Deeper & Sparser Exploration

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Abstract

We address the problem of efficient exploration by proposing a new meta algorithm in the context of model-based online planning for Bayesian Reinforcement Learning (BRL). We beat the state-of-the-art, while staying computationally faster, in some cases by two orders of magnitude. This is the first Optimism free BRL algorithm to beat all previous state-of-the-art in tabular RL. The main novelty is the use of a candidate policy generator, to generate long-term options in the belief tree, which allows us to create much sparser and deeper trees. We present results on many standard environments and empirically prove its performance.

1. Introduction

The exploration/exploitation trade-off is the fundamental problem of online-learning. Bayesian RL (BRL) is an elegant but computationally intractable solution to this problem, which requires performing dynamic programming in an exponentially large tree of information states. Although various myopic or partial look-ahead approximations were made in the past to solve it (Duff, 2002), these methods don’t perform as well as simpler algorithms. The fundamental problem lies in the trade-off between the value-function approximation at the leaf-nodes vs the depth of backup performed (Kearns et al., 1999). The core idea is to use the sparse sampling technique of (Kearns et al., 1999) together with a candidate policy generator in the belief tree, similar to (Wang et al., 2005), but instead of branching at every step of the planning tree, we only branch every \(K\) steps. That is, candidate policies are generated at a given node and followed for \(K\) steps before branching again at the resulting nodes. Intuitively this should work because belief changes are slow, and it allows us to make Bellman backups at deeper depths.

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2. Preliminaries

Markov Decision Processes (MDP) provide an elegant way to mathematically express the problem of sequential decision-making.

Definition 2.1 A Markov decision process (MDP) \(\mu\) is a discrete-time stochastic process with: A state \(s_t \in S\) at time \(t\) and a reward \(r_t \in \mathbb{R}\), generated by the process \(\mu\), and an action \(a_t \in A\), chosen by the decision maker. We denote the distribution over next states \(s_{t+1}\), which only depends on \(s_t\) and \(a_t\), by \(\mu(s_{t+1}|s_t, a_t)\). Furthermore \(\mu(r_{t+1}|s_t, a_t)\) is a reward distribution conditioned on states and actions. Finally, \(\mu(r_{t+1}, s_{t+1}|s_t, a_t) = \mu(r_{t+1}|s_t, a_t)\mu(s_{t+1}|s_t, a_t)\).

The decision maker takes actions according to a policy \(\pi\), which defines a distribution \(p(a_t|s_t)\) over \(A\), conditioned on the state \(s_t\), i.e. a set of probability measures over \(A\) indexed by \(s_t\). The expected utility of a policy \(\pi\) selecting actions in the MDP, from time \(t\) to \(T\), can be written as the value function:

\[
V_{t,T}^{\pi,\mu}(s) = E_{\pi,\mu}\left(\sum_{k=0}^{T-1} \gamma^k r_{t+k} | s_t = s\right)
\]

where \(E_{\pi,\mu}\) denotes the expectation under the Markov chain arising from acting policy \(\pi\) on the MDP \(\mu\). The optimal value function will be denoted by \(V^\ast_{\mu} \triangleq \max_{\pi} V^{\pi}_{\mu}\). If the MDP is known, we can calculate the optimal value function policy (Puterman, 1994) via backwards induction (value iteration). Whenever it is clear from context, superscripts and subscripts shall be omitted for brevity.

2.1. Bayes-adaptive MDP (BAMDP)

The problem of RL is to find the optimal policy when the underlying MDP is unknown. This amounts to a trade-off between information seeking actions in hopes of performing better or acting optimally given our current knowledge. The Bayesian analysis (Dearden et al., 1999) provides an elegant framework to quantify this trade-off.

Following the ideas from (Duff, 2002), we use a Bayesian framework to represent our uncertainty. We maintain a belief \(\xi_t\) about which MDP \(\mu \in \mathcal{M}\) corresponds to reality. More precisely, we define a measurable space \((\mathcal{M}, \mathcal{M})\),
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where $\mathcal{M}$ is a (usually countable) set of MDPs, and $\mathfrak{B}$ is a suitable $\sigma$-algebra. With an appropriate initial density $\xi_0(\mu)$, we can obtain a sequence of densities $\xi_t(\mu)$, representing our subjective belief at time $t$, by conditioning $\xi_t(\mu)$ on the latest observations:

$$\xi_{t+1}(\mu) \triangleq \frac{\mu(r_{t+1}, s_{t+1}|s_t, a_t) \xi_t(\mu)}{\int_{\mathcal{M}} \mu(r_{t+1}, s_{t+1}|s_t, a_t) \xi_t(\mu')d\mu'}$$

we will denote $E_\xi$ as expectation with respect to any belief $\xi$.

**Definition 2.2 (Bayesian value function)** For any belief $\xi$ over $\mathcal{M}$, we define the Bayesian value function of a policy $\pi$ to be:

$$V^\pi_\xi(s) \triangleq \int_{\mathcal{M}} V^\pi_\mu(s|\xi(\mu))d\mu.$$ 

Finding the policy attaining the maximum Bayesian value can be reduced to solving a Bayes-adaptive Markov decision process.

**Definition 2.3 (Bayes-adaptive MDP (BAMDP))** A BAMDP is an MDP with a set of hyper-states $\Omega = S \times \mathfrak{B}$, where $\mathfrak{B}$ is an appropriate set of probability measures on $\mathcal{M}$, and $S$ and $A$ are the common state and action sets of all $\mu \in \mathcal{M}$. At time $t$, the agent observes the hyper-state $\omega_t = (s_t, \xi_t) \in \Omega$ and takes action $a_t \in A$. We write the transition distribution as $\nu(\omega_{t+1}|\omega_t, a_t)$ and the reward distribution as $\tau(r_t|\omega_t)$.

The hyper-state $\omega_t$ has the Markov property. This allows us to treat the BAMDP as an infinite-state MDP with transitions $\nu(\omega_{t+1}|\omega_t, a_t)$, and rewards $\tau(r_t|\omega_t)$. When the horizon $T$ is finite, we only need to expand the tree up to depth $T - t$. Thus, backwards induction starting from the set of terminal hyper-states $\Omega_T$ and proceeding backwards to $T - 1, \ldots, t$ provides a solution:

$$V^*_{t}(\omega) = E_\tau(r_t|\omega) + \gamma \max_{a \in A} \sum_{\omega' \in \Omega_{t+1}} \nu(\omega'|\omega, a)V^*_{t+1}(\omega')$$

where $\Omega_t$ is the set of hyper-states at time $t$. We can approximately solve infinite horizon problems by expanding the tree to some finite depth and using bounds on the value function at leaf nodes.

### 2.2. Related Work

The work in Bayesian exploration has roots in works of Ronald A. Howard’s students (Silver, 1963; Martin, 1967). Modern research was initiated by (Dearden et al., 1998; 1999) and (Duff, 2002) gives one of the most comprehensive introduction to the subject. The sparse-sampling algorithm of Kearns et al. (1999), applied to BAMDP, is optimal (Asmuth, 2013) given infinite computation time. This would entail planning in the space of primitive actions, as shown in Fig.1. Most previous algorithms assumed an approximation to the optimal approach as infeasible, with exception of (Wang et al., 2005), which only samples a subset of promising actions, thereby utilizing the computational budget more efficiently. However, the high branching factor of the tree still makes planning with deep horizon a very difficult task. We now broadly divide the algorithms in two categories.

**Myopic:** Thompson sampling (TS) (called Bayesian DP in Strens, 2000) maintains a posterior distribution over transition models, samples an MDP and chooses the optimal policy for the sample. The Best Of Sampled Set (BOSS; Asmuth et al., 2009) algorithm generalizes this idea to a multi sample optimistic approach. MCBRL or Monte-Carlo Utility Estimates for BRL (Dimitrakakis, 2011; 2013) generalizes these ideas to lower bound policies and gradient based value function estimates for improved performance.

**Lookahead:** Beyond (Kearns & Singh, 1999; Wang et al., 2005), recent state-of-the-art algorithms include BFS3 (Asmuth & Littman, 2011) and BAMCP (Guez et al., 2012). The former comes closest to (Wang et al., 2005) but with an optimistic action selection strategy; the backups are still performed using backwards induction. The latter takes a Monte-Carlo approach to sparse lookahead in BAMDPs, but still using optimism for action selection, something that can sometimes yield suboptimal strategy (Coquelin & Munos, 2007). Furthermore, BAMCP utilises root sampling\(^1\) to reduce the burden of posterior inference. The algorithm further introduces lazy sampling and rollout policy inspired by previous applications of UCT.

**Our contribution.** Our main contribution is to propose a deeper sparse-sampling belief tree planning. The core idea is to use a policy generator to select a $K$-step policy rather than single actions in the belief tree. This allows us to build a tree that only branches every $K$ steps. The freedom to choose a policy generator internally, lets the algorithm scale quite elegantly: We choose Policy Iteration (PI) and a variant of Real Time Dynamic Programming (RTDP) by Barto et al. (1995) for different sizes of environments respectively. Our second contribution is a thorough, unbiased experimental comparison of state-of-the-art Bayesian reinforcement learning algorithms. As our results show, our algorithm finds significantly improved policies with much less computation compared to the current state of the art.

\(^1\)This means MDPs are sampled at the root node of the tree, and everything else is sampled conditional on the sampled MDP. For any given fixed policy, the corresponding tree distribution remains the same.
3. Deeper & Sparser Sampling

In order to illustrate our algorithm, we begin with full look-ahead tree search in a BAMDP. Starting from some initial hyper-state \( \omega \), we can expand the BAMDP horizon \( H \). The value of each terminal node \( \omega_H \) at the horizon is set to some approximate value, i.e. \( V_H(\omega) = 0 \), while the remaining values are obtained through the backwards induction equation (1). In practice calculating the value of each hyper-state requires to first construct them by posterior updates, as shown in Alg. 1.\(^2\)

\begin{algorithm}[H]
\caption{FLA: Full look-ahead tree search}
\textbf{Input:} current hyper-state \( \omega \) and depth \( h \).
\begin{algorithmic}
\If{\( h = 0 \)}
\State \( V(\omega_h) = \text{leaf-value approximation} \)
\EndIf
\ForAll{actions \( a \)}
\ForAll{next states \( s' \)}
\State \( \xi_{t+1} = \text{UpdatePosterior}(\omega_h, s', a) \)
\State \( \omega' = (s', \xi_{t+1}) \)
\State \( V(\omega') = \text{FLA}(\omega', h - 1) \)
\EndFor
\EndFor
\ForAll{\( \omega', a \)}
\State \( Q(\omega_h, a) = \nu(\omega'|\omega_h, a) \times V(\omega') \)
\EndFor
\State \( \text{return } \max_a Q(\omega_h, a) \)
\end{algorithmic}
\end{algorithm}

However, even with a relatively small horizon, this algorithm is impractical as there is a branching factor proportional to the number of actions. If we branch \(|A|\) times at each step, then we create \( O(|S| \times |A|^H) \) nodes. Our main idea is instead draw samples of complete policies, to reduce the branching factor at each step, and run them for

\[ K \] steps before branching again, so as to reduce the number of times we branch until the horizon. Since we are only looking at a subset of policies, the backwards induction procedure equation now becomes:

\[ V_t^*(\omega_t) = E(\omega_t | \omega_t) + \gamma \max_{\pi \in \Pi(\omega_t)} \sum_{\omega_{t+K} \in \Omega_{t+K}} \nu(\omega_{t+K} | \omega_t, \pi) V_{t+K}^*(\omega_{t+K}), \]

where \( \nu(\omega_{t+K} | \omega, \pi) \) is the marginal distribution of hyper-states \( K \) steps after \( \omega_t \). Fig. 2 shows how our algo. 2 expands the tree. To calculate (2), we maintain values for each \( K \)-step policy through \( Q \)-values, which are now mappings \( Q : \Omega \times \Pi \rightarrow \mathbb{R} \).

Since we now have policies at any given tree node, we
choose to rebranch only after running those policies for $K$ steps. Hence we can increase the effective depth of the belief-tree up to $HK$ for the same computational budget. The allows us to have a deeper lookahead, since then the approximation error propagated will be much smaller ($\gamma^H K$ instead of $\gamma^H$) and we will hopefully not lose much since the beliefs don’t change much in these $K$ steps.

The policies at each step are drawn from a policy generator $P$. In the experiments reported here, this is the Thompson sampling algorithm. In particular, a policy is generated as follows, for a given hyperstate $\omega = (\xi, s)$:

$$\hat{\mu} \sim \xi(\mu), \quad \hat{\pi} = \arg\max_{\pi} V^\pi_{\hat{\mu}}.$$

This algorithm has two steps: the first samples an MDP from the belief $\xi$ and the second calculates the optimal policy for that MDP. However, in large state spaces this latter computation is prohibitive. For that reason, in our experiments we are comparing against a real-time dynamic programming approximation to the optimisation problem.

4. Experiments

We perform experiments on many standard discrete state and action MDPs. They are:

1. Chain: Its the classical 5-state linear chain environment. Refer Dearden et al. (1998). Reward collected for first 1000 steps.

2. DoubleLoop: Its the classical 9-state, 2-action problem. Refer Dearden et al. (1998). Reward collected for first 1000 steps.

3. Grid5: A 5x5 grid, with start-state diagonally opposite to end-state, and no reward elsewhere. It has 4 directional actions and a slip-probability of 0.2. Reward collected for first 1000 steps.

4. Grid10: Similar to 5x5 grid, but with bigger state-space. Reward collected for first 2000 steps.
Following (Guez et al., 2012), we impose a limit 0.25sec/step for chain,loop and 1sec/step for the grid environments. We cross-validated hyperparameters of each algorithm for every environment for 10 experiments, before running them for a total of 100 experiments.

We cross-validate for various algorithms as follows:

1. SPARSER: Our meta-algorithm, we use two versions: PI and RTDP, i.e with PI and RTDP as policy generators. We finetune for $K, N, M, H$ as used in Algo.2. Most importantly, we finetune for lookahead depth $K$, which takes value in $\{\lfloor S/2, \lfloor S/2, 2 \times \lfloor S/2\rfloor\}$ for each environment. $M, N, H$ take values in sets $\{2, 4, 8\}, \{2, 4\}, \{1, 2\}$ respectively.

2. BAMCP: which is the current state-of-the-art (Guez et al., 2012)\(^3\). We finetune across no. of simulations $\{10, 100, 1000, 10000\}$ and for different lookahead depths $\{5, 10, 15, 50, auto\}$. Presumably, the original paper just kept it auto\(^5\).

3. SBOSS: A variant of BOSS, described in Castro & Precup (2010).\(^3\) We finetune for no. of sampled MDPs $\{2, 4, 8, 16, 32\}$ and sampling threshold $\{3, 5, 7\}$. We found sampling threshold 3 to be better, which is obvious as a new policy will be calculated more often.

4. BFS3: Similar to Guez et al. (2012)\(^3\) we finetune for branching factor $\{2, 5, 10, 15\}$ and no. of simulations (10 to 2000).

Our algorithm, like BFS3 and Wang et al. (2005), is a forward search algorithm, while SBOSS is myopically optimistic and BAMCP employs Monte-Carlo tree search.

As an underlying model, we use the hierarchical Dirichlet prior introduced by Friedman & Singer (1999) on the transition probabilities that is biased towards sparse models\(^4\). This was essentially for a fair comparison with previous implementations\(^3\). Moreover, for our SPARSER-PI version of algorithm, this prior is beneficial, as it is much faster to solve a sparse MDP using PI.

### 4.1. Evaluation

For a fair comparison with the previous state-of-the-art (BAMCP), we followed closely the finetuning process followed by (Guez et al., 2012). In brief, we finetuned only those parameters which the authors in BAMCP found amenable to performance vs computation comparison. We assume a known reward and discount factor $\gamma=0.95$, with $T \to \infty$. The initial belief parameters of our model are identical to those used BAMCP. In all cases, we used the same environment simulators\(^5\).

Figures 3 and 4 show the performance of various algorithms during the 10 cross-validation experiments in Grid10 environment. Each point is a single experiment. For a given algorithm, spread on y-axis denotes variance across experiments for the same set of hyperparameters. We also encode the single most important hyperparameter for each algorithm, i.e. the parameter which most reflects the performance vs. computation trade-off; they are no. of simulations, no. of sampled MDPs, branching factor and $K$ for BAMCP, SBOSS, BFS3 and SPARSER respectively\(^6\).

We also show individual performance of BAMCP and SPARSER in Fig.5,6 in Grid5 environment due to two different reasons. Firstly, we wanted to analyze the effect of depth (lookahead) parameter in BAMCP and SPARSER algorithms. Hence the marker encoding is different from previous figures (encoding depth). Secondly, we want to emphasize the skewness of results in Fig.4: see time scale in Fig.5 vs Fig.6.

After having selected the optimal hyperparameter for each problem and algorithm combination, we evaluated them for another 100 runs. These results are summarised in Table 4. We report the mean and standard error.

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\(^3\)We use the author’s implementation for all these remaining algorithms.  
\(^4\)More details can be found in the supplementary material.  
\(^5\)Available in https://github.com/acguez/bamcp  
\(^6\)Markers are thresholded at appropriate value of hyperparameter, with filled points indicating a higher value of hyperparameter.
The row *TIMES FASTER* shows computational advantage\(^7\) of SPARSER over the 2\(^{nd}\) best, while being in the respective limit of \(0.25\text{sec}\) and \(1\text{sec}\) per step for the first two (Chain, DoubleLoop) and last two (Grid5,10) environments.

4.2. Discussion

Besides clear performance benefit and low variance for SPARSER, we draw following observations from our experiments:

1. The computational cost of Bayesian posterior updating for building the belief-tree is not as prohibitive as thought before (the main motivation of using root sampling in BAMCP).

2. The cost computational cost of using TS for generating policies, while sometimes thought as prohibitive (Asmuth & Littman, 2011) can be overcome. We do this in two ways:
   
   (a) Use a sparse prior instead of Full Dirichlet prior, so that VI/PI can be performed efficiently to calculate candidate TS policy. This has been done many times before (BFS3, SBOSS, BAMCP).
   
   (b) Use any simulation based policy generator (like RTDP). This would still have been infeasible, had there been a need to calculate policy at every internal node, but not for SPARSER.

3. In our algorithm, for horizon parameter H=1, we obtain Bayes-optimal stationary policy (i.e. stationary with respect to BAMDP). This policy wins out in Grid5, and performs closely in Grid10. This has implications, since in such case, we end-up finding a bunch of candidate policies at root only (which is much faster).

Finally, SBOSS performs much better than expected, in Chain and DoubleLoop compared to BAMCP, but this may simply be due to a more optimistic strategy being taken by it (which actually reduces its performance in Grid5,10).

5. Conclusion

We beat the state-of-the-art in tabular RL. We presented an approximate model-based Bayesian planner to tackle the problem of exploration. It internally uses a candidate policy generator, and can therefore potentially be extended to deal with large state-action spaces. First extension to this work would be to provide its non-trivial Value function bounds, which seems hard because:

1. Sparser sampling implies non uniform sampling at each internal node, therefore analysis of (Kearns et al., 1999) is not directly applicable.

2. We don’t explicitly rely on optimism for belief-tree generation, so neither is the classical optimism-dependent upper bound analysis of Value function applicable here.

Even though the proposed technique is a meta-algorithm, we are hopeful that analysis on lines of (Russo & Van Roy, 2014) will prove useful. Therefore, future work includes both testing the algorithm on larger domains and proving useful bounds on Value function.

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