NUMERICAL EVALUATION OF THE OSCILLATORY INTEGRAL OVER $\exp(i\pi x)x^{1/x}$ BETWEEN 1 AND INFINITY

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ABSTRACT. Real and imaginary part of the limit $2N \to \infty$ of the integral $\int_1^{2N} \exp(i\pi x)x^{1/x}dx$ are evaluated to 20 digits with brute force methods after multiple partial integration, or combining a standard Simpson integration over the first half wave with series acceleration techniques for the alternating series co-phased to each of its points. The integrand is of the logarithmic kind; its branch cut limits the performance of integration techniques that rely on smooth higher order derivatives.

1. Scope

1.1. M. R. Burns’ Constant.

Definition 1. The MRB constant is the sum of the series [18 A037077]

\[ M \equiv \lim_{N \to \infty} \sum_{n=1}^{2N} (-1)^n \sqrt[n]{n} = \sum_{k=1}^{\infty} (-1)^k (k^{1/k} - 1) \approx 0.18785964. \]

Direct summation of the alternating series is slow and generates roughly 3 valid digits after ten thousand terms (Table 1). Euler summation [1 (3.6.27)] [10] is successful in accelerating the convergence, witnessed in Table 2.

Table 1. Partial sums of (1) as a function of the upper limit of summation.

| $10^k$ | $\sum_{k=1}^{10^k} (-1)^k (k^{1/k} - 1)$ |
|-------|----------------------------------------|
| 10    | 0.313231759254…                      |
| $10^2$ | 0.211329543346…                      |
| $10^3$ | 0.191323989712…                      |
| $10^4$ | 0.188320351076…                      |
| $10^5$ | 0.187917210140…                      |

More efficient methods lead to even quicker convergence, as demonstrated in Table 3. An accuracy of 60 digits is reached after 100 terms and will be sufficient for all purposes of this script.

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Table 2. Approximations to (1) after Euler resummation of the first \( \hat{k} \) terms.

| \( \hat{k} \) | \( E(1) \) |
|---|---|
| 10 | 0.1878858661380073035138243846468024292327407645248188116946... |
| 20 | 0.187859649854050194658445181421685949651095964112113134589... |
| 40 | 0.187859642462067120248517934054273142151463271236583869269... |
| 100 | 0.187859642462067120248517934054273230055903094900138786171986... |
| 200 | 0.187859642462067120248517934054273230055903094900138786171986... |

Table 3. Approximations to (1) with the first Cohen-Villegas-Zagier algorithm using terms up to \( \hat{k} \).

| \( \hat{k} \) | \( M \) |
|---|---|
| 10 | 0.187859642389333316574574761130167270485999309932998191180459... |
| 20 | 0.187859642462067119674255755542904758484982176117045969528027... |
| 40 | 0.187859642462067120248517934054273140023454509840554949525330... |
| 100 | 0.187859642462067120248517934054273230055903094900138786171986... |

1.2. Oscillatory Integral. The integrated analog of the series is a complex-valued integral of oscillatory character, which is difficult to evaluate by direct integration if the upper limit becomes large, illustrated by Figure 1.

Definition 2. (Sequence of oscillatory integrals)

\[
I(2N) \equiv \int_{1}^{2N} (-1)^x \sqrt{x} dx = \int_{1}^{2N} e^{i\pi x \frac{1}{x}} dx, \quad N \in \mathbb{Z}.
\]

Not convergent in the continuum limit as \( N \to \infty \), the limit of the sequence of integrals with an integral difference in the upper limits \( 2N \) exists. The objective of this work is to evaluate this limit \( M_I \).

Definition 3. (Ultraviolet limit of the sequence)

\[
M_I \equiv \lim_{N \to \infty} I(2N).
\]

Remark 1. The absolute value \( |M_I| \approx 0.6876523689 \) is close to \( M + \frac{1}{2} \) [A157852]. Changing the upper limit to \( 2N + 1 \) increases \( M_I \) by \( 2i/\pi \).

To compute \( M_I \), the manuscript looks at repeated partial integration to quench the integrand at large \( x \) in preparation for standard methods of sampling along the abscissa (Sections 2.1 and 2.2), investigates splitting the integral into an alternating series and a base interval (Section 2.3), expansion of \( x^{1/x} \) into a series over \( (\log x/x)^n \) (Section 2.4), changing the path of integration in the complex plane (Section 2.5), and considers reverse application of the Euler-Maclaurin integral formula (Appendix C).

2. Numerical Analysis

2.1. Iterated Partial Integration. A partial integration of (2) yields

\[
\int_{1}^{2N} e^{i\pi \frac{1}{x} x^{1/x}} dx = -\frac{i}{\pi} e^{i\pi \frac{1}{x} x^{1/x}} \bigg|_{1}^{2N} + \frac{i}{\pi} \int_{1}^{2N} e^{i\pi \frac{1}{x} x^{1/x}} \frac{1 - \log x}{x^2} dx.
\]
The limit $N \to \infty$ can be performed in the pre-integrated term,

\begin{equation}
M_I = -\frac{2i}{\pi} + \frac{i}{\pi} \int_1^{\infty} e^{i\pi x} x^{1/3} \frac{1 - \log x}{x^2} dx,
\end{equation}

which essentially compresses the oscillations with a factor $\propto \log(x)/x^2$, as shown in Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{Real and imaginary part of the integrand in (5).}
\end{figure}
A second partial integration of (5) may follow,

\[
\int_1^{2N} e^{i\pi x} x^{1/s} \frac{1 - \log x}{x^2} dx
= -\frac{i}{\pi} e^{i\pi x} x^{1/s} \frac{1 - \log x}{x^2} \left[ \frac{2N}{1} + \frac{i}{\pi} \int_1^{2N} e^{i\pi x} x^{1/s} \frac{1 - 3x + 2(x - 1) \log x + \log^2 x}{x^4} dx \right]
\]

with the limit

\[
M_I = -\frac{2i}{\pi} + \frac{1}{\pi^2} - \frac{1}{\pi^2} \int_1^{\infty} e^{i\pi x} x^{1/s} \frac{1 - 3x + 2(x - 1) \log x + \log^2 x}{x^4} dx.
\]

Repeating, a simple scheme for the \(n\)-th derivative of the base function \(f\),

\[
f^{(0)}(x) \equiv x^{1/s}; \quad f^{(n+1)}(x) \equiv \frac{d}{dx} f^{(n)}(x),
\]

can be phrased as a set of coefficients \(\alpha\),

\[
f^{(n)}(x) \equiv x^{1/s} \sum_{r,s \geq 0} \alpha_{n,r,s} \frac{\log^s(x)}{x^r}.
\]

Explicit computation with the chain rule establishes the recurrence

\[
\alpha_{n+1,r,s} = \alpha_{n,r-2,-s} - \alpha_{n,-r-2,-s-1} + (s + 1)\alpha_{n,-r-1,-s+1} - (r - 1)\alpha_{n,-r,-s}.
\]

The initial conditions are

\[
\alpha_{n,r,s} = 0 \quad \text{if} \quad r < 0 \quad \text{or} \quad s < 0; \quad \alpha_{0,0,0} = 1.
\]

Equations (5), (7) and the representation through \(n\)-fold partial integration are summarized with

\[
M_I = b(n) + \left( \frac{i}{\pi} \right)^n \int_1^{\infty} e^{i\pi x} f^{(n)}(x) dx,
\]

defining pre-integrated terms

\[
b(n+1) = b(n) - \frac{i}{\pi} f^{(n-1)}(1); \quad b(0) = 0; \quad b(1) = -\frac{2i}{\pi}.
\]

The infinite interval from 1 to \(\infty\) can be mapped onto the interval from 0 to \(1/2\) with the substitution (one of a family of rational maps)

\[
y = \frac{1}{1 + x}; \quad x = \frac{1}{y} - 1; \quad dx = -\frac{dy}{y^2};
\]

\[
\int_1^{\infty} e^{i\pi x} f^{(n)}(x) dx = \int_0^{1/2} e^{i\pi y} f^{(n)}(x(y)) \frac{dy}{y^2}.
\]

A combination of (12) and (15) yields:

**Algorithm 1.** Compute \(M_I\) with a Simpson integration on \(s\) points in the interval \(0 \leq y \leq 1/2:\)

\[
M_I = b(n) + \left( \frac{i}{\pi} \right)^n \int_0^{1/2} e^{i\pi y} f^{(n)}(x(y)) \frac{dy}{y^2}.
\]
Table 4. Algorithm 1: Simpson integration with \( s \) points in the interval \( 0 \leq y \leq 1/2 \) and the \( n \)-th partial integration \((12)\) inserted in \((15)\).

| \( n \) | \( s \)  | \( \Re(M_I) \)   | \( \Im(M_I) \)   |
|-------|-------|----------------|-----------------|
| 2     | 2000  | 0.0707787379203467823 | -0.6840084933091320239 |
| 2     | 4000  | 0.07077826545794147869 | -0.68400115445281602856 |
| 3     | 2000  | 0.07077594388618778788 | -0.684000397091691789 |
| 3     | 4000  | 0.0707760472567395562114 | -0.6840040978730247448 |
| 3     | 8000  | 0.07077604693793108739 | -0.6840003896832537849 |
| 3     | 16000 | 0.07077603931155570888 | -0.68400038943655207372 |
| 3     | 32000 | 0.07077603931154254337 | -0.68400038943792386069 |
| 3     | 64000 | 0.07077603931152730519 | -0.68400038943793211168 |
| 5     | 2000  | 0.07077603931150518881 | -0.68400038943840850746 |
| 5     | 4000  | 0.07077603931156321151 | -0.68400038943793231267 |
| 5     | 8000  | 0.07077603931154254337 | -0.68400038943793211168 |
| 5     | 16000 | 0.07077603931152730519 | -0.68400038943793204308 |
| 5     | 32000 | 0.07077603931155557088 | -0.68400038943793204308 |
| 5     | 64000 | 0.07077603931154254337 | -0.68400038943793204308 |
| 6     | 2000  | 0.07077603931150518881 | -0.68400038943840850746 |
| 6     | 4000  | 0.07077603931156321151 | -0.68400038943793231267 |
| 6     | 8000  | 0.07077603931154254337 | -0.68400038943793204308 |
| 6     | 16000 | 0.07077603931152730519 | -0.68400038943793204308 |
| 6     | 32000 | 0.07077603931155557088 | -0.68400038943793204308 |
| 6     | 64000 | 0.07077603931154254337 | -0.68400038943793204308 |

Table 4 illustrates that a choice of \( n \) near 5 or 6 yields optimum convergence (because \( b(n) \) then approximate \( M_I \) best), and that integration with \( s \approx 60000 \) abscissa points generates of the order of 13 valid digits. Note that Romberg (Richardson) extrapolation with the standard 15 : 1 weighting of step width halving does not work as the integrand is not in the polynomial class.

**Remark 2.** In a variant of this mapping on a finite interval, the transformation \( x = 1/u \) in the \( n \)-th partial integration yields

\[
M_I = b(n) + (i/\pi)^n \int_0^1 e^{ix/u} f^{(n)}(1/u) \frac{du}{u^2}.
\]

The precision is worse than with Algorithm 1 by approximately one digit. I have not looked into advanced schemes for this type of oscillatory integrals \([7, 11]\) or Sidi’s generalized methods of extrapolation.

2.2. Exponential Scaling. A characteristic of Algorithm 1 is that the integrand is basically reduced by another factor \( 1/x \) for each additional partial integration. The variable transformation \( \log x = z, \ x = e^z, \ dx = e^z dz \), helps to achieve exponential scaling as the integration variable heads towards infinity, at the cost of an irregular
chirp factor in the complex exponential:

\begin{equation}
F_m = \int_m^\infty e^{i\pi x} x^{1/x} \frac{1 - \log x}{x^2} \, dx = \int_{\log m}^\infty (1 - z) e^{i\pi \exp(z)} e^z \exp(-z) \, dz.
\end{equation}

Based on

\begin{equation}
\int e^{i\pi \exp(z)} + z \, dz = -i e^{i\pi \exp(z)},
\end{equation}

and “borrowing” a factor $e^z$ in (18) in the integrand,

\begin{equation}
F_m = \int_{\log m}^\infty \log m (1 - z) e^{i\pi \exp(z)} + z e^z \exp\left(-\frac{2z}{\sqrt{m}}\right) \, dz,
\end{equation}

a partial integration generates

\begin{equation}
F_m = \frac{i(-)^m}{\pi} \log m \frac{1 - \log m}{m^2} + \frac{i}{\pi} \int_{\log m}^\infty e^{i\pi \exp(z)} e^z \exp(-z) - 2z \left[2z - 3 + e^{-z}(1 - 2z + z^2)\right] \, dz.
\end{equation}

Alternatively, this results applying the substitution $\log x = z$ to (12). After this scheme of partial integrations has been repeated $n$ times, the non-oscillating exponential factor in the integrand is $\exp\left[z \exp(-z) - (n+1)z\right]$.

**Algorithm 2.** Perform $n$ partial integrations of (18), then use another transformation $u = e^{-z}$, $z = -\log u$ to map the range $0 \leq z \leq \infty$ to $0 \leq u \leq 1$ in the remaining integral, and integrate this over $u$ with a Simpson method. Eventually insert this $F_1$ in

\begin{equation}
M_I = -2i + i F_1
\end{equation}
as seen in (13).

An accuracy of $10^{-21}$ can be reached sampling two million points (Table 5) and is reported in the summary. Comparison with Table 4 shows that one to two digits are gained relative to Algorithm 1.

2.3. **Longman’s Method.** An integral $F$ with an undulating trigonometric factor multiplied by a monotonous $g(x)$ may be split into an integral over the half period with an alternating series attached to each point in that interval \[14, 13, 5\].

\begin{equation}
F_m = \int_m^\infty e^{i\pi x} g(x) \, dx = \sum_{l \geq 0} \int_0^1 e^{i\pi (m + l + y)} g(m + l + y) \, dy
\end{equation}

The $l$-series is only alternating if the function $g(x)$ is monotonous and does not change sign. In addition, Euler resummation assumes that the series converges. With $M_I$, $x^{1/x}$ has a maximum at $x = e$, so we integrate over $1 \leq x \leq 3$ with any other method, setting $m = 3$, and relay by one partial integration, $g(x) = f^{(1)}(x)$, to feature a $g(x)$ that has a single sign with decreasing $|g'(x)|$ in $[m, \infty)$. 
Table 5. Convergence of Algorithm 2: integration of the $n$-th partial integration of $F_1$ with $s$ equidistant points in the interval $0 \leq u \leq 1$.

| $n$ | $s$ | $\Re(M_I)$ | $\Im(M_I)$ |
|-----|-----|-------------|------------|
| 2   | 4000 | 0.07077612979610666804 | -0.68400038256040301228 |
|     | 8000 | 0.0707764264870771610 | -0.6840003741801246748 |
| 3   | 4000 | 0.07077603954212465077 | -0.684000385784542350 |
|     | 8000 | 0.07077603950170386538 | -0.68400038943788982953 |
| 4   | 4000 | 0.0707760392878287271 | -0.68400038944795068337 |
|     | 8000 | 0.07077603918586254 | -0.684000389441710684 |
| 5   | 4000 | 0.0707760393112220214 | -0.68400038943794252308 |
|     | 8000 | 0.07077603931148860789 | -0.68400038943793214262 |
|     | 16000 | 0.07077603931152840681 | -0.68400038943792930582 |
|     | 32000 | 0.07077603931152865655 | -0.6840003894379214262 |
|     | 64000 | 0.07077603931152880374 | -0.6840003894379212992 |
|     | 128000 | 0.07077603931152880345 | -0.6840003894379212926 |
|     | 256000 | 0.07077603931152880358 | -0.684000389437921291922485 |
|     | 512000 | 0.07077603931152880358 | -0.684000389437921291820339 |
|     | 1024000 | 0.07077603931152880359 | -0.68400038943792129182037 |
|     | 2048000 | 0.070776039311528803538 | -0.684000389437921291827445 |
| 6   | 4000 | 0.0707760393115613745 | -0.68400038943793535638 |
|     | 8000 | 0.07077603931152840681 | -0.68400038943793535638 |
|     | 16000 | 0.07077603931152865655 | -0.68400038943793535638 |
|     | 32000 | 0.07077603931152880374 | -0.68400038943793535638 |
|     | 64000 | 0.07077603931152880345 | -0.68400038943793535638 |
|     | 128000 | 0.07077603931152880345 | -0.68400038943793535638 |
|     | 256000 | 0.07077603931152880358 | -0.68400038943793535638 |
|     | 512000 | 0.07077603931152880358 | -0.68400038943793535638 |
|     | 1024000 | 0.07077603931152880359 | -0.68400038943793535638 |
|     | 2048000 | 0.070776039311528803538 | -0.68400038943793535638 |
|     | 4096000 | 0.070776039311528803538 | -0.68400038943793535638 |
| 8   | 4000 | 0.0707760393135438266 | -0.68400038943809843545 |
|     | 8000 | 0.07077603931152840681 | -0.68400038943809843545 |
|     | 16000 | 0.07077603931152865655 | -0.68400038943809843545 |
|     | 32000 | 0.07077603931152880374 | -0.68400038943809843545 |
|     | 64000 | 0.07077603931152880345 | -0.68400038943809843545 |
|     | 128000 | 0.07077603931152880345 | -0.68400038943809843545 |
|     | 256000 | 0.07077603931152880358 | -0.68400038943809843545 |
|     | 512000 | 0.07077603931152880358 | -0.68400038943809843545 |
|     | 1024000 | 0.07077603931152880359 | -0.68400038943809843545 |
|     | 2048000 | 0.070776039311528803538 | -0.68400038943809843545 |

Algorithm 3. Compute $M_I$ by Filon-Simpson-integration of $F_m$ in (23) over the interval $0 \leq y \leq 1$ (Appendix A) and (24)

\[ M_I = \int_1^m e^{ix} x^{1/2} dx + \frac{i}{\pi} \left[ (-)^m m^{1/m} - 1 \right] + \frac{i}{\pi} F_m; \quad g(x) = x^{1/2} \frac{1 - \log x}{x^2}; \quad (m = 3). \]

With this choice, $g(x)$ has a maximum near $x = 4.3$, associated with the zero of $f^{(2)}(x)$, which (after detailed inspection) does not destroy the alternating property—if higher order $f^{(n)}(x)$ were employed, $m$ would have to be chosen differently.

The speed of convergence of the method is demonstrated with Table 6. As the number of evaluations of $g$ is the product of $n$ and $l$, it turns out effectively to be slower than Algorithm 2.

2.4. Taylor Series the Logarithmic Term. The most advanced method expands the non-oscillatory term of (2) into the Taylor series of the exponential:
Table 6. Convergence of Algorithm 3 with a Simpson integration on \( n \) abscissa points over \([0, 1]\), truncating the alternating series after the \( l \)-th term followed by extrapolation [3].

| \( n \) | \( l \) | \( \Re(M_I) \) | \( \Im(M_I) \) |
|-------|-------|----------------|----------------|
| 32    | 60    | 0.07077603721021819390 | -0.68400038753980049654 |
| 32    | 70    | 0.07077603721021819390 | -0.68400038753980049654 |
| 64    | 70    | 0.07077603918043104863 | -0.684000389316813605 |
| 128   | 70    | 0.070776039333884842   | -0.68400038943052296458 |
| 256   | 70    | 0.070776039311461859   | -0.6840003894379038486 |
| 512   | 70    | 0.07077603931152680433 | -0.68400038943793032039 |
| 1024  | 70    | 0.07077603931152879572 | -0.68400038943793212212 |
| 2048  | 70    | 0.0707760393115287959  | -0.68400038943793212874 |
| 4096  | 70    | 0.07077603931152879572 | -0.68400038943793212212 |
| 8192  | 70    | 0.0707760393115287959  | -0.68400038943793212874 |
| 16384 | 70    | 0.0707760393115287959  | -0.68400038943793212874 |
| 8192  | 60    | 0.07077603931152880305 | -0.68400038943793212874 |
| 16384 | 60    | 0.07077603931152880305 | -0.68400038943793212874 |

Table 7. Convergence of the partial sum of Algorithm 4.

| \( \max n \) | \( \Re(M_I) \) | \( \Im(M_I) \) |
|------------|----------------|----------------|
| 1          | 0.05762490298863188764 | -0.68331060191932132015 |
| 2          | 0.06935454902524610824 | -0.68451362283943263006 |
| 3          | 0.07066781734932533318 | -0.68408393964817446557 |
| 4          | 0.0707697873632627015  | -0.68400839361204835470 |
| 5          | 0.07077575770475264223 | -0.68400096184084805332 |
| 6          | 0.07077602957520227901 | -0.68400042266805211811 |
| 7          | 0.07077603908225272182 | -0.6840003910740896452 |
| 8          | 0.0707760393105059528  | -0.6840003895469850 |
| 9          | 0.07077603931177817100 | -0.684000389446977421 |

Algorithm 4. (Logarithmic expansion of \( x^{1/x} \))

\[
\int_1^{2N} e^{i\pi x x^{1/x}} dx = \int_1^{2N} e^{i\pi x} e^{i\pi x \frac{\log x}{x^n}} dx = \int_1^{2N} e^{i\pi x} \left[ 1 + \sum_{n \geq 1} \frac{1}{n!} \frac{\log^n x}{x^n} \right] dx \\
= -\frac{2i}{\pi} + \sum_{n \geq 1} \frac{1}{n!} \int_1^{2N} e^{i\pi x} \log^n x dx.
\]

The integrals are the \( V(\pi, n, n) \) defined in equation (31) in Appendix B. Summing over values taken from Tables [9][11] produces Table 7. An accuracy of \( 10^{-19} \) in real and imaginary part of \( M_I \) requires summation up to \( n = 15 \)—an estimation derived from results of Remark 4—, and has not been worked out.

2.5. Contour Deformation. The path of the integration may be deformed to a straight line (hypotenuse) from \( z = 1 \) to \( z = 2N(1 + \tau i) \) with adjustable slope \( \tau > 0 \) towards the real axis, and a straight line (short leg) parallel to the imaginary axis back to \( 2N \) on the real axis. The contribution of \( \Im(z) \) to \( \exp(i\pi z) \) leads to a
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Table 8. Algorithm 5: Simpson integration along the line from $z = 1$ to $z = 2N(1 + \tau i)$ with $s$ abscissa points.

| $s$  | $N$  | $\tau$ | $\Re(M_I)$  | $\Im(M_I)$  |
|-----|-----|-------|-------------|-------------|
| 64000 | 20  | 0.1   | 0.7077602653194263984 | -0.6840020550941763076 |
| 64000 | 20  | 0.2   | 0.7077603931143220832 | -0.6840038944591701234 |
| 64000 | 20  | 0.3   | 0.7077603931156096762 | -0.6840038943797230450 |
| 128000 | 20  | 0.3   | 0.7077603931153081310 | -0.6840038943793467559 |
| 256000 | 20  | 0.3   | 0.7077603931152892845 | -0.6840038943793232378 |
| 64000 | 40  | 0.1   | 0.7077603931142945824 | -0.684003894426949930 |
| 128000 | 40  | 0.1   | 0.7077603931149682353 | -0.684003894357500783 |
| 64000 | 40  | 0.2   | 0.7077603931176127606 | -0.684003894386543692 |
| 128000 | 40  | 0.2   | 0.7077603931154332966 | -0.68400389437921085 |
| 64000 | 80  | 0.1   | 0.7077603931104131100 | -0.6840038945008174688 |
| 128000 | 80  | 0.1   | 0.7077603931145906913 | -0.6840038943869147372 |
| 256000 | 80  | 0.1   | 0.7077603931152445008 | -0.68400389437958811 |
| 64000 | 80  | 0.2   | 0.7077603931153454347 | -0.6840038945029316088 |
| 128000 | 80  | 0.2   | 0.7077603931176217587 | -0.684003894387046898 |
| 256000 | 80  | 0.2   | 0.7077603931154370430 | -0.684003894379041416 |
| 512000 | 80  | 0.2   | 0.7077603931152973483 | -0.684003894379351469 |
| 1024000 | 80  | 0.2   | 0.7077603931152886174 | -0.6840038943793231779 |
| 128000 | 160 | 0.1   | 0.7077603931039901029 | -0.684003894502529554 |
| 256000 | 160 | 0.1   | 0.7077603931145818783 | -0.6840038943870107039 |
| 128000 | 160 | 0.15  | 0.7077603931285988807 | -0.684003894502460466 |
| 256000 | 160 | 0.15  | 0.7077603931161990700 | -0.684003894387191535 |
| 512000 | 160 | 0.15  | 0.7077603931153400264 | -0.684003894379813810 |

exponential reduction of the integrand as the distance to the real axis grows. (More complicated paths appear not to be more efficient.) The short leg of this triangle contributes

$$
\lim_{N \to \infty} \int_{2N(1+\tau i)}^{2N(1+\tau i)} e^{i\pi z + \log z / z} \, dz = -\frac{i}{\pi}
$$

(26) to the integral.

**Algorithm 5.** Compute

$$
M_I = -\frac{i}{\pi} + \lim_{N \to \infty} \int_{1}^{2N(1+\tau i)} e^{i\pi z + \log z / z} \, dz
$$

(27) with a Simpson integration on $s$ points $z_j = 1 + j(1 + \tau i)\Delta t$, $j = 0, \ldots, s$.

The dependence on three configuration parameters makes quality assessment more difficult than with the other methods, illustrated by Table 8. The contribution missing for any finite $N$ is estimated by

$$
\int_{2N(1+\tau i)}^{\infty} e^{i\pi z + \log z / z} \, dz \approx \int_{2N(1+\tau i)}^{\infty} e^{i\pi z} \, dz = \frac{i}{\pi} e^{2\pi i N(1+\tau)}
$$

(28) which serves as a guideline how large $\tau$ ought be made given a targeted accuracy and an upper limit $N$. 

3. Summary

The value of $M_I$ is

\[
\lim_{N \to \infty} \int_1^{2N} e^{i\pi x} x^{1/x} \, dx \\
\approx 0.0707760393115288035395 - 0.6840038943793212918i.
\]

The integral features highly oscillatory behavior and a logarithmic factor in the main integrand, which sets up an interesting test case outside the range of methods that assume “nice” analytic properties in the complex plane.

Appendix A. Filon-Simpson Rule

The Simpson rule of integration of a function $G(y)$ is an interpolation between three abscissa points $(y_0, G(0))$, $(y_1, G(1))$ and $(y_2, G(2))$ by a quadratic polynomial

\[
G_m(y) = \frac{(y - y_1)(y - y_2)}{(y_0 - y_1)(y_0 - y_2)} G(0) + \frac{(y - y_0)(y - y_2)}{(y_1 - y_0)(y_1 - y_2)} G(1) + \frac{(y - y_0)(y - y_1)}{(y_2 - y_0)(y_2 - y_1)} G(2)
\]

followed by integration of this polynomial in the limits $y_0 \leq y \leq y_2$. A refinement in our case, based on the same quadratic interpolation, is

\[
\int_{y_0 + 2h}^{y_0 + 2h} e^{i\pi y} G_m(y) \, dy = \frac{i e^{y_0 \pi i}}{2h^2 \pi^3} \left[ -2 + 2h^2 \pi^2 - 3ih\pi + (2 - h\pi)e^{2i\pi h} \right] G(0) \\
- \frac{2e^{y_0 \pi i}}{h^2 \pi^3} \left[ -i + h\pi + (i + h\pi)e^{2i\pi h} \right] G(1) \\
- \frac{ie^{y_0 \pi i}}{2h^2 \pi} \left[ 2 + h\pi i + (-2 + 2h^2 \pi^2 + 3ih\pi)e^{2i\pi h} \right] G(2),
\]

supposing equidistant abscissa points $y_1 = y_0 + h$ and $y_2 = y_0 + 2h$. This explicit recognition of the exponential factor gains roughly one additional digit in accuracy in Table 6 compared to the evaluation with $e^{i\pi y}$ incorporated in the value of $G$.

Appendix B. Fichtenholz Integrals

B.1. Fundamental Form. In this appendix, a set of integrals $V(a, k, s)$ is targeted as an aid to Algorithm 4. This is basically evaluating the generalized integro-exponential function [16, 12] for a complex-valued parameter $z = -ia$.

Definition 4. (Generalized Integro-Exponential Function)

\[
V(a, k, s) \equiv \int_1^\infty e^{iax} \frac{\log^k x}{x^s} \, dx.
\]

Remark 3. There are three simple extensions:

- **Cases with a scaling factor $b$ in the logarithm can be reduced to this fundamental form by binomial expansion of $(\log b + \log x)^k$,**

\[
\int_1^\infty e^{iax} \frac{\log^k (bx)}{x^s} \, dx = \sum_{l=0}^{k} \binom{k}{l} \log^{k-l}(b) V(a, l, s).
\]
• Reading

(33) \[ \int_1^\infty e^{iax} \frac{\log^k (bx)}{x^s} \, dx = b^{s-1} \int_b^\infty e^{iax/y} \frac{\log^k y}{y^{s+1}} \, dy, \]

(obtained through the substitution \( bx \to y \)) from right to left shows that other lower limits than 1 are also accessible once the \( V \) are known for general \( a \).

• Integer powers of sines or cosines at the place of the exponential lead back to the fundamental form via Euler’s formula:

(34) \[ \int_1^\infty \sin^m(ax) \frac{\log^k x}{x^s} \, dx = \frac{1}{(2i)^m} \sum_{l=0}^{m} \binom{m}{l} (-1)^l V[a(m-2l), k, s]; \]

(35) \[ \int_1^\infty \cos^m(ax) \frac{\log^k x}{x^s} \, dx = \frac{1}{2^m} \sum_{l=0}^{m} \binom{m}{l} V[a(m-2l), k, s]. \]

Real and imaginary part of the value \( V(a, 1, 1) \)

(36) \[ \int_1^\infty e^{iax} \frac{\log x}{x^s} \, dx = \int_1^\infty \cos(ax) \frac{\log x}{x^s} \, dx + i \int_1^\infty \sin(ax) \frac{\log x}{x^s} \, dx \]

determined separately. The imaginary part is

(37) \[ \int_1^\infty \sin(ax) \frac{\log x}{x^s} \, dx = \int_0^\infty \sin(ax) \frac{\log x}{x^s} \, dx - \int_0^1 \sin(ax) \frac{\log x}{x^s} \, dx \]

with one constituent \[ \text{[9, (4.421.1)] [4, (865.63)]} \]

(38) \[ \int_0^\infty \sin(ax) \frac{\log x}{x^s} \, dx = -\frac{\pi}{2} (\gamma + \log a) \approx -2.704825746060380848849568 \quad (a = \pi). \]

The integral over \([0, 1]\) is evaluated by Taylor expansion of the sine which leads to the well converging representation

(39) \[ \int_0^1 \sin(ax) \frac{\log x}{x^s} \, dx = -a \sum_{n \geq 0} (-1)^n \frac{a^{2n}}{(2n+1)!(1+2n)^2} \]

\[ = -a \, {}_2F_3 \left( \begin{array}{c} 1/2, 1/2 \\ 3/2, 3/2, 3/2 \end{array} \right| -\frac{a^2}{4} \right) \approx -2.6581349165086 \quad (a = \pi). \]

The difference between this value and \[ \text{[38]} \] represents \[ \text{[37]} \].

(40) \[ \int_1^\infty \sin(ax) \frac{\log x}{x^s} \, dx = -\frac{\pi}{2} (\gamma + \log a) + a \sum_{n \geq 0} (-1)^n \frac{a^{2n}}{(2n+1)!(1+2n)^2} \]

The value at \( a = \pi \) is the head entry in Table \[ \text{[9]} \]

The real part is started from \[ \text{[4, (3.761.9)]} \]

(41) \[ \int_0^\infty \cos(ax) \frac{\log x}{x^{1-\mu}} \, dx = \frac{\Gamma(\mu)}{a^\mu} \cos\left( \frac{\mu \pi}{2} \right), \]

which is differentiated with respect to \( \mu \) with the product rule,

(42) \[ \int_0^\infty \cos(ax) \frac{\log x}{x^{1-\mu}} \, dx = \frac{\Gamma(\mu)}{a^\mu} \left( \psi(\mu) \cos\left( \frac{\mu \pi}{2} \right) - \ln a \cos\left( \frac{\mu \pi}{2} \right) - \frac{\pi}{2} \sin\left( \frac{\mu \pi}{2} \right) \right). \]
Table 9. Table of the imaginary part of $V(\pi, 1, s)$.

| s   | $\Im V(\pi, 1, s)$ |
|-----|-------------------|
| 1   | -0.046690829551739977074516092264 |
| 2   | -0.05040059937483879223041567776 |
| 3   | -0.0447367779764192936988882199  |
| 4   | -0.036181141255873216997609321919 |
| 5   | -0.027931519676734642467423612590 |
| 6   | -0.021096986691682143229642825070 |

Table 10. Table of the real part of $V(\pi, 1, s)$.

| s   | $\Re V(\pi, 1, s)$ |
|-----|--------------------|
| 1   | 0.057624902988631887643485542240 |
| 2   | 0.029913203983934978439301792236 |
| 3   | 0.010937363639874260291206201403 |
| 4   | -0.000250069139610211209861368961 |
| 5   | -0.006024230915536561502482189260 |
| 6   | -0.00850891881202475146200953761 |

The reflection formulas for the $\Gamma$-function and Digamma function $\psi$ show that in the limit $\mu \to 0$ [1, (6.1.17),(6.3.7)]

\[
\int_0^\infty \cos(ax) \frac{\log x}{x^{1-\mu}}\,dx \to -\frac{1}{\mu^2} - \frac{\pi^2}{24} + \gamma \left(\frac{\gamma}{2} + \ln a\right) + \frac{\ln^2 a}{2} + O(\mu)
\]

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant [15, A001620]. Expansion of the cosine in the standard Taylor series [9, (1.411.3)] and interchange of integration and summation yields

\[
\int_0^1 \frac{\cos(ax)}{x^{1-\mu}}\,dx = \sum_{n \geq 0} \frac{(-a^2)^n}{(2n)!(2n+\mu)}.
\]

Differentiation with respect to $\mu$ introduces the logarithm,

\[
\int_0^1 \cos(ax) \frac{\log x}{x^{1-\mu}}\,dx = -\sum_{n \geq 0} \frac{(-a^2)^n}{(2n)!(2n+\mu)^2}.
\]

We subtract this from (43) and notice that the $\sim 1/\mu^2$-singularity cancels with the term $n = 0$ of the series as $\mu \to 0$,

\[
\int_1^\infty \cos(ax) \frac{\log x}{x}\,dx = -\frac{\pi^2}{24} + \gamma \left(\frac{\gamma}{2} + \ln a\right) + \frac{\ln^2 a}{2} + \sum_{n \geq 1} \frac{(-a^2)^n}{(2n)!}(2n)^2.
\]

The sum converges quickly, the value at $a = \pi$ is the first entry in Table 10.

Combined with (40) this reads

\[
\int_1^\infty e^{iax} \frac{\log x}{x}\,dx = -\frac{\pi^2}{24} + \gamma \left(\frac{\gamma}{2} + \ln a\right) + \frac{\ln^2 a}{2} - \frac{\pi}{2}(\gamma + \log a) + \sum_{n \geq 1} \frac{(ia)^n}{n!n^2}.
\]
which constitutes a “root” value $V(a, 1, 1)$ in the tree of integrals (31). A summary of (12) and the associated imaginary part is
\begin{equation}
\int_1^\infty e^{iax} \log x \frac{1}{x^{1+\mu}} dx = V(a, 1, 1-\mu) = \frac{\Gamma(\mu)}{a^\mu} e^{i\pi \mu/2} \left( \psi(\mu) - \ln a + \frac{i\pi}{2} \right) + \sum_{n \geq 0} \frac{(ia)^n}{n!(n + \mu)^2}.
\end{equation}

B.2. Higher Powers of the Rational. Integration of (47) with respect to $a$ is a measure to increase the parameter $s$, the power of $x$ in the denominator, by one:
\begin{equation}
\int_1^\infty e^{iax} \log^k x \frac{1}{x^{1+s}} dx = \frac{1}{i} \int_1^\infty e^{iax} \log x \frac{1}{x^{2}} dx - \frac{1}{i}.
\end{equation}

\begin{equation}
\int_1^\infty e^{iax} \log x \frac{1}{x^{2}} dx = 1 + ia \left[ -\frac{\pi^2}{24} + \gamma \left( \frac{\gamma}{2} + \ln a - 1 \right) + \frac{\ln^2 a}{2} - \ln a + 1 - \frac{i\pi}{2} (\gamma + \ln a - 1) \right] + \sum_{n \geq 1} \frac{(ia)^{n+1}}{(n+1)!} n^2.
\end{equation}

where (17)
\begin{equation}
\sum_{n \geq 1} \frac{(ia)^{n+j}}{(n+j)!} n^2 = \frac{(ai)^{1+j}}{(1+j)!} \sum_{n \geq 1} \frac{(ia)^{n+1}}{(n+1)!}.
\end{equation}

Inserting $a = \pi$ yields the second lines in Table 10 and 9. The concept of (49) generalizes to higher powers $k$ of the logarithm,
\begin{equation}
\int_0^a da' V(a', k, s) = \frac{1}{i} \int_1^\infty [e^{iax} - 1] \log^k x \frac{1}{x^{1+s}} dx = -iV(a, k, 1 + s) + iV(0, k, 1 + s).
\end{equation}

The last term, the non-oscillatory integrals at $a = 0$, are known (4.272.6)],
\begin{equation}
V(0, 0, s) = \frac{1}{s - 1}, \quad V(0, k, s) = \frac{\Gamma(k + 1)}{(s - 1)^{k+1}}.
\end{equation}

Milgram’s equation (2.29) [10]. Iterated application of this rule computes the chain of $V(a, 1, 1) \rightarrow V(a, 1, 2) \rightarrow V(a, 1, 3) \rightarrow \ldots$ as follows:
\begin{equation}
\int_1^\infty e^{iax} \log x \frac{1}{x^{3}} dx = \frac{\pi^2 a^2}{48} - \frac{\gamma a}{4} \left( \gamma a + 2a \ln a - 3a \right) - \frac{a^2}{4} \left( \ln^2 a - 3 \ln a + 7/2 \right) + \frac{1}{4} + \frac{\pi a^2}{4} \left( \gamma + \ln - \frac{3}{2} \right) + \sum_{n \geq 1} \frac{(ia)^{n+2}}{(n+2)!} n^2.
\end{equation}

\begin{equation}
\int_1^\infty e^{iax} \log x \frac{1}{x^{4}} dx = \frac{\pi^2 a^3}{144} - \frac{\gamma a^3}{12} i(\gamma + 2 \ln a - 11/3) + \frac{a^3}{3} \left( \frac{85}{72} + \frac{11}{12} \ln a - \frac{1}{4} \ln^2 a + \frac{1}{4} \right) - \frac{\pi a^3}{12} \left( \gamma + \ln a - 11/6 \right) - \frac{a^2}{2} + \frac{1}{9} + \sum_{n \geq 1} \frac{(ia)^{n+3}}{(n+3)!} n^2.
\end{equation}

The cases of $a = \pi$ are in Tables 9 and 10.
B.3. **Higher Powers of the Logarithm.** As seen in Section B.1 differentiation with respect to the parameter $\mu$ increases the power of the logarithm:

$$
\frac{d}{ds} V(a, k, s) = - \int_1^\infty e^{iax} \frac{\ln^{k+1} x}{x^s} dx = - V(a, k+1, s).
$$

To support differentiation with respect to the $s$-parameter, we recompute $V(a, 1, 2 - \mu)$, which we re-integrate over $a$ as in (42):

$$
\int_0^1 \int_0^\infty \cos(a') \frac{\log x}{x^{2-\mu}} dx = \int_0^\infty \sin(ax) \frac{\log x}{x^{2-\mu}} dx
$$

$$
= \frac{\Gamma(\mu)}{a^{\mu-1}(1-\mu)} \left[ (\psi(\mu) + \frac{1}{1-\mu} - \ln a) \cos \frac{\mu \pi}{2} - \frac{\pi}{2} \sin \frac{\mu \pi}{2} \right].
$$

The integration is applied in parallel to the complementary interval $0 \leq x \leq 1$,

$$
\int_0^1 \sin(ax) \frac{\log x}{x^{2-\mu}} dx = - \sum_{n\geq 0} \frac{(-)^n a^{2n+1}}{(2n+1)!(2n+\mu)^2},
$$

and the difference is

$$
\int_1^\infty \sin(ax) \frac{\log x}{x^{2-\mu}} dx = \Im V(a, 1, 2 - \mu)
$$

$$
= \frac{\Gamma(\mu)}{a^{\mu-1}(1-\mu)} \left[ (\psi(\mu) + \frac{1}{1-\mu} - \ln a) \cos \frac{\mu \pi}{2} - \frac{\pi}{2} \sin \frac{\mu \pi}{2} \right] + \sum_{n\geq 0} \frac{(-)^n a^{2n+1}}{(2n+1)!(2n+\mu)^2}.
$$

This is differentiated with respect to $\mu$, and the singularities $\sim \frac{2a}{\mu^3}$ from the incomplete Gamma-function and the $n$-sum cancel in the limit $\mu \to 0$:

$$
\int_1^\infty \sin(ax) \frac{\log^2 x}{x^2} dx = a \left[ \gamma (-\gamma + 2 - \ln a) \ln a + \frac{1}{12} \pi^2 (\ln a - 1 + \gamma) - \frac{2}{3} \zeta(3) 
\right.

- 2 \ln a + \ln^2 a - \frac{1}{3} \ln^3 a - \frac{\gamma^3}{3} - 2\gamma + \gamma^2 + 2

- \left. 2a \sum_{n\geq 1} (-a^2)^n \frac{(2n+1)!}{(2n+\mu)^3} \right].
$$

The explicit value at $a = \pi$ is

$$
\int_1^\infty \sin(\pi x) \frac{\log^2 x}{x^2} dx = \Im V(\pi, 2, 2) \approx -0.00240604184022261982751961704408.
$$

This demonstrates the technique. Starting from $V(a, 2, 2)$, a ladder of integrals is constructed according to (62). Each time, a term $\sim (-1)^k k! (\pi a)^{k-1}/\mu^{k+1}$ cancels—carried over from the simple pole of the $\Gamma$-function through $k$ differentiations and $s - 1$ integrations—when the complementary integrals of $0 \leq x < \infty$ and $0 \leq x \leq 1$ are combined. Numerical examples are gathered in Table 11.

**Remark 4.** Partial integration

$$
\int e^{iax} \frac{\log^k x}{x^s} dx = \int e^{iax} \frac{\log x}{x^{s-1}} \frac{1}{x} dx
$$

$$
e^{iax} \frac{\log^{k+1} x}{x^{s-1}} - \int \left[ iae^{iax} \frac{\log^k x}{x^{s-1}} + e^{iax} k \frac{\log^{k-1} x}{x^s} + e^{iax} (1-s) \frac{\log^k x}{x^s} \right] \log x dx
$$
OSCILLATORY INTEGRAL OVER $\exp(\pi x) x^{1/15}$

Table 11. Table of $V(\pi, k, s)$.

| $k$ | $s$ | $\Re V(\pi, k, s)$ | $\Im V(\pi, k, s)$ |
|-----|-----|---------------------|---------------------|
| 2   | 2   | 0.0234592920732284411497929 | -0.002406041840240226198275196 |
| 3   | 3   | 0.01471676532407107850628908 | -0.007873907979083225509317 |
| 4   | 4   | 0.0080788243950624846223234 | -0.0087094002295813942921939 |
| 5   | 5   | 0.0038282314868382690834801 | -0.0076206659960747444416239 |
| 6   | 6   | 0.001384638136936091659413 | -0.0060334458936144149013900 |
| 7   | 7   | 0.00099871035655597496592 | -0.0045501919692521268703906 |
| 8   | 8   | -0.000509788978001596357674 | -0.003354839064072548162349 |
| 3   | 3   | 0.007879609444753496396155 | 0.00257809914754867966813 |
| 4   | 4   | 0.005379045281007149354663 | -0.000457891991342486140616 |
| 5   | 5   | 0.003230811441634551812315 | -0.001490807212523131998936 |
| 6   | 6   | 0.0017674846795167938450865 | -0.0015900553882259670812147 |
| 7   | 7   | 0.0008825074830997475214707 | -0.00134963170541886490445 |
| 8   | 8   | 0.00038723747938897353530 | -0.001041721813275702486827 |
| 4   | 4   | 0.0024472803443498967214338 | 0.00181304867026068466124 |
| 5   | 5   | 0.0018067911153741057224 | 0.0004312822675128498086496 |
| 6   | 6   | 0.001143001029650516946321 | -0.0001368354074002352164914 |
| 7   | 7   | 0.00065998517397927930692718 | -0.000301286041279483958833 |
| 8   | 8   | 0.0003564954008069045751359 | -0.0002990689545146756726797 |
| 5   | 5   | 0.0007164409787722497618537 | 0.0008918125440361658853376 |
| 6   | 6   | 0.0005826627556204301506092 | 0.000313286041279483958833 |
| 7   | 7   | 0.0003859348621737006208241 | -0.000040413520429189446708 |
| 8   | 8   | 0.00023040002848926492480918 | -0.0000473426202252359704144 |

yields a contiguous relation for three points in a triangle in the square grid of $(k,s)$-pairs (Milgram’s equation (2.4) [16]):

(63) $V(a,k,s) = -\frac{ia}{1+k}V(a,k+1,s-1) + \frac{s-1}{1+k}V(a,k+1,s)$.

Since the calculation of values at large $k$ is a laborious task, this formula offers a route to cheaper calculation by (i) tabulation of $V(a,k,s)$ at some small $k$ up to a rather large $\hat{s}$, involving only integrations of powers of $\ln a$ multiplied by powers of $a$ [9, (2.722)], (ii) numerical calculation of the $V(a,k+\Delta k, \hat{s})$ up to the desired $k$ with some brute force method like the $1/x$ mapping (17), which converges well since $\hat{s}$ is large, (iii) telescoping from $\hat{s}$ backwards with (63) to fill the table for increasing $k$ and decreasing $s$.

Remark 5. The last term in (63) can be eliminated via (56),

(64) $(1+k)V(a,k,s) = -iaV(a,k+1,s-1) + (1-s)\frac{d}{ds}V(a,k,s)$.

The solution of this inhomogeneous differential equation is

(65) $V(a,k,s) = \frac{1}{(s-1)^{1+k}}\left(-ia\int V(a,k+1,s-1)(s-1)^kd\hat{s} + \text{const}\right)$.
Appendix C. Inverse Euler-Maclaurin

A standard idea of integration is to split the integral over intervals that are commensurable with the frequency of the oscillation, to replace the generic factor $g$ in the integral by some approximation which allows integration in closed form—assuming that a massive cancellation can be obtained—, then to gather the sum over the intervals with some Euler-Maclaurin approach [2]. Applied to (23),

$$F_m = \int_m^\infty e^{i\pi x} g(x) dx = \sum_{k=m+1,m+3,m+5,...}^\infty \int_{k-1}^{k+1} e^{i\pi x} g(x) dx,$$

$g$ is approximated by its Taylor series, [15, 8],

$$g(x) = \sum_{d=0}^\infty \frac{1}{d!} g^{(d)}(k)(x-k)^d.$$

$$F_m = \sum_{k=m+1,m+3,m+5,...}^\infty e^{i\pi k} \int_{-1}^1 dy e^{i\pi y} \sum_{d=0}^\infty \frac{1}{d!} g^{(d)}(k)y^d.$$

Definition 5. (Moments of Filon Quadratures)

$$S(d) \equiv \int_{-1}^1 e^{i\pi y} y^d dy; \quad \bar{S}(d) \equiv \frac{S(d)}{d!}; \quad d = 0, 1, 2, 3, \ldots$$

Initial values are

$$S(0) = 0; \quad S(1) = \frac{2i}{\pi}.$$

The recurrence is

$$S(d+1) = \frac{i}{\pi} \left\{ [1 + (-1)^d] + (d+1)S(d) \right\}.$$

With [9] (3.761.5), (3.761.10) the cases for even and odd indices are

$$\bar{S}(2d+1) = \frac{2i}{\pi^{2d+1}} \sum_{j=0}^d \frac{1}{(2j+1)!} (-\pi^2)^j.$$  

$$\bar{S}(2d) = \frac{2}{\pi^{2d}} \sum_{j=0}^{d-1} \frac{1}{(2j+1)!} (-\pi^2)^j.$$  

This rewrites (68).

$$F_m = \sum_{k=m+1,m+3,m+5,...}^\infty e^{i\pi k} \sum_{d=0}^\infty \bar{S}(d) g^{(d)}(k) = (-1)^{m+1} \sum_{l=0}^\infty \sum_{d=1}^\infty \bar{S}(d) g^{(d)}(m+2l+1).$$

The Euler-Maclaurin formula proposes to replace the sum over the $d$-th derivatives by

$$\sum_{l=0}^\infty g^{(d)}(m+2l+1) = \frac{1}{2} g^{(d)}(m+1) + \int_{m+1}^\infty g^{(d)}(x) dx - \sum_{D=1,3,5,...}^\infty \frac{B_{D+1}}{(D+1)!} 2^{D+1} g^{(d+D)}(m+1).$$
OSCILLATORY INTEGRAL OVER $\exp(i\pi x) x^{1/x}$

$F_m$ becomes a double sum over $d$ and $D$, and resummation proposes Algorithm 6 to accumulate the derivatives of the (smooth) function $g$ to calculate

$$M_I = \frac{i}{\pi} (F - 2); \quad g \equiv x^{1/x} \frac{1 - \log x}{x^2}.$$

Algorithm 6. (1-sided Fourier-Euler-Maclaurin)

$$2(-)^{m+1} F = -\bar{S}(1) g^{(0)} (m+1)$$

$$+ \sum_{d=1}^\infty \left( \bar{S}(d) - \bar{S}(d+1) - \sum_{l=0}^{\lfloor (d-1)/2 \rfloor} \bar{S}(d - 1 - 2l) \frac{B_{2l+2} x^{2+2l}}{(2+2l)!} \right) g^{(d)} (m+1).$$

Implementation of this approach reveals that the sum over the $d$-th derivatives shows converging behavior only up to $d \approx 6$. I attribute this to the same logarithmic branch cut that constrains the useful depths of the partial integrations in Algorithms 1 and 2.

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