The Tautness Property of Homology Theory

Anzor Beridze and Leonard Mdzinarishvili

Department of Mathematics, Faculty of Exact Sciences and Education, Batumi Shota Rustaveli State University, 35, Ninoshvili St., Batumi, Georgia;
School of Mathematics, Kutaisi International University, Youth Avenue, 5th Lane, Kutaisi 4600, Georgia;
e-mail: a.beridze@bsu.edu.ge anzor.beridze@kiu.edu.ge

Department of Mathematics, Faculty of Informatics and Control Systems, Georgian Technical University, 77, Kostava St., Tbilisi, Georgia;
e-mail: l.mdzinarishvili@gtu.ge

Abstract

The tautness for a cohomology theory is formulated and studied by various authors. However, the analogous property is not considered for a homology theory. In this paper, we will define and study this very property for the Massey homology theory. Moreover, we will prove that the Kolmogoroff and the Massey homologies are isomorphic on the category of locally compact, paracompact spaces and proper maps. Therefore, we will obtain the same result for the Kolmogoroff homology theory.

Keywords: Functional Space, Finite Exact Sequence, Massey homology, Kolmogoroff Homology, Steenrod Homology, Milnor Homology

2000 MSC: 55N10

Introduction. Let $A$ be a closed subspace of a topological space $X$ and $\{U\}$ be a system of neighborhoods $U$ of $A$, directed by inclusion. Then for each cohomology theory $h^*$ there is a natural homomorphism

\[ i^* : \lim\to h^*(U) \rightarrow h^*(A). \]  

(1)

It is said that $A$ is tautly embedded in the space $X$ if the homomorphism $i^*$ is an isomorphism [Sp, §6.1]. The Alexander-Spanier cohomology on the category of paracompact Hausdorff spaces and continuous maps [Sp, Theorem 2 §6.6] and the Massey cohomology $H^*_c$ on the category of locally compact Hausdorff spaces and proper maps [Mas, Theorem 6.4, §6.4] are the examples of cohomologies for which any closed subspace $A$ is tautly embedded in $X$. It is natural to ask whether or not the analogous property fulfills for an exact homology theory as well. Therefore, the aim is to investigate a natural homomorphism

\[ i_* : h_*(A) \rightarrow \lim\to h_*(U) \]  

(2)

for a homology theory. In this paper, it is proved that for the Massey homology $\tilde{H}_*$, there exists an infinite exact sequence on the category of locally compact Hausdorff spaces $X$, which includes the homomorphism $i_*$. In particular, we have the following main theorem.

**Theorem 2.** The system $\{N\}$ of closed neighborhoods $N$ of closed subspace $A$ of a locally compact Hausdorff space $X$, directed by inclusion, induces the following exact sequence

\[ \cdots \rightarrow \lim\to \tilde{H}_{n+k+1}(N) \rightarrow \cdots \rightarrow \lim\to \tilde{H}_{n+2}(N) \rightarrow \lim\to \tilde{H}_{n+1}(N) \rightarrow \tilde{H}_n(A, G) \rightarrow \lim\to \tilde{H}_n(N) \rightarrow \lim\to \tilde{H}_{n+1}(N) \rightarrow \cdots \rightarrow \lim\to \tilde{H}_{n+k}(N) \rightarrow \cdots, \]

where $\tilde{H}_*(N) = \tilde{H}_*(N, G)$ is the Massey homology [Mas, §4.6] of closed neighborhood $N$ with coefficient in an abelian group $G$.

It is natural to study the same property for other exact homology theories [Kol], [St], [CH], [Mil]. Consequently, in the second part of the paper, it is proved that the Kolmogoroff [Kol, Mdz] and the Massey [Mas] homologies are isomorphic on the category of locally compact, paracompact spaces and proper maps.
isomorphic on the category of locally compact, paracompact spaces and proper maps. Using the obtained result, we will show that for the Kolmogoroff [Kol], the Milnor [Mil] and the Steenrod [St] homology theories the following properties hold:

**Corollary 5.** a) If $X$ is a locally compact, paracompact Hausdorff space, then for the system $\{N\}$ of closed neighborhoods $N$ of a closed subspace $A$ of $X$, there is an infinite exact sequence

\[
\cdots \longrightarrow \lim_{\leftarrow} (2k+1)^{\ast} \hat{H}_{n+k+1}(N) \longrightarrow \cdots \longrightarrow \lim_{\leftarrow} (3)^{\ast} \hat{H}_{n+2}(N) \longrightarrow \lim_{\leftarrow} (1)^{\ast} \hat{H}_{n+1}(N) \longrightarrow \hat{H}_n(A, G) \longrightarrow \cdots \]

where $\hat{H}_n(N) = \hat{H}_n(N, G)$ is the Kolmogoroff homology.

b) If $X$ is a compact Hausdorff space, then for the system $\{N\}$ of closed neighborhoods $N$ of a closed subspace $A$ of $X$, there is an infinite exact sequence

\[
\cdots \longrightarrow \lim_{\leftarrow} (2k+1)^{\ast} \bar{H}_{n+k+1}(N) \longrightarrow \cdots \longrightarrow \lim_{\leftarrow} (1)^{\ast} \bar{H}_{n+1}(N) \longrightarrow \bar{H}_n(A) \longrightarrow \cdots \]

where $\bar{H}_n(N) = \bar{H}_n(N, G)$ is the Milnor homology [Mil].

**Corollary 6.** a) If $X$ is a locally compact Hausdorff space with second countable axiom, then for each countable system $\{N_i\}$ of closed neighborhoods of a closed subspace $A$ of $X$ there is a short exact sequence

\[
0 \longrightarrow \lim_{\leftarrow} (1)^{\ast} \hat{H}_{n+1}(N_i) \longrightarrow \hat{H}_n(A, G) \longrightarrow \lim_{\leftarrow} \hat{H}_n(N_i) \longrightarrow 0,
\]

where $\hat{H}_n$ is the Massey homology [Mas].

b) If $X$ is a locally compact, paracompact Hausdorff space with second countable axiom, then for each countable system $\{N_i\}$ of closed neighborhoods of a closed subspace $A$ of $X$ there is a short exact sequence

\[
0 \longrightarrow \lim_{\leftarrow} (1)^{\ast} \bar{H}_{n+1}(N_i) \longrightarrow \bar{H}_n(A, G) \longrightarrow \lim_{\leftarrow} \bar{H}_n(N_i) \longrightarrow 0,
\]

where $\bar{H}_n$ is the Kolmogoroff homology [Kol].

c) If $X$ is a compact Hausdorff space with second countable axiom, then for each countable system $\{N_i\}$ of closed neighborhoods of a closed subspace $A$ of $X$ there is a short exact sequence

\[
0 \longrightarrow \lim_{\leftarrow} (1)^{\ast} \bar{H}_{n+1}(N_i) \longrightarrow \bar{H}_n(A, G) \longrightarrow \lim_{\leftarrow} \bar{H}_n(N_i) \longrightarrow 0,
\]

where $\bar{H}_n$ is the Milnor homology [Mil].

d) If $X$ is a compact metric space, then for each countable system $\{N_i\}$ of closed neighborhoods of closed subspace $A$ of $X$ there is a short exact sequence

\[
0 \longrightarrow \lim_{\leftarrow} (1)^{\ast} \bar{H}_{n+1}(N_i) \longrightarrow \bar{H}_n(A, G) \longrightarrow \lim_{\leftarrow} \bar{H}_n(N_i) \longrightarrow 0,
\]

where $\bar{H}_n$ is the Steenrod homology [St].

**1 Tautness.** In the paper [Mas, §1.1] W. Massey defined the cochain complex $C^\ast(X, G)$ for any locally compact Hausdorff spaces $X$ and any abelian group $G$. By Theorem 4.1 [Mas, §4.4], for each locally compact Hausdorff space $X$ and each integer $n$ the cochain group $C^n(X, \mathbb{Z})$ with integer coefficient is a free abelian group. The chain complex $C_\ast(X, G) = \text{Hom}(C^\ast(X, G), G)$ is completely defined by the cochain complex $C^\ast(X)$ with coefficient group $\mathbb{Z}$ of integers and therefore, by Theorem 4.1 [Mas, §4.4] and Theorem 4.1 (Universal Coefficients) [ML, §III.4], there is an exact sequence [Mas, Corollary 4.18, §4.8]
0 \rightarrow \text{Ext}(H^p_n(A), G) \rightarrow \check{H}_n(A, G) \rightarrow \text{Hom}(H^p_n(A), G) \rightarrow 0,
\quad (1)

where \( \check{H}_n(X, G) \) is the Massey homology group and \( H^p_{n+1}(X, G) \) is the Massey cohomology group, respectively [Mas, §4.6], i.e \( \check{H}_n(X, G) = H_n(\text{Hom}(C^*_c(X), G)) \) and \( H^p_{n+1}(X) = H_p(C^*_c(X, \mathbb{Z})) \). Moreover, this sequence is split. However, the splitting is only natural with respect to coefficient homomorphisms.

Let \( X \) be a locally compact space and \( A \) be a closed subspace of \( X \). In this case, for each closed neighborhood \( N \) of \( A \) there is a homomorphism \( i_N : h^p(N) \rightarrow h^p(A) \). If \( N_1 \subset N_2 \), then there is a homomorphism \( i_{N_1, N_2} : h^p(N_2) \rightarrow h^p(N_1) \). Therefore, there is the direct system \( \{h^p(N)\} \) of abelian groups and homomorphisms \( \{i_{N_1, N_2}\} \). Consequently, there exists a natural homomorphism

\[
i^n : \lim h^p(N) \rightarrow h^p(A).
\]

If \( h^p = H^p \) is the Massey cohomology [Mas, §4.6], then (see Theorem 6.4 [Mas, §6.4]) there is an isomorphism

\[
i^n : \lim H^p_n(N, G) \rightarrow H^p_n(A, G),
\quad (2)

In this case, a subspace \( A \) is said to be taut with respect to cohomology theory \( H^*(\cdot, G) \).

Let \( h_n \) be a homology theory on the category of some topological spaces. Let \( A \) be a closed subspace of \( X \). In this case, for a neighborhood \( N \) of \( A \) there is a homomorphism \( i_N : h_n(A) \rightarrow h_n(N) \). If \( N_1 \subset N_2 \), then there is a homomorphism \( i_{N_1, N_2} : h_n(N_1) \rightarrow h_n(N_2) \). Therefore, there is the inverse system \( \{h_n(N)\} \) of abelian groups and homomorphisms \( \{i_{N_1, N_2}\} \). Consequently, there exists a natural homomorphism

\[
i_n : h_n(A) \rightarrow \lim h_n(N).
\]

**Definition 1.** A closed subspace \( A \) of a space \( X \) is said to be tautly embedded in \( X \), if for some set \( N \) of neighborhoods there exists a long exact sequence

\[
\cdots \rightarrow \lim (2k+1)h_{n+k+1}(N) \rightarrow \cdots \rightarrow \lim (3)h_{n+2}(N) \rightarrow \lim (1)h_{n+1}(N) \rightarrow \\
\rightarrow h_n(A, G) \rightarrow \lim h_n(N) \rightarrow \lim (2)h_{n+1}(N) \rightarrow \cdots \rightarrow \lim (2k)h_{n+k}(N) \rightarrow \cdots
\]

which contains the homomorphism \( h_n(A) \rightarrow \lim h_n(N) \).

Let \( \check{H}_n(X, G) = H_n(\text{Hom}(C^*_c(X, G))) \) be the Massey homology group of locally compact Hausdorff spaces. Let \( A \) be a closed subspace of \( X \) and \( N \) be the set of all closed neighborhoods of \( A \). Then each homomorphism \( i_{N_1, N_2} : N_1 \rightarrow N_2 \) is a proper map (a map is proper if it is continuous and if inverse image of any compact subspace is compact) and induces a homomorphism \( i_{N_1, N_2} : \check{H}_n(N_1) \rightarrow \check{H}_n(N_2) \), which defines the homomorphism

\[
i_n : \check{H}_n(A, G) \rightarrow \lim \check{H}_n(N, G).
\]

Since the short exact sequence (1) is natural, there is a commutative diagram

\[
\begin{array}{cccccc}
0 & \rightarrow & \text{Ext}(H^p_{n+1}(A), G) & \rightarrow & \check{H}_n(A, G) & \rightarrow & \text{Hom}(H^p_n(A), G) & \rightarrow & 0 \\
& & \downarrow \rho' & & \downarrow i_n & & \downarrow \rho'' & & \\
0 & \rightarrow & \lim \text{Ext}(H^p_{n+1}(N), G) & \rightarrow & \lim \check{H}_n(N, G) & \rightarrow & \lim \text{Hom}(H^p_n(N), G) & \rightarrow & \\
& & & & \rightarrow \lim (1) \text{Ext}(H^p_{n+1}(N), G) & \rightarrow & \lim (1) \check{H}_n(N, G) & \rightarrow & \lim (1) \text{Hom}(H^p_n(N), G) & \rightarrow & \cdots
\end{array}
\quad (3)

with exact arrows.
Lemma 1 [Mdz]

where have a dual version. In particular,

Lemma 2 [Mdz]

there is a short exact sequence

Let

Theorem 1.

ff

Proof.

Using the split sequence (1) and commutative diagram (6), we obtain the following commutative diagram with

Therefore, a homomorphism \( \rho'' \) is an isomorphism.

Using the isomorphism (4) and the commutative diagram (3), we obtain the following commutative diagram

By Lemma 1 [Mdz], if a complex \( C_* \) is free, then there is an exact sequence

where \( B_{n-1} = \text{Im} \partial_n \), \( \partial : C_n \to C_{n-1} \) and \( Z^n = \ker \delta^{n+1} \), \( \delta^{n+1} : C^n \to C^{n+1} \), where \( C^* = \text{Hom}(C_*, G) \). In our case, we have a dual version. In particular, the cochain complex \( C^*_c(\cdot) \) is free and hence, there is an exact sequence

where \( Z_n = \ker \partial_n \), \( \partial_n : C_n \to C_{n-1} \) and \( B^{n+1}_c = \ker \delta^n \), \( \delta^n : C^c_n \to C^c_{n+1} \), where \( C_* = \text{Hom}(C^*_c, G) \). Consequently, using Lemma 2 [Mdz], for each \( A \in \mathbb{N} \) there is a commutative diagram with exact arrows

**Theorem 1.** Let \( \{C^n_c(N)\} \) be a direct system of free chain complexes \( C^n_c(N) \) of closed neighborhoods \( N \) of a closed subspace \( A \) of locally compact Hausdorff spaces \( X \) and \( G \) be an abelian group. In this case, for each \( n \in \mathbb{Z} \), and \( i \geq 1 \) there is a short exact sequence

which splits for \( i \geq 2 \).

**Proof.** Using the split sequence (1) and commutative diagram (6), we obtain the following commutative diagram with the exact arrows

\[
\cdots \xrightarrow{\text{lim}(i)} \text{Hom}(B^{n+1}_c(N), G) \xrightarrow{\text{lim}(i)} Z_n \xrightarrow{\text{lim}(i)} \text{Hom}(H^c_i(N), G) \xrightarrow{\text{lim}(i)} \cdots
\]

\[
\cdots \xrightarrow{\text{lim}(i)} \text{Ext}(H^{n+1}_c(N), G) \xrightarrow{\text{lim}(i)} \text{Hom}(H^c_i(N), G) \xrightarrow{\text{lim}(i)} \cdots
\]
In the paper [HM] it is shown that for each direct system \( \{A_n\} \) of abelian groups \( A_n \) there exists a short exact sequence

\[
0 \longrightarrow \lim_{\longrightarrow}^{(1)} \text{Hom}(A_n, G) \longrightarrow \text{Ext}(\lim_{\longrightarrow} A_n, G) \longrightarrow \lim_{\longleftarrow} \text{Ext}(A_n, G) \longrightarrow \lim_{\longleftarrow}^{(2)} \text{Hom}(A_n, G) \longrightarrow 0,
\]

and for each \( i \geq 1 \) there is an isomorphism

\[
\lim_{\longrightarrow}^{(i)} \text{Ext}(A_n, G) \cong \lim_{\longleftarrow}^{(i+2)} \text{Hom}(A_n, G).
\]

Consider a direct system \( \{B^{n+1}_c(N)\} \) of free groups \( B^{n+1}_c(N) \). In this case, by the exact sequence (9) and the isomorphism (10) we have

\[
\lim_{\longrightarrow}^{(i)} \text{Hom}(B^{n+1}_c(N), G) = 0 \quad \text{for} \quad i \geq 2.
\]

By the diagram (8) and the equality (11) we have

a) an isomorphism \( \lim_{\longrightarrow}^{(i)} Z_n \cong \lim_{\longrightarrow}^{(i)} \text{Hom}(H^n_c(N), G) \) for each \( i \geq 2 \);

b) an epimorphism \( \lim_{\longrightarrow}^{(i)} H^n_c(N, G) \twoheadrightarrow \lim_{\longrightarrow}^{(i)} \text{Hom}(H^n_c(N), G) \) for each \( i \geq 1 \);

c) a monomorphism \( \text{Ext}(H^n_c(N), G) \hookrightarrow \lim_{\longleftarrow}^{(i)} \text{Hom}(H^n_c(N), G) \) for each \( i \geq 2 \);

d) the trivial homomorphism \( \text{Hom}(H^n_c(N), G) \twoheadrightarrow \lim_{\longleftarrow}^{(i+1)} \text{Ext}(H^n_c(N), G) \) for each \( i \geq 1 \).

By b) and c) for each \( i \geq 2 \) we have a short exact sequence

\[
0 \longrightarrow \lim_{\longrightarrow}^{(i)} \text{Ext}(H^n_c(N), G) \longrightarrow \lim_{\longrightarrow}^{(i)} H^n_c(N, G) \longrightarrow \lim_{\longleftarrow}^{(i)} \text{Hom}(H^n_c(N), G) \longrightarrow 0.
\]

On the other hand, by a) for each \( i \geq 2 \) we can define a homomorphism

\[
\lim_{\longrightarrow}^{(i)} \text{Hom}(H^n_c(N), G) \twoheadrightarrow \lim_{\longrightarrow}^{(i)} Z_n \twoheadrightarrow \lim_{\longrightarrow}^{(i)} \text{Hom}(H^n_c(N), G).
\]

It is clear that the composition

\[
\lim_{\longrightarrow}^{(i)} \text{Hom}(H^n_c(N), G) \twoheadrightarrow \lim_{\longrightarrow}^{(i)} Z_n \twoheadrightarrow \lim_{\longrightarrow}^{(i)} H^n_c(N, G) \twoheadrightarrow \lim_{\longleftarrow}^{(i)} \text{Hom}(H^n_c(N), G)
\]

is the identity map. Therefore, for each \( i \geq 2 \) the sequence (12) splits.

**Theorem 2.** The system \( \{N\} \) of closed neighborhoods \( N \) of closed subspace \( A \) of a locally compact Hausdorff space \( X \), directed by inclusion, induces the following exact sequence

\[
\cdots \longrightarrow \lim_{\longrightarrow}^{(2k+1)} H^{n+k+1}_n(A) \longrightarrow \cdots \longrightarrow \lim_{\longrightarrow}^{(3)} H^{n+2}_n(A) \longrightarrow \lim_{\longrightarrow}^{(1)} H^{n+1}_n(A) \longrightarrow \lim_{\longrightarrow}^{(1)} H^{n+1}_n(N) \longrightarrow \lim_{\longrightarrow}^{(1)} H^{n+1}_n(N, G) \longrightarrow \lim_{\longrightarrow}^{(1)} \text{Hom}(H^n_c(N, G)) \longrightarrow \lim_{\longrightarrow}^{(1)} \text{Hom}(H^n_c(N, G)) \longrightarrow \lim_{\longrightarrow}^{(1)} \text{Hom}(H^n_c(N, G)) \longrightarrow \cdots
\]

where \( \hat{H}_n(A) = \hat{H}_n(-, G) \) is the Massey homology with a coefficient abelian group \( G \).

**Proof.** By the diagram (5) and the property d) from Theorem 1 we have the following exact sequence

\[
0 \longrightarrow \lim_{\longrightarrow} \text{Ext}(H^{n+1}_c(N), G) \longrightarrow \lim_{\longrightarrow} H^n(N, G) \longrightarrow \lim_{\longrightarrow} \text{Hom}(H^n_c(N), G) \longrightarrow \lim_{\longrightarrow}^{(1)} \text{Ext}(H^{n+1}_c(N), G) \longrightarrow \lim_{\longrightarrow}^{(1)} H^n(N, G) \longrightarrow \lim_{\longrightarrow}^{(1)} \text{Hom}(H^n_c(N), G) \longrightarrow 0.
\]

By the isomorphism (4) and exact commutative diagram (3), we will obtain the following exact sequence

\[
0 \longrightarrow \lim_{\longrightarrow}^{(1)} \text{Ext}(H^{n+1}_c(N), G) \longrightarrow \lim_{\longrightarrow}^{(1)} H^n(N, G) \longrightarrow \lim_{\longrightarrow}^{(1)} \text{Hom}(H^n_c(N, G)) \longrightarrow 0,
\]
where by Theorem \text{[1]} the sequence \text{[14]} splits for each \( i \geq 2 \).

Note that, by (5) and (9), there is an exact sequence

\[
0 \to \lim_{\to} \text{Hom}(H^{n+1}_c(N), G) \to \tilde{H}_n(A, G) \to \lim_{\to} \tilde{H}_n(N, G) \to \\
\lim_{\to} \text{Hom}(H^{n+1}_c(N), G) \to 0.
\] (15)

By the exact sequences \text{[7]}, \text{[14]} and \text{[15]} for \( i \geq 1 \), an isomorphism \text{[10]} and Theorem \text{[1]} we have the exact sequence

\[
\begin{array}{ccccccccc}
0 & \to & \lim_{\to} \text{Hom}(H^{n+1}_c(N), G) & \to & \tilde{H}_n(N) & \to & \lim_{\to} \text{Hom}(H^{n+2}_c(N), G) & \to & 0\\
& & & & & & & & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \to \\
0 & \to & \lim_{\to} \text{Ext}(H^{n+1}_c(N), G) & \to & \tilde{H}_{n+1}(N) & \to & \lim_{\to} \text{Ext}(H^{n+2}_c(N), G) & \to & 0\\
& & & & & & & & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \to \\
0 & \to & \lim_{\to} \text{Hom}(H^{n+1}_c(N), G) & \to & \tilde{H}_n(A, G) & \to & \lim_{\to} \tilde{H}_n(N, G) & \to & \lim_{\to} \text{Hom}(H^{n+1}_c(N), G) & \to & 0\\
& & & & & & & & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \to \\
0 & \to & \lim_{\to} \text{Ext}(H^{n+2}_c(N), G) & \to & \tilde{H}_{n+1}(N) & \to & \lim_{\to} \text{Ext}(H^{n+2}_c(N), G) & \to & 0\\
& & & & & & & & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \to \\
0 & \to & \lim_{\to} \text{Hom}(H^{n+2}_c(N), G) & \to & \tilde{H}_{n+2}(N) & \to & \lim_{\to} \text{Ext}(H^{n+3}_c(N), G) & \to & 0\\
& & & & & & & & \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \to \\
& & & & & & & & \\
& & & & & & & & \\
& & & & & & & & \\
\end{array}
\]

2 The Kolmogoroff homology. Our aim is to study the tautness property for other exact homology theories \text{[Kol], [St], [CH], [Ml]}. Among them, one of the main places is taken by the Kolmogoroff homology, which was defined as earlier as in 1936 \text{[Kol], [Mdz]}. A. N. Kolmogoroff defined homology on the category of locally compact Hausdorff spaces and proper maps with a compact coefficient group \text{[Kol], [Mdz]}. Using the homology defined by all finite partitions, in the paper \text{[CH]} G. S. Chogoshvili proved that Kolmogoroff homology and the Alexandroff-Čech homology groups are isomorphic on the category of compact Hausdorff spaces for a compact coefficient group \text{[Mdz]}. Since the Steenrod and the Alexandroff-Čech homologies are isomorphic on the category of compact metric spaces for a compact coefficient group \text{[St]}, we have the isomorphisms

\[
\begin{array}{ccccccccc}
\hat{H}_c(X, G) & \approx & \text{ch}_c(H_c(X, p, G)) & \approx & H_c(X, sp, G) & \approx & \check{H}_c(X, G) & \approx & \check{H}_c(X, G),
\end{array}
\] (16)

where \( \hat{H}_c(\cdot, G) \) is the Kolmogoroff \text{[Kol], [Mdz]}, \( \check{H}_c(\cdot, G) \) is the Chogoshvili projective \text{[CH], [Mdz]}, \( \text{ch}_c(\cdot, G) \) is the Chogoshvili spectral \text{[CH], [Mdz]}, \( \check{H}_c(\cdot, G) \) is the Alexandroff-Čech \text{[ES]} and \( \check{H}_c(\cdot, G) \) is the Steenrod \text{[St]}.\]
homology theories. Later the Kolmogoroff and the Chogoshvili homology theories were generalized and defined even for a discrete coefficient groups \([Mdz]\). However, there are no isomorphisms 2 and 4 as in \([16] [Mdz]\). Consequently, there was a natural interest to find the connection between the Kolmogoroff and Steenrod homology groups for any discrete groups. Using the Uniqueness Theorem given by Milnor \([Mil]\), in the paper \([Mdz]\) it is proved that on the category of compact metric spaces the Kolmogoro and the Steenrod homologies are isomorphic even for any discrete coefficient groups \([Mdz]\). Therefore, to study tautness properties for an exact homology theory, it is crucial to find a connection between the Kolmogoroff and the Massey homology theories.

By Theorem 2.8 \([Mas, §2.2]\), if \(X\) be a locally compact Hausdorff noncompact space and \(\breve{X}\) one point Alexander compactification, then the inclusion \(\mu : X \rightarrow \breve{X}\) induces an isomorphism

\[
\mu^* : H^n_c(X, G) \overset{\sim}{\longrightarrow} H^n_c(\breve{X}, *, G).
\]

**Corollary 1.** The inclusion \(\rho : X \rightarrow (\breve{X}, *)\), where \(\breve{X}\) is the one point Alexandroff compactification of locally compact Hausdorff space \(X\), indices an isomorphism

\[
\rho_* : \check{H}_*(X, G) \overset{\sim}{\longrightarrow} \check{H}_*(\breve{X}, *, G).
\]

**Proof.** The inclusion \(\rho : X \rightarrow (\breve{X}, *)\) induces a commutative diagram with the exact sequences

\[
\begin{array}{ccccccc}
0 & \rightarrow & \text{Ext}(H^{n+1}_c(X, G), G) & \rightarrow & H^n_c(X, G) & \rightarrow & \text{Hom}(H^n_c(X, G), G) & \rightarrow & 0 \\
\rho' & \downarrow & \rho' & \downarrow & \rho' & \downarrow & \rho' & \downarrow & \\
0 & \rightarrow & \text{Ext}(H^{n+1}_c(\breve{X}, *, G), G) & \rightarrow & H^n_c(\breve{X}, *, G) & \rightarrow & \text{Hom}(H^n_c(\breve{X}, *, G), G) & \rightarrow & 0.
\end{array}
\]

By the isomorphism \((17)\), the homomorphisms \(\rho'\) and \(\rho''\) are isomorphisms as well. Therefore, by the Lemma of Five Homomorphisms we obtain the required statement.

Now we will define the Kolmogoroff homology theory and using the isomorphism \((18)\), we will find the connection of it with the Massey homology theory.

Let \(X\) be a locally compact Hausdorff space. A subset \(A\) of space \(X\) is called bounded if \(\bar{A}\) is compact \([ES\ Definition 6.1, §X.6]\).

**Definition 2.** Let \(X\) be a locally compact Hausdorff space, \(E_X\) be the set of all bounded subsets \(E_i\) of \(X\), \(G\) be an abelian group. Denote by \(E^{n+1}_X = E_X \times E_X \times \cdots \times E_X\) - a direct product of \(E_X\). An \(n\)-dimension Kolmogoroff chain of the space \(X\) is called a function \(f_n : E^{n+1}_X \rightarrow G\) satisfying the following conditions:

- **K1)** If \(E_i = E'_i \cup E''_i\) and \(E'_i \cap E''_i = \emptyset\), then
  \[f_n(E_0, \ldots, E_i, \ldots, E_n) = f_n(E_0, \ldots, E'_i, \ldots, E_n) + f_n(E_0, \ldots, E''_i, \ldots, E_n);\]
  
- **K2)** \(f_n\) will not change by even permutation and will changes just the sign by odd permutation of argument; \(f_n = 0\), if two arguments are the same;

- **K3)** If \(\bar{E}_0 \cap \cdots \cap \bar{E}_n = \emptyset\), then \(f_n(E_0, \ldots, E_n) = 0\).

The sum \(f'_n + f''_n\) of two \(f'_n, f''_n\) functions is defined by the following equation

\[(f'_n + f''_n)(E_0, \ldots, E_n) = f'_n(E_0, \ldots, E_n) + f''_n(E_0, \ldots, E_n).\]

It is clear that the set of all \(n\)-dimensional functions \(f_n\) is an abelian group, which is denoted by \(K_n(X,G)\). The boundary operator \(\Delta : K_n(X,G) \rightarrow K_{n-1}(X,G)\) is defined by the equation

\[\Delta f_n(E_0, \ldots, E_{n-1}) = f_n(U, E_0, \ldots, E_{n-1}),\]

where \(U\) is an open bounded subset which includes \(\bigcup_{i=1}^{n-1} \bar{E}_i\). Since the space \(X\) is locally compact, such \(U\) exists and the boundary operator \(\Delta\) does not depend on the choice of \(U\).

The homology of the chain complex \(K_*(X,G) = (K_*(X,G), \Delta)\) is called the Kolmogoroff homology of a locally compact space \(X\) and it is denoted by \(\check{H}_*(X,G)\).
Definition 3. A locally finite system of bounded subspaces $e_i$ of space $X$, which are pairwise non-intersecting and the sum of them that gives the whole space $X = \cup e_i$ is called a regular partition.

Lemma 1. For each locally compact, paracompact space $X$ there exists a regular partition.

Proof. Since $X$ is a locally compact space, for each point $x \in X$ there exists a bounded neighborhood $U_x$. Since space $X$ is paracompact as well, an open covering $\{U_x\}_{x \in X}$ has a locally finite refinement $\{O_i\}$, which is contained with bounded subspaces $O_i$. If we write the elements of the covering $\{O_i\}$ as a transfinite sequence $O_1, O_2, \ldots, O_\lambda, \ldots$, then we construct a regular covering in the following way: $O_1, O_2 \setminus O_1, \ldots, O_\lambda \setminus \bigcup_{i < \lambda} O_i$, where $O_i$ runs through all the ordinal numbers preceding $\lambda$.

Denote by $S = \{S_\alpha\}$ the system of all regular partitions $S_\alpha$ of a space $X$.

Lemma 2. Each compact subspace $F$ of a locally compact space has a nonempty intersection only with finite number of closures $e_i^\alpha \in S_\alpha$.

Proof. Since $S_\alpha$ is a locally finite system, for each point $x \in F$ there exists neighborhood $U_x$, which has a nonempty intersection only with finite many elements $e_i^\alpha \in S_\alpha$. From the collection $\{U_x\}_{x \in F}$ of the neighborhoods a finite subsystem can be chosen, union of which covers the space $F$. Since for each open subspace $U$ and subspace $B$ there is an equivalence $U \cap B \neq \varnothing \iff U \cap \overline{B} \neq \varnothing$, we obtain truthfulness of the lemma.

Denote by $N_\alpha$ the nerve of a regular partition $S_\alpha \in S$, which consists of simplexes $\sigma_n = (e_i^0, \ldots, e_i^n)$, for which $\cap e_i^\alpha \neq \varnothing$. By the lemma the nerve $N_\alpha$ is locally finite [CH], [Mdz].

If $S_\alpha < S_\beta$, i.e. $S_\beta$ is refinement of $S_\alpha$ and if for each vertex $e_i^\beta \in N_\beta$ we take the uniquely defined vertex $e_i^\alpha \in N_\alpha$, which contains $e_i^\beta$, then we obtain a simplicial map $\pi_{\beta\alpha} : N_\beta \rightarrow N_\alpha$. By Lemma[2] the map $\pi_{\beta\alpha}$ will be locally finite [ES], i.e. image inverse of each simplex contains only finite many numbers of simplexes.

If we take for each $S_\alpha \in S$ the group of the infinite chains $C_n^\inf (N_\alpha, G)$ of nerve $N_\alpha$ and homomorphisms $\pi_{\beta\alpha}^*: C_n^\inf (N_\beta, G) \rightarrow C_n^\inf (N_\alpha, G)$, induced by simplicial maps $\pi_{\beta\alpha}$, then we obtain an inverse system $\{C_n^\inf (N_\alpha, G), \pi_{\beta\alpha}^*\}$, the inverse limit group of which is denoted by

$$C_n^\inf (X, G) = \lim_{\rightarrow} \{C_n^\inf (N_\alpha, G), \pi_{\beta\alpha}^*\}.$$

The boundary operator $\partial : C_n^\inf (X, G) \rightarrow C_{n-1}^\inf (X, G)$ is defined by the boundary operators $\partial_n : C_n^\inf (N_\alpha, G) \rightarrow C_{n-1}^\inf (N_\alpha, G)$, which commute with homomorphisms $\pi_{\beta\alpha}^*$. The homology group of the obtained complex $C^\inf (X, G)$ is called the Chogoshvili projection homology group and s denoted by $H_s(X, p, G)$.

Definition 4. Let $A = \{A_i\}$ and $B = \{B_j\}$ be finite systems of sets such that $B = \{B_j\}$ consists of pairwise non-intersecting sets. We will say that a system $B$ is a mosaic of the system $A$, if for each $B_j \in B$ there exists $A_i \in A$ such that $B_j \subset A_i$ and $A_i = \bigcup_j B_j$, where $B_j \in B$.

Lemma 3. For each finite system $A = \{A_i\}$, $i = 0, \ldots, n$ of sets $A_i$ there exists a mosaic.

Proof. The system consisting of subspaces $\bigcap_{i \neq 0} A_i, \bigcap_{i \neq 1} A_i \setminus A_i, \ldots, i_1, \ldots, i_p - p$ are different indexes from the system $i = 0, \ldots, n, 1 \leq p \leq n$ and $i_p$ obtains all value in the same system, except $i_1, \ldots, i_p$ which is a mosaic.

Lemma 4. If $f_n$ is a function on the directed system $e_0, \ldots, e_n$ mutually non-intersecting bounded subspaces $e_i$ of locally compact space $X$, which satisfies the conditions K1)–K3), then it can be extended to the function $f_n \in K_n(X, G)$.

Proof. By the lemma[3] for each directed system $E_0, \ldots, E_n$ of bounded subspaces $E_i$ there exists a mosaic $\{e_i\}$ such that $E_i = \cup e_i$. Therefore, the function $f_n(E_0, \ldots, E_n) = \sum_{0 \leq i \leq n} f_n(e_0, \ldots, e_i)$ does not depend on the choice of mosaic. Indeed, let $\{e_i'\}$ be another mosaic of the system $E_0, \ldots, E_n$ and $f_n'(E_0, \ldots, E_n) = \sum_{0 \leq i \leq n} f_n'(e_0, \ldots, e_i')$. It is clear that the intersection $\{e_i\} \cap \{e_i'\} = \{e_i''\}$ is a mosaic not only for $\{E_i\}$, but for each mosaic. Therefore, we have

$$f_n(E_0, \ldots, E_n) = \sum f_n(e_0, \ldots, e_n) = \sum f_n(e_0', \ldots, e_n') = \sum f_n(e_0', \ldots, e_n') = f_n'(e_0, \ldots, e_n).$$
Thus, the defined function $f_n$, satisfies the conditions K1–K3) and so $f_n \in K_n(X, G)$.

**Theorem 3.** Let $X$ be a locally compact, paracompact Hausdorff space. Then the Kolmogorov homology $\tilde{H}_*(X, G)$ is isomorphic to the Chogoshvili projection homology $H_*(X, p, G)$.

**Proof.** We will prove much stronger statement. In particular, there is an isomorphism of chain complexes $K_*(X, G)$ and $C_{\text{inf}}^*(X, G)$.

For each $S_\alpha \in S$ define a homomorphism $\xi_\alpha : K_*(X, G) \to C^\text{inf}_*(N_\alpha, G)$ by the formula $\xi_\alpha f_n(e_0^\alpha, \ldots, e_n^\alpha) = f_n(e_0^\alpha, \ldots, e_n^\alpha)$, where $f_n \in K_n(X, G), (e_0^\alpha, \ldots, e_n^\alpha) \in N_\alpha$. Therefore

$$\xi_\alpha \Delta f_n(e_0^\alpha, \ldots, e_n^\alpha-1) = \Delta f_n(e_0^\alpha, \ldots, e_n^\alpha-1) = f_n(U, e_0^\alpha, \ldots, e_n^\alpha-1).$$

By Lemma 2 and the property of uniqueness of a function $f_n$ (property K1), we have

$$f_n(U, e_0^\alpha, \ldots, e_n^\alpha-1) = f_n(\cup e_i^\alpha, e_0^\alpha, \ldots, e_n^\alpha-1) = \sum_{i} f_n(e_i^\alpha, e_0^\alpha, \ldots, e_n^\alpha-1).$$

Therefore,

$$\xi_\alpha \Delta f_n(e_0^\alpha, \ldots, e_n^\alpha-1) = \sum_{i} f_n(e_i^\alpha, e_0^\alpha, \ldots, e_n^\alpha-1) = \partial_n f_n(e_0^\alpha, \ldots, e_n^\alpha-1),$$

i.e. $\xi_\alpha \Delta = \partial_n \xi_\alpha$.

A homomorphism $\xi_\alpha$ induces an isomorphism

$$\xi : K_*(X, G) \to C^\text{inf}_*(X, G).$$

Let $c_\alpha = \{c_{\alpha, \beta} \in C^\text{inf}_n(X, G) \text{ and } E_0, \ldots, E_\alpha \text{ be a system of mutually non-intersecting bounded subspaces.}$ If we add to this system the subspace $X \setminus \cup_{i} E_i$, then we obtain a finite partition $D$ of space $X$. Let $S_\alpha \in S$, then $D \setminus S_\alpha = S_\alpha' \in S$ and for each $E_i = \cup e_i^\alpha$, where $e_i^\alpha \in S_\alpha'$.

Let $\tilde{f}_n$ be a function on the system $E_0, \ldots, E_\alpha$ which is defined by

$$\tilde{f}_n(E_0, \ldots, E_\alpha) = \sum_{0_j, \ldots, n_j} c_{\alpha, \beta}(e_0^\alpha, \ldots, e_n^\alpha),$$

where $0_j, \ldots, n_j$ obtain all values, where $(e_0^\alpha, \ldots, e_n^\alpha)$ denotes a simplex in $N_\alpha'$. It is easy to show that such defined function $\tilde{f}_n$ does not depend on the choice of $S_\alpha$ (and it satisfies the properties K1–K3). By Lemma 2 a function $\tilde{f}_n$ can be extended to a function $f_n \in K_n(X, G)$. If we define a homomorphism

$$\eta : C^\text{inf}_*(X, G) \to K_*(X, G)$$

by $\eta(c_\alpha) = f_n$, then it will be inverse of the homomorphism $\xi$.

**Theorem 4.** Let $\{G_\alpha, p_{\beta \alpha}\}_{\alpha \in \Lambda}$ be a direct system of free abelian groups $G_\alpha$, which satisfies the following conditions:

1) For each group $G_\alpha$ there exists a base $B = \{g_{\alpha, 1}, g_{\alpha, 2}, \ldots, g_{\alpha, n}\}$;

2) For each pair $\alpha < \beta$, $\alpha, \beta \in \Lambda$, a set of indexes $\{1, 2, \ldots, \tau(\beta), \ldots\}$ of elements of base $B_\beta$ can be decomposed with non-intersecting finite subspaces $I_1^\beta, I_2^\beta, \ldots, I_{\tau(\alpha)}^\beta$ such that

$$p_{\beta \alpha}(g_{\alpha, i}^\alpha) = \begin{cases} 1, & \text{if } I_i^\beta \neq \emptyset, \\ 0, & \text{if } I_i^\beta = \emptyset, \end{cases}$$

for $i = 1, 2, \ldots, \tau(\alpha)$.

Then the limit of the direct system $\{G_\alpha, p_{\beta \alpha}\}_{\alpha \in \Lambda}$ is a free group.
Proof. Denote by \( \overline{B}_a \) the set of all finite subspaces \( \alpha_t \) of a base \( B_a \) of a group \( G_a \) and by \( G^a_\alpha \) a subgroup of group \( G_a \), generated by all elements \( \alpha_t \in \overline{B}_a \). It is possible to prove that such a group \( G^a_\alpha \) is the direct limit group of the direct system of subgroups \( G^\alpha_\beta \).

Let \( \Lambda \) be a set \( \{(\alpha, t) | \alpha_t \in \overline{B}_a \} \). It is considered that \( (\alpha', t') > (\alpha, t) \), if \( \alpha' > \alpha \) and \( p_{\alpha' \alpha} G^\alpha_\alpha < G^\alpha_\alpha \). It is clear that \( \Lambda \) is a directed set and if we take \( G^\alpha_\alpha = G^\alpha_\alpha \) for each pair \( (\alpha, t) \in \Lambda \), we obtain a direct system \( \{G^\alpha_\alpha, p_{\alpha' \alpha}\} \) which satisfies the condition of Theorem 3 \[KK]\. Therefore, the direct limit of the given system is a free abelian group.

Let \( G^a_\infty = \lim \ G^a_\alpha \) and \( G^\alpha_\infty = \lim \ G^\alpha_\alpha \). Define a homomorphism \( \varphi : G^a_\infty \rightarrow G^\alpha_\infty \). Since \( (\alpha', t') > (\alpha, t) \) we have the following commutative diagram

\[
\begin{array}{ccc}
G^a_\alpha & \xrightarrow{p_{\alpha' \alpha}} & G^\alpha_\alpha \\
\downarrow{p_{\alpha' \alpha}} & & \downarrow{p_{\alpha' \alpha}} \\
G^\alpha_\alpha & \xrightarrow{p_{\alpha' \alpha}} & G^\alpha_\alpha
\end{array}
\]

A homomorphism \( \varphi \) is induced by \( \varphi_{\alpha, t} = p_{\alpha' \alpha} p_{\alpha', \alpha} \).

a) \( \varphi \) is an epimorphism. Let \( x \in G^\alpha_\infty \) and \( x_{\alpha, t} \in G^\alpha_\alpha \) be their representatives. Since \( G^\alpha_\alpha = \lim \ G^\alpha_\alpha \), there is a representative \( x_{\alpha, t} \in G^\alpha_\alpha \) of an element \( x_{\alpha, t} \). It is clear that a class \( x^* \in G^\alpha_\infty \), a representative of which is \( x_{\alpha, t} \), satisfies the properties \( \varphi(x^*) = x \).

b) \( \varphi \) is a monomorphism. Let \( \varphi(x^*) = 0 \). Since \( p_{\alpha' \alpha} p_{\alpha', \alpha}(x_{\alpha, t}) = 0 \), where \( x_{\alpha, t} \) is a representative of an element \( x^* \), there is such \( \beta > \alpha \), that \( p_{\alpha' \alpha}(p_{\alpha', \alpha}(x_{\alpha, t})) = 0 \). Let \( G^\beta_\alpha \) be the subgroup of a group \( G^\beta \), which is generated by all \( g^\beta_{\alpha} \in B^\beta \) such that \( p_{\alpha' \alpha}(g^\beta_{\alpha}) = \sum g^\beta_{\alpha} \), when \( g^\beta_{\alpha} \) runs through the base \( G^\alpha_\alpha \). Since \( \rho_{\alpha' \alpha} p_{\alpha' \alpha}(x_{\alpha, t}) = p_{\alpha' \alpha} p_{\alpha', \alpha}(x_{\alpha, t}) = 0 \) and \( \rho_{\alpha' \alpha} \) are monomorphisms, \( p_{\alpha' \alpha}(x_{\alpha, t}) = 0 \) and so \( x^* = 0 \).

\[\Box\]

Remark 1. Theorem 4 is a generalization of Theorem 3 \[KK\], which is proved in the case when the base \( B_a \) is a finite set.

Theorem 5. Let \( X \) be a locally compact, paracompact Hausdorff space, then there exists the universal coefficient formula for the Kolmogoroff homology group:

\[ 0 \rightarrow \text{Ext}(\mathcal{H}^{n+1}(\hat{X}, *), G) \rightarrow \hat{H}^n(X, G) \rightarrow \text{Hom}(\mathcal{H}^n(\hat{X}, *), G) \rightarrow 0. \]

Proof. It is easy to see that the direct system \( \{C^\alpha_\alpha(N_a), \pi^\alpha_{\beta}G^\alpha_\alpha\} \) of groups \( C^\alpha_\alpha(N_a) \) of cochains with integer coefficient group of nerves \( N_a \), where \( S_a \in S \), satisfies the condition of Theorem 4. Therefore, the direct limit \( C^\alpha_\alpha(X) = \lim \ (C^\alpha_\alpha(N_a), \pi^\alpha_{\beta}G^\alpha_\alpha) \) is a free group. By Theorem 4.1 \[ML\, §III.4\], for the homology group \( H_n(\text{Hom}(C^\alpha_\alpha(X), G)) \) there exists the Universal Coefficient Formula:

\[ 0 \rightarrow \text{Ext}(H^{n+1}_J(X, G) \rightarrow H_n(\text{Hom}(C^\alpha_\alpha(X), G)) \rightarrow \text{Hom}(H^n_J(X, G)), G) \rightarrow 0. \]  

Since \( \text{Hom}(C^\alpha_\alpha(X), G) \approx C^\alpha_\alpha_j(X, G) \), by an isomorphism \[19\] and Theorem 3 there exists an isomorphism

\[ H_n(\text{Hom}(C^\alpha_\alpha(X), G)) \approx H^\alpha_n(X, G) \approx \hat{H}^\alpha_n(X, G). \]  

On the other hand, by Theorem 2.1.1 \[CH\] and Theorem 6.9 \[ES\, §X.6\] we obtain isomorphisms

\[ H^\alpha_n(X, G) \approx H^\alpha_n(X, G) \approx \hat{H}^\alpha_n(X, *), G), \]

where \( H^\alpha_n \) is the Alexandroff homology with proper subcomplexes. Using the exact sequence \[20\], by the isomorphisms \[21\] and \[22\] we obtain the required statement.

\[\Box\]

Corollary 2. An inclusion \( \rho : X \rightarrow (\hat{X}, *) \), where \( \hat{X} \) is the one-point Alexandroff compactification of locally compact, paracompact Hausdorff space \( X \), induces an isomorphism

\[ \hat{H}_n(X, G) \approx \hat{H}_n(\hat{X}, *), G). \]
Corollary 3. Since the Kolmogorov and the Massey homology theories satisfy the condition of uniqueness, in particular the Universal Coefficient Formula \cite{Be}, Theorem 4.4, \cite{BM}, Theorem 1.5, they are isomorphisms on the category of compact spaces.

Corollary 4. By the corollaries \[1 \text{ and } 2 \] the Kolmogorov and the Massey homologies of locally compact, paracompact Hausdorff spaces are isomorphic the Kolmogorov and the Massey homologies of compact space, which is the one-point Alexadroff compactification of the given space. Therefore, by corollary \[3 \] there is an isomorphism

\[ \tilde{H}_\ast(X, G) \cong \hat{H}_\ast(X, G) \]

on the category of locally compact, paracompact Hausdorff spaces and proper maps.

Corollary 5. a) If \( X \) is a locally compact, paracompact Hausdorff space, then for the system \( \{N\} \) of closed neighborhoods \( N \) of a closed subspace \( A \) of \( X \), there is an infinite exact sequence

\[ \cdots \rightarrow \lim_{\rightarrow} (2k+1) \tilde{H}_{n+k+1}(N) \rightarrow \cdots \rightarrow \lim_{\rightarrow} (3) \tilde{H}_{n+2}(N) \rightarrow \lim_{\rightarrow} (1) \tilde{H}_{n+1}(N) \rightarrow \tilde{H}_n(A, G) \rightarrow \cdots \]

where \( \tilde{H}_\ast(N) = \tilde{H}_\ast(N, G) \) is the Kolmogorov homology.

b) If \( X \) is a compact Hausdorff space, then for the system \( \{N\} \) of closed neighborhoods \( N \) of a closed subspace \( A \) of \( X \), there is an infinite exact sequence

\[ \cdots \rightarrow \lim_{\rightarrow} (2k+1) \hat{H}_{n+k+1}(N) \rightarrow \cdots \rightarrow \lim_{\rightarrow} (1) \hat{H}_{n+1}(N) \rightarrow \hat{H}_n(A) \rightarrow \cdots \]

where \( \hat{H}_\ast(N) = \hat{H}_\ast(N, G) \) is the Milnor homology \cite{Mi}.

As it is known \cite{HM}, for each countable inverse system \( \{A_i\} \) of abelian groups \( A_k \) there is \( \lim_{\rightarrow} (1) \{A_k\} = 0 \) for \( i \geq 2 \).

By this fact and Theorem \[2 \] and Corollary \[5 \] we have

Corollary 6. a) If \( X \) is a locally compact Hausdorff space with second countable axiom, then for each countable system \( \{N_i\} \) of closed neighborhoods of a closed subspace \( A \) of \( X \) there is a short exact sequence

\[ 0 \rightarrow \lim_{\rightarrow} (1) \hat{H}_{n+1}(N_i) \rightarrow \hat{H}_n(A, G) \rightarrow \tilde{H}_n(N_i) \rightarrow 0, \]

where \( \hat{H}_\ast \) is the Massey homology \cite{Mas}.

b) If \( X \) is a locally compact, paracompact Hausdorff space with second countable axiom, then for each countable system \( \{N_i\} \) of closed neighborhoods of a closed subspace \( A \) of \( X \) there is a short exact sequence

\[ 0 \rightarrow \lim_{\rightarrow} (1) \tilde{H}_{n+1}(N_i) \rightarrow \tilde{H}_n(A, G) \rightarrow \hat{H}_n(N_i) \rightarrow 0, \]

where \( \tilde{H}_\ast \) is the Kolmogorov homology \cite{Kol}.

c) If \( X \) is a compact Hausdorff space with second countable axiom, then for each countable system \( \{N_i\} \) of closed neighborhoods of a closed subspace \( A \) of \( X \) there is a short exact sequence

\[ 0 \rightarrow \lim_{\rightarrow} (1) \hat{H}_{n+1}(N_i) \rightarrow \hat{H}_n(A, G) \rightarrow \tilde{H}_n(N_i) \rightarrow 0, \]

where \( \hat{H}_\ast \) is the Milnor homology \cite{Mi}.

d) If \( X \) is a compact metric space, then for each countable system \( \{N_i\} \) of closed neighborhoods of closed subspace \( A \) of \( X \) there is a short exact sequence

\[ 0 \rightarrow \lim_{\rightarrow} (1) \hat{H}_{n+1}(N_i) \rightarrow \hat{H}_n(A, G) \rightarrow \tilde{H}_n(N_i) \rightarrow 0, \]

where \( \hat{H}_\ast \) is the Steenrod homology \cite{St}.
References

[BM] A. Beridze, L. Mdzinarishvili, On the axiomatic systems of Steenrod homology theory of compact spaces. Topology Appl. 249 (2018), 73–82

[Ber] Berikashvili N., Axiomatics of the Steenrod-Sitnikov homology theory on the category of compact Hausdorff spaces. (Russian) Topology (Moscow, 1979). Trudy Mat. Inst. Steklov. 154 (1983), 24–37.

[CH] Chogoshvili G., On the equivalence of the functional and spectral theory of homology. (Russian) Izvestiya Akad. Nauk SSSR. Ser. Mat. 15, (1951). 421–438.

[ES] Eilenberg S., Steenrod N., Foundations of algebraic topology. Princeton, New Jersey: Princeton University Press, 1952.

[Gor] Gordon W. L., Locally-finitely-valued cohomology groups. Proc. Amer. Math. Soc. 1955. V. 6. P. 656–662.

[HM] Huber M., Meier W., Cohomology theories and infinite CW-complexes. Comment. Math. Helv. 1978. V. 53, no. 2. P. 239–257.

[KK] Kaup L., Keane M. S., Induktive Limiten endlich erzeugter freier Moduln. Manuscripta Math. 1969. V. 1. P. 9–21.

[Kol] Kolmogoroff A. N., Les groupes de Betti des espaces localement bicompacts. C. R. Acad. Sci., Paris. 1936. V. 202. P. 1144–1147; Propriétés des groupes de Betti des espaces localement bicompacts. ibid. 1936. V. 202. P. 1325–1327; Cycles relatifs. Théorèmes de dualité de M. Alexander, ibid. 1936. V. 202. P. 1641–1643.

[ML] Mac Lane S., Homology. Die Grundlehren der mathematischen Wissenschaften, Bd. 114. New York: Academic Press, Inc., Publishers; Berlin-Göttingen-Heidelberg: Springer-Verlag, 1963.

[Mas] Massey W. S., Homology and Cohomology Theory. An Approach Based on Alexander-Spanier Cochains. Monographs and Textbooks in Pure and Applied Mathematics, Vol. 46. New York-Basel: Marcel Dekker, Inc., 1978.

[Mdz1] Mdzinarishvili L., The relation between the homology theories of Kolmogorov and Steenrod. (Russian) Dokl. Akad. Nauk SSSR 203 (1972), 528–531

[Mdz2] Mdzinarishvili L., L. D. On exact homology. Geometric topology and shape theory (Dubrovnik, 1986), 164–182, Lecture Notes in Math., 1283, Springer, Berlin, 1987

[Mdz3] Mdzinarishvili L., The uniqueness theorem for cohomologies on the category of polyhedral pairs, Trans. A. Razmadze Math. Inst. 2018. V. 172, no. 2. P. 265–275.

[Mil] J. Milnor, On the Steenrod homology theory, Mimeographed Note, Princeton, 1960, in: Novikov Conjectures, Index Theorems and Rigidity, vol. 1, in: Lond. Math. Soc. Lect. Note Ser., vol. 226, Oberwolfach, 1993, pp. 79–96.

[Sp] Spanier E. H., Algebraic Topology. Corrected reprint of the 1966 original. New York: Springer-Verlag, 1966.

[St] Steenrod N. E., Regular cycles of compact metric spaces, Ann. of Math. (2). 1940. V. 41. P. 833–851.