OPERATORS GENERATED BY WAVELETS AND THEIR BOUNDEDNESS FROM $H^p(\mathbb{R}^n)$ INTO $L^p(\mathbb{R}^n)$

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Abstract. We consider the following operator generated by wavelets

$$T_\mu f(x) = \sum_{(l,j,k) \in \Lambda} \omega_{j,k}^l \langle \mu_j * f, \psi_{j,k}^l \rangle \phi_{j,k}^l(x),$$

where $\Lambda = \{(l,j,k) : l = 1, \ldots, 2^n - 1, j \in \mathbb{Z}, k \in \mathbb{Z}^n\}$, $\{\omega_{j,k}^l\} \in \ell^2(\Lambda)$, the $\mu_j$'s are measures defined by $\bar{\mu}_j(\xi) = \hat{\mu}(2^j \xi)$ where $\mu$ is certain Borel measure on $\mathbb{R}^n$ and $\{\psi_{j,k}^l\}_{(l,j,k) \in \Lambda}$ are orthonormal bases of $L^2(\mathbb{R}^n)$. In this work we obtain the $H^p(\mathbb{R}^n) - L^p(\mathbb{R}^n)$ boundedness of the operator $T_\mu$ for certain Borel measures $\mu$.

1. Introduction

The first mention of the name "wavelets" seems to be in 1909, in the Alfred Haar’s thesis. The concept of wavelets in its present theoretical form was first proposed in mid 80’s by Jean Morlet and Alex Grossmann. Since then, the subject has received considerable attention (see [4], [6], [7], [9], [14], and the references therein). Besides the intrinsic mathematical interest, the wavelets theory is also relevant by their many applications in physics, computer science, and engineering. An interesting historical point of view of this topic can be found in [5].

The purpose of this work is to study the behavior of certain operators generated by wavelets on the classical Hardy spaces $H^p(\mathbb{R}^n)$. More precisely, given a Borel measure $\mu$ on $\mathbb{R}^n$ ($n \geq 1$) we define formally the operator $T_\mu$ by

$$T_\mu f(x) = \sum_{(l,j,k) \in \Lambda} \omega_{j,k}^l \langle \mu_j * f, \psi_{j,k}^l \rangle \phi_{j,k}^l(x),$$

where $\langle f, g \rangle = \int f \cdot \overline{g}$, the index set $\Lambda$ is defined by

$$\Lambda = \{(l,j,k) : l = 1, \ldots, 2^n - 1, j \in \mathbb{Z}, k \in \mathbb{Z}^n\},$$

and $\omega_{j,k}^l$ is a sequence of complex numbers in $\ell^2(\Lambda)$, the measures $\mu_j$ are defined for each $j \in \mathbb{Z}$ by $\bar{\mu}_j(\xi) = \hat{\mu}(2^j \xi)$, where $\mu$ is certain Borel measure on $\mathbb{R}^n$, and the functions

$$\psi_{j,k}^l(x) := 2^{-nj} \psi_l(2^{-j}x - k), \quad \phi_{j,k}^l(x) := 2^{-nj} \phi_l(2^{-j}x - k),$$

$1 \leq l \leq 2^n - 1, k \in \mathbb{Z}^n, j \in \mathbb{Z}$, form two orthonormal bases of $L^2(\mathbb{R}^n)$. If we take $\mu = \delta$, the Dirac measure at the origin on $\mathbb{R}$, $\omega_{j,k}^l = \pm 1$ and $\phi_l = \psi_l$ in [11], we recover the operator studied in [4] (see p. 296) which results bounded on $L^p(\mathbb{R})$, $1 < p < +\infty$. From this follows that $\{\psi_{j,k} : j,k \in \mathbb{Z}\}$ is an unconditional basis for all the $L^p$-spaces, $1 < p < +\infty$ (see Theorem 9.1.6 p. 298 in [4]). In [2], M. Bownik extends the above result for $n \geq 1$ and he also prove that the operator $T_3$ is bounded on the anisotropic Hardy spaces $H^p_{\alpha}(\mathbb{R}^n)$, such spaces are a generalization of the classical Hardy spaces $H^p$. These results are achieved showing that the operator $T_3$...
admits a Calderón-Zygmund representation under certain additional assumptions on the wavelets functions $\psi$ and $\phi$.

We recall that each positive Borel measure $\mu$ that is finite on compact subsets can be decomposed in a unique way as the sum of three measures $\mu = \mu_L + \mu_D + \mu_S$ that are mutually singular. The measure $\mu_L$ is an absolutely continuous measure with respect to the Lebesgue measure, that is $d\mu_L = g(x)dx$, where $g \in L^1(\mathbb{R}^n)$ and $dx$ denotes the Lebesgue measure on $\mathbb{R}^n$. The measure $\mu_D$ is a discrete measure, that is 
\[ \mu_D = \sum_{i=1}^{\infty} \delta_{x_i} \quad \text{such that} \quad \delta_{x_i} \text{ is the Dirac measure at } x_i \in \mathbb{R}^n. \]

The measure $\mu_S$ is a singular continuous measure, this means that it is concentrated on a zero Lebesgue measure subset and each point has null measure $\mu_S$.

In this work we prove that the operator defined by (1) can be extended to a bounded operator on $H^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, for $\frac{N}{N+n} < p < +\infty$ ($N \in \mathbb{N}$) and all Borel measure $d\mu$ on $\mathbb{R}^n$ of the form:
\[ d\mu = \begin{cases} 
  g(x)dx + \sum_{i=1}^{\infty} c_i d\delta_{x_i}, & \text{if } n = 1 \\
  g(x)dx + \sum_{i=1}^{\infty} c_i d\delta_{x_i} + d\lambda, & \text{if } n \geq 2 
\end{cases} \]

where, for some $\epsilon > 0$, $(1 + |x|)^{2n+2N+2\epsilon}|g(x)| \leq C < +\infty$ for all $x \in \mathbb{R}^n$, the sequences $\{c_i\} \subset \mathbb{R}$ and $\{x_i\} \subset \mathbb{R}^n$ are such that \( \sum_{i=1}^{\infty} |c_i|(1 + |x_i|)^{2n+2N+2\epsilon} < \infty, \) and $\lambda$ is the singular Borel measure on $\mathbb{R}^n$ ($n \geq 2$) defined by
\[ \lambda(E) = \int_{\mathbb{R}^n} \chi_E(y,0)h(y)dy \]

where $1 \leq m < n$, $E$ is a Borel subset of $\mathbb{R}^n$ and $(1 + |y|)^{2n+2N+2\epsilon}|h(y)| \leq C$ for all $y \in \mathbb{R}^m$.

If $U_\lambda$ is the convolution operator with the singular Borel measure $\lambda$, i.e.:
\[ U_\lambda f(x,t) = (\lambda * f)(x,t) = \int_{\mathbb{R}^m} f(x-y,t)h(y)dy, \quad (x,t) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \]

then $\|U_\lambda f\|_p \leq \|h\|_{L^1(\mathbb{R}^m)}\|f\|_p$ for all Borel function $f$ in $L^p(\mathbb{R}^n)$, $1 \leq p \leq +\infty$; so the operator $U_\lambda$ can be extended to a bounded operator on $L^p(\mathbb{R}^n)$. Moreover, this operator can be extended to an $H^1(\mathbb{R}^n) - L^1(\mathbb{R}^n)$ bounded operator. As far as the authors know, it is unknown in the literature the $H^p - L^p$ boundedness of $U_\lambda$, for $0 < p < 1$. In this article we give an alternative result to this problem. More precisely, we prove that the operator $T_\lambda$, the "wavelet version" of the operator $U_\lambda$, results bounded from $H^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for $\frac{N}{N+n} < p \leq 1$.

Section 2 is meant to give a motivation of the problem that is the subject of our study. In Section 3 we state the good definition of the operator $T_\mu$ and give two known results about the wavelets theory. We also recall the definition and atomic decomposition of the Hardy spaces. In Section 4 we state some auxiliary lemmas and propositions to get the main results proved in Section 5.
Notation: We use the following convention for the Fourier transform in $\mathbb{R}^n$
\[ \hat{f}(\xi) = \int f(x)e^{-i\xi \cdot x} \, dx. \] We denote by $B(x_0, r)$ the ball centered at $x_0 \in \mathbb{R}^n$ of radius $r$. For a measurable subset $E \subset \mathbb{R}^n$ we denote $|E|$ and $\chi_E$ the Lebesgue measure of $E$ and the characteristic function of $E$ respectively. Given a real number $s \geq 0$, we write $[s]$ for the integer part of $s$. As usual we denote with $\mathcal{S}('\mathbb{R}^n)$ the space of all rapidly decreasing functions on $\mathbb{R}^n$, called Schwartz functions; with $\mathcal{S}'(\mathbb{R}^n)$ the dual space. If $\alpha$ is the multi-index $\alpha = (\alpha_1, ..., \alpha_n)$, then $|\alpha| = \alpha_1 + ... + \alpha_n$.

$x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$.

Throughout this paper, $C$ will denote a positive constant, not necessarily the same at each occurrence.

2. Motivation

In this section we give some motivation of considering the operator given in \( (1) \). For each $x \in \mathbb{R}^n$, let $\tau_x$ denote the translation operator that maps a function $f$ on $\mathbb{R}^n$ to the function $(\tau_x f)(y) := f(y - x)$. Let $\hat{f}(x) := f(-x)$ denote the reflection operator. Let $\mathcal{S}(\mathbb{R}^n)$ be the space of Schwartz functions and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. It is well known (as an application of the Schwartz kernel theorem) that every linear continuous operator $T : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ invariant by translations (i.e. $T(\tau_x f) = \tau_x(Tf) \forall x \in \mathbb{R}^n, \forall f \in \mathcal{S}(\mathbb{R}^n)$) is of convolution type. That is, there exists a unique distribution $\Phi_T \in \mathcal{S}'(\mathbb{R}^n)$ such that $Tf = \Phi_T \ast f$ for all $f \in \mathcal{S}(\mathbb{R}^n)$. We will restrict ourselves to the case where $\Phi_T$ is a Borel measure $\Phi_T = \mu$.

Apart from that, it is well known that if $\mu$ is a Borel measure and $V$ is a Banach space of functions on $\mathbb{R}^n$ such that

i. $V$ is invariant under translations and reflection, i.e., if $f \in V$, then $\tau_x f \in V$ and $\hat{f} \in V$,

ii. $||f||_V = ||\tau_x f||_V = ||\hat{f}||_V$ for all $f \in V$ and $x \in \mathbb{R}^n$,

iii. the translation is a continuous representations on $V$, i.e., for each $f \in V$ the map $x \to \tau_x f$ from $\mathbb{R}^n$ to $V$ is continuous,

then the Bochner integral
\[ \int_{\mathbb{R}^n} \tau_y \hat{f} d\mu(y) = \mu \ast f \]
is well defined for each $f \in V$ and it holds that
\[ ||\mu \ast f||_V \leq ||\mu||_1 ||f||_V \]
(where $|| \cdot ||_1$ denotes the total variation of the measure $\mu$). The Lebesgue spaces $L^p(\mathbb{R}^n)$, with $1 \leq p < \infty$, the Hardy space $H^1(\mathbb{R}^n)$ are examples of such space $V$.

Then it follows that, for $V = L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, or $V = H^1(\mathbb{R}^n)$, the operator
\[ T : V \to V \]
\[ Tf := \mu \ast f, \]
is well defined and is continuous with $||T|| \leq ||\mu||_1$.

Let $\{\psi_{j,k}\}_{(j,k) \in \Lambda}$ a Hilbert basis for $L^2(\mathbb{R}^n)$. We will be interested on the case where it is a wavelet basis with mother wavelet $\psi$. There are two important operators associated with $\{\psi_{j,k}\}$, the analysis operator
\[ L^2(\mathbb{R}^n) \to \ell^2(\Lambda) \]
\[ f \mapsto (\langle f, \psi_{j,k} \rangle)_{j,k} \]
and the synthesis operator
\[ \ell^2(\Lambda) \rightarrow L^2(\mathbb{R}^n) \]
\[ (c_j, k) \rightarrow \sum_{j, k} c_{j, k} \psi_{j, k}. \]

Since \( \{\psi_{j, k}\} \) is an orthonormal basis, we can represent the operator \( T \) given in (4), for \( V = L^2(\mathbb{R}^n) \), as the operator from \( L^2(\mathbb{R}^n) \) to \( L^2(\mathbb{R}^n) \) given by
\[ (5) \quad Tf = \sum_{j, k} \langle \mu \ast f, \psi_{j, k} \rangle \psi_{j, k}. \]

The passage to a multi-valued version of an analytic object is a useful device that arises in many situations. So, it is natural to replace in (5) the map \( f \rightarrow \mu \ast f \) by its multi-valued version \( f \rightarrow \{\omega_{j, k}(\mu \ast f)\} \), where \( \omega_{j, k} \in \mathbb{C} \) and \( \tilde{\mu}_j(\xi) = \tilde{\mu}(|2^j \xi|) \), arising in this way (at least formally) the operator \( T_\mu \) given in (1). Thus, our operator is the composition between the synthesis operator associated to \( \phi_{j, k} \), the analysis operator associated to \( \psi_{j, k} \) and the multi-valued version of the operator \( T \) defined in (1).

3. Preliminaries

We start recalling some fact about multiple series. A multiple series is a series of the form
\[ (6) \quad \sum_{k \in \mathbb{Z}^n} a_k, \]
where \( a_k \in \mathbb{C} \) for each \( k \in \mathbb{Z}^n \). There are many different ways to define the sum of a multiple series by mean of partial sums. In the literature the following two are the most common (see e.g. [1], [8], [11] and [13]):

The \( N \)th-quadratic partial sum of the series in (6) is defined by
\[ S_N = \sum_{|k|_\infty \leq N} a_k, \]
where \( k = (k_1, ..., k_n) \in \mathbb{Z}^n \) and \( |k|_\infty = \max\{|k_i| : i = 1, ..., n\} \).

If \( \lim_{N \to \infty} S_N \) exists we say that the series in (6) is \textit{quadratically convergent}.

The \( N \)th-circular partial sum of the series in (6) is defined by
\[ \tilde{S}_N = \sum_{|k| \leq N} a_k, \]
where \( k = (k_1, ..., k_n) \in \mathbb{Z}^n \) and \( |k| = (k_1^2 + \cdots + k_n^2)^{1/2} \). The series in (6) is \textit{circularly convergent} if \( \lim_{N \to \infty} \tilde{S}_N \) exists.

In general, the circular convergence and the quadratic convergence are not equivalent (see [1], pp. 7-8). However, if a series is absolutely convergent in the sense circular or quadratic, then both convergence are equivalent and their sums coincide. Since our results only involve absolutely convergent series we can use one or another definition as it suits.

\textbf{Lemma 1.} If \( \epsilon > 0 \), then the multiple series
\[ (7) \quad \sum_{k \in \mathbb{Z}^n \setminus \{0\}} \frac{1}{|k|^{n+\epsilon}} \]
converges.
Finally, letting $N$ be the $N$th-quadratic partial sum of the series in (7). A computation gives

$$S_N = \sum_{0 < |x| \leq N} \frac{1}{|k|^{n+\epsilon}} = 2^n \sum_{k_0=0}^N \cdots \sum_{k_{n-1}=0}^N \sum_{k_n=0}^N \frac{1}{(k_1^2 + \cdots + k_n^2)^{\frac{n}{2}}},$$

since $|k_1| + |k_2| + \cdots + |k_n| \leq n(k_1^2 + \cdots + k_n^2)^{1/2}$, by Multinomial Theorem results

$$(|k_1| + |k_2| + \cdots + |k_n|)^{n+\epsilon} \geq \max\{1, |k_1|^{1+\frac{\epsilon}{n}}\} \cdots \max\{1, |k_n|^{1+\frac{\epsilon}{n}}\}, \quad \forall k \in \mathbb{Z}^n\setminus\{0\}. $$

So,

$$\sum_{0 < |x| \leq N} \frac{1}{|k|^{n+\epsilon}} \leq 2^n \sum_{k_0=0}^N \cdots \sum_{k_{n-1}=0}^N \sum_{k_n=0}^N \frac{n}{\max\{1, |k_1|^{1+\frac{\epsilon}{n}}\} \cdots \max\{1, |k_n|^{1+\frac{\epsilon}{n}}\}}$$

$$\leq 2^n n \left(1 + \sum_{s=1}^n \frac{1}{s^{1+\frac{\epsilon}{n}}}\right)^n, \quad \forall l \geq 2.$$

Finally, letting $N$ tend to infinity, we obtain

$$\sum_{k \in \mathbb{Z}^n\setminus\{0\}} \frac{1}{|k|^{n+\epsilon}} := \lim_{N \to \infty} S_N \leq 2^n n \left(1 + \sum_{s=1}^n \frac{1}{s^{1+\frac{\epsilon}{n}}}\right)^n < \infty.$$

\[\Box\]

**Lemma 2.** Let $\epsilon$ be a positive constant. Then the multiple series

$$\sum_{k \in \mathbb{Z}^n} \frac{1}{(1 + |x - k|)^{n+\epsilon}} \leq C$$

for some positive constant $C$ independent of $x \in \mathbb{R}^n$. Moreover, such a series converges uniformly on compact subsets of $\mathbb{R}^n$.

**Proof.** Given $x \in \mathbb{R}^n$, there exists a cube $Q$ with vertices in $\mathbb{Z}^n$ such that $x \in Q$. Thus, there exists $k_0 = k_0(x) \in \mathbb{Z}^n$ such that $\sqrt{n} + |x - k| \geq |k - k_0|$ for all $k \in \mathbb{Z}^n$. So,

$$\sum_{k \in \mathbb{Z}^n} \frac{1}{(1 + |x - k|)^{n+\epsilon}} \leq 1 + \sum_{k \neq k_0} \frac{\frac{n+\epsilon}{|k-k_0|^{n+\epsilon}}}{1 + \frac{n+\epsilon}{|k|^{n+\epsilon}}},$$

by Lemma 1 the last series converges. So, it is enough to take $C = 1 + \sum_{k \neq k_0} \frac{n+\epsilon}{|k|^{n+\epsilon}}$. Finally, since $(1 + |x|)(1 + |x - k|) \geq 1 + |k|$ for any $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}^n$, then it follows the uniform convergence of the series on compact subsets. \[\Box\]

The following result establish the good definition of the operator $T_\mu$ on $L^2(\mathbb{R}^n)$.

**Proposition 3.** If $\mu$ is a measure as in (2), then the operator $T_\mu$ given by (1) is well defined on the space of all Borel functions in $L^2$. Moreover, $T_\mu$ can be extended to a bounded operator on $L^2(\mathbb{R}^n)$.

**Proof.** For each $j$, we define the measures $\mu_j$ by $\hat{\mu}_j(\xi) = \hat{\mu}(2^j \xi)$, where $\mu$ is a measure as in (2). To prove the good definition of the operator $T_\mu$, by the Riesz-Fischer property, it is enough to show that

$$\sum_{(l,j,k) \in \Lambda} |\omega_{l,j,k} \mu_j \ast f, \psi_{l,j,k} \phi_{l,j,k}|^2 < +\infty.$$
We study each case separately. Since \( \{ \psi_{j,k}^l \}_{(l,j,k) \in \Lambda} \) and \( \{ \phi_{j,k}^l \}_{(l,j,k) \in \Lambda} \) are orthonormal bases of \( L^2 \), we have that
\[
\sum_{(l,j,k) \in \Lambda} \| \omega_{j,k}^l \langle \mu_j * f, \psi_{j,k}^l \rangle \phi_{j,k}^l \|_2^2 = \sum_{(l,j,k) \in \Lambda} |\omega_{j,k}^l|^2 \| \langle \mu_j * f, \psi_{j,k}^l \rangle \|_2^2.
\]
If \( \mu = g(x)dx \), then \( \hat{\mu}_j(\xi) = \hat{g}(2^j \xi) \). From Cauchy-Schwarz inequality and Plancherel’s Theorem and since \( g \in L^1(\mathbb{R}^n) \) we have
\[
\sum_{(l,j,k) \in \Lambda} |\omega_{j,k}^l|^2 \| \langle \mu_j * f, \psi_{j,k}^l \rangle \|_2^2 \leq \sum_{(l,j,k) \in \Lambda} |\omega_{j,k}^l|^2 \| \hat{\mu}_j \hat{f} \|_2^2
\]
\[
= \sum_{(l,j,k) \in \Lambda} |\omega_{j,k}^l|^2 \int_{\mathbb{R}^n} |\hat{g}(2^j \xi)|^2 |\hat{f}(\xi)|^2 d\xi
\]
\[
\leq \sum_{(l,j,k) \in \Lambda} |\omega_{j,k}^l|^2 \|g\|_2^2 \|f\|_2^2 < +\infty.
\]
If \( \mu = \sum_{i=1}^{+\infty} c_i \delta_{x_i} \), then \( \mu_j = \sum_{i=1}^{+\infty} c_i \delta_{2^j x_i} \). Since \( \sum_{i=1}^{+\infty} |c_i| < +\infty \), we have \( \mu_j * f = \sum_i c_i f(\cdot - 2^j x_i) \in L^2 \) for each \( j \in \mathbb{Z} \), then one obtains
\[
\sum_{(l,j,k) \in \Lambda} |\omega_{j,k}^l|^2 \| \langle \mu_j * f, \psi_{j,k}^l \rangle \|_2^2 \leq \left( \sum_{(l,j,k) \in \Lambda} |\omega_{j,k}^l|^2 \right)^2 \left( \sum_{i=1}^{+\infty} |c_i| \right)^2 \|f\|_2^2.
\]
If \( \mu = \lambda \) is the singular Borel measure on \( \mathbb{R}^n \) given by (3), then a computation gives \( \hat{\mu}_j(\xi, \xi') = \hat{h}(2^j \xi) \) where \( (\xi, \xi') \in \mathbb{R}^m \times \mathbb{R}^{n-m} \), so
\[
\sum_{(l,j,k) \in \Lambda} |\omega_{j,k}^l|^2 \| \langle \mu_j * f, \psi_{j,k}^l \rangle \|_2^2 \leq \sum_{(l,j,k) \in \Lambda} |\omega_{j,k}^l|^2 \int_{\mathbb{R}^m \times \mathbb{R}^{n-m}} |\hat{h}(2^j \xi)|^2 |\hat{f}(\xi, \xi')|^2 d\xi d\xi'
\]
\[
\leq \sum_{(l,j,k) \in \Lambda} |\omega_{j,k}^l|^2 \|h\|_2^2 \|f\|^2_2.
\]
Thus, for each Borel function \( f \in L^2 \) there exists an unique function \( T_\mu f \in L^2 \) such that
\[
T_\mu f = \sum_{(l,j,k) \in \Lambda} \omega_{j,k}^l \langle \mu_j * f, \psi_{j,k}^l \rangle \phi_{j,k}^l,
\]
where the convergence is in the \( L^2 \)-norm. Moreover, we have
\[
\|T_\mu f\|_2 \leq C \|\omega_{j,k}^l\|_{L^2(\mathbb{R}^n)} \|f\|_2,
\]
for each Borel function \( f \in L^2(\mathbb{R}^n) \). So the operator \( T_\mu \) can be extended to a bounded operator on \( L^2(\mathbb{R}^n) \).

\[\square\]

**Remark 4.** We observe that the following identity
\[ (8) \]
\[ \langle \delta_{x_0} * f, \phi \rangle = \int f(x - x_0) \phi(x) dx, \]
holds for \( f \in \mathcal{S}(\mathbb{R}^n) \) and \( \phi \in L^2(\mathbb{R}^n) \). Since the right-side of (3) \[\|f\|_2 \]
has sense for \( f \in L^2(\mathbb{R}^n) \), we can redefine the operator \( T_\mu \) via the right-side of (3), so \( T_\mu \) results well defined on \( L^2 \).

The following Theorem can be found in [9], page 93.
Theorem 5. There exist \( q = 2^n - 1 \) functions \( \psi^1, \ldots, \psi^q \) in \( V_1 \) having the following two properties:
\[
|\phi^{2^j} \psi^i(x)| \leq C_N (1 + |x|)^{-N}
\]
for every multi-index \( \alpha \in \mathbb{N}_0^n \) such that \( |\alpha| \leq r \), each \( x \in \mathbb{R}^n \) and each every \( N \); and
\[
\{ \psi^i(x - k) : 1 \leq i \leq q, k \in \mathbb{Z}^n \}
\]
is an orthonormal basis of \( W_0 \).

For the definition of the spaces \( V_1 \) and \( W_0 \) see \([9]\).  

Corollary 6. The functions \( \psi^i_{j,k}(x) := 2^{-nj/2} \phi^i(2^{-j}x - k) \), \( 1 \leq j \leq q, k \in \mathbb{Z}^n \), \( j \in \mathbb{Z} \), form an orthonormal basis of \( L^2(\mathbb{R}^n) \). Thus \( \|f\|_{L^2(\mathbb{R}^n)} = \sum_{j,k \in \mathbb{Z}^n} \left| \langle f, \psi^i_{j,k} \rangle \right|^2. \)

We recall the notion of a Calderón-Zygmund operator. Let \( \Delta : \{ (x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y \} \).

Definition 7. A function \( K : \mathbb{R}^n \times \mathbb{R}^n \setminus \Delta \to \mathbb{C} \) is called a standard kernel if there exist positive constants \( \epsilon \) and \( C \) such that
\[
|K(x, y)| \leq \frac{C}{|x - y|^n} \quad \forall x \neq y
\]
\[
|K(x, y) - K(x', y)| \leq C \frac{|x - x'|^\epsilon}{(|x - y| + |x' - y|)^{n+\epsilon}} \quad \text{if } |x - x'| \leq \frac{1}{2} \min \{|x - y|, |x' - y|\}
\]
\[
|K(x, y) - K(x, y')| \leq C \frac{|y - y'|^\epsilon}{(|x - y| + |x - y'|)^{n+\epsilon}} \quad \text{if } |y - y'| \leq \frac{1}{2} \min \{|x - y|, |x - y'|\}
\]
A Calderón-Zygmund operator associated to \( K \) is a linear operator \( T \) defined on \( \mathcal{S}(\mathbb{R}^n) \) which admits a continuous extension to \( L^2(\mathbb{R}^n) \) (i.e., \( \|Tf\|_2 \leq \|f\|_2 \)) and satisfies
\[
Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy
\]
for all \( f \in C_c^\infty \) and \( x \notin \operatorname{supp}(f) \).

It is well known that a Calderón-Zygmund operator can be extended to a bounded operator on \( L^p(\mathbb{R}^n), 1 < p < \infty \), and from \( L^1(\mathbb{R}^n) \) to \( L^1_{\text{weak}}(\mathbb{R}^n) \).

We observe that the condition
\[
\sum_{i=1}^n (|\hat{\psi}_x, K(x, y)| + |\hat{\psi}_y, K(x, y)|) \leq \frac{C}{|x - y|^{n+1}}, \quad \forall x \neq y.
\]
implies (10) and (11) (cf. \([8]\) p. 211).

We conclude these preliminaries with the definition of Hardy space and \((p, \infty)\)-atomic operators.

The Hardy space \( H^p(\mathbb{R}^n) \) (for \( 0 < p < \infty \)) consists of tempered distributions \( f \in \mathcal{S}'(\mathbb{R}^n) \) such that for some Schwartz function \( \varphi \) with \( \int \varphi = 1 \), the maximal operator
\[
(M_\varphi f)(x) = \sup_{t > 0} (|\varphi_t * f|(x))
\]
is in \( L^p(\mathbb{R}^n) \), where \( \varphi_l(x) := \frac{1}{r_l} \varphi \left( \frac{x}{r_l} \right) \). In this case we define \( \|f\|_{H^p} := \|M\varphi\|_p \) as the \( H^p \) “norm”. It can be shown that this definition does not depend on the choice of the function \( \varphi \). For \( 1 < p < \infty \), it is well known that \( H^p(\mathbb{R}^n) \cong L^p(\mathbb{R}^n) \), \( H^1(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) \) strictly, and for \( 0 < p < 1 \) the spaces \( H^p(\mathbb{R}^n) \) and \( L^p(\mathbb{R}^n) \) are not comparable. One of the principal interest of \( H^p(\mathbb{R}^n) \) theory is that it gives a natural extension of the results for singular integrals, originally developed for \( L^p \) (\( p > 1 \)), to the range \( 0 < p \leq 1 \). This is achieved to decompose elements in \( H^p \) as sums of \((p, \infty)\)-atoms.

For \( 0 < p \leq 1 \), an \((p, \infty)\)-atom is a measurable function \( a \) supported on a ball \( B \) of \( \mathbb{R}^n \) satisfying

\[
(i) \|a\|_\infty \leq |B|^{-\frac{1}{p}}, \\
(ii) \int x^\alpha a(x)dx = 0, \text{ for all multi-index } \alpha \text{ with } |\alpha| \leq [n(p^{-1} - 1)].
\]

We observe that an \((p, \infty)\)-atom is an element of \( H^p \). Moreover, for each \((p, \infty)\)-atom \( a \) we have that \( \|a\|_{H^p} \leq C < \infty \), where \( C \) is independent of the atom \( a \).

Let \( 0 < p < 1 < s < +\infty \). We can decompose elements in \( H^p \cap L^s \) as sums of \( p \)-atoms (see [12] p. 107). Moreover, such a decomposition also converges in \( L^s \) for all \( 1 < s < +\infty \), this result was proved in [10]. More precisely, we have the following:

**Proposition 8.** Let \( f \in H^p(\mathbb{R}^n) \cap L^s(\mathbb{R}^n) \), with \( 0 < p \leq 1 < s < \infty \). Then there is a sequence of \((p, \infty)\)-atoms \( \{a_j\} \) and a sequence of scalars \( \{\lambda_j\} \) with \( \sum_j |\lambda_j|^p \leq c\|f\|_{H^p}^p \) such that \( f = \sum_j \lambda_j a_j \), where the series converges to \( f \) in \( L^s(\mathbb{R}^n) \).

**Theorem 9.** Let \( N \in \mathbb{N} \) fixed. If \( T \) is a Calderón-Zygmund bounded operator on \( L^2(\mathbb{R}^n) \) such that

\[
Tf(x) = \int K(x, y)f(y)dy, \; x \notin \text{supp}(f),
\]

where the kernel \( K \) satisfy

\[
(13) \; |\partial_x^\alpha K(x, y)| \leq C|x-y|^{-n-|\alpha|}, \text{ for all } x \neq y \text{ and all multi-index } 0 \leq |\alpha| \leq 1,
\]

and

\[
(14) \; |\partial_y^\beta K(x, y)| \leq C|x-y|^{-n-|\beta|}, \text{ for all } x \neq y \text{ and all multi-index } 1 \leq |\beta| \leq N,
\]

then \( T \) can be extended to a bounded operator from \( H^p(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \) for all \( \frac{n}{N+n} < p < +\infty \).

**Proof.** From (13) and (14) with \( N = 1 \) follow that \( T \) can be extended to bounded operator on \( L^p(\mathbb{R}^n) \) for each \( 1 < p < \infty \).

In view of the Bownik’s example (see [3]) it not suffice to check that an operator maps atoms into bounded elements of \( L^p \) (\( 0 < p \leq 1 \)) to establish that this operator extends to a bounded operator from \( H^p \) into \( L^p \). To address this problem we use Proposition 8 and a convergence argument.

Let \( 0 < p \leq 1 \). Given \( f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \), by Proposition 8 there exists an atomic decomposition for \( f \) such that \( f = \sum_j \lambda_j a_j \) in \( L^2 \) and \( \sum_j |\lambda_j|^p \leq c\|f\|_{H^p}^p \).

Since the operator \( T \) is bounded on \( L^2 \), we have that the sum \( \sum_j \lambda_j Ta_j \) converges to \( Tf \) in \( L^2 \), thus there exists a subsequence of natural numbers \( \{k_N\}_{N \in \mathbb{N}} \) such that
For \(x\) there exists \(x\) with \(C\) supported on a ball \(B\) where \(q\) and (14) we have, for each \(x\) for all \(p\) we get

\[
|Tf(x)| \leq \sum_j |\lambda_j||Ta_j(x)|, \text{ for a.e. } x \in \mathbb{R}^n.
\]

So, it is sufficient to show that there exists a positive constant \(C_p\) independent of the \(p\)-atom \(a\) such that

\[
\|Ta\|_{L^p} \leq C_p.
\]

Indeed, if \(\|Ta\|_{L^p} \leq C_p\) for all \((p, \infty)\)-atom, then from (15) and since \(0 < p \leq 1\), we get

\[
|Tf|_{L^p} \leq \sum_j |\lambda_j|^p\|Ta_j\|_{L^p}^p \leq C_p \sum_j |\lambda_j|^p \leq C_p \|f\|_{L^p}^p
\]

for all \(f \in H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\). Thus the theorem will follow from the density of \(H^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)\) in \(H^p(\mathbb{R}^n)\).

We will prove the estimate in (16) for each \(\frac{n}{N+n} < p \leq 1\). Let \(a\) be a \(p\)-atom supported on a ball \(B = B(x_0, r)\), and let \(B^* = B(x_0, 2r)\). Since \(T\) is bounded on \(L^2\) the Hölder inequality gives

\[
\int_{B^*} |Ta(x)|^p dx \leq |B^*|^{1-\frac{p}{q}}\|Ta\|_p^p \leq C|B^*|^{1-\frac{p}{q}}\|a\|_q^p \leq C|B^*|^{1-\frac{p}{q}}|B|^{-\frac{p}{q}} |B|^{\frac{p}{q}} \leq C_p.
\]

For each \(\frac{n}{N+n} < p \leq 1\) there exists an unique natural number \(1 \leq k \leq N\) such that \(\frac{n}{k+n} < p \leq \frac{n}{k-1+n}\), so \(k - 1 = [n(p-1) - 1]\). In view of the moment condition of \(a\) and (14) we have, for each \(x \in \mathbb{R}^n\setminus B^*\), that

\[
Ta(x) = \int_B K(x, y) a(y) dy = \int_B (K(x, y) - q_k(x, y)) a(y) dy,
\]

where \(q_k\) is the degree \(k-1\) Taylor polynomial of the function \(y \rightarrow K(x, y)\) expanded around \(x_0\). By the standard estimate of the remainder term in the Taylor expansion, there exists \(\xi\) between \(y\) and \(x_0\) such that

\[
|K(x, y) - q_k(x, y)| \leq C|y - x_0|^k \sum_{|\beta| = k} |\partial^\beta K(x, \xi)| \leq C|y - x_0|^k |x - \xi|^{-n-k}.
\]

For \(x \in \mathbb{R}^n\setminus B^*\) we have that \(|x - x_0| \geq 2r\), so \(|y - \xi| \leq \frac{|x - x_0|}{2}\) and thus \(|x - \xi| \geq |x - x_0| - |x_0 - \xi| \geq \frac{|x - x_0|}{2}\) for all \(x \in \mathbb{R}^n\setminus B^*\). Then

\[
|K(x, y) - q_k(x, y)| \leq C|y - x_0|^k |x - x_0|^{-n-k}
\]

for all \(x \in \mathbb{R}^n\setminus B^*\) and all \(y \in B\). This inequality gives

\[
\int_{\mathbb{R}^n \setminus B^*} \left| \int_B K(x, y) a(y) dy \right|^p dx \leq \int_{\mathbb{R}^n \setminus B^*} \left| \int_B (K(x, y) - q_k(x, y)) a(y) dy \right|^p dx
\]

\[
\leq C \left( \int_{\mathbb{R}^n \setminus B^*} |x - x_0|^{-pn-pk} dx \right)^p \left( \int_B |y - x_0|^k |a(y)| dy \right)^p
\]

\[
\leq C \left( \int_{2r}^{+\infty} t^{-pn-pk+n-1} dt \right)^{pk-n+n} = C_p r^{-pn-pk+n} r^{kp-n+n} = C_p,
\]

with \(C_p\) independent of the \(p\)-atom \(a\) since \(-pn-pk+n < 0\). Then (16) follows. \(\Box\)
4. Auxiliary Results

Let \( f \) be a function on \( \mathbb{R}^n \). For \( j \in \mathbb{Z} \) and \( k \in \mathbb{Z}^n \) we define \( f_{j,k} \) by
\[
f_{j,k}(x) = 2^{-\frac{nj}{2}} f(2^{-j}x - k).
\]

**Lemma 10.** Let \( \phi \) and \( \psi \) continuous functions defined on \( \mathbb{R}^n \) with continuous partial derivatives up to order \( M \geq 1 \) and \( N \geq 1 \) respectively such that
\[
|\partial^\alpha \phi(x)| \leq \frac{C}{(1 + |x|)^{2n+2|\alpha|+2\epsilon}}, \tag{17}
\]
\[
|\partial^\beta \psi(y)| \leq \frac{C}{(1 + |y|)^{2n+2|\beta|+2\epsilon}} \tag{18}
\]
for all \( x, y \in \mathbb{R}^n \), and all multi-index \( 0 \leq |\alpha| \leq M \) and \( 0 \leq |\beta| \leq N \).

Then the series \( \sum_{j,k} \phi_{j,k}(x) \psi_{j,k}(y) \) converges absolutely for all \( x \neq y \). Moreover, if
\[
K(x,y) = \sum_{j,k} \phi_{j,k}(x) \psi_{j,k}(y),
\]
then there exists a positive constant \( C \) such that
\[
|K(x,y)| \leq \frac{C}{|x-y|^n} \quad \forall x \neq y, \tag{19}
\]
\[
|\partial_2^\alpha K(x,y)| \leq \frac{C}{|x-y|^{n+|\alpha|}}, \quad \forall x \neq y, \tag{20}
\]
for all multi-index \( 1 \leq |\alpha| \leq M \), and
\[
|\partial_2^\beta K(x,y)| \leq \frac{C}{|x-y|^{n+|\beta|}}, \quad \forall x \neq y, \tag{21}
\]
for all multi-index \( 1 \leq |\beta| \leq N \).

**Proof.** The proof is based on the ideas of the proof of Lemma 5.5 in [2] p. 96.

We will prove first the absolute convergence of the series showing also that
\[
|K(x,y)| \leq \frac{C}{|x-y|^n} \quad \forall x \neq y.
\]

Let \( x \neq y \), we take \( \ell \in \mathbb{Z} \) such that \( 2^\ell \leq |x-y| \leq 2^{\ell+1} \).

If \( j \geq \ell \):
\[
\sum_{j \geq \ell} \sum_{k \in \mathbb{Z}^n} |\phi_{j,k}(x) \psi_{j,k}(y)| = \sum_{j \geq \ell} \sum_{k \in \mathbb{Z}^n} 2^{-nj} |\phi(2^{-j}x - k) \psi(2^{-j}y - k)|
\]
\[
\leq C \sum_{j \geq \ell} \left( \frac{1}{2^n} \right)^j \left( \sum_{k \in \mathbb{Z}^n} (1 + |2^{-j}x - k|)^{-n-\epsilon} \right)
\]
\[
\leq C \sum_{j \geq \ell} \left( \frac{1}{2^n} \right)^j C_2^n \frac{2^n}{2^n - 1} 2^{2\ell} = C \frac{2^n}{2^n - 1} \frac{2^n}{2^n - 1} \frac{2^n}{2^n - 1} 2^{2\ell+1}
\]
\[
\leq \left( C \frac{4^n}{2^n - 1} \right) \frac{1}{|x-y|^n} = \frac{C}{|x-y|^n},
\]
where the first inequality follows from (17) and (18) with \( \alpha = 0 \) and \( \beta = 0 \), the second one is a consequence of Lemma [2], and the rest is followed from the fact of having a geometric series and from the choice of such integer \( \ell \).
If \( j < \ell \); From (17), (18) and since \((1 + |z|)(1 + |z'|) \geq 1 + |z - z'|\) holds for any \(z, z' \in \mathbb{R}^n\) we have
\[
|\phi(2^{-j}x - k)\psi(2^{-j}y - k)| \leq C \frac{1}{(1 + |2^{-j}x - k|)^{2n+2\ell}} \frac{1}{(1 + |2^{-j}y - k|)^{n+\ell}} \leq C \frac{1}{(1 + |2^{-j}x - k|)^{n+\ell}} \frac{1}{(1 + |2^{-j}x - y|)^{n+\ell}}.
\]

By Lemma 2 results
\[
\sum_{j < \ell} \sum_{k \in \mathbb{Z}^n} |\phi_{\ell,k}(x)\psi_{\ell,k}(y)| \leq C \sum_{j < \ell} 2^{-n\ell} \frac{1}{(1 + 2^{-j}|x - y|)^{n+\ell}} \leq C \sum_{j < \ell} 2^{-n\ell} \frac{1}{2^{-j(n-\ell)}} = C \frac{2^{n-\ell}}{2^{(\ell+1)}} \sum_{j < \ell} (2^\ell)^j = C \frac{1}{2^{n(\ell+1)}} \leq C \frac{1}{|x - y|^n}.
\]
So, we obtain the inequality in (19). Now we will prove that there exists \( \partial_{x_1} K(x, y) \) for each \( x \neq y \). Let \( x_0, y_0 \in \mathbb{R}^n \) be fixed such that \( x_0 \neq y_0 \). So, \( |x - y_0| \geq \frac{|x_0 - y_0|}{2} \)
for all \( x \in B_{\frac{|x_0 - y_0|}{2}}(x_0) \). We take the largest \( \ell \in \mathbb{Z} \) such that \( 2^\ell \leq |x_0 - y_0| \). For \( M, N \in \mathbb{N} \), with \( |\ell| \leq M \), we consider
\[
S_{M,N}(x, y_0) = \sum_{|j| \leq M} \sum_{|k| \leq N} 2^{-jn} \phi(2^{-j}x - k)\psi(2^{-j}y_0 - k),
\]
to take the partial derivative \( \partial_{x_1} \) in this expression we obtain
\[
(\partial_{x_1} S_{M,N})(x, y_0) = \sum_{|j| \leq M} \sum_{|k| \leq N} 2^{-jn} \phi(2^{-j}x - k)\psi(2^{-j}y_0 - k).
\]
Since \( S_{M,N}(x, y_0) \to K(x, y) \) for each \( x \neq y_0 \), it is enough to show that the partial sum \((\partial_{x_1} S_{M,N})(x, y_0)\) converges uniformly (in the variable \( x \)) on \( B_{\frac{|x_0 - y_0|}{2}}(x_0) \). From this, it will follow that there exists \( \partial_{x_1} K(x_0, y_0) \). Being \( x_0 \neq y_0 \) arbitrary points of \( \mathbb{R}^n \), we will have guaranteed the existence of \( \partial_{x_1} K(x, y) \) for all \( x \neq y \).

To prove the uniform convergence of \((\partial_{x_1} S_{M,N})(x, y_0)\) on \( B_{\frac{|x_0 - y_0|}{2}}(x_0) \) we split the partial sum as follows
\[
(\partial_{x_1} S_{M,N})(x, y_0) = \sum_{|j| \leq M} \sum_{|k| \leq N} 2^{-jn} \phi(2^{-j}x - k)\psi(2^{-j}y_0 - k) = \left( \sum_{-M \leq j \leq \ell} \sum_{\ell < j \leq M} 2^{-jn} \phi(2^{-j}(x - k))\psi(2^{-j}y_0 - k) \right) + \left( \sum_{-M \leq j \leq \ell} \sum_{\ell < j \leq M} 2^{-jn} \phi(2^{-j}(x - k))\psi(2^{-j}y_0 - k) \right)
\]
\[
\leq \sum_{-M \leq j \leq \ell} \sum_{\ell < j \leq M} 2^{-jn} \phi(2^{-j}(x - k))\psi(2^{-j}y_0 - k) \leq \sum_{-M \leq j \leq \ell} \sum_{\ell < j \leq M} 2^{-jn} \phi(2^{-j}(x - k))\psi(2^{-j}y_0 - k)
\]
\[
\leq \sum_{-M \leq j \leq \ell} \sum_{\ell < j \leq M} 2^{-jn} \phi(2^{-j}(x - k))\psi(2^{-j}y_0 - k) \leq \sum_{-M \leq j \leq \ell} \sum_{\ell < j \leq M} 2^{-jn} \phi(2^{-j}(x - k))\psi(2^{-j}y_0 - k)
\]
for all \( x \in B_{\frac{|x_0 - y_0|}{2}}(x_0) \). Since the last numerical series converges, by the Weierstrass's test, we obtain the uniform convergence of \((\partial_{x_1} S_{M,N})(x, y_0)\) on \( B_{\frac{|x_0 - y_0|}{2}}(x_0) \).

The same argument can be used to prove the existence of \( \partial_{x_1} K(x, y) \) and of \( \partial_{y_1} K(x, y) \), for each \( x \neq y \) and each multi-index \( 1 \leq |\alpha| \leq M \) and \( 1 \leq |\beta| \leq N \).
Finally, the same technique used to obtain (19) allows us to get (20) and (21), since
\[
|\partial_\alpha^\beta K(x, y)| \leq \sum_{j,k} 2^{-j(n+|\alpha|)}|\partial_\alpha \phi(2^{-j}x - k)\phi(2^{-j}y - k)|,
\]
and
\[
|\partial_\alpha^\beta K(x, y)| \leq \sum_{j,k} 2^{-j(n+|\beta|)}|\phi(2^{-j}x - k)\partial_\beta \psi(2^{-j}y - k)|.
\]
Then (20) and (21) follow and the proof of the lemma is therefore concluded. □

**Lemma 11.** Let $\epsilon > 0$. Suppose that $g$ and $\psi$ satisfy
\[
|g(x)| + |\psi(x)| \leq \frac{C_1}{(1 + |x|)^{n+\epsilon}} \quad \text{for all } x \in \mathbb{R}^n,
\]
with $C_1$ independent of $x \in \mathbb{R}^n$. Then, there exists a positive constant $C$ such that for all $j, k, l, m \in \mathbb{Z}$ and $l \leq j$ we have that
\[
|\psi_{j,k}(x)| \leq \frac{C_{2}2^{-\frac{j}{2}}}{(1 + 2^{-j}|x - 2^j k - 2^l m|)^{n+\epsilon}} \quad \text{for all } x \in \mathbb{R}^n.
\]

*Proof.* The proof of this lemma can be followed from [6, Lemma 3.12 Ch. 6]. □

**Corollary 12.** Let $\epsilon > 0$. If $g$ and $\psi$ satisfy
\[
|g(x)| + |\partial_\beta \psi(x)| \leq \frac{C_1}{(1 + |x|)^{n+\epsilon}} \quad \text{for all } x \in \mathbb{R}^n,
\]
and all multi-index $0 \leq |\beta| \leq N$, with $C_1$ independent of $x \in \mathbb{R}^n$, then there exists a positive constant $C$ such that for all multi-index $0 \leq |\beta| \leq N$.
\[
|\partial_\beta (g \ast \psi)(x)| \leq \frac{C}{(1 + |x|)^{n+\epsilon}} \quad \text{for all } x \in \mathbb{R}^n.
\]

*Proof.* Since $g \ast \psi = g_{0,0} \ast \psi_{0,0}$ and $\partial_\beta (g \ast \psi) = g \ast \partial_\beta \psi$, the corollary follows from (23) and Lemma 11. □

**Proposition 13.** Let $g$ be a function and let $\phi$ and $\psi$ continuous functions defined on $\mathbb{R}^n$ with continuous partial derivatives up to order $M \geq 1$ and order $N \geq 1$ respectively such that
\[
|\partial_\alpha^\beta \phi(x)| \leq \frac{C}{(1 + |x|)^{2n+2M+2\epsilon}} \quad \text{for all } x \in \mathbb{R}^n,
\]
and all multi-index $0 \leq |\alpha| \leq M$, and
\[
|g(x)| + |\partial_\beta \psi(x)| \leq \frac{C}{(1 + |x|)^{2n+2N+2\epsilon}} \quad \text{for all } x \in \mathbb{R}^n,
\]
and all multi-index $0 \leq |\beta| \leq N$. If $K(x, y) = \sum_{j,k} \phi_{j,k}(x)(g \ast \psi)_{j,k}(y)$, then there exists a positive constant $C$ such that
\[
|K(x, y)| \leq \frac{C}{|x - y|^{n}}, \quad \forall x \neq y,
\]
\[
|\partial_\alpha^\beta K(x, y)| \leq \frac{C}{|x - y|^{n+|\alpha|}}, \quad \forall x \neq y,
\]
for all multi-index $1 \leq |\alpha| \leq M$, and
\[
|\partial_\alpha^\beta K(x, y)| \leq \frac{C}{|x - y|^{n+|\beta|}}, \quad \forall x \neq y,
\]
for all multi-index $1 \leq |\beta| \leq N$. 

Proof. From (24), (25) and the corollary [12] it follows that the pair of functions $\phi$ and $g * \psi$ satisfy the hypotheses of Lemma [10] then the proposition follows. \hfill $\square$

Lemma 14. Let $\psi$ be a function on $\mathbb{R}^n$ such that
\begin{equation}
|\partial^\beta \psi(y)| \leq \frac{C}{(1 + |y|)^{n+|\beta|+\epsilon}},
\end{equation}
for all $y \in \mathbb{R}^n$ and all multi-index $0 \leq |\beta| \leq N$, and let $\{c_i\} \subset \mathbb{R}$ and $\{x_i\} \subset \mathbb{R}^n$ such that
\begin{equation}
\sum_i |c_i|(1 + |x_i|)^{n+N+\epsilon} < +\infty.
\end{equation}
Then there exists a positive constant $C$ such that
\begin{equation}
|\partial^\beta \left( \sum_i c_i \tau_{x_i} \psi \right)(y) | \leq \frac{C}{(1 + |y|)^{n+|\beta|+\epsilon}},
\end{equation}
for all $y \in \mathbb{R}^n$ and all multi-index $0 \leq |\beta| \leq N$.

Proof. From (20) and since $(1 + |z|)(1 + |z'|) \geq 1 + |z - z'|$ holds for any $z, z' \in \mathbb{R}^n$, we have
\begin{equation}
|\partial^\beta (\tau_{-x} \psi)(y) | = |\partial^\beta \psi(y + x)| \leq C \frac{(1 + |y + x|)^{n+N+\epsilon}}{(1 + |y|)^{n+N+\epsilon}} \leq C(1 + |x_i|)^{n+N+\epsilon},
\end{equation}
from this inequality and (27) we get
\begin{equation}
|\partial^\beta \left( \sum_i c_i \tau_{-x_i} \psi \right)(y) | \leq C \sum_i \frac{|c_i|(1 + |x_i|)^{n+N+\epsilon}}{(1 + |y|)^{n+N+\epsilon}} \leq \frac{C}{(1 + |y|)^{n+|\beta|+\epsilon}},
\end{equation}
for all $y \in \mathbb{R}^n$ and all $0 \leq |\beta| \leq N$. \hfill $\square$

Remark 15. If $\{c_i\} \in \ell^1(\mathbb{N})$ and the sequence $\{x_i\}$ is included in a compact subset of $\mathbb{R}^n$, then
\begin{equation}
\sum_i |c_i|(1 + |x_i|)^{\gamma} < +\infty, \quad \forall \gamma > 0.
\end{equation}

The lemma [14] and the lemma [10] allow us to obtain the following proposition.

Proposition 16. If $\phi$ and $\psi$ are functions on $\mathbb{R}^n$ which satisfy (17) and (18) and
\begin{equation}
K(x, y) = \sum_{j,k} \phi_j(x) \left( \sum_i c_i \tau_{x_i} \psi \right)_{j,k} (y),
\end{equation}
where $\sum_i |c_i|(1 + |x_i|)^{2n+2N+2\epsilon} < +\infty$, then (19), (20) and (21) hold.

For the next result we need to introduce a bit of notation. Let $1 \leq m < n$. Each multi-index $\beta = (\beta_1, ..., \beta_n)$ can be written as $\beta = (\beta', \beta'')$ where $\beta'$ and $\beta''$ are the multi-indices $\beta' = (\beta_1, ..., \beta_m)$ and $\beta'' = (\beta_{m+1}, ..., \beta_n)$, so $|\beta| = |\beta'| + |\beta''|$. Analogously, given $k \in \mathbb{Z}^n$ we can write $k = (k', k'')$ with $k' \in \mathbb{Z}^m$ and $k'' \in \mathbb{Z}^{n-m}$.

Proposition 17. Let $\phi$ be a continuous function defined on $\mathbb{R}^n$ with continuous partial derivatives up to order $M \geq 1$ such that
\begin{equation}
|\partial^\alpha \phi(x)| \leq \frac{C}{(1 + |x|)^{2n+2M+2\epsilon}} \quad \text{for all } x \in \mathbb{R}^n,
\end{equation}
and all multi-index $0 \leq |\alpha| \leq M$, let $h$ and $\psi$ functions defined on $\mathbb{R}^m$ and let $\tilde{\psi}$ be a function defined on $\mathbb{R}^{n-m}$ with continuous partial derivatives up to order $N \geq 1$ such that
\begin{equation}
|h(y')| + |\partial^\beta \psi(y')| \leq \frac{C}{(1 + |y'|)^{2n+2N+2\epsilon}}, \quad (y' \in \mathbb{R}^m)
\end{equation}
and

\[ |\partial^{\beta''} \tilde{\psi}(y'')| \leq \frac{C}{(1 + |y''|)^{2n+2N+2\varepsilon}}, \quad (y'' \in \mathbb{R}^{n-m}) \]

for all multi-index \( \beta'' \) and all \((n-m)\)-tuples \( \beta'' \) such that \( 0 \leq |\beta'| + |\beta''| \leq N \).

If

\[ K(x, y) = \sum_{j,k} \phi_{j,k}(x)(h \ast \psi)_{j,k}(y') \tilde{\psi}_{j,k}(y''), \]

where \( y = (y',y'') \in \mathbb{R}^m \times \mathbb{R}^{n-m} \) and \( k = (k',k'') \in \mathbb{Z}^m \times \mathbb{Z}^{n-m} \), then there exists a positive constant \( C \) such that

\[ |K(x, y)|, \quad \forall x \neq y, \]

\[ |\partial^\alpha K(x, y)| \leq \frac{C}{|x - y|^{n+|\alpha|}}, \quad \forall x \neq y, \]

for all multi-index \( 1 \leq |\alpha| \leq M \), and

\[ |\partial^\beta K(x, y)| \leq \frac{C}{|x - y|^{n+|\beta|}}, \quad \forall x \neq y, \]

for all multi-index \( 1 \leq |\beta| \leq N \).

**Proof.** From (29) and Corollary (12) we have that

\[ |\partial^{\beta'} (h \ast \psi)(y')| \leq \frac{C}{(1 + |y'|)^{2n+2N+2\varepsilon}}, \]

for all \( y' \in \mathbb{R}^m \) and all multi-index \( 0 \leq |\beta'| \leq N \).

Given a multi-index \( n \)-dimensional \( \beta \in \mathbb{N}^n_0 \) we write \( \beta = (\beta', \beta'') \in \mathbb{N}^m_0 \times \mathbb{N}^{n-m}_0 \). We put \( \Psi(y) = (h \ast \psi)(y') \tilde{\psi}(y'') \) where \( y = (y',y'') \in \mathbb{R}^m \times \mathbb{R}^{n-m}, \) then (30) and (31) allow us to obtain

\[ |\partial^{\beta} \Psi(y)| = |\partial^{\beta'} (h \ast \psi)(y') \partial^{\beta''} \tilde{\psi}(y'')| \leq \frac{C}{(1 + |y'|)^{2n+2N+2\varepsilon}(1 + |y''|)^{2n+2N+2\varepsilon}} \]

\[ = \frac{C}{(1 + |y'|)^{2n+2N+2\varepsilon}(1 + |(0,y'')|)^{2n+2N+2\varepsilon}} \leq \frac{C}{|y'|^{2n+2N+2\varepsilon}}, \]

for all \( y \in \mathbb{R}^n \) and all multi-index \( 0 \leq |\beta| \leq N \). Since \( K(x, y) = \sum_{j,k} \phi_{j,k}(x) \Psi_{j,k}(y) \), the proposition follows from (28), (32) and Lemma (10). \( \square \)

5. Main Results

Theorem 5 guaranteed the existence of \( q := 2^n - 1 \) functions \( \phi^l \) and \( 2^n - 1 \) functions \( \psi^l \) which satisfy the decay conditions (17) and (18) of Lemma (10). Corollary 6 assert that the functions \( \phi^l_{j,k}(x) := 2^{-n/2} \phi^l(2^{-j}x - k) \) and \( \psi^l_{j,k}(x) := 2^{-n/2} \psi^l(2^{-j}x - k), 1 \leq l \leq q, k \in \mathbb{Z}^n, j \in \mathbb{Z} \), are two orthonormal bases of \( L^2(\mathbb{R}^n) \).

Our results are the following.

**Theorem 18.** If \( \mu \) is the measure defined by (2) with \( h = 0 \), then the operator \( T_\mu \), given by (1), where the functions \( \phi^l \) and \( \psi^l \) satisfy (17) with \( M = 1 \) and (18) with \( N \geq 1 \) for each \( l = 1, ..., 2^n - 1 \), can be extended to a bounded operator from \( H^p(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \) for \( \frac{1}{N+\frac{n}{m}} < p < \infty \).
Proof. For the sake of simplicity we are going to present the calculations only when \( l = 1 \). Then, we write
\[
T_\mu f(x) = \sum_{(j,k) \in \Lambda} \omega_{j,k} \langle \mu_j \ast f, \psi_{j,k} \rangle \phi_{j,k}(x).
\]
A computation gives that
\[
\langle \mu_j \ast f, \psi_{j,k} \rangle = \langle f, (\overline{\mu_j} \ast \overline{\psi_{j,k}}) \rangle,
\]
here \( \tilde{f}(x) = f(-x) \) and \( (\cdot) \) denotes complex conjugation. To apply the Fourier transform on the expression \( \overline{\mu_j} \ast \overline{\psi_{j,k}} \) we obtain
\[
(\overline{\mu_j} \ast \overline{\psi_{j,k}}) = (\overline{\mu} \ast \overline{\psi})_{j,k}.
\]
Replacing \( \mu \) by its equal we have that
\[
(\overline{\mu} \ast \overline{\psi})_{j,k} = (\overline{\mu} \ast \overline{\psi})_{j,k} + \left( \sum_i \overline{c_i} \tau_{-x_i} \psi \right)_{j,k}.
\]
Then the operator \( T_\mu \) can be expressed, for suitable functions \( f \), as
\[
T_\mu f(x) = \int K(x,y)f(y)dy, \quad x \notin \text{supp}(f),
\]
where
\[
K(x,y) = \sum_{j,k} \omega_{j,k} \phi(x) \overline{\psi}_{j,k}(y) + \sum_{j,k} \omega_{j,k} \overline{\phi}_{j,k}(x) \left( \sum_i \overline{c_i} \tau_{-x_i} \psi \right)_{j,k}.
\]
Being the kernels \( K_1 \) and \( K_2 \) of the type of Proposition 13 and Proposition 16 respectively, then using Theorem 9 we obtain the statement. \( \square \)

If \( n \geq 2 \) and \( \mu \) is the measure defined by (2), where \( h : \mathbb{R}^m \rightarrow \mathbb{C} \) is a function such that \((1 + |y'|)^{2n+2N+2r} |h(y')| \leq C < +\infty \) for all \( y' \in \mathbb{R}^m \), \( 1 \leq m < n \), we consider the operator \( T_\mu \) given by
\[
(33) \quad T_\mu f(x) = \sum_{(l,j,k) \in \Lambda} \omega_{l,j,k} \langle \mu_j \ast f, \Psi_{j,k} \rangle \phi_{j,k}(x),
\]
where the \( \phi^l \)'s satisfy (17) with \( M = 1 \) and, for each \( l = 1, \ldots, 2^n - 1 \), \( \Psi^l(y) = \psi^l(y) \psi^l(y') \), \( y = (y', y'') \in \mathbb{R}^m \times \mathbb{R}^{n-m} \), where \( \psi^l \) and \( \psi^l \) are functions defined on \( \mathbb{R}^m \) and \( \mathbb{R}^{n-m} \), which satisfy the decay conditions (20) and (31) of Proposition 19 with \( N \geq 1 \), and \( \{ \psi^l_{j,k} \}, \{ \phi^l_{j,k} \} \) are orthonormal bases of \( L^2(\mathbb{R}^m) \) and \( L^2(\mathbb{R}^{n-m}) \) respectively. With respect to this operator we have the following result.

**Theorem 19.** If \( T_\mu \) is the operator defined by (33), then \( T_\mu \) can be extended to a bounded operator from \( H^p(\mathbb{R}^n) \) into \( L^p(\mathbb{R}^n) \) for \( \frac{n}{n+m} < p < \infty \).

**Proof.** Since \( T_\mu = T_{\mu-\lambda} + T_\lambda \), where \( \lambda \) is the singular measure given by (3), we can apply Theorem 18 to the operator \( T_{\mu-\lambda} \). So, it is sufficient to prove the statement for the operator \( T_\lambda \). Without loss of generality, we can assume that \( l = 1 \) in (33). Thus
\[
T_\lambda f(x) = \sum_{j,k} \omega_{j,k} \langle \lambda_j \ast f, \Psi_{j,k} \rangle \phi_{j,k}(x).
\]
A computation similar to done in the proof of the theorem 18 allows us to write
\[
T_\lambda f(x) = \int K(x,y)f(y)dy, \quad x \notin \text{supp}(f),
\]
where
\[ K(x, y) = \sum_{j,k} w_{j,k} \phi_{j,k}(x)(\overline{\hat{h} * \psi}_j)_k \psi_{j,k}(y), \quad x \neq y, \]
y = (y', y'') \in \mathbb{R}^m \times \mathbb{R}^{n-m}, and \( k = (k', k'') \in \mathbb{Z}^m \times \mathbb{Z}^{n-m}. \) Since this kernel is of the type of Proposition \ref{prop:kernel-type}, the assertion of the theorem follows from this proposition and Theorem \ref{thm:kernel-bound}.

\[ \square \]

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