Materials with anisotropic material parameters can be utilized to fabricate many fascinating devices, such as hyperlenses, metasolids, and one-way waveguides. In this study, we analyze the effects of geometric anisotropy on a two-dimensional metamaterial composed of a rectangular array of elliptic cylinders and derive an effective medium theory for such a metamaterial. We find that it is possible to obtain a closed-form analytical solution for the anisotropic effective medium parameters, provided the aspect ratio of the lattice and the eccentricity of the elliptic cylinder satisfy certain conditions. The derived effective medium theory not only recovers the well-known Maxwell-Garnett results in the quasi-static regime, but is also valid beyond the long-wavelength limit, where the wavelength in the host medium is comparable to the size of the lattice so that previous anisotropic effective medium theories fail. Such an advance greatly broadens the applicable realm of the effective medium theory and introduces many possibilities in the design of structures with desired anisotropic material characteristics. A real sample of a recently theoretically proposed anisotropic medium, with a near-zero index to control the flux, is achieved using the derived effective medium theory, and control of the electromagnetic waves in the sample is clearly demonstrated.

Metamaterials, that is, artificial materials that possess unconventional material parameters, have been employed to achieve unprecedented functionality in the control of electromagnetic and acoustic waves, such as negative refraction and superlensing. One prominent class of metamaterials is anisotropic metamaterials, the material parameters of which are not scalars but tensors, with their principle components taking different values. This property causes the dispersion relations to display elliptic or hyperbolic shapes. Such anisotropic metamaterials exhibit distinctive properties, including negative refraction, super-resolution in the far-field through image magnification, and enhanced spontaneous emission. When one principle component in the material parameter tensor changes sign, a topological transition occurs. Earlier this year, Luo et al. proposed a method to arbitrarily control electromagnetic flux using a type of anisotropic medium. In this method, only one principle component is near zero and the other components take positive values. However, a real sample of such a medium is yet to be reported.
Very recently, a method based on reproducing the lowest orders of scattering amplitudes from a finite volume of metamaterials was proposed. This can give accurate predictions of the effective medium parameters over almost the entire Brillouin zone. These schemes work well for isotropic media in which both the scatterers and the lattice structures are isotropic. For anisotropic media, however, the homogenization scheme is more challenging, as it involves more degrees of freedom than in the case of isotropic media. Many conventional anisotropic EMTs are extensions of the Maxwell-Garnett theory and are consequently limited in application to the quasi-static regime. A multiple-scattering-based scheme has been introduced to study the effective medium properties of metamaterials with anisotropic lattices and isotropic scatterers yielding a scalar bulk modulus and tensorial mass density at finite frequencies in the long-wavelength regime. There exist other schemes that are also applicable to anisotropic scatterers, including the field-averaging, boundary-integration, and parameter-retrieving methods. The field-averaging and boundary-integration methods require prior knowledge of field distributions, while the parameter-retrieving method requires information about the transmission and reflection coefficients and may give non-unique solutions. More importantly, none of these three methods offer a closed-form analytical solution that directly predict reliable effective medium parameters from the material and geometric information of the system.

In this work, we consider a rectangular array of elliptic cylinders embedded in air and study its scattering properties. We discover that the special properties of elliptic coordinates and Mathieu functions (solutions to Helmholtz equations in elliptic coordinates) enable us to derive a closed-form analytical solution for the anisotropic effective medium parameters, provided the aspect ratio of the lattice and the eccentricity of the elliptic cylinder satisfy certain conditions. We verify the derived EMT by comparing its predictions with full-wave bandstructure simulations, and excellent agreements are found at finite frequencies beyond the long-wavelength limit. This new EMT suggests promising opportunities to expand the design of anisotropic metamaterials. We show that a recently theoretically proposed anisotropic zero-volume material, which can manipulate electromagnetic flux, can be achieved from the predictions of the derived EMT. The metamaterial is composed of common dielectric materials with simple structures, which makes the fabrication process feasible and would therefore greatly benefit the practical realization of the material.

**Results**

**Modeling and the analytical solution.** The system considered in our study is a two-dimensional (2D) metamaterial consisting of a periodic rectangular array of elliptic cylinders with permittivity, \( \varepsilon_0 \), and magnetic permeability, \( \mu_0 \), embedded in a background material with permittivity, \( \varepsilon_0 \), and magnetic permeability, \( \mu_0 \). A unit cell of the metamaterial is illustrated in Fig. 1(a). The elliptic cylinder’s semi-major and semi-minor axes are \( a \) and \( b \), respectively, and its filling ratio, \( r \), the ratio of the area of the elliptic cylinder to the area of the unit cell, is \( r = \frac{\pi a b}{a^2 + b^2} \). For the dispersion microstructure, in which the scatterers are always dispersed in the matrix, the CPA scheme considers the scattering of a coated cylinder in an effective medium (as shown in Fig. 1(b)). The inner elliptic cylinder represents the scatterer in the metamaterial and the coating layer is the background medium, while the semi-major and semi-minor axes of the outer elliptic cylinder are \( a_0 \) and \( b_0 \), respectively. Such a coated elliptic cylinder represents the microstructure of the metamaterial or the average cell in the CPA scheme, as the outside environment has been averaged as an effective medium. It is generalized from the circular (or spherical) average cells of isotropic lattices with isotropic scatterers. The cross-sectional area of the average cell should be identical to that of a unit cell, i.e., \( \pi a_0 b_0 = a b \), so that the filling fraction of the scatterer in the average cell is fixed to that of the metamaterial. The aspect ratio of the average cell should also equal that of the rectangular unit cell, i.e., \( a_0/b_0 = a/b \), in order to preserve the symmetry properties. With this average cell, the anisotropic property of the metamaterial is maintained and the scattering property in the effective medium is correctly produced. The effective medium parameters are obtained when the total scattering of the average cell vanishes in the limit \( \frac{1}{2} k_{ef} (a_0 + b_0) < 1 \).

In the Method section, we present detailed steps to derive the EMT of the metamaterial for a transverse-electric (TE)-polarized wave, in which the electric field is parallel to the elliptic cylinders. Here, we refer only to the final solutions, which are expressed as

\[
\begin{align}
\varepsilon_{\text{eff}} &+ 2 \varepsilon_0 \frac{J_{\varepsilon}(q; \zeta_0)}{k_{\varepsilon}^2 a_0 b_0 J_{\varepsilon}(q; \zeta_0)} = \frac{Y_{\varepsilon}(q; \zeta_0)}{i J_{\varepsilon}(q; \zeta_0)} 1 + D_{\varepsilon}(0), \\
\mu_{\text{eff}, x} &- \mu_0 \frac{a_0 J_{\mu}(q; \zeta_0)}{b_0 J_{\mu}(q; \zeta_0)} = \frac{Y_{\mu}(q; \zeta_0)}{i J_{\mu}(q; \zeta_0)} 1 + D_{\mu}(0), \\
\mu_{\text{eff}, y} &- \mu_0 \frac{a_0 J_{\mu}(q; \zeta_0)}{b_0 J_{\mu}(q; \zeta_0)} = \frac{Y_{\mu}(q; \zeta_0)}{i J_{\mu}(q; \zeta_0)} 1 + D_{\mu}(0),
\end{align}
\]

where the effective permittivity, \( \varepsilon_{\text{eff}} \), is a scalar related to the monopolar mode and the effective permeability is a tensor whose principle components \( \mu_{\text{eff}, x} \) and \( \mu_{\text{eff}, y} \) are associated with the y-polarized and x-polarized dipolar modes, respectively. The x- and y-axes are set in the directions of the semi-major and semi-minor axes of the elliptic cylinders, respectively. Note that all of the notations and subscripts appearing in Eq. (1) are defined or introduced in the Method section.

**Verification of the EMT.** In Fig. 2(a), we plot the band structure of a metamaterial obtained from a full-wave simulation using black dots. The metamaterial is composed of elliptic cylinders in a rectangular lattice embedded in air and the geometric sizes of the scatterer and the lattice are \( a_s = 0.26 r, b_s = 0.2 r, a = 1.16 r, b = 1.12 r \), where \( r \) is a normalized length unit. The material parameters are chosen as \( \varepsilon_0 = 1, \mu_0 = 1 \) for air. Also plotted in Fig. 2(a) (in red solid curves) are the band structures predicted by the EMT, i.e., Eq. (1). The corresponding effective permittivity and permeability are shown in Figs. 2(b) and 2(c), respectively. They provide us with a clear picture and understanding of the dispersion relations. We label three points on the band structure at the Brillouin
Figure 2 | Verification of derived EMT. (a) Band structure calculations (black dots) using COMSOL, compared with EMT predictions (red curves) from Eq. (1). (b) Corresponding effective permittivity and (c) permeability calculated from Eq. (1). (d) Transmission coefficient (in logarithmic scale) of a plane wave incident on a 9-layer metamaterial sample, compared with the EMT prediction (red curves). The left panel shows the result in the frequency regime \( \omega \approx (0, 0.11) \); the right panel shows the data for the \((0.48, 0.61)\) regime. (e) The same as (d) but in the \(y\)-direction. The left panel shows the \((0, 0.11)\) regime results; the right panel shows the \((0.53, 0.66)\) regime data. (f) Eigenfield patterns for points “A”, “B”, and “C” marked in (a). Dark red and dark blue represent the positive and negative maxima of the electric field and arrows indicate the magnetic fields. The geometric parameters are taken as \(a = 0.26r, b = 0.2r, a = 1.16r, \) and \(b = 1.12r\), while \(\varepsilon_i = 12, \mu_i = 1, c_0 = 1, \) and \(\mu_0 = 1\) are the material parameters.
where $\mu_{eff,x} = 0.002 - 0i$, $\mu_{eff,y} = 0.5637 \gg \mu_{eff,x}$, and $\varepsilon_{eff} = 0.1175$. This indicates that the system is an anisotropic zero-index material with only one component of the permeability tensor near zero. Very recently, such a medium was theoretically proposed and found to be capable of cloaking an arbitrarily shaped defect and of exciting evanescent waves near the defect boundaries, which therefore offers a new method of controlling the electromagnetic flux\textsuperscript{14}. Below, we provide simulated results of the wave transmission through such a metamaterial loaded with defects. Figure 3(a) illustrates a schematic picture of the sample, which is a waveguide filled with a metamaterial slab (composed of 12 × 10 previously mentioned unit cells). Three defects labeled “1”, “2”, and “3” are distributed within the slab, as shown in Fig. 3(a), with respective sizes of $2a \times 2b$, $2a \times b$, and $3a \times 2b$, and permeability $\mu = 1.5$, 0.4, and 2.1, respectively. The permittivity of the defects is set to 1. A TE-polarized plane wave with frequency $\omega_C$ is incident from the left.

As a comparison, we plot in Fig. 3(b) the electric field for the same sample shown in Fig. 3(a), but without the metamaterial. Strong scattered waves are excited by the defects, which significantly distort the incident wave fronts. However, the results are significantly altered in the presence of the metamaterial. Figures 3(c), 3(e), and 3(g) show, respectively, the electric field and the magnetic fields in the x- and y-directions. The field patterns at the outlet of the waveguide are almost the same as those of the incident wave, indicating the good cloaking effect of the metamaterial. From Fig. 3(c), we clearly observe an almost uniform field distribution in the y-direction (vertical direction) and an apparent phase change in the x-direction (horizontal direction) in the metamaterial, implying that the metamaterial is highly anisotropic. The wavelength is nearly infinite along the y-direction, but finite along the x-direction. The corresponding field distribution patterns for the same case, but with the metamaterial replaced by the effective medium, are plotted in Figs. 3(d), 3(f), and 3(h). Similar patterns to those shown in Figs. 3(c), 3(e), and 3(g) at the inlet and outlet of the waveguide are seen, suggesting that the EMT indeed describes the physical properties of the metamaterial. From Fig. 3(f), we find that evanescent waves around the defects are induced, which are essential for high transmittance\textsuperscript{14}.

Figure 3 demonstrates the functionality of the anisotropic zero-index metamaterial. Noting that the building blocks of the metamaterial are dielectric elliptic cylinders, which are easily attainable, and that there are no complex structures involved, we believe that the fabrication of such a metamaterial is feasible.

**Discussion**

We support the validity and application of our anisotropic EMT by illustrating a simulated example, in which a set of values of $a/b$, $\varepsilon_r$, $\mu_r$, and a filling ratio of $f = na/b/ab$ are chosen, and good agreements between the numerical simulations and the EMT predictions are observed. In the following, we conduct a systematic study of the manners in which the material and geometric parameters influence the accuracy of the EMT. In Fig. 4, we plot the frequencies at which zero effective medium parameters are obtained as functions of various parameters. The curves are obtained from Eq. (1) and the dots correspond to the frequencies of the lowest monopolar and dipolar states at the $\Gamma$ point, which are results of the band structure calculations. In Figs. 4(a)–4(c), we fix the permeability of the scatterers to 1, and change the aspect ratio, permittivity, and the filling ratio of the scatterers, respectively. In the lower panel of Fig. 4, we study similar cases to those in the upper panel but with the permittivity of the scatterers fixed at 1. Figure 4 demonstrates that the predictions of our EMT in general coincide with the band structure simulations. When the aspect ratio and filling ratio increase, the predictions deviate from the numerical results. This is reasonable as higher angular momentum terms, i.e., $m \geq 2$, contribute to the eigenmodes at low frequencies when the elliptic cylinder becomes flatter or larger. This effect leads to inaccurate predictions, because our derived effective medium scheme does not consider higher angular momentum terms.

In summary, we have derived an anisotropic EMT for a 2D electromagnetic metamaterial. This theory can provide closed-form analytical solutions for anisotropic effective medium parameters and reveal the link between the effective medium parameters and the resonant modes. It is found that the effective permittivity is related to the monopolar mode and the effective permeability tensor is associated with the dipolar modes. The validity of the theory is verified by band structure and transmission spectra calculations and we find that the theory is valid even when the wavelength in the background medium is comparable to the size of the lattice, which is beyond the long-wavelength limit. At the quasi-static limit, our EMT recovers the Maxwell-Garnett formula. We expect that the EMT developed here will facilitate the design of new metamaterials, and we show that a recently proposed anisotropic zero-index material can indeed be fabricated from a periodic structure. Additional anisotropic metamaterials with various desired properties may also be devised based on the predictions of our EMT. Although this theory is derived for electromagnetic metamaterials, it can be generalized to its acoustic counterpart because of the mathematical mapping between these two systems in two dimensions.
Figure 4 | Effects of different parameters on derived EMT predictions. The frequencies at which \( \varepsilon_{\text{eff}} = \mu_{\text{eff}} \) or \( \mu_{\text{eff}} \) become zero according to Eq. (1), as functions of various parameters are pictured as curves. For comparison, the frequencies of the lowest monopolar and dipolar states at the \( \Gamma \) point, which are obtained from the band structure calculations using COMSOL, are also plotted in dots. (a) Effects of changing \( a/b_2 \) with fixed \( \varepsilon_i = 12, \mu_i = 1 \), and \( f = 0.126 \). (b) Effects of changing \( \varepsilon_i \) with fixed \( a/b_2 = 1.3, \mu_i = 1 \), and \( f = 0.126 \). (c) Effects of changing \( f \) with fixed \( a/b_2 = 1.3, \varepsilon_i = 12 \), and \( \mu_i = 1 \). (d)–(f) Similar conditions as those shown in (a)–(c), but the dielectric cylinders (\( \mu_i = 1 \)) are replaced with magnetic cylinders (\( \varepsilon_i = 1 \)).

Methods

Solution of the Helmholtz equation in elliptic coordinates. Considering the microstructure shown in Fig. 1(b) for a TE-polarized wave, the electric field in the effective medium can be expressed as

\[
E_x = \sum_{m} z_m(\eta)S_m(q_{\text{eff}}; \eta)H_{0,m}(q_{\text{eff}}; \zeta) + \beta_m(\eta)S_m(q_{\text{eff}}; \eta)\sinh(q_{\text{eff}})J_{0,m}(q_{\text{eff}}; \zeta).
\]  

(3)

and similarly, the electric field in the background medium is given by

\[
E_x = \sum_{m} z_m(0)S_m(q_{\text{b}}; \eta)H_{0,m}(q_{\text{b}}; \zeta) + \beta_m(0)S_m(q_{\text{b}}; \eta)\sinh(q_{\text{b}})J_{0,m}(q_{\text{b}}; \zeta).
\]  

(4)

Here, \( \eta \) and \( \zeta \), where \( 0 \leq \eta < 2\pi \) and \( 0 \leq \zeta < \infty \), represent elliptic coordinates that can be transformed into Cartesian coordinates according to \( x = \epsilon \cos(\eta)\cosh(\zeta) \) and \( y = \epsilon \sin(\eta)\sinh(\zeta) \), where \( \epsilon = \sqrt{a^2 - b^2} = \sqrt{a_1^2 - b_1^2} \) represents the focal length of the elliptic coordinate system. In Eqs. (3) and (4), \( S_{m}(q_{\text{eff}}; \eta) \) denote the angular Mathieu functions of the first kind, while \( J_{0,m}(q_{\text{eff}}; \zeta) \) and \( H_{0,m}(q_{\text{eff}}; \zeta) \) are the radial Mathieu functions of the first and third kinds, respectively. The subscript \( m \) is an integer denoting the order of the Mathieu functions. The angular and radial Mathieu functions form solutions to the Helmholtz equation in elliptic coordinates, which split into decoupled even (denoted by subscript \( e \)) and odd modes (denoted by subscript \( o \)) with respect to the \( x \)-axis for non-zero \( m \). Here, the general notation \( \gamma = e, o \), or \( m \) is used. The variable \( q_{\text{eff}}(q_{\text{b}}) \) is a dimensionless quantity and is equal to \( \frac{k_0}{\sqrt{c_0^2 + c_2^2}} \), where \( k_0 = \sqrt{\varepsilon_0\mu_0} \) (\( k_{\text{eff}} = \sqrt{k_0^2 + k_{\text{b}}^2} \)) is the wave vector in the background (effective) medium.

Boundary Conditions. The expansion coefficients in Eqs. (3) and (4), i.e. \( z_m(0) \) and \( \beta_m(0) \) with \( \gamma = 0, \) or \( \text{eff} \), are related through the boundary conditions, which are the continuities of the tangential components of both electric and magnetic fields on the interface between the background and effective medium. The boundary conditions can be expressed as \( E_x(0) = E_x(\text{eff}) \) and \( H_z(0) = H_z(\text{eff}) \). The \( \rho_{\text{eff}} \) at \( \zeta = \zeta_{\text{bc}} \) and the outer boundary of the coated cylinder and \( H_z(\text{eff}) \) is expressed in the anisotropic effective medium as

\[
H_z(\text{eff}) = \frac{1}{i\alpha\sqrt{\Delta}} \left[ \frac{\cosh(\zeta_{\text{bc}})\sin(\eta)}{\rho_{\text{eff},x}} + \sin(\zeta_{\text{bc}})\cos(\eta) \right] E_x(\text{eff}) + \left[ \frac{\sinh(\zeta_{\text{bc}})\sin(\eta)}{\rho_{\text{eff},x}} + \cosh(\zeta_{\text{bc}})\cos(\eta) \right] \frac{E_z(\text{eff})}{\rho_{\text{eff},z}},
\]

with \( \Delta = \cosh(\zeta_{\text{bc}})\sin(\eta) + \sinh(\zeta_{\text{bc}})\cos(\eta) \). Substituting Eqs. (3) and (4) into the boundary conditions, we obtain

\[
\begin{bmatrix}
\begin{bmatrix}
\frac{z_m(0)(\text{eff})}{\beta_m(0)(\text{eff})}
\end{bmatrix}
\end{bmatrix} = F = \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} \begin{bmatrix}
\begin{bmatrix}
\frac{z_m(0)}{\beta_m(0)}
\end{bmatrix}
\end{bmatrix}
\]

(5)

where \( F \) is

\[
F = \frac{S_m(q_{\text{eff}}; \eta)}{\mu_0 S_m(q_{\text{eff}}; 0)} \left[ \frac{g_{\text{eff}}(q_{\text{eff}}; \zeta_{\text{bc}})}{g_{\text{bc}}(q_{\text{b}}; \zeta_{\text{bc}})} \right]^{-1},
\]

and

\[
A_{11} = S_m(q_{\text{bc}}; \eta) H_{0,m}(q_{\text{bc}}; \zeta_{\text{bc}}) - \mu_0 \Omega_{\text{bc}} J_{0,m}(q_{\text{bc}}; \zeta_{\text{bc}}),
\]

(6a)

\[
A_{12} = S_m(q_{\text{bc}}; \eta) H_{0,m}^{(1)}(q_{\text{bc}}; \zeta_{\text{bc}}) - \mu_0 \Omega_{\text{bc}} J_{0,m}^{(1)}(q_{\text{bc}}; \zeta_{\text{bc}}),
\]

(6b)

\[
A_{21} = \mu_0 \Omega_{\text{bc}} J_{0,m}(q_{\text{bc}}; \zeta_{\text{bc}}) - S_m(q_{\text{bc}}; \eta) J_{0,m}(q_{\text{bc}}; 0),
\]

(6c)

\[
A_{22} = \mu_0 \Omega_{\text{bc}} J_{0,m}^{(1)}(q_{\text{bc}}; \zeta_{\text{bc}}) - S_m(q_{\text{bc}}; \eta) J_{0,m}^{(1)}(q_{\text{bc}}; 0),
\]

(6d)

where

\[
\Omega_{\text{bc}} = \frac{1}{\Lambda} \left[ \frac{\cosh^2(\zeta_{\text{bc}})\sin^2(\eta)}{\rho_{\text{eff},x}} + \sinh^2(\zeta_{\text{bc}})\cos^2(\eta) \right] S_m(q_{\text{bc}}; \eta) H_{0,m}(q_{\text{bc}}; 0) + \frac{1}{\rho_{\text{eff},x}} S_m(q_{\text{bc}}; \eta) J_{0,m}(q_{\text{bc}}; 0),
\]

(6f)

and

\[
\Omega_{\text{bc}} = \frac{1}{\Lambda} \left[ \frac{\cosh^2(\zeta_{\text{bc}})\sin^2(\eta)}{\rho_{\text{eff},x}} + \sinh^2(\zeta_{\text{bc}})\cos^2(\eta) \right] S_m(q_{\text{bc}}; \eta) H_{0,m}^{(1)}(q_{\text{bc}}; 0) + \frac{1}{\rho_{\text{eff},x}} S_m(q_{\text{bc}}; \eta) J_{0,m}^{(1)}(q_{\text{bc}}; 0),
\]

(6f)

Analytical solution for the effective medium. The effective medium condition requires that the scattering of the coated cylinder vanishes. Since the scattered field of the coated cylinder is represented by \( H_{0,m}^{(1)}(q_{\text{eff}}; \zeta_{\text{bc}}) \), a vanishing scattered wave in the effective medium implies that \( \beta_m(\text{eff}) = 0 \). According to Eq. (5), such a condition leads to

\[
\begin{bmatrix}
A_{31} \\
A_{32}
\end{bmatrix} = \begin{bmatrix}
\beta_m(0) \\
\mu_m(0)
\end{bmatrix} = -D_m(0),
\]

(7)

where \( D_m(0) \) represent the Mie scattering coefficients of a scatterer of the metamaterial. These coefficients can be obtained by solving the Helmholtz equation and matching the boundary conditions between the scatterer and the background.
medium. They have the form
\[ D_{\alpha}(0) = \mu_{\alpha} J^q_{m}(q; \xi) \psi_{m}(\eta) + \mu_{\alpha} J^q_{m}(q; \xi) \psi_{m}(\eta), \]
\[ \mu_{\alpha} J^q_{m}(q; \xi) H^H_{m}(q; \xi) + \mu_{\alpha} J^q_{m}(q; \xi) H^H_{m}(q; \xi), \]
in which the subscript "x" means that the quantities take the corresponding values of the scatterer, while \( \xi \) indicates the boundary of the scatterer.

When the wavelength in the effective medium is much larger compared to the size of the coated cylinder, i.e., \( \frac{1}{k_c} (a_c + b_c) < 1 \), the scattering of the coated cylinder is dominated by monopolar (\( m = 0 \)) and dipolar (\( m = 1 \)) terms. Under this condition, we substitute Eqs. (6c) and (6d) into Eq. (7) and approximate the zero- and first-order Mathieu functions associated with the effective medium by \( S_{\alpha}(q; \eta) = 1 \). The effective permittivity and permeability are determined by \( S_{\alpha}(q; \eta) = 0, \) \( S_{\alpha}(q; \eta) = \cos(\eta), \) \( S_{\alpha}(q; \eta) = -\sin(\eta), \) \( S_{\alpha}(q; \eta) = \sin(\eta), \) respectively.

\[ \begin{aligned}
I_{\alpha}(q; \tilde{\eta}) &= \frac{\nu_1 - i \nu_2}{2} J_{m}(q; \tilde{\eta}) + \frac{\nu_1 + i \nu_2}{2} H^{H}_{m}(q; \tilde{\eta}) \quad (x = b_c), \\
I_{\alpha}(q; \tilde{\eta}) &= \frac{\nu_1 + i \nu_2}{2} J_{m}(q; \tilde{\eta}) + \frac{\nu_1 - i \nu_2}{2} H^{H}_{m}(q; \tilde{\eta}) \quad (x = a_c),
\end{aligned} \]

where \( \nu_{1,2} = q_{\alpha}^{(1,2)} \) are the Mathieu Neumann functions. Similar to the results for the isotropic media, the effective permittivity and permeability are determined by monopolar (\( m = 0 \)) and dipolar (\( m = 1 \)) modes, respectively. For the anisotropic case discussed here, however, the effective permeability is no longer a scalar, but a diagonalized tensor. It is interesting to note that the elements of the tensor, \( \mu_{\alpha \beta} \) and \( \mu_{\alpha \beta} \), correspond exactly to the scattering coefficients of the first-order y-polarized and z-polarized dipolar modes.

Results in the quasi-static limit. In the quasi-static limit, i.e., \( \frac{1}{k_c} (a_c + b_c) < 1 \) and \( \frac{1}{k_c} (a_c + b_c) < 1 \), the Mathieu functions in Eq. (1) can be approximated in the same manner as that used to obtain Eq. (1) and the Mathieu Neumann functions. When the wavelength in the effective medium is much larger compared to the size of the coated cylinder, the scattering of the coated cylinder is dominated by monopolar \( (m = 0) \) and dipolar \( (m = 1) \) terms. Under this condition, we substitute Eq. (6c) and (6d) into Eq. (7) and approximate the zero- and first-order Mathieu functions associated with the effective medium by \( S_{\alpha}(q; \eta) = 1 \). The effective permittivity and permeability are determined by \( S_{\alpha}(q; \eta) = 0, \) \( S_{\alpha}(q; \eta) = \cos(\eta), \) \( S_{\alpha}(q; \eta) = -\sin(\eta), \) \( S_{\alpha}(q; \eta) = \sin(\eta), \) respectively.
29. Mei, J., Wu, Y. & Liu, Z. Y. Effective medium of periodic fluid-solid composites. *Europhys. Lett.* **98**, 54001 (2012).

30. Torrent, D. & Sánchez-Dehesa, J. Multiple scattering formulation of two-dimensional acoustic and electromagnetic metamaterials. *New J. Phys.* **13**, 093018 (2011).

31. Wu, Y., Mei, J. & Sheng, P. Anisotropic dynamic mass density for fluid-solid composites. *Physica B: Cond. Matt.* **407**, 4093 (2012).

32. Smith, D. R. & Pendry, J. B. Homogenization of metamaterials by field averaging. *JOSA B* **23**, 391 (2006).

33. Chern, R. L. & Chen, Y. T. Effective parameters for photonic crystals with large dielectric contrast. *Phys. Rev. B* **80**, 075118 (2009).

34. Lai, Y., Wu, Y., Sheng, P. & Zhang, Z. Q. Hybrid Elastic Solids. *Nat. Mater.* **10**, 620 (2011).

35. Smith, D. R., Schultz, S., Markoš, P. & Soukoulis, C. M. Determination of effective permittivity and permeability of metamaterials from reflection and transmission coefficients. *Phys. Rev. B* **65**, 195104 (2002).

36. Fokin, V., Ambati, M., Sun, C. & Zhang, X. Method for retrieving effective properties of locally resonant acoustic metamaterials. *Phys. Rev. B* **76**, 144302 (2007).

37. Liu, X. X., Powell, D. A., & Alù, A. Correcting the Fabry-Perot artifacts in metamaterial retrieval procedures. *Phys. Rev. B* **84**, 235106 (2011).

38. Liu, X. X. & Alù, A. Generalized retrieval method for metamaterial constitutive parameters based on a physically driven homogenization approach. *Phys. Rev. B* **87**, 235136 (2013).

39. Jin, J. F., Liu, S. Y., Lin, Z. F. & Chui, S. T. Effective-medium theory for anisotropic magnetic metamaterials. *Phys. Rev. B* **80**, 115101 (2009).

40. Fouque, J.-P., Garnier, J., Papanicolaou, G. & Solna, K. *Wave Propagation and Time Reversal in Randomly Layered Media* (Springer Science & Business Media, 2007).

41. Moitra, P. et al. Realization of an all-dielectric zero-index optical metamaterial. *Nat. Photon.* **7**, 791 (2013).

42. Huang, X., Lai, Y., Hang, Z. H., Zheng, H. & Chan, C. T. Dirac cones induced by accidental degeneracy in photonic crystals and zero-refractive-index materials. *Nat. Mater.* **10**, 582 (2011).

43. Alù, A., Silveirinha, M. G., Salandrino, A. & Engheta, N. Epsilon-near-zero metamaterials and electromagnetic sources: Tailoring the radiation phase pattern. *Phys. Rev. B* **75**, 155410 (2007).

44. Cheng, Q., Jiang, W. X. & Cui, T. J. Spatial power combination for omnidirectional radiation via anisotropic metamaterials. *Phys. Rev. Lett.* **108**, 213903 (2012).

45. Luo, J. & Lai, Y. Anisotropic zero-index waveguide with arbitrary shapes. *Sci. Rep.* **4**, 5875 (2014).

46. Gutiérrez-Vega, J. C., Rodríguez-Dagnino, R. M., Meneses-Nava, M. A. & Chávez-Cerda, S. Mathieu functions, a visual approach. *Am. J. Phys.* **71**, 233 (2003)

47. Cohen, R. W., Cady, G. D., Coutts, M. D. & Abeles, B. Optical Properties of Granular Silver and Gold Films. *Phys. Rev. B* **8**, 3689 (1973).