ON THE UPPER BOUND OF THE CARDINALITY OF
SPECIAL LOCUS OF A FAMILY OF ALGEBRAIC CURVES
OVER POSITIVE CHARACTERISTIC WITH NONTRIVIAL
KODAIRA-SPENCER MAP

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1. Introduction

Let $k$ be an algebraically closed field of characteristic $p$. Let $M_{g,p}$ be the coarse moduli space of principally polarized Abelian varieties of dimension $g \geq 2$ defined over $k$. It is an interesting question in characteristic $p$ geometry to describe the various stratum in $M_{g,p}$ defined by the Newton polygon. By a theorem of Grothendieck, the sublocus $N_{g,p}$ in $M_{g,p}$ which underlies the isomorphism classes of the Abelian varieties with positive Newton slopes is naturally an closed algebraic subvariety.

In this paper we study the intersection behavior of an algebraic curve $C^0$ of $M_{g,p}$ with $N_{g,p}$, which is induced by a family of algebraic curves of genus $g$ over $k$ with nontrivial Kodaira-Spencer map and does not lie entirely in $N_{g,p}$. More concretely we let $f : X \to C$ be a semi-stable family of algebraic curves which compactifies the smooth family $f^0 : X^0 \to C^0$. Namely, one has the following commutative diagram:

\[
\begin{array}{ccc}
X^0 & \longrightarrow & X \\
\downarrow f^0 & & \downarrow f \\
C^0 & \longrightarrow & C.
\end{array}
\]

In this paper we shall call the sublocus of $C^0$ supporting the closed fibers whose Newton slopes are all positive special. It is also the locus where the $p$-rank of the

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Jacobian of the algebraic curves are equal to zero. For such a complete family as in the above diagram we will give an upper bound of the cardinality of the special locus by a function of the genus $b$ of the compactification $C$ of $C^0$, the degree $s$ of the divisor $C - C^0$, the genus $g$ and the prime number $p$.

2. Frobenius Morphism on the Second Hodge Bundle

Let $A$ be an $g$-dimensional Abelian variety defined over $\mathbb{F}_q$ with $p^r$ elements. Let $(F, H^1_{\text{cris}}(A))$ be the F-crystal associated with $A$. For a F-crystal one has the notion of Hodge polygon and Newton polygon (see [3] for the definitions). While the Hodge polygon of $(F, H^1_{\text{cris}}(A))$ remains the same for all $[A] \in M_{g,p}$, the Newton polygon of it does depend on the algebraic structure of $A$, which is actually equal to the Newton polygon of the characteristic polynomial $F^r$ on $H^1_{\text{cris}}(A)$. One knows that in general the Newton polygon lies on or above the Hodge polygon and both have the same end points. The two extreme cases are that $A$ is ordinary when the Newton polygon and the Hodge polygon coincides, and $A$ is supersingular when the Newton slopes are all equal.

Let $F_A : A \to A$ be the absolute Frobenius morphism defined by raising power $p$ on the structure sheaf of rings of $A$. It induces in turn the morphism

$$F^*_A : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A).$$

It is known that $A$ is ordinary if and only if $F^*_A$ is an isomorphism. On the other hand, we have the following simple

**Lemma 2.1.** Let $A$ be an $g$-dimensional Abelian variety defined over $k$. The Newton slopes of $A$ are all positive if and only if the morphism

$$(F^*_A)^g : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A)$$

is zero map.

**Proof:** Let $(F, H^1_{\text{cris}}(A, W(k)))$ be the F-crystal of $A$. It is known that the $k$-vector space $H^1(A, \mathcal{O}_A)$ is a natural quotient of $H^1_{\text{cris}}(A, W(k))$ modulo $p$, and
the map $F$ modulo $p$ induces a natural morphism on $H^1(A, \mathcal{O}_A)$, which is identical to $F_A^*$. By Katz 1.3.3 \[3\], all Newton slopes are positive if and only if $F^{2g}(H^1_{\text{crys}}(A, W(k))) \subset pH^1_{\text{crys}}(A, W(k))$, which implies in particular

$$(F_A^*)^{2g} : H^1(A, \mathcal{O}_A) \to H^1(A, \mathcal{O}_A)$$

is zero map. But since $\dim_k H^1(A, \mathcal{O}_A) = g$, the $g$ times iterate $(F^*_A)^g$ is already zero. Conversely, if $(F_A^*)^g$ is a zero morphism, then

$$(F_A^*)^{g+1} : H^1_{\text{dR}}(A/k) \to H^1_{\text{dR}}(A/k)$$

is zero. Since $2g \geq g + 1$, it implies that

$$F^{2g}(H^1_{\text{crys}}(A, W(k))) \subset pH^1_{\text{crys}}(A, W(k)).$$

Then still by Katz’s remark, the Newton slopes of $H^1_{\text{crys}}(A, W(k))$ are all positive. \hfill $\square$

Now we begin to consider the family situation. Let $f : X \to C$ over $k$ be a semi-stable family of $g$-dimensional Abelian varieties over a smooth projective base curve with nontrivial Kodaira-Spencer map. Let $S$ be the singular locus and $\Delta = f^{-1}(S)$ the simple normal crossing divisor of $X$ consisting of all semi-stable singular fibers of $f$. We introduce as usual the Hodge bundles of $f$:

$$E^{1,0} = f_*\Omega^1_{X/C}(\log \Delta), \quad E^{0,1} = R^1f_*\mathcal{O}_X.$$  

In characteristic $p$, we have the following commutative diagram induced by the absolute Frobenius morphisms of the base $C$ and the total space $X$:

$$
\begin{array}{ccc}
X & \xrightarrow{F_{\text{rel}}} & X' \\
\downarrow f & & \downarrow f' \\
\downarrow f & & \downarrow f \\
C & \xrightarrow{F_C} & C
\end{array}
$$

where $f' : X' \to C$ is the base change of $f$ under $F_C$ and $F_{\text{rel}} : X \to X'$ is the relative Frobenius morphism.
Lemma 2.2. The relative Frobenius morphism $F_{\text{rel}}$ induces a natural morphism of $\mathcal{O}_C$-modules

$$F_{\text{rel}}^*: F_C^* E^{0,1} \to E^{0,1}.$$  

Proof: The relative Frobenius morphism $F_{\text{rel}}$ gives the $\mathcal{O}_{X'}$-morphism of sheaves

$$\mathcal{O}_{X'} \to F_{\text{rel}*} \mathcal{O}_X.$$  

It induces the morphism on the direct images

$$R^1 f_{\ast}^* \mathcal{O}_{X'} \to R^1 f_{\ast}^* (F_{\text{rel}*} \mathcal{O}_X).$$  

The target sheaf has the natural morphism by spectral sequence

$$R^1 f_{\ast}^* (F_{\text{rel}*} \mathcal{O}_X) \to R^1 (f' \circ F_{\text{rel}})_\ast \mathcal{O}_X.$$  

Composition of the above two morphism yields the morphism

$$R^1 f_{\ast}^* \mathcal{O}_{X'} \to R^1 f_{\ast} \mathcal{O}_X.$$  

Finally, by the flat base change theorem(cf. Proposition 9.3 [2]) the source sheaf $R^1 f_{\ast}^* \mathcal{O}_{X'}$ is isomorphic to $F_C^* R^1 f_{\ast} \mathcal{O}_X$. The claim then follows.

\[\Box\]

3. An Arakelov-Yau Bound of Hodge Bundles for a Semi-stable Family of Algebraic Curves

Let $f : X \to C$ be a semi-stable family of algebraic curves of genus $g \geq 2$ with nontrivial Kodaira-Spencer map, where $C$ is a smooth projective algebraic curves over $k$. The non-triviality of Kodaira-Spencer map is equivalent to saying that the family $f$ is non-isotrivial and it is not the base change of another family $f' : X' \to C$ under Frobenius map $F_C$. Let $S$ be the singular locus. We have the following Arakelov-Yau bound of the slope of a coherent subsheaf in Hodge bundles:
Proposition 3.1. Let $f : X \to C$ be a semi-stable family of algebraic curves over $k$ with nontrivial Kodaira-Spencer map. For a coherent subsheaf $\mathcal{E}$ of $f_*\omega_{X/C}^{\otimes \nu}$, $\nu \geq 1$ we have the upper bound of the slope of $\mathcal{E}$:

$$\mu(\mathcal{E}) \leq 2\nu g(2b - 2 + s)$$

where $g$ is the genus of a general fiber, $b$ is the genus of the base curve $C$ and $s = |S|$ the number of the singular fibers. For a coherent subsheaf $\mathcal{F}$ of $R^1 f_*\mathcal{O}_X$ we have the upper bound of the slope of $\mathcal{F}$:

$$\mu(\mathcal{F}) \leq 2g(g - 1)(2b - 2 + s).$$

Proof: We start with the Harder-Narasimhan filtration of $f_*\omega_{X/C}^{\otimes \nu}$:

$$f_*\omega_{X/C}^{\otimes \nu} = \mathcal{E}_m \supseteq \mathcal{E}_{m-1} \supseteq \cdots \supseteq \mathcal{E}_1 \supseteq \mathcal{E}_0 = 0.$$  

It suffices to show the first inequality for $\mathcal{E}_1$.

One notes first that the image sheaf of the natural sheaf morphism $\alpha : f^*\mathcal{E}_1 \to \omega_{X/C}^{\otimes \nu}$ can be expressed by $\mathcal{I}_Z(\nu K_{X/C} - D)$, where $\mathcal{I}_Z$ is the ideal sheaf of a zero-dimensional subscheme $Z$, $K_{X/C}$ a relative canonical divisor and $D$ is an effective divisor. Now let $H$ be an ample divisor of $X$. Since $\omega_{X/C}$ is big and nef by Szpiro (see Theorem 1 and Proposition 3 in [6]), $m\nu K_{X/C} + H$ is ample for $m \geq 0$ by Nakai’s criterion (See Theorem 1 in [4]). By Bertini’s theorem, one can find a smooth curve

$$\Gamma \in |n(m\nu K_{X/C} + H)|$$

for $n$ large enough such that $\Gamma$ intersects transversally with a fixed smooth closed fiber $F$ of $f$, and $\Gamma$ is not contained in the support the kernel of $\alpha$. We put $\pi = f|_{\Gamma}$. By construction, $\pi$ is a separable finite morphism and the restriction of $\alpha$ to $\Gamma$

$$\alpha|_{\Gamma} : \pi^*\mathcal{E}_1 \to \mathcal{O}_\Gamma(\nu K_{X/C} - D)$$

is non-trivial. Because $\mathcal{E}_1$ is semi-stable and $\pi$ is separable, $\pi^*\mathcal{E}_1$ is still semi-stable. Hence we have

$$\mu(\pi^*(\mathcal{E}_1)) \leq \text{deg} \mathcal{O}_\Gamma(\nu K_{X/C} - D) = (\nu K_{X/C} - D)\Gamma.$$
We put \( N = \nu K_{X/C} - D - \mu(\mathcal{E}_1)F \). Then
\[
N \Gamma = (\nu K_{X/C} - D) \Gamma - \mu(\mathcal{E}_1)F \Gamma
\]
\[
= (\nu K_{X/C} - D) \Gamma - \mu(\pi^*(\mathcal{E}_1))
\]
\[
\geq 0.
\]

It implies that \( N(m\nu K_{X/C} + H) \geq 0 \). Since \( m \) can be arbitrarily large, we must have \( NK_{X/C} \geq 0 \). Therefore, we see that
\[
\nu K_{X/C}^2 = (N + D + \mu(\mathcal{E}_1)F)K_{X/C}
\]
\[
\geq \mu(\mathcal{E}_1)FK_{X/C}
\]
\[
= (2g - 2)\mu(\mathcal{E}_1).
\]

Now we recall the Szpiro’s inequality (see Proposition 4.2 in [6])
\[
K_{X/C}^2 < (4g - 4)g(2b - 2 + s).
\]

By combining the above two inequalities, we obtain the first inequality
\[
\mu(\mathcal{E}_1) \leq 2g\nu(2b - 2 + s).
\]

Now we let \( \nu = 1 \) and note that for a semi-stable family \( f_*\omega_{X/C} \) is simply the first Hodge bundle \( E^{1,0} \) as introduced in the last section. By the relative Serre duality one has the isomorphism
\[
E^{1,0} \simeq (E^{0,1})^*.
\]

So for a given coherent subsheaf \( \mathcal{F} \) of \( R^1f_*\mathcal{O}_X = E^{0,1} \) of rank \( r \), the dual of the quotient sheaf \( E^{0,1}/\mathcal{F} \) is a coherent subsheaf of \( E^{1,0} \). Furthermore, because the family \( f \) is non-isotrivial, \( \deg E^{1,0} > 0 \)(cf. Proposition 3 [6]). It follows that
\[
\deg(\mathcal{F}) = -\deg E^{1,0} + \deg (E^{0,1}/\mathcal{F})^*
\]
\[
\leq \deg (E^{0,1}/\mathcal{F})^*
\]
\[
\leq 2g(g - r)(2b - 2 + s).
\]
Therefore
\[ \mu(\mathcal{F}) \leq 2g(\frac{g-r}{r})(2b-2+s) \leq 2g(g-1)(2b-2+s). \]

The proof is completed. \(\square\)

**Remark 3.2.** In the above proposition we have a better upper bound of the slopes of coherent subsheaves in \(R^1f_*\mathcal{O}_X\) in the case that \(f\) contains an ordinary closed fiber. Namely, for \(\mathcal{F} \subset R^1f_*\mathcal{O}_X\) any coherent subsheaf, it holds
\[ \mu(\mathcal{F}) \leq 0. \]

This is because otherwise the image of \(F^*_c\mathcal{F}\) under \(F^*_\text{rel}\) with \(\mu(\mathcal{F})\) positive and maximal has larger slope than \(\mu(\mathcal{F})\).

Thanks to the main theorem of X. Sun in [3], we are also able to obtain an upper bound of the slope of a coherent subsheaf in \((F^*_c)^nE_{i,1}^{i,i}\) for \(i = 0, 1, n \geq 1\). To our purpose we shall state only the statement for that of \(E^{0,1}\).

**Proposition 3.3.** Let \(f : X \to C\) be a semi-stable family of algebraic curves over \(k\) with nontrivial Kodaira-Spencer map. Let \(\mathcal{F}\) be a coherent subsheaf of \((F^*_c)^nR^1f_*\mathcal{O}_X\) for \(n \geq 1\). Then the following inequality holds
\[ \mu(\mathcal{F}) \leq 2g(g-1)(2b-2+s)p^n + 2(g-1)(b-1). \]

**Proof:** For brevity we denote the \(n\)-th iterates \((F^*_c)^n\) of the Frobenius morphism \(F^*_c\) by \(F^*_n\). Let \(\mathcal{F}\) be a coherent subsheaf of \(F^*_nE^{0,1}\). We claim that
\[ \mu(\mathcal{F}) \leq (g-1)(2b-2) + p^n\mu_{\text{max}}(E^{0,1}). \]

where \(\mu_{\text{max}}(E^{0,1})\) is the maximal slope of the coherent subsheaves contained in \(E^{0,1}\). We consider the Harder-Narasimhan filtration of \(E^{0,1}\):
\[ E^{0,1} = \mathcal{E}_m \supseteq \mathcal{E}_{m-1} \supseteq \cdots \supseteq \mathcal{E}_1 \supseteq \mathcal{E}_0 = 0. \]
One has a natural map $\alpha : \mathcal{F} \to F_n^*\mathcal{E}_m/F_n^*\mathcal{E}_{m-1}$ which is the composition of the morphisms

$$\mathcal{F} \hookrightarrow F_n^*\mathcal{E}_m \to F_n^*\mathcal{E}_m/F_n^*\mathcal{E}_{m-1}.$$ 

The kernel of $\alpha$ is denoted by $\mathcal{K}$.

Theorem 3.1 of X. Sun in [5] says that

$$\mu_{\max}(F_n^*(\mathcal{E}_m/\mathcal{E}_{m-1})) - \mu_{\min}(F_n^*(\mathcal{E}_m/\mathcal{E}_{m-1})) \leq (g - 1)(2b - 2).$$

Since one has

$$\mu(\mathcal{F}/\mathcal{K}) \leq \mu_{\max}(F_n^*\mathcal{E}_m/F_n^*\mathcal{E}_{m-1}) = \mu_{\max}(F_n^*(\mathcal{E}_m/\mathcal{E}_{m-1}))$$

and

$$\mu_{\min}(F_n^*(\mathcal{E}_m/\mathcal{E}_{m-1})) \leq \mu(F_n^*(\mathcal{E}_m/\mathcal{E}_{m-1})) = p^n\mu(\mathcal{E}_m/\mathcal{E}_{m-1}),$$

we get

$$\mu(\mathcal{F}/\mathcal{K}) \leq (g - 1)(2b - 2) + p^n\mu(\mathcal{E}_m/\mathcal{E}_{m-1}) \leq (g - 1)(2b - 2) + p^n\mu_{\max}(\mathcal{E}^{0,1}).$$

One notes that $\mathcal{K}$ is a coherent subsheaf of $F_n^*\mathcal{E}_{m-1}$, and in case that $\mathcal{K}$ is not zero sheaf one can study further the projection of $\mathcal{K}$ into the quotient sheaf $F_n^*\mathcal{E}_{m-1}/F_n^*\mathcal{E}_{m-2}$. Because for all $1 \leq i \leq m$ one has the inequality

$$\mu(F_n^*\mathcal{E}_i/F_n^*\mathcal{E}_{i-1}) \leq p^n\mu_{\max}(\mathcal{E}^{0,1}),$$

we can assume that

$$\mu(\mathcal{K}) \leq (g - 1)(2b - 2) + p^n\mu_{\max}(\mathcal{E}^{0,1}).$$

by an inductive argument. Actually, one has

$$\deg(\mathcal{F}) = \deg \mathcal{F}/\mathcal{K} + \deg \mathcal{K} \leq ((g - 1)(2b - 2) + p^n\mu_{\max}(\mathcal{E}^{0,1}))(\text{rank}(\mathcal{F}/\mathcal{K}) + \text{rank}(\mathcal{K})) = ((g - 1)(2b - 2) + p^n\mu_{\max}(\mathcal{E}^{0,1}))\text{rank}(\mathcal{F}),$$

hence it follows that $\mu(\mathcal{F})$ satisfies the claimed inequality. Now we combine the inequality in Proposition 3.1 and thus obtain the inequality in the proposition.
4. Main Result and The Proof

Let $f : X \to C$ a semi-stable family of algebraic curves of genus $g \geq 1$ with nontrivial Kodaira-Spencer map, where $C$ is a smooth projective algebraic curves over $k$. Let $X_t$ be a smooth closed fiber of $f$. It is known that the F-crystal $H^{1\text{cris}}(X_t)$ is canonically isomorphic to the F-crystal $H^{1\text{cris}}(\text{Jac}(X_t))$ of its Jacobian. We assume that for a general $t \in C(k)$ the first Newton slope of $X_t$ is not positive. In this section we shall give an upper bound of the cardinality of the special locus in $C(k)$.

We proceed by considering first a relatively simple situation. Namely, we assume first that there exists an ordinary closed fiber in $f$. In this case our result is to provide an upper bound of the cardinality of Hasse locus of the family $f$. Since the special locus is clearly contained in the Hasse locus, the result gives also an upper bound for that of special locus. Precisely, we have the following

**Theorem 4.1.** Let $f : X \to C$ be a semi-stable family of algebraic curves of genus $g \geq 1$ over smooth projective algebraic curves $C$ of genus $b$ defined over $k$. We assume that the Kodaira-Spencer map of $f$ is non-trivial. Let $s$ be the number of singular fibers of $f$. If there is an ordinary closed fiber in $f$, then we have the following inequality for the degree of the Hasse locus of $f$:

$$\deg H(f) \leq 2g^2(p - 1)(2b - 2 + s).$$

where $H(f)$ is the Hasse locus of $f$.

**Proof:** We note first that in the $g = 1$ case the assumption about the existence of an ordinary fiber is void, because it is classically known that there exists only finitely many supersingular points in the $j$-line of elliptic curves over $k$. 
By taking the wedge $g$ power of $F^*_{rel}$, we obtain a non-trivial morphism of invertible sheaves over $C$:

$$\det(F^*_{rel}) : \bigwedge^g(F^*_C(E^{0,1})) \to \bigwedge^g(E^{0,1}).$$

Hence it must be injective and we have an short exact sequence of coherent sheaves over $C$:

$$0 \to \bigwedge^g(F^*_C(E^{0,1})) \xrightarrow{\det(F^*_{rel})} \bigwedge^g(E^{0,1}) \to Q \to 0$$

where $Q$ is a torsion sheaf. Obviously, $\deg H(f) \leq \deg Q$.

On the other hand, we have the Arakelov-Yau bound in Proposition 3.1 for $g \geq 2$ fibration:

$$\deg(E^{1,0}) \leq 2g^2(2b - 2 + s).$$

The above inequality holds also for elliptic fibration. Actually, the Kodaira-Spencer map induces a non-trivial morphism

$$\theta : E^{1,0} \to E^{0,1} \otimes \Omega_C(S).$$

Then there is an embedding

$$(E^{1,0})^\otimes 2 \to \Omega_C(S).$$

This implies that

$$\deg E^{1,0} \leq \frac{1}{2}(2b - 2 + s).$$

Therefore it follows that

$$\deg H(f) \leq \deg(Q)$$

$$= \deg(E^{0,1}) - \deg(F^*_C(E^{0,1}))$$

$$= (p - 1) \deg(E^{1,0})$$

$$\leq 2(p - 1)g^2(2b - 2 + s).$$

When $g = 1$, the bound can be replaced by $\frac{p-1}{2}(2b - 2 + s)$.
When the family $f$ does not contain an ordinary fiber, the morphism $F^*_\text{rel} : F_C^*E^{0,1} \to E^{0,1}$ is not an isomorphism at the generic point. So the above proof will not apply in a general situation. However, we still have an upper bound of the cardinality of the special locus of $f$, which is the main result of this paper.

**Theorem 4.2.** Let $f : X \to C$ be a semi-stable family of algebraic curves of genus $g \geq 2$ over smooth projective algebraic curves $C$ over $k$. We assume that the Kodaira-Spencer map of $f$ is non-trivial. The notations is as those in Theorem 4.1. If the Newton slopes of a general closed fiber of $f$ are not all positive, then the cardinality of the special locus of $f$ is bounded above by the function

$$P(p, g, b, s) = p^g[2g(g^2 - g + 1)(2b - 2 + s)] + 2(g - 1)(3g - 1)(b - 1) + 2g(g - 1)s.$$

**Proof:** We consider the composition of the following morphisms:

$$(F_C^*)^gE^{0,1} \xrightarrow{(F_C^*)^{g-1}F^*_\text{rel}} (F_C^*)^{g-1}E^{0,1} \xrightarrow{(F_C^*)^{g-2}F^*_\text{rel}} \cdots \xrightarrow{F^*_\text{rel}} E^{0,1},$$

and we denote it by $\Phi : (F_C^*E^{0,1} \to E^{0,1})$. By Lemma 2.1, the morphism $\Phi$ is not zero map under the assumption about $f$. Thus we obtain the factorization of $\Phi$

$$(F_C^*E^{0,1} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\phi} \mathcal{E} \xrightarrow{\phi} E^{0,1})$$

such that $\phi$ is an isomorphism at the generic point.

Then we have a short exact sequence of coherent sheaves over $C$

$$0 \to \text{det } \mathcal{F} \xrightarrow{\text{det } \phi} \text{det } \mathcal{E} \to Q \to 0$$

where $Q$ is a torsion sheaf. Again applying Lemma 2.1 we know that the special locus of $f$ is contained in the support of $Q$. Now the estimation of $\deg Q$ goes as in the proof of Theorem 4.1.

$$\deg Q = \deg \mathcal{E} - \deg \mathcal{F}$$

$$= \deg \mathcal{E} - \deg ((F_C^*E^{0,1}) + \deg(\ker \Phi))$$

$$= \deg \mathcal{E} + \deg(\ker \Phi) + p^g \deg(E^{1,0})$$

$$\leq p^g[2g(g^2 - g + 1)(2b - 2 + s)] + 2(g - 1)(3g - 1)(b - 1) + 2g(g - 1)s.$$
Theorem is therefore proved.

\[ \square \]

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