ESSENTIAL COMMUTATIVE CARTAN SUBALGEBRAS OF $C^*$-ALGEBRAS

JONATHAN TAYLOR

Abstract. We define essential commutative Cartan pairs of $C^*$-algebras generalising the definition of Renault [19] and show that such pairs are given by essential twisted groupoid $C^*$-algebras as defined by Kwaśniewski and Meyer [11]. We show that the underlying twisted groupoid is effective, and is unique up to isomorphism among twists over effective groupoids giving rise to the essential commutative Cartan pair. We also show that for twists over effective groupoids giving rise to such pairs, the automorphism group of the twist is isomorphic to the automorphism group of the induced essential Cartan pair via explicit constructions.

1. Introduction

The class of $C^*$-algebras arising from groupoid constructions contains many important examples, including all commutative $C^*$-algebras and all group $C^*$-algebras. Renault [19] defined a Cartan subalgebra of a $C^*$-algebra $B$ as a regular, non-degenerate, maximal abelian subalgebra with a faithful conditional expectation. When $B$ is separable Renault showed that $B$ is isomorphic to a twisted groupoid $C^*$-algebra of a twist over a second countable étale effective locally compact Hausdorff groupoid, via an isomorphism mapping the Cartan subalgebra to the algebra of continuous functions on the unit space of the groupoid. Moreover, this groupoid and twist are unique up to isomorphism among such groupoids.

The construction of this Weyl twist led to an inductive limit construction by Li in [14]. Li showed that every Elliott-classifiable $C^*$-algebra has a Cartan subalgebra in the sense of Renault, and so every classifiable $C^*$-algebra is a twisted groupoid $C^*$-algebra for a twist over a second countable étale effective locally compact Hausdorff groupoid. This deeply intertwines the research of classification to the field of groupoids and their algebras, and allows for more geometric approaches to classification problems.

After Renault gave his definition of commutative Cartan pairs, Exel [2] generalised the definition to consider a non-commutative analogue. In place of $A$ being a maximal abelian subalgebra, Exel considers subalgebras with the property that every bounded $A$-bimodule map (or virtual commutant) from an ideal $I$ in $A$ taking values in $B$ has range contained in $A$. The main result of [2] states that each separable noncommutative Cartan pair is isomorphic to a pair coming from a Fell bundle over an inverse semigroup. Kwaśniewski and Meyer [12] were able to remove the assumption of separability from Exel’s results, and further showed that noncommutative Cartan pairs are given by crossed products of the subalgebra by an inverse semigroup action. Independently, Raad [18] was also able to remove the separability assumption for commutative Cartan subalgebras. Kwaśniewski and Meyer were also able to show that all slices in one such Cartan pair consist of a trivial part intersecting the subalgebra, and the orthogonal complement which is purely outer. They also showed that the corresponding dual groupoid to the action is unique up to isomorphism for any choice of inverse subsemigroup of slices, hence allowing one to
reconstruct any slice for a noncommutative Cartan inclusion from a spanning family of slices.

The generalisation to consider $C^*$-inclusions with local multiplier algebra valued conditional expectations arose in [11] for defining the essential crossed product of inverse semigroup actions. For such crossed products, there is always a canonical faithful conditional expectation taking values in the local multiplier algebra of subalgebra. Kwaśniewski and Meyer also introduce the notion of aperiodic inclusions and aperiodic actions, and prove a number of results for such inclusions in [10], [11], and [13]. In particular, they define the essential $C^*$-algebra for a Fell bundle over an étale groupoid, and are able to characterise the ideal of the reduced $C^*$-algebra that gives rise to the essential groupoid $C^*$-algebra quotient. We show that aperiodicity is automatically satisfied for masa inclusions, and so we may utilise these results without loss of generality.

The results in this article mirror some noncommutative counterparts explored by the author in [20]. The proof techniques given here however rely heavily on commutativity of the subalgebra $A$, but in turn we are able to attain stronger results. With these techniques, the proofs and results should be more accessible to readers better acquainted with the commutative setting, and there is no reliance on their noncommutative counterparts.

In place of a crossed product by an inverse semigroup action, we are able to classify essential commutative Cartan pairs as essential twisted groupoid $C^*$-algebras as defined by Kwaśniewski and Meyer in [11]. Using the same groupoid construction as Renault [19], we show that the Weyl groupoid and Weyl twist associated to an essential commutative Cartan pair is unique up to canonical isomorphism among twists over effective étale groupoids with locally compact Hausdorff unit space that give rise to the original essential Cartan pair.

We also show that if a twist over an étale groupoid with locally compact Hausdorff unit space gives rise to an essential Cartan pair, then associated Weyl twist is a twist over the effective quotient of this groupoid. Hence we show that twists over such groupoids descend to their effective quotients. Moreover, we show that any twist over an effective étale groupoid with locally compact Hausdorff unit space gives rise to an essential Cartan pair.

Lastly, we provide constructions of automorphisms of twisted groupoids from automorphisms of the induced essential Cartan pairs and vice versa. Moreover, if the twist is over an effective groupoid, these constructions are mutually inverse and the resulting respective automorphism groups are isomorphic.

Acknowledgements. The author would like to thank his doctoral advisor Ralf Meyer for all the assistance and expertise he provided. The article consists of results from the author’s PhD thesis [21].

2. $C^*$-algebras of twists over groupoids

Let $G$ be an étale groupoid. A twist over $G$ is a central extension $\Sigma$ of $G$ by $G^{(0)} \times T$; the unit space times the circle group

$$G^{(0)} \times T \hookrightarrow \Sigma \twoheadrightarrow G.$$  

By definition, $\Sigma$ carries a central action of $T$. From this one may construct a canonical line bundle associated to the twist $(G, \Sigma)$ as $L := \Sigma \times \mathbb{C}$, where the quotient is by the action $z(\sigma, \lambda) = (z\sigma, \overline{z}\lambda)$ for $z \in T$, $\sigma \in \Sigma$, and $\lambda \in \mathbb{C}$. Sections of this bundle are then equivalent to functions $\Sigma \to \mathbb{C}$ satisfying $f(z\sigma) = \overline{z}f(\sigma)$, and we may often identify the two.

Renault defines the Weyl groupoid and Weyl twist in [19] for commutative non-degenerate inclusions of $C^*$-algebras. For our purposes we need not alter the construction of the Weyl groupoid and twist to accommodate for our more general definition of essential Cartan pairs.
Definition 2.1 ([9] 1.1]). Let $A \subseteq B$ be an inclusion of $C^*$-algebras. A normaliser of $A$ in $B$ is an element $n \in B$ satisfying $n^*A_n, An^* \subseteq A$. We call the inclusion $A \subseteq B$ regular if the collection $N(A, B) := \{n \in B : n$ is a normaliser of $A \subseteq B\}$ of normalisers of $A$ inside of $B$ spans a dense subspace of $B$.

We say an inclusion $A \subseteq B$ of $C^*$-algebras is non-degenerate if $A$ contains an approximate unit for $B$. It follows that $n^*n$ belongs to $A$ for any normaliser $n \in N(A, B)$ (see, for example, [19] Lemma 4.5). If $A = C_0(X)$ is a commutative non-degenerate $C^*$-subalgebra of $B$, then (following Renault [19]) for any normaliser $n \in N(A, B)$ we can define $\text{dom}(n) := \{x \in X : n^*nx > 0\}$ and $\text{ran}(n) := \{x \in X : nn^*(x) > 0\}$.

Kumjian [9] describes how normalisers of an inclusion $A \subseteq B$ of a commutative $C^*$-algebra implement partial homeomorphisms on the Gelfand dual $X$ of $A = C_0(X)$.

Proposition 2.2 ([9] 1.6]). Let $A = C_0(X)$ be a commutative $C^*$-subalgebra of $B$ containing an approximate unit for $B$. Let $n \in N(A, B)$ be a normaliser. There exists a unique homeomorphism $\alpha_n : \text{dom}(n) \to \text{ran}(n)$ satisfying

\[
n^*an(x) = a(\alpha_n(x))n^*(x),
\]

for all $x \in \text{dom}(n)$ and $a \in A$.

Lemma 2.3 ([9] 1.7]). Let $A$ be a commutative $C^*$-subalgebra of $B$ containing an approximate unit for $B$. Then

(1) if $a \in A$ then $\alpha_n = \text{id}_{\text{dom}(n)}$;

(2) for $m, n \in N(A, B)$ we have $\alpha_{mn} = \alpha_m \circ \alpha_n$ (wherever the composition is defined) and $\alpha_n^{-1} = \alpha_n^*$.

Let $W \subseteq N(A, B)$ be a closed linear subspace. We say $W$ is a slice for the inclusion $A \subseteq B$ if $AW + WA \subseteq W$ and $W^*W + WW^* \subseteq A$. We observe that $W$ inherits a Hilbert $A$-bimodule structure from the ambient $C^*$-algebra $B$. Any two normalisers belonging to the same slice must implement the same partial homeomorphism of $X$ on the intersection of their domains.

Corollary 2.4. Let $A$ be a commutative $C^*$-subalgebra of $B$ containing an approximate unit for $B$. Let $W \subseteq B$ be a slice for the inclusion. For $m, n \in W$, let $U := \text{dom}(m) \cap \text{dom}(n)$. Then $\alpha_m|_U = \alpha_n|_U$

Proof. Since $W$ is a slice, the element $m^*n$ belongs to $A$, and so the associated partial homeomorphism $\alpha_{m,n}$ is the identity map on $\text{dom}(m^*n)$ by Lemma 2.3. The same lemma implies that $\text{id}_{\text{dom}(m^*n)} = \alpha_{m,n} = \alpha_m^{-1} \circ \alpha_n$, whereby $\text{id}_{\text{dom}(m^*n)}$ and $\alpha_m^{-1} \circ \alpha_n$ have the same domain, namely $U$. Thus, on $U$ we have

\[
\alpha_m|_U = \alpha_m|_U \circ \text{id}_{\text{dom}(m^*n)} = \alpha_m|_U \circ \alpha_m^{-1} \circ \alpha_n|_U = \alpha_n|_U.
\]

The collection of partial homeomorphism $\mathcal{G}(A) := \{\alpha_n : n \in N(A, B)\}$ forms a pseudogroup acting on $X = \hat{A}$.

Definition 2.5 ([19] Definition 4.2]). We call $\mathcal{G}(A)$ the Weyl pseudogroup of the pair $(A, B)$. Define the Weyl groupoid $\mathcal{G}(A, B)$ of $(A, B)$ as the groupoid of germs of $\mathcal{G}(A)$ (see [19] Section 3]).

Concretely, the Weyl groupoid is defined as the quotient of $D = \{(\alpha_n, x) : n \in N(A, B), x \in \text{dom}(n)\}$ by the equivalence relation

\[
(\alpha_n, x) \sim (\alpha_m, y) \iff x = y \text{ and there exists open } U \subseteq X \text{ with } x \in U \text{ such that } \alpha_n|_U = \alpha_m|_U.
\]
The topology on $G(A,B)$ is given by basic open sets of the form $U_n := \{[\alpha_n, x] : x \in \text{dom}(n)\}$. The composition of two equivalence classes $[\alpha_n, x]$ and $[\alpha_m, y]$ in $G(A,B)$ is defined if $x = \alpha_m(y)$, and the product is given by $[\alpha_n, x] \cdot [\alpha_m, y] = [\alpha_n \circ \alpha_m, y]$, which by Lemma 2.3 is equal to $[\alpha_{nm}, y]$. The inverse is given by $[\alpha_n, x]^{-1} = [\alpha_{n^{-1}}, \alpha_n(x)]$.

Renault also defines the Weyl twist $\Sigma(A, B)$ by considering pairs $(n, x)$ where $n \in N(A, B)$ and $x \in \text{dom}(n)$ (see [19, Section 4]). Two pairs $(n, x)$ and $(m, y)$ are equivalent in the twist if $x = y$ and there are functions $a, b \in C_0(X) = A$ with $a(x), b(x) > 0$ and $an = bm$. The product of equivalence classes $[n, x]$ and $[m, y]$ in the twist is defined whenever the product $[\alpha_n, x]$ and $[\alpha_m, y]$ is defined in the Weyl groupoid, and is given by $[n, x] \cdot [m, y] = [mn, y]$. Similarly to the Weyl groupoid, the inverse of $[n, x]$ in the Weyl twist is $[n^*, \alpha_n(x)]$. The canonical surjection $\Sigma(A, B) \to G(A, B)$ is given by $[n, x] \mapsto [\alpha_n, x]$, which Renault shows to indeed be a twist over $G(A, B)$ if $A \subseteq B$ is a maximal abelian and non-degenerate inclusion (cf. [19, Proposition 4.12]).

Recall that an étale groupoid $G$ is effective if the interior of the isotropy of $G$ is the unit space. The Weyl groupoid is effective by construction. This is shown in, for example, [17, Corollary 3.2.7], and we provide a quick proof here for convenience.

**Lemma 2.6.** The Weyl groupoid $G(A, B)$ is effective.

**Proof.** It suffices to consider basic open sets of the form $U_n = \{[\alpha_n, x] : x \in \text{dom}(n)\}$ for $n \in N(A, B)$, since such open sets form a base for the topology on $G(A, B)$, so suppose $U_n$ is contained in isotropy for some $n \in N(A, B)$. Then for all $x \in \text{dom}(n)$ we have

$$\alpha_n(x) = r[\alpha_n, x] = s[\alpha_n, x] = x,$$

whereby $\alpha_n = \text{id}_{\text{dom}(n)}$. The germ relation then gives $[\alpha_n, x] = [\text{id}_{\text{dom}(n)}, x] \in G(A, B)^{(0)}$ for all $x \in \text{dom}(n)$, whereby $U_n \subseteq G(A, B)^{(0)}$, and $G(A, B)$ is effective. \hfill $\square$

Kwaśniewski and Meyer define the essential (twisted) groupoid $C^*$-algebra as follows (cf. [11, Section 7.2]). Let $(G, \Sigma)$ be a twist over an étale groupoid with locally compact Hausdorff unit space. There is a canonical line bundle $L := \mathbb{C}_\Sigma \Sigma$, where the quotient is by the diagonal action of $\mathbb{T}$ on $\mathbb{C} \times \Sigma$ given by $\lambda(z, \sigma) = (\lambda z, \lambda \sigma)$. For each open bisection $U \subseteq G$ let $C_c(U, \Sigma)$ be the compactly supported continuous sections $U \to L$ of the restriction of $L$ to $U$. Let $C_c(U, \Sigma)$ be the collection of sections in $C_c(U, \Sigma)$, extended by zero to sections $G \to L$. That is, an element $f \in C_c(U, \Sigma)$ is a section $G \to L$ with compact support contained in $U$, and the restriction $f|_U$ is continuous on $U$. Let $C_c(G, \Sigma) = \bigcup_{U \subseteq G} C_c(U, \Sigma)$, where $U$ ranges over open bisections of $G$. The space $C_c(G, \Sigma)$ carries a convolution product and involution given by

$$f \ast g(\gamma) := \sum_{r(\gamma) = r(\eta)} f(\eta) \cdot g(\eta^{-1} \gamma), \quad f^*(\gamma) = f(\gamma^{-1}),$$

for all $f, g \in C_c(G, \Sigma)$ and $\gamma \in G$. Note that the sum in this convolution product is finite since the $f$ and $g$ considered have compact support, and so have finite support over each $r$-fibre in $G$. The full twisted groupoid $C^*$-algebra $C^*_t(G, \Sigma)$ is defined as the maximal $C^*$-completion of the $*$-algebra $C_c(G, \Sigma)$.

The reduced twisted groupoid $C^*$-algebra $C^*_r(G, \Sigma)$ is defined by constructing a family of regular representations and taking the norm induced from them. Importantly, by [11, Proposition 7.10] one may embed $C^*_r(G, \Sigma)$ into algebra of Borel sections of the line bundle $L$ (denoted $\mathfrak{B}(G, \Sigma)$) via an injective and contractive $*$-homomorphism, and there is a generalised expectation $C^*_r(G, \Sigma) \to \mathfrak{B}(G^{(0)})$ given by restricting these sections to the unit space. Here we may identify section $G^{(0)} \to L$ over $G^{(0)}$ with functions $G \to \mathbb{C}$, since the line bundle $L$ is trivial over the unit space $G^{(0)}$. We will often identify $C^*_r(G, \Sigma)$ with its image in $\mathfrak{B}(G, \Sigma)$, so that we may consider elements of $C^*_r(G, \Sigma)$ as functions $\Sigma \to \mathbb{C}$.
The desired quotient of $C^*_r(G, \Sigma)$ consists of functions $\Sigma \to G$ with ‘small’ support. A subset of a topological space is \textit{nowhere dense} if its closure has empty interior. A subset of a topological space is \textit{meagre} if it is the union of countably many nowhere dense sets.

**Definition 2.7** ([11] Proposition 7.18). The \textit{essential twisted groupoid $C^*$-algebra} $C^*_{\text{ess}}(G, \Sigma)$ is defined as the quotient of $C^*_r(G, \Sigma)$ by the ideal

$$J_{\text{sing}} := \{ f \in C^*_r(G, \Sigma) : (f^* f)|_{G(0)} \text{ has meagre support} \}.$$  

**Remark 2.8.** One may note that we have defined $C_r(G, \Sigma)$ differently to simply considering continuous sections $G \to \Sigma$ with compact support. This is only a necessary step when the groupoid $G$ is not globally Hausdorff. If $G$ is Hausdorff, then $C_c(G, \Sigma)$ is exactly the space of continuous compactly supported sections $G \to \Sigma$. However, if $G$ is non-Hausdorff, then there is a net in $G$ converging to two distinct points $\gamma, \eta \in G$. A function $f$ compactly supported on a bisection neighbourhood of $\gamma$ will then satisfy $f(\gamma) \neq 0$, but $f(\eta) = 0$ as the two points $\gamma$ and $\eta$ will not lie on the same bisection. Thus $f$ is discontinuous on $G$, but may very well be continuous on the bisection neighbourhood of $\gamma$, and we wish to consider such functions in our groupoid $C^*$-algebras.

At this point it should be mentioned that Exel and Pitts [3] also define the “essential” (twisted) groupoid $C^*$-algebra, and their construction slightly predates the construction of Kwaśniewski and Meyer. Exel and Pitts’ construction is limited to groupoids that are topologically principal: groupoids with trivial isotropy on a dense subset of units. Exel and Pitts later prove that the two definitions of essential groupoid $C^*$-algebra agree for any groupoid with a second-countable unit space and a dense $G_\delta$-subset of points in the unit space with trivial isotropy. Since we wish to consider groupoids that may not be topologically principal, we shall adopt the definitions and conventions of Kwaśniewski and Meyer for the essential groupoid $C^*$-algebra. This is particularly notable when considering groupoids that are not second countable, as these may be effective without having any points with trivial isotropy, hence can never be topologically principal.

### 2.1. Aperiodic vs Masa inclusions.

The concept of aperiodicity for $C^*$-inclusions was first introduced by Kwaśniewski and Meyer in [11], inspired by the work of Kishimoto [7]. This condition implies several useful properties such as the generalised intersection property and uniqueness of pseudo-expectation (cf. [11,13]). The concept of aperiodicity for bimodules was given in [10] and we shall recall it here. Since we are considering inclusions of commutative $C^*$-algebras and their slices, aperiodicity is equivalent to both topological non-triviality and pure outerness for Hilbert bimodules (see [10] Theorem 8.1).

Throughout the later parts of this article we shall consider inclusion $A \subseteq B$ of $C^*$-algebras, where $A = C_0(X)$ is a maximal commutative subalgebra of $B$. We call such a subalgebra a \textit{masa} (acronym for ‘maximal abelian subalgebra’), and refer to inclusions of masas as \textit{masa inclusions}. In our analysis of masa inclusions we shall use results from [11,13,10] regarding \textit{aperiodic} bimodules and inclusions. We give the following definition of aperiodicity from [10] as stated there, not assuming that the $C^*$-algebra $A$ is commutative. It is worth pointing out that when $A$ is commutative, all hereditary subalgebras of $A$ are ideals, and one may (in this case) make that substitution in the following definition.

**Definition 2.9** ([10], [11] Definitions 5.9,5.14). Let $A$ be a $C^*$-algebra. A Banach $A$-bimodule $X$ is \textit{aperiodic} if for all $\varepsilon > 0$, non-zero hereditary subalgebras $D \subseteq A$, and elements $x \in X$ there is $a \in D$ with $a \geq 0$, $\|a\| = 1$ such that

$$\|axa\| < \varepsilon.$$  

An inclusion $A \subseteq B$ of $C^*$-algebras is \textit{aperiodic} if the Banach $A$-bimodule $B/A$ is an aperiodic bimodule.
Lemma 2.10. Let $A \subseteq B$ be a non-degenerate inclusion of $C^*$-algebras and let $I \triangleleft A$ be an essential ideal. An element $b \in B$ commutes with $A$ if and only if it commutes with $I$.

Proof. If $b \in B$ commutes with $A$ then it certainly commutes with $I$, so we prove the other implication. Suppose $ba = ab$ for all $a \in I$. For all $c \in A$ and $a \in I$ we have $(bc - cb)a = b(ca) - c(ba) = b(ca) - (ca)b = 0$ since $ca \in I$ and $b$ commutes with $I$. Hence $bc - cb$ annihilates $I$, whereby $bc - cb = 0$ since $I$ is an essential ideal in $A$ and $A \subseteq B$ is a non-degenerate inclusion. □

Lemma 2.11. Let $A \subseteq B$ be a regular non-degenerate inclusion with $A = C_0(X)$ commutative and let $n \in N(A, B)$ be a normaliser. Let $M_n := \overline{\text{span}} \text{AnA}$ be the slice of $A \subseteq B$ generated by $n$, and let $V \subseteq \text{dom}(n)$ be the interior of the set of points in $\text{dom}(n)$ that are fixed by $\alpha_n$. Then all $m \in M_n \cdot C_0(V)$ lie in the commutant of $A$.

Proof. Fix $m \in M_n \cdot C_0(V)$. Let $W := (X \setminus \text{dom}(n))^0$ be the interior of the complement of $\text{dom}(n)$ in $X$. Then $\text{dom}(m) \cup W$ is a dense open subset of $X$ so $C_0(\text{dom}(m) \cup W) \triangleleft C_0(X)$ is an essential ideal. By Lemma 2.10 it suffices to show that $m$ commutes with the ideal $C_0(\text{dom}(m) \cup W)$. Since $\text{dom}(m) \cap W = \emptyset$ we have $C_0(\text{dom}(m) \cup W) = C_0(\text{dom}(m)) \oplus C_0(W)$, so we shall show that $m$ commutes with the summand ideals. For $f \in C_0(\text{dom}(m))$ we have
\[
|mf - fm|^2 = |(mf - fm)^*(mf - fm)| = \sup_{x \in X} |f^*m^*mf(x) - m^*f^*mf(x) - f^*m^*fm(x) + m^*f^*fm(x)|,
\]
noting that each of the terms in the expansion on the right belong to $C_0(X)$ since $m$ is a normaliser. If $x \in \text{dom}(m) \subseteq V$ we have $m^*fm(x) = m^*mf(x)$ as $\alpha_n(x) = x$, and so
\[
(mf - fm)^*(mf - fm)(x) = f^*m^*mf(x) - m^*f^*mf(x) - f^*m^*fm(x) + m^*f^*fm(x)
\]
\[
= 2f^*f(x)m^*m(x) - 2f^*f(x)m(x)
\]
\[
= 0.
\]
Otherwise, if $x \notin \text{dom}(m)$ we have
\[
(mf - fm)^*(mf - fm)(x) = f^*m^*mf(x) - m^*f^*mf(x) - f^*m^*fm(x) + m^*f^*fm(x)
\]
\[
= m^*f^*fm(x)
\]
since $f(x) = 0$. By the Cohen-Hewitt factorisation theorem, we may write $m = m'g$ for some $m' \in M_n$ and $g \in C_0(\text{dom}(n))$. We then have $m^*f^*fm(x) = m'^*f^*fm'g(x) = 0$ since $x \notin \text{dom}(n)$, so $m$ commutes with $C_0(\text{dom}(n))$.

If $f \in C_0(W)$ then, using Cohen-Hewitt factorisation again, we may write $m = m'g$ for some $m' \in M_n$ and $g \in C_0(\text{dom}(n))$. It follows that $mf = m'gf = 0$ since $\text{dom}(m) \cap W = \emptyset$, so it remains to show that $fm = 0$. We have $f^*m^*mf(x) = |f(x)|m^*m(x) = 0$ for all $x \in \text{dom}(m) \cup W$, since $f$ has support contained in $W$, $m^*m$ has support contained in $\text{dom}(m)$, and the two sets $W$ and $\text{dom}(m)$ are disjoint. Hence $|fm|^2 = ||f^*m^*mf|| = 0$ whereby $mf = 0 = fm$ as required. □

Lemma 2.12. Let $A \subseteq B$ be a regular masa inclusion and let $n \in N(A, B)$ be a normaliser. Let $M_n := \overline{\text{span}} \text{AnA}$ be the slice of $A \subseteq B$ generated by $n$. Then the bimodule $M_n \cdot (M_n \cap A)^\perp$ is aperiodic.

Proof. Note first that the inclusion is non-degenerate by [16] Theorem 2.6. Let $V \subseteq \text{dom}(n)$ be the interior of the set of points $x \in \text{dom}(n)$ with $\alpha_n(x) = x$. By Lemma 2.11 we see that $M_n \cdot C_0(V)$ commutes with $A$, whereby $M_n \cdot C_0(V) \subseteq A$ since $A$ is maximal commutative. Hence $M_n \cdot C_0(V) = C_0(V) \subseteq M_n \cap A$ as $C_0(V) \subseteq s(M_n)$. Conversely for any $a \in M_n \cap A$ we have $\alpha_a = \text{id}_{\text{dom}(a)}$, whereby $\text{dom}(a) \subseteq V$ and $a \in C_0(V)$. Thus $C_0(V) = M_n \cap A$ and so $C_0(V)^\perp = C_0(X \setminus V)$. Let $W := X \setminus V$, so that $(M_n \cap A)^\perp = C_0(W)$.
Recall that hereditary subalgebras of commutative $C^*$-algebras are exactly ideals, so fix an open subset $U \subseteq X$. If $U \cap W = \emptyset$ then we have $C_0(U) (M_n \cdot (M_n \cap A)^+) C_0(U) = C_0(U) M_n C_0(W) C_0(U) = \{0\}$. Else if $U \cap W \neq \emptyset$, fix an element $x \in U \cap W$ with $\alpha_n(x) \neq x$, which exists since $U \cap W$ is open and disjoint from $V$. Fix neighbourhoods $U_1, U_2 \subseteq X$ of $x$ and $\alpha_n(x)$ with $U_1 \cap U_2 = \emptyset$. By restricting $U_1$ to $U_1 \cap \alpha_n^{-1}(U_2)$, we can assume that $U_1 \cap \alpha_n(U_1)$ is empty. Then for any $a \in C_0(U_1)$ we have

$$(an)(an)(x) = a(x)(n^*a^n)(x)a(x) = (a^*a)(x)(a^*a)(\alpha_n(x))(n^*n)(x) = 0.$$ 

If $b \in A$ and $c \in C_0(W)$ then $ab, ca \in C_0(U_1)$ and so $||a(bnc)|| = 0$. By [13] Lemma 4.2 the closed linear span of such elements $bnc$ for $b \in A$, $c \in C_0(W)$ is aperiodic, and so the bimodule $\text{span} A \cdot n \cdot C_0(W) = M_n \cdot (M_n \cap A)^+$ is aperiodic. □

The proof of Lemma 2.12 shows that a slice $M$ with trivial intersection with the subalgebra is topologically non-trivial. That is, the induced partial homeomorphism $s(M) \to r(M)$ of the spectrum of $A$ does not restrict to the identity map on any non-empty open subset of $s(M)$. In general this may be stronger than the bimodule $M$ being aperiodic (see [13] Theorem 4.7), but for bimodules over Type I $C^*$-algebras (in particular, commutative $C^*$-algebras) these conditions are equivalent by [10] Theorem 8.1.

Lemma 2.12 allows us to show that regular masa inclusions are aperiodic.

**Lemma 2.13.** Let $A \subseteq B$ be a non-degenerate inclusion of $C^*$-algebras. If $B$ is densely spanned by slices $W \subseteq B$ with the property that $W \cdot (W \cap A)^+$ is an aperiodic $A$-bimodule, then the inclusion $A \subseteq B$ is aperiodic.

**Proof.** The slices $W$ densely span $B$, so slices $W/(W \cap A)$ have densely spanning image in the $A$-bimodule $B/A$. The subbimodule $W/(W \cap A) \cdot (W \cap A)^+ \subseteq W \cdot (W \cap A)^+$ is then aperiodic by assumption, and so [11] Lemma 5.12 implies that $W/(W \cap A)$ is aperiodic. The set of points satisfying Kishimoto’s condition is a closed subspace of $B/A$ by [10] Lemma 4.2, hence the inclusion $A \subseteq B$ is aperiodic. □

**Corollary 2.14.** Let $A \subseteq B$ be a regular masa inclusion. Then $A \subseteq B$ is an aperiodic inclusion.

**Proof.** Combine Lemmata 2.12 and 2.13. □

### 3. Generalised conditional expectations

By a result of Frank, for commutative $C^*$-algebras Hamana’s injective hull (see [6]) and the local multiplier algebra agree [1] Theorem 1, so commutative inclusions $A \subseteq B$ always have a local multiplier algebra valued conditional expectation. By results of Dixmier [1] and later Gonshor [2], the local multiplier algebra of $C_0(X)$ (and hence the injective hull of $C_0(X)$) is isomorphic to

$$M_{\text{loc}}(C_0(X)) \cong \mathcal{B}(X)/\mathfrak{M}(X),$$

where $\mathfrak{B}(X)$ is the algebra of bounded Borel-measurable functions $X \to \mathbb{C}$, and $\mathfrak{M}(X)$ is the ideal of such functions that vanish on a comeagre subset of $X$ (i.e. functions with ‘meagre support’).

A generalised conditional expectation for an inclusion $A \subseteq B$ consists of an algebra $\hat{A}$ containing $A$, and a completely positive contractive linear map $E : B \to \hat{A}$ such that $E|_A = \text{id}_A$ (see [11] Definition 3.1). The particular generalised expectations we are interested in are those taking values in the local multiplier algebra of $A$. Since we consider the case where $A$ is commutative, the paragraph above implies this is the same as considering generalised expectations taking values in the injective hull.

Our motivations for considering such conditional expectations are twofold. Firstly, all inclusions of $C^*$-algebras have at least one generalised expectation taking values in the
injective hull by the universal property of the injective hull. Secondly, there is a natural choice of such an expectation for (twisted) groupoid $C^*$-algebras. Consider a twist $\Sigma$ over an étale groupoid $G$ with locally compact Hausdorff unit space $X$. Forming the algebra $C_c(G, \Sigma)$, there is a natural way to "cut down" sections over $G$ to sections over $G^{(0)}$: namely restriction. Since the twist is trivial over the unit space, such functions are equivalent to functions $X \to \mathbb{C}$. However, if $G$ is not Hausdorff, the functions in $C_c(G, \Sigma)$ and their restrictions to $G^{(0)}$ may not be continuous, and so the conditional expectation that Renault uses in [19] does not take values in $C_0(G^{(0)})$. These restrictions are however represented by measurable functions $G^{(0)} \to \mathbb{C}$, and so restriction induces a map $C_c(G, \Sigma) \to \mathfrak{B}(G^{(0)}/\mathfrak{M}(G^{(0)})$, which then extends to the essential groupoid $C^*$-algebra $C^\text{ess}_c(G, \Sigma) \to M_{\text{loc}}(C_0(G^{(0)})) = \mathfrak{B}(G^{(0)})/\mathfrak{M}(G^{(0)})$.

4. Essential commutative Cartan subalgebras of $C^*$-algebras

We now define an essential commutative Cartan pair in analogue to Renault’s definition [19] Definition 5.1).

**Definition 4.1.** Let $A \subseteq B$ be an inclusion of $C^*$-algebras with $A$ commutative. We say the pair $(A, B)$ is an essential commutative Cartan pair if the following conditions hold:

1. $(\text{ECC}1)$ $A$ is a regular subalgebra of $B$;
2. $(\text{ECC}2)$ $A$ is a maximal abelian subalgebra of $B$ (masa);
3. $(\text{ECC}3)$ there exists a faithful local expectation $E : B \to M_{\text{loc}}(A)$.

We may also use the phrase essential commutative Cartan inclusion. to refer to an essential commutative Cartan pair. There is also a notion of noncommutative Cartan pair (see [2], [12]), motivating the appearance of the word “commutative” in this definition. We shall not consider such objects in this article, and so we may for brevity drop the term ‘commutative’ and simply refer to an essential Cartan pair or inclusion.

There are some notable differences in this definition to the definition of Renault in [19]. Firstly, the condition that $A$ is a non-degenerate subalgebra of $B$ has been shown to be redundant by Pitts [16] Theorem 2.7, as all regular masa inclusions have this property. Secondly, the conditional expectation need not take values in $A$, but may take values in $M_{\text{loc}}(A)$. Since $A \subseteq B$ is assumed to be regular and masa, Corollary 2.14 implies that $A \subseteq B$ is an aperiodic inclusion, and so $E$ is the unique pseudo-expectation for the inclusion by [13] Proposition 8.2. Lastly, the condition that $B$ be a separable $C^*$-algebra was shown to be unnecessary by Kwaśniewski and Meyer in [12], and also by Raad in [18].

**Remark 4.2.** Pitts defines virtual Cartan inclusions as regular masa inclusions $A \subseteq B$ such that $A$ detects ideals in $B$ (see [15] Definition 1.1]). Such inclusions have a unique pseudo-expectation $E$ by combining Corollary 2.14 and [11] Proposition 8.2. This pseudo-expectation is symmetric since it must agree with the canonical $M_{\text{loc}}(A)$-expectation on $A \rtimes S$, the full crossed product arising from the canonical action of the slice inverse semigroup $S := S(A, B)$ on $A$. If $A$ detects ideals in $B$ then the kernel of $E$ cannot contain any non-zero ideal, so $E$ is faithful by [11] Corollary 3.8. Thus virtual Cartan inclusions are essential commutative Cartan inclusions. The converse is also true: [11] Proposition 5.8 together with [11] Corollary 3.8 give that essential Cartan pairs must detect ideals, hence are virtual Cartan inclusions.

The terminology of Pitts predates the earliest version of this article. To avoid potential confusion for returning readers, we keep the terminology of essential Cartan pair in this and other articles, the first versions of which used this terminology.

Throughout let $A \subseteq B$ be an essential commutative Cartan pair with faithful local expectation $E : B \to M_{\text{loc}}(A)$. 

4.1. The evaluation map. Renault defines an evaluation map \( B \to C^*_e(G, \Sigma) \) directly taking elements of \( B \) to sections of the canonical line bundle associated to \( \Sigma \). An integral part to this construction is that the conditional expectation take values in the subalgebra \( A = C_0(X) \), and so elements may be considered as functions on \( X \). This approach breaks down in our setting: Gonshor [5, Theorem 1] showed that the local multiplier algebra of \( C_0(X) \) is isomorphic to the algebra \( B(X)/2R(X) \) of Borel measurable functions quotient by meagrely supported functions. Thus elements of \( M_{\text{loc}}(C_0(X)) \) are not functions on \( X \), but rather equivalence classes of functions up to meagre support. Since singletons are (often) meagre in \( X \), elements of \( M_{\text{loc}}(A) \) cannot be evaluated at points. We circumvent this problem by showing that normalisers for the inclusion \( A \subseteq B \) can be represented by functions defined on a dense open subset, and so give a well defined class in the Borel-mod-meagre setting since \( X \) is a Baire space.

**Lemma 4.3.** Let \( A \subseteq B \) be regular masa inclusion. Then the inclusion \( A \subseteq B \) is aperiodic, and hence there is at most one local conditional expectation \( E : B \to M_{\text{loc}}(A) \), and \( E(W) \subseteq M(W \cap A) \) for any slice \( W \subseteq B \) of the inclusion.

**Proof.** Let \( E : B \to M_{\text{loc}}(A) \) be a local conditional expectation. By Corollary 2.4.1 the inclusion \( A \subseteq B \) is aperiodic, so has at most one pseudo-expectation by [13] Propositions 3.9,3.16. Lemma 2.12 states that for any slice \( W \subseteq B \), the subslice \( W \cdot (W \cap A)^\perp \) is an aperiodic \( A \)-bimodule. Then [11] Lemma 5.10 gives that \( M_{\text{loc}}(A) \) contains no non-zero aperiodic bimodules, and the image of an aperiodic bimodule under a bounded bimodule map is aperiodic by [11] Lemma 5.12. Thus \( E(W \cdot (W \cap A)^\perp) = \{0\} \).

For a normaliser \( n \in N(A,B) \) and a slice \( W \subseteq B \) with \( n \in W \), the local multiplier \( E(n) \) is determined by how it acts on the essential ideal \( (W \cap A) \cdot (W \cap A)^\perp \). Fix \( a \in W \cap A \) and \( a^+ \in (W \cap A)^\perp \). Then \( E(n) = E(na) + E(na^+) \). Since \( na \in W \cdot (W \cap A) = W \cap A \subseteq A \), we have \( E(na) = na \), and since \( na^+ \in W \cdot (W \cap A)^\perp \), we have \( E(na^+) = 0 \). Thus \( E(n) \) is a multiplier on \( W \cap A \), identifying \( M(W \cap A) \subseteq M_{\text{loc}}(W \cap A) \subseteq M_{\text{loc}}(A) \) via the canonical inclusions.

**Corollary 4.4.** For each normaliser \( n \in N(A,B) \) the element \( E(n) \in M_{\text{loc}}(A) \) is represented by a continuous bounded function on a dense open subset of \( X = A \). Moreover, the images of finite sums of normalisers under \( E \) are represented by continuous bounded functions on dense open subsets of \( X \).

**Proof.** By Lemma 4.3 we have that \( E(n) \) belongs to \( M(I) \) for some essential ideal \( I \triangleleft A \). Essential ideals of \( A = C_0(X) \) are of the form \( I = C_0(\mathcal{U}) \) for dense open subsets \( \mathcal{U} \subseteq X \). This, coupled with the identification of \( M(C_0(\mathcal{U})) \) with the algebra \( C_b(\mathcal{U}) \) of bounded continuous functions on \( \mathcal{U} \) allows us to express \( E(n) \) as a bounded continuous function on \( \mathcal{U} \).

For a finite linear collection of normalisers \( n_i \in N(A,B) \), we have \( E(\sum_i n_i) = \sum_i E(n_i) \), where each \( E(n_i) \) is a function on a dense open subset \( U_i \subseteq X \). The sum of these functions defines a continuous and bounded function on the intersection \( \bigcap_i U_i \), which is again dense and open as the intersection of finitely many dense open subsets.

Given a normaliser \( n \in N(A,B) \), the following corollary allows us to canonically pick a function on \( X = A \) to represent \( EL(n) \) in the quotient \( M_{\text{loc}}(A) = B(X)/2R(X) \) of Frolík decompositions of normalisers in the injective hull of \( A \) (which, in the commutative setting, coincides with the local multiplier algebra). We offer an alternative proof that does not require the machinery of Frolík decompositions.

**Corollary 4.5 (cf. [15] Corollary 3.6).** Let \( n \in N(A,B) \) be a normaliser, and let \( V_n := \{ x \in X : \alpha_n(x) = x \}^\circ \); the interior of the set of fixed points for \( \alpha_n \). Then \( E(n) \) is represented in \( M_{\text{loc}}(A) \) by a bounded continuous function on \( V_n^{\text{ess}} := V_n \cup (X \setminus V_n)^{\circ} \) which
we denote by \( n|_{V^{\text{ess}}_n} \), and for any \( g \in C_0(V_n) \) we have \( n|_{V^{\text{ess}}_n} g = E(ng) = ng \), and for any \( g^\perp \in C_0(V_n)^\perp \) we have \( n|_{V^{\text{ess}}_n} g = E(ng) = 0 \). If \( b = \sum_i n_i \) is a sum of finitely many normalisers \( n_i \in N(A,B) \), then \( E(b) \) is represented in \( M_{\text{loc}}(A) \) by a function defined on a dense open subset \( V^{\text{ess}}_b := \bigcap_i V^{\text{ess}}_{n_i} \).

**Proof.** Let \( W_n := \overline{\text{span}} \text{An} A \) be the smallest slice containing \( n \). Then \( s(W_n) \) is exactly \( \overline{\text{span}} \text{An}^* \text{An} \) in \( C_0(\text{dom}(n)) \), in which \( C_0(V_n) \oplus C_0((X \setminus \overline{V}_n) \cap \text{dom}(n)) \) is an essential ideal. Since \( \alpha_n|_{V_n} = \text{id}_{V_n} \), we have that \( n \) commutes with \( C_0(V_n) \), whereby \( n \) restricts to a multiplier of \( C_0(V_n) \) which is represented by a continuous and bounded function on \( V_n \). Conversely for all \( x \in (X \setminus \overline{V}_n)^c \), we have that \( \alpha_n(x) = x \), whereby \( W_n \cdot C_0((X \setminus \overline{V}_n) \cap \text{dom}(n)) \) is a topologically non-trivial Hilbert \( A \)-bimodule. The submodule \( W_n \cdot C_0((X \setminus \overline{V}_n) \cap A) \) is then purely outer by [10, Theorem 8.1], whereby \( W_n \cdot C_0((X \setminus \overline{V}_n) \cap A) = \{0\} \). Thus \( E(n) \) is a multiplier of \( C_0(V_n) \) by Lemma [13] which extends by zero to a multiplier of \( C_0(V_n) \oplus C_0((X \setminus \overline{V}_n) \cap \text{dom}(n)) \), which in turn gives by a bounded function \( n|_{V^{\text{ess}}_n} \) on \( V_n \). If \( g \in C_0(V_n) \) we have \( n|_{V^{\text{ess}}_n} g = E(ng) = ng \) and for any \( g^\perp \in C_0(V_n)^\perp \) we have \( n|_{V^{\text{ess}}_n} g^\perp = E(ng^\perp) = 0 \).

The result about finite sums of normalisers follows since the intersection of finitely many dense open subsets of a topological space is again dense and open. \( \square \)

In Corollary [13] a different choice of normalisers giving rise to \( b \in \overline{\text{span}} N(A,B) \) may give a different dense open subset \( V^{\text{ess}}_b \subseteq X \). However a differing choice of normalisers summing to \( b \) can only change the resulting set \( V^{\text{ess}}_b \) on a meagre set, since \( V^{\text{ess}}_n \) is always dense and open. For our purposes we only need that at least one such \( V^{\text{ess}}_b \) exists.

**Lemma 4.6.** Let \( n \in N(A,B) \) and let \( U_n := \{[\alpha_n, x] : x \in \text{dom}(n)\} \) be the open bisection of \( G(A,B) \) determined by \( n \). There is a section \( \hat{n} \in C_0(U_n, \Sigma(A,B)) \) defined by

\[
\hat{n}[m, x] = \begin{cases} \frac{(m^n)|_{V^{\text{ess}}_m}(x)}{\sqrt{m^*m}(x)}, & x \in V^{\text{ess}}_{m^n} \\ 0, & \text{otherwise.} \end{cases}
\]

If \( b = \sum_i n_i \) is a sum of finitely many normalisers then we define \( \hat{b} \in C_0(G(A,B), \Sigma(A,B)) \) by

\[
\hat{b}[m, x] = \begin{cases} \frac{(m^b)|_{V^{\text{ess}}_{m^bn}}(x)}{\sqrt{m^*m}(x)}, & x \in V^{\text{ess}}_{m^b} \\ 0, & \text{otherwise.} \end{cases}
\]

Moreover \( \hat{b} \) and \( \sum_i \hat{n}_i \) agree on \( U_m \cdot V^{\text{ess}}_{m^b} \) for each normaliser \( m \in N(A,B) \), and so agree on a dense open subset of \( G(A,B) \).

**Proof.** If \( [m, x] = [m', x] \) for some \( m, m' \in N(A,B) \) then either both \( \hat{n}[m, x] = \hat{n}[m', x] = 0 \) or \( x \in V^{\text{ess}}_{m^n} \cap V^{\text{ess}}_{m'^n} \) and there exist functions \( a, a' \in A = C_0(X) \) with \( a(x), a'(x) > 0 \) and \( ma = m'a' \). Then

\[
\frac{(m^n)|_{V^{\text{ess}}_m}(x)}{\sqrt{m^*m}(x)} = \frac{a'(x)(m^*n)|_{V^{\text{ess}}_{m^n}}(x)}{\sqrt{m^*m}(x)} = \frac{a(x)(m^n)|_{V^{\text{ess}}_m}(x)}{\sqrt{m^*m}(x)} = \frac{(m^n)|_{V^{\text{ess}}_m}(x)}{\sqrt{m^*m}(x)},
\]

so the function \( \hat{n} \) gives a well defined section of the canonical line bundle associated to the twist \( (G(A,B), \Sigma(A,B)) \). One notes also that the support of \( \hat{n} \) is contained in \( U_n \), as outside this set either \( x \) does not belong to \( V^{\text{ess}}_{m^n} \) or \( (m^n)|_{V^{\text{ess}}_m}(x) = 0 \).

To see the section \( \hat{n} \) is quasi-continuous, that is, its restriction to \( U_n \) is continuous, we note that for \( [\alpha_n, x] \in U_n \) we have \( [m, x] = [zn, x] \) for some \( z \in \mathbb{T} \). Thus, on \( U_n \) the section \( \hat{n} \) reduces to \( \hat{n}[zn, x] = \frac{\hat{n}[zn, x]}{\sqrt{n^*n}(x)} = \frac{\hat{n}[zn, x]}{\sqrt{n^*n}(x)} \) which is continuous and vanishing at topological infinity of \( U_n \) since \( n^*n \) vanishes at topological infinity of \( \text{dom}(n) = s(U_n) \).
The proof that \( \hat{b} \) is well defined follows since \((m^*n_i)\mid_{V^\text{res}_{m^*b}} + (m^*n_j)\mid_{V^\text{res}_{m^*b}} \) is exactly \((m^*(n_i + n_j))\mid_{V^\text{res}_{m^*b}} \) for all \( m \in N(A,B) \) and for any summands \( n_i, n_j \) describing \( b \). Since \( V^\text{res}_{m^*b} \subseteq V^\text{res}_{m^*n_i} \) for each \( i \) we see that \((m^*b)\mid_{V^\text{res}_{m^*b}} (x) = \sum_i (m^*n_i)\mid_{V^\text{res}_{m^*n_i}} \) whereby \( \hat{b} \) and \( \hat{\mathcal{N}} \) agree for values of \( x \in V^\text{res}_{m^*b} \). The union of \( U_m \cdot V^\text{res}_{m^*b} \) over all normalisers \( m \in N(A,B) \) is then a dense open subset of \( G(A,B) \) since the sets \( U_m \) form a basis for the topology on \( G(A,B) \), and each \( U_m \cdot V^\text{res}_{m^*b} \) is open. \( \square \)

**Lemma 4.7.** Let \( n \in N(A,B) \) be a normaliser. Any section \( f \in C_c(U_n, \Sigma(A,B)) \) is of the form \( \hat{n} \cdot h_f \) for a function \( h_f \in C_0(X) \). Moreover, the assignment \( \phi_n : f = \hat{n} \cdot h_f \mapsto nh_f \) is a well defined \( C_0(X) \)-bimodule map.

**Proof.** Existence of \( h_f \in C_0(X) \) with \( f = \hat{n} \cdot h_f \) follows since \( \hat{n} \) is a non-vanishing section over \( U_n \), and so since the support of \( f \) is contained in \( U_n \), there exists \( h_f \in C_0(X) \) with compact support contained in \( s(U_n) \) satisfying \( \hat{n}h = f \).

To see that the map \( \hat{n} \cdot h_f \mapsto nh_f \) is well defined, suppose \( f = \hat{n}h_f = \hat{n}g_f \). Then \( \hat{n}(h_f - g_f) = 0 \), so for all \([m, x] \in \Sigma(A,B)\) with \([\alpha_n, x] \in U_n\) we have

\[
0 = \hat{n}[m, x](h_f - g_f)(x),
\]

which implies that \( h_f(x) = g_f(x) \) for all \( x \in s(U_n) = \text{dom}(n) \) as \( \hat{n} \) is a non-vanishing section on \( U_n \). Then

\[
E((n(h_f - g_f)) \ast n(h_f - g_f)) = (h_f - g_f)E(n \ast n)(h_f - g_f) = (h_f - g_f)n^\ast n(h_f - g_f)
\]

and for all \( x \in X \) we have

\[
(h_f - g_f)n^\ast n(h_f - g_f)(x) = |h_f(x) - g_f(x)| n^\ast n(x) = 0,
\]

as either \( x \in \text{dom}(n) = \text{supp}(n^\ast n) \) where \( h_f(x) - g_f(x) = 0 \), or \( x \notin \text{dom}(n) \) so \( n^\ast n(x) = 0 \).

For \( f, g \in C_0(X) \), \( \hat{n}h \in C_c(U_n, \Sigma(A,B)) \), \( m \in N(A,B) \), and \( x \in \text{dom}(m) \cap s(U_n) \) we have

\[
(f \hat{\alpha}_n g)[m, x] = f(\alpha_n(x)) (m^\ast n)\mid_{V^\text{res}_{m^\ast n}}(x) h(x) g(x).
\]

Since \( h \) has compact support \( K \subseteq X \), the function \( (f \circ \alpha_n)\mid_{\text{dom}(n) \cap K} \) extends to some \( F \in C_0(X) \) by Tietze’s extension theorem, and we have \( \hat{n}h = \hat{\alpha}_n F \). We claim that \( f \hat{n}h = f \hat{n}h \). To see this, we note that for \( x \in \text{dom}(n) \cap K \) we have

\[
(f \hat{n}h - nhF)^* (f \hat{n}h - nhF)(x)
\]

\[
= (h^\ast n^\ast f^\ast f F^\ast h^\ast n f F - h^\ast n^\ast f^\ast nhF F + F^\ast h^\ast n^\ast nhF F)(x)
\]

\[
= 2|f(\alpha_n(x))|^2 n^\ast n(x) |h(x)|^2 - 2|f(\alpha_n(x))|^2 n^\ast n(x) |h(x)|^2
\]

\[
= 0.
\]

Alternatively if \( x \notin \text{dom}(n) \cap K \) then all the terms in the above calculation evaluate to zero as \( n^\ast n(x) \) is a common factor in each term. Thus

\[
\phi_n(f \hat{n}h g) = \phi_n(n F g) = nhF g = f n h g = f \phi_n(\hat{n}h) g,
\]

so \( \phi_n \) is a \( C_0(X) \)-bimodule homomorphism. \( \square \)

**Lemma 4.8.** For \( n \in N(A,B) \) let \( \phi_n \) be the \( C_0(X) \)-bimodule homomorphism in Lemma 4.7. Let \( D := \bigoplus_{n \in N(A,B)} C_c(U_n, \Sigma(A,B)) \) and define \( \Phi : D \to B \) by

\[
\Phi((f_n)_{n \in N(A,B)}) = \sum_{n \in N(A,B)} \phi_n(f_n).
\]

Let \( c : D \to C_c(G(A,B), \Sigma(A,B)) \) be the map \( c((f_n)_{n \in N(A,B)}) = \sum_{n \in N(A,B)} f_n \). There is a \( * \)-algebra structure on \( D \) such that both \( \Phi \) and \( c \) are \( * \)-homomorphisms. Moreover, \( c \) is surjective and \( \Phi \) has dense range in \( B \).
Proof. For \( f \in C_c(U_n, \Sigma(A, B)) \) and \( g \in C_c(U_m, \Sigma(A, B)) \) in \( D \), write \( f = \hat{n}f \) and \( g = \hat{m}g \) with \( j_f, k_g \in C_0(X) \) as in Lemma 1.7. Define the product \( fg = \hat{n}h \hat{m}k_g \in C_c(U_{nhkm}, \Sigma(A, B)) \), extending linearly to all of \( D \), and define the involution \( (\hat{n}f)^* = j_f^* \hat{n}^* \in C_c(U_{n^*}, \Sigma(A, B)) \), again extending anti-linearly. A brief computation shows that this gives \( D \) a \(*\)-algebra structure.

For \( f = \hat{n}f \in C_c(U_n, \Sigma(A, B)) \) and \( g = \hat{m}g \in C_c(U_m, \Sigma(A, B)) \), we have
\[
\Phi(\hat{n}f \hat{m}g) = \Phi(\hat{n}h \hat{m}k_g) = \phi_{nhkm}(\hat{m}h \hat{f}m k_g) = \hat{m}h \hat{f}m k_g = \phi_n(\hat{n}f)\phi_m(\hat{m}g) = \Phi(f)\Phi(g).
\]

Since \( f \) has compact support there is \( k \in C_0(X) \) with compact support such that \( f = \hat{k}h \), and so \( \Phi(f^*) = \phi_n^*(\hat{h} \hat{n}^* \hat{k}) = \hat{h} \phi_n^*(\hat{n}^* \hat{k}) = (k \hat{n}^*)^* = \Phi(f)^* \). Thus \( \Phi \) is a \(*\)-homomorphism.

To see that \( c \) is a \(*\)-homomorphism, let \( \hat{n}h \in C_c(U_n, \Sigma(A, B)) \) and \( \hat{m}k \in C_c(U_m, \Sigma(A, B)) \). Considering \( \hat{n}h \) and \( \hat{m}k \) as sections of the canonical line bundle associated to the twist \((G(A, B), \Sigma(A, B))\), for \([\alpha_p, x] \in G(A, B)\) we have
\[
(c(\hat{n}h) \cdot c(\hat{m}k))[\alpha_p, x] = \sum_{\alpha_q(\gamma) = \alpha_p(x)} \hat{n}h[\alpha_q, y]\hat{m}k[\alpha_q^* p, x]
\]
\[
= \hat{n}[\alpha_n, y]h(\gamma)\hat{m}[\alpha_{n^*} p, x]k(x)
\]
\[
= \begin{cases} 
\hat{n}[\alpha_n, \alpha_m(x)]h(\alpha_m(x))\hat{m}[\alpha_m, x]k(x), & [\alpha_p, x] = [\alpha_{nm}, x] \\
0, & \text{otherwise,}
\end{cases}
\]
\[
= c(\hat{n}h \hat{m}k)[\alpha_p, x]
\]
The second and third equalities hold since \( \hat{n} \) and \( \hat{m} \) have support contained in \( U_n \) and \( U_m \), respectively. For \( f = \hat{n}h \in C_c(U_n, \Sigma(A, B)) \) we have
\[
c(f^*)[\alpha_p, x] = \hat{n}h^*[\alpha_p, x]
\]
\[
= \langle \hat{n}h \rangle[\alpha_p^*, \alpha_p(x)]
\]
\[
= \begin{cases} 
h(x)\hat{n}[\alpha_n, \alpha_{n^*}(x)], & \alpha_p = \alpha_{n^*}, \\
0, & \text{otherwise,}
\end{cases}
\]
\[
= (\hat{n}h)^*[\alpha_p, x]
\]
\[
= c(f)^*[\alpha_p, x]
\]
thus \( c \) is a \(*\)-homomorphism.

That \( c \) is surjective is clear, since \( C_c(G(A, B), \Sigma(A, B)) \) is defined as the span of spaces \( C_c(U, \Sigma(A, B)) \) over all bisections \( U \subseteq G \), and bisections of the form \( U_n \) for normalisers \( n \in N(A, B) \) form a basis for the topology on \( G \).

To see that \( \Phi \) has dense range, it suffices to show that any normaliser \( n \in N(A, B) \) can be approximated by elements in the image of \( \Phi \) since \( A \subseteq B \) is a regular subalgebra. For \( n \in N(A, B) \) we can write \( n = mh \) for \( m \in N(A, B) \) and \( h \in C_0(X) = A \) by the Cohen-Hewitt factorisation theorem. Since \( C_c(X) \) is dense in \( C_0(X) \), we have compactly supported functions \( h_k \) with \( h_k \to h \) uniformly. Since \( n^*n \) has support contained in \( \text{dom}(n) = s(U_n) \), we can pick functions \( h_k \) with support contained in \( s(U_n) \). Then \( n = mh = \lim_{k \to \infty} mh_k \), and each \( mh_k \) is exactly the image of \( \hat{m}h_k \in C_c(U_m, \Sigma(A, B)) \). Thus the image of \( \Phi \) is dense in \( B \). \( \square \)
Proposition 4.9. Let $D$, $\Phi$, and $c$ be as in Lemma 4.8, and identify $A$ with $C_0(G(0))$. There is a linear map $R: C_c(G(A,B), \Sigma(A,B)) \to M_{\text{loc}}(C_0(G(0)))$ that restricts functions to the unit space $G(0)$ on $G$, such that the following diagram commutes:

$$
\begin{array}{ccc}
D & \xrightarrow{\Phi} & B \\
\downarrow c & & \downarrow E \\
C_c(G(A,B), \Sigma(A,B)) & \xrightarrow{R} & M_{\text{loc}}(C_0(G(0))) = M_{\text{loc}}(A)
\end{array}
$$

Proof. Let $f \in C_c(U_n, \Sigma(A,B))$. Then $f$ has support contained in $U_n$, so $f|_{G(0)}$ has support contained in $U_n \cap G(0)$. Since $f$ is continuous and compactly supported on $U_n$, $f|_{G(0)}$ is continuous and bounded on $U_n \cap G(0)$, so defines a multiplier of $C_0(U_n \cap G(0))$. As $f|_{G(0)}$ is zero outside of $U_n \cap G(0)$, we see that $f|_{G(0)}$ extends (by zero) to a multiplier on $C_0((U_n \cap G(0)) \cup (G(0) \setminus (U_n \cap G(0)))) = C_0(U_n \cap G(0)) \oplus C_0(U_n \cap G(0))^\perp$, which is an essential ideal in $C_0(G(0))$. Thus $f|_{G(0)}$ defines an element of $M_{\text{loc}}(C_0(G(0)))$, and the map $R$ is well defined.

Write $f = nh$ for some $h \in C_0(X)$ using Lemma 4.7. To show that $E(\Phi(f)) = c(f)|_{G(0)}$ it suffices to show that $E(\Phi(f))g = c(f)|_{G(0)}g$ for $g \in C_0(X)$ supported on a dense open subset. Let $V_n = \{x \in \text{dom}(n): \alpha_n(x) = x\}$ be as in Corollary 4.5. Then $U_n \cap G(0)$ is contained in $V_n$ since $\alpha_n(x) = x$ for all $x \in U_n \cap G(0)$ and $U_n \cap G(0)$ is an open set. Conversely, if $x \in V_n$ then there is a neighbourhood of $x$ contained in $\text{dom}(n)$ which is fixed by $\alpha_n$, and so $\alpha_n(x) = [id, x] \in U_n \cap G(0)$, hence $V_n = U_n \cap G(0)$. For $g \in C_0(V_n)$ we have $E(\Phi(f))g = E(nh)g = nhg = (nh)|_{V_n}g = (nh)|_{U_n \cap G(0)}g$ by Corollary 4.5. For $g^\perp \in C_0(V_n)^\perp$ we have $E(n)g^\perp = 0$ by Corollary 4.5 and $(nh)|_{G(0)}g^\perp = 0$ since the supports of $n|_{G(0)}$ and $g^\perp$ have zero intersection. Thus $E(\Phi(f)) = c(f)|_{G(0)}$ as both agree on an essential ideal of $C_0(G(0))$.

It is worth noting that the restriction map $R$ in Proposition 4.9 is the same as the conditional expectation $EL$ defined in [11] Proposition 4.3, and so is the canonical local multiplier algebra-valued expectation giving rise to the essential groupoid $C^*$-algebra.

Corollary 4.10. The map $\Phi$ descends via $c$ to a $*$-homomorphism

$$
\Psi_0 : C_c(G(A,B), \Sigma(A,B)) \to B
$$

satisfying $\Psi_0(c(f_n)) = \Phi(f_n)$ for all $(f_n) \in D$, and $R = E \circ \Psi_0$.

Proof. It suffices to show that the kernel of $c$ is contained in the kernel of $\Phi$. Contrapositively, we must show that if $\Phi(f_n) \neq 0$ for some $(f_n) \in D$ then $c(f_n) \neq 0$. Fix such $d \in D$ with $\Phi(d) \neq 0$. Since $c$ and $\Phi$ are $*$-homomorphisms and $P$ is faithful, we have

$$
R(c(d)^*c(d)) = R(c(d^*d)) = E(\Phi(d)^*\Phi(d)) = 0.
$$

In particular, $c(d)^*c(d)$ cannot be zero since $R$ is linear, whereby $c(d) \neq 0$.

Theorem 4.11. Let $A \subseteq B$ be an essential Cartan pair with faithful local conditional expectation $E : B \to M_{\text{loc}}(A)$. Let $(G(A,B), \Sigma(A,B))$ be the Weyl twist associated to $A \subseteq B$. There is a $*$-isomorphism $C^*_{\text{ess}}(G(A,B), \Sigma(A,B)) \cong B$ that restricts to an isomorphism $C_0(G(0)) \cong A$ and intertwines $E$ with the canonical local expectation $C^*_{\text{ess}}(G(A,B), \Sigma(A,B)) \to M_{\text{loc}}(C_0(G(0)))$, where $M_{\text{loc}}(A)$ is identified with $M_{\text{loc}}(C_0(G(0)))$.

Proof. Corollary 4.10 gives a $*$-homomorphism $\Psi_0 : C_c(G(A,B), \Sigma(A,B)) \to B$, and so this extends to a $*$-homomorphism $\Psi : C^*(G(A,B), \Sigma(A,B)) \to B$ by the universal property of the full twisted groupoid $C^*$-algebra. Since the canonical local conditional expectation $EL : C^*(G(A,B), \Sigma(A,B)) \to M_{\text{loc}}(A)$ is the continuous extension of the restriction map $R : C_c(G(A,B), \Sigma(A,B)) \to M_{\text{loc}}(C_0(G(0)))$, Proposition 4.9 implies that $\Psi$ intertwines
Moreover, and Theorem 5.2. Let $E \ell \Sigma(A,B))$ we have $E(a^*a) = E(\Psi(a)^*\Psi(a)) = 0$ if and only if $a \in \ker(\Psi)$ as $E$ is a faithful. In particular, the kernel of $\Psi$ is contained in the kernel of $E \ell$, so $\Psi$ descends to a homomorphism $\psi : C^e(G(A,B), \Sigma(A,B)) \to B$. Moreover, $\psi$ is injective since it intertwines expectations, and both $E \ell$ and $E$ are faithful. The map $\psi$ is surjective since $\Phi$ has dense image in $B$ by Lemma 4.8.

5. Uniqueness of the Weyl groupoid and twist

The Weyl groupoid and twist associated to an essential Cartan inclusion $A \subseteq B$ are not unique among the class of all étale groupoids giving rise to an isomorphic essential Cartan pair. The Weyl groupoid is however ‘minimal’ in a certain sense, as we can show that any other twist over a groupoid giving the same essential Cartan pair will have the Weyl groupoid and twist as a quotient. We also show that the Weyl pair is unique among twists over effective groupoids.

Definition 5.1. Let $H, \Omega$ and $G, \Sigma$ be twists over étale groupoids $H$ and $G$. A homomorphism of twists $(\beta, \tilde{\beta}) : (H, \Omega) \to (G, \Sigma)$ consists of groupoid homomorphisms $\beta : H \to G$ and $\tilde{\beta} : \Omega \to \Sigma$ such that the following diagram commutes:

$$
\begin{array}{ccc}
H^{(0)} \times \mathbb{T} & \longrightarrow & \Omega \\
\downarrow \beta_{H^{(0)}} \times \text{id} & & \downarrow \beta \\
G^{(0)} \times \mathbb{T} & \longrightarrow & \Sigma
\end{array}
$$

Theorem 5.2. Let $A \subseteq B$ be an essential Cartan pair and let $(G, \Sigma)$ be the associated Weyl twist. Let $(H, \Omega)$ be a twist over an étale groupoid with locally compact Hausdorff unit space. Suppose there is an isomorphism $\varphi : C^e(H, \Omega) \to C^e(G, \Sigma)$ that restricts to an isomorphism $\varphi_{C_0(H^{(0)})} : C_0(H^{(0)}) \to C_0(G^{(0)})$ and intertwines conditional expectations. Then there is a homomorphism of twists $(\beta_\varphi, \tilde{\beta}_\varphi) : (H, \Omega) \to (G, \Sigma)$ that restricts to a homeomorphism of unit spaces, and $H^{(0)}$ is a dense subset of the kernel of $\beta_\varphi$. Moreover $\tilde{\beta}_\varphi$ is a local homeomorphism.

Lemma 5.3. Let $U, V \subseteq H$ be open bisections in a twisted groupoid $(H, \Omega)$. Suppose that $U \cap V = \emptyset$. Then for $f \in C_c(U, \Omega)$ and $g \in C_c(V, \Omega)$ we have $||f|| \leq ||f + g||$ in the essential groupoid $C^*$-algebra.

Proof. The map $P : \mathfrak{B}(H, \Omega) \to \mathfrak{B}(H^{(0)}, \Omega)$ taking sections and restricting them to the unit space is a generalised expectation by Proposition 7.10, hence $P$ is contractive. The functions $f^*g$ and $g^*f$ have supports contained in $U^{-1}V$ and $V^{-1}U$ respectively, which do not intersect the unit space $H^{(0)}$ since $U \cap V = \emptyset$. Since $f$ and $g$ are supported on bisections, $f^*f$ and $g^*g$ are functions on the unit space, so $P(f^*f) = f^*f$ and $P(g^*g) = g^*g$. Moreover, $C_0(H^{(0)})$ is a commutative $C^*$-algebra so $||f^*f + g^*g|| \geq ||f^*f||$. Thus we have

$$
||f + g||^2 = ||(f + g)^*(f + g)|| \\
= ||f^*f + f^*g + g^*f + g^*g|| \\
\geq ||P(f^*f + f^*g + g^*f + g^*g)|| \\
= ||P(f^*f) + P(g^*g)|| \\
= ||f^*f + g^*g|| \\
\geq ||f^*f|| \\
= ||f||^2.
$$

□
Lemma 5.4. Let $f \in C^*_\text{ess}(H,\Omega)$ and suppose that $f$ is represented by a function $g \in C^*_0(H,\Omega)$, which we consider as a subspace of $\mathcal{B}(H,\Omega)$ by [11 Proposition 7.10]. If $g$ has continuous restriction to some open bisection $U \subseteq H$, then $\sup_{x \in U} |g(\gamma)| \leq ||f||$.

Proof. Since $g$ is continuous on $U$ the supremum $\sup_{\gamma \in U} |g(\gamma)|$ is equal to $\sup_{\gamma \subseteq C} |g(\gamma)|$ for any comeagre subset $C \subseteq U$. Thus taking an infimum over comeagre subsets of $U$ changes nothing, and we see

$$\sup_{\gamma \in U} |g(\gamma)|^2 = \inf_{C \subseteq U} \sup_{\gamma \subseteq C} |g(\gamma)|^2 \leq \inf_{D \subseteq H(0)} \sup_{x \in D} |g^*(g(x))| = ||EL(g^*g)|| = ||EL(f^*f)|| \leq ||f||^2,$$

where $D \subseteq H(0)$ ranges over all comeagre subsets of $H(0)$.

Lemma 5.5. Let $G$ be an effective groupoid with locally compact Hausdorff unit space. Let $U \subseteq G$ be a bisection, and let $\overline{U}^\circ$ be the interior of the closure of $U$. Then $U = \overline{U}^\circ \cdot s(U) = r(U) \cdot \overline{U}^\circ$.

Proof. The inclusion $U \subseteq \overline{U}^\circ \cdot s(U)$ is clear. Fix $\gamma \in \overline{U}^\circ \cdot s(U)$. Then there exists $\eta \in U$ with $s(\eta) = s(\gamma)$, and we have $\gamma \eta^{-1} \in \overline{U}^\circ \cdot s(U) \cdot U^{-1} \subseteq G(0)$. Since $G$ is effective and the closure of the unit space consists of isotropy, we see that $G(0)^\circ = G(0)$ and so $\gamma \eta^{-1} \in G(0)$. Thus $\gamma = \eta \in U$, and so $\overline{U}^\circ \cdot s(U) = U$. A similar argument shows $r(U) \cdot \overline{U}^\circ = U$.

Lemma 5.6. Let $X$ be a Baire space. Let $x \in X$ be a point and let $U \subseteq X$ be a neighbourhood of $x$. Let $V_1, ..., V_\ell \subseteq X$ be open sets such that $x \notin V_k$ for $1 \leq k \leq \ell$. That is, $x$ does not lie in the interior of the closure of any of the $V_k$. Then $Z := U \setminus \bigcup_{k=1}^\ell V_k$ is a non-empty open set, and $x$ lies in the closure of $Z$.

Proof. Since $Z$ is the intersection of $U$ with the complements of finitely many open sets, we see that $Z$ is open. To see that $Z$ is non-empty, we first note that $Z$ can be expressed as

$$Z = U \setminus \bigcup_{k=1}^\ell V_k = \left(U \setminus \bigcup_{k=1}^\ell \partial(V_k^\circ)\right) \cap \left(U \setminus \bigcup_{k=1}^\ell V_k^\circ\right).$$

Since each $\partial(V_k^\circ)$ is the boundary of an open set, it is meagre. Thus $Z$ is a comeagre subset of $U \setminus \bigcup_{k=1}^\ell V_k$, and hence is dense since $X$ is a Baire space. The set $U \setminus \bigcup_{k=1}^\ell V_k$ is non-empty since $x$ is not contained in any $V_k^\circ$ by hypothesis, so $x$ lies in the closure of $Z$ and in particular $Z$ is non-empty.

Proof of Theorem 5.2. First we shall establish that a map $\tilde{\beta}_z : \Omega \to \Sigma$ exists. Fix $\sigma \in \Omega$ and let $U \subseteq H$ be a bisection with $\sigma \in \Omega|_{U}$. Let $\theta_\varphi$ be the inverse to $\varphi|_{H(0)}^* : G(0) \to H(0)$; the homeomorphism induced by considering the restricted isomorphism of commutative $C^*$-algebras $\varphi|_{H(0)} : C_0(H(0)) \to C_0(G(0))$. Viewing elements of $C_c(U,\Omega)$ as continuous functions $\Omega|_{U} \to \mathbb{C}$ satisfying $f(\tau z) = f(\tau)z$ for $z \in \mathbb{T}$, let $f_\sigma : \Omega|_{U} \to \mathbb{C}$ be such a continuous compactly supported function with $f(\sigma) = 1$. Then $\varphi(f_\sigma)$ is a normaliser of $C_0(G(0)) \subseteq C^\ast_{\text{ess}}(G,\Sigma)$, and $\varphi(f_\sigma^* f_\sigma)(\theta_\varphi(s(\sigma))) = (f_\sigma^* f_\sigma)(s(\sigma)) = |f_\sigma(\sigma)|^2 = 1$, so $\theta_\varphi(s(\sigma)) \in \text{dom}(\varphi(f_\sigma))$. Thus $\tilde{\beta}_z(\sigma, U, f_\sigma) := [\varphi(f_\sigma), \theta_\varphi(s(\sigma))]$ is an element of the Weyl twist $\Sigma$.

We claim that $\tilde{\beta}_z(\sigma, U, f_\sigma)$ does not depend on the choice of $U \subseteq H$ or $f_\sigma \in C_c(U,\Sigma)$. To see this, fix another bisection $U' \subseteq H$ with $\sigma \in \Omega|_{U'}$ and $g_\sigma \in C_c(U',\Omega)$ with $g_\sigma(\sigma) = 1$.

Then $U \cap U'$ is non-empty since it contains $g(\sigma)$, and there exists $h \in C_0(X)$ with $\text{supp}(h) \subseteq s(U \cap U')$ and $h|_K = 1$ on a compact neighbourhood $K$ of $s(\sigma)$. The functions $f_\sigma \cdot h$ and $g_\sigma \cdot h$ are both compactly supported sections with support contained in $U \cap U'$. Thus there are functions $\alpha, \alpha' \in C_0(X)$ with $\alpha(s(\sigma)) = 1 = \alpha'(s(\sigma))$ and $f_\sigma \cdot h = g_\sigma \cdot h'$. We then have $\varphi(f_\sigma) \varphi(ha) = \varphi(g_\sigma) \varphi(h'a)$ and $\varphi(ha)(\theta_\varphi(s(\sigma))) = ha(s(\sigma)) = 1 = ha'(s(\sigma)) = \varphi(ha') (\theta_\varphi(s(\sigma)))$, so $[\varphi(f_\sigma), \theta_\varphi(s(\sigma))] = [\varphi(g_\sigma), \theta_\varphi(s(\sigma))]$ and...
\[ \tilde{\beta}_\varphi \] depends only on \( \sigma \in \Omega \). By overloading notation we write \( \tilde{\beta}_\varphi : \Omega \to \Sigma \) for the map \( \sigma \mapsto \tilde{\beta}_\varphi(\sigma) := \tilde{\beta}_\varphi(\sigma, U, f_\sigma) \).

To see that \( \tilde{\beta}_\varphi \) is a groupoid homomorphism, fix a composable pair \((\sigma, \tau) \in \Omega(2)\). For all \( a \in C_0(G(0)) \) we have

\[
a(r(\tilde{\beta}_\varphi(\tau))) = a(\alpha_\varphi(f_\tau)(\theta_\varphi(s(\tau)))) \\
a(\alpha_\varphi(f_\tau)(\theta_\varphi(s(\tau)))) \varphi(f^*_\tau f_\tau)(\theta_\varphi(s(\tau))) \\
= (\varphi(f_\tau)^* a \varphi(f_\tau))(\theta_\varphi(s(\tau))) \\
= f^*_\varphi \varphi^{-1}(a) f_\tau(s(\tau)) \\
= \varphi^{-1}(a)(r(\tau)) f^*_\varphi f_\tau(s(\tau)) \\
= a(\theta_\varphi(r(\tau))) \\
= a(\theta_\varphi(s(\sigma))) \\
= a(s(\tilde{\beta}_\varphi(\sigma))).
\]

Since this holds for all \( a \in C_0(G(0)) \), we see that \( r(\tilde{\beta}_\varphi(\tau)) = s(\tilde{\beta}_\varphi(\sigma)) \), so the pair \((\tilde{\beta}_\varphi(\sigma), \tilde{\beta}_\varphi(\tau))\) is composable.

Let \( U, V \subseteq H \) be bisections with \( \sigma \in \Omega|_U \) and \( \tau \in \Omega|_V \), and let \( f_\sigma \in C_c(U, \Omega) \) and \( f_\tau \in C_c(V, \Omega) \) be functions with \( f_\sigma(\sigma) = f_\tau(\tau) = 1 \). The product \( f_\sigma \cdot f_\tau \) belongs to \( C_c(U V, \Omega) \), and since both functions are supported on bisections we have \( (f_\sigma \cdot f_\tau)(\sigma \tau) = f_\sigma(\sigma)f_\tau(\tau) = 1 \), whereby

\[
\tilde{\beta}_\varphi(\sigma \tau) = [\varphi(f_\sigma f_\tau), \theta_\varphi(s(\sigma \tau))] \\
= [\varphi(f_\sigma), \theta_\varphi(s(\sigma))] \cdot [\varphi(f_\tau), \theta_\varphi(s(\tau))] \\
= \tilde{\beta}_\varphi(\sigma) \tilde{\beta}_\varphi(\tau).
\]

We now show that \( \tilde{\beta}_\varphi \) is a local homeomorphism. The topology on \( \Sigma \) is generated by open sets specified by normalisers \( n \in N(A, B) \) and the homeomorphisms

\[
G_n : \mathbb{T} \times \text{dom}(n) \to \Sigma|_{U_n}, \quad G_n(t, x) = [tn, x].
\]

For \( \sigma \in \Omega \) and an open bisection \( U \subseteq H \) with \( \sigma \in \Omega|_U \), pick \( f_\sigma \in C_c(U, \Omega) \) with \( f_\sigma(\sigma) = 1 \). Let \( U_\sigma \subseteq \Omega \) be the open support of \( f_\sigma \). Define the map \( H_\sigma : \mathbb{T} \times s(U_\sigma) \to U_\sigma \) by \( H_\sigma(z, x) = z \tau_x \), where \( \tau_x \in U_\sigma \) is the unique element of \( U_\sigma \) with \( s(\tau_x) = x \) and \( f_\sigma(\tau_x) > 0 \). Then \( H_\sigma \) is a homeomorphism since \( f_\sigma \) is continuous and non-zero on \( U_\sigma \). Note now that \( f_\sigma(z \tau_x) = z f_\sigma(\tau_x) \), so \( z f_\sigma(z \tau_x) > 0 \) for any \( \tau_x \in U_\sigma \) with \( f_\sigma(\tau_x) > 0 \), giving \( \beta(z \tau_x) = [z \varphi(f_\sigma), \theta_\varphi(s(\tau_x))] \). Letting \( n_\sigma := \varphi(f_\sigma) \), for \((z, x) \in \mathbb{T} \times s(U_\sigma) \) we compute

\[
G_{n_\sigma}^{-1} \circ \tilde{\beta}_\varphi \circ H_\sigma(z, x) = G_{n_\sigma}^{-1} \circ \tilde{\beta}_\varphi(z \tau_x) \\
= G_{n_\sigma}^{-1}[z \varphi(f_\sigma), \theta_\varphi(s(\tau_x))] \\
= G_{n_\sigma}^{-1}[zn_\sigma, \theta_\varphi(s(\tau_x))] \\
= (z, \theta_\varphi(s(\tau_x))).
\]

Thus the composition \( G_{n_\sigma}^{-1} \circ \tilde{\beta}_\varphi \circ H_\sigma \) agrees with the homeomorphism \( \text{id}_{\mathbb{T} \times \theta_\varphi} \) on its domain, implying that \( \tilde{\beta}_\varphi \) is a homeomorphism on \( U_\sigma \). Hence \( \beta_\varphi \) is a local homeomorphism. This argument also shows that \( \tilde{\beta}_\varphi \) descends to a local homeomorphism \( \beta_\varphi : H \to G \), as one need only ignore the \( \mathbb{T} \) component in the homeomorphisms \( H_\sigma \) and \( G_{n_\sigma} \). Hence \((\beta_\varphi, \tilde{\beta}_\varphi)\) forms a homomorphism of twists.

For contradiction, suppose that \( \tilde{\beta}_\varphi \) is not surjective. Fix \([n, x] \in \Sigma \) with \([n, x] \neq \tilde{\beta}_\varphi(\sigma)\) for any \( \sigma \in \Omega \). Then for any open bisection \( U \subseteq H \) and \( f \in C_c(U, \Omega) \), the element
\( \varphi(f) \) is a normaliser in \( C^*_\text{es}(G, \Sigma) \) and so we may consider \( \varphi(f) \) as a function \( \Sigma \to \mathbb{C} \) via Lemma 4.6. We then have \( \text{supp}(\varphi(f)) = \{ [z\varphi(f), y] : z \in \mathbb{T}, y \in \text{dom}(\varphi(f)) \} = \{ [z\varphi(f), \theta(\varphi(x)) : z \in \mathbb{T}x \in \text{supp}(f^*) \} \), which is exactly the image of the support of \( f \) under \( \hat{\beta}_\varphi \). Thus \( \varphi(f)[n, x] = 0 \) for all \( f \in C_c(V, \Omega) \), and since each \( f \) has compact support, the function \( \varphi(f) \) must vanish on an open bisection neighbourhood of \([n, x]\).

For \( f \in C_c(U, \Omega) \) let \( U_f := \text{supp}(f) \subseteq H \) be the open support of \( f \) in \( H \). Note that since \( \beta_\varphi \) restricts to a homeomorphism of unit spaces and is a local homeomorphism, the image of a bisection is a bisection under this map, so \( \beta_\varphi(U_f) \) is an open bisection in \( G \). We claim that \([\alpha_n, x]\) does not belong to the interior of the closure of \( \beta_\varphi(U_f) \). To see this, suppose that \([\alpha_n, x] \in \beta_\varphi(U_f) \). Then there exists an open neighbourhood \( V_{[\alpha_n, x]} \subseteq \beta_\varphi(U_f) \) with \([\alpha_n, x] \in V_{[\alpha_n, x]} \). Then the set

\[
V_{[\alpha_n, x]}^{-1} \beta_\varphi(U_f) = \{ v^{-1}u : v \in V_{[\alpha_n, x]}, u \in \beta_\varphi(U_f)^0, r(v) = r(u) \},
\]

is an open set contained in \( \beta_\varphi(U_f)^{-1} \beta_\varphi(U_f) \subseteq \varphi(U_f^{-1} U_f) \subseteq G(0) \). Thus \( V_{[\alpha_n, x]}^{-1} \beta_\varphi(U_f)^0 \) is contained in \( G(0) \) since \( G \) is effective by Lemma 2.6. This implies

\[
V_{[\alpha_n, x]} = V_{[\alpha_n, x]}^{-1} \beta_\varphi(U_f)^0 = r(V_{[\alpha_n, x]}) \cdot \beta_\varphi(U_f)^0 \subseteq r(\beta_\varphi(U_f)) \cdot \beta_\varphi(U_f)^0,
\]

and Lemma 5.6 then shows that \( V_{[\alpha_n, x]} \subseteq \beta_\varphi(U_f) \), contradicting that \([n, x]\) does not belong to the image of \( \beta_\varphi \).

Let \( f_k \in C_c(U_k, \Omega) \) for \( k = 1, \ldots, \ell \) be a finite collection of such normalisers, so that their sum is a generic element of \( C_c(H, \Omega) \). For each \( k = 1, \ldots, \ell \) let \( U_k \) be the open support of \( f_k \) in \( H \). Let \( \hat{n} \) be the function from Lemma 4.6 associated to the normaliser \( n \). Let \( W \) be an open bisection neighbourhood of \([\alpha_n, x]\) such that the function \( \hat{n} : \Sigma \to \mathbb{C} \) satisfies \( |\hat{n}[m, y]| \geq \hat{n}(n, x)/2 \) for all \([m, y] \in \Sigma|_W\). Note that we can do this since \( \hat{n} \) restricts to a continuous function on a bisection of \( G \). Define \( Z \subseteq G \) by

\[
Z := W \setminus \left( \bigcup_{k=1}^{\ell} \beta_\varphi(U_k) \right).
\]

Lemma 5.6 asserts that \( Z \) is non-empty and open. By Lemma 5.4 we then have

\[
\left| \hat{n} - \sum_{k=1}^{\ell} \varphi(f_k) \right| \geq \sup_{[m, x] \in \Sigma|_Z} \hat{n}[m, y] - \sum_{k=1}^{\ell} \varphi(f_k)[m, y] = \sup_{[m, y] \in \Sigma|_W} |\hat{n}[m, y]| \geq |\hat{n}[n, x]|/2 = \sqrt{n^* n(x)/2}.
\]

Since \( \sqrt{n^* n(x)} \) is positive, we see that \( n \) does not lie in the closure of the image of \( C_c(H, \Omega) \), which is \( C^*_\text{es}(G, \Sigma) \), rendering a contradiction. It follows that \( \beta_\varphi \) and hence \( \hat{\beta}_\varphi \) are surjective.

Finally we shall show that the (twist over the) unit space \( H(0) \) is dense in the kernel of \( \hat{\beta}_\varphi \). The kernel of \( \beta_\varphi \), that is, the preimage of the unit space \( G(0) \) under \( \beta_\varphi \) is open since \( G \) is étale. Let \( U \subseteq \text{ker}(\beta_\varphi) \) be an open bisection and suppose that \( U \cap H(0) = \emptyset \). Then for any \( f \in C_c(U, \Omega) \) we have \([\varphi(f), \theta(\varphi(x))] \in G(0) \times \mathbb{T} \) for any \( x \in \text{dom}(f) \), so \( \varphi(f) \in C_0(G(0)) \).

Since \( \varphi \) restricts to an isomorphism of subalgebras \( C_0(H(0)) \cong C_0(G(0)) \), the function \( f \) must represent an element of \( C_0(H(0)) \) in \( C^*_\text{es}(H, \Omega) \). So there is a function \( g \in C_0(H(0)) \) such that \( f = g \) in \( C^*_\text{es}(G) \). Consider \( f \) and \( g \) as sections of the line bundle associated to \((H, \Omega)\). Since \( U \cap H(0) = \emptyset \), Lemma 5.3 gives \( ||f|| = ||f - g|| = 0 \), implying that
\( f = 0 \). Thus \( U \) must be empty, so every non-empty open subset of \( \ker(\beta_\varphi) \) has non-empty intersection with \( H(0) \); equivalently \( H(0) \) is dense in \( \ker(\beta_\varphi) \).

**Corollary 5.7.** Let \( A \subseteq B \) be an essential Cartan pair and let \((G, \Sigma)\) be the associated Weyl twist. Let \((H, \Omega)\) be a twist over an étale groupoid with locally compact Hausdorff unit space. Suppose there is an isomorphism \( \varphi : C^*_{\text{ess}}(H, \Omega) \to C^*_{\text{ess}}(G, \Sigma) \) that intertwines conditional expectations and restricts to an isomorphism \( \varphi|_{C_0(H(0))} : C_0(H(0)) \to C_0(G(0)) \).

If \( H \) is effective, then the homomorphism \((\beta_\varphi, \tilde{\beta}_\varphi) : (H, \Omega) \to (G, \Sigma)\) in Theorem 5.2 is an isomorphism of twists.

**Proof.** The kernel of \( \beta_\varphi \) is an open normal isotropy group bundle in \( H \) by Theorem 5.2 and so is contained in the interior of the isotropy of \( H \). Since \( H \) is effective, the interior of the isotropy is exactly the unit space of \( H \) and so the map \( \beta_\varphi \) is injective, hence an isomorphism between \( H \) and \( G \).

Suppose \( \tilde{\beta}_\varphi(\omega) \in \Omega(0) \) for some \( \omega \in \Omega \). Let \( q_\Omega : \Omega \to H \) and \( q_\Sigma : \Sigma \to G \) be the canonical quotient maps associated to the twists. Since \((\beta_\varphi, \tilde{\beta}_\varphi)\) is a homomorphism of twists, we have \( \beta_\varphi(q_\Omega(\omega)) = q_\Sigma(\tilde{\beta}_\varphi(\omega)) \in G(0) \), so \( q_\Omega(\omega) \in H(0) \) as \( \beta_\varphi \) is an isomorphism. Hence \( \omega \in H(0) \times \mathbb{T} \), so \( \omega = (s(\omega), z) \) for some \( z \in \mathbb{T} \). Since \( \tilde{\beta}_\varphi(\omega) \in \Sigma(0) \) we have \( \tilde{\beta}_\varphi(\omega) = \tilde{\beta}_\varphi(s(\omega)) \).

Thus for any function \( f \in C_c(H(0), \Omega) \) with \( f(\omega) = 1 \) we have \( f(s(\omega)) = 1 \). Identifying \( \Omega(0) = H(0) \times \{1\} \) with \( H(0) \) we see \( 1 = f(\omega) = f(s(\omega), z) = z f(s(\omega), 1) = z f(s(\omega)) = z \).

Thus \( \omega = (s(\omega), 1) = s(\omega) \), so \( \omega \in \Omega(0) \). Hence \( \tilde{\beta}_\varphi \) and \( \beta_\varphi \) are injective, so give an isomorphism of twists. \( \square \)

Lastly, we show that twists over effective groupoids give rise to essential Cartan pairs. All that remains to show is that \( C_0(G(0)) \) is a maximal commutative subalgebra of \( C^*_{\text{ess}}(G) \) if \( G \) is effective.

**Proposition 5.8.** Let \( G \) be an effective groupoid and let \( \Sigma \) be a twist over \( G \). The inclusion \( C_0(G(0)) \subseteq C^*_{\text{ess}}(G, \Sigma) \) is maximal commutative.

**Proof.** Let \( L \) be the Fell line bundle associated to the twist \((G, \Sigma)\). Using the \( j \)-map in [11] Proposition 7.12] we view \( C^*_\Sigma(G) \) as subalgebra of \( \mathfrak{B}(G, L) \). Fix \( f \in C^*_{\text{ess}}(G, \Sigma) \) commuting with \( C_0(G(0)) \) and pick \( \tilde{f} \in C^*_\Sigma(G, \Sigma) \) with image equal to \( f \) in the essential quotient.

Since \( f \) is the (countable) limit of elements of \( C_c(G, \Sigma) \), and elements of \( C_c(G, \Sigma) \) have support contained in finitely many bisections, the support of \( \tilde{f} g - g \tilde{f} \) is contained in countably many bisections. The finite products and inverse of this family of bisections generates an open subgroupoid \( H \subseteq G \) that itself is covered by countably many bisections, and we may (by including \( G(0) \) into this generating family of bisections) assume that \( H(0) = G(0) \). We then have \( f \in C^*_\Sigma(H, \Sigma|_H) \) and it suffices to show that \( f \) belongs to \( C_0(H(0)) \).

Applying [11] Lemma 7.13] we see that there is a comeagre subset \( C \subseteq H \) on which \( \tilde{f} \) is continuous. If the support of \( \tilde{f} \) does not intersect \( C \), then \( f = 0 \) by [11] Proposition 7.18].

Else, for \( \gamma \in C \) with \( \tilde{f}(\gamma) \neq 0 \), there is \( \varepsilon > 0 \) and a neighbourhood bisection \( U \subseteq H \) of \( \gamma \) with \( |\tilde{f}(\gamma)| > \varepsilon \) for all \( \gamma \in U \cap C \). Then, for all \( \eta \in U \cap C \) and all \( g \in C_0(G(0)) \) we have

\[
g(r(\eta)) \tilde{f}(\eta) = (g \tilde{f})(\eta) = (\tilde{f} g)(\eta) = \tilde{f}(\eta) g(s(\eta)).
\]

Since \( \tilde{f}(\eta) \) is non-zero for all \( \eta \in U \cap C \) and functions \( g \in C_0(G(0)) \) separate points in \( G(0) \), we see that \( r(\eta) = s(\eta) \). Hence \( U \cap C \) is contained in the isotropy of \( H \), and so is \( U \) since \( C \subseteq H \) is comeagre (hence dense). Then \( U \) is a bisection contained in the interior of the
isotropy of $G$, therefore $U \subseteq G^{(0)}$ since $G$ is effective. Thus $\bar{f}$ is continuous on a comeagre subset of $G^{(0)}$ and (up to a cameagre subset) is supported there. Hence $f \in C_0(G^{(0)})$. 

**Remark 5.9.** For a twist $\Sigma$ topologically principal second countable étale groupoid $G$ with locally compact Hausdorff unit space, Exel and Pitts define their essential twisted groupoid $C^*$-algebra as the quotient of $C^*_\gamma(G, \Sigma)$ by the grey ideal. They define weak Cartan inclusions [3] Definition 2.11.5 and show that any such inclusion is isomorphic one arising from a twist over a topologically principal groupoid with their essential twisted groupoid $C^*$-algebra [3] Corollary 3.9.5. Combining this with [3] Proposition 3.4.8, if the Weyl groupoid $G$ of an essential Cartan inclusion $A \subseteq B$ is topologically free, then this inclusion is also a weak Cartan inclusion, hence $B$ is isomorphic to the Exel-Pitts essential twisted groupoid $C^*$-algebra by [3] Theorem 3.9.4. Of course, by Theorem 4.11 we have that $B$ is isomorphic to the Kwaśniewski-Meyer essential twisted groupoid $C^*$-algebra. Hence, if $G$ is a second countable, effective, topologically free étale groupoid with locally compact Hausdorff unit space and $\Sigma$ is a twist over $G$, then the essential twisted groupoid $C^*$-algebras of Exel-Pitts and Kwaśniewski-Meyer coincide. This extends the previously known case [3] Theorem 3.6.8 of these two essential $C^*$-algebras coinciding, swapping in effectivity of the groupoid in place of existence of a dense $G_\delta$ subset of units in $G^{(0)}$ with trivial isotropy.

### 5.1. Non-uniqueness for non-effective groupoids.

One may then ask the question of when two étale groupoids give rise to isomorphic essential groupoid $C^*$-algebras. We are not able to answer this question in full generality, but we can provide this for the condition when one groupoid is contained in another as an open subgroupoid. We answer this question in the case where the twist over the groupoid is trivial.

If $G$ is an étale groupoid and $H$ is an open subgroupoid then there is an inclusion $C_c(H) \hookrightarrow C_c(G)$ given by extending functions on $H$ by zero to $G$. This map is a *-homomorphism, so induces a *-homomorphism between the (full) groupoid $C^*$-algebras $\iota : C^*(H) \to C^*(G)$. This map intertwines conditional expectations since it does so on $C_c(H)$, hence $\iota$ descends to a homomorphism $C^*_{\text{ess}}(H) \to C^*_{\text{ess}}(G)$. This homomorphism is injective on the essential groupoid $C^*$-algebras since it intertwines expectations and the expectations are faithful on the essential quotients. Hence, we may always consider $C^*_{\text{ess}}(H) \subseteq C^*_{\text{ess}}(G)$ for an open subgroupoid $H \subseteq G$.

**Proposition 5.10.** Let $H \subseteq G$ be an open subgroupoid of an étale groupoid with locally compact Hausdorff unit space. Consider $C^*_{\text{ess}}(H) \subseteq C^*_{\text{ess}}(G)$. Then $C^*_{\text{ess}}(H) = C^*_{\text{ess}}(G)$ if and only if for every $\gamma \in G$ there exists a bisection $V \subseteq H$ with $\gamma \in V \cdot s(V)$. In particular, $H$ is dense in $G$ and $H^{(0)} = G^{(0)}$.

**Proof.** Suppose first that for every $\gamma \in G$ there is some bisection $V \subseteq H$ with $\gamma \in V \cdot s(V)$. Let $U \subseteq G$ be a bisection neighbourhood of $\gamma$ contained in $V \cdot s(V)$; such a bisection exists since $V \cdot s(V)$ is open. We claim that $C_c(U)$ and $C_c(V \cdot s(U))$ are equal in the essential groupoid $C^*$-algebra. To show this, fix $f \in C_c(U)$. Note that $s|_V^{-1} \circ s : U \to V \cdot s(U)$ is a homeomorphism, so we can define $\tilde{f} := f \circ s|_V^{-1} \circ s$ on $V$, and extend by zero to a function on $H$. Then $\tilde{f}$ is a continuous and compactly supported function on $V$ precisely since $f$ is, and $s|_V^{-1} \circ s$ is a homeomorphism between $U$ and $V \cdot s(U)$.

We claim $f - \tilde{f}$ has meagre support, which by [11] Proposition 7.18 shows that $f$ and $\tilde{f}$ represent the same element of $C^*_{\text{ess}}(G)$. Since $s|_V^{-1} \circ s$ restricts to the identity on $U \cap V$, the open support of $f - \tilde{f}$ must be contained in $(\text{supp}^\circ(f) \setminus (U \cap V)) \cup (\text{supp}^\circ(\tilde{f}) \setminus (U \cap V))$. Since the open supports of $f$ and $\tilde{f}$ are contained in $U$ and $V$ respectively, this reduces to $\text{supp}^\circ(f) \setminus V \cup \text{supp}^\circ(\tilde{f}) \setminus U$. We shall show that each of these components of the union is meagre.
Note that $\text{supp}^\circ (f) \subseteq U \subseteq \overline{\nabla}$, and $\nabla \setminus V$ is meagre as it is closed and contains no open subsets. Hence $\text{supp}^\circ (f) \setminus V$ is meagre.

We claim that $U$ is dense in the support of $\tilde{f}$. Then by a similar argument as above the set $\text{supp}^\circ (\tilde{f}) \subseteq \overline{U} \setminus U$ is meagre. Fix an open subset $W \subseteq \text{supp}^\circ (\tilde{f})$. Note that $W$ is open in $G$ since $\text{supp}^\circ (\tilde{f})$ is. Suppose $W \cap U = \emptyset$. We shall show that $W$ is then empty, whereby $\text{supp}^\circ (\tilde{f})$ is contained in the closure of $U$. Note that since $s(W) \subseteq s(\text{supp}^\circ (\tilde{f})) \subseteq s(U)$, it suffices to show that $U \cdot s(W)$ is empty. We then note that $(U \cdot s(W)) \cap V = U \cap V \cdot s(W) = U \cap W = \emptyset$, whereby $U \cdot s(W)$ is empty since $U$ lies in the closure of $V$, so every non-empty open subset of $U$ has non-empty intersection with $V$. Thus $W \cap U \neq \emptyset$ for all non-empty open subsets $W \subseteq \text{supp}^\circ (\tilde{f})$, whereby $\text{supp}^\circ (\tilde{f})$ is contained in the closure of $U$ so $\text{supp}^\circ (\tilde{f}) \setminus U$ is meagre. It follows that $\text{supp}^\circ (f - \tilde{f})$ is a meagre set and the homeomorphism $s|_\overline{V}^{-1} \circ s$ induces an identification of $C_c(U)$ and $C_c(V)$ in $C^*_{\text{ess}}(G)$. We can do this on neighbourhoods around any point in $G$, so a partition of unity argument shows that $C^*_{\text{ess}}(H)$ is dense in $C^*_{\text{ess}}(G)$, whereby they are equal.

We shall show the converse via contrapositive. Suppose there exists $\gamma \in G \setminus H$ with the property that, for any bisection $V \subseteq H$ we have $\gamma \notin \nabla^\circ \cdot s(V)$. That is, for any bisection $V \subseteq H$ either no open neighbourhood of $\gamma$ is contained in $\nabla$, or $s(\gamma) \notin s(V)$. Let $U$ be a bisection containing $\gamma$ and let $K \subseteq U$ be a compact neighbourhood also containing $\gamma$, and denote the interior of $K$ by $K^\circ$. Let $f \in C_0(U)$ be a function with $f|_K = 1$. We claim that $f$ does not belong to $C^*_{\text{ess}}(H)$. Recall that the subspaces $C_c(V)$ for bisections $V \subseteq H$ span a dense subalgebra of $C^*_{\text{ess}}(H)$, so it suffices to show that $f$ cannot be approximated by any finite sum of elements belonging to subspaces of this form. Fix bisections $V_i \subseteq H$ and functions $g_i \in C_c(V_i)$ for $i = 1, \ldots, n$. By assumption, for each $i \leq 1$, at least one of the following two statements is true: that $s(\gamma) \notin s(V_i)$ or $\gamma \notin V_i^\circ$.

Let $I$ denote the set of values of $i$ for which $s(\gamma) \notin s(V_i)$. For $i \in I$, $g_i$ has compact support $K_i \subseteq V_i$, and so $s(K_i)$ is closed in $G(0)$. Since $G(0)$ is regular, there are disjoint open neighbourhoods $V'_i$ and $W'_i$ separating $s(K_i)$ and $\{s(\gamma)\}$. Let $W_i = s|_U^{-1}(W'_i \cap s(U))$, that is, the lift of $W'_i$ to $U$ under the source map. Then $g_i|_{W_i} = 0$, since $g_i$ is only non-zero on its support, and the source of $g_i$'s support does not intersect the source of $W_i$. Let $W := \bigcap_{i \in I} W_i$, and note that this is an open neighbourhood of $\gamma$.

For the second case, let $J$ denote the set of values of $i$ such that $s(\gamma) \in s(V_i)$ but $\gamma \notin V_i^\circ$. The set $U'_J := K^\circ \setminus \bigcup_{i \in J} V_i$ is then open, non-empty, and has $\gamma$ as a limit by Lemma 5.5.

The set $Z := U'_J \cap W$ is non-empty since $\gamma \in W \cap U'_J$, and $W$ is open so intersects $U'_J$. Moreover, $Z$ is open as both $U'_J$ and $W$ are, and contained in $K^\circ$ since $U'_J$ is. Thus each $g_i$ is zero on $Z$ as $g_i$ is zero on either $W_i$ (if $i \in I$) or is zero on $U'_J$ (if $i \in J$), since $U'_J$ lies in the complement of each $V_i$, which in turn contains the support of $g_i$. Thus the sum of the $g_i$ is again zero on $Z$, and so

$$\left( f - \sum_{i=1}^n g_i \right)|_Z = f|_Z = 1,$$

as $Z \subseteq K$. By Lemma 5.4 we see that $\|f - \sum_i g_i\| \geq 1$ and so $f$ does not lie in the closure of the span of $C_c(V)$ for bisections $V \subseteq H$. \qed

Remark 5.11. If $(G, \Sigma)$ is a twist over an effective étale Hausdorff locally compact second countable groupoid, then [19 Proposition 4.11] states that the Weyl groupoid associated to the pair $C_0(G(0)) \subseteq C^*_\Gamma(G, \Sigma)$ is canonically isomorphic to $G$. When $G$ is globally Hausdorff and second countable, effectivity is equivalent to the condition that $G$ is topologically principal: that there is a dense set of points in the unit space of $G$ that have trivial
isotropy. If $G$ is Hausdorff, the conditions of Proposition 5.10 imply $G = H$, so this is only an interesting result for non-Hausdorff groupoids. Notably, if one has a groupoid $G$ satisfying the conditions of Proposition 5.10 such that the unit space $H := G^{(0)}$ is dense in $G$, then $C^*_\text{ess}(G) = C^*_\text{ess}(H) = C_0(G^{(0)})$. However, $G$ need not consist only of units since the unit space is not closed in a non-Hausdorff groupoid.

Example 5.12. Consider the line with two origins $G = (0, 1) \cup \{0_1, 0_2\}$. Define the range and source maps $r, s : G \to (0, 1) \cup \{0_1, 0_2\}$ by $r(t) = s(t) = t$ for $t > 0$ and $r(0_i) = s(0_i) = 0_0$ for $i = 0, 1$. Define a multiplication $G_x \times_r G \to G$ by $t \cdot t = t$ for $t > 0$ and $0_i \cdot 0_j = 0_{i + j \mod 2}$.

Then the unit space of $G$ is homeomorphic to the interval $[0, 1]$, and the groupoid $G$ satisfies the conditions of Proposition 5.10 with the dense open subgroupoid $H = G^{(0)}$. Thus $C^*_\text{ess}(G) \cong C[0, 1]$, but clearly $G$ is not isomorphic to $[0, 1]$ as a groupoid, since it is both non-Hausdorff and contains non-trivial isotropy (although, very little such isotropy).

Example 5.12 illustrates that sometimes removing ‘small’ amounts of isotropy does not change the resulting essential groupoid $C^*$-algebra. The next result states that when $G$ is effective, then removing any more isotropy will change the resulting essential groupoid $C^*$-algebra, so essential groupoids are optimal in a certain sense.

Corollary 5.13. Let $G$ be an effective étale groupoid with locally compact Hausdorff unit space. Then the only open subgroupoid $H \subseteq G$ with $C^*_\text{ess}(H) = C^*_\text{ess}(G)$ is $H = G$.

Proof. Fix $\gamma \in G$. By Proposition 5.10 there exists a bisection $U \subseteq H$ with $\gamma \in \overline{U}^{-r} \cdot s(U)$. Let $\eta \in U$ be the unique element of $U$ with $s(\eta) = s(\gamma)$. Then $\gamma \eta^{-1} \in \overline{U}^{-s(\eta)} \cdot s(U) \cdot U^{-1}$ which is an open set contained in the closure of the unit space of $G$, so is contained in $G^{(0)}$. Since $G$ has Hausdorff unit space, the closure of the unit space is consists only of isotropy, so $G^{(0)}$ is contained in the interior of the isotropy of $G$. Since $G$ is effective, the interior of the isotropy is again the unit space, and so we have $\gamma \eta^{-1} \in G^{(0)}$ implying $\gamma = \eta \in U \subseteq H$. This holds for all $\gamma \in G$ so we see $G = H$. □

Remark 5.14. As mentioned earlier, twists over effective groupoids are not the only twists that give rise to essential Cartan inclusions. Theorem 5.2 implies that such groupoids have effective twisted quotients giving rise to the same essential Cartan pair, and in particular twists descend to the effective quotients of these groupoids.

Unfortunately it is not known what properties of a groupoid $H$ and a twist $\Omega$ over $H$ ensure that the inclusion $C_0(H^{(0)}) \subseteq C^*_\text{ess}(H, \Omega)$ is maximal abelian. As shown, the Weyl groupoid associated to an essential commutative Cartan pair is effective, but Example 5.12 shows that these are not the only such groupoids.

6. AUTOMORPHISMS OF TWISTS AND CARTAN AUTOMORPHISMS

Having established the link between twists over certain étale groupoids and essential Cartan pairs, one may ask whether automorphisms of one object induce automorphisms of the other. We provide constructions in both directions: Cartan automorphisms from twisted groupoid automorphisms and vice versa. We also show that these constructions are mutually inverse if the twist is over an effective groupoid, giving an isomorphism of the automorphism group of an essential Cartan pair $A \subseteq B$ to the automorphism group of the corresponding Weyl pair $(G(A, B), \Sigma(A, B))$. In particular we show that all automorphisms of twists of effective groupoids that induce essential commutative Cartan pairs come from automorphisms of the induced essential Cartan pair and vice versa.

Throughout this section we assume $(G, \Sigma)$ is a twisted groupoid giving rise to an essential Cartan pair $C_0(G^{(0)}) \subseteq C^*_\text{ess}(G, \Sigma)$.

Definition 6.1. Let $(A_1, B_1)$ and $(A_2, B_2)$ be essential Cartan inclusions with local conditional expectations $E_i : B_i \to M_{\text{loc}}(A_i)$ for $i = 1, 2$. A Cartan homomorphism or
homomorphism of Cartan pairs \((A_1, B_1) \to (A_2, B_2)\) is a \(*\)-homomorphism \(\varphi : B_1 \to B_2\) such that

1. \(\varphi(N(A_1, B_1)) \subseteq N(A_2, B_2)\); and
2. \(E_2 \circ \varphi(b) = \varphi \circ E_1(b)\) for all \(b \in B_1\) with \(E_1(b) \in A_1\).

An isomorphism of Cartan pairs or Cartan isomorphism is a Cartan morphism that is a \(*\)-isomorphism.

One may expect that an appropriate definition of morphisms of inclusions of \(C^*\)-algebras should preserve the inclusion itself. While this is not directly required in the definition of a Cartan homomorphism, it does follow from condition (1) of Definition 6.1. The author thanks an anonymous reviewer for pointing this out and providing the following short proof.

**Lemma 6.2.** Let \(A_1 \subseteq B_1\) and \(A_2 \subseteq B_2\) be regular non-degenerate inclusions, and let \(\varphi : B_1 \to B_2\) be a \(*\)-homomorphism with \(\varphi(N(A_1, B_1)) \subseteq N(A_2, B_2)\). Then \(\varphi(A_1) \subseteq A_2\).

**Proof.** Since \(A_1\) is spanned by its positive elements, it suffices to show that positive elements of \(A_1\) are mapped into \(A_2\). Fix \(a \in A_1\) positive and let \((u_1) \subseteq A_2\) be an approximate identity. Since \(a\) is positive, its square root belongs to \(A_1\), and \(A_1\) is of course contained in the normalisers \(N(A_1, B_1)\). Hence we have \(\varphi(a^\frac{1}{2})^* u_1 \varphi(a^\frac{1}{2}) \in A_2\) for all \(\lambda\), and taking the limit gives \(\varphi(a^\frac{1}{2})^* \varphi(a^\frac{1}{2}) = \varphi(a) \in A_2\).

If \(\varphi : (A_1, B_1) \to (A_2, B_2)\) is an isomorphism of essential Cartan pairs as above, then \(\varphi(A_1) = A_2\) will hold since \(A_1\) is masa in \(B_1\). That is, \(\varphi\) restricts to an isomorphism \(A_1 \cong A_2\).

**Lemma 6.3.** Let \((A_1, B_1)\) and \((A_2, B_2)\) be essential Cartan inclusions with local conditional expectations \(E_i : B_i \to M_{\text{loc}}(A_i)\) for \(i = 1, 2\) and let \(\varphi : B_1 \to B_2\) be a \(*\)-homomorphism. If \(\varphi\) intertwines conditional expectations and \(\varphi(A_1) = A_2\), then \(\varphi\) is a Cartan morphism.

**Proof.** We need only show that \(\varphi\) maps normalisers to normalisers. Let \(n \in N(A_1, B_1)\) be a normaliser and fix \(a \in A_2\). Since \(\varphi(A_1) = A_2\), there exists \(a' \in A_1\) with \(\varphi(a') = a\). Then \(\varphi(n)^* \varphi(a) \varphi(n) = \varphi(n^* a' n) \in \varphi(A_1) = A_2\).

If the pairs \((A_1, B_1)\) and \((A_2, B_2)\) are aperiodic inclusions then the conditional expectations are unique by [10 Theorem 3.6]. Thus the expectations are intertwined by any \(*\)-homomorphism that restricts to an isomorphism of the subalgebras \(A_1\) and \(A_2\).

**Lemma 6.4.** The assignment \((A, B)\) to \((G(A, B), \Sigma(A, B))\) taking an essential Cartan pair to its Weyl groupoid and twist is functorial for Cartan isomorphisms. That is, if \(\varphi\) and \(\psi\) are automorphisms of \((A, B)\) then \(\tilde{\beta}_\varphi \circ \tilde{\beta}_\psi = \tilde{\beta}_\varphi \circ \tilde{\beta}_\psi\).

**Proof.** We observe that \(\tilde{\beta}_\varphi \circ \tilde{\beta}_\psi[n, x] = [\varphi(\psi(n)), ((\varphi \circ \psi))^*_\Sigma(G(0))^{-1}(x)] = \tilde{\beta}_\varphi \circ \tilde{\beta}_\psi[n, x]\).

**Proposition 6.5.** Let \((G, \Sigma)\) be a twist over an étale groupoid giving rise to an essential Cartan pair. Let \((\beta, \tilde{\beta}) : (G, \Sigma) \to (G, \Sigma)\) be an automorphism of twists. Then there is an automorphism of Cartan pairs \(\Phi_{(\beta, \tilde{\beta})} : (C_0(G(0)), C^*_\text{ess}(G, \Sigma)) \to (C_0(G(0)), C^*_\text{ess}(G, \Sigma))\) mapping a section \(f \in C_c(G, \Sigma)\) to \(f \circ \beta^{-1}\).

**Proof.** Since \((\beta, \tilde{\beta})\) is a twisted automorphism, so is \((\beta^{-1}, \tilde{\beta}^{-1})\), and in particular \(\beta^{-1}\) is a homeomorphism mapping bisections to bisections, so maps sums of compactly supported sections to sums of compactly supported sections. Let \(\Phi^0_{(\beta, \tilde{\beta})} : C_c(G, \Sigma) \to C_c(G, \Sigma)\) be the map \(\Phi^0_{(\beta, \tilde{\beta})}(f) = f \circ \beta^{-1}\). That \(\Phi^0_{(\beta, \tilde{\beta})}\) is a \(*\)-isomorphism follows from the fact that \((\beta, \tilde{\beta})\) is an isomorphism of twists.
The isomorphism $\Phi_0^{\gamma}(\beta,\tilde{\beta})$ then extends to an automorphism $\Phi_{(\beta,\tilde{\beta})}$ of the full twisted groupoid $C^*\text{-}algebra C^*(G,\Sigma)$. Let $R$ be the restriction map Proposition 4.3 and note that $R$ extends to the canonical conditional expectation $EL: C^*(G,\Sigma) \to M_0(C_0(G(0)))$. Then clearly $R \circ \Phi_0^{\gamma}(\beta,\tilde{\beta}) = \Phi_0^{\gamma}(\beta,\tilde{\beta}) \circ R$, so $\Phi_{(\beta,\tilde{\beta})}$ descends to an automorphism of the quotient $C^*_{\text{ess}}(G,\Sigma)$.

To see that $\Phi_{(\beta,\tilde{\beta})}$ is an automorphism of essential Cartan pairs, note that $\beta$ restricts to a homeomorphism of the unit space $G^{(0)}$, and so for $f \in C_0(G^{(0)})$ we have $\Phi_{(\beta,\tilde{\beta})}(f) = f \circ \beta^{-1}$ which belongs to $C_0(G^{(0)})$. That $\Phi_{(\beta,\tilde{\beta})}$ maps normalisers to normalisers follows from Lemma 6.3.

\textbf{Corollary 6.6.} The assignment $(G,\Sigma)$ to $(C_0(G^{(0)}), C^*_{\text{ess}}(G,\Sigma))$ is functorial lifting isomorphisms of twists to Cartan isomorphisms. That is, the map $(\beta,\tilde{\beta}) \mapsto \Phi_{(\beta,\tilde{\beta})}$ from automorphisms of the twist $(G,\Sigma)$ to Cartan automorphisms of $(C_0(G^{(0)}), C^*_{\text{ess}}(G,\Sigma))$ is a group homomorphism.

\textbf{Proof.} Let $(\beta,\tilde{\beta}),(\beta',\tilde{\beta}') \in \text{Aut}(G,\Sigma)$. For each slice $U \subseteq G$ and $f \in C_c(U,\Sigma)$ we have $\Phi_{(\beta,\tilde{\beta})}(\Phi_{(\beta',\tilde{\beta}')} f) = f \circ (\beta^{-1} \circ \beta) = \Phi_{(\beta,\tilde{\beta})} \circ \Phi_{(\beta',\tilde{\beta}')} f$. Such normalisers span a dense subalgebra of $C^*_{\text{ess}}(G,\Sigma)$ and both $\Phi_{(\beta,\tilde{\beta})} \circ \Phi_{(\beta',\tilde{\beta}')} \Phi_{(\beta,\tilde{\beta})} \Phi_{(\beta',\tilde{\beta}')}$ are linear and isometric so the equality extends to all elements of $C^*_{\text{ess}}(G,\Sigma)$.

\textbf{Proposition 6.7.} Let $(G,\Sigma)$ be a twist over an étale groupoid giving rise to an essential Cartan pair. Let $(\beta,\tilde{\beta})$ be an automorphism of the twist $(G,\Sigma)$. If the induced Cartan automorphism $\Phi_{(\beta,\tilde{\beta})}$ is equal to the identity on $C^*_{\text{ess}}(G,\Sigma)$, then there is a dense subset $D \subseteq G$ with $\beta|_D = \text{id}_D$. Moreover, if $\beta \neq \text{id}_G$ then there exists $\eta \in \overline{G^{(0)}} \setminus G^{(0)}$, and in particular $G^{(0)} \neq G^{(0)}$.

\textbf{Proof.} Suppose that $U \subseteq G$ is an open bisection with $\beta(\gamma) \neq \gamma$ for all $\gamma \in U$. We claim that there is a non-empty open subset $V \subseteq U$ with $\beta^{-1}(V) \cap V = \emptyset$. If $\beta^{-1}(U) \cap U = \emptyset$ then the choice of $V := U$ suffices. Else, if $\beta^{-1}(U) \cap U = \emptyset$ then there exists some $\gamma \in U$ with $\beta^{-1}(\gamma) \in U$. Since $U$ is Hausdorff, we can pick open $V_1, V_2 \subseteq U$ with $\gamma \in V_1, \beta^{-1}(\gamma) \in V_2$, and $V_1 \cap V_2 = \emptyset$. Set $V := V_1 \cup \beta(V_2)$. Then $V \cap \beta^{-1}(V) = V_1 \cap \beta(V_2) \cap \beta^{-1}(V_1) \cap V_2 \subseteq V_1 \cap V_2 = \emptyset$ as required. In particular, since $V$ and $\beta(V)$ belong to the same bisection and are disjoint the product slices $V^{-1}\beta^{-1}(V)$ and $\beta^{-1}(V)^{-1}V$ are empty. Thus for any $f \in C_c(V,\Sigma)$ the compositions $f \ast \Phi_{(\beta,\tilde{\beta})}(f)$ and $\Phi_{(\beta,\tilde{\beta})}(f) \ast f$ are zero as their supports are $V^{-1}\beta(V)$ and $\beta^{-1}(V)^{-1}V$, respectively. Since $\Phi_{(\beta,\tilde{\beta})}$ is the identity on $C^*_{\text{ess}}(G,\Sigma)$, we then have $0 = ||f - \Phi_{(\beta,\tilde{\beta})}(f)||^2 = ||f \ast \Phi_{(\beta,\tilde{\beta})}(f) - \Phi_{(\beta,\tilde{\beta})}(f) \ast f + \Phi_{(\beta,\tilde{\beta})}(f) \ast f|| = ||f \ast \Phi_{(\beta,\tilde{\beta})}(f) \ast f||$, and so $f \ast f = -\Phi_{(\beta,\tilde{\beta})}(f) \ast f$. But both $f \ast f$ and $\Phi_{(\beta,\tilde{\beta})}(f) \ast f$ are positive elements of $C^*_{\text{ess}}(G,\Sigma)$, so $f \ast f = 0$ must hold. Thus $f = 0$ for all $f \in C_c(V,\Sigma)$, whereby $V = \emptyset$, which is a contradiction. Hence there are no open bisections $U \subseteq G$ with $\beta(\gamma) \neq \gamma$ for all $\gamma \in U$, and so the subset $D$ of $G$ on which $\beta$ acts trivially is dense in $G$.

Suppose $\beta \neq \text{id}_G$. Then there exists $\gamma \in G$ with $\beta(\gamma) \neq \gamma$. Let $U \subseteq G$ be an open bisection with $\gamma \in U$. Then $U \cap D$ is dense in $U$, and so $(U \cap D)^{-1}\beta(U \cap D) = (U \cap D)^{-1}(U \cap D) = s(U \cap D)$ is dense in the bisection $U^{-1}\beta(U)$. In particular $U^{-1}\beta(U)$ is an open bisection contained in the closure of the unit space $G^{(0)}$, so in particular $U^{-1}\beta(U) \subseteq \overline{G^{(0)}}$. Since $\beta(\gamma) \neq \gamma$, we have that $\eta := \gamma^{-1}\beta(\gamma)$ is not a unit, and so $\eta \in \overline{G^{(0)}} \setminus G^{(0)}$.\hfill $\square$
Corollary 6.8. Let \((G, \Sigma)\) be the Weyl twist of an essential Cartan pair. An automorphism \((\beta, \tilde{\beta})\) of \((G, \Sigma)\) gives rise to the identity automorphism \(\Phi_{(\beta, \tilde{\beta})}\) of \(C^*_\text{ess}(G, \Sigma)\) if and only if \((\beta, \tilde{\beta}) = (\text{id}_G, \text{id}_\Sigma)\).

Proof. Since \(G\) has Hausdorff unit space, the closure of the unit space is contained in the isotropy of \(G\), and hence so is the interior. The interior of the isotropy is again the unit space of \(G\) since \(G\) is effective. If \(\tilde{\beta}\) is not the identity but \(\Phi_{(\beta, \tilde{\beta})} = \text{id}_{C^*_\text{ess}(G, \Sigma)}\) then Proposition 6.7 gives that \(\overline{G^{(0)}}\) is strictly larger than \(G^{(0)}\), leading to a contradiction. \(\Box\)

Corollary 6.9. If \(G\) is effective then the homomorphism

\[
\Phi: \text{Aut}(G, \Sigma) \to \text{Aut}(C_0(G^{(0)}), C^*_\text{ess}(G, \Sigma))
\]

is injective.

Theorem 6.10. For the Weyl twist \((G, \Sigma)\) of an essential Cartan pair \((A, B)\) the constructions in Theorem 5.2 and Proposition 6.9 are mutually inverse. That is, for a twisted groupoid automorphism \((\beta, \tilde{\beta}) : (G, \Sigma) \to (G, \Sigma)\) we have \(\beta_\Phi(\beta, \tilde{\beta}) = \beta\) and \(\tilde{\beta}_\Phi(\beta, \tilde{\beta}) = \tilde{\beta}\).

For a Cartan automorphism \(\Phi : (C_0(G^{(0)}), C^*_\text{ess}(G, \Sigma)) \to (C_0(G^{(0)}), C^*_\text{ess}(G, \Sigma))\) we have \(\Phi_{(\beta_\Phi, \tilde{\beta}_\Phi)} = \Phi\). Hence \(\text{Aut}(G, \Sigma)\) and \(\text{Aut}(C_0(G^{(0)}), C^*_\text{ess}(G, \Sigma))\) are isomorphic as groups via these constructions.

Proof. Let \(\Phi : (C_0(G^{(0)}), C^*_\text{ess}(G, \Sigma)) \to (C_0(G^{(0)}), C^*_\text{ess}(G, \Sigma))\) be a Cartan automorphism. By Lemma 6.4 we have \(\beta_\Phi(n, x) = [\Phi(n), \theta_\Phi(x)]\), and so \(\tilde{\beta}_\Phi^{-1}(n, x) = [\Phi^{-1}(n), \theta_\Phi^{-1}(x)]\). For a normaliser \(n \in N(C_0(G^{(0)}), C^*_\text{ess}(G, \Sigma))\) let \(U_n := \{[\alpha, x] : x \in \text{dom}(n)\}\) be the bisection specified by \(n\), and fix \(f \in C_c(U_n, \Sigma)\). Then \(f = \tilde{n}h\) for some \(h \in C_0(G^{(0)})\) by Lemma 4.7 and moreover for all \([n, x]\) in the open support of \(f\) we have \([n, x] = [\tilde{n}h, x]\) for some fixed \(z \in \mathbb{T}\) (namely \(z = h(x)\)).

Thus

\[
\Phi(f)[\Phi(f), \theta_\Phi(x)] = \Phi(\tilde{n}h)[\Phi(\tilde{n}h), \theta_\Phi(x)]
\]

\[
= \sqrt{\Phi((nh)^*(nh))}[\theta_\Phi(x)]
\]

\[
= \sqrt{(nh)^*(nh)(x)}
\]

\[
= \sqrt{f^*f(x)}
\]

\[
= f[f, x],
\]

and

\[
\Phi_{(\beta_\Phi, \tilde{\beta}_\Phi)}(f)[\Phi(f), \theta_\Phi(x)] = f(\tilde{\beta}_\Phi^{-1}[\Phi(f), \theta_\Phi(x)])
\]

\[
= f[\Phi^{-1}\Phi(f), \theta_\Phi^{-1}\theta_\Phi(x)]
\]

\[
= f[f, x].
\]

Hence \(\Phi = \Phi_{(\beta_\Phi, \tilde{\beta}_\Phi)}\), so the map \(\text{Aut}(G, \Sigma) \to \text{Aut}(C_0(G^{(0)}), C^*_\text{ess}(G, \Sigma))\) is surjective.

The Weyl groupoid is effective by Lemma 2.6 and so Corollary 6.9 gives that this map \(\text{Aut}(G, \Sigma) \to \text{Aut}(C_0(G^{(0)}), C^*_\text{ess}(G, \Sigma))\) is injective, hence an isomorphism. The above calculation shows that the construction in Theorem 5.2 is a one-sided inverse to this isomorphism, hence is the two-sided inverse and is itself an isomorphism. \(\Box\)

The isomorphism of Theorem 6.10 generalises a similar result of Raad [18], where the isomorphism is found in the case of Renault’s Cartan pairs and reduced twisted groupoid \(C^*\)-algebras.

In a recent preprint, Komura [8] Proposition 2.2.2] characterises the Cartan automorphism group of \(C^*_r(G)\) for Hausdorff étale effective groupoids \(G\) as the crossed product of the automorphism group of \(G\) by the group of continuous homomorphisms \(G \to \mathbb{T}\). Komura’s proof does not explicitly use the Cartan structure of the inclusion \(C_0(G^{(0)}) \subseteq \)
Corollary 6.11. The category of twists of effective étale groupoids with locally compact Hausdorff unit spaces with isomorphisms of twists as morphisms is equivalent to the category of essential Cartan pairs with Cartan isomorphisms via the functor taking a twist \((G, \Sigma)\) to the pair \((C_0(G(0)), C^{\ast}_{\text{ess}}(G, \Sigma))\).

Proof. Corollary 6.6 gives that this assignment is a functor, Theorem 6.10 gives that it is full and faithful. Proposition 5.8 implies that effective groupoids gives masa, hence essential Cartan inclusions, and Theorem 4.11 implies that all essential Cartan inclusions are of this form. Hence the functor is essentially surjective. \(\square\)

References

[1] Jacques Dixmier, Sur certains espaces considérés par M. H. Stone, Summa Bras. Math. 2 (1951), 151–181, doi: 10.24033/bsmf.1545 (French).
[2] R. Exel, Noncommutative Cartan sub-algebras of \(C^\ast\)-algebras, New York Journal of Mathematics, 17 (2011), 331–382.
[3] R. Exel, D.R. Pitts, Characterizing groupoid \(C^\ast\)-algebras of non-Hausdorff étale groupoids (2022) (Vol. 2306). Springer Nature.
[4] M. Frank, Injective envelopes and local multiplier algebras of \(C^\ast\)-algebras, Int. Math. J. 1 (2002), no. 6, 611–620. arXiv:math/9910109v2. MR 1860642.
[5] H. Gonshor, Injective hulls of \(C^\ast\)-algebras. II, Proc. Amer. Math. Soc. 24 (1970), 486–491, doi: 10.2307/2037393.
[6] M. Hamana, Injective envelopes of \(C^\ast\)-algebras, J. Math. Soc. Japan 31 (1979), no. 1, 181–197, doi: 10.2969/jmsj/03110181. MR 519044.
[7] A. Kishimoto, Outer automorphisms and reduced crossed products of simple \(C^\ast\)-algebras, Comm. Math. Phys. 81 (1981), no. 3, 429–435.
[8] F. Komura, \(^*\)-homomorphisms between groupoid \(C^\ast\)-algebras, arXiv:2302.10405 [math.OA], (2023).
[9] A. Kumjian, On \(C^\ast\)-diagonals, Canadian Journal of Mathematics, 38(4), 969–1008, doi:10.4153/CJM-1986-048-0
[10] B. K. Kwaśniewski, R. Meyer, Aperiodicity, topological freeness and pure outerness: from group actions to Fell bundles, Studia Math. 241 (2018), no. 3, 257–303, doi:10.4064/sm8762-5-2017. MR 3756105
[11] B. K. Kwaśniewski, R. Meyer, On Noncommutative Cartan \(C^\ast\)-subalgebras, Trans. Amer. Math. Soc. 373, no. 12, 8697–8724, doi: 10.1090/tran/8174 (2020)
[12] B. K. Kwaśniewski, R. Meyer, Aperiodicity, the almost extension property and uniqueness of pseudo-expectations, International Mathematics Research Notices (IMRN), published online 07 June 2021
[13] X. Li, Every classifiable simple \(C^\ast\)-algebra has a Cartan subalgebra. Invent. math. 219, 653–699 (2020). https://doi.org/10.1007/s00222-019-09914-0
[14] D. Pitts, Structure for Regular Inclusions. I, Journal of Operator Theory 78:2 (2017), 357–416 doi: 10.7900/jot.2016sep15.2128
[15] D. Pitts, Normalizers and Approximate Units for Inclusions of \(C^\ast\)-Algebras, arXiv:2109.00856 [math.OA], (2022), (To appear in Indiana University Mathematics Journal)
[16] A. I. Raad, Existence and uniqueness of inductive limit Cartan subalgebras in inductive limit \(C^\ast\)-algebras., PhD Thesis, University of Glasgow, Glasgow, (2021), 10.5525/gla.thesis.82456
[17] A. I. Raad, A Generalization of Renault’s Theorem for Cartan Subalgebras, arXiv:2101.03265 [math.OA], (2021), (Accepted for publication in Proc. Amer. Math. Soc)
[18] J. Renault, Cartan subalgebras in \(C^\ast\)-algebras, Irish Mathematical Society Bulletin, (2008).
[19] J. Taylor, Aperiodic dynamical inclusions of \(C^\ast\)-algebras, arXiv:2303.10095 [math.OA], (2023).
[20] J. Taylor, Aperiodic dynamical inclusions of \(C^\ast\)-algebras, PhD Thesis, Georg-August-Universität Göttingen (2022), http://dx.doi.org/10.53846/goediss-9727

Institut für Mathematik, University of Potsdam, Campus Golm, Haus 9, Karl-Liebknecht-Str. 24-25, 14476, Germany

Email address: jonathan.taylor@uni-potsdam.de