Abstract. We provide inverse semigroup and groupoid models for the Toeplitz and Cuntz-Krieger algebras of finitely aligned higher-rank graphs. Using these models, we prove a uniqueness theorem for the Cuntz-Krieger algebra.

1. Introduction

A higher-rank graph is a countable category $\Lambda$ endowed with a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the unique factorization property: For all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. The rank of $\Lambda$ is $k$ and for this reason, $(\Lambda, d)$ is also called a $k$-graph. A 1-graph is simply the finite-path category generated freely by an ordinary directed graph.\(^1\) In [2], Kumjian and Pask introduced the notion of a higher-rank graph in order to capture the essential features of the $\mathrm{C}^*$-algebras that Robertson and Steger associated to buildings \cite{14, 15, 16, 17} and to provide links between these and higher order shift dynamical systems. (See [3] also.)

The $\mathrm{C}^*$-algebras associated to higher-rank graphs are generalizations of ordinary graph $\mathrm{C}^*$-algebras in that they are generated by families of partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ that satisfy certain relations that have received a lot of attention in recent years. Kumjian and Pask defined and studied the $\mathrm{C}^*$-algebra of $(\Lambda, d)$, $\mathrm{C}^*(\Lambda)$, in terms of a certain type of groupoid that encodes the graph. They were motivated by, and generalized, the theory in [5] which has been the source of considerable inspiration in our subject. However, just as in the setting of ordinary graphs, where the groupoid techniques of [5] require hypotheses that rule out many interesting examples, the work of Kumjian and Pask requires hypotheses that place significant limitations on the nature of the $k$-graphs that may be analyzed. Our first objective in this paper, then, is to overcome the limitations that Kumjian and Pask place on their $k$-graphs and to show how to build a groupoid that gives the $\mathrm{C}^*$-algebra of an arbitrary $k$-graph subject only to the condition that it is “finitely aligned” (see Definition 3.4). This condition seems to lie at the natural “boundary” of the subject. That is, with or without the use of groupoids, little can be said about $k$-graphs that are not finitely aligned.

\(^1\)For the purpose of motivation, we discuss the theory of 1-graphs at some length in the next section. For a very readable account of graph $\mathrm{C}^*$-algebras, including the rudiments of the $\mathrm{C}^*$-algebras of $k$-graphs, we recommend the CBMS lectures by Iain Raeburn \cite{9}.
We were motivated in part by the important contribution of Paterson \cite{8} in which he successfully circumvented the limitations of \cite{5} that involve finiteness hypotheses on the graphs under consideration by first introducing an inverse semigroup that is naturally attached to the graph. Once this inverse semigroup is identified, he constructed a groupoid from it using technology that he and others have developed and which he exposed thoroughly in \cite{7}. Paterson’s success inspired our approach here. Given a $k$-graph $(\Lambda, d)$, we first build a natural inverse semigroup from $\Lambda$, $S_\Lambda$. However, in contrast to the rank-1 setting of \cite{8}, the groupoid we want is fairly far removed from the universal groupoid of $S_\Lambda$. Rather, it is obtained directly from a natural action of $S_\Lambda$ on a certain “infinite-path space” and is realized in terms of the sheaf of germs of the action (cf. \cite{1}).

One extra benefit of our analysis is that we obtain a groupoid presentation of the Toeplitz algebra of $(\Lambda, d)$, $TC^*\Lambda$. Another is that we overcome the limitation of \cite{8} whereby the graphs considered are free of sources. That is, our theory gives an extension of Paterson’s analysis even when restricted to the context of ordinary graphs.

We note, too, that under the hypotheses invoked by Kumjian and Pask \cite{2}, our analysis is different from theirs. They build a groupoid directly from the $k$-graph; we obtain a groupoid by first considering an inverse semigroup. Their groupoid and ours are the same, however, under their hypotheses. (See Remarks \cite{6,11}.)

The paper is arranged as follows. In the next section, we discuss some of the features of 1-graphs that inspired our analysis. The discussion here is informal and incomplete. Detailed work begins in Section 3 where some of the basic facts about $k$-graphs are exposed, notation is set up, and basic facts about the $C^*$-algebras we will study are presented. Also, two propositions regarding the structure of $k$-graphs are presented to be used in Section 5. As will be seen in due course, they lie at the heart of our analysis.

In Section 4, we restrict our attention to finitely aligned $k$-graphs $(\Lambda, d)$ (Definition \cite{3}), and define our inverse semigroup $S_\Lambda$ from the structure of paths in $\Lambda$. We then define a second countable, locally compact, Hausdorff space $X_\Lambda$ and an inverse semigroup action $\theta$ of $S_\Lambda$ on $X_\Lambda$. The space $X_\Lambda$ comprises all paths constructed on $\Lambda$ — finite, infinite and partially infinite — and $S_\Lambda$ acts naturally by “removing and adding initial segments”.

Section 5 defines the groupoid of germs (cf. \cite{1}) $G_\Lambda$ of the system $(X_\Lambda, S_\Lambda, \theta)$. With a topology naturally arising from the action and topology on $X_\Lambda$, $G_\Lambda$ becomes an ample groupoid with unit space $X_\Lambda$, and $S_\Lambda$ is identified with an inverse semigroup of ample subsets of $G_\Lambda$. It is then shown that $C^*(G_\Lambda)$ is isomorphic to the Toeplitz algebra $TC^*(\Lambda)$. A closed invariant subset $\partial \Lambda$ of the unit space $G_\Lambda^{(0)}$ is then identified and it is shown that $C^*(G_\Lambda|_{\partial \Lambda})$ is isomorphic to the $C^*$-algebra $C^*(\Lambda)$.

Finally, in Section 7 we present a Cuntz-Krieger uniqueness theorem for finitely aligned $k$-graphs. A similar theorem was given in \cite{12}, and we compare the two results here.

2. Motivation

In this section, we call attention to the salient features of graph $C^*$-algebras that are the inspiration for the present work. Our intention is to provide an outline of the crucial points of our analysis, especially to help those unfamiliar with groupoid
and inverse semigroup methods in operator algebra. Those familiar with the theory of graph C*-algebras, and with the associated theories of inverse semigroups and groupoids, may skip directly to the next section.

Let \( E = (E^0, E^1, r, s) \) be an ordinary directed graph. This means that \( E^0 \) and \( E^1 \) are sets, called respectively the set of vertices and the set of edges, and that \( r \) and \( s \) are functions from \( E^1 \) to \( E^0 \), called respectively the range and source maps. We will assume here that our graphs have countable vertex and edge sets. From \( E \) we build the associated finite-path category \( E^* \). This is just the free category generated by \( E \). It may and will be viewed as the collection of finite words \((e_1, e_2, \ldots, e_n)\) over \( E^1 \), where \( s(e_i) = r(e_{i+1}) \) for all \( i \leq n - 1 \), together with the vertices \( v \in E^0 \). The range and source maps extend in the obvious way to \( E^* \) so that \( E^* \) is also a graph. Composition in \( E^* \) is defined through concatenation. Thus, paths \((e_1, e_2, \ldots, e_n)\) and \((f_1, f_2, \ldots, f_m)\) are composable if and only if \( s(e_n) = r(f_1) \), and in this event, their product is \((e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_m)\); if \( v \) is a vertex and \( e \) is of the form \((e_1, e_2, \ldots, e_n)\), then \( v \) and \( e \) are composable if and only if \( v = r(e) \), and in this event, \( ve \) is just \( e \).

The category \( E^* \) is countable and is endowed with a degree functor \( d \) from \( E^* \) to the semigroup, or small category, of non-negative integers, \( N \). Namely, \( d \) describes the length of paths in \( E^* \), so \( d(v) = 0 \) for all vertices \( v \), and \( d(e) = n \) for \( e \) of the form \((e_1, \ldots, e_n)\). Further, the “freeness” of the construction of \( E^* \) is expressed by the fact that if \( d(e) = n_1 + n_2 \), \( n_1, n_2 \in N \), then there are unique paths \( e_1 \) and \( e_2 \) in \( E^* \) such that \( d(e_i) = n_i \), \( i = 1, 2 \), and \( e = e_1 e_2 \). Thus \( E^* \) and \( d \) satisfy the unique factorization property that we mentioned at the outset, and \((E^*, d)\) is a 1-graph. The innocuous-seeming factorization property is essential for the role graphs play in operator algebra.

Given a graph \( E \), one would like to build a \( C^* \)-algebra \( C^*(E) \) that is generated by a family \( \{s_e\}_{e \in E} \) which, at the very least, consists of partial isometries (i.e., \( s_e^* s_e \) is a projection for each \( e \)) satisfying \( s_e s_f = s_{e f} \) for all composable \( e, f \in E^* \). That is, one would like to build a \( C^* \)-algebra that codifies the representation theory of \( E^* \) by partial isometries. Experimental investigation reveals that the crucial elements in \( C^*(E) \) are or ought to be \( s_e s_f^*, e, f \in E^* \), and that one would like the products \( s_e s_f^* \) to behave like matrix units when \( e \) and \( f \) have the same degree. Further, in order for the matrix units \( s_e s_f^* \) of one degree to be linked nicely with the matrix units of a different degree, one is led naturally to require the partial isometries \( s_e \) satisfy the so-called Cuntz-Krieger condition:

\[
 s_v = \sum_{\{e \in E^1 : r(e) = v\}} s_e s_e^* 
\]

Note that since \( s_e \) must be an idempotent partial isometry, it is a projection, and customarily, one writes \( p_v \) instead of \( s_v \) to highlight this. Note, too, that for equation (2.1) to make sense, one requires that \( v \) must be the range of at least one edge, so, as one says, \( v \) is not a source. Also, since an infinite sum of projections cannot converge in a \( C^* \)-algebra, one requires also that the sum is finite; i.e., one requires that \( v \) is not an infinite receiver. Problems with sources and infinite receivers play an important role in the subject, and this paper contributes to their solution, but for this discussion, we will assume that our graph has no sources and that there are no infinite receivers.

The history and theory of graph \( C^* \)-algebras is fairly complex and involved, but to keep matters short, we jump to Paterson’s wonderful insight [8] that it is very
helpful to embed $E^*$ into a certain inverse semigroup $S_E$, and then to use the theory he and others have been developing to realize the $C^*$-algebra $C^*(E)$ as the $C^*$-algebra universal for particular representations of $S_E$ (see [8, Theorem 2]). In fact, in [7] Paterson advocates that the $C^*$-algebras universal for particular representations of an inverse semigroup may be effectively studied as the $C^*$-algebras of groupoids naturally associated to the semigroup. This is the tack we take here. Any inverse semigroup acts by partially defined homeomorphisms on the semicharacter space of its idempotent subsemigroup, and the collection of germs of these maps forms a groupoid. In our setting, there is a closed subset of the semicharacter space which is invariant under the action, and so we consider the groupoid of germs associated to the restricted action.

In the context of our graph $E$, the inverse semigroup $S_E$ consists of a zero element $z$ together with all pairs $(\alpha, \beta) \in E^* \times E^*$ such that $s(\alpha) = s(\beta)$. Multiplication in $S_E$ is given by the formula

$$ (\mu, \nu)(\alpha, \beta) = \begin{cases} (\mu \alpha', \beta) & \text{if } \alpha = \nu \alpha' \\ (\mu, \beta \nu') & \text{if } \nu = \alpha \nu' \\ z & \text{otherwise,} \end{cases} $$

and involution is given by $z^* = z$ and $(e, f)^* = (f, e)$. The path category $E^*$ is embedded in $S_E$ via the formula $e \mapsto (e, s(e))$.

As we just mentioned, we want to think of $S_E$ acting by partially defined homeomorphisms on the semicharacter space of the idempotent semigroup $E$ of $S_E$. Clearly $E$ consists of $z$ together with all the pairs $(e, e)$, $e \in E^*$. On the other hand, by definition, $\hat{E}$ consists of all (nonzero) semigroup homomorphisms from $E$ to the multiplicative semigroup $\{0,1\}$. With respect to the topology of pointwise convergence, $\hat{E}$ is a locally compact Hausdorff space. The semicharacter space $\hat{E}$ may be identified as the disjoint union of $\{z\}$, $E^*$ and the infinite-path space $E^\infty$; that is, $\hat{E}$ comprises $z$, the finite paths $(e_1, e_2, \ldots, e_n)$ of $E^*$, and the infinite sequences $(e_1, e_2, \ldots)$ satisfying $s(e_i) = r(e_{i+1})$ for $i \geq 1$. The infinite-path space is a closed subset of $\hat{E}$ which is invariant under the action of $S_E$. Under the identification of $\hat{E}$ with $\{z\} \cup E^* \cup E^\infty$, the restricted action of $S_E$ on $E^\infty$ is given by removing and affixing initial segments of infinite paths according to the formula

$$(\alpha, \beta) \cdot x = \alpha y, \quad \text{where } x = \beta y.$$
$k$-graphs with $k \geq 2$. Our way around this problem is to look at finite sets of such pairs satisfying certain conditions to be described in Section 4. In a natural way, this collection of finite sets turns out to be an inverse semigroup $S_\Lambda$ containing a natural embedding of $\Lambda$. Indeed, the image of $\Lambda$ generates $S_\Lambda$ as an inverse semigroup with a partially defined addition structure. Further, when $\Lambda = E^*$ is a 1-graph, $S_\Lambda$ contains $S_E$ as an inverse subsemigroup, and there is a one-to-one correspondence between representations of $S_E$ and additive representations$^2$ of $S_\Lambda$.

The second hurdle concerns the semicharacter space of the subsemigroup $E(S_\Lambda)$ of idempotents in $S_\Lambda$: it appears to be much larger than the space on which we want $S_\Lambda$ to act. We pass instead directly to an analogue of the infinite-path space. More accurately, we consider two analogues. The first, denoted $X$, acts by partially defined homeomorphisms on $X_\Lambda$. Further, we show in Section 5 that the groupoid of germs of this action, $G_\Lambda$, parameterizes the Toeplitz algebra of $\Lambda$, $T^*(\Lambda)$; this turns out to be new even in the 1-graph setting. The second analogue of $E^\infty$ is the subset of $X_\Lambda$ called the space of boundary paths, denoted $\partial \Lambda$. In the setting of a graph $E$ with no sources and no infinite receivers, $\partial \Lambda$ is $E^\infty$. In general, $\partial \Lambda$ is a closed subset of $X_\Lambda$ that is invariant under the action of $G_\Lambda$. The reduction of $G_\Lambda$ to $\partial \Lambda$, $G_\Lambda|_{\partial \Lambda}$, is our choice for the groupoid that parameterizes the Cuntz-Krieger algebra of $\Lambda$, $C^*(\Lambda)$.

3. Preliminaries

3.1. Higher-Rank Graphs and their $C^*$-algebras. Throughout the remainder of this paper, $\Lambda$ will denote a fixed $k$-graph and $d : \Lambda \to \mathbb{N}^k$ will be the associated degree functor. As we have spelled out above, the crucial property of $d$ is the unique factorization property: If $d(\lambda) = m + n$ in $\mathbb{N}^k$, then there are unique $\mu$ and $\nu$ in $\Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. Ordinarily a category $\Lambda$ is viewed as a system $\langle \text{Obj}(\Lambda), \text{Mor}(\Lambda), r, s \rangle$ where $\text{Obj}(\Lambda)$ and $\text{Mor}(\Lambda)$ are separate sets and $r$ and $s$ are maps from the second set to the first. However, it is convenient here to take the “arrows only” approach to categories $\mathfrak{G}$. So all elements of $\Lambda$ are morphisms and $\text{Obj}(\Lambda)$ is distinguished by virtue of being the idempotent morphisms. This perspective is especially appropriate because of our use of the degree functor. The unique factorization property allows us to identify $\text{Obj}(\Lambda)$ with the elements of $\Lambda$ that have degree zero. Because of our desire to generalize 1-graphs, we will also call the elements of $\Lambda$ (finite) paths. For $m \in \mathbb{N}^k$, we define $\Lambda^m := d^{-1}\{\{m\}\}$, so as we just mentioned, $\text{Obj}(\Lambda) = \Lambda^0$.

We follow $[11]$ Section 2 for the basic facts about $k$-graphs that we shall use.

Notation 3.1. (1) For $v \in \Lambda^0$ define $v\Lambda := r^{-1}(v)$ and $\Lambda v := s^{-1}(v)$, and for $n \in \mathbb{N}^k$ define $v^n := \Lambda^n \cap v\Lambda$. For $\lambda, \mu \in \Lambda$ define

$$\Lambda^{\min}(\lambda, \mu) := \{(\alpha, \beta) \in \Lambda \times \Lambda : \lambda\alpha = \mu\beta \text{ and } d(\lambda\alpha) = d(\lambda) \lor d(\mu)\}.$$

Here, and throughout, given $m, n \in \mathbb{N}^k$ we write $m \lor n$ for the coordinate-wise maximum of $m$ and $n$. That is, the $i$th coordinate of $m \lor n$ is the maximum of the $i$th coordinates of $m$ and $n$.

(2) $\Lambda \ast \Lambda = \{(\lambda, \mu) \in \Lambda \times \Lambda : s(\lambda) = s(\mu)\}$.

$^2$These are the representations of $S_\Lambda$ which preserve its partially defined addition structure; cf. [7] page 190].
Let \( \lambda \in \Lambda \) and let \( m \) and \( n \) satisfy the inequality \( 0 \leq m \leq n \leq d(\lambda) \). Then the unique factorization property guarantees that there are unique paths \( \lambda_i, i = 1, 2, 3 \), such that \( d(\lambda_1) = m, d(\lambda_2) = n - m, d(\lambda_3) = d(\lambda) - n \) and \( \lambda = \lambda_1 \lambda_2 \lambda_3 \). We shall write \( \lambda(0, m) \) for \( \lambda_1 \), \( \lambda(m, n - m) \) for \( \lambda_2 \) and \( \lambda(n, d(\lambda)) \) for \( \lambda_3 \).

Example 3.2. For \( m \in (\mathbb{N} \cup \{\infty\})^k \), define \( \Omega_{k,m} \) to be the \( k \)-graph with

\[
\text{Obj}(\Omega_{k,m}) = \{ p \in \mathbb{N}^k : p \leq m \},
\]

\[
\text{Mor}(\Omega_{k,m}) = \{ (p, q) \in \text{Obj}(\Omega_{k,m}) \times \text{Obj}(\Omega_{k,m}) : p \leq q \},
\]

\[
r(p, q) = p, \quad s(p, q) = q, \quad d(p, q) = q - p.
\]

Drawn below are \( \Omega_{2,(\infty,\infty)} \) and \( \Omega_{2,(1,2)} \). In the diagrams, edges of degree \((1,0)\) are solid; edges of degree \((0,1)\) are dashed. In each diagram \( \lambda = ((0,2), (1,2)) \) and \( \mu = ((0,0), (0,1)) \).

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & \downarrow & \downarrow & \downarrow \\
\vdots & \vdots & \vdots & \downarrow & \downarrow & \downarrow \\
\ast & \ast & \ast & \ast & \ast & \ast \\
\vdots & \vdots & \vdots & \downarrow & \downarrow & \downarrow \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
(0,0) & (0,1) & (1,0) & (1,1) & (1,2) & (2,2) \\
\end{array}
\]

\( \Omega_{2,(\infty,\infty)} \)

\[
\begin{array}{cccccc}
(0,2) & \lambda & (1,2) & \downarrow & \downarrow & \downarrow \\
\vdots & \vdots & \vdots & \downarrow & \downarrow & \downarrow \\
(0,1) & (1,1) & \mu & \downarrow & \downarrow & \downarrow \\
(0,0) & (1,0) & \end{array}
\]

\( \Omega_{2,(1,2)} \)

Definition 3.3. A morphism between two \( k \)-graphs \((\Lambda_1, d_1)\) and \((\Lambda_2, d_2)\) is a functor \( f : \Lambda_1 \to \Lambda_2 \) satisfying \( d_2(f(\lambda)) = d_1(\lambda) \) for all \( \lambda \in \Lambda_1 \).

Definition 3.4. A \( k \)-graph \((\Lambda, d)\) is finitely aligned if \( \Lambda^{\text{min}}(\lambda, \mu) \) is at most finite for all \( \lambda, \mu \in \Lambda \).

Remark 3.5. If \( \Lambda \) is a 1-graph, then \( \Lambda \) is automatically finitely aligned: for \( \lambda, \mu \in \Lambda \), \( \Lambda^{\text{min}}(\lambda, \mu) \) is either empty or a singleton. Also every row-finite \( k \)-graph is finitely aligned, where a \( k \)-graph is called row-finite if \( v\Lambda^n \) is finite for each \( v \in \Lambda^0 \) and \( n \in \mathbb{N}^k \).
Definition 3.6. Let $\Lambda$ be a $k$-graph and let $v \in \Lambda^0$. We say a subset $E \subseteq v\Lambda$ is exhaustive if for every $\mu \in v\Lambda$ there exists a $\lambda \in E$ such that $\Lambda^{\min}(\lambda, \mu) \neq \emptyset$. We denote the set of all finite exhaustive subsets of $\Lambda$ by $\mathcal{FE}(\Lambda)$, and for $v \in \Lambda^0$, we define $v\mathcal{FE}(\Lambda) := \{ E \in \mathcal{FE}(\Lambda) : E \subseteq v\Lambda \}$.

Examples 3.7.

1. For all $m \in (\mathbb{N} \cup \{\infty\})^k$ and $v \in \Omega_{k,m}^0$, any nonempty finite subset of $v\Omega_{k,m}$ is finite exhaustive.
2. Consider the $k$-graph $\Lambda$ below:

```
\[\begin{array}{c}
  \bullet \\
  | \alpha | | \eta \\
  \downarrow \downarrow \downarrow \\
  \bullet \bullet \bullet \\
  | \lambda | | \omega \\
  \downarrow \downarrow \downarrow \\
  \bullet \bullet \bullet \\
  | \xi | | \zeta \\
  \downarrow \downarrow \downarrow \\
  \bullet \bullet \bullet \\
  | \beta | | \mu \\
  \downarrow \downarrow \downarrow \\
  \bullet \bullet \bullet \\
  | \tau_i, i \in \mathbb{N} \\
  \downarrow \downarrow \downarrow \\
  \bullet \bullet \bullet
\end{array}\]
```

Dashed edges represent edges of degree $(0,1)$ and solid edges represent edges of degree $(1,0)$. The edges $\tau_i$ where $i \in \mathbb{N}$ each have degree $(1,0)$. Any finite exhaustive subset of $w\Lambda$ must contain $w$. The set $\{\mu\}$ is a finite exhaustive subset of $v\Lambda$, whereas $\{\lambda\}$ is not because $\Lambda^{\min}(\lambda, \mu, T_i) = \emptyset$ for any $i \in \mathbb{N}$.

Definition 3.8. Let $\Lambda$ be a finitely aligned $k$-graph. A Toeplitz-Cuntz-Krieger $\Lambda$-family in a $C^*$-algebra $B$ is a collection $\{t_\lambda : \lambda \in \Lambda\}$ of partial isometries in $B$ satisfying

1. $\{t_v : v \in \Lambda^0\}$ consists of mutually orthogonal projections;
2. $t_\lambda t_\mu = t_{\lambda \mu}$ whenever $s(\lambda) = r(\mu)$; and
3. $t^*_\lambda t_\mu = \sum_{(\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu)} t^*_\alpha t_\beta$ for all $\lambda, \mu \in \Lambda$.

A Cuntz-Krieger $\Lambda$-family is a Toeplitz-Cuntz-Krieger $\Lambda$-family $\{t_\lambda : \lambda \in \Lambda\}$ which satisfies

\[\text{(CK)} \prod_{\lambda \in E} (t_v - t^*_\lambda) = 0 \text{ for every } v \in \Lambda^0 \text{ and } E \in v\mathcal{FE}(\Lambda).\]

Of course, the hypothesis that $(\Lambda, d)$ is finitely aligned guarantees that the sums in Definition 3.8 are finite sums, and hence make sense in any $C^*$-algebra. The following remark summarizes the ontological properties of these relations and the $C^*$-algebras that codify them.

Remark 3.9. Let $(\Lambda, d)$ be a finitely aligned $k$-graph. Then there is a $C^*$-algebra $TC^*(\Lambda)$, called the Toeplitz algebra of $\Lambda$, generated by a Toeplitz-Cuntz-Krieger $\Lambda$-family $\{s_\lambda^T : \lambda \in \Lambda\}$, which is universal in the sense that if $\{t_\lambda : \lambda \in \Lambda\}$ is a Toeplitz-Cuntz-Krieger $\Lambda$-family in a $C^*$-algebra $B$, then there exists a $C^*$-homomorphism $\pi : TC^*(\Lambda) \to B$ such that $\pi(s_\lambda^T) = t_\lambda$ for all $\lambda \in \Lambda$. The Cuntz-Krieger algebra of $\Lambda$ is the $C^*$-algebra $C^*(\Lambda)$, generated by a Cuntz-Krieger $\Lambda$-family $\{s_\lambda : \lambda \in \Lambda\}$, which is universal for Cuntz-Krieger $\Lambda$-families. If we let $I$ be the ideal in $TC^*(\Lambda)$ generated by products $\prod_{\lambda \in E} (s_v - s^*_\lambda(s_\lambda^T)^*)$ where $v \in \Lambda^0$ and $E \in v\mathcal{FE}(\Lambda)$, then $C^*(\Lambda)$ can be identified with $TC^*(\Lambda)/I$. Furthermore, \cite[Proposition 2.12]{12} says that the $s_\lambda^T$ and $s_\lambda$ are nonzero for all $\lambda \in \Lambda$, implying that $TC^*(\Lambda)$ and $C^*(\Lambda)$ are nontrivial.
We want to call special attention to [12, Appendix A] for a thorough explanation of the Cuntz-Krieger relations of finitely aligned \( k \)-graphs, and to [12, Appendix B] for an account of how the theory encompasses the Cuntz-Krieger relations described in Equation (24).

3.2. Extending paths. In the study of graph \( C^* \)-algebras, i.e., in the study of \( C^* \)-algebras defined by 1-graphs, problems arise if the graph has sources or sinks. Recall that a source in a graph is a vertex \( v \) that does not receive any edges, i.e., \( r^{-1}(v) = \emptyset \), while \( v \) is a sink if \( v \) does not emit any edges, i.e., if \( s^{-1}(v) = \emptyset \). There is now a substantial literature on how to tackle these. In a \( k \)-graph \( \Lambda \), a vertex \( v \) is a source if \( v \Lambda^{e_i} = 0 \) for some \( i \in \{1, \ldots, k\} \) where \( e_i \) is the element in \( \mathbb{N}^k \) with 1 in the \( i \)-th coordinate and 0 in every other coordinate. Therefore in the setting of higher-rank graphs, the situations can be considerably more complicated. A vertex may receive edges from some directions and not from others, while emitting edges in still other directions, but not in all. The highly ramified collection of possibilities creates numerous difficulties. One of the achievements of our approach to the analysis of \( C^* \)-algebras associated to higher-rank graphs is to circumvent difficulties that sources and sinks can cause. We do not eliminate all problems, of course. Rather, we show that sources and sinks do not prevent one from defining and analyzing a groupoid associated to such \( k \)-graphs.

A key tool in our analysis are the two propositions of this subsection, Propositions 3.11 and 3.12.

**Definition 3.10.** Given \( \lambda \in \Lambda \) and \( E \subseteq r(\lambda)\Lambda \), write \( \text{Ext}(\lambda; E) \) for the set

\[
\bigcup_{\mu \in E} \{ \alpha \in \Lambda : (\alpha, \beta) \in \Lambda^{\min}(\lambda, \mu) \text{ for some } \beta \in \Lambda \}.
\]

**Proposition 3.11.** [12, Lemma C.4] Let \( (\Lambda, d) \) be a finitely aligned \( k \)-graph, let \( v \in \Lambda^0 \), let \( \lambda \in v\Lambda \), and suppose \( E \in vFE(\Lambda) \). Then \( \text{Ext}(\lambda; E) \in s(\lambda)FE(\Lambda) \).

**Proposition 3.12.** If \( (\Lambda, d) \) is a \( k \)-graph, then for \( v \in \Lambda^0 \), \( E \subseteq v\Lambda \), \( \lambda_1 \in v\Lambda \) and \( \lambda_2 \in s(\lambda_1)\Lambda \), the following equation holds:

\[
\text{Ext}(\lambda_2; \text{Ext}(\lambda_1; E)) = \text{Ext}(\lambda_1 \lambda_2; E).
\]

**Proof.** Let \( \alpha \in \text{Ext}(\lambda_2; \text{Ext}(\lambda_1; E)) \), so \( (\alpha, \beta) \in \Lambda^{\min}(\lambda_2, \xi) \) for some \( \xi \in \text{Ext}(\lambda_1; E) \) and \( \beta \in \Lambda \). Then \( \lambda_2 \alpha = \xi \beta \) and \( d(\lambda_2 \alpha) = d(\lambda_2) \vee d(\xi) \). Since \( \xi \in \text{Ext}(\lambda_1; E) \), we have \( (\xi, \eta) \in \Lambda^{\min}(\lambda_1, \mu) \) for some \( \mu \in E \) and \( \eta \in \Lambda \). So \( \lambda_1 \xi = \mu \eta \) and \( d(\lambda_1 \xi) = d(\lambda_1) \vee d(\mu) \). Thus

\[
\lambda_1 \lambda_2 \alpha = \lambda_1 \xi \beta = \mu \eta \beta
\]

and

\[
d(\lambda_1 \lambda_2 \alpha) = d(\lambda_1) + d(\lambda_2 \alpha)
\]

\[
= d(\lambda_1) + d(\lambda_2) \vee d(\xi)
\]

\[
= d(\lambda_1 \lambda_2) \vee d(\lambda_1 \xi)
\]

\[
= d(\lambda_1 \lambda_2) \vee (d(\lambda_1) \vee d(\mu))
\]

\[
= d(\lambda_1 \lambda_2) \vee d(\mu),
\]

so \( (\alpha, \eta \beta) \in \Lambda^{\min}(\lambda_1 \lambda_2, \mu) \). Therefore \( \alpha \in \text{Ext}(\lambda_1 \lambda_2; E) \), giving

\[
\text{Ext}(\lambda_2; \text{Ext}(\lambda_1; E)) \subseteq \text{Ext}(\lambda_1 \lambda_2; E).
\]
Now let $\alpha \in \text{Ext}(\lambda_1 \lambda_2; E)$, so $(\alpha, \beta) \in \Lambda^{\text{min}}(\lambda_1 \lambda_2, \mu)$ for some $\mu \in E$ and $\beta \in \Lambda$. Then
\[
\lambda_1 \lambda_2 \alpha = \mu \beta \quad \text{and} \quad d(\lambda_1 \lambda_2 \alpha) = d(\lambda_1 \lambda_2) \lor d(\mu).
\]
(3.1)
It follows that
\[
\begin{align*}
\lambda_2 \alpha &= (\mu \beta)(d(\lambda_1), d(\lambda_1 \lambda_2) \lor d(\mu)) \\
&= (\mu \beta)(d(\lambda_1), d(\lambda_1) \lor d(\mu))(\mu \beta)(d(\lambda_1) \lor d(\mu), d(\lambda_1 \lambda_2) \lor d(\mu)).
\end{align*}
\]
(3.2)
Defining $\xi := (\mu \beta)(d(\lambda_1), d(\lambda_1) \lor d(\mu))$, we then have
\[
\lambda_2 \alpha = \xi(\mu \beta)(d(\lambda_1) \lor d(\mu), d(\lambda_1 \lambda_2) \lor d(\mu)).
\]
(3.3)
To show that $\alpha \in \text{Ext}(\lambda_2; \text{Ext}(\lambda_1; E))$, we need to show that $\xi \in \text{Ext}(\lambda_1; E)$ and $d(\lambda_2 \alpha) = d(\lambda_2) \lor d(\xi)$. To this end, we calculate
\[
\begin{align*}
\lambda_1 \xi &= \lambda_1(\mu \beta)(d(\lambda_1), d(\lambda_1) \lor d(\mu)) \\
&= (\mu \beta)(0, d(\lambda_1) \lor d(\mu)) \quad \text{by (3.1)} \\
&= \mu \beta(0, (d(\lambda_1) \lor d(\mu)) - d(\mu)).
\end{align*}
\]
Furthermore, by (3.2) we have
\[
(3.2)
\]
\[
d(\lambda_1 \xi) = d(\lambda_1) \lor d(\mu).
\]
Hence $\xi \in \text{Ext}(\lambda_1; E)$. It remains to show that $d(\lambda_2 \alpha) = d(\lambda_2) \lor d(\xi)$. On the one hand
\[
\begin{align*}
d(\lambda_2 \alpha) &= (d(\lambda_1 \lambda_2) \lor d(\mu)) - d(\lambda_1) \\
&= d(\lambda_2) \lor (d(\mu) - d(\lambda_1)),
\end{align*}
\]
and on the other hand
\[
\begin{align*}
d(\lambda_2) \lor d(\xi) &= d(\lambda_2) \lor ((d(\lambda_1) \lor d(\mu)) - d(\lambda_1)) \\
&= d(\lambda_2) \lor (0 \lor (d(\mu) - d(\lambda_1))) \\
&= d(\lambda_2) \lor (d(\mu) - d(\lambda_1)).
\end{align*}
\]
So $d(\lambda_2 \alpha) = d(\lambda_2) \lor d(\xi)$, as required. Therefore $\alpha \in \text{Ext}(\lambda_2; \text{Ext}(\lambda_1; E))$, and we have
\[
\text{Ext}(\lambda_2; \text{Ext}(\lambda_1; E)) = \text{Ext}(\lambda_1 \lambda_2; E).
\]

4. INVERSE SEMIGROUPS OF HIGHER-RANK GRAPHS

For the remainder of the paper $(\Lambda, d)$ will be a finitely aligned $k$-graph. To build the groupoids associated to $\Lambda$, we first construct an inverse semigroup $S_\Lambda$ associated to $\Lambda$ that captures the salient features of Toeplitz-Cuntz-Krieger families. Recall that a semigroup $S$ is an inverse semigroup if for all $s \in S$ there exists a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. We follow [7] for the general theory and notation concerning inverse semigroups. In particular, we denote the semilattice of idempotents of $S$ by $E(S)$.

For $(\lambda, \mu), (\xi, \eta) \in \Lambda_* \Lambda$, we write $(\lambda, \mu) \perp (\xi, \eta)$ in case $\Lambda^{\text{min}}(\lambda, \xi) = \emptyset$ and $\Lambda^{\text{min}}(\mu, \eta) = \emptyset$.

**Definition 4.1.** Define $S_\Lambda$ to be the collection of all finite subsets $F$ of $\Lambda_* \Lambda$ such that for distinct $(\lambda, \mu)$ and $(\nu, \omega) \in F$, we have $(\lambda, \mu) \perp (\nu, \omega)$.
Remark 4.2. The empty subset of $\Lambda \ast_s \Lambda$ is an element of $S_\Lambda$.

Proposition 4.3. For elements $F, G \in S_\Lambda$, the equation

$$FG := \bigcup_{(\lambda,\mu) \in F, (\xi,\eta) \in G} \{(\lambda\alpha, \eta\beta) : (\alpha, \beta) \in \Lambda^{\min}(\mu, \xi)\}$$

defines an associative multiplication on $S_\Lambda$.

Proof. The product of two finite sets is a finite set since $\Lambda$ is finitely aligned. It is clear that the product also satisfies Definition 4.1, so $S_\Lambda$ is closed under multiplication.

For associativity, fix $(\lambda, \mu), (\xi, \eta), (\tau, \omega) \in \Lambda \ast_s \Lambda$. It suffices to show that

$$\left(\left(\{\lambda\alpha\} \cdot \{(\xi, \eta)\}\right) \cdot \{(\tau, \omega)\}\right) = \{\{\lambda\alpha\} \cdot \{(\xi, \eta)\}\} \cdot \{(\tau, \omega)\}.$$ 

The left-hand side equals

$$\bigcup_{(\alpha, \beta) \in \Lambda^{\min}(\mu, \xi)} \bigcup_{(\nu, \zeta) \in \Lambda^{\min}(\eta, \tau)} \{(\lambda\alpha\nu, \omega\zeta)\}$$

and the right-hand side equals

$$\bigcup_{(\nu', \zeta') \in \Lambda^{\min}(\eta, \tau)} \bigcup_{(\alpha', \beta') \in \Lambda^{\min}(\mu, \xi\nu')} \{(\lambda\alpha'\nu', \omega\zeta'\beta')\}.$$ 

So we need to show that

$$\{\{\lambda\alpha\nu, \omega\zeta\} : (\alpha, \beta) \in \Lambda^{\min}(\mu, \xi), (\nu, \zeta) \in \Lambda^{\min}(\eta, \tau)\}$$

$$= \{\{\lambda\alpha'\nu', \omega\zeta'\beta'\} : (\alpha', \beta') \in \Lambda^{\min}(\mu, \xi\nu')\}. \quad (\text{(4.2)})$$

Suppose $(\alpha\nu, \zeta)$ is an element of the left-hand side of (4.2), so $(\alpha, \beta) \in \Lambda^{\min}(\mu, \xi)$ and $(\nu, \zeta) \in \Lambda^{\min}(\eta, \tau)$. We will show that $(\alpha\nu, \zeta)$ is of the form $(\alpha', \zeta'\beta')$ where $(\nu', \zeta') \in \Lambda^{\min}(\eta, \tau)$ and $(\alpha', \beta') \in \Lambda^{\min}(\mu, \xi\nu')$. To begin,

$$\mu\alpha = \xi\beta \quad \text{and} \quad \eta\beta\nu = \tau\zeta \quad (\text{(4.3)})$$

where

$$\mu\alpha = d(\mu) \lor d(\xi) \quad \text{and} \quad \eta\beta\nu = d(\eta) \lor d(\tau), \quad (\text{(4.4)})$$

so we set

$$\nu' := (\beta\nu)(0, (d(\eta) \lor d(\tau)) - d(\eta)) \quad \text{and} \quad \zeta' := (\zeta(0, (d(\eta) \lor d(\tau)) - d(\tau)).$$

Then $(\nu', \zeta') \in \Lambda^{\min}(\eta, \tau)$. By (4.3),

$$\mu\alpha\nu = \xi\beta\nu = \xi\nu'(\beta\nu)((d(\eta) \lor d(\tau)) - d(\eta), d(\beta\nu)), \quad (\text{(4.5)})$$

so we set

$$\alpha' := \alpha\nu \quad \text{and} \quad \beta' := (\beta\nu)((d(\eta) \lor d(\tau)) - d(\eta), d(\beta\nu)).$$

This gives $\mu\alpha' = \xi\nu'\beta'$ and $\zeta'\beta' = \zeta$.

We now have $(\alpha\nu, \zeta) = (\alpha', \zeta'\beta')$ where $(\nu', \zeta') \in \Lambda^{\min}(\eta, \tau)$ and $\mu\alpha' = \xi\nu'\beta'$, so it remains to show that $(\alpha', \beta') \in \Lambda^{\min}(\mu, \xi\nu')$; that is, $d(\mu\alpha') = d(\mu) \lor d(\xi\nu')$. For
this we calculate
\[
d(\mu \alpha') = d(\mu \alpha) + d(\nu) = (d(\mu) \vee d(\xi)) + (d(\eta \beta) \vee d(\tau)) - d(\eta) - d(\beta) \]
by \ref{leq}
\[
= (d(\mu) \vee d(\xi)) + (0 \vee (d(\tau) - d(\eta) - d(\beta))) \]
by \ref{leq}
\[
= (d(\mu) \vee d(\xi)) + (0 \vee (d(\tau) - d(\eta) - (d(\mu) \vee d(\xi)) + d(\eta))) \]
by \ref{leq}
\[
= d(\mu) \vee (d(\xi) \vee d(\tau) - d(\eta) + d(\xi)) \]
\[
= d(\mu) \vee (d(\xi) \vee d(\eta) \vee d(\tau)) - d(\eta) \]
\[
= d(\mu) \vee (d(\xi) + d(\nu')) \quad \text{since } (\nu', \zeta') \in \Lambda^{\text{min}}(\eta, \tau) \]
\[
= d(\mu) \vee d(\xi),
\]
as required. Therefore \((\alpha \nu, \zeta)\) is of the form \((\alpha', \zeta' \beta')\) where \((\nu', \zeta') \in \Lambda^{\text{min}}(\eta, \tau)\)
and \((\alpha', \beta') \in \Lambda^{\text{min}}(\mu, \nu', \zeta')\), and so the left-hand side of \ref{leq}
is contained in the right-hand side of \ref{leq}.

The reverse containment can be proved in a similar way, giving the result. \(\square\)

**Proposition 4.4.** \(S_\Lambda\) is an inverse semigroup with involution defined for \(F \in S_\Lambda\)
by \(F^*: \{ (\mu, \lambda) : (\lambda, \mu) \in F \} \).

**Proof.** Fix \(F \in S_\Lambda\). We need to show that \(F^*\) is the unique element of \(S_\Lambda\) such
that \(FF^*F = F\) and \(F^*FF^* = F^*\). Using \ref{leq} we have
\[
F(F^*F) = F\{ (\mu, \lambda) : (\lambda, \mu) \in F \} = F,
\]
and similarly, \(F^*FF^* = F^*\). For uniqueness, suppose \(FGF = F\) and \(GFG = G\)
for some \(G \in S_\Lambda\). The product \(FGF\) can be written as
\[
\bigcup_{(\lambda, \mu), (\tau, \omega) \in F} \bigcup_{(\xi, \eta) \in G} \{ (\alpha \nu, \omega \zeta) : (\alpha, \beta) \in \Lambda^{\text{min}}(\mu, \xi), (\nu, \zeta) \in \Lambda^{\text{min}}(\eta, \tau, \zeta) \}.
\]
Since \(FGF = F\), for each \((\lambda', \mu') \in F\), there exist \((\lambda, \mu), (\tau, \omega) \in F\), \((\xi, \eta) \in G\),
\((\alpha, \beta) \in \Lambda^{\text{min}}(\mu, \xi)\) and \((\nu, \zeta) \in \Lambda^{\text{min}}(\eta, \tau)\) such that
\((\lambda', \mu') = (\lambda \nu, \omega \zeta)\). This
shows that \((\lambda', \mu') \not\in (\lambda, \mu)\) and \((\lambda', \mu') \not\in (\tau, \omega)\). Thus Definition \ref{leq}
implies \((\lambda', \mu') = (\lambda, \mu) = (\tau, \omega)\), so \(\alpha = \nu = \zeta = \sigma(\lambda)\). Hence \((s(\lambda), \beta) \in \Lambda^{\text{min}}(\mu, \xi)\) and
\((s(\lambda), s(\lambda)) \in \Lambda^{\text{min}}(\eta, \tau, \zeta)\), which gives
\[
(4.5) \quad \lambda = \eta \beta \quad \text{and} \quad \mu = \xi \beta.
\]
We claim that \(\beta = s(\xi)\). Since \((\xi, \eta) \in G\), \((\lambda, \mu) \in F\), \((\beta, s(\lambda)) \in \Lambda^{\text{min}}(\eta, \lambda)\) and
\((s(\lambda), \beta) \in \Lambda^{\text{min}}(\mu, \xi)\), it follows from the definition of multiplication that \((\xi, \eta, \beta) \in GFG\).
Since \(GFG = G\), it then follows from Definition \ref{leq} that \((\xi, \eta, \beta) = (\xi, \eta)\),
which gives \(\beta = s(\xi)\). Therefore \ref{leq} gives \(\lambda = \eta\) and \(\mu = \xi\). Thus for every
\((\lambda, \mu) \in F\), we have \((\mu, \lambda) \in G\), so \(F^* \subseteq G\). A similar argument gives \(G^* \subseteq F\) which
is equivalent to \(G \subseteq F^*\). Therefore \(G = F^*\), and \(S_\Lambda\) is an inverse semigroup. \(\square\)

**Remark 4.5.** As we noted in Section 2 the inverse semigroup Paterson used in
\[ \Lambda = \Lambda \ast \Lambda \cup \{ z \} \] The multiplication defined on Paterson’s inverse semigroup
coincides with the multiplication of singleton sets in \(S_\Lambda\). If \(\Lambda\) is a \(k\)-graph with \(k \geq 2\) and \(\lambda, \mu \in \Lambda\), then \(\Lambda^{\text{min}}(\lambda, \mu)\) generally contains more than one element.
Thus \(S_\Lambda\) was chosen to consist of finite subsets of \(\Lambda \ast \Lambda\) in order to obtain an
inverse semigroup that is closed under a multiplication and which captures the
properties of the $k$-graph. We note, too, that in one has to add a zero element to $\Lambda \ast_s \Lambda$, but this is not necessary in our setting, since the empty subset of $\Lambda \ast_s \Lambda$ is included in our $S_\Lambda$ and serves as a zero.

5. THE ACTION OF $S_\Lambda$ ON THE PATH SPACE $X_\Lambda$

**Definition 5.1.** Let $X_\Lambda$ be the set of all graph morphisms $x : \Omega_{k,m} \to \Lambda$ where $m \in (\mathbb{N} \cup \{\infty\})^k$ (See Definition 3.3 and Example 3.2). We extend range and degree maps to all $x : \Omega_{k,m} \to \Lambda$ in $X_\Lambda$ by setting $r(x) := x(0)$ and $d(x) := m$.

**Notation 5.2.** Let $x \in X_\Lambda$.

1. For $m \in \mathbb{N}^k$ with $m \leq d(x)$, we write $\sigma^m(x)$ for the graph morphism $\sigma^m(x) : \Omega_{k,d(x) - m} \to \Lambda$ defined by the equation
   
   $$\sigma^m(x)(p, q) = x(p + m, q + m)$$

   for all $p, q \in \mathbb{N}^k$, $p \leq q \leq d(x) - m$.

2. For $\lambda \in \Lambda_{\infty}(0)$, we write $\lambda x$ for the graph morphism $\lambda x : \Omega_{k,d(x) + d(\lambda)} \to \Lambda$ defined by $(\lambda x)(0, m) = \lambda x(0, m - d(\lambda))$ for all $m \in \mathbb{N}^k$, $d(\lambda) \leq m \leq d(x) + d(\lambda)$.

**Remark 5.3.** The set $X_\Lambda$ contains $\Lambda$ as a subset if we identify each path $\lambda \in \Lambda$ with the unique graph morphism $x_\lambda : \Omega_{k,d(\lambda)} \to \Lambda$ given by $x_\lambda(0, d(\lambda)) = \lambda$.

For $F \in S_\Lambda$ define

$$D_F := \{x \in X_\Lambda : \text{there exists } (\lambda, \mu) \in F \text{ such that } x(0, d(\mu)) = \mu\}$$

and define $\theta_F : D_F \to D_F$ by $\theta_F(x) := \lambda \sigma^{d(\mu)}(x)$ where $(\lambda, \mu)$ is the unique element of $F$ such that $x(0, d(\mu)) = \mu$. For $\lambda \in \Lambda$, we write $D_\lambda$ instead of $D_{((\lambda, \lambda))}$.

**Proposition 5.4.** The family of sets $\{D_F, \theta_F^* : F \in S_\Lambda\}$ is a subbasis for a Hausdorff topology on $X_\Lambda$.

**Proof.** Since $X_\Lambda = \bigcup_{c \in \Lambda^0} D_c$, the family $\{D_F, \theta_F^* : F \in S_\Lambda\}$ forms a subbasis for a topology on $X_\Lambda$. To show that the topology is Hausdorff, fix $x, y \in X_\Lambda$, $x \neq y$. If $x(0) \neq y(0)$, then $x \in D_{x(0)}$, $y \in D_{y(0)}$ and $D_{x(0)} \cap D_{y(0)} = \emptyset$. So assume that $x(0) = y(0)$, and let $n \in \mathbb{N}^k$ be minimal with respect to the condition

$$x(0, n) = y(0, n) \quad \text{and} \quad x(0, n + e_i) \neq y(0, n + e_i) \quad \text{for some } i \in \{1, \ldots, k\}.$$

Without loss of generality assume that $n + e_i \leq d(x)$. Then $x \in D_{x(0, n + e_i)}$ and $y \in D_{x(0, n + e_i)}^* \cap D_{x(0, n + e_i)}$, as required. \hfill $\Box$

**Remark 5.5.** For $F_1, \ldots, F_n, G_1, \ldots, G_m \in S_\Lambda$, letting $F = \prod_{i=1}^n F_i^* F_i$, we have

$$D_{F_1} \cap \cdots \cap D_{F_n} \cap D_{G_1}^c \cap \cdots \cap D_{G_m}^c = D_F \cap \bigcap_{i=1}^m D_{F_i G_i G_i^*}.$$

Since $D_F$ is the disjoint union of the $D_\lambda$ where $(\lambda, \lambda) \in F$, the family of sets

$$\{D_\lambda \cap D_{\lambda_1}^c \cap \cdots \cap D_{\lambda_t}^c : \lambda \in \Lambda \text{ and } \nu_1, \ldots, \nu_t \in s(\lambda)\}$$

is a basis for the given topology on $X_\Lambda$.

**Remark 5.6.** An infinite sequence of paths in $\Lambda$ is called wandering if for any finite set $E \subseteq \Lambda$, the sequence is eventually in $\Lambda \setminus E$. Convergence in $X_\Lambda$ is then given by: A sequence $\langle x_i \rangle$ converges to $x$ if and only if the following two conditions occur:

1. For all $n \in \mathbb{N}^k$ such that $n \leq d(x)$, there exists $I \in \mathbb{N}$ such that $i \geq I$ implies $x_i(0, n) = x(0, n)$, and
For each Proposition 5.7. locally compact Hausdorff space.

Proof. Let \( \langle x_i \rangle_{i=1}^{\infty} \) be a sequence in \( D_v \). We construct an element \( x \in X_\Lambda \) and a subsequence \( \langle y_i \rangle_{i=1}^{\infty} \) such that \( y_i \rightarrow x \).

Let \( \{ E_n \}_{n=1}^{\infty} \) be a listing of all finite subsets of \( v \Lambda \) containing \( v \). There is at least one \( \lambda \in E_1 \) such that \( x_i(0, d(\lambda)) = \lambda \) for infinitely many \( i \in \mathbb{N} \) (namely \( \lambda = v \)). Let \( \lambda_1 \) be such a \( \lambda \) of maximal degree.

Suppose that \( \lambda_1, \ldots, \lambda_n \) have already been defined so that \( \lambda_1 \cdots \lambda_n \in v \Lambda \) and \( x_i(0, d(\lambda_1 \cdots \lambda_j)) = \lambda_1 \cdots \lambda_n \) for infinitely many \( i \in \mathbb{N} \). There is at least one \( \lambda \in \text{Ext}(\lambda_1 \cdots \lambda_n; E_{n+1}) \) such that \( x_i(0, d(\lambda_1 \cdots \lambda_j \lambda)) = \lambda_1 \cdots \lambda_n \lambda \) for infinitely many \( i \in \mathbb{N} \) (namely \( \lambda = s(\lambda_n) \), since \( v \in E_{n+1} \)). Let \( \lambda_{n+1} \) be such a \( \lambda \) of maximal degree.

Define \( m \in (\mathbb{N} \cup \{ \infty \})^k \) by \( m := \lim_{n \to \infty} d(\lambda_1 \cdots \lambda_n) \). There is a unique \( x \in X_\Lambda \) such that \( d(x) = m \) and \( x(0, d(\lambda_1 \cdots \lambda_n)) = \lambda_1 \cdots \lambda_n \) for all \( n \in \mathbb{N} \). Let \( \langle y_i \rangle_{i=1}^{\infty} \) be a subsequence of \( \langle x_i \rangle_{i=1}^{\infty} \) such that for all \( n \in \mathbb{N} \),

\[
y_i(0, d(\lambda_1 \cdots \lambda_n)) = \lambda_1 \cdots \lambda_n \text{ for all } i \geq n.
\]

We claim that \( \lim_{i \to \infty} y_i = x \).

Fix a neighborhood \( D_\Lambda \cap D_{\lambda_1}^c \cap \cdots \cap D_{\lambda_l}^c \) of \( x \), so we have \( x(0, d(\lambda)) = \lambda \) and \( x(0, d(\lambda_{\nu_j})) \neq \lambda_{\nu_j} \) for \( j = 1, \ldots, l \). There exists \( n \in \mathbb{N} \) such that

\[
E_n = \{ v \} \cup \{ \lambda_{\nu_1}, \ldots, \lambda_{\nu_l} \}
\]

and

\[
\lambda_n \in \text{Ext}(\lambda_1 \cdots \lambda_{n-1}; E_n)
= \{ s(\lambda_{n-1}) \} \cup \bigcup_{j=1}^{l} \{ \alpha : (\alpha, \beta) \in \Lambda^{\min}(\lambda_1 \cdots \lambda_{n-1}, \lambda_{\nu_j}) \text{ for some } \beta \in \Lambda \}.
\]

For \( i \geq n-1 \), if \( y_i(0, d(\lambda_{\nu_j})) = \lambda_{\nu_j} \) for some \( j \in \{ 1, \ldots, l \} \), then by (5.1)

\[
y_i(d(\lambda_1 \cdots \lambda_{n-1}), d(\lambda_1 \cdots \lambda_{n-1}) \lor d(\lambda_{\nu_j})) \in \{ \alpha : (\alpha, \beta) \in \Lambda^{\min}(\lambda_1 \cdots \lambda_{n-1}, \lambda_{\nu_j}) \text{ for some } \beta \in \Lambda \}.
\]

Suppose, for contradiction, that there are infinitely many \( i \geq n-1 \) such that \( y_i(0, d(\lambda_{\nu_j})) = \lambda_{\nu_j} \) for some \( j \in \{ 1, \ldots, l \} \). Then there are infinitely many \( i \geq n-1 \) such that (5.2) holds for some \( j \in \{ 1, \ldots, l \} \). Since \( \lambda_n \) is chosen to be of maximal degree, we must then have

\[
\lambda_n \in \bigcup_{j=1}^{l} \{ \alpha : (\alpha, \beta) \in \Lambda^{\min}(\lambda_1 \cdots \lambda_{n-1}, \lambda_{\nu_j}) \}
\]

But then \( x(0, d(\lambda_1 \cdots \lambda_n)) = \lambda_1 \cdots \lambda_n = \lambda_{\nu_j} \beta \) for some \( j \in \{ 1, \ldots, l \} \) and \( \beta \in \Lambda \), contradicting the inequality \( x(0, d(\lambda_{\nu_j})) \neq \lambda_{\nu_j} \). Hence there are only finitely many \( i \in \mathbb{N} \) such that \( y_i \notin D_\Lambda \cap D_{\lambda_1}^c \cap \cdots \cap D_{\lambda_l}^c \). Therefore \( \lim_{i \to \infty} y_i = x \), and \( D_v \) is compact. \( \square \)

**Corollary 5.8.** For each \( F \in S_\Lambda \), \( D_F \) is compact.
Proof. For each \( \lambda \in \Lambda \), \( D_\lambda \) is a closed subset of \( D_{r(\lambda)} \), and hence is compact by Proposition 5.7. The result then follows since \( D_F = \bigcup_{(\lambda, \mu) \in F} D_\mu \) — a finite union.

Remark 5.9. In the next section we will construct two topological groupoids associated to a finitely aligned \( k \)-graph \( \Lambda \). The unit space of the first groupoid \( G_\Lambda \) is homeomorphic to \( X_\Lambda \), and \( C^*(G_\Lambda) \) is isomorphic to the Toeplitz algebra \( TC^*(\Lambda) \). The second groupoid is a reduction of \( G_\Lambda \) to a closed invariant subset \( \partial \Lambda \) of \( (G_\Lambda)^{(0)} \); we describe \( \partial \Lambda \) as a subspace of \( X_\Lambda \). The \( C^* \)-algebra of \( G_\Lambda|_{\partial \Lambda} \) is isomorphic to the Cuntz-Krieger algebra \( C^*(\Lambda) \).

Definition 5.10. An element \( x \in X_\Lambda \) is called a boundary path if for all \( n \in \mathbb{N} \) with \( n \leq d(x) \), and for all \( E \in x(n) \mathcal{F}(\Lambda) \), there exists \( \lambda \in E \) such that \( x(n, n + d(\lambda)) = \lambda \). We write \( \partial \Lambda \) for the set of all boundary paths, and for \( v \in \Lambda^0 \), write \( v(\partial \Lambda) \) for \( \{ x \in \partial \Lambda : r(x) = v \} \).

Examples 5.11. (1) If \( \Lambda \) is a row-finite \( k \)-graph with no sources, \( \partial \Lambda = \Lambda^\infty \) where \( \Lambda^\infty \) is the set of all graph morphisms from \( \Omega_{k,(\infty, \cdots, \infty, \infty)} \) to \( \Lambda \).
(2) For the \( k \)-graph in Example 5.7 (2), any path starting with \( \gamma, \eta, \xi, \omega \) or \( \tau_i \) for some \( i \) is a boundary path. Furthermore, since any finite exhaustive subset of \( w \Lambda \) must contain \( w \), every element in \( \Lambda w \) is also a boundary path.

Lemma 5.12. \( \partial \Lambda \) is closed in \( X_\Lambda \).

Proof. \( \partial \Lambda \) is closed since it is the complement in \( X_\Lambda \) of the open set

\[
\bigcup_{\lambda \in \Lambda} \bigcup_{E \in \sigma(\lambda)} D_\lambda \cap \bigcap_{v \in E} D_{\lambda v}^c.
\]

Straightforward calculations give the following lemma.

Lemma 5.13. Let \( (\Lambda, d) \) be a finitely aligned \( k \)-graph and let \( x \in \partial \Lambda \).

(1) If \( m \in \mathbb{N}^k \) and \( m \leq d(x) \), then \( \sigma^m(x) \in \partial \Lambda \).

(2) If \( \lambda = \Lambda x(0) \), then \( \lambda x \in \partial \Lambda \).

Remark 5.14. Lemma 5.12 and Lemma 5.13 imply that \( \partial \Lambda \) is a locally compact Hausdorff space which is invariant under the action of \( S_\Lambda \). Following on from Remark 5.9 this fact will give us the required closed invariant subset of the unit space of \( G_\Lambda \). The next lemma ensures that the partial isometries inside \( C^*(G_\Lambda|_{\partial \Lambda}) \) giving a Cuntz-Krieger \( \Lambda \)-family are nonzero.

Lemma 5.15. For all \( v \in \Lambda^0 \), \( v(\partial \Lambda) \) is nonempty.

Proof. We construct a sequence of paths \( \lambda_1, \lambda_2, \ldots \) such that \( r(\lambda_1) = v \) and \( s(\lambda_n) = r(\lambda_{n+1}) \). We then show that the sequence of paths defines an element of \( v(\partial \Lambda) \).

(1) Let \( \{ E_{1,i} \}_{i=1}^\infty \) be a listing of \( v \mathcal{F}(\Lambda) \). Choose \( \lambda_1 \in E_{1,1} \).

(2) Let \( \{ E_{2,i} \}_{i=1}^\infty \) be a listing of \( s(\lambda_1) \mathcal{F}(\Lambda) \). Choose \( \lambda_2 \in \text{Ext}(\lambda_1; E_{1,1}) \).

(3) Let \( \{ E_{3,i} \}_{i=1}^\infty \) be a listing of \( s(\lambda_2) \mathcal{F}(\Lambda) \). Choose \( \lambda_3 \in \text{Ext}(\lambda_2; E_{1,1}) \).

(4) Let \( \{ E_{4,i} \}_{i=1}^\infty \) be a listing of \( s(\lambda_3) \mathcal{F}(\Lambda) \). Choose \( \lambda_4 \in \text{Ext}(\lambda_3 \lambda_2 \lambda_3; E_{1,3}) \).

(5) Let \( \{ E_{5,i} \}_{i=1}^\infty \) be a listing of \( s(\lambda_5) \mathcal{F}(\Lambda) \). Choose \( \lambda_5 \in \text{Ext}(\lambda_5 \lambda_3 \lambda_4; E_{1,3}) \).

(6) Let \( \{ E_{6,i} \}_{i=1}^\infty \) be a listing of \( s(\lambda_6) \mathcal{F}(\Lambda) \). Choose \( \lambda_6 \in \text{Ext}(\lambda_6 \lambda_5 \lambda_4; E_{3,1}) \).
Thus by construction of Paterson’s book [7]. Let $G = \mu$ for some □ required.

Let $p \in \mathbb{N}^+$, $p \leq d(x)$, and let $E \in x(p)\mathcal{FE}(\Lambda)$. We will show that there exists $\mu \in E$ such that $x(p, p + d(\mu)) = \mu$. Let $n \in \mathbb{N}$ be the smallest number such that $d(\lambda_1 \cdots \lambda_n) \geq p$. Then $\text{Ext}(x(p, d(\lambda_1 \cdots \lambda_n)); E) \in s(\lambda_n)\mathcal{FE}(\Lambda)$ by Proposition 3.11, so there exists $i \in \mathbb{N}$ such that

$$E_{n+1,i} = \text{Ext}(x(p, d(\lambda_1 \cdots \lambda_n)); E).$$

By construction of $x$, there exists $m \in \mathbb{N}$ such that

$$\lambda_m \in \text{Ext}(\lambda_{n+1} \cdots \lambda_{m-1}; E_{n+1,i})$$

$$= \text{Ext}(\lambda_{n+1} \cdots \lambda_{m-1}; \text{Ext}(x(p, d(\lambda_1 \cdots \lambda_n)); E))$$

$$= \text{Ext}(x(p, d(\lambda_1 \cdots \lambda_{m-1})); E) \text{ by Proposition 3.12}.$$ 

Thus

$$x(p, d(\lambda_1 \cdots \lambda_m)) = x(p, d(\lambda_1 \cdots \lambda_{m-1}))\lambda_m$$

$$= \mu \alpha$$

for some $\mu \in E$ and $\alpha \in \Lambda$, so $x(p, p + d(\mu)) = \mu$. Therefore $x \in v(\partial \Lambda)$, as required.

6. THE GROUPOID OF THE SYSTEM $(X_\Lambda, S_\Lambda, \theta)$

We begin this section by recalling notions from groupoid theory. We follow Paterson’s book [7]. Let $G$ be a topological groupoid. The unit space of $G$ is denoted $G^{(0)}$ and the space of composable pairs is denoted $G^{(2)}$. We use $r$ and $s$ for the range and source maps $r(g) = gg^{-1}$ and $s(g) = g^{-1}g$ (for $g \in G$). A subset $U \subseteq G^{(0)}$ is invariant if $r(s^{-1}(U))$ is contained in $U$. We denote by $G^{op}$ the family of open Hausdorff subsets $A$ of $G$ such that $r|_A$ and $s|_A$ are homeomorphisms. A locally compact groupoid $G$ is called $r$-discrete if $G^{(0)}$ is open in $G$. In fact, a locally compact groupoid $G$ is $r$-discrete and admits a left Haar system if and only if $G^{op}$ is a basis for the topology of $G$, which happens if and only if $r : G \to G^{(0)}$ is a local homeomorphism (see [13] Proposition 2.8). We define

$$G^a := \{ A \in G^{op} : A \text{ is compact} \}.$$ 

An $r$-discrete groupoid $G$ is called ample if $G^a$ forms a basis for the topology of $G$.

We now construct the groupoid of germs of the system $(X_\Lambda, S_\Lambda, \theta)$. Define

$$\mathfrak{X}_\Lambda := \{(F, x) : F \in S_\Lambda, x \in D_F \}$$

and define a relation $\sim$ on $\mathfrak{X}_\Lambda$ by requiring that $(F, x) \sim (G, y)$ if and only if $x = y$ and there exists $P \in E(S_\Lambda)$ such that $x \in D_P$ and $FP = GP$.

Lemma 6.1. The relation $\sim$ is an equivalence relation on $\mathfrak{X}_\Lambda$.

Proof. The reflexivity and symmetry of $\sim$ are obvious. Suppose $(F, x) \sim (G, x)$ and $(G, x) \sim (H, x)$. Then there exist $P_1, P_2 \in E(S_\Lambda)$ such that $x \in D_{P_1}, x \in D_{P_2}, FP_1 = GP_1$ and $GP_2 = HP_2$. Thus $x \in D_{P_1}P_2$ and

$$FP_1P_2 = GP_1P_2 = GP_2P_1 = HP_2P_1 = HP_1P_2,$$

so $(F, x) \sim (H, x)$, and $\sim$ is an equivalence relation. 

□
The equivalence class of \((F, x)\) will be denoted \([F, x]\). It is called the germ of \(F\) at \(x\).

Write \(G_\Lambda\) for the set \(\Xi_\Lambda/\sim\). Then \(G_\Lambda\) becomes a groupoid where the composable pairs are of the form \(
\left([F, \theta_G(x)], [G, x]\right)\), and product and inversion are given by the formulas

\[
[F, \theta_G(x)] \cdot [G, x] = [FG, x] \quad \text{and} \quad [F, x]^{-1} = [F^*, \theta_F(x)].
\]

The unit space \(G_\Lambda^{(0)}\) will be identified with \(X_\Lambda\) via the map \(\left\{\{(r(x), r(x))\}, x\right\} \mapsto x\). Then for \([F, x] \in G_\Lambda\), we have \(r([F, x]) = \theta_G(x)\) and \(s([F, x]) = x\).

**Remark 6.2.** For \(F \in S_\Lambda\) and \(x \in D_F\), let \((\lambda, \mu) \in F\) be the unique element of \(F\) such that \(x(0, d(\mu)) = \mu\). Then \([F, x] = \left\{\{(\lambda, \mu)\}, x\right\}\), so \(G_\Lambda\) can be expressed as

\[
G_\Lambda = \left\{\left\{\{(\lambda, \mu)\}, x\right\} : (\lambda, \mu) \in \Lambda \ast_s \Lambda, x \in D_\mu\right\}.
\]

Using this description of \(G_\Lambda\), for \(\left\{\{(\lambda, \mu)\}, x\right\}, \left\{\{(\xi, \eta)\}, x\right\} \in G_\Lambda\),

\[
\left\{\{(\lambda, \mu)\}, x\right\} \cdot \left\{\{(\xi, \eta)\}, x\right\} \iff \lambda x(d(\mu), d(\mu) \lor d(\eta)) = \xi x(d(\eta), d(\mu) \lor d(\eta)).
\]

Furthermore, if \(\left\{\{(\lambda, \mu)\}, x\right\}, \left\{\{(\xi, \eta)\}, y\right\} \in G_\Lambda^{(2)}\), then setting

\[
\alpha := x(d(\mu), d(\mu) \lor d(\xi)) \text{ and } \beta := \sigma d(\mu)(0, d(\mu) \lor d(\xi)) - d(\xi),
\]

we have \((\alpha, \beta) \in \Lambda_{\min}(\mu, \xi)\) and

\[
\left\{\{(\lambda, \mu)\}, x\right\} \cdot \left\{\{(\xi, \eta)\}, y\right\} = \left\{\{(\lambda\alpha, \eta\beta)\}, y\right\}
\]

For \(F \in S_\Lambda\), define

\[
\Psi(F) := \left\{\left\{F, x\right\} : x \in D_F\right\} \subseteq G_\Lambda.
\]

**Proposition 6.3.** The set \(\{\Psi(F), \Psi(F)^c : F \in S_\Lambda\}\) is a subbasis for a locally compact Hausdorff topology on \(G_\Lambda\) in which each \(\Psi(F)\) is compact.

**Proof.** Since \(G_\Lambda = \bigcup_{(\lambda, \mu) \in \Lambda \ast_s \Lambda} \Psi\left(\{(\lambda, \mu)\}\right)\), the set \(\{\Psi(F), \Psi(F)^c : F \in S_\Lambda\}\) is a subbasis for a topology on \(G_\Lambda\).

For \((\lambda, \mu) \in \Lambda \ast_s \Lambda\), the source map in \(G_\Lambda\) restricted to \(\Psi\left(\{(\lambda, \mu)\}\right)\) is injective with image \(D_\mu\). The topology on \(D_\mu\) is generated by a subbasis comprising elements of the form \(D_\mu^c\) and \(D_\mu \cap D_\mu^c\). The inverse images of these subbasis elements,

\[
\left(s|\Psi\left(\{(\lambda, \mu)\}\right)\right)^{-1}(D_\mu) = \Psi\left(\{(\lambda\alpha, \mu\alpha)\}\right)
\]

and

\[
\left(s|\Psi\left(\{(\lambda, \mu)\}\right)\right)^{-1}(D_\mu \cap D_\mu^c) = \Psi\left(\{(\lambda, \mu)\}\right) \cap \Psi\left(\{(\lambda\alpha, \mu\alpha)\}\right)^c,
\]

form a subbasis for the topology on \(\Psi\left(\{(\lambda, \mu)\}\right)\). Therefore \(s|\Psi\left(\{(\lambda, \mu)\}\right)\) is a homeomorphism, and so Proposition 6.4 and Corollary 5.8 imply that \(\Psi\left(\{(\lambda, \mu)\}\right)\) is Hausdorff and compact. The result follows.

**Remark 6.4.** The family of sets of the form

\[
\Psi\left(\{(\lambda, \mu)\}\right) \cap \Psi\left(\{(\nu_1, \mu_1)\}\right)^c \cap \cdots \cap \Psi\left(\{(\nu_l, \mu_l)\}\right)^c,
\]

where \((\lambda, \mu) \in \Lambda \ast_s \Lambda\) and \(\nu_1, \ldots, \nu_l \in s(\lambda)\Lambda\), is a basis for the topology on \(G_\Lambda\).

**Proposition 6.5.** \(G_\Lambda\) is an \(r\)-discrete topological groupoid.
Proof. To show that composition is continuous, fix a composable pair
\[ \{(\lambda, \mu), [x], [(\xi, \eta)], [y]\} \in G_A^{(2)}. \]

Remark 6.2 says that the composable pair is of the form
\[ \{(\lambda, \mu), [x], [(\xi, \eta)], [y]\} \in G_A^{(2)}, \]
where \(\alpha, \beta \in \Lambda_{\text{min}}(\mu, \xi)\), and has product \(\{(\lambda, \eta)\}, [y]z\]. Let
\[ A := \Psi\{(\lambda \alpha, \eta \beta\}) \cap \bigcap_{i=1}^{l} \Psi\{(\lambda \alpha \tau/v, \eta \beta \tau/v\}\}\]
be a neighborhood of \(\{(\lambda, \eta)\}, [y]z\}. Then for neighborhoods
\[ B := \Psi\{(\lambda \alpha, \eta \beta\}) \cap \bigcap_{i=1}^{l} \Psi\{(\lambda \alpha \tau/v, \eta \beta \tau/v\}\}\]
and
\[ C := \Psi\{(\xi \beta \tau, \eta \beta \tau\}) \cap \bigcap_{i=1}^{l} \Psi\{(\xi \beta \tau/v, \eta \beta \tau/v\}\}\]
of \(\{(\lambda, \mu)\}, [x]z\) and \(\{(\xi, \eta)\}, [y]z\), respectively, we have \(BC \subseteq A\), which gives continuity of composition.

For \(F \in S_A\), \(\Psi(F) = \Psi(F^*)^{-1}\) and \(\Psi(F)^c = (\Psi(F^*)^{-1})\), so inversion is continuous.

Finally, \(G_A\) is \(r\)-discrete since \(G_A^{(0)} = \bigcup_{v \in A^0} \Psi\{(v, v)\}\) is open in \(G_A\). \(\square\)

Remark 6.6. The identification of \(X_A\) with \(G_A^{(0)}\) is a homeomorphism, taking \(D_A\) to \(\Psi\{(\lambda, \lambda)\}\) and \(D_A^*\) to \(G_A^{(0)} \cap \Psi\{(\lambda, \lambda)\}\).

Proposition 6.7. The map \(\Psi\) defined by (6.3) is an injective \(*\)-homomorphism of \(S_A\) into \(G_A^{op}\).

Proof. First we show that for \(F, G \in S_A\), \(\Psi(F)\Psi(G) = \Psi(FG)\). Composition in \(G_A\) gives \(\Psi(F)\Psi(G) \subseteq \Psi(FG)\). For the other containment, let \([FG, x] \in \Psi(FG)\).

Then there exists \((\tau, \omega) \in FG\) such that \(x(0, d(\omega)) = \omega\). From the definition of multiplication in \(S_A\), there exist \((\lambda, \mu) \in F\), \((\xi, \eta) \in G\) and \((\alpha, \beta) \in \Lambda_{\text{min}}(\mu, \xi)\) such that \((\lambda, \eta) = (\tau, \omega)\). But then \(x(0, d(\beta)) = \eta\), so \(x \in D_G\) and
\[ \theta_G(x) = \xi \sigma(d(\eta)) \theta_G(x) = \xi \beta \sigma(d(\eta)) \theta_G(x) = \mu \sigma(d(\beta)) \theta_G(x) \in D_F. \]

Thus, \([FG, x] = [F, \theta_G(x)] \cdot [G, x]\), which gives \(\Psi(FG) \subseteq \Psi(F)\Psi(G)\).

For each \(F \in S_A\), we have \(\Psi(F) \in G_A^{op}\) since \(\theta_F : D_F \to D_{F^*}\) is injective. Therefore \(\Psi\) is a homomorphism between inverse semigroups \(S_A\) and \(G_A^{op}\), and so preserves involution by Proposition 2.1.1(iv)].

Finally we show that \(\Psi\) is injective. Fix \(F, G \in S_A\) with \(F \neq G\). Assume without loss of generality that there exists \((\lambda, \mu) \in F\) such that \((\lambda, \mu) \notin G\). Suppose for contradiction that \(\Psi(F) = \Psi(G)\). Then in particular, regarding the path \(\mu\) as an element of \(X_A\), we have \(\mu \in D_F = D_G\), so there exists \((\xi, \eta) \in G\) such that \(\mu(0, d(\eta)) = \eta\) and
\[ [(\lambda, \mu), [\mu] = [F, [\mu] = [G, [\mu] = [(\xi, [\eta]), [\mu]. \]

By (6.3), we then have
\[ (6.4) \quad \xi \mu(d(\eta), d(\mu)) = \lambda. \]

We claim that \((\lambda, \mu) = (\xi, \eta)\), which contradicts the assumption that \((\lambda, \mu) \notin G\).
Regarding \( \eta \) as an element of \( X_\Lambda \), we have \( \eta \in D_G = D_F \), so there exists \( (\tau, \omega) \in F \) such that \( \eta(0, d(\omega)) = \omega \). But then \( \mu \) and \( \omega \) have a common extension, namely
\[
(6.5) \quad \mu = \eta \mu(d(\eta), d(\mu)) = \omega \eta(d(\omega), d(\eta)) \mu(d(\eta), d(\mu)).
\]
Hence by the definition of \( S_\Lambda \) we must have \( (\lambda, \mu) = (\tau, \omega) \), from which \( (6.3) \) gives \( \mu(d(\eta), d(\mu)) = s(\omega) \). Thus \( (6.4) \) and \( (6.5) \) give \( \lambda = \xi \) and \( \mu = \eta \), contradicting our assumption that \( (\lambda, \mu) \notin G \). \( \square \)

**Proposition 6.8.** \( G_\Lambda \) is an ample groupoid.

**Proof.** For \( G \in S_\Lambda \), \( \Psi(G)^c = \bigcup_{F \in S_\Lambda} \Psi(F) \cap \Psi(G)^c \). Hence \( \{ \Psi(F) \cap \Psi(G)^c : F, G \in S_\Lambda \} \) is a subsbasis for the topology of \( G_\Lambda \). Each \( \Psi(F) \) is compact by Proposition 6.3 and is an element of \( G_\Lambda^{op} \) by Proposition 6.7. Hence each \( \Psi(F) \cap \Psi(G)^c \) is a compact element of \( G_\Lambda^{op} \). The result follows. \( \square \)

**Theorem 6.9.** Let \( (\Lambda, d) \) be a finitely aligned \( k \)-graph. Then the set of characteristic functions \( \{ \Psi((\lambda, \mu)) : \lambda \in \Lambda \} \) is a Toeplitz-Cuntz-Krieger \( \Lambda \)-family in \( C^*(G_\Lambda) \) which gives a canonical isomorphism
\[
TC^*(\Lambda) \cong C^*(G_\Lambda).
\]

**Proof.** Propositions 6.3 and Proposition 2.2.6 imply that the map \( \{(\lambda, \mu)\} \mapsto 1_{\Psi((\lambda, \mu))} \) defines a \(*\)-homomorphism of \( S_\Lambda \) into \( C^*(G_\Lambda) \), and Definition 5.3 (1)–(3) follow from this.

By the universal property of \( TC^*(\Lambda) \) (Remark 3.3), there is a homomorphism \( \pi_T : TC^*(\Lambda) \to C^*(G_\Lambda) \) such that \( \pi_T(s^T_{\lambda}) = 1_{\Psi((\lambda, s(\lambda)))} \).

Next we show that \( \pi_T \) is surjective; we do this in two steps, first showing that \( C_c(G_\Lambda^{(0)}) \) is in the image of \( \pi_T \).

Let \( W = \text{span}\{1_{\Psi((\lambda, s(\lambda)))} : \lambda \in \Lambda \} = \text{span}\{\pi_T(s_\lambda s^*_\lambda) : \lambda \in \Lambda \} \). It is easy to see that \( W \) is a \(*\)-subalgebra of \( C_c(G_\Lambda^{(0)}) \). Furthermore, \( W \) separates points in \( G_\Lambda^{(0)} \) and does not vanish identically at any point of \( G_\Lambda^{(0)} \). Therefore by the Stone-Weierstrass Theorem, \( W \) is uniformly dense in \( C_c(G_\Lambda^{(0)}) \), and it follows that \( C_c(G_\Lambda^{(0)}) \) is in the image of \( \pi_T \).

Now fix \( f \in C_c(G_\Lambda) \). Let \( \{ \Psi((\lambda_i, \mu_i)) \}_{i=1}^n \) be a covering of \( \text{supp} f \) and let \( \{ \phi_i \}_{i=1}^n \) be a partition of unity subordinate to \( \{ \Psi((\lambda_i, \mu_i)) \}_{i=1}^n \); that is, \( \phi_i : G_\Lambda \to [0, 1] \), \( \text{supp} \phi_i \subseteq \Psi((\lambda_i, \mu_i)) \) and \( \sum_{i=1}^n \phi_i([F, x]) = 1 \) for all \( [F, x] \in \bigcup_{i=1}^n \Psi((\lambda_i, \mu_i)) \). We then have \( f = \sum_{i=1}^n \phi_i f \) (where the product is pointwise) and \( \text{supp}(\phi_i f) \subseteq \Psi((\lambda_i, \mu_i)) \); fixing \( i \in \{1, \ldots, n\} \), we will show that \( \phi_i f \) is in the image of \( \pi_T \). Define \( g \in C_c(G_\Lambda^{(0)}) \) by
\[
g(x) := \begin{cases} 
(\phi_i f)([\{(\lambda_i, \mu_i)\}, \mu_i x]) & \text{if } r(x) = s(\mu_i) \\
0 & \text{otherwise}.
\end{cases}
\]

Then there exists \( a \in TC^*(\Lambda) \) such that \( \pi_T(a) = g \). Furthermore,
\[
\phi_i f = 1_{\Psi((\lambda_i, s(\lambda_i)))} \pi_T(a) 1_{\Psi((s(\mu_i), \mu_i))} = \pi_T(s_{\lambda_i}) \pi_T(a) \pi_T(s^*_{\mu_i}),
\]
as required. Thus \( C_c(G_\Lambda) \) is in the image of \( \pi_T \). Since \( \pi_T \) is a \( C^* \)-homomorphism, and hence closed, it follows that \( \pi_T \) is surjective.
To see that \( \pi_T \) is injective, it suffices, by [14] Theorem 8.1, to show that for all \( v \in \Lambda^0 \) and all finite \( E \subseteq \langle v \Lambda \rangle \setminus \{v\} \),
\[
\prod_{\lambda \in E} (1_{\Psi((v,v))} - 1_{\Psi((\lambda, \lambda))} 1^*_{\Psi((\lambda, \lambda))}) > 0,
\]
where the product is convolution. For \( v \in \Lambda^0 \) and finite \( E \subseteq \langle v \Lambda \rangle \setminus \{v\} \), regarding \( v \) as an element of \( X_\Lambda \), we have \( [(v,v)], (v,v) \not\in \Psi((\lambda, \lambda)) \) for all \( \lambda \in E \). Thus the product in (6.6) is bounded below by 1, which is nonzero in \( C^*(G_\Lambda) \) since the left Haar system on \( G_\Lambda \) is given by the counting measures.

**Proposition 6.10.** \( \partial \Lambda \) is a nonempty closed invariant subset of \( G_\Lambda^{(0)} \).

**Proof.** Since \( G_\Lambda^{(0)} \) is homeomorphic to \( X_\Lambda \) (see Remark 6.10) and since \( r([F,x]) = \theta_F(x) \), it follows that Remark 5.14 implies that \( \partial \Lambda \) is a closed invariant subset of \( G_\Lambda^{(0)} \). It is nonempty by Lemma 5.15.

Proposition 6.10 implies that \( G_\Lambda|_{\partial \Lambda} \) is a locally compact, Hausdorff and ample groupoid.

For \( F \in S_\Lambda \), define \( \Psi_* : S_\Lambda \rightarrow (G_\Lambda|_{\partial \Lambda})^a \) by
\[
(6.7) \quad \Psi_*(F) := \Psi(F) \cap G_\Lambda|_{\partial \Lambda} = \{ [F,x] : x \in D_F \cap \partial \Lambda \}
\]

**Remarks 6.11.** (1) Unlike \( \Psi \), \( \Psi_* \) is not injective in general. For example, for the 1-graph given by
\[
\bullet e_1 \bullet e_2 \bullet \ldots
\]
we have for \( \Psi_*([v,v]) = \Psi_*([e_1, e_1]) = \Psi_*([e_1 \cdots e_n, e_1 \cdots e_n]) \). In general, given a finitely aligned \( k \)-graph \( \Lambda \), for all \( v \in \Lambda^0 \) and \( E \in v \mathcal{F}(\Lambda) \),
\[
(6.8) \quad \Psi_*([v,v]) = \bigcup_{\lambda \in E} \Psi_*([\lambda, \lambda]).
\]

(2) The family of sets of the form
\[
\Psi_*([\lambda, \mu]) \cap \Psi_*([\lambda v_1, \mu v_1]) \cap \cdots \cap \Psi_*([\lambda v_n, \mu v_1]) \cap \cdots
\]
where \( (\lambda, \mu) \in \Lambda \), \( \lambda v_1, \ldots, v_n \in s(\lambda) \Lambda \), is a basis for the topology on \( G_\Lambda|_{\partial \Lambda} \).

(3) In [2], the authors consider the \( C^* \)-algebras arising from \( k \)-graphs which are row-finite and have no sources (Recall that this means \( 0 < |v \Lambda^n| < \infty \) for all \( v \in \Lambda^0 \) and \( n \in \mathbb{N}_k \)) by constructing and analyzing a groupoid we call \( G_\Lambda^{KP} \) (see [2] Definition 2.7). In this setting, the boundary paths \( \partial \Lambda \) are precisely the paths \( x \in X_\Lambda \) such that \( d(x) = (\infty, \ldots, \infty) \) (that is, \( x \) is infinite in all \( k \) directions), and using the description of \( G_\Lambda \) given in Remark 6.2 and Remark 5.14 (2), one can see that the reduction \( G_\Lambda|_{\partial \Lambda} \) is isomorphic to a topological groupoid to \( G_\Lambda^{KP} \).

For \( z \in \mathbb{T}^k \) and \( m \in \mathbb{N}_k \), define \( z^m := z_1^{m_1} z_2^{m_2} \cdots z_k^{m_k} \in \mathbb{T} \). There is a strongly continuous gauge action \( \gamma \) of \( \mathbb{T}^k \) on \( C^*(\Lambda) \) defined by \( \gamma_z(s_\lambda s_\mu^*) = z^{d(\lambda) - d(\mu)} s_\lambda s_\mu^* \) for \( z \in \mathbb{T}^k \). The gauge-invariant uniqueness theorem for finitely aligned \( k \)-graphs [12] Corollary 4.3] says that if \( A \) is a \( C^* \)-algebra generated by a Cuntz-Krieger \( \Lambda \)-family, and if \( A \) carries a strongly continuous action of \( \mathbb{T}^k \) which is equivariant to the gauge action on \( C^*(\Lambda) \), then \( A \) is isomorphic to \( C^*(\Lambda) \). The next lemma shows that \( C^*(G_\Lambda|_{\partial \Lambda}) \) admits such an action.
Lemma 6.12. There is a strongly continuous action $\beta : \mathbb{T}^k \to \operatorname{Aut}(C^*(G_\Lambda|_{\partial \Lambda}))$ such that $\beta_\lambda(1_{\Psi_*(\{(\lambda,s(\lambda))\})}) = z^{d(\Lambda)}1_{\Psi_*(\{(\lambda,s(\lambda))\})}$ for all $\lambda \in \Lambda$.

Proof. Define a map $c : G_\Lambda|_{\partial \Lambda} \to \mathbb{Z}^k$ by $\{(\lambda,\mu), x\} \mapsto d(\lambda) - d(\mu)$. To see that $c$ is well-defined, suppose $\{(\lambda,\mu), x\} = \{(\xi,\eta), x\}$. Then by (6.11) we have $\lambda x(d(\mu), d(\mu) \vee d(\eta)) = \xi x(d(\eta), d(\mu) \vee d(\eta))$, so

$$
d(\lambda) - d(\mu) = d(\lambda) + (d(\mu) \vee d(\eta)) - d(\mu) - (d(\mu) \vee d(\eta)) = d(\xi) + (d(\mu) \vee d(\xi)) - d(\eta) - (d(\mu) \vee d(\eta)) = d(\xi) - d(\eta).
$$

To see that $c$ is a 1-cocycle, take $\{(\lambda,\mu), \xi \sigma^{d(\eta)}(x)\}, \{(\xi,\eta), x\} \in G_\Lambda|_{\partial \Lambda}$. Let

$$
\alpha := x(d(\eta), d(\eta) + (d(\mu) \vee d(\xi)) - d(\mu)) = (\sigma^{d(\eta)}(x))(0, (d(\mu) \vee d(\xi)) - d(\mu))
$$

and

$$
\beta := x(d(\eta), d(\eta) + (d(\mu) \vee d(\xi)) - d(\xi)) = (\sigma^{d(\eta)}(x))(0, (d(\mu) \vee d(\xi)) - d(\xi)).
$$

Then $(\alpha, \beta) \in \Lambda^{\min}(\mu, \xi)$, and we have

$$
c(\{(\lambda,\mu), \xi \sigma^{d(\eta)}(x)\}) + c(\{(\xi,\eta), x\}) = c(\{(\lambda,\mu), \xi \sigma^{d(\eta)}(x)\}) = c(\{(\xi,\eta), x\}).
$$

Hence $c$ is a $\mathbb{Z}^k$-valued 1-cocycle. To see that $c$ is continuous, simply observe that for $n \in \mathbb{Z}^k$

$$
c^{-1}(\{n\}) = \bigcup_{\{(\lambda,\mu)\in S_\Lambda \atop d(\lambda)-d(\mu)=n}} \Psi_*(\{(\lambda,\mu)\})
$$

is open in $G_\Lambda|_{\partial \Lambda}$. By [13] II.5.1, there is a strongly continuous action $\beta : \mathbb{T}^k \to \operatorname{Aut}(C^*(G_\Lambda|_{\partial \Lambda}))$ such that

$$
\beta_\lambda(1_{\Psi_*(\{(\lambda,\mu), x\})}) = z^{d(\lambda)-d(\mu)}1_{\Psi_*(\{(\lambda,\mu), x\})}
$$

for all $\{(\lambda,\mu), x\} \in G_\Lambda|_{\partial \Lambda}$. The result follows. \hfill \Box

Theorem 6.13. Let $(\Lambda, d)$ be a finitely aligned $k$-graph. Then the set of characteristic functions $\{1_{\Psi_*(\{(\lambda,s(\lambda))\})} : \lambda \in \Lambda\}$ is a Cuntz-Krieger $\Lambda$-family in $C^*(G_\Lambda|_{\partial \Lambda})$ that gives a canonical isomorphism

$$
C^*(\Lambda) \cong C^*(G_\Lambda|_{\partial \Lambda}).
$$

Proof. Proposition [13] and [19] Proposition 2.2.6 imply that $\{\lambda,s(\lambda)\} \mapsto 1_{\Psi_*(\{(\lambda,s(\lambda))\})}$ defines a $\ast$-homomorphism of $S_\Lambda$ into $C^*(G_\Lambda|_{\partial \Lambda})$, and (1)–(3) of Definition [13] follow from this. To show Definition [13] (CK) holds, we fix $v \in \Lambda^0$, $E \in v\mathcal{FE}(\Lambda)$ and $[F,x] \in G_\Lambda|_{\partial \Lambda}$, and evaluate

$$
(6.8) \quad \left( \prod_{\lambda \in E} (1_{\Psi_*(\{(v,v)\})} - 1_{\Psi_*(\{(\lambda,v)\})}) \right) [F,x]
$$

and evaluate
where the product is convolution.

For $\lambda \in E$, $\Psi_*(\{(\lambda, \lambda)\})$ is a subset of $\Psi_*(\{(v, v)\})$, so (6.8) can only be nonzero if $[F, x] = \{(v, v), x\}$, in which case $x \in v(\partial \Lambda)$. Since (6.8) is a product of characteristic functions of subsets of $(G_{\Lambda|\partial \Lambda})^{(0)}$, a straightforward calculation shows that (6.3) can be written as the pointwise product

$$
\prod_{\lambda \in E} \left( (1 - \psi_*(\{(v, v)\}))\right). 
$$

Since $E \in vF \mathcal{E}(\Lambda)$, there exists $\mu \in E$ such that $x(0, d(\mu)) = \mu$. We then have $[\{(v, v), x\}] = [\{(\mu, \mu), x\}]$, so the term $(1 - \psi_*(\{(v, v)\}))\{(\mu, \mu), x\}$ from (6.4) is zero. Therefore (6.8) is zero, and $\{\psi_*(\{(\lambda, \lambda)\}) : \lambda \in \Lambda\}$ is a Cuntz-Krieger $\Lambda$-family. By the universal property of $C^*(\Lambda)$ (Remark 3.9), there is a unique homomorphism $\pi : C^*(\Lambda) \to \mathcal{A}^*(G_{\Lambda|\partial \Lambda})$ such that $\pi(s_{\lambda}) = \psi_*(\{(\lambda, \lambda)\})$ for all $\lambda \in \Lambda$. Furthermore, since each $v(\partial \Lambda)$ is nonempty by Proposition 6.15, and since the left Haar system on $G_{\Lambda|\partial \Lambda}$ is given by counting measures, it follows that $\pi(s_{\nu}) = 1_{\psi_*(\{(v, v)\})}$ is nonzero for every $\nu \in \Lambda^0$.

Lemma 6.12 gives a strongly continuous action $\beta : T^k \to \text{Aut}(C^*(G_{\Lambda|\partial \Lambda}))$ such that $\beta_z \circ \gamma_v = \pi \circ \gamma_v$ for all $z \in T^k$. Therefore [12, Theorem 4.2] implies that $\pi$ is injective. We can see that $\pi$ is surjective in the same way that we saw that $\pi_T$ was surjective in the proof of Theorem 6.4. The result follows. 

7. The Cuntz-Krieger Uniqueness Theorem

In this section we present a Cuntz-Krieger uniqueness theorem for finitely aligned $k$-graphs. The theorem generalizes [2, Theorem 4.6] but differs from the Cuntz-Krieger uniqueness theorem [12, Theorem 4.5] obtained by direct methods. These differences will be discussed in Remarks 6.3.

**Theorem 7.1.** Let $(\Lambda, d)$ be a finitely aligned $k$-graph, and suppose that

$$[\text{for each } v \in \Lambda^0 \text{ there exists } x \in v(\partial \Lambda) \text{ such that} ]$$

$$\sigma^m(x) = \sigma^n(x) \implies m = n \text{ for all } m, n \in \mathbb{N}^k, m, n \leq d(x).$$

Suppose that $\pi : C^*(\Lambda) \to \mathcal{B}$ is a homomorphism such that $\pi(s_{\nu})$ is nonzero for all $\nu \in \Lambda^0$. Then $\pi$ is injective.

The proof of Theorem 6.11 relies on the following proposition.

**Proposition 7.2.** Let $(\Lambda, d)$ be a finitely aligned $k$-graph. Then $\Lambda$ satisfies condition (A) if and only if $G_{\Lambda|\partial \Lambda}$ is essentially free.

**Proof.** First, observe that $x \in (G_{\Lambda|\partial \Lambda})^{(0)}$ has trivial isotropy if and only if

$$[\text{for all } m, n \in \mathbb{N}^k \text{ with } m, n \leq d(x), \sigma^m x = \sigma^n x \implies m = n. ]$$

If $G_{\Lambda|\partial \Lambda}$ is essentially free, then since each $D_v$ is nonempty by Lemma 5.15, it follows that each $D_v = v(\partial \Lambda)$ contains a boundary path $x$ satisfying (A). Therefore $\Lambda$ satisfies Condition (A).

Conversely, suppose $\Lambda$ satisfies Condition (A), and let $x \in D_\Lambda \cap D_{\nu_1} \cap \cdots \cap D_{\nu_l} \subseteq (G_{\Lambda|\partial \Lambda})^{(0)}$.

Since $x$ is a boundary path, for any finite exhaustive set $E \subseteq \nu(\Lambda)\Lambda$ there exists $\xi \in E$ such that $x(d(\lambda), d(\lambda) + d(\xi)) = \xi$. Hence $\{\nu_1, \ldots, \nu_l\}$ cannot be exhaustive, so there exists $\eta \in s(\Lambda)\Lambda$ such that $\Lambda^{\min}(\eta, \nu_j) = 0$ for all $j \in \{1, \ldots, l\}$. 

Furthermore, since $\Lambda$ satisfies Condition $[A]$, there exists $y \in s(\eta)(\partial \Lambda)$ satisfying

$$\sigma^m y = \sigma^n y \text{ implies } m = n \text{ for all } m, n \leq d(y).$$

Therefore $\lambda \eta y$ is an element of $D_\Lambda \cap D^c_{\lambda \eta} \cap \cdots \cap D^c_{\lambda \eta}$, satisfying

$$\sigma^m (\lambda \eta y) = \sigma^n (\lambda \eta y) \text{ implies } m = n \text{ for all } m, n \leq d(\lambda \eta y).$$

Since $D_\Lambda \cap D^c_{\lambda \eta} \cap \cdots \cap D^c_{\lambda \eta}$ was an arbitrary basis set containing $x$, it follows that $G_\Lambda|_{\partial \Lambda}$ is essentially free.

\[\square\]

Proof of Theorem 7.1 Let $\pi_* : C^*(\Lambda) \to C^*(G_\Lambda|_{\partial \Lambda})$ be the canonical isomorphism of Theorem 6.13. Then $\pi$ is injective if and only if $\pi \circ \pi_*^{-1} : C^*(G_\Lambda|_{\partial \Lambda}) \to B$ is injective. By Proposition 7.2, $\pi \circ \pi_*^{-1}$ is injective on $C_0((G_\Lambda|_{\partial \Lambda})^{(0)})$. If the kernel of the restriction of $\pi \circ \pi_*^{-1}$ to $C_0((G_\Lambda|_{\partial \Lambda})^{(0)})$ is nonzero, it must contain a characteristic function $1_{\partial \lambda} = 1_{\pi_*^{-1}((\lambda, \lambda))}$ for some $\lambda \in \Lambda$. It follows that $\pi(s_\lambda s^*_\lambda) = 0$, which implies $\pi(s_\lambda) = 0$, a contradiction.

\[\square\]

Remarks 7.3. There are a number of issues to point out here. Firstly, when the $k$-graph is row-finite and has no sources, our condition $[A]$ is equivalent to the aperiodicity condition of [2] Definition 4.3, and Theorem 6.13 gives [2] Theorem 4.6.

In [12], a Cuntz-Krieger uniqueness theorem is given for finitely aligned $k$-graphs. [5] Theorem 4.5. To compare [12] Theorem 4.5 with Theorem 7.1, define $\Lambda^{\leq \infty}$ to the subset of $\Lambda$ consisting of all $x \in X_\Lambda$ for which there exists $n_x \in \mathbb{N}^k$, $n_x \leq d(x)$, satisfying

$$n \in \mathbb{N}^k, n_x \leq n \leq d(x), \text{ and } n_i = d(x)_i \text{ imply that } x(n)\Lambda^{\leq i} = \emptyset,$$

and for $v \in \Lambda^0$, define $v\Lambda^{\leq \infty} := \{x \in \Lambda^{\leq \infty} : r(x) = v\}$. Then [12] Theorem 4.5 differs from Theorem 7.1 in that our condition $[A]$ is replaced with the condition

for each $v \in \Lambda^0$ there exists $x \in v\Lambda^{\leq \infty}$ such that

$$\lambda, \mu \in \Lambda v \text{ and } \lambda \neq \mu \text{ implies } \lambda x \neq \mu x.$$

The two conditions $[A]$ and $[B]$ do not seem equivalent a priori; when the $k$-graph $\Lambda$ is row-finite and has no sources, the set $\Lambda^{\leq \infty}$ is precisely $\partial \Lambda$, and condition $[A]$ implies condition $[B]$ (see [11] Remark 4.4), making $[B]$ seem the weaker condition. However, even in the row-finite and no sources setting, it remains unclear whether $[B]$ is strictly weaker than $[A]$.

When the $k$-graph $\Lambda$ is finitely aligned, it is not clear that either condition implies the other. The set $\Lambda^{\leq \infty}$ used in [12] is in general a proper subset of $\partial \Lambda$. If $\Lambda^{\leq \infty}$ is replaced with the set $\partial \Lambda$ in $[B]$ to give

for each $v \in \Lambda^0$ there exists $x \in v(\partial \Lambda)$ such that

$$\lambda, \mu \in \Lambda v \text{ and } \lambda \neq \mu \text{ implies } \lambda x \neq \mu x,$$

then the resulting property of the groupoid $G_\Lambda|_{\partial \Lambda}$ is not the essential freeness implied by $[A]$, rather the curious property:

for all $v \in \Lambda^0$, there exists $x \in v(\partial \Lambda)$ such that

$$r([\{\lambda, v\}, x]) = r([\{\mu, v\}, x]) \text{ implies } [\{\lambda, v\}, x] = [\{\mu, v\}, x].$$
Although this property of $\mathcal{G}_\Lambda|_{\partial\Lambda}$ is quite different to essential freeness, it may still yield similar consequences to essential freeness. In particular, the conclusion [41, Lemma 3.5] may hold, the key result in the proof of [2, Theorem 4.6] and Theorem [74]. If this were the case, and if condition [44] was not equivalent to [55], then we would obtain a generalization of [12, Theorem 4.5] using groupoid methods (since [44] asks for an element of $v(\partial\Lambda)$ for each $v \in \Lambda^0$, whereas [55] asks for an element of the smaller set $v\Lambda^{\leq\infty}$).

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