DELAUNAY DECOMPOSITIONS IN DIMENSION 4

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Abstract. We prove that any Delaunay decomposition in dimension 4 is simplicially generating.

0. Introduction

The Voronoi cone decompositions have been attracting our attention in the compactification problem of the moduli scheme of abelian varieties since Namikawa’s work [NY76]. The objects to add as the boundary of the moduli scheme are stable quasi-abelian schemes, reduced or nonreduced, which are described in terms of Delaunay decompositions [NI75, NY76, AN99, NI99, NI10]. The purpose of this note is to prove that any Delaunay decomposition is simplicially generating if the dimension is less than 5 (Theorem 1.3). As a corollary, if the dimension is less than 5, a kind of stable quasi-abelian schemes called projectively stable quasi-abelian schemes are reduced, and therefore two kinds of stable quasi-abelian schemes, projectively stable quasi-abelian schemes [NI99] and torically stable quasi-abelian varieties [NI10], are the same class of varieties.

1. Delaunay decompositions

1.1. Definitions and Notation. Let $\mathbb{Z}_0$ (resp. $\mathbb{Q}_0, \mathbb{R}_0$) be the set of all nonnegative integers (resp. nonnegative rational numbers, nonnegative real numbers). Let $X$ be a lattice of rank $g$, $X_\mathbb{R} = X \otimes \mathbb{R}$, and let $B : X \times X \to \mathbb{Z}$ be a real positive definite symmetric bilinear form $B : X \times X \to \mathbb{Z}$, which determines the inner product $B(\cdot, \cdot)$ and a distance $\| \|$ on the Euclidean space $X_\mathbb{R}$ by $\|x\| := \sqrt{B(x, x)}$ ($x \in X_\mathbb{R}$) respectively. In what follows we fix $B$ once for all. For any $\alpha \in X_\mathbb{R}$ we say that $a \in X$ is nearest to $\alpha$ if

$$\|a - \alpha\| = \min\{\|b - \alpha\| : b \in X\}$$

We define a (closed) $B$-Delaunay cell $\sigma$ (or simply a D-cell if $B$ is understood) to be the closed convex closure of all lattice elements which are nearest to $\alpha$ for some $\alpha \in X_\mathbb{R}$. Note that for a given D-cell $\sigma$, $\alpha$ is uniquely determined as the center of $\sigma$.

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defined only if \( \sigma \) has the maximal possible dimension, equal to \( g \). In this case we call \( \alpha \) the hole or the center of \( \sigma \). Together all the \( B \)-D-cells constitute a locally finite decomposition of \( X_R \) into infinitely many bounded convex polyhedra which we call the \( B \)-Delaunay decomposition \( \mathrm{Del}_B \). It is clear from the definition that the Delaunay decomposition is invariant under translation by the lattice \( X \) and that the 0-dimensional cells are precisely the elements of \( X \). Let \( \mathrm{Del} := \mathrm{Del}_B \), and \( \mathrm{Del}(c) \) the set of all the D-cells containing \( c \in X \).

For any closed subset \( S \) of \( X_R \), we define \( \langle S \rangle \) be the convex closure of \( S \). If \( 0 \in S \), then we define \( C(0, S) \) (resp. \( \mathrm{Semi}(0, S) \)) to be the closed cone generated by \( S \) over \( R_0 \) (resp. the semigroup generated by \( S \) over \( Z_0 \)).

A closed convex bounded subset \( \tau \) of \( X_R \) is called \emph{integral} if \( \tau \) is the convex closure of \( \tau \cap X \). So any D-cell is integral. Let \( \tau \) be a \( g \)-dimensional integral (closed convex) subset of \( X_R \) which contains the origin 0. Then \( \tau \) is called \emph{totally generating} if \( C(0, \tau) \cap X = \mathrm{Semi}(0, \tau \cap X) \), while \( \tau \) is called \emph{simplicially generating} if there exist subsets \( \tau_i (i \in I) \) of \( \tau \):

- \( 0 \notin \tau_i \) and \( \tau_j \) have no common interiors for any \( i \neq j \);
- \( C(0, \tau_i) \cap X = \mathrm{Semi}(0, \tau_i \cap X) \);
- \( C(0, \tau) = \cup_{i \in I} C(0, \tau_i) \).

We need to modify the notion "simplicially generating" in [NI99, 1.1, pp. 662-663] into the above in order to prove Theorem 1.3 because there might exist \( \tau_i \) such that \( 0 \notin \tau_i \). See [NI99, 1.6, pp. 662-663] and Remark 2.1.

1.2. The Voronoi cones. Let \( X \) be a lattice of rank \( g \), and \( Y_g \) (resp. \( Y_g^+ \)) the space of real positive semi-definite symmetric bilinear forms on \( X_R \). Let \( s_i (i \in [1, g]) \) be a \( Z \)-basis of \( X \). Then we identify \( Y_g \) with the space of real positive semi-definite symmetric \( g \times g \) matrices \( B \) in an obvious manner: \( B = (b_{ij}) = (B(s_i, s_j)) \). For a subset \( I \) of \( [1, g] \), we set \( s_I = \sum_{i \in I} s_i \).

A closed subset \( V \) of \( Y_g \) is called a Voronoi cone if there exists a unique Delaunay decomposition of \( X_R \), which we denote by \( \mathrm{Del}_V \), such that

(a) \( \mathrm{Del}_V = \mathrm{Del}_B \) for any \( B \in V^0 \);

(b) for \( B \in Y_g \), \( B \in V^0 \) iff \( \mathrm{Del}_B = \mathrm{Del}_V \),

where \( V^0 \) stands for the relative interior of \( V \). The Voronoi cones form a closed cone decomposition of \( Y_g \), which is called the \emph{(second) Voronoi decomposition} of \( Y_g \). By [Vo09], the (second) Voronoi decomposition of \( Y_g \) is admissible [AMRT, p. 252], [NY76, 2.4].

The following is known by [Vo09] pp. 157-175.

(i) For \( g \leq 3 \), there is a unique \( \mathrm{GL}(g, Z) \)-equivalence class \( V \) of \( \frac{1}{2} g(g+1) \)-dimensional Voronoi cone, and any \( g \)-dimensional D-cell of \( \mathrm{Del}_V (0) \) is a simplex with \( g \) vertices besides 0, which is simplicially generating.

(ii) There are only three \( \mathrm{GL}(4, Z) \)-equivalence classes of 10-dimensional Voronoi cones \( V_1, V_2 \) and \( V_3 \). See Subsec. 3.2 for \( V_i \).

(iii) Any 4-dimensional D-cell of \( \mathrm{Del}_V (0) \) \( (i = 1, 2, 3) \) is a simplex with 4 vertices besides 0, which is simplicially generating.
A proof of (iii) is outlined in Sec. 4. See Theorem 4.2.

1.3. Fusion of Delaunay $g$-cells. Let $V$ be a Voronoi cone, $V^0$ the relative interior of $V$, and $V'$ a face of $V$, that is, a Voronoi cone contained in $V$ which is a proper subset of $V$. Let $B : [0, 1] \to Y_g$ be a continuous map such that $B([0, 1]) \subset V^0$ and $B(0) \in V'$. Let $\text{Del}_V$ (resp. $\text{Del}_{V'}$) be the Delaunay decomposition of $V$ (resp. $V'$).

The following may be rather implicit in [Vo09], [ER88] and [Va03].

Lemma 1.1. Let $V$ and $V'$ be Voronoi cones such that $V' \subset V$. Then any Delaunay $g$-cell of $V$ is contained in some Delaunay $g$-cell of $V'$. Conversely any Delaunay $g$-cell of $V'$ is the union of some Delaunay $g$-cells of $V$.

Proof. The Delaunay decomposition of $V$ (resp. $V'$) is the same as that of $B(t)$ ($t \in (0, 1]$ (resp. that of $B(0)$). Let $\sigma$ be a Delaunay $g$-cell of $V$, and $\alpha$ the center of $\sigma$. Though $\sigma$ is constant, the center $\alpha = \alpha(t)$ is a continuous function of $t \in (0, 1]$, which is a polynomial function of the coefficients of $B(t)$ divided by $|B(t)|$. Since $|B(0)| \neq 0$, $\alpha(0) := \lim_{t \to 0} \alpha(t)$ exists. Since there is no element of $X$ in the interior of $\sigma$, $\sigma \cap X$ is the set of all vertices of $\sigma$. Let $\sigma \cap X := \{a_i; i \in [1, n]\}$. For $t \in (0, 1]$,

$$B(t)(a_i - \alpha(t), a_i - \alpha(t)) = \min_{a \in X} B(t)(a - \alpha(t), a - \alpha(t)).$$

Hence

$$B(0)(a_i - \alpha(0), a_i - \alpha(0)) = \min_{a \in X} B(0)(a - \alpha(0), a - \alpha(0)).$$

This implies that $\alpha(0)$ is a center of a Delaunay cell $\sigma'$ which contains $\sigma$. Since $X_R$ is the union of Delaunay cells of $B(t)$, this shows that any Delaunay $g$-cell of $V'$ is the union of some Delaunay $g$-cells of $V$. \hfill $\square$

Corollary 1.2. Let $\sigma' \in \text{Del}_{V'}(0)$ such that $\sigma' = \bigcup_{\lambda \in \Lambda} \sigma_{\lambda}$ for some Delaunay cells $\sigma_{\lambda} \in \text{Del}_V$. Then

1. $C(0, \sigma', 0) = \bigcup_{0 \in \sigma \subset \sigma'} C(0, \sigma_{\lambda})$.

2. If $C(0, \sigma_{\lambda}) \cap X = \text{Semi} (0, \sigma_{\lambda} \cap X)$ for any $\lambda$, then $C(0, \sigma') \cap X = \text{Semi} (0, \sigma' \cap X)$.

Proof. Let $D(r)$ be a ball of radius $r$ with the origin as center. Since $\sigma'$ is convex, $C(0, \sigma') = C(0, D(r) \cap \sigma')$ for a sufficiently small $r > 0$. By Lemma 1.1

$$C(0, D(r) \cap \sigma') = C(0, D(r) \cap \bigcup_{\lambda \in \Lambda} \sigma_{\lambda}) = \bigcup_{0 \in \sigma_{\lambda}} C(0, \sigma_{\lambda}),$$

where we caution that there might exist $\lambda \in \Lambda$ such that $0 \notin \sigma_{\lambda}$. This proves (1). \hfill $\square$

(2) follows from (1). \hfill $\square$

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1Not all $\sigma_{\lambda}$ contains 0. See Remark 2.1.
1.4. Simplicial generation. The purpose of this note is to prove the following theorem, which seems to be known but only implicit in [Vo09]:

**Theorem 1.3.** If the dimension is less than five, any Delaunay decomposition is simplicially generating, in particular, the nilpotency [AN99, 1.15] of any Delaunay decomposition is equal to one.

See [AN99, 1.14] and [NI99, 1.6].

**Proof.** Clear from Corollary 1.2 and Subsec. 1.2 (i)-(iii).

□

**Corollary 1.4.** Any projectively stable quasi-abelian scheme (abbr. PSQAS) over a field is reduced if its dimension is less than 5.

**Proof.** Let $Q_0$ be a $g$-dimensional PSQAS over $k$. See [NI99]. It suffices to prove the corollary in the case where that it is a special fiber of a totally degenerate flat projective family $(Q, \mathcal{L}_Q)$ over $\text{Spec } R$ for some complete discrete valuation ring $R$ with residue field $k$. The family $(Q, \mathcal{L}_Q)$ is given by [NI99, 5.8]. Let $(P, \mathcal{L})$ be the normalization of $(Q, \mathcal{L}_Q)$, and $\mathcal{L}$ the pullback of $\mathcal{L}_Q$ to $P$ [NI99, 3.1, 5.1]. The family $(Q, \mathcal{L}_Q)$ (resp. $(P, \mathcal{L})$) is an algebraization of the formal quotient of the formal completion of an $R$-scheme $(\tilde{Q}, \tilde{\mathcal{L}}_Q)$ (resp. $(\tilde{P}, \tilde{\mathcal{L}})$) locally of finite type, each of which is given in [NI99, 5.3] (resp. a bit implicitly in [NI99, 3.4, 3.7, 3.8, 4.10]). The local affine chart $U(c)$ of $\tilde{P}$ ($c \in X$) is the normalization of a local affine chart $W(c)$ of $\tilde{Q}$, where $X$ is a lattice of rank $g$. Let $R(c)$ (resp. $S(c)$) be the coordinate ring of $U(c)$ (resp. $W(c)$), where each $R(c)$ (resp. $S(c)$) is isomorphic to $R(0)$ (resp. $S(0)$). Moreover there are some monomials $\zeta_{x,0}$ and $\xi_{x,0}$ such that

$$R(0) = R[\zeta_{x,0}, x \in C(0, \sigma) \cap X, \sigma \in \text{Del}(0)],$$

$$S(0) = R[\xi_{x,0}, x \in \text{Semi}(0, \sigma), \sigma \in \text{Del}(0)],$$

where by [NI99] Definition 3.4], $\zeta_{x,0} = \zeta_{x,0}$ if $x \in \sigma \cap X$ for $\sigma \in \text{Del}(0)$.

By Theorem 1.3, $C(0, \sigma) \cap X = \text{Semi}(0, \sigma)$ for any $g$-dimensional Delaunay cell $\sigma$ if $g \leq 4$. It follows that $\tilde{Q} = \tilde{P}$, $Q = P$ and $Q_0 = P_0$.

It follows from [AN99, 3.12] that $P_0$ is generically reduced (along any irreducible component of it) if $g \leq 4$. Hence there is no nontrivial torsion of the structure sheaf $O_{P_0}$ whose support contains an irreducible component of $P_0$. Since $P$ is Cohen-Macaulay, so is $P_0$ [AN99, 4.1], hence there is no torsion of $O_{P_0}$ whose support is nonempty but at most $(g - 1)$-dimensional. This completes the proof. □

2. Degree two and three

2.1. Degree two. Let $s_1$ and $s_2$ be a basis of $X$, $s_{12} := s_1 + s_2$ and $Y_2$ the space of real symmetric positive semi-definite bilinear forms on $X_R$, each bilinear form being identified with a $2 \times 2$ matrix. We define $V_1$, $V_2$ and
Consider the limit \( \lim_{t \to 0} V_t \). Let \( V_1 \cap V_2 = R_0 e_{13} + R_0 e_{23} + R_0 e_{12} \),
\( V_2 = R_0 e_{13} + R_0 e_{23} + R_0 f_{12} \),
\( V_1 \cap V_2 = R_0 e_{13} + R_0 e_{23} \),
where
\[
\begin{align*}
e_{13} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},
\quad e_{23} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
\quad e_{12} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},
\quad f_{12} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\end{align*}
\]

Then \( V_t \) (\( t = 1, 2 \)) and \( V_1 \cap V_2 \) are Voronoi cones, each of whose relative interior determines a unique Delaunay decomposition of \( X_R = \mathbb{R}^2 \):
\[
\begin{align*}
\text{Del}_{V_t} &= \{ \sigma_1, \sigma_2 \text{ their translates by } X \text{ and their faces} \},
\text{Del}_{V_2} &= \{ \sigma_3, \sigma_4 \text{ their translates by } X \text{ and their faces} \},
\text{Del}_{V_1 \cap V_2} &= \{ \sigma_5 \text{ their translates by } X \text{ and their faces} \}
\end{align*}
\]
where
\[
\begin{align*}
\sigma_1 &= \langle 0, s_1, s_{12} \rangle,
\sigma_2 &= \langle 0, s_2, s_{12} \rangle,
\sigma_3 &= \langle 0, s_1, s_2 \rangle,
\sigma_4 &= \langle s_1, s_2, s_{12} \rangle,
\sigma_5 &= \langle 0, s_1, s_2, s_{12} \rangle.
\end{align*}
\]
Let \( B \in V_1^0 \) and let \( c_i \) be the center of \( \sigma_i \) (\( i = 1, 2 \)):
\[
\begin{align*}
B(c_1, c_1) &= B(s_1 - c_1, s_1 - c_1) = B(s_{12} - c_1, s_{12} - c_1),
B(c_2, c_2) &= B(s_2 - c_2, s_2 - c_2) = B(s_{12} - c_2, s_{12} - c_2).
\end{align*}
\]
Equivalently,
\[
\begin{align*}
B(s_1, s_1) &= 2B(s_1, c_1),
B(s_2, s_2) &= 2B(s_2, c_1),
B(s_2, s_2) &= 2B(s_2, c_2),
B(s_1, s_1) + 2B(s_1, s_2) &= 2B(s_1, c_2).
\end{align*}
\]
Let \( V_1^0 \) be the relative interior of \( V_1 \), \( t := b_{12}, B(t) := (b_{ij}) \in V_1^0 \), and consider the limit \( t \to 0 \). Let \( B(0) = \lim_{t \to 0} B(t) = \text{diag}(b_{11}, b_{22}) \). Since
\[
B(0)(s_1, s_2) = 0, B(0)(\bullet, \lim c_1) = B(0)(\bullet, \lim c_2),
\]
hence \( \lim c_1 = \lim c_2 \). This implies that \( \sigma_1 \) and \( \sigma_2 \) fuse together into a D-cell of \( B(0) \):
\[
\sigma_5 = \sigma_1 \cup \sigma_2.
\]
In other words, the Delaunay decomposition of \( \lim B \) is obtained as a suitable fusion of the Delaunay decomposition of \( B \).
Similarly as we take the limit from \( V_2 \) to \( V_1 \cap V_2 \), we have another fusion
\[
\sigma_5 = \sigma_3 \cup \sigma_4.
\]
**Remark 2.1.** Note that \( C(0, \sigma_5) = C(0, \sigma_3) \) and \( 0 \notin \sigma_4 \).
2.2. **Degree three.** There is, up to $GL(3,\mathbb{Z})$-equivalence, a unique Voronoi cone of dimension 6 in $Y_3$:

$$V = \Sigma_{1 \leq i < j \leq 4} R_0 e_{ij}.$$  

The Delaunay decomposition of $V$ consists of 3 dimensional cells

$$\sigma_{ijk} := \langle 0, s_i, s_{ij}, s_{ijk} \rangle, \quad 1 \leq i, j, k \leq 3, \text{ all distinct}$$

and $GL(3,\mathbb{Z})$-transforms and $\mathbb{Z}^3$-translates.

3. **Degree four**

3.1. **Notation.** We define

$$e_{ij} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\frac{i}{i} & \frac{j}{j} & 0 & 0 \\
0 & 1 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad e_{i5} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
\frac{i}{i} & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

$$e_{12,345} = \begin{pmatrix}
2 & 1 & -1 & -1 & 0 \\
1 & 2 & -1 & -1 & 0 \\
-1 & -1 & 2 & 0 & 2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad f_{1234} = \begin{pmatrix}
1 & 1 & -1 & -1 \\
1 & 1 & -1 & -1 & 0 \\
-1 & -1 & 1 & 1 & 0 \\
-1 & -1 & 1 & 1 & 0
\end{pmatrix}$$

$$g_{123} = \begin{pmatrix}
1 & 1 & -1 & 0 \\
1 & 1 & -1 & 0 \\
-1 & -1 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \quad g_{124} = \begin{pmatrix}
1 & 1 & 0 & -1 \\
1 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
-1 & -1 & 0 & 1
\end{pmatrix}$$

where

$$\omega = e_{12,345} = \frac{1}{3} \left( \sum_{(i,j) \neq (1,2)} e_{ij} + f_{1234} + g_{123} + g_{124} \right).$$

We also define

$$V_1 = \Sigma_{i,j} R_0 e_{ij},$$

$$V_1 \cap V_2 = \Sigma_{(i,j) \neq (12)} R_0 e_{ij}$$

$$V_2 = \Sigma_{(i,j) \neq (1,2)} R_0 e_{ij} + R_0 e_{12345},$$

$$V_2 \cap V_3 = \Sigma_{(i,j) \neq (1,2),(3,4)} R_0 e_{ij} + R_0 e_{12345},$$

$$V_3 = \Sigma_{(i,j) \neq (1,2),(3,4)} R_0 e_{ij} + R_0 e_{12345} + R_0 f_{1234}$$

$$W_0 = \Sigma_{(i,j) \neq (1,2),(3,4)} R_0 e_{ij} + R_0 f_{1234}$$

$$K = \Sigma_{(i,j) \neq (1,2)} R_0 e_{ij} + R_0 g_{123} + R_0 g_{124} + R_0 f_{1234},$$

where $V_i \subset K \ (i = 2, 3)$. It is clear that $V_i$ is a 10-dimensional cone, which is the same as the Delaunay triangulation $\mathcal{D}_i$ in the sense of [Va03, p. 51].
Remark 3.1. With the notation in [67] p. 234, we define the chambers
\[ F_{ab} := \sum_{(i,j) \neq (a,b)} R_0 e_{ij} + R_0 e_{abcde}, \]
\[ G_{abcd} := \sum_{(i,j) \neq (a,b),(c,d)} R_0 e_{ij} + R_0 e_{abcde} + R_0 e_{cdab} \]
\[ F_{abcd} = \{ Y = (y_{ij}) \in G_{abcd}; y_{ab} \geq y_{cd} \}, \]
\[ F_{cdab} = \{ Y = (y_{ij}) \in G_{abcd}; y_{cd} \geq y_{ab} \} \]
for any subset \( \{a, b\} \subset [1, 4] \) and \( \{c, d\} = [1, 4] \setminus \{a, b\} \). This supplements the definition of \( F_{ab} \) and \( F_{abcd} \) in [N99] p. 664.

Note that \( F_{12} = V_2 \) is a Voronoi cone. Another chamber \( G_{1234} \) is the union of two \( \text{GL}(4, \mathbb{Z}) \)-equivalent Voronoi cones \( V_3 \) and \( V_4 \) (of type III) where
\[ V_3 = \sum_{(i,j) \neq (1,2),(3,4)} R_0 e_{ij} + R_0 e_{12345} + R_0 f_{1234} = F_{1234}, \]
\[ V_4 := \sum_{(i,j) \neq (1,2),(3,4)} R_0 e_{ij} + R_0 e_{34125} + R_0 f_{1234} = F_{3412}, \]
\[ V_3 \cap V_4 = \sum_{(i,j) \neq (1,2),(3,4)} R_0 e_{ij} + R_0 f_{1234} = W_0. \]

because
\[ e_{12345} + e_{34125} = f_{1234} + \sum_{(i,j) \neq (1,2),(3,4)} e_{ij}. \]

3.2. Black forks and red triangles. In what follows we identify the space of real symmetric bilinear forms on \( \mathbb{R}^4 \) with the space of real quadratic polynomials of 4 variables in an obvious way. There are two perfect cones \( V_1 \) and \( K \) in 4 variables by [V09] Part I, p. 172]. The cone \( V_1 \) is a 10-dimensional cone generated by
\[ x_i^2, (x_i - x_j)^2, (i, j \in [1, 4]) \]
where \( x_i^2 \) (resp. \( (x_i - x_j)^2 \)) corresponds to \( e_{i5} \) (resp. \( e_{ij} \)).

The cone \( K \) is also a 10-dimensional cone generated by
\[ x_i^2, (x_i - x_j)^2, (i, j \in [1, 4], (i, j) \neq (1, 2)), \]
\[ (x_1 + x_2 - x_3)^2, (x_1 + x_2 - x_4)^2, (x_1 + x_2 - x_3 - x_4)^2. \]

(1)

Let \( \omega \) be
\[ \omega = \frac{1}{3} \left( \sum_{i=1}^{4} x_i^2 + \sum_{(i,j) \neq (1,2)} (x_i - x_j)^2 \right. \]
\[ + (x_1 + x_2 - x_3)^2 + (x_1 + x_2 - x_4)^2 + (x_1 + x_2 - x_3 - x_4)^2 \]
\[ = 2 \sum_{i=1}^{4} x_i^2 + 2x_1x_2 - 2 \sum_{i=1,2;j=3,4} x_ix_j. \]

Note that \( \omega = e_{12345} \) in matrix form.

By the transformation (which we refer to as Voronoi transformation)
\[ (x_1, x_2, x_3, x_4) \mapsto (x_1 + x_2, x_1 - x_2, x_1 - x_3, x_1 - x_4) \]

(2)
every generator \((1)\) of \(K\) is transformed into either \((x_i + x_j)^2\) or \((x_i - x_j)^2\). While \((x_1 - x_2)^2\) is transformed into \(4x_2^2\). Hence

\[
K = \{ x = \sum_{i<j} \beta_{ij}(x_i + x_j)^2 + \sum_{i<j} \rho_{ij}(x_i - x_j)^2; \beta_{ij} \geq 0, \rho_{ij} \geq 0 \}.
\]

Let \(W\) be a face of \(K\) of codimension one. By [ER88, p. 1060], there exist precisely three pairs \((i, j)\) such that \(x \in W\) iff either \(\beta_{ij} = 0\) or \(\rho_{ij} = 0\). If \(\beta_{pq} = 0\) (resp. \(\rho_{pq} = 0\)), we connect the vertices \(p\) and \(q\) by a black edge (resp. a red edge). Then

(i) to each face \(W\), we can associate a connected graph \(\Gamma(W)\) with three colored edges, which is either forked or triangular.

(ii) Conversely, for any such graph \(\Gamma\) there is a unique face \(W\) of \(K\) of codimension one such that \(\Gamma(W) = \Gamma\) (as colored graphs).

(iii) There are 32 forked graphs and 32 triangular graphs by [ER88, 2.1, p. 1061]. (Easy)

(iv) There are two \(\text{GL}(4, \mathbb{Z})\)-equivalence classes of 64 faces, one being the equivalence class of black forked faces, and the other being the equivalence class of red triangular faces. We call the first equivalence class (resp. the second) BF (resp. RT).

(v) By [ER88, 2.5, p. 1063] any cone generated by \(e_{12345}\) and BF (resp. \(e_{12345}\) and RT) is a domain of type II (resp. type III) in the sense of Voronoi [Vo09].

(vi) By [ER88, 2.2, 2.3, p. 1060] BF (resp. RT) consists of 48 faces (resp. 16 faces), and each equivalence class is an orbit of a group \(G\) of order 1152, where the group \(G\) is generated by three types of elements:

\[
\tau_i : x_i \mapsto -x_i, \quad x_j \mapsto x_j \ (j \neq i) \\
\tau_{ij} : x_i \mapsto x_j, \quad x_j \mapsto x_i, \quad x_k \mapsto x_k \ (k \neq i, j) \\
\sigma : x_i \mapsto -x_i + \frac{1}{2} \sum_{k=1}^{4} x_k,
\]

where \(i, j, k \in [1, 4]\).

3.3. The faces \(V_1 \cap V_2\) and \(V_3 \cap V_4\). The following lemma follows easily from [ER88, § 2, pp.1060-1064].

Lemma 3.2. The cone \(W_0 = V_3 \cap V_4\) is \(\text{GL}(4, \mathbb{Z})\)-equivalent to a red triangle face of \(K\). Moreover \(V_1 \cap V_2\) is a black triangle face of \(K\), hence \(\text{GL}(4, \mathbb{Z})\)-equivalent to a black fork face.

Proof. It is obvious from Subsec. 3.2 that \(V_1 \cap V_2\) is a black triangle with vertices 1, 3, 4. Hence it is \(\text{GL}(4, \mathbb{Z})\)-equivalent to a black fork (BF) by [ER88, § 2, p.1062]. On the other hand we note that \(W_0\) is a red triangle
face (RT). Indeed, after Voronoi transformation (2), \( W_0 \) is spanned by
\[
(x_1 + x_2)^2, (x_2 + x_3)^2, (x_2 + x_4)^2, (x_3 + x_4)^2,
(x_1 - x_2)^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2,
\]
Hence the missing terms are
\[
(x_1 + x_3)^2, (x_1 + x_4)^2, (x_3 - x_4)^2.
\]
Therefore the graph of \( W_0 \) is a triangle with two black edges and a red edge. The transformation \( \tau_1: x_1 \mapsto -x_1, x_j \mapsto x_j (j \neq 1) \) belongs to the group \( G \), by which the missing terms in \( W_0 \) are transformed into
\[
(x_1 - x_3)^2, (x_1 - x_4)^2, (x_3 - x_4)^2,
\]
whence the graph of \( \tau_1(W_0) \) is a red triangle. Hence the face \( W_0 \) is \( \text{GL}(4, \mathbb{Z}) \)-equivalent to a red triangle. The rest is clear.

\[ \square \]

**Corollary 3.3.** The cone \( V_2 \) (resp. \( V_3 \)) is a 10-dimensional cone of type II (resp. type III) in the sense of Voronoi \([Vo09]\).

**Proof.** Since \( V_3 \) is generated by \( W_0 \) and \( e_{12345} \) and \( W_0 \) is RT, \( V_3 \) is of type III by \([ER88], 2.5, p. 1063\]. Meanwhile since \( V_2 \) is generated by \( V_1 \cap V_2 \) and \( e_{12345} \), and \( V_1 \cap V_2 \) is BF, \( V_2 \) is of type II.

\[ \square \]

4. **The Delaunay decompositions in dimension 4**

4.1. **The Delaunay decomposition of \( V_1 \).** For \( \{a, b, c, d\} = \{1, 2, 3, 4\} \) we define
\[
\sigma_{abcd} := \langle 0, s_a, s_{ab}, s_{abc}, s_{1234} \rangle.
\]
It is easy to see that the Delaunay decomposition of \( V_1 \) is given by \( \sigma_{abcd} \), their faces and their translates by \( \mathbb{Z}^4 \).

Let \( B \in V_1^0, \sigma = \sigma_{1234} \) and \( c \) the center of the D-cell \( \sigma \). Then we have
\[
B(e - c, e - c) = B(c, c) (\forall e \in \sigma \cap X),
\]
hence
\[
B(s_1, s_1) = 2B(s_1, c),
B(s_2, s_2) + 2B(s_1, s_2) = 2B(s_2, c),
B(s_3, s_3) + 2B(s_12, s_3) = 2B(s_3, c),
B(s_4, s_4) + 2B(s_{123}, s_4) = 2B(s_4, c),
B(e, e) - 2B(e, c) > 0 \text{ (otherwise)}.
\]
If \( B(s_1, s_2) = 0 \), then the second equation reduces to \( B(s_2, s_2) = 2B(s_2, c) \), which implies that \( \langle \sigma, s_2 \rangle \) is in the same D-cell of \( B \in V_1 \cap V_2 \).
4.2. The fusion of D-cells of \( V_1 \). Let \( B_0 \in (V_1 \cap V_2)^0 \). Since \( B_0 \in Y_1^+ \), \( B_0 \) determines a Delaunay decomposition of \( X_{R} \). Then there exists a unique D-cell \( \sigma \) of \( B_0 \) such that \( \sigma_{1234} \subseteq \sigma \). Let \( c \) be the center of \( \sigma \). Hence

\[
B_0(s_1, s_1) = 2B_0(s_1, c_0), \\
B_0(s_2, s_2) = 2B_0(s_2, c_0), \\
B_0(s_3, s_3) + 2B_0(s_{12}, s_3) = 2B_0(s_3, c_0), \\
B_0(s_4, s_4) + 2B_0(s_{123}, s_4) = 2B_0(s_4, c_0), \\
B_0(e, e) - 2B_0(e, c_0) \geq 0 \text{ (otherwise)}.
\]

**Lemma 4.1.** Let \( \sigma \) be the unique D-cell \( \sigma \) of \( V_1 \cap V_2 \) such that \( \sigma_{1234} \subseteq \sigma \). Then \( \sigma = \sigma_{1234} \cup \sigma_{2134} \). Similarly \( \sigma_{1243} \cup \sigma_{2143} \) is also a D-cell of \( V_1 \cap V_2 \).

**Proof.** By definition there exists \( r_{ij} > 0 \) such that

\[
B_0 = \sum_{(i, j) \neq (1, 2)} r_{ij} e_{ij}.
\]

It easy to see that if \( x = \sum_{i=1}^{4} x_i s_i \), then by (3)

\[
B_0(x, x) - 2B_0(x, c_0) \\
= \sum_{i=1}^{4} r_{i5}(x_i^2 - x_i) + \sum_{1 \leq i < j \leq 4, \ (ij) \neq (12)} r_{ij}((x_i - x_j)^2 - (x_i - x_j)) \geq 0.
\]

Since every \( r_{ij} > 0 \) for \( (ij) \neq (12) \), \( B_0(x, x) - 2B_0(x, c_0) = 0 \) iff every term in the rhs is equal to 0:

\[
x_i^2 - x_i = 0, (x_i - x_j)^2 - (x_i - x_j) = 0 \forall i, j.
\]

Hence \( x_i = 0, 1 \) and \( x_i - x_j = 0, 1 \) for \( i < j \ (i, j) \neq (1, 2) \). It follows that \( B_0(x, x) - 2B_0(x, c_0) = 0 \) iff

\[
x = 0, (1, 0, 0, 0), (0, 1, 0, 0), (1, 1, 0, 0), (1, 1, 1, 0), (1, 1, 1, 1).
\]

This shows \( \sigma = \sigma_{1234} \cup \sigma_{2134} \). In summary, \( \sigma_{1234} \) and \( \sigma_{2134} \) fuse together into a unique D-cell of \( V_1 \cap V_2 \) when \( B \in V_1^0 \) approaches a point \( B_0 \) of \( V_1 \cap V_2 \).

Similarly \( \sigma_{1243} \cup \sigma_{2143} \) is a D-cell of \( V_1 \cap V_2 \).

4.3. Tables 1 and 2. By similar computations, we obtain Tables 1 and 2. Let us explain what the tables show. The D-cells \( \sigma_{1234} \) and \( \sigma_{2134} \) fuse together into a D-cell of \( V_1 \cap V_2 \), which divides into two D-cells of \( V_2 \)

\[
[0, s_1, s_2, s_{123}, s_{1234}], [s_1, s_2, s_{12}, s_{123}, s_{1234}],
\]

which are no longer D-cells of \( V_1 \). These are numbered 1 and 2 respectively. The D-cells numbered from 3 to 12 are understood similarly. The D-cells of \( V_1 \) numbered from 13 to 24 are D-cells of both \( V_1 \cap V_2 \) and \( V_2 \). This is what Table 1 shows.
Next the D-cells of \( V_2 \) numbered 2 and 4 fuse together into a D-cell of \( V_2 \cap V_3 \), which decomposes into two D-cells of \( V_3 \):
\[
[ s_1, s_2, s_{12}, s_{123}, s_{124} ], [ s_1, s_2, s_{123}, s_{124}, s_{1234} ]
\]
Similarly the D-cells of \( V_2 \) numbered 9 and 11 (resp. 17 and 18, 19 and 20) fuse together into a D-cell of \( V_2 \cap V_3 \), which divides into two D-cells of \( V_3 \) which are no longer D-cells of \( V_2 \). However the other D-cells of \( V_2 \) are also D-cells of both \( V_2 \cap V_3 \) and \( V_3 \). This is what Table 2 shows.

The following is clear from Tables:

**Theorem 4.2.** ([Vo09]) The Delaunay decomposition of \( V_i \) \((i = 1, 2, 3)\) consists of 4-dimensional integral simplicies and their faces, where we mean by a 4-dimensional integral simplex a convex closure \( \langle a_0, \ldots, a_4 \rangle \) such that \( a_i \in X \) and \( a_i - a_0 \) is a \( \mathbb{Z} \)-basis of \( X \). In particular, the Delaunay decomposition of \( V_i \) is simplicially generating.

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Table 1. Fusion and division (1)

| $V_1$ | $V_1 \cap V_2$ | $V_2$ | no |
|-------|-----------------|-------|----|
| $\sigma_{1234}$ | $\sigma_{1234} \cup \sigma_{2134}$ | $(0, s_1, s_2, s_{123}, s_{1234})$ | 1 |
| $\sigma_{2134}$ | | $(s_1, s_2, s_{12}, s_{123}, s_{1234})$ | 2 |
| $\sigma_{1234}$ | $\sigma_{1243} \cup \sigma_{2143}$ | $(0, s_1, s_2, s_{123}, s_{1234})$ | 3 |
| $\sigma_{1243}$ | | $(s_1, s_2, s_{12}, s_{123}, s_{1234})$ | 4 |
| $\sigma_{3124}$ | $\sigma_{3124} \cup \sigma_{3214}$ | $(0, s_3, s_{13}, s_{23}, s_{1234})$ | 5 |
| $\sigma_{3214}$ | | $(0, s_{13}, s_{23}, s_{123}, s_{1234})$ | 6 |
| $\sigma_{4123}$ | $\sigma_{4123} \cup \sigma_{4213}$ | $(0, s_4, s_{14}, s_{24}, s_{1234})$ | 7 |
| $\sigma_{4123}$ | | $(0, s_{14}, s_{24}, s_{123}, s_{1234})$ | 8 |
| $\sigma_{3412}$ | $\sigma_{3412} \cup \sigma_{3421}$ | $(0, s_3, s_{34}, s_{134}, s_{234})$ | 9 |
| $\sigma_{3421}$ | | $(0, s_3, s_{134}, s_{234}, s_{1234})$ | 10 |
| $\sigma_{4312}$ | $\sigma_{4312} \cup \sigma_{4321}$ | $(0, s_4, s_{34}, s_{134}, s_{234})$ | 11 |
| $\sigma_{4321}$ | | $(0, s_4, s_{134}, s_{234}, s_{1234})$ | 12 |
| $\sigma_{1324}$ | $\sigma_{1324}$ | $\sigma_{1324}$ | 13 |
| $\sigma_{2134}$ | $\sigma_{2134}$ | $\sigma_{2134}$ | 14 |
| $\sigma_{2314}$ | $\sigma_{2314}$ | $\sigma_{2314}$ | 15 |
| $\sigma_{2413}$ | $\sigma_{2413}$ | $\sigma_{2413}$ | 16 |
| $\sigma_{1342}$ | $\sigma_{1342}$ | $\sigma_{1342}$ | 17 |
| $\sigma_{1432}$ | $\sigma_{1432}$ | $\sigma_{1432}$ | 18 |
| $\sigma_{2341}$ | $\sigma_{2341}$ | $\sigma_{2341}$ | 19 |
| $\sigma_{2431}$ | $\sigma_{2431}$ | $\sigma_{2431}$ | 20 |
| $\sigma_{3142}$ | $\sigma_{3142}$ | $\sigma_{3142}$ | 21 |
| $\sigma_{4132}$ | $\sigma_{4132}$ | $\sigma_{4132}$ | 22 |
| $\sigma_{3241}$ | $\sigma_{3241}$ | $\sigma_{3241}$ | 23 |
| $\sigma_{4231}$ | $\sigma_{4231}$ | $\sigma_{4231}$ | 24 |

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| no | $V_2$                                                                 | $V_2 \cap V_3$                                                                 | $V_3$                                                                 |
|----|---------------------------------------------------------------------|--------------------------------------------------------------------------------|----------------------------------------------------------------------|
| 2  | $\langle s_1, s_2, s_{12}, s_{13}, s_{123}, s_{1234} \rangle$       | $\langle s_1, s_2, s_{12}, s_{13}, s_{123}, s_{1234} \rangle$                  | $\langle s_1, s_2, s_{12}, s_{13}, s_{123}, s_{1234} \rangle$          |
| 4  | $\langle s_1, s_2, s_{12}, s_{13}, s_{123}, s_{1234} \rangle$       | $\cup \langle s_1, s_2, s_{12}, s_{13}, s_{123}, s_{1234} \rangle$            | $\langle s_1, s_2, s_{12}, s_{13}, s_{123}, s_{1234} \rangle$          |
| 9  | $\langle 0, s_3, s_{34}, s_{134}, s_{234} \rangle$                  | $\langle 0, s_3, s_{34}, s_{134}, s_{234} \rangle$                              | $\langle 0, s_3, s_{34}, s_{134}, s_{234} \rangle$                      |
| 11 | $\langle 0, s_4, s_{34}, s_{134}, s_{234} \rangle$                  | $\cup \langle 0, s_4, s_{34}, s_{134}, s_{234} \rangle$                        | $\langle s_3, s_4, s_{34}, s_{134}, s_{234} \rangle$                    |
| 17 | $\sigma_{1342}$                                                    | $\sigma_{1342} \cup \sigma_{1432}$                                            | $\langle 0, s_1, s_{13}, s_{14}, s_{1234} \rangle$                     |
| 18 | $\sigma_{1432}$                                                    | $\sigma_{2341} \cup \sigma_{2431}$                                            | $\langle 0, s_1, s_{13}, s_{14}, s_{1234} \rangle$                     |
| 19 | $\sigma_{2341}$                                                    |                                                                      |                                                                      |
| 20 | $\sigma_{2431}$                                                    |                                                                      |                                                                      |
| 1  | $\langle 0, s_1, s_{2}, s_{12}, s_{13}, s_{123} \rangle$             | $\langle 0, s_1, s_2, s_{12}, s_{13}, s_{123} \rangle$                         | $\langle 0, s_1, s_2, s_{12}, s_{13}, s_{123} \rangle$                  |
| 3  | $\langle 0, s_1, s_2, s_{12}, s_{13}, s_{123} \rangle$               | $\langle 0, s_1, s_2, s_{12}, s_{13}, s_{123} \rangle$                         | $\langle 0, s_1, s_2, s_{12}, s_{13}, s_{123} \rangle$                  |
| 5  | $\langle 0, s_3, s_{13}, s_{23}, s_{123} \rangle$                    | $\langle 0, s_3, s_13, s_{23}, s_{123} \rangle$                                | $\langle 0, s_3, s_13, s_{23}, s_{123} \rangle$                         |
| 6  | $\langle 0, s_4, s_{14}, s_{24}, s_{123} \rangle$                    | $\langle 0, s_4, s_14, s_{24}, s_{123} \rangle$                                | $\langle 0, s_4, s_14, s_{24}, s_{123} \rangle$                         |
| 7  | $\langle 0, s_4, s_{14}, s_{24}, s_{123} \rangle$                    | $\langle 0, s_4, s_14, s_{24}, s_{123} \rangle$                                | $\langle 0, s_4, s_14, s_{24}, s_{123} \rangle$                         |
| 8  | $\langle 0, s_4, s_{14}, s_{24}, s_{123} \rangle$                    | $\langle 0, s_4, s_14, s_{24}, s_{123} \rangle$                                | $\langle 0, s_4, s_14, s_{24}, s_{123} \rangle$                         |
| 10 | $\langle 0, s_{13}, s_{14}, s_{234}, s_{123} \rangle$                | $\langle 0, s_{13}, s_{14}, s_{234}, s_{123} \rangle$                          | $\langle 0, s_{13}, s_{14}, s_{234}, s_{123} \rangle$                   |
| 12 | $\langle 0, s_{14}, s_{314}, s_{234}, s_{123} \rangle$               | $\langle 0, s_{14}, s_{314}, s_{234}, s_{123} \rangle$                         | $\langle 0, s_{14}, s_{314}, s_{234}, s_{123} \rangle$                  |
| 13 | $\sigma_{1324}$                                                    | $\sigma_{1324}$                                                              | $\sigma_{1324}$                                                         |
| 14 | $\sigma_{1423}$                                                    | $\sigma_{1423}$                                                              | $\sigma_{1423}$                                                         |
| 15 | $\sigma_{2314}$                                                    | $\sigma_{2314}$                                                              | $\sigma_{2314}$                                                         |
| 16 | $\sigma_{2413}$                                                    | $\sigma_{2413}$                                                              | $\sigma_{2413}$                                                         |
| 21 | $\sigma_{3142}$                                                    | $\sigma_{3142}$                                                              | $\sigma_{3142}$                                                         |
| 22 | $\sigma_{4132}$                                                    | $\sigma_{4132}$                                                              | $\sigma_{4132}$                                                         |
| 23 | $\sigma_{3241}$                                                    | $\sigma_{3241}$                                                              | $\sigma_{3241}$                                                         |
| 24 | $\sigma_{4231}$                                                    | $\sigma_{4231}$                                                              | $\sigma_{4231}$                                                         |

Table 2. Fusion and division (2)