We calculate the current-pressure relation for pinholes connecting two volumes of bulk superfluid $^3$He-B. The theory of multiple Andreev reflections, adapted from superconducting weak links, leads to a nonlinear dependence of the dc current on pressure bias. In arrays of pinholes one has to take into account oscillations of the texture at the Josephson frequency. The associated radiation of spin waves from the junction leads to an additional dissipative current at small biases, in agreement with measurements.

PACS numbers: 67.57.De, 67.57.Fg, 67.57.Np
FIG. 1: The bound state energies $\epsilon_{\delta n \sigma}$ as a function of the phase difference $\phi$. Neglecting gap suppression $\epsilon_0 = -\delta \Delta \cos(\phi/2)$ (dotted line). Parameters: quasiparticle direction cosine 0.93, temperature $0.6T_c$ and $R^l = R(\mp \hat{e}, \theta_c)$. Numerically. One example is shown in Fig. 1. Second, since we consider the B phase, the bulk order parameter is of the form $\Delta R \exp(i\phi)$, where $R$ is a rotation matrix. Depending on the matrices on the left (l) and right (r) hand sides, $R(l,r)$, there is spin-splitting: $\epsilon_{\delta n \sigma}(\phi) \approx \epsilon_{\delta n \sigma}(\phi - \psi)$. Third, the only important source of inelastic processes is quasiparticle-quasiparticle collisions. Near a surface the order parameter is suppressed and therefore we approximate the scattering rate by the normal Fermi-liquid form $\Gamma = a[\pi k_B T^2 + \tilde{c}^2]/(\pi \sigma^2 k_B^2)$. Here $T_0 = 1.41 \mu$mK$^2$ is obtained from viscosity measurements and $a$ is a coefficient on the order of unity. The low-energy $\Gamma$ is denoted by $\Gamma_0 = a T^2/\tau_0$.

We are now equipped to analyze Fig. 2. Similar to the experiments, we study the low-bias region, $U \ll k_B T_c$. In this region we can neglect the normal current from energies above $\Delta$, excluding only a narrow temperature slice near the superfluid transition temperature $T_c$. A characteristic scale for $U$ is set by the scattering strength $\hbar \Gamma$. At $U = 0$, the distribution function is in equilibrium, $p_{\delta n \sigma} = f(\epsilon_{\delta n \sigma})$. Equation (5) then gives the equilibrium current-phase relation $I(\phi)$, dominated by $\sin(\phi)$ (at least for $R^l = R^r$) but with smaller admixtures of $\sin(n\phi)$ where $n > 1$. At $U \ll \hbar \Gamma$ the kinetic equation (8) can easily be solved by linearization. This leads to a time-independent component $I_0$ that is linear in $U$ and independent of $R^{l,r}$.

$$I_0 = g(T)(\Delta/\hbar \Gamma_0) G_0 U.$$  (4)

Here $G_0$ is the normal-state conductance and $g(T)$ a factor on the order of unity. Neglecting gap suppression, $g(T) = \int_{-1}^{1} \tanh(\Delta x/2k_B T)(x/\sqrt{1 - x^2}) dx$. The validity region of the linear dependence (8) vanishes as $T \to 0$ and is replaced by $I_0 \propto U^{1/3}$. Finally, at $U \gg \hbar \Gamma$ (but still $U \ll \Delta$) the scattering can be neglected and $p$ is determined by thermalization at gap edges, $p_{\delta n \sigma} = f(-\delta \Delta)$. Here $I_0$ approaches a constant value that is on the same order as the critical current. For a self-consistent order parameter all higher harmonics are effectively damped by the smooth $\epsilon_{\delta n \sigma}(\phi)$.

The $I_0(P)$ in Fig. 2 can be compared with the measurements in Ref. 8. They have the same shape and a good order-of-magnitude agreement on both axes. The agreement is surprisingly good taking into account that the aperture sizes are on the order of $\xi_0$ or larger, and thus our pinhole approximation is not justified.

The more recent experiment of Ref. 2 was done with an array of apertures. In contrast to Ref. 2, this experiment should be well in the region where the linear approximation (8) is valid. However, a clearly nonlinear $I$ vs $P$ curve was measured. The experimental results are also different for the two possible states of the weak link, the “H” and the “L” states, which have previously been identified as two nearly degenerate textural states with different $R^{l,r}$s. Yet, our numerical calculations confirm that $I_0(P)$ curves in Fig. 2 remain practically unchanged for all $R^{l,r}$. This can also be seen in the adiabatic model as follows. The spin splitting $\epsilon_{\delta(\phi)} \approx \epsilon_{\delta(\phi - \psi)}$ has an essential effect on the critical current so that $I_0^{\mp}$ can even vanish (10). In $I_0$, however, the relative phase shift is unimportant since the phase runs through all values.

We conclude that the nonlinearity and the texture-dependence in Ref. 2 are either large-aperture effects, and/or result form something other than MAR. In the following we demonstrate that at least part of the H-L difference can be accounted for by the anisotextural effect, which exists in array-type weak links of $^3$He-B.

The anisotextural model.—As discussed above, the current–phase relation $I(\phi)$ depends on the rotation ma-
The basic idea of the anisotextural effect is that \( R^\perp \) are not fixed, but tend to move toward their \( \phi \)-dependent equilibrium configuration. This effect was previously used to explain the so-called \( \pi \) states, where a local minimum of energy appears at \( \phi = \pi \). Below we generalize this theory to the dynamical case where \( \phi(t) = \omega t \).

A simple but still realistic model for the anisotextural effect in an array of apertures is based on the energy functional

\[
F[\eta] = F_J(r_0, \phi) + \frac{1}{2}K \int d^3r |\nabla \eta|^2.
\]

Here \( F_J \) is the Josephson coupling energy and the second term is due to the bending of the rotation axis \( \hat{n} \) which parameterizes the matrix \( R(\hat{n}, \theta_L) \). The rotation angle is fixed to \( \theta_L \approx 104^\circ \) by the bulk dipole-dipole interaction \([11]\). Due to the geometry of the experiment \([3]\), we assume \( \hat{n} \) to be fixed parallel to the wall normal \( \hat{z} \) on one side of the array. On the other side \( \hat{n} \) makes an angle \( \eta(r) \) with \( \hat{z} \), which depends on the distance \( r \) from the center of the array. We choose a cutoff at \( r = r_0 \), below which \( \eta(r) \) is assumed constant (see inset in Fig. 3). The \( F_J \) term depends on the angle \( \eta_0 \equiv \eta(r_0) \) near the junction, and the bending term is nonzero if \( \eta_0 \) differs from \( \eta_{\infty} \equiv \eta(\infty) \) preferred by walls and other textural interactions. At temperatures near \( T_c \), the energy \( F_J \) can be approximated by

\[
F_J = -E_J \cos \phi \sin \theta \eta + \frac{1}{2}R_{ij} R_{ij} + \frac{1}{4}R_{ij} R_{ij}.
\]

We choose to fix the \( r \) side \( \hat{n} = \hat{z} \), and on the \( l \) side it is convenient to take \( 0 < \eta_{\infty} < \pi/2 \). The \( H \) and \( L \) states are then identified as the states with \( \hat{n}^l = \pm \hat{z} \), respectively \([3]\). Linearizing around \( \eta \approx \eta_{\infty} \), we find \( E_J \approx E_{J_{\infty}} - J_{sp} \theta(r_0) \), where \( E_{J_{\infty}} \) is the energy density \( \gamma S^2 / 2 \chi \). The Leggett equations are canonical equations for a Hamiltonian where the spin energy density \( \mu_0 \gamma S^2 / 2 \) is added to \( F \) in Eq. \([3]\). The dynamical equations reduce to a wave equation in the bulk and \( F_J \) determines a boundary condition at the junction. The solution at constant \( P \) consists of waves that propagate radially out of the junction. This implies that the supercurrent \( I_s = (2m_3 / \hbar) \partial \eta F_J(r_0, \phi) \) has a nonzero average, the anisotextural current

\[
I_{ai} = \frac{2m_3}{\hbar} \frac{|J_{sp}(\eta_{\infty})|^2}{\gamma} \frac{\omega_J r_0 / c}{1 + (\omega_J r_0 / c)^2}.
\]

Here \( c = \sqrt{2 \mu_0 \gamma S^2 / (5 \chi)} \) is the spin-wave velocity and \( \gamma = b \pi K r_0 \). We estimate \( r_0 = A / \pi \), where \( A \) is the surface area of the array. We also take into account that the actual geometry around the junction differs from a half-space and use \( b = 0.31 \) instead of unity \([5]\).

The crucial features of the anisotextural current \( I_{ai} \) are the dependence on the textural configuration via \( J_{sp}(\eta_{\infty}) \), and the nonlinear dependence on pressure via \( \omega_J \). The origin of \( I_{ai} \) is the oscillating part in Eq. \([3]\), which was here approximated by the \( I_{ai}^0 \) term alone. The total dc current \( I_{dc} \) is the sum of \( I_{ai} \) and the constant component due to MAR, \( I_{dc} = I_0 + I_{ai} \).

**Comparison to experiments.**—It turns out that the \( L \) state current, which is rather linear \([3]\), can almost completely be attributed to MAR. The comparison of the \( L \) state data with \( I_0 \) is presented in Fig. 3. In the case of the \( L \) state approximation, which is not strictly valid for the experimental apertures (100 nm x 100 nm squares in a wall of thickness 50 nm are not small compared to \( \xi_0 = 77 \) nm).

Since both the \( H \) and \( L \) state data are assumed to con-
tain the same contribution $I_0$, it is convenient to subtract the H and L state data taken at equal temperatures. In this way one may directly compare the texture-dependent parts with $I_{\alpha i}$, regardless of any uncertainty that may be present in $I_0$. This is done in Fig. 3(b). The theoretical result corresponds to the difference between Eq. (9) calculated for the two states. There are no adjustable parameters, since the only free parameter is the textural angle $\eta_{\infty}$, which was previously found to be approximately $0.3\pi$ based on static properties. The anisotextural model, as presented above, can explain roughly one third of the observed H-L difference. [The theoretical current in Fig. 3(b) is arbitrarily multiplied by factor 3 in order to make it better visible.] Also, the curvature in the theoretical lines is at higher biases than in the experimental results. There are several possible sources for the differences, in particular the oversimplifications used in our anisotextural model. Unfortunately, it would be very demanding to improve upon the pinhole approximation, or to calculate the texture and propagation of spin waves in the complicated geometry of the experiment. There is also uncertainty in the experimental parameters, for example in the diameter of the apertures, which appears in its fourth power in $I_{\alpha i}$. As a result, the true reason for the factor-of-three discrepancy remains open.

Conclusion.—We have presented a theory of dissipative currents in $^3$He weak links. It shows that the non-linearities in the measurements of Refs. 4 and 5 have different origins, and both can semi-quantitatively be explained by natural extensions of existing theories. The extension of static anisotextural phenomena to dynamics gives further support for the theory, and provides the energy loss mechanism that is required for a weak link to become trapped in a $\pi$ state, as seen in experiments 1. The anisotextural phenomena are sensitive to the experimental cell and magnetic field, for example, and thus can be tested in detail in future experiments. Also the oscillating components $I_{\alpha i}^0$ and $I_{\alpha i}^\pm$ in Fig. 1 as well as the spin waves might be observable in experiments.

Appendix: Calculation of currents.—Consider a pinhole with open area $A_0$. Assume the l-side chemical potential to be shifted by $U$ with respect to the r side, and take the $z$ axis to point from l to r. We define $G_n$ in Eq. (1) as $G_n = \frac{1}{2}m_3v_F N(0)A_0$, where $N(0)$ is the single-spin density of states in the normal state and $v_F$ the Fermi velocity. The mass current may then be written as $I(t) = G_n \langle \hat{k}_z I(\hat{k}, t) \rangle_{\hat{k}_z > 0}$ where $\langle \cdots \rangle_{\hat{k}_z > 0} = \int_{\hat{k}_z > 0} (d\Omega_k/4\pi) \cdots$ denotes an average over the Fermi-surface points $\hat{k}$. Since $I(t)$ is periodic with period $T_J = 2\pi/\omega_J$, we expand $I(\hat{k}, t) = \sum_{n=-\infty} I_n(\hat{k}) e^{i\omega_J n t}$ such that $I_n(\hat{k}) = I^*_n(\hat{k})$. Using the $\gamma$ matrices of Ref. 12 to expand the Keldysh function, the amplitudes for $n \geq 0$ may be written as $I_n(\hat{k}) = \text{Tr} C \{ 2U_0 - \sum_{m=0}^{\infty} P \int d\epsilon \{ E_{mn}^{l,m+n}(\hat{k}, \epsilon, U) - E_{mn}^{m+n,l}(-\hat{k}, \epsilon, -U) \} \}$. The differences, in particular the oversimplifications used in our anisotextural model. 

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