Using Catalan words and a $q$-shuffle algebra to describe the Beck PBW basis for the positive part of $U_q(\hat{sl}_2)$

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Abstract

We consider the positive part $U^+_q$ of the quantized enveloping algebra $U_q(\hat{sl}_2)$. The algebra $U^+_q$ has a presentation involving two generators and two relations, called the $q$-Serre relations. There is a PBW basis for $U^+_q$ due to Damiani, and a PBW basis for $U^+_q$ due to Beck. In 2019 we used Catalan words and a $q$-shuffle algebra to express the Damiani PBW basis in closed form. In this paper we use a similar approach to express the Beck PBW basis in closed form. We also consider how the Damiani PBW basis and the Beck PBW basis are related to the alternating PBW basis for $U^+_q$.

Keywords. Catalan word, $q$-shuffle algebra, PBW basis; $q$-Serre relations.

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1 Introduction

The quantized enveloping algebra $U_q(\hat{sl}_2)$ appears in representation theory [7], statistical mechanics [11,18], combinatorics [12,16,30], and the theory of tridiagonal pairs [13,15,17,25]. In the present paper we consider a well known subalgebra $U^+_q$ of $U_q(\hat{sl}_2)$, called the positive part [2,5,6,20,26,32]. The algebra $U^+_q$ has a presentation involving two generators $A$, $B$ and two relations, called the $q$-Serre relations:

$$[A, [A, [A, B]_{q^{-1}}] = 0, \quad [B, [B, [B, A]_{q^{-1}}] = 0. $$

In [8] I. Damiani obtained a PBW basis for $U^+_q$, consisting of some elements $\{E_{n\delta+\alpha_0}\}_{n=0}^\infty$, $\{E_{n\delta+\alpha_1}\}_{n=0}^\infty$, $\{E_{n\delta}\}_{n=1}^\infty$ that are defined recursively. In [27] we expressed these elements in closed form, using Catalan words and a $q$-shuffle algebra. In Section 6 we will review this result in detail, and for now give a brief summary. Start with a free associative algebra $V$ on two generators $x, y$. These generators are called letters. For an integer $n \geq 0$, a word of length $n$ in $V$ is a product of letters $u_1 u_2 \cdots u_n$. The vector space $V$ has a basis consisting of its words; this basis is called standard. In [23,24] M. Rosso introduced an associative algebra structure on $V$, called a $q$-shuffle algebra. For letters $u, v$ their $q$-shuffle product is $u \star v = uv + q^{(u,v)}vu$, where $(u,v) = 2$ (resp. $(u,v) = -2$) if $u = v$ (resp. $u \neq v$).
In [24, Theorem 15] Rosso gave an injective algebra homomorphism $\langle \rangle$ from $U_q^+$ into the $q$-shuffle algebra $\mathbb{V}$, that sends $A \mapsto x$ and $B \mapsto y$. In [27] we applied $\langle \rangle$ to the Damiani PBW basis, and expressed the image in the standard basis for $\mathbb{V}$. This image involves words of the following type. Define $x = 1$ and $y = -1$. A word $u_1 u_2 \cdots u_n$ in $\mathbb{V}$ is said to be Catalan whenever $u_1 + u_2 + \cdots + u_i$ is nonnegative for $1 \leq i \leq n - 1$ and zero for $i = n$. In this case $n$ is even. For $n \geq 0$ define

$$C_n = \sum u_1 u_2 \cdots u_{2n}[1]_q[1 + \overline{u}_1]_q[1 + \overline{u}_1 + \overline{u}_2]_q \cdots [1 + \overline{u}_1 + \overline{u}_2 + \cdots + \overline{u}_{2n}]_q,$$

where the sum is over all the Catalan words $u_1 u_2 \cdots u_{2n}$ in $\mathbb{V}$ that have length $2n$. In [27, Theorem 1.7] we showed that the map $\langle \rangle$ sends $E_{n\delta + \alpha_0} \mapsto q^{-2n}(q - q^{-1})^{2}n xC_n$, $E_{n\delta + \alpha_1} \mapsto q^{-2n}(q - q^{-1})^{2}n yC_n$ for $n \geq 0$, and

$$E_{n\delta} \mapsto -q^{-2n}(q - q^{-1})^{2n-1}C_n$$

for $n \geq 1$. In [3] Proposition 6.1 J. Beck obtained a PBW basis for $U_q^+$ by adjusting the Damiani PBW basis as follows. The elements $\{E_{n\delta}\}_{n=1}^{\infty}$ are replaced by some elements $\{E_{n\delta}^{\text{Beck}}\}_{n=1}^{\infty}$ that satisfy the generating function identity below, see [6, p. 6]. Referring to the exponential function and an indeterminate $t$,

$$\exp\left((q - q^{-1}) \sum_{k=1}^{\infty} E_{k\delta}^{\text{Beck}} t^k\right) = 1 - (q - q^{-1}) \sum_{k=1}^{\infty} E_{k\delta} t^k.$$

The main result of the present paper is that $\langle \rangle$ sends

$$E_{n\delta}^{\text{Beck}} \mapsto \frac{[2n]_q}{n} q^{-2n}(q - q^{-1})^{2n-1} xC_{n-1}$$

for $n \geq 1$. In the above line, the notation $xC_{n-1}y$ refers to the free product.

We use our main result to obtain a number of corollaries and subsidiary results. For instance, we show that the following holds in the $q$-shuffle algebra $\mathbb{V}$:

$$\exp\left(\sum_{k=1}^{\infty} \frac{[2k]_q}{k} xC_{k-1} y t^k\right) = 1 + \sum_{k=1}^{\infty} C_k t^k.$$

In the above line, the exponential function is with respect to the $q$-shuffle product, and the notation $xC_{k-1}y$ refers to the free product.

In [28] we introduced the alternating words in $\mathbb{V}$, and used them to obtain the alternating PBW basis for $U_q^+$ [28, Theorem 10.1]. The following words are alternating:

$$\tilde{G}_1 = xy, \quad \tilde{G}_2 = xyxy, \quad \tilde{G}_3 = xyxyxy, \quad \ldots$$
Using our main result and [28, Proposition 11.8], we show that the following holds in the $q$-shuffle algebra $V$:

$$
\exp \left( -\sum_{k=1}^{\infty} \frac{(-1)^k[k]}{k} xC_{k-1}yt^k \right) = 1 + \sum_{k=1}^{\infty} \tilde{G}_kt^k.
$$

In the above line, the exponential function is with respect to the $q$-shuffle product, and the notation $xC_{k-1}y$ refers to the free product.

The paper is organized as follows. Section 2 contains some preliminaries. In Section 3 we recall the algebra $U_q^+$. In Section 4, we review the PBW bases for $U_q^+$ due to Damiani and Beck. In Sections 5, 6 we review the embedding of $U_q^+$ into the $q$-shuffle algebra $V$. Sections 7, 8 contain our main result and some corollaries. In Section 9, we apply our main result to the alternating words in $V$. In Appendix A we give some examples that illustrate certain results from the main body of the paper.

2 Preliminaries

We now begin our formal argument. Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $F$ denote a field with characteristic zero. Throughout this paper, every vector space we discuss is over $\mathbb{F}$. Every algebra we discuss is associative, over $\mathbb{F}$, and has a multiplicative identity. A subalgebra has the same multiplicative identity as the parent algebra.

**Definition 2.1.** (See [8, p. 299].) Let $A$ denote an algebra. A **Poincaré-Birkhoff-Witt** (or **PBW**) basis for $A$ consists of a subset $\Omega \subseteq A$ and a linear order $\prec$ on $\Omega$ such that the following is a basis for the vector space $A$:

$$ a_1a_2\cdots a_n \quad n \in \mathbb{N}, \quad a_1, a_2, \ldots, a_n \in \Omega, \quad a_1 \prec a_2 \prec \cdots \prec a_n. $$

We interpret the empty product as the multiplicative identity in $A$.

We will be discussing generating functions. Let $A$ denote an algebra and let $t$ denote an indeterminate. For a sequence $\{a_k\}_{k \in \mathbb{N}}$ of elements in $A$, the corresponding generating function is

$$ a(t) = \sum_{k \in \mathbb{N}} a_k t^k. $$

The above sum is formal; issues of convergence are not considered. We call $a(t)$ the **generating function over $A$ with coefficients** $\{a_k\}_{k \in \mathbb{N}}$. The coefficient $a_0$ is called the **constant coefficient**.

For generating functions $a(t) = \sum_{k \in \mathbb{N}} a_k t^k$ and $b(t) = \sum_{k \in \mathbb{N}} b_k t^k$ over $A$, their product $a(t)b(t)$ is the generating function $\sum_{k \in \mathbb{N}} c_k t^k$ such that $c_k = \sum_{i=0}^{k} a_i b_{k-i}$ for $k \in \mathbb{N}$. The set of generating functions over $A$ forms an algebra. Consider a generating function $a(t) = \sum_{k=1}^{\infty} a_k t^k$ over $A$ with constant coefficient 0. Then the exponential

$$ \exp a(t) = \sum_{n \in \mathbb{N}} \frac{(a(t))^n}{n!} $$

is a generating function over $A$. The exponential function is defined as

$$ e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}. $$

We will also use the notation $e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$.
is a generating function over $\mathcal{A}$ with constant coefficient 1. Moreover the natural logarithm
\[ \ln(1 + a(t)) = \sum_{n \in \mathbb{N}} \frac{(-1)^n(a(t))^{n+1}}{n+1} \]
is a generating function over $\mathcal{A}$ with constant coefficient 0. Let $a(t) = \sum_{k=1}^{\infty} a_k t^k$ and $b(t) = \sum_{k=1}^{\infty} b_k t^k$ denote generating functions over $\mathcal{A}$ that have constant coefficient 0. Then $\exp a(t) = 1 + b(t)$ if and only if $a(t) = \ln(1 + b(t))$.

**Definition 2.2.** A grading of an algebra $\mathcal{A}$ is a sequence $\{\mathcal{A}_n\}_{n \in \mathbb{N}}$ of subspaces of $\mathcal{A}$ such that (i) $1 \in \mathcal{A}_0$; (ii) the sum $\mathcal{A} = \sum_{n \in \mathbb{N}} \mathcal{A}_n$ is direct; (iii) $\mathcal{A}_r \mathcal{A}_s \subseteq \mathcal{A}_{r+s}$ for $r, s \in \mathbb{N}$. For $n \in \mathbb{N}$ the subspace $\mathcal{A}_n$ is called the $n$-homogeneous component of the grading.

Throughout the paper, fix a nonzero $q \in \mathbb{F}$ that is not a root of unity. Recall the notation
\[ [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad n \in \mathbb{N}. \]

### 3 The algebra $U^+_q$

In this section we recall the algebra $U^+_q$.

For elements $X, Y$ in any algebra, define their commutator and $q$-commutator by
\[ [X, Y] = XY - YX, \quad [X, Y]_q = qXY - q^{-1}YX. \]

Note that
\[ [X, [X, [X, Y]_q]_{q^{-1}}] = X^3Y - [3]_q X^2YX + [3]_q XYX^2 - YX^3. \tag{1} \]

**Definition 3.1.** (See [20, Corollary 3.2.6].) Define the algebra $U^+_q$ by generators $A, B$ and relations
\[ [A, [A, [A, B]_{q^{-1}}]] = 0, \tag{2} \]
\[ [B, [B, [B, A]_{q^{-1}}]] = 0. \tag{3} \]

We call $U^+_q$ the positive part of $U_q(\mathfrak{sl}_2)$. The relations (2), (3) are called the $q$-Serre relations.

For the moment abbreviate $U^+_q = U^+_q$. Since the $q$-Serre relations are homogeneous, the algebra $U^+_q$ has a grading $\{U^+_n\}_{n \in \mathbb{N}}$ with the following property: for $n \in \mathbb{N}$ the subspace $U^+_n$ is spanned by the products $g_1 g_2 \cdots g_n$ such that $g_i$ is among $A, B$ for $1 \leq i \leq n$. In particular $U^+_0 = \mathbb{F}1$ and $U^+_1$ is spanned by $A, B$.  

4
4 Two PBW bases for $U_q^+$

In [8], Damiani obtained a PBW basis for $U_q^+$ that involves some elements

$$\{E_{n\delta+a_0}\}_{n=0}^{\infty}, \quad \{E_{n\delta+a_1}\}_{n=0}^{\infty}, \quad \{E_{n\delta}\}_{n=1}^{\infty}. \quad (4)$$

These elements are recursively defined as follows.

$$E_{\alpha_0} = A, \quad E_{\alpha_1} = B, \quad E_{\delta} = q^{-2}BA - AB \quad (5)$$

and for $n \geq 1$,

$$E_{n\delta+a_0} = \frac{[E_{\delta}, E_{(n-1)\delta+a_0}]}{q + q^{-1}}, \quad E_{n\delta+a_1} = \frac{[E_{(n-1)\delta+a_1}, E_{\delta}]}{q + q^{-1}}, \quad (6)$$

$$E_{n\delta} = q^{-2}E_{(n-1)\delta+a_1}A - AE_{(n-1)\delta+a_1}. \quad (7)$$

**Proposition 4.1.** (See [8, p. 308].) A PBW basis for $U_q^+$ is obtained by the elements (4) in the linear order

$$E_{\alpha_0} < E_{\delta+a_0} < E_{2\delta+a_0} < \cdots < E_{\delta} < E_{2\delta} < E_{3\delta} < \cdots < E_{2\delta+a_1} < E_{\delta+a_1} < E_{\alpha_1}. \quad (8)$$

The PBW basis in Proposition 4.1 will be called the *Damiani PBW basis*.

**Lemma 4.2.** For the grading $\{U_n^+\}_{n \in \mathbb{N}}$ of $U^+ = U_q^+$, we have $E_{n\delta+a_0}, E_{n\delta+a_1} \in U_{2n+1}^+$ for $n \geq 0$ and $E_{n\delta} \in U_{2n}^+$ for $n \geq 1$.

**Proof.** Use (5)–(7). \qed

Next we recall some relations satisfied by the elements of the Damiani PBW basis.

**Lemma 4.3.** (See [8, p. 307].) The elements $\{E_{n\delta}\}_{n=1}^{\infty}$ mutually commute.

**Lemma 4.4.** (See [8, p. 307].) For $i, j \in \mathbb{N}$,

$$[E_{i\delta+a_0}, E_{j\delta+a_1}]_q = -qE_{(i+j+1)\delta}. \quad (9)$$

We just gave some relations involving the elements of the Damiani PBW basis. Additional relations involving these elements can be found in [8], see also [27, Section 3].

We have been discussing the Damiani PBW basis. Next we discuss a variation on this PBW basis, due to J. Beck [5]. Our discussion will involve some generating functions in an indeterminate $t$.

**Definition 4.5.** (See [6, p. 6].) Define the elements $\{E_{n\delta}^{\text{Beck}}\}_{n=1}^{\infty}$ in $U_q^+$ such that

$$\exp\left((q - q^{-1})\sum_{k=1}^{\infty} E_{k\delta}^{\text{Beck}} t^k\right) = 1 - (q - q^{-1})\sum_{k=1}^{\infty} E_{k\delta} t^k. \quad (10)$$
Example 4.6. We have
\[ E_\delta = -E_\delta^{\text{Beck}}, \quad E_{2\delta} = -E_{2\delta}^{\text{Beck}} - \frac{q - q^{-1}}{2} (E_\delta^{\text{Beck}})^2, \]
\[ E_{3\delta} = -E_{3\delta}^{\text{Beck}} - (q - q^{-1}) E_\delta^{\text{Beck}} E_{2\delta}^{\text{Beck}} - \frac{(q - q^{-1})^2}{6} (E_\delta^{\text{Beck}})^3. \]
Moreover
\[ E_\delta^{\text{Beck}} = -E_\delta, \quad E_{2\delta}^{\text{Beck}} = -E_{2\delta} - \frac{q - q^{-1}}{2} E_\delta^{2}, \]
\[ E_{3\delta}^{\text{Beck}} = -E_{3\delta} - (q - q^{-1}) E_\delta E_{2\delta} - \frac{(q - q^{-1})^2}{3} E_\delta^{3}. \]
We clarify how the elements \( \{E_{n\delta}\}_{n=1}^\infty \) and \( \{E_{n\delta}^{\text{Beck}}\}_{n=1}^\infty \) are related.

**Lemma 4.7.** The following hold for \( n \geq 1 \):

(i) \( E_{n\delta} \) is a homogeneous polynomial in \( E_\delta^{\text{Beck}}, E_{2\delta}^{\text{Beck}}, \ldots, E_{n\delta}^{\text{Beck}} \) that has total degree \( n \), where we view \( E_k^{\text{Beck}} \) as having degree \( k \) for \( 1 \leq k \leq n \);

(ii) \( E_{n\delta}^{\text{Beck}} \) is a homogeneous polynomial in \( E_\delta, E_{2\delta}, \ldots, E_{n\delta} \) that has total degree \( n \), where we view \( E_k \) as having degree \( k \) for \( 1 \leq k \leq n \).

**Proof.** Use (8) and induction on \( n \).

**Lemma 4.8.** For the grading \( \{U_n^+\}_{n \in \mathbb{N}} \) of \( U^+ = U_q^+ \), we have \( E_{n\delta}^{\text{Beck}} \in U_{2n}^+ \) for \( n \geq 1 \).

**Proof.** By Lemma 4.2 and Lemma 4.7(ii).

**Proposition 4.9.** (See [5, Proposition 6.1].) A PBW basis for \( U_q^+ \) is obtained by the elements
\[ \{E_{n\delta+\alpha_0}\}_{n=0}^\infty, \quad \{E_{n\delta+\alpha_1}\}_{n=0}^\infty, \quad \{E_{n\delta}^{\text{Beck}}\}_{n=1}^\infty \]
in the linear order
\[ E_{\alpha_0} < E_{\delta+\alpha_0} < E_{2\delta+\alpha_0} < \cdots < E_{\delta}^{\text{Beck}} < E_{2\delta}^{\text{Beck}} < E_{3\delta}^{\text{Beck}} < \cdots < E_{2\delta+\alpha_1} < E_{\delta+\alpha_1} < E_{\alpha_1}. \]

The PBW basis in Proposition 4.9 will be called the **Beck PBW basis**. Next we recall some relations satisfied by the elements of the Beck PBW basis.

**Lemma 4.10.** (See [6, Proposition 1.2].) The elements \( \{E_{n\delta}^{\text{Beck}}\}_{n=1}^\infty \) mutually commute.

**Lemma 4.11.** (See [6, Proposition 1.2].) For \( k \geq 1 \) and \( \ell \geq 0 \),
\[ [E_{\ell\delta+\alpha_0}, E_{k\delta}^{\text{Beck}}] = \frac{[2k]_q}{k} E_{(k+\ell)\delta+\alpha_0}, \tag{9} \]
\[ [E_{k\delta}^{\text{Beck}}, E_{\ell\delta+\alpha_1}] = \frac{[2k]_q}{k} E_{(k+\ell)\delta+\alpha_1}. \tag{10} \]

In Section 6 we will return our attention to the Damiani PBW basis and the Beck PBW basis. In the meantime, we will discuss an embedding, due to Rosso [23,24], of the algebra \( U_q^+ \) into a \( q \)-shuffle algebra. For this \( q \)-shuffle algebra, the underlying vector space is a free algebra on two generators. We denote this free algebra by \( \mathbb{V} \).
5 The free algebra $\mathcal{V}$

Let $x, y$ denote noncommuting indeterminates. Let $\mathcal{V}$ denote the free algebra generated by $x$ and $y$. By a *letter* in $\mathcal{V}$ we mean $x$ or $y$. For $n \in \mathbb{N}$, by a *word of length* $n$ in $\mathcal{V}$ we mean a product of letters $a_1a_2 \cdots a_n$. We interpret the word of length 0 to be the multiplicative identity in $\mathcal{V}$; this word is called *trivial* and denoted by 1. The vector space $\mathcal{V}$ has a (linear) basis consisting of its words; this basis is called *standard*. We endow the vector space $\mathcal{V}$ with a bilinear form $\langle \cdot, \cdot \rangle : \mathcal{V} \times \mathcal{V} \to F$ with respect to which the standard basis is orthonormal. This bilinear form is symmetric and nondegenerate. For a subspace $W \subseteq \mathcal{V}$, recall its orthogonal complement $W^\perp = \{ v \in \mathcal{V} | \langle v, w \rangle = 0 \ \forall w \in W \}$.

For $n \in \mathbb{N}$ let $\mathcal{V}_n$ denote the subspace of $\mathcal{V}$ spanned by the words of length $n$. The sum $\mathcal{V} = \sum_{n \in \mathbb{N}} \mathcal{V}_n$ is direct and the summands are mutually orthogonal. We have $\mathcal{V}_0 = F1$. We have $\mathcal{V}_r \mathcal{V}_s \subseteq \mathcal{V}_{r+s}$ for $r, s \in \mathbb{N}$. By these comments the sequence $\{ \mathcal{V}_n \}_{n \in \mathbb{N}}$ is a grading of the algebra $\mathcal{V}$.

**Lemma 5.1.** For $r, s \in \mathbb{N}$ and $X, X' \in \mathcal{V}_r$ and $Y, Y' \in \mathcal{V}_s$ we have

$$\langle XY, X'Y' \rangle = \langle X, X' \rangle \langle Y, Y' \rangle.$$ 

**Proof.** Since the standard basis for $\mathcal{V}$ is orthonormal. \hfill \Box

**Definition 5.2.** Define $J^+, J^- \in \mathcal{V}$ by

$$J^+ = xxyy - [3]_q xxyx + [3]_q xyxx - yxxx,$$

$$J^- = yyyx - [3]_q yyyx + [3]_q yxyy - xyyy.$$

**Definition 5.3.** Let $J$ denote the 2-sided ideal of the free algebra $\mathcal{V}$ generated by $J^+, J^-$. Consider the quotient algebra $\mathcal{V}/J$. Since the algebra $\mathcal{V}$ is freely generated by $x$ and $y$, there exists an algebra homomorphism $\xi : \mathcal{V} \to U_+^q$ that sends $x \mapsto A$ and $y \mapsto B$. The kernel of $\xi$ is equal to $J$, in view of (1) and Definition 3.1. Therefore, $\xi$ induces an algebra isomorphism $\mathcal{V}/J \to U_+^q$ that sends $x + J \mapsto A$ and $y + J \mapsto B$.

We just described a connection between $J$ and $U_+^q$. In Proposition 6.2 we will describe another connection between $J$ and $U_+^q$.

6 The $q$-shuffle algebra $\mathcal{V}$

In the previous section we discussed the free algebra $\mathcal{V}$. There is another algebra structure on $\mathcal{V}$, called the $q$-shuffle algebra. This algebra was introduced by Rosso [23, 24] and described further by Green [9]. We will adopt the approach of [9], which is suited to our purpose. The $q$-shuffle product is denoted by $\star$. To describe this product, we start with some special cases. We have $1 \star v = v \star 1 = v$ for $v \in \mathcal{V}$. For letters $u, v$ we have

$$u \star v = uv + vuq^{(u,v)}$$
where

\[
\begin{array}{c|cc}
(,) & x & y \\
\hline
x & 2 & -2 \\
y & -2 & 2 \\
\end{array}
\]

Thus

\[
x \star y = xy + q^{-2}yx, \quad y \star x = yx + q^{-2}xy, \\
x \star x = (1 + q^2)xx \quad y \star y = (1 + q^2)yy.
\]

For a letter \( u \) and a nontrivial word \( v = v_1v_2 \cdots v_n \) in \( V \),

\[
u \star v = \sum_{i=0}^{n} v_1 \cdots v_i uv_{i+1} \cdots v_n q^{(v_1,u)+(v_2,u)+\cdots+(v_n,u)},
\]

\[
v \star u = \sum_{i=0}^{n} v_1 \cdots v_i uv_{i+1} \cdots v_n q^{(v_n,u)+(v_{n-1},u)+\cdots+(v_1,u+1,u)}.
\]

For example

\[
x \star (yyy) = xyyy + q^{-2}yxyy + q^{-4}yyxy + q^{-6}yyyy,
\]

\[
(yyy) \star x = q^{-6}xxyy + q^{-4}xyyy + q^{-2}yxyy + yyyy.
\]

For nontrivial words \( u = u_1u_2 \cdots u_r \) and \( v = v_1v_2 \cdots v_s \) in \( V \),

\[
u \star v = u_1 \left( (u_2 \cdots u_r) \star v \right) + v_1 \left( u \star (v_2 \cdots v_s) \right) q^{(u_1,v_1)+(u_2,v_1)+\cdots+(u_r,v_1)}, \quad (11)
\]

\[
u \star v = \left( u \star (v_1 \cdots v_{s-1}) \right) v_s + \left( (u_1 \cdots u_{r-1}) \star v \right) u_r q^{(u_r,v_1)+(u_r,v_2)+\cdots+(u_r,v_s)}. \quad (12)
\]

For example, take \( r = 2 \) and \( s = 2 \). We have

\[
u \star v = u_1u_2v_1v_2
\]

\[
+ u_1v_1u_2v_2 q^{(u_2,v_1)}
\]

\[
+ u_1v_1u_2v_2 q^{(u_2,v_1)+(u_2,v_2)}
\]

\[
+ v_1u_1u_2v_2 q^{(u_1,v_1)+(u_2,v_1)}
\]

\[
+ v_1u_1u_2v_2 q^{(u_1,v_1)+(u_2,v_1)+(u_2,v_2)}
\]

\[
+ u_1v_1u_2v_2 q^{(u_1,v_1)+(u_1,v_2)+(u_2,v_1)+(u_2,v_2)}.
\]

Above Lemma 5.1 we mentioned a grading of the free algebra \( V \). This is also a grading for the \( q \)-shuffle algebra \( V \).

**Definition 6.1.** Let \( U \) denote the subalgebra of the \( q \)-shuffle algebra \( V \) generated by \( x, y \).

The algebra \( U \) is described as follows. With some effort (or by [23, Theorem 13], [9, p. 10]) one obtains

\[
x \star x \star x \star y - [3]_q x \star x \star y \star x + [3]_q x \star y \star x \star x - y \star x \star x \star x = 0, \quad (13)
\]

\[
y \star y \star y \star x - [3]_q y \star y \star y \star x + [3]_q y \star x \star y \star y - x \star y \star y \star y = 0. \quad (14)
\]
So in the \( q \)-shuffle algebra \( \mathcal{V} \) the elements \( x, y \) satisfy the \( q \)-Serre relations. Consequently there exists an algebra homomorphism \( \natural \) from \( U_q^+ \) to the \( q \)-shuffle algebra \( \mathcal{V} \), that sends \( A \mapsto x \) and \( B \mapsto y \). The map \( \natural \) has image \( U \) by Definition 6.1 and is injective by [24, Theorem 15]. Therefore \( \natural : U_q^+ \to U \) is an algebra isomorphism. See [10, 19, 21, 22] for more information about the \( q \)-shuffle algebra \( \mathcal{V} \) and its relationship to \( U_q^+ \).

Earlier we mentioned a grading for both \( U_q^+ \) and the \( q \)-shuffle algebra \( \mathcal{V} \). These gradings are related as follows. The algebra \( U \) inherits the grading of \( U_q^+ \) via \( \natural \). With respect to this grading, for \( n \in \mathbb{N} \) the \( n \)-homogeneous component of \( U \) is the \( \natural \)-image of the \( n \)-homogeneous component of \( U_q^+ \). This homogeneous component is equal to \( \mathcal{V}_n \cap U \).

The following result is a variation on [19, Theorem 5].

**Proposition 6.2.** (See [22, Lemma 6.5].) The ideal \( J \) from Definition 5.3 and the subalgebra \( U \) from Definition 6.1 are orthogonal complements with respect to the bilinear form \( \langle \cdot, \cdot \rangle \).

In [27, Theorem 1.7] we applied the map \( \natural \) to each element in the Damiani PBW basis for \( U_q^+ \), and expressed the image in the standard basis for \( \mathcal{V} \). We will review this result in Proposition 6.7 below. In order to prepare for Proposition 6.7 we make some comments.

**Definition 6.3.** Define \( \overline{e} = 1 \) and \( \overline{f} = -1 \). A word \( u_1 u_2 \cdots u_n \) in \( \mathcal{V} \) is said to be Catalan whenever \( \overline{u}_1 + \overline{u}_2 + \cdots + \overline{u}_i \) is nonnegative for \( 1 \leq i \leq n - 1 \) and zero for \( i = n \). In this case \( n \) is even.

**Example 6.4.** For \( 0 \leq n \leq 3 \) we display the Catalan words of length \( 2n \).

| \( n \) | Catalan words of length \( 2n \) |
|---|---|
| 0 | 1 |
| 1 | \( xy \) |
| 2 | \( xxxyy \) |
| 3 | \( xxyxyy, xyyxy, xyxyy, xxxyyy, xxyyy, xxyy \) |

**Definition 6.5.** (See [27, Definition 1.5].) For \( n \in \mathbb{N} \) define

\[
C_n = \sum u_1 u_2 \cdots u_n [1]_q \overline{u}_1 [1 + \overline{u}_1]_q [1 + \overline{u}_1 + \overline{u}_2]_q \cdots [1 + \overline{u}_1 + \overline{u}_2 + \cdots + \overline{u}_{2n}]_q, \tag{15}
\]

where the sum is over all the Catalan words \( u_1 u_2 \cdots u_n \) in \( \mathcal{V} \) that have length \( 2n \). We call \( C_n \) the \( n \)-th Catalan element in \( \mathcal{V} \). Note that \( C_n \in \mathcal{V}_{2n} \).

**Example 6.6.** We have

\[
C_0 = 1, \quad C_1 = [2]_q xy, \quad C_2 = [2]_q^2 xxyy + [3]_q^2 [2]_q^2 xyy, \quad C_3 = [2]_q^3 xxyxy + [3]_q^2 [2]_q^3 xyyxy + [3]_q^2 [2]_q^3 xxyyy + [4]_q [3]_q^2 [2]_q^3 xxyyy.
\]

**Proposition 6.7.** (See [27, Theorem 1.7].) The map \( \natural \) sends

\[
E_{n \delta + \alpha_0} \mapsto q^{-2n} (q - q^{-1})^{2n} x C_n, \quad E_{n \delta + \alpha_1} \mapsto q^{-2n} (q - q^{-1})^{2n} C_n y \tag{16}
\]

for \( n \geq 0 \), and

\[
E_{n \delta} \mapsto -q^{-2n} (q - q^{-1})^{2n-1} C_n \tag{17}
\]

for \( n \geq 1 \).
We emphasize that in (16), the notations $xC_n$ and $C_ny$ refer to the free product.

We mention three consequences of Proposition 6.7

**Corollary 6.8.** A PBW basis for $U$ is obtained by the elements

$$\{xC_n\}_{n=0}^{\infty}, \quad \{C_ny\}_{n=0}^{\infty}, \quad \{C_n\}_{n=1}^{\infty}$$

in the linear order

$$x < xC_1 < xC_2 < \cdots < C_1 < C_2 < C_3 < \cdots < C_2y < C_1y < y.$$ 

**Proof.** By Propositions 4.1, 6.7

**Corollary 6.9.** (See [27, Corollary 1.8].) For $i, j \in \mathbb{N}$,

$$C_i \star C_j = C_j \star C_i.$$  

(18)

**Proof.** By Lemma 4.3 and Proposition 6.7

**Corollary 6.10.** (See [27, Corollary 3.6].) For $i, j \in \mathbb{N}$,

$$q^{-1}C_{i+j+1} = q \frac{(xC_i) \star (C_jy) - q^{-1}(C_jy) \star (xC_i)}{q - q^{-1}}.$$  

(19)

**Proof.** By Lemma 4.4 and Proposition 6.7

We just displayed some relations involving the Catalan elements. Additional relations involving the Catalan elements can be found in [27, Section 3].

### 7 The main result

In this section we prove our main result, which is Theorem 7.1. Recall the map $\natural$ from below Definition 6.1

**Theorem 7.1.** The map $\natural$ sends

$$E_{n\delta}^{\text{Beck}} \mapsto \frac{[2n]_q}{n} q^{-2n} (q - q^{-1})^{2n-1} xC_{n-1}y$$  

(20)

for $n \geq 1$.

We emphasize that in (20) the notation $xC_{n-1}y$ refers to the free product. This notation is illustrated in Example 11.1

We will prove Theorem 7.1 after two preliminary lemmas.

**Lemma 7.2.** For $k \in \mathbb{N}$ we have $xC_ky \in U$. 

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Proof. By Proposition 6.2, it suffices to show that \( xC_ky \) is orthogonal to everything in \( J \). By Definition 5.3 and the construction, the vector space \( J \) is spanned by the elements of the form \( w_1J^\pm w_2 \), where \( w_1, w_2 \) are words in \( V \). Let \( w_1, w_2 \) denote words in \( V \). We will show that

\[
\langle xC_ky, w_1J^+w_2 \rangle = 0, \quad \langle xC_ky, w_1J^-w_2 \rangle = 0. \tag{21}
\]

Observe that \( xC_ky \in \mathbb{V}_{2k+2} \) and \( J^\pm \in \mathbb{V}_4 \). We may assume that \( k \geq 1 \) and \( \text{length}(w_1) + \text{length}(w_2) = 2k - 2 \); otherwise \( xC_ky \) and \( w_1J^\pm w_2 \) are in different homogeneous components of \( \mathbb{V} \), in which case (21) holds. We now investigate four cases. First assume that \( w_1 \) and \( w_2 \) are trivial. We have \( k = 1 \). We have \( C_1 = [2]_qxy \) by Example 6.6 and \( \langle xxyy, J^\pm \rangle = 0 \) by Definition 5.2. By these comments (21) holds. Next assume that \( w_1 \) is trivial and \( w_2 \) is nontrivial. We have \( k \geq 2 \). By Definition 6.5 \( C_k \) is a linear combination of the Catalan words in \( V \) that have length \( 2k \). For such a word \( w \) its first three letters form one of the words \( xxx \) or \( xxy \) or \( xyx \). By this and the coefficient formula (15), we obtain

\[
C_k = xxxR_1 + ([3]_q xxy + xyx)R_2, \quad R_1, R_2 \in \mathbb{V}_{2k-3}.
\]

By Definition 5.2 we obtain

\[
\langle xxx, J^+ \rangle = 0, \quad \langle xxy, J^+ \rangle = 1, \quad \langle xyx, J^+ \rangle = -[3]_q, \quad \langle xxx, J^- \rangle = 0, \quad \langle xxy, J^- \rangle = 0, \quad \langle xyx, J^- \rangle = 0.
\]

By this and Lemma 5.1

\[
\langle xC_ky, w_1J^\pm w_2 \rangle = \langle xC_ky, J^\pm w_2 \rangle = \langle xxxR_1y + ([3]_q xxy + xyx)R_2y, J^\pm w_2 \rangle = \langle xxxR_1y, J^\pm w_2 \rangle + \langle [3]_q xxy + xyxR_2y, J^\pm w_2 \rangle = \langle xxx, J^\pm \rangle \langle R_1y, w_2 \rangle + \langle [3]_q xxy + xyx, J^\pm \rangle \langle R_2y, w_2 \rangle = 0.
\]

We have established (21) for this case. Next assume that \( w_1 \) is nontrivial and \( w_2 \) is trivial. We have \( k \geq 2 \). Adjusting the argument of the previous case, we obtain

\[
C_k = L_1 yyyy + L_2 ([3]_q xyy + xyx), \quad L_1, L_2 \in \mathbb{V}_{2k-3}.
\]

By Definition 5.2 we obtain

\[
\langle yyyy, J^+ \rangle = 0, \quad \langle xyy, J^+ \rangle = 0, \quad \langle xyy, J^+ \rangle = 0, \quad \langle yyyy, J^- \rangle = 0, \quad \langle xyy, J^- \rangle = 0, \quad \langle xyy, J^- \rangle = 0.
\]

By this and Lemma 5.1

\[
\langle xC_ky, w_1J^\pm w_2 \rangle = \langle xC_ky, w_1J^\pm \rangle = \langle xL_1 yyyy + xL_2 ([3]_q xyy + xyx), w_1J^\pm \rangle = \langle xL_1 yyyy, w_1J^\pm \rangle + \langle xL_2 ([3]_q xyy + xyx), w_1J^\pm \rangle = \langle xL_1, w_1 \rangle \langle yyyy, J^\pm \rangle + \langle xL_2, w_1 \rangle \langle [3]_q xyy + xyx, J^\pm \rangle = 0.
\]
We have established (21) for this case. Next assume that each of \( w_1, w_2 \) is nontrivial. There exist letters \( a, b \) and words \( w'_1, w'_2 \) such that \( w_1 = aw'_1 \) and \( w_2 = w'_2b \). We have \( C_k \in U \) and \( w'_1J^\pm w'_2 \in J \) and \( \langle U, J \rangle = 0 \), so

\[
\langle C_k, w'_1J^\pm w'_2 \rangle = 0.
\]

By this and Lemma 5.1,

\[
\langle xC_ky, w_1J^\pm w_2 \rangle = \langle xC_ky, aw'_1J^\pm w'_2b \rangle = \langle x, a \rangle \langle C_k, w'_1J^\pm w'_2 \rangle \langle y, b \rangle = 0.
\]

We have established (21) for this case. The condition (21) holds in all four cases, and the result follows.

**Lemma 7.3.** For \( k \in \mathbb{N} \) we have

\[
xC_{k+1} = \frac{x \ast (xC_k y) - (xC_k y) \ast x}{q - q^{-1}}, \tag{22}
\]

\[
C_{k+1} y = \frac{(xC_k y) \ast y - y \ast (xC_k y)}{q - q^{-1}}. \tag{23}
\]

**Proof.** We first verify (22). Consider the left-hand side of (22). Setting \( i = 0 \) and \( j = k \) in (19), we obtain

\[
C_{k+1} = q^2 x \ast (C_k y) - (C_k y) \ast x
\]

Therefore

\[
xC_{k+1} = \frac{q^2 x \ast (C_k y) - (C_k y) \ast x}{q - q^{-1}}. \tag{24}
\]

Now consider the right-hand side of (22). Using (11) we obtain \( x \ast (xC_k y) = x((xC_k y) + q^2 x \ast (C_k y)) \) and \( (xC_k y) \ast x = x((C_k y) \ast x + xC_k y) \). By these comments, the right-hand side of (22) is equal to the right-hand side of (24). We have verified (22). The equation (23) is verified in a similar way.

**Proof of Theorem 7.1** Define \( C_n \in U \) such that the map \( \sharp \) sends

\[
E_{n, \delta}^{\text{Beck}} \mapsto \frac{2n}{n} q^{-2n} (q - q^{-1})^{2n-1} C_n. \tag{25}
\]

We show that \( C_n = xC_{n-1}y \). For (9) and (10), apply \( \sharp \) to each side and evaluate the result using (16), (25). The result is

\[
xC_{k+\ell} = \frac{(xC_{\ell} \ast C_k - C_k \ast (xC_{\ell}))}{q - q^{-1}}, \quad C_{k+\ell} y = \frac{C_k \ast (C_{\ell} y) - (C_{\ell} y) \ast C_k}{q - q^{-1}}. \tag{26}
\]

Setting \( \ell = 0 \) and \( k = n \) in (26), we obtain

\[
xC_n = \frac{x \ast C_n - C_n \ast x}{q - q^{-1}}, \quad C_n y = \frac{C_n \ast y - y \ast C_n}{q - q^{-1}}. \tag{27}
\]

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Setting \( k = n - 1 \) in Lemma 7.3, we obtain

\[
xC_n = \frac{x \ast (xC_{n-1}y) - (xC_{n-1}y) \ast x}{q - q^{-1}}, \quad C_n y = \frac{(xC_{n-1}y) \ast y - y \ast (xC_{n-1}y)}{q - q^{-1}}.
\] (28)

Consider the element \( C_n - xC_{n-1}y \). By Lemma 7.2 and the construction, \( C_n - xC_{n-1}y \in U \). By the equations on the left in (27) and (28), \( C_n - xC_{n-1}y \) commutes with \( x \) with respect to \( \ast \). By the equations on the right in (27) and (28), \( C_n - xC_{n-1}y \) commutes with \( y \) with respect to \( \ast \). By these comments \( C_n - xC_{n-1}y \) is contained in the center of \( U \). The algebra \( U \) is isomorphic to \( U^+_q \), so by [29, Lemma 6.1] the center of \( U \) is equal to \( F_1 \). Therefore, there exists \( \alpha_n \in F \) such that \( C_n - xC_{n-1}y = \alpha_n 1 \). We show that \( \alpha_n = 0 \). By Lemma 4.8 and (25) along with our comments above Proposition 6.2, we obtain \( C_n \in V_{2n} \). We have \( C_n - 1 \in V_{2n-2} \) so \( xC_n - 1 \in V_{2n} \). By these comments \( \alpha_n 1 = C_n - xC_{n-1}y \in V_{2n} \). However \( \alpha_n 1 \in V_0 \) and \( V_0 \cap V_{2n} = 0 \) since \( n \geq 1 \), so \( \alpha_n 1 = 0 \). Therefore \( \alpha_n = 0 \). We have shown that \( C_n = xC_{n-1}y \), as desired.

\[ \blacksquare \]

8 Some consequences of the main result

In this section we give some consequences of our main result Theorem 7.1.

**Corollary 8.1.** The following holds in the \( q \)-shuffle algebra \( V \):

\[
\exp \left( \sum_{k=1}^{\infty} \frac{[2k]_q}{k} xC_{k-1}yt^k \right) = 1 + \sum_{k=1}^{\infty} C_k t^k.
\] (29)

We emphasize that in (29) the exponential function is with respect to the \( q \)-shuffle product, and the notation \( xC_{k-1}y \) refers to the free product.

**Proof.** Apply the map \( \natural \) to each side of (8), and evaluate the result using (17), (20). This yields (29) after a change of variables in which \( t \) is replaced by \( q^2(q - q^{-1})^{-2}t \). \[ \blacksquare \]

Using (29) the elements \( \{xC_n y\}_{n=0}^{\infty} \) and the elements \( \{C_n\}_{n=1}^{\infty} \) can be recursively obtained from each other. This is illustrated in Example 11.2.

**Corollary 8.2.** For \( n \geq 1 \) the following hold in the \( q \)-shuffle algebra \( V \):

(i) \( C_n \) is a homogeneous polynomial in \( xC_0y, xC_1y, \ldots, xC_{n-1}y \) that has total degree \( n \), where we view \( xC_{k-1}y \) as having degree \( k \) for \( 1 \leq k \leq n \);

(ii) \( xC_{n-1}y \) is a homogeneous polynomial in \( C_1, C_2, \ldots, C_n \) that has total degree \( n \), where we view \( C_k \) as having degree \( k \) for \( 1 \leq k \leq n \).

We emphasize that the above homogeneous polynomials are with respect to the \( q \)-shuffle product.

**Proof.** Use (29) and induction on \( n \). \[ \blacksquare \]
Corollary 8.3. A PBW basis for $U$ is obtained by the elements

$$\{xC_n\}_{n \in \mathbb{N}}, \quad \{C_ny\}_{n \in \mathbb{N}}, \quad \{xC_ny\}_{n \in \mathbb{N}}$$

in the linear order

$$x < xC_1 < xC_2 < \cdots < xy < xC_1y < xC_2y < \cdots < C_2y < C_1y < y.$$

Proof. Apply the map $\natural$ to everything in Proposition 4.9 and evaluate the result using (16), (20).

Corollary 8.4. The elements $\{xC_ny\}_{n \in \mathbb{N}}$ mutually commute with respect to the $q$-shuffle product.

Proof. By Lemma 4.10 and (20).

Corollary 8.5. For $k, \ell \in \mathbb{N}$,

$$xC_{k+\ell+1} = \frac{(xC_\ell) \star (xC_ky) - (xC_ky) \star (xC_\ell)}{q - q^{-1}},$$

(30)

$$C_{k+\ell+1}y = \frac{(xC_ky) \star (C_\ell y) - (C_\ell y) \star (xC_ky)}{q - q^{-1}}.$$  (31)

Proof. For (9) and (10), apply the map $\natural$ to each side, and evaluate the results using (17), (20). This yields (30), (31) after a change of variables in which $k$ is replaced by $k + 1$.

9 The alternating words and the Catalan elements

In [28] we introduced the alternating PBW basis for $U_+^q$. In this section we discuss how certain elements of this PBW basis are related to $\{C_n\}_{n \in \mathbb{N}}$ and $\{xC_ny\}_{n \in \mathbb{N}}$.

We recall the alternating words in $V$.

Definition 9.1. (See [28, Definition 5.1].) A word $u_1u_2 \cdots u_n$ in $V$ is called alternating whenever $n \geq 1$ and $u_{i-1} \neq u_i$ for $2 \leq i \leq n$. Thus an alternating word has the form $\cdots xyxy \cdots$.

Definition 9.2. (See [28, Definition 5.2].) We name the alternating words as follows:

$$W_0 = x, \quad W_{-1} = xyx, \quad W_{-2} = xyxyx, \quad \cdots$$

$$W_1 = y, \quad W_2 = yxy, \quad W_3 = yxyxy, \quad \cdots$$

$$G_1 = yx, \quad G_2 = yxyx, \quad G_3 = yxyxyx, \quad \cdots$$

$$\tilde{G}_1 = xy, \quad \tilde{G}_2 = xyxy, \quad \tilde{G}_3 = xyxyxy, \quad \cdots$$

For notational convenience, define $G_0 = 1$ and $\tilde{G}_0 = 1$. 

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In [28, Propositions 5.7, 5.10, 5.11, 6.3, 8.1] we displayed many relations involving the alternating words. These relations show how the alternating words are related to each other, with respect to the \( q \)-shuffle product. Using these relations and referring to the \( q \)-shuffle product, in [28, Theorem 10.1] we recursively obtained each alternating word as a polynomial in \( x, y \). This result has the following consequence.

**Lemma 9.3.** (See [28, Theorem 8.3].) Each alternating word in \( V \) is contained in \( U \).

Next we describe the alternating PBW basis for \( U \).

**Proposition 9.4.** (See [28, Theorem 10.1].) A PBW basis for \( U \) is obtained by the elements

\[
\{W_i\}_{i \in \mathbb{N}}, \quad \{\tilde{G}_{j+1}\}_{j \in \mathbb{N}}, \quad \{W_{k+1}\}_{k \in \mathbb{N}}
\]

in any linear order \( < \) that satisfies

\[
W_i < \tilde{G}_{j+1} < W_{k+1} \quad i, j, k \in \mathbb{N}.
\]

For the rest of this section, we focus on the elements \( \{\tilde{G}_n\}_{n \in \mathbb{N}} \). In [28, Section 11] we described how the elements \( \{C_n\}_{n \in \mathbb{N}} \) are related to the elements \( \{\tilde{G}_n\}_{n \in \mathbb{N}} \). We will review this description, and then describe how the elements \( \{xC_ny\}_{n \in \mathbb{N}} \) are related to the elements \( \{\tilde{G}_n\}_{n \in \mathbb{N}} \). Our main result on this topic is Proposition 9.11 below.

**Lemma 9.5.** (See [28, Proposition 5.10].) The elements \( \{\tilde{G}_n\}_{n \in \mathbb{N}} \) mutually commute with respect to the \( q \)-shuffle product.

**Definition 9.6.** We define some generating functions in the indeterminate \( t \):

\[
C(t) = \sum_{n \in \mathbb{N}} C_n t^n, \quad \tilde{G}(t) = \sum_{n \in \mathbb{N}} \tilde{G}_n t^n.
\]

**Proposition 9.7.** (See [28, Lemma 9.12 and Proposition 11.8].) We have

\[
\tilde{G}(qt) * C(-t) * \tilde{G}(q^{-1}t) = 1.
\]

**Corollary 9.8.** (See [28, Theorem 11.14].) For \( n \in \mathbb{N} \),

\[
0 = \sum_{i=0}^{n} (-1)^i [2n - i]_q C_i \ast \tilde{G}_{n-i}.
\]

The following result is obtained by rearranging the terms in (33).

**Corollary 9.9.** For \( n \geq 1 \),

\[
C_n = \frac{-1}{[n]_q} \sum_{i=0}^{n-1} (-1)^{n-i} [2n - i]_q C_i \ast \tilde{G}_{n-i}.
\]

\[
\tilde{G}_n = \frac{-1}{[2n]_q} \sum_{i=1}^{n} (-1)^i [2n - i]_q C_i \ast \tilde{G}_{n-i}.
\]
Using Corollary 9.9 the elements \( \{C_n\}_{n=1}^{\infty} \) and the elements \( \{\tilde{G}_n\}_{n=1}^{\infty} \) can be recursively obtained from each other. This is illustrated in Example 11.3.

**Corollary 9.10.** (See [28, Corollary 11.11].) For \( n \geq 1 \) the following hold in the \( q \)-shuffle algebra \( V \).

(i) \( C_n \) is a homogeneous polynomial in \( \tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n \) that has total degree \( n \), where we view \( \tilde{G}_k \) as having degree \( k \) for \( 1 \leq k \leq n \);

(ii) \( \tilde{G}_n \) is a homogeneous polynomial in \( C_1, C_2, \ldots, C_n \) that has total degree \( n \), where we view \( C_k \) as having degree \( k \) for \( 1 \leq k \leq n \).

We emphasize that the above homogeneous polynomials are with respect to the \( q \)-shuffle product.

Next we describe how the elements \( \{xC_ny\}_{n \in \mathbb{N}} \) are related to the elements \( \{\tilde{G}_n\}_{n \in \mathbb{N}} \). Our description gives a variation on a formula in [1, Proposition 5.27] involving a certain generating function \( g_+(u) \) that corresponds to \( \tilde{G}(t) \).

**Proposition 9.11.** The following holds in the \( q \)-shuffle algebra \( V \):

\[
\exp \left( -\sum_{k=1}^{\infty} \frac{(-1)^k[k]}{k} xC_{k-1}yt^k \right) = 1 + \sum_{k=1}^{\infty} \tilde{G}_kt^k. \tag{34}
\]

We emphasize that in (34) the exponential function is with respect to the \( q \)-shuffle product, and the notation \( xC_{k-1}y \) refers to the free product.

**Proof.** Recall that \( \tilde{G}_0 = 1 \). Define the generating function \( \tilde{g}(t) = \sum_{k=1}^{\infty} \tilde{G}_kt^k \) and note that \( \tilde{G}(t) = 1 + \tilde{g}(t) \). We will be discussing the natural logarithm \( \ln \) with respect to \( \star \). Define \( h(t) = \ln(\tilde{G}(t)) \), and note that

\[
h(t) = \ln(1 + \tilde{g}(t)) = \tilde{g}(t) - \frac{\tilde{g}(t) \star \tilde{g}(t)}{2} + \frac{\tilde{g}(t) \star \tilde{g}(t) \star \tilde{g}(t)}{3} - \ldots
\]

We have \( \exp h(t) = \tilde{G}(t) \). To establish (34), it suffices to show that

\[
h(t) = -\sum_{k=1}^{\infty} \frac{(-1)^k[k]}{k} xC_{k-1}yt^k. \tag{35}
\]

By construction, \( h(t) \) has constant coefficient 0. Write \( h(t) = \sum_{k=1}^{\infty} h_k t^k \) with \( h_k \in U \) for \( k \geq 1 \). Applying \( \ln \) to each side of (32), we obtain

\[
\ln(\tilde{G}(qt)) + \ln(C(-t)) + \ln(\tilde{G}(q^{-1}t)) = 0. \tag{36}
\]

By construction

\[
\ln(\tilde{G}(qt)) = h(qt) = \sum_{k=1}^{\infty} h_k q^k t^k, \tag{37}
\]

\[
\ln(\tilde{G}(q^{-1}t)) = h(q^{-1}t) = \sum_{k=1}^{\infty} h_k q^{-k} t^k. \tag{38}
\]

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By Corollary 8.1,
\[
\ln(C(-t)) = \sum_{k=1}^{\infty} \frac{(-1)^k[2k]_q}{k} xC_{k-1}y t^k. 
\] (39)

Evaluating (36) using (37)–(39), we obtain

\[
h_k(q^k + q^{-k}) + \frac{(-1)^k[2k]_q}{k} xC_{k-1}y = 0, \quad k \geq 1. 
\] (40)

By (40) and \([2k]_q = [k]_q(q^k + q^{-k})\),

\[
h_k = -\frac{(-1)^k[k]_q}{k} xC_{k-1}y, \quad k \geq 1. 
\]

This implies (35), and the result follows.

Using (34) the elements \(\{xC_ny\}_{n=0}^{\infty}\) and the elements \(\{\tilde{G}_n\}_{n=1}^{\infty}\) can be recursively obtained from each other. This is illustrated in Example 11.4.

**Corollary 9.12.** For \(n \geq 1\) the following hold in the \(q\)-shuffle algebra \(V\).

(i) \(xC_{n-1}y\) is a homogeneous polynomial in \(\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_n\) that has total degree \(n\), where we view \(\tilde{G}_k\) as having degree \(k\) for \(1 \leq k \leq n\);

(ii) \(\tilde{G}_n\) is a homogeneous polynomial in \(xC_0y, xC_1y, \ldots, xC_{n-1}y\) that has total degree \(n\), where we view \(xC_{k-1}y\) as having degree \(k\) for \(1 \leq k \leq n\).

We emphasize that the above homogeneous polynomials are with respect to the \(q\)-shuffle product.

**Proof.** Use (34) and induction on \(n\). \(\square\)

**Corollary 9.13.** The following (i)–(iii) coincide:

(i) the subalgebra of the \(q\)-shuffle algebra \(V\) generated by \(\{C_n\}_{n=1}^{\infty}\);

(ii) the subalgebra of the \(q\)-shuffle algebra \(V\) generated by \(\{xC_ny\}_{n=0}^{\infty}\);

(iii) the subalgebra of the \(q\)-shuffle algebra \(V\) generated by \(\{\tilde{G}_n\}_{n=1}^{\infty}\).

**Proof.** By Corollaries 8.2, 9.10, 9.12 \(\square\)

For more information about the alternating words and related topics, see [1, 3, 4, 29–31].

10 Acknowledgement

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11 Appendix A

Recall the Catalan elements \( \{C_n\}_{n \in \mathbb{N}} \) from Definition 6.5 and the alternating elements \( \{\tilde{G}_n\}_{n \in \mathbb{N}} \) from Definition 9.2. Recall that \( C_0 = 1 \) and \( \tilde{G}_0 = 1 \). In this appendix we compare \( \{xC_ny\}_{n=0}^3 \) and \( \{C_n\}_{n=1}^4 \) and \( \{\tilde{G}_n\}_{n=1}^4 \).

First we display \( xC_ny \) for \( 0 \leq n \leq 3 \).

Example 11.1. We have

\[
xC_0y = xy, \quad xC_1y = [2]_q xyy, \quad xC_2y = [2]^2_q xxyyy + [3]_q [2]_q^2 xxyyy, \\
xC_3y = [2]^3_q xxyxyyy + [3]_q [2]_q^3 xxyxyyy + [3]_q [2]_q^2 xxyxyyy + [4]_q [3]_q [2]_q^2 xxyxyyy.
\]

Next we compare \( \{xC_ny\}_{n=0}^3 \) and \( \{C_n\}_{n=1}^4 \).

Example 11.2. We have

\[
xC_0y = \frac{C_1}{[2]_q}, \quad xC_1y = \frac{2C_2 - C_1 \ast C_1}{[4]_q}, \\
xC_2y = \frac{3C_3 - 3C_2 \ast C_1 + C_1 \ast C_1 \ast C_1}{[6]_q}, \\
xC_3y = \frac{4C_4 - 4C_3 \ast C_1 - 2C_2 \ast C_2 + 4C_2 \ast C_1 \ast C_1 - C_1 \ast C_1 \ast C_1 \ast C_1}{[8]_q}.
\]

Moreover

\[
C_1 = [2]_q xC_0y, \quad C_2 = \frac{[4]_q xC_1y + [2]_q^2 (xC_0y) \ast (xC_0y)}{2}, \\
C_3 = \frac{2[6]_q xC_2y + 3[2]_q [4]_q (xC_1y) \ast (xC_0y) + [2]_q^3 (xC_0y) \ast (xC_0y) \ast (xC_0y)}{6}, \\
C_4 = \frac{6[8]_q xC_3y + 8[6]_q [2]_q (xC_2y) \ast (xC_0y) + 3[4]_q^2 (xC_1y) \ast (xC_1y)}{24} \\
+ \frac{6[4]_q [2]_q^2 (xC_1y) \ast (xC_0y) \ast (xC_0y) + [2]_q^4 (xC_0y) \ast (xC_0y) \ast (xC_0y) \ast (xC_0y)}{24}.
\]

Next we compare \( \{C_n\}_{n=1}^4 \) and \( \{\tilde{G}_n\}_{n=1}^4 \).

Example 11.3. We have

\[
C_1 = [2]_q \tilde{G}_1, \quad C_2 = \frac{[2]_q [3]_q \tilde{G}_1 \ast \tilde{G}_1 - [4]_q \tilde{G}_2}{[2]_q}, \\
C_3 = \frac{[2]_q [6]_q \tilde{G}_3 - ([4]_q^2 + [2]_q^2 [5]_q) \tilde{G}_2 \ast \tilde{G}_1 + [2]_q [3]_q [4]_q \tilde{G}_1 \ast \tilde{G}_1 \ast \tilde{G}_1}{[2]_q [3]_q}, \\
C_4 = [2]_q^{-1} [3]_q^{-1} [4]_q^{-1} \text{ times a weighted sum with the following terms and coefficients.}
\]
Moreover

\[
\tilde{G}_1 = \frac{C_1}{[2]_q}, \quad \tilde{G}_2 = \frac{[3]_q C_1 \cdot C_1 - [2]_q^2 C_2}{[2]_q [4]_q},
\]

\[
\tilde{G}_3 = \frac{[2]_q [3]_q [4]_q C_3 - ([4]_q^2 + [2]_q [5]_q) C_2 \cdot C_1 + [3]_q [5]_q C_1 \cdot C_1 \cdot C_1}{[2]_q [4]_q [6]_q},
\]

\[
\tilde{G}_4 = [2]_q^{-1}[4]_q^{-1}[6]_q^{-1}[8]_q^{-1}
\]
times a weighted sum with the following terms and coefficients:

\[
\begin{array}{c|c}
\text{term} & \text{coefficient} \\
\hline
G_4 & -[2]_q [3]_q [8]_q \\
\tilde{G}_3 \ast \tilde{G}_1 & [2]_q^2 [3]_q [7]_q + [2]_q [5]_q [6]_q \\
\tilde{G}_2 \ast \tilde{G}_2 & [3]_q [4]_q [6]_q \\
\tilde{G}_2 \ast \tilde{G}_1 \ast \tilde{G}_1 & -[2]_q [3]_q [6]_q - [2]_q [5]_q^2 - [4]_q^2 [5]_q \\
\tilde{G}_1 \ast \tilde{G}_1 \ast \tilde{G}_1 \ast \tilde{G}_1 & [2]_q [3]_q [4]_q [5]_q \\
\end{array}
\]

Next we compare \( \{xC_n y\}_n^3 \) and \( \{\tilde{G}_n\}_n^4 \).

**Example 11.4.** We have

\[
x C_0 y = \tilde{G}_1, \quad x C_1 y = \frac{\tilde{G}_1 \ast \tilde{G}_1 - 2 \tilde{G}_2}{[2]_q},
\]

\[
x C_2 y = \frac{\tilde{G}_1 \ast \tilde{G}_1 \ast \tilde{G}_1 - 3 \tilde{G}_1 \ast \tilde{G}_2 + 3 \tilde{G}_3}{[3]_q},
\]

\[
x C_3 y = \frac{\tilde{G}_1 \ast \tilde{G}_1 \ast \tilde{G}_1 \ast \tilde{G}_1 - 4 \tilde{G}_1 \ast \tilde{G}_1 \ast \tilde{G}_2 + 2 \tilde{G}_2 \ast \tilde{G}_2 + 4 \tilde{G}_1 \ast \tilde{G}_3 - 4 \tilde{G}_4}{[4]_q}.
\]

Moreover

\[
\tilde{G}_1 = x C_0 y, \quad \tilde{G}_2 = \frac{(x C_0 y) \ast (x C_0 y) - [2]_q x C_1 y}{2},
\]

\[
\tilde{G}_3 = \frac{(x C_0 y) \ast (x C_0 y) \ast (x C_0 y) - 3 [2]_q (x C_0 y) \ast (x C_1 y) + 2 [3]_q x C_2 y}{6},
\]

\[
\tilde{G}_4 = \frac{(x C_0 y) \ast (x C_0 y) \ast (x C_0 y) \ast (x C_0 y) - 6 [2]_q (x C_0 y) \ast (x C_0 y) \ast (x C_0 y) \ast (x C_1 y)}{24} + \frac{3 [2]_q^2 (x C_1 y) \ast (x C_1 y) + 8 [3]_q (x C_0 y) \ast (x C_2 y) - 6 [4]_q x C_3 y}{24}.
\]
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