Reconstructing Gaussian sources by spatial sampling

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Abstract

Consider a Gaussian memoryless multiple source with $m$ components with joint probability distribution known only to lie in a given class of distributions. A subset of $k \leq m$ components are sampled and compressed with the objective of reconstructing all the $m$ components within a specified level of distortion under a mean-squared error criterion. In Bayesian and nonBayesian settings, the notion of universal sampling rate distortion function for Gaussian sources is introduced to capture the optimal tradeoffs among sampling, compression rate and distortion level. Single-letter characterizations are provided for the universal sampling rate distortion function. Our achievability proofs highlight the following structural property: it is optimal to compress and reconstruct first the sampled components of the GMMS alone, and then form estimates for the unsampled components based on the former.

Index Terms

Fixed-set sampling, Gaussian memoryless multiple source, sampling rate distortion function, universal sampling rate distortion function.

I. INTRODUCTION

Consider a set $\mathcal{M}$ of $m$ jointly Gaussian memoryless sources with joint probability density function (pdf) known only to belong to a given family of pdfs. A fixed subset of $k \leq m$ sources are sampled at each time instant and compressed jointly by a (block) source code, with the objective of reconstructing all the $m$ sources within a specified level of distortion under a mean-squared error criterion. “Universality” requires that the sampling and lossy compression code be designed without precise knowledge of the underlying pdf. In this paper we study the tradeoffs – under optimal processing – among sampling, compression rate and distortion level. This study builds on our prior works [3], [4] on sampling rate distortion for multiple discrete sources with known joint pmf and universal sampling rate distortion for multiple discrete sources with joint pmf known only to lie in a finite class of pmfs, respectively. Here, we do not assume the class of pdfs to be finite.

Problems of combined sampling and compression have been studied extensively in diverse contexts for discrete and Gaussian sources. Relevant works include lossless compression of analog sources in an information theoretic setting [38]; compressed sensing with an allowed detection error rate or quantization distortion [29]; sub-Nyquist temporal sampling of Gaussian sources followed by lossy reconstruction [15]; and rate distortion function for multiple sources with time-shared sampling [22]. See also [13], [33].

Closer to our approach that entails spatial sampling, in a setting of distributed acoustic sensing and reconstruction, centralized as well as distributed coding schemes and sampling lattices are studied in [16]. The rate distortion function has been characterized when multiple Gaussian signals from a random field are sampled and quantized (centralized or distributed) in [27], [24], [25]. In [14], a Gaussian field on the interval $[0, 1]$ and i.i.d. in time, is reconstructed from compressed versions of $k$-sampled sequences under a mean-squared error criterion.

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The rate distortion function is studied for schemes that reconstruct only the sampled sources first and then reconstruct the unsampled sources by forming minimum mean-squared error (MMSE) estimates based on the reconstructions for the sampled sources. All the sampling problems above assume a knowledge of the underlying distribution.

In the realm of rate distortion theory where a complete knowledge of the signal statistics is unknown, there is a rich literature that considers various formulations of universal coding: only a sampling is listed here. Directions include classical Bayesian and non-Bayesian methods [23], [26], [30], [41]; “individual sequences” studies [35], [36], [42]; redundancy in quantization rate or distortion [18]–[20]; and lossy compression of noisy or remote signals [6], [21], [34]. These works propose a variety of distortion measures to investigate universal reconstruction performance.

Our work differs materially from the approaches above. Sampling is spatial rather than temporal. Our notion of universality involves a lack of specific knowledge of the underlying pdf in a given compact family of pdfs. Accordingly, in Bayesian and nonBayesian settings, we consider average and peak distortion criteria, respectively, with an emphasis on the former.

Our technical contributions are as follows. In Bayesian and nonBayesian settings, we extend the notion of a universal sampling rate distortion function (USRDF) [4] to Gaussian memoryless sources, with the objective of characterizing the tradeoffs among sampling, compression rate and distortion level. To this end, we consider first the setting with known underlying pdf, and characterize its sampling rate distortion function (SRDF). This uses as an ingredient the rate distortion function for a discrete “remote” source-receiver model with known distribution [1], [2], [8], [39]. When the underlying pdf is known, we show that the overall reconstruction can be performed – optimally – in two steps: the sampled sources are reconstructed first under a modified weighted mean-squared error criterion and then MMSE estimates are formed for the unsampled sources based on the reconstructions for the sampled sources. This is akin to the structure observed in [3] for reconstructing discrete sources from subsets of sources under the probability of error criterion and in [37] for reconstructing remote Gaussian sources. The USRDF for Gaussian memoryless sources with known pdf will serve as a key ingredient in characterizing the USRDF for the Gaussian field, with known distribution, previously studied in [14] in a restricted setting. Building on the ideas developed, for the SRDF (with known pdf), we characterize next the USRDF for Gaussian memoryless sources in the Bayesian and nonBayesian settings and show that it remains optimal to reconstruct first the sampled sources and then form estimates for the unsampled sources based on the reconstructions of the sampled sources.

Our model is described in Section II and our main results and illustrative examples are presented in Section III. In Section IV, we present achievability proofs first when the pdf is known and then, building on it, the achievability proof for the universal setting, with an emphasis on the Bayesian setting. A unified converse proof is presented thereafter.

II. PRELIMINARIES

Denote \( \mathcal{M} = \{1, \ldots, m\} \) and let

\[
X_\mathcal{M} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_m \end{bmatrix}
\]

be a \( \mathbb{R}^m \)-valued zero-mean (jointly) Gaussian random vector with a positive-definite covariance matrix. For a nonempty set \( A \subseteq \mathcal{M} \) with \( |A| = k \), we denote by \( X_A \) the random (column) vector \((X_i, \ i \in A)^T\), with values in \( \mathbb{R}^k \). Denote \( n \) repetitions of \( X_A \), with values in \( \mathbb{R}^{nk} \), by \( X_A^n = (X_i^n, \ i \in A)^T \). Each \( X_i^n = (X_{i1}, \ldots, X_{im})^T, \ i \in \mathcal{M} \),
A, takes values in \( \mathbb{R}^m \). Let \( A^c = \mathcal{M} \setminus A \) and let \( \mathbb{R}^m \) be the reproduction alphabet for \( X_M \). All logarithms and exponentiations are with respect to the base 2 and all norms are \( \ell_2 \)-norms.

Let \( \Theta = \{ \Sigma_{M_t} \}_t \) be a set of \( m \times m \)-positive-definite matrices, and assume \( \Theta \) to be convex and compact in the Euclidean topology on \( \mathbb{R}^{m \times m} \). For instance, for \( m = 2 \),

\[
\Theta = \left\{ \begin{pmatrix} \sigma_1^2 & r \sigma_1 \sigma_2 \\ r \sigma_1 \sigma_2 & \sigma_2^2 \end{pmatrix}, \quad c_1 \leq \sigma_1^2, \sigma_2^2 \leq c_2, \quad -d_1 \leq r \leq d_1 \right\},
\]

with \( 0 < c_1 \leq c_2 \) and \( 0 \leq d_1 < 1 \). Hereafter, all covariance matrices under consideration will be taken as being positive-definite without explicit mention. We assume \( \theta \) to be a \( \Theta \)-valued rv with a pdf \( \nu_\theta \) that is absolutely continuous with respect to the Lebesgue measure on \( \mathbb{R}^{m^2} \). We assume

\[ \nu_\theta(\tau) > 0, \quad \tau \in \Theta, \]

and that \( \nu_\theta \) is continuous on \( \Theta \). We consider a jointly Gaussian memoryless multiple source (GMMS) \( \{ X_{M_t} \}_{t=1}^\infty \) consisting of i.i.d. repetitions of the rv \( X_M \) with pdf known only to the extent of belonging to the family of pdfs \( \mathcal{P} = \{ \nu_{X_M|\theta=\tau} = \mathcal{N}(0, \Sigma_{M_t}) \} \), \( \tau \in \Theta \). Two settings are studied: in a Bayesian formulation, the pdf \( \nu_\theta \) is taken to be known, while in a nonBayesian formulation \( \theta \) is an unknown constant in \( \Theta \).

**Definition 1.** For a fixed \( A \subseteq \mathcal{M} \) with \( |A| = k \), a \( k \)-fixed-set sampler (k-FS), \( 1 \leq k \leq m \), collects at each \( t \geq 1 \), \( X_{M_t} \) from \( X_M \). The output of the k-FS is \( \{ X_{M_t} \}_{t=1}^\infty \).

**Definition 2.** For \( n \geq 1 \), an \( n \)-length block code with k-FS for a GMMS \( \{ X_{M_t} \}_{t=1}^\infty \) with reproduction alphabet \( \mathbb{R}^m \) is the pair \((f_n, \varphi_n)\) where the encoder \( f_n \) maps the k-FS output \( X_{M_t}^n \) into some finite set \( \mathcal{J} = \{1, \ldots, J\} \) and the decoder \( \varphi_n \) maps \( \mathcal{J} \) into \( \mathbb{R}^{m-n} \). We shall use the compact notation \((f, \varphi)\), suppressing \( n \). The rate of the code \((f, \varphi)\) with k-FS is \( 1/n \log J \).

Our objective is to reconstruct all the components of a GMMS from the compressed representations of the sampled GMMS components under a suitable distortion criterion with (single-letter) mean-squared error (MSE) distortion measure

\[
||x_M - y_M||^2 = \sum_{i=1}^m (x_i - y_i)^2, \quad x_M, y_M \in \mathbb{R}^m.
\]

For threshold \( \Delta \geq 0 \), an \( n \)-length block code \((f, \varphi)\) with k-FS will be required to satisfy one of the following \((|| \cdot ||^2, \Delta)\) distortion criterion depending on the setting.

(i) **Bayesian:** The expected distortion criterion is

\[
\mathbb{E}\left[ \left| \left| X_M^n - \varphi(f(X_M^n)) \right| \right|^2 \right] = \mathbb{E}\left[ \frac{1}{n} \sum_{t=1}^n \left| \left| X_{M_t} - \varphi(f(X_M^n)) \right| \right|^2 \right] \\
\leq \Delta.
\]

(ii) **NonBayesian:** The peak distortion criterion is

\[
\sup_{\tau \in \Theta} \mathbb{E}\left[ \left| \left| X_M^n - \varphi(f(X_M^n)) \right| \right|^2 \theta = \tau \right] \leq \Delta,
\]

where \( \mathbb{E}[\cdot | \theta = \tau] \) denotes \( \mathbb{E}_{\nu_{X_M|\theta=\tau}}[\cdot] \).

\( ^1 \Theta \) is a collection of covariance matrices indexed by \( \tau \). By an abuse of notation, we shall use \( \tau \) to refer to the covariance matrix \( \Sigma_{M_t} \) itself.

\( ^2 \)Throughout this paper, \( \mathcal{N}(0, \Sigma) \) is used to denote the pdf of a Gaussian random vector with mean 0 and covariance matrix \( \Sigma \).
Definition 3. A number \( R \geq 0 \) is an achievable universal \( k \)-sample coding rate at distortion level \( \Delta \) if for every \( \epsilon > 0 \) and sufficiently large \( n \), there exist \( n \)-length block codes with \( k \)-FS of rate less than \( R + \epsilon \) and satisfying the fidelity criterion \( \| \cdot \|^2, \Delta + \epsilon \) in (1) or (2) above; \( (R, \Delta) \) will be termed an achievable universal \( k \)-sample rate distortion pair under the expected or peak distortion criterion. The infimum of such achievable rates for each fixed \( \Delta \) is denoted by \( R_A(\Delta) \). We shall refer to \( R_A(\Delta) \) as the \textit{universal sampling rate distortion function} (USRDF), suppressing the dependence on \( k \). For \( |\Theta| = 1 \), the USRDf is termed simply the \textit{sampling rate distortion function} (SRDf), denoted by \( \rho_A(\Delta) \).

Remarks: (i) The USRDf under (1) is no larger than that under (2).
(ii) When \( |\Theta| = 1 \), the pdf of the GMMS is, in effect, known.

Below, we recall (Chapter 1, [12]) the definition of mutual information between two random variables.

Definition 4. For real-valued rvs \( X \) and \( Y \) with a joint probability distribution \( \mu_{XY} \), the mutual information between the rvs \( X \) and \( Y \) is given by

\[
I(X \wedge Y) = \begin{cases} 
\mathbb{E}_{\mu_{XY}} \left[ \log \frac{d\mu_{XY}}{d\mu_X \times d\mu_Y}(X,Y) \right], & \text{if } \mu_{XY} \ll \mu_X \times \mu_Y \\
\infty, & \text{otherwise},
\end{cases}
\]

where \( \mu_{XY} \ll \mu_X \times \mu_Y \) denotes that \( \mu_{XY} \) is absolutely continuous with respect to \( \mu_X \times \mu_Y \) and \( \frac{d\mu_{XY}}{d\mu_X \times d\mu_Y} \) is the Radon-Nikodym derivative of \( \mu_{XY} \) with respect to \( \mu_X \times \mu_Y \).

III. Results

We begin with a setting where the pdf of \( X_M \) is known and provide a (single-letter) characterization for the SRDf. Next, in a brief detour, we introduce an extension of GMMS, namely a Gaussian memoryless field (GMF) and show how the ideas developed for a GMMS can be used to characterize the SRDf for a GMF. Finally, building on the SRDf for a GMMS, a (single-letter) characterization of the USRDf is provided for a GMMS in the Bayesian and non-Bayesian settings.

Throughout this paper, a recurring structural property of our achievability proofs is this: it is optimal to reconstruct the \textit{sampled} GMMS components first under a \textit{(modified) weighted} MSE criterion with \textit{reduced threshold} and then form deterministic (MMSE) estimates of the unsampled components based on the reconstruction of the former.

Before we present our first result, we recall that for a GMMS \( \{X_{M,t}\}_{t=1}^\infty \) with pdf \( \mathcal{N}(0, \Sigma_M) \) reconstructed under the MSE distortion criterion, the \textit{standard} rate distortion function (RDF) is

\[
R(\Delta) = \min_{\mu_{X_M,Y_M} \ll \mu_{X_M} \times \mu_{Y_M}} \mathbb{E}_{\|X_M-Y_M\|^2 \leq \Delta} I(X_M \wedge Y_M), \quad 0 < \Delta \leq \sum_{i=1}^m \mathbb{E}[X_i^2] \quad (3)
\]

where \( \lambda_i \)s are the eigenvalues of \( \Sigma_M \), and \( \alpha \) is chosen to satisfy \( \sum_{i=1}^m \min(\alpha, \lambda_i) = \Delta \).

A. \( |\Theta| = 1 \): Known pdf

Starting with \( |\Theta| = 1 \), for a GMMS \( \{X_{M,t}\}_{t=1}^\infty \) with (known) pdf \( \mathcal{N}(0, \Sigma_M) \), our first result shows that the fixed-set SRDf \( \rho_A(\Delta) \) for a GMMS is, in effect, the RDF of a GMMS \( \{X_{A,t}\}_{t=1}^\infty \) with a weighted MSE distortion measure \( d_A \) and a reduced threshold; here \( d_A : \mathbb{R}^k \times \mathbb{R}^k \rightarrow \mathbb{R}^+ \cup \{0\} \) is given by

\[
d_A(x_A, y_A) \triangleq (x_A - y_A)^T G_A(x_A - y_A), \quad x_A, y_A \in \mathbb{R}^k
\]
where

\[ \mathbf{G}_A = \mathbf{I} + \Sigma_{A^{-1}} \Sigma_{AA} \mathbf{A}^T \mathbf{A}^{-1}, \]  \hspace{1cm} (4) \]

Theorem 1. For a GMMS \( \{X_{M_t}\}_{t=1}^{\infty} \) with pdf \( \mathcal{N}(0, \Sigma_M) \) and fixed \( A \subseteq M \), the SRDf is

\[ \rho_A(\Delta) = \min_{\mu_X, \mu_Y, I(X_A \land Y_A), \Delta \leq \Delta_{\text{min},A}} \int \mathbb{E}[d(X_A, Y_A)] \leq \Delta - \Delta_{\text{min},A} \]

\[ = \frac{1}{2} \sum_{i=1}^{k} \left( \log \frac{\lambda_i}{\alpha} \right)^+, \quad \Delta_{\text{min},A} < \Delta \leq \Delta_{\text{max}} \]  \hspace{1cm} (5) \]

where

\[ \Delta_{\text{min},A} = \sum_{i \in A^c} \left( \mathbb{E}[X_i^2] - \mathbb{E}[X_i X_i^T] \Sigma_A^{-1} \mathbb{E}[X_A X_i] \right), \quad \Delta_{\text{max}} = \sum_{i \in M} \mathbb{E}[X_i^2] \]

and \( \lambda_i \)s are the eigenvalues of \( \mathbf{G}_A \Sigma_A \), and \( \alpha \) is chosen to satisfy \( \sum_{i=1}^{k} \min(\alpha, \lambda_i) = \Delta - \Delta_{\text{min},A} \).

Comparing (5) with (3), it can be seen that (5) is, in effect, the RDf for a GMMS with weighted MSE distortion measure. In contrast to the RDf (3), in (5) the minimization involves only \( X_A \) (and not \( X_M \)) under a weighted MSE criterion with reduced threshold level. For \( k = m \), i.e., \( A = M \), however this reduces to the RDf (3). Also, for every feasible distortion level the SRDf for any \( A \subseteq M \) is no smaller than that with \( A = M \).

In Section IV, the achievability proof of the theorem above involves reconstructing the sampled components of the GMMS first, and then forming MMSE estimates for the unsampled components based on the former. Accordingly, in (5), the MSE in the reconstruction of the entire GMMS is captured jointly by the weighted MSE (with weight-matrix \( \mathbf{G}_A \)) in the reconstructions of the sampled components and the minimum distortion \( \Delta_{\text{min},A} \).

Observing that (5) is equivalent to the RDf of a GMMS with a weighted MSE distortion measure enables us to provide an analytic expression for the SRDf using the standard reverse water-filling solution (6) [12]. An instance of this is shown in the example below.

Example 1. For a GMMS with a k-FS with \( k = 1 \), this example illustrates the effect of the choice of the sampling set on SRDf. Consider a GMMS \( \{X_{M_t}\}_{t=1}^{\infty} \) with covariance matrix \( \Sigma_M \) given by

\[ \Sigma_M = \begin{pmatrix}
\sigma_1^2 & r_{12} \sigma_1 \sigma_2 & \cdots & r_{1m} \sigma_1 \sigma_m \\
r_{21} \sigma_1 \sigma_2 & \sigma_2^2 & \cdots & r_{2m} \sigma_2 \sigma_m \\
\vdots & \vdots & \ddots & \vdots \\
r_{m1} \sigma_1 \sigma_m & r_{m2} \sigma_2 \sigma_m & \cdots & \sigma_m^2
\end{pmatrix}, \]

where\( r_{ij} = r_{ji}, 1 \leq i, j \leq m \). For \( A = \{j\}, \ j = 1, \ldots, m \), we have

\[ \mathbf{G}_{\{j\}} \Sigma_{\{j\}} = \left( 1 + \sum_{i \neq j} \frac{r_{ij}^2 \sigma_i^2}{\sigma_j^2} \right) \sigma_j^2 = \sigma_j^2 + \sum_{i \neq j} r_{ij}^2 \sigma_i^2 \]

and hence from (6), the SRDf is

\[ \rho_{\{j\}}(\Delta) = \frac{1}{2} \log \left( \frac{\sigma_j^2 + \sum_{i \neq j} r_{ij}^2 \sigma_i^2}{\Delta - \Delta_{\text{min},\{j\}}} \right) \]
for $\Delta_{\min\{j\}} < \Delta \leq \sum_{i=1}^{m} \sigma_i^2$, where $\Delta_{\min\{j\}} = \sum_{i \neq j} \sigma_i^2(1 - r_{ij}^2)$. Observe that every SRDf $\rho_{\{j\}}(\Delta)$ is a monotonically increasing function of $\Delta_{\min\{j\}}$ and that the SRDfs are translations of each other and hence decrease at the same rate. Thus, the SRDf with the smallest $\Delta_{\min\{j\}}$ is uniformly best among all fixed-set SRDfs. For $k > 2$ however, there may not be any $A \subset M$, $|A| = k$, whose fixed-set SRDf is uniformly best for all distortion levels.

Before turning to the USRDf for a GMMS, the ideas involved in Theorem 1 are used to study sampling and lossy compression of a Gaussian field which affords greater flexibility in the choice of sampling set. While Gaussian fields have been studied extensively under different formulations, we consider a Gaussian memoryless field (GMF) as in [14], which is described next. In lieu of $M$ and Gaussian r.v $X_M$ in Section II consider $I = [0, 1] \subset \mathbb{R}$ and let $X_I = \{X_u, u \in I\}$ be a $\mathbb{R}^I \triangleq \{\mathbb{R}, u \in I\}$-valued zero-mean Gaussian process† with a bounded covariance function $r(s_1, s_2) = \mathbb{E}[X_{s_1}X_{s_2}]$, $s_1, s_2 \in I$, such that, for any finite $B \subset I$

$$
\mathbb{E}[X_B X_B^T]
$$
is a positive-definite matrix and

$$
\int_I \int_I |r(u, v)| \, du \, dv < \infty. \quad (7)
$$

A GMF† $\{X_{I_t}\}_{t=1}^{\infty}$ consists of i.i.d. repetitions of $X_I$. We consider a GMF sampled finitely by a $k$-FS at $A \subset I$, with $|A| = k$, and with a reconstruction alphabet $\mathbb{R}^I$.

For a GMF with fixed-set sampler and MSE distortion measure

$$
||x_I - y_I||^2 = \int_I (x_u - y_u)^2 \, du, \quad x_I, y_I \in \mathbb{R}^I, \quad (8)
$$

the sampling rate distortion function is defined as in Definitions 2 and 3 with the decoder $\varphi$ characterized by a collection of mappings $\varphi = \{\varphi_u\}_{u \in I}$ with $\varphi_u : \{1, \ldots, J\} \rightarrow \mathbb{R}^n, \ u \in I$.

Analogous to a GMMS, for a GMF sampled at $A = \{a_1, \ldots, a_k\}, \ 0 \leq a_i \leq 1, \ i = 1, \ldots, k$, our next result shows that the SRDf is, in effect, the RDF of a GMMS $\{X_{A_l}\}_{l=1}^{\infty}$ with a weighted MSE distortion measure with weight-matrix given by

$$
G_{A,l} = \Sigma_A^{-1} \left( \int_I \mathbb{E}[X_A X_u] \mathbb{E}[X_u X_A^T] \, du \right) \Sigma_A^{-1}, \quad (9)
$$

with $\int$ connoting element-wise integration. Note that for every $0 \leq s_1, s_2 \leq 1$, (7) and the boundedness of $r(\cdot)$ imply that the integral

$$
\int_I r(u, s_1)r(u, s_2) \, du
$$

†A Gaussian process on an interval $[0, 1]$ means that any finite collection of rvs $(X_{s_1}, \ldots, X_{s_l}), \ s_i \in [0, 1], \ i \in \{1, \ldots, l\}, \ l \in \mathbb{N}$, are jointly Gaussian.

‡Extensive studies of memoryless repetitions of a Gaussian process exist, cf. [14], [24], under various terminologies.
exists and hence (9) is well-defined.

**Proposition 2.** For a GMF \( \{X_I\}_{I=1}^\infty \) with \( A \subset I \), the SRDf is

\[
\rho_A(\Delta) = \min_{\mu_{X,A}\ll\mu_{X,A}} I(X_A \land Y_A), \quad \Delta_{\min,A} < \Delta \leq \Delta_{\max}
\]

\[
= \frac{1}{2} \sum_{i=1}^k \left( \log \frac{\lambda_i}{\alpha} \right)^+, \quad \Delta_{\min,A} < \Delta \leq \Delta_{\max}
\]

where

\[
\Delta_{\min,A} = \int_I (\mathbb{E}[X_u^2] - \mathbb{E}[X_u X_{\Delta}] \Sigma_A^{-1} \mathbb{E}[X_A X_u]) \, du \quad \text{and} \quad \Delta_{\max} = \int_I \mathbb{E}[X_u^2] \, du,
\]

and \( \lambda_i \)'s are the eigenvalues of \( \Sigma_A \), and \( \alpha \) satisfies \( \sum_{i=1}^k \min(\alpha, \lambda_i) = \Delta - \Delta_{\min,A} \).

The SRDf for a GMF (10) and its equivalent form (11) can be seen as counterparts of (5) and (6), with (11) being the reverse water-filling solution for (10). As before, the expression (10) is the RDf of a GMMS with a weighted MSE distortion measure. In Section [IV] an achievability proof for the proposition above is provided involving a set of techniques different from the converse proof provided for Theorem I.

In contrast to a GMMS with a discrete set \( \mathcal{M} \), for a GMF, \( I \) being an interval affords greater flexibility in the choice of the sampling set allowing for a better understanding of the structural properties of the “best” sampling set. In contrast to Example I in the example below, considering a GMF with a stationary Gauss-Markov process, we show the structure of the optimal set for minimum distortion for \( k > 2 \) as well. In general, the optimal sampling set is a function of the threshold \( \Delta \).

**Example 2.** Consider a GMF with a zero-mean, stationary Gauss-Markov process \( X_I \) over \( I = [0,1] \) with covariance function

\[
r(s, u) = p^{|s-u|}, \quad 0 \leq s, u \leq 1,
\]

and \( 0 < p < 1 \). Note that the correlation between any two points in the interval depends only on the distance between them. For the Gauss-Markov process \( X_I \), for any \( 0 \leq u_1 < u_2 < \cdots < u_l \leq 1, \ l > 2, \) it holds that

\[
X_{u_1} \leadsto X_{u_2} \leadsto \cdots \leadsto X_{u_l}.
\]

For a \( k \)-FS with \( k = 1 \) and \( A = \{a\} \), \( 0 \leq a \leq 1 \),

\[
G_{\{a\},I} = 1 - \Delta_{\min,\{a\}}
\]

and \( \mathbb{E}[X_a^2] = 1 \). In (11), the eigenvalue \( \lambda_1 \) is \( G_{\{a\},I} \Sigma_{\{a\}} = 1 - \Delta_{\min,\{a\}} \) itself and hence, the SRDf is

\[
\rho_{\{a\}}(\Delta) = \frac{1}{2} \log \frac{1 - \Delta_{\min,\{a\}}}{\Delta - \Delta_{\min,\{a\}}}
\]

for \( \Delta_{\min,\{a\}} < \Delta \leq 1 \), where

\[
\Delta_{\min,\{a\}} = \int_0^a \left( \mathbb{E}[X_u^2] - \frac{\mathbb{E}[X_u X_a]}{\mathbb{E}[X_a^2]} \right) \, du + \int_a^1 \left( \mathbb{E}[X_u^2] - \frac{\mathbb{E}[X_u X_a]}{\mathbb{E}[X_a^2]} \right) \, du
\]

\[
= \int_0^a (1 - p^2(a-u)) \, du + \int_a^1 (1 - p^2(u-a)) \, du
\]
$$= 1 - \frac{p^{2a} - 1 + p^{2(1-a)} - 1}{\ln p}.$$ 

Note that the SRDf $\rho_{\{a\}}(\Delta)$ is a monotonically increasing function of $\Delta_{\min,\{a\}}$, which in turn is a monotonically increasing function of $|a - 0.5|$. Thus, $\rho_{\{0.5\}}(\Delta)$ is uniformly best among all SRDfs $\rho_{\{a\}}(\Delta)$, $0 \leq a \leq 1$, for all distortion levels. Now, for a $k$-FS with $k \geq 2$ and $A = \{a_1 = 0, a_2, \ldots, a_{k-1}, a_k = 1\}$, with $a_i \leq a_{i+1}$, $i = 1, \ldots, k - 1$, the minimum distortion $\Delta_{\min,A}$ admits a simple form

$$\Delta_{\min,A} = 1 - \sum_{i=1}^{k-1} \gamma(a_{i+1} - a_i),$$

where $\gamma(a_{i+1} - a_i)$ is according to

$$\gamma(a) \triangleq \frac{1}{1 - p^{2a}} \left( \frac{p^{2a}(1 - 2a \log p) - 1}{\log p} \right), \quad 0 < a < 1.$$ 

The minimum reconstruction error $\Delta_{\min,A}$ is the “sum” of the minimum error in reconstructing each segment $[a_i, a_{i+1}]$ of the GMF. Now, the Markov property (12) implies that the minimum error in reproducing each component $u \in I$ is determined by its nearest sampled points and hence the minimum error in reconstructing each segment $[a_i, a_{i+1}]$ of the GMF is independent of the location of sampling points other than $a_i$, $a_{i+1}$ and is given by

$$(a_{i+1} - a_i) - \gamma(a_{i+1} - a_i).$$

The stationarity of the field means that this minimum error depends on the length $|a_{i+1} - a_i|$ alone. Observing that $\gamma(a)$ is a concave function of $a$ over $[0,1]$. $\Delta_{\min,A}$ above is seen to be minimized when $a_{i+1} - a_i = \frac{1}{k-1}$, $i = 1, \ldots, k - 1$, i.e., when the sampling points are spaced uniformly. However, such a placement is not optimal for all distortion levels.

### B. Universal setting

Turning to the universal setting with a GMMS, consider a set $\Theta_1 = \{\Sigma_{A\tau}, \tau \in \Theta\} \subset \mathbb{R}^{k^2}$ with $\tau_1 \in \Theta_1$ indexing the members of $\Theta_1$, i.e., $\Theta_1 = \{\Sigma_{A\tau_1}\}_{\tau_1}$. An encoder $f$ associated with a $k$-FS observes $X^\nu_{A}$ alone and cannot distinguish among jointly Gaussian pdfs in $\mathcal{P}$ that have the same marginal pdf $\nu_{X_A|\theta=\tau}$. Accordingly (and akin to [4]), consider a partition of $\Theta$ comprising “ambiguity” atoms, with each atom of the partition comprising $\tau$s with identical $\nu_{X_A|\theta=\tau}$, i.e., identical $\Sigma_{A\tau}$ and for each $\tau_1 \in \Theta_1$, $\Lambda(\tau_1)$ is the collection of $\tau$s in the ambiguity atom indexed by $\tau_1$, i.e.,

$$\Sigma_{A\tau_1} \triangleq \Sigma_{A\tau}, \quad \tau \in \Lambda(\tau_1).$$

Let $\theta_1$ be a $\Theta_1$-valued rv induced by $\theta$. It is easy to see that $\Theta_1$ and $\Lambda(\tau_1)$, $\tau_1 \in \Theta_1$, are convex, compact subsets of $\mathbb{R}^{k^2}$ and the rv $\theta_1$ admits a pdf $\nu_{\theta_1}$ induced by $\nu_\theta$.

In the Bayesian setting,

$$\nu_{X_A|\theta_1=\tau_1} = \nu_{X_A|\theta=\tau} = \mathcal{N}(0, \Sigma_{A\tau_1}), \quad \tau \in \Lambda(\tau_1).$$

In the nonBayesian setting, in order to retain the same notation, we choose $\nu_{X_A|\theta_1=\tau_1}$ to be the right-side above.

Our characterization of the USRDf builds on the structure of the SRDf for a GMMS. Accordingly, in the Bayesian setting, consider the set of (constrained) probability measures

$$\kappa^B_{\{a\}}(\delta, \tau_1) \triangleq \mu_{X_{MN}\theta_1} : \theta, X_{M} \leftarrow \theta_1, X_{A} \leftarrow Y_{M}, \mu_{X_{A}Y_{M}|\theta_1=\tau_1} \ll \mu_{X_{A}|\theta_1=\tau_1} \times \mu_{Y_{M}|\theta_1=\tau_1},$$

where $\mathbb{E}[||X_{MN} - Y_{M}||^2|\theta_1=\tau_1] \leq \delta$.

$^\dagger$ The collection of covariance matrices $\Sigma_{A\tau}$ are indexed by $\tau_1$ and by an abuse $\tau_1$ will also be used to refer to $\Sigma_{A\tau_1}$ itself.
and (constraint) minimized mutual information
\[ \rho_A^B(\delta, \tau_1) \triangleq \min_{\kappa_A(\delta, \tau_1)} I(X_A \wedge Y_M|\theta_1 = \tau_1). \] (13)

Correspondingly, in the nonBayesian setting, consider
\[ \kappa^B_A(\delta, \tau_1) \triangleq \{ \mu_{X_M|X_A,\theta=\tau} = \mu_{Y_M|X_A,\theta=\tau}, \mu_{X_A,Y_M|\theta_1=\tau_1} \ll \mu_{X_A|\theta_1=\tau_1} \times \mu_{Y_M|\theta_1=\tau_1}, \mathbb{E}[||X_M - Y_M||^2|\theta = \tau] \leq \delta, \tau \in \Lambda(\tau_1) \} \]
and
\[ \rho_A^B(\delta, \tau_1) \triangleq \inf_{\kappa_A^B(\delta, \tau_1)} I(X_A \wedge Y_M|\theta_1 = \tau_1). \] (14)

Remark: In (13) and (14), the minimization is with respect to the conditional measure \( \mu_{Y_M|X_A,\theta_1=\tau_1} \).

The minimized conditional mutual informations above will be a key ingredient in the characterization of USRDf. First, we show in the proposition below that (13) and (14) admit simpler forms involving rvs corresponding to the sampled components of the GMMS and their reconstruction alone. In the Bayesian setting, for each \( \tau_1 \in \Theta_1 \), the mentioned simpler form involves a weighted MSE distortion measure \( d_{A\tau_1} \) with weight-matrix \( \Sigma_{A,\tau_1} \), defined as in (4) with \( \Sigma_{A,A^c} \) replaced by \( \mathbb{E}[X_A X_A^T|\theta_1 = \tau_1] \)
\[ d_{A\tau_1}(x_A, y_A) \triangleq (x_A - y_A)^T \Sigma_{A,\tau_1}(x_A - y_A), \quad x_A, y_A \in \mathbb{R}^k. \]

In the Bayesian setting, the modified distortion measure \( d_{A\tau_1} \) plays a role similar to that of \( d_A \).

Remark: Clearly, \( \rho_A^B(\delta, \tau_1) \) is a nonincreasing function of \( \delta > \Delta_{\min,A,\tau_1} \). Convexity of \( \rho^B_A(\delta, \tau_1) \) can be shown as in (32), and convexity implies the continuity of \( \rho^B_A(\delta, \tau_1) \). Now, to show the convexity, pick any \( \delta_1, \delta_2 > \Delta_{\min,A,\tau_1} \) and \( \epsilon > 0 \). For \( i = 1, 2 \), let \( \mu^i \in \kappa^B_A(\delta_i, \tau_1) \) be such that
\[ I_{\mu^i}(X_A \wedge Y_M|\theta_1 = \tau_1) \leq \rho^B_A(\delta_i) + \epsilon. \]
For \( \alpha > 0 \), by the standard convexity arguments, it can be seen that \( \alpha \mu^1 + (1 - \alpha) \mu^2 \in \kappa^B_A(\alpha \delta_1 + (1 - \alpha) \delta_2, \tau_1) \)
and
\[ I_{\alpha \mu^1 + (1 - \alpha) \mu^2}(X_A \wedge Y_M|\theta_1 = \tau_1) \leq \alpha \rho^B_A(\delta_1) + (1 - \alpha) \rho^B_A(\delta_2) + \epsilon. \] (15)
Since (15) holds for any \( \epsilon > 0 \), in the limit, we have
\[ \rho^B_A(\alpha \delta_1 + (1 - \alpha) \delta_2) \leq \alpha \rho^B_A(\delta_1) + (1 - \alpha) \rho^B_A(\delta_2). \]

Proposition 3. For each \( \tau_1 \in \Theta_1 \), in the Bayesian setting
\[ \rho_A^B(\delta, \tau_1) = \min_{\kappa_A(\delta, \tau_1)} I(X_A \wedge Y_A|\theta_1 = \tau_1) \] (16)
for \( \delta > \Delta_{\min,A,\tau_1} \), where
\[ \Delta_{\min,A,\tau_1} = \mathbb{E} \left[ \min_{y_{A^c} \in \mathbb{R}^{m-k}} \sum_{i \in A^c} (X_i - y_i)^2 | X_A, \theta_1 = \tau_1 | \theta_1 = \tau_1 \right]. \]
For each \( \tau_1 \in \Theta_1 \), in the nonBayesian setting
\[ \rho_A^B(\delta, \tau_1) = \inf_{\kappa^B_A(\delta, \tau_1)} I(X_A \wedge Y_A|\theta_1 = \tau_1), \delta > \Delta_{\min,A,\tau_1}, \] (17)
where the infimum in (17) is over \( \mu_{Y_M|X_A,\theta=\tau} \), such that
\[ \mu_{Y_M|X_A,\theta=\tau} = \mu_{Y_A|X_A,\theta_1=\tau_1} \times \mu_{Y_A|Y_A,\theta_1=\tau_1}, \tau \in \Lambda(\tau_1), \quad \text{and} \]
\[ \mu_{X_A,Y_A|\theta_1=\tau_1} \ll \mu_{X_A|\theta_1=\tau_1} \times \mu_{Y_A|\theta_1=\tau_1}. \]
and
\[
\Delta_{\text{min}, A, \tau_1} = \inf_{\mu_{Y_A|X_A, \theta = \tau_1} = \mu_{Y_A|X_A, \theta = \tau_1}} \max_{\tau_1 \in A} \sum_{i \in A} \mathbb{E}[(X_i - Y_i)^2|\theta = \tau].
\]

**Remark:** From (16), notice that \(R^R_A(\delta, \tau_1)\) is, in effect, the rate distortion function for a GMMS with pdf \(\nu_{X_A|\theta = \tau_1}\) and weighted MSE distortion measure. Hence, the minimum in (16) and ergo that in (13) exist and the standard properties of a rate distortion function are applicable to \(\rho^R_A(\delta, \tau_1)\) as well, i.e., \(\rho^R_A(\delta, \tau_1)\) is a convex, nonincreasing, continuous function of \(\delta > \Delta_{\text{min}, A, \tau_1}\).

**Theorem 4.** For a GMMS \(\{X_M\}_{i=1}^\infty\) with fixed \(A \subseteq \mathcal{M}\), the Bayesian USRDf is
\[
R_A(\Delta) = \min_{\Delta_{\text{min}, A, \tau_1}} \max_{\Theta} \rho^R_A(\Delta_{\text{min}, A, \tau_1})
\]
for \(\Delta_{\text{min}, A} < \Delta \leq \Delta_{\text{max}}\), where
\[
\Delta_{\text{min}, A} = \mathbb{E}\left[ \min_{y_A \in \mathbb{R}^{n-k}} \sum_{i \in A^c} (X_i - y_i)^2 | X_A, \theta_1 \right] \quad \text{and} \quad \Delta_{\text{max}} = \sum_{i \in M} \mathbb{E}[X_i^2].
\]
The nonBayesian USRDf is
\[
R_A(\Delta) = \max_{\tau_1 \in \Theta} \rho^B_A(\Delta, \tau_1)
\]
for \(\Delta_{\text{min}, A} < \Delta \leq \Delta_{\text{max}}\), where
\[
\Delta_{\text{min}, A} = \sup_{\tau_1 \in \Theta} \inf_{\mu_{Y_A|X_A, \theta = \tau_1} = \mu_{Y_A|X_A, \theta = \tau_1}} \max_{\tau_1 \in A} \sum_{i \in A^c} \mathbb{E}[(X_i - Y_i)^2|\theta = \tau] \quad \text{and} \quad \Delta_{\text{max}} = \max_{\tau \in \Theta} \sum_{i = 1}^m \mathbb{E}[X_i^2|\theta = \tau].
\]

**Remark:** In Appendix C a simple proof (using contradiction arguments) is provided to show the existence of \(\{\Delta_{\text{min}, A, \tau_1} \in \Theta_1\}\), with \(\Delta_{\text{min}, A, \tau_1}\) being continuous in \(\tau_1\), that attains the minimum and the maximum in (18).

Notice that \(\rho^B_A(\delta, \tau_1)\) and \(\rho^B_A(\delta, \tau_1)\) are reminiscent of the SRdf for a GMMS and, in fact, reduce to the SRdf for a GMMS with \(\nu_{X_M|\theta = \tau}\) for \(\tau \in \Lambda(\tau_1)\) when \(\Lambda(\tau_1)\) is a singleton. Thus, the equivalent forms (16) and (17) can be seen as counterparts of (5). Additionally, in Section IV we show that \(\rho^B_A(\delta, \tau_1)\) and \(\rho^B_A(\delta, \tau_1)\) are continuous in \(\tau_1 \in \Theta_1\).

The Bayesian USRDf with an outer minimization over \(\{\Delta_{\text{min}, A, \tau_1} \in \Theta_1\}\) can be strictly smaller than its nonBayesian counterpart. An illustration of the comparison of the Bayesian and nonBayesian USRDfs is provided in the example below.

**Example 3.** For \(\mathcal{M} = \{1, 2\}\) and fixed \(\sigma^2 > 0\), \(r_{\text{min}} > 0\) and \(r_{\text{max}} < 1\), consider a GMMS with pdf in \(\Theta\), where each \(\Theta = \{\Sigma_M\}_{\tau}\), where each \(\Sigma_M\) is given by
\[
\Sigma_M = \begin{pmatrix}
\sigma^2 & \sigma \sigma^2 \\
\sigma \sigma^2 & \sigma^2
\end{pmatrix}
\]
for \(r_{\text{min}} \leq r_{\tau} \leq r_{\text{max}}, \tau \in \Theta\). Let \(\theta\) be a \(\Theta\)-valued rv with pdf \(\nu_\theta\) continuous on \(\Theta\). For a \(k\)-FS with \(k = 1\), for both \(A = \{1\}\) and \(A = \{2\}\), \(\Theta_1\) is a singleton. Hence, in the Bayesian setting, the minimum and maximum in (18) are vacuous. For \(A = \{1\}\), \(\{2\}\), in the Bayesian setting we have
\[
G_{A, \tau_1} = 1 + \mathbb{E}[r_\theta],
\]
\[
\Delta_{\text{min}, A, \tau_1} = \sigma^2 (1 - \mathbb{E}[r_\theta]),
\]
and
\[
\Delta_{\text{min}, A, \tau_1} = \sigma^2 (1 - \mathbb{E}[r_\theta]),
\]
and (18) now yields the Bayesian USRDf to be

\[ R_{(1)}(\Delta) = R_{(2)}(\Delta) = \frac{1}{2} \log \frac{\sigma^2(1 + \mathbb{E}^2[r_\theta])}{\Delta - \sigma^2(1 - \mathbb{E}^2[r_\theta])}, \quad \sigma^2(1 - \mathbb{E}^2[r_\theta]) \leq \Delta \leq 2\sigma^2. \]  

(20)

Evaluating (19), the nonBayesian USRDf is

\[ R_{(1)}(\Delta) = R_{(2)}(\Delta) = \frac{1}{2} \log \frac{\sigma^2(1 + r_{\min}^2)}{\Delta - \sigma^2(1 - r_{\min}^2)}, \quad \sigma^2(1 - r_{\min}^2) \leq \Delta \leq 2\sigma^2. \]  

(21)

A simple comparison of (20) and (21) shows that the nonBayesian USRDf is strictly larger than its Bayesian counterpart. Also, it is seen from (20) and (21) above that when \( r_\tau > 0 \) for all \( \tau \in \Theta \), the average correlation, \( \mathbb{E}[r_\theta] \), and the smallest correlation, \( r_{\min} \), play similar roles in the expressions for Bayesian and nonBayesian USRDf, respectively.

Lastly, the standard properties of the SRDf and the USRDf for GMMS and GMF with fixed-set samplers are summarized in the lemma below, with the proof provided in Appendix E.

**Lemma 5.** The right-sides of (5), (10), (18) and (19) are finite-valued, decreasing, convex, continuous functions of \( \Delta_{\min}, A < \Delta \leq \Delta_{\max} \).

### IV. PROOFS

**A. Achievability proofs**

We present first the achievability proof of Theorem 1 where the sampled components of the GMMS are reconstructed first with a weighted MSE distortion measure under a reduced threshold, and then MMSE estimates are formed for the unsampled components based on the former. An achievability proof for Proposition 2 is along similar lines. Building on this, we present next an achievability proof for Theorem 4 with an emphasis on the Bayesian setting. All our achievability proofs emphasize the modular structure of the reconstruction mechanism, which allows GMMS reconstruction to be performed in two steps.

**Theorem 1** First, observe that

\[ \Delta_{\min, A} = \min_{X_{Ac} \rightarrow X_A \rightarrow Y_M} \mathbb{E}[\|X_M - Y_M\|^2] \]

\[ = \min_{X_{Ac} \rightarrow X_A \rightarrow Y_A} \sum_{i \in A^c} \mathbb{E}[(X_i - Y_i)^2] \quad \text{with} \quad Y_i = X_i, \; i \in A \]

\[ = \sum_{i \in A^c} \mathbb{E}[(X_i - \mathbb{E}[X_i|X_A])^2] \]

\[ = \sum_{i \in A^c} (\mathbb{E}[X_i^2] - \mathbb{E}[X_iX_A^T] \Sigma_A^{-1} \mathbb{E}[X_AX_i]) \]

and

\[ \Delta_{\max} = \min_{X_{Ac} \rightarrow X_A \rightarrow Y_M} \mathbb{E}[\|X_M - Y_M\|^2] \]

\[ = \min_{y_M} \mathbb{E}[\|X_M - y_M\|^2] \]

\[ = \sum_{i=1}^m \mathbb{E}[X_i^2], \]

where \( \Sigma_A^{-1} \) exists by the assumed positive-definiteness of \( \Sigma_M \).

Given \( \epsilon > 0 \), for the GMMS \( \{X_{At}\}_{t=1}^\infty \) with pdf \( \mathcal{N}(0, \Sigma_A) \) and weighted MSE distortion measure \( d_A \), consider a (standard) rate distortion code \( (f_A, \varphi_A), \; f_A : \mathbb{R}^{nk} \rightarrow \{1, \ldots, J\} \) and \( \varphi_A : \{1, \ldots, J\} \rightarrow \mathbb{R}^{nk} \) of rate...
\[
\frac{1}{n} \log J \leq \rho_A(\Delta) + \epsilon \quad \text{and with}
\]
\[
\mathbb{E}[d_A(X^a_A, Y^a_A)] \leq \Delta - \Delta_{\min,A} + \epsilon,
\]
for \(n \geq N_\epsilon\), say.

A code \((f, \varphi)\) is devised as follows. The encoder \(f\) is chosen to be \(f_A\), i.e.,
\[
f(x^a_A) \triangleq f_A(x^a_A), \quad x^a_A \in \mathbb{R}^{nk}
\]
and the decoder \(\varphi\) is given by
\[
\varphi(j) \triangleq (\varphi_A(j), \mathbb{E}[X^a_A | X^a_A = \varphi_A(j)]), \quad j \in \{1, \ldots, J\}.
\]
The rate of the code \((f, \varphi)\) is
\[
\frac{1}{n} \log J \leq \rho_A(\Delta) + \epsilon.
\]
Denote the output of the decoder \(\varphi(f(X^a_A))\) by \(Y^a_M = (Y^a_n, Y^a_1)\). Then, \(Y^a_n = \Sigma_{A^cA} \Sigma_A Y^a_A\) and by the standard properties of an MMSE estimate, for \(t = 1, \ldots, n\), it holds that
\[
(X^a_{At} - \Sigma_{A^cA} \Sigma_A^{-1} X_{At}) \perp X_{At}. \tag{22}
\]
The code \((f, \varphi)\) has expected distortion
\[
\mathbb{E}[||X_M^n - Y_M^n||^2] = \mathbb{E}[||X_A^n - \Sigma_{A^cA}^{-1}Y_A^n||^2]
\]
\[
= \mathbb{E}[||X_A^n - \Sigma_{A^cA}^{-1}Y_A^n||^2] + \mathbb{E}[||X_{A^c}^n - \Sigma_{A^cA}^{-1}Y_{A^c}^n||^2] + \mathbb{E}[||X_{A^c}^n - \Sigma_{A^cA}^{-1}Y_{A^c}^n||^2]
\]
\[
= \mathbb{E}[||X_A^n - \Sigma_{A^cA}^{-1}Y_A^n||^2] + \mathbb{E}[||X_{A^c}^n - \Sigma_{A^cA}^{-1}Y_{A^c}^n||^2] + \mathbb{E}[||X_{A^c}^n - \Sigma_{A^cA}^{-1}Y_{A^c}^n||^2]
\]
\[
= \Delta_{\min,A} + \frac{1}{n} \sum_{t=1}^n \mathbb{E}[(X_{At} - Y_{At})^T (I + \Sigma_{A^cA}^{-1} \Sigma_{AA^c} \Sigma_{A^cA}^{-1}) (X_{At} - Y_{At})]
\]
\[
= \Delta_{\min,A} + \mathbb{E}[d_A(X^a_A, Y^a_A)]
\]
\[
\leq \Delta + \epsilon, \tag{26}
\]
where (24) is by the orthogonality principle of MMSE estimates (22) and since for \(t = 1, \ldots, n\),
\[
\mathbb{E}[(X_{At} - \Sigma_{A^cA} \Sigma_A^{-1} X_{At})^T \Sigma_{A^cA} \Sigma_A^{-1} Y_{At}] = \mathbb{E}[(X_{At}^T \Sigma_{A^cA} \Sigma_A^{-1} Y_{At})]
\]
\[
= \mathbb{E}[X_{At}^T \Sigma_{A^cA} \Sigma_A^{-1} Y_{At} | X^n_A] - \mathbb{E}[X_{At}^T \Sigma_{A^cA} \Sigma_A^{-1} Y_{At}]
\]
\[
= \mathbb{E}[X_{At}^T \Sigma_{A^cA} \Sigma_A^{-1} Y_{At} | X^n_A] - \mathbb{E}[X_{At}^T \Sigma_{A^cA} \Sigma_A^{-1} Y_{At}]
\]
\[
= \mathbb{E}[X_{At}^T \Sigma_{A^cA} \Sigma_A^{-1} Y_{At} | X^n_A] - \mathbb{E}[X_{At}^T \Sigma_{A^cA} \Sigma_A^{-1} Y_{At}]
\]
\[
= 0.
\]

**Proposition 2**: The achievability proof of Proposition 2 is along the lines of Theorem 1. For a given \(\Delta_{\min} < \Delta \leq \Delta_{\max}\) and \(\epsilon > 0\), for the GMMS \(\{X_{At}\}_{t=1}^\infty\) with weighted MSE distortion measure
\[
\beta_A(x_A, y_A) = (x_A - y_A)^T G_A(x_A - y_A), \quad x_A, y_A \in \mathbb{R}^k,
\]
consider a rate distortion code \((f_A, \varphi_A)\), \(f_A: \mathbb{R}^{nk} \to \{1, \ldots, J\}\) and \(\varphi_A: \{1, \ldots, J\} \to \mathbb{R}^{nk}\) of rate \(\frac{1}{n} \log J \leq\)
\( \rho_A(\Delta) + \epsilon \) and with
\[
\mathbb{E}[d_A(X^n_A, Y^n_A)] \leq \Delta - \Delta_{\min,A} + \epsilon,
\]
for \( n \geq N_\epsilon. \)

A code \((f, \varphi)\) is then constructed as follows. The encoder \( f \) is chosen to be\( f(x^n_A) = f_A(x^n_A), \quad x^n_A \in \mathbb{R}^{nk}. \)
The output of decoder \( \varphi \), corresponding to each \( u \in I \), is given by
\[
(\varphi(j))_u = \mathbb{E}[X^n_u | X^n_A = \varphi_A(j)], \quad j \in \{1, \ldots, J\}.
\]
Denoting the output of the decoder \( \varphi(f(X^n_A)) \) by \( Y^n_t \), for \( u \in I, \ t = 1, \ldots, n, \)
\[
Y_{ut} = \Sigma_{\{u \}A}^{-1} Y_{At},
\]
where \( \Sigma_{\{u \}A} = \mathbb{E}[X_u X^T_A] \) and \( \Sigma_A = \mathbb{E}[X_A X^T_A] \). The rate of the code \((f, \varphi)\) is
\[
\frac{1}{n} \log J \leq \rho_A(\Delta) + \epsilon.
\]
The code \((f, \varphi)\) has expected distortion
\[
\mathbb{E}[\|X^n_I - Y^n_I\|] = \int \mathbb{E}[\|X^n_u - Y^n_u\|] \, du
\]
\[
= \int \mathbb{E}[\|X^n_u - \Sigma_{\{u \}A}^{-1} X^n_A\|] \, du
\]
\[
= \int \mathbb{E}[\|X^n_u - \Sigma_{\{u \}A}^{-1} X^n_A + \Sigma_{\{u \}A}^{-1} X^n_A - \Sigma_{\{u \}A}^{-1} X^n_A\|] \, du
\]
\[
= \int \mathbb{E}[\|X^n_u - \Sigma_{\{u \}A}^{-1} X^n_A\|^2] + \mathbb{E}[\|\Sigma_{\{u \}A}^{-1} X^n_A - \Sigma_{\{u \}A}^{-1} X^n_A\|^2] \, du
\]
\[
\leq \Delta_{\min,A} + \int \mathbb{E}[\|X^n_A - Y^n_A\|^2 \Sigma_{\{u \}A}^{-1} \Sigma_{\{u \}A}^{-1} (X^n_A - Y^n_A)\|^2] \, du
\]
\[
\leq \Delta_{\min,A} + \int \mathbb{E}[\|X^n_A - Y^n_A\|^2 G_{A,I}(X^n_A - Y^n_A)\|^2] \quad \text{by } (27)
\]
\[
\leq \Delta + \epsilon,
\]
where (27) is by the orthogonality principle of the MMSE estimates as in (24), (25).

Before we present the achievability proof of Theorem 4, we present pertinent technical results. We state first a standard technical result, a Vitali covering lemma (Theorem 17.1 in [7]), without proof. For any \( \epsilon > 0 \), this lemma guarantees the existence of a finite number of nonoverlapping Euclidean “balls” of radius \( \leq \epsilon \) such that the Lebesgue measure of the set of members of \( \Theta_1 \) not covered by the Euclidean balls is \( \leq \epsilon \). In the achievability proof of Theorem 4 the centers of such balls will be used to approximate \( \Theta_1 \) and (approximately) estimate \( \theta_1 \). For \( \tau_1 \in \Theta_1 \), let \( B_{\tau_1,\epsilon} \subset \mathbb{R}^{k^2} \) denote a standard Euclidean \( \ell_2 \) ball with center \( \tau_1 \) and radius \( \epsilon \).

**Lemma 6.** For every \( \epsilon > 0 \), there exists an \( N_\epsilon > 0 \) and a finite disjoint collection of balls \( \{B_{\tau_{1,i},\epsilon_i}\}_{i=1}^{N_\epsilon} \) such that \( \max_i \epsilon_i \leq \epsilon \) and
\[
\mu(\Theta_1 \setminus \bigcup_{i} B_{\tau_{1,i},\epsilon_i}) < \epsilon,
\]
(28)
where \( \mu \) is the Lebesgue measure on \( \mathbb{R}^{k^2} \) and \( \Delta \) is the standard set difference.

Remarks: i) The lemma above relies on \( \Theta_1 \) being a compact subset of \( \mathbb{R}^{k^2} \).

ii) For \( \epsilon > 0 \) and \( \{B_{\tau_i,\epsilon}^\prime\}_{i=1}^{N_\epsilon} \) as in the lemma above, let \( \Theta_{1,\epsilon} \subset \Theta_1 \) be the collection of “centers” \( \{\tau_i\}_{i=1}^{N_\epsilon} \).

While the lemma above is pertinent to the Bayesian and non-Bayesian parts of Theorem \( 4 \), Lemmas \( 7 \) and \( 8 \) below are pertinent to the Bayesian and non-Bayesian settings respectively.

Lemma 7. In the Bayesian setting, for every \( x_M \in \mathbb{R}^m \),

\[
\nu_{X_M|\theta_1}(x_M|\tau_1)
\]

is continuous in \( \tau_1 \). For any code \( (f, \varphi) \), \( f : \mathbb{R}^{nk} \rightarrow \{1, \ldots, J\}, \ \varphi : \{1, \ldots, J\} \rightarrow \mathbb{R}^{nm} \),

\[
\mathbb{E}[|X_M^1 - \varphi(f(X_M^n))|^2|\theta_1 = \tau_1]
\]

is continuous in \( \tau_1 \).

Proof: See Appendix \( A \).

Remarks: (i) Since \( \Theta_1 \) is a compact set, for every \( x_M \in \mathbb{R}^m \), the pdf \( \nu_{X_M|\theta_1}(x_M|\tau_1) \) and \( \mathbb{E}[|X_M^1 - \varphi(f(X_M^n))|^2|\theta_1 = \tau_1] \) are, in fact, uniformly continuous in \( \tau_1 \). Thus, for every \( x_M \in \mathbb{R}^m \) and \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for \( \tau_{1,1}, \tau_{1,2} \in \Theta_1 \) with \( ||\tau_{1,1} - \tau_{1,2}|| \leq \delta \), it holds that

\[
|\nu_{X_M|\theta_1}(x_M|\tau_{1,1}) - \nu_{X_M|\theta_1}(x_M|\tau_{1,2})| \leq \epsilon,
\]

and

\[
|\mathbb{E}[|X_M^1 - \varphi(f(X_M^n))|^2|\theta_1 = \tau_{1,1}] - \mathbb{E}[|X_M^1 - \varphi(f(X_M^n))|^2|\theta_1 = \tau_{1,2}]| \leq \epsilon.
\]

(ii) The claim (29) implies that

\[
\mathbb{E}[X_A X_A^T|\theta_1 = \tau_1] \text{ and } \mathbb{E}[X_A X_A^T|\theta_1 = \tau_1]
\]

are continuous in \( \tau_1 \) and hence,

\[
G_{A,\tau_1} = I + (\mathbb{E}[X_A X_A^T|\theta_1 = \tau_1])^{-1} \mathbb{E}[X_A X_A^T|\theta_1 = \tau_1] \mathbb{E}[X_A X_A^T|\theta_1 = \tau_1] (\mathbb{E}[X_A X_A^T|\theta_1 = \tau_1])^{-1}
\]

is continuous in \( \tau_1 \). Thus, from (16), for every \( \delta > \Delta_{\min,A,\tau_1} \), \( P_{A}^B(\delta, \tau_1) \) is continuous in \( \tau_1 \).

The following lemma implies that if \( \tau_{1,1}, \tau_{1,2} \in \Theta_1 \) are “close,” then there exist \( \hat{\tau} \) and \( \tilde{\tau} \) in the ambiguity atoms of \( \tau_{1,1} \) and \( \tau_{1,2} \), respectively, which too are “close.”

Lemma 8. For every \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that for every \( \tau_{1,1}, \tau_{1,2} \in \Theta_1 \) with \( ||\tau_{1,1} - \tau_{1,2}|| \leq \delta \), it holds that

\[
\min_{\hat{\tau} \in \Lambda(\tau_{1,1}), \tilde{\tau} \in \Lambda(\tau_{1,2})} ||\hat{\tau} - \tilde{\tau}|| \leq \epsilon.
\]

Proof: See Appendix \( B \).

Theorem 4: Consider \( \Theta_1 \) as in Section \( III \). Based on the output of the fixed-set sampler \( X_M^n \), the encoder forms a maximum-likelihood (ML) estimate for the covariance-matrix \( \Sigma_{A\tau_1} \) as

\[
\hat{\theta}_{1,n} = \hat{\theta}_{1,n}(X^n_M) = \frac{1}{n} \sum_{t=1}^{n} X_{At} X_{At}^T.
\]

Observe that \( \{X_{At}\}_{t=1}^{\infty} \) is a GMMS with pdf \( N(0, \Sigma_{A\tau_1}) \) and \( \nu_{X_A|\theta_1 = \tau_1} \) is continuous in \( \tau_1 \). Compactness of \( \Theta_1 \), the boundedness and continuity of \( \nu_{X_A|\theta_1 = \tau_1} \) in \( \tau_1 \) imply, by the law of large numbers [17], that

\[
\hat{\theta}_{1,n} \xrightarrow{a.s.} \tau_1,
\]
where the convergence is elementwise and is under $\nu_{X_\delta|\theta_1=\tau_1}$, and that for every $\varepsilon > 0$, there exists a $\delta$ and $N_{\varepsilon_1}$ such that for every $\tau_1 \in \Theta_1$

$$P_{\tau_1}(||\tau_1 - \hat{\theta}_{1,n}|| > \delta) \leq \varepsilon_1, \quad n \geq N_{\varepsilon_1}. \quad (31)$$

Now, considering a subset $\Theta_{1,\delta}$ of $\Theta_1$ as in the remark following Lemma 6, define $\hat{\theta}_{1,n}$ as

$$\hat{\theta}_{1,n} \triangleq \arg \min_{\tau_1 \in \Theta_{1,\delta}} ||\hat{\theta}_{1,n} - \tau_1||. \quad (32)$$

Fixing $\varepsilon > 0$ and $0 < \varepsilon_1 < \varepsilon$, from (31), (32) and Lemma 6 it follows that there exists a $\delta$ and $N_{\varepsilon_1}$ such that

$$P(||\theta_1 - \hat{\theta}_{1,n}|| > 2\delta) \leq \varepsilon_1, \quad n \geq N_{\varepsilon_1}. \quad (33)$$

Notice that while $\hat{\theta}_{1,n}$ may lie outside $\Theta_1$, $\hat{\theta}_{1,n}$ is an estimate of $\theta_1$ that takes values in a finite subset of $\Theta_1$. The estimate $\hat{\theta}_{1,n}$ (of $\theta_1$) will be used in the next part of the proof to select sampling rate distortion codes.

For a fixed $\Delta_{\min,A} < \Delta \leq \Delta_{\max}$, let $\{\Delta_{\tau_1}, \tau_1 \in \Theta_1\}$ be such that it attains the minimum in (18) and $\Delta_{\tau_1}$ is continuous in $\tau_1$ (see the remark below Theorem 4). Recall that for each $\tau_1 \in \Theta_{1,\delta}$, $\rho^B_\Delta(\Delta_{\tau_1}, \tau_1)$ is, in effect, the RDF for a GMMS $\{X_{At}\}_{t=1}^\infty$ with pdf $\nu_{X_\delta|\theta_1=\tau_1}$ under a weighted MSE distortion measure $d_{At}$. Thus, for each $\tau_1 \in \Theta_{1,\delta}$, there exists a (standard) rate distortion code $(f_{\tau_1}, \varphi_{\tau_1})$, $f_{\tau_1} : \mathbb{R} \rightarrow \{1, \ldots, J\}$ and $\varphi_{\tau_1} : \{1, \ldots, J\} \rightarrow \mathbb{R}^{nk}$ of rate $\frac{1}{n} \log J \leq \rho^B_\Delta(\Delta_{\tau_1}, \tau_1) + \varepsilon_1 \leq R_\Delta(\Delta) + \varepsilon_1$ and with

$$\mathbb{E}[d_{At}(X_{At}^n, \varphi_{\tau_1}(f_{\tau_1}(X_{At}^n))))|\theta_1 = \tau_1] \leq \Delta_{\tau_1} - \Delta_{\min,A,\tau_1} + \varepsilon_1$$

for all $n \geq N_{\varepsilon_1}$.

Now, consider a code $(f, \varphi)$ with $f$ taking values in $\mathcal{J} \triangleq \{1, \ldots, |\Theta_{1,\delta}|\} \times \{1, \ldots, J\}$ as follows. Order (in any manner) the elements of $\Theta_{1,\delta}$. The encoder $f$, dictated by the estimate $\hat{\theta}_{1,n}$, is given by

$$f(x^n_A) \triangleq (\hat{\theta}_{1,n}(x^n_A), f_{\hat{\theta}_{1,n}}(x^n_A)), \quad x^n_A \in \mathbb{R}^{nk}$$

and the decoder $\varphi$ is given by

$$\varphi(\tau_1, j) \triangleq (\varphi_{\tau_1}(j), \mathbb{E}[X^n_A|x^n_A = \varphi_{\tau_1}(j), \theta_1 = \tau_1)], \quad (\tau_1, j) \in \mathcal{J}. \quad (34)$$

By the finiteness of $\Theta_{1,\delta}$, the rate of the code $(f, \varphi)$ is

$$\frac{1}{n} \log |\mathcal{J}| = \frac{1}{n} \log |\Theta_{1,\delta}| + \frac{1}{n} \log J \leq R_\Delta(\Delta) + 2\varepsilon_1,$$

for $n$ large enough. Denoting the output of the decoder by $Y^n_M$ with $Y^n_A = \varphi_{\tau_1}(j)$ and $Y^n_A = \mathbb{E}[X_A^n | X_A^n = \varphi_{\tau_1}(j), \theta_1 = \tau_1]$, we have that

$$\mathbb{E}[||X^n_M - Y^n_M||^2] = \mathbb{E}[1(||\hat{\theta}_{1,n} - \theta_1|| \leq 2\delta)||X^n_M - Y^n_M||^2] + \mathbb{E}[1(||\hat{\theta}_{1,n} - \theta_1|| > 2\delta)||X^n_M - Y^n_M||^2]. \quad (35)$$

Using Lemma 7, it is shown in Appendix D that the first term in the right-side of (35) is

$$\mathbb{E}[1(||\hat{\theta}_{1,n} - \theta_1|| \leq 2\delta)||X^n_M - Y^n_M||^2] \leq \Delta + 4\varepsilon_1. \quad (36)$$

Next, we show that the second term in the right-side of (35) is “small.” First, that the finiteness of $\Theta_{1,\delta}$ implies the existence of an $M_1$ such that, for $t = 1, \ldots, n$,

$$|\varphi_{\tau_1}(f_{\tau_1}(x^n_A))|_{i,t} \leq M_1, \quad i \in A, \quad \tau_1 \in \Theta_{1,\delta}, \quad x^n_A \in \mathbb{R}^{nk}$$

and hence, from (34), there exists an $M_2 > 0$ such that, for $t = 1, \ldots, n$,

$$|\varphi(f(x^n_A))|_{i,t} \leq M_2, \quad i \in M, \quad x^n_A \in \mathbb{R}^{nk}. \quad (37)$$
For \( i \in \mathcal{M} \), from (37), Cauchy-Schwarz inequality, and the fact that \( \mathbb{E}[X_i^2] \) is bounded, there exists an \( M \) such that

\[
\mathbb{E}[(X_{it} - Y_{it})^4] \leq M, \quad t = 1, \ldots, n. \tag{38}
\]

Now, the second term on the right-side of (35),

\[
\mathbb{E}[1(||\tilde{\theta}_{1,n} - \theta_1|| > 2\delta)|| X_M^n - Y_M^n||^2] = \frac{1}{n} \sum_{t=1}^{n} \sum_{i=1}^{m} \mathbb{E}[1(||\tilde{\theta}_{1,n} - \theta_1|| > 2\delta)(X_{it} - Y_{it})^2]
\]

\[
\leq \frac{1}{n} \sum_{t=1}^{n} \sum_{i=1}^{m} \sqrt{\mathbb{E}[1^2(||\tilde{\theta}_{1,n} - \theta_1|| > 2\delta)] \mathbb{E}[(X_{it} - Y_{it})^4]}
\]

\[
\leq \frac{1}{n} \sum_{t=1}^{n} \sum_{i=1}^{m} \epsilon_1 M \quad \text{from (33) and (38)}
\]

\[
\leq \sqrt{\epsilon_1 M}, \quad \text{for } \epsilon_1 \text{ small enough.}
\]

In the nonBayesian setting, as a first step, Lemma 8 is used to show that \( \rho^B_A(\delta, \tau_1) \) is a continuous function of \( \tau_1 \). Then, the maximum in (19) is seen to exist as a continuous function over a compact set attains its supremum. Next, the achievability proof follows by adapting the steps above with the following differences. For each \( \tau_1 \in \Theta_{1,\delta} \), sampling rate distortion codes \( (f_{\tau_1}, \varphi_{\tau_1}) \), \( f_{\tau_1} : \mathbb{R}^{nk} \rightarrow \{1, \ldots, J\} \), \( \varphi_{\tau_1} : \{1, \ldots, J\} \rightarrow \mathbb{R}^{nm} \) are chosen to satisfy

\[
\mathbb{E}[||X_M^n - \varphi_{\tau_1}(f_{\tau_1}(X_A^n))||^2|\theta = \tau] \leq \Delta, \quad \tau \in \Lambda(\tau_1),
\]

with rate \( \frac{1}{n} \log ||f_{\tau_1}|| \leq R_A(\Delta) + \epsilon \), where \( R_A(\Delta) \) is the nonBayesian USRDf. A code \( (f, \varphi) \) with \( f \) taking values in \( \mathcal{J} = \{1, \ldots, |\Theta_{1,\delta}|\} \times \{1, \ldots, J\} \) is constructed based on the codes \( (f_{\tau_1}, \varphi_{\tau_1}) \) as before. While counterparts of (36) and (40) can be shown for each \( \tau_1 \in \Theta_1 \) using a similar set of ideas, a key distinction in the analysis is that Lemma 8 is used in lieu of Lemma 7 to show that

\[
\mathbb{E}[1(||\tilde{\theta}_{1,n} - \tau_1|| \leq 2\epsilon_1)||X_M^n - \varphi(\tilde{\theta}_{1,n}, f_{\tilde{\theta}_{1,n}}(X_A^n))||^2|\theta = \tau] \leq \Delta + \epsilon_1, \quad \tau \in \Lambda(\tau_1), \quad \tau_1 \in \Theta_1,
\]

the counterpart of (36). \( \square \)

### B. Converse proof

In contrast to the achievability proofs, we present a converse proof for Theorem 4 first, with an emphasis on the Bayesian setting; this is then adapted to Theorem 1. Prior to this, we prove the equivalence of expressions in (41), that will be pertinent to Theorem 1. Building on this, we show the equivalence of the simplified forms for \( \rho^B_A(\delta, \tau_1) \) and \( \rho^B_A(\delta, \tau_1) \) in Proposition 3. Next, we shall present a technical lemma. These will be used subsequently in the unified converse proof for Theorems 1 and 4. The converse proof for Proposition 2 uses an approach that does not rely on Lemma 9 and is presented last.

**Equivalence for Theorem 1** The following equality will be relevant in the proof of converse for Theorem 1:

\[
\min_{X_A \in \mathcal{C}, \mu_{X_A} \ll \mu_X \times \mu_A} \mathbb{E}[||X_M^n - Y_M^n||^2] \leq \Delta \implies I(X_A \land Y_A) = \min_{\mu_{X_A} \ll \mu_X \times \mu_A} I(X_A \land Y_A).
\]

\[
\min_{X_A \in \mathcal{C}, \mu_{X_A} \ll \mu_X \times \mu_A} \mathbb{E}[||X_M^n - Y_M^n||^2] \leq \Delta \implies I(X_A \land Y_A) = \min_{\mu_{X_A} \ll \mu_X \times \mu_A} I(X_A \land Y_A).
\]

\[
\min_{X_A \in \mathcal{C}, \mu_{X_A} \ll \mu_X \times \mu_A} \mathbb{E}[||X_M^n - Y_M^n||^2] \leq \Delta \implies I(X_A \land Y_A) = \min_{\mu_{X_A} \ll \mu_X \times \mu_A} I(X_A \land Y_A).
\]
For any pair of rvs $X_M, Y_M$ satisfying the constraints on the left-side of (41), consider
\[
\hat{Y}_M \triangleq \mathbb{E}[X_M | Y_M].
\] (42)

Now,
\[
\hat{Y}_A \triangleq \mathbb{E}[X_A | Y_M] = \mathbb{E}[\mathbb{E}[X_A | X_A, Y_M] | Y_M] = \mathbb{E}[\mathbb{E}[X_A | X_A] | Y_M] = \mathbb{E}[\Sigma_{A,A}^{-1}X_A | Y_M] = \Sigma_{A,A}^{-1}\hat{Y}_A.
\] (43)

By the optimality of the MMSE estimate,
\[
\mathbb{E}[\|X_M - \hat{Y}_M\|^2] \leq \mathbb{E}[\|X_M - Y_M\|^2] \leq \Delta.
\] (44)

It is readily checked (along the lines of (23)-(26)) that
\[
\mathbb{E}[\|X_M - \hat{Y}_M\|^2] = \mathbb{E}[d_A(X_A, \hat{Y}_A)] + \Delta_{min,A}.
\] (45)

Putting together (42)-(45), completes the proof of (41).

Proposition 3 The proof of (16) and (17) is along the lines of proof of (41), with the distinction that in the nonBayesian setting, $\hat{Y}_A$ is chosen to satisfy the orthogonality principle and $\hat{Y}_{A^c}$ is chosen to be a linear function of $\hat{Y}_A$.

The following technical lemma is the counterpart of Lemma 6 in [4].

Lemma 9. In the Bayesian setting, for any n-length k-FS code $(f, \varphi)$ with $f : \mathbb{R}^{nk} \rightarrow \{1, \ldots, J\}$, $\varphi : \{1, \ldots, J\} \rightarrow \mathbb{R}^{nm}$, for $t = 1, \ldots, n$, denoting $\varphi(f(X^n_A))$ by $Y^n_M$, it holds that
\[
\theta, X_{A^c} \rightarrow \theta_1, X_{At} \rightarrow Y_{Mt}.
\] (46)

Proof: First, note that
\[
\theta, X^n_{A^c} \rightarrow X^n_A \rightarrow Y^n_M
\] (47)
holds by code construction. From (47) (and since $Y^n_M$ above is a finite-valued rv), we have
\[
0 = I(\theta, X^n_{A^c} \wedge Y^n_M | X^n_A) = I(\theta \wedge Y^n_M | X^n_A) + I(X^n_{A^c} \wedge Y^n_M | X^n_A, \theta)
\]
\[
= I(\theta, \theta_1 \wedge Y^n_M | X^n_A) + I(X^n_{A^c} \wedge Y^n_M | X^n_A, \theta)
\]
\[
\geq I(\theta \wedge Y^n_M | X^n_A, \theta_1) + I(X^n_{A^c} \wedge Y^n_M | X^n_A, \theta),
\] (48)
\[
\geq I(\theta \wedge Y^n_M | X^n_A, \theta) + I(X^n_{A^c} \wedge Y^n_M | X^n_A, \theta),
\] (49)

where (48) is since $\theta_1$ is a function of $\theta$. Now, the second term on the right-side of (49) is
\[
0 = I(X^n_{A^c} \wedge Y^n_M | X^n_A, \theta) = \sum_{t=1}^{n} I(X_{A^c} | Y^n_M, X^n_A, \theta)
\]
\[
= \sum_{t=1}^{n} I(X_{A^c} | X_{A^c}^{t-1}, X^n_A, Y^n_M | X_{At}, \theta) - I(X_{A^c} | X_{A^c}^{t-1}, X^n_A | X_{At}, \theta)
\]
\[
= \sum_{t=1}^{n} I(X_{A^c} | X_{A^c}^{t-1}, X^n_A, Y^n_M | X_{At}, \theta), \quad \text{since } \nu_{X^n_{Mt} | \theta} = \prod_{t=1}^{n} \nu_{X_{Mt} | \theta}
\]
\[
\geq \sum_{t=1}^{n} I(X_{At} | Y_{Mt}, X_{At}, \theta).
\] (50)

Next, (49) and the fact
\[
\theta \rightarrow \theta_1 \rightarrow X^n_A
\]
imply
\[ 0 = I(\theta \land X^n_A|\theta_1) + I(\theta \land Y^n_M|X^n_A, \theta_1) \]

\[ = I(\theta \land X^n_A, Y^n_M|\theta_1) \]

and hence, for \( t = 1, \ldots, n \),
\[ I(\theta \land X_{At}, Y_{Mt}|\theta_1) = 0. \] (51)

Now, by (50) and (51), for \( t = 1, \ldots, n \),
\[ I(\theta \land Y_{Mt}|X_{At}, \theta_1) + I(X_{At^t} \land Y_{Mt}|X_{At}, \theta) = I(\theta, X_{At^t} \land Y_{Mt}|X_{At}, \theta_1) = 0, \]

hence, the claim of the lemma (46).

Converse: We provide first a converse proof for the Bayesian setting in Theorem 4, which is then refashioned to provide converse proofs for the nonBayesian setting and Theorem 1.

Let \((f, \varphi)\) be an \( n \)-length \( k \)-FS code of rate \( R \) and with decoder output \( Y^n_M = \varphi(f(X^n_A)) \) satisfying \( E[||X^n_M - Y^n_M||^2] \leq \Delta \). By lemma 9 for \( t = 1, \ldots, n \), we have
\[
\theta, X_{At^t} \rightarrow \theta_1, X_{At} \rightarrow Y_{Mt}. \] (52)

For \( t = 1, \ldots, n \), and \( \tau_1 \in \Theta_1 \), let \( \Delta_{\tau_1,t} \) denote \( E[||X^n_{Mt} - Y^n_{Mt}||^2|\theta_1 = \tau_1] \) and \( \Delta_{\tau_1} \triangleq \frac{1}{n} \sum_{t=1}^n E[||X^n_{Mt} - Y^n_{Mt}||^2|\theta_1 = \tau_1] \). Along the lines of proof of Theorem 9.6.1 in [10], for every \( \tau_1 \in \Theta_1 \),
\[
R = \frac{1}{n} \log |f| \geq \frac{1}{n} H(f(X^n_A)|\theta_1 = \tau_1) \geq \frac{1}{n} H(Y^n_M|\theta_1 = \tau_1) \geq \frac{1}{n} I(X^n_M|\theta_1 = \tau_1) \geq \frac{1}{n} \sum_{t=1}^n I(X_{At} \land X_{At^t-1}^t, Y_{At}^t|\theta_1 = \tau_1) \geq \frac{1}{n} \sum_{t=1}^n I(X_{At} \land Y_{At}|\theta_1 = \tau_1) \geq \frac{1}{n} \sum_{t=1}^n \min_{\theta, X_{At^t} \rightarrow \theta_1, X_{At} \rightarrow Y_{Mt}} I(X_{At} \land Y_{At}|\theta_1 = \tau_1) \]
by (52)
\[ \geq \frac{1}{n} \sum_{t=1}^n \min_{\mu_{X_{At}Y_{At}:\theta_1 = \tau_1}} I(X_{At} \land Y_{At}|\theta_1 = \tau_1) \]
by Proposition 3
\[ = \frac{1}{n} \sum_{t=1}^n \rho^n_A(\Delta_{\tau_1,t}, \tau_1) \]
\[ \geq \rho^n_A\left(\frac{1}{n} \sum_{t=1}^n \Delta_{\tau_1,t}, \tau_1\right) \geq \rho^n_A(\Delta_{\tau_1}, \tau_1). \] (53)
Now, (53) holds for every $\tau_1 \in \Theta$, hence
\begin{align*}
R \geq & \sup_{\tau_1 \in \Theta} \rho_A^B(\Delta_{\tau_1}, \tau_1) \\
\geq & \inf_{\Delta \in \Theta} \sup_{\tau_1 \in \Theta} \rho_A^B(\Delta_{\tau_1}, \tau_1) \\
= & R_A(\Delta)
\end{align*}
for $\Delta > \Delta_{\min,A}$.

In the nonBayesian setting, the analog of Lemma 9 is obtained similarly with $\theta = \tau$, $\theta_1 = \tau_1$ and (47), (46) replaced with appropriate conditional measures. The proof of the converse is along the lines of the proof above, but with $\rho_A^B(\Delta, \tau_1)$ in place of $\rho_A^B(\Delta_{\tau_1}, \tau_1)$, and without the outer minimization with respect to $\{\Delta_{\tau_1}, \tau_1 \in \Theta\}$.

The converse proof for Theorem 1 obtains immediately from the Bayesian setting with the following changes: $\Theta_1$ and $\Lambda(\tau_1)$, $\tau_1 \in \Theta_1$, are taken to be singletons (rendering the infimum and supremum in (54) superfluous) and (41) is used in place of Proposition 3.

The converse proof for Proposition 2 involves an approach which does not rely on Lemma 9 and is presented next.

**Converse proof for Proposition 2.** Let $(f, \varphi)$ be an $n$-length $k$-FS code of code $R$ with $E[|X_n^n - \varphi(f(X^n_A))|^2] \leq \Delta$. For $u \in I$ and $t = 1, \ldots, n$, define
\begin{align*}
\hat{Y}_{ut} & = E[X_{ut}|f(X^n_A)] \\
& = E[E[X_{ut}|X^n_A, f(X^n_A)]|f(X^n_A)] \\
& = E[E[X_{ut}|X^n_A]|f(X^n_A)] \\
& = E[E[X_{ut}|X At]|f(X^n_A)], \text{ since } X At, X_{ut} \perp \perp X^{\nu(t)}_A, X^{\nu(t)}_n \\
& = E[X_{(u,t)} A] \Sigma^{-1} A E[X At|f(X^n_A)].
\end{align*}
Notice that for $u \in I \setminus A$,
\begin{align*}
\hat{Y}_{ut} & = E[X_{(u,t)} A] \Sigma^{-1} A \hat{Y}_A, \quad t = 1, \ldots, n.
\end{align*}
By the optimality of the MMSE estimate
\begin{align*}
\Delta \geq E[|X_n^n - \varphi(f(X^n_A))|^2] \geq E[|X_n^n - \hat{Y}_A^n|^2] = E[(X_n^n - \hat{Y}_A^n)^T G_{A,I}(X_n^n - \hat{Y}_A^n)] + \Delta_{\min,A}. \quad (55)
\end{align*}
The equality in (55) can be seen to hold along the lines of (23)--(26). Now,
\begin{align*}
R & = \frac{1}{n} \log |f| \geq \frac{1}{n} H(f(X^n_A)) \\
& = \frac{1}{n} I(X^n_A \land f(X^n_A)) \\
\geq & \min_{\|X^n_A - f(X^n_A)\|^2 \leq \Delta} \frac{1}{n} I(X^n_A \land f(X^n_A)) \\
\geq & \min_{\|X^n_A - Y^n_A\|^2 G_{A,I}(X^n_A - Y^n_A) \leq \Delta - \Delta_{\min,A}} \frac{1}{n} I(X^n_A \land Y^n_A) \quad \text{by (55)} \\
= & \min_{\|X^n_A - Y^n_A\|^2 G_{A,I}(X^n_A - Y^n_A) \leq \Delta - \Delta_{\min,A}} \frac{1}{n} \sum_{t=1}^n \left( I(X At \land X^{t-1}_A, Y^n_A) - I(X At \land X^{t-1}_A) \right)
\end{align*}
Considering a  

\[ \mu_{X^n Y^n} \equiv \mu_{X^n} \times \mu_{Y^n} \]  

since  

\[ \mu_{X^n Y^n} \equiv \mu_{X^n} \times \mu_{Y^n} \]  

but  

\[ 2. \]  

However, we prefer the current manner of presentation which provides for unity of ideas.

where (56) is since  

\[ \mu_{X^n Y^n} \ll \mu_{X^n} \times \mu_{Y^n} \Rightarrow \mu_{X^n Y^n} \ll \mu_{X^n} \times \mu_{Y^n}, \quad t = 1, \ldots, n. \]  

The claim (57) is easy to see by contradiction. Consider any real-valued rvs  

\[ Z_1, Z_2, Z_3 \]  

with probability distribution  

\[ \mu_{Z_1 Z_2 Z_3} \ll \mu_{Z_1} \times \mu_{Z_2} \times \mu_{Z_3}. \]  

Suppose, if possible,  

\[ \mu_{Z_1 Z_2} \]  

is not absolutely continuous with respect to  

\[ \mu_{Z_1} \times \mu_{Z_2}, \text{i.e., there exist } E_1, E_2 \in \mathcal{B}(\mathbb{R}) \text{ such that} \]  

\[ \mu_{Z_1}(E_1) \times \mu_{Z_2}(E_2) = 0 \quad \text{and} \quad \mu_{Z_1 Z_2}(E_1 \times E_2) \neq 0. \]  

Considering a  

\[ E = E_1 \times E_2 \]  

by (58) we have  

\[ \mu_{Z_1 Z_2}(E_1 \times E_2) = 0 \]  

but  

\[ \mu_{Z_1 Z_2 Z_3}(E) \neq 0, \]  

since  

\[ \mu_{Z_1 Z_2}(E_1 \times E_2) \neq 0, \]  

a contradiction, since  

\[ \mu_{Z_1 Z_2 Z_3} \ll \mu_{Z_1} \times \mu_{Z_2} \times \mu_{Z_3}. \]  

\[ \square \]  

Note that a converse proof for Theorem 1 can be provided along the lines of the converse proof for Proposition 2. However, we prefer the current manner of presentation which provides for unity of ideas.

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Recall that the elements of the compact sets $\Theta$ and $\Theta_1$ are indexed by $\tau$ and $\tau_1$, which take values in $\mathbb{R}^{m^2}$ and $\mathbb{R}^{k^2}$ respectively. Now, every $\tau \in \Theta$ can be seen as $\tau = (\tau_1, \tau_2)$ with $\tau_2$ taking values in $\Theta_2$, a bounded subset of $\mathbb{R}^{m^2-k^2}$. A continuous function over a compact set is uniformly continuous, hence, for every $x_{\mathcal{M}} \in \mathbb{R}^m$,

$$\nu_{\chi_{\mathcal{M}}|\theta}(x_{\mathcal{M}}|\tau_1, \tau_2) \text{ and } \nu_{\theta}(\tau_1, \tau_2)$$

**APPENDIX**

**A. Proof of Lemma**

Recall that the elements of the compact sets $\Theta$ and $\Theta_1$ are indexed by $\tau$ and $\tau_1$, which take values in $\mathbb{R}^{m^2}$ and $\mathbb{R}^{k^2}$ respectively. Now, every $\tau \in \Theta$ can be seen as $\tau = (\tau_1, \tau_2)$ with $\tau_2$ taking values in $\Theta_2$, a bounded subset of $\mathbb{R}^{m^2-k^2}$. A continuous function over a compact set is uniformly continuous, hence, for every $x_{\mathcal{M}} \in \mathbb{R}^m$,
are uniformly continuous in \((\tau_1, \tau_2)\). Furthermore, as a function of \(\tau_2\), \(\nu_{X_M|\theta}(x_M|\tau_1, \tau_2)\) and \(\nu_\theta(\tau_1, \tau_2)\) are bounded functions over bounded set \(\Theta_2\) and hence so is \(\nu_{X_M|\theta}(x_M|\tau_1, \tau_2)\nu_\theta(\tau_1, \tau_2)\). By the Bounded Convergence Theorem, for every \(x_M \in \mathbb{R}^{\mathbf{n}}\) and \(\tau_1 \in \Theta_1\)

\[
\lim_{\tau_1 \to \tau_1} \nu_{\theta}(\tau_1) = \lim_{\tau_1 \to \tau_1} \int_{\Theta_2} \nu_\theta(\tau_1, \tau_2) d\tau_2 = \int_{\Theta_2} \nu_\theta(\tau_1, \tau_2) d\tau_2 = \nu_{\theta}(\tau_1) \tag{59}
\]

and

\[
\lim_{\tau_1 \to \tau_1} \int_{\Theta_2} \nu_{X_M|\theta}(x_M|\tau_1, \tau_2)\nu_\theta(\tau_1, \tau_2) d\tau_2 = \int_{\Theta_2} \nu_{X_M|\theta}(x_M|\tau_1, \tau_2)\nu_\theta(\tau_1, \tau_2) d\tau_2, \tag{60}
\]

and thus from (59) and (60),

\[
\lim_{\tau_1 \to \tau_1} \nu_{X_M|\theta}(x_M|\tau_1) = \nu_{X_M|\theta}(x_M|\tau_1).
\]

Continuity of \(\nu_{X_M|\theta}(x_M|\tau_1)\) in \(\tau_1\), implies that for \(i = 1, \ldots, m\), and \(t = 1, \ldots, n\),

\[
E\left[\left((X_i - (\varphi(f(X_M)))_{i,t}\right)^2 | \theta_1 = \tau_1\right]
\]

is continuous in \(\tau_1\). The continuity of

\[
E\left[||X_M^n - \varphi(f(X_M^n))||^2 | \theta_1 = \tau_1\right]
\]

in \(\tau_1\) is now immediate. Since \(\Theta_1\) is a compact set, (61) is uniformly continuous in \(\tau_1\).

\[\square\]

**B. Proof of Lemma**

First, observe that for every \(\tau_1 \in \Theta_1\), \(\Lambda(\tau_1)\) is a convex, compact set. Now, the minimum in (30) exists as that of a continuous function over a compact set. It is seen in a standard manner that the convexity of \(\Theta\) and \(\Theta_1\) imply the convexity of

\[
g(\tau_{1,1}, \tau_{1,2}) \triangleq \min_{\tau \in \Lambda(\tau_{1,2})} \min_{\tau \in \Lambda(\tau_{1,1})} \|\hat{\tau} - \bar{\tau}\|
\]

in \((\tau_{1,1}, \tau_{1,2})\). Consequently, \(g(\tau_{1,1}, \tau_{1,2})\) is continuous in \((\tau_{1,1}, \tau_{1,2})\). Define

\[
D(\delta) \triangleq \max_{\|\tau_{1,1} - \tau_{1,2}\| \leq \delta} g(\tau_{1,1}, \tau_{1,2}).
\]

Clearly, \(D(0) = 0\) and \(D(\delta)\) is a continuous nondecreasing function of \(\delta\) (Chapter 20, [31]).

Now, we prove the lemma by contradiction. Suppose if possible, there exists an \(\epsilon > 0\) such that for every \(\delta > 0\) there exist \(\tau_{1,1,\delta}, \tau_{1,2,\delta} \in \Theta_1\) with \(\|\tau_{1,1,\delta} - \tau_{1,2,\delta}\| \leq \delta\) and

\[
g(\tau_{1,1,\delta}, \tau_{1,2,\delta}) > \epsilon.
\]

Then,

\[
0 = D(0) = \lim_{\delta \to 0} D(\delta) = \lim_{\delta \to 0} \max_{\|\tau_{1,1,\delta}, \tau_{1,2,\delta} \| \leq \delta} g(\tau_{1,1,\delta}, \tau_{1,2,\delta}) \geq \epsilon,
\]

a contradiction. Hence, the lemma.

\[\square\]

**C. Proof of existence of the minimum and maximum in (18)**

For every \(\tau_1 \in \Theta_1\), recall that \(\rho_A^R(\delta, \tau_1)\) is, in effect, a rate distortion function, hence its inverse \(D_A^R(R, \tau_1)\) is well defined over \(R \geq 0\). Continuity of \(\nu_{X_M|\theta}(x_M|\tau_1)\) in \(\tau_1\) for every \(x_M \in \mathbb{R}^{\mathbf{n}}\) implies the continuity of
$D_A^B(R, \tau_1)$ in $\tau_1$.

We now show the existence of the minimum and maximum on the right-side of (18), i.e.,

$$\inf_{\Delta \tau_1 \in R} \sup_{\tau_1 \in \Theta_1} \rho_A^B(\Delta, \tau_1) = \min_{\Delta \tau_1 \in R} \max_{\tau_1 \in \Theta_1} \rho_A^B(\Delta, \tau_1).$$  (62)

Denote the left-side of (62) by $r$ and choose

$$\Delta_{\tau_1}^* = D_A^B(r, \tau_1), \quad \tau_1 \in \Theta_1.$$

The continuity of $D_A^B(r, \tau_1)$ in $\tau_1$ implies the continuity of $\Delta_{\tau_1}^*$ in $\tau_1$ and hence $\mathbb{E}[\Delta_{\tau_1}^*]$ exists. A simple proof of contradiction can be used to show that $\mathbb{E}[\Delta_{\tau_1}^*] \leq \Delta$. Thus, $\{\Delta_{\tau_1}, \tau_1 \in \Theta_1\}$ satisfies the constraint on the left-side of (62) and for every $\tau_1 \in \Theta_1$, $\rho_A^B(\Delta_{\tau_1}, \tau_1) = r$, with

$$\sup_{\tau_1 \in \Theta_1} \rho_A^B(\Delta_{\tau_1}, \tau_1) = r$$

and hence (62) holds.

\[ \Box \]

D. Proof of (66)

Noting that $\hat{\theta}_{1,n}(X^n_A)$ is a deterministic function of $X^n_A$, for $\tau_1 \in \Theta_1$ and $\tau_{i,1} \in \Theta_{1,\delta}$ with $||\tau_1 - \tau_{i,1}|| \leq 2\delta$ and $P_{\theta_1,n,|\theta_1(\tau_{i,1})|} > 0$,

$$\mathbb{E}[[|X^n_M - \varphi(\tau_{i,1}, f_{\tau_{i,1}}(X^n_A))|]^2 | \theta_1 = \tau_1, \hat{\theta}_1,n = \tau_{i,1}]$$

$$= \frac{1}{P_{\theta_1,n,|\theta_1(\tau_{i,1})|}} \mathbb{E}
\left[ I(\hat{\theta}_1,n(X^n_A) = \tau_{i,1}) \left| |X^n_M - \varphi(\tau_{i,1}, f_{\tau_{i,1}}(X^n_A))|\right|^2 | \theta_1 = \tau_1 \right]$$

$$\leq \frac{1}{P_{\theta_1,n,|\theta_1(\tau_{i,1})|}} \mathbb{E}
\left[ |X^n_M - \varphi(\tau_{i,1}, f_{\tau_{i,1}}(X^n_A))| \left| | \theta_1 = \tau_{i,1} \right| + \epsilon_1 \right]$$

by Lemma 7

$$\leq \frac{1}{P_{\theta_1,n,|\theta_1(\tau_{i,1})|}} \left( \Delta_{\tau_{i,1}} + 2\epsilon_1 \right)$$

(64)

$$\leq \frac{1}{P_{\theta_1,n,|\theta_1(\tau_{i,1})|}} \left( \Delta_{\tau_1} + 3\epsilon_1 \right)$$

(65)

where

(i) (63) is since $\hat{\theta}_{1,n}(X^n_A)$ is a deterministic function of $X^n_A$;
(ii) it is seen along the lines of the achievability proof of Theorem 1 that

$$\mathbb{E}[|X^n_M - \varphi(\tau_{i,1}, f_{\tau_{i,1}}(X^n_A))| | \theta_1 = \tau_{i,1}] \leq \Delta_{\tau_{i,1}} + \epsilon_1,$$

and hence (64) is obtained;
(iii) $\Delta_{\tau_1}$ is continuous in $\tau_1$ over the compact set $\Theta_1$, hence, $\Delta_{\tau_1}$ is in fact uniformly continuous in $\tau_1$; (65) now follows.

From (65), the first term on the right-side of (63) is

$$\mathbb{E}[I(\hat{\theta}_1,n - \theta_1 || \leq 2\delta) | X^n_M - \varphi(f(X^n_A))|2] \leq \sum_{\tilde{\tau}_i \in \Theta_{1,\delta}} \mathbb{E}[I |\theta_1 - \tilde{\tau}_i || \leq 2\delta | \Delta_{\theta_i} + 3\epsilon_1]$$

$$\leq \mathbb{E}[\Delta_{\theta_i}] + 3\epsilon_1$$

(28)

by (28)

$$\leq \Delta + 3\epsilon_1.$$
E. Proof of Lemma 5

The right-sides of (5) and (10) are, in effect, the RDf for GMMS with weighted MSE distortion criterion, and hence are finite-valued, decreasing, convex, continuous functions of $\Delta > \Delta_{\min, A}$ and $\Delta > \Delta_{\min, A, \tau_1}$, respectively.

The right-sides of (18) and (19) are clearly nonincreasing functions of $\Delta$. Convexity of the right-sides of (18) and (19) follows from the convexity of $\rho_B^R(\delta, \tau_1)$ and $\rho_{A,1}^R(\delta, \tau_1)$ using standard arguments; continuity for $\Delta > \Delta_{\min, A, \tau_1}$ is a consequence. Finite-valuedness of (18) and (19) follows from the finite-valuedness of $\rho_B^R(\delta, \tau_1)$ and $\rho_{A,1}^R(\delta, \tau_1)$ for $\delta > \Delta_{\min, A, \tau_1}$, respectively.

The convexity of the right-side of (18) can be shown explicitly as follows. Let $\tau_1(1)$ and $\tau_1(2)$ attain the maximum in (18) at $\Delta = \Delta_1$ and $\Delta = \Delta_2$, respectively, where $\Delta_1 < \Delta_2$. For $\Delta_1, \Delta_2 > \Delta_{\min, A}$, let $\{\Delta_1, \tau_1 \in \Theta_1\}$ and $\{\Delta_2, \tau_1 \in \Theta_1\}$, attain the minimum in (18), respectively and are as in Appendix C. For any $0 < \alpha < 1$, and $\tilde{\tau}_1 \in \Theta_1$,

$$\alpha R_A(\Delta_1) + (1 - \alpha) R_A(\Delta_2) = \alpha \rho_A^R(\Delta_1, \tilde{\tau}_1) + (1 - \alpha) \rho_A^R(\Delta_2, \tilde{\tau}_1) \geq \rho_A^R(\alpha \Delta_1 + (1 - \alpha) \Delta_2, \tilde{\tau}_1),$$

by the convexity of $\rho_A^R(\delta, \tau_1)$ in $\delta$. Now, (66) holds for every $\tilde{\tau}_1 \in \Theta_1$, hence

$$\alpha R_A(\Delta_1) + (1 - \alpha) R_A(\Delta_2) \geq \inf_{\tilde{\tau}_1 \in \Theta_1} \sup_{\delta > \Delta_{\min, A, \tau_1}} \rho_A^R(\Delta_1, \tilde{\tau}_1) \geq \inf_{\delta > \Delta_{\min, A, \tau_1}} \sup_{\Delta_1, \Delta_2} \rho_A^R(\Delta_1, \Delta_2, \tilde{\tau}_1) = R_A(\Delta_1, \Delta_2, \tilde{\tau}_1).$$