Orbital Evolution of Gas-driven Inspirals with Extreme Mass Ratios: Retrograde Eccentric Orbits

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Abstract

Using two-dimensional simulations, we compute the torque and rate of work (power) on a low-mass gravitational body, with softening length $R_{\text{soft}}$, embedded in a gaseous disk when its orbit is eccentric and retrograde with respect to the disk. We explore orbital eccentricities $e$ between 0 and 0.6. We find that the power has its maximum at $e \simeq 0.25(h/0.05)^{-3/2}$, where $h$ is the aspect ratio of the disk. We show that the power and the torque converge to the values predicted in the local (nonresonant) approximation of the dynamical friction (DF) when $R_{\text{soft}}$ tends to zero. For retrograde inspirals with mass ratios $\lesssim 5 \times 10^{-4}$ embedded in disks with $h \gtrsim 0.025$, our simulations suggest that (i) the rate of inspiral barely depends on the orbital eccentricity and (ii) the local approximation provides the value of this inspiral rate within a factor of 1.5. The implications of the results for the orbital evolution of extreme mass ratio inspirals are discussed.

Unified Astronomy Thesaurus concepts: Active galactic nuclei (16); Supermassive black holes (1663); Gravitational interaction (669); Dynamical friction (422); Tidal interaction (1699); Hydrodynamical simulations (767); Galaxy accretion disks (562); Compact objects (288)

1. Introduction

At the center of galaxies, stars can drain into the central supermassive black hole (SMBH) owing to two-body diffusion, resonant relaxation, and dynamical friction (DF) with the surrounding material (mainly dark matter and gas; e.g., Hopman & Alexander 2006). Compact objects (COs), such as stellar remnants and stellar-mass BHs, can inspiral into an SMBH and emit gravitational waves, which could be detected by the Laser Interferometer Space Antenna (LISA; e.g., Finn & Thorne 2000; Amaro-Seoane et al. 2017).

In the presence of accretion disks such as those in active galactic nuclei (AGNs), stars and COs can experience gravitational torques that can accelerate the radial migration toward the center (e.g., Armitage & Natarajan 2002; Kocsis et al. 2011). COs may belong to the nuclear cluster (McKernan et al. 2011) or may have formed inside the AGN star-forming disk (e.g., Levin 2007).

Nuclear cluster COs may have prograde as well as retrograde orbits with respect to the AGN accretion disk. COs born in the star-forming disk are expected to move on prograde orbits. Still, gravitational scattering between them or with other objects (including an SMBH binary companion) may excite large orbital eccentricities (e.g., Papaloizou & Terquem 2001; Breslau & Pfalzner 2019). In principle, it is plausible that some of the estimated $10^3$ BHs of mass (7–10)$M_\odot$ that reside within 0.1 pc of the central BH may be scattered to retrograde eccentric orbits and can even counterrotate with respect to the accretion disk.

Disk COs may counterrotate with respect to the AGN accretion disk if the AGN disk is rejuvenated with captured gas clouds having uncorrelated angular momentum (Imanishi et al. 2018; Impellizzeri et al. 2019), as occurs at galactic scales in some galaxies (e.g., García-Burillo et al. 2003; Corsini 2014; Martinsson et al. 2018).

COs and intermediate-mass BHs can also rotate with high inclinations, if they are brought to the galactic center anchored in an inclined stellar cluster (Portegies Zwart et al. 2003, 2006; Kim & Morris 2003; Gurkan & Rasio 2005; Antonini et al. 2012; Antonini 2014; Arca-Sedda & Gualandris 2018). After the stellar cluster is destroyed by tidal forces, all the COs and intermediate-mass BHs residing in the stellar cluster will be spread out in inclined orbits.

The evolution of the semimajor axis $a$, the eccentricity $e$, and the inclination $i$ of a perturber due to the tidal interaction with the disk has been studied intensively in the context of protoplanetary disks (e.g., Artymowicz 1993; Papaloizou & Larwood 2000; Goldreich & Sari 2003; Tanaka & Ward 2004; Cresswell et al. 2007; Marzari & Nelson 2009; Bitsch & Kley 2010, 2011; Bitsch et al. 2013). For $e$ or $i$ larger than the aspect ratio of the disk $h \equiv H/R$ (where $H$ is the scale height of the disk at radius $R$), the perturber moves supersonically with respect to the gas. For that reason, a DF approach has been invoked to describe the interaction between the disk and a body in inclined or eccentric orbits (e.g., Papaloizou 2002; Muto et al. 2011; Reif 2012; Amaro-Seoane et al. 2016). In Sánchez-Salcedo (2019), we find that a simple model based on DF describes the orbital evolution of bodies in coplanar ($i = 0$) eccentric orbits ($h < e \lesssim 0.6$), provided that the ratio between the mass of the perturber and the mass of the central object (denoted by $q$) is sufficiently small. For typical protoplanetary disks, this occurs for planets with $q \lesssim 10^{-3}$.

For highly inclined circular orbits, Rein (2012) considers the aerodynamical and gravitational drag forces on a planet when it crosses the protoplanetary disk. For orbits with $i = 45^\circ$, 90°, and 155°, he finds good agreement between the gravitational drag force measured in numerical simulations and the force predicted using a formula based on DF arguments. Xiang-Guess & Papaloizou (2013) carry out a set of numerical simulations of the orbital evolution of a gravitational perturber in a circular orbit, for the full range of inclinations. They argue that the qualitative behavior of the results can be interpreted using a simple formula based on DF.

The limiting case $i = 180^\circ$ (retrograde orbit) and $e = 0$ (circular orbit) was studied in Ivanov et al. (2015) and Sánchez-Salcedo et al. (2018). In this case, the perturber moves supersonically
(Mach numbers of ≈40–100). As a result, the perturber catches its own wake repeatedly, and, in fact, the pull imparted by the wake ahead of the perturber cannot be ignored unless the mass ratio $q$ is small enough (Sánchez-Salcedo et al. 2018).

In the general case (arbitrary values of $i$ and $e$, but larger than $h$), one expects that if $q$ is small enough, most of the contribution to the drag force arises from the portion of the wake just at the rear of the body, and therefore the DF approximation should be valid to quantify the components of the drag force and thereby the evolution of $a$, $e$, and $i$.

In order to complete our picture on the applicability and limitations of an approach based on DF, which is impulsive and nonresonant, we use numerical simulations to evaluate the components of the drag force when the orbit is retrograde ($i = 180^\circ$) and eccentric. Interestingly, for certain disk parameters typical for AGN disks, the DF formula predicts that the eccentricity may grow. This stands in sharp contrast to the rapid eccentricity damping seen in the prograde case. We wish to quantify to what extent the predictions based on DF considerations are reliable.

The structure of the paper is as follows. In Section 2, we describe our system and provide the basic equations. In Section 3, we present the DF framework in its local approximation (hereafter LA) and make some predictions. A comparison between predictions and the results of hydrodynamical simulations is given in Section 4. The implications for the evolution of COs embedded in AGN disks are discussed in Section 5. Finally, we summarize our conclusions in Section 6.

2. Description of the Model: Basic Equations

Our system consists of an accretion disk around a central SMBH with mass $M_* \approx 10^5$–$10^7 M_{\odot}$, plus a CO (e.g., a stellar BH), the perturber, with mass $M_p \approx 1$–$10 M_{\odot}$. Therefore, the mass ratio $q \equiv m/M_\odot$ is between 0.01 and 10. These systems are referred to as extreme mass ratio inspirals (EMRIs). The mass of the disk is assumed to be much smaller than $M_*$. The orbital plane of the CO is taken to be coplanar with the disk. The orbit can be prograde or retrograde.

The CO will exchange energy and angular momentum with the disk through the tidal interaction. As a consequence, the semimajor axis $a$ and the eccentricity $e$ of the CO will change with time. Let $P$ denote the power, i.e., the energy change of the CO per unit of time, and $T$ denote the torque imparted on the CO. The evolution equations for $a$ and $e$ are given by

$$\frac{da}{dt} = \frac{2P}{a\omega^2 M_p},$$

and

$$\frac{de}{dt} = \frac{\eta B}{ea^2\omega^2 M_p},$$

where $\eta \equiv \sqrt{1 - e^2}$, $\omega = \sqrt{GM_*/a^3}$, and $B \equiv P - \omega e^{-1}T$ (e.g., Murray & Dermott 1999). The bar over a variable denotes orbit-averaged values. In these equations, we have applied the sign convention that the torque is positive (negative) when the CO gains (loses) angular momentum.

Figure 1. Local Mach number $M$ vs. true anomaly $f$ along the Keplerian orbit described by a body in the midplane of a locally isothermal disk with aspect ratio $h = 0.05$, for prograde (lower curves) and retrograde motion (upper curves).

The migration timescale $t_a$, in units of the orbital period $t_{\text{orb}} \equiv 2\pi/\omega$, is

$$\frac{t_a}{t_{\text{orb}}} \equiv \left| \frac{a}{\frac{d}{dt}} \right| = \frac{a^2\omega^2 M_p}{4\pi|P|}.$$  \hspace{1cm} (3)

The orbital eccentricity changes on the timescale

$$\frac{e}{t_{\text{orb}}} \equiv \left| \frac{e}{\frac{d}{dt}} \right| = \frac{e^2a^2\omega^2 M_p}{2\pi|P|}.$$  \hspace{1cm} (4)

By their definitions, the timescales $t_a$ and $t_e$ are always positive. We anticipate that the eccentricity may be damped or excited, depending on the disk parameters. Therefore, we will give $t_e$ and specify the sign of $\dot{e}$.

The response of the disk to the presence of the CO depends on the relative velocity between the CO and the disk. We define the Mach number $M$ as the ratio $V_{\text{rel}}/c_s$, where $V_{\text{rel}}$ is the velocity of the perturber relative to the local gas and $c_s$ is the local sound speed. Figure 1 shows $M$ as a function of the true anomaly $f$ (the pericenter is at $f = 0$, and the apocenter is at $f = \pi$). We have assumed that the CO describes an elliptical orbit and the disk aspect ratio is constant $(h = 0.05)$, so that the isothermal sound speed $c_s = h\Omega/R_p$, where $\Omega$ is the Keplerian angular velocity $\Omega(R) = \sqrt{GM_*/R^3}$. From Figure 1, we see that the motion for retrograde orbits is always supersonic regardless of the value of $e$. A difference between prograde and retrograde rotation is the orbital position where $M$ achieves its maximum value. For retrograde orbits, the maximum of $M$ occurs at pericenter, whereas it occurs at $f \approx 2$ and 4.5 for prograde orbits. Another difference is that the orbital average $M$ increases with $e$ for prograde orbits, whereas it is essentially independent of $e$ for retrograde orbits.

Given their low $q$ and high $M$, EMRIs in retrograde orbits cannot open a gap in the disk (McKernan et al. 2014; Ivanov et al. 2015; Sánchez-Salcedo et al. 2018) and, in addition, their accretion radii $R_{\text{acc}}$ are generally much smaller than the vertical scale height $H$ of the disk. For instance, consider a retrograde EMRI at a radial distance $R_p$ embedded in a disk with constant
3. The Local Approximation in 3D Disks

In the LA, we apply the DF formula at every point of the orbit, ignoring the curvature of the spiral wave behind the body. In Sánchez-Salcedo et al. (2018), we have studied the range of validity of the LA for perturbers in retrograde and circular orbit. On the other hand, the case of prograde and eccentric orbits was presented in Sánchez-Salcedo (2019). These studies demonstrate that the LA can predict the power and the torque provided that \( q \) is small enough. We note that in the retrograde circular case there are no Lindblad resonances, but they appear when the orbit is eccentric (Ivanov et al. 2015; Nixon & Lubow 2015).

If the LA were also valid for retrograde and eccentric orbits, then it would be easy to find \( t_p \) and \( t_e \) as follows. The force \( F_{LA}^{(3D)} \) acting on a perfect accretor in the LA is

\[
F_{LA}^{(3D)} = \frac{\sqrt{8\pi} \Sigma_p (GM_p)^2 \ln \Lambda_p}{V_{rel}^3 H_p} \mathbf{V}_{rel}. \tag{5}
\]

where the subscript \( p \) indicates evaluation of the variable at the location of the perturber (Cantó et al. 2013; Sánchez-Salcedo et al. 2018). Here \( \Sigma \) is the unperturbed surface density of the disk, \( H \) is its vertical scale height, and \( \Lambda = 7.15H/R_{acc} \). In the derivation of Equation (5), it was assumed that the disk volume density is \( \rho(R, z) = \rho_0(R) \exp(-z^2/2H^2) \). The superscript 3D denotes that the 3D structure of the disk has been included.

We now assume that the equatorial plane of the disk is at \( z = 0 \) and that it rotates counterclockwise in a Keplerian fashion (we ignore deviations from the Keplerian rotation arising from the pressure gradient). The unperturbed velocity of the gas is \( \mathbf{v}_g = R \Omega \mathbf{e}_\phi \), and the relative velocity is \( \mathbf{V}_{rel} = \mathbf{v}_g - \mathbf{v}_p \), where

\[
\mathbf{v}_p = \frac{p \mathbf{a} \omega}{\eta} (\sin \phi \mathbf{e}_R + (1 + e \cos \phi) \mathbf{e}_\phi), \tag{6}
\]

and \( p = +1 \) for prograde orbits and \( p = -1 \) for retrograde orbits (recall that \( \eta \equiv (1 - e^2)^{-1/2} \)). We have assumed that the pericenter is at \( \phi = 0 \). Although we are mainly interested in the retrograde case, we give the expressions for both prograde and retrograde cases to highlight the differences.

Using Equations (5) and (6), the power and the torque are given by

\[
P_{LA}^{(3D)} = \mathbf{p}_p \cdot F_{LA}^{(3D)} = \frac{\sqrt{8\pi} \rho \eta q^2 \omega^2 a^3 \Sigma_p \ln \Lambda_p}{H_p} \times \frac{-p e^2 \sin^2 \phi + \xi \hat{\xi}}{[e^2 \sin^2 \phi + \xi^2]^{3/2}} \tag{7}
\]

and

\[
T_{LA}^{(3D)} = p \mathbf{e}_\phi \cdot (r_p \times F_{LA}^{(3D)}) = \frac{\sqrt{8\pi} \rho \eta q^2 \omega^2 a^3 \Sigma_p \ln \Lambda_p}{H_p} \times \frac{-e^2 \sin^2 \phi + \xi \hat{\xi}}{[e^2 \sin^2 \phi + \xi^2]^{3/2}}, \tag{8}
\]

where \( \xi(\phi) = 1 + e \cos \phi \) and \( \hat{\xi}(\phi) = \sqrt{\xi} - p \xi \). It is simple to show that the power and the torque are both negative at any orbital position if \( p = -1 \).

In the following we consider some disk models that have been used to describe protoplanetary disks and disks around the central BH in AGNs. These models assume that the surface density and the scale height of the disk are given by power laws. We suppose that

\[
\Sigma = \Sigma_a \left( \frac{R}{a} \right)^{-\alpha} \tag{9}
\]

and

\[
H = H_a \left( \frac{R}{a} \right)^{1+\lambda}, \tag{10}
\]

where \( \Sigma_a \) and \( H_a \) are the surface density and the scale height of the disk at \( R = a \), respectively.

Simplified models of the structure of Keplerian viscous disks around SMBHs suggest \( \alpha = 3/2 \) and \( \lambda = 1/2 \) at distances \( >10^3 R_{Sch} \), where \( R_{Sch} \) is the Schwarzschild radius of the central SMBH (e.g., Goodman 2003; Sirko & Goodman 2003). Figure 2 shows the predicted power and torque, in the LA, as a function of the orbital phase \( \phi \) when the EMRI is retrograde and has \( q_{e,0} = 0.1, a = 0.1 \) pc, and \( e = 0.3 \). The remainder of the parameters are \( M_s = 10^5 M_\odot, \Sigma_a = 5 \times 10^6 M_\odot \) pc\(^{-2} \), and \( H_a = 1.4 \times 10^{-3} \) pc. According to Figure 2, the eccentricity is excited at apocenter, because the torque is more negative than the power (\( B > 0 \)). On the contrary, the eccentricity decreases at pericenter (\( B < 0 \)). The orbital average \( de/dt \) is positive (albeit very small: \( \dot{B} = 0.0046 M_\odot \) km\(^2\) s\(^{-1}\)), implying that the eccentricity is excited.

For this model, we have computed how \( t_p \) and \( t_e \) depend on eccentricity (see Figure 3). We find that \( t_p \) and \( t_e \) are almost constant between \( e = 0 \) and \( e = 0.7 \). We note that \( t_p \sim 10 t_s \) for eccentricities in the range \( 0 < e < 0.8 \). Therefore, if the LA is correct, we expect that, in the retrograde case, the migration takes place at almost constant eccentricity. This is in sharp contrast with the prograde case, where the orbit circularizes on a timescale short compared to the migration timescale (typically \( t_e = 0.1 t_s \), e.g., Cresswell & Nelson 2008).

At \( R < 10^3 R_{Sch} \), the models of Sirko & Goodman (2003) predict \( \alpha = -1 \) and \( \lambda = -4/7 \). With the surface density being greater at apocenter, the positive value of \( de/dt \) at apocenter is enhanced in the retrograde case. Figure 3 shows \( t_p \) and \( t_e \) in this part of the disk for \( p = -1, M_s = 10^6 M_\odot, M_p = 10 M_\odot, a = 0.005 \) pc, \( \Sigma_a = 7 \times 10^6 M_\odot \) pc\(^{-2} \), and \( H_a = 8.5 \times 10^{-3} \) pc. We find again that \( \dot{e} > 0 \) (the eccentricity grows), but now \( t_e \sim t_s \).

The eccentricity may be damped for certain combinations of \( \alpha \) and \( \lambda \), if they are sufficiently large. Figure 4 compares the timescales for \( \alpha = 0.6 \) and \( \lambda = 0 \) with those for \( \alpha = 2 \) and \( \lambda = 0.8 \), with the remainder of the parameters \( (p, q, M_s, a, \Sigma_a, H_a) \) being the same. In the first case, \( \dot{e} \) is positive, whereas it is negative for the second set of parameters, but both cases have approximately the same \( t_e \) at \( e < 0.4 \).
It is now clear that the LA provides a very useful framework to predict and in a rather simple way. It is therefore crucial to study its validity domain.

4. Numerical Experiments

The aim of this section is to explore the conditions under which a local description can be used to estimate the tidal forces exerted on a retrograde perturber. Since the LA essentially ignores 2D effects, mainly the differential rotation and the curvature terms (the curvature of the wake and the curvature of perturber orbit), it is sufficient to consider 2D disks. In fact, once the range of validity of the LA is determined in 2D disks, the results can be extended to more realistic 3D disks. This will be done in Section 5.

The response of the disk to the gravitational potential $\Phi_p$ of the perturber (the secondary) is simulated using the code FARGO3D, which is a publicly available code (Benítez-Llambay & Masset 2016). The perturber is placed on a fixed retrograde orbit with eccentricity $e$. We use polar coordinates $(R, \phi)$, where $R$ is measured from the central object.

The potential $\Phi_p$ is modeled by introducing a softening length $R_{\text{soft}}$:

$$\Phi_p = -\frac{GM_p}{\sqrt{(r - r_p)^2 + R_{\text{soft}}^2}},$$

where $r_p$ is the position of the perturber. Strictly, we are not simulating a point-mass particle as a BH, but just an extended nonaccreting body. Nevertheless, it is simple to extend the results to accreting point-mass objects (see Section 5). For

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Figure 2. Predictions using the LA. The left panel shows the power (solid line) and the torque (dashed line), imparted on a CO with orbital eccentricity 0.3, as a function of the orbital phase in a disk with $\alpha = 3/2$ and $\lambda = 1/2$. The right panel shows $B = P - \omega \eta^{-1} T$.

Figure 3. Migration timescale (left panel) and eccentricity growth timescale (right panel) as a function of the orbital eccentricity for a 10$M_\odot$ BH in the outer parts of an AGN disk with a central SMBH of 10$^7M_\odot$ (solid lines) and in the inner parts of an AGN disk with a central SMBH of 10$^8M_\odot$ (dashed lines).
simplicity, we will take $R_{\text{soft}} = \mathcal{E} H$, where $H \equiv c_s / \Omega$ is the vertical scale height of the disk and $\mathcal{E}$ is a constant. We will also assume that the aspect ratio $h$ is constant over $R$; this condition fixes the radial profile of the temperature of the disk. Altogether, $R_{\text{soft}} \propto H \propto R$.

We will use dimensionless power $\mathcal{P}$ and torque $\mathcal{T}$ defined as

$$\mathcal{P} = \frac{\mathcal{E} h}{\pi q^2 \omega^2 \alpha^4 \Sigma_a} \mathcal{P}$$

and

$$\mathcal{T} = \frac{\mathcal{E} h}{\pi q^2 \omega^2 \alpha^4 \Sigma_a} \mathcal{T}.$$  

In terms of dimensionless quantities, the timescales are

$$t_a / t_{\text{orb}} = \frac{\mathcal{E} h}{4 \pi q q_d} \frac{1}{|\mathcal{P}|}$$

and

$$t_e / t_{\text{orb}} = \frac{e^2 \mathcal{E} h}{2 \pi \eta^2 q q_d} \frac{1}{|\mathcal{B}|},$$

with $q_d \equiv \pi \alpha^2 \Sigma_a / M$, and $\mathcal{B} \equiv \mathcal{P} - \eta^{-1} \mathcal{T}$.

In the razor-thin (2D) disk model, the LA predicts the following dimensionless power and torque:

$$\mathcal{P}^{(2D)}_{\text{LA}} = \frac{p \xi^{1+\alpha} (-p e^2 \sin^2 \phi + \xi^2)}{\eta^{1+2\alpha} (e^2 \sin^2 \phi + \xi^2)^{3/2}}$$

and

$$\mathcal{T}^{(2D)}_{\text{LA}} = \frac{p \eta^{2(1-\alpha)} e \xi \xi^2}{(e^2 \sin^2 \phi + \xi^2)^{3/2}}.$$  

Here we have used that the drag force on a body traveling supersonically in a rectilinear orbit inside a 2D layer of surface density $\Sigma$ is

$${\mathcal{P}}^{(2D)}_{\text{LA}} = \frac{\pi \alpha \Sigma G^2 M^2}{R_{\text{soft}} V_{\text{rel}}^3}.$$  

(Muto et al. 2011).

For illustration, Figure 5 shows $\mathcal{P}^{(2D)}_{\text{LA}}$ and $\mathcal{T}^{(2D)}_{\text{LA}}$ as a function of $e$, for $p = +1$ (prograde) and $p = -1$ (retrograde). As expected, $|\mathcal{P}^{(2D)}_{\text{LA}}|$ and $|\mathcal{T}^{(2D)}_{\text{LA}}|$ are smaller in the retrograde case,
especially at low eccentricities. It is remarkable that for \( p = -1 \) and \( \alpha = 0 \), \( \mathcal{P}_{\text{LA}}^{(2D)} \) is almost constant with \( e \).

4.1. Range of Parameters and Other Numerical Issues

We use values for \( q_{-5} \) between 1 and 50. Our reference value for \( h \) is 0.05, but we explore other values in Section 4.3. We vary the eccentricity between 0 and 0.6 and \( \mathcal{E} \) between 0.06 and 0.6. For these parameters, the accretion radius of the perturber is \( 2.5 \times 10^{-4} a \), which is much smaller than \( R_{\text{soft}} = \mathcal{E} h a = (3 \times 10^{-3} - 3 \times 10^{-2}) a \). In all our simulations we include a kinematic viscosity \( \nu = 10^{-5} \omega a^2 \) constant through the disk.

The computational domain extends from \( R_{\text{in}} \) to \( R_{\text{out}} \). Appendix A is devoted to assessing the importance of the finite size of the domain and to describing how the results depend on the boundary conditions. Unless otherwise specified, we employ wave-killing zones at \( R \in [R_{\text{in}}, 1.3 R_{\text{in}}] \) and at \( R \in [0.95 R_{\text{out}}, R_{\text{out}}] \), following the scheme described in de Val-Borro et al. (2006). Boundary effects are more pronounced for larger values of \( \mathcal{E} \). When wave-damping boundary conditions are used, we find that \( R_{\text{in}} = 0.2 a \) and \( R_{\text{out}} = 4 a \) are adequate to compute the power and the torque within 100 orbits even for \( \mathcal{E} = 0.6 \) (see Appendix A). In all the simulations presented in the remainder of the paper, we use \( R_{\text{out}} = 5.2 a \), and we take \( R_{\text{in}} = 0.2 a \) if \( e \leq 0.3 \) and \( R_{\text{in}} = 0.12 a \) if \( e > 0.3 \).

In all the simulations, the perturber is inserted suddenly at \( t = 0 \). In order to partially suppress transient effects during the relaxation process, Appendix B contains the results of simulations in which the mass of the perturber increases slowly over time until it reaches its final mass. In Appendix B, it is shown that those effects associated with relaxation are small.

We have studied the numerical convergence. For instance, for disks having \( h = 0.05 \), we found that the measured power and torque do not change for \( N_\phi \geq 2.5 \) and \( N_{r,\text{peri}} \geq 2.5 \), where

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**Figure 7.** Evolution of the power (left panel) and torque (right panel) using \( \alpha = 0, h = 0.05, q_{-5} = 1 \), and \( \mathcal{E} = 0.6 \). Different curves are for different eccentricities. The value of the eccentricity is given at each curve. The symbols on the right side of each panel indicate the values in the LA for \( e = 0 \) (stars) and \( e = 0.6 \) (squares). For intermediate eccentricities, the LA values lie in between.

**Figure 8.** Mean value of the power between the 5th and 140th orbits for the models in Figure 7. The dashed line indicates the power in the LA.

**Figure 9.** Temporal evolution of the dimensionless torque for various combinations of \( q \) and \( e \). In all cases we take \( \mathcal{E} = 0.6 \). The symbols at the right side indicate the corresponding values in the LA. For \( e = 0.1 \), the LA predicts \( 30\mathcal{T} = -7.59 \), but it is not shown because it is outside the range.
$N_0$ and $N_{r,peri}$ are the number of zones per $R_{soft}$ in the azimuthal and radial directions, respectively. We note that $N_{r,peri}$ is computed at pericenter. In all the simulations presented in this paper, both $N_0$ and $N_{r,peri}$ are larger than 3, typically ~5, to ensure that the resolution is adequate. We were especially careful to ensure that the resolution was enough to resolve the tightly wound density perturbations with small radial wavelength formed in the inner parts of the computational domain owing to the strong Keplerian shear.

4.2. Models with $\alpha = 0$ and $h = 0.05$

In this section we assume that $h = 0.05$ and $\alpha = 0$, i.e., the unperturbed surface density is constant along $R$, so that $\Sigma_{t=0} = \Sigma_0 =$ const. From a numerical point of view, an initial constant surface density reduces spurious reflections in the boundaries and preserves reasonably well the mass in our computational box.

Retrograde perturbers excite tightly wound density waves in the disk (see Figure 6). The perturbers repeatedly catch their own wakes with a frequency $2 \omega$. As a result, the surface density perturbation $\Sigma - \Sigma_0$ is very complex, changing from positive to negative values in the radial direction on a short spatial scale.

4.2.1. Models with $\mathcal{E} = 0.6$

In this section we fix the values of $\alpha$, $h$, and $\mathcal{E}$ and study how the power and the torque depend on the orbital eccentricity. We take $\alpha = 0$, $h = 0.05$, and $\mathcal{E} = 0.6$. Figure 7 shows $\mathcal{P}$ and $\mathcal{T}$ for a mass ratio $q_{.5} = 1$. We see that $\mathcal{P}$ remains fairly constant with time if $e \leq 0.3$. For $e \geq 0.45$, the shape of $\mathcal{P}$ versus time is not so flat, having maxima and minima.

Another remarkable feature is that $\langle \mathcal{P}(t) \rangle_{140}$, the mean value of $\mathcal{P}$ between $t = 5$ orbits and $t = 140$ orbits, changes from ~0.1 for $e = 0$ to ~0.4 for $e = 0.3$ (see Figure 8). For $e = 0.6$, $\langle \mathcal{P} \rangle$ takes a similar value to that for $e = 0$. The LA predicts $\langle \mathcal{P} \rangle = -0.25$. Thus, for eccentricities between the end values of our interval, the measured values of the power are a factor of 2.5 smaller than the LA value. On the other hand, for eccentricities between 0.15 and 0.37, the power in absolute value is larger than the LA value. This is likely a consequence of the Lindblad resonant effects, which are ignored in the LA.

In fact, in the case of retrograde circular orbits, for which there are no Lindblad resonances, the power is always less than or equal to the LA value.

On the other hand, the curves $\mathcal{T}$ versus time exhibit a deep valley at $t \simeq 30-50$ orbits for $e \geq 0.3$ (see right panel of Figure 7). In particular, in the case $e = 0.6$, $|\mathcal{T}|$ grows from $\sim 1.2$ at $t = 5$ orbits to $\sim 5$ after 48 orbits. These values are much larger than the value predicted in the LA (which is 0.38; see Figure 5). As long-term runs indicate (Figure 9), the torque does not converge asymptotically to a constant value but shows large variations over the runtime of our simulations. Therefore, we cannot establish well-defined values of $\langle \mathcal{T} \rangle$, at least when $\mathcal{E} = 0.6$.

The temporal variations in $\mathcal{P}$ but mainly in $\mathcal{T}$ reflect the fact that the flow properties are not periodic functions of time (in this sense we say that the disk has not reached a “steady state”). If the evolution of the disk could be described through the combination of linear density waves, it is expected that a steady state is reached in a few orbits. The temporal variations are a consequence of the secular evolution of the disk because of the deposition of angular momentum carried by the wake through shocks. A steady state will be reached on scales of the viscous time ($t_{\nu} \simeq e^{2} a^2/\nu \simeq 5 \times 10^3$ orbits, assuming $e = 0.6$), which is much longer than the crossing time. For perturbers in prograde and circular orbits, a description of the shock damping of waves in the weakly nonlinear regime (low-mass perturbers) can be found in Goodman & Rafikov (2001). In this regime, inviscid linear theory still predicts correctly the torques on the disk, although it implicitly assumes some dissipation. Here we find that for extended perturbers with $\mathcal{E} = 0.6$ in retrograde and eccentric orbit, the magnitude of the torque is sensitive to the shock propagation and wave damping, even if the excitation of the wake is linear.

While the curves $\mathcal{P}(t)$ and $\mathcal{T}(t)$ should not depend on the adopted value of $q$ if the density waves induced in the disk were strictly linear, some dependence on $q$ can be expected in the presence of wave damping. Figure 10 shows $\mathcal{P}$ and $\mathcal{T}$, as in Figure 7, but for $q_{.5} = 50$. The amplitude of the temporal variations of $\mathcal{P}$ for $e = 0.45$ and $e = 0.6$ increases when $q_{.5}$ is varied from 1 to 50. $\mathcal{T}(t)$ also changes in a comparable amount.
but they are less notorious because the fractional change is smaller.

### 4.2.2. Varying the Softening Radius

One expects that the LA will become more accurate as $\varepsilon$ decreases, because the main contribution to the drag force will arise from a closer vicinity of the body, admitting a local description. This holds true for prograde eccentric orbits (Sánchez-Salcedo 2019), as well as for retrograde circular orbits (Sánchez-Salcedo et al. 2018).

Figure 11 shows the power and the torque for $\varepsilon$ between 0.06 and 0.6. For $\varepsilon \leq 0.12$, the power remains fairly constant over time. Indeed, the temporal behavior of the power is already flat for $\varepsilon \simeq 0.3$ (not shown). In addition, the value of the power converges (from below or from above) to the value predicted in the LA as $\varepsilon$ decreases. This is more clearly seen in Figure 12, where we plot $\langle \mathcal{P} \rangle_{35}$ as a function of $\varepsilon$, where the brackets $\langle \ldots \rangle_{35}$ denote the mean value between $t = 5$ and $t = 35$ orbits. The choice of the values of $e$ in that figure is not completely arbitrary. We selected $e = 0.25$ because, as already mentioned in Section 4.2.1, the maximum value of the power (in absolute value) occurs at this critical eccentricity. This is more easily visualized in Figure 13, where we show $\langle \mathcal{P} \rangle_{35}$

![Figure 11](image1)

**Figure 11.** Evolution of the power (left panels) and torque (right panels) for $e = 0.1$ (top panels), $e = 0.3$ (middle panels), and $e = 0.45$ (bottom panels). Different curves are for different $\varepsilon$. The values of $\varepsilon$ are given at each curve. The squares indicate the value predicted in the LA. In all cases, the mass ratio is $q_{15} = 1$.

![Figure 12](image2)

**Figure 12.** Mean power between the 5th and 35th orbits as a function of $\varepsilon$ for simulations with different eccentricities. In these models $\alpha = 0$, $h = 0.05$, and $q_{15} = 1$. The vertical lines mark the value of $\varepsilon$ for which the error in the power introduced by the LA is a factor 1.5 (dashed line) or 1.25 (dotted–dashed line) from the values measured in the simulations. The open symbols on the left side of the figure indicate the values predicted in the LA.
versus eccentricity. The value $e = 0.6$ was selected because the power reaches its minimum value there (see Figure 13).

For our purposes, it is convenient to define $E_{p;1.5}$ as the maximum value required for the LA to give the power within a factor of $\sim 1.5$ from the values measured in the simulations. In other words, if $E \leq E_{p;1.5}$, then the ratio between the measured and the predicted power lies between 0.66 and 1.5. We find that $E_{p;1.5} = 0.25$. For $E = 0.12$, $\langle P \rangle_{35}$ as measured in the simulations lies between $-0.22$ and $-0.32$, in broad agreement with the value $-0.25$ derived in the LA.

Regarding the torque, the amplitude of its oscillations is reduced as $E$ is taken smaller (Figure 11). For $e = 0.45$, the amplitude of the temporal variations of the torque is still comparable to its mean value even for $E = 0.06$. For $E = 0.06$ and $e \leq 0.3$, the torque variations become relatively small. In these cases ($e \leq 0.3$ and $E = 0.06$), the discrepancy between the torque measured in the simulations and the predicted value in the LA is $\leq 25\%$ (see also Figure 14). Since $t_e$ depends on the difference between $P$ and $h^{-1}T$ (see Equation 15), it remains uncertain to determine whether $e$ grows or damps in these cases.

Given that the torque may oscillate on a timescale of 50–100 orbits, $\langle T \rangle_{35}$ should be interpreted with caution, as it reflects the depth of the first valley. Still, it is illustrative to see that $\langle T \rangle_{35}$ converges to the value predicted in the LA as $E$ decreases (Figure 14). An extrapolation of the curves in Figure 14 strongly suggests that $E_{p;1.5} \approx 0.04$, where $E_{p;1.5}$ is the equivalent to $E_{p;1.5}$ but for the torque.

4.3. Varying $h$

Our reference value for the aspect ratio, $h = 0.05$, is representative for protoplanetary disks. For AGN accretion disks, the aspect ratio is less constrained, but models suggest a range for $h$ between 0.01 and 0.1 (e.g., Sirko & Goodman 2003). Since the local Mach number for an object in retrograde orbit is $\sim 2/h$, the perturbed density in the disk depends on $h$. We have carried out a set of simulations with $h = 0.025$ and $h = 0.1$ (again with $\alpha = 0$), to check how the results depend on $h$.

Figure 15 plots the mean power $\langle P \rangle_{35}$ as a function of eccentricity. We see that the critical eccentricity depends on $h$. The critical eccentricity is 0.4 for $h = 0.1$ and 0.15 for $h = 0.025$.

Figure 16 shows $\langle P \rangle_{35}$ and $\langle T \rangle_{35}$ for $h = 0.1$. $\langle P \rangle_{35}$ presents a dispersion around the LA values similar to that found for $h = 0.05$. We find that $E_{p;1.5} = 0.2$, which is similar to the value found for $h = 0.05$.

For $h = 0.1$, the values of $\langle T \rangle_{35}$ get closer to the LA estimates than for $h = 0.05$. We infer $E_{p;1.5} \approx 0.05$. For $e \leq 0.4$ and $E = 0.12$, we run the simulations until 120 orbits and found that $\langle P \rangle$ and $\langle T \rangle$ are similar, implying that $B$ is significantly smaller than $P$.

Figure 17 shows the power for $h = 0.025$. Interestingly, the values of $\langle P \rangle_{35}$ spread apart from the values derived in the LA. In particular, we notice that the power is minimum (in absolute value) at $e = 0.45$ (i.e., $e/e_{\text{crit}} = 3$). If we restrict ourselves to orbital eccentricities $0 \leq e \leq 2e_{\text{crit}} = 0.3$, we obtain $E_{p;1.5} = 0.12$.
Putting together the results obtained for \( h = 0.025, 0.05, \) and \( 0.1 \), we find the following rules of thumb. The absolute value of the power is maximum at a critical eccentricity

\[ \varepsilon_{\text{crit}} \simeq 0.25 \left( \frac{h}{0.05} \right)^{2/3}. \]  

At eccentricities \( \varepsilon \simeq \varepsilon_{\text{crit}} \), the power is larger, in absolute value, than what the LA predicts. The LA predicts the power, with an error less than 30%, at eccentricities around \( \varepsilon \simeq 0.6 \varepsilon_{\text{crit}} \) provided that \( \varepsilon \leq 0.6 \). On the other hand, if we take \( \varepsilon \leq 0.12 \), the LA predicts the power in the range \( e \leq 2 \varepsilon_{\text{crit}} \) within a factor of less than 1.5. Finally, our results suggest that \( \varepsilon_{p,1.5} = 0.2 \min[1, (h/0.05)] \) in the range of eccentricities \( 0 < e < 0.6 \).

4.4. Varying \( \alpha \)

We have run models with \( q_{-5} = 10 \), \( h = 0.05 \), \( e = 0.45 \), and \( \varepsilon = 0.3 \), and different \( \alpha \). Figure 18 shows that the LA is equally well regardless of the value of \( \alpha \). This is expected because the unperturbed surface density changes on a radial scale of \( |\Sigma_0/\Sigma_0/dR| \approx a/|\alpha| \), which is much larger than the length of the wake that contributes most to the drag force.

5. Implications for the Evolution of Compact Objects in 3D Accretion Disks

Our simulations consider the response of a 2D disk to a softening potential, ignoring mass accretion onto the perturber. In real life, COs, such as BHs or neutron stars, have very small or null physical radii, and they are embedded in disks with
finite scale height. Following Sánchez-Salcedo (2019), we extend the results to the latter scenario.

Suppose that the LA predicts the power or the torque with some permissible error if \( R_{\text{soft}} \leq R_{\text{soft}} \equiv \varepsilon_{\max} H \). Once we know \( R_{\text{soft}} \) in a 2D disk, denoted by \( R_{\text{soft}}^{(2D)} \), we can obtain \( R_{\text{soft}}^{(3D)} \), the maximum softening radius in a 3D disk. In fact, Figure 13 in Sánchez-Salcedo (2019) shows the relationship between \( \varepsilon_{\max}^{(2D)} \) and \( \varepsilon_{\max}^{(3D)} \). In particular, for \( h \geq 0.025 \) and \( e \leq 0.6 \), we have found in Section 4.2.2 that \( \varepsilon_{r;1.5}^{(2D)} = 0.1 \). This translates into \( \varepsilon_{r;1.5}^{(3D)} = 0.01 \) for extended perturbers embedded in 3D disks. Note that \( \varepsilon_{r;1.5}^{(3D)} \) is smaller than \( \varepsilon_{r;1.5}^{(2D)} \) because the drag force depends logarithmically on \( R_{\text{soft}}^{-1} \) in a 3D disk, while it scales as \( R_{\text{soft}}^{-1} \) in a 2D disk.

For point-like objects, like COs, the LA will be valid as long as the accretion radius \( R_{\text{acc}} \) (which is the minimum effective scale of the interaction) is smaller than \( R_{\text{soft}} \). Note that \( R_{\text{acc}} \) of a body in eccentric orbit may vary along the orbit, being maximum at apocenter because the relative velocity is minimum. Thus, if we demand \( R_{\text{acc}} \leq R_{\text{soft}} \) at apocenter, we can derive an upper limit on the mass ratio of the inspiral. Using \( R_{\text{acc}} = 2 G M p / \sqrt{a} \), with \( \sqrt{a} \simeq 2 a / \sqrt{1 + e} \) at apocenter, the above condition can be cast, in terms of \( q \), as \( q < 2 \Delta e_{\max} \) for COs.

For a disk with \( h \geq 0.025 \) and for \( \varepsilon_{r;1.5}^{(3D)} = 0.01 \) (see above), we obtain that the LA predicts the power and therefore also the rate of inspiral \( t_{\nu} \), within a factor of 1.5, for inspirals having \( q_{-5} \leq 50 \). Interestingly, this range of masses includes EMRIs (see Section 2). We highlight that the estimates of the power and \( t_{\nu} \) are robust in the sense that they are weakly dependent on the orbital eccentricity.

The condition for the LA to predict the torque with the same error is much more restrictive. For instance, consider a disk with \( h = 0.05 \). For \( e \leq 0.6 \), our simulations suggest that \( \varepsilon_{r;1.5}^{(2D)} \simeq 0.04 \) (Section 4.2.2). This value corresponds to \( \varepsilon_{r;1.5}^{(3D)} \simeq 10^{-4} \), implying \( q_{-5} \lesssim 1 \). If we are only interested in orbital eccentricities smaller than 0.3, the corresponding condition is \( q_{-5} \lesssim 8 \).

Since \( t_{\nu} \) is inversely proportional to \( |\mathcal{F}| = |\mathcal{F} - \eta^{-2} T| \) (see Equation (4)), \( t_{\nu} \) remains very unconstrained unless \( \mathcal{F} \) and \( T \) are very dissimilar. In numerical simulations, it is difficult to determine the net evolution of the eccentricity because of alternating periods during which the eccentricity grows or damps.

### 6. Conclusions

A DF approach is commonly used to model the gravitational interaction between an accretion disk and an orbiter moving on an eccentric and/or inclined orbit. This approach assumes that the interaction is local, i.e., the gas ahead of the perturber remains unperturbed and, in addition, most of the contribution to the tidal forces arises from a region so close to the perturber that curvature terms are unimportant. In this paper, we have considered the orbital evolution of a low-mass perturber, having an eccentricity between 0 and 0.6 and an inclination of 180°, i.e., coplanar but retrograde orbit with respect to the gas disk. In such a situation, the perturber moves supersonically with Mach numbers 40–200 relative to the local gas, and it excites spiral waves that are wound tightly.

Notably, in typical accretion disk models, the local DF approach predicts that the eccentricity is excited. Nevertheless, in disks with a surface density that decays in the radial direction, the timescale for the growth of the eccentricity is larger than the timescale for the radially inward migration. Consequently, there exists the possibility that the inspiral could merge with a nonzero eccentricity. The DF approach also predicts that the rate of inspiral hardly depends on the orbital eccentricity. The purpose of this work was to assess when the local DF approximation can be applied to retrograde EMRIs.

We have computed the torque and the rate of work on a perturber on a fixed eccentric orbit in 2D simulations. The rate of energy loss by the perturber determines the rate of inspiral, whereas a combination of the power and the torque determines the evolution of the eccentricity.

We find that for eccentricities around the critical value \( e_{\text{crit}} \approx 0.25 (h/0.05)^{2/3} \), the power (in absolute value) is larger than predicted in the LA. Nevertheless, the orbital-averaged power and torque converge to the values predicted by the LA when \( \varepsilon \) tends to zero. This reflects the fact that curvature effects and resonances are less important for smaller bodies. For \( h \) between 0.025 and 0.1, and for \( e \leq 0.1 \), the LA predicts the power measured in 2D simulations within a factor of 1.5 or less. This condition for \( \varepsilon \), which was found for extended perturbers embedded in 2D disks, translates into \( q_{-5} \lesssim 50 \) for COs in 3D disks. This mass range includes EMRIs.

Numerical determinations of the mean torque require long-term simulations because the torque exhibits temporal variations, unless \( \varepsilon \) is taken to be very small. Such long-term simulations are a numerical challenge because of the spurious noise introduced through the boundaries. An extrapolation of our results indicates that the LA estimates of the torque are within a factor of 1.5 of the measured values if softened perturbers embedded in 2D disks with \( h \geq 0.05 \) have \( \varepsilon < 0.04 \). This implies \( q_{-5} \lesssim 1 \) for COs embedded in 3D disks. However, we should stress that even if the power and the torque are determined within a factor of 1.5, the error in the estimate of \( t_{\nu} \) using the LA might be larger because it depends on the difference \( \mathcal{F} - \eta T \).

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### Appendix A

#### Boundary Conditions and Radial Extent of the Computational Domain

The finite size of the computational domain may induce undesirable phenomena, leading to an inaccurate result. The domain should be taken as large as possible, and the boundary conditions should be chosen with the aim of minimizing spurious effects. Trusted results should be robust to reasonable changes of the size of the box domain. In this appendix, we study the sensitivity of the results to the location of the inner edge of the computational domain and on the adopted inner boundary condition. The outer edge is less problematic because it can always be placed so far away that its effects are comparably less important. We mainly focus on cases with the largest value of \( \varepsilon \) (i.e., \( \varepsilon = 0.6 \)) because the effects of the inner boundary are more prominent as \( \varepsilon \) increases. In addition, we...
set up $\alpha = 0$ and $h = 0.05$ in all the simulations presented in this appendix.

In the lack of resonances or collective modes, spurious phenomena generated at the boundaries are usually diminished when the extent of the domain increases. To illustrate this, Figure A1 shows $\overline{P}$ and $\overline{T}$ in simulations in which the boundaries behave as a rigid wall, where large reflections are expected. We take $R_{\text{out}} = 5.2a$ and three different values of $R_{\text{in}}$ ($0.07a$, $0.12a$, and $0.2a$). In the two runs having $R_{\text{in}} \leq 0.12a$, the power is approximately constant over time between $3t_{\text{orb}}$ and $40t_{\text{orb}}$. The mean value of the power between the 5th and 35th orbits, $\overline{(P)}_{35}$, in these two runs agrees within 7%.

In these simulations, the power is not perfectly smooth but presents some wiggles, with a small amplitude of $\sim 4\%$. These small oscillations are the consequence of the combination of two effects: (1) reflections at the inner boundary, and (2) the interaction of the perturber with its own wake ahead of it, which has memory that the perturber was turned on suddenly at $t = 0$. The timescale of the fluctuations caused by the memory effect is very small (the timescale is $t_{\text{orb}}$), and the amplitude of these wiggles is attenuated, especially at early times, if the perturber is inserted slowly in the simulations (see Appendix B). Reflections at the inner boundary, on the other hand, lead to temporal variations with a frequency determined by the sound-crossing times $t_{s}$, defined as the time a sound wave takes to travel from $R = a$ to the inner boundary, and getting back after reflection. In the run with $R_{\text{in}} = 0.07a$ we have that $t_{s} \simeq 4.2t_{\text{orb}}$, whereas $t_{s} \simeq 3.9t_{\text{orb}}$ in the run with $R_{\text{in}} = 0.2a$. This small difference in $t_{s}$ indicates that the wiggles caused by reflections are quite difficult to suppress just by reducing $R_{\text{in}}$.

On the other hand, $\overline{T}$ presents a local maximum and then a local minimum (see right panel of Figure A1). The minimum occurs about $\sim 15t_{\text{orb}}$ after the maximum. The locations of the maximum and minimum are not the same in the three simulations. The torque measured in the simulation with
\( R_{\text{in}} = 0.07a \) is similar in shape to the torque in the simulation using \( R_{\text{in}} = 0.12a \), though slightly shifted horizontally. The shift of about \( 3t_{\text{orb}} \) is much larger than the difference in \( t_s \) between the two simulations, which is only \( \sim 0.2t_{\text{orb}} \). In fact, the local maxima and minima in the torque are the result of global modes in the disk and cannot be interpreted in terms of boundary reflections that produce variations on a shorter timescale. The temporal shift in the torque does not affect much its mean value: the difference in the value of \( \langle T \rangle_{35} \) between these two simulations is less than 4%.

In addition to reflections, closed boundary conditions have the shortcoming that gas is piled up in the inner boundary.
because it cannot leave the computational domain, which is not realistic. Alternatively, open boundary conditions, which allow inflow and outflow through the boundaries, can be considered. The results of applying this condition at the inner and outer boundaries are shown in Figure A2. A weakness of using open boundaries is that the mass in the disk is not preserved.

To overcome the limitations of closed and open boundary conditions, it is common to implement buffer zones, where the density and velocity components are forced to gradually go back to their unperturbed values, as described in de Val-Borro et al. (2006). The timescale of this damping process is proportional to the local dynamical timescale. The resultant power and torque using wave-killing regions with different widths are depicted in Figure A3. The wave-damping ring extends from $R_{\text{in}}$ up to $R_{\text{end}}$. The three simulations with $R_{\text{end}} = 0.25a$ yield similar results, regardless of the adopted boundary condition (closed or outflow). In fact, the buffer regions are large enough to damp the waves before they reach $R = R_{\text{in}}$, so that the results only depend on $R_{\text{end}}$; they do not depend either on $R_{\text{in}}$ or on the boundary condition. It is apparent that the torque for $R_{\text{end}} = 0.17a$ is shifted to the right by $5t_{\text{orb}}$, but this shift has a minor effect on mean torque: $(\bar{T})_{35}$ agree within 6% in the four simulations shown in Figure A3.

For completeness, we have computed $\bar{P}$ and $\bar{T}$ for various combinations of $R_{\text{in}}$ and $R_{\text{out}}$ (see Figure A4). The wave-damping rings are $R \in [R_{\text{in}}, 1.3R_{\text{in}}]$ and $R \in [0.95R_{\text{out}}, R_{\text{out}}]$. We see that $R_{\text{in}} = 0.2a$ and $R_{\text{out}} = 4a$ are adequate to compute the power within 150 orbits. On the other hand, the torque in runs with $R_{\text{out}} \leq 4a$ is not reliable beyond 120 orbits. Again, it is worth noting that the torque in the simulations with $R_{\text{in}} = 0.12a$ and $R_{\text{out}} = 5.2a$ has the same shape as the torque in the simulation with $R_{\text{in}} = 0.2a$ and $R_{\text{out}} = 5.2a$, but they present a slight shift in time.

### Appendix B

#### Gradual Growth of the Mass of the Perturber

In models with $\varepsilon = 0.6$, the power exhibits small sawtooth variations during the first 20 orbits (see Figures 7, A3, and A4). For retrograde perturbers that are introduced instantaneously, Sánchez-Salcedo et al. (2018) showed that these variations in power occur when perturbers catch their own wake. If so, these transient features should be partially suppressed if the mass of the perturber gradually increases from 0 to $M_p$ during a time $t_M$ larger than $t_{\text{orb}}$. Figure B1 shows the power and the torque in simulations where we ramp up the mass of the perturber from 0 at $t = -10t_{\text{orb}}$ to $M_p$ at $t = 0$, so that $t_M = 10t_{\text{orb}}$. For comparison, the results for simulations in which the perturber is introduced suddenly at $t = 0$, so that $t_M = 0$, are also shown. The power and the torque in simulations with $t_M = 10t_{\text{orb}}$ are shifted in time relative to those in simulations with $t_M = 0$, because at $t = 0$ the perturbers have the same mass in both cases, but the perturber with $t_M = 10t_{\text{orb}}$ has been perturbing the disk during 10 orbital periods. As expected, the small wiggles in the power during the first 15 orbits are suppressed when the perturber is introduced smoothly (see Figure B2 for a zoomed-in view).

![Figure B1](image1.png)  
**Figure B1.** Power (left panel) and torque (right panel) when the mass of the perturber increases gradually from $t = -10t_{\text{orb}}$ to $t = 0$ (solid lines) and when the perturber is inserted suddenly at $t = 0$ (dashed lines). The parameters are $q, \varepsilon = 1, \varepsilon = 0.3, \varepsilon = 0.6$ (Set I) and $q, \varepsilon = 0.3, \varepsilon = 0.3$ (Set II). In all cases $R_{\text{in}} = 0.2a$ and $R_{\text{out}} = 5.2a$. 

![Figure B2](image2.png)
References

Amaro-Seoane, P., Audley, H., Babak, S., et al. 2017, arXiv:1702.00786
Amaro-Seoane, P., Maureira-Fredes, C., Dotti, M., & Colpi, M. 2016, A&A, 591, A114
Antonini, F. 2014, ApJ, 794, 106
Antonini, F., Capuzzo-Dolcetta, R., Mastrobuono-Battisti, A., & Merrit, D. 2012, ApJ, 750, 111
Arca-Sedda, M., & Gualandris, A. 2018, MNRAS, 477, 4423
Armitage, P. J., & Natarajan, P. 2002, ApJL, 567, L9
Artymowicz, P. 1993, ApJ, 419, 155
Benítez-Llambay, P., & Masset, F. S. 2016, ApJS, 223, 11
Bitsch, B., Crida, A., Libert, A.-S., & Lega, E. 2013, A&A, 555, 124
Bitsch, B., & Kley, W. 2010, A&A, 523, 30
Breslau, A., & Pfalzner, S. 2019, A&A, 621, 101
Cantó, J., Esquivel, A., Sánchez-Salcedo, F. J., & Raga, A. C. 2013, ApJ, 762, 21
Corsini, E. M. 2014, Multi-Spin Galaxies (San Francisco, CA: ASP)
Cresswell, P., Dirksen, G., Kley, W., & Nelson, R. P. 2007, A&A, 473, 329
Cresswell, P., & Nelson, R. P. 2008, A&A, 482, 677
de Val-Borro, M., Edgar, R. G., Artymowicz, P., et al. 2006, MNRAS, 370, 529
Finn, L. S., & Thorne, K. S. 2000, PhRvD, 62, 124021
García-Burillo, S., Combes, F., Hunt, L. K., et al. 2003, A&A, 407, 485
Goldreich, P., & Sari, R. 2003, ApJ, 585, 1024
Goodman, J. 2003, MNRAS, 339, 937
Goodman, J., & Rafikov, R. R. 2001, ApJ, 552, 793
Gurkan, M. A., & Kley, W. 2009, ApJ, 694, 236
Hopman, C., & Alexander, T. 2006, ApJL, 645, L133
Imanishi, M., Nakano, K., Izumi, T., & Wada, K. 2018, ApJL, 853, L25
Impellizzeri, C. M. V., Gallimore, J. F., Baum, S. A., et al. 2019, ApJL, 884, L28
Ivanov, P. B., Papaloizou, J. B., Paardekooper, S.-J., & Polnarev, A. G. 2015, A&A, 576, A29
Kim, S. S., & Morris, M. 2003, ApJ, 597, 312
Kocsis, B., Yunes, N., & Loeb, A. 2011, PRD, 84, 024032
Levin, Y. 2007, MNRAS, 374, 515
Martinsson, T. P. K., Sarzi, M., Knapen, J. H., et al. 2018, A&A, 612, 66
Marzari, F., & Nelson, A. F. 2009, ApJ, 705, 1575
McKernan, B., Ford, K. E. S., Kocsis, B., Lyra, W., & Winter, L. M. 2014, MNRAS, 441, 900
McKernan, B., Ford, K. E. S., Lyra, W., et al. 2011, MNRAS, 417, L103
Murray, C. D., & Dermott, S. F. 1999, Solar System Dynamics (Cambridge: Cambridge Univ. Press)
Muto, T., Takeuchi, T., & Ida, S. 2011, ApJ, 737, 37
Nixon, C. J., & Lubow, S. H. 2015, MNRAS, 448, 3472
Papaloizou, J. C. B. 2002, A&A, 388, 615
Papaloizou, J. C. B., & Larwood, J. D. 2000, MNRAS, 315, 823
Papaloizou, J. C. B., & Terquem, C. 2001, MNRAS, 325, 221
Portegies Zwart, S. F., Baumgardt, H., McMillan, S. L. W., et al. 2006, ApJ, 641, 319
Portegies Zwart, S. F., McMillan, S. L. W., & Gerhard, O. 2003, ApJ, 593, 352
Rein, H. 2012, MNRAS, 422, 3611
Sánchez-Salcedo, F. J. 2019, ApJ, 885, 152
Sánchez-Salcedo, F. J., Chametla, R. O., & Santillán, A. 2018, ApJ, 860, 129
Sirkko, E., & Goodman, J. 2003, MNRAS, 341, 501
Tanaka, H., & Ward, W. R. 2004, ApJ, 602, 388
Xiang-Gruess, M., & Papaloizou, J. C. B. 2013, MNRAS, 431, 1320