CHARACTERISTIC POTENTIALS FOR MESOSCOPIC RINGS THREADED BY AN AHARONOV-BOHM FLUX

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Electro-static potentials for samples with the topology of a ring and penetrated by an Aharonov-Bohm flux are discussed. The sensitivity of the electron-density distribution to small variations in the flux generates an effective electro-static potential which is itself a periodic function of flux. We investigate a simple model in which the flux sensitive potential leads to a persistent current which is enhanced compared to that of a loop of non-interacting electrons. For sample geometries with contacts the sensitivity of the electro-static potential to flux leads to a flux-induced capacitance. This capacitance gives the variation in charge due to an increment in flux. The flux-induced capacitance is contrasted with the electro-chemical capacitance which gives the variation in charge due to an increment in an electro-chemical potential. The discussion is formulated in terms of characteristic functions which give the variation of the electro-static potential in the interior of the conductor due to an increment in the external control parameters (flux, electro-chemical potentials).

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I. INTRODUCTION

In this work we discuss the electro-static potential in the interior of small mesoscopic conductors with the topology of a ring (see Fig. 1). An Aharonov-Bohm flux (AB-flux) penetrates the hole of the loop. Typically the electron-density distribution in such a loop is non-uniform and is a sensitive function of the AB-flux. The non-uniform density generates an electro-static potential which is also a function of the flux. We introduce a characteristic potential $v(r)$ which gives the variation of the electro-static potential in the interior of conductors in response to an increment of the AB-flux. We investigate a simple example of a one-dimensional loop structure (see Fig. 2) which permits a solution for the characteristic potential $v(r)$. This example provides a demonstration that interaction can enhance the persistent current above its value in the absence of interactions. The characteristic potential $v(r)$ is used to define a flux-induced capacitance which is the ratio of the charge increment on the conductor divided by the increment in flux. The flux-induced capacitance is an odd (periodic) function of flux. For samples with contacts we introduce a characteristic potential $u(r)$ which gives the variation of the local potential inside the conductor in response to an increment of an electro-chemical potential at a contact. This characteristic function allows the evaluation of the electro-chemical capacitance. The electro-chemical capacitance is an even periodic function of flux with period $\Phi_0 = \hbar c/e$. The electro-chemical capacitance can be measured with the help of small time-dependent, oscillating voltages applied to a capacitor. The flux-induced capacitance is measured by applying a small oscillatory flux superimposed on a steady state AB-flux.

Recent work [1] by Thomas, Prêtre and this author on the admittance of mesoscopic capacitors emphasized the need to distinguish between electric and electro-chemical capacitances. Capacitance coefficients are not electro-static entities determined by the geometry of the sample alone but are electro-chemical quantities which depend on the properties of the conductor [2]. The non-geometrical contributions to the capacitance coefficients arise due to the fact that electric fields impinging on a conductor are not screened immediately at the
surface of the conductor but penetrate over a distance of a screening length into the "bulk" of the conductor [3]. Clearly field penetration is very important for mesoscopic conductors [1] since one or more dimensions of such a small system might in fact be comparable to a screening length! A dramatic demonstration of the non-geometrical nature of capacitances is given by Chen et al. [4] who show that magnetic fields can completely quench certain elements of a capacitance tensor.

In samples without transmission the capacitance coefficients are an equilibrium phenomenon which reflects properties of the ground state. Another interesting ground state property of a small mesoscopic ring is the equilibrium persistent current. The existence of such currents in mesoscopic disordered normal rings was predicted by this author in collaboration with Imry and Landauer [5]. In earlier work persistent currents were discussed for large molecules [6,7], Landau diamagnetism [8], and in connection with superconductivity [8,10]. For many years equilibrium currents in mesoscopic conductors remained of interest to only a small community [11–15]. Experimental observation of such equilibrium currents in an ensemble of rings by Levy et al. [16], in single metallic loops by Chandrasekhar et al. [17], and single semiconductor rings by Mailly et al. [18] generated a considerable theoretical literature. We can refer the reader here to only a few representative contributions investigating the persistent current of non-interacting [19–21] and interacting [22–26] electrons. Persistent currents occur not only in isolated rings but also in rings connected via leads to electron reservoirs [12,14,21,27,28]. The experiment of Mailly et al. [18] in fact measures persistent currents in both closed and open rings of roughly the same amplitude. Since both capacitances and persistent currents reflect properties of the ground state it is intriguing to explore possible connections between the two phenomena.

II. SELF-CONSISTENT POTENTIAL OF A NORMAL LOOP

Consider the conductors in Fig. 1. The conductor of Fig. 1a is via a lead connected to an electron reservoir [12,27,28]. It is otherwise in an electrically isolating environment:
There are no electric field lines penetrating the surface of a volume which is at a sufficient distance from the loop and intersects the reservoir far enough from the connection to the lead. Typically conductors are not in such an electrically isolated environment but couple via long range Coulomb forces to other nearby metallic conductors. Such a situation is investigated for the conductor of Fig. 1b. Here the conductor of Fig. 1a is separated via an insulator from a second conductor, a gate, which is connected to an electron-reservoir at electro-chemical potential \( \mu_2 \). Again we will for clarity assume that there exists a volume which encloses both conductors and a portion of the reservoirs such that no electric-field lines penetrate its surface. In both structures one of the conductors forms a loop that is threaded by an AB-flux. The equilibrium electro-static potential \( U(\mu_1, \mu_2, \Phi, \mathbf{r}) \) for these conductors is a function of the electro-chemical potentials, the flux and a complicated function of position. The flux-dependence of the potential is generated by the electron-density of the loop which is flux dependent whenever the loop is not rotationally invariant. We are interested in the variation of the electro-static potential under small changes of the external control parameters \( \mu_1, \mu_2 \) and \( \Phi \). Small increases \( d\mu_1, d\mu_2, d\Phi \) in the electro-chemical potentials and or the flux will bring the conductor from the equilibrium state with potential \( U(\mu_1, \mu_2, \Phi, \mathbf{r}) \) to a new equilibrium state with potential \( U(\mu_1 + d\mu_1, \mu_2 + d\mu_2, \Phi + d\Phi, \mathbf{r}) \). The difference \( dU \) between these two equilibrium potentials can be expanded in powers of the increment in electro-chemical potential and the increment in flux. To linear order we have

\[
edU(\mu_1, \mu_2, \Phi, \mathbf{r}) = u_1(\mathbf{r})d\mu_1 + u_2(\mathbf{r})d\mu_2 + v(\mathbf{r})d\Phi.
\]

Here \( u_k(\mathbf{r}) = edU(\mu_1, \mu_2, \Phi, \mathbf{r})/d\mu_k|_{d\mu_k=0} \), with \( k = 1, 2 \) and \( v(\mathbf{r}) = edU(\mu_1, \mu_2, \Phi, \mathbf{r})/d\Phi|_{d\Phi=0} \) are the characteristic functions \([2]\). We will use the characteristic functions \( u_k(\mathbf{r}) \) to derive expressions for the electro-chemical capacitance \( C_\mu = dQ/d\mu \). In analogy to the electro-chemical capacitance we also derive with the help of the characteristic function \( v \) a flux-induced capacitance \( C_\Phi = dQ/d\Phi \). For the conductor of Fig. 1a which is an electrically isolating environment an increase in the electro-chemical potential of the reservoir or an increase in flux cannot change the overall charge. For this conductor both the electro-
chemical capacitance and the flux-induced capacitance vanish. The conductor of Fig. 1a is thus of interest only as a simple limiting case.

For the conductor of Fig. 2b, the characteristic potentials $u_1$ and $u_2$ have the following interesting properties:

1. For $r$ deep in reservoir $k$ the local potential must follow the electro-chemical potential of that reservoir and hence $u_k(r) = 1$.

2. A change in the electro-chemical potential of reservoir $k$ cannot affect the potential deep inside any other reservoir. Thus for $r$ deep inside reservoir $l \neq k$ the characteristic function must vanish, $u_k(r) = 0$.

3. If we change all electro-chemical potentials simultaneously and by an equal amount $d\mu_k = d\mu$ then we have only changed our (global) energy scale. Hence at every space point $r$ the potential $U$ must also change by $d\mu$. This implies that the sum of all characteristic functions at every space point is equal to 1,

$$\sum_k u_k(r) = 1.$$  \hspace{1cm} (2)

In contrast to the characteristic potentials $u_k$ which approach 1 in contact $k$, an increment in flux polarizes the sample but does not affect the state of the system deep inside a reservoir. Hence the characteristic function $v(r)$ vanishes deep inside the electron reservoirs.

To treat the Coulomb interactions it is useful to take the density functional theory as a guide. A discussion of persistent currents based on this approach is given in Ref. [25]. We are interested in the magnitude of the variation of the potential with flux. To this extend we will discuss the closed loop structure in Fig. 2. If the loop is taken to be in an electrically isolating environment the electro-static potential is a function of flux only. (There is no dependence on a chemical potential). The single particle wave functions of the ground state of the ring are thus determined by a Hamiltonian

$$H_{\text{eff}} = \frac{1}{2m}(p - \hbar \frac{2\pi}{L} \Phi_0)^2 + eU_{\text{eff}}(\Phi, r),$$  \hspace{1cm} (3)

Here $\Phi$ is the flux, $\Phi_0 = hc/e$ is the single charge flux quantum, and $L = 2\pi R$ is the circumference of the loop. The effective potential $eU_{\text{eff}}(\Phi, r)$ contains in addition to the
The total kinetic energy of the electrons in the loop is

\[ E_{\text{kin}} = \sum E_n(\Phi, U(\Phi, r)) - \int d^3r n(\Phi, r)eU_{\text{eff}}(\Phi, r), \]  

where \( E_n(\Phi, U(\Phi, r)) \) are the eigenvalues of Eq. (3) and \( n(\Phi, r) \) is the electron-density. The sum is over all occupied levels. The persistent current of this loop is the flux-derivative of the free energy. At \( kT = 0 \) the free energy is equal to the total internal energy \( E_{\text{int}} = E_{\text{kin}} + E_c + E_{\text{ex}} \), where \( E_c \) is the Coulomb interaction energy and \( E_{\text{ex}} \) is the exchange energy \([29,30]\). At \( kT = 0 \) the persistent current of a closed loop is determined by

\[ I_{\text{eq}}(\Phi) = -cdE_{\text{int}}/d\Phi. \]  

The persistent current does not in an explicit way depend on the characteristic function \( v(\Phi) \) introduced above. This is a consequence of the fact that for potential variations away from the true equilibrium potential the internal energy is stationary \([30]\). As remarked already in Ref. [25], the total flux derivative in Eq. (5) can be replaced by a partial derivative taken at constant potential \( U \). Nevertheless, the potential variation with flux is important: The spectrum of the interacting and the non-interacting system are not the same.

For the open conductors of Fig. 1 it is the grand canonical potential, \( \Omega = E_{\text{int}} - \sum_k N_k \mu_k \) that counts. The persistent current in these structures is

\[ I_{\text{eq}}(\Phi) = -cd\Omega/d\Phi. \]  

The electro-chemical capacitance coefficients are

\[ C_{\mu,k}(\Phi) = -e^2d^2\Omega/d\mu_k d\mu_l. \]  

and the flux-induced capacitances are

\[ C_{\Phi,k}(\Phi) = -ed^2\Omega/d\Phi d\mu_k. \]  

From this definition of the flux-induced capacitance another alternate interpretation of the second order mixed derivatives of the grand canonical potential becomes apparent. Since
$I_{eq}(\Phi) = -cd\Omega/d\Phi$ the flux induced capacitance is related to a "conductance" $G_{\Phi,k} = cC_{\Phi,k} = -ec(dI_{eq}/d\mu_k)$ which is a measure of the sensitivity of the persistent current to a small change in the electro-chemical potential. The flux-induced capacitances are thus connected to the gate voltage dependence of the persistent current.

We make a clear distinction between closed loops with a fixed number of carriers and open loops which are via leads connected to electron reservoirs. Much of the mesoscopic literature attempts to treat closed loops in a grand canonical ensemble and corrects this with a flux dependent chemical potential. Here electro-chemical potentials characterize metallic contacts. For interacting systems it is in addition necessary to distinguish the electrically isolated ring from rings which interact via long range Coulomb forces with other nearby conductors. We use as a reference state a self-consistent equilibrium state. The Coulomb interactions are determined by the actual sample specific charge distribution and not with respect to a flux averaged or ensemble averaged charge density.

III. COULOMB DRIVEN SUPPRESSION OF LEVEL HYBRIDIZATION

Consider a one-dimensional loop with a weak disorder potential. A stub, a wire of finite length, is via a barrier coupled to this loop \[31\] (see Fig. 2). This structure incorporates some features which are typically encountered in multi-channel rings with finite cross-sections \[14,15\]: As a function of flux a single particle energy exhibits regions in which the state is nearly flux-insensitive $dE_n/d\Phi \sim 0$ and behaves as if it were a localized state. These localized regions are interrupted by rapid changes as a function of flux and the state behaves as if it were highly mobile. We can view such a multichannel spectrum as a hybridization of a highly mobile subsystem with a subsystem of localized states. The simple example of Fig. 2 allows to investigate the interplay of states which are localized in the stub with highly mobile states in the loop.

We assume that the electro-static potential can be taken uniform inside the stub and can be taken uniform inside the loop. Let $U_s$ denote the electro-static potential (bottom of the
conduction band) inside the stub (index $s$). The electrostatic potential inside the ring (index $r$) is $U_r$. If the barrier is not transparent the states in the stub have energy $E_{s,m} + eU_s$, with $m = 1, 2, ..., M$. The spectrum of the states in the ring is $E_{r,n}(\Phi) + eU_r$ with $n = 1, 2, ..., N$. A spectrum of such a system is shown in Fig. 3a. The energies of the flux sensitive (mobile) states of the loop are drawn as solid lines. The energies of the flux-insensitive (localized) states of the stub are shown as broken lines. The $N$ electrons in the loop give a persistent current

$$I_N(\Phi) = -c \sum_{n=1}^{N} dE_{r,n}(\Phi)/d\Phi$$

shown in Fig. 4.

Now let us make the barrier transparent. Now the stub and the ring form one combined system. Assume that the transparency of the barrier is very small. The spectrum of the combined system will undergo only very small changes where levels of the disconnected system intersect. In a first step we evaluate the spectrum of the combined system taking the potentials $U_s$ and $U_r$ to be known. In a second step we will include the Coulomb interaction to determine these potentials. Consider any two levels $E_s$ and $E_r$ which in the absence of transmission intersect. In the combined system the energy levels of these two states have the hybridized energies

$$E_{\pm}^h = (1/2)(E_s + eU_s + E_r + eU_r \pm \Delta)$$

where the energy gap between the two levels is given by

$$\Delta = \left( (E_s + eU_s - E_r - eU_r)^2 + 4|t|^2 \right)^{1/2}.$$  

Here $|t|$ is the energy which couples the states of the ring and the stub. The spectrum $E_l^h$ with $l = 1, ..., M + N$ of the combined system is shown in Fig. 3b. It consists of pairs of states with eigenvalues given by Eq. (10) and consists of states $E_l^h = E_{s,m} + eU_s$ which do not intersect any level of the loop but fall into a gap. The persistent current of the hybridized system (in the absence of Coulomb interactions) is given by $I^h(\Phi) = -c \sum_l dE^h_l/d\Phi$. In
the absence of interactions the flux dependence still originates from the flux dependence
of the energies $E_{r,n}$. Since the hybridized states occur, except for the topmost occupied
state, in pairs the current is the sum of the persistent current of the uncoupled loop with
$N - 1$ electrons plus the contribution of the topmost occupied state with energy $E^h$ in the
hybridized system,

$$I^h(\Phi) = I_{N-1}(\Phi) - c dE^h/d\Phi.$$  \hspace{1cm} (12)

For a small flux we have $dE^h/d\Phi \sim dE_N/d\Phi$ and the persistent current of the combined
system is the same as that of the disconnected system. But for a flux far beyond the
hybridization point $\Phi_*$ we have $dE^h_\pm/d\Phi \sim 0$. Thus beyond the hybridization point the
persistent current is that of an $N - 1$ electron loop. This behavior is shown in Fig. 4. The
current of $N - 1$ electron states is typically of opposite sign and smaller in magnitude then
that of the $N$ electron loop. Thus the current of the hybridized system is typically smaller
in magnitude than the persistent current of the decoupled system. Below we will now show
that if the Coulomb interactions are taken into account and are of sufficient strength the
current of the combined system is in fact restored to a value close to that of the uncoupled
system.

The hybridized states with energies $E^h_\pm$ are coherent superpositions of a state that was
originally localized in the stub and a state that originally was confined to the ring. As a
consequence of the superposition the state $E^h_+$ has only a fractional charge $Q_{s+}$ inside the
stub and has a fractional charge $Q_{r+}$ inside the ring. Similarly the state with energy $E^h_-$
has a fractional charge $Q_{s-}$ inside the stub and has a fractional charge $Q_{r-}$ inside the ring.
These charges can be found by differentiating the energies of these states with respect to
the potentials $U_s$ and $U_r$. With $\alpha = r, s$ we find for the partial charges

$$Q_{\alpha,\pm} = dE^h_\pm/dU_\alpha.$$  \hspace{1cm} (13)

Of course the total charge in each filled state is $Q_{s\pm} + Q_{r\pm} = e$. From Eq. (13) we see
immediately that if both states $\pm$ are filled then the combined charge of these two states in
the stub is \( Q_{s+} + Q_{s-} = e \) and in the ring is \( Q_{r+} + Q_{r-} = e \). Consequently a net charge motion from the ring into the stub occurs only if the topmost occupied (empty) state in the ring intersects an empty (occupied) state in the stub as shown in Fig. 3b. Thus the Coulomb interaction is entirely determined by the charge motion in the topmost level \( E_{h} \).

Before we proceed to evaluate the potentials we calculate the variation of the charges of the topmost occupied level \( E_{h} \) in response to a small variation of the potentials. There is a Lindhard function \( \Pi_{\alpha\beta} \), with \( \alpha = r, s \) and \( \beta = r, s \) which gives the charge response in the stub or in the ring due to a variation of the potential in the stub or in the ring, \( dQ_{\alpha} = -\Pi_{\alpha\beta} dU_{\beta} \). From Eq. (13) we find \( \Pi_{\alpha\beta} = d^{2}E_{h}/dU_{\alpha}dU_{\beta} \). Using Eq. (10) we find after a little algebra \( \Pi_{ss} = \Pi_{rr} = -\Pi_{rs} = -\Pi_{sr} = -\Pi \) where

\[
\Pi = 2e^{2}|t|^{2}/\Delta^{3} \quad (14)
\]

In the absence of interactions, i.e. for fixed potentials \( U_{s} \) and \( U_{r} \), the polarization function exhibits a sharp peak of magnitude \( 1/|t| \) at the point of hybridization and is small as soon as the difference in energies of the two states exceeds \( |t|^{2} \). Next we want to characterize the variation in charge in the stub and in the ring as a function of flux. For fixed potentials an increment \( d\Phi \) in the flux causes a change in the charge on the stub given by

\[
(dQ_{s}/d\Phi)_{U} = -d(dE_{h}/dU_{s})/d\Phi \mid_{U}.
\]

Since at constant potential the flux dependence of the energy stems from \( E_{r,N} \) only we find

\[
(dQ_{s}/d\Phi)_{U} = -(d^{2}E_{h}/dU_{s}dE_{r,N})(dE_{r,N}/d\Phi).
\]

But \( d^{2}E_{h}/dU_{s}dE_{r,N} = (1/e)d^{2}E_{h}/dU_{s}dU_{r} \) and hence using Eq. (14) we find for the flux-induced charge variation

\[
(dQ_{s}/d\Phi)_{U} = (1/e)\Pi(dE_{r,N}/d\Phi) \quad (15)
\]

Since the total charge in each state is conserved an increment in flux leads to a charge

\[
(dQ_{r}/d\Phi)_{U} = -(1/e)\Pi(dE_{r,N}/d\Phi) \quad \text{in the ring.}
\]

We are now ready to determine the self-consistent potentials.

An increment \( d\Phi \) in the AB-flux changes the charge in the stub by an amount \( (dQ_{s}/d\Phi)_{U}d\Phi \). Screening of these charge causes a variation of the potentials by \( dU_{s} \) and \( dU_{r} \). According to the Lindhard function this causes an additional charge variation in the
stub given by \(-\Pi dU_s + \Pi dU_r\). The Coulomb interaction is taken into account with the help of a geometrical capacitance \(C\) between the stub and the ring. Taking the total charge variation on the stub to be equal to that permitted by the Coulomb interaction gives \[31\]

\[
(dQ_s/d\Phi)_U d\Phi - \Pi dU_s + \Pi dU_r = C(dU_s - dU_r).
\]

(16)

The charge in the ring obeys an equation which term by term is identical to Eq. (16) except for the sign of the charges. Thus Eq. (16) determines only the difference in potentials. The solution of Eq. (16) is the difference \(v = v_s - v_r\) of the characteristic potentials \(v_s = edU_s/d\Phi\), \(v_r = edU_r/d\Phi\),

\[
v = ed(U_s - U_r)/d\Phi = \Pi/(C + \Pi)(dE_{r,N}/d\Phi).
\]

(17)

Integration of Eq. (17) gives

\[
e(U_s - U_r) = \int^{\Phi} d\Phi'\Pi/(C + \Pi)(dE_{r,N}/d\Phi').
\]

(18)

The Lindhard function \(\Pi\) is also a function of the difference of the potentials \(e(U_s - U_r)\). Thus Eq. (20) is similar to the self-consistent equations encountered in other problems, for instance the BCS gap equation. We note that the Lindhard function modifies the geometrical capacitance and gives rise to a quantum correction. The charge induced in response to an external potential is determined by an effective capacitance \[1\]

\[1/C_{eff} = 1/C + 1/\Pi.\]

Fortunately there is a simple limiting case in which the solution of the self-consistent equation is obvious. For a very small geometrical capacitance the right hand side of Eq. (18) is independent of the potentials. The characteristic potential is then determined by the energy derivative of the topmost state of the completely decoupled system, \(v(\Phi) = dE_N/d\Phi\).

Integration gives

\[
e(U_s - U_r) = E_{r,N}(\Phi) - E_{r,N}(0) + eU_0
\]

(19)

where \(U_0\) is the potential difference between the conduction band bottoms of the ring and the stub at zero flux. Using Eq. (19) in Eq. (11) we find that for the topmost occupied level \(\Delta = ((E_s - E_{r,N}(0) - eU_0)^2 + 4|t|^2)^{1/2}\) is independent of flux.
Let us now show that in the limit of vanishing capacitance the persistent current of the interacting system is equal to that of the decoupled system. At $kT = 0$ the total energy for the interacting system is the sum of the kinetic energy (see Eq. (1)) and the interaction energy $E_c$,

$$E_{tot}(\Phi) = \sum_{n=1}^{N+M} E_n^h - Q_s U_s - eQ_r U_r + E_c. \quad (20)$$

Evaluating this energy we find

$$E_{tot}(\Phi) = \sum_{m=1}^{M} E_{s,m} + \sum_{n=1}^{N-1} E_{r,n} (\Phi) + E^h_r (\Phi) - eU_r (\Phi) - Q(U_s - U_r) + E_c. \quad (21)$$

But for very small capacitance the charge imbalance $Q$ vanishes and the interaction energy is a flux independent constant. Furthermore, the difference of the hybridized energy of the topmost state and $eU_r (\Phi)$ is $E^h_r (\Phi) - eU_r (\Phi) = E_{r,N} (\Phi) + E_0$ with $E_0$ a flux independent energy. Up to flux independent terms the total energy is that of the completely decoupled ring. Thus in the limit of small capacitance the persistent current of the interacting system is the same as that of the original, completely decoupled system. This is a consequence of the suppression of level hybridization through Coulomb interaction.

Below we investigate the characteristic potential $\nu$ for conductors which are open (see Fig. 1), i.e. connected to electron-reservoirs. This permits us to investigate the capacitances and permits us to investigate the effect on the characteristic potential of other nearby metallic bodies.

**IV. CHARACTERISTIC POTENTIALS FOR AN OPEN NORMAL LOOP**

Consider the open conductors of Fig. 1. We now relate the characteristic potentials for these conductors to electron densities. A variation in the electro-chemical potential $\mu_k = E_{F,k}(r) + eU(r)$ by $d\mu_k$ can be accomplished in two ways: We can either increase the Fermi energy by $dE_{F,k} = d\mu_k$ or the electric potential by $edU_k = d\mu_k$. We imagine a two step process: in the first step we increase the chemical potential in reservoir $k$ by
keeping the electro-static potential fixed. As a consequence, an additional charge density \( (dn(r, k)/dE)_U d\mu_k \) is injected into the conductor. In a second step we switch on the Coulomb interaction. The added charges create an additional induced electrical potential which in turn gives rise to an induced charge distribution \( dn_{ind}(r) \). Thus the total change in charge density due to an electro-chemical potential variation in conductor \( k \) is \( dn_k(r) = (dn(r, k)/dE)_U d\mu_k + dn_{ind,k}(r) \). Similarly, the variation of the electron density due to a change in flux consists of two contributions: We first evaluate the change in electron density keeping the electric potential fixed. In a second step we evaluate the potential due the polarization of the sample caused by the increase in flux and calculate the contribution of this potential to the variation in charge density \( dn_k(r) = (dn(r, k)/d\Phi)_U d\Phi + dn_{ind,k}(r) \).

In the presence of a variation of the electro-chemical potential and a variation in flux we have thus

\[
dn_k(r) = \left( \frac{dn_k(r)}{dE} \right)_U d\mu_k + \left( \frac{dn_k(r)}{d\Phi} \right)_U d\Phi + dn_{ind,k}(r). \tag{22}
\]

The induced density \( dn_{ind,k}(r) \) in conductor \( k \) generated by a variation in the electro-static potential \( dU(r) \) can be specified by the Lindhard function (or polarization function) \( \Pi_k(r, r') \),

\[
dn_{ind,k}(r) = - \int d^3r' \Pi_k(r, r') dU(r') \tag{23}
\]

with a potential given by Eq. (2). In Eq. (23) the integral \( d^3r' \) can be taken over all space enclosed by a volume which contains both conductors including a portion of the reservoirs [2]. (This convention applies also to subsequent volume integrals in this work). To obtain an overall charge neutral system the volume has to be chosen so large that no electric field lines penetrate its surface. Eq. (23) is just the continuous space analog of the Lindhard function specified by Eq. (15) for the closed system. Invariance of the charge distribution under simultaneous changes in all electro-chemical potentials [1,2] implies that the integral over the first or the second spatial argument of the Lindhard function is equal to the density of states in conductor \( k \),

\[
(dn_k(r)/dE)_U = \int d^3r' \Pi_k(r, r'). \tag{24}
\]

13
The density response described by the Lindhard function is a consequence of a change in the equilibrium potential. Since an equilibrium density is an even function of magnetic flux, the Lindhard function is also an even function of the magnetic flux, $\Pi_k(\Phi, \mathbf{r}', \mathbf{r}) = \Pi_k(-\Phi, \mathbf{r}', \mathbf{r})$. This has the consequence that the electro-chemical capacitance is an even function of the magnetic flux. We emphasize that these symmetry properties are characteristic for conductors connected to a single reservoir: Reciprocity symmetries apply for multiprobe conductors [2,4].

In the insulator (index $k = 0$) separating the conductors, a potential variation can polarize the insulator and induce a charge density $dn_{\text{ind}0}(\mathbf{r}) = -\int d^3r' \Pi_0(\mathbf{r}, \mathbf{r}') dE(\mathbf{r}')$. No external charges reach the insulating region and thus instead of Eq. (24) we find that the volume integral of the Lindhard function over either the first or the second argument vanishes, $\int d^3r' \Pi_0(\mathbf{r}, \mathbf{r}') = \int d^3r' \Pi_0(\mathbf{r}', \mathbf{r}) = 0$. Note that the Lindhard function which we introduced for the closed loop (see Eq. (14)) also has this property.

Next we write down Poisson’s equation for the potential $U$. If we expand $U$ with respect to $d\mu_k$ we find that the characteristic function $u_k$ is determined by [2]

$$
-\Delta u_k(\mathbf{r}) + 4\pi e^2 \int d^3r' \sum_{l=0}^{l=3} \Pi_l(\mathbf{r}, \mathbf{r}') u_k(\mathbf{r}') = 4\pi e^2 (dn_k(\mathbf{r})/dE)_U. \tag{25}
$$

Eq. (25) contains the sum of all Lindhard functions of all the conductors $l = 1, 2$ and of the insulating region $l = 0$. The density of states of conductor $k$ plays the role of a source term for the characteristic function $u_k$. Similarly the characteristic function $v(\mathbf{r})$ is a solution of the Poisson equation

$$
-\Delta v(\mathbf{r}) + 4\pi e^2 \int d^3r' \sum_{l=0}^{l=3} \Pi_l(\mathbf{r}, \mathbf{r}') v(\mathbf{r}') = 4\pi e^2 (dn_1(\mathbf{r})/d\Phi)_U. \tag{26}
$$

The flux-sensitivity of the electron density in the conductor with the loop is the source term of the characteristic function $v$. If the source term is replaced by a test charge $e\delta(\mathbf{r} - \mathbf{r}_0)$ which is concentrated at one point $\mathbf{r}_0$ the solution to Eq. (25) and Eq. (26) is Green’s function $g(\mathbf{r}, \mathbf{r}_0)$. With the help of Green’s function we find for the characteristic function,

$$
u_k(\mathbf{r}) = \int d^3r' g(\mathbf{r}, \mathbf{r}') (dn_k(\mathbf{r}')/dE)_U. \tag{27}$$

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Similarly the characteristic function \( v(r) \) is given by

\[
v(r) = \int d^3r' g(r,r')(dn_1(r')/d\Phi)_U. \tag{28}
\]

Eq. (2) implies for Green’s function the property \( 2 \)

\[
\int d^3r' g(r,r') \sum_k (dn_k(r')/dE)_U = 1. \tag{29}
\]

The same relationship follows from the condition that the sum of all induced charge densities plus the test charge is zero. Eq. (29) will be used to demonstrate charge and current conservation of the results derived below.

V. ELECTRO-CHEMICAL AND FLUX-INDUCED CAPACITANCE

The characteristic potentials derived above can now be used to find the electro-chemical capacitance and a flux-induced capacitance. Using Eqs. (22), (23), and (27), the total charge in conductor \( k \) can be expressed in terms of density of states and Green’s function. Differentiating the total charge \( dQ_k \) with respect to the voltage \( dV_l = d\mu_l/e \) gives an electro-chemical capacitance \( 2 \)

\[
C_{kl} = e^2 \int d^3r \int d^3r' (dn_k(r)/dE)_U (\delta_{kl}\delta (r - r') - g(r,r')(dn_l(r')/dE))_U. \tag{30}
\]

Eq. (30) expresses the capacitances in terms of the density of states of the reference state and Green’s function which mediates the Coulomb interactions. Conservation of charge relates the capacitance coefficients of our two-terminal conductor as follows: \( C_{11} = C_{22} = -C_{12} = -C_{21} \). To see this one makes use of Eq. (29). Below we will use the abbreviation \( C_\mu = C_{11} \) where the index \( \mu \) reminds us that we deal not with an electrostatic capacitance but with an electro-chemical capacitance.

Next, in analogy to the electro-chemical capacitance coefficients just discussed we consider the flux induced capacitance \( C_{\Phi,k} = dQ_k/d\Phi \). This is the ratio of the piled up charge in conductor \( k \) and the increment in flux \( \Phi \). We find the flux-induced capacitances
\[ C_{\Phi,k} = e \int d^3r \int d^3r' (\delta_1 \delta(r - r') - (dn_k(r)/dE) g(r, r') \delta_1 (r) (dn_k(r')/d\Phi) \delta_1 (r) \] 

(31)

These two coefficients are related. Charge conservation implies that the sum of these two coefficients is zero. This can again be demonstrated by using Eq. (31). Hence we are left with one coefficient only and denote it by \( C_{\Phi} \), \( C_{\Phi} = C_{\Phi,1} = -C_{\Phi,2} \). Whereas the electro-chemical capacitance is an even (and periodic) function of flux the flux-induced capacitance is an odd (and periodic) function of flux.

Consider now for a moment the conductor of Fig. 1a which is in an electrically neutral environment. Then according to Eq. (29) the spatial integral over \( r \) of Green’s function and the density of states is just equal to 1. Hence in this case both the capacitance \( C_\mu \) and \( C_\Phi \) vanish.

In the presence of time-dependent chemical potentials and in the presence of a time-dependent oscillating flux the current \( I_1(t) \) measured at terminal 1 is determined by \( I_1(t) = dQ_1/dt \) and the current at terminal 2 (the gate) is determined by \( I_2 = dQ_2/dt \). Since charge is conserved, the current is conserved, \( I_1(t) + I_2(t) = 0 \). The current depends only on the difference of the chemical potentials. Taking the oscillating chemical potential difference to be \( d\mu_1 - d\mu_2 = eV_d \exp(-i\omega t) \) and keeping the flux fixed gives rise to a current \( dI_\omega = -i\omega C_\mu dV_\omega \). Measurement of this current in a zero-impedance external circuit determines the electro-chemical capacitance. An oscillatory flux with Fourier component \( d\Phi_\omega \exp(-i\omega t) \) gives a current \( dI_\omega = -i\omega C_\Phi d\Phi_\omega \) in a zero-impedance external circuit.

If the external circuit has a non-vanishing impedance an oscillatory flux will also lead to an oscillating voltage. In this case a total induced current

\[ dI_\omega = -i\omega C_\mu dV_\omega - i\omega C_\Phi d\Phi_\omega \]  

(32)

is generated. The voltage \( dV_\omega \) depends on the external impedance. For a circuit with impedance \( Z(\omega) = -dV_\omega/dI_\omega \) an oscillatory flux generates a voltage

\[ dV_\omega = -\frac{i\omega C_\Phi}{1/Z_\omega} d\Phi_\omega. \]  

(33)
Since we deal with a capacitive effect it is useful to measure the voltage capacitively and take the external impedance to be capacitive $Z(\omega) = 1/(-i\omega C_{\text{ext}})$. In this case the measured voltage is independent of frequency and given by

$$dV_{\omega} = \frac{C_\Phi}{C_\mu + C_{\text{ext}}} d\Phi_\omega.$$  (34)

Our discussion of the conductor in Fig. 2 has shown that the charge displaced by the flux is at best one electronic charge and is less if interactions are taken into account. If this is indicative also of the open conductor in Fig. 1 then the magnitude of the flux-induced capacitance is of the order of $e/\Phi_0$ or $C_\Phi = e^2/hc$. This is equivalent to a "conductance" $G_\Phi = cC_\Phi = e^2/h$ equal to the fundamental conductance unit. Screening will reduce these values but both the measurement of the electro-chemical capacitance and the flux-induced capacitance should be feasible.

In this work we have discussed the response of the internal electro-static potential of mesoscopic conductors to small changes in an external parameter (flux, electro-chemical potential) with the help of characteristic functions. The characteristic potential $v$ is responsible for the enhancement of the persistent current above a value achieved in the non-interacting system. This potential also determines a flux-induced capacitance. The characteristic function $u$ determines the electro- chemical capacitance. Measurement of these capacitances can, therefore, provide an experimental means to determine the sensitivity of the electric potential.

**VI. NOTE ADDED IN PROOF**

To obtain an enhancement of the persistent current in situations different from that shown in Fig. 3 it is in general necessary that the stub acts like an *acceptor* in the hybridized system. The stub acts like an acceptor (rather than a donor) if its density of states is much larger then that of the loop. This corresponds to the situation in diffusive multichannel rings where we have many localized states and only a few highly mobile states.
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REFERENCES

[1] Büttiker, M., Thomas, H., and Prêtre, A., Phys. Lett. A180, 364 (1993): Büttiker, M., Prêtre, A., and Thomas, H., Z. Phys. - Condens. Matter B94, 133 (1994).

[2] Büttiker, M., J. Phys.: Condens. Matter 5, 9631, (1993).

[3] Fomin, N. V., Sov. Phys. - Solid State 9, 474 (1967); Stern, F., ”Low Temperature Capacitance of Inverted Silicon MOS Devices in High Magnetic Fields”, IBM Research Report, RC 3758, March 2, (1972).

[4] Chen, W., Smith, T., Büttiker, M., and Shayegan, M., Phys. Rev. Lett. (unpublished).

[5] Büttiker, M., Imry, Y., Landauer, R., Phys. Lett. A96, 365 (1983).

[6] Pauling, L., J. of Chem. Phys. 4, 673 (1936).

[7] London, F. J., Phys. Radium 8, 397 (1937).

[8] Hund, F., Ann. Phys. (Leipzig), 32, 102, (1938).

[9] Beyers, N., Yang, C., N.,=20 Phys. Rev. Lett., 46, 7 (1961).

[10] Bloch, F., Phys. Rev. 137, 787 (1965).

[11] Landauer, R., Büttiker, M., Phys. Rev. Lett. 54, 2049 (1985).

[12] Büttiker, M., Phys. Rev. B32, 1846 (1985).

[13] Büttiker, M., Klapwijk, T. M., Phys. Rev. B33, 5114 (1986).

[14] Cheung, H. F., Gefen, Y., Riedel, E. K., IBM J. Res. Develop. 32, 379 (1988).

[15] Bouchiat, H., Montambaux, G., J. Phys., (Paris) 50, 2695 (1989).

[16] Levy, L. P., Dolan, G., Dunsmuir, J., and Bouchiat, H., Phys. Rev. Lett. 64, 2074 (1990).

[17] Chandrasekhar, V., Webb, R. A., Brady, M. J., Ketchen, M. B., Gallagher, W. J., and
Kleinsasser, A., Phys. Rev. Lett. 67, 3578 (1991).

[18] Mailly, D., Chapelier, C., Benoit, A., Phys. Rev. Lett. 70, 2020 (1993).

[19] Altshuler, B. L., Gefen, Y., Imry, Y., Phys. Rev. Lett. 66, 88 (1991).

[20] von Oppen, F., Riedel, E. K., Phys. Rev. Lett. 66, 84 (1991).

[21] Akkermans, E., Auerbach, A., Avron, J. E., and Shapiro, B., Phys. Rev. Lett. 66, 76 (1991).

[22] Schmid, A., Phys. Rev. Lett. 66, 80 (1991).

[23] Loss, D. and Martin, Th., Phys. Rev. B47, 4619 (1993).

[24] Kopietz, P., Phys. Rev. Lett. 70, 3123 (1993): G. Vignale, Phys. Rev. Lett. 72, 433 (1994).

[25] Argaman, N., and Imry, Y., Physica Scripta T49, 333 (1993).

[26] Müller-Groeling, A., and Weidenmüller, H. A., Phys. Rev. B49, 4752 (1993).

[27] Mello, P., Phys. Rev. B47, 16358 (1993).

[28] Singha Deo, P. and Jayannavar, A. M., Mod. Phys. Lett. B7, 1045 (1993): Jayannavar, A. M., and Singha Deo, P., Phys. Rev. B49, 13685 (1994).

[29] Kohn, W., and Sham, L. J., Phys. Rev. 140, A1133 (1965): Hohenberg, P., and Kohn, W., Phys. Rev. 136, B864 (1964).

[30] Lang, N. D., in "Theory of the Inhomogeneous Electron Gas", edited by S. Lundquist and N. H. March (Plenum Press, New York, 1983). p. 309.

[31] M. Büttiker, (unpublished).
FIGURES

FIG. 1. (a) Ring with a lead connected to an electron reservoir with an electro-chemical potential $\mu_1$. (b) The ring is separated by an insulator from a second conductor (gate) which is connected to an electron-reservoir at an electro-chemical potential $\mu_2$.

FIG. 2. Normal electron ring which is via a tunneling barrier with capacitance $C$ connected to a short wire (stub) of finite length.

FIG. 3. Energy spectrum as a function of flux for the conductor of Fig. 2. (a) The ring and the stub are completely disconnected. (b) The barrier between the ring and the stub is transparent. Coulomb interactions are not taken into account.

FIG. 4. Persistent current $I_N$ (solid line) and $I_{N-1}$ (dashed line) of the completely disconnected system with $N$ and $N - 1$ electrons in the ring. For a transparent barrier and in the absence of interactions the persistent current is $I^h$ (dotted line). $\Phi_*$ is the flux at which the topmost energy level of the loop hybridizes with a localized state of the stub. In the presence of interactions the persistent current $I_N$ is recovered even for a transparent barrier.