Radiative contributions to gravitational scattering

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The linear-order effects of radiation-reaction on the classical scattering of two point masses, in General Relativity, are derived by a variation-of-constants method. Explicit expressions for the radiation-reaction contributions to the changes of 4-momentum during scattering are given to linear order in the radiative losses of energy, linear-momentum and angular momentum. The polynomial dependence on the masses of the 4-momentum changes is shown to lead to non-trivial identities relating the various radiative losses. At order $G^3$ our results lead to a streamlined classical derivation of results recently derived within a quantum approach. At order $G^4$ we compute the needed radiative losses to next-to-next-to-leading-order in the post-Newtonian expansion, thereby reaching the absolute fourth and a half post-Newtonian level of accuracy in the 4-momentum changes. We also provide explicit expressions, at the absolute sixth post-Newtonian accuracy, for the radiative-graviton contribution to conservative $O(G^4)$ scattering. At orders $G^2$ and $G^3$ we derive explicit theoretical expressions for the last two hitherto undetermined parameters describing the fifth-post-Newtonian dynamics. Our results at the fifth-post-Newtonian level confirm results of [Nucl. Phys. B 965, 115352 (2021)] but exhibit some disagreements with results of [Phys. Rev. D 101, 064033 (2020)].

I. INTRODUCTION

A new angle of attack on the time-honored two-body problem in General Relativity has been recently the focus of active research, namely the study of the (classical or quantum) scattering of two, gravitationally interacting, massive bodies. Several complementary avenues towards this problem have been pursued, using various approaches: post-Minkowskian (PM), e.g., $^{[18]}$, post-Newtonian (PN), e.g., $^{[9]}$, Effective One Body (EOB), e.g., $^{[10]}$, Effective Field Theory (EFT), e.g., $^{[11]}$, Tutti Frutti (TF) $^{[13]}$, and scattering amplitudes (including generalized unitarity, the double copy, eikonal resummation and advanced multiloop integration methods), e.g., $^{[14]}$.

A new twist in this story has recently gained prominence: the issue of radiative corrections to scattering. Indeed, a paradoxical discrepancy between the high-energy limit of the $O(G^3)$ scattering result of Bern et al. $^{[14]}$ and the massless scattering result of Amati, Ciafaloni and Veneziano $^{[22]}$ was highlighted in Ref. $^{[23]}$. This discrepancy was confirmed in recalculation of massless scattering $^{[24]}$. It is only quite recently that the resolution of this paradox was understood to be rooted in radiative contributions to scattering $^{[25]}$.

In Ref. $^{[26]}$, one of us derived the (leading-order) $O(G^3)$ radiation-reaction contribution to the scattering angle of two (classical) bodies in General Relativity. This result was confirmed by scattering-amplitude computations, $^{[21]}$ $^{[28]}$ $^{[29]}$, using various techniques (eikonal resummation or an observable-based formalism $^{[30]}$). In addition, Ref. $^{[23]}$ completed the scattering-angle result of $^{[26]}$ by deriving (from the observable-based formalism of Ref. $^{[30]}$) the full expressions for the 4-momentum changes, $\Delta p_{\mu a} = p_{\mu a}^+ - p_{\mu a}^-$, $a = 1, 2$, of the two scattering bodies.

The first aim of the present paper is to give a purely classical derivation of the linear-order radiation-reaction contributions to $\Delta p_{\mu a}$ by using a (first-order) variation-of-constant approach. Our method generalizes the one introduced in Ref. $^{[31]}$ by including recoil effects. The application of our general result to the $G^3$ level will be shown to reproduce the result obtained in Ref. $^{[26]}$. In addition, we shall apply our results to the $G^4$ level, thereby completing the recent $G^3$-level result of Ref. $^{[26]}$ by showing how to compute $O(G^4)$ linear-order radiation-reaction contributions to scattering. Our limitation to linear-order in radiation-reaction implies that our $O(G^4)$ dissipative results will be complete only up to the $O(1/c^3)$ PN order included. The $O(G^4)$ potential-graviton dynamics derived in Ref. $^{[26]}$ has been partially checked in $^{[32]}$, and recently fully rederived within a classical approach in $^{[33]}$. Our theoretical result for the latter $O(G^4)$ radiation-reaction contribution notably involves the $O(G^3)$ radiative loss of angular momentum, which has not yet been computed in PM gravity. We have, however, computed it (as well as the other needed PM radiative losses) with 2PN fractional accuracy. We also show how the mass polynomiality of the radiation-reacted momentum changes $\Delta p_{\mu a}$ yield remarkable a priori constraints on several PM radiative losses.

Besides providing expressions for the $O(G^4)$ radiation-reaction contributions to scattering, we shall also complete the potential-graviton contribution to $O(G^4)$ conser-
II. DECOMPOSING THE VARIOUS RADIATIVE CONTRIBUTIONS TO THE IMPULSE

In the present work we focus on the total change

$$\Delta p_{a\mu} \equiv p_{a\mu}^{r} - p_{a\mu}^{c}, \quad (2.1)$$

between the infinite past and the infinite future, of the (classical) 4-momentum $p_{a\mu} = m_{a}u_{a\mu}$ of a particle experiencing a gravitational two-body scattering. Here, the subscript $a = 1, 2$ labels the particle (of mass $m_{a}$), $u_{a\mu}$ denotes its incoming 4-velocity, and $u_{a\mu}^{2}$ its outcoming 4-velocity. These 4-velocities are measured in the asymptotic incoming and outgoing Minkowski spacetimes. Following the terminology used in Ref. [30], the momentum change $\Delta p_{a\mu}$ will often be called the “impulse” of particle $a$.

In the following, we decompose $\Delta p_{a\mu}$ as the sum of two contributions, namely

$$\Delta p_{a\mu}(u_{1}^{+}, u_{2}^{-}, b) = \Delta p_{a\mu}^{c}(u_{1}^{-}, u_{2}^{+}, b) + \Delta p_{a\mu}^{rr, tot}(u_{1}^{-}, u_{2}^{+}, b). \quad (2.2)$$

Here, $\Delta p_{a\mu}^{c}(u_{1}^{-}, u_{2}^{+}, b)$ denotes the conservative contribution to the impulse, obtained by neglecting all radiative losses, and expressed as a function of the incoming 4-velocities and of the (vectorial) impact parameter. As in our previous works, the conservative impulse $\Delta p_{a\mu}^{c}(u_{1}^{-}, u_{2}^{+}, b)$ is defined as the impulse that follows from the Fokker-Wheeler-Feynman-type time-symmetric gravitational interaction of two masses (using a half-retarded-half-advanced Green’s function at each PM order). The complementary radiation reaction contribution, $\Delta p_{a\mu}^{rr, tot}(u_{1}^{-}, u_{2}^{+}, b)$, is linked to the radiative losses occurring when the two bodies interact via retarded gravitational interactions. It will be obtained, by a linear-response computation, as the sum of two separate radiative effects, namely

$$\Delta p_{a\mu}^{rr, tot}(u_{1}^{-}, u_{2}^{+}, b) = \Delta p_{a\mu}^{rr, rel} + \Delta p_{a\mu}^{rr, rec} + O(F_{rr}^{2}). \quad (2.3)$$

Here, the first contribution, $\Delta p_{a\mu}^{rr, rel}$, is linked to radiative effects acting on the relative motion of the binary system, while the second one, $\Delta p_{a\mu}^{rr, rec}$, is linked to the over-all recoil of the two-body system. We have added an error term, $O(F_{rr}^{2})$ in Eq. (2.3), as a reminder that our derivation of $\Delta p_{a\mu}^{rr, rel}$ and $\Delta p_{a\mu}^{rr, rec}$ is valid only to first order in the radiation-reaction force $F_{rr}$. We will also include a discussion of some of the effects that are quadratic in $F_{rr}$ (which start at order $G^{2}/c^{10}$).

The decomposition (2.3) does not exhaust all the radiation-related contributions to the impulse. Indeed, when following the EFT approach to binary dynamics, the conservative impulse $\Delta p_{a\mu}^{c}$ is, itself, naturally decomposed into two contributions:

$$\Delta p_{a\mu}^{c}(u_{1}^{-}, u_{2}^{+}, b) = \Delta p_{a\mu}^{pot} + \Delta p_{a\mu}^{rad}. \quad (2.4)$$

Here $\Delta p_{a\mu}^{pot}$ denotes (when using the method of regions) the part of the conservative impulse that is due to the mediation of potential graviton modes, while $\Delta p_{a\mu}^{rad}$ denotes the part of the conservative impulse that is due to the mediation of radiation graviton modes. In other words, there are three different impulse effects linked to soft (radiative-like) graviton modes: a conservative effect $\Delta p_{a\mu}^{cons}$ linked to the exchange of time-symmetric radiation gravitons, and two separate dissipative effects linked to time-antisymmetric radiation-reaction forces acting on the system: $\Delta p_{a\mu}^{rr, rec}$ and $\Delta p_{a\mu}^{rr, rel}$.

Finally, let us also note that, from the technical point of view, we will PM-expand each impulse (and also, each partial contribution to the impulse), say

$$\Delta p_{a\mu} = \sum_{n \geq 1} \Delta p_{a\mu}^{G_{n}}, \quad (2.5)$$

each PM contribution $\Delta p_{a\mu}^{G_{n}}$ being, eventually, further expanded in a PN series:

$$\Delta p_{a\mu}^{G_{n}} \sim G_{n} \left( \frac{1}{c^{2}} + \frac{1}{c^{4}} + \cdots \right). \quad (2.6)$$

Let us also mention that it will be often convenient to decompose the impulse of each particle, along an appropriate basis of 4-vectors. A first possible basis is $\hat{b}_{1\mu}$, $u_{1\mu}$ and $u_{2\mu}$. This decomposition involves three scalar coefficients (for each $a = 1, 2$), $c_{a}^{\mu}, c_{a1}^{\mu}, c_{a2}^{\mu}$ ($a = 1, 2$), namely

$$\Delta p_{a\mu}(u_{1}^{-}, u_{2}^{+}, b) = c_{a}^{\mu} \hat{b} + c_{a1}^{\mu} u_{1\mu} + c_{a2}^{\mu} u_{2\mu}, \quad (2.7)$$

where $\hat{b} \equiv b^{\mu}/b$ is the unit (spacelike) 4-vector in the direction of the vectorial impact parameter$^{2} b^{\mu}$. The decomposition of any impulse $\Delta p_{a\mu}$ along the vectors $\hat{b}_{a\mu}$,

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1. By recoil of the two-body system, we mean the loss of total mechanical linear momentum of the system via radiation of linear momentum in the form of gravitational waves.

2. We recall that $b^{\mu} = b^{\nu}b_{\nu}$ is orthogonal to both incoming 4-velocities $u_{1\mu}$, $u_{2\mu}$, and is taken in the direction going from body 2 towards body 1.
\[ u_{1}^{n} \text{ and } u_{2}^{n} \text{ is a linear operation which commutes with any other linear decomposition of } \Delta p_{\mu}. \text{ This means in particular that the decomposition of } (2.7) \text{ induces a corresponding linear decomposition of the corresponding coefficients in the basis (2.7), say (with a label } X = b, u_{1}, u_{2}) \]

\[ c_{X}^{\mu} = c_{X}^{\mu, \text{cons}} + c_{X}^{\mu, \text{rel, rec}} + O(F_{\mu}^{2}). \tag{2.8} \]

### III. APPLICATION OF THE METHOD OF VARIATION OF CONSTANTS TO THE RELATIVE MOTION, AND THE RECOIL, OF TWO-BODY SYSTEMS

In the present work, we assume that the PM-expanded, physical, retarded \(^3\) equations of motion of a binary system can be described by adding to the (Fokker-Wheeler-Feynman time-symmetric) Poincaré-invariant conservative dynamics additional radiation-reaction forces \(F_{\mu}^{\mu}\) which have (at least to first-order) a time-antisymmetric character, and which cause losses of the total mechanical Noetherian quantities \(p_{\mu}^{\mu}\) of the conservative dynamics of the two-body system that balance the corresponding radiated quantities \(p_{\mu}^{\mu, \text{rad}}, J_{\mu}^{\mu, \text{rad}}\). In the case of the linear momentum (for which PM-expanded interaction terms are better controlled in all Lorentz frames), we have also the property that \(p_{\mu}^{\mu}\) reduces to the usual total mechanical linear momentum \(p_{\mu}^{\mu} = m_{1}u_{1}^{\mu} + m_{2}u_{2}^{\mu}\) in the asymptotic incoming \((u_{1,2} \rightarrow u_{1,2}^{\infty})\), and outgoing \((u_{1,2} \rightarrow u_{1,2}^{\infty})\), states.

The balance law in this case states that the additional radiation-reaction forces \(F_{\mu}^{\mu}\) imply that

\[ p_{\mu}^{\mu, \text{system}} = p_{\mu}^{\mu, \text{system}} + p_{\mu}^{\mu, \text{rad}}, \tag{3.1} \]

i.e., explicitly,

\[ p_{1}^{\mu} - p_{2}^{\mu} = p_{1}^{\mu, +} + p_{2}^{\mu, +} + p_{\mu}^{\mu, \text{rad}}. \tag{3.2} \]

The early studies (at the 2PM and 3PM levels) of the PM-expanded two-body dynamics in the 1980’s \([39, 41]\) have explicitly checked, at the leading order in radiation reaction, the validity of the decomposition of the PM-expanded two-body dynamics in conservative radiation-reaction effects (balancing the radiative losses) \([39, 42, 43]\). [These early PM-based works were motivated by unsatisfactory aspects of earlier PN-based studies of radiation damping \([14, 15]\). Other approaches (notably Hamiltonian-based studies \([40]\), and PN-based ones \([17]\)] have confirmed the validity of balance laws between mechanical properties (energy, linear momentum, angular momentum) of the radiation-reacted system and corresponding fluxes radiated as gravitational waves at the next-to-leading-order (NLO) in PN-expanded radiation reaction. Recently, a quantum-based computation \([29]\) (using the formalism of Ref. \([30]\)) has explicitly checked the validity of the balance Eq. (3.2) at the 3PM order.

Formulating the time-retarded dynamics in terms of additional radiation-reaction forces \(p_{\mu}^{\mu}\) acting on the time-symmetric conservative dynamics allows one to apply the general method (due to Lagrange) of variation of constants. In a PN context, this was done for ellipticlike bound motions in Refs \([33, 38]\), and for hyperboliform scattering motions in \([31]\). We have generalized these treatments in two ways: (i) by working within a PM context; and (ii) by including the effects linked to the overall recoil of the binary system. [The previous treatments applied the method of varying constants only to the relative motion of the two bodies, considered in the c.m. system, using an approximation where the total linear momentum of the 2-body system was conserved.] In our PM context, it will be crucial to take into account the non-conservation of the the total linear momentum of the 2-body system, i.e., the non-zero value of \(P_{\mu}^{\mu}\) in Eqs. (3.1), (3.2). Indeed, \(P_{\mu}^{\mu, \text{rad}} = O(G^{3})\), so that a complete PM description of radiation-reaction effects beyond the 2PM \([O(G^{2})]\) level must take into account recoil effects. [In a PN context, recoil is viewed as being \(O(\frac{G}{r})\), i.e., of 3.5PN order, while the radiative energy loss is \(O(\frac{G^{3}}{r^{3}})\), i.e., of 2.5PN order.]

### A. Separating the relativistic two-body dynamics in relative, and center-of-mass, dynamics

In order to compute the effect of the overall recoil of the binary system,

\[ \Delta P_{\mu}^{\mu} = P_{\mu}^{\mu, +} - P_{\mu}^{\mu, -} = -P_{\mu}^{\mu, \text{rad}}, \tag{3.3} \]

on the impulse \(\Delta p_{\mu}\) of each particle, one can make use of results from the literature on the separation of the relativistic dynamics of two-body systems in relative dynamics and center-of-mass (c.m.) dynamics (see, e.g., \([49]\) for an introduction). In particular, Schäfer and collaborators \([50, 51]\) have given a very concrete, and directly relevant, application of this separation to the case of gravitationally interacting binary systems. More precisely, Refs. \([50, 51]\) showed how to perturbatively construct a canonical transformation between standard (Arnol’d-Deser-Misner-type) two-body phase-space variables \(x_{1}, x_{2}, p_{1}, p_{2}\) and new phase-space variables \(r, p, R, P\) (where \(r, p\) describe the relative dynamics, while \(R, P\) describe the dynamics of the c.m.) such that the square of the total Hamiltonian,

\[ H_{\text{tot}}(x_{1} - x_{2}, p_{1}, p_{2}) = \sqrt{m_{1}^{2}c^{4} + c^{2}p_{1}^{2}} + \sqrt{m_{2}^{2}c^{4} + c^{2}p_{2}^{2}} + \sqrt{m_{1}^{2}c^{4} + c^{2}p_{2}^{2}} \tag{3.4} \]

describing the conservative dynamics of a gravitationally interacting binary system takes, when reexpressed in terms of the new variables, a form neatly showing the
separation between the relative dynamics and the c.m.
one, namely
\[ H_{\text{tot}}^2(r, p, R, P) = H_{\text{rel}}^2(r, p) + c^2 P^2. \] (3.5)
Here the function \( H_{\text{rel}}(r, p) \) is the usual c.m.-reduced relative Hamiltonian, namely
\[ H_{\text{rel}}(r, p) \equiv [H_{\text{tot}}(x_1 - x_2, p_1, p_2)]_{p_1 \to p, p_2 \to -p}. \] (3.6)
The canonical transformation between \( x_1, x_2, p_1, p_2 \) and \( r, p, R, P \) is complicated and can only be constructed perturbatively, either in a PN-expansion starting from its usual Newtonian analog \([50, 51]\), or in a combined expansion in powers of \( P = p_1 + p_2 \), and in powers of \( G \). Some quantities, however, have simple (Newtonian-looking) expressions. Namely, the total linear momentum is given by
\[ p = p_1 + p_2, \] (3.7)
and the total angular momentum (for spinless particles) is given by
\[ J = x_1 \times p_1 + x_2 \times p_2 = r \times p + R \times P. \] (3.8)
In addition, using the definition of the c.m. position variable, \( R \) \([50, 51]\), namely (denoting \( \mathcal{M} c^2 \equiv \sqrt{H_{\text{tot}}^2 - c^2 P^2} \))
\[ R = \frac{c^2 G}{H_{\text{tot}}} + \frac{1}{\mathcal{M}(H_{\text{tot}} + \mathcal{M} c^2)} \left( J - \left( \frac{c^2 G}{H_{\text{tot}}} \times P \right) \times P \right), \] (3.9)
where \( G = t P_1 + K^1 \), with \( K^1 \equiv J^0 \), we have checked that the square of the relative (3-dimensional) angular momentum \( J^\text{rel} = r \times p \) is simply equal to the square of the (4-dimensional) Pauli-Lubanski spin 4-vector,
\[ S_\mu = \frac{1}{2 \mathcal{M} c \eta_{\alpha \beta \gamma \mu}} P^\alpha J^{\beta \gamma}, \] (3.10)
where \( \eta_{\alpha \beta \gamma \mu} \) (with \( \eta_{0123} = +1 \)) denotes the Levi-Civita tensor and \( J^{\beta \gamma} = (J^{01}, J^{11}) = (K^1, e^{jk} K^k) \). Explicitly, we have
\[ (r \times p)^2 = S_\mu S^\mu. \] (3.11)
The other expressions relating relative variables to the original phase-space variables \( x_1, x_2, p_1, p_2 \) are complicated, and contain interaction terms. E.g., one has
\[ p = \frac{E_2(p_2)}{E_1(p_1) + E_2(p_2)} p_1 - \frac{E_1(p_1)}{E_1(p_1) + E_2(p_2)} p_2 + O(p_1 + p_2) + O\left(\frac{G}{c^2}\right), \] (3.12)
where we used the shorthand notation
\[ E_\alpha(p) \equiv \sqrt{m_\alpha^2 c^4 + c^2 p^2}. \] (3.13)
The above results on the relativistic separation between relative and c.m. dynamics hold in the conservative case. However, the existence, in the conservative case, of such a relativistic decomposition between relative motion and c.m. motion offers a useful framework for applying the method of varying constants in the non-conservative case where the two-body system loses energy, momentum and angular momentum in the form of gravitational radiation. To do so, it is convenient to work within the incoming c.m. frame of the system, with time axis
\[ U^{\mu^-} = \pm \frac{p_1^\mu - p_2^-}{|p_1 + p_2|}. \] (3.14)
In addition, when needed, we will use the sharper c.m. condition that the incoming value of the time-space components \( J_0^- \) of the incoming relativistic angular momentum vanish: \( J_0^- = 0 \), or
\[ (P^\mu J_{\mu \nu})^- = 0. \] (3.15)
Working in this frame, and working to first order in radiation-reaction effects, we can separately consider (before adding them together, as indicated in Eq. (2.3)), the effects of radiative losses in energy, linear momentum and angular momentum.

B. Effect of radiative losses on the relative dynamics

If we first neglect the linear-momentum loss (i.e. the overall recoil of the two-body system), we can set \( P \) to zero in Eq. (3.5), and consider the effect of radiative losses on the relative dynamics, i.e., on the dynamics described by the variables \( r, p \) in the Hamiltonian (3.5). Let us give here a new derivation of the effect of radiative losses on the relative scattering angle.

Radiation damping can be described by adding to the relative Hamilton equations a relative radiation reaction force \( F^{\text{rr}} \) in the evolution equation for \( r \) (as done in the EOB formalism \([52]\)), say
\[ \dot{r} = -\frac{\partial H_{\text{tot}}}{\partial r}, \]
\[ \dot{p} = -\frac{\partial H_{\text{tot}}}{\partial r} + F^{\text{rr}}. \] (3.16)
After the neglect of \( P \approx P^- = 0 \) (in the incoming c.m. frame), \( H_{\text{tot}}(r, p, R, P) \) reduces to \( H_{\text{rel}}(r, p) \), as follows from Eq. (3.5). Writing the first set of Hamilton equations (3.10) in polar coordinates \( r, \phi \) (in the conserved plane of motion), say
\[ r = r(\cos \phi \mathbf{e}_x + \sin \phi \mathbf{e}_y), \] (3.17)
yield equations for \( \dot{r} \) and \( \dot{\phi} \) in terms of \( r, p_r \) and \( p_\phi \), where \( p_\phi \) is equal to the magnitude (and the \( z \) component of

\[ \] In the present spinless case, \( F^{\text{rr}} \) lies in the initial \( r-p \) plane.
the relative angular momentum $\mathbf{r} \times \mathbf{p}$. Eliminating the time by considering the ratio $\frac{\dot{\phi}}{\dot{r}} = \frac{d\phi}{dr}$ and solving the energy equation $E_{rel} = H_{rel}(r, p_r, p_\phi)$ in $p_r$ yields an equation for the trajectory of the form

$$\frac{d\phi}{dr} = \Phi[r, E_{rel}, p_\phi] = \Phi[r, E_{rel}, J_{rel}], \quad (3.18)$$

where we denoted by $J_{rel}$ the magnitude of the relative angular momentum $J_{rel} = \mathbf{r} \times \mathbf{p}$. A crucial feature of Eq. (3.18) is that the quantities $E_{rel}$, $J_{rel}$ entering the right-hand side (rhs) are not conserved in presence of the (relative) radiation-reaction force $\mathcal{F}^{rr}$ in Eqs. (3.16), but that they adiabatically evolve during the scattering to interpolate between their incoming values $E_{rel}^+\!, J_{rel}^+$ and their outgoing ones $E_{rel}^\times\!, J_{rel}^\times$. In other words, Eq. (3.18) is a technically precise way of proving that the relative variables $\mathbf{r}, \mathbf{p}$ feel the external radiative losses only through the adiabatic variation of the two “constants of motion” they depend upon in the conservative case: the relative energy $E_{rel} = H_{rel}(r, \mathbf{p})$ and the relative angular momentum $J_{rel}^2 = (r \times \mathbf{p})^2$, both quantities being considered in the (incoming) c.m. frame. At this stage, one should also remember that Eq. (3.11) shows that $J_{rel}^2$ is equal to the square of the Pauli-Lubanski 4-vector $S_\mu$. The varying-constant trajectory equation (3.18) yields a simple proof of the result obtained in Ref. 31 for the effect of radiation reaction on the (relative) scattering angle. Indeed, if we expand the right-hand side (rhs) of (3.18) to first order in the variation of $E_{rel}$ and $J_{rel}$, either around their initial values, $E_{rel}, J_{rel}$, or, more conveniently around their values at the moment of closest approach, say

$$E_{rel}(r) = E_{rel}(r_{min}) + \delta E(r), \quad J_{rel}(r) = J_{rel}(r_{min}) + \delta J(r), \quad (3.19)$$

we get (to first order in the radiative variations $\delta E(r)$ and $\delta J(r)$) a trajectory equation of the form

$$\frac{d\phi}{dr} = \Phi[r, E_{rel}(r_{min}), J_{rel}(r_{min})] + \frac{\partial \Phi}{\partial E_{rel}} \delta E(r) + \frac{\partial \Phi}{\partial J_{rel}} \delta J(r). \quad (3.20)$$

Here, we formally expressed the time evolution in terms of the (relative) radial distance $r$, with the usual understanding that $r$ decreases ($p_r < 0$) during the first (approaching) half of the scattering, while it increases ($p_r > 0$) during the second (receding) half. [As the (single-valued) function $\phi(t)$ is continuously increasing with time, the (multi-valued) function $\phi(r)$ should decrease (with, correspondingly, $\Phi[r] < 0$) during the approach and increase (with, correspondingly, $\Phi[r] > 0$) during the outgoing motion.] The advantage of having expressed Eq. (3.20) in terms of quantities referring to the closest-approach is that we can use the fact that the conservative motion is time-symmetric with respect to the closest approach, namely the solution $\phi^{cons}(r, E, J)$ of the conservative scattering equation,

$$\frac{d\phi^{cons}(r, E, J)}{dr} = \Phi[r, E, J], \quad (3.21)$$

defines a curve in the $(r, \phi)$ plane which is symmetric with respect to the line $\phi = \phi_{\min}$, corresponding to the point of closest approach.

The main physical observable extracted from the latter curve is the total conservative scattering angle defined as

$$\chi^{cons}(E, J) \equiv \phi^{cons+} - \phi^{cons-}, \quad (3.22)$$

where $\phi^{cons+}(E, J)$ and $\phi^{cons-}(E, J)$ denote the incoming and outgoing asymptotic values of the solution $\phi^{cons}(r, E, J)$ of Eq. (3.21).

The time symmetry of the conservative dynamics has two consequences when solving the (first-order) radiation-reacted scattering equation (3.20): the first term on the rhs, namely $\Phi[r, E_{rel}(r_{min}), J_{rel}(r_{min})]$, yields a time-symmetric solution; while the two terms on the second line yield a correction to the scattering angle which, when multiplied by $dr = i dt$ is time-antisymmetric\(^5\) with respect to the moment of closest approach. As a consequence, the integrated effect of the two terms $\frac{dE_{rel}}{dr} \delta E(r) + \frac{dJ_{rel}}{dr} \delta J(r)$ in Eq. (3.20) is zero. The final result is then that, to first order in radiation reaction, the relative scattering angle is given by

$$\chi^{rel} = \chi^{cons}(E_{rel}(r_{min}), J_{rel}(r_{min})). \quad (3.23)$$

Using again the time symmetry between the two halves of the scattering, one can further replace the midpoint values $E_{rel}(r_{min}), J_{rel}(r_{min})$ of the slowly varying constants by the averages of the corresponding slowly varying constants: $E_{rel}(r_{min}) \approx \frac{1}{2}(E_{rel}^- + E_{rel}^+)\!,$ and $J_{rel}(r_{min}) \approx \frac{1}{2}(J_{rel}^- + J_{rel}^+)\!$. Another form is obtained by Taylor expanding around the incoming values, with the (first-order-accurate) result

$$\chi^{rel} = \chi^{cons}(E_{rel}^-, J_{rel}^-) + \delta r \chi^{rel}, \quad (3.24)$$

where

$$\delta r \chi^{rel} = \frac{1}{2} \frac{\partial \chi^{cons}}{\partial E_{rel}} \delta E_{rel} + \frac{1}{2} \frac{\partial \chi^{cons}}{\partial J_{rel}} \delta J_{rel}. \quad (3.25)$$

Here

$$\delta r E_{rel} = E_{rel}^+ - E_{rel}^- = -E_{rad}, \quad \delta r J_{rel} = J_{rel}^+ - J_{rel}^- = -J_{rad} \quad (3.26)$$

denote the radiation-reaction-related changes in $E_{rel}$ and $J_{rel}$, i.e. (using $E$ and $J$ balance) minus the radiative losses of $E$ and $J$ (computed in the incoming c.m. system).

\(^5\) The latter time-antisymmetry (around the closest approach) of the integrand giving the radiation-reaction contribution to the scattering angle is a direct consequence of the assumed global time-antisymmetry of the radiation-reaction force.
C. Effects of radiative losses on the c.m. dynamics

The total linear momentum $P^\mu = p^\mu_1 + p^\mu_2$ of the two-body system will change, because of radiative losses, from its incoming value, $P^\mu_{i}$, in the infinite past, to a different outgoing value, $P^\mu_{o}$, in the infinite future. Let us compute now the contribution $\Delta p^\mu_{\mu} \equiv \Delta P^\mu = P^\mu_{o} - P^\mu_{i}$, i.e. the spatial part, $\delta^{\mu\nu}P^\nu$ (wrt the incoming c.m. frame) of the momentum loss of the binary system defined by the following orthogonal split,

$$\Delta P^\mu = P^\mu_{o} - P^\mu_{i} = \delta^{\mu\nu}P^\nu \quad \text{(3.27)}$$

Here, $\delta^{\mu\nu}E_{\text{rel}} = -U_{\mu}\Delta P^\mu$ is the radiation-reaction-induced energy change (measured in the incoming c.m. frame), while $\delta^{\mu\nu}P^\nu$ is orthogonal to $U_{\nu}$. When working in the incoming c.m. frame, so that the incoming 3-momentum $p^- = 0$, the 4-vector $\delta^{\mu\nu}P^\nu$ is purely spatial, and simply equal to the outgoing 3-momentum

$$p^+ = \delta^{\mu\nu}P^\nu = -p^\text{rad}. \quad \text{(3.28)}$$

The computation of $\Delta p^\mu_{\mu,\text{re}}$ can again be estimated by using the method of varying constants. Here, the constant that we vary is the total 3-momentum, $P$, which varies (in the incoming c.m. frame) between its initial value $P^- = 0$ and its final value $P^+$. In the two (incoming and outgoing) asymptotic regions the interactions terms in Eq. (3.14) vanish. Combining then Eq. (3.7) and Eq. (3.14) allows one to relate the asymptotic values of $p_1$ and $p_2$ to the asymptotic values of $p$ and $P$. When working in the incoming c.m. frame (i.e. when using $P^- = 0$), this yields

$$\begin{align*}
p_1^- &= p^- , \\
p_2^- &= -p^- , \\
p_1^+ &= p^+ + \frac{E_1^+}{(E_1 + E_2)^+}P^+ + O(P_2^+), \\
p_2^+ &= -p^+ + \frac{E_2^+}{(E_1 + E_2)^+}P^+ + O(P_2^+). \quad \text{(3.29)}
\end{align*}$$

The corresponding asymptotic kinetic energies are

$$\begin{align*}
E_1^\pm &\equiv \sqrt{m_1^2 + (p_1^\pm)^2}, \\
E_2^\pm &\equiv \sqrt{m_2^2 + (p_2^\pm)^2}, \\
E_1^+ &\equiv \sqrt{m_1^2 + (p_1^+)^2}, \\
E_2^+ &\equiv \sqrt{m_2^2 + (p_2^+)^2}. \quad \text{(3.30)}
\end{align*}$$

By expanding the latter relations to first order in $P^+$, one finds that the radiation-reacted impulses

$$\Delta p_{\mu} = p_{\mu}^+ - p_{\mu}^- = (E_a^+ - E_a^-)p_a^- + p_a^+, \quad \text{(3.31)}$$

read

$$\begin{align*}
\Delta p_1^0 &= \sqrt{m_1^2 + (p_1^+)^2} - \sqrt{m_1^2 + (p_1^-)^2} + \frac{p_1^+ \cdot P^+}{(E_1 + E_2)^+}, \\
\Delta p_1 &= p^+ - p^- + \frac{E_1^+}{(E_1 + E_2)^+}P^+, \\
\Delta p_2^0 &= \sqrt{m_2^2 + (p_2^+)^2} - \sqrt{m_2^2 + (p_2^-)^2} - \frac{p_2^+ \cdot P^+}{(E_1 + E_2)^+}, \\
\Delta p_2 &= -(p^+ - p^-) + \frac{E_2^+}{(E_1 + E_2)^+}P^+, \quad \text{(3.32)}
\end{align*}$$

where the temporal component refers to the incoming c.m. time axis $e_0 = U^-$. In these expressions the terms which do not involve $P^+$ exactly correspond to the impulses that would be derived by neglecting the recoil and only considering the effect of radiative losses on the relative motion. More precisely, they would correspond (when separating out the relative radiation-reaction effects from the conservative contribution) to the sum $\Delta p_{\mu,\text{cons}} + \Delta P_{\mu,\text{rel}}$ in Eq. (2.22), when inserting Eq. (2.23) and neglecting the last, recoil contribution. In other words, the terms involving $P^+$ in Eqs. (3.32) describe the looked-for recoil contributions $\Delta p_{\mu}^\text{re}$ to the impulses.

We conclude that the explicit expressions of the recoil contributions (viewed in the incoming c.m. frame) read

$$\begin{align*}
\Delta p_{a}^0 &= \frac{\Delta p_{a}^+ \cdot P^+}{E_+}, \\
\Delta p_{a}^+ &= \frac{E_a^+}{E_+}P^+, \quad \text{(3.33)}
\end{align*}$$

where $E_+ = (E_1 + E_2)^+$ denotes the total outgoing c.m. energy. Note that the recoil contributions can be simply interpreted as coming from the linearized effect of a small boost of velocity vector equal to the outgoing c.m. velocity (in the ingoing c.m. frame). Indeed, let us consider the recoil velocity

$$V = \frac{P^+}{E_+} \approx \frac{P^+}{E^-}, \quad \text{(3.34)}$$

where $E^- = (E_1 + E_2)^-$ denotes the total outgoing c.m. energy, which can be replaced (as indicated in the second equation) by the incoming one when neglecting terms bilinear in the radiative losses $\Delta P$ and $\Delta E = E^+ - E^-$. The usual Lorentz-transformation formula for a 4-momentum vector, under an infinitesimal boost, reads

$$\delta^\gamma E_a = p_a \cdot V, \quad \delta^\gamma p_a = E_a V. \quad \text{(3.35)}$$

The recoil contributions Eq. (3.33) are indeed obtained by applying this transformation formula to the outgoing momenta $p_{a}^+$.
D. Final results for the radiation-reacted impulses

Let us summarize our results for the radiation-reacted impulses. To get fully explicit results we need to choose a vectorial basis in the plane of motion (still working in the incoming c.m. frame). As a first basis we can use the two orthogonal (spatial) unit vectors $\mathbf{b}$ and $\mathbf{n}_-$, where $\mathbf{b} = \frac{\mathbf{b}}{\mathbf{b}}$ is along the incoming vectorial impact parameter $\mathbf{b}$, and where the unit vector $\mathbf{n}_-$ lies along the direction of $\mathbf{p}^+_1$, namely

$$
\begin{align*}
\mathbf{p}^+_1 &= p^- \mathbf{n}_- \equiv P^-_{\text{c.m.}} \mathbf{n}_-, \\
\mathbf{p}^-_2 &= -p^- \mathbf{n}_- \equiv -P^-_{\text{c.m.}} \mathbf{n}_-. \tag{3.36}
\end{align*}
$$

Here $p^- = P^-_{\text{c.m.}}$ is linked to the incoming c.m. energy by

$$
\begin{equation}
P^-_{\text{c.m.}} = \frac{m_1 m_2 \sqrt{(u^-_1 \cdot u^-_2)^2 - 1}}{E^-_{\text{c.m.}}} = \frac{m_1 m_2 p^-}{E^-_{\text{c.m.}}}, \tag{3.37}
\end{equation}
$$

where $p^\infty = \sqrt{\gamma^2 - 1}$, $\gamma \equiv -u^-_1 \cdot u^-_2$, and

$$
E^-_{\text{c.m.}} = (E_1 + E_2)^- = M h(\gamma, \nu) \equiv M \sqrt{1 + 2 \nu (\gamma - 1)}. \tag{3.38}
$$

We note that $\mathbf{n}_-$ can be expressed as a linear combination of the incoming four velocities of the two bodies,

$$
\mathbf{n}_- = \frac{m_1 m_2}{P^-_{\text{c.m.}} E^-_{\text{c.m.}}} \left( \frac{E^-_1}{m_1} u^-_1 - \frac{E^-_2}{m_2} u^-_2 \right), \tag{3.39}
$$

as follows, for example, by subtracting the relations

$$
\begin{align*}
\mathbf{p}^-_1 &= m_1 u^-_1 = E^-_1 U + p^- \mathbf{n}_-, \\
\mathbf{p}^-_2 &= m_2 u^-_2 = E^-_2 U - p^- \mathbf{n}_-. \tag{3.40}
\end{align*}
$$

With respect to this basis the outgoing 3-momentum $\mathbf{p}^+$ reads

$$
\mathbf{p}^+ = p^+ \left( \cos \chi \mathbf{n}_- - \sin \chi \mathbf{b} \right). \tag{3.41}
$$

In Eq. (3.41), one must insert both the radiation-reacted value of the magnitude $p^+$ of the outgoing relative momentum $\mathbf{p}^+$, and the radiation-reacted value of the (relative-motion) scattering angle, $\chi$. In the latter radiation-reacted scattering angle $\chi \equiv \chi^{\text{cons}} + \delta \chi^{\text{rel}}$, Eq. (3.24), the radiation-damping contribution is given by Eq. (3.25). Concerning the radiation-reacted value of the magnitude $p^+$ of the outgoing relative momentum it is determined by writing that the total outgoing relative energy $E^+_\text{rel} = \sqrt{m^2_1 + (p^+)^2}$ is equal to $E^-_\text{rel} + \delta \chi^{\text{rel}} E_{\text{rel}} = E^-_\text{rel} - E_{\text{rad}}$, where $E^-_\text{rel} = \sqrt{m^2_1 + (p^-)^2 + \sqrt{m^2_2 + (p^-)^2}}$. Computing $\Delta p \equiv p^+ - p^-$, from first order, from the latter condition yields

$$
\Delta p \equiv |p^+| - |p^-| = \frac{E^+_\text{rel} E^-_\text{rel} \delta \chi^{\text{rel}} E_{\text{c.m.}}}{|p^+| E^-_{\text{rel}}}, \tag{3.42}
$$

Inserting all those results in the above expressions for the impulses, Eqs. (3.32), finally yields explicit expressions for the impulses as the sum of three contributions: a conservative part, a relative-motion radiation-reaction part (computed from $E_{\text{rad}}$ and $J_{\text{rad}}$, Eq. (3.25)) and a recoil radiation-reaction part (computed from $\delta^{\text{rel}} p^\infty$, Eq. (3.33)), namely

$$
\Delta p_{\alpha \mu} = \Delta p_{\alpha \mu}^{\text{cons}}(E^-_{\text{rel}}, J^-_{\text{rel}}) + \Delta p_{\alpha \mu}^{\text{rel}} + \Delta p_{\alpha \mu}^{\text{rec}}. \tag{3.43}
$$

Following Eqs. (2.24) and (2.28), the decomposition of $\Delta p_{\alpha \mu}$ along the 4-vectors $b^\mu \equiv b^\mu/b$, $u^{\alpha\mu}_1$, and $u^{\alpha\mu}_2$, yields six scalar coefficients $c_{\alpha}^a$, $c_{\alpha}^a$, $c_{\alpha}^a$ ($a = 1, 2$). Each scalar coefficient is readily computed as

$$
\begin{align*}
c_{\alpha}^a &= \hat{b} \cdot \Delta p_{\alpha}^X, \\
c_{\beta u_1}^{a,X} &= \gamma - m, \Delta p_{\beta}^X, \\
c_{\beta u_2}^{a,X} &= \gamma - m, \Delta p_{\beta}^X, \tag{3.44}
\end{align*}
$$

with $X = \text{cons}$, rr rel, rr rec, the dot product is the Minkowski one, and we recall that $\gamma \equiv -u^-_1 \cdot u^-_2$.

All these coefficients, conservative and radiation-reaction, are listed in Table IV. In this table $P^+_b$ and $P^+_n$ denote the components of $\mathbf{P}^+$ along the $(\mathbf{b}, \mathbf{n}_-)$ basis in the plane of motion, i.e.,

$$
\mathbf{P}^+ = P^+_b \mathbf{b} + P^+_n \mathbf{n}_-. \tag{3.45}
$$

When inspecting the expressions of $c_{\beta u_1}^{a,X}$ and $c_{\beta u_2}^{a,X}$, one notices the presence of denominators involving powers of $P^-_{\text{c.m.}}$ or of $p^\infty$; these factors imply some loss of PN accuracy. It is useful to minimize the appearance of such small denominators by computing the impulse coefficients along a slightly different basis. For instance, we can decompose (when $a = 1$), $\Delta p^X_1$ along the new basis

$$
\begin{align*}
\hat{b}, u_1, u_{1 \perp}, u_{2 \perp} \equiv u_2 - \gamma u^-_1, \tag{3.46}
\end{align*}
$$

where $u^-_2$ is replaced by $u_{2 \perp} \equiv u_2 - \gamma u^-_1 = \Pi_{\perp}(u_2)$, with $\Pi_{\perp}(v) \equiv v \cdot (v \cdot u)u$ denoting the projection of the vector $v$ orthogonally to the (unit, timelike) vector $u$. [This basis is orthogonal, but the third vector is not a unit vector.] Mutatis mutandis, $\Delta p^X_2$ is conveniently decomposed along $b, u_2$ and $u_{1 \perp} \equiv u^-_1 - \gamma u^-_2 = \Pi_{\perp}(u_1)$. The corresponding expansion coefficients differ from the previous ones only for the second one. For instance, we have

$$
\Delta p^X_1 = c^1_{\beta} \hat{b} + c^{1,X}_{\beta u_1} u_1 + c^{1,X}_{\beta u_2} u_{2 \perp}. \tag{3.47}
$$

6 Spatial vectors, differently from spacetime vectors, are usually denoted by boldface symbols. Sometimes it is convenient to represent a spatial vector with the same symbol as a spacetime one (but not in boldface).
where $c_{1,2}^{1,X}$ and $c_{1,2}^{2,X}$ are the same as before, and where

$$ c_{1,2}^{2,X} = -u^{-1}_b \cdot \Delta p_1^{2,X} = c_{1,2}^{1,X} + \gamma c_{1,2}^{1,X}. \quad (3.48) $$

In addition, the time-symmetry of the conservative hyperbolic motion around closest approach implies that the radiated linear momentum (evaluated in a time-symmetric way along the conservative hyperbolic motion) is aligned along the special direction $e_y$ in the plane of motion, defined as the bisector between the incoming velocity and the outgoing one (taken in the conservative dynamics). The unit vector $e_y$ is part of a basis $e_x$, $e_y$ in the plane of motion defined as

$$ e_x = \cos \frac{\chi_{\text{cons}}}{2} b + \sin \frac{\chi_{\text{cons}}}{2} n, \quad e_y = -\sin \frac{\chi_{\text{cons}}}{2} b + \cos \frac{\chi_{\text{cons}}}{2} n. \quad (3.49) $$

with inverse relations

$$ b = \cos \frac{\chi_{\text{cons}}}{2} e_x - \sin \frac{\chi_{\text{cons}}}{2} e_y, \quad n = \sin \frac{\chi_{\text{cons}}}{2} e_x + \cos \frac{\chi_{\text{cons}}}{2} e_y. \quad (3.50) $$

The collinearity of the recoil with the $e_y$ direction, i.e., the fact that one can write $P^+_b = \frac{P_{\gamma}^+ e_y}$ yields the links

$$ P^+_b = -\sin \frac{\chi_{\text{cons}}}{2} P_y^+ \quad \text{and} \quad P^+_n = +\cos \frac{\chi_{\text{cons}}}{2} P_y^+. \quad (3.51) $$

Inserting the links \((3.51)\) in the expressions of $c_{1,2}^{1,\text{rr,rec}}$ and $c_{1,2}^{2,\text{rr,rec}}$, and using the new basis \((3.48)\), leads to simplified expressions for the radiation-reaction impulse coefficients (see again Table \[\text{I}\]).

### E. Mass polynomiality of the impulses

In the following sections we are going to use the present knowledge on radiative losses to explicitly compute the impulse coefficients at successive PM orders. PN related information about these coefficients are postponed to Appendix \[\text{I}\]. It will be especially convenient to express the impulse coefficients in terms of the impact parameter $b$, of the relative Lorentz factor $\gamma$ and of the two masses, $m_1$, $m_2$. Indeed, the argument given just below Eq. (2.9) of Ref. \[28\] is valid in the general non-conservative case considered here and shows that the 4-vectorial impulses $\Delta p^\mu$, when decomposed on the basis $b^\mu$, $u_{1,-}^\mu$, $u_{2,-}^\mu$, have the general structure (for $a = 1$)

$$ \Delta p_1^\mu = -2Gm_1m_2 \frac{2\gamma^2 - 1}{\gamma^2 - 1} \frac{b^\mu}{b^\nu} + \sum_{n \geq 2} \Delta p_{1\mu}^{n\text{PM}}. \quad (3.52) $$

---

7 When working to first-order in radiation reaction one can consistently evaluate the gravitational wave (GW) radiation absorbed and then emitted in a time-symmetric way by the conservative dynamics, with equal amounts of incoming radiation recorded on past null infinity and of outgoing radiation recorded on future null infinity. See Appendices \[\text{II}\] and \[\text{III}\] below for further discussion.
Here each term $\Delta p_{1\mu}^{nPM}$ is a combination of the three vectors $b^\mu/b$, $u_1^b$, and $u_2^b$, with coefficients that are, at each order in $G$, homogeneous polynomials in $m_1$ and $m_2$, containing the product $m_1m_2$ as an overall factor. In other words, it has the structure

$$
\Delta p_{1\mu}^{nPM} \sim \frac{Gm_1m_2}{b^n} \left[(Gm_1)^{n-1} + (Gm_1)^{n-2}Gm_2 + \cdots + (Gm_2)^{n-1}\right], \quad (3.53)
$$

where each term is a combination of the three vectors $b^\mu/b$, $u_1^b$, and $u_2^b$, with coefficients that are functions of $\gamma$.

This implies that the impulse coefficients $c_{\mu\nu}^{1PM}$, $c_{\mu\nu}^{2PM}$, $c_{\mu\nu}^{3PM}$ are (when expressed in terms of $b$ and $\gamma$) polynomials in $m_1$ and $m_2$ of the form $Gm_1m_2 [(Gm_1)^{n-1} + (Gm_1)^{n-2}Gm_2 + \cdots + (Gm_2)^{n-1}]$.

The polynomials in the masses of the impulses was shown to imply, in the conservative case, a corresponding polynomiality in $\nu = m_1m_2/(m_1 + m_2)^2$ of the scattering angle [29], which played an important role in determining the structure of the 5PN and 6PN Hamiltonians [13, 37, 53, 54]. We shall see below that the simple polynomiality rule satisfied by the conservative scattering angle is violated when considering the radiation-reacted relative scattering angle. However, we shall explicitly check that the more general property of mass polynomiality is restored when adding the effect of recoil, i.e. considering the total, radiation-reacted impulses

$$
\Delta p_{\mu\nu}(u_1, u_2, b) = \Delta p_{\mu\nu}^{cons}(u_1, u_2, b) + \Delta p_{\mu\nu}^{rr, tot}(u_1, u_2, b), \quad (3.54)
$$

with

$$
\Delta p_{\mu\nu}^{rr, tot}(u_1, u_2, b) = \Delta p_{\mu\nu}^{rr, rel} + \Delta p_{\mu\nu}^{rr, rec} + O(F_{\mu\nu}^2). \quad (3.55)
$$

As a consequence, the expansion coefficients $c_{\mu\nu}^{rr, tot}$ of the total radiation reaction contributions (including relative and recoil effects) to the impulses,

$$
c_{\mu\nu}^{rr, tot} = c_{\mu\nu}^{rr, rel} + c_{\mu\nu}^{rr, rec}, \quad X = b, u_1, u_2, \quad (3.56)
$$

must satisfy (as we shall check in the cases where they are known) the mass-polynomiality of the impulse coefficients.

IV. SCATTERING AT ORDERS $O(G)$ AND $O(G^2)$

For completeness, let us recall that the scattering at orders $O(G)$ and $O(G^2)$ is conservative\(^8\). Therefore, the knowledge of the scattering angle suffices to determine all the impulse coefficients. Inserting in the general expressions listed in Table I the 2PM-accurate scattering angle \(^{54}\) (given in Appendix I for convenience) yields PM-exact values for these impulse coefficients. As explained above, it is useful to express the impulse coefficients in terms of the impact parameter $b$, of the relative Lorentz factor $\gamma$ (or equivalently of $p_\infty = \sqrt{\gamma^2 - 1}$), and of the two masses, $m_1, m_2$. This will allow us to exhibit their polynomial structure in the masses.

As exhibited in Eq. (3.52), the only non-zero coefficients at 1PM are the $c_{b}^{1PM}$ ones, namely

$$
c_{b}^{1PM} = \frac{Gm_1m_2}{b}\left(-\frac{2}{p_\infty} - 4p_\infty\right), \quad (4.1)
$$

and $c_{b}^{2PM} = -c_{b}^{1PM}$. We have then

$$
\Delta p_{1\mu}^{1PM} = \frac{Gm_1m_2}{b}\left(1 + \frac{2p_\infty^2}{15 - 4p_\infty}\right)b. \quad (4.2)
$$

At the 2PM order, we have

$$
c_{b}^{2PM} = \frac{Gm_1m_2 GM}{b}\left(-\frac{3}{p_\infty} - \frac{15}{4p_\infty}\right), \quad (4.3)
$$

and

$$
c_{b}^{2PM} = \frac{-Gm_1m_2}{b}2\left(\frac{1}{p_\infty^2} + 2\right), \quad c_{b}^{2PM} = \frac{+Gm_1m_2}{b}2\left(\frac{1}{p_\infty^2} + 2\right), \quad (4.4)
$$

The polynomiality in the masses of these coefficients is clearly exhibited in these expressions. The complete expression for $\Delta p_{1\mu}^{12PM}$ then reads

$$
\Delta p_{1\mu}^{12PM} = \frac{-G^2m_1m_2}{b^2}\left[(m_1 + m_2) \frac{3\pi}{4p_\infty} (4 + 5p_\infty^2)b + \frac{2(p_\infty^2 + 1)^2}{p_\infty^2}(m_2(u_1 - \gamma u_2) - m_1(u_2 - \gamma u_1))\right]. \quad (4.5)
$$

V. CLASSICAL SCATTERING AT ORDER $O(G^3)$

The scattering at order $O(G^3)$ has been the topic of several recent works. First, the conservative part of the 3PM scattering has been computed in Refs. [14, 15, 55]. Second, the radiation-reaction contribution to the relative scattering angle has been computed both in supergravity [26], and in General Relativity [26, 27]. Ref. [29] has completed the determination of the 3PM scattering by computing the total impulses $\Delta p_\mu^{3PM}$ from the $O(G^3)$ quantum scattering amplitude, using the approach of Ref. [31]. The result of Ref. [29] was recently confirmed

---

\(^8\) Though the radiation-reaction force starts at order $O(G^2)$ and causes a $O(G^2)$ angular momentum loss [26, 42], the total linear momentum of the system is conserved at order $O(G^2)$ [54].
reads in Ref. [21]. Let us show here how our purely classical approach to radiation-reacted scattering leads to a quite simple rederivation of the quantum-derived result of [20].

It has been known for a long time [54] that nonlocal-time effects, related to radiation-graviton (a.k.a. soft-graviton) exchange, start to arise at the $O(\gamma^3)$ level, i.e. at the 4PM and 4PN level. As a consequence, the potential-graviton contribution to conservative scattering (which has been derived in Refs. [14, 55]) fully describes the conservative part of the scattering at the third PM order, $O(G^3)$. This yields a 3PM conservative impulse contributing to the conservative part of the scattering angle. To determine the additional $O(G^3)$ radiation-reaction contributions to scattering, one can start from the general expressions of the impulse coefficients listed in Table I above, and use the facts that the radiation-reaction contributions (coming either from the additional contribution (3.25) to the relative scattering angle, or from energy-loss effects), and recoil contributions. These radiative contributions enter differently the various components $c_{b, c}^{1, a}, c_{b, c}^{1, a, b, u}$ of the decomposition of $\Delta p_{\mu}^{rr}$ along the $b, u_1, u_2$ basis.

Let us start by discussing the $b$ component $c_{b}^{1, 3PM}$. It reads

$$c_{b}^{1, \text{tot, 3PM}} = c_{b}^{1, \text{cons, 3PM}} + c_{b}^{1, \text{rr, 3PM}},$$

where the conservative contribution is

$$c_{b}^{1, \text{cons, 3PM}} = -\mu \sqrt{\gamma^2 - 1} \left[ 2 \chi_3^{\text{cons}} - \frac{3}{4} \left( \chi_1^{\text{cons}} \right) \right],$$

$$= -\mu \hbar^2 \left( \frac{\gamma^2}{\gamma^2 - 1} \right) \left[ 2 \chi_3^{\text{cons}} - \frac{4}{3} \left( \chi_1^{\text{cons}} \right) \right] \left( \frac{GM}{b} \right)^3,$$

with $\chi_1^{\text{cons}}$ expanded as

$$\frac{1}{2} \chi_1^{\text{cons}} = \sum_{n \geq 1} \chi_n^{\text{cons}} \left( \gamma, \nu \right),$$

with coefficients $\chi_1^{\text{cons}}, \chi_2^{\text{cons}}$ and $\chi_3^{\text{cons}}$ given in Eq. (3.25), and where the radiation-reaction contribution only comes from the effect $\delta^rr^{\text{rel}}$ on the relative scattering angle. Moreover, at the $G^3$ order $\delta^rr^{\text{rel}}$ is entirely determined by the angular-momentum-loss effect in Eq. (3.25). Namely

$$\left[ \delta^rr^{\text{rel}} \right]^{\text{3PM}} = \frac{1}{2} \chi_1^{\text{cons, 1PM}} \left[ \frac{J^{\text{rad}}}{J} \right]^{\text{2PM}}.$$

Here,

$$\chi_1^{\text{cons, 1PM}} = \frac{2(\gamma - 2)}{(\gamma - 1)} \frac{\mu \hbar}{(\gamma^2 - 1)} \frac{2(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}}.$$

while the 2PM-accurate fractional angular-momentum loss was found in Ref. [26] to be

$$\left[ \frac{J^{\text{rad}}}{J} \right]^{\text{2PM}} = \frac{2(2\gamma^2 - 1)}{b^2} \frac{G^2 m_1 m_2}{I(v)} I(v),$$

where, denoting $v = \sqrt{1 - \frac{1}{\gamma^2}}$,

$$I(v) = -\frac{16}{3} + \frac{2}{v^2} + \frac{2(3v^2 - 1)}{v^3} \text{arctanh}(v).$$

Inserting these results in the general expressions listed in Table I yields the explicit result

$$c_{b}^{1, \text{rr, 3PM}} = -\frac{G^3 m_1^2 m_2^2}{b^3} \frac{2(2\gamma^2 - 1)^2}{(\gamma^2 - 1)^2} I(v).$$

The beginning of the PN expansion of $c_{b}^{1, \text{rr, 3PM}}$ reads

$$c_{b}^{1, \text{rr, 3PM}} = \frac{G^3 m_1^2 m_2^2}{b^3} \left[ \frac{16}{5} + \frac{80}{7} P_2 + \frac{512}{63} P_4 + \cdots \right],$$

Concerning $c_{u_1, u_2}$ the conservative part is given by

$$c_{u_1}^{1, \text{cons, 3PM}} = -\mu \left( \frac{GM}{b} \right)^3 \frac{4(m_1 \gamma + m_2) b \chi_2^{\text{cons}} \chi_1^{\text{cons}}}{M(\gamma^2 - 1)^{3/2}},$$

$$c_{u_2}^{1, \text{cons, 3PM}} = \mu \left( \frac{GM}{b} \right)^3 \frac{4(m_2 \gamma + m_1) b \chi_2^{\text{cons}} \chi_1^{\text{cons}}}{M(\gamma^2 - 1)^{3/2}}.$$

Contrary to $c_{b}^{a}$, which received radiation-reaction contributions only from $\delta^rr^{\text{rel}}$, the two other coefficients, $c_{u_1, u_2}$, of the decomposition of $\Delta p_{\mu}^{rr}$ receive contributions both from the relative-motion impulse, $\Delta p_{\mu}^{rr, \text{rel}}$, and from the recoil impulse, $\Delta p_{\mu}^{rr, \text{rec}}$. Expressing them in terms of the c.m.-frame radiated\(^9\) energy or momentum, we have

$$c_{u_1}^{1, \text{rr, 3PM}} = \frac{(m_1 \gamma + m_2) b}{(\gamma^2 - 1) E_{\text{c.m.}}},$$

$$c_{u_2}^{1, \text{rr, 3PM}} = \frac{1}{\gamma} c_{u_1}^{1, \text{rr, 3PM}},$$

and

$$c_{u_1}^{1, \text{rec, 3PM}} = -\frac{m_1 \gamma}{\sqrt{\gamma^2 - 1} E_{\text{c.m.}}},$$

$$c_{u_2}^{1, \text{rec, 3PM}} = -\frac{1}{\gamma} c_{u_1}^{1, \text{rec, 3PM}}.$$

---

\(^9\) Remember that the radiated quantities are opposite to the corresponding changes in the two-body quantities used in Table I above: e.g., $\delta^rr^{\text{rel}} \text{3PM} = -E_{\text{c.m.}}$. 

As was mentioned in~[57] (see Eq. (6.19)), the symmetry of the gravitational-radiation emission in the center-of-momentum frame proven by Kovacs and Thorne~[57] implies the following 4-vectorial structure for the radiated 4-momentum

\[ P_{\text{rad 3PM}}^\mu = \pi G^3 m_1^2 m_2^2 \gamma (u_1^- + u_2^-). \] (5.13)

This structure in turn implies simple links\(^{10}\) between the energy radiated in the rest frame of one of the particles (namely \(\pi G^3 m_1^2 m_2^2 \gamma \)), the energy radiated in the c.m. frame, and the linear momentum radiated in the c.m. frame:

\[ E_{\text{c.m.}}^{\text{rad 3PM}} = \frac{\pi G^3 m_1^2 m_2^2 \gamma}{\gamma} \frac{\gamma - 1}{\gamma + 1} \frac{m_2 - m_1}{M} n_-. \] (5.14)

Here we used the link~[5,33] between \(n_-, u_1^-\) and \(u_2^-\). Let us also note that we have \(e_{\gamma} = n_- + O(G)\) so that \(P_{\text{rad 3PM}} = P_{\text{c.m.}}^{\text{rad 3PM}}\) (at the 3PM approximation).

Inserting Eqs. (5.14) into Eqs. (5.11) above leads then to

\[ c_{u_1}^{1 \text{rr rel 3PM}} = -\pi \left( \frac{GM}{b} \right)^3 \frac{\nu^2}{h^2} \tilde{E}(\gamma) \frac{(m_1 + m_2) \gamma}{(\gamma^2 - 1)}, \]

\[ c_{u_1}^{1 \text{rr rec 3PM}} = \pi \left( \frac{GM}{b} \right)^3 \frac{\nu^2}{h^2} \tilde{E}(\gamma) \frac{(m_1 + m_2) \gamma}{M(\gamma + 1)}, \]

\[ c_{u_2}^{1 \text{rr rel 3PM}} = -\frac{1}{\gamma} c_{u_1}^{1 \text{rr rel 3PM}}, \]

\[ c_{u_2}^{1 \text{rr rec 3PM}} = -\frac{1}{\gamma} c_{u_1}^{1 \text{rr rec 3PM}}. \] (5.15)

A crucial point here is that the separate expressions of \(c_{u_1}^{1 \text{rr rel 3PM}}, c_{u_1}^{1 \text{rr rec 3PM}}, c_{u_2}^{1 \text{rr rel 3PM}}, c_{u_2}^{1 \text{rr rec 3PM}},\) are not polynomials in the two masses (both because of the factor \(\nu^2 = m_1^2 m_2^2/(m_1 + m_2)^4\), and of the presence of \(h^2 = 1 + 2\nu(\gamma - 1)\) in the denominators. However, one finds that the total radiation-reacted values \(c_{u_1}^{1 \text{rr tot 3PM}} = c_{u_1}^{1 \text{rr rel 3PM}} + c_{u_1}^{1 \text{rr rec 3PM}}\) obtained by adding the relativemotion and the recoil contributions, are polynomial in the masses (and actually simply proportional to \(m_1^2 m_2^2\)). This property follows from the (non-evident) identity

\[ m_1 \gamma + m_2 - (\gamma - 1) \frac{(m_1 - m_2) m_1}{M} \equiv M h^2 \]

\[ \equiv M [1 + 2\nu(\gamma - 1)]. \] (5.16)

Indeed,

\[ c_{u_1}^{1 \text{rr tot 3PM}} = -\pi \left( \frac{GM}{b} \right)^3 \frac{\nu^2}{h^2} \tilde{E}(\gamma) \frac{(m_1 + m_2) \gamma}{(\gamma^2 - 1)} \frac{(m_1 + m_2) m_1}{M} \]

\[ = \frac{\pi G^3 m_1^2 m_2^2 \gamma}{b^3} \frac{\gamma}{(\gamma^2 - 1)} \tilde{E}(\gamma), \]

\[ c_{u_2}^{1 \text{rr tot 3PM}} = -\frac{1}{\gamma} c_{u_1}^{1 \text{rr tot 3PM}} \]

\[ = +\frac{\pi G^3 m_1^2 m_2^2}{b^3} \frac{\gamma}{1 - (\gamma^2 - 1)} \tilde{E}(\gamma). \] (5.17)

This mass-polynomiality is a nice check of our derivation above of the two types of radiation-reaction effects.

Summarizing so far, our simple, linear-response computation of the radiation-reaction contributions to the impulse yield, when considered at the 3PM level, a result of the form

\[ \Delta p_{u_1}^{\text{rr 3PM}} = \frac{G^3 m_1^2 m_2^2}{b^3} \left[ \frac{2(2\gamma^2 - 1)^2}{(\gamma^2 - 1)} I(\nu b) \right] + \pi \left( \frac{GM}{b} \right)^3 \frac{\nu^2}{h^2} \tilde{E}(\gamma) \frac{1}{\gamma + 1} (u_2^- - u_1^-). \] (5.18)

This result agrees with Ref. [29]. To make the results Eqs. (5.14), (5.15), explicit, we need to insert the explicit expression of the radiated-energy function \(\tilde{E}(\gamma)\). Kovacs and Thorne~[57] have given explicit expressions for the time-domain radiation pattern at \(O(G^3)\) (see also Ref. [8] for explicit expressions of the frequency-domain radiation pattern at \(O(G^3)\)). We did not succeed in computing the exact total radiated energy \(\tilde{E}(\gamma)\) by integrating over time and angles the radiation pattern of Ref. [57], but we could compute its PN expansion up to order \(v^5\) included, namely:

\[ \tilde{E}(\gamma) = \frac{37}{15} + \frac{2393}{840} v^3 + \frac{61703}{10080} v^5 + \frac{3131839}{354816} v^7 + \frac{51318329}{46126080} v^9 + \frac{60697345}{46126080} v^{11} + \frac{588430385}{39207168} v^{13} + \frac{1275574094617}{7628146608} v^{15} + O(v^{17}). \] (5.19)

Recently Refs. [21, 27, 29] have succeeded in obtaining the exact expression of the the 3PM radiated (energy or) 4-momentum. It reads

\[ \tilde{E}(\gamma) = f_1(\gamma) + f_2(\gamma) \ln \left( \frac{\gamma + 1}{2} \right) \]

\[ + f_3(\gamma) \arccosh(\gamma), \] (5.20)
where

\[
\begin{align*}
f_1(\gamma) &= \frac{1}{48(\gamma^2 - 1)^{3/2}}(1151 - 3336\gamma + 3148\gamma^2 \\
&- 912\gamma^3 + 339\gamma^4 - 552\gamma^5 + 210\gamma^6), \\
f_2(\gamma) &= -\frac{(76\gamma - 150\gamma^2 + 60\gamma^3 + 35\gamma^4 - 5)}{16(\gamma^2 - 1)^2}, \\
f_3(\gamma) &= \frac{\gamma(-3 + 2\gamma^2)(11 - 30\gamma^2 + 35\gamma^4)}{16(\gamma^2 - 1)^2}.
\end{align*}
\] (5.21)

Its high-energy limit, \(\gamma \to \infty\), is \(\hat{E}(\gamma) \approx \hat{C}\gamma^3\), with \(\hat{C} = \frac{5}{8}(1 + 2\ln(2)) \approx 10.4400\), i.e., \(E(\gamma) \approx C\gamma^3\), with \(C = \pi\hat{C} \approx 32.7983\).

The fact that \(\hat{E}(\gamma)\) grows faster than \(\gamma\) in the high-energy limit, i.e., that the energy loss at fixed impact parameter can exceed the energy of the system, shows that the domain of validity of PM gravity is limited. For discussions of the domain of physical validity of the PM expansion see e.g. Eqs. (6.46)-(6.50) in Ref. [23] (and references therein).

VI. RADIATION-GRAVITON CONTRIBUTION TO THE CONSERVATIVE \(O(G^4)\) SCATTERING ANGLE

Within the EFT approach [59], the conservative scattering angle \(\chi_{\text{cons},4\text{PM}}\) (which encodes the full \(O(G^4)\) conservative impulse, \(\Delta p_{\text{cons},4\text{PM}}\)) can be decomposed into two contributions: a potential-graviton contribution, and a radiation-graviton one:

\[
\chi_{G^4\text{cons}} = \chi_{G^4\text{cons, pot}} + \chi_{G^4\text{cons, radgrav}}.
\] (6.1)

These two conservative contributions derive from corresponding (subtracted) radial actions, say

\[
\chi_{G^4,\text{cons, pot}} = -\frac{\partial I_{r,4}^{\text{pot}}(J)}{\partial J},
\] (6.2)

\[
\chi_{G^4,\text{cons, radgrav}} = -\frac{\partial I_{r,4}^{\text{radgrav}}(J)}{\partial J}.
\] (6.3)

Here \(I_{r,4}^{\text{pot}}(J)\) is linked to the exchange of potential gravitons (with momenta \(k^{0} \sim v|k|, \ |k| \sim \frac{1}{\ell}\)), while \(I_{r,4}^{\text{radgrav}}(J)\) is linked to the time-symmetric exchange of soft gravitons propagating on large spatiotemporal distances \((k^{0} \sim |k| \ll \frac{1}{\ell})\).

Recently, Bern et al. [20] derived the potential-graviton contribution \(I_{r,4}^{\text{pot}}(J)\) to the radial action (see also [32]). The latter quantity is IR-divergent, and was written in Ref. [20] in the form

\[
I_{r,4}^{\text{pot}}(J, \gamma, m_1, m_2) = -f_{11} f_{13} [\mathcal{M}_{1}^{\dagger}(\gamma) \\
+ \nu \left(\frac{\mathcal{M}_{1}(\gamma)}{\epsilon} + \mathcal{M}_{1}^{\dagger}(\gamma)\right)] .
\] (6.4)

Here \(\epsilon \equiv -\frac{D-4}{2}\), the prefactors are defined as

\[
\begin{align*}f_{11} &= \frac{\pi G M^2 \mu^2 p^2}{8 E J^3}, \\
f_{13} &= \left(\frac{4 \rho^2 \epsilon \gamma J}{p^2}\right)^{-2c} .
\end{align*}
\] (6.5)

while the (amplitude-related) summands are

\[
\mathcal{M}_{1}^{\dagger}(\gamma) = -\frac{35}{8} \left(1 - 18\gamma^2 + 33\gamma^4\right),
\] (6.6)

and

\[
\mathcal{M}_{1}^{\dagger}(\gamma) = 4 p_{\infty} \hat{E}(\gamma),
\] (6.7)

Here, the term \(\mathcal{M}_{1}^{\dagger}(\gamma)\) encodes the test-particle (Schwarzschild background) dynamics, the IR-divergent, tail-related term \(\mathcal{M}_{1}^{\dagger}(\gamma)\) involves the \(O(G^3)\) radiated energy loss defined in Eq. (6.20), while the term \(\mathcal{M}_{1}(\gamma)\), entering \(\mathcal{M}_{1}^{\dagger}(\gamma)\), is a complicated, transcendental function of \(\gamma\) which encodes the finite piece of the potential-graviton amplitude. It is given in the last Eq. (6) of Ref. [20] (with the change of notation \(\sigma \to \gamma\)). For concreteness, the PN expansions of the various building blocks of the 4PM radial action can be found in Appendix B.

The aim of the present section is to complete the result of Ref. [20] by giving the explicit expression (at the 6PN accuracy) of the complementary radiation-graviton contributions to the scattering angle, \(\chi_{G^4\text{cons, radgrav}}\) or equivalently, to the radial action \(I_{r,4}^{\text{radgrav}}(J)\). For clarity, we shall express our results in terms of a radial action \(I_{r,4}^{\text{radgrav}}(J)\) written in a form closely connected with the form of the potential radial action used in Ref. [20]. Namely, we shall write the complementary radiation-graviton contribution to the radial action in the form

\[
I_{r,4}^{\text{radgrav}} = -f_{11} f_{13} \nu \left(\frac{\mathcal{M}_{1}(\gamma)}{\epsilon} + \mathcal{M}_{1}^{\dagger}(\gamma)\right) .
\] (6.9)

Here, we have conventionally separated out the same prefactors used in the potential-graviton contribution (6.3), and the opposite of the IR-divergent contribution \(+\nu \mathcal{M}_{1}(\gamma)\) present in Eq. (6.3). Indeed, past works on tail effects [60-62] have shown that the IR divergence exhibited in the near-zone (or potential-graviton) dynamics is cancelled by a (UV) divergence present in the multipolar expansion of the wave-zone (or radiation-graviton) dynamics.

---

11 The notations here are \(\mu_0 = \frac{1}{2G}, \mu^2 = 4\pi e^{-\gamma} \mu_0^2, \mu^2 e^{2\gamma} = 4\pi e^{\gamma} \mu_0^2, \mu^2 e^{2\gamma} = 4\pi e^{-\gamma} \mu_0^2\), with \(r_0 = \ell_0^{-2/\gamma^2} = \frac{\ell_0}{\sqrt{4\pi e^{-\gamma}}} \equiv \frac{\ell_0}{\sqrt{\nu}}\).
The recently developed TF formalism has allowed one to compute the $O(G^4)$ conservative, non-local (radiation-graviton) dynamics up to the 6PN accuracy included. In this formalism, the dynamics is decomposed into local-time-symmetric potential-graviton and radiation-graviton dynamics, one cannot simply identify the time-symmetric local-in-time dynamics of Refs. \cite{13, 38} to the EFT time-symmetric potential-graviton and radiation-graviton dynamics. However, one can identify the total, TF-derived conservative scattering angle to the total, conservative scattering angle that would result from computing the 6PN-accurate value of $\chi_4 - \chi_4^{\text{Schw}}$ is equal to

$$
\chi_4 - \chi_4^{\text{Schw}} = \pi\nu\tilde{A}(p_\infty) - \tilde{E}(p_\infty) \ln \left( \frac{p_\infty}{2} \right),
$$

where

$$
\tilde{A}(p_\infty) = -\frac{15}{4} + \frac{123}{256} - \frac{557}{16} p_\infty^4 + \left( \frac{33601}{16384} - \frac{6113}{96} \right) p_\infty^6 + \left( \frac{93031}{32768} - \frac{615581}{19200} \right) p_\infty^8 + \left( \frac{29201523}{33554432} - \frac{5824797}{627200} \right) p_\infty^{10}.
$$

Here we used the result, known from previous work, between the coefficient of the logarithmic term $\ln \left( \frac{p_\infty}{2} \right)$ and the energy radiated in gravitational waves (see, e.g., Ref. \cite{57}). The 6PN-accurate expression (6.17) applies to the conservative (in the sense Fokker-Wheeler-Feynman time-symmetric sense) $O(G^4)$ scattering angle. If we were considering instead the non-conservative, retarded scattering angle one should add to (6.17) radiation-reaction-related contributions. As we shall see, terms quadratic in radiation-reaction will start contributing at the 5PN order, namely at $O(G^4/c^6)$. We gave here the expression (6.17) for the purpose of comparison with forthcoming $O(G^4)$ conservative computations, under the assumption that the meaning of conservative is the same as the (Fokker-Wheeler-Feynman) one used here.

Identifying Eq. (6.10) with Eq. (6.17) finally leads to the following structure for the yet-uncomputed radiation-graviton contribution to the conservative 4PN radial action

$$
\mathcal{M}_4^{\text{radgrav}} = \mathcal{M}_4^{\text{radgrav, finite}} + 4\mathcal{M}_4^0 \ln \left( \frac{p_\infty}{2} \right) + 16p_\infty \hat{E} \ln \left( \frac{p_\infty}{2} \right),
$$

where the 6PN-accurate value of the non-logarithmic contribution is

$$
\mathcal{M}_4^{\text{radgrav, finite}} = \frac{12044}{75} p_\infty^2 + \frac{212077}{3675} p_\infty^4 + \frac{115917979}{793800} p_\infty^6.
$$

This result offers a benchmark to any future computation of the radiation-graviton contribution to conservative dynamics.

Our result (6.20) represents only the beginning of the PN expansion of the conservative part of the radiation-related contributions to the $O(G^4)$ scattering. The dissipative $O(G^4)$ contribution will be discussed below.
VII. MASS-POLYNOMIALITY OF THE $O(G^4)$ RADIATION-REACTION CONTRIBUTION TO THE IMPULSES AND ITS CONSEQUENCES

In the present section we shall explicitly show how the mass polynomiality of the 4-vectorial impulses $\Delta p^\mu_\rho$ (considered as functions of $u^\mu_{\rho1}$, $u^\mu_{\rho2}$, and $b^\mu = b b^\rho$) allows one to derive strong restrictions on the various theoretical building blocks entering our general radiation-reaction formulas derived in Section 11. For definiteness, we shall focus on the 4PM level, but it will be clear that our arguments extend to higher PM levels.

Let us recall the structure of the impulses that will constitute the starting point of our reasonings: the PM expansion of the impulse (say for $a = 1$) is of the form

$$\Delta p_{\mu a} = -2 G m_1 m_2 \frac{2 \gamma^2 - 1}{b^2} \sum_{n \geq 2} \Delta p^\mu_{\rho a} P_n^{PM}, \quad (7.1)$$

where the $n$-PM ($O(G^n)$) contribution $\Delta p^\mu_{\rho a} P_n^{PM}$ must be $\propto \frac{h_n}{n!}$, and be the product of $m_1 m_2$ by a homogeneous polynomial of degree $n - 1$ in the masses. See Eq. (3.53).

The structure (868) applies both to the total (radiation-reacted) impulse and to its conservative contribution. [Therefore, it also separately applies to the radiation-reaction contribution to the impulse.] Its consequences for the conservative scattering angle were discussed in Ref. 23. For the conservative part of the $O(G^4)$ (and $O(\frac{1}{B})$) coefficients $c^\text{cons}_{b,G^4}$, $c^\text{cons}_{u_1,G^4}$, and $c^\text{cons}_{u_2,G^4}$

$$c^\text{cons}_{b,G^4} = \frac{G^4 m_1 m_2 (m_1 + m_2)^3}{b^4} 4 \hbar^2 \zeta (\chi^2_1) (\chi^2_2) - \zeta (\chi^3),$$

$$c^\text{cons}_{u_1,G^4} = \frac{G^4 m_1 m_2 (m_1 + m_2)^2}{b^4} 2 (m_1 \gamma + m_2)$$

$$\times \left(-h^2 (\chi^4_1) + 3 (\chi^2_2) \chi^2_3\right),$$

$$c^\text{cons}_{u_2,G^4} = \frac{G^4 m_1 m_2 (m_1 + m_2)^2}{b^4} 2 (m_2 \gamma + m_1)$$

$$\times \left(-h^2 (\chi^4_1) + 3 (\chi^2_2) \chi^2_3\right),$$

(7.2)

We have here expressed them in terms of the $h$-rescaled scattering coefficients $\tilde{\chi}_n = h^{n-1} \chi_n$ (where $h = \sqrt{1 + 2 \nu (\gamma - 1)}$) which are polynomials in $\nu$ of order $\lfloor \frac{n}{2} \rfloor$. It is then easy to check (remembering that $\nu = \frac{m_1 m_2}{m_1 + m_2}$) that $c^\text{cons}_{b,G^4}$, $c^\text{cons}_{u_1,G^4}$, and $c^\text{cons}_{u_2,G^4}$ are polynomials in the masses.

Let us now discuss the consequences of such a structure for the radiation-reaction contribution to the impulse.

We have seen above that, at the $O(G^4)$ level, the mass dependence of the total radiation-reaction contribution to the impulse was $\Delta p^\mu_\rho \sim G^4 m_1^2 m_2^2 / b^4$. The presence of a factor $m_1 m_2$ in the LO contribution to $\Delta p^\mu_\rho$ is linked to the fact that radiation-reaction effects balance radiative fluxes that are quadratic in the emitted waveform, and that the (time-dependent part of the) emitted waveform contains a factor $m_1 m_2$. As a consequence, next-to-leading-order, and higher, contributions to $\Delta p^\mu_\rho$ must all contain at least a factor $m_1^2 m_2^2$. Polynomiality in the masses then shows that the radiation-reaction contribution to the impulses (and therefore to their coefficients $c^\text{cons}_{b}, c^\text{cons}_{u_1}, c^\text{cons}_{u_2}$) must have the structure

$$\Delta p^\mu_\rho \sim \frac{G^3 m_1^2 m_2^2}{b^3} \left[1 + \frac{G m_1}{b} + \frac{G m_2}{b} \right.\right.$$

$$\left.\left. + \frac{G^2 m_1^2 m_2}{b^2} \right] \right]$$

(7.3)

with (unwritten) coefficients depending only on $\gamma$. We now show how to use this mass-polynomiality, in conjunction with our general formulas above for the radiation-reaction effects, to restrict the mass dependence of the various radiative losses.

Previous studies have found convenient to express the PM expansion of the conservative scattering angle as a power expansion in the dimensionless variable $\frac{1}{J} = G m_1 m_2$, namely,

$$\chi^\text{cons} = \sum_{n=1}^{\infty} \frac{2 \chi^\text{cons}_n}{\gamma^n}.$$  

(7.4)

Remembering that (see Appendix B)

$$\frac{1}{J} = G M h$$

(7.5)

any expansion in powers of $\frac{1}{J}$ corresponds to an expansion in powers of $\frac{h}{\sqrt{\gamma}}$. However, the transcription between the two expansions involve powers of $M h = M \sqrt{1 + 2 \nu (\gamma - 1)}$. Keeping in mind the presence of such (non polynomial in masses) factors, we can similarly parametrize the PM expansions of the radiative losses by the coefficients of their power expansion in $\frac{1}{J}$, say

$$\delta_\text{rad} J = \frac{J_{c.m.}}{J_{c.m.}} = \pm \nu \sum_{n=2}^{\infty} \frac{J_n}{J},$$

$$\delta_\text{rad} E = \frac{E_{c.m.}}{M} = \pm \nu \sum_{n=2}^{\infty} \frac{E_n}{M},$$

$$\delta_\text{rad} P_y = \frac{P_{y,\text{c.m.}}}{M} = \pm \nu \sum_{n=3}^{\infty} \frac{P_n}{M}.$$  

(7.6)

Here we have adimensionalized the left-hand sides, and pulled out some powers of $\nu$ on the right-hand sides, to ensure that the expansion coefficients $J_n$, $E_n$, $P_n$ are dimensionless, and that their LO PN contribution is $\nu$-independent. [$J_{c.m.} = b_{c.m.} \gamma / h = G M^2 \nu_j$, in the denominator of the first equation contains a factor $\nu$ so that $J_{c.m.}$ actually contains the same factor $\nu^2$ as the other losses.]

Note the sign difference between the (negative) mechanical losses of the two-body system (denoted with $\delta_\text{rad}$) and the (positive) radiated quantities ($J_{\text{rad}} > 0$, $\nu_{\text{rad}} > 0$ and also, with our convention, $\nu_2^{-1} m_2 / m_1 > 0$, see below).

The quantities $J_{\text{rad}}$ and $E_{\text{rad}}$ are symmetric under the exchange of the two bodies. Therefore the expansion...
coefficients $J_n$ and $E_n$ are functions of $\gamma$ and of the symmetric mass-ratio $\nu$. The same is true for the expansion coefficients $P_n$ (obtained after factoring the antisymmetric mass ratio $\omega_{ij}$). It was shown in Ref. [37] that the $\nu$ dependence of $E_n$ is restricted by the property

$$h^{n+1} E_n(\gamma, \nu) = P^{\gamma}_{(n+1)}(\nu),$$  

(7.7)

where $P^{\gamma}_{(n)}(\nu)$ denotes a polynomial in $\nu$ of order $N$, with coefficients depending on $\gamma$. [The notation $[\cdots]$ indicates the integer part.] The mass-polyvanility of the radiation-reacted impulse implies analog restrictions on the $\nu$-dependence of the other expansion coefficients $J_n$ and $P_n$ entering the radiation losses (7.6). This is seen by using our previous results to write down explicit expressions for the PM-expansion contributions to the radiation-reacted impulse coefficients $c_{b}^{\nu}$, $c_{u1}^{\nu}$, and $c_{u2}^{\nu}$.

Using Eqs. (3.25), (7.4), and (7.5), the PM-expansion coefficients of the radiation-reaction contribution to the relative scattering angle, say

$$\delta^R\chi = \sum_{n=3}^{\infty} \frac{2 \chi^R_n}{\gamma^n},$$  

(7.8)

are found to read

$$2\chi^R_3 = \nu \chi^\nu_1 J_2,$$

$$2\chi^R_4 = \nu \left(2\chi^\nu_2 J_2 + \chi^\nu_1 J_3 - hE_3 \frac{d\chi^\nu_1}{d\gamma}\right),$$

$$2\chi^R_5 = \nu \left[\left(\chi^\nu_1 J_4 + 2\chi^\nu_2 J_3 + 3\chi^\nu_3 J_2\right)
- h \left(E_4 \frac{d\chi^\nu_1}{d\gamma} + E_3 \frac{d\chi^\nu_2}{d\gamma}\right)\right],$$  

(7.9)

eq etc. To make these PM results explicit we need to insert the explicit values of the various expansion coefficients $\chi^\nu_n$, $J_n$, and $E_n$. Among these PM expansion coefficients the current knowledge concerns: (i) the values of the $\chi^\nu_n$ up to $n = 3$ included, e.g., the first two read,

$$\chi^\nu_1 = \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}}, \quad \chi^\nu_2 = \frac{3}{8} \frac{5\gamma^2 - 1}{h};$$  

(7.10)

and the explicit values of $J_2$ [26] (see Eqs. (7.6) and (5.6) above), and of $E_3$ [26] (see Eqs. (5.14) and (5.20) above). Note, however, that the 3PM angular momentum loss $J_3$ has not yet been exactly computed. We, however, derived its 4.5PN-accurate value. See below and Appendix II.

The value of the radiation-reacted relative scattering angle $\delta^R\chi$ yields only a partial contribution to the radiation-reacted impulse. One needs to add the radiative-recoil contributions to get the total radiation-reacted impulse coefficients $c_{b}^{\nu}$, $c_{u1}^{\nu}$, and $c_{u2}^{\nu}$. Moreover, one needs to express their PM expansion in terms of the impact parameter $b$ in order to exhibit their polynomiality in the masses. Starting from the expansion of the impulse coefficients in powers of $G/b$, say

$$c_{b}^{\nu} = \sum_{n=3}^{\infty} c_{b}^{\nu,n} \left(\frac{G}{b}\right)^n,$$

$$c_{u1}^{\nu} = \sum_{n=3}^{\infty} c_{u1}^{\nu,n} \left(\frac{G}{b}\right)^n,$$

$$c_{u2}^{\nu} = \sum_{n=3}^{\infty} c_{u2}^{\nu,n} \left(\frac{G}{b}\right)^n,$$  

(7.11)

we can define dimensionless versions, $c_{b,\nu}^{\nu}$, $c_{u1,\nu}^{\nu}$, $c_{u2,\nu}^{\nu}$, of the expansion coefficients $c_{b}^{\nu}$, $c_{u1}^{\nu}$, $c_{u2}^{\nu}$, by writing

$$c_{b,\nu}^{\nu,n} = m_1 m_2 M^{n-1} c_{b}^{\nu,n},$$

$$c_{u1,\nu}^{\nu,n} = m_1 m_2 M^{n-1} c_{u1}^{\nu,n},$$

$$c_{u2,\nu}^{\nu,n} = m_1 m_2 M^{n-1} c_{u2}^{\nu,n}.$$  

(7.12)

Let us start by discussing the expression of the 4PM impulse coefficient $c_{b,\nu}^{\nu}$, which reads

$$c_{b,\nu}^{\nu} = \frac{\nu m_1 (m_1 - m_2)(m_1 \gamma + m_1)}{M^2(\gamma^2 - 1)^2} \chi_1^\nu, h^2 p_3$$

$$+ \frac{\nu h^3}{(\gamma^2 - 1)^{3/2}} \left[\frac{h^2 d\chi_1^\nu}{d\gamma} + \frac{2(m_2 \gamma + m_3)(m_1 \gamma + m_2)}{M^2(\gamma^2 - 1)} \chi_1^\nu\right] E_3$$

$$- \frac{2\nu h^3}{(\gamma^2 - 1)^{3/2}} \chi_1^\nu J_2$$

$$- \frac{\nu h^3}{(\gamma^2 - 1)^{3/2}} \chi_1^\nu J_3.$$  

(7.13)

The structure of this quantity can be simplified by using the explicit expressions of $\chi_1^\nu$ and $\chi_2^\nu$ recalled above, together with the $\nu$-dependences of $J_2$ and $E_3$, as well as the relation between $E_3$ and $P_3$. Defining

$$\tilde{J}_2(\gamma) \equiv 2(2\gamma^2 - 1) \sqrt{\gamma^2 - 1} \mathcal{I}(\nu),$$  

(7.14)

with $\mathcal{I}(\nu)$ defined in Eq. (5.14), and using Eq. (5.20) for the definition of $\tilde{E}$ and Eqs. (5.14) above, we get

$$h^2 J_2 = \tilde{J}_2(\gamma),$$

$$h^4 E_3 = \frac{\nu p_3^3}{p_3^3} \tilde{E}(\gamma),$$

$$\frac{P_3}{E_3} = \sqrt{\frac{\gamma - 1}{\gamma + 1}}.$$  

(7.15)

Note that in these relations, the right-hand sides depend only on $\gamma$ (and not on $\nu$). This leads to

$$c_{b,\nu}^{\nu} \left(\frac{G}{b}\right)^n = \frac{\nu p_3^3}{p_3^3} \left[6(\gamma^2 - 1)\gamma \tilde{E} - \frac{3(5\gamma^2 - 1)}{4} \tilde{J}_2\right]$$

$$- m_1 \left(\frac{(2\gamma^2 - 1)}{M} \tilde{E}\right)$$

$$- \frac{2\gamma^2 - 1}{p_3^3} \left(\frac{\nu p_3^3}{p_3^3} \tilde{E}^2 + h^2 J_3\right).$$  

(7.16)
The corresponding value of the dimensionfull impulse coefficient reads

$$e_{b,G^4}^{\text{tot}} = m_1 m_2 M^3 c_{b,G^4}^{\text{tot}} = (m_1 m_2)^2 M^{\frac{g_{b,G^4}}{\nu}}. \quad (7.17)$$

As one sees on the last member of these equations, the fact that $e_{b,G^4}^{\text{tot}}$ should be polynomial (and actually quintic) in the masses implies that the product $M^{\frac{g_{b,G^4}}{\nu}}$ should be linear in $m_1$ and $m_2$ (with $\gamma$-dependent coefficients). This property is clearly satisfied by the first three lines on the right-hand side of Eq. (7.16). By contrast, the last line on the right-hand side of Eq. (7.16) will violate this property because of its dependence on $\nu = \frac{m_1 m_2}{m_1 + m_2}$, unless the function $h^3 J_3(\gamma, \nu)$ has a special $\nu$-dependence which completely cancels the various $\nu$-dependencies contained in $\frac{\nu \nu_p \tilde{E}}{h^2}$. This proves that the quantity

$$\tilde{J}_3 \equiv h^3 J_3 + \frac{\nu \nu_p \tilde{E}}{h^2}, \quad (7.18)$$

must be independent of $\nu$, and be only a function of $\gamma$. In other words, by considering the limit $\nu \to 0$, we derive the remarkable identity

$$h(\gamma, \nu)^3 J_3(\gamma, \nu) + \frac{\nu \nu_p \tilde{E}(\gamma)}{h^2(\gamma, \nu)} = \tilde{J}_3(\gamma) = \lim_{\nu \to 0} J_3(\gamma, \nu). \quad (7.19)$$

Equivalently, one should have

$$\frac{h^3 J_3 - J_3|_{\nu=0}}{\nu} + \frac{\nu \nu_p \tilde{E}}{h^2} \equiv 0. \quad (7.20)$$

In other words, the $\nu$-dependent terms in the function $h^3 J_3$ must be entirely determined by the function $-\frac{\nu \nu_p \tilde{E}}{h^2}$.

We have computed $J_3(\gamma, \nu)$ with 2PN fractional accuracy (see Table XX). Our final result reads

$$\pi^{-1} J_3 = \frac{28}{5} \nu_p^2 \tilde{E}_\infty + \left( \frac{739}{84} - \frac{163}{15} \nu \right) \nu_p^4 \tilde{E}_\infty + \left( \frac{5777}{2520} - \frac{5339}{420} \nu + \frac{50}{3} \nu^2 \right) \nu_p^6 \tilde{E}_\infty + O(\nu^8). \quad (7.21)$$

We checked that it satisfies the rule \textsuperscript{[XX]}.

Let us now consider the two other 4PM impulse coefficients $c_{s_1,G^4}^{\text{tot}}, c_{s_2,G^4}^{\text{tot}}$. The initial expression for the first one reads

$$c_{s_1,G^4}^{\text{tot}} = \frac{\gamma(m_1 m_2)}{(\gamma^2 - 1)^{3/2} M^2} h^3 E_4 + \frac{m_1 (m_1 - m_2) \gamma}{(\gamma^2 - 1)^{1/2} M^2} h^3 P_4 - \frac{2(m_1 + m_2) (\chi_{\text{cons}})^2}{(\gamma^2 - 1)^2 M} h^2 J_2. \quad (7.22)$$

As in the case of $c_{b,G^4}^{\text{tot}}$, the $\nu$-dependence of the various building blocks, $E_4, P_4, J_2$, entering the latter expression is crucial to allow $c_{s_1,G^4}^{\text{tot}}$ to be a polynomial in the masses. We already know that the $\nu$-dependence of $J_2$ is determined by the fact that

$$h^2 J_2 = \tilde{J}_2(\gamma), \quad (7.23)$$

where $\tilde{J}_2(\gamma)$ is defined in the first line of Eq. (7.15), while the $\nu$-dependence of $E_4$ is restricted by Eq. (7.18). In the present case $n = 4$, the latter restriction says that $h^3 E_4$ must be linear in $\nu$:

$$h^3 E_4(\gamma, \nu) = E_4(\gamma) + \nu \tilde{E}_4(\gamma). \quad (7.24)$$

Given these $\nu$-dependences of $J_2$ and $E_4$, the $\nu$-dependence of the remaining building block $P_4$ must be such that it allows the corresponding dimensionfull impulse coefficient,

$$c_{s_1, G^4}^{\text{tot}} = m_1 m_2 M^3 c_{s_1, G^4}^{\text{tot}} = (m_1 m_2)^2 M^{\frac{g_{s_1, G^4}}{\nu}}, \quad (7.25)$$

to be a polynomial in the masses. In other words, $M^{\frac{g_{s_1, G^4}}{\nu}}$ must be linear in $m_1$ and $m_2$.

Combining the simple $\nu$ dependences of $h^2 J_2$ and $h^3 E_4$ with the crucial identity \textsuperscript{[XIX]}, one finds that the mass-polynomiality of $c_{s_1, G^4}^{\text{tot}}$ implies that the $\nu$-dependence of $P_4$ must be determined by the condition

$$h^5 P_4 = \frac{1}{p_\infty} \left( (\gamma - 1) \tilde{E}_4(\gamma) - \frac{1}{2} \tilde{E}_4(\gamma) \right). \quad (7.26)$$

This identity can be conveniently rewritten by replacing $\tilde{E}_4(\gamma) = h^3 E_4 - \nu \tilde{E}_4(\gamma)$, and using $p_\infty = \sqrt{\gamma^2 - 1}$. This yields the following (double) identity

$$h^3 \left[ P_4 - \frac{\gamma - 1}{\gamma + 1} E_4 \right] = \left[ P_4 - \frac{\gamma - 1}{\gamma + 1} E_4 \right]_{\nu=0} = \frac{\tilde{E}_4(\gamma)}{2 p_\infty}. \quad (7.27)$$

Let us mention in passing that the identities, between $E_\infty(\gamma, \nu)$ and $P_\infty(\gamma, \nu)$ (as well as the $\nu$-structure of $E_\nu$), derived here (in a somewhat indirect manner) by using the mass-polynomiality of $\Delta p_{\mu}^{\text{tot}}$ can also be directly derived from the mass-polynomiality structure of the large-$b$ expansion of the radiated 4-momentum $p_{\mu}^{\text{rad}}$. This is done by combining: (i) the $b^4, u^4, \bar{u}^4$ decomposition of $p_{\mu}^{\text{rad}}$; (ii) the property mentioned above that, in the c.m. frame, $p_{\mu}^{\text{rad}}$ is directed along the $e_\nu$ axis; and (iii) the $\nu$-structure of the conservative scattering angle. We leave this derivation (and its extension to higher PM orders) as an exercize to the reader. An extension of such a direct PM derivation to the case of $J_\mu(\gamma, \nu)$ is more challenging in view of the delicate issues linked to defining the radiation of relativistic angular momentum $J_{\mu\gamma}^{\text{rad}}$.

We checked that the 4.5PN-accurate PN expansions of $P_4$ and $E_4$ do indeed satisfy the relation \textsuperscript{[XX]} to the
Corresponding PN accuracy, while the PN expressions for $E_0^0$, $E_1^1$, and $J_3$ are given below in Eq. (8.3).

Concerning the third 4PM impulse coefficient, namely $c^{\text{rr tot}}_{u_2,G_4}$, one finds that (as was the case at the 3PM level) it is simply connected to $c^{\text{rr tot}}_{u_1,G_4}$ via

$$\nu^{-1} \left( c^{\text{rr tot}}_{u_2,G_4} + \frac{1}{\gamma} c^{\text{rr tot}}_{u_1,G_4} \right) = \frac{2(2\gamma^2 - 1)^2}{2M} J_2. \quad (7.28)$$

The latter relation implies the needed mass polynomials (namely the linearity in the masses of the product $\frac{M}{\nu^{\text{rr tot}}_{u_2,G_4}}$) without further restriction on the building blocks $E_4$, $P_4$, $J_2$.

Gathering our results, we conclude that the radiation-reaction contribution to the 4PM impulse can be written as (see Eq. (8.34.7))

$$\Delta p_1^{\text{rr 4PM}} = c^{\text{rr 4PM}}_0 + c^{\text{rr 4PM}}_{u_1} + c^{\text{rr 4PM}}_{u_21,1}, \quad (7.29)$$

where the coefficients have the following simple dependence on the masses

$$c^{\text{rr 4PM}}_0 = \frac{G^4 m_2^2 m_3^2}{b^2} \left[ C_{4PM} (\gamma) M + C_{b_{nm1}} (\gamma) m_1 \right],$$

$$c^{\text{rr 4PM}}_{u_1} = \frac{G^4 m_2^2 m_3^2}{b^4} C_{4PM} (\gamma) m_2,$$

$$c^{\text{rr 4PM}}_{u_2} = \frac{G^4 m_3^2 m_1^2}{b^4} \left[ C_{4PM} (\gamma) M + C_{b_{2m1}} (\gamma) m_1 \right]. \quad (7.30)$$

The $\gamma$-dependent coefficients entering the various impulse contributions read

$$C_{4PM} (\gamma) = \pi \frac{\gamma (6\gamma^2 - 5)}{(\gamma - 1)^2} - \frac{3}{4} \frac{J_2}{\gamma} \frac{(5\gamma^2 - 1)}{(\gamma^2 + 1)^2},$$

$$C_{b_{nm1}} (\gamma) = -\frac{2}{3} \frac{(2\gamma^2 - 1)}{(\gamma - 1)^2}, \quad (7.31)$$

The $\gamma$-dependent coefficients entering the various impulse contributions read

$$C_{b_{nm1}} (\gamma) = -\frac{2}{3} \frac{(2\gamma^2 - 1)}{(\gamma - 1)^2}, \quad (7.31)$$

$$C_{4PM} (\gamma) = \frac{4\gamma (2\gamma^2 - 1)^2 J_2 + 2E_1^0}{(\gamma^2 - 1)^2},$$

$$C_{u_{2m1}} (\gamma) = \frac{2(\gamma - 1)(2\gamma^2 - 1)^2 J_2 + E_1^0}{(\gamma^2 - 1)^3}. \quad (7.33)$$

Note that, in terms of our general notation (7.20), $\hat{E}$ could be replaced by

$$\hat{E}_3 (\gamma) \equiv h^4 E_3 = \pi \nu^3 \hat{E}. \quad (7.34)$$

Our general PM-expanded expressions for the impulse $\Delta p_4$ are given, up to order $G^5$, in Table I[6].

VIII. PN-EXPANDED IMPULSE COEFFICIENTS AT $O(G^5)$ AND $O(G^0)$

Our general results, displayed in Table I[6] for the radiation-reacted impulses involve the radiative losses (in the c.m. frame) of energy, angular momentum and linear momentum. These losses admit a double PM and PN expansion, which can be expressed as (with $j = J/(Gm_1 m_2)$, $p_\infty = \sqrt{\gamma^2 - 1}$ and $P^{\text{rad}} = P^{\text{rad}}_{\gamma_0}$)

$$E^{\text{rad}} = \nu^2 \left[ E_3 (p_\infty) \frac{j^3}{j^4} + E_4 (p_\infty) \frac{j^4}{j^4} + \cdots \right],$$

$$J^{\text{rad}} = \nu^4 \left[ J_2 (p_\infty) \frac{j^2}{j^4} + J_3 (p_\infty) \frac{j^4}{j^4} + \cdots \right],$$

$$P^{\text{rad}} = \nu^2 \left[ \frac{p_3 (p_\infty)}{j^3} + \frac{p_4 (p_\infty)}{j^4} + \cdots \right]. \quad (8.1)$$

Here the subscripts $n$ (e.g., in $E_n$) label the nPN order, i.e. $O(G^n)$. The subsequent expansion of the various PM coefficients, $E_n (p_\infty)$, $J_n (p_\infty)$, $P_n (p_\infty)$ in powers of $p_\infty$ then corresponds to the usual PN expansion.

The only radiative losses that are known in a PM-exact way are $J_2$ [27] and, $E_3$ and $P_4$ [29]. In order to compute the radiation-reacted impulses at PN orders $G^4$ and $G^5$, we also need to know the values of $J_3$, $E_4$ and $P_4$. Existing results in the PN literature (notably [63, 64]) provide the values of the needed radiative losses only at the NLO PN accuracy, i.e., with 1PN fractional accuracy beyond the LO PN result. Using techniques we already used in our previous works, we have computed the various needed radiative losses along hyperbolic motions at the NNLO accuracy, i.e. with 2PN fractional accuracy. In addition, we also evaluated the fractional 1.5PN correction coming from the time-antisymmetric tail contribution to radiation reaction. Details on our computations are given in Appendices I[11] and I[11]. We computed the complete, 2PN expressions for the radiative losses of energy, angular momentum and linear momentum in terms of two independent orbital parameters (namely, $\bar{a}$ and $e_p$). These results can then be easily expanded (to any needed order) in powers of $\frac{1}{\sqrt{\gamma}}$, i.e. in powers of $G$.

The explicit 2PN-accurate results for $E_n (p_\infty)$, $J_n (p_\infty)$ and $P_n (p_\infty)$ up to $n = 7$ will be found in Appendix I[11] Table IX. Let us only display here our final results for the PN expansion of the three functions of $\gamma$: $E_3 (\gamma)$, $E_4 (\gamma)$ and $J_3 (\gamma)$.

$$E_3 (\gamma) \equiv \frac{1568}{45} p_\infty^3 + \frac{28}{5} p_\infty^5 + \frac{739}{84} p_\infty^7,$$

$$E_4 (\gamma) \equiv \frac{352}{45} p_\infty^3 + \frac{1735}{2520} p_\infty^5 + \frac{739}{84} p_\infty^7,$$

$$J_3 (\gamma) \equiv \frac{1735}{2520} p_\infty^5 - \frac{28}{5} p_\infty^3,$$

In the following, we have used the PM-exact relation [17, 20], and our computation of the tail contribution to $P_4$ see Appendix...
TABLE II: PM-expanded expression for $\Delta p_{1}$ up to 5PM. We use the notation $\hat{u}_1 = \frac{u_1}{\gamma - 1}$, $\hat{u}_2 = \frac{u_2}{\gamma - 1}$, $\hat{J}_4 = h^4 J_4 + h^3 \nu E_4$, $P_6 = \frac{1}{4 \sqrt{\gamma - 1}} E_6 + \frac{2}{(\gamma - 1)^2} W_6$, where $P_6 \equiv h^6 P_0$, $E_5 \equiv h^5 E_5$ and $C_0 = p_\infty \left( \sqrt{\frac{\sqrt{2} P_6}{\gamma - 1}} \right)_{\nu=0} = \pi \left[ \frac{25}{17} P_\infty^4 + \frac{60797.6}{17000000000} + \frac{9392}{2800^2} \frac{7}{1} + \frac{97951}{80000000000} \right] P_\infty + O(p_\infty^6).

1PM

2PM

3PM cons

3PM rr, tot

4PM cons

4PM rr, tot

5PM cons

5PM rr, tot

\[ \Delta p_{\nu,\text{tot}}^{\text{4PM}} \sim \frac{1}{c^2} + \frac{1}{c^3} + \frac{1}{c^4} + \frac{1}{c^5}. \] (8.3)

Let us decompose the generic impulse coefficient $c_{X, \nu} = b, u_1, u_2$ (as well as $\hat{c}_{u_1}$) in the form

\[ c_{X} = \mu \sum_{n=1}^{\infty} \left( \frac{GM}{b} \right)^n c_{X, \nu^n}, \] (8.4)

where $X = b, u_1, u_2$, and

\[ c_{X, \nu^n} = c_{X, \nu^n}^{\text{cons}} + c_{X, \nu^n}^{\text{rr, rel}} + c_{X, \nu^n}^{\text{rr, rec}} \equiv c_{X, \nu^n}^{\text{cons}} + c_{X, \nu^n}^{\text{rr, tot}}, \] (8.5)

For $X = b$ at $O(G^4)$ we find (recalling the notation $\Delta \equiv \sqrt{1 - 4\nu}$)

\[ c_{b, \nu^4}^{\text{cons}} = \pi \left[ \frac{6}{p_\infty^8} + \left( \frac{21}{15} \frac{\Delta}{\nu} + \frac{27}{15} \frac{\nu}{\Delta} \right) + \left( -99 + \frac{797}{8} \right) \frac{1}{p_\infty^2} + O(p_\infty^4) \right], \]

\[ c_{b, \nu^4}^{\text{rr, rel}} = \pi \nu \left[ - \frac{47}{15} \frac{\Delta}{\nu} - \frac{37}{15} \nu + 10037 \frac{1}{p_\infty^6} + \left( \frac{5501}{840} \nu + \frac{37}{15} \nu^2 - \frac{221993}{10080} \right) \frac{1}{p_\infty^2} + O(p_\infty^4) \right], \]

\[ c_{b, \nu^4}^{\text{rr, rec}} = - \pi \nu \left[ - \frac{37}{60} \Delta - \frac{37}{60} \nu + 37 + \frac{5501}{840} \nu + \frac{37}{15} \nu^2 - \frac{1661}{1120} \Delta \frac{1}{p_\infty^2} + O(p_\infty^4) \right], \]

\[ c_{b, \nu^4}^{\text{rr, tot}} = \pi \nu \left[ \frac{37}{60} \Delta - \frac{2111}{168} \frac{1}{p_\infty^8} + \frac{118471}{5040} \frac{1}{p_\infty^6} + \frac{1661}{1120} \Delta \frac{1}{p_\infty^2} + O(p_\infty^4) \right], \] (8.6)
while at $O(G^5)$

\[ \begin{align*}
\epsilon_{b,G^5}^{\text{cons}} &= -\frac{2}{p_\infty} + \frac{(12 - 4\nu) + (9\pi^2 + 80 + 9\nu - 2\nu^2 + O\left(\frac{1}{p_\infty}\right)}{p_\infty}, \\
\epsilon_{b,G^5}^{\text{rr,rel}} &= \nu \left[ \frac{416}{45p_\infty} + \frac{123284}{45p_\infty} - \frac{47\pi^2}{3} - \frac{96\nu}{3} \right] - 896 + \frac{128\nu}{3} + \frac{37\nu^2}{10} - \frac{5697}{280\nu^2} - \frac{73592}{1575\nu} + \frac{116992}{1225} + O\left(p_\infty^2\right), \\
\epsilon_{b,G^5}^{\text{rr,rec}} &= \nu \left[ \frac{1}{3} - \frac{32\nu}{3} - \frac{128\nu}{3} + \frac{32\Delta}{3} \right] + \frac{1}{p_\infty} - \frac{13696}{525\nu} + \frac{128\nu^2}{3} + \frac{13696}{525} - \frac{37\nu^2}{10} + \frac{37\nu^2}{10} - \frac{20128}{175\nu} + \frac{37\nu^2}{40} \pi^2 \Delta \\
&+ O\left(p_\infty\right), \\
\epsilon_{b,G^5}^{\text{rr,tot}} &= \nu \left[ \frac{416}{45p_\infty} + \left( -\frac{47\pi^2}{5} + \frac{186464}{1575} + \frac{352\nu}{15} + \frac{32\Delta}{3} \right) \frac{1}{p_\infty} - \frac{896}{45p_\infty} + \frac{73592}{1575\nu} + \frac{116992}{1225} + O\left(p_\infty^2\right) \right].
\end{align*} \]

(8.7)

For $X = u_1$ at $O(G^4)$ we find

\[ \begin{align*}
\epsilon_{u_1,G^4}^{\text{cons}} &= \frac{2}{p_\infty} + \frac{(2\nu - \frac{1}{3}\Delta - \frac{15}{4})}{p_\infty} + \frac{(24\nu - \frac{91}{3} - \frac{9}{2}\pi^2 + \frac{47\nu}{3} - \frac{1}{4}\Delta\nu)}{p_\infty} + O\left(\frac{1}{p_\infty}\right), \\
\epsilon_{u_1,G^4}^{\text{rr,rel}} &= \nu \left[ -1856 + \frac{464\nu}{45p_\infty} + \frac{128\nu}{3} + \frac{12086}{175\nu} - \frac{3136}{45} \right] + \frac{1}{p_\infty} + \left( -\frac{32}{3} \pi \nu^2 - \frac{1266484}{11025} - \frac{128\nu^2}{3} + \frac{30796}{1575\nu} + \frac{8928}{175\nu} \right) p_\infty \\
&+ O\left(p_\infty^2\right), \\
\epsilon_{u_1,G^4}^{\text{rr,rec}} &= \nu \left[ -\frac{32}{3} + \frac{128\nu}{3} + \frac{32\Delta}{3} \right] + \frac{8928}{175\nu} - \frac{128\nu^2}{3} - \frac{5296}{525} + \frac{5296}{525} - \frac{32\Delta}{3} p_\infty + O\left(p_\infty^2\right), \\
\epsilon_{u_1,G^4}^{\text{rr,tot}} &= \nu \left[ -\frac{3136}{45} - \frac{1856}{45p_\infty} + \left( -\frac{46288}{525} + \frac{16\Delta}{3} \right) \frac{1}{p_\infty} + \left( -\frac{1155268}{11025} + \frac{14908}{1575\nu} \right) p_\infty + O\left(p_\infty^2\right) \right].
\end{align*} \]

(8.8)

while at $O(G^5)$

\[ \begin{align*}
\epsilon_{u_1,G^5}^{\text{cons}} &= \frac{6}{p_\infty} + \frac{(87\pi^2 + 21\nu - \frac{3}{2}\Delta)}{p_\infty} + \frac{2961}{8} + \frac{93}{8}\Delta + \frac{457}{2}\nu^2 - \frac{21}{4}\Delta\nu - \frac{123}{64}\nu^2}{p_\infty} + O\left(\frac{1}{p_\infty}\right), \\
\epsilon_{u_1,G^5}^{\text{rr,rel}} &= \frac{178}{5\nu} + \frac{89\nu}{20\nu} + \frac{3407}{22p_\infty} - \frac{297\nu^2}{20p_\infty} + \frac{1}{p_\infty} + \left( -\frac{89}{8}\Delta\nu - \frac{14899}{48p_\infty} - \frac{89}{2}\nu^2 + \frac{941}{14908}\nu - \frac{254119}{11025} \right) + O\left(p_\infty^0\right), \\
\epsilon_{u_1,G^5}^{\text{rr,rec}} &= \pi\nu \left[ \frac{106\nu}{3} + \frac{3\nu}{2} + \frac{33\Delta}{2} \right] + \frac{105083}{191040\nu} + \frac{89\Delta\nu}{2699} + \frac{65650}{2029\nu} + \frac{89}{2}\nu^2 - \frac{105083}{191040\nu} + O\left(p_\infty^0\right), \\
\epsilon_{u_1,G^5}^{\text{rr,tot}} &= \nu \left[ \frac{178}{5\nu} + \left( -\frac{1065}{8} + \frac{155}{6}\nu \right) \frac{1}{p_\infty} - \frac{297}{20p_\infty}\pi^2 + \left( \frac{51949}{2520} + \frac{28223}{5040} - \frac{545947}{2520} \right) \frac{1}{p_\infty} + O\left(p_\infty^2\right) \right].
\end{align*} \]

(8.9)

For $X = u_2$ at $O(G^4)$ we find

\[ \begin{align*}
\epsilon_{u_2,G^4}^{\text{cons}} &= -\frac{2}{p_\infty} + \frac{(15 - \frac{1}{3}\Delta - 2\nu)}{p_\infty} + \frac{(12\nu + 2\pi^2 + \nu^2 + 2\pi^2 - 2\Delta\nu - 2\nu + \frac{413}{3})}{p_\infty} + O\left(\frac{1}{p_\infty}\right), \\
\epsilon_{u_2,G^4}^{\text{rr,rel}} &= \nu \left[ 1856 + \left( -\frac{128\nu}{3} + \frac{282242}{1575\nu} - \frac{64\Delta}{3} \right) \right] + \frac{3136}{45} + \left( \frac{128}{3}\nu^2 + \frac{42844}{441} - \frac{15584}{525}\nu + \frac{32}{3}\Delta\nu + \frac{412}{225}\Delta \right) p_\infty \\
&+ O\left(p_\infty^2\right), \\
\epsilon_{u_2,G^4}^{\text{rr,rec}} &= \nu \left[ -\frac{1}{3}\Delta - \frac{128\nu}{3} + \frac{32\Delta}{3} \right] + \left( \frac{832}{175} + \frac{32}{3}\Delta\nu + \frac{832}{175}\nu^2 - \frac{15584}{525}\nu \right) p_\infty + O\left(p_\infty^2\right), \\
\epsilon_{u_2,G^4}^{\text{rr,tot}} &= \nu \left[ 1856 + \left( \frac{32\Delta}{9} + \frac{111424}{1575} \right) \frac{1}{p_\infty} + \frac{3136}{45} + \left( \frac{1018684}{11025} + \frac{10372}{1575}\Delta \right) p_\infty + O\left(p_\infty^2\right) \right].
\end{align*} \]

(8.10)
while at $O(G^5)$

\[ c_{u_2, G^5}^{\text{cons}} = \pi \left[ \frac{6}{p_\infty^6} - \frac{3}{2} \frac{-29 + 14\nu + \Delta}{p_\infty^5} \right] + \left( \frac{2961 + 424\Delta - 457\nu - 241\Delta \nu + 122\pi^2 \nu}{p_\infty^4} \right) + O\left( \frac{1}{p_\infty^3} \right), \]

\[ c_{u_2, G^5}^{\text{rr, rel}} = \pi \nu \left[ \frac{178}{5p_\infty^4} + \frac{(89 - 33\Delta)\nu - 15571}{120p_\infty^3} + \frac{297}{20p_\infty^2} \nu^2 + \left( \frac{60773}{1080} \nu + \frac{89}{20} \Delta \nu + \frac{21307}{3360} \Delta + \frac{230057}{1120} + \frac{89}{20} \nu^2 \right) \right] + O(p_\infty^0), \]

\[ c_{u_2, G^5}^{\text{rr, rec}} = -\pi \nu \left[ \left( \frac{53}{5} \frac{106 \nu - 53 \Delta}{p_\infty^4} + \frac{89}{20} \nu^2 - \frac{24229}{1200} \nu + \frac{89}{20} \Delta \nu - \frac{60563}{1080} \Delta \right) \right] + O(p_\infty^0), \]

\[ c_{u_2, G^5}^{\text{rr, tot}} = \pi \nu \left[ \frac{178}{5p_\infty^4} + \left( \frac{4837}{40} + \frac{83}{15} \Delta - \frac{55}{6} \nu \right) \frac{1}{p_\infty^3} + \frac{297}{20p_\infty^2} \nu^2 + \left( \frac{31121}{2520} \Delta - \frac{5123}{5040} \nu + \frac{197935}{1008} \right) \frac{1}{p_\infty} + O(p_\infty^0) \right]. \]

Finally, for $c_{u_1} = c_{u_1} + \gamma c_{u_2}$ at $O(G^4)$ we find

\[ c_{u_1, G^4}^{\text{cons}} = (1 + \Delta) \left[ \frac{1}{p_\infty^2} + \frac{(4 - \nu)}{p_\infty^3} + \left( \frac{9}{4} \pi^2 + \frac{56}{47} \nu \right) \right] + O(p_\infty^0), \]

\[ c_{u_1, G^4}^{\text{rr, rel}} = (1 + \Delta) \nu \left[ \frac{16}{5p_\infty} + \frac{624}{35} \frac{1}{p_\infty^2} + \frac{1952}{63} \frac{3}{p_\infty} + O(p_\infty^0) \right], \]

\[ c_{u_1, G^4}^{\text{rr, rec}} = 0, \]

\[ c_{u_1, G^4}^{\text{rr, tot}} = c_{u_1, G^4}^{\text{rr, rel}}, \]

while at $O(G^5)$

\[ c_{u_1, G^5}^{\text{cons}} = \pi (1 + \Delta) \left[ \frac{3}{p_\infty^2} + \frac{-21\nu + 45}{p_\infty^3} \right] + \left( \frac{717}{3} \frac{893}{p_\infty^2} + \frac{123}{13} \pi^2 \nu \right) + O(p_\infty^0), \]

\[ c_{u_1, G^5}^{\text{rr, rel}} = \pi (1 + \Delta) \nu \left( \frac{28}{5p_\infty^2} + \frac{15007}{420p_\infty^3} + \frac{235867}{2520} \frac{1}{p_\infty} + O(p_\infty^0) \right), \]

\[ c_{u_1, G^5}^{\text{rr, rec}} = 0, \]

\[ c_{u_1, G^5}^{\text{rr, tot}} = c_{u_1, G^5}^{\text{rr, rel}}. \]

It is easily checked that all the above coefficients are (when multiplied by $(GM)^n$) polynomials in the masses.

**IX. CONSERVATIVE SCATTERING BEYOND $O(G^4)$ AT 5PN ACCURACY: COMPARISON WITH EFT RESULTS**

To complete the results given in previous sections, we shall now discuss the (conservative) radiation-graviton contribution to scattering at PM orders $G^4$, $G^5$ and $G^6$, when working at the 5PN accuracy, i.e. the next-to-leading-order (NLO) in the tail-related, radiation-graviton contribution. Indeed, on the one hand, a recent EFT study of Blümlein et al. [34] has computed the potential-graviton contribution to the 5PN two-body Hamiltonian, while, on the other hand, a pioneering work of Foffa and Sturani [55, 56] has derived, in the EFT approach, semi-explicit expressions for the nonlocal-in-time NLO (5PN-level) conservative contributions coming from the exchange of radiation gravitons. The PN-expanded, EFT computation of potential gravitons to the conservative dynamics was found to agree, both at 5PN [34] and at 6PN [52] with the resummed, $O(G^4)$ potential-graviton dynamics of Ref. [28].

On the other hand, the TF approach has independently computed the full, local-plus-nonlocal, conservative dynamics at the 6PN accuracy, in terms of a small number of undetermined coefficients that enter at order $G^5$, $G^6$ and $G^7$. In particular, if we fix on the 5PN accuracy, the TF approach involves only two undetermined coefficients: one of them, denoted $d_h^5$, enters at the $G^5$ level,
while the other one, \( a_6^2 \), enters at the \( G^6 \) level. In addition, the TF approach has allowed one to compute the (full, local-plus-nonlocal) conservative scattering angle at the 5PN accuracy, and up to the sixth PM order, i.e, the 5PN-accurate values of the coefficients \( \chi_4 \), \( \chi_5 \), \( \chi_6 \) in the corresponding results obtained by adding to the corresponding (local) potential-graviton contributions \( \chi_{n,\text{cons},\text{pot}} \), computable from Ref. [34], the additional, radiation-graviton contributions \( \chi_{n,\text{cons},\text{rad}} \), computable from the results of Ref. [35]. Let us briefly sketch how we performed these various computations, and what results we got.

Using results already derived in our previous TF work [13, 37, 38, 53, 65], we find that the PN expansions of the energy-rescaled scattering coefficients \( \bar{\chi}_n^{\text{cons}} = h^{n-1} \chi_n^{\text{cons}} \) read

\[
\frac{1}{2} \chi_{n,\text{cons},\text{TF}}(j, \gamma, \nu) = \sum_{n \geq 1} \chi_{n,\text{cons},\text{tot}}^{\text{cons}}(\gamma, \nu). \tag{9.1}
\]

We can then compare the TF-computed values of \( \chi_{\text{cons},\text{tot}}^{\text{cons}} \) and the yet-undetermined \( \chi_{n,\text{cons},\text{rad}}^{\text{cons}} \) at the 5PN accuracy.

The TF result for \( \chi_{4,\text{cons},\text{TF}}^{\text{cons}} \) is fully explicit and does not involve any undetermined coefficient. By contrast, the TF result for \( \chi_{5,\text{cons},\text{TF}}^{\text{cons}} \) involve the yet-undetermined \( G^5 \)-level theory coefficient \( b_5^2 \). Similarly \( \chi_{6,\text{cons},\text{TF}}^{\text{cons}} \) involves both \( b_5^2 \) and the yet-undetermined \( G^8 \)-level theory coefficient \( a_6^2 \).

Concerning the EFT side of the calculation, we proceeded as follows. The additional term in the conservative effective action due to radiation-graviton exchange was derived by Foffa and Sturani [35, 36] in the form

\[
\chi_{n,\text{cons},\text{tot}}^{\text{cons}}(\gamma, \nu) = \sum_{n \geq 1} \chi_{n,\text{cons},\text{tot}}^{\text{cons}}(\gamma, \nu). \tag{9.1}
\]

We can then compare the TF-computed values of

\[
\chi_4^{\text{cons,TF}} - \chi_4^{\text{Schw}} = \pi \nu \left[ -\frac{15}{4} + \frac{123}{256} \pi^2 - \frac{557}{16} \right] p_\infty^2 + \left( -\frac{6113}{96} + \frac{33601}{16384} \right) \ln \left( \frac{p_\infty}{2} \right) p_\infty^4 + \left( -\frac{615581}{19200} + \frac{93031}{32768} \pi^2 - \frac{1357}{250} \ln \left( \frac{p_\infty}{2} \right) \right) p_\infty^5 + O(p_\infty^6), \tag{9.2}
\]

and

\[
\chi_5^{\text{cons,TF}} - \chi_5^{\text{Schw}} = \nu \left[ \frac{2}{5 \nu_\infty} + \left( -\frac{121}{10} + \frac{1}{5} \right) \nu_\infty^2 + \left( \frac{41}{8} \pi^2 - \frac{19457}{60} + \frac{59}{10} \right) \nu_\infty \right] p_\infty^3 + \left( \frac{782142451}{504000} - \frac{365555}{6048} \pi^2 - \frac{2816}{45} \nu \ln(2) + \frac{111049}{960} \pi^2 + \frac{23407}{5760} \nu^2 + \frac{1408}{45} \nu \ln \left( \frac{p_\infty}{2} \right) \right) p_\infty^5 + O(p_\infty^6), \tag{9.3}
\]

at the 5PN accuracy.

The results obtained for \( \chi_{n,\text{cons},\text{TF}}^{\text{cons}} \) are given in Table 1, and the results obtained for \( \chi_{n,\text{cons},\text{rad}}^{\text{cons}} \) are given in Table 2. The results obtained for \( \chi_{n,\text{cons},\text{tot}}^{\text{cons}} \) are given in Table 3.

\[
S_{\text{rad}} = \eta^8 S_{I_1} + \eta^{10} S_{I_3} + \eta^{12} S_{I_5} + \eta^{14} S_{I_7}, \tag{9.5}
\]

where (with \( \eta \equiv \frac{1}{\sqrt{G}} \)) \( S_{I_5} \) is the contribution of the (1PN-accurate, \( d \)-dimensional) mass-type (or electric-type) quadrupole moment \( I_{ij}^{(d)} \), \( S_{I_5} \) is the contribution of

\[\text{using the notation } D = d + 1 = 4 + \epsilon, \text{ and reserve } \epsilon \text{ for } D = 4 - 2 \epsilon.\]
the (0PN-accurate, d-dimensional) mass-type (electric-type) octupole moment \( I_{ijk}^{(d)} \), and \( S_{ij} \) is the contribution of the (0PN-accurate, d-dimensional) spin-type (or magnetic-type) quadrupolar moment \( J_{ij}^{(d)} \). When working in the frequency domain, the various (nonlocal) action contributions read (with \( E \equiv E_{\text{c.m.}} \))

\[
S_{I_{ij}} = -\frac{GE}{5} \int \frac{d\omega}{2\pi} \left( \frac{1}{e} - R_{\text{quad},e} + \mathcal{L} \right) \left( I_{ij}^{d,(3)} \right)^2,
\]

\[
S_{I_{ij}} = -\frac{GE}{189} \int \frac{d\omega}{2\pi} \left( \frac{1}{e} - R_{\text{oct},e} + \mathcal{L} \right) \left( J_{ij}^{d,(2)} \right)^2,
\]

\[
S_{I_{ij}} = -\frac{16GE}{45} \int \frac{d\omega}{2\pi} \left( \frac{1}{e} - R_{\text{quad},m} + \mathcal{L} \right) \left( J_{ij}^{d,(3)} \right)^2,
\]

\[
(9.6)
\]

where \( \left( I_{L}^{d,(3)} \right)^2 \equiv I_{L}^{d,(3)}(\omega)I_{L}^{d,(3)}(-\omega) \) and

\[
\mathcal{L} \equiv \ln \left( \frac{\omega^2 e^\gamma e}{\pi \mu_0^2} \right).
\]

Here, \( \gamma \) denotes Euler’s constant, \( \mu_0 = \frac{1}{4\pi} \) is the mass scale entering dimensional regularization, i.e. \( G_{[d]} = G\ell_5^d \), where \( G \) denotes the 4-dimensional gravitational constant. The terms \( R_{\text{quad},e}, R_{\text{oct},e}, R_{\text{quad},m} \) denote some rational numbers. These numbers have been evaluated by Foffa and Sturani to be

\[
\begin{align*}
R_{\text{quad},e} &= \frac{41}{30}, \\
R_{\text{oct},e} &= \frac{82}{35}, \\
R_{\text{quad},m} &= \frac{127}{60}.
\end{align*}
\]

The first, electric-quadrupole, term \( R_{\text{quad},e} \) was first derived in Ref. [66], and has been verified via the recent re-derivations of the 4PN dynamics [64, 65]. By contrast, there are no published, detailed rederivations of the values of the (5PN-level) electric-octupole and magnetic-quadrupole rational terms \( R_{\text{oct},e} \) and \( R_{\text{quad},m} \). In our computations, we shall therefore use the numerical values \( R_{\text{quad},e} = \frac{41}{30} \) of \( R_{\text{oct},e} \), but leave the other two rational terms as unspecified parameters.

The additional (local-in-time) contributions in the second line of the effective action Eq. (9.5), derived in Refs. [35, 36], are defined in dimension \( d = 3 \), and read (with \( \varepsilon_{ijk} = \varepsilon_{[jki]}, \varepsilon_{123} = +1 \), and \( L_i \) denoting the total Newtonian angular momentum of the system)

\[
\begin{align*}
S_{QQL} &= -C_{QQL} G^2 \int dt I_{3s}^{(4)} I_{s}^{(3)} \varepsilon_{ijk} L_k, \\
S_{QQQ_1} &= -C_{QQQ_1} G^2 \int dt I_{3s}^{(4)} I_{s}^{(4)} I_{ij}, \\
S_{QQQ_2} &= -C_{QQQ_2} G^2 \int dt I_{3s}^{(3)} I_{s}^{(2)} I_{ij}.
\end{align*}
\]

Here, similarly to what we did for the rational coefficients \( R_{\text{quad},e}, R_{\text{oct},e} \) and \( R_{\text{quad},m} \), we have not specified the values of the rational numerical coefficients \( C_{QQL}, C_{QQQ_1}, C_{QQQ_2} \). The estimated values of \( C_{QQQ_1} \), \( C_{QQQ_2} \) have varied between the published and erratum versions of Ref. [35]. The final estimates given by Foffa and Sturani for these coefficients are

\[
C_{QQL} = \frac{8}{15}, \quad C_{QQQ_1} = \frac{1}{15}, \quad C_{QQQ_2} = \frac{4}{105}.
\]

[Here we took into account the unconventional definition \( L_i^{\text{Ref.}} = -L_i^{\text{standard}} \), and the sign conventions of Eqs. (9.9), (9.10), (9.11).] We shall later compare these values to the results derived from the TF approach.

The presence of \( \frac{1}{d} \) (UV) poles in the first three contributions \( S_{I_{ij}}, S_{I_{ij}}, S_{I_{ij}} \) require that the multipole moments \( I_{ij}^{d,(3)}, J_{ij}^{d,(2)} \) be computed in \( d \) dimensions. General \( d \)-dimensional expressions for the mass-type multipole moments \( I_{L}^{d,(3)} \) have been derived some time ago in Ref. [68], see Eq. (3.50) there. We have used the latter expressions (together with the \( d \)-dimensional gravitational field expressions derived in Refs. [64, 65] to derive the 1PN-accurate values of the \( d \)-dimensional mass quadrupole and mass octupole of a binary system. For instance, we find, in the \((d \text{-dimensional})\) c.m. frame and at the 1PN level of accuracy (denoting \( x^1 = x_1^1 - x_2^1 \))

\[
\begin{align*}
\mu^{-1} I_{ij}^{d,(3)} &= (1 + \eta^2 C_1) x_{ij} + \eta^2 C_2 v_{ij} + \eta^2 C_3 x_{ij} v_{ij},
\end{align*}
\]

where \( C_1 = C_1(x^2, v \cdot n) \) [using the standard notation \( r^2 = x^2, (v \cdot n) = f, \text{etc.} \) are given by

\[
\begin{align*}
C_1 &= (1 - 3\nu) \frac{d^2(d^2 + 2d - 8(d - 1))}{2(d^2 + 4d - 4)(d - 2)} [d^2 + 2d - 8(d - 1)] U, \\
C_2 &= (1 - 3\nu) \frac{d^2(d + 2)^2}{2(d + 4)(d - 2)}, \\
C_3 &= -\frac{4(1 - 3\nu)r^2}{(d + 4)(d - 2)},
\end{align*}
\]

where

\[
U = f G_{[d]} M \hat{k} |y_1 - y_2|^{2-d} = G_{[d]} f \hat{k} M r^{2-d},
\]

with

\[
G_{[d]} = G \ell_6^{-d-3} = G(1 + \epsilon \ln(\ell_0)) + O(\epsilon^2),
\]

[The relations among the various scale factors is given in footnote 11.] and

\[
\tilde{k} = \frac{1}{f} \left( \frac{d^2}{2d^2} \right) f = \frac{2(d - 2)}{(d - 1)}.
\]

When \( d = 3 \), \( U = GM/r \) and the coefficients \( C_1 \) reduce to

\[
\begin{align*}
C_1 &= -\frac{29}{42} (-1 + 3\nu)r^2 + \frac{1}{7} (5 + 8\nu) U, \\
C_2 &= -\frac{11}{21} r^2 (-1 + 3\nu), \\
C_3 &= \frac{4}{7} (-1 + 3\nu) r(v \cdot n).
\end{align*}
\]
Note that the needed third time derivative of $I^{(d)}_{ijk}$ must also be computed by using the 1PN-accurate, $d$-dimensional equations of motion (see Ref. [68, 69]).

The Newtonian-level accurate expression for $I^{(d)}_{ijk}$ is simply (in the c.m. frame)

$$I^{(d)}_{ijk}(t) = \nu(m_2 - m_1) x_{ijk}.$$  \hspace{1cm} (9.19)

It was shown in Ref. [68] that spin-type multipoles in $d$ dimensions must be described by non-symmetric, mixed Young tableaux. In particular, the $d$-dimensional avatar of the spin-type quadrupole must be described by a rank-three tensor $J_{iab}$ that is antisymmetric with respect to (say) $i$ and $b$, and that satisfies the cyclic identity $J_{iab} + J_{abi} + J_{bij} = 0$. The explicit $d$-dimensional value of the spin-type quadrupole $J_{ij}$ has only been computed very recently [70]. Here, we only need its value at the Newtonian level. It reads

$$J_{iab} = \nu(m_2 - m_1) \left[ \left( x^{ia} - \frac{x^2}{d-1} \delta^{ia} \right) v_b - \left( x^{ab} - \frac{x^2}{d-1} \delta^{ab} \right) v_i - \frac{x \cdot v}{d-1} (x^b \delta^{ab} - x^a \delta^{ia}) \right].$$  \hspace{1cm} (9.20)

Though Ref. [35] did not use the needed $d$-dimensional magnetic quadrupole $J_{iab}$, their derivation rests on the coupling $\frac{1}{2} R_{iab} e_{ij} J_{ij}$ which could have been expressed (in $d$ dimensions) in terms of the coupling of $R_{iab}$ to the relevant spin quadrupole $e_{ij} J_{ij} \equiv J_{bij}$. This implies that one should simply re-interpret their action contribution $J^{(3)}_{ij} J^{(3)}_{ij}$ by the replacement

$$J^{(3)}_{ij} J^{(3)}_{ij} \rightarrow \frac{1}{2} J^{(3)}_{iab} J^{(3)}_{iab},$$  \hspace{1cm} (9.21)

where the right-hand side is to be evaluated (including the time derivatives) in $d$ dimensions.

A simple way to by-pass the subtle issues linked to the spin quadrupole would be to focus on the equal-mass case, $m_1 = m_2$, or $\nu = \frac{1}{2}$. For symmetry reasons, the spin-quadrupole (as well as the mass-octupole) would then not contribute, in any dimension, to the problematic nonlocal terms entering $S_{L_2}$ (and $S_{L_3}$). Both terms are proportional to $\nu^2 (1 - 4 \nu)$. In focussing on the effective action for the specific value $\nu = \frac{1}{2}$, we would, however, reduce the number of constraints obtained below by comparing TF and EFT.

The multipolar terms entering Eq. (9.18) should be time-differentiated an appropriate number of times. This is done by using the 1PN accurate $d$-dimensional Hamiltonian

$$\bar{H}^{[d]}_{1PN,h} = \frac{1}{2} p^2 - U - \nu^2 \left\{ \frac{1}{8} (1 - 3 \nu) p^4 + \frac{1}{2} \left( -U + \frac{d}{d-2} + \nu \right)^2 \right\}.$$  \hspace{1cm} (9.22)

Introducing the notation

$$F_{I_2} = \left( I^{(3)}_{ijk} \right)^2,$$

$$F_{I_3} = \left( I^{(4)}_{ijk} \right)^2,$$

$$F_{I_2} = \frac{1}{2} \left( I^{(3)}_{iab} \right)^2,$$

the structure of these terms, after replacing $d = 3 + \epsilon$ and expanding in $\epsilon$ is such that

$$F_{I_2} = \nu^2 \frac{G^2 M^4}{r^4} \left[ F_{I_2}^{00} + F_{I_2}^{01} \epsilon + \eta^2 \left( F_{I_2}^{20} + F_{I_2}^{21} \epsilon \right) \right],$$

$$F_{I_3} = \nu^2 \frac{G^2 M^4}{r^4} (1 - 4 \nu) \left[ F_{I_3}^{00} + F_{I_3}^{01} \epsilon \right],$$

$$F_{I_3} = \nu^2 \frac{G^2 M^4}{r^4} (1 - 4 \nu) \left[ F_{I_3}^{00} + F_{I_3}^{01} \epsilon \right],$$  \hspace{1cm} (9.24)

with each ($\epsilon$) term containing a no-log part and a ln$(r/r_0)$ part, e.g.,

$$F_{I_2}^{11} = F_{I_2}^{11 \text{no-log}} + F_{I_2}^{11 \text{ln}} \ln \left( \frac{r}{r_0} \right),$$  \hspace{1cm} (9.25)

where we recall the notation $r_0 \equiv \epsilon^{-2/3}$. The complete results are listed in Table III below.

It is easily seen that the (frequency-domain) terms involving the integral (in $d = 3$) of $L = \ln \left( \frac{r^2}{r_0^2} \right)$ exactly correspond to the part of the nonlocal action denoted $-W_{\text{nonloc}}^{\text{1PN}}$ in Ref. [37], i.e. the time-domain contribution

$$-W_{\text{nonloc}}^{\text{1PN}} = +2GE \int dt \mathcal{P}_{f_2} \int \frac{dt'}{|t-t'|} \mathcal{F}_{\text{split.GW}}^{[d=3]}(t, t').$$  \hspace{1cm} (9.26)

When adding the (UV-divergent) 5PN-accurate radiation-graviton EFT action \[5.3\]

$$S_{\text{rad}} = - \int dt H^\text{rad},$$  \hspace{1cm} (9.27)

to the (IR-divergent) 5PN-accurate potential-graviton (Hamiltonian) action derived in Ref. \[3.1\]

$$S_{\text{pot}} = - \int dt H^\text{pot},$$  \hspace{1cm} (9.28)

the $\frac{1}{2}$ poles cancel and one can (using the techniques explained in our previous works) compute the large-impact-parameter (or large-$j$) expansion of the (conservative) scattering angle predicted by the EFT dynamics. Namely, one uses the $d$-dimensional energy conservation

$$\dot{E} = H^\text{pot} + \dot{H}^\text{rad},$$  \hspace{1cm} (9.29)

to obtain the radial momentum $p_r = p_r(E, j; r)$ (expanded to the appropriate PN order, and also expanded to first order in $\epsilon$). Here $H^\text{pot}$ is the (harmonic gauge) potential-graviton Hamiltonian $H^\text{pot}$ of Ref. [37], while
TABLE III: Coefficients of the first order \( \varepsilon \)-expansion \([9.24]\) of the square of the \( d \)-dimensional multipoles \([9.28]\) with \( d = 3 + \varepsilon \).

\[
\begin{align*}
F_{I2}^{(2)} & = 32p^2 - \frac{88}{5}p_\nu^2 \ln \left( \frac{\nu}{r_0} \right) + 90\nu^2 - \frac{436}{5}p_\nu^2, \\
F_{I2}^{(3)} & = \left( -64p^2 + \frac{122}{7}p_\nu^2 \right) \ln \left( \frac{\nu}{r_0} \right) + 90\nu^2 - \frac{436}{5}p_\nu^2, \\
F_{I2}^{(4)} & = \left( \frac{998}{21} \frac{80}{7} \nu^4 \right) \frac{GM}{r_0} p_\nu^2 + \left( 168 - 64 \nu \right) p^4 + \left( -\frac{9880}{21} - \frac{64}{5} \nu \right) \frac{GM}{r_0} p^2 + \left( \frac{208}{21} + \frac{1024}{5} \nu \right) \nu^4 + \left( -\frac{1552}{7} - \frac{1384}{21} \nu \right) p^2 p_\nu^2, \\
F_{I2}^{(5)} & = \left( \frac{620}{21} + \frac{192}{7} \nu \right) \frac{GM}{r_0} p_\nu^2 + \left( -\frac{44}{7} - \frac{24}{5} \nu \right) \nu^4 + \left( \frac{2704}{21} + \frac{228}{7} \nu \right) p^2 p_\nu^2 + \left( -\frac{9880}{21} + \frac{84}{5} \nu \right) \frac{GM}{r_0} p_\nu^2, \\
F_{I2}^{(6)} & = \left( -336 + 128 \nu \right) \nu^2 \ln \left( \frac{\nu}{r_0} \right) + \left( \frac{1194}{7} + \frac{1722}{7} \nu \right) p_\nu^2 + \left( \frac{605528}{441} - \frac{704}{7} \nu \right) \frac{GM}{r_0} p_\nu^2 + \left( -\frac{22224}{441} - \frac{24780}{441} \nu \right) p_\nu^2 + \left( \frac{64448}{441} - \frac{19024}{441} \nu \right) \frac{GM}{r_0} p_\nu^2, \\
F_{I2}^{(7)} & = \left( \frac{1744}{21} + \frac{65248}{126} \nu \right) \nu^2 \nu^4, \\
P_{I2}^{(8)} & = \frac{2p^4}{\nu^2} + \frac{3p_\nu^4}{\nu^2} - \frac{2p_\nu^4}{\nu^2} \ln \left( \frac{\nu}{r_0} \right) - \frac{2}{5}p_\nu^4 + \frac{8}{11}p^2 p_\nu^2 + \frac{2}{7}p_\nu^4.
\end{align*}
\]

\( H_{\text{rad}}^{(d)} \) is the additional radiation-graviton Hamiltonian, deduced from the nonlocal action \([8.5]\). The scattering angle is then obtained as the sum of the potential-graviton (half) scattering angle \( \chi_{\text{tot}}^{(d)} \) (derived from the local Hamiltonian \( H_{\text{rad}}^{(d)} \)) and the correction coming from the nonlocal term \( H_{\text{rad}} \), obtained as

\[
\chi_{\text{rad}}^{(d)} = \frac{1}{2\nu^2} \int dt \tilde{\mathcal{H}}_{\text{rad}}^{(d=3+\varepsilon)} \big|_{p_r=p_r(E,j,r)} \cdot \chi_{\text{Schw}}^{(d=3+\varepsilon)} + \chi_{\text{EFT}}^{(d=3+\varepsilon)}.
\]

This method gave us the following results for the PM expansion of the 5PN-accurate total scattering angle,

\[
\chi_{\text{cons,EFT}}^{\text{F}} = \chi_{\text{pot}}^{\text{F}} + \chi_{\text{rad}}^{\text{F}} = \sum_n 2 \frac{\chi_{\text{cons,EFT}}^{(n)} p_{\nu^n}}{n!} = \sum_n \frac{\chi_{\text{cons,EFT}}^{(n)} p_{\nu^n}}{n!}, \quad \text{for } n \geq 1, \quad (9.31)
\]

predicted by the EFT approach, in terms of the various coefficients entering \([9.5]\) (namely \( R_{\text{oct},c}, R_{\text{quad},m}, C_{\text{QQL}}, C_{\text{QQQ}}, C_{\text{QQQ}} \)).
angles yields six equations, namely
\[ \chi_{6, \text{cons,EFT}} - \chi_{6, \text{Schw}} = \pi \nu \left[ \frac{615}{256} \pi^2 - \frac{625}{4} + \frac{105}{16} \nu \right] + \left( -\frac{228865}{192} - \frac{1845 \pi^2}{512} + \frac{257195 \pi^2}{8192} - \frac{2079}{8} \zeta(3) + \frac{10065}{64} \nu - 122 \ln \left( \frac{p_{\infty}}{2} \right) \right) p_{\infty}^2 \\
+ \left( \left( \frac{55}{3} R_{\text{quad,m}} + \frac{201}{2} \ln \left( \frac{p_{\infty}}{2} \right) - \frac{8645}{24} C_{\text{QQQ}}, + \frac{7575}{8} C_{\text{QQL}}, + \frac{2817}{16} \zeta(3) - \frac{38621}{84} R_{\text{oct,e}} \right) \right) \nu \right) \nu^2 + \frac{3136}{32768} - \frac{695}{4} \left( C_{\text{QQQ}}, \right) \nu \\
+ \frac{38621}{336} R_{\text{oct,e}} + \frac{2321185}{16384} \nu^2 + \frac{55}{12} R_{\text{quad,m}} - \frac{173486591}{47040} - \frac{13831}{56} \ln \left( \frac{p_{\infty}}{2} \right) \right) \nu \right) \nu^2 + \frac{49941}{64} \zeta(3) - \frac{9216}{7} \ln(2) \right] \right) p_{\infty}^4. \tag{9.34}

We can now compare, term by term, these expressions (which are 5PN accurate and respectively belong to the 4PM, 5PM and 6PM approximations) with the corresponding ones derived from the TF approach, and displayed in Eqs. (9.2), (9.3) and (9.4) above.

Let us denote as \( \Delta C_{\chi_i} (\nu^2 p_{\infty}^i) \) the coefficient of \( \nu^2 p_{\infty}^i \) in the difference \( \chi_{6, \text{cons,EFT}} - \chi_{6, \text{cons,TF}} \). The comparison, at the 5PN level, between the TF and EFT scattering angles yields six equations, namely
\[ \Delta C_{\chi_4} (\nu^2 p_{\infty}^4) = 0, \quad \Delta C_{\chi_5} (\nu^2 p_{\infty}^5) = 0, \quad \Delta C_{\chi_6} (\nu^2 p_{\infty}^6) = 0. \tag{9.35} \]

These six equations are linear in the seven variables \( R_{\text{oct,e}}, R_{\text{quad,m}}, C_{\text{QQQ}}, C_{\text{QQQ}}, C_{\text{QQL}}, d_{\text{oct}}^2 \) and \( a_{\text{oct}}^2 \). Their explicit form reads
\begin{align*}
\Delta C_{\chi_4} (\nu^2 p_{\infty}^4) &= \pi \left( \frac{1697}{560} R_{\text{oct,e}} + \frac{3}{20} R_{\text{quad,m}} - \frac{146357}{19600} \right), \\
\Delta C_{\chi_5} (\nu^2 p_{\infty}^5) &= \pi \left( \frac{253}{24} C_{\text{QQQ}}, - \frac{85}{12} C_{\text{QQQ}}, + \frac{207}{8} C_{\text{QQL}}, - \frac{1697}{140} R_{\text{oct,e}} - \frac{3}{5} R_{\text{quad,m}} + \frac{230281}{9800} \right), \\
\Delta C_{\chi_6} (\nu^2 p_{\infty}^6) &= \frac{81856}{945} R_{\text{oct,e}} + \frac{512}{135} R_{\text{quad,m}} - \frac{467968}{2205}, \\
\Delta C_{\chi_5} (\nu^2 p_{\infty}^5) &= \frac{4}{15} \nu^2 - \frac{38272}{135} C_{\text{QQQ}}, - \frac{7168}{45} C_{\text{QQQ}}, + \frac{2176}{3} C_{\text{QQQ}}, - \frac{327424}{945} R_{\text{oct,e}} - \frac{2048}{135} R_{\text{quad,m}} \\
&- \frac{61309 \pi^2}{117735492} + \frac{384}{33075}, \\
\Delta C_{\chi_6} (\nu^2 p_{\infty}^6) &= \pi \left( \frac{38621}{336} R_{\text{oct,e}} + \frac{55}{12} R_{\text{quad,m}} - \frac{1099659}{3920} \right), \\
\Delta C_{\chi_5} (\nu^2 p_{\infty}^5) &= \pi \left( \frac{15}{32} \nu^2 + \frac{15}{32} d_{\text{oct}}^2 - \frac{8645}{24} C_{\text{QQQ}}, - \frac{695}{4} C_{\text{QQQ}}, + \frac{7575}{8} C_{\text{QQQ}}, - \frac{38621}{84} R_{\text{oct,e}} - \frac{55}{3} R_{\text{quad,m}} \\
&- \frac{5375505 \pi^2}{16384} + \frac{10220575}{2352} \right). \tag{9.36}
\end{align*}

It is important to keep in mind the physical origin of these equations. The three equations involving the first
power of \(\nu\) correspond to the first-order self-force computations that constitute one of the important (and well-tested) building blocks of the TF approach. The TF side of these equations is therefore fully predicted. Their EFT side contain only \(R_{\text{act,e}}\) and \(R_{\text{quad,m}}\). We therefore obtain three linear equations involving \(R_{\text{act,e}}\) and \(R_{\text{quad,m}}\). One finds that there are only two independent equations among these three \(O(\nu^4)\) equations. This allows us to uniquely determine the values of \(R_{\text{act,e}}\) and \(R_{\text{quad,m}}\) from the TF results. We find

\[
[R_{\text{act,e}}]_{\text{from TF}} = \frac{82}{35} ,
\]

and

\[
[R_{\text{quad,m}}]_{\text{from TF}} = \frac{49}{20} = \frac{147}{60} .
\]

The TF-deduced value of \(R_{\text{act,e}}\), Eq. (9.37), satisfactorily coincides with the result of Foffa and Sturani [35]. By contrast, the TF-deduced value of the magnetic-quadrupole term, \(R_{\text{quad,m}}\), Eq. (9.38), disagrees with the one given in Ref. [35], which is instead \([R_{\text{quad,m}}]_{\text{Ref. 35}} = \frac{127}{60}\). The difference is

\[
[R_{\text{quad,m}}]_{\text{from TF}} - [R_{\text{quad,m}}]_{\text{Ref. 35}} = \frac{49}{20} - \frac{127}{60} = \frac{2}{3} .
\]

Ref. [34] similarly reported the need for correcting the spin-quadrupole contribution \((J_{ij}^{d(3)})^2\) derived in Ref. [35] by a finite-renormalization factor \(Z_J = 1 + \frac{3}{2} \epsilon\) (in our notation). A direct translation of this factor in terms of a change in \(R_{\text{quad,m}}\) would correspond to the change

\[
[R_{\text{quad,m}}]_{\text{Ref. 34}} - [R_{\text{quad,m}}]_{\text{Ref. 35}} = -\frac{1}{6} .
\]

which differs from our TF-derived result, Eq. (9.39). This difference is expected to come from the fact that Ref. [34] used a different estimate for the \(d\)-dimensional value of \((J_{ij}^{d(3)})^2\). Their final result for the nonlocal contributions, \(S_L\) and \(S_J\), in Eq. (9.36) to the 5PN dynamics must, however, agree with our corresponding result, both results being, essentially, calibrated on the 5PN \(O(\nu^2)\) self-force result first derived in Ref. [35].

Let us now go beyond the terms linear in \(\nu\) in the TF/EFT comparison. At the \(\nu^2\) level we get three new equations among Eqs. (9.35). Of particular importance among these equations is the equation \(\Delta C_{\chi}\nu^2 p_\infty = 0\).

From the TF side, this equation contains no undetermined parameters because the TF approach reached a complete determination of the 5PN (and 6PN) dynamics at the \(G^4\) level. On the other hand, from the EFT side this equation involves a linear combination of \(R_{\text{act,e}}\), \(R_{\text{quad,m}}\), \(C_{\text{qql}}, C_{\text{qqq}},\) and \(C_{\text{qqq}}\). After inserting the \(O(\nu^2)\)-based unique determinations of \(R_{\text{act,e}}\) and \(R_{\text{quad,m}}\), Eqs. (9.37), (9.38), one gets one linear equation involving \(C_{\text{qql}}, C_{\text{qqq}},\) and \(C_{\text{qqq}}\). Namely we get the following constraint on the coefficients \(C_{\text{qql}}, C_{\text{qqq}},\) and \(C_{\text{qqq}}\) of the 5PN EFT action Eq. (9.39):

\[
0 = \frac{2973}{350} - \frac{69}{2} C_{\text{qql}} + \frac{253}{18} C_{\text{qqq}} + \frac{85}{9} C_{\text{qqq}} .
\]

When inserting the rational values of \(C_{\text{qql}}, C_{\text{qqq}}\), derived in Ref. [35], namely Eqs. (9.12), one finds that the constraint (9.41) is not satisfied. [Nor is Eq. (9.41) satisfied when using the values of the published version of Ref. [35].] For instance, when using Eqs. (9.12), the right-hand side of Eq. (9.41) is equal to \(-1937/225 \approx -8.6\). [The corresponding value when using the published values \(C_{\text{qql}} = \frac{27}{5}, C_{\text{qqq}} = \frac{14}{5}, C_{\text{qqq}} = \frac{1}{3}\) is equal to \(19069/6300 \approx 3.0\).]

We shall discuss below possible subtleties underlying this discrepancy.

Our TF/EFT comparison (9.35) yields two more equations at the \(\nu^2\) level, namely \(\Delta C_{\chi}\nu^2 p_\infty = 0\) and \(\Delta C_{\chi}\nu^2 p_\infty = 0\). These two equations now involve, besides the EFT coefficients \(R_{\text{act,e}}\), \(R_{\text{quad,m}}\), \(C_{\text{qql}}, C_{\text{qqq}},\) the two yet-undetermined 5PN-level TF parameters \(d_{\nu^2}\) and \(a_{\nu^2}\). [As indicated by their subscripts, the latter EOB parameters respectively belong to the \(G^5\) and \(G^6\) PM levels.] One can solve these two constraints in terms of \(d_{\nu^2}\) and \(a_{\nu^2}\). This yields the two equations

\[
a_{\nu^2}^2 = \frac{25911}{256} \pi^2 + \frac{26429}{512} \pi^2 + R_{\text{ds}}(C_{\text{qql}}, C_{\text{qqq}}, C_{\text{qqq}}) ,
\]

\[
d_{\nu^2} = \frac{306545}{512} \pi^2 + R_{\text{ds}}(C_{\text{qql}}, C_{\text{qqq}}, C_{\text{qqq}}) ,
\]

where \(R_{\text{as}}\) and \(R_{\text{ds}}\) denote the following inhomogeneous linear combinations of \(C_{\text{qql}}, C_{\text{qqq}},\) and \(C_{\text{qqq}}\):

\[
R_{\text{as}} = -\frac{654389}{525} + \frac{700}{3} C_{\text{qql}} - \frac{848}{3} C_{\text{qqq}},
\]

\[
R_{\text{ds}} = -\frac{1773479}{315} - \frac{2720}{9} C_{\text{qql}} + \frac{9568}{9} C_{\text{qqq}} .
\]

As the work of Ref. [35] establishes that the coefficients \(C_{\text{qql}}, C_{\text{qqq}}, C_{\text{qqq}}\) must be rational, the results Eqs. (9.42) uniquely determine the irrational contributions to \(a_{\nu^2}\) and \(d_{\nu^2}\), namely the \(\pi^2\) terms entering Eqs. (9.42). These \(\pi^2\) terms, here directly derived by comparing TF results to the combination of EFT results in Refs. [34] and [35], satisfactorily agree with the analog results first derived by Blümlein et al. [35]. [However, Ref. [34] did not provide any results similar to our explicit expressions (9.43) for the rational contributions \(R_{\text{as}}\) and \(R_{\text{ds}}\).]

As the currently computed values for the rational coefficients \(C_{\text{qql}}, C_{\text{qqq}}, C_{\text{qqq}}\) are not consistent with our constraint (9.41), we looked for rational solutions of the
constraint \( (9.41) \) having smallish denominators involving only small powers of 2, 3, 5 and 7 (as suggested by the denominators appearing in Appendix B of [35]). The simplest\(^{14}\) such solutions we found, for possible triplets \( C_{QQL}, C_{QQQ_1}, C_{QQQ_2} \), are listed in Table IV. On the other hand, if we impose that \( C_{QQL} \) takes the value computed by Foffa and Sturani, namely \( C_{QQL} = \frac{8}{15} \), the simplest solutions we found all had larger denominators. The simplest of them involved \( 175 = 5^2 \cdot 7 \) and was

\[
\begin{bmatrix}
8 \\
81 \\
175 \\
25 \\
\end{bmatrix}.
\]

(9.44)

We added this possible solution in Table IV.

Let us also note that the expressions for the rational numbers \( R_{ae} \) and \( R_{de} \) can be reduced if we make use of the constraint \( (9.41) \). This can be done in many ways (depending on which coefficient we wish to eliminate). For instance, if we eliminate \( C_{QQQ_1} \), we get

\[
\begin{align*}
R_{ae} &= \frac{141907229}{132825} - \frac{256}{11} C_{QQL} - \frac{21760}{759} C_{QQL}, \\
R_{de} &= -\frac{90672257}{17325} - \frac{1216}{11} C_{QQL} - \frac{11584}{99} C_{QQL},
\end{align*}
\]

(9.45)

If we assume the value \( C_{QQL} = \frac{8}{15} = C^{*}_{QQL} \), Eq. \( (9.45) \) yields

\[
\begin{align*}
R^*_{ae} &= -\frac{143555869}{132825} - \frac{21760}{759} C_{QQL}, \\
R^*_{de} &= -\frac{109603697}{17325} - \frac{11584}{99} C_{QQL}.
\end{align*}
\]

(9.46)

In absence of computed values of \( C_{QQL}, C_{QQQ_1}, C_{QQQ_2} \), consistent with our constraint \( (9.41) \), we cannot derive any precise values for the two 5PN TF parameters \( a^6 \) and \( d^5 \). However, in view of the fact that all the currently published values of \( C_{QQL}, C_{QQQ_1}, C_{QQQ_2} \), and notably the latest ones, Eqs. \( (9.12) \), have an absolute magnitude which is smaller than 1, it seems reasonable to assume that \( C_{QQL}, C_{QQQ_1}, C_{QQQ_2} \) are all contained in the interval \([-1, 1]\). Making this assumption (say, “assumption \( A_1 \)”) then yields constraints on the possible values of \( R_{ae}, R_{de} \), and, therefore on \( a^6 \) and \( d^5 \). More precisely, we find that, under the assumption \( A_1 \) (constrained by \( (9.41) \))\(^{15}\), the ranges of possible values of \( a^6 \) and \( d^5 \) are

\[
\begin{align*}
-119.68 &\leq a^6 \leq -30.64, \\
-582.96 &\leq d^5 \leq -198.34.
\end{align*}
\]

(9.47)

Let us finally consider the less constraining assumption \( A^1 \) that only \( C_{QQL} \) and \( C_{QQQ_2} \) are within the interval \([-1, 1]\). Under this assumption, we find (using Eq. \( (9.45) \)) that the possible ranges for \( a^6 \) and \( d^5 \) become

\[
\begin{align*}
-121.37 &\leq a^6 \leq -17.48, \\
-590.99 &\leq d^5 \leq -135.88.
\end{align*}
\]

(9.48)

Inserting the estimates \( (9.47) \) or \( (9.48) \) in Eqs. (8.26)-(8.29) of Ref. [35] will yield 5PN-level numerical estimates for both the binding energy and the periastron advance along circular orbits which might eventually be compared with forthcoming second order self-force results for these quantities.

To illustrate these estimated ranges, we list in Table IV the values of \( a^6 \) and \( d^5 \) corresponding to a sample of smallish-denominator solutions of \( (9.41) \), respecting the assumption \( A_1 \), together with the solution \( (9.44) \). The listed values are compatible (as they should) with the range \( (9.47) \).

\[
\begin{array}{|c|c|}
\hline
C_{QQL}, C_{QQQ_1}, C_{QQQ_2} & a^6 \\
\hline
-\frac{2}{3}, -\frac{3}{4}, -\frac{4}{5} & -59.0019 \leq -322.1289 \\
-\frac{3}{4}, -\frac{4}{5}, -\frac{5}{6} & -69.2419 \leq -370.7689 \\
-\frac{4}{5}, -\frac{5}{6}, -\frac{6}{7} & -55.5886 \leq -305.9156 \\
-\frac{5}{6}, -\frac{6}{7}, -\frac{7}{8} & -53.6923 \leq -296.9082 \\
-\frac{6}{7}, -\frac{7}{8}, -\frac{8}{9} & -63.1738 \leq -341.9452 \\
-\frac{7}{8}, -\frac{8}{9}, -\frac{9}{10} & -109.7143 \leq -537.8191 \\
-\frac{8}{9}, -\frac{9}{10}, -\frac{10}{11} & -109.2267 \leq -535.5029 \\
-\frac{9}{10}, -\frac{10}{11}, -\frac{11}{12} & -76.8000 \leq -400.3714 \\
-\frac{10}{11}, -\frac{11}{12}, -\frac{12}{13} & -85.3333 \leq -432.0870 \\
-\frac{11}{12}, -\frac{12}{13}, -\frac{13}{14} & -92.1600 \leq -464.5137 \\
\hline
\end{array}
\]

X. NONLINEAR RADIATION-REACTION CONTRIBUTIONS TO SCATTERING

The derivation we gave above of radiation-reaction contributions to scattering was systematically based on a first-order treatment, linear in the radiation-reaction force \( F_{rr} \). In this final Section, we wish to point out some of the subtleties that arise when trying to go beyond such linear-in-\( F_{rr} \) effects.

Let us start by recalling that the PM-PN order of magnitude of \( F_{rr} \) is \( O(\alpha^2 m^2) \), i.e. \( F_{rr} \) is at the 2PM level and the 2.5PN level. Indeed, the leading-order value of the radiation-reaction force \( F_{rr}^{I\alpha} \) (to be added to the conservative equations of motion of the first body in a 2-body system) is given, in harmonic coordinates, by (see Eq.
where \( \mathbf{v}_{12} = \mathbf{v}_1 - \mathbf{v}_2 \), and \( \mathbf{n}_{12} = (\mathbf{x}_1 - \mathbf{x}_2)/r_{12} \).

In view of the expression (10.1), one would a priori expect that the contributions that are of second-order in radiation-reaction will be of order \( \frac{m_1}{c^6} \), i.e. at the 4PN level and the 5PN level. In addition, these contributions will necessarily contains (at least) a factor \( (m_1m_2)^2 \), i.e. a factor \( \nu^2 \). In particular, they are expected to contribute to the scattering angle (of each body in the c.m. frame) a term of order (using dimensional analysis)

\[
\mathcal{F}^{r/\chi} \sim \frac{G^4m_1^2m_2^2\nu^2}{c^6b^2}.
\] (10.2)

We wish to emphasize that such a contribution is at the same (PM, PN and \( \nu \)) levels as the terms on the second line of Eqs. (9.12) and (9.29) for \( \chi_1 \). Note that these terms are, in particular, comparable to the scattering effects generated by \( C_{QQL} \), \( C_{QQQ_1} \), and \( C_{QQQ_2} \). In addition, one expects (when considering the higher-order-in-\( G \) corrections in Eq. (10.1)) that quadratic-in-\( F_\nu \) effects will also yield contributions to \( \chi_5 \) and \( \chi_6 \) that will be again similar to those generated by \( C_{QQL} \), \( C_{QQQ_1} \), and \( C_{QQQ_2} \) in Eqs. (9.33), (9.34). Such terms are also similar to those generated by the conservative TF parameters \( \delta_2^c \) and \( \delta_3^c \).

Summarizing so far: scattering effects quadratic-in-\( F_\nu \) look like conservative 5PN-level effects quadratic in \( \nu \) (second-order self-force level), and of PM orders 4, 5 and 6.

Before further discussing this issue, let us explicitly check that a second-order treatment of \( F_{\nu \nu} \) does indeed induce (nonvanishing) contributions to the scattering of the expected order (10.2). We consider the (PN-expanded) radiation-reaction corrected equation of motion of body 1 (in harmonic coordinates\(^{16}\)), namely

\[
\frac{d^2\mathbf{x}_1}{dt^2} = -Gm_2 \frac{\mathbf{x}_1 - \mathbf{x}_2}{r_{12}^2} - \epsilon_{rr} \frac{v_{12}^2}{r_{12}^2} [\mathbf{v}_{12} - 3(\mathbf{v}_{12} \cdot \mathbf{n}_{12})\mathbf{n}_{12}] ,
\] (10.3)

where

\[
\epsilon_{rr} \equiv \frac{4}{5} \frac{G^2m_1m_2}{c^5} .
\] (10.4)

We can then integrate Eq. (10.3) by using second-order perturbation theory in \( \epsilon_{rr} \), i.e.

\[
\mathbf{x}_1(t) = \mathbf{x}_1^{(0)}(t) + \epsilon_{rr} \mathbf{x}_1^{(1)}(t) + \epsilon_{rr}^2 \mathbf{x}_1^{(2)}(t) + O(\epsilon_{rr}^3) .
\] (10.5)"
would be replaced by the kernel of the retarded Green’s function, i.e.

\[ \frac{1}{k^2 - \text{sign}(k^0) i 0} = \frac{1}{k^2} + i \pi \text{sign}(k^0) \delta(k^2). \]  

(10.8)

When considering a source (say \( T_{\mu\nu} \)) which is linearly coupled to a (linearized) gravitational field \( h_{\mu\nu} \), it is well known (since the classic work of Dirac) that the radiation-reaction force needed to ensure balance is derived from the reaction field \( h_{\nu\tau}^{\text{rec}} \), generated by the source via the reaction Green’s function

\[ G_{\text{rec}} \equiv \frac{1}{2} [G_{\text{ret}} - G_{\text{ad}}], \]

(10.9)

whose Fourier kernel involves the second term on the r.h.s. of Eq. 10.3, i.e.

\[ G_{\text{rec}}(k) = i \pi \text{sign}(k^0) \delta(k^2). \]

(10.10)

[See, e.g., Section IV F of \[74\] for a detailed discussion.] However, when considering the 3-loop diagrams representing the \( G^4 \) retarded gravitational interaction of two worldlines, it becomes technically unclear how to derive the needed radiation-reaction force by separating out the on-shell term \( G_{\text{rec}}(k) \) from the many intermediate interactions involving \( G_{\text{ret}}(k) = G_{\text{sym}}(k) + G_{\text{rec}}(k) \) so as to relate the retarded interaction to the time-symmetric one.

The situation is even more involved when considering (beyond the linear interaction of sources) a quantum scattering computation, involving the Feynman Green’s function, namely

\[ G_F(k) = \frac{1}{k^2 - i 0} = \frac{1}{k^2} + i \pi \delta(k^2). \]

(10.11)

Observables can be computed by using the formalism of Ref. \[39\], but the resulting expressions for the impulses are rather complex and make it unclear how to read off nonlinear radiation-reaction effects by separating out the on-shell term in \( G_F(k) = G_{\text{sym}}(k) + i \pi \delta(k^2) \).

Let us also note that these technical difficulties in trying to directly relate a retarded (or Feynman) scattering computation to a time-symmetric-plus-radiation-reaction one are compounded when the \( k \)-space integrals are computed by the method of expansion by regions \[74\]. The latter method has been quite effective in relating conservative and radiation-reaction effects up to order \( G^3 \), but it might encounter difficulties at the \( G^4 \) level because of the \( O(F_F^2) \) scattering effects discussed above.

Let us end this discussion by commenting on the discrepancy we found above between the constraint Eq. 9.41 derived from the TF approach, and the explicit values 9.12 derived in Refs. \[35, 50\]. This discrepancy concerns conservative-like 5PN terms that are of order \( \nu^2 \), and \( G^4 \), or beyond. The root of this discrepancy might be due to the choice made in Refs. \[35, 50\] of Green’s functions for the nonlinear interactions giving rise to the action contributions Eq. 9.39. Indeed, Refs. \[35, 50\] argued that the computation of these (initially nonlocal-in-time) interaction terms should involve intermediate causal Green’s functions (advanced or retarded), combined in a way to end up with a final time-symmetric action. In the case of the first, \( S_{QQL} \), action contribution, it seems that their computation would agree with a computation involving (as assumed in the TF approach) a time-symmetric propagator in intermediate interactions. [This is why, we are ready to assume that their estimate of \( C_{QQL} \) should apply when comparing TF to EFT.] By contrast, we do not understand the meaning of the a posteriori time-averaged action contributions \( S_{QQL}^{\text{avg}} \) and \( S_{QQL}^{\text{rel}} \) that they compute. We leave to future work a computation of the analogs of these terms when using time-symmetric propagators in intermediate interactions.

XI. SUMMARY OF RESULTS AND CONCLUDING REMARKS

We have presented several new results on radiative contributions to the classical two-body scattering in General Relativity.

We gave general formulas, valid only to linear order in radiation reaction, expressing the effect of radiative losses of energy, linear momentum and angular momentum on the 4-momentum changes (a.k.a., impulses), \( \Delta p_{\mu\nu} \equiv p_{\mu\nu}^+-p_{\mu\nu}^- \), i.e. the radiation-reacted contribution \( \Delta p_{\mu\nu}^{\text{rr, tot}} \) in

\[ \Delta p_{\mu\nu} = \Delta p_{\mu\nu}^{\text{cons}}(u_1^-, u_2^+, b) + \Delta p_{\mu\nu}^{\text{rr, tot}}(u_1^-, u_2^-, b). \]

(11.1)

We obtained this contribution as the sum of a relative term and a recoil one:

\[ \Delta p_{\mu\nu}^{\text{rr, tot}}(u_1^-, u_2^-, b) = \Delta p_{\mu\nu}^{\text{rr, rel}} + \Delta p_{\mu\nu}^{\text{rr, rec}} + O(F_F^2). \]

(11.2)

Our results are summarized in Table I.

We emphasized how the polynomial dependence of \( \Delta p_{\mu\nu}^{\text{rr, tot}}(u_1^-, u_2^-, b) \) can be exploited to yield some identity relating the various radiative losses. See Eqs. (7.19), (7.26).

We showed how the application of our general formulas at the \( O(G^3) \) level led to a streamlined classical derivation of the \( O(G^3) \) radiative contribution to \( \Delta p_{\mu\nu} \), which was recently derived within a quantum approach in Ref. \[29\].

Our general formulas involve the radiative losses (in the c.m. frame) of energy, angular momentum and linear momentum. These losses admit a double PM and PN expansion, which can be expressed as (with
Here the subscripts \(n\) (e.g., in \(E_n\)) label the \(n\)PM order, i.e., \(O(G^n)\). The subsequent expansion of the various PM coefficients, \(E_n(p_\infty), J_n(p_\infty), P_n(p_\infty)\) in powers of \(p_\infty\) then corresponds to the usual PN expansion. The only radiative losses that are known in a PM-exact way are \(J_2\) [20] and, \(E_3\) and \(P_3\) [20]. In the present paper, we have computed the fractionally 2PN-accurate expansions of the higher PM radiative losses \(E_n(p_\infty), J_n(p_\infty)\) and \(P_n(p_\infty)\) up to \(n = 7\), including the contribution of tails (see Table IX). Our new results are given in Eqs. (C11), (C12), (C13) for the expressions of the (instantaneous) radiated energy, Eqs. (E8), (E9), (E10) for the (instantaneous) radiated angular momentum and Eqs. (G8), (G9) for the (instantaneous) radiated linear momentum. [We have also confirmed the values of \(E_3\) and \(P_3\) at the 15th order in \(p_\infty\) (see Eqs. (7.19) and (7.20)).] At the 2PN level of accuracy, we have also given the general expressions for the radiative losses of energy, angular momentum and linear momentum along hyperbolic-like orbits in terms of two independent orbital parameters (namely, \(\bar{a}_i\) and \(\bar{e}_i\)) which, once re-expressed in terms of energy and angular momentum, can allow one to reach any order in a large-c expansion limit. Tail terms have instead been computed in this limit only up to \(O(G^7)\) (see Eqs. (11.4), (12), (13)).

Inserting the latter 2PN-accurate results in our general formulas for radiative scattering effects allowed us to derive the \(O(G^4)\) and \(O(G^5)\) contribution to \(\Delta P^{\text{rad, tot}}\) with absolute 4.5PN accuracy, i.e. with terms of PN order

\[
\Delta \mathbf{P}_{\text{rad, tot}}^{4.5\text{PN}} \sim \frac{1}{c^5} + \frac{1}{c^7} + \frac{1}{c^8} + \frac{1}{c^9}.
\]

This absolute 4.5PN accuracy is consistent with the limitation of our \(O(F^4)\) treatment, because contributions nonlinear in \(F^4\) start at order \(O(G^4/e^{10})\).

Our explicit results are contained in Section VIII. Let us note that, in the latter expansions, the terms of absolute PN order \(\frac{1}{c^7}\) (i.e. 4PN order) come from taking into account the leading-order tail contribution to the radiative multipole moments. Our computation of the relevant time-symmetric projections of these tail contributions is discussed in Appendices D, F, and H.

We also completed the recently derived potential-graviton contribution to conservative scattering [20] (see also [32]) by extracting from our recent TF results the (conservative) radiation-graviton contribution to 6PN accuracy, see Eq. (6.20).

Finally, we went beyond the \(G^4\) level and combined information from our TF results and from 5PN EFT results of Refs. [34] [35] to derive explicit theoretical expressions for the two hitherto undetermined \(O(G^5)\) and \(O(G^6)\) parameters, dubbed \(d_5^\nu\) and \(d_6^\nu\), entering the 5PN dynamics (as described through the TF approach). We expressed \(d_5^\nu\) and \(d_6^\nu\) as explicit linear combinations of \(\pi^2\) and of the (rational) coefficients entering various (local-in-time) nonlinear contributions to the conservative effective 5PN action induced by radiation-gravitons [33], [34]. Our results are given in Eqs. (6.12) and (6.13). These results confirm (for the \(\pi^2\) contributions) and extend (by providing the dependence on \(C_{QQL}, C_{QQQ}, C_{QQQQ}\)) results of Blümlein et al. [34]. On the other hand, our results exhibit several disagreements with results of Foffa and Sturani [33], [34]. We point out that part of these disagreements might be rooted in subtleties linked to contributions that are nonlinear in radiation-reaction effects (see Section XI).

We leave to future work a deeper study of the latter nonlinear radiation-reaction effects, and emphasize here that all our PN-expanded results, Eq. (11.4), for the radiative-losses related contribution to the scattering impulses only depend on linear-in-radiation-reaction effects, because, as discussed in Section XI, the leading-order nonlinear-in-radiation-reaction effect is of order \(G^6/c^8\). We also recall that in the TF framework, and in the present work, the conservative scattering angle is defined as the one coming from the time-symmetric Fokker-Wheeler-Feynman dynamics. Any comparison with forthcoming \(O(G^5)\) “conservative” computations should check that the meaning of conservative is the same, and any comparison with forthcoming \(O(G^4)\) “physical, retarded” computations should take into account all the needed contributions of order \(F^4\).

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**Appendix A: Notation and useful relations**

We collect here some definitions and relations which are often used.

The masses of the two bodies are denoted as \(m_1\) and \(m_2\), with the convention \(m_1 \leq m_2\). The symmetric mass ratio \(\nu\) is the ratio of the reduced mass \(\mu \equiv m_1m_2/(m_1 + m_2)\).
\( m_2 \) to the total mass \( M = m_1 + m_2 \),
\[
\nu \equiv \frac{m_1 m_2}{(m_1 + m_2)^2} = \frac{\mu}{M} \quad \text{ (A1)}
\]

We also define \( \Delta \equiv \sqrt{1 - 4\nu} \) which enters the mass ratios
\[
X_1 = \frac{m_1}{M} = \frac{1}{2}(1 - \Delta), \quad X_2 = \frac{m_2}{M} = \frac{1}{2}(1 + \Delta) \quad \text{ (A2)}
\]

with the property \( X_1 + X_2 = 1 \).

The asymptotic 4-momenta are denoted by \( p_a^\pm = m_a u_a^\pm \), with \( a = 1, 2 \), and the asymptotic energies by \( \varepsilon_a^\pm \). We often work within the incoming c.m. frame of the system, with time axis
\[
U^- \equiv \frac{p_1^- + p_2^-}{|p_1^- + p_2^-|} = \frac{m_1 u_1^- + m_2 u_2^-}{E_{c.m.}} \quad \text{ (A3)}
\]

where \( E_{c.m.}^- = (E_1 + E_2)^- \) is the incoming c.m. energy, which is related to the incoming value of the momentum by
\[
P_{c.m.}^- = \frac{m_1 m_2}{E_{c.m.}^-} \sqrt{\gamma^2 - 1} = \frac{m_1 m_2}{E_{c.m.}^-} p_\infty \quad \text{ (A4)}
\]

with
\[
\gamma \equiv -u_1^- \cdot u_2^- = \frac{p_1^- \cdot p_2^-}{m_1 m_2} = p_\infty \equiv \sqrt{\gamma^2 - 1} \quad \text{ (A5)}
\]

so that \( E_{c.m.}^- P_{c.m.}^- = m_1 m_2 p_\infty \). The total incoming energy can also be written as
\[
\frac{E_{c.m.}^-}{Mc^2} = h(\gamma, \nu) = \sqrt{1 + 2\nu(\gamma - 1)} \quad \text{ (A6)}
\]

implying that \( P_{c.m.}^- = \mu p_\infty / h \) and
\[
\frac{GE_{c.m.}^-}{b} = \frac{GM h}{b} = \frac{p_\infty}{j} \quad \text{ (A7)}
\]

where \( b \) is the impact parameter and
\[
j \equiv \frac{cJ_{c.m.}}{Gm_1 m_2} = \frac{cJ_{c.m.}}{GM\mu} \quad \text{ (A8)}
\]

is a dimensionless rescaled version of the total center-of-mass angular momentum \( J_{c.m.} \). The incoming energies
\[
E_1^- = \frac{m_2 \gamma + m_1}{E_{c.m.}} \quad \text{ and } \quad E_2^- = \frac{m_1 \gamma + m_2}{E_{c.m.}} \quad \text{ (A9)}
\]

can also be cast in the form
\[
E_1^- = \frac{E_{c.m.}^-}{2}(1 - \sqrt{1 - 4\xi}) \quad \text{ and } \quad E_2^- = \frac{E_{c.m.}^-}{2}(1 + \sqrt{1 - 4\xi}) \quad \text{ (A10)}
\]

where \( \xi \) denotes the symmetric energy ratio (simply related to the symmetric mass ratio and the c.m. energy)
\[
\xi = \frac{E_1^- E_2^-}{(E_{c.m.}^-)^2} = 1 - 4\xi = \frac{1 - 4\nu}{h^4} \quad \text{ (A11)}
\]

In the following Appendices we mostly use units where \( c \) and \( G \) are set to unity for simplicity.

**Appendix B: PM expansion of main quantities**

It is convenient to express the scattering angle as well as the radiative losses as power series expansions in the dimensionless variable \( \frac{1}{\gamma} = \frac{cJ_{c.m.}}{GM\mu} \), with coefficients depending on \( \gamma \) and the symmetric mass-ratio \( \nu \). The latter exhibit a simple \( \nu \)-structure when suitably rescaled by powers of \( h \).

The relative scattering angle is given by
\[
\chi^{rel} = \chi^{cons} + \delta^{rr} \chi^{rel} \quad \text{ (B1)}
\]

The PM expansion of the conservative part is
\[
\chi^{cons} = \sum_{n=1}^\infty \frac{2\chi_n^{cons}}{j^n} \quad \text{ (B2)}
\]

The only coefficients which are currently exactly known are
\[
\chi_1^{cons}(\gamma) = \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}},
\]
\[
\chi_2^{cons}(\gamma, \nu) = \frac{3}{8}\pi \frac{5\gamma^2 - 1}{h},
\]
\[
\chi_3^{cons}(\gamma, \nu) = \frac{64\gamma^6 + 120\gamma^4 + 60\gamma^2 - 5}{3(\gamma^2 - 1)^{3/2}} - \frac{2\nu}{h^2} \chi^{cons}(\gamma) \quad \text{ (B3)}
\]

with
\[
\bar{\delta}^{cons}(\gamma) = \frac{2}{3}\gamma(14\gamma^2 + 25) \quad \text{ (B4)}
\]
\[
+ 2\frac{4\gamma^4 - 12\gamma^2 - 3}{\sqrt{\gamma^2 - 1}} \arctanh \left( \frac{\sqrt{\gamma^2 - 1}}{\gamma} \right)
\]

Similarly, the expansion of the radiation-reaction part \( \delta^{rr} \chi^{rel} \) reads
\[
\delta^{rr} \chi^{rel} = \sum_{n=3}^\infty \frac{2\chi_n^{rr}}{j^n} \quad \text{ (B5)}
\]
with coefficients

\[ \nu^{-1} \lambda_{\gamma}^2 = \lambda_1 \text{cons}_J_2, \]

\[ \nu^{-1} \lambda_{\gamma}^4 = \lambda_1 \text{cons} J_3 + 2 \lambda_2 \text{cons}_J_2 - \hbar E_3 \frac{d \lambda_1 \text{cons}_J_2}{d \gamma}, \]

\[ \nu^{-1} \lambda_{\gamma}^6 = \lambda_1 \text{cons} J_5 + 2 \lambda_2 \text{cons}_J_4 + 3 \lambda_3 \text{cons}_J_3 + 4 \lambda_4 \text{cons}_J_2 \]

\[ \nu^{-1} \lambda_{\gamma}^8 = \lambda_1 \text{cons} J_7 + 2 \lambda_2 \text{cons}_J_6 + 3 \lambda_3 \text{cons}_J_5 + 4 \lambda_4 \text{cons}_J_4 + 5 \lambda_5 \text{cons}_J_3 + 6 \lambda_6 \text{cons}_J_2 \]

\[ \nu^{-1} \lambda_{\gamma}^{10} = \lambda_1 \text{cons} J_9 + 2 \lambda_2 \text{cons}_J_8 + 3 \lambda_3 \text{cons}_J_7 + 4 \lambda_4 \text{cons}_J_6 + 5 \lambda_5 \text{cons}_J_5 + 6 \lambda_6 \text{cons}_J_4 + 7 \lambda_7 \text{cons}_J_3 + 8 \lambda_8 \text{cons}_J_2. \]

etc. Here, \( E_0 \) and \( J_0 \) denote the coefficients of the PM expansion of the energy and angular momentum radiative losses

\[ -\frac{\delta^{\text{rr}} J}{J_{\text{c.m.}}} = \frac{J_{\text{rad}}}{J_{\text{c.m.}}} = \nu \sum_{n=2}^{\infty} J_n j^n, \]

\[ -\frac{\delta^{\text{rr}} E}{M} = \frac{E_{\text{rad}}}{M} = \nu^2 \sum_{n=3}^{\infty} E_n j^n, \]

respectively. The only coefficients of the above PM expansions which are currently exactly known are \( J_2 \) and \( E_3 \). They are given by

\[ \tilde{J}_2(\gamma) = \hbar^2 J_2 = 2(2\gamma^2 - 1)\sqrt{\gamma^2 - 1} I(\nu), \]

with \( I(\nu) \) defined in Eq. (5.7), and

\[ \tilde{E}_3(\gamma) = \hbar^4 E_3 = \pi \nu^3 \tilde{E}(\gamma), \]

with \( \tilde{E}(\gamma) \) defined in Eq. (5.20). The remaining coefficients are known only in PN sense. Their explicit expressions up to 2PN order are given in Appendix \[\text{C}\] the energy-rescaled coefficients

\[ \tilde{\lambda}_n^X = \hbar^{n-1} \lambda_n^X = P_{\frac{n-1}{2}}^\gamma(\nu), \]

with \( X = \text{cons}, \text{rr} \), exhibit a simple \( \nu \)-structure, i.e., polynomial of degree \( \lceil \frac{n-1}{2} \rceil \), with coefficients depending on \( \gamma \).

Similarly, the \( \nu \) dependence of \( E_n \) satisfies the property

\[ \tilde{E}_n(\gamma) = \hbar^{n+1} E_n(\gamma, \nu) = \nu P_{\frac{n+1}{2}}^\gamma(\nu), \]

where \( P_{\frac{n+1}{2}}^\gamma(\nu) \) denotes a polynomial in \( \nu \) of order \( N = \lfloor \frac{n+1}{2} \rfloor \), with coefficients depending on \( \gamma \). Concerning the angular momentum coefficients, we have shown that the quantity

\[ \tilde{J}_3 = \hbar^3 J_3 + \hbar^2 \nu E_3 = \hbar^3 J_3 + \frac{\nu \pi \nu^3 \tilde{E}}{\hbar^2}, \]

must be independent of \( \nu \), and be only a function of \( \gamma \).

We have checked this result by explicitly computing \( J_3 \) at the 2PN fractional accuracy. This is also true for the quantity

\[ \tilde{J}_4 = \hbar^3 J_4 + \hbar^3 \nu E_4. \]

Finally, the PM expansion of the radiative linear momentum loss (3.28) is

\[ -\frac{\delta^{\text{rr}} P_y}{M} = \frac{P_y^{\text{rad}}}{M} = \nu m_2 - m_1 \nu^2 \sum_{n=3}^{\infty} P_n, \]

the only PM-known coefficient being

\[ P_3 = \sqrt{\frac{\gamma - 1}{\gamma + 1}} E_3. \]

We found that the quantities

\[ \hbar^3 \left[ P_4 - \sqrt{\frac{\gamma - 1}{\gamma + 1}} E_4 \right] = \left[ P_4 - \sqrt{\frac{\gamma - 1}{\gamma + 1}} E_4 \right]_{\nu=0}, \]

\[ \hbar^4 \left[ P_5 - \sqrt{\frac{\gamma - 1}{\gamma + 1}} E_5 \right] = \left[ P_5 - \sqrt{\frac{\gamma - 1}{\gamma + 1}} E_5 \right]_{\nu=0}, \]

do not depend on \( \nu \), by explicitly computing the coefficients at the 2PN fractional accuracy.

Concerning the impulse coefficients \( c_{bX}^{1X}, c_{b1X}^{1X}, c_{b2X}^{1X} \), we have given their PM expansion in powers of \( GM/b \) instead of \( 1/j \) by using the relation \( j = \frac{2GM}{b^2} \) between the dimensionless angular momentum \( j \) and the impact parameter \( b \), namely

\[ c_{bX}^1 = \sum_{n=3}^{\infty} \frac{c_{bX}^{1n} G^n}{b^n}, \]

\[ c_{b1X} = \sum_{n=3}^{\infty} \frac{c_{b1X}^{1n} G^n}{b^n}, \]

\[ c_{b2X} = \sum_{n=3}^{\infty} \frac{c_{b2X}^{1n} G^n}{b^n}, \]

with \( X = \text{cons}, \text{rr} \). The expressions for the coefficients up to \( n = 5 \) can be easily deduced from the 5PM result for \( \Delta p_1 \) given in Table \[\text{II}\] fully known up to 3PM and only in PN sense up to 2PN at higher PM orders. The fractionally 2PN-accurate results at orders \( O(G^{2\pm \nu}) \) are listed in Appendix \[\text{C}\] whereas the new \( O(G^3) \) and \( O(G^5) \) results are given in Section \[\text{VIII}\].

Appendix C: Fractionally 2PN-accurate radiated energy along hyperboliclike orbits (without the 1.5PN tail contribution)

Most of the literature on radiated fluxes of energy (or angular momentum, or linear momentum) focusses on
ellipticlike orbits. Little attention has been given to radiative losses along hyperbolic motions. Exceptions are the fractionally 1PN-accurate computations of radiative energy loss \[77\] and angular momentum loss \[63\] in Ref. \[81\]. In addition, the latter reference gave the leading-order radiative loss of linear momentum along an hyperbolic motion.

For our present purposes, we improved these results by computing with fractional 2PN accuracy the energy and angular momentum losses along hyperbolic motions. This was done by inserting in the standard energy-momentum and angular momentum fluxes \[78\], the knowledge of the 2PN-accurate source multipole moments \[79\] with a 2PN-accurate quasi-Keplerian solution of hyperbolic motions \[80\]. More precisely, the gravitational wave energy flux is given by \( \mathcal{F}_{GW} = \frac{dE_{rad}}{dt} \) (where \( U \) denotes the retarded time in radiative coordinates; see Eq. (68a) of Ref. \[81\])

\[
\mathcal{F}_{GW} = \sum_{l=2}^{\infty} \frac{G}{c^{2l+1}} \left\{ \frac{l+1}{l(l-1)(l+1)!} U^{(1)}_{L} U^{(1)}_{L} + \frac{4l(l+2)}{c^{2}(l-1)(l+1)!} V^{(1)}_{L} V^{(1)}_{L} \right\}.
\]

(C1)

Here \( U_{L} \) and \( V_{L} \) denote the radiative multipole moments that parametrize the asymptotic waveform. These moments are perturbatively computed in terms of the source variables, and involve both instantaneous and past-hereditary contributions. For example, at the 2PN accuracy, one has (with some rational constants \( \kappa_{l} \) and \( \pi_{l} \); see \[81\])

\[
\begin{align*}
U_{L}(U) &= U^{\text{inst}}_{L}(U) + U^{\text{tail}}_{L}(U) \\
&= J^{(1)}_{L}(U) + \frac{2GE_{c.m.}}{c^{2}} \int_{0}^{\infty} d\tau J^{(l+2)}_{L}(U - \tau) \left[ \ln \left( \frac{c\tau}{2\nu_{0}} \right) + \kappa_{l} \right], \\
V_{L}(U) &= V^{\text{inst}}_{L}(U) + V^{\text{tail}}_{L}(U) \\
&= J^{(0)}_{L}(U) + \frac{2GE_{c.m.}}{c^{3}} \int_{0}^{\infty} d\tau J^{(l+2)}_{L}(U - \tau) \left[ \ln \left( \frac{c\tau}{2\nu_{0}} \right) + \pi_{l} \right],
\end{align*}
\]

(C2)

the superscript in parenthesis denoting repeated time-derivatives. When aiming at the 2PN fractional accuracy for \( \mathcal{F}_{GW} \) one can use

\[
\begin{align*}
\mathcal{F}_{GW} &= \frac{G}{c^{3}} \left( \frac{1}{5} U_{ij}^{(1)} U_{ij}^{(1)} + \frac{1}{180} U_{ijk}^{(1)} U_{ijk}^{(1)} \right) \\
&+ \frac{1}{c^{2}} \left[ \frac{1}{180} U_{ijk}^{(1)} U_{ijk}^{(1)} + \frac{16}{45} V_{ij}^{(1)} V_{ij}^{(1)} \right] \\
&+ \frac{1}{c} \left[ \frac{1}{9072} U_{ijk}^{(1)} U_{ijk}^{(1)} + \frac{1}{84} V_{ij}^{(1)} V_{ij}^{(1)} \right],
\end{align*}
\]

(C3)

in which one must insert Eqs. (C2) with the appropriately PN-accurate multipoles (notably the 2PN-accurate quadrupole \( U_{ij} \), for which \( \kappa_{2} = \frac{1}{12} \)).

To evaluate the total radiated energy \[20\]

\[
E_{rad} = \int_{-\infty}^{\infty} dt \mathcal{F}_{GW},
\]

one needs to explicitly evaluate the multipole moments along the hyperbolic orbit. This is conveniently done by using the 2PN-accurate quasi-Keplerian parametrization of the hyperbolic motion \[82\] \[83\].

\[
\begin{align*}
r &= \bar{a}_{r}(e_{r} \cosh v - 1), \\
\bar{\nu} &= e_{r} \sinh v + f_{t} V + g_{t} \sin V, \\
\phi &= K[V + f_{\phi} \sin 2V + g_{\phi} \sin 3V],
\end{align*}
\]

(C5)

with

\[
V(v) = 2 \arctan \left[ \sqrt{\frac{e_{r} + 1}{e_{\phi} - 1} \tanh \frac{v}{2}} \right].
\]

(C6)

Here we use dimensionless variables \( t = c^{3}t_{\text{phys}}/(GM) \) and \( r = c^{2}r_{\text{phys}}/(GM) \) as well as dimensionless rescaled orbital parameters, such as a dimensionless semi-major axis \( a_{r} = c^{2}a_{\text{phys}}/(GM) \). The expressions of the orbital parameters \( \bar{a}_{r}, \bar{a}_{t}, K, e_{r}, e_{\phi}, f_{t}, g_{t}, f_{\phi}, g_{\phi} \) are given, e.g., in Table VIII of Ref. \[81\] in harmonic coordinates as functions of the conserved energy and angular momentum of the system.

We computed \( E_{rad} \) with the 2PN (fractional) accuracy, including the 1.5PN tail contribution coming from the hereditary integral terms in Eqs. (C2). The tail contribution is discussed in the following Appendix. Let us display here our results for the 2PN-accurate energy loss coming from the instantaneous multipole contributions, starting with the quadrupolar one: \( U_{ij}^{\text{inst}} = I_{ij}^{(2)} \).

We first display \( E_{rad,inst}^{2PN} \) as an exact expression in terms of the harmonic-coordinate orbital parameters \( e_{r} \) and \( a_{r} \) (instead of \( e_{r} \) and \( j \) as in our previous work \[57\]).

---

19 Note that in Eq. (42), second line there is a missing term of \(+42120\nu(e_{\nu} - 1)^{2}\) inside the bracketed expression proportional to the arccos.

20 We use henceforth the fact that the (radiative) retarded time \( U \) only differs by an additive contribution from the (harmonic) coordinate time \( t \).
Eq. (D2), which generalizes the 1PN accurate results of 
Ref. [75], Eq. (5.7))

\[ E^{\text{rad,inst}}_{2PN} = \frac{\nu^2}{a r^2} \left( \Delta E^{\text{N}}_{\text{resc}} + \frac{\eta^2}{a r} \Delta E^{\text{1PN}}_{\text{resc}} + \frac{\eta^4}{a r^4} \Delta E^{\text{2PN}}_{\text{resc}} \right) , \]

where we have set \( G = M = c = 1 \) for simplicity, and

\[ \Delta E^{\text{2PN}}_{\text{resc}} = \frac{A^{\text{2PN}}}{\sqrt{c r^2 - 1}} \arccos \left( -\frac{1}{c r} \right) + B^{\text{2PN}}_E , \] (C8)

with coefficients

\[
\begin{align*}
(e_r^2 - 1)^3 A^N_E &= \frac{2}{15} (292 \nu^2 + 96 + 37 \nu^4), \\
(e_r^2 - 1)^3 B^N_E &= \frac{2}{45} (673 \nu^2 + 602), \\
(e_r^2 - 1)^4 A^{1\text{PN}}_E &= -\frac{112}{9} \nu - 1132 \frac{\nu}{21} + \left( \frac{37318}{105} - \frac{3 \nu}{32} \right) e_r^2 + \left( \frac{-233}{2} - \frac{249 \nu}{5} \right) e_r^4 + \left( \frac{1143}{140} - \frac{37 \nu}{15} \right) \nu e_r^6, \\
(e_r^2 - 1)^4 B^{1\text{PN}}_E &= -\frac{446}{9} \nu - \frac{210811}{1575} + \left( -105 \nu - \frac{592573}{1575} \right) e_r^2 + \left( -\frac{205 \nu}{9} - \frac{47659}{6300} \right) \nu e_r^4, \\
(e_r^2 - 1)^5 A^{2\text{PN}}_E &= \frac{105146}{315} \nu + \frac{32 \nu^2}{5} + \frac{44134}{405} + \left( \frac{387220}{567} + \frac{1089412 \nu}{315} + \frac{623 \nu^2}{15} \right) e_r^2 \\
&\quad + \left( -\frac{108326}{945} + \frac{622831 \nu}{210} + \frac{587 \nu^2}{12} \right) e_r^4 + \left( -\frac{424337}{2520} + \frac{273 \nu^2}{10} + \frac{27875 \nu^2}{168} \right) \nu e_r^6 \\
&\quad + \left( \frac{37 \nu^2}{20} + \frac{114101 \nu}{5040} - \frac{1411 \nu^2}{168} \right) \nu e_r^8, \\
(e_r^2 - 1)^5 B^{2\text{PN}}_E &= \frac{607888}{675} \nu + \frac{151 \nu^2}{10} + \frac{7846496}{297675} + \left( \frac{9047}{180} \nu^2 + \frac{84265357}{18900} + \frac{164159833}{238140} \right) e_r^2 \\
&\quad + \left( \frac{5968863}{37800} \nu - \frac{520075147}{1190700} + \frac{2048}{45} \nu^2 \right) e_r^4 + \left( -\frac{6299}{280} + \frac{2711041}{176400} + \frac{2723 \nu^2}{180} \right) \nu e_r^6. \quad \text{(C9)}
\end{align*}
\]

Expanding then the above results in inverse powers of \( j \) (once \( (a_r, e_r) \) have been reexpressed in terms of \( (p_\infty, j) \)), yields the following \( G^7 \)-accurate (2PN) results

\[ E^{\text{rad,inst}}_{2PN} = \nu^2 \left( E_N + \eta^2 E^{1\text{PN}}_E + \eta^4 E^{2\text{PN}}_E \right), \] (C10)

\[ E^{1\text{PN}}_N = \left( \frac{1357}{840} \nu + \frac{74}{15} \nu^3 \right) \frac{p^6_\infty}{j^3} \\
+ \left( \frac{18608}{525} - \frac{1424 \nu}{15} \right) \nu \frac{p^4_\infty}{j^4} \\
+ \left( \frac{13831}{280} - \frac{933 \nu}{10} \right) \nu \frac{p^2_\infty}{j^5} \\
+ \left( \frac{142112}{315} - \frac{26464 \nu}{45} \right) \nu \frac{p^0_\infty}{j^6} \\
+ \left( \frac{2259}{8} - 265 \nu \right) \nu \frac{p^0_\infty}{j^7} + O \left( \frac{1}{j^8} \right), \] (C12)

where

\[
\begin{align*}
E_N &= \frac{37 \pi}{15} \frac{p^4_\infty}{j^3} + \frac{1568}{45} \frac{p^2_\infty}{j^5} + \frac{122}{5} \frac{p^2_\infty}{j^3} \\
&\quad + \frac{4672}{45} \frac{p_\infty}{j^9} + \frac{85}{3} \nu \frac{1}{j^7} + O \left( \frac{1}{j^9} \right), \quad \text{(C11)}
\end{align*}
\]
and

\[
E_{2\text{PN}} = \left( \frac{27953}{10080} - \frac{839}{420} + \frac{37}{5} \nu^2 \right) \pi \frac{p^{6\infty}_{\infty}}{j^3} \\
+ \left( \frac{220348}{11025} - \frac{31036}{525} + 172\nu^2 \right) \pi \frac{p^{6\infty}_{\infty}}{j^3} \\
+ \left( -\frac{64579}{5040} + \frac{187559}{1680} + \frac{2067}{10} \nu^2 \right) \pi \frac{p^{6\infty}_{\infty}}{j^3} \\
+ \left( -\frac{293992}{1701} - \frac{673278}{4725} + \frac{24424}{15} \nu^2 \right) \pi \frac{p^{6\infty}_{\infty}}{j^6} \\
+ \left( 19319 - \frac{432805}{336} + \frac{7605}{8} \nu^2 \right) \pi \frac{p^{6\infty}_{\infty}}{j^3} \\
+ O\left( \frac{1}{j^8} \right). \quad \text{(C13)}
\]

From the above expressions one can easily get the PN expansion of the PM coefficients \(E_n\) (see Eq. (17)). For example

\[
E_3 = \frac{37}{15} \pi p^{4\infty}_{\infty} + \left( \frac{1357}{840} - \frac{74}{15} \nu \right) \pi p^{6\infty}_{\infty} \\
+ \left( \frac{27953}{10080} + \frac{839}{420} + \frac{37}{5} \nu^2 \right) \pi p^{6\infty}_{\infty} \\
+ O(p^{0\infty}_{\infty}). \quad \text{(C14)}
\]

Note that no tails appear at the level of \(E_3\) (see the next section).

Appendix D: Fractionally 1.5PN tail contribution to the radiated energy along hyperboliclike orbits

It was shown in Ref. \[84\] that the 4PN-level tail contribution to the radiation-reaction force is given by the following time-antisymmetric force

\[
\mathcal{F}_a^{\text{tail}} = -\frac{4\pi G^2 M}{c^5} m_a x_a^i H[I_{ij}^{(6)}](t). \quad \text{(D1)}
\]

Here, the superscript on the quadrupole moment \(I_{ij}\) denotes the sixth time-derivative, while \(H\) denotes the Hilbert transform, defined as (with \(P\) denoting the principal value)

\[
H[f](t) \equiv \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{dt'}{t-t'} f(t'). \quad \text{(D2)}
\]

The time-integrated, tail-related loss of mechanical energy of the two-body system induced by the radiation-reaction force \[\text{(D1)}\] is \(-\delta^\tau E_{\text{system}}\) \(-\int dt \sum_a x_a^i \mathcal{F}_a^{\text{tail}}\). Replacing the symmetric-trace-free projection of \(\sum_a m_a x_a^i x_a^j\) by (half) the time derivative of the quadrupole moment \(I_{ij}\), and integrating by parts, one easily finds that

\[
-\delta^\tau E_{\text{system}} = \frac{2\pi G^2 M}{5} \int dt I_{ij}^{(3)}(t) H[I_{ij}^{(4)}](t). \quad \text{(D3)}
\]

Let us compare this tail-related loss of mechanical energy to the usually considered expression for the leading-order (LO) tail contribution to the radiated energy, coming from including in Eq. \[\text{(C1)}\] the \(U_{ij}^{\text{past}}\) hereditary contribution of Eq. \[\text{(C2)}\], namely

\[
E_{\text{past tail}}^{\text{rad}} = \frac{4}{5} \frac{G^2 M}{c^5} \int dt I_{ij}^{(3)}(t) I_{ij}^{(5)}(t). \quad \text{(D4)}
\]

Here we used the notation of Ref. \[85\] (applicable to a generic multipole moment of electric or magnetic type)

\[
I_{L \text{ past}}^{(n)}(t) \equiv \int_0^{+\infty} d\tau \ln \left( \frac{\pi}{C_{L \text{ past}}} \right) I_{L \text{ past}}^{(n)}(t - \tau), \quad \text{(D5)}
\]

where the constant \(C_{L \text{ past}}\) depends on the considered multipole. Namely

\[
C_{L \text{ past}} = 2r_0 e^{-\kappa t}, \quad C_{J \text{ past}} = 2r_0 e^{-\pi t}, \quad \text{(D6)}
\]

where \(\kappa t\) and \(\pi t\) are those of Eq. \[\text{(C2)}\], and where the length scale \(r_0\) coincides with the one introduced in the multipolar post-Minkowskian formalism \[85\]; explicitly

\[
[C_{L \text{ past}}, C_{J \text{ past}}] = [2r_0 e^{-11/12}, 2r_0 e^{-97/60}, 2r_0 e^{-59/30}], \quad [C_{J \text{ past}}, C_{J \text{ past}}] = [2r_0 e^{-7/6}, 2r_0 e^{-5/3}]. \quad \text{(D7)}
\]

The radiated energy Eq. \[\text{(D4)}\] for a priori looks different from the energy loss \[\text{(D3)}\], notably because the tail-induced quadrupole moment \(I_{ij}^{(5)}(t)\) entering Eq. \[\text{(D4)}\] is past-hereditary (and therefore asymmetric under time-reversal), while the integrand of \[\text{(D3)}\] is time-symmetric. [The Hilbert transform being time-odd.] However, it is easily seen that if one decomposes the integral transform entering \(I_{ij}^{(5)}(t)\) into its time-even and time-odd parts, by replacing \(I_{ij}^{(5)}(t - \tau)\) by \(\frac{1}{2}(I_{ij}^{(5)}(t - \tau) + I_{ij}^{(5)}(t + \tau)) + \frac{1}{2}(I_{ij}^{(5)}(t - \tau) - I_{ij}^{(5)}(t + \tau)),\) the time-odd part will yield (after integrating by parts) a vanishing contribution to the double integral

\[
\int_{-\infty}^{+\infty} dt \int_0^{+\infty} d\tau I_{ij}^{(3)}(t) I_{ij}^{(5)}(t - \tau) - I_{ij}^{(5)}(t + \tau) \ln \tau = 0. \quad \text{(D8)}
\]

After a last integration by parts, one finds that \(E_{\text{past tail}}^{\text{rad}}\) is simply equal to the mechanical energy loss \(-\delta^\tau E_{\text{system}}\) given by the time-even expression \[\text{(D3)}\] involving the Hilbert transform \(H[I_{ij}^{(4)}].\)

This time-symmetric evaluation of tail effects is the concrete realization of the time-symmetric evaluation of GW radiation mentioned in footnote 7, and valid when treating radiation-reaction effects to linear order. Let us recall that the treatment used in this paper is only intended to be valid to first order in \(\mathcal{F}_a^{\text{tail}}.\)

A more transparent way to understand the equality between \(E_{\text{past tail}}^{\text{rad}}\) and \(-\delta^\tau E_{\text{system}}\) is to work in the frequency domain. The time-domain convolutions entering both \(H[I_{ij}^{(4)}]\) and \(I_{ij}^{(5) \text{ past}}(t)\) become multiplications
by the Fourier transforms of their kernels, say \( K(\omega) = \int dt e^{i\omega t} K(t) \). It is convenient to factor out some of the factors \((-i\omega)^n\) corresponding to a \( n \)th time derivative so as to write both integrals as bilinear forms in the Fourier transform of \( I_{ij}^{(3)}(t) \). More precisely, we shall write the various tail integrals in the form

\[
E_X^{\text{rad}} = \frac{2\pi G^2 M}{\epsilon^s} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} I_{ij}^{(3)}(-\omega) K_X(\omega) I_{ij}^{(3)}(\omega),
\]

where the label \( X \) is either \( r \) or past tail, or sym tail, to be defined below.

Using the fact that the kernel of the Hilbert transform is \(-i \text{sign}(\omega)\), the kernel \( K^{rr}(\omega) \) entering the mechanical energy loss \(-\delta^{rr} E_{\text{system}}\) is obtained as

\[
K^{rr}(\omega) = -(-i \text{sign}(\omega))(-i\omega) = +|\omega|.
\]

(D10)

On the other hand, using the integral (See, e.g., Refs. 64, 88)

\[
\int_0^\infty du e^{iu \omega} \ln \left( \frac{u}{\epsilon} \right) = -\frac{\pi}{2|\omega|} - \frac{i}{\omega} \ln(C|\omega|e^{\gamma\pi}),
\]

the kernel entering the past-tail radiated energy reads

\[
K^{\text{past tail}}(\omega) = |\omega| + \frac{2i}{\pi} \omega \ln(C|\omega|e^{\gamma\pi}).
\]

(D12)

The crucial point here is that the second contribution on the r.h.s. of the past-tail kernel, Eq. (D12), is odd under frequency reversal, \( \omega \rightarrow -\omega \) (corresponding to time-reversal). The corresponding integrated term (sandwiched between \( I_{ij}^{(3)}(-\omega) \) and \( I_{ij}^{(3)}(\omega) \) then vanishes.

In other words, if we define the symmetric tail as being the time-symmetric projection of the hereditary past tail, so that its frequency kernel is the frequency-symmetric projection of \( K^{\text{past tail}}(\omega) \),

\[
K^{\text{sym tail}}(\omega) = \frac{1}{2} \left[ K^{\text{past tail}}(\omega) + K^{\text{past tail}}(-\omega) \right] = |\omega| = K^{rr}(\omega),
\]

(D13)

we have the double result that the tail contribution to the total radiated energy is automatically equal to its time-symmetric projection, and that it balances the radiation-reaction energy loss

\[
E_{\text{past tail}}^{\text{rad}} = E_{\text{sym tail}}^{\text{rad}} = -\delta^{rr} E_{\text{system}}.
\]

(D14)

Besides being conceptually useful for clarifying the properties of tail integrals, the frequency-domain representation is very useful for computing the explicit values of the various needed tail integrals along hyperbolic motions. Similarly to the various hyperbolic-motion integrals we encountered in our previous works [13, 57, 58, 59], we could evaluate the large-impact-parameter expansion of the needed tail integrals of the type \( (D15) \) by starting from the Keplerian parametrization of the hyperbolic motion. The appropriate Newtonian-level limit of Eqs. (C5) reads

\[
\begin{align*}
 r &= a_r (e_r \cosh v - 1), \\
 \dot{n} t &= e_r \sinh v + v, \\
 \phi &= 2 \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \frac{v}{2} \right],
\end{align*}
\]

(D16)

with orbital parameters \( \bar{n} = p^3_\infty, \bar{a}_r = 1/(2\bar{E}) = 1/p^2_\infty, e_r = \sqrt{1 + p^2_\infty} \).

The first step consists in Fourier transforming the multipole moments entering Eq. (D15), i.e.,

\[
\hat{I}_{ab}(\omega) = \frac{1}{\bar{E}} \int \frac{dt}{dv} e^{i\omega(t-v)} I_{ab} |_{t=t(v)} dv,
\]

(D17)

and similarly for the other moments. This is done by using the integral representation of the Hankel functions of the first kind of order \( p \equiv \frac{a}{e_r} \) and argument \( q \equiv i u \), where

\[
u \equiv \omega e_r a_r^{3/2},
\]

(D18)

namely

\[
H_p^{(1)}(q) = \frac{1}{i\pi} \int_{-\infty}^{\infty} e^{u \sinh v - pv} dv.
\]

(D19)

As the argument \( q = i u \) of the Hankel function is purely imaginary, the Hankel function becomes converted into a Bessel K function, according to the relation

\[
H_p^{(1)}(iu) = \frac{2}{\pi} e^{-\frac{i\pi}{2} (p+1)} K_p(u).
\]

(D20)

Using standard identities valid for Bessel functions we could express the results in terms of only two orders: \( p = \frac{3}{2} \) and \( p+1 = 1+\frac{3}{2} \). For instance, the averaged energy tail turns out to read (in scaled variables, provisionally setting \( G = M = c = 1 \))

\[
E_{\text{tail}}^{\text{rad}} = \frac{1}{e_r^{3/2} a_r^{3/2}} \int_0^{\infty} du K_{\text{tail}}(u),
\]

(D21)

where we suppressed the qualification (past or sym) of the tail.
\[ K_{\text{tail}}(u) = -\frac{64}{5} \nu^2 \frac{p^2}{u^3} e^{-ip\nu} \left\{ u^2(p^2 + u^2 + 1)(p^2 + u^2)K_{p+1}(u) \right\} - 2u \left[ \left( p - \frac{3}{2} \right) u^2 + p(p - 1)^2 \right] (p^2 + u^2)K_p(u)K_{p+1}(u) + 2 \left[\frac{1}{2}u^6 + \left( 2p^2 - \frac{3}{2}p + \frac{1}{6} \right) u^4 + \left( \frac{5}{2}p^4 - \frac{7}{2}p^3 + p^2 \right) u^2 + p^4(p - 1)^2 \right] K_p^2(u) \} . \] (D22)

The integral (D21) cannot be performed in closed analytical form because the orders \( p \) or \( p + 1 \) of the Bessel \( K \) functions depend on the \( u \) integration variable. However, as \( p = \frac{2}{2n} \) tends to zero when \( e_r \to \infty \), the use of a large-eccentricity expansion allows to express the integral (D21) in terms of computable Bessel-function integrals. Indeed, Taylor-expanding to second order in \( p \),

\[ K_p(u) = K_0(u) + \frac{1}{2} p^2 \frac{\partial^2 K_v(u)}{\partial^2 v} \bigg|_{v=0} + O(p^3), \]
\[ K_{p+1}(u) = K_1(u) + \frac{p}{u} K_0(u) \]
\[ + \frac{1}{2} p^2 \frac{\partial^2 K_v(u)}{\partial^2 v} \bigg|_{v=1} + O(p^3), \] (D23) where

\[ \tilde{K}_{\text{NNLO}}^{\text{LO}}(u) = u \tilde{K}_{\text{NNLO}}^{\text{LO}}(u), \]
\[ \tilde{K}_{\text{NNLO}}^{\text{LO}}(u) = \frac{\nu^2}{e_r^2} \left[ \tilde{K}_{\text{tail}}^{\text{LO}}(u) + \frac{\pi}{e_r} \tilde{K}_{\text{NNLO}}^{\text{LO}}(u) \right] + \frac{1}{e_r^2} \tilde{K}_{\text{NNLO}}^{\text{LO}}(u) \] (D24)

\[ K_{\text{tail}}^{\text{LO}}(u) = \frac{64}{5} \nu^3 \left[ \left( \frac{1}{3} + u^2 \right) K_0^2(u) + 3uK_0(u)K_1(u) + (1 + u^2)K_1^2(u) \right], \]
\[ K_{\text{tail}}^{\text{NLO}}(u) = uK_{\text{tail}}^{\text{LO}}(u), \]
\[ K_{\text{tail}}^{\text{NNLO}}(u) = \frac{\pi^2}{2} u^2 K_{\text{tail}}^{\text{LO}}(u) - \frac{64}{5} \nu^3 \left\{ (1 + 3u^2)K_0^2(u) + 7uK_0(u)K_1(u) + (1 + 2u^2)K_1^2(u) \right\}
\[ + u^2 \left[ \left( u^2 + \frac{1}{3} \right) K_0(u) + \frac{3}{2} uK_1(u) \right] \frac{\partial^2 K_v(u)}{\partial^2 v} \bigg|_{v=0} + u^2 \left[ \frac{3}{2} uK_0(u) + (u^2 + 1)K_1(u) \right] \frac{\partial^2 K_v(u)}{\partial^2 v} \bigg|_{v=1} \} , \]
\[ K_{\text{tail}}^{\text{NNLO}}(u) = uK_{\text{tail}}^{\text{NNLO}}(u) - \frac{\pi^2}{3} u^3 K_{\text{tail}}^{\text{LO}}(u) . \] (D25)

We could then evaluate all the needed integrals. Converting the large-\( e_r \) expansion into a large-\( j \) expansion finally leads to the following explicit result
\[ E_{\text{tail}}^{\text{PN}} = \nu^2 \left[ \frac{3136}{45} \frac{p^6}{j^6} + \frac{2972}{20} \frac{p^6}{j^6} \right] + \left( \frac{9344}{45} + \frac{8857}{675} \pi^2 \right) \frac{p^6}{j^6} + \pi \left( \frac{2755}{64} + \pi^4 + \frac{1579}{3} \pi^2 \right) \frac{p^6}{j^6} \]
\[ + O \left( \frac{1}{j^6} \right) . \] (D26)

From the above expression, including the results of the previous section, one easily gets the PN expansion of the PM coefficients \( E_n \) (see Eq. (B7)). For example
\[ E_4 = \frac{1568}{45} \frac{p^3}{j^3} + \left( \frac{1680}{525} + \frac{1424}{15} \right) \frac{p^5}{j^5} + \frac{3136}{45} \frac{p^6}{j^6} + \frac{220348}{11025} - \frac{31036}{525} \nu + 172\nu^2 \right) \frac{p^7}{j^7} + O(p^8), \] (D27) which incorporates the 1.5PN tail contribution.
Appendix E: Fractionally 2PN-accurate radiated angular momentum along hyperbolic-like orbits (without the 1.5PN tail contribution)

The radiated angular momentum flux $\mathcal{G}_i$ in terms of radiative multipole moments reads

$$\mathcal{G}_i = \epsilon_{iab} \sum_{l=2}^{\infty} \frac{G}{c^2(l+1)!} \left\{ \frac{(l+1)(l+2)}{(l-1)!(2l+1)!} U_{aL-1} V_{bL-1}^{(1)} \right. 
+ \frac{4l(l+2)}{c^2(l-1)(l+1)!(2l+1)!!} V_{aL-1} V_{bL-1}^{(1)} \right\}, \quad \text{(E1)}$$

By replacing both $U_L$ and $V_L$ by their corresponding expressions (E2) in Eq. (E1), one finds for the instantaneous contribution to the radiated angular momentum

$$J_i^{\text{rad,inst}} = \int_{-\infty}^{\infty} dt \mathcal{G}_i^{\text{rad,inst}} , \quad \text{(E2)}$$

with

$$\mathcal{G}_i^{\text{rad,inst}} = \frac{G}{c^6} \epsilon_{iab} \left\{ \frac{2}{5} J_{a}^{(2)} I_{b}^{(3)} + \frac{1}{c^2} \left[ \frac{1}{63} J_{a}^{(4)} I_{b}^{(4)} + \frac{32}{45} J_{a}^{(2)} J_{b}^{(3)} \right] + \frac{1}{c^2} \left[ \frac{1}{2268} J_{a}^{(4)} I_{b}^{(5)} + \frac{1}{28} J_{a}^{(3)} J_{b}^{(4)} \right] \right\} , \quad \text{(E3)}$$

with coefficients

$$(c_r^2 - 1)^{3/2} A_j^N = \frac{64}{5} + \frac{56}{5} c_r^2 ,$$

$$(c_r^2 - 1)^{3/2} B_j^N = \frac{104}{5} + \frac{16}{5} c_r^2 ,$$

$$(c_r^2 - 1)^{5/2} A_j^{PN} = -\frac{184}{45} c_r^2 - \frac{3644}{105} + \left( \frac{208}{3} c_r^2 - \frac{9506}{105} \right) c_r^2 + \left( \frac{158}{15} c_r^2 + \frac{739}{42} \right) c_r^4 ,$$

$$(c_r^2 - 1)^{5/2} B_j^{PN} = -\frac{2524}{45} c_r^2 - \frac{8080}{105} + \left( \frac{2294}{45} c_r^2 - \frac{9007}{210} \right) c_r^2 + \left( \frac{8}{5} c_r^2 + \frac{80}{7} \right) c_r^4 ,$$

$$(c_r^2 - 1)^{3/2} A_j^{PN} = \frac{93316}{2835} c_r^2 + \frac{73316}{315} c_r^2 + \frac{16}{15} c_r^2 + \left( \frac{32636}{21} c_r^2 - \frac{25518}{945} + \frac{488}{5} c_r^2 \right) c_r^4 + \left( \frac{58}{5} c_r^2 - \frac{3328}{105} + \frac{5509}{180} \right) c_r^6 ,$$

$$(c_r^2 - 1)^{3/2} B_j^{PN} = \frac{1628}{45} c_r^2 + \frac{294773}{4725} c_r^2 + \frac{1297277}{42525} + \left( \frac{2823452}{6075} + \frac{1816}{15} c_r^2 + \frac{1419439}{9450} \right) c_r^2 + \left( \frac{5686}{1575} c_r^2 - \frac{137559}{700} + \frac{578}{9} c_r^2 \right) c_r^4 + \left( \frac{6}{5} c_r^2 - \frac{298}{35} + \frac{1952}{63} \right) c_r^6 . \quad \text{(E6)}$$

In the large-$j$ expansion limit one has

$$J_{2PN}^{\text{rad,inst}} = \nu^2 \left( J_N + \eta^2 J_{1PN} + \eta^4 J_{2PN} \right) , \quad \text{(E7)}$$

at the 2PN level of accuracy. The only nonvanishing component is the $z$ one. Suppressing the $z$ label we find

$$J_{2PN}^{\text{resc}} = \frac{\eta^2}{\nu^2} \left( \nu^2 \Delta J_{\text{resc}}^{\text{NP}} + \frac{\eta^2}{\nu^2} \Delta J_{\text{resc}}^{\text{1PN}} + \frac{\eta^4}{\nu^2} \Delta J_{\text{resc}}^{\text{2PN}} \right) , \quad \text{(E4)}$$

where

$$\Delta J_{\text{resc}}^{\text{2PN}} = \frac{A_{j}^{PN}}{\sqrt{\nu^2 - 1}} \arccos \left( -\frac{1}{\nu} \right) + B_{j}^{PN} , \quad \text{(E5)}$$

with

$$J_N = \frac{16}{5} \frac{p_{\infty}^3}{j^6} + \frac{28}{5} \frac{p_{\infty}^3}{j^4} + \frac{176}{5} \frac{p_{\infty}^3}{j^3} + 12\frac{1}{j^2} + \frac{304}{15p_{\infty}j^5} + \frac{144}{25p_{\infty}^4j^2} + O \left( \frac{1}{j^8} \right) . \quad \text{(E8)}$$
$J_{\text{IPN}} = \left( -\frac{16}{5}\nu + \frac{176}{35}\nu_\infty^{5} \right) \frac{p_\infty^{5}}{j} + 739\left( -\frac{163}{9}\nu_\infty^{2} \right) \frac{p_\infty^{2}}{j^{2}} + \left( \frac{814}{105} - \frac{1727}{9}\nu_\infty^{3} \right) \frac{p_\infty^{3}}{j^{3}} + 107\left( -\frac{374}{5}\nu_\infty^{4} \right) \frac{p_\infty^{4}}{j^{4}} + \left( \frac{84368}{315} - \frac{13456}{45}\nu_\infty \right) \frac{p_\infty^{6}}{j^{5}} + \frac{359}{4} - \frac{235}{3}\nu_\infty \frac{p_\infty}{j^{6}} + \left( \frac{16816}{105} - \frac{8336}{75}\nu_\infty \right) \frac{1}{j_{\text{inst}}} + O\left( \frac{1}{j^{5}} \right), \quad (E9)$

and

$J_{\text{2PN}} = \left( -\frac{608}{315}\nu + \frac{148}{35}\nu^{2} + \frac{16}{5}\nu \right) \frac{p_\infty^{5}}{j} + \left( -\frac{5777}{2520}\nu + \frac{5339}{420}\nu^{2} \right) \frac{p_\infty^{6}}{j^{2}} + \left( -\frac{93664}{1575}\nu + \frac{247724}{1575}\nu^{2} \right) \frac{p_\infty^{3}}{j^{3}} + \left( -\frac{101219}{1512}\nu + \frac{15856}{105}\nu^{2} + 209\nu \right) \frac{p_\infty^{4}}{j^{4}} + \left( -\frac{552320}{1705}\nu + \frac{31828}{27}\nu^{2} + 3568\nu \right) \frac{p_\infty^{5}}{j^{5}} + \left( -\frac{2999}{216}\nu + \frac{8497}{12}\nu^{2} + \frac{1484}{3}\nu \right) \frac{p_\infty^{6}}{j^{6}} + \left( -\frac{1000568}{14175}\nu \right) \frac{1}{j_{\text{inst}}} + O\left( \frac{1}{j^{7}} \right). \quad (E10)$

Appendix F: Fractionally 1.5PN tail contribution to the radiated angular momentum along hyperboliclike orbits

The tail contribution to the radiated angular momentum is obtained as in the case of the radiated energy. A direct calculation shows again that the integrated past-tail is equal to its time-symmetric projection:

$\Delta J_{\text{past tail}} = \Delta J_{\text{sym tail}} = \frac{G^{2}M}{c^{3}} \int_{0}^{\infty} d\omega \omega^{6} I_{jl}(\omega) I_{kl}(-\omega), \quad (F1)$

with only surviving component along the z-axis, with value

$\Delta J_{\text{sym tail}} = \nu^{2} \left( \frac{448}{5}\frac{p_\infty^{4}}{j^{3}} + \pi \frac{69\nu^{2} p_\infty^{6}}{5 j} \right) + \left( \frac{128}{15} + \frac{4352}{45}\nu^{2} \right) \frac{p_\infty^{4}}{j^{3}} + \pi \left( \frac{423}{16}\nu^{4} + 303\nu^{2} \right) \frac{p_\infty^{6}}{j^{5}} + O\left( \frac{1}{j^{7}} \right), \quad (F2)$

where we have suppressed the z label and the large-j expansion limit has been considered as usual.

Appendix G: Fractionally 2PN-accurate radiated linear momentum along hyperboliclike orbits (without the 1.5PN tail contribution)

The general expression for the linear momentum flux at the 2PN level of accuracy in terms of the radiative multipole moments is

$\frac{dP_{\text{rad}}^{j}}{dU} = \sum_{i=2}^{\infty} \left[ \frac{G}{c^{2l+3}} \frac{2(l+2)(l+3)}{l(l+1)(2l+3)!!} U_{L}^{(2)}(\nu) U_{L}^{(1)} + \frac{G}{c^{2l+5}} \frac{8(l+3)}{(l+1)(2l+5)!!} U_{L}^{(2)} \right], \quad (G1)$

The 2PN-accurate instantaneous contribution to the above expression reads

$\frac{dP_{\text{rad,inst}}^{j}}{dU} = \frac{G}{c^{j}} \left[ \frac{2}{63} f_{ipq}^{(4)} f_{ij}^{(3)} \nu_{pq} + \frac{16}{45} c_{jipq} f_{i}^{(3)} j_{pq}^{(3)} \right] + \frac{1}{c^{2}} \left[ \frac{4}{63} f_{ipq}^{(4)} f_{ij}^{(3)} \right] + \frac{1}{1134} f_{ipq}^{(5)} f_{i}^{(4)} j_{pq}^{(4)} + \frac{1}{126} \epsilon_{jipq} f_{ipq}^{(4)} f_{ij}^{(4)} \right] + \frac{1}{c^{3}} \left[ \frac{2}{945} f_{ipq}^{(5)} f_{ij}^{(4)} \right] + \frac{1}{14175} \epsilon_{jipq} f_{ipq}^{(5)} f_{ij}^{(4)} \right] . \quad (G2)$
Integrating Eq. (G2) along the hyperbolic orbit we find \( \Delta P_{\text{rad, inst}} = 0 = \Delta P_{\text{rad, inst}} \), and

\[
\Delta P_{\text{rad, inst}} = \nu^2 \sqrt{1 - 4\nu} \left( \frac{\eta^2}{(e_r^2 - 1)e_r a_r} Q_0 + \frac{\nu \eta^4}{(e_r^2 - 1)^2 a_r^2} Q_4 \right),
\]

\( (G3) \)

where

\[
Q_0 = A_1 e_r \arccos \left( -\frac{1}{e_r} \right) + \frac{(e_r^2 - 1)^{1/2}}{e_r} A_0,
\]

\[
Q_2 = B_1 \arccos \left( -\frac{1}{e_r} \right) + (e_r^2 - 1)^{1/2} B_0,
\]

\[
Q_4 = C_3 \arccos^3 \left( -\frac{1}{e_r} \right) + C_2 (e_r^2 - 1)^{1/2} \arccos^2 \left( -\frac{1}{e_r} \right) + C_1 \arccos \left( -\frac{1}{e_r} \right) + C_0 (e_r^2 - 1)^{1/2},
\]

\( (G4) \)

with coefficients

\[
A_0 = \frac{64}{45} e_r^2 + \frac{1502}{45} e_r^4 + \frac{283}{15} e_r^6,
\]

\[
A_1 = \frac{104}{5} + \frac{152}{5} e_r^2 + \frac{37}{5} e_r^4,
\]

\[
B_0 = -\frac{7786}{4725} - \frac{8}{5} \nu + \left( -\frac{1778}{45} \nu - \frac{341864}{945} \right) e_r^2 + \left( -\frac{1768}{45} \nu - \frac{7837243}{18900} \right) e_r^4 + \left( \frac{283}{30} \nu + \frac{5007}{1400} \right) e_r^6,
\]

\[
B_1 = \frac{64}{15} + \left( -\frac{12692}{63} - 24 \nu \right) e_r^2 + \left( -\frac{232}{15} \nu - \frac{7544}{945} \right) e_r^4 + \left( -\frac{54577}{630} - \frac{91}{5} \nu \right) e_r^6 + \left( \frac{1661}{280} - \frac{37}{30} \nu \right) e_r^8,
\]

\[
C_0 = \frac{443180371}{5953500} + \frac{3194}{63} \nu + \left( \frac{4}{45} \nu^2 + \frac{17629369}{9450} \nu + \frac{803259908}{496125} \nu \right) e_r^2 + \left( \frac{146}{45} \nu^2 + \frac{70624283}{18900} \nu + \frac{6582331379}{3175200} \right) e_r^4
\]

\[
+ \left( -\frac{4695130141}{13608000} + \frac{863}{60} \nu^2 + \frac{1104757}{1512} \nu \right) e_r^6 + \left( \frac{229891}{40} \nu^2 + \frac{153691}{840} \nu + \frac{495593}{529200} \right) e_r^8,
\]

\[
C_1 = \frac{54466}{1575} + \frac{40}{3} \nu + \left( \frac{42026}{45} \nu + \frac{4103647}{4050} \nu \right) e_r^2 + \left( \frac{229891}{63} \nu + \frac{8}{5} \nu^2 + \frac{6491512}{2835} \nu \right) e_r^4
\]

\[
+ \left( \frac{10538077}{45360} + \frac{34489}{20} \nu + \frac{53}{5} \nu^2 \right) e_r^6 + \left( \frac{353}{3} \nu^2 - \frac{3136361}{20160} \nu + \frac{63887}{1260} \nu \right) e_r^8 + \left( -\frac{8609}{1680} \nu + \frac{8359}{5040} \nu + \frac{37}{40} \nu^2 \right) e_r^{10},
\]

\( (G5) \)

\[
P_{\text{rad, inst}} = \nu^2 (m_2 - m_1) \left[ P_{\text{rad, N}} + \nu^2 P_{\text{rad, inst}} + \nu^4 P_{\text{rad, 2PN}} \right] e_y ,
\]

\( (G6) \)

where

\[
P_{\text{rad, N}} = \frac{37}{30} \nu^5 \frac{p_{\infty}^5}{j^3} + \frac{64}{3} \nu^4 \frac{p_{\infty}^4}{j^4} + \frac{1097}{60} \nu \frac{p_{\infty}^3}{j^5}
\]

\[
+ \frac{4384}{45} \nu^2 \frac{p_{\infty}^2}{j^6} + \frac{2841}{80} \nu \frac{p_{\infty}}{j^7} + O \left( \frac{1}{j^8} \right),
\]

\( (G7) \)

The large-\( j \) expansion of this result yields

\[
P_{\text{rad, 1PN}} = \left( \frac{839}{1680} + \frac{37}{15} \right) \pi \frac{p_{\infty}^7}{j^3}
\]

\[
+ \left( \frac{1664}{175} - \frac{160}{3} \right) \pi \frac{p_{\infty}^5}{j^4}
\]

\[
+ \left( \frac{148507}{10080} - \frac{3529}{60} \nu \right) \pi \frac{p_{\infty}^3}{j^5}
\]

\[
+ \left( \frac{813248}{4725} - \frac{18896}{45} \nu \right) \pi \frac{p_{\infty}^1}{j^6}
\]

\[
+ \left( \frac{5666883}{40320} - \frac{4373}{20} \nu \right) \pi \frac{p_{\infty}^0}{j^7},
\]

\( (G8) \)
and

\[ P_{\text{rad,2PN}}^y = \left( \frac{2699}{2016} - \frac{107}{280} \nu + \frac{37}{10} \nu^2 \right) \frac{\pi p_0^3}{j^4} + \left( \frac{22777}{33075} - \frac{1096}{105} \nu + \frac{280}{3} \nu^2 \right) \frac{p_\infty^8}{j^4} + \left( -\frac{1131443}{40320} - \frac{55009}{2016} \nu + \frac{608}{3} \nu^2 \right) \frac{\pi p_\infty^7}{j^4} + \left( -\frac{115852624}{212625} - \frac{2269324}{4725} \nu + \frac{15652}{15} \nu^2 \right) \frac{p_\infty^6}{j^6} + \left( -\frac{178354019}{362880} - \frac{37}{80} \pi^2 \nu \right) + \left( -\frac{5607509}{10080} + \frac{26789}{40} \nu^2 \right) \frac{\pi p_\infty^5}{j^4} + O \left( \frac{1}{j^8} \right). \] (G9)

Performing the integration in the large-\( j \) expansion limit we find that the only nonvanishing component is along the \( y \) axis (\( P_{\text{sym tail}}^x \equiv 0 = P_{\text{sym tail}}^z \)):

\[ P_{\text{sym tail}}^y = \frac{G^2 M}{c^{10}} \left[ \frac{2i}{63} \int_0^\infty d\omega \omega^9 \left[ I_{ijk}(-\omega)I_{jk}(\omega) - I_{ijk}(\omega)I_{jk}(-\omega) \right] + \frac{16}{45} \int_0^\infty d\omega \omega^7 \left[ I_{jkl}(-\omega)J_{kl}(-\omega) + I_{jkl}(\omega)J_{kl}(\omega) \right] \right]. \] (H1)

**Appendix H: Fractionally 1.5PN tail contribution to the radiated linear momentum along hyperbolicike orbits**

Contrary to what happened for the tail contributions to the energy and the angular momentum (for which the integration automatically projected the hereditary past tail into its time-symmetric projection), a subtlety arises for the tail contribution to the radiated linear momentum.

The time-symmetric tail contribution to the radiated linear momentum (which, in view of the radiation-reaction derivation of energy loss presented above, should be the only relevant quantity) has the following Fourier-domain expression (up to linear momentum no longer vanishes (because of the dis-symmetry between the scales \( C_{J_3} \) and \( C_{J_2} \) entering the tail logarithms). Its value is equal to

\[ P_{\text{past tail}}^x = -\nu^2 \sqrt{1 - 4\nu} \left[ \frac{1491}{400} \frac{p_\infty^7}{j^4} + \frac{20608}{225} \frac{p_\infty^6}{j^6} \right] + \frac{267583}{2400} \frac{p_\infty^5}{j^6} + \frac{64576}{75} \frac{p_\infty^4}{j^7} + O \left( \frac{1}{j^9} \right). \] (H3)

This past-tail contribution is along the same direction as the LO impulse, and therefore conservative-like. The reasoning (coming from \[ ^6 \] recalled in Appendix \[ ^6 \]) indicates that such a contribution (which is at the absolute 5PN level) is already included in the time-symmetric conservative dynamics and should not be considered among the radiation-reaction effects.

**Appendix I: PN evaluation of the impulse coefficients and the radiative losses and reminders on the PN-expanded scattering angle (conservative and radiative)**

We give below, in the form of various tables, the fractionally 2PN-expanded form of the various impulse coefficients \( c_b^X, c_u^X, c_{u_r}^X \) and \( c_{u_r}^X \), with \( X = \text{cons}, \text{rr rel}, \text{rr rec} \), entering the decomposition of \( \Delta p_\mu^i \), up to 3PM.
TABLE V: PN expansion of the various coefficients of \( c_{\text{cons}} \), \( c_{\text{rr,rel}} \), \( c_{\text{rr,rec}} \) (\( \frac{GM}{b} \))^n, with \( n \leq 3 \).

\[
G^3 \\
| \text{cons} | \frac{-\frac{2}{p_\infty}}{p_\infty} - 4p_\infty \\
| \text{rr, rel} | - \\
| \text{rr, rec} | - \\
\]

\[
G^2 \\
| \text{cons} | \pi \left( -\frac{3}{p_\infty} - \frac{15}{4} p_\infty \right) \\
| \text{rr, rel} | - \\
| \text{rr, rec} | - \\
\]

\[
G^3 \\
| \text{cons} | \frac{2}{p_\infty} + \frac{2n}{p_\infty} + \frac{\left( \frac{12}{p_\infty} + 32 \right)}{p_\infty} + O(p_\infty) \\
| \text{rr, rel} | \nu \left( -\frac{12}{p_\infty} - \frac{27}{p_\infty} - \frac{112}{27} p_\infty + O(p_\infty) \right) \\
| \text{rr, rec} | - \\
\]

TABLE VI: PN expansion of the various coefficients of \( c_{\text{cons}} \), \( c_{\text{rr,rel}} \), \( c_{\text{rr,rec}} \) (\( \frac{GM}{b} \))^n, with \( n \leq 3 \).

\[
G^2 \\
| \text{cons} | -\frac{\frac{6}{p_\infty}}{p_\infty} + \left( \frac{15}{p_\infty} + \frac{\Delta}{p_\infty} \right) - 2p_\infty + \frac{9}{2} \Delta + O(p_\infty^2) \\
| \text{rr, rel} | - \\
| \text{rr, rec} | - \\
\]

\[
G^3 \\
| \text{cons} | \pi \nu \left\{ \frac{-\frac{6}{p_\infty}}{p_\infty} + \frac{\left( \frac{12}{p_\infty} + 21 \right)}{p_\infty} - \frac{37}{8} + \frac{9}{2} \Delta + O(p_\infty^2) \right\} \\
| \text{rr, rel} | - \pi \nu \left[ \frac{15}{p_\infty} \nu - \frac{37}{8} \Delta + \frac{37}{8} \nu \right] + \left( \frac{125}{28} \nu^2 + \frac{253}{8} \nu \nu \Delta + \frac{253}{8} \nu^2 + O(p_\infty^3) \right) + O(p_\infty^5) \\
| \text{rr, rec} | - \pi \nu \left[ \frac{15}{p_\infty} \nu - \frac{37}{8} \Delta + \frac{37}{8} \nu \right] + \left( \frac{125}{28} \nu^2 + \frac{253}{8} \nu \nu \Delta + \frac{253}{8} \nu^2 + O(p_\infty^3) \right) + O(p_\infty^5) \\
| \text{rr, tot} | - \pi \nu \left[ \frac{15}{p_\infty} \nu - \frac{37}{8} \Delta + \frac{37}{8} \nu \right] + \left( \frac{125}{28} \nu^2 + \frac{253}{8} \nu \nu \Delta + \frac{253}{8} \nu^2 + O(p_\infty^3) \right) + O(p_\infty^5) \\
\]

order (see Table VII and VIII respectively). This is completed by writing down some useful PN relations concerning the conservative and radiative parts of the scattering angle (see Table X and XI respectively) as well as the coefficients of the PN expansion of the energy and momentum radiative losses (see Table XIII).

The PN expansion of \( c_{\text{cons}}(p_\infty \nu) \) is known at present up to the 6PN level \( \left[ \frac{37}{32}, \frac{37}{60}, \frac{125}{81}, \frac{37}{8}, \frac{839}{81} \right] \). However, since the PN knowledge of the radiated energy and angular momentum is only at the fractional 2PN level, it is enough to use \( \chi_{\text{cons}} \) (which starts at the Newtonian level) with an absolute 2PN accuracy. Below \( \eta = \frac{1}{6} \) is a place-holder for keeping track of (absolute or relative) PN expansions. Beware that most of the time we use the phrase 2PN accuracy in a fractional sense. One should keep in mind that radiation-reaction effects start at the 2.5PN level, i.e., \( \eta^2 \).

TABLE VII: PN expansion of the various coefficients of \( c_{\text{cons}} \), \( c_{\text{rr,rel}} \), \( c_{\text{rr,rec}} \) (\( \frac{GM}{b} \))^n, with \( n \leq 3 \).

\[
G^2 \\
| \text{cons} | \frac{2}{p_\infty} + \frac{\left( \frac{12}{p_\infty} + \frac{15}{4} \Delta \right)}{p_\infty} + \frac{79}{8} + \frac{15}{8} \Delta + O(p_\infty^2) \\
| \text{rr, rel} | - \\
| \text{rr, rec} | - \\
\]

\[
G^3 \\
| \text{cons} | \pi \nu \left[ \frac{6}{p_\infty} + \frac{\left( \frac{12}{p_\infty} + 21 \right)}{p_\infty} - \frac{37}{8} + \frac{9}{2} \Delta + O(p_\infty^2) \right] \\
| \text{rr, rel} | - \pi \nu \left[ \frac{15}{p_\infty} \nu - \frac{37}{8} \nu \Delta + \frac{37}{8} \nu \right] + \left( \frac{125}{28} \nu^2 + \frac{253}{8} \nu \nu \Delta + \frac{253}{8} \nu^2 + O(p_\infty^3) \right) + O(p_\infty^5) \\
| \text{rr, rec} | - \pi \nu \left[ \frac{15}{p_\infty} \nu - \frac{37}{8} \nu \Delta + \frac{37}{8} \nu \right] + \left( \frac{125}{28} \nu^2 + \frac{253}{8} \nu \nu \Delta + \frac{253}{8} \nu^2 + O(p_\infty^3) \right) + O(p_\infty^5) \\
| \text{rr, tot} | - \pi \nu \left[ \frac{15}{p_\infty} \nu - \frac{37}{8} \nu \Delta + \frac{37}{8} \nu \right] + \left( \frac{125}{28} \nu^2 + \frac{253}{8} \nu \nu \Delta + \frac{253}{8} \nu^2 + O(p_\infty^3) \right) + O(p_\infty^5) \\
\]
TABLE VIII: PN expansion of the various coefficients of $\chi^2_{n+1} = \mu \sum_{n=2}^{\infty} \left[ C_{\mu_1,\mu_2}^{(\text{cons})} + C_{\mu_1,\mu_2}^{(\text{rel})} + C_{\mu_1,\mu_2}^{(\text{rec})} \right] \left( \frac{GM}{r^3} \right)^n$, with $n \leq 3$.

| $G^2$ | cons | $1 + \Delta \left( \frac{1}{p^{\infty}} + 2p^{\infty} \right)^2$ |
|-------|------|--------------------------------------------------|
| rr, rel | -    | -                                                |
| rr, rec | -    | -                                                |

| $G^3$ | cons | $1 + \Delta \left( \frac{1}{p^{\infty}} + \frac{4}{p^2} \right)$ |
|-------|------|---------------------------------------------------------------|
| rr, rel | -    | 0                                                            |
| rr, rec | -    | 0                                                            |

TABLE IX: Fractionally 2PN-accurate expansion of the coefficients $E_\nu$, $J_\nu$ and $P_\nu$ of the PM expansions of the energy, angular momentum and linear momentum radiative losses, Eqs. (B7) and (B14), up to $n = 7$.

| $E_3$ | $\pi \left( \frac{37}{64} + \frac{157}{240} \right) + \left( \frac{1357}{840} - \frac{216}{2700} \right) p^{\infty} + \left( \frac{37}{64} + \frac{157}{240} \right) p^{\infty} + O(p^{10})$ |
|-------|---------------------------------------------------------------|
| $E_4$ | $\pi \left( \frac{37}{64} + \frac{157}{240} \right) + \left( \frac{1357}{840} - \frac{216}{2700} \right) p^{\infty} + \left( \frac{37}{64} + \frac{157}{240} \right) p^{\infty} + O(p^{10})$ |
| $E_5$ | $\pi \left( \frac{37}{64} + \frac{157}{240} \right) + \left( \frac{1357}{840} - \frac{216}{2700} \right) p^{\infty} + \left( \frac{37}{64} + \frac{157}{240} \right) p^{\infty} + O(p^{10})$ |
| $E_6$ | $\pi \left( \frac{37}{64} + \frac{157}{240} \right) + \left( \frac{1357}{840} - \frac{216}{2700} \right) p^{\infty} + \left( \frac{37}{64} + \frac{157}{240} \right) p^{\infty} + O(p^{10})$ |
| $E_7$ | $\pi \left( \frac{37}{64} + \frac{157}{240} \right) + \left( \frac{1357}{840} - \frac{216}{2700} \right) p^{\infty} + \left( \frac{37}{64} + \frac{157}{240} \right) p^{\infty} + O(p^{10})$ |

TABLE X: Expansion coefficients of $\hat{\chi}_{\nu}^{\text{cons}} = \sum_{n>2} \chi_{\nu}^{\text{cons}} p^{n}$ up to $n = 7$, evaluated at the (absolute) 2PN level of accuracy.

| $\chi_{\nu}^{\text{cons}}$ | $\chi_{\nu}^{\text{cons}}$ | $\chi_{\nu}^{\text{cons}}$ | $\chi_{\nu}^{\text{cons}}$ | $\chi_{\nu}^{\text{cons}}$ | $\chi_{\nu}^{\text{cons}}$ | $\chi_{\nu}^{\text{cons}}$ | $\chi_{\nu}^{\text{cons}}$ | $\chi_{\nu}^{\text{cons}}$ |
|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|---------------------------|
| $\chi_{\nu}^{\text{cons}}$ | $\frac{1}{p^{\infty}} + 2p^{\infty}\eta^2$ | $\pi \left( \frac{4}{p^{\infty}} + \frac{8}{p^{\infty}} \right)$ | $-\frac{1}{p^{\infty}} + 4p^{\infty}\eta^2$ | $(-8\nu + 24) p^{\infty}\eta^4$ | $\pi - \frac{4}{p^{\infty}} \frac{4\nu}{p^{\infty}}$ | $\frac{1}{p^{\infty}} - \frac{8}{p^{\infty}} \eta^2 + (32 - 8\nu) \frac{\eta^4}{p^{\infty}}$ | $0$ | $-\frac{1}{p^{\infty}} + \frac{8}{p^{\infty}} \eta^2 - (16 - 16\nu) \frac{\eta^4}{p^{\infty}}$ |

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TABLE XI: Expansion coefficients of $\frac{\chi_{\nu}^{rel}}{2} = \sum_{n \geq 3} \frac{\chi_{n}^{\nu}}{2}$ up to $n = 7$, evaluated at the fractional 2PN level of accuracy.

| $\chi_d^{\nu}$ | $\frac{1}{2} \left( 211 \nu^2 + \left( \frac{40}{9} - \frac{10}{9} \nu^2 \right) \eta \nu^2 + \left( \frac{256}{63} \nu^2 - \frac{19}{9} \nu^4 \right) \eta^2 \right) \nu^4$ |
|-----------------|--------------------------------------------------------------------------------------|
| $\chi_4^{\nu}$ | $\frac{1}{2} \left( 27253 \nu^2 + \left( \frac{48}{49} - \frac{11}{49} \nu^2 \right) \eta \nu^2 + \left( \frac{27217}{4} \nu^2 - \frac{11025}{4} \nu^4 \right) \eta^2 \right) \nu^4$ |
| $\chi_5^{\nu}$ | $\frac{1}{2} \left( 1449 \nu^2 + \left( \frac{3360}{11} - \frac{428}{11} \nu^2 \right) \eta \nu^2 + \left( \frac{2715}{8} \nu^2 - \frac{5895}{8} \nu^4 \right) \eta^2 \right) \nu^4$ |
| $\chi_6^{\nu}$ | $\frac{1}{2} \left( 45 \nu^2 + \left( \frac{90}{7} - \frac{18}{7} \nu^2 \right) \eta \nu^2 + \left( \frac{525}{8} \nu^2 - \frac{1260}{8} \nu^4 \right) \eta^2 \right) \nu^4$ |

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