Magnetohydrodynamic turbulence in a strongly magnetised plasma

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ABSTRACT

I present a review of incompressible magnetohydrodynamic (MHD) turbulence in a strongly magnetised plasma. The approach is phenomenological even where a more rigorous theory is available, so that a reader armed with paper, pencil and some determination may be able to work through most of the physics. The focus is on the inertial–range spectra for very large (fluid and magnetic) Reynolds numbers. These theories of the inertial–range are built on two important facts: (i) Kraichnan’s insight that the turbulent cascades are a result of nonlinear interactions between oppositely directed wavepackets of Elsasser fields; (ii) these oppositely directed wavepackets do not exchange energy, but contribute only to changing each other’s spatial structures. I begin with a description and critique of the Iroshnikov–Kraichnan theory, and explore the fundamental departures necessitated by the anisotropic nature of the turbulence. Derivations of the inertial–range spectra of four regimes of MHD turbulence — the balanced weak, balanced strong, imbalanced weak and the imbalanced strong cascades — are then presented. The need for studying the spectra of imbalanced turbulence when the waves on the outer scale have a short correlation time is briefly discussed.

Subject headings: MHD — turbulence

1. Introduction

Astrophysical systems offer numerous settings for turbulence in magnetised plasmas, such as the Sun, the solar wind, accretion discs, the ionised interstellar medium, molecular clouds and clusters of galaxies (Armstrong, Rickett & Spangler 1995, Horbury 1999,
Magnetohydrodynamics (MHD) offers the simplest description of the dynamics of a magnetised plasma. MHD is a single fluid description of a conducting magnetised plasma. Sources for the magnetic field, $B(x,t)$, could be external or due to currents in the plasma. Lorentz forces act on the fluid and affect the evolution of the velocity field, $v(x,t)$. If fluid motion is subsonic, incompressibility is a good approximation and the system can be described by the two vector fields, $B$ and $v$.

Stirring a magnetised plasma on spatial scale $L$ creates velocity and magnetic field perturbations whose nonlinear evolution gives rise to a turbulent cascade to smaller spatial scales and is ultimately dissipated by viscosity and resistivity into heat. Naturally occurring processes can be somewhat different from stirring; for instance, the energy sources could be distributed in some way in space. In the case of the solar wind, the ultimate source of energy for MHD waves is the Sun, whereas in the interstellar medium the energy sources could be supernova remnants located in the spiral arms of our Galaxy. In these cases, MHD waves are generated in some regions of space, from which they propagate away, interact and make the plasma turbulent. Modelling this inhomogeneous situation is not easy. So, to simplify the physics of a difficult problem, we make the standard assumption that we are dealing with a statistically homogeneous and stationary situation. Thus stirring is included as a source term in the Navier–Stokes equation for the velocity field, with appropriate statistical properties. Moreover, stirring is indispensable in numerical simulations, which complement analytical studies.

This review aims to be a self-contained description of MHD turbulence. However, there could be a couple of places where the reader may have to refer to other sources, and I hope I have made this clear in the text. This review could also seem somewhat one-sided in that much of it is based on work by me and my collaborators. There are, of course, other points of view and the interested reader should consult the literature; these could be Iroshnikov (1963), Kraichnan (1965), Shebalin, Matthaeus & Montgomery (1983), Goldreich & Sridhar (1995, 1997), Ng & Bhattacharjee (1996), Cho & Vishniac (2000), Biskamp & Müller (2000), Maron & Goldreich (2001), Cho, Lazarian & Vishniac (2002), Galtier et al. (2000, 2002), Lithwick & Goldreich (2003), Müller, Biskamp & Grappin (2003), Galtier, Pouquet & Mangeney (2005), Boldyrev (2005), Müller & Grappin (2005), Lithwick, Goldreich & Sridhar (2007), Beresnyak & Lazarian (2008), Chandran (2008), Perez & Boldyrev (2009) and Podesta & Bhattacharjee (2009).

1A topic on which I have not worked is imbalanced weak MHD turbulence, and the account in § 6 is based on my reading of Lithwick & Goldreich (2003).
In this review the focus is on the inertial–range spectra of the magnetic and velocity fields in a strongly magnetized incompressible fluid, in the limit of very large (fluid and magnetic) Reynolds numbers. § 2 begins with the equations of incompressible MHD which govern the dynamics of the magnetic field, $B(x, t)$, and the velocity field, $v(x, t)$. Both stirring and dissipation are included in the equations but will be dropped later. This is because in many cases they do not directly influence the inertial–range spectra for very large Reynolds numbers\(^2\). Alfvén and Slow waves emerge as linear perturbations of a uniformly magnetised plasma, and their polarisations and dispersion relations are discussed. Nonlinear waves are introduced and their role in enabling a broad physical picture of MHD turbulence is sketched. § 3 discusses the Iroshnikov–Kraichnan theory of MHD turbulence and the problems affecting this theory. Attempts at remedying the defects lead to new physics and the next few sections are devoted to the theories that have emerged over the last fifteen years. However, I have dispensed with the chronological in favour of a more logical presentation: thus § 4 and 5 are on balanced weak and strong cascades, respectively, followed by imbalanced weak and cascades in § 6 and 7, respectively. In § 8 I briefly discuss the need for studying the spectra of imbalanced turbulence when the waves on the outer scale have a short correlation time. § 9 offers comments on matters that have not been discussed in the text.

2. The physical system and MHD turbulence

2.1. Incompressible MHD

The equations describing incompressible MHD are given by,

$$\nabla \cdot B = 0, \quad \nabla \cdot v = 0$$  \hspace{1cm} (1)

$$\partial_t B - \nabla \times (v \times B) = \eta \nabla^2 B$$  \hspace{1cm} (2)

$$\partial_t v + (v \cdot \nabla) v = - \nabla p + \frac{(B \cdot \nabla) B}{4\pi \rho} + \nu \nabla^2 v + f$$  \hspace{1cm} (3)

where $\rho$ is the uniform and constant mass density of the fluid, $\eta$ is the resistivity, and $\nu$ is the kinematic viscosity. $f(x, t)$ is the rate of velocity stirring, assumed incompressible, $\nabla \cdot f = 0$. In many cases of interest to problems in turbulence, $f$ is stochastic, and its

\(^2\)Dissipation does play an important role in the case of imbalanced weak MHD turbulence and this is discussed in § 6. Stirring can affect the spectra in the imbalanced strong cascade and this is discussed in § 8.
statistics is assumed to be given. \( p \) is the ratio of the total pressure to the mass density, which is determined by requiring \( \nabla \cdot \mathbf{v} = 0 \). Thus we need to solve a Poisson equation,

\[
\nabla^2 p = \nabla \cdot \left[ \frac{(\mathbf{B} \cdot \nabla) \mathbf{B}}{4\pi \rho} - (\mathbf{v} \cdot \nabla) \mathbf{v} \right]
\]

with appropriate boundary conditions to determine \( p \). Note that, in this incompressible limit, the thermodynamic origins of \( p \) are not so apparent. Equations (1)–(4), together with initial and boundary conditions, completely determine the evolution of the vector fields, \( \mathbf{B}(x,t) \) and \( \mathbf{v}(x,t) \). Equivalently, the system can also be described by the Elsasser fields, \( \mathbf{W}^\pm \), defined by

\[
\mathbf{W}^+ = \mathbf{v} - \frac{\mathbf{B}}{\sqrt{4\pi \rho}} , \quad \mathbf{W}^- = \mathbf{v} + \frac{\mathbf{B}}{\sqrt{4\pi \rho}}
\]

When \( \rho \) is uniform, we have \( \nabla \cdot \mathbf{W}^\pm = 0 \). It turns out to be very useful to describe incompressible MHD turbulence in terms of the Elsasser fields, instead of the magnetic and velocity fields.

Below we ignore explicit consideration of stirring and dissipative terms, because our focus is on the description of the inertial–range at high Reynolds numbers.

### 2.2. Transverse waves

Uniformly magnetised static solution:

\[
\mathbf{v}_0 = \mathbf{0} , \quad \mathbf{B}_0 = B_0 \hat{z}
\]

In the uniformly magnetised state the Elsasser fields, \( \mathbf{W}^\pm_0 = \mp V_A \hat{z} \), where \( V_A = B_0 / \sqrt{4\pi \rho} \) is the Alfvén speed. When this state is perturbed, we can write,

\[
\mathbf{W}^+ = -V_A \hat{z} + \mathbf{w}^+ , \quad \mathbf{W}^- = V_A \hat{z} + \mathbf{w}^-.
\]

It must be noted that the perturbations are not necessarily small. The perturbations obey the equations

\[
(\partial_t + V_A \partial_z) \mathbf{w}^+ + (\mathbf{w}^- \cdot \nabla) \mathbf{w}^+ = -\nabla p
\]
\[
(\partial_t - V_A \partial_z) w^- + (w^+ \cdot \nabla) w^- = -\nabla p
\]  

(8)

where \( p \) is determined by requiring that \( \nabla \cdot w^\pm = 0 \). The linearised equations are

\[
(\partial_t + V_A \partial_z) w^+ = -\nabla p
\]

\[
(\partial_t - V_A \partial_z) w^- = -\nabla p
\]  

(9)

If we consider solutions that vanish at infinity, \( \nabla^2 p = 0 \) implies that \( p = \text{constant} \) for the linearised problem, and we have linear waves with

\[
w^+ = F^+(x, y, z - V_A t)
\]

\[
w^- = F^-(x, y, z + V_A t)
\]  

(10)

where \( F^\pm \) are arbitrary vector functions with \( \nabla \cdot F^\pm = 0 \). It is clear that \( w^+ \) is a wave that translates in the positive \( z \)-direction with speed \( V_A \), and \( w^- \) is a wave that translates in the negative \( z \)-direction with speed \( V_A \); wave propagation is along the magnetic field lines. We can restate this in \( k \)-space by assigning frequencies, \( \omega^\pm(k) \) to the \( w^\pm \) waves with dispersion relations,

\[
\omega^+(k) = V_A k_z, \quad \omega^-(k) = -V_A k_z
\]  

(11)

Being transverse waves, we must have \( k \cdot \tilde{F}^\pm(k) = 0 \), where \( \tilde{F}^\pm \) is the Fourier transform of \( F^\pm \). In other words, for any given \( k \), the waves have two independent degrees of polarisation, constrained to lie in the plane perpendicular to \( k \). The component perpendicular to \( \hat{z} \) is called the \textit{Alfvén wave}, and the other orthogonal component is called the \textit{Slow} (magnetosonic) \textit{wave}.

\[\text{2.3. Nonlinear solutions and the general nature of MHD turbulence}\]

It is remarkable fact that equations (8) admit exact, nonlinear solutions: it may be verified that, when either \( w^+ \) or \( w^- \) is equal to 0 at the initial time, the nonlinear terms, \( (w^- \cdot \nabla) w^+ \) and \( (w^+ \cdot \nabla) w^- \), vanish for all time. Therefore there are exact, nonlinear solutions of the form,
\[ w^\pm = F^\pm(x, y, z \mp V_A t), \quad w^\mp = 0 \]  

In other words, wavepackets that travel in one direction (i.e. either in the positive or negative \(z\)-direction) do not interact nonlinearly among themselves. Nonlinear interactions occur only between oppositely directed wavepackets. From this fact, Kraichnan (1965) came to an important conclusion:

(i) *Incompressible MHD turbulence in a strongly magnetised plasma can be described as being the result of (nonlinear) interactions between oppositely directed wavepackets.*

It is worth noting another important result that we can prove by manipulating equations (8), namely

\[ \frac{d}{dt} \int \frac{|w^\pm|^2}{2} d^3x = 0 \]  

This result is true even in the presence of nonlinearity, and is equivalent to the conservation of total energy and cross-helicity. Our second important conclusion now follows from equation (13):

(ii) *Wavepacket collisions conserve \(\pm\)ve energies separately, and collisions can only redistribute the respective energies in \(k\)-space. This is true regardless of how imbalanced the situation is. For instance, if stirring puts much more energy into \(+\)ve waves than \(-\)ve waves, the \(+\)ve waves cannot transfer any part of their energy to \(-\)ve waves. They can only scatter the \(-\)ve waves and redistribute them (and vice-versa) in \(k\)-space.*

### 3. Iroshnikov–Kraichnan (IK) Theory

It is very useful to begin with an account of a theory of incompressible MHD turbulence in a strongly magnetised plasma due to Iroshnikov (1963) and Kraichnan (1965), even if current views hold it to be incorrect. The physical picture is more transparent in the latter work and our description below follows this. Assume statistically steady, isotropic excitation of \(\pm\)ve waves, both with root–mean–squared (rms) amplitudes \(w_L \ll V_A\), correlated on spatial scale \(L\), which can be referred to as the *stirring scale* or *outer scale*. Nonlinear interactions between oppositely directed wavepackets create \(\pm\)ve wavepackets on smaller spatial scales. So let us consider the nature of collisions between wavepackets of scale \(\lambda \ll L\), where \(\lambda\) is still much larger than the dissipation scale. We will assume that the most effective collisions occur between wavepackets of similar sizes, an assumption that is also referred to
as locality of interactions. From equations (8) it follows that, in one collision between a +ve packet and a −ve packet, the rms amplitude of either packet is perturbed by an amount,

$$\delta w_\lambda \sim \frac{w_\lambda^2}{V_A}$$

(14)

We imagine that a +ve wavepacket of size $\lambda$ goes on to suffer a number of collisions with −ve wavepackets of similar scale, and vice versa. Successive perturbations add with random phases, so that in $n$ collisions the rms amplitude of the perturbation will increase to $n^{1/2}(w_\lambda^2/V_A)$. Therefore, the number of collisions for perturbations to build up to order unity is

$$N_\lambda \sim \left(\frac{V_A}{w_\lambda}\right)^2$$

(15)

The cascade time is the time taken for collisions to build up to order unity. Since each collision takes time $\sim \lambda/V_A$, the cascade time is,

$$t_\lambda \sim N_\lambda \frac{\lambda}{V_A}$$

(16)

We now use Kolmogorov’s hypothesis that the flux of energy cascading through the inertial–range is independent of the particular spatial scale belonging to the inertial–range. i.e. we assume that the energy flux,

$$\varepsilon \sim \frac{w_\lambda^2}{t_\lambda}$$

(17)

is independent of $\lambda$. This implies that in the inertial–range,

$$w_\lambda \sim w_L \left(\frac{\lambda}{L}\right)^{1/4}$$

(18)

Thus we have determined the $\lambda$–dependence of the rms amplitude in the inertial–range. The same result can also be expressed in $k$–space in terms of the (isotropic) energy spectrum, $E(k)k^3 \sim w_\lambda^2$, where $k \sim 1/\lambda$:

$$E(k) \sim \frac{w_L^2}{L^{1/2}k^{7/2}}$$

(19)
However, we are not quite finished with our task. In our estimates, we assumed that $N_\lambda \gg 1$ and we need to verify if this is true. Using equations (15) and (18), we can estimate that $N_\lambda \propto \lambda^{-1/2}$. Therefore $N_\lambda$ increases as $\lambda$ decreases; in other words, $N_\lambda$ increases as we go deeper into the inertial–range. So, it looks as if our phenomenological theory of the IK cascade is self–consistent, and we might be tempted to conclude that the inertial–range spectrum of MHD turbulence is given by equation (19). However, the physics is not so simple.

3.1. Problems with the IK theory

When $N_\lambda \gg 1$, nonlinear interactions between oppositely directed wavepackets are weak, and we should be able to calculate better than we have done above. There is a well–developed theory, called weak turbulence, that describes weakly nonlinear interactions between nearly linear waves — see Zakharov, L’vov & Falkovich (1992). In this approach, perturbation theory is used to derive kinetic equations in $k$–space describing wave–wave interactions. Then the inertial–range energy spectra emerge as stationary solutions carrying a flux of energy to large $k$. The lowest order of perturbation theory, corresponding to what are called 3–wave interactions in weak turbulence, is what is needed to make rigorous the IK cascade. Here, the rate of change of energy (or wave action) at a given $k$ is due to either (i) the coalescence of two other waves with wavevectors $k_1$ and $k_2$, or (i) decay of $k$ into two waves with wavevectors $k_1$ and $k_2$. For any given $k$, all $k_1$ and $k_2$ contribute, so long as they satisfy certain resonance conditions that depend on the dispersion relations satisfied by the linear waves. In the case of incompressible MHD, these 3–wave resonance conditions are somewhat special, because of the existence of the exact, nonlinear solutions given in equation (12). Since there are nonlinear interactions only between oppositely directed wavepackets, it means that $k_1$ and $k_2$ must belong to oppositely directed wavepackets. Then the 3–wave resonance conditions may be written as,

$$ k_1 + k_2 = k_3, \quad \omega_{1}^+ + \omega_{2}^- = \omega_{3}^+ \quad \text{or} \quad \omega_{3}^- $$

where $\omega_{1}^+ = \pm V_A k_{1z}$, $\omega_{2}^- = \mp V_A k_{2z}$, $\omega_{3}^+ = \pm V_A k_z$ and $\omega_{3}^- = \mp V_A k_z$. Note the resemblance to “momentum–energy” relations in quasi–particle interactions in quantum field theory or condensed matter physics. These resonance conditions have a remarkable property, noted by Shebalin, Matthaeus & Montgomery (1983), namely that one of $k_{1z}$ or $k_{2z}$ must be zero. This fact may be readily seen by considering the $z$–component of the wavevector resonance condition, $k_{1z} + k_{2z} = k_z$, together with the frequency resonance condition. If, for instance, we choose $k_1$ and $k$ to be $+$ waves and $k_2$ to be a $-$ wave, then the frequency resonance
condition is, $k_1 \zeta - k_2 \zeta = \zeta$. In this case, it is clear that $k_2 \zeta = 0$, implying that $k_1 \zeta = k_\zeta$.

Hence waves with values of $k_\zeta$ not present initially cannot be created by nonlinear interactions during wavepacket collisions. In other words, there is no cascade of energy to smaller spatial scales in the $\zeta$–direction, and hence the turbulence must be anisotropic.

The somewhat surprising fact is that, attempting to create an anisotropic version of the IK theory, leads us very far from the IK theory and to entirely new physics. Before we get into details, it is useful to introduce some terminology. When the random stirring on the large scale $L$ generates $+\zeta$ and $-\zeta$ waves with equal rms amplitudes (or energy input), we will refer to the situation as balanced; otherwise, the turbulence is called imbalanced. The case of imbalanced turbulence is also referred to as turbulence with non zero cross helicity, or non zero $\langle \mathbf{v} \cdot \mathbf{b} \rangle$, where $\mathbf{b}$ is the magnetic field perturbation. Also, the turbulence will be called weak when perturbation theory is applicable (when the cascade time is larger than the collision time); otherwise the turbulence will be called strong. Thus there are four cases to consider: balanced & weak, balanced & strong, imbalanced & weak and imbalanced & strong. Weak turbulence, whether balanced or imbalanced, can be described rigorously using the kinetic equations of weak turbulence. However strong turbulence, balanced or imbalanced, does not permit such a rigorous description and we are limited to phenomenological descriptions and numerical simulations.

4. Balanced Weak MHD Turbulence

The cascade is due to the resonant 3–wave interactions discussed above; see Ng & Bhattacharjee (1996), Goldreich & Sridhar (1997), Galtier et al. (2000) and Lithwick & Goldreich (2003). Consider again steady, balanced and isotropic excitation of $\pm \zeta$ waves, with amplitudes $w_L \ll V_A$ and scale $L$, which initiates a weak cascade to smaller spatial scales. As we have discussed above, there is no transfer of energy to smaller spatial scales in the direction parallel to the mean magnetic field. No parallel cascade implies that wavepackets of transverse scale $\lambda \ll L$ have parallel scales $L$.

It turns out that the different polarisations of the Alfvén waves and Slow waves play a very important role in the turbulent cascade. Viewed physically, we may think of the $(w^\pm \cdot \nabla) w^\mp$ terms as being responsible for the nonlinear cascade.\footnote{This statement is not entirely accurate, because the $\nabla p$ term also contributes a quantity of the same order of magnitude which does not have this precise form. However, the argument advanced in the text suffices for our phenomenological account in this review.} During collisions be-
between two oppositely directed wavepackets, we may think of \((w^\pm \cdot \nabla)\) as coming from the wavepacket that offers a perturbation, and the \(w^\mp\) wavepacket as the one that suffers the perturbation. For \(\lambda \ll L\), the polarisation of an Alfvén wave is such that its \(w^\pm\) is very nearly perpendicular to \(\hat{z}\), whereas \(w^\pm\) in a Slow wave is almost parallel to \(\hat{z}\). Then we estimate that \((w^\pm \cdot \nabla) \sim (w_\lambda/\lambda)\) when the perturbing wavepacket is an Alfvén wave, and \((w^\pm \cdot \nabla) \sim (w_\lambda/L)\) when the perturbing wavepacket is a Slow wave. _Clearly an Alfvén wave is a much stronger perturber than a Slow wave._ Thus, it comes as no surprise that Slow waves are mostly scattered by Alfvén waves, but do not significantly perturb Alfvén waves; _the cascade of Slow waves is slaved to the Alfvén waves_. Hence in our estimates below we ignore Slow waves, and restrict attention to Alfvén waves.

A collision between a +ve Alfvén wavepacket and a −ve Alfvén wavepacket now takes time \(\sim L/V_A\), because the wavepackets have parallel scales \(\sim L\). In one collision a wavepacket is perturbed by amount,

\[ \delta w_\lambda \sim \frac{Lw_\lambda^2}{\lambda V_A} < w_\lambda \tag{21} \]

The perturbations add with random phases. The number of collisions for perturbations to build up to order unity is

\[ N_\lambda \sim \left(\frac{\lambda V_A}{Lw_\lambda}\right)^2 \tag{22} \]

The cascade time is

\[ t_\lambda \sim N_\lambda \frac{L}{V_A} \tag{23} \]

Kolmogorov’s hypothesis of the \(\lambda\)-independence of the energy flux

\[ \varepsilon \sim \frac{w_\lambda^2}{t_\lambda} \tag{24} \]

implies that in the inertial-range, the rms amplitudes are given by,

\[ w_\lambda \sim w_L \left(\frac{\lambda}{L}\right)^{1/2} \tag{25} \]
The energy spectrum, $E(k)$, is highly anisotropic in $k$–space, and now depends on both $k_z$ and $k_\perp$. The precise dependence of $E$ on $k_z$ is not so interesting, because there is no parallel cascade; it suffices to know that $E$ is largely confined to the region $|k_z| < L^{-1}$. However, the dependence of $E$ on $k_\perp$ is of great interest, because this has been established by the cascade in the transverse directions, described above. Accounting for this anisotropy, we can estimate, $E(k_\perp, k_z) k_\perp^2 L^{-1} \sim w_\lambda^2$, where $k_\perp \sim \lambda^{-1}$. Now, using equation (25), we have

$$E(k_\perp, k_z) \sim \frac{w_\lambda^2}{k_\perp^3}, \quad \text{for} \quad |k_z| < L^{-1}$$

(26)

Being a theory based on weak turbulence, the turbulent cascade can be described rigorously using kinetic equations for energy transfer; see Galtier et al. (2000) and Lithwick & Goldreich (2003).

Having obtained the inertial–range energy spectrum of the balanced weak cascade, we need to perform checks of self consistency regarding the assumed weakness of the cascade. Using equations (22) and (25), we estimate that

$$N_\lambda \sim \left(\frac{V_A}{w_L}\right)^2 \frac{\lambda}{L}$$

(27)

We must first verify that the cascade initiated at the stirring scale $L$ is weak to begin with, which implies that we must have $N_\lambda \gg 1$. From equation (27), we see that this can be satisfied if $w_L \ll V_A$. In other words isotropic, balanced stirring initiates a weak cascade if the rms amplitudes at the stirring scale is much less than the Alfvén speed. A more geometric way of stating this is that a weak cascade is initiated if stirring bends field lines (of the mean magnetic field) only by small angles.

Equation (27) implies more interesting consequences: the transverse cascade strengthens as $\lambda$ decreases, because $N_\lambda$ decreases when $\lambda$ decreases. Therefore the assumption of the weakness of interactions must break down when $N_\lambda \sim 1$, which happens when $\lambda \sim \lambda_* \sim L(w_L/V_A)^2$. Therefore, the inertial–range spectrum of equation (26) is valid only for transverse scales larger than $\lambda_*$.

5. Balanced Strong MHD Turbulence

When $N_\lambda \sim 1$, the assumption of the weakness of the cascade is no longer even approximately valid. The turbulent cascade turns strong, and was first described by Goldreich &
Sridhar (1995). They conjectured that $N_\lambda$ remains of order unity for smaller values of $\lambda$ all the way down to the dissipation scale. This is equivalent to assuming that the cascade time remains of order the wave period, a condition that may be referred to as critical balance. The balanced weak cascade described above turns into a balanced strong cascade for $\lambda < \lambda^*$. Instead of following the transition from weak to strong, we prefer to describe the inertial–range of the balanced strong cascade when it is initiated at the stirring scale itself because there is less clutter in the description.

Consider steady, balanced and isotropic excitation of $\pm$ve waves, with amplitudes $w_L \sim V_A$ and scale $L$, which initiates a strong cascade to smaller spatial scales. Wavepackets of transverse scale $\lambda < L$ can now possess parallel scales that are smaller than $L$. This is because the resonance conditions are not in force when nonlinear interactions are strong so, in addition to a transverse cascade, a parallel cascade can also occur. Let eddies of transverse scale $\lambda$ possess parallel scale $\Lambda_\lambda$, which is as yet an unknown function of the transverse scale $\lambda$. Critical balance implies that

$$t_\lambda \sim \frac{\lambda}{w_\lambda} \sim \frac{\Lambda_\lambda}{V_A}$$

(28)

As before we invoke Kolmogorov’s hypothesis of the $\lambda$–independence of the energy flux

$$\varepsilon \sim \frac{w_\lambda^2}{t_\lambda}$$

(29)

which implies that in the inertial–range

$$w_\lambda \sim V_A \left(\frac{\lambda}{L}\right)^{1/3}, \quad \Lambda_\lambda \sim L^{1/3} \lambda^{2/3}$$

(30)

The energy spectrum is again anisotropic and may be estimated as, $E(k_\perp, k_z) k_\perp^2 (\Lambda_\lambda)^{-1} \sim w_\lambda^2$, where $k_\perp \sim \lambda^{-1}$. Now, using equation (30), we have

$$E(k_\perp, k_z) \sim \frac{w_\lambda^2}{L^{1/3} k_\perp^{11/3}}, \quad \text{for} \quad |k_z| < \frac{k_\perp^{2/3}}{L^{1/3}}$$

(31)

Summary of the balanced strong cascade

1. The cascade is critically balanced in that the cascade time is of order of the wave period throughout the inertial–range.
2. The rms amplitudes have a scaling with the transverse scale which is of the Kolmogorov form.

3. The turbulent cascade occurs in the transverse as well as parallel directions. However, the cascade occurs predominantly in the transverse directions, because the parallel correlation lengths scale as the 2/3rd power of the transverse correlation lengths.

6. Imbalanced Weak MHD Turbulence

The cascade is due to the resonant 3–wave interactions, see Galtier et al. (2000) and Lithwick & Goldreich (2003). Consider steady, imbalanced and isotropic excitation of ±ve Alfvén waves, with rms amplitudes $w_L^- \leq w_L^+ \ll V_A$ and scale $L$, which initiates weak cascades to smaller spatial scales. As we have discussed earlier, the 3–wave resonance conditions forbid a parallel cascade, so wavepackets of transverse scale $\lambda \ll L$ have parallel scales $L$.

The cascade times for the ±ve wavepackets can be estimated by considerations similar to those for the balanced case. Consider collisions suffered by a +ve wavepacket of transverse scale $\lambda$ and parallel scale $L$, with −ve wavepackets of similar spatial scales. A single collision occurs over time $\sim L/V_A$, during which the +ve wavepacket is perturbed by amount, $\delta w_\lambda^+ \sim (Lw_\lambda^- w_\lambda^+/\lambda V_A)$. Successive perturbations add with random phases, so the number of collisions for perturbations to build up to order unity is $N_\lambda^+ \sim (\lambda V_A/L w_\lambda^-)^2$. Similar considerations apply to collisions suffered by a −ve wavepacket with many +ve wavepackets, and hence $N_\lambda^- \sim (\lambda V_A/L w_\lambda^+)^2$. Therefore the cascade times for the ±ve waves are,

$$t_\lambda^+ \sim \frac{V_A}{L} \left( \frac{\lambda}{w_\lambda^-} \right)^2, \quad t_\lambda^- \sim \frac{V_A}{L} \left( \frac{\lambda}{w_\lambda^+} \right)^2$$

(32)

Kolmogorov’s hypothesis of the $\lambda$–independence of the energy fluxes

$$\varepsilon^+ \sim \frac{(w_\lambda^+)^2}{t_\lambda^+}, \quad \varepsilon^- \sim \frac{(w_\lambda^-)^2}{t_\lambda^-}$$

(33)

implies that

$$w_\lambda^+ w_\lambda^- \propto \lambda$$

(34)

In other words, the requirement that both $\varepsilon^+$ and $\varepsilon^-$ be independent of $\lambda$ leads to just one relation for the two quantities, $w_\lambda^+$ and $w_\lambda^-$. 

$$w_\lambda^+ w_\lambda^- \propto \lambda$$
Hence scaling arguments are, by themselves, insufficient for the determination of the inertial–range spectra of the imbalanced cascade. The new physics that determines the spectra is the fact that the $\pm$ energies are forced to equalise at the dissipation scale; this pinning completely determines the spectra.

Let the dissipation scale be at $\lambda_{\text{dis}}$, and the common rms amplitude of $w_\Lambda^+$ and $w_\Lambda^-$ at this scale be $w_{\text{dis}}$. Then equation (34) implies that we can write,

$$w_\Lambda^+ = w_{\text{dis}} \left( \frac{\lambda}{\lambda_{\text{dis}}} \right)^{(1+\alpha)/2}$$

$$w_\Lambda^- = w_{\text{dis}} \left( \frac{\lambda}{\lambda_{\text{dis}}} \right)^{(1-\alpha)/2}$$

(35)

where the parameters, $w_{\text{dis}}$, $\lambda_{\text{dis}}$ and $\alpha$ are to be determined. When the dissipation is caused by diffusive processes and $\nu \sim \eta$ (i.e. the Prandtl number is close to unity), the dissipation timescale is

$$t_{\text{dis}} \sim \frac{\lambda_{\text{dis}}^2}{\nu}$$

(36)

At the dissipation scale, the cascade time and the dissipation time are comparable. Setting $\lambda \sim \lambda_{\text{dis}}$ in equation (32) and setting it equal to $t_{\text{dis}}$, we can estimate that,

$$w_{\text{dis}} \sim \left( \frac{\nu V_A}{L} \right)^{1/2}$$

(37)

Note that $w_{\text{dis}}$ is independent of the the energy fluxes, $\varepsilon^\pm$. However, the other two parameters, $\lambda_{\text{dis}}$ and $\alpha$ depend on the energy fluxes, and these can be determined only by use of the kinetic equations of the weak turbulence theory of the cascade, as shown by Galtier et al. (2000) and Lithwick & Goldreich (2003). So, in contrast to the balanced weak cascade, dissipation plays a direct role in determining the inertial–range spectra of imbalanced weak MHD turbulence.

In common with the balanced weak case, we still need to verify that the turbulence is indeed weak in the inertial–range, So we need to verify that the quantities,

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$^4$It should be noted that $\lambda_{\text{dis}}$ and $\alpha$ can be expressed in terms of the rms amplitudes at the stirring scale, $w_L^+$ and $w_L^-$, but it is the energy fluxes, $\varepsilon^\pm$, that are the more interesting physical quantities.
\[ \chi_{\ell}^{\pm} \equiv \frac{Lw_{\ell}^{\pm}}{\lambda V_A} \ll 1 \quad (38) \]

throughout the inertial–range, \( L > \lambda > \lambda_* \). To do this, we use \( w_{\ell}^{\pm}/\lambda = w_{\text{dis}}^{2}/\lambda_{\text{dis}} \) and equation (37) to first work out \( \lambda_{\text{dis}} \):

\[ \lambda_{\text{dis}} \sim \frac{\nu V_A}{w_{\ell}^{+}w_{\ell}^{-}} \quad (39) \]

For the inertial–range to be of large enough extent we must have \( L \gg \lambda_{\text{dis}} \), which implies that

\[ \frac{w_{\ell}^{+}w_{\ell}^{-}}{V_A^2} \gg \frac{\nu}{LV_A} \quad (40) \]

Now, requiring that \( \chi_{\ell}^{\pm} \ll 1 \) when \( \lambda \sim \lambda_* \) gives us another condition:

\[ \frac{w_{\ell}^{+}w_{\ell}^{-}}{V_A^2} \ll \left( \frac{\nu}{LV_A} \right)^{1/2} \quad (41) \]

When \( \nu \ll LV_A \), the above inequalities can be satisfied with appropriately chosen \( w_{\ell}^{+}w_{\ell}^{-} \), and an imbalanced weak cascade with a substantial inertial–range is possible.

### 7. Imbalanced Strong MHD Turbulence

This is clearly the more general and generic case of MHD turbulence, and the account below is based on Lithwick, Goldreich & Sridhar (2007). Consider steady, imbalanced and isotropic excitation of ±ve waves, with rms amplitudes \( w_{\ell}^{\pm} \leq V_A \) and scale \( L \).

At some transverse scale \( \lambda \ll L \), let the amplitudes be \( w_{\lambda}^{\pm} \geq w_{\lambda}^{-} \), and parallel correlation lengths be \( \Lambda_{\pm}^{\lambda} \). Note that we allow the parallel correlation lengths to be functions of the transverse scale, because resonance conditions are not in force in strong turbulence. +ve waves bend field lines by angle \( (w_{\lambda}^{+}/V_A) \), hence a +ve wavepacket of parallel scale \( \Lambda_{\lambda}^{+} \) will displace field lines in the transverse directions by distances \( (w_{\lambda}^{+}\Lambda_{\lambda}^{+}/V_A) \). If this distance exceeds the transverse correlation length, \( \lambda \), then the −ve waves will suffer strong perturbations. In other words, the −ve cascade is strong if
\( \chi_{\lambda}^+ \equiv \frac{\Lambda_{\lambda}^+ w_{\lambda}^+}{\lambda V_A} \geq 1 \) \hspace{1cm} (42)

Then the \(-ve\) wave cascade time is

\[ t_{\lambda}^- \sim \frac{\lambda}{w_{\lambda}^+} \] \hspace{1cm} (43)

and the \(-ve\) wave parallel correlation length is

\[ \Lambda_{\lambda}^- \sim V_A t_{\lambda}^- \sim \lambda \left( \frac{V_A}{w_{\lambda}^+} \right) \] \hspace{1cm} (44)

From equations (42) and (44), we note that

\[ \Lambda_{\lambda}^+ \geq \Lambda_{\lambda}^- \] \hspace{1cm} (45)

We can infer a deeper result. Any two regions in the \(+ve\) packet that are separated by parallel lengths \( > \Lambda_{\lambda}^- \) will be cascaded by statistically uncorrelated \(-ve\) waves. Hence the \(-ve\) waves will imprint their parallel scales on the \(+ve\) waves, and both \(\pm\)\(ve\) waves will have the same parallel correlation lengths, \(\Lambda_{\lambda}^:\)

\[ \Lambda_{\lambda}^+ \sim \Lambda_{\lambda}^- \sim \Lambda_{\lambda} \sim \lambda \left( \frac{V_A}{w_{\lambda}^+} \right) \] \hspace{1cm} (46)

From equations (42) and (46) we have,

\[ \chi_{\lambda}^+ \sim 1 \] \hspace{1cm} (47)

Therefore the \(-ve\) cascade is critically balanced.

It turns out that the \(+ve\) wave cascade time is,

\[ t_{\lambda}^+ \sim \frac{\lambda}{w_{\lambda}^-} \] \hspace{1cm} (48)

This means that the straining rate imposed by the \(-ve\) waves on the \(+ve\) waves, \( (w_{\lambda}^-/\lambda) \), is imposed coherently over time \( (\lambda/w_{\lambda}^-) \). Then
How can the coherence time of the $-ve$ waves exceed their wave period? The answer is involved and is worked out in the Appendix of Lithwick, Goldreich & Sridhar (2007); here we merely summarise the result. Now let us look at the interaction between $\pm ve$ waves of transverse scale $\lambda$ and parallel scale $\Lambda_\lambda$, \textit{in the rest frame of the $+ve$ waves}. In this frame, the $-ve$ waves alter $+ve$ waves which react back onto the $-ve$ waves. Let $z'$ measure distance along the parallel direction in the rest frame of the $+ve$ waves. At a fixed $z'$ location the $+ve$ waves are changing on their cascade time scale $t^{+}_\lambda$. Hence, over times separated by $t^{+}_\lambda$, the $-ve$ waves crossing $z'$ are cascaded by entirely different $+ve$ waves. This implies that $t^{-}_{\text{corr}} \sim t^{+}_\lambda$. Because the $+ve$ waves are strained at rate $(w^-/\lambda)$, it follows that $t^{+}_\lambda \sim (\lambda/w^-)$.

We are now in a position to work out the scalings for the imbalanced strong cascade. Kolmogorov’s hypothesis of the $\lambda$–independence of the energy fluxes

\begin{equation}
\epsilon^- \sim \frac{(w^-)^2}{t^-_\lambda} \sim \frac{(w^-)^2 w^+_\lambda}{\lambda} \quad \epsilon^+ \sim \frac{(w^+_\lambda)^2}{t^+_\lambda} \sim \frac{(w^+_\lambda)^2 w^-}{\lambda}
\end{equation}

implies that in the inertial range,

\begin{align*}
w^\pm_\lambda & \sim \frac{(\epsilon^\pm)^{2/3}}{(\epsilon^\mp)^{1/3}} \lambda^{1/3} \\
\Lambda_\lambda & \sim \frac{(\epsilon^-)^{1/3}}{(\epsilon^+)^{2/3}} V_\lambda \lambda^{2/3}
\end{align*}

Equations (50) are given in Verma et al. (1996) and Verma (2004), although isotropy is assumed.

**Summary of the imbalanced strong cascade**

1. The ratio of the Elsasser amplitudes is independent of scale, and is equal to the ratio of the corresponding energy fluxes. Thus we can infer turbulent flux ratios from the amplitude ratios, thus providing insight into the origin of the turbulence.
2. In common with the balanced strong cascade, the energy spectra of both Elsasser waves are of the anisotropic Kolmogorov form, with their parallel correlation lengths equal to each other on all scales, and proportional to the 2/3rd power of the transverse correlation length.

3. The equality of cascade time and waveperiod (critical balance) that characterizes the strong balanced cascade does not apply to the Elsasser field with the larger amplitude. Instead, the more general criterion that always applies to both Elsasser fields is that the cascade time is equal to the correlation time of the straining imposed by oppositely-directed waves.

4. In the limit that the energy fluxes are equal, the turbulence corresponds to the balanced strong cascade.

We note that there are other theories of imbalanced cascades which differ from the above account; to the best of my knowledge, these are given in Beresnyak & Lazarian (2008), Chandran (2008), Perez & Boldyrev (2009) and Podesta & Bhattacharjee (2009).

8. Stirring–affected scales in the imbalanced strong cascade

Consider again steady, imbalanced and isotropic excitation of $\pm$ve waves, with rms amplitudes $w_L^- \leq w_L^+ < V_A$, on scale $L$. The wave periods of the $\pm$ve waves at the stirring scale $L/V_A$. Let the coherence time of the waves on the stirring scale be $T$; we expect that $T \geq L/V_A$, but otherwise $T$ is unconstrained and determined only by the physics of however the $\pm$ve waves are generated on scales $\sim L$. The theory of Lithwick, Goldreich & Sridhar (2007) applies when $T > t_L^+$; otherwise, when $T < t_L^+$, the coherent straining of $+$ve waves by the $-$ve waves will be interrupted before the $+$ve waves can cascade. Using,

$$t_L^+ \sim t_{\text{corr}}^\lambda \sim \frac{(\varepsilon^+)^{1/3}}{(\varepsilon^-)^{2/3}} \lambda^{2/3}$$

we can infer that $T < t_L^+$ implies,

$$\ell \equiv \frac{(\varepsilon^-) T^{3/2}}{(\varepsilon^+)^{1/2}} < L$$

Then, the results of Lithwick, Goldreich & Sridhar (2007) are valid for only for transverse scales $\lambda < \ell$.

The range of transverse scales, $\ell < \lambda < L$, can be referred to as stirring–affected scales.
For these scales, the +ve wave cascade is no longer strong and ±ve spectra are, as yet, unknown. It would also be interesting to see if there is any signature of these stirring–affected scales in solar wind turbulence.

9. Some comments

We have assumed that the strong large–scale magnetic field is uniform in magnitude and direction, but this need not be the case, as Kraichnan (1965) realised. We could consider a more general setting in which the magnetic field is disordered on large scales, with the largest spatial correlation length, $L$, being smaller than the system size. Let the magnetic energy in eddies of size $L$ be larger than the magnetic energy on smaller scales. Then the magnetic field of the eddies of size $L$ will act as the mean magnetic field for Alfvén and Slow waves of smaller scales, and all of what we have derived in this review for the inertial–range of MHD turbulence in a strong magnetic field may be expected to be still valid.\footnote{There is an interesting detail here that is, perhaps, worth noting. In the case of hydrodynamic turbulence, the velocity field of the large–scale eddies (of size $L$) will “sweep” eddies of smaller sizes, and this effect gives rise to observational consequences for the velocity time correlation function measured by a probe fixed at some location in the fluid. However, sweeping cannot affect the dynamics of nonlinear interactions, because a large–scale velocity field can be effectively transformed away by an appropriate Galilean boost. On the other hand, a large–scale magnetic field cannot be similarly transformed away; one way to physically see this is to note that such a field enables (Alfvén and Slow) waves that travel both up and down the field lines, and these cannot be wished away by any Galilean boost.}

The opposite case of the large–scale magnetic field being weak often occurs in problems concerning dynamo processes, and turns out to be harder to analyse.

We have considered incompressible MHD turbulence, but the general case in astrophysics involves compressive fluids. In compressible MHD, there are four types of waves: Alfvén, Fast, Slow and Entropy. The inertial–range spectra we have derived are applicable to cascades of Alfvén waves, which are always incompressible. These are valid, so long as the other three waves (which are compressive) have negligible effect on the Alfvén wave. That this is indeed so has been discussed in the following papers: Lithwick & Goldreich (2001), Maron & Goldreich (2001) and Cho & Lazarian (2003).

In the context of Navier–Stokes turbulence of a neutral fluid, it is well–known that a triple correlation function of velocities is related to the rate of energy input; see Karman & Howarth (1938), Kolmogorov (1941) and Monin (1959). Similar relations in the context of MHD turbulence have been obtained by Politano & Pouquet (1998a,b) and Podesta (2008). However, the relation to the theories of the inertial–range spectra of MHD turbulence has
not received as much attention as in the case of neutral fluids.

It has sometimes been argued that there is a “dynamical alignment” of velocity and magnetic fields, resulting in spectra that are flatter than Kolmogorov — see Boldyrev (2005) and Beresnyak & Lazarian (2006). However, observations of solar wind turbulence show spectra that are consistent with Kolmogorov — see Horbury, Forman & Oughton (2005).

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