Finiteness of Logarithmic Crystalline Representations II

Raju Krishnamoorthy, Jinbang Yang, and Kang Zuo

Abstract. Let $K$ be an unramified $p$-adic local field and let $W$ be the ring of integers of $K$. Let $(X, S)/W$ be a smooth proper scheme together with a simple normal crossings divisor and fix positive integers $r$ and $f$. We show that the set of absolutely irreducible representations $\pi_1(X_K) \to GL_r(\mathbb{Z}_p)\) that come from log crystalline $\mathbb{Z}_p$-local systems over $(X_K, S_K)$ of rank $r$ is finite. The proof uses $p$-adic nonabelian Hodge theory and a finiteness result due Abe/Lafforgue.

1. Introduction

To state our main theorem, the following setup will be convenient.

Setup 1.1. Let $r$ and $f$ be positive integers. Let $p$ be an odd prime and let $k$ be a finite field containing $\mathbb{F}_p$. Set $W := W(k)$ to be the ring of Witt vectors of $k$ and $K := \text{Frac}(W)$. Let $(X, S)/W$ be a smooth projective scheme together with a relative simple normal crossings divisor over $W$. Set $U := X \setminus S$. Let $x$ be a $\bar{K}$ point of $U$. For a positive integer $n \geq 1$ and a scheme $T$ over $W$, the notation $T_n$ refers to the reduction of $T$ modulo $p^n$.

The following is our main result, in which the base point is suppressed.

Theorem 1.2. Notation as in Setup 1.1. Then the following set

\[ \{ \rho : \pi_1^e(U_K) \to GL_r(\mathbb{Z}_p) \mid \rho \text{ is log crystalline} \] with HT weights in $[a, a + p - 1]$ for some $a \in \mathbb{Z}$ and $\rho^{\text{nc}} : \pi_1^e(U_K) \to GL_r(\mathbb{Q}_p)$ absolutely irreducible. \]

is finite.

Note that $\rho_1 \sim \rho_2$ in Theorem 1.2 if and only if there exists a character $\chi : \text{Gal}(\bar{K}/K) \to \mathbb{Z}_p^\times$ such that $\rho_1 \cong \rho_2 \otimes \chi$ because $\rho_1$ and $\rho_2$ are assumed to be geometrically absolutely irreducible.

Crystalline representations are a $p$-adic analog of polarized variations of Hodge structures. Therefore Theorem 1.2 is an arithmetic analog of a theorem of Deligne [Del87]. See also the very recent work of Litt for a finiteness result in a different spirit [Lit18].

2. Preliminaries

First of all, we reduce Theorem 1.2 to the case of curves.

Lemma 2.1. Notation as in Setup 1.1. Then there exists a smooth projective relative curve $C \subset X$ over $W$ that intersects $S$ transversely, with the property that $\pi_1(C_K \cap U_K) \to \pi_1(U_K)$ is surjective. Therefore, to prove Theorem 1.2 it suffices to consider the case when $X/W$ has relative dimension 1.

Proof. We claim that there exists a smooth ample relative divisor $D \subset X$ over $W$ that intersects $S$ transversely. Indeed, pick some ample line bundle $L$ on $X$; then for all $m \gg 0$, the map $H^0(X, L^m) \to H^0(X_1, L_1^m)$ is surjective. On the other hand, for $m \gg 0$, the vector space $H^0(X_1, L_1^m)$ has a section $s_1$ whose zero locus $V(s_1)$ is smooth and intersects $S_1$ transversely by Poonen’s Bertini theorem [Poo12, Theorem 1.3]. Take any lift $s \in H^0(X, L^m)$ of $s_1$; then the zero locus $V(s)$ is smooth over $W$ and intersects $S$ transversely. Finally, it...
is well known that the map on fundamental groups $\pi_1(D_K \cap U_K) \to \pi_1(U_K)$ is surjective because $D_K \subset X_K$ is ample and $D_K$ intersects $S_K$ transversely. Proceed by induction.

Now, as $\pi_1(C_K \cap U_K) \to \pi_1(U_K)$ is surjective, it follows that to prove Theorem \ref{thm:main} it suffices to prove it for the pair $(C, S \cap C)$, i.e., we may reduce to the case of curves.

To a logarithmic crystalline representation $\rho: \pi_1(U_K) \to \mathrm{GL}_r(\mathbb{Z}_p)$, we may attach an overconvergent $F$-isocrystal\footnote{For details, see \cite{KYZ20}, Section 2.} We now show that a logarithmic crystalline representation being irreducible implies that the attached overconvergent $F$-isocrystal is also irreducible. While this is not strictly useful for the rest of the article, it seems to be of independent interest.

**Lemma 2.2.** Notation as in Setup \ref{setup}. Let $\rho: \pi_1(U_K) \to \mathrm{GL}_r(\mathbb{Z}_p)$ be a crystalline representation with associated logarithmic Fontaine-Faltings module $(V, \nabla, \Fil, \varphi, \iota)$.

If $p\mathbb{Q}$ is irreducible then the overconvergent $F$-isocrystal $(V, \nabla, \varphi, \iota)_\mathbb{Q}$ in $\mathbf{F-Isoc}^\dagger(U_1)_{\mathbb{Q}_p}$ is irreducible.

**Proof.** First of all, it follows from \cite[Theorem 6.4.5]{Ked07} that it suffices to check that $(V, \nabla, \varphi, \iota)_\mathbb{Q}$ is irreducible in $\mathbf{F-Isoc}_{\log}(X_1, S_1)_{\mathbb{Q}_p}$. Our proof will proceed by contradiction.

Let $\Phi$ be a local lifting of the absolute Frobenius on $(X_1, S_1)$. For the logarithmic Fontaine-Faltings module, locally the $\varphi$-structure can be represented as an isomorphism

$$\varphi: \widetilde{\Phi^*(V, \nabla, \Fil)} \xrightarrow{\sim} (V, \nabla),$$

where $(\cdot)$ is Faltings’ tilde functor. In the case when $V$ is $p$-torsion free, one may describe this as follows:

$$(V, \nabla, \Fil) = \sum_i \Fil^i(V, \nabla)_{p^i} \subset (V, \nabla, \Fil)_{\mathbb{Q}}.$$

That $\varphi$ is an isomorphism encodes the strong divisibility in the definition of a Fontaine-Faltings module. After shifting the filtration, we assume $\Fil^0V = V$. In this case, $(V, \nabla) \subset (\widetilde{V}, \nabla, \Fil)$. and $\varphi$ can be restricted on $\Phi^*(V, \nabla)$,

$$\varphi: \Phi^*(V, \nabla) \to (V, \nabla).$$

This yields the underlying logarithmic $F$-crystal.

Suppose the $F$-isocrystal $(V, \nabla, \varphi, \iota)_\mathbb{Q}$ is not irreducible in $\mathbf{F-Isoc}_{\log}(X, S)_{\mathbb{Q}_p}$. Let $(V', \nabla', \varphi', \iota')$ be a proper sub $F$-isocrystal of $(V, \nabla, \varphi, \iota)_\mathbb{Q}$ in $\mathbf{F-Isoc}_{\log}(X, S)_{\mathbb{Q}_p}$. In particular, $V'$ is stable under the $\mathbb{Q}_p$-action, the connection $\nabla$, and $\varphi$. There is a natural choice of lattice:

$$V' = V' \cap V$$

Then the restriction of $\varphi$ induces a map

$$\varphi': \Phi^*(V', \nabla') \to (V', \nabla'),$$

since $\varphi(\Phi^*((V', \nabla'))) \subset \varphi(\Phi^*((V, \nabla))) \cap \varphi(\Phi^*((V, \nabla))) \subset (V', \nabla') \cap (V, \nabla) = (V', \nabla')$. Denote $\Fil'$ the restriction of $\Fil$ on $V'$. The endomorphism structure clearly restricts on the quadruple $(V', \nabla', \Fil', \varphi')$. In the following, we show that $(V', \nabla', \Fil', \varphi')$ forms a sub-Fontaine-Faltings module of $(V, \nabla, \Fil, \varphi)$. As the triple $(V', \nabla', \varphi')$ is a logarithmic $F$-crystal in finite, locally free modules, we must check that the pair $(\Fil', \varphi')$ is strongly divisible, i.e., that the isogeny $\varphi: \Phi^*(V', \nabla') \to (V', \nabla')$ extends to an isomorphism

$$\Phi^*(V', \nabla', \Fil') \xrightarrow{\sim} (V', \nabla')$$

Denote

$$(V'', \nabla'', \Fil'') := (V, \nabla, \Fil)/(V', \nabla', \Fil').$$
We constructed the embedding \((V', \nabla', \Fil') \to (V, \nabla, \Fil)\) to be saturated and strict with respect to the filtrations. Therefore the triple \((V'', \nabla'', \Fil'')\) is a filtered logarithmic de Rham bundle.

Applying Faltings’ tilde functor, one has short exact sequence
\[
0 \to \tilde{V}' \to \tilde{V} \to \tilde{V}'' \to 0.
\]
Locally, one has the following commutative diagram
\[
\begin{array}{cccccc}
0 & \to & \Phi^* \tilde{V}' & \to & \Phi^* \tilde{V} & \to & \Phi^* \tilde{V}'' & \to & 0 \\
\downarrow \varphi' & & \downarrow \sim & & \downarrow \varphi & & \downarrow \exists \varphi'' & & \\
0 & \to & V' & \to & V & \to & V'' & \to & 0
\end{array}
\]
where \(\varphi'\) (by abusing notation as in (2.2.1)) is the restriction of \(\varphi\) on \(\Phi^* \tilde{V}'\), which extends the \(\varphi'\) in (2.2.1). The image \(\varphi'(\Phi^* V')\) is contained in \(V'\), because
\[
\varphi(\Phi^* (\tilde{V}')) \subseteq \varphi(\Phi^* (V')) \cap \varphi(\Phi^* (\tilde{V})) \subseteq V' \cap V = V'.
\]
Since \(\varphi\) is surjective, \(\varphi''\) is also surjective. On the other hand \(\Phi^* \tilde{V}''\) and \(V''\) are bundles with the same rank, so \(\varphi''\) is actually an isomorphism. By the snake lemma \(\varphi'\) is also an isomorphism. This proves the strong divisibility, as desired.

By the Fontaine-Lafaille-Faltings correspondence, to \((V', \nabla', \Fil', \varphi', \iota')\) one may attach a (log crystalline) subrepresentation \(\rho'\) of \(\rho\) of strictly smaller rank [Fal89, Theorem 2.6* (i)]. It follows that \(\rho_Q\) is not irreducible, contradicting our hypothesis.

The following is a version of Lemma 2.2 with the stronger hypothesis that \(\rho_Q\) is geometrically absolutely irreducible. With these assumptions, we show something much stronger than the conclusion of Lemma 2.2

This will be essential in the proof of Theorem 1.2.

**Lemma 2.3.** Notation as in Setup 1.1 and suppose \(X/W\) is a curve. Let \(\rho: \pi_1(U_K) \to GL_r(\mathbb{Z}_{p})\) be a logarithmic crystalline representation such that \(\rho_Q\) is geometrically absolutely irreducible. Let \((V, \nabla, \Fil, \Phi, \iota)\) be the associated logarithmic Fontaine-Faltings module with endomorphism structure \(\iota\). Let \((V, \nabla)^{(0)}\) be the identity eigenspace of the \(\iota\)-action. Then the following holds.

1. The logarithmic de Rham bundle \((V, \nabla)^{(0)}\) on \((X, S)_{\mathbb{Q}_{p}}\) admits no proper de Rham subbundle.
2. The object \((V, \nabla, \Fil, \iota)_{Q} \in F_{\text{Isoc}}^1(\mathbb{Q}_{p})\) is absolutely irreducible.

**Proof.** We first prove the first statement. First of all, \((V, \nabla)^{(0)}\) is a semistable de Rham bundle of degree 0. Suppose for contradiction that there is a logarithmic de Rham subbundle \((V', \nabla')_{K'}\) of \((V, \nabla)_{K}\) for some finite unramified extension \(K'/K\). Set \(\mathcal{O}'\) to be the ring of integers and \(k'\) to be the residue field. Then \((V', \nabla')_{K'}\) automatically has nilpotent residues and hence also has degree 0 and is therefore semistable.

We claim that we may find a logarithmic de Rham subbundle \((W, \nabla)_{K'}\) of \((V, \nabla)_{K}\) such that \((W, \nabla)_{K'} \cong (V', \nabla')_{K'}\) and \((W, \nabla)_{k'}\) is semistable of degree 0. First of all, there is clearly an extension to a torsion-free logarithmic de Rham subsheaf \((W, \nabla)\). By [KY20c, Section 6], the degree of \(W_{k'}\) is 0; therefore \((W, \nabla)_{k'}\) is a degree 0 logarithmic de Rham subsheaf of \((V, \nabla)_{k'}\). Note that \((W, \nabla)_{k'} \subset (V, \nabla)^{(0)}\); as the latter is semistable of degree 0, it follows that the inclusion \((W, \nabla)_{k'} \subset (V, \nabla)^{(0)}\) is saturated and hence is a de Rham subbundle. As both de Rham bundles have degree 0 and \((V, \nabla)_{k'}\) is semistable, it follows that \((W, \nabla)_{k'}\) is semistable.

Let \(\mathcal{HDF}\) be the (logarithmic) Higgs-de Rham flow attached to \(\rho\). Run the Higgs-de Rham flow over \(X\) with initial term \((W, \nabla)\), where the Hodge filtrations are chosen to be the restrictions of \(\Fil\) on \(V\). One

\[\text{Note that because } X_k \text{ is a smooth curve, any torsion-free sheaf is automatically a vector bundle.}\]
obtains a sub Higgs-de Rham flow $\mathcal{HDF}'$ of $\mathcal{HDF}$. In general, this sub Higgs-de Rham is not preperiodic. Nonetheless, we claim that $\mathcal{HDF}'$ is preperiodic over each truncated level $W_m(k')$. This holds because $(V, \nabla)_{W_m(k')}$ has only finitely many subbundles of degree 0 for each $m$.

Note that because there are only finitely many (isomorphism classes of) Higgs terms in $\mathcal{HDF}$ by the periodicity. Therefore we may inductively shift the index of $\mathcal{HDF}'$, to find a sequence of sub Higgs-de Rham flows $\mathcal{HDF}'_{W_m(k')} \subset \mathcal{HDF}_{W_m(k')}$ which are periodic with periodicity $f_m$, and satisfying $\mathcal{HDF}'_{W_{m+1}(k')} \equiv \mathcal{HDF}'_{W_m(k')} \mod p^m$ and $f_m | f_{m+1}$.

To each of these truncated periodic Higgs-de Rham flows, there is an associated torsion logarithmic crystalline representation

$$\rho'_m : \pi_1(U_{Q_{unr}}) \to \text{GL}_L(W_m(k)).$$

Recall that $\overline{O}_{Q_{unr}}$, the $p$-adic completion of $O_{Q_{unr}}$, is equal to $W(k)$. Taking the inverse limit over $m$, one obtains a sub-representation

$$\rho' : \pi_1(U_{Q_{unr}}) \to \text{GL}_L(\overline{O}_{Q_{unr}})$$

of $\rho : \pi_1(U_{Q_{unr}}) \to \text{GL}_{\mathcal{L}_r}(\overline{O}_{Q_{unr}})$. We claim this is in contradiction with the fact that $\rho \otimes Q_p$ is geometrically absolutely irreducible. Indeed, $\rho' \mid_{\pi_1(U_K)} \otimes C_p$ is a non-trivial sub-representation of $\rho \mid_{\pi_1(U_K)} \otimes C_p$; on the other hand, the fact that $\rho \mid_{\pi_1(U_K)} \otimes C_p$ is irreducible implies that $\rho \mid_{\pi_1(U_K)} \otimes C_p$ is also irreducible. □

We come to the following crucial definition.

**Definition 2.4.** Notation as in Setup 1.1. Let $(V, \nabla, \varphi, \iota)$ be an object of $\text{F-Isoc}_{\text{nilp}}^\text{r}((X_1, S_1)_{Q_{pf}})$. An extension of $(V, \nabla, \varphi, \iota)$ is an logarithmic $F$-crystal in finite, locally free modules $(V, \nabla, \varphi, \iota)$ with $Z_{pf}$-structure such that $(V, \nabla, \varphi, \iota)_{Q_{pf}} \cong (V, \nabla, \varphi, \iota)$. An extension $(V, \nabla, \varphi, \iota)$ is said to be semistable if the logarithmic flat connection $(V, \nabla)_1$ on $(X_1, S_1)$ is semistable.

Recall that a rank-1 $F$-crystal over $k$ with $Z_{pf}$-structure is a pair $(L, \varphi)$ where $L$ is a finite free $W \otimes_{Z_p} Z_{pf}$-module and $\varphi : L \to L$ is an injective $\sigma \otimes 1$-semi-linear map where $\sigma : W \to W$ is the canonical lift of Frobenius. For any element $r \in (K \otimes_{Q_p} Q_{pf})^\times \cap (W \otimes_{Z_p} Z_{pf})$, we denote $L_r = W \otimes_{Z_p} Z_{pf} \cdot e$ with $\varphi_{L_r}(e) = re$. Conversely, for any rank-1 $F$-crystal over $W$ with $Z_{pf}$-structure is isomorphic to some $L_r$.

By tensoring $Q_p$, one gets an rank-1 $F$-isocrystal $L_r = L \otimes_{Z_p} Q_p$ over $k$ with $Q_{pf}$-structure.

Let $(V, \nabla, \varphi, \iota)$ be a logarithmic $F$-crystal in finite, locally free modules over $(X_1, S_1)$ with $Z_{pf}$-structure. Locally one can view $\varphi$ as a $\Phi \otimes 1$-semi-linear map of the $O_X \otimes Z_{pf}$-modules. We define the twist of $(V, \nabla, \varphi, \iota)$ by $L_r$ to be: $(V, \nabla, \varphi, \iota) \otimes L_r = (V, \nabla, r \cdot \varphi, \iota)$.

**Remark 2.5.** Twisting by a constant rank 1 object does not change the underlying de Rham bundle.

**Lemma 2.6.** Let $(V, \nabla, \varphi, \iota)$ be an object of $\text{F-Isoc}_{\text{nilp}}^\text{r}((X_1, S_1)_{Q_{pf}})$. Let $L$ be a rank-1 $F$-crystal over $k$ with $Z_{pf}$-structure and let $L = L \otimes_{Z_p} Q_p$. Denote $\varphi' = \varphi \otimes \varphi_L$. Then tensoring $L$ induces an injection

$$\{\text{extension of } (V, \nabla, \varphi, \iota)\} \to \{\text{extension of } (V, \nabla, \varphi', \iota)\}$$

**Proof.** Since an extension of an $F$-isocrystal is uniquely determined by the extension of the underlying de Rham bundle and twisting a constant rank-1 object doesn’t change the underlying de Rham bundle, the map is injective. □

**Lemma 2.7.** Notation as in Setup 1.1 and suppose $X/W$ is a curve. Let $(V, \nabla, \varphi, \iota)$ be an irreducible object of $\text{F-Isoc}_{\text{nilp}}^\text{r}((X_1, S_1)_{Q_{pf}})$. Then there exists only finitely many isomorphism classes of semistable extensions $(V, \nabla, \varphi, \iota)$ of $(V, \nabla, \varphi, \iota)$.

**Proof.** Assume there are infinitely many isomorphism classes of semistable extensions, choose a representative from each isomorphism class, and enumerate them as follows $\{T_i = (V, \nabla, \varphi, \iota) \mid i \in I\}$. We will construct
an infinite descending chain \( T_{i_0} \supset T_{i_1} \supset T_{i_2} \ldots \) such that the intersection yields a proper logarithmic sub-\( F \)-isocrystal, contradicting out original assumption.

Fix one element \( i_0 \in I \), and set \( I_0 = I \) and \( J_0 = I_0 - \{ i_0 \} \). By assumption, we may embed each \( (V, \nabla, \varphi, \iota) \) as a lattice in \( (V, \nabla, \varphi, \iota) \). By multiplying by a suitable power of \( p \) on each \( T_i \), for \( i \in J_0 \), we may assume that
\[
T_i \subset T_{i_0} \quad \text{and} \quad T_i \not\subset pT_{i_0}.
\]

This is equivalent to saying that the image of \( V_i \) in \( V_{i_0}/pV_{i_0} \) is a proper submodule. In fact, we claim that the image, namely \( (V_i + pV_{i_0})/pV_{i_0} \), together with the induced logarithmic flat connection, is a semistable logarithmic de Rham bundle on \( (X_1, S_1) \). Indeed, both \( V_{i_0}/pV_{i_0} \) and \( V_i/pV_i \) admit semistable flat connections of degree zero. Therefore the image \( (V_i + pV_{i_0})/pV_{i_0} \) has degree \( 0 \) and hence, when equipped with the induced connection, is semistable. Finally, we claim that \( (V_i + pV_{i_0})/pV_{i_0} \) is a submodule of \( V_{i_0}/pV_{i_0} \) (as opposed to merely a subsheaf). If not, the saturation would be a subsheaf; but any non-trivial saturation increases the degree. As \( (V_i + pV_{i_0})/pV_{i_0} \) with the induced flat connection is semi-stable, so is the saturation; this contradicts semistability of \( V_{i_0}/pV_{i_0} \).

Consider the map
\[
f_0: J_0 \rightarrow \Sigma_0 := \{ \text{proper sub bundles of } V_{i_0}/p \cdot V_{i_0} \text{ of degree } 0 \}
\]
\[
i \longmapsto (V_i + pV_{i_0})/pV_{i_0}
\]

The initial set is infinite by assumption. On the other hand, the terminal set is finite; indeed, this follows because the set all subbundles with fixed degree of a given bundle forms a bounded family and our base field is finite.

Thus there exists a submodule \( M_0 \) of \( V_{i_0}/pV_{i_0} \) such that \( I_1 := f_0^{-1}(M_0) \) is infinite. For any fixed \( i \in I_1 \), the submodule \( V_i + pV_{i_0} \) is the inverse image of \( M_0 \) under surjective map \( V_{i_0} \rightarrow V_{i_0}/pV_{i_0} \); hence, the submodule \( V_i + pV_{i_0} \) does not depend on the choice of \( i \in I_1 \). We further claim that for each \( i \in I_0 \), the module \( V_i + pV_{i_0} \) together with the induced logarithmic flat connection, Frobenius structure, and endomorphism structure, yields a semistable extension of \( (V, \nabla, \varphi, \iota) \). This follows from the fact that \( V_i/pV_{i_0} \) and \( V_{i_0}/pV_{i_0} \), equipped with their flat connections, are semistable de Rham bundles of degree \( 0 \). Thus there exists \( i_1 \in I_1 \) such that
\[
T_{i_1} = T_i + pT_{i_0} \quad \text{for all } i \in I_1.
\]

Denote \( J_1 = I_1 \setminus \{ i_1 \} \). Then for all \( i \in J_1 \) one has
\[
T_i \subsetneq T_{i_1} \subsetneq T_{i_0} \quad \text{and} \quad T_i \not\subset pT_{i_1}.
\]

Repeating the process, one can find a sequence of extensions
\[
\cdots \subsetneq T_{i_3} \subsetneq T_{i_2} \subsetneq T_{i_1} \subsetneq T_{i_0}
\]
satisfying \( T_{i_m} \not\subset pT_{i_0} \) for all \( m, n \geq 0 \).

Denote \( T_\infty = \bigcap_{k=0}^\infty T_{i_k} \). In the following we show that \( (T_\infty)_{\Omega} \) is a proper logarithmic sub-\( F \)-isocrystal of \( (V, \nabla, \varphi, \iota) \). Thus we get a contradiction with the irreducibility of \( (V, \nabla, \varphi, \iota) \in \mathbf{F-Isoc}_{\log}^\text{nilp}(X_1, S_1)_{\Omega_f} \).

Locally, we may assume \( T_{i_k} \) are free modules of the same rank \( r \) over a regular local ring \( R \) satisfying
\[
\cdots \subsetneq T_{i_3} \subsetneq T_{i_2} \subsetneq T_{i_1} \subsetneq T_{i_0}.
\]

Since \( T_\infty = \bigcap_{k=0}^\infty T_{i_k} \) is torsion-free and finitely generated over \( R \), we may choose a free sub-\( R \)-module of \( T'_\infty \subset T_\infty \) with maximal rank \( r_{\infty} \leq r \). We only need to show
\[
T_\infty \neq 0 \quad \text{and} \quad r_\infty \neq r.
\]
We first show that $T_\infty \neq 0$. For a given positive integer $n$, consider the descending sequence

$$\left(\left( T_{i_k} + p^nT_{i_0} \right)/p^nT_{i_0} \right)_k.$$

We claim the sequence stabilizes for $k \gg 0$. Each term is contained in $T_{i_0}/p^nT_{i_0}$. Let’s consider the index between $T_{i_k}$ and $p^nT_{i_0}$, which has only $p$-primary part and is finite. Thus the increasing sequence of the index $[T_{i_0} : T_{i_k} + p^nT_{i_0}]$ with upper bound $[T_{i_0} : p^nT_{i_0}]$ is stable. This implies $[T_{i_0} : T_{i_k} + p^nT_{i_0}] = [T_{i_0} : T_{i_k+1} + p^nT_{i_0}]$ for sufficiently large $k$, so $T_{i_k} + p^nT_{i_0} = T_{i_{k+1}} + p^nT_{i_0}$ for $k \gg 0$. Denote

$$\mathcal{T}_0^{(n)} := \bigcap_{k=0}^{\infty} \left( T_{i_k} + p^nT_0 \right)/p^nT_0 = \left( T_N + p^nT_0 \right)/p^nT_0 \neq 0, \text{ for } N \gg 0.$$

Thus one has an surjective inverse system

$$\cdots \rightarrow \mathcal{T}_0^{(4)} \rightarrow \mathcal{T}_0^{(2)} \rightarrow \mathcal{T}_0^{(0)} \rightarrow \mathcal{T}_0^{(0)}$$

whose inverse limit $\lim_n \mathcal{T}_0^{(n)}$ is non-empty. By the left exactness of inverse limits, the inclusion maps

$$\left( \mathcal{T}_0^{(n)} \right) \hookrightarrow \left( T_{i_k} + p^nT_0 \right)/p^nT_0$$

induce an injective map

$$\lim_n \mathcal{T}_0^{(n)} \hookrightarrow \lim_n \left( T_{i_k} + p^nT_0 \right)/p^nT_0 = T_i.$$

Thus $\lim_n \mathcal{T}_0^{(n)} \subset T_\infty = \bigcap_i T_{i_k}$. This implies that $T_\infty \neq 0$. The fact that $T_\infty \neq 0$ immediately implies that $T_\infty$ yields a logarithmic $F$-crystal in finite modules on $(X_1, S_1)$.

We now show that $r_\infty \neq r$. By étale localization, we reduce to the following setup in linear algebra. Let $A = W < x >$ be the $p$-adic completion of a polynomial ring in a single variable over $W$, and let $M_0 \supseteq M_1 \supseteq \ldots$ be an infinite nested collection of finite free modules of fixed rank $r$, that is strictly decreasing and such that $M_j \not\subset pM_k$ for $j, k \geq 0$. Set $M_\infty := \bigcap_{j=0}^{\infty} M_j$. We wish to prove that $M_\infty$ has rank smaller than $r$; equivalently, that it does not contain a lattice $L_\infty$ in $M_0 \otimes \text{Frac}(A)$. If it did, then $M_\infty$ would have finite, $p$-primary index in $M_0$. However, the index of $M_j$ in $M_0$ gets arbitrarily large; indeed, if $[M_j : M_k] = 1$, then $M_j = M_k$. As index is multiplicative, we obtain a contradiction. \[\square\]

**Lemma 2.8.** Notation as in Setup \[\square\] Let $(V, \nabla, \varphi, \iota)$ be an irreducible object of $\mathbf{F-Isoc}_{\text{log}}^\text{nilp}(X_1, S_1)_{\mathbb{Q}_p}$. Let $(V, \nabla, \varphi, \iota)$ be an extension $(V, \nabla, \varphi, \iota)$. Then there exists only finitely many Hodge filtrations $\text{Fil}_1$ on $(V, \nabla, \varphi, \iota)$ with $\text{Fil}_1^0 \mathcal{V} = \mathcal{V}$ and $\text{Fil}_1^1 \mathcal{V} = 0$ such that there exists $\varphi_1$ rendering the quintuple $(V, \nabla, \text{Fil}, \varphi, \iota)_1$ a logarithmic Fontaine-Faltings module with endomorphism structure over $(X_1, S_1)$.

**Proof.** By the Fontaine-Laffaille-Faltings correspondence [Fal89 Theorem 2.6*(i)], the category of $p$-torsion logarithmic Fontaine-Faltings modules (with endomorphism structure) on $(X_1, S_1)$ is equivalent to the category of logarithmic crystalline representations of $\pi_1^\text{et}(U_K)$ with coefficients in $\mathbb{F}_p$. The étale fundamental group $\pi_1^\text{et}(U_K)$ is topologically finitely generated. Therefore the set of isomorphism classes of $\text{GL}_r(\mathbb{F}_p)$ representations of $\pi_1^\text{et}(U_K)$ is finite. In particular, there are only finitely many isomorphism classes of crystalline $\text{GL}_r(\mathbb{F}_p)$ representations. Forgetting the $\varphi$-structure, it follows that the set of isomorphism classes of de Rham bundles (with endomorphism structure) which underlie a Fontaine-Faltings module (with endomorphism structure) over $(X_1, S_1)$ is also finite.

Suppose there are infinitely many distinct Hodge filtrations $\text{Fil}_1^{(i)} (i = 1, 2, \cdots)$ on $(V, \nabla, \varphi, \iota)_1$ such that for each $i$, there exists $\varphi^{(i)}$ rendering the quintuple $(V, \nabla, \text{Fil}, \varphi, \iota)_1$ a Fontaine-Faltings module. By the pigeonhole principle, there are infinitely $i$ such that there exists a log Fontaine-Faltings module $(V, \nabla, \text{Fil}^{(i)}, \varphi^{(i)}, \iota)_1$.
whose isomorphism class is independent of $i$. In particular, one deduces $\text{Aut}(V_i)$ is an infinite set. But this contradicts the finiteness of $\text{Aut}(M)$ for any vector bundle $M$ over $X_1$, as our base field is finite.

\[\square\]

**Lemma 2.9.** Notation as in [1.1]. Let $(V, \nabla, \varphi, \iota)$ be an irreducible object of $\mathbf{F}_{\text{-Isoc}}^{\text{nisp}}(X_1, S_1)_{\mathbb{Q}_p}$. Let $(V, \nabla, \varphi, \iota)$ be an extension $(V, \nabla, \varphi, \iota)$. Let $\text{Fil}_1$ be a Hodge filtration on $(V, \nabla, \iota)_1 := (V, \nabla, \iota) \pmod{p}$ with $\text{Fil}_1\varphi V_1 = V_1$ and $\text{Fil}_1^2 V_1 = 0$ such that there exists $\varphi_1$ rendering the quintuple $(V, \nabla, \text{Fil}, \varphi, \iota)_1$ a logarithmic Fontaine-Faltings module with endomorphism structure over $X_1$. Assume there exists two liftings $\text{Fil}$ and $\text{Fil}'$ of the Hodge filtration $\text{Fil}_1$.

Then there exists an automorphism $f: (V, \nabla, \iota) \to (V, \nabla, \iota)$ such that

$$f^*(\text{Fil}') = \text{Fil}.$$

In other words, one has an isomorphism $f: (V, \nabla, \text{Fil}, \iota) \to (V, \nabla, \text{Fil}', \iota)$.

**Proof.** The Hodge-de Rham spectral sequence associated to $(V, \nabla, \text{Fil}, \varphi, \iota)_1$ degenerates at $E_1$ [KYZ20a, Lemma 6.1]. Then this follows from [KYZ20a, Theorem 1.6(2)] \[\square\]

### 3. The Proof

The proof of the main theorem of this article is diagrammatically sketched below; the definition of the various terms will follow. Here is a two sentence summary of the proof. The Langlands correspondence implies that $\text{MF}^{\text{twist}}_{\mathbf{F}_{\text{-Isoc}}}$ is finite. By following the diagram, it follows that $\text{Rep}_{\text{GL}_r(\mathbb{Z}_p)}^{\text{irr, crys}}(\Gamma, \bar{\Gamma})$ is finite.

---

We explain all of the terms in the above diagram.

- $\Gamma = \pi_1^\text{et}(U_K, x)$ and $\bar{\Gamma} = \pi_1^\text{et}(U_K, x)$.
- $\text{Rep}_{\text{GL}_r(\mathbb{Z}_p)}^{\text{irr, crys}}(\Gamma, \bar{\Gamma})$ is the set of isomorphism classes of logarithmic crystalline representations $\rho: \Gamma \to \text{GL}_r(\mathbb{Z}_p)$ whose Hodge Tate weights are located in $[0, p - 1]$ such that $\rho_Q: \Gamma \to \text{GL}_r(\mathbb{Q}_p)$ is geometrically absolutely irreducible.

---

3While [KYZ20a, Theorem 1.6(2)] is written for vector bundles with a (logarithmic) flat connection, it easily generalizes to the case with endomorphism structure.
One may replace all objects above with the identity eigenspaces of $\iota$ which will yield equivalent categories because the identity eigenspace of the action of $\iota$ on $(V, \nabla, \Fil)$ in the additive category of filtered de Rham bundles equipped with an endomorphism structure: $[(V, \nabla, \Fil, \phi)] \mapsto [(V, \nabla, \Fil, \phi)]$. Thus one has a surjective map
$$\text{Rep}^{\text{irr,crys}}_{\text{GL}_r}(\mathbb{Z}_{p_f}) (\Gamma) \twoheadrightarrow \text{Rep}^{\text{irr,crys}}_{\text{GL}_r}(\overline{\mathbb{Z}_{p_f}}) (\Gamma, \overline{\Gamma}).$$

- **MF_{FF}** is the set of isomorphism classes of logarithmic Fontaine-Faltings modules $(V, \nabla, \Fil, \phi)$ with endomorphism structure $\iota: \mathbb{Z}_{p_f} \hookrightarrow \text{End}(V, \nabla, \Fil, \phi)$ associated to representations in $\text{Rep}^{\text{irr,crys}}_{\text{GL}_r}(\overline{\mathbb{Z}_{p_f}}) (\Gamma)$ via $\text{[Fal89]}$. Thus one has an bijection
$$\text{Rep}^{\text{irr,crys}}_{\text{GL}_r}(\overline{\mathbb{Z}_{p_f}}) (\Gamma) \xrightarrow{1:1} \text{MF}_{FF}.$$  

- **MF_{FdR}** is the set of isomorphism classes of logarithmic Fontaine-Faltings modules $[(V, \nabla, \Fil, \phi)]$ to the isomorphism class of the underlying logarithmic de Rham bundle with endomorphism structure: $[(V, \nabla, \Fil, \phi)] \mapsto [(V, \nabla, \Fil, \phi)]$. Thus one has a surjective map
$$\text{MF}_{FF} \twoheadrightarrow \text{MF}_{FdR}.$$  

- **MF_{F,Cris}** is the image of **MF_{FF}** under the map that sends an isomorphism class of a logarithmic Fontaine-Faltings module to the isomorphism class of the underlying logarithmic $F$-crystal in locally free modules with endomorphism structure: $[(V, \nabla, \Fil, \phi, \iota)] \mapsto [(V, \nabla, \Fil, \phi, \iota)]$. Thus one has a surjective map
$$\text{MF}_{FF} \twoheadrightarrow \text{MF}_{F,Cris}.$$  

- **MF_{F,Cris}^{\text{twist}}** is the set of equivalence classes of the set $\text{MF}_{F,Cris}$ modulo the equivalence relations defined by twisting $a$ by constant rank 1 Fontaine-Faltings modules (with endomorphism structure).

- **MF_{FdR}^{\text{twist}}** is the image of $\text{MF}_{F,Cris}^{\text{twist}}$ under the map that sends an isomorphism class of a logarithmic $F$-crystal to the isomorphism class of the underlying logarithmic de Rham bundle with endomorphism structure: $[(V, \nabla, \Fil, \phi)] \mapsto [(V, \nabla, \Fil, \phi)]$. Thus one has surjective maps
$$\text{MF}_{FdR} \twoheadrightarrow \text{MF}_{FdR}^{\text{twist}}.$$  

By Remark 2.5 the second surjective map factors through $\text{MF}_{F,Cris}^{\text{twist}}$.

- **MF_{Higgs}** is the image of **MF_{FdR}** under the map that sends an isomorphism class of filtered de Rham bundle with endomorphism structure to the isomorphism class of the associated Higgs bundle with endomorphism structure: $[(V, \nabla, \Fil, \iota)] \mapsto [\text{Gr}(V, \nabla, \Fil, \iota)]$. Thus one has a surjective map
$$\text{MF}_{FdR} \twoheadrightarrow \text{MF}_{Higgs}.$$  

- **MF_{F,Isoc}** is the image of **MF_{FF}** under the map that sends an isomorphism class of a logarithmic Fontaine-Faltings module to the isomorphism class of the associated overconvergent $F$-isocrystal and multiplication by $\mathbb{Q}_{p_f}$, i.e., the isomorphism class of an object of $\text{F-Isoc}^! (U_1)_{\mathbb{Q}_{p_f}}$.

- **MF_{F,Isoc}^{\text{twist}}** is the set of equivalence classes of the set $\text{MF}_{F,Isoc}$ modulo the equivalence relations defined by twisting a constant rank-1 $F$-isocrystal.

Every constant rank 1 $F$-isocrystal comes from a constant rank 1 Fontaine-Faltings module. Therefore the natural map $\text{MF}_{F,Cris}^{\text{twist}} \rightarrow \text{MF}_{F,Isoc}^{\text{twist}}$ is surjective.

If the reader is uncomfortable with carrying around the endomorphism structure $\iota$, we introduce the following notation: if $(V, \nabla, \Fil, \phi, \iota)$ is a logarithmic Fontaine-Faltings modules, then $(V, \nabla, \Fil)^{(0)}$ denotes the identity eigenspace of the action of $\iota$ on $(V, \nabla, \Fil)$, i.e., for any $v \in V$,
$$v \in V^{(0)} \iff \iota(r)v = rv \text{ for all } r \in \mathbb{Z}_{p_f}.$$  

One may replace all objects above with the identity eigenspaces of $\iota$ (together with $\varphi^{pf}$, if $\varphi$ shows up). This will yield equivalent categories because $\mathbb{F}_{pf} \subset k$ and hence $\mathbb{Z}_{p_f} \subset W(k)$. 

We have one final preliminary result, using the above notation.

**Lemma 3.1.** The map

\[ MF_{\text{F-Cris}}^{\text{twist}} \rightarrow MF_{\text{F-Isoc}}^{\text{twist}} \]

is finite-to-one.

**Proof.** Fix an object \( E = (V, \nabla, \varphi, \iota) \) in \( MF_{\text{F-Isoc}} \). Recall that any twisting of \( E \) by a constant rank-1 \( F \)-isocrystal is of the form \( E_\lambda = (V, \nabla, \lambda \varphi, \iota) \) for some \( \lambda \in (K \otimes \mathbb{Q}_p)^\times \). Let \( V = \bigoplus_{i=0}^{f-1} V_i \) be the eigen decomposition of \( \iota : \mathbb{Q}_p \rightarrow \text{End}(V, \nabla, \varphi) \). Then the semi-linear map \( \varphi \) can be decomposed as semi-linear maps

\[ \varphi_i : V_i \rightarrow V_{i+1} \text{ for } i = 0, \ldots, f - 2 \text{ and } \varphi_{f-1} : V_{f-1} \rightarrow V_0. \]

Analogously, for any \( \lambda = (\lambda_i) \in (K \otimes \mathbb{Q}_p)^\times = \prod_{i=0}^{f-1} K^\times \), the isogeny \( \lambda \cdot \varphi \) decomposes as

\[ \lambda \varphi_i : V_i \rightarrow V_{i+1} \text{ for } i = 0, \ldots, f - 2 \text{ and } \lambda_{f-1} \varphi_{f-1} : V_{f-1} \rightarrow V_0. \]

We denote

\[ v_p(\lambda) := \frac{1}{f} \sum_{i=0}^{f-1} v_p(\lambda_i) \in \frac{1}{f} \mathbb{Z}. \]

Set

\[ T_\lambda := \{ T \in MF_{\text{F-Cris}} : T \otimes_{\mathbb{Q}_p} \mathbb{Q}_p \simeq E_\lambda \}, \]

i.e., the set of \( F \)-crystals in finite, locally free modules which underlie \((V, \nabla, \varphi, \iota)\) up to isomorphism. This is a finite set by Lemma 2.7 for the following reason: if \((V, \nabla, \varphi, \iota)\) comes from a Fontaine-Faltings module, then \( V \) is automatically semistable. Under this notation, the lemma is claims that the following set

\[ \bigcup_{\lambda} T_\lambda / \sim \]

is finite, where the equivalence relation \( \sim \) is given as follows: \((V, \nabla, \varphi, \iota) \sim (V', \nabla', \varphi', \iota')\) if and only if they are twists by a constant, rank 1 Fontaine-Faltings module (with endomorphism structure). The finiteness will then follow from the following two claims:

**Claim 1.** \( T_\lambda = \emptyset \) if and only if \( v_p(\lambda) > p - 1 \) or \( v_p(\lambda) < 1 - p \).

**Claim 2.** \( T_\lambda = T_{\lambda'} \) in \( MF_{\text{F-Cris}}^{\text{twist}} \), if \( v_p(\lambda) = v_p(\lambda') \).

Let \((V, \nabla, \text{Fil}, \varphi, \iota) \in MF_{\mathbb{F}_p} \) be an object mapping to \( E \). By strong divisibility, one has

\[ p^{p-1} V_0 \subset \langle \varphi_{f-1}(V_{f-1}) \rangle \subset V_0 \text{ and } p^{p-1} V_{i+1} \subset \langle \varphi_i(V_i) \rangle \subset V_{i+1} \text{ for } i = 0, 1, \ldots, f - 2. \]

Considering the composition, one gets

\[ p^{(p-1)/f} V_0 \subset \langle \varphi_{f-1} \circ \cdots \circ \varphi_0(V_0) \rangle \subset V_0. \]

Choose a basis of \( V_0 \) and write \( \varphi_{f-1} \circ \cdots \circ \varphi_0 \) in terms of this basis. Then the \( p \)-adic valuation of the determinant of this matrix is well-defined; one has

\[ 0 \leq v_p(\det(\varphi_{f-1} \circ \cdots \circ \varphi_0)) \leq (p - 1) f \cdot \text{rank}(V_0). \]

Suppose \( T_\lambda \neq \emptyset \) for some \( \lambda = (\lambda_i) \in (K \otimes \mathbb{Q}_p)^\times = \prod_{i=0}^{f-1} K^\times \). Then by precisely analogus reasoning, one has

\[ 0 \leq v_p(\det((\lambda_{f-1} \varphi_{f-1}) \circ \cdots \circ (\lambda_0 \varphi_0))) \leq (p - 1) f \cdot \text{rank}(V_0) \]

4One way of seeing this is that a strict \( p \)-torsion Fontaine-Faltings module corresponds to a periodic Higgs-de Rham flow. The Higgs bundles in the flow are all semistable by [LSZ19, Proposition 6.3]. Because \( C^{-1} \) preserves semistability, this implies that \((V, \nabla)\) is semistable.
Since 
\[
v_p \left( \det((\lambda f_{-1} \varphi_{f_{-1}}) \circ \cdots \circ (\lambda_0 \varphi_0)) \right) = v_p(\lambda) \cdot f \cdot \operatorname{rank}(V_0) + v_p \left( \det(\varphi_{f_{-1}} \circ \cdots \circ \varphi_0) \right),
\]
by \(3.1.1\) and \(3.1.2\), one has
\[
1 - p \leq v_p(\lambda) \leq p - 1.
\]
Thus the Claim 1 follows.

We now show Claim 2. By replacing \(\varphi\) with \(\lambda'\varphi\), one may reduce the claim to case \(\lambda' = 1\); in this setting, as \(v_p(\lambda) = v_p(\lambda')\), it follows that \(v_p(\lambda) = 0\). Denote \(n_i = v_p(\lambda_i), c_i = \lambda_i/p^{n_i} \in W^x\) and \(m_i = n_1 + n_2 + \cdots + n_{i-1}\) for all \(i = 0, 1, \cdots, f - 1\). Consider the following map
\[
F_{i, j} : T_j \rightarrow T_\lambda
\]
which maps \((\oplus_i V_i, \oplus_i \nabla_i, \oplus_i \varphi_i, t)\) to \((\oplus_i p^{m_i} V_i, \oplus_i \nabla_i, \oplus_i \lambda_i \varphi_i, t)\). The Claim 2 is reduced to showing that this map

1. is well-defined;
2. is bijective and
3. preserves the twisted classes.

We first show it is well-defined. Suppose \((\lambda_i)\) are \(K^\times\) with \(\sum_{i=0}^{f-1} f - 1 v_p(\lambda_i) = 0\). Set \(n_i, c_i,\) and \(m_i\) as above. Suppose \(\text{Fil}_i\) is a Hodge filtration on \((V_i, \nabla_i)\) such that \((\oplus_i V_i, \oplus_i \nabla_i, \oplus_i \text{Fil}_i, \oplus_i \varphi_i, t)\) forms an object in \(\text{MF}_{\text{FF}}\). Then \((\oplus_i p^{m_i} V_i, \oplus_i \nabla_i, \oplus_i \text{Fil}_i, \oplus_i \lambda_i \varphi_i, t)\) is also contained in \(\text{MF}_{\text{FF}}\), because the pair \((\oplus_i \text{Fil}_i, \oplus_i \lambda_i \varphi_i)\) also satisfies strong divisibility on \(\oplus_i p^{m_i} V_i:\)
\[
(\lambda_i \varphi_i)(p^{m_i} V_i) = \lambda_i p^{m_i} \varphi_i(V_i) = \lambda_i p^{m_i} V_{i+1} = p^{m_{i+1}} V_{i+1}
\]
and
\[
(\lambda_{f-1} \varphi_{f-1})(p^{m_{j-1}} V_{f-1}) = \lambda_{f-1} p^{m_{j-1}} \varphi_{f-1}(V_{f-1}) = \lambda_{f-1} p^{m_{j-1}} V_0 = p^{m_0} V_0,
\]
in the last equality we used the fact that \(n_{f-1} + m_{f-1} = f \cdot v_p(\lambda) = 0 = m_0\). Thus \(F_{i, j}\) is well-defined. The map \(F_{i, j}\) is bijective, because one can define its inverse map \(T_\lambda \rightarrow T_j\) in similar manner by sending \((\oplus_i V_i, \oplus_i \nabla_i, \oplus_i \varphi_i, t)\) to \((\oplus_i p^{-m_i} V_i, \oplus_i \nabla_i, \oplus_i \lambda_i^{-1} \varphi_i, t)\).

Now we only need to show \((\oplus_i V_i, \oplus_i \nabla_i, \oplus_i \varphi_i, t)\) and \((\oplus_i p^{m_i} V_i, \oplus_i \nabla_i, \oplus_i \lambda_i \varphi_i, t)\) differ by twisting a rank 1 Fontaine-Faltings module. By the commutativity of following diagram

one gets an isomorphism \(\oplus_i (p^{m_i} \text{id}) : (\oplus_i V_i, \oplus_i \nabla_i, \oplus_i \varphi_i, t) \rightarrow (\oplus_i p^{m_i} V_i, \oplus_i \nabla_i, \oplus_i \lambda_i \varphi_i, t)\). We consider the rank-1 Fontaine-Faltings module with endomorphism structure
\[
\mathcal{L} = \left( \bigoplus_{i=0}^{f} W \cdot e_i, \text{Fil}_{tri}, \phi, t \right)
\]
where \(\phi(\sum_i a_i e_i) = c_0 a_0 e_1 + \cdots + c_{f-2} a_{f-2} e_{f-1} + c_{f-1} a_{f-1} e_0\). Since \(v_p(c_i) = 0\) for all \(i = 0, 1, \cdots, f-1\), this is a well-defined constant Fontaine-Faltings module. Then consider the twisting of \((\oplus_i p^{m_i} V_i, \oplus_i \nabla_i, \oplus_i \lambda_i \varphi_i, t)\) by \(\mathcal{L}\), which is nothing just equal to \((\oplus_i p^{m_i} V_i, \oplus_i \nabla_i, \oplus_i \lambda_i \varphi_i, t)\). Thus \((\oplus_i V_i, \oplus_i \nabla_i, \oplus_i \varphi_i, t)\) and \((\oplus_i p^{m_i} V_i, \oplus_i \nabla_i, \oplus_i \lambda_i \varphi_i, t)\) are differed by twisting a rank-1 Fontaine-Faltings module. \(\square\)
Proof of Theorem 1.2. By Faltings’ definition of a crystalline representation, the Hodge-Tate weights are in an interval of length $p - 1$. Since Tate twisting of a crystalline representation $\rho$ does not change the isomorphism class of $\rho | F$, it suffices to prove the finiteness of set

$$\left\{ \rho: \pi_1^{et}(U_K) \to \text{GL}_r(\mathbb{Z}_p) \bigg| \begin{array}{l}
\rho \text{ is log crystalline} \\
\text{with HT weights in } [0, p - 1] \\
\rho^{\text{tw}} : \pi_1^{et}(\overline{U}_K) \to \text{GL}_r(\mathbb{Q}_p) \text{ absolutely irreducible.}
\end{array} \right\} / \rho_1 \sim \rho_2 \text{ if } \rho_1|_{\pi_1^{et}(U_K)} \cong \rho_2|_{\pi_1^{et}(\overline{U}_K)}$$

Equivalently, to prove Theorem 1.2, we will show that $\text{Rep}^{\text{irr.crys}}_{\text{GL}_r(\mathbb{Z}_p)}(\Gamma, \bar{\Gamma})$ is finite.

Since the Faltings $p$-adic Simpson’s correspondence $\text{Fal05}$ is compatible with his $\mathbb{D}$-functor $\text{Fal89}$, one has following commutative diagram of surjective maps between sets

$$
\begin{array}{ccc}
\text{MF}_{\text{FF}} & \xrightarrow{} & \text{MF}_{\text{Higgs}} \\
\text{rep}_{\text{GL}_r(\mathbb{Z}_p)}^{\text{irr.crys}}(\Gamma) & \xrightarrow{\text{restriction}} & \text{rep}_{\text{GL}_r(\mathbb{Z}_p)}^{\text{irr.crys}}(\Gamma, \bar{\Gamma}) \\
\end{array}
$$

Since the two horizontal arrows and the left vertical arrow are surjective, the right vertical arrow is also surjective. One has a surjective composition

$$\text{MF}_{\text{dR}} \to \text{MF}_{\text{Higgs}} \to \text{rep}_{\text{GL}_r(\mathbb{Z}_p)}^{\text{irr.crys}}(\Gamma, \bar{\Gamma}).$$

To prove Theorem 1.2, we only need to show the finiteness of $\text{MF}_{\text{dR}}$.

Firstly, we claim that $\text{MF}_{\text{F-isoc}}^{\text{twist}}$ is finite. By Lemma 2.6 all elements in $\text{MF}_{\text{F-isoc}}$ are of absolutely irreducible. Then the set of equivalence classes of absolutely irreducible objects of $\text{F-isoc}^1(U_1)_{Q_p}$ up to twisting by a constant rank 1 $F$-isocrystal is finite by $[\text{Ked18}, \text{Corollary 2.1.5}]$.

Secondly, we claim that $\text{MF}_{\text{F-Cris}}^{\text{twist}}$ is finite. This follows from Lemma 2.4.

Finally, the map $\text{MF}_{\text{dR}} \to \text{MF}_{\text{dR}}$ is finite-to-one by Lemma 2.8 and Lemma 2.9. Thus $\text{MF}_{\text{dR}}$ is finite.

References

[Del87] Pierre Deligne. Un théorème de finitude pour la monodromie. Discrete groups in geometry and analysis. Pap. Hon. G. D. Mostow 60th Birthday, 1987.

[Fal89] Gerd Faltings. Crystalline cohomology and $p$-adic Galois-representations. In Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), pages 25–80. Johns Hopkins Univ. Press, Baltimore, MD, 1989.

[Fal05] Gerd Faltings. A $p$-adic Simpson correspondence. Adv. Math. 198(2):847–862, 2005.

[Ked07] Kiran S. Kedlaya. Semistable reduction for overconvergent $F$-isocrystals. I: Unipotence and logarithmic extensions. Compos. Math., 143(5):1164–1212, 2007.

[Ked18] Kiran S Kedlaya. Étale and crystalline companions I. arXiv:1811.02043v2, 2018.

[KYZ20a] Raju Krishnamoorthy, Jinhang Yang, and Kang Zuo. Deformations of periodic Higgs-de Rham flows. arXiv preprint arXiv:2005.00770, 2020.

[KYZ20b] Raju Krishnamoorthy, Jinhang Yang, and Kang Zuo. Finiteness of logarithmic crystalline representation. arXiv preprint arXiv:2005.13472, 2020.

[KYZ20c] Raju Krishnamoorthy, Jinhang Yang, and Kang Zuo. A Lefschetz theorem for crystalline representations. arXiv preprint arXiv:2003.08996v2, 2020.

[Lit18] Daniel Litt. Arithmetic representations of fundamental groups II: finiteness. arXiv preprint arXiv:1809.03521, 2018.

[LSZ19] Guihang Lan, Mao Sheng, and Kang Zuo. Semistable Higgs bundles, periodic Higgs bundles and representations of algebraic fundamental groups. J. Eur. Math. Soc. (JEMS), 21(10):3053–3112, 2019.

[Poo04] Bjorn Poonen. Bertini theorems over finite fields. Ann. of Math. (2), 160(3):1099–1127, 2004.

E-mail address: raju@uga.edu
Department of Mathematics, University of Georgia, Athens, GA 30605, USA

E-mail address: yjb@mail.ustc.edu.cn

Institut für Mathematik, Universität Mainz, Mainz 55099, Germany

E-mail address: zuok@uni-mainz.de

Institut für Mathematik, Universität Mainz, Mainz 55099, Germany