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Cryptanalysis of Public Key Cryptosystems Based on Non-Abelian Factorization Problems

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Cryptanalysis of Public Key Cryptosystems Based on Non-Abelian Factorization Problems

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Abstract: Advances in quantum computers threaten to break public-key cryptosystems (e.g., RSA, ECC, and EIGamal), based on the hardness of factoring or taking a discrete logarithm. However, no quantum algorithms have yet been found for solving certain mathematical problems in non-commutative algebraic structures. Recently, two novel public-key encryption schemes, BKT-B cryptosystem and BKT-FO cryptosystem, based on factorization problems have been proposed at Security and Communication Networks in 2013. In this paper we show that these two schemes are vulnerable to structural attacks and linearization equations attacks, and that they only require polynomial time complexity to obtain messages from associated public keys. We conduct a detailed analysis of the two attack methods and show corresponding algorithmic descriptions and efficiency analyses. In addition, we provide some improvement suggestions for the two public-key encryption schemes.

Key words: cryptography; post-quantum cryptography; public key encryption; cryptanalysis; linear equations

1 Introduction

Most public key cryptosystems used today rely on the assumed difficulty of either integer factorization problem or discrete logarithm problem. However, there exists polynomial time quantum algorithms that solve the two problems. These reasons motivate researchers to develop a new family of cryptosystems that can resist quantum computer attacks, and are more efficient in terms of computation. At present, many cryptographers are working on the area of post-quantum computational cryptography[1-5], and are constructing alternative post-quantum (i.e., quantum-resistant) public-key cryptosystems from other mathematically intractable problems based on non-commutative algebraic structures[6-12].

Before going into details we would like to mention that non-Abelian algebraic structures have already been used in a cryptographic context. In recent decades, many public-key cryptosystems based on non-Abelian algebraic structures were proposed, such as cryptosystems based on inner automorphism groups, general linear groups, and braid groups[1, 11, 12]. However, this area is immature, and there are few practical and efficient non-commutative cryptosystems in use at present. In 2013, Gu et al.[12] proposed two public-key encryption schemes that use non-Abelian algebraic structures and gave affirmative answers to open questions proposed by Myasnikov et al.[13]

In this paper we analyze these two public-key cryptosystems based on non-Abelian factorization problems over non-commutative rings proposed in Ref. [12], i.e., the BKT-B cryptosystem and the BKT-FO cryptosystem. The two schemes are both vulnerable to a structural attack and a linearization equations attack. Using the two attack methods, we can extract the message from associated public keys with significant probability in a reasonable time, and we provide corresponding algorithmic descriptions and efficiency analyses for the two schemes.

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The rest of this paper is organized as follows. Section 2 reviews necessary material for this paper, including the concepts of general linear groups and Kronecker products, as well as the factorization problem. Section 3 gives an overview of public-key cryptosystems based on non-Abelian factorization problems. Section 4 proposes two attack methods, shows corresponding algorithmic descriptions and efficiency analyses, and provides a toy example. At the end, Section 5 provides some concluding remarks and discusses possible lines of future work.

2 Preliminaries

In this paper, we use the following notations.

- $\mathbb{F}_q$: a finite field of order $q$.
- For an integer $k \geq 1$,
  - $GL_k(\mathbb{F}_q)$: the set of $k \times k$ invertible matrices of $\mathbb{F}_q$-entries.
  - $M_k(\mathbb{F}_q)$: the set of $k \times k$ matrices of $\mathbb{F}_q$-entries.
  - $I_k \in GL_k(\mathbb{F}_q)$: the identity matrix.

For every matrix $P$,
- $P^T$: the transpose of $P$.

For $P = \left( \begin{array}{ccc} p_{11} & \cdots & p_{1k} \\ \vdots & \ddots & \vdots \\ p_{k1} & \cdots & p_{kk} \end{array} \right) \in M_{k_1}(\mathbb{F}_q)$, $S \in M_{k_2}(\mathbb{F}_q)$,
- $P \otimes S = \left( \begin{array}{ccc} p_{11}S & \cdots & p_{1k}S \\ \vdots & \ddots & \vdots \\ p_{k1}S & \cdots & p_{kk}S \end{array} \right)$.

The Kronecker product $\otimes$ has the following properties for matrices $P, S, T, O, I, Z$:
- $(P \otimes S) \otimes T = P \otimes (S \otimes T)$;
- $P \otimes (S + T) = (P \otimes S) + (P \otimes T)$;
- $(P \otimes S)^T = P^T \otimes S^T$;
- $(P \otimes S)(T \otimes Z) = PT \otimes SZ$.

Stretching the row of a matrix into one long row vector $\vec{v}$ results in it having the following properties:
- $\vec{aP} + \vec{bS} = \vec{aP} + \vec{bS}$;
- $(\vec{P} \otimes \vec{T})^T = (\vec{P} \otimes \vec{T})^T$;
- $(\vec{O} \vec{S})^T = (\vec{I} \otimes \vec{S})^T$;
- $(\vec{P} \circ \vec{S})^T = (\vec{P} \circ \vec{S})^T$.

Definition 1 Factorization Problem (FP) Let $G = GL_n(\mathbb{F}_q)$ (or $SL_n(\mathbb{F}_q)$) be a finite group with identity $I_n$. Let $M, H \in G$ be two random elements so that $<H> \cap <M> = I_n$ and $HM \neq MH$. The factorization problem denoted by $FP^{G,H,M}$ with respect to $G, H$, and $M$ is to split the given product $N = H^xM^y \in G$ into a pair $(H^{x'},M^{y'}) \in G^2$, where $x$ and $y$ are picked randomly.

3 Description of Schemes Based on the FP

In this section, we review the two schemes based on the FP proposed in Ref. [12].

3.1 BKT-B cryptosystem

KeyGen $\mathcal{K}$: Let $\kappa$ be a system security. Let $G = GL_n(\mathbb{F}_q)$ be a non-Abelian group with identity $I_n$ and $|G| = O(2^{2n})$. Suppose that $H, M \in G$ are two non-commuting elements with orders $k, l = O(2^n)$ respectively and $<H> \cap <M> = I_n$. Let the message space $m = \{0,1\}^\ell \in \mathcal{M}$ be the set of all bit strings of length $\ell$ for any fixed $\ell \geq 2^k$, three collision-resistant hash functions $\pi : \mathcal{M} \rightarrow \mathbb{Z} \times \mathbb{Z}$, $\Phi : G \times G \rightarrow \mathcal{M}$, $\Psi : G \rightarrow \mathcal{M}$. Pick two integers $x$ and $y$ at random.

**public key:**

$G = GL_n(\mathbb{F}_q), \pi, \Phi, H, M, N = H^xM^y$.

**private key:** $(x, y)$.

**Enc:** (1) Alice chooses randomly $x_r, y_r \in \{0,1\}^x$ and computes $A = H^{x_r}NM^{y_r}$, $B = \Phi(A) \oplus m, (x_B, y_B) = \pi(B)$, $C = H^{x_B}AM^{y_B}$, $D = H^{x_B+y_B}M^{y_B}$, $E = \Phi(D)$, $C \oplus B$. Then she sends $(D, E)$ to Bob.

**Dec:** After receiving the ciphertext tuple $(D, E)$, Bob performs the following steps with the private key pair $(H^x, M^y)$: $C = H^xM^y \oplus E = \Phi(D, C) \oplus E$, $(x_H, y_H) = \pi(B')$, $A = H^{-x_H}C^M^y$, $m = \Psi(A) \oplus B'$.

3.2 BKT-FO cryptosystem

KeyGen $\mathcal{K}$: Let $\kappa$ be a system security. Let $G = GL_n(\mathbb{F}_q)$ be a non-Abelian group with identity $I_n$ and $|G| = O(2^{2n})$. Suppose that $H, M \in G$ are two non-commuting elements with orders $k, l = O(2^n)$ respectively and $<H> \cap <M> = I_n$. Let the message space $m = \{0,1\}^\ell \in \mathcal{M}$ be the set of all bit strings of length $\ell$ for any fixed $\ell \geq 2^k$, two random oracles $h_1 : G \rightarrow \{0,1\}^\ell$, $h_2 : \mathcal{M} \times G \rightarrow \{0,1\}^{2k}$. Pick randomly two integers $x$ and $y$.

**public key:** $(GL_n(\mathbb{F}_q), h_1, h_2, H, M, N = H^xM^y)$.

**private key:** $(x, y)$.

**Enc:** Alice chooses randomly $R \in G$ and computes $x'|y| = h_2(m, R)$, $\tau_1 = H^{x'}NM^{y'}$, $\tau_2 = H^{x'}M^{y'}$, $\tau_3 = m \oplus h_1(R)$. Then she sends $(\tau_1, \tau_2, \tau_3) \in G \times \mathcal{M}$ to Bob.
Dec: After receiving the ciphertext triple \( c = (r_1, r_2, r_3) \), Bob performs the following steps:

1. Compute \( R = M^r r_1 H^r r_2 \), \( m = r_3 \oplus h_1(R) \), \( x' \mid y' = h_2(m, R) \).
2. If \( r_2 = H^2 M^{y'} R \), then output \( m \); otherwise, output \( \bot \), which indicates the ciphertext is invalid.

4 Weakness of Schemes Based on the FP

In this section, we analyze two public key cryptosystems based on non-Abelian factorization problems.

4.1 Structural attack

The attack is able to get the information: \((M, H, D = H^{x_H + x_B} M^{y_H + y_B}, N = H^x M^y, E, \Phi, \Psi, \pi)\). If an adversary can find matrices \((X, Y)\) such that

\[
\begin{align*}
XH &= HX; \\
YM &= MY; \\
N &= XY
\end{align*}
\]

(1)

the proposed BKT-B cryptosystem always had weakness.

Using Proposition 2, the linear Eq. (1) can be transformed into

\[
\begin{pmatrix}
I_n \otimes H^T - H \otimes I_n & x^{-1} \\
(M \otimes I_n) - (I_n \otimes M^T) & Y \\
(I_n \otimes N^T) x^{-1} - (I_n \otimes I_n) & Y = 0
\end{pmatrix}
\]

(2)

Let \( Q = \begin{pmatrix} I_n \otimes H^T - H \otimes I_n & x^{-1} \\
0 & (M \otimes I_n) - (I_n \otimes M^T) \\
I_n \otimes N^T & -(I_n \otimes I_n) \end{pmatrix} \). Let \( x = (X^{-1}, Y)^T \), where \( \det(X) \neq 0 \). \( X^{-1} \) and \( Y \) are the stretches of matrices \( X^{-1} \) and \( Y \) respectively, \( \otimes \) represents the Kronecker product, and \( I_n \) is an \( n \times n \) identity. It remains to analyze the BKT-B cryptosystem as follows.

Proposition 1

If an adversary can find any matrices \( X \) and \( Y \) satisfying Eq. (1), the BKT-B cryptosystem can be broken.

Proof

If \( A \) finds matrices \( \bar{X} \) and \( \bar{Y} \) satisfying Eq. (1), according to \( \bar{X}H = H\bar{X} \) and \( \bar{Y}M = M\bar{Y} \), \( A \) has

\[
\bar{Y}M^2 = M\bar{Y}M = M(\bar{Y}M) = M(M\bar{Y}) = M^2\bar{Y},
\]

hence \( \bar{Y}M^k = M^k\bar{Y}, \bar{X}H^k = H^k\bar{X} \) for every integer \( k \) and

\[
\bar{X}D\bar{Y} = \bar{X}H^{x_H + x_B} M^{y_H + y_B} \bar{Y} = H^{x_H + x_B} M^{y_H + y_B} \bar{Y} = H^{x_H + x_B} N^M \bar{Y} = \]

Because \((M, H, D, N, E, \Phi, \Psi, \pi)\) are known information, \( B, x_B \), and \( y_B \) can be solved in the following form:

\[
B = E \oplus \Phi(D, \bar{X}D\bar{Y}),
\]

The matrix \( A \) can be computed as follows:

\[
A = H^{-x_B} CM^{-y_B}.
\]

Hence the message

\[
m = B \oplus \Psi(A)
\]

is obtained, that is, the BKT-B cryptosystem is broken.

This completes the proof.

Algorithm 1

Solve the message \( m \) of the BKT-B cryptosystem

1: Input \((M, H, D, N, E, \Phi, \Psi, \pi)\);
2: Solve the homogeneous linear equations in the \( 2n^2 \) entries of \( x : Qx = O \);
3: Fix a basis for the solution space \( x = (X^{-1}, Y)^T \), pick a random vector \( x \) and transform \( X^{-1} \) and \( Y \) to the matrices \( X^{-1} \) and \( Y \) until \( X^{-1} \) is invertible;
4: Compute \( C = XD\bar{Y} \);
5: Reckon \( B = E \oplus \Phi(D, C) \);
6: Compute \( (x_B, y_B) = \pi(B) \);
7: Calculate \( A = H^{-x_B} CM^{-y_B} \) and the message \( m = B \oplus \Psi(A) \);
8: Return \( m \).
It remains to analyze the BKT-FO cryptosystem as proposed. Scheme BKT-FO is vulnerable. The cryptosystem can be finished with a bit. Algorithm 2: Solve the message

\[ H^{-y}x \] 

Input: \( m \)

Output: \( R \)

1. \( \hat{x} = \hat{x}_1 \hat{x}_2 = \hat{y} \hat{y}_1 \hat{y}_2 \), where \( \hat{y}_1 = \frac{1}{y-x} \)
2. \( \hat{y}_2 = \hat{y}_1 \hat{y}_2 = \frac{1}{y-x} \)
3. \( \hat{y}_3 = \hat{y}_1 \hat{y}_2 = \frac{1}{y-x} \)
4. \( \hat{y}_4 = \hat{y}_1 \hat{y}_2 = \frac{1}{y-x} \)
5. \( \hat{y}_5 = \hat{y}_1 \hat{y}_2 = \frac{1}{y-x} \)

\[ R = R \]

This completes the proof.

4.2 Linearization equations attack

For any \( V \in \text{GL}_n(\mathbb{F}_q) \), there exists a corresponding minimum polynomial \( f_{\min}(V) \) such that \( f_{\min}(V) = 0 \). The characteristic polynomial \( f_t(x) \) of \( V \) satisfies

\[ f_t(x) = \text{det}(xI_n - V) = v_n x^n + v_{n-1} x^{n-1} + \cdots + v_1 x + v_0, \]

where \( v_i \in \mathbb{F}_q \). According to the Hamilton-Cayley Theorem,

\[ f_t(V) = v_n V^n + v_{n-1} V^{n-1} + \cdots + v_1 V + v_0 I_n = 0, \]

and

\[ V^{-1} = -\frac{v_n}{v_0} V^{n-1} - \frac{v_{n-1}}{v_0} V^{n-2} - \cdots - \frac{v_1}{v_0} I_n. \]

Hence for the matrix \( A, A^k (k \in \mathbb{Z}) \) can be linearly represented by the set \( B = \{ I, A, \ldots, A^{n-1} \} \). That means, \( H^{-x} \) can be linearly represented by the set \( \{ I, H, \ldots, H^{n-1} \} \) and \( M^y \) can be linearly represented by the set \( \{ I, M, \ldots, M^{n-1} \} \).

Let

\[ \begin{align*}
X &= H^{-x} = \sum_{i=0}^{n-1} a_i H^i, \\
Y &= M^y = \sum_{j=0}^{n-1} b_j M^j,
\end{align*} \]

\[ \begin{align*}
\sum_{i=0}^{n-1} a_i H^i N &= \sum_{j=0}^{n-1} b_j M^j,
\end{align*} \]

where \( H^{-x} N = M^y, N = X^{-1} Y, a_i, b_j \in \mathbb{F}_q \), and \( X \) is invertible.

Equation (5) is equivalent to solving a system of \( n^2 \) equations with \( 2n \) variables and can be expressed by

\[ Lx = 0 \]

where

\[ L = ((H^0 N)^T, \ldots, (H^{n-1} N)^T, \ldots, (-M^{n-1})^T), \]

\[ x = (a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1})^T \in \mathbb{F}_q^{2n \times 1}. \]

Then the proposed BKT-B cryptosystem and BKT-FO cryptosystem always are vulnerable, which can be analyzed as follows.

Proposition 3: If an adversary can find matrices \( X \) and \( Y \) satisfying Eq. (5), then the BKT-B and BKT-FO cryptosystems can be broken.

Proof: If an adversary can find matrices \( \overline{X} = \sum_{i=0}^{n-1} a_i H^i \) and \( \overline{Y} = \sum_{j=0}^{n-1} b_j M^j \) satisfying Eq. (5), then

\[ \overline{Y} M^y = \overline{X}^{-1} H, \overline{X}^{-1} H = H^{-1} \overline{X}^{-1} H = H^{-1}, \]

and

\[ \overline{Y} M^y = \overline{M} \overline{Y}, \overline{X}^{-1} H = H^{-1} \overline{X}^{-1} H = H^{-1}, \]

for every integers \( i \) and \( j \).

For the BKT-B cryptosystem,

\[ \overline{X}^{-1} D \overline{Y} = \left( \sum_{i=0}^{n-1} a_i H^i \right) H^{-x} + x (H^{-x} + \lambda (M^{-1} \overline{Y})) \]

\[ = \left( \sum_{i=0}^{n-1} a_i H^i \right) H^{-x} + x \left( \sum_{j=0}^{n-1} b_j M^j \right). \]
The complexity for Algorithm 3 can be concluded in Table 3.

Table 3 Complexity of algorithms.

| q | n | Bit complexity | Attack time of Algorithms 1 and 2 (s) |
|---|---|----------------|---------------------------------------|
| 2^8 | 10 | 2.30.0 | 2.31.3 | 0.016 |
| 2^{10} | 10 | 2.31.0 | 2.22.4 | 0.031 |
| 2^{10} | 20 | 2.27.2 | 2.21.2 | 1.912 |
| 2^{11} | 50 | 2.33.7 | 2.26.0 | 387.782 |
| 2^{11} | 100 | 2.38.5 | 2.29.3 | 30.345.860 |

The method to calculate the message \( m \) of the BKT-FO cryptosystem can be concluded in Algorithm 4.

Table 2 indicates that a linearization equations attack against the BKT-FO cryptosystem can be finished with a complexity of \( \mathcal{O}(n^{w+1} \log^2 q) \) without small constant factors.

4.3 Practical implementations for algorithms and a toy example

The computation costs of Algorithms 1 and 2 are almost the same, ignoring some small constants, and the computation costs of Algorithms 3 and 4 are smaller. We conduct our experiments on a 2.6 GHz Intel processor PC by Magma, with different parameters for Algorithm 1, the analysis results are given in Table 3.

We use a toy example to illustrate the attack methods over \( GL_3(\mathbb{F}_{2^l}) \) in our cryptanalysis and the common information:

\[
H = \begin{pmatrix}
3421 & 14524 & 29952 \\
48988 & 24774 & 4993 \\
23688 & 29935 & 32266
\end{pmatrix},
\]

\[
M = \begin{pmatrix}
24771 & 63557 & 612 \\
33158 & 24788 & 681 \\
962 & 57792 & 6149
\end{pmatrix}.
\]

Algorithm 4 Solve the message \( m \) of the BKT-FO cryptosystem

1: Input \( (h_1, h_2, H, M, N, \tau_1, \tau_2, \tau_3) \);
2: Solve \( X^{-1}, Y \) by steps 2-3 in Algorithm 3;
3: Compute \( R = Y \tau_1^{-1} X^{-1} \tau_2 \);
4: Calculate the message \( m = \tau_3 \oplus h_1(R) \);
5: Return \( m \).
Because 

4.3.1 A toy example for Algorithm 1

Hence and 

4.3.2 A toy example for Algorithm 2

Then gets 

4.3.3 A toy example for Algorithm 3

Due to Eq. (6),

### 4.3.2 A toy example for Algorithm 2

According to Algorithm 1 and known information \((h_1, h_2, \tau_1, \tau_2, \tau_3)\), where
Solving \( Lx = 0 \) by employing the method of Gauss elimination, \( A \) gets a basis for the solution space \( \mathcal{E} = \{11657, 51808, 25403, 21630, 21033, 1\} \).

Then
\[
X^{-1} = \begin{bmatrix}
11657 & 51808 & 25403 & 21630 & 21033 & 1
\end{bmatrix}
\]

Before having \( C \), \( A \) computes the message \( m = B \oplus \Psi(A) \) which is the same as Algorithm 1.

**4.3.4 A toy example for Algorithm 4**

The known information \((h_1, h_2, \tau_1, \tau_2, \tau_3)\) is the same as Algorithm 2. By using \( \bar{X}^{-1} \) and \( \bar{Y} \) of Algorithm 3, \( A \) computes

\[
\bar{X} = X^{-1} = \begin{bmatrix}
38828 & 9569 & 53184
31740 & 6994 & 27780
1025 & 23489 & 51390
\end{bmatrix}
\]

\[
\bar{Y} = 21630X + 21033M + M^2 = \begin{bmatrix}
33004 & 24540 & 13628
36484 & 24540 & 13628
65186 & 39086 & 51808
\end{bmatrix}
\]

Hence \( A \) gets

\[
C = \bar{X}D\bar{Y} = \begin{bmatrix}
3418 & 10960 & 52722
58639 & 22085 & 44909
56051 & 26831 & 10092
\end{bmatrix}
\]

After having \( C \), \( A \) computes the message \( m = B \oplus \Psi(A) \) which is the same as Algorithm 1.

**5 Conclusion**

In this paper, we showed that two public key cryptosystems based on non-Abelian factorization problems are insecure in the sense that an attacker who is able to solve the homogeneous linear equations with high efficiency in a given general linear group, is also able to break the two schemes. Moreover, other attack methods can also be of use in studying the two schemes. The question whether there exist other groups on which the two schemes are secure remains open.

When designing a public key cryptosystem based on non-Abelian factorization problems on other groups, the considerations of the previous section must be taken into account. How to make a public key cryptosystem based on non-Abelian algebraic structures, which has the potential to resist known quantum attacks, also remains open.

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