About the problem of flow past around thin profiles

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Abstract. In this paper some approaches to solving the problems of flow past around thin profiles are presented. We consider the problem of flow past around a contour defined by a set of segments, each of which is close to the segment (−a, a). A formula that gives an approximate velocities distribution in a flow flowing around a system of thin profiles is obtained. A mixed boundary-value problem for an analytic function in the upper half-plane is studied. A system of singular integral equations having a unique solution up to two arbitrary constants in the class of functions that are boundary values of analytic functions in the upper half-plane is obtained. Considered the generalized problem of flow past around thin profiles for infinite numbers of thin profiles.

1. Introduction
The mixed boundary value problems play an important role in mathematics and its applications. Numerous works are devoted to the study of these problems. So in the papers by P. N. Ivanshin, E. A. Shirokova [1] and E. A. Shirokova, A. El-Shenawy [2], the Cauchy integral method for the Laplace equation in multiply connected domains is considered. The article by Gary M. Lieberman [3] is concerned with the solution of a mixed boundary value problem for elliptic and parabolic differential equations of the second order. Boundary value problems are closely related to interpolation problems. Interpolation problems in spaces of finite order functions in the upper half-plane were considered in papers by K. G. Malyutin and A. L. Gusev [4, 5], O. A. Bozhenko and K. G. Malyutin [6]. We also note the monograph by N. V. Govorov [7] which considers boundary value problems with infinite index.

The following mixed boundary value problem is important in applications.

Let G be a simply connected region and ∂D be the boundary of D. Let points a₁, b₁, a₂, b₂, …, aₙ, bₙ belong ∂D. It is required to obtain a function f(z) analytic on the region D such that ℜf accepts given values on arcs a_kb_k and ℑf accepts given values on arcs b_ka_{k+1} (k = 1, 2, …, n; a_{n+1} = a₁).

In 1937, M. V. Keldysh and L. I. Sedov [8] gave a full investigation of this problem. They proved that, generally speaking, this problem has no solutions bounded near the ends of arcs a_k and b_k. If to refuse the condition of boundedness f(z) and require only the boundedness of the integral of f(z), then the problem has a solution with n + 1 an arbitrary constant. M. V. Keldysh and L. I. Sedov also proved that the problem has a unique solution if to require that f(z) is bounded near any n of the ends and has a fixed value of f(z) at some point of the boundary.

In this paper, we consider the last case. Assume that the region D is the upper half-plane C₊ = {z : ℑz > 0}. Any case can be reduced to this one using a conformal mapping. So, let
the points
\[-\infty < a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n < +\infty\]
are given on the real axis $x$. Let the functions
\[u : (a_k, b_k) \to (-\infty, +\infty), \quad k = 1, 2, \ldots, n,\]
\[v : (b_k, a_{k+1}) \to (-\infty, +\infty), \quad k = 1, 2, \ldots, n - 1,\]
have a finite number of discontinuity points of the first kind. It is required to obtain a function $f(z)$ analytic on the half-plane $C_+$ such that
\[\Re \lim_{y \to +0} f(x + iy) = u(x), \quad x \in (a_k, b_k),\]
\[\Im \lim_{y \to +0} f(x + iy) = v(x), \quad x \in (b_k, a_{k+1}).\]

The following theorem is true.

**Theorem (Keldysh, Sedov.)** The mixed boundary value problem for the upper half-plane $C_+$ has a unique solution $f(z)$ such that
1) the function $f(z)$ is bounded near all points $a_k$;
2) the integral $\int f(z) \, dz$ is bounded near all points $b_k$;
3) there exists a finite limit $\lim_{z \to \infty} f(z) = f(\infty)$ such that $f(\infty)$ is real number.

2. Solution of the problem of flow past around thin profiles
As was shown in the book [9, Chapter III, § 3], the problem of flow around an arbitrary profile is reduced to the problem of conformally mapping the exterior of this profile to the exterior of a circle. However, the actual construction of such a conformal mapping is often difficult and therefore one has to confine oneself to approximate solutions to the problem. In this paper, we consider the problem of flow around a contour defined by a set of segments, each of which is close to the segment $[-a, a]$. Thus, the aim of our work is to study the problem in the case when the contour $\tilde{N}$ is determined by a finite number of contours

\[C = \bigcup_{j=1}^{n} C_j.\]

Let us consider the solution of the problem of flow around thin profiles by the method of L. I. Sedov. Suppose that the contours of the profiles $C_j$, $j = 1, \ldots, n$, are determined by the equations
\[y = F_j^\pm(x), \quad a_j \leq x \leq b_j, \quad j = 1, \ldots, n,\]
and are close to segments
\[[a_j, b_j], \quad a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n, \quad j = 1, \ldots, n.\]

Let the profiles are flowed around with a translational flow which has the velocity $v_\infty$ at infinity. Let the velocity $v_\infty$ tilted to the segment $a_j, b_j$ under small angles of attack $\alpha_j$, $j = 1, \ldots, n$. In accordance with this, we will search the complex potential of the flow in the form
\[w = v_\infty e^{i\alpha} z + W\]
where $W = U + iV$ is an unknown function. We have
\[\Im w = v_\infty (y \cos \alpha - x \sin \alpha) + V(x, y).\]
Since $C_j, j = 1, \ldots, n,$ coincide with the segments, then $\Im w$ should take constant values on them, equal to zero for definiteness. Therefore on $C_j, j = 1, \ldots, n,$

$$\Im w = v_\infty (F_\pm^j (x) \cos \alpha - x \sin \alpha) + V(x, F_\pm^j (x)).$$

Using the assumptions made about the proximity of $C_j, j = 1, \ldots, n,$ to the segments $[a_j, b_j]$ and small angles $\alpha_j,$ we equivalently replace under this condition $\cos \alpha_j$ by 1, $\sin \alpha_j$ by $\alpha_j \in [a_j, b_j],$ we get the conditions on two edges of segments $[a_j, b_j]$ as

$$V(x, 0) = v_\infty (x \alpha - F_\pm^j (x)),$$

where $j = 1, \ldots, n.$

Since the complex potential of a field may be proved to be a multi-valued function, it is more convenient to consider its derivative

$$\frac{dW}{dz} = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x} = u + iv,$$

which is obviously one-valued.

Therefore, the problem reduced to the following problem: to obtain an analytical function outside the segments $[a_j, b_j], j = 1, \ldots, n,$ such that this function is equal to zero at infinity and the imaginary part of which $v(x, y)$ on the upper and lower edges of these segments takes the given values:

$$v = \frac{\partial V}{\partial x} = \begin{cases} v_\infty (\alpha_j - F_\pm^j (x)) = v_j^+(x) \\
 v_\infty (\alpha_j - F_\pm^j (x)) = v_j^-(x) \
\end{cases}, \quad x \in [a_j, b_j], \quad j = 1, \ldots, n.$$

This problem is solved by a method similar to the one used to derive the Keldysh-Sedov formula. We set

$$\frac{dW}{dz} = f_1(z) + f_2(z),$$

where

$$f_1(z) = u^{(1)}(z) + v^{(1)}(z), \quad f_2(z) = u^{(2)}(z) + v^{(2)}(z)$$

are functions analytic outside the segments $[a_j, b_j], j = 1, \ldots, n,$ and equal to zero at infinity.

Imaginary parts of the functions $f_1(z)$ and $f_2(z)$ satisfy the boundary conditions:

$$\begin{align*}
 v^{(1)+}(x) &= -v^{(1)-}(x) = \frac{v_j^+(x) - v_j^-(x)}{2}, \\
 v^{(2)+}(x) &= v^{(2)-}(x) = \frac{v_j^+(x) + v_j^-(x)}{2},
\end{align*}$$

where $x \in [a_j, b_j], j = 1, \ldots, n.$ (1)

Next, we consider the function

$$g(z) = \prod_{k=1}^n \sqrt{\frac{z - b_k}{z - a_k}}.$$

In this case, we consider a positive branch of the function $\sqrt{\frac{z - b_k}{z - a_k}}, k = 1, 2, \ldots, n,$ on the interval $(a_k, +\infty).$ This branch, obviously, is a one-valued function outside of the consider
segments. On the edges of slits, this function takes purely imaginary values that differ in sign, consequently:

\[ g^+(x) = -g^-(x) = i \prod_{k=1}^{n} \sqrt{b_k - x} \frac{b_k - x}{x - a_k}, \quad x \in [a_j, b_j], \quad j = 1, \ldots, n. \tag{2} \]

We construct a circle \( L \) centred at the origin, which has a sufficiently large radius \( R \). To the \((n + 1)\)-connected region bounded by this circle and the curves \( l_k, \; k = 1, \ldots, n \), spanning the segments and disjoint, we apply the Cauchy integral formula:

\[ f_1(z)g(z) = \frac{1}{2\pi i} \int_{L + \sum_{k=1}^{n} l_k} \frac{f_1(\zeta)g(\zeta)}{\zeta - z} \, d\zeta. \tag{3} \]

Since \( \lim_{z \to \infty} f_1(z) = 0, \lim_{z \to \infty} g(z) = 1 \), then the integral along \( L \) tends to zero as \( R \to \infty \).

By conditions (1) and (2) the product \( f_1(z)g(z) \) takes the same value on opposite edges of slits. Consequently, the integrals from this product along these edges are canceled. Therefore, we obtain from (3) as \( R \to \infty \) and \( l_k \) are coincided with segments:

\[ f_1(z) = \frac{1}{2\pi i g(z)} \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{u^{(1)}+(x) + u^{(1)}-(x)}{x - z} \prod_{k=1}^{n} \sqrt{b_k - x} \frac{b_k - x}{x - a_k} \, dx. \tag{4} \]

This formula shows that for a real \( z \) not belonging to the segments \([a_k, b_k], \; k = 1, \ldots, n\), the function \( f_1(z) \) takes real values. By the Schwarz reflection principle, it follows from this that the following reflection conditions hold:

\[ u^{(1)}(x, -y) = u^{(1)}(x, y), \quad v^{(1)}(x, -y) = -v^{(1)}(x, y). \tag{5} \]

Now apply the Cauchy formula to the same contour

\[ L + \sum_{k=1}^{n} l_k \]

and to the function \( f_1(z) \):

\[ f_1(z) = \frac{1}{2\pi i} \int_{L + \sum_{k=1}^{n} l_k} \frac{f_1(\zeta)}{\zeta - z} \, d\zeta. \]

By conditions (4) and (1), we obtain from this as \( R \to \infty \) and \( l_k \) are coincided with segments \([a_k, b_k], \; k = 1, \ldots, n\):

\[ f_1(z) = \frac{1}{2\pi} \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{v_k^+(x) - v_k^-(x)}{x - z} \, dx. \tag{6} \]

In exactly the same way, applying the Cauchy formula to the function \( f_2(z) \), we obtain that on the real axis, outside the segments \([a_k, b_k], \; k = 1, \ldots, n\), this function takes pure imaginary values. Therefore, the symmetry conditions hold for the function \( f_2(z) \):

\[ u^{(2)}(x, -y) = -u^{(2)}(x, y), \quad v^{(2)}(x, -y) = v^{(2)}(x, y). \tag{7} \]
By the formula
\[ f_2(z)g(z) = \frac{1}{2\pi i} \int_{L + \sum_{k=1}^{n} i_k} \frac{f_2(\zeta)g(\zeta)}{\zeta - z} d\zeta \]
and conditions (6), we obtain, as above:
\[ f_2(z) = -\frac{1}{2} ig(z) \sum_{k=1}^{n} \int_{a_k}^{b_k} \left( v_k^+(x) + v_k^-(x) \right) \prod_{k=1}^{n} \sqrt{\frac{b_k - x}{x - a_k}} dx. \]  

Adding (7) and (5), we obtain the solution to the problem in the form
\[ \frac{dW}{dz} = \frac{1}{2\pi} \sum_{k=1}^{n} \int_{a_k}^{b_k} \frac{v_k^+(x) - v_k^-(x)}{x - z} dx - \frac{1}{2\pi i} \prod_{k=1}^{n} \sqrt{\frac{z - a_k}{z - b_k}} \sum_{k=1}^{n} \int_{a_k}^{b_k} \left( v_k^+(x) + v_k^-(x) \right) \prod_{k=1}^{n} \sqrt{\frac{b_k - x}{x - a_k}} dx. \]

This formula gives an approximate distribution of the velocity in a stream flowing around a system of thin profiles. It can be seen from this formula, in particular, that in the vicinity of an infinitely distant point there is a decomposition
\[ \frac{dW}{dz} = \Gamma + iN + \ldots, \]
where
\[ \Gamma = \sum_{k=1}^{n} \int_{a_k}^{b_k} \left( v_k^+(x) + v_k^-(x) \right) \prod_{k=1}^{n} \sqrt{\frac{b_k - x}{x - a_k}} dx, \]
and
\[ N = -\sum_{k=1}^{n} \int_{a_k}^{b_k} (v_k^+(x) - v_k^-(x)) dx, \]
physically means the circulation and flow respectively, calculated for any contour surrounding the profiles.

3. Numerical solution of some boundary value problem

The boundary value problem described below and its numerical solution are closely related to the problem considered above (see [10]).

As we noted above, the standard mixed boundary-value problem for the function $F(z)$ analytic in the upper half-plane has a unique solution satisfying the given boundary conditions. More precisely.

It is required to obtain an analytic function $F(z)$ on the half-plane $C_+$ such that:
1) on the segments $(a_k, b_k)$ the real part $\Re \left( \lim_{y \to +0} F(x + iy) \right) = u(x)$ is known;
2) on the segments $(b_k, a_{k+1})$, the imaginary part $\Im \left( \lim_{y \to +0} F(x + iy) \right) = v(x)$ is known;
3) the function $F(z)$ is bounded near all points $a_k$:
4) the integral \( \int F(z) \, dz \) is bounded near all points \( b_k \);
5) the limit on the points of the upper half-plane is equal to \( \lim_{z \to \infty, z \in C_+} F(z) := u(\infty) \).

The solution of this problem was obtained by M. V. Keldysh and L. I. Sedov and has the form:

\[
F(z) = \frac{1}{i\pi g(z)} \left( \sum_{k=1}^{n} \int_{a_k}^{b_k} g(\tau)u(\tau) \frac{d\tau}{\tau - z} + i \sum_{k=1}^{n} \int_{b_k}^{a_k+1} g(\tau)v(\tau) \frac{d\tau}{\tau - z} \right) + h(z) + \frac{u(\infty)}{g(z)},
\]

where

\[
g(z) = \prod_{k=1}^{n} \sqrt{\frac{z - b_k}{z - a_k}}.
\]

We consider here a positive branch of the function \( \sqrt{\frac{z - b_k}{z - a_k}}, k = 1, 2, \ldots, n \), on the interval \((a_k, +\infty)\).

Let \( D \) be a simply connected region in the complex plane \( \mathbb{C} \) which is bounded by a piecewise-smooth curve \( \gamma \). On this curve we set the points \( w_1, w_1^*, w_2, w_2^*, \ldots, w_n, w_n^* \).

In further, we will assume that \( F(\infty) = u(\infty) = 0 \). This means that the arc \( w_n^*, w_1^* \) of the curve \( \gamma \) passes through the origin. Such a restriction does not reduce the generality of proofs. If a curve does not have this property, it can always be reduced to it by applying a linear fractional transformation.

The Keldysh-Sedov formula gives an explicit representation of a function conformally mapping the upper half-plane on an arbitrary simply connected region bounded by a piecewise-smooth contour (see [8]). However, the quantities included into the Keldysh-Sedov formula are unknown in advance. Thus, it is necessary to derive a system of equations from which these quantities can be found.

We set

\[
f_k(z) = \frac{1}{i\pi g(z)} \int_{a_k}^{b_k} g(\tau)u(\tau) \frac{d\tau}{\tau - z}, \quad f_k^*(z) = \frac{1}{\pi g(z)} \int_{b_k}^{a_k+1} g(\tau)v(\tau) \frac{d\tau}{\tau - z}.
\]

Then the Keldysh-Sedov formula can be written as:

\[
F(z) = \sum_{k=1}^{n} f_k(z) + \sum_{k=1}^{n} f_k^*(z), \quad \text{(8)}
\]

Let a point \( z_0 \in \mathbb{C}_+ \) belongs to the arc \( a_{\nu}, b_{\nu} \). Let the variable \( z \) tends to the point \( z_0 \). Then the function \( F(z) \) tends to value \( F(z_0) = u_{\nu}(z_0) + iv_{\nu}(z_0) \) and the limit values of the right-hand side of the expression (8) for \( k \neq \nu \) satisfy the relations:

\[
\lim_{z \to z_0} f_k(z) = \frac{1}{i\pi g(z_0)} \int_{a_k}^{b_k} g(\tau)u_k(\tau) \frac{d\tau}{\tau - z_0}, \quad \lim_{z \to z_0} f_k^*(z) = \frac{1}{\pi g(z_0)} \int_{b_k}^{a_k+1} g(\tau)v_k(\tau) \frac{d\tau}{\tau - z_0};
\]

and for \( k = \nu \) they satisfy the relations:

\[
\lim_{z \to z_0} f_\nu(z) = f_\nu(z_0) = \frac{1}{i\pi g(z_0)} \int_{a_\nu}^{b_\nu} g(\tau)u_\nu(\tau) \frac{d\tau}{\tau - z_0} = u_\nu(z_0) + \]

Then reasoning, similar to those made above, leads to the equation:

\[ \Phi(0) \frac{u(0)}{g(0)} = \Phi(0) \frac{u(0)}{g(0)} + \frac{u(0)}{\pi i} \ln \frac{b - z}{a - a_u}. \]

Note that the function \( g(\tau) \) takes purely imaginary values on the segments \((a_k, b_k)\) and only real values on the segments \((b_k, a_{k+1})\). We substitute into the relation (8) instead of functions \( F(z), f_k(z), f_k^*(z) \) their limiting values. In the resulting expression, we separate the real and imaginary parts. We obtain the equation

\[
v_\nu(z_0) = \sum_{k=1}^{n} \frac{1}{i \pi g(z_0)} \int_{b_k}^{a_{\nu+1}} g(\tau) v_k(\tau) d\tau - \sum_{k=1, k \neq \nu}^{n} \frac{1}{\pi g(z_0)} \frac{b_k}{a_k} \int_{a_k}^{a_{\nu}} g(\tau) u_k(\tau) d\tau \]

\[
- \frac{1}{\pi} \int_{a_\nu}^{b_\nu} \left( \frac{u(\tau)}{g(z_0)} - \frac{u(0)}{g(z_0)} \right) \frac{d\tau}{\tau - z_0} + \frac{u(0)}{\pi i} \ln \frac{b - z}{a - a_u}.
\]

Let now a point \( z_0 \in (b_k, a_{k+1}) \). Let the variable \( \tau \) tends to the point \( z_0 \) such that \( \tau \in \mathbb{C}_+ \). Then reasoning, similar to those made above, leads to the equation:

\[
u_\nu(z_0) = \frac{1}{\pi} \int_{b_\nu}^{a_{\nu+1}} \frac{v_k^*(\tau)}{R(\tau)} R(z_0) - v_\nu^*(z_0)\frac{d\tau}{\tau - z_0} +
\]

\[
+ \frac{v_\nu^*(z_0)}{\pi} \ln \frac{a_{\nu+1} - z_0}{b_\nu} + \sum_{k=1}^{n} \frac{R(z_0)}{i \pi} \int_{a_k}^{b_k} \frac{u_k(\tau)}{R(\tau)} \frac{d\tau}{\tau - z_0} +
\]

\[
+ \sum_{k=1, k \neq \nu}^{n} \frac{R(z_0)}{i \pi} \int_{b_k}^{a_{\nu+1}} \frac{v_k^*(\tau)}{R(\tau)} \frac{d\tau}{\tau - z_0}, \quad \nu = 1, \ldots, n.
\]

Remark, if the point \( z_0 \in (a_k, b_k) \) then the point that corresponds to \( z_0 \) by mapping \( \Phi(z_0) = u_k(z_0) + iv_k(z_0) \) belongs to the arc \( w_k, w_k^* \).

This means that the functions \( u_k(z_0) \) and \( v_k(z_0) \) are related by the equation:

\[ F_k(u_k(z_0), v_k(z_0)) = 0, \quad k = 1, \ldots, n. \]

Similarly, if the point \( z_0 \in (b_k, a_{k+1}) \) then the point that corresponds to \( z_0 \) by mapping \( \Phi(z_0) = u_k^*(z_0) + iv_k^*(z_0) \) belongs to the arc \( w_k^*, w_{k+1} \).

Then the functions \( u_k^*(z_0) \) and \( v_k^*(z_0) \) are related by the equation:

\[ F_k^*(u_k^*(z_0), v_k^*(z_0)) = 0, \quad k = 1, \ldots, n. \]

At the same time, it is easy to see that the following relations hold:

\[ u_k^*(a_k) = u_k(a_k) = \Re w_k, v_k^*(b_k) = v_k(b_k) = \Im w_k. \]
Thus, to define the functions \( u_k(z_0), v_k(z_0), u'_k(z_0), v'_k(z_0) \) in the amount to the equal \( 4n \) and \( 2n, n = 1, 2, \ldots \), respectively, it is necessary to solve a system of \( 6n \) equations:

\[
\begin{align*}
\nu &= 1, n; \quad (1) \\
\nu &= 1, n; \quad (2)
\end{align*}
\]

From equations (3) and (4) of the resulting system, the functions \( v_\nu(z) \) and \( u'_\nu(z) \) can be obtained through the functions \( u_\nu(z) \) and \( v'_\nu(z) \) respectively.

The expressions obtained for the functions \( v_\nu(z) \) and \( u'_\nu(z) \) can be substituted into equations (1) and (2) of the system.

In this case, we obtain equations depending only on the functions \( u_\nu(z) \) and \( v'_\nu(z) \). Thus, the number of equations in the system can be reduced to \( 4n \).

The resulting system of singular integral equations has a unique solution up to two arbitrary constants in the class of functions, which are the boundary values of the functions analytic in the upper half-plane (see [11]). This statement easily follows from the Riemann theorem on the existence of a unique function realizing a conformal mapping of one simply connected region to another (see [12]).

4. The generalized problem of flow past around thin profiles

Now, our aim is to study the problem of flow past around thin profiles in the case when the contour \( \tilde{N} \) is determined by an infinite number of contours

\[
C = \bigcup_{j=1}^{\infty} C_j.
\]

Suppose that the contours of the profiles \( C_j, j = 1, 2, \ldots, n, \ldots \), are determined by an infinite number of the equations

\[
y = F_\pm^{(j)}(x), a_j \leq x \leq b_j, \quad j = 1, 2, \ldots, n, \ldots
\]

and let these contours are close to segments

\[
[a_j, b_j], \quad a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n < \ldots, \quad j = 1, 2, \ldots, n, \ldots
\]

Further reasoning almost literally repeats the reasoning of the second section. Therefore, we repeat them briefly.
Let the profiles are flowed around a translational flow which at infinity has velocity $v_\infty$. Let the velocity $v_\infty$ is tilted to the segments $[a_j, b_j]$ under small angles of attack $\alpha_j$, $j = 1, \ldots, n, \ldots$. In accordance with this, we will search the complex potential of the flow in the form

$$w = v_\infty e^{ia} z + W$$

where $W = U + iV$ is unknown function. We have

$$\Im w = v_\infty (y \cos \alpha - x \sin \alpha) + V(x, y).$$

Since $C_j$, $j = 1, \ldots, n, \ldots$, coincide with the segments, then $\Im w$ should take constant values on them, equal to zero for definiteness. Therefore on $C_j$, $j = 1, \ldots, n, \ldots$,

$$\Im w = v_\infty (F^+_j(x) \cos \alpha - x \sin \alpha) + V(x, F^+_j(x)).$$

Using the assumptions made about the proximity of $C_j$, $j = 1, \ldots, n, \ldots$, to the segments $[a_j, b_j]$ and small angles $\alpha_j$, we equivalently replace $\cos \alpha_j$ by 1, $\sin \alpha_j$ by $\alpha_j$ in this condition and we transfer the conditions to the segments. In particular, replacing $V(x, F^+_j(x))$ by $V(x, 0)$, $x \in [a_j, b_j]$, we get conditions on two edges of segments $[a_j, b_j]$

$$V(x, 0) = v_\infty (x \alpha - F^+_j(x)),$$

where $j = 1, \ldots, n, \ldots$.

Since the complex potential of a field may be proved to be a multi-valued function, it is more convenient to consider its derivative

$$\frac{dW}{dz} = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x} = u + iv,$$

which is obviously one-valued.

Therefore the problem is reduced to the following problem: to obtain an analytical function outside the segments $[a_j, b_j]$, $j = 1, \ldots, n, \ldots$, such that this function equal to zero at infinity and the imaginary part of which $v(x, y)$ on the upper and lower edges of these segments takes the given values:

$$v = \frac{\partial V}{\partial x} = \begin{cases} v_\infty (\alpha_j - F^+_j(x)) = v^+_j(x), & x \in [a_j, b_j], \quad j = 1, \ldots, n, \ldots, \\ v_\infty (\alpha_j - F^-_j(x)) = v^-_j(x), & x \in [a_j, b_j], \quad j = 1, \ldots, n, \ldots, \end{cases}$$

such that

$$\sum_{j=1}^{\infty} \int_{b_j}^{a_j} \left| \frac{v^+_j(x) \pm v^-_j(x)}{x - z} dx \right| < \infty.$$

We set

$$\frac{dW}{dz} = f_1(z) + f_2(z),$$

where

$$f_1(z) = u^{(1)}(z) + v^{(1)}(z), \quad f_2(z) = u^{(2)}(z) + v^{(2)}(z)$$

functions are analytic outside the segments $[a_j, b_j]$, $j = 1, \ldots, n, \ldots$, and satisfy the conditions

$$\lim_{r \to \infty} \sup_{|z| = r} \frac{1}{r} \sup_{|z| = r} |f_k(z)| = 0, \quad k = 1, 2.$$  \hspace{1cm} (9)
Imaginary parts of the functions \( f_1(z) \) and \( f_2(z) \) satisfy the boundary conditions:

\[
\begin{align*}
&v^{(1)+}(x) = -v^{(1)-}(x) = \frac{v^+_j(x) - v^-_j(x)}{2}, \quad x \in [a_j, b_j], \quad j = 1, \ldots, n, \\
&v^{(2)+}(x) = v^{(2)-}(x) = \frac{v^+_j(x) + v^-_j(x)}{2}, \quad x \in [a_j, b_j].
\end{align*}
\]

Let the numbers \( \{a_n\} \) and \( \{b_n\} \) satisfy the conditions:

\[
\sum_{n=1}^{\infty} \frac{1}{a_n} < \infty, \quad \sum_{n=1}^{\infty} \frac{1}{b_n} < \infty.
\]

Then the function

\[
P(z) = \prod_{n=1}^{\infty} \frac{z - b_n}{z - a_n}
\]

is a meromorphic function of the first order [13], i.e.

\[
\sup_{r>0} \frac{1}{r} \left\{ \int_0^{2\pi} \log^+ |P(re^{i\theta})| \, d\theta + \sum_{|a_n| \leq r} \log \frac{r}{|a_n|} \right\} < \infty,
\]

where \( \ln^+ a = \begin{cases} 
\ln a, & a > 1, \\
0, & a \leq 1.
\end{cases} \)

We consider the function

\[
g(z) = \sqrt{P(z)}.
\]

In this case, we consider a positive branch of the function \( \sqrt{\frac{z - b_n}{z - a_n}}, n = 1, 2, \ldots, \) on the interval \((a_n, +\infty)\). This branch, obviously, is one-valued outside of the consider segments. On the edges of slits, this function takes purely imaginary values that differ in sign, consequently:

\[
g^+(x) = -g^-(x) = i \prod_{k=1}^{n} \sqrt{\frac{b_k - x}{x - a_k}}, \quad x \in [a_j, b_j], \quad j = 1, \ldots, n, \ldots
\]

We construct a circle \( L(R) \) centred at the origin, which has a sufficiently large radius \( R \). To the \((n+1)\)-connected region bounded by this circle and the curves \( l_k, k = 1, \ldots, n, \) spanning the segments \([a_k, b_k], k = 1, \ldots, n, \) and disjoint, we apply the Cauchy integral formula:

\[
f_1(z)g(z) = \frac{1}{2\pi i} \int_{L(R)+\sum_{k=1}^{n} l_k} \frac{f_1(\zeta)g(\zeta)}{\zeta - z} \, d\zeta.
\]

By conditions (9) and (10), the integral along \( L(R) \) tends to zero as \( R \to \infty \).

On opposite edges of slits, the product \( f_1(z)g(z) \) takes the same value. Consequently, the integrals from this product along these edges are cancelled. Therefore we obtain from this as \( R \to \infty \) and \( l_k \) are coincided with segments:

\[
f_1(z) = \frac{1}{2\pi g(z)} \sum_{k=1}^{\infty} \int_{[1,a_k]} \frac{b_k - x}{x - a_k} \prod_{k=1}^{n} \sqrt{\frac{b_k - x}{x - a_k}} \, dx.
\]
This formula shows that for \( z \) not belonging to the segments \([a_k, b_k], k = 1, \ldots, n, \ldots\), the function \( f_1(z) \) takes real values. By the Schwarz reflection principle, it follows from this that the following reflection conditions hold:

\[
u^{(1)}(x, -y) = u^{(1)}(x, y), \quad v^{(1)}(x, -y) = -v^{(1)}(x, y).
\]

Now apply the Cauchy formula to the same contour \( L(R) + \sum_{k=1}^{n} l_k \) and to the function \( f_1(z) \):

\[
f_1(z) = \frac{1}{2\pi i} \int_{L(R) + \sum_{k=1}^{n} l_k} \frac{f_1(\zeta)}{\zeta - z} d\zeta.
\]

We obtain from this as \( R \to \infty \) and \( l_k \) are coincided with segments \([a_k, b_k], k = 1, \ldots, n, \ldots\):

\[
f_1(z) = \frac{1}{2\pi} \sum_{k=1}^{\infty} \int_{a_k}^{b_k} \frac{v_k^{+}(x) - v_k^{-}(x)}{x - z} dx.
\]

In exactly the same way, applying the Cauchy formula to the function \( f_2(z) \), we obtain that on the real axis, outside the segments \([a_k, b_k], k = 1, \ldots, n, \ldots\), this function takes pure imaginary values. Therefore, the symmetry conditions hold for the function \( f_2(z) \):

\[
u^{(2)}(x, -y) = -u^{(2)}(x, y), \quad v^{(2)}(x, -y) = v^{(2)}(x, y).
\]

By the formula

\[
f_2(z)g(z) = \frac{1}{2\pi i} \int_{L(R) + \sum_{k=1}^{n} l_k} \frac{f_2(\zeta)g(\zeta)}{\zeta - z} d\zeta,
\]

we obtain, as above:

\[
f_2(z) = -\frac{1}{2\pi ig(z)} \sum_{k=1}^{\infty} \int_{a_k}^{b_k} \frac{v_k^{+}(x) + v_k^{-}(x)}{x - z} \prod_{k=1}^{\infty} \sqrt{\frac{b_k - x}{x - a_k}} dx.
\]

Further, we obtain the solution to the problem in the form

\[
\frac{dW}{dz} = \frac{1}{2\pi} \sum_{k=1}^{b_k} \int_{a_k}^{b_k} \frac{v_k^{+}(x) - v_k^{-}(x)}{x - z} dx - \frac{1}{2\pi ig(z)} \sum_{k=1}^{\infty} \int_{a_k}^{b_k} \frac{v_k^{+}(x) + v_k^{-}(x)}{x - z} \prod_{k=1}^{\infty} \sqrt{\frac{b_k - x}{x - a_k}} dx.
\]

This formula gives an approximate distribution of the velocity in a stream flowing around an infinite system of thin profiles.
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