Utility maximization in the large markets

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Abstract

In the large financial market, which is described by a model with countably many traded assets, we formulate the problem of the expected utility maximization. Assuming that the preferences of an economic agent are modeled with a stochastic utility and that the consumption occurs according to a stochastic clock, we obtain the “usual” conclusions of the utility maximization theory. We also give a characterization of the value function in the large market in terms of a sequence of the value functions in the finite-dimensional models.

1 Introduction

In the mathematical finance literature, the notion of the large security market was introduced by Kabanov and Kramkov [13] as a sequence of probability spaces with the corresponding time horizons and the semimartingales representing the traded assets. Investigation of the no-arbitrage conditions in the large market settings has naturally attracted the attention of the research community and is done in [14, 16, 17, 18, 19, 20], whereas the questions related to completeness are considered in [2, 3, 6, 7, 25].

In contrast to [13, 14], Björk and Naslund [1] assumed that a large market consists of one probability space, but the number of traded assets is countable, and among other contributions developed the arbitrage pricing theory
results in such settings. Note that the models with countably many assets embrace the ones with the stochastic dimension of the stock price process (considered e.g. in [24]). De Donno, Guasoni, and Pratelli [9] extended the formulation in [1] to a model driven by a sequence of semimartingales and established the standard conclusions of the theory for the utility maximization from terminal wealth problem as well as obtained the dual characterization of the superreplicable claims. Their results are based on the notion of a stochastic integral with respect to a sequence of semimartingales from De Donno and Pratelli [8]. The Merton portfolio problem in the settings with infinitely many traded zero-coupon bonds is investigated in [11, 23]. Other applications of the large market models in the analysis of the fixed income securities are considered in [2, 3, 4, 5, 7, 25].

We consider a market with countably many traded assets driven by a sequence of a semimartingales (as in [9]). In such settings, we formulate Merton’s portfolio problem for a rational economic agent whose preferences are specified via a stochastic utility of Inada’s type defined on the positive real line and whose consumption follows a stochastic clock. We establish the standard existence and uniqueness results for the primal and dual optimization problems under the condition of finiteness of both primal and dual value functions. We also characterize the primal and dual value functions in terms of the appropriate limits of the sequences of the value functions in the finite-dimensional models. In particular, we extend the utility maximization results in [9] by adding the intermediate consumption and assuming randomness of the agent’s preferences.

The proof of our results hinges on the dual characterization of the admissible consumption processes given in Proposition 3.1 which allows to link the present model with the abstract theorems of [22]. Note that our formulation of the admissible consumptions and trading strategies relies on the notion of the stochastic integral with respect to a sequence of semimartingales in the sense of [8].

We believe that our results provide a convenient set of conditions for analyzing other problems in the settings of the large markets with or without the presence of the intermediate consumption, such as robust utility maximization, optimal investment with random endowment, utility-based pricing, and existence of equilibria.

The remainder of the paper is organized as follows. Section 2 contains the model formulation and the main results, which are formulated in Theorem 2.2 and Lemma 2.4. Their proofs are given in section 3.
2 The model and the main result

We consider a filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})\), where the filtration \((\mathcal{F}_t)_{t\in[0,T]}\) satisfies the usual conditions, \(\mathcal{F}_0\) is the completion of the trivial \(\sigma\)-algebra. As in [1, 9], we assume that there is one fixed market which consists of a riskless bond and a sequence of semimartingales \(S = (S^n)_{n\geq 1} = ((S^i_t)_{t\in[0,T]})_{i=1}^\infty\) that describes the evolution of the stocks. The price of the bond is supposed to be equal to 1 at all times.

The notion of a strategy on the large market relies on the finite-dimensional counterparts, whose definitions we specify first. For \(n \in \mathbb{N}\), an \(n\)-elementary strategy is an \(\mathbb{R}^n\)-valued, predictable process, which is integrable with respect to \((S^i)_{i \leq n}\). An elementary strategy is a strategy which is \(n\)-elementary for some \(n\). For \(x \geq 0\), an \(n\)-elementary strategy \(H\) is \(x\)-admissible if \(H \cdot S = \sum_{i \leq n} H^i \cdot S^i\) is uniformly bounded from below by the constant \(-x\) \(\mathbb{P}\)-a.s. Let \(\mathcal{H}^n\) denote the set of \(n\)-elementary strategies that are also \(x\)-admissible for some \(x \geq 0\).

In the present settings specification of the admissible wealth processes and trading strategies is based on integration with respect to a sequence of semimartingales in the sense of [8]. Thus we recall several definitions from [8], upon which the formulation of the set of admissible consumptions is based. The reader that is familiar with this construction might proceed to definition of an \(x\)-admissible generalized strategy. Recall that \(\mathbb{R}^N\) is the space of all real sequences. An unbounded functional on \(\mathbb{R}^N\) is a linear functional \(F\), whose domain \(\text{Dom}(F)\) is a subspace of \(\mathbb{R}^N\). A simple integrand is a finite sequence of bounded predictable processes of the form \(\sum_{i \leq n} h^i e^i\), where \((e^i)\) is the canonical basis for \(\mathbb{R}^N\) and \(h^i\)’s are one-dimensional bounded and predictable processes.

A process \(H\) with values in the set of unbounded functionals on \(\mathbb{R}^N\) is predictable if there exists a sequence of simple integrands \((H^n)\), such that \(H = \lim_{n \to \infty} H^n\) \(\mathbb{P}\)-a.s., which means that \(x \in \text{Dom}(H)\) if the sequence \((H^n)\) converges and \(\lim_{n \to \infty} H^n(x) = H(x)\).

A predictable process \(H\) with values in the set of unbounded functionals on \(\mathbb{R}^N\) is integrable with respect to \(S\) if there exists a sequence \((H^n)\) of simple integrands, such that \((H^n)\) converges to \(H\) and the sequence of semimartingales \((H^n \cdot S)\) converges to a semimartingale \(Y\) in the semimartingale topology. In this case, we define the stochastic integral \(H \cdot S\) to be \(Y\).
For every $x \geq 0$, a process $H$ is an $x$-admissible generalized strategy if $H$ is integrable with respect to the semimartingale $S$ and there exists an approximating sequence $(H^n)$ of $x$-admissible elementary strategies, such that $(H^n \cdot S)$ converges to $H \cdot S$ in the semimartingale topology. Note that this is Definition 2.5 from [9].

Let us define a portfolio $\Pi$ as a triple $(x, H, c)$, where the constant $x$ is an initial value, $H$ is a predictable and $S$-integrable process (with the values in the set of unbounded functionals on $\mathbb{R}^N$) specifying the amount of each asset held in the portfolio, and $c = (c_t)_{t \in [0,T]}$ is a nonnegative and optional process that specifies the consumption rate in the units of the bond.

Hereafter we fix a stochastic clock $\kappa = (\kappa_t)_{t \in [0,T]}$, which is a non-decreasing, càdlàg, adapted process such that

\[(2.1)\]
\[\kappa_0 = 0, \quad \mathbb{P}[\kappa_T > 0] > 0, \quad \text{and} \quad \kappa_T \leq A\]

for some finite constant $A$. Stochastic clock represents the notion of time according to which consumption occurs.

We will use the following notation: for arbitrary constants $x$ and $y$ and processes $X$ and $Y$, $(x + yXY)$ denotes the process $(x + yX_tY_t)_{t \in [0,T]}$. For a portfolio $(x, H, c)$, we define the wealth process as

\[X = x + H \cdot S - c \cdot \kappa.\]

Note that the closure of the sets of wealth processes in the semimartingale topology is investigated in [4, 15] (with the corresponding definitions of a wealth process being different from the one here). For $x \geq 0$, we define the set of $x$-admissible consumptions as

\[\mathcal{A}(x) \triangleq \{ c \geq 0 : c \text{ is optional, and there exists an } x\text{-admissible generalized strategy } H, \text{ s.t. } x + H \cdot S - c \cdot \kappa \geq 0 \}.\]

Thus a constant strictly positive consumption $c^*_t \triangleq x/A, \ t \in [0,T]$, belongs to $\mathcal{A}(x)$ for every $x > 0$.

For $n \geq 1$, let $\mathcal{Z}^n$ denote the set of càdlàg densities of equivalent martingale measure for $n$-elementary strategies, i.e.

\[\mathcal{Z}^n \triangleq \{ Z > 0 : Z \text{ is a càdlàg martingale, s.t. } Z_0 = 1 \text{ and } (1 + H \cdot S)Z \text{ is a local martingale for every } H \in \mathcal{H}^n \}.\]
Note that \( Z^{n+1} \subseteq Z^n, n \geq 1 \). We also define
\[
Z^\Delta = \bigcap_{n \geq 1} Z^n,
\]
and assume that
\[
(2.2) \quad Z^\Delta \neq \emptyset,
\]
which coincides with the no-arbitrage condition in [9].

The preferences of an economic agent are modeled via a stochastic utility
\( U : [0, T] \times \Omega \times [0, \infty) \to \mathbb{R} \cup \{-\infty\} \) that satisfies the conditions below.

**Assumption 2.1.** For every \((t, \omega) \in [0, T] \times \Omega\) the function \( x \to U(t, \omega, x) \) is
strictly concave, increasing, continuously differentiable on \((0, \infty)\) and satisfies
the Inada conditions:
\[
\lim_{x \downarrow 0} U'(t, \omega, x) = +\infty \quad \text{and} \quad \lim_{x \to \infty} U'(t, \omega, x) \triangleq 0,
\]
where \( U' \) denotes the partial derivative with respect to the third argument.
At \( x = 0 \) we suppose, by continuity, \( U(t, \omega, 0) = \lim_{x \downarrow 0} U(t, \omega, x) \), this value
may be \(-\infty\). For every \( x \geq 0 \) the stochastic process \( U(\cdot, \cdot, x) \) is optional.

The conditions on \( U \) coincide with the ones in [22] (on the finite time horizon). For simplicity of notations for a nonnegative optional process \( c \), the processes with trajectories \((U(t, \omega, c_t(\omega)))_{t \in [0, T]}, (U'(t, \omega, c_t(\omega)))_{t \in [0, T]}, (U^-(t, \omega, c_t(\omega)))_{t \in [0, T]} \) (where \( U^- \) designates the negative part of \( U \)) will be
denoted by \( U(c), U'(c), \) and \( U^-(c) \) respectively.

For a given initial capital \( x > 0 \) the goal of the agent is to maximize his
expected utility. The value function of this problem is denoted by
\[
(2.3) \quad u(x) \triangleq \sup_{c \in \mathcal{A}(x)} \mathbb{E}[U(c) \cdot \kappa_T], \quad x > 0.
\]
We use the convention
\[
(2.4) \quad \mathbb{E}[U(c) \cdot \kappa_T] \triangleq -\infty \quad \text{if} \quad \mathbb{E}[U^-(c) \cdot \kappa_T] = +\infty.
\]

To study (2.3) we employ standard duality arguments as in [21] and [26]
and define the **conjugate stochastic field** \( V \) to \( U \) as
\[
V(t, \omega, y) \triangleq \sup_{x > 0} (U(t, \omega, x) - xy), \quad (t, \omega, y) \in [0, T] \times \Omega \times [0, \infty).
\]
It is well-known that $-V$ satisfies Assumption 2.1. For $y \geq 0$, we also denote

$$Y(y) \triangleq \text{cl} \{Y : Y \text{ is càdlàg adapted and} \quad 0 \leq Y \leq yZ, \quad (d\kappa \times \mathbb{P}) \text{ a.e. for some } Z \in \mathcal{Z} \},$$

where the closure is taken in the topology of convergence in measure $(d\kappa \times \mathbb{P})$ on the space of finite-valued optional processes. We will denote this space $L^0(d\kappa \times \mathbb{P})$ or $L^1$ for brevity.

Similarly to composition of $U$ with $c$, for a nonnegative optional process $Y$, the stochastic processes, whose realizations are $(V(t, \omega, Y_t(\omega)))_{t \in [0,T]}$ and $(V^+(t, \omega, Y_t(\omega)))_{t \in [0,T]}$ (where $V^+$ is the positive part of $V$), will be denoted by $V(Y)$ and $V^+(Y)$ respectively. After these preparations, we define the value function of the dual optimization problem as

$$v(y) \triangleq \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}[V(Y) \cdot \kappa_T], \quad y > 0,$$

where we use the convention:

$$\mathbb{E}[V(Y) \cdot \kappa_T] \triangleq +\infty \quad \text{if} \quad \mathbb{E}[V^+(Y) \cdot \kappa_T] = +\infty.$$

The following theorem constitutes the main contribution of the present article.

**Theorem 2.2.** Assume that conditions (2.1) and (2.2) and Assumption 2.1 hold true and suppose

$$v(y) < \infty \quad \text{for all } y > 0 \quad \text{and} \quad u(x) > -\infty \quad \text{for all } x > 0.$$

Then we have:

1. $u(x) < \infty$ for all $x > 0$, $v(y) > -\infty$ for all $y > 0$. The functions $u$ and $v$ are conjugate, i.e.,

$$v(y) = \sup_{x > 0} (u(x) - xy), \quad y > 0,$$

$$u(x) = \inf_{y > 0} (v(y) + xy), \quad x > 0.$$

The functions $u$ and $-v$ are continuously differentiable on $(0, \infty)$, strictly increasing, strictly concave and satisfy the Inada conditions:

$$u'(0) \triangleq \lim_{x \downarrow 0} u'(x) = +\infty, \quad -v'(0) \triangleq \lim_{y \downarrow 0} -v'(y) = +\infty,$$

$$u'(\infty) \triangleq \lim_{x \to \infty} u'(x) = 0, \quad -v'(\infty) \triangleq \lim_{y \to \infty} -v'(y) = 0.$$
2. For every $x > 0$ and $y > 0$ the optimal solutions $\hat{c}(x)$ to (2.3) and $\hat{Y}(y)$ to (2.5) exist and are unique. Moreover, if $y = u'(x)$ we have the dual relations

\[ \hat{Y}(y) = U'(\hat{c}(x)), \quad (d\kappa \times \mathbb{P}) \text{ a.e.} \]

and

\[ \mathbb{E} \left[ \left( \hat{c}(x)\hat{Y}(y) \right) \cdot \kappa T \right] = xy. \]

3. We have,

\[ v(y) = \inf_{Z \in \mathcal{Z}} \mathbb{E} [V(yZ) \cdot \kappa_T], \quad y > 0, \]

2.1 Large market as a limit of a sequence of finite-dimensional markets

Motivated by the question of liquidity, we discuss the convergence of the value functions as the number of available traded securities increases. For this purpose, we need the following definitions. For every $n \geq 1$, we set

\[ \mathcal{A}^n(x) \triangleq \{ \text{optional } c \geq 0 : \exists H \in \mathcal{H}^n, \ x + H \cdot S_T - c \cdot \kappa_T \geq 0 \ \text{P-a.s.} \}, \]

(2.7)

\[ u^n(x) \triangleq \sup_{c \in \mathcal{A}^n(x)} \mathbb{E} [U(c) \cdot \kappa_T], \quad x > 0, \]

\[ \mathcal{Y}^n(y) \triangleq \text{cl} \{ Y : Y \text{ is càdlàg adapted and } 0 \leq Y \leq yZ \ (d\kappa \times \mathbb{P}) \text{ a.e. for some } Z \in \mathcal{Z}^n \}, \]

where the closure is taken in $\mathbb{L}^0$,

(2.8)

\[ v^n(y) \triangleq \inf_{Y \in \mathcal{Y}^n(y)} \mathbb{E} [V(Y) \cdot \kappa_T], \quad y > 0, \]

and assume the conventions (2.4) and (2.6). Note that for every $z > 0$, both $(u^n(z))$ and $(v^n(z))$ are increasing sequences. We suppose that

(2.9)

\[ \mathcal{A}(1 - \varepsilon) \subset \text{cl} \left( \bigcup_{n \geq 1} \mathcal{A}^n(1) \right) \quad \text{for every } \varepsilon \in (0, 1], \]

where the closure is taken in $\mathbb{L}^0$.

Let $1_E$ denotes the indicator function of a set $E$. 
Remark 2.3. It follows from Proposition 3.1 below and Fatou’s lemma that $\text{cl} \left( \bigcup_{n \geq 1} \mathcal{A}^n(1) \right) \subseteq \mathcal{A}(1)$. Assumption (2.9) gives a weaker version of the reverse inclusion. Note that (2.9) holds if either of the conditions below is valid.

1. $\kappa_t = 1_{T}(t)$, $t \in [0,T]$, i.e. if (2.3) defines the problem of optimal investment from terminal wealth. Then (2.9) follows from Lemma 3.4 in [9].

2. The processes $\kappa$ and $S$ are (componentwise) continuous. This is the subject of Lemma 3.7 below.

Lemma 2.4. Assume that there exists $n \in \mathbb{N}$, such that $u^n(x) > -\infty$ for every $x > 0$, $v(y) < +\infty$ for every $y > 0$.

Then, under conditions (2.1), (2.2), and (2.9) as well as Assumptions 2.1, we have

$$u(x) = \lim_{n \to \infty} u^n(x), \quad x > 0, \quad \text{and} \quad v(y) = \lim_{n \to \infty} v^n(y), \quad y > 0.$$  

3 Proofs

In the core of the proof of Theorem 2.2 lies the following result.

Proposition 3.1. Let conditions (2.1) and (2.2) hold. Then a nonnegative optional process $c$ belongs to $\mathcal{A}(1)$ if and only if

$$\sup_{Z \in \mathcal{Z}} \mathbb{E} \left[ ((cZ) \cdot \kappa)_T \right] \leq 1.$$  

The proof of Proposition 3.1 will be given via several lemmas.

Lemma 3.2. Let $H$ be a 1-admissible generalized integrand. Under the conditions Proposition 3.1, $X \triangleq 1 + H \cdot S$ is nonnegative $\mathbb{P}$-a.s. and for every $Z \in \mathcal{Z}$, $ZX$ is a supermartingale.

Proof. First, we show that $X$ is nonnegative. Let $(H^n)$ be a sequence of 1-admissible elementary strategies, such that $X^n \triangleq 1 + H^n \cdot S$, $n \geq 1$, converges to $X$ in the semimartingale topology. This implies that for every
\( t \geq 0, (X^n_t - X_t) \) converges to 0 in probability. Also note that \( X^n \)'s are nonnegative. Let us fix \( t \geq 0 \) and set
\[
A_m = \left\{ X_t \leq -\frac{1}{m} \right\}, \quad m \geq 1.
\]
Then for every \( k \geq 1 \) there exists \( n_k \in \mathbb{N} \), such that
\[
P\left[ |X_t^{n_k} - X_t| \geq \frac{1}{2m} \right] \leq \frac{1}{k}.
\]

Let us denote \( B_k \triangleq \{ |X_t^{n_k} - X_t| \geq \frac{1}{2m} \} \), then \( P[B_k] \leq \frac{1}{k} \) and \( P[A_m \cap B_k^c] = 0 \). Therefore, we have
\[
P[A_m] = P[A_m \cap B_k] + P[A_m \cap B_k^c] \leq P[B_k] \leq \frac{1}{k},
\]
which holds for every \( k \geq 1 \). Taking the limit as \( k \to \infty \), we obtain that \( P[A_m] = 0, m \geq 1 \). Since \( A \triangleq \{ X_t < 0 \} \subseteq \bigcup_{m \geq 1} A_m \), we deduce that \( P[A] = 0 \). Since \( t \) is arbitrary and \( X \) is a semimartingale (therefore it is right-continuous), we deduce that \( X \) is nonnegative \( P\)-a.s.

Second, we fix an arbitrary \( Z \in \mathcal{Z} \) and show that \( ZX \) is a supermartingale. Let \((H^n)\) be a sequence of 1-admissible elementary strategies, such that \((1 + H^n \cdot S)\) converges to \( X \) in the semimartingale topology. Fix \( t_1 < t_2 \). Then there exists a subsequence, which we still denote \((H^n)\), such that
\[
1 + \lim_{n \to \infty} H^n \cdot S_{t_i} = X_{t_i}, \quad i = 1, 2 \quad P\text{-a.s.}
\]

For every \( n \geq 1 \), since \( Z(1 + H^n \cdot S) \) is a nonnegative local martingale, it is a supermartingale. Now Fatou's lemma gives
\[
E[Z_{t_2} (1 + H \cdot S_{t_2}) | \mathcal{F}_{t_1}] \leq \liminf_{n \to \infty} E[Z_{t_2} (1 + H^n \cdot S_{t_2}) | \mathcal{F}_{t_1}]
\]
\[
\leq \liminf_{n \to \infty} Z_{t_1} (1 + H^n \cdot S_{t_1}) = Z_{t_1} (1 + H \cdot S_{t_1}).
\]

This concludes the proof of the lemma. \( \square \)

**Lemma 3.3.** Let \( H \) be a 1-admissible generalized strategy, \( c \) be a nonnegative optional process. Under the conditions Proposition 3.1, the following statements are equivalent.
\((i)\)
\[ c \cdot \kappa_T \leq 1 + H \cdot S_T, \quad \mathbb{P}-a.s., \]

\((ii)\)
\[ c \cdot \kappa \leq 1 + H \cdot S, \quad \mathbb{P}-a.s. \]
\((i.e. \quad c \cdot \kappa_t \leq 1 + H \cdot S_t \quad \text{for every } t \in [0, T], \quad \mathbb{P}-a.s.)\]

**Proof.** Let us assume that \((i)\) holds and fix \( Z \in \mathcal{Z} \). It follows from Lemma 3.2 that \( Z(1 + H \cdot S) \) is a supermartingale. Therefore, using monotonicity of \( c \cdot \kappa \), for every \( t \leq T \) we have

\[
Z_t(c \cdot \kappa_t) = \mathbb{E}[Z_T(c \cdot \kappa_T)|\mathcal{F}_t] \leq \mathbb{E}[Z_T(c \cdot \kappa_T)|\mathcal{F}_t] \\
\leq \mathbb{E}[Z_T(1 + H \cdot S_T)|\mathcal{F}_t] \leq Z_t(1 + H \cdot S_t),
\]

which implies \((ii)\) in view of the strict positivity of \( Z \) and the right-continuity of both \( (1 + H \cdot S) \) and \( (c \cdot \kappa) \), where the latter follows e.g. from Proposition I.3.5 in [12].

\[ \Box \]

**Proof of Proposition 3.1.** Let \( c \in \mathcal{A}(1) \). Fix \( Z \in \mathcal{Z} \) and \( T > 0 \). Then there exists a 1-admissible generalized strategy \( H \), such that

\[ 1 + H \cdot S_T \geq c \cdot \kappa_T. \]

Multiplying both sides by \( Z \) and taking the expectation, we get

\[
\mathbb{E}[Z_T(1 + H \cdot S_T)] \geq \mathbb{E}[Z_T(c \cdot \kappa_T)],
\]

where the right-hand side (via monotonicity of \( c \cdot \kappa \) and an application of Theorem I.4.49 in [12]) can be rewritten as

\[
\mathbb{E}[Z_T(c \cdot \kappa_T)] = \mathbb{E}[((Zc) \cdot \kappa)_T].
\]

By definition of \( H \), there exists a sequence \((H^n)\) of 1-admissible elementary strategies, such that

\[ (H^n \cdot S)_{n \geq 1} \quad \text{converges to} \quad H \cdot S \quad \text{in the semimartingale topology}. \]

Consequently, \((H^n \cdot S_T)\) converges to \( H \cdot S_T \) in probability, and therefore there exist a subsequence, which we still denote \((H^n \cdot S)\), such that \((H^n \cdot S_T)\)
converges to $H \cdot S_T$ $\mathbb{P}$-a.s. Therefore, for every $Z \in \mathcal{Z}$ we obtain from the definition of 1-admissibility and Fatou’s lemma

$$1 \geq \liminf_{n \to \infty} \mathbb{E} [Z_T(1 + H^n \cdot S_T)] \geq \mathbb{E} [Z_T(1 + H \cdot S_T)].$$

Combining this with (3.2) and (3.3), we conclude that

$$1 \geq \mathbb{E} [((Zc) \cdot \kappa)_T],$$

which holds for every $Z \in \mathcal{Z}$.

Conversely, let (3.1) holds. Using the same argument as in (3.3), we obtain from (3.1) that

$$1 \geq \sup_{Z \in \mathcal{Z}} \mathbb{E} [Z_T(c \cdot \kappa)_T].$$

Consequently, the random variable $c \cdot \kappa_T$ satisfies the assumption (i) of Theorem 3.1 in [9] with $x = 1$. Therefore, we obtain from this theorem that there exists a 1-admissible generalized strategy $H$ such that

$$c \cdot \kappa_T \leq 1 + H \cdot S_T.$$

By Lemma 3.3 this implies that $c \in \mathcal{A}(1)$. This concludes the proof of the proposition.

Let $\mathbb{L}_0^+$ denote the positive orthant of $\mathbb{L}_0^0$. We recall that a subset $A$ of $\mathbb{L}_0^+$ is called solid if $f \in A$, $g \in \mathbb{L}_0^+$, and $g \leq f$ implies that $g \in A$, a subset $B \subset \mathbb{L}_0^+$ is the polar of $A$, if $B = \{ h \in \mathbb{L}_0^+ : \mathbb{E} [((h f) \cdot \kappa)_T] \leq 1, \text{ for every } f \in A \}$, in this case we denote $B = A^\circ$.

**Lemma 3.4.** Under the conditions of Proposition 3.1, we have

(i) The sets $\mathcal{A}(1)$ and $\mathcal{Y}(1)$ are convex, solid, and closed subsets of $\mathbb{L}_0^0$.

(ii) $\mathcal{A}(1)$ and $\mathcal{Y}(1)$ satisfy the bipolar relations

$$c \in \mathcal{A}(1) \iff \mathbb{E} [((c Y) \cdot \kappa)_T] \leq 1, \text{ for every } Y \in \mathcal{Y}(1),$$

$$Y \in \mathcal{Y}(1) \iff \mathbb{E} [((c Y) \cdot \kappa)_T] \leq 1, \text{ for every } Y \in \mathcal{A}(1).$$

(iii) Both $\mathcal{A}(1)$ and $\mathcal{Y}(1)$ contain strictly positive elements.
Proof. Assertions of item (iii) follow from conditions (2.1) and (2.2) respectively. Now in view of Proposition 3.1, the proof of the remaining items goes along the lines of the proof of Proposition 4.4 in [22]. It is therefore omitted here.

**Lemma 3.5.** Under the conditions of Proposition 3.1, we have

(i) \( \sup_{Z \in \mathcal{Z}} \mathbb{E}[(cZ \cdot \kappa)_{T}] = \sup_{Y \in \mathcal{Y}^1} \mathbb{E}[(cY \cdot \kappa)_{T}] \) for every \( c \in \mathcal{A}(1) \),

(ii) the set \( \mathcal{Z} \) is closed under the countable convex combinations, i.e. for every sequence \( (Z^n) \) in \( \mathcal{Z} \) and a sequence of positive numbers \( (a^n) \) such that \( \sum_{m \geq 1} a^m = 1 \), the process \( Z \triangleq \sum_{m \geq 1} a^m Z^m \) belongs to \( \mathcal{Z} \).

**Proof.** For every \( n \geq 1 \), and \( H \in \mathcal{H}^n \), in view of the positivity of \( X \triangleq x + H^n \cdot S \) (for an appropriate \( x \geq 0 \)),

\[
\tau_k \triangleq \inf \{ t > 0 : X_t \geq k \} \wedge T, \quad k \geq 1,
\]

is a localizing sequence for \( XZ \) for every \( Z \in \mathcal{Z} \). This implies (ii), whereas (i) results from Fatou’s lemma and the definitions of the sets \( \mathcal{Z} \) and \( \mathcal{Y}(1) \).

**Proof of Theorem 2.2.** By Lemma 3.4, the sets \( \mathcal{A}(1) \) and \( \mathcal{Y}(1) \) satisfy the assumptions of Theorem 3.2 in [22] that implies the assertions (i) and (ii) of Theorem 2.2. The conclusions of item (iii) supervene from Lemma 3.5 and Theorem 3.3 in [22]. This completes the proof of Theorem 2.2.

For the proof of Lemma 2.4, we need the following technical result.

**Lemma 3.6.** Under the conditions of Lemma 2.4, for every \( \varepsilon \in (0, 1) \) we have

\[
\bigcap_{n \geq 1} \mathcal{Y}^n(1) \subset \mathcal{Y} \left( \frac{1}{1 - \varepsilon} \right).
\]

**Proof.** Observe that by Proposition 4.4 in [22], for every \( n \geq 1 \), the sets \( \mathcal{A}(1) \) and \( \mathcal{Y}(1) \) satisfy the bipolar relations, likewise by Lemma 3.4, we have \( \mathcal{A}(1)^o = \mathcal{Y}(1) \). Fix an \( \varepsilon \in (0, 1) \). From (2.9) using Fatou’s lemma we obtain

\[
\mathcal{A}(1 - \varepsilon)^o \supset \left( \bigcup_{n \geq 1} \mathcal{A}^n(1) \right)^o.
\]
Therefore we conclude

\[ Y \left( \frac{1}{1 - \varepsilon} \right) = \mathcal{Y}(1 - \varepsilon)^{o} \supset \left( \bigcup_{n \geq 1} \mathcal{Y}^{n}(1) \right)^{o} = \bigcap_{n \geq 1} \mathcal{Y}^{n}(1)^{o} = \bigcap_{n \geq 1} \mathcal{Y}^{n}(1). \]

This concludes the proof of the lemma. \( \square \)

**Proof of Lemma 2.4.** Without loss of generality, we will assume that \( u^{1}(x) > -\infty, \ x > 0 \). We will only show the second assertion, as the proof of the first one is entirely similar. Also, for convenience of notations, we will assume that \( y = 1 \). Let \( Z^{n} \) be a minimizer to the dual problem (2.8), \( n \geq 1 \), where the existence of the solutions to (2.8) follows from Theorem 2.3 in [22].

It follows from (2.1) that the set \( \mathcal{Y}^{1} \) is bounded in \( \mathbb{L}^{1}(d\kappa \times \mathbb{P}) \). This in particular implies that \( \mathcal{Y}^{1}(1) \) is bounded in \( \mathbb{L}^{0}(d\kappa \times \mathbb{P}) \). Therefore, by Lemma A1.1 in [10], there exists a sequence \( \tilde{Z}^{n} \in \text{conv} (Z^{n}, Z^{n+1}, \ldots), n \geq 1 \), and an element \( Z \in \mathbb{L}^{0}(d\kappa \times \mathbb{P}) \), such that \( (\tilde{Z}^{n}) \) converges to \( Z (d\kappa \times \mathbb{P}) \)-a.e.

We also have

\[ Z = \lim_{n \to \infty} \tilde{Z}^{n} \in \bigcap_{n \geq 1} \mathcal{Y}^{n}(1) \subset \mathcal{Y} \left( \frac{1}{1 - \varepsilon} \right) \quad \text{for every} \quad \varepsilon \in (0, 1), \]

where the latter inclusion follows from Lemma 3.6. By convexity of \( V \), we get

\[ (3.4) \quad \limsup_{n \to \infty} \mathbb{E} \left[ V(\tilde{Z}^{n}) \cdot \kappa_{T} \right] \leq \lim_{n \to \infty} v^{n}(1). \]

Note that \( (\tilde{Z}^{n}) \subset \mathcal{Y}^{1}(1) \). Consequently, using Lemma 3.5 in [22], we conclude that \( \left( V^{-} (\tilde{Z}^{n}) \right) \) in uniformly integrable (here \( V^{-} \) denotes the negative part of the stochastic field \( V \)). Therefore, from Fatou’s lemma and (3.4) we deduce

\[ v \left( \frac{1}{1 - \varepsilon} \right) \leq \mathbb{E} [V(Z) \cdot \kappa_{T}] \leq \liminf_{n \to \infty} \mathbb{E} \left[ V(\tilde{Z}^{n}) \cdot \kappa_{T} \right] \leq \lim_{n \to \infty} v^{n}(1) \]

for every \( \varepsilon \in (0, 1) \). Taking the limit as \( \varepsilon \downarrow 0 \) and using the continuity of \( v \) (by convexity, see Theorem 2.2), we obtain that

\[ v(1) \leq \lim_{n \to \infty} v^{n}(1). \]

Also, since \( \mathcal{Y}(1) \subset \mathcal{Y}^{n}(1) \) for every \( n \geq 1 \), we have

\[ v(1) \geq \lim_{n \to \infty} v^{n}(1). \]

Thus, \( v(1) = \lim_{n \to \infty} v^{n}(1) \). The proof of the lemma is now complete. \( \square \)
Lemma 3.7. Let \( \kappa \) and \( S \) are continuous processes (i.e. every component of \( S \) is continuous) that satisfy (2.1) and (2.2). Then (2.9) holds.

Proof. Fix an \( \varepsilon \in (0, 1] \) and \( c \in \mathcal{A}(1 - \varepsilon) \). Let \( H \) be a \((1 - \varepsilon)\)-admissible generalized strategy, such that

\[
c \cdot \kappa \leq 1 - \varepsilon + H \cdot S, \quad \mathbb{P}\text{-a.s.}
\]

Let \((H^n)\) be a sequence of \((1 - \varepsilon)\)-admissible elementary strategies, such that \(H^n \cdot S\) converges to \(H \cdot S\) in the semimartingale topology. Let us define a sequence of stopping times as

\[
\tau_n \triangleq \inf \{ t \in [0, T] : c \cdot \kappa_t \geq 1 + H^n \cdot S_t \} \wedge T.
\]

Then we have

\[
\mathbb{P}[\tau_n < T] \leq \mathbb{P} \left[ \sup_{t \in [0, T]} (c \cdot \kappa_t - 1 + \varepsilon - H^n \cdot S_t) \geq \varepsilon \right] \leq \mathbb{P} \left[ \sup_{t \in [0, T]} (H \cdot S_t - H^n \cdot S_t) \geq \varepsilon \right],
\]

which converges to 0 as \( n \to \infty \). Let us define a sequence of consumptions \((c^n)\) as follows

\[
c^n_t \triangleq c_t 1_{[0, \tau_n]}(t), \quad t \in [0, T], \quad n \geq 1.
\]

Then, by continuity of \( \kappa \) and \( S \) we get

\[
c^n \cdot \kappa \leq 1 + H^n \cdot S \quad \text{on } [0, \tau_n] \quad \mathbb{P}\text{-a.s., } \quad n \geq 1.
\]

Since \(H^n 1_{[0, \tau_n]}\) is a 1-admissible elementary strategy, we deduce that \(c^n \in \mathcal{A}^n(1), n \geq 1\). One can also see that \((c^n)\) converges to \(c\) in \(L^0\). This concludes the proof of the lemma. \(\square\)

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