Hadamard Multipliers on Spaces of Holomorphic Functions

Maria Trybula

Dedicated to the memory of Professor Paweł Domański.

Abstract. Several representation theorems of multipliers are derived: in terms of analytic functionals, and germs of holomorphic functions. The co-induced topology via the representation theorems is discussed. As an application of the representation theorems the density of non-invertible multipliers is proved. Moreover, Euler differential operators are distinguished among all multipliers.

Mathematics Subject Classification. 30H10, 46F15, 44A35, 30B40.

Keywords. Spaces of holomorphic functions, Multiplier, Analytic functional, Hadamard multiplication, Dilation set.

1. Notations and Main Statements

Let \( \Omega \subset \mathbb{C} \) be a Runge open set. A linear continuous map \( M : H(\Omega) \to H(\Omega) \) is called a Hadamard, or coefficient type multiplier if every monomial is an eigenvector of \( M \). The corresponding sequence of eigenvalues \((m_n)_{n \in \mathbb{N}}\) will be called the multiplier sequence.

The set of all multipliers on \( H(\Omega) \), denoted by \( M(\Omega) \), is a linear subspace of the space of all continuous operators \( L_b(H(\Omega)) \) on the Fréchet space \( H(\Omega) \). Recall that the space \( L_b(H(\Omega)) \) is endowed with the strong topology \( \tau_b \), i.e., the topology of uniform convergence on bounded subsets of \( H(\Omega) \). We equip \( M(\Omega) \) with the induced topology from \( L_b(H(\Omega)) \). Observe that \( M(\Omega) \) is a closed subspace of \( L_b(H(\Omega)) \), thus complete.

Abel’s Theorem shows that monomials form a Schauder basis of \( H(\Omega) \) if and only if \( \Omega = r\mathbb{D} \) for some \( r > 0 \), or \( \Omega = \mathbb{C} \). Thus, multipliers are not just diagonal operators. On the other hand, since \( \Omega \) is a Runge open set, every multiplier acting on \( H(\Omega) \) is unambiguously determined by its behaviour on monomials.

The research was supported by National Center of Science (Poland), Grant No. 2013/10/A/ST1/00091.
Example. Hadamard multiplication operators. (cf. [6,11,16–18], as well as the survey article [20]) For any function $f$ holomorphic around 0 let $G_f$ stand for the maximal starlike domain to which $f$ is analytically continuable. Following Hadamard define the Hadamard multiplication on the space of germs of holomorphic functions around the origin:

$$\ast : H(\{0\}) \times H(\{0\}) \to H(\{0\}), \quad \sum_n a_n z^n \ast \sum_n b_n z^n := \sum_n a_n b_n z^n.$$ 

Assume that $U, \tilde{U}$ are arbitrary domains, $0 \in U \cap \tilde{U}$. An operator $L_g : H(U) \to H(\tilde{U})$ given by the formula $L_g(f) = f \ast g$, for some $g \in H(\{0\})$ is called the Hadamard multiplication operator. If $U \subset G_{f \ast g}$ for every $f \in H(U)$, and $U$ is Runge open, then $L_g \in M(U)$. We shall prove later that every multiplier is in fact a Hadamard multiplication operator. However, this requires to extend the Hadamard product to a larger class of sets than those which contain the origin.

Example. Euler differential operators, i.e., operators of the form

$$\sum_{n=0}^\infty a_n \theta^n, \quad \text{where } \theta f(z) := zf'(z)$$

$a_n \in \mathbb{C}$. The interested reader is invited to get acquainted with the references given in [9].

In the last section we shall provide a characterization of Euler type differential among $M(\Omega)$.

Example. Hardy averaging operator $H : H(r\mathbb{D}) \to H(r\mathbb{D})$, $r \in (0, \infty]$, is the operator that corresponds to the multiplier sequence $\{1_{n+1}\}_n$. It is not hard to check that

$$H(f)(z) = \frac{1}{z} \int_{[0,z]} f(\zeta) d\zeta.$$ 

The main aim of the paper is to find a representation of all multipliers on the space $H(\Omega)$. Our work is motivated by the recent paper [9] where an analogous problem in the real analytic case has been considered. Similarly as Domanski and Langenbruch we give the description of all Hadamard multipliers on the space $H(\Omega)$ via analytic functionals $H(V(\Omega))'$ on the space of holomorphic germs on the dilation set of $\Omega$, i.e.,

$$V(\Omega) := \{z \in \mathbb{C} : z\Omega \subset \Omega\}.$$ 

In the one real variable case the core of the problem was hidden in two results: the so-called Köthe–Grothendieck Duality which represents an analytic functional as a holomorphic function on the complement of its support; and the fact which says that every analytic functional $T \in H(S)'$, $S \subset \mathbb{R}$ open, has the minimal real carrier contained in $S$. In general it does not make sense to talk about the minimal carrier, due to the simple reason: intersection of two carriers might be empty. Nevertheless, on the assumption that we made about $\Omega$, there is a simple way to omit this obstacle. Namely, intersection of two polynomially convex carriers is a carrier. Consequently, there is the
The smallest polynomially convex carrier for every $T \in H(\mathbb{C})'$ (see Sect. 3). Because of this and some other properties of dilation sets (all are collected in Sect. 2), we are able to conclude:

**Theorem 1.1.** (Representation theorem) Let $\Omega \subset \mathbb{C}$ be a Runge open set. Then the map

$$\Phi : H(V)' \to M(\Omega),$$

$$\Phi(T)(g)(z) := T(g_z), \quad g \in H(\Omega), \quad z \in \Omega,$$  \hspace{1cm} (1.1)

is an isomorphism, where $V = V(\Omega)$, $g_s(w) := g(sw)$. The inverse is defined as follows: for a given $M \in M(\Omega)$ and $z^{-1} \in \Omega^*$, the analytic functional

$$T_z := \delta_1 \circ M_z^{-1} \circ M \circ M_z : H(z\Omega) \to \mathbb{C},$$

(where $M_s(f)(w) := f(sw)$, $\delta_1(f) := f(1)$), does not depend on $z$ and has a polynomially convex carrier contained in $V(\Omega)$; furthermore, $T_z$ has a unique extension $T$ acting on $H(V)$ that solves the equation $\Phi(T) = M$.

The celebrated Hadamard’s Theorem provides the answer to the question concerning the domain of existence of the Hadamard product of Taylor series around the origin (for the classical formulation cf. [4, Chapter 1.4], or [19, Chapter 6.3]; for the extended one see [18]). However, it says nothing about the possible holomorphic extension of $f \ast g$ beyond its star product. On the other hand, we feel tempted to describe the set

$$\left\{ \varphi : \varphi \text{ is holomorphic around } 0, \ M_\varphi \in M(\Omega) \right\},$$  \hspace{1cm} (1.2)

where $M_\varphi h := h \ast \varphi$, $\Omega \subset \mathbb{C}$ is a Runge open set containing the origin, and $\ast$ denotes the Hadamard product. Surprisingly, (1.2) is well defined for every Runge open set if “$\ast$” is understood properly.

Let $f \in H(\Omega_f)$ and $g \in H(\Omega_g)$, where $\Omega_f$, $\Omega_g \subset \hat{\mathbb{C}}$ are open (!) sets such that $\Omega_f \ast \Omega_g \neq \emptyset$, where

$$\Omega_f \ast \Omega_g := \hat{\mathbb{C}} \setminus \left( (\hat{\mathbb{C}} \setminus \Omega_f) \cdot (\hat{\mathbb{C}} \setminus \Omega_g) \right).$$

For every $z \in \Omega_f \ast \Omega_g$ there exists $\gamma \subset \Omega_f \setminus \{ z(\hat{\mathbb{C}} \setminus \frac{1}{\Omega_g}) \cup \{ 0, \infty \} \}$, a finite union of closed curves such that $\gamma$ separates $\hat{\mathbb{C}} \setminus \Omega_f$ and $z(\hat{\mathbb{C}} \setminus \frac{1}{\Omega_g})$, and ind $\gamma$ satisfies certain conditions (see [18, Theorem 2.4]). Define

$$\left( f \ast g \right)(z) := \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta} g \left( \frac{z}{\zeta} \right) d\zeta.$$ \hspace{1cm} (1.3)

Then $f \ast g \in H(\Omega_f \ast \Omega_g)$ is called the (extended) Hadamard product of $f$ and $g$. In case $0 \in \Omega_f \cap \Omega_g$, (1.3) coincides with the classical definition of the Hadamard product. Thus, with no confusion we might simply write the Hadamard product. In particular, the Hadamard product is well defined for every pair $(\Omega_1, \Omega_2)$, $\Omega_1 \subset \hat{\mathbb{C}}_*$ an open set and $\Omega_2 \subset \mathbb{C}$ a Runge open set. Then $\gamma$ surrounds $z(\mathbb{C} \setminus \frac{1}{\Omega_2})$ (compact and connected) in $\Omega_1$. 
Theorem 1.2. ([18, Theorem 2.9(4)]) The map
\[ * : H(\Omega_1) \times H(\Omega_2) \to H(\Omega_1 \ast \Omega_2) \]
is bilinear continuous provided \( \Omega_1, \Omega_2 \) are open sets satisfying \( \Omega_1 \ast \Omega_2 \neq \emptyset \).

The preliminaries together with the Köthe–Grothendieck Duality lead to the following conclusion

Theorem 1.3. (Representation theorem') Let \( \Omega \) be a Runge open set, \( V = V(\Omega) \). Then \( H(\hat{\mathbb{C}} \setminus \frac{1}{V}) \) with the Hadamard multiplication is isomorphic (as algebras) with \( M(\Omega) \). The isomorphism is given as follows:
\[ \varphi : H(\hat{\mathbb{C}} \setminus \frac{1}{V}) \ni f \mapsto \left( H(\Omega) \ni g \to f \ast g \in H(\Omega) \right) \in M(\Omega). \]

Let \( S \subset \hat{\mathbb{C}}, \infty \in S \). Define
\[ H_0(S) := \{ h \in H(S) : \lim_{z \to \infty} h(z) = 0 \}. \]
Theorem 1.3 can be formulated equivalently as follows:

Theorem 1.4. (Representation theorem'') Let \( \Omega \subset \mathbb{C} \) be a Runge open set and \( V = V(\Omega) \). Then the map
\[ \tilde{\varphi} : M(\Omega) \ni M \mapsto f_M \in H_0(\hat{\mathbb{C}} \setminus V), \]
where \( f_M \) is defined by the equation:
\[ f_M(z) = \sum_{n \in \mathbb{N}} \frac{m_n}{z^{n+1}} \quad \text{if} \quad Mz^n = m_nz^n, \quad n \in \mathbb{N}, \]
is an algebra isomorphism between the multipliers on \( H(\Omega) \) with composition of operators, and the space of holomorphic germs on \( \hat{\mathbb{C}} \setminus V \) vanishing at \( \infty \) with multiplication of Laurent series, i.e.,:
\[ \sum_{n \in \mathbb{N}} \frac{a_n}{z^{n+1}} \ast \sum_{n \in \mathbb{N}} \frac{b_n}{z^{n+1}} := \sum_{n \in \mathbb{N}} \frac{a_n b_n}{z^{n+1}}. \]

The second problem that we look at is related to the co-induced topology on \( H(V)' \) by \( \Phi \) from \( M(\Omega) \), i.e., the topology on \( H(V)' \) that makes \( \Phi \) defined in Theorem 1.1 a topological isomorphism. Before we give the precise statement define
\[ \tau_k := \text{proj}_{K \subset \Omega, \text{compact}} H(V_K(\Omega))', \]
where \( V_K(\Omega) := \{ \xi \in \mathbb{C} : \xi K \subset \Omega \} \). Section 4 is entirely devoted to the proof of the fact that
\[ \Phi : (H(V)', \tau_k) \to (M(\Omega), \tau_b) \] (1.4)
defined by (1.1) is a topological isomorphism. The proof of (1.4) relies on methods explored in [10,21].

The paper ends with two applications of the results from Sects. 3 and 4: a density property of noninvertible multipliers and a characterization of Euler differential operators.

Before we proceed to the main part of the paper we explain some notation which we use throughout the paper.
\(C\) denotes the Riemann sphere.

For \(\Omega \subset C\) let \(H(\Omega)\) denote the space of holomorphic functions on \(\Omega\). The space \(H(\Omega)\) is endowed with the topology of uniform convergence on compact sets. For an arbitrary non-empty set \(S \subset C\) put \(H(S)\) for the space of germs of holomorphic functions on \(S\), i.e.,

\[
H(S) := \bigcup_{\substack{S \subset \Omega \text{ open}}} H(\Omega).
\]

The natural topology considered on \(H(S)\) is the finest locally convex topology for which all the restrictions \(H(\Omega) \ni f \mapsto f|_S \in H(S)\) are continuous. This topology is called *inductive*.

Unless otherwise stated, the dual space to \(H(\Omega)\) is equipped with the strong topology \(H(\Omega)'_b\), i.e., the topology of uniform convergence on bounded sets in \(H(\Omega)\).

Recall that \(\Omega \subset C\) is called a Runge open set if \(\hat{C}\setminus \Omega\) is connected. Monomials are linearly dense in \(H(\Omega)\) if and only if \(\Omega\) is Runge.

For \(S \subset C\) put \(S^* := S\setminus\{0\}\).

\[
D(z,r) := \{w : |w - z| < r\}, \quad D(0,r) := D(r), \quad D := D(1).
\]

We denote by \(M_z, z \in C\) the dilation operator

\[
M_z(f) := f_z,
\]

where \(f_z(w) := f(zw), w \in C\).

We assume that \(0 \in \mathbb{N}\).

For non-explained notions from Functional Analysis we refer to the book [15].

### 2. Dilation Set

For an open set \(\Omega \subset C\) define

\[
V(\Omega) := \{z \in C : z\Omega \subset \Omega\}.
\]

\(V(\Omega)\) will be called the *dilation set* of \(\Omega\). Clearly, it is always non-empty since \(1 \in V(\Omega)\).

The next proposition collects some basic facts about \(V(\Omega)\). In contrast to the real case we add one condition which seems to be the most important.

**Proposition 2.1.** *For an open set \(\Omega \subset C\) the dilation set \(V = V(\Omega)\) satisfies the following conditions:

\[d-1\) \((V_\ast, \cdot)\) is a multiplicative semigroup with unity;
\[d-2\) \(V_\ast \subset \subset C\); \(\bigcap V_\ast = \Omega\);
\[d-3\) if \(V\) is unbounded and contains zero, then \(V = \Omega = C\);
\[d-4\) if \(\Omega\) is a Runge open set, then \(\hat{C}\setminus V\) is connected.*

Before we indicate the proposition we make two remarks:

- \(0 \in V\) if and only if \(0 \in \Omega\); Consequently,

\[
V = \{z \in C : z\Omega_\ast \subset \Omega\} = \bigcap_{z \in \Omega_\ast} z^{-1}\Omega. \tag{2.1}
\]
being a Runge set is invariant under dilation, i.e., $\mathbb{C} \setminus \Omega$ has only one connected component if and only if $\mathbb{C} \setminus w\Omega$ has only one connected component for every $w \in \mathbb{C}_*$.

Proof. d-1) is clear. Proof of d-2): let $\{z_n\} \subset V_*$, $z_n \to z \in \mathbb{C}_*$. Fix $w \in \Omega$. Because $wz = \left(w \frac{z}{z_n}\right)z_n$, and the factor $w \frac{z}{z_n}$ is in $\Omega$ for $n \gg 1$, we conclude that $wz \in \Omega$.

d-3): Take $r > 0$ so that $D(r) \subset \Omega$, and $\{\alpha_n : n \in \mathbb{N}\} \subset V$, where $|\alpha_n| \to \infty$. The definition of the dilation set implies

$$\Omega \supset \bigcup_n \alpha_n D(r) = \mathbb{C}.$$ 

d-4): If $\Omega$ is a Runge open set, then by (2.1) set $\mathbb{C} \setminus V$ is a union of connected sets with non-empty intersection. Hence, $\mathbb{C} \setminus V$ is connected. □

Remark 1. (1) If $V = V(\Omega)$, where $\Omega \subset \mathbb{C}$ is open, and $0 \in V$, then $V = V_3$. (2) In the real case properties d-1) – d-3) give the precise characterization of dilation sets (see [9, Proposition 2.1]). However, in the complex case, it is no longer true as the example below indicates.

(3) If $V$ is connected, $0 \in V$, and satisfies all four conditions from Proposition 2.1, then there exists a Runge open set $\Omega$ such that $V(\Omega) = V$. Indeed, define

$$\Omega := \begin{cases} 
\mathbb{C} & \text{if } V = \mathbb{C}, \\
\mathbb{C} \setminus \frac{1}{a} & \text{otherwise.}
\end{cases}$$

Then $\Omega$ has the required properties. Indeed, by d-2) $\Omega$ is open. So, it suffices to show that $V(\Omega) = V$. Clearly, we may assume that $\Omega \neq \mathbb{C}$. First, let us observe that

$$a \notin V \text{ if and only if } \frac{1}{a} \in \Omega.$$ 

To get $V \subset V(\Omega)$, assume the contrary. Take $v \in V$, $y \in \Omega$ so that $vy \notin \Omega$. Hence, $vy = \frac{1}{z}$ for some $z \in V_*$. By d-1) $zv \in V$. From the observation we conclude that $y \notin \Omega$; a contradiction. Using once more the observation, the condition d-1), and $a \cdot \frac{1}{a} = 1 \in V$, we obtain: $a \notin V \Rightarrow a \notin V(\Omega)$, i.e., the opposite inclusion.

Example. (1) Let $V_1 := \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$. An easy observation shows that $V_1$ is a dilation set of a Runge open set $\Omega$, where

$$\Omega := \left\{z : 0 < \text{Arg} \, z < \frac{\pi}{2}, \, |z| < 2\right\}\setminus \left\{z : |z| = 2^{-n} \text{ for some } n \in \mathbb{N}, \, \frac{\pi}{4} \leq \text{Arg} \, z < \frac{\pi}{2}\right\}.$$ 

(2) Fix $\theta \in [0, 2\pi)$ such that $\frac{\theta}{\pi} \notin \mathbb{Q}$. As a candidate for a dilation set consider

$$V_2 := \bigcup_{n \in \mathbb{N}} \left(0, \left(\frac{e^{i\theta}}{2}\right)^n\right].$$

There is no Runge open set $\Omega$ with $V_2$ as its dilation set despite $V_2$ satisfies all the conclusions of Proposition 2.1. Indeed, assume that such $\Omega$ exists.
Because \((0, 1] \subset V_2\) and \(\Omega\) is Runge, some halfline \(l\) containing the origin is disjoint from \(\Omega\) (otherwise for every \(\psi \in [0, 2\pi)\) there is a non-empty segment \((0, r_\psi e^{i\psi}]\) contained in \(\Omega\), \(r_\psi > 0\); hence, \(\hat{\mathbb{C}} \setminus \Omega\) is a union of two disjoint closed non-empty sets: one that contains the origin, and the other containing infinity). But this contradicts the fact that \(\{e^{in\theta} : n \in \mathbb{N}\}\) is dense in \(\mathbb{T}\). Indeed, because \(\Omega\) is a non-empty open set, we may find a non-empty arc \(\{re^{i\xi} : \xi \in (\alpha, \beta)\} =: A\) contained in \(\Omega\) \((r > 0)\). Finally, notice that there is \(n_0 \in \mathbb{N}\) such that \(\emptyset \neq l \cap (\xi^{n_0} A \subset l \cap \Omega = \emptyset\). A contradiction.

(3) Let \(\mathbb{H}\) denote a halfplane. It is easy to check the following:

\[
V(\mathbb{H}) := \begin{cases} 
(0, \infty) & \text{if } 0 \in \partial \mathbb{H}, \\
[0, 1] & \text{if } 0 \in \mathbb{H}, \\
[1, \infty) & \text{if } 0 \notin \mathbb{H}.
\end{cases}
\]

(4) \(V(r\mathbb{D}) = \overline{B}\) for any \(r > 0\).

Proposition 2.2. Let \(V\) be a planar set. Then \(V\) has a neighborhood basis of Runge open sets if \(V^*\) is non-empty, closed in \(\hat{\mathbb{C}}\), and its complement in \(\hat{\mathbb{C}}\) is connected (e.g., \(V = V(\Omega)\) for some Runge open set \(\Omega\)).

Proof. We must show that for every choice of a neighborhood \(U\) of \(V\) we may find a Runge neighborhood \(W\) of \(V\) that is contained in \(U\).

First, let us assume that \(V = V\). Our aim is to construct an ascending sequence of Runge open sets \((W_n)_{n \in \mathbb{N}}\) contained in \(U\) whose union covers \(V\). If so, by Runge’s Theorem, for \(W\) we may take the union (cf. [1, Theorem 3.1.1]).

We proceed to the construction. There exists an open Runge set \(W_1\) such that

\[V \cap \overline{B} \subset W_1 \subset U \cap \mathbb{D}(2)\]

(cf. [2, Lemma 1.3.6]). In the \(n\)th step the compactness of \(V \cap \overline{B}(n)\) guarantees the existence of an open Runge set \(W_n\) such that

\[(V \cap \overline{B}(n)) \cup \overline{W}_{n-1} \subset W_n \subset U \cap \mathbb{D}(n + 1)\]

(we applied [2, Lemma 1.3.6] to the pair \((V \cap \overline{B}(n)) \cup \overline{W}_{n-1}, U \cap \mathbb{D}(n + 1)\)). The construction might stop after a finite number of steps. If it does not, we carry it on inductively.

Now, assume that \(V \neq V\) and \(V\) is bounded. Replacing \(V\) by its image under the map \(z \mapsto \frac{1}{z}\), we reduce the situation to the previous one.

Finally, only the case when \(V\) is neither bounded nor closed is left. We shall apply the former cases. Fix \(0 < \epsilon \ll \frac{1}{2}\). Observe that \(V \cap (\hat{\mathbb{C}} \setminus \mathbb{D}(1 - \epsilon))\) is closed, and \(V \cap \overline{\mathbb{D}}(1 + \epsilon)\) is bounded but closed. Thus, there exist Runge open sets \(U_\pm\) such that

\[V \cap \overline{\mathbb{D}}(1 + \epsilon) \subset U_+ \subset U \cap \mathbb{D}(1 + 2\epsilon),\]

and

\[V \cap (\hat{\mathbb{C}} \setminus \mathbb{D}(1 - \epsilon)) \subset U_- \subset U \setminus \overline{\mathbb{D}}(1 - 2\epsilon).\]
Directly from the construction we get that \( \infty \in (\hat{C} \setminus U_+) \cap (\hat{C} \setminus U_-) \). So, \( W := U_+ \cup U_- \) is a Runge open neighborhood of \( V \) contained in \( U \).

3. Topology on \( H(V) \) and its Dual

By an analytic functional we mean any element \( T \) in the dual space \( H(\mathbb{C})' \). \( T \) is said to be carried by a compact \( K \) if for every neighborhood \( \Omega \) of \( K \) there is a constant \( C_\Omega \) such that

\[
|Th| \leq C_\Omega \|h\|_\Omega, \quad h \in H(\mathbb{C}).
\]  

From the definition of the topology in \( H(\mathbb{C}) \) it follows that every analytic functional has a carrier.

Observe that if \( K \) carries \( T \) and \( S \) is any polynomally convex set containing \( K \), then \( T \) might actually be considered as an element of \( H(S)' \). Indeed, it is enough to take the extension of \( T \) to \( H(S) \). By the convexity the extension is unique.

For \( T \in H(\mathbb{C})' \) let \( Tz^n := t_n, \; n \in \mathbb{N} \). We call \( (t_n)_{n \in \mathbb{N}} \) the moment sequence of \( T \).

Every \( T \in H(K)' \), where \( K \subset \mathbb{C} \) compact, corresponds to a holomorphic function \( f_T \in H_0(\hat{\mathbb{C}} \setminus K) \) defined by the equation

\[
f_T(z) = T(\frac{1}{z-}) , \quad z \notin K.
\]  

The correspondence between \( f_T \) and \( T \) is given by the so-called Köthe–Grothendieck Duality

\[
T(h) = \frac{1}{2\pi i} \int_{\gamma} h(\zeta) f_T(\zeta) \, d\zeta, \quad h \in H(K),
\]  

where \( \gamma \) is a finite union of closed curves contained in \( \Omega \setminus K \) if \( h \in H(\Omega) \), where \( \Omega \supset K \) is open and such that the index \( n(\gamma, z) = 1 \) for any \( z \in K \) (cf. [2, Theorem 1.3.5]). In fact, the map

\[
H(K)' \ni T \mapsto f_T \in H_0(\hat{\mathbb{C}} \setminus K)
\]

is a topological isomorphism. Of course, since both \( H(K) \) and \( H(\Omega) \) are reflexive this gives also an analogous representation of the dual of \( H(\Omega) \). However, for our purposes we wish to allow the sets in the representations theorem to be not necessarily open nor compact.

Let \( \emptyset \neq S \subset \mathbb{C} \). Let us recall the definition of the inductive topology:

\[
H_i(S) := \text{ind}_{U \supset \text{open}} H(U).
\]

**Proposition 3.1.** Assume \( V \) is a dilation set of some Runge open set. Then

\[
H_i(V)' = H_0(\hat{\mathbb{C}} \setminus V)
\]

algebraically.

**Proof.** First observe that by Proposition 2.2

\[
\text{ind}_{U \supset \text{open}} H(U) = \text{ind}_{U \supset \text{Runge open}} H(U).
\]
Fix $T \in H_i(V)'$. Let $U$ be a Runge neighborhood of $V$. Since the composition

$$H(U) \xrightarrow{i_{U,V}} H(V) \xrightarrow{T} \mathbb{C}$$

is continuous we may apply the Köthe–Grothendieck Duality described above. We get a compact set $K_{U,V}$ (the intersection of all polynomially convex carriers of $T$) and $f_{U,V} \in H_0(\mathbb{C}\setminus K_{U,V})$ such that

$$T \circ i_{U,V}(h) = \frac{1}{2\pi i} \int_{\gamma} h(\zeta)f_{U,V}(\zeta) \, d\zeta, \quad h \in H(U), \quad (3.4)$$

for any function $f_{U,V}$ analytic on $\mathbb{C}\setminus (K_{U,V} \cup \tilde{K}_{U,V})$. Changing the roles of $U, \tilde{U}$, we obtain that $K_{U,V} = K_{\tilde{U},V}$. Consequently, $T$ has a minimal polynomially convex carrier $K_T$ contained in $V$, and there is $f_T \in H_0(\mathbb{C}\setminus K_T)$ such that (3.3) holds for $h \in H(V)$.

For the opposite inclusion, observe that (3.3) defines an analytic functional with carrier in $V$ for every $f_T \in H_0(\mathbb{C}\setminus V)$.

The linearity of the correspondence follows from the uniqueness. □

While the topology on the dilation set $V$ is clear, it is just the induced topology from $\mathbb{C}$, we have two candidates for a topology on $H(V)$:

- inductive $\text{ind}_{U \supset V \text{ open} \text{ Runge}} H(U) =: H_i(V)$ (see Proposition 2.2),
- projective $\text{proj}_{K \subset V \text{ compact}} H(K) =: H_p(V)$.

Define:

$$K_n := \begin{cases} V & \text{if } V \text{ compact}, \\ V \cap \mathbb{D}(n) & \text{if } V \text{ is unbounded, } 0 \in V, \quad n \in \mathbb{N}_+ \\ V \cap \mathbb{D}(n) \cap (\mathbb{C}\setminus \mathbb{D}(\frac{1}{n})) & \text{otherwise.} \end{cases} \quad (3.5)$$

Then the sequence $\{K_n\}_{n \geq 1}$ gives the exhaustion of $V$ by polynomially convex compact sets, i.e.,

$$K_n \in \text{int}_V K_{n+1}, \quad \bigcup_{n \in \mathbb{N}_+} K_n = V, \quad \mathbb{C}\setminus K_n \text{ connected.}$$

Hence, $H_p(V) = \text{proj}_n H(K_n)$. Moreover

**Proposition 3.2.**

$$H_p(V)' = \text{ind}_n H(K_n)' \quad \text{algebraically.}$$

**Proof.** Since polynomials are dense in every $H(K_n)$, it suffices to use the result on a reduced projective spectrum (cf. [6, p. 57]). □
In general the inductive topology is finer than the productive one. However, we will show that they coincide if \( V \) is a dilation set. For that purpose recall:

**Theorem 3.3.** ([14, Theorem 1.1]) Let \( S \subset \mathbb{C}^n \) be a locally closed set (i.e., every point in \( S \) has a neighborhood \( U \subset \mathbb{C}^n \) such that \( U \cap S \) is closed in \( U \)). The following conditions are equivalent:

1. \( H_p(S) = H_i(S) \);
2. \( H_p(S) \) is (ultra-)bornological;
3. \( H_p(S)' = H_i(S)' \) algebraically.

We will show that the third condition in Theorem 3.3 holds for \( V \).

**Theorem 3.4.** Let \( V = V(\Omega) \) where \( \Omega \) is a Runge open set. Then

\[
H_i(V) = H_p(V).
\]

**Proof.** To complete the demonstration let us note that:

\[
H_p(V)' \overset{\text{Prop. 3.2}}{=} H_0(\hat{\mathbb{C}} \setminus V) \overset{\text{Prop. 3.1}}{=} H_i(V)'
\]

in algebraic sense.

For a different proof see [14, Proposition 1.9]. \( \square \)

**Corollary 3.5.** Let \( V \) be as in Theorem 3.4. Then:

1. The inductive (projective) topology on \( H(V) \) is regular.
2. \( \text{proj}_{U \supset V} \) Runge open \( H(U)'_b = H(V)'_b \) algebraically and topologically.

**Proof.** Ad. 1: Fix a bounded set \( U \) in \( H_p(V) \). By the definition of projective topology \( U \) is contained in a bounded set \( B \) of the form

\[
B = \left\{ f : \sup_{U_n} |f| < C_n, n \in \mathbb{N}_* \right\},
\]

where \( U_n \subset \mathbb{C} \) is a Runge neighborhood of \( K_n \) and \( C_n > 0, n \in \mathbb{N}_* \) (see (3.5)). Observe that \( \bigcup_n U_n \) is a Runge open set (since \( \infty \notin U_n \) for every \( n \), we get \( \hat{\mathbb{C}} \setminus \bigcup_n U_n \) is connected), and obviously it contains \( V \). Thus, \( B \) is a locally bounded family of holomorphic functions on \( \bigcup_n U_n \). Ad. 2: It follows from (1) and [3, Proposition 2.1]. \( \square \)

4. Proofs of the Representation Theorems

**Proof of Representation Theorem.** Let \( T \in H(V)' \). Take an arbitrary \( g \in H(\Omega) \). By the openness of \( \Omega \) the function \( g_w \) is holomorphic near \( V \) provided \( w \in \Omega \). Therefore, \( \Phi(T)(g)(w) \) is well defined.

Fix \( w_0 \in \Omega \). By the proof of Proposition 3.1 we can find a polynomially convex compact set \( K_T \subset V \) and a holomorphic function \( f_T \in H_0(\hat{\mathbb{C}} \setminus K_T) \) such that (3.3) holds. In particular,

\[
T(g_{w_0}) = \frac{1}{2\pi i} \int_\gamma g_{w_0}(\zeta) f_T(\zeta) d\zeta \quad \text{if} \quad h_{w_0} \in H(\hat{\Omega}),
\]
where $\gamma \subset \tilde{\Omega} \setminus K_T$ is a suitably chosen union of curves and $\tilde{\Omega} \supset V$ is a Runge open set. Plainly, we might find open $U \ni w_0$ in a such way that $h_w$ is defined on $\gamma$ for $w \in U$. So,

$$T(g_w) = \frac{1}{2\pi i} \int_{\gamma} g_w(\zeta) f_T(\zeta) d\zeta, \quad w \in U,$$

and $\Phi(T)(g) \in H(\Omega)$.

**Φ is well-defined:** We shall show that $\Phi(T)$ is sequentially continuous. Because $H(\Omega)$ is a Frechet space it will imply the continuity of $\Phi(T)$. Let $g_n \to g$ in $H(\Omega)$ and $w_0 \in \Omega$. Choose $\gamma \subset \{ \zeta : w_0 \zeta \in \Omega \}$ as in the Köthe–Grothendieck Duality. There is $r > 0$ such that $\overline{D}(w_0, 2r) \cdot \gamma \subset \Omega$. Hence:

$$\Phi(T)(g_n)(w) = \frac{1}{2\pi i} \int_{\gamma} g_n(w\zeta) f_T(\zeta) d\zeta \quad \text{for} \quad w \in \overline{D}(w_0, r).$$

The conclusion follows from the compactness of $\overline{D}(w_0, r) \cdot \gamma \subset \Omega$. Furthermore,

$$\Phi(T)(z^n)(w) = \frac{1}{2\pi i} \int_{\gamma} (w\zeta)^n f_T(\zeta) d\zeta = w^n \frac{1}{2\pi i} \int_{\gamma} \zeta^n f_T(\zeta) d\zeta = w^n T(z^n).$$

So, $z^n$ is an eigenvector corresponding to the eigenvalue $T(z^n)$.

**Surjectivity of Φ:** Fix $M \in \mathcal{M}(\Omega)$. For $w \in \frac{1}{\Omega^*}$ define

$$T_w := \delta_1 \circ (M_w)^{-1} \circ M \circ M_w : H(w\Omega) \to \mathbb{C},$$

where

$$M_w(f) : H(w\Omega) \to H(\Omega), \quad M_w(f) := M(f_w), \quad f \in H(w\Omega).$$

Observe that $T_w$ is an analytic functional whose moment sequence is the same as the multiplier sequence of $M$, say $(m_n)_{n\in\mathbb{N}}$. Because of the proof of Proposition 3.1, $T_w$ has the minimal polynomially convex carrier $K_{T_w} \subset w\Omega$ and there is a holomorphic function $f_{T_w} \in H_0(\hat{\mathbb{C}} \setminus K_{T_w})$ such that

$$T_w(h) = \frac{-1}{2\pi i} \int_{-\gamma} h(\zeta) f_{T_w}(\zeta) d\zeta, \quad h \in H(w\Omega) \quad (4.1)$$

for a suitably chosen $\gamma$ ($w\Omega$ is Runge and $-\gamma$ means the negatively oriented curve). Taking in (4.1) $h = z^n$ for $n \in \mathbb{N}$ we see that

$$f_{T_w}(z) = \sum_{n=0}^{\infty} \frac{m_n}{z^{n+1}} \quad \text{near} \quad \infty.$$ 

Taking into account (3.2) and (3.3) we conclude that

$$f_T := \bigcup_{w^{-1} \in \Omega_*} f_{T_w}$$

is a well defined holomorphic function on $\hat{\mathbb{C}} \setminus K_T$, where

$$K_T := \bigcap_{w^{-1} \in \Omega_*} K_{T_w} \subset \bigcap_{w^{-1} \in \Omega_*} w^{-1}\Omega = V.$$
Hence, every $T_w$ extends to an analytic functional $T \in H(V)'$ given by the formula

$$T(g) = \frac{1}{2\pi i} \int_{\gamma} g(\zeta) f_T(\zeta) \, d\zeta.$$  

Further, Cauchy’s Theorem gives $\Phi(T) = M$. 

**Proof of Representation Theorem".** Proposition 3.1 with the proof of Theorem 1.1 give that $\phi$ is a bijection. It remains to show that $\phi$ is an algebra homomorphism. Fix $S, M \in M(\Omega)$. Clearly, $S \circ M \in M(\Omega)$ with $(s_n m_n)_{n \in \mathbb{N}}$ as its multiplier sequence. Because $f_S \ast f_M \in H_0(\{\infty\})$, and $f_S \ast f_M = f_{S \circ M}$ near $\infty$, we conclude that $f_{S \circ M}$ is a holomorphic extension of $f_S \ast f_M$ to $\hat{\mathbb{C}} \setminus K_{S \circ M}$. 

**Proof of Representation Theorem’.** The map defined by

$$\Gamma(f)(z) := \frac{1}{z} f\left(\frac{1}{z}\right)$$

determines an algebra isomorphism between $H_0(\hat{\mathbb{C}} \setminus V)$ and $H(\hat{\mathbb{C}} \setminus \frac{1}{V})$. Since $\infty \notin V$, it remains to apply Theorem 1.4. 

5. Continuity of $\Phi$ and $\Phi^{-1}$

A natural question is to ask what is the coinduced topology on $H(V)'$ by $\Phi$, i.e., topology that makes $\Phi$ an isomorphism while on $M(\Omega)$ we have the strong topology $t_b$. There is one natural candidate: $\tau_k$,

$$\tau_k := \text{proj}_{K \subset \Omega \text{ compact}} H(V^0_K(\Omega))'_b,$$

where $V^0_K(\Omega)$ is the union of all connected components of $V_K(\Omega)$ which have a nonempty intersection with $V(\Omega)$,

$$V_K(\Omega) := \{\zeta \in \mathbb{C} : \zeta K \subset \Omega\} = \bigcap_{z \in K^*} z^{-1} \Omega, \ K \text{ compact}.$$  

For the origin of the definition of $\tau_k$ we refer to [21].

The main goal of this section is to demonstrate the following:

**Theorem 5.1.** Let $\Omega \subset \mathbb{C}$ be a Runge open set. Then

$$\Phi : (H(V)', \tau_k) \to (M(\Omega), t_b)$$

defined in Theorem 1.1 is a topological isomorphism.

Since $\Omega$ is a Runge open set, we have

$$V_K(\Omega) = V_K \cup \{\text{bounded components of } \hat{\mathbb{C}} \setminus K\}(\Omega)$$

for every compact $K \subset \Omega$ (if $K'$ is a bounded component of $\hat{\mathbb{C}} \setminus K$ and $zK \subset \Omega$ for some $z \neq 0$, then $zK' \subset \Omega$ as well—here we have used the fact that $\hat{\mathbb{C}} \setminus \Omega$ is connected). And, finally by the same cause

$$\tau_k = \text{proj}_{K \subset \Omega \cap \mathbb{C}^* \text{ compact}} H(V^0_K(\Omega))'_b.$$  

(5.2)
Without loss of generality we might assume that $K \subset \mathbb{C}^*$ is compact. Moreover, since dilations do not change $V(\Omega)$ suppose $1 \in \Omega$ and $1 \in K$. Hence, $V_K(\Omega) \subset \Omega$ [by (5.1)].

Define

$$M(\Omega, K) := \{ M \in L_b(H(\Omega), H(K)) : M \text{ admits all monomials as 'eigenvectors'} \},$$

$$MC(\Omega, K) := \{ M \in L_b(H(\Omega), C(K)) : M \text{ admits all monomials as 'eigenvectors'} \}.$$

**Proposition 5.2.** $M(\Omega, K) = MC(\Omega, K)$ as sets and their equicontinuous sets coincide. The map $\Phi : T \mapsto M_T$ (as in Theorem 1.1) is a topological isomorphism from $H(V_K(\Omega))'$ onto $M(\Omega, K)$. Its inverse is given by $M \mapsto T_M$, $T_M f = M f(1)$. Moreover, both maps send equicontinuous sets to equicontinuous sets.

**Proof.** $\Phi$ is into: First of all, we point out that $\Phi$ is well defined, i.e., $\Phi(T)f$ is holomorphic near $K$ for every $f \in H(\Omega)$. Indeed, since $K$ is compact and all dilations are continuous, the map $K \ni z \mapsto T_I f \in C$ is well defined on some neighbourhood $U$ of $K$ such that $UV_K(\Omega) \subset \Omega$.

Consequently, by the Köthe–Grothendieck Duality $\Phi(T)f$ can be locally represented on $K$ by

$$\frac{1}{2\pi i} \int_{\gamma} f(z\zeta)f_T(\zeta) d\zeta,$$

and we might differentiate under the integral sign. On the other hand, $M_T z^n = (T w^n) z^n$. Only the continuity of $\Phi(T)$ is left. Once more we shall apply the Closed Graph Theorem, but this time in a more general context. $H(K)$ is a countable inductive limit of Fréchet spaces, so has a web, and de Wilde’s Closed Graph Theorem holds for $(H(\Omega), H(K))$ (cf. [15, Theorem 24.31]). Assume $(f_\alpha, M_T f_\alpha) \to (f, g)$. Then

$$g(z) = \lim_{\alpha} M_T (f_\alpha)(z) = \lim_{\alpha} \frac{1}{2\pi i} \int_{\gamma} f_\alpha(z\zeta)f_T(\zeta) d\zeta = T_I f = M_T f(z).$$

Thus, $\Phi(T)f = g$.

$\Phi$ is onto $MC(\Omega, K)$: Let $T_w$ be as in the proof of Theorem 1.1. Observe, that $T_w = T_{w'}$ on the set of polynomials for every $w, w' \in K^{-1}$, i.e., they represent the same analytic functional $T$, and the minimal polynomially convex carrier of $T$ is contained in every $\frac{1}{2} \Omega$, $z \in K$, so as well in $V_K(\Omega)$.

$\Phi, \Phi^{-1}$ sends equicontinuous sets into equicontinuous sets: The first part is quite obvious because if the minimal polynomially convex carrier of $T$ is contained in $U \subset V_K(\Omega)$ satisfying $|T f| \leq C\|f\|_U$ for some $C > 0$, then

$$\| M_T f \|_K = \sup_{z \in K} |T f_z| \leq C \sup_{z \in K} \| f_z \|_U = C\|f\|_{KU}.$$
Assume now that $\emptyset \neq M \subset MC(\Omega, K)$, and there are a positive constant $C$ and an open set $U \subset \Omega$, $1 \in U$ such that
\[ \|Mf\|_K \leq C\|f\|_U. \]
Fix $f \in H(\mathbb{C})$ and $\epsilon^{-1} \in K$. We have
\[ |T_Mf| = |Mf(1)| = |Mf(\epsilon)| \leq C\|f\|_U = C\|f\|_{\ell^U}. \]
Hence, the minimal polynomial convex carrier for $T_M$ is a compact set contained in $\overline{V_K(U)} \subset \bigcap_{\epsilon \in K} \epsilon^{-1}U \subset V_K(\Omega)$. Since every $\epsilon^{-1}U$ is compact, the minimal polynomially convex carrier of $T_M$ is contained in $\bigcap_{\epsilon \in K} \epsilon^{-1}U$, and the letter set does not depend on the choice of $M \in M$. Now, the equicontinuity of $\{T_M : M \in M\}$ easily follows.

**Continuity of $\Phi$:** Recall that $H(V_K(\Omega))'$ is a reflexive Fréchet space so ultrabonological (cf. [15, Chapter 23, 24]). Thus, every bounded set is equicontinuous. Consequently, $\Phi$ is locally bounded, hence continuous ([15, Proposition 23.27, 24.10]).

$MC(\Omega, K) = M(\Omega, K)$: We have proved that $i \circ \Phi$ is surjective and continuous, here $i : M(\Omega, K) \hookrightarrow MC(\Omega, K)$ is an embedding. Hence, $MC(\Omega, K) = M(\Omega, K)$.

**Continuity of $\Phi^{-1}$:** We shall recall the main result from [21]. Let $X$ be a Banach space and $d \in \mathbb{N}$. We put
\[ F_k := \{ (x_\alpha)_{\alpha \in \mathbb{N}^d} \in X^{\mathbb{N}^d} : \|x\|_k := \sup_{\alpha \in \mathbb{N}^d} \|x_\alpha\|_{k-|\alpha|} < \infty \} \]
and
\[ F := \text{ind}_k F_k, \]
where $|\alpha| := \alpha_1 + \ldots + \alpha_d$. On $F$ we consider for any positive null-sequence $\delta = (\delta_n)_{n \in \mathbb{N}}$ the continuous norm
\[ |x|_\delta := \sup_{\alpha \in \mathbb{N}^d} \|x_\alpha\|_{\delta^{(|\alpha|)}}, \]
where $\delta^{(|\alpha|)} := \delta_0 \delta_1 \cdots \delta_{|\alpha|}$.

**Lemma 5.3.** ([21]) The norms $\delta_n$ are a fundamental system of seminorms on $F$.

Lemma 5.3 was the key observation that led Vogt in [21] to discover a canonical system of seminorms on $H(K)$, where $K \subset \mathbb{R}$ compact. It turns out that after a small modification this proof works in our setting.

Since we are interested in the case $d = 1$, for simplicity we put $\delta^{(n)} := \delta^{(n)}_n$, $n \in \mathbb{N}$. 

**Proposition 5.4.** Let $V \subset \mathbb{C}$ be a non-empty polynomially convex open set. Then a fundamental system of seminorms on $H_0(\mathbb{C}\setminus V)$ is given by
\[ |f|_\delta := \max \left\{ \sup_{z \in \partial V, n \in \mathbb{N}} \left| f^{(n)}(z) \right| \delta^{(n)}, \sup_{n \in \mathbb{N}} \left| (f \circ g)^{(n)}(0) \right| \delta^{(n)} \right\}, \]
where $g(w) := \frac{1}{w}$, and $\delta$ is any strictly positive null sequence.
Proof. The case $V = \mathbb{C}$, is covered by Lemma 5.3. Let $V$ be a proper subset of $\mathbb{C}$.

Define $F$ as in Lemma 5.3 with

$$X := C_0(\partial V) \times \mathbb{C},$$

and the norm $| (f, x) | := \max \{ \sup_{\partial V} |f|, |x| \}$, where $C_0(\partial V) := \{ f \in C(\partial V) : \lim_{z \to \infty} f(z) = 0 \}$. Let

$$L : H_0(\hat{\mathbb{C}} \setminus V) \rightarrow F,$$

$$L(f) := \left( \frac{1}{n!} f^{(n)}|_{\partial V}, \frac{(f \circ g)^{(n)}(0)}{n!} \right)_{n \in \mathbb{N}}.$$

First of all, let us observe that $F$ is an imbedding spectrum of Banach spaces. Hence, in particular $F$ is a webbed (DF)-space, and the family

$$\{ B_n | n \in \mathbb{N} \}, B_n := \{ f \in F_n : |f|_n \leq 1 \}$$

is a fundamental system of bounded sets in $F$ (note that $B_n$ is closed in $F$ and apply [15, Lemma 25.16]). Next, because $\hat{\mathbb{C}} \setminus V$ is a compact set, the space $H_0(\hat{\mathbb{C}} \setminus V)$ is ultrabornological and Montel (because it is topologically isomorphic with $H(L)$ for some compact $L \subset \mathbb{C}$). Moreover, since $H_0(\hat{\mathbb{C}} \setminus V)$ is an inductive limit (even regular), continuity can be proved on the step spaces. Hence, let us fix an open set $U \supset \hat{\mathbb{C}} \setminus V$. It is enough to show that the graph of $L |_{H(U)}$ is sequentially closed, according to [13, Theorem 13.3.4]. But, this easily follows from the Weierstrass’s Theorem. Finally, we claim that $L$ is actually an injective topological embedding. By Baernstein’s Lemma this is so if $L^{-1}$ is locally bounded. But, the preimage of $B_n$ by Weierstrass’s Theorem is closed in the locally uniform topology since it consists all those $f \in H_0(\hat{\mathbb{C}} \setminus V)$ that are holomorphic near the set $g(\bar{D}(1/\epsilon)) \cup \bigcup_{z \in \partial V} D(z, 1/n)$. So, the functions in $L^{-1}(B_n)$ are uniformly bounded on $(\partial V + \bar{D}(\epsilon)) \cup (\hat{\mathbb{C}} \setminus \bar{D}(1/\epsilon))$ for some $\epsilon > 0$. \hfill \Box

Fix an arbitrary strictly positive null-sequence $\delta$. Let $f_T \in H_0(\hat{\mathbb{C}} \setminus V_K(\Omega))$ be the representation of $T \in H(V_K(\Omega))'_b$ given by the Köthe–Grothendieck Duality.

We check easily that

$$\sup_{n \in \mathbb{N}} \frac{|(f_T \circ g)^{(n)}(0)|}{n!} \delta(n) = \sup_{n \in \mathbb{N}_*} |T(z^{n-1})| \delta(n)$$

$$\leq \sup_{n \in \mathbb{N}_*, y \in K} |M_T(\delta(n) z^{n-1})(y)|.$$

Since $\delta$ is a null sequence, for every compact $S$ there is a positive number $C$ such that $\delta(n) \leq C ||z||_S^n$ for all $n \in \mathbb{N}$. Thus, the family $\{ z^n \delta(n+1) : n \in \mathbb{N} \}$ is bounded in $H(\mathbb{C})$. 


On the other hand, since for every $z \in \partial V_K(\Omega)$ there is $w_z \in K$ such that $zw_z \notin \Omega$. The remaining part of $|\delta|$ we might estimate as follows:

$$\sup_{k \in \mathbb{N}, z \in \partial V_K(\Omega)} \left| f^{(k)}(z) \right| \delta^{(k)} = \sup_{k \in \mathbb{N}, z \in \partial V_K(\Omega)} \left| \zeta T\left( \frac{1}{z - \zeta} \right)^{k+1} \delta^{(k)} \right|$$

$$= \sup_{k \in \mathbb{N}, z \in \partial V_K(\Omega)} \left| M_{\zeta} T\left( \frac{1}{z - \zeta} \right)^{k+1} \delta^{(k)} \right| \leq \sup_{h \in B, y \in K} |M_T h(y)|,$$

where

$$B := \left\{ h = h_{z,k} : \zeta \mapsto \frac{\delta^{(k)}}{(z - \zeta)^{k+1}} \mid z \in \partial V, k \in \mathbb{N} \right\}. \quad (5.3)$$

It remains to show that $B$ is bounded in $H(\Omega)$. For this fix a compact $J \subset \Omega$. The conclusion will follow if we show that

$$C_L := \inf_{z \in \partial V, \zeta \in J} \left| z - \frac{\zeta}{w_z} \right| > 0. \quad (5.4)$$

Assume this is not the case, i.e., there are $z_n \in \partial V$, $\zeta_n \in J$ such that $|z_n - \frac{\zeta_n}{w_{z_n}}| \to 0$, and $w_{z_n} \to w \in K \subset \mathbb{C}^*$, $\zeta_n \to \zeta \in J$. Clearly, we have the following

$$z_n w_{z_n} - \zeta_n = w_{z_n} \left( z_n - \frac{\zeta_n}{w_{z_n}} \right) \to 0. \quad (5.5)$$

The sequence $z_n$ must be bounded. Say $z_n \to z \in \partial V$. Hence,

$$L \ni \zeta = \lim_{n} \zeta_n = \lim_{n} z_n w_{z_n} \notin \Omega,$$

a contradiction, and the proof is completed. $\square$

Proof of Theorem 5.1. We have the following

$$M(\Omega) = \text{proj}_{K \in \Omega} M(\Omega, K) \Omega \text{ is Runge open } \Rightarrow \text{proj}_{K \in \Omega} M(\Omega, K)$$

Hence, by Proposition 5.2 $\tau_k$ is the co-induced topology. $\square$

6. Further Remarks and Comments

We open the section by presenting one application of Theorems 1.4 and 5.1. In the proof we shall apply some known facts concerning algebras on the spaces of holomorphic functions on admissible domains as well, i.e., these which are closed under Hadamard multiplication—cf. the survey article [20].

Proposition 6.1. Let $\Omega$ be a Runge open set. Then the set of non-invertible multipliers is dense in $M(\Omega)$.

Proof. Fix $f \in H_0(\hat{\mathbb{C}} \backslash V(\Omega))$. Without loss of generality we may assume that $f$ is invertible, equivalently, the equation $f \ast g = \frac{1}{1-z}$ has a solution in $H_0(\hat{\mathbb{C}} \backslash V(\Omega))$. Put $\gamma_n(z) := \frac{1}{1-z} - \frac{1}{n(1-z)^2}$, $n \in \mathbb{N}_*$. [18, Theorem 2.7] guarantees that the the Hadamard multiplication on $H_0(\hat{\mathbb{C}} \backslash V(\Omega))^2$ coincides with the Hadamard multiplication on $H_0(\hat{\mathbb{C}} \backslash \{1\})$ and $H_0(\hat{\mathbb{C}} \backslash V(\Omega))$. Moreover, for large $R$ the integral $\int_{\partial B(R)} \gamma_n(\zeta) \overline{\zeta}^n d\zeta$ is zero for all $n \in \mathbb{N}$. Hence, $\gamma_n$ is not invertible in $H_0(\hat{\mathbb{C}} \backslash V(\Omega))$. On the other hand, $\gamma_n \to \frac{1}{1-z}$ in $H_0(\hat{\mathbb{C}} \backslash \{1\})$. Thus,
by Theorem 1.2 we get: $\gamma_n * f \to f$ in $H_0((\hat{C}\setminus\{1\})*(\hat{C}\setminus V(\Omega))) = H_0(\hat{C}\setminus V(\Omega))$.

Now, because $\hat{\gamma} = -* \text{ on } H_0(\hat{C}\setminus V(\Omega))$, it remains to see that $\gamma_n * f$ does not have an inverse because $\gamma_n$ has no inverse.

So far we have shown that the set of non-invertible elements in $H_0(\hat{C}\setminus V(\Omega))$ endowed with the inductive topology is dense. By Theorem 5.1 the co-induced topology by $\varphi$ on $H_0(\hat{C}\setminus V(\Omega))$ is weaker than the inductive one (see Theorem 1.4), and that completes the proof. □

Secondly, we would like to single out Euler differential operators among multipliers. Recall that an operator of the form

$$\sum_{n=0}^{\infty} a_n \theta^n$$

is called an *Euler differential operator*, where $a_n \in \mathbb{C}$, and the differential operator $\theta$ is defined recursively as follows:

$$\begin{cases}
\theta^0(f) := f, \\
\theta(f) := zf', \\
\theta^{n+1} := \theta \circ \theta^n, \quad n \in \mathbb{N}.
\end{cases}$$

It acts on a class of functions whenever the series $\sum_{n=0}^{\infty} a_n \theta^n(f)(z)$ converges pointwise for functions $f$ from the considered class.

Domański and Langenbruch have shown that every Euler operator acting on $H(I)$, where $I \subset \mathbb{R}$ is open, is associated with a unique entire function of exponential type zero (see [9, Theorem 4.1]). Consequently, an Euler type operator acts on $H(I)$ if and only it acts on $H(\mathbb{R})$. As the example below indicates, the situation in $\mathbb{C}$ is slightly different.

**Example.** Let $M := \sum_{n=0}^{\infty} \frac{1}{n!} \theta^n$. Clearly, if $f$ is entire, then

$$\theta^n f(z) = \sum_{k=1}^{\infty} f^{(k)}(0) \frac{k^n z^k}{k!}.$$ 

Hence,

$$\sum_{n=0}^{m} \frac{1}{n!} \theta^n f(z) = \sum_{k=0}^{\infty} \left( \sum_{n \leq m} \frac{k^n}{n!} \right) \frac{f^{(k)}(0) z^k}{k!} \overset{C}{\to} \sum_{k=0}^{\infty} \frac{e^k f^{(k)}(0) z^k}{k!} = f(ze).$$

This shows that though $M$ is associated with a function which is not of exponential type zero, $M$ is a multiplier on $H(\mathbb{C})$.

In fact, the above example indicates something more:

**Proposition 6.2.** Let a series $\sum_{n \in \mathbb{N}} a_n z^n$ defines an entire function of exponential type 1, i.e., there exist $M, \tau > 0$ so that $|f(z)| \leq M \exp(\tau|z|)$ for all $z \in \mathbb{C}$. Then $M = \sum_{n=0}^{\infty} a_n \theta^n$ is an Euler differential operator. In particular, $M$ is a Hadamard multiplier on $H(\mathbb{C})$.

We presume that the converse to Proposition 6.2 is true. However, we are able to show only a partial result.
Remark 2. Assume $M = \sum_{n=0}^{\infty} a_n \theta^n$ is an Euler operator acting on $H(\mathbb{C})$ with values in $H(\mathbb{C})$. Then the function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of order at most 1, i.e., $|f(z)| \leq \exp(|z|^{1+\epsilon})$ as $|z| \gg 1$ for every $\epsilon > 0$. Indeed, let $P_n$ be defined by the equation
\[
\theta^n \exp(z) := P_n(z) \exp(z).
\]
($P_n$ is known as the $n$th Touchard polynomial.) Consider the numbers $P_n(1) = B_n, n \in \mathbb{N}$. The asymptotic behaviour of $B_n$ was established by de Bruijn ([8, Section 6.2]). Namely,
\[
\ln B_n \sim \ln n - \ln \ln n + \frac{1}{2} \frac{\ln \ln n}{\ln n} + O \left( \frac{\ln \ln n}{(\ln n)^2} \right) \quad (n \to \infty).
\]
Hence,
\[
\limsup_{n \to \infty} \frac{n \ln n}{n \ln |a_n|} \leq \limsup_{n \to \infty} \frac{n \ln n}{\ln B_n} = 1,
\]
or equivalently, $f$ is of order at most 1 (cf. [5, Chapter 2]).

On the other hand, considering the function $\sum_{n=3}^{\infty} (\frac{\ln n}{n})^n z^n$ and using (6.1) one might show that not every entire function of order 1 induces an Euler operator on $H(\mathbb{C})$.

**Theorem 6.3.** Let $M = \sum_{n=0}^{\infty} a_n \theta^n$ be an Euler differential operator. The following assertions are equivalent:

1. $M$ acts on $H(\Omega)$ for every Runge open set $\Omega \not\subset \mathbb{C}$.
2. $M$ acts on $H(I)$ for every open set $I \subset \mathbb{R}$.
3. $M$ acts on $H(\Omega)$ for some Runge open set $\Omega \not\subset \mathbb{C}$.
4. $M$ acts on $H(I)$ for some open set $I \subset \mathbb{R}$.
5. The function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of exponential type zero.
6. $M : H(I) \to H(I)$ is a continuous multiplier for every open set $I \subset \mathbb{R}$.
7. $M : H(\Omega) \to H(\Omega)$ is a continuous multiplier for every Runge open set $\Omega \not\subset \mathbb{C}$.

**Proof.** The equivalence of (2), (4), (5), and (6) was proven in [9, Theorem 4.1]. Thus, it remains to show that (3) $\Rightarrow$ (5) $\Rightarrow$ (7). First, since $\theta^n f = \sum_{1 \leq k \leq n} c_{n,k} z^k f^{(k)}$ for some $c_{n,k} \in \mathbb{N}$, notice that without loss of generality we might assume that $1 \in \partial \Omega$. So, for any $0 < \epsilon < \frac{1}{2}$ there is $\alpha_\epsilon \in \Omega \cap \mathbb{D}(1, \frac{1}{2})$. Furthermore, $\frac{1}{1-z} \in H(\Omega)$, and around the origin
\[
\theta^n \left( \frac{1}{1-z} \right) = \sum_{k=0}^{\infty} k^n z^k = \frac{A_n(z)}{(1-z)^{n+1}},
\]
where $A_n$ are the so-called Euler polynomials defined recursively by:
\[
A_0(z) = 1, \quad A_{n+1}(z) = A_n'(z)(z - z^2) + (n + 1)A_n(z)z.
\]
By (6.2) we easily deduce that $\deg A_n = n$ and $A_n(1) = n!$. Furthermore, $A_n$ has only real non-positive zeros (cf. [7, p. 292]). Thus,

$$|\theta^n\left(\frac{1}{1-z}\right)(\alpha_\epsilon)| = \left|\frac{A_n(\alpha_\epsilon)}{(1-\alpha_\epsilon)^{n+1}}\right| \geq \frac{2n!}{\epsilon^{n+1}},$$

where the last inequality follows from the fact that $|\alpha_\epsilon - z_j| \geq \frac{1}{2} |1 - z_j|$ for any zero $z_j$ of $A_n$, what completes the proof of (3) $\Rightarrow$ (5).

If (5) holds, then $M : H(\Omega) \to H(\Omega), M = \sum_n a_n \theta^n$ is well defined and has a closed graph for every domain $\Omega \subset \mathbb{C}$ (cf. [12, Theorem 11.2.3]). So, (7) holds (cf. [13, Theorem 13.3.4]).

□

Remark 3. (1) Theorem 1.3 rediscovers a well know fact:

$$M(\mathbb{D}) \cong H(\mathbb{D})$$

via the map

$$f \mapsto M_f.$$  (6.4)

It is not hard to see that the unit disc is the only open set for which (6.3) with (6.4) holds.

(2) $M(\mathbb{P}) \cong H(\mathbb{C}\setminus\{1\}), \mathbb{P} := \{z : |\Re z| < m\}, m > 0$, and the isomorphism is given by (6.4).

(3) Putting together Theorem 1.1 and [9, Theorem 2.4] we conclude that the algebra $M(\mathbb{C})$ ($M(\mathbb{R})$) is not isomorphic with $M(U)$, for any $U \subset \mathbb{R}$ open or $U \subset \subset \mathbb{C}$ Runge open set ($U \subset \subset \mathbb{C}$ Runge open set).

Acknowledgements

The author would like to thank Jose Bonet, Michał Goliński, and Michael Langenbruch for their comments which improved the presentation of the paper. She also thank the anonymous referee for many valuable hints and very careful reading of the manuscript.

Open Access. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

References

[1] Berenstein, C., Gay, R.: Complex Variables: An Introduction. Springer, Berlin (1991)
[2] Berenstein, C., Gay, R.: Complex Analysis and Special Topics in Harmonic Analysis. Springer, Berlin (1995)
[3] Bierstedt, K.D.: An introduction to locally convex inductive limits. In: Hogbe-Nlend, H. (ed.) Functional Analysis and Its Applications. World Science, Singapore, pp. 35–133 (1988)
[4] Bieberbach, L.: Analytische Fortsetzung. Springer, Berlin (1955)
[5] Boas, R.P.: Entire Functions. Academic Press Inc., Cambridge (1954)
[6] Brück, R., Render, H.: Invertibility of holomorphic functions with respect to the Hadamard product. Complex Var. 42, 207–223 (2000)
[7] Comptet, L.: Advanced Combinatorics. Reidel, Dordrecht (1974)
[8] De Bruijn, N.G.: Asymptotic Methods in Analysis. North Holland Publishing Co., Amsterdam (1958)
[9] Domański, P., Langenbruch, M.: Representation of multipliers on spaces of real analytic functions. Analysis 32, 137–162 (2012)
[10] Domański, P., Langenbruch, M., Vogt, D.: Hadamard type operators on spaces of real analytic functions in several variables. J. Funct. Anal. 269, 3868–3913 (2015)
[11] Frerick, L.: Coefficient multipliers with closed range. Note Mat. (Lecce) 17, 61–70 (1997)
[12] Hille, E.: Analytic Function Theory, vol. II. Chelsea, New York (1987)
[13] Jarchow, H.: Locally Convex Spaces. B.G. Teubner, Leipzig (1981)
[14] Martineau, A.: Sur la topologie des espaces de fonctions holomorphes. Math. Ann. 163, 62–88 (1966)
[15] Meise, R., Vogt, D.: Introduction to Functional Analysis. Clarendon Press, Oxford (2004)
[16] Müller, J.: The Hadamard multiplication theorem and applications in summability theory. Complex Var. 18, 155–166 (1992)
[17] Müller, J.: Coefficient multipliers from \( H(G_1) \) into \( H(G_2) \). Arch. Math. 61, 75–81 (1993)
[18] Müller, J., Pohlen, T.: The Hadamard product on open sets in the extended plane. Complex Anal. Oper. Theory 6(1), 257–274 (2012)
[19] Segal, S.: Nine Introductions in Complex Analysis, Revised edn. Elsevier, Amsterdam (2008)
[20] Render, H.: Hadamard’s multiplication theorem-recent developments. Colloq. Math. 74, 79–92 (1997)
[21] Vogt, D.: A Fundamental System of Seminorms for \( A(K) \), preprint 2013. arXiv:1309.6292

Maria Trybula
Faculty of Mathematics and Informatics
Adam Mickiewicz University
Umultowska 87
61-614 Poznan
Poland
e-mail: maria.h.trybula@gmail.com

Received: October 17, 2016.
Revised: March 31, 2017.