Calculating cohomology groups of moduli spaces of curves
via algebraic geometry

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In this paper we compute the first, second, third, and fifth rational cohomology groups of $\mathcal{M}_{g,n}$, the moduli space of stable $n$-pointed genus $g$ curves. It turns out that $H^1(\mathcal{M}_{g,n}, \mathbb{Q})$, $H^3(\mathcal{M}_{g,n}, \mathbb{Q})$, and $H^5(\mathcal{M}_{g,n}, \mathbb{Q})$ are zero for all values of $g$ and $n$, while $H^2(\mathcal{M}_{g,n}, \mathbb{Q})$ is generated by tautological classes, modulo relations that can be written down explicitly; the precise statements are given by Theorems (2.1) and (2.2). We are convinced that the computation of the fourth cohomology of all moduli spaces $\mathcal{M}_{g,n}$ should also be accessible to our methods.

It must be observed that some of these results are not new. In fact, it is known that $\mathcal{M}_{g,n}$ is simply connected (cf. [2], for instance), while Harer has determined $H^2(\mathcal{M}_{g,n}, \mathbb{Q})$ [8]; once this is known, it is not hard to compute the corresponding group for $\mathcal{M}_{g,n}$. Harer [10] has also shown that $H^3(\mathcal{M}_{g,n}, \mathbb{Q})$ vanishes, at least for large enough genus. What is really new here is the method of proof, which is mostly based on standard algebro-geometric techniques, rather than geometric topology. Especially for odd cohomology, this provides proofs that are quite short and, we hope, rather transparent. It should also be noticed that the odd cohomology of $\mathcal{M}_{g,n}$, at least in the range we can deal with, seems to be somewhat better behaved than the one of $\mathcal{M}_{g,n}$, for it is certainly not the case that the first and third cohomology groups of $\mathcal{M}_{g,n}$ are always zero.

Roughly speaking, the idea of the proof is as follows. If one could apply the Lefschetz hyperplane theorem, one might reduce the computation of $H^k(\mathcal{M}_{g,n}, \mathbb{Q})$ to the one of $H^k(\partial \mathcal{M}_{g,n}, \mathbb{Q})$, for low $k$. Although the standard Lefschetz theorem cannot be used, since $\partial \mathcal{M}_{g,n}$ is almost never ample, a foundational result of Harer, which is a direct consequence of the construction of a cellular decomposition of $\mathcal{M}_{g,n}$ by means of Strebel differentials, provides a suitable substitute. Once this is established, a little Hodge theory shows that, always for low enough $k$, $H^k(\mathcal{M}_{g,n}, \mathbb{Q})$ injects not only in $H^k(\partial \mathcal{M}_{g,n}, \mathbb{Q})$, but also in the $k$-th cohomology group of the normalization $N$ of $\partial \mathcal{M}_{g,n}$. Put otherwise, the combinatorics of the boundary does not contribute to $H^k(\mathcal{M}_{g,n}, \mathbb{Q})$. Since the components of $N$ are essentially products of moduli spaces $\mathcal{M}_{\gamma,\nu}$ such that either $\gamma < g$ or $\gamma = g$ and $\nu < n$, one may try to compute $H^k(\mathcal{M}_{g,n}, \mathbb{Q})$ by double induction on $g$ and $n$, starting from a few seed case to be handled directly. This turns out to be possible, and reduces to elementary linear algebra. An interesting, and somewhat unexpected, byproduct of the proof is that, for any $k$, the $k$-th cohomology group of $\mathcal{M}_{g,n}$ injects into the $k$-th cohomology of the normalization of the component of the boundary parametrizing irreducible singular curves, provided $g$ is large enough.

There are other cases, in addition to the ones mentioned above, in which the cohomology of moduli spaces of curves has been computed. First of all, Harer [11] has computed the fourth cohomology of $\mathcal{M}_{g,n}$ for large enough $g$. In a different direction, the entire coho-

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mology ring of $\overline{M}_{0,n}$ has been described for any $n$ by Keel [13] in terms of generators and relations. Likewise, Getzler [6] has announced the computation of the even cohomology ring $H^{\text{even}}(\overline{M}_{1,n}, \mathbb{Q})$ for any $n$. Mumford [18] and Faber [5] have computed the cohomology rings of $\overline{M}_2$ and $\overline{M}_{2,1}$, while Getzler [7] has computed the cohomology ring of $\overline{M}_{2,2}$ and announced the determination of the cohomology of $\overline{M}_{2,3}$. Finally, Looijenga [14] has computed the cohomology of $\overline{M}_3$ and $\overline{M}_{3,1}$.

We will assume Keel’s result, which is derived entirely via algebraic geometry, although the part of it that we use could be proved without much effort by our methods. Some of Getzler’s and Looijenga’s results will be needed, while computing $H^5$ to deal with some of the initial cases of the induction; except for this, our treatment will be self-contained and for the most part quite elementary.

We are grateful to Eduard Looijenga for indicating to us that Proposition (2.8) could be proved via Hodge theory; our original proof was based on a fairly involved combinatorial argument and worked with certainty only for $k \leq 2$.

1. Tautological classes

The purpose of this section is to fix notation and to collect a few results about tautological cohomology classes that are well known to specialists but for which a comprehensive reference seems to be lacking. All the varieties we shall consider will be over $\mathbb{C}$. Only rational cohomology will be used; when we omit mention of the coefficient group, we always implicitly assume rational coefficients.

Let $g$ and $n$ be non-negative integers such that $2g - 2 + n > 0$. We denote by $\overline{M}_{g,n}$ the moduli space of stable $n$-pointed genus $g$ curves and by $\overline{M}_{g,n}$ its subspace parametrizing smooth curves. More generally, if $P$ is a set with $n$ elements, it will be technically convenient to consider also stable $P$-pointed curves. These are simply stable curves whose marked points are indexed by $P$, and not by $\{1, \ldots, n\}$. We shall denote by $\overline{M}_{g,P}$ and $\overline{M}_{g,P}$ the corresponding moduli spaces. The boundary of $\overline{M}_{g,P}$ is $\partial \overline{M}_{g,P} = \overline{M}_{g,P} \setminus \overline{M}_{g,P}$.

By a graph we shall mean the datum $G$ of:
- a non-empty finite set $V = V(G)$ (the vertices of $G$),
- a non-negative integer $g_v$ for every $v \in V$,
- a finite set $L = L(G)$ (the half-edges of $G$),
- a partition $\mathcal{P}$ of $L$ in subsets with one or two elements,
- a subset $L_v \subset L$ for every $v \in V$,

with the property that
\[ L = \coprod_{v \in V} L_v. \]

We shall call the elements of $\mathcal{P}$ with one element legs of $G$ and those with two elements edges; the set of all the latter will be denoted $E = E(G)$. We also set $l_v = |L_v|$. In what follows we shall implicitly consider only graphs which are connected, in an obvious sense. If $P$ is a finite set, by a $P$-labelled graph we shall mean the datum of a graph $G$ plus a bijection between the set of its legs and $P$.

To every stable $P$-pointed genus $g$ curve $(C; \{q_p\}_{p \in P})$ we may associate a $P$-labelled graph $G$ as follows. Let $\pi : N \to C$ be the normalization of $C$. We let $V(G)$ be the set
of all components of $N$, and $L(G)$ the set of all points of $N$ which map either to nodes or
to marked points of $C$; two of these constitute an edge if they map to the same point (a
node) of $C$, while the remaining ones are legs. The indexing of the legs by $P$ is the obvious
one. We also set $g_v = \text{genus of } v$, and let $L_v$ be the set of all elements of $L$ belonging to
$v$. Notice that

$$g = \sum_{v \in V(G)} g_v + 1 - |V(G)| + |E(G)|.$$  

Conversely, this formula can be used to define the \textit{genus} of any connected $P$-labelled
graph. The graph associated to a stable $P$-pointed genus $g$ curve is \textit{stable} in the sense that
$2g_v - 2 + l_v > 0$ for every vertex $v$; more exactly, to say that a curve is stable is equivalent
to saying that its graph is.

Let $G$ be a connected stable $P$-labelled graph of genus $g$. We denote by $\mathcal{M}(G)$ the
moduli space of all $P$-pointed genus $g$ stable curves whose associated graph is $G$; it a
locally closed subspace of $\overline{\mathcal{M}}_{g, P}$ of codimension $|E(G)|$. We also denote by $\delta_G$ the orbifold
fundamental class of $\overline{\mathcal{M}(G)}$, that is, the crude fundamental class divided by the order
of the automorphism group of a general element of $\mathcal{M}(G)$. The degree two classes correspond
to graphs with one edge. These come in two kinds; there is the graph $G_{\text{irr}}$, with one vertex
of genus $g-1$, and there are the graphs $G_{a, A}$, which have two vertices, one of genus $a$, with
attached the legs indexed by $A$, and one of genus $g-a$, with attached the legs indexed by
$A^c = P \setminus A$. We shall normally write $\delta_{\text{irr}}$ and $\delta_{a, A}$ instead of $\delta_{G_{\text{irr}}}$ and $\delta_{G_{a, A}}$. We shall
also write $\Delta_{\text{irr}}$ and $\Delta_{a, A}$ to denote $\overline{\mathcal{M}(G_{\text{irr}})}$ and $\overline{\mathcal{M}(G_{a, A})}$, respectively. It is clear that

$$(1.1) \quad \delta_{a, A} = \delta_{g-a, A^c},$$

and also that $\delta_{a, A}$ does not make sense as defined unless $G_{a, A}$ is a stable graph, i.e., unless
$2a - 2 + |A| \geq 0$ and $2(g - a) - 2 + |A^c| \geq 0$. In practice, this means that $\delta_{a, A}$ is still
undefined if $a = 0$ and $|A| < 2$ or $a = g$ and $|A| > |P| - 2$. We will find it convenient to
set $\delta_{a, A}$ to zero if $a < 0$, $a > g$, $2a - 2 + |A| < 0$, or $2(g - a) - 2 + |A^c| < 0$. The class $\delta_{\text{irr}}$,
as defined above, also does not make sense in genus zero; we will set it to zero in this case.

The basic maps between moduli spaces are

$$\pi : \overline{\mathcal{M}}_{g, P \cup \{q\}} \to \overline{\mathcal{M}}_{g, P},$$

$$\xi : \overline{\mathcal{M}}_{g-1, P \cup \{q, r\}} \to \overline{\mathcal{M}}_{g, P},$$

$$\eta : \overline{\mathcal{M}}_{a, A \cup \{q\}} \times \overline{\mathcal{M}}_{g-a, A^c \cup \{r\}} \to \overline{\mathcal{M}}_{g, P},$$

which are defined as follows. The image under $\pi$ of a $P \cup \{q\}$-pointed curve is obtained by
"forgetting" the point labelled by $q$ and passing to the stable model. The image under $\xi$
of a $P \cup \{q, r\}$-pointed genus $g-1$ curve is obtained by identifying the points labelled by $q$
and $r$; likewise, the image under $\eta$ of a pair consisting of an $A \cup \{q\}$-pointed genus $a$ curve
and an $A^c \cup \{r\}$-pointed genus $g-a$ curve is the $P$-pointed curve of genus $g$ obtained by
identifying the points labelled by $q$ and $r$.

The map $\pi : \overline{\mathcal{M}}_{g, P \cup \{q\}} \to \overline{\mathcal{M}}_{g, P}$ is also called the \textit{universal curve} over $\overline{\mathcal{M}}_{g, P}$. It has
$|P|$ sections, indexed by $P$; the section $\sigma_p$ attaches to any $P$-pointed curve $(C; \{x_i\}_{i \in P})$
the $P \cup \{q\}$-pointed curve obtained by attaching to $C$ a copy of $\mathbb{P}^1$ by identifying $x_p$ and
0 \in \mathbb{P}^1$, and labelling the points 1 and $\infty$ by $p$ and $q$. One may use the universal curve to define further cohomology classes on $\overline{M}_{g,P}$ as follows. We denote by $\omega_\pi$ the relative dualizing sheaf and by $D_p$ the image of $\sigma_p$. One then sets
\[
\psi_p = \sigma_p^*(c_1(\omega_\pi)), \quad p \in P,
\]
\[
\kappa_i = \pi_*^*(c_1(\omega_\pi(\sum D_p))^{i+1}), \quad i \geq 0.
\]
The classes $\psi_p$ have degree 2, while $\kappa_i$ has degree $2i$. In the rest of this paper, whenever we speak of tautological or natural classes (of degree 2) on $\overline{M}_{g,P}$, we refer to $\kappa_1$, the $\psi_p$, $\delta_{irr}$, and the $\delta_{a,A}$. The classes $\delta_{irr}$ and $\delta_{a,A}$ will be called boundary classes.

Our next task is to describe how the natural classes pull back under $\pi$, $\xi$, and $\eta$.

**Lemma (1.2).**

i) $\pi^*(\kappa_1) = \kappa_1 - \psi_q$;

ii) $\pi^*(\psi_p) = \psi_p - \delta_{0,\{p,q\}}$ for any $p \in P$;

iii) $\pi^*(\delta_{irr}) = \delta_{irr}$;

iv) $\pi^*(\delta_{a,A}) = \delta_{a,A} + \delta_{a,A \cup \{q\}}$.

Part i) of the lemma is proved in [1], while iii) and iv) are clear. To prove ii) we reason as follows. Consider the diagram
\[
\begin{array}{ccc}
\overline{M}_{g,P \cup \{q,r\}} & \xrightarrow{\mu} & \overline{M}_{g,P \cup \{r\}} \\
\varphi' \downarrow & & \varphi' \downarrow \\
\overline{M}_{g,P \cup \{q\}} & \xrightarrow{\pi} & \overline{M}_{g,P} \\
\end{array}
\]
where $\varphi'$ and $\varphi$ are defined as “forgetting the point labelled by $r$” and $\mu$ as “forgetting the point labelled by $q$”. It is known (cf. [1], for instance) that
\[
\mu^*(\omega_\varphi) = \omega_{\varphi'}(-\sum_{x \in P} \Delta_{0,\{x,q,r\}}).
\]
Thus, if $\tau_x$, $x \in P$ (resp., $\tau'_x$, $x \in P \cup \{q\}$), are the canonical sections of $\varphi$ (resp., $\varphi'$), then
\[
\pi^*(\tau_p^*(\omega_\varphi)) = \tau'_p^*(\mu^*(\omega_\varphi)) = \tau'_p^*(\omega_{\varphi'}(-\sum_{x \in P} \Delta_{0,\{x,q,r\}}))
\]
for any $p \in P$. This translates into ii), finishing the proof of the lemma.

**Lemma (1.3).**

i) $\xi^*(\kappa_1) = \kappa_1$;

ii) $\xi^*(\psi_p) = \psi_p$ for any $p \in P$;

iii) $\xi^*(\delta_{irr}) = \delta_{irr} - \psi_q - \psi_r + \sum_{q \in B, r \notin B} \delta_{b,B}$;

iv) $\xi^*(\delta_{a,A}) = \begin{cases} 
\delta_{a,A} & \text{if } g = 2a, \ A = P = \emptyset, \\
\delta_{a,A} + \delta_{a-1,A \cup \{q,r\}} & \text{otherwise.}
\end{cases}$
On the other hand, as we announced, the first cohomology of \( \eta \) which associates to any \( A \cup \{q\} \)-pointed genus \( a \) curve the \( P \)-pointed genus \( g \) curve obtained by glueing to it a fixed \( A^c \cup \{r\} \)-pointed genus \( g - a \) curve \( C \) via identification of \( q \) and \( r \). On the other hand, as we announced, the first cohomology of \( \overline{M}_{g, P} \) always vanishes, by the Künneth formula, so the second cohomology of \( \overline{M}_{a, A} \times \overline{M}_{g-a, A^c} \) is the direct sum of \( H^2(\overline{M}_{a, A}) \) and \( H^2(\overline{M}_{g-a, A^c}) \). Thus knowing how the natural classes pull back under \( \vartheta \) actually tells us how they pull back under \( \eta \) (once the vanishing of the first cohomology has been proved). It is important to stress that, although of course \( \vartheta \) depends on the choice of \( C \), any two choices give rise to homotopic maps so that, in cohomology, the pullback map \( \vartheta^* \) is independent of the choice of \( C \).

LEMMA (1.4). i) \( \vartheta^*(\kappa_1) = \kappa_1 \); 

ii) \( \vartheta^*(\psi_p) = \begin{cases} \psi_p & \text{if } p \in A, \\ 0 & \text{if } p \in A^c; \end{cases} \)

iii) \( \vartheta^*(\delta_{irr}) = \delta_{irr} \).

Suppose \( A = P \). Then

iv) \( \vartheta^*(\delta_{b,B}) = \begin{cases} \delta_{2a-g, P \cup \{q\}} - \psi_q & \text{if } (b, B) = (a, P) \text{ or } (b, B) = (g - a, \emptyset), \\ \delta_{b,B} + \delta_{b+a-g, B \cup \{q\}} & \text{otherwise.} \end{cases} \)

Suppose \( A \neq P \). Then

iv') \( \vartheta^*(\delta_{b,B}) = \begin{cases} -\psi_q & \text{if } (b, B) = (a, A) \text{ or } (b, B) = (g - a, A^c), \\ \delta_{b,B} & \text{if } B \subset A \text{ and } (b, B) \neq (a, A), \\ \delta_{b+a-g,(B \setminus A^c) \cup \{q\}} & \text{if } B \supset A^c \text{ and } (b, B) \neq (g - a, A^c), \\ 0 & \text{otherwise.} \end{cases} \)

Again, the only part that needs justification is i), which is proved in [1].

We may now determine all relations among tautological classes in degree two. The answer depends on the genus. We begin with genus zero. In this case it has been observed by Keel [13] that, for any four distinct elements \( p, q, r, s \) of \( P \), the following relations hold among the classes \( \delta_{0,A} \) such that \( |A| \geq 2 \) and \( |A^c| \geq 2 \):

\[
\sum_{A \ni \{p, q\}} \delta_{0,A} = \sum_{A \ni \{r, s\}} \delta_{0,A} = \delta_{0,A}.
\]

What is more important, Keel proves that \( H^2(\overline{M}_{0, P}) \) is the quotient of the vector space generated by the \( \delta_{0,A} \) such that \( |A| \geq 2 \) and \( |A^c| \geq 2 \) modulo the trivial relations (1.1) and the relations (1.5) for all possible choices of \( p, q, r, s \).

PROPOSITION (1.6). For any choice of distinct elements \( x, y, z \in P \), the following relations hold in \( H^2(\overline{M}_{0, P}) \):

\[
\psi_z = \sum_{A \ni x, y} \delta_{0,A},
\]

\[
\kappa_1 = \sum_{A \ni x, y} (|A| - 1) \delta_{0,A}.
\]
These, together with the relation $\delta_{rr} = 0$ and relations (1.1) and (1.5), generate all relations in $H^2(\mathcal{M}_0, P)$ among the natural classes $\kappa_1$, $\psi_1$, $\delta_{rr}$, and $\delta_{0,A}$ with $|A| \geq 2$ and $|A^c| \geq 2$.

In view of Keel’s result, all that needs to be shown is that (1.7) and (1.8) hold. The proof is by induction on $|P|$, starting from the obvious remark that $0 = \kappa_1 = \psi_1 = \psi_2 = \psi_3$ on $\mathcal{M}_0, P$ when $|P| = 3$. The induction step is based on Lemma (1.2). Suppose that (1.7) and (1.8) hold in $\mathcal{M}_0, P$. By symmetry, it suffice to prove their analogues in $\mathcal{M}_{0,P,\cup\{q\}}$ for $x, y, z \in P$. Pulling back (1.7) via $\pi$ gives that

$$\psi_x - \delta_{0,\{z,q\}} = \sum_{z \in A \subset P, x,y \notin A} \delta_{0,A} + \delta_{0,A \cup \{q\}},$$

which is nothing but the analogue of (1.7) for $\mathcal{M}_{0,P,\cup\{q\}}$. Similarly, pulling back (1.8) yields

$$\kappa_1 - \psi_q = \sum_{x, y \notin A \subset P} (|A| - 1)(\delta_{0,A} + \delta_{0,A \cup \{q\}}),$$

that is, using (1.7) to express $\psi_q$ in terms of boundary classes,

$$\kappa_1 = \sum_{x, y \notin A \subset P} \delta_{0,A \cup \{q\}} + \sum_{x, y \notin A \subset P} (|A| - 1)(\delta_{0,A} + \delta_{0,A \cup \{q\}}),$$

which is exactly what had to be shown.

To state and prove the analogues of (1.6) in higher genus, it will be convenient to adopt the following notational conventions. We set $\psi = \sum \psi_i$ and, for any integer $a$, we let $\delta_a$ be the sum of all classes $\delta_{a,A}$; notice that, in case $g = 2a$, the summand $\delta_{a,A} = \delta_{a,A^c}$ occurs only once, and not twice, in this sum. We also let $\delta$ denote the sum of $\delta_{rr}$ and of all the $\delta_a$ with $2a \leq g$. Finally, as is customary, we denote by $\lambda$ the Hodge class, that is, $c_1(\pi_*(\omega_\pi))$.

**Proposition (1.9).** i) The following relations hold in $H^2(\mathcal{M}_{1,P})$, for any $p \in P$:

\begin{align*}
(1.10) \quad & \kappa_1 = \psi - \delta_0, \\
(1.11) \quad & 12\psi_p = \delta_{rr} + 12 \sum_{s \supset p, |s| \geq 2} \delta_{0,S}.
\end{align*}

These, together with the (1.1), generate all relations in $H^2(\mathcal{M}_{1,P})$ among the natural classes $\kappa_1$, $\psi_i$, $\delta_{rr}$, and $\delta_{a,A}$ with $0 \leq a \leq 1$ and $2 \leq |A| \leq |P| - 2$ if $a = 0$.

ii) The following relation holds in $H^2(\mathcal{M}_{2,P})$:

\begin{equation}
(1.12) \quad 5\kappa_1 = 5\psi + 5\delta_{rr} - 5\delta_0 + 7\delta_1.
\end{equation}

This relation and the (1.1) generate all relations in $H^2(\mathcal{M}_{2,P})$ among the natural classes $\kappa_1$, $\psi_i$, $\delta_{rr}$, and $\delta_{a,A}$ with $0 \leq a \leq 2$ and $2 \leq |A| \leq |P| - 2$ if $a = 0$.

iii) If $g \geq 3$, the (1.1) generate all relations in $H^2(\mathcal{M}_{g,P})$ among the classes $\kappa_1$, $\psi_i$, $\delta_{rr}$, and $\delta_{a,A}$ with $0 \leq a \leq g$ and $2 \leq |A| \leq |P| - 2$ if $a = 0$. 

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The proofs of (1.10), (1.11), and (1.12) are the exact analogues of those of (1.8) and (1.7). The initial cases of the induction are as follows. First of all, for any \( g \) and any \( P \), one has Mumford’s relation \[17][3]

\[(1.13) \quad \kappa_1 = 12 \lambda - \delta + \psi.\]

For \( g = |P| = 1 \), one knows that \( \psi = \lambda \), and that \( 12 \lambda - \delta = 0 \) \[12\], so that \( \kappa_1 = \psi \) and \( \delta = 12 \psi \). Since in this case \( \delta = \delta_{irr} \) and \( \delta_0 = 0 \), these identities are just the relations (1.10) and (1.11). For \( g = 2 \), \( P = \emptyset \), Mumford \[18\] has shown that \( 10 \lambda = \delta_{irr} + 2 \delta_1 \). Coupled with (1.13), this says that

\[
5\kappa_1 = 60 \lambda - 5 \delta + 5 \psi = 6 \delta_{irr} + 12 \delta_1 - 5 \delta_{irr} - 5 \delta_0 - 5 \delta_1 + 5 \psi
= 5 \psi + \delta_{irr} - 5 \delta_0 + 7 \delta_1,
\]

as desired.

What remains to be shown is that there are no relations in addition to the ones listed above. We begin with the case of genus 1. We need the following simple remark.

**Lemma (1.14).** The homomorphism \( \xi^* : H^2(\overline{M}_{1,1}, P) \to H^2(\overline{M}_{0, P \cup \{q, r\}}) \) maps \( \delta_{irr} \) to zero.

Let \( p \) be an element of \( P \), and let \( \rho : \overline{M}_{1,1} \to \overline{M}_{1,\{p\}} \) be the morphism defined by forgetting the points labelled by elements of \( P \) other than \( p \). The cycle \( \Delta_{irr} \) is the inverse image of the point at infinity \( x_0 \in \overline{M}_{1,\{p\}} \). Therefore the cohomology class \( \delta_{irr} \) is the fundamental class of \( \rho^{-1}(x) \), where \( x \) is any point of \( \overline{M}_{1,\{p\}} \). Since \( \rho^{-1}(x) \) does not touch \( \Delta_{irr} = \xi(\overline{M}_{0, P \cup \{q, r\}}) \) if \( x \neq x_0 \), it follows that \( \xi^*(\delta_{irr}) = 0 \), as desired.

What we need to do to finish the genus 1 case is to show that \( \delta_{irr} \) and the classes \( \delta_{1,S} \) are independent. First consider the inclusion \( \vartheta : \overline{M}_{1,1} \hookrightarrow \overline{M}_{1, P} \) obtained by sending any 1-pointed genus 1 curve \( C \) to the union of \( C \) with a fixed \( P \cup \{q\} \)-pointed smooth rational curve \( E \), with the marked point of \( C \) identified with the point of \( E \) labelled by \( q \). Notice that, by (1.4), \( \vartheta^*(\delta_{irr}) = \delta_{irr} \), so \( \delta_{irr} \in H^2(\overline{M}_{1, P}) \) is not zero since its pullback to \( H^2(\overline{M}_{1,1}) \) does not vanish.

Now look at the pullbacks of the boundary classes via the morphism \( \xi : \overline{M}_{0, P \cup \{q, r\}} \to \overline{M}_{1, P} \). We have seen that \( \delta_{irr} \) pulls back to zero. On the other hand it is clear that \( \xi^*(\delta_{1,S}) = \delta_{0, S \cup \{q, r\}} \). The independence of the classes \( \delta_{irr} \) and \( \delta_{1,S} \) will then follow from the remark that \( \delta_{irr} \neq 0 \) and from the following result.

**Lemma (1.15).** The classes \( \delta_{S \cup \{q, r\}} \), where \( S \) runs through all subsets of \( P \) with at most \(|P| - 2\) elements, are independent in \( H^2(\overline{M}_{0, P \cup \{q, r\}}) \).

This would follow, after some work, from Keel’s description of the cohomology ring of \( \overline{M}_{0, P \cup \{q, r\}} \), but we choose a different approach. To begin with, it suffices to do the proof when \( P = \{1, \ldots, n\} \), \( q = n + 1 \), \( r = n + 2 \). The lemma is clearly true for \( n \leq 2 \), so we can assume that \( n > 2 \). For each \( S \), we shall produce a 1-parameter family \( F_S \) of stable \((n + 2)\)-pointed genus zero curves and will show that the matrix whose entries are the degrees of the classes \( \delta_{T \cup \{n+1, n+2\}} \) on these families is non-degenerate.
Let \( f : X = \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \) be the projection onto the first factor, and let \( D_i', i \in \{1, \ldots, n+3\} \) be disjoint constant sections of \( f \). Suppose first that \( S \neq \emptyset \). We then let \( F'_S \) be the family of pointed curves over \( \mathbb{P}^1 \) consisting of the blowup of \( f : X \to \mathbb{P}^1 \) at the points of intersection of the diagonal with the sections \( D_i', i \in S \cup \{n+1, n+2, n+3\} \), together with the sections \( D_i, i \in S \cup \{n+1, n+2, n+4\} \), where \( D_i \) is the proper transform of \( D_i' \) for \( i \neq n+4 \), and \( D_{n+4} \) is the proper transform of the diagonal. We also let \( F''_S \) be the family of pointed curves over \( \mathbb{P}^1 \) consisting of \( f : X \to \mathbb{P}^1 \) together with the sections \( D_i = D_i', i \in \{1, \ldots, n, n+3\} - S \). When \( S = \emptyset \), the definitions of \( F'_S \) and \( F''_S \) are different. The family \( F'_S \) is just \( f : X \to \mathbb{P}^1 \), together with the sections \( D_i = D_i', i = n+1, n+2, n+3 \).

As for \( F''_S \), blow up \( f : X \to \mathbb{P}^1 \) at the points where the diagonal meets the sections \( D_i', i = 1, \ldots, n, \) and take as sections \( D_i, i = 1, \ldots, n, n+4, \) the proper transforms of the \( D_i' \), \( i \leq n \), and of the diagonal. In any case, \( F_S \) is defined to be the family of stable \((n+2)\)-pointed genus zero curves over \( \mathbb{P}^1 \) obtained from \( F'_S \) and \( F''_S \) by identifying the sections \( D_{n+3} \) and \( D_{n+4} \).

The degrees of the classes \( \delta_{T \cup \{n+1, n+2\}} \) are readily calculated. They are as follows.

- \( S \neq \emptyset \)
  
  \[
  \deg_{F_S} \delta_{S \cup \{n+1, n+2\}} = -|S|, \\
  \deg_{F_S} \delta_{S \setminus \{s\} \cup \{n+1, n+2\}} = 1 \text{ if } s \in S, \\
  \deg_{F_S} \delta_{T \cup \{n+1, n+2\}} = 0 \text{ otherwise.}
  \]

- \( S = \emptyset \)
  
  \[
  \deg_{F_S} \delta_{\{n+1, n+2\}} = 2 - n, \\
  \deg_{F_S} \delta_{\{s, n+1, n+2\}} = 1 \text{ if } s = 1, \ldots, n, \\
  \deg_{F_S} \delta_{T \cup \{n+1, n+2\}} = 0 \text{ otherwise.}
  \]

Now suppose there is a relation \( \sum_T a_T \delta_{T \cup \{n+1, n+2\}} = 0 \). Evaluating on \( F_{\emptyset} \) yields

\[
(2 - n)a_{\emptyset} + \sum_{s=1}^{n} a_{\{s\}} = 0,
\]

while evaluating on \( F_{\{s\}} \) gives

\[
-a_{\{s\}} + a_{\emptyset} = 0, \quad s = 1, \ldots, n.
\]

Combining these relations gives \( a_{\emptyset} = 0 \) and \( a_{\{s\}} = 0 \) for \( s = 1, \ldots, n \). To show that, in fact, \( a_S = 0 \) for any \( S \), one proceeds by induction on \( |S| \). We may assume that \( S \neq \emptyset \). Suppose we know that \( a_T = 0 \) whenever \( |T| < |S| \). Evaluating on \( F_S \) yields

\[
|S|a_S = \sum_{s \in S} a_{S \setminus \{s\}} = 0,
\]

so \( a_S = 0 \). This finishes the proof of part \( i) \) of (1.9).

To complete the proof of part \( ii) \) it remains to show that the boundary classes and the classes \( \psi_p, p \in P \), are independent in \( H^2(M_2, P) \) modulo the trivial relations (1.1). We distinguish several cases.
**Case 1:** $P = \emptyset$. Let $F'$ be a non-isotrivial family of stable 1-pointed genus 1 curves over a smooth complete curve; by attaching a fixed elliptic tail to the marked point of each fiber we get a family $F$ of stable genus 2 curves. We then have that

\begin{equation}
\begin{aligned}
\deg_F \delta_{\text{irr}} &= \deg_{F'} \delta_{\text{irr}} = 12 \deg_{F'} \lambda, \\
\deg_F \delta_{1,\emptyset} &= -\deg_{F'} \psi_1 = -\deg_{F'} \lambda.
\end{aligned}
\end{equation}

Next, let $C$ be a smooth elliptic curve, and let $x$ be a fixed point on $C$. Let $X$ be the surface obtained by blowing up $C \times C$ at the intersection point between the diagonal and $\{x\} \times C$, and let $X \to C$ be the composition of the blow-down map and of the projection from $C \times C$ onto the second factor. We get a family $E$ of stable genus 2 curves over $C$ by identifying the proper transforms of the diagonal of $C \times C$ and of $\{x\} \times C$. For this family we have

\[ \deg_E \delta_{\text{irr}} = -2, \quad \deg_E \delta_{1,\emptyset} = 1. \]

Putting this together with (1.16) shows that $\delta_{\text{irr}}$ and $\delta_{1,\emptyset}$ are independent; in fact, the non-isotriviality of $F'$ implies that $\deg_{F'} \lambda \neq 0$.

**Case 2:** $|P| = 1$. Denote by $p$ the only point of $P$. We construct two families of stable 1-pointed genus 2 curves by modifying slightly those constructed in case 1. Let $F_1$ be as $F'$, but with the addition of the section traced out by a fixed point on the constant elliptic tail. To construct $E_1$, choose a point $y \in C$ distinct from $x$, and blow up $E \times E$ at the intersections between the diagonal, $\{x\} \times C$, and $\{y\} \times C$. Then identify the proper transforms of the diagonal and $\{x\} \times C$; the canonical section is the proper transform of $\{y\} \times C$. A third family $G_1$ is $\Gamma \times \Gamma \to \Gamma$, where $\Gamma$ is a smooth genus 2 curve, with the diagonal as section. The invariants for these families are readily calculated. They are

\[ \begin{aligned}
\deg_{F_1} \delta_{\text{irr}} &= 12 \deg_{F'} \lambda, \\
\deg_{E_1} \delta_{\text{irr}} &= -2, \\
\deg_{G_1} \delta_{\text{irr}} &= 0, \\
\deg_{F_1} \delta_{1,\emptyset} &= -\deg_{F'} \lambda, \\
\deg_{E_1} \delta_{1,\emptyset} &= 1, \\
\deg_{G_1} \delta_{1,\emptyset} &= 0, \\
\deg_{E_1} \psi_p &= 1, \\
\deg_{G_1} \psi_p &= 2, \\
\end{aligned} \]

Thus $\delta_{\text{irr}}, \delta_{1,\emptyset}$, and $\psi_p$ are independent.

**Case 3:** $|P| > 1$. We proceed by induction on $|P|$. Let $p, q$ be distinct elements of $P$; set $Q = P - \{p, q\}$ and let $r, r'$ be distinct and not belonging to $P$. Let

\[ \vartheta : \overline{\mathcal{M}}_{2, Q \cup \{r\}} \simeq \overline{\mathcal{M}}_{2, Q \cup \{r\}} \times \overline{\mathcal{M}}_{0, \{r', p, q\}} \to \overline{\mathcal{M}}_{2, P} \]

be the map obtained by identifying the points labelled by $r$ and $r'$. It follows from (1.4) that

\begin{equation}
\begin{aligned}
\vartheta^*(\psi_i) &= \psi_i, \quad i \in Q, \\
\vartheta^*(\psi_p) &= \vartheta^*(\psi_q) = 0, \\
\vartheta^*(\delta_{\text{irr}}) &= \delta_{\text{irr}}, \\
\vartheta^*(\delta_{0, \{p, q\}}) &= -\psi_r, \\
\vartheta^*(\delta_{a, A}) &= \delta_{a, A}, \quad a \leq 1, \ A \subset Q, \\
\vartheta^*(\delta_{a, A \cup \{p, q\}}) &= \delta_{a, A \cup \{r\}}, \quad a \leq 1, \ A \subset Q, \ A \neq \emptyset \text{ if } a = 0, \\
\vartheta^*(\delta_{a, A}) &= 0, \quad \{p, q\} \notin A, \ \{p, q\} \not\subset P - A.
\end{aligned}
\end{equation}
Suppose there is a relation

\[(1.18) \quad \sum_{i \in P} a_i \psi_i + b \delta_{irr} + \sum_{|A| \geq 2} c_A \delta_{0,A} + \sum_{|A| \leq |P|/2}^' d_A \delta_{1,A} = 0,\]

where the quotation mark affixed to the last summation symbol means that, for each \(A \subset P\) such that \(|A| = |P|/2\), only one of the summands \(d_A \delta_{1,A}\), \(d_{P-A} \delta_{1,P-A}\) occurs. Then the induction hypothesis and (1.17) imply that \(a_i = 0\) unless \(i = p\) or \(i = q\), that \(b = 0\), and that \(c_A = d_A = 0\) unless \(\{p, q\} \not\subset A\), \(\{p, q\} \not\subset P - A\). Since \(p\) and \(q\) can be chosen at will, for \(|P| \geq 3\) this implies that all coefficients of (1.18) vanish. If \(P = \{p, q\}\), all we can say is that all the coefficients of (1.18) are zero, except possibly for \(a_p, a_q,\) and \(d_{\{p\}}\). To see that these are zero as well, let \(C\) be a smooth genus 2 curve, and let \(x\) be a point on it. Blowing up \(C \times C\) at the intersection point of \(\{x\} \times C\) and the diagonal yields a family of genus 2 curves with two sections, namely the proper transforms of \(\{x\} \times C\) and of the diagonal. Labelling the first of these with \(p\) and the second with \(q\), for the resulting family of stable \(\{p, q\}\)-pointed curves we have that

\[
\text{deg} \psi_p = -1, \quad \text{deg} \psi_q = -3, \quad \text{deg} \delta_{1,\{p\}} = 0.
\]

Since the roles of \(p\) and \(q\) can be interchanged, this implies that \(a_p = a_q = 0\). To conclude it suffices to produce a family of stable \(\{p, q\}\)-pointed curves of genus 2 for which \(\text{deg} \delta_{1,\{p\}}\) does not vanish. One such is obtained by attaching a fixed elliptic tail with two extra fixed marked points to the points of the canonical section of the family \(F'\) of elliptic curves considered in the analysis of Case 1 above. The proof of part \(ii)\) of (1.9) is now complete.

To do part \(iii)\) we proceed by induction on \(g\). When \(P \neq \emptyset\), fix an element \(p \in P\); we have to show that \(\kappa_1,\) the \(\psi_i,\) \(\delta_{irr}\) and the \(\delta_{a,A}\) such that \(p \in A\) are independent. When \(P = \emptyset\), instead, we have to show independence of \(\kappa_1,\) the \(\psi_i,\) \(\delta_{irr}\) and the \(\delta_a\) with \(2a \leq g\). For \(g > 3\), the formulas in Lemma (1.3), plus the induction hypothesis, show directly that the pullbacks of these classes via \(\xi: \overline{M}_{g,P} \to \overline{M}_{g-1,P \cup \{q,r\}}\) are already independent. For \(g = 3\), we argue as follows. Suppose there is a relation

\[
0 = a \kappa_1 + \sum b_i \psi_i + c \delta_{irr} + \cdots
\]

among them. Pulling back via \(\xi\), and using (1.3) and (1.12), we find a relation on \(\overline{M}_{2,P \cup \{q,r\}}\) or the form

\[
0 = (a - c) \psi_q + \cdots + (c + a/5) \delta_{irr} + \cdots
\]

By \(ii)\), all coefficients in this relation must vanish, so \(a = 0\). At this point we may proceed as for \(g > 3\). The proof of (1.9) is now complete.

2. The main results

In this section we state our main results and present the core of their proofs. The first theorem describes the first, third and fifth cohomology groups of the moduli spaces of stable curves.
Theorem (2.1). $H^k(\mathcal{M}_{g,n}) = 0$ for $k = 1, 3, 5$ and all $g$ and $n$ such that $2g - 2 + n > 0$.

The next result describes the second cohomology group of $\mathcal{M}_{g,n}$ in terms of generators and relations. It turns out that this group is always generated by the natural classes. The relations among these have already been determined in section 1.

Theorem (2.2). For any $g$ and $n$ such that $2g - 2 + n > 0$, $H^2(\mathcal{M}_{g,n})$ is generated by $\kappa_1$, the classes $\psi_i$, $\delta_{irr}$, and the classes $\delta_{a,A}$ such that $0 \leq a \leq g, 2a - 2 + |A| \geq 0$ and $2(g - a) - 2 + |A^c| \geq 0$. The relations among these classes are as follows.

a) If $g > 2$ all relations are generated by those of the form

\[ \delta_{a,A} = \delta_{g-a,A^c}. \]

b) If $g = 2$ all relations are generated by the (2.3) plus the following one

\[ 5\kappa_1 = 5\psi + \delta_{irr} - 5\delta_0 + 7\delta_1. \]

c) If $g = 1$ all relations are generated by the (2.3) plus the following ones

\[ \kappa_1 = \psi - \delta_0, \]

\[ 12\psi_p = \delta_{irr} + 12 \sum_{S \ni p, |S| \geq 2} \delta_{0,S}. \]

d) If $g = 0$ all relations are generated by the (2.3) plus the following ones

\[ \kappa_1 = \sum_{A \not\ni x,y} (|A| - 1)\delta_{0,A}, \]

\[ \psi_z = \sum_{A \ni z, A \not\ni x,y} \delta_{0,A}, \]

\[ \delta_{irr} = 0. \]

Observe, first of all, that the moduli spaces $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$, although in general not smooth, are orbifolds; in particular, Poincaré duality holds for them in rational cohomology.

The proof of (2.1) and (2.2) begins with a simple remark. Look first at $\mathcal{M}_{0,n}$; it can be viewed as the space of all $n$-tuples $(0, 1, \infty, z_4, \ldots, z_n)$ of distinct points of $\mathbb{P}^1$ or, which is the same, as the space of all $(n - 3)$-tuples $(z_4, \ldots, z_n)$ in $\mathbb{C}^{n-3}$ such that $z_i \neq 0, 1$ for all $i$ and $z_i \neq z_j$ for all $i \neq j$. In other words, $\mathcal{M}_{0,n}$ is nothing but $\mathbb{C}^{n-3}$ minus a bunch of hyperplanes, so in particular it is an $(n - 3)$-dimensional affine variety. It follows that $H_k(\mathcal{M}_{0,n}) = 0$ for $k > n - 3$. Things are similar in higher genus. In fact, when $g > 0$, $n > 0$, Harer [9] (see also [16]) constructs a $(4g - 4 + n)$-dimensional spine for $\mathcal{M}_{g,n}$; thus $H_k(\mathcal{M}_{g,n})$ vanishes for $k > 4g - 4 + n$. The spine in question is constructed starting from the cellular decomposition of $\mathcal{M}_{g,n}$ defined in terms of Strebel differentials. Harer [9] also shows, by a spectral sequence argument, that $H_k(\mathcal{M}_g) = 0$ for $k > 4g - 5$. By
Poincaré duality all this is equivalent to saying that the cohomology with compact support $H^k_c(M_{g,n})$ vanishes for $k \leq d(g,n)$, where

$$d(g,n) = \begin{cases} 
    n - 4 & \text{if } g = 0, \\
    2g - 2 & \text{if } n = 0, \\
    2g - 3 + n & \text{if } g > 0, n > 0.
\end{cases}$$

Looking at the exact sequence of cohomology with compact supports

$$\cdots \to H^k_c(M_{g,n}) \to H^k_c(M_{g,n}) \to H^k_c(M,\partial M_{g,n}) \to H^k_c(M_{g,n}) \to \cdots$$

then proves

**Lemma (2.5).** The homomorphism $H^k_c(M_{g,n}) \to H^k_c(M,\partial M_{g,n})$ is an isomorphism for $k < d(g,n)$ and is injective for $k = d(g,n)$.

Let $g$ and $n$ be non-negative integers such that $2g - 2 + n > 0$, and let $P$ be a set with $n$ elements. Denote by $D_1, D_2, \ldots$ the different irreducible components of $\partial M_{g,P}$. Each of these is the image of a map $\mu_i : X_i \to \overline{M}_{g,P}$, where $X_i$ can be of two different kinds. Either $X_i = \overline{M}_{g-1,P\cup(q,r)}$, or else $X_i = \overline{M}_{g,A\cup\{q\}} \times \overline{M}_{b,B\cup\{r\}}$, where $a + b = g$, $A \coprod B = P$, and both $2a - 2 + |A|$ and $2b - 2 + |B|$ are non-negative. In any case $q$ and $r$ are distinct points not belonging to $P$, and the map $\mu_i$ is gotten by identifying $q$ and $r$.

**Lemma (2.6).** The map $H^k_c(M_{g,P}) \to \bigoplus_i H^k(X_i)$ is injective whenever $H^k_c(M_{g,P}) \to H^k_c(M,\partial M_{g,P})$ is.

The proof uses a bit of Hodge theory, in the form of the following result of Deligne.

**Proposition (2.7) ([4], Proposition (8.2.5)).** Let $Y$ be proper. If $u : X \to Y$ is a proper surjective morphism, and $X$ is smooth, then the weight $k$ quotient of $H^k(Y,\mathbb{Q})$ is the image of $H^k(Y,\mathbb{Q})$ in $H^k(X,\mathbb{Q})$.

In our application, $Y$ is $\partial M_{g,P}$, and $X$ is the disjoint union of the $X_i$. Of course, Deligne’s result is stated for varieties and not for orbifolds, and the $X_i$ are smooth as orbifolds, but usually not as varieties. There are at least two ways out. One is to convince oneself that Deligne’s proof works also in the orbifold context. The other is to appeal to the results of Looijenga [15] and Boggi-Pikaart [2] which imply that each of the $X_i$ is the quotient of a smooth variety $Z_i$ by the action of a finite group; one may then take as $X$ the disjoint union of the $Z_i$ and prove injectivity of the map $H^k_c(M_{g,P}) \to H^k(X) = \bigoplus_i H^k(Z_i)$, which obviously implies the injectivity of $H^k_c(M_{g,P}) \to \bigoplus_i H^k(X_i)$.

Whatever road we choose, the proof proceeds as follows. The homomorphism

$$\rho : H^k_c(M_{g,P}) \to H^k_c(M,\partial M_{g,P})$$

is a morphism of mixed Hodge structures, and hence is strictly compatible with the filtrations. Thus

$$\rho(H^k_c(M_{g,P})) \cap W_{k-1}(H^k_c(M,\partial M_{g,P})) = \rho(W_{k-1}(H^k_c(M_{g,P}))) = \rho(\{0\}) = \{0\},$$

since $H^k_c(M_{g,P})$ is pure of weight $k$. As we are assuming that $\rho$ is injective, this shows that $H^k_c(M_{g,P})$ injects into $H^k_c(M,\partial M_{g,P})/W_{k-1}(H^k_c(M,\partial M_{g,P}))$. On the other hand (2.7) says that $H^k_c(M,\partial M_{g,P})/W_{k-1}(H^k_c(M,\partial M_{g,P}))$ injects into $H^k(X)$.

In view of (2.5), an immediate corollary of Lemma (2.6) is the following result.
Proposition (2.8). The map \( H^k(\mathcal{M}_{g,P}) \to \bigoplus_i H^k(X_i) \) is injective whenever \( k \leq d(g,n) \), where \( n = |P| \).

Proposition (2.8) makes it possible to give a quick inductive proof of (2.1), based on the following intermediate result.

Lemma (2.9). Let \( k \) be an odd integer, \( h \) a non-negative integer, and suppose \( H^q(\mathcal{M}_{g,n}) = 0 \) for all odd \( q \leq k \), all \( g \leq h \), and all \( n \) such that \( q > d(g,n) \). Then \( H^q(\mathcal{M}_{g,n}) = 0 \) for all odd \( q \leq k \), all \( g \leq h \), and all \( n \). If \( H^q(\mathcal{M}_{g,n}) = 0 \) for all odd \( q \leq k \) and all \( g \) and \( n \) such that \( q > d(g,n) \), then \( H^q(\mathcal{M}_{g,n}) = 0 \) for all odd \( q \leq k \) and all \( g \) and \( n \).

Clearly, it suffices to prove the first assertion. We argue by induction on \( k \). We may assume, inductively, that \( H^q(\mathcal{M}_{g,n}) = 0 \) for all odd \( q < k \), all \( g \leq h \) and all \( n \). If \( g \leq h \) and \( k \leq d(g,n) \), Proposition (2.8) says that \( H^k(\mathcal{M}_{g,n}) \) injects into a direct sum of vector spaces \( H^k(X_i) \), where \( X_i = \mathcal{M}_{g-1,n+2} \) or a product of two moduli spaces \( \mathcal{M}_{a,a} \) and \( \mathcal{M}_{b,b} \) such that \( a + b = g \), \( \alpha + \beta = n + 2 \); in the latter case either \( a < g \) or \( a = g \) and \( \alpha < n \), and similarly for \( b \) and \( \beta \). By the Künneth formula, \( H^k(\mathcal{M}_{g,n}) \) injects into the direct sum of \( H^k(\mathcal{M}_{g-1,n+2}) \) and of all the tensor products \( H^l(\mathcal{M}_{a,a}) \otimes H^m(\mathcal{M}_{b,b}) \) with \( l + m = k \).

Since either \( l \) or \( m \) must be odd, the induction hypothesis guarantees that all these tensor products vanish, except possibly those for which \( l \) or \( m \) is zero. This means that \( H^k(\mathcal{M}_{g,n}) \) injects into a direct sum of vector spaces \( H^k(\mathcal{M}_{g,n}) \) such that \( \gamma < g \) or \( \gamma = g \) and \( \nu < n \). But then the result follows by double induction on \( g \) and \( n \).

Lemma (2.9) reduces the proof of the vanishing of odd cohomology (so long as it does vanish!) to checking it explicitly for finitely many values of \( g \) and \( n \) in each odd degree \( k \), that is, those for which \( k > d(g,n) \). When \( k = 1 \) this means \( g = 0 \), \( n \leq 4 \) or \( g = n = 1 \). Now, \( \mathcal{M}_{0,3} \) is a point, while \( \mathcal{M}_{0,4} \) and \( \mathcal{M}_{1,1} \) are both isomorphic to the projective line, so the first cohomology groups of all three are zero. This concludes the proof of (2.1) in case \( k = 1 \). For \( k = 3,5 \) the initial cases of the induction are slightly more complicated and will be dealt with in section 3. Lemma (2.9) cannot be applied as such if \( k \geq 11 \), since it is known that \( H^{11}(\mathcal{M}_{1,1}) \) is not zero. So far as we know, the cases \( k = 7,9 \) are still open.

We now turn to the proof of (2.2). Clearly, all that has to be shown is that \( H^2(\mathcal{M}_{g,n}) \) is always generated by tautological classes, the relations among these having already been determined in section 1. As was the case for (2.1), the proof is by double induction on \( g \) and \( n \). Here we will describe the inductive step in genus 3 or more. The cases of lower genus are a bit more involved, and will be treated in section 4. The initial cases of the induction, that is, those for which \( 2 > d(g,n) \), will be dealt with in section 3.

Our strategy for the inductive step is quite simple. Suppose we want to show that \( H^2(\mathcal{M}_{g,n}) \) is generated by tautological classes, assuming the same is known to be true in genus less then \( g \), or in genus \( g \) but with fewer than \( n \) marked points. Proposition (2.8) shows that \( H^2(\mathcal{M}_{g,n}) \) injects into the direct sum of the second cohomology groups of the \( X_i \). By induction hypothesis, these are generated by tautological classes, all relations among which are known. By (1.3) and (1.4), we have complete control on the effect of each map \( H^2(\mathcal{M}_{g,n}) \to H^2(X_i) \) on tautological classes, so that, at least in principle, we can decide which classes in \( \bigoplus_i H^2(X_i) \) come from tautological classes on \( H^2(\mathcal{M}_{g,n}) \). On
the other hand, let α be any class in \( H^2(\mathcal{M}_{g,n}) \); if we denote by \( \alpha_i \) its pullback to \( X_i \), these classes satisfy obvious compatibility relations on the “intersections” of the \( X_i \). The subspace of \( \bigoplus H^2(X_i) \) defined by these compatibility relations can be completely described using (1.3) and (1.4), at least in principle, because the spaces \( H^2(X_i) \) are generated by tautological classes. What we will show, in essence, is that it coincides with the one generated by the images of the tautological classes of \( H^2(\mathcal{M}_{g,n}) \). By the injectivity of \( H^2(\mathcal{M}_{g,n}) \to \bigoplus H^2(X_i) \), this will conclude the proof.

A first step in the strategy outlined above is the following result, which is also of independent interest.

**Theorem (2.10).** Let \( g \geq 1 \) be an integer, let \( P \) be a finite set such that \( 2g - 2 + |P| > 0 \), and let \( q, r \) be distinct and not belonging to \( P \). Then, if \( \xi : \mathcal{M}_{g-1, P\cup\{q,r\}} \to \mathcal{M}_{g, P} \) is the morphism obtained by identifying the points labelled by \( q \) and \( r \), the pullback map \( \xi^* : H^k(\mathcal{M}_{g, P}) \to H^k(\mathcal{M}_{g-1, P\cup\{q,r\}}) \) is injective for any \( k \leq 2g - 2 \) if \( g \leq 7 \), and for any \( k \leq g + 5 \) if \( g \geq 7 \).

This we will prove by triple induction on \( k, g \) and \( n = |P| \). The statement is true when \( k = 0 \), and also when \( k = 1 \), since \( H^1(\mathcal{M}_{g,n}) = 0 \) for any \( g \) and \( n \). Suppose then that \( g \leq 7 \), that \( k \leq 2g - 2 \), and that the result is known to hold for all triples \( (k', g', n') \) such that either \( k' < k \), or \( k' = k \) and \( g' < g \), or \( k' = k \), \( g' = g \), and \( n' < n \). In view of (2.8), what we have to show is that, if \( x \) is any element of \( H^k(\mathcal{M}_{g, P}) \) such that \( \xi^*(x) = 0 \), then \( x \) pulls back to zero under any one of the maps

\[
\mathcal{M}_{a, A\cup\{s\}} \times \mathcal{M}_{b, B\cup\{t\}} \to \mathcal{M}_{g, P},
\]

where \( g = a + b \) and \( P = A \bigcup B \). By the Künneth formula, \( H^k(\mathcal{M}_{a, A\cup\{s\}} \times \mathcal{M}_{b, B\cup\{t\}}) \) breaks up into a direct sum of summands \( H^l(\mathcal{M}_{a, A\cup\{s\}}) \otimes H^m(\mathcal{M}_{b, B\cup\{t\}}) \), where \( k = l + m \). Thus we have to show that \( x \) goes to zero under any one of the maps

\[
\rho : H^k(\mathcal{M}_{g, P}) \to H^l(\mathcal{M}_{a, A\cup\{s\}}) \otimes H^m(\mathcal{M}_{b, B\cup\{t\}}).
\]

Suppose \( l \geq 2a - 1 \) and \( m \geq 2b - 1 \); then \( k = l + m \geq 2(a + b) - 2 = 2g - 2 \). The only possibility is that \( l = 2a - 1 \), \( m = 2b - 1 \); in particular, \( l \) and \( m \) are both odd. Since they add to \( k \leq 2g - 2 \leq 12 \), one of them must equal 5 or less. In view of (2.1), this implies that \( H^l(\mathcal{M}_{a, A\cup\{s\}}) \otimes H^m(\mathcal{M}_{b, B\cup\{t\}}) = 0 \), so we are done in this case. We may then suppose that either \( l \leq 2a - 2 \) or \( m \leq 2b - 2 \). Say \( l \leq 2a - 2 \); this implies, in particular, that \( a > 0 \). Then \( \rho \) fits into a commutative diagram

\[
\begin{array}{ccc}
H^k(\mathcal{M}_{g, P}) & \xrightarrow{\rho} & H^l(\mathcal{M}_{a, A\cup\{s\}}) \otimes H^m(\mathcal{M}_{b, B\cup\{t\}}) \\
\xi^* & & \lambda \\
H^k(\mathcal{M}_{g-1, P\cup\{q,r\}}) & \xrightarrow{\rho} & H^l(\mathcal{M}_{a-1, A\cup\{s,q,r\}}) \otimes H^m(\mathcal{M}_{b, B\cup\{t\}})
\end{array}
\]

If \( a = g \), that is, if \( b = 0 \), then \( |B| > 1 \), and hence \( |A\cup\{s\}| < n \). Thus \( \lambda \) is always injective, by induction hypothesis. Since \( \xi^*(x) = 0 \), and so \( \lambda \rho(x) = 0 \), this implies that \( \rho(x) = 0 \), as desired.
When \( g > 7 \) and \( k \leq g + 5 \), the argument is similar, but simpler. Set \( f(n) = 2n - 2 \) if \( n \leq 7 \) and \( f(n) = n + 5 \) if \( n \geq 7 \). To show that \( \rho(x) = 0 \) we may argue as in the previous case, provided we can show that either \( a > 0 \) and \( l \leq f(a) \) or \( b > 0 \) and \( m \leq f(b) \). If \( a = 0 \), then \( m \leq k \leq f(g) = f(b) \), and similarly if \( b = 0 \); we may therefore assume that \( a > 0 \) and \( b > 0 \). Three cases are possible. Suppose first that \( a \leq 7 \) and \( b \leq 7 \). If \( l > f(a) = 2a - 2 \), \( m > f(b) = 2b - 2 \), then \( k \geq 2g - 2 > g + 5 \), against the assumptions. Suppose next that \( a \leq 7 \) and \( b > 7 \). If \( l > f(a) = 2a - 2 \), \( m > f(b) = b + 5 \), then \( k \geq a + g + 5 > g + 5 \), contrary to what we have assumed. Suppose finally that \( a \geq 7 \) and \( b \geq 7 \). If \( l > f(a) = a + 5 \), \( m > f(b) = b + 5 \), then \( k > g + 10 > g + 5 \), again against the assumptions.

**Remark (2.11).** The above argument actually shows that, if \( v \) is an odd integer and we know that the odd cohomology of all the \( \overline{M}_{g,n} \) vanishes in degree not exceeding \( v \), then \( H^k(\overline{M}_{g,p}) \to H^k(\overline{M}_{g-1,P∪\{q,r\}}) \) is injective for \( k \leq 2g - 2 \) if \( g \leq v + 2 \), and for \( k \leq g + v \) if \( g \geq v + 2 \). Thus we could improve slightly on (2.10) if we could prove the vanishing of the odd cohomology in degree greater than 5. In a different direction, just knowing that the first cohomology vanishes, which is all that has been fully proved up to now, suffices to show that \( H^2(\overline{M}_{g,p}) \) injects into \( H^2(\overline{M}_{g-1,P∪\{q,r\}}) \) as soon as \( g \geq 2 \). This is the only consequence of (2.10) that we will need.

We are now in a position to describe the inductive step in the proof of (2.2), in genus 3 or more. As we announced, the cases of lower genus will be treated in section 4.

Let then \( g \geq 3 \) be an integer, and let \( P \) be a finite set. If \( P \) is not empty, let \( p \) be a fixed element of \( P \). Let \( q, r \) be distinct and not belonging to \( P \). Let \( \xi : M_{g-1,P∪\{q,r\}} \to M_{g,P} \) be the map that is obtained by identifying the points labelled by \( q \) and \( r \). We wish to show that \( H^2(\overline{M}_{g,p}) \) is generated by tautological classes, assuming the analogous statement is known to hold for \( \overline{M}_{g-1,v} \), whenever \( \gamma < g \) or \( \gamma = g \) and \( v < |P| \). We will do this only for \( P \neq \emptyset \), the argument for \( P = \emptyset \) being entirely similar. Let \( y \) be any element of \( H^2(\overline{M}_{g,p}) \).

The pullback \( \xi^*(y) \) is invariant under the operation of interchanging \( q \) and \( r \). Therefore, by the induction assumptions, it is a linear combination of \( \kappa_1 \), the \( \psi_i \), \( \psi_q + \psi_r \), \( \delta_{irr} \), and the classes \( \delta_{u,U} \), \( \delta_{u,U∪\{q,r\}} \) and \( \delta_{u,U∪\{q\}} + \delta_{u,U∪\{r\}} \), where \( u \) is any integer between 0 and \( g \) and \( U \) runs through all subsets of \( P \) containing \( p \); when \( g = 3 \), we can even do without \( \kappa_1 \). Formulas (1.3) tell us that there is a linear combination \( z \) of tautological classes such that the pullback of \( x = y - z \) is of the form

\[
\xi^*(x) = f(\psi_q + \psi_r) + \sum_{0 \leq u \leq g-2} g_{u,U} \delta_{u,U∪\{q,r\}} + \sum_{0 \leq u \leq g-1} h_{u,U}(\delta_{u,U∪\{q\}} + \delta_{u,U∪\{r\}})
\]

for suitable coefficients \( f, g_{u,U}, h_{u,U} \). In case \( g = 3 \), we may even assume, using (1.12), that \( f = 0 \). We will show that, in fact, \( \xi^*(x) = 0 \); (2.10) and (2.11) will then tell us that \( x \) itself vanishes, proving that \( y \) is a linear combination of tautological classes, as desired.

Suppose \( s \notin P∪\{q,r\} \), and let \( \varphi : M_{g-1,P∪\{s\}} \to M_{g,P} \) be the map that is obtained by attaching a fixed elliptic tail at the point labelled by \( s \). Look at the diagram

\[
\begin{align*}
\overline{M}_{g-1,P∪\{s\}} & \xrightarrow{\varphi} \overline{M}_{g,P} \\
\overline{M}_{g-1,P∪\{q,r\}} & \xrightarrow{\xi} \overline{M}_{g,P}
\end{align*}
\]
where $\varphi$ attaches the point labelled by $t$ of a sphere marked by $\{t, q, r\}$ to the point labelled by $s$ of a variable curve in $\overline{\mathcal{M}}_{g-1, P \cup \{s\}}$. This diagram is commutative up to homotopy. The identity $\varphi^* \xi^* (x) = \vartheta^* (x)$, together with formulas (2.12) and (1.4), applied to $\varphi$, implies that

$$
\vartheta^* (x) = \sum_{p \in U, 0 \leq u \leq g-2} g_{u, U} \delta_{u, U \cup \{s\}}.
$$

Now consider the commutative diagram

$$
\begin{array}{ccc}
\overline{\mathcal{M}}_{g-2, P \cup \{q, r, s\}} & \xrightarrow{\beta} & \overline{\mathcal{M}}_{g-1, P \cup \{q, r\}} \\
\gamma \downarrow & & \xi \downarrow \\
\overline{\mathcal{M}}_{g-1, P \cup \{s\}} & \xrightarrow{\vartheta} & \overline{\mathcal{M}}_{g, P}
\end{array}
$$

where $\gamma$ and $\beta$ are the analogues of $\xi$ and $\vartheta$, respectively. If we write down explicitly the identity $\gamma^* \vartheta^* (x) = \beta^* \xi^* (x)$ using formulas (2.12), (2.13), (1.3), and (1.4), we get a relation

$$
\sum_{p \in U, 0 \leq u \leq g-2} g_{u, U} \left( \delta_{u, U \cup \{s\}} + \delta_{u-1, U \cup \{s, q, r\}} \right)
$$

$$
= f(\psi_q + \psi_r) + \sum_{p \in U, 0 \leq u \leq g-2} g_{u, U} \left( \delta_{u, U \cup \{q, r\}} + \delta_{u-1, U \cup \{s, q, r\}} \right)
$$

$$
+ \sum_{p \in U, 0 \leq u \leq g-1} h_{u, U} \left( \delta_{u, U \cup \{q\}} + \delta_{u, U \cup \{r\}} + \delta_{u-1, U \cup \{s\}} + \delta_{u-1, U \cup \{r\}} \right)
$$

in $H^2(\overline{\mathcal{M}}_{g-2, P \cup \{q, r, s\}})$. If $g \geq 4$, all the tautological classes appearing in (2.15) are independent, so $f = g_{u, U} = h_{u, U} = 0$ for all $u$ and $U$. When $g = 3$, we already know that $f = 0$; since the boundary classes are independent in genus 1, we conclude that $g_{u, U} = h_{u, U} = 0$ for all $u$ and $U$ in this case as well. This shows that $\xi^* (x) = 0$, as desired.

3. The initial cases of the induction

In this section we calculate those cohomology groups which are needed to start the inductive proofs of (2.1) and (2.2). More exactly, we shall compute the $k$-th cohomology group of $\overline{\mathcal{M}}_{g, n}$ for all $k$, $g$ and $n$ such that $k \leq 3$ or $k = 5$ and $d(g, n) < k$. Our treatment will be elementary and self-contained for $k \leq 3$, while for $k = 5$ we shall use, directly or indirectly, some of the results of [6], [7], and [14].

We have already settled the case $k = 1$ in the body of the proof of (2.1). For $k = 2$, the values of $g$ and $n$ involved are $g = 0$ and $n \leq 5$, and $g = 1$ and $n \leq 2$. For $k = 3$ they are $g = 0$ and $n \leq 6$, $g = 1$ and $n \leq 3$, and $g = 2$ and $n \leq 1$, while for $k = 5$ they are $g = 0$ and $n \leq 8$, $g = 1$ and $n \leq 5$, $g = 2$ and $n \leq 3$, and $g = 3$ and $n \leq 1$.

As we said, in genus zero we rely on Keel’s results [13], although the computations could be easily done directly. What Keel shows, among other things, is that $H^k(\overline{\mathcal{M}}_{0, n})$
vanishes for all odd \( k \), and that \( H^2(\overline{M}_{0,n}) \) is generated by tautological classes, modulo the relations described in (1.6), and has dimension \( 2^n - \binom{n}{2} - 1 \). In the range we shall have to examine, that is, \( 3 \leq n \leq 6 \), the dimensions are \( 0, 1, 5, 16 \), respectively.

We now turn to higher genus. Since \( \overline{M}_{1,1} \) is isomorphic to \( \mathbb{P}^1 \), its second cohomology is one-dimensional (and generated by \( \delta_{\text{irr}} \)). We next show that there are surjective morphisms \( \alpha : \overline{M}_{0,6} \to \overline{M}_{2,0} \) and \( \beta : \overline{M}_{0,7} \to \overline{M}_{2,1} \). This implies that the third cohomology groups of \( \overline{M}_{2,0} \) and \( \overline{M}_{2,1} \) vanish. Let \( (C; p_1, \ldots, p_6) \) be a 6-pointed stable genus zero curve. The morphism \( \alpha \) associates to it the stable model of the double admissible covering of \( C \) branched at the \( p_i \). As for \( \beta \), the image under it of a 7-pointed stable genus zero curve \( (C; p_1, \ldots, p_7) \) is defined to be the stable model of \( (D; q) \), where \( D \) is the double admissible covering of \( C \) branched at \( p_1, \ldots, p_6 \), and \( q \) is one of the points lying above \( p_7 \). Notice that it is immaterial which of the two possible choices for \( q \) we make, since they yield isomorphic 1-pointed curves.

To complete our analysis for \( k = 2, 3 \) it remains to compute the second cohomology of \( \overline{M}_{1,2} \) and the third cohomology of \( \overline{M}_{1,3} \). It will suffice to prove the following.

**Lemma (3.1).** \( h^2(\overline{M}_{1,2}) = 2, h^3(\overline{M}_{1,3}) = 0. \)

In fact, there are exactly two boundary classes in \( H^2(\overline{M}_{1,2}) \), namely \( \delta_{\text{irr}} \) and \( \delta_{1,\emptyset} \), which are independent by part \( i) \) of (1.9). Thus the first part of (3.1) implies that \( \delta_{\text{irr}} \) and \( \delta_{1,\emptyset} \) generate \( H^2(\overline{M}_{1,2}) \).

The proof of the second part of (3.1) relies on Theorem (2.2), and hence also on the first part, in that we will use the fact that \( H^2(\overline{M}_{1,3}) \) is freely generated by boundary classes. Since the boundary classes are \( \delta_{\text{irr}}, \delta_{1,\emptyset}, \delta_{1,\{i\}}, \delta_{1,\{1\}}, \delta_{1,\{2\}}, \) and \( \delta_{1,\{3\}} \), this shows in particular that \( h^2(\overline{M}_{1,3}) = 5 \). As we know that \( h^1(\overline{M}_{1,2}) = h^1(\overline{M}_{1,3}) = 0 \), Poincaré duality implies that \( \chi(\overline{M}_{1,2}) = 2 + h^2(\overline{M}_{1,2}) \) and \( \chi(\overline{M}_{1,3}) = 12 - h^3(\overline{M}_{1,3}) \). Lemma (3.1) is then a consequence of the following result.

**Lemma (3.2).** \( \chi(\overline{M}_{1,2}) = 4, \chi(\overline{M}_{1,3}) = 12. \)

The proof of the lemma rests on the following simple remark. For any space \( X \) we denote by \( \chi_c(X) \) the Euler characteristic of \( X \) with compact supports, that is, the alternating sum of the dimensions of the \( \mathbb{Q} \)-cohomology groups of \( X \) with compact supports. Now suppose \( X \) is a quasi-projective algebraic variety, and let

\[
X = \overline{X}_d \supset \overline{X}_{d-1} \supset \cdots \supset \overline{X}_1 \supset \overline{X}_0
\]

be a filtration of \( X \) by closed subvarieties. Suppose that \( X_i = \overline{X}_i \setminus \overline{X}_{i-1} \) is of pure dimension \( i \) (or is empty) for every \( i \). Then

\[
\chi_c(X) = \sum \chi_c(X_i).
\]

This can be proved by induction on \( d \). There is nothing to prove when \( d = 0 \). Now assume the result known for \( \overline{X}_{d-1} \); thus

\[
\chi_c(\overline{X}_{d-1}) = \sum_{i < d} \chi_c(X_i).
\]
On the other hand the exact sequence of cohomology with compact supports

\[ \cdots \to H^j_c(X_d) \to H^j_c(X) \to H^j_c(X_{d-1}) \to \cdots \]

shows that

\[ \chi_c(X) = \chi_c(X_d) + \chi_c(X_{d-1}) = \chi_c(X_d) + \sum_{i<d} \chi_c(X_i) = \sum \chi_c(X_i). \]

There are two special cases when \( \chi_c(X) = \chi(X) \). The first is obviously the one when \( X \) is compact. The other is when \( X \) is an orbifold; in this case Poincaré duality implies that \( h^q_c(X) = h^{2d-q}(X) = h^{2d-q}(X) \), and hence

\[ \chi_c(X) = \sum (-1)^q h^q_c(X) = \sum (-1)^{2d-q} h^{2d-q}(X) = \chi(X). \]

Thus, when the \( X_i \) are orbifolds, (3.3) translates into

\[ \chi_c(X) = \sum \chi(X_i). \]

If \( X \) is compact, or an orbifold, this in turn yields

\[ \chi(X) = \sum \chi(X_i). \]

We will first use these formulas to compute the Euler characteristics of the spaces \( M_{0,n} \) for \( n \leq 6 \). Since \( M_{0,4} \) can be identified with the complex plane minus two points,

\[ \chi(M_{0,4}) = -1. \]

We next deal with \( M_{0,5} \). This space can be identified with the complement, inside the \( \mathbb{C}^2 \) with coordinates \( z_1, z_2 \), of the lines \( z_1 = 0, z_1 = z_1, z_2 = 0, z_2 = 1, z_1 = z_2 \). Put otherwise, if \( x_0, x_1, x_2 \) are homogeneous coordinates in \( \mathbb{P}^2 \), then \( M_{0,5} \) can be identified with the complement, inside \( \mathbb{P}^2 \), of the six projective lines

\[ x_0 = 0, x_1 = 0, x_2 = 0, x_0 = x_1, x_0 = x_2, x_1 = x_2. \]

Each of these lines has exactly 3 points which are in common with some of the others, and there are 7 points where two or more lines meet. Applying (3.5) to the filtration 7 points \( \subset \) union of 6 lines \( \subset \mathbb{P}^2 \) then gives

\[ 3 = \chi(\mathbb{P}^2) = \chi(M_{0,5}) + 6\chi(M_{0,4}) + 7 = \chi(M_{0,5}) - 6 + 7, \]

that is,

\[ \chi(M_{0,5}) = 2. \]
The same method can be applied to calculate the Euler characteristic of \( \mathcal{M}_{0,6} \). This space can be identified with the complement inside \( \mathbb{P}^3 \) of the 10 planes

\[
x_i = 0, \quad 0 \leq i \leq 3; \quad x_i = x_j, \quad 0 \leq i < j \leq 3.
\]

Each nine of these cut on the remaining one a configuration of 6 lines which is identical to the one occurring in the analysis of \( \mathcal{M}_{0,5} \) we just completed. In addition, three of these lines are common to three planes, and three only to two. Hence the total number of lines is 25. There are exactly 15 points where three independent planes meet, namely the points all of whose homogeneous coordinates are either 0 or 1. Applying formula (3.5) then gives

(3.8) \[ \chi(\mathcal{M}_{0,6}) = -6. \]

We next deal with the genus 1 case. Since \( \mathcal{M}_{1,1} \) can be identified with the complex plane,

(3.9) \[ \chi(\mathcal{M}_{1,1}) = 1. \]

We will now show that

(3.10) \[ \chi(\mathcal{M}_{1,2}) = 1. \]

The proof of this fact and the remainder of the computation of the Euler characteristics of \( \overline{\mathcal{M}}_{1,2} \) and \( \overline{\mathcal{M}}_{1,3} \) require the calculation of the Euler characteristics of the quotients of some spaces \( \mathcal{M}_{0,n} \) by certain group actions. We will denote by \( \mathcal{M}'_{0,4} \) and \( \mathcal{M}'_{0,5} \) the quotients of \( \mathcal{M}_{0,4} \) and \( \mathcal{M}_{0,5} \) modulo the operation of interchanging the labelling of two of the marked points, and by \( \mathcal{M}''_{0,5} \) the quotient of \( \mathcal{M}_{0,5} \) modulo permutations of the labellings of three of the marked points. We claim that

(3.11) \[ \chi(\mathcal{M}'_{0,4}) = 0; \quad \chi(\mathcal{M}'_{0,5}) = 1; \quad \chi(\mathcal{M}''_{0,5}) = 1. \]

The map \( \mathcal{M}_{0,4} \to \mathcal{M}'_{0,4} \) has degree 2. We claim that there is a unique fiber consisting of only one point, so that

\[-1 = \chi(\mathcal{M}_{0,4}) = 2\chi(\mathcal{M}'_{0,4}) - 1,\]

proving the first identity in (3.11). Suppose in fact that there is an isomorphism between the two 4-pointed curves \( (\mathbb{P}^1; 0, \infty, 1, x) \) and \( (\mathbb{P}^1; 0, \infty, x, 1) \). This means that there is an automorphism \( \alpha \) of \( \mathbb{P}^1 \) such that \( \alpha(0) = 0, \alpha(\infty) = \infty, \alpha(1) = x, \) and \( \alpha(x) = 1. \) The first two conditions imply that \( \alpha \) is of the form \( \alpha(z) = az \), for some nonzero complex number \( a \). The last two conditions say that \( x = a \) and \( a^2 = 1. \) Thus the curve in question, up to isomorphism, is \( (\mathbb{P}^1; 0, \infty, 1, -1) \). The same kind of argument proves the last identity in (3.11). In fact, let \( x, y \) be distinct and different from 0, 1, \( \infty \), and let \( \alpha \) be a non-trivial automorphism of \( \mathbb{P}^1 \) which fixes 0 and \( \infty \) and permutes \( 1, x, y \); observe that none of these last three points can be fixed. Clearly, \( \alpha \) is of the form \( \alpha(z) = az \) for some nonzero complex number \( a \). On the other hand, suppose for instance that \( \alpha(1) = x, \alpha(x) = y \) and \( \alpha(y) = 1. \) Then \( x = a, y = a^2 \) and \( a^3 = 1. \) This shows that the degree 6 morphism \( \mathcal{M}_{0,5} \to \mathcal{M}'_{0,5} \) has exactly one fiber which consists of fewer than 6 points, namely the fiber made up of
This proves the first statement in (3.2), and consequently also the first one in (3.1).

We now return to $M_{1,2}$. Let $U$ be the open subset of $M_{1,2}$ consisting of those curves $(C; p_1, p_2)$ such that, if $\tau$ stands for the $-1$ involution about the origin $p_2$, then $\tau(p_1) \neq p_1$. We may associate to each 5-pointed rational curve $(\mathbb{P}^1; q_1, \ldots, q_5)$ the 2-pointed genus one curve $(C; p_1, p_2)$, where $C$ is the double covering of $\mathbb{P}^1$ branched at $q_2, \ldots, q_5$, and $p_1, p_2$ map to $q_1$ and $q_2$, respectively (notice that the two possible choices of $p_2$ give isomorphic 2-pointed curves). This defines a morphism from $M_{0,5}$ to $M_{1,2}$, which clearly factors through an isomorphism between $M''_{0,5}$ and $U$. On the other hand the complement of $U$ in $M_{1,2}$ can be identified with $M'_{0,4}$, so that applying (3.5) yields

$$\chi(M_{1,2}) = \chi(M''_{0,5}) + \chi(M'_{0,4}) = 1,$$

which is just (3.10). To compute the Euler characteristic of $\overline{M}_{1,2}$ we use the stratification by graph type. The open strata other than $M_{1,2}$ are indexed by the graphs in Figure 1. Here, and elsewhere, we adopt the convention that a solid dot stands for a component of genus zero, and a hollow one for a component of genus one.

![Figure 1](image_url)

There are two one-dimensional strata $V_1$ and $V_2$, corresponding to the first two graphs, while the zero-dimensional strata are the two points corresponding to the last two graphs. It is evident that $V_1$ is isomorphic to $M'_{0,4}$, and $V_2$ to $M_{1,1}$. Thus

$$\chi(\overline{M}_{1,2}) = \chi(M_{1,2}) + \chi(M'_{0,4}) + \chi(M_{1,1}) + 2 = \chi(M_{1,2}) + 3 = 4.$$

This proves the first statement in (3.2), and consequently also the first one in (3.1).

The next step is to compute the Euler characteristic of $M_{1,3}$. We imitate the argument used for $M_{1,2}$. Let $U$ be the open subset of $M_{1,3}$ consisting of those curves $(C; p_1, p_2, p_3)$ such that, if $\tau$ stands for the $-1$ involution about the origin $p_3$, then $p_1 \neq \tau(p_1) \neq p_2 \neq \tau(p_2)$. Also, let $H$ be the subvariety of $M_{1,6}$ consisting of all curves $(C; p_1, p_2, p_3, p_4, p_5, p_6)$ such that, if $\tau$ is as above, then $p_4, p_5$ and $p_6$ are fixed by $\tau$ and $\tau(p_1) \neq p_2$. There are two dominant maps $\alpha : H \to M_{0,6}$ and $\beta : H \to U$. The first of them sends $(C; p_1, p_2, p_3, p_4, p_5, p_6)$ to $(C/\tau; \overline{p}_1, \overline{p}_2, \overline{p}_3, \overline{p}_4, \overline{p}_5, \overline{p}_6)$, where $\overline{p}_i$ stands for the image of $p_i$. The other map just “forgets” $p_4, p_5$ and $p_6$. The map $\alpha$ is clearly unramified and of degree
2, while $\beta$ has degree 6. We claim that $\beta$ is also unramified. To see this, all we must show is that, given any element $(C; p_1, p_2, p_3)$ of $U$, its only automorphism is the identity. Any automorphism $\sigma$ of $C$ fixing $p_3$ commutes with $\tau$, hence descends to an automorphism $\rho$ of $C/\tau = \mathbb{P}^1$. If, in addition, $\sigma$ fixes $p_1$ and $p_2$, then $\rho$ is an automorphism of $(C/\tau; \bar{p}_1, \bar{p}_2, \bar{p}_3)$. Since this is a 3-pointed rational curve, $\rho$ is the identity, hence $\sigma$ equals $\tau$ or the identity.

On the other hand, the definition of $U$ says that $\tau$ is not an automorphism of $(C; p_1, p_2, p_3)$. Having proved that $\beta$ is unramified, we can conclude that $\chi(U) = 2, \chi(M_{0,6}) = -2$.

At this point, to compute the Euler characteristic of $\mathcal{M}_{1,3}$ we observe that this moduli space is the union of $U$, of three two-dimensional strata $U_1$, $U_2$, and $U_3$, and a one-dimensional stratum $U_4$ which are defined as follows. The points of $U_1$ are the curves $(C; p_1, p_2, p_3)$ such that $\tau(p_1) = p_1$ and $\tau(p_2) \neq p_2$, while $U_2$ is like $U_1$, but with the roles of 1 and 2 interchanged. The points of $U_3$ are the curves $(C; p_1, p_2, p_3)$ such that $\tau(p_1) = p_2$, and those of $U_4$ are the curves $(C; p_1, p_2, p_3)$ such that $\tau(p_1) = p_1$ and $\tau(p_2) = p_2$. Clearly, $U_1$ and $U_2$ are isomorphic to $\mathcal{M}'_{0,5}$, $U_3$ is isomorphic to $\mathcal{M}''_{0,5}$ and $U_4$ to $\mathcal{M}_{0,4}$. Thus

$$\chi(\mathcal{M}_{1,3}) = \chi(U) + 2\chi(\mathcal{M}'_{0,5}) + \chi(\mathcal{M}''_{0,5}) + \chi(\mathcal{M}_{0,4}) = -2 + 2 + 1 - 1 = 0.$$  

To compute the Euler characteristic of $\overline{\mathcal{M}}_{1,3}$ we use the stratification by graph type. The open strata other than $\mathcal{M}_{1,3}$ correspond to the graphs in Figure 2 (plus a labelling of the legs by 1, 2, 3).

![Figure 2](image_url)

There are: one stratum corresponding to graph $A$, isomorphic to $\mathcal{M}'_{0,5}$, one corresponding to graph $B$, isomorphic to $\mathcal{M}_{1,1} \times \mathcal{M}_{0,4}$, three strata corresponding to graph $C$ and isomorphic to $\mathcal{M}_{1,2}$, one stratum corresponding to graph $D$, isomorphic to $\mathcal{M}_{0,4}$, three
corresponding to graph $E$ and isomorphic to $\mathcal{M}_{0,4}'$, three corresponding to graph $F$ and isomorphic to $\mathcal{M}_{1,1}'$, three corresponding to graph $G$ and isomorphic to $\mathcal{M}_{0,4}'$, and seven zero-dimensional strata, all points, three corresponding to graph $H$, three to graph $I$, and one to graph $J$. Putting everything together we conclude that

$$
\chi(\overline{\mathcal{M}}_{1,3}) = \chi(\mathcal{M}_{1,3}) + \chi(\mathcal{M}_{0,5}') + \chi(\mathcal{M}_{1,1})\chi(\mathcal{M}_{0,4}) + 3\chi(\mathcal{M}_{1,2}) \\
+ \chi(\mathcal{M}_{0,4}) + 6\chi(\mathcal{M}_{0,4}) + 3\chi(\mathcal{M}_{1,1}) + 7 = 12,
$$

as desired. Theorem (2.1) is now completely proved for $k = 1, 3$. It remains to examine the initial cases of the induction for $k = 5$, in positive genus. These are: $g = 1$ and $n \leq 5$, $g = 2$ and $n \leq 3$, $g = 3$ and $n \leq 1$. The group $H^5(\overline{\mathcal{M}}_{1,n})$ vanishes when $n \leq 2$ for dimension reasons, and when $n \leq 4$ by Poincaré duality; in fact, $H^5(\overline{\mathcal{M}}_{1,3})$ and $H^5(\overline{\mathcal{M}}_{1,4})$ are Poincaré dual to $H^1(\overline{\mathcal{M}}_{1,3})$ and $H^3(\overline{\mathcal{M}}_{1,4})$. Likewise, $H^5(\overline{\mathcal{M}}_{2,1})$ and $H^5(\overline{\mathcal{M}}_{2,1})$ are Poincaré dual to $H^1(\overline{\mathcal{M}}_{2,1})$ and $H^3(\overline{\mathcal{M}}_{2,1})$, which are both zero. On the other hand, Getzler has shown in [6] that $H^5(\overline{\mathcal{M}}_{1,5})$ vanishes, while in [7] he has proved that $H^5(\overline{\mathcal{M}}_{2,2})$ is zero and announced that $H^5(\overline{\mathcal{M}}_{2,3})$ vanishes as well. At this point (2.9) implies that $H^5(\overline{\mathcal{M}}_{g,n})$ vanishes for $g \leq 2$ and all $n$. In genus $3$ we may argue as follows. Looijenga [14] proves that $H^7(\mathcal{M}_3)$ and $H^9(\mathcal{M}_3,1)$ are zero. By Poincaré duality, this is the same as saying that $H^5_\epsilon(\mathcal{M}_3)$ and $H^5_\delta(\mathcal{M}_3,1)$ vanish, so the exact sequence of cohomology with compact supports shows that $H^5(\mathcal{M}_3)$ and $H^5(\mathcal{M}_3,1)$ inject into $H^5(\partial \mathcal{M}_3)$ and $H^5(\partial \mathcal{M}_3,1)$, respectively. Lemma (2.6) then says that both $H^5(\mathcal{M}_3)$ and $H^5(\mathcal{M}_3,1)$ inject into sums $\bigoplus H^5(X_i)$, where the $X_i$ are products of moduli spaces $\overline{\mathcal{M}}_{g,n}$ with $g < 3$. By what has already been proved, $H^5(X_i) = 0$ for all $i$, hence $H^5(\mathcal{M}_3) = H^5(\mathcal{M}_3,1) = 0$. This concludes the proof of (2.1).

4. The induction step in low genus

In this section we will complete the proof of (2.2); it remains to deal with the genus one and genus two cases.

**Genus 1.** We begin by improving on Lemma (1.14). Let $P$ be a finite set, set $n = |P|$ and let $y$ and $z$ be distinct and not belonging to $P$. As usual, we let

$$
\xi : \overline{\mathcal{M}}_{0,P \cup \{y,z\}} \to \overline{\mathcal{M}}_{1,P}
$$

be the map gotten by identifying the points labelled by $y$ and $z$.

**Lemma (4.1).** The kernel of $\xi^* : H^2(\overline{\mathcal{M}}_{1,P}) \to H^2(\overline{\mathcal{M}}_{0,P \cup \{y,z\}})$ is one-dimensional and is generated by $\delta_{\text{irr}}$.

Lemma (1.14) and part $i)$ of Proposition (1.9) say in particular that $\delta_{\text{irr}}$ is not zero and belongs to the kernel of $\xi^*$. It remains to show that any other element of $\ker(\xi^*)$ is a multiple of $\delta_{\text{irr}}$. The case $|P| = 1$ is trivial. We have seen in the previous section that, when $|P| = 2$, $H^2(\overline{\mathcal{M}}_{1,P})$ has dimension two. On the other hand, the class $\delta_0, \emptyset \in H^2(\overline{\mathcal{M}}_{1,P})$ maps to $\delta_{0,\{y,z\}} \in H^2(\overline{\mathcal{M}}_{0,\emptyset \cup \{y,z\}})$, which is not zero. This takes care of the case when $|P| = 2$. We then proceed by induction on $n = |P|$. Let $\alpha$ be an element of the kernel of
\(\xi^*\). Let \(s, t\) be distinct and not belonging to \(P \cup \{y, z\}\). For any subset \(S\) of \(P\) with at most \(n - 2\) elements consider the diagram

\[
\begin{array}{c}
\overline{\mathcal{M}}_{0, S \cup \{y, z, s\}} \times \overline{\mathcal{M}}_{0, S^c \cup \{t\}} \\
\downarrow \eta \\
\overline{\mathcal{M}}_{1, S \cup \{s\}} \times \overline{\mathcal{M}}_{0, S^c \cup \{t\}}
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow \xi \\
\overline{\mathcal{M}}_{1, P}
\end{array}
\]

where the vertical arrows are obtained by identifying the points labelled by \(y\) and \(z\), and the horizontal ones by identifying the points labelled by \(s\) and \(t\). Using the Künneth formula and the vanishing of \(H^1\) we may write \(\nu_S^*(\alpha) = (\beta, \gamma)\), where \(\beta \in H^2(\overline{\mathcal{M}}_{1, S \cup \{s\}})\) and \(\gamma \in H^2(\overline{\mathcal{M}}_{0, S^c \cup \{t\}})\). From \(\xi^*(\alpha) = 0\) we deduce that \(\eta^*(\beta, \gamma) = 0\). On the other hand \(\eta^*(\beta, \gamma) = (\beta', \gamma)\), where \(\beta'\) is the pullback of \(\beta\) to \(\overline{\mathcal{M}}_{0, S \cup \{y, z, s\}}\). It follows that \(\gamma = 0\) and, by induction hypothesis, that \(\beta = a_S \delta_{\text{irr}}\) for a suitable constant \(a_S\). In other words,

\[\nu_S^*(\alpha) = (a_S \delta_{\text{irr}}, 0)\,.
\]

We now wish to show that, actually, \(a = a_S\) does not depend on \(S\). This will conclude the proof, since then the difference \(a - a_S \delta_{\text{irr}}\) will restrict to zero on all components of \(\partial \overline{\mathcal{M}}_{1, P}\) and hence will be zero by (2.8). To show that \(a_S\) is independent of \(S\) we proceed as follows.

If \(S \neq \emptyset\) write \(S = T \cup \{w\}\), where \(w \not\in T\), and consider the diagram

\[
\begin{array}{c}
\overline{\mathcal{M}}_{1, T \cup \{w\}} \times \overline{\mathcal{M}}_{0, \{w, z, s\}} \times \overline{\mathcal{M}}_{0, S^c \cup \{t\}} \\
\downarrow \sigma \\
\overline{\mathcal{M}}_{1, S \cup \{s\}} \times \overline{\mathcal{M}}_{0, S^c \cup \{t\}}
\end{array} \quad \begin{array}{c}
\longrightarrow \\
\downarrow \nu_T \\
\overline{\mathcal{M}}_{1, P}
\end{array}
\]

where the vertical arrows are obtained by identifying the points labelled by \(y\) and \(z\), and the horizontal ones by identifying the points labelled by \(s\) and \(t\). We find that

\[(a_T \delta_{\text{irr}}, 0, 0) = \tau^*(a_T \delta_{\text{irr}}, 0) = \tau^*(\nu_T^*(\alpha)) = \sigma^*(\nu_S^*(\alpha)) = \sigma^*(a_S \delta_{\text{irr}}, 0) = (a_S \delta_{\text{irr}}, 0, 0)\,.
\]

and hence that \(a_S = a_T\). Repeated applications of this argument show that \(a_S = a_{\emptyset}\) for any \(S\). This proves the lemma.

Because of the relations between tautological classes given in Proposition (1.9), to prove (2.2) in genus one it suffices to show that \(H^2(\overline{\mathcal{M}}_{1, P})\) is generated by boundary classes. More precisely we will show that the classes \(\delta_{\text{irr}}\) and \(\delta_{1, S}\), where \(S\) runs through all subset of \(P\) with at most \(n - 2\) elements, generate \(H^2(\overline{\mathcal{M}}_{1, P})\). We prove this claim by induction on \(n\). The first case \(n = 2\) has already been checked. Denote by \(V = V_{1, P}\) the subspace of \(H^2(\overline{\mathcal{M}}_{1, P})\) generated by the elements \(\delta_{1, S}\), where \(S\) runs through all subset of \(P\) with at most \(n - 2\) elements. In view of Lemma (1.14), to prove our claim it suffices to show that the morphism \(\xi^*\) vanishes modulo \(V\). For this we shall use an explicit basis of \(H^2(\overline{\mathcal{M}}_{0, P \cup \{y, z\}})\) which we will presently describe.
Lemma (4.2). Let $Q$ be a finite set with at least four elements, and let $x, y, z \in Q$ be distinct. Then $\delta_{0,\{y,z\}}$ and all the classes $\delta_{0,S}$ such that $x \in S$ and $2 \leq |S| \leq |Q| - 3$ constitute a basis of $H^2(\overline{M}_{0,Q})$.

As we already recalled, Keel [13] shows that the dimension of $H^2(\overline{M}_{0,n})$ is $2^{n-1} - \binom{n}{2} - 1$. Now denote by $V$ the subspace spanned by the classes listed in the statement of the lemma. Since there are $2^{|Q|-1} - \binom{|Q|}{2} - 1$ of these, it suffices to show that all classes $\delta_{0,T}$ belong to $V$. The only classes that are not already present in the list are those of the form $\delta_{0,\{a,b\}}$, where $x \not\in \{a,b\}$ and $\{a,b\} \neq \{y,z\}$. Let $p, q, r$ be elements of $Q$, all different from $x$ and such that $p \neq q \neq r$. We claim that

$$\delta_{0,\{p,r\}} \equiv \delta_{0,\{q,r\}} \mod V.$$  \tag{4.3}

If $p = q$ there is nothing to prove. If $p \neq q$ then, in view of (1.1), one of Keel’s relations (1.5) is

$$\delta_{0,\{x,p\}} + \sum_{x, p \in S \not\ni q, r \atop 3 \leq |S| \leq |Q| - 3} \delta_{0,\{x,S\}} = \delta_{0,\{x,q\}} + \sum_{x, q \in S \not\ni p, r \atop 3 \leq |S| \leq |Q| - 3} \delta_{0,\{x,S\}},$$

which implies (4.3). One among $y$ and $z$, say $z$, is different from both $a$ and $b$. Two applications of (4.3) then give

$$\delta_{0,\{a,b\}} \equiv \delta_{0,\{a,z\}} \equiv \delta_{0,\{y,z\}} \equiv 0 \mod V,$$

proving the lemma.

We will call a basis such as the one constructed in Lemma (4.2) a canonical basis (with respect to $x, y$ and $z$). To simplify notation, from now on in the genus zero case we shall write $\delta_S$ instead of $\delta_{0,S}$. Let $Q = P \cup \{y, z\}$. Let $\mathcal{B}$ be the canonical basis of $H^2(\overline{M}_{0,Q})$ relative to $x, y, z$. Let $\alpha \in H^2(\overline{M}_{1,P})$. Using the fact that $\xi$ is invariant under the involution exchanging $y$ and $z$, we can write $\xi^*\alpha$ in terms of $\mathcal{B}$:

$$\xi^*\alpha = a_{\{y,z\}}\delta_{\{y,z\}} + \sum_{S \subseteq X, x \in S, |S| \geq 2, |X \setminus S| \geq 1} a_S\delta_S$$

$$+ \sum_{S \subseteq X, x \in S, |X \setminus S| \geq 3} b_S\delta_{S \cup \{y,z\}} + \sum_{S \subseteq X, x \in S, |X \setminus S| \geq 2} c_S(\delta_{S \cup \{y\}} + \delta_{S \cup \{z\}}).$$  \tag{4.4}

Consider a subset $R$ of $P$ such that $|P \setminus R| \geq 2$, and look at the morphism

$$\vartheta_R : \overline{M}_{1, R \cup \{u\}} \to \overline{M}_{1,P}$$

defined by taking a varying stable, genus 1, $R \cup \{u\}$-pointed curve, a fixed stable, genus zero, $(P \setminus R) \cup \{v\}$-pointed curve $C_0$, and identifying the points labelled by $u$ and $v$. As
we already noticed before stating Lemma (1.4), the homomorphism $\vartheta^*_R$ does not depend
on the choice of $C_0$. By induction hypothesis, we have

\[(4.5) \quad \vartheta^*_R(\alpha) = \sum_{S \subset R, |R \setminus S| \geq 1} f^R_S \delta_{1,S} + \sum_{S \subset R, |R \setminus S| \geq 2} g^R_S \delta_{1,S \cup \{u\}} + f^R \delta_{\text{irr}}. \]

Consider the elements $\delta_{1,S} \in H^2(\overline{M}_{1,P})$ with $|P \setminus S| \geq 2$. Recalling the convention about
the symbols $\delta_{a,A}$, Lemma (1.3) says that

\[(4.6) \quad \xi^* \delta_{1,X \setminus \{p,q\}} = \delta_{\{p,q\}} \notin \mathcal{B}, \quad \text{if } p \neq x, q \neq x, \]

\[(4.6) \quad \xi^* \delta_{1,S} = \begin{cases} \delta_{S \cup \{y,z\}} \in \mathcal{B}, & \text{if } x \in S, S \neq P \setminus \{p,q\}, p \neq x, q \neq x, \\ \delta_{P \setminus S} \in \mathcal{B}, & \text{if } x \notin S. \end{cases} \]

So all the elements $\delta_{1,S}$ restrict to elements in $\mathcal{B}$ except when $S = P \setminus \{p,q\}$. We also
have, by lemma (1.4), that

\[(4.7) \quad \vartheta^*_R \delta_{1,S} = \begin{cases} \delta_{1,S} & \text{if } S \subset R, S \neq R, \\ -\psi_u = -\sum_{S' \subset R, |R \setminus S'| \geq 1} \delta_{1,S'} - \frac{1}{12} \delta_{\text{irr}} & \text{if } S = R, \\ 0 & \text{if } S \supset P \setminus R, \end{cases} \]

\text{otherwise.}

Remark (4.8). The only element $\delta_{1,S}$ with $S \subset P$ and $|P \setminus S| \geq 2$ such that, in the
expression of $\vartheta^*_R \delta_{1,S}$, the element $\delta_{1,(R \setminus \{p,q\}) \cup \{u\}}$ appears with non zero coefficient, is
$\delta_{1,P \setminus \{p,q\}}$ and in this case $\vartheta^*_R \delta_{1,X \setminus \{p,q\}} = \delta_{1,R \setminus \{p,q\} \cup \{u\}}$

Let us go back to the expression (4.5). Let $\{r,s\} \subset R' \subset R$. We claim that

\[(4.9) \quad g^R_{R \setminus \{r,s\}} = g^R_{R \setminus \{r,s\}}. \]

Clearly, it suffices to prove the claim in case $R' = R \setminus \{q\}$. Look at the diagram

\[
\begin{array}{ccc}
\overline{M}_{1,R \cup \{u\}} & \xrightarrow{\xi_R} & \overline{M}_{1,P} \\
\varphi \downarrow & & \downarrow \xi_{R \setminus \{q\}} \\
\overline{M}_{1,(R \setminus \{q\}) \cup \{w\}} & & & \end{array}
\]

where the maps are defined in the obvious way: we take the smooth rational curve $C_0$
pointed by $(P \setminus R) \cup \{v\}$, which we used in the definition of $\vartheta_R$, a smooth rational curve
$C'_\eta$ pointed by $\{u, q, t\}$ and we define $C'_\eta$ by identifying the points labelled by $u$ and $v$ in
$C'_\eta$ and $C_0$ respectively; then $\varphi$ is defined by identifying the point labelled by $w$ with the
point of $C'_\eta$ labelled by $t$, while $\vartheta_{R \setminus \{q\}}$ is defined by identifying the point labelled by $w$
with the point of $C'_\eta$ labelled by $t$. To prove the claim just use the preceding remark with
$P = R \cup \{u\}$, together with the fact that $\varphi^* \vartheta^*_R \alpha = \vartheta^*_{R \setminus \{q\}} \alpha$. 25
After this preparation we are ready to modify $\alpha$. Suppose first that $P = \{x, p, q\}$. The first move consists in adding to $\alpha$ a suitable multiple of $\delta_{\{y, z\}}$ so as to make $a_{\{x, p\}} = 0$. The second move consists in adding to $\alpha$ a suitable multiple of $\delta_{\{x\}}$ so as to make $f^R_{\{x\}} = 0$. The third move consists in adding to $\alpha$ a suitable linear combination of $\delta_{\{p\}}$ and $\delta_{\{q\}}$ so as to make $a_{\{x, p\}} = a_{\{x, q\}} = 0$. The three moves, taken in that order, do not interfere with each other. As a result

\begin{equation}
\xi^*\alpha = c_{\{x\}}(\delta_{\{x, y\}} + \delta_{\{x, z\}}),
\end{equation}

\begin{equation}
g^i_{\{x\}}\alpha = f^i_{\{x\}}\delta_{irr}
\end{equation}

Assume now that $|P| \geq 4$. Let $p \neq x$ and $q \neq y$. Observe that, by (4.9), whenever $|P \setminus R| \geq 2$, $|P \setminus S| \geq 2$ and $\{p, q\} \in R \cap S$, we have

\begin{equation}
g^R_{\{p, q\}} = g^R_{R \cap S \setminus \{p, q\}} = g^S_{S \setminus \{p, q\}},
\end{equation}

so that $g^R_{R \setminus \{p, q\}} = \gamma_{p, q}$ does not depend on $R$. Therefore subtracting from $\alpha$ the class $\sum_{X \supset \{p, q\}} \gamma_{p, q}\delta_{\{x \setminus \{p, q\}\}}$, we get that $g^R_{R \setminus \{p, q\}} = 0$, for all $p$, $q$ and $R$ such that $p \neq x$, $q \neq x$, $|P \setminus R| \geq 2$ and $\{p, q\} \in R$.

The second move consists in adding to $\alpha$ a linear combination of elements of type $\delta_{\{y, z\}}$ with $S \neq P \setminus \{p, q\}$, $p \neq x$, $q \neq x$, in such a way that

\begin{equation}
\xi^*\alpha = \sum_{x \in S \subset X \setminus \{X \setminus S \geq 1\}} c_S(\delta_{S \cup \{y\}} + \delta_{S \cup \{z\}}).
\end{equation}

By the above remark, the second move does not alter what has been accomplished by the preceding one. For convenience we shall set $c_T = 0$ when $x \notin T$.

To prove our initial claim we must prove that all the $c_S$ are equal to 0. Consider the square

\begin{equation}
\begin{array}{ccc}
\overline{\mathcal{M}}_{0, R \cup \{y, z, u\}} & \xrightarrow{\eta R} & \overline{\mathcal{M}}_{0, P \cup \{y, z\}} \\
\eta \downarrow & & \xi \uparrow \\
\overline{\mathcal{M}}_{1, R \cup \{u\}} & \xrightarrow{\xi R} & \overline{\mathcal{M}}_{1, P}
\end{array}
\end{equation}

where $\eta$ is the morphism obtained by identifying the points labelled by $y$ and $z$, while $\eta_R$ is obtained by identifying the point labelled by $u$ on the varying curve in $\overline{\mathcal{M}}_{0, R \cup \{y, z, u\}}$, with the point labelled by $v$ on the fixed curve $C_0$. We will examine the consequences of the equalities

\begin{equation}
\eta^*_R \xi^* \alpha = \eta^* \xi^*_R \alpha, \quad R \subset P, \quad |P \setminus R| \geq 2.
\end{equation}

We have

\begin{equation}
\eta^*_R \xi^*_R \alpha = \sum_{S \subset R, |R \setminus S| \geq 1} f^R_S \delta_{S \cup \{y, z\}} + \sum_{S \subset R, |R \setminus S| \geq 2} g^R_S \delta_{S \cup \{y, z, u\}}.
\end{equation}
Let us look at \( \eta_R^*\xi^*\alpha \). We have, for \( S \subset P, \ |P \setminus S| \geq 2 \),

\[
\eta_R^*\delta_{S \cup \{y\}} = \begin{cases} \\
\delta_{S\cup\{y\}} = \delta_{(R \setminus S)\cup\{z,u\}} & \text{if } S \subset R, \\
\delta_{P \setminus S \cup \{z\}} = \delta_{R \setminus (P \setminus S) \cup \{y,u\}} & \text{if } P \setminus S \subset R, \\
0 & \text{otherwise},
\end{cases}
\]

and a similar relation holds for \( \eta_R^*\delta_{S \cup \{z\}} \). Thus

\[
\eta_R^*\xi^*\alpha = \sum_{S \subset R, |P \setminus S| \geq 2} c_S(\delta_{S \cup \{y\}} + \delta_{S \cup \{z\}})
\]

\[
+ \sum_{P \setminus S \subset R, |P \setminus S| \geq 2} c_S(\delta_{P \setminus S \cup \{z\}} + \delta_{P \setminus S \cup \{y\}}).
\]

(4.14)

We are going to prove that \( c_T = 0 \) by descending induction on \( |T| \). The first non-trivial case occurs when \( |T| = n - 2 \).

If \( P = \{x, p, q\} \) this amounts to showing that \( c_{\{x\}} = 0 \). Looking at (4.10), relation (4.12) for \( R = \{x\} \) says that

\[
0 = c_{\{x\}}(\delta_{\{x,y\}} + \delta_{\{x,z\}}),
\]

and we are done in this case.

Suppose \( |P| \geq 4 \), and set \( P \setminus \{p, q\} = T \). We can assume that \( p \neq x \) and \( q \neq x \), otherwise \( c_T = 0 \). For \( R = \{p, q\} \) the equality (4.12) now reads

\[
f_\emptyset^{\{p, q\}}(\delta_{\{y, z\}}) + f_{\{p\}}^{\{p, q\}}(\delta_{\{p, y, z\}}) + f_{\{q\}}^{\{p, q\}}(\delta_{\{q, y, z\}}) =
\]

\[
c_{P \setminus \{p, q\}}(\delta_{\{p, q, y\}} + \delta_{\{p, q, z\}}),
\]

where on the left-hand side we used the fact that, by our first move, \( g_\emptyset^{\{p, q\}} = 0 \) while, to simplify the right-hand side, we used again the fact that \( c_S = 0 \) if \( x \notin S \). The above is an equality in \( H^2(\mathcal{M}_0, \{p, q, y, z, u\}) \), where as a canonical basis we take \( \delta_{\{p, u\}}, \delta_{\{q, u\}}, \delta_{\{y, u\}}, \delta_{\{z, u\}}, \delta_{\{y, z\}} \). It follows that

\[
f_\emptyset^{\{p, q\}} = f_{\{p\}}^{\{p, q\}} = f_{\{q\}}^{\{p, q\}} = c_{P \setminus \{p, q\}} = 0.
\]

(4.15)

The first step in the induction is completed.

Now let \( r \geq 3 \), assume that \( c_S = 0 \) if \( |P \setminus S| < r \), let \( T \) be such that \( |P \setminus T| = r \), and set \( R = P \setminus T \). As usual we can assume that \( x \notin R \). Let us first assume that \( |T| = |P \setminus R| \geq 2 \). Relation (4.12) reads

\[
\sum_{S \subset R, |R \setminus S| \geq 1} f_S^R(\delta_{R \setminus S \cup \{u\}}) + \sum_{S \subset R, |R \setminus S| \geq 2} g_S^R(\delta_{S \cup \{y, z, u\}}) = c_T(\delta_{\{y, u\}} + \delta_{\{z, u\}}),
\]

where, to simplify the right-hand side, we used the inductive hypothesis together with the fact that \( c_S = 0 \) if \( x \notin S \). This is an equality in \( H^2(\mathcal{M}_0, R \cup \{y, z, u\}) \). As a basis
for $H^2(\mathcal{M}_{0,R}\cup\{y,z,u\})$ we take a canonical basis where now the role of $Q$ is played by $R\cup\{y,z,u\}$ and the role of $x$ is played by $u$. Look at the elements appearing in the above equality; since $x \notin R$, and $g^R_{R\setminus\{p,q\}} = 0$, these elements belong to the canonical basis and they are all distinct; so we are done.

Let us finally assume that $|T| = 1$, or what is the same, that $T = \{x\}$. We start with a general remark. Pulling back the class

$$
\xi^*_\{p,q\} = f_\{p,q\}^{\{p,q\}} \delta_{1,\emptyset} + f_\{p,q\}^{\{p,q\}} \delta_{1,\{p\}} + f_\{q\}^{\{p,q\}} \delta_{1,\{q\}} + g_\{p,q\}^{\{p,q\}} \delta_{1,\{u\}} + f_\{p,q\}^{\{p,q\}} \delta_{\text{irr}}
$$

from $H^2(\mathcal{M}_{1,\{p,q,u\}})$ to $H^2(\mathcal{M}_{1,\{q,w\}})$, comparing it with

$$
\xi^*_q = f_\emptyset^{\{q\}} \delta_{1,\emptyset} + f_\emptyset^{\{q\}} \delta_{\text{irr}}
$$

and looking at the coefficient of $\delta_{1,\emptyset}$ we get

$$
(4.16) \quad f_\emptyset^{\{p,q\}} - f_\emptyset^{\{q\}} = f_\emptyset^{\{q\}} \quad \forall \ p, q.
$$

But if $|T| = 1$ (and $|P| \geq 4$), then relations (4.15) have already been proved so that, in particular, $f_\emptyset^{\{q\}} = 0$ for $q \neq x$. Using (4.16) again we get

$$
(4.17) \quad f_\emptyset^{\{x,q\}} = f_\emptyset^{\{x,q\}}.
$$

Now look at relation (4.12) for $R = \{x, q\}$. Using the induction hypothesis to simplify the right-hand side one then gets

$$
f_\emptyset^{\{x,q\}} \delta_{(y,z)} + f_\emptyset^{\{x\}} \delta_{(x,y,z)} + f_\emptyset^{\{q\}} \delta_{(q,y,z)} + g_\emptyset^{\{x,q\}} \delta_{(x,q)} = c_x(\delta_{(x,z)} + \delta_{(x,y)})
$$

which, by Keel’s relations, can be written as

$$
f_\emptyset^{\{x,q\}} \delta_{(y,z)} + f_\emptyset^{\{x\}} \delta_{(x,y,z)} + f_\emptyset^{\{q\}} \delta_{(x,u)} + g_\emptyset^{\{x,q\}} \delta_{(x,q)} = c_x(\delta_{(x,z)} + \delta_{(x,y)}).
$$

Take

$$
\delta_{(x,q)}, \delta_{(x,u)}, \delta_{(x,y)}, \delta_{(x,z)}, \delta_{(y,z)}
$$

as a canonical basis for $H^2(\mathcal{M}_{0,\{x,q,y,z,u\}})$. Looking at the coefficient of $\delta_{(y,z)}$ we get $f_\emptyset^{\{x,q\}} = -f_\emptyset^{\{x,q\}}$ which, together with (4.17), implies that all coefficients in the above identity must vanish, concluding the proof of the lemma.

**Genus 2.** In order to prove (2.2) in genus two we must show that, for any finite set $P$, the space $H^2(\mathcal{M}_{2,P})$ is generated by the classes $\psi_q$, with $q \in P$, and by the boundary classes $\delta_{\text{irr}}, \delta_{1,A}, \delta_{2,B}$, where $A$ and $B$ run through all subset of $P$ such that $|B^c| \geq 2$, and such that, if $P \neq \emptyset$, then $A$ contains a preassigned point $p \in P$. We set $n = |P|$.
We first consider the cases \( n = 0 \) and \( n = 1 \). By Theorem (2.10), \( H^2(\mathcal{M}_2) \) injects in \( H^2(\mathcal{M}_{1,2}) \) via \( \xi^* \). On the other hand, \( H^2(\mathcal{M}_{1,2}) \) is two-dimensional and the classes \( \delta_{\text{irr}} \) and \( \delta_{1,0} \) are independent in \( H^2(\mathcal{M}_2) \), so that \( \xi^* \) is an isomorphism, and the result follows in this case.

We next consider the case \( n = 1 \). Again \( H^2(\mathcal{M}_{2,p}) \) injects in \( H^2(\mathcal{M}_{1,p,x,y}) \) via \( \xi^* \).

Look at the diagrams

\[
\begin{align*}
\mathcal{M}_0,{x,y,z} & \xrightarrow{\nu} \mathcal{M}_1,{p,z} \\
\mathcal{M}_1,{p,x,y} & \xrightarrow{\xi} \mathcal{M}_2,p \\
\mathcal{M}_1,{p,z} & \xrightarrow{\varphi} \mathcal{M}_2,p \\
\mathcal{M}_1,{p,x,y} & \xrightarrow{\varphi} \mathcal{M}_1,{p,x,y}
\end{align*}
\]

(4.18)

The map \( \vartheta \) consists in attaching a fixed one-pointed smooth elliptic curve \( C_0 \) to the point labelled by \( z \) of a variable 2-pointed elliptic curve. The map \( \varphi \) consists in attaching the point labelled by \( w \) of a fixed smooth rational curve \( C'_0 \), marked by the set \( \{ x, y, w \} \), to the point labelled by \( z \) of a variable 2-pointed elliptic curve. The two diagrams are commutative up to homotopy.

Using Lemma (1.3), Lemma (1.4) and Proposition (1.9), we find that

\[
\begin{align*}
\xi^*(\psi_p) &= \psi_p = \frac{1}{12} \delta_{\text{irr}} + \delta_{1,\{x\}} + \delta_{1,\{y\}} + \delta_{1,\emptyset}, \\
\xi^*(\delta_{1,0}) &= \delta_{1,0} + \delta_{1,\{p\}}, \\
\xi^*(\delta_{\text{irr}}) &= \delta_{\text{irr}} - \psi_x - \psi_y + \delta_{1,\{x\}} + \delta_{1,\{y\}} = \frac{5}{6} \delta_{\text{irr}} - 2 \delta_{1,\{p\}} - 2 \delta_{1,\emptyset}, \\
\vartheta^*(\psi_p) &= \psi_p = \frac{1}{12} \delta_{\text{irr}} + \delta_{1,0}, \\
\vartheta^*(\delta_{1,0}) &= \delta_{1,0} - \psi_z = -\frac{1}{12} \delta_{\text{irr}}, \\
\vartheta^*(\delta_{\text{irr}}) &= \delta_{\text{irr}}.
\end{align*}
\]

A priori, given a class \( \alpha \) in \( H^2(\mathcal{M}_{2,p}) \), we have

\[
\begin{align*}
\xi^*(\alpha) &= a\delta_{\text{irr}} + b(\delta_{1,\{x\}} + \delta_{1,\{y\}}) + c\delta_{1,\{p\}} + d\delta_{1,\emptyset}, \\
\vartheta^*(\alpha) &= t\delta_{1,0} + s\delta_{\text{irr}},
\end{align*}
\]

so that, by adding to \( \alpha \) a suitable linear combination of \( \psi_p, \delta_{\text{irr}}, \) and \( \delta_{1,\{p\}} \), one can assume that \( s = b = d = 0 \). Then the equality \( \nu^* \vartheta^*(\alpha) = \mu^* \xi^*(\alpha) \) gives \( c = t \), while the equality \( \varphi^* \xi^*(\alpha) = \vartheta^*(\alpha) \) gives

\[
a\delta_{\text{irr}} - t\psi_z = a\delta_{\text{irr}} - t\left( \frac{1}{12} \delta_{\text{irr}} + \delta_{1,0} \right) = t\delta_{1,0},
\]

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so that \( a = t = c = 0 \), and the result is proved also in this case.

We now turn to the general case of \( \overline{M}_{2,p} \) with \( |P| = n \geq 2 \), and proceed by induction on \( n \). Fix, once and for all, a point \( p \) in \( P \). Consider subsets \( R \subset P \) such that \( p \in R^c \) and such that \( R^c = P \backslash R \) contains two or more points. We look at the maps

\[
\xi : \overline{M}_{1,P \cup \{x,y\}} \to \overline{M}_{2,p}, \\
\vartheta_R : \overline{M}_{2,R \cup \{z\}} \to \overline{M}_{2,p}, \\
\vartheta : \overline{M}_{1,P \cup \{s\}} \to \overline{M}_{2,p}.
\]

The map \( \vartheta_R \) is defined by identifying the point labelled by \( w \) of a fixed irreducible smooth rational curve marked by the set \( R^c \cup \{w\} \) with the point labelled by \( z \) of a variable curve in \( \overline{M}_{2,R \cup \{z\}} \). The map \( \vartheta \) is defined by identifying the point labelled by \( t \) of a fixed 1-pointed elliptic curve with the point labelled by \( s \) of a variable curve in \( \overline{M}_{1,P \cup \{s\}} \). Given a class \( \alpha \in H^2(\overline{M}_{2,p}) \), a priori one has

\[
\xi^*(\alpha) = a^r \delta_{irr} + \sum_{S \subset P} a_S \delta_{1,S} + \sum_{S \subset P, |S| \geq 2} b_S \delta_{1,S \cup \{x,y\}} + \sum_{S \subset P, |S| \geq 1} c_S (\delta_{1,S \cup \{x\}} + \delta_{1,S \cup \{y\}}),
\]

\[
\vartheta_R^*(\alpha) = a^R \delta_{irr} + \sum_{r \in R} b^R_r \psi_r + b^R_z \psi_z + \sum_{S \subset R, |S| \geq 2} c^R_S \delta_{0,S} + \sum_{S \subset R, |S| \geq 1} d^R_S \delta_{0,S \cup \{z\}} + \sum_{S \subset R} h^R_S \delta_{1,S}.
\]

Let us show that, by adding to \( \alpha \) a suitable linear combination of \( \delta_{irr} \) and of the \( \psi_i \) with \( i \neq p \), one can assume that \( a^R = b^R_r = 0 \) for all \( r \in R \) and for all \( R \subset P \) such that \( |R^c| \geq 2 \). For each proper subset \( R' \) of \( R \) there is an obvious diagram

\[
\begin{array}{ccc}
\overline{M}_{2,R' \cup \{z'\}} & \xrightarrow{\vartheta_{R'}} & \overline{M}_{2,p} \\
\downarrow{\zeta} & & \\
\overline{M}_{2,R \cup \{z\}} & \xrightarrow{\vartheta_R} & \overline{M}_{2,p}
\end{array}
\]

(4.20)

which is commutative up to homotopy, and it is evident that \( a^R = a^R' \) and \( b^R_r = b^R_r' \) whenever \( r \in R' \). But then it suffices to annihilate \( a^\theta \) and \( b^\theta_r \), for every \( r \in P \backslash \{p\} \), which can be achieved by adding to \( \alpha \) a suitable linear combination of \( \delta_{irr} \) and \( \psi_r \) for \( r \in P \backslash \{p\} \). Next, recall that

\[
\xi^*(\psi_p) = \frac{1}{12} \delta_{irr} + \sum_{T \subset P \cup \{x,y\}, p \in P} \delta_{0,T},
\]

so that, by adding to \( \alpha \) a suitable multiple of \( \psi_p \), we can assume that \( a = 0 \). Finally, since

\[
\xi^*(\delta_{2,S}) = \delta_{1,S \cup \{x,y\}}, \\
\xi^*(\delta_{1,S}) = \delta_{1,S} + \delta_{0,S \cup \{x,y\}} = \delta_{1,S} + \delta_{1,S^c},
\]

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we can assume that $b_S = 0$ and that $a_S = 0$ if $p \notin S$. In conclusion

$$
(4.21) \quad \xi^*(\alpha) = \sum_{S \subseteq P, p \in S} a_S \delta_{1,S} + \sum_{S \subseteq P, |S| \geq 1} c_S (\delta_{1,S \cup \{x\}} + \delta_{1,S \cup \{y\}}),
$$

$$
(4.22) \quad \varphi^*_R(\alpha) = b_z^R \psi_z + \sum_{S \subseteq R, |S| \geq 2} c_S^R \delta_{0,S} + \sum_{S \subseteq R, |S| \geq 1} d_S^R \delta_{0,S \cup \{z\}} + \sum_{S \subseteq R} h_S^R \delta_{1,S}.
$$

Now set $R = P \setminus \{p, q\}$, for some $q \in P$ with $q \neq p$, and look at the diagram

$$
\begin{array}{ccc}
\mathcal{M}_{1,R \cup \{x,y,z\}} & \xrightarrow{\varphi} & \mathcal{M}_{1,P \cup \{x,y\}} \\
\mu \downarrow & & \xi \downarrow \\
\mathcal{M}_{2,R \cup \{z\}} & \xrightarrow{\varphi_R} & \mathcal{M}_{2,P}
\end{array}
$$

Using Lemma (1.4), we get

$$
(4.23) \quad \nu^* \xi^*(\alpha) = \sum_{\{p,q\} \subseteq S \subseteq P} a_S \delta_{1, (S \setminus \{p,q\}) \cup \{z\}} + \sum_{S \subseteq R} c_S (\delta_{1,S \cup \{x\}} + \delta_{1,S \cup \{y\}})
$$

$$
+ \sum_{\{p,q\} \subseteq S \subseteq P, |S| \geq 1} c_S (\delta_{1, (S \setminus \{p,q\}) \cup \{x,z\}} + \delta_{1, (S \setminus \{p,q\}) \cup \{y,z\}}).
$$

$$
(4.24) \quad \mu^* \varphi_R^*(\alpha) = b_z^R \psi_z + \sum_{S \subseteq R, |S| \geq 2} c_S^R \delta_{1, (R \setminus S) \cup \{x,y,z\}}
$$

$$
+ \sum_{S \subseteq R, |S| \geq 1} d_S^R \delta_{1, (R \setminus S) \cup \{x,y\}} + \sum_{S \subseteq R} h_S^R (\delta_{1,S} + \delta_{1, (R \setminus S) \cup \{z\}}),
$$

The equality $\nu^* \xi^*(\alpha) = \mu^* \varphi_R^*(\alpha)$ provides a relation of linear dependence between elements of the following types

$$
\delta_{1,T \cup \{z\}}, \delta_{1,T \cup \{x\}}, \delta_{1,T \cup \{y\}}, \delta_{1,T \cup \{x,z\}}, \delta_{1,T \cup \{y,z\}},
$$

$$
\psi_z, \delta_{1,T \cup \{x,y,z\}}, \delta_{1,T \cup \{x,y\}}, \delta_{1,T} + \delta_{1,R \setminus T \cup \{z\}},
$$

where $T \subseteq R$. By the results on $H^2(\overline{\mathcal{M}_{1,\nu}})$, and since $\psi_z = (1/12) \delta_{\text{irr}} + \cdots$, the above elements are linearly independent, so that all the coefficient in (4.23) and (4.24) are zero. As $q$ is any element in $P$ different from $p$, this means that (4.21) becomes

$$
(4.25) \quad \xi^*(\alpha) = a_{\{p\}} \delta_{1,\{p\}} + c_{\{p\}} (\delta_{1,\{p,x\}} + \delta_{1,\{p,y\}})
$$

$$
+ c_{\{P \setminus \{p\}\}} (\delta_{1,(P \setminus \{p\}) \cup \{x\}} + \delta_{1,(P \setminus \{p\}) \cup \{y\}}),
$$

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while $\vartheta_R^*(\alpha) = 0$. We now look at the following diagram, which is the analogue of (4.19).

\[
\begin{array}{ccc}
\mathcal{M}_{1, P \cup \{s\}} & \xrightarrow{\vartheta} & \mathcal{M}_{2, P} \\
\varphi \downarrow & & \downarrow \xi \\
\mathcal{M}_{1, P \cup \{x, y\}} & & \\
\end{array}
\]

The identity $\varphi^* \xi^*(\alpha) = \vartheta^*(\alpha)$, together with (1.4), applied to $\varphi$, gives

(4.26) $\vartheta^*(\alpha) = a_{\{p\}} \delta_{1, \{p\}}$.

Finally, we consider the diagram

\[
\begin{array}{ccc}
\mathcal{M}_{0, P \cup \{x, y, s\}} & \xrightarrow{A} & \mathcal{M}_{1, P \cup \{x, y\}} \\
B \downarrow & & \downarrow \xi \\
\mathcal{M}_{1, P \cup \{s\}} & \xrightarrow{\vartheta} & \mathcal{M}_{2, P} \\
\end{array}
\]

The identity $B^* \vartheta^*(\alpha) = A^* \xi^*(\alpha)$ gives

\[
a_{\{p\}} \delta_{\{p, x, y\}} = a_{\{p\}} \delta_{\{p, s\}} + c_{\{p\}} (\delta_{\{p, x, s\}} + \delta_{\{p, y, s\}}) + c_{\{p\}} (\delta_{\{P \setminus \{p\}\} \cup \{x, s\}} + \delta_{\{P \setminus \{p\}\} \cup \{y, s\}}).
\]

As long as $|P| \geq 3$, the boundary classes appearing in the above relation belong to the canonical basis of $H^2(\mathcal{M}_{0, P \cup \{x, y, s\}})$, relative to $p, x, y$, so that all coefficients must vanish and we are done. If $P = \{p, q\}$, we further simplify notation and rewrite the above relation as

\[
a_{\{p\}} \delta_{\{q, s\}} = a_{\{p\}} \delta_{\{q, x\}} + c_{\{q\}} (\delta_{\{q, y\}} + \delta_{\{q, x\}}) + d_{\{q\}} (\delta_{\{p, y\}} + \delta_{\{p, x\}}).
\]

We choose $\delta_{\{q, s\}}, \delta_{\{q, x\}}, \delta_{\{q, y\}}, \delta_{\{p, s\}}$ as a basis for $\mathcal{M}_{0, 5}$ and, using the relations

\[
\delta_{\{p, y\}} = \delta_{\{p, s\}} + \delta_{\{q, y\}} - \delta_{\{q, s\}},
\]

\[
\delta_{\{p, x\}} = \delta_{\{p, s\}} + \delta_{\{q, x\}} - \delta_{\{q, s\}},
\]

we get $a = -2d$ and $c = -d$. Thus

\[
\xi^*(\alpha) = c (2 \delta_{1, \{p\}} + \delta_{1, \{p, x\}} + \delta_{1, \{p, y\}} - \delta_{1, \{q, x\}} - \delta_{1, \{q, y\}}).
\]

On the other hand, using (1.3) and (1.11), one finds that

\[
2 \delta_{1, \{p\}} + \delta_{1, \{p, x\}} + \delta_{1, \{p, y\}} - \delta_{1, \{q, x\}} - \delta_{1, \{q, y\}} = \xi^* (\delta_{1, \{p\}} + \psi_{q} - \psi_{p}),
\]

so (2.10) implies that $\alpha = c (\delta_{1, \{p\}} + \psi_{q} - \psi_{p})$. The proof of (2.2) is now complete.
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