V.A. Karmanov · J. Carbonell

Current conservation in electrodisintegration of a bound system in the Bethe-Salpeter approach

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Abstract Using our solutions of the Bethe-Salpeter equation with OBE kernel in Minkowski space both for the bound and scattering states, we calculate the transition form factors for electrodisintegration of the bound system which determine the electromagnetic current $J$ of this process. Special emphasis is put on verifying the gauge invariance which should manifest itself in the current conservation. We find that for any value of the momentum transfer $q$ the contributions of the plane wave and the final state interaction to the quantity $J \cdot q$ cancel each other thus providing $J \cdot q = 0$. However, this cancellation is obtained only if the initial Bethe-Salpeter amplitude (bound state), the final one (scattering state) and the current operator are strictly consistent with each other. A reliable result for the transition form factor can be found only in this case.

Keywords Bethe-Salpeter equation · Electromagnetic current · Electromagnetic form factors

1 Introduction

Bethe-Salpeter (BS) equation provides an efficient theoretical framework to describe bound and scattering states of a relativistic system. Finding its solution is complicated by the singularities in the integrand of the equation as well as by the singular character of the amplitude itself. To avoid this difficulty, one can perform Wick rotation and transform the BS equation in the Euclidean form. However, the Wick rotation cannot be performed in the integral for electromagnetic (e.m.) form factors (see e.g. [3]). Therefore, to calculate form factors, we need the BS solution in Minkowski space.

In finding these solutions, an important progress was achieved in the recent years using different and independent methods (see [4] for a brief review).

In one of these methods the kernel of the BS equation is approximately represented in a separable form. This allows to perform more advanced analytical calculations and therefore simplifies finding solutions and form factors. Another method is based on representing the BS amplitude via the Nakanishi integral both for bound [7,8,9,10] and scattering [12] states. The elastic e.m. form factor in this method was calculated in [13].

Recently we developed a method based on the direct treatment of singularities of the BS equation. Both the bound and scattering state amplitudes in Minkowski space were found. They allow to calculate the electrodisintegration of the bound system, i.e., form factor of the transition: bound $\rightarrow$ scattering state. The contribution of the final state interaction (FSI) to this form factor is given by

V.A. Karmanov
Lebedev Physical Institute, Leninsky Prospekt 53, 119991 Moscow, Russia
E-mail: karmanov@sci.lebedev.ru

J. Carbonell
Institut de Physique Nucléaire, Université Paris-Sud, IN2P3-CNRS, 91406 Orsay Cedex, France
the Feynman graph shown in fig. 1 (left panel). The left and right vertices in this graph are just the Minkowski BS amplitudes for the bound (left) and scattering (right) states.

In this paper we will analyze the conservation of the calculated e.m. current in the inelastic transition. We will see that it is indeed conserved (as it should be), however this conservation is due to rather delicate cancellation between the plane wave (PW) contribution (right panel of fig. 1) and FSI (left panel of fig. 1) which requires consistency between the bound and scattering state solutions and also the e.m. current operator. All of them should be found in the same dynamical framework. This provides a strong test of all these quantities simultaneously and, especially, an important test of the transition form factor.

The need for an internal consistency required to ensure the gauge invariance was extensively discussed in [16] in the framework of the BS and the Gross spectator equations. Our numerical results are in agreement with this general expectation. If this consistency and, hence, the gauge invariance are violated, the consequence of that is not reduced only to appearance in the current of a non-conserved part (which, however, drops out from the cross section at all). We emphasize that the transition form factors, extracted from the conserved part of the non-conserved current, are also deficient.

We consider in this work spinless particles. The generalization to the fermion case is straightforward and does not contain new problems, since the fermionic and spinless propagators have the same singularities.

2 Transition form factors

For spinless particles, the general decomposition of the transition current $J_\mu$ in terms of form factors, non-assuming current conservation, has the form:

$$J_\mu = \left[ (p_\mu + p'_\mu) + (p'_\mu - p_\mu) \frac{Q^2}{Q'^2} \right] F(Q^2) - (p'_\mu - p_\mu) \frac{Q^2}{Q'^2} F'(Q'^2),$$

(1)

where $Q^2 = -(p - p')^2$ and $Q'^2 = M'^2 - M^2$. $M$ is the bound state mass, $M'$ is the invariant mass of the final (scattering) state. The factor $\frac{Q^2}{Q'^2}$ in the last term is introduced for convenience, to ensure some scale properties to the form factor $F'(Q'^2)$. The decomposition (1) implies that the initial and final states have total zero angular momenta, i.e. they are composed from the S-waves only.

The first term in (1) satisfies the current conservation, whereas the second one does not, that is:

$$q \cdot J = Q^2 F'(Q'^2) \neq 0, \quad \text{if} \quad F'(Q'^2) \neq 0. \quad (2)$$

In the elastic case the last term in (1) is forbidden by the symmetry between initial and final states. In the inelastic case, this term should be zero due to the current conservation.

Below we will calculate both form factors $F(Q^2)$ and $F'(Q'^2)$. Note that the sum of the two graphs shown in fig. 1 exhausts all the contributions to the e.m. current in the model with OBE kernel, i.e., it provides the full e.m. current. We will check by numerical calculations whether this full current is conserved or not, i.e., whether $F'(Q'^2) = 0$. We will see that the current conservation is a rather subtle property which is violated by approximations in the solutions or by inconsistencies in the input quantities, like e.g. initial and final BS amplitudes and/or e.m. current operator.

3 Calculating form factors

It is convenient to carry out calculations in the reference system where $p'_0 = p_0$ (i.e., $q_0 = 0$) and $p$ and $p'$ are collinear (i.e., they are either parallel or anti-parallel to each other, depending on the value of the momentum transfer $Q^2$). In the elastic case this system coincides with the Breit frame $p + p' = 0$ (and so $|p| = |p'|$, $p'_0 = p_0$). In inelastic case and in the frame with $p'_0 = p_0$ we have $|p| \neq |p'|$. If in this frame we take the zero component of the current to find the form factor $F(Q^2)$

$$J_0 = 2p_0 F(Q^2).$$

(3)
We first calculate the FSI contribution. The current $J_\mu$ is obtained by applying the Feynman rules to the graph in left panel of fig. 1. We follow the Feynman rules in the convention of Itzykson-Zuber \cite{17}. Taking the component $J_0$ from eq. \ref{eq:17} we find:

$$F_{FSI}(Q^2) = i \int \frac{d^4 k}{(2\pi)^4} \frac{(p_0 - k_0)}{p_0} \frac{\Gamma_i \left( \frac{i}{2} p - k, p \right) \Gamma_f \left( \frac{i}{2} p' - k, p' \right)}{(k^2 - m^2 + i\epsilon)(p - k)^2 - m^2 + i\epsilon}.$$  \hfill (4)

The factor $(p_0 - k_0)/p_0$ comes from the zero component of the four-vector $p_\mu + p'_\mu - 2k_\mu$, corresponding to the upper (e.m.) vertex of the FSI graph in fig. 1 under the condition $q_0 = 0$ and after dividing this component by $2p_0$ from eq. \ref{eq:13}. $\Gamma_i$ is the initial (bound state) vertex, $\Gamma_f$ is the final vertex (half-off-shell scattering BS amplitude). As mentioned, both S-wave vertices were found numerically in \cite{14}. More precisely, in the assumed normalization convention, the function $\Gamma_f$ is related to the S-wave scattering solution $F_0$ calculated in \cite{14} by $\Gamma_f = 16\pi F_0$.

If the integral \eqref{eq:13} would not contain $\Gamma_i$, $\Gamma_f$, it can be calculated by applying to the product of propagators the Feynman parametrization and then the Wick rotation. However, since the initial and final BS amplitudes $\Gamma_i, \Gamma_f$ in \eqref{eq:13} are known numerically, the Feynman parametrization cannot be applied to the full integrand in \eqref{eq:13} and this 4D singular integral must be computed numerically as well. This calculation requires some treatment of the pole singularities of propagators. For this aim, each pole contribution (two for each propagator, six in total) is represented as sum of a principal value and a delta-function. The integrals containing the delta-functions are evaluated analytically. For the terms having the pole singularities (and also other, weaker singularities in the vertex functions $\Gamma_i, \Gamma_f$) we use the subtraction technique. The terms, where the pole singularities are canceled by subtractions, are integrated numerically, whereas the integrals from the added (pole) terms are calculated analytically. After that, the pole singularities in the added terms turn into the logarithmic ones. The latter ones can be safely integrated using appropriate numerical methods (as we did in \cite{14}) or by "brut force".

The details of this calculation will be published in a future paper.

The PW contribution to the current $J_\mu$ is obtained by applying the Feynman rules to the right panel of fig. 1. Like in the case of FSI, the corresponding form factor is extracted from the component $J_0$ by eq. \ref{eq:17} in the reference system where $q_0 = 0$. That is:

$$F_{PW} = - \int d^4 k \frac{(p_0 - k_0)}{p_0} \frac{\Gamma_i \left( \frac{i}{2} p - k, p \right)}{[(p - k)^2 - m^2 + i\epsilon]} \int \delta^{(4)} \left( k - p_s - p'_s \right) \frac{d\Omega_{p_s}}{4\pi}.$$  \hfill (5)

We keep the integral over $d^4 k$ and the delta-function for the four-momenta conservation. This delta-function replaces now the final BS amplitude $\Gamma_f$. When calculating the FSI contribution, we decomposed the final state BS amplitude in partial waves and took into account the S-wave only. This is equivalent to averaging $\Gamma_f$ over the solid angle of the vector $p_s$ in the rest frame $p'_s = 0$. To extract the S-wave from the final plane wave we introduced in eq. \ref{eq:13} the integration over $\frac{d\Omega_{p_s}}{4\pi}$.

The total form factor $F(Q^2)$ is then obtained as a sum of the two contributions \eqref{eq:4} and \eqref{eq:5}:

$$F(Q^2) = F_{FSI}(Q^2) + F_{PW}(Q^2).$$
Now we will find the form factor \( F'(Q^2) \). According to eq. (1), this form factor is extracted from the e.m. current by:

\[
F'(Q^2) = \frac{J \cdot q}{Q_c^2}.
\]  \( \text{(6)} \)

As mentioned above the expression for \( J_\mu \) contains the four-vector \((p_\mu + p'_\mu - 2k_\mu)\). After multiplying it by \(\frac{q_\mu}{Q_c^2} \), we get the factor

\[
\left. \frac{1}{Q_c^2} (p' - p) \cdot (p + p' - 2k) \right|_{p'_\mu = p_\mu} = \left( 1 - 2 \sqrt{\frac{Q^2 z k}{Q_c^2}} \right),
\]

where \( z \) is cosine of the angle between the integration variable \( k \) and the momentum \( p \) of the initial (bound state) system in the reference frame where \( q_0 = 0 \). This factor appears instead of \((p_0 - k_0)/p_0\) in (4) and (5). Therefore, two contributions \( F'_{FSI}(Q^2) \) and \( F'_{PW}(Q^2) \) in the form factor \( F'(Q^2) = F'_{FSI}(Q^2) + F'_{PW}(Q^2) \) are determined by eqs. (4) and (5) with the replacement

\[
\frac{(p_0 - k_0)}{p_0} \rightarrow \left( 1 - 2 \sqrt{\frac{Q^2 z k}{Q_c^2}} \right). \]

\( \text{(7)} \)

Fig. 2 (color online). Transition e.m. form factor \( F(Q^2) \) as a function of \( Q^2 \). Initial (bound) state corresponds to the binding energy \( B = 0.01 \) m; final (scattering) state corresponds to the relative momentum \( p_s = 0.1 \) m \((M' = 2.00998)\). FSI contribution is shown by the dashed curve, PW one by dotted curve and total form factor by solid curve. Left panel is the real part of form factor. Right panel is the imaginary part.

Fig. 3 (color online). Transition e.m. form factor \( F'(Q^2) \) as a function of \( Q^2 \) for \( p_s = 0.1 \). Parameters and notations are the same as in fig. 2.
4 Numerical results

Contrary to the elastic scattering, the inelastic (transition) form factor is complex. Its real and imaginary parts as function of $Q^2$ are shown in fig. 2 at the left and right panels correspondingly. The initial (bound) state corresponds to a binding energy $B = 0.01 \, m$ ($M = 1.99 \, m$), the final (scattering) state corresponds to a relative momentum $p_s = 0.1 \, m$ ($M' = 2\sqrt{p_s^2 + m^2} = 2.00998 \, m$). The coupling constant in the OBE kernel is $\alpha = 1.437$. One can see that at relatively small momentum transfer $Q^2 < 1$ both contributions – FSI and PW – are important and they considerably cancel each other. Whereas, at high momentum transfer the re-scattering (i.e., FSI) determines the tail of form factor and dominates over PW. Calculations for larger relative momentum $p_s = 0.5 \, m$ (the final state mass: $M' = 2.336$) do not indicate on the predominance of FSI at large momentum transfer.

As mentioned above, the form factor $F'(Q^2)$ (equal to zero if the current is conserved) is obtained from $F(Q^2)$ by the replacement 7 in the integrands 4 and 5. Calculated in this way, the form factor $F'(Q^2)$ for $p_s = 0.1$ is shown in fig. 3. We see that the value of $F'(Q^2)$, in comparison to $F(Q^2)$ (fig. 2) is practically indistinguishable from zero. This zero value is however obtained as cancellation of FSI and PW contributions. The accuracy of our calculations provides that the sum of two contributions $F_{FSI}'(Q^2) + F_{PW}'(Q^2)$ is by two-three orders of magnitude smaller than each contribution. We conclude that in the considered model the current is conserved with the precision of the numerical solutions.

Fig. 4 (color online). Transition e.m. form factor $F(Q^2)$ as a function of $Q^2$ for $p_s = 0.1$ calculated with the "phenomenological" FSI function 8. The notations are the same as in fig. 2.

Fig. 5 (color online). Transition e.m. form factor $F'(Q^2)$ as a function of $Q^2$ for $p_s = 0.1$ calculated with the "phenomenological" FSI function 8. The notations are the same as in fig. 2.

Though the current conservation is required from the first principles, the cancellation of FSI and PW contributions is a rather delicate point and therefore can serve as a strong test of the quantities involved. Indeed, both FSI and PW contributions contain the same initial bound state BS amplitude. At the
same time, FSI contribution contains the final state BS amplitude, whereas the PW contribution – does not. The cancellation of FSI and PW takes place provided both BS amplitudes and the current operator are consistent with each other, i.e. if they are correctly found in the same dynamical framework. To demonstrate that violation of this consistency indeed violates the current conservation, we replaced the final BS amplitude, found for the OBE kernel, by an “ad-hoc” function

\[ G_f(k_0, k) \sim \frac{1}{(k_0^2 + a^2)(k_0^2 + b^2)} \]  

(8)

without changing the initial BS amplitude and the current. The form factor \( F(Q^2) \) calculated with the function [8] is shown in fig. 4. It has a typical behavior which, in itself, does not arouse any concern. The transition form factor \( F'(Q^2) \) calculated with the function [9] for the parameters \( a^2 = 1.5 \), \( b^2 = 1 \) is shown in fig. 4. We see that it is not zero. This means that the e.m. current calculated with the “phenomenological” FSI function [8] is not correct. Therefore the form factor \( F(Q^2) \), fig. 4, extracted from this current, is also incorrect.

5 Conclusion

Using the OBE model, we have found that if the bound and scattering state BS amplitudes as well as the operator of e.m. current are exact solutions of the dynamical equations and consistent with each other, the transition e.m. current is conserved. If this consistency is destroyed, the conservation is violated. This violation results in two consequences. (i) The decomposition of the e.m. current in form factors obtains and additional contribution (the second term in eq. 14 with the non-zero form factor \( F'(Q^2) \)) which does not satisfy the equality \( J \cdot q = 0 \). However, the appearance of this term itself does not make any influence on observables, since \( F'(Q^2) \) does not contribute in the cross section, due to conservation of the e.m. current of the incident electron. (ii) The most important consequence is the fact that the non-conservation of the calculated e.m. current means its deficiency at all which prevents us from extracting a reliable transition form factor \( F(Q^2) \). It is therefore mandatory, in practical calculations, like, for instance, in the deuteron electrodisintegration, to fulfill the current conservation. If the form factor \( F'(Q^2) \) responsible for non-conserved e.m. current is comparable to the physical ones \( (F(Q^2)^2 \) in the spineless case), one can hardly trust to the calculated physical form factors too.

The popular trick, consisting in the replacement of the non-conserved current \( J_\mu \) by the conserved combination \( J_\mu \rightarrow \tilde{J}_\mu = J_\mu - q_\mu(J \cdot q)/q^2 \) does not solve the problem but only hides it. This new current \( \tilde{J}_\mu \) satisfies the current conservation for any arbitrary \( J_\mu \), not only for the correct one. If \( J \cdot q \neq 0 \), this indicates that the current \( J_\mu \) is defective at all. If so, one can extract the correct transition form factor neither from \( J_\mu \), nor from \( \tilde{J}_\mu \).

References

1. Salpeter, E.E., Bethe, H.A.: A Relativistic Equation for Bound-State Problems. Phys. Rev. 84, 1232-1242 (1951)
2. Wick, G.C.: Properties of Bethe-Salpeter Wave Functions. Phys. Rev. 96, 1124 (1954)
3. Carbonell, J., Karmanov, V.A.: Solutions of the Bethe-Salpeter equation in Minkowski space and applications to electromagnetic form factors. Few-Body Syst. 49, 205-222 (2011)
4. Karmanov, V.A.: Present status of the Bethe-Salpeter approach in Minkowski space. Few-Body Syst. 55, 545-554 (2014)
5. Bondarenko, S.S., Burov, V.V., Rogochaya, E.P.: Covariant relativistic separable kernel approach for electrodisintegration of the deuteron at high momentum transfer. Few-Body Syst. 49, 121-128 (2011); Relativistic complex separable potential of the neutron-proton system. Phys. Lett. B 705, 264-268 (2011); Final state interaction effects in electrodisintegration of the deuteron within the Bethe-Salpeter approach. JETP Lett. 94, 738-743 (2012)
6. Nakanishi, N.: Partial-Wave Bethe-Salpeter Equation. Phys. Rev. 130, 1230 (1963)
7. Kusaka, K., Williams, A.G.: Solving the Bethe-Salpeter equation for scalar theories in Minkowski space. Phys. Rev. D 51, 7026 (1995)
8. Kusaka, K., Simpson, K., Williams, A.G.: Solving the Bethe-Salpeter equation for bound states of scalar theories in Minkowski space. Phys. Rev. D 56, 5071 (1997)
9. Karmanov, V.A., Carbonell, J.: Solving Bethe-Salpeter equation in Minkowski space. Eur. Phys. J. A 27, 1-9 (2006)
10. Carbonell, J., Karmanov, V.A.: Cross-ladder effects in Bethe-Salpeter and Light-Front equations. Eur. Phys. J. A 27, 11-21 (2006)
11. Sauli, V.: Solving the Bethe-Salpeter equation for a pseudoscalar meson in Minkowski space. J. Phys. G 35, 035005 (2008)
12. Frederico, T., Salmè, G., Viviani, M.: Two-body scattering states in Minkowski space and the Nakanishi integral representation onto the null plane. Phys. Rev. D 85, 036009 (2012)
13. Carbonell, J., Karmanov, V.A., Mangin-Brinet, M.: Electromagnetic form factor via Bethe-Salpeter amplitude in Minkowski space. Eur. Phys. J. A 39, 53-60 (2009)
14. Carbonell, J., Karmanov, V.A.: Bethe-Salpeter scattering amplitude in Minkowski space. Phys. Lett. B 727, 319-324 (2013); Solving Bethe-Salpeter scattering state equation in Minkowski space. Phys. Rev. D 90, 056002 (2014)
15. Carbonell, J., Karmanov, V.A.: Transition electromagnetic form factor in the Minkowski space Bethe-Salpeter approach. Few-Body Syst. 55, 687-691 (2014)
16. Adam, J., Jr, Van Orden, J. W., Gross, F.: Electromagnetic interactions for the two-body spectator equations. Nucl. Phys. A 640, 391-434 (1998)
17. Itzykson, C., Zuber, J.-B. (1980): Quantum field theory. (Dover Publications, New York)
