On representations of a chiral alternative to vierbein

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ABSTRACT

In an attempt to facilitate the construction of a quantum theory of gravity, 't Hooft has considered a chiral alternative to the vierbein field in general theory of relativity. These objects, \( f^a_{\mu\nu} \), behave like the “cube root” of the metric tensor. We try to construct specific representations of these tensors in terms of Dirac \( \gamma \) matrices in Euclidean and Minkowski space and promote these to curved space through Penrose-Newman formalism. We conjecture that these new objects, with physical significance, are the analog of Killing-Yano tensors.
1 Introduction

A consistent and complete quantum formulation of gravity is yet to be realised. Leaving aside renormalization, just quantizing the theory brings in many problems. Even for pure gravity, besides the problems arising at the Planck scale, many conceptual and calculational difficulties are encountered in quantising gravity. This is because diffeomorphism invariance is to be applied to space-time itself and the background independence of the action [1]. To overcome these, there have been attempts through different approaches and a variety of methods have been used to tackle each of the problems encountered. These have resulted in varying amount of successes in addressing problems dealing with particular aspects [2]. The problems with quantizing even pure gravity have been analyzed by Nicolai and Peeters [3].

The very existence of matter fields, and in particular the fermions, necessitates their incorporation in a complete quantisation programme. Unlike tensors, there is no spinor representation for the diffeomorphism invariance corresponding to the $GL(4)$ general coordinate transformation of general relativity [4]. The only way to incorporate spinors in general relativity is through the introduction of vierbeins or tetrads. A tetrad behaves like the “square root” of the metric tensor $g_{\mu\nu}$. Using this it has been possible to incorporate the spinor, which is “square root” of a vector or tensor of rank one, into general relativity. The construction of both fermions and bosons out of spinors and their various combinations has been exploited through this approach. Tetrads were introduced by Weyl [5] and are useful in some cases in calculating Riemann curvature and Ricci tensors. Use of tetrads simplifies calculations and the results expressed in terms of the tetrads are in more easily understandable form [6]. By the choice of tetrads, when tensors are written via tetrad components, the tensors can also be expressed in a coordinate independent fashion and their algebraic properties become more transparent and the tensor components can be simplified [7]. The variation of the action with respect to tetrads or their higher dimensional incarnations, and the spin connections, gives the Einstein field equations. Such an action is the basis for quantum formulation of gravity in string theory as well as in other covariant formulations, and also in the canonical approach. The application of tetrad formalism has been further explored in [8]. In the canonical approach to quantum gravity these appear ultimately through Ashtekar variables [9].

Another class of objects, closely connected to the vierbein, are the self-
dual or anti-self-dual solutions to the Yang-Mills equations of the gauge theories in the form of monopoles, instantons etc.

Pure quantum gravity was found to be one-loop finite on shell by ’t Hooft and Veltman. However, adding scalar matter field made the theory nonrenormalizable [10]. Goroff and Sagnotti found that the conventional quantum field theoretic methods, that has been reasonably successful in describing the electro-weak and strong interactions at energy scales presently accessible, fail in the case of gravity [11]. In the context of string theory formulation, the programme to obtain all the interactions in a unique manner appears to have too many choices due to the landscape of the ground state [12].

It would be interesting to have a theory where the SU(3) symmetry of QCD comes out naturally in a manner similar to the Dirac’s relativistic theory of electrons (fermions) where the spin, magnetic moments and other physical consequences come out automatically. In the chiral alternative to vierbein [13], there is a rare natural appearance of an SU(3) symmetry, although it does not seem to be related to Quantum Chromodynamics. However, it is tempting to speculate that the elementary fields could have various symmetries under \( U(1), SU(2) \) and \( SU(3) \) representations, somehow arising naturally from the chiral alternative. A chiral alternative to the vierbein was formulated by ’t Hooft as an object with interesting characteristics that reflect some properties of the dual Yang-Mills solutions. Another motivation for looking at the chiral alternative to the vierbein is to analyse the ”cube root of the metric tensor” so as to obtain some objects analogous to the Killing-Yano tensors. One would expect these to lead to symmetries, conservation laws and separability of the equation [14].

The canonical approach was initiated by trying to put the Einstein equation into a Hamiltonian form [17, 16, 15]. As is well known, even in some classical Lagrangians there appears problems in finding the canonically conjugate momenta, such as the gauge freedom in electrodynamics and massive vector particles [18], and in case of gravity these are further exacerbated by complications due to the full diffeomorphism invariance. One approach in formulating quantum gravity is through respecting this invariance, which is a non-perturbative background independent approach. This analogue of gauge invariance of electrodynamics is a huge freedom and to tackle this the formalism of constraints was developed by Dirac in his attempt to quantize gravity. In canonical approach, the diffeomorphism invariance for the spatial part is manifestly kept. The \( D + 1 \) dimensional manifold, \( M \) is assumed to have the special topology of \( M = \mathbb{R} \times \sigma \), where \( \sigma \) is a fixed, \( D \) dimensional
(where $D$ is taken to be three), compact manifold without boundary. The first and the second fundamental forms, pulled back to $\sigma$, are the “spatial” tensors used as the basic ingredients of the action and the Legendre transformed Hamiltonian. These relate to the curvature and Christoffel connection of the four dimensional space via the Gauss-Codacci equation and also serve as the basic dynamical variables along with the lapse function and shift vector. This results in a singular Lagrangian. The vanishing of conjugate momenta for the lapse and shift are the primary Dirac constraints and the consistency of the equations of motion leads to the secondary constraints which are the diffeomorphism constraint and the Hamiltonian constraint. This results in making the Hamiltonian constrained to vanish. Treating the lapse function and shift vector as Lagrange multipliers, one arrives at the Arnowitt-Deser-Misner action [19]. Analysis of the ADM action shows that the spacetime diffeomorphisms generated by the Lie algebra of a symmetry group is implemented in the canonical framework. The basic variables, namely, the first fundamental form [20] and the corresponding momentum are not observables as they are not gauge invariant. The classical constraint functions depend nonpolynomially (also nonanalytically) on the metric which is identified as the first fundamental form. The curvature scalar depends on the inverse of this metric. Since the products of the metric and the conjugate momentum also appear in the constraint equations, this leads to difficulties in quantizing the theory. By breaking up the above $D$ dimensional metric in terms of an $su(2)$-valued one form, which are D-bein fields on $\sigma$ and by introducing the Sen variables [21], Ashtekar with a subtle choice of the spin connection variables was able to modify the theory in such a way so as to get the constraints in a polynomial form [9]. It has been shown that general relativity arises out of a connection dynamics which parallel transports the chiral fermions. Here, triads (analogue of tetrads in three dimensions) and the connections are the conjugate variables and the constraint equations of general relativity looks similar to those of Yang-Mills theories. In early works Einstein and Schroedinger had used affine connections as basic variables and simplification occurs for the chiral fermions [22]. Further generalisation by Immirzi, Barbero, Rovelli, Smolin and others resulted in the loop formulation in terms of spin network [1]. However, a satisfactory quantisation of gravity has yet to be achieved in any formulation. Hence, alternate approaches are needed to be explored.

One such approach is the alternate to chiral vierbein by ’t Hooft. Though the formulated Lagrangian is not renormalizable, the chiral alternative has
many interesting features, such as, emergence of new symmetry, self(anti)-
duality etc. In this paper, we have tried to construct the representations of
the chiral alternatives in terms of Dirac γ matrices in curved space. In section
2, we briefly introduce the chiral alternative. In section 3, we work out the
representations in flat Minkowski as well as Euclidean space and we give a
generalization to curved space. In section 4, we conjecture a generalization
of Killing-Yano tensor and discuss the future prospective.

2 A chiral alternative

Vier(l)beins and chiral fermions play a fundamental role in most formulations
of quantum gravity, such as, covariant, canonical, loop and string formul-
tions. In an attempt to better understand quantum gravity, ’t Hooft has
introduced an alternate new interesting object that behaves like the “cube
root” of the metric tensor, instead of the vierbein. Following ’t Hooft, we
briefly review the role of different fields leading to the chiral alternative and
in setting up the basic formalism [13].

For introduction of Dirac field to general relativity, as well as an alter-
native to the metric tensor as the fundamental variable in the covariant
formalism, it is useful to introduce the “square root” of the metric tensor
$g_{\mu\nu}$, the vierbein field $e^a_\mu,$

$$g_{\mu\nu} = e^a_\mu e^a_\nu. \quad (1)$$

Here $\mu, \nu$ are four vector indices, $a$ represents “internal” indices. $e^a_\mu$ has
16 components but $g_{\mu\nu}$ has 10 independent ones. So 6 degrees of freedom
should reside in internal $O(3,1).$ For the covariant derivative of the vierbein,
a connection field $A^{ab}_\mu$ corresponding to this local symmetry is introduced.
The metric being covariantly constant, we expect the vierbein to be so and require

$$D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^a_{\mu\nu} e^a_\lambda + A^{ab}_\mu e^b_\nu = 0. \quad (2)$$

$\Gamma^a_{\mu\nu}$ has 40, and $A^{ab}_\mu$ has 24 degrees of freedom, respectively.
$R^a_{\beta\mu\nu}$ can be expressed in terms of $\Gamma$ and $A$ fields as

$$[D_\mu, D_\nu] e^a_\beta = -R^a_{\beta\mu\nu} e^a_\alpha + F^{ab}_{\mu\nu} e^b_\beta = 0, \quad (3)$$

where

$$F^{ab}_{\mu\nu} = \partial_\mu A^{ab}_\nu - \partial_\nu A^{ab}_\mu + [A_\mu, A_\nu]^{ab} \quad (4)$$
implying

\[ R^{ab\mu\nu} = F^{ab\mu\nu}. \]  

(5)

In terms of \( F^{ab\mu\nu} \) and \( e^\mu_a \) the Lagrangian leading to Einstein equation becomes

\[ \mathcal{L} = \sqrt{g} \, R = \text{det}(e) \, F^{ab\mu\nu} \, e^\mu_a \, e^\nu_b \]  

(6)

So a simple polynomial Lagrangian is obtained, with independent variations of \( A^{ab\mu} \) and \( e^a_\mu \) giving the Einstein equation.

Using the 't Hooft invariant self-dual tensor, introduced in the monopole like solutions\[23\],

\[ \eta^a_{\mu\nu} = -\eta^a_{\nu\mu} = \varepsilon_{a\mu\nu} + \delta_{a\mu} \delta_{\nu4} - \delta_{a\nu} \delta_{\mu4}, \]  

(7)

where \( a = 1, 2 \) or \( 3 \), \( \varepsilon \) is 3-dim Levi-Civita symbol, 't Hooft has introduced a field \( e^a_\mu \) in curved space-time that takes the values \( \eta^a_{\mu\nu} \) in a locally flat coordinate frame. It satisfies,

\[ \varepsilon_{abc} \, e^a_{\mu\nu} \, e^b_{\kappa\lambda} \, e^c_{\rho\sigma} \, e^{\mu\nu\kappa\rho} = 24 \sqrt{g} \, g_{\lambda\rho}. \]  

(8)

This is invariant under any transformation of the form

\[ e^a_{\mu\nu} \rightarrow S^a_b \, e^b_{\mu\nu} \]  

(9)

where, \( S^a_b \in SL(3) \), for the Euclidean space and \( \text{det} \, S = 1 \). Analogous to vierbein field, one introduces an \( SL(3) \) connection field \( A^a_{by} \) by demanding

\[ D_{\mu} \, e^a_{\alpha\beta} = \partial_{\mu} e^a_{\alpha\beta} - \Gamma^\lambda_{\mu\alpha} \, e^a_{\lambda\beta} - \Gamma^\lambda_{\mu\beta} \, e^a_{\alpha\lambda} + A^a_{by} \, e^b_{\alpha\beta} = 0. \]  

(10)

This leads to \[13\],

\[ A^a_{\alpha\mu} = 0 \]  

(11)

and a bilinear expression in \( e^a_{\alpha\beta} \):

\[ K^{ab} = \frac{1}{8} \, \varepsilon^{a\beta\mu\nu} \, e^a_{\alpha\beta} \, e^b_{\mu\nu} \]  

(12)

which has an inverse \( K_{ab} \).
By redefining
\[ e^a_{\mu\nu} (\text{det} K)^{-\frac{1}{2}} = f^a_{\mu\nu} \] (13)
equation (8) can be rewritten as,
\[ \epsilon_{abc} f^a_{\mu\nu} f^b_{\kappa\lambda} f^c_{\rho\sigma} \epsilon^{\mu\nu\kappa\rho} = 24 g_{\lambda\rho} \] (14)
where, \( f^a_{\mu\nu} \) is the chiral alternative. The Lagrangian now becomes
\[ \mathcal{L} = \frac{1}{32} \epsilon^{\kappa\lambda\sigma} f^c_{\kappa\lambda} f^b_{\rho\sigma} f^a_{\mu\nu} \epsilon_{abd} f^d_{\alpha\beta} \epsilon^{\mu\nu\alpha\beta} \] (15)
with
\[ F^a_{b\mu\nu} = \partial_\mu A^a_{b\nu} - \partial_\nu A^a_{b\mu} + [A_\mu, A_\nu]^a_{b} \]
\[ = \frac{1}{2} \epsilon_{abd} \eta^d_{\lambda\alpha} R_{\lambda\mu\nu\alpha} \] (16)
This gives a relation between \( f \) and the metric \( g \), i.e., for “cube root” of \( g_{\kappa\tau} \),
\[ \epsilon_{abc} \epsilon^{\mu\nu\lambda\rho} f^a_{\mu\nu} f^b_{\lambda\kappa} f^c_{\rho\tau} = 24 g_{\kappa\tau} \] (17)
The imposition of the gauge condition and the consequent constraint term added to the Lagrangian makes it non-renormalizable. In Minkowski space, putting a reality condition
\[ \hat{f}^a_{\mu\nu} = (f^a_{\mu\nu})^* \] (18)
converts the internal \( SL(3) \) to \( SU(3) \) symmetry.

3 An attempt to obtain a representation in flat space and generalization to curved space

Here, we try to construct \( f^a_{\mu\nu} \) in terms of Dirac \( \gamma^\mu \) matrices to get a relation related to (17) for flat Euclidean space and also for Minkowski space. Then, as for the generalisation of Dirac equation to curved space, we can promote the \( \gamma^\mu \) matrices, and hence \( f^a_{\mu\nu} \) to the curved space.
3.1 Euclidean space

The 't Hooft tensor, introduced in the context of monopole-like solutions is given by,

\[ \eta^a_{\mu\nu} = \epsilon_{a\mu\nu} + g_{4\mu} g_{\nu a} - g_{4\nu} g_{\mu a} \]  

(19)

where, \((a = 1, 2, 3)\) and it has the property of being antisymmetric and self-dual.

If we express \( f^a_{\mu\nu} \) as

\[ f^a_{\mu\nu} = \sigma^a \gamma_5 \sigma_{\mu\nu} \]  

(20)

with \( \gamma^\mu \) matrices in Pauli-Dirac representation [24],

\[ \sigma_{\mu\nu} = \frac{1}{2i} [\gamma_{\mu}, \gamma_{\nu}], \]

\[ \gamma_{\mu} \gamma_{\nu} + \gamma_{\nu} \gamma_{\mu} = 2\delta_{\mu\nu}, \]

\[ \gamma_5 = \gamma_1 \gamma_2 \gamma_3 \gamma_4 = \frac{1}{4} \epsilon_{\mu\nu\lambda\rho} \gamma_{\mu} \gamma_{\nu} \gamma_{\lambda} \gamma_{\rho}, \]

\[ \gamma_{\mu}^\dagger = \gamma_\mu \]  

(21)

then \( f^a_{\mu\nu} \) is antisymmetric in \( \mu \) and \( \nu \) and is anti-self-dual modulo \( \gamma_5 \). Here \( f^a_{\mu\nu} \) appears as an object in the direct product of two spaces, the internal space indices \( a, b, c \) and the 4-space Greek indices \( \mu, \nu \ldots \) etc.

For \( \kappa = \tau \), one can show that,

\[ \epsilon_{abc} \epsilon_{\mu\nu\lambda\rho} f^a_{\mu\nu} f^b_{\lambda\kappa} f^c_{\rho\tau} \]

\[ = \epsilon_{abc} \epsilon_{\mu\nu\lambda\rho} \sigma^a \sigma^b \sigma^c \gamma_5 \gamma_{\mu} \gamma_{\nu} \gamma_5 \gamma_{\lambda} \gamma_{\kappa} \gamma_5 \gamma_{\rho} \gamma_{\tau} \]

\[ = 6i \ I_2 \times 12 \ I_4 \]  

(22)

This expression must vanish for \( \kappa \neq \tau \). To make the right hand side vanish for \( \kappa \neq \tau \), we may take either the trace or the determinant of \( \epsilon_{abc} \epsilon_{\mu\nu\lambda\rho} f^a_{\mu\nu} f^b_{\lambda\kappa} f^c_{\rho\tau} \), so that

\[ tr \left( \epsilon_{abc} \epsilon_{\mu\nu\lambda\rho} f^a_{\mu\nu} f^b_{\lambda\kappa} f^c_{\rho\tau} \right) \propto g_{\kappa\tau} \]  

(23)

Here, the expression on r.h.s. means value of that component. One can also use,

\[ det \left( \epsilon_{abc} \epsilon_{\mu\nu\lambda\rho} f^a_{\mu\nu} f^b_{\lambda\kappa} f^c_{\rho\tau} \right) \propto g_{\kappa\tau} \]  

(24)

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where \( g_{\kappa\tau} \) is the Euclidean flat space metric.

We can also get the general relation,

\[
\varepsilon_{abc} \varepsilon_{\mu\nu\lambda\rho} f^a_{\mu\nu} f^b_{\lambda\kappa} f^c_{\rho\tau} + (\kappa \equiv \tau) = 6i I_2 \times 12 I_4 g_{\kappa\tau}
\]

(25)

So the relation (17) is not produced exactly, but equation (17) satisfies equation (25) modulo a constant and identity matrices. Since we have considered in the simplest case \( \sigma^a \) to be two dimensional, and the space to be a direct product space, it would be more appropriate to call our construct as an attempt to a realisation rather than a representation. It is also intriguing that due to the presence of \( \gamma \)'s in equation (20), \( f_{\mu\nu} \) can act on spinors.

### 3.2 Minkowski space

In the Minkowskian case, we again define

\[
f^a_{\mu\nu} = \sigma^a \gamma_5 \sigma_{\mu\nu}
\]

(26)

which is antisymmetric in \( \mu \) and \( \nu \) and is anti-self-dual modulo \( i\gamma_5 \). Here we use the \( \gamma_\mu \) matrices of Bjorken and Drell [25]

\[
\sigma_{\mu\nu} = \frac{1}{2i} [\gamma^\mu, \gamma^\nu], \\
\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2g_{\mu\nu}, \\
g_{\mu\nu} = (1, -1, -1, -1) \\
\gamma_5 = -i\gamma_0 \gamma_1 \gamma_2 \gamma_3.
\]

(27)

We get, for \( \kappa = \tau \),

\[
\varepsilon_{abc} \varepsilon_{\mu\nu\lambda\rho} f^a_{\mu\nu} f^b_{\lambda\kappa} f^c_{\rho\tau} = 6i I_2 \times (-i) 12 I_4 g_{\kappa\tau}
\]

(28)

To make it vanish for \( \kappa \neq \tau \), we consider like in the Euclidean case and obtain,

\[
\varepsilon_{\mu\nu\lambda\rho} f^a_{\mu\nu} f^b_{\lambda\kappa} f^c_{\rho\tau} + (\kappa \equiv \tau) = 144 I_2 \times I_4 g_{\kappa\tau}
\]

(29)

or take the trace of the left hand side to obtain,

\[
tr \left( \varepsilon_{abc} \varepsilon_{\mu\nu\lambda\rho} f^a_{\mu\nu} f^b_{\lambda\kappa} f^c_{\rho\tau} \right) \propto g_{\kappa\tau}.
\]

(30)
We can also construct another object

\[ \tilde{f}^a_{\mu\nu} = \sigma^a (g_{\mu\mu} g_{\nu\nu} + i\gamma_5) \sigma_{\mu\nu} \]  

(31)

which is antisymmetric in \( \mu \) and \( \nu \) and is anti-self-dual. Its properties are similar to that of \( f^a_{\mu\nu} \). However, promoting this to curved space will have problem as both the left and right side would contain the metric tensor \( g_{\mu\nu} \), unless we write

\[ \tilde{f}^a_{ij} = \sigma^a (1 + i\gamma_5) \sigma_{ij}, \]

\[ \tilde{f}^a_{i0} = \sigma^a (-1 + i\gamma_5) \sigma_{i0}. \]  

(32)

### 3.3 Curved space

To go from flat Minkowski space to curved space-time we follow Chandrasekhar [8] in making use of Penrose-Newman formalism. For this, the constant Pauli matrices, \( \sigma_i \), constituting the \( \gamma_\mu \) matrices are replaced by,

\[ \sigma^i_{AB'} = \frac{1}{\sqrt{2}} \begin{vmatrix} l^i & m^i \\ \bar{m}^i & n^i \end{vmatrix} \]  

(33)

where \( l, m, \bar{m}, n \) are the null basis vectors [8] and \( A, B' \) are the spinor indices. This way we obtain \( f^a_{\mu\nu} \) for a curved space.

### 4 Conclusion and outlook

Killing tensors and Killing-Yano tensors give constants of geodesic motion that are in evolution. By enumeration of the constants in involution, the geodesic motion becomes completely integrable. These are also related to the constants in the separation of the Hamilton-Jacobi and Klein-Gordon equations [14]. The “square root” of a Killing tensor of order two, the Killing-Yano tensor encode the symmetries of the theory and is related to the quadratic first integrals of the geodesic motion, as well as, to the separability of the partial differential equation.

For matter coupled to gravity, where there is coupling to higher spin states, the acceleration is expected to depend on higher powers of the four velocity and this may give rise to new type of conserved quantities [20]. Symmetries of spinning particles in Schwarzschild and Kerr-Newman type space times
have been analysed by Gibbons, Rietdijk and van Holten [27]. They found new nontrivial supersymmetry corresponding to the Killing-Yano tensor for the black-hole space-time. This also plays an important role in solving the Dirac equation in these black-hole metrics. The fermionic symmetries are generated by the square root of bosonic constants of motion other than the Hamiltonian.

The new object, the ’t Hooft tensor $f_{a \mu \nu}$ itself has many interesting properties. The use of twistors in producing anti-self-dual solutions of the Yang-Mills equations indicates these are related to ’t Hooft tensor $f_{a \mu \nu}$. We also note that an internal $SU(3)$ transformation of $f_{a \mu \nu}$ appears for the Minkowski space. In another context, an $SU(3)$ arises naturally from the Runge-Lenz type of symmetry in harmonic oscillator potential [25], which is a consequence of a dynamical symmetry. However, here the $SU(3)$ is not the consequence of a potential and its origin is geometrical. It is natural to expect conserved quantities arising out of the $SU(3)$ symmetry of equation (18). It would be interesting to relate these to the Killing-Yano type symmetries.

Therefore, one would like to conjecture that the “cube root” $f^{a \mu \nu}$ would have analogous relation with the symmetries and the conserved quantities. This Killing-’t Hooft tensor $f_{a \mu \nu}$ would be a new mathematically interesting object to analyse. Establishing such a connection is expected to lead to a better understanding of the structure of space-time.

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