Choice of Gauge in Quantum Gravity

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Abstract

This paper is an extended version of the talk given at 19th Texas Symposium of Relativistic Astrophysics and Cosmology, Paris, 1998. It reviews some recent work; mathematical details are skipped. It is well-known that a choice of gauge in generally covariant models has a twofold purpose: not only to render the dynamics unique, but also to define the spacetime points. A geometric way of choosing gauge that is not based on coordinate conditions—the so-called covariant gauge fixing—is described. After a covariant gauge fixing, the dynamics is unique and the background manifold points are well-defined, but the description remains invariant with respect to all diffeomorphisms of the background manifold. Transformations between different covariant gauge fixings form the well-known Bergmann-Komar group. Each covariant gauge fixing determines a so-called Kuchař decomposition. The construction of the quantum theory is based on the Kuchař form of the action and the Dirac method of operator constraints. It is demonstrated that the Bergmann-Komar group is too large to be implementable by unitary maps in the quantum domain.
1 Introduction

In this paper, we are considering a broad class of diffeomorphism invariant models similar to general relativity. Thus, the dynamical equations will be generally covariant and the spacetime will be dynamical.

In such a situation, one might be tempted to view the system as a dynamics of some fields and objects on a naked manifold, the manifold consisting of well-defined and distinguishable points. This point of view is, however, afflicted with well-known difficulties and paradoxes. One old example of such difficulties is Einstein’s ‘hole’ argument [1], whereas a more recent example is due to Fredenhagen and Haag [2]. The way out of the difficulties was known already to Einstein [1]: spacetime points can only be defined and distinguished by values of physical fields or positions of physical objects.

This principle is not spectacular if one does not leave the realm of a fixed classical solution, but it is rather awkward from the point of view of the whole dynamics of the model, especially if one is interested in its quantization. Various methods can be found in the literature that help to circumvent the problem. One is a WKB expansion around a (classical) solution; this enables one to define the spacetime points by means of the classical metric and fields of the solution similarly as it is done in Minkowski spacetime. Of course, this method will work only if the WKB approximation is applicable. Another method is to add some material that breaks the diffeomorphism invariance; this way has been quite systematically explored by Kuchař [3]. Finally, one can, so to speak, fasten the coordinates to particular bumps of the fields in the model, that is, one chooses a gauge. This last method seems to be, unlike the first one, generally applicable, and it does not, like the second one, violate the diffeomorphism (gauge) invariance, provided one can prove that the measurable results are independent of the gauge choice.

In the present paper, we shall concentrate on the last method and study the question of how much it can be used in the quantum theory of the generally covariant models. Our results will suggest that quantum theories constructed on the basis of different gauges are not unitarily equivalent. The reason is that their gauge group is huge; it is not just the diffeomorphism group of one manifold, but a cartesian product of diffeomorphism groups, one group for the spacetime of each solution; this is the well-known Bergmann-Komar group [4].

2 Covariant Gauge Fixing

In this section, we explain the origin of the Bergmann-Komar group and sketch the idea of our definition of the covariant gauge fixing.

It is clear that each fixed classical spacetime solution defines a nice spacetime; problems can only arise if one considers more than one solution. To see this, we study a simplified example: the family of Schwarzschild spacetimes. Let us first
consider the metric in the Eddington-Finkelstein coordinates:

\[ ds^2 = -\left(1 - \frac{2M}{R}\right)dW^2 + 2dWdR + R^2d\Omega^2, \] (1)

where \(d\Omega^2\) is the metric of the unit sphere in the spherical coordinates \(\vartheta \in (0, \pi)\) and \(\varphi \in (0, 2\pi)\); \(W \in (-\infty, \infty)\) and \(R \in (0, \infty)\). \(M\) is the Schwarzschild mass: For each value of \(M \in (0, \infty)\), there is one spacetime manifold and a metric on it. Eq. (1) can also be interpreted as a family of metric fields on a fixed background manifold \(\mathcal{M}_{EF}\); this is simply the manifold \(\mathbb{R}^2 \times S^2\) covered by the coordinates \(W, R, \vartheta\) and \(\varphi\) with the above ranges.

Another possibility is to choose the Kruskal coordinates. Then, the metric has the form

\[ ds^2 = -\frac{16M^2}{\kappa(-UV)}e^{-\kappa(-UV)}dUdV + 4M^2(\kappa(-UV))^2d\Omega^2, \] (2)

where \(\kappa : (-1, \infty) \mapsto (0, \infty)\) is the well-known Kruskal function defined by its inverse, \(\kappa^{-1}(x) := (x - 1)e^x\) for all \(x \in (0, \infty)\); \(U \in (-\infty, \infty)\) and \(V \in (0, \infty)\). Again, we can interpret Eq. (2) as a family of metric fields on a background manifold \(\mathcal{M}_K\); it is \(\mathbb{R}^2 \times S^2\) with coordinates \(U, V, \vartheta\) and \(\varphi\) in the above ranges.

The crucial observation is that the transformation between the Eddington-Finkelstein and Kruskal coordinates,

\[ U = \left(\frac{R}{2M} - 1\right)e^{R/2M}e^{-W/4M}, \]
\[ V = e^{W/4M}, \]

is not a map from \(\mathcal{M}_{EF}\) to \(\mathcal{M}_K\) because it depends on \(M\). The conclusion from this observation is that the background manifold is not defined by any unique and natural way, but by some non-trivial method of identifying all solution manifolds. In our case, we have introduced a particular geometric coordinate system in each solution manifold, and then identified those points of different solution manifolds that have the same values of these coordinates. The choice of particular geometric coordinates in all solutions is usually a consequence of a gauge choice (or coordinate condition). Hence, it is the gauge fixing that defines points of a background manifold in general relativity. Moreover, a transformation between two different gauge fixings is not a transformation of coordinates on the background manifold, but a whole family of coordinate transformations, one for each solution. That is the origin of Bergmann-Komar group.

Turning to a general case, we need some additional notions. Our description of a model is based on the canonical formalism. Each model possesses an extended phase space \(\Gamma'\) that contains a constraint surface \(\Gamma\). Each point of \(\Gamma\) represents a possible initial datum for the dynamical equations of the model. Points at \(\Gamma\) that represent initial data of one and the same solution form a surface in \(\Gamma\) that we call \(c\)-orbit. The \(c\)-orbits are in one-to-one correspondence with classical solutions. The
Bergmann-Komar group acts only at $\Gamma$. It is generated by vector fields at $\Gamma$ that are tangential to c-orbits.

The discussion above of the Schwarzschild family seems to imply that it is not necessary to choose coordinates in order to define the background manifold; one can also directly identify the points of different solution spacetimes. An advantage thereof is that no solution has to be covered by just one chart; another one is that everything can be done in a way covariant with respect to the coordinate transformation on the resulting background manifold. A crucial condition for such a construction to work is the absence of the points at the constraint surface that correspond to spacetime solutions with any symmetry (even a discrete one). We cut these points out; then there is a one-to-one relation between remaining points and Cauchy surfaces in the solutions. Suppose that, for each solution, there is a smooth injection sending this solution into a fixed background manifold. Then, for each point of the constraint surface, there is a unique embedding of the Cauchy manifold into the background manifold: these embeddings are made to functions on the constraint surface. One can then show that the embeddings together with any complete system of functions that are constant along the c-orbits form a coordinate system at $\Gamma$. The set of the differentiable injections described above is called covariant gauge fixing.

Such a construction has been performed in detail for a system that can be called “extended shell model” [5]. It consists of a null-dust thin shell surrounded by its own gravitational field; everything is spherically symmetric. This is a system of one degree of freedom (the radius of the shell, say), and one usually reduces the action correspondingly, see, e.g. [6]. The word “extended” in the name of the model refers to the spacetime outside. Such extension is necessary in order that a generally covariant model with a dynamical spacetime results.

An important question about covariant gauge fixing is that of its global existence. This existence has been shown as yet for the following models: the minisuperspace cosmological models, 2+1 gravity [7], the cylindrical waves [8], the Schwarzschild family [9], a dilatonic model [10], and the extended shell model [5]. For general relativity, the existence may be a problem. However, if a global covariant gauge fixing does not exist for some model, then the model cannot be reformulated as a field theory on a background manifold even in its classical version and one does not need to pass to quantum theory in order to prove this negative result.

3 The Kuchař Decomposition

The Kuchař decomposition will play an important role in our argument. Let us briefly introduce it using the Hamiltonian formulation of general relativity.

The Hamiltonian formalism starts with the so-called ADM action:

$$ S = \int dt \int_{\Sigma} d^3x \left[ \pi^{kl}(x) \dot{q}_{kl}(x) - \mathcal{N}(x) \mathcal{H}(x) - \mathcal{N}^k(x) \mathcal{H}_k(x) \right], $$

where $\Sigma$ is a three-dimensional initial value manifold (Cauchy surface), the pair of fields $(q_{kl}(x), \pi^{kl}(x))$ on $\Sigma$ determines a point of $\Gamma'$, $\mathcal{N}(x)$ and $\mathcal{N}^k(x)$ are Lagrange
multipliers and $\mathcal{H}(x)$ and $\mathcal{H}^k(x)$ are the constraints; the constraints are functionals of the fields $q_{kl}(x)$ and $\pi^{kl}(x)$. $\Gamma$ is determined by the constraint equations $\mathcal{H}(x) = 0$ and $\mathcal{H}^k(x) = 0$.

From the above action, it follows that the dynamics is generated by the constraints; the Lagrange multipliers can be interpreted as components of the vector fields determining the direction in which the dynamics proceeds (for details, see [11]). The functionals $F[q_{kl}(x), \pi^{kl}(x)]$ that satisfy the conditions

$$\{ F, \mathcal{H} \}|_\Gamma = 0, \quad \{ F, \mathcal{H}_k \}|_\Gamma = 0,$$

are gauge invariants and simultaneously integrals of motion, so they are constant along the $c$-orbits; we call them perennials.

Kuchař observed (for some simplified cases) [8, 12] that there is a canonical transformation from the variables $(q_{kl}(x), \pi^{kl}(x))$ to new variables $(X^\mu(x), P_\mu(x), q_\alpha, p^\alpha)$ such that the action becomes

$$S = \int dt \left[ \int d^3 x \left( P_\mu(x) \dot{X}^\mu(x) - \mathcal{N}^\mu(x) P_\mu(x) \right) + \sum_\alpha p_\alpha \dot{q}_\alpha \right], \quad (3)$$

where $X^\mu(x)$ is a coordinate description of an embedding of the Cauchy surface $\Sigma$ into some background manifold $\mathcal{M}$, $X^\mu$ being some coordinates on $\mathcal{M}$, $x^k$ those on $\Sigma$, and $(q^\alpha, p_\alpha)$ represent some complete system of perennials; the index $\alpha$ can run through discrete and/or continuous ranges, and the sum in the action is, therefore, to be understood as a kind of Stieltjes integral.

The new variables can be cleanly split into pure kinematical variables $X^\mu(x)$ and $P_\mu(x)$ on one hand, and true dynamical variables $q_\alpha$ and $p^\alpha$ on the other—this is what we call Kuchař decomposition.

Each value of the pair $(q^\alpha, p_\alpha)$ from some range determines a unique solution; the metric of the solution can be written as a $(q^\alpha, p_\alpha)$-dependent metric field on the background manifold $\mathcal{M}$:

$$ds^2 = g_{\mu\nu}(q, p; X)dX^\mu dX^\nu. \quad (4)$$

This metric appeared first in [10] and we call it Kuchař-Romano-Varadarajan (KRV) metric. Every Kuchař decomposition must clearly be associated with some gauge fixing.

## 4 Extension Theorem

In Sec. 4 we have seen that a covariant gauge fixing leads to a definition of the coordinates $X^\mu(x)$, $q^\alpha$ and $p_\alpha$ on the constraint surface $\Gamma$. According to the previous section, this is a Kuchař decomposition at $\Gamma$. In the present section, we show that there is a Kuchař decomposition in $\Gamma'$ for each covariant gauge fixing.

One would like that the following properties hold:
1. The functions $X^\mu(x)$, $q^\alpha$ and $p_\alpha$ as constructed by a covariant gauge fixing in Sec. 2 can be extended to a neighbourhood $U$ of $\Gamma$ in $\Gamma'$ so that their Poisson brackets in $U$ are

$$\{X^\mu(x), X^\nu(y)\} = 0, \quad \forall \mu, \nu, x, y,$$

$$\{X^\mu(x), q^\alpha\} = \{X^\mu(x), p_\alpha\} = 0, \quad \forall \mu, x, \alpha,$$

$$\{q^\alpha, p_\beta\} = \delta^\alpha_\beta, \quad \{q^\alpha, q^\beta\} = \{p_\alpha, p_\beta\} = 0, \quad \forall \alpha, \beta.$$

2. There are functions $P_\mu(x)$ in $U$ such that the equations $P_\mu(x) = 0$ define $\Gamma$, and such that

$$\{P_\mu(x), P_\nu(y)\} = 0, \quad \forall \mu, \nu, x, y,$$

$$\{P_\mu(x), q^\alpha\} = \{P_\mu(x), p_\alpha\} = 0, \quad \forall \mu, x, \alpha,$$

and

$$\{X^\mu(x), P_\nu(y)\} = \delta^\mu_\nu \delta(x, y), \quad \forall \mu, \nu, x, y.$$

The proof of these properties is based on the Darboux-Weinstein theorem (see, e.g. [13]). A formal proof is easy; “formal” means that the subtleties of infinite-dimensional spaces are neglected. A full (rigorous) proof is given for the extended shell model [8]; it uses the ideas about weak symplectic forms and the associated weak metrics as described in [14]. Extension of the proof to other models seems to be straightforward, if some quite plausible assumptions about the submanifold structure of the constraint set $\Gamma$ and of the quotient manifold structure of the set $\Gamma/c$-orbits (true degrees of freedom) are satisfied.

The theorem is a pure existence theorem, even if its proof provides a method of how the extensions of $X^\mu(x)$, $p_\alpha$ and $q^\alpha$ out of the constraint surface can be constructed, at least in principle; the method is practically viable only in very simple cases. Still, the result is useful because no explicit knowledge of the extension is, in fact, needed for the construction of the quantum theory. An explicit calculation of the extension by any known method is very difficult. Kuchař and his collaborators managed to find such extensions only in the cases of cylindrical gravitational waves [8], spherically symmetric vacuum gravitational field [9], and a dilatonic model of gravitational collapse [10].

The extensions guaranteed by the theorem enable us to extend the covariant gauge fixing and the Bergmann-Komar group out of the constraint surface. This, however, is not really necessary. Further, it is even not unique, because the extensions are not (this follows from the proof). An important property of the Bergmann-Komar transformation is that it does not change the values of the perennials $q^\alpha$ and $p_\alpha$ at the constraint surface $\Gamma$; as it is well-known, the Poisson algebra of perennials is determined by their restrictions to $\Gamma$ [14], so this algebra is also gauge invariant.
5 Quantum Theory

The construction of the quantum theory based on the action in the Kuchař form \( \mathcal{A} \) and the Dirac method of operator constraints is rather straightforward and we shall only briefly sketch it.

The states are described by wave functions \( \Psi(X, q) \) of the embeddings \( X^\mu(x) \) and the perennials \( q^\alpha \). The functional Schrödinger equation (see, e.g. [15]) then reads:

\[
\hat{P}_\mu(x)\Psi = \frac{1}{i} \frac{\delta \Psi}{\delta X^\mu(x)} = 0.
\]

Hence, physical state \( \Psi \) is simply independent of the embeddings.

The resulting quantum theory possesses what we can call a gauge invariant core. This consists of (i) the states described by wave functions \( \Psi(q) \), (ii) the scalar product,

\[
(\Psi, \Phi) := \int d\mu(q) \Psi^*(q)\Phi(q),
\]

where \( d\mu(q) \) is some measure on the space with coordinates \( q^\alpha \), and (iii) the observables \( \hat{q}^\alpha \) and \( \hat{p}_\alpha \) defined by

\[
\hat{q}^\alpha\Psi(q) := q^\alpha\Psi(q), \quad \hat{p}_\alpha \Psi(q) := \frac{1}{i} \frac{\partial \Psi}{\partial q^\alpha}.
\]

This structure is completely independent of any choice of gauge (or covariant gauge fixing), and it is manifestly invariant with respect to the Bergmann-Komar group.

In principle, there is also additional information about the points and geometry of the spacetime. At least in the classical theory, the spacetime points could be defined by a covariant gauge fixing, and the KRV metric \( \mathcal{A} \) determined the corresponding geometric properties. In the classical theory, this description of the properties is gauge dependent, but the results for really observable properties are gauge independent. This part of the classical theory would correspond to the method of gauge fixing known from other gauge theories. For example the values of coordinates after a gauge has been fixed are, in principle, measurable quantities. In our quantum theory, the embeddings \( X^\mu(x) \) are trivial operators: they commute with every other variable and with each other (the conjugate quantities \( P_\mu(x) \) have been excluded from the quantum theory). This corresponds to the situation that is usual in the quantum field theory: the spacetime coordinates are just parameters. We assume that the KRV metric can be transferred to the quantum theory by some suitable factor ordering. The resulting operator of geometry \( \hat{g}_{\mu\nu}(\hat{q}, X) \) depends on the four parameters \( X^0, X^1, X^2 \) and \( X^3 \) and represents the “quantum geometry” at the point \((X^0, X^1, X^2, X^3)\) of the background manifold \( \mathcal{M} \). This formalism can be made manifestly invariant with respect to the group of diffeomorphisms of \( \mathcal{M} \), or with respect to any coordinate transformation \( X'^\mu = X'^\mu(X) \) on \( \mathcal{M} \), but not with respect to the full Bergmann-Komar group. In fact, we show that general Bergmann-Komar transformations cannot be unitarily implemented for our quantum theories.
Let us choose two different covariant gauge fixings; these lead to two different descriptions $X_1^\mu(x)$ and $X_2^\mu(x)$ of embeddings, and to two Kuchař decompositions. Using these decompositions, we can construct the corresponding quantum theories $QT_1$ and $QT_2$ by the method given above. Within the theory $QT_1$, $X_1^\mu$ are $c$-numbers, within $QT_2$, $X_2^\mu$ are. However, $X_2$ in $QT_1$ is given by the Bergmann-Komar transform:

$$X_2^\mu = X_2^\mu(X_1^\mu, q^\alpha, p_\alpha).$$

Hence, within $QT_1$, $X_2$ is a genuine operator; formally,

$$\hat{X}_2^\mu := X_2^\mu(X_1^\mu, \hat{q}^\alpha, \hat{p}_\alpha)$$

(we suppose that the corresponding factor ordering problem can be solved satisfactorily). Thus, $X_2$ is a $q$-number in $QT_1$ and a $c$-number in $QT_2$. If $QT_1$ and $QT_2$ are unitarily equivalent, then the unitary map between the corresponding Hilbert spaces must send the physical quantities with the same meaning into each other. For example, $\hat{X}_2^\mu$ of $QT_2$ is to be sent into $\hat{X}_2^\mu$ of $QT_1$. However, no unitary map can send $c$-numbers in $q$-numbers.

6 Conclusion

We have studied the symplectic structure of diffeomorphically invariant models, especially the aspects associated with gauge (that is, coordinate) choice. There were two main results.

The first one is the existence of the Kuchař decomposition for each covariant gauge fixing. The corresponding quantum theory also decomposes, namely into a gauge-invariant core in the form of a representation of an algebra of observables on one hand, and into the additional information about geometry containing the definition of background manifold and its points, and the quantum geometry at these points, on the other.

The second main result is that the information in the second part of the quantum theory depends of the gauge. This suggests that, if we insist on gauge invariance in quantum gravity, there may be no spacetime points, and that gauge fixing methods may be more precarious than, say, in quantum Yang-Mills field theories.

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