Tides and Dumbbell Dynamics

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Received July 12, 2021; revised February 04, 2022; accepted March 08, 2022

Abstract—We discuss a model describing the effects of tidal dissipation on the satellite’s orbit in the two-body problem. Tidal bulges are described in terms of a dumbbell, coupled to the rotation by a dissipative interaction. The assumptions on this dissipative coupling turn out to be crucial in the evolution of the system.

MSC2010 numbers: 70F05, 70F15
DOI: 10.1134/S1560354722030078

Keywords: two-body problem, tidal dissipation, tides, dumbbell dynamics

1. INTRODUCTION

Celestial bodies such as the Earth or some of the Jovian satellites consist of several layers of materials having different densities and interacting differently with other celestial bodies (e.g., the Moon, the Sun, Jupiter or other satellites). As a consequence of this interaction and the nonhomogeneity of the inner structure, tidal bulges are created on the surface of these celestial bodies. The rotational velocity of the bulges is, in general, not the same as that of the inner layers. It is reasonable to think that the friction between the bulges and the remainder of the body depends on their mutual angular velocity.

In this framework we introduce a simple model in order to compute the effects of tidal dissipation in the two-body problem. In particular, we want to investigate the effects that this dissipation has on the orbits of both celestial bodies. We will assume that the two bodies have very different masses, say $M \gg m$, and we call planet the body with mass $M$ and satellite the body with mass $m$. To keep the model as simple as possible, we assume the axes of rotation of both the planet and the satellite to be perpendicular to the orbital plane. We study the effects of the tides formed by the satellite on the planet and, with the same approach, the effects of the tides formed by the planet on the satellite, assuming that the satellite is in 1:1 spin-orbit resonance with the planet.

In our model, we describe the system in terms of the dumbbell dynamics: the planet and the satellite are described in terms of a point $P$ of mass $M - \mu$ and a mechanical dumbbell centered in $P$, i.e., a system of two points, each having mass $\mu/2$, constrained to be at a fixed mutual distance $2r$, having $P$ as the center of mass. The idea is to substitute the study of the two tidal bulges with the study of their respective centers of mass. There is a vast literature concerning the study of the shape and the rheological properties of the celestial body. In these works, following Darwin’s classical approach, the tidal potential is written in terms of its Fourier components, each having its own rheological parameters, e.g., its own dissipation function. Relevant references include [1–5]. Though a detailed analysis of this kind allows one to study the properties of celestial bodies in terms of the theory of viscoelastic bodies, our simple model has a different focus: we set aside a

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detailed discussion concerning the viscoelastic inner structure of the bodies, but, nevertheless, we take into account the effects of the tidal torque exerted on the bulges, obtaining, for instance, in a clear way, its classical expression, see the appendix below. This approach helps to clarify that the classical expression of the tidal torque may be derived independently of the model of friction chosen to describe the rheology of the celestial body. We will assume that the motion of the tidal dumbbell and the rotation of the related heavenly body (planet or satellite) are coupled by a viscous dissipative friction, and we will use the simplest friction model involving an explicit dependence on the angular velocities.

With these assumptions we will use an approach based on the Rayleigh dissipation function, obtaining in a unified and relatively standard way the equations of motion in the two cases mentioned above. We will call the study of the influence on the satellite’s orbit of the tides on the planet Earth–Moon system, while the study of the effect of the tides on the satellite, supposed in 1 : 1 resonance, will be called Jupiter–Io system. All the computations will be performed to the lowest order in the small parameters (the ratio between the radius of the bodies and the orbit of the satellite and the eccentricity).

The effects of tidal dissipation on the orbital parameters of both bodies are, in this model, quite clear. We hope that the model will serve as a stepping stone for the study of the effects of tidal dissipation in systems involving several celestial objects such as the case of Jovian satellites where resonance and tidal dissipation seem to be closely related.

Note that the dynamics of the dumbbells in celestial mechanics has been already studied in different contexts as in [6] and [7] in which the dumbbell is used to study satellites’ attitudes. To our knowledge, however, the idea of applying the dumbbell dynamics to the dissipative tidal effects is not present in the literature yet.

The work is organized as follows. In Section 3 we present the equations of motion of the Earth–Moon system, i.e., considering the tidal torque exerted by the satellite on a dumbbell centered on the planet. Then we evaluate the evolution of the orbital parameter of the Moon due to this interaction and show that the dissipation tends to circularize the orbit of the Moon. In Section 4 we study the Jupiter–Io system, showing that also in this case the orbit tends to be circularized. In both cases we prove that the eccentricity tends to zero exponentially. Finally, Section 4 is devoted to the conclusion. In the appendix we compute directly the torque between the dumbbell and the other body.

2. EARTH–MOON SYSTEM

In this section we want to study the evolution of the Earth–Moon system. We derive the equations of motion in a Lagrangian formalism. To this end, as we show in Fig. 1, we imagine the Earth, of total mass $M_E$, as a sphere of radius $R_E$ plus a symmetric dumbbell of diameter $2r$ and mass $\mu$. $\mu$ depends on the semimajor axis $a$ of the orbit of the satellite that varies in time, as we show in the appendix, see (A.4) and (A.8). However, in our model we consider $\mu$ to be constant. This assumption is justified a posteriori by the fact that we are interested in the study of the effects of dissipation on orbital parameters, such as, for example, the circularization of the orbit. These effects have characteristic timescales that are faster than the characteristic timescale of elongation of the semimajor axis. Hence, if we study the evolution of the system for a time much shorter than the latter timescale, we can consider $a$ and $\mu$ as constants. The calculation concerning the comparison between the characteristic time of the circularization of the orbit and the characteristic time of elongation of the semimajor axis is presented in the appendix. We imagine the Moon, denoted by $S$, as a point of mass $m$. We indicate with $\rho$ the distance between the Moon and the center of the Earth: we observe that in an elliptical orbit this distance varies with time. Finally, we call $\varphi$, $\vartheta$ and $(\vartheta + \varepsilon)$ the angular positions of the Earth, the Moon and the dumbbell with respect to a fixed direction ($x$-axis), respectively. So $\dot{\varphi}$, $\dot{\vartheta}$ and $(\dot{\vartheta} + \dot{\varepsilon})$ are the angular velocities of the Earth, the Moon and the dumbbell, respectively.

Given $M_E \gg m$, we assume the Earth to be fixed at the origin of the reference system. The total kinetic energy is the sum of the kinetic energies of the Moon, the Earth and the dumbbell:

$$T = \frac{1}{2}m(\dot{\rho}^2 + \rho^2 \dot{\vartheta}^2) + \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}\mu r^2(\dot{\vartheta} + \dot{\varepsilon})^2,$$

where $I$ is the Earth’s moment of inertia.
The potential energy is the sum of three pieces of gravitational attraction: that between the Earth (deprived of the dumbbell) and the Moon; that between point $P$ (a bulge of the dumbbell) and the Moon and that between point $P'$ (the other bulge of dumbbell) and the Moon:

$$V = -\frac{k(M_E - \mu)m}{\rho} - \frac{k\mu m}{\sqrt{r^2 + \rho^2 - 2 \rho\rho \cos \epsilon}} - \frac{k\mu m}{\sqrt{r^2 + \rho^2 + 2 \rho\rho \cos \epsilon}},$$

where $k$ is the universal gravitational constant. If we now expand the potential up to the second order in $\frac{\rho}{\rho}$ (which is a dimensionless small parameter), we obtain the following expression:

$$V = -\frac{gm\rho}{\rho} \left[ 1 + \frac{\mu}{M_E \rho^2} \left( \frac{3}{2} \cos^2 \epsilon - \frac{1}{2} \right) \right],$$

where $g = kM_E$.

So, the Lagrangian of the system is, with all the aforementioned assumptions,

$$\mathcal{L} = T - V = \frac{1}{2} m(\dot{\rho}^2 + \rho^2 \dot{\varphi}^2) + \frac{1}{2} I \dot{\varphi}^2 + \frac{1}{2} \mu r^2 (\dot{\vartheta} + \dot{\epsilon})^2 + \frac{gm}{\rho} \left[ 1 + \frac{\mu}{M_E \rho^2} \left( \frac{3}{2} \cos^2 \epsilon - \frac{1}{2} \right) \right].$$

Now we want to take into account the dissipation of energy. The reasonable mechanism of such dissipation arises from the fact that a friction between the dumbbell and the underlying Earth is present. Since both the ocean and the Earth can be considered fluids (by assuming that the Earth is a highly viscous fluid), it is reasonable to assume a Stokes-type friction both for the oceanic and the solid tides: namely, a friction proportional to the difference between the angular velocity of the Earth (that is $\dot{\varphi}$) and that of the ocean’s bulges (that is $(\dot{\vartheta} + \dot{\epsilon})$).

So we assume a frictional force\(^1\) of the form:

$$f = -\alpha(\dot{\varphi} - \dot{\vartheta} - \dot{\epsilon}),$$

with $\alpha$ a small friction coefficient. Although $\alpha$ depends on $\mu$, we can consider it constant as long as we consider $\mu$ constant too, as we argued above.

A standard approach to treating viscous friction in Lagrangian formalism is to use the Rayleigh dissipation function $R$, defined as a function such that $\frac{\partial R}{\partial q_i} = f_i$, where $f_i$ is the frictional force acting on the $i$th variable.

In our case, the Rayleigh dissipation function assumes the form

$$R = -\frac{1}{2} \alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\epsilon} \right)^2.$$

\(^1\)More precisely, a frictional torque
The Euler–Lagrange equations become
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i}.
\]
(2.7)

Finally, it is easy show that, \( R \) being of the form \( R = -\frac{1}{2}v^2(\dot{q}) \) with \( v(\dot{q}) = \sum_j a_j \dot{q}_j \) (that is \( v(\dot{q}) \) linear in \( \dot{q} \)), the energy dissipation rate is
\[
\dot{E} = 2R = -\alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right)^2.
\]
(2.8)

Notice that in the classical literature, see, for instance, [8, (4.151)], the dissipation is assumed to be linear in the difference of angular velocities, implying nondifferentiable behavior in \( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \).

The Euler–Lagrange equations (2.7) lead to the following equations for the dynamical variables \( \rho, \vartheta, \varphi, \varepsilon \):
\[
m\ddot{\rho} = \frac{\partial L}{\partial \rho},
\]
(2.9)
\[
\frac{d}{dt} \left( m\rho^2 \dot{\rho} \right) + \mu r^2 \left( \ddot{\vartheta} + \ddot{\varepsilon} \right) = \frac{\partial R}{\partial \dot{\vartheta}} = \alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right),
\]
(2.10)
\[
I \ddot{\varphi} = \frac{\partial R}{\partial \dot{\varphi}} = -\alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right),
\]
(2.11)
\[
\mu r^2 \left( \ddot{\vartheta} + \ddot{\varepsilon} \right) = \frac{\partial L}{\partial \varepsilon} + \frac{\partial R}{\partial \dot{\varepsilon}} = \frac{\partial L}{\partial \varepsilon} + \alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right).
\]
(2.12)

From these four equations, we can write down two interesting relations.

First, from (2.10) and (2.11) we obtain the conservation of angular momentum:
\[
\frac{d}{dt} \left( m\rho^2 \dot{\rho} \right) + \mu r^2 \left( \ddot{\vartheta} + \ddot{\varepsilon} \right) + I \ddot{\varphi} = 0 \implies m\rho^2 \dot{\rho} + \mu r^2 \left( \ddot{\vartheta} + \ddot{\varepsilon} \right) + I \ddot{\varphi} = J = \text{const.}
\]

Second, from (2.10) and (2.12) we obtain the equation that determines the evolution of orbital angular momentum \( J^{(O)} \):
\[
\frac{d}{dt} \left( m\rho^2 \dot{\rho} \right) = -\frac{\partial L}{\partial \varepsilon}.
\]
(2.13)

The explicit expression of \( \frac{\partial L}{\partial \varepsilon} \) and its comparison with the classical form of the tidal torque are briefly presented in the first part of the appendix.

In Eqs. (2.9) to (2.12), there are some negligible terms. In fact, assuming small eccentricity, \( \rho \sim a \), we observe that the results presented in the appendix, namely, (A.2), imply that \( \frac{\partial L}{\partial \varepsilon} \propto \frac{\mu r^2}{M_E a^2} \), and Eq. (2.13) becomes
\[
ma^2 \ddot{\vartheta} \propto \frac{1}{a} \left( \frac{\mu}{M_E a^2} \right) r^2.
\]

We want to study the system keeping the lowest order in \( \frac{\mu r^2}{M_E a^2} \), which is a very small quantity. Since
\[
\mu r^2 \ddot{\vartheta} \propto M_E \frac{1}{m a} \left( \frac{\mu}{M_E a^2} \right)^2 r^2,
\]
the term \( \mu r^2 \ddot{\vartheta} \) in (2.10) and (2.12) can be neglected. Moreover, it is also reasonable to assume initial conditions such that \( \varepsilon = O(\ddot{\vartheta}) \), namely, the variation of angular velocity of the bulges is of the same order as the variation of the angular velocity of the Moon. Hence, the term \( \mu r^2 \varepsilon \) in (2.10) and (2.12) can be neglected too.
Consequently, the simplified equations of motion are

\[ m\ddot{\rho} = \frac{\partial L}{\partial \dot{\rho}} \quad (2.14) \]

\[ \frac{d}{dt}(m\rho^2\dot{\vartheta}) = \alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right) \quad (2.15) \]

\[ I\ddot{\varphi} = -\alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right) \quad (2.16) \]

\[ \frac{\partial L}{\partial \varepsilon} + \alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right) = 0 \quad (2.17) \]

We call \( G \) the orbital angular momentum of the Moon (that is, using Delaunay’s canonical variable):

\[ G = m\rho^2 \dot{\vartheta} \quad (2.18) \]

from (2.15) we have

\[ \dot{G} = \alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right) \quad (2.19) \]

Moreover, the rate of dissipation of energy is given by (2.8). The energy variation of the system is made of three contributions: the variation of energy of the Earth, the variation of energy of the Moon (namely, the variation of orbital energy \( E^{(O)} \)) and the variation of energy of the bulges. The latter can be neglected for the same reason as why we have neglected the variation of the angular momentum of the bulges. So (2.8) becomes

\[ -\alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right)^2 = \frac{d}{dt} \left( \frac{1}{2} I\dot{\varphi}^2 \right) + \frac{dE^{(O)}}{dt}, \quad (2.20) \]

where \( E^{(O)} \) is the orbital energy of the Moon:

\[ E^{(O)} = \frac{1}{2} m(\dot{\rho}^2 + \rho^2 \dot{\vartheta}^2) - \frac{gm}{\rho}. \]

From (2.16), (2.19) and (2.20), we have

\[ -\alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right)^2 = -\alpha\dot{\varphi} \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right) + \frac{dE^{(O)}}{dt} \implies \frac{dE^{(O)}}{dt} = \left( \dot{\vartheta} + \dot{\varepsilon} \right) \dot{G}. \]

Consider now the other canonical Delaunay variable \( L \), defined as

\[ L = \frac{m^2 g}{\sqrt{-2E^{(O)}}}; \]

hence

\[ \dot{L} = \frac{gE^{(O)}}{\left( -\frac{2E^{(O)}}{m} \right)^{\frac{3}{2}}}. \quad (2.22) \]

Now, on a Keplerian orbit, we have

\[ E^{(O)} = -\frac{mg}{2a} \implies -\frac{2E^{(O)}}{m} = \frac{g}{a}, \]

and Kepler’s third law

\[ \omega^2 a^3 = g. \]
Hence, (2.22) becomes
\[
\dot{L} = \frac{\dot{E}^{(O)}}{\omega} = \frac{\dot{\varphi} + \dot{\varepsilon}}{\omega} \dot{G},
\] (2.23)
where we have used (2.21).

We now observe that several quantities, such as \(\langle \dot{\varphi} \rangle = \Omega, \langle \dot{\varepsilon} \rangle = \omega\) and both the orbital energy and the orbital angular momentum (and hence \(L\) and \(G\)) vary very slowly. We can therefore assume that these quantities remain constant on one orbit and undergo a very small increase (or decrease) at the end of each revolution.

Therefore, we can compute the average on one orbit of \(\dot{G}\) and \(\dot{L}\), obtaining
\[
\langle \dot{G} \rangle = \langle \alpha (\dot{\varphi} - \dot{\varepsilon}) \rangle = \alpha (\Omega - \omega)
\] (2.24)
and
\[
\langle \dot{L} \rangle = \langle \frac{\dot{\varphi} + \dot{\varepsilon}}{\omega} \dot{G} \rangle = \frac{\alpha}{\omega} \left( \dot{\varphi} - \dot{\varepsilon} \right).
\] (2.25)
But:
\[
\dot{\varphi} = \omega t + 2e \sin(\omega t) \implies \dot{\varphi} = \omega + 2e \omega \cos(\omega t).
\]
And (see (A.10) for the definition of \(A, B\) and \(\delta\)):
\[
\varepsilon = A + eB \sin(\omega t + \delta) \implies \dot{\varepsilon} = eB \omega \cos(\omega t + \delta).
\]
Therefore,
\[
\langle \dot{L} \rangle = \frac{\alpha}{\omega} \left( [\omega + 2e \cos(\omega t) + eB \omega \cos(\omega t + \delta)] [\Omega - \omega - 2e \cos(\omega t) - eB \omega \cos(\omega t + \delta)] \right)
\]
\[
= \frac{\alpha}{\omega} \left( \omega (\Omega - \omega) + we (\Omega - 2\omega) (2 \cos(\omega t) + B \cos(\omega t + \delta)) - \omega^2 e^2 ([2 \cos(\omega t) + B \cos(\omega t + \delta)]^2) \right)
\]
\[
= \langle \dot{G} \rangle + \omega^2 \varepsilon^2 ([2 \cos(\omega t) + B \cos(\omega t + \delta)]^2),
\]
where we have used (2.24).

The computation of the remaining averages leads to
\[
\langle 2 \cos(\omega t) + B \cos(\omega t + \delta) \rangle = 0
\]
and
\[
\langle [2 \cos(\omega t) - B \cos(\omega t + \delta)]^2 \rangle = \langle 4 \cos^2(\omega t) + 4B \cos(\omega t) \cos(\omega t + \delta) + B^2 \cos^2(\omega t + \delta) \rangle
\]
\[
= 2 + B^2 \langle \cos^2(\omega t) \cos(\delta) - \sin(\omega t) \cos(\omega t) \sin(\delta) \rangle + \frac{B^2}{2}
\]
\[
= 2 + 2B \cos(\delta) + \frac{B^2}{2}.
\]
Hence,
\[
\langle \dot{L} \rangle = \langle \dot{G} \rangle - \omega \varepsilon^2 \left( 2 + 2B \cos(\delta) + \frac{B^2}{2} \right) = \langle \dot{G} \rangle - \varepsilon^2 C,
\] (2.26)
with \(C = \omega \left( 2 + 2B \cos(\delta) + \frac{B^2}{2} \right) > 0\).

Now, if we assume initially \(L \sim G\), with \(L > G\), we have
\[
\frac{d}{dt} \frac{C^2}{L^2} = \frac{2GG^2 - 2L \dot{L} \dot{G}^2}{L^4} = \frac{2GG^2 - 2L \dot{G}^2 (G - \varepsilon^2 C)}{L^4} = \frac{2GG^2 - 2G \dot{G} C^2 + 2 \dot{L} \dot{G} C}{L^4}
\]
\[
= \frac{2GG^2}{L^4} - \frac{2L \dot{G}^2 C}{L^4} + \frac{2 \dot{L} \dot{G} e^2 C}{L^4} = \frac{2GG^2 + 2 \dot{L} \dot{G} e^2 C}{L^4}
\]
\[
= \frac{2}{\tau_M} e^2,
\] (2.27)
with \(\tau_M = \frac{L^4}{GG + L^2 C} > 0\).
Finally, the Delaunay variables $L$ and $G$ satisfy the relation $G^2 = (1 - e^2)L^2$, hence we have

$$\frac{d}{dt} \frac{G^2}{L^2} = \frac{d}{dt}(1 - e^2) = -2e\dot{e}. \quad (2.28)$$

Putting together (2.27) and (2.28), we obtain

$$-2e\dot{e} = \frac{2}{\tau_M} e^2 \quad \Rightarrow \quad \dot{e} = -\frac{e}{\tau_M} \quad \Rightarrow \quad e(t) = e_0 \exp \left( -\frac{t}{\tau_M} \right). \quad (2.29)$$

Therefore, the eccentricity tends exponentially to zero as $t \to \infty$: the orbit becomes circular.

Consider now the limiting case of a circular orbit: $e = 0$. In this case, on each single orbit $\dot{\phi} = \Omega = \text{const}$ and $\dot{\theta} = \omega = \text{const}$ and then in Eq. (2.17) we have a constant forcing. Hence, the solution of such an equation is asymptotic to a constant: $e = \text{const}$.

From (2.17) and (2.19) we then have

$$\dot{G} = \alpha \left( \dot{\phi} - \dot{\theta} \right) = \alpha (\Omega - \omega) = -\frac{\partial L}{\partial \varepsilon} = \Gamma. \quad (2.30)$$

We assume, as discussed before, that the very slow variation on $\Gamma$ is applied as a final kick after an unperturbed Keplerian orbit on which $\Gamma$ is assumed constant. We have

$$G(T) - G(0) = \int_0^T \frac{dG}{dt} dt = \Gamma T. \quad (2.31)$$

Hence, from (2.18) and using Kepler’s third law $a^3\omega^2 = g$, we have

$$G(T) = G(0) + \Gamma T = m\sqrt{ga(0)} + \Gamma T$$

and

$$G(T) = m\sqrt{ga(T)} = m\sqrt{g[a(0) + \dot{a}T]};$$

therefore,

$$m\sqrt{g[a(0) + \dot{a}T]} = m\sqrt{ga(0)} + \Gamma T \quad \Rightarrow \quad ga(0) + \dot{g}aT = ga(0) + \frac{2\Gamma}{m}T\sqrt{ga(0)} \quad \Rightarrow \quad \dot{a} = \frac{2\Gamma}{m\sqrt{a(0)/g}},$$

where we have neglected the term $(\frac{\Gamma}{m})^2$.

Finally, using again Kepler’s third law,

$$\dot{a} = \frac{2\alpha}{m\omega a} (\Omega - \omega). \quad (2.32)$$

Note that in (2.32) the dependence of $\dot{a}$ on $(\Omega - \omega)$ is regular (namely linear), while in the literature, see [8, (4.160)], the dependence has a singularity for $(\Omega - \omega) = 0$, since it depends on $\text{sign}(\Omega - \omega)$. Note also that in this context the results one finds in the literature correspond to a different choice of the friction law in (2.5), i.e., the choice $f = -\text{const}$. This choice is equivalent to considering the system as if it were made up of two solid surfaces that slide over each other.

We want to point out that we are not the first to notice this nonsingularity, in fact, there are several models in the literature of tides in which a Rayleigh dissipation function is used and no singularity appears in the equation for $\dot{a}$, see, for instance, [9, (62)].

3. JUPITER–IO SYSTEM

In this section we want to study the evolution of the Jupiter–Io system using the same formalism developed above.

To this end, as we show in Fig. 2, we imagine Io, of total mass $m$, as a sphere plus a symmetric dumbbell of diameter $2r$ and mass $\mu$. We imagine Jupiter, indicated with $J$, as a point of mass $M_J$ placed at the origin of the reference frame. We indicate with $\rho$ the distance between Jupiter and the center of Io: we observe that in an elliptical orbit this distance varies with time. Finally,
we call $\varphi$, $\vartheta$ and $(\vartheta + \varepsilon)$ the angular positions of the rotation of Io, the angular positions of the revolution of Io and the angular position of the dumbbell with respect to a fixed direction ($x$-axis), respectively. So $\dot{\varphi}$, $\dot{\vartheta}$ and $(\dot{\vartheta} + \dot{\varepsilon})$ are the angular velocity of the rotation of Io, the angular velocity of the revolution of Io and the angular velocity of the dumbbell, respectively.

In this case, the kinetic energy of this system is the sum of kinetic energies of the dumbbell and the kinetic energy of Io, which has two pieces: one due to the revolution around Jupiter and one due to the rotation around its own axis. The potential energy is the sum of three pieces of gravitational attraction: that between Io (deprived of the dumbbell) and Jupiter, that between point $P$ (a bulge of the dumbbell) and Jupiter, and that between point $P'$ (the other bulge of dumbbell) and Jupiter. The potential can be expanded up to the second order in $r\rho$ (which is a dimensionless small parameter).

So, the Lagrangian of the system is very similar to that of the previous section:

$$\mathcal{L} = \frac{1}{2}m(\rho^2 + \rho^2\dot{\vartheta}^2) + \frac{1}{2}I\dot{\varphi}^2 + \frac{1}{2}\mu r^2(\dot{\vartheta} + \dot{\varepsilon})^2 + \frac{g m}{\rho} \left[ 1 + \frac{\mu}{m} \frac{r^2}{\rho^2} \left( \frac{3}{2} \cos^2 \varepsilon - \frac{1}{2} \right) \right],$$ \hspace{1cm} (3.1)

where $I$ represents the moment of inertia of Io, while in (2.4) it represents the moment of inertia of the Earth. Moreover, now $g = k M_J$.

Assuming the same dissipation mechanism of the previous section, justified by the fact that it is known that Io is made of molten material, we have the same Rayleigh dissipation function as in the previous section:

$$R = -\frac{1}{2}\alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right)^2.$$ \hspace{1cm} (3.2)

So the equations of motion are the same as those in the previous section:

$$m\ddot{\rho} = \frac{\partial \mathcal{L}}{\partial \rho},$$ \hspace{1cm} (3.3)

$$\frac{d}{dt} \left( m\rho^2\dot{\vartheta} \right) = \alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right),$$ \hspace{1cm} (3.4)

$$I\ddot{\varphi} = -\alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right),$$ \hspace{1cm} (3.5)

$$\frac{\partial \mathcal{L}}{\partial \varepsilon} + \alpha \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right) = 0.$$ \hspace{1cm} (3.6)
However, in this case the initial conditions are different. In fact, while in the Earth–Moon system we have $\dot{\varphi} \gg \dot{\vartheta}$ (in fact, the period of rotation of the Earth is much smaller than the period of revolution of the Moon), in the Jupiter–Io system we have $\langle \dot{\varphi} \rangle = \langle \dot{\vartheta} \rangle = \omega$ (in fact, the period of rotation of Io is the same as its period of revolution around Jupiter).

Equation (3.6) admits a solution of the form $\varepsilon(t) = eB \sin(\omega t + \delta)$, with $B$ and $\delta$ constants. This is evident if one performs all the steps seen in the appendix in the case of the Earth–Moon system.

Although $\varepsilon(t)$ is different from that of the previous section (in fact, here it does not contain the constant term $A$), all the results up to (2.23) remain still valid.

The average on one orbit of $\dot{G}$ and $\dot{\vartheta}$ are slightly different from those in the previous section:

$$\langle \dot{G} \rangle = 0$$

and

$$\langle \dot{\vartheta} \rangle = \langle \dot{\varphi} \rangle - \frac{\alpha}{\omega} \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right)$$

$$\langle \dot{\varphi} \rangle = \dot{\vartheta}$$

From now on, all the steps done in the previous section are the same, with $\langle \varphi \rangle = \omega$. Therefore,

$$\langle \dot{\vartheta} \rangle = \langle \dot{\varphi} \rangle = \omega$$

$$\langle \dot{G} \rangle = 0$$

and

$$\langle \dot{\vartheta} \rangle = \langle \dot{\varphi} \rangle - \frac{\alpha}{\omega} \left( \dot{\varphi} - \dot{\vartheta} - \dot{\varepsilon} \right)$$

Putting together (3.10) and (3.11), we obtain

$$\frac{d}{dt} \frac{G^2}{L^2} = \frac{d}{dt} \left( 1 - e^2 \right) = -2e\dot{e}$$

Putting together (3.10) and (3.11), we obtain

$$-2e\dot{e} = \frac{2}{\tau_I} e^2 \implies \dot{e} = -\frac{e}{\tau_I} \implies e(t) = e_0 \exp \left( -\frac{t}{\tau_I} \right)$$

Therefore, the eccentricity tends exponentially to zero as $t \to \infty$: the orbit becomes circular even in the Jupiter–Io case.

4. CONCLUSION

In this short note we have proposed a simplified model to describe the effects of tidal dissipation on the orbital parameters of the two-body problem. We think that the main virtue of the model lies in the clarification of the relevance of the model of friction used in order to describe the interaction between the body and the tidal bulges. We have pointed out that some of the classical results assume tacitly that this friction is velocity-independent, like the solid-on-solid friction, while viscous friction seems to be more realistic and solves some difficulties present in the classical theory.
APPENDIX A. COMPUTATION OF $\varepsilon(t)$

In this appendix we want to show the detailed computation of $\varepsilon(t)$ starting from Eq. (2.17), which we rewrite in this form:

$$\alpha \ddot{\varepsilon}(t) - \frac{\partial L}{\partial \varepsilon} - \alpha \left( \dot{\varphi} - \dot{\varphi} \right) = 0. \quad (A.1)$$

First of all, we compute $\frac{\partial L}{\partial \varepsilon}$ from Eq. (2.4):

$$\frac{\partial L}{\partial \varepsilon} = -\frac{g \mu}{\rho} \frac{r^2}{M_E} 3 \cos \varepsilon \sin \varepsilon \simeq -3 \frac{k m \mu r^2}{\rho^2} \varepsilon, \quad (A.2)$$

where we have used $g = k M_E$ after developing $\cos \varepsilon \sin \varepsilon$ in power series of $\varepsilon$ and keeping only the linear terms in $\varepsilon$.

Recall that $\mu$ represents the mass of the ocean’s bulges. These bulges can be imagined as an ellipsoid of radii $R_E$, $R_E$ and $R_E + h$ deprived of a sphere of radius $R_E$ concentric to it, $h$ being the tidal height.

If we replace $h$ with the Newton formula for tidal height:

$$h = \frac{3}{2} \frac{m}{M_E} \left( \frac{R_E}{\rho} \right)^3 R_E, \quad (A.3)$$

then $\mu$ becomes

$$\mu = \delta_W \frac{4}{3} \pi R_E^2 h = \delta_W \frac{4}{3} \pi R_E^2 3 \frac{m}{2} \frac{m}{M_E} \left( \frac{R_E}{\rho} \right)^3 R_E = \frac{3}{2} \frac{\delta_W}{\delta_E} \left( \frac{R_E}{\rho} \right)^3 w, \quad (A.4)$$

where $\delta_W$ and $\delta_E$ represent the densities of the liquid part and the solid part of the planet. Finally, in (A.2), $r$ is the distance between the center of the Earth and the center of mass of each bulge, that is, the center of mass of the aforementioned ellipsoid deprived of a sphere. It is a standard calculation to show that $r = \frac{3}{4} R_E$ up to terms that go to zero as $\frac{h}{R_E}$.

Therefore,

$$\frac{\partial L}{\partial \varepsilon} = -\frac{81}{32} \frac{\delta_W}{\delta_E} k m^2 R_E^5 \rho^5 \varepsilon = -D k m^2 R_E^5 \rho^5 \varepsilon, \quad (A.5)$$

with $D = \frac{81}{32} \frac{\delta_W}{\delta_E}$ a dimensionless constant.

Equation (A.1) thus becomes

$$\alpha \ddot{\varepsilon} + D k m^2 \frac{R_E^5}{\rho^5} \varepsilon - \alpha \left( \dot{\varphi} - \dot{\varphi} \right) = 0. \quad (A.6)$$

Finally, we have to consider the time dependence of $\rho$ and $\vartheta$. We also assume that $\Omega$, $\omega$ and $a$ remain constant during each revolution and change their values only at the end of each revolution. This assumption yields

$$\vartheta \simeq \lambda + 2 \varepsilon \sin \lambda = \omega t + 2 \varepsilon \sin (\omega t) \implies \dot{\vartheta} \simeq \omega + 2 \omega \varepsilon \cos (\omega t) \quad (A.7)$$

and

$$\rho(t) = \frac{p}{1 + e \cos(\omega t)} = \frac{a(1 - e^2)}{1 + e \cos(\omega t)} \simeq a [1 - e \cos(\omega t)] \implies \frac{1}{\rho} \simeq \frac{1}{a^6} [1 + 6 e \cos(\omega t)]. \quad (A.8)$$

So, Eq. (A.6) becomes

$$\ddot{\varepsilon} + \frac{\gamma_c}{\alpha} \left[ 1 + 6 e \cos(\omega t) \right] \varepsilon - \Omega + \omega + 2 \omega e \cos(\omega t) = 0, \quad (A.9)$$

where $\gamma_c = D k m^2 \frac{R_E^5}{\rho^{56}}$.

A trivial solution of this equation is

$$\varepsilon(t) = \frac{\alpha}{\gamma_c} (\Omega - \omega) - e \frac{6 \Omega - 4 \omega}{\omega} \cos \delta \sin(\omega t + \delta) = A + e B \sin(\omega t + \delta), \quad (A.10)$$

with $A = \frac{\alpha}{\gamma_c} (\Omega - \omega),\ B = \frac{6 \Omega - 4 \omega}{\omega} \cos \delta$ and $\tan \delta = \frac{\gamma_c}{\alpha \omega}$. 

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Remark 1. The ratio \( \frac{\alpha}{\gamma} \) contains, as a whole, the information about the quantities appearing in the system. This suggests, for instance, that the friction coefficient \( \alpha \) has to decay very fast for increasing \( a \), the semimajor axis of the orbit, in order to balance the dependence of \( \gamma_c \) on \( a \) (see the comments immediately following (2.5)). This sounds reasonable, since the friction should depend on the total amount of liquid involved in the motion of the bulges, and this clearly depends on \( a \). However, we do not have a detailed model of the tidal currents inside the oceans (in the Earth–Moon case) or in the inner mantle (Io–Jupiter case). We think that one of the virtues of the simple model we presented in this work is the fact that it suggests the correct relations among the various elements of the orbits, while their quantitative evaluation has to be based on empirical observations.

APPENDIX B. CHARACTERISTIC TIMESCALES OF THE SYSTEM

In this appendix we want to compare the characteristic timescales of the system in order to show that the assumptions we made in our model are justified.

In particular, as we explained above, we are interested in the comparison between the timescale of elongation of the semimajor axis, which we call \( \tau_a \), and the timescale of circularization of the orbit, that is, \( \tau_M \) or \( \tau_I \) in Eqs. (2.29) and (3.12), respectively.

From Eq. (2.32) we see that \( \tau_a \simeq \frac{\tau}{\tau_\Omega} \), with \( \tau_\Omega \) the characteristic timescale of evolution of \( \Omega \). Moreover, we can rewrite Eq. (2.16) as

\[
\dot{\Omega} = -\frac{\alpha}{\Omega},
\]

hence we have \( \tau_\Omega \simeq \frac{L}{\alpha} \).

On the other hand,

\[
\tau_M = \frac{L^4}{GGL^2 + LG^2C} \simeq \frac{L}{G + C} \simeq \frac{L}{\alpha 18\Omega^2}. \tag{B.2}
\]

Therefore, in the case of the Earth–Moon system, \( \tau_M \simeq \frac{L}{500\Omega} \), and hence

\[
\frac{\tau_a}{\tau_M} \simeq 500\frac{L}{\Omega} \simeq 500 \cdot \frac{2}{\omega} \frac{M_E R_E^2}{m R_{EM}^2} \simeq 500 \cdot 28 \cdot \frac{2}{5} \frac{M_E}{m} \left( \frac{R_E}{R_{EM}} \right)^2 \simeq 120 > 1. \tag{B.3}
\]

Conversely, in the case of the Jupiter–Io system, the analogue of Eq. (B.2) is \( \tau_I \simeq \frac{L}{80\Omega} \), yielding

\[
\frac{\tau_a}{\tau_I} \simeq 80\frac{L}{\Omega} \simeq 80 \cdot \frac{2}{\omega} \frac{M_J R_J^2}{m R_{JJ}^2} = 32 \cdot 1 \cdot \frac{2}{5} \frac{M_J}{m} \left( \frac{R_J}{R_{JJ}} \right)^2 \simeq 1.9 \times 10^4 \gg 1. \tag{B.4}
\]

These calculations show that the assumption of a constant \( \mu \) appears to be very reasonable already in the case of the Earth–Moon system. In the case of Jupiter and Io the approximation is much better and the difference between a variable and a constant \( \mu \) is really negligible.

ACKNOWLEDGMENTS

The authors want to thank the anonymous referees whose contributions have helped to improve the readability of the paper. We are indebted to Ugo Locatelli for many useful discussions. We have also benefited from several comments of Giuseppe Pucacco and Gabriella Pinzari.

FUNDING

This work has been supported by PRIN-CELMECH. BS acknowledges the support of the Italian MIUR Department of Excellence grant (CUP E83C18000100006). AT has been supported through the H2020 Project Stable and Chaotic Motions in the Planetary Problem (Grant 677793 StableChaoticPlanetM of the European Research Council).
CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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