Continuous permanent unobserved heterogeneity in
dynamic discrete choice models

Jackson Bunting*

August 15, 2023

Abstract

In dynamic discrete choice (DDC) analysis, it is common to use mixture models to control for unobserved heterogeneity. However, consistent estimation typically requires both restrictions on the support of unobserved heterogeneity and a high-level injectivity condition that is difficult to verify. This paper provides primitive conditions for point identification of a broad class of DDC models with multivariate continuous permanent unobserved heterogeneity. The results apply to both finite- and infinite-horizon DDC models, do not require a full support assumption, nor a long panel, and place no parametric restriction on the distribution of unobserved heterogeneity. In addition, I propose a seminonparametric estimator that is computationally attractive and can be implemented using familiar parametric methods.

Keywords: Mixture models, dynamic discrete choice problems, nonparametric identification, unobserved heterogeneity.

JEL Classification Codes: C14, C61

*Texas A&M University (jbunting@tamu.edu). I am grateful to Federico Bugni, Adam Rosen, Matt Masten, Arnaud Maurel, Stéphane Bonhomme, Giovanni Compiani, Paul Diegert, Amanda Fratrik, Craig Fratrik, Dave Kaplan and Takuya Ura as well as seminar audiences at UC Leuven, U Chicago, Duke, Georgetown U, CUHK, NUS, Notre Dame, Ohio State University, Texas A&M University, U Washington, and TSE.


1 Introduction

In dynamic discrete choice (DDC) analysis, it is common to use mixture models to control for permanent unobserved heterogeneity. For instance, Keane and Wolpin (1997) and Cameron and Heckman (1998) model the observed distribution of schooling and work decisions as a mixture of individuals with varying unobserved abilities, which differ across occupations.

However, the use of mixture models in DDC analysis has limitations. First, existing identification results restrict the permanent unobserved heterogeneity to be either discrete (Kasahara and Shimotsu 2009) or a scalar random variable (Hu and Shum 2012). In the schooling and work example, this limitation may mean the mixture model does not capture the full richness of ability types and patterns of comparative advantage across occupations.

Second, identification of mixture DDC models depends on having ‘enough variation’ in agent behaviour (Kasahara and Shimotsu 2009; Hu and Shum 2012), a condition that is typically assumed at a high level. In the context of the schooling and work example, ‘enough variation’ might require that agents with different unobserved abilities respond adequately differently to changes in wages. Concretely, ‘enough variation’ is an injectivity condition. To express the condition formally, let $P_t(a, x, b)$ represent the probability that an agent chooses action $a$ in period $t$ given observed covariates $x$ and persistent unobserved heterogeneity $b$. The ‘enough variation’ assumption states for any signed measure $\mu$ on the support of persistent unobserved heterogeneity

$$\forall (a, x), \int P_t(a, x, b)d\mu(b) = 0 \iff \forall b, \mu(b) = 0.$$  \hspace{1cm} (1)

An injectivity condition of this style is imposed in the existing indentification literature\textsuperscript{1}. Yet, despite the crucial role of the injectivity assumption to identification\textsuperscript{2}, there appear to be few results in the literature on whether it holds in a given DDC model. Gaining insights into the conditions under which injectivity holds is particularly significant given that the assumption, as stated in Kasahara and Shimotsu (2009, p. 151), “is not empirically testable from the observed data”. Moreover, verifying injectivity of an integral operator is known to be a challenging problem in general (e.g., Andrews 2017).

The main contribution of this paper is to propose a general class of DDC models with permanent unobserved heterogeneity that is both continuous and multivariate, and provide

\textsuperscript{1}Specifically, Equation (1) generalizes the rank condition assumed in Proposition 1 Kasahara and Shimotsu (2009), and is a specialization of Assumption 2 Hu and Shum (2012).

\textsuperscript{2}Under some conditions, injectivity is equivalent to identification. See the discussion of Theorem 1.
low-level conditions for its identification. Applied to the schooling and work example, the class of DDC models in this paper would allow abilities to vary continuously across individuals and to be occupation-specific. I provide sufficient conditions for point identification of all model parameters, including the distribution of agent types (i.e., the distribution of permanent unobserved heterogeneity) and the type-specific choice model. By establishing low-level conditions for identification, the paper provides affirmation of the injectivity assumption for DDC models, demonstrating that it holds at least within one broad class of DDC models.

The paper contains two main results on identification of multinomial DDC models. The first result (Section 2) pertains to DDC models with random coefficients. The second (Section 3.1) relates to DDC models with random intercepts. I also prove several extensions to these main results, encompassing both stationary (i.e., infinite horizon) and non-stationary (i.e., finite horizon) DDC models. Furthermore, I show an important implication of the results under the additional restriction that permanent unobserved heterogeneity is discrete — an assumption that is standard in applied work. In this case, a key modeling decision is the number of agent types (i.e., the number of support points of permanent unobserved heterogeneity)\(^3\), which may be a challenging decision if there is no theoretical guidance on the number of agent types. My identification results imply a solution to this problem: namely, that that the number of agent types is identified if it is assumed to be finite.

Within a standard DDC model in the style of Rust (1987) and Magnac and Thesmar (2002), the low-level conditions for identification can be broadly categorized into two groups. First, I assume a short panel of observations with some continuous variation in the observed covariates, which is a natural prerequisite for nonparametric identification of a continuous latent distribution. Importantly, the results do not require the covariates to have full support, nor place parametric restrictions on the distribution of the permanent unobserved heterogeneity. Second, restrictions on the model primitives are used to ensure injectivity holds. These restrictions have three components: a distributional assumption on the random utility shock, a functional form assumption on the per-period payoffs, and a relevance condition on the covariates. The restrictions have the advantage of being low-level and interpretable. For example, the relevance condition can be interpreted as requiring (at least one) covariate to have a non-zero effect on the agent’s utility. Moreover, and notably, many of the restrictions

\(^3\)In general, only a lower bound on the number of mixture components is identified (e.g., Kasahara and Shimotsu (2009, Proposition 3)) so identification of finite mixture models requires knowledge of an upper bound (e.g., Freyberger (2018, Theorem S.1)). See Section 3.4 for discussion.
are commonly made in the literature. For example, it is common to make distributional assumptions on the random utility shock and functional form assumptions on the per-period payoffs (Aguirregabiria and Mira 2010). In this way, the results of this paper demonstrate that commonly made assumptions impose structure on DDC models that is useful for proving the (otherwise high-level) injectivity condition.

To implement the identification results, I propose a novel estimation method. Existing DDC estimation methods which focus on the parametric case (Aguirregabiria and Mira 2002; Arcidiacono and Miller 2011) do not apply to the model of this paper, as the distribution of unobserved heterogeneity may be an infinite dimensional parameter of interest. Similarly, the computational complexity of DDC models means that immediately available nonparametric methods (such as sieve likelihood estimation) may be impractical. To address these issues I propose a two-step sieve M-estimator, and show it is consistent for the model parameters. I also propose a computationally convenient sieve space based on Heckman and Singer (1984). Intuitively, the estimator approximates the possibly continuous distribution of permanent unobserved heterogeneity by a discrete distribution. In this setup, the ‘fixed grid’ of support points of the approximating distribution is a tuning parameter of the sieve estimator. Computationally, this estimator is identical to an estimator for a model with finite types, but instead of the number of support points being an identifying assumption, it is simply a tuning parameter.

I illustrate the theory with two empirical applications. The first estimates an optimal stopping model of taxi driver shift length (Frechette, Lizzeri, and Salz 2019) where the marginal utility of earnings may be heterogeneous. The second revisits the labor supply model of Altuğ and Miller (1998), in which workers may have individual-specific labor productivity. In addition, I consider a suite of Monte Carlo simulations based on a simplified version of the model of Altuğ and Miller (1998).

After discussing related literature, I introduce the model and provide one main identification result (Section 2). Section 3 contains the second main identification result (Section 3.1) and other extensions, including to non-stationary DDC problems. Section 4 proposes the two-step sieve M-estimator and shows its consistency. Section 5 considers the applications, Section 6 presents simulation results.

---

4In principle, standard DDC models may be semiparametric in the presence of continuous covariates, however, in practice, continuous covariates are often discretized and treated as such for estimation.
Related literature. This paper is closely related to the literature on point identification of DDC models with persistent unobserved heterogeneity (Kasahara and Shimotsu 2009; Hu and Shum 2012). These papers use a short panel to identify type-specific conditional choice probabilities and the distribution of unobserved heterogeneity via an eigendecomposition of the observed data. As mentioned earlier, these papers consider persistent unobserved heterogeneity that is either discrete (Kasahara and Shimotsu 2009) or a scalar random variable (Hu and Shum 2012). Relative to these papers, I allow for permanent unobserved heterogeneity that is both continuous and multivariate. As previously mentioned, another important difference is that I provide low-level conditions for the injectivity condition. On the other hand, their approach allows unobserved heterogeneity to enter the model very flexibly, restricted only by certain high-level assumptions. For example, my assumptions rule out type-specific transition functions (e.g., Kasahara and Shimotsu (2009, Section 3.2)) or unobserved heterogeneity that is first-order Markov (e.g., Hu and Shum (2012)). Williams (2020) adopts a similar approach for models with lagged dependent variables and a ‘special regressor’. For a general review of mixture models in econometrics see Compiani and Kitamura (2016).

Several other papers have analyzed persistent unobserved heterogeneity in DDC models from a partial identification perspective. For instance, Aguirregabiria, Gu, and Luo (2021) focuses on (point) identification of a subvector of the model parameters, treating permanent unobserved heterogeneity as nuisance parameter. Some general approaches that allow for set identification include Chernozhukov et al. (2013) and Berry and Compiani (2022). Compared to these papers, I provide conditions for point identification of the DDC model.

The seminonparametric estimator I propose is based on Heckman and Singer (1984). Similar ‘fixed grid’ estimators have been analyzed for both the parametric and non-dynamic models (Fox et al. 2011; Fox, Kim, and Yang 2016), and are increasingly used in applied work (e.g., Nevo, Turner, and Williams 2016; Illanes and Padi 2019).

However, it is worth noting that Hu and Shum (2012) do not allow for identification of permanent unobserved heterogeneity from variation in choice behavior alone. Specifically, Hu and Shum (2012) Assumption 3(ii) requires variation in the state transition by type. To see this, in their notation let $W_t = (Y_t, X_t)$ be observed and $X_t^* = X^*$ latent, then their equation (11) becomes

$$k(w_t, \bar{w}_{t-1}, w_{t-1}, X^*) = \frac{f_{X_t|X_{t-1}, Y_{t-1}, X^*}(x_t|X_{t-1}, Y_{t-1}, X^*)}{f_{X_t|X_{t-1}, Y_{t-1}, X^*}(x_t|X_{t-1}, Y_{t-1}, X^*)} \frac{f_{X_t|X_{t-1}, Y_{t-1}, X^*}(x_t|X_{t-1}, Y_{t-1}, X^*)}{f_{X_t|X_{t-1}, Y_{t-1}, X^*}(x_t|X_{t-1}, Y_{t-1}, X^*)},$$

and thus their Assumption 3(ii) which requires $k(w_t, \bar{w}_t, w_{t-1}, \bar{w}_{t-1}, X^*)$ to vary in $X^*$ fails if the state transition $f_{X_t|X_{t-1}, Y_{t-1}, X^*}$ does not depend on $X^*$. Williams (2020) also makes this point.
Notation: For a random variable $X$, $\text{Supp}(X)$ and $f_X$ denote the support and probability density (or mass) function, respectively. For $x \in \mathbb{R}^K$ and $1 \leq k \leq K$, $x_k \in \mathbb{R}$ represents the $k$'th element, $x_{[k]} \in \mathbb{R}^k$ represents the first $k$ elements and $x_{[-k]} \in \mathbb{R}^{K-k}$ represent the final $K-k$ elements.

2 Model and identification

I consider a standard single-agent dynamic discrete choice structural model as described in Aguirregabiria and Mira (2010). In each period $t = 1, \ldots, T = \infty$, a single agent observes a vector of state variables $(S_t, \epsilon_t)$ and chooses an action $A_t$ from a finite set of actions $A \equiv \{0, 1, \ldots, |A|\}$ (with $|A| > 0$) to maximize their expected discounted utility. The action denoted by 0 is referred to as the outside option, and I denote $\bar{A} \equiv \{1, \ldots, |A|\}$.

I assume $\epsilon_t = (\epsilon_{ta} : a \in A)$ is independent of $(\epsilon_{\tau}, A_{\tau}, S_{\tau+1})$ for $\tau < t$, and is identically distributed according to $dF_\epsilon(e) = \prod_a dF_{a}(e_a)$. In addition, conditional on $(A_t, S_t) = (a_t, s_t)$, $S_{t+1}$ is independent of $(\epsilon_{\tau}, A_{\tau-1}, S_{\tau-1})$ for $\tau \leq t$, with probability distribution $dF_s(s_{t+1} \mid a_t, s_t)$. It then follows that $(S_{t+1}, \epsilon_{t+1})$ is a Markov process with a probability density that satisfies

$$d\Pr(S_{t+1} = s', \epsilon_{t+1} = e' \mid S_t = s, \epsilon_t = e, A_t = a) = dF_e(e') \times dF_s(s' \mid a, s).$$

(2)

The agent has a time-separable utility and discounts future payoffs by $\rho \in [0, 1)$. The period $t$ payoff is $u_t(S_t, \epsilon_t, A_t)$. Under these conditions, the agent’s choice in time $t$ satisfies

$$a_t = \arg \max_{a \in \bar{A}} \{u_t(s_t, \epsilon_t, a) + \rho E[v_{t+1}(S_{t+1}) \mid S_t = s_t, A_t = a]\},$$

(3)

where $v_t$ is the so-called integrated value function:

$$v_t(s_t) = E\left[\max_{a \in \bar{A}} \{u_t(s_t, \epsilon_t, a) + \rho E[v_{t+1}(S_{t+1}) \mid S_t = s_t, A_t = a]\}\right].$$

(4)

In this section I present conditions for identification of the distribution of continuous unobserved heterogeneity within the above model. The first assumption imposes restrictions that are standard for stationary DDC models without permanent unobserved heterogeneity.

Assumption II. (i) $u_t(S_t, \epsilon_t, A_t) = u(S_t, A_t) + \sum_{a \in A} \epsilon_{ta} 1[a = A_t]$. (ii) $\rho \in [0, 1)$ is known. (iii) Equation (2) (iv) $u(S_t, 0) = 0$. (v) $\epsilon_{ta}$ is independently distributed extreme value type I. (vi) $\text{Supp}(S_t)$ is bounded.
Assumption I1 contains standard identifying assumptions for DDC models (Magnac and Thesmar 2002; Aguirregabiria and Mira 2010), including additive separability of the flow utility, that the discount factor is known, a conditional independence assumption, and the outside good. These assumptions are not innocuous — for example, Norets and Tang (2014) show that the choice of outside good may affect predicted counterfactual outcomes. Nevertheless, it is standard to assume the unobserved state variables have a known distribution, of which normal and extreme value type I are common choices. It is also common to assume that \( S_t \) lies in a compact set, which helps ensure the integrated value function is a bounded function of \( S_t \) (Rust 1987; Kristensen et al. 2021).

The next assumption introduces permanent unobserved heterogeneity into the model as an unobserved state variable.

**Assumption I2.** (i) \( S_t = (X_t, \beta) \in \mathbb{R}^k \times \mathbb{R}^{|A|} \). \( \beta \mid X_1 = x \) is either discrete or absolutely continuous and \( f_{\beta|x} \) is bounded. (ii) For \( \gamma = \{ \gamma_a \in \mathbb{R}^{k-1} : a \in A \} \), \( u(s,a) = x^\top (\beta_a, \gamma_a) \). (iii) \( d\Pr(X_{t+1} = x' \mid A_t = a, X_t = x, \beta = b) = dF_z(x' \mid x, a) \). (iv) \( \Gamma_A = (\gamma_1[|A|], \gamma_2[|A|], \ldots, \gamma_{|A||A|][|A|]) \in \mathbb{R}^{|A|\times|A|} \) is full rank. (v) The probability distribution of \( X_{t+1} \) conditional upon \( (A_t, X_t) = (a, x) \) has no singular components, and the associated probability density and mass functions are real analytic functions of \( x_{|A|+1} \) with bounded analytic continuations to \( \mathbb{R}^{|A|+1} \).

Assumption I2(i) states that permanent unobserved heterogeneity enters the model as an unobserved state variable. The restrictions placed on its distribution are mild. First, it allows, but does not require, the distribution to have uncountable support. Intuitively this means there may be infinitely many types of agents, while allowing for the typical assumption of finitely many types as a special case. Second, there may be arbitrary dependence between the initial state variable and permanent unobserved heterogeneity.

Parts (ii) and (iii) of Assumption I2 control how permanent unobserved heterogeneity enters the model. Part (ii) states that the permanent unobserved heterogeneity enters the model as a random coefficient in the per-period payoff. Importantly, the (possibly) continuous \( \beta \) is vector-valued, allowing its effect to differ across different choice alternatives. For example, if \( \beta_a \) represents an agent’s ability in occupation \( a \in A \), some agents may be high ability in all occupations, other agents may be high in some occupations and low in others. Part (iii) requires that the transition of the state variable not depend on the unobserved state variable. As explained below (Remark 2), this assumption enables conditions on the model primitives to be used for identification.
The next condition (Assumption I2(iv)) imposes that the state variable cannot affect payoffs for each choice in a similar fashion. For example, in the binary choice case ($|A| = 1$), the assumption states that $\gamma_1 \neq 0 \in \mathbb{R}^{k-1}$.

Assumption I2(v) allows the state transition to be a mixture of an absolutely continuous and discrete random variable, but restricts the probability distribution to be a smooth function of the conditioning state variable. In particular, the component probability density and mass functions must be real analytic functions — that is, functions that have a convergent power series representation. An example of a state transition satisfying Assumption I2(v) is a mixture of a mass point at $x_{t+1} = 0$ and a truncated normal:

$$F_x(x'; x, a) = \pi_1(x' = 0) + (1 - \pi)F_x(x'; x, a),$$

where $F_x(x'; x, a)$ is a truncated normal whose mean and variance are real analytic functions of $(x, a)$. Other examples of real analytic functions include polynomials, the logistic function, trigonometric functions, the Gaussian function, in addition to compositions, products and linear combinations of these functions. This class of functions is known to include good approximators to square-integrable functions (e.g., Chen 2007, Section 2.3), and can therefore approximate many density functions arbitrarily well.

Define the conditional choice probability (CCP) function $P(a, x, b) = \Pr(A_t = a \mid X_t = x, \beta = b)$. The first main theorem states that under the above conditions, the integral operator defined by the CCP function is injective.

**Theorem 1 (Injectivity). Assume I1 and I2. For any $\mathcal{X} \subsetneq \mathbb{R}^k$ for which $\{x_{|A|+1} : x \in \mathcal{X}\}$ contains a non-empty set and any finite signed measure $\mu$ over set $\text{Supp}(\beta)$,

$$\forall (a, x) \in A \times \mathcal{X}, \int P(a, x, b)d\mu(b) = 0 \implies \forall b \in \text{Supp}(\beta), \mu(b) = 0$$

The injectivity condition in Theorem 1 is fundamental to identification of mixture models. To explain, consider the simple case that $\beta$ is independent of $X_t$, the CCP function is known (i.e., $\gamma$ is known), and the distribution of $(A_t, X_t)$ is observed. In this case, the data satisfies $\Pr(A_t = a \mid X_t = x) = \int P(a, x, b)dF_\beta(b)$ and the only unknown model parameter is $dF_\beta$, the distribution of permanent unobserved heterogeneity. Then, the injectivity condition is equivalent to identification of the distribution of unobserved heterogeneity: it states that if two distributions $dF_\beta$ and $d\tilde{F}_\beta$ are observationally equivalent, i.e.,

$$\forall (a, x) \in \text{Supp}(A) \times \text{Supp}(X), \int P(a, x, b)dF_\beta(b) = \int P(a, x, b)d\tilde{F}_\beta(b),$$

then the two distributions are the same, i.e., $dF_\beta = d\tilde{F}_\beta$. 
More generally, the injectivity condition in Theorem 1 is an example of the injectivity assumption in the measurement error literature (Hu and Schennach 2008, Assumption 3), with analogs in the context of DDC models (Kasahara and Shimotsu 2009, Proposition 1; Hu and Shum 2012, Assumption 2).

Theorem 1 is proved in Appendix A.1, but now I provide an overview of the argument. The first step is to notice that the CCP function inherits the smoothness properties of $F_x$ and $F_{\xi}$: in particular, $x_{[\left|A\right|+1]} \mapsto P(a, x, b)$ are real analytic functions that extend to $\mathbb{R}^{\left|A\right|+1}$. By Lemma A.3, this allows us to characterize the injectivity condition as

$$\forall (a, x) \in A \times \overline{X}, \quad \int P(a, x, b) d\mu(b) = 0 \implies \forall b \in \text{Supp}(\beta), \mu(b) = 0, \quad (5)$$

where $\overline{X}$ is a superset of $X$. I then show an equivalence result from Stinchcombe and White (1998) can be applied to characterize condition (5) in terms of the approximation properties of the set of functions $\{b \mapsto P(a, x, b) : (a, x) \in A \times \overline{X}\}$. For intuition of this characterization, consider that in the case that $\beta$ has $R < \infty$ support points, the full (row) rank condition is that the collection of vectors $\{(P(a, x, b) : b = 1, \ldots, R) : (a, x) \in A \times \overline{X}\} \span \mathbb{R}^R$. Finally, adapting methods from the classical neural network literature (Hornik, Stinchcombe, and White 1989; Hornik 1993), I show $\{b \mapsto P(a, x, b) : (a, x) \in A \times \overline{X}\}$ is dense in all square integrable functions of $b \in \text{Supp}(\beta)$.

Remark 1. Theorem 1 relies on having some continuous variation in $X$: namely that $\{x_{[\left|A\right|+1]} : x \in X\}$ contains a non-empty open set. Given that injectivity is equivalent to the set $\{b \mapsto P(a, x, b) : (a, x) \in A \times X\}$ being dense in all square integrable functions, it is natural to require that the set has infinitely many elements. However, importantly $\{x_{[\left|A\right|+1]} : x \in X\}$ may be arbitrarily small so long as it contains a non-empty open set. Moreover, there are no limitations on the support of $X_{[-(\left|A\right|+1)]}$. For instance $X_{[-(\left|A\right|+1)]}$ may contain discrete variables such as a constant or indicator functions.

Remark 2. In the case that the state transition also depends on permanent unobserved heterogeneity (i.e., if Assumption I2(iii) did not apply), then the kernel of the integral operator useful for identification would depend on both the CCP $P(a, x, b)$ and the state transition $F_x(x' ; x, a, b)$. In this case, without a behavioral model of $F_x(x' ; x, a, b)$ it appears to be challenging to provide low level conditions for injectivity of the integral operator. Kasahara and Shimotsu (2009, Proposition 6) and Hu and Shum (2012, Theorem 1) provide an identification result for this case, using a high level injectivity assumption.

In order to invoke Theorem 1 in identification of the DDC model, we require the support
of the state variable to contain an open set:

**Assumption I3.** For all \( x \in \text{Supp}(X_1) \), \( \exists a \in A \) such that: (i) \( \text{Supp}(X_2 | A | t+1) \mid X_1 = x, A_1 = a \) and \( \text{Supp}(X_3 | A | t+1) \mid X_2 \in \text{Supp}(X_2 | X_1 = x, A_1 = a), A_2 = 0 \) contain a non-empty open set; (ii) \( S_3 \equiv \text{Supp}(X_3 \mid X_2 \in \text{Supp}(X_2 | X_1 = x, A_1 = a), A_2 = 0) \) and \( \cap_{a_3 \in \text{Supp}(A_3)} \text{Supp}(X_4 \mid X_3 \in S_3, A_3 = a_3) \) span \( \mathbb{R}^k \).

Assumption I3 places restrictions on the support of the observed state variable \( X_t \in \mathbb{R}^k \). Part (i) requires that the state variable contain at least \( |A| + 1 \) continuous elements. Part (ii) requires that the supports contain \( k \) linearly independent elements, a mild rank condition which is standard in linear models. As discussed in Example 1, Assumption I3 allows for renewal models like Rust (1987). However, it rules out lagged dependent variables (i.e., when \( X_t \) contains the lagged choice \( A_{t-1} \)). In particular, lagged dependent variables contradict Assumption I3(ii) since \( \text{Supp}(X_4 \mid X_3 = x, A_3 = a) \) and \( \text{Supp}(X_4 \mid X_3 = x, A_3 = \hat{a}) \) are disjoint for \( a \neq \hat{a} \). Finally, unlike some results in the literature\(^6\), Assumption I3 does not require that the support be ‘rectangular’ — which requires that, starting from any sequence of choices and past state variables, any state can be reached (i.e., for all \( t \) and \( (a, x) \in \text{Supp}(A_t, X_t) \), \( \text{Supp}(X_{t+1} \mid X_t = x, A_t = a) = \text{Supp}(X_{t+1}) = \text{Supp}(X_1) \)).

**Example 1 (Renewal model).** Consider a bivariate state variable \( X_t \in \mathbb{R}^k \), where action \( A_t = 0 \) ‘regenerates’ the state variable to its baseline as in Rust (1987, p. 1006). As in Kristensen et al. (2021, Section 6.1), the transition kernel may be a mixture of a point mass and a continuous random variable:

\[
F_x(x_{t+1}; x_t, a_t) = \pi 1(x_{t+1} = a_t x_t) + (1 - \pi) F_\gamma(x_{t+1}; x_t, a_t),
\]

for \( \pi \in [0, 1] \) and where \( F_\gamma(x; x, a) \) has support \( \text{Supp}(X_{t+1} \mid X_t = x, A_t = a_t) = \times_{k' = 1}^k [a_t x_{tk'}, K_{k'}] \). When \( \pi < 1 \), \( \text{Supp}(X_{t+1} \mid X_t = x, A_t = 0) = \times_{k' = 1}^k [0, K_{k'}] \) so Assumptions I3(i) is satisfied. It follows that I3(ii) is satisfied with \( \cap_{a_3 \in \text{Supp}(A_3)} \text{Supp}(X_4 \mid X_3 \in S_3, A_3 = a_3) = \times_{k' = 1}^k [0, K_{k'}] \).

The model parameters are \( (F_x, \gamma, f_{\gamma|X_1}) \): the state transition, the homogeneous payoff parameter, and the conditional distribution of permanent unobserved heterogeneity. As the state transition is identified by direct observation, the following result (proved in Section A.1.2) handles the remaining parameters:

\(^6\)For example, this is Assumption 1(c)-(e) used in Kasahara and Shimotsu (2009) Propositions 1-9 and subsequently relaxed in Propositions 10 and 11.
Theorem 2 (Identification). Assume the distribution of \((X_t, A_t)_{t=1}^T\) is observed for \(T \geq 4\), generated from agents solving the model of equation (3) satisfying assumptions I1-I3. Then \((\gamma, f_{\beta|X_1})\) is point identified.

Remark 3. Theorem 2 requires at least four observations per individual. In contrast Kasahara and Shimotsu (2009) require only \(T = 3\). With three periods, identification of the model in Theorem 2 is possible under a high-level assumption on the joint distribution of permanent unobserved heterogeneity and the first period state variable\(^7\). However, the advantage of \(T = 4\) is to avoid this type of high level condition on the distribution of \((X_1, \beta)\), instead using low level conditions on the choice model.

3 Extensions

In this section I provide identification results for a number of variations on the model in Section 2.

3.1 Non-stationary conditional choice probabilities

In many contexts, the agent's decision rule may change between periods: for example, if the agent has a finite time-horizon, or if the state variables are subject to structural breaks. In these cases, it is natural to allow the per-period utility function and state transitions to be non-stationary, i.e., to be time-dependent. In this section I consider a finite horizon dynamic discrete choice model in which the terminal decision period is observed. For example, in a model of retirement from the labor force (Rust and Phelan 1997), we may eventually observe all individuals retire. Similarly, in a model of educational attainment, we may observe all individuals reach a terminal state (Heckman, Humphries, and Veramendi 2018). By definition, the decision-maker has no strategic influence over future utility flows to consider in the terminal period and thus a different proof strategy is adopted. This argument allows for identification of random intercepts ('fixed effects'), which was not the case in Section 2.

I begin by adapting Assumptions I1 and I2 to the non-stationary context. In particular, by allowing the flow utility and state transition to be time-dependent.

\(^7\)For example, Kasahara and Shimotsu (2009, Proposition 1) assumes that for some \(x \in \text{Supp}(X_1)\), \(\Pr(A_1 = 1, X_1 = x, \beta = b) = \Pr(A_1 = 1|X_1 = x, \beta = b) \Pr(\beta = b|X_1 = x_1) \Pr(X_1 = x_1) > 0\) is injective in \(b\).
Assumption F1. (i) \( u_t(S_t, \epsilon_t, A_t) = u_t(S_t, A_t) + \sum_{a \in A} \epsilon_{ta} 1[a = A_t] \). (ii) \( \rho \in [0, 1) \) is known. (iii) \( d\Pr(S_{t+1} = s', \epsilon_{t+1} = \epsilon' \mid S_t = s, \epsilon_t = \epsilon, A_t = a) = dF'_\epsilon(\epsilon') \times dF_{s_t}(s' \mid a, s) \). (iv) \( u_t(S_t, 0) = 0 \). (v) \( \epsilon_{ta} \) is independently distributed extreme value type I. (vi) \( \text{Supp}(S_t) \) is bounded.

Assumption F2. (i) \( S_t = (X_t, \beta) \in \mathbb{R}^k \times \mathbb{R}^{(1+p)|A|} \). If \( \beta \mid X_t = x \) admits a density \( f_{\beta|x} \), it is bounded. (ii) For \( \gamma_t = \{ \gamma_{ta} \in \mathbb{R}^{k-p} : a \in A \} \), \( u_t(s, a) = \beta_{a1} + x^\top (\beta_{a[-1]}, \gamma_{ta}) \). (iii) \( d\Pr(X_{t+1} = x_{t+1} \mid A_t = a_t, X_t = x_t, \beta = b) = dF_{x_t}(x_{t+1} \mid x_t, a_t) \). (iv) \( \Gamma_{AT} \equiv (\gamma_{T1[|A|]}, \gamma_{T2[|A|]}, \ldots, \gamma_{T|A|[|A|]} ) \in \mathbb{R}^{|A| \times |A|} \) is full rank.

Assumption F2 states that permanent unobserved heterogeneity enters the model as a state variable. The restrictions are weaker than those in the infinite horizon model (Assumption I2). Specifically, the permanent unobserved heterogeneity can include a random intercept. As was the case for the infinite horizon model, the support of permanent unobserved heterogeneity may be finite, but it need not be. Like Assumption I2(iv), Assumption F2(iv) imposes that the state variable cannot affect payoffs for each choice in a similar fashion. Since identification is attained from the terminal period, we place weaker restrictions on the transition \( F_{x_t} \) relative to Assumption I2(v).

To describe the injectivity result for the finite horizon model, denote the CCP function \( P_t(a, x, b) = \Pr(A_t = a \mid X_t = x, \beta = b) \) and let \( T \) denote the decision horizon of the agent.

**Theorem 3** (Injectivity). Assume F1 and F2. For any \( \mathcal{X} \subseteq \mathbb{R}^k \) for which \( \{ x_{p+[|A|]} : x \in \mathcal{X} \} \) contains a non-empty set and finite signed measure \( \mu \) over set \( \text{Supp}(\beta) \),

\[
\forall (a, x) \in A \times \mathcal{X}, \quad \int P_T(a, x, b)d\mu(b) = 0 \quad \Longrightarrow \quad \forall b \in \text{Supp}(\beta), \quad \mu(b) = 0.
\]

The proof of Theorem 3 is contained in Section A.2.

As for the time stationary model, we require further restrictions on the state variable \( X_t \) for identification of the DDC model. First, Assumption F3 requires there be some continuous variation in \( X_T \) after conditioning upon each history of actions and state variables.

**Assumption F3.** For each \( x_1 \in \text{Supp}(X_1) \) and \( (a_1, a_2, \ldots, a_{T-1}) \in A^{T-1} \), there is \( (x_2, x_3, \ldots, x_{T-1}) \in \times_{t=2}^{T-1} \text{Supp}(X_t) \) such that

\[
\text{Supp}(X_{T[p+[|A|]} \mid A_{T-1} = a_{T-1}, X_{T-1} = x_{T-1}, \ldots, A_1 = a_1, X_1 = x_1)
\]

contains a non-empty open set. Moreover, for each \( t \), \( \text{Supp}((1, X_t)) \) spans \( \mathbb{R}^{k+1} \).
To introduce the final assumption, define $S_T \equiv \text{Supp}(X_T | A_{T-1} = a_{t-1}, X_{T-1} = x_{t-1}, \ldots, A_1 = a_1, X_1 = x_1)$ and let $E \subset S_T \times A$, $P_T(a; x, b, \gamma)$ be the model implied probability of $A_T = a$ conditional upon $X_T = x$ evaluated at $\beta = b$ and $\gamma_T = \gamma$, and $\mathcal{L}_A$ be the set of bounded functions on $A$. Then define the operator

$$L_{T, \beta}^{E, \gamma} : \mathcal{L}_{\text{Supp} (\gamma)} \rightarrow \mathcal{L}_E \quad [L_{T, \beta}^{E, \gamma} m](x, a) = \int P_T(a; x, b, \gamma)m(b) db.$$ 

Denote $(L_{T, \beta}^{E, \gamma})^{-1}$ as the left inverse of $L_{T, \beta}^{E, \gamma}$.

**Assumption F4.** For every $\gamma \neq \tilde{\gamma}$, there exists $E, \tilde{E} \subseteq S_T \times A$ for which $\{e_{[p, + | A]}; e \in E\}$ and $\{e_{[p, + | A]}; e \in \tilde{E}\}$ contain non-empty open sets and such that the operator

$$L_{T, \beta}^{E, \gamma, E, \tilde{\gamma}} : \mathcal{L}_{\text{Supp} (\beta)} \rightarrow \mathcal{L}_{\text{Supp} (\beta)} \quad [L_{T, \beta}^{E, \gamma, E, \tilde{\gamma}} m](b) = \left[\left((L_{T, \beta}^{E, \gamma})^{-1} L_{T, \beta}^{E, \tilde{\gamma}} - (L_{T, \beta}^{E, \gamma})^{-1} L_{T, \beta}^{E, \tilde{\gamma}}\right) m\right](b)$$

is injective.

This high-level condition ensures that the parameter $\gamma_T$ can be identified without knowledge of the distribution of unobserved heterogeneity. A few comments on Assumption F4 are in order. First, given Theorem 3, Assumptions F1-F3 imply that, for any $E$ for which $\{e_{[p, + | A]}; e \in E\}$ contains a non-empty open set, $L_{T, \beta}^{E, \gamma}$ is injective so that $L_{T, \beta}^{E, \gamma, E, \tilde{\gamma}}$ exists. Second, the condition is stated in terms of observed objects, and thus the operator defined in Assumption F4 is identified by direct observation. Third, should Assumption F4 not hold, I show in the Appendix (Lemma A.4) that under Assumptions F1-F3 and a scale restriction on $\gamma_T$, that $\gamma_T$ and the distribution of unobserved heterogeneity are identified.

Finally, the condition can be related to the high-level necessary conditions for identification of a common parameter in discrete choice panel data given in Johnson (2004) and Chamberlain (2010). To describe their result, fix $x \equiv (x_1, x_2, \ldots, x_T)$ and for convenience let $A = \{0, 1\}$ and $\gamma$ be time-invariant. Let $p(\beta; x, \gamma)$ be the length $2^T$ vector of choice probabilities $\{\prod_{t=1}^T P_t(a_t, x_t, b; \gamma) : (a_t)_{t=1}^T \in \{0, 1\}^T \setminus \{0_T\}\}$ in the $(2^T - 1)$-dimensional hypercube. Johnson (2004, Theorem 2.2) states that the common parameter $\gamma$ will not be identified if the set $\{p(\beta; x, \gamma) : \beta \in S_\beta\}$ does not lie in a hyperplane for some $x$. For the static binary choice model with $T = 2$, Chamberlain (2010) shows that the hyperplane restriction is satisfied if and only if the unobserved state variables are iid extreme-value type I. Given the remarkable result of Chamberlain (2010), one may conjecture that the $T = 2$ dynamic binary choice model does not satisfy Johnson (2004)’s condition and therefore $\gamma$ is not identified. If
this is the case, then \( \forall x_2 \in \text{Supp}(X_2) \) and \( \gamma \neq \tilde{\gamma} \), there exist some \( f_{|X_1,X_2} \neq \tilde{f}_{|X_1,X_2} \) such that

\[
\left[ L_{2,\beta}^{\text{Supp}(X_2),\gamma} f_{|X_1,X_2} (\cdot, x_1, x_2) \right](x_2) = \left[ L_{2,\beta}^{\text{Supp}(X_2),\tilde{\gamma}} \tilde{f}_{|X_1,X_2} (\cdot, x_1, x_2) \right](x_2),
\]

where the distribution of unobserved heterogeneity \( f_{|X_1,X_2} \) is allowed to depend on \( x_2 \) as in Johnson (2004) and Chamberlain (2010). If the distribution is restricted to be the same for all \( x_2 \in \text{Supp}(X_2) \), the above condition implies that for each \( \gamma \neq \tilde{\gamma} \), \( x_2 \in \text{Supp}(X_2) \),

\[
\left[ L_{2,\beta}^{\text{Supp}(X_2),\gamma} f_{|X_1,X_2} (\cdot, x_1, x_2) \right](x_2) = \left[ L_{2,\beta}^{\text{Supp}(X_2),\tilde{\gamma}} \tilde{f}_{|X_1,X_2} (\cdot, x_1, x_2) \right](x_2),
\]

However, since the distribution of unobserved heterogeneity is required to be the same for all \( x_2 \), there may be some other \( \tilde{x}_2 \in \text{Supp}(X_2) \) such that

\[
\left[ L_{2,\beta}^{\text{Supp}(X_2),\gamma} f_{|X_1,X_2} (\cdot, x_1, x_2) \right](\tilde{x}_2) \neq \left[ L_{2,\beta}^{\text{Supp}(X_2),\tilde{\gamma}} \tilde{f}_{|X_1,X_2} (\cdot, x_1, x_2) \right](\tilde{x}_2).
\]

Let \( E, \tilde{E} \) be neighborhoods of \((x_2, \tilde{x}_2)\), respectively. In the proof to Theorem 4 it is shown that, without knowledge of \( f_{|X_1} \) or \( \tilde{f}_{|X_1} \), there does exist such an \( \tilde{x}_2 \) if the operator defined in equation (6) is injective. This can be viewed as a partial converse to Johnson (2004)'s high-level condition: in that case, without knowledge of \( f_{|X_1} \) or \( \tilde{f}_{|X_1} \), one can show there does not exist such an \( \tilde{x}_2 \) if their ‘rank’ condition does not apply. In principle, the logic of Assumption F4 can be extended to the general discrete choice panel model of Johnson (2004), if the distribution of unobserved heterogeneity is required to be independent of covariates.

To state the theorem denote \( \gamma = \{ \gamma_t : t = 1, \ldots, T \} \).

**Theorem 4** (Identification). Assume the distribution of \((X_t, A_t)_{t=1}^{T} \) is observed for \( T \geq 2 \), generated from agents solving the model of equation (3) satisfying assumptions F1-F4. Then \((\gamma_T, f_{|X_1}) \) is point identified.

Section A.2 contains the proof of Theorem 4.

### 3.2 Non-stationary conditional choice probabilities without the terminal period

In many empirical settings, the decision horizon of the agent extends beyond the period of observation. For example, a worker’s labor force participation decisions may not be observed
for their entire working life. This poses an issue for identification since in-sample decisions reflect payoff parameters for both in- and out-of-sample time periods. This section provides two solutions for this issue. The first approach is to impose restrictions on out-of-sample payoffs. Section 3.2.1 adopts this approach and shows that the model without random intercepts is identified.

The second approach is to use a property of the state transition known as ‘finite dependence’, which occurs if multiple sequences of actions leads to the same distribution of the state variable (Arcidiacono and Ellickson 2011). Finite dependence limits the number of out-of-sample time periods that affect in-sample decisions. Section 3.2.2 considers a model that exhibits finite dependence, and shows a binary choice model with random coefficients is identified.

For both approaches, I consider a model that satisfies the following condition:

**Assumption F2′.** (i) $S_t = (X_t, \beta) \in \mathbb{R}^k \times \mathbb{R}^{|A|}$, $\beta \mid X_1 = x$ is either discrete or absolutely continuous and $f_{\beta|x}$ is bounded. (ii) For $\gamma_t = \{\gamma_{ta} \in \mathbb{R}^{k-1} : a \in \bar{A}\}$, $u_t(s,a) = x^\top(\beta_a, \gamma_{ta})$. (iii) $d\Pr(X_{t+1} = x' \mid A_t = a, X_t = x, \beta = b) = dF_{x_t}(x_{t+1} \mid x_t, a_t)$. (iv) $\Gamma_{At} \equiv (\gamma_{t[A]}, \gamma_{t2[A]}, \ldots, \gamma_{t|A|[A]}) \in \mathbb{R}^{|A|^{|A|}}$ is full rank. (v) The probability distribution of $X_{t+1}$ conditional upon $(A_t, X_t)$ has no singular components, and the associated probability density and mass functions are real analytic functions of $x_{t+1}$ with bounded analytic continuations to $\mathbb{R}^{|A|+1}$.

Analogously to the Sections 2 and 3.1, Assumptions F1 and F2′ are sufficient for injectivity of the integral operator with kernel function $P_t(a, x, b) = \Pr(A_t = a \mid X_t = x, \beta = b)$.

### 3.2.1 Out of sample restrictions

Let $T$ denote the final observed period and $T_1 > T$ denote the final decision period of the agent. Since we do not observe behavior in periods $(T + 1, \ldots, T_1)$, the following restriction is placed on out-of-sample behavior:

**Assumption F5.** For all $t \in (T + 1, \ldots, T_1)$, $\gamma_t = \gamma_T$ and $dF_{x_{t-1}}(x' \mid x, a) = dF_{x_{T-1}}(x' \mid x, a)$

Finally, as the eigendecomposition is different in the non-stationary case, the following support condition is changed from Assumption I3:
**Assumption F3’.** For each \( x \in \text{Supp}(X_1) \), \( \exists a \in A \) such that \( \forall a_2, a_3 \in A \) (i) \( \text{Supp}(X_1) \), \( S_2 = \text{Supp}(X_2 \mid X_1 = x, A_1 = a) \), \( S_3 = \text{Supp}(X_3 \mid X_2 \in S_2, A_2 = a_2) \) and \( \cap_{a_3 \in A} \text{Supp}(X_4 \mid X_3 \in S_3, A_3 = a_3) \) span \( \mathbb{R}^k \). (ii) \( \text{Supp}(X_3[|A|+1] \mid X_2 \in S_2, A_2 = a_2) \) and \( \text{Supp}(X_4[|A|+1] \mid X_3 \in S_3, A_3 = a_3) \) contain a non-empty open set.

With these conditions, identification as described in the following result is a Corollary of Theorem 2. The proof is found in Section B.1.1.

**Corollary 1.** Assume the distribution of \((X_t, A_t)_{t=1}^T\) is observed for \( T = 4 \), generated from agents solving the model of equation (3) satisfying Assumptions F1, F2∗, F3′ and F5. Then \((\gamma, f_{\beta|x_1})\) is point identified.

### 3.2.2 Finite dependence

A DDC model exhibits finite dependence if there are multiple sequences of actions that yield the same distribution over the state variable. Finite dependence is useful for estimation as it allows the continuation value to be expressed in terms of CCPs (Arcidiacono and Ellickson 2011). This fact also makes finite dependence useful for identification in models without permanent unobserved heterogeneity, as it reduces the number of periods of out-of-sample behavior that must be assumed known (Arcidiacono and Miller 2020, Section 3.3).

In this section I show a similar feature is present for models with continuous permanent unobserved heterogeneity. In particular, I assume the transition function exhibits a special case of finite-dependence: the renewal action. The canonical example of renewal is machine replacement, but models of turnover and job matching also display this pattern (Arcidiacono and Miller 2020). The following assumption imposes this restriction:

**Assumption F6.** For each \( t \), \( \exists a \in A \) such that \( dF_{x_t}(x'|x,a) = dF_{x_t}(x'|\bar{x},a) \) for all \( x, \bar{x} \in \text{Supp}(X_t) \).

The result uses the following support condition:

**Assumption F3′′.** For each \( x \in \text{Supp}(X_1) \), \( \exists a \in A \) such that (i) \( \text{Supp}(X_1) \), \( S_2 = \text{Supp}(X_2 \mid X_1 = x, A_1 = a) \), \( S_3 = \cap_{a_2 \in A} \text{Supp}(X_3 \mid X_2 \in S_2, A_2 = a_2) \) and \( \text{Supp}(X_4 \mid X_3 \in S_3, A_3 = 1) \) span \( \mathbb{R}^k \). (ii) \( \cap_{a_2 \in A} \text{Supp}(X_3[|A|+1] \mid X_2 \in S_2, A_2 = a_2) \) and \( \text{Supp}(X_4[|A|+1] \mid X_3 \in S_3, A_3 = 1) \) contain a non-empty open set.
Corollary 2. Assume the distribution of \((X_t, A_t)_{t=1}^T\) is observed, generated from agents solving the model of equation (3) with \(|A| = 1\) and satisfying assumptions \(F1, F2', F3'', \) and \(F6\). Then \((\gamma, f_{\beta|x_1})\) is point identified.

Section B.1.2 contains the proof to corollary 2. The most substantial steps follow the proof of Theorem 2. The key difference is in showing identification of the finite parameter \(\gamma\).

3.3 Random intercepts in a stationary model

This section considers identification of an infinite-horizon DDC model with random intercepts. It shows point identification can be attained under an additional restriction on the state transition. Specifically, there must be some point in the support of \(X_t\) for which the state transition is not choice dependent. For instance, the machine replacement model of Kasahara and Shimotsu (2009, Example 9) displays this property. Before introducing the restriction on the state transition, the next assumption states that the permanent unobserved heterogeneity enters the model as a random intercept:

**Assumption I2'.** (i) \(S_t = (X_t, \beta) \in \mathbb{R}^k \times \mathbb{R}^{|A|}\). \(\beta \mid X_1 = x\) is either discrete or absolutely continuous and \(f_{\beta|x}\) is bounded. (ii) For \(\gamma = \{\gamma_a \in \mathbb{R}^{k-1} : a \in \bar{A}\}\), \(u(s, a) = \beta_a + x^\top \gamma_a\). (iii) \(d\Pr(X_{t+1} = x_{t+1} \mid A_t = a_t, X_t = x_t, \beta = b) = dF_x(x_{t+1} \mid a_t, x_t)\). (iv) \(\Gamma_A \equiv (\gamma_1[[A]], \gamma_2[[A]], \ldots, \gamma_{|A|}[[A]]) \in \mathbb{R}^{|A| \times |A|}\) is full rank.

The next assumption strengthens Assumption I3 by requiring the state transition to be constant across choices:

**Assumption I3'.** For all \(x_1 \in \text{Supp}(X_1)\), \(\exists a_1 \in \text{Supp}(A_1)\) such that: (i) \(\text{Supp}(X_2[[A]] \mid X_1 = x_1, A_1 = a_1)\) and \(\text{Supp}(X_3[[A]] \mid X_2 \in \text{Supp}(X_2 \mid X_1 = x_1, A_1 = a_1), A_2 = 0)\) contain a non-empty open set on which \(dF_x(x' \mid \tilde{a}, x) = dF_x(x' \mid x, a)\); (ii) \(S_3 \equiv \text{Supp}((1, X_3) \mid X_2 \in \text{Supp}(X_2 \mid X_1 = x_1, A_1 = a_1), A_2 = 0)\) and \(\cap_{a_3 \in \text{Supp}(A_3)} \text{Supp}((1, X_4) \mid X_3 \in S_3, A_3 = a_3)\) span \(\mathbb{R}^{k+1}\).

**Corollary 3.** Assume the distribution of \((x_t, a_t)_{t=1}^T\) is observed for \(T \geq 4\), generated from agents solving the model of equation (3) satisfying assumptions I1, I2' and I3'. Then \((\gamma, f_{\beta|x_1})\) is point identified.
The proof to Corollary 3 is contained in Section B.1.3. It follows from the proofs of Theorems 2 and 3.

3.4 Identifying the number of mixture components

In the existing DDC literature, it is common to assume permanent unobserved heterogeneity is discrete. When this assumption is made, a key parameter is the number of support points of permanent unobserved heterogeneity. In practice, it is common to assume the number of support points is known, although there are methods to identify a lower bound on the number of support points (Kasahara and Shimotsu 2009; Kasahara and Shimotsu 2014; Kwon and Mbakop 2021) which have been applied in economics (Igami and Yang 2016). However, in general, these methods can only identify the number of support points if an upper bound is known. This is because there is no guarantee a priori that there is enough variation in the data and structure on the model to to identify any arbitrarily large number of types. Intuitively, the population likelihood may be flat as a mixture component is added, but this may be because the initial likelihood had the true number of mixture components or because the models with and without an additional mixture component are observationally equivalent. Technically, this issue can be resolved by imposing an injectivity condition, i.e., a rank assumption on an unobserved matrix (Kasahara and Shimotsu 2009, Proposition 3; Kwon and Mbakop 2021, Assumption 2.1).

The purpose of this section is to show the models of Theorem 2 and Corollary 1 satisfy a condition equivalent to Kwon and Mbakop (2021, Assumption 2.1) when the distribution of unobserved heterogeneity is discrete. This means the number of types is identified, without knowledge of an upper bound on the number of types.

**Corollary 4.** Assume the distribution of \( Y = (X_t, A_t)_{t=1}^T \) is observed for \( T \geq 3 \), generated from the DDC model satisfying either Assumptions I1-I3 or Assumptions F1, F2', F3' and F5. In addition, suppose that the support of \( \beta \mid X_1 \) has \( R < \infty \) points of support. Then \( R \) is identified as the rank of the operator

\[
[Lu](x_3) = \int u(x_2) \frac{f_{A_3A_2A_1X_1X_2|X_1}(a_3,a_2,a_1,x_3,x_2,x_1)}{F_{x_3}(x_3,a_2)F_{x_2}(x_2|a_1)} \, dx_2.
\]

The proof to Corollary 4 is found in Section B.1.4. The result means that the techniques of Kasahara and Shimotsu (2014) and Kwon and Mbakop (2021) can be used to consistently
estimate the number of types should the applied econometrician wish to maintain the standard assumption that permanent unobserved heterogeneity is discrete. Broadly speaking, these estimators consist of forming a matrix of observed choice probabilities with values of $X_3$ varying over the rows, and $X_2$ over the columns. The identification result means that, at the population level, the rank of the matrix equals the true number of types.

4 Estimation

This section considers consistent estimation of the model parameters $(F_x, \gamma, f_{\beta|x_1})$ in a short panel. The distribution of $Y = ((A_t, X_t)^T_{t=2}, A_1)$ conditional upon $X_1 = x_1$ can be written as

$$f_{Y|x_1}(y, x_1) = \int \prod_{t=2}^{T} (P_t(a_t, x_t, b; \gamma, F_x)F_{x_1}(x_t|x_{t-1}, a_{t-1})) P_1(a_1, x_1, b; \gamma, F_x)d f_{\beta|x_1}(b, x_1),$$

where the dependence of the CCPs on $(\gamma, F_x)$ is made explicit. I propose two-step sieve M-estimation based on the above expression. The first step consists of estimating the state transition $F_x = \{F_{x_t} : t = 1, \ldots, T\}$. The second step consists of forming the pseudo-likelihood function using the fact that the CCPs $P_t$ are known up to the state transition and payoff parameter $(F_x, \gamma)$, and using sieve M-estimation methods to estimate $(\gamma, f_{\beta|x_1})$.

It is of course possible to estimate the model in a single step as a sieve maximum likelihood problem. The advantage of the proposed two-step approach is computational: for example, in the infinite horizon case, the integrated value function does not have to be recomputed within the second step optimization. This is similar to the idea of using the Hotz and Miller (1993) inversion to avoid full solution estimation of parametric DDC models.

Although I show consistency for a general sieve space, this may be computationally burdensome to implement, since estimation requires computing the CCPs for every point in the support of the sieve. To circumvent this issue, I suggest a ‘fixed grid’ estimator (Heckman and Singer 1984) which reduces the computational burden by having a finite number of support points.

In this section, I focus on estimating the cumulative distribution function of $\beta$. While it would be possible to present conditions for consistent estimation of the density function, smoothness restrictions would rule out the possibility that the type distribution has discrete support, which is the standard assumption in the literature.
As a final comment, in practice there will be an approximation error in the evaluation of the CCPs. This problem is inherent to dynamic discrete processes with large state spaces, and has received significant attention in the recent literature (Rust 2008; Kristensen et al. 2021). I assume away the effect of these errors on estimation — that is, that the approximation error is negligible relative to sampling error. In principle, the results of Kristensen et al. (2021) could be used to explicitly consider the effect of value function approximation error on estimation, though I do not pursue this here. Of course, the approximation error can be made arbitrarily small at increased computational cost.

4.1 A general two-step seminonparametric estimator

In this section, I briefly outline the two-step sieve M-estimator and present the general consistency result. Denote the true parameters as \( \theta_0 = (F_x, \gamma, f_{\beta|x_1}) \in \Theta = \mathcal{F} \times \Gamma \times \mathcal{M} \), where \( \mathcal{F} \) is the space of state transitions, \( \Gamma \subseteq \mathbb{R}^{\text{dim} \gamma} \), and \( \mathcal{M} \) is the space of distribution functions on \( \text{Supp}(\beta) \) conditional upon \( x \in \text{Supp}(X_1) \). The first step consists of forming a consistent estimator \( \hat{F}_x \) for the state transition \( F_x \). Since the state transition is directly observed, standard non-parametric methods are available. For the second step, the log-likelihood contribution of the \( i \)-th observation is

\[
\psi(y_i, \hat{F}_x, \gamma, f_{\beta|x_1}) \equiv \log \int \prod_{t=1}^T P_t(a_{it}, x_{it}, b; \hat{F}_x, \gamma) df_{\beta|x_1}(b, x_{i1})
\]

Given a sieve space \( \mathcal{M}_n \), which approximates \( \mathcal{M} \) arbitrarily well for large \( n \), the second step estimator is defined as

\[
\frac{1}{n} \sum_{i=1}^n \psi(y_i, \hat{F}_x, \gamma, f_{\beta|x_1}) \geq \sup_{(\gamma, f) \in \Gamma \times \mathcal{M}_n} \frac{1}{n} \sum_{i=1}^n \psi(y_i, \hat{F}_x, \gamma, f) - o_p(1/n) \tag{7}
\]

The following result states that under standard regularity conditions, the estimator is consistent.

**Theorem 5.** Let \( (A_{it}, X_{it} : t = 1, \ldots, T)_{i=1}^n \) be iid data generated from the DDC model satisfying either Assumptions I1-I3 or Assumptions F1-F4. If Assumptions E1-E4 hold, then the estimator \( (\hat{\gamma}, \hat{f}_{\beta|x_1}) \) defined in equation (7) is consistent for \( (\gamma, f_{\beta|x_1}) \).

The full statement of Theorem 5 and its proof are contained in Appendix B.2.1.
4.2 Fixed grid estimation

In this section I propose a particular choice of sieve which has the advantage of being simple to implement: the first-order monotone spline sieve. This is a popular choice of sieve for seminonparametric models, see for example Heckman and Singer (1984), Chen (2007), and Fox, Kim, and Yang (2016). To define the sieve, let $B_n = \{b_j : j = 1, \ldots, B(n)\}$ be a set of knots that partition $\text{Supp}(\beta)$ and $X_n = \{X_{k,n} : k = 1, \ldots X(n)\}$ be a partition of $\text{Supp}(X_1)$. The sieve space $M_n$ is defined as follows:

$$\left\{ f : S_\beta \times \text{Supp}(X_1) \rightarrow [0,1] : f(b, x_1) = \sum_{j=1}^{B(n)} \sum_{k=1}^{X(n)} P_{j,k} 1(b_j \leq b) 1(x_1 \in X_{k,n}), P_{j,k} \geq 0, \sum_{j=1}^{B(n)} P_{j,k} = 1 \right\},$$

where the sets $(B_n, X_n)$ are tuning parameters. For given choice of tuning parameters, an element of $M_n$ is a piecewise constant function with jumps of size $P_{j,k}$ at point $b_j$. The computational advantages of this sieve are clear: to find the supremum in (7), the CCP functions need only be evaluated for the values $b_j \in B_n$. This would not be the case if the sieve space consisted of functions that were continuous in $b$.

A theoretical advantage of this sieve space is that many of the high-level conditions for consistency are attained as long as the number of knots does not grow too fast. See Appendix B.2.2 for details.

**Theorem 6.** Let $(A_{it}, X_{it} : t = 1, \ldots, T)_{i=1}^n$ be iid data generated from the DDC model satisfying either Assumptions I1-I3 or Assumptions F1-F4. If Assumptions E1, E3 and E4 hold, then the estimator $(\hat{\gamma}, \hat{f}_{\beta|x_1})$ defined in equation (7) is consistent for $(\gamma, f_{\beta|x_1})$.

To implement the estimator, the number and location of grid points must be chosen. For consistency, it is enough that $B(n) X(n) \log(B(n) X(n)) = o(n)$ and that the grid points become dense in the support of $(\beta, X_1)$. In principle, convergence rates for this estimator could be derived to determine optimal growth rates for $B(n), X(n)$.

For computation, it may be attractive to use profiling. In particular, to form $(\hat{\gamma}, \hat{f}_{\beta|x_1})$, fix $\gamma$ and let

$$\hat{f}_{\beta|x_1}(\gamma) = \arg \sup_{f \in M_n} \frac{1}{n} \sum_{i=1}^n \psi(y_i, \hat{F}_x, \gamma, f).$$

For the sieve space (8) this is a convex optimization problem, with a unique global optimum and that can be computed efficiently (e.g., Koenker and Mizera (2014)). The profile estimator
is formed as
\[ \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \hat{F}_x, \gamma, \hat{f}_\beta | x_1(\gamma)) \geq \sup_{\gamma \in \Gamma} \frac{1}{n} \sum_{i=1}^{n} \psi(y_i, \hat{F}_x, \gamma, \hat{f}_\beta | x_1(\gamma)) - o_p(1/n). \]

5 Applications

5.1 An optimal stopping problem

In this section, I estimate a model of New York City taxi driver’s labor supply where drivers may have heterogeneous returns to earnings. Following Frechette, Lizzeri, and Salz (2019), I model taxi driver’s labor supply decision as an optimal stopping problem. That is, after a driver concludes each trip \( t = 1, 2, \ldots, T \equiv \infty \), they observe a set of state variables \((x_t, \epsilon_t)\) and choose whether or not to end their shift. The choice is encoded in \( a_t \in \{0, 1\} \), which takes value 1 if the driver decides to end their shift after fare \( t \), and \( a_t = 0 \) otherwise. The publicly-observed state variables \( x_t \) consist of two components: cumulative hours worked \( h_t \) and cumulative earnings \( y_t \) in the shift (i.e., \( x_t = (y_t, h_t) \)).

I specify the flow utility function as follows. If a driver concludes their shift after fare \( t \), they realize their utility from earnings and disutility from time working:
\[ \beta y_t - (\gamma_0 + \gamma_1 h_t + \gamma_2 1(h_t > 5) + \epsilon_{t1}). \]

On the other hand, if the driver chooses to accept another fare, their flow utility is \( \epsilon_{t0} \). The privately-observed state variable \( \epsilon_t = (\epsilon_{t0}, \epsilon_{t1}) \in \mathbb{R}^2 \) is assumed to have the cumulative distribution function \( dF_\epsilon(\epsilon_t = e_t \mid x_t) = \prod_{a=0}^{1} \exp(-e_{ta}) \). That is, \( \epsilon_{ta} \) is independently drawn from an extreme value distribution with unit dispersion. It is assumed that drivers discount the future at rate \( \rho = 0.95 \).

The dataset is the universe of ‘yellow cab’ trips from 2011 and 2012 released by the New York City Taxi and Limousine Commission. For each trip, the dataset contains information on the fare earnings, distance, duration, time and date, in addition to car and driver unique identifiers. I use these identifiers to construct a subsample of cars that are driven by a single driver, and drivers that drive a single car, which I refer to ‘owner-operator’ trips. The total sample has 930 drivers and 14,070 trips (i.e., on average 15 trips per driver).

I estimate the structural parameter \( \theta = (\gamma, f_\beta) \) where \( \gamma = (\gamma_0, \gamma_1, \gamma_2) \) and \( f_\beta \) is the cumulative distribution function of \( \beta \) via the sieve maximum likelihood estimator of Section
4.2. The sieve space is chosen to have choose 51 knots equally spaced between 0 and 60. For comparison, I present the maximum likelihood estimate under the assumption that $\beta$ is degenerate (i.e., homogeneous returns to earnings).

| Estimator: | Parametric MLE | Sieve MLE |
|------------|----------------|-----------|
| Variable   | Estimate       | Standard errors | Estimate | Standard errors |
| Intercept  | 1.804          | 0.068      | 2.192    | 0.109 |
| Hours worked | 2.796          | 0.162      | 2.844    | 0.476 |
| 1(Hours worked > 5) | -0.233         | 0.146      | 0.231    | 0.282 |

Table 1: Estimation results for $\gamma$ for the optimal stopping problem of Section 5.1. “Parametric MLE” refers to the maximum likelihood estimate assuming that $\beta$ is a degenerate random variable. “Sieve MLE” refers to the maximum likelihood estimate under the assumption $\beta$ has an unknown distribution with support contained within $[0,60]$ using the fixed grid estimator of Section 4.2. “Estimate” refers to the point estimate and “Standard errors” refers to standard errors calculated as the standard deviation of the estimator over 1,000 bootstrap samples.

Table 1 presents the estimates for the homogeneous utility parameter $\gamma$. In the model with heterogeneous returns to earnings, the estimated coefficient on hours worked is non-zero with relatively small standard errors. This is important since identification of the DDC model relies on this parameter being non-zero (Assumption I2(iv)). As expected, the coefficient on hours worked is positive, indicating disutility from time spent working. Comparing the parametric and sieve estimates, we observe that the parametric model yields smaller standard errors than the nonparametric model. Qualitatively, the estimated intercept and effect of hours worked are similar between models. Conversely, the estimates of $\gamma_3$, which represents the additive utility effect of working a long shift (i.e., longer than 5 hours), change dramatically between the two models. Namely, in parametric model the additive effect is to increase utility, whereas in the model with heterogeneous returns to earnings the additive effect is to decrease utility.

Figure 1 presents the estimated distribution of $\beta$ from the fixed grid estimator. The estimated distribution has eight points of support, with mean 9.04, median 4.81, standard deviation 12.2 and skewness 3.21, indicating substantial heterogeneity in returns to earnings. For comparison, in the model where $\beta$ is assumed to be homogeneous, the maximum likelihood estimate is $\beta = 6.30$. 

23
5.2 A labor force participation model

This section revisits the female labor supply model of Altuğ and Miller (1998). I combine the life-cycle model of Altuğ and Miller (1998) with the identification results of Section 2 to estimate the distribution of labor productivity from data on labor force participation. Before introducing the econometric model used in this section, I discuss the approach of Altuğ and Miller (1998).

Altuğ and Miller (1998) introduces a framework to understand female labor supply that takes into account aggregate shocks and time non-separable preferences. In their model, agents gain utility from consumption and leisure. Under their specification of consumption and Pareto optimality, individual $i$ at time $t$ generates utility from consumption as:

$$\eta_i \lambda_t \beta_t \omega_t \exp(\gamma_3^T x_{Wit}) l_{it}.$$  

(9)
The term \((\eta \lambda_t)\) is the shadow value of consumption, which is estimated from data on consumption. The term \((\beta \omega_t \exp(\gamma'x_{Wit})l_{it})\) represents an individual’s predicted earnings\(^8\), which is equal to the amount of time they spend working conditional on participating, \(l_{it}\), multiplied by their marginal product. The individual-specific marginal product of labor consists of unobserved aggregate and individual productivity effects \((\omega_t, \beta_i)\) in addition to a component that depends on covariates \(x_{it}\). These terms are estimated from the wage equation, which is as follows:

\[
\tilde{w}_{it} = \omega_t \beta_i \exp(\gamma'x_{Wit}) \exp(\tilde{\epsilon}_{it}).
\]

Altuğ and Miller (1998) consider two estimators for the individual-specific productivity \(\beta_i\). First, they use the fixed effects estimator from the wage equation above. Of course, in the asymptotic framework considered in this paper where \(n\) is large but \(T\) is fixed, this estimator is subject to the incidental parameters problem and is not consistent in general. For the second estimator, the authors assume that the fixed effect is an unknown function of observables, and then estimate that function non-parametrically. The observed variables consists of demographic data such as race, marital status and education levels. This estimator will be inconsistent if the set of observed variables is misspecified—that is, if individual productivity cannot be written as a function of observed data. Furthermore, both estimators require that \(\tilde{\epsilon}_{it} - \tilde{\epsilon}_{i1}\) is mean independent of \(\beta_i\), which restricts how an individual’s wage can vary over time.

The identification results of Section 2 obviate the need to estimate individual-specific productivity from the wage equation. Instead, \(\beta_i\) can be interpreted as a random coefficient in the discrete choice model of labor force participation. In particular, suppose the per-period payoff from entering the labor market for individual of type \(\beta_i\) is:

\[
x_{it}^\top (\beta_i, \gamma) + \epsilon_{it1}
\]

with \(x_{it} = (\tilde{z}_{it}, 1, \text{hinc}_{it}, \text{age}_{it}, \text{nkids}_{it}, \text{educ}_{it})\). Here \(\tilde{z}_{it}\) is constructed following the schema of Altuğ and Miller (1998). Precisely, \(\tilde{z}_{it} = \hat{\eta}_i \lambda_t \omega_t \exp(\gamma'x_{it})\tilde{l}_{it}\). The remaining observed state variables are, respectively, a constant term, annual head-of-household income, age, and a dummy variable for the presence of children in the household.

Relative to the participation model in Altuğ and Miller (1998, Equation (6.7)), \(\beta_i\) is treated as an unobserved random variable. In their model \(\beta_i\) is replaced by first-stage

\(^8\)For clarity, in this section I will denote permanent unobserved heterogeneity as \(\beta_i\).
estimator $\hat{\beta}_i$, so that $\beta_i z_{it}$ is treated as a known constant. Like Altuğ and Miller (1998), I make the outside good assumption and assume that $\epsilon_{ita}$ is independently distributed extreme value type I. For simplicity, the agents’ time horizon is assumed to be infinite.

As in Altuğ and Miller (1998), the labor force participation model is estimated using a subset of data from the PSID. The construction of the subset largely followed the details in Altuğ and Miller (1998, Appendix B). The final data set contains 3084 individuals, each of whom have between four and ten panel observations, with an average close to eight. For the sieve space, I choose 71 knots equally spaced between 0 and 15. As in Section 5.1, for comparison I present maximum likelihood estimates assuming that $\beta_i$ is constant.

| Estimator: Parameter | Parametric MLE | Sieve MLE |
|----------------------|----------------|-----------|
|                      | Estimate | Standard errors | Estimate | Standard errors |
| Intercept            | -2.494  | 0.136     | -2.490  | 0.124   |
| Head-of-household income | -0.268  | 0.018     | -0.312  | 0.027   |
| Number of children   | 0.345   | 0.044     | 0.051   | 0.079   |
| Age                  | -0.979  | 0.041     | -0.607  | 0.077   |
| Education            | -0.045  | 0.048     | 0.333   | 0.075   |

Table 2: Estimation results for $\gamma$ for the participation model of Section 5.2. “Parametric MLE” refers to the maximum likelihood estimate assuming that $\beta_i$ is a point mass. “Sieve MLE” refers to the maximum likelihood estimate under the assumption $\beta_i$ has an unknown distribution with support contained within $[0, 15]$ using the fixed grid estimator of Section 4.2. “Estimate” refers to the point estimate and “Standard errors” refers to standard errors calculated as the standard deviation of the estimator over 633 bootstrap samples.

Table 2 presents point estimates of the finite dimensional parameter $\gamma$ alongside bootstrapped standard errors. Estimates of the model with heterogeneous labor productivity indicate the utility from working increases with education, but decreases with head-of-household income and age. The number of children is estimated to have a negligible effect on utility from working. This is unlike estimates of the model with constant labor productivity, in which the number of children increase the utility of working but education has a small negative effect.

Figure 2 presents the estimated distribution of $\beta_i$ from the fixed grid estimator. The estimated distribution has 48 points of support, with mean 3.17, median 3.00, standard deviation 1.43 and skewness 2.03, indicating substantial heterogeneity in labor productiv-
ity. For comparison, in the model where $\beta_i$ is assumed to be homogeneous, the maximum likelihood estimate is 2.92.

![Graph showing estimated distribution of $\beta_i$ for participation model of Section 5.2.](image)

**Figure 2:** Estimated distribution of $\beta_i$ for the participation model of Section 5.2. The black curve represents the point estimate, the red curves represent bootstrapped 95% pointwise confidence intervals. The ticks on the x-axis represent the grid points.

## 6 Monte Carlo Simulations

This section investigates the fixed grid estimator in a Monte Carlo simulation. The main goals of this section are threefold: first, to explore the finite sample performance of the estimator; second, to demonstrate the computational requirements; and, third, to verify the asymptotic results of Section 4. I simulate data using a simple labor force participation model based on Altuğ and Miller (1998, Section 6).

In each period, each individual decides whether or not to enter the labor force, upon observation of the state variable. Thus $A = \{0, 1\}$, with $a_t = 1$ representing an individual
decision to enter the labor force at time $t$. The state variables are $s_t = (x_t, \epsilon_t)$ where $\epsilon_t = (\epsilon_{t0}, \epsilon_{t1})$ is unobserved and $x_t \in X \subseteq \mathbb{R}^2$ is observed. The period period payoff from entering the labor market depends on individual-specific labor productivity $\beta$ as follows:

$$\beta x_{t1} + \gamma x_{t2} + \epsilon_{t1}$$

Following the model of Altu˘g and Miller (1998), $x_{t1}$ can be interpreted as an average consumption value (see Section 5 for details) and $x_{t2}$ is equal to the income of the primary earner in the household. The period payoff from not entering is $\epsilon_{t0}$. The random preference shock $\epsilon_{ta}$ is assumed to be independently distributed extreme value type I, and the agents’ time horizon is assumed to be infinite. In addition, I assume that $\beta$ is independent of $X_t$ and follows a mixture of three truncated normal distributions. In particular

$$\beta \sim \begin{cases} 
N_{tr}(1.5, 1) & \text{with prob. } 1/3 \\
N_{tr}(2.5, 0.25) & \text{with prob. } 1/3 \\
N_{tr}(3.5, 1) & \text{with prob. } 1/3 
\end{cases}$$

Where $N_{tr}(\mu, \sigma)$ is the truncated normal distribution with parameters $(\mu, \sigma)$, minimum value 0 and maximum value 50. The simulation results are the average of 1,000 i.i.d. datasets $(a_{it}, x_{it} : t = 1, \ldots, 8)_{i=1}^n$ drawn from this model. Results are presented for four sample sizes: $n = 100$, $n = 500$, $n = 1,000$ and $n = 10,000$. For estimation I choose the number of grid points equal to $4n^{1/4}$, which satisfies the rate conditions required for Theorem 6.

Table 3 presents results for the estimator of $(\gamma, f_{\beta})$, in addition to computation times. First consider results for $\gamma$. Here, empirical variance is significantly larger than empirical bias, which diminishes with sample size. Scaled empirical mean squared error is largely flat across sample sizes. In terms of computational burden, the fixed grid estimator takes around 30 seconds to run for the smaller sample sizes, though it takes around 2 minutes for $n = 10,000$.

Turning to results for the estimation of $f_{\beta}$, both measures of integrated error diminish with sample size. The number of grid points increases slowly with sample size — indeed slower than the growth of the number of support points selected by the estimator. For $n = 100$, on average 3.8 points are selected. This increases to 10.1 for the large sample size. This pattern is broadly similar to previous simulation results for a parametric variant of this estimator (Fox et al. 2011).

---

9Integrated absolute error for simulation run $m$ with estimate $\hat{f}_{\beta,m}$ is $\int |\hat{f}_{\beta,m}(b) - f_{\beta}(b)| \, db$ and mean integrated squared error is equal to $\frac{1}{M} \sum_{m=1}^{M} \int \left( \hat{f}_{\beta,m}(b) - f_{\beta}(b) \right)^2 \, db$ where $M = 1,000$. 

28
Table 3: Simulation results for estimation of $\gamma$ and $f_\beta$. “$\gamma$” denotes results for estimation of $\gamma$, which includes scaled average empirical bias (“Bias”), variance (“Var”) and mean-squared error (“MSE”). “Time” denotes median computation time in seconds. “MISE” denotes empirical mean integrated squared error, “IAE” denotes empirical integrated absolute error, and “No. types” denotes the number of support points.

|        | $n = 100$ | $n = 500$ | $n = 1,000$ | $n = 10,000$ |
|--------|------------|------------|-------------|--------------|
| $\gamma$ | Bias       | -0.328     | -0.211      | -0.093       | 0.074        |
|        | Var        | 2.750      | 2.890       | 2.840        | 2.720        |
|        | MSE        | 2.860      | 2.930       | 2.850        | 2.730        |
|        | Time       | 27.3       | 31.9        | 41.4         | 131.9        |
|        | MISE       | 0.0754     | 0.040       | 0.032        | 0.020        |
|        | Mean       | 0.458      | 0.334       | 0.302        | 0.240        |
|        | Min        | 0.255      | 0.199       | 0.199        | 0.18         |
|        | Max        | 0.925      | 0.520       | 0.482        | 0.337        |
| $4n^{1/4}$ |            |            |             |              |
|        | Mean       | 5.2        | 6.8         | 7.6          | 10.1         |
|        | Min        | 2          | 4           | 5            | 6            |
|        | Max        | 9          | 9           | 11           | 24           |

Figure 3 presents empirical quantiles for the estimator of $f_\beta$. For each sample size the median estimate (the black curve) falls close to the true distribution (the blue curve). The empirical pointwise confidence bands are substantially narrower for the larger sample sizes.

7 Conclusion

In this paper I show point identification of a broad class of multinomial dynamic discrete choice models with multivariate continuous permanent unobserved heterogeneity. Relative to the existing literature, I allow for permanent unobserved heterogeneity that is both multivariate and continuous, and provide low-level conditions for point identification. My results encompass both finite and infinite horizon models, and do not rely on a full support condition, nor parametric assumptions on the distribution on permanent unobserved heterogeneity.

I propose a seminonparametric estimator for the distribution of continuous permanent
Figure 3: Simulation results for estimation of $f_\beta$ for each sample size. The black curve represents the median estimate, the red curves pointwise 97.5%, 2.5% quantiles, and the blue curve the true distribution. The ticks on the x-axis represent the grid points.

unobserved heterogeneity in the style of Heckman and Singer (1984). The estimator is computationally simple, and coincides with the estimator for a semiparametric model. As a result, the applied econometrician can proceed as they would for discrete permanent unobserved heterogeneity, providing they commit to increasing the number of support points as the sample size grows.

References

Aguirregabiria, V., Gu, J., and Luo, Y. (2021). “Sufficient statistics for unobserved heterogeneity in structural dynamic logit models”. *Journal of Econometrics* 223.2, pp. 280–311.
Aguirregabiria, V. and Mira, P. (2002). “Swapping the nested fixed point algorithm: A class of estimators for discrete Markov decision models”. *Econometrica* 70.4, pp. 1519–1543.
— (2010). “Dynamic discrete choice structural models: A survey”. *Journal of Econometrics* 156.1, pp. 38–67.

Altuğ, S. and Miller, R. A. (1998). “The effect of work experience on female wages and labour supply”. *The Review of Economic Studies* 65.1, pp. 45–85.

Andrews, D. W. (2017). “Examples of L2-complete and boundedly-complete distributions”. *Journal of Econometrics* 199.2, pp. 213–220.

Arcidiacono, P. and Ellickson, P. B. (2011). “Practical methods for estimation of dynamic discrete choice models”. *Annu. Rev. Econ.* 3.1, pp. 363–394.

Arcidiacono, P. and Miller, R. A. (2011). “Conditional choice probability estimation of dynamic discrete choice models with unobserved heterogeneity”. *Econometrica* 79.6, pp. 1823–1867.
— (2020). “Identifying dynamic discrete choice models off short panels”. *Journal of Econometrics* 215.2, pp. 473–485.

Bajari, P., Chernozhukov, V., Hong, H., and Nekipelov, D. (2015). *Identification and efficient semiparametric estimation of a dynamic discrete game*. Tech. rep. National Bureau of Economic Research.

Berry, S. T. and Compiani, G. (2022). “An Instrumental Variable Approach to Dynamic Models”. *The Review of Economic Studies*.

Cameron, S. V. and Heckman, J. J. (1998). “Life cycle schooling and dynamic selection bias: Models and evidence for five cohorts of American males”. *Journal of Political economy* 106.2, pp. 262–333.

Carrasco, M., Florens, J.-P., and Renault, E. (2007). “Linear inverse problems in structural econometrics estimation based on spectral decomposition and regularization”. *Handbook of Econometrics* 6, pp. 5633–5751.

Chamberlain, G. (2010). “Binary response models for panel data: Identification and information”. *Econometrica* 78.1, pp. 159–168.

Chen, X. (2007). “Large sample sieve estimation of semi-nonparametric models”. *Handbook of Econometrics* 6, pp. 5549–5632.

Chernozhukov, V., Fernández-Val, I., Hahn, J., and Newey, W. (2013). “Average and quantile effects in nonseparable panel models”. *Econometrica* 81.2, pp. 535–580.

Compiani, G. and Kitamura, Y. (2016). “Using mixtures in econometric models: a brief review and some new results”. *The Econometrics Journal* 19.3, pp. C95–C127.
Fox, J., Kim, K., Ryan, S., and Bajari, P. (2011). “A simple estimator for the distribution of random coefficients”. *Quantitative Economics* 2.3, pp. 381–418.

Fox, J. T., Kim, K. I., and Yang, C. (2016). “A simple nonparametric approach to estimating the distribution of random coefficients in structural models”. *Journal of Econometrics* 195.2, pp. 236–254.

Frechette, G. R., Lizzieri, A., and Salz, T. (2019). “Frictions in a competitive, regulated market: Evidence from taxis”. *American Economic Review* 109.8, pp. 2954–92.

Freyberger, J. (2018). “Non-parametric Panel Data Models with Interactive Fixed Effects”. *The Review of Economic Studies* 85.3, pp. 1824–1851.

Heckman, J. and Singer, B. (1984). “A method for minimizing the impact of distributional assumptions in econometric models for duration data”. *Econometrica*, pp. 271–320.

Heckman, J. J., Humphries, J. E., and Veramendi, G. (2018). “Returns to education: The causal effects of education on earnings, health, and smoking”. *Journal of Political Economy* 126.S1, S197–S246.

Hornik, K. (1993). “Some new results on neural network approximation”. *Neural Networks* 6.8, pp. 1069–1072.

Hornik, K., Stinchcombe, M., and White, H. (1989). “Multilayer feedforward networks are universal approximators.” *Neural Networks* 2.5, pp. 359–366.

Hotz, V. J. and Miller, R. A. (1993). “Conditional choice probabilities and the estimation of dynamic models”. *The Review of Economic Studies* 60.3, pp. 497–529.

Hu, Y. and Schennach, S. M. (2008). “Instrumental variable treatment of nonclassical measurement error models”. *Econometrica* 76.1, pp. 195–216.

Hu, Y. and Shum, M. (2012). “Nonparametric identification of dynamic models with unobserved state variables”. *Journal of Econometrics* 171.1, pp. 32–44.

Igami, M. and Yang, N. (2016). “Unobserved heterogeneity in dynamic games: Cannibalization and preemptive entry of hamburger chains in Canada”. *Quantitative Economics* 7.2, pp. 483–521.

Illanes, G. and Padi, M. (2019). *Retirement policy and annuity market equilibria: Evidence from chile*. Tech. rep. National Bureau of Economic Research.

Johnson, E. G. (2004). “Identification in discrete choice models with fixed effects”. *Working paper, Bureau of Labor Statistics*. Citeseer.

Kasahara, H. and Shimotsu, K. (2009). “Nonparametric identification of finite mixture models of dynamic discrete choices”. *Econometrica* 77.1, pp. 135–175.
Kasahara, H. and Shimotsu, K. (2014). “Non-parametric identification and estimation of the number of components in multivariate mixtures”. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 76.1, pp. 97–111.

Keane, M. P. and Wolpin, K. I. (1997). “The career decisions of young men”. *Journal of Political Economy* 105.3, pp. 473–522.

Koenker, R. and Mizera, I. (2014). “Convex optimization, shape constraints, compound decisions, and empirical Bayes rules”. *Journal of the American Statistical Association* 109.506, pp. 674–685.

Krantz, S. G. and Parks, H. R. (2002). *A primer of real analytic functions*. Springer Science & Business Media.

Kristensen, D., Mogensen, P. K., Moon, J. M., and Schjerning, B. (2021). “Solving dynamic discrete choice models using smoothing and sieve methods”. *Journal of Econometrics* 223.2, pp. 328–360.

Kwon, C. and Mbakop, E. (2021). “Estimation of the number of components of nonparametric multivariate finite mixture models”. *The Annals of Statistics* 49.4, pp. 2178–2205.

Magnac, T. and Thesmar, D. (2002). “Identifying dynamic discrete decision processes”. *Econometrica* 70.2, pp. 801–816.

Masten, M. A. (2018). “Random coefficients on endogenous variables in simultaneous equations models”. *The Review of Economic Studies* 85.2, pp. 1193–1250.

Mattner, L. (1999). *Complex differentiation under the integral*. Universität Hamburg. Institut für Mathematische Stochastik.

Nevo, A., Turner, J. L., and Williams, J. W. (2016). “Usage-based pricing and demand for residential broadband”. *Econometrica* 84.2, pp. 411–443.

Norets, A. and Tang, X. (2014). “Semiparametric inference in dynamic binary choice models”. *Review of Economic Studies* 81.3, pp. 1229–1262.

Rust, J. (1987). “Optimal replacement of GMC bus engines: An empirical model of Harold Zurcher”. *Econometrica*, pp. 999–1033.

— (2008). “Dynamic programming”. *The New Palgrave Dictionary of Economics* 1, p. 8.

Rust, J. and Phelan, C. (1997). “How social security and medicare affect retirement behavior in a world of incomplete markets”. *Econometrica*, pp. 781–831.

Stinchcombe, M. and White, H. (1998). “Consistent specification testing with nuisance parameters present only under the alternative”. *Econometric Theory* 14.3, pp. 295–325.

Williams, B. (2020). “Nonparametric identification of discrete choice models with lagged dependent variables”. *Journal of Econometrics* 215.1, pp. 286–304.
A. Proofs

Throughout this appendix I use the following notations: \( S_\beta = \text{Supp}(\beta) \); for \( \lambda \) the Lebesgue measure, \( L^2_A \) is the usual \( L^2 \) space \( L^2(A, \lambda) \) and \( L_A \) is the usual \( L^\infty \) space \( L^\infty(A, \lambda) \); \( \text{sp}\, A \) indicates the linear span of set \( A \), and \( \overline{\text{sp}}\, A \) indicate its closure in the \( L^2 \) norm.

A.1 Proof of results in Section 2

A.1.1 Proof of Theorem 1

Proof. Suppose the measure is absolutely continuous. For \( \mathcal{V} \subset \mathbb{R}^k \), define

\[
L^*_{\beta, \mathcal{V}} : L_{S_\beta} \to \mathcal{L}_\mathcal{V} \quad [L^*_{\beta, \mathcal{V}} m](v) = \int P(0, x, b) m(b) db.
\]

We wish to prove \( L^*_{\beta, \mathcal{X}} \) is injective. Define \( \overline{\mathcal{X}} = \mathbb{R}^{|A|+1} \times \{ x_{[-(|A|+1)]} : x \in \mathcal{X} \} \). By Lemma A.1, \( x_{[|A|+1]} \mapsto P(a, x, b) \) is a real analytic function on \( \mathbb{R}^{|A|+1} \) for any fixed \( (a, x_{[-(|A|+1)]}, b) \).

Since \( S_\beta \) is compact and \( \{ x_{[|A|+1]} : x \in \mathcal{X} \} \) contains a non-empty open set, by Lemma A.3 injectivity of \( L^*_{\beta, \overline{\mathcal{X}}} \) implies injectivity of \( L^*_{\beta, \mathcal{X}} \). It remains to show \( L^*_{\beta, \overline{\mathcal{X}}} \) is injective. Define

\[
\tilde{\mathcal{H}} = \{ h : S_\beta \to [0, 1] : h(b) = P(0, x, b), x \in \overline{\mathcal{X}} \}.
\]

By Lemma A.1 and Theorem 3.1 in Stinchcombe and White (1998), \( L^*_{\beta, \overline{\mathcal{X}}} \) is injective if \( \text{sp}\tilde{\mathcal{H}} \) is dense in \( L^2_{S_\beta} \). First, by Lemma A.2,

\[
\overline{\text{sp}}\tilde{\mathcal{H}} \supset \mathcal{H}_1 = \{ h : S_\beta \to [0, 1] : h(b) = 1\{ \alpha_1 b > \alpha_0 \}, (\alpha_1, \alpha_0) \in \mathbb{R}^{|A|+1} \}.
\]

Then, Hornik, Stinchcombe, and White (1989, Corollary 2.2) (i.e., with \( r = |A| \) and \( \Psi(\lambda) = 1\{ \lambda > 0 \} \)) implies that \( \overline{\text{sp}}\mathcal{H}_1 \supset L^2_{S_\beta} \) and thus \( \overline{\text{sp}}\tilde{\mathcal{H}} \supset L^2_{S_\beta} \) as required.

Now suppose the measure has \( R < \infty \) points of support. In this case, the operator \( L^*_{\beta, \overline{\mathcal{X}}} : \mathbb{R}^R \to \mathcal{L}_\overline{\mathcal{X}} \). From the above approximation result, for each \( r \), a sequence with elements \( x_{r,n} = (x_{[|A|+1],r,n}, x_{[-(|A|+1)]}) \in \overline{\mathcal{X}} \) can be found such that \( \lim P(0, x_{r,n}, b_r) = 1 \) for \( r \geq r \) and \( \lim P(0, x_{r,n}, b_r) = 0 \) for \( r < r \). Define a sequence of \( R \times R \) matrices whose \( r \)-th row is \( \tilde{P}(x_{r,n}) \equiv (P(0, x_{r,n}, b_r) : \bar{r} = 1, \ldots, R) \). Since the limit of the sequence of matrices is full rank, for any \( m \in \mathbb{R}^R \), for \( n \) large enough \( \forall r = 1, \ldots, R, \tilde{P}(x_{r,n}) \bar{m} = 0 \) implies \( m = 0 \).

\[ \square \]
A.1.2 Proof of Theorem 2

Proof of Theorem 2. By Assumptions I1 and I2,
\[ f_{A_4A_3A_2A_1A_4A_3A_2A_1}(a_4, a_3, 0, a_1, x_4, x_3, x_2, x_1) = \int P(a_4, x_4, b) F_{x}(x_4|x_3, a_3) P(a_3, x_3, b) \times F_x(x_3|x_2, 0) P(0, x_2, b) F_x(x_2|x_1, a_1) P(a_1, x_2, b) df_{\beta|x_1}(b, x_1). \]

Where the transition kernel has positive measure, we can write
\[ \frac{f_{A_4A_3A_2A_1A_4A_3A_2A_1}(a_4, a_3, 0, a_1, x_4, x_3, x_2, x_1)}{F_x(x_4|x_3, a_3) F_x(x_3|x_2, 0) F_x(x_2|x_1, a_1)} = \int P(a_4, x_4, b) P(a_3, x_3, b) P(0, x_2, b) P(a_1, x_2, b) df_{\beta|x_1}(b, x_1). \]

Fix \( x_1 \in \text{Supp}(X_1) \) and let \( a_1 \in \text{Supp}(A_1) \) satisfy Assumption I3. Let \( S_2 = \text{Supp}(X_2|X_1 = x_1, A_1 = a_1) \) and \( S_4 = \cap_{a_1 \in A} \text{Supp}(X_4|X_3 \in S_3, A_3 = a_3) \) and define the operators \( L_{3,4,2} : \mathcal{L}_{S_2} \to A \times \mathcal{L}_{S_3} \) and \( L_{3,2} : \mathcal{L}_{S_2} \to A \times \mathcal{L}_{S_3} \) as follows:
\[
[L_{3,4,2}] (a_3, x_3) = \int \frac{f_{A_4A_3A_2A_1A_4A_3A_2A_1}(a_4, a_3, 0, a_1, x_4, x_3, x_2, x_1)}{F_x(x_4|x_3, a_3) F_x(x_3|x_2, 0) F_x(x_2|x_1, a_1)} m(x_2) dx_2,
\]
\[
[L_{3,2}] (a_3, x_3) = \int \frac{f_{A_3A_2A_1A_4A_3A_2A_1}(a_3, 0, a_1, x_3, x_2, x_1)}{F_x(x_3|x_2, 0) F_x(x_2|x_1, a_1)} m(x_2) dx_2.
\]

Under Assumption I3 the above operators are observed and well-defined for some fixed \((x_4, a_4)\). The operators can be decomposed into constituent parts. For this purpose define
\[
L_{3,\beta} : \mathcal{L}_{\beta} \to A \times \mathcal{L}_{\beta}, \quad [L_{3,\beta}] (a_3, x_3) = \int P(a_3, x_3, b) m(b) db,
\]
\[
D_{\beta}^4 : \mathcal{L}_{\beta} \to \mathcal{L}_{\beta}, \quad [D_{\beta}^4] (b) = P(a_4, x_4, b) m(b),
\]
\[
D_{\beta} : \mathcal{L}_{\beta} \to \mathcal{L}_{\beta}, \quad [D_{\beta}] (b) = P(a_1, x_1, b) f_{\beta|x_1}(b, x_1) m(b),
\]
\[
L_{\beta,2} : \mathcal{L}_{S_2} \to \mathcal{L}_{\beta}, \quad [L_{\beta,2}] (b) = \int P(0, x_2, b) m(x_2) dx_2.
\]

It is straightforward to derive that \( L_{3,4,2} = L_{3,\beta} D_{\beta}^4 D_{\beta} L_{\beta,2} \) and \( L_{3,2} = L_{3,\beta} L_{\beta,2} \).

By Theorem 1, \( L_{3,\beta} \) and \( L_{\beta,2}^* \) are injective where \( L_{\beta,2}^* \) is the adjoint\(^{10}\) of \( L_{\beta,2} \). Then, since \( D_{\beta} \) is invertible (as \( P(a_1, x_1, b) f_{\beta|x_1}(b, x_1) > 0 \) almost surely-S\( \beta \)) and \( L_{3,\beta} \) and \( L_{\beta,2}^* \) are injective, \( L_{3,2} \) has a right inverse, the equivalence
\[
L_{4,3,2} L_{3,2}^{-1} = L_{3,\beta} D_{\beta}^4 L_{3,\beta}
\]
\(^{10}\)The adjoint of a linear operator between Hilbert Spaces \( L : U \to V \) is the operator \( L^* : V \to U \) that satisfies \( \langle Lu, v \rangle_V = \langle u, L^* v \rangle_U \) where \( \langle \cdot, \cdot \rangle_W \) is the inner product on \( W \). See Carrasco, Florens, and Renault (2007) for further discussion.
holds and $L_{3,\beta}D_{\beta}^4L_{3,\beta}^{-1}$ is the eigendecomposition of the known operator $L_{4,3,2}^4L_{3,2}^{-1}$ (Williams 2020, Lemma A.1). Each $b$ indexes an eigenvalue $P(a_4, x_4, b)$ of $L_{4,3,2}^4L_{3,2}^{-1}$, with corresponding eigenfunction $(a_3, x_3) \mapsto P(a_3, x_3, b)$. As in Hu and Schennach (2008), the decomposition is unique up to (1) scaling of the eigenfunctions, (2) uniqueness of the eigenvalues, and (3) reindexing of the eigenvalues (“ordering”).

First, the scale of the eigenfunctions $(a_3, x_3) \mapsto P(a_3, x_3, b)$ is fixed since they are probabilities that must satisfy $\sum_{a_3 \in A} P(a_3, x_3, b) = 1$. Second, for eigenvalue uniqueness, as shown in Hu and Schennach (2008, p. 213), it is sufficient that for each $b \neq \tilde{b} \in S_\beta$, there exist some $(a_4, x_4) \in A \times S_4$ such that $P(a_4, x_4, b) \neq P(a_4, x_4, \tilde{b})$. To show this, suppose for all $(a_4, x_4) \in A \times S_4$, $P(a_4, x_4, b) = P(a_4, x_4, \tilde{b})$. Then, by standard arguments for identification of homogenous parameters in DDC models (e.g., Bajari et al. 2015, Section 3.5), it follows that for each $a \in A$

$$\left(\tilde{b}_a \ 	ilde{\gamma}_a^\top\right) x_4 = \left(b_a \ \gamma_a^\top\right) x_4.$$

Then, since $S_4$ contains $k$ linearly independent elements, $\tilde{b}_a = b_a$ and thus $\tilde{b} = b$ as required.

Finally, the problem of ordering arises because any injective function $R$ may be used to redefine the latent variable $\beta = R(\tilde{\beta})$ while satisfying $L_{3,\tilde{\beta}}D_{\tilde{\beta}}^4L_{3,\tilde{\beta}}^{-1} = L_{3,\beta}D_{\beta}^4L_{3,\beta}^{-1}$\footnote{This equality is shown explicitly in Hu and Schennach (2008, Supplement S.3).} where

$$L_{3,\tilde{\beta}} : \mathcal{L}_{S_{\tilde{\beta}}} \to A \times \mathcal{L}_{S_3} \quad [L_{3,\tilde{\beta}}m](a, x) = \int \Pr(A_3 = a \mid X_3 = x, \tilde{\beta} = b)m(b)db,$$

$$D_{\tilde{\beta}}^4 : \mathcal{L}_{S_{\tilde{\beta}}} \to \mathcal{L}_{S_{\tilde{\beta}}} \quad [D_{\tilde{\beta}}^4m](b) = \Pr(A_4 = a_4 \mid X_4 = x_4, \tilde{\beta} = b)m(b).$$

Notice that $\Pr(A_3 = a \mid X_3 = x, \tilde{\beta} = b) = \Pr(A_3 = a \mid X_3 = x, \beta = R(b)) = P(a, x, R(b))$. I show the only admissible reordering function is identity. For this purpose, suppose that for all $(a_3, x_3) \in A \times S_3$, $P(a_3, x_3, R(b)) = P(a_3, x_3, b)$. By standard arguments for identification of homogenous parameters in DDC models (e.g., Bajari et al. 2015, Section 3.5), it follows that for each $a \in A$,

$$\left(R(b_a) \ 	ilde{\gamma}_a^\top\right) x_3 = \left(b_a \ \gamma_a^\top\right) x_3.$$

Under Assumption I3(ii) $S_3$ contains $k$ linearly independent vectors, so it follows that $(R(b_a), \tilde{\gamma}_a) = (b_a, \gamma_a)$ and thus $\gamma$ and $P(a_3, x_3, \beta)$ are identified.

To identify $f_{\beta|x_1}$, notice that

$$\frac{f_{a_2a_1|x_2|x_1}(0, a_1, x_2, x_1)}{F_x(x_2|x_1, a_1)} = \left[L_{3,\beta}^2(P(a_1, x_1, \cdot) f_{\beta|x_1}(\cdot | x_1))\right](x_2).$$
$L_{\beta, 2}^*$ is injective and identified, since its kernel (the CCP function) is identified. Applying the left inverse of $L_{\beta, 2}^*$, $P(a, x_1, b) f_{\beta|x_1}(b, x_1)$ and thus $f_{\beta|x_1}(b, x_1)$ is identified.

In the case that $\beta$ conditional upon $X_1 = x_1$ has $R < \infty$ points of support, the above arguments apply directly with matrices replacing integral operators where appropriate. □

### A.1.3 Supporting lemmas

**Lemma A.1** (Properties of the CCP function). **Assume I1 and I2.** For any $x \in \mathcal{X}$ for which $\{x_{[|A|+1]} : x \in \mathcal{X}\}$ contains a non-empty set and $\overline{\mathcal{X}} = \mathbb{R}^{[|A|+1]} \times \{x_{[-(|A|+1)}} : x \in \mathcal{X}\}$, the set \( \mathcal{H} = \{h : S_\beta \to [0, 1] : h(b) = P(0, x, b), x \in \overline{\mathcal{X}}\} \) is a norm bounded subset of $L_{S_\beta}^2$. The function $x_{[|A|+1]} \mapsto P(a, x, b)$ are real analytic functions on $\mathbb{R}^{[|A|+1]}$ for any fixed $(a, x_{[-(|A|+1)], b})$.

**Proof.** Under Assumptions I1 and I2, for any $(a, x, b) \in A \times \mathcal{X} \times S_\beta$,

$$P(a, x, b) = \frac{\exp(x^\top(b, \gamma_a) + \rho \int v(x', b) dF_x(x'|x, a))}{\sum_{a \in A} \exp(x^\top(b, \gamma_a) + \rho \int v(x', b) dF_x(x'|x, a))}. \quad (14)$$

Since $\{x_{[|A|+1]} : x \in \mathcal{X}\}$ contains an open set and the analytic continuation of a vanishing function on an open set is vanishing everywhere, the analytic continuation of $x_{[|A|+1]} \mapsto F_x(x'|x, a)$ to $\mathbb{R}^{[|A|+1]}$ satisfies $\{x' : \exists x \in \mathcal{X}, dF_x(x'|x, a) > 0\} = \{x' : \exists x \in \overline{\mathcal{X}}, dF_x(x'|x, a) > 0\}$. Therefore $P$ in equation (14) is well-defined on $A \times \overline{\mathcal{X}} \times S_\beta$.

Since the set $S_\beta$ is a compact subset of $\mathbb{R}^{|A|}$ and $|P(a, x, b)| \leq 1$ for all $(a, x, b) \in A \times \overline{\mathcal{X}} \times S_\beta$,

$$\|P(a, x, \cdot)\|^2_2 = \int_{S_\beta} P(a, x, b)^2 d\lambda(b) \leq \int_{S_\beta} d\lambda(b) < \infty,$$

and thus $b \mapsto P(a, x, b)$ is an element of $L_{S_\beta}^2$.

To show $x_{[|A|+1]} \mapsto P(a, x, b)$ is real analytic, consider that since the sum, composition and ratio of strictly positive real analytic functions are real analytic (Krantz and Parks 2002) it is sufficient to show $x_{[|A|+1]} \mapsto \int v(x', b) dF(x'|x, a)$ is real analytic. By Assumption I2(v),

$$\int v(x', b) dF(x'|x, a) = \int v(x', b) f_c(x'|x, a) dx' + \sum_{i=1}^N v(i; b) f_d(i|x, a)$$

where $f_c(\cdot|x, a)$ is a p.d.f. and $f_d(\cdot|x, a)$ is a p.m.f. with $N$ points of support. Since $f_d$ is a real analytic function of $x_{[|A|+1]}$, it is sufficient to show $\int v(x', b) f_c(x'|x, a) dx'$ is real analytic. By assumption I2(v), $f_c(x'|x, a)$ is real analytic on $x_{[|A|+1]} \in \mathbb{R}^{[|A|+1]}$. That is, for
each fixed \((a, x_{[-([A]+1)]}, x')\), there is a unique power series representation, such that for all \(x_{[|A|+1]} \in \mathbb{R}^{|A|+1}\),
\[
f_c(x'|x, a) = \sum_{n \in \mathbb{N}^{|A|+1}} \alpha_n(a, x_{[-([A]+1)]}, x') x^n_{[|A|+1]}.\]
For any \(x'\) outside its bounded support and any \((a, x_{[-([A]+1)]})\), since \(f_c(x'|x, a) = 0\) for \(x \in \mathcal{X}\), \(f_c(x'|x, a) = 0\) for \(x \in \overline{\mathcal{X}}\) since \(\{x_{[|A|+1]} : x \in \mathcal{X}\}\) contains an open set. We are now in a position to show the result.
\[
\int v(x'; b) f_c(x'|x, a) dx' = \int v(x'; b) \sum_{n \in \mathbb{N}^{|A|+1}} \alpha_n(a, x_{[-([A]+1)]}, x') x^n_{[|A|+1]} dx'
= \int \sum_{n \in \mathbb{N}^{|A|+1}} \tilde{\alpha}_n(a, x_{[-([A]+1)]}, x') x^n_{[|A|+1]} dx'
= \sum_{n \in \mathbb{N}^{|A|+1}} \left( \int \tilde{\alpha}_n(a, x_{[-([A]+1)]}, x') dx' \right) x^n_{[|A|+1]} - \sum_{n \in \mathbb{N}^{|A|+1}} \tilde{\alpha}_n x^n_{[|A|+1]}
\]
The first equality holds by definition. The second holds from defining \(\tilde{\alpha}_n(a, x_{[-([A]+1)]}, x') = v(x'; b) \alpha_n(a, x_{[-([A]+1)]}, x')\). The third equality holds from the bounded convergence theorem because, the integral being supported on a bounded set, \(\tilde{\alpha}_n(a, x_{[-([A]+1)]}, x')\) is dominated by its supremum taken over its bounded support. The final equality is by definition of \(\tilde{\alpha}_n = \int \tilde{\alpha}_n(a, x_{[-([A]+1)]}, x') dx'\), which exists since the defining integral is supported on a bounded set.

**Lemma A.2** (Approximation). Assume I1 and I2 and let \(\overline{\mathcal{X}} \subset \mathbb{R}^k\) such that \(\mathbb{R}^{|A|+1} = \{x_{[|A|+1]} : x \in \overline{\mathcal{X}}\}\). Then for any \((\alpha_1, \alpha_0) \in \mathbb{R}^{|A|+1}\),
\[
\forall \epsilon > 0 \exists f \in \text{sp} \{h : S_{\beta} \to [0, 1] : h(b) = P(0, x, b), x \in \overline{\mathcal{X}}\}\quad (15)
\]
such that \(\int (1\{\alpha_1^T b > \alpha_0\} - f(b))^2 db < \epsilon\).

**Proof.** Under Assumptions I1 and I2,
\[
P(a, x, b) = \frac{\exp(x^T(b_a, \gamma_a)) + \rho \int v(x'; b) dF_x(x'|x, a))}{\sum_{a \in A} \exp(x^T(b_{\bar{a}}, \gamma_{\bar{a}}) + \rho \int v(x'; b) dF_x(x'|x, \bar{a}))}.\quad (16)
\]
The proof will proceed in two steps. First, I show that for any \(c \in \mathbb{R}^{|A|}\) there is a sequence of functions in \(\tilde{\mathcal{H}} \equiv \{h : S_{\beta} \to [0, 1] : h(b) = P(0, x, b), x \in \overline{\mathcal{X}}\}\) whose limit function is \(\prod_{a=1}^{|A|} 1\{c_a < b_a\}\). Second, I use this result to construct a function in \(\text{sp} \tilde{\mathcal{H}}\) that is arbitrarily close to \(1\{\alpha_1^T b > \alpha_0\}\) in the \(L^2\) norm.
For the first step, given \( c \in \mathbb{R}^{|A|} \) and \( n \in \mathbb{N} \), let \( \bar{x}_n \) be a solution to the system of equations
\[
nc_a = \gamma_a x_{-1}, \quad a = 1, \ldots, |A|,
\]
which exists due to Assumption I2(iv). Denote \( x_n = (-n, \bar{x}_n) \). If
\[
\lim_{n \to \infty} \left( x_n^\top (b, \gamma_a) + \rho \int v(x'; b) dF_x(x'|x, a) \right) = \lim_{n \to \infty} (x_n^\top (b, \gamma_a)) ,
\]
then \( P(0, x_n, b) \to \prod_{a=1}^{|A|} 1 \{ c_a < b_a \} \) uniformly on \( b \in \mathbb{R}^{|A|} \setminus \bigcup_{a=1}^{|A|} \{ b \in \mathbb{R}^{|A|} : |b_a - c_a| < \delta \} \) for any \( \delta > 0 \). Since \( x_n^\top (b, \gamma_a) = -n(b_a - c_a) \) diverges when \( b_a \neq c_a \), for equation (17) it is sufficient that \( \int v(x'; b) dF_x(x'|x, a) \) is uniformly bounded. Denote \( S_{x'} \) as the support of the state transition kernel and consider that
\[
\left| \int v(x'; b) dF_x(x'|x, a) \right| \leq \int |v(x'; b)||dF_x(x'|x, a)|
\]
\[
= \int_{x' \in S_{x'}} |v(x'; b)||dF_x(x'|x, a)| + \int_{x' \not\in S_{x'}} |v(x'; b)||dF_x(x'|x, a)|
\]
\[
\leq M_1(b) \int_{x' \in S_{x'}} M_2(a, x_{[-(|A|)+1]}, x') dx' + 0
\]
\[
\leq M(a, x_{[-(|A|)+1]}, b) < \infty
\]
The second inequality follows because (a) the value function \( v(x; b) \) is bounded when the state space is contained in a compact set (Kristensen et al. 2021), (b) \( F_x \) is a bounded function of \( x_{[|A|]+1} \) (Assumption I2(v)) and, (c) as argued in Lemma A.1, \( F_x \) is vanishing for \( x' \) outside its bounded support. The final inequality follows from the existence of the integral over \( S_{x'} \), a bounded set. The uniform bound is attained as the maximum of \( M(a, x_{[-(|A|)+1]}, b) \) over its bounded support.

For the second step, let \( k(b) = 1\{ \alpha^1_b > \alpha^0 \} \) and \( \epsilon > 0 \) be given. If \( \sup_{b \in S_{x'}} \{ \alpha^1_b - \alpha^0 \} < 0 \), from the above step there exists \( x \in \bar{X} \) such that \( \sup_{b \in S_{x'}} \{ P(0, x, b) - k(b) \} \) is arbitrarily small. Similarly if \( \inf_{b \in S_{x'}} \{ \alpha^1_b - \alpha^0 \} > 0 \). Otherwise, let \( \mathcal{S} = [\underline{k}, \bar{k}]^{|A|} \subset \mathbb{R}^{|A|} \) contain \( S_{x'} \) and intersect the hyperplane \( \{ c : \alpha^1 c = \alpha^0 \} \). Given the first step, there is a sequence \( (h_n)_{n \in \mathbb{N}} \in \text{sp}\bar{H} \), each element formed by adding and subtracting \( 2^{|A|} \) elements of \( \mathcal{H} \), such that \( h_n \to \prod_{a=1}^{|A|} 1 \{ c_a < b_a < c_a + \Delta \} \) uniformly on \( b \in \mathbb{R}^{|A|} \setminus \bigcup_{a=1}^{|A|} \{ b \in \mathbb{R}^{|A|} : \min(|b_a - c_a|, |b_a - c_a - \Delta|) < \delta \} \) for any \( 0 < \delta << \Delta \) and \( c \in \mathbb{R}^{|A|} \). Therefore, for any \( \eta > 0 \) and \( n \in \mathbb{N} \), there exists \( h(b; c, n) \in \text{sp}\bar{H} \) such that
\[
h(b; c, n) \in \begin{cases} 
(1 - n^{-1}, 1 + n^{-1}) & \text{if } \sum_{a=1}^{|A|} 1 \{ c_a + \frac{k_a - k_{a-1}}{n^{1/2}} < b_a < c_a + \left( \frac{k_a - k_{a-1}}{n^{1/2}} \right) \} = |A| \\
(-2^{|A|}, 2^{|A|}) & \text{if } \sum_{a=1}^{|A|} 1 \{ \min(|b_a - c_a - \frac{k_a - k_{a-1}}{n^{1/2}}|, |b_a - c_a - \frac{k_a - k_{a-1}}{n^{1/2}}|) < \frac{k_a - k_{a-1}}{n^{1/2}} \} \geq 1 \\
(-n^{-|A|-1}, n^{-|A|-1}) & \text{otherwise.}
\end{cases}
\]
Let \( k_i = \bar{k} + (i - 1) \times (\bar{k} - \underline{k})/n \) for \( i = 1, \ldots, n+1 \), and \( k_{\alpha} = (k_{\alpha_1}, k_{\alpha_2}, \ldots, k_{\alpha_{|A|}}) \) for \( \alpha \in \mathbb{N}^{|A|} \),
then set $\beta_\alpha = 1\{\forall b \in x_{\alpha_1}^{\lfloor |A| / \alpha_0 \rfloor}, \alpha_1 b > \alpha_0 \}$ and define the function $\tilde{k}_n(b) \in \text{sp}\mathcal{H}$ as

$$\tilde{k}_n(b) = \sum_{\alpha \in (1, \ldots, n)^{|A|}} \beta_\alpha \times h(b, k_\alpha, n).$$

To complete the proof, it remains to show that for some $n$, $\sqrt{\int_{S_\delta} |\tilde{k}_n(b) - k(b)|^2 db} < \epsilon$. Let $\{S_\delta, S_0, S_{1/2}, S_1\}$ partition $S$ as follows:

$$S_\delta = \bigcup_{\alpha \in (1, \ldots, n+1)^{|A|}} \bigcup_{a=1}^{|A|} \{s \in S : |s_a - k_{\alpha_a}| < \frac{\delta}{n^{3+\eta}} \},$$

$$S_1 = \bigcup_{\{\alpha \in (1, \ldots, n)^{|A|} : \beta_\alpha = 1\}} x_{\alpha_1}^{\lfloor |A| \rfloor} \backslash S_\delta,$$

$$S_0 = \bigcup_{\{\alpha \in (1, \ldots, n)^{|A|} : \forall b \in x_{\alpha_1}^{\lfloor |A| / \alpha_0 \rfloor}, \alpha_1 b > \alpha_0 \}} x_{\alpha_1}^{\lfloor |A| \rfloor} \backslash S_\delta,$$

$$S_{1/2} = S \backslash (S_1 \cup S_0 \cup S_\delta).$$

Therefore

$$\sqrt{\int_{S_\delta} |\tilde{k}_n(b) - k(b)|^2 db} \leq \sqrt{\int_S |\tilde{k}_n(b) - k(b)|^2 db} \leq \sqrt{\int_{S_1 \cup S_0} |\tilde{k}_n(b) - k(b)|^2 db} + \sqrt{\int_{S_{1/2}} |\tilde{k}_n(b) - k(b)|^2 db} + \sqrt{\int_{S_\delta} |\tilde{k}_n(b) - k(b)|^2 db}.$$

Denote $N_1 = |\{\alpha \in (1, \ldots, n)^{|A|} : \beta_\alpha = 1\}|$. For $b \in S_0$, $k(b) = 0$ and $|\tilde{k}_n(b)| < N_1 n^{-|A|-1} < n^{-1}$. Similarly, if $b \in S_1$, $k(b) = 1$ and $|\tilde{k}_n(b) - k(b)| < |1 + n^{-1} + N_1 n^{-|A|-1} - 1| < 2n^{-1}$. Therefore

$$\sqrt{\int_{S_1 \cup S_0} |\tilde{k}_n(b) - k(b)|^2 db} < 2n^{-1} \sqrt{\int_{S_1 \cup S_0} 1db} \rightarrow 0.$$

For any $b \in S_{1/2}$, $k(b) \in \{0, 1\}$ and $|\tilde{k}_n(b)| < N_1 n^{-|A|-1} < n^{-1}$ so that

$$\sqrt{\int_{S_{1/2}} |\tilde{k}_n(b) - k(b)|^2 db} \leq (1 + n^{-1}) \sqrt{\int_{S_{1/2}} 1db} \leq (1 + n^{-1}) \sqrt{(k - k)|A|n^{-1}} \rightarrow 0.$$

Finally, for $b \in S_\delta$, $|\tilde{k}_n(b)| < 2^{|A|} + |A|(n - 1)2^{|A|} + N_1 n^{-|A|-1} \leq |A| n 2^{|A|} + n^{-1}$, so

$$\sqrt{\int_{S_\delta} |\tilde{k}_n(b) - k(b)|^2 db} < (1 + |A| n 2^{|A|} + n^{-1}) \times \sqrt{(k - k)|A|(1 - (1 - 2n^{-2})|A|)} \rightarrow 0.$$

We conclude $\|\tilde{k}_n(b) - k(b)\|_2 \to 0$ as $n \to \infty$. So for $n$ large enough, $\|\tilde{k}_n(b) - k(b)\|_2 < \epsilon$. □
Lemma A.3 is a straightforward generalization of Stinchcombe and White (1998, Theorem 3.8) that allows for non-linear kernel functions and the domain of the functions in the image of the integral transform to be a strict subset of the Euclidean space.

Lemma A.3. Let $F$ be a signed measure with compact support $\mathcal{Y}$ and $\mathcal{X} \subseteq \mathbb{R}^k$. If

$$\forall x \in \mathcal{X}, \int f(x,y) dF(y) = 0 \Rightarrow \forall y \in \mathcal{Y}, F(y) = 0$$

(18)

and $x \mapsto f(x,y)$ is a real analytic function on $\mathcal{X}$ for each $y \in \mathcal{Y}$, then for any $T \subseteq \mathcal{X}$ open and non-empty,

$$\forall x \in T, \int f(x,y) dF(y) = 0 \Rightarrow \forall y \in \mathcal{Y}, F(y) = 0$$

Proof. Suppose that equation (18) holds and that $\forall x \in T, \int f(x,y) dF(y) = 0$, for some $T \subseteq \mathbb{R}^k$ open and non-empty. Since $f$ is real analytic for each $y$ and $\mathcal{Y}$ is bounded, $\int f(x,y) dF(y)$ is a real analytic function of $x$ (Mattner 1999). Since $\int f(x,y) dF(y)$ is zero on an open set, it is zero on the Euclidean space and by equation (18), $F$ vanishes on $\mathcal{Y}$.

A.2 Proof of results in Section 3.1

A.2.1 Proof of Theorem 3

Proof. Under Assumptions F1 and F2,

$$P_T(a, x, b) = \frac{\exp\left(b_{a1} + x^\top(b_{a[-1]}; \gamma_T a)^\top\right)}{\sum_{\tilde{a} \in A} \exp\left(b_{\tilde{a}1} + x^\top(b_{\tilde{a}[-1]}; \gamma_{\tilde{a}} a)^\top\right)}.$$  

Denote $z = x_p$ and $w = x_{-p}$, and observe $(z, w_{|\mathcal{A}|}) \mapsto P_T(a, x, b)$ is real analytic. Since $S_\beta$ is compact, Lemma A.3 applies and the result holds if, for any signed measure $\mu$,

$$\forall (a, x) \in A \times \mathcal{X}, \int P_T(a, x, b) d\mu(b) = 0 \implies \forall b \in \text{Supp}(\beta), \mu(b) = 0,$$

where $\mathcal{X} = \mathbb{R}^{p+|\mathcal{A}|} \times \{x_{-p+|\mathcal{A}|} : x \in \mathcal{X}\}$. I show this condition directly. Begin by assuming $\mu(b)$ is a finite signed measure satisfying

$$\forall (a, z) \in A \times \mathbb{R}^p, \int P_T(a, x, b) \mu(b) db = 0$$

(19)

for any fixed $w$. Viewed as a function of a $w_{|\mathcal{A}|} \in \mathbb{R}^{|\mathcal{A}|}$ this object is infinitely differentiable and since it is identically zero, all of its derivatives are zero. Furthermore, since both $P_T$
and \( \mu \) are bounded, we can exchange the order of differentiation and integration, so that for any \( 1 \leq i \leq |A| \),
\[
\forall n \in \mathbb{N}^+, \forall (a, z) \in A \times \mathbb{R}^p, \int \frac{\partial^n}{\partial w_i^n} P_T(a, x, b) \mu(b) db = 0.
\]
Fix \( a \) and consider the first-order partial derivative \( (n = 1) \) with respect to \( w_i \):
\[
\forall z \in \mathbb{R}^p, \gamma aT_i \int P_T(a, x, b) \mu(b) db - \sum_{j \in A} \gamma jT_i \int P_T(a, x, b) P_T(j, x, b) \mu(b) db = 0.
\]
From the above equalities, it follows that,
\[
\forall (a, z) \in A \times \mathbb{R}^p, \sum_{j \in A} \gamma jT_i \int P_T(a, x, b) P_T(j, x, b) \mu(b) db = 0.
\]
Repeating the argument for all \( i \in \tilde{A} \) yields the system of linear equations
\[
\Gamma^\top_{AT} \int P_T(a, x, b) \otimes \tilde{P}_T(x, b) = 0^\top_{|A|}
\]
where \( \tilde{P}_T(x, b) \) is the vector \( (P_T(a, x, b): a \in \tilde{A}) \), \( \otimes \) is the Kronecker product and \( 0_{|A|} \in \mathbb{R}^{|A|} \) is the zero vector. By Assumption F2(iv), \( \Gamma_{AT} \) is full rank and thus \( \int P_T(a, x, b) \otimes \tilde{P}_T(x, b) = 0^\top_{|A|} \). Repeating the argument for each \( a \),
\[
\forall z \in \mathbb{R}^p, \int \tilde{P}_T(x, b)^\alpha \mu(b) db = 0
\]
for multi-indices \( \alpha \in \{1, 2\}^{|A|} \). Repeating the argument for higher order derivatives,
\[
\forall z \in \mathbb{R}^p, \int \tilde{P}_T(x, b)^\alpha \mu(b) db = 0
\]
for all \( \alpha \in \mathbb{N}^{|A|} \). Let \( \mu_z \) be the signed measure induced by \( \beta \rightarrow \tilde{P}_T(x, \beta) \), i.e.,
\[
\mu_z(B) = \int \mu(b) 1\{\tilde{P}_T(x, b) \in B\} db.
\]
That is, \( \mu_z \) is the measure of the random variable \( \tilde{P}_T(x, \beta) \). Thus from equation (20),
\[
\forall z \in \mathbb{R}^p, \int x^\alpha \mu_z(x) dx = 0
\]
for all \( \alpha \in \mathbb{N}^{|A|} \). It follows that the Fourier transform of \( \tilde{P}_T(x, \beta) \) is identically zero, and thus the measure \( \mu_z \) is zero for each \( z \in \mathbb{R}^p \) (Hornik 1993, Theorem 1 Proof). Since \( \tilde{P}_T(x, \beta) = 0 \) implies \( \beta a[1] + x^\top (\beta a[-1], \gamma Ta) = 0 \) for all \( a \in A \), \( \mu_z(B) = 0 \) implies for all \( z \in \mathbb{R}^p \),
\[
\int \mu(b) 1\{b a[1] + x^\top (b a[-1], \gamma Ta): a \in \tilde{A}\} \in B\} db = 0.
\]
From here standard arguments (Masten 2018, Lemma 1) give that the characteristic function of \( \beta \) is zero and thus the signed measure \( \mu(b) = 0 \).
A.2.2 Proof of Theorem 4

Proof. Let $Y = ((A_t, X_t)_{t=2}^T, A_1)$. By Assumption F1, the distribution of $Y$ conditional upon $X_1 = x$ is

$$f_{y|x_1}(y, x_1) = \int \prod_{t=2}^T (P_t(a_t, x_t, b) F_{x_t}(x_{t-1}, a_{t-1})) P_1(a_1, x_1, b) df_{y|x_1}(b, x_1).$$

Fix $x_1 \in \text{Supp}(X_1)$ and $(a_t)_{t=1}^{T-1} \in A^{T-1}$. By Assumption F3,

$$\frac{f_{y|x_1}(y, x_1)}{\prod_{t=2}^T F_{x_t}(x_{t-1}, a_{t-1})} = \int \prod_{t=1}^T P_t(a_t, x_t, b) df_{y|x_1}(b, x_1).$$

Define $g(b;(a_t)_{t=1}^{T-1}) = \prod_{t=1}^{T-1} P_t(a_t, x_t, b) f_{y|x_1}(b, x_1)$, and define the operator

$$L_{T,\beta} : L_{S_T} \to A \times L_{S_T} \quad [L_{T,\beta} m](a_T, x_T) = \int P_T(a_T, x_T, b) m(b) db.$$

Under Assumption F1-F3, Theorem 3 implies $L_{T,\beta}$ is injective and that the operator defined in F4 exists. Suppose $\gamma_T, \tilde{\gamma}_T$ are observationally equivalent, i.e.,

$$(x_T, a_T) \in S_T \times A, \int P_T(a_T, x_T, b; \gamma_T) g(b;(a_t)_{t=1}^{T-1}) db = \int P_T(a_T, x_T, b; \tilde{\gamma}_T) \tilde{g}(b;(a_t)_{t=1}^{T-1}) db.$$

In particular for $E$ as in Assumption F4, $[L_{T,\beta}^{E,\gamma_T} g](a_T, x_T) = [L_{T,\beta}^{E,\tilde{\gamma}_T} \tilde{g}](a_T, x_T)$ for all $(x_T, a_T) \in E$. Since $L_{T,\beta}^{E,\gamma_T}$ is injective, $g(b;(a_t)_{t=1}^{T-1}) = [(L_{T,\beta}^{E,\gamma_T})^{-1} L_{T,\beta}^{E,\tilde{\gamma}_T} \tilde{g}](b)$. Similarly, by Assumption F4, for some $\tilde{E}$, $g(b;(a_t)_{t=1}^{T-1}) = [(L_{T,\beta}^{E,\gamma_T})^{-1} L_{T,\beta}^{E,\tilde{\gamma}_T} \tilde{g}](b)$. It follows that

$$0 = \left[\left((L_{T,\beta}^{E,\gamma_T})^{-1} L_{T,\beta}^{E,\tilde{\gamma}_T} \tilde{g} \right)^T (L_{T,\beta}^{E,\gamma_T})^{-1} L_{T,\beta}^{E,\tilde{\gamma}_T} \tilde{g} \right](b),$$

but $\tilde{g}(b;(a_t)_{t=1}^{T-1}) \neq 0$. Under Assumption F4, if $\gamma_T \neq \tilde{\gamma}_T$ then $L_{T,\beta}^{E,\gamma_T} L_{T,\beta}^{E,\tilde{\gamma}_T} = (L_{T,\beta}^{E,\gamma_T})^{-1} L_{T,\beta}^{E,\tilde{\gamma}_T} - (L_{T,\beta}^{E,\gamma_T})^{-1} L_{T,\beta}^{E,\gamma_T}$ is injective, so we conclude $\gamma_T = \tilde{\gamma}_T$. Next, $g(b;(a_t)_{t=1}^{T-1})$ is identified as

$$g(b;(a_t)_{t=1}^{T-1}) = \left[ L_{T,\beta}^{-1} \prod_{t=2}^T f_{y|x_1}(y, x_1) \right] (b),$$

which is possible since $L_{T,\beta}$ is identified and injective. Repeating this argument for each choice sequence $(a_t)_{t=1}^T$, $f_{y|x_1}(b, x_1)$ is identified as $\sum_{a \in A_T} g(b; a)$. Similarly, $P_t(a_t, x_t, b)$ is identified as the sum of $g(b,(a_t)_{t=1}^{T-1}) f_{y|x_1}(b, x_1)$ over the support of $(a_t)_{t=1}^{T-1}$ for all periods except the $t$th. Finally, given identification of $\gamma_{t+1}, \gamma_{t+2}, \ldots \gamma_T$, under Assumption F3, $\gamma_t$ is identified. \qed
A.2.3 Proof without rank condition

**Lemma A.4** (Result without rank condition). Assume the distribution of \((X_t, A_t)_{t=1}^T\) is observed for \(T \geq 2\), generated from agents solving the model of equation (3) with \(|A| = 1\) satisfying assumptions F1-F3. Furthermore, \(S_T\) contains no isolated points. Under the scale assumption \(\gamma_{T1} = 1\), \((\gamma_T, f_{\beta|X_1})\) is point identified.

**Proof.** Proceed as in the proof to Theorem 4. For identification of \(\gamma_T\), suppose for all \(x \in S_T\),

\[
\int \Lambda\left(b_1 + x^\top (b_{[-1]}, \gamma_T)\right) g(b; a_1) db = \int \Lambda\left(b_1 + x^\top (b_{[-1]}, \tilde{\gamma}_T)\right) \tilde{g}(b; a_1) db.
\]

Since \(S_T\) contains no isolated points, we can differentiate the above equation with respect to \(x \in S_T\). Furthermore, as both \(\Lambda\) and \(g\) are bounded, the limits defining differentiation and integration may be exchanged, so that for all \(x \in S_T\) and \(p < k' \leq k\),

\[
\int \frac{\partial}{\partial x_{k'}} \Lambda\left(b_1 + x^\top (b_{[-1]}, \gamma_T)\right) g(b; a_1) db = \int \frac{\partial}{\partial x_{k'}} \Lambda\left(b_1 + x^\top (b_{[-1]}, \tilde{\gamma}_T)\right) \tilde{g}(b; a_1) db.
\]

Since the derivative of \(\Lambda(x)\) is \(\Lambda(x)(1 - \Lambda(x))\), the above display is equivalent to

\[
\gamma_{Tk'} \int [\Lambda(1-\Lambda)]\left(b_1 + x^\top (b_{[-1]}, \gamma_T)\right) g(b; a_1) db = \tilde{\gamma}_{Tk'} \int [\Lambda(1-\Lambda)]\left(b_1 + x^\top (b_{[-1]}, \tilde{\gamma}_T)\right) \tilde{g}(b; a_1) db.
\]

By assumption \(\gamma_{T1} = \tilde{\gamma}_{T1} = 1\), so for all \(x \in S_T\),

\[
\int [\Lambda(1-\Lambda)]\left(b_1 + x^\top (b_{[-1]}, \gamma_T)\right) g(b; a_1) db = \int [\Lambda(1-\Lambda)]\left(b_1 + x^\top (b_{[-1]}, \tilde{\gamma}_T)\right) \tilde{g}(b; a_1) db.
\]

Therefore, for any \(k'\)

\[
(\gamma_{Tk'} - \tilde{\gamma}_{Tk'}) \int [\Lambda(1-\Lambda)]\left(b_1 + x^\top (b_{[-1]}, \gamma_T)\right) g(b; a_1) db = 0,
\]

and since the logistic function takes values in \((0,1)\), \(\gamma_{Tk'} = \tilde{\gamma}_{Tk'}\) and \(\gamma_T\) is identified. Given identification of \(\gamma_T\), \(f_{\beta|x_1}\) is identified by the argument in the proof to Theorem 4.
B Online appendix

Throughout this appendix I use the following notations: \( S_\beta = \text{Supp}(\beta); \mathcal{L}_A \) denotes the usual \( L^\infty \) space \( L^\infty(A, \lambda) \) where \( \lambda \) the Lebesgue measure.

B.1 Additional proofs for Section 3

B.1.1 Proof of Corollary 1

**Proof.** Fix \( x_1 \in \text{Supp}(X_1) \) and denote \( S_4 = \text{Supp}(X_4 | X_3 \in S_3, A_3 = a_3) \) which satisfies Assumption F3'. The operators \( L_{4,2,3} : \mathcal{L}_{S_3} \rightarrow A \times \mathcal{L}_{S_4} \) and \( L_{4,3} : \mathcal{L}_{S_3} \rightarrow A \times \mathcal{L}_{S_4} \) defined as

\[
[L_{4,2,3}^m](a_4, x_4) = \int \frac{f(a_4x_4,x_4x_3|x_1)(a_4,a_3,a_2,a_1,x_4,x_2,x_1)}{F_{x_1}(x_4|x_3,a_3)F_{x_3}(x_3|x_2,a_2)F_{x_2}(x_2|x_1,a_1)}m(x_3)dx_3
\]

\[
[L_{4,3}^m](a_4, x_4) = \int \sum_{a_2 \in A} \frac{f(a_4,a_3,a_2,a_1,x_4,x_3,x_2,x_1)}{F_{x_4}(x_4|x_3,a_3)F_{x_3}(x_3|x_2,a_2)F_{x_2}(x_2|x_1,a_1)}m(x_3)dx_3
\]

are well-defined and observed for \( x_2 \in S_2 \). Define the following operators:

\[
L_{4,\beta} : \mathcal{L}_{S_3} \rightarrow A \times \mathcal{L}_{S_4} \hspace{1cm} [L_{4,\beta}^m](a_4, x_4) = \int P_4(a_4,x_4,b)m(b)db
\]

\[
D_{\beta}^2 : \mathcal{L}_{S_3} \rightarrow \mathcal{L}_{S_3} \hspace{1cm} [D_{\beta}^2 m](b) = P_2(a_2,x_2,b)m(b)
\]

\[
D_{\beta} : \mathcal{L}_{S_3} \rightarrow \mathcal{L}_{S_3} \hspace{1cm} [D_{\beta} m](b) = P_1(a_1,x_1,b)\int f_{\beta|x_1}(b,x_1)m(b)
\]

\[
L_{\beta,3} : \mathcal{L}_{S_3} \rightarrow \mathcal{L}_{S_3} \hspace{1cm} [L_{\beta,3} m](b) = \int P_3(a_3,x_3,b)m(x_3)dx_3
\]

It is straightforward to show \( L_{4,2,3} = L_{4,\beta}D_{\beta}^2L_{\beta,3} \) and \( L_{4,3} = L_{4,\beta}D_{\beta}L_{\beta,3} \). We begin by showing injectivity of \( L_{4,\beta} \) and \( L_{\beta,3}^* \). Notice

\[
P_t(a_t, x_t, b) = \frac{\exp(x^\top(b_a, \gamma_t a) + \rho \int v_{t+1}(x'; b) dF_{x_t}(x'|x,a))}{\sum_{\tilde{a} \in A} \exp(x^\top(\tilde{b}_a, \gamma_{t\tilde{a}}) + \rho \int v_{t+1}(x'; b) dF_{x_t}(x'|x,\tilde{a}))}
\]

differs from equation (14) only by the time-dependence of \( \gamma_t, v_t \) and \( F_{x_t} \). Since Assumption F2' places restrictions on \( (\gamma_t, F_{x_t}) \) that are analogous to restrictions placed by Assumption I2 on \( (\gamma, F_x) \) in the stationary model, injectivity will result from the arguments of Lemmas A.1 and A.2. The arguments of Lemma A.1 apply directly. The arguments of Lemma A.2 do not directly apply since in the non-stationary model the value function \( v_t \) is defined recursively, so we cannot use the uniform bound on \( v_t \) from Lemma A.2. To develop the uniform bound on \( v_t \), I proceed recursively.
First define \( e(a, x) = E[\epsilon_t | x, a] \). Under Assumption F1, the function \( e(a, x) \) is known and bounded (Aguirregabiria and Mira 2007). Now consider the terminal value function (i.e., \( t = T_1 \)),

\[
v_{T_1}(x;b) = \sum_{a \in A} P_{T_1}(a, x, b) \left( x^\top (\beta_a, \gamma_{Ta}) + e(a, x) \right),
\]

which is bounded because the CCP functions are. For \( t < T_1 \), suppose that \( v_{t+1} \) is finite. Since

\[
v_t(x;b) = \sum_{a \in A} P_t(a, x, b) \left( x^\top (\beta_a, \gamma_{ta}) + \int v_{t+1}(x';b) dF_{x_1}(x'|x,a) \right),
\]

\( v_t(x;b) \) is finite also. So for any \( t \), \( v_t(x;b) \) is finite for any \( (x,b) \) and a uniform bound is given by the supremum over the support. Therefore the remaining steps in Lemma A.2 go through directly.

The arguments in the proof to Theorem 2 imply that \( L_{4,3,2} = L_{4,3}D_{\beta}^2D_{\beta}L_{\beta,3} \) and \( L_{4,3} = L_{4,\beta}D_{\beta}L_{\beta,3} \), and that the spectral decomposition

\[
L_{4,2,3}L_{4,3}^{-1} = L_{4,\beta}D_{\beta}^2L_{4,\beta}^{-1}
\]

identifies \( P_1(a, x, b) \) and thus \( \gamma_4 \). Exchanging the role of \( L_{4,\beta} \) and \( L_{3,\beta} \) yields identification of \( P_3(a, x, b) \) and thus \( \gamma_3 \). Given identification of \( D_{\beta}^2 \), \( \gamma_4 \) and \( \gamma_3 \), \( \gamma_2 \) is identified under Assumption F3'. Finally, given \( D_\beta = L_{4,\beta}^{-1}L_{4,3}L_{\beta,3}^{-1} \), \( f_{\beta|x_1} \) and \( P_1(a, x, b) \) (and thus \( \gamma_1 \)) are identified.

\[\hfill\]

\subsection*{B.1.2 Proof of Corollary 2}

\textit{Proof.} Define the following operators:

\[
\begin{align*}
L_{3,4,2} & : \mathcal{L}_2 \to \mathcal{L}_3 & [L_{3,4,2}m](x_3) &= \int \frac{f_{A_1A_2A_3A_4X_4X_3X_2|X_1}(1,1,1,a_1,x_4,x_3,x_2,x_1)}{F_{x_4}(x_4|x_3,1)F_{x_3}(x_3|x_2,1)F_{x_1}(x_2|x_1,a_1)} m(x_2) dx_2 \\
L_{3,2} & : \mathcal{L}_2 \to \mathcal{L}_3 & [L_{3,2}m](x_3) &= \int \sum_{a_2 \in A} \frac{f_{A_1A_2A_3A_4X_4X_3X_2|X_1}(1,1,a_2,a_1,x_4,x_3,x_2,x_1)}{F_{x_4}(x_4|x_3,1)F_{x_3}(x_3|x_2,1)F_{x_1}(x_2|x_1,a_1)} m(x_2) dx_2 \\
L_{3,\beta} & : \mathcal{L}_{\beta} \to \mathcal{L}_3 & [L_{3,\beta}m](x_3) &= \int P_3(1,x_3,b) m(b) db \\
D_{\beta}^4 & : \mathcal{L}_{\beta} \to \mathcal{L}_{\beta} & [D_{\beta}^4m](b) &= P_4(1,x_4,b) m(b) \\
D_{\beta} & : \mathcal{L}_{\beta} \to \mathcal{L}_{\beta} & [D_{\beta}m](b) &= P_1(a_1,x_1,b) f_{\beta|x_1}(b,x_1) m(b) \\
L_{\beta,2} & : \mathcal{L}_2 \to \mathcal{L}_{\beta} & [L_{\beta,2}m](b) &= \int P_2(1,x_2,b) m(x_2) dx_2
\end{align*}
\]
Under Assumptions F1 and F3′′ these operators are well-defined and observed. One can show \( L_{4,3,2} = L_{3,\beta} D_{\beta}^4 D_{\beta} L_{\beta,2} \) and \( L_{3,2} = L_{3,\beta} D_{\beta} L_{\beta,2} \). Under Assumptions F1, F2′ and F3′′, \( L_{3,\beta} \) and \( L_{\beta,2}^* \) are injective and thus the observed operator \( L_{4,3,2}^{-1} \) has the eigendecomposition \( L_{3,\beta} D_{\beta}^4 L_{\beta,2}^{-1} \). I now show the eigenvalue-eigenfunction representation is unique. Since the model is binary choice with real valued \( \beta \), the function \( P_4(1, x_4, b) \) is injective in \( b \). It follows that the eigenvalues are unique, and, up to the ordering function \( R, P_4(1, x_4, R(b)) \) is identified. The eigenfunctions of the decomposition identify \( P_3(1, x_3, R(b)) \), which equal

\[
\Lambda \left( x_3^T(R(b), \gamma_3) + \int v_4(x'; R(b)) \left( dF_{x_3}(x' | x_3, 1) - dF_{x_3}(x' | x_3, 0) \right) \right).
\]

Under Assumption F6, \( v_4(x'; R(b)) \) can be expressed in terms of \( P_4(1, x_4, R(b)) \), and is therefore identified. Therefore identification consists of showing that \( (R(b), \gamma_3) \) can be identified from \( x_3^T(R(b), \gamma_3) \), which follows from the support assumption. Given identification of \( P_4(a, x, b) \), identification of \( \gamma \) and \( f_{\beta | X_1} \) are attained under Assumption F3′′ by a sequential argument as in Corollary 1.

\[\Box\]

### B.1.3 Proof of Corollary 3

**Proof.** The proof follows closely the structure of the proof to Theorem 2. As in that proof, Assumptions I1 and I3′ enable the decompositions \( L_{3,4,2} = L_{3,\beta} D_{\beta}^4 D_{\beta} L_{\beta,2} \) and \( L_{3,2} = L_{3,\beta} D_{\beta} L_{\beta,2} \) where the operators are defined in proof to Theorem 2. I first show injectivity of \( L_{3,\beta} \) and \( L_{\beta,2}^* \). By Assumption I3′, for \( t = 2, 3 \), the conditional supports of \( X_{t[A]} \) contains a non-empty open set for which

\[
P(a, x, b) = \frac{\exp (\beta_a + x' \gamma_a)}{\sum_{\tilde{a} \in \tilde{A}} \exp (\beta_{\tilde{a}} + x' \gamma_{\tilde{a}})}.
\]

Given this functional form, the arguments of Theorem 3 give that

\[
\int \tilde{P}(x; b) \alpha d\mu(b) = 0
\]

for all multi-indices \( \alpha \in \mathbb{N}^{[A]} \) where \( \tilde{P}(x; b) = \{ P(a, x, b) : a \in \tilde{A} \} \). It follows that the measure induced by the mapping \( \beta \to \tilde{P}(x; \beta) \) is identically zero. Because this mapping is injective, the measure \( \mu(b) \) is identically zero and thus \( L_{3,\beta} \) and \( L_{\beta,2}^* \) are injective. Then, under Assumption I3′, identification follows from the proof to Theorem 2. \(\Box\)
B.1.4 Proof of Corollary 4

Proof. From the definitions in the proof to Theorem 2 and Corollary 1, it is immediate that 
\[ L = L_{3,\beta}D_\beta L_{\beta,2}. \]
From those proofs, \( L_{3,\beta}, D_\beta \) and \( L_{\beta,2} \) are matrices with rank \( R \).

\[ \square \]

B.2 Appendix to Section 4

B.2.1 Theorem 5

This section details the assumptions of Theorem 5 that provide for consistent estimation of 
\[ \theta_0 = (F_x, \gamma, f_{\beta|x_1}) \in \Theta = \mathcal{F} \times \Gamma \times \mathcal{M} \] 
where \( \mathcal{F} \) is the space of state transitions, \( \Gamma \subseteq \mathbb{R}^{\dim \gamma} \), and 
\( \mathcal{M} = \{ f : S_\beta \times \text{Supp}(X_1) \to [0, 1] : b \mapsto f(b, x) \text{ is càdlàg} \}. \) The first assumption supposes the existence of a consistent estimator for the state transition \( F_x \):

**Assumption E1.** There exists an estimator \( \hat{F}_{x,n} \) that satisfies \( \| \hat{F}_{x,n} - F_x \|_\mathcal{F} = o_p(1) \), where \( \| \cdot \|_\mathcal{F} \) is a norm on \( \mathcal{F} \).

One such estimator that satisfies Assumption E1 is the kernel estimator of the conditional density:

\[
\hat{F}_{x,n}(x'|x, a) = \frac{\sum_{i=1}^N K_{X',h_X}(x'-x_{t,i})K_{X,h_X}(x-x_{t,i})1\{a_{it} = a\}}{\sum_{i=1}^N K_{X,h_X}(x-x_{t,i})1\{a_{it} = a\}}
\]

(21)

where \( K_{Z,h_Z} \) are multivariate kernel functions with bandwidth \( h_Z \). Let \( \mathcal{M}_n \) be a sieve space that approximates \( \mathcal{M} \), and denote \( d_{\mathcal{M}}(\cdot, \cdot) \) as the Prokhorov metric. The Prokhorov distance between two measures \( f, \tilde{f} \) on \( S_\beta \) is

\[
\inf \left\{ \delta > 0 : \forall B \in \mathcal{B}(S_\beta), (f(B) \leq \tilde{f}(B_{\delta}) + \delta) \lor (\tilde{f}(B) \leq f(B_{\delta}) + \delta) \right\},
\]

where \( B_{\delta} \) is the \( \delta \) neighborhood of \( B \subseteq S_\beta \) and \( \mathcal{B}(S_\beta) \) is the Borel sigma field. The next assumption requires that the true parameter is a well-separated maximum.

**Assumption E2.** For all \( \epsilon > 0 \) there exists some decreasing sequence of positive numbers \( c_n(\epsilon) \) satisfying \( \lim \inf c_n(\epsilon) > 0 \) such that

\[
E[\psi(Y, F_x, \gamma, f_{\beta|x_1})] - \sup_{\{(\tilde{\gamma}, \tilde{f}) \in \Gamma \times \mathcal{M}_n \mid \text{\supp}(X_1), d_{\mathcal{M}}(\tilde{f}, f_{\beta|x_1}) \leq \epsilon\}} E[\psi(Y, F_x, \tilde{\gamma}, \tilde{f})] \geq c_n(\epsilon).
\]

Assumption E2 is the condition of Remark 3.1(2) in Chen (2007) that strengthens their Condition 3.1. If the strict inequality restriction on \( c_n \) were replaced by a weak inequality, then the assumption would be implied by the identification result.
**Assumption E3.** The sieve space (i) satisfies $\mathcal{M}_n \subseteq \mathcal{M}_{n+1} \subseteq \mathcal{M}$ and (ii) is such that there exists a sequence $f_n \in \mathcal{M}_n$ that converges to $f_{\beta|x_1}$ and satisfies
\[\left| E[\psi(Y, F_x, \gamma, f_n)] - E[\psi(Y, F_x, \gamma, f_{\beta|x_1})] \right| = o(1).\]

These are standard restrictions on the sieve space and the population criterion function (Chen 2007, Condition 3.2, 3.3(ii)). The second condition is a local continuity assumption. As per Chen (2007, Remark 2.1), it is implied by compactness of the sieve space and continuity of the population criterion function on $\mathcal{M}_n$.

Define $\mathcal{F}_n$ to be the set of possible values that the estimator $\hat{f}_n$ can take. For example, if the conditional density kernel estimator is chosen, then an element of the set $\mathcal{F}_n$ takes the form in equation (21) and the set $\mathcal{F}_n$ is defined by ranging $(X_{t+1}, X_t, A_t)$ over its support.

**Assumption E4.** The following two conditions hold
\[\sup_{(\bar{F}_x, \tilde{\gamma}, \tilde{f}) \in \mathcal{N}_{\bar{F}_x,n} \times \Gamma \times \mathcal{M}_n} \left| \frac{1}{n} \sum_{i=1}^n \psi(y_i, \bar{F}_x, \tilde{\gamma}, \tilde{f}) - E[\psi(Y, \bar{F}_x, \tilde{\gamma}, \tilde{f})] \right| = o_p(1),\]
\[\sup_{(\bar{F}_x, \tilde{\gamma}, \tilde{f}) \in \mathcal{N}_{\bar{F}_x,n} \times \Gamma \times \mathcal{M}_n} \left| E[\psi(Y, \bar{F}_x, \tilde{\gamma}, \tilde{f})] - E[\psi(Y, F_x, \tilde{\gamma}, \tilde{f})] \right| = o(1).\]

This is similar to Hahn, Liao, and Ridder (2018, Assumption 5.3), which is based on Chen (2007, Condition 3.5) but includes an additional condition to account for the presence of a first-step estimator.

Theorem 5 is a direct consequence of Hahn, Liao, and Ridder (2018, Theorem 5.1), so the proof is omitted. In the proof, by consistency it is meant that $\|\hat{\gamma} - \gamma\| + d_{\mathcal{M}}(\hat{f}_{\beta|x_1}, f_{\beta|x_1}) = o_p(1)$.

**B.2.2 Theorem 6**

The choice of tuning parameters must satisfy the following condition:

**Assumption E3’.** $\mathcal{M}_n$ defined in equation (8) is such that (i) $\mathcal{M}_n \subseteq \mathcal{M}_{n+1}$ and as $n \to \infty$, (ii) $\mathcal{B}_n \times \mathcal{X}_n$ becomes dense in $S_{\beta} \times \text{Supp}(X_1)$ and (iii) $I(n) \log I(n) = o(n)$ for $I(n) = B(n)X(n)$. 
We also place some restrictions on the complexity of $\mathcal{N}_{F_x,n}$, the neighborhood to which the estimator $\hat{F}_{x,n}$ belongs with probability approaching one. For this purpose define $N(w, \mathcal{G}, \| \cdot \|_{\mathcal{G}})$ as the covering number of set $\mathcal{G}$ with balls of radius $w$ under the norm $\| \cdot \|_{\mathcal{G}}$.

**Assumption E4’.** (i) $(\mathcal{N}_{F_x,n}, \| \cdot \|_x)$ and $\Gamma$ are compact. (ii) $P_t$ is Lipschitz continuous in $\gamma \in \Gamma$ and continuous in $F_x \in \mathcal{N}_{F_x,n}$. (iii) $\log N(w, \sqrt{I(n), \mathcal{N}_{F_x,n}, \| \cdot \|_x}) = o(n)$ with $I(n)$ as in Assumption E3’.

**Proof of Theorem 6.** The proof consists of verifying the assumptions of Theorem 6 imply those of Theorem 5. Assumption E1 is assumed. To verify assumption E2, suppose that (i) $\mathcal{M}_n$ and $\mathcal{M}$ are compact in the weak topology and (ii) that $E[(Y, F_x, \gamma, \tilde{f}_{\beta|x_1})]$ is continuous in $\tilde{f}_{\beta|x_1} \in \mathcal{M} \supset \mathcal{M}_n$ in the weak topology and $\gamma \in \Gamma$. Then, since $\theta_0$ is identified, for any $(\tilde{\gamma}, \tilde{f}_{\beta|x_1}) \neq (\gamma, f_{\beta|x_1})$,

$$E[\psi(Y, F_x, \gamma, \tilde{f}_{\beta|x_1})] - E[\psi(Y, F_x, \tilde{\gamma}, \tilde{f}_{\beta|x_1})] > 0$$

Because $\{(\tilde{\gamma}, \tilde{f}) \in \Gamma \times \mathcal{M}_n : \|\tilde{\gamma} - \gamma\| + d_{\mathcal{M}}(\tilde{f}, f_{\beta|x_1}) \geq \epsilon\}$ is closed in the compact set $\mathcal{M}_n \times \Gamma$, it is compact and the infimum

$$E[\psi(Y, F_x, \gamma, f_{\beta|x_1})] - \sup\left\{ (\tilde{\gamma}, \tilde{f}) \in \Gamma \times F_x, f_{\beta|x_1} : \|\tilde{\gamma} - \gamma\| + d_{\mathcal{M}}(\tilde{f}, f_{\beta|x_1}) \geq \epsilon \right\} E[\psi(Y, F_x, \tilde{\gamma}, \tilde{f}_{\beta|x_1})]$$

is attained for each $(\epsilon, n)$. Set this difference to $c_n(\epsilon)$. It remains to show that $\lim\inf c_n(\epsilon) > 0$. Consider that

$$c_n(\epsilon) = E[\psi(Y, F_x, \gamma, f_{\beta|x_1})] - \sup\left\{ (\tilde{\gamma}, \tilde{f}) \in \Gamma \times \mathcal{M}_n : \|\tilde{\gamma} - \gamma\| + d_{\mathcal{M}}(\tilde{f}, f_{\beta|x_1}) \geq \epsilon \right\} E[\psi(Y, F_x, \tilde{\gamma}, \tilde{f}_{\beta|x_1})]$$

$$\geq E[\psi(Y, F_x, \gamma, f_{\beta|x_1})] - \sup\left\{ (\tilde{\gamma}, \tilde{f}) \in \Gamma \times \mathcal{M} : \|\tilde{\gamma} - \gamma\| + d_{\mathcal{M}}(\tilde{f}, f_{\beta|x_1}) \geq \epsilon \right\} E[\psi(Y, F_x, \tilde{\gamma}, \tilde{f}_{\beta|x_1})] > 0$$

The weak inequality is because $\mathcal{M}_n \subseteq \mathcal{M}$. The strict inequality is because the set $\{(\tilde{\gamma}, \tilde{f}) \in \Gamma \times \mathcal{M}_n : \|\tilde{\gamma} - \gamma\| + d_{\mathcal{M}}(\tilde{f}, f_{\beta|x_1}) \geq \epsilon\}$ is compact and $E[(Y, F_x, \gamma, f_{\beta|x_1})]$ is continuous. Since $c_n(\epsilon)$ is bounded away from zero uniformly in $n$, its limit inferior is strictly positive.

To complete the argument, it must be shown that (i) $\mathcal{M}_n$ and $\mathcal{M}$ are compact in the weak topology and (ii) that $E[\psi(Y, F_x, \gamma, f_{\beta|x_1})]$ is continuous on $\mathcal{M} \supset \mathcal{M}_n$ in the weak topology and $\gamma \in \Gamma$. Compactness of $\mathcal{M}$ and $\mathcal{M}_n$ in the weak topology is shown in Fox, Kim, and Yang (2016, pp. 240, 247). Since the CCP functions $P_t$ are continuous in $(b, \gamma)$ (Norets
the argument of Fox, Kim, and Yang (2016, Remark 2) implies the function \( f_{\beta|x_1} \mapsto \int \prod_{t=1}^T P_t(a_{it}, x_{it}, b; F_x, \gamma) df_{\beta|x_1}(b, x_{i1}) \) is continuous. Since it is bounded away from zero, \( f_{\beta|x_1} \mapsto \log \int \prod_{t=1}^T P_t(a_{it}, x_{it}, b; F_x, \gamma) df_{\beta|x_1}(b, x_{i1}) \) is also continuous. And since this function is bounded away from negative infinity, \( f_{\beta|x_1} \mapsto E[\log \int \prod_{t=1}^T P_t(a_{it}, x_{it}, b; F_x, \gamma) df_{\beta|x_1}(b, x_{i1})] \) is continuous by the bounded convergence theorem.

Assumption E3(i) is guaranteed by Assumption E3′(i). For Assumption E3(ii), Fox, Kim, and Yang (2016, p. 247) show the existence of a sequence \((f_n)_{n \in \mathbb{N}} \subseteq M\) that converges to \( f_{\beta|x_1} \in M \). Since the sequence \((f_n)_{n \in \mathbb{N}}\) takes values in \( M \) and \( E[\psi(YF_x, \gamma, f_{\beta|x_1})] \) is continuous on \( M \), we have that

\[ |E[\psi(Y, F_x, \gamma, f_n)] - E[\psi(Y, F_x, \gamma, f_{\beta|x_1})]| = o(1). \]

For the first part of Assumption E4, note that

\[
E[\psi(Y, F_x, \gamma, f_{\beta|x_1})] \leq E[|\psi(Y, F_x, \gamma, f_{\beta|x_1})|] \\
\leq E \left[ \sum_{t=2}^T \log F_{x_t}(X_t, X_{t-1}, A_{t-1}) \right] + E \left[ \log \int \prod_{t=1}^T P_t(A_t, X_t, b; F_x, \gamma) df_{\beta|x_1}(b, x_{i1}) \right] < \infty.
\]

The left term in the sum is finite by construction. Since \( N_{f,n} \times \Gamma \times S_\beta \) is compact and \( P_t \) is strictly positive for each \((b, F_x, \gamma)\), \( P_t \) is uniformly bounded away from zero, so the right term is finite also. Then by (Chen 2007, p. 5592), \( \log N(w, \{\psi(\cdot, F_x, \gamma, f_{\beta|x_1}) : (F_x, \gamma, f_{\beta|x_1}) \in N_{f,n} \times \Gamma \times M_n \}, \| \cdot \|_1) = o_p(n) \) implies the first part of Assumption E4. This entropy is bounded above by the sum of the entropies associated with \( N_{F_x,n}, \Gamma \) and \( M_n \). Fox, Kim, and Yang (2016, p. 248) show the entropies associated with \( \Gamma \) and \( M_n \) are \( o_p(n) \) under Assumption E3′(iii). By Assumption E4′(iii), the entropy associated with \( N_{F_x,n} \) is \( o_p(n) \). The second part of Assumption E4 follows easily from the continuity of the population criterion function on the compact set \( N_{F_x,n} \times \Gamma \times M_n \).

References for Online Appendix

Aguirregabiria, V. and Mira, P. (2007). “Sequential estimation of dynamic discrete games”. *Econometrica* 75.1, pp. 1–53.

Chen, X. (2007). “Large sample sieve estimation of semi-nonparametric models”. *Handbook of Econometrics* 6, pp. 5549–5632.
Fox, J. T., Kim, K. I., and Yang, C. (2016). “A simple nonparametric approach to estimating the distribution of random coefficients in structural models”. *Journal of Econometrics* 195.2, pp. 236–254.

Hahn, J., Liao, Z., and Ridder, G. (2018). “Nonparametric two-step sieve M estimation and inference”. *Econometric Theory* 34.6, pp. 1281–1324.

Norets, A. (2010). “Continuity and differentiability of expected value functions in dynamic discrete choice models”. *Quantitative economics* 1.2, pp. 305–322.