Global Classical Solutions to Full Compressible Magnetohydrodynamic System with Large Oscillations and Vacuum in 3D Exterior Domains

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Abstract

The full compressible magnetohydrodynamic system in three-dimensional exterior domains is investigated. For the initial-boundary-value problem of this system with slip boundary condition for the velocity, adiabatic one for the temperature, and perfect one for the magnetic field, the global existence and uniqueness of classical solutions is established, under the condition that the initial data are of small energy but possibly large oscillations. In particular, the initial density and temperature are both allowed to vanish. Moreover, the large-time behavior of the classical solutions is also obtained.

Keywords: full compressible magnetohydrodynamic system; vacuum; large oscillations; exterior domains; classical solutions.

1 Introduction

The motion of a viscous, compressible, and heat conducting magnetohydrodynamic (MHD) flow in a three-dimensional (3D) spatial domain $\Omega \subset \mathbb{R}^3$ can be described by the full compressible MHD system (see [18]):

\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P &= \mu \Delta u + (\mu + \lambda) \nabla \text{div} u + (\nabla \times H) \times H, \\
\frac{\gamma - 1}{\gamma - 1} (\rho \theta)_t + \text{div}(\rho \theta u) + P \text{div} u &= \kappa \Delta \theta + \lambda (\text{div} u)^2 + 2 \mu |\mathbb{D}(u)|^2 + \nu |\text{curl} H|^2, \\
H_t - \nabla \times (u \times H) &= -\nu \nabla \times (\nabla \times H), \\
\text{div} H &= 0,
\end{align*}

where $t \geq 0$ is time, $x \in \Omega$ is the spatial coordinate, and $\rho$, $u = (u^1, u^2, u^3)^{\text{tr}}$, $\theta$, $P = R \rho \theta (R > 0)$, and $H = (H^1, H^2, H^3)^{\text{tr}}$ represent respectively the fluid density,
velocity, absolute temperature, pressure, and magnetic field. The viscosity coefficients \( \mu \) and \( \lambda \) are constants satisfying the physical restrictions:

\[
\mu > 0, \quad 2\mu + 3\lambda \geq 0.
\] (1.2)

The heat-conductivity coefficient \( \kappa \) and magnetic resistivity coefficient \( \nu \) both are positive constants. \( \gamma > 1 \) is the adiabatic constant.

Let \( \Omega = \mathbb{R}^3 - \bar{D} \) be the exterior of a simply connected bounded smooth domain \( D \subset \mathbb{R}^3 \), we impose on the system (1.1) the following initial data:

\[
(\rho, \rho u, \rho \theta, H)(x, t = 0) = (\rho_0, \rho_0 u_0, \rho_0 \theta_0, H_0)(x), \quad x \in \Omega,
\] (1.3)

the boundary conditions:

\[
\begin{aligned}
&u \cdot n = 0, \; \text{curl} u \times n = 0, \quad \text{on} \; \partial \Omega \times (0, T), \\
&\nabla \theta \cdot n = 0, \; H \cdot n = 0, \; \text{curl} H \times n = 0 \quad \text{on} \; \partial \Omega \times (0, T),
\end{aligned}
\] (1.4)

and the far field behavior:

\[
(\rho, u, \theta, H)(x, t) \to (1, 0, 1, 0) \text{ as } |x| \to \infty,
\] (1.5)

where \( n = (n^1, n^2, n^3)^{tr} \) is the unit outward normal vector on \( \partial \Omega \). Indeed, for fixed \( \Omega \), there exists a positive constant \( d \) such that \( \bar{D} \subset B_d \), then one can extend the unit outer normal \( n \) to \( \bar{\Omega} \) such that \( n \in C^\infty(\bar{\Omega}) \) and \( n \equiv 0 \) on \( \mathbb{R}^3 \setminus B_{2d} \). The method of extension is not unique, and we fix one through out the paper.

There are a lot of literatures on the well-posedness and dynamic behavior of the solutions to the compressible MHD system due to its physical importance and mathematical challenges. The one-dimensional problem has been studied extensively by many people, see [4, 20, 25, 47] and the references therein. For the multi-dimensional case, the local existence of strong or classical solutions with initial vacuum for isentropic or full compressible MHD system are showed in [13, 36, 37, 44, 49]. The global classical solutions to the 2D Cauchy problem of compressible MHD system were first proved by Kawashima [24] for the initial data close to a non-vacuum equilibrium in \( H^3 \)-norm. Chen-Tan [5] extended this result to the 3D Cauchy problem. See also the generalizations to the Cauchy problem [41] or the exterior domains [30] for the full compressible MHD system.

On the other hand, when vacuum state appears, the problem becomes more complicated and difficult. Hu-Wang [18, 19] gave the global existence of renormalized weak solutions with large data in 3D bounded domains. See the relevant results in [12, 31, 42] for isentropic case or non-isentropic case. The global existence and uniqueness of classical solutions to the Cauchy problem of 3D isentropic compressible MHD system was established by Li et al. [26], where the initial energy is small but oscillations maybe large and the vacuum state is allowed. Later, the result was generalized by Hong et al. [16] to the large initial data with \( \gamma - 1 \) and \( \nu^{-1} \) are suitably small. For the 2D case, Lv et al. [38] obtained the global existence and uniqueness of classical solutions and some better a priori decay rates. Recently, Chen et al. [8] derived the global classical solutions to isentropic compressible MHD system with slip boundary conditions in 3D bounded domains for the regular initial data with small energy but possibly large oscillations and vacuum. More recently, the global well-posedness of strong and weak solutions of the isentropic compressible MHD system in 2D bounded domains with large
initial data and vacuum was investigated by Chen et al. [7]. As for the full compressible MHD system, the global strong solutions to the 3D Cauchy problem was investigated by Liu-Zhong [32, 34] and Hou et al. [17] under some certain small conditions, referring to [14, 28–30, 43–48, 50] for more results to compressible non-resistive or inviscid MHD equations. Considering the exterior domains problem with slip boundary conditions, Chen et al. [9] obtained the global existence of classical solutions with small energy but possibly large oscillations to barotropic compressible MHD system. Then the main aim of this paper is to extend this result to full compressible MHD system [1, 11], which in fact is based [3, 27] for compressible Navier-Stokes equations with slip boundary conditions in exterior domains and [6] for full compressible MHD system with slip boundary conditions in bounded domains.

Before stating the main result, we first introduce the notations and conventions used throughout this paper. For $1 \leq p \leq \infty$ and integer $k \geq 0$, we adopt the following notations for the standard homogeneous and inhomogeneous Sobolev spaces:

\[
\begin{align*}
L^p = L^p(\Omega), & \quad W^{k,p} = W^{k,p}(\Omega), & \quad H^k = W^{k,2}, \\
D^{k,p} = \{ f \in L^1_{loc}(\Omega) | \nabla^k f \in L^p(\Omega) \}, & \quad D^k = D^{k,2}, \\
H^s_w = \{ f \in H^2 | f \cdot n = 0, \text{curl} f \times n = 0 \text{ on } \partial \Omega \}.
\end{align*}
\]

Define the initial energy $C_0$ as follows:

\[
C_0 = \int_\Omega \left( \frac{1}{2} \rho_0 |u_0|^2 + R(1 + \rho_0 \log \rho_0 - \rho_0) + \frac{R}{\gamma - 1} \rho_0 (\theta_0 - \log \theta_0 - 1) + \frac{1}{2} |H_0|^2 \right) dx. \tag{1.6}
\]

Then the main result in this paper can be stated as follows:

**Theorem 1.1.** Let $\Omega = \mathbb{R}^3 - \hat{D}$ be the exterior of a simply connected bounded smooth domain $D \subset \mathbb{R}^3$. For given numbers $M > 0$ (not necessarily small), $q \in (3, 6)$, $\hat{\rho} > 2$, and $\hat{\theta} > 1$, suppose that the initial data $(\rho_0, u_0, \theta_0, H_0)$ satisfies

\[
\rho_0 - 1 \in H^2 \cap W^{2,q}, \quad (u_0, H_0) \in H^2_w, \quad \theta_0 - 1 \in H^1, \quad \text{div} H_0 = 0, \quad \nabla \theta_0 \cdot n|_{\partial \Omega} = 0,
\]

\[
0 \leq \inf \rho_0 \leq \sup \rho_0 < \hat{\rho}, \quad 0 \leq \inf \theta_0 \leq \sup \theta_0 \leq \hat{\theta}, \quad \| \nabla u_0 \|_{L^2} + \| \nabla H_0 \|_{L^2} \leq M, \tag{1.7}
\]

and the compatibility condition

\[
- \mu \Delta u_0 + (\mu + \lambda) \nabla \text{div} u_0 + R \nabla (\rho_0 \theta_0) - (\nabla \times H_0) \times H_0 = \sqrt{\rho_0} g \tag{1.8}
\]

with $g \in L^2$. Then there exists a positive constant $\varepsilon$ depending only on $\mu$, $\lambda$, $\nu$, $\kappa$, $R$, $\gamma$, $\hat{\rho}$, $\hat{\theta}$, $\Omega$, and $M$ such that

\[
C_0 \leq \varepsilon, \tag{1.9}
\]

the problem (1.1)–(1.5) admits a unique global classical solution $(\rho, u, \theta, H)$ in $\Omega \times (0, \infty)$ satisfying

\[
0 \leq \rho(x, t) \leq 2\hat{\rho}, \quad \theta(x, t) \geq 0, \quad x \in \Omega, \quad t \geq 0, \tag{1.10}
\]

and

\[
\begin{align*}
\rho - 1 & \in C([0, T]; H^2 \cap W^{2,q}), \\
(u, H) & \in C([0, T]; H^2), \quad \theta - 1 \in C((0, T]; H^2), \\
u & \in L^2(0, T; H^3) \cap L^\infty(\tau, T; H^3 \cap W^{3,q}), \\
\theta - 1 & \in L^\infty(0, T; H^1) \cap L^2(0, T; H^2) \cap L^\infty(\tau, T; H^4), \\
H & \in L^2(0, T; H^3) \cap L^\infty(\tau, T; H^4), \\
(u, \theta, H_t) & \in L^\infty(\tau, T; H^2) \cap H^1(\tau, T; H^1),
\end{align*} \tag{1.11}
\]
for any $0 < \tau < T < \infty$. Moreover, the following large-time behavior holds:

$$\lim_{t \to \infty} (\|\rho(\cdot, t) - 1\|_{L^p} + \|\nabla u(\cdot, t)\|_{L^r} + \|\nabla \theta(\cdot, t)\|_{L^r} + \|\nabla H(\cdot, t)\|_{L^r}) = 0,$$

(1.13) for any $p \in (2, \infty)$ and $r \in [2, 6]$.

Next, as a direct application of (1.13), the following Corollary 1.2 indicates the large-time behavior of the gradient of density when the vacuum state appears initially. The proof is analogue to that of [23, Theorem 1.2].

Corollary 1.2. In addition to the conditions of Theorem 1.1, assume further that there exists some point $x_0 \in \Omega$ such that $\rho_0(x_0) = 0$. Then the unique global classical solution $(\rho, u, \theta, H)$ to the problem (1.1)–(1.5) obtained in Theorem 1.1 has to blow up as $t \to \infty$, in the sense that for any $r > 3$,

$$\lim_{t \to \infty} \|\nabla \rho(\cdot, t)\|_{L^r} = \infty.$$

A few remarks are in order:

Remark 1.1. One can deduce from (1.12) and Sobolev imbedding theorem that for any $0 < \tau < T < \infty$,

$$(\rho - 1, \nabla \rho, u, H) \in C(\overline{\Omega} \times [0, T]), \quad \theta - 1 \in C(\overline{\Omega} \times [\tau, T]),$$

(1.14) and

$$(\nabla u, \nabla^2 u) \in C([0, T]; L^2) \cap L^{\infty}(\tau, T; W^{1, q}) \hookrightarrow C(\overline{\Omega} \times [\tau, T]),$$

$$(\nabla \theta, \nabla^2 \theta, \nabla H, \nabla^2 H, u_t, \theta_t, H_t) \in C([\tau, T]; L^2) \cap L^{\infty}(\tau, T; H^2) \hookrightarrow C(\overline{\Omega} \times [\tau, T]),$$

which combined with (1.1) and (1.14) implies

$$\rho_t \in C(\overline{\Omega} \times [\tau, T]).$$

Therefore, the solution $(\rho, u, \theta, H)$ obtained in Theorem 1.1 is a classical one to the problem (1.1)–(1.5) in $\Omega \times (0, \infty)$.

Remark 1.2. Theorem 1.1 extends the global existence result of the barotropic compressible MHD system studied in Chen-Huang-Shi [9] to the full compressible MHD system, which is the first result concerning the global existence of classical solutions with initial vacuum to problem (1.1)–(1.5) in the exterior domains. Although it’s energy is small, the oscillations could be arbitrarily large.

We now comment on the analysis of this paper. First we extend the local classical solutions away from vacuum (see Lemma 2.1) globally in all time provided that the initial energy is suitably small (see Proposition 5.1), then let the lower bound of the initial density tend to zero to obtain the global classical solutions with vacuum. It turns out that the key issue in this paper is to derive both the uniform (in time) upper bound for the density and the time-dependent higher-order estimates of the smooth solution $(\rho, u, \theta, H)$, which are all independent of the lower bound of density.

Compared to the isentropic case [9], the first difficulty in this paper lies in the fact that the basic energy $E(t)$ defined in (3.14) is unavailable due to the slip boundary
To overcome this difficulty, making an a priori assumption on $A_2(T)$ (see (3.9)), we then derive a “weaker” basic energy estimate (see Lemma 3.1)

$$E(t) \leq CC_0^{1/4},$$

which will cause some essential difficulties in later calculations.

Next, based on the elaborate analysis of the maximum of temperature and the structure of (1.1), we succeed in re-establishing the basic energy in small time $[0, \sigma(T)]$ and then deriving the energy-like estimate on $A_2(T)$, which includes the key bounds on the $L^2$-norm (in both time and space) of $(\nabla u, \nabla \theta, \nabla H)$, provided the initial energy is small (see Lemma 3.6). Here, we make use of some estimates on $(\dot{u}, \dot{\theta}, H_t)$ due to Hoff [15] (see also Huang-Li [21]), which contributes to deal with the a priori estimates $A_3(T)$ as well. However, in this process slip boundary conditions lead to a difficulty once again, i.e. the boundary integral terms (see (3.56)). In fact, they can be solved by the ideas in [2] (see (3.58)), i.e. using the fact $u = u^\perp \times n$ on $\partial \Omega$ with $u^\perp = -u \times n$, the boundary integral terms can only be bounded by the gradients of velocity and efficient viscous flux. On the other hand, we adopt some ideas in Cai-Li-Lü [3] to re-establish the estimates on $(\nabla u, \text{div} u, \text{curl} u)$ and $(\nabla H, \text{curl} H)$, which are used repeatedly in the whole analysis.

Thus, with the help of the lower estimates, we can derive the time-independent upper bound of the density smoothly (see Lemma 3.7) and then obtain the higher-order estimates (see Section 4) under the compatibility condition on the velocity. It’s noted that all the a priori estimates are independent of the lower bound of the initial density, thus after a standard approximation procedure, we can arrive at the global existence of classical solutions with vacuum.

The rest of this paper is organized as follows: In Section 2, some elementary facts and inequalities will be exhibited. To extend the local solutions to all time, we establish the lower-order a priori estimates on classical solutions in Section 3 and the higher-order estimates in Section 4. Finally, with all a priori estimates at hand, the main result Theorem 1.1 is proved in Section 5.

### 2 Preliminaries

We begin with the following well-known local existence theory with strictly positive initial density, which can be proved by the standard contraction mapping argument as in [10, 39, 45].

**Lemma 2.1.** Let $\Omega$ be as in Theorem 1.1. Assume that $(\rho_0, u_0, \theta_0, H_0)$ satisfies

$$\begin{align*}
\left\{ \begin{array}{l}
(\rho_0 - 1, u_0, \theta_0 - 1, H_0) \in H^3, \\
u_0 \cdot n = 0, \quad \text{curl} u_0 \times n = 0, \quad \nabla \theta_0 \cdot n = 0 \quad \text{on} \partial \Omega,
\end{array} \right.
\end{align*}$$

(2.1)

Then there exist a small time $T_0 > 0$ and a unique classical solution $(\rho, u, \theta, H)$ to the problem (1.1)–(1.5) on $\Omega \times (0, T_0]$ satisfying

$$\inf_{(x,t) \in \Omega \times (0, T_0]} \rho(x,t) \geq \frac{1}{2} \inf_{x \in \Omega} \rho_0(x),$$

(2.2)
and
\[
\begin{cases}
(r - 1, u, \theta - 1, H) \in C([0, T_0]; H^3), & \rho_t \in C([0, T_0]; H^2), \\
(tu_t, \theta_t, \rho_t) \in C([0, T_0]; H^1), & (u, \theta - 1, H) \in L^2(0, T_0; H^4).
\end{cases}
\] (2.3)

**Remark 2.1.** Applying the same arguments as in [27, Lemma 2.1] to the classical solution \((\rho, u, \theta, H)\) obtained in Lemma [2.1], one gets
\[
\begin{cases}
(tu_t, \theta_t, \rho_t) \in L^2(0, T_0; H^3), & (tu_t, \theta_t, \rho_t) \in L^2(0, T_0; H^1), \\
(t^2 u_{tt}, t^2 \theta_{tt}, t^2 H_{tt}) \in L^2(0, T_0; H^3), & (t^2 u_{tt}, t^2 \theta_{tt}, t^2 H_{tt}) \in L^2(0, T_0; L^2).
\end{cases}
\]

Moreover, for any \((x, t) \in \Omega \times [0, T_0]\), the following estimate holds:
\[
\theta(x, t) \geq \inf_{x \in \Omega} \theta_0(x) \exp \left\{- (\gamma - 1) \int_0^{T_0} \| \text{div} u \|_{L^\infty} dt \right\}.
\] (2.4)

The following classical Gagliardo-Nirenberg-Sobolev-type inequality (see [11]) will be used later frequently.

**Lemma 2.2.** Assume that \(\Omega\) is the exterior of a simply connected Lipschitz domain \(D\) in \(\mathbb{R}^3\). For \(r \in [2, 6]\), \(p \in (1, \infty)\), and \(q \in (3, \infty)\), there exist positive constants \(C\) which may depend on \(r, p,\) and \(q\), such that for \(f \in H^1(\Omega)\), \(g \in L^p(\Omega) \cap D^{1,q}(\Omega)\), and \(\varphi, \psi \in H^2(\Omega),\)
\[
\|f\|_{L^r} \leq C\|f\|_{L^2}^{(3-r)/2r}\|
\n\|g\|_{C(\overline{\Omega})} \leq C\|g\|_{L^p}^{p/(3q+p(q-3))}\|
\n\|\varphi\|_{H^2} \leq C\|\varphi\|_{H^2}\|
\]
\[
(2.5)
\]
\[
(2.6)
\]
\[
(2.7)
\]

Then, as a result of (2.5), one has the following Sobolev-type inequality (see [27, Lemma 2.3]), which provides an estimate on \(\theta - 1\) (see (3.98)).

**Lemma 2.3.** Let the function \(g(x)\) defined in \(\Omega\) be non-negative and satisfy \(g(\cdot) - 1 \in L^2(\Omega)\). Then there exists a universal positive constant \(C\) such that for \(s \in [1, 2]\) and any open set \(\Sigma \subset \Omega\), the following estimate holds
\[
\int_{\Sigma} |f|^s dx \leq C \int_{\Sigma} g |f|^s dx + C\|g - 1\|_{L^2(\Omega)}^{(6-s)/3}\|
\]
\[
(2.8)
\]
for all \(f \in \{D^1(\Omega) \mid \|g\|_{L^1(\Sigma)} \leq 1\} \).

Next, to estimate \(\|\nabla u\|_{L^\infty}\) for the further higher order estimates, we need the following Beale-Kato-Majda-type inequality [1], which in fact is a critical case of Sobolev inequality, whose detailed proof is similar to that of the case of slip boundary condition in [2, Lemma 2.7] (see also [22]).

**Lemma 2.4.** Let \(\Omega = \mathbb{R}^3 - \bar{D}\) be as in Theorem [1.1]. For \(3 < q < \infty\), assume that \(u \in \{f \in L^1_{\text{loc}} \mid \nabla f \in L^2(\Omega) \cap D^{1,q}(\Omega)\} \) and \(f \cdot n = 0, \text{curl} f \times n = 0 \) on \(\partial \Omega\), then there is a constant \(C = C(q)\) such that
\[
\|\nabla u\|_{L^\infty} \leq C \left(\|\text{div} u\|_{L^\infty} + \|\text{curl} u\|_{L^\infty}\right) \ln(e + \|\nabla^2 u\|_{L^q}) + C\|\nabla u\|_{L^2} + C.
\]

The div-curl-type estimates of \(\nabla v\) with boundary condition \(v \cdot n = 0\) or \(v \times n = 0\) on \(\partial \Omega\) are obtained in the following Lemmas [2.5, 2.7], whose proof can be found in [46, Theorem 3.2], [35, Theorem 5.1], and [3, Lemma 2.9], respectively.
Lemma 2.5. \[ \text{[35, Theorem 3.2]} \] Let \( \Omega = \mathbb{R}^3 - \bar{D} \) be the exterior of a simply connected bounded domain \( D \subset \mathbb{R}^3 \) with \( C^{1,1} \) boundary. For \( v \in D^{1,q}(\Omega) \) with \( v \cdot n = 0 \) on \( \partial \Omega \), it holds that

\[
\| \nabla v \|_{L^q} \leq C(q, \Omega) (\| \text{div} v \|_{L^q} + \| \text{curl} v \|_{L^q}), \quad \text{for any} \ 1 < q < 3, \quad (2.9)
\]

\[
\| \nabla v \|_{L^q} \leq C(q, \Omega) (\| \text{div} v \|_{L^q} + \| \text{curl} v \|_{L^q} + \| \nabla v \|_{L^2}), \quad \text{for any} \ 3 \leq q < +\infty. \quad (2.10)
\]

Lemma 2.6. \[ \text{[35, Theorem 5.1]} \] Let \( \Omega \) be as in Lemma 2.5, for any \( v \in W^{1,q}(\Omega) \) \( 1 < q < +\infty \) with \( v \times n = 0 \) on \( \partial \Omega \), it holds that

\[
\| \nabla v \|_{L^q} \leq C(q, \Omega) (\| v \|_{L^q} + \| \text{div} v \|_{L^q} + \| \text{curl} v \|_{L^q}). \quad (2.11)
\]

Lemma 2.7. \[ \text{[35, Lemma 2.9]} \] Let \( \Omega \) be as in Theorem 1.1. For any \( q \in [2, 4) \), if \( v \in \{ D^{1,2}(\Omega) \mid v(x) \to 0 \ as \ |x| \to \infty \} \), then

\[
\| v \|_{L^q(\partial \Omega)} \leq C(q, \Omega) \| \nabla v \|_{L^2(\Omega)}. \quad (2.12)
\]

Moreover, for \( p \in [2, 6], \ k \geq 1 \), every \( v \in \{ f \in D^{1,2}(\Omega) \mid \nabla f \in W^{k,p}(\Omega), \ f(x) \to 0 \ as \ |x| \to \infty \} \) with \( v \cdot n|_{\partial \Omega} = 0 \) or \( v \times n|_{\partial \Omega} = 0 \) satisfies

\[
\| \nabla v \|_{W^{k,p}} \leq C(k, p, \Omega) (\| \text{div} v \|_{W^{k,p}} + \| \text{curl} v \|_{W^{k,p}} + \| \nabla v \|_{L^2}). \quad (2.13)
\]

Next, we state the following estimate on \( \nabla \dot{u} \) with \( u \cdot n|_{\partial \Omega} = 0 \) (see [27, Lemma 2.9]).

Lemma 2.8. Let \( \Omega = \mathbb{R}^3 - \bar{D} \) be as in Theorem 1.1. Assume that \( u \) is smooth enough and \( u \cdot n|_{\partial \Omega} = 0 \), then there exists a generic positive constant \( C = C(\Omega) \) such that

\[
\| \nabla \dot{u} \|_{L^2} \leq C (\| \text{div} \dot{u} \|_{L^2} + \| \text{curl} \dot{u} \|_{L^2} + \| \nabla \dot{u} \|_{L^4}^2 + \| \nabla \dot{u} \|_{L^2}^2). \quad (2.14)
\]

The following Grönwall-type inequality will be used to get the uniform (in time) upper bound of the density (see [21, Lemma 2.5]).

Lemma 2.9. Let the function \( y \in W^{1,1}(0, T) \) satisfy

\[
y'(t) + \alpha y(t) \leq g(t) \ on \ [0, T], \quad y(0) = y_0,
\]

where \( \alpha \) is a positive constant and \( g \in L^p(0, T_1) \cap L^q(T_1, T) \) for some \( p, q \geq 1, T_1 \in [0, T] \). Then

\[
\sup_{0 \leq t \leq T} y(t) \leq |y_0| + (1 + \alpha^{-1}) (\| g \|_{L^p(0, T_1)} + \| g \|_{L^q(T_1, T)}). \quad (2.15)
\]

Finally, we mainly show some elliptic estimates, which are the important ingredients of the whole analysis.

For the Neumann boundary value problem

\[
\begin{cases}
-\Delta v = \text{div} f, & \text{in } \Omega, \\
\frac{\partial v}{\partial n} = -f \cdot n, & \text{on } \partial \Omega, \\
\nabla v \to 0, & \text{as } |x| \to \infty,
\end{cases}
\quad (2.16)
\]

by virtue of [40, Lemma 5.6], we have the following conclusion.
Lemma 2.10. Considering the system (2.16), for \( q \in (1, \infty) \), it holds

1. If \( f \in L^q(\Omega) \), then there exists a unique (modulo constants) solution \( v \in D^{1,q}(\Omega) \) such that
   \[
   \| \nabla v \|_{L^q} \leq C(q, \Omega) \| f \|_{L^q}.
   \]

2. If \( f \in W^{k,q}(\Omega) \) with \( k \geq 1 \), then \( \nabla v \in W^{k,q}(\Omega) \) and
   \[
   \| \nabla v \|_{W^{k,q}} \leq C(k, q, \Omega) \| f \|_{W^{k,q}}.
   \]

In particular, if \( f \cdot n = 0 \) on \( \partial\Omega \), one gets
   \[
   \| \nabla^2 v \|_{L^q} \leq C(q, \Omega) \| \text{div} f \|_{L^q}.
   \]

Denoting the material derivative of \( f \), the effective viscous flux \( G \), and the vorticity \( \omega \) respectively by

\[
\begin{align*}
D_t f & \triangleq \dot{f} \triangleq f_t + u \cdot \nabla f, \\
G & \triangleq (2\mu + \lambda) \text{div} u - R(\rho \theta - 1) - |H|^2/2, \\
\omega & \triangleq \text{curl} u.
\end{align*}
\]

Now, we establish some necessary estimates for them. In fact, the standard \( L^p \)-estimate for the following elliptic equation:

\[
\begin{aligned}
\Delta G &= \text{div}(\rho \dot{u} - H \cdot \nabla H), \quad \text{in } \Omega, \\
\frac{\partial G}{\partial n} &= (\rho \dot{u} - H \cdot \nabla H) \cdot n, \quad \text{on } \partial\Omega, \\
\nabla G &\to 0, \quad \text{as } |x| \to \infty.
\end{aligned}
\]

combined with Lemmas 2.2, 2.3, and 2.10 yields the following estimates, whose proof is analogue to that of Lemma 2.8 (see also Lemma 2.9).

Lemma 2.11. Assume \( \Omega = \mathbb{R}^3 - \bar{D} \) is the same as in Theorem 1.1. Let \( (p, u, \theta, H) \) be a smooth solution of (1.1)–(1.5) on \( \Omega \times (0, T] \). Then for any \( p \in [2, 6] \), there exists a positive constant \( C \) depending only on \( p, \mu, \lambda, R, \) and \( \Omega \) such that

\[
\begin{align*}
\| \nabla H \|_{L^p} &\leq C(\| \text{curl} H \|_{L^p} + \| \text{curl} H \|_{L^2}), \\
\| \nabla G \|_{L^p} &\leq C(\| \rho \dot{u} \|_{L^p} + \| H \cdot \nabla H \|_{L^p}), \\
\| \nabla \omega \|_{L^p} &\leq C(\| \rho \dot{u} \|_{L^p} + \| H \cdot \nabla H \|_{L^p} + \| \rho \dot{u} \|_{L^2} + \| H \cdot \nabla H \|_{L^2} + \| \nabla u \|_{L^2}), \\
\| G \|_{L^p} &\leq C(\| \rho \dot{u} \|_{L^2} + \| H \cdot \nabla H \|_{L^2}^2)^{3p-6}/2p + \| \rho \theta - 1 \|_{L^2}^2 + \| H \|_{L^2}^2)^{6-p}/2p, \\
\| \omega \|_{L^p} &\leq C(\| \rho \dot{u} \|_{L^2} + \| H \cdot \nabla H \|_{L^2}^2)^{3p-6}/2p + \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^2}, \\
\| \nabla u \|_{L^p} &\leq C(\| \rho \dot{u} \|_{L^2} + \| H \|_{L^2} \| \nabla H \|_{L^2} + \| \rho \theta - 1 \|_{L^2})^{3p-6}/2p + \| \nabla u \|_{L^2}^2)^{6-p}/2p + C \| \nabla u \|_{L^2}^2.
\end{align*}
\]

Remark 2.2. By virtue of (2.11) and (2.13), we will obtain the estimate on \( \nabla^{k+2} H \). Precisely, for \( p \in [2, 6] \) and \( k \geq 0 \), there exists a positive constant \( C = C(p, k, \Omega) \) such that

\[
\begin{align*}
\| \nabla^{k+2} H \|_{L^p} &\leq C \| \text{curl} H \|_{W^{k+1,p}} + C \| \nabla H \|_{L^2} \\
&\leq C(\| \text{curl} H \|_{W^{k,p}} + \| \nabla H \|_{L^2}).
\end{align*}
\]
On the other hand, we can get the estimate on $\nabla^{k+2} u$, which will be devoted to getting the higher order estimates in Section 4. In fact, applying Lemma 2.10 to elliptic equation (2.18) along with Lemmas 2.6, 2.7, and 2.11 yields that for $p \in [2, 6]$ and $k \geq 0$, there exists a positive constant $C = C(p, k, \Omega)$ such that

$$
\|\nabla^{k+2} u\|_{L^p} \\
\leq C(\|\text{div} u\|_{W^{k+1, p}} + \|\omega\|_{W^{k+1, p}} + \|\nabla u\|_{L^2}) \\
\leq C(\|\rho \hat{u}\|_{W^{k, p}} + \|H \cdot \nabla H\|_{W^{k, p}} + \|\nabla P\|_{W^{k, p}} + \|\nabla H\|_{W^{k, p}}^2) \\
+ C(\|\rho \hat{u}\|_{L^2} + \|H \cdot \nabla H\|_{L^2} + \|\rho \theta - 1\|_{L^2} + \|\nabla H\|_{L^2}^2 + \|\nabla u\|_{L^2}).
$$

(2.26)

3 A priori estimates (I): lower-order estimates

In this section, we aim to derive the time-independent a priori estimates for the local-in-time smooth solution to the problem (1.1)–(1.5) obtained in Lemma 2.1. Let $(\rho, u, \theta, H)$ be a smooth solution to the problem (1.1)–(1.5) on $\Omega \times (0, T)$ for some fixed time $T > 0$, with the initial data $(\rho_0, u_0, \theta_0, H_0)$ satisfying (2.1).

For $\sigma(t) \triangleq \min\{1, t\}$, we define $A_i(T)$ ($i = 1, \cdots, 4$) as follows:

$$
A_1(T) \triangleq \sup_{t \in [0, T]} \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \int_0^T (\|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \|\text{curl} H\|_{L^2}^2 + \|H_t\|_{L^2}^2) dt, \right. \\
$$

(3.1)

$$
A_2(T) \triangleq \frac{R}{2(\gamma - 1)} \sup_{t \in [0, T]} \int_0^T \rho(\theta - 1)^2 dx + \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) dt, \\
$$

(3.2)

$$
A_3(T) \triangleq \sup_{t \in [0, T]} \sigma \left( \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) \\
+ \int_0^T \sigma (\|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \|\text{curl} H\|_{L^2}^2 + \|H_t\|_{L^2}^2) dt, \\
$$

(3.3)

$$
A_4(T) \triangleq \sup_{t \in [0, T]} \sigma^2 (\|\sqrt{\rho} \hat{u}\|_{L^2}^2 + \|\text{curl} H\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) \\
+ \int_0^T \sigma^2 (\|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 + \|\sqrt{\rho} \hat{u}\|_{L^2}^2) dt.
$$

(3.4)

Then we have the following key a priori estimates on $(\rho, u, \theta, H)$.

**Proposition 3.1.** For given numbers $M > 0$, $\hat{\rho} > 2$, and $\hat{\theta} > 1$, assume further that $(\rho_0, u_0, \theta_0, H_0)$ satisfies

$$
0 < \inf \rho_0 \leq \sup \rho_0 < \hat{\rho}, \quad 0 < \inf \theta_0 \leq \sup \theta_0 \leq \hat{\theta}, \quad \|\nabla u_0\|_{L^2} + \|\nabla H_0\|_{L^2} \leq M. \quad (3.5)
$$

Then there exist positive constants $K$ and $\varepsilon_0$ both depending on $\mu, \nu, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega,$ and $M$ such that if $(\rho, u, \theta, H)$ is a smooth solution to the problem (1.1)–(1.5) on $\Omega \times (0, T)$ satisfying

$$
0 < \rho \leq 2\hat{\rho}, \quad A_1(\sigma(T)) \leq 3K, \quad A_2(T) \leq 2C_0^{1/4}, \quad A_3(T) + A_4(T) \leq 2C_0^{1/6}, \quad (3.6)
$$

the following estimates hold:

$$
0 < \rho \leq 3\hat{\rho}/2, \quad A_1(\sigma(T)) \leq 2K, \quad A_2(T) \leq C_0^{1/4}, \quad A_3(T) + A_4(T) \leq C_0^{1/6}, \quad (3.7)
$$

provided

$$
C_0 \leq \varepsilon_0. \quad (3.8)
$$
Proof. Indeed, we conclude Proposition 3.1 as a result of the following Lemmas 3.2, 3.4, 3.6, and 3.7, with \( \varepsilon_0 \) as in (3.129).

In this section, we always assume that \( C_0 \leq 1 \) and let \( C \) denote some generic positive constant depending only on \( \mu, \lambda, \nu, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega, \) and \( M \), and we write \( C(\alpha) \) to emphasize that \( C \) may depend on \( \alpha \).

We start with the following estimate on the basic energy.

**Lemma 3.1.** Under the conditions of Proposition 3.1, there exists a positive constant \( C \) depending only on \( \mu, R, \) and \( \hat{\rho} \) such that if \( (\rho, u, \theta, H) \) is a smooth solution to the problem (1.1)–(1.5) on \( \Omega \times (0, T) \) satisfying

\[
0 < \rho \leq 2\hat{\rho}, \quad A_2(T) \leq 2C_0^{1/4},
\]

the following estimate holds:

\[
\sup_{0 \leq t \leq T} \int (\rho|u|^2 + (\rho - 1)^2 + |H|^2) \, dx \leq CC_0^{1/4}.
\]

Proof. First, by virtue of (3.5) and (2.4), we have that for all \( (x, t) \in \Omega \times (0, T) \),

\[
\theta(x, t) > 0.
\]

Next, note that

\[
\Delta u = \nabla \text{div} u - \nabla \times \omega, \quad (\nabla \times H) \times H = H \cdot \nabla H - \nabla |H|^2/2,
\]

one can rewrite (1.1) as

\[
\rho(u_t + u \cdot \nabla u) = (2\mu + \lambda)\nabla \text{div} u - \mu \nabla \times \omega - \nabla P + H \cdot \nabla H - \nabla |H|^2/2.
\]

Similarly, some simple calculations show that (1.1) is equivalent to

\[
H_t + u \cdot \nabla H = H \cdot \nabla u + H \text{ div} u + \nu \nabla \times \text{curl} H = 0.
\]

Denote the basic energy by

\[
E(t) \triangleq \int \left( \frac{1}{2} \rho|u|^2 + R(1 + \rho \log \rho - \rho) + \frac{R}{\gamma - 1} \rho(\theta - \log \theta - 1) + \frac{1}{2}|H|^2 \right) \, dx.
\]

Then, multiplying (3.12), (1.1), and (3.13) by \( u, 1 - \theta^{-1}, \) and \( H \) respectively, summing them up, and integrating the resulting equality over \( \Omega \) by parts, we deduce from (1.1), (1.4), and (1.5) that

\[
E'(t) + \int \left( \frac{\lambda(d\text{div} u)}{\theta} + 2\mu \left| \nabla (\text{div} u) \right|^2 + \nu \left| \text{curl} H \right|^2 \right) \, dx \\
= -\mu \int \left( |\omega|^2 + 2(d\text{div} u)^2 - 2|\nabla (\text{div} u)|^2 \right) \, dx \\
\leq 2\mu \int |\nabla u|^2 \, dx.
\]

Integrating (3.15) with respect to \( t \) over \( (0, T) \) and using (3.9), (1.2), and (3.11) yield

\[
\sup_{0 \leq t \leq T} E(t) \leq C_0 + 2\mu \int_0^T \int |\nabla u|^2 \, dx \, dt \leq CC_0^{1/4}.
\]
which as well as
\[(\rho - 1)^2 \geq 1 + \rho \log \rho - \rho = (\rho - 1)^2 \int_0^1 \frac{1 - \alpha}{\alpha(\rho - 1) + 1} d\alpha \geq \frac{(\rho - 1)^2}{2(2\rho + 1)} \quad (3.17)\]
infers (3.10). The proof of Lemma 3.1 is completed.

The following lemma establishes the estimate on \(A_1(\sigma(T))\).

**Lemma 3.2.** Under the conditions of Proposition 3.1, there exist positive constants \(K\) and \(\varepsilon_1\) both depending only on \(\mu, \lambda, \nu, R, \gamma, \hat{\rho}, \Omega,\) and \(M\) such that if \((\rho, u, \theta, H)\) is a smooth solution to the problem \((1.1) - (1.5)\) on \(\Omega \times (0, T)\) satisfying
\[0 < \rho \leq 2\hat{\rho}, \quad A_2(T) \leq 2C_0^{1/4}, \quad A_1(\sigma(T)) \leq 3K, \quad (3.18)\]
the following estimate holds:
\[A_1(\sigma(T)) \leq 2K, \quad (3.19)\]
provided \(C_0 \leq \varepsilon_1\).

**Proof.** First, it follows from (2.5), (2.6), and (2.25) that
\[\|H\|_{L^\infty} \leq C \|H\|_{L^6}^{1/2} \|\nabla H\|_{L^6}^{1/2} \leq C \|\nabla H\|_{L^2} + C \|\nabla H\|_{L^6}^{1/2} \|\text{curl} H\|_{L^2}^{1/2}, \quad (3.20)\]
which combined with (3.13), (2.5), and (2.25) shows that
\[\nu \|\text{curl} H\|_{L^2} \leq \|H_t\|_{L^2} + \|H \cdot \nabla u - u \cdot \nabla H - H \text{div} u\|_{L^2} \leq \|H_t\|_{L^2} + C \|\nabla u\|_{L^2} \|H\|_{L^\infty} + C \|u\|_{L^6} \|\nabla H\|_{L^3} \leq \|H_t\|_{L^2} + C \|\nabla u\|_{L^2} \|\nabla H\|_{L^2}^{1/2} (\|\text{curl} H\|_{L^2} + \|\nabla H\|_{L^2})^{1/2} \quad (3.21)\]
Thus we have further estimates on \(H\). More precisely, by (3.20) and (3.21), it holds
\[\|H\|_{L^\infty} \leq C \|\nabla H\|_{L^2} (\|\nabla u\|_{L^2} + 1) + C \|\nabla H\|_{L^2}^{1/2} \|H_t\|_{L^2}^{1/2}, \quad (3.22)\]
\[\|H\|_{L^\infty} \|\nabla H\|_{L^2} \leq C \|\nabla H\|_{L^2}^{3/2} (\|\nabla u\|_{L^2} + 1) + C \|\nabla H\|_{L^2}^{3/2} \|H_t\|_{L^2}^{1/2}. \quad (3.23)\]
Next, by virtue of (3.18), (2.5), and (3.10), we have that for any \(p \in [2, 6],
\[\|\rho \theta - 1\|_{L^p} = \|\rho(\theta - 1) + (\rho - 1)\|_{L^p} \leq \|\rho(\theta - 1)\|_{L^p}^{(6-p)/(2p)} \|\rho(\theta - 1)\|_{L^6}^{3(6-p)/(2p)} + \|\rho - 1\|_{L^p} \quad (3.24)\]
which along with (2.24) and (3.23) indicates
\[\|\nabla u\|_{L^6} \leq C \left( \|\rho \hat{u}\|_{L^2} + \|H_t\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} + C_0^{1/24} \right), \quad (3.25)\]
\[+ C \|\nabla H\|_{L^2} (\|\nabla H\|_{L^2} + \|\nabla u\|_{L^2} + 1).\]
Now, integrating (3.13) multiplied by $2H_t$ over $\Omega$ by parts, we deduce from (1.4) and (3.21) that

\[
(\nu \|\text{curl} H \|^2_{L^2})_t + \frac{3}{2} \|H_t\|^2_{L^2} \\
\leq C \int |H \cdot \nabla u - u \cdot \nabla H - H\text{div}u|^2 dx \\
\leq C \|\nabla u\|^2_{L^2} \|\nabla H\|_{L^2} (\|\text{curl}^2 H\|_{L^2} + \|\nabla H\|_{L^2}) \\
\leq \frac{1}{4} \|H_t\|^2_{L^2} + C(\|\nabla u\|^2_{L^2} + \|\nabla u\|^4_{L^2}) \|\nabla H\|^2_{L^2}.
\]  

(3.26)

Multiplying (3.12) by $2u_t$ and integrating the resulting equality by parts yield

\[
\frac{d}{dt} \int (\mu|\omega|^2 + (2\mu + \lambda)(\text{div}u)^2) \, dx + \int \rho|\dot{u}|^2 dx \\
\leq 2 \int P_d \text{div} u dx + \int \rho |u \cdot \nabla u|^2 dx + \int (2H \cdot \nabla H - \nabla |H|^2) \cdot u_t dx \\
= 2R \frac{d}{dt} \int (\rho \theta - 1) \text{div} u dx - 2 \int P_2 \text{div} u dx + \int \rho |u \cdot \nabla u|^2 dx \\
+ \frac{d}{dt} \int (2H \cdot \nabla H - \nabla |H|^2) \cdot u dx - \int (2H \cdot \nabla H - \nabla |H|^2)_t \cdot u dx \\
= 2R \frac{d}{dt} \int (\rho \theta - 1) \text{div} u dx - \frac{R^2}{2\mu + \lambda} \frac{d}{dt} \int (\rho \theta - 1)^2 dx \\
- \frac{4}{2\mu + \lambda} \int P_1 (2G + |H|^2) dx + \int \rho |u \cdot \nabla u|^2 dx \\
+ \frac{d}{dt} \int (2H \cdot \nabla H - \nabla |H|^2) \cdot u dx - \int (2H \cdot \nabla H - \nabla |H|^2)_t \cdot u dx,
\]

(3.27)

where in the last equality we have used (2.17).1.

Denote

\[
B_0(t) \triangleq \int \left( \mu|\omega|^2 + (2\mu + \lambda)(\text{div}u)^2 + \frac{R^2}{2\mu + \lambda} (\rho \theta - 1)^2 \right) dx \\
- \int (2R (\rho \theta - 1) \text{div} u + (2H \cdot \nabla H - \nabla |H|^2) \cdot u) \, dx,
\]

(3.28)

then (3.27) can be rewritten as

\[
B'_0(t) + \int \rho|\dot{u}|^2 dx \leq - \frac{4}{2\mu + \lambda} \int P_1 (2G + |H|^2) dx + \int \rho |u \cdot \nabla u|^2 dx \\
- \int (2H \cdot \nabla H - \nabla |H|^2)_t \cdot u dx \triangleq \sum_{i=1}^{3} I_i.
\]

(3.29)

$I_i (i = 1, 2, 3)$ can be estimated respectively.

Noticing that (1.3) implies

\[
P_1 = - \text{div}(Pu) - (\gamma - 1)P\text{div}u + (\gamma - 1)\kappa \Delta \theta \\
+ (\gamma - 1) \left( \lambda(\text{div}u)^2 + 2\mu |\mathcal{D}(u)|^2 + \nu |\text{curl} H|^2 \right),
\]

(3.30)
we thus obtain after using integration by parts, (2.20), (2.25), (3.24), (3.25), (3.18), (3.21), (3.23), and (3.10) that

\[
|I_1| \leq C \int (P|u| + |\nabla \theta|)(|\nabla G| + |\nabla |H|^2|)dx \\
+ C \int P|\nabla u||2G + |H|^2|dx + C \int (|\nabla u|^2 + |\nabla |H|^2|)2G + |H|^2|dx
\]

\[
\leq C(||\rho u||L^2 + ||\nabla \theta||L^2)(||\nabla G||L^2 + ||\nabla |H|^2||L^2)
\]

\[
+ C(||\rho(\theta - 1)||L^2 + ||\nabla \theta||L^2)(||\nabla G||L^2 + ||\nabla |H|^2||L^2)||\nabla u||L^2
\]

\[
+ C(||\nabla u||L^2 + ||\rho \theta - 1||L^2)||\nabla u||L^2
\]

(3.31)

\[
\leq \delta(||\nabla G||L^2 + ||\nabla |H|^2||L^2)(||\nabla u||L^2 + ||\nabla \theta||L^2 + ||\nabla |H||L^2 + 1)
\]

\[
\leq C\delta(||\rho^{1/2} \dot{u}||L^2 + ||H||L^2)
\]

\[
+ C\delta(||\nabla u||L^2 + ||\nabla \theta||L^2 + ||\nabla |H||L^2 + 1).
\]

Then, one deduces from (2.5), (3.18), and (3.25) that

\[
|I_2| \leq C||u||L^2 + ||\nabla u||L^2 + ||\nabla u||L^6
\]

\[
\leq \delta(||\rho^{1/2} \dot{u}||L^2 + ||H||L^2) + C(\delta)(||\nabla u||L^2 + ||\nabla \theta||L^2 + ||\nabla |H||L^2 + 1).
\]

(3.32)

Combining integration by parts with (1.1) and (3.22) leads to

\[
|I_3| = \left| \int (2(H \otimes H)_t : \nabla u - (|H|^2)_{\text{div}} u) dx \right|
\]

\[
\leq C \int |H| ||\nabla u||dx
\]

\[
\leq C||H||L^2 ||\nabla u||L^2
\]

\[
\leq \delta||H||L^2 + C(\delta)(||\nabla u||L^2 + ||\nabla \theta||L^2 + 1).
\]

(3.33)

Substituting (3.31)-(3.33) into (3.29) and adding the resulting inequality to (3.26), one derives after choosing \( \delta \) suitably small and using (3.21) that

\[
(\nu||\text{curl}H||L^2 + B_0)_t + \frac{2}{9} \nu^2||\text{curl}^2H||L^2 + \frac{1}{2} (||H||L^2 + ||\rho^{1/2} \dot{u}||L^2)
\]

\[
\leq C(||\nabla u||L^2 + ||\nabla \theta||L^2 + ||\nabla |H||L^2 + 1).
\]

(3.34)

Finally, by (1.1), (2.5), and (3.10), one arrives at

\[
\left| \int (2H \cdot \nabla H - |H|^2) \cdot u dx \right| = \left| \int (2(H \otimes H) : \nabla u - |H|^2_{\text{div}} u) dx \right|
\]

\[
\leq C||\nabla u||L^2 ||H||L^2 ||\nabla H||L^2
\]

\[
\leq CC_0^{1/16} ||\nabla u||L^2 ||\nabla H||L^2.
\]

(3.35)
which together with (3.34), (2.9), (3.18), (3.28), (3.24), and H"older's inequality yields
\[
\sup_{0 \leq t \leq \sigma(T)} (\|\nabla u\|_L^2 + \|\nabla H\|_L^2) + \int_0^{\sigma(T)} \left( \|\rho^{1/2} \dot{u}\|_L^2 + \|H_t\|_L^2 + \|\text{curl}^2 H\|_L^2 \right) dt \\
\leq CM^2 + CC_0^{1/4} + \hat{C}_1 C_0^{1/4} \sup_{0 \leq t \leq \sigma(T)} (\|\nabla u\|_L^4 + \|\nabla H\|_L^4) \\
\leq K + 9K^2 \hat{C}_1 C_0^{1/4} \\
\leq 2K,
\]
with \(K \triangleq CM^2 + C + 1\), provided
\[
C_0 \leq \varepsilon_1 \triangleq \min \left\{ 1, \left(9\hat{C}_1 K\right)^{-4} \right\}.
\]
The proof of Lemma 3.2 is completed. \(\square\)

Next, we adopt the approach due to Hoff \cite{15} (see also Huang-Li \cite{21}) to establish the elementary estimates on \(\dot{u}, \dot{\theta}, \text{ and } H_t\), which are the footstone for the estimates on \(A_3(T)\) and \(A_2(T)\), and the ideas in Cai-Li \cite{2} to deal with the emerging boundary terms.

**Lemma 3.3.** Under the conditions of Proposition 3.1, let \((\rho, u, \theta, H)\) be a smooth solution to the problem \((1.1)-(1.5)\) on \(\Omega \times (0, T)\) satisfying (3.4) with \(K\) as in Lemma 3.2. Then there exist positive constants \(C, C_1, \text{ and } C_2\) depending only on \(\mu, \lambda, \nu, k, R, \gamma, \hat{\rho}, \Omega, \text{ and } M\) such that, for any \(\beta, \eta \in (0, 1)\) and \(m \geq 0\), the following estimates hold:

\[
(\sigma B_1)'(t) + \frac{3}{2} \sigma \|\rho^{1/2} \dot{u}\|_L^2 \\
\leq CC_0^{1/4} \sigma' + 2\beta \sigma^2 \|\rho^{1/2} \dot{\theta}\|_L^2 + C \sigma^2 \|\nabla u\|_L^4 + C \sigma \|H_t\|_L^2 \\
+ C \beta^{-1} \left( \|\nabla u\|_L^2 + \|\nabla H\|_L^2 + \|\nabla \theta\|_L^2 \right), \tag{3.36}
\]

\[
(\sigma^m B_2)'(t) + C_1 \sigma^m \left( \|\nabla \dot{u}\|_L^2 + \|\nabla H_t\|_L^2 \right) \\
\leq C_2 \sigma^m \|\rho^{1/2} \dot{\theta}\|_L^2 + C \sigma^{-m} \sigma' \sigma^m \left( \|\rho^{1/2} \dot{u}\|_L^2 + \|H_t\|_L^2 \right) \\
+ C \left( \|\nabla u\|_L^2 + \|\nabla H\|_L^2 + \|\nabla \theta\|_L^2 \right) \\
+ C \sigma^m \left( \|\nabla u\|_L^4 + \|H_t\|_L^4 + \|\nabla \theta\|_L^4 \right), \tag{3.37}
\]

and

\[
(\sigma^m B_3)'(t) + \sigma^m \|\rho^{1/2} \dot{\theta}\|_L^2 \\
\leq C_\eta \sigma^m \left( \|\nabla \dot{u}\|_L^2 + \|\nabla H_t\|_L^2 \right) + C \sigma \|\nabla \theta\|_L^2 + C \sigma^m \|\nabla u\|_L^4 \\
+ C \eta^{-1} \sigma^m \left( \|\nabla u\|_L^2 + \|H_t\|_L^2 + \|\nabla \theta\|_L^2 + \|\nabla H\|_L^2 \right), \tag{3.38}
\]

where
\[
B_1(t) \triangleq \mu \|\omega\|_L^2 + (2\mu + \lambda) \|\text{div} u\|_L^2 - 2R \int \text{div}(\rho \theta - 1) dx - \int (2H \cdot \nabla H - \nabla |H|^2) \cdot u dx, \tag{3.39}
\]

\[
B_2(t) \triangleq \|\rho^{1/2} \dot{u}\|_L^2 + \|H_t\|_L^2 + 2 \int_{\partial \Omega} (u \cdot \nabla n \cdot u) GdS, \tag{3.40}
\]

and
\[
B_3(t) \triangleq \frac{\gamma - 1}{R} \left( \kappa \|\nabla \theta\|_L^2 - 2 \int (\lambda (\text{div} u)^2 + 2\mu |\nabla u|^2 + \nu |\text{curl} H|^2) \theta dx \right). \tag{3.41}
\]
Proof. The proof is divided into the following three parts.

Part I: The proof of (3.36).

First, we can rewrite (3.12) as
\[ \rho \dot{u} + \mu \nabla \times \text{curl} u - \nabla G - H \cdot \nabla H = 0. \] (3.42)

Multiplying (3.42) by \( \sigma \dot{u} \) and integrating the resulting equality by parts yield
\[
\int \sigma \rho |\dot{u}|^2 dx = \int (\sigma \dot{u} \cdot \nabla G - \sigma \mu \nabla \times \text{curl} u \cdot \dot{u} + \sigma H \cdot \nabla H \cdot \dot{u}) dx \\
= \int_{\partial \Omega} \sigma (u \cdot \nabla u \cdot n) G ds - \int \sigma \text{div} \dot{u} G dx - \mu \int \sigma \text{curl} u \cdot \text{curl} \dot{u} dx \\
+ \int \sigma H \cdot \nabla H \cdot \dot{u} dx \triangleq \sum_{i=1}^{4} M_i. \] (3.43)

Note that (1.4) implies
\[ u \cdot \nabla u \cdot n = -u \cdot \nabla n \cdot u \quad \text{on} \ \partial \Omega, \] (3.44)

and
\[ u = u^\perp \times n \quad \text{on} \ \partial \Omega, \] (3.45)

with \( u^\perp = -u \times n \). Then, the combination of (2.12) and (3.44) leads to
\[
M_1 = -\int_{\partial \Omega} \sigma (u \cdot \nabla n \cdot u) G ds \\
\leq C \sigma \|u\|_{L^2(\partial \Omega)}^2 \|G\|_{L^2(\partial \Omega)} \\
\leq C \sigma \|\nabla u\|_{L^2}^2 \|\nabla G\|_{L^2} \\
\leq \delta \sigma \left( \|\rho \|^1/2 \|\dot{u}\|_{L^2}^2 + \sigma \|H_t\|_{L^2}^2 + C(\delta) \sigma \|\nabla u\|_{L^2}^2 \right), \] (3.46)

where in the last inequality we have used the following simple facts:
\[
\sup_{t \in [0,T]} (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) \leq A_1(\sigma(T)) + A_3(T) \leq C, \] (3.47)

and
\[
\|\nabla G\|_{L^2} \leq C (\|\rho \dot{u}\|_{L^2} + \|H_t\|_{L^2} + \|\nabla H\|_{L^2}) \] (3.48)

owing to (2.20), (3.23), and (3.47).

Notice that
\[ P_t = (R \rho \theta)_t = R \rho \dot{\theta} - \text{div} (Pu), \] (3.49)
which as well as direct calculations shows
\[ \text{div}(\hat{u}G) = (\text{div}(u \cdot \nabla u))(2\mu + \lambda)\text{div}u - R(\rho \theta - 1) - \frac{1}{2}|H|^2 \]
\[ = \frac{2\mu + \lambda}{2}(\text{div}(u \cdot \nabla u))_t - (R(\rho \theta - 1)\text{div}u)_t + R\rho \dot{\theta} \text{div}u - \text{div}(Pu)\text{div}u \]
\[ + (2\mu + \lambda)\text{div}(u \cdot \nabla u)\text{div}u - R(\rho \theta - 1)\text{div}(u \cdot \nabla u) \]
\[ - \frac{1}{2}|H|^2\text{div}u)_t + H \cdot H_t \text{div}u - \frac{1}{2}|H|^2\text{div}(u \cdot \nabla u) \]
\[ = \frac{2\mu + \lambda}{2}(\text{div}(u \cdot \nabla u))_t - (R(\rho \theta - 1)\text{div}u)_t + R\rho \dot{\theta} \text{div}u \]
\[ + (2\mu + \lambda)\nabla u : (\nabla u)^\text{tr} \text{div}u + \frac{2\mu + \lambda}{2}u \cdot \nabla(\text{div}(u \cdot \nabla u))^2 \]
\[ - \text{div}(R(\rho \theta - 1)\text{div}u) - R(\rho \theta - 1)\nabla u : (\nabla u)^\text{tr} - R(\text{div}u)^2 \]
\[ - \frac{1}{2}|H|^2\text{div}(u \cdot \nabla u) = (\nabla u)^\text{tr} \text{div}u - \frac{1}{2}|H|^2u \cdot \nabla u. \]

Combining this with integration by parts, (2.5), and (3.6) gives that for any \( \beta \in (0, 1] \),
\[ M_2 = -\frac{1}{2} \left( \int \sigma ((2\mu + \lambda)(\text{div}u)^2 - 2R(\rho \theta - 1)\text{div}u - |H|^2\text{div}u) \, dx \right)_t \]
\[ + \frac{1}{2}\sigma' \int ((2\mu + \lambda)(\text{div}u)^2 - 2R(\rho \theta - 1)\text{div}u - |H|^2\text{div}u) \, dx \]
\[ - R\sigma \int \rho \dot{\theta} \text{div}udx - (2\mu + \lambda)\int \nabla u : (\nabla u)^\text{tr} \text{div}udx \]
\[ + \frac{2\mu + \lambda}{2}\sigma \int (\text{div}u)^3 \, dx + R\sigma \int (\rho \theta - 1)\nabla u : (\nabla u)^\text{tr} \, dx \]
\[ + R\sigma \int (\text{div}u)^2 \, dx - \sigma \int H \cdot H_t \text{div}udx + \frac{1}{2}\sigma \int |H|^2\nabla u : (\nabla u)^\text{tr} \, dx \]
\[ + \frac{1}{2}\sigma \int |H|^2u \cdot \nabla \text{div}udx \]
\[ \leq -\frac{1}{2} \left( \int \sigma ((2\mu + \lambda)(\text{div}u)^2 - 2R(\rho \theta - 1)\text{div}u - |H|^2\text{div}u) \, dx \right)_t \]
\[ + C\sigma' |\rho \theta - 1|^2_{L^2} + \beta \sigma^2 |\rho^{1/2} \dot{\theta}|^2_{L^2} + C\beta^{-1}||\nabla u||^2_{L^2} + \sigma ||H_t||^2_{L^2} + C||H||^4_{L^4} \]
\[ + C\sigma^2 ||\nabla u||^4_{L^4} + C\sigma \int \theta |\nabla u|^2 \, dx + C\sigma ||u||^2_{L^2} ||\nabla u||_{L^2} ||H||_{L^6} ||\nabla H||_{L^6} \]
\[ \leq -\frac{1}{2} \left( \int \sigma ((2\mu + \lambda)(\text{div}u)^2 - 2R(\rho \theta - 1)\text{div}u - |H|^2\text{div}u) \, dx \right)_t \]
\[ + CC_0^{1/4}\sigma' + \beta \sigma^2 |\rho^{1/2} \dot{\theta}|^2_{L^2} + C\delta \sigma |\rho^{1/2} \dot{u}|^2_{L^2} + C\sigma ||H_t||^2_{L^2} \]
\[ + C\sigma^2 ||\nabla u||^4_{L^4} + C(\delta)\beta^{-1}||\nabla u||^2_{L^2} + ||\nabla H||^2_{L^2} + ||\nabla \theta||^2_{L^2} , \]
where in the last inequality we have used (3.47), (3.24), and the following facts:
\[ \int \theta |\nabla u|^2 \, dx \leq \int |\theta - 1||\nabla u|^2 \, dx + \int |\nabla u|^2 \, dx \]
\[ \leq C ||\theta - 1||_{L^6} ||\nabla u||^{3/2}_{L^2} ||\nabla u||^{1/2}_{L^2} + ||\nabla u||^2_{L^2} \]
\[ \leq ||\nabla u||^2_{L^2} + C||\nabla \theta||_{L^2} ||\nabla u||^{3/2}_{L^2} \]
\[ \cdot (||\rho \dot{u}||_{L^2} + ||H_t||_{L^2} + ||\nabla H||_{L^2} + ||\nabla u||_{L^2} + ||\nabla \theta||_{L^2} + 1)^{1/2} \]
\[ \leq \delta \left( ||\nabla u||^2_{L^2} + ||\rho^{1/2} \dot{u}||^2_{L^2} + ||H_t||^2_{L^2} \right) + C(\delta) ||\nabla u||^2_{L^2} \]
due to (2.5), (3.25), (3.6), (3.47), and
\[ \| \nabla^2 H \|_{L^2} \leq C(\| H_t \|_{L^2} + \| \nabla H \|_{L^2}) \] (3.52)
due to (2.25), (3.21), and (3.47).

Analogously, we can deal with the term \( M_4 \) as follows
\[
M_4 = \left( \int \sigma H \cdot \nabla H \cdot u \, dx \right)_t - \sigma' \int \text{div}(H \otimes H) \cdot u \, dx
- \sigma \int \text{div}(H \otimes H) u \, dx + \int \sigma H \cdot \nabla H \cdot (u \cdot \nabla u) \, dx
\]
\[
\leq \left( \int \sigma H \cdot \nabla H \cdot u \, dx \right)_t + C\sigma' \| \nabla u \|_{L^2} \| H \|_{L^2}^{1/2} \| \nabla H \|_{L^2}^{3/2}
+ C\sigma \| H_t \|_{L^2} \| \nabla H \|_{L^2} \| u \|_{L^2} \| \nabla u \|_{L^2} \| H \|_{L^6} \| \nabla H \|_{L^6}
\leq \left( \int \sigma H \cdot \nabla H \cdot u \, dx \right)_t + C\| H_t \|_{L^2}^2 + \sigma^2 \| \nabla u \|_{L^4}^4
+ C(\| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2}^2).
\] (3.53)

Finally, some direct calculations infer that
\[
M_3 = -\frac{\mu}{2} \int \sigma |\text{curl} u|^2 \, dx - \mu \sigma \int \text{curl} u \cdot \text{curl}(u \cdot \nabla u) \, dx
= -\frac{\mu}{2} (\sigma \| \text{curl} u \|_{L^2})_t + \frac{\mu}{2} \sigma' \| \text{curl} u \|_{L^2}^2 - \mu \sigma \int \text{curl} u \cdot (\nabla u \times \nabla u) \, dx
+ \frac{\mu}{2} \sigma \int |\text{curl} u|^2 \text{div} u \, dx
\leq -\frac{\mu}{2} (\sigma \| \text{curl} u \|_{L^2})_t + C\| \nabla u \|_{L^2}^2 + C\sigma^2 \| \nabla u \|_{L^4}^4.
\] (3.54)

Putting (3.46), (3.50), (3.53), and (3.54) into (3.43) and choosing \( \delta \) suitably small, we arrive at (3.36) directly.

Part II: The proof of (3.37).

For \( m \geq 0 \), operating \( \sigma^m \hat{\omega}^j \left[ \partial_t + \text{div}(u) \right] \) to (3.42)\(^j\) and integrating the resulting equality with respect to \( x \) over \( \Omega \) by parts, one has
\[
\left( \frac{\sigma^m}{2} \int \rho |\hat{\omega}|^2 \, dx \right)_t - \frac{m}{2} \sigma^{m-1} \sigma' \int \rho |\hat{\omega}|^2 \, dx
= \int_{\partial \Omega} \sigma^m \hat{\omega} \cdot n G_t \, dS - \int \sigma^m [\text{div} \hat{\omega} G_t + u \cdot \nabla \hat{\omega} \cdot \nabla G] \, dx
- \mu \int \sigma^m \hat{\omega}^j \left[ (\nabla \times \text{curl} u)^j_t + \text{div}(u(\nabla \times \text{curl} u)^j) \right] \, dx
+ \int \sigma^m \left[ \hat{\omega} \cdot \text{div}(H \otimes H)_t + \hat{\omega}^j \text{div}(H \cdot \nabla H^j u) \right] \triangleq \sum_{i=1}^4 N_i.
\] (3.55)
First, we need to deal with the boundary integral $N_1$. By (1.4) and (3.44), it holds

$$N_1 = - \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) G_t dS$$

$$= - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) GdS \right)_t + m \sigma^{m-1} \sigma' \int_{\partial \Omega} (u \cdot \nabla n \cdot u) GdS + \int_{\partial \Omega} \sigma^m (\tilde{u} \cdot \nabla n \cdot \tilde{u}) GdS$$

$$- \int_{\partial \Omega} \sigma^m G (u \cdot \nabla) u \cdot \nabla n \cdot udS - \int_{\partial \Omega} \sigma^m G u \cdot \nabla n \cdot (u \cdot \nabla) udS$$

$$\leq - \left( \int_{\partial \Omega} \sigma^m (u \cdot \nabla n \cdot u) GdS \right)_t + C \sigma^{m-1} \sigma' \| \nabla u \|_{L^2}^2 \| \nabla G \|_{L^2}$$

$$+ \delta \sigma^m \| \nabla \tilde{u} \|_{L^2}^2 + C(\delta) \sigma^m \| \nabla u \|_{L^2}^2 \| \nabla G \|_{L^2}^2$$

$$- \int_{\partial \Omega} \sigma^m G (u \cdot \nabla) u \cdot \nabla n \cdot udS - \int_{\partial \Omega} \sigma^m G u \cdot \nabla n \cdot (u \cdot \nabla) udS,$$

where we have used the following estimates:

$$\left| \int_{\partial \Omega} (\tilde{u} \cdot \nabla n \cdot u + u \cdot \nabla n \cdot \tilde{u}) GdS \right| \leq C \| \tilde{u} \|_{L^1(\partial \Omega)} \| u \|_{L^2(\partial \Omega)} \| G \|_{L^4(\partial \Omega)}$$

$$\leq C \| \tilde{u} \|_{L^2} \| u \|_{L^2} \| \nabla G \|_{L^2},$$

and

$$\int_{\partial \Omega} (u \cdot \nabla n \cdot u) GdS \leq C \| \nabla u \|_{L^2}^2 \| \nabla G \|_{L^2}$$

(3.57) owing to (2.12).

For the last two boundary terms in (3.56), we adopt the ideas in (2) to handle. In fact, combining (3.45) with (2.5) and integration by parts infers

$$- \int_{\partial \Omega} G (u \cdot \nabla) u \cdot \nabla n \cdot udS$$

$$= - \int_{\partial \Omega} G u^1 \times n \cdot \nabla u^1 \nabla_i n \cdot udS$$

$$= - \int_{\partial \Omega} G n \cdot (\nabla u^1 \times u^1) \nabla_i n \cdot udS$$

$$= - \int \text{div}(G (\nabla u^1 \times u^1) \nabla_i n \cdot u) dx$$

$$= - \int \nabla (\nabla_i n \cdot u G) \cdot (\nabla u^1 \times u^1) dx - \int \text{div}(\nabla u^1 \times u^1) \nabla_i n \cdot u G dx$$

$$= - \int \nabla (\nabla_i n \cdot u G) \cdot (\nabla u^1 \times u^1) dx + \int G \nabla u^1 \cdot \nabla \times u^1 \nabla_i n \cdot udx$$

$$\leq C \int | \nabla G | | \nabla u | | u |^2 dx + C \int |G| (| \nabla u |^2 | u | + | \nabla u | | u |^2) dx$$

$$\leq C \| \nabla G \|_{L^6} \| \nabla u \|_{L^2} \| u \|_{L^6}^2 + C \| G \|_{L^6} \| \nabla u \|_{L^2}^2 \| u \|_{L^6} + C \| G \|_{L^6} \| \nabla u \|_{L^2} \| u \|_{L^6}^2$$

$$\leq \delta \| \nabla G \|_{L^6}^2 + C(\delta) \| \nabla u \|_{L^2}^4 + C \| \nabla u \|_{L^2}^4 + C \| \nabla G \|_{L^2}^2 (\| \nabla u \|_{L^2}^2 + 1).$$

Similarly,

$$- \int_{\partial \Omega} G u \cdot \nabla n \cdot (u \cdot \nabla) udS$$

$$\leq \delta \| \nabla G \|_{L^6}^2 + C(\delta) \| \nabla u \|_{L^2}^4 + C \| \nabla u \|_{L^2}^4 + C \| \nabla G \|_{L^2}^2 (\| \nabla u \|_{L^2}^2 + 1).$$

(3.59)
Next, by virtue of (2.17) and (3.49), we obtain
\[ G_t = (2\mu + \lambda)\text{div}\dot{u} - (2\mu + \lambda)\text{div}(u \cdot \nabla u) - R\rho\dot{\theta} + \text{div}(Pu) - H \cdot H_t \]
\[ = (2\mu + \lambda)\text{div}\dot{u} - (2\mu + \lambda)\nabla u : (\nabla u)^{\text{tr}} - u \cdot \nabla G + P\text{div}\dot{u} - R\rho\dot{\theta} - u \cdot \nabla H \cdot H - H \cdot H_t,\]
which together with (3.6) leads to
\[ N_2 = -(2\mu + \lambda) \int \sigma^m(\text{div}\dot{u})^2 dx + (2\mu + \lambda) \int \sigma^m\text{div}\dot{u} : (\nabla u)^{\text{tr}} dx \]
\[ + \int \sigma^m\text{div}\dot{u} \cdot \nabla G dx - \int \sigma^m\text{div}\dot{u} P\text{div}\dot{u} dx \]
\[ + R \int \sigma^m\text{div}\dot{u} \rho\dot{\theta} dx - \int \sigma^m u \cdot \nabla \dot{u} \cdot \nabla G dx \]
\[ + \int \sigma^m\text{div}\dot{u} \cdot \nabla H \cdot H dx + \int \sigma^m\text{div}\dot{u} H \cdot H_t dx \]
\[ \leq -(2\mu + \lambda) \int \sigma^m(\text{div}\dot{u})^2 dx \]
\[ + C\sigma^m \|\nabla \dot{u}\|_{L^2} \|\nabla u\|_{L^2}^4 + C\sigma^m \|\nabla \dot{u}\|_{L^2} \|\nabla G\|_{L^6}^{1/2} \|\nabla G\|_{L^6}^{1/2} \|u\|_{L^6} \]
\[ + C\sigma^m \|\nabla \dot{u}\|_{L^2} \|\theta\nabla u\|_{L^2} + C\sigma^m \|\nabla \dot{u}\|_{L^2} \|\rho^{1/2}\dot{\theta}\|_{L^2} \]
\[ + C\sigma^m \|\nabla \dot{u}\|_{L^2} \|u\|_{L^6} \|\nabla H\|_{L^6} + C\sigma^m \|\nabla \dot{u}\|_{L^2} \|H_t\|_{L^2} \|H\|_{L^\infty}. \tag{3.60} \]

Analogously, it holds that
\[ N_4 = - \int \sigma^m (H \otimes H)_t : \nabla \dot{u} dx - \int \sigma^m u \cdot \nabla \dot{u} H \cdot \nabla H^i dx \]
\[ \leq C\sigma^m \|\nabla \dot{u}\|_{L^2} \|u\|_{L^6} \|\nabla H\|_{L^6} + C\sigma^m \|\nabla \dot{u}\|_{L^2} \|H_t\|_{L^2} \|H\|_{L^\infty}. \tag{3.61} \]

Observe that
\[ \text{curl}\dot{u}_t = \text{curl}\dot{u} - u \cdot \nabla \text{curl}\dot{u} - \nabla u^i \times \nabla_i u, \]
which as well as some direct calculations shows that
\[ N_3 = -\mu \int \sigma^m |\text{curl}\dot{u}|^2 dx + \mu \int \sigma^m \text{curl}\dot{u} \cdot (\nabla u^i \times \nabla_i u) dx \]
\[ + \mu \int \sigma^m u \cdot \nabla \text{curl}\dot{u} \cdot \text{curl}\dot{u} dx + \mu \int \sigma^m u \cdot \nabla \dot{u} \cdot (\nabla \times \text{curl}\dot{u}) dx \]
\[ \leq -\mu \int \sigma^m |\text{curl}\dot{u}|^2 dx + \delta\sigma^m (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla \text{curl}\dot{u}\|^2_{L^2}) \]
\[ + C(\delta)\sigma^m \|\nabla u\|_{L^4}^4 + C(\delta)\sigma^m \|\nabla u\|_{L^2}^4 \|\nabla \text{curl}\dot{u}\|_{L^2}^2. \tag{3.62} \]

Finally, combining Lemma 2.11 with (2.5), (2.6), (3.23), (3.47), and (3.52) yields
\[ \|\nabla G\|_{L^6} \leq C\|\rho\dot{u}\|_{L^6} + C\|H \cdot \nabla H\|_{L^6} \]
\[ \leq C\|\nabla \dot{u}\|_{L^2} + C\|\nabla H\|_{L^2}^{3/2} \|\nabla^2 H\|_{L^2}^{3/2}, \tag{3.63} \]
\[ \|\nabla \omega\|_{L^6} \leq C(\|\nabla \dot{u}\|_{L^2} + \|\rho\dot{u}\|_{L^2} + \|\nabla u\|_{L^2} + \|\nabla H\|_{L^2}) + C \|H_t\|_{L^2}^{3/2}. \tag{3.64} \]
Thus, substituting (3.56) and (3.60)–(3.62) into (3.55), it follows from (3.58), (3.59), (3.6), (3.10), (3.63), (3.64), (3.22), (3.47), (3.48), and (3.52) that

\[
\left(\frac{\sigma_m}{2}\|\rho^{1/2}\dot{u}\|_{L^2}^2\right)_t + (2\mu + \lambda)\sigma_m\|\text{div}\dot{u}\|_{L^2}^2 + \mu\sigma_m\|\text{curl}\dot{u}\|_{L^2}^2 \leq -\left(\int_{\partial\Omega} \sigma_m^m(u \cdot \nabla_n \cdot u)GdS\right)_t + C\delta\sigma_m\|\nabla\dot{u}\|_{L^2}^2 + C(\delta)\sigma_m\|\rho^{1/2}\dot{\theta}\|_{L^2}^2
\]

\[
+ C(\delta)(\sigma_m^{-1}\sigma' + \sigma_m)(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2)
\]

\[
+ C(\delta)(\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) + C(\delta)\sigma_m(\|\nabla u\|_{L^1}^2 + \|H_t\|_{L^2}^2 + \|\nabla H\|_{L^2}^2).
\]

Next, it remains to estimate \(\|\nabla H_t\|_{L^2}\). Notice that (3.13) implies

\[
\begin{align*}
H_t + \nu \nabla \times (\text{curl} H_t) &= (H \cdot \nabla u - \dot{u} \cdot \nabla H - H \text{div} u)_t, & \text{in } \Omega, \\
H_t \cdot n &= 0, & \text{curl} H_t \times n &= 0, & \text{on } \partial\Omega, \\
H_t \to 0, & \text{as } |x| \to 0.
\end{align*}
\]

Multiplying (3.66) by \(\sigma^m H_t\) with \(m \geq 0\) and integrating by parts, one has

\[
\begin{align*}
&\left(\frac{\sigma^m}{2}\|H_t\|_{L^2}^2\right)_t + \nu\sigma^m\|\text{curl} H_t\|_{L^2}^2 - \frac{m}{2}\sigma^m\|\dot{H}_t\|_{L^2}^2 \\
&= \int \sigma^m(H_t \cdot \nabla u - \dot{u} \cdot \nabla H - H \text{div} u) \cdot H_t dx \\
&+ \int \sigma^m(H \cdot \nabla \dot{u} - \dot{u} \cdot \nabla H - H \text{div} \dot{u}) \cdot H_t dx \\
&- \int \sigma^m(H \cdot \nabla (u \cdot \nabla u) - u \cdot \nabla u \cdot \nabla H - H \text{div}(u \cdot \nabla u)) \cdot H_t dx \\
&= \frac{3}{2} \sum_{i=1}^3 K_i.
\end{align*}
\]

Combining (2.5) with direct calculations and (3.44) leads to

\[
K_1 \leq C\sigma^m\|H_t\|_{L^2}^{1/2}\|\nabla H_t\|_{L^2}^{1/2}(\|H_t\|_{L^6}\|\nabla u\|_{L^2} + \|\nabla H_t\|_{L^2}\|u\|_{L^6})
\]

\[
\leq \delta\sigma^m\|\nabla H_t\|_{L^2}^2 + C(\delta)\sigma^m\|\nabla u\|_{L^2}^4\|H_t\|_{L^2}^2,
\]

\[
K_2 \leq C\sigma^m\|H_t\|_{L^2}^{1/2}\|\nabla H_t\|_{L^2}^{1/2}(\|H_t\|_{L^6}\|\nabla \dot{u}\|_{L^2} + \|\nabla H\|_{L^2}\|\dot{u}\|_{L^6})
\]

\[
\leq \delta\sigma^m(\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) + C(\delta)\sigma^m\|\nabla H\|_{L^2}^4\|H_t\|_{L^2}^2,
\]

and

\[
K_3 = \int \sigma^m H \cdot \nabla H_t \cdot (u \cdot \nabla u) dx + \int_{\partial\Omega} \sigma^m H \cdot H_t (u \cdot \nabla u \cdot n)dS
\]

\[
- \int \sigma^m u \cdot \nabla u \cdot \nabla H_t \cdot H dx
\]

\[
\leq - \int_{\partial\Omega} \sigma^m H \cdot H_t (u \cdot \nabla n \cdot u)dS + C\sigma^m\|H\|_{L^6}\|\nabla H_t\|_{L^2}\|\nabla u\|_{L^6}\|u\|_{L^6}
\]

\[
\leq \delta\sigma^m\|\nabla H_t\|_{L^2}^2 + C(\delta)\sigma^m(\|\nabla H\|_{L^2}^2 + \|\nabla H\|_{L^6}^8)(\|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^2}^4)
\]

\[
+ C(\delta)\sigma^m\|\nabla H\|_{L^2}^2\|\nabla u\|_{L^2}^2(\|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2),
\]

where in the last inequality we have used (3.25) and the following fact:

\[
\int_{\partial\Omega} H \cdot H_t (u \cdot \nabla n \cdot u)dS \leq C\|H\|_{L^6(\partial\Omega)}\|H_t\|_{L^6(\partial\Omega)}\|u\|_{L^6(\partial\Omega)}^2
\]

\[
\leq C(\|\nabla H\|_{L^2}\|\nabla H_t\|_{L^2}\|\nabla u\|_{L^2}^2).
\]
owing to (2.12). Putting (3.68)–(3.70) into (3.67) and using (3.47), we derive
\[
\left( \frac{\sigma^m}{2} \left\| H_t \right\|_{L^2}^2 \right)_t + \nu \sigma^m \left\| \text{curl} H_t \right\|_{L^2}^2 \\
\leq 3 \delta \sigma^m (\left\| \nabla u \right\|_{L^2}^2 + \left\| \nabla H_t \right\|_{L^2}^2) + C(\delta)(\sigma^m - 1 \sigma^\prime + \sigma^m) \| H_t \|_{L^2}^2 \\
+ C(\delta) \sigma^m \| \rho^{1/2} u \|_{L^2}^2 + C(\delta)(\| \nabla u \|_{L^2}^2 + \| \nabla \theta \|_{L^2}^2).
\] (3.71)

Adding (3.65) to (3.71) and using (2.14), (2.9), and (3.47), one concludes (3.37) after choosing \( \delta \) small enough.

**Part III: The proof of (3.38).**

For \( m \geq 0 \), multiplying (1.13) by \( \sigma^m \theta \) and integrating the resulting equality over \( \Omega \), it holds that
\[
\frac{\kappa \sigma^m}{2} \left\| \nabla \theta \right\|_{L^2}^2 + \frac{R \sigma^m}{\gamma - 1} \int \rho \| \dot{\theta} \|_{L^2}^2 dx \\
= -\kappa \sigma^m \int \nabla \theta \cdot \nabla (u \cdot \nabla \theta) dx + \lambda \sigma^m \int (\text{div} u)^2 \theta dx + 2 \mu \sigma^m \int |\mathcal{D}(u)|^2 \dot{\theta} dx \\
+ \nu \sigma^m \int |\text{curl} H|^2 \dot{\theta} dx - R \sigma^m \int \rho \theta \text{div} u \dot{\theta} dx \triangleq \sum_{i=1}^5 J_i.
\] (3.72)

First, by (2.5), one gets
\[
J_1 \leq C \sigma^m \| \nabla u \|_{L^2} \| \nabla \theta \|_{L^2} \| \nabla^2 \theta \|_{L^2}^{3/2} \\
\leq \delta \sigma^m \| \rho^{1/2} \dot{\theta} \|_{L^2}^2 + C \sigma^m \left( \| \nabla u \|_{L^4}^4 + \| \nabla H \|_{L^2} \| \nabla^2 H \|_{L^2}^{3/2} + \| \theta \nabla u \|_{L^2}^2 \right) \\
+ C(\delta) \sigma^m \| \nabla u \|_{L^2}^2 \| \nabla \theta \|_{L^2}^2,
\] (3.73)

where in the last inequality we have used the following estimate:
\[
\| \nabla^2 \theta \|_{L^2} \leq C \left( \| \rho \dot{\theta} \|_{L^2} + \| \nabla u \|_{L^2}^2 + \| \nabla H \|_{L^2} \| \nabla^2 H \|_{L^2}^{3/2} + \| \theta \nabla u \|_{L^2} \right),
\] (3.74)

which is derived from the combination of Lemma 2.10 and the following elliptic problem:
\[
\begin{cases}
\kappa \Delta \theta - \frac{R}{\gamma - 1} \rho \dot{\theta} \\
= R \rho \text{div} u - \lambda (\text{div} u)^2 - 2 \mu |\mathcal{D}(u)|^2 - \nu |\text{curl} H|^2, \quad \text{in } \Omega \times [0, T], \\
\nabla \theta \cdot n = 0, \quad \text{on } \partial \Omega \times [0, T], \\
\nabla \theta \rightarrow 0, \quad \text{as } |x| \rightarrow \infty.
\end{cases}
\] (3.75)

Next, by (2.5), we obtain that for any \( \eta, \delta \in (0, 1] \),
\[
J_2 = \lambda \sigma^m \int (\text{div} u)^2 \theta_t dx + \lambda \sigma^m \int (\text{div} u)^2 u \cdot \nabla \theta dx \\
= \lambda \sigma^m \left( \int (\text{div} u)^2 \theta_t dx \right)_t - 2 \lambda \sigma^m \int \theta \text{div} \text{div}( \dot{u} - u \cdot \nabla u) dx \\
+ \lambda \sigma^m \int (\text{div} u)^2 u \cdot \nabla \theta dx \\
= \lambda \sigma^m \left( \int (\text{div} u)^2 \theta_t dx \right)_t - 2 \lambda \sigma^m \int \theta \text{div} \text{div} \dot{u} dx \\
+ 2 \lambda \sigma^m \int \theta \text{div} \partial_i \dot{u} \partial_j u_t dx + \lambda \sigma^m \int u \cdot \nabla (\theta (\text{div} u)^2) dx \\
\leq \lambda \left( \sigma^m \left( \int (\text{div} u)^2 \theta_t dx \right)_t - \lambda m \sigma^{m-1} \sigma^\prime \int (\text{div} u)^2 \theta dx \right) \\
+ \eta \lambda \sigma^m \| \nabla u \|_{L^2}^2 + C \eta^{-1} \sigma^m \| \theta \nabla u \|_{L^2}^2 + C \sigma^m \| \nabla u \|_{L^4}^4,
\] (3.76)
we deduce from (1.2), (3.47), and (3.52) that (3.38) holds.

Under the conditions of Proposition 3.1, there exists a positive constant $\varepsilon_2$ depending only on $\mu$, $\lambda$, $\nu$, $\kappa$, $R$, $\gamma$, $\hat{\rho}$, $\Omega$, and $M$ such that if $(\rho, u, \theta, H)$ is a smooth solution to the problem (1.1)–(1.5) on $\Omega \times (0, T]$ satisfying (3.6) with $K$ as in Lemma 3.2, the following estimate holds:

$$A_3(T) + A_4(T) \leq C_0^{1/6},$$

provided $C_0 \leq \varepsilon_2$.

**Lemma 3.4.** Under the conditions of Proposition 3.1, there exists a positive constant $\varepsilon_2$ depending only on $\mu$, $\lambda$, $\nu$, $\kappa$, $R$, $\gamma$, $\hat{\rho}$, $\Omega$, and $M$ such that if $(\rho, u, \theta, H)$ is a smooth solution to the problem (1.1)–(1.5) on $\Omega \times (0, T]$ satisfying (3.6) with $K$ as in Lemma 3.2, the following estimate holds:

$$A_3(T) + A_4(T) \leq C_0^{1/6},$$

provided $C_0 \leq \varepsilon_2$.

**Proof.** First, multiplying (3.26) by $\sigma$ and integrating the resulting inequality with respect to $t$, one gets after using (2.19), (3.6), (3.47), and (3.21) that

$$\sup_{0 \leq t \leq T} \sigma \|\nabla H\|_{L^2}^2 + \int_0^T \sigma (\|\text{curl}^2 H\|_{L^2}^2 + \|H_t\|_{L^2}^2) dt \leq C C_0^{1/4}. \tag{3.81}$$

Note that by virtue of (2.10), (2.22), (2.23), (3.24), (2.5), (3.10), (3.23), and (3.47), it holds

$$\|\nabla u\|_{L^4}^4 \leq C \|G\|_{L^4} + C \|\text{curl} u\|_{L^4}^4 + C \|\rho \theta - 1\|_{L^4}^4 + C \|H^2\|_{L^4}^4 + C \|\nabla u\|_{L^2}^4.$$
which along with (3.6) leads to
\[
\sigma \|\nabla u\|_{L^4}^4 \leq C C_0^{1/12} \|\rho^{1/2} \hat{u}\|_{L^2}^2 + C \sigma \|\rho - 1\|_{L^4}^4 \\
+ C (\|\nabla \theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla H\|_{L^2}^2).
\] (3.83)

Combining this with (3.36) shows that
\[
(\sigma B_1)'(t) + \sigma \|\rho^{1/2} \hat{u}\|_{L^2}^2 \\
\leq C C_0^{1/4} \sigma' + 2 \beta \sigma^2 \|\rho^{1/2} \hat{\theta}\|_{L^2}^2 + C \sigma^2 \|\rho - 1\|_{L^4}^4 \\
+ C \sigma \|H_t\|_{L^2}^2 + C \beta^{-1} (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2),
\] provided that \(C_0 \leq \varepsilon_{2.1} \equiv \min\{1, (2C)^{-12}\} \).

Next, for \(C_2\) as in (3.37), adding (3.38) multiplied by \(C_2 + 1\) to (3.37) and choosing \(\eta\) suitably small, one derives
\[
(\sigma^m \varphi)'(t) + C_1 \sigma^m (\|\nabla \hat{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) + \sigma^m \|\rho^{1/2} \hat{\theta}\|_{L^2}^2 \\
\leq C (\sigma^{m-1} \sigma' + \sigma^m)(\|\rho^{1/2} \hat{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2) \\
+ C (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) \\
+ C \sigma^m (\|\nabla \theta\|_{L^2}^4 + \|H_t\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\theta \nabla u\|_{L^2}^2),
\] where \(\varphi(t)\) is defined by
\[
\varphi(t) \triangleq B_2(t) + (C_2 + 1) B_3(t).
\] (3.86)

Taking \(m = 2\) in (3.85) and using (3.83) and (3.6) lead to
\[
(\sigma^2 \varphi)'(t) + C_1 \sigma^2 (\|\nabla \hat{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) + \sigma^2 \|\rho^{1/2} \hat{\theta}\|_{L^2}^2 \\
\leq C_3 \sigma \|\rho^{1/2} \hat{u}\|_{L^2}^2 + C \sigma \|H_t\|_{L^2}^2 + C \sigma^2 \|\rho - 1\|_{L^4}^4 \\
+ C (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2),
\] (3.87)

where we have used the following estimate:
\[
\|\theta \nabla u\|_{L^2}^2 \leq \|\theta - 1\|_{L^6}^6 \|\nabla u\|_{L^6} \|\nabla u\|_{L^6} + \|\nabla u\|_{L^2}^2 \\
\leq C (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) (\|\rho \hat{u}\|_{L^2} + \|H_t\|_{L^2} + \|\nabla \theta\|_{L^2} + 1)
\] (3.88)
due to (2.5), (3.25), and (3.47).

Notice that by (2.5), (3.52), and (3.47), we have
\[
\int |\text{curl} H|^2 \theta dx \leq C \|\nabla H\|_{L^2}^2 + C \|\theta - 1\|_{L^6} \|\nabla \theta\|_{L^2}^{3/2} \|\nabla^2 H\|_{L^2}^{1/2} \\
\leq \delta (\|H_t\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2) + C(\delta) \|\nabla H\|_{L^2}^2.
\] (3.89)

which together with (3.40), (3.41), (3.57), (3.48), (3.51), and (3.47) indicates
\[
\varphi(t) \geq \frac{1}{2} (\|\rho^{1/2} \hat{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2) + \frac{k(\gamma - 1)}{R} \|\nabla \theta\|_{L^2}^2 - C_4 (\|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2).
\] (3.90)

For \(B_1\) defined in (3.39), it holds
\[
B_1 \geq C_5 \|\nabla u\|_{L^2}^2 - C \|\nabla H\|_{L^2}^2 - CC_0^{1/4}
\] (3.91)
Hence, integrating (3.92) with respect to $t$ which implies (3.80), provided (3.21), and (3.6) yields

$$(C_0\sigma B_1 + \sigma^2 \varphi)'(t) + \sigma^2 \rho^{1/2} \dot{u}^2_{L^2} + \frac{C_1}{2} \sigma^2 (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) + \frac{1}{2} \sigma^2 \theta^2 + \frac{1}{2} \sigma^2 \rho^{1/2} \hat{\varphi}^2_{L^2}$$

(3.92)

$$\leq CC_0^{1/4} \sigma' + C \sigma \|H_t\|_{L^2}^2 + C \sigma^2 \rho - 1 \|\dot{u}\|_{L^4}^4 + C(\|H_t\|_{L^2}^2 + \|\dot{\varphi}\|_{L^2}^2).$$

Finally, we claim

$$\int_0^T \sigma^2 \rho - 1 \|\dot{u}\|_{L^4}^4 dt \leq CC_0^{1/4}. \quad (3.93)$$

Hence, integrating (3.92) with respect to $t$ together with (3.90), (3.91), (3.93), (3.81), (3.21), and (3.6) yields

$$A_3(T) + A_4(T) \leq \dot{C} C_0^{1/4} \leq C_0^{1/6},$$

which implies (3.80), provided

$$C_0 \leq \varepsilon_2 \triangleq \min\{\varepsilon_2, \dot{C}^{-12}_2\}.$$

It remains to prove (3.93). It follows from (1.1) and (2.17) that $\rho - 1$ satisfies

$$(\rho - 1)_t + \frac{R}{2\mu + \lambda} (\rho - 1)$$

$$= - u \cdot \nabla (\rho - 1) - (\rho - 1) \text{div} u - \frac{G}{2\mu + \lambda} - \frac{R\rho (\theta - 1)}{2\mu + \lambda} - \frac{|H|^2}{2(2\mu + \lambda)}.$$

(3.94)

Multiplying (3.94) by $4(\rho - 1)^3$ and integrating the resulting equality over $\Omega$ by parts, we obtain after using (2.5), (3.24), (3.6), (3.47), (3.10), (3.48), and (3.23) that

$$\left(\|\rho - 1\|_{L^4}^4\right)_t + \frac{4R}{2\mu + \lambda} \|\rho - 1\|_{L^4}^4$$

$$= -3 \int (\rho - 1)^4 \text{div} u dx - \frac{4}{2\mu + \lambda} \int (\rho - 1)^3 G dx$$

$$- \frac{4R}{2\mu + \lambda} \int (\rho - 1)^3 \rho (\theta - 1) dx - \frac{2}{2\mu + \lambda} \int (\rho - 1)^3 |H|^2 dx$$

$$\leq \frac{2R}{2\mu + \lambda} \|\rho - 1\|_{L^4}^4 + C \|\nabla u\|_{L^2}^2 + C \|\rho - 1\|_{L^4}^4 \|G\|_{L^2}^{1/4} \|\nabla G\|_{L^2}^{3/4}$$

$$+ C \|\rho - 1\|_{L^4}^4 \|\rho (\theta - 1)\|_{L^4}^{1/4} \|\nabla \theta\|_{L^2}^{3/4} + \|H_t\|_{L^2}^{1/4} \|\nabla |H|^2\|_{L^2}^{3/4}$$

(3.95)

$$\leq \frac{3R}{2\mu + \lambda} \|\rho - 1\|_{L^4}^4 + C \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2$$

$$+ C \|\rho^{1/2} \dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^3 + \|\nabla \theta\|_{L^2}^3.$$
respect to \( t \), we deduce from (3.6) and (3.10) that
\[
\int_0^T \sigma^n \| \rho - 1 \|^4_{\text{L}^4} \, dt \\
\leq CA_3^{1/2}(T) \int_0^T \sigma^{n-1} (\| \rho^{1/2}u \|^2_{\text{L}^2} + \| H_t \|^2_{\text{L}^2} + \| \nabla \theta \|^2_{\text{L}^2}) \, dt \\
+ CC_0^{1/4} + C \int_0^{\sigma(T)} \| \rho - 1 \|^4_{\text{L}^4} \, dt \\
\leq CC_0^{1/4} + CC_0^{1/12} \int_0^T \sigma^{n-1} (\| \rho^{1/2}u \|^2_{\text{L}^2} + \| H_t \|^2_{\text{L}^2}) \, dt.
\]
Taking \( n = 2 \) in (3.96) as well as (3.6) infers (3.97) directly.

The proof of Lemma 3.4 is completed. \( \square \)

Since the basic energy estimate in Lemma 3.1 is “weaker”, which is not enough to control \( A_2(T) \), we need to re-establish the basic energy estimate for short time \([0, \sigma(T)]\) to overcome this difficulty, and then show that the spatial \( L^2 \)-norm of \( \theta - 1 \) could be bounded by the combination of the initial energy and the spatial \( L^2 \)-norm of \( \nabla \theta \).

**Lemma 3.5.** Under the conditions of Proposition 3.1, there exists positive constant \( C \) depending only on \( \mu, \lambda, \nu, \kappa, R, \gamma, \hat{\rho}, \Omega, \) and \( M \) such that if \((\rho, u, \theta, H)\) is a smooth solution to the problem (1.1)–(1.5) on \( \Omega \times (0, T) \) satisfying (3.6) with \( K \) as in Lemma 3.2, the following estimates hold:
\[
\sup_{0 \leq t \leq \sigma(T)} \int (\rho |u|^2 + (\rho - 1)^2 + \rho(\theta - \log \theta - 1) + |H|^2) \, dx \leq CC_0, \tag{3.97}
\]
and
\[
\| \theta(\cdot, t) - 1 \|_{L^2} \leq C \left( C_0^{1/2} + C_0^{1/3} \| \nabla \theta(\cdot, t) \|_{L^2} \right), \tag{3.98}
\]
for all \( t \in (0, \sigma(T)) \).

**Proof.** The proof is divided into the following two steps.

**Step I: The proof of (3.97).**

First, adding (3.12) multiplied by \( u \) to (3.13) multiplied by \( H \), we obtain after using integration by parts and (1.1) that
\[
\frac{d}{dt} \int \left( \frac{1}{2} \rho |u|^2 + R(1 + \rho \log \rho - \rho) + \frac{1}{2} |H|^2 \right) \, dx \\
+ \int (\mu |\omega|^2 + (2\mu + \lambda)(\text{div} u)^2 + \nu |\text{curl} H|^2) \, dx \\
= R \int \rho(\theta - 1) \text{div} u \, dx. \tag{3.99}
\]

Note that
\[
\theta - \log \theta - 1 = (\theta - 1)^2 \int_0^{\alpha} \frac{\alpha}{\alpha(\theta - 1) + 1} \, d\alpha \geq \frac{1}{2(\| \theta(\cdot, t) \|_{L^\infty} + 1)} (\theta - 1)^2, \tag{3.100}
\]
which as well as (3.99), (2.9), (3.6), and Cauthy inequality shows that
\[
\frac{d}{dt} \int \left( \frac{1}{2} \rho |u|^2 + R(1 + \rho \log \rho - \rho) + \frac{1}{2} |H|^2 \right) dx + C_7 \int (|\nabla u|^2 + |\nabla H|^2) dx \\
\leq C(\|\theta(\cdot, t)\|_{L^\infty} + 1) \int \rho(\theta - \log \theta - 1) dx.
\]
(3.101)

Then, adding (3.101) multiplied by \((2\mu + 1)C_7^{-1}\) to (3.15) leads to
\[
((2\mu + 1)C_7^{-1} + 1) \frac{d}{dt} \int \left( \frac{1}{2} \rho |u|^2 + R(1 + \rho \log \rho - \rho) + \frac{1}{2} |H|^2 \right) dx \\
+ \frac{R}{\gamma - 1} \frac{d}{dt} \int \rho(\theta - \log \theta - 1) dx + \int (|\nabla u|^2 + |\nabla H|^2) dx \\
\leq C(\|\theta(\cdot, t)\|_{L^\infty} + 1) \int \rho(\theta - \log \theta - 1) dx.
\]
(3.102)

Next, we claim that
\[
\int_0^{\sigma(T)} \|\theta\|_{L^\infty} dt \leq C.
\]
(3.103)

Combining this with (3.102), (3.17), and Grönwall inequality infers (3.97) directly.

Now, we prove the claim (3.103). Note that taking \(n = 1\) in (3.96) together with (3.6) implies
\[
\int_0^{\sigma(T)} \|\rho - 1\|_{L^4}^4 dt \leq C.
\]
(3.104)

Then considering (3.85) with \(m = 1\) and integrating the resulting inequality with respect to \(t\), it follows from (3.90), (3.88), (3.83), (3.6), and (3.104) that
\[
\sup_{0 \leq t \leq T} \sigma \left( \|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right) \\
+ \int_0^t \sigma \left( \|\nabla \dot{u}\|_{L^2}^2 + \|\rho^{1/2}\dot{\theta}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 \right) dt \\
\leq C \int_0^t \left( \|\rho^{1/2}\dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 \right) d\tau \\
+ C \int_0^t \sigma \|\rho - 1\|_{L^4}^4 d\tau \\
\leq C.
\]
(3.105)

Moreover, by virtue of (3.6), (3.74), (3.88), (3.83), (3.104), (3.52), and (3.105), it holds
\[
\int_0^T \sigma \|\nabla \theta\|_{L^2}^2 dt \leq C \int_0^T \left( \sigma \|\rho \dot{\theta}\|_{L^2}^2 + \|\rho \dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 \right) dt \\
+ C \int_0^T \left( \|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla H\|_{L^2}^2 + \sigma \|\rho - 1\|_{L^4} \right) dt \\
\leq C,
\]
(3.106)
which combined with (3.6) yields
\[
\int_0^{\sigma(T)} \|\theta - 1\|_{L^\infty} dt \leq C \int_0^{\sigma(T)} \|\nabla \theta\|_{L^2}^{1/2} (\|\nabla^2 \theta\|_{L^2}^1) \sigma^{-1/4} dt \\
\leq C \left( \int_0^{\sigma(T)} \|\nabla \theta\|_{L^2}^2 dt \int_0^{\sigma(T)} \sigma \|\nabla^2 \theta\|_{L^2}^2 dt \right)^{1/4} \\
\leq CC_0^{1/16},
\]
where we have used
\[
\|\theta - 1\|_{L^\infty} \leq C \|\theta - 1\|_{L^6}^{1/2} \|\nabla \theta\|_{L^6}^{1/2} \leq C \|\nabla \theta\|_{L^2} \|\nabla^2 \theta\|_{L^2}^{1/2} \|\theta\|_{L^2}^{1/2}
\]
due to (2.6) and (2.5). Then, (3.103) is a consequence of (3.107).

**Step II: The proof of (3.98).**

Denote
\[
(\theta(\cdot, t) > 2) \triangleq \{x \in \Omega | \theta(x, t) > 2\}, \quad (\theta(\cdot, t) < 3) \triangleq \{x \in \Omega | \theta(x, t) < 3\}.
\]
Direct calculations combined with (3.100) implies
\[
\theta - \log \theta - 1 \geq \frac{1}{8} (\theta - 1) 1_{(\theta(\cdot, t) > 2)} + \frac{1}{12} (\theta - 1)^2 1_{(\theta(\cdot, t) < 3)},
\]
which together with (3.97) shows that
\[
\sup_{0 \leq t \leq \sigma(T)} \int (\rho(\theta - 1) 1_{(\theta(\cdot, t) > 2)} + \rho(\theta - 1)^2 1_{(\theta(\cdot, t) < 3)}) dx \leq CC_0.
\]

Now, for \( t \in (0, \sigma(T)) \), taking \( g(x) = \rho(x, t) \), \( f(x) = \theta(x, t) - 1 \), \( s = 2 \) and \( \Sigma = (\theta(\cdot, t) < 3) \) in (2.8), it follows from (3.110) and (3.97) that
\[
\|\theta - 1\|_{L^2(\theta(\cdot, t) < 3)} \leq CC_0^{1/2} + CC_0^{1/3} \|\nabla \theta\|_{L^2}.
\]
Similarly, taking \( g(x) = \rho(x, t) \), \( f(x) = \theta(x, t) - 1 \), \( s = 1 \) and \( \Sigma = (\theta(\cdot, t) > 2) \) in (2.8), and using (3.110) and (3.97), it holds that
\[
\|\theta - 1\|_{L^1(\theta(\cdot, t) > 2)} \leq CC_0 + CC_0^{5/6} \|\nabla \theta\|_{L^2},
\]
which along with Hölder’s inequality and (2.5) leads to
\[
\|\theta - 1\|_{L^2(\theta(\cdot, t) > 2)} \leq \|\theta - 1\|_{L^1(\theta(\cdot, t) > 2)} \|\theta - 1\|_{L^6}^{2/5} \leq C \left( C_0^{2/5} + C_0^{1/3} \|\nabla \theta\|_{L^2}^{2/5} \right) \|\nabla \theta\|_{L^2}^{3/5} \\
\leq CC_0^{1/2} + CC_0^{1/3} \|\nabla \theta\|_{L^2}.
\]
Combining this with (3.111) concludes (3.98). The proof of Lemma 3.5 is completed.

**Remark 3.1.** One can deduce from (3.16) and (3.109) that (3.110) holds for \( 0 < t \leq T \) with \( C_0 \) replaced by \( C_0^{1/4} \). Thus for all \( t \in [0, T] \),
\[
\|\theta(\cdot, t) - 1\|_{L^2} \leq C \left( C_0^{1/8} + C_0^{1/12} \|\nabla \theta(\cdot, t)\|_{L^2} \right).
\]

(3.112)
With (3.98), which is better than (3.112) in small time, in mind, we can establish the estimate on $A_2(T)$.

**Lemma 3.6.** Under the conditions of Proposition 3.1, there exists a positive constant $\varepsilon_3$ depending only on $\mu$, $\lambda$, $\nu$, $\kappa$, $R$, $\gamma$, $\rho$, $\Theta$, $\Omega$, and $M$ such that if $(\rho, u, \theta, H)$ is a smooth solution to the problem (1.1)–(1.5) on $\Omega \times (0, T]$ satisfying (3.4) with $K$ as in Lemma 3.2, the following estimate holds:

$$A_2(T) \leq C_0^{1/4},$$

(3.113)

provided $C_0 \leq \varepsilon_3$.

**Proof.** First, multiplying (1.13) by $(\theta - 1)$ and integrating the resulting equality by parts yield

$$\frac{R}{2(\gamma - 1)} \frac{d}{dt} \int \rho(\theta - 1)^2 dx + \kappa \|\nabla \theta\|_{L^2}^2 = - \int R\rho \theta(\theta - 1)\text{div} u dx + \int (\theta - 1)(\lambda(\nabla \theta)^2 + 2\mu|\mathcal{D}(\theta)|^2 + \nu|\text{curl} H|^2) dx.$$  

(3.114)

Adding (3.114) to (3.99) infers that

$$\frac{d}{dt} \int \left(\frac{1}{2}\rho|u|^2 + R(1 + \rho \log \rho - \rho) + \frac{1}{2}|H|^2 + \frac{R}{2(\gamma - 1)}(\theta - 1)^{2}\right) dx
+ \mu\|\omega\|_{L^2}^2 + (2\mu + \lambda)|\text{div} u|^2_{L^2} + \kappa \|\nabla \theta\|_{L^2}^2 + \nu \|\text{curl} H\|_{L^2}^2
= - \int R\rho(\theta - 1)^2|\text{div} u|^2 dx + \int (\theta - 1)(\lambda|\nabla \theta|^2 + 2\mu|\mathcal{D}(\theta)|^2 + \nu|\text{curl} H|^2) dx.$$  

(3.115)

Note that by virtue of (2.5), (3.47), and (3.6), one has

$$\int \rho|\theta - 1|^2|\text{div} u|^2 dx \leq C\|\rho^{1/2}(\theta - 1)|1/2\|_{L^2} \|\rho^{1/2}(\theta - 1)|3/2\|_{L^6} \|
\nabla u\|_{L^2}
\leq CA_2^{1/4}(T) \|
\nabla \theta\|_{L^2}^{3/2} \|
\nabla u\|_{L^2}
\leq CC_0^{1/16} (\|
\nabla \theta\|_{L^2}^2 + \|
\nabla u\|_{L^2}^2).$$  

(3.116)

For the second term on the right-hand side of (3.115), we deal with it for the short time $[0, \sigma(T)]$ and the large time $[\sigma(T), T]$, respectively.

For $t \in [0, \sigma(T)]$, it follows from (3.98), (2.5), (3.25), (3.47), and (3.52) that

$$\int |\theta - 1|(|\nabla u|^2 + |\nabla H|^2) dx
\leq C\|\theta - 1|^{1/2}\|_{L^6} \|
\nabla u\|_{L^6} \|
\nabla u\|_{L^6} + \|
\nabla H\|_{L^2} \|
\nabla H\|_{L^2}
\leq C \left(C_0^{1/4} \|
\nabla \theta\|_{L^2}^{1/2} + C_0^{1/6} \|
\nabla \theta\|_{L^2} \right) \|
\nabla u\|_{L^2} + \|
\nabla H\|_{L^2}
\cdot \left(\|\rho \dot{u}\|_{L^2} + \|
\nabla u\|_{L^2} + \|
\nabla H\|_{L^2} + \|
\nabla \theta\|_{L^2} + \|H_t\|_{L^2} + C_0^{1/24}\right)
\leq CC_0^{7/24} (\|
\rho^{1/2}\dot{u}\|_{L^2}^2 + \|H_t\|_{H^1}^2 + 1)
+ CC_0^{1/24} (\|
\nabla \theta\|_{L^2}^2 + \|
\nabla u\|_{L^2}^2 + \|
\nabla H\|_{L^2}^2).$$  

(3.117)
For $t \in [\sigma(T), T]$, one deduces from (2.5) and (3.6) that

$$
\int |\theta - 1| (|\nabla u|^2 + |\nabla H|^2) dx 
\leq C \|\theta - 1\|_{L^6} (\|\nabla u\|_{L^6}^{3/2} \|\nabla u\|_{L^6}^{1/2} + \|\nabla H\|_{L^2}^{3/2} \|\nabla^2 H\|_{L^2}^{1/2}) 
\leq CC_0^{1/16} (\|\nabla \theta\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla H\|_{L^2}^2),
$$

(3.118)

where one has used the following fact:

$$
\sup_{0 \leq t \leq T} \left( \sigma(\|\nabla u\|_{L^6} + \|\nabla^2 H\|_{L^2}) \right) \leq CC_0^{1/24}
$$

(3.119)

owing to (3.6), (3.52), and (3.25).

Submitting (3.116)–(3.118) into (3.115) and using (3.6), it holds that

$$
\sup_{0 \leq t \leq T} \int \left( \frac{1}{2} \rho |u|^2 + R(1 + \rho \log \rho - \rho) + \frac{1}{2} |H|^2 + \frac{R}{2(\gamma - 1)} \rho(\theta - 1)^2 \right) dx 
+ \int_0^T (\mu \|\omega\|_{L^2}^2 + (2\mu + \lambda) \|\text{div} u\|_{L^2}^2 + \nu \|\text{curl} H\|_{L^2}^2 + \kappa \|\nabla \theta\|_{L^2}^2) dt 
\leq CC_0 + CC_0^{1/24} \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|\nabla H\|_{L^2}^2) dt 
+ C C_0^{7/24} \int_{\sigma(T)}^T (\|\rho \dot{u}\|_{L^2}^2 + \|H_t\|_{L^2}^2 + 1) dt 
\leq CC_0^{7/24}.
$$

(3.120)

Then combining (3.120) with (2.9) indicates that

$$
A_2(T) \leq \hat{C}_3 C_0^{7/24} \leq C_0^{1/4},
$$

provided

$$
C_0 \leq \varepsilon_3 \triangleq \min \left\{ 1, \hat{C}_3^{-24} \right\}.
$$

(3.121)

The proof of Lemma 3.6 is completed.

Now, we are in position to obtain the time-independent upper bound for the density, which turns out to be the key to obtaining all the higher order estimates and thus to extending the classical solution globally.

**Lemma 3.7.** Under the conditions of Proposition 3.1, there exists a positive constant $\varepsilon_0$ depending only on $\mu, \lambda, \nu, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega$, and $M$ such that if $(\rho, u, \theta, H)$ is a smooth solution to the problem (1.1)–(1.5) on $\Omega \times (0, T]$ satisfying (3.6) with $K$ as in Lemma 3.2, the following estimate holds:

$$
\sup_{0 \leq t \leq T} \|\rho(\cdot, t)\|_{L^\infty} \leq \frac{3\hat{\rho}}{2},
$$

(3.122)

provided $C_0 \leq \varepsilon_0$. 

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Proof. First, by virtue of (3.108), (3.106), and (3.6), one derives
\[
\int_{\sigma(T)}^T \| \theta - 1 \|_{L^\infty}^2 dt \leq C \int_{\sigma(T)}^T \| \nabla \theta \|_{L^2} \| \nabla^2 \theta \|_{L^2} dt \\
\leq C \left( \int_{\sigma(T)}^T \| \nabla \theta \|_{L^2}^2 dt \right)^{1/2} \left( \int_{\sigma(T)}^T \| \nabla^2 \theta \|_{L^2}^2 dt \right)^{1/2} \tag{3.123}
\]
Next, it follows from (2.6), (3.48), (3.63), (3.105), and (3.6) that
\[
\int_{0}^{\sigma(T)} \| G \|_{L^\infty} dt \\
\leq C \int_{0}^{\sigma(T)} \| \nabla G \|_{L^2}^{1/2} \| \nabla G \|_{L^6}^{1/2} dt \\
\leq C \int_{0}^{\sigma(T)} (\| \rho \dot{u} \|_{L^2} + \| H_t \|_{L^2} + \| \nabla H \|_{L^2})^{1/2} (\| \nabla \dot{u} \|_{L^2} + 1)^{1/2} dt \\
+ C \int_{0}^{\sigma(T)} (\| \rho \dot{u} \|_{L^2} + \| H_t \|_{L^2} + \| \nabla H \|_{L^2})^{1/2} \| H_t \|_{L^2}^{3/4} dt \\
\leq C \sup_{0 \leq t \leq T} (\sigma(\| \rho \dot{u} \|_{L^2} + \| H_t \|_{L^2} + \| \nabla H \|_{L^2}))^{1/4} \\
\int_{0}^{\sigma(T)} (\sigma(\| \rho \dot{u} \|_{L^2}^2 + \| H_t \|_{L^2}^2 + \| \nabla H \|_{L^2}^2))^{1/8} (\sigma(\| \nabla \dot{u} \|_{L^2}^2 + 1))^{1/4} \sigma^{-5/8} dt \\
+ C \sup_{0 \leq t \leq T} (\sigma(\| \rho \dot{u} \|_{L^2} + \| H_t \|_{L^2} + \| \nabla H \|_{L^2}))^{1/2} \int_{0}^{\sigma(T)} (\sigma(\| H_t \|_{L^2}^2))^{3/8} \sigma^{-7/8} dt \\
\leq CC_0^{1/48}
\]
and
\[
\int_{\sigma(T)}^T \| G \|_{L^\infty}^2 dt \\
\leq C \int_{\sigma(T)}^T \| \nabla G \|_{L^2} \| \nabla G \|_{L^6} dt \\
\leq C \int_{\sigma(T)}^T \left( \| \rho^{1/2} \dot{u} \|_{L^2}^2 + \| \nabla \dot{u} \|_{L^2}^2 + \| \nabla H \|_{L^2}^2 + \| H_t \|_{L^2}^2 + \| H_t \|_{L^2}^2 \right) dt \\
\leq CC_0^{1/6}
\]
Analogously, by (2.5), (2.6), (3.6), and (3.52), it holds that
\[
\int_{0}^{\sigma(T)} \| H \|_{L^\infty}^2 dt \\
\leq C \int_{0}^{\sigma(T)} \| \nabla H \|_{L^2} \| \nabla^2 H \|_{L^2} dt \\
\leq C \left( \int_{0}^{\sigma(T)} \| \nabla H \|_{L^2}^2 dt \right)^{1/2} \left( \int_{0}^{\sigma(T)} (\| \nabla H \|_{L^2}^2 + \| H_t \|_{L^2}^2) dt \right)^{1/2} \tag{3.126}
\]
\[
\leq CC_0^{1/8}
\]
and
\[ \int_{\sigma(T)}^{T} \|H\|_{L^\infty}^4 \, dt \leq C \int_{\sigma(T)}^{T} \|\nabla H\|_{L^2}^2 \|\nabla^2 H\|_{L^2}^2 \, dt \]
\[ \leq C \int_{\sigma(T)}^{T} (\|\nabla H\|_{L^2}^2 + \|H_t\|_{L^2}^2) \, dt \]
\[ \leq CC_0^{1/6}. \tag{3.127} \]

Note that (1.1) can be rewritten in terms of the Lagrangian coordinates as follows
\[ (2\mu + \lambda)D_t \rho = -R\rho(\rho - 1) - R\rho^2(\rho - 1) - \rho G - \frac{1}{2}\rho|H|^2 \]
\[ \leq -R(\rho - 1) + C(\|\rho - 1\|_{L^\infty} + \|G\|_{L^\infty} + \|H\|_{L^\infty}^2), \]
which leads to
\[ D_t(\rho - 1) + \frac{R}{2\mu + \lambda}(\rho - 1) \leq C(\|\rho - 1\|_{L^\infty} + \|G\|_{L^\infty} + \|H\|_{L^\infty}^2). \tag{3.128} \]

Next, taking
\[ y = \rho - 1, \quad \alpha = \frac{R}{2\mu + \lambda}, \quad g = C(\|\rho - 1\|_{L^\infty} + \|G\|_{L^\infty} + \|H\|_{L^\infty}^2), \quad T_1 = \sigma(T), \]
in Lemma 2.9, we thus obtain after using (3.107), (3.123)–(3.128), and (2.15) that
\[ \rho \leq \hat{\rho} + 1 + C \left( \|g\|_{L^1(0, \sigma(T))} + \|g\|_{L^2(\sigma(T), T)} \right) \leq \hat{\rho} + 1 + \hat{C}_4 C_0^{1/48}, \]
which implies (3.122), provided
\[ C_0 \leq \varepsilon_0 \triangleq \min \{\varepsilon_1, \ldots, \varepsilon_4\} \tag{3.129} \]
with \( \varepsilon_4 \triangleq \left( \frac{\hat{\rho} - 2}{2\hat{C}_4} \right)^{48} \). The proof of Lemma 3.7 is finished. \( \square \)

At the end of this section, we summarize some uniform estimates on \((\rho, u, \theta, H)\), which guarantees the large-time behavior of the classical solutions and the higher-order estimates in next section.

**Lemma 3.8.** In addition to the conditions of Proposition 3.1, assume \((\rho_0, u_0, \theta_0, H_0)\) satisfies (3.8) with \(\varepsilon_0\) as in Proposition 3.1. Then there exists a positive constant \(C\) depending only on \(\mu, \lambda, \nu, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \hat{\Omega}, \) and \(M\) such that if \((\rho, u, \theta, H)\) is a smooth solution to the problem (1.1)–(1.5) on \(\Omega \times (0, T]\) satisfying (3.6) with \(K\) as in Lemma 3.2, it holds:
\[ \sup_{0 \leq t \leq T} \sigma^2 \int \rho|\theta|^2 \, dx + \int_{0}^{T} \sigma^2 \|\nabla \theta\|_{L^2}^2 \, dt \leq C. \tag{3.130} \]
Moreover, it holds that
\[ \sup_{0 \leq t \leq T} \left( \sigma \|\nabla u\|_{L^4}^4 + \sigma \|H\|_{H^1}^2 + \sigma^2 \|\nabla \theta\|_{H^1}^2 \right) \]
\[ + \int_{0}^{T} \left( \sigma \|\nabla u\|_{L^4}^4 + \sigma \|\nabla \theta\|_{H^1}^2 + \sigma \|H_t\|_{H^1}^2 + \sigma^2 \|\nabla \theta_t\|_{L^2}^2 + \sigma \|\rho - 1\|_{L^4}^4 \right) \, dt \leq C. \tag{3.131} \]
Proof. First, operating the operator $\partial_t + \text{div}(u \cdot \cdot)$ to (1.13) and using (1.11), one has

\[
\begin{align*}
\frac{R}{\gamma - 1} \rho \left( \partial_t \theta + u \cdot \nabla \theta \right) \\
= \kappa \Delta \theta_t + \kappa \text{div}(\Delta \theta u) + (\lambda(\text{div}u)^2 + 2\mu|\nabla (u)|^2 + \nu|\text{curl}H|^2) \text{div}u \\
+ R \rho \theta \partial_t u^1 \partial_t u^k - R \rho \partial_t \text{div}u - R \rho \text{div}\bar{u} + 2\lambda \left( \text{div}\bar{u} - \partial_k \partial_t u^k \right) \text{div}u \\
+ \mu (\partial_i u^3 + \partial_j u^i) \left( \partial_i \bar{u}^j + \partial_j \bar{u}^i - \partial_i u^k \partial_k \bar{u}^j - \partial_j u^k \partial_k \bar{u}^i \right) \\
+ 2\nu (\text{curl}H_t + u \cdot \nabla \text{curl}H) \cdot \text{curl}H.
\end{align*}
\] (3.132)

Multiplying (3.132) by $\bar{\theta}$ and integrating the resulting equality over $\Omega$ yield

\[
\begin{align*}
\frac{R}{2(\gamma - 1)} \left( \int \rho \theta^2 dx \right)_{\Omega} + \kappa \|
abla \theta\|_{L^2}^2 \\
\leq C \int |\nabla \bar{\theta}| \left( |\nabla \theta||u| + |\nabla \theta||\nabla u| \right) dx + C \int \rho \theta - 1||\nabla \bar{u}||\theta|dx \\
+ C \int |\nabla u|^2 \bar{\theta} \left( |\nabla u| + |\theta - 1| \right) dx + C \int |\nabla \bar{u}| \rho \theta |dx \\
+ C \int \left( |\nabla u|^2 \bar{\theta} + \rho \theta^2 |\nabla u| + |\nabla u||\nabla \bar{\theta}| \right) dx \\
+ C \int |\nabla H||\bar{\theta}||\nabla H| + |\nabla H_t| + |u||\nabla^2 H|) dx \\
\leq C \|
abla u\|_{L^2}^{1/2} \|
abla u\|_{L^6}^{1/2} \|
abla^2 \theta\|_{L^2} \|
abla \theta\|_{L^2} \\
+ C \|ho(\theta - 1)\|_{L^2}^{1/2} \|
abla \theta\|_{L^2}^{1/2} \|
abla \bar{u}\|_{L^2} \|
abla \theta\|_{L^6} \\
+ C \|
abla u\|_{L^6} \|
abla u\|_{L^6} \|
abla \theta\|_{L^6} \|
abla \bar{u}\|_{L^2} \|
abla \theta\|_{L^2} \\
+ C \|
abla \theta\|_{L^6} \|
abla H\|_{L^2} \|
abla H\|_{L^2} \|
abla H_t\|_{L^2} \\
\leq \frac{K}{2} \|
abla \bar{\theta}\|_{L^2}^2 + C \|
abla u\|_{L^6}^2 \|
abla \theta\|_{L^2}^2 + C \|
abla u\|_{L^2}^2 \|
abla u\|_{L^6}^4
\end{align*}
\] (3.133)

where we have used (2.5), (2.6), (3.6), and the following fact:

\[
\int (\Delta \theta_t + \text{div}(\Delta \theta u)) \bar{\theta} dx = - \int (\nabla \theta_t \cdot \nabla \bar{\theta} + \Delta \theta_t \cdot \nabla u \cdot \nabla \bar{\theta}) dx
\]

\[
= - \int |\nabla \bar{\theta}|^2 dx + \int (\nabla (u \cdot \nabla \theta) \cdot \nabla \bar{\theta} - \Delta \theta_t u \cdot \nabla \bar{\theta}) dx.
\]

Multiplying (3.133) by $\sigma^2$ and integrating the resulting inequality over $(0, T)$, it
follows from integration by parts, (3.6), (3.119), (3.105), (3.106), and (3.52) that

\[ \sup_{0 \leq t \leq T} \sigma^2 \int_0^T \rho |\dot{\theta}|^2 \, dx + \int_0^T \sigma^2 \| \nabla \theta \|^2_{L^2} \, dt \leq C \sup_{0 \leq t \leq T} \left( \sigma^2 \| \nabla u \|^2_{L^6} \right) \int_0^T \left( \| \nabla u \|^2_{L^6} \| \nabla \theta \|^2_{L^2} + \| \nabla \theta \|^2_{L^2} \right) \, dt + C \sup_{0 \leq t \leq T} (\sigma (1 + \| \nabla u \|_{L^6} + \| \nabla \theta \|_{L^2})) \cdot \int_0^T \sigma \left( \| \nabla^2 \theta \|^2_{L^2} + \| \nabla \dot{u} \|^2_{L^2} + \| \rho^{1/2} \dot{\theta} \|^2_{L^2} \right) \, dt \]

\[ + C \sup_{0 \leq t \leq T} \sigma \| \nabla u \|_{L^6} \int_0^T \| \nabla u \|^2_{L^2} \, dt + C \sup_{0 \leq t \leq T} \sigma \| \nabla^2 H \|_{L^2} \int_0^T \| \nabla^2 H \|^2_{L^2} \, dt \]

\[ + C \int_0^T \sigma \| \rho^{1/2} \dot{\theta} \|^2_{L^2} \, dt \]

\[ \leq C, \]

where in the last inequality we have used

\[ \int_0^T \| \nabla u \|^2_{L^2} \, dt \leq C \int_0^T \| \nabla u \|^2_{L^2} \left( \| \rho^{1/2} \dot{u} \|^2_{L^2} + \| \nabla u \|^2_{L^2} + \| \nabla \theta \|^2_{L^2} + \| \nabla H \|^2_{L^2} + \| \dot{H} \|^2_{L^2} + 1 \right) \, dt \]

\[ \leq C \int_0^T \left( \| \rho^{1/2} \dot{u} \|^2_{L^2} + \| \nabla u \|^2_{L^2} + \| \nabla \theta \|^2_{L^2} + \| \nabla H \|^2_{L^2} + \| H_t \|^2_{L^2} \right) \, dt \]

\[ \leq C \]

due to (3.25), (3.6), and (3.47).

Finally, we deduce from (3.6), (3.25), (3.47), (3.10), (3.52), (3.74), (3.88), (3.82), (3.130), and (3.104)–(3.106) that

\[ \sup_{0 \leq t \leq T} \sigma \| \nabla u \|^2_{L^2} + \sigma \| H \|^2_{H^2} + \sigma^2 \| \nabla \theta \|^2_{H^1} \]

\[ + \int_0^T (\sigma \| \nabla u \|^4_{L^4} + \sigma \| \nabla \theta \|^2_{H^1} + \sigma \| H_t \|^2_{H^1} + \sigma \| \rho - 1 \|^2_{L^4}) \, dt \leq C, \]

which together with (3.6) and (3.130) shows that

\[ \int_0^T \sigma^2 \| \nabla \dot{t} \|^2_{L^2} \, dt \leq C \int_0^T \sigma^2 \| \nabla \theta \|^2_{L^2} \, dt + C \int_0^T \sigma^2 \| \nabla (u \cdot \nabla \theta) \|^2_{L^2} \, dt \]

\[ \leq C + C \int_0^T \sigma^2 \| \nabla u \|_{L^2} \| \nabla u \|_{L^6} \| \nabla \theta \|^2_{L^2} \, dt \]

\[ \leq C. \]

The combination of (3.134) and (3.135) leads to (3.131) immediately.

The proof of Lemma 3.8 is completed. \qed
4 A priori estimates (II): higher-order estimates

This section is devoted to establishing the higher-order estimates of the classical solution \((\rho, u, \theta, H)\) to the problem \((1.1) - (1.5)\) on \(\Omega \times (0, T)\) with initial data \((\rho_0, u_0, \theta_0, H_0)\) satisfying \((2.1)\) and \((3.5)\). Moreover, we assume that both \((3.6)\) and \((3.8)\) hold as well. To proceed, we define \(\tilde{g}\) as

\[
\tilde{g} \triangleq \rho_0^{-1/2} (-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + R \nabla(\rho_0 \theta_0) - (\nabla \times H_0) \times H_0).
\]

Thus we deduce from \((2.1)\) and \((3.5)\) that

\[
\tilde{g} \in L^2.
\]

From now on, the generic constant \(C\) will depend only on

\[
T, \|\tilde{g}\|_{L^2}, \|\rho_0 - 1\|_{H^{2r} W^{2,r}}, \|u_0\|_{H^2}, \|\theta_0 - 1\|_{H^1}, \|H_0\|_{H^2},
\]

besides \(\mu, \lambda, \nu, \kappa, R, \gamma, \hat{\rho}, \hat{\theta}, \Omega,\) and \(M\).

To begin with, we derive the following estimates on the spatial gradient of the smooth solution \((\rho, u, \theta, H)\).

Lemma 4.1. The following estimates hold:

\[
\sup_{0 \leq t \leq T} \left( \|\rho^{1/2} \tilde{u}\|_{L^2}^2 + \sigma \|\rho^{1/2} \tilde{\theta}\|_{L^2}^2 + \|\theta - 1\|_{H^1}^2 + \|H\|_{H^2}^2 + \sigma \|\nabla^2 \theta\|_{L^2}^2 + \|H_1\|_{L^2}^2 \right)
\]

\[
+ \int_0^T \left( \|\nabla \tilde{u}\|_{L^2}^2 + \|\rho^{1/2} \tilde{\theta}\|_{L^2}^2 + \|\nabla \tilde{\theta}\|_{L^2}^2 + \sigma \|\nabla \tilde{\theta}\|_{L^2}^2 + \|H_1\|_{L^2}^2 \right) dt \leq C,
\]

\[
\sup_{0 \leq t \leq T} \left( \|u\|_{H^2} + \|\rho - 1\|_{H^2} \right)
\]

\[
+ \int_0^T \left( \|\nabla u\|_{L^\infty}^{3/2} + \sigma \|\nabla^2 \theta\|_{L^2}^2 + \|u\|_{H^3}^2 + \|H\|_{H^3}^2 \right) dt \leq C.
\]

Proof. First, for \(\varphi(t)\) as in \((3.86)\), taking \(m = 0\) in \((3.85)\), we obtain after using \((3.6)\), \((3.10)\), \((3.90)\), \((3.82)\), and \((3.88)\) that

\[
\varphi'(t) + \frac{C_1}{2} \left( \|\nabla \tilde{u}\|_{L^2}^2 + \|\nabla \tilde{H}\|_{L^2}^2 \right) + \|\rho^{1/2} \tilde{\theta}\|_{L^2}^2
\]

\[
\leq C \left( \|\rho^{1/2} \tilde{u}\|_{L^2}^2 + \|\tilde{H}_1\|_{L^2}^2 + \|\nabla \tilde{u}\|_{L^2}^2 + \|\nabla \tilde{H}\|_{L^2}^2 + \|\nabla \tilde{\theta}\|_{L^2}^2 \right)
\]

\[
+ C \left( \|\rho \tilde{u}\|_{L^2}^2 + \|\nabla \tilde{\theta}\|_{L^2}^2 + \|\tilde{H}_1\|_{L^2}^2 + \|\rho - 1\|_{L^4}^4 \right)
\]

\[
\leq C \left( \|\rho^{1/2} \tilde{u}\|_{L^2}^2 + \|\tilde{H}_1\|_{L^2}^2 + \|\nabla \tilde{\theta}\|_{L^2}^2 \right) (\varphi(t) + 1) + C.
\]

Taking into account \((1.1)\) and \((4.1)\), we can define

\[
\sqrt{\rho} \tilde{u}(x, t = 0) \triangleq -\tilde{g},
\]

and

\[
H_i(x, t = 0) \triangleq -u_0 \cdot \nabla H_0 + H_0 \cdot \nabla u_0 - H_0 \text{div} u_0 - \nu \nabla \times \text{curl} H_0,
\]

which as well as \((3.86)\), \((3.40)\), \((3.41)\), \((3.51)\), \((4.2)\), \((3.57)\), \((3.89)\), and \((3.48)\) yields that

\[
|\varphi(0)| \leq C \|\tilde{g}\|_{L^2}^2 + C \leq C.
\]
Then, integrating (4.5) with respect to $t$ and applying Grönwall’s inequality to the
resulting inequality, we deduce from (3.6), (4.7), and (3.90) that
\[
\sup_{0 \leq t \leq T} \left( \|\rho^{1/2}\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla\|_{L^2}^2 \right) \\
+ \int_0^T \left( \|\nabla u\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 + \|\rho^{1/2}\|_{L^2}^2 \right) dt \leq C, \tag{4.8}
\]
which together with (3.112) leads to
\[
\sup_{0 \leq t \leq T} \|\theta - 1\|_{L^2} \leq C. \tag{4.9}
\]

Next, multiplying (3.133) by $\sigma$ and integrating over $(0, T)$ show that
\[
\sup_{0 \leq t \leq T} \sigma \int_0^T \rho|\dot{\theta}|^2 dx + \int_0^T \sigma \|\nabla\|_{L^2}^2 dt \\
\leq C \int_0^T \left( \|\nabla^2\|_{L^2}^2 + \|\rho^{1/2}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 \right) dt + C \tag{4.10}
\]
\[
\leq C,
\]
where we have used (4.8), (3.6), (3.68), (3.74), and the following fact:
\[
\sup_{0 \leq t \leq T} (\|\nabla u\|_{L^6} + \|H\|_{H^2}) \leq C \tag{4.11}
\]
owing to (3.25), (3.52), (3.6), (3.10), and (4.8). Hence, it follows from (4.8), (4.10),
(3.74), (3.88), (3.6), and (4.11) that
\[
\sup_{0 \leq t \leq T} \sigma \|\nabla^2\|_{L^2}^2 + \int_0^T \|\nabla^2\|_{L^2}^2 dt \leq C,
\]
which together with (4.8)–(4.11) concludes (4.3).

It remains to prove (4.4).

First, note that by virtue of (2.26), (3.6), (3.10), and (4.3), it holds
\[
\|\nabla^2 u\|_{L^p} \leq C \left( \|\rho\dot{u}\|_{L^p} + \|\nabla(\rho\theta)\|_{L^p} + \|H\|_{H^1} \|\nabla H\|_{L^p} \right) + C \\
\leq C \left( \|\nabla u\|_{L^2}^2 + \|\nabla^2\|_{L^2}^2 \right) + C(\|\nabla^2\|_{L^2}^2 + 1)\|\nabla\|_{L^p}^2 + C. \tag{4.12}
\]

Combining this with standard calculations infers that for $2 \leq p \leq 6$,
\[
\partial_t \|\nabla\|_{L^p} \leq C \|\nabla u\|_{L^\infty} \|\nabla\|_{L^p} + C \|\nabla^2 u\|_{L^p} \\
\leq C \left( 1 + \|\nabla u\|_{L^\infty} + \|\nabla^2\|_{L^2} \right) \|\nabla\|_{L^p} \\
+ C \left( 1 + \|\nabla u\|_{L^2}^2 + \|\nabla^2\|_{L^2}^2 \right). \tag{4.13}
\]

To deal with $\|\nabla u\|_{L^\infty}$, we deduce from Lemma 2.4 (3.6), and (4.12) that
\[
\|\nabla u\|_{L^\infty} \leq C \left( \|\text{div} u\|_{L^\infty} + \|\text{curl} u\|_{L^\infty} \right) \log(e + \|\nabla^2 u\|_{L^6}) + C\|\nabla u\|_{L^2} + C \\
\leq C \left( \|\text{div} u\|_{L^\infty} + \|\text{curl} u\|_{L^\infty} \right) \log(e + \|\nabla u\|_{L^2} + \|\nabla^2\|_{L^2}) \\
+ C \left( \|\text{div} u\|_{L^\infty} + \|\text{curl} u\|_{L^\infty} \right) \log(e + \|\nabla\|_{L^6}) + C, \tag{4.14}
\]
where
\[
\int_0^T \left( \| \text{div}u \|_{L^\infty}^2 + \| \text{curl}u \|_{L^\infty}^2 \right) dt \\
\leq C \int_0^T \left( \| G \|_{L^2}^2 + \| \text{curl}u \|_{L^\infty}^2 \right) dt \\
\leq C \int_0^T \left( \| \nabla G \|_{L^2}^2 + \| \text{curl}u \|_{W^{1,6}}^2 + \| \theta - 1 \|_{L^2}^2 \right) dt + C \\
\leq C \int_0^T \left( \| \nabla \dot{u} \|_{L^2}^2 + \| \nabla^2 \theta \|_{L^2}^2 \right) dt + C \\
\leq C
\]

owing to (2.17), (2.5), (2.6), (4.3), (3.48), (4.11), (3.6), (3.63), and (3.64).

Denote
\[
\begin{align*}
    f(t) & \triangleq e + \| \nabla \rho \|_{L^6} \\
    \tilde{f}(t) & \triangleq 1 + \| \text{div}u \|_{L^\infty}^2 + \| \text{curl}u \|_{L^\infty}^2 + \| \nabla \dot{u} \|_{L^2}^2 + \| \nabla^2 \theta \|_{L^2}^2,
\end{align*}
\]

then, in light of (4.14), (4.13) with \( p = 6 \) is equivalent to
\[
f'(t) \leq C \tilde{f}(t) f(t) \ln f(t),
\]

which implies
\[
(\ln(\ln f(t)))' \leq C \tilde{f}(t),
\]

which along with Grönwall’s inequality, (4.15), and (4.3) leads to
\[
\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^6} \leq C. \tag{4.16}
\]

This together with (4.14), (4.15), and (4.3) gives
\[
\int_0^T \| \nabla u \|_{L^2}^{3/2} dt \leq C. \tag{4.17}
\]

On the other hand, taking \( p = 2 \) in (4.13) combined with (4.17), (4.3), and Grönwall’s inequality implies
\[
\sup_{0 \leq t \leq T} \| \nabla \rho \|_{L^2} \leq C, \tag{4.18}
\]

which gives
\[
\sup_{0 \leq t \leq T} \| \nabla P \|_{L^2} \leq C \sup_{0 \leq t \leq T} \left( \| \nabla \theta \|_{L^2} + \| \nabla \rho \|_{L^2} + \| \theta - 1 \|_{L^2} \right) \| \nabla \rho \|_{L^6} \leq C \tag{4.19}
\]

owing to (3.6), (4.3), and (4.16). Then it follows from (4.19), (2.26), (3.6), (4.3), and (3.10) that
\[
\sup_{0 \leq t \leq T} \| u \|_{H^2} \leq C \sup_{0 \leq t \leq T} \left( \| \rho \dot{u} \|_{L^2} + \| \nabla P \|_{L^2} + \| H \| \| \nabla H \|_{L^2} + 1 \right) \leq C. \tag{4.20}
\]

Next, direct calculations show that
\[
\frac{d}{dt} \| \nabla^2 \rho \|_{L^2}^2 \leq C \left( 1 + \| \nabla u \|_{L^\infty} \right) \| \nabla^2 \rho \|_{L^2}^2 + C \| \nabla u \|_{H^2}^2 \\
\leq C \left( 1 + \| \nabla u \|_{L^\infty} + \| \nabla^2 \theta \|_{L^2} \right) \left( 1 + \| \nabla^2 \rho \|_{L^2}^2 \right) + C \| \nabla \dot{u} \|_{L^2}^2, \tag{4.21}
\]

(4.15)
where one has used (3.6), (4.16), and the following estimate:

\[
\|\nabla u\|_{H^2} \leq C (\|\rho\|_{H^1} + \|H\|_{H^1} + \|\rho - 1\|_{H^2}) + C \\
\leq C \|
abla \rho\|_{L^2} + C \|\nabla u\|_{L^2} + C \|\rho - 1\|_{H^2} + 1 \\
+ C \|\nabla^2 \theta\|_{L^2} + C \\
\leq C \|
abla \nabla u\|_{L^2} + C (1 + \|\nabla^2 \theta\|_{L^2})(1 + \|\nabla^2 \rho\|_{L^2})
\]

(4.22)
due to (2.26), (3.10), (4.3), (4.16), (4.18), and (3.6). Then applying Grönwall’s inequality to (4.21) and using (4.3) and (4.17), it holds

\[
\sup_{0 \leq t \leq T} \|\nabla^2 \rho\|_{L^2} \leq C,
\]

(4.23)
which as well as (4.22), (4.20), and (4.3) yields

\[
\int_0^T (\|\nabla u\|^2_{H^3} + \|H\|^2_{H^3}) dt \leq C,
\]

(4.24)
where we have used the following fact:

\[
\|\nabla^3 H\|_{L^2} \leq C \|\text{curl}^2 H\|_{H^1} + C \|\nabla H\|_{L^2} \\
\leq C (\|H_t\|_{H^1} + \|H\|_{H^1} + \|\nabla u\|_{H^1} + \|\nabla H\|^2_{H^1} + \|\nabla \theta\|_{L^2}) \\
\leq C \|\nabla H\|_{H^2} + 1 \\
\]

(4.25)
owing to (2.25), (4.3), (3.13), and (4.20). Finally, combining the standard $H^1$-estimate to elliptic problem (3.75) with (3.6), (4.3), (4.16), (4.18), and (4.20), we obtain that

\[
\|\nabla^2 \theta\|_{H^1} \leq C \left( \|\rho \nabla \theta\|_{H^1} + \|\rho \nabla \nabla \theta\|_{H^1} + \|\nabla u\|_{H^1} + \|\nabla H\|^2_{H^1} + \|\nabla \theta\|_{L^2} \right) \\
\leq C \left( 1 + \|\nabla \theta\|_{L^2} + \|\rho^{1/2} \theta\|_{L^2} + \|\nabla (\rho \nabla \theta u)\|_{L^2} + \|\nabla u\|_{L^2} \right) \\
+ C \|\nabla H\|_{H^2} \|\nabla^2 \theta\|_{L^2} \\
\leq C \left( 1 + \|\nabla \theta\|_{L^2} + \|\rho^{1/2} \theta\|_{L^2} + C \|\nabla^3 u\|_{L^2} + \|\nabla^3 H\|_{L^2} \right).
\]

Then (4.4) is a consequence of (4.3), (3.10), (4.23), (4.24), (4.16)–(4.18), and (4.20) immediately. The proof of Lemma 4.1 is finished. □

**Lemma 4.2.** The following estimates hold:

\[
\sup_{0 \leq t \leq T} \|\rho_t\|_{H^1} + \int_0^T (\|u_t\|^2_{H^3} + \|\theta_t\|^2_{H^1} + \|\rho u_t\|^2_{H^3} + \|\rho \theta_t\|^2_{H^1}) dt \leq C,
\]

(4.27)

\[
\sup_{0 \leq t \leq T} \sigma (\|\nabla H_t\|^2_{L^2} + \|H_t\|^2_{H^3}) + \int_0^T \sigma (\|H_{tt}\|^2_{L^2} + \|\nabla H_t\|^2_{H^1}) dt \leq C,
\]

(4.28)

\[
\int_0^T \sigma (\|(\rho u)_t\|^2_{H^{-1}} + \|(\rho \theta)_t\|^2_{H^{-1}}) dt \leq C.
\]

(4.29)

**Proof.** First, one deduces from (1.1), (4.4), and (2.7) that

\[
\|\rho_t\|_{H^1} \leq \|\text{div}(\rho u)\|_{H^1} \leq C \|u\|_{H^2} (\|\rho - 1\|_{H^2} + 1) \leq C.
\]

(4.30)
Next, it follows from (4.3) and (4.4) that
\[
\sup_{0 \leq t \leq T} \int \left( \rho |u_t|^2 + \sigma \rho \theta_t^2 \right) dt + \int_0^T \left( \|\nabla u_t\|^2_{L^2} + \sigma \|\nabla \theta_t\|^2_{L^2} \right) dt \\
\leq \sup_{0 \leq t \leq T} \int \left( \rho |\dot{u}|^2 + \sigma |\rho \dot{\theta}|^2 \right) dt + \int_0^T \left( \|\nabla \dot{u}\|^2_{L^2} + \sigma \|\nabla \dot{\theta}\|^2_{L^2} \right) dt \\
+ \sup_{0 \leq t \leq T} \int \rho \left( |u \cdot \nabla u|^2 + |u \cdot \nabla \theta|^2 \right) dx \\
+ \int_0^T \left( \|\nabla (u \cdot \nabla u)\|^2_{L^2} + \sigma \|\nabla (u \cdot \nabla \theta)\|^2_{L^2} \right) dt \\
\leq C,
\] which along with (4.4) shows that
\[
\int_0^T \left( \|\nabla (\rho u_t)\|^2_{L^2} + \sigma \|\nabla (\rho \theta_t)\|^2_{L^2} \right) dt \\
\leq C \int_0^T \left( \|\nabla u_t\|^2_{L^2} + \|\nabla \rho\|^2_{L^2} \|u_t\|^2_{L^6} + \sigma \|\nabla \theta_t\|^2_{L^2} + \sigma \|\nabla \rho\|^2_{L^2} \|\theta_t\|^2_{L^6} \right) dt \\
\leq C.
\]
This together with (4.31), (4.30), (4.4), and (2.8) infers (4.27).

Note that Lemma (4.1) leads to
\[
\begin{aligned}
\| (H \cdot \nabla u - u \cdot \nabla H - H \div u_t) \|_{L^2} \\
&\leq C(\|H_t\|_{L^6} \|\nabla u\|_{L^4} + \|H\|_{L^6} \|\nabla u_t\|_{L^4} + \|u_t\|_{L^6} \||\nabla H\|_{L^4} + \|\nabla H_t\|_{L^2} \|u\|_{L^6}) \\
&\leq C(\|\nabla H_t\|_{L^2} + \|\nabla u_t\|_{L^2}).
\end{aligned}
\] (4.32)

Multiplying (3.66) by 2H_{tt} and integrating by parts, one has after using (4.32) that
\[
\nu(\|\text{curl} H_t\|^2_{L^2})' + 2H_{tt}^2 \leq C(\|\nabla H_t\|^2_{L^2} + \|\nabla u_t\|^2_{L^2}),
\] (4.33)
which as well as (4.3), (4.27), and (2.9) yields
\[
\sup_{0 \leq t \leq T} (\sigma \|\nabla H_t\|^2_{L^2}) + \int_0^T \sigma \|H_{tt}\|^2_{L^2} dt \leq C.
\] (4.34)

Next, notice that (2.25) holds as well for H_t. Combining this with (3.66) and (4.32) leads to
\[
\|\nabla^2 H_t\|_{L^2} \leq C(\|\text{curl}^2 H_t\|_{L^2} + \|\nabla H_t\|_{L^2}) \\
\leq C(\|H_{tt}\|_{L^2} + \|H \cdot \nabla u - u \cdot \nabla H - H \div u_t\|_{L^2} + \|\nabla H_t\|_{L^2}) \\
\leq C(\|H_{tt}\|_{L^2} + \|\nabla H_t\|_{L^2} + \|\nabla u_t\|_{L^2}),
\] (4.35)
which combined with (4.34), (4.3), (4.25), and (4.27) gives (4.28).

Finally, differentiating (3.12) with respect to t yields that
\[
(pu_t)_t = - (pu \cdot \nabla u)_t + ((2\mu + \lambda)\nabla \text{div} u - \mu \nabla \times \omega)_t - \nabla P_t \\
+ (H \cdot \nabla H - \nabla H \cdot H)_t,
\] (4.36)
It follows from (4.27), (4.4), (4.3), and (3.74) that
\[
\| (\rho u \cdot \nabla u)_t \|_{L^2} = \| \rho u \cdot \nabla u_\tau + \rho u_t \cdot \nabla u + \rho u \cdot \nabla u_\tau \|_{L^2} \\
\leq C\| \rho u \|_{L^6} \| \nabla u \|_{L^3} + C\| u_t \|_{L^6} \| \nabla u \|_{L^3} + C\| u \|_{L^\infty} \| \nabla u_\tau \|_{L^2} \tag{4.37}
\]

\[
\| \nabla P_t \|_{L^2} = R\| \rho_t \nabla \theta + \rho \nabla \theta_t + \nabla \rho_t \theta + \nabla \rho \theta_t \|_{L^2} \\
\leq C (\| \rho_t \|_{L^6} \| \nabla \theta \|_{L^3} + \| \nabla \theta_t \|_{L^2} + \| \theta \|_{L^\infty} \| \nabla \rho_t \|_{L^2} + \| \nabla \rho \|_{L^3} \| \theta_t \|_{L^6}) \tag{4.38}
\]

and
\[
\| (H \cdot \nabla H - \nabla H \cdot H)_t \|_{L^2} \leq C \| H_t \|_{L^6} \| \nabla H \|_{L^3} + C \| H \|_{L^\infty} \| \nabla H_t \|_{L^2} \\
\leq C \| \nabla H_t \|_{L^2}. \tag{4.39}
\]

The combination of (4.36)–(4.39) and (4.27) concludes
\[
\int_0^T \sigma \| (\rho u_t)_t \|_{H^{-1}}^2 dt \leq C. \tag{4.40}
\]

Analogously,
\[
\int_0^T \sigma \| (\rho \theta_t)_t \|_{H^{-1}}^2 dt \leq C,
\]

which together with (4.40) implies (4.29). The proof of Lemma 4.2 is completed. \qed

**Lemma 4.3.** The following estimate holds:
\[
\sup_{0 \leq t \leq T} \sigma (\| \nabla u_t \|_{L^2}^2 + \| \rho u_t \|_{L^2}^2 + \| u \|_{H^3}^2) + \int_0^T \sigma \left( \| \rho^{1/2} u_{tt} \|_{L^2}^2 + \| \nabla u_t \|_{H^1}^2 \right) dt \leq C. \tag{4.41}
\]

**Proof.** First, differentiating (3.12) with respect to \( t \) leads to
\[
\begin{cases}
(2\mu + \lambda) \nabla \text{div} u_t - \mu \nabla \times \text{curl} u_t + (H \cdot \nabla H - \frac{1}{2} \nabla |H|^2)_t \\
= \rho u_{tt} + \rho_t u_t + \rho u \cdot \nabla u + \rho u_t \cdot \nabla u + \rho u \cdot \nabla u_t + \nabla P_t, & \text{in } \Omega \times [0, T], \\
u_t \cdot n = 0, \ \text{curl} u_t \times n = 0, & \text{on } \partial \Omega \times [0, T], \\
u_t \rightarrow 0, & \text{as } |x| \rightarrow \infty.
\end{cases} \tag{4.42}
\]
Multiplying (4.42) by \(u_{tt}\) and integrating the resulting equality by parts yield

\[
\frac{1}{2} \frac{d}{dt} \int (\mu |\text{curl} u_t|^2 + (2\mu + \lambda)(\text{div} u_t)^2) \, dx + \int \rho |u_{tt}|^2 dx
\]

\[
= \frac{d}{dt} \left( -\frac{1}{2} \int \rho_t |u_t|^2 dx - \int \rho_t u \cdot \nabla u \cdot u_t dx + \int P_t \text{div} u_t dx \\
- \int \left( (H \otimes H)_t : \nabla u_t - \frac{1}{2} |H|^2 \text{div} u_t \right) dx \right) \\
+ \frac{1}{2} \int \rho_t |u_t|^2 dx + \int (\rho_t u \cdot \nabla u_t) \cdot u_t dx - \int \rho u \cdot \nabla u \cdot u_t dx \\
- \int \rho u \cdot \nabla u_t \cdot u_t dx - \int (P_t - \kappa(\gamma - 1) \Delta \theta_t) \text{div} u_t dx \\
+ \kappa(\gamma - 1) \int \nabla \theta_t \cdot \nabla \text{div} u_t dx \\
+ \frac{1}{2} \int (2(H \otimes H)_t : \nabla u_t - |H|^2 \text{div} u_t) \, dx \right) \triangleq \frac{d}{dt} \tilde{I}_0 + \sum_{i=1}^7 \tilde{I}_i.
\]

Each term \(\tilde{I}_i (i = 0, \cdots, 7)\) can be estimated as follows:

We deduce from simple calculations, (1.11), (4.27), (4.4), (4.3), (2.8), and (4.31) that

\[
|\tilde{I}_0| = \left| -\frac{1}{2} \int \rho_t |u_t|^2 dx \right|
\]

\[
\leq C \|\rho_t\|_{L^2} \|u_t\|_{L^6} \|\nabla u\|_{L^2} \|u_t\|_{L^6} \\
+ C \|\rho t\|_{L^2} \|\nabla u_t\|_{L^2} + C \|H\|_{L^\infty} \|H_t\|_{L^2} \|\nabla u_t\|_{L^2}
\]

\[
\leq C \int \rho |u_t|^2 dx + C \|\nabla u_t\|_{L^2} \] (4.44)

\[
2|\tilde{I}_1| = \left| \int (\rho_t u \cdot \nabla u_t) \cdot u_t dx \right|
\]

\[
\leq C \|\rho u_t\|_{L^2} \|u_t\|_{L^4} \frac{1}{2} \|u_t\|_{L^6}^{3/2} \\
\leq C \|\rho u_t\|_{L^2} \|\nabla u_t\|_{L^4} \|u_t\|_{L^6}^{3/2} \\
\leq C \|\rho u_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^4 + C \] (4.45)

\[
|\tilde{I}_2| = \left| \int \rho_t u \cdot \nabla u \cdot u_t dx \right|
\]

\[
\leq C \|\rho_t\|_{L^2} \|\nabla u\|_{L^2} \|u_t\|_{L^6} \|u_t\|_{L^6} + C \|\rho_t\|_{L^2} \|u_t\|_{L^6}^2 \|\nabla u_t\|_{L^6} \] (4.46)

\[
\leq C \|\rho u_t\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2 \\
\]

\[
|\tilde{I}_3| + |\tilde{I}_4| = \left| \int \rho u_t \cdot \nabla u \cdot u_{tt} dx \right| + \left| \int \rho u \cdot \nabla u_t \cdot u_{tt} dx \right|
\]

\[
\leq C \|\rho^{1/2} u_{tt}\|_{L^2} \left( \|u_t\|_{L^6} \|\nabla u\|_{L^3} + \|u\|_{L^\infty} \|\nabla u_t\|_{L^2} \right) \\
\leq \frac{1}{4} \|\rho^{1/2} u_{tt}\|_{L^2}^2 + C \|\nabla u_t\|_{L^2}^2 \] (4.47)
and
\[ |\tilde{I}_7| = \frac{1}{2} \int (2(H \otimes H)_{tt} : \nabla u_t - |H|^2_t \text{div} u_t) \, dx \]
\[ \leq C\|\nabla u_t\|_{L^2} (\|H\|_{L^\infty} \|H_{tt}\|_{L^2} + \|H_t\|_{L^2}^{1/2} \|\nabla H_t\|_{L^2}^{3/2}) \]  
\[ \leq C(\|\nabla u_t\|_{L^2}^2 + \|H_{tt}\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^3). \tag{4.48} \]

Then it follows from (3.30), (4.38), (4.25), and Lemma 4.1 that
\[ \|P_{tt} - \kappa(\gamma - 1)\Delta \theta_t\|_{L^2} \]
\[ \leq C\|(u \cdot \nabla P)_t\|_{L^2} + C\|(P \text{div} u)_t\|_{L^2} + C\|\nabla u\|\nabla u_t\|_{L^2} + C\|\nabla H\|\nabla H_t\|_{L^2} \]
\[ \leq C\|u_t\|_{L^6} \|\nabla P\|_{L^3} + C\|u\|_{L^\infty} \|\nabla P_t\|_{L^2} + C\|P_t\|_{L^6} \|\nabla u\|_{L^3} \]
\[ + C\|P\|_{L^\infty} \|\nabla u_t\|_{L^2} + C\|\nabla u\|_{L^\infty} \|\nabla u_t\|_{L^2} + C\|\nabla H\|_{L^\infty} \|\nabla H_t\|_{L^2} \]
\[ \leq C \left(1 + \|\nabla u\|_{L^\infty} + \|\nabla^2 \theta\|_{L^2}\right) \|\nabla u_t\|_{L^2} \]
\[ + C \left(1 + \|\nabla \theta_t\|_{L^2}^2 + \|\rho^{1/2} \theta_t\|_{L^2}^2 + \|H_t\|_{L^2}^4\right). \]

which implies
\[ |\tilde{I}_5| = \left|\int (P_{tt} - \kappa(\gamma - 1)\Delta \theta_t) \text{div} u_t \, dx \right| \]
\[ \leq C\|P_{tt} - \kappa(\gamma - 1)\Delta \theta_t\|_{L^2} \|\nabla u_t\|_{L^2} \]
\[ \leq C \left(1 + \|\nabla u\|_{L^\infty} + \|\nabla^2 \theta\|_{L^2}\right) \|\nabla u_t\|_{L^2}^2 \]
\[ + C \left(1 + \|\nabla \theta_t\|_{L^2}^2 + \|\rho^{1/2} \theta_t\|_{L^2}^2 + \|H_t\|_{L^2}^4\right). \tag{4.49} \]

Next, for the term \( \tilde{I}_6 \), Cauchy’s inequality implies
\[ |\tilde{I}_6| = \left|\kappa(\gamma - 1) \int \nabla \theta_t \cdot \nabla \text{div} u_t \, dx \right| \]
\[ \leq C\|\nabla^2 u_t\|_{L^2} \|\nabla \theta_t\|_{L^2} \]
\[ \leq \frac{1}{4}\|\rho^{1/2} u_t\|_{L^2}^2 + C \left(1 + \|\nabla u_t\|_{L^2}^2 + \|\rho^{1/2} \theta_t\|_{L^2}^2 + \|\nabla \theta_t\|_{L^2}^2 + \|H_t\|_{L^2}^4\right), \tag{4.50} \]

where we have used the following standard elliptic estimate to Lamé’s system (4.42):
\[ \|\nabla^2 u_t\|_{L^2} \]
\[ \leq C\|\rho u_t\|_{L^2} + C\|\rho t\|_{L^3} \|u_t\|_{L^6} + C\|\rho_t\|_{L^3} \|\nabla u_t\|_{L^6} \|u\|_{L^\infty} \]
\[ + C\|u_t\|_{L^6} \|\nabla u_t\|_{L^2} + C\|u_t\|_{L^6} \|u_t\|_{L^\infty} + C\|\nabla P_t\|_{L^2} \]
\[ + C\|H_t\|_{L^6} \|\nabla H_t\|_{L^2} + C\|H\|_{L^\infty} \|\nabla H_t\|_{L^2} \]
\[ \leq C \left(\|\rho u_t\|_{L^2} + \|\nabla u_t\|_{L^2} + \|\rho^{1/2} \theta_t\|_{L^2} + \|\nabla \theta_t\|_{L^2} + \|H_t\|_{L^2} + 1\right) \]  
\[ \tag{4.51} \]

due to Lemma 4.1 (4.27), and (4.38).

Submitting (4.45)–(4.50) into (4.43) arrives at
\[ \frac{d}{dt} \int \left(\mu |\text{curl} u|^2 + (2\mu + \lambda)(\text{div} u_t)^2 - 2\tilde{I}_0\right) \, dx + \int \rho |u_t|^2 \, dx \]
\[ \leq C \left(1 + \|\nabla u\|_{L^\infty} + \|\nabla u_t\|_{L^2}^2 + \|\nabla^2 \theta\|_{L^2}^2\right) \|\nabla u_t\|_{L^2}^2 \]
\[ + C \left(1 + \|\rho u\|_{L^2}^2 + \|\rho^{1/2} \theta\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 + \|H_t\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^4\right). \tag{4.52} \]
Multiplying \((4.52)\) by \(\sigma\) and integrating the resulting inequality over \((0, T)\), it follows from \((2.9), (4.3), (4.4), (4.31), (4.44), (4.28), (4.27)\), and Grönwall’s inequality that
\[
\sup_{0 \leq t \leq T} \sigma \|\nabla u_t\|_{L^2}^2 + \int_0^T \sigma \|\rho_{tt}\|^2 \, dx \, dt \leq C, \quad (4.53)
\]
where we have used the following estimate:
\[
\|\rho_{tt}\|_{L^2} = \|\text{div}(\rho u_t)\|_{L^2} \\
\leq C \left( \|\rho_t\|_{L^6} \|\nabla u\|_{L^3} + \|\nabla u_t\|_{L^2} + \|u_t\|_{L^6} \|\nabla \rho\|_{L^3} + \|\nabla \rho_t\|_{L^2} \right) \quad (4.54)
\]
owing to \((1.1)_{11}, (4.4),\) and \((4.27)\). Finally, we deduce from Lemma \(4.1\), \((4.54), (4.22), (4.51), (4.27)\), and \((4.53)\) that
\[
\sup_{0 \leq t \leq T} \sigma \left( \|\rho_{tt}\|_{L^2}^2 + \|u_t\|_{H^3}^2 \right) + \int_0^T \sigma \|\nabla u_t\|_{H^1}^2 \, dt \leq C,
\]
which combined with \((4.53)\) shows \((4.41)\). The proof of Lemma \(4.3\) is finished. \(\square\)

**Lemma 4.4.** For \(q \in (3, 6)\) as in Theorem 1.1, it holds that
\[
\sup_{0 \leq t \leq T} \|\rho - 1\|_{W^{2,q}} + \int_0^T \|\nabla^2 u\|_{W^{1,q}} \, dt \leq C, \quad (4.55)
\]
where
\[
1 < p_0 < 4q/(5q - 6) \in (1/4, 3). \quad (4.56)
\]

**Proof.** First, it follows from \((2.26)\) and Lemma 4.1 that
\[
\|\nabla^2 u\|_{W^{1,q}} \leq C \left( \|\rho u\|_{W^{1,q}} + \|\nabla P\|_{W^{1,q}} + \|H\|_{W^{1,q}} \right) + C \|\nabla \rho\|_{L^q} \|\nabla \theta\|_{L^3} + \|\theta \nabla^2 \rho\|_{L^p} \\
+ \|\nabla \theta\|_{L^3} + \|\nabla^2 \theta\|_{L^3} + \|\nabla H\|_{L^q} \|\nabla^2 H\|_{L^3} + 1 \quad (4.57)
\]
which combined with \((1.1)_{11}, (4.4),\) and \((4.27)\) yields
\[
\frac{d}{dt} \|\nabla^2 \rho\|_{L^q}^q \\
\leq C \|\nabla u\|_{L^q} \|\nabla^2 \rho\|_{L^q}^q + C \|\nabla^2 u\|_{W^{1,q}} \|\nabla^2 \rho\|_{L^q}^{q-1} \|\nabla \rho\|_{L^q} + 1 \quad (4.58)
\]
\[
\leq C \|u\|_{H^3} \|\nabla^2 \rho\|_{L^q}^q + C \|\nabla^2 u\|_{W^{1,q}} \|\nabla^2 \rho\|_{L^q}^{q-1} \\
\leq C \left( \|u\|_{H^3} + \|\nabla u\|_{L^q} + \|\nabla (\rho u)\|_{L^q} + \|\nabla \rho\|_{L^q} + \|\nabla^2 H\|_{L^2} + 1 \right) \cdot \|\nabla^2 \rho\|_{L^q}^q + 1.
\]

Note that by Lemma 4.1, \((2.5)\), and \((4.41)\), one has
\[
\|\nabla (\rho u)\|_{L^q} \leq C \|\rho\|_{L^q} \|\nabla u\|_{L^6}^{q/(3(q-2))} \|\nabla u\|_{L^q}^{2(q-3)/3(q-2)} + C \|\nabla \rho\|_{L^q} \\
\leq C \|\nabla u\|_{L^q} + C \|\nabla \rho\|_{L^q} \\
\leq C \|\nabla u_t\|_{L^q} + C \|\nabla u_t\|_{L^q} + C \|\nabla (u \cdot \nabla u)\|_{L^q} + C \|\nabla (u \cdot \nabla u)\|_{L^q} \quad (4.59)
\]
\[
\leq C \sigma^{-1/2} \|\nabla u_t\|_{L^q}^{(6q)/2q} \|\nabla u_t\|_{L^q}^{3(q-2)/2q} + C \|\nabla u\|_{L^q} \|\nabla^2 u\|_{L^q} \\
\leq C \sigma^{-1/2} \left( \sigma \|\nabla u_t\|_{H^1}^{3(q-2)/4q} + C \|u\|_{H^3} + C \sigma^{-1/2},
\right.
\]
and
\[
\|\nabla^2 \theta\|_{L^q} \leq C \|\nabla^2 \theta\|_{L^2}^{(6-q)/2q} \|\nabla^3 \theta\|_{L^2}^{3(q-2)/2q}
\]
\[
\leq C \sigma^{-1/2} \left( \sigma \|\nabla^2 \theta\|_{L^2}^2 \right)^{(q-2)/4q}.
\]
Combining these with Lemma 4.1 and (4.41) shows that, for \( p_0 \) as in (4.56),
\[
\int_0^T \left( \|\nabla (\rho \dot{u})\|_{L^p}^{p_0} + \|\nabla^2 \theta\|_{L^p}^{p_0} \right) \, dt \leq C.
\] (4.60)

Finally, applying Grönwall’s inequality to (4.58), we deduce from (4.3), (4.4), and (4.60) that
\[
\sup_{0 \leq t \leq T} \|\nabla^2 \rho\|_{L^q} \leq C,
\]
which along with Lemma 4.1, (4.60), and (4.57) infers (4.55). The proof of Lemma 4.4 is finished.

**Lemma 4.5.** For \( q \in (3, 6) \) as in Theorem 1.1, the following estimate holds:
\[
\sup_{0 \leq t \leq T} \sigma \left( \|\theta_t\|_{H^1} + \|H_{tt}\|_{L^2} + \|u_t\|_{H^2} + \|H_t\|_{H^2} \right)
\]
\[
+ \sup_{0 \leq t \leq T} \sigma \left( \|\nabla^2 \theta\|_{H^1} + \|u\|_{W^{3,q}} + \|H\|_{H^4} \right)
\]
\[
+ \int_0^T \sigma^2 \|\nabla u_t\|_{L^2}^2 + \|\nabla H_{tt}\|_{L^2}^2 \, dt \leq C.
\] (4.61)

**Proof.** First, differentiating (4.42) and (3.66) with respect to \( t \) respectively yields
\[
\begin{cases}
\rho u_{ttt} + \rho u \cdot \nabla u_{tt} - (2\mu + \lambda) \nabla (\text{div} u_{tt}) + \mu \nabla \times \text{curl} u_{tt} \\
= 2 \text{div}(\rho u) u_{tt} + \text{div}(\rho u_t u_t - 2\rho u_t) \cdot \nabla u_{tt} - \rho u_t \cdot \nabla u \\
- (\rho_t u + 2\rho u_t) \cdot \nabla u - \nabla P_{tt} + (H \cdot \nabla H - \frac{1}{2} \nabla |H|^2)_{tt}, \\
\text{in } \Omega \times [0, T], \\
u_{tt} \cdot n = 0, \quad \text{curl} u_{tt} \times n = 0, \\
u_{tt} \to 0, \\
\end{cases}
\]
\[
\begin{cases}
H_{ttt} + \nu \nabla \times (\text{curl} H_{tt}) = (H \cdot \nabla u - u \cdot \nabla H - H \text{div} u)_{tt}, \\
\text{in } \Omega \times [0, T], \\
H_{tt} \cdot n = 0, \quad \text{curl} H_{tt} \times n = 0, \\
H_{tt} \to 0, \\
\end{cases}
\] (4.62)

(4.63)

Multiplying (4.62)_1 and (4.63)_1 by \( u_{tt} \) and \( H_{tt} \) respectively and integrating the resulting equality over \( \Omega \) by parts lead to
\[
\frac{1}{2} \frac{d}{dt} \int (\rho |u_{tt}|^2 + |H_{tt}|^2) \, dx
\]
\[
+ \int \left( (2\mu + \lambda)(\text{div} u_{tt})^2 + \mu |\text{curl} u_{tt}|^2 + \nu |\text{curl} H_{tt}|^2 \right) \, dx
\]
\[
= -4 \int u_{tt} \rho u \cdot \nabla u_{tt} \, dx - \int (\rho u)_{tt} \cdot (\nabla (u_t \cdot u_{tt}) + 2 \nabla u_t \cdot u_{tt}) \, dx
\]
\[
- \int (\rho_t u + 2\rho u_t) \cdot \nabla u \cdot u_{tt} \, dx - \int \rho u_{tt} \cdot \nabla u \cdot u_{tt} \, dx
\]
\[
+ \int P_{tt} \text{div} u_{tt} \, dx - \frac{1}{2} \int (2(H \otimes H)_{tt} : \nabla u_{tt} - |H_{tt}|^2 \text{div} u_{tt}) \, dx
\]
\[
+ \int (H \cdot \nabla u - u \cdot \nabla H - H \text{div} u)_{tt} H_{tt} \, dx \triangleq \sum_{i=1}^7 J_i.
\] (4.64)
We deduce from Lemmas 4.1–4.3, (4.31), (2.8), and (4.54) that, for \( \eta \in (0, 1) \),
\[
|\mathcal{J}_1| \leq C\|\rho^{1/2}u_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2} \|u\|_{L^\infty} \leq \eta\|\nabla u_{tt}\|_{L^2}^2 + C(\eta)\|\rho^{1/2}u_{tt}\|_{L^2}^2, \tag{4.65}
\]
\[
|\mathcal{J}_2| \leq C\left(\|\rho u_{tt}\|_{L^3} + \|\rho_{t} u\|_{L^3}\right)\left(\|\nabla u_{tt}\|_{L^2} \|u\|_{L^6} + \|u_{tt}\|_{L^6} \|\nabla u_{tt}\|_{L^2}\right)
\leq C\left(\|\rho^{1/2}u_{tt}\|_{L^2}^2 \|u_{tt}\|_{L^6}^2 + \|\rho_{t} u\|_{L^6} \|u\|_{L^6}\right) \|\nabla u_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2}
\leq \eta\|\nabla u_{tt}\|_{L^2}^2 + C(\eta)\|\nabla u_{tt}\|_{L^2}^3 + C(\eta)
\leq \eta\|\nabla u_{tt}\|_{L^2}^2 + C(\eta)\sigma^{-3/2}, \tag{4.66}
\]
\[
\begin{align*}
|\mathcal{J}_3| & \leq C\left(\|\rho u\|_{L^2} \|u\|_{L^6} + \|\rho_{t} u\|_{L^2} \|u\|_{L^6} + \|u_{tt}\|_{L^6}\right) \|\nabla u\|_{L^6} \|u_{tt}\|_{L^6}
\leq \eta\|\nabla u_{tt}\|_{L^2}^2 + C(\eta)\sigma^{-1},
\end{align*}
\tag{4.67}
\]
\[
|\mathcal{J}_4| + |\mathcal{J}_5|
\leq C\|\rho u_{tt}\|_{L^2} \|\nabla u\|_{L^3} \|u_{tt}\|_{L^6} + C\|\rho\theta + \rho_{t}\|_{L^2} \|\nabla u_{tt}\|_{L^2}
\leq \eta\|\nabla u_{tt}\|_{L^2}^2 + C(\eta)\left(\|\rho^{1/2}u_{tt}\|_{L^2}^2 + \|\rho\theta\|_{L^2}^2 + \|\rho_{t} \theta\|_{L^2}^2 + \|\rho^{1/2}\theta_{tt}\|_{L^2}^2\right)
\leq \eta\|\nabla u_{tt}\|_{L^2}^2 + C(\eta)\left(\|\rho^{1/2}u_{tt}\|_{L^2}^2 + \|\nabla \theta_{t}\|_{L^2}^2 + \|\rho^{1/2}\theta_{tt}\|_{L^2}^2 + \sigma^{-2}\right),
\tag{4.68}
\]
\[
|\mathcal{J}_6| \leq C\|\nabla u_{tt}\|_{L^2}(\|H\|_{L^\infty} \|H_{tt}\|_{L^2} + \|H_{tt}\|_{L^2}^1/2 \|\nabla H_{tt}\|_{L^2}^3/2)
\leq \eta\|\nabla u_{tt}\|_{L^2}^2 + C(\eta)(\|H_{tt}\|_{L^2}^2 + \|\nabla H_{tt}\|_{L^2}^3)
\leq \eta\|\nabla u_{tt}\|_{L^2}^2 + C(\eta)(\|H_{tt}\|_{L^2}^2 + \sigma^{-3/2}),
\tag{4.69}
\]
and
\[
|\mathcal{J}_7| \leq C\|H_{tt}\|_{L^2}(\|H\|_{L^\infty} \|\nabla u_{tt}\|_{L^2} + \|u\|_{L^\infty} \|\nabla H_{tt}\|_{L^2})
+ C\|H_{tt}\|_{L^2}(\|H_{tt}\|_{L^5} \|\nabla u_{tt}\|_{L^3} + \|u_{tt}\|_{L^6} \|\nabla H\|_{L^3})
+ C\|H_{tt}\|_{L^2} \|H_{tt}\|_{L^3} \|\nabla u_{tt}\|_{L^2} + \|u_{tt}\|_{L^3} \|\nabla H_{tt}\|_{L^2}
\leq \eta(\|\nabla u_{tt}\|_{L^2}^2 + \|\nabla H_{tt}\|_{L^2}^2) + C(\eta)(\|H_{tt}\|_{L^2}^2 + \|H_{tt}\|_{L^2}^2/2 + \|u_{tt}\|_{L^2}^2))
\leq \eta(\|\nabla u_{tt}\|_{L^2}^2 + \|\nabla H_{tt}\|_{L^2}^2) + C(\eta)(\|H_{tt}\|_{L^2}^2 + \sigma^{-2}).
\tag{4.70}
\]
Putting (4.65)–(4.70) into (4.64), we obtain after using (2.9) and choosing \( \eta \) suitably small that
\[
\frac{d}{dt} \int (\rho |u_{tt}|^2 + |H_{tt}|^2) dx + C_8 \int (\|\nabla u_{tt}\|^2 + |\nabla H_{tt}|^2) dx
\leq C\sigma^{-2} + C\|\rho^{1/2}u_{tt}\|_{L^2}^2 + C\|\nabla \theta_{t}\|_{L^2}^2 + C\|H_{tt}\|_{L^2}^2 + C_9\|\rho^{1/2}\theta_{tt}\|_{L^2}^2.
\tag{4.71}
\]
Next, differentiating (3.75) with respect to \( t \) gives
\[
\begin{cases}
-\frac{\kappa(\gamma-1)}{\rho} \Delta \theta_{t} + \rho \theta_{tt} \\
= -\rho_{t} \theta_{t} - \rho (u \cdot \nabla \theta + (\gamma - 1) \theta \text{div} u) - \rho (u \cdot \nabla \theta + (\gamma - 1) \theta \text{div} u)_{t}
+ \frac{\kappa(\gamma-1)}{\rho} \left(\lambda(\text{div} u)^2 + 2\mu \|\nabla u\|^2 + \nu \|\text{curl} H\|^2\right)_{t},
& \text{in } \Omega \times [0, T),

\nabla \theta_{t} \cdot n = 0, & \text{on } \partial \Omega \times [0, T],

\nabla \theta_{t} \to 0, & \text{as } |x| \to \infty.
\end{cases}
\tag{4.72}
\]
\[ (\kappa(\gamma - 1) \left\| \nabla \theta_t \right\|_{L^2}^2 + H_0) + \int \rho \theta_t^2 \, dx \]
\begin{align*}
&= \frac{1}{2} \int \rho \theta_t^2 \, dx + \int \rho \left( u \cdot \nabla \theta + (\gamma - 1) \partial \Omega \partial u \right) \theta_t \, dx \\
&+ \int \rho \left( u \cdot \nabla \theta + (\gamma - 1) \partial \Omega \partial u \right) \, \theta_t \, dx \\
&- \int \rho \left( u \cdot \nabla \theta + (\gamma - 1) \partial \Omega \partial u \right) \, \theta_{tt} \, dx \\
&- \frac{\gamma - 1}{R} \int (\lambda(\partial \Omega \partial u)^2 + 2\mu(\partial \Omega \partial u)^2 + \nu |\partial \Omega \partial u|^2) \, \theta_{tt} \, dx \triangleq \sum_{i=1}^{4} H_i,
\end{align*}

where

\[ H_0 \triangleq \frac{1}{2} \int \rho \theta_t^2 \, dx + \int \rho \left( u \cdot \nabla \theta + (\gamma - 1) \partial \Omega \partial u \right) \, \theta_t \, dx \]

It follows from (4.74), (4.31), (4.54), and Lemmas 4.1 and 4.3 that

\[ |H_0| \leq C \int \rho |u||\theta_t||\nabla \theta_t|_L^2 + C \left( \left\| \nabla u \right\|_{L^2} \left\| \nabla u_t \right\|_{L^2} + \left\| \nabla H \right\|_{L^2} \left\| \nabla H_t \right\|_{L^2} \right) \left\| \theta_t \right\|_{L^6} \]

\[ + C \left( \left\| \theta_t \right\|_{L^5} \left\| \nabla \theta \right\|_{L^3} \left\| \nabla \theta_t \right\|_{L^2} + \left\| \nabla u \right\|_{L^2} + \left\| \theta - 1 \right\|_{L^6} \left\| \nabla u \right\|_{L^3} \right) \]

\[ \leq C \left\| \nabla \theta_t \right\|_{L^2} \left( \left\| \theta_t \right\|_{L^5} \left\| \nabla \theta \right\|_{L^3} \left\| \nabla u_t \right\|_{L^2} + \left\| \nabla u \right\|_{L^2} + \left\| \nabla H \right\|_{L^2} + 1 \right) \]

\[ \leq \frac{\kappa(\gamma - 1)}{4R} \left\| \nabla \theta_t \right\|_{L^2}^2 + C \sigma^{-1}, \]

\[ |H_1| \leq C \left( \left( \left\| \nabla u \right\|_{L^2} + \left\| \theta_t \right\|_{L^5} \left\| \nabla \theta \right\|_{L^3} + \left\| \nabla u_t \right\|_{L^3} + \left\| \theta - 1 \right\|_{L^6} \left\| \nabla u \right\|_{L^3} \right) \right) \]

\[ \leq C \left( \left\| \nabla \theta_t \right\|_{L^2} \left( \left\| \theta_t \right\|_{L^5} \left\| \nabla \theta \right\|_{L^3} \left\| \nabla u_t \right\|_{L^2} + \left\| \nabla u \right\|_{L^2} + \left\| \nabla H \right\|_{L^2} + 1 \right) \right) \]

\[ \leq C \left( 1 + \left\| \nabla u_t \right\|_{L^2} \right) \left\| \nabla \theta_t \right\|_{L^2}^2 + C \sigma^{-3/2}, \]

\[ |H_4| \leq C \int \left( \left| \nabla u \right|^2 + \left| \nabla u_t \right|^2 + \left| \nabla H_t \right|^2 + \left| \nabla H \right| \left| \nabla H_{tt} \right| \right) \left| \theta_t \right| \, dx \]

\[ \leq C \left( \left\| \nabla u \right\|_{L^2}^{3/2} \left\| \nabla u_t \right\|_{L^2}^{1/2} + \left\| \nabla u \right\|_{L^2} \left\| \nabla u_t \right\|_{L^2} \right) \left\| \theta_t \right\|_{L^6} \]

\[ + C \left( \left\| \nabla H_t \right\|_{L^2}^{3/2} \left\| \nabla H \right\|_{L^2}^{1/2} + \left\| \nabla H \right\|_{L^2} \left\| \nabla H_{tt} \right\|_{L^2} \right) \left\| \theta_t \right\|_{L^6} \]

\[ \leq \delta \left( \left\| \nabla u_t \right\|_{L^2}^2 + \left\| \nabla H_{tt} \right\|_{L^2}^2 \right) + C \left( \left\| \nabla u_t \right\|_{L^2}^2 + \left\| \nabla H_t \right\|_{L^2}^2 \right) \]

\[ + C\left( \delta \left\| \nabla \theta_t \right\|_{L^2}^2 + C \sigma^{-1} \left( \left\| \nabla u_t \right\|_{L^2}^2 + \left\| \nabla H_t \right\|_{L^2}^2 \right) \right) \]

and

\[ |H_2| + |H_3| \leq C \left( \sigma^{-1/2} \left\| \nabla \theta_t \right\|_{L^2} + \left\| \nabla \theta_t \right\|_{L^2} \right) \left( \left\| \rho \right\|_{L^5} \left\| \theta_t \right\|_{L^6} + \left\| \rho \theta_t \right\|_{L^2} \right) \]

\[ \leq \frac{1}{2} \int \rho \theta_t^2 \, dx + C \left| \nabla \theta_t \right|_{L^2}^2 + C \sigma^{-1} \left| \nabla u_t \right|_{L^2}^2, \]
where in the last inequality we have used the following fact:

\[
\| (u \cdot \nabla \theta + (\gamma - 1) \theta \div u)_{t} \|_{L^{2}} \\
\leq C (\| u_{t} \|_{L^{6}} \| \nabla \theta \|_{L^{3}} + \| \nabla u \|_{L^{6}} \| \theta \|_{L^{\infty}} + \| \nabla u_{t} \|_{L^{2}} + \| \theta \|_{L^{\infty}} \| \nabla u_{t} \|_{L^{2}})
\]

(4.78)
due to Lemma 4.1.

Then, substituting (4.75)–(4.77) into (4.73) infers

\[
\left( \frac{k(\gamma - 1)}{2R} \| \nabla \theta \|_{L^{2}}^{2} + H_{0} \right)_{t} + \frac{1}{2} \int \rho \theta_{tt}^{2} dx
\leq \delta (\| \nabla u_{tt} \|_{L^{2}}^{2} + \| \nabla H_{tt} \|_{L^{2}}^{2}) + C(\delta) ((1 + \| \nabla u_{t} \|_{L^{2}}) \| \nabla \theta \|_{L^{2}}^{2} + \sigma^{-3/2})
\]

(4.79)

\[+ C(\| \nabla^{2} u_{t} \|_{L^{2}}^{2} + \| \nabla^{2} H_{t} \|_{L^{2}}^{2}) + C\sigma^{-2}(\| \nabla u_{t} \|_{L^{2}}^{2} + \| \nabla H_{t} \|_{L^{2}}^{2}).\]

Finally, for \( C_{0} \) as in (4.71), adding (4.79) multiplied by \( 2(C_{0} + 1) \) to (4.71) and choosing \( \delta \) suitably small, one derives

\[
\begin{align*}
2(C_{0} + 1) \left( \frac{k(\gamma - 1)}{2R} \| \nabla \theta \|_{L^{2}}^{2} + H_{0} \right)_{t} + \int (\rho |u_{tt}|^{2} + |H_{tt}|^{2}) dx \\
+ \int \rho \theta_{tt} dx + C_{8} \int (\| \nabla u_{tt} \|^{2} + \| \nabla H_{tt} \|^{2}) dx \\
\leq C(1 + \| \nabla u_{t} \|_{L^{2}}^{2} + \| \nabla H_{t} \|_{L^{2}}^{2}) (\sigma^{-2} + \| \nabla \theta \|_{L^{2}}^{2}) + C(\| \nabla u_{t} \|_{L^{2}}^{2} + \| \nabla u_{tt} \|_{L^{2}}^{2}) \]
\[+ C(\| H_{tt} \|_{L^{2}}^{2} + C(\| \nabla^{2} u_{t} \|_{L^{2}}^{2} + \| \nabla^{2} H_{t} \|_{L^{2}}^{2}).\]
\]

(4.80)

Multiplying (4.80) by \( \sigma^{2} \) and integrating the resulting inequality over \((0, T)\), we obtain after using (4.74), (4.3), (4.41), (4.28), (4.27), and Grönwall’s inequality that

\[
\sup_{0 \leq t \leq T} \sigma^{2} \int (\| \nabla \theta \|^{2} + \rho |u_{tt}|^{2} + |H_{tt}|^{2}) dx \\
+ \int_{0}^{T} \sigma^{2} \int (\rho \theta_{tt}^{2} + |\nabla u_{tt}|^{2} + |\nabla H_{tt}|^{2}) dx dt \leq C,
\]

(4.81)

which as well as Lemmas 4.1, 4.4, 4.51, 4.26, 4.31, 4.57, 4.35, and 4.59 yields

\[
\sup_{0 \leq t \leq T} \sigma \left( \| \nabla u_{t} \|_{H^{1}} + \| \nabla H_{t} \|_{H^{2}} + \| |H| \|_{H^{1}} + \| \nabla^{2} \theta \|_{H^{1}} + \| \nabla^{2} u \|_{W^{1,2}} \right) \leq C,
\]

(4.82)

where we have used the following estimate:

\[
\| \nabla^{4} H \|_{L^{2}} \leq C(\| \text{curl}^{2} H \|_{H^{2}} + \| \nabla H \|_{L^{2}})
\]

\[\leq C(\| H_{t} \|_{H^{2}} + \| H \cdot \nabla u - u \cdot \nabla H - H \div u \|_{H^{2}} + 1)
\]

\[\leq C(\| H_{t} \|_{H^{2}} + \| H \|_{H^{2}} \| \nabla u \|_{H^{2}} + \| u \|_{H^{2}} \| \nabla H \|_{H^{2}} + 1)
\]

owing to Lemma 4.1, 2.25, 3.13, and (2.7).

Hence, one concludes (4.61) by using (4.81), (4.82), (4.31), (2.8), and (4.4). The proof of Lemma 4.5 is completed.

**Lemma 4.6.** The following estimate holds:

\[
\sup_{0 \leq t \leq T} \sigma^{2} \left( \| \nabla^{2} \theta \|_{H^{2}} + \| \theta_{t} \|_{H^{2}} + \| \rho^{1/2} \theta_{tt} \|_{L^{2}} \right) + \int_{0}^{T} \sigma^{4} \| \nabla \theta_{tt} \|_{L^{2}}^{2} dt \leq C.
\]

(4.83)
Proof. First, differentiating (4.72) with respect to $t$ gives

\[
\begin{aligned}
\begin{cases}
\rho_{tt} \tau - \frac{\kappa (\gamma - 1)}{R} \Delta \theta_{tt} \\
= -\rho u \cdot \nabla \theta_{tt} - \rho_{tt} (\theta_t + u \cdot \nabla \theta + (\gamma - 1) \theta \text{div} u) \\
+ 2 \text{div} (\rho u) \theta_{tt} - 2 \rho_t (u \cdot \nabla \theta + (\gamma - 1) \theta \text{div} u) + \\
- \rho (u_{tt} \cdot \nabla \theta + 2 u_t \cdot \nabla \theta_t + (\gamma - 1) (\theta \text{div} u)_{tt}) \\
+ \frac{\gamma - 1}{R} \left( \lambda (\text{div} u)^2 + 2 \mu \mathcal{D}(u)^2 + \nu |\text{curl} H|^2 \right)_{tt},
\end{cases}
\end{aligned}
\]

in $\Omega \times [0, T]$, \( \nabla \theta_{tt} \cdot n = 0 \), on $\partial \Omega \times [0, T]$, \( \nabla \theta_{tt} \to 0 \) as $|x| \to \infty$.

Multiplying (4.84)$_1$ by $\theta_{tt}$ and integrating the resulting equality over $\Omega$ lead to

\[
\frac{1}{2} \frac{d}{dt} \int \rho |\theta_{tt}|^2 dx + \frac{\kappa (\gamma - 1)}{R} \int |\nabla \theta_{tt}|^2 dx = -4 \int \theta_{tt} \rho u \cdot \nabla \theta_{tt} dx - \int \rho_{tt} (\theta_t + u \cdot \nabla \theta + (\gamma - 1) \theta \text{div} u) \theta_{tt} dx \\
- 2 \int \rho_t (u \cdot \nabla \theta + (\gamma - 1) \theta \text{div} u) \theta_{tt} dx \\
- \int \rho (u_{tt} \cdot \nabla \theta + 2 u_t \cdot \nabla \theta_t + (\gamma - 1) (\theta \text{div} u)_{tt}) \theta_{tt} dx \\
+ \frac{\gamma - 1}{R} \int (\lambda (\text{div} u)^2 + 2 \mu \mathcal{D}(u)^2 + \nu |\text{curl} H|^2)_{tt} \theta_{tt} dx \leq \sum_{i=1}^5 K_i.
\]

It follows from Lemmas 4.1, 4.3, 4.5 and (2.8) that

\[
\sigma^4 |K_1| \leq C \sigma^4 \|\rho^{1/2} \theta_{tt}\|_{L^2} \|\nabla \theta_{tt}\|_{L^2} \|u\|_{L^\infty} \\
\leq \delta \sigma^4 \|\nabla \theta_{tt}\|_{L^2}^2 + C(\delta) \sigma^4 \|\rho^{1/2} \theta_{tt}\|_{L^2}^2,
\]

(4.86)

\[
\sigma^4 |K_2| \leq C \sigma^4 \|\rho_{tt}\|_{L^2} \|\theta_{tt}\|_{L^6} (\|\theta_{tt}\|_{H^1} + \|\nabla \theta\|_{L^4} + 1) \\
\leq \delta \sigma^4 \|\nabla \theta_{tt}\|_{L^2}^2 + C(\delta),
\]

(4.87)

\[
\sigma^4 |K_3| \leq C \sigma^4 \|\theta_{tt}\|_{L^2} \left( \|\nabla \theta\|_{L^3} \|\rho u_{tt}\|_{L^2} + \|\nabla \theta_t\|_{L^2} \|\nabla u_t\|_{L^2} \right) \\
+ C \sigma^4 \|\nabla u_{tt}\|_{L^6} \left( \|\nabla u\|_{L^3} \|\rho \theta_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} \|\theta_{tt}\|_{L^3} \right) \\
+ C \sigma^4 \|\theta\|_{L^\infty} \|\rho \theta_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2} \\
\leq \delta \sigma^4 \|\nabla \theta_{tt}\|_{L^2}^2 + C(\delta) \left( \sigma^4 \|\rho^{1/2} \theta_{tt}\|_{L^2}^2 + \sigma^3 \|\nabla u_{tt}\|_{L^2}^2 \right) + C(\delta),
\]

(4.88)

\[
\sigma^4 |K_4| \leq C \sigma^4 \|\theta_{tt}\|_{L^6} \left( \|\nabla u_t\|_{L^3} \|\nabla \theta\|_{L^2} + \|\nabla u\|_{L^3} \|\rho u_{tt}\|_{L^2} + \|\nabla u_{tt}\|_{L^2} \|\nabla u_t\|_{L^2} \right) \\
+ C \sigma^4 \|\theta_{tt}\|_{L^6} \left( \|\nabla u\|_{L^3} \|\rho \theta_{tt}\|_{L^2} + \|\nabla u_t\|_{L^3} \|\theta_{tt}\|_{L^3} \right) \\
+ C \sigma^4 \|\theta\|_{L^\infty} \|\rho \theta_{tt}\|_{L^2} \|\nabla u_{tt}\|_{L^2} \\
\leq \delta \sigma^4 \|\nabla \theta_{tt}\|_{L^2}^2 + C(\delta) \left( \sigma^4 \|\rho^{1/2} \theta_{tt}\|_{L^2}^2 + \sigma^3 \|\nabla u_{tt}\|_{L^2}^2 \right) + C(\delta),
\]

(4.89)

\[
\sigma^4 |K_5| \leq C \sigma^4 \|\theta_{tt}\|_{L^6} \left( \|\nabla u_{tt}\|_{L^2}^{3/2} \|\nabla u_t\|_{L^2}^{1/2} + \|\nabla u\|_{L^3} \|\nabla u_{tt}\|_{L^2} \right) \\
+ C \sigma^4 \|\theta_{tt}\|_{L^6} \left( \|\nabla H\|_{L^6}^{3/2} \|\nabla H_t\|_{L^2}^{1/2} + \|\nabla H\|_{L^3} \|\nabla H_{tt}\|_{L^2} \right) \\
\leq \delta \sigma^4 \|\nabla \theta_{tt}\|_{L^2}^2 + C(\delta) \sigma^4 \left( \|\nabla u_{tt}\|_{L^2}^2 + \|\nabla H_{tt}\|_{L^2}^2 \right) + C(\delta),
\]

and

\[
\sigma^4 |K_3| \leq C \sigma^4 \|\rho_t\|_{L^3} \|\theta_{tt}\|_{L^6} \left( \sigma^{-1/2} \|\nabla u_t\|_{L^2} + \|\nabla \theta_t\|_{L^2} \right) \\
\leq \delta \sigma^4 \|\nabla \theta_{tt}\|_{L^2}^2 + C(\delta),
\]

(4.90)

where in the last inequality we have used (4.78).
Next, multiplying (4.85) by $\sigma^4$ and substituting (4.86)–(4.90) into the resulting inequality, one obtains after choosing $\delta$ suitably small that
\[
\frac{d}{dt} \int \sigma^4 \rho |\theta_t|^2 \, dx + \frac{\kappa(\gamma - 1)}{R} \int \sigma^4 |\nabla \theta_t|^2 \, dx \\
\leq C \sigma^2 \left( \|\rho^{1/2} \theta_t\|^2_{L^2} + \|\nabla u_t\|^2_{L^2} + \|\nabla H_t\|^2_{L^2} \right) + C,
\]
which along with (4.81) gives
\[
\sup_{0 \leq t \leq T} \sigma^4 \int \rho |\theta_t|^2 \, dx + \int_0^T \sigma^4 \int |\nabla \theta_t|^2 \, dx \, dt \leq C. \tag{4.91}
\]

Finally, applying the standard $L^2$-estimate to (4.72), we deduce from Lemmas 4.1, 4.3, 4.31, (4.91), and (4.61) that
\[
\sup_{0 \leq t \leq T} \sigma^2 \|\nabla^2 \theta_t\|_{L^2} \\
\leq C \sup_{0 \leq t \leq T} \sigma^2 \left( \|\rho \theta_t\|_{L^2} + \|\rho t\|_{L^3} \|\theta_t\|_{L^6} + \|\rho t\|_{L^6} \left( \|\nabla \theta\|_{L^3} + 1 \right) \right) \\
+ C \sup_{0 \leq t \leq T} \sigma^2 \left( \|\rho^{1/2} \theta_t\|_{L^2} + \|\nabla \theta_t\|_{L^2} + (1 + \|\nabla^2 \theta\|_{L^2}) \|\nabla u_t\|_{L^2} \right) \tag{4.92} \\
+ C \sup_{0 \leq t \leq T} \sigma^2 (\|\nabla u_t\|_{L^6} + \|\nabla H_t\|_{L^6}) \\
\leq C.
\]

Moreover, it follows from the standard $H^2$-estimate of (3.75), (2.7), (4.28), (4.41), and Lemma 4.1 that
\[
\|\nabla^2 \theta\|_{H^2} \\
\leq C \left( \|\rho \theta_t\|_{H^2} + \|\rho u \cdot \nabla \theta\|_{H^2} + \|\rho \theta \text{div} u\|_{H^2} + \|\nabla u^2\|_{H^2} + \|\nabla H^2\|_{H^2} \right) \\
\leq C \left( (1 + \|\rho - 1\|_{H^2}) \|\theta_t\|_{H^2} + ((\|\rho - 1\|_{H^2} + 1) \|u\|_{H^2} \|\nabla \theta\|_{H^2}) \right) \\
+ C(1 + \|\rho - 1\|_{H^2})(1 + \|\theta - 1\|_{H^2}) \|\text{div} u\|_{H^2} + C(\|\nabla u^2\|_{H^2} + \|\nabla H^2\|_{H^2}) \\
\leq C \sigma^{-1} + C \|\nabla^3 \theta\|_{L^2} + C \|\theta_t\|_{H^2}.
\]

Combining this with (4.61), (4.92), and (4.91) concludes (4.83). We finish the proof of Lemma 4.6.

\section{Proof of Theorem 1.1}

With all the a priori estimates in Sections 3 and 4 at hand, we are in a position to prove the main result of this paper in this section.

\textbf{Proposition 5.1.} For given numbers $M > 0$ (not necessarily small), $\hat{\rho} > 2$, and $\hat{\theta} > 1$, assume that $(\rho_0, u_0, \theta_0, H_0)$ satisfies (2.1), (3.5), and (3.8). Then there exists a unique classical solution $(\rho, u, \theta, H)$ of problem (1.1)–(1.5) in $\Omega \times (0, \infty)$ satisfying (2.3) and (2.4) with $T_0$ replaced by any $T \in (0, \infty)$. Moreover, (3.7), (3.10), (3.130), and (3.131) hold for any $T \in (0, \infty)$.\hfill $\square$
Proof. First, with the help of the standard local existence result (Lemma 2.1), there exists a small \( T_0 > 0 \) which may depend on \( \inf_{x \in \Omega} \rho_0(x) \), such that the problem (1.1)–(1.5) with initial data \((\rho_0, u_0, \theta_0, H_0)\) has a unique classical solution \((\rho, u, \theta, H)\) on \( \Omega \times (0, T_0] \), which satisfies (2.2)–(2.4). Now we will use the a priori estimates in Proposition 3.1 to extend the local classical solution to all time. Since

\[
A_1(0) \leq M^2, \quad A_2(0) \leq C_0^{1/4}, \quad A_3(0) + A_4(0) = 0, \quad \rho_0 < \hat{\rho}, \quad \theta_0 \leq \hat{\theta},
\]

then there exists a \( T_1 \in (0, T_0] \) such that (3.6) holds for \( T = T_1 \).

We set

\[
T^* = \sup \left\{ T \left| \sup_{t \in [0, T]} \| (\rho - 1, u, \theta - 1, H) \|_{H^3} < \infty \right. \right\},
\]

and

\[
T_* = \sup \{ T \leq T^* \mid (3.6) \text{ holds} \}. \tag{5.1}
\]

Then \( T^* \geq T_* \geq T_1 > 0 \). Next, we claim that

\[
T_* = \infty. \tag{5.2}
\]

Otherwise, \( T_* < \infty \). Hence, by (3.8), Proposition 3.1 tells us (3.7) holds for all \( 0 < T < T_* \), which implies Lemmas 4.1–4.6 still hold for all \( 0 < T < T_* \). Note here that all constants \( C \) in Lemmas 4.1–4.6 depend on \( T_* \) and \( \inf_{x \in \Omega} \rho_0(x) \), are in fact independent of \( T \). Then, we claim that there exists a positive constant \( \tilde{C} \) which may depend on \( T_* \) and \( \inf_{x \in \Omega} \rho_0(x) \) such that, for all \( 0 < T < T_* \),

\[
\sup_{0 \leq t \leq T} \| \rho - 1 \|_{H^3} \leq \tilde{C}, \tag{5.3}
\]

which together with Lemmas 4.2, 4.3, 4.5, 2.3, 5.4, and 3.5 infers

\[
\inf_{x \in \Omega} \rho(x, T_*), \theta(x, T_*), 0 \geq 0.
\]

Lemma 2.1 thus implies that there exists some \( T^{**} > T_* \), such that (3.6) holds for \( T = T^{**} \), which contradicts (5.1). Hence, (5.2) holds. This as well as Lemmas 2.1, 3.1, 3.8, and Proposition 3.1 thus concludes the proof of Proposition 5.1.

Finally, it remains to prove (5.3). By (1.1)–3 and (2.1), one can define

\[
\sqrt{\rho} \theta(\cdot, 0) \triangleq - \frac{\gamma - 1}{R} \rho_0^{-1/2} (\kappa \Delta \theta_0 - R \rho_0 \theta_0 \text{div} u_0)
\]

\[
+ \frac{\gamma - 1}{R} \rho_0^{-1/2} \left( \lambda (\text{div} u_0)^2 + 2 \mu \| \mathcal{D}(u_0) \|^2 + \nu |\text{curl} H_0|^2 \right),
\]

which along with (2.1) yields

\[
\| \sqrt{\rho} \theta(\cdot, 0) \|_{L^3} \leq \tilde{C}. \tag{5.4}
\]

Then it follows from (5.4), (3.133), and Lemma 4.1 that

\[
\sup_{0 \leq t \leq T} \int \rho |\theta|^2 \, dx + \int_0^T \| \nabla \theta \|_{L^2}^2 \, dt \leq \tilde{C}, \tag{5.5}
\]
which combined with (3.74) and Lemma 4.1 shows that
\[
\sup_{0 \leq t \leq T} \| \nabla^2 \theta \|_{L^2} \leq \bar{C}.
\] (5.6)

Next, by (1.12) and (2.1), one can define
\[
u_t(\cdot, 0) \triangleq -u_0 \cdot \nabla u_0 - R\rho_0^{-1} \nabla (\rho_0 \theta_0)
+ \rho_0^{-1} \left( \mu \Delta u_0 + (\mu + \lambda) \nabla \text{div} u_0 + H_0 \cdot \nabla H_0 - \frac{1}{2} \nabla |H_0|^2 \right),
\]
which as well as (2.1) and (4.6) leads to
\[
\| \nabla u_t(\cdot, 0) \|_{L^2} + \| \nabla H_t(\cdot, 0) \|_{L^2} \leq \bar{C}.
\] (5.7)

Thus, one deduces from Grönwall's inequality, Lemmas 4.1, 4.2, (2.8), (4.31), (4.54), (4.33), and (5.5)–(5.7) that
\[
\sup_{0 \leq t \leq T} (\| u_t \|_{H^1} + \| \nabla H_t \|_{L^2}) + \int_0^T (\rho |u_t|^2 + |H_{tt}|^2) \, dt \leq \bar{C},
\] (5.8)

which together with (4.22), (4.25), (5.6), and (4.4) implies
\[
\sup_{0 \leq t \leq T} (\| u \|_{H^3} + \| H \|_{H^3}) \leq \bar{C}.
\] (5.9)

This combined with Lemma 4.1, 4.26, 4.51, (5.5), (5.6), (5.8), and (5.9) gives
\[
\int_0^T (\| \nabla^3 \theta \|_{L^2}^2 + \| \nabla u_t \|_{H^1}^2) \, dt \leq \bar{C}.
\] (5.10)

Now, some standard calculations lead to
\[
(\| \nabla^3 \rho \|_{L^2})_t
\leq \bar{C} \left( \| \nabla^3 u \|_{L^2} + \| \nabla^2 u \|_{L^2} + \| \nabla \theta \|_{L^2} + \| \nabla u \|_{L^2} + \| \nabla \rho \|_{L^2} + \| \nabla^4 u \|_{L^2} \right)
\leq \bar{C} \left( \| \nabla^3 u \|_{L^2} + \| \nabla^2 u \|_{L^2} + \| \nabla \rho \|_{L^2} + \| \nabla u \|_{L^2} + \| \nabla \rho \|_{L^2} + \| \nabla^4 u \|_{L^2} \right)
\leq \bar{C} (1 + \| \nabla^3 \theta \|_{L^2} + \| \nabla^2 u_t \|_{L^2}^2 + \| \nabla^3 \theta \|_{L^2}^2),
\] (5.11)

where we have used (5.9), Lemma 4.1 and the following fact:
\[
\| \nabla^2 u \|_{H^2} \leq \bar{C} \left( \| \rho \hat{u} \|_{H^2} + \| H \|_{H^2} + \| \nabla P \|_{H^2} + 1 \right)
\leq \bar{C} (1 + \| \rho - 1 \|_{H^2}) (\| u_t \|_{H^2} + \| u \|_{H^2}) + \bar{C} \| \nabla \theta \|_{H^2} + \| \nabla \rho \|_{H^2}
\leq \bar{C} (1 + \| \rho - 1 \|_{H^2} + \| \theta - 1 \|_{H^2}) (\| \nabla \rho \|_{H^2} + \| \nabla \theta \|_{H^2}) + \bar{C}.
\]

Applying Grönwall's inequality to (5.11) and using (5.10) show that
\[
\sup_{0 \leq t \leq T} \| \nabla^3 \rho \|_{L^2} \leq \bar{C},
\]
which together with (4.4) gives (5.3). The proof of Proposition 5.1 is completed. □
With Proposition 5.1 at hand, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let \((\rho_0, u_0, \theta_0, H_0)\) satisfying (1.7), (1.9) be the initial data in Theorem 1.1. Assume that \(C_0\) satisfies (1.10) with 
\[
\varepsilon \triangleq \varepsilon_0/2,
\]
where \(\varepsilon_0\) is given in Proposition 3.1.

To begin with, we construct the approximate initial data \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta}, H_0^{m,\eta})\).

For constants 
\[
m \in \mathbb{Z}^+ \text{, } \eta \in (0, \eta_0) \text{, } \eta_0 \triangleq \min \left\{ 1, \frac{1}{2}(\hat{\rho} - \sup_{x \in \Omega}\rho_0(x)) \right\},
\]
we set 
\[
\rho_0^{m,\eta} = \frac{\rho_0^m + \eta}{1 + \eta}, \quad u_0^{m,\eta} = \frac{u_0^m}{1 + \eta}, \quad \theta_0^{m,\eta} = \frac{\theta_0^m + \eta}{1 + \eta}, \quad H_0^{m,\eta} = \frac{H_0^m}{1 + \eta},
\]
where \(\rho_0^m, \ u_0^m, \ \theta_0^m, \) and \(H_0^m\) satisfy that 
\[
0 \leq \rho_0^m \in C^\infty, \quad \lim_{m \to \infty} \|\rho_0^m - \rho_0\|_{H^{2 \cap W^{2,q}}} = 0,
\]
\[
u_0^m \in C^\infty, \quad \nu_0^m \cdot n = 0 \text{ on } \partial \Omega, \quad \lim_{m \to \infty} \|\nu_0^m - u_0\|_{H^2} = 0,
\]
\[
H_0^m \in C^\infty, \quad \text{div}H_0^m = 0, \quad H_0^m \cdot n = 0, \quad \text{curl}H_0^m \times n = 0 \text{ on } \partial \Omega,
\]
\[
\lim_{m \to \infty} \|\tilde{H}_0^m - H_0\|_{H^2} = 0,
\]
and \(\theta_0^m\) is the unique smooth solution to the following Poisson equation:
\[
\begin{aligned}
\Delta \theta_0^m &= \Delta \tilde{\theta}_0^m, \quad \text{in } \Omega, \\
\nabla \theta_0^m \cdot n &= 0, \quad \text{on } \partial \Omega, \\
\theta_0^m &\to 1, \quad \text{as } |x| \to \infty,
\end{aligned}
\]
with \(\tilde{\theta}_0^m = \tilde{\theta}_0 \ast j_{m^{-1}}, \tilde{\theta}_0 - 1\) is the \(H^1\)-extension of \(\theta_0 - 1, \) and \(j_{m^{-1}}(x)\) is the standard mollifying kernel of width \(m^{-1}\).

Then for any \(\eta \in (0, \eta_0), \) there exists \(m_1(\eta) \geq 0\) such that for \(m \geq m_1(\eta), \) the approximate initial data \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta}, H_0^{m,\eta})\) satisfies
\[
\begin{cases}
(\rho_0^{m,\eta} - 1, u_0^{m,\eta}, \theta_0^{m,\eta} - 1, H_0^{m,\eta}) \in C^\infty, \\
\frac{\eta}{2} \leq \rho_0^{m,\eta} < \hat{\rho}, \quad \frac{\eta}{4} \leq \theta_0^{m,\eta} \leq \hat{\theta}, \\
u_0^{m,\eta} \cdot n = 0, \quad \text{curl}u_0^{m,\eta} \times n = 0, \\
H_0^{m,\eta} \cdot n = 0, \quad \text{curl}H_0^{m,\eta} \times n = 0 \text{ on } \partial \Omega,
\end{cases}
\]
(5.12)
\[
\begin{aligned}
\lim_{\eta \to 0} \lim_{m \to \infty} (\|\rho_0^{m,\eta} - \rho_0\|_{H^{2 \cap W^{2,q}}} + \|u_0^{m,\eta} - u_0\|_{H^2}) &= 0, \\
\lim_{\eta \to 0} \lim_{m \to \infty} (\|\theta_0^{m,\eta} - \theta_0\|_{H^1} + \|H_0^{m,\eta} - H_0\|_{H^2}) &= 0.
\end{aligned}
\]
(5.13)
Moreover, the initial energy \(C_0^{m,\eta}\) for \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta}, H_0^{m,\eta})\), which is defined by the right-hand side of (1.6) with \((\rho_0, u_0, \theta_0, H_0)\) replaced by \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta}, H_0^{m,\eta})\), satisfies
\[
\lim_{\eta \to 0} \lim_{m \to \infty} C_0^{m,\eta} = C_0.
\]
Therefore, there exists an \( \eta_1 \in (0, \eta_0) \) such that, for any \( \eta \in (0, \eta_1) \), we can find some \( m_2(\eta) \geq m_1(\eta) \) such that

\[
C_0^{m_2(\eta)} \leq C_0 + \varepsilon_0/2 \leq \varepsilon_0,
\]

provided that

\[
0 < \eta < \eta_1, \ m \geq m_2(\eta).
\]

Now, assume that \( m, \eta \) satisfy (5.15), Proposition 5.1 combined with (5.14) and (5.12) thus implies that the problem (1.1)–(1.5) with initial data \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta}, H_0^{m,\eta})\) has a smooth solution \((\rho^{m,\eta}, u^{m,\eta}, \theta^{m,\eta}, H^{m,\eta})\) on \( \Omega \times (0,T] \) for all \( T > 0 \). Moreover, (1.11), (3.7), (3.10), (3.130), and (3.131) with \((\rho, \theta, H)\) being replaced by \((\rho^{m,\eta}, u^{m,\eta}, \theta^{m,\eta}, H^{m,\eta})\) all hold.

Next, for the initial data \((\rho_0^{m,\eta}, u_0^{m,\eta}, \theta_0^{m,\eta}, H_0^{m,\eta})\), the function \( \tilde{g} \) in (4.1) is

\[
\tilde{g} \triangleq (\rho_0^{m,\eta})^{-1/2} (-\mu \Delta u_0^{m,\eta} - (\mu + \lambda)\nabla \text{div} u_0^{m,\eta} + R \nabla (\rho_0^{m,\eta} \theta_0^{m,\eta}))
\]

\[
- (\rho_0^{m,\eta})^{-1/2} (\nabla \times H_0^{m,\eta}) \times H_0^{m,\eta}
\]

\[
= (\rho_0^{m,\eta})^{-1/2} \sqrt{\rho_0} g + \mu (\rho_0^{m,\eta})^{-1/2} \Delta (u_0 - u_0^{m,\eta})
\]

\[
+ (\mu + \lambda) (\rho_0^{m,\eta})^{-1/2} \nabla \text{div} (u_0 - u_0^{m,\eta}) + R (\rho_0^{m,\eta})^{-1/2} \nabla (\rho_0^{m,\eta} \theta_0^{m,\eta} - \rho_0 \theta_0)
\]

\[
+ (\rho_0^{m,\eta})^{-1/2} ((\nabla \times H_0) \times H_0 - (\nabla \times H_0^{m,\eta}) \times H_0^{m,\eta}),
\]

where in the second equality we have used (1.9). Since \( g \in L^2 \), it follows from (5.16), (5.12), (5.13), and (1.7) that for any \( \eta \in (0, \eta_1) \), there exist some \( m_3(\eta) \geq m_2(\eta) \) and a positive constant \( C \) independent of \( m \) and \( \eta \) such that

\[
\| \tilde{g} \|_{L^2} \leq (1 + \eta)^{1/2} \| g \|_{L^2} + C \eta^{-1/2} \delta(m) + C \eta^{1/2},
\]

with \( 0 \leq \delta(m) \to 0 \) as \( m \to \infty \). Hence, for any \( \eta \in (0, \eta_1) \), there exists some \( m_4(\eta) \geq m_3(\eta) \) such that for any \( m \geq m_4(\eta) \),

\[
\delta(m) < \eta.
\]

We thus obtain from (5.17) and (5.18) that there exists some positive constant \( C \) independent of \( m \) and \( \eta \) such that

\[
\| \tilde{g} \|_{L^2} \leq \| g \|_{L^2} + C,
\]

provided that

\[
0 < \eta < \eta_1, \ m \geq m_4(\eta).
\]

Now, we assume that \( m, \eta \) satisfy (5.20). We thus deduce from (5.12)–(5.14), (5.19), Proposition 3.1 and Lemmas 3.8, 4.1, 4.6 that for any \( T > 0 \), there exists some positive constant \( C \) independent of \( m \) and \( \eta \) such that (1.11), (3.7), (3.10), (3.130), (3.131), (4.3), (4.4), (4.27), (4.29), (4.41), (4.55), (4.61), and (4.83) hold for \((\rho^{m,\eta}, u^{m,\eta}, \theta^{m,\eta}, H^{m,\eta})\).

Then passing to the limit first \( m \to \infty \), then \( \eta \to 0 \), along with standard arguments leads to that there exists a solution \((\rho, u, \theta, H)\) of the problem (1.1)–(1.5) on \( \Omega \times (0,T] \) for all \( T > 0 \), which satisfies (1.11), (3.10), (3.130), (3.131), (4.3), (4.4), (4.27)–(4.29), (4.41), (4.55), (4.61), (4.83), and the estimates of \( A_i(T) \) \((i = 1, \cdots, 4)\) in (3.7). Therefore, \((\rho, u, \theta, H)\) satisfies (1.11) and (1.12).

Finally, since the proof of the uniqueness of \((\rho, u, \theta, H)\) is similar to that of \( \text{Theorem 1} \), we omit it here for simplicity. To finish the proof of Theorem 1.1, it remains to prove (1.13). On the one hand, one can rewrite (1.1) as

\[
(\rho - 1)_t + \text{div}((\rho - 1)u) + \text{div} u = 0.
\]

(5.21)
Multiplying (5.21) by $4(\rho - 1)^3$ and integrating by parts, one derives that, for $t \geq 1$,
$$
(\|\rho - 1\|_{L^4}^4)'(t) = -3 \int (\rho - 1)^4 \text{div} u dx - 4 \int (\rho - 1)^3 \text{div} u dx,
$$
which combined with (3.131) implies that
$$
\int_1^\infty |(\|\rho - 1\|_{L^4}^4)'(t)| dt \leq C \int_1^\infty \|\rho - 1\|_{L^4} dt + C \int_1^\infty \|\nabla u\|_{L^4}^4 dt \leq C. \tag{5.22}
$$
On the other hand, we deduce from $A_i(T)(i = 2, 3, 4)$ in (3.7) and (3.131) that
$$
\int_1^\infty |(\|\nabla u\|_{L^2}^2)'(t)| dt = 2 \int_1^\infty \left| \int \partial_j u^i \partial_j u^i dx \right| dt
= 2 \int_1^\infty \left| \int \partial_j u^i \partial_j (\dot{u}^i - u^k \partial_k u^i) dx \right| dt
= \int_1^\infty \left| \int (2\partial_j u^i \partial_j u^i - 2\partial_j u^i \partial_j u^k \partial_k u^i + |\nabla u|^2 \text{div} u) dx \right| dt \tag{5.23}
\leq C \int_1^\infty (\|\nabla u\|_{L^2} \|\nabla \dot{u}\|_{L^2} + \|\nabla u\|_{L^4}^3) dt
\leq C \int_1^\infty (\|\nabla \dot{u}\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{L^4}^4) dt \leq C,
$$
$$
\int_1^\infty |(\|\nabla \theta\|_{L^2}^2)'(t)| dt = 2 \int_1^\infty \left| \int \nabla \theta \cdot \nabla \theta_t dx \right| dt
\leq C \int_1^\infty (\|\nabla \theta\|_{L^2}^2 + \|\nabla \theta_t\|_{L^2}^2) dt \leq C, \tag{5.24}
$$
and
$$
\int_1^\infty |(\|\nabla H\|_{L^2}^2)'(t)| dt = 2 \int_1^\infty \left| \int \nabla H \cdot \nabla H_t dx \right| dt
\leq C \int_1^\infty (\|\nabla H\|_{L^2}^2 + \|\nabla H_t\|_{L^2}^2) dt \leq C. \tag{5.25}
$$
It thus follows from (3.131), (5.22)–(5.25), and $A_2(T)$ in (3.7) that
$$
\lim_{t \to \infty} (\|\rho - 1\|_{L^1} + \|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\nabla H\|_{L^2}) = 0,
$$
which together with (3.10), (1.11), and (3.131) concludes (1.13). The proof of Theorem 1.1 is completed.

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