Dynamics of a Stratified Population of Optimum Seeking Agents on a Network - Part II: Steady State Analysis

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Abstract—In this second part of our work, we study the steady state of the population and the social utility for the three dynamics SSD, NBRD and NRPM; which were introduced in the first part. We provide sufficient conditions on the network based on a maximum payoff density parameter of each node under which there exists a unique Nash equilibrium. We then utilize positive correlation properties of the dynamics to reduce the flow graph in order to provide an upper bound on the steady state social utility. Finally we extend the idea behind the sufficient condition for the existence of a unique Nash equilibrium to partition the graph appropriately in order to provide a lower bound on the steady state social utility. We also illustrate interesting cases as well as our results using simulations.

Index Terms—Multi-agent systems, population dynamics on networks, collective behavior, bounds on steady state social utility.

I. INTRODUCTION

A primary goal in the analysis of evolutionary dynamics is to characterize the set of equilibrium points and to study their stability. Many applications might also require the knowledge of the steady state, given the initial condition of the population, for higher level control and planning problems. Similarly, knowledge of the steady state value of the social utility as a function of the initial population state can be useful for many purposes. In this paper, we seek to obtain efficiently computable bounds on the steady state social utility under stratified smith dynamics (SSD), nodal best response dynamics (NBRD) and network restricted payoff maximization (NRPM) as a function of the initial population state.

A. Literature Survey

Population games and evolutionary dynamics [1] find application in problems related to distributed control and formation control [2]–[4] as well as in social or socio-technical systems such as transportation and opinion dynamics. In the context of network games, [5]–[9] model a finite population of agents as nodes and models the interactions using the graph. Other works of literature [3], [10]–[13] consider the nodes of the graph to represent choices with the state of the population being composed of the fractions of population choosing different nodes. In these works, the network plays a major role in the evolution of the population.

In part one of this work [14], we modeled the game with stratified populations with their choices as the nodes of a graph, modeled the three dynamics: SSD, NBRD and NRPM. We also existence and uniqueness of solutions of these dynamics and showed that their solutions converge to the set of Nash equilibria.

B. Contribution

In this second part of our work, we first provide sufficient conditions on the graph under which the stratified population game has a unique Nash equilibrium. Then, for general graphs, we provide a computationally efficient method for computing bounds on the steady state social utility. This method is particularly useful in the case of NBRD and NRPM, as these dynamics rely on an underlying optimization problem, which makes simulating the full dynamics in order to determine these steady state values computationally expensive. Indeed, the method we propose depends only on the initial population state, the network and the node parameters.

References [3], [11]–[13] have a setup for the underlying game closest to ours. However, all these papers assume that Nash equilibrium is unique and in the relative interior of the $n$-dimensional probability simplex and results are local. Moreover, in all these works, the population is not stratified. Our preliminary work on this topic [15] considers only quadratic cumulative payoff functions as opposed to strictly concave functions and is concerned only with the convergence analysis. Here, we provide a sufficient condition on the graph and node parameters under which the Nash equilibrium is unique and for a general graph provide a computationally efficient method for computing bounds on the steady state social utility, given the initial population state.

C. Organization

The rest of the paper is organized as follows. In Section II, we list a few important results from the first part of this work and outline the problem we wish to address in this paper. In
Section III, we give sufficient conditions on the graph under which the population game has a unique Nash equilibrium. In Section IV we give algorithms to provide meaningful bounds on the steady state social utility. In Section I we provide simulation results and some interesting examples.

D. Notation and Definitions

We denote the set of real numbers, the set of non-negative real numbers and the set of integers with \( R, R_+, \) and \( Z, \) respectively. We let \([p, q]_Z\) be the set of all integers between \( p \) and \( q \) (inclusive), i.e., \([p, q]_Z := \{ x \in Z | p \leq x \leq q \}. \) \( \mathbb{R}^n \) (similary \( \mathbb{R}^n_+ \)) is the cartesian product of \( \mathbb{R} \) (equivalently \( \mathbb{R}^n_+ \)) with itself \( n \) times. If \( v \) is a vector in \( \mathbb{R}^n, \) we denote \( v_i \) as the \( i^{th} \) component of \( v \) and for a vector \( v \in \mathbb{R}^n, \) we let \( \text{supp}(v) := \{ i \in [1, n]_Z | v_i \neq 0 \}. \) We let \( 1 \) be the vector, of appropriate size, with all its elements equal to \( 1 \) and we let \( e_i \) be the vector, again of appropriate size, with its \( i^{th} \) element equal to \( 1 \) and 0 for all other elements. The empty set is denoted by \( \emptyset. \) If \( Q \) is an ordered countable set, then \( Q_i \) denotes the \( i^{th} \) member of \( Q \) and \( \#Q \) is used to represent the cardinality of \( Q. \)

For two sets \( U, V \subset Q, \) the set subtraction operation is denoted by \( U \setminus V = U \cap V^c, \) where \( V^c \) is the set complement of \( V \in Q. \) \( \{i, j\} \) is used to denote an unordered pair while \((i, j)\) is used to denote an ordered pair. For a vector \( v \in \mathbb{R}^n, \) \( v \geq 0 \) is used to denote term wise inequalities. By \( [.]_+ : \mathbb{R} \rightarrow \mathbb{R}_+ \) we denote the function that is defined as \( [x]_+ := \max \{ x, 0 \}. \) For a function \( f(x) : \mathbb{R}^n \rightarrow \mathbb{R}, \) \( \nabla_x f \) is used to denote the gradient of \( f(\cdot) \) with respect to \( x, \) i.e., the \( j^{th} \) component of \( \nabla_x f \) is \( \frac{\partial f}{\partial x_j}. \) We denote by \( S^n_+: \{ v \in \mathbb{R}^n | v \geq 0 \}, \) the \( n \)-dimensional simplex \( S^n_+: = \{ v \in \mathbb{R}^n | v \geq 0, 1^T v = \gamma \}. \)

II. Preliminaries

In this section, we recollect the framework and the essential aspects about the three dynamics SSD, NBRD, NRPM that we developed in [14]. We let the population is composed of a continuum of agents and the network of choices is given by an undirected connected graph \( G := (V, E) \), where \( V \) is a set of nodes and \( E \subseteq V \times V \) is a set of edges. The fraction of population in node \( i \) is represented by \( x_i \in [0, 1] \) and \( \sum_{i \in V} x_i = 1. \) Thus \( x \in S^{|V|}_1 \) represents the population state.

Let \( p_i(\cdot) : [0, 1] \rightarrow \mathbb{R} \) be the function that models the cumulative payoff of the fraction \( x_i. \) We assume that the fraction in each node is stratified and the agents in different strata of a given node receive different payoffs. Let \([a, b] \subseteq [0, x_i] \) be an arbitrary interval. Then the agents of node \( i \) that are in the strata \([a, b]\) get an average payoff of \( \frac{p_i(b) - p_i(a)}{b - a}. \)

For a node \( i \) if \( a \in [0, x_i], \) then \( u_i(a) := \frac{dp_i}{dy}(a), \) is the average payoff that the agents in the strata \([a]\) of node \( i \) receive. By strata \([a]\) we mean the collection of infinitesimal strata around \( a \) in \([0, x_i]. \) We call \( u_i(\cdot) \) as the payoff density function of node \( i. \) We let \( u_i(0) \) be the right derivative of \( p_i(\cdot) \) at zero and \( u_i(1) \) be the left derivative of \( p_i(\cdot) \) at one. We let \( u(\cdot) \) be the vector whose \( i^{th} \) element is \( u_i(\cdot). \) Through out this paper, we make the following assumption.

(A1) For all \( i \in V, \) \( p_i(\cdot) \) is twice continuously differentiable and strictly concave. Hence, \( \forall i \in V, u_i(\cdot) \) is a strictly decreasing function.

The function,

\[
U(x) := \sum_{i \in V} [p_i(x_i) - p_i(0)],
\]

which we call as the social utility function represents the aggregate payoff of the population as a whole. Note that \( U(\cdot) \) is a strictly concave function and \( e_i^T \nabla U(x) = u_i(x_i), \) \( \forall x \in S^{|V|}_1. \) Let \( N^i \) be the set of all neighbors of node \( i \) in the graph \( G \) and let \( \bar{N}^i = N^i \cup \{ i \}. \) Given the undirected graph \( G, \) we let

\[
A := \bigcup_{\{i,j\} \in E} \{(i,j), (j,i)\}.
\]

When an infinitesimal agent in node \( i \in V \) decides to switch to a node \( j \in N^i, \) it gets to enter the strata \([x_j]\) of node \( j. \) The dynamics that govern the evolution of the stratified population can be described by the general equation

\[
x = A \Delta(x) := F(x).
\]

Here \( A \in \mathbb{R}^{|V| \times |A|} \) is the incidence matrix of the directed graph \( \mathcal{F} := (V, A) \) and \( \Delta(x) \in \mathbb{R}^{|A|} \) is the vector that accumulates the outflows \( \delta_{ij} \) along each arc \( (i, j) \in A \) into a vector. Such a class of dynamics is referred to as flow balanced dynamics. The following discussion holds for a general case where the total population is \( \rho \in \mathbb{R}_+, \) i.e \( \sum_{i \in V} x_i = \rho. \)

The sub class of flow balanced dynamics that additionally satisfies the following definition of strong positive correlation are referred to as strongly positively correlated flow balanced dynamics.

Definition 2.1: (Strong positive correlation). We say that \( F(\cdot) : \mathbb{R}^{|V|} \rightarrow \mathbb{R}^{|V|} \) in (3) is strongly positively correlated with \( u(\cdot) \) if \( \forall x \in S^{|V|}_\rho, \) \( u_i(x_i) \geq u_j(x_j) \) implies \( \delta_{ij} = 0, \) \( \forall (i, j) \in A. \)

The following lemma enumerates the key result regarding this type of dynamics.

Lemma 2.2: (Non-emptiness of set of Nash equilibrium for strongly positively correlated dynamics). Suppose \( F(\cdot) \) is strongly positively correlated with \( u(\cdot). \) Then the set

\[
\mathcal{NE}_\rho := \{ x \in S^{|V|}_\rho | u_i(x_i) \geq u_j(x_j), \forall j \in N^i, \forall \bar{v} \in \text{supp}(x) \},
\]

is non-empty. Furthermore, \( \mathcal{NE}_\rho \) is a subset of the equilibrium points of \( F(\cdot) \) in \( S^{|V|}_\rho \) and the unique optimizer of

\[
P_1(V', \rho) : f_{V'}(\rho) := \max_{x} U(x) \text{ s.t. } x \in S^{|V'|}_\rho,
\]

with \( V' \subset V \) belongs to the set \( \mathcal{NE}_\rho. \)

The set \( \mathcal{NE}_\rho \) in (4) is the set of Nash equilibria of the population game.

The three dynamics, namely, SSD, NBRD and NRPM, which we have introduced in [14] vary in the levels of
coordination displayed by the infinitesimal agents. The $\Delta$ vector is hence calculated in different ways for each of them. We discuss these dynamics briefly and state key results related to them.

(a) Stratified Smith Dynamics (SSD): Here, each agent is selfish and revises its choice at independent and random time instants by comparing its payoff with the payoff of a neighboring agent. Thus the individual components of $\Delta$ are given by
\[
\delta_{ij} = [u_j(x_j)(x_i - y_{ij}) - (p_i(x_i) - p_i(y_{ij}))]_+ ,
\]
where,
\[
y_{ij} := \begin{cases} 
0, & \text{if } u_i^{-1}(u_j(x_j)) < 0 \\
u_i^{-1}(u_j(x_j)), & \text{if } 0 \leq u_i^{-1}(u_j(x_j)) \leq x_i \\
x_i, & \text{if } u_i^{-1}(u_j(x_j)) > x_i.
\end{cases}
\]  

(6)

The key results regarding SSD are also presented in the following theorem.

Theorem 2.3: (Strong positive correlation of SSD and asymptotic convergence to the set of Nash equilibria.) Suppose that for all $i \in V$, $u_i(.)$ is strictly decreasing. Let the evolution of $x$ be governed by SSD, dynamics in (3) with $\delta_{ij}$ defined in (6), with an initial condition $x(0) \in S^{|V|}_p$. Then SSD is strongly positively correlated with $u(.)$ and the set of equilibrium points of SSD in $S^{|V|}_p$ is the set $N^C_{p}^{\rho}$ defined in (4). Furthermore, as $t \to \infty$, $x(t)$ approaches $N^C_{p}^{\rho}$ defined in (4) and $U(x(t))$ converges to a constant.

(b) Nodal Best Response Dynamics (NBRD): Here, the agents coordinate with each other at the nodal level. The fraction of population in each node decides to redistribute itself among its neighbors by computing its best response to the current configuration of $x$; while assuming that the fraction in the neighboring nodes do not change. Thus $\delta_{ij}$ for NBRD is computed from the following optimization problem.

\[
P_3^j : \max_{\{d_{ij} \in N^+ \}} \sum_{j \in N^+} [p_j(x_j + d_{ij}) - p_j(x_j)] + [p_i(d_{ii}) - p_i(0)]
\]
\[\text{s.t. } d_{ii} + \sum_{j \in N^+} d_{ij} = x_i, \quad d_{ij} \geq 0, \quad \forall j \in N^+.
\]

(8)

Thus if $\{d_{ij}^*\}_{(i,j) \in A}$ is the subset of optimizers of $P_3^j_{i \in V}$, then $\delta_{ij} = d_{ij}^* \forall (i,j) \in A$. The key results related to NBRD are listed below.

Theorem 2.4: (Strong positive correlation; existence and uniqueness of solutions and asymptotic convergence to the set of Nash equilibria for NBRD). Let $\{d_{ij}^*\}_{(i,j) \in A}$ be the subset of optimizers of $\{P_3^j\}_{i \in V}$. Let NBRD be the dynamics in (3) with $\delta_{ij} = d_{ij}^* \forall (i,j) \in A$. NBRD is strongly positively correlated with $u(.)$. For each $x(0) \in S^{|V|}_p$, NBRD has a unique solution that exists for $\forall t \geq 0$. The set of equilibrium points of NBRD in $S^{|V|}_p$ is $N^C_{p}^{\rho}$ defined in (4). Further, if $x(t)$ evolves according to NBRD, then as $t \to \infty$, $U(x(t))$ converges to a constant and $x(t)$ approaches $N^C_{p}^{\rho}$.

(c) Network Restricted Payoff Maximization (NRPM): Here, the agents coordinate across the entire population and evolve under a centralized decision scheme. They do so in order to maximize the social utility of the entire population.

\[
P_3 : \max_{z,d} \sum_{i \in V} [p_i(z_i) - p_i(0)] = \max_{z,d} U(z)
\]
\[\text{s.t. } z_i = x_i + \sum_{j \in N^+} (d_{ji} - d_{ij}), \quad \forall i \in V,
\]
\[
\sum_{j \in N^+} d_{ij} = x_i, \quad \forall i \in V,
\]
\[
d_{ij} \geq 0, \quad \forall (i,j) \in \bar{A} := A \cup \{(i,i) \mid i \in V\}.
\]

(9)

Thus, if $(z^*, d^*)$ is an optimizer of $P_3$, then the dynamics is given by
\[
x = A \Delta(x) = z^*(x) - x.
\]

(10)

Key results regarding NRPM are listed next.

Lemma 2.5: (The resultant node fractions in any optimal solution of $P_3$ are unique). Consider the problem $P_3$ in (9) with $x \in S^{|V|}_p$. Let $\mathcal{OP}$ be the set of all optimizers of $P_3$. Then $\forall (z^*, d^*) \in \mathcal{OP}$, $z^* = x + A d^*$ is unique.

Lemma 2.6: (Properties of optimizers of $P_3$). Suppose $(z^*, d^*)$ is an optimizer of $P_3$. Then
\[
u_j(z_j^*) \geq u_i(z_i^*), \quad \forall (i,j) \in A \text{ s.t. } d_{ij}^* > 0.
\]

(11)

Theorem 2.7: (Existence and uniqueness of solutions of NRPM and asymptotic convergence to the set of Nash equilibria). Let NRPM be the dynamics in (10). For each $x(0) \in S^{|V|}_p$, NRPM has a unique solution that exists for $\forall t \geq 0$. The set of equilibrium points of NRPM in $S^{|V|}_p$ is $N^C_{p}^{\rho}$ defined in (4). Further, if $x$ evolves according to NRPM, then as $t \to \infty$, $U(x(t))$ converges to a constant and $x(t)$ approaches $N^C_{p}^{\rho}$.

We provide another definition and a remark next to conclude the preliminaries.

Definition 2.8: (Nash convergent dynamics). Let $G' = (V', A')$ be the underlying graph. Any flow balanced dynamics $F(.)$ on $G'$ that asymptotically approaches set of Nash equilibria $N^C_{p}^{\rho}$ from any initial condition in $S^{|V'|}_p$ is termed as Nash convergent dynamics.

Remark 2.9: (SSD, NBRD and NRPM are Nash Convergent). By Theorems 2.3, 2.4 and 2.7; SSD, NBRD and NRPM are Nash convergent on $G$.

A. Problem Setup

In general SSD, NBRD and NRPM may not converge to the same state. In this paper, we seek to study the steady state and the steady state social utility with the knowledge of the structure of the graph $G = (V, E)$ and the initial condition, which we denote by $x^0 := x(0)$.

From Remark 2.9 SSD, NBRD and NRPM are all Nash convergent dynamics. Furthermore, for all three dynamics,
the social utility converges to a constant value. We define $U_{\text{SSD}}(x^0)$, $U_{\text{NBRD}}(x^0)$ and $U_{\text{NRPM}}(x^0)$ to be the values to which the social utility converges to under the three different dynamics from an initial condition $x^0$.

In Section III, we provide a sufficient condition under which the steady state state of any Nash convergent dynamics is the same. Then in Section IV, we provide bounds on the steady state social utility for the three dynamics.

III. UNIQUE NASH EQUILIBRIUM

In this section, we give sufficient conditions on the graph $G$ and the payoff density functions, under which the set of Nash equilibria is a singleton. We define for each node $i \in V$, the maximum payoff density parameter (MPDP) as

$$a_i := \max_{y \in [0,1]} u_i(y) = u_i(0),$$  \hspace{1cm} (12)

where we use the fact that $u_i(\cdot)$ is a strictly decreasing function in $[0,1]$ for all $i \in V$.

We formally define a path in a graph next.

Definition 3.1: (Path in graph $G = (V, E)$). A path, in graph $G = (V, E)$, between $i \in V$ and $j \in V$ is given by an ordered set of non-repeating elements $P(i,j) \subseteq V$ such that $P_1(i,j) = i$, $P_n(i,j) = j$ and $\{P_k(i,j), P_{k+1}(i,j)\} \notin E$, $\forall k \in [1, n-1]$, where $P_k(i,j)$ denotes the $k$th element in $P(i,j)$ and $n = |P(i,j)|$.

Using this definition and (12), we define a path with quasi-concave MPDP’s next.

Definition 3.2: (Path with quasi-concave MPDP’s). Suppose $P(i,j)$ is a path between $i \in V$ and $j \in V$ and let $n := |P(i,j)|$. Let $\pi(i,j) := P_k(i,j)$. We say $P(i,j)$ is a path with quasi-concave MPDP’s if and only if $\forall k, l \in [1, n]$ such that $l \geq k$ and $\forall m \in [k, l]$, $a_{\pi(m)} \geq \min \{a_{\pi(k)}, a_{\pi(l)}\}$.

Definition 3.2 can be interpreted in the following way. For a path with quasi-concave MPDP’s, if we arrange the MPDP’s in order of the nodes visited in the path, then the MPDP’s form a quasi-concave function. Using this interpretation and Definition 3.2, the following result can be immediately stated regarding the nature of MPDP’s in such a path.

Lemma 3.3: (Monotonicity of MPDP’s in a path with quasi-concave MPDP’s). Let $P(i,j)$ be a path with quasi-concave MPDP’s between $i$ and $j$, where $i, j \in V$. Let $\pi(k) := P_k(i,j)$ and $n = |P(i,j)|$. Then $\forall k \in [1, n]$, either $a_i = a_{\pi(1)} \leq a_{\pi(2)} \leq \cdots a_{\pi(k)}$ or $a_{\pi(k)} \geq \cdots \geq a_{\pi(n-1)} \geq a_{\pi(n)} = a_j$.

A proof of this is provided in Appendix II-A.1.

Using Definition 3.2, we define a quasi-concave hill next.

Definition 3.4: (Quasi-concave hill or QCH). We call $G = (V, E)$ to be a quasi-concave hill or QCH if and only if there exists a path with quasi-concave MPDP’s between every two distinct nodes (i.e. $i, j \in V$ and $i \neq j$).

Note that Definition 3.4 requires the graph $G$ to be connected. The ‘hill’ in the name of a QCH is given to evoke the interpretation that the agents are always seeking to climb a hill to increase their payoffs. The following lemma shows that in a QCH, if the population is in an equilibrium (in the set of Nash equilibria), then the payoff densities across the non-empty nodes is the same.

Lemma 3.5: (Same payoff density across non-empty nodes in a population at equilibrium in a QCH). Let $G$ be a QCH and let $x \in N_{\rho}^{\pi}$. Then $\forall i, j \in \text{supp}(x)$, $u_i(x_i) = u_j(x_j)$.

We present the proof in Appendix II-A.2.

Finally, in this section, we show that for a QCH the Nash equilibrium is unique. We illustrate the consequence of this fact in the remark following the theorem, proof of which is presented in Appendix II-A.3

Theorem 3.6: (Uniqueness of Nash equilibrium for a QCH). Suppose $\rho \geq 0$. If $G = (V, E)$ is a QCH then $N_{\rho}^{\pi}$ in (4) is a singleton and the unique $x \in N_{\rho}^{\pi}$ is the unique optimizer of $P_1$ in (5).

Remark 3.7: (Same steady state state of Nash convergent dynamics for QCH). The immediate consequence of Theorem 3.6 is that $\forall(x(0)) \in S_{\rho}^{\pi}$ the steady state of any Nash convergent dynamics is the same. Thus, from Remark 2.9, we can say that if the population dynamics is governed by any one of SSD, NBRD and NRPM then $\forall x(0) \in S_{\rho}^{\pi}$, the population converges to the unique Nash equilibrium.

The utility of Theorem 3.6 and Remark 3.7 extends to the case in which $G$ is not a QCH. In particular, we can use them to provide bounds on the steady state social utility of SSD, NBRD and NRPM. This is the topic of the next section.

IV. BOUNDS ON THE STEADY STATE SOCIAL UTILITY

In general, the steady state for SSD, NBRD and NRPM may not be the same. The graph $G$ being a QCH is a sufficient condition for all these dynamics to share a common steady state irrespective of the initial condition in $S_{\rho}^{\pi}$. In this section, we provide meaningful bounds on the steady state social utility of SSD, NBRD and NRPM in the case where $G$ is not a QCH. In the construction of these bounds, the results stated in Section III serve as important building blocks.

Recall that $G = (V, E)$ is the underlying undirected graph and $A$ the associated arc set where for each undirected edge $\{i, j\} \in E$, we have directed arcs $(i, j)$ and $(j, i)$ in $A$. Moreover, $A$ contains no other arcs. Based on the payoff functions $u_i(\cdot)$, it is possible to reduce $A$ based on the fact that some $x_i(\cdot)$’s will remain zero for SSD, NBRD and NRPM no matter what the initial condition is. We characterize this in the following lemma.

Lemma 4.1: (Reduction of flow graph for SSD, NBRD and NRPM). Let $u_i(\cdot)$’s be strictly decreasing functions. Let $\theta \in \mathbb{R}^{\pi}$ be a vector with individual components $\theta_i \geq 0$. If $x_i \in [0, \theta_i]$, then

$$\delta_{ij}(x) = 0, \forall (i, j) \in A \text{ s.t. } u_i(\theta_i) \geq u_j(0),$$  \hspace{1cm} (13)

for SSD and NBRD. Further if $z^*_i(x) \in [0, \theta_i]$, then (13) holds for NRPM as well.

A proof for the same is provided in Appendix II-B.1. Thus, for a given initial state $x(0) = x^0$, if we are able to find a uniform (in time) upper bound $\theta_i$ on $x_i$ and $z^*_i(x)$ for each $i \in V$ then for the arcs $(i, j) \in A$ with the property $u_i(\theta_i) \geq u_j(0)$, the flow $\delta_{ij} = 0$ for all time $t \geq 0$ and hence the arc $(i, j)$ can be removed without modifying the evolution of $x$. Now as $\forall i \in V, x_i \in [0,1]; 1 \geq \theta_i, \forall i \in V$. Thus one is a good initial
estimate of $\theta_i$. Using this idea we can remove the arcs in the set
\[ \{(i, j) \in A \mid u_i(1) \geq u_j(0)\} \tag{14} \]
without affecting the dynamics. We call this reduction as a static reduction of the arc set as this reduction is carried out solely based on the properties of $G$ and the functions $p_i(.)$. Note that almost one arc per pair of nodes is removed in this process as $u_i(1) \leq u_j(0)$ and $u_i(1) \leq u_i(0)$ are not simultaneously possible due to the strictly decreasing nature of $u_i(.)$’s.

In order to reduce the graph further, the initial condition $x^0$ must be taken into account. In fact, only the nodes which have a directed path in $\hat{F}$ from some node in $\text{supp}(x^0)$ are the ones that may participate in the evolution of $x$. The following remark describes how Algorithm 1 utilizes Lemma 4.1 to carry out such a reduction.

**Algorithm 1**: reduceGraph$(F, x^0)$

Data: $F = (V, A), x^0$

Result: $\hat{F}$

1. $\hat{F} \leftarrow (V, A)$
2. $F_{\text{temp}} \leftarrow (\emptyset, \emptyset)$
3. while $\hat{F} \neq F_{\text{temp}}$ do
   4. $F_{\text{temp}} \leftarrow \hat{F}$
   5. $\hat{F} \leftarrow \text{repeatReduction}(\hat{F}, x^0)$
4. end
5. return $\hat{F}$

**Algorithm 2**: repeatReduction$(\hat{F}, x^0)$

Data: $F = (V, A), x^0$

Result: Reduced graph

1. for $i \in V$ do
   2. $R^i \leftarrow \{j \in \text{supp}(x^0) \mid \exists$ a directed path in $\hat{F}$ from $j$ to $i\}$
   3. $\theta_i \leftarrow \sum_{j \in R^i} x_j$
      \hspace{1cm} $\triangleright$ estimates the maximum population fraction that can visit $i$
4. end
5. $A_{\text{temp}} \leftarrow A \setminus \{(i, j) \in A \mid u_i(\theta_i) \geq u_j(0)\}$
6. $\hat{F}_{\text{temp}} \leftarrow (V, A_{\text{temp}})$
7. $\text{supp}(x^0)$
   \hspace{1cm} $\triangleright$ $\text{supp}(x^0)$ is always a subset of the reduced node set
8. $\hat{V}_{\text{temp}} \leftarrow \{i \in V \mid \exists$ a directed path in $\hat{F}_{\text{temp}}$ from some $j \in \text{supp}(x^0)$ to $i\}$
9. $A_{\text{temp}} \leftarrow A_{\text{temp}} \setminus \{(i, j) \in A_{\text{temp}} \mid i \notin \hat{V}_{\text{temp}}$ or $j \notin \hat{V}_{\text{temp}}\}$
   \hspace{1cm} $\triangleright$ remove hanging arcs
10. return $\hat{F}_{\text{temp}} = (\hat{V}_{\text{temp}}, A_{\text{temp}})$

**Remark 4.2**: (Graph reduction using initial state). The function reduceGraph() described in Algorithm 1 takes in the graph $F$ and $x^0$ as inputs and returns the reduced graph $\hat{F} = (\hat{V}, \hat{A})$. Algorithm 2 simultaneously estimates $\theta_i$ and removes nodes and arcs of $\hat{F}$ until the graph cannot be reduced further. Steps 2-3 are used to estimate $\theta_i$ by setting it as the sum of all possible population fractions that can reach $i$, respecting the structure of the graph and the node parameters. Then Steps 7-9 only consider $\text{supp}(x^0)$ and those nodes in $\hat{F}$ that have a directed path from some node in $\text{supp}(x^0)$. The other nodes and hanging arcs are removed. Note that the first pass of Algorithm 1 performs the static reduction by removing the arcs in the set in (14). This is because $\hat{F}$ is strongly connected and hence the $\theta$ estimated in the first pass will be $\rho\theta$. Algorithm 1, thus, returns the reduced graph $\hat{F} = (\hat{V}, \hat{A})$ with the following properties:
   - $\text{supp}(x^0)$ is contained in $\hat{V}$;
   - $\hat{A}$ does not contain arcs $(i, j)$ of $A$ with the property $u_i(\theta_i) \geq u_j(0)$ for the estimated $\theta$.
   $\hat{F}$ is called the initial condition reduced graph or ICRG of $G$.

Now, the fact that the Steps 2-3 of Algorithm 2 refine, with each pass, the upper bound on both $x(t)$ and $z^s(x(t))$ can be easily seen from the fact that $z^s(x)$ is just a simple rearrangement of the population state $x$ among $V$.

The ICRG is useful in providing reasonable upper and lower bounds on the steady state social utility. We demonstrate this in the next two sections.

**A. Upper Bound on Steady State Social Utility**

In order to compute an upper bound on the steady state social utility, we try to compute the best social utility that the population as a whole can receive starting from $x^0$. This best social utility is in the space of all possible dynamics that have the property that the evolution is same under $G$ and $\hat{F}$. From Remark 4.2, we see that $\theta_i$ provided by reduceGraph$(F, x^0)$, Algorithm 1, is an upper bound on $x_i(t)$ and $z^s(x(t))$ for all $t \geq 0$. Thus, if $\hat{F}$ is the ICRG of $G$ and $(i, j) \notin \hat{A}$ then by Lemma 4.1 $\delta_{ij}(x(t)) = 0$ for all $t \geq 0$ in the case of SSD, NBRD and NRPM. Moreover if $i \in G$ but $i \notin \hat{F}$, then $x_i(t) = 0$, for all $t \geq 0$. For a node $i \in V$ of $G$, we define the set of in-reachable nodes of $i$ as $\text{in}R^i := \{j \in V \mid \exists$ a directed path from $j$ to $i$ in $\hat{F}\} \cup \{i\}$ and the set of out-reachable nodes of $i$ as $\text{out}R^i := \{j \in V \mid \exists$ a directed path from $i$ to $j$ in $\hat{F}\} \cup \{i\}$. We use $r_{ij}$ to represent the outflow from $i$ to $j$ in $\text{out}R^i$ and let $r$ be the vector that accumulates all the $r_{ij}$ into a vector. Then the optimum value of the optimization problem $P_4$ can be used as an upper bound on the steady state utility.

\[
U_{\text{max}} := \max_{w, r} \sum_{i=1}^{|V|} \left[ p_i(w_i) - p_i(0) \right] \\
\text{s.t. } w_i = \sum_{j \in \text{in}R^i} r_{ij}, \forall i \in V \\
r_{ij} \geq 0, \forall i \in V, \forall j \in \text{out}R^i \\
\sum_{j \in \text{out}R^i} r_{ij} = x_0^i, \forall i \in V.
\tag{15}
\]

Note that $P_4$ is visually similar to $P_3$ of NRPM but they are very different problems. In $P_3$, the decision variables in $d$
restricts the movement of the population fraction in each node to itself and among its neighbors in \( G \), i.e., \( d_{ij} \) captures the outflow form node \( i \in \mathcal{V} \) to node \( j \in \mathcal{N}^i \) (the neighbor set only). In \( P_A \), on the other hand, the decision variables in \( r \) allows the movement of the population fraction from each node to any node that is out-reachable from it in \( \mathcal{F} \). Thus, while \( P_A \) gives us the instantaneous flows in NRPM, \( P_A \) gives us the socially optimal longterm redistribution of the population starting from \( x^0 \) and under the path constraints imposed by \( \mathcal{F} \).

We present the main result of this section next and show the proof in Appendix II-B.2.

Theorem 4.3: (Upper bound on steady state social utility). Suppose \( x^0 \in S^0_\mathcal{V} \). Then
\[
U_{\text{SSD}}(x^0) \leq U_{\text{max}}; \quad U_{\text{NBRD}}(x^0) \leq U_{\text{max}}; \quad U_{\text{NRPM}}(x^0) \leq U_{\text{max}}.
\]

In the next subsection, we provide algorithms to compute lower bounds on the steady state social utility.

B. Lower Bound on Steady State Social Utility

In order to provide a meaningful lower bound on the steady state social utility, we partition the graph into certain subgraphs, each of which is a directed QCH, which we define formally in the sequel. Then, using the properties in Section III, we utilize (5) to compute the social utility of the population among each group independently for different allocations of population fractions in the group. Then, the problem is converted to one of optimal allocations to the collection of directed QCHs so that social utility is minimized.

We now formally define a directed QCH and subsequently, we also define other useful definitions. Note: up to the very end of this subsection, we present in the context of an arbitrary graph and in the end we apply it to ICRG \( \mathcal{F} \).

Definition 4.4: (Directed quasi-concave hill). We call a di-graph to be a directed quasi-concave hill (DQCH) if and only if its corresponding undirected graph is a QCH.

Definition 4.5: (Directed quasi-concave hill component or DQCH component). A subgraph \( \mathcal{H} = (\mathcal{W}, \mathcal{Q}) \) of the graph \( \mathcal{F} = (\mathcal{V}, A) \) is said to be a directed quasi-concave hill component or DQCH component if and only if \( \mathcal{H} \) is a DQCH.

Each DQCH component can be classified into one of two types: attractive or non-attractive. We define these next and also describe their significance.

Definition 4.6: (Attractive DQCH component or A-DQCH component). Let \( \mathcal{F} = (\mathcal{V}, A) \) be a directed graph and let \( \text{out} \mathcal{N}^i(\mathcal{F}) \) be the set of out-neighbors of a node \( i \in \mathcal{V} \) in \( \mathcal{F} \). A subgraph \( \mathcal{H} = (\mathcal{W}, \mathcal{Q}) \) of the graph \( \mathcal{F} \) is said to be an attractive directed quasi-concave hill component or A-DQCH component of \( \mathcal{F} \) if and only if
\[
\mathcal{H} \text{ is a DQCH}; \quad \forall i \in \mathcal{W}, \quad \text{out} \mathcal{N}^i(\mathcal{F}) \subseteq \mathcal{W} \quad \text{and} \quad \mathcal{Q} = \{(i,j) | j \in \text{out} \mathcal{N}^i(\mathcal{F})\}.
\]

Definition 4.7: (Non attractive DQCH component or NA-DQCH component). Let \( \mathcal{F} = (\mathcal{V}, A) \) be a directed graph and let \( \text{out} \mathcal{N}^i(\mathcal{F}) \) be the set of out-neighbors of a node \( i \in \mathcal{V} \) in \( \mathcal{F} \). A subgraph \( \mathcal{H} = (\mathcal{W}, \mathcal{Q}) \) of the graph \( \mathcal{F} \) is said to be a non-attractive directed quasi-concave hill component or NA-DQCH component of \( \mathcal{F} \) if and only if
\[
\mathcal{H} \text{ is a DQCH}; \quad \exists i \in \mathcal{W} \text{ and } j \in \text{out} \mathcal{N}^i(\mathcal{F}) \text{ such that } j \notin \mathcal{W}.
\]

From the definition of an A-DQCH component, it is clear that if some fraction of population starts inside it, then the fraction remains there forever as there are no outgoing arcs from such a component. This is the main reasoning for the nomenclature. Moreover, for every DQCH component, if the fraction of population in the said component is known a priori, we can use (5) to calculate the steady state social utility for that population fraction. Thus, the graph \( \mathcal{F} \) can be partitioned into such DQCH components in order to formulate a suitable optimization problem (like in Section IV-A) for providing a lower bound on the steady state social utility. Note that such a partition is not unique. We use the following partition to tighten the lower bound as much as possible.

Definition 4.8: (Maximal attractive component partition or MAC partition). A collection of subgraphs \( \{\mathcal{H}^q = (\mathcal{W}^q, \mathcal{Q}^q)\}_{q \in [1,n]} \) is said to be a maximal attractive component partition or MAC partition of \( \mathcal{F} = (\mathcal{V}, A) \) if and only if
\[
\forall q, r \in [1,n], \text{ such that } q \neq r, \mathcal{W}^q \cap \mathcal{W}^r = \emptyset, \mathcal{Q}^q \cap \mathcal{Q}^r = \emptyset;
\]
\[
\bigcup_{q \in [1,n]} \mathcal{W}^q = \mathcal{V}, \bigcup_{q \in [1,n]} \mathcal{Q}^q \subseteq A;
\]
\[
\forall q \in [1,n], \mathcal{H}^q \text{ is a DQCH component of } \mathcal{F};
\]
\[
\forall q \in [1,n], \text{ if } \mathcal{H}^q \text{ is an NA-DQCH component, then } \mathcal{H}^q \text{ does not contain any A-DQCH component of };
\]
\[
\forall q \in [1,n], \text{ if } \mathcal{H}^q \text{ is an A-DQCH component, then there does not exist any } I \subseteq [1,n], \text{ such that } T^I := \left( \bigcup_{q \in I} \mathcal{W}^q, \bigcup_{q \in I} \mathcal{Q}^q \right) \text{ is an A-DQCH component}.
\]

Note that a MAC partition of a directed graph always exists as a node in itself is a DQCH component of a graph. So is a combination of two neighboring nodes of the graph. Thus, a way to find a MAC partition would be to locate all the A-DQCH components of \( \mathcal{F} \) and then club the remaining nodes and arcs into different NA-DQCH components. This is the main logic behind Algorithm 3 and the supporting Algorithm 4. In particular, the function MACPartition(), which we describe in Algorithm 3, takes \( \mathcal{F} \) as an input and returns a MAC partition of the same.

Remark 4.9: (Computation of MAC partitions of \( \mathcal{F} \)). The function MACPartition(\( \mathcal{F}, a \)) in Algorithm 3 computes a MAC partition of \( \mathcal{F} \). The function makeQCHComp() in Algorithm 4 aids in this process. Notice that if any directed graph is passed to makeQCHComp(), it returns a collection of components. This is because in Step 5, a node \( i \) with highest value of \( a_i \) is chosen. This definitely belongs to a QCH component as every node in itself is a QCH component. Then in Step 6, all nodes that have a path with quasi-concave MPDP parameters from this node is included into this component. Thus \( (V_{\text{temp}}, A_{\text{temp}}) \) at the end of a pass of the while loop is a DQCH by definition. The algorithm then repeats the process by considering the nodes which have not been visited in this process.
Algorithm 3: MACPartition(, a)

Data: = (, , a)
Result: MAC Partition

H ← makeQCHComp(, a)
Vregroup ← ∅
Aregroup ← ∅
for ∈ do
  tempVar ← ∅
  while ∅ ≠ tempVar
do
  for ∈ do
    if ∃ ∈ such that then
      remove nodes with out-neighbors not in and try to make an A-DQCH
      Q ← Q \ {(, ) ∈ | ∈ Q , }\{(, ) ∈ Q} • remove corresponding arcs
      Vregroup ← Vregroup ∪ {i}
      Aregroup ← Aregroup ∪ {(, ) ∈ A}
      Remove a from a
  end
end
Fregroup ← (Vregroup, Aregroup)
H ← H ∪ makeQCHComp(Fregroup, a) • regroup the remaining nodes and arcs into NA-DQCH components

return: H

Algorithm 4: makeQCHComp(, a)

Data: = (, , a)
Result: QCH components

visitSet ← ∅ • keeps track of nodes already visited
H ← ∅
i ← arg max(a) • node with highest MPDP is definitely in a QCH
Vtemp ← {i}
Vtemp ← Vtemp ∪ { ∈ | ∈ V \ {j} | ∃ a path with quasi-concave MPDP’s between j and i in }
Atemp ← {(i, ) ∈ | ∈ V temporarily ∪ Vtemp}
visitSet ← visitSet ∪ Vtemp
Remove indices {i ∈ visitSet} from a
H ← H ∪ (Vtemp, Atemp)

end
return: H

Now once such a collection of DQCH components is returned by makeQCHComp() to MACPartition() in Step 1, Algorithm 3 checks every component and removes nodes that have out-neighbors not within that component (Step 10). Thus the nodes and arcs remaining (if any) at the end (Step 17) of each pass of the for-loop forms and A-DQCH component. The nodes and arcs added to Vregroup and Aregroup in Steps 12, 13 have the property that they have outgoing arcs to some other components. Thus they are regrouped into NA-DQCH components in Step 20. Note that any NA-DQCH component formed in this process cannot contain any A-DQCH because of the aforementioned property. Thus, the partition obtained at the end of Algorithm 3 satisfies all the requirements of Definition 4.8.

Using such a partition, we can create a super graph of in order to find a lower bound on the steady state social utility of the dynamics. This is also useful as it reduces the number of variables (significantly in some cases) to consider as we can group multiple nodes into a super node and consider the super node as a whole rather than considering the individual nodes that constitute it.

Definition 4.10: (Maximal attractive component super graph or MAC-SG). Suppose = (, , a) is a MAC partition of = (, , ). Construct a graph ∈ (, ) with the following property:
  • ∈ \ {i ∈ \ l\}:
  • (, ) ∈ ℐ if and only if ∃ ∈ , ∈ \ such that (, ) ∈ .

Such a graph ∈ is called a Maximal Attractive Component Super Graph or MAC-SG of the MAC partition = (, , a) of . The s are called super nodes.

The function MACSG() takes in a MAC partition of and returns the corresponding MAC-SG.

Let ∈ (, ℐ) be such a MAC-SG of a MAC partition of . Similar to the nodes, for a super node ∈ , we define the set of in-reachable super nodes of as = \ { ∈ | ∃ a directed path from r to q in ∈ \} and the set of out-reachable super nodes of as = \ { ∈ | ∃ a directed path from q to r in ∈ \}. We then let denote the fraction of population moving from super node ∈ to ∈ ∈ ∪ ℐ. We then let denote the fraction of population moving to ∈ ∈ ∪ ℐ. We then let denote the fraction of population moving from super node ∈ to ∈ ∈ \ ∪ ℐ. We then let denote the fraction of population moving to ∈ ∈ ∪ ℐ.

Now, if we allow the entire population to fit in every super node, then we might end up with a conservative lower bound. To compute a more realistic lower bound, the fact that some nodes will not contain any population fraction in the steady state state must be taken into account. We formally define such nodes next and then provide a result to identify such nodes.

Definition 4.11: (Eventually empty nodes). Let the evolution of be governed by a Nash convergent dynamics from
an initial condition $x^0$. Let $L^+$ be the positive limit set of the trajectory. Then $i \in V$ is said to be an eventually empty node if and only if $x_i = 0, \forall x \in L^+$.  

**Lemma 4.12:** (Sufficient condition for being eventually empty for SSD, NBRD and NRPM). Let $x^0 \in S_p(V)$ and let $\tilde{F} = (V, A)$ be an ICRG of $G = (V, E)$. Let the evolution of $x$ be governed by SSD, NBRD or NRPM. For a node $i \in V$, if $\exists j \in N^i$ such that $(i, j) \in A$ and $(j, i) \notin A$, then $i$ is an eventually empty node.

A proof of this is provided in Appendix II-B.3. Thus, if a super node contains an eventually empty node, there exists an inherent bound on the fraction of population that can stay in that super node in the steady state. We discuss this in the following result.

**Lemma 4.13:** (Upper bound on population fraction in super nodes for a Nash convergent dynamics). Suppose $x^0$ is the initial condition and $\tilde{F}$ is an ICRG of $G$. Let $\{H^q = (W^q, Q^q)\}_{q \in [1,n],z}$ be a MAC partition of $\tilde{F}$ and let $\Gamma = (\Lambda, \Pi)$ be the corresponding MAC-SG. Let $L^+$ be the positive limit set of the trajectories of a Nash convergent dynamics. For a super node $q \in \Lambda$, let the set of eventually empty nodes in $q$ be denoted by $M^q := \{i \in W^q \mid i \text{ is eventually empty}\}$. Let $\rho^q_{\text{max}} := \max_{i \in M^q} a_i$, if $M^q \neq \emptyset$, and finally, let

$$\rho^q_{\text{max}} := \begin{cases} 0, & \text{if } M^q = W^q \\ \sum_{i \in W^q \setminus M^q} u_i^{-1}(a^q_{\text{max}}), & \text{if } W^q \supset M^q \neq \emptyset \\ 1, & \text{if } M^q = \emptyset. \end{cases} $$

Then, $\forall x \in L^+$

$$\sum_{i \in W^q} x_i \leq \rho^q_{\text{max}}, \forall q \in \Lambda. \quad (19)$$

We present a proof in Appendix II-B.4.

Recall that $f_{V'}(\rho)$ is the solution of the optimization problem $P_1(V', \rho)$ in (5). The optimum of the following optimization problem then provides a lower bound to the steady state social utility.

$$U_{\text{min}} := \min_{\xi, \rho} \sum_{q \in \Lambda} f_{W^q}(\xi_q) \quad \text{s.t.} \quad \xi_q = \sum_{r \in \Pi q} \rho_{r q}, \forall q \in \Lambda,$$

$$\rho_{r q} \leq \rho^q_{\text{max}}, \forall q \in \Lambda. \quad (20)$$

The following lemma, proof of which is given in Appendix II-B.5 shows concavity of the cost function in $P_5$.

**Lemma 4.14:** Consider a fixed $V'$ and the function $f_{V'}(\rho)$ which is the optimum of $P_1(V', \rho)$ in (5). Then $f_{V'}(\rho)$ is concave in $\rho \in \mathbb{R}_+$. The constraints set of $P_5$ is a polyhedron. Thus it is a well known fact that the global optimum will occur at an extreme point. Any standard method such as [16]–[18] can be used to solve this problem.

The main sequence of steps, required to compute a lower bound on the steady state social utility, described in this section is listed in Algorithm 5 and the result regarding the same is stated in the next to conclude the section.

**Algorithm 5:** Compute lower bound for steady state social utility

- **Data:** $x^0, \tilde{F}$
- **Result:** $U_{\text{min}}$

1. $\tilde{F} \leftarrow \text{reduceGraph}(\tilde{F}, x^0)$
2. $\mathcal{H} \leftarrow \text{MACPartition}(\tilde{F}_0)$
   - individual components are $H^q = (W^q, Q^q)$
   - $\Gamma = (\Lambda, \Pi)$
3. $U_{\text{min}} \leftarrow \text{solution of } P_5$

**Theorem 4.15:** (Lower bound on steady state social utility for SSD, NBRD and NRPM). Let $x^0 \in S_p(V)$. Then

$$U_{\text{min}} \leq U_{\text{SSD}}(x^0); \quad U_{\text{min}} \leq U_{\text{NBRD}}(x^0); \quad U_{\text{min}} \leq U_{\text{NRPM}}(x^0).$$

A proof of this is presented in Appendix II-B.6

**V. Conclusion**

We provided sufficient conditions on the network under which there exists a unique Nash equilibrium for the stratified population game. We also provided algorithms to reduce the graph using the initial condition without affecting the population evolution. We then provided algorithms to partition the reduced graph and utilized the conditions for unique Nash equilibrium to provide upper and lower bounds on the steady state social utility for SSD, NBRD and NRPM.

Future work includes utilizing the dynamics to further reduce the graph and refine the bounds on the steady state social utility. We would also like to extend the ideas further to compute conditions under which the bounds coincide and utilize this knowledge to compute the steady state state.

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Fig. 1: Graph structure, initial state and MPDP’s for showing that coordination is not always good. The tuple \((i, j)\) near each node \(i\) represents \((x^0_i, a_i)\). (a) SSD outperforms NBRD. (b) NBRD outperforms NRPM.

**APPENDIX I**

**SIMULATIONS AND ANALYSIS**

In this section, we highlight some properties of SSD, NBRD and NRPM and illustrate some results stated in Parts I and II using some simulations. We used CVXPY [19], [20] for solving the optimization problems. We performed all simulations in a python3 programming language environment on a standard laptop with intel 10th generation Core i5 processor.

**A. Myopic Coordination can be Worse than Myopic Selfish Behavior**

Even though SSD, NBRD and NRPM converge asymptotically to some point in the set of Nash equilibria, they may not converge to the same state in general. Moreover, even though SSD, NBRD and NRPM display increasing levels of coordination among the agents, it is not always true that NBRD performs better than SSD nor that NRPM performs better than NBRD. This is because even though there is coordination within a node in NBRD and coordination of the entire population in NRPM, they are still essentially network restricted gradient ascent dynamics.

In this section we illustrate two situations where the act of coordination does not result in better social utility. For this section we assume the cumulative payoff functions to be of the form \(p_i(x_i) := -0.5x_i^2 - ax_i\). This is the uniform water tank model presented in [15].

1) **SSD outperforms NBRD:** To demonstrate such a phenomenon we consider a four node graph with initial state and MPDP’s described in Figure 1(a). Note that in the case of NBRD, the outflow from node 3 is always directed only to node 4. Hence the population misses out on a better payoff in node 1. For SSD, on the other hand, there are outflow from node 3 both towards nodes 2 and 4. Thus a fraction of the population gets a chance to move to node 1. Hence the population as a whole receives a higher utility. The simulation results are shown in Figure 2.

2) **NBRD outperforms NRPM:** To demonstrate this phenomenon we consider a five node graph with initial state and MPDP’s described in Figure 1(b). Note that in the case of NBRD, the population fraction in node 3 is unaware that fraction in node 2 is also simultaneously revising its choices. Thus, there is a greater outflow from node 3 to node 4 than to node 3. However, in the case of NRPM, the...
fraction in node 3 is aware of the fact that the fraction in node 2 will move to node 1. Hence there are equal outflows from node 3 to nodes 2 and 4. Thus a larger fraction of population gets to visit node 5 in case of NBRD than for NRPM. This causes the steady state social utility of NBRD to be greater than NRPM. Simulation results illustrating this are provided in Figure 3.

B. Bounds on Steady-State Social Utility

Here we consider an 18 node graph and quadratic cumulative payoff functions and verify the upper and lower bounds on the steady-state social utility. The graph structure \( G \) and cumulative payoff function details are provided in Figure 4(a). Initial population is distributed between nodes 4, 10, 17 and 18 as \( x^0_4 = x^0_{10} = 0.1, x^0_{17} = 0.3, x^0_{18} = 0.5 \) and \( x^0_i = 0, \forall i \in [1, 18] \setminus \{4, 10, 17, 18\} \). The corresponding ICRG \( \mathcal{F} \) is provided in Figure 4(b). Note that \( \mathcal{F} \) has only 17 nodes and not 18 (node 7 has been removed). The only bi-directional arc in \( \mathcal{F} \) is between nodes 4 and 11. Rest of the arcs are all unidirectional. Thus there is over 50% reduction of \( \delta_{ij} \) variables from \( G \) to \( \mathcal{F} \). The A-DQCH components are made up of the node sets \( \{2, 8, 9, 14, 17\} \) and corresponding arcs. The NA-DQCH components are made up of the node sets \( \{1, 3, 4, 5, 6, 11, 12, 15\}, \{13\}, \{18\} \) and corresponding arcs. To compute the upper and lower bounds the total time taken was \( \approx 3.48 \) sec.

The simulation results are provided in Figures 4(c) - 4(f). The time axis in each of these diagrams is represented as a log scale, i.e., each tick on the horizontal axis represents \( \log(k^{-t} + 1) \) rather than \( t \) (the value of \( k \) is given in Figure 4). This is done to magnify the initial time frame where the main redistribution occurs and shrink the later time frame. The average time taken to solve the optimization problems for NBRD and NRPM were \( \approx 0.32 \) sec and \( \approx 0.16 \) sec respectively. The total time required to complete the simulations for SSD, NBRD and NRPM were \( \approx 108.43 \) sec, \( \approx 3.19 \times 10^4 \) sec and \( \approx 1.65 \times 10^4 \) sec respectively. The upper and lower bounds obtained were \(-3.17 \) and \(-7.58 \) respectively while the actual steady-state social utility for SSD, NBRD and NRPM were \(-5.16, -5.01 \) and \(-4.17 \) respectively.

A. Proof of Results on Unique Nash Equilibrium

1) Proof of Lemma 3.3: (By contradiction) Suppose \( \exists k \in [1, n] \setminus \{p\}, p \in [1, k - 1] \), \( q \in [k + 1, n] \), such that \( a_{\pi(p)} > a_{\pi(p+1)} \) and \( a_{\pi(q-1)} < a_{\pi(q)} \). But \( \pi(p), \pi(q) \in \mathcal{P}(i,j) \) which is a path of quasi-concave MPDP’s and one of the previous inequalities violates the condition that \( \forall r \in [p, q], a_{\pi(r)} \geq \min\{a_{\pi(p)}, a_{\pi(q)}\} \). This is a contradiction and hence the assumption was incorrect.

2) Proof of Lemma 3.5: Choose any \( i, j \in \text{supp}(x) \) and let \( \mathcal{P}(i, j) \) be a path between \( i \) and \( j \) with quasi-concave MPDP’s. Existence of such a path is guaranteed as the graph \( G \) is a QCH. Let \( \pi(k) := \mathcal{P}(i, j) \). As \( \pi(k) \in \mathcal{P}(i, j) \), by Lemma 3.3 \( a_l \leq a_{\pi(2)} \leq \cdots a_{\pi(k)} \) or \( a_{\pi(k)} \geq \cdots \geq a_{\pi(1)} \geq a_i \). Without loss of generality, suppose that \( a_l \leq a_{\pi(2)} \leq \cdots a_{\pi(k)} \).

Now, consider the nodes \( \pi(1) = i \) and \( \pi(2) \). As \( x_1 > 0 \) and \( a_i \leq a_{\pi(2)} \), we have

\[
 u_{\pi(2)}(0) = u_i(0) > u_i(x_i) \geq u_{\pi(2)}(x_{\pi(2)}),
\]

where we have used the fact that \( u_k(\cdot) \) is a strictly decreasing function \( \forall k \), which implies \( u_k(0) > u_k(y) \) if \( y > 0 \). Further, the last inequality comes from the fact that \( x \in \mathcal{N}_{\rho} [V] \). Thus \( x \in \mathcal{N}_{\rho} [V] \), \( i \in \text{supp}(x) \) and \( a_l \leq a_{\pi(2)} \). Together implies \( x_{\pi(2)} > 0 \). Now, repeating this argument for every pair of nodes \( (\pi(r), \pi(r + 1)) \) for \( r \in \{1, \ldots, k - 1\} \), we conclude that \( x_{\pi(k)} > 0 \), that is \( \pi(k) \in \text{supp}(x) \).

Thus, \( \forall \pi(k) \in \mathcal{P}(i, j) \), \( \pi(k) \in \text{supp}(x) \). Then as \( x \in \mathcal{N}_{\rho} [V] \), \( u_{\pi(k)}(x_{\pi(k)}) = u_{\pi(1)}(x_{\pi(1)}) \forall \pi(k), \pi(l) \in \mathcal{P}(i, j) \). Thus \( u_i(x_i) = u_j(x_j) \). As \( i \) and \( j \) were chosen arbitrarily, the proof of the lemma is complete.

3) Proof of Theorem 3.6: First note that \( U(x) \) is a strictly concave function in \( x \) and \( P_1 \) is always feasible as \( \mathcal{S}_\rho [V] \) is non-empty. Also \( P_1 \) is a strictly convex program and hence has a unique optimizer [21]. Thus, it suffices to show that if \( x \in \mathcal{N}_{\rho} [V] \) then \( x \) also optimizes \( P_1 \).

If \( \rho = 0 \) then the result is trivially true. So, now suppose \( \rho > 0 \). The Lagrangian for \( P_1 \) can be written as

\[
 L_1 = \sum_{i \in V} \rho_i(x_i) - \lambda \left( \sum_{i \in V} x_i - \rho \right) + \sum_{i \in V} \mu_i x_i,
\]
Fig. 3: NBRD performs better than NRPM. The first two plots share a common label and a common legend. (a) Evolution of population state under NBRD. (b) Evolution of population state under NRPM. (c) Evolution of social utility.

Fig. 4: Simulation with $p_i(x_i) := -\alpha_i x_i^2 - \beta_i x_i$. The last three plots share a common label and a common legend. Here $k \approx 5 \times 10^{-5}$. (a) Graph structure with 18 nodes. The tuple (.,.) around each node $i$ represents ($\alpha_i$, $\beta_i$). (b) Corresponding ICRG. Nodes in same D-QCH components have same color and have been grouped together. Nodes in different D-QCH components are colored differently. (c) Evolution of social utility and bounds on steady-state social utility. (d) Evolution of population state under SSD. (e) Evolution of population state under NBRD. (f) Evolution of population state under NRPM.
where \( \lambda \) and \( \{ \mu_i \geq 0 \}_{i \in \mathcal{V}} \) are the Lagrange multipliers. The KKT conditions for \( P_1 \) include

\[
\begin{align}
  u_i(x_i) - \lambda + \mu_i &= 0, \quad \forall i \in \mathcal{V}, \tag{21a} \\
  \mu_i x_i &= 0, \quad \forall i \in \mathcal{V}. \tag{21b}
\end{align}
\]

Now, let \( x \in \mathcal{NE}_\rho^{[\mathcal{V}]} \). Since the graph is a QCH, we know from Lemma 3.5 that

\[
u_i(x_i) = H, \quad \forall i \in \text{supp}(x), \tag{22}
\]

for some \( H \). Clearly, if \( \rho > 0 \), problem \( P_1 \) satisfies Slater’s condition. Thus, we will show that \( x \in \mathcal{NE}_\rho^{[\mathcal{V}]} \) is the unique optimizer of \( P_1 \) by showing that for \( x \) there exist Lagrange multipliers that satisfy (21) and the feasibility constraints. Note that as \( x \in \mathcal{NE}_\rho^{[\mathcal{V}]} \), \( x \) is a feasible solution for \( P_1 \). Now, we set

\[
\lambda^* = H, \quad \mu^*_i = 0, \quad \forall i \in \text{supp}(x), \quad \mu^*_j = H - u_j(0), \quad \forall j \notin \text{supp}(x).
\]

As \( x \in \mathcal{NE}_\rho^{[\mathcal{V}]} \), we can say from (22) that \( \mu^*_j \geq 0, \quad \forall j \notin \text{supp}(x) \). Thus, for each \( i \in \mathcal{V}, \mu^*_i \geq 0 \) and satisfies (21b).

5) Proof of Lemma 4.14: By contradiction Suppose \( \exists \rho_1, \rho_2 \in \mathbb{R}_+ \) and \( \sigma \in (0, 1) \) such that

\[
\left| \sigma \rho_1 + (1 - \sigma) \rho_2 \right| < \sigma f_{\psi'}(\rho_1) + (1 - \sigma) f_{\psi'}(\rho_2). \tag{23}
\]

We ignore the trivial cases where \( \sigma = 0 \) or \( \sigma = 1 \). Let \( x^1 \) and \( x^2 \) be the optimizers of \( P_1(\mathcal{V}', \rho_1) \) and \( P_1(\mathcal{V}', \rho_2) \) respectively. Thus, \( f_{\psi'}(\rho_1) = U(x^1) \) and \( f_{\psi'}(\rho_2) = U(x^2) \). Then it is easy to see that \( \sigma x^1 + (1 - \sigma) x^2 \) is a feasible solution of \( P_1(\mathcal{V}', \sigma \rho_1 + (1 - \sigma) \rho_2) \).

Therefore,\( U\left( \sigma x^1 + (1 - \sigma) x^2 \right) \leq f_{\psi'}\left( \sigma \rho_1 + (1 - \sigma) \rho_2 \right) < \sigma f_{\psi'}(\rho_1) + (1 - \sigma) f_{\psi'}(\rho_2) = \sigma U(x^1) + (1 - \sigma) U(x^2) \).

Here the first inequality comes from the fact that \( f_{\psi'}\left( \sigma \rho_1 + (1 - \sigma) \rho_2 \right) \) is the optimum of \( P_1(\mathcal{V}', \sigma \rho_1 + (1 - \sigma) \rho_2) \). The second inequality comes from (23). This contradicts the strict concave nature of \( U(.) \) and hence completes the proof.

6) Proof of Theorem 4.15: Note that the main steps of the process are illustrated in Algorithm 5. By Remark 4.2, we know that \( \mathcal{F} \) in Step 1 computes an ICGR of \( \mathcal{G} \) and that the evolution of the population is unaffected by it. Next Step 2 computes a MAC partition of \( \mathcal{F} \), the accuracy of which is demonstrated in Remark 4.9. \( \Gamma \) in Step 3 is the corresponding MAC-SG. By Lemma 4.13, \( \rho^\max_q \) is an upper bound on the population fraction in each super node for every state in the positive limit set (say \( \mathcal{L}^+ \)) of the trajectory starting from \( x^0 \). Thus, every \( x \in \mathcal{L}^+ \) can be written as a feasible solution for \( P_5 \). Moreover, inside each super node, the steady state social utility is given by \( P_1 \). Hence the lower bound holds.