Logic of Machine Learning

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Abstract

The main question is: why and how can we ever predict based on a finite sample? The question is not answered by statistical learning theory. Here, I suggest that prediction requires belief in “predictability” of the underlying dependence, and learning involves search for a hypothesis where these beliefs are violated the least given the observations. The measure of these violations (“errors”) for given data, hypothesis and particular type of predictability beliefs is formalized as concept of incongruity in modal Logic of Observations and Hypotheses (LOH). I show on examples of many popular textbook learners (from hierarchical clustering to k-NN and SVM) that each of them minimizes its own version of incongruity. In addition, the concept of incongruity is shown to be flexible enough for formalization of some important data analysis problems, not considered as part of ML.

Introduction

ML is usually associated with making predictions to improve decisions. Of course, the future is unknown, so prediction got to be difficult. But we can’t know much about the past either.

An applied ML scientist (practitioner) is aware that nothing is known for sure about the reality we model. For example, the features may not define completely the feedback we are trying to model, there is uncertainty in measurements, random mis-classification and so on. Objects with the same features may have different feedback, and the same object evaluated twice may have different features or even feedback.

This is not a bad luck, but a inevitability. Indeed, if there is no exact theory explaining the phenomenon we are trying to predict, we do not know what it depends on or how to measure it. If there is a theory, one does not need ML. ML deals with real raw life, not an abstraction.

More the over, the time is critical when it concerns prediction based decision making. It means the less time we spend on accumulating the data for predictions, and the less data we use for prediction, the better. So, data shortage is not a bug, it is a feature of ML.

Thus to predict future we need to model nondeterministic dependence with as little data as possible.

The first question should be: what does it mean to model a nondeterministic underlying dependence, and how do we, actually, do it?

The answer proposed here is that modeling is possible because (i) we implicitly rely on “predictability” of the underlying dependence: close or identical data points shall correspond to close feedback. Violation of this principle is called here “incongruity”. And (ii) we search for the least incongruent model of the dependence.

A Logic of Observations and Hypotheses as well as concept of “proper aggregation” are introduced here to formalize the idea of incongruity of a hypothesis given the predictability beliefs and data.

The main conjecture of this work is that each learner has its own version of incongruity, and its loss criterion evaluates this incongruity for a given hypothesis and the training set. As a part of this main conjecture, I outlined general steps each learner performs, exposing inner similarity of diverse learners from k-NN to K Means.

To the best of my knowledge, here it is shown for the first time that the large variety of learners can be described in common terms and that they have common logical justification.

1 Traditional views on ML

Here I describe a “a naive” idea of ML and the issues with this idea. I pose the questions the theory needs to answer. Then I present the only commonly accepted theory of automatic
learning and show that it does not really answer these questions.

1.1 Prediction problem

Denote \( \Omega \) the set of real life objects of interest. For example, this may be patients with skin cancer, or bank clients or engine failures. There is a hidden essential quality of the objects we would like to find out (may be, a diagnosis or prognosis). Some properties (features) of the objects \( \Omega \) can be always evaluated and numerically expressed. Some of them are expected to be relevant to the hidden property. Suppose, there are \( n \) such features. Denote \( X \in \mathbb{R}^n \) domain of feature vectors for objects in \( \Omega \). The hidden essential quality also has numerical expression from domain \( Y \in \mathbb{R} \). The value of the hidden essence in a given object is called “feedback”. We assume there is an “underlying dependence” \( \varphi : X \rightarrow Y \) between feature vectors and the feedback. Yet, we can not assume that the dependence is deterministic.

The information about the underlying dependence \( \varphi \) is given as (imprecise) observations about values of feedback in certain data points and can be recorded as set of formulas \( \{ \varphi(x) \approx y \} \) or as set of tuples \( \{ (x, y) \} \).

The set of recorded observations is called training set. In a Prediction problem, the goal is to find a function \( f : X \rightarrow Y \), which is “close enough” to the underlying dependence \( \varphi \), in the sense that the probability of large errors on future observations of objects in \( \Omega \) is low enough:

\[
P[|\varphi(x) - f(x)| \geq \delta] < \epsilon.
\]

The Prediction problem is ill posed: generally, knowing a finite set of observations with non repeating data points does not imply anything about expected values of feedback \( \varphi(x) \) in the same data points, let alone in other points. Strictly speaking, the givens and goal of the problem are not related.

Any theory of machine learning needs to answer at least two fundamental questions:

1. Q1: What shall be done with the training set for learning?
2. Q2: When and why can a decision predict well enough?

In the next subsection, I show how statistical learning theory understand machine learning.

1.2 Statistical Learning Theory Approach

Statistical Learning (SL) theory is the only commonly accepted theory of ML. The most popular version of this theory is also called “VC-theory”, because VC-dimension plays important role here.

There is some confusion about the term “learner” in learning theory textbooks. For example, in [18], the term is understood as a procedure for solving a problem in a finite number of steps - when actual procedures are discussed. But in chapters talking about statistical learning theory (PAC learning, VC-theory) the terms “learner” and “algorithm” mean “functional that takes in a finite training set \( S \) and outputs a function \( h : X \rightarrow Y \)” [1]: “VC-theory did not impose any requirement on the learners actually being implementable by algorithms”. For disambiguation, talking about SL, I will use the term “learning functional”.

Thus, strictly speaking, statistical learning theory does not talk about the main subject of this work, the learning algorithms. Still, I will describe its main results here because of its unique importance.

Only binary labels \( Y \) are considered here.

Denote \( D \) the distribution on \( X \times Y \) from which the training set \( S \) is drawn. Denote

\[
L(f, S) = \sum_S I(f(x_i) \neq y_i),
\]

empirc risk of a a hypothesis \( f : \chi \rightarrow Y \) on the training set \( S \),

\[
L(f, D) = P[f(x) \neq y]|(x,y) \sim D],
\]

generalized risk, or error on the general population,

\[
L(H, D) = \min_{f \in H} L(f, D),
\]
optimal generalized risk in the class of hypotheses $H$.

The hypothesis $h = \alpha(S)$ output by a learning functional $\alpha$ on a given training set will be called a **decision** of this functional on $S$.

For a given $\varepsilon > 0$ a hypothesis $h$ is **approximately correct** for the class of functions $H$ if $L(h, D) < L(H, D) + \varepsilon$. Denote $A_\varepsilon(h, D)$ this property

$$A_\varepsilon(h, D) \overset{\text{def}}{=} (L(h, D) < L(H, D) + \varepsilon).$$

As the work [1] clarifies, the approximately correct for the class $H$ decision $h$ does not have to belong to the class $H$, and it does not have to be in such form that one could use it to calculate the function $h$ on any arguments.

Instead of considering a single decision on a given training set $S$, statistical learning is interested in all the decisions obtained on different training sets with a given lower bound $m$ on their size. Denote $Z(\alpha, m, D)$ set of all the decisions by the learning functional $\alpha$ on all the training sets of the size $m$ or larger randomly generated by the same distribution $D$.

For a given constant $\delta$, if

$$P_{h \in Z(\alpha, m, D)}[A_\varepsilon(h, D)] > 1 - \delta,$$

every decision $h \in Z(\alpha, m, D)$ is called **probably approximately correct**. Denote this property $B(\alpha, m, D, \varepsilon, \delta)$

$$B(\alpha, m, D, \varepsilon, \delta) \overset{\text{def}}{=} \left( P_{h \in Z(\alpha, m, D)}[A_\varepsilon(h, D)] > 1 - \delta \right).$$

For a given class $H$, a learning functional $\alpha$ is called **successful learner**, if for every $\varepsilon, \delta$, regardless of distribution $D$, for a large enough $m(\varepsilon, \delta)$ predicate $B(\alpha, m(\varepsilon, \delta), D, \varepsilon, \delta)$ is true.

The main focus in the theory is classes of PAC-learnable functions. The class $H$ is called **PAC (probably approximately correct) learnable** if there exists a successful learning functional for this class.

“Our current goal is to figure out which classes $H$ are PAC learnable, and to characterize exactly the sample complexity of learning a given hypothesis class.” [18] (sample complexity is politically correct name for minimal size of the training sample.)

The main result of the theory is that classes of functions which have finite VC dimension and only such classes are PAC learnable. The theory claims there is a function $m(\varepsilon, \delta, VCD(H))$, which gives lower bound of the parameter $m$ in the set $Z(\alpha, m, D)$ of PAC decisions for PAC-learnable classes. Also, it states that if class is PAC learnable, then a functional which picks a functions from the class $H$ with minimal empiric risk is a successful PAC learner.

Here are some issues with SL.

1. **The theory solves an irrelevant problem.** The main result of the theory

   - expresses required size of the training set through VC dimension of the function class $H$, yet the decisions may be outside of the class $H$; the relevance of the class $H$ is not clear;
   - makes statement about an arbitrary distribution, yet most of the distributions are of no interest because they do not support the existence of an underlying dependence; it should be easier to find a dependence when it exists.
   - evaluates probability of having “approximately correct” hypothesis in $Z(\alpha, m, D)$. This set of hypotheses is pure speculation: there is only one training set. But even we imagine $Z(\alpha, m, D)$, infinitely many hypotheses in this set are obtained on training sets arbitrary larger than $m$ and non is obtained on sets smaller than $m$. Therefore even if probability of failure on $Z(\alpha, m, D)$ is low, it implies nothing about likelihood of failure on a given training sample of the size $m$.
   - recommends empiric risk minimizing functional of a class $H$ as a successful PAC learner, but, if the class $H$ is infinite, there may not be an algorithm implementing this functional in a finite number of steps.
   - talks about relative generalized risk, while practical applications are interested only in absolute generalized risk.
2. Indefinitely increasing training set size contradicts the idea of learning.

- **The training set does not increase.** "Intuitively, it seems reasonable to request that a learning algorithm, when presented more and more training examples, should eventually “converge” to an optimal solution." [14] There is nothing intuitive or reasonable about such a request, because training set is fixed.

- **The bounds of the sizes of the training sets are unreasonable.** The theory gives upper bounds for the desired training sample size. These sizes are grotesque. There are no training sets of such sizes for most of ML problems.

- **Large training sets are counterproductive.** Timing of decisions which are supposed to be made on the basis of ML is critical, therefore smaller training sets are preferable. Large training sets as the theory requires would describe the master distribution in very fine details, making ML pointless. People do ML exactly because it allows one to compensate for lack of accumulated knowledge.

3. SL ignores both critical questions

- The theory deals with learning functionals and not algorithms, so it cannot answer the first critical question (“What shall be done with the training set for learning?”).

- Being interested only in relative loss, SL cannot answer the second critical question: “When and why can learner’s decision predict well enough?”

Theoreticians usually do not dwell on these issues. But it does not mean that they do not notice them.

One of fathers of SL, V. Vapnik formulated the justification for SL theory in the most direct way [19]:

> Why do we need an asymptotic theory (⋯) if the goal is to construct algorithms from a limited number of observations? The answer is as follows: To construct any theory one has to use some concepts in terms of which the theory is developed (⋯).

I other words, SL has to use the statistics toolkit. Statistics has laws of large numbers, and it is what one uses for deduction in statistical learning theory.

Perhaps, lack of the suitable apparatus to understand the true problem with fixed finite data and un-quantifiable uncertainty is the root of this divergence between the theory and needs of the applications.

The theory so remote from real life applications cannot help practitioners, does not answer the most common questions they raise [17].

Therefore, there is a need in a learning theory which would make sense of the actual practice of ML, I show below that, based on popular learners, learning is possible not because of ever increasing training set, but because the underlying dependence is expected to be “predictable” or “congruent”.

2 Predictability

I am not trying, vainly, to forecast accuracy of a decision on the general population. Rather, I want to answer the first critical question: what do we do with the training set in applied ML, and what is the logic behind these manipulations?

2.1 Existing Approaches to Logics of Uncertainty

There are plenty of well established logical approaches to reasoning under uncertainty, as well as to study of nondeterministic dependencies and modeling inconsistent data.

Modal logics are introduced to take into account some subjectivity and uncertainty. Yet, they would not tolerate inconsistency.

Fuzzy logic [15] and Subjective Logic [11] would not help either, because they assume there is an objective omnipotent observer, who can quantity degrees of certainty or belief about given statements. The systematic review [9] describes various approaches to reasoning
under uncertainty as ways to quantify and exactly measure uncertainty of statements and sets of statements. Exactness about uncertainty of empirical observations appears to be a contradiction in terms.

A typical approach to resolve inconsistency of knowledge is to assign some kind of “certainty” or “preference” for each formula, and then select a the most preferable (“probable”, “certain”, “reliable”) subset of consistent formulas. One of the first works of this type was [8]. The main idea there is to assign reliability to each statement and remove the least reliable ones to avoid a contradiction. Here are the main issues with this approach:

1. When we are not certain about the knowledge, we can not be certain about comparative reliability of it.

2. For a nondeterministic dependence, contradictory observations are the rule, not an exception. Together they create more complete picture of reality than any non-contradictory subset.

3. In case of ML contradictions between a hypothesis and noisy observations shall be present always: exact fit of noise is not desirable. It means, excluding inconsistency is not an option.

Nondeterministic logics [3] and logical operations with nondeterministic tables are introduced to derive logical functions from exact but incomplete data, which is different from the situation of inconsistent observations or inconsistency between hypothesis and the training set.

There are several approaches to describe logic of learning. For example, [7] considers asymptotic learning: precise observations are presented indefinitely, the “nature” has in mind particular function from the given class, and the learner has to chose the correct hypothesis. Unfortunately, in applied ML, all the good assumptions about this learning idea are false: the training set is finite and too small, observations are known to be tentative, and whatever nature has in mind, is far from the selected class of functions or, rather, is not a function of given features at all.

The researchers already noticed that logic where all formulas have truth values does not describe certain types of logical reasoning [2], particularly legal reasoning about existing norms. The logics with modalities like “it ought to be”, “you ought to do” are introduced, but to the best of my knowledge, epistemic modalities which can not be associated with truth values were not explored.

2.2 Informal description of the new approach

Let us go back to the original prediction problem: Given imprecise observations of nondeterministic dependence \( \varphi \), to find a function \( f \) to model \( \varphi \) and predict its values on new data points. I noticed that the problem is incorrect, can not be solved as is.

In real life, the solution of a ML problem is possible, because we believe in “predictability” of \( \varphi \): the dependence has to have similar values in close data points. If we do not believe it, there is no problem.

And we, usually, have good reasons for predictability belief. ML problems do not appear from nowhere. They are thoroughly prepared by the same specialists who solve them. Preparation includes posing a meaningful question and developing the features to be predictive.

From philosophy point of view, predictability beliefs are founded on the fundamental belief in inner congruence, “consistency” of the reality: the world does not, usually, change sharply. Otherwise, the prediction would not be possible and we could not exist as animals, let alone humans: the brain is an organ of prediction in all animals [13]. The fundamental belief leads us further to prefer models which appear to be more “consistent” with the the available data: if the model was more “agreeable” in the past, we have expectations that it will continue to be. Thus, we usually assume that optimal agreement with available data is a desirable property of the model.

Instead of searching for the model which will work good in the future - which is impossible - we search for a model which works on accumulated data both well enough and the best among others. And this problem is, often, tractable, at least approximately.

As predictability may take different forms, depending on the problems, so does incongruity. To understand the process of learning we would need a general concept of incongruity.
In the simplest case, for each observation \( (x, y) \), we take a hypothetical instance \( h(x) = y_1 \) where the value of the underlying dependence \( \varphi \) are evaluated in the same point \( x \), and measure how the values \( y_1, y \) are different.

In general, the concept of incongruity requires (i) identifying associations between hypothetical instances and observations and (2) evaluating the disagreement, or deviation, for each such associated pair.

This will give us the set of deviations for each hypothesis. So, on top of this, one would need to aggregate these deviations, so each hypothesis can be characterized by a single number.

One may view “deviation” as a fuzzy measure of contradiction. If \( y_1 = y \) in the above example, there is no contradiction between the observation and the associated hypothetical case. Yet, the values \( y, y_1 \) are not expected to be identical. The concept of logical contradiction is meaningless for a nondeterministic function and its model. So, deviations evaluate degree, to which the desired “consistency” is violated. Then value of incongruity is a fuzzy measure replacing rigid concept of inconsistency of set of formulas.

The conjecture of this work is that every learner has its own concept of incongruity as a loss criterion to compare hypotheses and derive the decision. The conjecture will be corroborated on examples of many learners.

To avoid inconsistency in reasoning about nondeterministic dependencies, I introduce subjective modalities of perceptions or assumptions. These modalities allow contradictions. For example, different subjects (or even the same subject) can observe things differently in different times. This makes the concept of inconsistency of our knowledge irrelevant.

3 Formal definition of the approach

Modal logic of observations and hypotheses (LOH) formalizes reasoning about predictability and deviations from predictability belief. The instances are presented as first order modal formulas. The second order relations on the first order formulas will be used to describe predictability.

3.1 Logic of observations and hypotheses (LOH)

The first order signature has four sorts:

| Sort | Meaning | Variables | Constants |
|------|---------|-----------|-----------|
| 1 X  | finite numeric set from \( \mathbb{R}^n \) | \( x, x_1, x_2, \ldots \) | |
| 2 Y  | finite numeric set | \( y, y_1, y_2, \ldots \) | |
| 3 \( \Theta \) | symbols of modalities | \( s, s_1, s_2, \ldots \) | |
| 4 R  | real numbers | \( r, r_1, r_2, \ldots \) | \( a, b, a_1, a_2, a_3, \ldots \) |

We assume the domains of the sorts \( X \) and \( Y \) are subsets of some metric spaces. For example, the set \( Y \) with two values \{0, 1\} may be considered a metric space with the norm

\[
\|y_1 - y_2\| = \begin{cases} 
0, & \text{if } y_1 = y_2 \\
1, & \text{otherwise}.
\end{cases}
\]

It is obvious that the norm satisfies the axioms of metric spaces.

We will consider the next symbols of modalities \{\( \approx \), \( \approx_1 \), \ldots, \( \approx_{i} \), \ldots \}. The symbol \( \approx \) is interpreted as \textit{Assume that}, it indicates the hypothetical instances. The modalities \( \approx \) are interpreted as \textit{It appears that} and indicate observations. The index in a notation of modality are used to separate groups of hypothetical instances and groups of observations with different context. The number of modalities may be different between interpretations.

There is one dedicated first order unary functional symbol \( \varphi : X \to Y \), which denotes the underlying dependence, and the only first order relation = with standard interpretation.

Interpretations of this logic may have optional other sorts, first order functional symbols and operations, which will be specified in formalization of each problem as needed.
All the first order formulas have the form $\Box(\varphi(x) = y)$ where $\Box \in \Theta$ is one of the modalities. The formula $\approx (\varphi(x), y)$ corresponds to an observation $(x, y)$, the formula $\equiv (\varphi(x) = y)$ corresponds to the hypothetical instance $(x, y)$.

The formulas are not assigned truth values: they reflect subjective, uncertain knowledge. Nothing can be deduced from these formulas. There is no logical connectives, no first order inference.

The set of all first order formulas of a LOH model $\mathcal{M}$ will be denoted as $\Upsilon(\mathcal{M})$. The set $\Upsilon(\mathcal{M})$ is the domain of the second order functions and predicates. The variables $\alpha, \alpha_1, \ldots$ will denote formulas from $\Upsilon(\mathcal{M})$.

### Table 2: Second order function symbols

| Symbol | Arity | Sorts | Semantic |
|--------|-------|-------|----------|
| $x$    | 1     | $\Upsilon(\mathcal{M}) \to X$ | the first variable |
| $y$    | 1     | $\Upsilon(\mathcal{M}) \to Y$ | the second variable |
| $s$    | 1     | $\Upsilon(\mathcal{M}) \to \Theta$ | modality symbol |
| $\rho_x$ | 2    | $\Upsilon(\mathcal{M}) \times \Upsilon(\mathcal{M}) \to \mathbb{R}$ | distance between first variables |
| $\rho_y$ | 2    | $\Upsilon(\mathcal{M}) \times \Upsilon(\mathcal{M}) \to \mathbb{R}$ | distance between second variables |

The second order functions are defined by the axiom

$$A_1 : \forall \alpha \left( s(\alpha) \left( \varphi(x(\alpha)) = y(\alpha) \right) \right) = \alpha.$$  

$$A_2 : \forall \alpha_1, \forall \alpha_2 \rho_x(\alpha_1, \alpha_2) = \|x(\alpha_1) - x(\alpha_2)\|.$$  

$$A_2 : \forall \alpha_1, \forall \alpha_2 \rho_y(\alpha_1, \alpha_2) = \|y(\alpha_1) - y(\alpha_2)\|.$$  

The second order formulas will have

- relations $\leq, <, >, \geq, =$, on real numbers,
- regular logical connectives ($\lor, \land$)
- real valued constants.

### 3.2 Incongruity Concept

Informally, incongruity means that for associated pairs of formulas, when arguments of the underlying dependence in them are “close”, so are their feedback.

#### 3.2.1 Main Definitions

**Definition 1** (Collision condition). Any second order LOH statement with two free variables over first order formulas will be called an **collision condition**.

Figuratively speaking, an collision relation identifies pairs of formulas which potentially can be “vaguely contradictory”. For such a pair of formulas, “deviation” will determine the degree of its “vague contradiction”, or collision.

**Definition 2** (Deviations). **Deviation function** is a function $t(r_1, r_2) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ isotone by $r_1$ and antitone by $r_2$.

For a pair of first order formulas $\alpha_1, \alpha_2$ and a deviation function $t$ their **deviation degree** is

$$\delta(\alpha_1, \alpha_2) = t(\rho_y(\alpha_1, \alpha_2), \rho_x(\alpha_1, \alpha_2)).$$

Deviation degrees are often called “errors” in ML.

There may be many aspects of incongruity. This gives rise to the concept of incongruity theory.
**Definition 3** (Incongruity theory). A sequence of collision conditions and corresponding deviation functions

\[ \{ (\pi_i, t_i), i = 1 : k \}, k \geq 1 \]

is called a **incongruity theory**. Each pair \( (\pi_i, t_i) \) is called incongruity aspect.

For a incongruity theory \( T \) the notation \( T \vdash (\alpha_1, \alpha_2) \) means the theory includes a collision condition \( \pi \) such that \( \vdash \pi (\alpha_1, \alpha_2) \), the pair \( \alpha_1, \alpha_2 \) is in the collision condition \( \pi \).

#### 3.2.2 Example of an incongruity theory

This example of a incongruity theory is called **Point-Wise** incongruity theory, \( T_{pw} \). It has only one aspect with the collision condition

\[ \pi_{pw}(\alpha_1, \alpha_2) = \left( (s(\alpha_1) = \varepsilon) \& (s(\alpha_2) = \approx) \& (x(\alpha_1) = x(\alpha_2)) \right) \]  

(1)

and the deviation function \( t(r_1, r_2) = r_1 \). The function \( t \) is obviously isotone by the the \( r_1 \). The function does not depend on \( r_2 \), so for any \( r_1, r_2, r_2' \) \( t(r_1, r_2') = t(r_1, r_2) \). Therefore, the condition of antitony by the second variable is not violated.

By definition, hypothetical instance \( \alpha_1 \) and an observation \( \alpha_2 \) satisfy the point-wise collision condition \( \pi_{pw}(\alpha_1, \alpha_2) \) when \( x(\alpha_1) = x(\alpha_2) \). For any pair of first order formulas \( \alpha_1, \alpha_2 : \vdash \pi_{pw}(\alpha_1, \alpha_2) \) their deviation \( \delta(\alpha_1, \alpha_2) = \rho_{pw}(\alpha_1, \alpha_2) \).

#### 3.2.3 Full Model

Various models of LOH will have various sets of first order formulas. The same theory may have different set of deviations depending on the model. There needs to be an agreement about a LOH model used to evaluate deviations for a given theory, hypothesis and observations.

**Definition 4** (Full model). Given an incongruity theory \( T = \{ (\pi_i, t_i), i = 1 : k \} \), hypothesis \( h \) and the set of observations \( S \), a model \( \mathcal{M} \) is a **full model** for \( h, S, T \) if

- \( S \subseteq \Upsilon(\mathcal{M}) \)
- For any hypothetical formula \( \alpha \) and for any observation formula \( \beta \in \Upsilon(\mathcal{M}) \) if

\[ (T \vdash (\alpha, \beta)) \lor (T \vdash (\beta, \alpha)) \]

then \( \alpha \in \Upsilon(\mathcal{M}) \)
- For any hypothetical formulas \( \alpha_1, \alpha_2 \) such that \( T \vdash (\alpha_1, \alpha_2) \) the formulas \( \alpha_1, \alpha_2 \) are in \( \Upsilon(\mathcal{M}) \).
- \( \Upsilon(\mathcal{M}) \) is the **minimal set of formulas**, satisfying these conditions.

Given the hypothesis, observations and incongruity theory, the definition determines the set of first order formulas of a full model uniquely.

#### 3.2.4 Example of a full model

For example, for the \( T_{pw} \) theory, any hypothesis \( h \) and the set of observations \( S = \{ \beta_1, \ldots, \beta_m \} \) the set of formulas \( \Upsilon(\mathcal{M}(h, S, T_{pw})) \) will consist of the next two subsets

- \( S \)
- \( \{ \alpha \mid (s(\alpha) = \varepsilon) \& (\exists \beta (\beta \in S) \& (x(\alpha) = x(\beta))) \} \).

For a given set of \( m \) observations \( S \) and a hypothesis \( h \) full model of the theory \( T_{pw} \) will have all the first order formulas of the observations \( S \), and for every observation \( \beta \in S \) there will be a hypothetical formula

\[ (\varepsilon (x(\beta)) = h(x(\beta))) \]  

. So, there will be exactly \( m \) pairs of first order formulas \( \alpha_1, \alpha_2 \) satisfying the condition \( \pi_{pw} \).
3.2.5 Regularization

Usually, predictability of a dependence means that it has close values on close data points. When a hypothesis is a known differentiable function, there are ways to evaluate some aspects of its “predictability” independently of data. A good, predictable hypothesis shall be smooth, it has to have uniformly low derivatives.

This method of including some measure of derivatives in the loss criterion of a learner is called “regularization”. Regularization is used sometimes in addition to incongruity to measure violation of predictability.

3.3 Proper aggregation

An aggregation operation maps a multiset of real numbers into a real number.

The operation of aggregation $\text{TOT}(G) : 2^\mathbb{R} \rightarrow \mathbb{R}$ defined on all finite multiset in $\mathbb{R}$ is called proper aggregation, if it satisfies three axioms.

1. **Monotony**: For two multisets in $\mathbb{R}$ If there exists isomorphism $q : G_1 \rightarrow G_2$ such that

\[
\left( \forall x \; (q(x) \geq x) \Rightarrow \text{TOT}(G_2) \geq \text{TOT}(G_1) \right) \& \left( \forall x \; (q(x) > x) \Rightarrow \text{TOT}(G_2) > \text{TOT}(G_1) \right)
\]

2. **Idempotence**: $\text{TOT}(G \cup \{\text{TOT}(G)\}) = \text{TOT}(G)$.

3. **Tautology**: If $G = \{x\}$ then $\text{TOT}(G) = x$.

Some natural properties of proper aggregation follow from the axioms.

**Statement 1.** Any proper aggregation $\text{TOT}(G)$ has the next properties:

1. If the multiset $G$ consists of $n$ identical elements $x$, then $\text{TOT}(G) = x$.

2. $\min(G) \leq \text{TOT}(G) \leq \max(G)$.

**Proof.**

1. Let us prove it by induction by $n = \|G\|$. It $n = 1$, it follows from the axiom Tautology. Suppose, the statement is proven for $n = k$. Then for $n = k + 1$ it follows from the axiom Idempotence.

2. Let us prove by contradiction. Suppose

\[\exists G \forall x \; (x \in G) \Rightarrow (\text{TOT}(G) > x)\].

Denote $n = \|G\|$. By the previous property, if the set $G_1$ consists of $n$ elements $\text{TOT}(G)$ then $\text{TOT}(G_1) = \text{TOT}(G)$. It contradicts the axiom of Monotony since every element of $G_1$ is larger than all elements of $G$. The same way we can prove that $\text{TOT}(G)$ can not be lower than all elements of $G$.

One example of proper aggregation operation is $\mu(G)$, median on $G \subset \mathbb{R}$.

**Statement 2.** Operation $\mu(G)$ is a proper aggregation.

**Proof.** Let us prove monotony. Denote $q$ isomorphism $G_1 \rightarrow G_2 : q(x) \geq x$ and

\[\rho_i = \mu(G_i), \; i = 1, 2\].

Because the sets are isomorphic, they have the same power $\|G_1\| = \|G_2\| = n$.

For some integer $k : n = 2k$ or $n = 2k + 1$. The number of elements in $G_2$ which are larger than $\rho_2$ is the same as the number of elements larger than $\rho_1$ in $G_1$. In both cases and for both sets the number is equal $k$.

Denote

\[G^-_1 = \{x \mid (x \in G_1) & (x < \rho_i)\}, \; \text{for } i = 1, 2\].

\[G^+_1 = \{x \mid (x \in G_1) & (x > \rho_i)\}, \; \text{for } i = 1, 2\].

By definition of $q$, for any $x \in G^+_1$, $q(x) \in G^+_2$. So, $\|G_2^+\| \geq \|G_1^+\|$.
Suppose, $\|G_1^+\| > \|G_1^-\|$. This means, $\|G_1^+\| > k$. Therefore $\rho_2 \in G_1^+$ and $\rho_2 > \rho_1$. It proves the theorem for the case $q(x) \geq x$.

Suppose, $\|G_1^+\| = \|G_1^-\| = k$ In this case, $\|G_2^-\| = \|G_1^-\| = k$ and for every element $x \in G_1^-$, $q(x) \in G_2^-$. First, suppose $n = 2k + 1$. Then $\rho_1 \in G_1$. The only element of $G_2$ which does not belong to $G_2$, $G_2^+$ is $q(\rho_1)$. Therefore $q(\rho_1) = \rho_2$, and $\rho_2 \geq \rho_1$.

Now, suppose $n = 2k$ In this case, for $i = 1, 2$

$$\rho_i = \mu(G_i) = \frac{\min(G_i^+) + \max(G_i^-)}{2}.$$ 

Since $q(\max(G_1^-)) \geq \max(G_1^+)\) and $q(\max(G_1^-)) \in G_2^+$ then $\max(G_2^-) \geq \max(G_1^+)$. Let us notice that $x \in G_1^+$ if and only if $q(x) \in G_2^+$. For any $x \in G_1^-$, $q(x) \geq x \geq \min(G_2^+)$ therefore $\min(G_2^+) \geq \min(G_1^+)$.

This proves the monotony for the case, when $q(x) \geq x$. The case when for every $x : q(x) > x$ is proven similarly.

Let us prove the idempotence. Denote $\rho = \mu(G)$, $b < \rho < c$ are two closest elements in $G$ to $\rho$. Suppose, $\|G\| = 2k$ and $\rho = (b + c)/2$. Then $\mu(G \cup \{\rho\}) = \rho$. Suppose $\|G\| = 2k + 1$. Then $G \cup \{\rho\}$ has has two identical elements equal $\rho$ in the middle. And $\mu(G \cup \{\rho\}) = \rho$.

Tautology is trivial, because median of $\{x\}$ is $x$.

The statement could be proven not only for median, but for any percentile. So any percentile can be used as a proper aggregation.

### 3.4 Total proper incongruity

If an incongruity theory has $k$ aspects, then each hypotheses will be characterized by $k$ sets of deviations. To compare hypotheses, the deviations need to be aggregated. For this purpose, we use two step procedure: first deviations for each aspect are aggregated using its own proper aggregation, then the results of these operations are further aggregated (along with some regularization components, possibly) to have a single number representing incongruity for a given hypothesis.

The result of the aggregation of deviations for a single aspect will be called aspect incongruity. Given training set $S$, hypothesis $h$ and incongruity theory $T$, a procedure $TOT(G)$ will be called total proper aggregation procedure if it satisfies three conditions:

1. The set $G$ contains all aspect incongruity, each obtained with a proper aggregation on the full model of $S, h, T$.
2. In addition, the set $G$ may contain regularization components.
3. The operator $TOT(G)$ is isotone: For any multisets $G_1, G_2$ and real numbers $x, y$

$$(G_1 = (G \setminus \{x\}) \cup \{y\}) \& (y \geq x) \Rightarrow TOT(G_1) \geq TOT(G).$$

The result of applying a total proper aggregation procedure on aspects of incongruity and regularization components will be called total proper incongruity.

### 3.5 Logic of recursive aggregation

One drawback of using percentiles for aggregation is, perhaps, the non-linear calculation complexity. Learners usually prefer to use aggregation which requires going through all the elements of the multiset $G$ only once. To express recursive aggregation, I will use extension of the first order logic with added counting quantifiers [12] $\exists^c x$, where $x$ is a variable, and $c$ may be a natural number or variable with values in $\mathbb{N}$. The quantifier means: there exists exactly $c$ of $x$.

There are three sorts.

There is total order $\prec$ on the domain $G$. The functions in the language of aggregation are described in the next table
The table shows axioms characterizing properties of other functions in the language:

| Sort | Meaning | Variables       |
|------|---------|-----------------|
| 1    | $\mathbb{G}$ | finite set of real numbers | $x, y, z$ |
| 2    | $\mathbb{N}$ | $N = \{1, \ldots, n\}, \ n = \|\mathbb{G}\|$ | $i, n, i_1, n_1, \ldots$ |
| 3    | $\mathbb{R}$ | space of real numbers | $r, r_1, r_2, \ldots$ |

3.5.1 Theory of recursive aggregation

The order $\prec$ on $\mathbb{G}$ is defined as a strict total order (with axioms of irreflexivity, transitivity, anti-symmetry and total order). The relationships $\{<, \leq, \geq, =\}$ are defined in usual way on real numbers. The functions $\text{get}(i), \text{count}()$ are defined uniquely as $i$-th element in the order $\prec$ and the cardinality of $G$ when the domain $G$ of the sort $\mathbb{G}$ and the order $\prec$ on it are known:

$$\forall x \forall i \ (\text{get}(i) = x) \iff (\exists^{i-1} y \ y \prec x)$$

$$\forall n \ \text{count}() = n \iff \text{count}().$$

The table shows axioms characterizing properties of other functions in the language:

| Axiom | Commentary |
|-------|------------|
| $B_1$ | $(x_1 > x) \Rightarrow (\text{scale}(x_1) \geq \text{scale}(x))$ | monotony |
| $B_2$ | $\text{plus}(x, y) = \text{plus}(y, x)$ | symmetry |
| $B_3$ | $(x_1 > x) \Rightarrow (\text{plus}(x_1, y) \geq \text{plus}(x, y))$ | monotony |
| $B_4$ | $\text{plus}(x, \text{plus}(y, z)) = \text{plus}(\text{plus}(x, y), z)$ | associativity |
| $B_5$ | $\forall i \ (\text{agg}(i) = \text{scale}(\text{get}(i)))$ & $\text{agg}(i+1) = \text{plus}(\text{agg}(i), \text{scale}(\text{get}(i+1)))$ | recursive aggregation |
| $B_6$ | $(x_1 > x) \Rightarrow (\text{norm}(x_1, n) \geq \text{norm}(x, n))$ & $\text{norm}(x, n_1) \leq \text{norm}(x, n)$ | norm |
| $B_7$ | $\forall i \ (\text{get}(i+1) = \text{norm}(\text{agg}(i), i)) \Rightarrow \text{idempotence}$ |
| $B_8$ | $\text{norm}(\text{scale}(x), 1) = x$ | tautology |
| $B_9$ | $(x_1 \neq x_2) \Rightarrow (\text{scale}(x_1) \neq \text{scale}(x_2))$ & $\text{norm}(x_1, i) \neq \text{norm}(x_2, i)$ | strict |
| $B_{10}$ | $(x_1 < x_2) \& (y_1 < y_2) \Rightarrow \text{plus}(x_1, y_1) < \text{plus}(x_2, y_2)$ | strict |

Typical examples of the function $\text{plus}$ are:

- $\text{plus}(x, y) = x + y$
- $\text{plus}(x, y) = x \cdot y$
- $\text{plus}(x, y) = \text{max}(x, y)$

All these functions are used by popular learners, as I will demonstrate. Each model $\mathcal{M}$ uniquely defines an operation

$$\text{TOT}(\mathcal{M}) = \text{norm}(\text{agg}(\text{count}), \text{count})$$
Given an interpretation of functions \( \text{scale, plus agg, norm} \), \( \text{TOT}(M) \) is defined by its finite domain \( G \) and the strict total order \( \prec \) on it.

The next theorem shows that \( \text{TOT}(M) \) does not depend on the order \( \prec \).

**Theorem 1.** Suppose \( M_0, M_1 \) models of recursive aggregation language are different by the orders \( \prec \) only: domains of the sort \( G \) consist of the same elements and interpretations of all the functions of the language are identical. Then \( \text{TOT}(M_0) = \text{TOT}(M_1) \).

**Proof.** Suppose, the models are different by the orders on domains \( G_0, G_1 \) of sort \( G \). For the finite domain \( G \), the order \( \prec_1 \) may be considered as a permutation of order \( \prec \). Each permutation can be obtained by finite number of simple transpositions (transpositions of neighboring elements). Suppose, the order \( \prec_1 \) is obtained from order \( \prec_0 \) by \( K \) simple transpositions. Let us prove the theorem by induction by \( K \). First, suppose \( K = 1 \). Denote \( x_1, x_2, \ldots, x_i, x_{i+1}, \ldots, x_n \) elements of \( G \) ordered by \( \prec_0 \). Suppose, the order \( \prec_1 \) transposes elements \( x_i, x_{i+1} \). Denote \( \text{agg}(l), \text{agg}(l) \) values of the recursive aggregation function obtained on the step \( l \) with the orders \( \prec_0, \prec_1 \) respectively. Since all the elements prior to \( i \) are identical in these orders, \( \text{agg}(i - 1) = \text{agg}^0(i - 1) \). By definition

\[
\begin{align*}
\text{agg}(i) &= \text{plus}(\text{agg}(i - 1), x_i) \\
\text{agg}(i + 1) &= \text{plus}(\text{agg}(i), x_{i+1}) \\
&= \text{plus}(\text{plus}(\text{agg}(i - 1), x_i), x_{i+1}) \\
\text{agg}(i) &= \text{plus}(\text{agg}(i - 1), x_{i+1}) \\
\text{agg}(i + 1) &= \text{plus}(\text{agg}(i - 1), x_{i+1})
\end{align*}
\]

Using symmetry and associativity of the function \( \text{plus} \) \( (B_2, B_4) \) we get

\[
\begin{align*}
\text{agg}(i + 1) &= \text{plus}(\text{plus}(\text{agg}(i - 1), x_{i+1}), x_i) \\
&= \text{plus}(x_i, \text{plus}(\text{agg}(i - 1), x_{i+1})) \\
&= \text{plus}(\text{plus}(x_i, \text{agg}(i - 1)), x_{i+1}) \\
&= \text{plus}(\text{plus}(\text{agg}(i - 1), x_i), x_{i+1}) \\
&= \text{plus}(\text{agg}(i), x_{i+1}) \\
&= \text{agg}(i+1).
\end{align*}
\]

All the elements in the orders \( \prec_0, \prec_1 \) after \((i + 1)\)-th are identical. Therefore, \( \text{agg}_1(n) = \text{agg}^0(n) \). So, the constants \( \text{out}(M) \) and \( \text{out}(M_1) \) will be identical in this case.

Suppose, we proved the theorem for \( K = k \). Let us prove it for \( k + 1 \). Suppose, the first \( k \) simple transpositions involve elements with the indices below \( i - 1 \), and the last simple transposition involves elements \( x_i, x_{i+1} \). Then, the same considerations apply again.

\[\square\]

Every interpretation of the functions \{scale, plus agg, norm\} will have potentially infinite number of models different by the domains of the sort \( G \).

The theorem means that, given interpretation of the functions of the recursive aggregation language, the operation \( \text{TOT}(M) \) is an aggregation operation on the domain \( G \) of sort \( G : \text{TOT}(M) = \text{TOT}(G) \).

An aggregation defined by an interpretation of the recursive aggregation language may be called recursive aggregation.

The next theorem shows recursive aggregation is a proper aggregation.

**Theorem 2.** For any interpretation of the recursive aggregation language, the operation \( \text{TOT}(G) \) is a proper aggregation.

**Proof.** Let us prove tautology. If \( G = \{x\} \), then

\[
\text{TOT}(G) = \text{norm(agg(1), 1)} = \text{norm(scale(get(1)), 1)} = \text{norm(scale(x), 1)} = x,
\]

using the axiom \( B_8 \).

Let us prove monotony. Suppose, two models with domains \( G_1, G_2 \) of the sort \( G \) belong to the same interpretation, and \( q : G_1 \rightarrow G_2 \) is isomorphism such that \( q(x) \geq x \). Suppose they are ordered in such a way that \( q \) maps \( i \)-th element of \( G_1 \) into \( i \)-th element of \( G_2 \).
Let us prove it by induction by \( n = \|G_1\| = \|G_2\| \). For \( n = 1 \) it is true based on the axiom \( \mathcal{B}_8 \). Suppose, the statement is proven for \( n = k \). Denote \( \text{agg}_1(i), \text{agg}_2(i) \) results of aggregation on the domains \( G_1, G_2 \) on the step \( i \), and denote corresponding elements of \( G_1, G_2 : x_j, y_j = q(x_j), j = 1, \ldots, n \). By the assumption of induction, \( \text{norm}(\text{scale} \circ \text{agg}_2(k)), k) \geq \text{norm}(\text{scale}(\text{agg}_1(k)), k) \)

Let us prove the statement for \( n = k + 1 \).

\[
\text{norm}(\text{scale}(\text{agg}_1(k+1)), k+1) = \text{norm}(\text{scale}(\text{plus}(\text{agg}_2(k)), \text{scale}(x_{k+1}), k+1)). \\
\text{norm}(\text{scale}(\text{agg}_2(k+1)), k+1) = \text{norm}(\text{scale}(\text{plus}(\text{agg}_1(k)), \text{scale}(y_{k+1}), k+1)).
\]

We need to show that

\[
\text{norm}(\text{scale}(\text{plus}(\text{agg}_2(k)), \text{scale}(y_{k+1}), k+1), k+1) \geq \text{norm}(\text{scale}(\text{plus}(\text{agg}_1(k)), \text{scale}(x_{k+1}), k+1)).
\]

The function \( \text{norm} \) is isotone by the first argument (\( \mathcal{B}_6 \)). The function \( \text{scale} \) is isotone (\( \mathcal{B}_1 \)). The function \( \text{plus} \) is isotone by the first argument (\( \mathcal{B}_3 \)) and symmetric (\( \mathcal{B}_2 \)), therefore it is isotone by both arguments. It follows that inequality holds. The strict monotony follows from monotony and the axioms \( \mathcal{B}_3, \mathcal{B}_5, \mathcal{B}_6 \).

Denote \( G_0, G_1 = G_0 \cup r \). To prove idempotence, assume \( r = \text{TOT}(G_0) \). For \( G_1 \), get \( n+1 = r = \text{norm}(\text{agg}(n), n) \). Using axiom \( \mathcal{B}_7 \) we get

\[
\text{TOT}(G_1) = \text{norm}(\text{agg}(i+1), i+1) = r = \text{TOT}(G_0).
\]

It is easy to show that each of the next combinations of functions satisfies all the axioms \( \mathcal{B}_1 - \mathcal{B}_{10} \).

| \( \text{plus}(x, y) \) | \( \text{scale}(x) \) | \( \text{norm}(x, i) \) | \( \text{TOT}(G) = \) |
|----------------|----------------|----------------|----------------|
| 1 \( x + y \) | \( x \) | \( x/i \) | \( \frac{1}{n} \sum x_i \) |
| 2 \( x + y \) | \( x^2 \) | \( \sqrt{x/i} \) | \( \sqrt{\frac{n}{i} \sum x_i^2} \) |
| 3 \( \text{max}(x, y) \) | \( x \) | \( x \) | \( \text{max}(G) \) |
| 4 \( x \cdot y \) | \( x \) | \( x^{1/i} \) | \( (\prod x_i)^{1/n} \) |

\( G \) is domain of the sort \( \mathcal{G} \), \( n = \|G\| \).

**Statement 3.** Each combination of functions in the table 6 is an interpretations of the recursive aggregation language.

**Proof.** Monotonicity of all the functions is obvious. Symmetry of the function \( \text{plus} \) in all the combinations is obvious. We need to show that axioms of tautology and idempotence are true for all combinations. Let us show it for combinations from the lines (2) and (4), where it may be not obvious. If \( G = \{x\}, \text{TOT}(G) = \text{norm}(\text{agg}(1), 1) = \text{norm}(\text{scale}(x), 1) \). For the combination (2) \( \text{TOT}(G) = \sqrt{x^2/1} = x \). For the combination (4) \( \text{TOT}(G) = (\prod x)^{1/n} = x \).

Let us prove idempotence. For combinations (1) and (3) it is obvious. Let us prove it for combinations (2) and (4) again. Suppose, \( z = x_{n+1} = \text{TOT}\{x_1, \ldots, x_n\} \).
For combination (2)

\[
z = TOT(\{x_1, \ldots, x_n\})
\]

\[
TOT(\{x_1, \ldots, x_n, z\}) = \sqrt{\frac{\sum_{i=1}^{n} x_i^2 + (\sum_{i=1}^{n} x_i^2)/n}{n+1}} = TOT(\{x_1, \ldots, x_n\})
\]

For combination (4)

\[
z = TOT(\{x_1, \ldots, x_n\}) = (\prod_i x_i)^{1/n}
\]

\[
TOT(\{x_1, \ldots, x_n, z\}) = \left((\prod_i x_i)^{1/n}\right)^{1/n+1} = \left(\prod_i x_i^{(n+1)/n}\right)^{1/n+1} = \prod_i x_i^{1/n} = (\prod_i x_i)^{1/n} = z.
\]

In the next section, I show examples of how to evaluate incongruity in some real life situations.

4 Incongruity evaluation for data analysis

The concept of incongruity was developed for evaluation of predictability, to give logical foundation for learning in machine learning. The next examples show that the concept may be used for wide array of data analysis problems, where we evaluate various assumptions about dependencies of interest.

4.1 Incongruity of scales

Many people have body weight scales. I have two. The assumption is that they measure the same weight with small errors. Disagreements between the imprecise scales may be formalized as incongruity to evaluate validity of this assumption. As in the case of ML, the estimate of incongruity may be used for decision making: shall I go on with these scales, or buy a new, more reliable one.

In this case, the underlying dependence \(\varphi\) is the dependence of my “true” weight on time. The Language of observations and hypotheses (LOH) has two modalities \(\approx_1, \approx_2\) corresponding to readings from the first and the second scale respectively. So, all the formulas of the observations have the form

\[
\approx_i (\varphi(x) = y),
\]

where \(i = 1, 2\) is the index of the scale, \(x\) is time, \(y\) is weight.

The collision condition shall identify pairs of measurements of two scales taken within small interval of time. For these pairs larger differences of weight mean larger errors. The collision condition is defined by the formula:

\[
\pi(\alpha_1, \alpha_2) = ((s(\alpha_1) = \approx_1) \& (s(\alpha_2) = \approx_2) \& (\rho_x(\alpha_1, \alpha_2) \leq 5\text{min})).
\]
with the deviation function

\[ t(\alpha_1, \alpha_2) = \begin{cases} 0, & \text{if } \rho_y(\alpha_1, \alpha_2) < 1 \\ \rho_y(\alpha_1, \alpha_2) - 1, & \text{otherwise.} \end{cases} \]

For the proper aggregation of the deviations I use 80-th percentile. If 20% of deviations are positive, the scales can not be used. I may also use maximum. If the maximal deviation is more than 1 pound, the scales are useless.

### 4.2 Is there a dependence?

Suppose, I want to check an assumption that my weight is a non-decreasing function of the amount of consumed calories. The goal is to discover the actionable pattern. Again, we can use incongruity to make a conclusion.

Let us assume that when the amount of calories from day to day changes less than 100 calories, it may not affect on my weight; and the weight is evaluated with precision 1 pound.

In this case, the underlying dependence \( \varphi(x) \) is a dependence of the recorded weight on the consumed calories.

There is only one modality \( \approx \) so all the formulas in the language of observations and hypotheses have the form \( \approx (\varphi(x) = y) \).

Since the assumption is that my weight is a monotone function of consumed calories, the next situations violate the assumptions

- when the weight changes in one direction, but consumed calories change in opposite direction;
- when I consume about the same amount of calories, but the weight changes.

In both cases, the larger are the differences in weight (variable \( y \)), the larger shall be deviations.

Accordingly, there shall be two aspects of incongruity with these collision conditions:

\[ \pi_1(\alpha_1, \alpha_2) = \left( (x(\alpha_1) < x(\alpha_2)) \& (y(\alpha_1) > y(\alpha_2)) \& (\rho_x(\alpha_1, \alpha_2) > 100) \right) \]

\[ \pi_2(\alpha_1, \alpha_2) = \left( (y(\alpha_1) > y(\alpha_2)) \& (\rho_y(\alpha_1, \alpha_2) < 100) \right). \]

In both cases the deviation function is

\[ t(\alpha_1, \alpha_2) = \begin{cases} 0, & \text{if } \rho_y(\alpha_1, \alpha_2) < 1 \\ \rho_y(\alpha_1, \alpha_2) - 1, & \text{otherwise.} \end{cases} \]

I would use the first combinations of functions from the table 6 of typical interpretations of the Language of recursive aggregation to get a handle on how big are the deviations, and if the dependence is strong enough.

I may study various independent variables and their combinations to see if some of them are associated with the weight better. If I cannot assume monotonicity of the dependence, only the second aspect of incongruity will be used. The lower is the total incongruity of a hypothesis, the more likely there is the dependence which can be used to control weight.

Usually, the correlation is evaluated using coefficient of correlation in statistics. Statistics answer the next question: how likely is that the correlation exists in general population? First of all, the concept of “general population” does not make sense for my weight measurements. Then, my question is not about existence of dependence, but if there is strong enough dependence to use for prediction of my weight changes. This is completely different question. The values of the regression coefficient depend on the sample size, which is extremely important for the statistical question, and irrelevant for prediction.
4.3 Tracking Oswald

Many witnesses reported seeing Lee Harvey Oswald during the day of Kennedy assassination. The investigators may have several theories about his movements on this day. Incongruity evaluation may be used to find the theory maximally reconciled with witnesses testimonies, even though some of them may be unreliable.

The underlying dependence \( \varphi \) reflects the true movements of Oswald. It maps times (variable \( x \)) into locations (variable \( y \)) with particular memorable names (such as “the sixth floor of the Texas School Book Depository”).

The distance in time (by variable \( x \)) is measured in minutes. The distance between locations is also measured in minutes: the time required to get from one place to another. Formulas (instances) of observations describe locations and times of Oswald’s sightings by witnesses and have form \( \approx (\varphi(x) = y) \). Formulas of hypotheses (hypothetical instances) reflect the investigator’s theory, and have the form \( \approx (\varphi(x) = y) \).

So, for two instances (formulas of LOH) \( \alpha_1, \alpha_2 \), \( \rho_x(\alpha_1, \alpha_2) \) is the time which elapsed between the (hypothetical, observed) sightings, and \( \rho_y(\alpha_1, \alpha_2) \) is the minimal time, required to travel between the locations \( y(\alpha_1), y(\alpha_2) \). The times between sightings in any two locations shall not be less than the minimal time required to travel between these locations: \( \rho_y(\alpha_1, \alpha_2) \leq \rho_x(\alpha_1, \alpha_2) \).

For example, if Oswald was seen in the location \( A \) in the time \( t_1 \) (observation \( \alpha_1 \)), and hypothetically he was in some location \( B \) in time \( t_2 \) (hypothetical instance \( \alpha_2 \)) and the time to travel between \( A \) and \( B \) is \( z \), then for the observation and the hypothesis to be both true it is required that \( |t_1 - t_2| > z \).

Accordingly, the incongruity theory has only aspect with the collision condition:

\[
\pi(\alpha_1, \alpha_2) = \left[ (s(\alpha_1) = \approx) \land (s(\alpha_2) = \approx) \right],
\]

which simply identifies the formula \( \alpha_1 \) as a hypothetical instance, and the formula \( \alpha_2 \) as an observations.

The deviations are calculated by formula

\[
\delta(\alpha_1, \alpha_2) = \begin{cases} 
0, & \text{if } \rho_y(\alpha_1, \alpha_2) \leq \rho_x(\alpha_1, \alpha_2) \\
\rho_y(\alpha_1, \alpha_2) - \rho_x(\alpha_1, \alpha_2), & \text{otherwise}.
\end{cases}
\]

For a given hypothesis (investigators theory) the full model will include all the witnesses observations and hypothetical formulas with times of Oswald’s whereabouts in all the locations mentioned by the witnesses.

For proper aggregation of deviations, I would use the first combination of functions from the table 6 of typical interpretations of the language of recursive aggregation.

The theory with the lowest incongruity may be considered optimal. The advantage of this approach is that there is no subjectivity in evaluation of witnesses testimonies and theories of Oswald’s movements.

The same way, as witnesses testimonies are evaluated for incongruity with the theories, the testimonial of one witness can be compared with testimonies of all others. Incongruity of each witness with other witnesses can be used to, may be, exclude exceptionally contradictory witnesses.

5 Structure of learners

Now I concentrate on the learners.

5.1 The Main Conjecture

In practical applications, the training set is the set of given observations \( \{\beta_i, i = 1 : m\} \). The procedures of \( k \)-NN, Naive Bayes, SVM, hierarchical clustering, for example, appear to have nothing in common - because they are formulated in different terms.

Here I propose a common language to describe procedures used by learners.
1. Let $F$ be a class of hypothesis for a learner $Z$. There exists a loss criterion $L_Z(h, S)$, $h \in F$, such that, given a training set $S$ and parameters $q$, the learner $Z$ performs

Proper training
- Optional, Focusing: transformation $U : S \rightarrow S_q$
- Fitting: generation of the hypotheses $h \in F$ and evaluation of $L_Z(h, S_q)$
- Optimal selection: selecting a hypothesis $h'(q)$ with lowest loss $L_Z(h, S_q)$ as a decision.

If $Z$ is a wrapper-type learner, it has the next steps performed in a loop by $i$:
- Generating parameters $q_i$
- Proper training with parameters $q_i$
- Calculating weight $W(q_i)$
- Combining decisions: $d = \Psi(\{h'(q_i), W(q_i)\})$.

2. There exists an incongruity theory $T$, regularization functional $R(h)$ and a total proper aggregation procedure $\tau$ such that for a hypothesis $h$ and observations $S$:

$$L_Z(h, S) = \tau(h, S, T, R(h)),$$

total proper incongruity of $h, S, T$.

On a Focusing step transformation $U$ may be non-linear transformation of data prior to building a model.

Yet, typically, focusing is used to select observations or features or emphasize some of them with weights.

The “lowest loss” is, usually, a minimal loss among the tested hypotheses. It may coincide with the lowest loss on $F$ or not.

The procedures may use sequence control operators: loops, breaks and so on.

The main conjecture answers the first fundamental question: **What shall be done with the training set for learning?** The main conjecture defines the steps used by each learner, describes a loss criterion as incongruity.

### 6 Popular learners support the Main Conjecture

#### 6.1 ERM-type learners

Denote $\beta_i = (\approx (\varphi(x_i) = y_i))$ $i$-th observation in the training set $S$.

In this case, class of functions is not specified and the procedure is not described. All we have is a loss criterion

$$L(h, S) = \frac{1}{m} \sum_i |h(x(\beta_i)) - y(\beta_i)|.$$

Let us demonstrate that the loss criterion is the incongruity of the hypothesis $h$, training set and the **Point-Wise predictability theory**, $T_{pw}$ (see (1)).

For any hypothesis $h$, denote $\Upsilon$ all the formulas of the full model $\mathcal{M}(h, S, T_{pw})$. By the definition of the full model for the theory $T_{pw}$, $S \subseteq \Upsilon$, and $\Upsilon$ contains the hypothetical instances of the hypothesis $h$ defined in the same data points $x_i$ as observations.

So, for each pair of formulas $\alpha_1, \alpha_2$ from $\Upsilon$ satisfying the collision condition $\pi_{pw}$, the deviation is

$$\delta(\alpha_1, \alpha_2) = |y(\alpha_1) - y(\alpha_2)| = |h(x(\alpha_2)) - y(\alpha_2)|,$$

where $\alpha_2 \in S$.

Total proper aggregation here is proper aggregation for the only aspect of incongruity. If we use the proper aggregation operation defined in the first line of the table 6, then
\[ L(h, S) = \sum_{x \in S} |h(\beta) - y(\beta)| = \gamma(h, S, T_{ph}). \]

Thus, the loss criterion empiric risk is a total proper incongruity, and it supports the main conjecture.

### 6.2 Linkage-based clustering

The learner is also popularly known as hierarchical clustering.

Intuitively, clustering is a learning problem, because it is about modeling of a predictable dependence: close data points shall belong to the same cluster.

In [18], a general concept of linkage-based clustering is introduced this way:

These algorithms proceed in a sequence of rounds. They start from trivial clustering that has each data point in a single-point cluster. Then, repeatedly, these algorithms merge “closest” clusters of the previous clustering. (…) Input to a clustering algorithm is between-point distance, \( d \). There are many ways of extending \( d \) to a measure of distance between domain subsets (or clusters). The most common ways are.

1. Single Linkage clustering, in which the between-clusters distance is defined by the minimum distance between members of the two clusters (…)
2. Average Linkage clustering, in which the distance between two clusters is defined to be average distance between a point in one of the clusters and a point in another (…)  
3. Max Linkage clustering, in which the distance between two clusters is defined as maximum distance between their elements (…)

The last option clearly contradicts declared goal “merge ‘closest’ clusters”. But I will consider it too.

Close observations shall belong to the same cluster. The opposite is also true: observations of the same cluster shall be some-what close to each other. The last dependence is used for clustering. So, we consider cluster number as an independent variable \( x \), and the observation vector as dependent variable, \( y \).

The training set is a sequence of formulas

\[ \{ \approx (\varphi(x_i) = y_i), i = 1 : m \}, \]

where \( c_i \) is a cluster if \( i \)-th observation, and \( y_i \) is the observed vector of the same observation.

Denote

\[ C_i = \{ y \mid \exists \alpha(\alpha \in S) \& (y = y(\alpha)) \& (i = x(\alpha)) \}; \]

the set of elements of the \( i \)-th cluster.

The clustering consists of repeated rounds: two “closest” clusters are found, combined, and the procedure repeats until there is only one cluster left. Proper learning happens when we search for the “closest” clusters. On this step, for the each cluster \( i \), we check each of the clusters \( j > i \) and evaluate their “distance”; then the two “closest” clusters are selected for combining.

When we evaluate the “distance” between clusters \( A, B \), it is convenient to see one cluster (say, \( A \)) as a target, and another (\( B \)) as a candidate to combine with the target. In other words, we evaluate the hypothesis that elements of the cluster \( B \) are a “good fit” to belong to \( A \). Thus, the notation \( h^{ij} \) will indicate a hypothesis that elements of the cluster \( j \) are a “good fit” for the cluster \( i \).

Let \( H_k = \{ h^{ij} \mid i < j \leq k \} \) denote the class of all the hypotheses for the case, when there are \( k \) clusters.

The instances of the hypothesis \( h^{ij} \) have the form \( H^{ij} = \{ \approx (\varphi(C_i) = y) \mid y \in C_j \} \). For a hypothetical instance \( \alpha \in H^{ij} \) and the observation \( \beta \in C_i \) their distance is \( d(\alpha, \beta) = p_0(\alpha, \beta) \), distance between data points \( y(\alpha), y(\beta) \) of these two formulas.

The mis-fit between clusters \( i, j \) defined in the textbook [18] may be evaluated as \( \gamma(G(i, j)) \), where \( \gamma \) is some aggregation operation (minimum, average or maximum), and \( G(i, j) \) is the set of pairwise distances for elements of \( C_i, C_j \). The learning procedure is searching for a hypothesis \( h^{ij} \in H_k \) with the lowest mis-fit criterion \( \gamma(G(i, j)) \).

Thus, the function \( \gamma(G(i, j)) \) can be considered a loss criterion of the learner.
Hierarchical Clustering

- Loop by all \(i, j: i < j \leq k\)
  - Fitting: For the hypothesis \(h^{ij}\) evaluate the loss criterion \(\gamma(G(i, j))\)
- Optimal selection: Select a hypothesis \(h^{ij} \in H_k\) with the minimal value of the loss criterion \(\gamma(G(i, j))\)

The steps of this procedure are as described in the main conjecture. To see that the learner agrees with the main conjecture completely, we just need to show that for some incongruity theory, the loss criterion \(\gamma(G(i, j))\) is a total proper incongruity for any aggregating procedure \(\gamma\) mentioned in the book.

In this case the incongruity theory is the point-wise theory \(T_{pw}\) again.

For two formulas \(\alpha_1, \alpha_2\) such that \(\vdash \pi_{pw}(\alpha_1, \alpha_2)\), the deviation is \(\delta(\alpha_1, \alpha_2) = \rho_y(\alpha_1, \alpha_2)\).

Every aggregation operation \(\gamma\), mentioned in [18], satisfies axioms of the proper aggregation. Therefore, in every case, the loss criterion \(\gamma(G(i, j)) = \gamma(h, S, T_{pw})\) is the proper total incongruity.

This proves that linkage-based clustering agrees with the main conjecture. It would agree with the main conjecture not only for the aggregation operations mentioned in the book, but also for any other proper aggregation operation.

6.3 k-NN

The observations have binary feedback in \(Y = \{0, 1\}\). Given a new data point \(x \in \chi\), the goal is to output prediction \(f(x)\) of the underlying dependence \(\varphi: \chi \rightarrow Y\). Thus, feedback is defined in one point \(x\) each time.

The procedure can be described in these steps.

- **Focusing**: selecting focus training set \(Q(x)\) of \(k\) observations with data points closest to \(x\).
- **Fitting**: evaluate error rate of each of the constant functions 0, 1 on \(Q(x)\).
- **Optimal selection**: Selection of the constant function with minimal error rate.

The learner minimizes error rate, which is the same as empiric risk \(L(f, S)\) on functions \(f \in \{0, 1\}\) defined on focus training sample \(Q(x)\). We have already demonstrated that empiric risk is total proper incongruity for the point-wise incongruity theory.

Thus, k-NN supports the main conjecture.

6.4 Two k-NN learners with adaptive choice of \(k\)

The parameter \(k\) defines the size of the focus training sample. Optimally, for most of data points \(\xi \in \chi\), the neighborhood \(Q_k(\xi)\) shall be small enough to have majority of the points of the same class as the point \(\xi\) due to the "predictability" of the underlying dependence, and large enough of that random outliers did not confuse us.

Here I discuss two approaches to select \(k\) for every new data point. The first is described in [4], the second is my new algorithm. Both learners find prevalent class \(y\) in the focus sample, calculate its frequency \(p_k(y)\) and the error rate \(r_k(y) = 1 - p_k(y)\) the same as k-NN.

The work [4] proposes, given a data point \(x\), start with a small \(k\) and gradually increase it while calculating bias \(t_k(y) = p_k(y) - 0.5\) of the prevalent class with every \(k\). The procedure stops when the bias reaches certain threshold. If the threshold was not ever reached, they don’t output any answer.

The threshold they propose to use is:

\[
\Delta(n, k, \delta, c_1) = c_1 \sqrt{\frac{\log(n) + \log(\frac{1}{k})}{k}}.
\]
where \( n \) is size of the training sample, \( \delta \) and \( c_1 \) are some user-selected parameters. The learner uses the same criterion as \( k \)-NN.

The procedure can be described like this:

| Ada \( k \)-NN |
|----------------|
| • Loop by \( k \) for \( k_0 \leq k \leq n \) |
| • **Proper training:** |
| * Focusing: Select focus training set \( Q_k(x) \) of \( k \) observations with data points closest to \( x \). |
| * Fitting: Evaluate error rate \( r_k(c) \) of two constant functions \( c \in \{0, 1\} \) in \( Q_k(x) \). |
| * Optimal selection: Select constant \( c' \in \{0, 1\} \) with minimal error rate \( r_k(c') \). |
| • **Break the loop by \( k \):** If \( r_k(c') > \Delta(S, k, \delta, c_1) \) stop. |
| • **Combining decisions:** If \( k < n \), output \( c' \) as decision. Otherwise, refuse to output the decision. |

Thus, the learner uses the same steps as described in the main conjecture, and it uses the same loss criterion, error rate, as original \( k \)-NN. Therefore, the learner agrees with the main conjecture.

This learner is developed within the statistical learning paradigm, where the training set is expected to grow to infinity fast. As \( n \) increases, so does the threshold \( \Delta(n, k, \delta, c_1) \). Therefore, the selected value \( k \), the size of the focus training set, will go to infinity with \( n \). And thus, by the law of (very) large numbers, the solution will converge asymptotically to the expectation of the class in the given neighborhood. At the same time, the ratio of \( k \) to \( n \) is expected to decrease, thus the size the \( k \)-neighborhood will tend to 0. If the distribution is continuous in \( x \), then the learner will likely find the solution as \( n \) tends to infinity.

The issue here is that \( n \) is not going to infinity or anywhere. For a fixed \( n \), the learner favors smaller \( k \), where the evaluation of prevalent class is subject to random fluctuations caused by small sample.

To alleviate this issue, I propose an alternative approach which uses Hoeffding inequality [18] to select \( k \).

The Hoeffding inequality can be written as

\[
P[| p - E | > t ] \leq 2 \exp(-2k t^2),
\]  

where \( p \) is observed frequency of an event, \( E \) is the expected frequency (probability) of the same event, and \( t \) is an arbitrary threshold, and \( k \) is the sample size.

Suppose, \( p \) evaluates observed frequency of class 1 (rate of the class 1 among the neighbors), \( E \) is the probability of the class 1 in the neighborhood of a given point. If \( p \) is above 0.5, then observations of the class 1 prevail, and we pick hypothesis 1 out of two. Otherwise, the we pick hypothesis 0.

Let \( t = |0.5 - p| \). If \( | p - E | > t \) the expected prevalent class is different from the observed prevalent class. If it is the case, we selected the wrong hypothesis. In this case, the right side of the inequality gives us an upper limit of probability that we picked the prevalent class wrong.

For selection of \( k \) we use the weight, calculated as the right part of (2):

\[
W(y, S, k) = 2 \cdot \exp(-2 k | p - 0.5 |^2).
\]

Obviously, the larger is \( k \), and the further is the frequency \( p \) from 0.5, the lower is the weight. The weight will serve well for the selection of the parameters \( k \), because we need to find the neighborhood where \( p \) is far from uncertainty, 0.5, yet, the size of the neighborhood is not too small.

Here is the description of the learner’s procedure for the given data point \( x \).
Hoeffding k-NN

• Loop by $k$ from $k_0$ to $n - 1$
  – Proper training:
    • Focusing: Select focus training set $Q_k(x)$ of $k$ observations with data points closest to $x$.
    • Fitting: Evaluate error rate $r_k(c)$ of hypotheses $c \in \{0, 1\}$ in $Q_k(x)$
    • Optimal selection: Select the hypothesis $c'(k)$ with minimal error rate $r_k(c'(k))$.
  – Calculate weight $W(x, S, k)$.
• Wrapper decision: Select $k'$ and the decision $c'(k')$ with minimal weight $k' = \arg \min W(x, S, k)$.

The proper learning procedure in both $k$-NN wrappers minimizes empiric risk, the same as original $k$-NN, and this criterion is demonstrated to be a incongruity for the $T_{pw}$ incongruity theory. Thus, this modification of $k$-NN also corroborates the main conjecture.

6.5 Decision trees

For this learner, the features are expected to be “ordinal”: every feature has finite number of ordered values; there are no operations on feature values. The feedback of observations is binary.

The learner starts with whole domain, split it in two subdomains by a value of some feature. Then, the procedure is repeated for every of the subdomains until a subdomain called "leaf" is reached. The decision is selected for this subdomain. The navigation over the tree of subdomains continues until some stopping criterion is reached. The algorithm has a precise rule for generating the parameters of the next subdomain based on the previous trajectory and the obtained results.

There are two criteria of a leaf:

1. Number of observations in the subdomain is below a threshold $N$.
2. Percentage of observations of the prevalent class in the subdomain is above the threshold $q$.

The procedure may be described as a wrapper algorithm:

Decision Tree:

• Generating parameters $g$ of the next subdomain
  – Proper training:
    • Focusing: select subdomain $G(g)$ with parameters $g$
    • Fitting: evaluate error rate of constant hypotheses $\{0, 1\}$ in $G(g)$
    • Optimal selection: if the leaf criteria in $G(g)$ are satisfied, select the hypothesis with minimal error rate.
  • End of loop by parameters $g$

In this case, we do not need to analyze whole tree before we create a wrapper decision: the decision is prevalent value on each leaf. For the points, which do not belong to any leaf, the decision is not defined.

The error of a constant function in a subdomain $Q$ is defined as empiric risk. And we demonstrated that empiric risk is total proper incongruity of the point-wise incongruity theory $T_{pw}$. Therefore, this description of the procedure agrees with the main conjecture.
6.6 Naive Bayes

The algorithm works as if it deals with nominal data: the only relationship between data points is equivalence of feature values. The feedback of observations is binary, and so is feedback of the solution.

The procedure defines decision function on one data point at the time. For a given data point $z = \langle z_1, \ldots, z_n \rangle$ the procedure selects $n$ subsets of the training set. Subset $S_j$ includes all the observations with $j$-th coordinate of the data point equal $z_j$. For each subset $S_j$, the learner evaluates error rate $e_{j,c}$ of each hypothesis $c \in \{0, 1\}$. Then for each hypothesis it calculates loss

$$\Delta(c, S) = 1 - \prod_{j}(1 - e_{j,c}).$$

The learner selects a hypothesis with the minimal loss as a decision.

Let us define the incongruity theory $T_{nb}$ for this learner. LOH Language of the theory has an additional finite sort $N$ with values $1, 2, \ldots, n$, and variables $i, j, i_1, \ldots$, where $n$ is the dimensionality of the metric space for the sort $X$. The language has an additional first order function $X \times N \rightarrow \mathbb{R}$, denoted $(x)_i$, the $i$ coordinate of a vector $x \in \mathbb{R}$.

The theory $T_{nb}$ has $n$ aspects. For aspect $i \in 1 : n$ the collision condition is

$$\pi_i(a_1, a_2) = (s(a_1) = \approx) & (s(a_2) = \approx) & \left((x(a_1))_i = (x(a_2))_i\right).$$

The deviation function is the same for every aspect

$$\delta_i(a_1, a_2) = |y(a_1) - y(a_2)|.$$

To properly aggregate all deviations of an aspect of incongruity we use error rate:

$$e_{j,c} = \frac{1}{k_j} \sum \delta_i(a_1, a_2),$$

where $k_j$ is the number of deviations for the $j$-th aspect in the full model.

To combine aspect incongruities $\{e_{j,c}\}$ into total proper incongruity, we use the aggregation operation

$$\Delta(c, S) = 1 - \prod_{i}(1 - e_{j,c}).$$

The function is isotone.

We can conclude that the function $\Delta(c, S)$ satisfies the requirements on total proper aggregation. At the same time it is the loss criterion of this learner used to select the hypothesis with minimal value of this criterion.

Now the procedure of the learner may be described as very simple

| Naive Bayes |
|-------------|
| **Fitting**: generating hypotheses $c \in \{0, 1\}$ and calculating the loss criterion $\Delta(c, S)$: |
| - Loop by aspects $i \in 1 : n$ |
| - Calculating error rate $e_{i,c}$ for the aspect $i$ |
| - Calculating the criterion $\Delta(c, S)$ |
| **Optimal Selection**: selection of the hypothesis $c$ with the lowest criterion $\Delta(c, S)$. |

This proves that Naive Bayes supports the main conjecture.

A product in the aggregation of the aspects in the loss function is chosen in Naive Bayes because it is sensitive to the low frequencies of class: if some value $1 - e_{i,c}$ is close to 0, the product will be affected much more than the sum of the frequencies, for example. If some feature value almost never happens in a given class $c$, the hypothesis $c$ will have no chance of being selected, regardless of other feature values of $z$. It justifies choice of product for aggregation.

The products of frequencies are traditionally interpreted as evaluation of posterior probabilities with “naive” assumption that the feature values are independent. There are several
issues with this narrative. The first is its uniqueness. Only this learner is based on Bayes rule. Other learners would need different foundations. Another issue is that it creates an impression that the learner needs an improvement, is not sophisticated enough. It means, the narrative does not, really, explain or justify this learner.

I hope, I demonstrated that interpretation of the learner as “naïve” and “Bayesian” misses the point. The procedure is driven by its specific data type, not by naïve fondness for Bayes theorem.

6.7 Logistic Regression

This learner assumes the features are continuous, the feedback of the observations is binary, but the feedback of the decision is continuous. The decision is defined on the domain $\chi$. The procedure of generating the hypotheses is not specified.

The class of functions associated with logistic regression is $F = \{ \frac{1}{1 + \exp(-\langle w, x \rangle)} \}$.

The functions have values in the interval $(0, 1)$.

The learner minimizes criterion $\Delta(f, S) = \frac{1}{m} \sum_{s \in S} \log(|y(s) - f(x(s))|)$.

Let us define the incongruity theory $T_{lr}$ for this learner. There is one collision condition which coincides with the condition $\pi_{pw}$ of $T_{pw}$.

$\pi(\alpha_1, \alpha_2) = (s(\alpha_1) = \approx) \& (s(\alpha_1) = \approx) \& (x(\alpha_1) = x(\alpha_2))$.

The deviation function is $\delta(\alpha_1, \alpha_2) = \log(\rho_y(\alpha_1, \alpha_2))$.

The aggregation uses recursive aggregation functions from the line 1 of the table (6). Thus the loss criterion $\Delta(f, S)$ coincides with total proper incongruity for the training set $S$, hypothesis $f$ and the incongruity theory $T_{lr}$.

So, the logistic regression supports the main conjecture as well.

6.8 Linear SVM for classification

All the previous learners belong to machine learning “folklore”. Their authors are not known, or, at least, not famous.

SVM is one of the first learners associated with a known author: it is invented by V. Vapnik. His earliest English publications on this subject appeared in early nineties [6], [5].

Let us start with linear SVM for binary classification. The observations $S = \{ \beta_i, i = 1 : m \}$ have two class labels: $\{-1, 1\}$ with data points $x \in \mathbb{R}^n$.

The class of hypotheses $F$ consists of linear functions $f(x)$ with $n$ variables. For a $f \in F$, $f(x) = x^T \beta + \beta_0$, denote $w(f) = \beta, b(f) = \beta_0$.

The problem is formulated as minimization of the criterion

\[
\begin{align*}
L(f, S, \xi) &= \alpha \|w(f)\|^2 + \frac{1}{m} \sum_{\beta \in S} \xi(\beta) \\
\text{s.t.} \quad \text{for all} \ \beta \in S, \ y(\beta) \cdot f(x(\beta)) &\geq 1 - \xi(\beta) \quad \text{and} \ \xi(\beta) \geq 0.
\end{align*}
\]

The criterion may be simplified though. For this, we want to switch to narrower class of functions, which shall contain all the same decisions.

The observations $\beta \in S$ satisfying condition $y(\beta) \cdot f(x(\beta)) > 0$ are considered correctly classified by the function $f$. Denote $S^{\circ}(f)$ all correctly classified observations by the function $f$, and $S^\circ(f) = S \setminus S^\circ(f)$ the rest of the observations.
Let us consider all the functions $f \in F$ such that $S^\varnothing(f) \neq \emptyset$ and
\[
\min_{S^\varnothing(f)} |f(x(\beta))| = 1.
\]
Denote this class of function $F'(S)$. The class $F'(S)$ is not empty. Indeed, if for some $f$, $f \neq 0$, $S^\varnothing(f) = \emptyset$, then, $S^\varnothing(-f) = S$. If
\[
q = \min_{S^\varnothing(f)} |f(x(\beta))| \neq 1,
\]
then the function $f' = \frac{1}{q} f$ satisfies the condition
\[
\min_{S^\varnothing(f)} |f'(x(\beta))| = 1.
\]

The last consideration implies that if $f$ is the decision of the problem, then the problem has a decision $f'$ in the class $F'(S)$ with the same set of correctly recognized observations $S^\varnothing(f') = S^\varnothing(f)$.

Therefore, we can restrict the search for a decision in the class $F'(S)$ only.

**Theorem 3.** The linear SVM classification problem minimizes the loss criterion
\[
L_{\text{svm}}(f, S) = \alpha \|w(f)\|^2 + \frac{1}{m} \sum_{\beta \in S^\varnothing(f)} |y(\beta) - f(x(\beta))|,
\]
for $f \in F'(S)$.

**Proof.** The conditions (4) can be rewritten as $\forall \beta, \beta' \in S$:
\[
\begin{cases}
\xi(\beta) \geq 1 - y(\beta) \cdot f(x(\beta)) \\
\xi(\beta) \geq 0.
\end{cases}
\]
(5)

or
\[
\xi(\beta) \geq \max \{1 - y(\beta) \cdot f(x(\beta)), 0\}.
\]

The values $\xi(\beta), \beta \in S$ do not depend on each other, so the minimum of their sum is achieved when every variable $\xi(\beta)$ equals its lowest possible value. Let us find these lowest values for $\xi(\beta)$ depending on if $\beta \in S^\varnothing(f)$ or $\beta \in S^\varnothing(f)$.

If $\beta \in S^\varnothing(f)$,
\[
y(\beta) \cdot f(x(\beta)) = |f(x(s))|.
\]

By definition of $F'(S)$, $|f(x(s))| \geq 1$. Then
\[
\xi(\beta) \geq \max \{1 - y(\beta) \cdot f(x(\beta)), 0\} = 0.
\]

In this case, the lowest possible value for $\xi(\beta)$ is 0.

If $\beta \in S^\varnothing(f)$,
\[
y(\beta) \cdot f(x(\beta)) = -|f(x(\beta))|.
\]

Then
\[
\xi(\beta) \geq \max \{1 - y(\beta) \cdot f(x(\beta)), 0\} = 1 + |f(x(\beta))|.
\]

In this case, the lowest possible value for $\xi(\beta)$ is $1 + |f(x(\beta))|$.

So,
\[
\min_{\xi} \frac{1}{m} \sum_{S} \xi(\beta) = \sum_{\beta \in S^\varnothing(f)} (1 + |f(x(\beta))|).
\]
(6)

We still need to prove that for $\beta \in S^\varnothing(f)$
\[
1 + |f(x(\beta))| = |y(\beta) - f(x(\beta))|.
\]

Let us take $\beta \in S^\varnothing(f)$. If $y(\beta) = 1$, then $f(x(\beta)) < 0$ and $|f(x(\beta))| = -f(x(\beta))$. So,
\[
(1 + |f(x(\beta))|) = 1 - f(x(\beta)) = |y(\beta) - f(x(\beta))|.
\]

If $y(\beta) = -1$, then $f(x(\beta)) > 0$ and $|f(x(\beta))| = f(x(\beta))$. So,
\[
1 + |f(x(\beta))| = 1 + f(x(\beta)) = -y(\beta) + f(x(\beta)) = |y(\beta) - f(x(\beta))|.
\]
The part $\|w(f)\|_2^2$ of the criterion is a regularization component: $w(f)$ is the gradient of the hypothesis $f$, and $\|w(f)\|_2^2$ is the square of its norm. Minimizing this component, we reduce the speed of the hypothesis change and make the model more “predictable”.

Now to prove that the learner agrees with the main conjecture, I just need to define the incongruity theory which explains the second component of the loss criterion $L(f, S) = \frac{1}{m} \sum_{\beta \in S} |y(\beta) - f(x(\beta))|$. 

For this, we need to define how the distance is measured between the feedback of observations and the function value. The rule is: for $\alpha_1 = (\approx (\varphi(x_1) = y_1)), \alpha_2 = (\approx (\varphi(x_2) = y_2))$

$$\rho_y(\alpha_1, \alpha_2) = \begin{cases} 0, & \text{if } y_1 \cdot y_2 \geq 0 \\ |y_2 - y_1|, & \text{otherwise} \end{cases}$$

Then the incongruity theory $T_{svm}$ coincides with point-wise incongruity theory $T_{pw}$. The total proper incongruity is constructed using proper recursive aggregation defined in the first line of the table 6.

### 6.9 Linear Support vector regression

The learner minimizes criterion [10]

$$L_{svr}(f, S) = \sum_{i=1}^{m} V(|y(\beta_i) - f(x(\beta_i))|) + \lambda \|w(f)\|_2^2,$$

where

$$V(|r|) = \begin{cases} 0, & \text{if } |r| < \epsilon \\ |r| - \epsilon, & \text{otherwise} \end{cases}$$

and $S = \{\beta_1, \ldots, \beta_m\}$.

The second component of the loss criterion is regularization, the same as in the SVM. The distance between feedback of an observation and the value of a hypothesis is defined through the function $V$: for $\alpha_1 = (\approx (\varphi(x_1) = y_1)), \alpha_2 = (\approx (\varphi(x_2) = y_2))$

$$\rho_y(\alpha_1, \alpha_2) = V(y_1 - y_2).$$

Then the incongruity theory $T_{svr}$ coincides with the point wise theory $T_{pw}$. The total proper aggregation is defined again as the first line in the table 6.

So, the linear support vector regression supports the main conjecture as well.

### 6.10 Support Vector Regression with Kernels

Suppose ([10]) we have a set of basis functions $H = \{h_i(x), i = 1, \ldots, k\}$. We are looking for hypotheses

$$f(x) = \sum_{i=1}^{k} w_i h_i(x) + b.$$ 

The loss criterion used here is

$$L(f, S) = \sum_{i=1}^{m} V(|y(\beta_i) - f(x(\beta_i))|) + \lambda \|w(f)\|_2^2,$$

where

$$V(|r|) = \begin{cases} 0, & \text{if } |r| < \epsilon \\ |r| - \epsilon, & \text{otherwise} \end{cases}$$

Here the transformation $x \rightarrow \langle h_1(x), \ldots, h_k(x) \rangle$ from a $n$-dimensional space $R^n$ into $k$-dimensional space $H(x)$ may be called focusing. Then the problem is reduced to solving a linear SVM regression in the transformed space. Thus, SVR with kernel supports the main conjecture as well.
6.11 Ridge Regression

The learner finds the solution in the same class of linear hyperplanes $F = \{ f : f = wx + b \}$ as linear SVM for classification, and it has the criterion

$$L_{rr}(f, S) = \alpha \|w(f)\|^2 + \frac{1}{m} \sum_{\beta \in S} (f(x(\beta)) - y(\beta))^2.$$  

The first component of the loss criterion is regularization, the same as in SVM, SVR. The second component can be explained as total proper incongruity where the theory’s only collision condition coincides with the condition $\pi_{pw}$ of point-wise incongruity theory $T_{pw}$, the deviation is defined by the formula $\delta(\alpha_1, \alpha_2) = (y(\alpha_1) - y(\alpha_2))^2$, and the recursive aggregation is defined in the line 1 of the table 6.

Thus, Ridge regression corroborates the main conjecture too.

6.12 Neural Network (NN)

Let us consider single hidden layer NN for two class classification as it is described in [10].

First, the learner transforms $n$-dimensional metric space of inputs $\mathbb{R}$ into $k$-dimensional space $\mathbb{Z}$ using non-linear transformation;

$$Z_i(x) = \delta(g_i(x)), i = 1, \ldots, k,$$

where $\delta(r)$ is delta function and $g_i$ are linear functions. Denote $z(x)$ the vector with coordinates $\langle Z_1(x), \ldots, Z_k(x) \rangle$.

Then, for each class $c \in \{0, 1\}$, the learner builds linear voting function $f_c(z(x))$. Denote $G = \{g_1, \ldots, g_k\}$, and $F = \{f_0, f_1\}$.

For each $x \in \mathbb{R}^n$ the class is selected as $C(x, G, F) = \arg\max_c f_c(z(x))$.

The learner uses the loss criterion

$$L_{nn}(G, F, S) = \sum_{\beta \in S} (y(\beta) - C(x(\beta), G, F)).$$

It is obvious that he loss criterion is a total proper incongruity for by the point-wise incongruity theory. The learner optimizes simultaneously parameters of the functions $G$ and $F$. For selection of parameters of these functions the learner uses gradient descent, which is called “back propagation” in this case. The learner uses some additional stopping criterion.

So, the procedure does not have a focusing stage. If calculates loss for given set of parameters, evaluates gradients by each parameter, and then updates parameters based on the gradients. After the stopping criterion is achieved, the algorithm outputs the decision with the lowest loss criterion.

The procedure has only two types of steps:

1. fitting, which includes
   - generation of the $C(x, G, F)$ hypothesis based on previous value of loss criterion and gradients
   - evaluation of loss criterion of the current hypothesis $L_{nn}(G, F, S)$.

2. optimal selection: selection of the hypothesis with the lowest loss criterion.

Thus, NN also agrees with the main conjecture too.

6.13 K Means Clustering

The learner is different from hierarchical clustering in that it does not combine clusters, rather, for each observation, it chooses the proper cluster. It is assumed that the distance on the domain of data points is Euclidean.

Here is the description of the learner from [10].

1. Given the current set of means of clusters $M = \{m_1, \ldots, m_K\}$, each observation is assigned to the cluster with the closest mean.
2. The rounds of assignment of all observations are repeated until clusters do not change.

The proper learning happens when we search for the cluster for the given observed data point. Denote \( C(x) \) the assignment of a cluster to a data point \( x \). Given the set of observed data points \( S_x = \{x_1, \ldots, x_n\} \), \( K \) clusters with cluster centers \( M \) of the sizes \( \{l_1, \ldots, l_K\} \) the procedure assigns a new class to an observed data point to minimize sum of all pairwise distances within each cluster

\[
W(C, S) = \frac{1}{2} \sum_{k=1}^{K} \sum_{C(\xi) = k} \sum_{C(\zeta) = k} ||\xi - \zeta||^2,
\]

where \( \xi, \zeta \in S \), and \( \bar{x}_k \) is mean of the \( k \)-th cluster. I use the (7) to prove that the learner agrees with the main conjecture.

Denote \( x_0 \) a data point, \( x_0 \in S_x \), which we need to assign a cluster on this step.

As in the case of hierarchical clustering, we consider underlying dependence \( \varphi \) as a function from cluster index \( k \) to the observed data point \( x \). There are \( K \) hypotheses \( H(x_0) = \{h_1, \ldots, h_K\} \). Each hypothesis \( h_i \) has a single hypothetical case \( x \in \{\varphi(i) = x_0\} \).

We assume, before current run of the learner, the clusters are already assigned to each observed data point besides \( x_0 \). So, the run starts with the training set having observations

\[ S = \{ (\approx (\varphi(i) = x)) \mid i \in 1, \ldots, K; x \in S_x \setminus \{x_0\} \}. \]

There is only one aspect incongruity with the collision condition

\[
\pi(\alpha_1, \alpha_2) = (x(\alpha_1) = x(\alpha_2)),
\]

which says that we evaluate deviation for each pair of formulas with the same argument \( x \), the same cluster, regardless of modality. The collision condition is symmetrical, therefore for each pair of formulas \( \alpha_1, \alpha_2 \) which satisfies the condition, the pair \( \alpha_2, \alpha_1 \) satisfies the condition as well. In effect, every pair is counted twice.

The deviation function is

\[
\delta(\alpha_1, \alpha_2) = \rho_y(\alpha_1, \alpha_2)^2
\]

For the proper aggregation we use the averaging. The formula of loss criterion \( W(C, S) \) does not explicitly have the scaling coefficient \( \frac{1}{m} \) because it would be the same for every hypothesis. Otherwise, the loss criterion in this case is the total proper incongruity for the described theory.

The learner generates all hypotheses \( H(x_0) \), evaluates the loss criterion for each of them and selects the hypothesis with the lowest loss criterion. Thus this learner corroborates the main conjecture as well.

### 7 Conclusions

Here I propose a modal logic LOH to explain the learning in machine learning. The logic generalizes existing learners to explain, what we do, when we learn.

The underlying dependence we are learning is assumed to be non-deterministic. The first order formulas of LOH (Logic of hypotheses and observations) are statements about values of the underlying dependence in some points. The formulas always have modalities “it appears” or “assume that”: they describe observations and hypotheses respectively. Being subjective, modal formulas can not have truth values, so they can not have contradictions, inconsistencies in the strict logical sense.

The underlying dependence is expected to be predictable in the vague sense that “close” data points shall correspond to “close” feedback. The implied “closeness” depends on the task: its data types, precision of measurement, goals and so on. Instead of the predictability, I formalize the opposite concept: incongruity. It is defined to be flexible to match the tasks as well. Each version of incongruity is defined by its own “incongruity theory”, where “collision conditions” are binary predicates expressed as second order formulas of LOH, and the “deviation” functions evaluate disagreement between “colliding” first order formulas.
The main conjecture of this work is that every learner has a loss criterion which can be presented as incongruity in some incongruity theory and, given the observations, the learner performs certain steps to find the hypothesis minimizing this loss criterion.

The main conjecture is illustrated on large number of popular learners, including SVM, SVR, hierarchical clustering, \( K \) mean clustering, neural network, Naive Bayes and others. Each of these learners corroborates the main conjecture.

Here are some of the advantages of the proposed ML paradigm over traditional statistical one.

1. The framework provides unified logical justification and explanation for large variety for real life learners used by practitioners. It explains how and why we can learn from fixed finite data, while statistical learning theory is not able to do it.
2. I demonstrated inner similarity of the regression, classification and clustering methods: all of them are shown to corroborate the main conjecture. Statistical learning theory can not include clustering in their concept of learning with ever increasing training set.
3. The proposed approach allows to understand “regularization” component of loss criteria as an aspect of incongruity.
4. Described here a general structure of a learner shall facilitate classification, selection, customization and design of new learners. The proposed language can express much wider variety of learners than are being commonly used. New varieties of learners may be especially advantageous when the available data are limited.
5. As an example of such learner customization, I proposed a version of adaptive \( k \)-NN learner based on Hoeffding inequality. The learner shall have advantages over ADA \( k \)-NN for small data.
6. In addition, the concept of incongruity is demonstrated to be helpful for some of common data analysis problems, where statistics approach appears to be inadequate also.

I want to point out philosophical implications of these results. The learning is usually considered to be an inductive process: they say, the decision “generalizes” observations. Philosophers \([16]\) noticed logical contradictions of the concept of induction: how can a decision “follow” from the data?

The main conjecture suggests a possible explanation. Suppose, a class of hypotheses is fixed, and we need a certain type of agreement between observations and a hypothesis. We also believe that the world is somewhat predictable: usually, it does not change sharply. Experience of our and other species teaches us that, otherwise we could not survive in a rapidly changing, unpredictable environment. This belief is called here fundamental.

If we rely on the fundamental belief, the winning strategy is to find the hypothesis, which violates the agreement with the observations the least: it has the best chance to be good in the future too. And this is exactly what we do in machine learning, minimizing the incongruity.

If the class of hypotheses is infinite, we may not find the optimal decision, we may only approximate it.

Thus, the main conjecture shows that learning (and induction) work as “approximate deduction”. The (approximate) decision is “deducted” from the incongruity theory and the observations.

We believe in predictability, knowing that the dependencies we learn are non-deterministic. This makes testing a critical part of learning cycle. I plan to talk about it in the following work.

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