ON (WEAK) FPC GENERATORS

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Abstract. Corrado Bhm once observed that if $Y$ is any fixed point combinator (fpc), then $Y(\lambda y x. x(yx))$ is again fpc. He thus discovered the first “fpc generating scheme” — a generic way to build new fpcs from old. Continuing this idea, define an fpc generator to be any sequence of terms $G_1, \ldots, G_n$ such that

$$Y \text{ is fpc } \implies \text{YG}_1 \cdots \text{G}_n \text{ is fpc}$$

In this contribution, we take first steps in studying the structure of (weak) fpc generators. We isolate several classes of such generators, and examine elementary properties like injectivity and constancy. We provide sufficient conditions for existence of fixed points of a given generator $(G_1, \ldots, G_n)$: an fpc $Y$ such that $Y = \text{YG}_1 \cdots \text{G}_n$. We conjecture that weak constancy is a necessary condition for existence of such (higher-order) fixed points. This generalizes Statman’s conjecture on the non-existence of “double fpcs”: fixed points of the generator $(G) = (\lambda y x. x(yx))$ discovered by Bhm.

Dedicated to Corrado Böhm, a pioneer of the lambda calculus.

1. Introduction

Fixed point combinators (fpcs) are a fascinating class of lambda terms. Arising in the proof of the Fixed Point Theorem, their dynamical character affects the global structure of the Lambda Calculus in a fundamental way. Being a mechanism of unrestricted recursion, they are directly responsible for the Turing-completeness of the lambda calculus as a programming language. And when lambda terms are used as the computational basis of a logical system — whether based on the Curry–Howard isomorphism or illative combinatory logic — fixed point combinators appear unexpectedly as the (untyped) skeletons of paradoxes, heralding inconsistency of the supervenient logic. [2] [8] [3] [7] [4] [12]

It is an elementary fact that a term $Y$ is a fixed point combinator if and only if $Y$ is itself a fixed point of the combinator $\delta = \lambda y \lambda x. x(yx)$. This can even be taken as the definition of fpcs: $Y \in \Lambda$ is fpc iff $Y = \delta Y$. Corrado Böhm noticed that also $Y \delta$ is fpc whenever $Y$ is. For example, if $Y = \gamma$ is Curry’s fpc, then $Y \delta = \Theta$ is Turing’s fpc. A major open problem in

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1 In fact, from the historical point of view, it is the apparent possibility to encode computational processes of unlimited complexity in the lambda calculus that originally led Church to formulate his thesis, and thus to the concept of Turing-completeness.
the Lambda Calculus asks whether there exists a “double fpc” \( Y \) satisfying \( \delta Y = Y = Y \delta \). Statman [14] conjectures that no such \( Y \) exists.\(^2\)

Böhm’s observations revealed that fpcs themselves have compositional structure, where one constructs new fpcs from old by applying them to \( \delta \). Since then, other “fpc generating schemes” have been discovered and investigated by several authors. [13] [5] These contributions have confirmed that fpcs have a rich mathematical structure indeed.

In this note, we will explore such fpc generators “in the abstract”, studying their general properties and providing a basic taxonomy. We formulate several new problems, including a vast generalization of Statman’s conjecture.

2. Notations and definitions

**Notation 2.1.** We assume the reader is familiar with the basic notions of lambda calculus: \( \lambda \)-terms, free variables, substitution, and beta-conversion. We refer to [1] for background on these matters. Here we shall employ the following symbols and notions.

- \( \Lambda \) is the set of \( \lambda \)-terms. \( \Lambda^0 = \{ M \in \Lambda \mid \text{FV}(M) = \emptyset \} \) is the set of closed \( \lambda \)-terms.
- \( \text{FV}(M) \) is the set of free variables of \( M \in \Lambda \).
- \( M[x := N] \) is the result of capture-avoiding substitution of \( N \) for \( x \) in \( M \).
- If \( \vec{N} = (N_1, \ldots, N_k) \) is a sequence of \( \lambda \)-terms, then \( M\vec{N} = Mn_1\cdots N_k \).
- \( F^k(z) := F(F(\cdots F(z)\cdots)) \), with \( k \) Fs.
- \( I = \lambda x.x, K = \lambda xy.x, c_k = \lambda xy.x^k(y), \delta = \lambda yx.y(x) \).
- \( M = N \) denotes beta conversion between \( M \) and \( N \).
- \( M \rightarrow N \) denotes beta reduction from \( M \) to \( N \).
- \( M \) is solvable, if \( M\vec{N} = I \) for some \( \vec{N} \). Otherwise, \( M \) is unsolvable.
- \( M =^\infty N \) if \( M \) and \( N \) have the same Böhm tree. This relation is defined using one axiom and one inference rule, which is to be understood coinductively (see [10, 6]):

\[
\begin{array}{c|c|c|c|c|c}
M, N \text{ unsolvable} & M = \lambda \vec{x}. y \vec{M} & N = \lambda \vec{x}. y \vec{N} & M_1 =^\infty N_1 & \cdots & M_k =^\infty N_k \\
\hline
M =^\infty N & M =^\infty N
\end{array}
\]

- \( z \# M \) means \( z \notin \text{FV}(M) \). For \( S \subseteq \Lambda \), \( z \# S \) means \( z \# M \) for each \( M \in S \).
- \( z \notin M \) if there exists \( N = M \) such that \( z \# N \).
- \( z \notin^\infty M \) if there exists \( M =^\infty N \) such that \( z \# N \). Otherwise, \( z \in^\infty M \).

**Definition 2.2.** \( Y \in \Lambda \) is a fixed point combinator (fpc) if \( Yx = x(Yx) \) for \( x \# Y \).

**Definition 2.3.** \( Y \in \Lambda \) is a weak fixed point combinator (wfpc) if \( Yx =^\infty x(Yx) \) for \( x \# Y \).

Notice that every fpc is a wfpc.

All (w)fpcs have the same Böhm tree, so \( Y \in \Lambda \) is wfpc iff \( Y =^\infty Y_0 \) for some fpc \( Y_0 \).

An wfpc \( Y \) can equivalently be given by a sequence of terms \( (Y_n) \) with \( Y = Y_0 \) and \( Y_{n+1} x = x(Y_{n+1} x) \), with \( x \# Y_n \). [11, Prop. 2.9]

**Notation 2.4.** We write FPC (WFPC) for the set of fpcs (weak fpcs).

**Definition 2.5.** An fpc generating vector, or fgv, is a sequence of terms \( \vec{G} \) satisfying

\[
Y \in \text{FPC} \implies Y\vec{G} \in \text{FPC}
\]

\(^2\)An early attack on this problem was undertaken by Intrigila [9]. Unfortunately, Endrullis discovered a gap in the argument which seems difficult to overcome. For recent developments, see [5], [11].
Definition 2.6. A weak fpc generating vector, or wfgv, is a sequence of terms $\vec{G}$ satisfying

$$Y \in \text{WFPC} \implies Y\vec{G} \in \text{WFPC}$$

Proposition 2.7. TFAE:

(i) $\vec{G}$ is wfgv.
(ii) $Y \in \text{FPC} \implies Y\vec{G} \in \text{WFPC}$.
(iii) $Y\vec{G} \in \text{WFPC}$ for some $Y \in \text{FPC}$.

Proof. (i) $\implies$ (ii). Let $\vec{G}$ be wfgv, $Y$ be fpc. Then $Y$ is wfpc, and $Y\vec{G}$ is wfpc.

(ii) $\implies$ (iii). Trivial.

(iii) $\implies$ (i). Let $Z$ be wfpc. Then $Z =^\infty Y$. Also $Z\vec{G} =^\infty Y\vec{G}$ is wfpc. $$\square$$

Corollary 2.8. Every fpc generator is wfpc generator.

Proof. Let $\vec{G}$ be fpc generator. Pick $Y \in \text{FPC}$. Then $Y\vec{G}$ is fpc, hence $\vec{G}$ is wfgv. $$\square$$

Proposition 2.9. Consider the following conditions on $\vec{G}$.

(i) $Y \text{ fpc} \implies Y\vec{G} \text{ fpc}$
(ii) $Y \text{ wfpc} \implies Y\vec{G} \text{ wfpc}$
(iii) $Y \text{ fpc} \implies Y\vec{G} \text{ wfpc}$
(iv) $Y \text{ wfpc} \implies Y\vec{G} \text{ fpc}$

The following relations are valid:

(iv) $\implies$ (i) $\implies$ (ii) $\iff$ (iii)

Proof. These relations simply summarize the facts noted above. $$\square$$

3. Examples and first observations

Examples 3.1.

- Turing’s fpc. Let $\Theta_x = VVx$, where $V = \lambda vx.x(vvx)$. Then $\Theta \in \text{FPC}$.
- Parametrized Turing’s fpc. For $M \in \Lambda$, let $\Theta_Mx = VVMx$, where $V = \lambda vmx.x(vvmx)$. Then $\Theta_M \in \text{FPC}$.
- Let $z$ be a variable. Put $\Psi_z = W_zW_z1$, where $W_z = \lambda wpx.x(ww(zp)x)$. Then $\Psi_z \in \text{WFPC} \setminus \text{FPC}$

Proposition 3.2. $\Theta_M = \Theta_N \implies M = N$.

Proof. This is manifest upon inspecting the reduction graph of $\Theta_z$ — the set of reducts of $\Theta_z$. For a precise proof, see [11, Lemma 3.1]. $$\square$$

Examples 3.3.

- Let $\vec{G} = ()$, the empty vector. Obviously, $Y \in (W)\text{FPC} \implies Y\vec{G} = Y \in (W)\text{FPC}$.
  We call this generator trivial. In subsequent sections, we will tacitly assume all generators to be non-trivial.
- Fix a (w)fpc $Y_0$, and let $\vec{G} = (KY_0)$. Then $(KY_0)$ yields the same (w)fpc on every input:

  $$Y \in \text{WFPC} \implies Y(KY_0) = KY_0(Y'(KY_0)) = Y_0 \in (W)\text{FPC}$$

  We call such generators constant. They are not very interesting either.
Recall that $\delta y x = x(y x)$. It is easy to verify the following:
- $\delta^k(z) x = x^k(z x)$.
- If $Y$ is fpc, then $Y = \delta Y = \delta^k(Y)$.
- If $Y = (Y_0, \ldots, Y_k, \ldots)$ is wfp, then $Y_0 = \delta^k(Y_k)$.

Let $\bar{G} = (\delta)$. Then $Y \in \text{FPC} \implies Y \delta x = \delta(Y \delta) x = x(Y \delta x) \in \text{FPC}$.

It is an open problem whether there exists $Y \in (W)\text{FPC}$ such that $Y = Y \delta$.

• Let $\bar{G} = (\lambda y. \Theta_y)$. Then

$$Y \in \text{FPC} \implies Y \bar{G} = Y(\lambda y. \Theta_y) = (\lambda y. \Theta_y)(Y(\lambda y. \Theta_y)) = \Theta_Y \bar{G} \in \text{FPC}$$

Furthermore, there exists fpc $Y$ such that $Y = Y \bar{G}$.

Indeed, take $Y = \Theta(\lambda x. \Theta_x(\lambda y. \Theta_y)) = \Theta_Y(\lambda y. \Theta_y)$. Then $Y \in \text{FPC}$, and

$$Y(\lambda y. \Theta_y) = (\lambda y. \Theta_y)(Y(\lambda y. \Theta_y)) = \Theta_Y(\lambda y. \Theta_y) = Y$$

• The set of (w)fgvs is closed under composition: if $\bar{G}$ and $\bar{G}'$ are fgvs, then

$$Y \in \text{FPC} \implies Y \bar{G} \in \text{FPC} \implies Y \bar{G} \bar{G}' \in \text{FPC}$$

Thus, $(\delta, \lambda y. \Theta_y)$ and $(\lambda y. \Theta_y, \delta)$ are both fgvs.

• Many other examples of fpcs and fgvs can be found in [5] and [11].

**Definition 3.4.** A (w)fgv $\bar{G}$ is injective if for all (w)fpcs $Y, Y'$, $Y \bar{G} = Y' \bar{G}$ implies $Y = Y'$.

**Proposition 3.5.** No non-trivial (w)fgv is injective.

**Proof.** Suppose $\bar{G} = (G_0, \ldots, G_n)$, for $n \geq 0$, is injective.

Since $\Theta \bar{G} = G_0(\Theta G_0)G_1 \cdots G_n$ is wfp, $G_0$ must be solvable.

That is, $G_0 \bar{P} = I$ for some closed $\bar{P}$. Define fpcs $Y, Y'$ by

$$Y x = \Theta x \bar{P} x$$
$$Y' x = \Theta x \bar{P} x$$

By Proposition 3.2, $Y \neq Y'$. Yet

$$Y \bar{G} = \Theta G_0 \bar{P} G_0 \bar{G} = \Theta I G_0 \bar{G} = \Theta G_0 \bar{P} G_0 \bar{G} = Y' \bar{G}$$

Thus, $\bar{G}$ is not injective.

(Notice that $Y$ and $Y'$ are closed, so even restricting to closed terms, no non-trivial wfp generator is injective.)

**Corollary 3.6.** Suppose wfgv $\bar{G}$ fixes every fpc: $Y \bar{G} = Y$ for all fpc $Y$. Then $\bar{G}$ is trivial.

An interesting consequence of these observations is that there is no uniform way to “Böhm out” an inner level of a wfp.

**Proposition 3.7.** For $m > 0$, it is not possible to find terms $(M_0, \ldots, M_n)$ such that

$$Z = (Z_0, Z_1, \ldots) \text{ wfp} \implies Z_0 \bar{M} = Z_{m+1} \quad (3.1)$$

**Proof.** Suppose such $\bar{M} = (M_0, \ldots, M_n)$ exists.

Then $\bar{M}$ is a wfg.

For every fpc $Y$, we have $Y \bar{M} = Y$, so every fpc is fixed by $\bar{M}$.

(In particular, $\bar{G}$ is a fgv.)

By Corollary 3.6, $\bar{G}$ is trivial: $\bar{G} = ()$.

But then $\bar{G}$ fixes every wfp as well, and thus cannot satisfy the hypothesis in (3.1).
4. Constant and compact generators

For the next definition, note that for every (w)fpc $Y$ and $k \geq 0$, there exists $Y' = (\infty) Y$ with

$$\forall \tilde{G}. \quad Y\tilde{G} = G_{\tilde{G}}^k(Y'G_0)G_1\cdots G_n = \delta^k(Y')\tilde{G}$$

**Definition 4.1.** Let $\tilde{G}$ be a (w)fgv. Fix $z\#\tilde{G}$.

- $\tilde{G}$ is constant if there is a $k$ such that $z \notin G_0^k(z)G_1\cdots G_n$.
- $\tilde{G}$ is weakly constant if there is a $k$ s.t. $z \notin G_0^k(z)G_1\cdots G_n$.
- $\tilde{G}$ is compact if there is a $k$ such that $G_0^k(z)G_1\cdots G_n \in \text{FPC}$.
- $\tilde{G}$ is weakly compact if there is a $k$ s.t. $G_0^k(z)G_1\cdots G_n \in \text{WFPC}$.

The least $k$ satisfying one of these conditions is then called the *modulus of constancy*, or *modulus of compactness*, accordingly.

From now on, let $\tilde{G}$ be a possibly weak fgv. We will omit freshness conditions $x\#Y$, $z\#\tilde{G}$ etc., as they will always be obvious from the context.

**Proposition 4.2.** Let $\tilde{G}$ be constant. There is a term $Z$ such that

$$Y\tilde{G} = Z$$

for all wfpc $Y$. Hence $Z$ is (w)fpc.

**Proof.** Let $\tilde{G}$ be constant, and let $k$ be such that $z \notin G_0^k(z)G_1\cdots G_n$.

That is, $z \notin \text{FV}(Z)$ for some $Z \in \Lambda$ convertible to $G_0^k(z)G_1\cdots G_n$.

Then for any wfpc $Y = (Y_0, Y_1, \ldots)$, we have

$$Y\tilde{G} = Y_0\tilde{G} = Y_0G_0\cdots G_n$$

$$= G_0(Y_1G_0)G_1\cdots G_n$$

$$= \vdots$$

$$= G_0^k(Y_kG_0)G_1\cdots G_n$$

$$= G_0^k(z)G_1\cdots G_n[Y_kG_0/z]$$

$$= Z[Y_kG_0/z]$$

$$= Z \square$$

**Proposition 4.3.** The following observations are immediate.

1. Every constant fgv is compact.
2. Every constant (w)fgv is weakly constant.
3. Every compact (w)fgv is weakly compact.

**Proposition 4.4.** $\tilde{G}$ is weakly constant iff $\tilde{G}$ is weakly compact.

**Proof.** Let $\tilde{G}$ be a wfgv. Then

$$z \notin G_0^k(z)G_1\cdots G_n \implies G_0^k(z)G_1\cdots G_n = G_0^k(\Theta G_0)G_1\cdots G_n = \Theta\tilde{G} \in \text{WFPC}$$

$$G_0^k(z)G_1\cdots G_n \in \text{WFPC} \implies G_0^k(z)G_1\cdots G_n = \Theta \implies z \notin G_0^k(z)G_1\cdots G_n \square$$

**Proposition 4.5.** Every (weakly) compact generator has a fixed point:

1. If $\tilde{G}$ is compact fgv, there exists fpc $Y$ with $Y\tilde{G} = Y$.
2. If $\tilde{G}$ is weakly compact wfgv, there exists wfpc $Y$ with $Y\tilde{G} = Y$. 


Proof. The construction is the same for both claims. We will first treat the weak case, and then specialize the proof to the first claim as well.

Let be a weakly compact wfgv.

Let be the modulus of weak compactness, so that . Since is wfpc, write

Define , and . Now compute

Since is wfpc, so is , proving the second claim.

Now suppose that was actually compact fgv. Then would be fpc, while all of the steps above would remain valid, with for each .

Then would be fpc as well, proving the first claim.

We believe the converse to the second statement in Proposition 4.5 holds as well.

Conjecture 4.6. If there is wfpc such that then is weakly constant.

Intuition. If is not weakly constant, it must bring back to the stem of the constructed wfgv Böhm tree infinitely often. But would be slower to normalize at any node than would on its own, so any conversion between and must happen after all such nodes have been fully developed. (See [5] for an exploration of this idea.)

Remark 4.7. The converse to the first statement in Proposition 4.5 is consistent with known information, and we find it plausible. However, at the time of this writing, we do not yet have any compelling reasons to believe it, so we do not assert it as a formal conjecture.

Remark 4.8. Conjecture 4.6 is a vast generalization of Statman’s conjecture on nonexistence of “double fpcs” \( Y \) satisfying \( \delta Y = Y = Y \delta \).

Indeed, since \( \delta^k(z)x = x^k(\delta z) \), clearly \( z \in \delta^k(z) \) for all \( k \geq 0 \). Thus, the fgv \( \tilde{G} = (\delta) \) is not weakly compact. By Conjecture 4.6, \( (\delta) \) has no fixed point.

5. Rectifying generators

Definition 5.1. A vector \( \tilde{G} \) is rectifying if it satisfies condition \((iv)\) of Proposition 2.9:

\[ Y \in \text{WFPC} \implies Y\tilde{G} \in \text{FPC} \]

Example 5.2. \( \tilde{G} = (\lambda y.\Theta_y) \) is rectifying:

\[ Y \in \text{WFPC} \implies Y(\lambda y.\Theta_y) = (\lambda y.\Theta_y)(Y'(\lambda y.\Theta_y)) = \Theta_{Y'(\lambda y.\Theta_y)} \in \text{FPC} \]
In Example 3.3, we saw that \((\lambda y. \Theta y)\) has a fpc fixed point. We shall presently see that so does every rectifying fgv.

Our original proof of this fact first showed that if \(\vec{G}\) is rectifying, then \(\vec{G}\) is weakly constant, and thus has a wfpc fixed point \(Y\). But then \(Y = \vec{Y} \in \text{FPC}\) because \(\vec{G}\) is rectifying, hence \(Y\) is fpc.

Considering that compactness provides another sufficient condition for existence of fpc fixed points, it was natural to wonder whether rectifying and compact fgvs are related. This led us to the following result.

**Theorem 5.3.** A fgv \(\vec{G}\) is compact iff it is rectifying.

**Proof.** \((\Rightarrow)\) Suppose \(G^0(z)G_1\ldots G_n \in \text{FPC}\). Then

\[
Y = (Y_0, Y_1, \ldots) \in \text{WFPC}
\]

\[
\implies \quad \vec{Y} \vec{G} = Y_0G_0\ldots G_n = C^0_k(Y_kG_0)G_1\ldots G_n = C^k_0(z)G_1\ldots G_n[z := Y_kG_0] \in \text{FPC}
\]

\((\Leftarrow)\) The intuition for this direction is that, although the Böhm tree of a wfpc \(Y\) is infinite, only a finite part of it can be used in any conversion \(\rho: Y\vec{G}x = x(Y\vec{G}x)\). Thus writing \(Yx = x^k(Y_kx) = \delta^k(Y_k)x\) for large enough \(k\) will ensure that \(Y_k\) is not touched by any redex contractions. Then the whole conversion \(\rho\) could be lifted to \(\rho = \sigma[z := Y_k]\), where \(\sigma: \delta^k(z)\vec{G}x = x(\delta^k(z)\vec{G}x)\).

To formalize this intuition, suppose \(\vec{G}\) is rectifying.

Fix \(c\#\vec{G}\). Define

\[
W_{x,p} = \lambda v. x(v(cp)v) \\
V_x = \lambda p. W_{x,p} \\
\Upsilon = \lambda x. V_x \text{IV}_x
\]

That is, \(V_x = \lambda pv. x(v(cp)v)\). Note that \(W_{x,p}\) and \(V_x\) are normal forms.

Let \(\Upsilon_x^k = V_x \delta^k(\text{I})V_x\). The term \(\Upsilon_x\) reduces as follows:

\[
\Upsilon x \rightarrow \Upsilon_x^0 \equiv V_x \text{IV}_x \rightarrow W_{x,1}V_x \rightarrow x(V_x(c\text{I})V_x) \equiv x(\Upsilon_x^1) \\
\rightarrow x(W_{x,c\text{I}}V_x) \rightarrow x(x(V_xc^2(\text{I})V_x)) \equiv x^2(\Upsilon_x^2) \\
\rightarrow \ldots \\
\rightarrow x^k(\Upsilon_x^k) \\
\rightarrow \ldots
\]

Since each term appearing in the above reduction sequence has a unique redex, the reduction is completely deterministic. That is — the above sequence actually comprises the entire reduction graph of \(\Upsilon x\).

The sequence also shows that \(\Upsilon\) is a wfpc.

It is not a fpc however, since \(\Upsilon_x^0\) obviously has no reducts in common with \(\Upsilon_x^1\).

But \(\vec{G}\) is rectifying, so \(\Upsilon \vec{G}\) is fpc.

By the Church–Rosser theorem, let \(X\) be a common reduct

\[
\Upsilon \vec{G}x \rightarrow X \leftarrow x(\Upsilon \vec{G}x) \quad (5.1)
\]

We will use these reductions to show that \(\delta^k(z)\vec{G} \in \text{FPC}\) for large enough \(k\).

We proceed with the following sequence of claims, which are hopefully sufficiently clear not to warrant additional elaboration.
(1) If \( \Upsilon M \to X \), then \( X \to X' \equiv C[V_{M_1}c^k_1(I)\ldots,Vc^k_m(I)\ldots], \) with \( M \to M_i \), \( M' \to M_i' \), and every occurrence of \( c \) in \( X' \) being displayed in the subterm \( c^k(I) \) in one of the holes in \( C[] \).

(2) If \( \Upsilon M \to X \), then \( X \to X' \equiv C[\Upsilon_{M_1}^k,\ldots,\Upsilon_{M_m}^k], \) with \( M \to M_i \) and every occurrence of \( c \) being uniquely determined by its occurrence in some \( \Upsilon_{M_i}^k \).

This is obtained from above by finding a common reduct for each \( M_i, M_i' \).

(3) If \( C[\Upsilon M] \to X \), then \( X \to C'[\Upsilon_{M_1}^k,\ldots,\Upsilon_{M_m}^k] \), with the same conditions on \( M_i \) and occurrences of \( c \) as in the previous point.

(4) If \( C[\Upsilon M] \to X \), then \( X \to C'[\Upsilon_{N}^k,\ldots,\Upsilon_{N}^k] \), where \( M \to N \) and each occurrence of \( c \) being uniquely determined by its occurrence in some \( \Upsilon_{N}^k \).

This is obtained from the previous claim by “bumping all \( \Upsilon^k \)'s along” to stage \( k \geq \max\{k_i\} \), and letting \( N \) be a common reduct of all the \( M_i \)’s.

(5) If the reduction \( \rho : C[\Upsilon M] \to C'[\Upsilon_{N}^k,\ldots,\Upsilon_{N}^k] \) is obtained by the algorithm given in the previous steps, then \( \rho \) lifts to \( \rho = \sigma[ux := \Upsilon_{N}^k] \), where

\[ \sigma : C[\delta^k(u)M] \to C'[uN,\ldots,uN] \]

And now we are done! The common reductions in (5.1) can be continued to

\[ \Upsilon_{G}X \to C'[\Upsilon_{N}^k,\ldots,\Upsilon_{N}^k] \iff x(\Upsilon_{G}X) \]

such that, for both reductions, all of the descendants of \( \Upsilon \) are displayed in the context. (This follows from the fact that every occurrence of \( c \) is witnessed in some \( \Upsilon_{N}^k \), and \( c \) was chosen to be fresh. The variable \( c \) acts as a “label” for the unfolding depth of \( \Upsilon \).)

The conclusion of the last step therefore holds for both of these reductions, so

\[ C[\delta^k(u)M] = [\delta^k(u)G_0]G_1\ldots G_n \to C'[uN,\ldots,uN] \iff x(\delta^k(u)G_0\ldots G_n) \]

That is, \( G_0^k(uG_0)G_1\ldots G_n \in \text{FPC} \). Since \( u \) is free, so is \( G_0^k(u)G_1\ldots G_n \).

\[ \square \]

**Corollary 5.4.** Every rectifying fgv has a fixed point in fpcs.

**Remark 5.5.** The proof of the nontrivial direction of Theorem 5.3 suggests a deeper connection between uniform properties (finite conversions) and terms obeying a coinductive pattern (such as wfpcs). While we were not able to isolate the general “continuity principle” that seems to be at work here, we will see a different application of the same argument in the next section.

We finish this section with an example of a weakly constant fgv which is not rectifying. It follows that compactness is indeed stronger than weak compactness.

**Proposition 5.6.** There exist weakly constant fpc generators which are not rectifying.

**Proof.** Consider the following combinators:

\[
\begin{align*}
P_{xy} & = yx \\
Q_{yz} & = z(yQz) \\
W_{wpz} & = z(ww(zp)z) \\
R_{yz} & = WW(yQz)z
\end{align*}
\]

First we observe that \( (P,Q) \) is an fgv: for \( Y \) fpc, we have

\[ Y PQx = P(Y P)Qx = Q(Y P)x = x(Y PQx) \quad (5.2) \]
We claim that \((P, Q)\) is not rectifying.
If it did, then by the previous theorem, it would be compact, hence weakly compact, hence weakly constant.
Which it’s not. (Inspection.)
Next, we verify that \((P, R)\) is again fgv:
\[
YPRx = P(YP)Rx = R(YP)x = WW(YPQ)x
\]
\[
= x(WW(x(YPQ)x)x)
\]
\[
=_{(5.2)} x(WW(YPQ)x)
\]
\[
= x(YPRx)
\]
At the same time, we see that \(z \in P^1(z)R\):
\[
P^1(z)Rx = PzRx = Rz = WW(zQx)
\]
\[
= x(WW(\cdots)x)
\]
\[
= x^2(WW(\cdots)x)
\]
\[
= \cdots
\]
\[
= x^n(\cdots)
\]
The variable \(z\) is being pushed to infinity, and does not appear on the Böhm tree of \(PzRx\) — nor on the Böhm tree of \(PzR = \lambda x.PzRx\).
Thus \(\hat{G} = (P, R)\) is weakly constant. We claim it is not rectifying.
For a wfpc \(Z\), the term \(ZPR\) reduces as follows:
\[
ZPRx \rightarrow P(ZP)Rx
\]
\[
\rightarrow^2 R(ZP)x
\]
\[
\rightarrow^2 WW(ZPQ)x
\]
\[
\rightarrow^3 x(WW(x(ZPQ)x)x)
\]
\[
\rightarrow^3 x^2(WW(x^2(ZPQ)x)x)
\]
\[
\rightarrow \vdots
\]
\[
ZPRx \rightarrow x^{n+2+2+3n}x^n(WW(x^n(ZPQ)x))x
\]
From this analysis, it is manifest that any common reduction
\[
ZPRx \rightarrow \cdot \leftrightarrow x(ZPRx)
\]
must contain a common reduction between
\[
x^n(ZPQx) \rightarrow \cdot \leftrightarrow x^{n+1}(ZPQx)
\]
As we observed earlier, \((P, Q)\) is not rectifying, so there exist wfcps \(Z\) for which such conversion is not possible.
Thus \((P, R)\) is not rectifying either.

**Remark 5.7.** By changing the term slightly, we can get a non-rectifying weakly constant fpc generator with an arbitrary modulus of constancy. This is achieved by passing the argument of the generator into the head position \(k\) times before pushing it to infinity.
6. THE MONOID OF WFGVS

Thewfpc and fpc generators have an obvious monoid structure:

\[(G_1, \ldots, G_n) \odot (G'_1, \ldots, G'_m) \coloneqq (G_1, \ldots, G_n, G'_1, \ldots, G'_m)\]

The identity is the trivial generator (1).

The concatenation operation is associative, and satisfies the identity laws.

We thus have a monoid \((G, \odot, (\_))\) of wfgvs, containing fgvs as a submonoid.

Since there are infinitely many constant (w)fgvs of arbitrary complexity, neither of the
monoids is finitely generated.

Definition 6.1. A two-sided ideal in a monoid \((M, \cdot)\) is a set \(I \subseteq M\) such that

\[i \in I, m \in M \implies i \cdot m \in I, m \cdot i \in I\]

Proposition 6.2.

1. The constant wfgvs form a two-sided ideal in \(G\).
2. The weakly constant wfgvs form a two-sided ideal in \(G\).
3. The compact fgvs form a two-sided ideal in the submonoid \(F\) of fgvs.

Proof. (1) This point is rather obvious.
(2) Let \(\vec{G} \in G\) be weakly constant with modulus \(k\): \(z \notin \infty G_k^0(z)G_1\cdots G_n\).

Let \(\vec{G}' \in G\) be arbitrary.

Since \(z\) cannot be Böhm-out, clearly

\[z \notin \infty G_k^0(z)G_1\cdots G_nG'_1\cdots G'_m\]

— and \(\vec{G} \odot \vec{G}'\) is weakly constant.

On the other hand, we know that \(\vec{G}'\) maps wfpcs to themselves:

\[(\lambda y. y\vec{G}') : WFPC \to WFPC\]

All wfpcs have the same Böhm tree, and in the tree topology, its neighborhood basis
consists of the set \(\{\lambda x.x^n(\Omega) \mid n \geq 0\}\).

By Continuity Theorem (Barendregt 1984, 14.3.22), there exists a \(k_0 \geq 0\) such that

\[(\lambda y. y\vec{G}')(\lambda x.x^{k_0}(\Omega)) = x^k(Z)\]

Hence, \((\lambda x.x^{k_0}(\Omega))\vec{G}'\vec{G}\) has the full Böhm tree of a weak fpc.

So \(\vec{G}' \odot \vec{G}\) is weakly constant with modulus \(k_0\).

(3) It is immediate that the rectifying fgvs form a two-sided ideal. By Theorem 5.3, so do
the compact ones. \(\square\)

Extensional equality. Since the primary interest in (w)fgvs is in their ability to generate new (w)fpcs from old, it is natural to identify generators having the same functional behavior.

Definition 6.3. We say a fgv or wfgv \(\vec{G}\) is extensionally equal to \(\vec{G}'\), written \(G \simeq G'\), if for every fpc \(Y\), \(Y\vec{G} = Y\vec{G}'\).

Examples 6.4. • If \(\vec{G}\) is a constant generator, say, \(Y\vec{G} = Z\) for all \(Y\), then \(\vec{G} \simeq (KZ)\):

\[Y(KZ) = KZ(Y'(KZ)) = Z = Y\vec{G}\]
• Recall the combinator $\mathcal{C} = \lambda f x y. f y x$.

Let $Gyz = z(y(Cz)))(\delta(y(Cz)))$.

Then $(G, K)$, and $(\ell G, \ell K)$ are fgvs, and $(G, K) = (G, \ell C)$:

$$\begin{align*}
Y \in \text{FPC} \implies YGK &= G(YG)k = k(YG(\ell C))(\delta(YG(\ell C))) \\
&= YG(\ell C) = G(YG)(\ell K) = (\ell K)(YG(\ell C))(\delta(YG(\ell C))) \\
&= \ell(\delta(YG(\ell C))) = \delta(YGK) \in \text{FPC}
\end{align*}$$

The reason that in the definition of $\equiv$ the quantifier ranges over fpcs both in the case of fgvs as well as wfgvs is that, when the quantifier is taken over all wfgcs, it makes the resulting notion of equality much more restrictive. (The following proposition will demonstrate this fact rather concretely.)

Since we obviously want equal fgvs to remain equal as wfgvs, the definition of extensional equality is expressed in terms of behavior on fpcs.

**Proposition 6.5.** If $Y \overline{G} = Y \overline{G'}$ for every wFPC $Y$, then for some $k$, $\delta^k(z)\overline{G} = \delta^k(z)\overline{G'}$

**Proof.** This statement follows by the same reasoning as used in Theorem 5.3.

Take $z \# \overline{G}, \ell \overline{G}$, and let $\Upsilon = \Upsilon_z$ be the canonical wFPC defined there with a deterministic reduction graph that uses the variable $z$ to track its unfolding history.

The argument subsequently showed how every conversion $C[\Upsilon M] \rightarrow X \leftrightarrow C'[\Upsilon M]$ can be extended through $X \rightarrow X'$, such that $X' = D[\Upsilon_k, \ldots, \Upsilon_N]$ with $M \rightarrow N$ and every occurrence of $z$ in $X'$ to be found among the displayed $\Upsilon_k$. We could then conclude that the common reduction may be lifted to a finite truncation of $\Upsilon$.

In the present case, our starting conversion has the form

$$C[\Upsilon G_0] = [\Upsilon G_0]G_1 \cdots G_n = [\Upsilon G'_0]G'_1 \cdots G'_{n'} = C'[\Upsilon G'_0] \quad (6.1)$$

We should thus argue why $G_0 = G'_0$.

Let $X$ be a reduct of $C[\Upsilon G_0]$. By recalling the reduction graph of $\Upsilon$, it is evident that every innermost occurrence of $z$ in $X$ is applied to a reduct of $G_0$.

If $X$ is also a reduct of $C'[\Upsilon G'_0]$, then the same conclusion will hold, with $G'_0$ in place of $G_0$. Thus, the very fact of occurrence of $z$ in $X$ forces $G_0$ and $G'_0$ to be convertible.

Of course, if $z$ does not occur in $X$ at all, that only means that all descendants of $\Upsilon$ have already been erased, in which case we have nothing left to prove.

So $G_0 = G'_0$. We can thus adjust conversion in (6.1) to

$$C[\Upsilon G_0] \equiv [\Upsilon G_0]G_1 \cdots G_n = [\Upsilon G_0]G'_1 \cdots G'_{n'} \equiv C'[\Upsilon G'_0] = C'[\Upsilon G'_0]$$

where the conversion on the right takes place inside the subterm immediately to the right of $\Upsilon$.

Now we extend the other conversion to a common reduct

$$[\Upsilon G_0]G_1 \cdots G_n \rightarrow D[\Upsilon_k, \ldots, \Upsilon_N] \leftrightarrow [\Upsilon G_0]G'_1 \cdots G'_{n'}$$

and proceed to lift these reductions to

$$[\delta^k(u)G_0]G_1 \cdots G_n \rightarrow D[uN, \ldots, uN] \leftrightarrow [\delta^k(u)G_0]G'_1 \cdots G'_{n'}$$

Converting $G_0$ in the right term to $G'_0$, we obtain the desired result.

From now on, we will consider the monoid $\mathcal{G}$ up to extensional equality.

We will also write concatenation of vectors by juxtaposition: $\vec{F} \mathcal{G} = \vec{F} \odot \mathcal{G}$.
Green’s relations.

**Definition 6.6.** For \( \tilde{G}, \tilde{G}' \in \mathcal{G} \), put

\[
\mathcal{L}(\tilde{G}) = \{ \tilde{H} \tilde{G} \mid \tilde{H} \in \mathcal{G} \}
\]

\[
\mathcal{R}(\tilde{G}) = \{ \tilde{G} \tilde{H} \mid \tilde{H} \in \mathcal{G} \}
\]

\[
\tilde{G} \leq_{\mathcal{L}} \tilde{G}' \iff \mathcal{L}(\tilde{G}) \subseteq \mathcal{L}(\tilde{G}')
\]

\[
\tilde{G} \leq_{\mathcal{R}} \tilde{G}' \iff \mathcal{R}(\tilde{G}) \subseteq \mathcal{R}(\tilde{G}')
\]

\[
\tilde{G} \sim_{\mathcal{L}} \tilde{G}' \iff \mathcal{L}(\tilde{G}) = \mathcal{L}(\tilde{G}')
\]

\[
\tilde{G} \sim_{\mathcal{R}} \tilde{G}' \iff \mathcal{R}(\tilde{G}) = \mathcal{R}(\tilde{G}')
\]

Here we record several observations about the relations above. These shed light on the structure of the monoid \( \mathcal{G} \).

(1) If \( \tilde{G} \simeq (KZ) \) is a constant generator, then \( \mathcal{L}(\tilde{G}) = \{ KZ \} \), so all constant generators are each in their own left class.

That is, \( \tilde{G} \sim_{\mathcal{L}} (KZ) \) implies \( \tilde{G} \simeq (KZ) \).

(2) On the other hand, \( (KZ, KZ') \simeq (KZ') \), thus \( KZ \preceq KZ' \). Since the choice of \( Z, Z' \) was arbitrary, \( KZ \sim_{\mathcal{R}} KZ' \) for all \( Z \) and \( Z' \). That is, constant generators are all in the same right class.

Since constant generators form an ideal, \( \tilde{G} \sim_{\mathcal{R}} (KZ) \) or \( \tilde{G} \preceq_{\mathcal{R}} (KZ) \) imply \( \tilde{G} \simeq (KZ') \). So \( \mathcal{R}(KZ) = \{ (KZ') \mid Z' \in \text{WFPC} \} \).

(3) By the same token, if \( \tilde{G} \) is (weakly) compact, then so is every element of \( \mathcal{L}(\tilde{G}) \) and \( \mathcal{R}(\tilde{G}) \).

That is, the only (w)fgvs that can be congruent to \( \tilde{G} \) modulo \( \sim_{\mathcal{L}} \) or \( \sim_{\mathcal{R}} \) are again (weakly) compact.

(4) Suppose \( \tilde{G} \sim_{\mathcal{R}} \tilde{G}' \). Then we can find \( \tilde{F}, \tilde{F}' \in \mathcal{G} \) such that \( \tilde{G} \simeq \tilde{G}' \tilde{F} \), and \( \tilde{G}' \simeq \tilde{G} \tilde{F}' \).

But then \( \tilde{G} \simeq \tilde{G} \tilde{F}' \tilde{F} \), and \( \tilde{G}' \simeq \tilde{G}' \tilde{F} \tilde{F}' \).

If \( \tilde{G} \simeq \tilde{G} \tilde{F}' \tilde{F} \), then for every \( Y, Y \tilde{G} = Y \tilde{G} \tilde{F}' \tilde{F} \) is a fixed point of \( \tilde{F}' \tilde{F} \).

By Conjecture 4.6, \( \tilde{F}' \tilde{F} \) is weakly constant. By Proposition 6.2, so is \( \tilde{G} \tilde{F}' \).

But \( \tilde{G} \tilde{F}' \tilde{F} \simeq \tilde{G} \). So \( \tilde{G} \) is weakly constant.

Of course, everything we just said applies to \( \tilde{G}' \) as well.

We conclude that, modulo Conjecture 4.6, nontrivial \( \sim_{\mathcal{R}} \)-relations can only exist between weakly constant wfgvs.

The last example motivates the following.

**Conjecture 6.7.** If \( \tilde{G} \sim_{\mathcal{L}} \tilde{G}' \) or \( \tilde{G} \sim_{\mathcal{R}} \tilde{G}' \) with neither wfgv weakly constant, then \( \tilde{G} \simeq \tilde{G}' \).

Ultimately, we would like to see that the monoid of wfgvs is “freely generated”, in the sense that every fgv can be written as a composition of “atomic” fgvs, such that this decomposition is unique up to extensional equality. However, the presence of weakly constant generators complicates the precise formulation of this property, since these generators may have non-trivial relations between each other. In the following examples, we show that it is possible to have \( \tilde{F} \tilde{G} \simeq \tilde{G} \) or \( \tilde{F} \tilde{G} \simeq \tilde{F} \) under certain conditions. In both cases, (weak) compactness plays an essential role.

**Proposition 6.8.** For \( \tilde{G} \) weakly constant, there exists non-constant \( \tilde{F} \) with \( \tilde{F} \tilde{G} \simeq \tilde{F} \).

(In particular, \( \tilde{F} \leq_{\mathcal{L}} \tilde{G} \).)
Proof. The idea is to make \( \tilde{F} \) generate the fixed points of \( \tilde{G} \) according to the scheme in Proposition 4.5.

Let \( k, F_0, F_k \) be chosen as in the proof of that proposition.

Put \( A = \lambda y b(y\delta), B = \lambda y F_0[y(\lambda u F_k[uG_0])G_0] \), and \( \tilde{F} = (A, B) \).

Observe that that

\[
Y \in \text{FPC} \quad \rightarrow \quad YA\delta = A(YA)\delta = \delta(YA\delta) \\
YAB = A(YA)B = B(YA\delta) = F_0[(YA\delta)(\lambda u F_k[uG_0])G_0]
\]

Since \( YA\delta \) is thereby forced to be fpc, it follows that \( YAB = F_0[UG_0] \), where \( U = F_k[UG_0] \).

This allows us to calculate as in the proof of Proposition 4.5 that \( YAB \) is a fixed point of \( \tilde{G} \):

\[
Y\tilde{F}\tilde{G} = YAB\tilde{G} = YAB
\]

Note however, that \( \tilde{F} \) will not be constant in general, because it uses its fpc argument to define \( U \). \( \Box \)

**Proposition 6.9.** Let \( \tilde{F} = (F_0, \ldots, F_n) \) be wfgv with \( n \geq 1 \).

There exists a compact fgv \( \tilde{G} \) such that \( F\tilde{G} \simeq \tilde{G} \).

(In particular, \( \tilde{G} \not\in \tilde{F} \).

Proof. First, recall that \( F_0 = \lambda v_0 \ldots v_l v_k \tilde{P} \) is solvable. Since \( Y\tilde{F} = F_0(Y'F_0)F_1 \ldots F_n \), we also know that the head variable of \( v_h \) cannot be \( v_0 \), for otherwise the result would be unsolvable, while it must be fpc.

We let \( \tilde{G} = (F_0, G_1, \ldots, G_{n+1}) \). We will only need to specify a couple of \( G_i \)s.

Set \( G_h = \lambda \tilde{P}, \lambda g_{l+1}, \ldots, g_{n+1} : G_{n+1}(F_0v_0F_1 \ldots F_n), G_{n+1}y = \Theta(\lambda g y. g(y\tilde{F})) = G_{n+1}(y\tilde{F}) \).

\[
Y\tilde{G} = G_0(Y'G_0)G_1 \ldots G_{n+1} \\
= F_0(Y'F_0)G_1 \ldots G_{n+1} \\
= G_h\tilde{P}[v_0 := Y'F_0][v_i := G_i]_{1 \leq i \leq l} G_{l+1} \ldots G_{n+1} \\
= \Theta_{G_{n+1}}(F_0(Y'F_0)F_1 \ldots F_n) \\
= \Theta_{G_{n+1}}(YF_0 \ldots F_n) \\
= \Theta_{G_{n+1}}(Y\tilde{F}) \tag{*} \\
= \Theta_{G_{n+1}}(Y\tilde{F}) \\
= \Theta_{G_{n+1}}(Y\tilde{F}) \tilde{G} \quad \text{by \textit{(*)}, with } Y := Y\tilde{F} \quad \Box
\]

Our final observation is a corollary to one of the first ones.

**Proposition 6.10.** The monoid \( G \) is zerosum-free: If \( \tilde{F}\tilde{G} \simeq (\cdot) \), then \( \tilde{F} \simeq (\cdot) \simeq \tilde{G} \).

Proof. Suppose \( \tilde{F}\tilde{G} \simeq (\cdot) \). Then, considered as endofunctions on \( \text{WFPC} / \approx _\beta \), \( \tilde{G} \) acts as a left inverse of \( \tilde{F} \), making \( \tilde{F} \) a split mono (modulo beta).

But we have seen in Proposition 3.5 that no wfgv is injective, so no wfgv can be monic. Specifically, take \( Y \not\simeq Y' \) such that \( Y\tilde{F} = Y'\tilde{F} \).

Since \( (\cdot) \simeq \tilde{F}\tilde{G} \), we have \( Y = Y\tilde{F}\tilde{G} = Y'\tilde{F}\tilde{G} = Y' \), a contradiction. \( \Box \)
7. Concluding remarks

In this paper, we have broached the topic of abstract fpc generators. Our first investigations revealed that these operators naturally fall into a few robust classes. We established elementary relationships between these classes.

What becomes clear from our investigations is that there is yet much to be uncovered about the structure of fixed point combinators. Some of the possible future research directions include the following.

(1) The most pressing issue is the status of Conjecture 4.6. All the evidence available points to this conjecture being true, yet current techniques in untyped lambda calculus decidedly come up short in settling the question. However it will be decided, the insights to be gathered from the new approaches will greatly deepen our understanding of lambda terms.

(2) Of course, one could take the next step and ask whether the converse to the first claim in Proposition 4.5 is also valid. Considering how difficult the former question is, this one will likely remain out of reach for the foreseeable future.

(3) What is the structure of the monoid $G$? Do non-compact wfgvs “freely generate” it, modulo extensional equality? Does every non-compact wfgv have a unique representation as a composition of “prime” elements?

(4) Do Green’s relations trivialize outside weakly compact wfgvs? What is the status of Conjecture 6.7?

(5) Since the monoid of (w)fgvs naturally acts on the set of (w)fpcs, how much of the structure of fpcs is captured by this monoidal action? Does every fpc have a representation in terms of the prime elements of the monoid — again, modulo extensional equality, and the ideal of compact generators?

(6) Finally, while not directly relevant to the earlier discussion, an answer to the following question could also shed light on recursion-theoretic properties of FPCs:

Let $Y$ be Curry’s simplest fpc. Is $\{\#M \mid M = Y_0\}$ a decidable subset of FPC? Specifically, does there exist a term $\Delta_Y$ satisfying, for all $Y \in \text{FPC}$, the following:

$$\Delta_Y xy = \begin{cases} x & Y = y \\ y & Y \neq y \end{cases}$$

Notice that Scott’s theorem does not yet apply because FPC is not all of $\Lambda$, but is only a computably enumerable subset of it. $\Delta_Y$ can diverge outside this set.

An upcoming paper [11] proposes another approach to Statman’s conjecture based on simple types. We note that the generalization of the conjecture stated there is consistent with ours, since no simply-typed generator can be weakly constant.

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ON (WEAK) FPC GENERATORS

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