Non-Equilibrium Quantum Field Theory and Entangled Commutation Relations *

L. Accardi †, I. Ya. Aref’eva ‡, I. V. Volovich §

Centro Vito Volterra
Università di Roma Tor Vergata

January 19, 2018

Abstract

Non-equilibrium quantum field theory studies time dependence of processes which are not available for the S-matrix description. One of the new methods of investigation in non-equilibrium quantum theory is the stochastic limit method. This method is an extension of the works by Bogoliubov, van Hove and Prigogine and it permits to study not only the system but also the reservoir degrees of freedom. We consider the stochastic limit of translation invariant Hamiltonians in quantum field theory and show that the master field satisfies a new type of commutation relations, the so called entangled (or interacting) commutation relations. These relations extend the interacting Fock relations established earlier in non-relativistic QED and the free (or Boltzmann) commutation relations which have been found in the large N limit of QCD. As an application of the stochastic limit method we consider the photon splitting cascades in magnetic field and show that photons in cascades form entangled states (“triphons”) and they obey not Bose but a new type of statistics corresponding to the entangled commutation relations.

*To be published in the Special Issue of Proc. of the Steklov Mathematical Institute dedicated to the 90th birthday of N.N. Bogoliubov.

†Graduate School of Polymathematics, Nagoya University, accardi@math.nagoya-u.ac.jp
‡Steklov Mathematical Institute, Gubkin St.8, GSP-1, 117966, Moscow, Russia, arefeva@mi.ras.ru
§Steklov Mathematical Institute, Gubkin St.8, GSP-1, 117966, Moscow, Russia, volovich@mi.ras.ru
1 Introduction

The basic object to study in quantum field theory is the $S$-matrix introduced by Heisenberg. Bogoliubov and Shirkov have developed the $S$-matrix formalism which includes all the quantities considered in quantum field theory [1]. The physical idea behind the $S$-matrix approach is that in the scattering processes there exists a characteristic time scale such that in a time regime larger than this time scale one can neglect interaction and particles evolve according to the free regime.

The situation in statistical physics is different because here one has not just one but several relevant time scales and as a result we don’t have here a universal method comparable with the $S$-matrix approach in quantum field theory. One can say that the role of $S$-matrix approach in non-equilibrium statistical physics is played by various master and kinetic equations. It was the fundamental Bogoliubov idea about the existence of two time scales which lead to the modern progress in the microscopic derivation of kinetic equations [2].

Methods of quantum field theory, in particularly Green functions, are widely used in equilibrium and non-equilibrium statistical physics [3, 4, 5, 6]. It is well known that the so-called double-time thermodynamic Green functions which were used by Bogoliubov and Taublikov have had great success especially when applied to magnetic problems. From the other side there are not so many investigations in quantum field theory where results and methods of non-equilibrium statistical physics are exploited. N.N. Bogoliubov has made great contributions to both of these sciences. As far as we know his works in non-equilibrium statistical physics and in quantum field theory were performed completely separately. Probably the reason was that important problems in quantum field theory in that time were related with the scattering processes described by the $S$-matrix and they were really very different from typical problems in non-equilibrium statistical physics such as the derivation of kinetic equations.

We would like to point out that there are important problems in quantum field theory where the standard $S$-matrix description is not very convenient or even not applicable. These include not only investigation of bound states and spectral problems (see [1]) but also processes with unstable particles [7, 8] (in fact almost all particles are unstable), atom-photon interactions [9], elementary particles in "semidressed states" with non-equilibrium proper fields [10], electroweak baryogenesis and phase transitions in the early Universe and in high-energy collisions [11], quantum optics [12] etc. In the consideration of such processes we are interested in the time regime smaller than the "infinite" time when the $S$-matrix description becomes applicable. One can say that the consideration of such processes belongs to non-equilibrium quantum field theory. One believes that the method of $S$-matrix in quantum field theory is analogous to the Gibbs distribution in equilibrium statistical physics and that there exists a general method (the stochastic limit, see below) in non-equilibrium quantum field theory which provides a description of quantum phenomena depending on time. One of the first works on the systematic application of methods of non-equilibrium statistical physics in quantum field theory is the work of Prigogine in which kinetic equations for the Lee model have been derived.

A general method in non-equilibrium quantum field theory is the method of stochastic limit [14, 15]. The idea of this method is the systematic application of the $\lambda^2t$-limit and quantum stochastic differential equations. One considers the evolution operator $U(t)$ of quantum system for small coupling constant $\lambda$ and large time $t$. Intuitively, the weak coupling – long time limit means that we are looking at times in which the particle has already weakly interacted many times with the field (long time cumulative effects). The net average effect of these interactions amounts
to a loss of memory (Markovian approximation). Therefore in this limit we can expect to be able to approximate the microscopic time evolution, which contains complicated memory effects, with a simpler markovian evolution.

When we say that the $S$-matrix method is not sufficient in non-equilibrium quantum field theory we mean that the standard dynamical definition of the $S$-matrix in real time is given for example in terms of wave operators (including dressing [13, 17, 18]) or LSZ-formalism. This definition can not be applied immediately to the processes with unstable particles. The flexible Bogoliubov–Shirkov approach [1] to $S$-matrix in principle can be applied to the description of unstable particles. There exists a phenomenological approach to $S$-matrix which is not based on a Hamiltonian formalism. In this approach unstable particles are described by the Breit-Wigner complex poles of the scattering amplitudes [13]. The dynamical justification of this phenomenological approach is given in the Weisskopf-Wigner resolvent method, for a discussion of the resolvent method see for example [8, 20]. The resolvent method is usually used for the investigation of the degrees of freedom of the system interacting with reservoir. The stochastic limit method is conveniently used for the consideration of degrees of freedom not only of the system but also of the reservoir.

The first rigorous result about the interaction of a system with a reservoir where the role of a scaling limit involving $\lambda^2 t$ begun to emerge, is due to Bogoliubov [21] (see the next section). Friedrichs, in the context of the now well known Friedrichs model [22], was lead to consider the scaling limit

$$\lambda \to 0, \quad t \to \infty, \quad \lambda^2 t = \text{constant}$$

by second order perturbation theory.

In the mid 1950’s van Hove [14] used this scaling as a device to extract the effects of a small perturbation of the global Hamiltonian of a composite system, on the reduced evolution of a subsystem and to derive a Pauli–type master equation describing the irreversible time evolution of the observables of the sub–system. Therefore, in the quantum theory of open systems, the limit (1) is known as the van Hove or the $\lambda^2 t$ limit.

Using the perturbative development of the dynamics in powers of $\lambda$, van Hove argued that the terms of order $2n$ should behave as $\lambda^{2n} t^n$ (in contrast with the rough estimate $\lambda^{2n} t^{2n}$). Furthermore he made plausible that almost all the terms of order $2n$ should behave as $\lambda^{2n} t^n - \varepsilon$, for some $\varepsilon > 0$ and therefore vanish in the limit (1), while the remaining terms should sum so to give a transport equation.

In the stochastic limit we study the behaviour of the system in the large time and small coupling constant regime. If $G(x_1, ... x_n)$ is the Green function then we want to investigate the asymptotic behaviour of the expression $G(x_1, x_1^0/\lambda^2, ..., x_n, x_n^0/\lambda^2)$ when $\lambda \to 0$. This can be performed by using the anisotropic renormalization group method [1, 23, 24].

Notice that the $\lambda^{2n} t^n$–behaviour of the terms of order $2n$ of the iterated series is exactly of the same order of magnitude of the behaviour of the moments of order $2n$ of a Brownian motion with time parameter rescaled as $\lambda^2 t$. Therefore a posteriori we can interpret van Hove’s perturbative result as a fist indication that the limit (1) should evidentiate a kind of quantum Brownian motion or quantum white noise underlying the dynamics of the quantum system. To describe and to solve the dynamical equations after the stochastic limit one has to derive the stochastic limit for the collective operators, the so called master field. For simple models the master field is the quantum white noise whose creation and annihilation operators satisfy the relations

$$[b(t), b^+(t')] = \delta(t - t')$$
In the stochastic limit one gets for the evolution operator the equation

\[ \frac{dU(t)}{dt} = (F^+(t)b(t) + b^+(t)F(t))U(t) \]

Here the white noise operators \( b(t), b^+(t) \) are singular operator functions of \( t \) and \( F(t) \) is a regular operator function of \( t \). This singular equation represents the Hamiltonian form of quantum stochastic differential equations. This equation can be explicitly solved for many models. The crucial role for the possibility of the explicit solution plays the commutation relation (1).

For more complex models the master fields are more complex and one gets the entangled or interacting commutation relations. The program of investigation of models of quantum field theory in the stochastic limit consists from two parts. First we have to find the commutation relations for the master field and then study the singular differential equation for the evolution operator. In this paper we shall discuss only the first part of the program. For the consideration of the evolution operator see for example \([15, 25, 26]\).

In recent years various modifications and deformations of the algebra of canonical commutation relations have been discussed, see for example \([27, 28, 29]\) and references therein. In particular, in the large \( N \) limit for \( SU(N) \) invariant gauge theory (as well as for \( NxN \) matrix models), the following relations

\[ b(k)b^+(k') = \delta(k - k') \]  

appear naturally \([24]\). Here \( k, k' \) are momentum variables. The algebra generated by the operators \( b(k), b^+(k') \) satisfying (3) is called the free (or Boltzmann) algebra.

In this paper we prove that the stochastic limit of interacting fields, under the only constraint of momentum conservation, leads to a new algebra of commutation relations. We get the relations of the form

\[ B(p|k_1, ..., k_m)B^+(p'|k'_1, ..., k'_m) = n(p)\delta(E(p, k_1, ..., k_m))\delta(p - p')\delta(k_1 - k'_1)\delta(k_2 - k'_2) ... \delta(k_n - k'_n) \]  

where \( B^\pm(p|k_1, ..., k_m) \) is the master field, obtained as the stochastic limit of a translation invariant Hamiltonian, \( n(p) \) is the operator density of particles, and \( E(p, k_1, ..., k_m) \) is the energy associated with the interaction vertex. These relations are called the entangled (or interacting) commutation relations. Notice that the equality (4) extends the Boltzmann algebra (3) because the right hand side is an operator (in the particle space) rather than a scalar: in this sense one speaks of Hilbert module [30] (rather than Hilbert space) commutation relations. \( B(p|k_1, ..., k_m) \) and the density \( n(p) \) satisfy

\[ [n(p'), B(p|k_1, ..., k_m)] = (\delta(p' - p) - \delta(p' - p + \sum k_i))B(p|k_1, ..., k_m) \]  

As one of physical applications of the above ideas we will argue that photon splitting cascades in the magnetic field create entangled states and that photons in cascades obey not Bose but a new type of statistics – infinite or quantum Boltzmann statistics. Therefore, this statistics has a physical meaning since it describes photons in cascades and more generally the dominant diagrams in the long time/weak coupling limit in quantum field theory. The states in cascades are formed from triples of entangled photons and may be called triphons. They belong to an interacting Fock space [30, 31]. Interacting Cuntz algebra has been considered in [32].

The standard definition of the stochastic limit is given as the limit of the Wightman correlation function. For some models the limit of these correlation functions is equal to zero. We show that the stochastic limit for the Green functions is non-trivial even in these cases.
The paper is organized as follows. In Sect. 2 we remind the Bogoliubov theorem and compare it with the stochastic limit. In Sect. 3 we discuss the stochastic limit. Sections 4 and 5 are devoted to the derivation of the entangled commutation relations. In Sect. 6 we discuss the stochastic limit for the Green functions and in Sect. 7 the universality classes of the stochastic limit and decay processes are considered. Finally Sect. 8 is devoted to applications of the stochastic limit to photon splitting cascades.

2 Bogoliubov’s Theorem

The first rigorous result, about the interaction of a system with a reservoir, where the role of a scaling limit involving \( \lambda^2 t \) begun to emerge, is due to Bogoliubov \([21]\) who considered a classical system with Hamiltonian

\[
H = H_S + H_R + H_{int}
\]

where

\[
H_S = \frac{1}{2}(p^2 + \omega^2 q^2), \quad H_R = \frac{1}{2} \sum_{n=1}^{N} (p_n^2 + \omega_n^2 q_n^2), \quad H_{int} = \lambda q \sum_{n=1}^{N} (\alpha_n q_n) \tag{7}
\]

He assumed that \( \sum_{\sigma<\omega_n} \frac{\alpha_n^2}{\omega_n^2} \rightarrow \int_0^\infty J(\nu) d\nu < \infty \) as \( N \rightarrow \infty \) with \( J(\nu) \) being a non-negative continuous function. Supposing that, at \( t = 0 \) the system is in the state \( q = q_0, p = p_0 \) and that the state of the reservoir is a random variable with the distribution \( \rho_R = \exp\{-H_R/T\} \) he proved that:

- As \( N \rightarrow \infty \) the limit distribution \( \rho_S = \rho_S(t, p, q) \) of the random variables \( p = p_t, q = q_t \), of the system, exists at any time \( t > 0 \).
- If \( H_S \) is given by (2), define the function \( \rho^0_S = \rho_0(t, H_S) = \rho_0(t, p, q) \), by:

\[
\rho^0_S(t, H_S) = \frac{\omega}{4\pi^2 T (1 - e^{-2a\lambda^2 t})} \int_0^{2\pi} d\phi \exp\{-\frac{H_S + E_0 e^{-2a\lambda^2 t} - 2\sqrt{H_S E_0} e^{-2a\lambda^2 t} \cos \phi}{T (1 - e^{-2a\lambda^2 t})}\}
\]

where \( a = \frac{\pi}{4} J(\omega) \) and \( E_0 = (p_0^2 + q_0^2)/2 \) where \( (p_0, q_0) \) is the initial state. Then for small \( \lambda \) the function \( \rho_S \) can be approximated by the function \( \rho^0_S \), in the sense that, for any positive \( \alpha, \beta \) as \( \lambda \rightarrow 0 \), one has, uniformly in the interval \( \frac{\alpha}{\lambda^2} < t < \frac{\beta}{\lambda^2} \):

\[
\frac{1}{\Delta t_\lambda} \int_t^{t+\Delta t_\lambda} (\rho_S - \rho^0_S) dt \rightarrow 0
\]

for any subsequence \( \{\Delta t_\lambda\} \) such that \( \lambda^2 \Delta t_\lambda \rightarrow 0, \Delta t_\lambda \rightarrow \infty \)

He gave a rigorous proof of this result using a Volterra integro-differential equation.

Bogoliubov’s condition \( \lambda^2 \Delta t_\lambda \rightarrow 0 \) is different from the one used in the stochastic limit (\( \lambda^2 t \rightarrow \) constant), notice however that \( \rho^0_S(t) \) depends only on \( \lambda^2 t \) and also that for \( t \rightarrow \infty \) one gets the Gibbs distribution,

\[
\rho^0_S(t) \rightarrow \frac{\omega}{2\pi T} e^{-H_S/T}
\]
3 The Stochastic Limit

The stochastic limit is now widely used in the consideration of the long time/weak coupling behaviour of quantum dynamical systems with dissipation, see for example [15].

Let be given a quantum system described by the Hamiltonian

\[ H = H_0 + \lambda V \]

where \( \lambda \) is the coupling constant. The starting point of the stochastic limit is the equation for the evolution operator in interaction picture

\[ \frac{dU^{(\lambda)}(t)}{dt} = -i\lambda V(t)U^{(\lambda)}(t) \]

where

\[ V(t) = e^{itH_0}V e^{-itH_0} \]

The main idea is that there exist a new quantum field (master field) and a new evolution operator \( U(t) \) (they both live on a space different from the original one) which approximates the old one

\[ U^{(\lambda)}(t) \approx U(\lambda^2 t) \]

and the approximation is meant in the sense of appropriately chosen matrix elements. The above approximation suggests a natural interpretation of the van Hove rescaling \([16]\) \( \lambda \rightarrow 0, \ t \rightarrow \infty \) so that \( \lambda^2 t \sim \text{constant} = \tau \) (new time scale): it means that we measure time in units of \( 1/\lambda^2 \) where \( \lambda \) measures the strength of the self–interaction. By putting \( \tau = 1 \) we see that the van Hove rescaling is equivalent to the time rescaling \( t \rightarrow t/\lambda^2 \), and therefore the limit \( \lambda \rightarrow 0 \) will capture the dominating contributions to the dynamics in the new time scale (the error can be estimated to be of order \( \lambda^2 \)). It is remarkable that in this limit the dominating contributions can be explicitly resummed giving rise to a new unitary operator.

A simple change of variables shows that the time rescaling \( t \rightarrow t/\lambda^2 \) is equivalent to the following rescaling of the Schrödinger equation for the evolution operator:

\[ \frac{dU^{(\lambda)}(t/\lambda^2)}{dt} = -i\frac{1}{\lambda} V(t/\lambda^2)U^{(\lambda)}(t/\lambda^2) \] (8)

The unitary operator \( U(t) \) is then obtained by taking the limit \( \lambda \rightarrow 0 \):

\[ U(t) = \lim_{\lambda \rightarrow 0} U^{(\lambda)}(t/\lambda^2) \] (9)

and the corresponding limit equation is

\[ \frac{dU(t)}{dt} = -iV(t)U(t) \]

where

\[ V(t) = \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} V \left( \frac{t}{\lambda^2} \right) \]

For a number important models the interaction Hamiltonian has the form

\[ V = A + A^+ \]
where $A$ is a monomial in the creation and annihilation operators. The master field is given by the asymptotic behaviour of the collective operator

$$A_\lambda(t) = \frac{1}{\lambda} A\left(\frac{t}{\lambda^2}\right)$$

and its Hermitian conjugate. Here

$$A(t) = e^{itH_0} A e^{-itH_0}$$

The stochastic limit is meant in the sense of the convergence of correlation functions $<A^{\epsilon_1}(t_1/\lambda) \ldots A^{\epsilon_n}(t_n/\lambda)>$. Here $\epsilon_i = \pm$.

4 Entangled Commutation Relations

In recent years there has been an interest in various modifications and deformations of the algebra of canonical commutation relations. Especially the simple relations (3) were discussed. An extension of the algebra (3) has been given in [31] as the algebra describing the interacting Fock space [30] obtained in the stochastic limit for non-relativistic QED.

In this paper we prove that the stochastic limit of interacting fields, under the only constraint of momentum conservation leads to a generalization of the algebra (3). We obtain that the new algebra has as its generators the master field $B(p,k)$ depending on two momenta $p$ and $k$ and the operator density of particles $n(k)$ which satisfy the relations

$$B(p,k)B^+(p',k') = n(p)\delta(E(p,k))\delta(p - p')\delta(k - k'),$$

$$[n(p'), B(p,k)] = (\delta(p' - p)) - \delta(p' - p + k))B(p,k),$$

$$[n(p), n(p')] = 0$$

Here $E(p,k)$ is the energy associated to the interaction vertex.

We call the relations (10)-(12) entangled commutation relations because, on one hand they allow to calculate correlations of any order among the field and, on the other hand they show that the master fields are not kinematically independent.

In the construction of these operators we use the Van Hove time rescaling $t \to t/\lambda^2$, $\lambda \to 0$ where $\lambda$ is the coupling constant.

One can get a generalization of the algebra (3) for the multiparticle master field $B(p|k_1, \ldots, k_n)$. In [31] a generalization of the algebra (3) of the following form has been obtained:

$$B(k)B^+(k') = \delta(k - k')\delta(\omega(k) + \hat{P} \cdot k), \quad [\hat{P}, B(k)] = kB(k)$$

with $\hat{P}$ being the operator of momenta of particles. This algebra is not realized in the usual Fock space but in the interacting Fock space. One can get the algebra (13) from the algebra (10)-(12) if we set

$$\hat{P} = \int pn(p)dp, \quad B(k) = \int B(p,k)dp$$

To avoid a discussion of renormalization procedure we assume that there is an ultra-violet cut-off.
These relations show that, contrarily to what happens before the limit, the observables of the particle do not commute with the master field. In other words: before the stochastic limit the particle and the field are kinematically independent but nonlinearly related by the dynamics; after the stochastic limit the dynamics is simplified but particle and field are no longer kinematically independent: this is what we call entanglement. These new features imply that the algebra (10)-(12) is not realized in the usual Fock space but in the interacting Fock space [30].

Let us consider the Hamiltonian

\[ H_\lambda = H_0 + \lambda V \]

where the free Hamiltonian

\[ H_0 = \int \varepsilon(p)c^+(p)c(p)dp + \int \omega(k)a^+(k)a(k)d^3k, \]

\[ \{c(p), c^+(p')\} = \delta(p-p'), [a(k), a^+(k')] = \delta(k-k') \]

and the interaction Hamiltonian:

\[ V = \int d^3kd^3pg(k,p)(c^+(p)c(p-k)a(k) + h.c.) \quad (14) \]

Here \( g(k,p) \) is a test function and \( \varepsilon(p) \) and \( \omega(k) \) are one-particle dispersion laws, for example \( \varepsilon(p) = p^2/2, \omega(k) = |k| \).

The rescaled collective fields in this case have the form

\[ A_\lambda(p,k,t) = \frac{1}{\lambda}e^{itH_0/\lambda^2}c^+(p)a(k)c(p-k)e^{-itH_0/\lambda^2} = \frac{1}{\lambda}c^+(p)a(k)c(p-k)e^{itE(p,k)/\lambda^2} \quad (15) \]

\[ A^+_\lambda(p,k,t) = \frac{1}{\lambda}e^{itH_0/\lambda^2}c^+(p-k)a^+(k)c(p)e^{-itH_0/\lambda^2} = \frac{1}{\lambda}c^+(p-k)a^+(k)c(p)e^{-itE(p,k)/\lambda^2} \quad (16) \]

where

\[ E(p,k) = \varepsilon(p) - \omega(k) - \varepsilon(p-k) \quad (17) \]

is the corresponding energy. One has the following main theorem.

**Theorem 1.** The stochastic limit

\[ \lim_{\lambda \to 0} A_\lambda(p,k,t) = B^-(p,k,t), \quad \lim_{\lambda \to 0} A^+_\lambda(p,k,t) = B^+(p,k,t) \]

exists in the sense of the convergence of the matrix elements ( \( \epsilon_i = \pm \))

\[ \lim_{\lambda \to 0} <0|c(q)A^i_\lambda(p_1,k_1,t_1)\cdots A^n_\lambda(p_n,k_n,t_n)c^+(q')|0 >= \]

\[ (\Psi_0, c(q)B^i_\lambda(p_1,k_1,t_1)\cdots B^n_\lambda(p_n,k_n,t_n)c^+(q')\Psi_0) \quad (18) \]

as distributions and the limiting operators \( B^- = B \) and \( B^+ \) satisfy the entangled commutation relations

\[ B(p,k,t)B^+(p',k',t') = 2\pi\delta(t-t')\delta(p-p')\delta(k-k') \cdot \delta(E(p,k))\delta(n(p)) \quad (19) \]

\[ [n(p'), B^+(p,k,t)] = (\pm)(\delta(p' - p) - \delta(p' - p + k))B^+(p,k,t) \quad (20) \]
\[ [n(p), n(p')] = 0 \]  \hspace{1cm} (21)

Here \( \Psi_0 \) is the vacuum in the new Hilbert space, \( B(p,k,t) \prod_i c^+(q_i) \Psi_0 = 0 \). We use the same notations for the creation and annihilation operators of \( c \)-particles in the original and in the new Hilbert spaces. \( n(p) \) is the operator density of the \( c \)-particles, \( n(p) = c^+(p)c(p) \).

If we set

\[
B(p,k,t) = b(t) \otimes B(p,k)
\]

where

\[
b(t)b^+(t') = 2\pi\delta(t-t')
\]

then we get the relations (11)

\[
B(p,k)B^+(p',k') = n(p)\delta(E(p,k))\delta(p-p')\delta(k-k')
\]

The theorem will be proved in the next section. Note that to get non-zero in the RHS of (19) we have to chose suitable dispersion relations, so that there are non-trivial solutions of equation \( E(p,k) = 0 \). We will discuss this point in Sect. 6.

5 Proof of the Theorem

Let us consider the matrix elements

\[
\langle 0 \vert c(q) \prod_{i=1}^\nu A_k^{\pm}(p_i, k_i, t_i) c^+(q') \vert 0 \rangle
\]  \hspace{1cm} (22)

To evaluate (22) we apply Wick’s theorem [1] and consider only the corresponding connected diagrams. Each vertex contains 3 lines which are characterized by two momenta \( (k_i, p_i) \). We find the momentum corresponding to the 3rd line using the momentum conservation. If a diagram contains \( L \) loops then only \( L + A + B - 1 \) momenta are independent (\( L \) momenta for loop variables and \( A + B - 1 \) momenta for the exterior lines).

For every vertex there is the corresponding energy exponent. These energies depend on the momenta of the lines that enter to the given vertex,

\[
E^\pm_i = E^i_\pm(k_i, p_i)
\]

via dispersion laws,

\[
E^\pm_i = [\epsilon(p_i) - \epsilon(p_i \pm k_i) \pm \omega(k_i)]
\]  \hspace{1cm} (23)

In the proof of the theorem we will use the following lemma.

**Lemma 1.** One has the following relation in the sense of distributions

\[
\lim_{\lambda \to 0} \frac{1}{\lambda^2} e^{iE(p,k)/\lambda^2} = 2\pi\delta(t)\delta(E(p,k))
\]

The proof of the lemma is standard, see [33].
Let us consider a connected diagram corresponding to the matrix element (22). The proof of the theorem consists of three parts. First we prove that only diagrams consisting of pairs of conjugated vertices don’t vanish in the limit $\lambda \to 0$. Next we prove that such diagrams are in fact non-crossing or half-planar diagrams. And finally we show that the non-crossing diagrams are described by the entangled commutation relations.

Generally, the sets of momenta corresponding to different vertices are different. However, it may happens that the same set of momenta corresponds to two different vertices. More precisely, momenta which come in the first vertex come out from the second vertex and vice versa.

**Definition 1.** We say that two incident vertices of a given connected diagram are *conjugated* if the momenta coming in the first vertex come out from the second vertex, i.e. the vertices have the same momenta but with the opposite orientation.

If the $i$–vertex has a conjugated vertex then we denote the latter by $\hat{i}$. A typical example of diagrams containing at least one pair of conjugated vertices is a diagram with a mass insertion such that this insertion contains a line that does not cross others lines of the diagram (see Fig.1). Let us prove the main

**Lemma 2.** If a connected diagram doesn’t consist only from pairs of conjugated vertices then it vanishes in the limit $\lambda \to 0$ (in the sense of distributions).

**Proof.** To a given diagram, representing a matrix element (22) being integrated over $t_1, \ldots, t_v$ with test functions, corresponds the expression that schematically can be written as

$$
\frac{1}{\lambda^v} \int e^{\sum_{i=1}^v E_i t_i / \lambda^2} \phi(t, p, q) \prod_{a} dp_a \prod_{l=1}^L dq_l \prod_{i=1}^v dt_i
$$

(24)

here by $t$ we mean $t_1, \ldots, t_v$, by $p$ we mean $p_1, \ldots, p_A, p'_1, \ldots, p'_B$ and $q$ denotes the set of independent momenta associated with the diagram under consideration. $E_i$ are given by (23) and $\phi(t, p, q)$ is a test function.

To evaluate the asymptotic behaviour of this expression when $\lambda \to 0$ we will make the change of variables corresponding to the conjugated vertices. Notice that for the diagram doesn’t vanish, the number of vertices $v$ should be even, $v = 2n$. Suppose that there are $n_0$ pairs of conjugated vertices which are denoted $\{i_1, \hat{i}_1, \ldots, i_{n_0}, \hat{i}_{n_0}\}$. Let us divide the set of all vertices $\{1, \ldots, 2n\}$ into
two disjoint subsets \( \{i_1, i_2, \ldots, i_{n_0}, i_{n_0+1}, \ldots, i_n\} \) and \( \{\hat{i}_1, \hat{i}_2, \ldots, \hat{i}_{n_0}, \hat{i}_{n_0+n_0+1}, \ldots, \hat{i}_{2n}\} \) such that in every subset there are no conjugated vertices. We denote the corresponding set of time variables \( \{t_{i_1}, \ldots t_{i_n}\} \) by \( t^{(1)} \) and the set \( \{t_{\hat{i}_1}, \ldots t_{\hat{i}_{2n}}\} \) by \( t^{(2)} \).

Now we perform the following change of variables

\[
(t^{(1)}, t^{(2)}) \rightarrow (\tau, t^{(2)})
\]

or more precisely

\[
(t^{(1)}, t^{(2)}) = (t_1, \ldots, t_{2n}) \rightarrow (\tau, t^{(2)}) = (t_1, \ldots, t_n; \ t_{i_j}, j = 1 \ldots, n_0; \ t_{i_{n_0+j}}, r = n_0 + 1, \ldots, n)
\]

(26)

\[
t_{i_j} = t_{i_j} + \lambda^2 \tau, \quad 1 \leq j \leq n_0
\]

(27)

\[
t_{i_j} = t_{i_{n_0+j}} + \lambda^2 \tau, \quad n_0 < j \leq n
\]

(28)

After this the integral (24) takes the following form

\[
\int e^{i \sum_{j=1}^{n} \tau_j E_{i_j} e^{i \sum_{j=1}^{n_0} (E_{i_j} + E_{i_j}) \tau t_{i_j} / \lambda^2}} \cdot e^{i \sum_{j=n_0+1}^{n} (E_{i_j} + E_{i_{n_0+j}}) \tau t_{i_{n_0+j}} / \lambda^2}
\]

\[
\phi(t^{(2)} + \lambda^2 \tau, t^{(2)}, p, q) \cdot \prod_{j=1}^{n} dt_j \prod_{j=1}^{n_0} dt_{i_j} \prod_{j=n_0+1}^{n} dt_{i_{n_0+j}} \prod dp \prod dq
\]

(29)

By definition of conjugated vertices \( E_{i_j} + E_{\hat{i}_j} = 0 \) and we left with

\[
\int e^{i \sum_{j=1}^{n} \tau_j E_{i_j} e^{i \sum_{j=n_0+1}^{n} (E_{i_j} + E_{i_{n_0+j}}) \tau t_{i_{n_0+j}} / \lambda^2}}
\]

\[
\phi(t^{(2)} + \lambda^2 \tau, t^{(2)}, p, q) \cdot \prod_{j=1}^{n} dt_j \prod_{j=1}^{n_0} dt_{i_j} \prod_{j=n_0+1}^{n} dt_{i_{n_0+j}} \prod dp \prod dq
\]

(30)

Here \( t^{(2)} + \lambda^2 \tau \) schematically represents the dependence of the half of the \( t \)-variables on \( \lambda \). When \( \lambda \to 0 \) we can neglect the dependence of \( \phi \) on \( \lambda \) and the integration over \( \tau \) gives the product of \( \delta(E_{i_j}) \)

\[
\int \prod_{j=1}^{n} \delta(E_{i_j}) e^{i \sum_{j=n_0+1}^{n} (E_{i_j} + E_{i_{n_0+j}}) \tau t_{i_{n_0+j}} / \lambda^2} \phi(t, t, p, q) \prod_{j=n_0+1}^{n} dt_{i_{n_0+j}} \prod dp \prod dq
\]

(31)

Note that the second exponent in the expression (30) vanishes since the energies in the conjugated vertices are equal.

Suppose that in the diagram there are non-conjugated vertices, i.e. \( n \neq n_0 \). When \( \lambda \to 0 \) the expression (30) goes to zero since, according to our assumption the set of momenta in vertices \( \hat{i}_j \) and \( i_{n+j}, \ n_0 < j \leq n \) do not coincide and therefore the functions \( E_{i_j} + E_{i_{n+j}} \) as functions of momenta don’t vanish and therefore according the Riemann-Lebesgue lemma we get zero in the limit \( \lambda \to 0 \).

In the case when \( n = n_0 \) the exponent in (30) vanishes and generally we get the non–zero answer:

\[
\int \left( \prod_{i=1}^{n} \delta(E_{i_j}) \right) \phi(\{t_{i_j}\}, \{t_{i_j}\}, p, q) \prod dp \prod dq
\]

(32)
The lemma is proved.

**Lemma 3.** If a connected diagram with two external lines consists only from pairs of conjugated vertices then it is half-planar, i.e. it can be drawn in the half-plane without self intersections.

We will not present here the simple proof of this lemma.

**Lemma 4.** The limiting expression (32) is equal to the RHS of the relation (18).

The proof is obtained by using the algebra (19), (21).

Now the theorem follows from the above four lemma.

**Remark 1.** We have considered only matrix elements with two external particles and as a result the matrix element doesn’t vanish only if \( v \) is even, \( v = 2n \). In the general case of matrix elements with an arbitrary number of external lines the number of vertices \( v \) can be odd. For the case of odd \( v \), \( v = 2n - 1 \), we once again select \( n \) vertices and make the following change of variables

\[
(t_1, \ldots, t_{2n-1}) \rightarrow (\tau_1, \ldots, \tau_n; t_{ij}, j = 1, \ldots, n_0; t_{in+r}, r = 1, \ldots, n - 1 - n_0)
\]

with the \( \tau_i \) as before (see (27)).

\[
t_{ij} = t_{ij} + \lambda^2 \tau_j,
\]

\[
t_{ij} = t_{i+n+j} + \lambda^2 \tau_j \quad n_0 + 1 \leq j \leq n - 1
\]

\[
t_{in} = \lambda^2 \tau_n
\]

The difference with the case of even \( v \) is that we get an extra factor \( \lambda \) in front of the integral

\[
\lambda \int e^{i \sum_{j=1}^{n} \tau_j E_{ij}} e^{i \sum_{j=1}^{n_0} (E_{ij} + E_{i+n,j}) t_{ij} / \lambda^2} e^{i \sum_{j=n_0+1}^{n-1} (E_{ij} + E_{i+n,j}) t_{i+n+j} / \lambda^2}
\]

\[
\phi(t + \lambda \tau, t, p, q) \prod_{j=1}^{n} dp dq \prod_{j=n_0}^{n} dt_i \prod_{j=n_0}^{n} dt_{i+n+j}
\]

(34)

Here we use the same schematical notations as in (5). When \( \lambda \rightarrow 0 \) we neglect the \( \lambda \)-dependence of \( \phi \) and we get a product of \( \delta \)-functions. The second exponent goes out. The third exponent disappears in the case then \( n_0 = n - 1 \), but since we have an extra factor \( \lambda \) the expression (8) always goes to zero as \( \lambda \rightarrow 0 \).

**Remark 2.** The theorem admits a generalization to the case of an arbitrary number of external particles

\[
< 0 | \prod_i c(q_i) \mathcal{A}^{c_i}_X(p_1, k_1, t_1) \cdots \mathcal{A}^{c_n}_X(p_n, k_n, t_n) \prod_j c^+(q'_j) | 0 >
\]

**Remark 3.** For the composite operators

\[
\mathcal{A}_X(t, p, k_1, \ldots, k_m) = \frac{1}{\lambda} e^+(p)c(p - k_1)a^{c_1}(k_1)\cdots a^{c_m}(k_m)e^{itE(p,k_1,\ldots,k_m)/\lambda^2}
\]

in the stochastic limit one gets the entangled commutation relations (4).
6 The stochastic limit for Green functions

The master field in the standard formulation of the stochastic limit is defined by the limit at \( \lambda \to 0 \) of the Wightman correlation functions

\[
< 0|A_\lambda^{(1)}(t_1)\ldots A_\lambda^{(n)}(t_n)|0 >
\]

(35)

This limit defines the Hilbert space in which the master field lives. For some models the limit of these correlation functions is trivial (equal to zero). However this doesn’t mean that the stochastic limit of such the model is trivial because we can consider another natural formulation of the stochastic limit which is defined by the convergence of the chronologically ordered correlation functions (Green functions)

\[
< 0|T(A_\lambda^{(1)}(t_1)\ldots A_\lambda^{(n)}(t_n))|0 >
\]

(36)

For the evolution operator one has to consider

\[
< 0|T(A_\lambda^{(1)}(t_1)\ldots A_\lambda^{(n)}(t_n)U(t))|0 >
\]

(37)

Here the \( T \)-product is defined as

\[
T(A_\lambda^{(1)}(t_1)\ldots A_\lambda^{(n)}(t_n)) = A_\lambda^{(i_1)}(t_{i_1})\ldots A_\lambda^{(i_n)}(t_{i_n})
\]

if \( t_{i_1} \geq \ldots \geq t_{i_n} \).

The limit of the Green functions can be nontrivial even if the limit of the Wightman functions vanishes. In the simplest case in the first formulation we use the relation

\[
\lim_{\lambda \to 0} \frac{1}{\lambda^2} e^{itx/\lambda^2} = 2\pi \delta(t)\delta(x)
\]

(38)

and in the second

\[
\lim_{\lambda \to 0} \frac{1}{\lambda^2} \theta(t) e^{itx/\lambda^2} = i\delta(t) \frac{1}{x + i0}
\]

(39)

Here \( \theta(t) = 1, t \geq 0; \theta(t) = 0, t < 0 \). In particular if

\[
A_\lambda(t,p) = \frac{1}{\lambda} e^{i\omega(p)/\lambda^2} a(p)
\]

where \([a(p), a^+(p')] = \delta(p - p'), p, p' \in \mathbb{R}^3\) then the stochastic limit of the Wightman functions is

\[
< 0|A_\lambda(t,p)A_\lambda^{+}(t',p')|0 > = \frac{1}{\lambda^2} e^{i(t-t')\omega(p)/\lambda^2} \delta(p - p') \to 2\pi \delta(t - t')\delta(\omega(p))\delta(p - p')
\]

and the stochastic limit of the Green functions is

\[
< 0|T(A_\lambda(t,p)A_\lambda^{+}(t',p'))|0 > = \frac{1}{\lambda^2} \theta(t - t') e^{i(t-t')\omega(p)/\lambda^2} \delta(p - p') \to i\delta(t - t') \frac{1}{\omega(p) + i0} \delta(p - p')
\]

It is important to notice that in the last formula one gets a nontrivial limit even if \( \delta(\omega(p)) = 0 \) when in the former formulation the limit is trivial.
7 Stochastic Limit and Decay

In the formulation of Theorem 1 we assumed the special form of the interaction Hamiltonian (14). In local quantum field theory the typical Hamiltonian is more complicated than (14). In particular, for the Yukawa interaction of fields $\psi$ and $\phi$ the Hamiltonian has the form

$$H_\lambda = H_0 + \lambda V$$

where $H_0$ is the sum of the free Hamiltonians for the fermionic field $\psi$ and for the scalar field $\phi$ with relativistic dispersion laws,

$$\omega_a(k) = \sqrt{m_a^2 + k^2}, \quad \omega_b(k) = \sqrt{m_b^2 + k^2}$$  \hspace{1cm} (40)

and

$$V = \int d^3x g\bar{\psi}\psi\phi = \int d^3k d^3p g(k,p)(c^+(p)c(p-k)a(k) + c^+(p)c^+(k-p)a(k) + c^+(p)a^+(k)c^+(-p-k) + h.c.)$$  \hspace{1cm} (41)

Here $g(k,p)$ is a test function. This expression is not a well defined operator in the Fock space but it defines a bilinear form. We have the following collective operators

$$\frac{1}{\lambda} c^+(p)c(p-k)a(k) \exp \frac{it}{\lambda^2} (\varepsilon(p) - \varepsilon(p-k) - \omega(k))$$  \hspace{1cm} (42)

$$\frac{1}{\lambda} c^+(p)c^+(k)a(p+k) \exp \frac{it}{\lambda^2} (\varepsilon(p) + \varepsilon(k) - \omega(p+k))$$  \hspace{1cm} (43)

$$\frac{1}{\lambda} c^+(p)c^+(k)a^+(p-k) \exp \frac{it}{\lambda^2} (\varepsilon(p) + \varepsilon(k) + \omega(p+k))$$  \hspace{1cm} (44)

In the limit $\lambda \to 0$ the operator (44) vanishes because in the correlation functions one gets

$$\delta(\omega_c(p) + \omega_a(k) + \omega_c(p+k)) = 0$$

due to the positivity of energy.

The limit of the operator (42) is also zero since for relativistic dispersion laws (40)

$$\delta(\omega_c(p) + \omega_a(k) - \omega_c(p+k)) = 0$$

Only operator (43) has a chance to be non-zero. In this case we have to find non-trivial solutions of

$$\omega_c(p) + \omega_c(k-p) = \omega_a(k)$$  \hspace{1cm} (45)

There are solutions if

$$m_a^2 > 2m_c^2,$$

i.e. if one has the decay. Therefore we obtain that the relativistic interaction Hamiltonian (41) is within the same stochastic universality class as the interaction Hamiltonian (14)

$$V = \int d^3k d^3p g(k,p)(c^+(p)c^+(k-p)a(k) + h.c.)$$  \hspace{1cm} (46)
In quantum electrodynamics (QED) the interaction has the form

\[ V = \int d^3x \bar{\psi} \gamma^\mu \psi A_\mu \]

If one neglects the spinor and polarization indices then the free Hamiltonian has the form (40) and the interaction Hamiltonian has the form (41). We see that the stochastic limit of QED Hamiltonian is reduced to the form of the above discussed Hamiltonian and as a result we obtain the trivial limit, since there is no decay in the standard QED. However, if we consider QED in external field we can get a non-zero result due to the change of dispersion law. One of such examples we will consider in the next section.

8 Photon splitting cascades and a new statistics

In the previous section we have seen that the stochastic limit of Wightman functions in relativistic QED is trivial and that it can be non-trivial in the presence of external fields. It is known that there is the splitting of a photon into two in an external magnetic field. This splitting is one of the most interesting manifestations of the non–linearity of Maxwell’s equations with radiative corrections in an external magnetic field. In a constant uniform field, this process occurs with conservation of energy and momentum. The process was considered by Adler et al. in the early ’70s by using the Heisenberg–Euler effective Lagrangian [34, 35, 36]. Photon splitting was considered as a possible mechanism for the production of linearly polarized gamma–rays in a pulsar field. Recently the splitting of photons has found astrophysical applications in the study of annihilation line suppression in gamma ray pulsars and spectral formation of gamma ray bursts from neutron stars [37, 38]. Photon splitting cascades have also been used in models of soft gamma-ray repeaters, where they soften the photon spectrum [39, 40]. The process of photon splitting is potentially important in applications to a possible explanation of the origin of high energy cosmic rays from Active Galactic Nuclei [41]. A recalculation of the amplitude for photon splitting in a strong magnetic field has been performed recently in [42, 43, 44].

We will start from a discussion of the theory of photon splitting cascades and show the emergence of infinite statistics in this theory and then discuss its connection with the stochastic limit of quantum field theory. In the decay of a photon with momentum \( k \) into photons with momentum \( k_1 \) and \( k_2 \), we have conservation of momentum and energy \( k = k_1 + k_2 \), \( \omega(k) = \omega_1(k_1) + \omega_2(k_2) \). For photons in vacuum, in the absence of external fields, \( \omega = \omega_1 = \omega_2 = k \) and although these two equations have a solution the decay is forbidden by the invariance under charge conjugation (Furry’s theorem).

In a constant uniform magnetic field \( B_0 \) there are only two decay processes kinematically allowed, \( \gamma_\parallel \rightarrow \gamma_\perp + \gamma_\perp \) and \( \gamma_\parallel + \gamma_\perp \rightarrow \gamma_\parallel + \gamma_\perp \) [33]. Here the subscripts \( \perp \) and \( \parallel \) will denote polarizations of the photon with respect to the vector \( B_0 \). More precisely, in presence of a magnetic field one has a distinctive plane, namely the \( kB_0 \) plane. One takes the linear polarization of the magnetic field of the photon parallel and orthogonal to this plane as the two independent polarizations of the photon, \( \parallel \) and \( \perp \), respectively.

The vacuum in the presence of the field \( B_0 \) acquires an index of refraction \( n \), and the photon dispersion relation is modified from \( k/\omega = 1 \) to \( k/\omega = n \). The indices of refraction \( n_\parallel, n_\perp \) can be calculated from the Heisenberg–Euler effective lagrangian. Adler showed that for subcritical fields in the limit of weak vacuum dispersion only the splitting mode \( \parallel \rightarrow \perp + \perp \) operates below pair production threshold. For weak dispersion \( n_\perp = 1 + \frac{7}{90} \beta \) and \( n_\parallel = 1 + \frac{2}{45} \beta \), where \( \beta = \frac{e^2}{m^2 c^2} B_0^2 \sin^2 \theta \).
and $\theta$ is the angle between $k$ and $B_0$. It is mentioned by Harding et al. that in magnetar models of soft gamma repeaters, where supercritical fields are employed, moderate vacuum dispersion arises. In such a regime, it is not clear whether Adler’s selection rules still endure since in his analysis higher order contributions to the vacuum polarization are omitted. In [38] photon cascades are considered for the case where all three photon splitting modes allowed by CP invariance are operating. Baier et al. [36] have found that there is only one allowed transition ($\parallel \rightarrow \perp + \perp$) for any magnetic field. They suggested that a photon cascade could develop only if magnetic field changes its direction. It seems that the question on the validity of Adler’s rule for a non weak vacuum dispersion deserves a further study. In this work we consider photon cascades when both kinematically allowed modes ($\parallel \rightarrow \perp + \perp$ and $\parallel \rightarrow || + \perp$) operate.

The interaction operator for the decay $\parallel \rightarrow \perp + \perp$ is known to be [36]

$$V_{\parallel}(t) = \lambda_1 \int (B_0 E_1)(B_0 E_2)(B_0 B) d^3 x,$$  \hspace{1cm} (47)

where the coupling constant $\lambda_1 = 13e^6/315\pi^2 m^8$ and magnetic and electric parts of photon field are

$$B = i(4\pi)^{1/2}k \times E_{\parallel} e^{-i(kr - \omega t)}a_{\parallel}(k), \hspace{0.5cm} E_{1} = -i(4\pi)^{1/2}\omega_{1} E_{\perp} e^{i(k_{1}r - \omega_{1} t)} a_{\perp}^{+}(k_1)$$ \hspace{1cm} (48)

and similarly for $E_2$, here $\omega = \omega_{\parallel}(k)$ and $\omega_{i} = \omega_{\perp}(k_i), \ i = 1, 2$. Here $B, B_0, E_i$ are three-dimensional fields and $k_i$ are three dimensional vectors.

For the decay $\parallel \rightarrow || + \perp$ one has a similar interaction operator with the operator structure

$$A^+(t) = \lambda a_{\parallel}^{+}(p-k) a_{\perp}(k) a_{\parallel}(p) e^{-itE}$$ \hspace{1cm} (49)

where $E = \omega_{\parallel}(p) - \omega_{\parallel}(p-k) - \omega_{\perp}(k)$. The coupling constant $\lambda$ in this case can be estimated as $\lambda/\lambda_1 = \alpha (B_0/B_{cr})^2$, where $\alpha = e^2/hc$ and $B_{cr} = m^2e^3/eh \simeq 4.4 \times 10^{13}$ Gauss.

Let us consider a photon cascade created by a photon with momentum $p$ and polarization $\parallel$. The photon splits as $\gamma_{\parallel}(p) \rightarrow \gamma_{\perp}(k_1) + \gamma_{\parallel}(p-k_1)$. Then one has the next generation of splitting: $\gamma_{\parallel}(p-k_1) \rightarrow \gamma_{\perp}(k_2) + \gamma_{\parallel}(p-k_1-k_2)$ etc. After $N$ generations of splitting one gets a cascade with $N$ photons with $\perp$ polarization and momenta $k_1, k_2 \ldots, k_N$ and also one photon with $\parallel$ polarization and momentum $k - k_1 - \ldots - k_N$. An example of a cascade with two generations is shown in Fig.2.

Our goal is to consider cascades with real photons (i.e. on the mass shell) including the intermediate states (compare with what one sees in the Wilson camera). These cascades can be

![Figure 2: Cascade: $\gamma_{\parallel}(p) \rightarrow \gamma_{\perp}(k_1) + \gamma_{\parallel}(p-k_1) \rightarrow \gamma_{\perp}(k_1) + \gamma_{\parallel}(p-k_2) + \gamma_{\parallel}(p-k_1-k_2)$]
drawn as shown in Fig. 2. This diagram is not a Feynman one because all the lines (including an intermediate one) correspond to real particles on the mass shell and not to virtual states. More precisely all the lines in the diagram are “dressed” lines on the mass shell and the initial photon $\gamma_\parallel(k)$ is prepared in a special way such that it undergoes the decay in a finite time. So we cannot use the standard $S$-matrix approach and the standard Feynman diagram technique to describe this process. The diagram is also not a diagram in the non–covariant diagram technique [9] because we have the conservation of energy at every vertex. The cascade in Fig. 2 may be intuitively described by the following state

$$|\psi(p, k_1, k_2)\rangle = f(p, k_1) f(p - k_1, k_2) a_\parallel^+(p - k_1 - k_2) a_\perp^+(k_2) a_\perp^+(k_1)|0\rangle$$

(50)

where momentum conservation is built in creation and annihilation operators and energy conservation is accounted for by the factor $f(p, k) = f(\omega_\parallel(p) - \omega_\perp(k) - \omega_\parallel(p - k))$ where $f(\omega)$ is a function with support at $\omega = 0$. As we shall see below this is not the $\delta$–function but roughly speaking its “square root”. Indeed, the transition amplitude between two cascade states is given by scalar product

$$\langle \psi(p', k'_1, k'_2)|\psi(p, k_1, k_2)\rangle = |f(p, k_1)|^2 |f(p - k_1, k_2)|^2 \delta(p - p') \delta(k_1 - k'_1) \delta(k_2 - k'_2)$$

(51)

Notice that in the scalar product (51) only the non–crossing diagram Fig. 3a contributes. In fact the contribution from the crossing diagram in Fig. 3b is negligible because of conservation of energy and momentum. This is the crucial point where the difference between our diagrams describing real particles in intermediate states and the Feynman diagrams having virtual particles in intermediate states is manifested. In the Feynman diagram technique the amplitude of emission of the two photons is represented by a sum of two diagrams differing by the order in which the two photons are emitted. Here we have only one diagram, Fig. 2.

Now let us observe that if in (50) we replace the operators $a_\perp^+(k_1)$ and $a_\perp^+(k_2)$ by the quantum Boltzmann operators $b_\perp(k_1)$ and $b_\perp(k_2)$ satisfying the relations (1), i.e. $b_\perp(k)b_\perp^+(p) = \delta(k - p)$, then it will be automatically guaranteed that only the non-crossing diagrams survive. Therefore it is natural to describe cascade wave functions in terms of these operators. It is well known that standard free photons are bosons. Therefore to see the quantum Boltzmann statistics we have to prepare a special state depending on the interaction. In fact it is natural to expect that the cascades with physical intermediate states occur at a time scale slower than the one occurring in the standard $S$-matrix approach to multiparticle production. A natural method, leading to this result, is suggested by the stochastic limit technique.

Now let us consider the question how one can prepare a state which exhibits the new statistics for photons in cascade. If we would deal with the scattering of 2–particle states at infinite time ($S$–matrix) we simply have to consider two Feynman diagrams to take into account the Bose statistics
of photons. However in the cascade we deal with evolution in finite time and the states of photons $\gamma_{\perp}(k_1)$ and $\gamma_{\perp}(k_2)$ are prepared in a special way because they are emitted at times $t_1$ and $t_2$, respectively. Therefore, there is a reason not to add the second diagram. A special procedure which is adequate to this situation is the stochastic limit technique described in Sect.3. In our case the master field is given by the asymptotic behaviour of the following collective operator

$$\lim_{\lambda \to 0} \frac{1}{\lambda} a^\dagger_\| (p - k) a^\dagger_\| (k) a_\| (p) e^{-iE/\lambda^2} = B^\dagger (p, k)$$

where $E$ is the same as in (19) and as in Sect.3 – Sect.5 the limit is meant in the sense of the Wightman correlation functions.

As it follows from Theorem 1 the master field $B^{\pm}(p, k)$ satisfies the following commutation relations

$$\mathcal{B}(p, k, t)B^\dagger (p', k', t') = 2\pi \delta (t - t') \delta (E) \delta (k - k') \delta (p - p') b^\dagger_\| (k) b_\| (k)$$

and $b_\| (p), b^\dagger_\| (p')$ satisfy the relations

$$b_\| (p) b^\dagger_\| (p') = \delta (p - p')$$

The presence of the $\delta(E)$-factor ($E = \omega_\| (p) - \omega_\| (p - k) - \omega_\perp (k) )$ has two important physical consequences. First, the commutation relations for the $B^\#$ are not a consequence of the corresponding relations for $b^\#_\|$ and $b^\#_\perp$: the three photons are entangled into a single new object (triphon). Second, the triphon creation and annihilation operators $B^\#$ operate not on the usual Fock space but in interacting Fock space.

By introducing the auxiliary creation and annihilation operators $b(t), b^\dagger (t), b_\perp (p), b^\dagger_\perp (p)$, satisfying the quantum Boltzmann relations

$$b(t)b^\dagger (t') = \delta (t - t'),$$

$$b_\perp (p) b^\dagger_\perp (p') = \delta (p - p')$$

and introducing the symbolic relation

$$\mathcal{B}^\dagger (p, k, t) = b^\dagger (t) b^\dagger_\| (k_1) b^\dagger_\perp (k_2) b_\| (k) (2\pi)^{1/2} \delta_{1/2} (E)$$

we can disentangle the master field by expressing it as a product of individual Boltzmannian fields. Here the notation $\delta_{1/2} (E)$ is purely symbolic and it simply means that, since the new commutation relation (53) is quadratic in the master creation and annihilation operators, if in the above symbolic relation we consider the symbol $\delta_{1/2} (E)$ as a scalar function satisfying the formal relation $[\delta_{1/2} (E)]^2 = \delta (E)$, and if we use the right hand side of (51) to express the left hand side of (53), then the standard Boltzmannian relations (56) and (54) will reduce (53) to an identity. The intuitive understanding of the "disentangling" relation (57) is that the triphon master field $\mathcal{B}^\dagger (p, k, t)$ can be expressed as the product of three "Boltzmannian photons": one can think that each photon of the triple has its own (Boltzmannian) creation and annihilation operators depending on its own momentum, however in the master field these operators can only appear in the combination given by the right hand side of (57) and there is a constraint among the three momenta expressed by energy conservation.

Notice the Boltzmannian white noise relation (53), which makes our model particularly suitable for Monte Carlo simulations. The origin for these new commutation relations lies in the fact that
the crossing diagrams in the computation of the matrix element (50) are suppressed in the weak coupling/large time limit.

A photon splits into two not only in a magnetic field but also in a nonlinear medium. In fact such processes are well known in nonlinear quantum optics, see for example [12]. In the nonlinear process of parametric down conversion a high frequency photon splits into two photons with frequencies such that their sum equals that of the high-energy photon. The two photons produced in this process possess quantum correlations and have identical intensity fluctuations.

In conclusion, in this paper we have argued that photon cascades in a strong magnetic field might create a new type of entangled states (triphons) which obey not Bose but the quantum Boltzmann statistics. This prediction is based on the assumption that both kinematically allowed photon splitting modes operate.

Given the validity of this assumption we prove that, in the stochastic regime the intermediate photons in a cascade are real and virtual particles. The dominating contributions to the dynamics come from these entangled triples of photons which behave like single new particles (triphons) whose statistics can be experimentally observed by counting the emitted photons in the corresponding cascades.

As explained in the introduction, the time scale in which our predictions are true is long compared to the strength of the coupling but short if compared to the time scale of the $S$–matrix approach. This remark should be kept into account in a possible experimental verification of these predictions. A better theoretical understanding of the photon splitting with a non weak dispersion is required. From the experimental side new more precise devices such as the planned Integral mission [37] might significantly advance our understanding of the fundamental problem of photon statistics.

Acknowledgments. I.Ya.A. and I.V.V. are grateful to the Centro Vito Volterra Universita di Roma Tor Vergata for the kind hospitality. This work is supported in part by INTAS grant 96-0698, I.Ya.A. is supported also by RFFI-99-01-00166 and I.V.V. by RFFI-99-01-00105

References

[1] N.N. Bogoliubov and D.V. Shirkov, Introduction to the theory of quantum fields, Nauka, 1973
[2] N.N. Bogoliubov, Problems of dynamical theory in statistical physics, Gostehizdat, 1946
[3] V.L. Bonch-Bruevich and S.V. Tyablikov, Method of Green functions in statistical physics, Nauka, Moscow, 1961
[4] A.A. Abrikosov, L.P. Gorkov and I.E. Dzyaloshinskii, Methods of quantum field theory in statistical physics, Nauka, Moscow, 1962
[5] D.N. Zubarev, Non-equilibrium statistical thermodynamics, Nauka, Moscow, 1971
[6] N.N. Bogoliubov and S.V. Tyablikov, Dokladi AN USSR, 126(1959)53-57
[7] J. Schwinger, Field theory of unstable particles, Ann. Phys. 9(1960)169-193
[8] M.L. Goldberger and K.M. Watson, Collision theory, John Wiley & Sons, Inc, New York-London-Sydney, 1964
[9] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grinberg, Atom-Photon Interactions, Basic Processes and Applications, John Wiley & Sons, Inc., 1992

[10] E.L. Feinberg, A particle with non-equilibrium proper field, in: Problems of theoretical physics, Memorial volume to Igor E. Tamm, Nauka, Moscow, 1972

[11] V.A. Rubakov and M.E. Shaposhnikov, Electroweak baryon number non-conservation in the early Universe and in high-energy collisions, Uspehi Fisicheskikh nauk, 166 (1996) 493-538

[12] D.F. Walls and G.J. Milburn, Quantum Optics, Springer-Verlag, 1994.

[13] I. Prigogine, In: Fundamental problems in elementary particle physics, XIV Conseil de Physique solvay, Brussels, October 1967, Interscience, 1968

[14] L. Accardi, A. Frigerio and Y.G. Lu, On the weak coupling limit problem, in: Quantum Probability and Applications IV, Springer LNM, N1396 (1987) 20-58

[15] L. Accardi, Y.G. Lu and I.V.Volovich, Quantum theory and its stochastic limit, Oxford Univ.Press, (in press)

[16] L. van Hove, Quantum mechanical perturbations giving rise to a transport equation, Physica, 21(1955)517-540

[17] L.D. Faddeev, Dokladi AN USSR, 152 (1963) 573

[18] I.Ya. Aref’eva, Teor. Mat. Fis. 14 (1973) 3

[19] G.F. Chew, The analytic S-matrix. A basis for nuclear democracy, W.A.Benjamin, Inc. New York-Amsterdam, 1966

[20] I. Prigogine, F. Henin, Kinetic theory and subdynamics, In: Problems of theoretical physics, Essays dedicated to Nikolai N. Bogolubov on the occasion of his sixtieth birthday, Nauka, Moscow, 1969, p.356-364

[21] N.N. Bogoliubov, An elementary example of the statistical equilibrium in a system related with reservoir, in: On certain statistical methods in mathematical physics, Academy of Sciences of Ukraine SSR, 1946.

[22] K.O. Friedrichs, On the perturbation of continuous spectra, Comm. Pure Appl. Math. 1(1948)361-406

[23] I.Ya. Aref’eva, Phys.Lett. B325 (1994) 171; I.Ya. Aref’eva and I.V. Volovich, Anisotropic asymptotics and high energy scattering in QCD, in: “Quarks 94”, eds, D.Yu. Grigoriev et al., World Scientific, 1995, p.155-170

[24] V.A. Matveev, Origin of the quark counting laws, in: “Quarks 94”, eds, D.Yu. Grigoriev et al., World Scientific, 1995, p.41-51

[25] L. Accardi, S.V. Kozyrev, I.V. Volovich, Dynamics of dissipative two-state systems in the stochastic approximation, Phys. Rev. A 57 N. 3 (1997); quant-ph/9706021

[26] L. Accardi, S.V. Kozyrev, I.V. Volovich, On the non-exponential decay in the polaron model
[27] I.Ya. Aref’eva and I.V. Volovich, Quantum group particles and non-archimedean geometry, Phys. Lett. B268(1991)179-187

[28] O. Greenberg, Phys.Rev.Lett. 64 (1990) 705.

[29] I.Ya. Aref’eva and I.V. Volovich, Nucl.Phys. B462(1996)600-615

[30] L. Accardi, Y.G. Lu, Comm. Math. Phys. 180 (1996) 605

[31] L. Accardi, Y.G. Lu and I.V. Volovich, Interacting Fock spaces and Hilbert module extensions of the Heisenberg commutation relations, Publications of IAS, Kyoto,1997

[32] M. B. Halpern and C. Schwartz, The Algebras of Large N Matrix Mechanics, hep-th/9809197.

[33] V.S. Vladimirov, Equations of mathematical physics, Nauka, Moscow, 1981

[34] S.L. Adler, J.N. Bahcall, C.G. Callan and M.N. Rosenbluth, Phys. Rev. Lett., 25(1970)1061.

[35] S.L. Adler, Ann. Phys. 67(1971)599

[36] V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, Quantum Electrodynamics, Pergamon Press, 1982.

[37] M.G. Baring, A.K. Harding, and P.L. Gonthier, The attenuation of gamma-ray emission in strongly-magnetized pulsars, astrop-ph/9704210.

[38] A.K. Harding, M.G. Baring and P.L. Gonthier, Photon splitting cascades in gamma-ray pulsars and the spectrum of PSR1509-58, astro-ph/9609167.

[39] M.G. Baring, Astrophys. Journ. Lett., 440 (1995) L 69.

[40] A.K. Harding and M.G. Baring, Photon Splitting in Soft Gamma Repeaters, astro-ph/9603095.

[41] R.J. Protheroe, Origin and propagation of the highest energy cosmic rays, astro-ph/9610100

[42] S.L. Adler and C. Schubert, Photon splitting in a strong magnetic field: recalculation and comparison with previous calculations, hep-th/9605033.

[43] V.N. Baier, A.I. Milstein and R.Zh. Shaisultanov, Phys. Rev. Lett., 77(1996)1691.

[44] J.S. Heyl and L. Hernquist, Birefringence and Dichroism of the QED Vacuum, hep-ph/9705367.