Bounce from Inflation

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We construct a class of viable bouncing models that are conformally related to cosmological inflation. There are three main difficulties in constructing such a model: (i) A stable (attractor) solution, (ii) A non-singular bounce, and (iii) to bypass the no-go theorem that states that simultaneously maintaining the observational bounds on the tensor-to-scalar ratio and the non-Gaussian scalar spectrum are not possible. We show that a non-minimal coupling of the scalar field helps to bypass these difficulties and provides viable bouncing models. We also obtain a naturally occurring reheating epoch briefly after the bouncing phase.

Introduction: The inflationary paradigm remains the most successful paradigm in explaining the early Universe as the theoretical predictions are in excellent agreement with the recent observations. However, there is an attempt of finding other alternatives to inflation as, despite the ever-tightening observational constraints, there seems to exist many inflationary models that continue remain consistent with the data [1]. A popular alternative is the classical bouncing scenario where, the Universe undergoes a phase of contraction until the scale factor reaches a minimum value, before it enters the expanding phase [2].

However, it also suffers difficulties, arguably even more than the inflationary paradigm. First, most of the bouncing solutions (except for ekpyrotic bounce) are not stable and therefore, in these cases, the anisotropic energy density can grow faster than the energy density responsible for the bounce (contracting phase), and hence may potentially break the homogeneous background. This is known as the Belinsky-Khalatnikov-Lifshitz (BKL) instability [3]. Second, while constructing the contracting phase is rather easy to achieve, obtaining the non-singular bouncing phase is extremely difficult as it requires to violate the null energy condition [4, 5]. Third, and most importantly, even if one may construct a model evading the first two difficulties, these models fails to be in line with the observational constraints: a small tensor-to-scalar ratio \( r \lesssim 0.06 \) and simultaneously, very small scalar non-Gaussianity parameter \( f_{NL} \sim \mathcal{O}(1) \). This is referred to as the no-go theorem [4, 6]. Apart from these main three issues, in general, bouncing models possess another, rather weak, difficulty as well: a natural exit mechanism from the bouncing phase to enter into the conventional reheating phase.

In solving these problems, it is soon realized that one needs to go beyond the canonical theories and consider non-minimal couplings, viz. the Horndeski theories or even beyond Horndeski theories [3–7]. However, it is still an open issue to construct a viable bouncing model that can evade all the difficulties simultaneously. This is because: there is not a single mechanism known that can be used to solve all the difficulties at the same time. However, our recent works suggest that conformal coupling can play a pivotal role in resolving those issues [8]. With that keeping in mind, in this letter, we intend to construct a model that is stable and at the same time, may provide viable perturbed spectra with the help of conformal coupling. The approach is the following: since most of the slow-roll inflationary models satisfy all the observational constraints, we shall conformally transform the inflationary scale factor solution into a bouncing solution. Since the conformal transformation preserves the stability and the perturbations, the reconstructed bouncing model holds the stability as well as perturbations, similar to the conformal inflationary model. Later, we shall show that this kind of reconstruction helps us to automatically achieve a non-singular bouncing phase as well as a naturally occurring exit mechanism from bounce to reheating epoch, hinting towards a class of the first viable bouncing models.

Constructing the model: Slow-roll inflationary model can easily be constructed by using a single canonical scalar field \( \phi \) minimally coupled to the gravity, i.e., the Einstein gravity:

\[
S_I = \frac{1}{2} \int d^4x \sqrt{-g_I} \left[ M_{Pl}^2 R - g_I^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - 2 V_I(\phi) \right] \tag{1}
\]

The sub(super)script ‘I’ (not to be confused with the power or any indices) denotes the quantity in the minimal Einstein theory (similarly, we shall reserve ‘b’ for the yet to be constructed bouncing theory). \( R^I \) is the Ricci scalar for the metric \( g^I_{\mu \nu} \) and \( V_I(\phi) \) is the potential responsible for the inflationary solution. Assuming the above theory is responsible for the slow-roll inflationary dynamics, the scale factor solution during inflation \( a_I(\eta) \) in comoving time \( \eta \) can be approximated as \( a_I(\eta) \approx -1/(H_I \eta) \), where \( H_I \) is the inflationary Hubble parameter (nearly constant). It also can approximately be written as a function of the scalar field \( \phi \) as

\[
a_I(\phi) \propto \exp \left( - \int^{-\phi}_0 \frac{d\phi}{M_{Pl}^2 V_I(\phi)} \right), \tag{2}
\]

with \( V_{I,\phi} \equiv \frac{\partial V_I}{\partial \phi} \). Now we shall construct a model which is conformal to the above inflationary action (1) in such

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We desire the bouncing solution of the form
\[ a_b = f(\phi) a_0(\eta). \tag{3} \]

\( a_b(\eta) \) and \( a_0(\eta) \) are the inflationary metric and the scale factor solutions for the action (1), respectively, and \( a^b_{\mu \nu} \) and \( a_0(\eta) \) are the required bouncing metric and scale factor solutions, respectively, along with \( f(\phi) \) being the coupling function. Using the above conformal transformation along with the action (1), we can construct the action responsible for such bouncing solution as
\[
S_b = \frac{1}{2} \int d^4x \sqrt{-g_b} \left[ M^2_{\text{Pl}} f^2(\phi) R^b - \omega(\phi) g^b_{\mu \nu} \partial_\mu \phi \partial_\nu \phi - 2 V_b(\phi) \right]. \tag{4}
\]

\( R^b \) is the new Ricci scalar for the bouncing metric \( g^b_{\mu \nu} \) and the solution to the above action leads to the required scale factor solution (6). However, keep in mind that the comoving solutions of the scalar field in these required scale factor solution (6). In this epoch, the large scales of cosmological interests leave the Hubble horizon. However, when the field \( \phi \) approaches close to the minima of the potential \( V_I(\phi) \), the solution of the scale factor (6) does not hold and \((1 - f_N/f)\) starts increasing, and at the minima of the potential, it becomes positive infinity. Therefore, before the field reaches to the minimum of the potential, at one point, it crosses the value zero (when \( (\alpha + 1) \phi_N = -M^2_{\text{Pl}} V_I(\phi)/V_I \)) where
\[
H_b = f H_I \left( 1 - \frac{f_N}{f} \right), \quad \epsilon_b = \left( \frac{\epsilon_I - \epsilon_N}{1 - \frac{f_N}{f}} \right) - \frac{f_b^2}{1 - \frac{f_N}{f}}. \tag{8}
\]

\( N \) is the e-fold time convention in the minimal inflation theory: \( a_I(N) \propto e^N \). The subscript \( N \) denotes the derivative with respect to it. When the field \( \phi \) is high and far away from the minima of the potential \( V_I(\phi) \), it slowly rolls down the potential towards minima with the solution \( \phi_N \simeq -M^2_{\text{Pl}} V_I/\omega(\phi) \). This immediately implies \( (1 - f_N/f) \simeq 1 - \alpha \) with \( \epsilon_b \simeq (\alpha + 1)/\alpha \) which corresponds to \( H_b < 0 \) for \( \alpha > 0 \). This is represented by the solutions (6) and (7) and it ensures that the Universe in the non-minimal theory is contracting with the scale factor solution (6). In this epoch, the large scales of cosmological interests leave the Hubble horizon. However, when the field \( \phi \) approaches close to the minima of the potential \( V_I(\phi) \), the solution of the scale factor (6) does not hold and \((1 - f_N/f)\) starts increasing, and at the minima of the potential, it becomes positive infinity. Therefore, before the field reaches to the minimum of the potential, at one point, it crosses the value zero (when \( (\alpha + 1) \phi_N = -M^2_{\text{Pl}} V_I(\phi)/V_I \)) where \( H_b \) vanishes and \( \epsilon_b \) diverges to negative infinity. This point corresponds to the required non-singular bouncing point where the null-energy condition is violated, which we achieve automatically by choosing the coupling function as Eq. (7).

Shortly after the bounce, as field \( \phi \) reaches the minima, it starts oscillating around it and decays. Since, at the minima of the potential, the coupling function \( f(\phi) \) (as well as \( \omega(\phi) \)) becomes unity, the non-minimal action (4) as well as its solution also merges with the minimal action (1) and its solution, i.e., \( \{a_0, H_b\} \simeq \{a_I, H_I\} \) Therefore, as the Universe in the minimal theory undergoes through the reheating epoch, the non-minimal bouncing theory also experiences similar phenomena and the difference between these two conformal theories eventually vanishes. Also, note that the Hubble parameter \( H_b \) remains positive after the bounce. Hence, the newly constructed non-minimal theory also possesses a naturally
occurring exit mechanism from bounce into the reheating epoch, similar to an inflationary model. These results suggest that the constructed bouncing model is free from the difficulties mentioned earlier and at the same time, it can be in agreement with the observations.

Example - chaotic bounce: Consider the chaotic inflation with the potential $V_I(\phi) = \frac{1}{2} m^2 \phi^2$. If we require scale factor solution in the non-minimal theory to be the matter bounce, i.e., $\alpha = 2$, the coupling function $f(\phi)$ as well as $\omega(\phi)$ take the form

$$f(\phi) = \exp \left( -\frac{3}{4} \frac{\phi^2}{M_{Pl}^2} \right), \quad \omega(\phi) = f^2(\phi) \left( 1 - \frac{27}{2} \frac{\phi^2}{M_{Pl}^2} \right).$$

We numerically solve the non-minimal theory (4) with e-N-fold time convention $N$, which is defined as $a_b(N) = a_0 \exp \left( \frac{N^2}{2} \right)$. Note that the bounce occurs at $N = 0$, with negative and positive values of $N$ corresponding to the contracting and the expanding phases, respectively. In Fig. 1, we plot the slow-roll parameter $\epsilon_b$ as a function of $\phi$ to demonstrate the attractor behavior as well as the required matter contracting solution. Different colors correspond to different initial conditions, however, it quickly approaches the desired solution with $\epsilon_b \simeq 3/2$ as

FIG. 1. Slow-roll parameter $\epsilon_b$ is plotted for the chaotic bouncing model with $\alpha = 2$ with respect to the field $\phi$. Different colors signify different initial conditions which is far away from the desired value of the slow-roll parameter. However, all solutions quickly approaches to the value of $\epsilon_b \simeq 3/2$ which is the solution for $\alpha = 2$. This clearly indicates that our bouncing model is an attractor.

FIG. 2. Slow-roll parameter $\epsilon_b$ is plotted for the chaotic bouncing model with $\alpha = 2$ with respect to the e-N-fold time convention $N$. As it can be clearly seen, at the bounce, it diverges, as expected. Shortly after the bounce, it starts oscillating (around 1.5). This behavior is similar to the conventional reheating epoch.

FIG. 3. Hubble parameter $H_b$ is plotted for the chaotic bouncing model with $\alpha = 2$ before and after the bounce with respect to the e-N-fold time convention $N$. Note that, shortly after bounce, it starts oscillating which represents the reheating epoch.

FIG. 4. The coupling function $f(\phi)$ is plotted for the chaotic bouncing model with $\alpha = 2$ with respect to the e-N-fold time convention $N$ before and after the bounce. Shortly after the bounce, during reheating, it quickly approaches to unity.
\( \phi \) approaches the minima of the potential at \( \phi = 0 \), confirming that the model is indeed stable and the Universe is matter contracting. In Figs. 2, 3 and 4, we plot the slow-roll parameter \( \epsilon_b \), the Hubble parameter \( H_b \) and the coupling function \( f(\phi) \) as functions of \( N \) around and after the bouncing point. From these figures, it can clearly be seen that, shortly after the bounce, the scalar field starts oscillating and decays, and the coupling function \( f(\phi) \) (as well as \( \omega_b(\phi) \)) approaches unity. This is similar to the conventional reheating phase, as mentioned before.

To compare the merging of these two theories into one during the reheating phase, let us compare the Hubble parameters in two different theories. By solving the chaotic inflation model with e-fold time convention \( N \) and by using (3), one can easily obtain a one-to-one correspondence in these time parameters \( \{ N \leftrightarrow \mathcal{N} \} \). Therefore, any parameters can be expressed in any time conventions \( N \) or \( \mathcal{N} \). In Fig. 5, we plot the Hubble parameters in these two theories in e-fold \( N \) time convention. The blue curve is the Hubble parameter in the non-minimal frame (this is identical to Fig. 3, except the time parameter is now changed to \( N \)), while red denotes the same in the minimal inflationary theory. \( H_b \) vanishes at \( N \approx 73.2 \) which represents \( \mathcal{N} = 0 \), i.e., the non-singular bouncing phase, and as \( N \) increases, the difference between \( H_b \) and \( H_I \) vanishes.

However, recent observations ruled out many models including the chaotic inflation, and therefore, the reconstructed chaotic bounce is also not in line with the constrained data. Despite that, a significant number of inflationary models are still in good agreement with the tighter constraints. For example, the Starobinsky model with \( V_I(\phi) = \frac{3}{4} m^2 M_{Pl}^2 \left( 1 - e^{\sqrt{2} \frac{\phi}{M_{Pl}}} \right)^2 \) is one of them and similar to chaotic bounce, one can also construct a Starobinsky bouncing model with the coupling function

\[
\begin{align*}
 f(\phi) &= f_0 \exp \left[ -\frac{3}{4} (\alpha + 1) \left( e^{\sqrt{2} \frac{\phi}{M_{Pl}}} - \sqrt{\frac{2}{3}} \frac{\phi}{M_{Pl}} \right) \right].
\end{align*}
\]

Using the above coupling function, one can easily obtain the non-minimal theory (4) responsible for the required bouncing solution. Using this solution, one can similarly solve the non-minimal dynamics and achieve similar nature of the solutions, as shown in the case of chaotic inflation. In this case, as it is obvious, the bouncing models evade all the difficulties including the no-go theorem.

**Conclusions:** While we successfully resolve the above issues of bouncing problems by constructing such a class of models which is induced from the minimal inflationary theories, we should also stress that, since the above reconstruction is solely based on the conformal transformation, it may not be possible to distinguish between an inflationary model and the corresponding conformal bouncing model. There are other significant things to note. First, while the value of \( \alpha \) we consider as positive, in theory, we can create any form of the scale factor using \( f(\phi) \). Second, the above procedure holds even for arbitrary field redefinition (e.g., disformal transformation). Third, during reheating, while the average behavior in these conformal theories is the same, in Fig. 5, one can clearly see the difference in amplitude of the oscillation in the Hubble parameter (and in the slow-roll parameter as well) which needs further attention in the future. Fourth and most importantly, while we provide a class of viable bouncing models, we bring a similar problem that plagues the inflationary paradigm: there are many inflationary models, and hence our proposed conformal bouncing models, that remain consistent even with the data.

However, while distinguishing conformal frames is mostly believed to be not possible as they are ‘equivalent’, the ‘equivalence’ is still debatable as has been pointed out in several articles in the literature [8, 9]. Recently, it has also been shown that the reheating dynamics in different conformal theories may behave differently which can help to differentiate the frames [10]. Therefore, it remains an open problem. However, the ‘equivalence’ can be broken during and after the reheating epoch as it also depends on how the newly created relativistic particles coupled with the gravity, and therefore, the viability of these models can be verified from the future experiments. We reserve the analysis for our future work.

**Acknowledgements:** The author thanks L. Sriramkumar and S. Shankaranarayanan for useful discussions and their valuable comments. The author also wishes to thank the Indian Institute of Technology Madras, Chennai, India, for support through the Exploratory Research Project PHY/17-18/874/RFER/LSRI.
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