Tenfold Way for Holography : AdS/CFT and Beyond

V.K. Dobrev

Institute for Nuclear Research and Nuclear Energy,
Bulgarian Academy of Sciences,
72 Tsarigradsko Chaussee, 1784 Sofia, Bulgaria

Abstract

The main purpose of the present paper is to lay the foundations of generalizing the AdS/CFT (holography) idea beyond the conformal setting. The main tool is to find suitable realizations of the bulk and boundary via group theory. We use all ten families of classical real semisimple Lie groups $G$ and Lie algebras $\mathfrak{g}$. For this are used several group and algebra decompositions: the global Iwasawa decomposition and the local Bruhat and Sekiguchi-like decompositions. The same analysis is applied to the exceptional real semisimple Lie algebras.
1 Introduction

For the last twenty years due to the remarkable proposal of [1] the AdS/CFT correspondence is a dominant subject in string theory and conformal field theory. Actually the possible relation of field theory on anti de Sitter space to conformal field theory on boundary Minkowski space-time was studied also before, cf., e.g., [2–7]. The proposal of [1] was further elaborated in [8] and [9]. After that there was an explosion of related research which continues also currently.

Let us recall that the AdS/CFT correspondence has 2 ingredients [1,8,9]:

1. the holography principle, which is very old, and means the reconstruction of some objects in the bulk (that may be classical or quantum) from some objects on the boundary;
2. the reconstruction of quantum objects, like 2-point functions on the boundary, from appropriate actions on the bulk.

Our focus here is on the first ingredient. For further reference we note that in most applications to physics the boundary is interpreted as the surface of the bulk, (see e.g. [1,8–10]), thus their dimensions differ by one and we shall follow this.

We note that the first explicit presentation of the holography principle was realized in the Euclidean case, [9], i.e., for the Euclidean conformal group $G_E = SO_0(d+1,1)$. In this case the bulk space $S_0$ is isomorphic to the factor-space:

$$S_0 = G_E/K = SO_0(d+1,1)/SO(d+1)$$

(1)

where $K = SO(d+1)$ is the maximal compact subgroup of $G_E$. It is important that in this case we use the so-called Iwasawa decomposition $G_E = K A_0 N_0$ (explained mathematically below, see also [11–13]), where the subgroups $A_0$, $N_0$, are important from the physics point of view, namely $A_0$ is the subgroup of dilatations, $N_0$ is the subgroup of Euclidean translations (isomorphic to the subgroup of special conformal transformations, and also to the $d$-dimensional Euclidean space, $R^d$). Note that the dimension $r_0$ of $A_0$ is called the split rank of $G$.

Thus, we have for the bulk: $S_0 \cong A_0 N_0$, while the boundary is isomorphic to $N_0$ [14]. Note that this realization of the bulk is very suitable for the applications since the parametrization of $A_0$ provides and easy limit from the bulk to the boundary.

As historical Remark we mention that the problem is related to the construction of the discrete series of unitary representations in [15,16], which was then applied in [17] for the Euclidean conformal group $SO_0(4,1)$ (also for $SO_0(2,1)$). The approach applied to the Euclidean case $G_E$ in [14] is different. We should mention that the nonrelativistic Schrödinger case was
considered in [18] using the representation theory developed in [19] (for an invite review see [20]).

The next task was to show the holography principle for Minkowski space-time, i.e., for the conformal group $G_M = SO_0(d, 2)$. Initially, this was done relying on Wick rotations of the final results, cf., e.g. [9], (see also some other early papers [21–39]). For some more recent papers and reviews see, e.g., [40–58].

Of course, it is desirable to apply group theory tools directly to $G_M$. The problem here is that the Iwasawa decomposition $SO_0(d, 2) = K_M A_M N_M$, is not suitable for the physics applications since $A_M$ is two-dimensional, (thus, there is no natural parametrization of the limit from the bulk to the boundary), $K_M = SO(d) \times SO(2)$, and $N_M$ has dimension $2d - 2$ (bigger than $d$ for $d > 2$). Fortunately, there is a suitable group decomposition, called Bruhat decomposition, which has the necessary group-theory properties and useful physical interpretation. Namely, it is:

$$G_M = SO_0(d, 2) =_{\text{loc}} \tilde{N} MAN$$

where $N, \tilde{N}$ are isomorphic $d$-dimensional spaces, also isomorphic to $d$-dimensional Minkowski space-time $M$, $A$ represents the one-dimensional dilatations, $M = SO_0(d - 1, 1)$ are the Lorentz transformations of $M$. It is easy to see that in this setting the role of bulk space is played by $S_M = AN$ which can be obtained by Wick rotation from the Euclidean $S_0 \cong A_0 N_0$, while the boundary is Wick rotation from $N_0$ to $N$. This decomposition was used in our setting for $d = 3$ in [59].

[On the technical side : the designation $\text{loc}$ above means that the subgroup $\tilde{N} MAN$ is an open dense set of $G_M$.]

Remark: One may ask why not use the Bruhat decomposition also in the split rank one cases. Indeed, this is possible and is a matter of choice. Our choice is motivated by our experience that calculations for split rank one case are easier with the Iwasawa decomposition. ♦

One more important ingredient in the present paper is the fact that the space $S_M$ may arise also by using another decomposition first introduced for some groups by Sekiguchi [60], though with no relation to our setting. For the conformal group this decomposition is:

$$SO_0(d, 2) =_{\text{loc}} HAN, \quad H = SO_0(d, 1)$$

Note that this decomposition may be obtained via Wick rotation from the Iwasawa decomposition: $SO_0(d + 1, 1) \rightarrow SO_0(d, 2), \quad SO(d + 1) \rightarrow SO_0(d, 1)$. We initiated the use of this decomposition in our setting in [61]. There it was
shown that although the bulk space $S_M = AN$ obtained via $SO_0(d, 2)/\tilde{N}M$ is isomorphic to the one obtained via $SO_0(d, 2)/H$, the actual parametrizations in terms of the groups elements of $SO_0(d, 2)$ do not coincide. Furthermore, the parametrization obtained from (3) is simpler which is important for the applications.

This decomposition is not universal as it does not exist for all real semisimple Lie groups. In [60] the Sekiguchi decomposition was defined for $SO_0(p, q)$, $SU(p, q)$, $Sp(p, q)$. Following the idea of [60] we define Sekiguchi-like decomposition for a real semisimple Lie group $G$ as follows. Let there be a Bruhat decomposition
\[ G =_{loc} \tilde{N}MAN, \]
(4)
such that there exists a subgroup $H$ of $G$, so that the Sekiguchi-like decomposition
\[ G =_{loc} HAN \]
(5)
exists with the same subgroups $AN$. Following the case of the conformal group $SO_0(d, 2)$ we expect that in the cases when (5) exists it will be simpler to use than the Bruhat decomposition. Below we give many examples of Sekiguchi-like decomposition beyond the list of [60].

Now we can state the main purpose of the present paper: to lay the foundations of generalizing the holography (AdS/CFT) idea beyond the conformal setting.

What we do is to consider the real Lie groups and to explicate for each of them their subgroups mentioned above in the conformal setting.

The words tenfold way in the title refer to the fact that there are ten classical real Lie groups when we include the complex classical groups considered over the reals. We should note that the tenfold way has many other manifestations in mathematics and physics which are well described on John Baez’ web-page [62] (see especially [63–65] for applications to solid state physics). Our approach differs from others since we use noncompact Lie groups which is essential in the holography applications.

Let us summarize what we got as guidelines from the above. In the cases of $G$ of split rank $r_0 = 1$ most suitable is the Iwasawa decomposition $G = KA_0N_0$ in which case the bulk space is $S_0 = A_0N_0$ and the boundary space is $N_0$. In the cases of split rank $r_0 > 1$ we would use the Sekiguchi-like decomposition (5) when it exists, otherwise we shall use the Bruhat decomposition (4). In the latter cases the bulk space is $S = AN$ and the boundary space is $N$.

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1We reserve the term ‘Sekiguchi decomposition’ for the cases contained in [60].
The paper is organized as follows. In Section 2 we introduce more systematically the necessary group theory prerequisites. In Section 3 we consider the bulk-boundary correspondence for the special case of groups of split rank 1. In Section 4 we consider the bulk-boundary correspondence for the cases of split rank > 1. There are five Tables containing our analysis and results of the relevant group-theory data. The tables are placed at the end of the paper in order not to interrupt the exposition.

2 Preliminaries

2.1 Lie group and algebra decompositions

We need some well-known preliminaries to set up our notation and conventions (cf. e.g., [11], see also [12, 13]). Let $G$ be a noncompact semisimple Lie group. Let $K$ denote a maximal compact subgroup of $G$. Then we have the global Iwasawa decomposition:

$$G = KA_0N_0,$$

(6)

where $A_0$ is abelian simply connected, a vector subgroup of $G$, $N_0$ is a nilpotent simply connected subgroup of $G$ preserved by the action of $A_0$. This decomposition is called global since every element $g \in G$ may be represented as the product of three elements of the corresponding subgroups, namely, $g = ka_0n_0$, $k \in K$, $a_0 \in A_0$, $n_0 \in N_0$.\(^2\) Note that there is another nilpotent subgroup $\tilde{N}_0$, which is isomorphic to $N_0$, and that there is analogous Iwasawa decomposition $G = KA_0\tilde{N}_0$.

Further, let $M_0$ be the centralizer of $A_0$ in $K$. Then the subgroup $P_0 = M_0A_0N_0$ is a minimal parabolic subgroup of $G$.

Further, let $M'_0 \supset M_0$ be the normalizer of $A_0$ in $K$. The finite group $W = W(G, A_0) = M'_0/M_0$ is called the Weyl group for the pair $(G, A_0)$, or a restricted Weyl group. It has two elements $W = \{1, w\}$. The nilpotent subgroups $N_0$ and $\tilde{N}_0$ are conjugate under the Weyl transformation: $wNw^{-1} = \tilde{N}_0$.

A parabolic subgroup $P = MAN$ is any subgroup of $G$ which contains a minimal parabolic subgroup. The number of non-conjugate parabolic subgroups is $2^{r_0} - 1$, where $r_0 = \text{rank } A_0$, called the split rank of $G$.\(^3\)

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\(^2\)Note that the order of the three factors of the Iwasawa decomposition may vary, then of course the elements representing the subgroups changes.

\(^3\)In some expositions authors are counting as a parabolic subgroup also the group $G$, then the number of non-conjugate parabolic subgroups would be $2^{r_0}$. 

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Note that in general $M$ is a reductive Lie group with structure: $M = M_d M_s M_a$, where $M_d$ is a finite group, $M_s$ is a semisimple Lie group, $M_a$ is an abelian Lie group central in $M$.

For further use we note that the Abelian group $A$ may be used as the product of one-dimensional subgroups:

$$A = A_1 \cdots A_r, \quad a \in A, \quad a = a_1 \cdots a_r, \quad a_k \in A_k$$

Another important decomposition is the local Bruhat decomposition which exists for every parabolic subgroup $P = MAN$:

$$G = |_{\text{loc}} \tilde{N} MAN,$$

where $\tilde{N}$ is a nilpotent simply connected subgroup of $G$ preserved by the action of $A$, conjugate to $N$. (This decomposition is called local since there is a subset of elements of $G$ of lower dimension which can not be represented as the product of four elements of the corresponding subgroups.)

An important class of parabolic subgroups are the maximal parabolic subgroups which are defined by the property that $r = \text{rank } A = 1$. In that case the restricted Weyl group $W(G, A)$ has also two elements.

We need also the corresponding Lie algebras. Thus, $\mathcal{G}, \mathcal{K}, \mathcal{A}_0, \mathcal{M}_0, \mathcal{N}_0, \tilde{\mathcal{N}}_0, \mathcal{A}, \mathcal{M}, \mathcal{N}, \tilde{\mathcal{N}}$, denote the Lie algebras of $G, K, A_0, M_0, N_0, \tilde{N}_0, A, M, N, \tilde{N}$, resp. We also have the Lie algebra versions of the Iwasawa decomposition:

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{A}_0 \oplus \mathcal{N}_0$$

and the Bruhat decompositions:

$$\mathcal{G} = \tilde{\mathcal{N}} \oplus \mathcal{M} \oplus \mathcal{A} \oplus \mathcal{N}$$

### 2.2 Elementary representations

We start with the most general representations called (in the representation theory of semisimple Lie groups) generalized principal series representations (cf. [66]) (see also [67]). In [17, 68, 69] they were called elementary representations (ERs). They are obtained by induction from parabolic subgroups $P = MAN$. The induction is from finite-dimensional (nonunitary in general) irreps of $M$, from arbitrary (non-unitary) characters of $A$, and trivially from $N$. There are several realizations of these representations. We give now the

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Note that when the split rank $r_0 = 1$, then the minimal parabolic subgroup is also maximal.
so-called *noncompact picture* of the ERs - it is the one most often used in physics.

The representation space of these induced representations consists of smooth functions on $\tilde{N}$ with values in the corresponding finite-dimensional representation space of $M$, i.e.:

$$C_\chi = \{ f \in C^\infty(\tilde{N}, V_\mu) \}$$

(11)

where $\chi = [\mu, \nu]$, $\nu$ is a character function on $A$, $\nu(a) = \nu_1(a_1) \cdots \nu_r(a_r)$, $\mu$ is a irrep of $M$, $V_\mu$ is the finite-dimensional representation space of $\mu$. 5

The representation $T^\chi$ acts in $C_\chi$ by:

$$(T^\chi(g)f)(\tilde{n}) = \prod_{k=1}^r |a_k|^{-\nu_k(a_k)} \cdot D^\mu(m) f(\tilde{n}')$$

(12)

where the nonglobal Bruhat decomposition $g = \tilde{n} m a n$ is used:

$$g^{-1}\tilde{n} = \tilde{n}' m^{-1} a^{-1} n^{-1}, \quad g \in G, \tilde{n}, \tilde{n}' \in \tilde{N}, m \in M, a \in A, n \in N, \quad |a_k|$$

is a suitable positive function on $A_k$, $D^\mu(m)$ is the representation matrix of $\mu$ in $V_\mu$. 6 7

The importance of the elementary representations comes also from the remarkable result of Langlands-Knapp-Zuckerman [70, 71] stating that every irreducible admissible representation of a real connected semisimple Lie group $G$ with finite centre is equivalent to a subrepresentation of an elementary representation of $G$. 8 To obtain a subrepresentation of a topologically reducible ER one has to solve certain invariant differential equations, cf. [17, 68, 69].

Finally, we recall that Casimir operators $C_i$ of $G$ have constant values on the ERs:

$$C_i(\{X\}) \varphi(x) = \chi_i(\mu, \nu) \varphi(x), \quad i = 1, \ldots, \text{rank } G$$

(14)

where $\{X\}$ denotes symbolically the generators of the Lie algebra $G$ of $G$. 8

5In addition, these functions have special asymptotic expansion as suitable parameter(s) on $\tilde{N}$ tend to $\infty$.

6For the cases with measure zero for which $g^{-1}\tilde{n}_x$ does not have a Bruhat decomposition of the form $\tilde{n} m a n$ the action is defined separately, and the passage from (12) to these special cases is ensured to be smooth by the asymptotic properties mentioned above.

7The representation space $C_\chi$ can be thought of as the space of smooth sections of the homogeneous vector bundle (called also vector $G$-bundle) with base space $G/P$ and fibre $V_\lambda$, (which is an associated bundle to the principal $P$-bundle with total space $G$). Actually, we do not need this description, but following [69] we replace the above homogeneous vector bundle with a line bundle again with base space $G/P$. The resulting functions $\hat{\varphi}$ can be thought of as smooth sections of this line bundle.

8Subrepresentations are irreducible representations realized on invariant subspaces of the ER spaces (in particular, the irreducible ERs themselves). The admissibility condition is fulfilled in the physically interesting examples.
2.3 Bulk representations via Iwasawa decomposition

In the previous subsection we discussed representations on $\tilde{N}$ induced from the parabolic subgroup $MAN$ which is natural since the subgroup $\tilde{N}$ is locally isomorphic to the factor space $G/MAN$ (via the Bruhat decomposition). Similarly, it is natural to discuss representations on the bulk space $S_0 \cong \tilde{N}_0A_0$ which are induced from the maximal compact subgroup $K$ since the solvable group $\tilde{N}_0A_0$ is isomorphic to the factor space $G/K$ (via the Iwasawa decomposition in the version $G = \tilde{N}_0A_0K$). Namely, we consider the representation space:

$$\hat{C}_\tau = \{ \phi \in C^\infty(S_0, U_\tau) \}$$  \hspace{1cm} (15)

where $\tau$ is an arbitrary unitary irrep of $K$, $U_\tau$ is the finite-dimensional representation space of $\tau$, with representation action:

$$(\hat{T}_\tau(g)\phi)(\tilde{n}a) = \tilde{D}_\tau(k)\phi(\tilde{n}'a')$$  \hspace{1cm} (16)

where the Iwasawa decomposition is used:

$$g^{-1}\tilde{n}a = \tilde{n}'a'k^{-1}, \quad g \in G, k \in K, \tilde{n}, \tilde{n}' \in \tilde{N}_0, a, a' \in A_0$$  \hspace{1cm} (17)

and $\tilde{D}_\tau(k)$ is the representation matrix of $\tau$ in $U_\tau$. However, unlike the ERs, these representations are reducible, and to single out an irrep equivalent, say, a subrepresentation of an ER, one has to look for solutions of the eigenvalue problem related to the Casimir operators [17], [12].

2.4 Bulk representations via Bruhat decomposition

In subsection 2.2 we discussed representations on $\tilde{N}$ induced from the parabolic subgroup $P = MAN$ which is natural since the subgroup $\tilde{N}$ is locally isomorphic to the factor space $G/MAN$ (via the Bruhat decomposition). Now we shall discuss representations on the bulk space $S \cong \tilde{N}A$ which are induced from the parabolic subgroup $P$ similarly to (11)

$$\check{C}_\chi = \{ f \in C^\infty(\tilde{N}A, V_\mu) \}$$  \hspace{1cm} (18)

The representation $\hat{T}_\chi$ acts in $\check{C}_\chi$ by:

$$(\hat{T}_\chi(g)f)(\tilde{n}a) = \prod_{k=1}^{r} |a_k|^{-\nu_k(a_k')} \cdot D^\mu(m) f(\tilde{n}'a')$$  \hspace{1cm} (19)

where the Bruhat decomposition is used:

$$g^{-1}\tilde{n}a = \tilde{n}'a'm^{-1}n^{-1}$$  \hspace{1cm} (20)
2.5 Bulk representations via Sekiguchi-like decomposition

These representations are introduced similarly to those using the Iwasawa decomposition. Namely, we consider the representation space:

\[ \hat{C}_\sigma = \{ \phi \in C^\infty(S, W_\sigma) \}, \quad S = \tilde{N}A, \]

where \( \sigma \) is a finite-dimensional irrep of \( H \), \( W_\sigma \) is the representation space of \( \sigma \), with representation action:

\[ (\hat{T}_\sigma(g)\phi)(\tilde{n}a) = \hat{D}_\sigma(h) \phi(\tilde{n}'a') \]

where the Sekiguchi-like decomposition is used:

\[ g^{-1}\tilde{n}a = \tilde{n}'a'h^{-1}, \quad g \in G, h \in H, \tilde{n}, \tilde{n}' \in \tilde{N}, a, a' \in A \]

and \( \hat{D}_\sigma(h) \) is the representation matrix of \( \sigma \) in \( W_\sigma \).

2.6 Table of the ten classical real Lie groups

Before proceeding further we present a table of the classical real Lie groups in Table 1 containing our analysis of the relevant for our purposes data. (Tables are given at the end of the paper in order not to interrupt the exposition.) Note that we have included the classical complex Lie groups but considered as real - these are denoted as types \( A, BD, C \). Note also that for split real forms the subgroup \( M_0 \) is trivial and this is designated by the unit element \( e \).

3 The split rank one case

We start with the cases of split rank 1. This is natural since this class includes the very important for applications Euclidean conformal group \( SO_0(p, 1) \). Furthermore, this was the first explicit AdS/CFT case considered in [9] (see also [17] for \( SO_0(4, 1) \) and \( SO_0(2, 1) \)).

Our results on the structure of the real Lie algebras of split rank 1 are given in Table 2. Note that most of these cases are of low dimension and they are conjugate to special cases of \( SO_0(p, 1) \) for \( p = 2, 3, 5 \). For completeness besides the classical cases we have included also the only exceptional real Lie algebra of split rank 1: \( F_4(-20) \).

As stated in the introduction we are interested in the group-theoretic aspect of the AdS/CFT correspondence. More precisely, we consider the
relation between the representations on the bulk space $S_0 \cong \tilde{N}_0 A_0$ and
the elementary representations on the boundary space $\tilde{N}_0$. In the case
of $SO_0(p, 1)$ these operators for low dimensional representations of $K, M_0$
were given in [9], while the general case was given in [14]. We recall briefly
the main results, introducing some additional notation. We shall use the
following group decomposition for every $k \in K$:

$$k = m(k)k_c$$  \hspace{1cm} (24)

where $m(k)$ parametrizes the subgroup $M_0$, while $k_c$ parametrizes the coset
$K/M_0$, thus, (24) represents the decomposition of $K$ into its subgroup
$M_0$ and the coset $K/M_0 : K \cong M_0 K/M_0$. We shall use also the
relation $K/M_0 \cong \tilde{N}_0$ following from:

$$G = KA_0 N_0 \cong \tilde{N}_0 M_0 A_0 N_0 \Rightarrow K \cong \tilde{N}_0 M_0 \Rightarrow K/M_0 \cong \tilde{N}_0 \hspace{1cm} (25)$$

Now we can state the Bulk - boundary intertwining relations:

**Theorem 1 : [14]**

1. **Bulk-to-boundary intertwining relation**: Let us define the operator:

$$L^\tau_\chi : \hat{C}_\tau \rightarrow C_\chi ,$$  \hspace{1cm} (26)

with the following action:

$$(L^\tau_\chi \phi)(\tilde{n}) = \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi^\tau_\mu \phi(\tilde{n}a)$$  \hspace{1cm} (27)

where $\Delta = \nu(a)$, $\Pi^\tau_\mu$ is the standard projection operator from the $K$-
representation space $\hat{U}_\tau$ to the $M$-representation space $V_\mu$, which acts in
the following way on the $K$-representation matrices:

$$\Pi^\tau_\mu \tilde{D}^\tau(k) = D^\mu(m(k)) \Pi^\tau_\mu \tilde{D}^\tau(k_c)$$  \hspace{1cm} (28)

where we have used (24). Then $L^\tau_\chi$ is an intertwining operator, i.e.:

$$L^\tau_\chi \circ \hat{T}^\tau(g) = T^\chi(g) \circ L^\tau_\chi , \hspace{1cm} \forall g \in G .$$  \hspace{1cm} (29)

In addition, in (27) the operator $\Pi^\tau_\mu$ acts in the following truncated way:

$$\Pi^\tau_\mu \tilde{D}^\tau(k) = D^\mu(m(k)) \Pi^\tau_\mu$$  \hspace{1cm} (30)

2. **Boundary-to-bulk intertwining relation**: The operator which is gener-
ically inverse to $L^\tau_\chi$ and which restores a function on de Sitter space $S_0$
from its boundary value is given as follows:

$$\tilde{L}^\tau_\chi : C_\chi \rightarrow \hat{C}_\tau ,$$  \hspace{1cm} (31)
\[ \tilde{T}^\tau(g) \circ \tilde{L}^\tau \chi = \tilde{L}^\tau \circ T^\chi(g), \quad \forall g \in G, \quad (32) \]

\[ (\tilde{L}^\tau \chi f)(\tilde{n}_x a) = \int K^\chi(\tilde{n}_x, |a|; \tilde{n}_x') f(\tilde{n}_x') d^d x', \quad (33) \]

where \( K^\chi(\tilde{n}_x, |a|; \tilde{n}_x') \) is a linear operator acting from the space \( V_\mu \) to the space \( U_{\tau \chi} \), and we have used the fact that \( \tilde{n}_x \in \tilde{N} \) may be parametrized by \( R^d \), i.e., \( x, x' \in R^d \), further \( d^d x \) is the Haar measure on \( R^d \). Actually, the integral kernel depends only on the difference \( z = x - x' \) and is explicitly given by:

\[ K^\chi(z, |a|) = N^\chi \left( \frac{|a|}{b(z) + |a|^2} \right)^{d-\Delta} \tilde{D}^\tau(k_c(-\frac{z}{|a|})) \Pi^\mu, \quad z = x - x', \quad (34) \]

where \( b(z) \) is a bilinear form on \( z \in R^d \), \( b(z) = z_1^2 + \cdots + z_d^2 \), and we use (25), thus \( k_c \) from (24) is written explicitly as \( k_c(z) \).

In (34) \( N^\chi \) is arbitrary for the moment and should be fixed from the requirement that \( \tilde{L}^\tau \chi \) is inverse to \( L^\tau \chi \) (the latter being true except on a parameter subspace of \( (\chi, \tau) \) of lower dimensionality). \( \diamond \)

Remark 1: The Theorem was proved in [14] for the case \( G = SO_0(p, 1) \), with \( d = p - 1 \). For the other real rank 1 cases \( G = SU(r, 1), G = Sp(r, 1), F_4(-20) \), the details will be given in [72]. In those cases the dimension of \( \tilde{N}_0 \) is \( 2r - 1, 4r - 1, 15 \), resp. \( \diamond \)

Remark 2: Note that the Theorem is valid also for the replacement of the representation \( \chi = [\mu, \nu] \) by the conjugate representation (called ”shadow” in the physics applications) \( \tilde{\chi} = [\tilde{\mu}, \tilde{\nu}] \), where \( \tilde{\mu} \) (called ”mirror”) is the Weyl conjugate of \( \mu \), while \( \tilde{\nu}(a) = d - \Delta \). \( \diamond \)

Furthermore, on the elementary representations \( \chi \) are defined the integral Knapp-Stein \( G_\chi \) operators which intertwine the representation \( \chi \) with the representation \( \tilde{\chi} \):

\[ G_\chi : C_\chi \rightarrow C_{\tilde{\chi}}, \quad G_\chi \circ T_\chi(g) = T_{\tilde{\chi}} \circ G_\chi \quad (35) \]

The operators \( G_\chi, G_{\tilde{\chi}} \) have integral kernels that are given by the corresponding two-point functions, cf. [17], [12]. The representations \( \chi, \tilde{\chi} \) are called partially equivalent due to the existence of the above intertwining operators. The representations are called equivalent if the latter intertwining operators are onto and invertible.

We also recall that the Casimirs \( C_i \) have the same values on the partially equivalent ERs:

\[ C_i(\mu, \nu) = C_i(\tilde{\mu}, \tilde{\nu}) \quad (36) \]

Thus, a bulk representation has two elementary representations as boundaries!
4 The split rank $> 1$ cases

4.1 The split rank two cases

We restrict first the exposition to the cases of split rank 2. This is natural since this includes the very important for applications Minkowskian conformal group $SO_0(p, 2)$ in $p$-dimensions. (For $p=4$ see, e.g., [73], for $p=3$ [74].) Furthermore, these cases are indicative for the general cases. Our results on the structure of the real Lie algebras of split rank 2 are given in Table 3. For completeness besides the classical cases we have included also the three exceptional real Lie algebra of split rank 2: $E_{6(-14)}$, $E_{6(-26)}$ and $G_{2(2)}$.

For our considerations we shall use first the universal elementary representations introduced from a suitable maximal parabolic $P = MAN$ so that the subgroup $N$ is of maximal dimension w.r.t. other possible choices. For split rank 2 we have possibly two maximal parabolics shown as $M_1$ and $M_2$ in Table 3. Sometimes the two maximal parabolics are isomorphic and in that case there is only one entry for $M$ in the table. We first consider the elementary representations induced from a maximal parabolic $P = MAN$, so that in the case when two such parabolics are available we designate the chosen one again by $M$. Thus, we have:

- Bulk-boundary via Bruhat decomposition:
  This intertwining relation is similar to Theorem 1 above but relations (26), (27), (29) are replaced by:

$$L_{\chi} : \tilde{C}_{\chi} \longrightarrow C_{\chi},$$

$$L_{\chi}(\phi)(\tilde{n}) = \lim_{|a| \to 0} |a|^{-\nu(a)} \phi(\tilde{n}a)$$

$$L_{\chi} \circ \tilde{T}^\chi(g) = T^\chi(g) \circ L_{\chi},$$

while relations (28), (30) are not relevant as there is no factor $\Pi_\mu$. Furthermore, relations (31), (32), (33), (34) are replaced by:

$$\tilde{L}_{\chi} : C_{\chi} \longrightarrow \tilde{C}_{\chi},$$

$$\tilde{T}^\chi(g) \circ \tilde{L}_{\chi} = \tilde{L}_{\chi} \circ T^\chi(g), \quad \forall g \in G,$$

$$(\tilde{L}_{\chi} f)(\tilde{n}x a) = \int K_{\chi}(\tilde{n}x, |a|; \tilde{n}x') f(\tilde{n}x') d^d x'$$

$$K_{\chi}(z, |a|) = N_{\chi}\left(\frac{|a|}{b'(z) + |a|^2}\right)^{d-\Delta} D_\mu(r(-\frac{z}{|a|})),$$

where $z = x - x'$, $r(y) \in M$. 

12
where the explicit form of \( b'(z) \) depends on the concrete \( N \), \( r(y) \) depends on the concrete \( M \) and the concrete representation \( \mu \) (see [72] where some cases will be considered).

- **Bulk-boundary via Sekiguchi-like decomposition:**
  This intertwining relation is similar to Theorem 1 above. Actually, we just need to replace \( K \) with \( H \) - when it exists, and then replace \( M_0 \) with \( M \) (which is a subgroup of \( H \)). Thus, we have
  \[
  L_\chi^\sigma : \hat{C}_\sigma \rightarrow C_\chi ,
  \]
  with the following action:
  \[
  (L_\chi^\sigma \phi)(\bar{n}) = \lim_{|a| \rightarrow 0} |a|^{-\Delta} \Pi_\mu^\sigma \phi(\bar{n}a)
  \]
  where \( \Delta = \nu(a) \), \( \Pi_\mu^\sigma \) is the standard projection operator from the \( H \)-representation space \( \hat{W}_\sigma \) to the \( M \)-representation space \( \hat{V}_\mu \), which acts in the following way on the \( H \)-representation matrices:
  \[
  \Pi_\mu^\sigma \hat{D}^\sigma(H) = D^\mu(m(H)) \Pi_\mu^\sigma \hat{D}^\sigma(h_c)
  \]
  where we have used group decomposition for every \( h \in H \):
  \[
  h = m(h)h_c
  \]
  where \( m(h) \) parametrizes the subgroup \( M \subset H \), while \( h_c \) parametrizes the coset \( H/M \), thus, (47) represents the decomposition of \( H \) into its subgroup \( M \) and the coset \( H/M : H \cong M K/M \). We shall use also the relation \( H/M \cong \tilde{N} \) following from:
  \[
  G = HAN \cong \tilde{N}MAN \Rightarrow H \cong \tilde{N}M \Rightarrow H/M \cong \tilde{N}
  \]
  Then \( L_\chi^\sigma \) is an intertwining operator, i.e.:
  \[
  L_\chi^\sigma \circ \hat{T}^\sigma(g) = T^\chi(g) \circ L_\chi^\sigma , \quad \forall g \in G .
  \]
  In addition, in (45) the operator \( \Pi_\mu^\sigma \) acts in the following truncated way:
  \[
  \Pi_\mu^\sigma \hat{D}^\sigma(h) = D^\mu(m(h)) \Pi_\mu^\sigma
  \]
  The operator inverse to \( L_\chi^\sigma \), which would restore a function on de Sitter space \( S \) from its boundary value is given as follows:
  \[
  \tilde{L}_\chi^\sigma : C_\chi \rightarrow \hat{C}_\sigma ,
  \]
\[ \hat{T}^\sigma(g) \circ \hat{L}^\sigma = \hat{L}^\sigma \circ T^\chi(g), \quad \forall g \in G, \quad (52) \]

\[ (\hat{L}^\sigma f)(\tilde{n}_x a) = \int K^\sigma(x,|a|; \tilde{n}_x) f(\tilde{n}_x') d^d x', \quad (53) \]

where \( K^\sigma(\tilde{n}_x,|a|; \tilde{n}_x') \) is a linear operator acting from the space \( V_\mu \) to the space \( W_\sigma \), and we have used the fact that \( \tilde{n}_x \in \tilde{N} \) may be parametrized by \( R^d \), i.e., \( x, x' \in R^d \), further \( d^d x \) is the Haar measure on \( R^d \). Actually, the integral kernel depends only on the difference \( z = x - x' \) and is given by:

\[ K^\sigma(z,|a|) = N^\sigma_x \left( \frac{|a|}{b''(z) + |a|^2} \right)^{d-\Delta} \hat{D}^\sigma(h_c(-\frac{z}{|a|})) \prod^\mu_x. \quad (54) \]

where the explicit form of \( b''(z) \) depends on the concrete \( N \), \( h_c(g) \) depends on the concrete \( H \) and the concrete representation \( \sigma \).

Remark: Sekiguchi [60] introduced this decomposition for the cases AIII, BDI, CII, though not in our context. In the AdS/CFT context in the case BDI for \( G = SO_0(d,2) \) the local coordinates of \( \tilde{N}A \) are [61]:

\[ x_\mu = \frac{g_{\mu,d+1}}{g_{d+1,d} + g_{d+1,d+1}}, \quad \mu = 0, \ldots, d-1, \]

\[ |a| = |g_{d+1,d} + g_{d+1,d+1}|, \quad (55) \]

where the matrix \( g \in G \) is represented explicitly by \( g_{\alpha\beta}, \alpha, \beta = 0, 1, \ldots, d, d+1 \). ◊

### 4.2 Split rank > 2 cases

Our results on the structure of the cases of higher split rank > 2 are given in Tables 4 for classical real semisimple Lie groups and in Table 5 for exceptional real semisimple Lie groups and algebras.

In Table 4 we give the important factors when using maximal parabolics for classical real semisimple Lie groups. The various parabolics are enumerated by giving explicitly the factors \( M_j \) in column 4. We also give the Sekiguchi-like factors \( H \) when available. In some cases \( H \) coincides with some \( M_j \) factor and this is pointed out in Column 3. In other cases it is important to record which \( H \) factor is consistent with some factor \( M_j : M_H = M_j \), so that the decompositions hold:

\[ G \cong \tilde{N}A_m H \cong \tilde{N}A_m M_H N \quad (56) \]

In Table 5 we give the important factors when using maximal parabolics exceptional real semisimple Lie algebras. In the case there is no parametric
enumeration of the $M_j$ factors, so the possible cases are given explicitly. Here there are less occurrences of Sekiguchi-like factors $H_j$ and they are given next to the consistent with them $M_j$ factors:

$$G \cong \tilde{N}A_mH \cong \tilde{N}A_mM,HN$$ (57)

The bulk-boundary correspondence is given similarly to the Split rank 2 cases. In the exceptional cases we use the language of Lie algebras. This could be important for the subtle differences between the Lie groups $M,H$ and their Lie algebras $M, H$, yet there is no problem since we use finite-dimensional representations of $M$ and $H$ for the induction process.

## 5 Summary and Outlook

In the present paper we have laid down the foundations of generalizing the AdS/CFT (holography) idea beyond the conformal setting. The main tool is to find suitable realizations of the bulk and boundary via group theory. We use all ten families of classical real semisimple Lie groups $G$ and Lie algebras $G$. On the boundaries we use the notion of elementary representations since these provide as subrepresentations all possible irreducible admissible representations of $G$. In the bulk we use several group and algebra decompositions choosing what is most simple to use. Thus, we use the global Iwasawa decomposition in the cases when $G$ is of split rank one. In the cases when $G$ is of split rank $>1$ we use the local Sekiguchi-like decompositions when it exists, otherwise we use the local Bruhat decomposition. All results are given in separate tables. The same analysis is applied to the exceptional real semisimple Lie algebras.

We stress that these are only the foundations. Further investigations for explicit applications would require separate work on each member of the ten families of classical real semisimple Lie groups, also separately for the cases of split rank one and split rank $>1$, also taking into account the availability or not of the Sekiguchi-like decompositions.

Among further more remote possible applications we would mention the following. One may look to accommodate in some cases of our setting a time direction and energy associated with it, maybe using the fact that in some cases there exist positive energy representations via holomorphic highest weight representations. Certainly, similar ideas from the present paper may be applied to more general symmetry objects such as: quantum groups, supergroups, Kac-Moody groups, especially, if suitable for applications to string theory.
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Table 1: Tenfold list of classical real semisimple Lie groups $G$

| Type of $G$ | $G = KA_0 N_0$ | $K$ maximal compact subgroup | dim $A_0 N_0$ | split rank | $M_0$ |
|------------|-----------------|------------------------------|--------------|-----------|-------|
| A          | $SL(n, \mathbb{C})_\mathbb{R}$ | $SU(n)$ | $n^2 - 1$ | $n - 1$ | $U(1) \times \ldots \times U(1)$ $n - 1$ times |
| AI         | $SL(n, \mathbb{R})$ | $SO(n)$ | $\frac{1}{2}(n^2 + n - 2)$ | $n - 1$ | $e$ |
| AII        | $SU^*(2n)$ | $Sp(n)$ | $2n^2 - 2n + 1$ | $n - 1$ | $SU(2) \times \ldots \times SU(2)$ $n$ times |
| AIII       | $SU(p, q)$ | $S(U(p) \times U(q))$ | $2pq$ | $q$ | $SU(p-q) \times U(1) \times \ldots \times U(1)$ $p > q$, $q$ times $U(1) \times \ldots \times U(1)$, $q - 1$ times $p = q$ |
| BD         | $SO(n, \mathbb{C})_\mathbb{R}$ | $SO(n)$ | $\frac{1}{2}n(n - 1)$ | $\lceil \frac{n}{2} \rceil$ | $U(1) \times \ldots \times U(1)$ $\lceil \frac{n}{2} \rceil$ times |
| BDI        | $SO_0(p, q)$ | $SO(p) \times SO(q)$ | $pq$ | $q$ | $SO(p-q)$ |
| DIII       | $SO^*(2n)$ | $U(n)$ | $n(n - 1)$ | $\lceil \frac{n}{2} \rceil$ | $SO(3) \times \ldots \times SO(3)$ $n = 2r$, $r$ times $SO(2) \times SO(3) \times \ldots \times SO(3)$ $n = 2r + 1$, $r$ times |
| C          | $Sp(n, \mathbb{C})_\mathbb{R}$ | $Sp(n)$ | $n(2n + 1)$ | $n$ | $U(1) \times \ldots \times U(1)$ $n$ times |
| CI         | $Sp(n, \mathbb{R})$ | $U(n)$ | $n(n + 1)$ | $n$ | $e$ |
| CII        | $Sp(p, q)$ | $Sp(p) \oplus Sp(q)$ | $4pq$ | $q$ | $sp(p-q) \times sp(1) \times \ldots \times sp(1)$ $q$ times |
| Type of $G$ | $G = KA_0N_0 \cong \tilde{N}_0M_0A_0N_0$ | $K$ maximal compact subgroup | $\dim A_0N_0$ | $M_0$ |
|-------------|---------------------------------|-----------------|----------------|-------|
| A$\cong$BDI $p = 3$ | $SL(2, \mathbb{C})_\mathbb{R}$ | $SU(2)$ | 3 | $U(1)$ |
| A$\cong$BDI $p = 2$ | $SL(2, \mathbb{R})$ | $SO(2)$ | 2 | e |
| AII$\cong$BDI $p = 5$ | $SU^*(4)$ | $Sp(2)$ | 5 | $SU(2) \times SU(2)$ |
| AIII | $SU(r, 1)$ | $U(r)$ | $2r$ | $U(r - 1)$ |
| BD$\cong$BDI $p = 3$ | $SO(3, \mathbb{C})_\mathbb{R}$ | $SO(3)$ | 3 | $SO(2)$ |
| BDI $p \geq 2$ | $SO_0(p, 1)$ | $SO(p)$ | $p$ | $SO(p - 1)$ |
| DIII $p \geq 2$ | $SO^*(4)$ | $SO(3) \times SO(2)$ | 2 | $SO(3)$ |
| C$\cong$BDI $p = 3$ | $Sp(1, \mathbb{C})_\mathbb{R}$ | $Sp(1)$ | 3 | $U(1)$ |
| CI$\cong$BDI $p = 2$ | $Sp(1, \mathbb{R})$ | $U(1)$ | 2 | e |
| CII $r \geq 2$ | $Sp(r, 1)$ | $Sp(r) \oplus Sp(1)$ | $4r$ | $Sp(r - 1) \oplus Sp(1)$ |
| FII | $F_{4(-20)} = F'_4$ | $so(9)$ | 16 | $so(7)$ |
Table 3: Tenfold Table of real semisimple Lie groups $G$ of split rank 2, ($\dim A_0 = 2$), showing maximal parabolic subgroups $P = MA_m N$, ($\dim A_m = 1$), showing also Sekiguchi(-like) subgroups $H$

| Type of $G$ | $G = KA_0N_0 \cong \tilde{N}A_m MN \cong \tilde{N}A_m H$ | $K$ | $M$ | $\dim NA_m = \dim G/MN$ ($= \dim G/H$) = bulk dim. = $d+1$ |
|-------------|-------------------------------------------------|-----|-----|-------------------|
| A           | $SL(3, \mathbb{C})_\mathbb{R}$ 16              | $SU(3)$ | $U(1) \times U(1)$ | $M = U(1) \times SL(2, \mathbb{C})_\mathbb{R}$ | 5 |
| AI          | $SL(3, \mathbb{R})$ 8                           | $SO(3)$ | $e$               | $M = SL(2, \mathbb{R})$ | 3 |
| AII         | $SU^*(6)$ 35                                     | $Sp(3)$ | $SU(2) \times SU(2) \times SU(2)$ | $M = SU(2) \times SU^*(4)$ | 9 |
| AIII        | $SU(p, 2)$ $p^2 + 4p + 3$ $p \geq 2$            | $U(p) \times SU(2)$ | $U(p - 2) \times U(1)$ | $M_1 = M = U(p - 1, 1)$, $M_2 = U(p - 2) \times SL(2, \mathbb{C})_\mathbb{R}$, $H = U(p, 1)$ | $2(p + 1)$ |
| BD          | $SO(4, \mathbb{C})_\mathbb{R}$ 12              | $SO(4)$ | $SO(2) \times SO(2)$ | $M_1 = M = SO(2) \times SO(2, \mathbb{C})_\mathbb{R}$, $M_2 = SO(2) \times SL(2, \mathbb{C})_\mathbb{R}$, $H = SO(2) \times SO(3, \mathbb{C})_\mathbb{R}$ | 5 |
| BD          | $SO(5, \mathbb{C})_\mathbb{R}$ 20              | $SO(5)$ | $SO(2) \times SO(2)$ | $M_1 = M = SO(2) \times SO(3, \mathbb{C})_\mathbb{R}$, $M_2 = SO(2) \times SO(4, \mathbb{C})_\mathbb{R}$ | 7 |
| BD          | $SO_0(p, 2)$ $(p + 2)(p + 1)/2$ $p \geq 2$      | $SO(p) \times SO(2)$ | $SO(p - 2)$ | $M_1 = M = SO_0(p - 1, 1)$, $M_2 = SO(p - 2) \times SL(2, \mathbb{R})$, $H = SO_0(p, 1)$ | $p + 1$ |
| DIII        | $SO^*(8)$ 28                                     | $U(4)$ | $SO(3) \times SO(3)$ | $M_1 = M = SO(3) \times SO^*(4)$, $M_2 = SO_0(5, 1)$, $H = SO(3) \times SO^*(6)$ | 10 |
| DIII        | $SO^*(10)$ 45                                    | $U(5)$ | $SO(2) \times SO(3) \times SO(3)$ | $M_1 = M = SO(3) \times SO^*(6)$, $M_2 = SO_0(5, 1) \times SO(2)$, $H = SO(3) \times SO^*(8)$ | 14 |
| C           | $Sp(2, \mathbb{C})_\mathbb{R}$ 20               | $Sp(2)$ | $U(1) \times U(1)$ | $M = U(1) \times Sp(1, \mathbb{C})_\mathbb{R}$, $H = U(1) \times Sp(1, \mathbb{C})_\mathbb{R} \times Sp(1, \mathbb{C})_\mathbb{R}$ | 7 |
| CI          | $Sp(2, \mathbb{R})$ 10                           | $U(2)$ | $e$               | $M = Sp(1, \mathbb{R})$, $H = Sp(1, \mathbb{R}) \times Sp(1, \mathbb{R})$ | 4 |
| CII         | $Sp(p, 2)$ $(p + 2)(2p + 5)$ $p \geq 2$        | $Sp(p) \times Sp(2)$ | $Sp(p - 2) \times Sp(1) \times Sp(1)$ | $M_1 = M = Sp(1) \times Sp(p - 1, 1)$, $M_2 = Sp(p - 2) \times SU^*(4)$, $H = Sp(1) \times Sp(p, 1)$ | $4(p + 1)$ |
| EIII        | $E_{6(-14)} = E_6^{11}$ 78                      | $so(10) \oplus so(2)$ | $so(6) \oplus so(2)$ | $M_1 = so(7, 1) \oplus so(2)$, $M_2 = su(5, 1)$ | 25 |
| EIV         | $E_{6(-26)} = E_6^{10}$ 78                      | $f_4$ | $so(8)$ | $M = so(9, 1)$ | 17 |
| G           | $G_2(2) = G_2^*$ 14                             | $so(3) \oplus so(3)$ | $e$               | $M = sl(2, \mathbb{R})$, $H = sl(3, \mathbb{R})$ | 6 |
Table 4 : Tenfold Table of classical real semisimple Lie groups $G$ showing maximal parabolic subgroups $P = MA_mN$, $\dim A_m = 1$, also Sekiguchi(-like) subgroups

| Type of $G$ | $G \cong N A_m H \cong N A_m M N$ | $H$ Sekiguchi(-like) subgroup | $M_j$ | bulk dim.: $\dim \bar{N} j A_m = d + 1 = \dim G/H$ |
|-------------|----------------------------------|-------------------------------|-------|---------------------------------|
| A           | $SL(n, \mathbb{C})_{\mathbb{R}}$ | $n = 7 \downarrow H \cong M_1$ | $U(1) \times SL(j, \mathbb{C})_{\mathbb{R}} \times SL(n-j, \mathbb{C})_{\mathbb{R}}$ | dim $M_j = 2(n^2 + 2j^2 - 2nj) - 3$ | $2j(n-j)+1$ |
|             | $2(n^2 - 1)$ | $H = U(1) \times SL(6, \mathbb{C})_{\mathbb{R}}$, $\dim H = 71$ | | $n = 7, j = 3, 4 \downarrow$ | $\dim \bar{N} j A_m = 25$ |
| AI          | $SL(n, \mathbb{R})$ | $n = 7 \downarrow H \cong M_1$ | $SL(j, \mathbb{R}) \oplus SL(n-j, \mathbb{R})$ | dim $M_j = n^2 + 2j^2 - 2nj - 2$ | $j(n-j)+1$ |
|             | $n^2 - 1$ | $H = SL(6, \mathbb{R})$, $\dim H = 35$ | | $n = 7, j = 3, 4 \downarrow$ | $\dim \bar{N} j A_m = 13$ |
| AI II       | $SU^*(2n)$ | | $SU^*(2j) \times SU^*(2n-2j)$ | dim $M_j = 4(n^2 - 2nj + 2j^2) - 2$ | $4j(n-j)+1$ |
|             | $4n^2 - 1$ | | | | |
| AI II       | $SU(p, q)$ | $U(p, q - 1)$ | $M_j = U(p-j, q-j) \times SL(j, \mathbb{C})_{\mathbb{R}}$ | for $j \leq q$, $M_H = M_1$ | $j(2(p+q)-3j)+1$ |
|             | $(p + q)^2 - 1$ | $p \geq q > 1$ | $n^2 + 6j^2 - 4nj - 2, n = p+q$ | | |
| BD          | $SO(n, \mathbb{C})_{\mathbb{R}}$ | $SO(2) \times SO(n-1, \mathbb{C})_{\mathbb{R}}$ | $M_j = SO(2) \times SL(j, \mathbb{C})_{\mathbb{R}} \times SO(n-2j, \mathbb{C})_{\mathbb{R}}, j \leq \frac{n}{2}$ | $n^2 + 6j^2 - 4jn - n - 2j - 1$ | $2jn - 3j^2 - j + 1$ |
|             | $n(n-1), n > 2$ | $n^2 - 3n + 3$ | | | |
| BDI         | $SO_0(p, q)$ | $SO_0(p, q - 1)$ | $M_{j-3} = SL(q-3, \mathbb{R}) \times SL(4, \mathbb{R})$ for $p=q \geq 4$ | $q^2 - 6q + 23 \rightarrow \rightarrow$ | |
|             | $(p + q)(p + q - 1)/2$ | $p > q > 1, p = q > 2$ | $M_H = M_1$ | | |
|             | | | | | |
| DIII        | $SO^*(2n)$ | $SO^*(2n-2) \times SU(2)$ | $M_j = SO^*(2n-4j) \times SU^*(2j)$ | $j \leq \frac{n}{2}$ | $M_H = M_1$ | $4j(n-j)-2j^2 - j + 1$ |
|             | $n(2n-1)$ | | | | |
| C           | $Sp(n, \mathbb{C})_{\mathbb{R}}$ | $U(1) \times Sp(1, \mathbb{C})_{\mathbb{R}} \times Sp(n-1, \mathbb{C})_{\mathbb{R}}$ | $U(1) \times SL(j, \mathbb{C})_{\mathbb{R}} \times Sp(n-j, \mathbb{C})_{\mathbb{R}}$ | dim $M_H = M_2$ | $4nj + j - 3j^2 + 1$ |
|             | $2n(2n+1)$ | | | | $\dim \bar{N} j A_m = 8n-9$ |
| CI          | $Sp(n, \mathbb{R})$ | $Sp(1, \mathbb{R}) \times Sp(n-1, \mathbb{R})$ | $M_j = SL(j, \mathbb{R}) \times Sp(n-j, \mathbb{R})$ | $2(n^2 - 2nj) + 3j^2 + n - j - 1$ | $2nj + \frac{j^2 - 3j}{2} + 1$ |
|             | $n(2n+1)$ | $dim H = 2n^2 - 3n + 4$ | | | $\dim \bar{N} A_m = 4n-4$ |
| CII         | $Sp(p, q)$ | $Sp(1) \times Sp(p, q - 1)$ | $Sp(p-j, q-j) \times SU^*(2j)$, $j \leq q$ | $2(n^2 - 4nj + 6j^2) + n - 2j - 1$ | $j(4(p+q)-6j)+j+1$ |
|             | $(p + q)(2(p+q) + 1)$ | $p \geq q > 1$ | | | |
### Table 5

Exceptional real semisimple Lie groups $G$, resp. algebras $\mathcal{G}$, of split rank $> 2$

| Type | $G = KA_0N_0$ | $\mathcal{G} = \mathcal{K} \oplus A_0 \oplus N_0$ | $\mathcal{G} = \mathcal{N} \oplus A_m \oplus M \oplus N'$ | $\dim N_0A_0 \mathcal{M}$ | $\dim N A_m = d+1$ |
|------|----------------|---------------------------------|---------------------------------|-----------------|-----------------|
| EI   | $E_{6(6)} = E_6'$ | $sp(4)$ e                      | 42                              | 6               | $M_1 = so(5, 5)$ |
|      |                |                                 |                                 |                 | $M_2 = sl(5, \mathbb{R}) \oplus sl(2, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $M_3 = sl(3, \mathbb{R}) \oplus sl(3, \mathbb{R}) \oplus sl(2, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $H_3 = so(5, 5) \oplus sl(2, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $M_4 = sl(6, \mathbb{R})$ |
| EII  | $E_{6(2)} = E_6''$ | $su(6) \oplus su(2)$ u(1) \oplus u(1) | 40                              | 4               | $M_1 = so(5, 3) \oplus so(2)$ |
|      |                |                                 |                                 |                 | $M_2 = sl(3, \mathbb{R}) \oplus u(1) \oplus sl(2, \mathbb{C})$ |
|      |                |                                 |                                 |                 | $M_3 = sl(2, \mathbb{R}) \oplus sl(3, \mathbb{C})$ |
|      |                |                                 |                                 |                 | $M_4 = su(3, 3)$ |
| EV   | $E_{7(7)} = E_7'$ | $su(8)$ e                       | 70                              | 7               | $M_1 = so(6, 6)$ |
|      |                |                                 |                                 |                 | $M_2 = sl(6, \mathbb{R}) \oplus sl(2, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $M_3 = sl(4, \mathbb{R}) \oplus sl(3, \mathbb{R}) \oplus sl(2, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $H_3 = E_6' \oplus so(2)$ |
|      |                |                                 |                                 |                 | $M_4 = sl(5, \mathbb{R}) \oplus sl(3, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $M_5 = so(5, 5) \oplus sl(2, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $M_6 = E_6''$ |
|      |                |                                 |                                 |                 | $M_7 = sl(7, \mathbb{R})$ |
| EVI  | $E_{7(-5)} = E_7''$ | $so(12) \oplus so(3)$ so(3) \oplus so(3) \oplus so(3) | 64                              | 4               | $M_1 = so(7, 3) \oplus su(2)$ |
|      |                |                                 |                                 |                 | $M_2 = sl(3, \mathbb{R}) \oplus su^*(4) \oplus su(2)$ |
|      |                |                                 |                                 |                 | $M_3 = sl(2, \mathbb{R}) \oplus su^*(6)$ |
|      |                |                                 |                                 |                 | $M_4 = so^*(12)$ |
| EVII | $E_{7(-25)} = E_7'''$ | $e_6 \oplus so(2)$ so(8) | 54                              | 3               | $M_1 = e_6''$ |
|      |                |                                 |                                 |                 | $M_2 = sl(2, \mathbb{R}) \oplus so(9, 1)$ |
|      |                |                                 |                                 |                 | $M_3 = so(10, 2)$ |
| EVIII | $E_{8(8)} = E_8'$ | $so(16)$ e                      | 128                             | 8               | $M_1 = so(7, 7)$ |
|      |                |                                 |                                 |                 | $M_2 = sl(7, \mathbb{R}) \oplus sl(2, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $M_3 = sl(5, \mathbb{R}) \oplus sl(3, \mathbb{R}) \oplus sl(2, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $M_4 = sl(5, \mathbb{R}) \oplus sl(4, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $M_5 = so(5, 5) \oplus sl(3, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $M_6 = E_6' \oplus sl(2, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $M_7 = E_7'$ |
|      |                |                                 |                                 |                 | $M_8 = sl(8, \mathbb{R})$ |
| EIX  | $E_{8(-24)} = E_8''$ | $e_7 \oplus so(3)$ so(8) | 112                             | 4               | $M_1 = so(11, 3)$ |
|      |                |                                 |                                 |                 | $M_2 = sl(3, \mathbb{R}) \oplus so(9, 1)$ |
|      |                |                                 |                                 |                 | $M_3 = sl(2, \mathbb{R}) \oplus e_6^{iv}$ |
|      |                |                                 |                                 |                 | $M_4 = e_7^{iv}$ |
| FI   | $F_{4(4)} = F_4'$ | $sp(3) \oplus so(3)$ e          | 28                              | 4               | $M_1 = sl(3, \mathbb{R}) \oplus sl(2, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $M_2 = so(4, 3)$ |
|      |                |                                 |                                 |                 | $H_2 = so(5, 4)$ |
|      |                |                                 |                                 |                 | $M_3 = sp(3, \mathbb{R})$ |
|      |                |                                 |                                 |                 | $H_3 = sp(4, \mathbb{R})$ |