Piecewise interpolation solution of the Cauchy problem for the transport equation with iterative refinement

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Abstract. This article deals with the method of approximate solution of the Cauchy problem for the transport equation based on the Newton interpolation polynomial of two variables with iterative refinement. In each element of the division of a rectangular domain into subdomains, a polynomial approximation of the partial time derivative is constructed. The interpolation polynomial is transformed into the form of an algebraic polynomial with numerical coefficients. In the integrated form, the polynomial is substituted for the dependent variable in the right side of the expression of the partial time derivative. Iterative renewal of the process at a fixed polynomial degree leads to the analogue of Picard's method of successive approximations.

1. Introduction and formulation of the question
For numerical and computer simulation of physical, chemical and technological processes, the reduction of the error of the approximate solution of partial differential equations is an actual problem. In particular, this can be attributed to the models of electroconvection, the processes of transfer of ternary electrolyte in electrochemical systems, the processes of pressure-pulse propagation and blood velocity in the blood vessel, to the models of wave propagation and tsunami. For the transport equation, this problem is discussed in [1], where the value of the transport equation for the numerical solution of non-stationary problems in continuum mechanics and gas dynamics problems is covered. The error of the existing methods in a rectangular domain of a small size, in the smoothness conditions of the solution, as a rule, are within the limits $10^{-11} - 10^{-8}$ [2, 3]. In order to reduce the error, a piecewise interpolation solution of the Cauchy problem for the transport equation with iterative refinement is proposed. The prototype is the method presented in [4] for the case of ordinary differential equations. In the case of partial differential equations, the method is constructed on the basis of the Newton interpolation polynomial of two variables, which is converted to the form of an algebraic polynomial with numerical coefficients, which makes it possible to construct successive approximations to clarify the solution. In the case of the model one-dimensional Cauchy problem for the linear transport equation, the uniform convergence of the method is proved and the rate of convergence is estimated. In [5] the program is given and the numerical experiment is described, where the considered problem in a rectangle of unit height is solved with an error $10^{-19} - 10^{-18}$, the article contains the main results of the experiment.
2. Piecewise interpolation calculation of functions of two variables

In Cartesian coordinate system \( UXT \) piecewise interpolation approximation of the real function \( u = u(x, t) \) of two real variables in a rectangular domain

\[
G = \{ (x, t) \mid x \in [a, b], t \in [c, d] \}
\]

is constructed as follows. Domain (1) is divided into rectangular subdomains \( G_{ij} \) with intersecting boundaries:

\[
G = \bigcup_{j=0}^{p-1} \bigcup_{i=0}^{p-1} G_{ij},
\]

\[
G_{ij} = \{ (x, t) \mid x \in [x_i, x_{i+1}], t \in [t_j, t_{j+1}] \}, P_x = 2^k, P_t = 2^{k_t}, k, k_t \in \{ 0, 1, \ldots \}.
\]

The Newton interpolation polynomial of two variables in the subdomain (3) uses \((n+1)(n+2)/2\) nodes with a triangular arrangement. In this case, the function is interpolated in the lower triangular part \( G_{ij} \), the upper part actually needs extrapolation, which is taken into account in the future. Let the boundary \( \varepsilon \) of the absolute error of the function approximation \( u(x, t) \) be arbitrarily given. In \( G_{ij} \) the Newton interpolation polynomial \( \Psi_{n}^{ij}(z, w) \) is constructed with equidistant per steps \( h_x, h_t \), in the directions of the coordinates of the nodes \( x_i, t_m \) where

\[
h_x = (x_{i+1} - x_i)/n, \ h_t = (t_{j+1} - t_j)/n, \ z = (x-x_i)/h_x, \ w = (t-t_j)/h_t, \ (x, t) \in G_{ij},
\]

\[
x_{i+m} = x_i + m h_x, \ t_{j+m} = t_j + m h_t, \ \ell = 0, n, \ m = 0, n - \ell.
\]

The sought polynomial will take the form [6]

\[
\Psi_{n}^{ij}(z, w) = u(x_i, t_{j0}) + \sum_{\ell=0}^{n} \sum_{m=0}^{k} \frac{\Delta_{x_i}^{k} u(x_{i0}, t_{j0})}{s! (k-s)!} \prod_{j=0}^{s-1} (z-\ell) \prod_{m=0}^{k-s-1} (w-m),
\]

where \( \Delta_{x_i}^{k} u(x_{i0}, t_{j0}) \) – finite differences of \( k \) order. The degree of the polynomial \( n \) is chosen equal for all \( G_{ij} \) and minimal provided that \( |u(x, t) - \Psi_{n}^{ij}(z, w)| \leq \varepsilon \ \forall (x, t) \in G_{ij}, \ i = 0, P_x - 1, j = 0, P_t - 1. \)

It is assumed that \( n \leq N_0, \ N_0 = \text{const} \), but the values \( k_x \) and \( k_t \) are not abstractly limited. The polynomial (5) with the transformation of Vieta’s formulae [7] is equivalently converted to the algebraic polynomial form with numeric coefficients

\[
\Psi_{n}^{ij}(z, w) = \sum_{\ell=0}^{n} \sum_{m=0}^{n-\ell} a_{i+m}^{\ell} z^\ell w^m.
\]

Due to the triangular arrangement of the nodes (4), the remainder term of the interpolation [8, 9] actually refers only to the lower triangular part, and not to the entire rectangular domain \( G \). To estimate the error in the entire domain \( G \) existence and continuity of all partial derivatives up to the order of \( 2n+1 \) inclusively is assumed. In these assumptions \( C_0 = \max_{i} \left| \frac{\partial^{2n+1} u(x, t)}{\partial x^{i} \partial t^{2n+1}} \right| \), \( C_0 = \text{const} \ \forall i, 0 \leq i \leq 2n + 1, \ \forall n \leq N_0 \). It is assumed that the dimensions \( G \) allow us to consider the distances between the nodes to be less than one for all the \( n \) considered. Nodes are equidistant along the axis directions, then there exist \( h, q, p \) such that
\[ h_x = qh, \quad q = \text{const}, \quad h_t = ph, \quad p = \text{const}, \quad h < 1, \quad q < 1, \quad p < q. \quad (7) \]

It is evident that \( \sum_{i=0}^{2^n-1} q^{2^n} \leq c_0, \quad c_0 = \text{const}. \) In the notation \( C = C_0c_0 \) introduces the estimate

\[ |R_G(x,t)| \leq C h^{2n+1}, \quad C = \text{const}. \quad (8) \]

Now let according to the same scheme in each subdomain \( G_{ij} \) the interpolation polynomial (5) with the replacement of the degree \( n \) by \( 2n \) is constructed. If we choose \( 2^k = 2^k = 2^k \), then \( h \) in (8) is inversely proportional to \( 2^k \), as a result we get:

\[ |R_{G_{ij}}(x,t)| \leq C 2^{-k(2n+1)} h^{2n+1} \forall (x,t) \in G_{ij} \forall i, j; \quad i = \text{const}, \quad j = \text{const}. \quad (9) \]

In (9) \( C, h \) are from (8), \( h < 1 \). \( i, j \) are from (2). Thus, we have

**Lemma 1.** Let the function \( u(x,t) \) be defined in \( G \) from (1), where all its partial derivatives up to and including continuous up to the order of \( 2n+1 \) exist. Then, on condition that \( G \) is divided into \( 2^{2k} \) subdomains (2), (3) in case \( 2^k = 2^k = 2^k \), piecewise interpolation approximation of this function in \( G \) by polynomials of the form (5) of degree \( 2n \), can be performed with an absolute error (9), where the interpolation steps \( h_s, h_t \) from (4) are related to \( h < 1 \) from (8) by relations (7).

This approximation under study is invariant with respect to \( i, j \) from (2), (3), therefore in the left part of (9) there can be taken the maximum \( \max_{\forall (x,t) \in G_{ij}, \forall i, j = 0,2^{k-1}} |R_{G_{ij}}(x,t)| \) for all \( i, j : \)

\[ \max_{\forall (x,t) \in G_{ij}, \forall i, j = 0,2^{k-1}} |R_{G_{ij}}(x,t)| \leq C 2^{-k(2n+1)} h^{2n+1}, \quad C = \text{const}. \quad (10) \]

Hence the following theorem:

**Theorem 1.** Under Lemma 1, the piecewise interpolation approximation converges uniformly to the function \( u(x,t) \) in the domain \( G \) when \( k \to \infty \) at the rate of convergence (10).

In each subdomain, the polynomial (5) can be transformed to form (6) with the degree \( 2n \) without changing the estimates (9), (10). Further the relations are used

\[ u(x,t) \approx \sum_{i=0}^{2^n} \sum_{m=0}^{2n-\ell} a_{i,m} z^\ell w^m, \quad z = (x-x_{10})h^{-1}, \quad w = (t-t_{10})h^{-1}, \quad (11) \]

\[ u_s \approx h_s^{-1} \sum_{i=1}^{2^n} \sum_{m=0}^{2n-\ell} a_{i,m} z^\ell w^m, \quad f(x,t) \approx \overline{c} + h \sum_{i=0}^{2^n} \sum_{m=0}^{2n-\ell} a_{i,m} z^\ell w^{m+1}/(m+1), \quad \overline{c} = \text{const}. \quad (12) \]

Constants and parameters are determined in the course of presentation, \( h_s, h_t, \) in (11), (12) are proportional to \( h \) according to (7).

### 3. Piecewise interpolation solution of the Cauchy problem for the transport equation

#### 3.1. Initial assumptions

At first, the Cauchy problem for the linear equation is considered

\[ u_t + a(x,t)u_x = f(x,t), \quad u(x,0) = \varphi(x), \quad (13) \]

where \( a(x,t), f(x,t) \) are the given functions considered in the half-plane \( \{(x,t) \mid x \in R, t \geq 0\} \), \( \varphi(x) \) is the given function \( x \in R \). To construct an approximate solution method, a rectangular domain
for specifying these functions and one of its boundaries, defined directly below, are selected. Due to the use of piecewise interpolation with an estimate in (10), the following constraints are used for the convergence analysis.

I. Approximate solution of the problem (13) is constructed in the domain $G$ from (1), uniting the $G_{i,j}$ subdomains from (2), (3), when $2^{k_i} = 2^{k_j} = 2^k$, the area of $\varphi(x)$ definition is the base $G$ on the axis $OX$. The values $a$ and $b$ are not specified, however, everywhere below $c = 0, \; d = T, \; t \in [0, T]$. It is assumed that $u(x,t)$ belongs to the domain $\{ |u(x,t) - u(x,0)| \leq \tilde{b}; (x,t) \in G \}$, where $\tilde{b}$ – constant, the value of which can be set arbitrarily, if necessary, it is specified.

II. It is assumed that in the domain $G_{\Delta} = \{ (x,t) \mid x \in [a-\Delta, b+\Delta], t \in [0, T] \}$, when $\forall \; \Delta: \; 0 < \Delta < 2^{-1}(b-a), \; \Delta = \text{const}, \; \text{all partial derivatives} \; u(x,t) \; \text{exist and are continuous up to an order of} \; 2n+1 \; \text{inclusively} \; \forall n \leq N_0, N_0 = \text{const}, \; \text{with the same order are} \; a(x,t) \; \text{and} \; f(x,t)$ continuously differentiable, the function $\varphi(x)$ is defined and continuously differentiable by the factor of $2n+1$ on the segment $x \in [a-\Delta, b+\Delta]$.

III. It is assumed that in the subdomain $G_{\Delta(i,j)} = \{ (x,t) \mid x \in [x_i - \Delta, x_{i+1} + \Delta], t \in [t_j, t_{j+1}] \}$ the value $\Delta$ can be arbitrary within the boundaries $0 < \Delta < 2^{-k-i}(b-a), \; \Delta = \text{const}$.

The existence and uniqueness, as well as the stability of the problem solution (13) with respect to the perturbation of the initial data [1] are determined in assumptions I, II.

In $G_{i,j}$ the approximate solution is constructed by using an interpolation polynomial with iterative refinement, the transition from $G_{i,j}$ to $G_{i(j+1)}$ is performed sequentially according to $j$. For the initial conditions in $G_{i(j+1)}$, the approximation from $G_{i,j}$ on the adjacent with $G_{i(j+1)}$ boundary is taken. In $G_{00}$ the nodal values of the interpolation the function $\varphi(x)$ sets. Specifically interpolated is $f(x,t) - a(x,t) u_x$, which gives an approximation $u$. The time integral of the interpolation polynomial is taken as an approximation of the solution, which is substituted in the expression $u$, from (13). The resulting approximation $u$ is again interpolated, and the described process is repeated with the usage of (11), (12). In practice, iterations are performed up to the required approximation accuracy, abstractly their number is assumed to be unlimited. The process is reproduced in each subdomain until the complete passage of the $G$ domain.

3.2. Iterative refinement with regard for remainder terms of interpolation

In $G_{i,j}$

$$\int_{t_0}^{t_1} u \, dt = \int_{t_0}^{t_1} (f(x,t) - a(x,t) u_x) \, dt, \; (x,t) \in G_{i,j}, \; t_0 = t_0(i,j), \; u(x,t_0) = \varphi_j(x,t_0). \quad (14)$$

The notations are adjusted to distinguish them from the ones used in the interpolation function description. An index $2k$ according to the number of subdomains $2^{2k}$ is added to the lower indices of the polynomial $\Psi_{2n}^{ij}(z,w)$ from (11), a similar indexing is applied to other expressions. The polynomial $\Psi_{2k,2n}^{ij}(z,w)$ interpolates not $u(x,t)$, but the integrand function of the right part (14), which is marked by the unified index $\partial u$, the remainder term of the interpolation $c_{2k,2n}$ is estimated from (10),

$$\Psi_{\partial u}^{ij}_{2k,2n}(z,w) = f(x,t) - a(x,t) \, \partial u + c_{2k,2n}, \; |c_{2k,2n}| \leq C \, 2^{-k-i} \, h^{2n+1}, \quad (15)$$
where \( z, w \) are from (11), \( \tilde{u}(x, t) = u(x, t) \) is the approximation of the solution given in the preceding iteration. Similarly in (11), \( \Psi_{\partial u 2k2n}^{ij}(z, w) \) is converted to form (6). The solution at the current iteration is approximated by the integral of \( \Psi_{\partial u 2k2n}^{ij}(z, w) \), defined similarly in (12). The resulting polynomial is denoted \( \Gamma P_{u 2k2n}(x, t) \) – by degree \( 2n \), by the number of subdomains \( 2^k \), and approximation \( u(x, t) \) by integration of polynomial \( \Psi_{\partial u 2k2n}^{ij}(z, w) \) over \( t \). The iterative process will take the form

\[
\Psi_{\partial u 2k2n}^{ij}(z, w) = f(x, t) - a(x, t) \frac{\partial \Gamma P_{u 2k2n(r-1)}(x, t)}{\partial x} + c_{2k2n},
\]

where

\[
\Gamma P_{u 2k2n r}(x, t) = \int_{t_0}^{t} \Psi_{\partial u 2k2n(r-1)}^{ij}(z, w) dt + \Gamma c_{u 2k2n}, \quad \Gamma c_{u 2k2n} = \int_{t_0}^{t} c_{2k2n} dt - \text{the remainder term from the } u(x, t) \text{ approximation by polynomial}
\]

\[
\Gamma P_{u 2k2n r}(x, t) \text{ taken on a single iteration. With regard for (15) }
\]

\[
\int_{t_0}^{t} c_{2k2n} dt \leq C 2^{-k(2n+1)} h^{2n+1} (t - t_0),
\]

according to the assumption II \( \Gamma c_{u 2k2n} \leq C 2^{-k(2n+1)} h^{2n+1} \times 2^{-k} T \). Without an explicit expression

\[
\Psi_{\partial u 2k2n}^{ij}(z, w), \text{ the same process can be written as}
\]

\[
\Gamma P_{u 2k2n r}(x, t) = \Gamma P_{u 2k2n r}(x, t_0) + \int_{t_0}^{t} \left( f(x, t) - a(x, t) \frac{\partial \Gamma P_{u 2k2n(r-1)}(x, t)}{\partial x} \right) dt + \Gamma c_{u 2k2n}, \quad (x, t) \in G_{ij}, \quad t_0 = t_0(i, j),
\]

where \( r = 1, 2, … \), coefficients \( \frac{\partial \Gamma P_{u 2k2n(r-1)}(x, t)}{\partial x} \) are related to coefficients \( \Gamma P_{u 2k2n(r-1)}(x, t) \) according to (13), the remainder term is estimated from the relation

\[
\Gamma c_{u 2k2n} \leq \tilde{C} 2^{-k(2n+2)} h^{2n+1}, \quad \tilde{C} = CT, \quad \tilde{C} = \text{const} \forall (x, t) \in G_{ij}, \forall G_{ij} \subseteq G.
\]

Let the rectangle \( G_{ij} \) be arbitrarily fixed and the equation (14) is considered in it with the same initial conditions on the boundary with which the iterations are performed (16). In this case, the solution is not exact, which is marked by a line above, similarly marked approximating it polynomial, from (16) follows that

\[
\Gamma P_{u 2k2n r}(x, t) = \Gamma P_{u 2k2n r}(x, t_0) + \int_{t_0}^{t} \left( f(x, t) - a(x, t) \frac{\partial \Gamma P_{u 2k2n(r-1)}(x, t)}{\partial x} + c_{2k2n} \right) dt, \quad (x, t) \in G_{ij},
\]

where \( \Gamma c_{u 2k2n} = \int_{t_0}^{t} c_{2k2n}, \quad \Gamma u(x, t) \approx u(x, t), \quad \Gamma u(x, t_0) = \Gamma \varphi_{ij}(x, t_0), \quad \Gamma \varphi_{ij}(x, t_0) = \Gamma P_{u 2k2n r}(x, t_0), \quad r = 1, 2, … \)

is used. In assumptions I, II \( u(x, t) \) satisfies the relation

\[
\forall \varepsilon_0 > 0 \exists \Delta > 0, \Delta = \Delta(\varepsilon_0), \Delta = \text{const}:
\]
\[ \forall \Delta x, 0 < |\Delta x| \leq \Delta, \quad \left| \frac{\partial u(x,t)}{\partial x} - \frac{u(x+\Delta x,t) - u(x,t)}{\Delta} \right| \leq \varepsilon_0 \quad \forall (x,t) \in G, \quad (x+\Delta x,t) \in G_\Delta, \quad (19) \]

in particular, (19) is executed in the case \( \Delta x = \Delta \).

An auxiliary problem is considered for the study of convergence (18)

\[
\begin{align*}
\frac{\partial u_\Delta(x,t)}{\partial t} + a(x,t) \frac{u_\Delta(x+\Delta t,t) - u_\Delta(x,t)}{\Delta} &= f_\Delta(x,u_\Delta,t), \\
\end{align*}
\]

(20)

where \( f_\Delta(x,u_\Delta,t) = f(x,t) - a(x,t) \left( \frac{\partial u_\Delta(x,t)}{\partial x} - \frac{u_\Delta(x+\Delta t,t) - u_\Delta(x,t)}{\Delta} \right) \), \( f(x,t) \) from (13), \( \Delta \) is arbitrarily selected according to (19) and fixed. The problem (20) is an equivalent transformation (13), in the domain \( G \) the solving of the problems coincide and at the same time are resistant to the perturbation of the initial data. In assumptions I, II \( f_\Delta(x,u_\Delta,t) \) satisfies a Lipschitz condition with respect to \( u_\Delta \). In the problem situations (20) \( u_\Delta(x,t) \equiv u(x,t) \), \( \frac{\partial u_\Delta(x,t)}{\partial x} \equiv \frac{\partial u(x,t)}{\partial x} \), is performed. In \( G_{ij} \) the solution of the problem can be given in the form

\[
\begin{align*}
\phi_j(i_0) + \int_{i_0}^t f_\Delta(x,u_\Delta,t) - a(x,t) \frac{u_\Delta(x+\Delta t,t) - u_\Delta(x,t)}{\Delta} \, dt,
\end{align*}
\]

(21)

where \( (x,t) \in G_{ij}, \quad (x+\Delta t,t) \in G_{\Delta ij} \). The approximating \( u_\Delta(x,t) \) polynomial will be denoted as \( P_{u_\Delta 2k2n}(x,t) \). The polynomial approximating \( f_\Delta(x,u_\Delta,t) - a(x,t) \frac{u_\Delta(x+\Delta t,t) - u_\Delta(x,t)}{\Delta} \), is denoted by \( \Psi_{\Delta 2k2n}(z,w) \), \( z \) and \( w \) from (11). The values of the remainder interpolation terms will change, but their designations will remain, which should not lead to misunderstandings. In these notations, the solution (21) approximates the sequence

\[
\begin{align*}
\tilde{P}_{u_\Delta 2k2n}(x,t) = \tilde{P}_{u_\Delta 2k2n}(x,0) + \int_0^t \left( f_\Delta(x,\tilde{P}_{u_\Delta 2k2n}(x,\tau)) - a(x,\tau) \frac{\tilde{P}_{u_\Delta 2k2n}(x+\Delta \tau,\tau) - \tilde{P}_{u_\Delta 2k2n}(x,\tau)}{\Delta} \right) \, d\tau
\end{align*}
\]

(22)

where \( \tilde{P}_{u_\Delta 2k2n}(x,t_0) = \overline{u}_\Delta(x,t_0), \quad r = 0, 1, \ldots \), due to the inaccuracy of the initial data, the solution is marked by a line. Relation (21) will go into the relation

\[
\begin{align*}
\overline{u}_\Delta(x,t) = \overline{u}_\Delta(x,t_0) + \int_{t_0}^t \left( f_\Delta(x,\overline{u}_\Delta,\tau) - a(x,\tau) \frac{\overline{u}_\Delta(x+\Delta \tau,\tau) - \overline{u}_\Delta(x,\tau)}{\Delta} \right) \, d\tau, \quad \overline{u}_\Delta(x,t_0) = \phi_j(i_0, j_0),
\end{align*}
\]

(23)

where \( (x,t) \in G_{ij}, \quad (x+\Delta t,t) \in G_{\Delta ij}, \quad t_0 = t_0(i,j), \quad \phi_j(i_0, j_0) = \overline{P}_{u_\Delta 2k2n}(x,0) \).

Here and subsequently, the value \( \Delta \) in (20) - (23) is considered as a parameter whose choice within the condition (19) will allow to estimate the convergence of the sequence (18) based on the convergence estimate (22). From (22), (23)
\( \forall (x,t) \in G_{ij}, (x + \Delta t) \in G_{\Delta ij}, \max_{G_{ij}} \left| \bar{u}_\lambda(x,t) - \bar{u}_{a_{2k2n}}(x,t) \right| \leq \int_{t_0}^{t} \left( \bar{L} + 2M^\lambda \right) \max_{G_{ij}} \left| \bar{u}_\lambda(x,t) - \bar{u}_{a_{2k2n}(r-1)}(x,t) \right| + \max_{G} c_{2k2n} \, dt. \) (24)

Here and below \( M = \max_{G} \left| a(x,t) \right|, M = \text{const}, \bar{L} = \text{Lipschitz constant.} \) First, we consider the case when in (24) the error of a single interpolation does not exceed the error \( r - 1 \) of iterations with some constant coefficient, more precisely,

\[ \exists Q > 0, Q = \text{const} : 0 < \max_{G} c_{2k2n} \leq Q \max_{G_{ij}} \left| \bar{u}_\lambda(x,t) - \bar{u}_{a_{2k2n}(r-1)}(x,t) \right|, r = 1, 2, \ldots \] (25)

The value \( Q \) in (25) can be arbitrarily increased, and for \( \forall e_0 \) from (19) it is chosen so that

\[ Q^{-1} \max_{G} c_{2k2n} \leq e_0. \] (26)

In what follows the case is analyzed when (25) is violated: for some \( e_0 > r \), the following relation will be performed

\[ \max_{G_{ij}} \left| \bar{u}_\lambda(x,t) - \bar{u}_{a_{2k2n}(r-1)}(x,t) \right| < Q^{-1} \max_{G} c_{2k2n}, r_0 \geq r + 1, \] (27)

in this case, as a result of (27) and (26) the violation (25) entails

\[ \max_{G_{ij}} \left| \bar{u}_\lambda(x,t) - \bar{u}_{a_{2k2n}(r-1)}(x,t) \right| < e_0. \] Let us assume that (25) is not violated first, in this case according to (24), (25)

\[ \max_{G_{ij}} \left| \bar{u}_\lambda(x,t) - \bar{u}_{a_{2k2n}(r)}(x,t) \right| \leq N \int_{t_0}^{t} \max_{G_{ij}} \left| \bar{u}_\lambda(x,t) - \bar{u}_{a_{2k2n}(r-1)}(x,t) \right| dt, N = (\bar{L} + 2M + Q)\Delta^\lambda, \Delta \leq 1. \]

If \( \varepsilon_{k\ell} = \max_{G_{ij}} \left| \bar{u}_\lambda(x,t) - \bar{u}_{a_{2k2n}(r)}(x,t) \right| \), the inequality will take the form

\[ \varepsilon_{k\ell} \leq N \int_{t_0}^{t} \varepsilon_{k-1}(t), \ell = 1, 2, \ldots \] (28)

where \( \varepsilon_{k0} = \max_{G_{ij}} \left| \phi_j(x,t_0) - \bar{u}_{a_{2k2n}(r)}(x,t_0) \right| \leq \max_{G_{ij}} \left| c_{a_{2k2n}} \right|, \phi_j(x,t_0) = \text{function on the adjacent with } G_{ij} \text{ boundary } G_{i,j-1}, \phi_j(x,t_0) = \bar{u}_{a_{2k2n}(r)}(x,t), \ell = \text{number of the final iteration performed in } G_{i,j-1}. \) From (28) and (17) \( \varepsilon_{k1} \leq \tilde{C}2^{-k(2n+2)}h^{2n+1}N \int_{t_0}^{t} dt, \text{ or, } \varepsilon_{k1} \leq \tilde{C}khN(t-t_0), \text{ here and below } \tilde{C}_{kh} = \tilde{C}2^{-k(2n+2)}h^{2n+1}. \) Then \( \varepsilon_{k2} \leq \tilde{C}h^{-1}N^2(t-t_0)^2. \) By induction \( \varepsilon_{k\ell} \leq \tilde{C}_{kh} N^\ell (t-t_0)^\ell, \ell = 3, 4, \ldots \) According to assumption I, with regard for \( G_{ij} \) dimensions, the inequality \( t-t_0 \leq 2^\lambda T \) is true, resulting in

\[ \varepsilon_{k\ell} \leq \tilde{C}_{kh} \frac{(2^\lambda NT)^\ell}{\ell!}, \ell = 1, 2, \ldots \] (29)
In (29) \((2^{-k} NT)^\ell / \ell! \to 0, \ell \to \infty\), therefore \(e_k \ell \to 0, \ell \to \infty\). This implies that (25) is expected to be violated,  
\[ \exists L = L(i, j, k, Q, \Delta, \varepsilon_0) \colon \tilde{c}_{kh} \frac{(2^{-k} NT)^L}{L!} < Q^{-1} \max_G |c_{2k2n}|, \quad e_k \ell < Q^{-1} \max_G |c_{2k2n}|, \quad e_k \ell < \varepsilon_0. \]
In this case by induction it is proved that
\[ \max_{G_{ij}} |\tilde{u}_\Delta(x, t) - \tilde{P}_{u_{2k2n}(r+1)}(x, t)| < \tilde{c}_{kh} \frac{(2^{-k} NT)^L}{L!} < \max_G |c_{2k2n}| Q^{-1} \]
and
\[ \exists r = r(i, j, k, Q, \Delta, \varepsilon_0) : \max_{G_{ij}} |\tilde{u}_\Delta(x, t) - \tilde{P}_{u_{2k2n}(r+1)}(x, t)| < \varepsilon_0 \quad \forall \ell = 0, 1, \ldots \quad (30) \]

On this basis Lemma 2 is proved.

**Lemma 2.** Let in (1) through (3) \(2^{2k}\) of the subdomains are fixed and the subdomain \(G_{ij}\) is arbitrarily selected. Provided that \(\tilde{u}_\Delta(x, t)\) from (23) is considered in \(G_{ij}\), \((x, t) \in G_{ij}\), \((x + \Delta, t) \in G_{\Delta ij}\), with the same initial conditions at the boundary with \(G_{i(j-1)}\), with which the iterations are performed (22), \(\tilde{P}_{u_{2k2n}}(x, t_0) = \tilde{u}_\Delta(x, t_0), r = 0, 1, \ldots, t_n = t_n(i, j)\), the sequence (22) uniformly \(\forall (x, t) \in G_{ij}\) converges to \(\tilde{u}_\Delta(x, t)\). As long as the relation (25) is not violated, the rate of convergence is estimated from (29). In any case takes place (30).

The considered approximation is uniformly continuous in a closed subdomain \(G_{ij}\). Considering that for both (23) and (22) in \(G_{ij}\), \(\tilde{u}_\Delta(x, t_0)\) \((x, t) \in G_{ij}, t_n = t_n(i, j)\) is performed, where \(\tilde{\ell}^2\) is the number of the final iteration performed in \(G_{i(j-1)}\), then the uniform continuity of approximation (22) is preserved in the transition from \(G_{i(j-1)}\) to \(G_{ij}\) and thus in the entire time layer \(\bigcup_{j=0}^{2^{k-1}} G_{ij}\) for each \(i = \text{const}\).

This yields the Corollaries.

**Corollary 1.** Piecewise interpolation approximation of the problem solution (20) with iterative refinement in each subdomain \(G_{ij}\) for any number of iterations is a uniformly continuous function. In particular, when \(k = 0\) the approximation is uniformly continuous in \(G\), when \(k > 0\) it is piecewise continuous in \(G\) and retains uniform continuity in the layer of subdomains \(\bigcup_{j=0}^{2^{k-1}} G_{ij}\) for each \(i = \text{const}\).

It will remain for \(\forall \varepsilon > 0\) (30), if we take \(\varepsilon_0 \leq 2^{-k-1}\varepsilon\), and a priori also take in (19) the corresponding value \(\Delta\), and specify depending on them \(r\) in (30):
\[ \exists e_\Delta = \text{const}, \forall \Delta, 0 < \Delta \leq \Delta_\text{const}, e_\Delta = \text{const}, \exists r = r(i, j, k, Q, \Delta, \varepsilon) : \max_{G_{ij}} |\tilde{u}_\Delta(x, t) - \tilde{P}_{u_{2k2n}(r+1)}(x, t)| < 2^{-k-1}\varepsilon \quad \forall \ell = 0, 1, \ldots \quad (31) \]

From the fact that at the output from \(G_{ij}\) the polynomial \(\tilde{P}_{u_{2k2n}(r+1)}(x, t)\) sets the initial conditions \(\tilde{u}_\Delta(x, t_0) = \varphi_{i(j-1)}(x, t_0), t_0 = t_0(i, j+1)\), in \(G_{i(j+1)}\), and at the same time (31) is performed, hence the assessment of changes in the initial conditions during the transition from one subdomain to another.
Corollary 2. Under conditions of Lemma 2 \( \forall \varepsilon > 0 \) there is \( \Delta_1 = \text{const} \), such that \( \forall \Delta \) from (19), \( 0 < \Delta \leq \Delta_1 \), the maximum deviation of the sequence (22) in \( G_{ij} \) from \( \bar{u}_\alpha(x,t) \) is estimated from (31). The same value will not exceed the maximum deviation of the initial conditions in \( G_{i(j+1)} \) from those initial conditions \( \phi_{i(j+1)}(x,t_0) \), which would be set in the transition from \( G_{ij} \) into \( G_{i(j+1)} \) by the exact solution, taken with the initial conditions from \( G_{ij} \) in the form of \( \phi_{ij}(x,t_0) = \bar{P}_{u_\alpha} \), \( 2^{k+2n} \), \( t \rightarrow t_0(i,j) \).

The stability of the problem solution (20) is retained in each subdomain and means that \( \forall \varepsilon_0 > 0 \) \( \exists \delta > 0 \) is so that if \( \left| \bar{u}_\alpha(x,t_0) - \bar{u}_\alpha(x,t_0) \right| < \delta \), \( t_0 = t_0(i,j) \), then \( \left| \bar{u}_\alpha(x,t) - \bar{u}_\alpha(x,t) \right| \leq \varepsilon_0 \) \( \forall (x,t) \in G_{ij} \). With regard for the stability, we can estimate the maximum deviation of the piecewise interpolation approximation with iterative refinement from the exact unperturbed solution of the problem (20) in this whole layer:

\[
\max_{\mathcal{G}_{ij}, \mathcal{J} \equiv 0, 2^k - 1} \left| u(x,t) - \bar{P}_{u_\alpha} \right| \leq \varepsilon \quad \forall t \geq r_{\max}^{\mathcal{G}_{ij}}, \Delta = \text{const}, \ 0 < \Delta \leq \Delta_{\min}.
\]

According to the maximum selection used in these estimates, they are retained for any layer \( \bigcup_{j=0}^{2^{j-1}} \mathcal{G}_{ij} \) with the constant \( i \in (0, 2^k - 1) \) from the domain \( G \). Hence, and also from the fact that according to the construction of an auxiliary problem (20) for any \( \Delta \) from (19) \( u_\alpha(x, t) = u(x, t) \) \( \forall (x,t) \in G_{ij} \), where \( u(x, t) \) – the solution of equation (13), follows

Lemma 3. Under conditions of Lemma 2, the piecewise interpolation approximation with iterative refinement (22) of the problem solutions (20) converge uniformly in the domain \( G \) to the solution \( u(x,t) \) of the problem (13). Herewith,

\[
\forall \varepsilon > 0 \ \exists \Delta_{\min} = \text{const}, \forall \Delta, \ 0 < \Delta \leq \Delta_{\min}, \Delta = \text{const}, \exists r_{\max} = \max_{\forall t, G_{ij} \in G} r(i, j, k, Q, \Delta, \varepsilon) :
\]

\[
\max_{\forall t, G_{ij} \in G} \left| u(x,t) - \bar{P}_{u_\alpha} \right| < \varepsilon \quad \forall t \geq r_{\max}.
\]

The statement of Corollary 1 is repeated relatively uniform and piecewise continuity of successive approximations without changes, with the proviso that it applies not only to the solution of the problem (20), but also to the solution of the problem (13).

4. Convergence of the main sequence to the solution of the transport equation
For the transition from (22) to (18), under conditions of Lemma 3 and with regard for (32), the difference is preestimated \( D_{r=1} = \frac{\partial}{\partial x} \bar{P}_{u_\alpha} \frac{2k+2n(r-1)}{(x+\Delta,t)} - \bar{P}_{u_\alpha} \frac{2k+2n(r-1)}{(x,t)} \). With the application of the mean-value theorem,

\[
\left| D_{r=1} \right| = \left| \frac{\partial}{\partial x} \bar{P}_{u_\alpha} \frac{2k+2n(r-1)}{(x,t)} - \frac{\partial}{\partial x} \bar{P}_{u_\alpha} \frac{2k+2n(r-1)}{(x,t)} \right|_{x_0 = \bar{u}} \leq \bar{\alpha} \Delta, \ 0 < \bar{\alpha} < 1.
\]

With the repeated application of the theorem,
The right part of (34) is limited by virtue of the construction of the interpolation. Let the sequence be considered along with (22)

\[ \frac{\partial^2 \tilde{P}_{u_{2k2n}}(r,x,t)}{\partial x^2} \bigg|_{x=t} \leq \max_{G_{ij}} \frac{1}{h_x} \left| \frac{\partial^2 \tilde{\psi}_{\Delta x}^{ij}}{\partial z^2} \right| (z,w) \]  

(34)

From (23) and (35), under equal initial conditions in \( G_{ij} \), the relation is proved

\[ \forall \varepsilon > 0 \ \exists \Delta_{\min} = \text{const}, \forall \Delta, \ 0 < \Delta \leq \Delta_{\min} \left( \Delta_{\min}, \ C_{0}^{-1} M^{-1} \max_{G} \left| c_{2k2n} \right| \right), \ \Delta = \text{const} , \]

\[ \exists R_{\max} = \max_{\forall i,j,G_{ij} \subseteq G} r(i,j,k,Q,\Delta,\varepsilon) : \max_{\forall i,j,G_{ij} \subseteq G} \left| u_{\Delta}(x,t) - \tilde{P}_{u_{2k2n}}(r,x,t) \right|_{k,j,G_{ij}} \leq \varepsilon \ \forall r \geq R_{\max} , \]  

(36)

where \( \tilde{P}_{u_{2k2n}}(r,x,t) \) from (35), \( u_{\Delta}(x,t) \) is the solution of the problem (20). Since \( u_{\Delta}(x,t) \equiv u(x,t) \ \forall (x,t) \in G \), the sequence (35) with the estimate (36) approximates the solution of the problem (13). Thus, we get

**Lemma 4.** Under conditions of Lemma 2, the piecewise interpolation approximation with iterative refinement (35) of the problem solution (20) converge uniformly in the domain \( G \) to the solution \( u(x,t) \) of the problem (13). Herewith \( \forall \varepsilon > 0 \) and the relation (36) is performed, where \( u_{\Delta}(x,t) \equiv u(x,t) \ \forall (x,t) \in G \). The statement of Lemma 3 is repeated relatively uniform and piecewise continuity of successive approximations (35) without changes.

On the basis of Lemma 4 the following theorem is proved.

**Theorem 2.** Let in domain \( G \) from (1) – (3), \( 2^{2k} \) subdomains are fixed. Suppose that in each subdomain \( G_{ij} \), the equation (14) is considered with the same initial conditions on the boundary with \( G_{i(j-1)} \), with which the iterations are performed (18), with that, they are retained with the change of the iteration number: \( \tilde{u}(x,t_0) = \phi_{ij}(x,t_0) \), \( \tilde{u}_{0}^{2k2n}(x,t_0) = \phi_{ij}(x,t_0) \), \( r = 0, 1, \ldots, t_0 = r_0(i,j) \). Then the piecewise interpolation approximation with iterative refinement (18) of the problem solution (13) converges uniformly in \( G \) to the solution \( u(x,t) \) with the estimate

\[ \forall \varepsilon > 0 \ \exists r_{\Delta} : \max_{\forall(x,t) \in G} \left| u(x,t) - \tilde{u}_{0}^{2k2n}(x,t) \right|_{k,j,G_{ij}} \leq \varepsilon \ \forall r \geq r_{\Delta} , \]  

(37)

With respect to the uniform and piecewise continuity of successive approximations (18) without fundamental changes, the analogue of Lemma 4 statement given with respect to approximations (35) is formulated.

**5. Numerical experiment**

The described method is computer implemented, the program code and a detailed numerical experiment are given in [5]. The implementation contains deviations from the original description. That is, interpolation is performed on the basis of (5) with the transition in the form (6) without doubling the degree of the polynomial. The analog (18)
\[ \Psi_{ij}^{(r-1)}(z,w) \approx f(x,t) - a(x,t) \frac{\partial}{\partial x} \Psi_{ij}^{(r-1)}(x,t) = \int_{t_0}^{t} P_{2kn+r}^{(r-1)}(z,w) dt, \]

is used for iterative refinement, the coefficients are converted taking into account (11), (12).

The initial value problem for the linear transport equation [10]:

\[ u'_x + u'_t = 0, \quad u(x,0) = \sin(x) \quad (38) \]

has an analytical solution \( u(x,t) = \sin(x-t) \). Its approximate solution is given below in \( G = \{ (x,t) \mid x \in [0,1], t \in [0,1] \} \). In Table 1 the values of the absolute error of the piecewise interpolation solution with iterative refinement (implementation in Delphi, extended data type) in 10 uniformly distributed points on the segment \( x \in [0,1] \) with the value \( t = 0.1 \) and \( t = 1 \) (the degree of the interpolation polynomial \( n = 13 \), the number of iterations \( r = 50 \), the \( G \) domain is not divided into subdomains) are given. In the first column of the table – the decision time on the personal computer (for the compared methods the decision time on the same computer is given in table 2).

**Table 1. Absolute Error of Approximate Problem Solution (38).**

| s | \( h_s \approx 0.08 \) | \( h_s \approx 0.04 \) | \( t = 0.1 \) | \( 9.4 \times 10^{-10} \) | \( 9.2 \times 10^{-10} \) | \( 2.2 \times 10^{-10} \) | \( 1.1 \times 10^{-10} \) |
|---|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| 11 | \( 1.6 \times 10^{-10} \) | \( 6.5 \times 10^{-10} \) | \( 7.6 \times 10^{-10} \) | \( 6.8 \times 10^{-10} \) | \( 3.0 \times 10^{-10} \) |

For the unity of the comparison conditions, the solution of the initial boundary value problem in MathCAD, Maple systems and the proposed method without its fundamental changes is given below. The initial-boundary value problem for the quasi-linear transport equation [11]:

\[ u'_x + u'_t = -u^2 + 2e^{r+t} + e^{2(r+t)}, \quad u(x,0) = e^r, \quad u(0, t) = e^t \quad (39) \]

has an analytical solution \( e^{r+x} \). In Table 2 the values of the absolute error of the piecewise interpolation solution with iterative refinement in 10 uniformly distributed points on the segment \( x \in [0,1] \) with the value \( t = 1 \) (the degree of the interpolation polynomial \( n = 13 \), the number of iterations \( r = 40 \), the \( G \) domain is not divided into subdomains) are given.

**Table 2. Absolute Error of Approximate Problem Solution (39).**

| \( s \) | \( h_s \approx 8.0 \times 10^{-2} \) | \( h_s \approx 4.0 \times 10^{-2} \) | \( 6.5 \times 10^{-10} \) | \( 4.3 \times 10^{-10} \) | \( 8.7 \times 10^{-10} \) | \( 5.2 \times 10^{-10} \) | \( 3.5 \times 10^{-10} \) |
|---|------------------|------------------|------------------|------------------|------------------|------------------|------------------|
| 16 | \( 1.6 \times 10^{-10} \) | \( 6.1 \times 10^{-10} \) | \( 8.6 \times 10^{-10} \) | \( 9.9 \times 10^{-10} \) | \( 1.0 \times 10^{-9} \) |
| 47 | \( 2.5 \times 10^{-10} \) | \( 5.8 \times 10^{-10} \) | \( 8.6 \times 10^{-10} \) | \( 1.0 \times 10^{-9} \) |
| 51 | \( 8.0 \times 10^{-10} \) | \( 1.4 \times 10^{-9} \) | \( 2.4 \times 10^{-9} \) | \( 6.4 \times 10^{-9} \) | \( 7.1 \times 10^{-9} \) |

The row (1) corresponds to the proposed method, (2) – to the pdesolve function of MathCAD program [2] (the choice of the difference method corresponds to the best approximation), (3) – to the pdsolve function of Maple program [3]. The relation of the steps satisfies the convergence condition of the finite difference method for hyperbolic-type equations. The relation of the accuracy of the proposed method with the known, as a rule, is similar to the data in Table 2 in terms of smoothness: in well-known studies, the accuracy of these numerical experiments usually does not exceed the accuracy in rows 2, 3 of Table 2. So, in [12] the solution of the second-kind hyperbolic equation by means of Taylor polynomials from two variables is formulated. The absolute error of this method in the square the size of [0,1]×[0,1] with the increase of \( t \) reaches \( 7.61 \times 10^{-9} \). In [13] the same problem is solved on the basis of cubic trigonometric spline interpolation. The absolute approximation error in \( G = \{ (x,t) \mid x \in [0,2\pi], t \in [0,3] \} \) is not lower than \( 5.43 \times 10^{-7} \). In [14], a piecewise interpolation
solution of the transport equation is constructed using the cubic B-spline of quasi-interpolation, its absolute error in the square the size of \([0, 1] \times [0, 1]\) is not lower than \(3.01 \times 10^{-10}\) with \(t = 1\).

The comparative accuracy of the proposed method is achieved by iterative refinement (not used in the works with which the comparison was made). To reduce the error, the horizontal and vertical shifts of the approximation domain are used, with which the extrapolation error zone outside the triangular arrangement of the interpolation polynomial nodes is covered.

The presented method is transferred to the quasi-linear transport equation, the analogs of the method can be formulated for equations of other classes, including integral and integro-differential equations.

6. Conclusion

The method of approximate solution of the Cauchy problem for the transport equation, with a relatively high accuracy and piecewise continuous nature of the approximation is expounded. The method is based on piecewise interpolation by Newton polynomials for two variables and iterative refinement by analogy with Picard's method of successive approximations. On the basis of the transformation of interpolation polynomials, formulated for the solution and its partial derivatives, into the form of algebraic polynomials with numerical coefficients, a sequential refinement of the solution similar to the two-dimensional analog of Picard's successive approximations is formulated. In the case of a linear transport equation, the method converges uniformly to the solution in the rectangular domain. The approximation of the solution is uniformly continuous in a given domain, piecewise continuous, when it is divided into subdomains.

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