Equivalence of Markov’s Symbolic Sequences to Two-Sided Chains

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A new object of the probability theory, two-sided chain of events (symbols), is introduced. A theory of multi-steps Markov chains with long-range memory, proposed earlier in Phys. Rev. E 68, 06117 (2003), is developed and used to establish the correspondence between these chains and two-sided ones. The Markov chain is proved to be statistically equivalent to the definite two-sided one and vice versa. The results obtained for the binary chains are generalized to the chains taking on the arbitrary number of states.

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I. INTRODUCTION

The problem of long-range correlated random symbolic systems (LRCS) has been under study for a long time in many areas of contemporary physics 1, 2, 3, 4, 5, 6, biology 7, 8, 9, 10, 11, 12, economics 13, 14, linguistics 15, 16, 17, 18, 19, etc.

Among the ways to get a correct insight into the nature of correlations of complex dynamic systems the use of the multi-step Markov chains is one of the most important because they give a possibility to construct a random sequence with necessary correlated properties in the most natural way. This was demonstrated in Ref. 20, where the concept of Markov chain with the step-wise memory function, which consist in coordinate independence of the conditional probability, was introduced. The concept of additive chains turned out to be very useful due to the ability to evaluate the binary correlation function of the chain through the memory function (see for the details Refs. 24, 25). The correlation properties of some dynamic systems (coarse-grained sequences of the Eukarya’s DNA and dictionaries) can be well described by this model 20.

Another important reason for the study of Markov chains is its application to the various physical objects 21, 22, 23, e.g., to the Ising chains of spins. The problem of thermodynamics description of the Ising chains with long-range spin interaction is opened even for the 1D case. However, the association of such systems with the Markov chains can shed light on the non-extensive thermodynamics of the LRCS.

Multi-step Markov chains are characterized by the probability that each symbol of the sequence takes on the definite value under condition that some previous symbols are fixed. This chains can be easily constructed by the consequent generation using prescribed conditional probability function. Besides, the statistical properties of Markov chains can be determined in some simple cases. At the same time, there is another class of correlated sequences, the so-called two-sided chains. They are determined by the probability that each symbol of the sequence takes on the definite value under the condition that some symbols at the both sides of the chosen symbol are fixed. An example of systems with such property is the above-mentioned Ising chain. But the approach, used for the finding of Markov chains properties (the probability of concrete ”word” occurring, the correlation functions, and so on) unfortunately cannot be used in this case. In this paper we prove, that such mathematical objects, determined in the Sec. II A as two-sided chains, are the Markov chains. So, the statistical properties of Markov chains and the method of their constructing can be used for the studying the two-sided chains.

The paper is organized as follows. In the first Section we give the definition of Markov and two-sided chains. The next Section is devoted to the proof of the main statement: the first Subsection contains the proof of the direct statement, that every binary Markov chain is in the same time the binary two-sided one; the second Subsection shows, that the classes of these two chains coincide. Finally, in the last Subsection we generalize this results to the case of non-binary chains.

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II. BASIC NOTIONS

A. General definitions

Let us determine the \( N \)-step Markov chain. This is a sequence of random variables \( a_i, i = -M, -M + 1, \ldots, M \) \((M \gg N)\), referred to as the symbols, which have the following property: the probability of symbol \( a_i \) to have a certain value under the condition that the values of all previous symbols are fixed depends on the values of \( N \) previous symbols only,

\[
P(a_i = a | \ldots, a_{i-2}, a_{i-1}) = P(a_i = a | a_{i-N}, \ldots, a_{i-2}, a_{i-1}).
\]  

(1)

Such defined chain is a stationary one, because the conditional probability does not depend explicitly on \( i \), i.e., does not depend on the position of symbols \( a_{i-N}, \ldots, a_{i-1}, a_i \) in the chain and depends on the values of the symbols and their positional relationship only.

The chain under consideration is defined for arbitrary but finite length \( M \). Nevertheless, all results that will be obtained do not depend on \( M \). Thus, they are correct in the infinite limit provided that the conditional probability function is fixed.

In different mathematical and physical problems we confront with the sequences for which the probability of symbol \( a_i \) to have certain value, under the condition that the values of the rest of symbols are fixed, depends on a value of \( N \) previous and \( N \) next symbols only,

\[
P(a_i = a | \ldots, a_{i-2}, a_{i-1}, a_{i+1}, a_{i+2}, \ldots) =
\]

\[
= P(a_i = a | a_{i-N}, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{i+N}).
\]  

(2)

Let us name these chains as \( N \)-two-sided ones. By the same reason as above, see Eq. (1), this chain is a stationary one.

An important class of the random sequences is the binary chains. If each symbol \( a_i \) of the chain can take on only two values, \( s_0 \) and \( s_1 \), then we refer to this chain as a binary. It is convenient to change a value of \( a_i \) to 0 and 1 using a linear transformation,

\[
a_i := \frac{a_i - s_0}{s_1 - s_0}.
\]

Now, we will describe the ways of the constructing of the defined chains.

B. Constructing of the chains

The Markov chain defined in such way is simple for numerical simulations. There are two basic approaches for this. In both of them we find successively the each next generated symbol by \( N \) previous ones. But these approaches differ in the method of constructing for the first \( N \)-word, the set of \( N \) sequent symbols.

For first approach one needs to calculate in addition some conditional probabilities. They can be found from the compatibility equation for the conditional probabilities:

\[
P(a_i = a | a_{i-k}, \ldots, a_{i-1}) = \sum_{a_{i-N}} \ldots \sum_{a_{i-k-1}} P(a_i = a | a_{i-N}, \ldots, a_{i-1}) P(a_{i-N}, \ldots, a_{i-1})
\]

(3)

Here \( k = 0, \ldots, N-1 \) and the sign \( \sum_{a_j} \) means summation (or integration) over all possible values of symbol \( a_j \). The probabilities of \( N \)-words occurring, \( P(a_{i-N}, \ldots, a_{i-1}) \), should be obtained from the following linear system,

\[
\begin{align*}
P(a_1, a_2, \ldots, a_N) &= \sum_{a_0} P(a_N | a_0, a_1, \ldots, a_{N-1}) P(a_0, a_1, \ldots, a_{N-1}), \\
\sum_{a_1} \ldots \sum_{a_N} P(a_1, a_2, \ldots, a_N) &= 1.
\end{align*}
\]  

(4)
The following formula can be easily obtained,

\[ P(a_1), P(a_2|a_1), P(a_3|a_1, a_2), \ldots, P(a_N|a_1, \ldots, a_{N-1}). \]

The second approach is based on the random choice of N-word. The second approach is simpler than the first one because it does not make the calculation of additional probabilities. But it does not allow to get the stationary chain, as it is possible in the first method. For generation of the same chain using the second approach we must construct as many symbols as one needs to get the stationary chain (the initial part of the chain should be removed).

There is no simple method for two-sided chains constructing. The best known and simple approach is Metropolis’ algorithm, but it needs much more operations than constructing of the Markov chain. Therefore it is very important to prove the equivalence of the Markov and two-sided chains.

III. EQUIVALENCE OF THE MARKOV AND TWO-SIDED CHAINS

In this section we prove an equivalence of two random sequences, the Markov and two-sided chains. The proof is produced for a binary chain, but it can be directly generalized for arbitrary chains (see Subsection III.C for details). This proof requires some formulas for a conditional probability. Its definition is

\[ P(A|B) = \frac{P(A, B)}{P(B)}. \]  

(5)

Here and below comma between the symbols-events means that both of these events occur simultaneously, it is a product of two events, \((A, B) = A \cap B\). Using evident equation,

\[ P(A, B|C) = P(A|B, C)P(B|C), \]  

(6)

the following formula can be easily obtained,

\[ P(A|B, C) = \frac{P(A, B|C)}{P(A, B|C) + P(A, B|C)}, \]  

(7)

where \(\overline{A}\) is an event opposite to \(A\).

A. From the Markov to two-sided chain

Let us demonstrate that a Markov chain is a two-sided one. For this purpose using Eq. (7) we rewrite the probability for symbol \(a_i\) to be equal unity, under the condition that the values of the rest of symbols are fixed, in following form:

\[ P(a_i = 1|A_i^-, A_i^+ =) = \frac{P(a_i = 1, A_i^+|A_i^-)}{P(a_i = 1, A_i^+|A_i^-) + P(a_i = 0, A_i^+|A_i^-)}. \]  

(8)

where \(A_i^- = (\ldots, a_{i-2}, a_{i-1})\) and \(A_i^+ = (a_{i+1}, a_{i+2}, \ldots)\).

To obtain the value of \(P(a_i = 1, A_i^+|A_i^-)\) one needs to use Eq. (8) many times to express \(P(\cdot|\cdot)\) as the product:

\[ P(a_i = 1, A_i^+|A_i^-) = P(a_i = 1|A_i^-)P(a_{i+1}, A_i^{+1}|A_i^-) = \]  

\[ = P(a_i = 1|A_i^-)P(a_{i+1}|a_i = 1, A_i^-)P(a_{i+2}, A_i^{+2}|A_i^-) = \]  

\[ \ldots = \prod_{i=1}^{M} P(a_i|A_i^-). \]  

(9)

However the chain under consideration is the \(N\)-step Markov one and, according to definition (1), the probability of symbol \(a_i\), under the condition that the values of all previous symbols are fixed, depends on the values of \(N\) previous
symbols only. So, the factors of the product for \( r > i + N \) in Eq. (9) do not depend on \( a_i \). Substituting expression (9) for \( P(a_i = 1, A_i^+ | A_i^-) \) into Eq. (8) we derive the following equation,

\[
P(a_i = 1 | T_i^-, T_i^+) = \frac{\prod_{r=0}^{N} P(a_{i+r} | T_{i+r}^-)}{\prod_{r=0}^{N} P(a_{i+r} | T_{i+r}^-) + \prod_{r=0}^{N} P(a_{i+r} | T_{i+r}^+)}.
\]  

(10)

Here \( T_j^- = (a_{j-N}, \ldots, a_{j-1}) \) and \( T_j^+ = (a_{j+1}, \ldots, a_{j+N}) \) are previous and next words of the length \( N \) with respect to symbol \( a_j \).

Equation (10) is the fundamental relation for association of Markov and two-sided chains. One can see from it that the probability of symbol \( a_j \) under the condition of fixed values of the rest of symbols is determined only by two words of the length \( N \), \( T_i^- \) and \( T_i^+ \). So, according to definition (2), the Markov chain is the two-sided one, quod erat demonstrandum.

**B. From two-sided to the Markov chain**

Now we prove the opposite statement: the two-sided chain is a Markov one. I.e., we prove that the probability of the symbol \( a_i \) to be equal to unity, under the condition that all \( \text{previous} \) symbols are fixed, depends on the values of \( N \) previous symbols only. Thereto, let us take two sets of symbols \( A' \) and \( A'' \) which are two variants of the word \( A_{i-N}^- \) and differ only by one symbol \( a_{i-N-k} \) at arbitrary value of \( k > 0 \).

Using definition of the conditional probability (4) we obtain

\[
P(a_i | A', T_i^-) = \frac{P(A', T_i^- \cap A_i)}{P(A', T_i^-)} = \frac{P(a_{i-N-k}' \cap A_i | A', T_i^-) P(\hat{A}_i | A', T_i^-) P(A_{i-N-k}'|A_{i-N-k} | A', T_i^-) P(A_{i-N-k}'|A_{i-N-k} | A', T_i^-)}{P(a_{i-N-k}' | A', T_i^-) P(\hat{A}_i | A', T_i^-) P(A_{i-N-k}'|A_{i-N-k} | A', T_i^-) P(A_{i-N-k}'|A_{i-N-k} | A', T_i^-)}
\]

where \( \hat{A} \) is a set of symbols \( A' \) (or \( A'' \)) except for symbol \( a_{i-N-k} \). However, according to the definition of two-sided chain, conditional probability \( P(a_{i-N-k}' | \hat{A}_i, T_i^-, a_i) \) does not depend on symbol \( a_i \) since the latter is situated at a distance more than \( N \) from \( a_{i-N-k} \). Hence one gets

\[
P(a_i | A', T_i^-) = P(a_i | \hat{A}, T_i^-) = P(a_i | A'', T_i^-).
\]

So, we find that probability \( P(a_i | A_i^-) \) takes on the same value for any arbitrary word \( A_{i-N}^- \). We conclude that the probability does not depend on \( A_{i-N}^- \). Thus we attest ourselves that

\[
P(a_i | A_i^-) = P(a_i | T_i^-).
\]

In other words, according to definition (11), the two-sided chain is a Markov one, quod erat demonstrandum.

It should be emphasized that every two-sided chain is equivalent to the single Markov one though it is not evident because of the non-linear structure of Eq. (10). Using trivial expression of Eq. (5),

\[
P(a_i | T_i^-) = \frac{P(T_i^- \cap a_i)}{P(T_i^-)}
\]

(11)

one can easily make sure that a single chain cannot have two different conditional probabilities. The matter is that the probabilities of \( N- \) and \( (N + 1)- \)words occurring determines uniquely the conditional probability according to Eq. (11). Hence, for the chain under study the Markov conditional probability is determined uniquely.
C. The case of non-binary chain

The results obtained in previous Secs. III A and III B can be generalized to non-binary chains. And we can develop the similar proof and get the following equation connecting the conditional probability functions,

\[ P(a_i = a | T_{i-}, T_{i+}) = \frac{\prod_{r=0}^{N} P(a_{i+r} | T_{i+r-})}{\sum_{\xi \in A} \prod_{r=0}^{N} P(a_{i+r} | T_{i+r-})}, \]

that is analogue of Eq. (10).

In this formula we used the following notations:
- if symbol \( a \) takes on the finite set of values \( A \) then we use the conditional probabilities \( P(a | \ldots) \);
- if symbol \( a \) takes on the continuous set of values \( A \) then we used conditional probability density \( P(a | \ldots) \) and sign \( \sum_{\xi \in A} \int_A d\xi \).

Thus, the equivalence between the N-two-sided and N-step Markov chains is proved for non-binary chains also. We found the very important formula for the conversion the Markov’s conditional probability to the two-sided one and inversely. This method can be used for numerical and analytic calculations of the conditional probabilities.

IV. CONCLUSION

Thus, we proved that the classes of the “one-sided” Markov chains and two-sided ones coincide. The obtained relationship between the conditional probabilities (or its densities in the case of continuous distribution of values taking on by the elements of the chains) allows to construct numerically the Markov chain possessing the same statistical properties as the initial two-sided one. So, two-sided sequence can be easily reproduced numerically with conservation of all statistical properties but not binary correlation function as it was done in the papers [24, 25].

Besides, found Eq. (12) allows to use results of analytical studies of Markov chains (for example, see [25]) for the two-sided sequences. This can be very useful for the study of physical systems. The example is the Ising chain, that is the two-sided one.

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