A transient phase in cosmological evolution: A multi-fluid approximation for a quasi-thermodynamical equilibrium

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\(\star\) (Dated: June 19, 2019)

In this article, we examine the dynamics of a multi-fluid system in which (i) different fluids interact, and (ii) there exists a limit in which the multiple fluids evolve into a mixture that satisfies single-fluid approximation. We consider the potential application of this modelling to studies of a cosmological transition which is marked by one material domination replaced by another. The thermodynamical implications of such fluid dynamics are explored where we find that the second law of thermodynamics holds given an emergent Rindler horizon. We argue that such transient periods exist in cosmological evolutions and are best modelled using multi-fluid approximation. Our application of the modelling to an interacting dark-sector leads to the modification of the equation of state of a third non-interacting constituent in a 3-species multi-fluid system.

I. INTRODUCTION

Cosmology is dominated by studies of segmented or discretized history of the universe, eras, whose dynamics are characterized by a single dominant material. Examples of these include inflaton [1, 2], radiation [3], matter [4] and dark energy [5]. Such studies are generally carried out using single-fluid approximation in the sense that the modelling is based on a single observer world-line. To a large extent, cosmological observations have yielded results that agree with the predictions from the building blocks of the standard Model as seen, for example, in the analyses of the cosmic microwave background (CMB) radiation and the anisotropy thereof [6]. Although not all predictions are confirmed, what has been achieved has enabled us to build a probable-model of the evolving universe based on the scaffolding of the knowledge of the different eras. Nevertheless, the interplay between cosmological theory (theories) and observations have not always been smooth, resulting in a number of unanswered questions, with some observations leading to questions that demand the re-examination of the underlying theory or theories. Examples of these are "axis of evil" in the CMB [7] and late time acceleration [8, 9], just to mention two. The first line of attack has been to tweak the existing theory or improve technology with the hope that this could help explain the anomalous observation. These attempts have had limited success, forcing some to suggest a complete overhaul of the underlying theories whether they are of gravity or of the material content of the universe. But there remains a yet to be explored alternative approach that may ease some of the tensions between theory and observation. The modelling of transition between eras does not feature much in literature but has the potential to resolve some of these issues. Such modelling is applicable to epochal studies, where we think of a cosmological epoch as the event signifying when a change has taken place to an extent that it marks the beginning of a new era. As mentioned earlier, the analysis is often performed with the assumption that the dynamics of the universe is dominated by one type of material, but a transition is required from one domination to the next. This transition period is, predictably, complex and requires a completely different approach. Conceivably, the first line of attack is to assume that the transition is not instantaneous but occurs gradually allowing for a transient period that is not dominated by a single-fluid but momentarily by multiple fluids. The purpose of this paper is to model the transition period or epoch between dominant eras and to analyze the dynamics of such a phase.

A study of multiple fluids requires a way of approximating aggregated fluid properties, for example using the multi-fluid approximation on one hand and ways of dealing with how the different fluids interact in cases where they do, on the other hand. This indirectly demands knowledge or ways of handling thermodynamics. In order to carry out such a study, one needs to link theories for single fluid dynamics and thermodynamics [10–13] to relativistic multi-fluid dynamics [14–16] and thermodynamics [17–19]. It also requires that one goes beyond perfect fluids to consider fluids that exhibit dissipation and fluids in which bulk viscosity plays a role [20].

This paper is arranged as follows, section (II) discusses the interaction between dark energy and dark matter in a multiple-fluid environment. Section (III) discusses two
formalisms for multi-fluid thermodynamics. Section (IV) discusses generalized second law of thermodynamics for a multi-fluid system, and section (V) gives the general discussions and conclusions.

II. DARK ENERGY (DE) INTERACTING WITH MATTER (DM) IN PRESENCE OF RADIATION(χ)

Although we have three particle species, we have a single observer world-line and hence \( t \equiv u^a \nabla_a \) where \( u^a \) is the common 4-velocity. Let this 4-velocity be the determinant of a word-line of a fiducial frame of reference. This is single-fluid approximation. The metric is given by

\[
ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega^2 \right),
\]

where \( a(t) \) is the scale-factor such that \( \dot{a}(t)/a(t) = 3H. \) \( H \) is the Hubble parameter.

\[
H^2 + \frac{\kappa}{a^2} = \frac{1}{3}(\rho_{DE} + \rho_{DM} + \rho_\chi),
\]

where \( \rho \) is the energy density while the subscripts \( DE, DM \) and \( \chi \) represent dark energy, dark matter and radiation respectively. We have set \( 8\pi G = 1. \) It is easy to show that the corresponding Friedman equation is

\[
\frac{d}{dt} \left( H^2 + \frac{\kappa}{a^2} \right) = \frac{1}{3}(\dot{\rho}_{DE} + \dot{\rho}_{DM} + \dot{\rho}_\chi).
\]

This could be reformulated in terms of individual species world-lines but which are related to a single predominant world-line (given here by \( t \)). Let \( \tau, \eta, \zeta \) be the respective time parameter along these world-lines. This means that

\[
\frac{d}{dt} \left( H^2 + \frac{\kappa}{a^2} \right) = \frac{1}{3}(\frac{d\rho_{DE}}{d\tau} \alpha + \frac{d\rho_{DM}}{d\eta} \beta + \frac{d\rho_\chi}{d\zeta} \gamma),
\]

where

\[
\alpha = \frac{d\tau}{dt}, \beta = \frac{d\eta}{dt}, \gamma = \frac{d\zeta}{dt}.
\]

We will require that \( \lim_{t \rightarrow t_{\text{equil}}} (\alpha, \beta, \gamma) \rightarrow (1, 1, 1) \), where \( t_{\text{equil}} \) represents the time when thermodynamics equilibrium is attained. We will return to this in the section of thermodynamics, but for now, this means that the parameters \( \tau, \eta, \zeta \) are not constants but evolve towards a constant time parameter \( t \). This has the implication that an observer in the rest-frame of one species will in the interim (i.e. \( t < t_{\text{equil}} \)) record measurements that differ from those measured by another observer sitting in a separate species rest-frame. It is only at \( t \geq t_{\text{equil}} \) that their recordings will be the same. The evolution of time parameters can be ascribed bulk viscosities and dissipative effects such as entrainment [19], a subject that we will return to later in this article. In particular, it is known that Einstein’s definition of global time is not applicable in curved spacetimes since inertial frames exist locally [21]. We note that any shared time one introduces to synchronize events are not coincident as neighbouring events can only make sense in limited regions. It is however possible to define an extended reference frame in the presence of gravity that extends over a congruence of time like curves [22–25] and it is in this frame we will make our approximation a subject that will be addressed in [26]. It suffices to say that it is the interim period, leading up to equilibrium designating the extended reference frame, that we investigate using a relativistic multi-fluid approximation. The conservation equations for the species energy-densities are given by:

\[
\frac{d\rho_{DE}}{d\tau} + 3H(\rho_{DE} + p_{DE}) = -\frac{Q'}{\alpha},
\]

\[
\frac{d\rho_{DM}}{d\eta} + 3H(\rho_{DM} + p_{DM}) = \frac{Q}{\beta},
\]

\[
\frac{d\rho_\chi}{d\zeta} + 3H(\rho_\chi + p_\chi) = \frac{Q - Q'}{\gamma},
\]

where \( Q \) and \( Q' [27] \) are the interaction terms. The requirement that the sum of interaction terms on the right-hand side vanish leads to \( \alpha = \beta = \gamma \neq 0 \). Again this is the precise condition for interacting fluids in thermodynamic equilibrium. The implication of \( Q' \neq Q \) may have a profound implication for the evolution of the interacting fluids, not least of which might include a possible change
in the mass of standard particles as we understand them or even include a fifth force[38]. In general these may require the analysing of corresponding Boltzmann equations for the system and invoking of the screening mechanism respectively, however we will analysis the system when \( Q' - Q \approx 0 \). This means that the coupling will not induce changes in the baryonic fluid as might be feared. We will nevertheless keep the distinction \( Q' \) and \( Q \) for the sake of developing a general formalism.

Let’s turn to relativistic thermodynamics in order to lay a proper ground for the physics of the transient period that we are interested in. The definition of energy to be introduced in the next section, as much as its subsequent thermodynamic treatment, rests on a foliation of space-time involving a time-like vector field \( u^\mu \) defined in [21, 28]. But the multi-fluid approximation where there is more than one such time-like vector requires more careful consideration as comparison of the rest frames of the multiple observers introduce ambiguity related to gauge transformations as will be discussed in [26]. In addition, the definition of properties such as heat and work are not straightforward. For example, if two systems \( I \) and \( II \) representing the rest frame of two observers which only interact with each other then \( \text{Internal energy}^{I} + \text{Internal energy}^{II} \) is constant and \( dW^{I} + dW^{II} = 0 \) which would imply that \( \text{heat}^{I} + \text{heat}^{II} = 0 \) where the systems have the same velocity otherwise \( dQ^{I} + dQ^{II} \neq 0 \). Where the velocities are different, the thermal energy and momentum transfers as the heat lost by one system is not necessarily equal to the heat gained by the other system. This is because the heat contents of the transmitting agency such as electromagnetic waves are not the same for all observers. At the heart of this is the way heat, work and volume are defined and handled in [29]. In particular, heat and work may be represented by two inertial frames; the frame in which the decomposition of heat and work is defined, and the frame in which three volume is defined could yield different results. This is the subject of discussion in the next section.

III. THERMODYNAMICS

Studies of single-fluid relativistic systems and thermodynamics systems have had separate historical development and efforts are being made to forge a merged development leading to a number of controversies (see for example the account given in [29]). Great advancement has been made though, for example, it was shown in [30], that one could derive relativistic equations of motion from thermodynamics quantities. In particular, Einstein equation \( G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \), where \( \kappa \) is a constant in terms of \( \hbar \), is derived from the proportionality of entropy and horizon area together with the thermodynamic equilibrium or reversible relations \( dS = \delta Q/T \). The key idea is to demand that this relation hold for all the local Rindler[31] causal horizons through each space-time point, with \( \delta Q \) and \( T \) interpreted as the energy flux and Unruh temperature respectively as seen by an accelerated observer[32] just inside the horizon. In standard thermodynamics, heat is defined as the energy that flows between degrees of freedom; degrees that are not macroscopically observable. In space-time dynamics, we shall define heat as the energy flowing across causal horizons. Such energy can be detected via the gravitational field it generates. Its form or nature will, however, be unobservable from outside the horizon. This indirectly links gravitation to thermodynamics. In fact, it was conjectured [33] that temperature has weight and effectively mass which could exert a gravitational effect. But we are interested in a multi-fluid system and therefore we need a way to model its thermodynamical properties.

Consider the three observers in the previous section at the point leading to a causal connection. For our study, we take the horizons to be the causal boundaries to be the Rindler horizons for individual observers which are subject to cosmic censorship. Let’s assume that the individual species entropy is proportional to the corresponding horizon area. These causal horizons are embedded in an expanding volume constituting a system. Since like volume, the area scales with the system, it will be taken as an extensive parameter for the horizon. It is important to note that not all physical properties can be classified as either extensive or intensive and indeed dynamical behaviour such as species interactions may void such classification. These two categories are therefore not all-inclusive [35, 36]. The system we consider is, thermodynamically speaking, in quasi-equilibrium because the different horizons will invariably experience expansion, contraction or shear as the volume scales. The relevant question to ask is what approximations would enable one to define thermodynamics equilibrium conditions given the present scenario. To this end, we need to be clear about how space-time events are perceived given the different observer world-lines.

It is always possible to define a flat hyper-surface along each world-line, subject to the equivalence principle, such that the expansion and shear vanish in the neighbourhood of these events. Given that the observers’ velocities are evolving toward a common velocity, by extension, it will be possible to define a common event (label it \( P \)) in whose neighbourhood a flat hyper-surface exists such that both expansion and shear locally vanish. This limit is reached precisely at \( t_{\text{equil}} \) given in FIG.1.

We have in mind a mathematical expression of neighbourhood that takes the form, \( \mathcal{P} : |P - \mathcal{P}| < \delta \), where \( \delta \) is some event-scale defining the extent of an open-disc (the cross-section of an open-ball) centred at \( \mathcal{P} \) (a past horizon referred to as local Rindler horizon). The temperature associated with the merged-observers just inside this common horizon is the Unruh temperature (see (FIG. 2)). We need a thermodynamics theory for such an interaction and this we do in the next section.
FIG. 2. The schematic drawing of the trajectories of three individual worldlines that are, phenomenologically, causally connected. The drawing depicts trajectories and growing apparent horizons before and after the worldlines merge. We are interested in physics just before and during the merger. In particular, there may be residual effects from the pre-merger that show up after the merger and which cannot be accounted for by the single fluid (fundamental observer) approximation? This might be of interest to studies of late time acceleration of the universe.

III.1. Müller-Israel-Stewart (MIS) theory

We present, in this section, a stripped down version of the extended MIS theory. The original MIS theory is given in [16, 17, 37] while the complete version of the extended MIS theory for a multi-fluid system will appear in [26]. Drawing from kinetic theory and assuming that the temperature is expressible as a scalar quantity, it is shown that one can develop a covariant relativistic theory consistent with causality. This does not imply that a theory can not be developed where temperature is a scalar part of a locally defined geometrical tensorial object [39]. In particular, if $\mathcal{T}_{ab}$ is such a tensor object then $\mathcal{T}_{ab} = \mathcal{T}^S_{ab} + \mathcal{T}^V_{ab} + \mathcal{T}^T_{ab}$, where the first term represents a pure tensor, the second a pure vector and the last a pure tensor such that $\mathcal{T}^S_{ab} = \nabla_a \nabla_b T$, $\mathcal{T}^V_{ab} = \nabla_a \mathcal{T}_b$, and $\mathcal{T}^T_{ab} = \nabla^a \nabla^b T_{ab}$. In this regard Stewart’s theory is based on associating temperature with locally defined scalar (i.e. $T = T^T$). Attempts to develop axiomatic theories in which temperature transforms as a vector have led to some problems which will be examined elsewhere [26].

In the extended MIS theory, the primary extensive parameters are taken to be the species number flux current $N^\mu_{(i)}$, the stress momentum tensor $T^{\mu\nu}_{(i)}$, and the entropy flux vector $S^\mu_{(i)}$. The subscript $(i)$ is not the same as the subscript $A$ in $N_A$ used in [16] where it represents the components of the same fluid. In our case the subscripts denote different fluids (as in a multi-fluid context). Looked at differently, [16] uses single-fluid approximation while we use multi-fluid approximation. We choose to present the general form where $i = X, Y, Z$ rather than limiting the presentation to $i = DE, DM, \chi$ but the matching should be straightforward. It is important to note that the total stress-energy momentum is conserved but not the individual i.e. $\nabla_\mu T^{\mu\nu} = \nabla_\mu (\sum_i T^{\mu\nu}_{(i)} + T^{\mu\nu}) = 0 \neq \nabla_\mu T^{\mu\nu}_{(i)}$, where $T^{\mu\nu}$ is the momentum due to the interactions. This allows for the inclusion of the bulk viscosity and dissipative effects in a predictively more realistic scenario, even though phenomenological. The full formalism will be discussed in a separate article [26]. In our case the parameters are defined subject to the individual velocity vectors $u^\mu_{(i)}$, such that

$$N^\mu_{(i)} = n_i u^\mu_{(i)} + n^\mu_{(i)}, \quad (6)$$

where $n_i$ is the species number and $n^\mu_{(i)}$ is the species diffusion current. Similarly

$$T^{\mu\nu}_{(i)} = \rho_{(i)} u^\mu_{(i)} u^\nu_{(i)} + p_{(i)} \Pi^{\mu\nu}_{(i)} + 2u^\mu_{(i)} u^\nu_{(i)} + \pi_{(i)}^{\mu\nu}, \quad (7)$$

where $\Pi^{\mu\nu}_{(i)} = q_{(i)}^\mu - u^\mu_{(i)} u^\nu_{(i)}$ is the projection tensor unique to the individual observer. $\rho_{(i)} = u_{(i)} \pi_{(i)} T^{\mu\nu}_{(i)}$ is the energy density, $p_{(i)}$ is the isotropic pressure, $\pi_{(i)}^{\mu\nu}$ is the anisotropic pressure, $q_{(i)}^\mu$ is the heat flux vector. The use of different velocities have also been considered in single-fluid approximation, for example in the comparison of the energy and the particle frames in [16], or the rest frame and the boosted frame in [40]. But we remind the reader that we, in contrast, are looking at multi-fluid approximation. Some of the dissipative effects we will encounter will indeed recover those seen in the single fluid cases. This raises the question of how one defines a common reference frame, without which the parameters are meaningless. The entropy density current for the multi-fluid with dissipative terms takes the form:

$$S^\mu_{(i)} = s_{(i)} u^\mu_{(i)} + s_{(i)}^\mu$$

$$= s_{(i)} u^\mu_{(i)} + \frac{q_{(i)}^\mu}{T_{(i)}}$$

$$- \left( \beta_{(i)} \Pi_{(i)}^2 + \beta_{(i)} \rho_{(i)} u^\mu_{(i)} u_{(i)} + \beta_{(i)} \pi_{(i)}^\mu \delta_{(i)} \pi_{(i)}^{\nu \gamma \delta} \right) \frac{u_{(i)}^\mu}{2 T_{(i)}}$$

$$+ \left( \alpha_{(i)} \Pi_{(i)} q_{(i)}^\mu + \alpha_{(i)} \pi_{(i)}^{\mu \nu} q_{(i)}^\nu \right) \frac{1}{T_{(i)}}. \quad (9)$$

$s_{(i)}$ is the entropy density, $s_{(i)}^\mu$ is the entropy flux with respect to $u^\mu_{(i)}$ such that $s_{(i)} u^\mu_{(i)} = 0$. $\Pi_{(i)}$ is the bulk viscosity. The complexity of the detailed interactions is immense but tractable as will be demonstrated in [20]. To illustrate this, consider the case of one species

\[ \text{The reader will note that } [33, 34] \text{ uses temperature transformation of the form } T^T = T/\sqrt{1 - \beta^2}, \text{ which is different to what we do in this article.} \]
change in entropy $\Delta s^\mu_{(i)}$, as described by an observer moving with the $u^\mu_{(i)}$ and where the components of velocity are treated as thermodynamics parameters alongside the temperature[41]. If the species were isolated, the change in internal energy would be from heat supplied and may be represented by

$$\Delta s^\mu_{(i)} \approx \frac{u^\mu_{(i)}}{T_{(i)}} (\Delta G^\mu_{(i)} - p_{(i)} \Delta V^\mu_{(i)}), \quad (10)$$

as seen by the observer moving with $u^\mu_{(i)}$, in a quasi-equilibrium, with the effect of volume change given by the last term and where $G^\mu_{(i)}$ is a momentum vector and $p_{(i)}$ is the pressure. This form is, collectively, similar to the second term in Eq. (9). As explained in [29], heat supplied in a co-moving frame may result in a momentum change and hence do work in another frame. This means that the effect of momentum must be separated in the decomposition of heat and work. In our context, the possibility of momentum change having an effect on a different frame suggests a possible coupling or interaction which should not be neglected in the multi-fluid approximation. In this regard, the total entropy vector takes the phenomenological expressions

$$S^\mu = \sum_i S^\mu_i + \bar{S}^\mu, \quad (11)$$

where, as in the stress-energy-momentum tensor, the term with the bar denotes interaction effects. These may be set to vanish where no interactions take place. It is important to reflect on the dynamics and changes that take place with regard to our realm of approximation. We postulate that there is a gradual change that sees the terms with the bars moving from sub-dominant to the dominant role as the fluids flow. This has the consequence that single-fluid approximation becomes the more appropriate tool in the latter stage. This, as previously mentioned, is ascribed to the interactions. There is a comparably formalism in which the interactions are made more explicit. We present this in the next section.

III.2. Convective Variational Formalism or Carter’s theory

This formalism was originally conceived in [14, 18, 47], reformulated for multi-fluids in [10] and expanded to incorporate entrainment and thermal effects in [19, 43, 44, 46] and applied to cosmology in [45]. We first give the case of this formulation involving three fluids species. The fundamental variables are taken to be the number fluxes denoted by $n^\mu_{(i)}|i = X, Y, Z$ and their corresponding entropy fluxes denoted by $s^\mu_{(i)}|i = X, Y, Z$ for non-entrained species. We will retain individual notation i.e. $n^\mu_X, n^\mu_Y$ and $n^\mu_Z$ in order to allow for clear notation of entrainment terms. Next one formulates a master function made of a fundamental scalar derived from the fluxes. In particular $\Lambda(n_{(i)}, s_{(i)})$ where $n_{(i)} = \sqrt{n^\mu_{(i)} n_{\mu(i)}}$ and similarly, $s_{(i)} = \sqrt{s^\mu_{(i)} s_{\mu(i)}}$ for the non-entrained fluxes. It is clear that the entrained number flux between species of types $X$ and $Y$ is given by $n_{XY} = \sqrt{n_X n_Y}$ compared to the non-entrained flux scalar for species of type $X$ which is given by $n_X = \sqrt{m^\mu_x n_{\mu X}}$[43, 46]. The master function is then taken as the density in the Lagrangian formulation of the matter action.

$$S_M = \int d\Omega \Lambda(n_{(i)}, s_{(i)}), \quad (12)$$

where $i = X, Y, Z, XY$ assuming only fluids of types $X$ and $Y$ are entrained. The unconstrained variation takes the form

$$\delta \Lambda = \frac{\partial \Lambda}{\partial n_{(i)}} \delta n_{(i)} + \frac{\partial \Lambda}{\partial s_{(i)}} \delta s_{(i)} \quad (13)$$

where [45]

$$\delta n_{(i)} = -\frac{1}{2n} \left(2g_{\mu\nu} n^\mu_{(i)} \delta n^\nu_{(i)} + n^\mu_{(i)} n^\nu_{(i)} \delta g_{\mu\nu}\right) \quad (14)$$

$$\delta s_{(i)} = -\frac{1}{2s} \left(2g_{\mu\nu} s^\mu_{(i)} \delta s^\nu_{(i)} + s^\mu_{(i)} s^\nu_{(i)} \delta g_{\mu\nu}\right). \quad (15)$$

The conjugate momentum associated with these fluxes are

$$\mu_{\mu(i)} = \frac{\partial \Lambda}{\partial n^\mu_{(i)}} = -2g_{\mu\nu} \left(\frac{\partial \Lambda}{\partial n^\nu_{(i)}}\right) n^\nu_{(i)} \quad (16)$$

$$\theta_{\mu(i)} = \frac{\partial \Lambda}{\partial s^\mu_{(i)}} = -2g_{\mu\nu} \left(\frac{\partial \Lambda}{\partial s^\nu_{(i)}}\right) s^\nu_{(i)}. \quad (17)$$

Since $i = X, Y, Z, XY$, the entrainment terms are embedded in Eqs. (16) and (17). Unlike in [19, 43, 45] the entrainment presented here is not between particle number and entropy but rather between separate fluid species. It follows from the variation of Eq. (12) that the stress energy momentum tensor takes the form

$$T^\nu_{\mu} = \sum_i \left(\mu_{\mu(i)} n^\nu_{(i)} + \theta_{\mu(i)} s^\nu_{(i)} + \Psi_{(i)} \delta^\nu_{\mu}\right), \quad (18)$$

where

$$\Psi_{(i)} = \Delta - \nu_{\mu(i)} n^\nu_{(i)} - \tilde{\theta}_{\mu(i)} s^\nu_{(i)} \quad (19)$$

such that $\Psi$ denotes the generalised pressure. These equations are structurally similar to those of [43] but with the added requirement that the stress momentum tensor be given by a sum of individual species contribution. Embedded in this is the entrainment which is the case given by setting $i = XY$, so that $T^\nu_{\mu} = \sum_i = X, Y T_{\mu(i)} + T_{\mu(XY)}$. The last term is the phenomenological equivalent of $T^\nu_{\mu}$ in the previous section. It is also instructive to note that Eq.(18) can be separated into number and entropy contributions. In general the equations of motion result from the conservation of the stress momentum tensor i.e. $\nabla^\nu T^\mu_{\nu} = \nabla^\nu (T^\mu_{\nu}|a + T^\mu_{\nu}|s) = 0$, where we have
separated the contribution by particular numbers and entropies with each term having entrainment included. The total particle number flux is conserved but not the individual, i.e. $\nabla_\nu (\sum_i n_i^\nu) = 0 \neq \nabla_\nu n_i^\nu$, allowing for particle creation and annihilation. The entropy is not conserved on the other hand. But this presentation is generic, requiring a choice of a reference frame if we are to define parameters such as temperature and 4-velocities and the corresponding entropy densities. There are different ways to achieve this but because we have considered several species it makes sense to choose the centre of mass frame [13] rather than the matter (or Eckart) frame. In particular the conservation of number flux suggests the orthogonality condition $n_{\mu(1)} (\nabla_\nu T^\nu_{\mu(1)} |_x) = 0$ and similarly $n_{\mu(2)} (\nabla_\nu T^\nu_{\mu(2)} |_s) = 0$. There exists 4-velocities $u^\mu_{\nu(1)}$ such that $n^\nu = nu^\mu_{\nu(1)}$ satisfying the requirement $u^\mu_{\nu(1)} u_{\nu(1)} = -1$ and a centre of mass velocity $u^\mu_{\nu(2)}$ such that $u^\mu_{\nu(2)} u_{\mu(2)} = -1$.

$$T_{\mu\nu} = -(\Lambda - p_\nu \sigma^\nu) u^\mu_{\nu u} + 2u_{(\mu \rho) u} + P_{\mu\nu}$$

(20)

The attempt, in [15], to match the variational formalism by Carter and the Israel-Stewart theory of dissipative fluids (i.e. first order version of the MIS theory) found that the two theories are not equivalent to all orders but are members of a set of related theories. It was found that the two theories lead to the same causal connections when subjected to perturbations about a thermodynamic equilibrium. It follows that in the thermal equilibrium limit, the two theories manifest similar characteristic surfaces and causality properties. Because of these similarities, we choose to examine the second law of thermodynamics in the extended MIS theory for multi-fluids i.e. we examine $\nabla_\mu S^\mu$, where $\nabla$ is defined with respect to the rest frame of an observer moving with the merged velocity $u^\nu$. It should be possible to carry out the same analysis in the variational approach. The variational formalism first developed in [14] is where a viscous fluid described by means of an entropy current, a particle current, and one viscosity tensor was analyzed. This has been extended in [10, 19, 43, 50] and results compared to those found in the MIS formalism [17] in the limit of linearized perturbation about thermal equilibrium [15].

IV. GENERALIZED SECOND LAW OF THERMODYNAMICS FOR A MULTI-FUID SYSTEM

In this section, we focus on a system of fluids made up of three species, $DM, DE$ and $\chi$. Instead of the generic subscript, $i = X,Y,Z$, we now restrict our notation to $i = DE, DM, \chi$. It is known that perfect fluids in equilibrium state do not generate entropy or heating due to friction as their dynamics is devoid of dissipation and is reversible. However, perfect fluid models are inadequate for modelling most astrophysical and cosmological processes. Such processes are best modelled using more realistic fluids which exhibit irreversible properties. Indeed, some processes in astrophysics and cosmology can only be understood as dissipative processes thereby requiring a relativistic theory of dissipative fluids [20]. It has been shown that for single-fluid approximation, irreversible thermodynamics implies that the entropy is no longer conserved but grows in accordance with the second law of thermodynamics. We need to examine if the law holds in our multi-fluid approximation.

IV.1. Irreversible thermodynamics

We here consider the limit, in the evolution of the fluids, where the observer world-line are just about to merge. In this approximation, these observers share a common Rindler horizon. Using spherical symmetry, the metric (1) can be expressed as

$$ds^2 = \gamma_{ab} dx^a dx^b + \tilde{r}^2 d\Omega_2^2,$$

(21)

where $\tilde{r} = a(t) r, x^0 = t, x^1 = r$ and the 2D metric $\gamma_{ab} = \text{diag}(-1, a^2/(1 - kr^2))$. The dynamical apparent horizon is determined by the relation $\gamma^{ab} \partial_a \tilde{r} \partial_b \tilde{r} = 0$ implying the vector $\nabla \tilde{r}$ is null on the apparent horizon surface. The apparent horizon radius for the FLRW [42] is

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + \frac{\kappa}{3}}}.$$

(22)

This apparent horizon can also be construed as a causal horizon [48, 49, 51]. The apparent horizon has two regions; the trapped or the inside region and the anti-trapped or the outside region. It follows that a change of this radius results in the change in the size of the two regions. In principle, this horizon evolves in time. The inside-region remains inaccessible if this radius decreases over time, otherwise it comes into view if the radius grows. This is similar to what is thought to happen to the black hole where the trapped region may become non-trapped should the apparent horizon decrease as a result of Hawking radiation [49].

It follows from Eq. (2) as shown in section (A1) that this radius evolves in times as

$$\dot{\tilde{r}}_A = -\frac{\tilde{r}_A^3}{6} \left( \frac{d\rho_{DE}}{d\tau} + \frac{d\rho_M}{d\eta} + \frac{d\rho_{\chi}}{d\zeta} \right),$$

(23)

where individual species contribution to the time evolution is captured. It is also easy to show from, Eq. (5), that

$$\dot{\tilde{r}}_A = \frac{H\tilde{r}_A^3}{2} \sum_i (\rho_i + p_i),$$

(24)

where $i = DE, DM, \chi$, note that we have not included $i = DE - DM$ since the interaction term is encoded in the $Q$ and $Q'$ in Eq. (5). At almost-equilibrium, it follows
where we define quasi-equilibrium by demanding the species temperature difference negligible. In this regards \( T_{(i)} = T \). Of course this assumption is not necessary and the analysis of the full system can still be performed. However, the full detailed analysis is involved and will not be pursued here as it would distract from the primary goal of this study. We would like to check how the total entropy evolves in time as mediated by the different species contributions in our quasi-equilibrium description. The individual entropy evolution, with respect to own world line, to takes the form

\[
\frac{\partial S_{DE}}{\partial \tau} = \frac{1}{T} \left( 4\pi^2 \rho_{DE}(\partial \rho_{DE})^2 + 2\pi^2 \partial \rho_{DE} \partial E_{DE} - \mu_{DE} \partial N_{DE} \right),
\]

\[
\frac{\partial S_{DM}}{\partial \eta} = \frac{1}{2} \left( 4\pi^2 \rho_{DM}(\partial \rho_{DM})^2 + 2\pi^2 \partial \rho_{DM} \partial E_{DM} - \mu_{DM} \partial N_{DM} \right),
\]

\[
\frac{\partial S_{X}}{\partial \gamma} = \frac{1}{2\pi^2} \left( 48 \rho X (\partial \rho X)^2 + 2\pi^2 \partial \rho X \partial E_{X} - \mu X \partial N_{X} \right),
\]

where \( i = DE, DM, \chi \). It is clear from Euler’s relation that

\[
p_i = \rho_i(\partial \rho_i, s_i, n_i),
\]

where \( \rho_i = E_i/V, s_i = S_i/V \) and \( n_i = N_i/V \). If one ignores the transfer of energy due to the internal degrees of freedom but one, one could assume the barotropic equation of state consistent with adiabatic pressure. \( p_i = \omega \rho_i \). We will return this this spacial case later. It is known that the temperature of a horizon is related to its radius \([30, 42, 57–59]\) when black hole thermodynamics is extended to cosmology i.e.

\[
T_h = \frac{1}{2\pi \tilde{r}_A},
\]

The entropy of the horizon can be defined as \( S_h = 4\pi \tilde{r}_A^2/4G = 8\pi^2 \tilde{r}_A^3 \), where \( 8\pi G = 1 \). The total entropy is then given by

\[
S_{Tot} \simeq \sum_i S_i + S_h.
\]

The time evolution of the total entropy takes the form

\[
\dot{S}_{Tot} \simeq \frac{\partial S_{DE}}{\partial \tau} \alpha + \frac{\partial S_{DM}}{\partial \eta} \beta + \frac{\partial S_{X}}{\partial \gamma} \gamma + \dot{S}_h,
\]

where the overdot is the derivative with respect to the time parameter \( t \) and where

\[
\frac{\partial S_{DE}}{\partial \tau} = \frac{4\pi^2}{T} \left[ (\rho_{DE} + \rho_{DE})(\partial \rho_{DE})^2 + \tilde{r}_A \tilde{H} \tilde{A} - \frac{3\tilde{A}}{\tilde{A}} \right] - \frac{1}{T\mu_{DE}} \partial N_{DE}.
\]

\[
\frac{\partial S_{DM}}{\partial \eta} = \frac{4\pi^2}{T} \left[ (\rho_{DM} + \rho_{DM})(\partial \rho_{DM})^2 + \tilde{r}_A \tilde{H} \tilde{A} - \frac{3\tilde{A}}{\tilde{A}} \right] - \frac{1}{T\mu_{DM}} \partial N_{DM}.
\]

\[
\frac{\partial S_{X}}{\partial \gamma} = \frac{4\pi^2}{T} \left[ (\rho X + \rho X)(\partial \rho X)^2 + \tilde{r}_A \tilde{H} \tilde{A} - \frac{3\tilde{A}}{\tilde{A}} \right] - \frac{1}{T\mu_X} \partial N_{X}.
\]

The horizon entropy evolves as \( \dot{S}_h = 16\pi^2 \tilde{r}_A \dot{\tilde{r}}_A \) and the total entropy therefore obeys the evolution equation

\[
\dot{S}_{Tot} \simeq 8\pi^2 \tilde{r}_A \sum_i [(\rho_i + p_i)(\dot{\rho}_i - \dot{A} \tilde{H})] - \sum_i 2\pi \tilde{r}_A \dot{\tilde{r}}_A \dot{N}_i + 16\pi^2 \tilde{r}_A \dot{\tilde{r}}_A.
\]
\( \dot{r}_A \) can be substituted using Eq.(24) to give

\[
\dot{S}_{Tot} \simeq 4\pi^2 r_A^5 H \left( \sum_i (\rho_i + p_i) \right)^2 - 2\pi r_A \sum_i \mu_i \dot{N}_i (35)
\]

It is clear that this finding holds regardless of the nature of gravitational interaction \( Q \). This result modifies the finding in [27] where the new equation of state of one of the species ( e.g. \( DE \)) emerges for the critical \( \dot{S}_{Tot} = 0 \), implying

\[
\left( \sum_i (\rho_i + p_i) \right)^2 = \frac{1}{2\pi r_A^5 H} \sum_i \mu_i \dot{N}_i (36)
\]

and on expanding the left-hand side

\[
\frac{p_{DE(i)} - \rho_{DE(i)}}{\rho_{DE}} = -1 - (\rho_{DM} + \rho_{DM}) \frac{1}{\rho_{DE}} - (\rho_X + \rho_X) \frac{1}{\rho_{DE}} + \frac{1}{\rho_{DE}} \sqrt{\frac{1}{2\pi r_A^5 H} \sum_i \mu_i \dot{N}_i} (37)
\]

More importantly, entrainment between \( DE \) and \( DM \) is straightforwardly incorporated by setting \( i = DE-DM \). The effect of the entrainment on the critical equation of state can then be monitored via Eq.(37).

V. DISCUSSIONS AND CONCLUSIONS

Let’s examine Eq. (34). We know that \( \dot{r}_A, H, p_i, \rho_i \) are by definition positive. Eq. (23) guarantees that \( \dot{r}_A > 0 \). The result of the summation will be positive since the horizon radius is greater than the Hubble parameter. This is confirmed by setting

\[
(\rho_i + p_i) (\dot{r}_A - \dot{r}_A H) > 0 (38)
\]

which implies

\[
\frac{\dot{r}_A}{r_A} > H = \frac{\dot{a}}{a} (39)
\]

as expected for the case where the surface term is neglected. Finally, \( \dot{S}_{Tot} > 0 \) is achieved if \( \sum_i \mu_i \dot{N}_i < 0 \), which is exactly what is expected of Gibbs free energy for negative chemical potentials. If we label Gibbs free energy using the letter \( E_G \), then \( E_G < 0 \) implies \( \dot{S}_G > 0 \). This establishes the generalised second law of thermodynamics for interacting dark-sector and radiation where the interaction goes beyond previous studies involving gravitational interactions. We note that the inclusion of the chemical interaction in a multi-fluid approximation conserves the second law of thermodynamics. Effects of non-zero chemical potential on the equation of state of the dark energy in single-fluid approximation were examined in [52] where it was found that the equation of state depended heavily on the magnitude and the sign of the chemical potential. Eq. (37) modifies those findings and has the potential to lift the \( \omega_{cr} \) into the non-phantom state. We hesitate to provide an estimate as this would require the accurate estimation of \( \dot{r}_A \) and \( \sum_i \mu_i \dot{N}_i \) in the quasi-equilibrium state. Using a multi-fluid approximation, we have investigated a cosmological scenario involving three particles species with two of these interacting both gravitationally and chemically. Each of the three world-lines have apparent horizons that evolve in time toward a shared common apparent horizon. On contact with neighbouring apparent horizons, causal connections form. It is known, in the case of black holes, that the horizon may evolve in two ways: either smoothly, in a space-like manner, or in a discontinuous-jump allowing new or emergent horizons to form around old horizons [53]. Nothing stops this from occurring for Rindler horizons.

After examining how a common 4-velocity (single-fluid approximation) emerges from multi-velocities (multi-fluid approximation) and how this gives rise to a cosmological model, we investigated the generalized second law of thermodynamics in the context of this approximation. We then considered the universe as a thermodynamical system enclosed by the dynamical apparent horizon emerging from a set of Rindler horizons, and calculated separately the entropy variation for each fluid species. The sum of these entropy variations together with that of the common horizon gives the total entropy of the universe. We find that the generalised second law of thermodynamics holds. It is important to note that we used dynamical apparent horizons and did check cases involving other types of horizons. Although we have shown, in the present work, that the generalized second law of thermodynamics in the interaction scenario involving dark energy (\( DE \)) and dark matter(\( DM \)) and radiation(\( \chi \)) further investigation is still needed in order to make the findings applicable to quantitative or numerical cosmological analysis.

Appendix A: An evolving apparent horizon

We gave a definition of the horizon radius

\[
\dot{r}_A = \frac{1}{\sqrt{H^2 + \frac{\kappa}{a^2}}} (A1)
\]

in section (II). It will be noticed that one can replace the radius using Eq.(3). In particular,

\[
H^2 + \frac{\kappa}{a^2} = \frac{1}{\dot{r}_A^2} = \frac{1}{3} (\rho_{DE} + \rho_{DM} + \rho_{X}). (A2)
\]

The evolution with respect to proper time, \( t \), yields

\[
\frac{d}{dt} \left( H^2 + \frac{\kappa}{a^2} \right) = -2 \frac{\dot{r}_A}{r_A^2} \frac{d}{d\tau} \left( H^2 + \frac{\kappa}{a^2} \right) = \frac{1}{3} (\dot{\rho}_{DE} + \dot{\rho}_{DM} + \dot{\rho}_{X})(A3)
\]
from which it follows that
\begin{equation}
\dot{\rho}_A = -\frac{\dot{r}_A^3}{6} (\dot{\rho}_{DE} + \dot{\rho}_{DM} + \dot{\rho}_\chi) = -\frac{\dot{r}_A^3}{6} \left( \frac{d\rho_{DE}}{d\tau} \alpha + \frac{d\rho_{DM}}{d\eta} \beta + \frac{d\rho_\chi}{d\zeta} \gamma \right)
\end{equation}

(A4)

Appendix B: References

[1] A. Linde in Particle Physics and Inflationary Cosmology (Harwood, Chur, Switzerland, 1990).
[2] A. R. Liddle and D. H. Lyth in Cosmological inflation and large-scale structure (Cambridge University Press, 2000).
[3] C. L. Bennett et al, The Astrophysical Journal Supplement Series 208 2.
[4] B. Ryden in Introduction to Cosmology (Cambridge University Press, 2017).
[5] J. A. Frieman, M. S. Turner and D. Huterer, Annual Review of Astronomy and Astrophysics, 46 (2008) 1.
[6] P. A. R. Ade et al, Astronomy & Astrophysics 594 (2015) 13.
[7] K. Land and J. Magueijo, Physical Review Letters 95 (2005) 7.
[8] S. Perlmutter et al., Astrophys. J. 517 (1999) 565.
[9] A. G. Riess et al., Astron. J. 116 (1998) 1009.
[10] R. Prix, Phys. Rev. D 69 (2004) 43001.
[11] C. Eckart, Phys. Rev. 58 (1940) 267.
[12] W. A. Hiscock and L. Lindblom, Phys. Rev. D311 (1985) 725.
[13] L. Landau and E. M. Lifshitz, in Fluid Mechanics (Addison-Wesley, Reading, MA, 1958) 127.
[14] B. Carter, in Relativistic Fluid Dynamics (Nato, 1987), eds. A. Anile and M. Choquet-Bruhat (Springer-Verlag, Heidelberg, Germany, 1989).
[15] D. Priou, Phys. Rev. D 43 (1991) 4.
[16] W. Israel, Ann. of Phys. 100 (1976) 310.
[17] W. Israel and J. M. Stewart, Ann. Phys. 118 J M (1979) 341.
[18] B. Carter B, in Journées Relativistes, eds. M. Cahen, R. Debever, J. Gegeniau (Universite Libre de Bruxelles, 1976).
[19] N. Andersson and G. L. Comer, Living Reviews in Relativity, 10 (2007) 1.
[20] R. Maartens, astro-ph/9609119.
[21] R. Tresguerres, Int. J. Geom. Meth. Mod. Phys. 5 (2008): 905.
[22] E. Minguzzi, Phys. Lett. 15 (2002).
[23] E. Minguzzi, Class. Quant. Grav. 20 (2003).
[24] E. Minguzzi, gr-qc/0501125.
[25] E. Minguzzi, gr-qc/0506177.
[26] B. Osano B, (2019) Relativistic Extended Thermodynamics for Multiple Interacting Fluids, in preparation.
[27] M. Jamil et al, Phys. Rev. D81 (2010) 023007.
[28] R. Tresguerres, Phys. Rev. D 89 (2014) 064032.
[29] T. K. Nakamura, Progress of Theoretical Physics, Vol. 128, No. 3, September 2012.
[30] T. Jacobson T, Phys. Rev. Lett. 75 (1995).
[31] W. Rindler, MNRAS 116 (1956) 662.
[32] Y. Friedman and T. Scarr, Physica Scripta 87 (2013) 055004.
[33] R. C. Tolman, Phys. Rev. 35 (1930).
[34] R. C. Tolman, in the Tenth Josiah Willard Gibbs Lecture, American Mathematical Society (1932).
[35] B. Osano (2019). Fundamental Thermodynamics Relation: A perturbation approach, in preparation.
[36] O. Redlich, J. Chem. Educ. 47 (1970) 2.
[37] I. Müller, Living Rev. Relativity 2 (1999) 1.
[38] Thanks to the anonymous reviewer for pointing these out.
[39] C. Clarkson C and B. Osano, Class. Quant. Grav. 28 (2011) 22.
[40] T. Padmanabhan, Phys. Rev. D 83 (2011) 044048.
[41] N. G. Van Kampen, Phys. Rev. 173 (1968) 295.
[42] M. Akbar and R-G. Cai, Phys. Rev. D75 (2007) 084003.
[43] C. S. Lopez-Monsalvo and N. Andersson, Proc. Roy. Soc. Lond. A (2010) 476.
[44] N. Andersson et al, Class. Quantum Grav. 34 (2017) 125001.
[45] G. L. Comer, Peter and N. Andersson Phys. Rev. D 85 (2007) 10.
[46] B. Osano and T. Oreta T, Inter. Jour. Mod. Phys.s D 28 (2019) 1950078.
[47] B. Carter B, in Highlight in gravitation and cosmology, AIP Conf. Proc. AIP Conf. Proc. 1160 (2009) 347.
[48] S. A. Hayward, S. Mukohyama and M. C. Ashworth, Phys. Lett. A 256 (1999), 347.
[49] D. Bak and S. J. Rey, Class. Quantum Grav. 17 (2000) L83.
[50] S. A. Hayward, gr-qc/9803071.
[51] S. A. Hayward, Class. Quantum Grav. 15 S A (1998) 3147.
[52] J. A. S. Lima and S. H. Pereira, Phys. Rev. D78 (2008) 083504.
[53] J. A. S. Lima and S. H. Pereira, Phys. Rev. D78 (2008) 104015.
[54] S. A. Hayward, Class. Quantum Grav. 24 (2007) 124023.
[55] T. Padmanabhan, AIP Conf. Proc. (2010) 1241.
[56] T. Padmanabhan, Phys. Rep. 49 (2005) 406.
[57] A. Paranjape, S. Sarkar and T. Padmanabhan, Phys. Rev. D74 (2006) 104015.
[58] R. G. Cai and S. P. Kim, J. High Energy Phys. 2 (2005) 050.