Towards a Background Independent Formulation of Perturbative Quantum Gravity

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Abstract. The recent formulation of locally covariant quantum field theory may open the way towards a background independent perturbative formulation of Quantum Gravity.

1. Problems of perturbative Quantum Gravity

In quantum field theory the fields are defined as operator-valued distributions on a given spacetime, and many of their properties, in particular the commutativity at spacelike separated points, depend in a crucial way on properties of the background. In a perturbative approach to quantum gravity, one decomposes the metric $g_{\mu\nu}$ into a background metric $\eta_{\mu\nu}$ and a quantum field $h_{\mu\nu}$, which is treated according to standard methods in perturbation theory. The (up to now observed) effects of this quantum field are very small, hence a perturbative approach seems to be appropriate. There are, however, several obstructions which raise doubts on the validity of the perturbative approach:

1. The arising quantum field theory is nonrenormalizable [10]. Hence, infinitely many counterterms occur in the process of renormalization, and it is unclear how the arising ambiguities can be fixed [9].

2. The perturbatively constructed theory depends on the choice of the background. It is unlikely that a perturbative formulation can describe a drastic change of the background.

The two main approaches to quantum gravity try to cope with these difficulties in different ways. String theory, in a first attempt, accepts the choice of a fixed background, and aims at a more general theory where the perturbation series is finite in every order. Loop quantum gravity, on the other hand, uses a background free formulation where the degrees of freedom of gravity are directly quantized. Problems with renormalizability do not occur, but it seems to be difficult to check whether such theories describe the world as we see it.
Instead of following these routes one may take a more conservative approach and study first the influence of classical gravitational fields on quantum fields. Because of the weakness of gravitational forces this approximation is expected to have a huge range of validity. Surprisingly, this seemingly modest approach leads to many conceptual insight, and it may even lead to a new approach to quantum gravity itself [8].

In this paper we want to review the recently developped new formulation of quantum field theory on curved spacetimes, which satisfies the conditions of general covariance [3,11,12,16]. We will show that the arising structure has great similarities with Segal’s concept of topological quantum field theories [15] and its generalization to Riemannian spaces. It may be considered as a Lorentzian version of this approach. It is gratifying that the axiom of local commutativity is implied in this framework by the tensorial structure of the theory, while the time slice axiom (i.e., a form of dynamics) is related to cobordisms.

It is remarkable that the new structures emerged from the (finally successful) attempt to construct interacting theories in the sense of renormalized perturbation theory.

Up to now the complete proofs apply only to scalar field theories. The extension to gauge theories requires the control of BRST invariance. Preliminary steps in this direction have been performed [4,5,6], and no obstruction is visible.

More or less, the same construction then should apply to quantum gravity, treated in a background formulation. The leading idea is that background independence can be reached from a background dependent formulation provided the change of background amounts to a symmetry of the theory.

2. Locally covariant quantum field theory

We adopt the point of view [3] of algebraic quantum field theory and identify physical systems with $C^*$-algebras with unit (if possible, $C^*$-algebras) and subsystems with subalgebras sharing the same unit. In quantum field theory the subsystems can be associated to spacetime regions. Every such region may be considered as a spacetime in its own right, in particular it may be embedded into different spacetimes. It is crucial that the algebra of the region does not depend on the way it is embedded into a larger spacetime. For instance, in a Schwarzschild spacetime the physics outside the horizon should not depend on a possible extension to a Kruskal spacetime.

We formulate our requirements in form of five axioms:

1. Systems: To each time oriented globally hyperbolic spacetime $M$ we associate a unital $*$-algebra $\mathcal{A}(M)$.

2. Subsystems: Let $\chi : M \to N$ be an isometric causality preserving embedding of globally hyperbolic spacetimes. Then, there exists a uniquely defined (injective) $*$-homomorphism $\alpha_\chi : \mathcal{A}(M) \to \mathcal{A}(N)$. 

3. Covariance: If $\chi : M_1 \to M_2$ and $\chi' : M_2 \to M_3$ are embeddings as above, then $\alpha_{\chi\chi'} = \alpha_\chi \alpha_{\chi'}$.

4. Causality: If $\chi_1 : M_1 \to M$ and $\chi_2 : M_2 \to M$ are embeddings as above, such that $\chi_1(M_1)$ and $\chi_2(M_2)$ cannot be connected by a causal curve in $M$, then

$$\alpha_{\chi_1}(\mathcal{A}(M_1)) \vee \alpha_{\chi_2}(\mathcal{A}(M_2)) \simeq \alpha_{\chi_1}(\mathcal{A}(M_1)) \otimes \alpha_{\chi_2}(\mathcal{A}(M_2))$$

where $\vee$ indicates the generated subalgebra of $\mathcal{A}(M)$.

5. Dynamics: Let $\chi : M \to N$ be an embedding as above such that $\chi(M)$ contains a Cauchy surface of $N$. Then $\alpha_\chi(\mathcal{A}(M)) = \mathcal{A}(N)$.

The axioms above describe a functor $\mathcal{A}$ from the category $\text{Loc}$ (the localization category) whose objects are time-oriented globally hyperbolic spacetimes and whose arrows are the causal isometric embeddings, to the category $\text{Obs}$ (the observables category) whose objects are unital $*$-algebras and whose arrows are (injective) $*$-homomorphisms.

Axiom 1 is similar to the usual axiom in local quantum theories on a fixed background, where the arrow has specific (un)bounded regions on that background as a domain. Here, it is imperative to quantize simultaneously on all globally hyperbolic spacetimes (of the given type).

Axiom 2 may be pictured in the form

$$
\begin{array}{ccc}
M & \xrightarrow{\chi} & N \\
\mathcal{A} & & \mathcal{A} \\
\alpha_M & \xrightarrow{\alpha_\chi} & \alpha_N
\end{array}
$$

where $\alpha_\chi = \mathcal{A}\chi$.

Axiom 3 says that the functor $\mathcal{A}$ is covariant.

Axiom 4 may be reformulated in terms of a tensor structure. Namely, require for disjoint unions,

$$\mathcal{A}(M_1 \amalg M_2) = \mathcal{A}(M_1) \otimes \mathcal{A}(M_2), \quad \mathcal{A}(\emptyset) = \mathbb{C},$$

with $\chi_i : M_i \to M$, $i = 1, 2$ for which $\alpha_{\chi_1\chi_2} = \alpha_{\chi_1} \otimes \alpha_{\chi_2}$. Let $\chi$ be a causal embedding of $M_1 \amalg M_2$ into $M$. Then $\chi(M_1)$ and $\chi(M_2)$ are spacelike separated, hence with $i_k : M_k \to M_1 \amalg M_2$, and with $\chi_k = \chi \circ i_k$ (see fig.1), we see that $\alpha_\chi(\mathcal{A}(M_1) \otimes \mathcal{A}(M_2))$ is equal to the algebra generated by $\alpha_{\chi_1}(\mathcal{A}(M_1))$ and $\alpha_{\chi_2}(\mathcal{A}(M_2))$, hence the causality axiom is satisfied. In short, the functor $\mathcal{A}$ is promoted to a tensor functor. This is very reminiscent of G. Segal’s approach [15].

Axiom 5 may be interpreted as a description of motion of a system from one Cauchy surface to another. Namely, let $N_+$ and $N_-$ be two spacetimes that embed into two other spacetimes $M_1$ and $M_2$ around Cauchy surfaces, via causal embeddings given by $\chi_{k, \pm}, k = 1, 2$. Figure 2 gives a hint.

Then $\beta = \alpha_{\chi_1}, \alpha_{\chi_2}^{-1}, \alpha_{\chi_2}, \alpha_{\chi_1}^{-1}$ is an automorphism of $\mathcal{A}(M_1)$. One may say that in case $M_1$ and $M_2$ are equal as topological manifold but their metrics differ by a (compactly supported) symmetric tensor $h_{\mu\nu}$ with $\text{supp} h \cap J^+(N_+) \cap J^-(N_-) = $
Figure 1. Causal Embedding

Figure 2. Evolution
the automorphism depends only on the spacetime between the two Cauchy surfaces, hence in particular, on the tensor $h$. It can then be shown that

$$\Theta_{\mu,\nu}(x) \doteq \frac{\delta \beta_h}{\delta h_{\mu,\nu}(x)}|_{h=0}$$

is a derivation valued distribution which is covariantly conserved, an effect of the diffeomorphism invariance of the automorphism $\beta_h$, and may be interpreted as the commutator with the energy-momentum tensor. Indeed, in the theory of the free scalar field this has been explicitly verified \cite{3}, and it remains true in perturbatively constructed interacting theories \cite{14}.

The structure described may also be understood as a version of cobordism. Namely, one may associate to a Cauchy surface $\Sigma$ of the globally hyperbolic spacetime $M$, the algebra

$$\mathcal{A}(\Sigma) \doteq \lim_{\rightarrow} \mathcal{A}(N)_{N \supset \Sigma},$$

where the inverse limit extends over the globally hyperbolic neighborhoods $N$ of $\Sigma$. Clearly, $\mathcal{A}(\Sigma)$ depends only on the germ of $\Sigma$ as a submanifold of $M$. The elements of $\mathcal{A}(\Sigma)$ are germs of families $(A_N)_{N \supset \Sigma}$ with the coherence condition $\alpha_{N_1 N_2}(A_{N_2}) = A_{N_1}$, where $\alpha_{N_1 N_2}$ is the homomorphism associated to the inclusion $N_2 \subset N_1$.

We then define a homomorphism

$$\alpha_{\Sigma_1 \Sigma_2} : \mathcal{A}(\Sigma_2) \to \mathcal{A}(\Sigma_1),$$

by $\alpha_{\Sigma_1 \Sigma_2}(A) = \alpha_{\Sigma_1 N}(A_N)$, $N \supset \Sigma$, where the r.h.s. is independent of the choice of the neighborhood $N$. By the time slice axiom, $\alpha_{\Sigma_1 \Sigma_2}$ is invertible, hence for a choice of two Cauchy hypersurfaces $\Sigma_1$, $\Sigma_2$ of $M$ we find a homomorphism

$$\alpha_{\Sigma_1 \Sigma_2}^M : \mathcal{A}(\Sigma_2) \to \mathcal{A}(\Sigma_1),$$

with $\alpha_{\Sigma_1 \Sigma_2}^{-1} \doteq \alpha_{\Sigma_1 \Sigma_2}^M$. We may interpret $\Sigma_1$ and $\Sigma_2$ as past and future boundaries, respectively, of $M$ and obtain for any spacetime $M$ connecting $\Sigma_1$ and $\Sigma_2$ a homomorphism of the corresponding algebras.

3. Locally covariant fields

One problem with a theory on a generic spacetime is that it is not clear what it means to do the same experiment at different spacetime points. In quantum theory we are however forced to repeat the experiments in order to obtain a probability distribution. In a spacetime with a large symmetry group one may use these symmetries to compare measurements on different spacetime points. In the framework of locally covariant quantum field theory, as described above, quantum fields can serve as means for comparison of observables at different points. Namely, we define a locally covariant quantum field $A$ as a family of algebra valued distributions
(AM) indexed by the objects M ∈ Loc which satisfy the following covariance condition
\[ \alpha_\chi(AM(x)) = AN(\chi(x)) , \quad \chi : M \to N . \]

More formally, one may define a locally covariant quantum field as a natural transformation from the functor A, that associates to each manifold its test-function space, to the functor A. Actually, we pinpoint that fields, differently from the traditional point of view, are now objects as fundamental as observables. Not only, they might even be more fundamental in cases like that of quantum gravity where local observables would be difficult to find (if they exist at all).

Let us look at an example. We define the theory of a real Klein-Gordon field in terms of the algebras A0(M) which are generated by elements ϕM(f), f ∈ D(M), satisfying the relations
\[ \begin{align*}
(i) & \ f \mapsto \varphi_M(f) \text{ is linear;} \\
(ii) & \ \varphi_M(f)^* = \varphi_M(f) \\
(iii) & \ \varphi_M((\Box_M + m^2)f) = 0; \\
(iv) & \ [\varphi_M(f), \varphi_M(g)] = i(f, \Delta_M g)1
\end{align*} \]
where \( \Delta_M = \Delta_M^{\text{ret}} - \Delta_M^{\text{adv}} \) with the retarded and advanced propagators of the Klein-Gordon operator, respectively. The homomorphism \( \alpha_\chi, \chi : M \to N \), is induced by
\[ \alpha_\chi(\varphi_M(f)) = \varphi_N(\chi^*(f)) \]
where \( \chi^*f \) is the push-forward of the test function \( f \). We see immediately that \( \varphi = (\varphi_M) \) is a locally covariant quantum field associated to the functor A0.

To find also other locally covariant fields it is convenient to enlarge the previously constructed algebra by neglecting the field equation (iii). Notice that the new functor A00 does no longer satisfy the time slice axiom. We then introduce localized polynomial functionals \( \mathcal{F}(M) \) on the space of classical field configurations \( \phi \in \mathcal{C}^\infty(M) \),
\[ \mathcal{F}(M) \ni F(\phi) = \sum_{n=0}^{\text{ord}(F)} \langle f_n, \phi \otimes^n \rangle , \quad \phi \in \mathcal{C}^\infty(M) \]
where \( f_n \in \mathcal{E}'(\Gamma_n)(M^n) \), i.e., \( f_n \) is a distribution of compact support whose wave front set WF(\( f_n \)) ⊂ Γ_n and \( \Gamma_n \cap \{(x, k) \in T^*M^n, k \in \overline{V_+} \cup \overline{V_-} \} = \emptyset \). We choose a decomposition of \( \Delta_M, \Delta_M(x, y) = H(x, y) - H(y, x) \), such that WF(\( H \)) = \{\( (x, k) \in \WF(\Delta_M), k \in \overline{V_+ \times V_-} \)\}. (This is a microlocal version of the decomposition into positive and negative frequencies, for explanations see, for instance, \cite{2}.) Then, we define a product on \( \mathcal{F}(M) \) by
\[ F \ast_H G = \sum_n \frac{1}{n!} \left( \frac{\delta^n F}{\delta \phi^n} \otimes \frac{\delta^n G}{\delta \phi^n}, H^{\otimes n} \right), \]
which makes \( \mathcal{F}(M) \) to an associative algebra \( (\mathcal{F}(M), \ast_H) \).

But \( H \) is not unique. If we change \( H \) to \( H' = H + w, w \in \mathcal{C}^\infty_{\text{symm}}(M^2) \) (since the difference between two \( H \)'s is always a smooth symmetric function), we find
\[\gamma_w(F \ast_H G) = \gamma_w(F) \ast_H \gamma_w(G)\]

with

\[\gamma_w(F) = \sum_n \frac{1}{2^n n!} \left\langle \frac{\delta^{2n} F}{\delta \phi^{2n}}, w^{\otimes n} \right\rangle.\]

The algebra \(\mathcal{A}_{00}(M)\) may be embedded into \((\mathcal{F}(M), \ast_H)\) by

\[\alpha_H(\varphi_M(f)) = (f, \phi),\]

hence, in particular

\[\alpha_H(\varphi_M(f) \varphi_M(g)) = (f \otimes g, \phi^{\otimes 2}) + \langle f, Hg \rangle.\]

Then \(\alpha_H(\mathcal{A}_{00}(M))\) is a subalgebra of \((\mathcal{F}(M), \ast_H)\) with coefficients \(f_n \in \mathcal{D}(M^n)\). Since the space \(\mathcal{F}(M)\) is uniquely determined by the coefficients, we may equip it with the inductive topology of the direct sum of the spaces \(\mathcal{E}'_{\Gamma_n}(M^n)\). Since \(\mathcal{D}(M^n)\) is dense in \(\mathcal{E}'_{\Gamma_n}(M^n)\), \(\alpha_H(\mathcal{A}_{00}(M))\) is dense in \(\mathcal{F}(M)\). We may now equip the algebra \(\mathcal{A}_{00}(M)\) with the initial topologies of \(\alpha_H\). Since \(\gamma_w\) is a homeomorphism, it turns out that all induced topologies coincide, and the completion is identified with our sought algebra \(\mathcal{A}(M)\).

This algebra contains, besides the usual fields, also their normal ordered products.

Now, for the sake of constructing interacting quantum fields one may use the following steps:

1. Construction of locally covariant Wick polynomials; this turns out to the solution of a cohomological problem \(3\) (see also \(12\)) which can be solved in terms of an explicit Hadamard parametrix of the Klein Gordon equation.

2. Construction of locally covariant retarded and time ordered products; this requires the generalization of the Epstein-Glaser renormalization scheme to curved spacetime \(11\) and again a solution of a cohomological problem in order to be able to impose the same renormalization conditions on every point of a given spacetime and even on different spacetimes.

3. Construction of the algebras of interacting fields, together with a family of locally covariant fields \(1, 5, 11, 12\).

We refrain from giving details of these steps and refer to the original publications \(1, 3, 5, 11, 12, 13, 14\).

4. Quantization of the background

As usual, we view the metric as a background plus a fluctuation, namely,

\[g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\]
and we look at the fluctuation as a quantum field. Note that differently from
other approaches the background metric does not need to be Minkowskian, we only
restrict to backgrounds complying with the requirements of the previous sections.
So we have a quantum field $h$ that propagates via the linearized Einstein
equations on a fixed background $\eta$.

One may now proceed by the general strategy for constructing gauge theories
by the BRST method. It is crucial that this method can be adapted to localized
interactions, as was done in [4, 6]. Furthermore, the freedom in renormalization
can be used to arrive at quantized metric and curvature fields satisfying Einstein’s
equation.

The condition of background independence may be formulated as the condi-
tion that the automorphism $\beta_\kappa$ describing the relative Cauchy evolution corre-
sponding to a change of the background between two Cauchy surfaces must be
trivial. In perturbation theory, it is sufficient to check the infinitesimal version
of this condition. This amounts to the equation

$$\frac{\delta \beta_\kappa}{\delta \kappa_{\mu\nu}(x)} = 0 .$$

In contrast to the situation for an unquantized background metric the left hand
side involves in addition to the energy momentum tensor also the Einstein tensor.
Hence the validity of Einstein’s equation for the quantized fields should imply
background independence.

There remain, of course, several open questions. First of all, the details of
the proposal above have to be elaborated, and in particular the question of BRST
invariance has to be checked. A possible obstruction could be that locally the
cohomology of the BRST operator is trivial, corresponding to the absence of local
observables in quantum gravity. Another problem is the fact that the theory is
not renormalizable by power counting. Thus the theory will have the status of
an effective theory. Nevertheless, due to the expected smallness of higher order
counter terms the theory should still have predictive power.

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