α-Admissibility for Ritt Operators

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Abstract Let $T : X \to X$ be a power bounded operator on Banach space. An operator $C : X \to Y$ is called admissible for $T$ if it satisfies an estimate $\sum_k \|CT_k(x)\|^2 \leq M^2\|x\|^2$. Following Harper and Wynn, we study the validity of a certain Weiss conjecture in this discrete setting. We show that when $X$ is reflexive and $T$ is a Ritt operator satisfying an appropriate square function estimate, $C$ is admissible for $T$ if and only if it satisfies a uniform estimate $(1 - |\omega|^2)^{1/2} \|C(I - \omega T)^{-1}\| \leq K$ for $\omega \in \mathbb{C}, |\omega| < 1$. We extend this result to the more general setting of $\alpha$-admissibility. Then we investigate a natural variant of admissibility involving $R$-boundedness and provide examples to which our general results apply.

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1 Introduction

Admissibility of observation operators for semigroups has attracted a lot of attention in the last fifteen years. In general, one starts with a bounded $C_0$-semigroup $(T_t)_{t \geq 0}$ on some Banach space $X$, with generator $-A$, one considers an operator $C$ defined and continuous on the domain of $A$, taking values in another Banach space $Y$ ($C$ is the so-called observation operator) and one wonders whether there exists an estimate

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valid for any $x$ belonging to the domain of $A$. In this case $C$ is called admissible for $(T_t)_{t \geq 0}$, and this property is important for the study of certain linear control systems. A general question then is to find criteria ensuring admissibility. We refer the reader to [13] for background, various results around this question, and applications.

In the paper [17], we exhibited such a criterion by showing that if $(T_t)_{t \geq 0}$ is a bounded analytic semigroup satisfying a square function estimate

$$
\int_0^\infty \|A^{1/2}T_t(x)\|^2 dt \leq \kappa^2 \|x\|^2,
$$

(1.1)

then an observation operator $C$ is admissible for $(T_t)_{t \geq 0}$ if and only if there exists a constant $K \geq 0$ such that

$$
(1-|\omega|^2)^{1/2} \|C(I-\omega T)^{-1}\| \leq K, \quad \omega \in \mathbb{D}.
$$

(1.3)

Again this is particularly interesting when $X$ is a Hilbert space, since (1.2) holds true whenever $T$ is a Ritt operator and $\|T\| \leq 1$ (see [18, Thm. 8.1]). Thus in the
terminology of [7, 20–22], Ritt contractions on Hilbert space satisfy a strong form of the discrete Weiss conjecture.

The above results will be established in Sect. 3, however we place them in a broader context. We consider a weighted form of admissibility, called $\alpha$-admissibility. That concept was introduced in [11] in the classical semigroup setting and then in [20] in the discrete setting. See the above two papers for motivation. All the necessary background is provided in Sect. 2. In this more general framework, we show that under the assumption (1.2), $\alpha$-admissibility is equivalent to a certain resolvent estimate.

In the last Sect. 4, we consider a variant of the above discussed notion, called $R$-admissibility. This name refers to $R$-boundedness. The study of $R$-admissibility in the semigroup setting goes back to [7, 9, 10, 17], and turns out to have more applications than classical admissibility when one deals with non Hilbertian Banach spaces. We introduce a relevant definition of discrete $R$-admissibility and again, we give sufficient conditions under which $R$-admissibility can be characterized by a resolvent estimate.

2 $\alpha$-Admissibility, Resolvent Estimates and Square Functions

We start with a few definitions and preliminary results which either come from Wynn’s paper [20] or generalize it in a simple way.

Let $X$ be a Banach space and let $T : X \to X$ be a power bounded operator, that is, there exists a constant $c_0 \geq 0$ such that

$$\|T^k\| \leq c_0, \quad k \geq 0.$$  \hfill (2.1)

Let $Y$ be another Banach space, let $C : X \to Y$ be a bounded operator and let $\alpha > -1$ be a real number. We say that $C$ is $\alpha$-admissible for $T$ if there is a constant $M \geq 0$ such that

$$\sum_{k=0}^{\infty} (k + 1)^{\alpha} \|CT^k(x)\|^2 \leq M^2 \|x\|^2, \quad x \in X.$$  \hfill (2.2)

Admissibility (as discussed in Sect. 1) simply coincides with 0-admissibility.

The following elementary result connects $\alpha$-admissibility to resolvent estimates. Note that the power boundedness assumption ensures that the spectrum of $T$ is included in $\overline{\mathbb{D}}$.

**Proposition 2.1** Let $\alpha, \beta > -1$ be real numbers such that $m = \frac{\alpha + \beta}{2}$ is a nonnegative integer. If $C$ is $\alpha$-admissible for $T$, then there exists a constant $K \geq 0$ such that

$$\left(1 - |\omega|^2\right)^{\frac{1+\beta}{2}} \|C(I - \omega T)^{-(m+1)}\| \leq K, \quad \omega \in \overline{\mathbb{D}}.$$  \hfill (2.3)
Proof We assume that $T$ satisfies (2.2). We will use the existence of a constant $c > 0$ such that

$$\sum_{k=1}^{\infty} k^\beta s^{k-1} \leq \frac{c}{(1-s)^{\beta+1}}, \quad s \in (0, 1).$$

(2.4)

See e.g. [20, Lemma 2.2] for a proof of this uniform estimate.

For any integer $k \geq 1$, we set

$$c_k = \frac{k(k+1) \cdots (k+m-1)}{k^{\frac{\alpha}{2}}},$$

(2.5)

with the convention that the above numerator equals 1 when $m = 0$. For any $z \in \mathbb{D}$, we have

$$\frac{m!}{(1-z)^{m+1}} = \sum_{k=1}^{\infty} k(k+1) \cdots (k+m-1)z^{k-1},$$

taking $z = \omega^2$ for any $\omega \in \mathbb{D}$ we have

$$m!(I - \omega T)^{-(m+1)} = \sum_{k=1}^{\infty} c_k \omega^{k-1} k^{\frac{\beta}{2}} T^{k-1}.$$  

(2.6)

Composing with $C$ and using the Cauchy–Schwarz inequality, we deduce that for any $x \in X$,

$$\| C(I - \omega T)^{-(m+1)} x \| \leq \frac{1}{m!} \left( \sum_{k=1}^{\infty} k^{\alpha} \| CT^{k-1}(x) \|^2 \right)^{\frac{1}{2}} \left( \sum_{k=1}^{\infty} c_k^2 |\omega|^{2(k-1)} \right)^{\frac{1}{2}} \leq \frac{M}{m!} \| x \| \left( \sum_{k=1}^{\infty} c_k^2 |\omega|^{2(k-1)} \right)^{\frac{1}{2}}.$$

Now observe that $c_k \sim k^{m-\frac{\alpha}{2}}$, so that $c_k^2 \sim k^\beta$. Hence for a suitable constant $M' \geq 0$, we actually have an estimate

$$\| C(I - \omega T)^{-(m+1)} x \| \leq M' \| x \| \left( \sum_{k=1}^{\infty} k^\beta |\omega|^{2(k-1)} \right)^{\frac{1}{2}}.$$

Applying (2.4) with $s = |\omega|^2$, we deduce (2.3) with $K = c_2 M'$. \qed

We will focus on the specific class of Ritt operators. We shall use results and ideas from [1,18] to which we refer for background and relevant information. We recall
that a power bounded operator $T : X \to X$ is called a Ritt operator if there exists a constant $c_1 \geq 0$ such that

$$k \left\| T^k - T^{k-1} \right\| \leq c_1, \quad k \geq 1.$$  \hfill (2.7)

In this case, the operator $I - T$ is sectorial and we may therefore define its fractional powers $(I - T)^a$ for any $a > 0$ (see e.g. [18, Sect. 2] and [12, Chap. 3] for details). Then we define a ‘square function’ $\| \cdot \|_{T,a}$ on $X$ by letting

$$\| x \|_{T,a} = \left( \sum_{k=1}^{\infty} k^{2a-1} \left\| T^{k-1} (I - T)^a x \right\|^2 \right)^{\frac{1}{2}}, \quad x \in X. \hfill (2.8)$$

Note that $\| x \|_{T,a}$ may be infinite. For the sake of clarity we indicate that when $X$ is a Hilbert space, the above square functions coincide with the ones defined in [1,18]. However this is no longer the case when $X$ is not Hilbertian. In that situation, the square functions from [1,18] coincide with the ones which will be considered later on in Sect. 4.

When $X$ is reflexive, the Mean Ergodic Theorem ensures that we have a direct sum decomposition

$$X = \text{Ker}(I - T) \oplus \text{Ran}(I - T).$$ \hfill (2.9)

Then the argument in the proof of [1, Thm. 3.3] shows that the above square functions are pairwise equivalent. We state that result for further use.

**Lemma 2.2** Assume that $X$ is reflexive and let $T : X \to X$ be a Ritt operator. For any positive real numbers $a, a' > 0$, there exists a constant $c > 0$ such that

$$c^{-1} \| x \|_{T,a'} \leq \| x \|_{T,a} \leq c \| x \|_{T,a'}$$

for any $x \in X$.

In the sequel we say that $T$ satisfies a square function estimate if it satisfies an inequality $\| x \|_{T,a} \leq \kappa \| x \|$ for one (equivalently for all) $a > 0$. To be more specific ($a = 1$), $T$ satisfies a square function estimate when there exists a constant $\kappa \geq 0$ such that

$$\sum_{k=1}^{\infty} k \left\| T^k(x) - T^{k-1}(x) \right\|^2 \leq \kappa^2 \| x \|^2, \quad x \in X. \hfill (2.10)$$

### 3 Ritt Operators Satisfying the Discrete Weiss Conjecture

The general question considered in this main section is whether the estimate (2.3) implies that $C$ is $\alpha$-admissible for $T$. That question actually has several variants. In [20–22], the case when $\alpha \in (-1,1)$, $\beta = -\alpha$ and $X, Y$ are Hilbert spaces is
considered. In this situation, \( T : X \to X \) is said to satisfy the discrete Weiss conjecture if any \( C : X \to Y \) satisfying (2.3) is \( \alpha \)-admissible for \( T \). This is shown to be the case when \( \alpha \in (0, 1) \), \( Y = \mathbb{C} \) and \( T \) is normal. Moreover it is shown in [7] that when \( \alpha = 0 \) and \( Y = \mathbb{C} \), any contraction \( T \) on Hilbert space also satisfies the discrete Weiss conjecture. As far as we know, all other published results are counterexamples. In the case \( Y = \mathbb{C} \), it is shown in [21] that for any \( \alpha \in (-1, 0) \), there exist normal operators not satisfying the discrete Weiss conjecture, whereas it is shown in [22] that for any \( \alpha \in (0, 1) \), there exist contractions not satisfying the discrete Weiss conjecture.

The main interest of the next result (and of its extensions in Sect. 4) is to provide positive results, and hence large classes of operators satisfying a discrete Weiss conjecture. In particular Corollary 3.4 should be compared to the above mentioned results.

**Theorem 3.1** Let \( X, Y \) be Banach spaces, assume that \( X \) is reflexive and let \( T : X \to X \) be a Ritt operator. Assume that \( T \) satisfies the square function estimate (2.10). Let \( \alpha > -1 \) and \( \beta \in (-1, 3) \) be real numbers such that \( m = \frac{\alpha + \beta}{2} \) is a nonnegative integer. Then a bounded operator \( C : X \to Y \) is \( \alpha \)-admissible for \( T \) if (and only if) there exists a constant \( K \geq 0 \) such that

\[
(1 - |\omega|^2)^{\frac{1+\beta}{2}} \| C(I - \omega T)^{-\left(m+1\right)} \| \leq K, \quad \omega \in \mathbb{D}.
\]

(3.1)

Before writing the proof of this theorem, we need some background on sectorial operators. For any \( \nu \in (0, \pi) \), we let

\[
\Sigma_\nu = \{ \xi \in \mathbb{C}^* : |\text{Arg}(\xi)| < \nu \}
\]

be the sector of angle \( 2\nu \) around the positive real axis. By definition a closed operator \( B : D(B) \to X \) with dense domain \( D(B) \subset X \) is called sectorial if there exists an angle \( \nu \in (0, \pi) \) such that \( \xi - B \) is invertible for any \( \xi \in \mathbb{C} \setminus \Sigma_\nu \) and there exists a constant \( K_{\nu} \geq 0 \) such that

\[
|\xi| \| (\xi - B)^{-1} \| \leq K_{\nu}, \quad \xi \notin \Sigma_\nu.
\]

We say that \( B \) is of type \( \sigma \in (0, \pi) \) if this holds true for any \( \nu \in (\sigma, \pi) \).

In the sequel, we let

\[
\phi_\theta(z) = \frac{z^\theta}{1 + z}, \quad z \in \mathbb{C} \setminus \mathbb{R}_-,
\]

for any \( \theta \in (0, 1) \). The holomorphic functional calculus and the definition of fractional powers of sectorial operators lead to

\[
\phi_\theta(B) = B^\theta (I + B)^{-1},
\]

see e.g. [12] for details.
Any sectorial operator with a dense range is 1-1 and in this case, the resulting operator $B^{-1}$ is a well-defined closed operator whose domain is equal to the range of $B$ (see [5, Thm. 3.8]).

Another point is that whenever $T : X \to X$ is a Ritt operator, then the spectrum of $T$ is included in $\mathbb{D} \cup \{1\}$ and there exists a constant $c_2 \geq 0$ such that

$$\| (\lambda - T)^{-1} \| \leq \frac{c_2}{|\lambda - 1|}, \quad |\lambda| > 1,$$

(3.2)

see e.g. [2]. This implies that $B = I - T$ is a (bounded) sectorial operator of type $< \frac{\pi}{2}$. We refer the reader to [2,3,18] for more on the relationships between Ritt and sectorial operators.

Theorem 3.1 is the discrete analog of [11, Thm. 4.2]. In the course of the proof of the latter ‘sectorial’ result, we established the following property that we state for later use.

**Lemma 3.2** Let $\alpha, \beta, m$ as in Theorem 3.1 above, and let $B : D(B) \to X$ be a sectorial operator. Assume that $B$ has a dense range. Let $\Delta : D(B^{m+1}) \to Y$ be a continuous operator such that the set

$$\left\{ t^{1+\beta} \Delta(t + B)^{-(m+1)} : t > 0 \right\}$$

(3.3)

is bounded. Let $\theta \in (0, 1)$ such that $\theta < \frac{1+\beta}{2} < 1 + \theta$. Then the set

$$\left\{ t^{1+\alpha-m} \Delta B^{-m} \varphi_\theta(tB) : t > 0 \right\}$$

is bounded as well.

**Proof of Theorem 3.1** By Proposition 2.1 we only need to prove the ‘if’ assertion. Assume (3.1) and let $B = I - T$. We fix some $\theta \in (0, 1)$ such that $\theta < \frac{1+\beta}{2} < 1 + \theta$. We aim at proving that the set (3.3) for the operator $\Delta = C$ is bounded. Consider an arbitrary $t > 0$ and set $\omega = \frac{1}{1+t^2}$. On the one hand, we have

$$t + B = t + 1 - T = (t + 1)(I - \omega T),$$

hence

$$(t + B)^{-(m+1)} = \frac{1}{(t + 1)^{m+1}} (I - \omega T)^{-(m+1)}.$$

On the other hand,

$$\left(1 - |\omega|^2\right)^{1+\beta} = \left(1 - \frac{1}{(1+t)^2}\right)^{1+\beta} = \frac{(t^2 + 2t)^{1+\beta}}{(1+t)^{1+\beta}} = t^{\frac{1+\beta}{2}} \frac{(2 + t)^{1+\beta}}{(1+t)^{1+\beta}}.$$
Consequently,
\[ t^{1+\beta/2} C(t + B)^{-(m+1)} = (1 - |\omega|^2)^{1+\beta/2} C(I - \omega T)^{-(m+1)} \rho(t), \]
with
\[ \rho(t) = (1 + t)^{\beta - m} (2 + t)^{-1 + \beta/2}. \]

Since \( \alpha + \beta = 2m \), the order of \( \rho(t) \) at \( \infty \) is \( t^{-1+\alpha/2} \). Since \( \alpha > -1 \), this implies that \( \rho \) is bounded on \( (0, \infty) \). This implies the boundedness of (3.3). For any \( x \in \text{Ker}(B) \),
\[ t^{1+\beta/2} C(t + B)^{-(m+1)}(x) = t^{1+\beta/2 -(m+1)} C(x) = t^{-1+\alpha/2} C(x). \]

Since \( \alpha > -1 \), the boundedness of (3.3) implies that \( C(x) = 0 \) in this case. This shows that (2.2) holds true on \( \text{Ker}(B) \). By the reflexivity assumption and (2.9), it therefore suffices to show (2.2) on \( \text{Ran}(B) \).

Then we may (and do) assume that \( B = I - T \) has a dense range. Applying Lemma 3.2, we deduce the existence of a constant \( K' \geq 0 \) such that
\[ t^{1+\alpha/2 - m} \| CB^{-m} \varphi_\theta(tB) \| \leq K', \quad t > 0. \quad (3.4) \]

Assume that \( m \geq 1 \) (the case \( m = 0 \) is treated later). For any integer \( k \geq 1 \), we define an operator \( U_k : X \to X \) by setting
\[ U_k = k^{m-1/2 - \theta} (1 + kB) B^{m-\theta} T^{k-1}. \]

Then
\[ k^{\alpha/2} CT^{k-1} = k^{\alpha/2 - m} CB^{-m} k^\theta (1 + kB)^{-1} U_k \]
\[ = k^{\alpha/2 - m} CB^{-m} \varphi_\theta(kB) U_k. \]

Applying (3.4), we deduce that for any \( x \in X \),
\[ \sum_{k=1}^{\infty} k^\alpha \| CT^{k-1}(x) \|^2 \leq K^2 \sum_{k=1}^{\infty} \| U_k(x) \|^2. \quad (3.5) \]

Now observe that
\[ U_k = k^{m-1/2 - \theta} B^{m-\theta} T^{k-1} + k^{m+1/2 - \theta} B^{m+1-\theta} T^{k-1}, \quad (3.6) \]
and hence
\[ \| U_k(x) \|^2 \leq 2(k^{2m-2\theta-1} \| B^{m-\theta} T^{k-1} x \|^2 + k^{2m-2\theta+1} \| B^{m+1-\theta} T^{k-1} x \|^2). \]
According to (2.8) this yields
\[
\sum_{k=1}^{\infty} \|U_k(x)\|^2 \leq 2 \left( \|x\|_{T,m-\theta}^2 + \|x\|_{T,m+1-\theta}^2 \right)
\]
for any \( x \in X \). (This is the place where assuming that \( m \geq 1 \) is useful.) Applying Lemma 2.2 twice together with the estimate (3.5), we deduce
\[
\sum_{k=1}^{\infty} k^\alpha \|CT^{k-1}(x)\|^2 \leq K''2 \|x\|^2, \quad x \in X,
\]
for an appropriate \( K'' \geq 0 \), and hence the \( \alpha \)-admissibility of \( C \).

When \( m = 0 \), we have \( \alpha \in (-1, 1) \) and \( \beta = -\alpha \). Then (3.1) means that for \( \omega \) varying in \( \mathbb{D} \),
\[
(1 - |\omega|^2)^{1-\alpha} C(I - \omega T)^{-1}
\]
is uniformly bounded. By a series expansion, we have
\[
\| (I - \omega T)^{-1} \| \leq \frac{c_0}{1 - |\omega|}, \quad \omega \in \mathbb{D},
\]
where \( c_0 \) is given by (2.1). Since \( 1 - |\omega|^2 \leq 2(1 - |\omega|) \) when \( \omega \in \mathbb{D} \), we deduce that \( (1 - |\omega|^2) \| (I - \omega T)^{-1} \| \leq 2c_0 \) on \( \mathbb{D} \). This implies the existence of a constant \( K \geq 0 \) such that
\[
(1 - |\omega|^2)^{\frac{3-\alpha}{2}} \| C(I - \omega T)^{-2} \| \leq K, \quad \omega \in \mathbb{D}.
\]
We may therefore apply the preceding part of the proof with \( \beta = 2 - \alpha \) and \( m = 1 \) to obtain that \( C \) is \( \alpha \)-admissible for \( T \).

Let \( \alpha > -1 \) be a real number and let \( T : X \to X \) be a Ritt operator. Applying (2.8) with \( a = \frac{1+\alpha}{2} \) and Lemma 2.2, we see that \( C = (I - T)^{\frac{1+\alpha}{2}} \) is \( \alpha \)-admissible for \( T \) if and only if \( T \) satisfies a square function estimate. The next result shows that \( C = (I - T)^{\frac{1+\alpha}{2}} \) always fulfils the Weiss condition (3.1). Thus the square function estimate assumption in Theorem 3.1 cannot be omitted and is the ‘right’ assumption to make in that statement.

**Proposition 3.3** Let \( T \) be a Ritt operator, and let \( \alpha, \beta > -1 \) be real numbers such that \( m = \frac{\alpha + \beta}{2} \) is a nonnegative integer. Then there exists a constant \( K \geq 0 \) such that
\[
(1 - |\omega|^2)^{\frac{1+\beta}{2}} \| (I - T)^{\frac{1+\alpha}{2}} (I - \omega T)^{-(m+1)} \| \leq K, \quad \omega \in \mathbb{D}.
\]
We let \( B = I - T \) as before. Since \( B \) is sectorial of type \( < \frac{\pi}{2} \), there exists \( \gamma < \frac{\pi}{2} \) and a constant \( K_\gamma \) such that \( \xi - B \) is invertible and \( ||\xi\|| ||(\xi - B)^{-1}|| \leq K \) for any \( \xi \in \mathbb{C}^* \) with \( \text{Arg}(\xi) \geq \gamma \). Consider an auxiliary angle \( \nu \in (\frac{\pi}{2}, \pi - \gamma) \). Clearly \( z + B \) is invertible for any \( z \in \Sigma_\nu \). We claim that

\[
\sup \left\{ \frac{|z|}{2} \| B^{-\frac{1+\beta}{2}} (z + B)^{-(m+1)} \| : z \in \Sigma_\nu \right\} < \infty. \tag{3.8}
\]

Indeed let \( \Gamma_\gamma \) be the boundary of \( \Sigma_\gamma \) oriented counterclockwise. For any \( z \in \Sigma_\nu \), we have

\[
B^{-\frac{1+\beta}{2}} (z + B)^{-(m+1)} = \frac{1}{2\pi i} \int_{\Gamma_\gamma} \xi^{-\frac{1+\beta}{2}} (z + \xi)^{m+1} (\xi - B)^{-1} d\xi,
\]

see e.g. [12] for details. We immediately deduce that

\[
\| B^{-\frac{1+\beta}{2}} (z + B)^{-(m+1)} \| \leq \frac{K_\gamma}{2\pi} \int_{\Gamma_\gamma} \frac{\xi^{-\frac{1+\beta}{2}}}{|z + \xi|^{m+1}} |d\xi|.
\]

Writing \( z = |z|e^{i\varphi} \) and changing \( \xi \) into \( |z|\xi \), we obtain that the above integral is equal to

\[
\frac{|z|^{-\frac{1+\beta}{2}}}{|z|^{m+1}} \int_{\Gamma_\gamma} \frac{\xi^{-\frac{1+\beta}{2}}}{|e^{i\varphi} + \xi|^{m+1}} \left| \frac{d\xi}{\xi} \right|.
\]

The latter integral remains bounded when \( \varphi \) varies from \( \pi - \nu \) to \( \pi + \nu \). Moreover we have \( \frac{1+\beta}{2} - (m + 1) = -\frac{1+\beta}{2} \), hence (3.8) follows at once.

When \( \omega \in \mathbb{D} \setminus \{0\} \) and \( \lambda = \frac{1}{\omega} \), we have

\[
(1 - |\omega|^2)^{-\frac{1+\beta}{2}} B^{-\frac{1+\beta}{2}} (I - \omega T)^{-(m+1)} = \frac{\lambda^{m+1}}{|\lambda|^{1+\beta}} (|\lambda|^2 - 1)^{-\frac{1+\beta}{2}} B^{-\frac{1+\beta}{2}} (\lambda - T)^{-(m+1)}.
\]

Further the norms of these operators are bounded when \( |\omega| \) is away from 1 (equivalently, when \( |\lambda| \to \infty \)). Hence it suffices to show that

\[
\{ |\lambda|^{m-\beta} (|\lambda|^2 - 1)^{-\frac{1+\beta}{2}} B^{-\frac{1+\beta}{2}} (\lambda - T)^{-(m+1)} : 1 < |\lambda| < 2 \}
\]

is bounded. Writing \( |\lambda|^2 - 1 = (|\lambda| - 1)(|\lambda| + 1) \), we see that this is equivalent to showing that the set

\[
\{ (|\lambda| - 1)^{\frac{1+\beta}{2}} B^{-\frac{1+\beta}{2}} (\lambda - T)^{-(m+1)} : 1 < |\lambda| < 2 \}
\]
is bounded. Since $T$ is a Ritt operator and $\nu > \frac{\pi}{2}$, $(\lambda - T)^{-1}$ is bounded on the set $\{\lambda \notin 1 + \Sigma_\nu\} \cap \{1 < |\lambda| < 2\}$, by (3.2). Hence $(|\lambda| - 1)^{\frac{1+\beta}{2}} B^{\frac{1+\alpha}{2}} (\lambda - T)^{-(m+1)}$ is bounded on that set. It therefore suffices to show that

$$\left\{ \left( |\lambda| - 1 \right)^{\frac{1+\beta}{2}} B^{\frac{1+\alpha}{2}} (\lambda - T)^{-(m+1)} : \lambda \in 1 + \Sigma_\nu, |\lambda| > 1 \right\}$$

is bounded. Writing $\lambda = 1 + z$, we have $\lambda - T = z + B$ and $|\lambda| - 1 \leq |z|$. Hence

$$\left( |\lambda| - 1 \right)^{\frac{1+\beta}{2}} \left\| B^{\frac{1+\alpha}{2}} (\lambda - T)^{-(m+1)} \right\| \leq |z|^{\frac{1+\beta}{2}} \left\| B^{\frac{1+\alpha}{2}} (z + B)^{-(m+1)} \right\|.$$

The boundedness of (3.3) therefore follows from (3.8). $\square$

To apply Theorem 3.1, one needs to know which Ritt operators satisfy a square function estimate. According to [18, Thm. 8.1], Hilbert space contractions have this property. This leads to the following.

**Corollary 3.4** Let $X$ be a Hilbert space, let $T : X \to X$ be a contraction and assume that $T$ is a Ritt operator. Then for any $\alpha > -1$ and $\beta \in (-1, 3)$ such that $m = \frac{\alpha + \beta}{2}$ is a nonnegative integer, and for any Banach space $Y$, a bounded operator $C : X \to Y$ is admissible for $T$ if and only if there exists a constant $K \geq 0$ such that

$$\left( 1 - |\omega|^2 \right)^{\frac{1+\beta}{2}} \left\| C (I - \omega T)^{-(m+1)} \right\| \leq K, \quad \omega \in \mathbb{D}.$$

Note that by [18, Prop. 8.2], there exist Ritt operators on Hilbert space with a square function estimate (hence satisfying the above corollary) without being similar to a contraction.

**Remark 3.5** When $X$ and $Y$ are Hilbert spaces, one may deduce Theorem 3.1 and Corollary 3.4 from [11,17] using a Cayley transform argument similar to the ones used in [7,20]. However this cannot be carried out in the non Hilbertian case.

## 4 R-Admissibility

In this section we give an alternate set of results, similar to those established in Sects. 2 and 3, but using square functions different from the ones in (2.2) or (2.8). The $\ell^2$-norms appearing in these formulas will be replaced by Rademacher averages. This approach is very fruitful when dealing with non Hilbertian spaces, see in particular Corollaries 4.4 and 4.5 below. The use of Rademacher averages in the context of admissibility for a $c_0$-semigroup was initiated in [16] in the framework of $L^p$-spaces and then extended to a much broader context by Haak and Kunstmann [7,9,10].

We start with a little background on Rademacher sums. Throughout we let $(\varepsilon_k)_{k \geq 1}$ be a sequence of independent Rademacher variables on some probability space $(\mathcal{M}, d\mathbb{P})$ and for any Banach space $X$, we let $\text{Rad}(X)$ denote the closed subspace
of the Bochner space $L^2(\mathcal{M}; X)$ spanned by the set $\{\varepsilon_k \otimes x : k \geq 1, x \in X\}$. Thus for any finite family $(x_k)_{k \geq 1}$ of elements of $X$,

$$\left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)} = \left( \int \left\| \sum_k \varepsilon_k(u)x_k \right\|^2_X d\mathbb{P}(u) \right)^{\frac{1}{2}}. \quad (4.1)$$

Elements of $\text{Rad}(X)$ are sums of convergent series of the form $\sum_{k=1}^{\infty} \varepsilon_k \otimes x_k$. Moreover when $X$ does not contain $c_0$ (as an isomorphic subspace), then a series $\sum_k \varepsilon_k \otimes x_k$ converges in $L^2(\mathcal{M}; X)$ if and only its partial sums are uniformly bounded [15].

We now turn to admissibility. As before we consider a power bounded operator $T : X \to X$ and we let $\alpha > -1$. We say that an operator $C : X \to Y$ is $\alpha$-$R$-admissible for $T$ if the series $\sum_{k \geq 1} k^\alpha \varepsilon_k \otimes CT^{k-1}(x)$ converges in $\text{Rad}(Y)$ for any $x \in X$ and there is a constant $M \geq 0$ such that

$$\left\| \sum_{k=1}^{\infty} k^\alpha \varepsilon_k \otimes CT^{k-1}(x) \right\|_{\text{Rad}(Y)} \leq M \|x\|, \quad x \in X. \quad (4.2)$$

When $Y$ is a Hilbert space, $\left\| \sum_k \varepsilon_k \otimes y_k \right\|_{\text{Rad}(Y)}$ is equal to $\left( \sum_k \|y_k\|^2 \right)^{\frac{1}{2}}$ for any $(y_k)_k$ in $Y$. Thus in this case, $\alpha$-$R$-admissibility (4.2) coincides with $\alpha$-admissibility (2.2). However in general, these two notions are quite different.

We will make use of a few notions from Banach space theory such as cotype and $K$-convexity, for which we refer e.g. to [6]. We recall that $X$ being $K$-convex means that the space $\text{Rad}(X)^*$ is canonically isomorphic to $\text{Rad}(X^*)$.

The following is the ‘$R$-analog’ of Proposition 2.1.

**Proposition 4.1** Let $X, Y$ be Banach spaces and assume that $Y$ is $K$-convex. Let $\alpha, \beta > -1$ be real numbers such that $m = \frac{\alpha + \beta}{2}$ is a nonnegative integer. If $C$ is $\alpha$-$R$-admissible for $T$, then the set

$$\left\{ (1 - |\omega|^2)^{\frac{1 + \beta}{2}} C(I - \omega T)^{-(m+1)} : \omega \in \mathbb{D} \right\} \subset B(X, Y)$$

is $R$-bounded.
\textbf{Proof} Let \((\omega_k)_k\) be a finite family of \(D\), let \(V_k = (1 - |\omega_k|^2)^{1+\beta} C (I - \omega_k T)^{-(m+1)}\) for any \(k\) and let \((x_k)_k\) be a finite family of \(X\). By the \(K\)-convexity assumption, there exists a finite family \((z_k)_k\) of \(Y^*\) such that

\[
\left\| \sum_k \varepsilon_k \otimes V_k(x_k) \right\|_{\text{Rad}(Y)} = \sum_k \langle z_k, V_k(x_k) \rangle \tag{4.3}
\]

and

\[
\left\| \sum_k \varepsilon_k \otimes z_k \right\|_{\text{Rad}(Y^*)} \leq K_0, \tag{4.4}
\]

where \(K_0\) is a numerical constant only depending on \(Y\).

For any \(k\) and \(j \geq 1\), set

\[
a_{jk} = \frac{1}{m!} \left(1 - |\omega_k|^2\right)^{1+\beta} c_j \omega_k^{-j-1},
\]

where \(c_j\) is defined by (2.5). The computation at the end of the proof of Proposition 2.1 shows that the \(\sum_j |a_{jk}|^2\) are uniformly bounded, so that we have a constant \(K_1 \geq 0\) such that

\[
\left( \sum_j |a_{jk}|^2 \right)^{1/2} \leq K_1, \quad k \geq 1. \tag{4.5}
\]

According to (2.6),

\[
V_k(x_k) = \sum_{j=1}^{\infty} a_{jk} j^{\frac{\alpha}{2}} CT^{j-1}(x_k)
\]

for any \(k\), hence

\[
\sum_k \langle z_k, V_k(x_k) \rangle = \sum_{j,k} \langle a_{jk} z_k, j^{\frac{\alpha}{2}} CT^{j-1}(x_k) \rangle
\]

\[
\leq \left\| \sum_{j,k} a_{jk} \varepsilon_k \otimes \varepsilon_j \otimes z_k \right\|_{\text{Rad}(\text{Rad}(Y^*))} \left\| \sum_{j,k} j^{\frac{\alpha}{2}} \varepsilon_k \otimes \varepsilon_j \otimes CT^{j-1}(x_k) \right\|_{\text{Rad}(\text{Rad}(Y))}.
\]

Using (4.1) and the \(\alpha\)-R-admissibility assumption, we obtain that

\[
\left\| \sum_{j,k} j^{\frac{\alpha}{2}} \varepsilon_k \otimes \varepsilon_j \otimes CT^{j-1}(x_k) \right\|_{\text{Rad}(\text{Rad}(Y))}^2
\]

\[
= \int \int \left\| \sum_{j,k} \varepsilon_k(u) \varepsilon_j(v) j^{\frac{\alpha}{2}} CT^{j-1}(x_k) \right\|_Y^2 d\mathbb{P}(v) d\mathbb{P}(u)
\]
Further since \( Y \) is \( K \)-convex, the space \( Y^* \) has a finite cotype. Hence it follows from [14, Cor. 3.4] that for some constant \( K_2 \geq 0 \) (only depending on \( Y^* \)), we have

\[
\left\| \sum_{j,k} a_{jk} \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)} \leq K_2 \sup_k \left( \sum_j |a_{jk}|^2 \right)^{1/2} \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)}.
\]

Inserting (4.3), (4.4) and (4.5) in the above computation, we obtain

\[
\left\| \sum_k \varepsilon_k \otimes V_k(x_k) \right\|_{\text{Rad}(Y)} \leq MK_0 K_1 K_2 \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\text{Rad}(X)},
\]

which yields the result. \( \square \)

We say that \( T : X \to X \) is an \( R \)-Ritt operator if the two sets

\[
\{ T^k : k \geq 0 \} \quad \text{and} \quad \{ k(T^k - T^{k-1}) : k \geq 1 \}
\]

are both \( R \)-bounded. This is a strengthening of (2.1) and (2.7). This notion was introduced by Blunck [2,3], see also [18] for information.

For any Ritt operator \( T \) and any \( a > 0 \), let us consider the abstract square function \( \text{SF}_{T,a} \) defined by

\[
\text{SF}_{T,a}(x) = \left\| \sum_{k=1}^{\infty} k^{a-1/2} \varepsilon_k \otimes T^{k-1}(I - T)^a(x) \right\|_{\text{Rad}(X)}, \quad x \in X
\]

(with the convention that \( \text{SF}_{T,a}(x) = \infty \) if the above series diverges). These square functions coincide with the ones defined in [1, Section 6]. The following is the ‘\( R \)-analog’ of Lemma 2.2.

**Lemma 4.2** [1, Thm. 6.1] Assume that \( X \) is reflexive with a finite cotype and let \( T : X \to X \) be an \( R \)-Ritt operator. Then the square functions \( \text{SF}_{T,a} \) are pairwise equivalent.

In the sequel we say that \( T \) satisfies an \( R \)-square function estimate if there exists a constant \( \kappa \geq 0 \) such that

\[
\left\| \sum_{k=1}^{\infty} \varepsilon_k \otimes (T^k(x) - T^{k-1}(x)) \right\|_{\text{Rad}(X)} \leq \kappa \|x\|, \quad x \in X.
\]
We can now state the main result of this section, which is the analog of Theorem 3.1 for $R$-admissibility.

**Theorem 4.3** Let $X, Y$ be Banach spaces and assume that $X$ is reflexive and has a finite cotype. Let $T : X \to X$ be an $R$-Ritt operator satisfying the $R$-square function estimate (4.6). Let $\alpha > -1$ and $\beta \in (-1, 3)$ be real numbers such that $m = \frac{\alpha + \beta}{2}$ is a nonnegative integer. Then a bounded operator $C : X \to Y$ is $\alpha$-$R$-admissible for $T$ if the set

$$
\left\{ \left(1 - |\omega|^2\right)^{\frac{1 + \beta}{2}} C(I - \omega T)^{-m} : \omega \in \mathbb{D} \right\} \subset B(X, Y) \tag{4.7}
$$

is $R$-bounded.

**Proof** This is a simple adaptation of the proof of Theorem 3.1 so we will be deliberately sketchy. We consider $B = I - T$, we may assume that it has a dense range, and we take some $\theta \in (0, 1)$ such that $\theta < \frac{1 + \beta}{2} < 1 + \theta$. Under the assumption that (4.7) is $R$-bounded, the computation in the proof of Theorem 3.1 shows that the set

$$
\left\{ t^{\frac{1 + \beta}{2}} C(t + B)^{-m} : t > 0 \right\}
$$

is $R$-bounded. Then the computation from [11] leading to Lemma 3.2 shows that in turn, the set

$$
\left\{ t^{\frac{1 + \alpha}{2} - m} CB^{-m} \varphi_T(tB) : t > 0 \right\}
$$

is $R$-bounded. (4.8)

Assume that $m \geq 1$ and let

$$
V_k = k^{\frac{1 + \alpha}{2} - m} CB^{-m} \varphi_T(kB)
$$

for any $k \geq 1$. According to the proof of Theorem 3.1, we have

$$
k^{\frac{1}{2}} C T^{k-1} = V_k \left(k^{m - \frac{1}{2} - \theta} B^{-m - \theta} T^{k-1}\right) + V_k \left(k^{m + \frac{1}{2} - \theta} B^{-m + 1 - \theta} T^{k-1}\right)
$$

for any $k \geq 1$. Then for any $x \in X$, and for any integers $n_1 > n_0 \geq 0$, we have

$$
\left\| \sum_{k=n_0}^{n_1} k^{\frac{1}{2}} \varepsilon_k \otimes C T^{k-1}(x) \right\|_{\text{Rad}(Y)} \leq \left\| \sum_{k=n_0}^{n_1} \varepsilon_k \otimes V_k \left(k^{m - \frac{1}{2} - \theta} B^{-m - \theta} T^{k-1}(x)\right) \right\|_{\text{Rad}(Y)}
$$

$$
+ \left\| \sum_{k=n_0}^{n_1} \varepsilon_k \otimes V_k \left(k^{m + \frac{1}{2} - \theta} B^{-m + 1 - \theta} T^{k-1}(x)\right) \right\|_{\text{Rad}(Y)}.
$$
By the definition of $R$-boundedness and (4.8), this implies an estimate
\[
\left\| \sum_{k=n_0}^{n_1} k^\frac{\alpha}{2} \varepsilon_k \otimes C T^{k-1}(x) \right\|_{\text{Rad}(Y)} \leq K \left( \left\| \sum_{k=n_0}^{n_1} \varepsilon_k \otimes k^{m-\frac{1}{2}-\theta} B^{m-\theta} T^{k-1}(x) \right\|_{\text{Rad}(X)} + \left\| \sum_{k=n_0}^{n_1} \varepsilon_k \otimes k^{m+\frac{1}{2}-\theta} B^{m+1-\theta} T^{k-1}(x) \right\|_{\text{Rad}(X)} \right).
\]

Taking into account our square function estimate assumption and Lemma 4.2, this shows that $C$ is $\alpha$-$R$-admissible for $T$, with
\[
\left\| \sum_{k=0}^{\infty} k^\frac{\alpha}{2} \varepsilon_k \otimes C T^{k-1}(x) \right\|_{\text{Rad}(Y)} \leq K \left( \text{SF}_{T,m-\theta}(x) + \text{SF}_{T,m+1-\theta}(x) \right).
\]

The case $m = 0$ can be deduced from the case $m = 1$ as in the proof of Theorem 3.1.

The main motivation for considering $R$-admissibility lies in the existence of classical classes of Ritt operators satisfying an $R$-square function estimate (4.6). We refer the reader to [18] for various results concerning this property, and its relationships with the so-called $H^\infty(B_\gamma)$ functional calculus that we are going to use in the next statement. Combining Proposition 4.1 and Theorem 4.3 with [18, Corollary 6.9], we obtain the following.

**Corollary 4.4** Let $X, Y$ be Banach spaces, assume that $X$ is reflexive and has a finite cotype, and assume that $Y$ is $K$-convex. Let $T: X \to X$ be a Ritt operator which admits a bounded $H^\infty(B_\gamma)$ functional calculus for some $\gamma < \frac{\pi}{2}$. Then for any $\alpha > -1$ and $\beta \in (-1, 3)$ such that $m = \frac{\alpha+\beta}{2}$ is a nonnegative integer, a bounded operator $C: X \to Y$ is $\alpha$-$R$-admissible for $T$ if and only if the set (4.7) is $R$-bounded.

On $L^q$-spaces, Rademacher averages are equivalent to genuine square functions. Indeed whenever $(y_k)_k$ is a finite sequence of $L^q$ (with $q < \infty$), then $\left\| \sum_k \varepsilon_k \otimes y_k \right\|_{\text{Rad}(L^q)}$ is equivalent to $\left\| \left( \sum_k |y_k|^2 \right)^{\frac{1}{2}} \right\|_{L^q}$. Thus when $Y$ is an $L^q$-space, the above statements can be written in a more concrete form. Here is an illustration.

**Corollary 4.5** Let $(\Omega, \mu)$ be a measure space and let $T: L^p(\Omega) \to L^p(\Omega)$ be a positive contraction, with $1 < p < \infty$. Assume that $T$ is a Ritt operator. Let $(\Omega', \mu')$ be another measure space and let $1 < q < \infty$. Then for any $\alpha > -1$ and $\beta \in (-1, 3)$ such that $m = \frac{\alpha+\beta}{2}$ is a nonnegative integer, and for any operator $C: L^p(\Omega) \to L^q(\Omega')$, the set (4.7) is $R$-bounded if and only if there exist a constant $M \geq 0$ such that
\[
\left\| \left( \sum_{k=0}^{\infty} (k+1)^{\alpha/2} \left| C T^k(x) \right|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \leq M \|x\|_p, \quad x \in L^p(\Omega).
\]
Proof By [19, Prop. 3.2], \( T \) admits a bounded \( \mathcal{H}^{\infty}(B_{\gamma}) \) functional calculus for some \( \gamma < \frac{\pi}{2} \). Further \( X = L^p(\Omega) \) is reflexive with a finite cotype and \( Y = L^q(\Omega') \) is \( K \)-convex. Hence Corollary 4.4 ensures that the set (4.7) is \( R \)-bounded if and only if \( C \) is \( \alpha-R \)-admissible for \( T \). According to the discussion before the statement of Corollary 4.5, this is equivalent to (4.9).

\[ \square \]

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