Covering versus partitioning with Polish spaces

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BLAST

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Covering and partitioning numbers

Given a completely metrizable space $X$, define

$$\text{cov}(X) = \min\{|C| : C \text{ is a covering of } X \text{ with Polish spaces}\},$$

$$\text{par}(X) = \min\{|\mathcal{P}| : \mathcal{P} \text{ is a partition of } X \text{ into Polish spaces}\}.$$
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Motivating question:

*Is it possible to have \( \text{cov}(X) < \text{par}(X) \)?*
Partitioning spaces of weight $< \aleph_\omega$
Structure beyond $\aleph_\omega$: SSH and $\Box$
Chaos at $\aleph_\omega$: $(\aleph_\omega+1, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$

The main results

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**Theorem**

*It is consistent relative to a huge cardinal that* \( \text{cov}(X) < \text{par}(X) \)* for some completely metrizable space \( X \).
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Furthermore, large cardinal hypotheses are required for obtaining this inequality:

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We aim to sketch some of the main ideas involved in proving these two theorems, beginning with the second. The first step is a ZFC theorem concerning spaces of weight \( < \aleph_\omega \).
Partitioning spaces of uncountable weight

Lemma

If $X$ is completely metrizable and $w(X) = \kappa$ has uncountable cofinality, then $X$ can be partitioned into $\leq \kappa$ completely metrizable spaces of strictly smaller weight.
Partitioning spaces of uncountable weight

Lemma

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Proof.

Let $\mathcal{B}$ be a basis for $X$ such that $|\mathcal{B}| = w(X) = \kappa$, and every point of $X$ is contained in only countably many members of $\mathcal{B}$. 
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Let $\mathcal{B}$ be a basis for $X$ such that $|\mathcal{B}| = w(X) = \kappa$, and every point of $X$ is contained in only countably many members of $\mathcal{B}$. Write $\mathcal{B} = \{ U_\alpha : \alpha < \kappa \}$. For each $\alpha < \kappa$, let

$X_\alpha = \{ x \in X : \text{if } x \in U_\beta \text{ then } \beta < \alpha \}$

and

$Y_\alpha = X_\alpha \setminus \bigcup_{\beta < \alpha} X_\beta.$
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$X_\alpha = \{x \in X : \text{if } x \in U_\beta \text{ then } \beta < \alpha\}$ and $Y_\alpha = X_\alpha \setminus \bigcup_{\beta < \alpha} X_\beta$.

Because $\text{cf}(\kappa) > \omega$, and because of our choice of $\mathcal{B}$, every $x \in X$ is in some $X_\alpha$, and therefore the $Y_\alpha$ partition $X$. 
Partitioning spaces of weight $< \mathfrak{w}$
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# Partitioning spaces of uncountable weight

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$$X_\alpha = \{x \in X : \text{if } x \in U_\beta \text{ then } \beta < \alpha\} \text{ and } Y_\alpha = X_\alpha \setminus \bigcup_{\beta < \alpha} X_\beta.$$  

Because $cf(\kappa) > \omega$, and because of our choice of $\mathcal{B}$, every $x \in X$ is in some $X_\alpha$, and therefore the $Y_\alpha$ partition $X$. Clearly each $X_\alpha$, and therefore each $Y_\alpha$, has weight $< \kappa$. 

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If $\langle x_n : n \in \omega \rangle$ is a sequence in $X_\alpha$ with limit $p$, then every neighborhood $U_\beta$ of $p$
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If $\langle x_n : n \in \omega \rangle$ is a sequence in $X_\alpha$ with limit $p$, then every neighborhood $U_\beta$ of $p$ is also a neighborhood of some $x_n$. 
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**Lemma**

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If $\langle x_n : n \in \omega \rangle$ is a sequence in $X_\alpha$ with limit $p$, then every neighborhood $U_\beta$ of $p$ is also a neighborhood of some $x_n$, and therefore $\beta < \alpha$ (as $x_n \in X_\alpha$).
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If $\langle x_n : n \in \omega \rangle$ is a sequence in $X_\alpha$ with limit $p$, then every neighborhood $U_\beta$ of $p$ is also a neighborhood of some $x_n$, and therefore $\beta < \alpha$ (as $x_n \in X_\alpha$). So $\beta < \alpha$ whenever $U_\beta \ni p$, which means that $p \in X_\alpha$. 
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It is now straightforward to show the $Y_\alpha$ are completely metrizable: If $\alpha = \beta + 1$, then $Y_\alpha = X_\alpha \setminus X_\beta$ is the difference of two closed sets, hence $G_\delta$. 

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- If $\alpha$ has countable cofinality, write $\alpha = \sup\langle \beta_n : n \in \omega \rangle$, and then $Y_\alpha = X_\alpha \setminus \bigcup_{n \in \omega} X_{\beta_n}$ is again $G_\delta$. 
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If $\alpha$ has uncountable cofinality, then $Y_\alpha = \emptyset$, because $X_\alpha = \bigcup_{\beta < \alpha} X_\beta$ by our choice of $B$. 

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Theorem

If $X$ is a completely metrizable space and $w(X) = \aleph_n < \aleph_\omega$, then $\text{cov}(X) = \text{par}(X) = \aleph_n$. 

Proof. The previous theorem, plus a simple induction argument, shows $\text{par}(X) \leq w(X)$ whenever $w(X) < \aleph_\omega$. It is not difficult to prove that $w(X) \leq \text{cov}(X) \leq \text{par}(X)$ for any $X$. 

Corollary

Suppose $\text{cov}(X) < \text{par}(X)$ for some completely metrizable space $X$. If $\kappa$ is the minimum possible weight of such a space $X$, then $\kappa$ is a singular cardinal with $\text{cf}(\kappa) = \omega$. Furthermore, $\aleph_\omega \leq \kappa < \mathfrak{c}$. 

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**Corollary**

Suppose $\text{cov}(X) < \text{par}(X)$ for some completely metrizable space $X$. If $\kappa$ is the minimum possible weight of such a space $X$, then $\kappa$ is a singular cardinal with $\text{cf}(\kappa) = \omega$. Furthermore, $\aleph_\omega \leq \kappa < c$. 
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Assume $\square_\kappa$ holds for all singular cardinals $\kappa < \text{c}$,
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**Theorem**

Assume $\square_\kappa$ holds for all singular cardinals $\kappa < \frak{c}$, and (SSH): if $\kappa$ is a singular cardinal with cofinality $\omega$, then the poset $([\kappa]^{\omega}, \subseteq)$ has cofinality $\kappa^+$. 

Then $\text{cov}(X) = \text{par}(X)$ for all completely metrizable spaces $X$. 

**Corollary**

If $\text{cov}(X) < \text{par}(X)$ for some completely metrizable space $X$, then $0^+$ exists. 

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For every model $M$ for a countable language $\mathcal{L}$ that contains a unary predicate $A$, if $|M| = \aleph_{\omega+1}$ and $|A| = \aleph_\omega$ then there is an elementary submodel $M' \prec M$ such that $|M'| = \aleph_1$ and $|M' \cap A| = \aleph_0$. 

Levinsky, Magidor, and Shelah proved $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ is consistent relative to a large cardinal hypothesis a little stronger than a huge cardinal. This was improved recently by Eskew and Hayut, who showed $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ is consistent relative to a huge cardinal.
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Suppose \(M\) is a collection of \(\geq \aleph_{\omega+1}\) structures (molecules), each built from countably many members of an \(\aleph_\omega\)-sized set \(A\) (atoms). Then there is a single countable \(A_0 \subseteq A\) that was used to build uncountably many members of \(M\).
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So, $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ implies that uncountably many members of $M$ are defined from some countable $A_0 \subseteq A$. 
how to get $\text{cov}(X) < \text{par}(X)$

**Theorem**

Let $D$ be the discrete space of cardinality $\aleph_\omega$. It is consistent, relative to a huge cardinal, that $\text{cov}(D^\omega) < \text{par}(D^\omega)$. 

Proof sketch.

Begin with a model of GCH $\leftrightarrow (\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$. (This is the part that requires a huge cardinal.)

Then force with any ccc poset $P$ makes $\non(M) \geq \aleph_{\omega+2}$. For example, the forcing to add $\aleph_{\omega+2}$ Cohen reals will do.

In the extension, we have $\text{cov}(D^\omega) = \aleph_{\omega+1}$. (This is where the GCH of the ground model comes in.)

Forcing with a ccc poset preserves $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$, so $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ holds in the extension.

Working in the extension, suppose $P$ is a partition of $D^\omega$. 

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Begin with a model of GCH + $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$. (This is the part that requires a huge cardinal.) Then force with any ccc poset that makes $\text{non}(M) \geq \aleph_{\omega+2}$. For example, the forcing to add $\aleph_{\omega+2}$ Cohen reals will do.

In the extension, we have $\text{cov}(D^{\omega}) = \aleph_{\omega+1}$. (This is where the GCH of the ground model comes in.)

Forcing with a ccc poset preserves $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$, so $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$ holds in the extension.
how to get $\text{cov}(X) < \text{par}(X)$

**Theorem**

*Let $D$ be the discrete space of cardinality $\aleph_\omega$. It is consistent, relative to a huge cardinal, that $\text{cov}(D^\omega) < \text{par}(D^\omega)$.*

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Begin with a model of GCH $+ (\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$. (This is the part that requires a huge cardinal.) Then force with any ccc poset that makes $\text{non}(\mathcal{M}) \geq \aleph_{\omega+2}$. For example, the forcing to add $\aleph_{\omega+2}$ Cohen reals will do.

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Working in the extension, suppose $P$ is a partition of $D^\omega$. 
Theorem

Let $D$ be the discrete space of cardinality $\aleph_\omega$. It is consistent, relative to a huge cardinal, that $\text{cov}(D^\omega) < \text{par}(D^\omega)$.

Proof sketch.

As $\text{par}(D^\omega) \geq \text{cov}(D^\omega) = \aleph_{\omega+1}$, $\mathcal{P}$ is a collection of $\geq \aleph_{\omega+1}$ Polish spaces.
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As $\text{par}(D^\omega) \geq \text{cov}(D^\omega) = \aleph_{\omega+1}$, $\mathcal{P}$ is a collection of $\geq \aleph_{\omega+1}$ Polish spaces. Applying $(\aleph_{\omega+1}, \aleph_\omega) \mapsto (\aleph_1, \aleph_0)$, there are uncountably many members of $\mathcal{P}$ that are all "built from" the same countable collection of basic open sets.
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Will Brian

Covering versus partitioning with Polish spaces
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how to get $\text{cov}(X) < \text{par}(X)$
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Let $D$ be the discrete space of cardinality $\aleph_\omega$. It is consistent, relative to a huge cardinal, that $\cov(D^\omega) < \par(D^\omega)$.

**Proof sketch.**

As $\par(D^\omega) \geq \cov(D^\omega) = \aleph_{\omega+1}$, $\mathcal{P}$ is a collection of $\geq \aleph_{\omega+1}$ Polish spaces. Applying $(\aleph_{\omega+1}, \aleph_\omega) \rightarrow (\aleph_1, \aleph_0)$, there are uncountably many members of $\mathcal{P}$ that are all "built from" the same countable collection of basic open sets. Consequently, there is a countable $A \subseteq D$ such that $A^\omega$ contains uncountably many members of $\mathcal{P}$. Let $X = A^\omega \subseteq D^\omega$, and note that $\mathcal{P} \upharpoonright X$ is an uncountable partition of $X$. By a result of Fremlin and Shelah, every uncountable partition of a Polish space into $G_\delta$'s has size $\geq \non(M)$. Hence $|\mathcal{P}| \geq |\mathcal{P} \upharpoonright X| \geq \non(M) \geq \aleph_{\omega+2} > \cov(D^\omega)$. 

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Open questions

Question

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*What is the consistency strength of the failure of SSH?*
Thank you for listening