On the graded algebras associated with Hecke symmetries, II. The Hilbert series

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Abstract

Hecke symmetries give rise to a family of graded algebras which represent quantum groups and spaces of noncommutative geometry. The present paper continues the work aiming to understand general properties of these algebras without a restriction on the parameter $q$ of Hecke relation used in earlier results. However, if $q$ is a root of 1, we need a restriction on the indecomposable modules for the Hecke algebras of type $A$ that can occur as direct summands of representations in the tensor powers of the initial vector space $V$. In this setting, we generalize known results on rationality of Hilbert series. The combinatorial nature of this problem stems from a relationship between the Grothendieck ring of the category of comodules for the Faddeev–Reshetikhin–Takhtajan bialgebra $A(R)$ associated with a Hecke symmetry $R$ and the ring of symmetric functions. We then improve two results on monoidal equivalences of corepresentation categories and on Gorensteinness of graded algebras from a previous article.

Keywords  Hecke symmetries · Graded algebras · Hilbert series

Mathematics Subject Classification  16S37 · 16T20

1 Introduction

With a Hecke symmetry $R$ on a finite-dimensional vector space $V$, one associates the $R$-symmetric algebra $S(V, R)$, the $R$-skewsymmetric algebra $Λ(V, R)$, and the bialgebra $A(R)$ given by the Faddeev–Reshetikhin–Takhtajan construction [24]. The first two algebras are noncommutative analogs of the symmetric and the exterior algebras of $V$. The algebra $A(R)$ is a noncommutative analog of the ring of polynomial functions

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on the space of $r \times r$ matrices. By localizing $A(R)$, one obtains a Hopf algebra which represents a nonstandard quantum group (see [15]).

For two Hecke symmetries $R, R'$ with the same parameter $q$ of the Hecke relation, there is also an algebra $A(R', R)$ generalizing $A(R)$. This algebra introduced by Hai [18] in a different notation represents a “quantum hom-space.” All these algebras are quadratic graded algebras.

The present paper continues the work started in [27] which attempts to understand general properties of the graded algebras associated with Hecke symmetries without a restriction on the parameter $q$ of the Hecke relation imposed in earlier results. The known results rely heavily on semisimplicity of the Hecke algebras $H_n = H_n(q)$ of type $A$ which operate in the tensor powers of the initial space $V$. This semisimple case occurs precisely when

$$1 + q + \cdots + q^{n-1} \neq 0 \text{ for all } n > 0.$$ 

Here, we will be concerned with the determination of Hilbert series and several related results. For a graded algebra $A = A_0 \oplus A_1 \oplus A_2 \oplus \ldots$ with finite-dimensional homogeneous components, its Hilbert series is a formal power series in one indeterminate defined as

$$H_A(t) = \sum (\dim A_n) t^n.$$ 

The question as to what are possible Hilbert series of the algebras $S(V, R)$ and $\Lambda(V, R)$ arose in the work of Gurevich [15]. About 10 years later Hai [16] and, independently, Davydov [5] observed that in the semisimple case, the dimensions of homogeneous components of the two algebras form totally positive sequences. From this, they deduced that the Hilbert series of these algebras are rational functions with negative roots and positive poles. This conclusion is based on analytic results obtained by Aissen et al. [1] and Edrei [12] which describe the generating series of totally positive sequences.

As we have seen in [27], good properties may be lost when $q$ is a root of 1. However, it was shown there that several previously known results extend to the case of an arbitrary $q$ provided that a certain additional condition is imposed. Recall that an indecomposable $\mathcal{H}_n$-module is said to have a one-dimensional source if it is a direct summand of an $\mathcal{H}_n$-module induced from a one-dimensional representation of a parabolic subalgebra, and we say that $R$ satisfies the one-dimensional source condition if for each $n > 0$ all indecomposable direct summands of $V^\otimes n$ regarded as an $\mathcal{H}_n$-module with respect to the representation arising from $R$ have one-dimensional sources. This condition is satisfied automatically in the semisimple case mentioned earlier. Our main result in the new paper is

**Theorem 4.8** Suppose that $R$ satisfies the one-dimensional source condition. Then,

$$H_{\Lambda(V, R)}(t) = f_0(-t)/f_1(t), \quad H_{S(V, R)}(t) = f_1(-t)/f_0(t)$$

with integer polynomials $f_0, f_1 \in \mathbb{Z}[t]$ whose constant terms are equal to 1, and all roots are positive real numbers.
The pair $(r_0, r_1)$ where $r_i = \deg f_i$ for $i = 0, 1$ is called the \textit{birank} of $R$. Thus, the Hilbert series of the two algebras can be written as

$$
\prod_{i=1}^{r_0} (1 + \alpha_i t) \cdot \prod_{j=1}^{r_1} (1 - \beta_j t)^{-1}
$$

and

$$
\prod_{j=1}^{r_1} (1 + \beta_j t) \cdot \prod_{i=1}^{r_0} (1 - \alpha_i t)^{-1}
$$

where $\alpha_i$ and $\beta_j$ are positive algebraic integers.

The already mentioned results of Hai and Davydov may be viewed as a nice application of the theory of symmetric functions. We will use a nonstandard notation Sym for the ring of symmetric functions defined as in Macdonald [22]. (In our paper, the letter $\Lambda$ is reserved for a different ring.) Consider the Grothendieck ring $\text{Grot}(R)$ of the category of finite-dimensional right $A(R)$-comodules. In the semisimple case, one can use a quantum version of the Schur–Weyl duality to obtain a ring homomorphism Sym $\rightarrow$ $\text{Grot}(R)$ under which each Schur function $s_\lambda$ is sent either to 0 or to the class of a simple comodule.

Since Sym is a polynomial ring in a countable set of indeterminates, a homomorphism $\varphi : \text{Sym} \rightarrow \text{Grot}(R)$ can be easily constructed in the case of arbitrary $q$ by specifying its values on the generators. The main obstacle we encounter is to show that the element $\varphi(s_\lambda) \in \text{Grot}(R)$ is \textit{positive} in the sense that $\varphi(s_\lambda)$ is the class of an actual comodule $V^\otimes$, in general not defined uniquely, of course. What is needed here can be reformulated in terms of the Grothendieck group of the category of finite-dimensional $\text{End}\mathcal{H}_n X$-modules where $X = V^\otimes n$ with the $\mathcal{H}_n$-module structure arising from $R$. The necessary property is stated in Corollary 3.8, and Sect. 3 is devoted to its proof. A key role is played by a version of the decomposition map which provides a bridge between the Grothendieck groups in the semisimple and nonsemisimple cases.

Total positivity of the sequence $\left(\dim S_n(V, R)\right)$ is an immediate consequence of positivity of the images of Schur functions under $\varphi$. Once it is known, we can invoke the analytic result of [1,12]. Actually it will be shown in Sect. 4 that rationality of Hilbert series can be explained by purely algebraic arguments, and the remainder of the proof is then much shorter than in general. In this way, we present a self-contained proof of Theorem 4.8.

Under the same assumption about $R$, it will be shown in Sect. 5 that the class of $V^\otimes n$ in the Grothendieck group of the category of finite-dimensional $\mathcal{H}_n$-modules is completely determined by the Hilbert series of $\mathcal{S}(V, R)$. Moreover, we describe in Theorem 5.5 a certain element $\text{ch}(V^\otimes n) \in \text{Sym}$ which contains full information about this class $[V^\otimes n]$. However, it is not clear whether $V^\otimes n$ can always be determined as an $\mathcal{H}_n$-module up to isomorphism.

If the algebra $A(V, R)$ is finite dimensional and $R$ satisfies the \textit{trivial source condition} in the sense that for each $n > 0$, the indecomposable $\mathcal{H}_n$-module direct summands of $V^\otimes n$ are induced from the \textit{trivial} one-dimensional representations of parabolic subalgebras of $\mathcal{H}_n$, then our results are much more complete. Indeed, the $\mathcal{H}_n$-module $V^\otimes n$ is described in Theorem 7.1. As a consequence, in this case, the algebra $A_n(R)^*\otimes$ dual to the subcoalgebra $A_n(R) \subset A(R)$ is Morita equivalent to the $q$-Schur algebra of Dipper and James $S_q(r, n)$ [8] where $r = r_0$ is the \textit{rank} of $R$ (we denote by $A_n(R)$ the degree $n$ homogeneous component of $A(R)$). Therefore, the category of $A_n(R)$-comodules
is equivalent to the well-studied category of $S_q(r, n)$-modules. In particular, this category depends only on $q$, $r$, and $n$, but not on $R$ itself. It is a highest weight category (see Donkin [9]).

It has been known for a long time that $A_n(R)^\ast \cong S_q(r, n)$ for many different quantizations of the semigroup of $r \times r$ matrices. This phenomenon was first observed by Du et al. [11] in the case of Takeuchi’s 2-parameter family of deformations. In an equivalent formulation, two bialgebras $A(R)$ and $A(R')$ in this family are isomorphic as coalgebras whenever the corresponding parameters satisfy a certain relation, and then their corepresentation categories are obviously equivalent. As seen from [11, (2.7)], it was not clear at that time whether these two categories are monoidally equivalent. On the level of Hopf envelopes, a general result on braided monoidal equivalence was obtained later by Hai in the semisimple case [19].

Theorem 7.1 is used to strengthen two results from the previous paper [27]. Keeping the previous assumption about $R$, let $R'$ be a second Hecke symmetry satisfying the same conditions. Theorem 7.3 states that there is a braided monoidal equivalence between the categories of $A(R)$-comodules and $A(R')$-comodules provided that the two Hecke symmetries have the same parameter $q$ and the same rank $r$. This equivalence is obtained by cotensoring right $A(R)$-comodules with the bicomodule algebra $A(R', R)$ (see [27, Th. 7.2]). By Theorem 7.4, the graded algebra $A(R', R)$ is Gorenstein under similar assumptions, and this time the equality of ranks is not required.

The trivial source indecomposable $\mathcal{H}_n$-modules are known as the Young modules [7]. As shown in [8], they are parameterized by partitions of $n$. Arbitrary indecomposable $\mathcal{H}_n$-modules with a one-dimensional source may be called signed Young modules as in the case of representations of symmetric groups [10,14]. However, an earlier text of Donkin [9] uses this term in a more restricted sense. There may be more such modules than partitions of $n$, and then the $\mathcal{H}_n$-modules are not distinguished by their images in Sym. Because of this, we cannot generalize Theorem 7.1 to Hecke symmetries satisfying the one-dimensional source condition.

In the semisimple case, the algebras $A(R)$ and $A(R', R)$ also have rational Hilbert series. Moreover, Hai gives a formula for $H_{A(R', R)}$ in terms of $H_{S(V, R)}$ and $H_{S(V', R')}$ [18, Th. 3.1]. In Sect. 6, these results are extended to the case of arbitrary $q$ under the assumption that both $R$ and $R'$ satisfy the one-dimensional source condition.

I would like to thank the referee for comments and several suggestions. Following the referee’s advice, I have added at the end of Sect. 2 a discussion of two families of Hecke symmetries on a two-dimensional space, verifying the one-dimensional source condition in these examples. The referee asked an interesting question concerning this condition which I am not able to answer. It has also been stated there.

2 Preliminaries

We fix an arbitrary field $\mathbb{k}$. Unless specified otherwise, algebras and coalgebras will be considered over $\mathbb{k}$. Let $V$ be a finite-dimensional vector space over $\mathbb{k}$. A Hecke symmetry on $V$ is a linear operator $R : V \otimes V \rightarrow V \otimes V$ satisfying the braid equation

$$(R \otimes \text{Id})(\text{Id} \otimes R)(R \otimes \text{Id}) = (\text{Id} \otimes R)(R \otimes \text{Id})(\text{Id} \otimes R)$$
and the quadratic Hecke relation

$$(R - q \cdot \text{Id})(R + \text{Id}) = 0 \quad \text{where} \quad 0 \neq q \in \mathbb{k}.$$ 

Denote by $\mathcal{H}_n(q)$ the Hecke algebra of type $A_{n-1}$ with the same parameter $q$ as in the quadratic relation imposed on $R$. Since $R$ and $q$ will generally be fixed, we do not indicate $q$ in the notation $\mathcal{H}_n$ when there is no danger of confusion. The algebra $\mathcal{H}_n$ is generated by $n - 1$ elements $T_1, \ldots, T_{n-1}$ subject to the defining relations

$$T_i T_j = T_j T_i \quad \text{whenever} \quad |i - j| > 1,$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{for} \quad i = 1, \ldots, n - 2,$$

$$(T_i - q)(T_i + 1) = 0 \quad \text{for} \quad i = 1, \ldots, n - 1.$$ 

Let $\mathfrak{S}_n$ be the symmetric group of permutations of the set $\{1, \ldots, n\}$. It is generated by basic transpositions $\tau_i = (i, i + 1)$, $0 < i < n$. Denote by $\ell(\sigma)$ the length of a permutation $\sigma \in \mathfrak{S}_n$ with respect to these generators. There is a standard basis $\{T_\sigma\}$ of $\mathcal{H}_n$ characterized by the properties that $T_\tau_i = T_i$ for each $i$ and $T_\pi \sigma = T_\pi T_\sigma$ for $\pi, \sigma \in \mathfrak{S}_n$ whenever $\ell(\pi \sigma) = \ell(\pi) + \ell(\sigma)$. We adopt the convention that $\mathcal{H}_0 = \mathcal{H}_1 = \mathbb{k}$.

The Hecke symmetry $R$ gives rise to a representation of $\mathcal{H}_n$ in the $n$th tensor power of $V$ such that $T_i$ acts on $V \otimes^n$ as the linear operator

$$R_i^{(n)} = \text{Id}^{\otimes(i-1)} \otimes R \otimes \text{Id}^{\otimes(n-i-1)}.$$ 

In this way, $V \otimes^n$ becomes a left $\mathcal{H}_n$-module.

Denote by $A(R)$ the $R$-matrix bialgebra. It decomposes as a direct sum of subcoalgebras

$$A(R) = \bigoplus_{n=0}^{\infty} A_n(R)$$

where $A_n(R)$ is the coalgebra dual to the finite-dimensional algebra $\text{End}_{\mathcal{H}_n} V \otimes^n$.

Let $\text{Coend} V$ be the coalgebra dual to $\text{End}_{\mathbb{k}} V$. For each $n \geq 0$, we may identify $(\text{Coend} V) \otimes^n$ with the dual of the algebra $(\text{End}_{\mathbb{k}} V) \otimes^n \cong \text{End}_{\mathbb{k}} V \otimes^n$ Since $\text{End}_{\mathcal{H}_n} V \otimes^n$ is a subalgebra of $\text{End}_{\mathbb{k}} V \otimes^n$, we have

$$A_n(R) \cong (\text{Coend} V) \otimes^n / I_n$$

where $I_n = (\text{End}_{\mathcal{H}_n} V \otimes^n)^\perp = \{ f \in (\text{Coend} V) \otimes^n \mid \langle f, \text{End}_{\mathcal{H}_n} V \otimes^n \rangle = 0 \}$ is a coideal of $(\text{Coend} V) \otimes^n$. Clearly, $I_n = 0$ for $n = 0, 1$. For $n > 1$, there is an equality

$$\text{End}_{\mathcal{H}_n} V \otimes^n = \bigcap_{i=1}^{n-1} E_i^{(n)}$$

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where $E_i^{(n)}$ stands for the centralizer of $R_i^{(n)}$ in $\text{End}_k V^{\otimes n}$. Since

$$E_i^{(n)} = (\text{End}_k V)^{(i-1)} \otimes (\text{End}_k^2 V^{\otimes 2}) \otimes (\text{End}_k V)^{(n-i-1)}$$

for each $i$, we get

$$I_n = \sum_{i=1}^{n-1} E_i^{(n)}^\perp = \sum_{i=1}^{n-1} (\text{Coend } V)^{(i-1)} \otimes I_2 \otimes (\text{Coend } V)^{(n-i-1)}.$$ 

This shows that $I = \bigoplus_{n=0}^{\infty} I_n$ is an ideal of the tensor algebra

$$\mathbb{T}(\text{Coend } V) = \bigoplus_{n=0}^{\infty} (\text{Coend } V)^{\otimes n}$$

generated by the homogeneous component $I_2$ of degree 2. By an earlier observation, $I$ is also a coideal. Therefore, $A(R) \cong \mathbb{T}(\text{Coend } V)/I$ gets the structure of a factor bialgebra of $\mathbb{T}(\text{Coend } V)$. This bialgebra coacts on $V$ universally with respect to the property that the induced coaction on $V^{\otimes 2}$ commutes with $R$. As observed in [21], this property characterizes the bialgebra arising from the FRT construction.

For an associative algebra $\mathfrak{A}$ over some field, we denote by $\text{Grot } \mathfrak{A}$ the Grothendieck group of the category of finite-dimensional left $\mathfrak{A}$-modules. To each finite-dimensional left $\mathfrak{A}$-module $X$, there corresponds an element $[X] \in \text{Grot } \mathfrak{A}$, and to each short exact sequence $0 \to X' \to X \to X'' \to 0$ of finite-dimensional left $\mathfrak{A}$-modules, there corresponds a relation $[X] = [X'] + [X'']$ in this group. The elements corresponding to finite-dimensional left $\mathfrak{A}$-modules form a subsemigroup of $\text{Grot } \mathfrak{A}$. Given $\xi \in \text{Grot } \mathfrak{A}$, we write $\xi \geq 0$ if $\xi$ lies in that subsemigroup, i.e., if $\xi = [X]$ for some finite-dimensional left $\mathfrak{A}$-module $X$.

By the definition we have given above

$$A_n(R) = \left(\text{End}_{\mathfrak{H}_n} V^{\otimes n}\right)^*.$$ 

Therefore, right $A_n(R)$-comodules may be identified with left modules for the algebra $\text{End}_{\mathfrak{H}_n} V^{\otimes n}$. The Grothendieck group $\text{Grot}_n(R)$ of the category of finite-dimensional right $A_n(R)$-comodules is identified with the group $\text{Grot}(\text{End}_{\mathfrak{H}_n} V^{\otimes n})$. The Grothendieck group of the category of finite-dimensional right $A(R)$-comodules is the direct sum

$$\text{Grot}(R) = \bigoplus_{n=0}^{\infty} \text{Grot}_n(R).$$

Moreover, $\text{Grot}(R)$ is a graded ring with respect to the multiplication induced by tensor products of comodules.

The algebras $S(V, R)$ and $A(V, R)$ are defined as the factor algebras of the tensor algebra $\mathbb{T}(V) = \bigoplus_{n=0}^{\infty} V^{\otimes n}$ by the ideals generated, respectively, by the subspaces

$$\text{Im } (R - q \cdot \text{Id}) \subset V^{\otimes 2} \quad \text{and} \quad \text{Ker } (R - q \cdot \text{Id}) \subset V^{\otimes 2}.$$ 

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These ideals are stable under the coaction of $A(R)$ on $\mathbb{T}(V)$, and therefore $S(V, R)$ and $A(V, R)$ are right $A(R)$-comodule algebras in a natural way. The homogeneous components $S_n(V, R)$ and $A_n(V, R)$ of these algebras are right $A_n(R)$-comodules for each $n \geq 0$.

A composition of $n$ is any finite sequences of positive integers $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $|\lambda| = n$ where the weight of $\lambda$ is defined as $|\lambda| = \sum \lambda_i$. The length of $\lambda$ is the number $\ell(\lambda) = k$ of its parts $\lambda_i$. As is done customarily, we extend $\lambda$ by putting $\lambda_i = 0$ for $i > \ell(\lambda)$. If $\lambda_i \geq \lambda_{i+1}$ for all $i$, then $\lambda$ is called a partition of $n$. Denote by $\mathcal{P}(n)$ the set of all partitions of $n$ and by $\mathcal{P}$ the disjoint union of the sets $\mathcal{P}(0), \mathcal{P}(1), \mathcal{P}(2), \ldots$ where $\mathcal{P}(0)$ is regarded as a single element set consisting of the zero partition.

We denote by $\text{Sym}$ the ring of symmetric functions in a countable set of commuting indeterminates $x_1, x_2, \ldots$ defined as in Macdonald [22]. Thus, if $\text{Sym}(r)$ stands for the subring of symmetric polynomials in the ring $\mathbb{Z}[x_1, \ldots, x_r]$, then $\text{Sym}$ is the limit of the inverse system

$$\cdots \rightarrow \text{Sym}(3) \rightarrow \text{Sym}(2) \rightarrow \text{Sym}(1) \rightarrow \mathbb{Z}$$

in the category of graded rings. If $u \in \text{Sym}$ and $\alpha = (\alpha_1, \ldots, \alpha_r) \in K^r$ where $K$ is any commutative ring, then $u(\alpha) \in K$ is defined as the value at $\alpha$ of the polynomial obtained by projecting $u$ to $\text{Sym}(r)$.

We conform to standard notation in regard to several families of symmetric functions [22]. The elementary and complete symmetric functions will be $e_n$ and $h_n$ ($n = 0, 1, \ldots$) with $e_0 = h_0 = 1$. For each partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, one defines $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$ and $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$. The monomial functions $m_\lambda$ and the Schur functions $s_\lambda$ labeled by partitions $\lambda$ form two other well-known $\mathbb{Z}$-bases of Sym. On each homogeneous component $\text{Sym}_\mu$ of the graded ring Sym, there is a standard scalar product $\langle \cdot, \cdot \rangle$ with respect to which $\{s_\lambda \mid \lambda \in \mathcal{P}(n)\}$ is an orthonormal basis.

To each composition $\lambda$ of $n$, there corresponds a parabolic subalgebra $\mathcal{H}_\lambda = \mathcal{H}_n(q)$ of the Hecke algebra $\mathcal{H}_n = \mathcal{H}_n(q)$. It is generated by the set $\{T_i \mid i \in \mathcal{I}_\lambda\}$ where

$$\mathcal{I}_\lambda = \{i \in \mathbb{N} \mid 1 \leq i < |\lambda| \text{ and } i \neq \lambda_1 + \cdots + \lambda_j \text{ for each } j = 1, \ldots, \ell(\lambda)\}$$

and has a basis $\{T_\sigma \mid \sigma \in \mathcal{S}_\lambda\}$ where $\mathcal{S}_\lambda$ is the Young subgroup of $\mathcal{S}_n$ generated by $\{\tau_i \mid i \in \mathcal{I}_\lambda\}$.

Each homomorphism $\chi : \mathcal{H}_\lambda \rightarrow \mathbb{K}$ is completely determined by its values on the generators. It follows from the Hecke relations that $\chi(T_i) \in \{-1, q\}$ for each $i \in \mathcal{I}_\lambda$. If both $i$ and $i + 1$ lie in $\mathcal{I}_\lambda$, then $\chi(T_i) = \chi(T_{i+1})$ by the braid relations. In other words, $\chi$ is constant on each of the $\ell(\lambda)$ contiguous (possibly empty) segments of lengths $\lambda_i - 1, i = 1, \ldots, \ell(\lambda)$, which comprise the set $\mathcal{I}_\lambda$.

If $\nu$ is a composition of $n$ obtained from $\lambda$ by permuting its components in an arbitrary order, then the subalgebras $\mathcal{H}_\lambda$ and $\mathcal{H}_\nu$ are conjugate by an inner automorphism of $\mathcal{H}_n$. In this case, the induction functors from $\mathcal{H}_\lambda$ and from $\mathcal{H}_\nu$ produce isomorphic $\mathcal{H}_n$-modules. Given a homomorphism $\chi : \mathcal{H}_\lambda \rightarrow \mathbb{K}$, it is possible to pass to an equivalent homomorphism $\chi' : \mathcal{H}_\nu \rightarrow \mathbb{K}$ such that all segments on which $\chi'$ takes value $q$ precede all segments on which $\chi'$ takes value $-1$, and any pair of
segments on which $\chi'$ takes the same value follow in nonincreasing order of their lengths.

When forming the $\mathcal{H}_n$-modules induced from one-dimensional representations of parabolic subalgebras, it suffices to consider only the pairs $(\mathcal{H}_v, \chi')$ satisfying the previous conditions. This leads to a parameterization of such modules by the set

$$\mathcal{P}^2(n) = \{(\lambda, \mu) \in \mathcal{P} \times \mathcal{P} | \lambda + |\mu| = n\}.$$  

Each pair $(\lambda, \mu) \in \mathcal{P}^2(n)$ determines a composition $(\lambda_1, \ldots, \lambda_{\ell(\lambda)}, \mu_1, \ldots, \mu_{\ell(\mu)})$.

We will denote by $\mathcal{H}_{\lambda, \mu}$ the corresponding parabolic subalgebra of $\mathcal{H}_n$, by $S_{\lambda, \mu}$ the corresponding Young subgroup of $S_n$, and by $I_{\lambda, \mu}$ the index set for the generators $T_i \in \mathcal{H}_{\lambda, \mu}$ and $t_i \in S_{\lambda, \mu}$. Then,

$$I_{\lambda, \mu} = I_{\lambda, \mu}^0 \cup I_{\lambda, \mu}^1 \quad \text{where} \quad I_{\lambda, \mu}^0 = \{i \in I_{\lambda, \mu} | i < |\lambda|\} = I_{\lambda},$$

$$I_{\lambda, \mu}^1 = \{i \in I_{\lambda, \mu} | i > |\lambda|\} = \{i \in I_{\lambda} | i < |\mu|\}.$$  

Note that $\mathcal{H}_{\lambda, \mu} \cong \mathcal{H}_{\lambda} \otimes \mathcal{H}_{\mu}$. Define a homomorphism $\chi_{\lambda, \mu} : \mathcal{H}_{\lambda, \mu} \to \mathbb{k}$ by the rule

$$\chi_{\lambda, \mu}(T_i) = \begin{cases} q & \text{for } i \in I_{\lambda, \mu}^0, \\ -1 & \text{for } i \in I_{\lambda, \mu}^1, \end{cases}$$

and denote by $\mathbb{k}_{\lambda, \mu}$ the corresponding one-dimensional $\mathcal{H}_{\lambda, \mu}$-module.

If $\mu = 0$, then $\mathcal{H}_{\lambda, 0} = \mathcal{H}_\lambda$ and $\mathbb{k}_{\lambda, 0}$ is the trivial $\mathcal{H}_\lambda$-module $\mathbb{k}_{\text{triv}}$ on which each generator $T_i \in \mathcal{H}_\lambda$ operates as multiplication by $q$. If $\lambda = 0$, then $\mathcal{H}_{0, \mu} = \mathcal{H}_\mu$ and $\mathbb{k}_{0, \mu}$ is the alternating $\mathcal{H}_\mu$-module $\mathbb{k}_{\text{alt}}$ on which each $T_i \in \mathcal{H}_\mu$ operates as multiplication by $-1$.

From the preceding discussion it follows that an indecomposable left $\mathcal{H}_n$-module has a one-dimensional (respectively, trivial) source if and only if it is isomorphic to a direct summand of the induced module $\mathcal{H}_n \otimes \mathcal{H}_{\lambda, \mu} \mathbb{k}_{\lambda, \mu}$ (respectively, $\mathcal{H}_n \otimes \mathcal{H}_{\lambda} \mathbb{k}_{\text{triv}}$) for some $(\lambda, \mu) \in \mathcal{P}^2(n)$ (respectively, $\lambda \in \mathcal{P}(n)$).

The Specht $\mathcal{H}_n$-modules $S^\lambda$ labeled by partitions $\lambda \in \mathcal{P}(n)$ were constructed by Dipper and James [6]. We will use this notation for left modules and sometimes also for right modules as in [6]. The dimension of $S^\lambda$ depends neither on the field $\mathbb{k}$ nor on the parameter $q$. In particular, it is the same as the dimension of the respective Specht module for the symmetric group $S_n$.

The Hecke algebras $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2, \ldots$ are all semisimple if and only if $[n]_q \neq 0$ for all $n > 0$ where

$$[n]_q = 1 + q + \cdots + q^{n-1}.$$  

We now recall briefly what happens in the semisimple case. The simple $\mathcal{H}_n$-modules are precisely the Specht modules. Moreover, each $S^\lambda$ is absolutely irreducible, so that
End $\mathcal{H}_n S^\lambda \cong \mathbb{k}$. By semisimplicity of $\mathcal{H}_n$, an arbitrary $\mathcal{H}_n$-module $N$ is a direct sum of its isotypic components, and the isotypic component of type $S^\lambda$ can be expressed as $\text{Hom}_{\mathcal{H}_n} (S^\lambda, N) \otimes S^\lambda$. Taking $N = V^\otimes n$, we get

$$V^\otimes n \cong \bigoplus_{\lambda \in \mathcal{P}(n)} V^\lambda_R \otimes S^\lambda$$

where $V^\lambda_R = \text{Hom}_{\mathcal{H}_n} (S^\lambda, V^\otimes n)$.

It follows from this decomposition that

$$\text{End}_{\mathcal{H}_n} V^\otimes n \cong \prod_{\lambda \in \mathcal{P}(n)} \text{End}_{\mathbb{k}} V^\lambda_R.$$}

Thus, the algebra $\text{End}_{\mathcal{H}_n} V^\otimes n$ is semisimple with $\{V^\lambda_R | \lambda \in \mathcal{P}(n), V^\lambda_R \neq 0\}$ being a full set of pairwise nonisomorphic simple left modules. Note that $V^\lambda_R \neq 0$ if and only if $S^\lambda$ embeds in $V^\otimes n$ as an $\mathcal{H}_n$-submodule. We see that all right $A(R)$-comodules are semisimple, which means that $A(R)$ is a cosemisimple bialgebra. Moreover,

$$\{V^\lambda_R | \lambda \in \mathcal{P}, V^\lambda_R \neq 0\}$$

is a full set of pairwise nonisomorphic simple right $A(R)$-comodules. Their isomorphism classes $[V^\lambda]$ form a $\mathbb{Z}$-basis of the Grothendieck group $\text{Grot}(R)$. Note also that $\dim S^\lambda$ is the multiplicity of $V^\lambda_R$ as an $A(R)$-comodule summand of $V^\otimes n$, while $\dim V^\lambda_R$ is the multiplicity of $S^\lambda$ as an $\mathcal{H}_n$-module summand of $V^\otimes n$.

Let $\mathcal{H}_{m,n}$ be the subalgebra of $\mathcal{H}_{m+n}$ generated by $\{T_i | 0 < i < m+n, i \neq m\}$. In other words, $\mathcal{H}_{m,n} = \mathcal{H}_{(m),(n)}$ for partitions $(m), (n)$ of length 1. Denote by $\text{ind}_{m,n}^{m+n}$ the induction functor from $\mathcal{H}_{m,n}$ to $\mathcal{H}_{m+n}$. There is a graded ring structure on the direct sum of Grothendieck groups $\bigoplus_{k=0}^{\infty} \text{Grot} \mathcal{H}_k$ defined by the rule

$$[X] \cdot [Y] = [\text{ind}_{m,n}^{m+n} X \otimes Y]$$

whenever $X$ is an $\mathcal{H}_m$-module and $Y$ an $\mathcal{H}_n$-module, both of finite dimension. Here, $X \otimes Y$ is viewed as an $\mathcal{H}_{m,n}$-module by means of the canonical isomorphism $\mathcal{H}_{m,n} \cong \mathcal{H}_m \otimes \mathcal{H}_n$. In this ring,

$$[S^\mu] \cdot [S^v] = \sum_{\lambda \in \mathcal{P}} c^\lambda_{\mu v} [S^\lambda]$$

for $\mu, v \in \mathcal{P}$

with the Littlewood–Richardson coefficients $c^\lambda_{\mu v}$ which also occur as structure constants for the multiplication in the ring of symmetric functions:

$$s_\mu s_v = \sum_{\lambda \in \mathcal{P}} c^\lambda_{\mu v} s_\lambda.$$

This means that there is an isomorphism of graded rings

$$\text{Sym} \cong \bigoplus_{k=0}^{\infty} \text{Grot} \mathcal{H}_k$$
under which $s_\lambda \in \text{Sym}$ corresponds to $[S^\lambda] \in \text{Grot } \mathcal{H}_{[\lambda]}$. Conceptual explanation of this isomorphism is provided by Zelevinsky’s approach [29]. With some additional structure, the direct sum of the groups $\text{Grot } \mathcal{H}_{k}$ satisfies the axioms of a connected positive selfadjoint Hopf algebra over $\mathbb{Z}$, and it contains only one irreducible primitive element. It was proved in [29] that such a Hopf algebra is unique up to isomorphism and is isomorphic to the ring of symmetric functions.

Lemma 2.1 For each $(\lambda, \mu) \in \mathcal{P}^2(n)$ the symmetric function $h_\lambda e_\mu$ maps to the class of the induced module $\mathcal{H}_n \otimes \mathcal{H}_{[\lambda, \mu]} k_{[\lambda, \mu]}$ in the Grothendieck group $\text{Grot } \mathcal{H}_n$. As a consequence, the following relations hold in this group:

$$\sum_{i=0}^{n} (-1)^i [\mathcal{H}_n \otimes \mathcal{H}_{[i, n-i]} k_{[i, n-i]}] = 0 \quad \text{if } n > 0,$$

$$[\mathcal{H}_n \otimes \mathcal{H}_\mu k_{\text{triv}}] = \sum_{\lambda \in \mathcal{P}(n)} K_{\lambda, \mu} [S^\lambda] \quad \text{for } \mu \in \mathcal{P}(n)$$

where $k_{[i, n-i]}$ is the one-dimensional $\mathcal{H}_{[i, n-i]}$-module associated with the pair of partitions $((i), (n-i)) \in \mathcal{P}^2(n)$ and $K_{\lambda, \mu}$ are the Kostka numbers.

Proof Since for each $p > 0$ the symmetric functions $h_p = s_{(p)}$ and $e_p = s_{(1^p)}$ map to the classes of the Specht modules $S^{(p)} = k_{\text{triv}}$ and $S^{(1^p)} = k_{\text{alt}}$ in the group $\text{Grot } \mathcal{H}_p$, it follows from the definition of the multiplication in the direct sum of the groups $\text{Grot } \mathcal{H}_{k}$ that $h_\lambda$ and $e_\lambda$, for any $\lambda \in \mathcal{P}$, map to the classes of induced modules $\mathcal{H}_{[\lambda]} \otimes \mathcal{H}_{\mu} k_{\text{triv}}$ and $\mathcal{H}_{[\lambda]} \otimes \mathcal{H}_{\mu} k_{\text{alt}}$ in the group $\text{Grot } \mathcal{H}_{[\lambda]}$. Hence,

$$h_\lambda e_\mu \mapsto [\mathcal{H}_{[\lambda]} \otimes \mathcal{H}_{\mu} k_{\text{triv}}] \cdot [\mathcal{H}_{[\lambda]} \otimes \mathcal{H}_{\mu} k_{\text{alt}}] = [\mathcal{H}_n \otimes \mathcal{H}_{[\lambda, \mu]} k_{[\lambda, \mu]}]$$

for $(\lambda, \mu) \in \mathcal{P}^2(n)$. The required equalities in the group $\text{Grot } \mathcal{H}_n$ are now immediate consequences of the well-known equalities

$$\sum_{i=0}^{n} (-1)^i h_i e_{n-i} = 0, \quad h_\mu = \sum_{\lambda \in \mathcal{P}(n)} K_{\lambda, \mu} s_\lambda$$

in the group $\text{Sym}_n$ (see [22, Ch. I]).

In the semisimple case, tensor products of simple $A(R)$-comodules are computed easily. If $\mu \in \mathcal{P}(m)$ and $\nu \in \mathcal{P}(n)$, then

$$V^\mu_R \otimes V^n_R = \text{Hom } \mathcal{H}_m (S^\mu, V^\otimes m) \otimes \text{Hom } \mathcal{H}_n (S^n, V^\otimes n)$$

$$\cong \text{Hom } \mathcal{H}_{m,n} (S^\mu \boxtimes S^n, V^\otimes (m+n))$$

$$\cong \text{Hom } \mathcal{H}_{m+n} (\text{ind}^{m+n}_{m,n} S^\mu \otimes S^n, V^\otimes (m+n)).$$
It follows that for each $\lambda \in \mathcal{P}(m+n)$, the multiplicity of $V^\lambda_R$ in $V^\mu_R \otimes V^\nu_R$ equals the multiplicity of the simple $\mathcal{H}_{m+n}$-module $S^\lambda$ in $\text{ind}_{m,n}^{m+n} S^\mu \otimes S^\nu$. Thus,

$$[V^\mu_R] \cdot [V^\nu_R] = \sum_{\lambda \in \mathcal{P}} C_{\mu,\nu}^{\lambda} [V^\lambda_R] \text{ in } \text{Grot}(R),$$

and so there is a surjective ring homomorphism $\varphi : \text{Sym} \to \text{Grot}(R)$ given by the assignments $s_{\lambda} \mapsto [V^\lambda_R]$, $\lambda \in \mathcal{P}$. Let

$$\Gamma(r_0, r_1) = \{\lambda \in \mathcal{P} | \lambda \leq r_1 \text{ for all } j > r_0\}$$

be the set of $(r_0, r_1)$-hook partitions. It was proved by Hai [16] that $V^\lambda_R \neq 0$ if and only if $\lambda \in \Gamma(r_0, r_1)$ where $(r_0, r_1)$ is the birank of $R$. Hence, the classes $[V^\lambda_R]$ with $\lambda \in \Gamma(r_0, r_1)$ form a $\mathbb{Z}$-basis of $\text{Grot}(R)$ and $\text{Ker} \varphi$ coincides with the $\mathbb{Z}$-linear span of $\{s_{\lambda} | \lambda \notin \Gamma(r_0, r_1)\}$.

In the present paper, it will be shown that some features of this situation extend to the nonsemisimple case provided that $R$ satisfies the one-dimensional source condition. I believe that this condition is responsible for good behavior of the associated graded algebras.

As an illustration, let us check that the one-dimensional source condition is satisfied for two known families of Hecke symmetries. Suppose that $\dim V = 2$, and let $x, y$ be a basis for $V$. The monomials $x, y$ form a basis for the tensor algebra $\mathbb{T}(V)$. We omit the sign $\otimes$ when writing these monomials. Consider first the standard Hecke symmetry defined by its action on the basis of $V^{\otimes 2}$ as follows:

$$R(x^2) = qx^2, \quad R(xy) = yx, \quad R(y^2) = qy^2, \quad R(yx) = qxy + (q - 1)yx.$$ 

For $n \geq 2$, it is easy to describe the respective action of generators $T_i \in \mathcal{H}_n$ on the basis of $V^{\otimes n}$. If $w \in V^{\otimes n}$ is one of basis monomials, then either $T_i w = qw$ or $T_i w$ is a linear combination of $w$ and another monomial obtained from $w$ by replacing one occurrence of either $xy$ or $yx$ by $yx$ or $xy$, respectively.

The linear span $W_{k,l}$ of all monomials of some weight $(k, l)$ where $k + l = n$ is stable under the action of $\mathcal{H}_n$. By the weight of a monomial $w$, we mean the pair of nonnegative integers which count the occurrences of $x$ and $y$ in $w$. It is moreover clear that the $\mathcal{H}_n$-module $W_{k,l}$ is generated by any monomial of weight $(k, l)$. In particular, $W_{k,l} = \mathcal{H}_n(x^k y^l)$. Note that $x^k y^l$ spans a one-dimensional $\mathcal{H}_{k,l}$-submodule with the trivial action of $\mathcal{H}_{k,l}$. Hence, $W_{k,l}$ is a homomorphic image of the induced module $\mathcal{H}_n \otimes \mathcal{H}_{k,l}$ $\mathbb{k}_{\text{triv}}$. Since $\dim W_{k,l} = \binom{n}{k}$, comparison of dimensions entails

$$W_{k,l} \cong \mathcal{H}_n \otimes \mathcal{H}_{k,l} \otimes \mathbb{k}_{\text{triv}}.$$ 

We have $V^{\otimes n} = \bigoplus_{k+l=n} W_{k,l}$. So it follows that all indecomposable direct summands of the $\mathcal{H}_n$-module $V^{\otimes n}$ have trivial sources.

Consider now the modified version of the Takeuchi–Tambara operator given in [17, p. 114]. With respect to a suitably rescaled basis of $V$ this Hecke symmetry is written
\[ R(x^2) = qx^2 + xy + yx, \quad R(xy) = -xy, \]
\[ R(y^2) = qy^2 + q(xy + yx), \quad R(yx) = -yx. \]

It has a rime matrix in the terminology of [23]. Note that \( Ru^2 = qu^2 \) where we put \( u = qx + y \). Define monomials \( a_k, b_k \) for \( k \geq 0 \) inductively, setting

\[
a_0 = b_0 = 1, \quad a_k = \begin{cases} x a_{k-1} & \text{if } k \text{ is odd}, \\ y a_{k-1} & \text{if } k \text{ is even}, \end{cases} \quad b_k = \begin{cases} y b_{k-1} & \text{if } k \text{ is odd}, \\ x b_{k-1} & \text{if } k \text{ is even}. \end{cases}
\]

For instance, \( a_4 = yxyx \). Then,

\[ T_i u^k = qu^k, \quad T_i a_k = -a_k, \quad T_i b_k = -b_k \quad (1 \leq i < k). \]

If \( k + l = n \), then \( u^k a_l \) spans a one-dimensional \( \mathcal{H}_{k,l} \)-submodule of \( V^\otimes n \) on which \( T_i \) acts as \( q \cdot \text{Id} \) when \( 1 \leq i < k \) and as \( -\text{Id} \) when \( k \leq i < n \). The \( \mathcal{H}_n \)-submodule \( W_{k,l} \) generated by \( u^k a_l \) is a homomorphic image of the induced module \( \mathcal{H}_n \otimes \mathcal{H}_{k,l} \triangleleft u^k a_l \).

Let us prove by induction on \( n \) that the \( \mathcal{H}_n \)-module \( V^\otimes n \) is generated by \( \{ u^k a_l \mid k + l = n \} \), as well as by \( \{ u^k b_l \mid k + l = n \} \). If \( n = 1 \), this assertion follows from the fact that \( u \) and \( x \), as well as \( u \) and \( y \) are linearly independent. Suppose that \( n > 1 \), and the claim holds for lower values of \( n \). Put

\[ X = \sum_{k+l=n} W_{k,l} = \sum_{k+l=n} \mathcal{H}_n(u^k a_l), \quad Y = \sum_{k+l=n} \mathcal{H}_n(u^k b_l). \]

Since \( a_l = b_{l-1} x \) for \( l > 0 \), we get \( Y'x \subset X \) where \( Y' = \sum_{k+l=n-1} \mathcal{H}_{n-1}(u^k b_l) \). By the induction hypothesis, \( Y' = V^{n-1} \). (We are omitting the sign \( \otimes \).) It is verified similarly that \( V^{n-1} y \subset Y \). Since

\[ T_{n-1}(tx^2) = qtx^2 + txy + ytx \quad (t \in V^{n-2}) \]

and both \( tx^2 \in X \) and \( txy \in X \), we deduce that \( txy \in X \) for all \( t \in V^{n-2} \) as well. In particular, \( u^k b_l \in X \) whenever \( k + l = n \) and \( l \geq 2 \). But \( u^n \in X \), and also

\[ u^{n-1} b_1 = u^{n-1} y = u^n - qua^{n-1}x \in X. \]

It follows that \( Y \subset X \). Hence \( V^n = V^{n-1} x + V^{n-1} y \subset X \), i.e., \( X = V^n \). It is proved similarly that \( X \subset Y \), and so \( Y = V^n \).

Thus, \( \sum_{k+l=n} W_{k,l} = V^\otimes n \). Since dim \( V^\otimes n = 2^n \), while dim \( W_{k,l} \leq \binom{n}{k} \), we must have

\[ V^\otimes n = \bigoplus_{k+l=n} W_{k,l} \quad \text{and} \quad W_{k,l} \cong \mathcal{H}_n \otimes \mathcal{H}_{k,l} \triangleleft u^k a_l. \]
This means that all indecomposable direct summands of the $H_n$-module $V^\otimes n$ have one-dimensional sources. If $q \neq -1$, then $R$ has birank $(1, 1)$. In contrast to this, $R$ is an even Hecke symmetry of rank 2 when $q = -1$. The corresponding quantum group was described in [26]. This example shows that properties of Hecke symmetries and the associated graded algebras may change drastically under specialization of parameter.

It can be expected that the one-dimensional source condition should hold for all Hecke symmetries which give rise to graded algebras with classical Hilbert series. However, explicit decomposition of $V^\otimes n$ as a direct sum of induced modules may be hard to write out in less standard examples. The referee asked whether that condition is preserved under specialization:

**Question** Let $O$ be a discrete valuation ring with residue field $\mathbb{k}$. Suppose that $R \in \text{Mat}_d(O)$ is a Hecke symmetry over $O$ satisfying the Hecke relation

$$(R - z \cdot \text{Id})(R + \text{Id}) = 0$$

where $z$ is an invertible element of $O$ such that all $z$-integers $1 + z + \cdots + z^{i-1}$ for $i > 0$ are nonzero. Does then its specialization $R_\mathbb{k} = R \otimes_O \mathbb{k}$ satisfy the one-dimensional source condition?

In my opinion, this question admits no easy solution because it is impossible to control how a finite-dimensional module behaves under specialization. The answer may still be “yes,” but it requires hidden properties of Hecke symmetries.

### 3 The decomposition map

The decomposition map is a standard tool in the modular representation theory of finite groups. More generally, such a map can be defined in the following situation (see, e.g., [13, 7.4.3]). Suppose that $O$ is a discrete valuation ring with residue field $\mathbb{k}$ and the field of fractions $Q$. Let $A$ be an associative unital algebra over $O$ whose underlying $O$-module is free of finite rank. Then, $A_\mathbb{k} = A \otimes_O \mathbb{k}$ and $A_Q = A \otimes_O Q$ are finite-dimensional algebras over the respective fields. By an $A$-lattice, we mean any finitely generated $O$-free $A$-module (a finitely generated $O$-module is free if and only if it is torsionfree). The decomposition map

$$d : \text{Grot } A_Q \longrightarrow \text{Grot } A_\mathbb{k}$$

is a homomorphism of groups characterized by the property that $d([L_Q]) = [L_\mathbb{k}]$ for each left $A$-lattice $L$ where $L_Q = L \otimes_O Q$ and $L_\mathbb{k} = L \otimes_O \mathbb{k}$.

This map is well-defined since each $A_Q$-module of finite dimension over $Q$ is isomorphic to $L_Q$ for some $A$-lattice $L$, and the image of $L_\mathbb{k}$ in $\text{Grot } A_\mathbb{k}$ does not depend on the choice of $L$. Every short exact sequence of $A_Q$-modules is the image under the functor $\otimes_Q Q$ of a short exact sequence of $A$-lattices $0 \to L' \to L \to L'' \to 0$. Since the latter splits as a sequence of $O$-modules, it induces a short exact
sequence $0 \rightarrow L'_k \rightarrow L_k \rightarrow L''_k \rightarrow 0$. Therefore, the defining relations of the group $\text{Grot } A_Q$ map to the defining relations of the group $\text{Grot } A_k$.

Let $z$ be an invertible element of $O$ and $\mathcal{H}_n(z)$ the Hecke algebra of type $A_{n-1}$ with parameter $z$ over the ring $O$. If $q$ is the image of $z$ in $k$, then $\mathcal{H}_n(z)_k \cong \mathcal{H}_n(q)$. We will assume that

$$1 + z + \cdots + z^{i-1} \neq 0 \quad \text{for all } i > 0.$$  

This can be achieved, e.g., by taking $z$ to be an indeterminate and $O$ the localization of the polynomial ring $k[z]$ at its maximal ideal generated by $z - q$. Then, $\mathcal{H}_n(z)_Q$ is a semisimple Hecke algebra of type $A_{n-1}$ over the field $Q$. By completing $O$, we may also assume that $O$ is a complete discrete valuation ring.

We will denote by $T_1, \ldots, T_{n-1}$ the canonical generators of $\mathcal{H}_n(z)$ and also their canonical images in either $\mathcal{H}_n(q)$ or $\mathcal{H}_n(z)_Q$. For $(\lambda, \mu) \in P^2(n)$ we have a parabolic subalgebra $\mathcal{H}_{\lambda, \mu}(z)$ generated by $\{T_i \mid i \in I_{\lambda, \mu}\}$ (see Sect. 2). Define a homomorphism of $O$-algebras $\chi_{\lambda, \mu} : \mathcal{H}_{\lambda, \mu}(z) \rightarrow O$ by the rule

$$\chi_{\lambda, \mu}(T_i) = \begin{cases} z & \text{for } i \in I_{\lambda, \mu}, \\ -1 & \text{for } i \notin I_{\lambda, \mu}, \end{cases}$$

and let $O_{\lambda, \mu}$ be $O$ with the $\mathcal{H}_{\lambda, \mu}(z)$-module structure arising from $\chi_{\lambda, \mu}$. Let $Q_{\lambda, \mu}$ be the one-dimensional $\mathcal{H}_{\lambda, \mu}(z)_Q$-module defined similarly. We will write $\mathcal{H}_{\lambda, \mu}(z)$, $O_{\text{triv}}$, and $Q_{\text{triv}}$ instead of $\mathcal{H}_{\lambda, \mu}(z)$, $O_{\lambda, \mu}$, and $Q_{\lambda, \mu}$ when $\lambda \in P(n)$ and $\mu = 0$.

**Notation** Denote by $\mathcal{R}ep^1$ (respectively, by $\mathcal{R}ep^0$) the class of all $\mathcal{H}_n(z)$-lattices isomorphic to direct summands of finite direct sums of the induced modules

$$M^{\lambda, \mu} = \mathcal{H}_n(z) \otimes \mathcal{H}_{\lambda, \mu}(z) O_{\lambda, \mu} \quad \text{(respectively, } M^\lambda = \mathcal{H}_n(z) \otimes \mathcal{H}_n(z) O_{\text{triv}})$$

for various $(\lambda, \mu) \in P^2(n)$ (respectively, $\lambda \in P(n)$).

Note that $M^\lambda = M^{\lambda, 0}$. Since $\mathcal{H}_n(z)$ is a free $\mathcal{H}_{\lambda, \mu}(z)$-module with respect to the action by right multiplications, $M^{\lambda, \mu}$ is an $\mathcal{H}_n(z)$-module. The functor $\otimes_O k$ takes $M^{\lambda, \mu}$ to

$$M^{\lambda, \mu}_k \cong \mathcal{H}_n(q) \otimes \mathcal{H}_{\lambda, \mu}(q) k_{\lambda, \mu}.$$ 

It follows that all indecomposable direct summands of the $\mathcal{H}_n(q)$-module $M_k$ have a one-dimensional (respectively, trivial) source whenever $M \in \mathcal{R}ep^1$ (respectively, $M \in \mathcal{R}ep^0$).

**Lemma 3.1** Let $M, N \in \mathcal{R}ep^1$. If $q = -1$ assume that $M, N \in \mathcal{R}ep^0$. Then,

$$\text{Hom}_{\mathcal{H}_n(q)}(N_k, M_k) \cong \text{Hom}_{\mathcal{H}_n(z)}(N, M) \otimes_O k,$$

$$\text{Hom}_{\mathcal{H}_n(z)_Q}(N_Q, M_Q) \cong \text{Hom}_{\mathcal{H}_n(z)}(N, M) \otimes_O Q.$$ 

Hence $\dim_k \text{Hom}_{\mathcal{H}_n(q)}(N_k, M_k) = \dim_Q \text{Hom}_{\mathcal{H}_n(z)_Q}(N_Q, M_Q).$
Proof The canonical maps $k_{XY} : \text{Hom} \mathcal{H}_\pi(z)(X, Y) \otimes O \xrightarrow{\cong} \text{Hom} \mathcal{H}_\pi(q)(X_k, Y_k)$ defined for arbitrary $\mathcal{H}_\pi(z)$-modules $X$ and $Y$ give a natural transformation of two functors additive in each argument. If $X \cong X' \oplus X''$ (respectively, $Y \cong Y' \oplus Y''$), then bijectivity of $k_{XY}$ is equivalent to bijectivity of $k_{X'Y'}$ and $k_{X''Y''}$ (respectively, $k_{XY'}$ and $k_{XY''}$). By the conditions on $M$ and $N$ in the statement of Lemma 3.1, it suffices therefore to prove that $k_{NM}$ is bijective when $q \neq -1$ and $N = M^{\lambda, \mu}$, $M = M^{\rho, \theta}$ for some pairs $(\lambda, \mu)$, $(\rho, \theta) \in \mathcal{D}^2(n)$ or when $q = -1$ and $N = M^\rho$, $M = M^\rho$ with $\lambda$, $\rho \in \mathcal{P}(n)$.

Consider the case $q \neq -1$. Denote by $\mathcal{D}$ the set of distinguished representatives of the $\mathcal{G}_{\lambda, \mu} - \mathcal{G}_{\rho, \theta}$ double cosets in $\mathcal{G}_n$. Then, $M = \bigoplus_{\pi \in \mathcal{D}} M(\pi)$ where $M(\pi)$ is the $\mathcal{H}_{\lambda, \mu}(z)$-submodule of $M$ generated by the element $T_{\pi} \otimes 1 \in M$. This is the Mackey decomposition of the induced module $M$ with respect to the parabolic subalgebra $\mathcal{H}_{\lambda, \mu}(z)$ (see [6, 2.7] and [13, 9.1.8]). For each $\pi \in \mathcal{D}$, let $v(\pi)$ be the composition of $n$ such that

$$I_{\nu(\pi)} = \{ i \in I_{\lambda, \mu} | \pi^{-1} \tau_i \pi \in \mathcal{G}_{\rho, \theta} \} = \{ i \in I_{\lambda, \mu} | \pi^{-1} \tau_i \pi = \tau_j \text{ for some } j \in I_{\rho, \theta} \}.$$  

If $i \in I_{\nu(\pi)}$, then $\ell(\tau_i \pi) > \ell(\pi)$ since $\pi$ is the shortest element in the coset $\mathcal{G}_{\lambda, \mu} \pi$, and the equality $\tau_i = \pi \tau_j \pi^{-1}$ means that $i = \pi(j)$ and $i + 1 = \pi(j + 1)$. Therefore, $T_i T_\pi = T_\pi T_i = T_{j + 1} T_j$ with $j = \pi^{-1}(i) \in I_{\rho, \theta}$, i.e., $T_{j + 1} T_j = T_{j + 1}(i) \in \mathcal{H}_{\rho, \theta}(z)$. By the Mackey formula

$$M(\pi) \cong \mathcal{H}_{\lambda, \mu}(z) \otimes \mathcal{H}_{\nu(\pi)}(z) O(\chi_{\nu(\pi)}')$$

where $O(\chi_{\nu(\pi)})$ is $O$ regarded as an $\mathcal{H}_{\nu(\pi)}(z)$-module by means of the ring homomorphism $\chi_{\nu(\pi)}' : \mathcal{H}_{\nu(\pi)}(z) \rightarrow O$ such that

$$\chi_{\nu(\pi)}'(h) = \chi_{\rho, \theta}(T_{\pi}^{-1} h T_\pi) \quad \text{for } h \in \mathcal{H}_{\nu(\pi)}(z).$$

Note that

$$\chi_{\nu(\pi)}'(T_i) = \chi_{\rho, \theta}(T_{\pi}^{-1}(i)) = \begin{cases} z & \text{for } i \in I_{\nu(\pi)} \text{ with } \pi^{-1}(i) \in I_{\rho, \theta}^0, \\ -1 & \text{for } i \in I_{\nu(\pi)} \text{ with } \pi^{-1}(i) \in I_{\rho, \theta}^1. \end{cases}$$

By the Frobenius reciprocity (see [6, 2.5, 2.6] and [13, 9.1.7]),

$$\text{Hom} \mathcal{H}_\pi(z)(N, M) \cong \text{Hom} \mathcal{H}_{\lambda, \mu}(z)(O_{\lambda, \mu}, M) \cong \bigoplus_{\pi \in \mathcal{D}} \text{Hom} \mathcal{H}_{\nu(\pi)}(z)(O_{\lambda, \mu}, O(\chi_{\nu(\pi)}')).$$

The $O$-module $\text{Hom} \mathcal{H}_{\nu(\pi)}(z)(O_{\lambda, \mu}, O(\chi_{\nu(\pi)}'))$ is nonzero precisely when $\chi_{\lambda, \mu}$ agrees with $\chi_{\nu(\pi)}'$ on $\mathcal{H}_{\nu(\pi)}(z)$, in which case this module is isomorphic to $O$. It follows that $\text{Hom} \mathcal{H}_\pi(z)(M, N)$ is a free $O$-module with a basis indexed by the set of those $\pi \in \mathcal{D}$ for which $I_{\nu(\pi)} \cap I_{\lambda, \mu} = I_{\nu(\pi)} \cap \pi(I_{\rho, \theta}^0)$.

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The respective homomorphisms $N \to M$ can be described explicitly as in \cite{6,3.4}. The vector space $\text{Hom}_{\mathcal{H}_n(q)}(N_k, M_k)$ has a similar description, and we see that the functor $\otimes_{O_k}$ takes the basic homomorphisms $N \to M$ to the basic homomorphisms $N_k \to M_k$. If $\mu = 0$ and $\theta = 0$, then the basic homomorphisms are parameterized by the whole set $\mathcal{D}$, and the previous arguments go through for $q = -1$ as well. This proves bijectivity of $k_{NM}$.

The second isomorphism in the statement of Lemma 3.1 can be explained in exactly the same way, but in fact it is almost obvious and holds more generally for arbitrary $\mathcal{H}_n(z)$-modules $M$ and $N$ with the only restriction that $N$ should be finitely generated. In the last assertion of Lemma 3.1, both dimensions are equal to the rank of the free $O$-module $\text{Hom}_{\mathcal{H}_n(z)}(N, M)$. $\square$

For the rest of this section with the exception of Lemma 3.7, we fix $M \in \mathcal{R}ep^1$, and moreover we will assume that $M \in \mathcal{R}triv$ when $q = -1$. The ring $A = \text{End}_{\mathcal{H}_n(z)} M$ is an algebra over $O$ whose underlying $O$-module is free of finite rank. By Lemma 3.1 $A_k \cong \text{End}_{\mathcal{H}_n(q)} M_k$ and $A_Q \cong \text{End}_{\mathcal{H}_n(z)} M_Q$. Thus, we have the decomposition map

$$d : \text{Grot}(\text{End}_{\mathcal{H}_n(z)} M_Q) \longrightarrow \text{Grot}(\text{End}_{\mathcal{H}_n(q)} M_k).$$

**Lemma 3.2** If $q \neq -1$ and $(\lambda, \mu) \in \mathcal{P}^2(n)$, then

$$d([Q_{\lambda, \mu} \otimes_{\mathcal{H}_n(z)} M_Q]) = [k_{\lambda, \mu} \otimes_{\mathcal{H}_n(q)} M_k].$$

In particular, $d([Q_{\text{triv}} \otimes_{\mathcal{H}_n(q)} M_Q]) = [k_{\text{triv}} \otimes_{\mathcal{H}_n(q)} M_k]$ for $\lambda \in \mathcal{P}(n)$, and this equality holds even when $q = -1$.

**Proof** Put $L = O_{\lambda, \mu} \otimes_{\mathcal{H}_n(z)} M$. This is an $\text{End}_{\mathcal{H}_n(z)} M$-module such that

$$L_k \cong k_{\lambda, \mu} \otimes_{\mathcal{H}_n(q)} M_k \quad \text{and} \quad L_Q \cong Q_{\lambda, \mu} \otimes_{\mathcal{H}_n(z)} M_Q.$$

We will check that $L$ is $O$-free of finite rank, i.e., $L$ is a lattice. It will follow then that $d([L_Q]) = [L_k]$ by the definition of $d$, and we will get the required equality.

Since the verifications can be done on direct summands, it suffices to consider the case when $M = M^\rho, \theta$ for some $(\rho, \theta) \in \mathcal{P}^2(n)$. Using the Mackey decomposition $M = \bigoplus_{\pi \in \mathcal{D}} M(\pi)$ with respect to $\mathcal{H}_n(\pi)$ as in the proof of Lemma 3.1, we get $L = \bigoplus_{\pi \in \mathcal{D}} L(\pi)$ with

$$L(\pi) = O_{\lambda, \mu} \otimes_{\mathcal{H}_n(z)} M(\pi) \cong O_{\lambda, \mu} \otimes_{\mathcal{H}_n(z)} O(\chi^\prime_\pi) \cong O/I_\pi$$

where $I_\pi$ is the ideal of $O$ generated by $\{\chi_{\lambda, \mu}(h) - \chi^\prime_\pi(h) \mid h \in \mathcal{H}_n(\pi)(z)\}$. Since the $O$-algebra $\mathcal{H}_n(\pi)(z)$ is generated by $\{T_i \mid i \in \mathcal{J}_n(\pi)\}$, the ideal $I_\pi$ is generated by the elements

$$\chi_{\lambda, \mu}(T_i) - \chi^\prime_\pi(T_i) \quad \text{with} \quad i \in \mathcal{J}_n(\pi).$$

$\square$ Springer
If $\chi_{\lambda,\mu}$ agrees with $\chi_{\pi}'$ on $\mathcal{H}_{\nu(\pi)}(z)$, then $I_{\pi} = 0$. Otherwise $\chi_{\lambda,\mu}(T_i) \neq \chi_{\pi}'(T_i)$ for at least one $i \in \mathcal{F}_{\nu(\pi)}$. Since $-1$ and $z$ are the only two possible values of the homomorphisms $\chi_{\lambda,\mu}$ and $\chi_{\pi}'$ on the generators $T_i$, we get $I_{\pi} = (z + 1)O$ in the latter case. The image of $z + 1$ in the residue field $\mathbb{k}$ of the local ring $O$ equals $q + 1$. If $q \neq -1$, then $z + 1$ is invertible in $O$, and so $I_{\pi} = O$. We see that $O/I_{\pi}$ equals either $O$ or $0$ for each $\pi \in \mathcal{D}$ when $q \neq -1$, i.e., each $O$-module in the direct sum decomposition of $L$ is free of rank $1$ or $0$.

Suppose now that $M \in \mathcal{D}_{\text{triv}}$. In this case, we may assume that $M = M^\rho$ for some $\rho \in \mathcal{D}(n)$, i.e., we can take $\theta = 0$. Then $\chi_{\pi}'$ is the trivial representation. If $\mu = 0$, then $\chi_{\lambda,\mu}$ is the trivial representation as well, whence $I_{\pi} = 0$ and $O/I_{\pi} \cong O$ for all $\pi$, even when $q = -1$. \hfill $\square$

**Notation** For any $\mathcal{H}_n(q)$-module $X$ and $(\lambda, \mu) \in \mathcal{P}^2(n)$ put

$$
\Sigma_{\lambda,\mu}(X) = \sum_{i \in \mathcal{F}_{\lambda,\mu}} (T_i - q)X + \sum_{i \in \mathcal{F}_{\lambda,\mu}^1} \{ x \in X \mid T_i x = qx \}.
$$

This is an $\text{End}_{\mathcal{H}_n(q)}$ $X$-submodule of $X$. In particular, we write

$$
\Sigma_{i,n-i}(X) = \sum_{0 < j < i} (T_j - q)X + \sum_{i < j < n} \{ x \in X \mid T_j x = qx \}
$$

when $\lambda = (i)$, $\mu = (n - i)$ for some $i = 0, \ldots, n$, and we write

$$
\Sigma_\lambda(X) = \Sigma_{\lambda,0}(X) = \sum_{i \in \mathcal{F}_\lambda} (T_i - q)X
$$

when $\lambda \in \mathcal{P}(n)$, $\mu = 0$.

The case $q = -1$ incurs technical complications, and we have to look deeper into the structure of induced modules. The conclusion of Lemma 3.2 is reformulated below in a form suitable for any $q$:

**Lemma 3.3** We have $d([Q_{\lambda,\mu} \otimes \mathcal{H}_{\lambda,\mu}(z)Q]M_Q)) = [M_{\mathbb{k}}/\Sigma_{\lambda,\mu}(M_{\mathbb{k}})]$.

**Proof** Put $L = O_{\lambda,\mu} \otimes \mathcal{H}_{\lambda,\mu}(z) M$ and $A = \text{End}_{\mathcal{H}_n(z)} M$. The assignment $m \mapsto 1 \otimes m$ defines an epimorphism of $A$-modules $\psi : M \rightarrow L$. The induced map $\psi_{\mathbb{k}} = \psi \otimes \mathbb{k}$ is an epimorphism of $A_{\mathbb{k}}$-modules $M_{\mathbb{k}} \rightarrow L_{\mathbb{k}} \cong \mathbb{k}{\lambda,\mu} \otimes \mathcal{H}_{\lambda,\mu}(q)M_{\mathbb{k}}$ such that

$$
\text{Ker} \psi_{\mathbb{k}} = \sum_{h \in \mathcal{H}_{\lambda,\mu}(q)} (h - \chi_{\lambda,\mu}(h))M_{\mathbb{k}} = \sum_{i \in \mathcal{F}_{\lambda,\mu}} (T_i - \chi_{\lambda,\mu}(T_i))M_{\mathbb{k}} = \sum_{i \in \mathcal{F}_{\lambda,\mu}^0} (T_i - q)M_{\mathbb{k}} + \sum_{i \in \mathcal{F}_{\lambda,\mu}^1} (T_i + 1)M_{\mathbb{k}}.
$$

\hfill $\square$ Springer
Suppose that \( q \neq -1 \). It follows then from the relation \((T_i - q)(T_i + 1) = 0\) that the equality \( T_i m = q m \) holds for an element \( m \in M_{\mathbb{k}} \) if and only if \( m \in (T_i + 1)M_{\mathbb{k}} \). Hence, \( \text{Ker} \psi_{\mathbb{k}} = \Sigma_{\lambda, \mu}(M_{\mathbb{k}}) \), and

\[
L_{\mathbb{k}} \cong M_{\mathbb{k}} / \text{Ker} \psi_{\mathbb{k}} = M_{\mathbb{k}} / \Sigma_{\lambda, \mu}(M_{\mathbb{k}}).
\]

The equality \( d([L_Q]) = [L_{\mathbb{k}}] \) of Lemma 3.2 can be rewritten as in the statement of Lemma 3.3.

Suppose further that \( q = -1 \). In this case, \( L \) may fail to be \( O \)-free. The set \( L^0 \) of all elements of \( L \) which have nonzero annihilators in \( O \) is an \( A \)-submodule of \( L \). The factor module \( \overline{L} = L/L^0 \) is \( O \)-torsionfree and \( \overline{L}_Q \cong L_Q \). Hence, \( \overline{L} \) is a lattice, which yields \( d([L_Q]) = [L_{\mathbb{k}}] \).

We have \( \overline{L} \cong M/M^0 \) where \( M^0 = \psi^{-1}(L^0) \subset M \). Consider \( M^0_{\mathbb{k}} = M^0 \otimes_O \mathbb{k} \).

The canonical map \( M^0_{\mathbb{k}} \to M_{\mathbb{k}} \) is injective since \( \overline{L} \) is \( O \)-free. Thus, \( M^0_{\mathbb{k}} \) is identified with an \( A_{\mathbb{k}} \)-submodule of \( M_{\mathbb{k}} \). By the right exactness of the functor \( ? \otimes_O \mathbb{k} \), we obtain \( \overline{L}_{\mathbb{k}} \cong M_{\mathbb{k}}/M^0_{\mathbb{k}} \), and it remains to show that \( M^0_{\mathbb{k}} = \Sigma_{\lambda, \mu}(M_{\mathbb{k}}) \).

Note that \( M^0_{\mathbb{k}} \) and \( \Sigma_{\lambda, \mu}(M_{\mathbb{k}}) \) are evaluations at \( M \) of two additive functors defined on the category of \( \mathcal{H}_n(z) \)-modules. Verification of the equality \( M^0_{\mathbb{k}} = \Sigma_{\lambda, \mu}(M_{\mathbb{k}}) \) can be done therefore on direct summands of \( M \). Since \( M \in \mathcal{T}_{\text{triv}} \), it suffices to consider the case when \( M = M^0 \) for some \( \rho \in \mathcal{P}(n) \). Let us assume this and take \( \theta = 0 \) in the proof of Lemma 3.2. We have seen there that \( L = \bigoplus_{\pi \in \mathcal{P}} L(\pi) \) with \( L(\pi) \cong O/I_\pi \).

Since \( \chi'_\pi \) is now the trivial representation for each \( \pi \in \mathcal{P} \), we have \( I_\pi \neq 0 \) if and only if \( \chi_{\lambda, \mu}(T_i) = -1 \) for at least one \( i \in \mathcal{I}_{v(\pi)} \). This condition on \( \pi \) means precisely that \( \mathcal{I}_{v(\pi)} \cap \mathcal{I}^1_{\lambda, \mu} \neq \emptyset \). Hence,

\[
L^0 = \bigoplus_{\pi \in \mathcal{P}, \mathcal{I}_{v(\pi)} \cap \mathcal{I}^1_{\lambda, \mu} \neq \emptyset} L(\pi).
\]

For each \( \sigma \in \mathcal{S}_\pi \) denote by \( a_{\sigma} \) the element \( T_\sigma \otimes 1 \in M = \mathcal{H}_n(z) \otimes \mathcal{H}_n(z) O_{\text{triv}} \) and by \( a'_\sigma \) a similar element of \( M_{\mathbb{k}} \). Then, \( a'_\sigma \) is the image of \( a_{\sigma} \) under the canonical map \( M \to M_{\mathbb{k}} \). Since \( L(\pi) \) is the cyclic \( O \)-submodule of \( L \) generated by \( \psi(a_{\pi}) \), we get

\[
M^0 = \sum_{\sigma \in \mathcal{S}_\pi \cap \mathcal{I}^1_{\lambda, \mu} \neq \emptyset} Oa_{\sigma} + \text{Ker} \psi_{\mathbb{k}}, \quad \text{and therefore}
\]

\[
M^0_{\mathbb{k}} = \sum_{\pi \in \mathcal{P}, \mathcal{I}_{v(\pi)} \cap \mathcal{I}^1_{\lambda, \mu} \neq \emptyset} \mathbb{k} a'_{\sigma} + \text{Ker} \psi_{\mathbb{k}}.
\]

Put \( U_i = \{ m \in M_{\mathbb{k}} \mid T_i m = -m \} \). Since \((T_i + 1)^2 = 0\) in the algebra \( \mathcal{H}_n(q) \), we have \((T_i + 1)M_{\mathbb{k}} \subset U_i \). Hence,

\[
\text{Ker} \psi_{\mathbb{k}} = \sum_{i \in \mathcal{I}_{\lambda, \mu}} (T_i + 1)M_{\mathbb{k}} \subset \sum_{i \in \mathcal{I}^0_{\lambda, \mu}} (T_i + 1)M_{\mathbb{k}} + \sum_{i \in \mathcal{I}^1_{\lambda, \mu}} U_i = \Sigma_{\lambda, \mu}(M_{\mathbb{k}}).
\]
If \( \pi \in \mathcal{D} \), then \( T_i a'_\pi = -a'_\pi \), i.e., \( a'_\pi \in U_1 \), for each \( i \in \mathcal{I}_v(\pi) \). This shows that \( a'_\pi \) lies in \( \Sigma_{\lambda,\mu}(M_k) \) whenever \( \mathcal{I}_v(\pi) \cap \mathcal{I}_{1,\mu} \neq \emptyset \). Hence, \( M_k^{0} \subset \Sigma_{\lambda,\mu}(M_k) \).

Conversely, we claim that \( U_i \subset M_k^{0} \) for each \( i \in \mathcal{I}_{1,\mu} \), which entails the opposite inclusion \( \Sigma_{\lambda,\mu}(M_k) \subset M_k^{0} \). Fix such an \( i \) and consider the Mackey decomposition \( M_k = \bigoplus_{\pi \in \mathcal{D}} M(\pi)_k \) where \( M(\pi)_k \) is the \( \mathcal{K}_{\lambda,\mu}(q) \)-submodule of \( M_k \) generated by \( a'_\pi \). The standard basis of \( M(\pi)_k \cong \mathcal{K}_{\lambda,\mu}(q) \otimes \mathcal{K}_{\nu(\pi)}(q) \) \( \mathcal{K}_{\text{triv}} \) is formed by the elements \( a'_{\sigma,\pi} = T_\sigma a'_\pi \) with \( \sigma \) in the set \( \mathcal{D}_{\lambda,\mu} \) of distinguished representatives of the cosets \( \mathcal{K}_{\lambda,\mu}/\mathcal{K}_v(\pi) \).

For each \( \sigma \in \mathcal{D}_{\lambda,\mu} \) either \( \tau_i \sigma \in \mathcal{D}_{\lambda,\mu} \) or, by Deodhar’s Lemma, \( \tau_i \sigma = \sigma \tau_j \) for some \( j \in \mathcal{I}_v(\pi) \). In the first case, \( a'_{\sigma,\pi} \) and \( a'_{\tau_i \sigma,\pi} \) span a two-dimensional \( T_i \)-invariant subspace whose intersection with \( U_i \) is spanned by a single element

\[
a'_{\tau_i \sigma,\pi} + a'_{\sigma,\pi} = \pm (T_i + 1)a'_{\sigma,\pi} \in \text{Ker } \psi_k \subset M_k^{0}.
\]

In the second case, \( T_i T_\sigma = T_\tau \sigma = T_\sigma T_j \) and \( T_j a'_\pi = -a'_\pi \), whence

\[
T_i a'_{\sigma,\pi} = T_i T_\sigma a'_{\pi} = T_\sigma T_j a'_\pi = -T_\sigma a'_{\pi} = -a'_{\sigma,\pi}.
\]

Thus, \( a'_{\sigma,\pi} \) spans a one-dimensional \( T_i \)-invariant subspace. Note that \( j \in \mathcal{I}_{1,\mu} \) since the transpositions \( \tau_i \) and \( \tau_j \) are conjugate in the group \( \mathcal{G}_{\lambda,\mu} \). Thus, \( \mathcal{I}_v(\pi) \cap \mathcal{I}_{1,\mu} \neq \emptyset \), which entails \( a'_\pi \in M_k^{0} \). But \( T_\sigma a'_\pi = (-1)^{f(\sigma)} a'_\pi \) modulo \( \text{Ker } \psi_k \) since \( \sigma \in \mathcal{G}_{\lambda,\mu} \) and \( (T_i + 1)M_k \subset \text{Ker } \psi_k \) for all \( i \in \mathcal{G}_{\lambda,\mu} \). Therefore, \( a'_{\tau_i \sigma,\pi} \in M_k^{0} \) too.

The whole \( M_k \) is thus a direct sum of \( T_i \)-invariant subspaces spanned by at most two basis elements. It follows that \( U_i \) is a direct sum of its intersections with those subspaces of \( M_k \). We have checked that \( M_k^{0} \) contains each of the summands in this decomposition of \( U_i \). Hence, \( U_i \subset M_k^{0} \), as claimed.

\( \square \)

**Lemma 3.4** The equality \( \sum_{i=0}^{n} (-1)^i [M_k/\Sigma_i.n-i(M_k)] = 0 \) holds in the Grothendieck group \( \text{Grot}(\text{End } \mathcal{H}_{n}(q), M_k) \).

**Proof** Consider the parabolic subalgebra \( \mathcal{H}_{i,n-i}(z)Q \) of \( \mathcal{H}_{n}(z)Q \) generated by all elements \( T_j \) with \( 0 < j < n \), \( j \neq i \). Let \( Q_{i,n-i} \) be the field \( Q \) regarded as the one-dimensional \( \mathcal{H}_{i,n-i}(z)Q \)-module on which \( T_j \) operates as multiplication by \( z \) for \( j < i \) and as \(- \text{Id} \) for \( j > i \). By Lemma 3.3,

\[
\sum_{i=0}^{n} (-1)^i [M_k/\Sigma_i.n-i(M_k)] = d \left( \sum_{i=0}^{n} (-1)^i [Q_{i,n-i} \otimes \mathcal{H}_{i,n-i}(z)Q M_Q] \right).
\]

Since the algebra \( \mathcal{H}_{n}(z)Q \) is semisimple, the functor \( \otimes \mathcal{H}_{n}(z)Q M_Q \) is exact, and therefore, this functor induces a group homomorphism

\[
g : \text{Grot} \mathcal{H}_{n}(z)Q \to \text{Grot}(\text{End } \mathcal{H}_{n}(z)Q M_Q)
\]

where we denote by \( \text{Grot} \mathcal{H}_{n}(z)Q \) the Grothendieck group of the category of finite-dimensional right \( \mathcal{H}_{n}(z)Q \)-modules. Lemma 2.1 in its equivalent formulation for right
modules shows that
\[
\sum_{i=0}^{n} (-1)^i [Q_{i,n-i} \otimes H_{i,n-i}(z) Q H_{n}(z) Q] = 0 \quad \text{in} \quad \text{Grot}^r H_n(z)_Q.
\]
Applying the above map $g$, we get
\[
\sum_{i=0}^{n} (-1)^i [Q_{i,n-i} \otimes H_{i,n-i}(z) Q M_Q] = 0 \quad \text{in} \quad \text{Grot}(\text{End} H_n(z)_Q M_Q),
\]
and the desired conclusion follows.

**Lemma 3.5** There are uniquely determined elements $\zeta_\lambda \in \text{Grot}(\text{End} H_n(q) M_k)$ with $\lambda \in \mathcal{P}(n)$ such that
\[
[K_{\text{triv}} \otimes H_n(q) M_k] = \sum_{\lambda \in \mathcal{P}(n)} K_{\lambda\mu} \zeta_\lambda \quad \text{for each} \quad \mu \in \mathcal{P}(n).
\]
Moreover, $\zeta_\lambda \geq 0$, i.e., $\zeta_\lambda$ represents an actual module, for each $\lambda \in \mathcal{P}(n)$.

**Proof** Let $S^\lambda$, $\lambda \in \mathcal{P}(n)$, be the right Specht modules for the algebra $H_n(z)_Q$, as defined by Dipper and James [6]. By Lemma 2.1,
\[
[Q_{\text{triv}} \otimes H_n(z)_Q H_n(z)_Q] = \sum_{\lambda \in \mathcal{P}(n)} K_{\lambda\mu} [S^\lambda] \quad \text{in} \quad \text{Grot}^r H_n(z)_Q.
\]
Applying the group homomorphism $g$ defined in the proof of Lemma 3.4, we get
\[
[Q_{\text{triv}} \otimes H_n(z)_Q M_Q] = \sum_{\lambda \in \mathcal{P}(n)} K_{\lambda\mu} [S^\lambda \otimes H_n(z)_Q M_Q] \quad \text{in} \quad \text{Grot}(\text{End} H_n(z)_Q M_Q).
\]
Applying now the decomposition map $d$ and making use of Lemma 3.2, we get the desired equalities in the group Grot(End $H_n(q) M_k$) with
\[
\zeta_\lambda = d([S^\lambda \otimes H_n(z)_Q M_Q]).
\]
We have $\zeta_\lambda \geq 0$ since $d$ preserves positivity by the construction. Uniqueness of this collection of elements follows from the fact that the Kostka matrix is invertible. Indeed, this matrix is even unitriangular with respect to a suitable ordering of partitions [22, Ch. I, (6.5)].

In the next lemma, the assumption that the discrete valuation ring $O$ is complete goes into action.

**Lemma 3.6** If $X$ is any direct summand of the $H_n(q)$-module $M_k$, then $X = N_k$ for some direct summand $N$ of the $H_n(z)$-module $M$. 

\[ \square \text{ Springer} \]
Proof There exists an idempotent \( e \in \text{End}\mathcal{H}_n(q)M_{\mathbb{k}} \) such that \( X = \text{Im} e \). By Lemma 3.1, \( \text{End}\mathcal{H}_n(q)M_{\mathbb{k}} \cong A_{\mathbb{k}} \cong A/mA \) where \( A = \text{End}\mathcal{H}_n(z)M \) and \( m \) is the maximal ideal of \( O \). The \( O \)-algebra \( A \) is free of finite rank as a module. It is known that in this situation every idempotent of \( A/mA \) lifts to an idempotent of \( A \). Hence, there exists an idempotent \( \tilde{e} \in A \) such that \( e = \tilde{e} \otimes O. \) We may take \( N = \text{Im} \tilde{e}. \) 

Lemma 3.7 Suppose that \( X \) is a finite-dimensional \( \mathcal{H}_n(q) \)-module whose indecomposable direct summands all have one-dimensional sources. Then, \( X = M_{\mathbb{k}} \) for some \( M \in \text{Rep}_{\mathbb{k}}. \) If \( q = -1 \), then we can even take \( M \in \text{Triv}. \)

Proof The class of \( \mathcal{H}_n(q) \)-modules for which the conclusion of this lemma holds is obviously closed under direct sums. By Lemma 3.6, this class is closed also under direct summands. Since for \( X = \mathcal{H}_n(q) \otimes \mathcal{H}_{\lambda,\mu}(q) \langle \lambda, \mu \rangle \), we can take \( M = M_{\lambda,\mu} \), the first assertion holds then in full generality provided \( q \neq -1 \). If \( q = -1 \), then any one-dimensional representation of a parabolic subalgebra of \( \mathcal{H}_n(q) \) is trivial, so that we need only to look at the modules \( X = \mathcal{H}_n(q) \otimes \mathcal{H}_{\lambda,\mu}(q) \langle \lambda, \mu \rangle \text{triv} \) for which \( M = M_{\lambda,\mu} \) will do.

Corollary 3.8 If \( X \) is as in Lemma 3.7, then:

(i) \( \sum_{i=0}^{n} (-1)^i [X/\Sigma_{i,n-i}(X)] = 0 \) in the Grothendieck group \( \text{Grot}(\text{End}\mathcal{H}_n(q)X), \)

(ii) there exist finite-dimensional \( \text{End}\mathcal{H}_n(q)X \)-modules \( V^\lambda, \lambda \in \mathcal{P}(n), \) such that in that group \( [X/\Sigma_{\mu}(X)] = \sum_{\lambda \in \mathcal{P}(n)} K_{\lambda,\mu} [V^\lambda] \) for each \( \mu \in \mathcal{P}(n). \)

This corollary repeats the conclusions of Lemmas 3.4 and 3.5. Note that \( X/\Sigma_{\mu}(X) \cong \text{Triv} \otimes \mathcal{H}_{\mu}(q)X. \)

4 The Hilbert series of the \( R \)-symmetric algebras

Let \( R \) be a Hecke symmetry with parameter \( q \) on a finite-dimensional vector space \( V \) over the ground field \( \mathbb{k}. \) For each \( n \geq 0 \), consider the \( \mathcal{H}_n(q) \)-module structure on \( V^\otimes n \) arising from \( R \). Our first goal in this section is to describe a ring homomorphism \( \varphi : \text{Sym} \rightarrow \text{Grot}(R) \). All essential arguments needed to establish properties of \( \varphi \) are provided by the results of Sect. 3.

As a preliminary step, we will determine certain quotients of \( V^\otimes n \). In accordance with the notation introduced in Sect. 3 we have for \( (\lambda, \mu) \in \mathcal{P}^2(n) \)

\[
\Sigma_{\lambda,\mu}(V^\otimes n) = \sum_{i \in \mathcal{I}_{\lambda,\mu}} \text{Im}(R_i^{(n)} - q \cdot \text{Id}) + \sum_{i \in \mathcal{J}_{\lambda,\mu}} \text{Ker}(R_i^{(n)} - q \cdot \text{Id}).
\]

This is an \( \text{End}\mathcal{H}_nV^\otimes n \)-submodule, i.e., an \( A_n(R) \)-subcomodule, of \( V^\otimes n \). For each partition, \( \lambda = (\lambda_1, \ldots, \lambda_k) \) put

\[
S^\lambda = S_{\lambda_1}(V, R) \otimes \ldots \otimes S_{\lambda_k}(V, R), \quad A^\lambda = A_{\lambda_1}(V, R) \otimes \ldots \otimes A_{\lambda_k}(V, R).
\]

It will be assumed that \( S^\lambda = A^\lambda = \mathbb{k} \) when \( \lambda = 0. \)
Lemma 4.1 For each $(\lambda, \mu) \in \mathcal{P}^2(n)$, there is an isomorphism of $A_n(R)$-comodules
\[ V^\otimes n / \Sigma_{\lambda, \mu} (V^\otimes n) \cong S^\lambda \otimes \Lambda^\mu. \]

Proof Put $l = |\lambda|$ and $m = |\mu|$. Writing $V^\otimes n$ as $V^\otimes l \otimes V^\otimes m$ and noting that
\[ \Sigma_{\lambda, \mu} (V^\otimes n) = \Sigma_{\lambda, 0} (V^\otimes l) \otimes V^\otimes m + V^\otimes l \otimes \Sigma_{0, \mu} (V^\otimes m), \]
we get
\[ V^\otimes n / \Sigma_{\lambda, \mu} (V^\otimes n) \cong V^\otimes l / \Sigma_{\lambda, 0} (V^\otimes l) \otimes V^\otimes m / \Sigma_{0, \mu} (V^\otimes m). \]
The ideal $I^S$ defining the factor algebra $S(V, R)$ of the tensor algebra $T(V)$ has homogeneous components $I^S_k = \sum 0 < i < k \text{ Im}(R^{(k)} - q \cdot \text{Id}) \subset V^\otimes k$. Hence,
\[ \Sigma_{\lambda, 0} (V^\otimes l) = \sum_{j=1}^{(\lambda) \otimes (\lambda_1 + \cdots + \lambda_{j-1}) \otimes I_{\lambda_j}^S \otimes V^\otimes (\lambda_{j+1} + \cdots + \lambda_l)}, \]
and it follows that $V^\otimes l / \Sigma_{\lambda, 0} (V^\otimes l) \cong S^\lambda$. On the other hand, the ideal $I^A$ defining $A(V, R)$ has homogeneous components $I^A_k = \sum 0 < i < k \text{ Ker}(R^{(k)} - q \cdot \text{Id})$. One obtains similarly $V^\otimes m / \Sigma_{0, \mu} (V^\otimes m) \cong \Lambda^\mu$. \hfill $\Box$

Proposition 4.2 Suppose that $R$ satisfies the one-dimensional source condition. Then, there is a ring homomorphism $\varphi : \text{Sym} \to \text{Grot}(R)$ such that
(i) $\varphi(h_n) = [S_n(V, R)]$ and $\varphi(e_n) = [A_n(V, R)]$ for all $n \geq 0$,
(ii) $\varphi(s_\lambda) \geq 0$ for all $\lambda \in \mathcal{P}$.

Proof Since $h_1, h_2, \ldots$ are algebraically independent generators of the ring of symmetric functions $\text{Sym}$ (see (22.8)), homomorphisms from $\text{Sym}$ to another ring are uniquely determined by their values on those elements. Thus, we can define $\varphi$ setting $\varphi(h_n) = [S_n(V, R)]$ for all $n > 0$. Since $h_0 = e_0 = 1$ and $S_0(V, R) = A_0(V, R) = \mathbb{R}$ is the trivial one-dimensional $A(R)$-comodule which represents the identity element of the ring $\text{Grot}(R)$, the two equalities in (i) are obvious for $n = 0$. If $n > 0$, then the relation $\sum_{i=0}^n (-1)^i h_i e_{n-i} = 0$ in $\text{Sym}$ yields $\sum_{i=0}^n (-1)^i \varphi(h_i) \varphi(e_{n-i}) = 0$. The value $\varphi(e_n)$ is determined by induction on $n$ from the latter equality. To show that the second equality in (i) holds for all $n > 0$, we have to check that
\[ \sum_{i=0}^n (-1)^i [S_i(V, R)] \cdot [A_{n-i}(V, R)] = \sum_{i=0}^n (-1)^i [S_i(V, R) \otimes A_{n-i}(V, R)] = 0. \]

(i')

If $\mu \in \mathcal{P}(n)$, then $\varphi(h_\mu) = S^\mu$. The equality $h_\mu = \sum_{\lambda \in \mathcal{P}(n)} K_{\lambda, \mu} s_\lambda$ in $\text{Sym}$ entails $[S^\mu] = \sum_{\lambda \in \mathcal{P}(n)} K_{\lambda, \mu} \varphi(s_\lambda)$. This determines the values of $\varphi$ on the Schur functions since the Kostka matrices are invertible. Part (ii) means that there exists a collection of finite-dimensional $A(R)$-comodules $V^\lambda, \lambda \in \mathcal{P}$, such that $\varphi(s_\lambda) = [V^\lambda]$ for each.
\[ \lambda \in \mathcal{P}. \] Those equalities are equivalent to the equalities
\[ \left[ S_\mu \right] = \sum_{\lambda \in \mathcal{P}(n)} K_{\lambda, \mu} [V^\lambda] \quad \text{for each } \mu \in \mathcal{P}(n). \tag{ii'} \]

The validity of both (i') and (ii') in the group \( \text{Grot}_n(R) \) is ensured by Corollary 3.8. Indeed, for \( X = V \otimes^n \) the group \( \text{End} \mathcal{H}_n(q) X \) has been identified with \( \text{Grot}_n(R) \), and we have
\[ X / \Sigma_{i, n-i}(X) \cong S_i(V, R) \otimes \Lambda_{n-i}(V, R) \quad \text{and} \quad X / \Sigma_\mu(X) \cong S_\mu \]
by Lemma 4.1.

Corollary 4.3 In the ring \( \text{Grot}(R)[[t]] \) consider the formal power series
\[ G_S = \sum_{i=0}^{\infty} [S_i(V, R)] t^i, \quad G_A = \sum_{i=0}^{\infty} [A_i(V, R)] t^i. \]

We have \( G_S(t)G_A(-t) = 1 \).

**Proof** The coefficient of \( t^n \) in the product \( G_S(t)G_A(-t) \) vanishes for each \( n > 0 \) according to equality (i') in the proof of Proposition 4.2. The constant term of this product is the identity element \([k]\) of the ring \( \text{Grot}(R) \). \qed

Corollary 4.3 strengthens the well-known relation \( H_S(t)H_A(-t) = 1 \) between the Hilbert series of the algebras \( S = S(V, R) \) and \( \Lambda = \Lambda(V, R) \). This relation was proved by Gurevich [15] in the semisimple case. The more general case when \( q \) is arbitrary and \( R \) satisfies the one-dimensional source condition was treated in [27].

For each \( \lambda \in \mathcal{P} \), we will denote by \( V^\lambda \) any finite-dimensional right \( A(R) \)-comodule such that \([V^\lambda] = \varphi(s_\lambda)\). In the nonsemisimple case, such a comodule is generally not unique, but we will be concerned with numeric characteristics such as the dimension of \( V^\lambda \) which depend only on the class of \( V^\lambda \) in the group \( \text{Grot}(R) \).

Let \( K \) be a commutative ring. For a formal power series \( f = \sum_{i=0}^{\infty} a_i t^i \in K[[t]] \) with \( a_0 = 1 \), we denote by
\[ \hat{f} : \text{Sym} \longrightarrow K \]
the ring homomorphism such that \( h_n \mapsto a_n \) for each \( n \geq 0 \). We will assume tacitly that \( a_i = 0 \) for all integers \( i < 0 \).

**Corollary 4.4** Let \( f = \sum_{i=0}^{\infty} a_i t^i \in Z[[t]] \) be the Hilbert series of the \( R \)-symmetric algebra \( S(V, R) \). Then,
\[ \dim V^\lambda = \hat{f}(s_\lambda) = \det (a_{\lambda_i-i+j})_{1 \leq i, j \leq k} \]
for each partition \( \lambda \) with \( k \) parts \( \lambda_1, \ldots, \lambda_k \).
There is a ring homomorphism \( \delta : \text{Grot}(R) \rightarrow \mathbb{Z} \) such that \( \delta([X]) = \dim X \) for each finite-dimensional right \( A(R) \)-comodule \( X \). We have \( \hat{f} = \delta \circ \varphi \) since the left- and right-hand sides of this equality are presented by ring homomorphisms with the same values \( a_n \) on the generators \( h_n \) of the ring \( \text{Sym} \).

Hence, \( \hat{f}(s_\lambda) = \delta(\varphi(s_\lambda)) = \dim V^\lambda \). The second equality follows from the Jacobi–Trudi identity \( s_\lambda = \det(h_{\lambda_i-i+j})_{1 \leq i,j \leq k} \).

An infinite sequence of real numbers \( a_0, a_1, a_2, \ldots \) with \( a_0 = 1 \) is called \emph{totally positive} or a \emph{Pólya frequency sequence} if all minors of the infinite Toeplitz matrix \( (a_{j-i})_{i,j \geq 0} \) are nonnegative. Denoting by \( f \) the generating series of the given sequence, we see that \( \hat{f}(s_\lambda) \) is one of these minors taken from a set of consecutive columns. Other minors are the values of \( \hat{f} \) on skew Schur functions \( s_{\nu/\mu} \). Since \( s_{\nu/\mu} = \sum_{\lambda \in \mathcal{P}} c_{\lambda/\mu} s_\lambda \) with the Littlewood–Richardson coefficients \( c_{\lambda/\mu} \geq 0 \), it follows that the sequence is totally positive if and only if

\[
\hat{f}(s_\lambda) \geq 0 \quad \text{for each} \quad \lambda \in \mathcal{P},
\]

i.e., there is no need to look at the other minors. This formulation of total positivity together with two other equivalent conditions is discussed by Stanley [28, Exercise 7.91e] in the case when \( f \) is a polynomial. The statement given above is equally valid for infinite series.

Since \( \dim V^\lambda \geq 0 \) for all \( \lambda \in \mathcal{P} \), it follows from Corollary 4.4 that the dimensions of the homogeneous components \( \mathbb{S}_n(V, R) \) form a totally positive sequence. All possibilities for the generating series \( f \) of a totally positive sequence were determined in the work of Aissen, Schoenberg, Whitney [1] and Edrei [12]. In particular, [1, Th. 1] states that \( f \) converges in a neighborhood of 0 in \( \mathbb{C} \) and extends to a meromorphic function with negative real zeros and positive real poles on the whole \( \mathbb{C} \). As it turns out, \( f \) is rational precisely when \( \hat{f}(s_\lambda) = 0 \) for at least one \( \lambda \in \mathcal{P} \).

Thus, rationality of the Hilbert series of \( \mathbb{S}(V, R) \) is equivalent to the existence of partitions \( \lambda \) for which \( V^\lambda = 0 \). As observed by Hai [16], in the semisimple case such \( \lambda \) do exist since otherwise the representation of \( \mathcal{H}_n \) in \( V^{\otimes n} \) would be faithful for each \( n \), but this is impossible since \( \dim \mathcal{H}_n = n! > (\dim V)^{2n} \) for large \( n \). We cannot use this argument directly when \( q \) is a root of 1. However, the next lemma provides a replacement.

For integers \( a, k, \lambda > 0 \) denote by \( (a^k) \) the partition \( (a, \ldots, a) \) with exactly \( k \) parts, each equal to \( a \).

**Lemma 4.5** Suppose that \( R \) satisfies the one-dimensional source condition. There exist integers \( n > 0 \) and \( k > 0 \) such that \( V^{\otimes k} = 0 \). Moreover, if \( V^{\otimes k} = 0 \), then \( V^{\otimes m} = 0 \) for all \( m > n \).

**Proof** Denote by \( d^\lambda \) the dimension of the Specht module \( S^\lambda \), i.e. \( d^\lambda \) is equal to the Kostka number \( K_{\lambda,(\rho)} \) counting the standard \( \lambda \)-tableaux. We first note that \( V^\lambda = 0 \) for each \( \lambda \in \mathcal{P}(n) \) with \( d^\lambda > (\dim V)^n \). To prove this, we start with the equality

\[
h^\rho_1 = h_{(\rho)} = \sum_{\rho \in \mathcal{P}(n)} d^\rho s_\rho \quad \text{in the ring} \quad \text{Sym}.
\]

\( \text{Springer} \)
An application of $\varphi$ yields $[V^\otimes n] = \sum_{\rho \in \mathcal{P}(n)} d^\rho [V^\rho]$ in $\text{Grot}(R)$. Hence,

$$(\dim V)^n = \dim V^\otimes n = \sum_{\rho \in \mathcal{P}(n)} d^\rho \dim V^\rho,$$

and therefore $d^\lambda \dim V^\lambda \leq (\dim V)^n$, which entails the previous claim.

By the hook length formula

$$d^{(nk)} = \frac{(kn)!}{\prod_{i=1}^{k} \prod_{j=1}^{n} (n + k - i - j + 1)} = (kn)! \prod_{i=1}^{k} \frac{(k-i)!}{(n+k-i)!} = \frac{(kn)!}{n!^{k}} \prod_{i=0}^{k-1} \left( \frac{n+i}{i} \right)^{-1}.$$

Using the Stirling asymptotic formula $n! \sim n^n e^{-n} \sqrt{2\pi n}$ and observing that $\binom{n+i}{i} \sim n^i/i!$ for each $i$, we deduce that

$$d^{(nk)} \sim k^{kn} n^{-(k^2-1)/2} \cdot \sqrt{k/(2\pi)^{k-1}} \prod_{i=0}^{k-1} i! \quad \text{as} \quad n \to \infty.$$  

If $k > \dim V$, then $d^{(nk)} > (\dim V)^{kn}$ for large $n$. As we have observed, this yields $V^{(nk)} = 0$, proving the first assertion of Lemma 4.5.

Next, if $\lambda \in \mathcal{P}$ is such that $V^\lambda = 0$, then $0 = [V^\lambda][V^\mu] = \sum_{\nu \in \mathcal{P}} c^\nu_{\lambda,\mu} [V^\nu]$ for each $\mu \in \mathcal{P}$. Since $c^\nu_{\lambda,\mu} \geq 0$ for all $\nu$, it follows that $V^\nu = 0$ whenever $c^\nu_{\lambda,\mu} \neq 0$. In particular, this applies in the case when $\lambda = (nk), \mu = ((m-n)k)$ and $\nu = (mk)$ with $m > n$. Indeed, for these partitions, we have $c^\nu_{\lambda,\mu} = 1$ by the Littlewood–Richardson rule. 

For the application to the Hilbert series of the $R$-symmetric algebras the full strength of the analytic result on totally positive sequences is actually not needed. For one thing rationality of the series follows from a purely algebraic fact formulated in terms of Hankel determinants (see [20, Th. 7.5f]). These determinants with the reversed order of rows are certain minors of the Toeplitz matrix. The next lemma gives a version of that result paying attention to the ring of coefficients.

**Lemma 4.6** Let $f = \sum_{i=0}^{\infty} a_i t^i$ be a formal power series with coefficients in a commutative Noetherian domain $K$. For each pair of integers $i \geq k > 0$ put

$$\Delta_i^{(k)} = \hat{f}(s_{(k+1)}^{(i)}) = \begin{vmatrix}
a_i & a_{i+1} & \cdots & a_{i+k-1} 
a_{i-1} & a_i & \cdots & a_{i+k-2} 
\vdots & \vdots & \ddots & \vdots 
a_{i-k+1} & a_{i-k+2} & \cdots & a_i
\end{vmatrix}.$$ 

Suppose that there are integers $n > r > 0$ such that $\Delta_i^{(r)} \neq 0$ and $\Delta_i^{(r+1)} = 0$ for all $i \geq n$. Then, $f = q^{-1} p$ for some polynomials $p, q \in K[t]$ with $q(0) = 1$. If $K$ is $\mathbb{C}$ then $\Delta_i^{(r)}(0) \neq 0$ for all $i \geq n$.
integ rally closed, then such an expression holds with $q$ of degree $r$, in which case $p$ and $q$ are relatively prime in the ring $Q[t]$ where $Q$ is the field of fractions of $K$.

**Proof**  Put $A_i^{(k)} = (a_{i-k+1}, a_{i-k+2}, \ldots, a_i)$. Part of the hypothesis means that for each $i \geq n$ the $r$ vectors $A_i^{(r)}, A_{i+1}^{(r)}, \ldots, A_{i+r-1}^{(r)}$ are linearly independent over $Q$ and so form a basis for the $r$-dimensional vector space $Q^r$. Then,

$$A_{i+r}^{(r)} = c_1 A_{i+r-1}^{(r)} + \cdots + c_{r-1} A_{i+1}^{(r)} + c_r A_i^{(r)} \quad \text{(Rel}_i)$$

with uniquely determined coefficients $c_1, \ldots, c_r \in Q$. We claim that these coefficients do not depend on $i$. This will follow once we show that for each $i > n$ the coefficients in (Rel$_i$) are the same as those in (Rel$_{i-1}$). But $A_{i+r}^{(r+1)}$ is a linear combination of vectors $A_j^{(r+1)}, A_{j+1}^{(r+1)}, \ldots, A_{j+r-1}^{(r+1)} \in Q^{r+1}$ because $A_j^{(r+1)} = 0$. Since the projection $Q^{r+1} \to Q^r$ onto the last $r$ components maps $A_j^{(r+1)}$ to $A_j^{(r)}$ for each $j$, we must have

$$A_{i+r}^{(r+1)} = c_1 A_{i+r-1}^{(r+1)} + \cdots + c_{r-1} A_{i+1}^{(r+1)} + c_r A_i^{(r+1)}$$

with coefficients from (Rel$_i$). Noting that the projection $Q^{r+1} \to Q^r$ onto the first $r$ components maps $A_j^{(r+1)}$ to $A_j^{(r)}$ for each $j$, we get (Rel$_i$) with the same coefficients, and the claim is proved.

Now, (Rel$_i$) shows that $a_{i+1} = \sum_{j=1}^r c_j a_{i+1-j}$ for each $i \geq n$. Setting

$$g = 1 - \sum_{j=1}^r c_j t^j \in Q[t],$$

we see that the coefficient of $t^{i+1}$ in the formal power series $gf$ vanishes whenever $i \geq n$. Hence, $gf$ is a polynomial with coefficients in $Q$.

Define a linear operator $\theta$ on the vector space $Q^r$ by the formula

$$\theta(x_1, x_2, \ldots, x_r) = (x'_1, x'_2, \ldots, x'_r)$$

where $x'_j = x_{j+1}$ for $0 < j < r$ and $x'_r = \sum_{j=1}^r c_j x_{r+1-j}$. We have $\theta(A_i^{(r)}) = A_{i+1}^{(r)}$ for each $i \geq n$, and it follows that

$$\theta^r - c_1 \theta^{r-1} - \cdots - c_{r-1} \theta - c_r \text{Id} = 0$$

since this operator annihilates all vectors $A_i^{(r)}$ with $i \geq n$ in view of (Rel$_i$). On the other hand, the operators $\theta, \ldots, \theta^{r-1}, \theta^r$ are linearly independent over $Q$ since so are the vectors $A_{i+1}^{(r)}, \ldots, A_{i+r-1}^{(r)}, A_{i+r}^{(r)}$ for $i \geq n$. Hence, $c_r \neq 0$, and $g$ is a scalar multiple of the minimal polynomial of the inverse operator $\theta^{-1}$.

Note that $\theta(M) \subset M$ where $M$ is the submodule of the free $K$-module $K^r$ generated by $\{A_i^{(r)} \mid i \geq n\}$. Since $K$ is Noetherian, $M$ has to be finitely generated. Hence, $\theta$ is
integral over $K$, i.e., for some $k > 0$ there exists a relation

$$\theta^k + e_1 \theta^{k-1} + \cdots + e_{k-1} \theta + e_k \text{Id} = 0$$

with coefficients $e_j \in K$. Take $q = 1 + \sum_{j=1}^{k} e_j t^j$. Then, $g$ divides $q$ since $q(\theta^{-1}) = 0$, and it follows that $qf$ is a polynomial whose coefficients are in $K$ since so are the coefficients of $f$ and $q$. With $p = qf$ the first conclusion is thus proved.

We can write $q = (1 - \xi_1 t) \cdots (1 - \xi_k t)$ with $\xi_1, \ldots, \xi_k$ in the algebraic closure of the field $Q$. Each $\xi_j$ is integral over $K$ since $q(\xi_j^{-1}) = 0$. Since $g$ is a divisor of $q$ with $g(0) = 1$, it is the product of some of these factors $1 - \xi_j t$. Hence, all coefficients of $g$ are integral over $K$ too. If $K$ is integrally closed, we get $g \in K[t]$, and so we may take $q = g$. Then, $p$ and $q$ cannot have a common divisor in $Q[t]$ since otherwise $hf \in Q[t]$ for some polynomial $h \in Q[t]$ of degree less than $r$, but this implies that the sequence $(a_{i})$ starting at some term satisfies a linear recurrence relation of order less than $r$, which contradicts the linear independence of the previously considered vectors $A_{i}^{(r)}, A_{i+1}^{(r)}, \ldots, A_{i+r-1}^{(r)}$. \hfill $\Box$

Knowing rationality of $f$, one needs only part of the arguments given in [1] to determine the location of zeros and poles. Moreover, Stanley’s criterion of total positivity provides further simplifications:

**Lemma 4.7** Let $f = \sum_{i=0}^{\infty} a_{i} t^i \in \mathbb{R}[t]$ be the generating series of a totally positive sequence. Suppose that $f$ represents a rational function of $t$. Then, all its zeroes are negative, while all its poles are positive real numbers.

**Proof** If $a_n \neq 0$ and $a_{n+1} = 0$ for some $n \geq 0$, then $f$ is a polynomial since for each $i > n$, the conditions $a_i a_{n+1} - a_{i+1} a_n \geq 0$ and $a_{i+1} a_n \geq 0$ entail $a_{i+1} = 0$.

Suppose that $a_n \neq 0$ for all $n \geq 0$. Then, the positive numbers $a_{n+1}/a_n$ form a monotone nonincreasing sequence. Hence, this sequence converges to some $\gamma \geq 0$. Rationality of $f$ implies that $\gamma^{-1}$ is one of its poles, i.e., $1 - \gamma t$ is a divisor of the denominator $q$ in the expression of $f$ as a fraction of two relatively prime polynomials. In particular, $\gamma \neq 0$. The power series $\hat{f}_0 = (1 - \gamma t)f$ has coefficients

$$a_i' = a_i - \gamma a_{i-1}$$

which form a totally positive sequence. Indeed, given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$, we have

$$\hat{f}_0(s_\lambda) = \det (a_{\lambda_i-i+j}')_{1 \leq i,j \leq k} = \begin{vmatrix} 1 & \gamma & & \gamma^k \\ a_{\lambda_1-1} & a_{\lambda_1} & \cdots & a_{\lambda_1+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\lambda_k-k+1} & a_{\lambda_k-k+2} & \cdots & a_{\lambda_k} \end{vmatrix}.$$ 

Since $\gamma^i = \lim_{n \to \infty} a_{n+i}/a_n$, this yields

$$\hat{f}_0(s_\lambda) = \lim_{n \to \infty} \hat{f}(s_{\lambda(n)})/a_n \geq 0$$
where we put \( \lambda^{(n)} = (n, \lambda_1, \ldots, \lambda_k) \). Thus, \( f_0 \) satisfies the same assumptions as \( f \).

We have seen that \( f \) has a pole \( \gamma^{-1} > 0 \). Proceeding by induction on the degree of the denominator \( q \), we conclude that all poles of \( f \) are positive.

The coefficients \( b_i \) of the power series \( g(t) = 1/f(-t) \) also form a totally positive sequence. This fact was proved in [1], but again it can be explained very easily within the theory of symmetric functions. Since \( \sum_{i=0}^n (-1)^i a_i b_{n-i} = 0 \) for \( n > 0 \), we have \( b_n = \hat{f}(e_n) \). Hence, \( \hat{g} = \hat{f} \circ \omega \) where \( \omega \) is the automorphism of the ring Sym such that \( h_n \mapsto e_n \) for each \( n \). Since \( \omega(s_\lambda) = s_{\lambda'} \) where \( \lambda' \) is the conjugate of \( \lambda \), we get \( \hat{g}(s_\lambda) = \hat{f}(s_{\lambda'}) \geq 0 \) for each \( \lambda \in \mathcal{P} \). Thus, all poles of \( f \) are positive, which means that all zeroes of \( f \) are negative. \( \square \)

**Theorem 4.8** Suppose that \( R \) satisfies the one-dimensional source condition. Then,

\[
H_{\Lambda(V, R)}(t) = f_0(-t)/f_1(t), \quad H_{\mathcal{S}(V, R)}(t) = f_1(-t)/f_0(t)
\]

with integer polynomials \( f_0, f_1 \in \mathbb{Z}[t] \) whose constant terms are equal to 1 and all roots are positive real numbers.

**Proof** Put \( f = H_{\mathcal{S}(V, R)} \). Since \( H_{\Lambda(V, R)}(t) = f(-t)^{-1} \), it suffices to prove only the formula for \( f \). Define \( \Delta^{(k)}_i \) as in Lemma 4.6. By Corollary 4.4, \( f \) is the generating series of a totally positive sequence, and also \( \Delta^{(k)}_i = \dim V^{(i_k)} \). Let \( r \) be the smallest nonnegative integer for which there exists \( n > 0 \) such that \( V^{(n+1)} \) is well-defined and \( V^{(m+1)} = 0 \) for all \( m > n \).

If \( r = 0 \), then \( \mathcal{S}_n(V, R) = 0 \), which means that \( f \) is a polynomial. Note that the constant term of \( f \) is equal to 1.

If \( r > 0 \), then the determinants \( \Delta^{(k)}_i \) satisfy the assumption of Lemma 4.6. Taking \( \mathcal{K} = \mathbb{Z} \), we deduce that \( f \) is a fraction of two relatively prime integer polynomials with constant terms equal to 1. By Lemma 4.7, \( f \) has negative zeros and positive poles. This ensures the desired properties of \( f_0 \) and \( f_1 \). \( \square \)

Now, we will extend to the present situation two additional results obtained by Hai in the semisimple case [16, Th. 5.1, Cor. 5.2].

**Corollary 4.9** Let \((r_0, r_1)\) be the birank of \( R \), i.e., \( r_0 = \deg f_0 \) and \( r_1 = \deg f_1 \). Then, \( r_0 + r_1 \leq \dim V \). Moreover, if \( r_0 + r_1 = \dim V \), then

\[
H_{\Lambda(V, R)}(t) = (1 + t)^{r_0}/(1 - t)^{r_1}, \quad H_{\mathcal{S}(V, R)}(t) = (1 + t)^{r_1}/(1 - t)^{r_0}
\]

**Proof** We have \( f_0(t) = \prod_{i=1}^{r_0} (1 - \alpha_i t) \) and \( f_1(t) = \prod_{i=1}^{r_1} (1 - \beta_i t) \) with \( \alpha_i, \beta_i > 0 \). The fact that all coefficients of these polynomials are integers entails

\[
\frac{\alpha_1 + \cdots + \alpha_{r_0}}{r_0} \geq r_0 \sqrt{\alpha_1 \cdots \alpha_{r_0}} \geq 1, \quad \frac{\beta_1 + \cdots + \beta_{r_1}}{r_1} \geq r_1 \sqrt{\beta_1 \cdots \beta_{r_1}} \geq 1.
\]

Since the coefficient of \( t \) in \( H_{\mathcal{S}(V, R)} \) is equal to the dimension of \( V \), we get

\[
\dim V = \sum \alpha_i + \sum \beta_j \geq r_0 + r_1.
\]
If the equality is attained here, then \( \sum \alpha_i = r_0 \) and \( \sum \beta_j = r_1 \), so that the equalities are attained also in the previously displayed formulas. This is only possible when all \( \alpha_i \) and \( \beta_j \) are equal to 1.

\[ \tag*{\Box} \]

**Corollary 4.10** Let \( (r_0, r_1) \) be the birank of \( R \). Then, \( V^\lambda \neq 0 \) if and only if \( \lambda_j \leq r_1 \) for all \( j > r_0 \), i.e., \( \lambda \in \Gamma(r_0, r_1) \).

**Proof** Let \( x_1, \ldots, x_{r_0}, y_1, \ldots, y_{r_1} \) be commuting indeterminates. Consider the ring homomorphism \( \text{Sym} \rightarrow \mathbb{Z}[x_1, \ldots, x_{r_0}, y_1, \ldots, y_{r_1}] \) under which the formal power series \( \sum_{n=0}^{\infty} e_n t^n \) and \( \sum_{n=0}^{\infty} h_n t^n \) specialize, respectively, to

\[
\prod_{i=1}^{r_0} (1 + x_i t) \cdot \prod_{j=1}^{r_1} (1 - y_j t)^{-1} \quad \text{and} \quad \prod_{j=1}^{r_1} (1 + y_j t) \cdot \prod_{i=1}^{r_0} (1 - x_i t)^{-1}.
\]

Denote by \( u(x/y) \) the image of \( u \in \text{Sym} \) in the ring \( \mathbb{Z}[x_1, \ldots, x_{r_0}, y_1, \ldots, y_{r_1}] \) and by \( u(\alpha/\beta) \) the value of this polynomial \( u(x/y) \) at the point

\[(\alpha_1, \ldots, \alpha_{r_0}, \beta_1, \ldots, \beta_{r_1}) \in \mathbb{R}^{r_0 + r_1}\]

where \( \alpha_i \) and \( \beta_j \) are as in the proof of Corollary 4.9. Then,

\[
\dim V^\lambda = s^\lambda(\alpha/\beta) \quad \text{for each} \quad \lambda \in \mathcal{P}.
\]

Since the left- and right-hand sides of this equality are evaluations at \( s^\lambda \) of two ring homomorphisms \( \text{Sym} \rightarrow \mathbb{Z} \), it suffices to check it on the generators \( s^{(n)} = h_n \) of the ring \( \text{Sym} \). But for \( \lambda = (n) \) we have \( V^\lambda = S_n(V, R) \), while \( s^\lambda(\alpha/\beta) = h_n(\alpha/\beta) \) is exactly the coefficient of \( t^n \) in the Hilbert series of the algebra \( S(V, R) \), so that the equality is indeed true.

The specialization \( u \mapsto u(x/y) \) defined above differs from that of Macdonald [22, Ch. I, Example 3.23] in that each \( y_j \) is changed to \(-y_j\). With this change formula (1) in [22, Ch. I, Example 5.23] reads as

\[
s^\lambda(x/y) = \sum_{\mu \in \mathcal{P}} s^{\mu}(x)s^\lambda/\mu'(y)
\]

where \( s^\lambda/\mu' \) is the skew Schur function corresponding to the pair \( \lambda', \mu' \) of partitions conjugate to \( \lambda \) and \( \mu \). Thus, \( s^\lambda(x/y) \) is precisely the hook Schur function \( HS^\lambda \) in the notation and terminology of Berele and Regev [3, Definition 6.3].

Recall that each skew Schur function is a linear combination of monomial symmetric functions with nonnegative integer coefficients. Hence, any monomial in the indeterminates \( x_1, \ldots, x_{r_0}, y_1, \ldots, y_{r_1} \) has nonnegative coefficient in \( s^\lambda(x/y) \). It is known also that \( s^\lambda(x/y) \neq 0 \) if and only if \( \lambda \in \Gamma(r_0, r_1) \) [3, Cor. 6.5]. Since all real numbers \( \alpha_i \) and \( \beta_j \) are positive, we deduce that \( \dim V^\lambda = s^\lambda(\alpha/\beta) > 0 \) if and only if \( \lambda \in \Gamma(r_0, r_1) \).
Remark By Corollary 4.10, the image of \( \varphi \) is the subgroup of \( \text{Grot}(R) \) generated by the classes \( [V^k] \) with \( \lambda \in \Gamma(r_0, r_1) \). It is not clear whether these classes are always linearly independent over \( \mathbb{Z} \). In any event for each \( n \geq 0 \) the rank of the free abelian group \( \varphi(\text{Sym}_n) \) does not exceed the cardinality of the set \( \Gamma(r_0, r_1) \cap \mathcal{P}(n) \). On the other hand, the group \( \text{Grot}_n(R) \) may have a larger rank. When this happens, \( \varphi \) is not surjective unlike what we have seen in the semisimple case.

As an example consider the supersymmetry \( R \) on a \( \mathbb{Z}/2\mathbb{Z} \)-graded vector space \( V = V_0 \oplus V_1 \). It is defined by the rule \( R(v \otimes w) = w \otimes v \) for homogeneous elements \( v, w \in V \) at least one of which is even, and \( R(v \otimes w) = -w \otimes v \) when both \( v \) and \( w \) are odd. We assume here that \( \text{char } k \neq 2 \). This operator is a Hecke symmetry with parameter \( q = 1 \), so that \( \mathcal{H}_n(q) \) is the group algebra \( R \mathbb{S}_n \). It is easy to see that the one-dimensional source condition is satisfied. Since \( S(V, R) \) is the tensor product \( S(V_0) \otimes \wedge(V_1) \) of the ordinary symmetric and exterior algebras, it has Hilbert series \( (1+t)^{r_1}/(1-t)^{r_0} \) where \( r_i = \dim V_i \) for \( i = 0, 1 \). Thus, the birank of \( R \) coincides with the superdimension of \( V \).

The right \( A_n(R) \)-comodules may be identified with the degree \( n \) polynomial representations of the supergroup \( GL_{r_0|r_1} \) over the field \( \mathbb{k} \). (In the framework of super theory, one endows \( A(R) = \bigoplus_{k=0}^{\infty} A_k(R) \) with a modified version of the multiplication described in Sect. 2. This modified multiplication makes \( A(R) \) into a super bialgebra rather than an ordinary bialgebra, and in this way \( A(R) \) is identified with the subalgebra of the coordinate algebra of \( GL_{r_0|r_1} \) generated by the coefficient functions of the natural representation on \( V \).)

Thus, the rank of \( \text{Grot}_n(R) \) equals the number of irreducible polynomial representations of \( GL_{r_0|r_1} \) of degree \( n \). Irreducible representations are classified by their highest weights with respect to a maximal torus \( T \) of the group \( GL_{r_0} \times GL_{r_1} \subset GL_{r_0|r_1} \). The highest weights of polynomial representations may be interpreted as pairs of partitions \( \lambda, \mu \) such that \( \ell(\lambda) \leq r_0 \) and \( \ell(\mu) \leq r_1 \). However, not all such pairs correspond to a polynomial representation.

If \( \text{char } k = 0 \), then the highest weights of irreducible polynomial representations of \( GL_{r_0|r_1} \) are selected by the additional condition \( \ell(\mu) \leq \lambda_{r_0} \) on the pair \( (\lambda, \mu) \) (see [25, Cor. 1 to Th. 2]). On the other hand, since the group algebras \( \mathbb{k} \mathbb{S}_n \) are semisimple, the irreducible polynomial \( GL_{r_0|r_1} \)-modules of degree \( n \) are precisely the simple \( A_n(R) \)-comodules

\[
V^v_R = \text{Hom}_{\mathbb{k} \mathbb{S}_n}(S^v, V^\otimes n)
\]

with \( v \in \Gamma(r_0, r_1) \cap \mathcal{P}(n) \), while \( V^v_R = 0 \) when \( v \notin \Gamma(r_0, r_1) \). In terms of representations of Lie superalgebras, this fact was established long ago independently by Sergeev [25] and Berele, Regev [3]. It should be noted also that the character of \( V^v_R \) defined in terms of weight spaces with respect to the torus \( T \) is exactly the hook Schur function \( s_v(x/y) \). This explains the properties of these functions referred to in the proof of Corollary 4.10.

Suppose now that \( \text{char } k = p > 0 \). In this case, the irreducible representation of \( GL_{r_0|r_1} \) with highest weight represented by the pair \( (\lambda, \mu) \) is polynomial if and only if \( j(\mu) \leq \lambda_{r_0} \) where \( j(\mu) \) is the cardinality of a combinatorially defined subset of
nodes of the Young diagram of $\mu$ which contains at most one node from each row of the diagram. This condition was found by Brundan and Kujawa [4, Th. 6.5]. The inequality $j(\mu) \leq \ell(\mu)$ holds for each $\mu$. At the same time there exist highest weights $(\lambda, \mu)$ such that $\ell(\mu) > \lambda r_0$ but $j(\mu) \leq \lambda r_0$. For example, $j(\mu) = 0$ if $\mu \equiv 0 \pmod{p}$ for all $i$. From this, it follows that the group $\text{Grot}_n(R)$ has larger rank than that in the case of a field of characteristic 0, and therefore $\varphi(\text{Sym}_n) \neq \text{Grot}_n(R)$, for infinitely many $n.$

5 Tensor powers $V^\otimes n$ as modules over Hecke algebras

Consider $V^\otimes n$ as an $H_n(q)$-module with respect to the representation arising from a Hecke symmetry $R$. Assuming that $R$ satisfies the one-dimensional source condition, we will associate with this module a symmetric function $\text{ch}([V^\otimes n]) \in \text{Sym}_n$ which encodes enough information to determine the image of $V^\otimes n$ in the Grothendieck group $\text{Grot}_n(q)$.

For an associative algebra $\mathfrak{A}$ over some field, say $F$, denote by $\text{Rep}\mathfrak{A}$ the abelian group generated by the isomorphism classes $[X]$ of finite-dimensional left $\mathfrak{A}$-modules with the defining relations $[X] = [X'] + [X'']$ for each triple of finite-dimensional left $\mathfrak{A}$-modules such that $X \cong X' \oplus X''$. It is a free abelian group with a basis consisting of the isomorphism classes of indecomposable finite-dimensional left $\mathfrak{A}$-modules. By abuse of notation $[X]$ will stand for an element of either $\text{Rep}\mathfrak{A}$ or $\text{Grot}\mathfrak{A}$, depending on the context. There is a canonical group homomorphism

$$c : \text{Rep}\mathfrak{A} \longrightarrow \text{Grot}\mathfrak{A}$$

sending the class of $X$ in $\text{Rep}\mathfrak{A}$ to the class of $X$ in $\text{Grot}\mathfrak{A}$. This map is an isomorphism when $\mathfrak{A}$ is semisimple. In general $X \cong Y$ whenever $[X] = [Y]$ in $\text{Rep}\mathfrak{A}$.

Consider a $\mathbb{Z}$-bilinear form on $\text{Rep}\mathfrak{A}$ defined by the formula

$$\langle [X], [Y] \rangle = \langle X, Y \rangle = \dim_F \text{Hom}_{\mathfrak{A}}(X, Y)$$

for each pair of finite-dimensional left $\mathfrak{A}$-modules $X$ and $Y$.

Denote by $\text{Rep}^1 \mathcal{H}_n(q)$ (respectively, $\text{Triv}\mathcal{H}_n(q)$) the subgroup of $\text{Rep}\mathcal{H}_n(q)$ generated by the isomorphism classes of indecomposable left $\mathcal{H}_n(q)$-modules which have a one-dimensional (respectively, trivial) source. Then, $\text{Triv}\mathcal{H}_n(q) \subset \text{Rep}^1 \mathcal{H}_n(q)$.

In the next lemma, we work in the settings of Sect. 3:

**Lemma 5.1** There is a group homomorphism $e : \text{Rep}^1 \mathcal{H}_n(q) \rightarrow \text{Grot}\mathcal{H}_n(z) \mathbb{Q}$ such that $e([M_M]) = [M_Q]$ for each lattice $M \in \mathcal{H}_n(q)$ if $q \neq -1$ and for $M \in \text{Triv}$ if $q = -1$. It makes commutative the diagram

$\begin{array}{ccc}
\text{Rep}^1 \mathcal{H}_n(q) & \xrightarrow{c} & \text{Grot}\mathcal{H}_n(q) \\
\downarrow e & & \downarrow d \\
\text{Grot}\mathcal{H}_n(z) \mathbb{Q} & & \\
\end{array}$
where \( d \) is the decomposition map and \( c \) is the canonical map.

**Proof** Since \( \mathcal{H}_n(z)_Q \) is semisimple, the classes of Specht modules \( S_\lambda^k, \lambda \in \mathcal{P}(n) \), for this algebra form a \( \mathbb{Z} \)-basis of \( \text{Grot} \mathcal{H}_n(z)_Q \). Since these modules are absolutely irreducible, we have \( \langle S_\lambda^k, S_\mu^\ell \rangle = \delta_{k,\ell} \) (the Kronecker symbol). In particular, the bilinear form on \( \text{Grot} \mathcal{H}_n(z)_Q \) is nondegenerate. The equality of dimensions in Lemma 3.1 can be restated by saying that

\[
\langle N_{h_k}, M_{h_k} \rangle = \langle N_Q, M_Q \rangle
\]

for any two lattices \( M, N \in \text{Rep}^1 \) if \( q \neq -1 \) and for \( M, N \in \text{Triv} \) if \( q = -1 \).

If \( M, M' \in \text{Rep}^1 \) are such that \( M_{h_k} \cong M'_{h_k} \) as \( \mathcal{H}_n(q) \)-modules, and if moreover \( M, M' \in \text{Triv} \) when \( q = -1 \), then it follows from the displayed equality that \( \langle X, M_Q \rangle = \langle X, M'_Q \rangle \) for each permutation \( \mathcal{H}_n(z)_Q \)-module \( X \), i.e., a module induced from the trivial one-dimensional representation of a parabolic subalgebra of \( \mathcal{H}_n(z)_Q \). Since the classes of permutation modules form a \( \mathbb{Z} \)-basis of \( \text{Grot} \mathcal{H}_n(z)_Q \), we conclude that \( M_Q \cong M'_Q \) in this case.

This shows that the map \( e \) is well-defined on the elements \( [M_{h_k}] \). By Lemma 3.7, each indecomposable \( \mathcal{H}_n(q) \)-module with a one-dimensional source is isomorphic to \( M_{h_k} \) for a suitable choice of \( M \). Hence, \( e \) is well-defined on the semigroup of positive elements in \( \text{Rep}^1 \mathcal{H}_n(q) \). Since both \( \otimes \mathbb{O} \) and \( \otimes \mathbb{Q} \) are additive functors, we have \( e(a + b) = e(a) + e(b) \) for any pair of positive elements \( a, b \in \text{Rep}^1 \mathcal{H}_n(q) \). It follows that \( e \) extends to a group homomorphism on the whole \( \text{Rep}^1 \mathcal{H}_n(q) \). Commutativity of the diagram is clear from the definition of \( d \) in Sect. 3.

**Proposition 5.2** There exist group homomorphisms \( \text{ch} : \text{Rep}^1 \mathcal{H}_n(q) \to \text{Sym}_n \) and \( \psi : \text{Sym}_n \to \text{Grot} \mathcal{H}_n(q) \) with the following properties:

(i) \( \psi \circ \text{ch} \) is equal to the canonical map \( c : \text{Rep}^1 \mathcal{H}_n(q) \to \text{Grot} \mathcal{H}_n(q) \),

(ii) \( \psi(h_\lambda e_\mu) = [\mathcal{H}_n(q) \otimes \mathcal{H}_{\lambda,\mu}(q) k_{\lambda,\mu}] \) for each pair \( (\lambda, \mu) \in \mathcal{P}^2(n) \),

(iii) if \( q \neq -1 \) then \( \text{ch}([\mathcal{H}_n(q) \otimes \mathcal{H}_{\lambda,\mu}(q) k_{\lambda,\mu}]) = h_\lambda e_\mu \) for each pair \( (\lambda, \mu) \in \mathcal{P}^2(n) \),

(iv) \( \text{ch}([\mathcal{H}_n(q) \otimes \mathcal{H}_{\lambda}(q) k_{\text{triv}}]) = h_\lambda \) for each \( \lambda \in \mathcal{P}(n) \),

(v) \( \langle \text{ch}(a), \text{ch}(b) \rangle = \langle a, b \rangle \) for all \( a, b \in \text{Rep}^1 \mathcal{H}_n(q) \),

(vi) \( \text{ch} \) maps the subgroup \( \text{Triv} \mathcal{H}_n(q) \) isomorphically onto \( \text{Sym}_n \).

**Proof** By the semisimple case recalled in Sect. 2, there is a canonical isomorphism

\[
\text{Grot} \mathcal{H}_n(z)_Q \cong \text{Sym}_n.
\]

Composing the group homomorphisms \( d \) and \( e \) of Lemma 5.1 with the previous isomorphism, we obtain \( \psi \) and \( \text{ch} \), respectively. The commutative diagram in Lemma 5.1 yields (i).

Now take \( M = M_{h_\lambda e_\mu}, \) i.e., \( M = \mathcal{H}_n(z) \otimes \mathcal{H}_{\lambda,\mu}(z)_Q o_{\lambda,\mu} \in \text{Rep}^1 \) (see Sect. 3). We have \( d([M_Q]) = [M_{h_k}] \) by the definition of \( d \). If \( q \neq -1 \) then \( e([M_{h_k}]) = [M_Q] \) by the definition of \( e \). If \( \mu = 0 \), so that \( M = M_k \in \text{Triv} \), then \( e([M_{h_k}]) = [M_Q] \) for any \( q \).

Since

\[
[M_Q] = [\mathcal{H}_n(z)_Q \otimes \mathcal{H}_{\lambda,\mu}(z)_Q Q_{\lambda,\mu}] \in \text{Grot} \mathcal{H}_n(z)_Q
\]
corresponds to \( h_\lambda e_\mu \in \text{Sym}_n \) by Lemma 2.1, we get (ii)–(iv).

By Lemma 3.7, the group \( \text{Rep}^1 \mathcal{H}_n(q) \) is generated by the classes \([N_k]\) for all lattices \( N \in \mathcal{H}_n \), and when \( q = -1 \) it suffices to use only the lattices \( N \in \mathcal{T}_{\text{triv}} \). Since \( e([N_k]) = [N_Q] \) for such lattices, it follows from Lemma 3.1 that

\[
\langle e(a), e(b) \rangle = \langle a, b \rangle
\]

for all \( a, b \in \text{Rep}^1 \mathcal{H}_n(q) \). Since the isomorphism \( \text{Sym}_n \cong \text{Grot} \mathcal{H}_n(z) \) is isometric with respect to the scalar products defined on these groups, we get (v).

The trivial source indecomposable \( \mathcal{H}_n(q) \)-modules are precisely the Young modules \( Y^\lambda \) parameterized by partitions \( \lambda \in \mathcal{P}(n) \). These modules were described by Dipper and James in [7, Lemma 2.5]. For each \( \lambda \), the “permutation” module

\[
\mathcal{H}_n(q) \otimes \mathcal{H}_\lambda(q) \otimes_{\text{triv}}
\]

has \( Y^\lambda \) as its direct summand of multiplicity 1. All other direct summands of this permutation module are the Young modules \( Y^\nu \) with \( \nu > \lambda \) with respect to the dominance order on partitions.

The isomorphism classes of Young modules \( Y^\lambda \) form a \( \mathbb{Z} \)-basis of \( \text{Triv} \mathcal{H}_n(q) \). It follows that the classes of permutation modules also form a \( \mathbb{Z} \)-basis of \( \text{Triv} \mathcal{H}_n(q) \). According to (iv), \( \text{ch} \) maps this basis of \( \text{Triv} \mathcal{H}_n(q) \) to the \( \mathbb{Z} \)-basis \( \{ h_\lambda \mid \lambda \in \mathcal{P}(n) \} \) of \( \text{Sym}_n \). This entails (vi).

\( \Box \)

**Remark** Actually, the fact that the permutation \( \mathcal{H}_n(q) \)-modules do not have any indecomposable direct summands other than the Young modules was not proved in [7] for arbitrary \( q \). The isomorphism classes of those indecomposable summands are in a bijection with the isomorphism classes of simple modules for the \( q \)-Schur algebra \( S_q(k, n) \) when \( k \geq n \). In a later paper, Dipper and James showed that the latter modules are parameterized by partitions of \( n \) [8, Th. 8.8]. This settled the question about the direct summands of permutation modules (see Donkin [9, 4.4]).

Let us return to consideration of the Hecke symmetry \( R \). Put \( \mathcal{H}_n = \mathcal{H}_n(q) \).

**Lemma 5.3** Let \( f \) be the Hilbert series of the \( R \)-symmetric algebra \( S(V, R) \). Then,

\[
\dim \text{Hom}_{\mathcal{H}_n}(V^\otimes_n, \mathcal{H}_n \otimes \mathcal{H}_v \otimes_{\text{triv}}) = \hat{f}(h_v) = \sum_{\lambda \in \mathcal{P}(n)} \hat{f}(m_\lambda) \langle h_\lambda, h_v \rangle
\]

for each \( v \in \mathcal{P}(n) \).

**Proof** By the Frobenius reciprocity

\[
\text{Hom}_{\mathcal{H}_n}(V^\otimes_n, \mathcal{H}_n \otimes \mathcal{H}_v \otimes_{\text{triv}}) \cong \text{Hom}_{\mathcal{H}_v}(V^\otimes_n, \otimes_{\text{triv}}) \cong \text{Hom}_{k}(V^\otimes_n/\Sigma_v(V^\otimes_n), k)
\]

where \( \Sigma_v(V^\otimes_n) = \sum_{i \in \mathcal{H}_v} (T_i - q)V^\otimes_n \). If \( v = (v_1, \ldots, v_k) \), then

\[
V^\otimes_n/\Sigma_v(V^\otimes_n) \cong S_{v_1}(V, R) \otimes \cdots \otimes S_{v_k}(V, R)
\]

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by Lemma 4.1. Since \( \dim \mathbb{S}_V(V, R) = \hat{f}(h_{\nu}) \), we get

\[
\dim V^\otimes n / \sum V^\otimes n = \prod_{i=1}^k \hat{f}(h_{\nu_i}) = \hat{f}(h_{v_1} \cdots h_{v_k}) = \hat{f}(h_{\nu}),
\]

which yields the first equality in the statement of Lemma 5.3. The second equality follows from the fact that the two bases \( \{ m_{\lambda} \mid \lambda \in \mathcal{P}(n) \} \) and \( \{ h_{\lambda} \mid \lambda \in \mathcal{P}(n) \} \) of the group \( \text{Sym}_n \) are dual to each other with respect to the scalar product on \( \text{Sym}_n \) [22, Ch. I, (4.5)]. Hence, \( u = \sum_{\lambda \in \mathcal{P}(n)} \langle h_{\lambda}, u \rangle m_{\lambda} \) for each \( u \in \text{Sym}_n \), and so this formula can be used for \( u = h_{\nu}. \)

\[ \square \]

Lemma 5.4

If \( f = \prod_{j=1}^{r_1} (1 + \beta_j t) \cdot \prod_{i=1}^{r_0} (1 - \alpha_i t)^{-1} \), then

\[
\hat{f}(h_{\nu}) = \sum_{(\lambda, \mu) \in \mathcal{P}_2(n)} N(\lambda, \mu, \nu) m_{\lambda, \alpha} m_{\mu, \beta}, \quad \nu \in \mathcal{P}(n),
\]

where we put \( u(\alpha) = u(\alpha_1, \ldots, \alpha_{r_0}), \ u(\beta) = u(\beta_1, \ldots, \beta_{r_1}) \) for each \( u \in \text{Sym} \) and \( N(\lambda, \mu, \nu) \) is the number of all pairs of nonnegative integer matrices

\[
A = (a_{ij})_{1 \leq i \leq \ell(\lambda), 1 \leq j \leq \ell(\mu)}, \quad B = (b_{lj})_{1 \leq l \leq \ell(\nu), 1 \leq j \leq \ell(\mu)}
\]

such that \( B \) has only entries equal to 0 or 1,

\[
\sum_{l=1}^{\ell(\nu)} a_{il} = \lambda_i \quad \text{for each } i = 1, \ldots, \ell(\lambda), \quad \sum_{l=1}^{\ell(\nu)} b_{lj} = \mu_j \quad \text{for each } j = 1, \ldots, \ell(\mu),
\]

\[
\text{and } \sum_{i=1}^{\ell(\lambda)} a_{il} + \sum_{j=1}^{\ell(\mu)} b_{lj} = v_l \quad \text{for each } l = 1, \ldots, \ell(\nu).
\]

Also, \( N(\lambda, \mu, \nu) = \langle h_{\lambda} e_\mu, h_{\nu} \rangle \).

**Proof** Recall that \( \prod (1 - \alpha \alpha) \hat{f}(h_{\nu}) = \hat{f}(h_{\nu}) \) by the definitions of complete and elementary symmetric functions. For each integer \( p \geq 0 \), the coefficient of \( t^p \) in \( f \) is

\[
\hat{f}(h_{\nu}) = \sum_{i=0}^{p} h_i(\alpha) e_{p-i}(\beta),
\]

i.e., \( \hat{f}(h_{\nu}) \) equals the sum of all monomials \( \alpha_1^{a_1} \cdots \alpha_{r_0}^{a_0} \beta_1^{b_1} \cdots \beta_{r_1}^{b_{r_1}} \) with integer exponents \( a_i \geq 0 \) and \( b_j \in \{0, 1\} \) such that \( \sum a_i + \sum b_j = p \). Hence,

\[
\hat{f}(h_{\nu}) = \prod_{l=1}^k \hat{f}(h_{\nu_l}) = \sum_{A, B} \prod_{l=1}^k (\alpha_1^{a_{1l}} \cdots \alpha_{r_0}^{a_{0l}} \beta_1^{b_{1l}} \cdots \beta_{r_1}^{b_{r_1}})
\]
where \( k = \ell(v) \) and the sum runs over all pairs of nonnegative integer matrices

\[
A = (a_{li})_{1 \leq l \leq k, 1 \leq i \leq r_0}, \quad B = (b_{lj})_{1 \leq l \leq k, 1 \leq j \leq r_1}
\]
such that \( B \) has only entries equal to 0 or 1 and

\[
\sum_{i=1}^{r_0} a_{li} + \sum_{j=1}^{r_1} b_{lj} = v_l \quad \text{for each} \quad l = 1, \ldots, k.
\]

For \( \lambda, \mu \in \mathcal{P} \) we have \( m_\lambda(\alpha) = 0 \) whenever \( \ell(\lambda) > r_0 \) and \( m_\mu(\beta) = 0 \) whenever \( \ell(\mu) > r_1 \). If \( \ell(\lambda) \leq r_0 \) and \( \ell(\mu) \leq r_1 \), then each occurrence of the monomial \( \alpha_1 \cdots \alpha_{r_0} \beta_1 \cdots \beta_{r_1} \) in the previous expression for \( \hat{f}(h_v) \) corresponds to a pair of matrices \( A, B \) used in the definition of \( N(\lambda, \mu), v \). Thus, \( N(\lambda, \mu), v \) is the total number of such occurrences. Since \( \hat{f}(h_v) \) is a symmetric function of \( \alpha_1, \ldots, \alpha_{r_0} \) and a symmetric function of \( \beta_1, \ldots, \beta_{r_1} \), it is a \( \mathbb{Z} \)-linear combination of the products of monomial symmetric functions in these sets of elements, and we obtain the desired formula.

Next, writing out the product \( h_\lambda e_\mu = h_{\lambda_1} \cdots h_{\lambda_{\ell(\lambda)}} e_{\mu_1} \cdots e_{\mu_{\ell(\mu)}} \) as a sum of monomials in the indeterminates \( x_1, x_2, \ldots \), we see that the coefficient of \( x_{v_1} \cdots x_{v_k} \) in \( h_\lambda e_\mu \) equals \( N(\lambda, \mu), v \) too. Hence,

\[
h_\lambda e_\mu = \sum_{\rho \in \mathcal{P}(n)} N(\lambda, \mu), \rho m_\rho \quad \text{for} \quad (\lambda, \mu) \in \mathcal{P}^2(n),
\]

and it follows that \( \langle h_\lambda e_\mu, h_v \rangle = N(\lambda, \mu), v \). \( \square \)

We will write \( \text{ch}(X) = \text{ch}([X]) \) for each finite-dimensional left \( \mathcal{H}_n \)-module \( X \) where \( \text{ch} \) is the map of Proposition 5.2. The property

\[
\langle \text{ch}(X), \text{ch}(Y) \rangle = \langle X, Y \rangle
\]

of this map enables us to determine \( \text{ch}(V^\otimes n) \):

**Theorem 5.5** Let \( R : V^\otimes 2 \rightarrow V^\otimes 2 \) be a Hecke symmetry of birank \( (r_0, r_1) \) with parameter \( q \). Suppose that \( R \) satisfies the one-dimensional source condition and

\[
H_{\Sigma(V, R)} = \prod_{j=1}^{r_1} (1 + \beta_j t) \cdot \prod_{i=1}^{r_0} (1 - \alpha_i t)^{-1}.
\]

Then, \( \text{ch}(V^\otimes n) = \sum_{(\lambda, \mu) \in \mathcal{P}^2(n)} m_\lambda(\alpha)m_\mu(\beta)h_\lambda e_\mu \). In particular,

\[
[V^\otimes n] = \sum_{(\lambda, \mu) \in \mathcal{P}^2(n)} m_\lambda(\alpha)m_\mu(\beta) [\mathcal{H}_n(q) \otimes \mathcal{H}_{\lambda, \mu}(q) \mathbb{1}_{\lambda, \mu}] \]

in the Grothendieck group \( \text{Grot} \mathcal{H}_n(q) \).

\( \square \) Springer
Proof By Lemmas 5.3 and 5.4, 

$$\langle V^\otimes n, H_n \otimes H_{\nu} \kappa_{\text{triv}} \rangle = \hat{f}(h_{\nu}) = \sum_{(\lambda, \mu) \in \mathcal{P}^2(n)} m_\lambda(\alpha) m_\mu(\beta) \langle h_\lambda e_\mu, h_{\nu} \rangle$$

for each $\nu \in \mathcal{P}(n)$. Also, $\langle \text{ch}(V^\otimes n), h_\nu \rangle = \langle V^\otimes n, H_n \otimes H_{\nu} \kappa_{\text{triv}} \rangle$ by (iv) and (v) of Proposition 5.2. This means that

$$\text{ch}(V^\otimes n) - \sum_{(\lambda, \mu) \in \mathcal{P}^2(n)} m_\lambda(\alpha) m_\mu(\beta) h_\lambda e_\mu$$

is orthogonal to all functions $h_\nu$, $\nu \in \mathcal{P}(n)$, which generate the whole group Sym$_n$. Nondegeneracy of the scalar product entails the required formula for ch$(V^\otimes n)$. The final equality is obtained then by applying the map $\psi : \text{Sym}_n \to \text{Grot} H_n$ described in Proposition 5.2. $\square$

Corollary 5.6 If $n$ is such that the Hecke algebra $H_n(q)$ is semisimple, then as an $H_n(q)$-module,

$$V^\otimes n \cong \bigoplus_{(\lambda, \mu) \in \mathcal{P}^2(n)} (H_n(q) \otimes H_{\lambda, \mu}(q) \kappa_{\lambda, \mu})^{m_\lambda(\alpha)m_\mu(\beta)}.$$ 

Proof In this case, the three maps $c, d, e$ of Lemma 5.1 are bijective. Hence, so too is the map $\text{ch} : \text{Rep} H_n(q) \to \text{Sym}_n$. The two modules in the statement are isomorphic since they have the same image in Sym$_n$. $\square$

6 The Hilbert series of intertwining algebras

Let $K$ be any commutative ring. Denote by $U_K$ the multiplicative subgroup of the ring $K[[t]]$ consisting of all formal power series with constant term equal to 1. In other words, $U_K = 1 + m$ where $m$ is the ideal of $K[[t]]$ generated by $t$. There is a well-known $\lambda$-ring structure on $U_K$ [2, Lemma 1.1]. Addition in this ring is given by the usual multiplication of power series, while the ring multiplication is another binary operation $\circ$ which has the property that $(1 + at) \circ (1 + bt) = 1 + abt$ for $a, b \in K$. However, Theorem 6.5 makes use of a different multiplication $\circ$ which gives an isomorphic ring structure on $U_K$ and satisfies

$$(1 + at) \circ (1 + bt) = (1 - at)^{-1} \circ (1 - bt)^{-1} = (1 - abt)^{-1}.$$ 

Now, we will give a formal definition of this binary operation using the theory of symmetric functions. For each $n \geq 0$, extend the scalar product on Sym$_n$ to a symmetric $K$-bilinear form on the $K$-module Sym$_n,K = K \otimes_Z \text{Sym}_n$. Since this bilinear form induces a bijection 

$$\text{Sym}_n,K \to \text{Hom}_K(\text{Sym}_n,K, K) \cong \text{Hom}_Z(\text{Sym}_n,K),$$

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to each \( f \in U_K \), there corresponds a uniquely determined element \( \xi_n(f) \in \text{Sym}_{n,K} \) such that
\[
\hat{f}(u) = \langle \xi_n(f), u \rangle \quad \text{for all} \quad u \in \text{Sym}_n
\]
where \( \hat{f} : \text{Sym} \to K \) is the ring homomorphism defined in Sect. 4, and we identify elements \( u \in \text{Sym}_n \) with their images \( 1 \otimes u \) in \( \text{Sym}_{n,K} \). Put
\[
f \odot g = \sum_{n=0}^{\infty} \langle \xi_n(f), \xi_n(g) \rangle t^n \quad \text{for} \quad f, g \in U_K.
\]

Denote by \( c_n(F) \) the coefficient of \( t^n \) in a formal power series \( F \). Considering the extensions of \( \hat{f}, \hat{g} \) to \( K \)-algebra homomorphisms \( K \otimes_{\mathbb{Z}} \text{Sym} \to K \), we have
\[
c_n(f \odot g) = \langle \xi_n(f), \xi_n(g) \rangle = \hat{f}(\xi_n(g))\hat{g}((\xi_n(f)).
\]
Lemma 6.2 Let \( f \) be the Hilbert series of the \( R \)-symmetric algebra \( \mathbb{S}(V, R) \). If \( R \) satisfies the one-dimensional source condition, then \( \xi_n(f) = \chi(V^\otimes n) \) where \( \chi \) is the map of Proposition 5.2.

Proof Here, \( K = \mathbb{Z} \), and so \( \xi_n(f) \in \text{Sym}_n \). As we have seen in the proof of Theorem 5.5,

\[
\langle \chi(V^\otimes n), h_\nu \rangle = \langle V^\otimes n, \mathcal{H}_n \otimes \mathcal{H}_\nu \xi_{\text{triv}} \rangle = \hat{f}(h_\nu)
\]

for all \( \nu \in \mathcal{P}(n) \). Hence \( \langle \chi(V^\otimes n), u \rangle = \hat{f}(u) \) for all \( u \in \text{Sym}_n \), and the conclusion follows from the definition of \( \xi_n(f) \).
Suppose now that $V'$ is a second finite-dimensional vector space over the same field $\mathbb{k}$, and $R'$ is a Hecke symmetry on $V'$ with the same parameter $q$ as the Hecke symmetry $R$ on $V$. We will define the algebra $A(R', R)$ (cf. [18, 27]).

Consider the tensor algebra of the vector space $\text{Hom}_\mathbb{k}(V, V')^*$ dual to $\text{Hom}_\mathbb{k}(V, V')$. Its homogeneous component of degree $n$ admits the following realization via canonical $\mathbb{k}$-linear bijections

$$T_n(\text{Hom}_\mathbb{k}(V, V')^*) \cong (\text{Hom}_\mathbb{k}(V, V')^\otimes n)^* \cong \text{Hom}_\mathbb{k}(V^\otimes n, V'^\otimes n)^*.$$ 

Denote by $\mathcal{R}$ the linear operator on the vector space $\text{Hom}_\mathbb{k}(V^\otimes 2, V'^\otimes 2)^*$ such that the dual operator $\mathcal{R}^*$ on $\text{Hom}_\mathbb{k}(V^\otimes 2, V'^\otimes 2)$ is defined by the formula

$$\mathcal{R}^*(h) = (R')^{-1} \circ h \circ R, \quad h \in \text{Hom}_\mathbb{k}(V^\otimes 2, V'^\otimes 2).$$

In other words, $\mathcal{R} = (R'^*)^{-1} \otimes R$ under the canonical identification

$$\text{Hom}_\mathbb{k}(V^\otimes 2, V'^\otimes 2)^* \cong (V'^\otimes 2)^* \otimes V^\otimes 2.$$

The algebra $A(R', R)$ is defined as the factor algebra of $\mathbb{T}(\text{Hom}_\mathbb{k}(V, V')^*)$ by the homogeneous ideal $I$ generated by

$$I_2 = \text{Im}(\mathcal{R} - \text{Id}) \subset \text{Hom}_\mathbb{k}(V^\otimes 2, V'^\otimes 2)^* \cong \mathbb{T}_2(\text{Hom}_\mathbb{k}(V, V')^*).$$

Under the canonical pairings between $T_n(\text{Hom}_\mathbb{k}(V, V')^*)$ and $\text{Hom}_\mathbb{k}(V^\otimes n, V'^\otimes n)$, we have

$$I_2^\perp = \{ h \in \text{Hom}_\mathbb{k}(V^\otimes 2, V'^\otimes 2) \mid \mathcal{R}^*(h) = h \}$$

$$= \{ h \in \text{Hom}_\mathbb{k}(V^\otimes 2, V'^\otimes 2) \mid h \circ R = (R' \circ h) \}.$$ for $n = 2$. In degree $n > 2$, the homogeneous component of the ideal $I$ is

$$I_n = \sum_{i=1}^{n-1} T_{i-1}(\text{Hom}_\mathbb{k}(V, V')^*) \otimes I_2 \otimes T_{n-i-1}(\text{Hom}_\mathbb{k}(V, V')^*).$$

Consider $V^\otimes n$ and $V'^\otimes n$ as $\mathcal{H}_n(q)$-modules with respect to representations arising from the Hecke symmetries $R$ and $R'$, respectively. It follows then that

$$I_n^\perp = \{ h \in \text{Hom}_\mathbb{k}(V^\otimes n, V'^\otimes n) \mid hT_i = T_i h \text{ for all } i = 1, \ldots n - 1 \}$$

$$= \text{Hom}_{\mathcal{H}_n}(V^\otimes n, V'^\otimes n).$$

This equality holds for all $n \geq 0$ since $I_n = 0$ when $n \leq 1$. Hence,

$$A_n(R', R) = \mathbb{T}_n(\text{Hom}_\mathbb{k}(V, V')^*)/I_n \cong \text{Hom}_{\mathcal{H}_n}(V^\otimes n, V'^\otimes n)^* \text{ for all } n.$$
Note that $A(R', R) = A(R)$ when $R' = R$. Another graded algebra $E(R', R)$ related to $A(R', R)$ is defined as the factor algebra of $\mathcal{T}(\text{Hom}_k(V, V'))^n$ by the ideal generated by $\text{Ker}(\mathcal{R} - \text{Id})$.

**Theorem 6.5** Let $R, R'$ be Hecke symmetries on finite-dimensional vector spaces $V, V'$ with the same parameter $q$ of the Hecke relation. Suppose that both $R$ and $R'$ satisfy the one-dimensional source condition. Let

$$f = \prod_{j=1}^{r_1}(1 + \beta_j t) \cdot \prod_{i=1}^{r_0}(1 - \alpha_i t)^{-1}, \quad g = \prod_{j=1}^{r_1'}(1 + \beta_j' t) \cdot \prod_{i=1}^{r_0'}(1 - \alpha_i' t)^{-1}$$

be the Hilbert series of the algebras $\mathcal{S}(V, R)$ and $\mathcal{S}(V', R')$. Then, the Hilbert series of the algebra $A(R', R)$ equals

$$\prod_{i=1}^{r_1} \prod_{j=1}^{r_0'} f(i, j) \cdot \prod_{i=1}^{r_1'} \prod_{j=1}^{r_0} g(i, j)$$

**Proof** The isomorphism classes of $\mathcal{H}_n(q)$-modules $V^\otimes n$ and $V'^\otimes n$ define elements of $\text{Rep}_1\mathcal{H}_n(q)$. We have $\xi_n(f) = \text{ch}(V^\otimes n)$ and $\xi_n(g) = \text{ch}(V'^\otimes n)$ by Lemma 6.4. Now,

$$\dim A_n(R', R) = \dim \text{Hom}_{\mathcal{H}_n(q)}(V^\otimes n, V'^\otimes n) = \langle V^\otimes n, V'^\otimes n \rangle.$$  

Proposition 5.2(v) and Lemma 6.4 yield

$$\dim A_n(R', R) = \langle \text{ch}(V^\otimes n), \text{ch}(V'^\otimes n) \rangle = \langle \xi_n(f), \xi_n(g) \rangle.$$  

Thus, the Hilbert series $\sum_{n \geq 0} (\dim A_n(R', R)) t^n$ of the algebra $A(R', R)$ coincides with $f \circ g$. Taking $K = \mathbb{R}$ and applying Lemmas 6.2 and 6.3, we get

$$f \circ g = \prod_{i=1}^{r_0'} f(i) \cdot \prod_{j=1}^{r_1'} f(-\beta_j t)^{-1},$$

which gives the required formula. \qed

### 7 Even symmetries satisfying the trivial source condition

If $R$ is a Hecke symmetry of birank $(r_0, r_1)$ with $r_1 = 0$, then $R$ is said to have rank $r = r_0$. In the situation of Theorem 4.8, this happens precisely when the Hilbert series of $A(V, R)$ is a polynomial of degree $r$, i.e., the algebra $A(V, R)$ is finite dimensional.
with the grading of length \( r \). In [15], Gurevich calls a closed Hecke symmetry of rank \( r \) even. We do not need the assumption of closedness, however. Put

\[
P(n, r) = \{ \lambda \in P(n) \mid \ell(\lambda) \leq r \}.
\]

**Theorem 7.1** Suppose that \( R \) is a Hecke symmetry of rank \( r \) satisfying the trivial source condition. Let

(i) \( V \otimes n \cong \bigoplus_{\lambda \in P(n)} (\mathcal{H}_n(q) \otimes \mathcal{H}_\lambda(q) \mathbb{K}_{\text{triv}})^{m_\lambda(\alpha)} \) as \( \mathcal{H}_n(q) \)-modules,

(ii) the algebra \( A_n(R)^* \) is Morita equivalent to the \( q \)-Schur algebra \( S_q(r, n) \),

(iii) \( \{ [V^\lambda] \mid \lambda \in P(n, r) \} \) is a \( \mathbb{Z} \)-basis of \( \text{Grot}_n(R) \).

There is an isomorphism of graded rings \( \text{Grot}(R) \cong \text{Sym} / I_r \) where \( I_r \) is the ideal of \( \text{Sym} \) with a \( \mathbb{Z} \)-basis \( \{ s_\lambda \mid \lambda \in P, \ell(\lambda) > r \} \).

**Proof** Under the present assumptions, the \( \mathcal{H}_n(q) \)-module \( V \otimes n \) defines an element of the group \( \text{Triv}_n \mathcal{H}_n(q) \), and so too does the \( \mathcal{H}_n(q) \)-module in the right-hand side of (i).

By Theorem 5.5,

\[
\text{ch}(V \otimes n) = \sum_{\lambda \in P(n)} m_{\lambda}(\alpha) h_{\lambda}.
\]

Parts (iv) and (vi) of Proposition 5.2 show that the second module has the same image under the isomorphism of groups \( \text{ch} : \text{Triv}_n \mathcal{H}_n(q) \rightarrow \text{Sym}_n \), and therefore the two modules are isomorphic. This is conclusion (i).

Recall that \( A_n(R)^* \cong E_n = \text{End}_n \mathcal{H}_n(q) V \otimes n \), while \( S_q(r, n) \) is the endomorphism ring of the \( \mathcal{H}_n(q) \)-module

\[
M(r, n) = \bigoplus_{\mu = (\mu_1, \ldots, \mu_r)} (\mathcal{H}_n(q) \otimes \mathcal{H}_\mu(q) \mathbb{K}_{\text{triv}})
\]

where the sum is taken over all weak compositions of \( n \) with \( r \) components. A weak composition is allowed to have zero components \( \mu_i \), and the corresponding parabolic subalgebra \( \mathcal{H}_\mu(q) \) is the same as one would get by removing from \( \mu \) all its zero components. If \( \mu'' \in P(n, r) \) is the partition obtained from \( \mu \) by rearranging its nonzero components in decreasing order, then

\[
\mathcal{H}_n(q) \otimes \mathcal{H}_\mu(q) \mathbb{K}_{\text{triv}} \cong \mathcal{H}_n(q) \otimes \mathcal{H}_{\mu''}(q) \mathbb{K}_{\text{triv}}.
\]

On the other hand, \( m_\lambda(\alpha) \geq 0 \) for each \( \lambda \in P(n) \), and \( m_\lambda(\alpha) = 0 \) if and only if \( \ell(\lambda) > r \). Therefore, nonzero summands in the right-hand side of (i) correspond precisely to partitions \( \lambda \in P(n, r) \). We see that the \( \mathcal{H}_n(q) \)-modules \( V \otimes n \) and \( M(r, n) \) have the same set of isomorphism classes of indecomposable direct summands. But then the endomorphism rings \( E_n \) and \( S_q(r, n) \) of the two modules are Morita equivalent. This entails (ii).

Although the direct sum decomposition of (i) in general does not come from an action of a torus, we can still use it to describe the structure of \( A(R) \)-comodules...
by means of a kind of weight spaces. Recall that right \( A_n(R) \)-comodules may be identified with left \( E_n \)-modules. Fix an isomorphism in (i). It gives a collection of \( H_n(q) \)-submodules

\[ M_{\lambda, i} \subset V^{\otimes n} \quad \text{with} \quad \lambda \in \mathcal{P}(n, r), \ 1 \leq i \leq m_\lambda(\alpha) \]

such that \( V^{\otimes n} = \bigoplus M_{\lambda, i} \) and \( M_{\lambda, i} \cong H_n(q) \otimes H_n(q) \) for each pair \( (\lambda, i) \). Denote by \( \xi_{\lambda, i} \) the projection onto \( M_{\lambda, i} \) with respect to this decomposition. Then,

\[ \{ \xi_{\lambda, i} | \lambda \in \mathcal{P}(n, r), \ 1 \leq i \leq m_\lambda(\alpha) \} \]

is a set of pairwise orthogonal idempotents in the ring \( E_n \) with \( \sum \xi_{\lambda, i} = 1 \). If \( X \) is any left \( E_n \)-module, then \( X = \bigoplus X_{\lambda, i} \) where \( X_{\lambda, i} = \xi_{\lambda, i} X \). Put \( X_\lambda = X_{\lambda, 1} \) and define the character of \( X \) by the formula

\[ \chi(R)(X) = \sum_{\lambda \in \mathcal{P}(n, r)} (\dim X_\lambda) m_\lambda(x_1, \ldots, x_r) \in \text{Sym}_n(r) \subset \mathbb{Z}[x_1, \ldots, x_r] \]

where \( \text{Sym}_n(r) \) is the group of all symmetric homogeneous polynomials of degree \( n \) in \( r \) indeterminates \( x_1, \ldots, x_r \).

Since \( M_{\lambda, i} \cong M_{\lambda, 1} \), the idempotent \( \xi_{\lambda, i} \) is conjugate to \( \xi_{\lambda, 1} \) by an inner automorphism of \( E_n \). Hence, \( \dim X_{\lambda, i} = \dim X_\lambda \) for all \( i \) such that \( 1 \leq i \leq m_\lambda(\alpha) \). In particular, it follows that \( X = 0 \) whenever \( \chi(R)(X) = 0 \).

If \( 0 \to X' \to X \to X'' \to 0 \) is an exact sequence of finite-dimensional left \( E_n \)-modules, then \( \dim X_\lambda = \dim X'_{\lambda} + \dim X''_{\lambda} \) for each \( \lambda \), whence

\[ \chi(R)(X) = \chi(R)(X') + \chi(R)(X'') \]

So \( \chi(R) \) gives rise to a group homomorphism

\[ \text{Grot}_n(R) \cong \text{Grot} E_n \longrightarrow \text{Sym}_n(r) \]

which will be denoted by the same symbol \( \chi(R) \).

It is well-known that the Schur polynomials \( s_\lambda(x_1, \ldots, x_r) \) with \( \lambda \in \mathcal{P}(n, r) \) form a \( \mathbb{Z} \)-basis of \( \text{Sym}_n(r) \). On the other hand, isomorphism classes of simple \( S_q(n, r) \)-modules are also parameterized by the set \( \mathcal{P}(n, r) \). The Morita equivalent algebra \( E_n \) has the same number of simple modules equal to the cardinality of \( \mathcal{P}(n, r) \). In other words, \( \text{Grot}_n(R) \) and \( \text{Sym}_n(r) \) are free abelian groups of equal ranks. As will be proved separately in Lemma 7.2, \( s_\lambda(x_1, \ldots, x_r) \) is the image of \( V^\lambda \) under \( \chi(R) \). Hence, \( \chi(R) \) is surjective, but then \( \chi(R) \) has to map the group \( \text{Grot}_n(R) \) isomorphically onto \( \text{Sym}_n(r) \), and (iii) is also clear.

The ring homomorphism \( \phi : \text{Sym} \to \text{Grot}(R) \) defined in Proposition 4.2 sends \( s_\lambda \) to \( [V^\lambda] \) for each \( \lambda \in \mathcal{P} \). It follows from (iii) that \( \phi \) maps the subgroup of \( \text{Sym} \) generated by \( \{ s_\lambda \mid \ell(\lambda) \leq r \} \) isomorphically onto \( \text{Grot}(R) \). In particular, \( \phi \) is surjective. If
Lemma 7.2 For each $\lambda \in \mathcal{P}(n)$, we have $\mathrm{ch}^R(V^\lambda) = s_\lambda(x_1, \ldots, x_r)$. 

Proof First, we evaluate $\mathrm{ch}^R(S^\mu)$ for $\mu \in \mathcal{P}(n)$. Recall that $S^\mu \cong \mathcal{K}_{\text{triv}} \otimes \mathcal{H}_{\mu}(q) V^\otimes n$. If $\lambda \in \mathcal{P}(n, r)$, then

$$S^\mu_\lambda \cong \mathcal{K}_{\text{triv}} \otimes \mathcal{H}_{\mu}(q) M_{\lambda, 1} \cong \mathcal{K}_{\text{triv}} \otimes \mathcal{H}_{\mu}(q) \mathcal{H}_n(q) \otimes \mathcal{H}_{\lambda}(q) \mathcal{K}_{\text{triv}}$$

since the $E_n$-module structure on $S^\mu$ comes from the action of $E_n$ on $V^\otimes n$. Hence, $\dim S^\mu_\lambda$ is equal to the number $N_{\mu\lambda}$ of all $\mathcal{S}_\mu - \mathcal{S}_\lambda$ double cosets in the group $\mathcal{S}_n$. There is also a combinatorial description of $N_{\mu\lambda} = N_{\lambda\mu}$ as the number of all nonnegative integer matrices of size $\ell(\lambda) \times \ell(\mu)$ having row sums $\lambda_i$, $i = 1, \ldots, \ell(\lambda)$, and column sums $\mu_j$, $j = 1, \ldots, \ell(\mu)$. By [22, Ch. I, (6.7)] $h_\mu = \sum_{\lambda \in \mathcal{P}(n)} N_{\mu\lambda} m_{\lambda}$. If $\ell(\lambda) > r$, then $m_{\lambda}(x_1, \ldots, x_r) = 0$. Therefore,

$$\mathrm{ch}^R(S^\mu) = \sum_{\lambda \in \mathcal{P}(n, r)} N_{\mu\lambda} m_{\lambda}(x_1, \ldots, x_r) = h_\mu(x_1, \ldots, x_r).$$

Let $\varphi : \text{Sym} \to \text{Grot}(R)$ be the ring homomorphism defined in Proposition 4.2. Since $\varphi(h_\mu) = [S^\mu]$, the previous equality can be rewritten as

$$\mathrm{ch}^R(\varphi(u)) = u(x_1, \ldots, x_r)$$

for $u = h_\mu$. Since the set $\{h_\mu \mid \mu \in \mathcal{P}(n)\}$ generates the whole group $\text{Sym}_n$, the formula above holds then for all $u \in \text{Sym}_n$. Taking $u = s_\lambda$, we get the required conclusion. \qed

Remark For each $\lambda \in \mathcal{P}(n, r)$ it follows from Lemma 7.2 that $\dim(V^\lambda)_\lambda = 1$ and $(V^\lambda)_\mu = 0$ unless $\mu \leq \lambda$. In this sense, $\lambda$ is the highest weight of $V^\lambda$. Since the function $X \mapsto \dim X_\lambda$ is additive on exact sequences of $E_n$-modules, there is exactly one composition factor $L^\lambda$ of $V^\lambda$ with nonzero $\lambda$-weight space. Clearly, $L^\lambda$ is also a module of the highest weight $\lambda$. In particular, $L^\lambda \not\cong L^\mu$ whenever $\lambda, \mu \in \mathcal{P}(n, r)$ and $\lambda \neq \mu$. This implies that $\{L^\lambda \mid \lambda \in \mathcal{P}(n, r)\}$ is the full set of pairwise nonisomorphic simple $E_n$-modules.

Let $H(R)$ stand for the Hopf envelope of $A(R)$. If $R$ and $R'$ are two closed Hecke symmetries with the same parameter $q$ and of the same birank $(r_0, r_1)$, then in the
semisimple case, by a theorem of Hai [19, Th. 4.3], there is a braided monoidal equivalence between the categories of right $H(R)$-comodules and right $H(R')$-comodules.

We are interested in the bialgebra version of this result. Let $R$ and $R'$ be not necessarily closed Hecke symmetries on vector spaces $V$ and $V'$, respectively, with the same parameter $q$ of the Hecke relation. Consider the $H_n(q)$-module structures on $V^\otimes n$ and $V'^\otimes n$ arising from $R$ and $R'$. It was proved in [27, Th. 7.2] that there is a braided monoidal equivalence between the categories of right $A(R)$-comodules and right $A(R')$-comodules provided that for each $n > 0$ the indecomposable $H_n(q)$-modules isomorphic to direct summands of $V^\otimes n$ are the same as those isomorphic to direct summands of $V'^\otimes n$.

If $R$ and $R'$ have the same birank $(r_0, r_1)$, then in the semisimple case the required condition on direct summands is satisfied since in each of the two $H_n(q)$-modules the simple submodules are precisely the Specht modules $S^\lambda$ with $\lambda \in \Gamma(r_0, r_1) \cap \mathcal{P}(n)$. Unfortunately, we are not able to extend this result to the nonsemisimple case because Theorem 5.5 does not provide enough information on the module structure of tensor powers. Therefore, we have to restrict the class of Hecke symmetries in the following statement:

**Theorem 7.3** Let $R$ and $R'$ be Hecke symmetries with the same parameter $q$, of the same rank $r$, and both satisfying the trivial source condition. Then, there is a braided monoidal equivalence between the categories of right (or left) $A(R)$-comodules and $A(R')$-comodules.

**Proof** By Theorem 7.1, the $H_n(q)$-modules $V^\otimes n$ and $V'^\otimes n$ have the same set of isomorphism classes of indecomposable direct summands. So [27, Th. 7.2] does apply.

In conclusion, we improve yet another result from [27]. Definitions of the algebras $A(R', R)$ and $E(R', R)$ are recalled in Sect. 6.

**Theorem 7.4** Let $R$ and $R'$ be Hecke symmetries with the same parameter $q$, both satisfying the trivial source condition. Suppose that the algebras $A(V, R)$ and $A(V', R')$ are Frobenius with the gradings of length $r$ and $r'$, respectively. Then, the algebra $E(R', R)$ is Frobenius with the grading of length $rr'$, while $A(R', R)$ is Gorenstein of global dimension $rr'$.

**Proof** The Hilbert series of the algebras $A(V, R)$ and $A(V', R')$ are polynomials of degree $r$ and $r'$, respectively. We can write them as

$$
\prod_{i=1}^{r}(1 + \alpha_i t) \quad \text{and} \quad \prod_{j=1}^{r'}(1 + \alpha'_j t).
$$

The Hilbert series of $\mathbb{S}(V, R)$ and $\mathbb{S}(V', R')$ are $\prod (1 - \alpha_i t)^{-1}$ and $\prod (1 - \alpha'_j t)^{-1}$ by Theorem 4.8. For the graded algebras $A(R', R)$ and $E(R', R)$, it holds then

$$
H_{A(R', R)} = \prod_{i=1}^{r} \prod_{j=1}^{r'} (1 - \alpha_i \alpha'_j t)^{-1}, \quad H_{E(R', R)} = \prod_{i=1}^{r} \prod_{j=1}^{r'} (1 + \alpha_i \alpha'_j t).
$$

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The first formula here follows from Theorem 6.5. The second formula is a consequence of the relation $H_{E(R', R)}(t) H_{A(R', R)}(-t) = 1$ proved in [27, Th. 6.2].

Since $\dim \Lambda_r(V, R) = \dim \Lambda_r'(V', R') = 1$, we have $\prod \alpha_i = \prod \alpha'_j = 1$. Thus $H_{E(R', R)}$ is a polynomial of degree $rr'$ with the leading coefficient equal to 1. The conclusion follows then from [27, Th. 6.6].

\[ \square \]

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