The initial inhomogeneity and halo formation in intense charged particle beams

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Abstract. Although undesired in many applications, the intrinsic and spurious spatial inhomogeneity that permeates real systems is the forerunner instability that leads high-intensity charged particle beams to its equilibrium. In general, this equilibrium is reached in a particular way, by the development of a tenuous particle population around the original beam, conventionally known as the halo. In this direction, the purpose of this work is to analyze the influence of the magnitude of the initial inhomogeneity over the dynamics of quasi-homogeneous mismatched beams. For that, all beam constituent particles, which are initially disposed in an equidistant form, suffer a progressive perturbation through a noise of a variable amplitude. Beam quantities are quantified as functions of the noise amplitude, which indirectly is assumed a consistent measure of the initial beam inhomogeneity. The results have been obtained by the means of full self-consistent \( N \)-particle beam numerical simulations and seem to be an important complement to the investigations already carried out in prior works.

1. Introduction

It is well known that perfectly round and homogeneous beams do not suffer any thermalization effect under its excursion inside the magnetic focusing system. This means that if the beam is assumed initially cold (all beam particles have initially a negligible velocity), during its linear path inside the magnetic focusing channel, the beam remains cold (particles’ velocity in magnitude are still unworthy). In essence, while confined inside the accelerator structure, the particles that compose the beam individually cannot earn extra mechanical energy through the increasing of its own kinetic energy. Using the jargon of the field, the beam particles are not heated as propagate inside the channel.

All of this is readily possible to show analytically by the calculus of the beam emittance. The emittance [1] is a macroscopic beam quantity that involves the velocity of each beam constituent. Emittance in average depends on a monotonic way of particle velocities: if the particle velocities in average increase during the confinement process, beam emittance also increases; if the particle velocities in average decrease, beam emittance in the same way also decreases. Analytical calculations show that perfectly round and homogeneous beams do not experiment any heating process. If the beam is initially cold, which implies emittance to be initially zero, after the beam confinement process the emittance conserves its initial value and thus remains zero. With this in mind, since emittance is a
good indicator of the macroscopic increasing of beam kinetic energy, it is possible to say that the particles that compose pretty homogeneous beams in average do not have their velocities increased. However, one will find a much different situation if, instead of considering an initial completely homogeneous beam, one consider a quasi-homogeneous beam. By a quasi-homogeneous beam it should be understood the situation in which the beam particles are not disposed equidistantly one from the other anymore. The beam becomes quasi-homogeneous because the equidistance property (between beam particles) is weakly broken. The direct result of this is that for a given spatial region of the beam transversal section, particles can be slightly closer when compared with other neighbor beam spatial regions, in which the particles can be slightly farther. The beam particles are not dispersed homogeneously anymore, giving rise to some kind of charge concentration and/or accumulation. This tenuous inhomogeneity introduced implies that the forces inside the beam become weakly nonlinear. Under the action of this kind of force, the beam particles pass to oscillate with a frequency that is thus dependent of its initial coordinate [2]. As a natural consequence of this new situation, synchronization problems can potentially occur inside the beam: during their dynamics, some beam particles can eventually collapse their orbits, being expelled and/or ejected from the beam core. From the beam phase space picture, this ejection looks like the breaking of density waves [3][4].

Once out of the core, the ejected particles become susceptible to the influences of the envelope mismatch. These particles are driven by the oscillation of the core and, as a consequence, progressively have their velocity improved. As the beam core one should understand the set of beam particles that have not been ejected. Since the velocity of the ejected particles increase by the excitation imposed by the oscillatory movement of the core, the beam emittance also increases. In this way, due to the initial small inhomogeneity, initially quasi-homogeneous beams suffer some heating and present an emittance growth that cannot be neglected. Note: the core movement just can excite beam particles because they are out of the beam [5]. And for the particles get out the beam, a nonlinear, even weak, force must act over them inside the beam. For this reason, one can say that while the beam envelope mismatch scales the emittance growth, the initial inhomogeneity is the forerunner mechanism that allows the emittance to increase [6]. The greater is the initial beam envelope mismatch, the greater is the emittance growth. But one must say that even in initially matched beams the inhomogeneity induces an emittance growth that should be considered.

Perfectly homogeneous beams are such an idealization pretty hard to achieve in the engineering sense. Always some inhomogeneity will be present and its influences over the beam dynamics are then something to be better investigated. Following this reasoning line, the purpose of this work is to evaluate how the time scale of the emittance growth depends on the magnitude of the initial inhomogeneity introduced in the beam density. In prior works [2][6], it has been found that the time scale of emittance growth has contributions from the initial inhomogeneity and from the beam envelope mismatch. Also, it is of interest to verify quantitatively how the beam quantities behave at equilibrium as a function of the initial inhomogeneity.

The present work is organized in the following way. In the section 2, some aspects related with the system of interest in this work are discussed. The scheme for generation of the initial beam density is shown. In the section 3, the approach developed to produce – in a controlled manner – the inhomogeneity is presented. The section 4 contains the results obtained with the analysis. Opportunely, this section also describes the method of numerical simulation employed. Finally, in the section 5, the conclusions and the perspectives of future works are addressed.

2. The system of interest

The system considered here is an initially cold and azimuthally symmetric beam of charged particles evolving in linear path inside an accelerator structure. To keep the particles confined, a constant magnetic field is applied in the direction of propagation of the beam. For mechanical and electrical reasons, a conducting pipe surrounds the linear channel. The beam particles propagate along the longitudinal direction, which is aligned with the pipe symmetry axis. The z axis of the coordinate system considered here is exactly over the pipe symmetry axis. The motion of beam particles
transversal to this longitudinal direction is accounted by the radial coordinate \( r = r(x, y) \). The time is referred as the quantity \( s \) hereafter.

Before starting the discussion about quasi-homogeneous beams, it is necessary to analyze the completely homogeneous case. The first step related to the problem is to generate the initial beam distribution for the numerical simulation. Considering that the beam is initially cold, the numerical generation of the initial beam distribution becomes a little bit simplified since just the initial position of each particle has to be specified. The velocity of all the beam particles is initially zero.

Suppose that the beam transversal section is composed by \( N \) particles disposed equidistantly one from the other. The beam transversal section is circular with an initial radius \( r_0 \). Using jargons of the field, the quantity \( r_0 \) is the initial beam envelope. For this case, the beam density \( n_b \) can be written in the form

\[
n_b = \frac{N}{\pi r_0^2}.
\]

Note that \( n_b = \text{constant} \). Observe also that \( n_b \) is a superficial density. In this way, to compose \( n_b \), roughly the particles would have to be numerically distributed in the plane. However, due to the cylindrical symmetry of the problem, it is pretty easier to generate a homogeneous linear density \( \lambda \) and then find a mathematical relation between \( n_b \) and \( \lambda \) to achieve the desired superficial beam density. This is much easier because numerically it is simpler to generate homogeneous linear densities than the superficial ones: instead of 2, it is just necessary to specify 1 spatial coordinate for each particle.

The relation between \( n_b \) and \( \lambda \) can be directly obtained by particle conservation. Suppose a small ring of width \( dr \) at the radial coordinate \( r \) of the beam superficial density \( n_b \). For the superficial density to be compatible with the linear density, the number of beam particles inside this ring must be the same that the one found in an element of length \( d\gamma \) of the linear density \( \lambda \). In this way, one finds

\[
n_b \cdot (2\pi r dr) = \lambda d\gamma
\]

which simplifies to the discrete version

\[
r_i = r_0 \sqrt{\frac{\lambda}{\gamma_i}},
\]

considering that the coordinates \( r_{\gamma i} \) can be expressed as \( r_{\gamma i} = r_0 \gamma_i \) with \( 0 \leq \gamma_i \leq 1 \). The subscript \( i \) is an integer and indexes each particle that composes the beam, \( 1 \leq i \leq N \). Observe that the set of \( \gamma_i \) coordinates are homogeneously generated in the interval \([0; 1]\) while \( r_i \) is the associated coordinate each particle \( i \) must have for its dispersion satisfies a homogeneous superficial density \( n_b \).

3. The approach

The key point to achieve the goal is how to turn the beam density inhomogeneous in a controlled way. That is, how to progressively introduce inhomogeneity in the beam density in a way that its effects over the beam dynamics could be identified and then quantified.

One interesting approach is to suppose that initially all the beam particles are disposed in an equidistant form. However, each beam particle at radial coordinate \( r_i \) starts to be affected by a noise \( \xi_i \). As a consequence, the beam as a whole starts to be a little bit inhomogeneous. The generated noise \( \xi_i \), comprised in the interval \([-1; 1]\) with some probability distribution \( P(\xi_i) \), has its intensity controlled by the parameter \( A \), here defined as the noise amplitude. To assure the noise \( \xi_i \) affects each particle with the same intensity, the noise amplitude \( A \) is considered a constant for all the beam particles.

From the Equation (3), it is possible to observe that the coordinate \( r_i \) is obtained from the homogeneous generation of coordinate \( r_{\gamma i} \). Thus, the noise \( \xi_i \) only equally affects all beam particles if it acts over the quantity \( r_{\gamma i} \). Thence, to make the beam quasi-homogeneous, it is possible to substitute \( r_{\gamma i} \) by \( r_{\gamma i} \), being \( r_{\gamma i} \) a quantity computed through the following equation

\[
r_{\gamma i} = (i + A\xi_i)\bar{r}.
\]

The quantity \( \bar{r} = 1/N \) is a characteristic spatial scale associated with the regular distance that initially each one of the beam particles is disposed from the others. For any particle, parameter \( A \) satisfies the condition \( A\xi_i \bar{r} \ll 1 \). If \( A = 0 \), in fact the noise \( \xi_i \) does not impacts over each particle coordinate \( r_i \) and
they are homogeneously distributed, composing what is called, for compactness, a crystalline beam. Nevertheless, if \( A \neq 0 \), the initial noise starts to perturb the spatial coordinate \( r \) of each particle, imposing that it fluctuates around \( \bar{r} \). Once probability distribution \( P(\xi_i) \) of \( \xi_i \) is not completely uniform and \( N \) — although usually large — is finite, the noise introduced inside the beam does not affect randomly each one of its particles. The generation of \( \xi_i \) is a little bit biased.

The noise breaks the equidistance characteristic between particles existent before, given rise to the concentration or accumulation of charge mentioned in the previous section. The beam loses its initial crystalline structure and starts to be inhomogeneous. Since parameter \( A \) controls the noise intensity, and the noise induces spatial fluctuations, indirectly \( A \) is a significant measure of the magnitude of inhomogeneity that exists in the beam density. As greater is \( A \), more inhomogeneity permeates the beam. Although the beam cannot be considered homogeneous anymore, because in fact \( A \neq 0 \), it is possible to say that the beam is quasi-homogeneous, since the condition \( A\xi_i \bar{r} \ll 1 \) is yet satisfied.

4. The results

The influences of the inhomogeneity over the beam dynamics have been analyzed through full self-consistent \( N \)-particle beam numerical simulations. The method employed to simulate numerically the system is based on Gauss’ Law: particles interact with each other just by the means of the generated electromagnetic fields. No collisions exist and thus just collective effects are accounted, such as desired. The angular momentum of each beam particle is conserved and thus only the radial set of ODEs must be integrated. All the results have been obtained considering a total number of \( N = 10000 \) beam particles, which has proportioned convergence to the results. The beam particles follow the ODE below [2]

\[
\frac{d^2}{ds^2}r + \kappa r - \frac{K}{r}Q(r) = 0.
\]

The quantity \( \kappa \) is the coefficient of magnetic focusing, which accounts how strength is the force originated by the applied magnetic field. The quantity \( K \) is known as the beam perveance, which measures how intense are the interactions between beam particles. \( Q(r) = (1/N) \int n_brdrd\theta \) is the fraction of charge trapped by a Gauss surface at \( r \). Note that in fact \( Q(r) \) accounts the effects of all the other particles with coordinate smaller than \( r \) in that at \( r \).

The initial beam envelope has been set to \( r_0 = 1.5 \) for all the numerical simulations carried out. This means a 50% mismatch, once the beam envelope of equilibrium has been adjusted to \( r_{\text{eq}} = 1 \), by the scaling of the set of ODEs. The equilibrium radius \( r_{\text{eq}} \) can be readily found imposing \( d^2r/ds^2 = 0 \) in Equation (5). Fixing the beam envelope mismatch, it is expected that any influence of the envelope mismatch is suppressed. This allows that one can better quantify the effects of the inhomogeneity induced by the noise over the initial beam density. The only parameter that is varied between all the numerical simulations is the amplitude of noise \( A \). As desired, with this approach, only the magnitude of the initial inhomogeneity that permeates \( n_b \) is changed.

The Figure 1 presents results that exemplify how the fluctuations introduced by the noise acts over the initial beam superficial density \( n_b \). The Panel (a) of this figure shows the result for the completely homogeneous – the crystalline beam – case given by \( A = 0 \). As expected, no spatial fluctuations can be seen, and \( n_b \) is strictly constant. However, as noise amplitude \( A \) is increased from zero, spatial fluctuations in \( n_b \) arise. The beam becomes weakly inhomogeneous. The effects of the introduced noise over the superficial beam density \( n_b \) are shown in Figure 1b. The noise amplitude employed is \( A = 100 \). Through a direct inspection of Figure 1b, the fluctuations along the radial coordinate \( r \) of the superficial density \( n_b \) can be clearly perceived. Both panels of Figure 1 show the superficial beam density \( n_b \) when \( s = 0 \).
The influences of the noise amplitude in the density \( n_b \) can be also visualized in Figure 2, which presents the initial beam transversal section for \( A = 100 \). The noise perturbs the beam crystalline structure so particles accumulate in some regions, living gaps in its original unperturbed positions, and consequently giving rise to the desired inhomogeneity in the superficial beam density \( n_b \). This is exactly the phenomenon of charge concentration and/or accumulation previously commented.

The envelope \( r_b \) and emittance \( \epsilon \) of the beam as it evolves inside the focusing channel is respectively presented in the panels (a) and (b) of Figure 3. While in black it is shown the results of the beam envelope and emittance for the noise amplitude \( A = 0 \), in blue the results of the same quantities for \( A = 0 \) is plotted. The values of the noise amplitude are \( A = 0 \), for the black curves, and \( A = 100 \), for the blue curves. It is important to note the pretty different behavior of both quantities when the noise amplitude is increased from \( A = 0 \) to \( A = 100 \). While for the crystalline case the beam oscillates with an invariant profile, for the quasi-homogeneous situation the beam suffers a pretty perceptible decay. From Figure 3a, for the \( A = 100 \) case, it is possible to observe that the beam envelope \( r_b \) starts to decrease when time \( s \cong 400 \). Concatenated with the envelope decay, considering also the situation for \( A = 100 \), approximately at this same time, from Figure 3b, the beam emittance suffers a sharp increasing. There is no envelope decay and emittance growth if one analyses the respective curves in Panels (a) and (b) of Figure 3 for the \( A = 0 \) situation. The numerical simulations have been carried out from \( s = 0 \) until \( s = 800 \). The initial beam envelope is \( r_0 = 1.5 \).
Figure 2. The spatial appearance of the initial beam density \( n_b(r, s = 0) \). The amplitude of the noise that perturbs the beam crystalline structure is \( A = 100 \). A red circle detaches the process of charge accumulation induced by the noise.

Figure 3. The beam (a) envelope \( n_b \) and the beam (b) emittance \( \epsilon \) as functions of the time \( s \) for \( A = 0 \) and \( A = 100 \). The results pertain to the interval \( 0 \leq s \leq 800 \).

As commented before, the noise introduced in the crystalline structure of the beam perturbs each one of its initial radial coordinates so that the beam density becomes a little bit inhomogeneous. Inhomogeneity implies that the forces inside the beam become nonlinear and the oscillation frequency of each particle become dependent of its initial position. In this way, in a finite time, many particles'
orbits will collapse. These particles will be violently ejected from the beam core and then driven by the envelope oscillations. From the energy conservation point of view, the macroscopic potential energy associated with the envelope oscillations is transformed into kinetic energy of this small group of ejected particles, known as the beam halo. Energy flows from the beam core to this small amount of ejected particles. Although the group of ejected particles is small, since the coupling of the halo particles with beam core is resonant, a very important amount of energy is transferred from the core to the halo. This is the reason by which the beam envelope decay and consequently the emittance growth are so prominent as the Figure 3 shows.

In the Figure 3b, one can observe that the beam emittance grows from zero to some equilibrium value. It is interesting to investigate how this equilibrium value of emittance depends on the magnitude of the initial beam inhomogeneity. With this objective in mind, numerical simulations have been carried out for a set of distinct initial beam inhomogeneity magnitudes, specified by the noise amplitude $A = \{0,5,25,125,625\}$. The results obtained are compactly presented in Figure 4. It is possible to observe that as greater is $A$, as earlier the beam emittance increases. However, although apparently the noise amplitude $A$ affects the time scale in which the emittance grows, the noise amplitude does not influence in its equilibrium value. In fact, the time scale of the emittance growth has contributions from the initial inhomogeneity that permeates the beam and from the initial beam mismatch [2]. When the noise amplitude $A$ is changed, exactly the contribution from the initial inhomogeneity is changing. Even in the presence of the envelope mismatch, if the beam is pretty homogenous, no emittance growth is observed. For this reason the initial beam inhomogeneity arises as the forerunner mechanism of emittance growth in nonneutral charged particle beams.

![Figure 4](image_url)

Figure 4. The emittance $\epsilon(s)$ for different values of noise amplitude $A$. Observe how the time scale of the emittance growth strongly depends on the noise amplitude $A$.

To finish the current analysis, in Figure 5 the equilibrium values of the beam envelope and emittance are shown for the same set of noise amplitude $A$ employed before. The beam envelope and emittance are respectively plotted in red and blue lines in Figure 5. It is promptly possible to find that the noise amplitude does not impact in the equilibrium values of these beam quantities. This is what one would expect, since the equilibrium values of beam quantities are related just with topological aspects of the problem. Although the beam envelope has a coadjuvant participation in the time scale of the emittance growth, it has greater importance and is fundamentally associated with the values of the beam quantities at the equilibrium. Large beam envelope mismatches does not only imply that the energy stored in the beam oscillations is large, but also that the interactions between the beam and individual particles are intense. This clearly appears if one analyses the phase space orbit of a particle excited by the beam oscillations: large envelope mismatches induces the formation of large resonant islands. When in this large resonances, some of beam particles can earn much kinetic energy from the core oscillations. Great initial envelope mismatches implies in great emittance growths and envelope decays.
5. Conclusions and future works
The initial inhomogeneity is determinant for the emittance growth observed in charged particle beams and has impact over its time scale. However, no influences over beam macroscopic quantities such as envelope and emittance have been observed at the equilibrium. In fact, beam quantities at equilibrium strongly depend on the initial beam envelope mismatch. Future works will contain more information about the issue.

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