1. Introduction

The purpose of these lecture notes is to give an overview of the theories of factorization and extrapolation for Muckenhoupt $A_p$ weights. The $A_p$ weights were introduced by Muckenhoupt [73] in the early 1970s and a wide ranging theory quickly developed: see [41, 46, 51] for details of this early history and extensive references.

Very early on the fine structure of $A_p$ weights—e.g. the $A_\infty$ condition, the reverse Hölder inequality and the fact that $A_p$ implies $A_{p-\epsilon}$—played an important role in the
theory. These properties were central to the proofs of the boundedness of maximal operators and singular integral operators on weighted spaces: see Coifman and Fefferman [13].

The deep structure revealed by the Jones factorization theorem—that every $A_p$ weight can be factored as the product of two $A_1$ weights—was conjectured by Muckenhoupt [74] at the Williamstown conference in 1979, and Jones [60] proved it at the same conference. His proof was highly technical and was soon overshadowed by simpler approaches.

A very simple proof of factorization was given by Coifman, Jones and Rubio de Francia [14]. At the heart of their proof were techniques developed by Rubio de Francia to prove his own fundamental contribution to the theory of weighted norm inequalities: the theory of extrapolation [83, 84, 85]. In its simplest form, this result says that if an operator $T$ satisfies

$$\int_{\mathbb{R}^n} |Tf|^2 w \, dx \leq C \int_{\mathbb{R}^n} |f|^2 w \, dx$$

for all weights $w \in A_2$, then for any $1 < p < \infty$ and any $w \in A_p$

$$\int_{\mathbb{R}^n} |Tf|^p w \, dx \leq C \int_{\mathbb{R}^n} |f|^p w \, dx.$$ 

Note in particular that this is true if we let $w = 1$, so (unweighted) $L^p$ estimates follow from weighted $L^2$ estimates. In other words, if a norm inequality holds at some point in a scale function spaces (in this case weighted Lebesgue spaces), then it holds at every point in this scale. Early on, Antonio Córdoba [50] summarized this by saying, “There are no $L^p$ spaces, only weighted $L^2$.”

The theory of Rubio de Francia extrapolation (as it is now called) has undergone a renaissance in the last twenty years. New and simpler proofs have been developed, including proofs that yield sharp constants. The theory has been extended to other settings and other classes of weights, and has been used to prove norm inequalities in a large class of Banach function spaces. It has found a number of applications, including the proof of the $A_2$ conjecture by Hytönen [57]. Extrapolation has also been extended to the setting of two weight norm inequalities. The latter theory is beyond the scope of our discussions here: see [27, 31] for further details. But here we want to note that it played a very surprising role in the disproof of the long standing Muckenhoupt-Wheeden conjectures for singular integral operators: see [32, 80, 81].

In these notes we survey the theories of factorization and extrapolation and we describe some of the many applications. They are organized as follows: in Section 2 we define the $A_p$ weights and examine their close relationship with the Hardy-Littlewood maximal operator. We do so because the maximal operator lies at the heart of the theories of factorization and extrapolation, with the connection coming from the Rubio de Francia iteration algorithm. In Section 3 we will consider the fine properties of
$A_p$ weights, and in particular we will prove the reverse Hölder inequality. Somewhat surprisingly, though no longer needed to prove the boundedness of the maximal operator and singular integrals, the reverse Hölder inequality still plays an important role in weighted theory. In Section 4 we prove the Jones factorization theorem and a generalization that shows that the factorization also encodes information about the reverse Hölder classes of weights. Here we introduce the iteration algorithm, which provides a tool for creating $A_1$ weights with very precise control of their size. In Section 5 we prove the Rubio de Francia extrapolation theorem. We adopt the abstract perspective of families of extrapolation pairs which lets us derive a number of corollaries as trivial consequences of the main extrapolation theorem. In Section 6 we give three applications of extrapolation; these have been chosen to illustrate some of the typical ways in which extrapolation can be applied. In Section 7 we discuss sharp constant extrapolation, which is used to prove weighted inequalities with optimal control of the constant in terms of the $A_p$ constant $[w]_{A_p}$. We illustrate this by sketching an elementary proof of the $A_2$ conjecture and describing its application to regularity results for the Beltrami operator. In Section 8 we give two variants of extrapolation which can be used to prove norm inequalities for a restricted range of exponents. Restricted range extrapolation arose in the study of operators related to second order elliptic PDEs and the Kato conjecture. In Section 9 we apply restricted range extrapolation to prove a bilinear extrapolation theorem. Finally, in Section 10 we briefly discuss the extension of Rubio de Francia extrapolation to other scales of Banach function spaces, and in particular to the variable Lebesgue spaces.

In writing these notes there is a tension between brevity and completeness, and in many instances brevity has won. We provide proofs of the central results on factorization and extrapolation, and sketch many of the other proofs. We provide extensive references for the missing details and also for the historical context in which these ideas were developed. These notes should be accessible to anyone who has completed a graduate course in measure theory (say from Royden [82] or Wheeden and Zygmund [91]), but some familiarity with the basics of harmonic analysis (say the first six chapters of Duoandikoetxea [41] or the first four chapters of Grafakos [52]) would be helpful. An earlier set of lecture notes [18] from a conference in Antequera, Spain, in 2014 is a useful complement to the current document. Though primarily concerned with fractional integral operators, it contains a fairly complete and detailed treatment of one weight norm inequalities from the perspective of dyadic operators. We will make extensive use of this “dyadic technology” in our applications.
2. The maximal operator and Muckenhoupt $A_p$ weights

We begin with some basic definitions. We will always be working on $\mathbb{R}^n$ and the underlying measure will be Lebesgue measure. We will denote this measure by $dx$, $dy$, etc. The variable $n$ will only be used to denote the dimension of the underlying space. By a weight $w$ we will mean a locally integrable, non-negative function and we define $L^p(w)$, $1 \leq p < \infty$, to be $L^p(\mathbb{R}^n, w \, dx)$. We will denote the set of bounded functions of compact support by $L^\infty_c$, and the set of smooth functions of compact support by $C^\infty_c$.

By a cube we will always mean a set of the form
$$Q = [a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n),$$
where $b_j - a_j = \ell(Q) > 0$ for $1 \leq j \leq n$. (In other words, we consider cubes whose edges are parallel to the coordinate axes.) Sometimes we will assume the cubes $Q$ are open and other times that they are closed. Since we will only be considering absolutely continuous measures on $\mathbb{R}^n$, this will generally not matter and we will take whatever is convenient.

We will work extensively with average integrals and we will use the notation
$$\frac{1}{|Q|} \int_Q w \, dx.$$
Though we will generally use this notation for cubes, it works equally well if we replace the cube $Q$ by a measurable set $E$ such that $0 < |E| < \infty$. We will apply the same notation for averages with respect to other (absolutely continuous) measures. Given a weight $\sigma$ that is positive a.e., define
$$\frac{1}{\sigma(Q)} \int_Q w \, \sigma \, dx.$$

Constants will be denoted by $C$, $c$, etc. and may change value at each appearance. Generally, constants will depend on the dimension $n$, the value $p$ of any associated $L^p$ space, and possibly the operator under consideration. For emphasis, we may denote this dependence by writing $C(n, p)$, etc. We will consider dependence on the weight $w$ more carefully as we will make clear below. If the underlying constant is not particularly important, we may use the notation $A \lesssim B$ to denote $A \leq cB$ for some constant $c > 0$.

We now define the fundamental weight classes we are interested in.

\footnote{Much of what we say can be extended to the more general setting of spaces of homogeneous type, but this is beyond the scope of these notes.}
**Definition 2.1.** Given $1 < p < \infty$, a weight $w$ is in the Muckenhoupt class $A_p$, denoted by $w \in A_p$, if $0 < w(x) < \infty$ a.e. and

$$[w]_{A_p} = \sup_Q \left( \frac{\int_Q w}{\int_Q w^{1-p'}} \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q$.

Since $p' - 1 = \frac{p}{p'}$, we can also write the $A_p$ condition in an equivalent form using $L^p$ and $L^{p'}$ norms: for any cube $Q$,

$$\left| Q \right|^{-1} \left\| w^{\frac{1}{p'}} \chi_Q \right\|_p \left\| w^{-\frac{1}{p'}} \chi_Q \right\|_{p'} \leq [w]_{A_p}^{\frac{1}{p'}}. \quad (2.1)$$

The definition of $A_p$ is symmetric: given $w \in A_p$, let $\sigma = w^{1-p'}$. Then $\sigma \in A_{p'}$ and $[\sigma]_{A_{p'}} = [w]_{A_p}^{p' - 1}$.

To understand the $A_p$ condition, it is helpful to note that by Hölder’s inequality, for every cube $Q$,

$$1 \leq \left( \frac{\int_Q w}{\int_Q w^{1-p'}} \right)^{p-1}.$$

Thus, the $A_p$ condition can be thought of as a kind of “reverse” Hölder inequality.

If we adopt the convention that $0 \cdot \infty = 0$, then in this definition we could omit the assumption that $0 < w(x) < \infty$ a.e. However, nothing is gained by doing so, since this assumption is actually a consequence of the definition: see [51, Section IV.1] for more details.

**Definition 2.2.** When $p = 1$, we say that a weight $w$ is in $A_1$, denoted by $w \in A_1$, if

$$[w]_{A_1} = \sup_Q \sup_{x \in Q} \frac{w(x)^{-1} \int_Q w}{\int_Q w} < \infty,$$

where again the supremum is taken over all cubes $Q$.

Equivalently, $w \in A_1$ if for every cube $Q$,

$$\int_Q w \, dy \leq [w]_{A_1} \operatorname{ess inf}_{x \in Q} w(x),$$

or if $Mw(x) \leq [w]_{A_1} \operatorname{ess inf}_{x \in Q} w(x)$, where $M$ denotes the Hardy-Littlewood maximal operator (see below). For a proof of this equivalence, see [51, Section IV.1]. The $A_1$ condition is the limit of the $A_p$ condition as $p \to 1$: see Rudin [86, pp. 73–4].

By Hölder’s inequality we have the following inclusions: for $1 < p < q < \infty$, $A_1 \subset A_p \subset A_q$, and $[w]_{A_q} \leq [w]_{A_p} \leq [w]_{A_1}$. These inclusions are proper, as is shown
by the family of weights \( w(x) = |x|^a \). For \( 1 < p < \infty \), \( w \in A_p \) if \( -n < a < (p-1)n \), and \( w \in A_1 \) if \( -n < a \leq 0 \). Define the overarching class \( A_\infty \) by

\[
A_\infty = \bigcup_{p \geq 1} A_p.
\]

The weights in \( A_\infty \) are characterized by a reverse Jensen inequality: there exists a constant \([w]_{A_\infty}\) such that for all cubes \( Q \),

\[
\int_Q w \, dx \leq [w]_{A_\infty} \exp \left( \int_Q \log(w) \, dx \right).
\]

For a proof, see [51, Section IV.2]. This inequality is the limit of the \( A_p \) condition as \( p \to \infty \); consequently, we have that \([w]_{A_\infty} \leq [w]_{A_p}\). (Again, see Rudin [86, p. 73].) But in fact, for any weight \( w \in A_\infty \),

\[
[w]_{A_\infty} = \lim_{p \to \infty} [w]_{A_p}.
\]

For a proof, see Sbordone and Wik [89].

There is a close connection between the Muckenhoupt \( A_p \) weights and the Hardy-Littlewood maximal operator. For \( f \in L^1_{\text{loc}} \) define

\[
Mf(x) = \sup_Q \int_Q |f| \, dy \cdot \chi_Q(x),
\]

where the supremum is taken over all cubes \( Q \). It is well known that for \( 1 \leq p < \infty \), \( M \) satisfies the weak \((p,p)\) inequality: there exists \( C > 0 \) such that for all \( f \) and all \( t > 0 \),

\[
|\{x \in \mathbb{R}^n : Mf(x) > t\}| \leq \frac{C}{t^p} \int_{\mathbb{R}^n} |f|^p \, dx;
\]

further, for \( 1 < p \leq \infty \) it satisfies the strong \((p,p)\) inequality: there exists \( C > 0 \) such that for all \( f \),

\[
\|Mf\|_p \leq C\|f\|_p.
\]

The \( A_p \) condition lets us prove the same inequalities in the weighted Lebesgue spaces \( L^p(w) \), \( 1 \leq p < \infty \).

**Theorem 2.3.** Given \( 1 \leq p < \infty \) and a weight \( w \), the following are equivalent:

1. \( w \in A_p \);
2. for all \( t > 0 \),

\[
w(\{x \in \mathbb{R}^n : Mf(x) > t\}) \leq C(n,p)[w]_{A_p} \frac{1}{t^p} \int_{\mathbb{R}^n} |f|^p \, w \, dx;
\]
(3) if in addition, $p > 1$,

$$\int_{\mathbb{R}^n} (Mf)^p w \, dx \leq C(n, p)[w]_{A_p}^p \int_{\mathbb{R}^n} |f|^p w \, dx.$$ 

For brevity, we will restrict ourselves to proving the equivalence of (1) and (3) when $1 < p < \infty$. Furthermore, we will restrict ourselves to the dyadic maximal operator. Recall that the set of dyadic cubes is the countable collection

$$\Delta = \bigcup_{k \in \mathbb{Z}} \Delta_k,$$

where

$$\Delta_k = \left\{ 2^{-k}([0,1]^n + m) : m \in \mathbb{Z}^n \right\}.$$ 

The dyadic maximal operator is defined by

$$M^d f(x) = \sup_{Q \in \Delta} \int_Q |f| \, dy \cdot \chi_Q(x).$$ 

The proof we will give below can be adapted to the general case in several ways; for this proof and for the proof of the weak type inequality, we refer the reader to [18, 41, 51]. We want to concentrate on the dyadic operator since it makes the main ideas of the proof clear while avoiding some technical difficulties.

The proof requires three lemmas. The first is a construction that yields a collection of dyadic cubes often referred to as Calderón-Zygmund cubes. For a proof, see [27, 41, 51].

**Lemma 2.4.** Let $f \in L^p$, $1 \leq p < \infty$. Then for any $\lambda > 0$, there exists a collection of pairwise disjoint dyadic cubes $\{Q_j\}$ such that

$$\{x \in \mathbb{R}^n : M^d f(x) > \lambda\} = \bigcup_j Q_j$$

and

$$\lambda < \int_{Q_j} |f| \, dx \leq 2^n \lambda.$$ 

Moreover, given $a \geq 2^{n+1}$, for each $k \in \mathbb{Z}$ let $\{Q_j^k\}_j$ be the cubes gotten by taking $\lambda = a^k$. Define

$$\Omega_k = \{x \in \mathbb{R}^n : M^d f(x) > a^k\} = \bigcup_j Q_j^k,$$

and let $E_j^k = Q_j^k \setminus \Omega_{k+1}$. Then the sets $E_j^k$ are pairwise disjoint and $|E_j^k| \geq \frac{1}{2}|Q_j^k|$.

The second lemma shows that, in some sense, the measure $dw = w \, dx$ behaves like Lebesgue measure uniformly at all scales.
Lemma 2.5. Let \( 1 \leq p < \infty \) and \( w \in A_p \). Then given any cube \( Q \) and any measurable set \( E \subset Q \),
\[
\frac{|E|}{|Q|} \leq [w]_{A_p}^\frac{1}{p} \left( \frac{w(E)}{w(Q)} \right)^{\frac{1}{p}}.
\]

Proof. When \( p > 1 \), this follows at once from Hölder’s inequality and the definition of \( A_p \):
\[
\frac{|E|}{|Q|} = \int_Q \chi_E w^\frac{1}{p} w^{-\frac{1}{p}} \; dx \leq \left( \int_Q w \chi_E \; dx \right)^\frac{1}{p} \left( \int_Q w^{1-p'} \; dx \right)^{\frac{1}{p'}} \\
\leq [w]_{A_p}^\frac{1}{p} \left( \int_Q w \chi_E \; dx \right)^\frac{1}{p} \left( \int_Q w \; dx \right)^{-\frac{1}{p}} = [w]_{A_p}^\frac{1}{p} \left( \frac{w(E)}{w(Q)} \right)^{\frac{1}{p}}.
\]

When \( p = 1 \) the proof follows directly from the definition of \( A_1 \). \( \square \)

For the third lemma, we introduce a weighted dyadic maximal operator. Given a weight \( \sigma \), let
\[
M^d_{\sigma} f(x) = \sup_{Q \in \Delta} \int_Q |f| \; d\sigma \cdot \chi_Q(x).
\]

Lemma 2.6. Given a weight \( \sigma \), then for all \( 1 < p \leq \infty \), there exists a constant \( C(p) > 0 \) such that for all \( f \), \( \| M^d_{\sigma} f \|_p \leq C(p) \| f \|_p \).

This inequality is proved exactly as the unweighted norm inequalities for \( M^d \). When \( p = \infty \) it is immediate. When \( p = 1 \), use Lemma 2.4 to prove the weak \( (1,1) \) inequality, and then apply Marcinkiewicz interpolation to get the desired inequality.

Proof of Theorem 2.3. As we indicated above, we will prove the equivalence of (1) and (3) when \( 1 < p < \infty \). To prove necessity, fix a cube \( Q \) and let \( f = w^{1-p'} \chi_Q \).

Then for \( x \in Q \),
\[
M(w^{1-p'} \chi_Q)(x) \geq \int_Q w^{1-p'} \; dx,
\]
and so by the strong type inequality,
\[
\left( \int_Q w^{1-p'} \; dx \right)^p \int_Q w \; dx \leq C \int_Q w^{1-p'} \; dx.
\]

The \( A_p \) condition follows at once.

To prove sufficiency we adapt a proof originally due to Christ and Fefferman [12]. Let \( \sigma = w^{1-p'} \). By a standard approximation argument, we may assume \( f \geq 0 \) and \( f \in L^\infty \). Fix \( a \geq 2^{n+1} \). Then, with the notation of Lemma 2.4, we have that
\[
\int_{\mathbb{R}^n} (M^d f)^p w \; dx \leq \sum_k \int_{Q_k \setminus Q_{k+1}} (M^d f)^p w \; dx
\]
\[ \sum_k a^{kp} w(\Omega_k) \]
\[ = \sum_{k,j} a^{kp} w(Q_j^k) \]
\[ \leq \sum_{k,j} \left( \int_{Q_j^k} f^{-1} \sigma \, dx \right)^p w(Q_j^k) \]
\[ = \sum_{k,j} \left( \int_{Q_j^k} f^{-1} \sigma \, d\sigma \right)^p \left( \int_{Q_j^k} w^{1-p'} \, dx \right)^{p-1} \int_{Q_j^k} w \, dx \, \sigma(Q_j^k); \]
by Lemma 2.5 applied to \( \sigma \in A_{p'} \) and by the definition of \( A_p \),
\[ \leq [w]_{A_p} [\sigma]_{A_{p'}} \sum_{k,j} \left( \int_{Q_j^k} f^{-1} \sigma \, d\sigma \right)^p \sigma(E_k^j) \]
\[ \leq [w]_{A_p}^{p'} \sum_{k,j} \int_{E_k^j} M^d_\sigma(f^{-1})^p \, d\sigma \]
\[ \leq [w]_{A_p}^{p'} \int_{\mathbb{R}^n} M^d_\sigma(f^{-1})^p \, d\sigma; \]
by Lemma 2.6,
\[ \leq [w]_{A_p}^{p'} \int_{\mathbb{R}^n} (f^{-1})^p \, d\sigma \]
\[ = [w]_{A_p}^{p'} \int_{\mathbb{R}^n} f^p w \, dx. \]

\[ \square \]

The constant we get in Theorem 2.3 for the strong \((p, p)\) inequality, in terms of the exponent on the \( A_p \) constant \([w]_{A_p}\), is sharp: see Buckley [11] for examples. Buckley also proved the strong \((p, p)\) inequality with this constant using a different proof. Yet another proof is due to Lerner [64]. The fact that the sharp constant was implicit in the proof of Christ and Fefferman [12] seems to have been overlooked for many years.\(^2\) The sharp constant for the maximal operator plays a role in the proof of sharp constant extrapolation discussed in Section 7 below.

\(^2\)I learned this fact from Kabe Moen, who in turn learned it from an anonymous referee.
3. The fine properties of $A_p$ weights

In this section we consider some of the fine properties of $A_p$ weights, particularly the reverse Hölder inequality, which yields another characterization of the class $A_{p\infty}$.

**Definition 3.1.** Given a weight $w$ and $s > 1$, we say that $w$ satisfies the reverse Hölder inequality with exponent $s$, denoted by $w \in RH_s$, if

$$[w]_{RH_s} = \sup_Q \left( \frac{1}{\int_Q w^{s} \, dx} \right)^{\frac{1}{s}} \left( \int_Q w \, dx \right)^{-1} < \infty,$$

where the supremum is taken over all cubes $Q$.

**Theorem 3.2.** If $w \in A_{\infty}$, then there exists $s > 1$ such that $w \in RH_s$. In fact, there exists $s > 1$ depending on $[w]_{A_p}$ such that for every cube $Q$,

$$\left( \frac{1}{\int_Q w^{s} \, dx} \right)^{\frac{1}{s}} \leq 2\int_Q w \, dx.$$

Conversely, if $w \in RH_s$ for some $s > 1$, then $w \in A_{\infty}$.

We will only prove the first half of Theorem 3.2. For the proof of the converse, which involves defining the $A_p$ and $RH_s$ classes with respect to arbitrary measures and showing a certain “duality” condition, see [51, Section IV.2].

Before proving Theorem 3.2 we give two corollaries. The first is important for historical reasons.

**Corollary 3.3.** Given $1 < p < \infty$, if $w \in A_p$, then there exists $\epsilon > 0$ such that $w \in A_{p-\epsilon}$.

As a consequence of this corollary, the strong $(p, p)$ inequality follows from the weak $(p, p)$ inequality by Marcinkiewicz interpolation: if $w \in A_p$, then $w \in A_{p\pm\epsilon}$. Moreover, by a covering lemma argument (using Lemma 2.4) we can prove the weak $(p \pm \epsilon, p \pm \epsilon)$ inequalities. For this classical approach, see [41, 51]. The advantage of the proof of Theorem 2.3 given above is that it shows that the reverse Hölder inequality is not required.

**Proof.** Given $w \in A_p$, $w^{1-p'} \in A_{p'} \subset A_{\infty}$, so $w^{1-p'} \in RH_s$ for some $s > 1$. Fix $\epsilon > 0$ such that

$$\frac{(p - \epsilon)'}{p'} - 1 = s.$$

Then, given any cube $Q$,

$$\left( \frac{1}{\int_Q w^{1-(p-\epsilon)'} \, dx} \right)^{\frac{p-1}{p'}} \leq [w]_{RH_s} \left( \int_Q w^{1-p'} \, dx \right)^{p-1};$$

it follows at once that $w \in A_{p-\epsilon}$. $\square$
The next corollary gives an inequality which is essentially the opposite of that in Lemma 2.5. Together, these two results show that $A_p$ weights behave, in some sense, like constants uniformly at all scales.

**Corollary 3.4.** If $w \in A_\infty$, then there exist constants $C, \delta > 0$ such that for any cube $Q$ and measurable set $E \subset Q$,

$$\frac{w(E)}{w(Q)} \leq C \left(\frac{|E|}{|Q|}\right)^\delta.$$

**Proof.** This follows immediately from Hölder’s inequality and the reverse Hölder inequality: since $w \in RH_s$ for some $s > 1$,

$$w(E) = \int_Q w(x) \chi_E(x) \, dx \leq \left(\int_Q w^s(x) \, dx\right)^{\frac{1}{s}} |E|^{\frac{1}{s'}} |Q|^{\frac{1}{s}}.$$

This gives the desired inequality with $C = [w]_{RH_s}$ and $\delta = \frac{1}{s'}$.

The inequality in Corollary 3.4 is often taken as the definition of the $A_\infty$ condition. There are many equivalent definitions: for a thorough treatment of them, see Duoandikoetxea, Martín-Reyes and Ombrosi [43].

To prove the reverse Hölder inequality we need two lemmas. The first lets us replace an $A_p$ weight by its bounded truncation.

**Lemma 3.5.** If $w \in A_p, 1 < p < \infty$, then for any $N > 0$, $w_N = \min(w, N) \in A_p$ and $[w_N]_{A_p} \leq 2^p[w]_{A_p}$.

**Proof.** Since $w_N^{-1} \leq N^{-1} + w^{-1}$, and since $(a + b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}}$, for any cube $Q$, by Minkowski’s inequality and (2.1),

$$\|w_N^{-\frac{1}{p}}\|_{p'} \|w_N^{-\frac{1}{p}}\chi_Q\|_{p'} \leq \|N^{-\frac{1}{p}}\|_{p'} \|N^{-\frac{1}{p}}\chi_Q\|_{p'} + \|w^{-\frac{1}{p}}\chi_Q\|_{p'} \|w^{-\frac{1}{p}}\chi_Q\|_{p'} \leq |Q| + [w]_{A_p}^\frac{1}{p}|Q| \leq 2[w]_{A_p}^\frac{1}{p}|Q|.$$

The second lemma is a local version of Lemma 2.4 that is proved in exactly the same way. Given a fixed cube $Q$, let $\Delta(Q)$ be the set of all cubes that are gotten by bisecting the sides of $Q$, and then repeating this process inductively on each sub-cube so formed. For $x \in Q$ define the local dyadic maximal operator by

$$M_Q^d f(x) = \sup_{P \in \Delta(Q)} \int_P |f| \, dy \cdot \chi_P(x).$$
Lemma 3.6. Given a cube $Q$, let $w$ be a weight such that $\int_Q w \, dx = 1$. Fix $a \geq 2^{n+1}$; then for each $k \geq 0$ we can write the set

$$\Omega_k = \{ x \in Q : M_Q^d w(x) > a^k \} = \bigcup_j Q_j^k,$$

where for each $k$ the cubes $Q_j^k \in \Delta(Q)$ are disjoint and satisfy

$$a^k < \int_{Q_j^k} w \, dx \leq 2^n a^k.$$

Further, if $E_j^k = Q_j^k \setminus \Omega_{k+1}$, then the $E_j^k$ are pairwise disjoint and $|E_j^k| \geq \frac{1}{2} |Q_j^k|$.

Proof of Theorem 3.2. Fix $w \in A_\infty$; we will assume for the moment that $w$ is bounded. Fix a cube $Q$; by homogeneity, without loss of generality we may assume that $\int_Q w \, dx = 1$. Let $0 < \epsilon < 1$; we will fix the precise value below. Then

$$\int_Q M_Q^d(w)\epsilon w \, dx = \int_0^\infty \epsilon t^{\epsilon-1} w(\{ x \in Q : M_Q^d w(x) > t \}) \, dt$$

$$= \int_0^1 \ldots + \int_1^\infty \ldots$$

$$\leq w(Q) + \epsilon \sum_{k=0}^\infty w(\Omega_k) \int_{a^k}^{a^{k+1}} t^{\epsilon-1} \, dt$$

$$\leq |Q| + \epsilon \sum_{k,j} a^{(k+1)}(Q_j^k) \int_{a^k}^{a^{k+1}} t^{-1} \, dt$$

$$= |Q| + \epsilon a\log(a) \sum_{k,j} a^k w(Q_j^k);$$

by Lemma 2.5,

$$\leq |Q| + \epsilon a\log(a) 2^p[w]_{A_p} \sum_{k,j} \left( \int_{Q_j^k} w \, dx \right)^\epsilon w(E_j^k)$$

$$\leq |Q| + C(a, p) \epsilon \sum_{k,j} \int_{E_j^k} M_Q^d(w)\epsilon w \, dx$$

$$\leq |Q| + C(a, p) \epsilon \int_Q M_Q^d(w)\epsilon w \, dx.$$

Now fix $\epsilon > 0$ sufficiently small that $C(\epsilon) = \frac{1}{2}$. Since $w$ is bounded,

$$\int_Q M_Q^d(w)\epsilon w \, dx < \infty.$$
Therefore, by rearranging terms and by the Lebesgue differentiation theorem we have that
\[
\frac{1}{2} \int_Q w^{1+\epsilon} \, dx \leq \frac{1}{2} \int_Q M^\epsilon_Q(w)^\epsilon w \, dx \leq |Q|.
\]
The desired inequality thus holds for bounded weights.

Finally, given an arbitrary weight \( w \), by Lemma 3.5 and the previous argument we have that the reverse Hölder inequality holds for \( w^N \) with a constant independent of \( N \). Hence, by the monotone convergence theorem it holds for \( w \).

It is possible to give a very sharp estimate of the exponent \( s \). To do so we need to introduce another condition equivalent to the \( A_\infty \) condition. We say that a weight \( w \) satisfies the Fujii-Wilson \( A_\infty \) condition if
\[
[w]'_{A_\infty} = \sup_Q w(Q)^{-1} \int_Q M(w\chi_Q) \, dx < \infty,
\]
where the supremum is taken over all cubes \( Q \). This condition is equivalent to \( w \in A_\infty \), a fact discovered independently by Fujii [49] and Wilson [92]. It has the advantage that it is generally much smaller than the other \( A_\infty \) constants: see Beznosova and Reznikov [10]. Using this definition, Hytönen and Pérez [58] showed that
\[
s = 1 + \frac{1}{c(n)[w]'_{A_\infty}}.
\]

Our proof of Theorem 3.2 is adapted from theirs; it is somewhat simpler since we do not get the sharp constant.

If \( w \in A_\infty \), then there exist \( 1 < p, s < \infty \) such that \( w \in A_p \) and \( w \in RH_s \). However, there is no direct connection between these two exponents: The example of power weights shows that given any pair of \( p, s \), there exists \( w \in A_p \cap RH_s \). However, as the next result shows, there is a weaker connection. This proposition will play a role in restricted range extrapolation: see Section 8 below.

**Proposition 3.7.** Given \( 1 < p, s < \infty \) and a weight \( w, w \in A_p \cap RH_s \) if and only if \( w^s \in A_q \), where \( q = s(p - 1) + 1 \).

**Proof.** Suppose first that \( w \in A_p \cap RH_s \). By the definition of \( q \) we have that \( p' - 1 = s(q' - 1) \). Hence, for any cube \( Q \),
\[
\left( \int_Q w^s \, dx \right) \left( \int_Q w^{s(1-q')} \, dx \right)^{q-1}
\leq [w]_{RH_s}^s \left( \int_Q w \, dx \right)^s \left( \int_Q w^{1-p'} \, dx \right)^{(p-1)} \leq [w]_{RH_s}^s [w]_{A_p}^s.
\]

Thus, \( w^s \in A_q \).
Conversely, if \( w^s \in A_q \), then essentially the same argument using Hölder’s inequality instead of the reverse Hölder inequality shows that \( w \in A_p \). Moreover, again given any cube \( Q \), by the definition of \( A_q \) and Hölder’s inequality,

\[
\int_Q w^s \, dx = \int_Q w^s \left( \int_Q w^{s(q')} \, dx \right)^{q-1} \left( \int_Q w^{1-p'} \, dx \right)^{-s(p-1)} \leq \left[ w^s \right]_{A_q} \left( \int_Q w \, dx \right)^s.
\]
Hence, \( w \in RH_s \). \( \square \)

As a final application of the reverse Hölder inequality we will prove a multilinear version. This inequality will be used in Section 9 below when we consider weighted norm inequalities for bilinear operators. This result was first proved in [30] in the bilinear case. Recently, a simpler proof for the general, multilinear case was given in [28]. To simplify the presentation, we give this proof in the bilinear case.

**Proposition 3.8.** Given \( w_1, w_2 \in A_\infty \), suppose \( w_1 \in RH_s \) and \( w_2 \in RH_{s'} \) for some \( 1 < s < \infty \). Then there exists \( C > 0 \) such that for every cube \( Q \),

\[
\left( \int_Q w_1^s \, dx \right)^{\frac{1}{s}} \left( \int_Q w_2^{s'} \, dx \right)^{\frac{1}{s'}} \leq C \int_Q w_1 w_2 \, dx.
\]

**Proof.** Since \( w_1, w_2 \in A_\infty \), by Proposition 3.7, \( w_1^s, w_2^{s'} \in A_\infty \). Moreover, since the \( A_p \) classes are nested, we may assume that they are both in \( A_q \) for some \( q > 1 \). Therefore, again by Proposition 3.7, there exists \( 0 < r < 1 \), such that \( w_1^{rs}, w_2^{rs'} \in A_2 \cap RH_r \). If we use these two conditions and then Hölder’s inequality three times, we get that for every cube \( Q \),

\[
\left( \int_Q w_1^s \, dx \right)^{\frac{1}{s}} \left( \int_Q w_2^{s'} \, dx \right)^{\frac{1}{s'}} \leq \left( \int_Q w_1^{rs} \, dx \right)^{\frac{1}{rs}} \left( \int_Q w_2^{rs'} \, dx \right)^{\frac{1}{rs'}} \leq \left( \int_Q w_1^{-rs} \, dx \right)^{-\frac{1}{rs}} \left( \int_Q w_2^{-rs'} \, dx \right)^{-\frac{1}{rs'}} \leq \left( \int_Q w_1^{-r} w_2^{-r} \, dx \right)^{-\frac{1}{r}} \leq \left( \int_Q w_1^r w_2^r \, dx \right)^{\frac{1}{r}} \leq \int_Q w_1 w_2 \, dx.
\]

\( \square \)
4. Factorization

In this section we prove the Jones factorization theorem. At the heart of the proof is the Rubio de Francia iteration algorithm, which allows us, given an arbitrary weight $u$, to construct an $A_1$ weight $w$ that is the “same size” as $u$ in a precisely specified way. The iteration algorithm also plays a central role in the proof of extrapolation as we will see in Section 5 below.

**Theorem 4.1.** Fix $1 < p < \infty$ and $w \in A_p$. For any non-negative function $h \in L^p(w)$, define

$$R h(x) = \sum_{k=0}^{\infty} \frac{M^k h(x)}{2^k \|M\|_{L^p(w)}^k},$$

where for $k > 0$, $M^k h = M \circ \cdots \circ M h$ denotes $k$ iterations of the maximal operator and $M^0 h = h$. Then:

1. $h(x) \leq R h(x)$;
2. $\|R h\|_{L^p(w)} \leq 2\|h\|_{L^p(w)}$;
3. $R h \in A_1$ and $[R h]_{A_1} \leq 2\|M\|_{L^p(w)}$.

**Proof.** If we take the first term in the sum, (1) is immediate. To prove (2) we apply Minkowski’s inequality:

$$\|R h\|_{L^p(w)} \leq \sum_{k=0}^{\infty} \frac{\|M^k h\|_{L^p(w)}}{2^k \|M\|_{L^p(w)}^k} \leq \sum_{k=0}^{\infty} 2^{-k} \|h\|_{L^p(w)} = 2\|h\|_{L^p(w)}.$$

Finally, (3) holds since the maximal operator is subadditive:

$$M(R h)(x) \leq \sum_{k=0}^{\infty} \frac{M^{k+1} h(x)}{2^k \|M\|_{L^p(w)}^k} \leq 2\|M\|_{L^p(w)} R h(x).$$

We note that the existence of an $A_1$ majorant for a function $h$ is, somewhat surprisingly, linked to $h$ being an element of the set $\bigcup_{p>1} L^p$. For a precise description of this connection, see Knese, McCarthy and Moen [61].

An important feature of the proof of Theorem 4.1 is that we only use the fact that the underlying operator is the maximal operator to prove that $R h \in A_1$. If we replace $M$ by a positive, sublinear operator $S$ that is bounded on $L^p(w)$, then the same proof yields (1) and (2) and the $A_1$-type property that $S(R h) \leq 2\|S\|_{L^p(w)} R h$. This simple generalization lets us prove the Jones factorization theorem.

**Theorem 4.2.** For $1 < p < \infty$, a weight $w$ is in $A_p$ if and only if there exist $w_1, w_2 \in A_1$ such that $w = w_1 w_2^{1-p}$. 
Proof. One direction is easy: in [27] we dubbed this fact “reverse factorization.”\footnote{Unfortunately, this terminology has not gained universal acceptance.} Fix $p$ and $w_1, w_2 \in A_1$. Then for any cube $Q$ and a.e. $x \in Q$,
\[
\int_Q w_i \, dy \leq [w_i]_{A_1} w_i(x), \quad i = 1, 2.
\]
Let $w = w_1 w_2^{1-p}$; then we have that
\[
\int_Q w \, dx \left( \int_Q w^{1-p'} \, dx \right)^{p-1} = \int_Q w_1^{1-p} \, dx \left( \int_Q [w_2 w_1^{1-p}]^{-p'} \, dx \right)^{p-1} \leq [w_1]_{A_1} [w_2]_{A_1}^{p-1} \int_Q w_1 \, dx \left( \int_Q w_2 \, dx \right)^{1-p} \left( \int_Q w_1 \, dx \right)^{-1} = [w_1]_{A_1} [w_2]_{A_1}^{p-1}.
\]

The difficult direction is the converse. Fix $w \in A_p$, $1 < p < \infty$, and let $q = pp' > 1$. Define the operator
\[
S_1 f(x) = w(x)^{\frac{1}{p'}} M(f^{p'} w^{-\frac{1}{p'}})(x)^{\frac{1}{p'}}.
\]
Then $S_1$ is sublinear and $S_1 : L^q \to L^q$ since
\[
\int_{\mathbb{R}^n} (S_1 f)^q \, dx = \int_{\mathbb{R}^n} M(f^{p'} w^{-\frac{1}{p'}})^p w \, dx \leq C[w]_{A_p}^{p'} \int_{\mathbb{R}^n} f^q \, dx.
\]
In particular, $\|S_1\|_{L^q} \lesssim [w]^{\frac{1}{p'}}_{A_p}$. Similarly, let $\sigma = w^{1-p'} \in A_{p'}$ and define
\[
S_2 f = \sigma^{\frac{1}{p'}} M(f^{p'} \sigma^{-\frac{1}{p'}})^{\frac{1}{p'}}.
\]
Then $S_2$ is sublinear, $S_2 : L^q \to L^q$, and $\|S_1\|_{L^q} \lesssim [\sigma]^{\frac{1}{p'}}_{A_{p'}} = [w]^{\frac{1}{p'}}_{A_p}$.

Define $S = S_1 + S_2$ and form the Rubio de Francia iteration algorithm
\[
R h(x) = \sum_{k=0}^{\infty} S^k h(x) 2^k \|S\|_{L^q}^{-k}.
\]
Then, by the proof of Theorem 4.1, $R : L^q \to L^q$. Fix any non-zero function $h \in L^q$; then $R h$ is finite almost everywhere. Moreover, $S(Rh)(x) \leq 2\|S\|_{L^q} R h(x)$. In particular, we have that
\[
w^{\frac{1}{p'}} M((R h)^{p'} w^{-\frac{1}{p'}})^{\frac{1}{p'}} = S_1(R h) \lesssim R h.
\]
Hence, if we let \( w_2 = (Rh)^p w^{-\frac{1}{p'}} \), then this inequality becomes \( Mw_2 \lesssim w_2 \), so \( w_2 \in A_1 \). Similarly, if we repeat this argument with \( S_2 \) in place of \( S_1 \), we get \( w_1 = (Rh)^p \sigma^{-\frac{1}{p'}} \in A_1 \). Moreover, it is immediate that \( w_1 w_2^{-1} = w_1 w_2^{1-p} = w \). \( \square \)

We note that in the proof of factorization, the function \( h \) is chosen essentially arbitrarily. It is an open question whether the choice of \( h \) can be used to optimise this factorization in some way.

The factorization in Theorem 4.2 also encodes information about the reverse Hölder class of the weight \( w \). The proof is fairly easy and mostly requires reinterpreting the terms in the Jones factorization theorem. This generalization was first proved in [30]. To state it, we need to introduce the class \( RH_\infty \), which is related to the reverse Hölder classes \( RH_s \) in a way that is analogous to the relationship between the \( A_1 \) and \( A_p \) classes.

**Definition 4.3.** Given a weight \( w \), we say \( w \in RH_\infty \) if

\[
[w]_{RH_\infty} = \sup_{Q} \text{ess sup}_{x \in Q} w(x) \left( \frac{1}{Q} \int w(y) \, dy \right)^{-1} < \infty,
\]

where the supremum is taken over all cubes \( Q \).

From the definition we have that for every cube \( Q \) and a.e. \( x \in Q \),

\[
w(x) \leq [w]_{RH_\infty} \frac{1}{Q} \int w \, dy.
\]

Raising both sides to the power \( s > 1 \) and integrating over \( Q \) shows that \( RH_\infty \subset RH_s \).

**Theorem 4.4.** For \( 1 < p, s < \infty \), given a weight \( w, w \in A_p \cap RH_s \) if and only if there exist weights \( v_1, v_2 \) such that \( w = v_1 v_2 \), \( v_1 \in A_1 \cap RH_s \) and \( v_2 \in A_p \cap RH_\infty \).

For the proof of Theorem 4.4 we need three lemmas. The first extends Proposition 3.7 to \( A_1 \) weights.

**Lemma 4.5.** Given a weight \( w \) and \( s > 1 \), \( w \in A_1 \cap RH_s \) if and only if \( w^s \in A_1 \).

**Proof.** Suppose first that \( w \in A_1 \cap RH_s \). Given any cube \( Q \),

\[
\int_Q w^s \, dy \lesssim \left( \int_Q w \, dy \right)^s \lesssim \text{ess inf}_{x \in Q} w(x)^s.
\]

Hence, \( w^s \in A_1 \).

Conversely, suppose \( w^s \in A_1 \). Given any cube \( Q \), by Hölder’s inequality,

\[
\int_Q w \, dy \leq \left( \int_Q w^s \, dy \right)^{\frac{1}{s}} \lesssim \text{ess inf}_{x \in Q} w(x) \leq \int_Q w \, dy.
\]

It follows at once that \( w \in A_1 \cap RH_s \). \( \square \)
The next two lemmas consider dilations of $A_1$ and $RH_\infty$ weights.

**Lemma 4.6.** If $w \in A_1$, then for any $r > 0$, $w^{-r} \in RH_\infty$.

*Proof.* Fix a cube $Q$. By Hölder’s inequality with exponent $p = 1 + r$,

$$1 = \int_Q w^{\frac{1}{r}} w^{-\frac{1}{r}} \, dx \leq \left( \int_Q w \, dy \right)^{\frac{1}{1+r}} \left( \int_Q w^{-r} \, dx \right)^{\frac{1}{1+r}}.$$ 

If we combine this with the fact that $w \in A_1$, we get that for a.e. $x \in Q$,

$$w(x)^{-r} \lesssim \left( \int_Q w \, dy \right)^{-r} \leq \int_Q w^{-r} \, dy.$$ 

Hence, $w^{-r} \in RH_\infty$. □

**Lemma 4.7.** If $w \in RH_\infty$, then for any $r > 0$, $w^r \in RH_\infty$.

*Proof.* If $r > 1$, this is follows from Hölder’s inequality: for any cube $Q$ and a.e. $x \in Q$,

$$w(x)^r \lesssim \left( \int_Q w \, dy \right)^r \leq \int_Q w^r \, dy.$$ 

If $r < 1$, then, since $w \in A_\infty$, by Proposition 3.7, $w^r \in RH_1/r$. Hence, we can repeat the above argument using the reverse Hölder inequality to get that $w^r \in RH_\infty$. □

Note that the analog of Lemma 4.6 is not true for $RH_\infty$ weights. Since $|x|^{-a} \in A_1$ for $0 \leq a < n$, by Lemma 4.6, $w(x) = |x|^b \in RH_\infty$ for any $b > 0$. But if $b > n$, then $w^{-1} \notin A_1$ since it is not locally integrable.

We also note in passing that the fact that $A_\infty$ is closed under the dilation $w^r$, $0 < r < 1$, seems to be particular to this class. For instance, there exists a doubling weight (i.e. $w$ such that $w(2Q) \leq w(Q)$ for all cubes $Q$) such that $w^r$ is not doubling for any $0 < r < 1$. See [17].

**Proof of Theorem 4.4.** We first fix $v_1 \in A_1 \cap RH_s$ and $v_2 \in A_p \cap RH_\infty$. By Lemmas 4.5 and 4.7, $v_1^s \in A_1$ and $v_2^s \in RH_\infty$. Then given any cube $Q$,

$$\int_Q w^s \, dx \lesssim \int_Q v_1^s \, dx \int_Q v_2^s \, dx \lesssim \int_Q v_1^s \, dx \left( \int_Q v_2 \, dx \right)^s \lesssim \left( \int_Q v_1 v_2 \, dx \right)^s.$$ 

Thus, $w \in RH_s$. Similarly, by Lemma 4.6, $v_1^{1-p'} \in RH_\infty$ and $v_1, v_2 \in A_p$, and so

$$\int_Q v_1 v_2 \, dx \left( \int_Q [v_1 v_2]^{1-p'} \, dx \right)^{p-1} \lesssim \int_Q v_1 \, dx \int_Q v_2 \, dx \left( \int_Q v_1^{1-p'} \, dx \right)^{p-1} \left( \int_Q v_2^{1-p'} \, dx \right)^{p-1} \leq [v_1]_{A_p} [v_2]_{A_p}.$$
Thus \( w \in A_p \).

To prove the converse, fix \( w \in A_p \cap RH_s \). Then by Proposition 3.7, \( w^s \in A_q \) with \( q = s(p - 1) + 1 \). But then by Theorem 4.2 there exist \( w_1, w_2 \in A_1 \) such that \( w^s = w_1 w_2^{1-q} \), or equivalently, \( w = w_1^q w_2^{1-p} = v_1 v_2 \). By Lemma 4.5, \( v_1 \in A_1 \cap RH_s \), and again by Theorem 4.2 and Lemma 4.6, \( v_2 \in A_p \cap RH_\infty \). \( \square \)

Finally, we note that the iteration algorithm and the Jones factorization theorem can be extended to other settings. For the factorization of the one-sided weights \( A^\pm_p \), see [27, 71]. For the extension of factorization to pairs of positive operators and to the two weight setting, see [27]. For reverse factorization for the variable \( A_p(\cdot) \) weights (the analog of the Muckenhoupt weights in the variable Lebesgue spaces [22]) see [35].

5. Rubio de Francia extrapolation

In this section we state and prove the Rubio de Francia extrapolation theorem. Our approach to extrapolation is based on the abstract formalism of families of extrapolation pairs. This approach was introduced (in passing) in [31] and first fully developed in [25]. (See also [27].) It was implicit from the beginning that in extrapolating from an inequality of the form

\[
\int_{\mathbb{R}^n} |Tf|^p w \, dx \lesssim \int_{\mathbb{R}^n} |f|^p w \, dx
\]

the operator \( T \) and its properties (positive, linear, etc.) played no role in the proof. Instead, all that mattered was that there existed a pair of non-negative functions \(|Tf|, |f|\) that satisfied a given collection of norm inequalities. Therefore, the proof goes through working with any pair \((f, g)\) of non-negative functions.

As a consequence, other kinds of inequalities can be proved using extrapolation. For example, if we take pairs of the form \(|Tf|, Mf\), where, for example, \( T \) is a Calderón-Zygmund singular integral operator, then we can prove Coifman-Fefferman type inequalities [13]:

\[
\int_{\mathbb{R}^n} |Tf|^p w \, dx \lesssim \int_{\mathbb{R}^n} (Mf)^p w \, dx.
\]

This was one of the reasons that this approach was adopted in [25]. We discuss this and other examples in detail below.

Hereafter, we will adopt the following conventions. A family of extrapolation pairs \( \mathcal{F} \) will consist of pairs of non-negative, measurable functions \((f, g)\) that are not equal to 0 a.e. When we write an inequality of the form

\[
\int_{\mathbb{R}^n} f^p w \, dx \leq C \int_{\mathbb{R}^n} g^p w \, dx, \quad (f, g) \in \mathcal{F},
\]
where $0 < p < \infty$ and $w \in A_p$, $1 \leq q \leq \infty$, we mean that this inequality holds for all pairs $(f, g) \in \mathcal{F}$ such that $\|f\|_{L^p(w)} < \infty$—i.e., that the left-hand side of the inequality is finite. We further assume that the constant $C$ can depend on $\mathcal{F}$, $p$, $q$, $n$, and the $[w]_{A_q}$ constant of $w$, but that it does not depend on the specific weight $w$. Note the assumption that $f$, $g$ are not identically 0 simply rules out trivial norm inequalities: since $A_\infty$ weights are positive a.e., we have that $\|f\|_{L^p(w)}$, $\|g\|_{L^p(w)} > 0$. Otherwise, if $f = 0$, then these inequalities hold for any $g$, and if $g = 0$, they only hold if $f = 0$.

If this seems mysterious, it may help to think of the particular family $\mathcal{F} = \{(|Tf|, |f|), f \in \mathcal{X}\}$, where $T$ is some operator we are interested in and $\mathcal{X}$ is some “nice” family of functions: $L^\infty_c$, $C^\infty_c$, etc. We will return to this point in Section 6 below when we consider applications of extrapolation.

**Theorem 5.1.** Given a family of extrapolation pairs $\mathcal{F}$, suppose that for some $p_0$, $1 \leq p_0 < \infty$, and every $w_0 \in A_{p_0}$,

$$\int_{\mathbb{R}^n} f^{p_0} w_0 \, dx \leq C \int_{\mathbb{R}^n} g^{p_0} w_0 \, dx, \quad (f, g) \in \mathcal{F}. \quad (5.1)$$

Then for every $p$, $1 < p < \infty$, and every $w \in A_p$,

$$\int_{\mathbb{R}^n} f^p w \, dx \leq C \int_{\mathbb{R}^n} g^p w \, dx, \quad (f, g) \in \mathcal{F}. \quad (5.2)$$

In the statement of Theorem 5.1 we want to call attention to the fact that while we can start with an endpoint inequality (i.e., with the assumption that $p_0 = 1$), we cannot use Rubio de Francia extrapolation to prove an endpoint inequality: we must assume $p > 1$. To see that this restriction is natural, note that the operator $M^2 = M \circ M$ is bounded on $L^p(w)$, $1 < p < \infty$, $w \in A_p$, but does not satisfy an unweighted weak $(1, 1)$ inequality. It is possible to prove endpoint estimates using generalizations of the extrapolation theorem, but much stronger, two weight hypotheses are required. See [27, Section 8.3].

**Proof.** Before giving the details of the proof, we first sketch the basic ideas underlying it. To prove (5.2) from (5.1) we need to pass between $L^p$ and $L^{p_0}$ inequalities. To do this we will use duality and Hölder’s inequality. The original proofs of extrapolation required two cases, depending on whether $p_0 < p$ or $p_0 > p$; we avoid this by first dualising to $L^1$ and then using Hölder’s inequality. (This comes with a cost: see the discussion of sharp constants in Section 7 below.)

Next, to apply (5.1) we need to construct an $A_{p_0}$ weight, using only that we have a weight in $A_p$. Here we will use the Rubio de Francia iteration algorithm to construct $A_1$ weights, and then use reverse factorization (the easy half of Theorem 4.2) to form the desired weight.
Fix $p$, $1 < p < \infty$, and $w \in A_p$. We begin with the iteration algorithms. Since $w \in A_p$, $\sigma = w^{1-p'} \in A_{p'}$. Therefore, by Theorem 4.1 we can define the two iteration algorithms

$$\mathcal{R}_1 h_1 = \sum_{k=0}^{\infty} \frac{M^k h_1}{2^k \|M\|^k_{L_p(w)}}, \quad \mathcal{R}_2 h_2 = \sum_{k=0}^{\infty} \frac{M^k h_2}{2^k \|M\|^k_{L'_{p'}(\sigma)}},$$

which satisfy the following properties:

- $(A_1)$ $h_1(x) \leq \mathcal{R}_1 h_1(x)$
- $(A_2)$ $h_2(x) \leq \mathcal{R}_2 h_2(x)$
- $(B_1)$ $\|\mathcal{R}_1 h_1\|_{L_p(w)} \leq 2\|h_1\|_{L_p(w)}$
- $(B_2)$ $\|\mathcal{R}_2 h_2\|_{L'_{p'}(\sigma)} \leq 2\|h_2\|_{L'_{p'}(\sigma)}$
- $(C_1)$ $[\mathcal{R}_1 h_1]_{A_1} \leq 2\|M\|_{L_p(w)}$
- $(C_2)$ $[\mathcal{R}_2 h_2]_{A_1} \leq 2\|M\|_{L'_{p'}(\sigma)}$.

We now define $h_1$. Fix $(f, g) \in F$ such that $\|f\|_{L_p(w)} < \infty$. We may also assume $\|g\|_{L_p(w)} < \infty$, since otherwise there is nothing to prove. Define

$$h_1 = \frac{f}{\|f\|_{L_p(w)}} + \frac{g}{\|g\|_{L_p(w)}},$$

then $h_1 \in L_p(w)$ and $\|h_1\|_{L_p(w)} \leq 2$.

We now prove the desired inequality. We will assume $1 < p_0 < \infty$; the case $p_0 = 1$ requires some minor modifications to the argument and we omit the details. Since $f \in L_p(w)$, there exists a non-negative function $h_2 \in L'_{p'}(w)$, $\|h_2\|_{L'_{p'}(w)} = 1$, such that

$$\|f\|_{L_p(w)} = \int_{\mathbb{R}^n} f h_2 w \, dx.$$

By $(A_2)$ and Hölder’s inequality,

$$\leq \int_{\mathbb{R}^n} f (\mathcal{R}_1 h_1)^{-\frac{1}{p_0}} (\mathcal{R}_1 h_1)^{\frac{1}{p_0}} \mathcal{R}_2(h_2 w) \, dx \leq \left( \int_{\mathbb{R}^n} f^{p_0} (\mathcal{R}_1 h_1)^{1-p_0} \mathcal{R}_2(h_2 w) \, dx \right)^{\frac{1}{p_0}} \left( \int_{\mathbb{R}^n} \mathcal{R}_1 h_1 \mathcal{R}_2(h_2 w) \, dx \right)^{\frac{1}{p_0}} = I_{f_0} \cdot I_{2_0}.$$ We first estimate $I_2$: by $(B_1)$ and $(B_2)$,

$$I_2 = \int_{\mathbb{R}^n} \mathcal{R}_1 h_1 w^{\frac{1}{p'}} \mathcal{R}_2(h_2 w) w^{-\frac{1}{p'}} \, dx \leq \|\mathcal{R}_1 h_1\|_{L_p(w)} \|\mathcal{R}_2(h_2 w)\|_{L'_{p'}(\sigma)} \leq 4\|h_1\|_{L_p(w)} \|h_2 w\|_{L'_{p'}(\sigma)} \leq 8\|h_2\|_{L'_{p'}(w)} = 8.$$
To estimate $I_1$ we want to apply (5.1). To do so, first note that by $(C_1)$, $(C_2)$ and Theorem 4.2,

$$w_0 = (R_1 h_1)^{1-p_0} R_2(h_2 w) \in A_{p_0}.$$ 

Further, we have that $I_1 < \infty$: by $(A_1)$,

$$\frac{f}{\|f\|_{L^p(w)}} \leq h_1 \leq R_1 h_1,$$

and so

$$I_1 \leq \left\| f \right\|_{L^p(w)} \int_{\mathbb{R}^n} R_1 h_1 R_2(h_2 w) \, dx < \infty.$$ 

Therefore, by (5.1) and since, again by $(A_1)$,

$$\frac{g}{\|g\|_{L^p(w)}} \leq h_1 \leq R_1 h_1,$$

$$I_1 \lesssim \int_{\mathbb{R}^n} g^{p_0} (R_1 h_1)^{1-p_0} R_2(h_2 w) \, dx \leq \left\| g \right\|_{L^p(w)}^{p_0} \int_{\mathbb{R}^n} R_1 h_1 R_2(h_2 w) \, dx \lesssim \|g\|_{L^p(w)}^{p_0}.$$ 

Combining these estimates we get (5.2) and this completes the proof. □

We will now prove three extensions of Rubio de Francia extrapolation that are immediate consequences of Theorem 5.1 and the formalism of extrapolation pairs.

**Corollary 5.2.** Given a family of extrapolation pairs $\mathcal{F}$, suppose that for some $p_0, 1 \leq p_0 < \infty$, and every $w_0 \in A_{p_0}$,

\[(5.3) \quad \left\| f \right\|_{L^{p_0, \infty}(w_0)} \leq C\|g\|_{L^{p_0}(w_0)}, \quad (f, g) \in \mathcal{F}.\]

Then for every $p, 1 < p < \infty$, and every $w \in A_p$,

\[(5.4) \quad \left\| f \right\|_{L^{p, \infty}(w)} \leq C\|g\|_{L^p(w)}, \quad (f, g) \in \mathcal{F}.\]

**Proof.** Define a new family

$$\mathcal{F}' = \{(f_t, g) = (t\chi_{\{x: f(x) > t\}}, g) : (f, g) \in \mathcal{F}, t > 0\}.$$ 

Then by our assumption (5.3),

$$\|f_t\|_{L^{p_0}(w_0)} = tw_0(\{x \in \mathbb{R}^n : f(x) > t\})^{1/p_0} \leq \|f\|_{L^{p_0, \infty}(w_0)} \leq C\|g\|_{L^{p_0}(w_0)}.$$ 

Therefore, (5.1) holds for the family $\mathcal{F}'$. Hence, for all $p$ and $w \in A_p$, (5.2) holds for $\mathcal{F}'$ with a constant independent of $t$, and this implies that (5.4) holds. □

Our second corollary shows that vector-valued inequalities are an immediate consequence of Rubio de Francia extrapolation.


Corollary 5.3. Given a family of extrapolation pairs \( F \), suppose that for some \( p_0 \), \( 1 \leq p_0 < \infty \), and every \( w_0 \in A_{p_0} \),

\[
\|f\|_{L^{p_0}(w_0)} \leq C \|g\|_{L^{p_0}(w_0)}, \quad (f, g) \in F.
\]

Then for every \( 1 < p, q < \infty \) and every \( w \in A_p \),

\[
\left\| \left( \sum_i f_i^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \leq C \left\| \left( \sum_i g_i^q \right)^{\frac{1}{q}} \right\|_{L^p(w)}, \quad \{(f_i, g_i) \} \subset F.
\]

Proof. Fix \( q, 1 < q < \infty \), and define the new family of extrapolation pairs

\[
F_q = \left\{ (F, G) = \left( \left( \sum_i f_i^q \right)^{\frac{1}{q}}, \left( \sum_i g_i^q \right)^{\frac{1}{q}} \right) : (f, g) \in F \right\},
\]

where all of the sums are taken to be finite. Since \( (5.5) \) holds, by Theorem 5.1, \( (5.2) \) holds with \( p = q \) and \( w \in A_q \). Hence, for all \( (F, G) \in F_q \),

\[
\left\|F\right\|_{L^p(w)}^q = \sum_i \int_{\mathbb{R}^n} f_i^q w \, dx \lesssim \sum_i \int_{\mathbb{R}^n} g_i^q w \, dx = \left\|G\right\|_{L^p(w)}^q.
\]

If we take this as our hypothesis, we can again apply Theorem 5.1 to conclude that for \( 1 < p < \infty \) and \( w \in A_p \),

\[
\|F\|_{L^p(w)} \lesssim \|G\|_{L^p(w)}, \quad (F, G) \in F_q.
\]

But this in turn is equivalent to \( (5.6) \) for all finite sums. By the monotone convergence theorem we may pass to arbitrary sums, which completes the proof. \( \Box \)

Our final corollary shows that we can rescale extrapolation families and so derive the \( A_{\infty} \) extrapolation theorem first proved in [25].

Corollary 5.4. Given a family of extrapolation pairs \( F \), suppose that for some \( p_0, 0 < p_0 < \infty \), and every \( w_0 \in A_{p_0} \),

\[
\|f\|_{L^{p_0}(w_0)} \leq C \|g\|_{L^{p_0}(w_0)}, \quad (f, g) \in F.
\]

Then for every \( p, 0 < p < \infty \), and every \( w \in A_{\infty} \),

\[
\|f\|_{L^p(w)} \leq C \|g\|_{L^p(w)}, \quad (f, g) \in F.
\]

Proof. Fix \( q_0, 1 < q_0 < \infty \), and define the new family

\[
F_0 = \left\{ (F, G) = \left( \left( \frac{f_i}{w_0} \right)^{\frac{p}{p_0}}, \left( \frac{g_i}{w_0} \right)^{\frac{p}{p_0}} \right) : (f, g) \in F \right\}.
\]

Then for every weight \( w_0 \in A_{q_0} \) and every pair \( (F, G) \in F_0 \),

\[
\int_{\mathbb{R}^n} F_{q_0} w_0 \, dx = \int_{\mathbb{R}^n} f_{q_0} w_0 \, dx \lesssim \int_{\mathbb{R}^n} g_{q_0} w_0 \, dx = \int_{\mathbb{R}^n} G_{q_0} w_0 \, dx.
\]
Therefore, (5.1) holds with \( p_0 = q_0 \) for the family \( \mathcal{F}_0 \), and so by Theorem 5.1, for any \( q, 1 < q < \infty \), and \( w \in A_q \), \( \|F\|_{L^q(w)} \lesssim \|G\|_{L^q(w)} \), \( (F, G) \in \mathcal{F}_0 \). Equivalently,

(5.9) \[ \int_{\mathbb{R}^n} f^{\frac{p_0}{q_0}} w \, dx \lesssim \int_{\mathbb{R}^n} g^{\frac{p_0}{q_0}} w \, dx, \quad (f, g) \in \mathcal{F}. \]

To complete the proof, we use that we can choose \( q_0 \) and \( q \) freely. Fix \( 0 < p < \infty \) and \( w \in A_\infty \). Then \( w \in A_q \) for some \( q > 1 \), and since the Muckenhoupt classes are nested, we may assume that \( q > \frac{p}{p_0} \). Therefore, we can fix \( q_0 > 1 \) such that \( q = \frac{p}{p_0} q_0 \), or \( \frac{q_0}{p_0} = p \). Then (5.9) gives us (5.8). \( \square \)

6. Applications of Rubio de Francia extrapolation

In this section we give three applications of Rubio de Francia extrapolation and the extensions proved in the last section. These examples are not exhaustive but should give some sense of the ways in which extrapolation can be used.

First, however, we consider further the technical hypothesis that we only work with extrapolation pairs \( (f, g) \) for which the left-hand side of the weighted norm inequality in question is finite. We can eliminate this hypothesis with the following approximation argument. Given a family \( \mathcal{F} \), we define a new family

\[ \mathcal{F}_0 = \{(F, G) = (\min(f, N) \chi_{B(0, N)}, g) : (f, g) \in \mathcal{F}, N \in \mathbb{N}\}. \]

Since a weight \( w \in A_\infty \) is locally integrable, we have that for any \( p, 0 < p < \infty \), and any pair \( (F, G) \in \mathcal{F}_0 \),

\[ \int_{\mathbb{R}^n} F^p w \, dx \leq N^p w(B(0, N)) < \infty. \]

Therefore, we can apply Theorem 5.1 to the family \( \mathcal{F}_0 \); the desired inequality for a given pair \( (f, g) \in \mathcal{F} \), whether or not \( \|f\|_{L^p(w)} \) is finite, follows from the monotone convergence theorem if we let \( N \to \infty \).

Given this reduction, it is now straightforward to prove weighted norm inequalities for an operator \( T \). Suppose, for instance, that for some \( p_0 \geq 1 \) and \( w_0 \in A_{p_0} \), we know that

\[ \|T f\|_{L^{p_0}(w_0)} \lesssim \|f\|_{L^{p_0}(w_0)}, \]

where the constant depends only on \( T, p_0, n, T \) and \([w]_{A_{p_0}}\). Then, in particular, it holds for some suitable dense subset \( \mathcal{X} \) of this space: e.g., \( \mathcal{X} = L^\infty, C^\infty \), etc. (Indeed, this inequality may only have been proved for functions in this dense family.) Then if we define the family of extrapolation pairs

\[ \mathcal{F} = \{([Tf], |f|) : f \in \mathcal{X}\}, \]

we have that the hypothesis (5.1) of Theorem 5.1 holds, and so we can conclude that for all \( p \) and \( w \in A_p \), (5.2) holds. If we do not know a priori that the left-hand side of this inequality is finite, then we can apply the theorem to a family \( \mathcal{F}_0 \) defined as
above, and get the desired conclusion via approximation. To prove that the operator is bounded on all $f \in L^p(w)$, it suffices to use another standard approximation argument.

We now turn to our examples. The first is the well-known vector-valued inequality for the maximal operator. In the unweighted case this was proved by Fefferman and Stein [47]; the weighted estimate is due to Andersen and John [1]. We want to emphasize that given the scalar inequality in Theorem 2.3, the vector-valued inequality is an immediate consequence of Corollary 5.3: no further work is required.

**Theorem 6.1.** For every $1 < p, q < \infty$ and every $w \in A_p$,

$$
\left\| \left( \sum_i (Mf_i)^q \right)^{\frac{1}{q}} \right\|_{L^p(w)} \lesssim \left\| \left( \sum_i |f_i|^q \right)^{\frac{1}{q}} \right\|_{L^p(w)}.
$$

Similar vector-valued inequalities hold for other operators, such as Calderón-Zygmund singular integral operators and commutators. We refer the reader to [25, 27] for further examples.

Our second example uses extrapolation to prove the Coifman-Fefferman inequality relating singular integrals and the maximal operator [13].

**Theorem 6.2.** Let $T$ be any Calderón-Zygmund singular integral operator. Then for $0 < p < \infty$, $w \in A_\infty$ and $f \in L^\infty_c$,

$$
(6.1) \quad \int_{\mathbb{R}^n} |Tf|^p w \, dx \lesssim \int_{\mathbb{R}^n} (Mf)^p w \, dx.
$$

**Proof.** By Corollary 5.4 it will suffice to prove (6.1) when $p = 1$. We will sketch an easy proof in this case using the theory of dyadic grids and sparse operators. In the past decade, this approach has come to play a central role in the theory of weighted norm inequalities in harmonic analysis, starting with Hytönen’s proof of the $A_2$ conjecture [57] (see also [16, 65, 67]). For an overview of these techniques (though from the perspective of fractional integral operators) see [18].

We begin by defining $3^n$ translates of the standard dyadic grid using the so-called “one-third” trick:

$$
\mathcal{D}^t = \{2^j((0,1)^n + m + t) : j \in \mathbb{Z}, m \in \mathbb{Z}^n \}, \quad t \in \{0, \pm 1/3\}^n.
$$

The translation by $t$ does not affect any of the underlying properties of the dyadic cubes. In particular, Lemmas 2.4 and 2.6 are still true, in the latter replacing $M^d_\sigma$ with $M^d_{\mathcal{D}^t}$, the dyadic maximal defined with respect to cubes in $\mathcal{D}^t$.

A set $S \subseteq \mathcal{D}^t$ is said to be sparse if for every $Q \in S$ there exists a measurable set $E_Q \subset Q$ such that $|E_Q| \geq \frac{1}{2}|Q|$ and the sets $E_Q$ are pairwise disjoint. A sparse
operator is a positive linear operator of the form

\[ T_S f(x) = \sum_{Q \in S} \int_Q f(y) \, dy \cdot \chi_Q(x). \]

These operators are dyadic models of Calderón-Zygmund singular integrals. More importantly, we have the following pointwise estimate: given a Calderón-Zygmund singular integral \( T \) and a function \( f \in L^\infty_c \), there exist sparse sets \( S_t \subset D^t \) such that

\[ |T f(x)| \lesssim \sum_{t \in \{0, \pm 1/3\}^n} T_{S_t}(|f|)(x). \tag{6.2} \]

This estimate was originally proved by Lerner and Nazarov [67] and independently by Conde-Alonso and Rey [16]. Since then there have been a number of new proofs and extensions: see, for instance, Lerner [66], Hytönen, et al. [59], Lacey [63], and Conde-Alonso, et al. [15].

Given inequality (6.2), to complete the proof it will suffice to show that given any sparse set \( S \subset D^t \) and \( w \in A^\infty \), for non-negative \( f \in L^\infty_c \),

\[ \int_{\mathbb{R}^n} T_S f \, w \, dx \lesssim \int_{\mathbb{R}^n} M f \, w \, dx; \]

in fact, we will prove this inequality with the Hardy-Littlewood maximal operator replaced by the smaller dyadic maximal operator \( M_{D^t} \) defined with respect to the cubes in \( D^t \). But this is almost trivial: by Lemma 2.5,

\[ \int_{\mathbb{R}^n} T_S f \, w \, dx = \sum_{Q \in S} \int_Q f(y) \, w(Q) \leq \sum_{Q \in S} \int_Q f(y) \, dy \cdot w(E_Q) \leq \sum_{Q \in S} \int_{E_Q} M_{D^t}(f) \, w \, dx \leq \int_{\mathbb{R}^n} M_{D^t}(f) \, w \, dx. \]

\[ \square \]

For our final application we consider weighted norm inequalities for rough singular integrals. Unlike the previous results which were originally proved without extrapolation, the following theorem was proved by Duoandikoetxea and Rubio de Francia [45] using extrapolation in a critical way. For a version of this result with quantitative estimates on the constants, see [59]. For a generalization to a larger class of rough singular integrals, see [15].

By a rough singular integral we mean the singular convolution operator

\[ T_{\Omega} f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(y/|y|)}{|y|^n} f(x - y) \, dy, \]

where \( \Omega \in L^\infty(S^{n-1}) \) and \( \int_{S^{n-1}} \Omega \, dx = 0. \)
Theorem 6.3. Given a rough singular integral $T_{\Omega}$, for every $1 < p < \infty$ and every $w \in A_p$, 

\begin{equation} \label{6.3} \int_{\mathbb{R}^n} |T_{\Omega} f|^p w \, dx \lesssim \int_{\mathbb{R}^n} |f|^p w \, dx. \end{equation}

**Proof.** We sketch the argument in [45], emphasizing those parts of the proof that are more widely applicable. We begin with the key reduction: by Theorem 5.1 it suffices to prove (6.3) when $p = 2$ and $w \in A_2$.

Using Fourier transform techniques and Littlewood-Paley theory, they showed that there exist operators $T_j$, $j \in \mathbb{Z}$, such that for all $f \in L^2$,

\begin{equation} \label{6.4} T_{\Omega} f(x) = \sum_j T_j f(x). \end{equation}

Moreover, they showed that there exist $C, \alpha > 0$ such that for all $j$

\begin{equation} \label{6.5} \|T_j f\|_{L^2(w)} \leq C 2^{-\alpha|j|} \|f\|_{L^2}. \end{equation}

Thus, in particular, the series decomposition of $T_{\Omega}$ converges in $L^2$.

To get estimates in $L^2(w)$, $w \in A_2$, they used weighted Littlewood-Paley theory [62, 88] to prove that for all $f \in C^\infty_c$,

\begin{equation} \label{6.6} \|T_j f\|_{L^2(w)} \leq C \|f\|_{L^2(w)}, \end{equation}

where the constant $C > 0$ is independent of $j$ and depends only on $[w]_{A_2}$ and not on the weight itself. However, the constant has no decay, so this inequality cannot be used to directly prove weighted norm inequalities for $T_{\Omega}$.

To overcome this, note that since $w \in A_2$, $w^{-1} \in A_2$, so by the reverse Hölder inequality (applied twice) there exists $\epsilon > 0$ such that $w^{1+\epsilon} \in A_2$, and in fact we can choose $\epsilon$ so that $[w^{1+\epsilon}]_{A_2} \leq 4[w]_{A_2}$. (See Theorem 3.2.) Hence, for all $f \in C^\infty_c$,

\begin{equation} \label{6.7} \|T_j f\|_{L^2(w^{1+\epsilon})} \leq C \|f\|_{L^2(w^{1+\epsilon})}, \end{equation}

and the constant is independent of $\epsilon$. Therefore, by the interpolation with change of measure theorem due to Stein and Weiss [90] (see also [9]) we can interpolate between (6.5) and (6.7) to get

$$\|T_j f\|_{L^2(w)} \leq C 2^{-\frac{\alpha}{1+\epsilon}|j|} \|f\|_{L^2(w)}. \|

Hence, if we combine this with (6.4), we have that for all $w \in A_2$ and $f \in C^\infty_c$,

$$\|T_{\Omega} f\|_{L^2(w)} \leq C \|f\|_{L^2(w)}, \|

which completes the proof. \qed

We want to highlight one feature of this proof. The use of extrapolation to reduce the problem to proving $L^2$ estimates makes it possible to more easily prove various square function and Littlewood-Paley estimates. For an application of this approach
to multiplier theory and Kato-Ponce inequalities, see [29]. For an application in a somewhat different context, see Fefferman and Pipher [48].

Further, by reducing the problem to $L^2$, the argument using interpolation with change of measure allows unweighted inequalities derived using Fourier transform estimates to be “imported” into weighted $L^2(w)$, overcoming the fact that there are no useful weighted estimates for the Fourier transform. For another application of this technique in the study of degenerate elliptic PDEs and the Kato problem, see [33].

7. Sharp constant extrapolation

In this section we consider the problem of the sharp constant, in terms of the $A_p$ constant, in Rubio de Francia extrapolation. Suppose that we know that for some $p_0$, $1 \leq p_0 < \infty$, and family of extrapolation pairs $F$, there exists a function $N_{p_0}$ such that for every $w_0 \in A_{p_0}$,

$$
\|f\|_{L^{p_0}(w_0)} \leq N_{p_0}([w]_{A_{p_0}})\|g\|_{L^{p_0}(w_0)}, \quad (f, g) \in F.
$$

Then for $1 < p < \infty$ the problem is to find the optimal function $N_p$ such that for all $w \in A_p$,

$$
\|f\|_{L^p(w)} \leq N_p([w]_{A_p})\|g\|_{L^p(w)}, \quad (f, g) \in F.
$$

A close examination of the proof of Theorem 5.1 shows that we get

$$
N_p([w]_{A_p}) = c_1 N_{p_0}(c_2 [w]_{A_p}^{1 + \frac{p_0 - 1}{p - 1}}),
$$

where $c_1, c_2 > 0$ depend on $n, p, p_0$. However, this can be improved.

**Theorem 7.1.** Given $1 \leq p_0 < \infty$ and a family of extrapolation pairs $F$, suppose that for every $w_0 \in A_{p_0}$,

$$
\|f\|_{L^{p_0}(w_0)} \leq N_{p_0}([w]_{A_{p_0}})\|g\|_{L^{p_0}(w_0)}, \quad (f, g) \in F.
$$

Then for every $1 < p < \infty$ and every $w \in A_p$,

$$
\|f\|_{L^p(w)} \leq N_p([w]_{A_p})\|g\|_{L^p(w)}, \quad (f, g) \in F,
$$

where

$$
N_p([w]_{A_p}) \leq C(p, p_0) N_{p_0}(C(n, p, p_0)[w]_{A_p}^{\max(1, \frac{p_0 - 1}{p - 1})}).
$$

As we will see below, this is the optimal result, since it yields sharp inequalities for singular integrals and other operators. For a complete proof, see [42] or [27, Theorem 3.22]. Here we will restrict ourselves to giving an idea of why the proof of Theorem 5.1 does not yield the best constant, and how the proof has to be modified to achieve this.

One of the main features of the proof of Theorem 5.1 that distinguishes it from previous proofs is that it only required a single case. However, as a consequence we
have to use both iteration algorithms \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \). Each one contributes a power of the \( A_p \) constant of \( w \), so we get the sum \( 1 + \frac{p_0 - 1}{p - 1} \) in the exponent in (7.1).

To avoid this, we need to modify the proof and treat two cases. If \( p < p_0 \), then we can apply Hölder’s inequality immediately and then argue only using the iteration algorithm \( \mathcal{R}_2 \). This yields the exponent \( \frac{p_0 - 1}{p - 1} \). On the other hand, if \( p > p_0 \), then, instead of using duality, we can fix \( h_1 \) so that \( f \leq \mathcal{R}_1 h_1 \) and write

\[
\int_{\mathbb{R}^n} f^p w \, dx = \int_{\mathbb{R}^n} f^{p_0}(\mathcal{R}_1 h_1)^{-(p_0 - p)} w \, dx.
\]

We can now modify the previous proof; this yields the exponent 1. In both cases we make use of the sharp constant in the weighted norm inequalities for the maximal operator from Theorem 2.3.

An interesting open question is to determine a sharp constant version of Corollary 5.4, \( A_\infty \) extrapolation. The precise constant may depend on which of the equivalent definitions of \( A_\infty \) is used.

We now want to consider two examples where the sharp constant, in terms of the \([w]_{A_p}\) constant, matters. The first is not a direct application of Theorem 7.1, but it uses some of the same ideas.

**Proposition 7.2.** Let \( T \) be an operator such that for some \( p_0, 1 \leq p_0 < \infty \), and every \( w_0 \in A_{p_0} \),

\[
\|Tf\|_{L^{p_0}(w_0)} \leq C[w_0]_{A_{p_0}}^\alpha \|f\|_{L^{p_0}(w_0)}.
\]

Then as \( p \to \infty \),

\[
\|Tf\|_p \leq C p^\alpha \|f\|_p.
\]

**Proof.** Our proof uses the Rubio de Francia iteration algorithm and is, in some sense, a special case of the proof of Theorem 7.1. Fix \( p > p_0 \) and define the iteration algorithm

\[
\mathcal{R} h = \sum_{k=0}^{\infty} \frac{M^k h}{2^k \|M\|_{(p/p_0)'}^k}.
\]

By the standard proofs of the boundedness of the maximal operator (using Marcinkiewicz interpolation), \( \|M\|_{(p/p_0)'} = C(n, p_0)p \). Therefore, by Theorem 4.1,

\[
[\mathcal{R} h]_{A_{p_0}} \leq [\mathcal{R} h]_{A_1} \leq 2 \|M\|_{(p/p_0)'} = C(n, p_0)p.
\]

We can now argue as follows: by duality there exists \( h \in L^{(p/p_0)'} \), \( \|h\|_{(p/p_0)'} = 1 \), such that

\[
\|Tf\|_{p_0}^p = \int_{\mathbb{R}^n} |Tf|^{p_0} h \, dx;
\]
by the majorant property of $\mathcal{R}$, our hypothesis, Hölder’s inequality and the boundedness of $\mathcal{R}$ on $L^{(p/p_0)'}$,

$$\leq \int_{\mathbb{R}^n} |Tf|^{p_0} \mathcal{R}h \, dx$$

$$\leq C(n, p_0) p^{\alpha p_0} \int_{\mathbb{R}^n} |f|^{p_0} \mathcal{R}h \, dx$$

$$\leq C(n, p_0) p^{\alpha p_0} \|f\|_{L^p(\mathcal{A})} \|\mathcal{R}h\|_{(p/p_0)'}$$

$$\leq C(n, p_0) p^{\alpha p_0} \|f\|_{p_0}.$$  \hfill \Box

Proposition 7.2 is implicit in Fefferman and Pipher [48] who used it to get estimates for multiparameter singular integrals. In [26] this argument was used to show that the exponent $\alpha$ obtained for the weighted norm inequality for the dyadic square function was the best possible. Luque, P´erez and Rela [69] developed this idea further to show the general relationship between the best exponent in the weighted inequalities and the behavior of the constant in the unweighted inequality as $p \to 1$ or $p \to \infty$.

A much deeper application of the optimal constant in extrapolation comes from the study of the Beltrami equation in the plane. Given a bounded, open set $\Omega \subset \mathbb{C}$, a map $f : \Omega \to \mathbb{C}$ is a weakly $K$-quasiregular map if $f \in W^{1,q}_{\text{loc}}(\Omega)$, $1 \leq q \leq 2$, and $f$ is a solution of the Beltrami equation,

$$\partial_z f(z) = \mu(z) \partial_{\bar{z}} f(z), \quad \text{a.e. } z \in \Omega,$$

where $\mu$ is a bounded, complex-valued function such that

$$\|\mu\|_{\infty} \leq k = \frac{K - 1}{K + 1} < 1.$$

If $f$ is also continuous, then we say that it is $K$-quasiregular. If $f \in W^{1,1+k+\epsilon}_{\text{loc}}(\Omega)$, $\epsilon > 0$, then it was shown that $f$ is continuous; if $f \in W^{1,1+k-\epsilon}_{\text{loc}}(\Omega)$, then there are examples of weakly $K$-quasiregular maps that are not $K$-quasiregular (see [3]). In the critical exponent case, that is, when $f \in W^{1,1+k}_{\text{loc}}(\Omega)$, Astala, Iwaniec and Saksman [3] showed that $f$ is continuous if the Beurling-Ahlfors transform,

$$Tf(z) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{f(w)}{(w - z)^2} dA(w),$$

satisfies a quantitative weighted norm inequality: for every $p \geq 2$ there exists $C > 0$ such that for every $w \in \mathcal{A}_p$, (7.2)

$$\|Tf\|_{L^p(\mathcal{A})} \leq C[w] \mathcal{A}_p \|f\|_{L^p(\mathcal{A})}.$$

The Beurling-Ahlfors transform is a two-dimensional Calderón-Zygmund singular integral operator. The original proofs of weighted norm inequalities for singular
integrals did not give quantitative bounds in terms of the $A_p$ constant: later, a close examination of the proofs showed that the constant was on the order of $\exp(c[w]_{A_p})$. Buckley [11] proved that for all $1 < p < \infty$ and any singular integral $T$,

$$\|Tf\|_{L^p(w)} \leq C[w]_{A_p} \frac{1}{p-1} \|f\|_{L^p(w)};$$

he also gave examples to show that in general, the smallest possible exponent was $\max(1, \frac{1}{p-1})$.

By Theorem 7.1, to prove that this is the sharp exponent, and, in particular, to prove (7.2) for the Beurling-Ahlfors transform, it suffices to prove that for $p = 2$ and $w \in A_2$,

$$\|Tf\|_{L^2(w)} \leq C[w]_{A_2} \|f\|_{L^2(w)}.$$

Because of this, the sharp constant problem for singular integrals became known as the $A_2$ conjecture.

For the Beurling-Ahlfors transform, this conjecture was proved by Petermichl and Volberg [79] using a Bellman function argument. Petermichl then extended these techniques to prove it for the Hilbert transform [77] and the Riesz transforms [78]. A number of partial results were obtained for more general singular integrals: see, for instance [26] and the references it contains. The problem was finally solved in full generality by Hytönen [57]. In all of these arguments extrapolation played a central role in reducing to the case $p = 2$.

The sparse domination inequality (6.2) was developed to simplify the original argument of Hytönen; here we give this proof.

Theorem 7.3. Given a Calderón-Zygmund singular integral operator $T$, for every $1 < p < \infty$ and every $w \in A_p$,

$$\|Tf\|_{L^p(w)} \leq C[w]_{A_p}^{\max(1, \frac{1}{p-1})} \|f\|_{L^p(w)}.$$

Proof. By Theorem 7.1 and inequality (6.2), it will suffice to show that if $S$ is a sparse subset of some dyadic grid $D^t$, then for all $w \in A_2$ and non-negative $f \in L^\infty_c$,

$$\|T_S f\|_{L^2(w)} \leq C[w]_{A_2} \|f\|_{L^2(w)}.$$

To prove this we will use an argument from [26]. Let $\sigma = w^{-1}$. Then by duality there exists $h \in L^2(\sigma)$, $\|h\|_{L^2(\sigma)} = 1$, such that

$$\|T_S f\|_{L^2(w)} = \int_{\mathbb{R}^n} T_S f h \, dx$$

$$= \sum_{Q \in S} \int_Q f \, dx \int_Q h \, dx |Q|;$$
by the definition of a sparse set and the definition of $A_2$,

$$
\leq 2 \left[ w \right]_{A_2} \sum_{Q \in S} w(Q) \frac{\sigma(Q)}{|Q|} \int_Q f w \, d\sigma \int_Q h \sigma \, dw \, |E_Q|
$$

by Hölder’s inequality and Lemma 2.6 (which holds for general dyadic grids with the same proof),

$$
\leq 2 \left[ w \right]_{A_2} \sum_{Q} \int_{E_Q} M_{\sigma}^{D_t}(f w) M_{w}^{D_t}(h \sigma) \, dx
$$

$$
\leq 2 \left[ w \right]_{A_2} \int_{\mathbb{R}^n} M_{\sigma}^{D_t}(f w) \sigma^{1/2} M_{w}^{D_t}(h \sigma) w^{1/2} \, dx;
$$

Finally, we note in passing that it is possible to prove Theorem 7.3 without using extrapolation. The $L^2$ estimate for sparse operators can be extended to weighted $L^p$, though the resulting proof is more complicated. See Moen [72] for the details.

8. Restricted range extrapolation

In this section we consider a second variation of Rubio de Francia extrapolation, restricted range extrapolation. Restricted range extrapolation was first proved by Auscher and Martell [7] (and also by Duoandikoetxea [44] but with a very different perspective). Auscher and Martell were considering families of operators associated with certain second order elliptic PDEs; these PDEs in turn were of interest because of their connection with the Kato conjecture (for a history of this problem, see [5] and the references it contains). Let $A$ be an $n \times n$ matrix of measurable, complex valued functions that for some $0 < \lambda < \Lambda < \infty$ satisfies the ellipticity conditions

$$
\lambda |\xi|^2 \leq \text{Re} \langle A \xi, \xi \rangle, \quad |\langle A \xi, \nu \rangle| \leq \Lambda |\xi||\nu|, \quad \xi, \nu \in \mathbb{C}^n.
$$

Define the differential operator $Lu = -\text{div} A \nabla u$. Then the Kato conjecture states that for all $u \in W^{1,2}(\mathbb{R}^n)$ (i.e., $u$ such that $u, \nabla u \in L^2$),

$$
\|L^{1/2} u\|_2 \approx \|\nabla u\|_2,
$$

where the operator $L^{1/2}$ is defined using the functional calculus. We can define (again, via the functional calculus) the associated Riesz transform $\nabla L^{-1/2}$; when $A$ is the
identity matrix, this is just the classical (vector) Riesz transform. It follows from (8.1) that
\[ \| \nabla L^{-1/2}u \|_2 \lesssim \| u \|_2. \]
These operators also satisfy $L^p$ inequalities, $p \neq 2$, but unlike the classical Riesz transforms, one cannot take $p \in (1, \infty)$. Rather, for each operator $L$ there exist $1 \leq p_- < 2 < p_+ \leq \infty$ such that if $p \in (p_-, p_+)$, then
\[ \| \nabla L^{-1/2}u \|_p \lesssim \| u \|_p. \]
In certain cases this estimate holds for all $p \in (1, \infty)$, but there exist operators such that $(p_-, p_+) = (2 - \delta, 2 + \epsilon)$ where $\epsilon, \delta > 0$ are small: see [4].

It is natural to ask under what conditions the corresponding weighted inequalities,
\[ \| \nabla L^{-1/2}u \|_{L^p(w)} \lesssim \| u \|_{L^p(w)}, \]
hold. Auscher and Martell [6] showed that for $p_- < p < p_+$, this inequality holds for all weights $w$ such that $w \in A_{p/p_- \cap RH(p_+/p)_}$, where we interpret $\infty' = 1$. (Note that by Theorem 4.4 this class is never empty.) As part of the (lengthy) proof of this inequality, they proved a restricted range extrapolation theorem.

**Theorem 8.1.** Given a family of extrapolation pairs $\mathcal{F}$, suppose there exist $1 \leq p_- < p_0 < p_+ \leq \infty$ such that for every $w_0 \in A_{p_0/p_- \cap RH(p_+/p_0)'},$
\[ \int_{\mathbb{R}^n} f_0^{p_0}w_0 \, dx \lesssim \int_{\mathbb{R}^n} g_0^{p_0}w_0 \, dx, \quad (f, g) \in \mathcal{F}. \]
Then for every $p_- < p < p_+$ and every $w \in A_{p/p_- \cap RH(p_+/p)_}$,
\[ \int_{\mathbb{R}^n} f^p w \, dx \lesssim \int_{\mathbb{R}^n} g^p w \, dx, \quad (f, g) \in \mathcal{F}. \]

We will not prove this theorem, as the proof is very long and technical, and we refer the reader to [27, Theorem 3.31] for the details. Instead, we will describe the heuristic argument that leads to the proof. This approach was used to find many of the proofs in [27] but was never made explicit and indeed, the traces were generally removed. A detailed explanation of it, in the context of proving extrapolation in the variable Lebesgue spaces, was given in [38, Section 4].

To expand upon the discussion at the beginning of the proof of Theorem 5.1, to prove Theorem 8.1 we have the following at our disposal:

- The boundedness of the maximal operator on $L^q(w)$ when $w \in A_q$. In this case, however, we will not take $q = p$ and $w \in A_p$. By our hypothesis and Proposition 3.7, we have $u = w^{(p_+/p)_'} \in A_\tau$, where
\[ \tau = \left( \frac{p_+}{p} \right)' \left( \frac{p}{p_-} - 1 \right) + 1 = \frac{1}{p_- - 1} - \frac{1}{p} + \frac{1}{p_-} + 1. \]
Though the final expression looks more complicated, in retrospect we believe that this is the correct way to write it: see the calculations in [24].

- Using the weights \( u \) and \( v = u^{1-r'} \) we can define Rubio de Francia iteration algorithms \( R_1 \) and \( R_2 \). However, these are no longer bounded on the space \( L^p(w) \) or its dual, so it is necessary to rescale. We do this by introducing functions of the form

\[
H_1 = R_1(h_1^\alpha w^\beta)^{\frac{1}{\alpha}} w^{-\frac{\beta}{\alpha}}, \quad H_2 = R_2(h_2^\gamma w^\delta)^{\frac{1}{\gamma}} w^{-\frac{\delta}{\gamma}}.
\]

- Finally, we can use duality, but dualising to \( p = 1 \) may no longer work. Therefore, we fix \( 1 \leq s < \min(p, p_0) \) and dualize to \( L^s \): for some \( h_2 \in L^{(p/s)'(w)} \), \( \|h_2\|_{L^{(p/s)'(w)}} = 1 \),

\[
\|f\|_{L^p(w)}^s = \int_{\mathbb{R}^n} f^s h_2 w \, dx.
\]

Given these tools, the goal is to follow the proof of Theorem 4.1, writing

\[
\int_{\mathbb{R}^n} f^s h_2 w \, dx \leq \int_{\mathbb{R}^n} f^s H_1^{-\epsilon} H_2^s w \, dx,
\]

applying Holder’s inequality, and then using Theorem 4.4 to create a weight \( W \in A_{p_0/p - \cap RH_{(p + 1)/p_0}} \). At each stage this imposes constraints on the constants \( \alpha, \beta, \gamma, \delta, \epsilon \) and \( s \), and it is the “miracle” of extrapolation that these constraints can all be satisfied simultaneously.

Very recently, Martell and I were interested in proving a bilinear version of Theorem 8.1, with the goal of proving weighted norm inequalities for the bilinear Hilbert transform, generalizing a result of Culiuc, di Plinio and Ou [36]. (See Section 9 below.) Using an idea from Duoandikoetxea [42] we showed that we could prove the desired bilinear extrapolation theorem if we could prove an off-diagonal version of Theorem 8.1.

An off-diagonal inequality is an inequality of the form \( \|f\|_{L^q(w^\beta)} \lesssim \|g\|_{L^p(w^\gamma)} \), \( p \neq q \); we write it in this way, with different powers on the weight on the left and right-hand sides, in order to make the inequality homogeneous in the weight. Off-diagonal inequalities are natural for operators such as the fractional integral operator

\[
I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy, \quad 0 < \alpha < n.
\]

Muckenhoupt and Wheeden [75] proved that for \( 1 < p < \frac{n}{\alpha} \) and \( 1 < q < \infty \) such that \( \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n} \), a necessary and sufficient condition for the inequality

\[
(8.2) \quad \|I_\alpha f\|_{L^q(w^\beta)} \lesssim \|f\|_{L^p(w^\gamma)}
\]
is that $w \in A_{p,q}$:

\[(8.3) \quad [w]_{A_{p,q}} = \sup_{Q} \left( \int_{Q} w^q \, dx \right)^{\frac{1}{q}} \left( \int_{Q} w^{-p'} \, dx \right)^{\frac{1}{p'}} < \infty, \]

where the supremum is taken over all cubes $Q$. When $p = q$, this is equivalent to assuming $w^p \in A_p$.

In [24] we proved the following limited range, off-diagonal extrapolation theorem.

**Theorem 8.2.** Given $0 \leq p_- < p_+ \leq \infty$ and a family of extrapolation pairs $F$, suppose that for some $p_0, q_0 \in (0, \infty)$ such that $p_- \leq p_0 \leq p_+ \leq \frac{1}{q_0} - \frac{1}{p_0} + \frac{1}{p_+} \geq 0$, and all $w$ such that $w^{p_0} \in A_{p_0/p_- \cap RH(p_+/p_0)^r}$,

\[(8.4) \quad \left( \int_{\mathbb{R}^n} f^{q_0} w^{p_0} \, dx \right)^{\frac{1}{q_0}} \lesssim \left( \int_{\mathbb{R}^n} g^{p_0} w^{p_0} \, dx \right)^{\frac{1}{p_0}}, \quad (f, g) \in F. \]

Then for every $p, q$ such that $p_- < p < p_+, 0 < q < \infty, \frac{1}{p} - \frac{1}{q} = \frac{1}{p_0} - \frac{1}{q_0}$, and every $w$ such that $w^p \in A_{p/p_- \cap RH(p_+/p)^r}$,

\[(8.5) \quad \left( \int_{\mathbb{R}^n} f^{q} w^{q} \, dx \right)^{\frac{1}{q}} \lesssim \left( \int_{\mathbb{R}^n} g^{p} w^{p} \, dx \right)^{\frac{1}{p}}, \quad (f, g) \in F. \]

The proof of Theorem 8.2 is similar to that of Theorem 8.2; following the heuristic argument laid out above, the central difficulty in the proof is determining the constraints on the constants and showing that they are consistent.

Theorem 8.2 generalizes almost all of the extrapolation theorems we have discussed as well as several others in the literature we have passed over.

- If we take $p_- = 1, p_+ = \infty$, and $p_0 = q_0$, then we get the classical Rubio de Francia extrapolation theorem, Theorem 4.1.
- If we take $p_- = 0, p_+ = \infty$, and $p_0 = q_0$, then we get $A_\infty$ extrapolation, Corollary 5.4.
- If we take $p_- = 1, p_0 < q_0, p_+ = \left( \frac{1}{p_0} - \frac{1}{q_0} \right)^{-1}$, then we get an off-diagonal extrapolation theorem due to Harboure, Macias and Segovia [56]. This result allows one to extrapolate inequalities of the form (8.2) using weights in $A_{p,q}$. To see that these are equivalent, note that by our assumptions and Proposition 3.7, $w \in A_{p,q}$ if and only if $w^p \in A_{p} \cap RH_{q/p} = A_{p/p_- \cap RH(p_+/p)^r}$. If we take $0 < p_- < p_+ < \infty$ and $p_0 = q_0$, we get the limited range extrapolation theorem of Auscher and Martell, Theorem 8.1.
- If we take $p_- = 0, p_+ = 1$ and $q_0 = p_0$, we get the extrapolation theorem for reverse Hölder weights discovered independently by Martell and Prisuelos [70] and [2]. In the first reference this was used to proved weighted norm inequalities for conical square functions associated with elliptic operators, and in the
second to prove weighted norm inequalities for the bilinear fractional integral operator.

We also note that there is significant overlap between Theorem 8.2 and an off-diagonal extrapolation theorem due to Duoandikoetxea \[42\].

**Theorem 8.3.** Given a family of extrapolation pairs $F$, suppose that for some $1 \leq p_0 < \infty$, $0 < q_0$, $r_0 < \infty$, and $w \in A_{p_0,r_0}$, inequality (8.4) holds. Then for all $1 < p < \infty$ and $0 < q, r < \infty$ such that

$$\frac{1}{q} - \frac{1}{q_0} = \frac{1}{r} - \frac{1}{r_0} = \frac{1}{p} - \frac{1}{p_0},$$

and all $w \in A_{p,r}$, inequality (8.5) holds.

Note that in the statement of Theorem 8.3, unlike in the classical definition (8.3), we do not assume $p_0 < r_0$ or $p < r$. If we assume that $r_0 \geq \min(p_0, q_0)$, then Theorem 8.3 can be gotten from Theorem 8.2 by taking $p_- = 1$ and $p_+ = \left(\frac{1}{p_0} - \frac{1}{r_0}\right)^{-1}$. For in this case, by Proposition 3.7 we have that $w \in A_{p_0,r_0}$ is equivalent to $w^{p_0} \in A_{p_0} \cap RH_{r_0/p_0} = A_{p_0/p_-} \cap RH_{(p_+/p_o)}$, and we have $\frac{1}{q_0} - \frac{1}{p_0} + \frac{1}{p_+} \geq 0$, since $r_0 \geq q_0$.

Despite this overlap, there are differences between these two theorems. In Theorem 8.2 we eliminate the restriction $p_0, p > 1$. And, for values of $p_- \neq 1$, it is not clear whether Theorem 8.2 can be gotten from Theorem 8.3 by rescaling. On the other hand, Theorem 8.2 does not seem to imply Theorem 8.3 when $r_0 < \min(p_0, q_0)$.

## 9. Bilinear extrapolation

In this section we introduce bilinear extrapolation and show how Theorem 8.2 can be used to prove it. All of the results we consider in this section are true in the multilinear case, but we restrict ourselves to bilinear inequalities to simplify the presentation.

We are interested in weighted, bilinear inequalities of the form

$$\|T(f, g)\|_{L^p(w^p)} \lesssim \|f\|_{L^{p_1}(w^{p_1})} \|g\|_{L^{p_2}(w^{p_2})},$$

where $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and $w = w_1w_2$.\footnote{It is also possible to consider endpoint inequalities where $p_1 = 1$ or $p_2 = 1$ and we replace $L^p(w^p)$ by $L^{p,\infty}(w^p)$, but for brevity we will not consider this case.} Weighted norm inequalities of this kind were first considered by Grafakos and Torres \[54\] and Grafakos and Martell \[53\] for bilinear Calderón-Zygmund singular integrals. Lerner, et al. \[68\] introduced a generalization of the Muckenhoupt $A_p$ condition. Given $\vec{p} = (p_1, p_2, p)$,
\( \vec{w} = (w_1, w_2, w) \), we say \( \vec{w} \in A_{\vec{p}} \) if

\[
[w]_{A_{\vec{p}}} = \sup_{Q} \left( \int_{Q} w_1^{p_1} dx \right)^{\frac{1}{p_1}} \left( \int_{Q} w_2^{p_2} dx \right)^{\frac{1}{p_2}} < \infty.
\]

They showed that a necessary and sufficient condition for the bilinear maximal operator\(^5\)

\[
M(f, g)(x) = \sup_{Q} |f| dy \int_{Q} |g| dy \cdot \chi_{Q}(x),
\]

is that \( \vec{w} \in A_{\vec{p}} \). Using this fact they showed that the \( A_{\vec{p}} \) condition is sufficient for a bilinear Calderón-Zygmund singular integral operator \( T \) to satisfy (9.1). (For the precise definition of these operators and their unweighted theory, see [55].)

It is natural to expect that there is a bilinear extrapolation theory for weights in \( A_{\vec{p}} \), but it is unknown whether this is possible. This remains a very important open question in the theory of bilinear weighted norm inequalities.

Therefore, to develop a theory of extrapolation we will work with a restricted class of weights \( \vec{w} \) where \( w_i^{p_i} \in A_{p_i}, i = 1, 2 \). By Hölder’s inequality, we have that in this case \( \vec{w} \in A_{\vec{p}} \), but it is relatively straightforward to construct examples of weights in \( A_{\vec{p}} \) such that \( w_i^{p_i} \not\in A_{p_i} \); see [28, 68].

We generalize the formalism of extrapolation pairs to the bilinear setting by defining a family \( \mathcal{F} \) of extrapolation triples: \( (f, g, h) \) such that each function is non-negative, measurable, and not identically 0. If we write

\[
||h||_{L^p(w^p)} \lesssim ||f||_{L^{p_1}(w_1^{p_1})} ||g||_{L^{p_2}(w_2^{p_2})}, \quad (f, g, h) \in \mathcal{F},
\]

then we mean that this inequality holds for every triple in \( \mathcal{F} \) such that \( ||h||_{L^p(w^p)} < \infty \).

As in the linear case, it is straightforward to prove weighted norm inequalities for operators: the ideas in Section 6 extend immediately to the bilinear setting. Similarly, we can use extrapolation to prove bilinear versions of Corollaries 5.2 and 5.3.

Bilinear extrapolation was first proved for operators by Grafakos and Martell [53]; the following theorem generalizes their result to families of extrapolation triples.

**Theorem 9.1.** Given a family \( \mathcal{F} \) of extrapolation triples, suppose that for some \( \vec{p} = (p_1, p_2, p) \), where \( 1 \leq p_1, p_2 < \infty \) and \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \), and weights \( \vec{w} = (w_1, w_2, w) \) such that \( w_i^{p_i} \in A_{p_i} \) and \( w = w_1w_2 \),

\[
||h||_{L^p(w^p)} \lesssim ||f||_{L^{p_1}(w_1^{p_1})} ||g||_{L^{p_2}(w_2^{p_2})}, \quad (f, g, h) \in \mathcal{F}.
\]

Then for every \( \vec{q} = (q_1, q_2, q) \), where \( 1 < q_1, q_2 < \infty \) and \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \), and weights \( \vec{w} = (w_1, w_2, w) \) such that \( w_i^{q_i} \in A_{q_i} \) and \( w = w_1w_2 \),

\[
||h||_{L^q(w^q)} \lesssim ||f||_{L^{q_1}(w_1^{q_1})} ||g||_{L^{q_2}(w_2^{q_2})}, \quad (f, g, h) \in \mathcal{F}.
\]

\(^{5}\)Properly, this operator should be called the “bi-sublinear” maximal operator, but it is common to abuse terminology and simply refer to it as a bilinear operator.
We can also prove a restricted range version of Theorem 9.1, but in order to make the main ideas of the proof clearer, we omit this generalization. For details, see [24].

**Proof.** Our proof is adapted from Duoandikoetxea [42]. Given \( 1 < p_1, p_2 < \infty \), fix \( w_2^{p_2} \in A_{p_2} \) and fix a function \( g \) such that there exist functions \( f, h \) with \((f, g, h) \in \mathcal{F}\). By assumption \( \|g\|_{L^{p_2}(w_2^{p_2})} > 0 \); assume for the moment that \( \|g\|_{L^{p_2}(w_2^{p_2})} < \infty \). Define a new family of extrapolation pairs

\[
\mathcal{F}_g = \{(F, f) = (hw_2\|g\|_{L^{p_2}(w_2^{p_2})}^{-1}, f) : (f, g, h) \in \mathcal{F}\}.
\]

Let \( p_1 = 1, p_+ = \infty \). Since \( \|F\|_{L^{p}(w^{p})} < \infty \) if and only if \( \|h\|_{L^{p}(w^{p})} < \infty \), for all \( w_1 \) such that \( w_1^{p_1} \in A_{p_1} = A_{p_1/p_+} \cap RH_{(p_+/p_1)'}, \)

\[
\|F\|_{L^{p}(w^{p})} \lesssim \|f\|_{L^{p_1}(w_1^{p_1})}, \quad (F, f) \in \mathcal{F}_g.
\]

Moreover, we have that \( \frac{1}{p} - \frac{1}{p_1} + \frac{1}{p_+} \geq 0 \) since \( \frac{1}{p} > \frac{1}{p_1} \). Therefore, by Theorem 8.2, for all \( q \) and \( q_1 \) such that

\[
(9.2) \quad \frac{1}{q} - \frac{1}{q_1} = \frac{1}{p} - \frac{1}{p_1},
\]

and all \( w_1 \) such that \( w_1^{q_1} \in A_{q_1}, \)

\[
\|F\|_{L^{q}(w^{q})} \lesssim \|f\|_{L^{q_1}(w_1^{q_1})}, \quad (F, f) \in \mathcal{F}_g.
\]

By the definition of \( \mathcal{F}_g \) we therefore have that

\[
(9.3) \quad \|h\|_{L^{q}(w^{q})} \leq \|f\|_{L^{q_1}(w_1^{q_1})}\|g\|_{L^{p_2}(w_2^{p_2})},
\]

provided that we assume that \( \|g\|_{L^{p_2}(w_2^{p_2})} < \infty \). However, if \( \|g\|_{L^{p_2}(w_2^{p_2})} = \infty \), then inequality (9.3) still holds. Since this is true for all \( g \) and \( w_2 \) with \( w_2^{p_2} \in A_{p_2} \), we must have that (9.3) holds for all \((f, g, h) \in \mathcal{F}\). Furthermore, note that (9.2) implies that

\[
\frac{1}{q} = \frac{1}{q_1} + \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{p_2}.
\]

We now repeat this argument: fix \( q \) and \( q_1 \) such that \( \frac{1}{q} = \frac{1}{q_1} + \frac{1}{p_2} \) and weight \( w_1 \) such that \( w_1^{q_1} \in A_{q_1} \). Fix a function \( f \) such that \( 0 < \|f\|_{L^{q_1}(w_1^{q_1})} < \infty \) and there exist \( g, h \) with \((f, g, h) \in \mathcal{F}\). Define the new family

\[
\mathcal{F}_f = \{(G, g) = (hw_1\|f\|_{L^{q_1}(w_1^{q_1})}^{-1}, g) : (f, g, h) \in \mathcal{F}\}.
\]

Then we can argue as above, applying Theorem 8.2 to conclude that for all \( 1 < q_1, q_2 < \infty \) and \( w_i^{q_i} \in A_{q_i}, i = 1, 2, \)

\[
\|h\|_{L^{q}(w^{q})} \lesssim \|f\|_{L^{q_1}(w_1^{q_1})}\|g\|_{L^{p_2}(w_2^{p_2})}, \quad (f, g, h) \in \mathcal{F}.
\]

\[\square\]
As an application of Theorem 9.1 we give an elementary proof of weighted norm inequalities for bilinear Calderón-Zygmund singular integral operators for this restricted class of weights.

**Theorem 9.2.** Let $T$ be a bilinear Calderón-Zygmund singular integral operator. Then for all $1 < p_1, p_2 < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, and weights $w_i^{p_i} \in A_{p_i}$, $i = 1, 2$, $w = w_1 w_2$,

$$\|T(f, g)\|_{L^p(w^p)} \lesssim \|f\|_{L^{p_1}(w_1^{p_1})} \|g\|_{L^{p_2}(w_2^{p_2})}.$$ 

**Proof.** Again, we use domination by sparse operators. If $T$ is a bilinear singular integral and $f, g \in L^\infty$, then, with the same notation for dyadic grids used in Section 6, there exist sparse sets $S_t$ such that

$$|T(f, g)(x)| \lesssim \sum_{t \in \{0, \pm \frac{1}{3}\}^n} T_{S_t}(|f|, |g|),$$

where for any sparse set $S$,

$$T_S(f, g)(x) = \sum_{Q \in S_t} \int_Q f \, dy \int_Q g \, dy \cdot \chi_Q(x).$$

By Theorem 9.1 it will suffice to show that given any dyadic grid $D_t$, sparse set $S \subset D_t$, and weights $w_i$ such that $w_i^2 \in A_2$, $i = 1, 2$, we have that for all non-negative functions $f, g \in L^\infty$,

$$\|T_S(f, g)\|_{L^1(w^1)} \lesssim \|f\|_{L^2(w_1^2)} \|g\|_{L^2(w_2^2)}.$$ 

The proof is nearly identical to the argument in the linear case given in the proof of Theorem 7.3. Let $\sigma_i = w_i^{-2}$, $i = 1, 2$. Then

$$\|T_S(f, g)\|_{L^1(w^1)} = \sum_{Q \in S_t} \int_Q f \, dx \int_Q g \, dx \int_Q w \, dx |Q|$$

$$= \sum_{Q \in S_t} \int_Q w_1 w_2 \, dx \int_Q \sigma_1 \, dx \int_Q \sigma_2 \, dx \int_Q f \, w_1^2 \, d\sigma_1 \int_Q g \, w_2^2 \, d\sigma_2 |Q|.$$ 

By assumption, $w_i^{-2} \in A_2$, and so by Proposition 3.7, $w_i \in A_2 \cap RH_2$. Therefore, by Hölder’s inequality and Proposition 3.8,

$$\int_Q w_1 w_2 \, dx \int_Q w_1^{-2} \, dx \int_Q w_2^{-2} \, dx$$

$$\lesssim \left( \int_Q w_1^2 \, dx \right)^\frac{1}{2} \left( \int_Q w_2^2 \, dx \right)^\frac{1}{2} \left( \int_Q w_1^{-2} \, dx \right)^\frac{1}{2} \left( \int_Q w_2^{-2} \, dx \right)^\frac{1}{2} \int_Q w_1^{-1} w_2^{-1} \, dx$$

$$\lesssim \int_Q w_1^{-1} w_2^{-1} \, dx \lesssim \frac{1}{|Q|} \int_{E_Q} w_1^{-1} w_2^{-1} \, dx.$$
The last two inequalities hold since $w_i^{-1}w_{i-1}^{-1} \in A_2$: this in turn follows from Hölder’s inequality since $w_i^{-2} \in A_2$, $i = 1, 2$. The final inequality then follows from Lemma 2.5.

Hence, we can continue the above estimate, getting

$$
\sum_{Q \in S} \int_Q w_1w_2 \, dx \int_Q \sigma_1 \, dx \int_Q \sigma_2 \, dx \int_Q f w_1^2 \, d\sigma_1 \int_Q g w_2^2 \, d\sigma_2 |Q|
\lesssim \sum_{Q \in S} \int_Q f w_1^2 \, d\sigma_1 \int_Q g w_2^2 \, d\sigma_2 \int_{E_Q} \sigma_1^{\frac{1}{2}} \sigma_2^{\frac{1}{2}} \, dx
\leq \int_{\mathbb{R}^n} M_{\sigma_1}^D (f w_1^2) M_{\sigma_2}^D (g w_2^2) \sigma_1^{\frac{1}{2}} \sigma_2^{\frac{1}{2}} \, dx
\leq \| M_{\sigma_1}^D (f w_1^2) \|_{L^2(\sigma_1)} \| M_{\sigma_2}^D (g w_2^2) \|_{L^2(\sigma_2)};
$$

by Lemma 2.6, which holds for arbitrary dyadic grids,

$$
\lesssim \| f w_1^2 \|_{L^2(\sigma_1)} \| g w_2^2 \|_{L^2(\sigma_2)}
= \| f \|_{L^2(w_1^2)} \| g \|_{L^2(w_2^2)}.
$$

\[ \square \]

10. Extrapolation on Banach function spaces

In this final section we discuss how extrapolation can be used to prove norm inequalities in Banach function spaces, starting from norm inequalities in weighted $L^p$. This lets us generalize the aphorism of Antonio Córdoba given in Section 1 and assert: “There are no Banach function spaces, only weighted $L^2$.” (Cf. [27, Chapter 1].)

We begin with some definitions. For more information on the theory of Banach function spaces, see Bennett and Sharpley [8]. Let $\mathcal{X}$ be a Banach space of Lebesgue measurable functions defined on $\mathbb{R}^n$ with norm $\| \cdot \|_X$. We say that $\mathcal{X}$ is a Banach function space if the norm satisfies the following properties:

- if $|f| \leq |g|$ a.e., then $\|f\|_X \leq \|g\|_X$;
- if $|f_k|$ increases pointwise a.e. to $|f|$, then $\|f_k\|_X \rightarrow \|f\|_X$;
- if $E \subset \mathbb{R}^n$, $|E| < \infty$, then $\|\chi_E\|_X < \infty$, and there exists $C(E) > 0$ such that for all $f \in \mathcal{X}$,

$$
\int_E |f| \, dx \leq C(E) \|f\|_X.
$$

Given a Banach function space $\mathcal{X}$, we define the associate space $\mathcal{X}'$ to be the set of measurable functions $g$ such that

$$
\|g\|_{\mathcal{X}'} = \sup \left\{ \int_{\mathbb{R}^n} fg \, dx : \|f\|_X \leq 1 \right\} < \infty.
$$
Then \( \| \cdot \|_{X'} \) is a norm and \( X' \) is itself a Banach function space. The two norms are related by the generalized Hölder’s inequality,

\[
\int_{\mathbb{R}^n} |fg| \, dx \leq \|f\|_{X} \|g\|_{X'}.
\]

The associate space embeds (up to an isomorphism) in the dual space \( X^* \), and in some (though not all) cases they are equal. However, the associate spaces are always reflexive: for any Banach function space \( X \), \((X')' = X\).

**Theorem 10.1.** Given a family of extrapolation pairs \( F \), suppose that for some \( 1 \leq p_0 < \infty \) and every \( w_0 \in A_{p_0} \),

(10.1) \[
\int_{\mathbb{R}^n} f^{p_0} w_0 \, dx \lesssim \int_{\mathbb{R}^n} g^{p_0} w_0 \, dx, \quad (f, g) \in F.
\]

Let \( X \) be a Banach function space such that the maximal operator satisfies \( M : X \to X \) and \( M : X' \to X' \). Then

(10.2) \[
\|f\|_X \lesssim \|g\|_X, \quad (f, g) \in F.
\]

**Proof.** The proof is actually a simple variation of the proof of Theorem 5.1. By this result, we may assume without loss of generality that (10.1) holds for \( p_0 = 2 \) and weights \( w_0 \in A_2 \). We define two iteration algorithms:

\[
\mathcal{R}_1 h_1 = \sum_{k=0}^{\infty} \frac{M^k h_1}{2^k \|M\|^k_{X}}, \quad \mathcal{R}_2 h_2 = \sum_{k=0}^{\infty} \frac{M^k h_2}{2^k \|M\|^k_{X'}}.
\]

Then the proof of Theorem 4.1 generalizes to give the following:

\[
(A_1) \quad h_1(x) \leq \mathcal{R}_1 h_1(x) \quad (A_2) \quad h_2(x) \leq \mathcal{R}_2 h_2(x)
\]

\[
(B_1) \quad \|\mathcal{R}_1 h_1\|_X \leq 2\|h_1\|_X \quad (B_2) \quad \|\mathcal{R}_2 h_2\|_{X'} \leq 2\|h_2\|_{X'}
\]

\[
(C_1) \quad [\mathcal{R}_1 h_1]_{A_1} \leq 2\|M\|_X \quad (C_2) \quad [\mathcal{R}_2 h_2]_{A_1} \leq 2\|M\|_{X'}.
\]

Now fix \((f, g) \in F\); without loss of generality \( 0 < \|f\|_X, \|g\|_X < \infty \). Define

\[
h_1 = \frac{f}{\|f\|_X} + \frac{g}{\|g\|_X};
\]

then \( \|h_1\|_X \leq 2 \). By the definition of the associate space and reflexivity, there exists \( h_2 \in X' \), \( \|h_2\|_{X'} = 1 \), such that

\[
\|f\|_X \lesssim \int_{\mathbb{R}^n} fh_2 \, dx;
\]
by \((A_2)\) and Hölder’s inequality,
\[
\begin{align*}
&\leq \int_{\mathbb{R}^n} f(R_1h_1)^{-\frac{1}{2}}(R_1h_1)^{\frac{1}{2}}R_2h_2 \, dx \\
&\leq \left( \int_{\mathbb{R}^n} f^2(R_1h_1)^{-1}R_2h_2 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^n} R_1h_1R_2h_2 \, dx \right)^{\frac{1}{2}} \\
&= I_1^\frac{1}{2} \cdot I_2^\frac{1}{2}.
\end{align*}
\]

To estimate \(I_2\) we use the generalized Hölder’s inequality and \((B_1)\) and \((B_2)\):
\[
I_2 \leq \|R_1h_1\|_x \|R_2h_2\|_{x'} \leq 4\|h_1\|_x \|h_2\|_{x'} \leq 8.
\]

To estimate \(I_1\), note first that by \((A_1)\), \(f \leq h_1\|f\|_x \leq \|f\|_{x} R_1h_1\), so \(I_1 \leq \|f\|^2_{x}I_2 < \infty\). Furthermore, by \((C_1)\), \((C_2)\) and Theorem 4.2, \((R_1h_1)^{-1}R_2h_2 \in A_2\). Therefore, by \((10.1)\) and again by \((A_1)\),
\[
I_1 \lesssim \int_{\mathbb{R}^n} g^2(R_1h_1)^{-1}R_2h_2 \, dx \leq \|g\|^2_{x'} \cdot I_2 \lesssim \|g\|^2_{x'}.
\]

If we combine all of these inequalities we get \((10.2)\) and our proof is complete. \(\Box\)

Extrapolation into Banach function spaces was first considered in [20] in the context of the variable Lebesgue spaces (see below). The result proved there is somewhat different, and only requires that \((10.1)\) holds for weights \(w_0 \in A_1\), though a version of Theorem 10.1 was proved as a corollary. Theorem 10.1 is a variant of the extrapolation theorem proved for the weighted variable Lebesgue spaces in [38]. Curbera, et al. [37] proved an extrapolation theorem into rearrangement invariant Banach function spaces such as Orlicz spaces. For a general treatment of extrapolation into Banach function spaces, see [27, Chapter 4]. Very recently in [23], extrapolation was extended to the Musielak-Orlicz spaces, a very general class of function spaces that include the Lebesgue spaces, Orlicz spaces, and the variable Lebesgue spaces as special cases. (For more information about these spaces, see [76].)

We conclude these notes by considering the application of extrapolation to the variable Lebesgue spaces. These spaces are a generalization of the classical Lebesgue spaces, replacing the constant exponent \(p\) with an exponent function \(p(\cdot)\). We begin with some definitions; for complete details and references on these spaces, see [19, 40]. Given a measurable function \(p(\cdot) : \mathbb{R}^n \to [1, \infty]\), let \(\mathbb{R}^n_\infty = \{x \in \mathbb{R}^n : p(x) = \infty\}\), and define
\[
p_- \equiv \text{ess inf}_x p(x), \quad p_+ \equiv \text{ess sup}_x p(x).
\]
Define $L^{p(\cdot)}$ to be the set of measurable functions $f$ such that for some $\lambda > 0$,

$$\rho_{p(\cdot)}(f/\lambda) = \int_{\mathbb{R}^n \setminus \mathbb{R}^n_n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx + \lambda^{-1} \|f\|_{L^\infty(\mathbb{R}^n)} < \infty.$$  

Then $L^{p(\cdot)}$ is a Banach function space with respect to the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}} = \|f\|_{p(\cdot)} = \inf \{ \lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1 \}.$$  

When $p(\cdot) = p$, $1 \leq p \leq \infty$, $L^{p(\cdot)} = L^p$ with equality of norms.

The associate space of $L^{p(\cdot)}$ equals $L^{p'(\cdot)}$ with an equivalent norm, where $p'(\cdot)$ is defined pointwise by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$$

with the convention $1/\infty = 0$. Consequently, we have the generalized Hölder’s inequality

$$\int_{\mathbb{R}^n} |fg| \, dx \leq 2 \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$  

The boundedness of the maximal operator on $L^{p(\cdot)}$ requires some regularity on the exponent $p(\cdot)$. A very useful sufficient condition is log-Hölder continuity, defined locally by

$$\left| \frac{1}{p(x)} - \frac{1}{p(y)} \right| \leq \frac{C_0}{-\log(|x-y|)}, \quad |x-y| < \frac{1}{2},$$

and at infinity by

$$\left| \frac{1}{p(x)} - \frac{1}{p_{\infty}} \right| \leq \frac{C_\infty}{\log(e + |x|)}.$$  

We denote this by writing $p(\cdot) \in LH$. The following result was first proved in [21]; for a simpler proof, see [19, Chapter 3].

**Theorem 10.2.** Given an exponent function $p(\cdot)$ such that $1 < p_- \leq p_+ < \infty$ and such that $p(\cdot) \in LH$, $\|Mf\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}$.

Clearly, if $p(\cdot) \in LH$, then $p'(\cdot) \in LH$, so if $1 < p_- \leq p_+ < \infty$ and $p(\cdot) \in LH$, the maximal operator is bounded on $L^{p(\cdot)}$ and $L^{p'(\cdot)}$. Moreover, Diening [39, 40] proved the following very deep result: given any exponent function $p(\cdot)$, if $1 < p_- \leq p_+ < \infty$, the maximal operator is bounded on $L^{p(\cdot)}$ if and only if it is bounded on $L^{p'(\cdot)}$.\footnote{Diening also showed that if $M$ is bounded on $L^{p(\cdot)}$, then there exists $s > 1$ such that it is also bounded on $L^{p(\cdot)/s}$. We used this fact instead of the boundedness of $M$ on both $L^{p(\cdot)}$ and $L^{p'(\cdot)}$ to prove our extrapolation theorem in [20]. In addition, we assumed an abstract version of this property to prove extrapolation for general Banach function spaces in [27].}

It follows from these facts that we can apply extrapolation to the variable Lebesgue spaces $L^{p(\cdot)}$, assuming only that $1 < p_- \leq p_+ < \infty$ and that the maximal operator is
bounded on $L^{p(\cdot)}$. As an immediate consequence, we get that in this case, if $T$ is a Calderón-Zygmund singular integral, then $\| Tf \|_{p(\cdot)} \lesssim \| f \|_{p(\cdot)}$ whenever $M$ is bounded on $L^{p(\cdot)}$. In [19] we conjectured that this was a necessary as well as sufficient condition. We recently learned that this conjecture was proved by Rutsky [87].

Similarly, many other norm inequalities can be extended to variable Lebesgue spaces using the corresponding weighted norm inequalities. For a number of examples, see [19, 20, 27]. For the application of extrapolation to develop the theory of variable Hardy spaces, see [34]. Finally, in [29] we developed a theory of bilinear extrapolation which we used to prove bilinear inequalities in variable Lebesgue spaces starting from weighted bilinear inequalities. This led to both new (and simpler) proofs of known results for bilinear operators on variable Lebesgue spaces and also to new results.

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