JOIN-IRREDUCIBLE BOOLEAN FUNCTIONS

MONCEF BOUAZIZ\textsuperscript{1}, MIGUEL COUCEIRO\textsuperscript{2}, AND MAURICE POUZET\textsuperscript{3}

Abstract. This paper is a contribution to the study of a quasi-order on the set $\Omega$ of Boolean functions, the simple minor quasi-order. We look at the join-irreducible members of the resulting poset $\check{\Omega}$. Using a two-way correspondence between Boolean functions and hypergraphs, join-irreducibility translates into a combinatorial property of hypergraphs. We observe that among Steiner systems, those which yield join-irreducible members of $\check{\Omega}$ are the $-2$-monomorphic Steiner systems. We also describe the graphs which correspond to join-irreducible members of $\check{\Omega}$.

1. Introduction

Two approaches to Boolean function definability have been considered in recent years; one in terms of functional equations \cite{EFHH}, and one other in terms of relational constraints \cite{Di}. As it turned out, these two approaches have the same expressive power in the sense that they specify exactly the same classes (or properties) of Boolean functions. The characterization of these classes was first obtained by Ekin, Foldes, Hammer and Hellerstein \cite{EFHH} who showed that equational classes of Boolean functions can be completely described in terms of a quasi-ordering $\leq$ of the set $\Omega$ of all Boolean functions, called identification minor in \cite{EFHH,He}, simple minor in \cite{Di,Ar,Co,Gu}, subfunction in \cite{Hu}, and simple variable substitution in \cite{Bo}. This quasi-order can be described as follows: for $f, g \in \Omega$, $g \leq f$ if $g$ can be obtained from $f$ by identification of variables, permutation of variables, and addition or deletion of dummy variables. As shown in \cite{EFHH}, equational classes of Boolean functions coincide exactly with the initial segments $\downarrow K = \{ g \in \Omega : g \leq f, \text{ for some } f \in K \}$ of this quasi-order, or equivalently, they correspond to antichains $A$ of Boolean functions in the sense that they constitute sets of the form $\Omega \setminus \uparrow A$. Moreover, those equational classes definable by finitely many equations correspond to finite antichains of Boolean functions. Since then, several investigations have appeared in this direction, to mention a few, see \cite{Bo,Ch,Di,Di1,Di2,Di3}.

The importance of Boolean function definability led to a greater emphasis on this quasi-ordering $\leq$ \cite{Ar,Co,Gu}. As any quasi-order, the simple minor relation $\leq$ induces a partial order $\sqsubseteq$ on the set $\check{\Omega}$ made of equivalence classes of Boolean functions. Several properties of the resulting poset $(\check{\Omega}, \sqsubseteq)$ were established in \cite{Ar}. In particular, it was shown that this poset is as complex as $(\omega^{<\omega}, \subseteq)$, the.

\textsuperscript{1}Institut Supérieur des Technologies Médicales de Tunis, \textsuperscript{2}University of Luxembourg, \textsuperscript{3}Université Claude-Bernard Lyon1 and The University of Calgary, Calgary, Alberta, Canada

\textit{Date}: March 23, 2009.

\textit{2000 Mathematics Subject Classification.} Combinatorics (05C75), (05C65), (05B05), (05B07), Order, lattices, ordered algebraic structures (06A07), (06E30), Information and communications, circuits (94C10).

\textit{Key words and phrases.} Boolean function, minor quasi-order, hypergraph, designs, Steiner systems, monomorphy.

The research of the first and last author has been supported by CMCU Franco-Tunisien "Outils mathématiques pour l’informatique".
In this paper we are interested in determining the join-irreducible members of the poset \((\Omega, \sqsubseteq)\), that is, those equivalence classes having a unique lower cover in \((\Omega, \sqsubseteq)\). Rather than taking a direct approach, we attack this problem by looking at hypergraphs. As it is well-known, every Boolean function can be represented by a unique multilinear polynomial over the two-element field \(GF(2)\), that is, a polynomial in which each variable has degree at most one (see Zhegalkin [24]). This polynomial representation of Boolean functions allows the two-way correspondence between Boolean functions and hypergraphs.

For any hypergraph \(H = (V, E)\) we associate the multilinear polynomial \(P_H \in GF(2)[X]\), where \(X = (x_i : i \in V)\), given by \(P_H = \sum_{E \in E} \prod_{i \in E} x_i\). In fact, every multilinear polynomial \(P \in GF(2)[X]\) is of the form \(P = P_H\) where \(H = (V, E)\) and \(E\) is the set of hyperedges corresponding to the monomials of \(P\). The simple minor relation translates into the realm of hypergraphs through the notion of quotient map. Say that a map \(h' : V' \to V\) is a quotient map from \(H' = (V', E')\) to \(H = (V, E)\) if for every \(E \subseteq V, E \in E\) \iff \(|\{E' \in E' : h'(E') = E\}|\) is odd. For two hypergraphs \(H\) and \(H'\) set \(H \preceq H'\) if there is a quotient map from \(H'\) to \(H\). As we are going to see \(\preceq\) constitutes a quasi-order between hypergraphs and two hypergraphs are related by \(\preceq\) if and only if the corresponding Boolean functions are related by \(\leq\) (see Lemma [13] and Theorem [18] resp.).

The fact that a Boolean function corresponds to a join-irreducible of the poset \((\Omega, \sqsubseteq)\) translates into a combinatorial property of the corresponding hypergraph. A description of all hypergraphs satisfying this property eludes us. However, among these hypergraphs some have been intensively studied for other purposes. The basic examples are the non-trivial hypergraphs whose automorphism group is 2-set transitive. We show that Steiner systems which yield join-irreducible members of \((\Omega, \sqsubseteq)\) are exactly those which are -2-monomorphic in the sense that the induced hypergraphs obtained by deleting any pair of two distinct vertices are isomorphic (Theorem [22]). Among Steiner triple systems those with a flag-transitive automorphism group enjoy this property. We do not know if there are other. We also describe those graphs corresponding to join-irreducible members of \((\Omega, \sqsubseteq)\) (Theorem [32]). In doing so, we show that all the lower covers of each Boolean function \(f\) have the same essential arity (Theorem [8]). By a result of A.Salomaa ([24], Theorem 4, p.7) it follows that this essential arity is either ess \(f - 1\) or ess \(f - 2\), where ess \(f\) is the essential arity of \(f\). In the latter case, \(f\) has (up to equivalence) a unique lower cover. This follows from Theorem [6] (first shown in [15]) which provides an explicit description of those functions whose arity gap is two.

Some of the results in this paper were presented at the ROGICS’08 conference May 12-17, Mahdia (Tunisia) [2]. The authors would like to express their gratitude to the organizers, Professors Y.Boudabbous and N.Zaguia.

2. Boolean functions

A Boolean function is simply a mapping \(f : \{0, 1\}^n \to \{0, 1\}\), where \(n \geq 1\). The integer \(n\) is called the arity of \(f\). As simple examples of Boolean functions we have the projections, i.e., mappings \((a_1, \ldots, a_n) \mapsto a_i\), for \(1 \leq i \leq n\) and \(a_1, \ldots, a_m \in \{0, 1\}\), and which we also refer to as variables. For each \(n \geq 1\), we denote by \(\Omega^{(n)} = \{0, 1\}^{(0,1)^n}\) the set of all \(n\)-ary Boolean functions and we denote by \(\Omega = \bigcup_{n \geq 1} \Omega^{(n)}\) the set of all Boolean functions.
Let $GF(2)$ be the two-element field $\{0,1\}$ and let $GF(2)[x_1,\ldots,x_n]$ be the commutative ring of polynomials in the indeterminates $x_1,\ldots,x_n$. To each polynomial $P \in GF(2)[x_1,\ldots,x_n]$ there corresponds an $n$-ary Boolean function $f_P : \{0,1\}^n \to \{0,1\}$ which is given as the evaluation of $P$, that is, for every $(a_1,\ldots,a_n) \in \{0,1\}^n$, $f_P(a_1,\ldots,a_n) = P(a_1,\ldots,a_n)$. The function $f_P$ is said to be represented by $P$. As it is well-known every Boolean function can be represented in this way. In fact:

**Theorem 1.** Every Boolean function $f : \{0,1\}^n \to \{0,1\}$, $n \geq 1$, is uniquely represented by a multilinear polynomial $P \in GF(2)[x_1,\ldots,x_n]$ in which each variable has degree at most one.

The multilinear polynomial $P$ is called Zhegalkin (or Reed–Muller) polynomial of $f$ [19, 22, 24].

A variable $x_i$ is an essential variable of a Boolean function $f$ if $f$ depends on its $i$-th argument, that is, if there are $a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n \in \{0,1\}$ such that the unary function $x \mapsto f(a_1,\ldots,a_{i-1},x,a_{i+1},\ldots,a_n)$ is nonconstant. By essential arity of a function $f \in \Omega^{(n)}$, denoted $\text{ess } f$, we simply mean the number of its essential variables. For instance, constant functions are exactly those functions with essential arity 0. Functions with essential arity 1 are either projections or negated projections. From Theorem 1 we have the following corollary.

**Corollary 2.** A variable $x_i$ is essential in $f \in \Omega^{(n)}$ if and only if $x_i$ appears in the Zhegalkin polynomial of $f$. In particular, $\text{ess } f$ is the number of variables appearing in the Zhegalkin polynomial of $f$.

2.1. **Simple minors of Boolean functions.** A Boolean function $g \in \Omega^{(m)}$ is said to be a simple minor of a Boolean function $f \in \Omega^{(n)}$ if there is a mapping $\sigma : \{1,\ldots,n\} \to \{1,\ldots,m\}$ such that

$$g(a_1,\ldots,a_m) = f(a_{\sigma(1)},\ldots,a_{\sigma(m)}),$$

for every $a_1,\ldots,a_m \in \{0,1\}$. If $\sigma$ is not injective, then we speak of identification of variables. If $\sigma$ is not surjective, then we speak of addition of inessential variables. If $\sigma$ is a bijection, then we speak of permutation of variables. As it is easy to verify, these Mal’cev operations are sufficient to completely describe the simple minor relation.

**Fact 3.** The simple minor relation between Boolean functions is a quasi-order.

Let $\leq$ denote the simple minor relation on the set $\Omega$ of all Boolean functions. If $g \leq f$ and $f \leq g$, then we say that $f$ and $g$ are equivalent, denoted $f \equiv g$. The equivalence class of $f$ is denoted by $\bar{f}$. If $g \leq f$ but $f \not\leq g$, then we use the notation $g < f$. The arity gap of $f$, denoted $\text{gap } f$, is defined by $\text{gap } f = \min\{\text{ess } f - \text{ess } g : g < f\}$. Note that equivalent functions may differ in arity, but not in essential arity nor in arity gap.

**Fact 4.** If $g \leq f$, then $\text{ess } g \leq \text{ess } f$, with equality if and only if $g \equiv f$.

By Corollary 2 in the case of polynomial expressions, to describe the simple minor relation we only need to consider identification and permutation of essential variables, since the operation of addition of inessential variables produces the same polynomial representations. Moreover, from Fact 3 it follows that the strict minors of a given function $f$ have Zhegalkin polynomials with strictly less variables, and that the Zhegalkin polynomials of functions equivalent to $f$ are obtained from the Zhegalkin polynomial of $f$ by permutation of its variables. For further developments see [7].
Let \((\tilde{\Omega}, \subseteq)\) denote the poset made of equivalence classes of Boolean functions associated with the simple minor relation, that is, \(\tilde{\Omega} = \Omega / \equiv\) together with the partial order \(\subseteq\) given by \(\tilde{g} \subseteq \tilde{f}\) if and only if \(g \leq f\). Several properties of this poset were established in [8]. For example, Fact 4 implies that each principal initial segment \(\downarrow f = \{\tilde{g} : \tilde{g} \subseteq f\}\) is finite. This means that \((\tilde{\Omega}, \subseteq)\) decomposes into levels \(\tilde{\Omega}_0, \ldots, \tilde{\Omega}_n, \ldots\), where \(\tilde{\Omega}_n\) is the set of minimal elements of \(\tilde{\Omega}\) \(\cup \{\tilde{\Omega}_m : m < n\}\). For instance, the first level \(\tilde{\Omega}_0\) comprises four equivalence classes, namely, those of constant 0 and 1 functions, and those of projections and negated projections. These four classes induce a partition of \((\tilde{\Omega}, \subseteq)\) into four different blocks with no comparabilities in between them. These facts were observed in [8] where it was shown that each level of \((\tilde{\Omega}, \subseteq)\) is finite ([8], Corollary 1, p.75). The latter is entailed by the following result by A. Salomaa [23].

**Theorem 5.** The arity gap of any Boolean function is at most 2.

The description of those Boolean functions with arity gap 2 is given below.

**Theorem 6.** (In [5]:) Let \(f : \{0, 1\}^n \rightarrow \{0, 1\}\) be a Boolean function with at least two essential variables. Then the arity gap of \(f\) is two if and only if it is equivalent to one of the following functions:

1. \(x_1 + x_2 + \cdots + x_m + c\) for some \(m \geq 2\),
2. \(x_1 x_2 + x_1 + c\),
3. \(x_1 x_2 + x_1 x_3 + x_2 x_3 + c\),
4. \(x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 + x_2 + c\),

where \(c \in \{0, 1\}\) and where \(+\) is taken modulo 2. Otherwise the arity gap of \(f\) is one.

Theorem 6 allows to work with polynomials rather than Boolean functions. This approach turns out to be quite useful when studying the poset \((\tilde{\Omega}, \subseteq)\). For instance, the four equivalence classes in \(\tilde{\Omega}_0\) mentioned above are represented by 0, 1, \(x_1\) and \(x_1 + 1\). As it is easy to verify, above the equivalence classes represented by the constant polynomials 0 or 1 we have the equivalence classes of those functions whose Zhegalkin polynomials are the sum of an even number of nonconstant monomials plus 0 or 1, respectively, and above the equivalence classes represented by \(x_1\) or \(x_1 + 1\) we have the equivalence classes of those functions whose Zhegalkin polynomials are the sum of an odd number of nonconstant monomials plus 0 or 1, respectively.

### 2.2. Join-irreducible Boolean functions.

We say that an element \(\tilde{f} \in \tilde{\Omega}\) is **join-irreducible** if there is \(\tilde{f}' \in \tilde{\Omega}\) such that \(\tilde{f}' \subseteq \tilde{f}\) and for every \(\tilde{g} \in \tilde{\Omega}\), if \(\tilde{g} \subseteq \tilde{f}\), then \(\tilde{g} \subseteq \tilde{f}'\). Since \(\tilde{\Omega}\) decomposes into levels this amounts to say that \(\tilde{f}\) has a unique lower cover. For the sake of simplicity, we say that a function \(f \in \Omega\) is **join-irreducible** if \(\tilde{f}\) is join-irreducible. Likewise, we say that \(g\) is a lower cover of \(f\) if \(\tilde{g}\) is a lower cover of \(\tilde{f}\).

To illustrate, consider the binary conjunction \(x_1 \land x_2\), the binary disjunction \(x_1 \lor x_2\). Both of these functions are join-irreducible since they have, up to equivalence, a unique strict simple minor, namely, a projection. This uniqueness clearly extends to any conjunction and disjunction of \(n \geq 2\) variables, showing that any of the latter functions also constitute join-irreducible functions. But this is not the case for the composite \((x_1 \lor x_2) \land x_3, x_1 \land x_3 \land x_4 < (x_1 \lor x_2) \land x_3 \land x_4, \) but \((x_1 \lor x_2) \land x_3 \neq x_1 \land x_3 \land x_4\). These observations lead to the following problem.

**Problem 1.** Describe the join-irreducible Boolean functions.
Let $f: \{0, 1\}^n \to \{0, 1\}$ be a Boolean function and let $i, j \in \{1, \ldots, n\}$. We denote by $f_{i=j}$ the function obtained from $f$ by identifying the variable $x_i$ to the variable $x_j$ with the convention that $f_{i=i} = f$.

**Lemma 7.** If $g < f$, then there are two distinct essential variables $i$ and $j$ of $f$ such that $g \leq f_{i=j} < f$.

Using Lemma 7 we see that each of the functions given in Theorem 8 is join-irreducible since each has, up to equivalence, a unique lower cover, namely, $c$ in case (2) and $x + c$ in cases (1), (3) and (4). Thus to solve Problem 11 we need to focus on those functions whose arity gap is equal to one. Towards this problem we will make use of the following result.

**Theorem 8.** All lower covers of a given function $f$ have the same essential arity which is either $\text{ess } f - 1$ or $\text{ess } f - 2$. In the latter case, $f$ is join-irreducible.

To prove Theorem 8 we make use of the following auxiliary properties of a Boolean function $f: \{0, 1\}^n \to \{0, 1\}$.

**Lemma 9.** Let $i_1, i_2, t \in \{1, \ldots, n\}$ such that $i_1 \neq i_2$. If $x_{t}$ is inessential in $f_{i_1=i_2}$, then $f_k \geq f_{i_1=i_2, k} = f_{i_1=i_2}$, for all $k \in \{1, \ldots, n\}$.

**Lemma 10.** If $f_{i_1=k} \equiv f_{i_2=k}$ and $x_{k}$ is essential in $f_{i_1=k}$ and in $f_{i_2=k}$, then $x_{i_1}$ is inessential in $f_{i_2=k}$ if and only if $x_{i_2}$ is inessential in $f_{i_1=k}$.

**Proof.** Suppose that $x_{i_1}$ is inessential in $f_{i_2=k}$. By Lemma 9, $f_{i_2=k, i_1=k} \equiv f_{i_2=k}$. Suppose for the sake of contradiction that $x_{i_2}$ is essential in $f_{i_1=k}$. Then, since $x_{k}$ is essential in $f_{i_1=k}$ and in $f_{i_2=k}$, we have $f_{i_1=k, i_2=k} < f_{i_2=k}$. Hence, $f_{i_2=k} < f_{i_1=k}$ which constitutes the desired contradiction. \qed

Note that the hypotheses of Lemma 10 are satisfied by $f = x_1x_2 + x_1x_3 + x_2x_3$.

**Lemma 11.** Suppose that $i_1, i_2, k$ are distinct indices in $\{1, \ldots, n\}$ such that $x_{i_1}$ is inessential in $f_{i_2=k}$ and $x_{i_2}$ is inessential in $f_{i_1=k}$. If $x_{i_1}$ is inessential in $f_{i_2=i_1}$, then $x_{i_1}$ is inessential in $f$.

**Proof.** Without loss of generality, suppose that $i_1 = 1, i_2 = 2$ and $k = 3$. Let $(a_1, \ldots, a_n) \in \{0, 1\}^{n-3}$ and let $g: \{0, 1\}^3 \to \{0, 1\}$ be the function given by $g(x_1, x_2, x_3) = f(x_1, x_2, x_3, a_1, \ldots, a_n)$. We show that

\begin{equation}
\tag{1}
g(a_1, a_2, a_3) = g(1 + a_1, a_2, a_3)
\end{equation}

for all $(a_1, a_2, a_3) \in \{0, 1, 2\}^3$. Indeed, if $a_2 = a_3$, then

\begin{equation}
\tag{1}
g(a_1, a_2, a_3) = g(1 + a_1, a_2, a_3) = g(1 + a_1, a_2, a_3)
\end{equation}

since $x_1$ is inessential in $f_{2=3}$.

Now suppose that $a_2 = 1 + a_3$. If $a_1 = a_3$, then

\begin{equation}
\tag{1}
g(a_1, a_2, a_3) = g(a_3, 1 + a_3, a_3) = g(a_3, a_3, a_3) = g(a_1, 1 + a_2, a_3)
\end{equation}

since $x_2$ is inessential in $f_{1=3}$, and we have

\begin{equation}
\tag{1}
g(a_1, 1 + a_2, a_3) = g(a_3, a_3, a_3) = g(1 + a_3, 1 + a_3, a_3) = g(1 + a_1, a_2, a_3)
\end{equation}

because $x_1$ is inessential in $f_{2=1}$. Thus (1) holds.

If $a_1 = 1 + a_3$, then

\begin{equation}
\tag{1}
g(a_1, a_2, a_3) = g(1 + a_1, 1 + a_3, a_3) = g(a_3, a_3, a_3) = g(1 + a_1, 1 + a_2, a_3)
\end{equation}

since $x_1$ is inessential in $f_{2=1}$, and we have

\begin{equation}
\tag{1}
g(1 + a_1, 1 + a_2, a_3) = g(a_3, a_3, a_3) = g(a_3, 1 + a_3, a_3) = g(1 + a_1, a_2, a_3)
\end{equation}

because $x_2$ is inessential in $f_{1=3}$. Thus (1) holds and the proof is now complete. \qed
Lemma 12. Let g be a lower cover of f. If \( \text{ess}(g) < \text{ess}(f) - 1 \), then g is, up to equivalence, the unique lower cover of f.

Proof. With no loss of generality, we may assume that every variable of f is essential. Let \( G := (V, \mathcal{E}) \), where \( V = \{1, \ldots, n\} \) and \( \mathcal{E} := \{\{i, j\} \in [V]^2 : f_{i=j} \equiv g\} \). Since g is a lower cover of f, G is not the empty graph. Our aim is to show that G is a complete graph. Let \( A := \{i \in V : \{i, k\} \in \mathcal{E} \text{ for all } k \in V \setminus \{i\}\} \). Our aim reduces to prove that \( A = V \).

Claim 1. Let \( \{i, j\} \in \mathcal{E} \). Then:

(a) If \( x_1 \) is inessential in \( f_{i=j} \), then \( t \) belongs to A.
(b) If \( j \notin A \) and if \( x_1 \) is inessential in \( f_{i=j} \), for some \( t \in V \setminus \{i\} \), then \( t \neq j \), \( \{t, j\} \in \mathcal{E} \) and \( x_1 \) is inessential in \( f_{t=j} \).
(c) \( i \) and \( j \) belong to A.

Proof of Claim 1. First we prove (a). Let \( k \in V \setminus \{t\} \). According to Lemma 9, \( f_{i=j} \leq f_{i=k} \). Since g is a lower cover of f and all variables of f are essential, \( f_{t=k} \equiv g \), that is, \( \{t, k\} \in \mathcal{E} \). Since this holds for every \( k \in V \setminus \{t\} \), we have \( t \in A \).

To prove (b), suppose that \( j \notin A \). Then from (a) it follows that \( x_1 \) is essential in \( f_{i=j} \). Since g is a lower cover of f and \( x_1 \) is inessential in \( f_{i=j} \), Lemma 9 yields \( f_{i=j} = f_{t=j} \), thus \( \{t, j\} \in \mathcal{E} \). Since \( j \notin A \), it follows from (a) that \( x_1 \) is essential in \( f_{t=j} \). Applying Lemma 10 to f, with \( k := j, i_1 := i, i_2 := t \), yields that \( x_1 \) is inessential in \( f_{t=j} \).

To prove (c) suppose, for the sake of a contradiction, that \( j \notin A \). Hence, \( x_1 \) is essential in \( f_{i=j} \). Since \( \text{ess}(g) < \text{ess}(f) - 1 \), there is some \( t \in V \setminus \{i, j\} \) such that \( x_1 \) is inessential in \( f_{i=j} \) and thus \( t \in A \). From (b), it follows that \( x_1 \) is inessential in \( f_{i=t} \). By Lemma 11, \( x_1 \) is essential in \( f_{i=t} \). Since \( x_1 \) is inessential in \( f_{t=j} \), it follows from Lemma 10 that \( x_1 \) is inessential in \( f_{i=t} \). This implies \( j \in A \), a contradiction.

From (c) it follows that \( A = V \) and the proof of the lemma is complete. \( \square \)

Proof of Theorem 8. By Theorem 5, the essential arity of the lower covers of f are either \( \text{ess } f - 1 \) or \( \text{ess } f - 2 \). By Lemma 12 if there is a lower cover with essential arity equal to \( \text{ess } f - 2 \), then it is, up to equivalence, unique and thus f is join-irreducible.

Set \( \text{Ess } f = \{i \in \{1, \ldots, n\} : x_i \text{ is an essential variable of } f\} \) and let \( [\text{Ess } f]^2 \) be the set of 2-element subsets of \( \text{Ess } f \). For \( e = \{i, j\}, e' = \{i', j'\} \in [\text{Ess } f]^2 \), define \( e \approx e' \) if \( f_{i=j} \equiv f_{i'=j'} \). Obviously, \( \approx \) is an equivalence relation on \( [\text{Ess } f]^2 \) and \( \text{ess } f_{i=j} = \text{ess } f_{i'=j'} \) for all \( e = \{i, j\}, e' = \{i', j'\} \) such that \( e \approx e' \). According to Lemma 7, if a Boolean function g is a lower cover of f, then there is some \( \{i, j\} \in [\text{Ess } f]^2 \) such that \( g \equiv f_{i=j} \). From this observation, we get the following fact.

Fact 13. A Boolean function f is join-irreducible provided that \( \text{ess } f \geq 2 \) and \( [\text{Ess } f]^2 \) is an equivalence class.

To provide our first criterion for join-irreducibility, let \( C_f = \{\{i, j\} \in [\text{Ess } f]^2 : \text{ess } f_{i=j} = \text{ess } f - \text{gap } f\} \). Clearly, for each pair \( \{i, j\} \in C_f \), we have that \( f_{i=j} \) is a lower cover of f, and Theorem 8 asserts that the converse also holds. Hence, \( C_f \) is the union of equivalence classes of pairs \( e = \{i, j\} \in [\text{Ess } f]^2 \) such that \( f_{i=j} \) is a lower cover of f. For example, if \( \text{gap } f = 2 \), then \( C_f \) is an equivalence class, namely, the whole set \( [\text{Ess } f]^2 \) (to see this, use the description given in Theorem 7). These observations yield our first criterion for join-irreducibility.
Lemma 16. Let \( E \) be a set of subsets of \( V \) called hyperedges. We write \([V]^m\) to denote the set of \( m\)-element subsets of \( V \).

Let \( H = (V,E) \) be an hypergraph with \( n \) vertices. To such an hypergraph \( H \) we associate a Zhegalkin polynomial \( P_H \in GF(2)[x_i : i \in V] \) which is given by \( P_H = \sum_{E \in \mathcal{E}} \prod_{i \in E} x_i \).

Example 1. Let \( H_1 = (\{1,2,3\},\emptyset) \), \( H_2 = (\{1,2,3\},\{\{1,2\},\emptyset\}) \) and \( H_3 = (\{1,2,3\},\{\{1,2\},\{1,3\},\{2,3\}\}) \). Then \( P_{H_1} = 0 \), \( P_{H_2} = x_1x_2 + 1 \) and \( P_{H_3} = x_1x_2 + x_1x_3 + x_2x_3 \), respectively.

Conversely, to each Zhegalkin polynomial \( P \in GF(2)[x_1,\ldots,x_n] \) is associated an hypergraph \( H_P = (V,E) \) where \( V = \{1,\ldots,n\} \) and \( E \) is the set of hyperedges corresponding to the monomials of \( P \). By Theorem 1 we have the following.

Theorem 15. For each Boolean function \( f : \{0,1\}^n \to \{0,1\} \), \( n \geq 1 \), there is a unique hypergraph \( H = (V,E) \), \( V = \{1,\ldots,n\} \), such that \( f = f_{H_P} \).

For the sake of simplicity, let \( f_H \) denote the function \( f_{P_H} \) determined by \( H \).

3. Boolean functions and hypergraphs

By an hypergraph we simply mean a pair \( H = (V,E) \) where \( V \) is a finite nonempty set whose elements are called vertices, and where \( E \) is a collection of subsets of \( V \) called hyperedges. We write \([V]^m\) to denote the set of \( m\)-element subsets of \( V \).

For each Boolean function \( f : \{0,1\}^n \to \{0,1\} \), \( n \geq 1 \), there is a unique hypergraph \( H = (V,E) \), \( V = \{1,\ldots,n\} \), such that \( f = f_{H_P} \).

For the sake of simplicity, let \( f_H \) denote the function \( f_{P_H} \) determined by \( H \).

3.1. Simple minors of hypergraphs. Let \( H' = (V',\mathcal{E}') \) be an hypergraph and let \( h' : V' \to V \) be a map. For each \( E \subseteq V \), set \( h^{-1}E = \{E' \in \mathcal{E} : h'(E') = E\} \), where \( h'(E') = \{h'(i') : i' \in E'\} \). If \( H = (V,E) \) is an hypergraph, then the map \( h' \) is said to be a quotient map from \( H' \) to \( H \), denoted \( h' : H' \to H \), if for every \( E \subseteq V \), the following condition holds: \( E \in \mathcal{E} \) if and only if the cardinality \( |h^{-1}E| \) is odd. We say that an hypergraph \( H \) is a simple minor of an hypergraph \( H' \), denoted \( H \preceq H' \), if there is a quotient map from \( H' \) to \( H \).

To illustrate, let \( H = (V,E) \) be an hypergraph with \( V = \{1,\ldots,n\} \). Let \( e = \{i,j\} \), \( i,j \in V \), and fix \( l_e \notin V \). Consider the hypergraph \( H_e = (V_e,\mathcal{E}_e) \) given as follows: \( V_e = (V \setminus e) \cup \{l_e\} \) and for each \( E \subseteq V_e \), we have \( E \in \mathcal{E}_e \) if either

(i) \( E \in \mathcal{E} \) and \( e \cap E = \emptyset \), or
(ii) \( l_e \in E \) and among \((E \setminus \{l_e\}) \cup e, (E \setminus \{l_e\}) \cup \{i\} \) and \((E \setminus \{l_e\}) \cup \{j\} \), either one or all of these sets belong to \( \mathcal{E} \).

Note that conditions (i) and (ii) guarantee that the map \( h : V \to V_e \) defined by \( h(i) = h(j) = l_e \) and \( h(k) = k \), for each \( k \neq i,j \), constitutes a quotient map from \( H \) to \( H_e \), thus showing that \( H_e \) is a simple minor of \( H \).

Lemma 16. The simple minor relation between hypergraphs is a quasi-order.

Proof. Let \( H = (V,E) \), \( H' = (V',\mathcal{E}') \) and \( H'' = (V'',\mathcal{E}'') \) be hypergraphs such that \( H \preceq H' \preceq H'' \). Let \( h' : H' \to H \) and \( h'' : H'' \to H' \) be the corresponding quotient maps. We claim that \( h = h'' \circ h' \) is a quotient map from \( H'' \) to \( H \), i.e., \( H \preceq H'' \).

Let \( E \subseteq V \). The set \( h^{-1}E = \{E' \in \mathcal{E} : h'(E') = E\} \) decomposes into two sets, namely, \( A = \bigcup \{h''^{-1}E' : E' \in \mathcal{E}', h'(E') = E\} \) and \( B = \bigcup \{h''^{-1}E' : E' \notin \mathcal{E}', h'(E') = E\} \). Now \( A \) is a disjoint union of sets of odd size and \( B \) is a
disjoint union of sets of even size and hence, \( B \) has even size. Thus the parity of \(|h^{-1}[E]|\) is the same as the parity of \(|A|\) which, in turn, is the same as the parity of \(|h^{-1}[E]|\). Since \( E \in \mathcal{E} \) if and only if \(|h^{-1}[E]|\) is odd, the proof is now complete. \( \square \)

A simpler proof of Lemma \[ \text{Lemma 10} \] can be obtained using the following construction. Given an hypergraph \( \mathcal{H}' = (V', \mathcal{E}') \), a set \( V \) and a map \( h' : V' \to V \), let \( \mathcal{H}_h : (V, \mathcal{E}_h) \) where \( \mathcal{E}_h = \{ E \subseteq V : |h^{-1}[E]|_2 = 1 \} \) and \(|h^{-1}[E]|_2 \) denotes the cardinality of \(|h^{-1}[E]|\) modulo 2.

**Lemma 17.** Let \( \mathcal{H}' = (V', \mathcal{E}') \) and a map \( h' : V' \to V \), with \( V = \{1, \ldots, n\} \) and \( V' = \{1, \ldots, m\} \). Then \( h' \) is a quotient map from \( \mathcal{H}' \) to \( \mathcal{H}_h \) and

\[
|f_{\mathcal{H}_h}(a_1, \ldots, a_n)| = |f_{\mathcal{H}}(h'((a_1, \ldots, a_n)))|
\]

for all \( a_1, \ldots, a_n \in \{0, 1\} \). Moreover, if \( h' \) is a quotient map from \( \mathcal{H}' \) to \( \mathcal{H} \), then \( \mathcal{H} = \mathcal{H}_h \).

**Proof.** By construction, \( h' \) is a quotient map from \( \mathcal{H}' \) to \( \mathcal{H}_h \). Using the fact that \( a^2 = a \), for every \( a \in \{0, 1\} \), we have

\[
|f_{\mathcal{H}_h}(a_1, \ldots, a_n)| = \sum_{E \in \mathcal{E}_h} \prod_{i \in E} a_i = \sum_{E \in \mathcal{E}_h} |h^{-1}[E]|_2 \prod_{i \in E} a_i = \sum_{E' \in \mathcal{E}} \sum_{i \in E'} a_i = \sum_{E' \in \mathcal{E}} \prod_{i' \in E'} a_{h'(i')} = |f_{\mathcal{H}}(h'((a_1, \ldots, a_n)))|
\]

for every \( a_1, \ldots, a_n \in \{0, 1\} \). The last claim is an immediate consequence of the construction of \( \mathcal{H}_h \). \( \square \)

The following theorem establishes the connection between the simple minor relation on Boolean functions and the simple minor relation on hypergraphs.

**Theorem 18.** Let \( \mathcal{H} = (V, \mathcal{E}) \) and \( \mathcal{H}' = (V', \mathcal{E}') \) be two hypergraphs, with \( V = \{1, \ldots, n\} \) and \( V' = \{1, \ldots, m\} \), respectively. Then \( \mathcal{H} \preceq \mathcal{H}' \) if and only if \( f_{\mathcal{H}} \leq f_{\mathcal{H}'} \).

**Proof.** Let \( h' : V' \to V \). If \( h' \) is a quotient from \( \mathcal{H}' \) to \( \mathcal{H} \), then \( \mathcal{H} = \mathcal{H}_h \) and thus

\[
|f_{\mathcal{H}}(a_1, \ldots, a_n)| = |f_{\mathcal{H}_h}(a_1, \ldots, a_n)| = |f_{\mathcal{H}}(h'((a_1, \ldots, a_n)))|
\]

for every \( a_1, \ldots, a_n \in \{0, 1\} \), by Lemma 17. Conversely, if

\[
|f_{\mathcal{H}}(a_1, \ldots, a_n)| = |f_{\mathcal{H}}(h'((a_1, \ldots, a_n)))|
\]

for every \( a_1, \ldots, a_n \in \{0, 1\} \), then \( f_{\mathcal{H}} = f_{\mathcal{H}_h} \), by Lemma 17. By uniqueness of the Zhegalkin polynomial representation, \( \mathcal{H} = \mathcal{H}_h \). Hence, \( h' \) is a quotient map from \( \mathcal{H}' \) to \( \mathcal{H} \). This completes the proof of the theorem. \( \square \)

### 3.2. Conditions for join-irreducibility

**Lemma 19.** Let \( \mathcal{H} = (V, \mathcal{E}) \) and \( \mathcal{H}' = (V', \mathcal{E}') \) be two hypergraphs. A map \( \phi : V \to V' \) is said to be an isomorphism from \( \mathcal{H} \) onto \( \mathcal{H}' \) if \( \phi \) is bijective and for every \( E \subseteq V, E \in \mathcal{E} \) if and only if \( \phi(E) \in \mathcal{E}' \). Two hypergraphs \( \mathcal{H} \) and \( \mathcal{H}' \) are said to be isomorphic, denoted \( \mathcal{H} \cong \mathcal{H}' \), if there is an isomorphism \( \phi \) from \( \mathcal{H} \) onto \( \mathcal{H}' \). If \( \mathcal{H} = \mathcal{H}' \), then \( \phi \) is called an automorphism of \( \mathcal{H} \). The group made of automorphisms of \( \mathcal{H} \) is denoted by \( \text{Aut}(\mathcal{H}) \).

Let \( \mathcal{H} = (V, \mathcal{E}) \) be an hypergraph. We set \( \hat{V} = \bigcup \mathcal{E} \). Since the essential variables of \( f_{\mathcal{H}} \) are those which appear in the Zhegalkin polynomial, \( \text{Ess} f_{\mathcal{H}} = \hat{V} \), and hence, \( \text{ess} f = |\hat{V}| \). Note that for every \( e, e' \in [V]^2 \), we have \( e \approx e' \) whenever
$\mathcal{H}_e \cong \mathcal{H}_{e'}$. Set $D_{\mathcal{H}} = \{ e \in [\hat{V}]^2 : \hat{V}_e = (\hat{V} \setminus e) \cup l_e \}$. Clearly, $e \in D_{\mathcal{H}}$ if and only if $\text{ess } f_{\mathcal{H}_e} = \text{ess } f_{\mathcal{H}} - 1$.

Given these observations, we obtain from Fact 13 our first criterion for join-irreducibility.

**Proposition 19.** Let $\mathcal{H} = (V, E)$ be an hypergraph. If $|\hat{V}| \geq 2$ and $D_{\mathcal{H}} = [\hat{V}]^2$ then $f_{\mathcal{H}}$ is join-irreducible.

We may translate Theorem 14 as follows.

**Theorem 20.** Let $\mathcal{H} = (V, E)$ be an hypergraph. Then $f_{\mathcal{H}}$ is join-irreducible if and only if $|\hat{V}| \geq 2$ and either $D_{\mathcal{H}}$ is an equivalence class or $\mathcal{H} = \mathcal{H}_P$ where $P$ is one of the polynomials given in Theorem 6.

In the search for hypergraphs $\mathcal{H} = (V, E)$ determining join-irreducible Boolean functions, Theorem 20 invites us to look at differences $\text{ess } f_{\mathcal{H}} - \text{ess } f_{\mathcal{H}_e}$, especially, at the cases when $\text{ess } f_{\mathcal{H}} - \text{ess } f_{\mathcal{H}_e} > 1$. For the latter to occur, there are two possibilities:

(i) the vertex $l_e$ becomes isolated and this is the case if and only if, for every $F$ disjoint from $e$, the number of $e' \subseteq V$ such that $\emptyset \neq e' \subseteq e$ and $e' \cup F \in E$, is even, or

(ii) another vertex, say $i \in V$, becomes isolated and this is the case if and only if, for every $e' \in E$, if $i \in e'$ then $e \cap e' \neq \emptyset$ and there is exactly one $e'' \in E$ distinct from $e'$ such that $i \in e''$ and $e \setminus e' = e'' \setminus e$.

Proposition 19 reveals some interesting connections with some well-known combinatorial properties of hypergraphs. A group $G$ acting on a set $V$ is 2-set transitive if for every $e, e' \in [V]^2$, there is some $g \in G$ such that $g(e) = e'$.

**Corollary 21.** Let $\mathcal{H} = (V, E)$ be an hypergraph. If $|V| \geq 2$, $\bigcup E = V$ and $\text{Aut}(\mathcal{H})$ is 2-set transitive, then $f_{\mathcal{H}}$ is join-irreducible.

**Proof.** Let $\varphi \in \text{Aut}(\mathcal{H})$. Take $e \in [V]^2$ and let $e' = \varphi(e) \in [V]^2$. Consider the mapping $\varphi : V_e \to V_{e'}$ defined by $\varphi(l_e) = l_{e'}$ and $\varphi(i) = \varphi(i)$ for every $i \in V_e \setminus \{ l_e \}$. Clearly, $\varphi$ constitutes the desired isomorphism from $\mathcal{H}_e$ to $\mathcal{H}_{e'}$. $\square$

Proposition 19 and Corollary 21 naturally give raise to the following questions:

**Problem 2.** For which hypergraphs $\mathcal{H}$

(i) $\text{Aut}(\mathcal{H})$ is 2-set transitive?

(ii) $\mathcal{H}_e \cong \mathcal{H}_{e'}$, for every $e, e' \in [V]^2$?

3.3. **Steiner Systems.** Let $\mathcal{H} = (V, E)$ be an hypergraph. We say that the hypergraph $\mathcal{H}$ is a 2-$(n, k, \lambda)$ design if $|V| = n$, $E \subseteq [V]^k$, and for every $e \in [V]^2$, $|\{ E \in E : e \subseteq E \}| = \lambda$. If $\lambda = 1$, then we say that $\mathcal{H}$ is a Steiner system and, in addition, if $k = 3$, then we say that $\mathcal{H}$ is a Steiner triple system.

For each $e \in [V]^2$, set $\mathcal{H}_{-e} = (V \setminus e, E \cap [V \setminus e]^2)$. If for every $e, e' \in [V]^2$, $\mathcal{H}_{-e} \cong \mathcal{H}_{-e'}$, then we say that $\mathcal{H}$ is 2-monomorphic. The following theorem shows that, in the case of Steiner systems, join-irreducibility is equivalent to 2-monomorphy.

**Theorem 22.** Let $\mathcal{H} = (V, E)$ be a Steiner system. The following are equivalent:

(i) $f_{\mathcal{H}}$ is join-irreducible;

(ii) $\mathcal{H}_e \cong \mathcal{H}_{e'}$, for every $e, e' \in [V]^2$;

(iii) $\mathcal{H}$ is 2-monomorphic.

Note that for $k = 2$ a Steiner system is a complete graph. In this case (i), (ii) and (iii) hold simultaneously. For the proof of Theorem 22 we suppose $k \geq 3$. We will need the following lemmas.
Lemma 23. Let \( H = (V, \mathcal{E}) \) be a Steiner system. Then \( D_H = |V|^2 \).

Proof. Observe first that \( \cup \mathcal{E} = V \). Next, let \( e = \{i, j\} \in |V|^2 \). Since \( H \) is a Steiner system and \( k \geq 3 \), a subset \( E \subseteq V_e \) is an hyperedge of \( H_e \) if and only if \( l_e \notin E \) and \( E \) is an hyperedge of \( H \), or \( E = \{l_e\} \cup T \setminus e \), where \( T \) is the unique hyperedge of \( H \) such that \( e \subseteq T \), or \( i \in T \) and \( j \notin T \) or \( j \in T \) and \( i \notin T \). Since \( \cup \mathcal{E} = V \), it follows that \( \cup \mathcal{E} = V_e \), hence \( e \in D_H \).

Lemma 24. Let \( H \) and \( H' \) be two Steiner systems and \( e, e' \in |V|^2 \). Then:

1. A map \( f : V_e \rightarrow V_{e'} \) is an isomorphism from \( H_e \) to \( H_{e'} \) if and only if the map \( \overline{f} : V_e \rightarrow V_{e'} \) given by \( \overline{f}(l_e) = l_{e'} \) and \( \overline{f}(k) = f(k) \) for \( k \neq l_e \), is an isomorphism from \( H_e \) to \( H_{e'} \).

2. Every isomorphism \( g : H_e \rightarrow H_{e'} \) is of the form \( \overline{f} \) for some isomorphism \( f : H_e \rightarrow H_{e'} \).

Proof. (1) Clearly, \( f \) is an isomorphism from \( H_e \) to \( H_{e'} \) whenever \( \overline{f} \) is an isomorphism from \( H_e \) to \( H_{e'} \).

To show that the converse also holds, suppose that \( f \) is an isomorphism. Let \( T \) and \( T' \) be the unique hyperedges of \( H \) containing \( e \) and \( e' \), respectively.

Claim 2. \( f[T \setminus e] = T' \setminus e' \). In particular, \( \overline{f}[\{l_e\} \cup (T \setminus e)] = \{l_{e'}\} \cup (T' \setminus e') \).

Proof of claim. Let \( i \in V \setminus e \); denote by \( d_{e}(i) \) the number of hyperedges in \( E_{e} \) containing \( i \) and define similarly \( d_{e}(i) \). Since \( H \) is a Steiner system, there are exactly one hyperedge containing \( i \) and intersecting \( e \) if \( i \in T \) (namely \( T \)) and exactly two hyperedges when \( i \notin T \). In other words, the difference \( \eta(e, i) := d_{e}(i) - d_{e}(i) \) is \( 1 \) if \( i \in T \) and \( 2 \) if \( i \in V \setminus T \). Furthermore \( d_{e}(i) = \frac{n-1}{k-1} \).

Hence, with obvious notations \( \eta(e, i) = \eta(e', f(i)) \). From this, \( i \in T \) if and only if \( f(i) \in T' \) and thus the claim follows.

Now, let \( S \) be an hyperedge of \( H \) such that \( S \cap e \) is a singleton.

Claim 3. There is an hyperedge \( S' \) of \( H \) such that \( S' \cap e' \) is a singleton and such that \( f[S \setminus e] = S' \setminus e' \). In particular, \( \overline{f}[\{l_e\} \cup (S \setminus e)] = \{l_{e'}\} \cup (S' \setminus e') \).

Proof of claim. Since \( k \geq 3 \), \( |S \setminus e| \geq 2 \). Let \( j, j' \in S \setminus e \), \( j \neq j' \) and let \( S' \) be the unique hyperedge of \( H \) containing \( \{f(j), f(j')\} \). Since \( f \) is an isomorphism, we have \( S' \cap e' \neq \emptyset \). By the previous claim we also have \( S' \cap e' \neq e' \), and thus \( S' \cap e' \) is a singleton.

Let \( i \in S \setminus (e \cup \{j, j'\}) \). We need to show that \( f(i) \in S' \). For that, let \( S_j' \) and \( S_{j'}' \) be the hyperedges of \( H \) containing \( \{f(j), f(i)\} \) and \( \{f(j'), f(i)\} \), respectively. Note first that \( S_j, S_j' \) and \( S_{j'} \) intersect pairwise over \( V \setminus e \). Moreover, since \( f \) is an isomorphism, replacing \( j, j' \) by \( i, j \) and \( S' \) by \( S_j' \) we obtain, via the argument above, that \( S_j' \cap e' \) is a singleton and, by the same token, that \( S_j' \cap e' \) is a singleton too. Since this also holds for \( S_j' \cap e' \) and \( |e| = 2 \), two members of \( S_j', S_j' \) and \( S_{j'} \) contain the same element of \( e \). Since \( H \) is a Steiner system, these two members coincide. Thus they contain \( f(j) \) and \( f(j') \). Again from the fact that \( H \) is a Steiner system, we have that they coincide with \( S' \). Hence \( f(i) \in S' \) and the proof of the claim is complete.

These two claim ensure that \( \overline{f} \) is an isomorphism.

(2) Clearly, the lemma holds whenever \( V \) is itself an hyperedge of \( H \). Thus, we may assume that this is not the case. Now to prove the lemma, it is enough to show \( g(l_e) = l_{e'} \) since in this case the restriction \( f = g |_{V_e} \) constitutes an isomorphism from \( H_{e} \) to \( H_{e'} \).

Let \( T \) be the unique hyperedges of \( H \) containing \( e \) and let \( T' = l_e \cup (T \setminus e) \). Define \( T' \) similarly and let \( \overline{T'} = l_{e'} \cup (T' \setminus e') \). Clearly \( \overline{T'} \) is the only edge of
\( \mathcal{H}_e \) having size \( k - 1 \). Since \( g \) is an isomorphism of \( \mathcal{H}_e \) on \( \mathcal{H}_e' \), it maps \( \overline{T} \) on \( \overline{T}' \). In particular, \( e' \in T' \). Now observe that for each \( i \notin T \) there are exactly two hyperedges of \( \mathcal{H}_e \) containing both \( k \) and \( l_e \), whereas for each \( k \in T \) there is exactly one, namely, \( \overline{T} \). Thus \( g(l_e) = l_{e'} \).

**Proof of Theorem 2.23** Implication \((i) \Rightarrow (ii)\) follows from Lemma 2.23 Implication \((ii) \Rightarrow (i)\) follows from Proposition 1.19. The implications \((ii) \Rightarrow (iii)\) and \((iii) \Rightarrow (ii)\) follow respectively from (2) and (1) of Lemma 2.24. The proof of the theorem is now complete. \( \Box \)

**Problem 3.** For a Steiner triple system \( \mathcal{H} = (V, \mathcal{E}) \) does the following hold: \( \mathcal{H} \) is a 2-monomorphic if and only if \( Aut(\mathcal{H}) \) is 2-set transitive?

Note that the automorphism group of a Steiner systems is flag-transitive whenever it is 2-set transitive. The converse holds for Steiner triple systems. There are several deep results about Steiner systems with a 2-transitive or a flag transitive automorphism group (see the survey by Kantor 17). For example, any Steiner triple system with a 2-transitive automorphism group must be a projective space over \( GF(2) \) or an affine space over \( GF(3) \), see e.g. 13, 18. The notion of monomorphy (with some of its variations) is due to R. Fraisse. His book 13 contains some important results concerning this notion.

## 4. Join-irreducible graphs

In this section, we answer Problem 1 in the particular case of functions which are determined by undirected graphs, possibly with loops, that is, graphs \( \mathcal{G} = (V, \mathcal{E}) \) where \( \mathcal{E} \subseteq [V]^2 \cup [V]^1 \). As in the case of hypergraphs, if we remove the isolated vertices of \( \mathcal{G} \), the resulting graph \( \hat{\mathcal{G}} \) yields an equivalent function. In the sequel, when we speak of a join-irreducible graph we simply mean a graph \( \mathcal{G} \) such that \( f_{\mathcal{G}} \) is join-irreducible. Note that \( \mathcal{G} \) is join-irreducible if and only if \( \hat{\mathcal{G}} \) is join-irreducible; note also that a join-irreducible graph must satisfy \( |V| \geq 2 \).

Given a graph \( \mathcal{G} = (V, \mathcal{E}) \), we write \( i \sim j \) if \( \{i, j\} \in \mathcal{E} \). Set \( V(i) = \{j \in V : i \sim j\} \cup \{i\} \). The degree of a vertex \( i \), denoted \( d(i) \), is the cardinality \( |V(i)| - 1 \). For example, in the complete graph \( K_n \) each vertex has degree \( n - 1 \), while in a cycle \( C_n \) each vertex has degree 2. Note that loops, i.e., singleton edges, do not contribute to the degree of vertices.

A graph \( \mathcal{G} \) is said to be connected if any two vertices of \( \mathcal{G} \) are connected by a path. We denote by \( \overline{\mathcal{G}} \) the complement of \( \mathcal{G} \), that is, \( \overline{\mathcal{G}} = (V, ([V]^2 \setminus \mathcal{E}) \cup ([V]^1 \setminus \mathcal{E})) \). The disjoint union of two graphs \( \mathcal{G}_1 = (V_1, \mathcal{E}_1) \), \( \mathcal{G}_2 = (V_2, \mathcal{E}_2) \), \( V_1 \cap V_2 = \emptyset \), is defined as the graph \( \mathcal{G} = (V_1 \cup V_2, \mathcal{E}_1 \cup \mathcal{E}_2) \). The graph join of \( \mathcal{G}_1 = (V_1, \mathcal{E}_1) \) and \( \mathcal{G}_2 = (V_2, \mathcal{E}_2) \), denoted \( \mathcal{G}_1 + \mathcal{G}_2 \), is defined as the disjoint union of \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) together with the edges \( \{i_1, i_2\} \) for all \( i_1 \in V_1 \) and \( i_2 \in V_2 \). For further background in graph theory, see e.g. 11, 12, 13.

### 4.1. Join-irreducible graphs: the loopless case

In this subsection, we present an explicit description of those join-irreducible loopless graphs, i.e., join-irreducible graphs \( \mathcal{G} = (V, \mathcal{E}) \) where \( \mathcal{E} \subseteq [V]^2 \). So throughout this subsection, we assume that \( \mathcal{G} \) is a loopless graph. Furthermore, since \( \mathcal{G} \) is join-irreducible if and only if \( \hat{\mathcal{G}} \) is join-irreducible, we also assume that \( \mathcal{G} \neq \overline{\mathcal{G}} \). We start with the disconnected case.

**Proposition 25.** Suppose that \( \mathcal{G} \) is disconnected. Then \( \mathcal{G} \) is join-irreducible if and only if \( \mathcal{G} \) is isomorphic to the disjoint union of \( n \) copies of \( K_3 \), for some \( n \geq 2 \).
Proof. Clearly, if $G$ is isomorphic to the disjoint union of $n$ copies of $K_3$, for some $n \geq 2$, then $G$ is join-irreducible. For the converse, let $G_1$ and $G_2$ be two connected components of $G$. Note that $|G_1|, |G_2| \geq 2$. Take $i \in G_1$ and $j \in G_2$, and let $e = \{i, j\}$. Clearly, $|G_e| = |G| - 1$, no vertex is isolated in $G_e$ and $G_e$ has one less connected component than $G$.

Now take $i, i' \in G_1$ and let $e' = \{i, i'\}$. Clearly, for every such choice of $e'$, we have $e \neq e'$. Since $G$ is join-irreducible, Theorem 20 implies that $\text{ess } f_{G_e} < \text{ess } f_{G_e'}$. In other words, for every $e' = \{i, i'\}$, $i, i' \in G_1$, $\text{ess } f_{G_e'} \leq \text{ess } f_{G_e} - 2$.

From Theorem 8 it follows that $G_1$ must be isomorphic to $K_3$. Since the choice of connected components was arbitrary, we conclude that $G$ is isomorphic to the disjoint union of $n$ copies of $K_3$, for some $n \geq 2$. 

To deal with the case of connected (loopless) graphs, we need to introduce some terminology. Let $G = (V, E)$ be a graph. A subset $S \subseteq V$ is said to be autonomous if for every $i, i' \in S$ and $j \in V \setminus S$, $i \sim j$ if and only if $i' \sim j$. Moreover, $S$ is said to be independent if for every $i, i' \in S$, $i \neq i'$. For simplicity, we refer to autonomous independent sets as ai-sets. We say that $G$ is ai-prime if its ai-sets are empty or singletons.

Fact 26. For each $i \in V$, the union of all ai-sets containing $i$ is an ai-set called the ai-component of $i$. Moreover, each graph $G$ decomposes into ai-components.

On the set of ai-components of $G$ there is a graph structure, denoted $G_{ai}$, in such a way that $G$ is the lexicographic sum of its ai-components and indexed by $G_{ai}$. Note that the graph $G_{ai}$ is ai-prime.

These constructions are variants of the classical notions of decomposition of graphs and prime graphs (see 10). The following technical lemma is proved and extended to arbitrary graphs (not necessarily loopless) in the next subsection (see Lemma 54).

Lemma 27. Let $G = (V, E)$ be a connected graph and suppose that there is $e \in [V]^2 \setminus E$ such that $G_e$ has no isolated vertices. Then there is $e' \in E$ such that $G_{e'}$ has no isolated vertices.

Thus, if $G = (V, E)$ is join-irreducible, $G_e$ has an isolated vertex for every $e \in [V]^2 \setminus E$. Moreover, a nonedge $e = \{i_1, i_2\} \in [V]^2 \setminus E$ must be in a ai-component or there is $j \in V$ such that $d(j) = 2$ and $i_1 \sim j \sim i_2$.

We say that a graph $G = (V, E)$ satisfies (P) if for every nonedge $e = \{i_1, i_2\} \in [V]^2 \setminus E$ there is $j \in V$ such that $d(j) = 2$ and $i_1 \sim j \sim i_2$.

Lemma 27 and the observation above yield the following.

Corollary 28. If a connected graph $G$ is join-irreducible, then $G_{ai}$ satisfies (P).

Our next proposition describes those graphs satisfying (P).

Proposition 29. A graph $G = (V, E)$ satisfies (P) if and only if $G$ is either isomorphic to $K_n$, for some $n \geq 2$, $C_5$, $C_4$ or to a 3-element path.

Proof. We observe that each member of the list satisfies (P). Conversely, suppose that $G$ satisfies (P).

Claim 4. Let $e = \{i_1, i_2\} \in [V]^2 \setminus E$ and $j \in V$ such that $d(j) = 2$ and $i_1 \sim j \sim i_2$. If $e' = \{j, j'\} \in [V]^2 \setminus E$ then either $e_1 := \{i_1, j'\} \in E$ and $d(i_1) = 2$ or $e_2 := \{i_2, j'\} \in E$ and $d(i_2) = 2$.

Proof of Claim 4. Let $k \in V$ be such that $j \sim k \sim j'$. Since $V(j) = \{i_1, i_2\}$, we have $k = i_1$ or $k = i_2$. By property (P), it follows that either $e_1 := \{i_1, j'\} \in E$ and $d(i_1) = 2$ or $e_2 := \{i_2, j'\} \in E$ and $d(i_2) = 2$. 

□
Claim 5. Let \( i \in V \). If \( d(i) \geq 3 \), then \( G(i) = (V(i), |V(i)|^2 \cap E) \) is isomorphic to \( K_n \), for some \( n \geq 3 \). Moreover, \( G(i) = G \).

Proof of Claim 5. Let \( i \in V \) be such that \( d(i) \geq 3 \) and, for the sake of contradiction, suppose that \( G(i) \) is not isomorphic to \( K_n \), for some \( n \geq 3 \). Take \( i_1, i_2 \in V(i) \) such that \( i_1 \neq i_2 \). By property (P), there is \( j \in V \) such that \( i_1 \sim j \sim i_2 \) and \( d(j) = 2 \). Since \( d(i) \geq 3 \) and \( d(j) = 2 \), \( j \notin V(i) \). Let \( j' \in V(i) \setminus \{i, i_1, i_2\} \) (such a \( j' \) exists because \( d(i) \geq 3 \)). Note that \( j' \sim j \).

By Claim 4, either \( e_1 := \{i_1, j'\} \in E \) and \( d(i_1) = 2 \) or \( e_2 := \{i_2, j'\} \in E \) and \( d(i_2) = 2 \). This yields the desired contradiction because both \( V(i_1) \) and \( V(i_2) \) contain \( \{i, j, j'\} \).

To see that the last claim holds, suppose that there is \( k \in V \setminus V(i) \). Take \( j \in V \) such that \( i \sim j \sim k \). Since \( G(i) \) isomorphic to \( K_n \), for some \( n \geq 3 \), we have that \( d(j) \geq 3 \), which contradicts property (P). \( \square \)

According to Claim 5, if \( G \) is not isomorphic to \( K_n \), then the degree of each vertex is at most 2. Since \( G \) satisfies (P), it must be isomorphic to \( C_5 \), \( C_4 \) or to a 3-element path. \( \square \)

As a corollary we get the following result.

Corollary 30. If \( G \) is connected and join-irreducible, then \( G_{ai} \) is isomorphic to \( K_n \), for some \( n \geq 2 \), or to \( C_5 \).

Clearly, each \( K_n \), \( n \geq 2 \), and \( C_5 \) are join-irreducible graphs. Thus, if \( G = (V, E) \) is an ai-prime graph, then \( G \) is join-irreducible if and only if \( G \) is isomorphic to \( K_n \), for some \( n \geq 2 \), or to \( C_5 \).

Now if a connected and join-irreducible graph \( G = (V, E) \) is not an ai-prime graph, then \( G_{ai} \) cannot be isomorphic to \( C_5 \). Indeed, for the sake of contradiction, suppose that \( G_{ai} \) is isomorphic to \( C_5 \). Let \( G_1, \ldots, G_5 \) be the ai-components of \( G \) such that \( G_i \) is connected to \( G_{i+1} \), for \( i = 1, 2, 3, 4 \) and \( G_5 \) is connected to \( G_1 \). Assume, without loss of generality, that \( |G_1| \geq 2 \). Consider \( i, i' \in G_1 \), \( i_2 \in G_2 \) and \( i_3 \in G_3 \) and let \( e = \{i, i_2\} \) and \( e' = \{i', i_3\} \). Clearly, \( e \neq e' \) and \( ess_f_{G_1} = ess_f_{G_2} \). By Theorem 20 it follows that \( G \) is not join-irreducible which constitutes the desired contradiction. Thus, by Corollary 30 it follows that, in the non ai-prime case, if \( G = (V, E) \) is join-irreducible, then \( G_{ai} \) is isomorphic to \( K_n \), for some \( n \geq 2 \).

Proposition 31. Let \( G = (V, E) \) be a connected graph which is not ai-prime. Then \( G \) is join-irreducible if and only if \( G \) is isomorphic to one of the following graphs:

(i) \( K_2 + \overline{K}_m \), for some \( m \geq 2 \);
(ii) \( \overline{K}_n + \overline{K}_m \), for some \( n, m \) with \( 1 \leq n < m \);
(iii) a graph join \( \overline{K}_n + \ldots + \overline{K}_n \) of \( r \) copies of \( K_n \), for some \( r, n \geq 2 \).

Proof. As observed if \( G \) is join-irreducible, then \( G_{ai} \) is isomorphic to \( K_r \), for some \( r \geq 2 \). If \( r = 2 \), then it is clear that \( G \) is isomorphic to \( \overline{K}_n + \overline{K}_m \), for some \( n, m \geq 1 \).

Now suppose that \( r = 3 \). If there is \( i \in V \) such that \( d(i) = 2 \), then \( G \) is isomorphic to \( K_2 + \overline{K}_m \), for some \( m \geq 2 \). To verify the latter claim, let \( G_1, G_2 \) and \( G_3 \) be the ai-components of \( G \) and suppose that there is \( i \in V \) such that \( d(i) = 2 \), say, \( i \in C_1 \). Then, \( G_2 \) and \( G_3 \) are singletons, and since \( G \) is non ai-prime and \( G_{ai} \) is isomorphic to \( K_3 \), it follows that \( G \) is isomorphic to \( K_2 + \overline{K}_m \), for some \( m \geq 2 \). If for every \( i \in V \) we have \( d(i) > 2 \), then \( G \) is isomorphic to \( \overline{K}_m_1 + \overline{K}_m_2 + \overline{K}_m_3 \), where at most one ai-component is a singleton. Since \( G \) is join-irreducible, it follows that \( m_1 = m_2 = m_3 = n \), for some \( n \geq 2 \).
If \( r > 3 \), then for every \( i \in V \) we have \( d(i) > 2 \). Again from join-irreducibility, it follows that \( G \) is isomorphic to a graph \( \overline{K}_{n} + \ldots + \overline{K}_{n} \) of \( r \) copies of \( K_{n} \) for some \( n \geq 2 \).

From these results, we obtain the description of the join-irreducible (loopless) graphs.

**Theorem 32.** Let \( G = (V, E) \) be a (loopless) graph. Then \( G \) is join-irreducible if and only if \( G \) is isomorphic to one of the following graphs:

(i) a disjoint union of \( n \) copies of \( K_{3} \), for some \( n \geq 2 \);
(ii) \( C_{5} \);
(iii) \( K_{2} + \overline{K}_{m} \), for some \( m \geq 2 \);
(iv) \( K_{n} \), for some \( n \geq 2 \);
(v) \( \overline{K}_{n} + \overline{K}_{m} \), for some \( n, m \) with \( 1 \leq n < m \);
(vi) a graph join-irreducible, then \( G \) is either a disjoint union of loops or the disjoint union of loops and a copy of \( K_{3} \), or a graph join-irreducible with no isolated vertices.

4.2. Join-irreducible graphs: the general case. In this subsection, we extend Theorem 32 to arbitrary (not necessarily loopless) graphs. The key result is the following. For a graph \( G = (V, E) \), let \( G^{0} = (V, E^{0}) \), where \( E^{0} = E \cap [V]^{2} \).

**Proposition 33.** Let \( G = (V, E) \) be any graph such that \( V = \bigcup E \). If \( G \) is join-irreducible, then \( G \) is either

(1) a disjoint union of loops or the disjoint union of loops and a copy of \( K_{3} \), or
(2) \( G^{0} \) is join-irreducible with no isolated vertices.

To prove Proposition 33 we need to first extend Lemma 27 to arbitrary graphs.

**Lemma 34.** Let \( G = (V, E) \) be a connected graph. Suppose that there are \( i, j \in V \) such that \( e = \{i, j\} \notin E \) and \( \text{ess}_{G_{e}} = \text{ess}_{G} - 1 \). Then there are \( i', j' \in V \) such that \( e' = \{i', j'\} \in E \) and \( \text{ess}_{G_{e'}} = \text{ess}_{G} - 1 \).

**Proof.** Let \( e = \{i, j\} \in [V]^{2} \setminus E \) such that \( \text{ess}_{G_{e}} = \text{ess}_{G} - 1 \). Since \( G \) is connected, there exists \( i' \in V \) such that \( i' \sim i \) or \( i' \sim j \). Without loss of generality, assume that \( d = \{i', i\} \in E \). If \( \text{ess}_{G_{d}} = \text{ess}_{G} - 1 \), then we are done. So suppose that \( \text{ess}_{G_{d}} < \text{ess}_{G} - 1 \).

If either \( \{i\}, \{i'\} \in E \) or \( \{i\}, \{i'\} \notin E \), then there is \( j' \in V \) such that \( i' \sim j' \sim i \) and \( d(j') = 2 \). Since \( G \) is connected, we have \( d(i) > 2 \) or \( d(i') > 2 \). Suppose that \( d(i) > 2 \). If \( \{i\}, \{i'\} \in E \), then \( \text{ess}_{G_{d'}} = \text{ess}_{G} - 1 \), where \( d' = \{i, j\} \). If \( \{i\}, \{i'\} \notin E \), then \( \text{ess}_{G_{e'}} = \text{ess}_{G} - 1 \), where \( e' = \{i', j'\} \). Similarly, the claim holds when \( d(i') > 2 \).

So we may assume that, say, \( \{i\} \in E \) and \( \{i'\} \notin E \). The case \( \{i'\} \in E \) and \( \{i\} \notin E \) can be verified similarly. If \( l_{d} \), where \( d = \{i, i'\} \), appears in some edge (singleton or pair) of \( G_{d} \), then there is \( j' \in V \) such that \( i' \sim j' \sim i \) and \( d(j') = 2 \), and we have that \( \text{ess}_{G_{d}} = \text{ess}_{G} - 1 \), where \( d' = \{i, j'\} \), or \( \text{ess}_{G_{e'}} = \text{ess}_{G} - 1 \), where \( e' = \{i', j'\} \), according to whether \( d(i) > 2 \) or \( d(i') > 2 \), respectively.

If \( l_{d} \) does not appear in any edge of \( G_{d} \), then for every \( k \in V \) we have that \( c = \{i, k\} \in E \) if and only if \( c' = \{i', k\} \in E \). By connectivity, there is at least one such \( k \). If there is exactly one, then \( \text{ess}_{G_{e'}} = \text{ess}_{G} - 1 \) because \( d(k) > 2 \). If there are at least two, then by choosing such a \( k \) of greatest degree, it follows that \( f_{G_{c}} = \text{ess}_{G} - 1 \), and the proof of the lemma is complete.

In the sequel, we will also need the fact below.

**Fact 35.** Let \( G = (V, E) \) be a connected loopless graph and let \( i, i' \in V \). Suppose that \( \{e_{e}\} \), where \( e = \{i, i'\} \), is not a member of \( E_{e} \). If \( \text{ess}_{G_{e}} = \text{ess}_{G} - 1 \), then...
for any $G^+ = (V, E^+)$, where $E^+ = E \cup \{\{j\}: j \in S\}$ for some $S \subseteq V$, we have $\text{ess } f_{G^+} = \text{ess } f_G - 1$.

Finally, we will make use of the following result which shows that in most cases, reducibility can be verified using an edge and a nonedge.

**Lemma 36.** Let $G = (V, E)$ be a join-reducible, connected loopless graph. If for every $e = \{i, i'\} \in [V]^2 \setminus E$, $G_e$ has at least one isolated vertex, then $G$ is isomorphic to the graph join
\[
K_{m_1} + K_{m_2} + \ldots + K_{m_n},
\]
where $n \geq 3$, $0 < m_1 \leq \ldots \leq m_n$ with $m_1 < m_n$, and for $n = 3$, $n_2 \neq 1$.

**Proof.** Necessarily, there is $e = \{i, i'\} \in [V]^2 \setminus E$, for otherwise $G$ is a complete graph and thus it is join-irreducible. Suppose first that there is such a pair $e = \{i, i'\}$ for which $l_e$ belongs to some member of $E_e$. Then there is $k \in V$ such that $i \sim k \sim i'$ and $d(k) = 2$. Since $G$ is join-reducible, $|V| > 3$ and hence there is $j \in V$ such that either $j \sim i$ but $j \not\sim i'$ or $j \sim i'$ but $j \not\sim i$.

Suppose $j \sim i$ but $j \not\sim i'$ and let $d = \{k, j\}$. Since $\text{ess } f_{G_j} < \text{ess } f_G - 1$, we must have $d(i) = 2$. Also, there must exist $l$ such that $i \sim l \sim j$ and $d(l) = 2$, otherwise $\text{ess } f_{G_l} = \text{ess } f_G - 1$, for $e = \{i', j\}$. Moreover, $\text{ess } f_{G_{i'}} < \text{ess } f_G - 1$, for $e = \{i, l\}, \{k, l\}$, and hence, $d(i') = d(j) = 2$. Since $G$ connected, it is isomorphic to $C_5$ which is a contradiction. Thus, there is no $j \in V$ such that $j \sim i$ but $j \not\sim i'$. Similarly, we can verify that there is no $j \in V$ such that $j \sim i'$ but $j \not\sim i$.

So we may assume that, for every $e = \{i, i'\} \in [V]^2 \setminus E$, $l_e$ does not belong to any member of $E_e$. Thus, for every $j \in V$ we have $j \sim i$ if and only if $j \sim i'$, i.e., $i$ and $i'$ belong to the same $ai$-component. Moreover, $G_{ai}$ is isomorphic to a complete graph $K_n$. Since $G$ is not join-irreducible, it has the form given in the lemma.

**Proof of Proposition 33** Suppose that $G$ is disconnected. Then each connected component $G'$ having at least two vertices is join-irreducible. If $G$ has an isolated loop, then $\text{gap } f_{G'} = 2$. Using Theorem 6 it can be verified that $G'$ is a $K_3$. Moreover, it is not difficult to see that there is no more than one such connected component. If there is no isolated loop, then by reasoning as in the proof of Proposition 25 we can be shown that $G$ is isomorphic to a disjoint union of $n \geq 2$ copies of $K_3^+ = (\{1, 2, 3\}, \{(1, 2, 3)\}^2 \cup \{j\}: j \in S))$, for some $S \subseteq \{1, 2, 3\}$. This case is considered in Proposition 37 below.

Suppose now that $G$ is connected and that $G^0$ is not join-irreducible. If $G^0$ isomorphic to the graph join
\[
(2) \quad K_{m_1} + K_{m_2} + \ldots + K_{m_n}
\]
as in Lemma 36 then it is easy to see that any graph obtained from $G^0$ by adding singletons $\{i\}, i \in V$, to the set of edges of $G^0$, is join-reducible, and thus $G$ is join-irreducible.

So suppose that $G^0$ is a join-irreducible graph nonisomorphic to (2). By Lemma 36 there is $e = \{i, i'\} \in [V]^2 \setminus E^0$ such that $G^0_e$ has no isolated vertices, that is, $\text{ess } f_{G^0} = \text{ess } f_{G^0} - 1$. Since $\{l_e\}$ is not a member of $E^0_e$, it follows from Fact 55 that $\text{ess } f_{G^0} = \text{ess } f_G - 1$. By Lemma 64 there is $e' = \{j, j'\} \in E$ such that $\text{ess } f_{G^0} = \text{ess } f_G - 1$. Since $e \notin E$, we have $e \neq e'$ and thus $G$ is join-irreducible.

□

By Proposition 35 to completely describe the join-irreducible graphs (possibly with loops) we only need to focus on those graphs $G$ without isolated loops such that $G^0$ is join-irreducible. The description of the latter graphs is given
In the sequel, we consider a graph $G = (V, E)$ without isolated loops.

The following proposition shows that, among those with no isolated loops, the disconnected join-irreducible graphs are exactly those which are loopless and join-irreducible.

**Proposition 37.** If $G^0$ is a disjoint union of $n$ copies of $K_3$, for some $n \geq 2$, then $G$ is join-irreducible if and only if $G = G^0$.

**Proof.** By Theorem 32, the condition $G = G^0$ is sufficient. To show that it is necessary, observe first that no connected component $G_i$ of $G$ has loops in each vertex, for otherwise, by taking $i_1, i_2 \in G_i$ and $j_1, j_2 \in G_j$, where $G_j$ is a connected component of $G$ different to $G_i$, we have for $e = \{i_1, i_2\}$ and $e' = \{i_1, j\}$,

$$\text{ess } f_{G_i} = \text{ess } f_{G_j} = \text{ess } f_G - 1$$

but $e \neq e'$ which contradicts join-irreducibility by Theorem 20.

We claim that no connected component of $G$ has a loop. Indeed, suppose for the sake of contradiction that there is a connected component $G_i$ of $G$ with a loop. As we observed, we can find $i_1, i_2 \in G_i$ such that $\{i_1\}$ is a loop, but not $\{i_2\}$. Now take $j_1 \in G_j$, where $G_j$ is a connected component of $G$ different to $G_i$. We have for $e = \{i_1, j\}$ and $e' = \{i_2, j\}$,

$$\text{ess } f_{G_i} = \text{ess } f_{G_j} = \text{ess } f_G - 1$$

but it is easy to verify that $e \neq e'$. This constitutes the desired contradiction. Thus $G$ is a loopless graph and $G = G^0$. \hfill $\square$

The following propositions provide explicit descriptions of the remaining join-irreducible graphs, i.e., those graphs $G$ for which $G^0$ is isomorphic to one of the loopless graphs given in $(ii) - (vi)$ of Theorem 32.

**Proposition 38.** If $G^0 = C_5$, then $G$ is join-irreducible if and only if $G = G^0$.

**Proof.** By Theorem 32, the condition $G = G^0$ is sufficient. Conversely, suppose for the sake of contradiction that $\{i\}$ is a loop in $G$. Take distinct $i_1, i_2 \in V$ such that $i_1 \sim i \sim i_2$. As before, we have for $e = \{i_1, i_2\}$ and $e' = \{i_1, i\}$

$$\text{ess } f_{G_i} = \text{ess } f_{G_j} = \text{ess } f_G - 1$$

but $e \neq e'$, contradicting the join-irreducibility of $G$. \hfill $\square$

**Proposition 39.** If $G^0 = K_n$, for $n \geq 2$, then $G$ is join-irreducible if and only if it has $0, 1, n - 1$ or $n$ loops.

**Proof.** If $G$ has either $0$ or $n$ loops, then its automorphism group is 2-set transitive and, by Corollary 21, it is join-irreducible. If $G$ has only one loop $\{i\}$, then for every distinct $i_1, i_2 \in V \setminus \{i\}$, we have for $e = \{i, i_1\}$ and $e' = \{i_1, i_2\}$

$$\text{ess } f_{G_i} < \text{ess } f_{G_j} = \text{ess } f_G - 1$$

and, by Theorem 20 $G$ is join-irreducible. Similarly, it can be verified that if $G$ has $n - 1$ loops, then it is join-irreducible.

Now suppose for the sake of contradiction that $G$ has $n - k$ loops for $2 \leq k \leq n - 2$. Take distinct $\{i_1\}, \{i_2\} \in E$ and $\{j_1\}, \{j_2\} \in [V] \setminus E$. It is easy to see that, for $e = \{i_1, i_2\}$ and $e' = \{j_1, j_2\}$

$$\text{ess } f_{G_i} = \text{ess } f_{G_j} = \text{ess } f_G - 1$$

but $e \neq e'$ since $G_e$ has 2 more loops than $G_{e'}$. Thus $G$ is not join-irreducible which constitutes the desired contradiction. \hfill $\square$
Proposition 40. If $G^0 = \overline{K}_n + \overline{K}_m$, for some $1 \leq n,m$, then $G$ is join-irreducible if and only if $\overline{K}_n$ and $\overline{K}_m$ have either no loops or loops in every vertex.

Proof. It is easy to verify that the conditions suffice to guarantee that $G$ is join-irreducible. Let us prove that the converse also holds. The case when $m \geq n = 1$ is straightforward. So let $m,n \geq 2$ and, for the sake of a contradiction, suppose that there exist $i_1$ and $i_2$ in $\overline{K}_n$ or $\overline{K}_m$ such that $\{i_1\}$ is a loop but $\{i_2\}$ is not a loop. Take $j$ in $\overline{K}_m$ or $\overline{K}_n$, according to whether $i_1$ and $i_2$ in $\overline{K}_n$ or $i_1$ and $i_2$ in $\overline{K}_n$, respectively. Clearly, we have for $e = \{i_1, j\}$ and $e' = \{i_2, j\}$,

$$\text{ess } f_{G_e} = \text{ess } f_{G_{e'}} = \text{ess } f_G - 1$$

but $e \neq e'$. Thus $G$ is not join-irreducible which yields the desired contradiction. \hfill $\square$

Proposition 41. If $G^0 = K_2 + \overline{K}_m$, for some $m \geq 2$, then $G$ is join-irreducible if and only if either $K_2$ has no loops or two loops and $\overline{K}_m$ has no loops, or $\overline{K}_m$ has $m$ loops and $K_2$ has exactly one loop.

Proof. It is easy to verify that the conditions suffice to guarantee that $G$ is join-irreducible. To prove the converse, suppose first that $K_2$ has exactly one loop, say, $i_1$ without and $i_2$ with a loop, and for the sake of a contradiction, suppose that $\overline{K}_m$ has at least one vertex $j$ without a loop. It is easy to see that we have, for $e = \{i_1, j\}$ and $e' = \{i_2, j\}$,

$$\text{ess } f_{G_e} = \text{ess } f_{G_{e'}} = \text{ess } f_G - 1$$

but $e \neq e'$ and thus $G$ is not join-irreducible which constitutes the desired contradiction.

Now suppose that $K_2$ has no loops or two loops and let $i_1$ and $i_2$ be its vertices. If every vertex of $\overline{K}_m$ has a loop then, for every vertex $j$ of $\overline{K}_m$, we have for $e = \{i_1, i_2\}$ and $e' = \{i_1, j\}$,

$$\text{ess } f_{G_e} = \text{ess } f_{G_{e'}} = \text{ess } f_G - 1$$

but $e \neq e'$. If there are vertices $j_1$ and $j_2$ of $\overline{K}_m$ such that $j_1$ has a loop but $j_2$ has no loop, then we have for $e = \{j_1, j_2\}$ and $e' = \{i_1, j_2\}$,

$$\text{ess } f_{G_e} = \text{ess } f_{G_{e'}} = \text{ess } f_G - 1$$

but $e \neq e'$. In either case, we have that $G$ is not join-irreducible and the proof of the proposition is complete. \hfill $\square$

Proposition 42. If $G^0 = \overline{K}_n + \ldots + \overline{K}_n$ of $r$ copies of $K_n$, for some $r \geq 3$ and $n \geq 2$, then $G$ is join-irreducible if and only if $G$ has no loops or loops in each vertex.

Proof. It is easy to verify that the conditions suffice to guarantee that $G$ is join-irreducible. To prove the converse, observe first that if there exist $i_1$ and $i_2$ in a $\overline{K}_n$ such that $\{i_1\}$ is a loop but not $\{i_2\}$, then by taking $j$ in another $\overline{K}_n$, we have $\{i_1, j\} \neq \{i_2, j\}$, and thus $G$ is not join-irreducible. Hence, we may assume that each $\overline{K}_n$ has either no loops or loops in each vertex.

Now, if there is one $\overline{K}_n$ with loops in each vertex, and another with no loops, then $G$ is not join-irreducible. Indeed, by taking $i_1$ in the former $\overline{K}_n$, $i_2$ in the latter $\overline{K}_n$, and another vertex $k$ such that $i_1 \sim k \sim i_2$, we have for $e = \{i_1, k\}$ and $e' = \{i_2, k\}$,  

$$\text{ess } f_{G_e} = \text{ess } f_{G_{e'}} = \text{ess } f_G - 1$$

but $e \neq e'$. This completes the proof of Proposition 42. \hfill $\square$
References

[1] A. Bondy, U.S.R. Murty, Graph Theory, Series: Graduate Texts in Mathematics, Vol. 244, 2008, XII, 652 p. 235 illus.

[2] M. Bouaziz, M. Couceiro, M. Pouzet, Join-irreducible Boolean functions, in Proceedings of International Conference on Relations, Orders and Graphs: Interaction with Computer Science (ROGICS’08), Mahdia, Tunisia, May 2008, pp 47-56. see arXiv:0801.2939.

[3] M. Couceiro, S. Foldes, On closed sets of relational constraints and classes of functions closed under variable substitutions, Algebra Universalis, 54 (2005) 149–165.

[4] M. Couceiro, S. Foldes, Functional equations, constraints, definability of function classes, and functions of Boolean variables, Acta Cybernet. 18 (2007) 61–75.

[5] M. Couceiro, E. Lehtonen, On the effect of variable identification on the essential arity of functions on finite sets, Int. J. Found. Comput. Sci. 18 (2007) 975–986.

[6] M. Couceiro, E. Lehtonen, Generalizations of Świerczkowski’s lemma and the arity gap of finite functions, arXiv:0712.1753v1.

[7] M. Couceiro, E. Lehtonen, On the arity gap of finite functions: results and applications, in Proceedings of International Conference on Relations, Orders and Graphs: Interaction with Computer Science (ROGICS’08), Mahdia, Tunisia, May 2008.

[8] M. Couceiro, M. Pouzet, On a quasi-ordering on Boolean functions, Theoret. Comput. Sci. (2008), doi:10.1016/j.tcs.2008.01.025.

[9] R. Diestel, Graph Theory, Third Edition. Springer-Verlag, Heidelberg, Graduate Texts in Mathematics, Volume 173, 2005.

[10] A. Ehrenfeucht, T. Harju and G. Rozenberg, The theory of 2-structures. A framework for decomposition and transformation of graphs, World Scientific, 1999.

[11] O. Ekin, S. Foldes, P. L. Hammer, L. Hellerstein, Equational characterizations of Boolean function classes, Discrete Math. 211 (2000) 27–51.

[12] S. Foldes, G. Pogosyan, Post classes characterized by functional terms, Discrete Applied Mathematics 142 (2004) 35–51.

[13] R. Fraïssé, Theory of relations, second edition, North-Holland Publishing Co., Amsterdam, p.ii+451, 2000.

[14] R.L. Graham, M. Grötzschel, L. Lovasz, (Editors), Handbook of Combinatorics vol. I, Elsevier, Amsterdam. The MIT Press, Cambridge, 1995.

[15] M. Hall, Jr., Steiner triple systems with doubly transitive automorphism group, J. Comb. Theory (A) 38(1985) 192–202.

[16] L. Hellerstein, On generalized constraints and certificates, Discrete Mathematics, 226 (2001) 211–232.

[17] W.M. Kantor, 2-transitive and flag-transitive designs, Coding theory, design theory, group theory (Burlington, VT, 1990),13-30, Wiley-Intersi.Publ., Wiley, Nex-York, 1993.

[18] J.D.Key and E.E.Shult, Steiner triple systems with doubly transitive automorphism groups: A corollary of the classification theorem for the finite simple groups, J. Comb. Theory (A) 36(1984) 105–110.

[19] D. E. Muller, Application of Boolean algebra to switching circuit design and to error correction, IRE Trans. Electron. Comput. 3(3) (1954) 6–12.

[20] N. Pippenger, Galois theory for minors of finite functions, Discrete Math. 254 (2002) 405–419.

[21] G. Pogosyan, Classes of Boolean functions defined by functional terms, Multiple Valued Logic, 7 (2002) 417–448.

[22] I. S. Reed, A class of multiple-error-correcting codes and the decoding scheme, IRE Trans. Inf. Theory 4(1954) 38–49.

[23] A. Saloma, On essential variables of functions, especially in the algebra of logic, Ann. Acad. Sci. Fenn. Ser. A I. Math. 339 (1963) 3–11.

[24] I. I. Zhegalkin, On the calculation of propositions in symbolic logic, Mat. Sb. 34 (1927) 9–28 (in Russian).

[25] I. E. Zverovich. Characterization of Closed Classes of Boolean Functions in Terms of Forbidden Subfunctions and Post Classes, Discrete Applied Mathematics 149 (2005) 200–218.
Institut Supérieur des Technologies Médicales de Tunis, 9 avenue Dr Zouihaïr Essafi, 1006 Tunis, Tunisie
E-mail address: Moncef.Bouaziz@istmt.rnu.tn

Department of Mathematics, University of Luxembourg, 162a, avenue de la Faïencerie, L-1511 Luxembourg
E-mail address: miguel.couceiro@uni.lu

ICJ, Department of Mathematics, Université Claude-Bernard Lyon1, 43 Bd 11 Novembre 1918, 68622 Villeurbanne Cedex, France, Department of Mathematics and Statistics, The University of Calgary, 2500 University Drive NW, Calgary, Alberta, Canada T2N 1N4
E-mail address: pouzet@univ-lyon1.fr, mpouzet@ucalgary.ca