Base matrices of various heights

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Abstract. A classical theorem of Balcar, Pelant, and Simon says that there is a base matrix of height $h$, where $h$ is the distributivity number of $\mathcal{P}(\omega)/\text{fin}$. We show that if the continuum $c$ is regular, then there is a base matrix of height $c$, and that there are base matrices of any regular uncountable height $\leq c$ in the Cohen and random models. This answers questions of Fischer, Koelbing, and Wohofsky.

1 Introduction

A collection $\mathfrak{A} = \{A_\gamma : \gamma < \vartheta\}$ of mad (maximal almost disjoint) families of subsets of the natural numbers $\omega$ is called a refining matrix of height $\vartheta$ if:

- $A_\delta$ refines $A_\gamma$ for $\delta \geq \gamma$, i.e., for all $A \in A_\delta$, there is $B \in A_\gamma$ with $A \subseteq^* B$, and
- there is no common refinement of the $A_\gamma$, i.e., no mad family $\mathcal{A}$ refining all the $A_\gamma$.

$\mathfrak{A}$ is a base matrix if it is a refining matrix and $\bigcup_{\gamma < \vartheta} A_\gamma$ is dense in $\mathcal{P}(\omega)/\text{fin}$, i.e., for all $B \in [\omega]^\omega$, there are $\gamma < \vartheta$ and $A \in A_\gamma$ with $A \subseteq^* B$. The distributivity number $h$ of $\mathcal{P}(\omega)/\text{fin}$ is the least cardinal $\kappa$ such that $\mathcal{P}(\omega)/\text{fin}$ as a forcing notion is not $\kappa$-distributive; equivalently, it is the least $\kappa$ such that there is a collection $\mathfrak{A}$ of size $\kappa$ of mad families without common refinement. Clearly, a refining matrix must have height at least $h$, and it is easy to see that there is one of height $h$ and none of regular height $> c$. Furthermore, if there is a refining matrix of height $\vartheta$, then there is one of height $cf(\vartheta)$ so that it suffices to consider regular heights. A famous theorem of Balcar, Pelant, and Simon [BPS] (see also [Bl, Theorem 6.20]) says that there is even a base matrix of height $h$. It is natural to ask whether there can consistently be refining (base) matrices of other heights, and in interesting recent work, Fischer, Koelbing, and Wohofsky [FKW1] proved that it is consistent that $h = \omega_1$ and there is a refining matrix of height $\vartheta \leq c$, where $\vartheta > \omega_1$ is regular, all of whose maximal branches are cofinal (see Section 2 for a formal definition). We show the following theorems.

**Theorem A** If $c$ is regular, then there is a base matrix of height $c$.

**Theorem B** In the Cohen and random models, there are base matrices of any regular uncountable height $\leq c$. 

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This answers Questions 7.5 and 7.7 of [FKW1]. Note that our results are incomparable with the one of the latter work. Their construction does not give a base matrix (in fact, by another result of Fischer, Koelbing, and Wohofsky [FKW2], a base matrix of height > \( \kappa \) always has some non-cofinal maximal branches, though one may still ask whether one can get such a base matrix in which some maximal branches are cofinal), whereas ours necessarily gives non-cofinal maximal branches. In fact, in the Cohen and random models, \( \kappa = \omega_1 \) is the only cardinal \( \vartheta \) for which there is a refining (base) matrix of height \( \vartheta \) all of whose maximal branches are cofinal, and higher refining matrices have no cofinal branches at all (this follows from Fact 1).

2 Preliminaries

The Cohen model (resp. random model) is the model obtained by adding at least \( \omega_2 \) many Cohen (resp. random) reals to a model of the continuum hypothesis CH [BJ].

For \( A, B \subseteq \omega \), we say \( A \) is almost contained in \( B \), and write \( A \subseteq^* B \), if \( A \setminus B \) is finite. \( A \subseteq B \) if \( A \subseteq^* B \) and \( B \setminus A \) is infinite. For an ordinal \( \vartheta_0 \), \( \{ A_y : y < \vartheta_0 \} \) is a \( \subseteq^* \)-decreasing chain of length \( \vartheta_0 \) if \( A_\delta \subseteq^* A_y \) for all \( y < \delta < \vartheta_0 \). \( \subseteq^* \)-decreasing chains are defined analogously. For a refining matrix \( \mathcal{A} = \{ A_y : y < \vartheta \} \) and an ordinal \( \vartheta_0 \leq \vartheta \), \( \{ A_y : y < \vartheta_0 \} \) is a branch in \( \mathcal{A} \) if it is a \( \subseteq^* \)-decreasing chain and \( A_y \in A_y \) for \( y < \vartheta_0 \). A branch is maximal if it cannot be properly extended to a longer branch. A branch is cofinal if \( \vartheta_0 = \vartheta \). Every cofinal branch is maximal, but there may be maximal branches that are not cofinal.

Fact 1 (Folklore) There are no \( \subseteq^* \)-decreasing chains of length \( \omega_2 \) in \( \mathcal{P}(\omega) \) in the Cohen and random models.

This is proved by an isomorphism-of-names argument using the homogeneity of the Cohen or random algebra.

For \( A, B \in [\omega]^{\omega} \), \( A \) splits \( B \) if both \( A \cap B \) and \( B \setminus A \) are infinite. \( X \subseteq [\omega]^{\omega} \) is a splitting family if every \( B \in [\omega]^{\omega} \) is split by a member of \( X \). The splitting number \( s \) is the least size of a splitting family. It is well known that \( h \leq s \) ([Bl] or [Ha]).

Fact 2 (Folklore [see [Bl]; see also [Ha, Proposition 22.13] for Cohen forcing]) After adding at least \( \omega_1 \) Cohen or random reals to a model of ZFC, \( s = \omega_1 \). (In fact, the first \( \omega_1 \) generics are a witness for \( s \).)

We will prove the following.

Main Theorem 3 Assume \( \vartheta \leq c \) is a regular cardinal and
(A) either there is no \( \subseteq^* \)-decreasing chain of length \( \vartheta \) in \( \mathcal{P}(\omega) \),
(B) or \( s \leq \vartheta \).
Then there is a base matrix of height \( \vartheta \).

Clearly, Theorem A follows from part (B) of the main theorem. (We note, however, that splitting families and \( s \leq c \) are not needed in this case [see the comment at the beginning of the proof of Main Claim 5].) Theorem B follows from either (A) or (B).
in view of Facts 1 and 2. Note that part (B) implies that in many other models of set theory there are base matrices of height $\mathfrak{g}$ for any regular $\mathfrak{g}$ between $\mathfrak{h}$ and $\mathfrak{c}$, e.g., in the Hechler model (this satisfies $s = \omega_1$ by [BD]; see also [Bl]), or in any extension by at least $\omega_1$ Cohen or random reals (Fact 2). The former is, and the latter may be (depending on the ground model), a model for the failure of (A). We do not know whether (A) $\rightarrow$ (B) is consistent but conjecture that it is. This clearly implies $s \geq b^{++}$, where $b$ is the unbounding number (which is known to be consistent; see [BF]).

3 Proof of main theorem

By recursion on $\alpha < \mathfrak{c}$, we shall construct sets $\Omega_\gamma \in \mathfrak{c}$ and families $\mathcal{A}_\gamma = \{ A_{\gamma,\alpha} : \alpha \in \Omega_\gamma \}$, $\gamma < \mathfrak{g}$, such that:

(I) All $\mathcal{A}_\gamma$ are mad.

(II) If $\gamma < \delta < \mathfrak{g}$ and $\delta \in \Omega_\delta$, then there is $\alpha \leq \beta$ in $\Omega_\gamma$ such that $A_{\delta,\beta} \subseteq^* A_{\gamma,\alpha}$.

(III) For all $B \subseteq [\omega]^\omega$, there are $\gamma < \delta$ and $\alpha \in \Omega_\gamma$ such that $A_{\gamma,\alpha} \subseteq^* B$.

This is clearly sufficient. In case (B), let $\{ S_\zeta : \zeta < \nu \}$ be a splitting family with $\nu < \mathfrak{g}$. Let $\{(X_\alpha, \xi_\alpha) : \alpha < \mathfrak{c}\}$ list all pairs $(X, \xi) \in [\omega]^\omega \times [\mathfrak{g}]$. At stage $\alpha$ of the construction, we will have sets $\{\Omega_{\gamma} \cap \alpha : \gamma < \mathfrak{g}\}$, ordinals $\{ \eta_\beta : \beta < \alpha \}$ below $\mathfrak{g}$, and families $\{ A_{\gamma,\beta} : \beta \in \Omega_{\gamma} \cap \alpha \} : \gamma < \mathfrak{g} \}$ such that:

(i) $A_{\alpha} := \{ A_{\gamma,\beta} : \beta \in \Omega_{\gamma} \cap \alpha \}$ is almost disjoint for $\gamma < \mathfrak{g}$.

(ii) For all $\beta < \alpha$, the set $\{ \gamma : \beta \in \Omega_{\gamma} \}$ is the interval of ordinals $[\eta_\beta, \max(\eta_\beta, \xi_\beta)]$ and

- for $\gamma \in [\eta_\beta, \max(\eta_\beta, \xi_\beta)]$, $A_{\gamma,\beta} = A_{\eta_\beta,\beta}$, and

- for $\gamma < \eta_\beta$, there is $\beta' < \beta$ in $\Omega_\gamma$ such that $A_{\eta_\beta,\beta'} \subseteq^* A_{\gamma,\beta}$.

(iii) For all $\beta < \alpha$, $A_{\eta_\beta,\beta'} \subseteq^* X_\beta$ and, in case (B), $A_{\eta_\beta,\beta'} \subseteq^* S_\zeta$ or $A_{\eta_\beta,\beta'} \subseteq^* \omega \setminus S_\zeta$, where $\zeta$ is minimal such that $S_\zeta$ splits $A_{\gamma,\beta'}$ whenever $\gamma < \eta_\beta$ and $\beta' \in \Omega_{\gamma} \cap \beta$ are such that $A_{\eta_\beta,\beta'} \subseteq^* A_{\gamma,\beta'}$.

Let us first see that this suffices for completing the proof: indeed, (II) and (III) follow from (ii) and (iii), respectively. To see (I), fix $\gamma < \mathfrak{g}$ and $Y \subseteq [\omega]^\omega$. Then there is $\alpha < \mathfrak{c}$ such that $(Y, Y') = (X_\alpha, \xi_\alpha)$. So $A_{\max(\eta_\alpha, \xi_\alpha), \alpha} = A_{\eta_\alpha, \alpha} \subseteq^* Y$ by (ii) and (iii) and $A_{\max(\eta_\alpha, \xi_\alpha), \alpha} \subseteq^* A_{\gamma,\beta}$ for some $\beta \leq \alpha$ by (ii). Thus $Y \cap A_{\gamma,\beta}$ is infinite, as required.

Next, we notice that, for $\alpha = 0$ and for limit $\alpha$, there is nothing to show. Hence it suffices to describe the successor step, that is, the construction at stage $\alpha + 1$, and to prove that (i) through (iii) still hold. Assume $Y \subseteq^* X_\alpha \cap A_{\gamma,\beta}$ for some $\gamma < \mathfrak{g}$ and $\beta \in \Omega_{\gamma} \cap \alpha$, and let $\delta$ be such that $\gamma < \delta < \mathfrak{g}$. We say that $Y$ splits at $\delta$ if:

- for all $\gamma'$ with $\gamma \leq \gamma' < \delta$, there is $\beta \in \Omega_{\gamma'} \cap \alpha$ such that $Y \subseteq^* A_{\gamma',\beta}$, and

- there is no $\beta \in \Omega_{\delta} \cap \alpha$ such that $Y \subseteq^* A_{\delta,\beta}$.

We say $Y$ splits below $\gamma_0 > \gamma$ if there is $\delta$ with $\gamma < \delta < \gamma_0$ such that $Y$ splits at $\delta$. For infinite $Y \subseteq X_\alpha$, call $A_{\alpha} \upharpoonright Y$ mad if $\{ Y \cap A_{\gamma,\beta} : \beta \in \Omega_{\gamma} \cap \alpha \}$ and $| Y \cap A_{\gamma,\beta}| = \aleph_0$ is a mad family below $Y$. The following is crucial for our construction.

Crucial Lemma 4 Let $\gamma_0 < \mathfrak{g}$ be an ordinal, and let $Y_0 \subseteq X_\alpha$ be infinite. Assume (mad) $A_{\alpha} \upharpoonright Y_0$ is mad for all $\gamma < \gamma_0$. 

Then there are \( y < \gamma_0, \beta \in \Omega_\gamma \cap \alpha \), and an infinite \( Y \subseteq^* Y_0 \cap A_{\gamma, \beta} \) that does not split below \( \gamma_0 \).

**Proof** We make a proof by contradiction. Assume 

(split) if \( Z \subseteq^* Y_0 \cap A_{\gamma, \beta} \) for some \( y < \gamma_0 \) and \( \beta \in \Omega_\gamma \cap \alpha \), then \( Z \) splits below \( \gamma_0 \).

By recursion on \( n \in \omega \), we construct infinite sets \( (Y_s^0 : s \in 2^{<\omega}) \) and \( (Y_s : s \in 2^{<\omega}) \), as well as ordinals \( (\delta_n^0 : s \in 2^{<\omega}) \) and \( (\delta_n : n \in \omega) \) such that:

(a) \( Y_s \subseteq Y_s^0 \) and \( Y_s^0 \subseteq Y_s \) for \( i \in \{0, 1\} \).

(b) \( \delta_n = \max\{\delta_n^0 : |s| = n\} < \gamma_0 \) and \( \delta_n^0 > \delta_{|s|} \) for \( i \in \{0, 1\} \).

(c) \( Y_s \) splits at \( \delta_n^0 \) and there are distinct \( \beta, \beta' \in \Omega_{\delta_0} \cap \alpha \) such that \( Y_s^0 = Y_s \cap A_{\delta_n^0, \beta} \) and \( Y_s^0 = Y_s \cap A_{\delta_n^0, \beta'} \) (in particular, \( Y_s^0 \cap Y_s^1 \) is finite).

(d) \( Y_s^0 = Y_s^0 \cap A_{\delta_n^0, \beta} \) for some \( \beta \in \Omega_{\delta_0} \cap \alpha \), for \( i \in \{0, 1\} \).

We verify that we can carry out the construction. In the basic step \( n = 0 \) and \( s = \{\} \), by (mad), let \( Y_{\{\}} = Y_0 = Y_0 \cap A_0 \) for some \( \beta \in \Omega_0 \cap \alpha \) such that this intersection is infinite. By clause (split), we know that there is \( \delta_0 = \delta_0^0 \) with \( 0 < \delta_0 < \gamma_0 \) such that \( Y_{\{\}} \) splits at \( \delta_0 \).

Suppose \( Y_0^0, Y_s, \) and \( \delta_n \) have been constructed for \( |s| = n \) and \( \delta_n = \max\{\delta_n^0 : |s| = n\} < \gamma_0 \) are such that (a) through (d) hold. We thus know that \( Y_s \) splits at \( \delta_n^0 \) and, by the definition of splitting and clause (mad), we can find distinct \( \beta, \beta' \in \Omega_{\delta_0} \cap \alpha \) such that \( Y_s^0 := Y_s \cap A_{\delta_n^0, \beta} \) and \( Y_s^0 := Y_s \cap A_{\delta_n^0, \beta'} \) are infinite. Using again (mad), we see that for \( i \in \{0, 1\} \) there is \( \beta \in \Omega_{\delta_n} \cap \alpha \) such that \( Y_s^0 := Y_s^i \cap A_{\delta_n^0, \beta} \) is infinite. Again by (split), there is \( \delta_n^0, i \in \{0, 1\} \), with \( \delta_n < \delta_n^0 \) such that \( Y_s^0 \cap Y_s^1 \) splits at \( \delta_n^0 \).

Finally, let \( \delta_{n+1} := \max\{\delta_n^0 : |s| = n \) and \( i \in \{0, 1\}\} < \gamma_0 \). This completes the construction.

Let \( \delta_\omega = \bigcup_n \delta_n \). Clearly \( \delta_\omega \subseteq \gamma_0 \) is a limit ordinal of countable cofinality. Next, for \( f \in 2^\omega \), let \( Y_f \) be a pseudointersection of the \( Y_{f|n}, n \in \omega \). If possible, choose \( \beta_f \in \Omega_{\delta_\omega} \cap \alpha \) such that \( Y_f \cap \alpha \) is infinite. By (a) and (c) in this construction and by (ii), we see that if \( f \neq f' \) then \( \beta_f \neq \beta_f' \). However, \( \Omega_{\delta_\omega} \cap \alpha \) has size strictly less than \( \omega \), and therefore there is \( f \in 2^\omega \) for which there is no such \( \beta_f \). Since \( Y_f \subseteq^* Y_0 \) by construction, this implies that \( A_{\delta_\omega} \cap Y_0 \) is not mad and, by (mad), \( \gamma_0 = \delta_\omega \). This means, however, that any \( Y_f \) contradicts (split). This completes the proof of the crucial lemma.

We next show:

**Main Claim 5** There is \( y < \theta \) such that \( A_\gamma^a \upharpoonright X_\alpha \) is not mad.

**Proof** Note that, in case \( \theta = \omega \), there is nothing to show because by (ii) we see that a tail of the sequence \( (\Omega_\gamma \cap \alpha : y < \theta) \) is empty, and therefore so is \( A_\gamma^a \) (in fact, the proof of Theorem A is quite a bit simpler than the general argument: there is no need to list the \( \xi_\alpha \), we may simply let \( \xi_\alpha = \alpha, \eta_\alpha \) will always be \( \leq \alpha \), and the splitting family is unnecessary).

Hence assume \( \theta < \omega \). By way of contradiction, suppose all \( A_\gamma^a \upharpoonright X_\alpha \) are mad. By the crucial lemma with \( \gamma_0 = \theta \) and \( Y_0 = X_\alpha \), we know that there are \( y < \theta, \beta \in \Omega_\gamma \cap \alpha \) and an infinite \( Y \subseteq^* X_\alpha \cap A_{\gamma, \beta} \) that does not split below \( \theta \). This means for all \( \delta \) with \( y \leq \delta < \theta \) there is \( \beta \in \Omega_\delta \cap \alpha \) such that \( Y \subseteq^* A_{\delta, \beta} \). By (ii) and (iii), we see that there
must be a strictly increasing sequence \((\beta_\varepsilon : \varepsilon < \vartheta)\) of ordinals below \(\alpha\) such that for \(\varepsilon' > \varepsilon\),
- \(\eta_\beta > \max(\eta_\beta, \xi_\beta)\) and \(Y \sqsubseteq ^* A_{\eta_\beta, \beta_\varepsilon} \sqsubseteq ^* A_{\eta_\beta, \beta_{\varepsilon'}}\).

In case (A), this contradicts the initial assumption that there are no \(\varepsilon^*\)-decreasing chains of length \(\vartheta\) in \(\mathcal{P}(\omega)\). So assume we are in case (B). Define a sequence \((\zeta_\varepsilon : \varepsilon < \vartheta)\) of ordinals below \(\nu\) such that
- \(\zeta_\varepsilon\) is minimal such that \(S_{\zeta_\varepsilon}\) splits all \(A_{\eta_\beta, \beta_{\varepsilon'}}\) for \(\varepsilon' < \varepsilon\).

Using (iii\(_a\)), we see that \(S_{\zeta_\varepsilon}\) does not split \(A_{\eta_\beta, \beta_\varepsilon}\). Therefore, the sequence must be strictly increasing, which is impossible (and thus contradictory) in case \(\nu < \vartheta\). If \(\nu = \vartheta\) note that there cannot be any \(\xi\) such that \(S_\xi\) splits \(Y\), contradicting the initial assumption that the \(S_\xi\) form a splitting family. This final contradiction establishes the main claim.

We now let \(\eta_a := \min\{\gamma : A^a_\gamma \upharpoonright X_a\) is not mad\} < \(\vartheta\). Choose \(Y_0 \subseteq X_a\) infinite and almost disjoint from all members of \(A^a_\eta\). Note that \(A^a_\gamma \upharpoonright Y_0\) is mad for all \(\gamma < \eta_a\). Thus, by the crucial lemma with \(\gamma_0 = \eta_a\), we know there are \(\gamma < \eta_a, \beta \in \Omega_\gamma \cap \alpha\), and an infinite \(Y \subseteq ^* Y_0 \cap A_{\eta, \beta}\) that does not split below \(\eta_a\). Then,
\[(*)\] for all \(\delta\) with \(\gamma \leq \delta < \eta_a\), there is \(\beta = \beta_\delta \in \Omega_\delta \cap \alpha\) such that \(Y \subseteq ^* A_{\delta, \beta}\).

Choose infinite \(A_{\eta_a, \alpha} \sqsubseteq ^* Y\). In case (B), choose \(\zeta < \nu\) minimal such that \(S_\zeta\) splits all \(A_{\delta, \beta_\delta}\) with \(\gamma < \delta < \eta_a\). If \(Y \cap S_\zeta\) is infinite, additionally require \(A_{\eta_a, \alpha} \sqsubseteq ^* Y \cap S_\zeta\). If not, we will automatically have \(A_{\eta_a, \alpha} \sqsubseteq ^* \omega \setminus S_\zeta\).

Next, for all \(\gamma\) with \(\eta_a \leq \gamma \leq \max(\eta_a, \xi_\alpha)\), we let \(A_{\gamma, \alpha} = A_{\eta_a, \alpha}\). Also put

\[
\Omega_\gamma \cap (\alpha + 1) = \begin{cases} \Omega_\gamma \cap \alpha, & \text{if } \gamma < \eta_a \text{ or } \gamma > \max(\eta_a, \xi_a), \\ (\Omega_\gamma \cap \alpha) \cup \{\alpha\}, & \text{if } \eta_a \leq \gamma \leq \max(\eta_a, \xi_a). \end{cases}
\]

Then clauses (i\(_{a+1}\)) and (iii\(_{a+1}\)) are immediate, and (ii\(_{a+1}\)) follows from (*) This completes the proof of the main theorem.

4 Further remarks and questions

Obviously, the main remaining problem is whether the spectrum of heights of base matrices can be non-convex on regular cardinals.

**Question 6** Is it consistent that for some regular \(\vartheta\) with \(h < \vartheta < c\) there is no base (refining) matrix of height \(\vartheta\)?

The simplest instance would be \(h = \omega_1\) and \(c = \omega_3\) with no base (refining) matrix of height \(\omega_2\). By (B) in Main Theorem 3, this would imply \(s = \omega_3\).

As the referee remarked, another constellation for a nontrivial spectrum, which would be convex, might be a model where \(s = c\) is singular and there is a regular cardinal \(\kappa \geq h\) with \(\kappa < c\) such that the spectrum consists exactly of the regular cardinals in the interval \([h, \kappa]\). It is unknown, however, whether \(s = c\) singular is consistent at all. The consistency of singular \(s\) was shown by Dow and Shelah [DS], but in their model, \(c\) is at least \(s^+\).
The proof of Main Theorem 3 may look a little like cheating because we do not refine our mad families everywhere when going to the next level. Thus, let us say \( \mathcal{A} = \{ A_\gamma : \gamma < \vartheta \} \) is a strict base (refining) matrix if it is a base (refining) matrix and for any \( \gamma < \delta < \vartheta \) and any \( A \in \mathcal{A}_\delta \) there is \( B \in \mathcal{A}_\gamma \) with \( A \not\prec^* B \). We then obtain the following.

**Proposition 7** Assume \( \vartheta \leq c \) is a regular cardinal such that there are \( \prec^* \)-decreasing chains of length \( \alpha \) in \( \mathcal{P}(\omega) \) for any \( \alpha < \vartheta \)

(A) either there is no \( \prec^* \)-decreasing chain of length \( \vartheta \) in \( \mathcal{P}(\omega) \),

(B) or \( s \leq \vartheta \).

Then there is a strict base matrix of height \( \vartheta \).

**Proof sketch** Modify the proof of Main Theorem 3 by attaching a \( \prec^* \)-decreasing chain of length \( \zeta_\beta + 1 \) to the set \( \{ \gamma : \beta \in \Omega_\gamma \} = [\eta_\beta, \max(\eta_\beta, \xi_\beta)] \), where \( \eta_\beta + \zeta_\beta = \max(\eta_\beta, \xi_\beta) \). This is clearly possible by assumption.

To analyze this a bit further, let \( \delta \) denote the least ordinal \( \alpha \) such that there is no \( \prec^* \)-decreasing chain of length \( \alpha \) in \( \mathcal{P}(\omega) \). It is easy to see that \( \delta \) is a regular cardinal with \( b^+ \leq \delta \leq c^+ \). Put \( \vartheta_0 = \min\{ \delta, c \} \), and assume \( \vartheta_0 \) is regular. Then:

(1) there are strict base matrices of heights \( h \) and \( \vartheta_0 \), and

(2) all strict refining matrices have height between \( h \) and \( \vartheta_0 \).

To see (1), use the previous proposition for height \( \vartheta_0 \), and note that the original construction of [BPS] gives a strict base matrix of height \( h \). (2) is obvious. We leave it to the reader to verify that Proposition 7 implies the corresponding versions of Theorems A and B.

**Corollary 8** If \( c \leq \omega_2 \), then there is a strict base matrix of height \( c \).

**Corollary 9** Let \( \vartheta \) be a regular uncountable cardinal. In the Cohen and random models, the following are equivalent:

(i) \( \vartheta \in \{ \omega_1, \omega_2 \} \).

(ii) There is a strict base matrix of height \( \vartheta \).

(iii) There is a strict refining matrix of height \( \vartheta \).

To see, e.g., Corollary 9, note that by Fact 1, \( \delta = \omega_2 \) in either model, and use (1) and (2) above.

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Base matrices of various heights

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