GLOBAL EXISTENCE OF A UNIFORMLY LOCAL ENERGY SOLUTION FOR THE INCOMPRESSIBLE FRACTIONAL NAVIER–STOKES EQUATIONS

BY

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Abstract. We introduce the concept of local Leray solutions starting from locally square-integrable initial data to the fractional Navier–Stokes equations with \( s \in \left(3/4, 1\right)\). Furthermore, we prove their local-in-time existence when \( s \in \left(3/4, 1\right)\). In particular, if a locally square-integrable initial datum vanishes at infinity, we show that the fractional Navier–Stokes equations admit a global-in-time local Leray solution when \( s \in \left[5/6, 1\right)\). For such local Leray solutions starting from locally square-integrable initial data vanishing at infinity, a singularity only occurs in \( B_R(0) \) for some \( R \).

1. Introduction. In this paper, we consider the following incompressible fractional Navier–Stokes equations in \( \mathbb{R}^3 \times (0, \infty) \) with \( s > 0 \):

\[
\begin{align*}
\partial_t u + A^{2s}u + u \cdot \nabla u + \nabla P &= 0, \\
\text{div } u &= 0,
\end{align*}
\]

with the initial data

\[
\begin{align*}
u(0, x) &= u_0(x) \quad \text{in } \mathbb{R}^3.
\end{align*}
\]

Here the vector field \( u \) and the scalar function \( P \) describe the velocity field and the associated pressure, respectively. The fractional Laplacian operator \( A^s \) is a nonlocal operator defined in terms of the Fourier transform:

\[
\mathcal{F}(A^s u)(\xi) = |\xi|^s \hat{u}(\xi).
\]

The fractional Laplacian operator appears in a wide class of hydrodynamical statistical-mechanical and physiological problems, including Lévy flights, stochastic interfaces and anomalous diffusion (see [22, 28]). So the fractional Navier–Stokes system (FNS) has important physical significance. For example, when \( s \in (0, 1) \), Zhang [30] described the stochastic Lagrangian particle approach via (FNS). For \( s = 1 \), (FNS) becomes the classical 3D incompressible Navier–Stokes equations.

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Recently, there are interesting results about weak and strong solutions of (FNS). The strong solutions of (FNS)–(FNSI) have been studied in a series of works using analytical tools (see e.g. [6, 10]).

In this paper, we focus on weak solutions to (FNS). This work can be traced back to [16], where Leray introduced the concept of Leray–Hopf weak solutions to (FNS)–(FNSI) with $s = 1$ and showed their global existence for each divergence free initial datum $u_0 \in L^2(\mathbb{R}^3)$. Following the argument in [16], for all $s > 1/2$, we prove that for any solenoidal vector $u_0 \in L^2(\mathbb{R}^3)$, (FNS)–(FNSI) admits at least one weak solution in $L^\infty([0, \infty); L^2(\mathbb{R}^3)) \cap L^2([0, \infty); \dot{H}^s(\mathbb{R}^3))$, which is continuity at $t = 0$ into $L^2(\mathbb{R}^3)$ and satisfies the energy estimate

\begin{equation}
\int_{\mathbb{R}^3} |u(t)|^2 \, dx + 2 \int_0^t \int_{\mathbb{R}^3} |A^s u|^2 \, dx \, d\tau \leq \int_{\mathbb{R}^3} |u_0|^2 \, dx.
\end{equation}

For simplicity, such weak solutions are also called Leray–Hopf weak solutions. For $s \geq 5/4$, Lions [18] proved that (FNS)–(FNSI) admits a unique global smooth solution for any prescribed smooth initial datum. However, for $s < 5/4$, the uniqueness and global regularity of Leray–Hopf weak solutions are still open problems. Instead, partial regularity has been put on the agenda. In this respect, Scheffer [23, 24, 25] first discussed a class of weak solutions to classical Navier–Stokes equations which satisfy the local energy inequality

\begin{equation}
2 \int_0^\infty \int_{\mathbb{R}^3} |\nabla u|^2 \varphi \, dx \, d\tau \leq \int_0^\infty \int_{\mathbb{R}^3} \left( |u|^2 (\partial_t \varphi + \Delta \varphi) + (|u|^2 + 2p) u \cdot \nabla \varphi \right) \, dx \, d\tau
\end{equation}

for every nonnegative function $\varphi \in \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^+)$. He proved that such a weak solution has a singular set with finite $5/3$-Hausdorff measure. Later, Scheffer’s result was improved by Caffarelli, Kohn and Nirenberg [2], who introduced the concept of suitable weak solutions and proved that the 1D parabolic Hausdorff measure of the associated singular set is zero. We refer to [17] for a simplified proof. For $1 < s < 5/4$, Katz and Pavlović [11] proved the first version of CKN theory, that is, the Hausdorff dimension of singular set at the first blow up time is at most $5-4s$. Recently, Colombo, De Lellis and Massaccesi [8] introduced the definition of suitable weak solution via harmonic extension of the higher order fractional Laplace operator due to Yang [29] and proved a stronger version of the results of Katz and Pavlović, which fully extends CKN theory. For $3/4 < s < 1$, via the harmonic extension established in [3], Tang and Yu [27] defined and proved the existence of suitable weak solutions to (FNS) and then generalized CKN theory to this case. CKN theory at the endpoint level $s = 3/4$ was studied by Ren, Wang and Wu [21].
To give a precise statement about questions discussed in this paper, we introduce Banach spaces of local measures. For any $p \in [1, \infty]$, we define

$$L^p_{uloc}(\mathbb{R}^n) := \left\{ f \in L^p_{loc}(\mathbb{R}^n) \mid \|f\|_{L^p_{uloc}(\mathbb{R}^n)} := \sup_{x_0 \in \mathbb{R}^n} \|f\|_{L^p(B_1(x_0))} < \infty \right\},$$

$$E^p(\mathbb{R}^n) := \left\{ f \in L^p_{uloc}(\mathbb{R}^n) \mid \lim_{|x_0| \to \infty} \|u\|_{L^p(B_1(x_0))} = 0 \right\}.$$

It is obvious that $E^p(\mathbb{R}^n)$ is the closure of $C_0^\infty(\mathbb{R}^n)$ in the $L^p_{uloc}(\mathbb{R}^n)$ norm when $1 \leq p < \infty$. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$ be a nonnegative function such that $\phi(x) = 1$ for all $|x| \leq 1$ and $\phi(x) = 0$ for all $|x| \geq 2$. Set $\phi_{x_0}(\cdot) = \phi(x_0 - \cdot)$ for $x_0 \in \mathbb{R}^n$. Then for all $s \in \mathbb{R}$, the corresponding Sobolev spaces $H^s_{uloc}(\mathbb{R}^n)$ and $\bar{H}^s_{uloc}(\mathbb{R}^n)$ can be defined as follows:

$$H^s_{uloc}(\mathbb{R}^n) := \left\{ u \in H^s_{loc}(\mathbb{R}^n) \mid \|u\|_{H^s_{uloc}(\mathbb{R}^n)} := \sup_{x_0 \in \mathbb{R}^n} \|\phi_{x_0} u\|_{H^s(\mathbb{R}^n)} < \infty \right\},$$

$$\bar{H}^s_{uloc}(\mathbb{R}^n) := \left\{ u \in H^s_{uloc}(\mathbb{R}^n) \mid \lim_{|x_0| \to \infty} \|\phi_{x_0} u\|_{H^s(\mathbb{R}^n)} = 0 \right\}.$$

Finally, we define the mixed time-space Banach spaces

$$L^{p,q}_{uloc}(T_1, T_2, \mathbb{R}^n) := \left\{ f \in L^p([T_1, T_2]; L^q_{loc}(\mathbb{R}^n)) \mid \|u\|_{L^{p,q}_{uloc}(T_1, T_2, \mathbb{R}^n)} < \infty \right\},$$

$$G^{p,q}(T_1, T_2, \mathbb{R}^n) := \left\{ f \in L^{p,q}_{uloc}(T_1, T_2, \mathbb{R}^n) \mid \lim_{|x_0| \to \infty} \|u\|_{L^p([T_1, T_2]; L^q(B_1(x_0)))} = 0 \right\}$$

with

$$\|f\|_{G^{p,q}(T_1, T_2, \mathbb{R}^n)} := \|f\|_{L^{p,q}_{uloc}(T_1, T_2, \mathbb{R}^n)} := \sup_{x_0 \in \mathbb{R}^n} \|u\|_{L^p([T_1, T_2]; L^q(B_1(x_0)))}.$$

For brevity, we adopt the following notations:

$$L^p := L^p(\mathbb{R}^3), \quad L^p_{uloc} := L^p_{uloc}(\mathbb{R}^3), \quad E^p := E^p(\mathbb{R}^3),$$

$$H^s := H^s(\mathbb{R}^3), \quad H^s_{uloc} := H^s_{uloc}(\mathbb{R}^3), \quad \bar{H}^s := \bar{H}^s_{uloc}(\mathbb{R}^3),$$

$$L^{p,q}_{uloc}(T) := L^{p,q}_{uloc}(0, T, \mathbb{R}^3), \quad G^{p,q}(T) := G^{p,q}(0, T, \mathbb{R}^3).$$

For $s = 1$, with the aid of $\varepsilon$-regularity theory established in [2], Lemarié-Rieusset [14] introduced a concept of local Leray solutions to (FNS), which satisfy the local energy inequality (1.2), and proved that for any divergence free initial datum $u_0$ in $E^2$, (FNS)–(FNSI) has a global-in-time local Leray solution.

In view of the partial regularity results in [8, 21, 27], a natural and interesting question arises:

(Q) Does there exist a global-in-time weak solution to (FNS)–(FNSI), with $s \in (3/4, 1)$ for each divergence free initial datum $u_0$ in $E^2$?
Investigating this question for the general case $s \neq 1$, we face a difficult point which does not appear in the classical Navier–Stokes equations ($s = 1$): Since $\Lambda^s$ is a nonlocal operator, we cannot deal with $\Lambda^{2s}u$ as for the classical Navier–Stokes equations. To overcome it, we use the nonlocal commutator $[\Lambda^s, \varphi]$ where $s \in (0, 1)$ and $\varphi \in C^1(\mathbb{R}^n)$. When $s \in (0, 1)$, we formally write

$$\langle \Lambda^{2s}u, w\varphi \rangle = \langle \varphi \Lambda^{2s}u, w\varphi \rangle = \langle [\varphi, \Lambda^s]\Lambda^s u, w\varphi \rangle + \langle \varphi \Lambda^s u, [\Lambda^s, \varphi]u \rangle + \langle \varphi \Lambda^s u, \varphi \Lambda^s u \rangle$$

for any $\varphi \in C^1(\mathbb{R}^n)$ satisfying $\varphi \varphi = 1$. For $s > 1$, the commutator $[\Lambda^s, \varphi]$ cannot be controlled and we first consider the equation

$$\langle \Lambda^{2s}u, w\varphi \rangle = \langle (-\Delta)\Lambda^{2s-2}u, w\varphi \rangle = -\langle \Lambda^{2s-2}\nabla u, \nabla (w\varphi) \rangle$$

and then use the commutator again. The work for $1 < s \leq 5/4$ may need more complicated calculations. Hence, in this paper, we only consider the case $s < 1$.

With the aid of $[\Lambda^s, \varphi]$, we can define the local Leray solutions to (FNS)–(FNSI) for $s < 1$.

**Definition 1.1** (Local Leray solutions). Let $s \in [3/4, 1)$ and $u_0$ be a solenoidal vector field in $L^2_{uloc}$. We call $u$ a local Leray solution to (FNS) on $\mathbb{R}^3 \times (0, T)$ starting from $u_0$ if $u$ satisfies the following conditions:

(i) $u \in L^\infty([0, T']; L^2_{uloc})$ and $\Lambda^s u \in L^2_{uloc}(T')$ for any $T' < T$;

(ii) there exists $P \in \mathcal{D}'(\mathbb{R}^3 \times (0, T))$ such that $\partial_t u + \Lambda^{2s}u + u \cdot \nabla u + \nabla P = 0$

in the sense of distributions;

(iii) for any compact subset $K$ of $\mathbb{R}^3$, $\lim_{t \to 0^+} \|u(u(t) - u_0\|_{L^2(K)} = 0$;

(iv) $u$ is suitable, that is, for every nonnegative function $\phi \in \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^+)$, the vector field $u$ satisfies the local energy inequality

$$\text{(1.3)} \quad \int_0^t \int_{\mathbb{R}^3} \psi^2 \Lambda^s u^2 \, dx \, d\tau \leq \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |u|^2 \partial_t \psi \, dx \, d\tau - \int_0^t \int_{\mathbb{R}^3} [\tilde{\psi}, \Lambda^s] \Lambda^s u \cdot u \psi \, dx \, d\tau$$

$$- \int_0^t \int_{\mathbb{R}^3} [\Lambda^s, \psi]u \cdot (\tilde{\psi} \Lambda^s u) \, dx \, d\tau$$

$$+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |u|^2 u \cdot \nabla \psi \, dx \, d\tau + \int_0^t \int_{\mathbb{R}^3} P \cdot u \nabla \psi \, dx \, d\tau$$

with $\nabla P = -\nabla \frac{\text{div} \, \text{div}}{\Delta} (u \otimes u)$. Here, $\tilde{\psi} \in \mathcal{D}(\mathbb{R}^3)$ satisfies $\tilde{\psi} \geq 0$ and $\tilde{\psi} \psi = \psi$.

**Remark 1.2.** The restriction on the pressure $P$ avoids the existence of a trivial local Leray solution, for example, $u = \rho(t)$ and $P = -\rho'(t) \cdot x$ where $\rho(t)$ is a smooth vector field with $\rho(t) \neq 0$. In addition, for all $x_0 \in \mathbb{R}^3$ and
r > 0, we can decompose \( P \) in \( B_r(x_0) \) as \( P_{x_0,r} + P_{x_0,r}(t) \) where \( P_{x_0,r}(t) \) is a function depending only on \( x_0, r \) and

\[
P_{x_0,2}(x,t) = -\Delta^{-1} \text{div} \text{div}(u(t) \otimes u(t)\psi_{x_0})
\]

\[
- \int_{\mathbb{R}^3} (k(x-y) - k(x_0-y))(u \otimes u)(y,t) (1 - \psi_{x_0}(y)) \, dy
\]

\[
=: P_{x_0,2}^1(x,t) + P_{x_0,2}^2(x,t)
\]

with \( \psi_{x_0} \in \mathcal{D}(B_{4r}(x_0)) \) satisfying \( 0 \leq \psi_{x_0} \leq 1 \) and \( \psi_{x_0} = 1 \) in \( B_{2r}(x_0) \). Here \( k(x) \) is the kernel of the Calderón–Zygmund operator \( \Delta^{-1} \text{div} \text{div} \), satisfying

\[
|k(x-y) - k(x_0-y)| \lesssim \frac{|x - x_0|}{|x-y|^4} \quad \text{if} \quad |x - x_0| \leq \frac{1}{2} |x - y|.
\]

**Remark 1.3.** The restriction on \( s \) ensures that the following term is well defined:

\[
\int_{0}^{t} \int_{\mathbb{R}^3} |u|^2 u \cdot \nabla \psi \, dx \, d\tau.
\]

We prove our first main result by the regularization method.

**Theorem 1.4.** Let \( s \in (3/4, 1) \). Then, for each solenoidal vector field \( u_0 \in L^2_{uloc} \), there exists a \( T > 0 \) and a local Leray solution to **(FNS)** starting from \( u_0 \) on \( \mathbb{R}^3 \times (0, T) \).

Due to the last two terms on the right side of (1.3), we cannot directly get the global-in-time existence of a local Leray solution \( u \). Inspired by [14], we can construct a local Leray solution on \( (0, \infty) \) via \( \varepsilon \)-regularity theory of [27] and a “weak-strong” uniqueness theorem. To do so, we first show the following additional regularities of local Leray solutions with initial data in \( E^2 \).

**Theorem 1.5.** If \( u \) is a local Leray solution to **(FNS)** on \( \mathbb{R}^3 \times (0, T) \) starting from \( u_0 \in E^2 \) with \( \text{div} u_0 = 0 \), then \( u \in L^\infty([0,T^*]; E^2) \) and \( A^s u \in G^{2,2}(T') \) for any \( T' \in (0, T) \), and

\[
\lim_{t' \to t^+} \|u(t') - u(t)\|_{L^2_{uloc}} = 0 \quad \text{for} \quad t = 0 \quad \text{and for almost every} \quad t \in (0, T).
\]

With the aid of Theorem 1.5, we find that inequality (1.3) is equivalent to the local energy inequality given in [27] via harmonic extension. This implies that the local Leray solution with initial data in \( E^2 \) is a suitable weak solution as defined in Section 4. Hence, by Proposition 4.4, we can prove Proposition 4.6 which says that the local Leray solution \( u \) starting from \( u_0 \in E^2 \) is regular in \( B_{R}^c \) away from \( t = 0 \) for some large enough \( R \) and there exists a \( t_0 \) close to \( T \) such that \( u(t_0) \in \tilde{H}^s_{uloc} \) and \( \lim_{t \to t_0^+} \|u(t) - u(t_0)\|_{E^2} = 0 \). Relying on these, invoking the well-posedness theory in Theorem 6.2 and the
we can prove the following theorem via the construction method of [14], which gives a positive answer to (Q).

**Theorem 1.6.** Let $s \in [5/6, 1)$. Then, for any solenoidal vector field $u_0 \in E^2$, there exists a local Leray solution $u$ to (FNS)–(FNSI) in $\mathbb{R}^3 \times (0, \infty)$.

**Remark 1.7.** To obtain a solution in $\bar{H}^s_{uloc}$ which ensures that the uniqueness in Proposition 5.1 holds, we need $s \geq 5/2 - 2s$, where $5/2 - 2s$ is the critical index. This gives $s \geq 5/6$.

**Remark 1.8.** From the above analysis, every local Leray solution $u$ to (FNS)–(FNSI) on $\mathbb{R}^3 \times (0, t_0)$ with $u_0$ in $E^2$ is regular in $B_R^{c}$ away from $t = 0$ for some large enough $R > 0$.

**Remark 1.9.** Given $\lambda > 0$, set

$$u_\lambda(x, t) = \lambda^{2s-1}u(\lambda x, \lambda^{2s}t), \quad P_\lambda(x, t) = \lambda^{4s-2}P(\lambda x, \lambda^{2s}t).$$

We call a solution on $\mathbb{R}^3 \times (0, \infty)$ a forward self-similar solution if $u(x, t) = u_\lambda(x, t)$ and $P(x, t) = P_\lambda(x, t)$ for all $\lambda > 0$. The existence of forward self-similar solutions to (FNS)–(FNSI) was first studied for $s = 1$ in [4, 7, 9]. Recently, Lai, Miao and Zheng [12] proved the existence of forward self-similar solutions to (FNS)–(FNSI) with $5/6 < s \leq 1$ for arbitrarily large self-similar initial data via a blow-up argument. Since $E^2$ contains nontrivial scale-invariant functions, for example $\sigma(x)/|x|^{2s-1}$, inspired by [9] we believe that, with the aid of the global-in-time existence of local Leray solutions (Theorem 1.6), the result of [12] can be proved in another way.

The remainder of this paper is structured as follows. In Section 2, we give some key estimates, which will be used repeatedly. In Section 3, by the regularization method, we prove Theorem 1.4. Section 4 is devoted to developing decay and regularity properties of local Leray solutions to (FNS)–(FNSI) with initial data in $E^2$. In Section 5, we complete the proof of Theorem 1.6 via Theorem 1.4 and the properties given in Section 4. Finally, in the Appendix, for the convenience of the readers, we prove some folklore results relevant to FNS, which have not formally appeared before.

**Notation.** We denote by $C$ absolute positive constants; $C_{\lambda, \gamma, \ldots}$ denotes a positive constant depending only on $\lambda, \gamma, \ldots$. We adopt the convention that a nonessential constant $C$ may change from line to line. Given two quantities $a$ and $b$, we write $a \lesssim b$ and $a \lesssim_{\lambda, \gamma, \ldots} b$ when $a \leq Cb$ and $a \leq C_{\lambda, \gamma, \ldots}b$ respectively. In addition, if $Cb \leq a \leq C^{-1}b$ or $C_{\lambda, \gamma, \ldots}b \leq a \leq C_{\lambda, \gamma, \ldots}^{-1}b$, we write $a \approx b$ resp. $a \approx_{\lambda, \gamma, \ldots} b$. For any $x_0 \in \mathbb{R}^3$ and $t_0 \in \mathbb{R}^+$, $B_R(x_0)$ is the (open) ball in $\mathbb{R}^3$ with radius $R$ centered at $x_0$, $B_R^{c}(x_0) = \mathbb{R}^3 \setminus B_R(x_0)$ and $Q_R(x_0, t_0) = B_R(x_0) \times (t_0 - R^2, t_0) \in \mathbb{R}^3 \times \mathbb{R}$.
2. Preliminaries. In this section, we mainly introduce some important results in $\mathbb{R}^n$, $n \geq 2$, which will be used in the following sections.

2.1. Estimates of the heat kernel and the Oseen kernel of the fractional Laplacian. In this subsection, we consider the following Cauchy problem for the linear fractional Stokes problem in $\mathbb{R}^n \times \mathbb{R}^+$:

$$
\begin{aligned}
\partial_t u + A^{2s} u + \nabla p &= \text{div} \, F, \quad \text{div} \, u = 0, \\
u(x, 0) &= u_0,
\end{aligned}
$$

where $F$ is a given second-order tensor field. Since $\text{div} \, u = 0$, applying to the first equation above the Leray projection operator $P = \text{Id} + R \otimes R$ where $R := (R_1, \ldots, R_n)$ and $R_j$ is the Riesz operator, by Duhamel’s formula we get

$$
\begin{aligned}
u(x, t) &= e^{-tA^{2s}} u_0 + \int_0^t e^{-(t-\tau)A^{2s}} P(\nabla \cdot F) \, d\tau \\
&= G_t \ast u_0 + \int_0^t \mathcal{O}_{j,k,t-\tau} \ast (\partial_t F_{ik})(\tau) \, d\tau
\end{aligned}
$$

where $G_t(x)$ and $\mathcal{O}_{j,k,t}$ are the kernel functions of $e^{-tA^{2s}}$ and $e^{-tA^{2s}}(\delta_{jk} + R_j R_k)$, respectively. It is obvious that

$$
\begin{aligned}
A^\alpha G_t(x) &= t^{-\frac{\alpha+n}{2s}} (A^\alpha G_1) \left( \frac{x}{t^{1/(2s)}} \right), \\
A^\alpha \mathcal{O}_{j,k,t}(x) &= t^{-\frac{\alpha+n}{2s}} (A^\alpha \mathcal{O}_{j,k,1}) \left( \frac{x}{t^{1/(2s)}} \right)
\end{aligned}
$$

and $\int_{\mathbb{R}^n} G_t(x) \, dx = 1$. In addition, $G_t$ and $\mathcal{O}_{j,k,t}$ satisfy the following pointwise and $L^p$-$L^q$ estimates.

**Lemma 2.1.** Let $s > 0$. Then for every $x \in \mathbb{R}^n$,

$$(2.1) \quad |G_1(x)| \leq C(1 + |x|)^{-n-2s}, \quad |[A^\alpha G_1](x)| \leq C(1 + |x|)^{-n-\alpha} \quad (\alpha > 0).$$

Moreover, for every $0 \leq \alpha_2 \leq \alpha_1$ and $1 \leq r \leq p \leq \infty$ we have

$$(2.2) \quad \|A^{\alpha_1} G_t \ast f\|_{L^p(\mathbb{R}^n)} \leq C_{s,\alpha_1-\alpha_2,p,r} t^{-\frac{\alpha_1-\alpha_2}{2s} - \frac{n}{2s} (\frac{1}{r} - \frac{1}{p})} \|A^{\alpha_2} f\|_{L^r(\mathbb{R}^n)},$$

and for every $1 \leq r < \infty$,

$$(2.3) \quad \|A^{\alpha_1} G_t \ast f\|_{L^r_{\text{uloc}}(\mathbb{R}^n)} \leq C_{s,\alpha_1-\alpha_2} t^{-\frac{\alpha_1-\alpha_2}{2s}} \|A^{\alpha_2} f\|_{L^r_{\text{uloc}}(\mathbb{R}^n)},$$

$$(2.4) \quad \|A^{\alpha_1} G_t \ast f\|_{L^\infty(\mathbb{R}^n)} \leq C_{s,\alpha_1-\alpha_2} t^{-\frac{\alpha_1-\alpha_2}{2s}} \max(1, t^{-\frac{3}{2s}}) \|A^{\alpha_2} f\|_{L^r_{\text{uloc}}(\mathbb{R}^n)}.$$

To prove Lemma 2.1 we use the following estimates.
Lemma 2.2 ([13, 15]). Let $1 \leq p < \infty$ and $\alpha > 0$. Then
\begin{align}
(2.5) \quad & \|f * g\|_{L^p_{uloc}(\mathbb{R}^n)} \leq \|f\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p_{uloc}(\mathbb{R}^n)}, \\
(2.6) \quad & \|f * g\|_{L^\infty(\mathbb{R}^n)} \leq C\alpha(1 + |x|)^{(n + \alpha)/2} \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2_{uloc}(\mathbb{R}^n)}.
\end{align}

Proof of Lemma 2.1. Since (2.1)–(2.2) have been proved in [20], we only need to prove (2.3)–(2.4). From (2.1), we easily deduce $A^\alpha G_1 \in L^1(\mathbb{R}^n)$ for every $\alpha \geq 0$. Hence, by (2.5),
\begin{align}
\|A^{\alpha_1}G_t * f\|_{L^r_{uloc}(\mathbb{R}^n)} &= \|A^{\alpha_1-\alpha_2}G_t * A^{\alpha_2}f\|_{L^r_{uloc}(\mathbb{R}^n)} \\
&\leq C_{s,\alpha_1-\alpha_2}t^{-\frac{\alpha_1-\alpha_2}{2s}} \|A^{\alpha_2}f\|_{L^r_{uloc}(\mathbb{R}^n)}, \quad 0 \leq \alpha_2 \leq \alpha_1.
\end{align}
This implies (2.3). On the other hand, in view of (2.1),
\begin{align}
\sum_{x_0 \in \mathbb{R}^n} \sup_{x \in B_1(x_0)} |A^\alpha G_1(x)| &\leq C_{s,\alpha}, \quad \alpha \geq 0.
\end{align}
Hence, for every $x \in \mathbb{R}^n$ and $t > 0$,
\begin{align}
\|[A^{\alpha_1}G_t * f](x)\| &= t^{-\frac{\alpha_1-\alpha_2}{2s}} \left| \int_{\mathbb{R}^n} A^{\alpha_1-\alpha_2}G_1(y)A^{\alpha_2}f(x - t^{1/(2s)}y) \, dy \right| \\
&\leq t^{-\frac{\alpha_1-\alpha_2}{2s}} \sum_{x_0 \in \mathbb{R}^n} \sup_{y \in B_1(x_0)} |A^{\alpha_1-\alpha_2}G_1(y)| \int_{y \in B_1(x_0)} |A^{\alpha_2}f(x - t^{1/(2s)}y)| \, dy \\
&\leq C_{s,\alpha_1-\alpha_2}t^{-\frac{\alpha_1-\alpha_2}{2s}} \sup_{z_0 \in \mathbb{R}^n} \int_{|z - z_0| \leq t^{1/(2s)}} |f(z)| \, dy \\
&\leq C_{s,\alpha_1-\alpha_2}t^{-\frac{\alpha_1-\alpha_2}{2s}} \max\{1, t^{-\frac{n}{2s}}\} \|f\|_{L^r_{uloc}(\mathbb{R}^n)},
\end{align}
which implies (2.4) and completes the proof of Lemma 2.1.

In the same way as for Lemma 2.1 we obtain

Lemma 2.3. Let $s > 0$. Then for every $x \in \mathbb{R}^n$,
\begin{align}
(2.7) \quad |A^\alpha O_{j,k,l}(x)| &\leq C(1 + |x|)^{-n-\alpha}, \quad \alpha \geq 0.
\end{align}
In addition, for every $0 \leq \alpha_2 \leq \alpha_1$ and $1 \leq r \leq p \leq \infty$,
\begin{align}
(2.8) \quad & \|A^{\alpha_1}O_{j,k,t} * f\|_{L^p(\mathbb{R}^n)} \leq C_{\alpha_1-\alpha_2,p}t^{-\frac{\alpha_1-\alpha_2}{2s} - \frac{n}{2s} \left(\frac{1}{r} - \frac{1}{p}\right)} \|A^{\alpha_2}f\|_{L^r(\mathbb{R}^n)}
\end{align}
extcept when $\alpha_1 = \alpha_2 = 0$ for $r = 1$ or $r = \infty$, and for every $0 \leq \alpha_2 < \alpha_1$ and $1 \leq r < \infty$,
\begin{align}
(2.9) \quad & \|A^{\alpha_1}O_{j,k,t} * f\|_{L^r_{uloc}(\mathbb{R}^n)} \leq C_{\alpha_1-\alpha_2}t^{-\frac{\alpha_1-\alpha_2}{2s}} \|f\|_{L^r_{uloc}(\mathbb{R}^n)}, \\
(2.10) \quad & \|A^{\alpha_1}O_{j,k,t} * f\|_{L^\infty(\mathbb{R}^n)} \leq C_{\alpha_1-\alpha_2}t^{-\frac{\alpha_1-\alpha_2}{2s}} \max\{1, t^{-\frac{3}{2s}}\} \|f\|_{L^r_{uloc}(\mathbb{R}^n)}.
\end{align}
2.2. Estimates of the nonlocal commutator $[A^s, \varphi]$. We shall introduce a nonlocal commutator $[A^s, \varphi]$ with $s \in (0, 1)$ and $\varphi \in C^1(\mathbb{R}^n)$, $\varphi \neq 0$, which will be used to deal with the nonlocal effect of $A^s$, first established by Lazar [13]. For the convenience of the readers, we prove it here again.

**Lemma 2.4.** Let $\varphi \in C^1(\mathbb{R}^n)$ satisfy $\varphi \neq 0$ and $|\nabla \varphi| \neq 0$. Then for every $s \in (0, 1)$ and $1 \leq p \leq \infty$,

\[
\|[A^s, \varphi]u\|_{L^p_{uloc}(\mathbb{R}^n)} \leq C_s M_\varphi \|u\|_{L^p_{uloc}(\mathbb{R}^n)},
\]

\[
\|[A^s, \varphi]u\|_{L^{p,p}_{uloc}(0,T;\mathbb{R}^n)} \leq C_s M_\varphi \|u\|_{L^{p,p}_{uloc}(0,T;\mathbb{R}^n)},
\]

where

\[M_\varphi := \|\varphi\|_{L^\infty(\mathbb{R}^n)}^{-1-s} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)}^s.\]

**Proof.** It is obvious that

\[
[A^s, \varphi]u = C_s \int_{\mathbb{R}^n} \varphi(x)(u(x) - \varphi(y)u(y)) \frac{1}{|x-y|^{n+s}} \, dy - \int_{\mathbb{R}^n} \varphi(x)u(x) - \varphi(x)u(y) \frac{1}{|x-y|^{n+s}} \, dy
\]

\[= C_s \int_{\mathbb{R}^n} \varphi(x)u(y) - \varphi(y)u(y) \frac{1}{|x-y|^{n+s}} \, dy \leq C_s \int_{\mathbb{R}^n} \min\{|\nabla \varphi|_{L^\infty}|x-y|, |\varphi|_{L^\infty(\mathbb{R}^n)}\} \frac{|u(y)|}{|x-y|^{n+s}} \, dy
\]

\[= (K * |u|)(x) \text{ where } K(x) = C_s \min\{|x| |\nabla \varphi|_{L^\infty(\mathbb{R}^n)}, |\varphi|_{L^\infty(\mathbb{R}^n)}\}.\]

By a simple calculation,

\[\|K\|_{L^1(\mathbb{R}^n)} \leq C_s \|\varphi\|_{L^\infty(\mathbb{R}^n)}^{1-s} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)}^s.\]

Thus, from (2.5) and Minkowski’s inequality, we deduce (2.11).

**Corollary 2.5.** Let $\varphi \in C^0(\mathbb{R}^n)$ be a nonnegative function satisfying $\varphi \neq 0$ and $|\nabla \varphi| \neq 0$. Then for every $s \in (0, 1)$ and $1 \leq p \leq \infty$,

\[
\|[\varphi, A^s]u\|_{L^p(\mathbb{R}^n)} \leq C_s d_\varphi (M_\varphi + 1) \|u\|_{L^p_{uloc}(\mathbb{R}^n)},
\]

\[
\|[\varphi, A^s]u\|_{L^p((0,T);L^p(\mathbb{R}^n))} \leq C_s d_\varphi (M_\varphi + 1) \|u\|_{L^{p,p}_{uloc}(0,T;\mathbb{R}^n)}
\]

where $M_\varphi$ is defined in Lemma 2.4 and $d_\varphi = \text{diam supp } \varphi$.

**Proof.** Let $0 \leq \psi \in \mathcal{D}(\mathbb{R}^n)$ satisfy $\psi = 1$ in $B_{d_\varphi+1}$. Obviously,

\[(1 - \psi)[\varphi, A^s]u = -(1 - \psi)A^s(\varphi u) = (1 - \psi)\left( C_s \int_{|x| \geq 1} u \varphi \right).\]
Hence, by Lemma 2.4 and Young’s inequality,

$$\| [\varphi, A^s] u \|_{L^p(\mathbb{R}^n)} \leq \| \psi [\varphi, A^s] u \|_{L^p(\mathbb{R}^n)} + \| (1 - \psi) [\varphi, A^s] u \|_{L^p(\mathbb{R}^n)}$$

$$\leq C_{s, d, \varphi} M_\varphi \| u \|_{L^p_{\text{uloc}}(\mathbb{R}^n)} + C_s \| u \varphi \|_{L^p(\mathbb{R}^n)}$$

$$\leq C_{s, d, \varphi} (M_\varphi + 1) \| u \|_{L^p_{\text{uloc}}(\mathbb{R}^n)}.$$ 

Similarly,

$$\| [\varphi, A^s] u \|_{L^p(0,T;\mathbb{R}^n)} \leq C_{s, d, \varphi} (M_\varphi + 1) \| u \|_{L^p(0,T;\mathbb{R}^n)}, \quad 1 \leq p \leq \infty.$$ 

As a consequence of Lemma 2.4 and Corollary 2.5 we obtain

**Lemma 2.6.** The norm \( \| \cdot \|_{H^s_{\text{uloc}}} \) satisfies the following equivalence relations:

$$\| u \|_{H^s_{\text{uloc}}(\mathbb{R}^n)} \simeq_s \| u \|_{L^2_{\text{uloc}}(\mathbb{R}^n)} + \| A^s u \|_{L^2_{\text{uloc}}(\mathbb{R}^n)}; \quad s \in (0, 1),$$

$$\| u \|_{H^s_{\text{uloc}}(\mathbb{R}^n)} \simeq \| u \|_{L^2_{\text{uloc}}(\mathbb{R}^n)} + \| \nabla u \|_{L^2_{\text{uloc}}(\mathbb{R}^n)};$$

$$\| u \|_{H^s_{\text{uloc}}(\mathbb{R}^n)} \simeq \| u \|_{H^1_{\text{uloc}}(\mathbb{R}^n)} + \| A^{s-1} \nabla u \|_{L^2_{\text{uloc}}(\mathbb{R}^n)}; \quad s \in (1, 2).$$

### 3. Local-in-time existence of local Leray solutions

In this section, we show in steps the local-in-time existence of local Leray solutions to \((\text{FNS})-(\text{FNSI})\) with initial data \(u_0 \in L^2_{\text{uloc}}\).

**Step 1: Approximate solution sequence.** In this step, we consider the following mollification of \((\text{FNS})-(\text{FNSI})\):

\[
\begin{align*}
(\text{MFNS}) \quad \left\{ \begin{array}{l}
\partial_t u_\varepsilon + A^{2s} u_\varepsilon + J_\varepsilon u_\varepsilon \cdot \nabla u_\varepsilon + \nabla P_\varepsilon = 0, \\
u_\varepsilon |_{t=0} = u_0,
\end{array} \right.
\end{align*}
\]

where \(J_\varepsilon f := \varepsilon^{-3} \int_{B_1} \eta(\frac{x-y}{\varepsilon}) f(y) \, dy\) with \(\eta \in \mathcal{D}(B_1)\) and \(0 \leq \eta \leq 1\). Applying the Leray projection operator \(\mathbb{P}\) to the first equation of \((\text{MFNS})\), by the Duhamel principle, we can rewrite \((\text{MFNS})\) as follows:

\[(3.1) \quad u_\varepsilon = e^{-t A^{2s}} u_0 - \int_0^t \nabla \cdot e^{-(t-\tau) A^{2s}} \mathbb{P} (J_\varepsilon u_\varepsilon \otimes u_\varepsilon)(\tau) \, d\tau \]

\[=: e^{-t A^{2s}} u_0 + B_\varepsilon(u_\varepsilon, u_\varepsilon).\]

Next, we will use the Banach fixed point theorem given in [5] to prove the existence of solutions to \((3.1)\) in the Banach space \((X_{\text{T}_\varepsilon}, \| \cdot \|_{X_{\text{T}_\varepsilon}})\) with \(T_\varepsilon \leq 1, \varepsilon \leq 1\) defined as follows:

\[X_{\text{T}_\varepsilon} := \{ u \in L^\infty([0, T_\varepsilon]; L^2_{\text{uloc}}) \mid A^s u \in L^2([0, T_\varepsilon]; L^2_{\text{uloc}}) \}; \]

\[\| u \|_{X_{\text{T}_\varepsilon}} := \| u \|_{L^\infty([0, T_\varepsilon]; L^2_{\text{uloc}})} + \| A^s u \|_{L^2([0, T_\varepsilon]; L^2_{\text{uloc}})}.\]

To do so, we first show \(e^{-t A^{2s}} u_0 \in X_{\text{T}_\varepsilon}\). By \((2.3)\), we have

\[\| e^{-t A^{2s}} u_0 \|_{L^\infty([0, T_\varepsilon]; L^2_{\text{uloc}})} \lesssim \| u_0 \|_{L^2_{\text{uloc}}}.\]
To prove
\begin{equation}
\|A^s e^{-tA^{2s}} u_0\|_{L^2([0,T\varepsilon);L^2_{uloc})} \lesssim_s \|u_0\|_{L^2_{uloc}},
\end{equation}
we split $u_0$ into $u_0^1 := \chi_{B_\varepsilon(x_0)} u_0$ and $u_0^2 := \chi_{B_\varepsilon^c(x_0)} u_0$. It is obvious that
\[ \|A^s e^{-tA^{2s}} u_0^1\|_{L^2([0,T\varepsilon];L^2)} \lesssim \|u_0^1\|_{L^2} \lesssim \|u_0\|_{L^2_{uloc}}. \]
To show $A^s e^{-tA^{2s}} u_0^2 \in L^2([0,T\varepsilon);L^2_{uloc})$, we will invoke the following lemma.

**Lemma 3.1.** Let
\[ [T_{x_0,R,\alpha} f](x) := \int_{B_R^c(x_0)} \frac{1}{|x - y|^{3+\alpha}} f(y) \, dy, \quad x_0 \in \mathbb{R}^3, \; R > 2. \]
Then for all $x_0 \in \mathbb{R}^3$,
\[ \|T_{x_0,R,\alpha} f\|_{L^\infty(B_1(x_0))} \leq C_\alpha R^{-\alpha} \|f\|_{L^1_{uloc}}. \]

**Proof.** Since $|x - y| \geq R|y - x_0|/2$ for $x \in B_1(x_0)$ and $y \in B_R^c(x_0)$, it follows that for all $x \in B_1(x_0)$,
\[ \|T_{x_0,R,\alpha} f(x)\| \leq C_\alpha \int_{|y-x_0| \geq R} \frac{1}{|x_0 - y|^{3+\alpha}} |f(y)| \, dy \]
\[ \leq C_\alpha \sum_{i=1}^{\infty} \int_{R \cdot 2^{i-1} < |y-x_0| \leq R \cdot 2^i} \frac{1}{|x_0 - y|^{3+\alpha}} |f(y)| \, dy \]
\[ \leq C_\alpha \sum_{i=1}^{\infty} \frac{(R \cdot 2^{i-1})^3}{(R \cdot 2^{i-1})^{3+\alpha}} \|f\|_{L^1_{uloc}} \leq C_\alpha R^{-\alpha} \|f\|_{L^1_{uloc}}. \]

From (2.1) and Lemma 3.1, we easily deduce that for all $x_0 \in \mathbb{R}^3$,
\[ \|A^s e^{-tA^{2s}} u_0^2\|_{L^\infty([0,T\varepsilon] \times B_1(x_0))} \lesssim_s \|u_0\|_{L^2_{uloc}}. \]
Collecting the above three estimates, we get
\begin{equation}
\|e^{-tA^{2s}} u_0\|_{X_{T\varepsilon}} \leq C_1 (1 + \sqrt{T\varepsilon}) \|u_0\|_{L^2_{uloc}}
\end{equation}
for some positive constant $C_1$ depending only on $s$. Next, we deal with $B_\varepsilon(v_1,v_2)$ with $v_1, v_2 \in X_{T\varepsilon}$. Setting $F = -J_\varepsilon v_1 \otimes v_2$, we have by (2.9),
\[ \|B_\varepsilon(v_1,v_2)\|_{L^2_{uloc}} + \|A^s B_\varepsilon(v_1,v_2)\|_{L^2_{uloc}} \]
\[ \lesssim \int_0^t (t - \tau)^{-\frac{1}{2s}} (\|F(\tau)\|_{L^2_{uloc}} + \|A^s F(\tau)\|_{L^2_{uloc}}) \, d\tau, \]
which shows that
\begin{equation}
\|B_\varepsilon(v_1,v_2)\|_{X_{T\varepsilon}} \lesssim_s T^{-1/2s}_\varepsilon \|F\|_{X_{T\varepsilon}}.
\end{equation}
To proceed from (3.4), by (2.6) with \( \alpha = 1 \), we observe that
\[
\|J_\varepsilon v_1\|_{L^\infty(\mathbb{R}^3 \times [0,T_\varepsilon])} \lesssim \varepsilon^{-3/2} \|v_1\|_{L^\infty([0,T_\varepsilon];L^2_{uloc})},
\]
(3.5)
\[
\|\nabla J_\varepsilon v_1\|_{L^\infty(\mathbb{R}^3 \times [0,T_\varepsilon])} \lesssim \varepsilon^{-5/2} \|v_1\|_{L^\infty([0,T_\varepsilon];L^2_{uloc})}.
\]
(3.6)
Hence from (3.5) we get
\[
\|F\|_{L^\infty([0,T_\varepsilon];L^2_{uloc})} \leq \|J_\varepsilon v_1\|_{L^\infty(\mathbb{R}^3 \times [0,T_\varepsilon])} \|v_2\|_{L^\infty([0,T_\varepsilon];L^2_{uloc})} \lesssim \varepsilon^{-3/2} \|v_1\|_{L^\infty([0,T_\varepsilon];L^2_{uloc})} \|v_2\|_{L^\infty([0,T_\varepsilon];L^2_{uloc})}.
\]
In addition, by Lemma 2.4 and (3.5)–(3.6),
\[
\|A^s F\|_{L^2([0,T_\varepsilon];L^2_{uloc})} \lesssim \|A^s J_\varepsilon v_1\|_{L^2([0,T_\varepsilon];L^2_{uloc})} + \|J_\varepsilon v_1\|_{L^\infty(\mathbb{R}^3 \times [0,T_\varepsilon])} \|A^s v_2\|_{L^2([0,T_\varepsilon];L^2_{uloc})} \lesssim (\varepsilon^{-3/2-s} T_\varepsilon^{1/2} \|v_2\|_{L^\infty([0,T_\varepsilon];L^2_{uloc})}) + \varepsilon^{-3/2} \|A^s v_2\|_{L^2([0,T_\varepsilon];L^2_{uloc})} \times \|v_1\|_{L^\infty([0,T_\varepsilon];L^2_{uloc})}.
\]
Substituting the above two estimates into (3.4), we obtain
\[
\|B_\varepsilon(v_1, v_2)\|_{X_{T_\varepsilon}} \leq C_2 \varepsilon^{-3} T_\varepsilon^{1/3} \|v_1\|_{X_{T_\varepsilon}} \|v_2\|_{X_{T_\varepsilon}}
\]
where we have used the fact that \( \varepsilon < 1, T_\varepsilon < 1 \) and \( 3/4 < s < 1 \).

Finally, combining (3.3) and (3.7), and applying the Banach fixed point theorem to (3.1), we get the existence of a unique mild solution to (MFNS) in \( X_{T_\varepsilon} \) provided that
\[
T_\varepsilon < \min\{1, \varepsilon^9 (4C_1 C_2 \|u_0\|_{L^2_{uloc}})^{-3}\}.
\]

**Step 2: Uniform existence time.** Since \( u_\varepsilon \in L^\infty((0,T_\varepsilon);L^2_{uloc}) \), the equation
\[
\nabla P_\varepsilon = -\nabla \text{div div} \left( J_\varepsilon u_\varepsilon \otimes u_\varepsilon \right)
\]
is well-defined. Thus, \( P_\varepsilon \) is defined up to a function which does not depend on \( x \). For each nonnegative function \( \varphi \in \mathcal{D}(\mathbb{R}^3) \), multiplying the first equation of (MFNS) by \( u_\varepsilon \varphi \) and then integrating over \( \mathbb{R}^3 \times [0,t] \), we get
\[
\int_{\mathbb{R}^3} \varphi |u_\varepsilon(t)|^2 \, dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi x_0 |A^s u_\varepsilon|^2 \, dx \, d\tau
\]
\[
= \int_{\mathbb{R}^3} \varphi |u_0|^2 \, dx - 2 \int_0^t \int_{\mathbb{R}^3} ([\bar{\varphi}, A^s] A^s u_\varepsilon \cdot u_\varepsilon \varphi + \tilde{\varphi} A^s u_\varepsilon \cdot [A^s, \varphi] u_\varepsilon) \, dx \, d\tau
\]
\[
+ \int_0^t \int_{\mathbb{R}^3} |u_\varepsilon|^2 J_\varepsilon u_\varepsilon \cdot \nabla \varphi \, dx \, d\tau - 2 \int_0^t \int_{\mathbb{R}^3} \nabla P_\varepsilon \cdot u_\varepsilon \varphi \, dx \, d\tau,
\]
where \( \tilde{\varphi} \in \mathcal{D}(\mathbb{R}^3) \) satisfies \( \varphi \tilde{\varphi} = 1 \). Let \( 0 \leq \phi \in \mathcal{D}(B_2(0)) \) satisfy \( \phi(x) = 1 \) in \( B_1(0) \) and \( 0 \leq \phi(x) \leq 1 \). Apply (3.8) for \( \varphi = \phi_{x_0} = \phi(\cdot - x_0) \) and \( \varphi = \phi((\cdot - x_0)/2) \), and then denote by I.1 to I.3 the last three terms on the right-hand side of (3.8). Set

\[
\alpha_\epsilon(t) = \sup_{x_0 \in \mathbb{R}^3} \left( \int_{B_1(x_0)} |u_\epsilon(t)|^2 \, dx \right)^{1/2}, \quad \gamma_\epsilon(t) = \sup_{x_0 \in \mathbb{R}^3} \left( \int_0^t \int_{0 B_1(x_0)} |u_\epsilon|^3 \, dx \, d\tau \right)^{1/3}, \quad \beta_\epsilon(t) = \sup_{x_0 \in \mathbb{R}^3} \left( \int_0^t \int_{0 B_1(x_0)} |A^s u_\epsilon|^2 \, dx \, d\tau \right)^{1/2}.
\]

Then we claim that

\[
\gamma_\epsilon(t)^3 \lesssim_s \beta_\epsilon(t) \frac{3}{27} \left( \int_0^t \alpha_\epsilon(\tau)^{\frac{6(2s-1)}{4s-3}} \, d\tau \right)^{1-s} + \frac{1}{6} \int_0^t \alpha_\epsilon(\tau)^3 \, d\tau.
\]

In fact, by interpolation between \( L^2 \) and \( L^{\frac{6}{3-2s}} \) and the Sobolev inequality \( \dot{H}^s \hookrightarrow L^{\frac{6}{3-2s}} \), we have

\[
\|\phi_{x_0} u_\epsilon\|_{L^3(\mathbb{R}^3)} \leq \|\phi_{x_0} u_\epsilon\|_{L^2(\mathbb{R}^3)}^{\frac{2s-1}{2s}} \|A^s(\phi_{x_0} u_\epsilon)\|_{L^2(\mathbb{R}^3)}^{\frac{1}{2s}}.
\]

Furthermore, by Corollary 2.5

\[
\|A^s(\phi_{x_0} u_\epsilon)\|_{L^2} \lesssim_s \|\phi_{x_0} A^s u_\epsilon\|_{L^2} + \|u_\epsilon(t)\|_{L_{3,loc}^2}.
\]

Combining the above two inequalities, we easily deduce (3.9).

Next, we estimate I.1–I.3 in turn. By Hölder’s and Young’s inequalities, and Lemma 2.4, we have

\[
|I.1| + |I.2| \leq \frac{1}{2} \beta_\epsilon(t)^2 + C_s \int_0^t \alpha_\epsilon(\tau)^2 \, d\tau + \gamma_\epsilon(t)^3.
\]

To estimate I.3, we only need information on \( P_\epsilon \) in \( B_2(x_0) \). Hence, for any fixed \( x \in B_2(x_0) \times (0, T) \), we decompose \( P_\epsilon(x, t) \) in the following way: take \( \psi_{x_0} := \psi(\cdot - x_0) \) with \( 0 \leq \psi \in \mathcal{D}(B_8(0)) \) and \( \psi = 1 \) in \( B_4(0) \). Then there exists a function \( P_{x_0,\epsilon}(t) \) depending only on \( x_0, t, \psi_{x_0} \) such that for \( (x, t) \in B_2(x_0) \times (0, T) \),

\[
P_\epsilon(x, t) = P_{x_0,\epsilon}(t) - \Delta^{-1} \text{div}(J_\epsilon u_\epsilon \otimes u_\epsilon \psi_{x_0})
- \int_{\mathbb{R}^3} (k(x - y) - k(x_0 - y)) (J_\epsilon u_\epsilon \otimes u_\epsilon)(y, t)(1 - \psi_{x_0}(y)) \, dy
= : P_{x_0,\epsilon}(t) + P_{1 x_0,\epsilon}(x, t) + P_{2 x_0,\epsilon}(x, t) = : P_{x_0,\epsilon}(t) + P_{x_0,\epsilon}(x, t).
\]

Then we can replace \( P_\epsilon \) by \( P_{x_0,\epsilon} \). Furthermore, by the Calderón–Zygmund estimate,

\[
\|P_{1 x_0,\epsilon}(t)\|_{L^{3/2}([0, t] \times \mathbb{R}^3)} \lesssim \|J_\epsilon u_\epsilon \otimes u_\epsilon\|_{L^{3/2}([0, t] \times B_8(x_0))} \lesssim \gamma_\epsilon(t)^2.
\]
For $P^2_{x_0,\varepsilon}(x,t)$, by (1.5) and Lemma 3.1 for all $x \in B_2(x_0)$ we have
$$|P^2_{x_0,\varepsilon}(x,t)| \lesssim \alpha_\varepsilon(t)^2.$$ Combining the above two estimates, we deduce that
$$|I.3| \lesssim \|P^1_{x_0,\varepsilon}\|_{L^{3/2}([0,T];L^{3/2}(\mathbb{R}^3))} \|u_\varepsilon\|_{L^3([0,T];L^3(B_4(x_0)))}$$
$$+ \int_0^t \|P^2_{x_0,\varepsilon}(\tau)\|_{L^2(B_4(x_0))} \|u_\varepsilon(\tau)\|_{L^2(B_4(x_0))} d\tau \lesssim \gamma_\varepsilon(t)^3 + \int_0^t \alpha_\varepsilon(\tau)^3 d\tau.$$ Collecting the estimates of I.1–I.3, we obtain
$$\alpha_\varepsilon(t)^2 + 2\beta_\varepsilon(\tau)^2 \leq 2\alpha_\varepsilon(0)^2 + \frac{1}{2} \beta_\varepsilon(t)^2 + C_S \int_0^t \left( \alpha_\varepsilon(\tau)^2 + \alpha_\varepsilon(\tau)^3 \right) d\tau + C_S \gamma_\varepsilon(t)^3.$$ Substituting (3.9) into the above inequality and then using Young’s and Hölder’s inequalities, we obtain
$$\alpha_\varepsilon(t)^2 + \beta_\varepsilon(t)^2 \leq 2\alpha_\varepsilon(0)^2 + C_S \left( \int_0^t \alpha_\varepsilon(\tau)^2 + \alpha_\varepsilon(\tau)^3 \right) d\tau + C_S \gamma_\varepsilon(t)^3.$$ Then, by the continuity method, we get
$$\alpha_\varepsilon(t)^2 + \beta_\varepsilon(t)^2 \leq 4\|u_0\|^2_{L^2_{uloc}}, \quad \forall 0 \leq t \leq T^*,$$
with
$$T^* = \min\left\{ 1, (4C_S)^{-1}, C_S^{-1} 2^{\frac{6(2s-1)}{4s-3}} \|u_0\|_{L^2_{uloc}}^{-\frac{4s}{4s-3}} \right\}.$$ **Step 3: Existence of a weak approximate solution.** From Step 2, we know that there exists an $M > 0$ depending only on $\|u_0\|_{L^2_{uloc}}$ and $T^*$ defined in (3.12) such that
$$\|u_\varepsilon\|_{L^\infty([0,T^*];L^2_{uloc})} + \|A^s u_\varepsilon\|_{L^2_{uloc}(T^*)} + \|P_{x_0,\varepsilon}\|_{L^{3/2,3/2}_{uloc}(T^*)} \leq M, \quad \forall \varepsilon \in (0,1).$$ This, together with Lemma 2.6, implies that for any nonnegative $f \in D(\mathbb{R}^3)$, $\{\phi u_\varepsilon\}$ is bounded in $L^\infty([0,T^*];L^2) \cap L^2([0,T^*];\dot{H}^s)$ and $\{\phi P_{x_0,2,\varepsilon}\}$ is bounded in $L^3(\mathbb{R}^3)$ for all $\varepsilon \in (0,1)$. According to the Aubin–Lions Lemma (see [26]), by the Cantor diagonal process we can find a subsequence of $(u_\varepsilon, P_{x_0,\varepsilon})$, still denoted by $(u_\varepsilon, P_{x_0,\varepsilon})$, and a pair $(u,P)$ with $P = P_{x_0} + P_{x_0}(t)$ defined in Remark 1.2 such that
$$\partial_t u_\varepsilon + A^s u_\varepsilon + \mathbb{P}(J_{\varepsilon} u_\varepsilon \cdot \nabla u_\varepsilon) = 0,$$ we conclude that $\phi \partial_t u_\varepsilon$ remains bounded in $L^{3,2}(\mathbb{R}^3;H^{-3,2})$. According to the Aubin–Lions Lemma (see [26]), by the Cantor diagonal process we can find a subsequence of $(u_\varepsilon, P_{x_0,\varepsilon})$, still denoted by $(u_\varepsilon, P_{x_0,\varepsilon})$, and a pair $(u,P)$ with $P = P_{x_0} + P_{x_0}(t)$ defined in Remark 1.2 such that
$$\begin{cases} u_\varepsilon \to u \text{ in } L^\infty([0,T^*];L^2_{uloc}) \text{ and } A^s u_\varepsilon \to A^s u \text{ in } L^2_{uloc}(T^*); \\ u_\varepsilon \to u \text{ in } L^3_{uloc}(T^*); \\ P_{x_0,\varepsilon} \to P_{x_0} \text{ in } L^{3/2}_{uloc}(T^*). \end{cases}$$
Using (3.13), we can easily verify that \((u, P)\) satisfies \([\text{FNS}]\) in the sense of distributions.

**STEP 4: Local energy inequality for the weak limit.** It is obvious that \((u_\varepsilon, P_{x_0,\varepsilon})\) satisfies the following equality:

\[
\int_0^t \int_{\mathbb{R}^3} \psi |A^s u_\varepsilon|^2 \, dx \, d\tau = \int_0^t \int_{\mathbb{R}^3} |u_\varepsilon|^2 \partial_t \psi \, dx \, d\tau - \int_0^t \int_{\mathbb{R}^3} [\tilde{\psi}, A^s] A^s u_\varepsilon \cdot u_\varepsilon \psi \, dx \, d\tau
\]

\[
- \int_0^t \int_{\mathbb{R}^3} \tilde{\psi} A^s u_\varepsilon \cdot [A^s, \psi] u_\varepsilon \, dx \, d\tau
\]

\[
+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |u_\varepsilon|^2 J_\varepsilon u_\varepsilon \cdot \nabla \psi \, dx \, d\tau + \int_0^t \int_{\mathbb{R}^3} p_{x_0,\varepsilon} \cdot u_\varepsilon \nabla \psi \, dx \, d\tau
\]

for every nonnegative \(\psi \in \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^+)\) and \(\tilde{\psi} \in \mathcal{D}(\mathbb{R}^3)\) with \(\tilde{\psi} \psi = \psi\). According to (3.13), we can easily get

\[
(3.14) \quad \int_0^t \int_{\mathbb{R}^3} |\psi A^s u|^2 \, dx \, d\tau \leq \int_0^t \int_{\mathbb{R}^3} |u|^2 \partial_t \psi \, dx \, d\tau - \int_0^t \int_{\mathbb{R}^3} [\tilde{\psi}, A^s] A^s u \cdot u \psi \, dx \, d\tau
\]

\[
- \int_0^t \int_{\mathbb{R}^3} \tilde{\psi} A^s u \cdot [A^s, \psi] u \, dx \, d\tau
\]

\[
+ \frac{1}{2} \int_0^t \int_{\mathbb{R}^3} |u|^2 J_\varepsilon u \cdot \nabla \psi \, dx \, d\tau + \int_0^t \int_{\mathbb{R}^3} p \cdot u \nabla \psi \, dx \, d\tau
\]

for every nonnegative \(\psi \in \mathcal{D}(\mathbb{R}^3 \times \mathbb{R}^+)\) and \(\tilde{\psi} \in \mathcal{D}(\mathbb{R}^3)\) with \(\tilde{\psi} \psi = \psi\).

**STEP 5: Strong convergence of \(u(t)\) to \(u_0\) in \(L^2_{\text{loc}}\).** Since \(\varphi \partial_t u \in L^{3/2}([0, T^*]; H^{-3/2})\) for each nonnegative \(\varphi \in \mathcal{D}(\mathbb{R}^3)\), we see that \([0, T^*] \ni t \mapsto u(t) \in \mathcal{D}'(\mathbb{R}^3)\) is continuous. This implies that

\[
(3.15) \quad \|u_0\|_{L^2(K)} \leq \liminf_{t \to 0^+} \|u(t)\|_{L^2(K)} \quad \text{for every compact } K \subset \mathbb{R}^3.
\]

On the other hand, applying (3.13) to (3.8), we deduce that for all \(t \in [0, T^*]\),

\[
\int_{\mathbb{R}^3} \varphi |u(t)|^2 \, dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi |A^s u(\tau)|^2 \, dx \, d\tau
\]

\[
\leq \int_{\mathbb{R}^3} \varphi |u_0|^2 \, dx - 2 \int_0^t \int_{\mathbb{R}^3} ([\tilde{\varphi}, A^s] A^s u \cdot u \varphi + \varphi A^s u \cdot [A^s, \varphi] u) \, dx \, d\tau
\]

\[
+ \int_0^t \int_{\mathbb{R}^3} |u|^2 u \cdot \nabla \varphi \, dx \, d\tau + 2 \int_0^t \int_{\mathbb{R}^3} p \cdot u \nabla \varphi \, dx \, d\tau
\]
for any nonnegative $\varphi, \tilde{\varphi} \in \mathcal{D}(\mathbb{R}^3)$ satisfying $\varphi \tilde{\varphi} = 1$. Letting $t \to 0+$, we have
\begin{equation}
\limsup_{t \to 0+} \|u(t)\|_{L^2(K)} \leq \|u_0\|_{L^2(K)} \quad \text{for every compact } K \subset \mathbb{R}^3.
\end{equation}
Combining (3.15) and (3.16), we get
\begin{equation}
\lim_{t \to 0+} \|u(t)\|_{L^2(K)} = \|u_0\|_{L^2(K)} \quad \text{for every compact } K \subset \mathbb{R}^3.
\end{equation}
This, together with the weak convergence of $u(t)$ to $u_0$ in $L^2_{uloc}$, shows that
\begin{equation}
\lim_{t \to 0+} \|u(t) - u_0\|_{L^2(K)} = 0 \quad \text{for every compact } K \subset \mathbb{R}^3.
\end{equation}
Summing up, we have proved Theorem 1.4.

4. Regularity of local Leray solutions with initial data in $E^2$.

In this section, we mainly study the additional regularity of local Leray solutions when the initial data not only belong to $L^2_{uloc}$ but also vanish at infinity.

**Theorem 4.1.** Suppose that $u$ is a local Leray solution to (FNS)–(FNSI) on $\mathbb{R}^3 \times (0, T)$ starting from $u_0 \in E^2$ with $\text{div} u_0 = 0$. Let $\chi \in C^\infty(\mathbb{R}^3)$ satisfy
\begin{equation}
\chi = 0 \quad \text{in } B_1(0), \quad \chi = 1 \quad \text{in } B_2^c(0), \quad 0 \leq \chi \leq 1.
\end{equation}
Define $\chi_R(x) := \chi(x/R)$ and $M_{T'} := \|u\|_{L^\infty((0, T'); L^2_{uloc})} + \|u\|_{L^2_{uloc}(T')}$ for $T' < T$. Then, for each $T' < T$, there exists a constant $C_{s, T', M_{T'}}$ such that for all $t \in (0, T')$,
\begin{equation}
\|u(t, \cdot)\chi_R\|_{L^2_{uloc}} + \|A^s u \chi_R\|_{L^2_{uloc}(t)} \leq C_{s, T', M_{T'}}(\|u_0\chi_R\|_{L^2_{uloc}} + R^{-1/4}).
\end{equation}

**Proof.** First, we simplify the local energy inequality (1.3). Let $0 \leq \theta \in \mathcal{D}(\mathbb{R})$ satisfy $\int_\mathbb{R} \theta \, dx = 1$ and $\text{supp} \theta \subset [-1, 1]$. Define
\begin{equation}
\alpha_\eta(t) = \frac{1}{\eta} \int_{-\infty}^{t} \left[ \theta \left( \frac{\tau - t_0}{\eta} \right) - \theta \left( \frac{\tau - t_1}{\eta} \right) \right] \, d\tau \quad \text{for } 0 < \eta < t_0 < t_1 \text{ and } t_1 + \eta < T.
\end{equation}
It is clear that $\alpha_\eta(\cdot) \in \mathcal{D}((0, T))$. Let $0 \leq \varphi \in \mathcal{D}(\mathbb{R}^3)$. Then, applying (1.3) for $\psi(x, t) := \alpha_\eta(t)\varphi(x)$, we get
\begin{align*}
2 \int_{0}^{t} \int_{\mathbb{R}^3} \alpha_\eta \varphi |A^s u|^2 \, dx \, d\tau &\leq \int_{0}^{t} \int_{\mathbb{R}^3} |u|^2 \partial_t \alpha_\eta \varphi \, dx \, d\tau \\
&\quad - 2 \int_{0}^{t} \int_{\mathbb{R}^3} \alpha_\eta ([\tilde{\varphi}, A^s] A^s u \cdot w \varphi + \tilde{\varphi} A^s u \cdot [A^s, \varphi] u) \, dx \, d\tau \\
&\quad + \int_{0}^{t} \int_{\mathbb{R}^3} \alpha_\eta |u|^2 \varphi \cdot \nabla \varphi \, dx \, d\tau + 2 \int_{0}^{t} \int_{\mathbb{R}^3} \alpha_\eta P_{x_0, 2} u \cdot \nabla \varphi \, dx \, d\tau.
\end{align*}
We denote the last three terms on the right side by I–III. Set definition of local Leray solutions, for any $M \phi$

\[
\lim_{\eta \to 0} \alpha_{\eta}(t) = \begin{cases} 
0, & 0 \leq t < t_0 \text{ or } t > t_1, \\
1/2, & t = t_0 \text{ or } t = t_1, \\
1, & t_0 < t < t_1.
\end{cases}
\]

Assume that $t_0$ and $t_1$ are two Lebesgue points of the map $t \mapsto \int_{\mathbb{R}^3} |u(t)|^2 \varphi \, dx$. Then, letting $\eta \to 0+$, in view of \((4.3)\), by the Lebesgue dominated convergence theorem,

\[
(4.3) \quad \int_{\mathbb{R}^3} \varphi |u(t)|^2 \, dx + 2 \int_{t_0}^{t_1} \int_{\mathbb{R}^3} \varphi |A^s u|^2 \, dx \, d\tau \\
\quad \leq \int_{\mathbb{R}^3} \varphi |u(t_0)|^2 \, dx - 2 \int_{\mathbb{R}^3} \int_{0}^{t} \left( [\tilde{\varphi}, A^s] A^s u \cdot u \varphi + \tilde{\varphi} A^s u \cdot [A^s, \varphi] u \right) \, dx \, d\tau \\
\quad + \int_{\mathbb{R}^3} \int_{0}^{t} |u|^2 u \cdot \nabla \varphi \, dx \, d\tau + 2 \int_{\mathbb{R}^3} \int_{0}^{t} P_{x_0,2} u \cdot \nabla \varphi \, dx \, d\tau.
\]

Since $\lim_{t \to 0^+} \|u(t) - u_0\|_{L^2_{\text{loc}}} = 0$ and the map $t \mapsto u(\cdot, t)$ is weakly continuous, we see that \((4.3)\) holds for all $t_1 \in (0, T)$ and $t_0 = 0$.

Next, we will prove the theorem with the aid of \((4.3)\). According to the definition of local Leray solutions, for any $T' < T$ there exists a constant $M_{T'} > 0$, which depends only on $T'$, such that

\[
\|u\|_{L^\infty([0,T']'; L^2_{\text{loc}})} + \|A^s u\|_{L^2_{\text{loc}}(T')} + \|u\|_{L^3_{\text{loc}}(T')} \leq C_{M_{T'}}.
\]

Let $\phi_{x_0}$ and $\tilde{\varphi}_{x_0}$ be as in Step 2 in Section 3. Applying \((4.3)\) for $\varphi := \phi_{x_0} x_R^2$ we get, for each $0 < t \leq T'$,

\[
\int_{\mathbb{R}^3} \phi_{x_0} |u(t)|^2 \, dx + 2 \int_{0}^{t} \int_{\mathbb{R}^3} \phi_{x_0} |\chi_R A^s u|^2 \, dx \, d\tau \leq \int_{\mathbb{R}^3} \phi_{x_0} |u_0\chi_R|^2 \, dx \\
- \int_{0}^{t} \int_{\mathbb{R}^3} \left( [\tilde{\phi}_{x_0}, A^s] A^s u \cdot u \phi_{x_0} x_R^2 + \tilde{\phi}_{x_0} A^s u \cdot [A^s, \phi_{x_0} x_R^2] u \right) \, dx \, d\tau \\
+ \int_{\mathbb{R}^3} \int_{0}^{t} |u|^2 u \cdot \nabla (\phi_{x_0} \chi_R^2) \, dx \, d\tau + 2 \int_{\mathbb{R}^3} \int_{0}^{t} P_{x_0,2} u \cdot \nabla (\phi_{x_0} \chi_R^2) \, dx \, d\tau.
\]

We denote the last three terms on the right side by I–III. Set

\[
\alpha_R(t) = \sup_{x_0 \in \mathbb{R}^3} \left( \int_{B_1(x_0)} |\chi_R u(t)|^2 \, dx \right)^{1/2},
\]

with $P_{x_0,2}$ defined in \((1.4)\) with $r = 2$ for all $\tilde{\varphi} \in \mathcal{D}(\mathbb{R}^3)$ satisfying $\tilde{\varphi} \geq 0$ and $\tilde{\varphi} \varphi = \varphi$. In view of the definition of $\alpha_{\eta}$,

\[
\partial_t \alpha_{\eta} = \frac{1}{\eta} \theta \left( \frac{t-t_0}{\eta} \right) - \frac{1}{\eta} \theta \left( \frac{t-t_1}{\eta} \right),
\]

\[
(4.2)
\]
Hence, by Lemma 2.4 and Young’s inequality, we have

\[
\gamma(t) = \sup_{x_0 \in \mathbb{R}^3} \left( \int_0^t \int_{B_1(x_0)} |\chi_R A^s u|^3 \, dx \, d\tau \right)^{1/3}.
\]

Obviously \( \alpha_R(t) \leq \|u(t)\|_{L^2_{\text{uloc}}} \) and \( \beta_R(t) \leq \|A^s u\|_{L^2_{\text{uloc}}(t)} \). Similar to (3.9), we have

\[
\gamma(t)^3 \lesssim_{s} \beta_R(t)^{3/2} \left( \int_0^t \alpha_R(\tau)^{6(2s-1)/4s-3} \, d\tau \right)^{1/4s} + \int_0^t \alpha_R(\tau)^3 \, d\tau,
\]

which follows from Lemma 2.4 and the fact that

\[
\|\phi_{x_0} \chi_R u(t)\|_{L^3} \leq \|\phi_{x_0} \chi_R u\|_{L^3}^{1-1/(2s)} \|A^s(\phi_{x_0} \chi_R u)\|_{L^2}^{1/(2s)},
\]

\[
\|A^s(\phi_{x_0} \chi_R u)\|_{L^2} \leq \|\phi_{x_0} \chi_R A^s u\|_{L^2} + \|\phi_{x_0} [A^s, \chi_R] u\|_{L^2} + \|[A^s, \phi_{x_0}] \chi_R u\|_{L^2}.
\]

Next, we deal with I–III in turn. To estimate I, we observe that

\[
\chi_R[\varphi_{x_0}, A^s] A^s u = \varphi_{x_0} [\chi_R, A^s] A^s u + [\varphi_{x_0}, A^s] \chi_R A^s u + [A^s, \chi_R] \varphi_{x_0} A^s u,
\]

\[
[A^s, \varphi_{x_0} \chi_R]^2 u = [A^s, \chi_R] \varphi_{x_0} \chi_R u + \chi_R[A^s, \varphi_{x_0}] \chi_R u + \chi_R \varphi_{x_0} [A^s, \chi_R] u.
\]

Hence, by Lemma 2.4 and Young’s inequality,

\[
|I| \leq C_s(\beta_R(t) + R^{-s}\beta(t)) \left( \int_0^t \alpha_R(\tau)^2 \, d\tau \right)^{1/2} + C_s \beta_R(t) \left( \int_0^t (\alpha_R(\tau)^2 + R^{-2s} \alpha(\tau)^2) \, d\tau \right)^{1/2} \leq C_s, T', M_T, R^{-s} + \frac{1}{4} \beta_R(t)^3 + C_s \sqrt{\int_0^t \alpha_R(\tau)^2 \, d\tau}.
\]

For II, by Hölder’s inequality we get

\[
|II| \leq C \|u\|_{L^3([0,t]; L^3(B_2(x_0)))} \times \left( \|u \chi_R\|_{L^2([0,t]; L^3(B_2(x_0)))}^2 + \frac{1}{R^1} \|u\|_{L^3([0,t]; L^3(B_2(x_0)))}^2 \right) \leq C_{M_T, R}^2 \gamma_R(t) + C_{M_T, R} R^{-1}.
\]

Finally, we consider III. Since \( \text{div} \, u = 0 \), we rewrite

\[
III = III.1 + III.2
\]

\[
:= \int_0^t \int_{\mathbb{R}^3} P_{x_0, 2} (u \cdot \nabla \varphi_{x_0}) \chi_R^2 \, dx \, d\tau + \int_0^t \int_{\mathbb{R}^3} P_{x_0, 2} (u \cdot \nabla \chi_R) \varphi_{x_0} \chi_R \, dx \, d\tau.
\]
By \([1.5]\), Lemma 3.1, Hölder’s inequality and the Calderón–Zygmund estimates, we have

\[
\text{(4.7) } \|\text{III.2}\| \leq CR^{-1}\|P_{x_0,2}^1\|_{L^{3/2}([0,t];L^{3/2}(B_2(x_0)))}\|u\|_{L^2([0,t];L^3(B_2(x_0)))} \\
+ R^{-1}\int_0^t\|P_{x_0,2}^2\|_{L^2(B_2(x_0)))}\|u\|_{L^2(B_2(x_0)))} \, dt \\
\leq CR^{-1}\|u\|^3_{L^3_{\text{uloc}}(t)} + tR^{-1}\|u\|^3_{L^\infty([0,t];L^2_{\text{uloc}})} \leq C_{T',M_T} R^{-1}.
\]

To estimate III.1, we observe that

\[
\chi_R(x)P_{x_0,2}^1(x) = -\left(\chi_R(x) - \chi_R(x_0)\right) \left[\frac{\text{div div}}{\Delta} u \otimes u \psi_{x_0}\right](x) - \left[\frac{\text{div div}}{\Delta} u \otimes u(\chi_R(x_0) - \chi_R)\psi_{x_0}\right](x) - \left[\frac{\text{div div}}{\Delta} u \otimes u \psi_{x_0}\chi_R\right](x).
\]

Thus, by the mean value theorem, Hölder’s inequality and the Calderón–Zygmund estimates, we obtain

\[
\text{(4.8) } \|\chi_R P_{x_0,2}^1\|_{L^{3/2}(B_2(x_0))} \leq CR^{-1}\|u\|^2_{L^3(B_8(x_0))} + C\|u\|_{L^3(B_8(x_0))}\|\chi_R u\|_{L^3(B_8(x_0))}.
\]

For \(\chi_R(x)P_{x_0,2}^2(x, t)\), we rewrite

\[
\chi_R(x)P_{x_0,2}^2(x, t) = \int_{B_{2\sqrt{R}(x_0)}} (k(x_0 - y) - k(x - y)) (\chi_R(x) - \chi_R(y)) ((1 - \psi_{x_0})u \otimes u)(y, t) \, dy \\
- \int_{|y - x_0| \leq 2\sqrt{R}} (k(x - y) - k(x_0 - y)) \chi_R(y) ((1 - \psi_{x_0})u \otimes u)(y, t) \, dy \\
- \chi_R(x) \int_{|y - x_0| \geq 2\sqrt{R}} (k(x - y) - k(x_0 - y)) ((1 - \psi_{x_0})u \otimes u)(y, t) \, dy.
\]

Hence, using the mean value theorem we get, for all \(x \in B_2(x_0)\),

\[
|\chi_R(x)P_{x_0,2}^2(x, t)| \leq C \left(\frac{1}{\sqrt{R}}\right) \int_{4 \leq |y - x_0| \leq 2\sqrt{R}} |k(x - y) - k(x_0 - y)| |u(y, t)|^2 \, dy \\
+ \int_{4 \leq |y - x_0| \leq 2\sqrt{R}} \chi_R(y)|k(x - y) - k(x_0 - y)| |u(y, t)|^2 \, dy \\
+ \int_{|y - x_0| \geq 2\sqrt{R}} |k(x - y) - k(x_0 - y)| |u(y)|^2 \, dy.
\]

Hence, in view of (1.5) and Lemma 3.1, for all \(x \in B_2(x_0)\),

\[
|\chi_R(x)P_{x_0,2}^2(x, t)| \leq CR^{-1/2}\|u(t)|\|_{L^2_{\text{uloc}}}^2 + \alpha R(t)\|u(t)|\|_{L^2_{\text{uloc}}}.
\]
Combining (4.8) and (4.9), we obtain

\[
(4.10) \quad |\supseteq \leq \int_0^t \| \chi R P_{x_0,2}^1 \|_{L^{3/2}(B_2(x_0))} \| \chi R u(\tau) \|_{L^3(B_2(x_0))} \, d\tau \\
+ \int_0^t (R^{-1/2} \| u(\tau) \|_{L^2_{uloc}}^3 + \alpha_R(\tau)^2 \| u(\tau) \|_{L^2_{uloc}}^2) \, d\tau \\
\leq C_{T',M_{T'}} R^{-1/2} + C_{M_{T'}} \gamma_R(t)^2 + C_{M_{T'}} \int_0^t \alpha_R(\tau)^2 \, d\tau.
\]

Summing (4.5)–(4.7) and (4.10) we have, for all \( t \in [0, T'] \),

\[
(4.11) \quad \alpha_R(t)^2 + \frac{3}{2} \beta_R(t)^2 \\
\leq 2 \alpha_R(0)^2 + C_{s,T',M_{T'}} R^{-1/2} + C_{s,M_{T'}} \left( \int_0^t \alpha_R(\tau)^2 \, d\tau + \gamma_R(t)^2 \right).
\]

Substituting (4.4) into (4.11), we obtain

\[
\alpha_R(t)^2 + \beta_R(t)^2 \\
\leq 2 \alpha_R(0)^2 + C_{s,T',M_{T'}} R^{-1/2} + C_{s,M_{T'}} \left( \int_0^t \alpha_R(\tau)^2 \, d\tau + \gamma_R(t)^2 \right) \gamma_R(0)^{-2}.
\]

This yields

\[
\alpha_R(t)^{6(2s-1)/4s-3} \leq 2 \alpha_R(0)^{6(2s-1)/4s-3} + C_{s,T',M_{T'}} R^{-3(2s-1)/(4s-3)} + C_{s,M_{T'}} \int_0^t \alpha_R(\tau)^{6(2s-1)/4s-3} \, d\tau.
\]

By Gronwall’s inequality we get, for all \( t \leq T' \),

\[
\alpha_R(t)^{6(2s-1)/4s-3} \leq \left( 2 \alpha_R(0)^{6(2s-1)/4s-3} + C_{s,T',M_{T'}} R^{-3(2s-1)/(4s-3)} \right)^{4s-3/(3s-2)} e^{C_{s,M_{T'}} T'}.
\]

This implies (4.1) and the proof of Theorem 4.1 is complete. \( \blacksquare \)

Next, we will use Theorem 4.1 to prove Theorem 1.5.

**Proof of Theorem 1.5.** It is obvious that for all \( T' < T, u \in L^\infty([0, T']; E^2) \) and \( A^\omega u \in G^{2,2}(T') \), which follows from Theorem 4.1. Thus, in what follows, we focus on the proof of (1.6).

Let \( Q_L \) be the set of all Lebesgue points of \( u(t) \) in \((0, T)\). Since \([0, T] \ni t \mapsto u(t) \in D'(\mathbb{R}^3)\) is continuous, and \( u \in L^\infty([0, T']; E^2)\) for any \( T' < T \), we obtain, for all \( t_0 \in Q_L \),

\[
\| u(t_0) \|_{L^2(K)} \leq \liminf_{t_1 \to t_0^+} \| u(t_1) \|_{L^2(K)}, \quad \forall K \subset \mathbb{R}^3,
\]

where \( K \) denotes a compact set here and in what follows. On the other hand,
the inequality (4.3) tells us that for all \( t_0 \in Q_L \),
\[
\limsup_{t_1 \to t_0^+} \|u(t_1)\|_{L^2(K)} \leq \|u(t_0)\|_{L^2(K)}, \quad \forall K \subset \mathbb{R}^3.
\]
Thus, for all \( t_0 \in Q_L \),
\[
\lim_{t_1 \to t_0^+} \|u(t_1)\|_{L^2(K)} = \|u(t_0)\|_{L^2(K)}, \quad \forall K \subset \mathbb{R}^3.
\]
Together with the weak continuity of \( t \mapsto u(\cdot, t) \) and the fact that
\[
\lim_{t_1 \to 0^+} \|u(t_1) - u(0)\|_{L^2(K)} = 0, \quad \forall K \subset \mathbb{R}^3,
\]
we obtain, for all \( t_0 \in Q_L \cup \{0\} \),
\[
(4.12) \quad \lim_{t_1 \to t_0^+} \|u(t_1) - u(t_0)\|_{L^2(K)} = 0, \quad \forall K \subset \mathbb{R}^3.
\]
In addition, \( u \in L^\infty([0,T'); E^2) \) implies that
\[
(4.13) \quad \lim_{|x_0| \to \infty} \|u\|_{L^\infty([0,T']; L^2(B_1(x_0)))} = 0, \quad \forall T' < T.
\]
Combining (4.12) and (4.13), we deduce (1.6), which completes the proof of Theorem 1.5. ■

In the light of Theorem 1.5, for each local Leray solution \( u \) to \((FNS) - (FNSI)\) with initial datum in \( E^2 \), we can define its harmonic extension \( u^* \):
\[
(4.14) \quad u^*(x, y) = C_s \int_{\mathbb{R}^3} \frac{y^{1-2s}}{(|x - \xi|^2 + |y|^2)^{(3+2s)/2}} u(\xi) \, d\xi.
\]
According to [3], \( u^* \) satisfies
\[
(4.15) \quad \begin{cases}
\text{div}(y^{1-2s} \nabla u^*) = 0, & (x, y) \in \mathbb{R}^3 \times \mathbb{R}^+, \\
u^*(x, 0) = u(x), & x \in \mathbb{R}^3, \\
-C_s \lim_{y \to 0^+} y^{1-2s} \partial_y u^* = (-\Delta)^s u(x), & x \in \mathbb{R}^3.
\end{cases}
\]
Hence, the local energy inequality (1.3) for \( u \in L^\infty((0,T); E^2) \) and \( \Lambda^s u \in G^{2,2}(T) \) is equivalent to the inequality
\[
(4.16) \quad \int_{\mathbb{R}^3 \times \{t\}} |u|^2 \psi \, dx + 2C_s \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^+} y^{1-2s} |\nabla u^*|^2 \psi \, d\bar{x} \, d\tau
\]
\[
\leq C_s \int_0^t \int_{\mathbb{R}^3 \times \mathbb{R}^+} |u^*|^2 \text{div}(y^{1-2s} \nabla \psi) \, d\bar{x} \, d\tau + \int_0^t \int_{\mathbb{R}^3} (u \cdot \nabla \psi)(2\rho + |u|^2) \, dx \, d\tau
\]
\[
+ \int_0^t \int_{\mathbb{R}^3} |u|^2 \left( \partial_t \psi + C_s \lim_{y \to 0^+} (y^{1-2s} \partial_y \psi) \right) \, dx \, d\tau
\]
for every nonnegative $\psi \in \mathcal{D}(\mathbb{R}^3 \times (0, T))$. Note that the $\psi$ of the integration on $\mathbb{R}^3 \times \mathbb{R}^+$ in the inequality (4.16) should be understood as any function in $H^1(\mathbb{R}^3 \times \mathbb{R}^+, y^{1-2s})$ which equals $\varphi$ on $\mathbb{R}^3 \times \{t = 0\}$.

On account of the above analysis, we can give another notion of suitable weak solution which is a generalization of the version introduced in [27].

**Definition 4.2.** Let $T > 0$ and $s \in [3/4, 1)$. We call $(u, P)$ a suitable weak solution to (FNS) in $\mathbb{R}^3 \times (0, T)$ if

(i) $u \in L^\infty((0, T); E^2)$, $\Lambda^s u \in G(0, T)$ and $P \in L^{3/2}((0, T); E^{3/2})$;

(ii) $u$ and $P$ satisfy (FNS) in the sense of distributions on $\mathbb{R}^3 \times (0, T)$;

(iii) $u$ and $P$ satisfy the generalized local energy inequality (1.3) on $(0, T)$ for any nonnegative $\psi \in \mathcal{D}(\mathbb{R}^3 \times (0, T))$.

**Remark 4.3.** In the light of Theorem 1.5 and Definition 4.2, a local Leray solution with initial data in $E^2$ is a suitable weak solution.

**Proposition 4.4.** Let $s \in [3/4, 1)$. If $(u, P)$ is a suitable weak solution to (FNS) on $\mathbb{R}^3 \times (0, T)$, then there exists a small positive constant $\varepsilon_0$, only depending on $s$, such that if

$$r^{4s-6} \int_{Q_r(x_0,t_0)} (|u|^3 + |P|^{3/2}) \, dx \, d\tau < \varepsilon_0,$$

then there exists a positive constant $C$ such that

$$\sup_{z \in Q_{r/2}(x_0,t_0)} |u| \leq Cr^{-1}.$$

**Remark 4.5.** In the range $s \in (3/4, 1)$, Proposition 4.4 was proved in [27] under the following condition replacing (4.17):

$$\lim_{r \to 0^+} r^{-5+4s} \int_{Q_r^*(x_0,t_0)} y^{1-2s} |\nabla u^*|^2 \, d\bar{x} \, d\tau < \varepsilon_1.$$

Here $Q_r^*(x_0,t_0) = B_r(x_0) \times (0, r) \times (t_0 - r^{2s}, t_0)$ and $\varepsilon_1$ is the constant in Theorem 1.2]. Later, Ren, Wang and Wu [21] proved Proposition 4.4 in the endpoint case $s = 3/4$. Following the method used in [21], we can prove Proposition 4.4 for all $s \in [3/4, 1)$. Since this goes through without any additional tricks, it is not necessary to reproduce that purely mechanical procedure here in detail.

Relying on Proposition 4.4, we prove that local Leray solutions to (FNS)–(FNSI) with initial data in $E^2$ satisfy the following additional regularity.

**Proposition 4.6.** Let $u$ be a local Leray solution to (FNS) on $\mathbb{R}^3 \times (0, T)$ starting from $u_0 \in E^2$ with $\text{div} \, u_0 = 0$. Then:

1. $u$ is regular in $B_R^\circ$ away from $t = 0$ for some large enough $R > 0$;
2. $\Lambda^s u \in L^2([\delta, t]; E^2)$ for all $0 < \delta < t < T$. 

Similarly, for \( (4.19) \) we know that \( u \in L^\infty((0,t);E^2) \) and \( A^s u \in G^{2,2}(t) \) for any \( t < T \). Hence, by interpolation, there exists an \( R > 0 \) such that for all \( |x_0| \geq R \) and \( t_0 \in (0,t] \),

\[
\int_{Q_r(x_0,t_0)} (|u|^3 + |P - P_{x_0}|^{3/2}) \, dx \, d\tau < r^{6-4s}\varepsilon_0
\]

where \( r^2 = \min\{(T-t_0)/2, t_0/2\} \) and \( \varepsilon_0 \) is the constant of Proposition 4.4. Using Proposition 4.4, we get \((1)\) is proved.

Next, we prove \((2)\). Since \( u \in L^\infty((0,t);E^2) \) and \( A^s u \in G^{2,2}(t) \) for any \( t < T \), we have, for almost every \( t \in (0,T) \),

\[
\lim_{|x_0| \to \infty} ||A^s u(t_0)||_{L^2(B_1(x_0))} = 0.
\]

Thus, by the definition of \( E^2 \), it suffices to prove the following.

First, invoking that \( ||u||_{L^\infty(B_R^c \times [\delta,t])} \leq C \) for all \( 0 < \delta < t \), with the use of nonhomogeneous Besov spaces \( B_{p,q}^s \) in \( \mathbb{R}^3 \) defined in \([1,19]\), by bootstrap, we will prove improved regularity of \( u \). In view of Theorem 6.1, for all \( 0 < t_0 \leq t < T \) we have

\[
u(t,x) = e^{-(t-t_0)A^2s} u(t_0) + \int_{t_0}^{t} \div e^{-(t-\tau)A^2s} \mathbb{P}(u \otimes u)(\tau) \, d\tau.
\]

Let \( \varphi_r \in D(\mathbb{R}^3) \) \((r \geq 1)\) satisfy \( 0 \leq \varphi_r \leq 1 \), \( \varphi_r = 1 \) in \( B_{3r/2} \) and \( \varphi_r = 1 \) in \( B_{2r}^c \). Set \( \phi_1 = 1 - \varphi_2 \) and \( \phi_2 = (1 - \varphi_r)^2 \). We split \( u \) into \( u_1 + u_2 \) with

\[
u_i(t,x) = e^{-(t-t_0)A^2s} \varphi_i u(t_0) + \int_{t_0}^{t} \div e^{-(t-\tau)A^2s} \mathbb{P}(\phi_i u \otimes u)(\tau) \, d\tau.
\]

Then, by Lemmas 2.1 and 2.3, for \( x \in B_{3r}^c \) and \( 0 < t_0 \leq t < T \),

\[
|u_1(t,x)| \lesssim_s \int_{|y| \leq 2r} \frac{t}{|x-y|^{3+2s}} |u(t_0,y)| \, dy + \int_{t_0}^{t} \frac{1}{|x-y|^{4}} |u(\tau,y)|^2 \, dy
\]

\[
\lesssim_s ||u(t_0)||_{L^2(B_{2r})} + (t-t_0) ||u||_{L^\infty([t_0,t];L^2(B_{2r}))}^2.
\]

Similarly, for \( x \in B_{3r}^c \) and \( 0 < t_0 \leq t < T \),

\[
|\nabla u_1(t,x)| \leq ||u(t_0)||_{L^2(B_{2r})} + (t-t_0) ||u||_{L^\infty([t_0,t];L^2(B_{2r}))}^2.
\]

Combining the above two estimates, we deduce that

\[
(1 - \varphi_r)u_1 \in B_{\infty,\infty}^c \leq ||(1 - \varphi_r)u_1||_{W^{1,\infty}} \leq ||u_1||_{W^{1,\infty}(B_{3r}^c)} < \infty
\]

where \( \varphi_r(x) = \varphi_r(x/2) \). To deal with \( u_2 \), we observe that
we deduce

\[(4.20) \| (1 - \varphi_r)^2 u \otimes u \|_{B^0_{\infty, \infty}} \leq \| (1 - \varphi_r) u \|^2_{L^\infty}, \]

\[(4.21) \| (1 - \varphi_r)^2 u \otimes u \|_{B^{1/2}_{\infty, \infty}} \leq \| (1 - \varphi_r) u \|^2_{B^{1/2}_{\infty, \infty}}, \]

\[(4.22) \| (1 - \tilde{\varphi}_r) u_2 \|_{B^{(k+1)/2}_{\infty, \infty}} \]

\[\leq \| 1 - \tilde{\varphi} \|_{L^\infty} \| u_2 \|_{B^{k/2}_{\infty, \infty}} + \| u_2 \|_{L^\infty} \| 1 - \tilde{\varphi} \|_{B^{k/2}_{\infty, \infty}}. \]

From (4.20)-(4.21) we get

\[(4.23) \| u_2 \|_{B^{1/2}_{\infty, \infty}} \lesssim_{s, t - t_0} \| (1 - \varphi_r) u(t_0) \|_{L^\infty} + \| (1 - \varphi_r) u \|^2_{L^\infty([t_0, t] \times \mathbb{R}^3)}, \]

\[(4.24) \| u_2 \|_{B^{1}_{\infty, \infty}} \lesssim_{s, t - t_0} \| (1 - \varphi_r) u(t_0) \|_{L^\infty} + \| (1 - \varphi_r) u \|^2_{L^\infty([t_0, t] ; B^{1/2}_{\infty, \infty})}. \]

First, letting \( r = R \), we deduce from (4.19), (4.22) with \( k = 0 \) and (4.23) that

\[(4.25) \| (1 - \tilde{\varphi}_r) u \|_{B^{1/2}_{\infty, \infty}} \leq C_R. \]

Next, letting \( r = 2R \), combining (4.19), (4.22) with \( k = 1 \) and (4.24)-(4.25), we deduce \( u_1 \in L^\infty([\delta, t]; W^{1, \infty}(B^c_{6R})) \) and \( u_2 \in L^\infty([\delta, t]; B^{1, \infty}) \).

Then, for every \( 0 < \delta < t < T \),

\[
\| A^s u_1 \|_{L^2([\delta, t]; L^2_{\text{uloc}})} \leq \| \varphi_{4R} u_1 \|_{L^2([\delta, t]; H^s_{\text{uloc}})} + \| (1 - \varphi_{4R}) u_1 \|_{L^2([\delta, t]; H^s_{\text{uloc}})} \\
\lesssim \| u_1 \|_{L^2([\delta, t]; L^2_{\text{uloc}})} + \| A^s u_1 \|_{L^2([\delta, t]; L^2(B_{8R}))} + \| (1 - \varphi_{4R}) u_1 \|_{L^2([\delta, t]; H^1_{\text{uloc}})} \\
\lesssim_{\delta, t} \| u_1 \|_{L^\infty([\delta, t]; L^2_{\text{uloc}})} + \| A^s u_1 \|_{L^2([\delta, t]; L^2(B_{8R}))} + \| u_1 \|_{L^\infty([\delta, t]; W^{1, \infty}(B^c_{6R}))} < \infty \]

and

\[
\| A^s u_2 \|_{L^2([\delta, t]; L^2_{\text{uloc}})} \lesssim_{\delta, t} \| A^s u_2 \|_{L^\infty([\delta, t]; L^\infty)} \lesssim_{\delta, t} \| u_2 \|_{L^\infty([\delta, t]; B^{1, \infty})} < \infty. \]

This completes the proof of Proposition 4.6.

5. Global-in-time existence of local Leray solutions. In this section, we will prove the global-in-time existence of local Leray solutions to (FNS)-(FNSI) with initial data in \( E^2 \). To do it, we need the following "weak-strong" uniqueness and decomposition lemma.

**Proposition 5.1 (Weak-strong uniqueness).** Let \((u, P), (v, \tilde{P})\) be two local Leray solutions to (FNS) on \( \mathbb{R}^3 \times (0, T) \) with the same initial data \( u_0 \). Suppose that \( \nabla v \in L^q([0, T']; E^p) \) for all \( T' < T \) with \( 2s/q + 3/p = 2s \) and \( p \geq 2 \). Then \( u = v \) in \( \mathbb{R}^3 \times (0, T) \) almost everywhere.

**Remark 5.2.** When \( s < 1 \), the properties of \( u \) provide no information about \( \nabla u \). Hence, unlike the case \( s = 1 \), to control \( \int_0^t ((u - v) \cdot \nabla) (u - v) \cdot v \varphi \, d\tau \), we need at least the additional information on \( \nabla v \).

**Proof.** Since \( L^\infty([0, T']; E^2) \cap L^2([0, T']; \tilde{H}^s_{\text{uloc}}) \hookrightarrow L^\alpha([0, T']; L^\beta_{\text{uloc}}) \) with \( 2s/\alpha + 3/\beta = 3/2 \), we have
\[ \int_{\mathbb{R}^3} v(t) \cdot u(t) \varphi \, dx + 2 \int_0^t \int_{\mathbb{R}^3} (A^s v \cdot A^s u) \varphi \, dx \, d\tau \]

\[ = \int_{\mathbb{R}^3} |w_0|^2 \varphi \, dx - \int_0^t \int_{\mathbb{R}^3} (\varphi, A^s \nabla \cdot w \varphi + \varphi A^s \cdot [A^s, \tilde{\varphi}] w) \, dx \, d\tau \]

\[ - \int_0^t \int_{\mathbb{R}^3} \left( \varphi, A^s u \cdot \nabla \varphi + \varphi A^s u \cdot [A^s, \tilde{\varphi}] v \right) \, dx \, d\tau \]

\[ - \int_0^t \int_{\mathbb{R}^3} \left( (u \cdot \nabla) u \cdot v + (v \cdot \nabla) v \cdot u \right) \varphi \, dx \, d\tau \]

\[ + \int_0^t \int_{\mathbb{R}^3} ((\bar{P} u + P v)) \cdot \nabla \varphi \, dx \, d\tau \]

for any nonnegative \( \varphi, \tilde{\varphi} \in \mathcal{D}(\mathbb{R}^3) \) satisfying \( \tilde{\varphi} \varphi = 1 \). Since \((u, P)\) and \((v, \bar{P})\) both satisfy the local energy inequality \((1.3)\), the function \( w = u - v \) satisfies

\[ \int_{\mathbb{R}^3} \varphi |w|^2 \, dx + 2 \int_0^t \int_{\mathbb{R}^3} \varphi |A^s w|^2 \, dx \, d\tau \]

\[ \leq -2 \int_0^t \int_{\mathbb{R}^3} (\varphi, A^s w) \nabla \varphi \, dx \, d\tau + 2 \int_0^t \int_{\mathbb{R}^3} \left( (w \cdot \nabla v) \cdot w \varphi \right) \, dx \, d\tau \]

\[ + 2 \int_0^t \int_{\mathbb{R}^3} (P - \bar{P}) w \cdot \nabla \varphi \, dx \, d\tau \]

for any nonnegative \( \varphi, \tilde{\varphi} \in \mathcal{D}(\mathbb{R}^3) \) satisfying \( \tilde{\varphi} \varphi = 1 \). Set

\[ \alpha(t) = \sup_{x_0 \in \mathbb{R}^3} \left( \int_{B_1(x_0)} |w(t)|^2 \, dx \right)^{1/2}, \quad \beta(t) = \sup_{x_0 \in \mathbb{R}^3} \left( \int_{0}^{t} \int_{B_1(x_0)} |A^s w|^2 \, dx \, d\tau \right)^{1/2}. \]

Then, along the lines of the proof of \((3.11)\), we deduce that

\[ \alpha(t)^2 + \beta(t)^2 \leq M \int_0^t \left( \alpha(\tau)^2 + \alpha(\tau)^{6(2s-1) / 4s-3} \right) \, d\tau \]

for some constant \( M \) depending only on \( s, T, \|u\|_{L^\infty((0,T);E^2)}, \|A^s u\|_{L^2((0,T);E^2)} \) and \( \|\nabla u(t)\|_{L^q((0,T);E^p)} \). This gives \( \alpha(t) = 0 \) on \((0, T)\), which implies that \( u = v \) on \((0, T)\).

**Lemma 5.3.** Let \( s > 0 \) and \( f \in \tilde{H}^s_{uloc} \). Then for all \( \varepsilon > 0 \), there exist \( g \in \tilde{H}^s_{uloc} \) and \( h \in L^2 \) such that \( f = g + h \) and \( \|g\|_{\tilde{H}^s_{uloc}} \leq \varepsilon. \)
REMARK 5.4. Since the proof is completely parallel to that of [17, Proposition 12.1], we omit it.

Next, we turn to the proof of Theorem 1.6.

Proof of Theorem 1.6 We will prove the global existence of local Leray solutions with initial values in $E^2$ by construction.

Step 1: Local existence in $E^2$. For each divergence free vector $u_0 \in E^2$, Theorem 1.4 tells us that there exists a local Leray solution $u^1$ to (FNS) on $\mathbb{R}^3 \times (0, T_1)$ starting from $u_0$.

Step 2: Construction of a local Leray solution from $(0, T_N)$ to $(0, T_{N+1})$ with $T_{N+1} - T_N > 3/4$. Assume that $u^N$ is a local Leray solution on $(0, T_N)$.

Proposition 4.6 yields $S_{N+1} \in (\sup\{0, T_N - 1/4\}, T_N)$ such that

$$u^N(S_{N+1}) \in \bar{H}_{uloc}^{5/2 - 2s}$$

and

$$\lim_{t \to S_{N+1}^+} \|u^N(t) - u^N(S_{N+1})\|_{E^2} = 0.$$

Next, we can find a solution $X^{N+1} \in C([S_{N+1}, U_{N+1}]; \bar{H}_{uloc}^{5/2 - 2s})$ with $S_{N+1} < U_{N+1} < T_N$ and $X^{N+1}(S_{N+1}) = u^N(S_{N+1})$ by Theorem 6.2. In addition,

$$X^{N+1} \in L^{4s/3}([S_{N+1}, U_{N+1}]; \bar{H}_{uloc}^1) \cap L^2([S_{N+1}, U_{N+1}]; \bar{H}_{uloc}^{5/2 - 2s}).$$

By Corollary 6.4, $X^{N+1}$ is a local Leray solution to (FNS) with initial data $u^N(S_{N+1})$. On the other hand, by Lemma 5.3, we can split $u^N(S_{N+1} + 1)$ into $v_0^{N+1} + w_0^{N+1}$ satisfying $\text{div} v_0^{N+1} = \text{div} w_0^{N+1} = 0$, $\|v_0^{N+1}\|_{\bar{H}_{uloc}^{5/2 - 2s}} < \varepsilon$ and $w_0^{N+1} \in L^2$, where $\varepsilon$ is so small that (FNS) admits a unique solution $v^{N+1} \in C([S_{N+1}, S_{N+1} + 1]; \bar{H}_{uloc}^{5/2 - 2s})$ with $v^{N+1}(S_{N+1}) = v_0^{N+1}$ satisfying

$$v^{N+1} \in L^2([S_{N+1}, S_{N+1} + 1]; \bar{H}_{uloc}^{5/2 - 2s}).$$

Then, by Theorem 6.5 there exists a weak solution $w^{N+1} \in L^\infty([S_{N+1}, S_{N+1} + 1]; L^2) \cap L^2([S_{N+1}, S_{N+1} + 1]; \bar{H}^s)$ satisfying (6.9) to the following perturbed system:

$$\begin{cases}
\partial_t w^{N+1} + A^{2s} w^{N+1} + w^{N+1} \cdot \nabla w^{N+1} \\
+ v^{N+1} \cdot \nabla w^{N+1} + w^{N+1} \cdot \nabla v^{N+1} + \nabla \Pi = 0, \\
\text{div} w^{N+1} = 0, \\
w^{N+1}(S_{N+1}, x) = w_0^{N+1}.
\end{cases}$$

Set $Y^{N+1} = v^{N+1} + w^{N+1}$. By Corollary 6.4, it is obvious that $Y^{N+1}$ is a local Leray solution to (FNS) on $(S_{N+1}, S_{N+1} + 1)$ starting from $u^N(S_{N+1})$.

Finally, since $X^{N+1} \in L^{4s/3}([S_{N+1}, U_{N+1}]; \bar{H}_{uloc}^1)$, from Proposition 5.1 we conclude that $u^N = X^{N+1} = Y^{N+1}$ on $(S_{N+1}, U_{N+1})$. Hence, defining $u^{N+1} = \chi_{[0, S_{N+1}]} u^N + Y^{N+1}$ and $T_{N+1} = S_{N+1} + 1$, we construct a local Leray solution to (FNS) on $(0, T_{N+1})$ with $T_{N+1} - T_N > 3/4$. 

STEP 3: Global existence in $E^2$. By induction on $N$, we can construct a sequence of Leray solutions $u^N$ on $(0, T_N)$ satisfying $u^N = u^{N+1}$ on $(0, T_N)$ and $T_{N+1} - T_N > 3/4$. Hence, $u = \lim_{N \to \infty} u^N$ is well defined on $(0, \infty)$. Furthermore, $u$ is a local Leray solution to (FNS) starting from $u_0$ on $\mathbb{R}^3 \times (0, \infty)$. This completes the proof of Theorem 1.6.

6. Appendix

6.1. Equivalence of differential and integral formulations for FNS

THEOREM 6.1. Let $u \in L^2([0, T']; L^2_{uloc})$ for any $T' < T$. Then the following two statements are equivalent:

1. $u$ is a solution of the differential fractional Navier–Stokes equations
   \[
   \begin{cases}
   \partial_t u + A^{2s}u + \mathbb{P} \text{div}(u \otimes u), \\
   \text{div } u = 0.
   \end{cases}
   \]

2. $u$ is a solution of the integral fractional Navier–Stokes equations
   \[
   \exists u_0 \in \mathcal{S}'(\mathbb{R}^3), \quad \begin{cases}
   u = e^{-tA^{2s}}u_0 - \int_0^t e^{-(t-s)A^{2s}} \mathbb{P} \text{div}(u \otimes u) \, ds, \\
   \text{div } u = 0.
   \end{cases}
   \]

In particular, if $u \in L^2([0, T']; E^2)$ for all $T' < T$, the above statements are equivalent to $u$ being a weak solution to (FNS).

Since the proof is completely parallel to that for the classical Navier–Stokes system, we omit it. For details, see [14, Theorems 11.1 and 11.2].

6.2. Kato’s theory of FNS in spaces of local measures.

Set $X_T := \{ f \in L^4([0, T]; \tilde{H}^{5/2 - 3s}_{uloc}) \cap L^{4s/3 - s}([0, T]; \tilde{H}^{1}_{uloc}) \mid t^{1-\frac{1}{2s}} f \in C([0, T]; L^\infty) \}$ with
\[
\|f\|_{X_T} := \|f\|_{L^4([0, T]; \tilde{H}^{5/2 - 3s}_{uloc})} + \|f\|_{L^{4s/3 - s}([0, T]; \tilde{H}^{1}_{uloc})} + \sup_{0 \leq t \leq T} t^{1-\frac{1}{2s}} \|f\|_{L^\infty}.
\]

Then, we can prove the existence and uniqueness for (FNS) in $X_T$.

THEOREM 6.2. Let $s \in [5/6, 1)$. Then for every divergence free vector field $u_0 \in \tilde{H}^{5/2 - 2s}_{uloc}$, there exists $0 < T \leq 1$ satisfying
\[4C_{\nu} \|e^{\nu tA^{2s}}u_0\|_{X_T} < 1\]
for some constant $C$ and a unique mild solution
\[u \in C([0, T]; \tilde{H}^{5/2 - 2s}_{uloc}) \cap X_T \cap L^2([0, T]; \tilde{H}^{5/2 - s}_{uloc})\]
to (FNS) with $u(\cdot, 0) = u_0$. In particular, there exists a constant $\eta > 0$ such that if $\|u_0\|_{\tilde{H}^{5/2 - 2s}_{uloc}} < \eta$, then $T = 1$.

To prove Theorem 6.2, we will invoke the following lemma.
Lemma 6.3. Let \( s \in [5/6, 1] \). Then

\[(6.1) \quad \|fg\|_{\dot{H}_{uloc}^{7/2 - 3\alpha}} \leq C \|f\|_{\dot{H}_{uloc}^{(5-3\alpha)/2}} \|g\|_{\dot{H}_{uloc}^{(5-3\alpha)/2}}.\]

Proof. It is well known that
\[\|fg\|_{H^{7/2 - 3\alpha}} \leq C \|f\|_{H^{(5-3\alpha)/2}} \|g\|_{H^{(5-3\alpha)/2}}.\]

Let \( \phi \in \mathcal{D}(B_2) \) be a nonnegative function satisfying \( \phi = 1 \) in \( B_1 \). Hence for all \( \phi_{x_0}(\cdot) := \phi(\cdot - x_0) \), we have
\[\|\phi_{x_0}^2 fg\|_{H^{7/2 - 3\alpha}} \leq C \|\phi_{x_0}f\|_{H^{(5-3\alpha)/2}} \|\phi_{x_0}g\|_{H^{(5-3\alpha)/2}}.\]

Then, from the definition of Sobolev spaces of local measures, we deduce (6.1). ■

Proof of Theorem 6.2. By the Duhamel principle, we have
\[u = e^{-tA^2s}u_0 + \int_0^t \nabla \cdot e^{-(t-\tau)A^2s} \mathbb{P}(-u \otimes u)(\tau) d\tau := e^{-\nu tA^2s}u_0 + B(u,u).\]

The proof is in two steps.

Step 1: Unique existence in \( X_T \). For any \( v_1, v_2 \in X_T \), by (2.9), we have
\[\|B(v_1, v_2)\|_{L^\infty} \lesssim_c \int_0^t (t-\tau)^{-\frac{1}{2s}} \tau^{-\frac{1}{2s}} d\tau \left( \sup_{0 \leq t < T} t^{1-\frac{1}{2s}} \|v_1\|_{L^\infty} \right) \left( \sup_{0 \leq t < T} t^{1-\frac{1}{2s}} \|v_2\|_{L^\infty} \right) \lesssim_s \left( \sup_{0 \leq t < T} t^{1-\frac{1}{2s}} \|v_1\|_{L^\infty} \right) \left( \sup_{0 \leq t < T} t^{1-\frac{1}{2s}} \|v_2\|_{L^\infty} \right).\]

This gives
\[(6.2) \quad \sup_{0 \leq t < T} t^{1-\frac{1}{2s}} \|B(v_1, v_2)\|_{L^\infty} \lesssim_s \left( \sup_{0 \leq t < T} t^{1-\frac{1}{2s}} \|v_1\|_{L^\infty} \right) \left( \sup_{0 \leq t < T} t^{1-\frac{1}{2s}} \|v_2\|_{L^\infty} \right).\]

In addition, by Lemmas 2.3, 2.6 and 6.3
\[\|B(v_1, v_2)\|_{\dot{H}_{uloc}^{1/2}} \lesssim_s \int_0^t \left( (t-\tau)^{-\frac{1}{2s}} + (t-\tau)^{-\frac{3}{4s}} \right) \|v_1 \otimes v_2\|_{\dot{H}_{uloc}^{1/2}} d\tau \lesssim_s \int_0^t \left( (t-\tau)^{-\frac{1}{2s}} + (t-\tau)^{-\frac{3}{4s}} \right) \|v_1\|_{\dot{H}_{uloc}^{1/2}} \|v_2\|_{\dot{H}_{uloc}^{1/2}} d\tau.\]
and
\[
\|A^{(3-3s)/2}\nabla B(v_1, v_2)\|_{E^2} \lesssim_s \int_0^t (t - \tau)^{-3/4} \|v_1 \otimes v_2\|_{\dot{H}^{5/2-3s}_{uloc}} \, d\tau
\]
\[
\lesssim_s \int_0^t (t - \tau)^{-3/4} \|v_1\|_{\dot{H}^{(5-3s)/2}_{uloc}} \|v_2\|_{\dot{H}^{(5-3s)/2}_{uloc}} \, d\tau
\]
where we use $7/2 - 3s < 1$. These estimates imply
\begin{equation}
\tag{6.3}
\|B(v_1, v_2)\|_{\dot{H}^{5-3s}_{uloc}(0,T)} \lesssim \|v_1\|_{\dot{H}^{4s-3}([0,T];\dot{H}^1_{uloc})} \|v_2\|_{\dot{H}^{4s-3}([0,T];\dot{H}^1_{uloc})}
\end{equation}
\begin{equation}
\tag{6.4}
\|A^{3-3s/2} \nabla B(v_1, v_2)\|_{L^4([0,T];E^2)} \lesssim_s \|v_1\|_{L^4([0,T];\dot{H}^{(5-3s)/2}_{uloc})} \|v_2\|_{L^4([0,T];\dot{H}^{(5-3s)/2}_{uloc})}.
\end{equation}
Combining (6.3) and (6.4), by Hölder’s inequality and Lemma 2.6 we have
\begin{equation}
\tag{6.5}
\|B(v_1, v_2)\|_{L^4([0,T];\dot{H}^{5-3s}_{uloc})} \leq \|A^{3-3s/2} \nabla B(v_1, v_2)\|_{L^4([0,T];E^2)} + \|B(v_1, v_2)\|_{L^4([0,T];\dot{H}^1_{uloc})}
\lesssim \|v_1\|_{L^4([0,T];\dot{H}^{5-3s}_{uloc})} \|v_2\|_{L^4([0,T];\dot{H}^1_{uloc})}
+ \|v_1\|_{L^4([0,T];\dot{H}^{5-3s}_{uloc})} \|v_2\|_{L^4([0,T];\dot{H}^{5-3s}_{uloc})}.
\end{equation}
Combining (6.2), (6.3) and (6.5) we get, for any $v_1, v_2 \in X_T$ with $0 < T \leq 1$,
\begin{equation}
\tag{6.6}
\|B(v_1, v_2)\|_{X_T} \leq C_1 \|v_1\|_{X_T} \|v_2\|_{X_T}.
\end{equation}
On the other hand, since $\dot{H}^{5/2-2s}_{uloc} \hookrightarrow E^{3/(2s-1)}$, by Lemma 2.1 we get
\[
\sup_{0 \leq t \leq T} t^{1 - \frac{1}{2s}} \|e^{-tA^{2s}} u_0\|_{L^\infty} \lesssim_s \|u_0\|_{\dot{H}^{(5-3s)/2}_{uloc}}, \quad \lim_{t \to 0^+} t^{1 - \frac{1}{2s}} \|e^{-tA^{2s}} u_0\|_{L^\infty} = 0.
\]
Moreover, adopting the splitting method used in the proof of Theorem 1.4 we deduce that for $T \leq 1$,
\[
\|e^{-tA^{2s}} u_0\|_{L^4([0,T];\dot{H}^{(5-3s)/2}_{uloc}) \cap L^{4s/(4s-3)}([0,T];\dot{H}^1_{uloc})} \lesssim_s \|u_0\|_{\dot{H}^{5/2-2s}_{uloc}}.
\]
Combining the above three estimates, we get
\begin{equation}
\tag{6.7}
\|e^{-tA^{2s}} u_0\|_{X_T} \leq C_2 \|u_0\|_{\dot{H}^{5/2-2s}_{uloc}}, \quad \lim_{T \to 0^+} \|e^{-tA^{2s}} u_0\|_{X_T} = 0.
\end{equation}
According to (6.7), we can find a time $T > 0$ such that $4C_1 \|e^{-tA^{2s}} u_0\|_{X_T} < 1$. Hence, in view of (6.6), by the Banach fixed point theorem, we conclude that (FNS)–(FNSI) admits a unique mild solution $u \in X_T$. In particular, if $u_0$ satisfies $4C_1C_2 \|u_0\|_{\dot{H}^{5/2-2s}} < 1$, we can choose $T = 1$. 
STEP 2: Improved regularity. We start to show $u \in L^2([0,T]; \tilde{H}^{5/2-\varepsilon}_{uloc})$. Obviously

$$\|e^{-tA^{2s}}u_0\|_{L^2([0,T]; \tilde{H}^{5/2-\varepsilon}_{uloc})} \leq C\|u_0\|_{\tilde{H}^{5/2-\varepsilon}_{uloc}}.$$  

On the other hand, exactly as in the proof of (3.2), we get

$$\|A^{3/2-s}\nabla B(u, u)\|_{E^2} \leq C \int_0^t (t - \tau)^{-1} \|u\|^2_{\tilde{H}^{(5-3s)/2}_{uloc}} d\tau \leq \|u\|^2_{L^4([0,T]; \tilde{H}^{(5-3s)/2}_{uloc})}.$$  

Combining the above two estimates and (6.3) yields $u \in L^2([0,T]; \tilde{H}^{5/2-\varepsilon}_{uloc})$.

Next, we prove $u \in C([0,T]; \tilde{H}^{5/2-\varepsilon}_{uloc})$ and $v_2 \in X_T$, by Lemma 2.2 we get

$$\|B(v_1, v_2)\|_{E^2} \leq C \int_0^t (t - \tau)^{-1/2} \tau^{1/2s} \|v_1\|_{L^\infty([0,T]; E^2)} \sup_{0 \leq t \leq T} t^{1-1/2s} \|v_2\|_{L^\infty}$$

and

$$\|A^{5/2-2s}B(v_1, v_2)\|_{E^2} \leq C \int_0^t (t - \tau)^{-1/2} \tau^{1/2} \|v_2\|_{L^\infty([0,T]; \tilde{H}^{5/2-2s}_{uloc})} \sup_{0 \leq t \leq T} t^{1-1/2s} \|v_1\|_{L^\infty}.$$

Hence

(6.8)  $$\|B(v_1, v_2)\|_{L^\infty([0,T]; \tilde{H}^{5/2-2s}_{uloc})} \leq C\|v_2\|_{L^\infty([0,T]; \tilde{H}^{5/2-2s}_{uloc})} \sup_{0 \leq t \leq T} t^{1-1/2s} \|v_2\|_{L^\infty},$$

which implies that $B(v_1, v_2) \in C([0,T]; \tilde{H}^{5/2-2s}_{uloc})$. By Lemma 2.1

$$\|e^{-tA^{2s}}u_0\|_{L^\infty([0,T]; \tilde{H}^{5/2-2s}_{uloc})} \leq C\|u_0\|_{\tilde{H}^{5/2-2s}_{uloc}}.$$  

Since $C_0^\infty(\mathbb{R}^3)$ is dense in $\tilde{H}^{5/2-2s}_{uloc}$, we have $e^{-tA^{2s}}u_0 \in C([0,T]; \tilde{H}^{5/2-2s}_{uloc})$. The above analysis yields $u \in C([0,T]; \tilde{H}^{5/2-2s}_{uloc})$.

**Corollary 6.4.** If $u$ is the mild solution to (FNS) with initial datum $u_0 \in \tilde{H}^{5/2-2s}_{uloc}$ given in Theorem 6.2, then $u$ is a local Leray solution to (FNS) starting from $u_0$.

**Proof.** From Theorem 6.2 we know that $u \in C^\infty(\mathbb{R}^3 \times (0,T))$. In addition, due to $u \in C([0,T]; E^s)$, we know that $P = \frac{\text{div} \div}{\Delta} (u \otimes u)$ is well-defined on $\mathbb{R}^3 \times (0,T)$, which implies that $P \in L^{3/2}((0,T'); L^{3/2}_{uloc})$ for any $T' < T$.  


Hence, by elliptic theory and the fact that \( u \in C^\infty(\mathbb{R}^3 \times (0, T)) \), we deduce that

\[
\|P(\cdot, t)\|_{C^{k,\alpha}(B_1)} \leq C_k \|u(\cdot, t) \otimes u(\cdot, t)\|_{C^{k,\alpha}(B_2)} + C_k \|P(\cdot, t)\|_{L^{3/2}(B_2)},
\]

which implies that \( P \) is smooth on \( \mathbb{R}^3 \times (0, T) \). So, multiplying both sides of the equation

\[
\partial_t u + \Lambda^{2s} u + u \cdot \nabla u + \nabla P = 0
\]

by \( u \varphi, \varphi \in \mathcal{D}(\mathbb{R}^3 \times (0, 1)) \), we derive that \( u \) satisfies the local energy inequality (1.3). This, together with \( u \in C_b([0, T']; E^p) \) and \( P \in L^{3/2}((0, T'); L^{3/2}_{uloc}) \), shows that \( u \in L^\infty((0, T'); L^2_{uloc}) \) and \( \Lambda^s u \in L^{2,2}_{uloc}(T') \) for any \( T' < T \). Hence, \( u \) is a local Leray solution to \( (\text{FNS})-\text{FNSI} \) on \( \mathbb{R}^3 \times (0, T) \) starting from \( u_0 \).

6.3. Leray theory for perturbed fractional Navier–Stokes equations. Similar to the Leray theory for perturbed classical Navier–Stokes equations developed in [14] Chapter 21, for the perturbed problem of the incompressible fractional Navier–Stokes equations:

\[
\begin{cases}
\partial_t w + \Lambda^{2s} w + w \cdot \nabla w + w \cdot \nabla v + v \cdot \nabla w + \nabla \tilde{P} = 0, \\
\text{div } w = 0, \\
w(x, 0) = w_0,
\end{cases}
\]

we also have a similar theory.

THEOREM 6.5. Let \( v \) be a divergence free vector field satisfying \( \nabla v \in L^q([0, T]; E^p) \) with \( 2s/q + 3/p = 2s - 1 \) and \( p > 3 \). Then, for any divergence free \( w_0 \in L^2 \), there exists a weak solution \( u \) to \( (\text{PFNS}) \) satisfying

\( (1) \) \( w \in L^\infty([0, T]; L^2) \cap L^2([0, T]; \dot{H}^s); \)

\( (2) \) \( \lim_{t \to 0} \|w(t) - w_0\|_{L^2} = 0; \)

\( (3) \) \( \|w(t)\|_{L^2}^2 + 2\int_0^t \|\Lambda^s w(\tau)\|_{L^2}^2 d\tau \leq \|w_0\|_{L^2}^2 - 2\int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla) v \cdot w dx d\tau; \)

\( (4) \) for any \( \varphi \in \mathcal{D}(\mathbb{R}^3 \times (0, T)) \) and \( \tilde{\varphi} \in \mathcal{D}(\mathbb{R}^3) \) satisfying \( \tilde{\varphi} \varphi = 1, \)

\[
2 \int_0^t \int_{\mathbb{R}^3} \psi |\Lambda^s w|^2 dx d\tau \leq \int_0^t \int_{\mathbb{R}^3} |w|^2 \partial_t \psi dx d\tau - 2\int_0^t \int_{\mathbb{R}^3} [\tilde{\psi}, \Lambda^s] \Lambda^s w \cdot w \psi dx d\tau
\]

\[
- 2\int_0^t \int_{\mathbb{R}^3} [\Lambda^s, \psi] w \cdot (\tilde{\psi} \Lambda^s w) dx d\tau + 2\int_0^t \int_{\mathbb{R}^3} |w|^2 (w + v) \cdot \nabla \psi dx d\tau
\]

\[
- 2\int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla) v \cdot w \psi dx d\tau + 2\int_0^t \int_{\mathbb{R}^3} \tilde{P} \cdot u \nabla \psi dx d\tau.
\]
Proof. We first give a key a priori estimate. Multiplying both sides of the first equation of \((PFNS)\) by \(w\) and integrating over \(\mathbb{R}^3 \times (0,t)\), we get
\[
\|w(t)\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\partial_t w|^2 \, dx \, d\tau = \|w_0\|_{L^2}^2 - 2 \int_0^t \int_{\mathbb{R}^3} (w \cdot \nabla) v \cdot w \, dx \, d\tau.
\]
Let \(X_r\) be the pointwise multiplier space of negative order defined in [14, Chapter 21]. By [14, Theorem 21.1], we know that \(E^{3/r} \hookrightarrow X_r\) for any \(r \in (0,1]\). So,
\[
\|w \cdot \nabla v\|_{L^2} \leq C \|w\|_{H^{3/p}} \|\nabla v\|_{E^p}.
\]
Hence,
\[
\|w(t)\|_{L^2}^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\partial_t w|^2 \, dx \, d\tau \leq \|w_0\|_{L^2}^2 + C \int_0^t \int_{\mathbb{R}^3} \|w\|_{L^2}^2 \|w\|_{H^{3/p}} \|\nabla v\|_{E^p} \, d\tau
\]
\[
\leq \|w_0\|_{L^2}^2 + C \int_0^t \int_{\mathbb{R}^3} \|w\|_{L^2}^2 \left( \|w\|_{L^2}^2 + \|w\|_{L^2}^{1-\frac{3}{ps}} \|w\|_{H^s}^{\frac{3}{ps}} \right) \|\nabla v\|_{E^p} \, d\tau
\]
\[
\leq \|w_0\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\partial_t w|^2 \, dx \, d\tau + C \int_0^t \int_{\mathbb{R}^3} \|w\|_{L^2}^2 \left( \|\nabla v\|_{E^p}^2 + \|\nabla v\|_{E^p}^{2ps-3} \right) \, d\tau.
\]
Then, by Gronwall’s inequality, for any \(0 \leq t \leq T\),
\[
(6.10) \quad \|w(t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} |\partial_t w|^2 \, dx \, d\tau
\]
\[
\lesssim \|w_0\|_{L^2}^2 \exp \left\{ t^{1-\frac{1}{p}} \left( \sup_{0 \leq t \leq T} \|\nabla v\|_{E^p} \right) + t^{1-\frac{p}{2ps-3}} \left( \sup_{0 \leq t \leq T} t^{\frac{1}{ps}} \|\nabla v\|_{E^p} \right)^{\frac{2ps}{2ps-3}} \right\}.
\]
Relying on the priori estimate (6.10) and following the argument used in [14, Chapter 21], we complete the proof.  

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