246        | 0.0524        | 0.0186        |
| 5   | 0.0766        | 0.0275        | -             | -             |
| 6   | 0.0980        | 0.0339        | -             | -             |

TABLE I: $\chi_{\max}$ and $\chi_{\min}$ values of the Holevo information of Figure 2 for all causal orders in $N = 3$. 


In contrast, for those combinations where $\chi_{Q3S} = \chi_{\min} = 0$, the quantum 3-switch is switching locally only two individual channels $N_j$ and $N_k$ instead of globally switching all channels. Indeed the $\chi_{Q3S} = \chi_{\min} = 0$ case is interesting to shed light on the seminal quantum 2-switch scheme. In fact, QSS incorporates an implicit channel linking the output of $N_2$ to the input of $N_1$. This identity channel has no loss in the $N = 2$ picture as it would be otherwise a third channel. The transmission of information reported in Ref. [11] for the Q2S could be attributed to this implicit channel. Our results for $N = 3$ give the proper framework to make this third channel explicit. If this identity channel is a fully depolarizing channel, our result for $N = 3$ and the corresponding $m = 2$ case shows that indeed no information is transmitted.

The behavior of Fig. 2 can be summarized by noticing that the more the quantum switch switches channels, the more information is transmitted. It seems that the $m$ dependence of the Holevo information can be tracked back comparing the number of the combinations in the control state and their nature that we label local and global. As detail below Table II illustrates our global and local switching denomination. It gives the explanation for the Holevo values obtained from selected combinations switching three channels with $m = 3$ and $m = 4$ orderings. We shall present a more formally proof using properties of the quantum switches matrix in a future work.

For $m = 3$, the combination $P_3P_1P_2$ yields a low value for the Holevo information $\chi_{\min}(m = 3)$. In 2 subsets of 2 causal orders we only locally switch the channels. We highlight this by putting colors on the fixed points which indicate this local switching. This $P_3P_1P_2$ combination only has 1 subset of 2 causal orders $\{P_3, P_2\}$ that globally switch the channels without fixed point. In contrast, in the $P_1P_3P_5$ superposition there are 3 subsets of 2 causal orders to globally exchange all channels thus yielding $\chi_{\max}(m = 3)$. The global switching has no fixed points.

For $m = 4$, the combination yielding $\chi_{\min}(m = 4)$ has 4 highlighted fixed points, i.e., 4 pairs of local switching, or 2 pairs that globally switch the channels among the possible 6 pairs. While there are only 3 fixed points and 3 possible combinations to globally switch the 3 channels for the $P_1P_3P_5P_4$ superposition yielding $\chi_{\max}(m = 4)$. Note that the number of combinations to globally switch the channels yielding the high values $\chi_{\max}(m)$ of $m = 3$ and $m = 4$ are equal, however they amount to all possibilities for $m = 3$, where some combinations only achieve local switching for $m=4$ which results in our interpretation in a decrease of transmitted information in the results shown in Fig. 2.

Our reasoning is independent of the dimension of the target state $d$. Note that indeed the Holevo information decreases as $d$ increases but the overall $m$ dependence is the same.

| $\chi_{\min}$ $(m = 3)$ | $\chi_{\max}$ $(m = 3)$ |
|-------------------------|--------------------------|
| $P_3 \ N_3 \ N_1 \ N_2$ | $P_1 \ N_3 \ N_2 \ N_1$ |
| $P_1 \ N_3 \ N_2 \ N_1$ | $P_4 \ N_1 \ N_3 \ N_2$ |
| $P_2 \ N_2 \ N_3 \ N_1$ | $P_5 \ N_2 \ N_1 \ N_3$ |

| $\chi_{\min}$ $(m = 4)$ | $\chi_{\max}$ $(m = 4)$ |
|-------------------------|--------------------------|
| $P_3 \ N_3 \ N_1 \ N_2$ | $P_1 \ N_3 \ N_2 \ N_1$ |
| $P_1 \ N_3 \ N_2 \ N_1$ | $P_5 \ N_2 \ N_1 \ N_3$ |
| $P_2 \ N_2 \ N_3 \ N_1$ | $P_2 \ N_2 \ N_3 \ N_1$ |
| $P_4 \ N_1 \ N_3 \ N_2$ | $P_4 \ N_1 \ N_3 \ N_2$ |

TABLE II: Example of evaluation of combinations switching three channels. We detail here the four examples of superposition of $m$ orders underlined in Table III and relate them to their high $\chi_{\max}(m)$ or low $\chi_{\min}(m)$ transfer of information, by evaluating the ratio of globally switching pairs among possible pairs in the superposition of $m$ orders. Colors indicate fixed points, see main text.

IV. IMPLEMENTATION

Fig. 3 sketches our optical proposal to implement the quantum switch for two (a) and three (b) channels respectively. To implement experimentally the quantum switch channel, two main ingredients are required: a control and target system. Implementation of a quantum switch for $N \geq 2$ thus faces several challenges: (i) Operations on the chosen quantum system should be applied only on the target system (dimension $d$), without disturbing the control system (ii) The dimension $d_c$ of the control system, an appropriate quantum system to coherently control the order of operations, must be adapted to route all orders of the $N$ operations and grows as the number of permutations $N!$ (iii) As the number of operations $N$ grows, the experiment requires coherent control in a robust and scalable manner of dimension $d_c \times d$ which a priori grows as $N! \times N!$. In the existing experiments [5][17][19][20], the control system has been realized using either the path or polarization degrees of freedom of a single photon, and for the target system, it has been implemented using either the polarization or the transverse spatial mode of the same photon. In all these implementations, fibered or in free space, the quantum switch is limited due to the encoding of the control system in a two dimensional space and thus does not scale up to more than $N = 2$ quantum channels, although in [20] the arrival time encodes a $d$-dimensional target system.

We suggest to scale up the quantum $N$-switch to $N \geq 2$ with the generation and manipulation of single photons at telecom-wavelength, in a frequency-comb structure [24][25]. We propose to use the frequency bin of a single photon from a comb as the control system for routing the causal order of the operation of the channels, i.e. each frequency bin of the single photon supports a different quantum state of $p_e$ and thus a different ordering of channels. For the target system, we propose to use the...
FIG. 3: Optical proposal for the quantum $N$-switch channel. In both proposals, a frequency-delocalized single photon with $N!$ frequencies is injected as an input to a series of wavelength division multiplexers and de-multiplexers (WDMs). Each frequency is used to route the order of operation of the channels. At the end of the quantum switch, the frequencies are coherently multiplexed into the output mode. All color lines represent optical links which connect the WDMs and the channels $N_j$. (a) For the quantum 2-switch, if the frequency is on mode 1 (black), the order to apply the channels will be $N_2 \circ N_1$. On the other hand, if the frequency is on mode 2 (blue), the order will be $N_1 \circ N_2$. (b) For the quantum 3-switch, if the frequency is on mode 1 (black), the order to apply the channels will be $N_3 \circ N_2 \circ N_1$. If the frequency is on mode 2 (blue), the order will be $N_1 \circ N_3 \circ N_2$. If the frequency is on mode 3 (red), the order will be $N_2 \circ N_1 \circ N_3$. If the frequency is on mode 4 (purple), the order will be $N_3 \circ N_2 \circ N_1$. If the frequency is on mode 5 (yellow), the order will be $N_2 \circ N_3 \circ N_1$. Finally, if the frequency is on mode 6 (green), the order will be $N_1 \circ N_2 \circ N_3$. By sending, in both cases (a) and (b), a single photon in a superposition of frequencies will have all causal orders simultaneously. Note that our optical proposal can be seen as the implementation for the architecture proposed in [11]. We propose frequency encoding using off-the-shelf and mature telecom components.

In our suggested implementation, see Fig. 3, the input mode, a frequency-delocalized single photon with $N!$ frequencies is injected. The photon is firstly demultiplexed using wavelength division multiplexers (WDM), and then depending on the frequency, it is guided by selective optical links through the corresponding causal order $k$, which is acting separately on the time-bin degree of freedom. At the end of the quantum switch, the frequencies of the single photon are coherently multiplexed into the output mode. Our scheme is feasible with telecom standard technology or in an integrated Silicon platform. It is only bounded by optical losses and has no fundamental limitation on implementing any arbitrary number $N$ of causal orders or increasing the dimensionality of the quantum systems involved. In practice this schemes requires robust and reliable filtering and perfect matching of fibered or integrated multiplexing and de-multiplexing to the frequency combs.

V. CONCLUSION.

We have investigated quantum control of three fully-noisy channels in the context of quantum Shannon theory with superpositions of trajectories in the specific case of superposition of causal orders [7, 13].

We exhibit two different behaviors of the information transmission in the arbitrary $N$ case for two specific combinations of two causal orders. Our work on $N = 3$ channels is a significant advance in quantum control of causal order: In contrast with the previous $N = 2$ studies where only one combination of orders is accessible for quantum control, getting to $N = 3$ provides 57 combinations. Beyond the mere increase of combinations, this opens-up a full quantum control through the game of local or global switches, something that is not possible for $N = 2$. We assessed them and exhibited the influence of the number and nature of the $m$ causal orders involved in those combinations on the Holevo information. We thus uncover new quantum features of indefinite causal structures with combinations that are more efficient than others. Finally, we propose an implementation using standard telecom
technology to test our findings experimentally. Our work is thus to our knowledge the first quantitative study of indefinite causal structures providing predictions in a multipartite scenario within a new paradigm for the quantum information and quantum communications fields.

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[1] Ebler, D., Salek, S. & Chiribella, G. Enhanced communication with the assistance of indefinite causal order. Physical review letters 120, 120502 (2018).
[2] Salek, S., Ebler, D. & Chiribella, G. Quantum communication in a superposition of causal orders. arXiv preprint arXiv:1809.06653 (2018).
[3] Chiribella, G. et al. Indefinite causal order enables perfect quantum communication with zero capacity channel. arXiv preprint arXiv:1810.10457 (2018).
[4] Goswami, K., Romero, J. & White, A. Communicating via ignorance. arXiv preprint arXiv:1807.07583 (2018).
[5] Guo, Y. et al. Experimental investigating communication in a superposition of causal orders. arXiv preprint arXiv:1811.07526 (2018).
[6] Abbott, A. A., Wechs, J., Horsman, D., Mhalla, M. & Branciard, C. Communication through coherent control of quantum channels. arXiv preprint arXiv:1810.09826 (2018).
[7] Chiribella, G. & Kristjánsson, H. Quantum shannon theory with superpositions of trajectories. Proceedings of the Royal Society A 475, 20180903 (2019).
[8] Guérin, P. A., Rubino, G. & Brukner, Č. Communication through quantum-controlled noise. Physical Review A 99, 062317 (2019).
[9] Chiribella, G., D’Ariano, G. M., Perinotti, P. & Valiron, B. Quantum computations without definite causal structure. Physical Review A 88, 022318 (2013).
[10] Chiribella, G. Perfect discrimination of no-signalling channels via quantum superposition of causal structures. Physical Review A 86, 040301 (2012).
[11] Araújo, M., Costa, F. & Brukner, Č. Computational advantage from quantum-controlled ordering of gates. Phys. Rev. Lett. 113, 250402 (2014).
[12] Guérin, P. A., Feix, A., Araújo, M. & Brukner, Č. Exponential communication complexity advantage from quantum superposition of the direction of communication. Physical review letters 117, 100502 (2016).
[13] Procopio, L. M., Delgado, F., Enríquez, M., Belabas, N. & Levenson, J. A. Communication enhancement through quantum coherent control of N channels in an indefinite causal-order scenario. Entropy 21, 1012 (2019).
[14] Oreshkov, O. & Giarmatzi, C. Causal and causally separable processes. New Journal of Physics 18, 093020 (2016).
[15] Zhao, X. & Giulio, C. Advantage of indefinite causal order in quantum metrology. In Quantum Information and Measurement, F5A–23 (Optical Society of America, 2019).
[16] Mukhopadhyay, C., Gupta, M. K. & Pati, A. K. Superposition of causal order as a metrological resource for quantum thermometry. arXiv preprint arXiv:1812.07508 (2018).
[17] Procopio, L. M. et al. Experimental superposition of orders of quantum gates. Nature communications 6, 7913 (2015).
[18] Rubino, G. et al. Experimental verification of an indefinite causal order. Science advances 3, e1602589 (2017).
[19] Goswami, K. et al. Indefinite causal order in a quantum switch. Physical review letters 121, 090503 (2018).
[20] Wei, K. et al. Experimental quantum switching for exponentially superior quantum communication complexity. Physical review letters 122, 120504 (2019).
[21] Holevo, A. S. The capacity of the quantum channel with general signal states. IEEE Transactions on Information Theory 44, 269–273 (1998).
[22] Nielsen, M. A. & Chuang, I. Quantum computation and quantum information (2002).
[23] Wilde, M. M. Quantum information theory (Cambridge University Press, 2013).
[24] Kues, M. et al. On-chip generation of high-dimensional entangled quantum states and their coherent control. Nature 546, 622 (2017).
[25] Wang, C. et al. Monolithic lithium niobate photonic circuits for kerr frequency comb generation and modulation. Nature communications 10, 978 (2019).
[26] Mazeas, F. et al. High-quality photonic entanglement for wavelength-multiplexed quantum communication based on a silicon chip. Optics express 24, 28731–28738 (2016).
### A. Combinations of superimposing $m$ causal orders

| $m = 1$ | $m = 2$ | $m = 3$ | $m = 4$ | $m = 5$ | $m = 6$ |
|---------|---------|---------|---------|---------|---------|
| $p_1$   | $p_1 p_2$ | $p_1 p_2 p_3$ | $p_1 p_2 p_3 p_4$ | $p_1 p_2 p_3 p_4 p_5$ | $p_1 p_2 p_3 p_4 p_5 p_6$ |
| $p_2$   | $p_2 p_3$ | $p_2 p_3 p_4$ | $p_2 p_3 p_4 p_5$ | $p_2 p_3 p_4 p_5 p_6$ | $p_2 p_3 p_4 p_5 p_6$ |
| $p_3$   | $p_3 p_4$ | $p_3 p_4 p_5$ | $p_3 p_4 p_5 p_6$ | $p_3 p_4 p_5 p_6$ | $p_3 p_4 p_5 p_6$ |
| $p_4$   | $p_4 p_5$ | $p_4 p_5 p_6$ | $p_4 p_5 p_6$ | $p_4 p_5 p_6$ | $p_4 p_5 p_6$ |
| $p_5$   | $p_5 p_6$ | $p_5 p_6$ | $p_5 p_6$ | $p_5 p_6$ | $p_5 p_6$ |
| $p_6$   | $p_6$ | $p_6$ | $p_6$ | $p_6$ | $p_6$ |

**TABLE III: Table of possible combinations of $m$ causal orders.** $\rho_1 = \sum_{k, k' = 1}^{N} \sqrt{P_k P_{k'}} |k\rangle \langle k'|$ involves a superposition of $m$ causal orders, with $m$ different values. $P_i$ are non-zero probabilities. There are $(\frac{3!}{m!})$ possible configurations $m$ causal orders. The green color indicates which combinations yield the maximum values $\chi_{max}^\alpha (m)$ in Fig. 2 for $d = 2$ and $d = 3$. For simplicity we set for our estimates the non zero $P_i$ to be $\frac{1}{m}$. The underlined terms correspond to the cases studied in Table II. In Figures 1(b) to (e) we show only one combination of $P_k$'s from this table corresponding to a specific partial superposition.