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Abstract: In this paper, a sufficient condition for the partial approximate controllability of semilinear deterministic control systems is proved. Generally, the theorems on controllability are formulated for control systems given as a first-order differential equation, while many systems can be written in this form only by enlarging the dimension of the state space. The ordinary controllability conditions for such systems are too strong because they involve the enlarged state space. Therefore, it becomes useful to define partial controllability concepts, which assume the original state space. The method of proof, given in this paper, differs from the traditional proofs by fixed point theorems. The obtained result is demonstrated on examples.

Keywords: approximate controllability, exact controllability, partial controllability, semilinear system

AMS subject classification: 93B05

1. Introduction
In 1960, Kalman (1960) defined the concept of controllability as a property of control systems to attain every point in the state space from every initial state point for a finite time. Further studies in this field resulted with a separation of this concept into two concepts: a stronger concept of exact controllability and a weaker concept of approximate controllability. The reason for this was the fact

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PUBLIC INTEREST STATEMENT
In the real life, the controllability means being able to manipulate completely or almost completely with the state of an object. For example, TVs are manipulated by remote controls and marionettes by actors. E-mail messaging is controllable if everyone is able to send a message to everyone. If some of e-mail accounts are blocked, then e-mail messaging becomes partially controllable as displayed in the cover image. In engineering, a control system is controllable if it is possible to reach its every (or almost every) state for a finite time. It has great applications, for example, in robotics. In this paper, conditions are proved under which a given system is controllable.
that many infinite dimensional control systems are approximately controllable while they are not completely controllable (see Fattorini, 1966; Russel, 1967). The controllability concepts for linear systems are discussed in Curtain and Zwart (1995), Bensoussan (1992), Bensoussan, Da Prato, Delfour, and Mitter (1993), Zabczyk (1995), Bashirov (2003), Klamka (1991), etc.

Recently, in Bashirov, Etikan, and Şemi (2010) and Bashirov, Mahmudov, Şemi, and Etikan (2007), the partial controllability concepts were introduced and the basic controllability conditions from Bashirov and Mahmudov (1999a) and Bashirov and Kerimov (1997) (see also Bashirov, 1996; Bashirov & Mahmudov, 1999b, 1999c) were extended to partial controllability concepts by a replacement of the controllability operator by its partial version. A study of partial controllability concepts is a significant part of the overall study in the area of controllability. This is motivated by the fact that the theorems on controllability are formulated for control systems given as a first-order differential equation, while many systems, such as higher order differential equations, wave equations, and delay equations, can be written in this form only by enlarging the dimension of the state space. Therefore, the ordinary controllability conditions for such systems are too strong because they involve the enlarged state space while controllability concepts require the original state space. This motivates to define the partial controllability concepts that differ from ordinary controllability concepts by an additional projection of the enlarged state to the original state. In more details, this is discussed in Section 2.

In this paper, we study the partial approximate controllability for semilinear systems. The controllability concepts for nonlinear systems are intensively studied in the literature (see, e.g. Balachandran & Sakthivel, 2001; Bashirov & Jneid, 2013; Klamka, 2000, 2001, 2002, 2008; Leiva, Merentes, & Sanchez, 2011, 2012, 2013; Ren, Dai, & Sakthivel, 2013; Sakthivel, Ganesh, & Suganya, 2012; Sakthivel, Ganesh, Ren, & Anthoni, 2013; Sakthivel, Mahmudov, & Kim, 2009; Sakthivel, Mahmudov, & Nieto, 2012; Sakthivel & Ren, 2013; Sakthivel, Suganya, & Anthoni, 2012, etc.). An underlying method of study in these works is fixed point theorems. In the present paper, we apply a different and more natural method. The idea of this method is that we divide the time interval \([0, T]\) into two subintervals \([0, T - \delta]\) and \([T - \delta, T]\). On the interval \([0, T - \delta]\), we choose any control and steer the initial state to some state at \(T - \delta\). Then on the interval \([T - \delta, T]\), we choose the sequence of controls steering the state at \(T - \delta\) arbitrarily close to target state at \(T\) along the linear part of the semilinear system. Using the fact that on the small time interval, the nonlinearity of the semilinear system disturbs its linear part for a small value, we obtain the partial approximate controllability of the semilinear system. This simple idea is realized in this paper and a sufficient condition of partial approximate controllability for the semilinear system is proved. A significant point in the proved sufficient condition is that instead of the positiveness of the controllability operator, we assume the positiveness of the partial controllability operator. This produces a weaker condition in comparison with the similar condition for ordinary approximate controllability. So, the main contribution of the paper is the proved sufficient condition as well as the method of the proof. The obtained result is demonstrated on examples.

One major notation is that we prefer to write the arguments of functions in the subscripts, for example, \(f_t\) instead of \(f(t)\). \(\mathbb{R}^n\) denotes an \(n\)-dimensional Euclidean space and \(\mathbb{R}^{m \times k}\) the space of \(m \times k\) matrices. As always, \(\mathbb{R} = \mathbb{R}^1\). The norm and scalar products in all considered spaces are denoted by \(\| \cdot \|\) and \(\langle \cdot, \cdot \rangle\), being clear from the context. \(I\) and 0 are the identity and zero matrices or operators independently on their dimensions. \(A^*\) is the adjoint of the linear closed operator \(A\). In the case when \(A \in \mathbb{R}^{m \times m}\), \(A^*\) becomes the transpose of \(A\). For a linear operator \(F\) on \(H\), we write \(F \geq 0\) (respectively, \(F > 0\)) if \(F^* = F\) and \(\langle Fh, h \rangle \geq 0\) (respectively, \(\langle Fh, h \rangle > 0\)) for all nonzero \(h \in H\). For \(H\)-valued functions on \([a, b]\), we use the symbols \(C[a, b; H]\) for continuous functions, \(PC[a, b; H]\) for piecewise continuous functions, and \(L_1[a, b; H]\) for square integrable functions. In the case \(H = \mathbb{R}\), \(H\) is dropped in these symbols.

2. Setting the problem and motivation

Consider the semilinear control system

\[
\dot{x}'_t = Ax_t + Bu_t + f(t, x_t, u_t)
\]

(1)
on the interval \([0, T]\) with \(T > 0\). Here, \(x\) is a state process and \(u\) is a control from \(U_{ad}\). We assume:

(A) \(X\) and \(U\) are separable Hilbert spaces, \(H\) is a closed subspace of \(X\), and \(L\) is a projection operator from \(X\) to \(H\).

(B) \(A\) is a densely defined closed linear operator on \(X\), generating a strongly continuous semigroup \(e^{At}\), \(t \geq 0\).

(C) \(B\) is a bounded linear operator from \(U\) to \(X\).

(D) \(f\) is a nonlinear function from \([0, T] \times X \times U\) to \(X\) and satisfies:
   • \(f\) is continuous on \([0, T] \times X \times U\),
   • \(f\) is bounded on \([0, T] \times X \times U\),
   • \(f\) satisfies Lipschitz condition with respect to \(x\).

(E) \(U_{ad} = PC(0, T; U)\).

Under these conditions, for every \(u \in U_{ad}\) and \(x_0 \in X\), Equation 1 admits a unique continuous mild solution (see Li & Yong, 1995), that is, there is a unique continuous function \(x_u, x_0\) from \([0, T]\) to \(X\) such that

\[
x^{u, x_0}_t = e^{At}x_0 + \int_0^t e^{A(t-s)} (Bu + f(s, x^{u, x_0}_s, u_s)) \, ds, \quad 0 \leq t \leq T
\]

Define the set

\[
D^x_T = \{ x \in X : \exists u \in U_{ad} \text{ such that } x = x^{u, x_0}_T \}
\]

According to Bashirov et al. (2010, 2007), the semilinear system in Equation 1 is said to be \(L\)-partially \(A\)-controllable on \(U_{ad}\) if \(L(D^x_T) = H\) for all \(x_0 \in X\), where \(D\) is the closure of \(D\) and \(L(D)\) is the image of \(D\) under \(L\). Similarly, the semilinear system in Equation (1) is said to be \(L\)-partially \(E\)-controllable on \(U_{ad}\) if \(L(D^x_T) = H\) for all \(x_0 \in X\). Here, \(A\) and \(E\) are abbreviations for the terms “approximate” and “exact,” respectively. In the case \(H = X\), these are just well-known approximate and exact controllability concepts, respectively. If \(H\) is the zero-dimensional subspace of \(X\), then the \(L\)-partial controllability concepts reduce to the null-controllability.

To motivate the partial controllability concepts, consider a few examples of systems which can be written in the form of Equation 1 after enlarging the state space.

**Example 1** Consider the nonlinear system

\[
x^{(n)}_t = f \left( t, x_t, x'_t, \ldots, x^{(n-1)}_t, u_t \right)
\]

with the one-dimensional state space \(X = \mathbb{R}\). Write this system as a first-order differential equation

\[
y'_t = Ay_t + F(t, y_t, u_t)
\]

for

\[
y = \begin{bmatrix}
x_t \\
x'_t \\
\vdots \\
x^{(n-2)}_t \\
x^{(n-1)}_t
\end{bmatrix}, \quad A = \begin{bmatrix}
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0
\end{bmatrix}
\]
and

\[
F(t, y, u) = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
f(t, x, x', \ldots, x^{(n-1)}, u)
\end{bmatrix}
\]

The state space of the system in Equation 3 is the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \) and, respectively, its attainable set is a subset of \( \mathbb{R}^n \). Therefore, the controllability concepts for the system in Equation 3 are stronger than the same for in Equation 2. But if we define the projection operator \( L \) by

\[
L = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} : \mathbb{R}^n \rightarrow \mathbb{R}
\]

then the \( L \)-partial controllability concepts for the system in Equation 3 become the same as the ordinary controllability concepts for the system in Equation 2.

**Example 2** Consider the nonlinear wave equation

\[
\frac{\partial^2 x_{t, \theta}}{\partial t^2} = \frac{\partial^2 x_{t, \theta}}{\partial \theta^2} + b_{t} u_{t} + f(t, x_{t, \theta}, \partial x_{t, \theta}/\partial \theta, u_{t})
\]

(4)

where \( x \) is a real-valued function of two variables \( t \geq 0 \) and \( 0 \leq \theta \leq 1 \). The state space of this system is \( L_2(0, 1) \). This system can also be written as the first-order abstract differential equation

\[
y'_{t} = Ay_{t} + Bu_{t} + F(t, y_{t}, u_{t})
\]

(5)

if

\[
y_{t} = \begin{bmatrix} x_{t, \theta} \\
\partial x_{t, \theta}/\partial \theta \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I \\
d^2/d\theta^2 & 0 \end{bmatrix}, \quad F(t, y, u) = \begin{bmatrix} 0 \\
f(t, y_{1}, y_{2}, u) \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\
b \end{bmatrix}
\]

where

\[
y = \begin{bmatrix} y_{1} \\
2 \end{bmatrix} \in L_2(0, 1) \times L_2(0, 1)
\]

The state space \( L_2(0, 1) \times L_2(0, 1) \) of the system in Equation 5 is the enlargement of the state space \( L_2(0, 1) \) of the system in Equation 4. This is a cost paid to bring the wave equation to a first-order differential equation. The ordinary controllability concepts for the system in Equation 5 are too strong for the system in Equation 4. But if the projection operator \( L \) is defined by

\[
L = \begin{bmatrix} I & 0 \end{bmatrix} : L_2(0, 1) \times L_2(0, 1) \rightarrow L_2(0, 1)
\]

then \( L \)-partial controllability concepts for the system in Equation 5 become the same as the ordinary controllability concepts for the system in Equation 4.

**Example 3** Delay equations form another class of systems suitable for application of partial controllability concepts. Consider the system

\[
x'_{t} = f\left(t, x_{t}, \int_{-\infty}^{0} x_{t+\theta} d\theta, u_{t}\right)
\]

(6)
which contains a simple distributed delay in the nonlinear term, assuming that \( x \) is a real-valued function. The state space of this system is \( \mathbb{R} \). To bring it to a system without delay, enlarge \( \mathbb{R} \) to \( \mathbb{R} \times L_2(-\varepsilon, 0) \) and define \( L_2(-\varepsilon, 0) \)-valued function

\[
[x_{t,\theta}]_{\theta \geq 0, \ -\varepsilon \leq \theta \leq 0}
\]

Then the above system can be written as an abstract system

\[
y'_t = Ay_t + f(t, y_t, u_t)
\]

if

\[
y_t = \begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 \\ 0 & d/d\theta \end{bmatrix}, \quad F(t, y, u) = \begin{bmatrix} f(t, x, \Gamma \dot{x}, u) \\ 0 \end{bmatrix}
\]

where \( \Gamma \) is the integral operator

\[
\Gamma \dot{x} = \int_{-\varepsilon}^{0} \dot{x}_\theta d\theta, \quad \dot{x} \in L_2(-\varepsilon, 0)
\]

One can easily observe that the ordinary controllability concepts for the system in Equation 7 are too strong for the system in Equation 6. But for

\[
L = [1 \quad 0] : \mathbb{R} \times L_2(0, 1) \rightarrow \mathbb{R}
\]

the \( L \)-partial controllability concepts for the system in Equation 7 are exactly the ordinary controllability concepts for the system in Equation 6.

For \( 0 < \delta < T \), we will associate with the semilinear system in Equation 1 the following linear system

\[
y'_t = Ay_t + Bu_t, \quad T - \delta < t \leq T
\]

where \( v \in V_{ad}^\delta = C(T - \delta, T; U) \). The solution of Equation 8 is again understood in the mild sense, that is, for every \( v \in V_{ad}^\delta \) and \( y_{T-\delta} \in X \), the function

\[
y_{v, y_{T-\delta}}^T = e^{A(t-T+\delta)}y_{T-\delta} + \int_{T-\delta}^{t} e^{A(t-s)}Bu_s ds, \quad T - \delta < t \leq T
\]

is a unique mild solution of Equation 8. The controllability operator for the linear system in Equation 8 is defined by

\[
Q_{\delta} = \int_{T-\delta}^{T} e^{A(T-t)}BB^* e^{A^*(T-t)} dt = \int_{0}^{\delta} e^{At}BB^* e^{A^*t} dt
\]

We let

\[
\tilde{Q}_{\delta} = LQ_{\delta}L^*
\]

and call it an \( L \)-partial controllability operator. In addition to preceding conditions (A–E), we will also assume that

\[
(F) \text{ For all } 0 < \delta \leq T, \tilde{Q}_{\delta} > 0.
\]

Note that \( Q_{\delta} > 0 \) implies \( \tilde{Q}_{\delta} > 0 \) but the converse is not true. Therefore, condition (F) is weaker that the same kind of condition involving \( Q_{\delta} \) instead of \( \tilde{Q}_{\delta} \). In this paper, we study the concept of \( L \)-partial A-controllability, while concerning the concept of \( L \)-partial E-controllability as well, for the semilinear system in Equation 1. We prove that under conditions (A–F), the system in Equation 1 is \( L \)-partially A-controllable.
3. Main result
The resolvent of \(-Q\), is defined by \( R(\lambda, -\vec{Q}_\delta) = (\lambda I, +\vec{Q}_\delta)^{-1} \). Obviously, \( R(\lambda, -Q_\delta) \) exists for all \( \lambda > 0 \) since \( \lambda I + \vec{Q}_\delta \) is coercive.

**Lemma 1**  Under the above conditions and notation, for given \( \lambda > 0 \) and \( h \in H \), there exists a unique optimal control \( v^* \in V^h_{ad} \) at which the functional

\[
J^*(v) = \|Ly_{T}^{v,T} - h\|^2 + \lambda \int_{T-\delta}^{T} \|v\|^2 \, dt
\]

along the linear system in Equation 8 takes its minimal value on \( V^h_{ad} \). Moreover,

\[
v^*_\delta = -\lambda^{-1}B^*e^{A(T-\delta)}L^* \left( Ly_{T}^{v,T} - h \right), \quad T - \delta \leq t \leq T
\]

and

\[
Ly_{T}^{v,T} - h = \lambda R(\lambda, -\vec{Q}_\delta) \left( Le^{AT}y_{T-\delta} - h \right)
\]

*Proof*  This lemma is a restatement of Lemma 1 for the case of the interval \([T - \delta, T]\), proved in Bashirov et al. (2007).

**Lemma 2**  Under the above conditions and notation, assume that the linear system in Equation 8 is \( L \)-partially \( A \)-controllable on \( V^h_{ad} \). Then for every initial value \( y_{T-\delta} \in X \) and \( h \in H \),

\[
\|Ly_{T}^{v,T} - h\| \to 0 \text{ as } \lambda \to 0
\]

where \( v^* \) is a control in \( V^h_{ad} \) defined by Equations 9–10.

*Proof*  By the resolvent condition for the \( L \)-partial \( A \)-controllability from Bashirov et al. (2007), \( \lambda R(\lambda, -\vec{Q}_\delta) \) converges to the zero operator as \( \lambda \to 0 \) in the strong operator topology. Therefore, by Equation 10, the convergence in Equation 11 takes place.

**Lemma 3**  Under the above conditions and notation, let the linear system in Equation 8 be \( L \)-partially \( E \)-controllable on \( V^h_{ad} \). Then for every \([T - \delta, T]\) and \( 0 < \lambda \leq \lambda_0 \) with some \( \lambda_0 > 0 \),

\[
\|v^*_\delta\| \leq c_1\|y_{T-\delta}\| + c_2\|h\|
\]

where \( c_1 \geq 0 \) and \( c_2 \geq 0 \) are constants and \( v^* \) is a control in \( V^h_{ad} \) defined by Equations 9–10.

*Proof*  From Equations 9–10,

\[
v^*_\delta = -B^*e^{A(T-\delta)}L^* R(\lambda, -\vec{Q}_\delta) \left( Le^{AT}y_{T-\delta} - h \right)
\]

Denote \( M = \sup_{[0,1]} \|e^{AT}\| \) By the resolvent condition for the \( L \)-partial \( E \)-controllability from Bashirov et al. (2007), \( R(\lambda, -\vec{Q}_\delta) \) converges as \( \lambda \to 0 \) in the uniform operator topology. Hence, \( \|R(\lambda, -\vec{Q}_\delta)\| \leq K \) for some \( K \geq 0 \). Also, \( \|L\| \leq 1 \) since \( L \) is a projection operator. Therefore, for \( c_1 = M^2K\|B\| \) and \( c_2 = MK\|B\| \), the inequality in Equation 12 holds.

**Theorem 1**  Under conditions (A–F) the semilinear system in Equation 1 is \( L \)-partially \( A \)-controllable on \( U^h_{ad} \).

*Proof*  Give arbitrary \( \varepsilon > 0 \). Take any \( x_\delta \in X \) and \( h \in H \). Let \( 0 < \delta < T \). Consider any function \( u \in C(0, T; U) \). For example, it may be zero function. Let \( x^*_T, x_\delta \) be the value of the mild solution of the semilinear system
in Equation 1 at $t$, corresponding $u$ and $x_0$. Define the control $u^{\hat{A}, \hat{\delta}}$ by letting it to be $u^{\hat{A}, \hat{\delta}}_t = u$ if $0 \leq t \leq T - \delta$ and

$$u^{\hat{A}, \hat{\delta}}_t = -B^*e^{A^T(T-s)}L^*R(A_{\hat{A}}, -\bar{Q}_\delta) (Le^{At}x_{T_{\delta-\hat{\delta}}} - h) \text{ if } T - \delta \leq t \leq T \quad (13)$$

Obviously, $u^{\hat{A}, \hat{\delta}}$ is piecewise continuous for all $\lambda > 0$ and $0 < \hat{\delta} < T$. So, $u^{\hat{A}, \hat{\delta}} \in U_a$. We can write $x_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}}$ as

$$x_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}} = e^{At}x_{0} + \int_{T_{\delta}}^{T} e^{A(T-s)} (Bu_s^* + f(s, x_s^{u^{\hat{A}, \hat{\delta}}}, u_s^*)) \, ds$$

Also, for the mild solution of the linear system in Equation 8, we have

$$y_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}} = e^{At}x_{0} + \int_{T_{\delta}}^{T} e^{A(T-s)}Bu_s^* \, ds$$

Therefore,

$$\|x_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}} - y_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}}\| \leq \int_{T_{\delta}}^{T} \|e^{A(T-s)}\| \|f(s, x_s^{u^{\hat{A}, \hat{\delta}}}, u_s^*)\| \, ds$$

Letting $M = \sup\{0, T\} \|e^{At}\|$ and $K = \sup\{0, T\} \|f(t, x, u)\|$, we obtain

$$\|x_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}} - y_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}}\| \leq MK\delta$$

This yields

$$\|Lx_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}} - h\| \leq \|Lx_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}} - Ly_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}}\| + \|Ly_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}} - h\| \leq MK\delta + \|Ly_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}} - h\|$$

Now let $0 < \hat{\delta} < \min\{T, \varepsilon / 2MK\}$. The condition $\bar{Q}_\delta > 0$ implies that the linear system in Equation 8 is $L$-partially $A$-controllable. Then, by Lemma 2, we can find sufficiently small $\lambda > 0$ such that

$$\|Ly_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}} - h\| < \varepsilon$$

For these $\delta$ and $\lambda$, the control $u^{\hat{A}, \hat{\delta}}$ satisfies

$$\|Lx_{T_{\delta}}^{u^{\hat{A}, \hat{\delta}}} - h\| < MK\frac{\varepsilon}{2MK} + \frac{\varepsilon}{2} = \varepsilon$$

From the arbitrariness of $x_0 \in X$, $h \in H$ and $\varepsilon > 0$, we arrive to the $L$-partial $A$-controllability of the semilinear system in Equation 1.

**Remark 1** According to the proof of Theorem 1, the control $u^{\hat{A}, \hat{\delta}}$ from Equation 13 does not completely determine a sequence of controls, steering the initial state $x_0$ arbitrarily close to $h \in H$ because the selection of $\lambda$ depends on $\delta$, that is, $\lambda = \lambda_\delta$. This means that the rate of the convergence $\lambda \to 0$ is subject to the rate of the convergence $\delta \to 0$.

**Remark 2** The condition on boundedness of $f$ in (D) can be dropped if $X = \mathbb{R}^n$ and $U = \mathbb{R}^m$. Indeed, the concepts of $L$-partial $A$- and $E$-controllability for the linear systems in finite dimensions coincide. Therefore, we can use Lemma 3. Fix $x_0 \in X$ and $h \in H$. For a fixed function $u \in PC(0, T; \mathbb{R}^m)$, the mild solution $x^{u, x_0}$ of Equation 1 is a continuous function on the compact interval $[0, T]$. Therefore, it is bounded. Let $\|x^{u, x_0}_t\| \leq c$ for some $c \geq 0$. Then from Lemma 3,

$$\|u^{\hat{A}, \hat{\delta}}_t\| \leq c_1 c + c_2 \|h\| = r_1$$

This and the Lipschitz condition in (D) implies that all $x^{u^{\hat{A}, \hat{\delta}}}$ range in a bounded set, that is,

$$\|x^{u^{\hat{A}, \hat{\delta}}}_t\| \leq r_2$$
for some \( r_2 \geq 0 \). Therefore, we can restrict the function \( f \) into the compact set \([0, T] \times B^n(r_2) \times B^m(r_1)\), where \( B^n(r_2) \) and \( B^m(r_1) \) are closed balls in \( \mathbb{R}^n \) and \( \mathbb{R}^m \) with radii \( r_2 \) and \( r_1 \), respectively, centered at the origin. Then the continuity condition on \( f \) in (D) implies the boundedness of \( f \) on \([0, T] \times B^n(r_2) \times B^m(r_1)\). So, the boundedness of \( f \) becomes a consequence from the other conditions.

Remark 3 The system in Equation 1 can also be written as

\[
x'_i = (A + A_1)x_i + (B + B_1)u_i + (f(t, x_i, u_i) - A_1x_i - B_1u_i), \quad 0 < t \leq T
\]

for some linear bounded operators \( A_i \) and \( B_i \). Therefore, in some circumstances, the boundedness of the nonlinear function \( f(t, x, u) \) in the condition (D) can be replaced by the boundedness of \( f(t, x, u) - A_i x - B_i u \) for suitable \( A_i \) and \( B_i \).

Remark 4 The used method of proof has three advantages in comparison to the method by fixed point theorems:

- There is no need to consider larger space \( L_2(0, T; U) \) as a set of admissible controls because the sequence of controls, used in the proof, are piecewise continuous.
- It suffices the Lipschitz continuity of \( f(t, x, u) \) just in \( x \) for the existence and uniqueness of the mild solution of Equation 1, that is, the condition on Lipschitz continuity in \( u \) can be removed.
- No need in unusual inequality, required in the method by fixed point theorems.

At the same time its disadvantage is that it is not applicable for study of exact controllability.

4. Examples
We demonstrate the features of Theorem 1 in the following examples of control systems.

Example 4 To demonstrate that the conditions of Theorem 1 are just sufficient for \( L \)-partial \( A \)-controllability but not necessary, let \( L = I \), reducing the \( L \)-partial \( A \)-controllability to approximate controllability. Consider any infinite-dimensional control system of the form

\[
x'_i = 2Ax_i + Bu_i, \quad x_0 \in \mathbb{R}
\]

where \( A \) is closed but not bounded and \( B \) is bounded operators. Assume that

\[
Q_i = \int_0^\delta e^{2At}BB^*e^{2A^t}dt > 0
\]

for all \( i > 0 \). Then the system in Equation 14 is approximately controllable. For example, such a system may be controllable heat equation studied in Bashirov and Mahmudov (1999d). Have another look to the system in Equation 14 by writing it as

\[
x'_i = Ax_i + Bu_i + f(x_i), \quad x_0 \in \mathbb{R}
\]

with \( f(x) = Ax \). Here, the function \( f \) is neither continuous nor bounded since \( A \) is an unbounded operator. Therefore, it does not satisfy the conditions of Theorem 1 while it is approximately controllable.

Example 5 To demonstrate that a system may not be approximately controllable while being \( L \)-partially \( A \)-controllable, consider the control system consisting of two one-dimensional differential equations

\[
\begin{cases}
  x'_1 = y_1 + bu_1, & x_0 \in \mathbb{R} \\
  y'_1 = f(t, x_1, y_1, u_1), & y_0 \in \mathbb{R}
\end{cases}
\]

on \([0, T]\), where \( u \in U_{ad} = PC(0, T; \mathbb{R})\). We can write the system in Equation 16 as the following semilinear system
where
\[ z_\tau = Az_\tau + Bu_\tau + F(t, z_\tau, u_\tau) \]

and \( z \in \mathbb{R}^2 \) is the vector
\[ z = \begin{bmatrix} x \\ y \end{bmatrix} \]

Solving the system of equations
\[
\begin{align*}
  x'_t &= y_t, \\ x_0 &= c_1 \\
  y'_t &= 0, \\ y_0 &= c_2
\end{align*}
\]
we find \( x_t = c_2 t + c_1 \) and \( y_t = c_2 \). Therefore,
\[
\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}
\]

implying
\[ e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \]

Hence, the controllability operator of the system in Equation 17 is
\[
Q_0 = \int_0^\infty e^{tBB^*} e^{A^*t} dt = b^2 \delta \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}
\]

Thus, the condition \( Q_0 > 0 \) fails for this example. Therefore, the linear part of the system in Equation 17 is not approximately controllable. Respectively, all approximate controllability results for the system in Equation 17 (or 16) that are based on approximate controllability of its linear part fail. Instead, we can investigate the \( L \)-partial \( A \)-controllability of the system in Equation 17 (or 16) related to the first component \( x_t \) of \( z_t \).

Letting \( L = \begin{bmatrix} 1 & 0 \end{bmatrix} \), we have
\[
\tilde{Q}_0 = LQ_0 L^* = b^2 \delta > 0, \quad 0 < \delta \leq T
\]

Therefore, by Theorem 1 and Remark 2, the semilinear system in Equation 17 (or 16) is \( L \)-partially \( A \)-controllable if \( f \) is continuous and satisfies the Lipschitz condition in \( x \) and \( y \).

**Example 6** Although the wave equation from Example 2 looks like suitable for application of partial controllability concepts, indeed, the system in 11 does not satisfy condition (F) of Theorem 1 since its linear part is \( L \)-partially \( A \)-controllable only for the time \( T \geq 2 \) (if the Fourier sine coefficients of \( b \) are nonzero), and, respectively, \( \tilde{Q}_0 > 0 \) only for \( \delta > 2 \). For this result, we refer to Zabczyk (1995) and Bashirov and Mahmudov (1999a). Making the nonlinear part of this system to be zero, we obtain another example of approximately controllable system (for the time \( T \geq 2 \) which does not satisfy condition (F)). This demonstrates that the condition of Theorem 1 is sufficient but not necessary.

**Example 7** Delay equations are typical for demonstration of partial controllability concepts. Consider a semilinear delay equation with distributed delays in the linear and nonlinear terms:
\[
\begin{align*}
  x'_t &= Ax_t + \int_{\tau-\theta}^{\tau} M_t x_{t+\theta} d\theta + Bu_t + f(t, x_t, \int_{\tau-\theta}^{\tau} N_t x_{t+\theta} d\theta, u_t) \\
  x_0 &= \xi, \quad x_0 = \eta_\theta, \quad -\varepsilon \leq \theta \leq 0
\end{align*}
\]
on \([0, T]\), where \(\epsilon > 0\), \(A \in \mathbb{R}^{n \times n}\), \(B \in \mathbb{R}^{n \times m}\), \(M, N \in C(-\epsilon, 0; \mathbb{R}^{n \times n})\), \(\xi \in \mathbb{R}^n\), \(\eta \in L_2(-\epsilon, 0; \mathbb{R}^n)\) and \(u \in U_{ad} = PC(0, T; \mathbb{R}^m)\).

Let \(\bar{x} : [0, T] \to L_2(-\epsilon, 0; \mathbb{R}^n)\) be a function defined by
\[
|\bar{x}_t|_\eta = x_{t, \eta}, \quad 0 \leq t \leq T, \quad -\epsilon \leq \theta \leq 0
\]
Then
\[
\bar{x}_t' = (d/d\theta) \bar{x}_t, \quad \bar{x}_0 = \eta, \quad 0 < t \leq T
\]
Let \(T_t, t \geq 0\), be the semigroup generated by the differential operator \(d/d\theta\) and let \(\Gamma_1\) and \(\Gamma_2\) be the integral operators from \(L_2(-\epsilon, 0; \mathbb{R}^n)\) to \(\mathbb{R}^n\), defined by
\[
\Gamma_1 h = \int_{-\epsilon}^{0} M_{\theta} h_{\theta} d\theta, \quad \Gamma_2 h = \int_{-\epsilon}^{0} N_{\theta} h_{\theta} d\theta, \quad h \in L_2(-\epsilon, 0; \mathbb{R}^n)
\]
Then for
\[
z_t = \left[ \begin{array}{c} x_t \\ \xi_t \end{array} \right], \quad \zeta = \left[ \begin{array}{c} \xi \\ \eta \end{array} \right] \in \mathbb{R}^n \times L_2(-\epsilon, 0; \mathbb{R}^n)
\]
we can write the system in Equation 18 as a semilinear system
\[
z_t' = \tilde{A} z_t + F(t, z_t, u_t) + \tilde{B} u_t, \quad z_0 = \zeta
\]
where
\[
\tilde{A} = \begin{bmatrix} A & \Gamma_1 \\ 0 & \partial/\partial \theta \end{bmatrix}, \quad F(t, z, u) = \begin{bmatrix} f(t, x, \Gamma_2 \bar{x}, u) \\ 0 \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B \\ 0 \end{bmatrix}
\]
and
\[
z = \left[ \begin{array}{c} x \\ \bar{x} \end{array} \right] \in \mathbb{R}^n \times L_2(-\epsilon, 0; \mathbb{R}^n)
\]
Define
\[
L = [ I \quad 0 ] : \mathbb{R}^n \times L_2(\epsilon, 0; \mathbb{R}^n) \to \mathbb{R}^n
\]
Then the approximate controllability of the system in Equation 18 is the same as the \(L\)-partial \(A\)-controllability of the system in Equation 19. The \(L\)-partial controllability operator of the system in Equation 19 is calculated in Bashirov et al. (2007) in the form
\[
\tilde{Q}_\delta = \int_{0}^{\delta} \mathcal{Y}_t BB^* \mathcal{Y}_t^* ds
\]
where \(\mathcal{Y}\) is a unique operator solution of the equation
\[
\mathcal{Y}_t = e^{A t} + \int_{0}^{\max(0, t-\epsilon)} \int_{-\epsilon}^{0} e^{A \theta} M_{\theta} \mathcal{Y}_{t-\theta} d\theta d\theta
\]
Hence, by Theorem 1, the system in Equation 19 is \(L\)-partially \(A\)-controllable and, respectively, the system in Equation 18 is approximately controllable if
\[
\int_{0}^{\delta} \mathcal{Y}_t BB^* \mathcal{Y}_t^* ds > 0 \quad \text{for all} \quad \delta > 0
\]
and the function \( f \) is continuous, bounded, and satisfies Lipschitz condition in its second and third variables.

In particular, if \( n = m = 1, A = a, B = b \) and \( M_\sigma \equiv 0 \), then in Bashirov and Jneid (2013) (see Equation 65) the \( L \)-partial controllability operator is calculated in the form

\[
\tilde{Q}_\sigma = \frac{b^2(e^{2\sigma a} - 1)}{2a} > 0
\]

for all \( \sigma > 0 \). So it just remains to assume the preceding conditions on \( f \) to obtain the approximate controllability of the system in Equation 18.

5. Conclusion
The basic contribution of this paper can be summarized in two items: (1) finding a sufficient condition for partial approximate controllability of a semilinear system and (2) proposing an alternative method for study of the controllability concepts.

The sufficient condition, given in Theorem 1, allows to get approximate controllability of one or several components of the state vector and becomes useful in the case when the total of the state vector is not approximately controllable. It is especially useful for systems which can be written as a first-order differential equation by enlarging the state space. Unfortunately, the important in applications wave equation does not fit to the frame of this sufficient condition (Example 6). At the same time, delay equations well suit to this frame (Example 7). Another kind of systems, for which partial controllability can be suitable are stochastic systems driven by wide band noises. In the linear case, these systems are investigated in Bashirov et al. (2007, 2010). This issue is not yet investigated for nonlinear stochastic systems and can be considered as a subject for a separate paper.

The proof method of Theorem 1 differs from the traditional method by fixed point theorems. We find this method natural and less complicated, although it has also disadvantages (Remarks 1 and 4). This method requires a separate consideration of the linear and nonlinear parts of a control system while the method by fixed point theorems combines the linear and nonlinear parts into one total. An interesting development may be a combination of these methods in the form: application of fixed point theorems on small intervals \([T - \delta, T]\) rather than on the total interval \([0, T]\). It seems a sufficient condition for the partial (or not) exact controllability can be proved by this combined method, in which the conditions on \( f \) can be relaxed.

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