MILNOR INVARIANTS OF COVERING LINKS

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Abstract. We consider Milnor invariants for certain covering links as a generalization of covering linkage invariants formulated by R. Hartley and K. Murasugi. A set of Milnor invariants for covering links is a cobordism invariant of a link, and that this invariant can distinguish some links for which the ordinary Milnor invariants coincide. Moreover, for a Brunnian link \( L \), the first non-vanishing Milnor invariants of \( L \) is modulo-2 congruent to a sum of Milnor invariants of covering links. As a consequence, a sum of linking numbers of ‘iterated’ covering links gives the first non-vanishing Milnor invariant of \( L \) modulo 2.

1. Introduction

For a link \( L \) in the 3-sphere \( S^3 \), we consider a branched cover of \( S^3 \) branched over components of \( L \). The set of linking numbers between 2-component sublinks of the preimage \( \tilde{L} \) of \( L \) had been recognized as a useful invariant for knots and links, see for example [2], [10], [16] and [17]. R. Hartley and K. Murasugi [9] called this invariant the covering linkage invariant.

J. Milnor [12], [13] defined a family of invariants for a link indexed by sequences of integers in \( \{1, 2, \ldots, n\} \), where \( n \) is the number of components of the link. Since the Milnor invariant of a link for a length two sequence \( ij \) coincides with the linking number of the \( i \)th and \( j \)th components of the link, we could regard the Milnor invariants for sequences with the length at least 3 as a kind of higher order linking numbers.

So it seems to be natural to consider the Milnor invariants for (sublinks of) \( \tilde{L} \) as a generalization of covering linkage invariants. In fact, for a prime number \( p \), T.D. Cochran and K.E. Orr [6] defined ‘mod \( p \) or \( p \)-adic versions’ Milnor invariants for links in the \( \mathbb{Z}_p \)-homology 3-sphere. Their Milnor invariants can be also defined for \( \tilde{L} \) in the \( p \)-fold cyclic branched cover of \( S^3 \) branched over a component of \( L \).

In order to investigate the ordinary Milnor invariants for \( \tilde{L} \), we only consider a simple case as follows: Let \( L \) be a link with a trivial component \( K \) such that the linking numbers between \( K \) and the other components are even. For the double branched cover \( \Sigma(K) \) branched over \( K \), a link in \( \Sigma(K) \) which consists of components of the preimage of each component of \( L \setminus K \) is said to be a covering link. Note that there are \( 2^{n-1} \) covering links in \( \Sigma(K) \), where \( n \) is the number of components of \( L \).

(We remark that in [6], they call the preimage \( \tilde{L} \) of \( L \) ‘the’ covering link.) Since \( \Sigma(K) \) is also \( S^3 \), we can define the ordinary Milnor invariants for each covering link.

It is known that the Milnor invariants are cobordism invariants [3]. It is also true that a set of the Milnor invariants for certain covering links are cobordism invariants of \( L \) (Theorem 3.1). In [6], Cochran and Orr show that their invariants are \( (p-) \)-cobordism invariants of the covering link \( \tilde{L} \).

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In [14], they do not make it clear if their Milnor invariants of the covering link are useful when the ordinary Milnor invariants are useless. Our invariants, the Milnor invariants of a covering link, can distinguish some links for which the ordinary Milnor invariants coincide, see Remark 7.2.

For a Brunnian link $L$ in $S^3$, we show that the first non-vanishing Milnor invariants of $L$ is modulo-2 congruent to a sum of Milnor invariants of certain covering links (Theorem 4.1). In [15], Murasugi proved that Milnor invariants of a link are linking numbers in appropriate nilpotent covering spaces of $S^3$ branched over the link, i.e., he described the exact correspondence between Milnor invariants and covering linkage invariants. While it looks that our result is weaker than the result of Murasugi, the authors believe that the result is worth mentioning because it is hard to treat nilpotent covers in general.

2. Milnor invariants

Let $L$ be an oriented ordered $n$-component link in $S^3$. Let $p$ be a prime number. Let $G$ be the fundamental group of the complement of $L$ and $G^p$ a normal subgroup of $G$ generated by $[x, y]^p$ for $x \in G$, $y, z \in G^{p-1}$, where $[x, y]$ is the commutator of $x$ and $y$, and $G^p_1$ means $G$. (We remark that in [18], $G$ is defined to be $G^{p_0}$.)

It is shown by similar to Theorem 4 in [13] that the quotient group $G/G^p_1$ is isomorphic to a group with the following presentation:

$$\langle \alpha_1, \alpha_2, \ldots, \alpha_n \mid [\alpha_i, \lambda_i^q](i = 1, 2, \ldots, n), F^p_q \rangle,$$

where $\alpha_i$, $\lambda_i^q$ represent $i$th meridian and longitude of $L$ respectively and $F$ is a free group generated by $\alpha_1, \alpha_2, \ldots, \alpha_n$. In particular, $\lambda_i^q$ can be chosen as a word in $\alpha_1, \alpha_2, \ldots, \alpha_n$ so that $\lambda_i^q = \lambda_i^0$.

We introduce the Magnus $\mathbb{Z}_p$-expansion of $\lambda_i^q$. The Magnus $\mathbb{Z}_p$-expansion $E^p$ is an embedding homomorphism of $F$ to the formal power series ring in non-commutative variables $X_1, X_2, \ldots, X_n$ with $\mathbb{Z}_p$ coefficients defined by $E(\alpha_i) = 1 + X_i$ and $E(\alpha_i^{-1}) = 1 - X_i + X_i^2 - X_i^3 + \cdots (i = 1, 2, \ldots, n)$ [18, 6.1 Lemma]. Then $E^p(\lambda_i^q)$ can be written in the form

$$E^p(\lambda_i^q) = 1 + \sum_{k<q} \mu^q_k(i_1i_2\cdots i_k)_{p}X_{i_1}X_{i_2}\cdots X_{i_k} + \text{(terms of degree } \geq q),$$

where a coefficient $\mu^q_k(i_1i_2\cdots i_k)$ is defined for each sequence $i_1i_2\cdots i_k$ of integers in $\{1, 2, \ldots, n\}$. Let $1 + f$ be the Magnus $\mathbb{Z}_p$-expansion of an element of $F^p_q$. Then the degree of any term in $f$ is at least $q$ (see [18, 6.2 Lemma]). This implies that for $q < q'$

$$\mu^q_k(i_1i_2\cdots i_k)_{p} = \mu^{q'}_k(i_1i_2\cdots i_k)_{p}.$$  

Taking $q$ sufficiently large, we may ignore $q$ and hence denote $\mu^q_k(i_1i_2\cdots i_k)_{p}$ by $\mu_k(i_1i_2\cdots i_k)_{p}$.

For a sequence $I = i_1i_2\cdots i_k$ ($k < q$), let $\Delta_L(I)_p$ be the ideal of $\mathbb{Z}_p$ generated by $\mu(J)_{p}$'s, where $J$ is obtained from a proper subsequence of $I$ by permuting cyclicly.

Then the Milnor invariant $\overline{\mu}_L(I)_p$ is defined by

$$\overline{\mu}_L(I)_p \equiv \mu_L(I)_p \mod \Delta_L(I)_p.$$ 

The length of $\overline{\mu}_L(I)_p$ means the length of $I$.

Remark 2.1. (1) The ordinary Milnor invariant $\overline{\mu}_L(I)$ given in [12], [13] is equal to $\overline{\mu}_L(I)_0$ (In [13], $\Delta_L(I)_0$ is defined as the greatest common divisor of $\mu_L(J)_0$'s.)

When $p$ is a prime number, $\Delta_L(I)_p$ is equal to either $\{0\}$ or $\mathbb{Z}_p$ since $\mathbb{Z}_p$ is a field. Hence we essentially consider the first non-vanishing of $\overline{\mu}_L(I)_p$ when $p$ is prime. Since $G^0_p \subset G^p_1$ for any prime number $p$, $\mu_L(I)_p$ is the modulo-$p$ reduction
of $\mu_L(I)_0$. Taking $p$ sufficiently large, we may regard that $\overline{p}_L(I)_p = \overline{p}_L(I)_0$ if $\Delta_L(I)_0 = \{0\}$.

(2) In order to state our results, we only need the original definition by Milnor [12, 13]. The definition in this section will be needed to apply a theorem of Stallings [18, 5.1 Theorem] in the proof of Theorem 3.1.

3. COVERING MILNOR INVARIANTS

Let $L = K_1 \cup K_2 \cup \cdots \cup K_{n+1}$ be an oriented $(n + 1)$-component link in $S^3$ with $K_{n+1}$ is trivial and linking numbers of $K_{n+1}$ and $K_i$ are even for all $i = 1, 2, \ldots, n$. Let $\Sigma(K_{n+1})$ be the double branched cover of $S^3$ branched over $K_{n+1}$ and $K_i \subset \Sigma(K_{n+1})$ a component of the preimage of $K_i$ ($\in \{0, 1\}$, $i = 1, 2, \ldots, n$). Then we denote by $L(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_n)$ a link $K_1^{\varepsilon_1} \cup K_2^{\varepsilon_2} \cup \cdots \cup K_n^{\varepsilon_n}$ ($\varepsilon_i \in \{0, 1\}$) in $\Sigma(K_{n+1})$ and call it a covering link of $L$. Since by the covering translation of $\Sigma(K_{n+1})$ there is an bijection from $\{L(0\varepsilon_2 \cdots \varepsilon_n) \mid \varepsilon_i \in \{0, 1\}\}$ to $\{L(1\delta_2 \cdots \delta_n) \mid \delta_i \in \{0, 1\}\}$, we only consider the set $\{L(0\varepsilon_2 \cdots \varepsilon_n) \mid \varepsilon_i \in \{0, 1\}\}$ in this section (and the last section). It is obvious that the set $\{L(0\varepsilon_2 \cdots \varepsilon_n) \mid \varepsilon_i \in \{0, 1\}\}$ is an invariant of $L$.

Note that $\Sigma(K_{n+1})$ is still $S^3$ because the branch set $K_{n+1}$ is trivial. Hence Milnor invariants are naturally defined for covering links. In particular for a sequence $I$ of $\{1, 2, \ldots, n\}$, $M_I(L) = \{L(0\varepsilon_2 \cdots \varepsilon_n) \mid \varepsilon_i \in \{0, 1\}\}$ is an invariant of $L$, where each element $i$ of $\{2, \ldots, n\}$ corresponds to the subindex of $K_i^{\varepsilon_i}$. We call $M_I(L)_p$ the covering Z$p$-Milnor invariant of $L$. In particular, we denote $M_I(L)_0$ by $M_L(I)$ and call it the covering Milnor invariant of $L$. It is easy to see that $M_I(L)_p$ is an ambient isotopy invariant for each $p$ and $M_L(I)$ is the strongest invariant of them. But it is not clear whether each $M_I(L)_p$ is a cobordism invariant. Here, two $n$-component links $L = K_1 \cup K_2 \cup \cdots \cup K_n$ and $L' = K'_1 \cup K'_2 \cup \cdots \cup K'_n$ in $S^3$ are cobordant if there is a disjoint union of annuli $A_1, A_2, \cdots, A_n$ in $S^3 \times [0, 1]$ such that the boundary $\partial A_i = K_i \cup (-K'_i)$ for each $i = 1, 2, \ldots, n$, where $-K'_i$ is the $i$th component $K'_i$ with the opposite orientation. It is known that Milnor invariants of links are cobordism invariants [5]. The same result holds for the first non-vanishing covering Milnor invariants as follows.

**Theorem 3.1.** Let $L = K_1 \cup K_2 \cup \cdots \cup K_{n+1}$ be an oriented $(n + 1)$-component link in $S^3$ with $K_{n+1}$ is trivial and linking numbers of $K_{n+1}$ and $K_i$ are even for $i = 1, 2, \ldots, n$. For a sequence $I$, if $\Delta_L(0\varepsilon_2 \cdots \varepsilon_n)(I)_0 = \{0\}$ for all $\varepsilon_i$ ($i \in \{2, \ldots, n\}$, $\varepsilon_i \in \{0, 1\}$), then $M_L(I)$ is a cobordism invariant of $L$. That is, the first non-vanishing covering Milnor invariants are cobordism invariants.

Let us first prove the following.

**Lemma 3.2.** Let $W$ be a connected 4-manifold with the first and second betti numbers are zero. There is a prime number $p$ such that the first and second homology groups $H_1(W; \mathbb{Z}_p)$, $H_2(W; \mathbb{Z}_p)$ of $W$ with $\mathbb{Z}_p$ coefficients are trivial. Moreover, for any prime number $p'$ with $p' > p$, $H_1(W; \mathbb{Z}_{p'}) \cong H_2(W; \mathbb{Z}_{p'}) \cong \{0\}$.

**Proof.** Since the first and second betti numbers are zero by the hypothesis, there is a prime number $p$ such that $H_1(W; \mathbb{Z}) \otimes \mathbb{Z}_p \cong H_2(W; \mathbb{Z}) \otimes \mathbb{Z}_p \cong \{0\}$.

Note that this is also true for any prime number which is greater than $p$.

Since $W$ is connected, $H_0(W; \mathbb{Z}) \cong \mathbb{Z}$. By the universal coefficient theorem for homology, we have the following two short exact sequences:

\[
\begin{align*}
\{0\} \rightarrow H_1(W; \mathbb{Z}) \otimes \mathbb{Z}_p &\rightarrow H_1(W; \mathbb{Z}_p) \rightarrow \text{Tor}(\mathbb{Z}, \mathbb{Z}_p) \rightarrow \{0\}, \\
\{0\} \rightarrow H_2(W; \mathbb{Z}) \otimes \mathbb{Z}_p &\rightarrow H_2(W; \mathbb{Z}_p) \rightarrow \text{Tor}(H_1(W; \mathbb{Z}), \mathbb{Z}_p) \rightarrow \{0\},
\end{align*}
\]
where \( \text{Tor}(G_1, G_2) \) is the torsion product of abelian groups \( G_1 \) and \( G_2 \). Now we have \( \text{Tor}(\mathbb{Z}, \mathbb{Z}_p) \cong \text{Tor}(H_1(W; \mathbb{Z}), \mathbb{Z}_p) \cong \{0\} \). This completes the proof. \( \square \)

**Proof of Theorem 3.1** Let \( L = K_1 \cup K_2 \cup \cdots \cup K_{n+1} \) (resp. \( L' = K'_1 \cup K'_2 \cup \cdots \cup K'_{n+1} \)) be an oriented \((n+1)\)-component link in \( S^3 \) with \( K_{n+1} \) (resp. \( K'_{n+1} \)) is trivial and linking numbers of \( K_{n+1} \) and \( K_1 \) (resp. \( K'_{n+1} \) and \( K'_1 \)) are even for \( i = 1, 2, \ldots, n \). Suppose that \( L \) and \( L' \) are cobordant, i.e., there is a disjoint union of annuli \( A_1, A_2, \ldots, A_{n+1} \) in \( S^3 \times [0, 1] \) such that each boundary \( \partial A_i = K_i \cup (-K'_i) \) for \( i = 1, 2, \ldots, n+1 \).

Let \( W \) be the double branched cover of \( S^3 \times [0, 1] \) branched over \( A_{n+1} \). Fix \( \varepsilon_2, \ldots, \varepsilon_n \), we may assume that the covering links \( L(0 \varepsilon_2 \cdots \varepsilon_n) \) and \( L'(0 \varepsilon_2 \cdots \varepsilon_n) \) bound a disjoint union \( \tilde{A} \) of annuli in \( W \) that are components of the preimage of \( A_1 \cup A_2 \cup \cdots \cup A_n \). We note that \( 4W = \Sigma(K_{n+1}) \cup (-\Sigma(K'_{n+1})) = S^3 \cup (-S^3) \). We may assume that \( K_{n+1} \) is in the boundary of the 4-ball \( B^4 \) and bounds a properly embedded 2-disk in \( B^4 \) which is obtained from a disk in \( \partial B^4 \) bounded by \( K_{n+1} \) by pushing it into \( B^4 \). Let \( \Sigma \) be the double branched cover of \( B^4 \) branched over the properly embedded disk. Let \( N(\tilde{A}) \) be a regular neighborhood of \( \tilde{A} \), and \( D = \Sigma \cup N(\tilde{A}) \). Let \( E \) be the closure of \( W \setminus N(\tilde{A}) \). Then we have \( D \cup E = W \cup \Sigma \) and note that \( D \cap E \) is homeomorphic to \( \Sigma(K_{n+1}) \cap N(L(0 \varepsilon_2 \cdots \varepsilon_n)) \). Applying Lemma 4.2 in [14] to \( D \cup E \), we have that the first and second homology groups of \( D \cup E \) with rational coefficients are trivial. Hence, by the universal coefficient theorem for homology, we have the first and second betti numbers of \( D \cup E \) are zero. Lemma 4.7 therefore implies that there is a prime number \( p \) such that \( H_1(D \cup E; \mathbb{Z}_p) \cong H_2(D \cup E; \mathbb{Z}_p) \cong \{0\} \). By the Mayer-Vietoris exact sequence, we have the following:

\[
\cdots \to H_2(D \cup E; \mathbb{Z}_p) \to H_2(D; \mathbb{Z}_p) \oplus H_2(E; \mathbb{Z}_p) \to \{0\} \to H_1(D \cap E; \mathbb{Z}_p) \to H_1(D; \mathbb{Z}_p) \oplus H_1(E; \mathbb{Z}_p) \to \{0\} \to \cdots.
\]

Since \( D \) is homotopic to a point, we have that \( H_1(D; \mathbb{Z}_p) \cong H_2(D; \mathbb{Z}_p) \cong \{0\} \). Therefore the homomorphism \( H_1(D \cap E; \mathbb{Z}_p) \to H_1(E; \mathbb{Z}_p) \) is a bijection and the homomorphism \( H_2(D \cap E; \mathbb{Z}_p) \to H_2(E; \mathbb{Z}_p) \) is a surjection. A theorem of Stallings [18, 5.1 Theorem] implies that the inclusion map \( D \cap E \to E \) induces the following isomorphism:

\[
\frac{\pi_1(D \cap E)}{(\pi_1(D \cap E))_q} \cong \frac{\pi_1(E)}{(\pi_1(E))_q}
\]

for any natural number \( q \). Hence the inclusion map \( \Sigma(K_{n+1}) \setminus L(0 \varepsilon_2 \cdots \varepsilon_n) \to W \setminus \tilde{A} \) induces the following isomorphism:

\[
\frac{\pi_1(\Sigma(K_{n+1}) \setminus L(0 \varepsilon_2 \cdots \varepsilon_n))}{(\pi_1(\Sigma(K_{n+1}) \setminus L(0 \varepsilon_2 \cdots \varepsilon_n)))_q} \cong \frac{\pi_1(W \setminus \tilde{A})}{(\pi_1(W \setminus \tilde{A}))_q}
\]

Similarly the inclusion map \( \Sigma(K'_n) \setminus L'(0 \varepsilon_2 \cdots \varepsilon_n) \to W \setminus \tilde{A} \) implies that

\[
\frac{\pi_1(\Sigma(K'_n) \setminus L'(0 \varepsilon_2 \cdots \varepsilon_n))}{(\pi_1(\Sigma(K'_n) \setminus L'(0 \varepsilon_2 \cdots \varepsilon_n)))_q} \cong \frac{\pi_1(W \setminus \tilde{A})}{(\pi_1(W \setminus \tilde{A}))_q}
\]

It follows that we have

\[
\frac{\pi_1(\Sigma(K_{n+1}) \setminus L(0 \varepsilon_2 \cdots \varepsilon_n))}{(\pi_1(\Sigma(K_{n+1}) \setminus L(0 \varepsilon_2 \cdots \varepsilon_n)))_q} \cong \frac{\pi_1(\Sigma(K'_n) \setminus L'(0 \varepsilon_2 \cdots \varepsilon_n))}{(\pi_1(\Sigma(K'_n) \setminus L'(0 \varepsilon_2 \cdots \varepsilon_n)))_q}
\]

Since \( L(0 \varepsilon_2 \cdots \varepsilon_n) \) and \( L'(0 \varepsilon_2 \cdots \varepsilon_n) \) bound \( \tilde{A} \), both peripheral structures of them are preserved by the isomorphism above. This implies that \( M_L(I)_p = M_{L'}(I)_p \). We note that the equation hold for any prime number which is greater than \( p \). This and the fact that \( \mu_L(I)_p \) is the modulo-\( p \) reduction of \( \mu_L(I)_0 \) (Remark 2.1 (1)) complete the proof. \( \square \)
4. Milnor invariants and covering Milnor invariants

From now on we denote $\overline{\mu}_L(I)_0$ by $\overline{\mu}_L(I)$ as usual. A link is Brunnian if any proper sublink is trivial.

**Theorem 4.1.** Let $L$ be an oriented ordered $(n+1)$-component Brunnian link in $S^3$ ($n \geq 2$). For a non-repeated sequence $I = i_1 i_2 \cdots i_{n+1}$ with $i_k = n+1$ ($2 \leq k \leq n$),

$$\overline{\mu}_L(I) \equiv \sum_{(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n}) \in \mathcal{E}(I)} \overline{\mu}_L(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{n}) (I \setminus \{n+1\}) \mod 2,$$

where $\mathcal{E}(I) = \{(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{n}) \in \mathbb{Z}_2^n \mid \varepsilon_{i_{k-1}} = \varepsilon_{i_{k+1}} = 0\}$ and $I \setminus \{n+1\}$ is a subsequence of $I$ obtained by deleting $n+1 (= i_k)$.

**Remark 4.2.** There is a 3-component Brunnian link (see Figure 7.1) such that $\overline{\mu}_L(132) = -1$ and $\overline{\mu}_L(00)(12) = 1$. Hence Theorem 4.1 does not hold without taking modulo 2.

**Remark 4.3.** Since the image of $L(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{n}) \setminus K^{1}_{i_{1}}$ is a trivial link that bounds a disjoint union of disks in $S^3 \setminus K_{n+1}$, $L(\varepsilon_1 \varepsilon_2 \cdots \varepsilon_{n})$ is also a Brunnian link in $\Sigma(K_{n+1}) (= S^3)$. Hence we can repeatedly apply Theorem 4.1 to covering links. Then we have that the length $n+1$ Milnor invariants for $L$ is modulo-2 congruent to a sum of linking numbers of ‘iterated’ covering links.

5. Claspers

We use clasper theory introduced by K. Habiro [7] to prove Theorem 4.1. In this section, we briefly recall from [7] the basic notions of clasper theory. In this paper, we essentially only need the notion of $C_k$-tree. For a general definition of claspers, we refer the reader to [7]. Let $L$ be a link in $S^3$.

**Definition 5.1.** An embedded disk $D$ in $S^3$ is called a tree clasper for $L$ if it satisfies the following (1), (2) and (3): (1) $D$ is decomposed into disks and bands, called edges, each of which connects two distinct disks.

(2) The disks have either 1 or 3 incident edges, called disk-leaves or nodes respectively.

(3) $D$ intersects $T$ transversely and the intersections are contained in the union of the interior of the disk-leaves.

The degree of a tree clasper is the number of the disk-leaves minus 1. (In [7], a tree clasper is called a strict tree clasper.) A degree $k$ tree clasper is called a $C_k$-tree. A $C_k$-tree is simple if each disk-leaf intersects $L$ at one point.

We will make use of the drawing convention for claspers of Fig. 4 in [7]. Given a $C_k$-tree $T$ for $L$, there is a procedure to construct a framed link $\gamma(T)$ in a regular neighborhood of $T$. Surgery along $T$ means surgery along $\gamma(T)$. Since surgery along $\gamma(T)$ preserves the ambient space, surgery along the $C_k$-tree $T$ can be regarded as a local move on $L$ in $S^3$. We will denote by $LT$ the link in $S^3$ which is obtained from $L$ by surgery along $T$. Similarly, for a disjoint union of tree claspers $T_1 \cup T_2 \cup \cdots \cup T_m$, we can define $L_{T_1 \cup T_2 \cup \cdots \cup T_m}$. A $C_k$-tree $T$ having the shape of tree clasper like Figure 5.1 is called a linear $C_k$-tree, and the leftmost and rightmost disk-leaves of $T$ are called the ends of $T$. In particular, surgery along a simple linear $C_k$-tree for $L$ is ambient isotopic to a band summing of $L$ and the $(k+1)$-component Milnor link (Fig. 7 in [12]), see Figure 5.1.

**Definition 5.2.** A simple $C_k$-tree $T$ for $L = K_1 \cup K_2 \cup \cdots \cup K_n$ is a $C_k^L$-tree if $\{i \mid |K_i \cap T \neq \emptyset, i = 1, 2, \ldots, n\} = k+1$. Let $T$ be a linear $C_k^L$-tree with the ends $f_0$ and $f_k$. Since $T$ is a disk, we can travel from $f_0$ to $f_k$ along the boundary of
T so that we meet all other disk-leaves $f_1, \ldots, f_{k-1}$ in this order. If $f_s$ intersects the $i_s$th component of $L$ ($s = 0, \ldots, k$), we can consider two vectors $(i_0, \ldots, i_k)$ and $(i_k, \ldots, i_0)$, and may assume that $(i_0, \ldots, i_k) \leq (i_k, \ldots, i_0)$, where ‘$\leq$’ is the lexicographic order in $\mathbb{Z}^{k+1}$. We call $(i_0, \ldots, i_k)$ the o-index of $T$.

The following theorem is essentially shown by Milnor [12].

**Theorem 5.3.** [12, Section 5] Let $O = O_1 \cup O_2 \cup \cdots \cup O_{n+1}$ be an oriented $(n+1)$-component trivial link in $S^3$, and $T$ a linear $C_n^d$-tree for $O$ with the o-index $(i_1, i_2, \ldots, i_{n+1})$. Then all Milnor invariants of $OT$ with the length $\leq n$ vanish and for a non-repeated sequence $i_1, j_2, \ldots, j_n, i_{n+1}$ of $\{1, 2, \ldots, n+1\}$

$$|\mu_{O_T}(i_1, j_2, \ldots, j_n, i_{n+1})| = \begin{cases} 1 & \text{if } j_2, j_3, \ldots, j_n = i_2, i_3, \ldots, i_n, \\
0 & \text{otherwise}. \end{cases}$$

The following lemma is shown similarly to Lemma 2.9 in [11].

**Lemma 5.4.** Let $L$ be a link, and let $T_I$, $T_H$ and $T_X$ be $C_k$-trees for $L$ which differ only in a small ball as illustrated in Figure 5.2. Then there are $C_k$-trees $T'_H$ and $T'_X$ such that $L \cup T'_H \cup T'_X$ is ambient isotopic to $L \cup T_H \cup T_X$ by changing crossings among edges of the claspers and $L$.

![Figure 5.2. The IHX relation for $C_k$-trees](image)

6. **Proof of Theorem**

Let $L = K_1 \cup K_2 \cup \cdots \cup K_{n+1}$ be an oriented $(n+1)$-component Brunnian link in $S^3$. Since for a non-repeated sequence $I = i_1, i_2, \ldots, i_{n+1}$ in Theorem 4.1 the essential condition is that $n+1$ is neither $i_1$ nor $i_{n+1}$, we may assume that $i_1 = 1$ and $i_{n+1} = 2$ to avoid complicated arguments in this section.

Let $\Sigma(K_{n+1})$ be the double branched cover of $S^3$ branched over $K_{n+1}$. It is shown in [14], [8] that there is a disjoint union of linear $C_n^d$-trees $T_1 \cup T_2 \cup \cdots \cup T_m$ for an $(n+1)$-component trivial link $O = O_1 \cup O_2 \cup \cdots \cup O_{n+1}$ such that $L$ and $OT_1 \cup T_2 \cup \cdots \cup T_m$ are ambient isotopic. Consider induction on the length of the path connecting two disk-leaves of each $T_r$ grasping $O_1$ and $O_2$, Lemma 5.4 implies the following.
Proposition 6.1. Let $L$ be an $(n+1)$-component Brunnian link in $S^3$. There is a disjoint union of linear $C^a_n$-trees $T_1 \cup T_2 \cup \cdots \cup T_m$ for an $(n+1)$-component trivial link $O = O_1 \cup O_2 \cup \cdots \cup O_{n+1}$ such that $L$ and $O_T \cup O_1 \cup O_2$ are ambient isotopic and that the ends of $T_r$ grasp $O_1$ and $O_2$ for each $r(=1,2,\ldots,m)$.

We identify $K_{n+1}$ and $O_{n+1}$. In the following, we will observe that the preimage of $K_1 \cup K_2 \cup \cdots \cup K_n$ is obtained from the preimage of $O_1 \cup O_2 \cup \cdots \cup O_n$ by surgery along claspers. There is a disjoint union of 3-balls $B_1 \cup B_2 \cup \cdots \cup B_m$ in $S^3$ such that $B_r \cap (T_1 \cup T_2 \cup \cdots \cup T_m) = T_r$ and $B_r \cap O$ is a trivial tangle. Let $\alpha = B_r \cap O_{n+1}$, see Figure 6.1 (a). We consider the double branched cover of $B_r$ branched over $\alpha$. By using move 9 in [7], we decompose $T_r$ into two tree claspers $T_{r+1}$ and $T_{r+2}$ (see Figure 6.1 (b)), where $T_{r+2}$ intersects $O_j$ ($j = 1,2$). Recall that the ends of $T_r$ grasp $O_1$ and $O_2$. Set $O_{j_1}^\delta$ (resp. $O_{j_1}^{\delta+1}$) be a preimage of $O_j$ (resp. $O_j$) for $s = 1,2,\ldots,n-2, \delta(= \varepsilon_{j_1}) \in \mathbb{Z}_2$ (resp. for $j = 1,2,\varepsilon_j \in \mathbb{Z}_2$). By using genus-1 surface $F$ as illustrated in Figure 6.1 (b), we have a surgery description of the double branched cover as illustrated in Figure 6.2 [1]. Let $T_{r+1}$ (resp. $T_{r+2}$) be a preimage of $T_{r+1}$ with $T_{r+1} \cap O_{j+1}^\delta \neq \emptyset$ (resp. $T_{r+2} \cap O_{j+1}^{\delta+1} \neq \emptyset$) for $\varepsilon \in \{0,1\}$. Then we have a new clasper $G_r$ with two boxes, see Figure 6.3. Here a box, given in [7], is a disk with 3 incident edges which is obtained by replacing a disjoint union of 3 disk-leaves as illustrated in Figure 6.4.

The following lemma is shown by Habiro [7 Proposition 3.4].

Lemma 6.2. [7 Proposition 3.4] Let $T$ be a tree clasper for a link $L$ such that $T$ has a disk-leaf without intersecting $L$. Then $L_T$ is ambient isotopic to $L$. 

![Figure 6.1](image1)

Figure 6.1. A surgery description of the double branched cover
Lemma 6.3. Let $T_r$ be a linear $C^d_n$-tree for $O = O_1 \cup O_2 \cup \cdots \cup O_{n+1}$ with the $o$-index $(1, j_2, \ldots, j_n, 2)$. Then for a sequence $I = i_2 \epsilon_2, \ldots, i_n 2$ with $i_k = n + 1$, the following holds:

$$
\sum_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in E(I)} |\bigcap_{O_{1}^{j_{1}} \cup O_{2}^{j_{2}} \cup \cdots \cup O_{n}^{j_{n}})(I \backslash \{n+1\})| = \begin{cases} 1 & \text{if } j_2 \cdots j_n 2 = I, \\ 0 & \text{otherwise.} \end{cases}
$$

Proof. Suppose that $1j_2 \cdots j_n 2 = I$. If $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in E(I)$, then $\epsilon_{k-1} = \epsilon_{k+1} = 0$, and hence by Lemma 6.2, $(O_{1}^{j_{1}} \cup O_{2}^{j_{2}} \cup \cdots \cup O_{n}^{j_{n}})_{G_r}$ is ambient isotopic to $(O_{1}^{j_{1}} \cup O_{2}^{j_{2}} \cup \cdots \cup O_{n}^{j_{n}})_{G_r \setminus (T_{r_{1}} \cup T_{r_{2}})}$. Applying move 2 in [7] to $G_r \setminus (T_{r_{1}} \cup T_{r_{2}})$, we have a linear $C_{n-1}^{d}$-tree $\overline{T_r}$ for $O_{1}^{j_{1}} \cup O_{2}^{j_{2}} \cup \cdots \cup O_{n}^{j_{n}}$. If there is an index $i \in \{1, 2, \ldots, n\}$ such that $O_{i}^{j_{i}} \cap \overline{T_r} = \emptyset$, then $(O_{1}^{j_{1}} \cup O_{2}^{j_{2}} \cup \cdots \cup O_{n}^{j_{n}})_{\overline{T_r}}$ is a trivial link by Lemma 6.2.

Since there is a unique choice $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in E(I)$ such that $O_{i}^{j_{i}} \cap \overline{T_r} \neq \emptyset$ for any $i$, and since the $o$-index of $\overline{T_r}$ is equal to $(1, i_2, \ldots, i_{k-1}, i_{k+1}, \ldots, i_n, 2)$, then by Theorem 5.3 we have that

$$
\sum_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in E(I)} |\bigcap_{O_{1}^{j_{1}} \cup O_{2}^{j_{2}} \cup \cdots \cup O_{n}^{j_{n}})(I \backslash \{n+1\})| = 1.
$$

Suppose that $1j_2 \cdots j_n 2 \neq I$ and $1j_2 \cdots j_n 2 \{n+1\} = I \{n+1\}$. If $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in E(I)$, then $G_r \cap (O_{k-1}^{0} \cup O_{k+1}^{0})$ is contained in $T_{r_{1}}^{0} \cup T_{r_{1}}^{1}$ or $T_{r_{2}}^{0} \cup T_{r_{2}}^{1}$. Without loss of generality, we may assume that $T_{r_{1}}^{0} \cup T_{r_{1}}^{1}$ contains $G_r \cap (O_{k-1}^{0} \cup O_{k+1}^{0})$. If both $T_{r_{1}}^{0}$ and $T_{r_{1}}^{1}$ intersect $O_{k-1}^{0} \cup O_{k+1}^{0}$, then by Lemma 6.2, $(O_{1}^{j_{1}} \cup O_{2}^{j_{2}} \cup \cdots \cup O_{n}^{j_{n}})_{G_r}$ is trivial. It follows that

$$
\sum_{(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in E(I)} |\bigcap_{O_{1}^{j_{1}} \cup O_{2}^{j_{2}} \cup \cdots \cup O_{n}^{j_{n}})(I \backslash \{n+1\})| = 0.
$$

Suppose that either $T_{r_{1}}^{0}$ or $T_{r_{1}}^{1}$ contains $G_r \cap (O_{k-1}^{0} \cup O_{k+1}^{0})$. Here we may assume that $G_r \cap (O_{k-1}^{0} \cup O_{k+1}^{0}) \subset T_{r_{1}}^{1}$. Then $(O_{1}^{j_{1}} \cup O_{2}^{j_{2}} \cup \cdots \cup O_{n}^{j_{n}})_{G_r}$ is ambient isotopic to $(O_{1}^{j_{1}} \cup O_{2}^{j_{2}} \cup \cdots \cup O_{n}^{j_{n}})_{G_r \setminus T_{r_{1}}^{1}}$. We note that by Lemma 6.2, there are exactly two possibilities $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) \in E(I)$ such that $(O_{1}^{j_{1}} \cup O_{2}^{j_{2}} \cup \cdots \cup O_{n}^{j_{n}})_{G_r \setminus T_{r_{1}}^{1}}$ is
a nontrivial link. These two choice give us \((O^1_r \cup O^2_r \cup \cdots \cup O^m_r)_{G_r \setminus T_r}^\prime\) which is ambient isotopic to either \((O^1_r \cup O^2_r \cup \cdots \cup O^m_r)_{G_r \setminus \{T^1_r \cup T^2_r\}}\) or \((O^1_r \cup O^2_r \cup \cdots \cup O^m_r)_{G_r \setminus \{T^1_r \cup T^2_r\}}\). Applying move 2 in \([7]\) to \(G_r \setminus \{T^1_r \cup T^2_r\}\) for \(\varepsilon \in \{0, 1\}\), we obtain two linear \(G_{n-1}\)-trees with the o-indexes \((1, i_2, \ldots, i_k, 1, i_{k+1}, \ldots, i_n, 2)\). Hence Theorem 6.3 implies that

\[
\sum_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in E(I)} |\delta(O^1_r \cup O^2_r \cup \cdots \cup O^m_r)_{G_r}(I \setminus \{n + 1\})| = 2.
\]

Suppose that \(1j_2 \cdots j_n 2 \setminus \{n + 1\} \neq I \setminus \{n + 1\}\). Let \((\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in E(I)\) be a choice such that \((O^1_r \cup O^2_r \cup \cdots \cup O^m_r)_{G_r}^\prime\) is nontrivial, that is, by Lemma 6.2 it is ambient isotopic to \((O^1_r \cup O^2_r \cup \cdots \cup O^m_r)_{G_r \setminus \{T^1_r \cup T^2_r\}}\) for some \(\varepsilon, \varepsilon' \in \mathbb{Z}_2\). Applying move 2 in \([7]\) to \(G_r \setminus \{T^1_r \cup T^2_r\}\), we obtain a linear \(G_{n-1}\)-tree whose the o-index is not equal to \((1, i_2, \ldots, i_k, 1, i_{k+1}, \ldots, i_n, 2)\). Hence Theorem 6.3 implies that

\[
\sum_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in E(I)} |\delta(O^1_r \cup O^2_r \cup \cdots \cup O^m_r)_{G_r}(I \setminus \{n + 1\})| = 0.
\]

\[\square\]

**Proof of Theorem 4.1** It is shown in \([3]\) that the length-\(l\) Milnor invariants are additive under the band sum operation if all Milnor invariants with the length \(\leq l - 1\) vanish. The boundary \(\partial B_r\) of the regular neighborhood \(B_r\) of \(T_r\) can be assumed to be decomposing sphere for a band sum operation between \(O_{T_r}\) and \(O_{(T_1 \cup T_2 \cup \cdots \cup T_m) \setminus T_r}\). Hence we may assume that \(L\) is a band sum of \(m + 1\) links \(O_{T_1}, O_{T_2}, \ldots, O_{T_m}\) and \(O\). Since \(O_{T_r}\) is an \((n + 1)\)-component Brunnian link for each \(r(= 1, 2, \ldots, m)\), all Milnor invariants with the length \(\leq n\) vanish. Therefore we have that

\[
\overline{\mu}_L(I) = \sum_{r=1}^{m} \overline{\mu}_{O_{T_r}}(I).
\]

By combining Theorem 6.3 and Lemma 6.3 we have

\[
\sum_{r=1}^{m} \overline{\mu}_{O_{T_r}}(I) \equiv \sum_{r=1}^{m} \sum_{(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in E(I)} |\delta(O^1_r \cup O^2_r \cup \cdots \cup O^m_r)_{G_r}(I \setminus \{n + 1\})| \mod 2.
\]

Since \(L(1\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)\) is also Brunnian, we have that

\[
\overline{\mu}_L(I \setminus \{n + 1\}) = \sum_{r=1}^{m} \delta(O^1_r \cup O^2_r \cup \cdots \cup O^m_r)_{G_r}(I \setminus \{n + 1\})).
\]

This completes the proof. \[\square\]

**7. Link-homotopy**

Link-homotopy is an equivalence relation generated by crossing changes on the same component of links. It is shown in \([12]\) that Milnor invariants for non-repeated sequences are link-homotopy invariants of links. However for covering Milnor invariants the same result does not hold.

**Example 7.1.** Let \(O = O_1 \cup O_2 \cup O_3\) be an oriented 3-component trivial link in \(S^3\). Let \(L\) be a link which is obtained from \(O\) by surgery along a linear \(G_{n-1}\)-tree \(T_1\) as illustrated in Figure 7.1(a), and \(L'\) a link which is obtained from \(O\) by surgery along a disjoint union of \(T_1\) and a simple linear \(C_3\)-tree \(T_2\) as illustrated in Figure 7.2(a). It is not hard to see that \(L\) and \(L'\) are link-homotopic.

Since the both \(L\) and \(L'\) are Brunnian links, we have the double branched cover of \(S^3\) branched over \(O_3\) which is a component of \(L\) (resp. \(L'\)). Moreover we have surgery descriptions as illustrated in Figure 7.3(b) and Figure 7.2(b). Then
covering links $L(00)$ and $L(01)$ of $L$ are links as illustrated in Figure 7.3 and hence we have $\overline{\mu}_L(00)(12) = 1, \overline{\mu}_L(01)(12) = -1$. We conclude that $M_L(12) = \{1, -1\}$. On the other hand, covering links $L'(00)$ and $L'(01)$ of $L'$ are links as illustrated in Figure 7.4. Since $\overline{\mu}_{L'}(00)(12) = 3, \overline{\mu}_{L'}(01)(12) = -3$, we have that $M_L'(12) = \{3, -3\}$. Therefore $L$ and $L'$ are link-homotopic, and $M_L(12) \neq M_{L'}(12)$.

Remark 7.2. Since $L$ and $L'$ are Brunnian links and they are link-homotopic, $\overline{\mu}_L(I) = \overline{\mu}_{L'}(I)$ for any sequence $I$ with the length at most 3. We note that $L$ is the Borromean rings. Hence $|\overline{\mu}_L(123)| = |\overline{\mu}_L(132)| = 1$. It follows that for any sequence $I$ with the length at least 4, $\Delta_L(I)_0 = \Delta_{L'}(I)_0 = \mathbb{Z}$ if each $i \in \{1, 2, 3\}$ appears in $I$, or $\mu_L(I) = \mu_{L'}(I) = 0$ otherwise. In both cases, we have $\overline{\mu}_L(I) = \overline{\mu}_{L'}(I) = 0$. This implies that $L$ and $L'$ have the same ordinary Milnor invariants, and different covering Milnor invariants.

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