VANISHING OF THE TOP LOCAL COHOMOLOGY MODULES OVER NOETHERIAN RINGS

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ABSTRACT. Let $R$ be a (not necessarily local) Noetherian ring and $M$ a finitely generated $R$-module of finite dimension $d$. Let $a$ be an ideal of $R$ and $\mathfrak{m}$ denote the intersection of all prime ideals $p \in \text{Supp}_R H^d_a(M)$. It is shown that

$$H^d_a(M) \simeq H^d_{\mathfrak{m}}(M) / \sum_{n \in \mathbb{N}} < \mathfrak{m} > (0 : H^d_{\mathfrak{m}}(M) a^n),$$

where for an Artinian $R$-module $A$ we put $< \mathfrak{m} > A = \cap_{n \in \mathbb{N}} \mathfrak{m}^n A$. As a consequence, it is proved that for all ideals $a$ of $R$, there are only finitely many non-isomorphic top local cohomology modules $H^d_a(M)$ having the same support. In addition, we establish an analogue of the Lichtenbaum-Hartshorne Vanishing Theorem over rings that need not be local.

1. INTRODUCTION

Throughout this paper, let $R$ denote a commutative Noetherian ring. Let $M$ be a finitely generated $R$-module of finite dimension $d$ and $a$ an ideal of $R$. The present article is concerned with the top local cohomology module $H^d_a(M)$. We refer the reader to [3] for more details about local cohomology. By Grothendieck’s Vanishing Theorem [3, Theorem 6.1.2], it is known that $H^i_a(M) = 0$ for all $i > \text{dim} M$. So $H^d_a(M)$ is the last possible non-vanishing local cohomology module of $M$. Also, by [3, Exercise 7.1.7] the top local cohomology module $H^d_a(M)$ is Artinian. There are many papers concerning the top local cohomology modules of finitely generated modules over local rings. But, according to the best knowledge of the author, [2] and [4] are the only existing articles studying such local cohomology modules over general Noetherian rings. In this paper, we investigate the structure of the top local cohomology modules of finitely generated modules over rings that need not be local.

When $R$ is local with the maximal ideal $\mathfrak{m}$, it is proved that there is a natural isomorphism $H^d_a(M) \simeq H^d_{\mathfrak{m}}(M) / \sum_{n \in \mathbb{N}} < \mathfrak{m} > (0 : H^d_{\mathfrak{m}}(M) a^n)$, see [10, Theorem 3.2]. As a result, in [10] a new proof is provided for the Lichtenbaum-Hartshorne Vanishing Theorem. In

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Section 2, we establish an analogue of the above isomorphism over rings that are not necessarily local. To be more precise, we will prove that if \( M \) denotes the intersection of all prime ideals \( p \in \text{Supp}_R H^d_a(M) \), then there is a natural isomorphism
\[
H^d_a(M) \simeq H^d_{\mathfrak{m}}(M) / \sum_{n \in \mathbb{N}} < \mathfrak{m} > (0 : H^d_{\mathfrak{m}}(M) a^n).
\]
This will be proved in Theorem 2.3.

Knowing more about \( \text{Att}_R H^d_a(M) \), the set of attached primes of \( H^d_a(M) \), could lead to better understanding of the structure of the top local cohomology module \( H^d_a(M) \). In particular, knowing \( \text{Att}_R H^d_a(M) \) implies vanishing results for \( H^d_a(M) \). In the case \( R \) is local, the set \( \text{Att}_R H^d_a(M) \) is already determined (see e.g. [18], [10] and [6]). In Theorem 2.5 below, we determine the set \( \text{Att}_R H^d_a(M) \) without the assumption that \( R \) is local, namely we show that
\[
\text{Att}_R H^d_a(M) = \{ p \in \text{Assh}_R M : \text{cd}_R(a, R/p) = d \}
\]
(here for an \( R \)-module \( N \), \( \text{cd}_R(a, N) \) denotes the cohomological dimension of \( N \) with respect to the ideal \( a \)). Then as an application, we provide an improvement of the main result of [2]. Next, for a finitely generated \( R \)-module \( N \) so that \( H^d_c(N) \), \( c := \text{cd}_R(a, N) \), is representable, we examine the set \( \text{Att}_R H^d_c(N) \).

In Section 3, first we show that for all ideals \( a \) of \( R \), there are only finitely many non-isomorphic top local cohomology modules \( H^d_a(M) \) having the same support. Next, as an application of Theorems 2.3 and 2.5, we extend the Lichtenbaum-Hartshorne Vanishing Theorem to (not necessarily local) Noetherian rings. Namely, we prove that if \( \mathfrak{m} \) is as above and \( T \) denotes the \( \mathfrak{m} \)-adic completion of \( R \), then the following are equivalent:

i) \( H^d_a(M) = 0 \).

ii) \( H^d_{\mathfrak{m}}(M) = \sum_{n \in \mathbb{N}} < \mathfrak{m} > (0 : H^d_{\mathfrak{m}}(M) a^n) \).

iii) For any integer \( l \in \mathbb{N} \), there exists an \( n = n(l) \in \mathbb{N} \) such that
\[
0 : H^d_{\mathfrak{m}}(M) a^l \subseteq < \mathfrak{m} > (0 : H^d_{\mathfrak{m}}(M) a^n).
\]

iv) \( \dim T / aT + p > 0 \) for all \( p \in \text{Assh}_T(M \otimes_R T) \).

v) \( \text{cd}_R(a, R/p) < d \) for all \( p \in \text{Assh}_R M \).

Throughout the paper, for an \( R \)-module \( M \), \( \text{Assh}_R M \) denotes the set of all associated prime ideals \( p \) of \( M \) such that \( \dim R / p = \dim M \). Also, for an Artinian \( R \)-module \( A \), we denote \( \cap_{n \in \mathbb{N}} a^n A \) by \( < a > A \).

2. ATTACHED PRIME IDEALS

A nonzero \( R \)-module \( S \) is called secondary if for each \( x \in R \) the multiplication map induced by \( x \) on \( S \) is either surjective or nilpotent. If \( S \) is secondary, then the ideal
\( p := \text{Rad}(\text{Ann}_R S) \) is a prime ideal and \( S \) is called \( p \)-secondary. For an \( R \)-module \( M \), a secondary representation of \( M \) is an expression for \( M \) as a sum of finitely many secondary submodules of \( M \). An \( R \)-module \( M \) is said to be representable if it has a secondary representation. From any secondary representation for an \( R \)-module \( M \), one can obtain another one as \( M = S_1 + \cdots + S_n \) such that the prime ideals \( p_i := \text{Rad}(\text{Ann}_R S_i), i = 1, \ldots, n \) are all distinct and \( S_j \not\subseteq \Sigma_{i \neq j} S_i \) for all \( j = 1, \ldots, n \). A such secondary representation for \( M \) is said to be minimal. It is shown that the set \( \{p_1, \ldots, p_n\} \) is independent of the chosen minimal secondary representation for \( M \). This set is denoted by \( \text{Att}_R M \) and each element of this set is said to be an attached prime ideal of \( M \). It is known that a representable \( R \)-module \( M \) is zero if and only if \( \text{Att}_R M = \emptyset \) and that if \( 0 \rightarrow N \rightarrow M \rightarrow L \rightarrow 0 \) is an exact sequence of representable \( R \)-modules and \( R \)-homomorphisms, then \( \text{Att}_R L \subseteq \text{Att}_R M \subseteq \text{Att}_R N \cup \text{Att}_R L \). Also, it is known that any Artinian \( R \)-module is representable. For more information about the theory of secondary representations see [12] or [14, Section 6, Appendix].

**Lemma 2.1.** i) Let \( f : R \rightarrow U \) be a ring homomorphism and \( M \) a representable \( U \)-module. Then \( M \) is also representable as an \( R \)-module and \( \text{Att}_R M = \{f^{-1}(p) : p \in \text{Att}_U M\} \).

ii) Let \( A \) be an Artinian \( R \)-module. Then \( \text{Supp}_R A \) equals \( \text{Ass}_R A \) and is a finite subset of \( \text{Max} \ R \). Moreover, if \( \text{Supp}_R A = \{m_1, \ldots, m_t\} \), then the natural \( R \)-homomorphism \( \psi : A \rightarrow \bigoplus_{i=1}^t A_{m_i} \) is an isomorphism. In particular, \( \text{Att}_R A = \bigcup_{i=1}^t \text{Att}_R A_{m_i} \).

iii) Let \( m_1, \ldots, m_t \) be distinct maximal ideals of \( R \) and \( A_1, \ldots, A_t \) Artinian \( R \)-modules so that \( \text{Supp}_R A_i = \{m_i\} \) for all \( i = 1, \ldots, t \). Let \( A = \bigoplus_{i=1}^t A_i \). Then for any ideal \( a \) of \( R \) such that \( a \subset \mathfrak{m} := \cap_{i=1}^t m_i \), there is a natural isomorphism

\[
\sum_{n \in \mathbb{N}} < \mathfrak{m} > (0 : A a^n) \cong \bigoplus_{i=1}^t \sum_{n \in \mathbb{N}} < m_i > (0 : A_a a^n).
\]

**Proof.** i) holds by [15, Proposition 4.1].

ii) The first assertion of (ii) holds by [17, Exercises 8.49 and 9.43]. Now, we are going to prove the second assertion of (ii). It follows by [17, Exercise 8.49], that \( A = \bigoplus_{i=1}^t \Gamma_{m_i}(A) \). This yields that for each \( i \), \( A_{m_i} \cong \Gamma_{m_i}(A) \), and so \( A_{m_i} \) as an \( R \)-module, supported only at the maximal ideal \( m_i \). So \( \psi_{m_i} : A_{m_i} \rightarrow (\bigoplus_{i=1}^t A_{m_i})_m \) is an isomorphism for any maximal ideal \( m \) of \( R \). Thus \( \psi \) is an isomorphism, as claimed. Finally, the last assertion of (ii) is immediate by (i) and the fact that for any given finitely many secondary representable \( R \)-modules \( M_1, \ldots, M_t \), it turns out that \( \bigoplus_{i=1}^t M_i \) is also representable and that

\[
\text{Att}_R (\bigoplus_{i=1}^t M_i) = \bigcup_{i=1}^t \text{Att}_R M_i.
\]
iii) First of all note that for any Artinian $R$-module $B$ and any two ideals $a, b$ of $R$, it is easy to see that \( \{ 0 :_B a^n \}_{n \in \mathbb{N}} \), with the natural maps induced by the identity map of $B$, is a direct system and that

\[
\sum_{n \in \mathbb{N}} (0 :_B a^n) < b > (0 :_B a^n)
\]

is its direct limit. In particular, if $a \subseteq \bigcap_{m \in \text{Supp} \hat{R} \hat{M}}$, then each element of $B$ is annihilated by some power of $a$, and so

\[
\lim_{n \to \infty} (0 :_B a^n) < b > (0 :_B a^n) = \sum_{n \in \mathbb{N}} (0 :_B a^n).
\]

Next, note that $A_m \cong A_i$ for all $i = 1, \ldots, t$. Thus in view of (ii), we have the following isomorphisms

\[
\sum_{n \in \mathbb{N}} < m > (0 :_A a^n) \cong \lim_{n \to \infty} (0 :_A a^n)
\]

\[
\cong \lim_{n \to \infty} [\bigoplus_{1 \leq i \leq t} (0 :_A a^n)]
\]

\[
\cong \bigoplus_{i=1}^t A_i \sum_{n \in \mathbb{N}} < m_i > (0 :_A a^n).
\]

**Remark 2.2.**

i) Let $a$ be an ideal of $R$. For a prime ideal $p$ of $R$, we say that $a$ is **formally isolated at** $p$ if $a \subseteq p$ and if there is some prime ideal $p^* \in \text{Supp} \hat{R} \hat{M}$ such that $\dim \hat{R}_p/p^* = \text{ht}(p)$ and $\dim \hat{R}_p/a \hat{R}_p + p^* = 0$. Assume that $R$ has finite dimension $d$, and let $\mathcal{P}_a$ denote the set of all prime ideals $p$ such that $\text{ht}(p) = d$ and such that $a$ is formally isolated at $p$.

Then, by [2, Theorem 3.3 (b)] for any finitely generated faithful $R$-module $M$, we have $\text{Supp} R H^d_a(M) = \mathcal{P}_a$.

ii) Let $M$ be a finitely generated $R$-module of finite dimension $d$. Let $\mathcal{P}_{a,M}$ denote the set of all $p \in \text{Var}(\text{Ann}_R M + a)$ so that there is some prime $p^* \in \text{Supp} \hat{R}_p \hat{M}_p$ such that $\dim \hat{R}_p/p^* = d$ and that $\dim \hat{R}_p/a \hat{R}_p + p^* = 0$. Then, by adapting the method of the proof of [2, Theorem 3.3(b)], one can easily deduce that $\text{Supp} R H^d_a(M) = \mathcal{P}_{a,M}$. Also, in Corollary 4.1 below, we establish another characterization of $\mathcal{P}_{a,M}$. 
In the remainder of the paper, for a finitely generated $R$-module $M$ of finite dimension $d$ and an ideal $a$ of $R$, let $\mathcal{P}_{a,M}$ be as in Remark 2.2 (ii).

**Theorem 2.3.** Let $a$ be an ideal of $R$, $M$ a finitely generated $R$-module of finite dimension $d$ and $\mathfrak{m} = \bigcap_{p \in \mathcal{P}_{a,M}} p$. There is a natural isomorphism

$$H^d_a(M) \cong H^d_{\mathfrak{m}}(M) / \sum_{a^n : H^d_{\mathfrak{m}}(M)}.$$  

**Proof.** By Remark 2.2 (ii), we have $\text{Supp}_R H^d_a(M) = \mathcal{P}_{a,M}$. Let $\text{Supp}_R H^d_a(M) = \{m_1, \ldots, m_t\}$ and for each $i$ denote the local ring $R_{m_i}$ by $R_i$.

Let $a$ be an ideal of a local ring $(U, n)$. By [10, Theorem 3.2], it turns out that for any finitely generated $U$-module $M$, there is a natural isomorphism

$$H^d_a(M) \cong H^d_{m_i}(M_{m_i}) / \sum_{n \in \mathbb{N}} < n > (0 : H^d_{m_i}(M)),$$

where $d = \dim M$. Observe that by the Flat Base Change Theorem [3, Theorem 4.3.2] and Lemma 2.1 (ii) the modules $H^d_{m_i, R_i}(M_{m_i})$ and $H^d_{m_i}(M)$ are isomorphic for all $1 \leq i \leq t$. Therefore applying Lemma 2.1 (ii) again, provides the following isomorphisms

$$H^d_a(M) \cong \bigoplus_{i=1}^t H^d_{m_i, R_i}(M_{m_i}) \cong \bigoplus_{i=1}^t \frac{H^d_{m_i, R_i}(M_{m_i})}{\sum_{m_i R_i < n > (0 : H^d_{m_i, R_i}(M_{m_i}))} \sum_{n \in \mathbb{N}} < m_i R_i > (0 : H^d_{m_i, R_i}(M_{m_i}))} \cong \bigoplus_{i=1}^t \frac{H^d_{m_i}(M)}{\sum_{m_i R_i < n > (0 : H^d_{m_i}(M))}}.$$  

On the other hand, the Mayer-Vietoris sequence for local cohomology [3, Theorem 3.2.3] yields the following isomorphism

$$H^d_{\mathfrak{m}}(M) \cong \bigoplus_{i=1}^t H^d_{m_i}(M).$$

This finishes the proof, by Lemma 2.1 (iii). □

Recall that for an $R$-module $M$, the cohomological dimension of $M$ with respect to an ideal $a$ of $R$ is defined as $\text{cd}_R(a, M) := \sup \{i \in \mathbb{N}_0 : H^i_a(M) \neq 0\}$. It is appropriate to list some basic properties of this notion. First of all note that, it is immediate by Grothendieck’s Vanishing Theorem, that $\text{cd}_R(a, M) \leq \dim M$. Next, note that if $V$ is a multiplicative subset of $R$, then it becomes clear by the Flat Base Change Theorem, that $\text{cd}_{VR^{-1}}(aV^{-1}R, V^{-1}M) \leq \text{cd}_R(a, M)$. Also, if $M$ and $L$ are two finitely generated $R$-modules so that $\text{Supp}_R L \subseteq \text{Supp}_R M$, then [9, Theorem 2.2] implies that
Lemma 2.4. Let \( a \) be an ideal of a local ring \((R, \mathfrak{m})\) and \( d \) a natural number. For any prime ideal \( p \) of \( R \) so that \( \dim R/p \leq d \), the following are equivalent:

i) \( \text{cd}_{R}(a, R/p) = d \).

ii) \( p \) is the contraction to \( R \) of a prime ideal \( p^* \) of \( \hat{R} \) such that \( \dim \hat{R}/p^* = d \) and \( \dim \hat{R}/a\hat{R} + p^* = 0 \).

**Proof.** Let \( M \) be a finitely generated \( R \)-module of dimension \( d \). Then by the Lichtenbaum-Hartshorne Vanishing Theorem, it turns out that \( H^d_a(M) \neq 0 \) if and only if there exists \( p^* \in \text{Ass}_R \hat{M} \) such that \( \dim \hat{R}/a\hat{R} + p^* = 0 \) (see e.g. [10, Corollary 3.4]). Assume that (i) holds. Then \( H^d_a(R/p) \neq 0 \), and so there exists \( p^* \in \text{Ass}_R (\hat{R}/p\hat{R}) \) such that \( \dim \hat{R}/a\hat{R} + p^* = 0 \). Since \( H^d_a(R/p) \neq 0 \), by Grothendieck’s Vanishing Theorem, we have \( \dim R/p = d \). Thus

\[
\dim \hat{R}/p^* = \dim \hat{R}/p\hat{R} = d.
\]

On the other hand, by [14, Theorem 23.2 (i)], we have

\[
\{p\} = \text{Ass}_R (R/p) = \{Q \cap R : Q \in \text{Ass}_R (\hat{R}/p\hat{R})\}.
\]

Hence \( p = p^* \cap R \), and so (ii) follows.

Now, assume that (ii) holds. We have

\[
d \geq \dim R/p = \dim \hat{R}/p\hat{R} \geq \dim \hat{R}/p^* = d.
\]

So \( \dim R/p = d \). In particular, \( p^* \) is minimal over \( p\hat{R} \), and so \( p^* \in \text{Ass}_R (\hat{R}/p\hat{R}) \). Thus \( H^d_a(R/p) \neq 0 \), by the Lichtenbaum-Hartshorne Vanishing Theorem (that we commented earlier its statement in the beginning of the proof). Therefore \( \text{cd}_{R}(a, R/p) = d \), as required. \( \Box \)

The following extends the main result of [6] to general Noetherian rings.

**Theorem 2.5.** (See [4, Theorem 1.2]) Let \( a \) be an ideal of \( R \) and \( M \) a finitely generated \( R \)-module of finite dimension \( d \). Then

\[
\text{Att}_R H^d_a(M) = \{p \in \text{Ass}_R M : \text{cd}_{R}(a, R/p) = d\}.
\]

**Proof.** Assume that \( \text{Supp}_R H^d_a(M) = \{m_1, \ldots, m_l\} \). Then by the Flat Base Change Theorem and Lemma 2.1 (ii), it follows that

\[
\text{Att}_R H^d_a(M) = \bigcup_{i=1}^l \text{Att}_R H^d_{a\hat{R}_{m_i}}(M_{m_i}).
\]

In the remainder of the proof, we will use this equality without further comment.
Let $M$ be a finitely generated module over a local ring $(U, n)$. Then by [10, Corollary 3.3] for any ideal $a$ of $U$, $\text{Att}_U H^d_{aU}(M)$ consists of all $p \in \text{Ass}_{U} \hat{M}$ such that $\dim \hat{U}/a\hat{U} + p = 0$. Fix $1 \leq i \leq t$. Since

$$H^d_{aR_m}(M_{m_i}) \simeq (H^d_{a}(M))_{m_i} \neq 0,$$

we have $\dim M_{m_i} = d$. It now follows, by Lemma 2.1 (i) and Lemma 2.4 that

$$\text{Att}_{R_{m_i}} H^d_{aR_{m_i}} (M_{m_i}) = \{ Q \cap R_{m_i} : Q \in \text{Assh}_{R_{m_i}} \hat{M}_{m_i}, \dim \hat{R}_{m_i}/a\hat{R}_{m_i} + Q = 0 \}$$

$$= \{ pR_{m_i} \in \text{Assh}_{R_{m_i}} M_{m_i} : \text{cd}_{R_{m_i}} (aR_{m_i}, R_{m_i}/pR_{m_i}) = d \}.$$

Because $\dim M_{m_i} = \dim M = d$ and

$$\text{Ass}_{R_{m_i}} M_{m_i} = \{ pR_{m_i} : p \subseteq m_i, \text{ and } p \in \text{Ass}_R M \},$$

it follows that $\text{Assh}_{m_i} M_{m_i}$ consists of all prime ideals $pR_{m_i} \in \text{Ass}_{R_{m_i}} M_{m_i}$, such that $p \in \text{Ass}_R M$. Hence, if $p \in \text{Att}_R H^d_{a}(M)$, then $p \in \text{Ass}_R M$ and $\text{cd}_R (a, R/p) = d$.

Conversely, assume that $p \in \text{Ass}_R M$ is such that $\text{cd}_R (a, R/p) = d$. Let $m \in \text{Supp}_R H^d_{a}(R/p)$. Then $H^d_{aR_m} (R_m/pR_m) \neq 0$, and so $\dim R_m/pR_m = d$. Hence, we have $\text{cd}_{R_m} (aR_m, R_m/pR_m) = d$ and $pR_m \in \text{Ass}_{R_m} M_{m_i}$. By Lemma 2.4, $pR_m$ is the contraction to $R_m$ of a prime ideal $p^*$ of $\hat{R}_m$ such that $\dim \hat{R}_m/p^* = \dim \hat{R}_m/a\hat{R}_m + p^* = 0$. It is easy to see that $p^* \in \text{Ass}_{R_m} \hat{M}_{m_i}$, and so by Lemma 2.1 (i) and the above mentioned result of [10], it turns out that $pR_m \in \text{Att}_{R_m} H^d_{aR_m} (M_{m_i})$. Hence $p \in \text{Att}_R H^d_{a}(M)$, by using Lemma 2.1 (i) again. Note that, since $\text{Att}_R H^d_{a}(M)_m$ is not empty, it follows that $m \in \text{Supp}_R H^d_{a}(M)$.

**Example 2.6.** In [8, Corollary 3.3], the fact that the top local cohomology modules of finitely generated modules of finite dimension are Artinian is extended to an strictly larger class of modules. Namely, it is shown that if $a$ is an ideal of $R$ and $M$ a $ZD$-module of finite dimension $d$ such that $a$-relative Goldie dimension of any quotient of $M$ is finite, then $H^d_{a}(M)$ is Artinian. It would be interesting to know whether the conclusion of Theorem 2.5 remains valid for this larger class of modules. Unfortunately, this is not the case, even if $R$ is local. To this end, let $(R, m)$ be a local ring with $\dim R > 0$. Take $a = m$ and $M = E(R/m)$, the injective envelop of the residue field of $R$. Then $M$ is a $ZD$-module and $a$-relative Goldie dimension of any quotient of $M$ is finite. We have

$$\text{Att}_R H^0_{a}(M) = \text{Att}_R M = \text{Ass}_R R,$$

while the maximal ideal $m$ is the only element of the set

$$\{ p \in \text{Ass}_R M : \text{cd}_R (a, R/p) = 0 \}.$$
As a corollary to Theorem 2.5, we present an improvement of the main result of [2]. In the sequel, let \( \mathcal{P}_a \) be as in Remark 2.2 (i).

**Corollary 2.7.** Let \( a \) and \( b \) be two ideals of \( R \) and assume that \( R/b \) has finite dimension \( d \). Then

i) \( \mathcal{P}_{a,R/b} = \{ m \in \text{Max } R : \exists p \in \text{Assh}_R(R/b) \text{ such that } p \subseteq m \text{ and } \text{cd}_{R_m}(aR_m, R_m/pR_m) = d \}. \) In particular, if \( R \) has finite dimension \( d \), then \( \mathcal{P}_a = \{ m \in \text{Max } R : \exists p \in \text{Assh}_R R \text{ such that } p \subseteq m \text{ and } \text{cd}_{R_m}(aR_m, R_m/pR_m) = d \}. \)

ii) For any finitely generated \( R \)-module \( M \) so that Assh \( R \) \( M = \text{Assh}_R(R/b) \), we have \( \text{Supp}_R H^d_a(M) = \mathcal{P}_{a,R/b} \). In particular, \( \mathcal{P}_{a,R/b} \) is a finite set.

iii) (See [4, Theorem 1.3 (g)]) If \( d > 0 \), then for any \( M \) as in (ii), the \( R_m \)-module \( (H^d_a(M))_m \) is not finitely generated for all \( m \in \mathcal{P}_{a,R/b} \).

**Proof.** First, it should be noted that \( \mathcal{P}_a = \mathcal{P}_{a,R} \). By Remark 2.2 (ii), we have \( \text{Supp}_R H^d_a(R/b) = \mathcal{P}_{a,R/b} \). Hence, to prove (i) and (ii), it will be enough to show that for any finitely generated \( R \)-module \( M \) with Assh \( R \) \( M = \text{Assh}_R(R/b) \), \( \text{Supp}_R H^d_a(M) \) consists of all maximal ideals \( m \) of \( R \) so that there exists a prime ideal \( p \in \text{Assh}_R(R/b) \) such that \( p \subseteq m \) and \( \text{cd}_{R_m}(aR_m, R_m/pR_m) = d \). Assume that \( M \) is a finitely generated \( R \)-module with Assh \( R \) \( M = \text{Assh}_R(R/b) \), and let \( m \in \text{Supp}_R H^d_a(M) \). Then \( H^d_{aR_m}(M_m) \neq 0 \), and so by Theorem 2.5, there exists a prime ideal \( Q \in \text{Assh}_{R_m} M_m \) such that \( \text{cd}_{R_m}(aR_m, R_m/Q) = d \). But, then there exists a prime ideal \( p \subseteq m \) of \( R \) such that \( Q = pR_m \). As we have seen in the proof of Theorem 2.5, \( Q \in \text{Assh}_{R_m} M_m \), implies that

\[ p \in \text{Assh}_R M = \text{Assh}_R(R/b). \]

Conversely, let \( m \) be a maximal ideal of \( R \) such that there exists a prime ideal \( p \in \text{Assh}_R(R/b) \) such that \( p \subseteq m \) and \( \text{cd}_{R_m}(aR_m, R_m/pR_m) = d \). Since \( \text{Var}(pR_m) \subseteq \text{Supp}_{R_m} M_m \), by [9, Theorem 2.2], it turns out that \( \text{cd}_{R_m}(aR_m, M_m) = d \). But, this implies that \( m \in \text{Supp}_R H^d_a(M) \).

iii) Let \( m \in \mathcal{P}_{a,R/b} \). Then by part (ii), we deduce that \( H^d_{aR_m}(M_m) \neq 0 \). Hence, [2, Lemma 2.1] yields that the \( R_m \)-module \( (H^d_a(M))_m \) is not finitely generated. \( \Box \)

**Remark 2.8.** i) Let \( M \) and \( N \) be two finitely generated \( R \)-modules of finite dimension \( d \) so that Assh \( R \) \( N = \text{Assh}_R M \). Having Theorem 2.5 in mind, it becomes clear that \( \text{Att}_R H^d_a(N) = \text{Att}_R H^d_a(M) \). Also, it follows by Corollary 2.7 (ii) that \( \text{Supp}_R H^d_a(N) = \text{Supp}_R H^d_a(M) \). In particular, \( H^d_a(N) = 0 \) if and only if \( H^d_a(M) = 0 \).

ii) Let \( R \) be a ring of finite dimension \( d \) and \( a \) an ideal of \( R \). Also, let \( M \) be a finitely generated \( R \)-module. If \( M \) is faithful, then it follows by [2, Theorem 3.3 (b)] that \( \text{Supp}_R H^d_a(M) = \mathcal{P}_a \). It is perhaps worth pointing out that by part (i), this conclusion for \( M \) remains valid under the weaker assumption that Assh \( R \) \( M = \text{Assh}_R R \).
The following lemma will be needed in the proof of our last result in this section.

**Lemma 2.9.** Let \( a \) and \( b \) be two ideals of \( R \) and \( c \) a natural number. Assume that \( M \) is a finitely generated \( R \)-module so that \( \text{cd}_R(a, M) \leq c \). Then there is a natural isomorphism

\[
H^c_a(M/bM) \simeq H^c_a(M)/bH^c_a(M).
\]

**Proof.** Let \( U = R/\text{Ann}_R M \). Since \( \text{Supp}_R U = \text{Supp}_R M \), it follows by [9, Theorem 2.2], that \( H^i_{aU}(U) = 0 \) for all \( i > c \). Hence \( H^i_{aU}(-) \) is a right exact functor on the category of \( U \)-modules and \( U \)-homomorphisms. Thus

\[
H^c_a(M/bM) \simeq H^c_{aU}(U) \otimes U M/bM \\
\simeq (H^c_{aU}(U) \otimes U M) \otimes R/b \\
\simeq H^c_a(M)/bH^c_a(M).
\]

**Theorem 2.10.** Let \( a \) be an ideal of \( R \) and \( M \) a finitely generated \( R \)-module such that \( c := \text{cd}_R(a, M) \neq -\infty \). Let \( \mathfrak{M} \) be the set of all \( p \in \text{Supp}_R M \) such that \( \text{dim } R/p = \text{cd}_R(a, R/p) = c \) and \( \mathfrak{X} := \mathfrak{M} \cap \text{Ass}_R M \).

i) If \( b := \bigcap_{p \in \mathfrak{X}} p \), then \( p_{a, R/b} \subseteq \text{Supp}_R H^c_{a}(M) \).

ii) If \( H^c_{a}(M) \) is representable, then \( \mathfrak{X} \subseteq \text{Att}_R H^c_{a}(M) \).

iii) Assume that \( H^c_{a}(M) \) is representable. If \( p \in \text{Att}_R H^c_{a}(M) \) is so that \( \text{dim } R/p = c \), then \( p \in \mathfrak{M} \).

**Proof.** By [1, p.263, Proposition 4], there is a submodule \( N \) of \( M \) such that \( \text{Ass}_R(M/N) = \mathfrak{X} \). In particular, \( \text{dim } M/N = c \). Since \( \text{Supp}_R N \subseteq \text{Supp}_R M \), by [9, Theorem 2.2], we have \( H^i_{a}(N) = 0 \) for all \( i > c \). Thus, the exact sequence

\[
0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0
\]

provides the following exact sequence of local cohomology modules

\[
\ldots \rightarrow H^c_{a}(N) \rightarrow H^c_{a}(M) \rightarrow H^c_{a}(M/N) \rightarrow 0.
\]

Thus \( \text{Supp}_R H^c_{a}(M/N) \subseteq \text{Supp}_R H^c_{a}(M) \), and so (i) follows by Corollary 2.7 (ii). If \( H^c_{a}(M) \) is representable, then the above exact sequence implies that \( \text{Att}_R H^c_{a}(M/N) \subseteq \text{Att}_R H^c_{a}(M) \), and so (ii) follows by Theorem 2.5.

Next, we prove (iii). Let \( p \in \text{Att}_R H^c_{a}(M) \) be so that \( \text{dim } R/p = c \). By [12, 2.5], there is a submodule \( N \) of \( H^c_{a}(M) \) such that \( p = N :_R H^c_{a}(M) \). Hence \( pH^c_{a}(M) \subseteq N \), and so by Lemma 2.9, it turns out that \( H^c_{a}(M)/N \) is isomorphic to a quotient of \( H^c_{a}(M/pM) \). Now, by the Independence Theorem [3, Theorem 4.2.1], we have the following isomorphisms

\[
H^c_{a}(M/pM) \simeq H^c_{aR/p}(M/pM) \\
\simeq H^c_{aR/p}(R/p) \otimes_R (M/pM) \\
\simeq H^c_{a}(R/p) \otimes_R M.
\]
Thus $H_\delta^d(M/pM)$ is Artinian and $p \in \text{Att}_R H_\delta^d(M/pM)$. Because, by [7, Corollary 3.3] for
an Artinian $R$-module $A$ and a finitely generated $R$-module $N$, we have
$$\text{Att}_R (A \otimes_R N) = \text{Att}_R A \cap \text{Supp}_R N,$$
the conclusion follows by Theorem 2.5. □

3. LICHTENBAUM-HARTSHORNE VANISHING THEOREM

Let the situation be as in Theorem 2.5. In the case that the ideal $a$ is the intersection of
finitely many maximal ideals of $R$, we can find a better description of the set $\text{Att}_R H_\delta^d(M)$.
We do this in the next result. The last assertion of this result might be considered as the
generalization of Grothendieck’s non-Vanishing Theorem to semi-local rings.

**Proposition 3.1.** Assume that $m_1, \ldots, m_t$ are maximal ideals of $R$ and $M$ a finitely generated
$R$-module of finite dimension $d$. Let $a = \bigcap_{i=1}^t m_i$. Then
$$\text{Att}_R H_\delta^d(M) = \{ p \in \text{Assh}_R M : \exists 1 \leq i \leq t \text{ such that } p \subseteq m_i \text{ and } \text{ht} \frac{m_i}{p} = d \}.$$  
In particular, if $R$ is semi-local with the only maximal ideals $m_1, \ldots, m_t$, then $\text{Att}_R H_\delta^d(M) = \text{Assh}_R M$, and so $H_\delta^d(M) \neq 0$ whenever $M$ is nonzero.

**Proof.** Let $1 \leq i \leq t$. Since $\text{Supp}_R H_{m_i}^d(M) \subseteq \{ m_i \}$, by Lemma 2.1 (ii) and the Flat Base
Change Theorem, it turns out that $H_{m_i}^d(M) \simeq H_{m_i, R_{m_i}}^d(M_{m_i})$. Hence, applying the Mayer-Vietoris sequence for local cohomology provides the following natural isomorphisms
$$H_\delta^d(M) \simeq \bigoplus_{i=1}^t H_{m_i}^d(M) \simeq \bigoplus_{i=1}^t H_{m_i, R_{m_i}}^d(M_{m_i}).$$
By [13, Theorem 2.2], for a finitely generated module $M$ over a local ring $(U, n)$, we have
$\text{Att}_U H_\delta^d(n) = \text{Assh}_U M$, where $d = \dim M$. Thus by Lemma 2.1 (i), we conclude that
$$\text{Att}_R H_\delta^d(M) = \bigcup_{i=1}^t \{ p \in \text{Assh}_R M : pR_{m_i} \subseteq \text{Assh}_{R_{m_i}} M_{m_i} \text{ and } \dim R_{m_i}/pR_{m_i} = d \}$$
$$= \{ p \in \text{Assh}_R M : \exists 1 \leq i \leq t \text{ such that } p \subseteq m_i \text{ and } \text{ht} \frac{m_i}{p} = d \}.$$  
The last assertion is immediate by the first one. □

**Remark 3.2.** Let $A$ be an Artinian $R$-module. Suppose that $\text{Supp}_R A = \{ m_1, \ldots, m_t \}$ and
put $\mathfrak{M} = \bigcap_{i=1}^t m_i$. Let $T$ denote the $\mathfrak{M}$-adic completion of $R$.

(i) Sharp [16] showed that $A$ has a natural structure as a module over $T$. Let $\theta : R \rightarrow T$
denote the natural ring homomorphism. The $T$-module structure of $A$ is such that for
any element $r \in R$ the multiplication by $r$ on $A$ has the same effect as the multiplication
of $\theta(r) \in T$. Furthermore a subset of $A$ is an $R$-submodule of $A$ if and only if it is a $T$-
submodule of $A$. 


The following are equivalent:

\[ \text{ii) Let } A \subseteq B \text{ denote two ideals of } R \text{ and } B := \Sigma_{n \in \mathbb{N}} < b > (0 :_A a^n). \text{ By [10, Theorem 2.4]}, the following are equivalent:} \\
\text{a) For any } l \in \mathbb{N}, \text{ there is an integer } n = n(l) \text{ such that } 0 :_A \mathfrak{M}^l \subseteq < b > (0 :_A a^n). \\
\text{b) } B = A. \\
\text{c) } \text{Rad}(p + aT) \subseteq \text{Rad}(p + bT) \text{ for all } p \in \text{Att}_T A.

\[ \text{iii) For all ideals } c \text{ of } R, \text{ there are at most } 2^{|\text{Assh}_R(M \otimes_{R^T})|} \text{ non-isomorphic top local cohomology modules } H^d_c(M) \text{ such that } \text{Supp}_R H^d_c(M) = \mathcal{P}. \]

\[ \text{Proof. Let } A = H^d_{\mathfrak{M}}(M), \]
\[ B_1 = \sum_{n \in \mathbb{N}} < \mathfrak{M} > (0 :_A a^n) \]
\[ B_2 = \sum_{n \in \mathbb{N}} < \mathfrak{M} > (0 :_A b^n). \]

Then, Theorem 2.3 yields the natural isomorphisms
\[ H^d_a(M) \cong A/B_1 \text{ and } H^d_b(M) \cong A/B_2. \]
Let \( A = S_1 + \cdots + S_n \) be a minimal secondary representation of \( A \) as a \( T \)-module and set
\[ J_j := \text{Att}_T A \setminus \text{Att}_T (A/B_j) \]
for \( j = 1, 2 \). Then by Remark 3.2 (iii), \( B_j = \Sigma_{p_j \in J_j} S_i \) for \( j = 1, 2 \). Thus, if either \( a \subseteq b \) or \( \text{Att}_T H^d_a(M) \subseteq \text{Att}_T H^d_b(M) \), then \( B_2 \subseteq B_1 \) and so \( H^d_a(M) \) is isomorphic to a quotient of \( H^d_b(M) \). Also, if \( \text{Att}_T H^d_a(M) = \text{Att}_T H^d_b(M) \), then \( B_1 = B_2 \), and so \( H^d_a(M) \cong H^d_b(M) \).

Next, we are going to prove (iii). Since \( T \) and \( \Pi_{m \in \mathcal{P}} \hat{R}_m \) are isomorphic as \( R \)-modules, by the Flat Base Change Theorem, we have the following isomorphisms
Let \( a \) and \( b \) be two ideals of a local ring \((R, m)\) and \( M \) a finitely generated \( R \)-module. Let \( d = \dim M \).

i) If either \( a \subseteq b \) or \( \text{Att}_R H^d_a(M) \subseteq \text{Att}_R H^d_b(M) \), then \( H^d_a(M) \) is isomorphic to a quotient of \( H^d_b(M) \).

ii) If \( \text{Att}_R H^d_a(M) = \text{Att}_R H^d_b(M) \), then \( H^d_a(M) \simeq H^d_b(M) \).

iii) The number of non-isomorphic top local cohomology modules \( H^d_c(M) \) is at most \( 2^{|\text{Assh}_R \hat{M}|} \) for all ideals \( c \) of \( R \).

Example 3.5. It might be of some interest to replace \( \hat{R} \) by \( R \) in Corollary 3.4 (ii). But, as we show in the sequel, this would not be the case. To this end, we use an example of Brodmann and Sharp (see [3, Exercise 8.2.9]). Let \( K \) be a field of characteristic 0. Let \( R' := K[X, Y, Z] \), \( m' := (X, Y, Z) \) and \( b = (Y^2 - X^2 - X^3) \). Set \( R := (R'/b)_{m'/b} \) and let \( p \) denote the extension of the ideal

\[
(X + Y - YZ, (Z - 1)^2(X + 1) - 1)
\]

of \( R' \) to \( R \). As it is mentioned in [3, Exercise 8.2.9], it follows that \( R \) is a 2-dimensional local domain and that \( p \hat{R} \) is a prime ideal of \( \hat{R} \) with \( \dim \hat{R}/p \hat{R} = 1 \). Also, it follows that \( H^2_p(R) \neq 0 \), (see again [3, Exercise 8.2.9]). So \( \text{Att}_R H^2_p(R) \) is not empty. Now, let \( p^* \) be a minimal associated prime ideal of \( \hat{R} \) such that \( p^* \subseteq p \hat{R} \). Then the inclusion must be strict,
because otherwise we would have

\[ p = p \hat{R} \cap R = p^* \cap R \in \text{Ass}_R \hat{R} = \{(0)\}, \]

a contradiction. This yields that \( \dim \hat{R}/p^* = 2 \), and so \( p^* \in \text{Ass}_R \hat{R} \). On the other hand, we have

\[ \dim \hat{R}/pR + p^* = \dim \hat{R}/pR = 1. \]

Hence \( p^* \) does not belong to \( \text{Att}_R H^2_p(R) \). Thus, if \( m \) denotes the maximal ideal of the local ring \( R \), then

\[ \emptyset \neq \text{Att}_R H^2_p(R) \subseteq \text{Ass}_R \hat{R}. \]

In particular, it becomes clear that \( H^2_p(R) \) and \( H^2_m(R) \) are not isomorphic. On the other hand, we have

\[ \text{Att}_R H^2_p(R) = \text{Att}_R H^2_m(R) = \{(0)\}. \]

We therefore conclude that, it is not possible to replace \( \hat{R} \) by \( R \) in Corollary 3.4 (ii).

The following is an analogue of the Lichtenbaum-Hartshorne Vanishing Theorem for general Noetherian rings.

**Theorem 3.6.** Let \( a \) be an ideal of \( R \) and \( M \) a finitely generated \( R \)-module of finite dimension \( d \).

Let \( \mathfrak{M} = \bigcap_{m \in P_{a,M}} m \) and \( T \) denote the \( \mathfrak{M} \)-adic completion of \( R \). Then the following are equivalent:

i) \( H^d_a(M) = 0 \).

ii) \( H^d_{\mathfrak{M}}(M) = \sum_{n \in \mathbb{N}} < \mathfrak{M} > (0 : H^d_{\mathfrak{M}}(a^n)). \)

iii) For any integer \( l \in \mathbb{N} \), there exists an \( n = n(l) \in \mathbb{N} \) such that

\[ 0 : H^d_{\mathfrak{M}}(a^l) \subseteq < \mathfrak{M} > (0 : H^d_{\mathfrak{M}}(a^n)). \]

iv) \( \dim T/aT + p > 0 \) for all \( p \in \text{Ass}_T(M \otimes_R T) \).

v) \( \text{cd}_R(a, R/p) < d \) for all \( p \in \text{Ass}_R M \).

**Proof.** Let \( p \in \text{Ass}_T(M \otimes_R T) \). Then, it is easy to see that \( \dim T/aT + p > 0 \) if and only if \( \text{Rad}(p + aT) \nsubseteq \text{Rad}(p + \mathfrak{M}T) \). Therefore, the equivalence of the conditions (i), (ii) and (iv) is clear by Theorem 2.3 and Remark 3.2 (ii). Note that in the proof of Theorem 3.3, we have seen that \( \text{Att}_T H^d_{\mathfrak{M}}(M) = \text{Ass}_T(M \otimes_R T) \).

Since \( a \subseteq \mathfrak{M} \), any element of \( H^d_{\mathfrak{M}}(M) \) is annihilated by some power of \( a \). Thus (iii) \( \Rightarrow \) (ii) becomes clear.

(ii) \( \Rightarrow \) (iii) Let \( A = H^d_{\mathfrak{M}}(M) \) and \( l \) a fixed natural number. Then \( (0 :_A a^l) / < \mathfrak{M} > (0 :_A a^l) \) is a Noetherian \( R \)-module and the sequence \( \{(0 :_A a^l) \cap < \mathfrak{M} > (0 :_A a^l)\}_{n \in \mathbb{N}} \) satisfies the ascending chain condition. Thus, it follows by [10, Lemma 2.1] that (ii) implies (iii).
By Grothendieck’s Vanishing Theorem, it turns out that $cd_R(a, R/p) \leq d$ for all $p \in \text{Supp}_R M$. Therefore, the equivalence (i) and (v) is immediate by Theorem 2.5. □

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