Motion of an object through a quantum fluid

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(received 13 June 2000; accepted in final form 15 September 2000)

PACS. 03.75.Fi – Phase coherent atomic ensembles; quantum condensation phenomena.
PACS. 47.37.+q – Hydrodynamic aspects of superfluidity.
PACS. 67.40.Vs – Vortices and turbulence.

Abstract. – We simulate the motion of a massive object through a dilute Bose-Einstein condensate by numerical solution of the Gross-Pitaevskii equation coupled to an equation of motion for the object. Under a constant applied force, the object accelerates up to a maximum velocity where a vortex ring is formed which slows the object down. If the applied force is less than a critical value, the object becomes trapped within the vortex core. We show that the motion follows the time-independent solutions, and use these solutions to predict the conditions required for vortex detachment.

Introduction. – One of the most elementary questions that can be asked about a fluid, is how will an object move through it? In quantum fluids, it is expected that the object moves without resistance at velocities, \( v \), up to a critical value, \( v_c \), where energy and momentum conservation allow excitations. If the object mass is large, the critical velocity is given by the Landau criterion, \( v_c = (\epsilon/p)_{\text{min}} \), where \( \epsilon \) and \( p \) are the energy and momentum of elementary excitations of the fluid (phonons) [1]. Experiments on the motion of objects in superfluid helium (HeII) suggest that the appearance of drag is often associated with the formation of vortices [2]; however, understanding the exact mechanism of vortex formation is impeded by the lack of a complete hydrodynamical model. In contrast, for dilute quantum fluids such as the recently discovered atomic vapour Bose-Einstein condensates [3], the dynamics can be accurately described by the Gross-Pitaevskii (GP) equation, a form of non-linear Schrödinger equation [4]. Consequently, this system provides a near ideal testing ground for advancing our knowledge of superfluid flow. Experimental measurements of the heating produced by a laser beam moving through an atomic condensate suggest that the dominant mechanism of momentum transfer is vortex shedding [5], in agreement with GP simulations of a two-dimensional homogeneous flow past a fixed object [6–8].

If the object has a finite mass, the fluid back-action is significant, and completely different dynamics can arise. For example, ions in HeII nucleate vortex rings and become trapped within the vortex core [2, 9]. Here, we study the general case of the motion of an object with finite mass moving through a dilute Bose-Einstein condensate. The time evolution is found by solving the GP equation coupled to an equation of motion for the object. We show...
that the object can be accelerated up to a maximum velocity where a vortex ring emerges encircling the object. If the applied force is not too large, the object subsequently becomes trapped within the core of the ring. We show that the motion can be predicted from the time-independent uniform flow states and apply this method to give a complete description of energy and momentum conservation during vortex detachment.

**Numerical method.** The dynamics of the fluid are described by a wave function, \( \Psi(r, t) \), whose time evolution in the fluid rest frame is given by

\[
i\hbar \frac{\partial}{\partial t} \Psi(r, t) = -\frac{\hbar^2}{2m} \nabla^2 \Psi(r, t) + V(r - r_o(t)) \Psi(r, t) + \frac{\hbar^2}{mn_0\xi^2} |\Psi(r, t)|^2 \Psi(r, t) ,
\]

where \( V \) is the object potential, \( n_0 \) is the asymptotic number density, \( \xi \) is the healing length, and \( m \) is the mass of a fluid atom. If distance and velocity are measured in terms of the healing length, \( \xi \), and the speed of sound, \( c = \hbar/m\xi \), and the asymptotic number density is rescaled to unity, \( \psi(r, t) = \Psi(r, t)/\sqrt{n_0} \), then the GP equation can be written in dimensionless form,

\[
i\partial_t \psi(r, t) = -\frac{1}{2} \nabla^2 \psi(r, t) + V(r - r_o(t)) \psi(r, t) + |\psi(r, t)|^2 \psi(r, t) .
\]

The position of the object is given by

\[
M \ddot{r}_o = F + \int d^3r \frac{dV}{dr} |\psi(r, t)|^2 ,
\]

where \( F \) is an external force and the second term is the force on the object due to the fluid. The computation is simplified by transforming into the frame of the object, where eq. (2) may be rewritten as

\[
i\partial_t \tilde{\psi}(r', t) = -\frac{1}{2} \nabla^2 \tilde{\psi}(r', t) + V(r') \tilde{\psi}(r', t) + |\tilde{\psi}(r', t)|^2 \tilde{\psi}(r', t) + i\mathbf{v} \cdot \nabla \tilde{\psi}(r', t) ,
\]

where \( \tilde{\psi}(r', t) = \psi(r, t) \) is the wave function in the fluid frame written in terms of the object frame coordinates, \( r' = r - r_o(t) \), and \( \mathbf{v} \) is the object velocity at time \( t \). The system is prepared in a time-independent laminar flow state, \( \tilde{\psi}(r', t) = \phi(r')e^{i\mu t} \), where \( \mu \) is the chemical potential, by solving eq. (4) using Newton’s method [10,11]. From this initial state, the time evolution due to an applied force, \( F \), is evaluated by integrating eq. (4) using a semi-implicit Crank-Nicholson formula. The conservation of momentum

\[
P_0 + Ft = M\mathbf{v} + \frac{i}{2} \int d^3r' [\psi \nabla' \psi^* - \psi^* \nabla' \psi] ,
\]

where \( P_0 \) is the initial momentum of the fluid, is imposed as an additional constraint. The equations are discretized on a three-dimensional grid using the non-linear transform \( \hat{x} = x/(D + |x|) \), where \( D \) is a scaling parameter, to map an infinite box onto the space \(-1 \rightarrow 1\). The grid contains 140 points in each dimension and we use a time step \( dt = 0.02 \). For the object we choose a penetrable sphere with mass \( M = 200 \) and radius \( R = 3.3 \) (\( V = 1.0 \) for \( |r'| \leq 3.3 \) and 0 elsewhere). To convert between dimensionless units and values for helium, we use the measured values of the number density, \( n_0 = 2.18 \times 10^{28} \text{ m}^{-3} \), the quantum of circulation, \( \kappa = \hbar/m = 9.98 \times 10^{-8} \text{ m}^2\text{s}^{-1} \), and the healing length \( \xi/\sqrt{2} = 0.128 \text{ nm} \) [9] leading to a mass unit, \( \hat{m} = mn_0\xi^3 = 0.13m \), where \( m \) is the mass of a helium atom. Our object parameters, \( R = 3.3, M = 200, \) correspond to 25 helium atoms with radius 0.6 nm, similar to the “snowball” that surrounds a positive ion [2]. For an atomic condensate our object would correspond to a small cluster of impurity atoms in a sphere with a diameter of a few microns.
Fig. 1 – The evolution of the object velocity due to a constant applied force, $F$. The velocity is plotted (bold line) against the total momentum $P = Ft$ for (a) $F = 0.05$, (b) $F = 2$ and (c) $F = 4$. The other curves show the time-independent solutions of the uniform flow equation, (4). The continuous curve (1)-(2) corresponds to laminar flow and encircling vortex ring solutions. The additional branch (3) corresponds to a pinned vortex ring. The dotted line in (a) shows the velocity vs. momentum curve for a free vortex ring. The dots in (b) and (c) correspond to the times of the isosurface plots shown in fig. 2. In each case, we begin with a laminar flow state with velocity $v = 0.5$. The applied force accelerates the object along the laminar flow solution (1) up to a peak velocity, $v_c = 0.68$, where an encircling vortex ring emerges and begins to slow the object down. In (a), an abrupt decrease in velocity occurs when the object moves into the vortex core, i.e., when the time-dependent curve switches from the encircling ring solution (2) to the pinned ring solution (3). This jump excites vibrations of the ring leading to large oscillations of the object velocity (inset). The oscillations are damped as the vortex ring grows. When the ring radius is large the object velocity tends to the velocity of a free vortex ring, indicated by the dotted line in (a). In (c), the force is sufficient to detach the object from the ring and the cycle repeats.

**Evolution of the object velocity.** – An important result is that the time-dependent evolution of the object velocity is found to follow the time-independent stationary solutions, $\psi(r', t) = \phi(r') e^{i\mu t}$, of the uniform flow equation, (4). The time-independent solutions for a three-dimensional flow past a spherical object are studied in detail in ref. [11]. For an object velocity below the critical velocity, one finds three solutions, which in order of increasing energy correspond to laminar flow, a vortex ring pinned to the object, and a vortex ring encircling the object [11]. The velocity vs. momentum for these solutions for a spherical object with mass $M = 200$ and radius $R = 3.3$ are plotted as thin lines in fig. 1(a)-(c). The laminar, encircling ring, and pinned ring branches are labelled (1), (2) and (3), respectively, in fig. 1(a).

The time-dependent evolution of the object velocity due to a constant applied force, $F$, is plotted as a bold line in fig. 1(a)-(c). With increasing momentum, $P = Ft$, the object velocity increases along the laminar flow branch, (1). Even though there is no drag, momentum is transferred from the object to the fluid. This momentum transfer can be described in terms of an increase in the effective or hydrodynamical mass of the object, $m_{\text{eff}} = (\partial P / \partial v)^{-1}$. The effective mass becomes infinite at the critical velocity, $v_c = 0.68$, and then negative as the object begins to slow down. At the peak velocity, $v_c$, a vortex ring emerges encircling the object. The encircling vortex ring is apparent in frame 2 of fig. 2(a) which shows an isosurface of constant fluid density. The object is weakly bound in the direction of motion, therefore as the ring grows, it decelerates the object.
Fig. 2 – Sequence of surface contour plots of the fluid density for (a) $F = 2$ and (b) $F = 4$. The motion is from left to right, and the real space deflection due to the attraction of the vortex core is indicated by the transverse position. The momentum (or time) of each frame is indicated by a dot in fig. 1(b) and (c), except for the last frame in (a), where $P = 1562$. Note that after detachment, (b), the ring size remains constant and the object and ring move at different velocities.

When the vortex core begins to separate from the object boundary, the encircling ring configuration, corresponding to the stationary solution (2), becomes unstable with respect to transverse motion, and stochastic fluctuations induce a transition to a pinned ring solution (3), where the object is bound within the vortex core as in frame 4 of fig. 2(a). In our simulations, defining the external force, $F$, at a slight angle to the numerical grid axis is sufficient to induce the transition. On moving into the core, the object acquires a transverse velocity thereby deflecting its trajectory (fig. 2). The deflection angle is a few degrees, so this effect could be observable. If the ring detaches, a second ring forms and the object is pulled back in the opposite direction. Consequently, vortices are emitted on alternating sides of the object, similar to the vortex shedding behaviour observed in classical fluids.

The jump into the core also leads to the excitation of oscillatory modes of the vortex ring fig. 2(a). One mode of oscillation dominates [12] and the frequency is independent of the applied force. As the fluid is compressible, an accelerating object creates sound waves which damp the motion. This damping is apparent in the oscillations of the object velocity in fig. 1(a) inset. If the applied force is maintained the vortex radius continues to increase and eventually the motion becomes indistinguishable from that of a free vortex ring, indicated by the dotted line in fig. 1(a).

From fig. 1(a), it follows that excluding the ring excitations, the motion closely follows the time-independent solutions, therefore these solutions may be used to predict the motion of more complicated objects. To test whether a spherical object favours the encircling vortex ring configuration, we performed calculations on a sphere ($R = 3.3$) with a hemispherical surface bump ($R = 1.5$). The largest effect occurs when the bump lies in the equatorial plane. In this case, the critical velocity is reduced from 0.68 to 0.65, and the vortex ring emerges asymmetrically with its axis pulled towards the bump. However, the initial ring radius is still similar to the no bump case. Subsequently, the object or ring rotates such that the vortex core is pinned to the bump.
Vortex detachment. – If the applied force is sufficient, the object can detach from the vortex ring as in fig 2(b). After detachment the size of the ring remains constant, and the object can accelerate again up to the critical velocity where another ring forms and so on. The beginning of this repetitive cycle is apparent in the time evolution shown in fig. 1(c). If detachment occurs, the initial encircling ring system evolves into an object and a free vortex ring which moves more slowly as in fig. 2(b). One can regard this as a “decay” of an encircling ring state into a free vortex ring and a new laminar flow state. In fig. 3, we plot the dispersion curves for a moving object and a free vortex ring [13] obtained from the time-independent solutions of eq. (4). The shaded region indicates the range of initial energies and momenta where vortex detachment may occur, i.e., for which there exists final vortex ring and laminar states which satisfy the conservation laws. In a time-dependent simulation with a constant external force, the object moves up the dispersion curve passing through the critical point where a vortex is formed and then continues along the encircling ring branch (2). When the system enters the shaded region (point A in fig. 3) energy and momentum conservation permit “decay” into an object dressed by laminar flow (B) and a free vortex ring (C). Note that the energy at point A is higher than the time-independent value because the object is dragged out of the plane of the ring (see fig. 2). If the applied force is low, the object moves into the vortex core and the system relaxes towards the lower branch of the dispersion curve (3) and detachment is forbidden.
Comparison with ions in HeII. – It is interesting to note that the velocity vs. momentum profile shown in fig. 1(a) is similar to the velocity vs. electric field profiles observed for ions in HeII (see, e.g., [14]). If we define the healing length, $\xi = 0.18$ nm [9], we obtain a critical velocity $v_c \sim 60$ ms$^{-1}$, in rough agreement with experiments [14]. As discussed above, any surface roughness will tend to reduce this value. However, it should be noted that the situation in the HeII experiments is very different to that modelled here due to the effects of roton and impurity ($^3$He) scattering. In the experiments, the ions are accelerated up to a constant velocity where the external force is equal to the drag force due to roton/impurity scattering. This would correspond to sitting at a fixed point on the laminar flow branch of the velocity-momentum curve (labelled (1) in fig. 1). From there, the transition to higher momentum (higher energy) pinned vortex state (labelled (3) in fig. 1) is driven by thermal or quantum fluctuations. For ions in HeII, there is strong experimental evidence that vortex nucleation involves an energy barrier [15]. In the GP model the energy barrier corresponds to the energy required to transfer from the laminar flow (1) to the pinned ring solution (3). This energy decreases with increasing flow velocity but remains finite at the critical velocity because the pinned ring branch (3) joins the laminar-encircling ring curve (1)-(2) at an energy above the critical point, as apparent in fig. 3. The size of the energy barrier depends on the object radius, mass, and penetrability. For an impenetrable sphere with radius $R = 3.3$ (see fig. 5 in ref. [11]), we find an energy barrier of approximately $8\hbar n_0\xi^2$. For HeII, this would correspond to $\sim 3.8$ K, similar to the value derived by Muirhead et al. using arguments based on classical fluid mechanics [16]. The advantage of the GP model is that the property of fluid healing is included explicitly making it possible to identify a complete path of vortex formation.

Summary. – In summary, we have studied the motion of an object with finite mass through a dilute quantum fluid. We include the back action of the fluid on the object and show that under a constant applied force there is a continuous transition from laminar flow to an encircling vortex ring followed by a jump where the object moves into the vortex core. This jump leads to a deflection of the object trajectory and excitations of the vortex ring. If the object has a surface bump near the equator, the encircling vortex emerges asymmetrically. If the applied force is large, the object evades capture by the ring leading to periodic vortex shedding. We show that the motion and the conditions required for vortex detachment can be predicted from the allowed time-independent states. This approach could be extended to provide useful insight into other complex problems in quantum fluid mechanics such as vortex reconnections and sound emission. Finally, we consider the applicability of the model to experiments on ions in HeII.

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We thank D. C. Samuels, C. F. Barenghi, B. Jackson and J. F. McCann for stimulating discussions. Financial support for this work is provided by the Engineering and Physical Sciences Research Council (EPSRC).

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