INDEPENDENCE COMPLEXES OF HYPERGRAPHS AND BOUNDED DEGREE COMPLEXES

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Abstract. The bounded degree complex $BD(G, \lambda)$ is a generalization of the matching complexes of a graph. In this paper, we show that the bounded degree complex of a forest is shellable, by using independence complexes of hypergraphs. We obtain a wedge decomposition result of bounded degree complexes when a graph $G$ has a leaf.

1. Introduction

Let $G$ be a simple graph and $\lambda: V(G) \to \mathbb{Z}_+$ a function. Here $\mathbb{Z}_+$ denotes the set of non-negative integers. The bounded degree complex $BD(G, \lambda)$ of $G$ with respect to $\lambda$ is the simplicial complex defined as follows: The underlying set of $BD(G, \lambda)$ is the edge set $E(G)$ of $G$, and the simplices are the subgraphs $H$ of $G$ such that the degree of $v$ in $H$ is not larger than $\lambda(v)$ for each vertex $v$ in $G$.

Recall that the matching complex $M(G)$ of a graph $G$ is the simplicial complex whose vertex set is $E(G)$ and whose simplices are matchings of $G$. The matching complex $M(G)$ is the bounded degree complex $BD(G, \lambda)$ in the case $\lambda$ is the constant function at 1. Matching complexes have been studied by several authors (see [4], [8], [10], and [13]), and the bounded degree complex is a natural generalization of it (see [6] and [11]). For a more comprehensive introduction to this subject, we refer to [8] and [13].

Marietti and Testa [10] showed that the matching complex of a forest is homotopy equivalent to a wedge of spheres, and Singh [12] recently generalize their result:

Theorem 1.1 (Singh [12]). Every bounded degree complex of a forest is homotopy equivalent to a wedge of spheres.

The purpose in this paper is to strengthen Singh’s result. Recall that a shellable simplicial complex is homotopy equivalent to a wedge of spheres, the following theorem implies Theorem 1.1.

Theorem 1.2. The bounded degree complex of a forest is shellable.

Recall that the vertex decomposability of simplicial complexes is a stronger condition than the shellability. In fact, the matching complex of a forest is vertex decomposable.

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Our method to deduce Theorem 1.2 is to regard the bounded degree complex as an independence complex of a hypergraph  \( \mathcal{L}(G, \lambda) \). Then we see that the hypergraph  \( \mathcal{L}(G, \lambda) \) is chordal in the sense of Woodroofe [15]. The main result of [15] asserts that the independence complex of a chordal hypergraph is shellable, so the bounded degree complex  \( BD(G, \lambda) = I(\mathcal{L}(G, \lambda)) \) of a forest \( G \) is shellable.

We also obtains a following wedge decomposition when the graph \( G \) has a leaf:

**Theorem 1.3.** Let \( G \) be a simple graph, \( e \) a leaf \( \{v, w\} \) with \( \text{deg}(v) = 1 \). Suppose that \( e \) is incident to another edge \( f = \{w, u\} \), and that \( \lambda(u), \lambda(v), \) and \( \lambda(w) \) are non-zero. Then there is a homotopy equivalence

\[
BD(G, \lambda) \simeq BD(G - f, \lambda) \lor \Sigma BD(G - f, \lambda_f).
\]

Here \( \lambda_f : V(G) \rightarrow \mathbb{Z}_+ \) is a function defined as follows:

\[
\lambda_f(x) = \begin{cases} 
\lambda(x) - 1 & (x \in f) \\
\lambda(x) & (x \notin f). 
\end{cases}
\]

Iterating this, we have a finer wedge decomposition of the bounded degree complex of a graph having a leaf (see Theorem 4.1). We will show that Theorem 1.3 yields the decomposition result (Theorem 4.2) by Singh that he used to prove Theorem 1.1.

This paper is organized as follows. In Section 2, we fix notations and terminology concerning independence complexes of hypergraphs. In Section 3, we regard the bounded degree complex \( BD(G, \lambda) \) as the independence complex of a certain hypergraph \( \mathcal{L}(G, \lambda) \), and show Theorem 1.2. In Section 3, we show Theorem 1.3 and that Theorem 1.3 yields a wedge decomposition result by Singh [12].

2. Independence complexes of hypergraphs

In this section, we recall some terminology and facts of independence complexes of hypergraphs.

A hypergraph is a pair \( \mathcal{H} = (X, E) \) consisting of a set \( X \) together with a subset \( E \) of \( 2^X \). We call \( X = V(\mathcal{H}) \) the vertex set and \( E = E(\mathcal{H}) \) the edge set. Throughout this paper, we assume that \( V(\mathcal{H}) \) is finite and the notation \( \mathcal{H} \) indicates a hypergraph.

A subset \( \sigma \) of \( X \) is independent if there is no edge \( \alpha \) of \( \mathcal{H} \) contained in \( \sigma \). The independent sets in \( \mathcal{H} \) form an abstract simplicial complex, and we call it the independence complex \( I(\mathcal{H}) \) of \( \mathcal{H} \). Independence complexes of hypergraphs have been considered in several references (see [7] and [14]). We call a vertex \( v \) in \( \mathcal{H} \) looped if \( \{v\} \) is an edge in \( \mathcal{H} \). The vertex set of \( I(\mathcal{H}) \) is the set of non-looped vertices in \( \mathcal{H} \).

**Remark 2.1.** Here we give a few remarks concerning \( I(\mathcal{H}) \).
yields many important results concerning independence complexes of simple graphs.

(2) Let \( \alpha \) be an edge in \( \mathcal{H} \). If there is \( \beta \in E \) such that \( \beta \subseteq \alpha \), then \( I(\mathcal{H}) = I(\mathcal{H} - \alpha) \).

Here \( \mathcal{H} - \alpha \) is the hypergraph \( \mathcal{H} = (V, E - \{\alpha\}) \). Thus if \( \mathcal{H}' = (X, E') \) is the hypergraph where \( E' \) is the minimal elements in \( E \), then implies \( I(\mathcal{H}) = I(\mathcal{H}') \).

(3) \( I(\mathcal{H}_1 \sqcup \mathcal{H}_2) = I(\mathcal{H}_1) * I(\mathcal{H}_2) \)

Let \( v \) be a vertex in \( \mathcal{H} = (X, E) \). We define the deletion \( \mathcal{H} \setminus v \) and contraction \( \mathcal{H}/v \) at \( v \) as follows:

- \( V(\mathcal{H} \setminus v) = X - \{v\} \) and \( E(\mathcal{H} \setminus v) = \{e \in E \mid v \notin e\} \).
- \( V(\mathcal{H}/v) = X - \{v\} \) and \( E(\mathcal{H}/v) = \{e - \{v\} \in E \mid e \in E\} \).

Then the following holds:

**Proposition 2.2** (see Woodroofe [14]). Let \( v \) be a non-looped vertex in \( \mathcal{H} \). Then \( I(\mathcal{H}) \setminus v = I(\mathcal{H} \setminus v) \) and \( \text{link}_{I(\mathcal{H})}(v) = I(\mathcal{H}/v) \).

In the case of a simple graph \( G \), this proposition says that \( I(G) \) is the mapping cylinder of the inclusion \( I(G \setminus N[v]) \hookrightarrow I(G \setminus v) \). Here \( N(v) \) is the set \( \{w \in V(G) \mid \{v, w\} \in E(G)\} \) and \( N[v] \) is the set \( N[v] = N(v) \cup \{v\} \). Adamaszek [1] mentioned that this observation yields many important results concerning independence complexes of simple graphs.

The following corollary will be used in the proof of the wedge decomposition of bounded degree complexes:

**Corollary 2.3.** Let \( v \) be a non-looped vertex in \( \mathcal{H} \). Then \( I(\mathcal{H}) \) is the unreduced mapping cone of the inclusion \( I(\mathcal{H}/v) \hookrightarrow I(\mathcal{H} \setminus v) \). Thus if the inclusion \( I(\mathcal{H}/v) \hookrightarrow I(\mathcal{H} \setminus v) \) is homotopic to the constant map at \( w \in V(\mathcal{H} \setminus v) \), then there is a homotopy equivalence

\[
I(\mathcal{H}) \simeq I(\mathcal{H} \setminus v) \vee_w \Sigma I(\mathcal{H}/v).
\]

Here \( \Sigma \) denotes the suspension. We consider that \( S^{-1} = \emptyset \) and \( \Sigma \emptyset = S^0 \). The notation \( \vee_w \) implies that we consider the basepoint of \( I(\mathcal{H} \setminus v) \) is \( w \). Note that \( \Sigma I(\mathcal{H}/v) \) is connected or \( S^0 \), it is not necessary to mention its basepoint as far as we discuss homotopy types.

### 3. Bounded degree complexes

In this section, we regard bounded degree complexes as independence complexes of hypergraphs, and prove Theorem [1.2]

Let \( G \) be a simple graph and \( \lambda: V(G) \to \mathbb{Z}_+ \) a function. For a vertex \( v \) in \( G \), we write \( E_v \) to mean the set \( \{e \in E(G) \mid v \in e\} \). Then a subset \( \sigma \) of \( E(G) \) is a simplex of \( BD(G, \lambda) \) if and only if \( |E_v \cap \sigma| \leq \lambda(v) \) for each \( v \in V(G) \).

Let \( v \) be a vertex in \( G \) and suppose \( \lambda(v) = 0 \). Then \( BD(G, \lambda) = BD(G - v, \lambda|V(G) - \{v\}) \).

Here \( G - v \) is the subgraph of \( G \) induced by \( V(G) - \{v\} \).

Define the hypergraph \( \mathcal{L}(G, \lambda) \) as follows:
Lemma 3.2. The bounded degree complex $BD(G, \lambda)$ is the independence complex of $\mathcal{L}(G, \lambda)$.

Proof. Let $\sigma$ be a subset of $E(G)$. Then $\sigma$ is an independent set in $\mathcal{L}(G, \lambda)$ if and only if $|E_v \cap \sigma| \leq \lambda(v)$ for every $v \in V(G)$, which means $\sigma \in BD(G, \lambda)$.

Remark 3.1. Suppose that $\lambda(v) > 0$ for every $v \in V(G)$. Then for each $\alpha \in \mathcal{L}(G, \lambda)$, there is only one vertex $v \in V(G)$ such that $\alpha \subset E_v$. In fact, if there are two distinct vertices $v$ and $w$ with $\alpha \subset E_v \cap E_w$, then we have $|\alpha| \leq 1$ since $E_v \cap E_w = \{\{v, w\}\}$. This is a contradiction since there is $x \in V(G)$ such that $|\alpha| = \lambda(x) + 1 > 1$.

Theorem 3.3 (Woodroofe [15]). If $\mathcal{H}$ is a chordal hypergraph, then $I(\mathcal{H})$ is shellable.

Thus to prove Theorem 1.2, it suffices to show that the hypergraph $\mathcal{L}(G, \lambda)$ is chordal.

Lemma 3.4. Let $e = \{v, w\} \in E(G)$ be a leaf of a simple graph $G$, and suppose that both $\lambda(v)$ and $\lambda(w)$ are non-zero. Then $e$ is a simplicial vertex of $\mathcal{L}(G, \lambda)$.

Proof. Let $\sigma_1, \sigma_2 \in E(\mathcal{L}(G, \lambda))$ satisfying $e \in \sigma_1 \cap \sigma_2$ and $\sigma_1 \neq \sigma_2$. It follows from $\lambda(v), \lambda(w) > 0$, we have that $\sigma_1, \sigma_2 \subset E_w$ and $|\sigma_1| = |\sigma_2| = \lambda(w)$. This means that
there is $\sigma_3$ satisfying $\sigma_3 \subset \sigma_1 \cup \sigma_2 - \{e\} \subset E_w$, and $|\sigma_3| = \lambda(w)$. This completes the proof. \qed

For an edge $e$ in a simple graph $G$, let $G - e$ denote the graph $(V(G), E(G) - \{e\})$.

**Lemma 3.5.** Let $e = \{v, w\}$ be an edge in $G$ and suppose that both $\lambda(v)$ and $\lambda(w)$ are non-zero. Then the following hold:

1. $\mathcal{L}(G, \lambda) \setminus e = \mathcal{L}(G - e, \lambda)$
2. The minimal edges of $\mathcal{L}(G)/e$ coincide with those of $\mathcal{L}(G - e, \lambda_e)$.

Here $\lambda_e : V(G) \to \mathbb{Z}_+$ is a function defined by

$$
\lambda_e(x) = \begin{cases} 
\lambda(x) - 1 & (x \in e) \\
\lambda(x) & (x \notin e).
\end{cases}
$$

**Proof.** Since $\mathcal{L}(B, \lambda) \setminus e = \mathcal{L}(G - e, \lambda)$, the former is trivial.

Let $\sigma \in \mathcal{L}(G - e, \lambda_e)$. Then there is $x \in V(G)$ such that $\sigma \subset E_x$ and $|\sigma| = \lambda_e(x) + 1$. If $e \notin \sigma$, then $\sigma \subset E_v$ or $\sigma \subset E_w$, and hence we have $x \in e$. Therefore $x \notin e$ implies $e \notin \sigma$ and $|\sigma| = \lambda(x) + 1$. Thus we have $\sigma \in \mathcal{L}(G, \lambda)$.

Suppose $x \in e$. Then $|\sigma| = \lambda(x)$. Since $\sigma \cup \{e\} \subset E_x$ and $|\sigma \cup \{e\}| = \lambda(x) + 1$, we have $\sigma \cup \{e\} \in \mathcal{L}(G, \lambda)$. Thus we have $\sigma \in \mathcal{L}(G, \lambda)/e$. Thus we have $E(\mathcal{L}(G - e, \lambda_e)) \subset E(\mathcal{L}(G, \lambda)/e)$.

To complete the proof of (2), it suffices to show that every edge in $E(\mathcal{L}(G, \lambda)/e)$ contains an edge in $E(\mathcal{L}(G - e, \lambda_e))$. Let $\sigma \in E(\mathcal{L}(G, \lambda)/e)$. Then there is $\sigma' \in \mathcal{L}(G, \lambda)$ such that $\sigma = \sigma' - \{e\}$. Since $\sigma' \in \mathcal{L}(G, \lambda)$, there is $x \in V(G)$ such that $\sigma' \subset E_x$ and $|\sigma'| = \lambda(x) + 1$. Then we consider the two cases:

1. Suppose $e \in \sigma'$. Then $x \in e$ and $\sigma' - \{e\} \subset E_x$. Since $|\sigma' - \{e\}| = \lambda(x) = \lambda_e(x) + 1$, and hence we have $\sigma \in \mathcal{L}(G - e, \lambda_e)$.
2. Suppose $e \notin \sigma'$. Then $|\sigma'| = \lambda(x) + 1 \geq \lambda_e(x) + 1$, and hence there is a subset $\sigma''$ of $\sigma'$ such that $|\sigma''| = \lambda_e(x) + 1$. Since $\sigma'' \subset \sigma' \subset E_x$, we have $\sigma'' \in \mathcal{L}(G - e, \lambda_e)$.

By (1) and (2), every edge in $\mathcal{L}(G, \lambda)/e$ contains an edge in $\mathcal{L}(G - e, \lambda_e)$. This means that an independent set in $\mathcal{L}(G - e, \lambda_e)$ is independent in $\mathcal{L}(G, \lambda)/e$. \qed

**Proof of Theorem 1.2.** We prove by the induction by the number of edges in $G$. Suppose that $\mathcal{L}(G, \lambda)$ is non-empty and let $e = \{v, w\}$ be a vertex in $\mathcal{L}(G, \lambda)$ (i.e. an edge in $G$ such that $\lambda(v)$ and $\lambda(w)$ are non-zero). It suffices to show that $\mathcal{L}(G, \lambda)/e$ and $\mathcal{L}(G, \lambda) \setminus e$ are chordal. We now show that $\mathcal{L}(G, \lambda)/e$ is chordal. Since the minimal edges in $\mathcal{L}(G, \lambda)/e$ and $\mathcal{L}(G - e, \lambda_e)$ coincide (Lemma 3.5), $\mathcal{L}(G, \lambda)/e$ is chordal if and only if $\mathcal{L}(G - e, \lambda_e)$ is.

The inductive hypothesis implies that $\mathcal{L}(G - e, \lambda_e)$ is chordal, so we have that $\mathcal{L}(G, \lambda)/e$ is chordal. The case of $\mathcal{L}(G, \lambda) \setminus e$ is similar. This completes the proof. \qed
Remark 3.6. The vertex decomposability of a simplicial complex is a stronger condition than the shellability. By the above observation we have that the matching complex of a forest is vertex decomposable. In fact, if $\lambda$ is the constant function at 1, then $L(G, \lambda)$ is a simple graph $L(G)$, which is called the line graph, and $M(G) = I(L(G))$. Then Lemma 3.5 implies that $L(G)$ is chordal. Of course, it is easy to check this directly from the definition. Since the independence complex of a chordal graph is vertex decomposable (see [5] and [14]), The matching complex $M(G) = I(L(G))$ is vertex decomposable.

4. Wedge decomposition of bounded degree complexes

In this section, we prove Theorem 1.3 and that Theorem 1.3 yields a wedge decomposition result shown by Singh [12].

Proof of Theorem 1.3. It suffices to show that the inclusion $BD(G - f, \lambda_f) \hookrightarrow BD(G - f, \lambda)$ is null-homotopic. To see this, it suffices to show that $\sigma \in BD(G - f, \lambda_f)$ implies $\sigma \cup \{e\} \in BD(G - f, \lambda)$. In fact, this assertion means that $BD(G - f, \lambda_e)$ is contained in the star at $e$ in $BD(G - f, \lambda)$, and hence the inclusion $BD(G - f, \lambda_e) \hookrightarrow BD(G - f, \lambda)$ factors through a contractible space.

Let $\sigma \in BD(G - f, \lambda_f)$. We want to show that $|(\sigma \cup e) \cap E_x| \leq \lambda(x)$ for every $x \in V(G)$. If $x = u, v, w$, then $|(\sigma \cup \{e\}) \cap E_x| = |\sigma \cap E_x| \leq \lambda_f(x) = \lambda(x)$. If $x = u, w$, then $|(\sigma \cup \{e\}) \cap E_x| \leq |\sigma \cap E_x| + 1 \leq \lambda_f(x) + 1 = \lambda(x)$. If $x = v$, then $|E_x| = 1$ and hence $|(\sigma \cup \{e\}) \cap E_x| \leq 1 \leq \lambda(x)$. Thus we have shown that $\sigma \cup \{e\} \in BD(G - f, \lambda)$. This completes the proof. □

Iterating Theorem 1.2, we have a finer wedge decomposition of a bounded degree complex having a leaf.

Theorem 4.1. Let $G$ be a simple graph having a leaf $e = \{v, w\}$ with $\text{deg}(v) = 1$. Suppose that $\lambda$ is non-zero at every vertex contained in $N[w]$. Put $N(w) = \{v, u_1, \cdots, u_n\}$ and suppose that $\lambda$ is non-zero at every vertex contained in $N[w]$. Let $G_e$ be the graph obtained by deleting the vertices $v$ and $w$ and edges incident to $w$. Then there is a homotopy equivalence

$$BD(G, \lambda) \simeq \bigvee_{T \subset N(w) - \{v\}, |T| = \lambda(w)} \sum_{T} \lambda^w(BD(G_e, \lambda_T)).$$

Here $\lambda_T : V(G_e) \to \mathbb{Z}_+$ is defined by

$$\lambda_T(v) = \begin{cases} 
\lambda(x) - 1 & (x \in T) \\
\lambda(x) & \text{(otherwise)} 
\end{cases}$$
Proof. Put $e_i = \{u_i, w\}$. We show this theorem by the induction of $n$. In the case $\lambda(w) = 1$, then there is a homotopy equivalence

$$BD(G, \lambda) \simeq BD(G - e_1, \lambda) \vee \Sigma BD(G - e_1, \lambda_{e_1}).$$

Since $L(G - e_1, \lambda)$ contains an exposed vertex $e_1$, we have that $BD(G - e_1, \lambda) \simeq \ast$. Thus this theorem holds for $n = 1$. In the case of $n \geq 2$, there is a homotopy equivalence

$$BD(G, \lambda) \simeq BD(G - e_n, \lambda) \vee \Sigma BD(G - e_n, \lambda_{e_n}).$$

Applying the inductive hypothesis to $BD(G - e_n, \lambda)$ and $\Sigma BD(G - e_n, \lambda_{e_n})$, we have the desired homotopy equivalence. □

In the rest of this section, we compare Theorem 1.2 with Singh's result. We call a vertex $v$ in a simple graph $G$ an interior point if the degree of $v$ is greater than 1. A corner point is an interior point $v$ which is adjacent to only one interior point. Singh [12] showed that $BD(G, \lambda)$ has a wedge decomposition of suspensions of bounded degree complexes if $G$ has a corner point. Namely, Singh [12] shows the following theorem:

**Theorem 4.2** (Singh [12]). Let $G$ be a graph, $v$ an interior vertex, and $\lambda: V(G) \to \mathbb{Z}_+$ a function such that $\lambda(x) \neq 0$ for every $x$. Suppose that the degree of $v$ is $n+1$ with a positive integer $n$, and that $v$ is adjacent to only one interior vertex $w$. Then there is a homotopy equivalence

$$BD(G, \lambda) \simeq \bigvee_{(\lambda(v))} \Sigma^{\lambda(v)} BD(G', \lambda|_{V(G')}) \vee \bigvee_{(\lambda(v)-1)} \Sigma^{\lambda(v)} BD(G', \hat{\lambda}).$$

Here $G'$ is the graph obtained by deleting $v$ and the leaves adjacent to $v$ from $G$, and $\hat{\lambda}: V(G') \to \mathbb{Z}_+$ is the function defined by

$$\hat{\lambda}(x) = \begin{cases} 
\lambda(x) & (x \neq w) \\
\lambda(x) - 1 & (x = w).
\end{cases}$$

We show that our decomposition result (Theorem 1.2) yields Theorem 4.2. We start with the observation of the bounded degree complex of $K_{1,n}$. Let $v_0$ denote the root vertex of $K_{1,n}$. For a positive integer $k$, let $\lambda_k$ denote the function $V(K_{1,n}) \to \mathbb{Z}_+$ defined by

$$\lambda_k(x) = \begin{cases} 
k & (x = v_0) \\
1 & (x \neq v_0).
\end{cases}$$

We write $BD(K_{1,n}, k)$ instead of $BD(K_{1,n}, \lambda_k)$. Then $BD(K_{1,n}, k)$ is isomorphic to the $(k-1)$-skeleton of the $(n-1)$-simplex. Then it is known (see [9] for example) that there is a homotopy equivalence

$$BD(K_{1,n}, k) \simeq \bigvee_{(n-1)} S^{k-1}.$$
Proof of Theorem 4.2. Let $e = \{v, w\}$. Then there is a homotopy equivalence

$$BD(G, \lambda) \simeq BD(G - e, \lambda) \vee \Sigma BD(G - e, \lambda')$$

$$\simeq \big( BD(G', \lambda|_{V(G')}\big) * BD(K_{1,n}, \lambda(v)) \big) \vee \Sigma \big( BD(G', \lambda'), BD(K_{1,n}, \lambda(v) - 1) \big)$$

$$\simeq \bigvee_{\lambda(v)} \Sigma \lambda(v) BD(G', \lambda|_{V(G')}) \vee \bigvee_{\lambda(v) - 1} \Sigma \lambda(v) BD(G', \lambda').$$

Here we use Theorem 1.3 to obtain the first homotopy equivalence. \qed

On the other hand, at least at first glance, Theorem 4.2 does not imply Theorem 1.2. In fact, there are many graphs having leaves but not having corner points. For example, consider adding a leaf to a cycle.

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