HÖLDER CONTINUITY AND DIMENSIONS OF FRAC TAL FOURIER SERIES

EFSTATHIOS-K. CHRONTSIOS-GARITSIS AND A.J. HILDEBRAND

ABSTRACT. Motivated by applications in number theory, analysis, and fractal geometry, we consider regularity properties and dimensions of graphs associated with Fourier series of the form
\[ F(t) = \sum_{n=1}^{\infty} f(n)e^{2\pi int}/n, \]
for general coefficient functions \( f \). Our main result states that if, for some constants \( C \) and \( \alpha \) with \( 0 < \alpha < 1 \), we have \( |\sum_{1 \leq n \leq x} f(n)e^{2\pi int}| \leq Cx^\alpha \) uniformly in \( x \geq 1 \) and \( t \in \mathbb{R} \), then the series \( F(t) \) is Hölder continuous with exponent \( 1 - \alpha \), and the graph of \( |F(t)| \) on the interval \([0,1]\) has box-counting dimension \( \leq 1 + \alpha \). As applications we recover
the best-possible uniform Hölder exponents for the Weierstrass functions \( \sum_{k=1}^{\infty} a^k \cos(2\pi b^k t) \) and the Riemann function \( \sum_{n=1}^{\infty} \sin(\pi n^2 t)/n^2 \). Moreover, under the assumption of the Generalized Riemann Hypothesis, we obtain nontrivial bounds for Hölder exponents and dimensions associated with series of the form \( \sum_{n=1}^{\infty} \mu(n)e^{2\pi in^2 t}/n^k \), where \( \mu \) is the Möbius function.

1. INTRODUCTION

Given an arithmetic function \( f : \mathbb{N} \to \mathbb{C} \), we consider the Fourier series
\[ F(t) = F(f; t) = \sum_{n=1}^{\infty} \frac{f(n)e^{2\pi int}}{n}. \]
We are interested in regularity properties of the function \( F(t) \) and in the dimensions of the following natural geometric objects associated with this function:

- The image \( F([0,1]) \) of the interval \([0,1]\) under \( F \), i.e., the path \( F(t) \), \( 0 \leq t \leq 1 \), in the complex plane (identified with \( \mathbb{R}^2 \)):
  \[ F([0,1]) = F(f; [0,1]) = \{(\text{Re} \, F(t), \text{Im} \, F(t)) \in \mathbb{R}^2 : t \in [0,1]\}. \]

- The graphs of the real-valued functions \(|F(t)|\), \( \text{Re} \, F(t) \), and \( \text{Im} \, F(t) \), \( 0 \leq t \leq 1 \), which we denote by \( G(F) \), \( G_R(F) \), and \( G_I(F) \), respectively; i.e., the sets
  \[ G(F) = G(f; F) = \{(t, |F(t)|) \in \mathbb{R}^2 : t \in [0,1]\}, \]
  \[ G_R(F) = G_{R}(f; F) = \{(t, \text{Re} \, F(t)) \in \mathbb{R}^2 : t \in [0,1]\}, \]
  \[ G_I(F) = G_{I}(f; F) = \{(t, \text{Im} \, F(t)) \in \mathbb{R}^2 : t \in [0,1]\}. \]

Motivated by applications in both number theory and analysis, we are interested in relating uniform bounds for the exponential sums
\[ S(x,t) = S(f; x,t) = \sum_{1 \leq n \leq x} f(n)e^{2\pi int} \]
to regularity conditions on the function \( F(t) \) and dimensions of the associated geometric objects defined above. Our key result is the following theorem, which provides such a relation between bounds for \( S(x,t) \) and the Hölder exponent of the associated Fourier series \( F(t) \). (See Definition 2.1 below for the definitions of Hölder continuity and Hölder exponent.)
Theorem 1.1. Assume that $S(x,t)$ satisfies, for some constants $C > 0$ and $0 < \alpha < 1$,

\begin{equation}
|S(x,t)| \leq Cx^{\alpha} \quad (x \geq 1, \ t \in \mathbb{R}).
\end{equation}

Then we have

\begin{equation}
|F(t + h) - F(t)| \leq C_1 h^{1-\alpha} \quad (t \in \mathbb{R}, \ h > 0),
\end{equation}

where

\begin{equation}
C_1 = C_1(C, \alpha) = \frac{6\pi C}{\alpha(1-\alpha)}.
\end{equation}

Moreover, the same bound holds for the functions $|F(t)|$, $\text{Re} \ F(t)$, and $\text{Im} \ F(t)$.

In particular, the functions $F(t)$, $|F(t)|$, $\text{Re} \ F(t)$, and $\text{Im} \ F(t)$ are Hölder continuous with exponent $1 - \alpha$.

Using known results relating the Hölder exponent of a function to the dimensions of its graph and image set (see Proposition 2.3 below), this yields the following corollary. (See Definition 2.2 below for the definition of box-counting dimension.)

Corollary 1.2. Under the assumptions of Theorem 1.1 we have:

(i) The upper box-counting dimension of the image $F([0,1])$ of the interval $[0,1]$ under $F$ satisfies

\begin{equation}
\overline{\dim}_B(F([0,1])) \leq \frac{1}{1-\alpha}.
\end{equation}

(ii) The upper box-counting dimension of the graph $G(F)$ defined in (1.3) satisfies

\begin{equation}
\overline{\dim}_B(G(F)) \leq 1 + \alpha.
\end{equation}

Moreover, the same bound holds for the upper box-counting dimensions of the graphs $G_R(F)$ and $G_I(F)$, defined in (1.4) and (1.5), respectively.

A key feature of Theorem 1.1 is its generality: aside from the exponential sum estimate (1.7), there are no restrictions on the coefficients $f(n)$. In particular, the function $f$ need not satisfy any regularity or smoothness properties, nor does it have to be bounded. This opens up the result to a wide range of potential applications.

Another feature of Theorem 1.1 is the explicit nature of the constant $C_1(C, \alpha) = 6\pi C/(\alpha(1-\alpha))$. We hope that this may be useful in future applications.

The intuition behind the exponent $1 - \alpha$ in (1.8) is as follows: Assume, for simplicity, that the function $f$ is bounded by 1. Then we have the trivial bound $|S(x,t)| \leq \sum_{1 \leq n \leq x} |f(n)| \leq x$, so (1.7) represents a saving of a factor $x^{1-\alpha}$ over this trivial bound. On the other hand, since $F(t)$, as a continuous periodic function, is bounded, we have trivially $|F(t + h) - F(t)| \leq c = h^\alpha$ for some constant $c$ and all $t$ and $h$. The estimate (1.8) thus represents a saving of the same power of $h$, namely $h^{1-\alpha}$, over this trivial bound. This heuristic also suggests that the exponent $1 - \alpha$ in (1.8) is best-possible. In fact, in Section 5 we present examples for which the exponent $1 - \alpha$ is indeed optimal (see Corollaries 5.1 and 5.2).

The proof of Theorem 1.1, given in Section 3, is based on the summation by parts formula, a standard technique in analytic number theory (see Lemma 3.1 below). The theorem can be generalized in a variety of directions, using essentially the same approach. For example, one can replace the weights $1/n$ in (1.1) by weights of the form $1/n^p$ for some $p > 0$, or even by a more general sequence of weights $\{w_n\}$ belonging to some $l_p$-space and satisfying appropriate regularity conditions. One can also generalize the frequencies $n$ in the exponential terms $e^{2\pi int}$ in (1.6) and (1.1) to more general frequencies $\phi(n)$ subject to some growth conditions.
Here we confine ourselves to proving a generalization to Fourier series of the form
\[(1.12)\quad F_{k,p}(t) = \sum_{n=1}^{\infty} \frac{f(n)e^{2\pi in^kt}}{n^p},\]
where \(k\) is an arbitrary positive integer and \(p\) an arbitrary positive real number. Let
\[(1.13)\quad S_k(x,t) = \sum_{1\leq n \leq x} f(n)e^{2\pi in^kt}\]
be the exponential sum associated with the series (1.12).

**Theorem 1.3.** Assume that for some positive constants \(C\) and \(\alpha\) satisfying
\[(1.14)\quad \max(0, p - k) < \alpha < p\]
we have
\[(1.15)\quad |S_k(x,t)| \leq Cx^\alpha \quad (x \geq 1, \ t \in \mathbb{R}).\]
Then we have
\[(1.16)\quad |F_{k,p}(t + h) - F_{k,p}(t)| \leq C_2 h^{(p - \alpha)/k} \quad (t \in \mathbb{R}, \ h > 0),\]
where \(C_2\) is the constant \(C_1\) of Theorem 1.1 with \(C\) and \(\alpha\) replaced by \(\tilde{C} = (1 + |k - p|/(\alpha + k - p))C\) and \(\tilde{\alpha} = (\alpha + k - p)/k\), respectively, i.e.,
\[(1.17)\quad C_2 = C_2(k, p, C, \alpha) = C_1 \left(1 + \frac{|k - p|}{\alpha + k - p}\right) C \left(1 + \frac{|k - p|}{\alpha + k - p}\right) = \frac{6\pi}{\alpha + k - p} \cdot \left(1 - \frac{\alpha + k - p}{k}\right) = \frac{6\pi C (\alpha + k - p + |k - p|) k^2}{(\alpha + k - p)^2 (p - \alpha)}.
\]
Moreover, the same conclusion holds for the functions \(|F_{k,p}(t)|\), \(\text{Re} F_{k,p}(t)\), and \(\text{Im} F_{k,p}(t)\).

In particular, the functions \(F_{k,p}(t)\), \(|F_{k,p}(t)|\), \(\text{Re} F_{k,p}(t)\), and \(\text{Im} F_{k,p}(t)\) are Hölder continuous with exponent \((p - \alpha)/k\).

Note that, when \(k = p = 1\), the functions \(F_{k,p}(t)\) and \(S_k(t)\) reduce to the functions \(F(t)\) and \(S(t)\) defined in (1.1) and (1.6), respectively, the condition (1.14) on \(\alpha\) reduces to \(0 < \alpha < 1\), and the bound \(\leq C_2 h^{(p - \alpha)/k}\) in (1.16) of Theorem 1.3 reduces to the bound \(\leq C_1 h^{1 - \alpha}\) in (1.8) of Theorem 1.1. Thus, Theorem 1.3 generalizes Theorem 1.1. However, in proving Theorem 1.3 we will make use of Theorem 1.1.

As in the case of Theorem 1.1, Theorem 1.3 implies bounds on the dimensions of the graphs \(G(F_{k,p})\), \(G_R(F_{k,p})\) and \(G_I(F_{k,p})\) associated with \(F_{k,p}(t)\), which are defined in the same way the graphs \(G(F)\), \(G_R(F)\), and \(G_I(F)\).

**Corollary 1.4.** Under the assumptions of Theorem 1.3 we have:
(i) The upper box-counting dimension of the image \(F_{k,p}([0,1])\) of the interval \([0,1]\) under \(F_{k,p}\) satisfies
\[(1.18)\quad \overline{\dim}_B(F_{k,p}([0,1])) \leq \frac{k}{p - \alpha},\]
(ii) The upper box-counting dimension of the graph \(G(F_{k,p})\) defined in (1.3) satisfies
\[(1.19)\quad \overline{\dim}_B(G(F_{k,p})) \leq 2 - \frac{p - \alpha}{k}.\]
Moreover, the same bound holds for the upper box-counting dimensions of the graphs \(G_R(F_{k,p})\) and \(G_I(F_{k,p})\), defined in (1.4) and (1.5), respectively.
Series of the form (1.1) and (1.12) arise in a wide variety of contexts in number theory, analysis, and fractal geometry, and graphs associated with such series often exhibit interesting fractal properties. A typical example is the series

\begin{equation}
F(\mu; t) = \sum_{n=1}^{\infty} \frac{\mu(n)e^{2\pi int}}{n},
\end{equation}

where \( \mu(n) \) is the Möbius function. The graphs \( F(\mu; [0, 1]) \) and \( G(\mu; F) \) associated with the Möbius series \( F(\mu; t) \) are shown in Figure 1.

Bohman and Fröberg [BF95] investigated the series (1.20) and some related series numerically, focusing on their fractal properties. In particular, these authors were the first to observe the peculiar shape of the path \( F(\mu; [0, 1]) \) shown on the left of Figure 1.

On the theoretical side, Bateman and Chowla [BC63] proved that the series \( F(\mu; t) \) converges uniformly and thus represents a continuous function of \( t \). More precise regularity properties for the series \( F(\mu; t) \) and similar series involving the Möbius function have been recently established by Veech [VFF18] under the assumption of the Generalized Riemann Hypothesis (GRH). As applications of Theorems 1.1 and 1.3 and their corollaries we will obtain, under the same assumption, non-trivial bounds for Hölder exponents and dimensions of graphs associated with the Möbius Fourier series \( F(\mu; t) \) and its generalizations \( F_{k,k}(\mu; t) \) defined in (1.12) (see Corollaries 5.3 and 5.4).

Another class of functions to which our results can be applied are the Riemann and Weierstrass functions, defined by

\begin{equation}
\sum_{n=1}^{\infty} \frac{\sin(\pi n^2 t)}{n^2},
\end{equation}

\begin{equation}
\sum_{k=1}^{\infty} a^k \cos(2\pi b^k t),
\end{equation}

respectively, where \( a \) and \( b \) are real numbers satisfying \( 0 < a < 1 \) and \( b > 1/a \). These functions are classical examples of continuous functions that are almost everywhere non-differentiable, and they have been extensively studied in the literature, both from a regularity point of view and from a geometric/fractal point of view; see, for example, [BBR14, Bar15, Dui91, Ece20, Har16, Hun98, Jaf96, KMPY84, She18].

In Section 5 we will see that the Weierstrass and Riemann functions can be viewed as special cases of the Fourier series (1.1) and (1.12), respectively. By applying Theorems 1.1 and 1.3 we will...
obtain bounds on the Hölder exponent that turn out to be the best-possible uniform bounds of this type; see Corollaries 5.1 and 5.2.

The Weierstrass and Riemann functions can be generalized in a variety of ways. For example, Chamizo, Cordoba, and Ubis [CC99, CU07] investigated functions of the form $\sum_{n=1}^{\infty} e^{2\pi in^k/n^p}$ for $k, p \geq 2$, which can be regarded as complex generalizations of the Riemann function, and more generally functions of the form (1.12) under certain regularity assumptions on the coefficients $f(n)$ (such as monotonicity). We note that our results apply a much broader class of coefficient sequences, though are not as precise as those obtained by Chamizo, Cordoba, and Ubis.

**Outline of the paper.** In the hope that the results of this paper will be of interest to researchers in both analysis and number theory, we tried to keep the exposition broadly accessible, recalling definitions and results that may be only known to specialists in the relevant area.

In Section 2 we recall the definitions of Hölder continuity, Hölder exponent, and box-counting dimension, and we cite a key result relating the latter two quantities. We then use this result to deduce the corollaries from Theorems 1.1 and 1.3. Section 3 contains the proof of Theorem 1.1, while Section 4 contains the proof of Theorem 1.3.

In Section 5 we present the applications of these results mentioned above to Riemann and Weierstrass type functions and to Fourier series associated with the Möbius function. In the final section, Section 6, we discuss possible extensions and generalizations of our results and open problems suggested by these results.

2. Background on Hölder continuity and dimensions and proof of the corollaries

We begin by defining Hölder continuity, Hölder exponents, and pointwise Hölder exponents.

**Definition 2.1** ([Fal14, p. 8] and [Ja96]). Let $I \subseteq \mathbb{R}$ be a non-trivial compact interval and let $f : I \to \mathbb{R}^d$.

(i) The function $f$ is called **Hölder continuous** if there are constants $C > 0$ and $\eta \in (0, 1)$ such that

$$|f(x) - f(y)| \leq C|x - y|^\eta$$

holds for all $x, y \in I$. We call $\eta$ a **Hölder exponent** of $f$.

(ii) Let $x_0 \in I$. The function $f$ is called **locally Hölder continuous at** $x_0$ if there are constants $C > 0$ and $\eta > 0$ and a polynomial $P$ of degree less than $\eta$ such that

$$|f(x) - P(x - x_0)| \leq C|x - x_0|^\eta$$

holds for all $x$ in some open neighborhood of $x_0$. The exponent $\eta$ is called a **local Hölder exponent** of $f$ at $x_0$. The supremum of all $\eta$ for which (2.2) holds is called the **pointwise Hölder exponent** of $f$ at $x_0$ and is denoted by $\eta(x_0)$.

We note that, if $0 < \eta < 1$, (2.2) reduces to

$$|f(x) - f(x_0)| \leq C|x - x_0|^\eta$$

as the polynomial $P(x)$ must be of degree 0 and hence be a constant.

We next recall the definition of the **box-counting dimension**, which is one of several standard concepts of dimensions in fractal geometry.

**Definition 2.2** ([Fra21, p. 6]). Let $E$ be a bounded subset of $\mathbb{R}^d$. For $r > 0$, denote by $N(E, r)$ the minimal number of sets of diameter at most $r$ needed to cover $E$. The **upper box-counting dimension** of $E$ is defined as

$$\dim_B E = \limsup_{r \to 0} \frac{\log N(E, r)}{\log(1/r)}.$$
Similarly, the lower box-counting dimension of $E$ is defined as

$\dim_B E = \liminf_{r \to 0} \frac{\log N(E, r)}{\log(1/r)}$. \hfill (2.5)

When the upper and lower box-counting dimensions coincide, we call the common value the box-counting dimension of $E$ and denote it by $\dim_B E$.

The following proposition relates the Hölder exponent of a function to the (upper) box-counting dimension of sets associated with this function.

**Proposition 2.3** ([Fra21, p. 49] and [Fal14, Cor. 11.2]). Let $I \subseteq \mathbb{R}$ be a nontrivial compact interval, and let $f : I \to \mathbb{R}^d$ be a Hölder continuous function with exponent $\eta > 0$. Then we have:

(i) The upper box-counting dimension of $f(I) \subseteq \mathbb{R}^d$ satisfies

$$\dim_B f(I) \leq \frac{1}{\eta}. \hfill (2.6)$$

(ii) In the case when $d = 1$, the upper box-counting dimension of the graph $G(f) = \{(x, f(x)) \in \mathbb{R}^2 : x \in I\}$ satisfies

$$\dim_B G(f) \leq 2 - \eta. \hfill (2.7)$$

**Proof of Corollaries 1.2 and 1.4.** Corollary 1.2 is the special case $k = p = 1$ of Corollary 1.4, so it suffices to prove the latter result.

Assume that $S_k(t)$ satisfies the assumptions of Corollary 1.4. By Theorem 1.3 it follows that the functions $F_{k,p}(t)$, $|F_{k,p}(t)|$, $\text{Re} F_{k,p}(t)$, and $\text{Im} F_{k,p}(t)$, are Hölder continuous with exponent $\eta = (p - \alpha)/k$. Applying Proposition 2.3 then shows that the box-counting dimension of $F_{k,p}([0, 1])$ is bounded above by $1/\eta = k/(p - \alpha)$, while the box-counting dimensions of the graphs $G(F_{k,p})$, $G(|F_{k,p}|)$, $G(\text{Re} F_{k,p})$, and $G(\text{Im} F_{k,p})$ are bounded above by $2 - \eta = 2 - (p - \alpha)/k$. These bounds are exactly the dimension bounds (1.18) and (1.19) of Corollary 1.4. \hfill $\square$

3. **Proof of Theorem 1.1**

The proof depends in a crucial way on the summation by parts formula (Abel’s identity), a standard technique in analytic number theory. For the convenience of the reader, we recall this formula in the following lemma.

**Lemma 3.1.** Let $a : \mathbb{N} \to \mathbb{C}$ be an arithmetic function, and let $A(x) = \sum_{1 \leq n \leq x} a(n)$, with the convention that $A(x) = 0$ if $x < 1$.

(i) Let $0 < y < x$ be real numbers and assume that $\phi(u)$ is defined on the closed interval $[y, x]$ and has a continuous derivative on this interval. Then

$$\sum_{y < n \leq x} a(n)\phi(n) = A(x)\phi(x) - A(y)\phi(y) - \int_y^x A(u)\phi'(u)du. \hfill (3.1)$$

(ii) Let $x \geq 1$ and assume that $\phi(u)$ is defined and has a continuous derivative on the interval $(0, x]$. Then

$$\sum_{1 \leq n \leq x} a(n)\phi(n) = A(x)\phi(x) - \int_1^x A(u)\phi'(u)du. \hfill (3.2)$$

**Proof.** Assertion (i) of the lemma is exactly Lemma 4.2 of [Apo76]. Assertion (ii) follows from (i) on letting $y = 1/2$ in (3.1) and noting that $A(u) = 0$ for $u < 1$. \hfill $\square$
For the remainder of this section, we assume that \( f : \mathbb{N} \to \mathbb{C} \) is an arithmetic function, we let \( S(x, t) \) be the associated exponential sum defined in (1.6), and we assume that \( S(x, t) \) satisfies the hypothesis (1.7) of Theorem 1.1 with some constants \( C \) and \( \alpha \) with \( 0 < \alpha < 1 \).

Given a positive integer \( N \), let \( F_N(t) \) denote the \( N \)th partial sum of \( F(t) \), i.e.,

\[
F_N(t) = \sum_{1 \leq n \leq N} \frac{f(n)e^{2\pi int}}{n}.
\]

**Lemma 3.2.** Uniformly in integers \( M > N \geq 1 \) and \( t \in \mathbb{R} \) we have

\[
|F_M(t) - F_N(t)| \leq C_{11} N^{\alpha - 1},
\]

where

\[
C_{11} = C_{11}(C, \alpha) = \frac{2C}{1 - \alpha}.
\]

**Proof.** Applying Lemma 3.1(i) with \( y = N, x = M, a(n) = f(n)e^{2\pi int} \) (so that \( A(u) = S(u, t) \)) and \( \phi(u) = 1/u \) along with the estimate (1.7), we obtain

\[
|F_M(t) - F_N(t)| = \left| \sum_{N < n \leq M} \frac{f(n)e^{2\pi int}}{n} \right|
\]

\[
= \left| \frac{S(M, t)}{M} - \frac{S(N, t)}{N} + \int_N^M \frac{S(u, t)}{u^2} du \right|
\]

\[
\leq C \left( M^{\alpha - 1} + N^{\alpha - 1} + \int_N^M u^{\alpha - 2} du \right)
\]

\[
= C \left( M^{\alpha - 1} + N^{\alpha - 1} + \frac{1}{1 - \alpha} \left( N^{\alpha - 1} - M^{\alpha - 1} \right) \right)
\]

\[
\leq C \left( N^{\alpha - 1} + \frac{N^{\alpha - 1}}{1 - \alpha} \right)
\]

\[
\leq \frac{2C}{1 - \alpha} N^{\alpha - 1} = C_{11} N^{\alpha - 1},
\]

as claimed. \( \square \)

**Lemma 3.3.** The series \( F(t) \) converges uniformly in \( t \), and we have

\[
F(t) - F_N(t) \leq C_{11} N^{\alpha - 1}.
\]

**Proof.** Both assertions follow from Lemma 3.2 on letting \( M \to \infty \). \( \square \)

**Lemma 3.4.** Uniformly for \( t \in \mathbb{R}, h > 0 \), and any positive integer \( N \) we have

\[
|F_N(t + h) - F_N(t)| \leq C_{12} h N^{\alpha},
\]

where

\[
C_{12} = C_{12}(C, \alpha) = \frac{6\pi C}{\alpha}.
\]

**Proof.** We have

\[
F_N(t + h) - F_N(t) = \sum_{n=1}^{N} \frac{f(n) \left( e^{2\pi int + h} - e^{2\pi int} \right)}{n} = \sum_{n=1}^{N} f(n)e^{2\pi int} \phi(n),
\]

where

\[
\phi(x) = \frac{e^{2\pi ihx} - 1}{x}.
\]
Applying Lemma 3.3 with \( a(n) = f(n)e^{2\pi int} \) and \( \phi(x) \) defined by (3.10), the sum on the right of (3.9) becomes

\[
\sum_{n=1}^{N} f(n)e^{2\pi int}\phi(n) = S(N, t)\phi(N) - \int_{1}^{N} S(x, t)\phi'(x)dx.
\]

Using the elementary inequality \(|e^{iu} - 1| \leq u\) for all \( u \in \mathbb{R} \), we see that the functions \( \phi(x) \) and \( \phi'(x) \) in (3.11) satisfy

\[
|\phi(x)| = \left| \frac{e^{2\pi ix} - 1}{x} \right| \leq \frac{|2\pi h x|}{x} = 2\pi h,
\]

\[
|\phi'(x)| \leq \frac{|e^{2\pi ihx} - 1|}{x} + \frac{|2\pi ih e^{2\pi ihx}|}{x} \leq 2\pi h - 2\pi h = 4\pi h.
\]

Substituting the bounds (3.12) and (3.13) along with our assumption (1.7) into the right-hand side of (3.9), we obtain

\[
\left| \sum_{n=1}^{N} f(n)e^{2\pi int}\phi(n) \right| \leq 2\pi hCN^{\alpha} + 4\pi hC \int_{1}^{N} x^\alpha dx
\]

\[
\leq ChN^{\alpha} \left( 2\pi + \frac{4\pi}{\alpha} \right)
\]

\[
\leq ChN^{\alpha} \frac{6\pi}{\alpha} = C_{12}hN^{\alpha}.
\]

Combining this with (3.9) yields the desired estimate (3.7).

**Proof of Theorem 1.1** For the proof of the bound (1.8), let \( t \in \mathbb{R} \) and \( h > 0 \) be given. Since \( F(t) \) is a periodic function with period 1, we may assume that

\[
0 < h \leq 1.
\]

Let \( N \) be a positive integer. Then

\[
|F(t + h) - F(t)| = \left| (F(t + h) - F_N(t + h)) - (F(t) - F_N(t)) + (F_N(t + h) - F_N(t)) \right|
\]

\[
\leq |F(t + h) - F_N(t + h)| + |F(t) - F_N(t)| + |F_N(t + h) - F_N(t)|.
\]

By Lemma 3.3, the first two terms on the right of (3.16) are bounded by \( \leq C_{11}N^{\alpha-1} \) each, and by Lemma 3.4 the third term is bounded by \( \leq C_{12}hN^{\alpha} \). It follows that

\[
|F(t + h) - F(t)| \leq 2C_{11}N^{\alpha-1} + C_{12}hN^{\alpha}
\]

for any positive integer \( N \). To (approximately) optimize this bound, we choose \( N \) as

\[
N = N_h = \lfloor 1/h \rfloor,
\]

where \( \lfloor \cdot \rfloor \) is the floor function. In view of our assumption (3.15), \( N \) is a positive integer satisfying \( 1/(2h) \leq N \leq 1/h \). Therefore we have

\[
2C_{11}N^{\alpha-1} + C_{12}hN^{\alpha} \leq 2C_{11}(2h)^{1-\alpha} + C_{12}h^{1-\alpha}
\]

\[
\leq (4C_{11} + C_{12})h^{1-\alpha}
\]

\[
= C \left( \frac{8}{1 - \alpha} + \frac{6\pi}{\alpha} \right) h^{1-\alpha}
\]

\[
\leq \frac{6\pi C}{\alpha(1 - \alpha)}h^{1-\alpha} = C_{11}h^{1-\alpha}.
\]

Combining this with (3.17) yields the inequality (1.8) for \( |F(t + h) - F(t)| \).
In view of the elementary inequalities
\[ ||F(t + h)| - |F(t)|| \leq |F(t + h) - F(t)|, \]
\[ |\text{Re } F(t + h) - \text{Re } F(t)| \leq |F(t + h) - F(t)|, \]
\[ |\text{Im } F(t + h) - \text{Im } F(t)| \leq |F(t + h) - F(t)| \]
the same conclusion holds for the functions $|F(t)|$, Re $F(t)$, and Im $F(t)$.

This completes the proof of Theorem 1.1. \hfill \Box

4. Proof of Theorem 1.3

Let $k$ be a positive integer and $p$ a positive real number. Let $f : \mathbb{N} \to \mathbb{C}$ be an arithmetic function and let $S_k(x, t)$ and $F_{k,p}(t)$ denote the associated exponential sums and Fourier series defined by (see (1.13) and (1.12))
\[
S_k(x, t) = \sum_{1 \leq n \leq x} f(n)e^{2\pi i nt},
\]
\[
F_{k,p}(t) = \sum_{n=1}^{\infty} \frac{f(n)e^{2\pi i nt}}{np}.
\]

Define an arithmetic function $\tilde{f}$ by
\[
\tilde{f}(n) = \begin{cases} 
  m^{k-p}f(m) & \text{if } n = mk \text{ for some } m \in \mathbb{N}, \\
  0 & \text{otherwise},
\end{cases}
\]
and let $\tilde{S}(x, t)$ and $\tilde{F}(t)$ be defined as in (1.6) and (1.1), but with respect to the arithmetic function $\tilde{f}(n)$. Then
\[
\tilde{S}(x, t) = \sum_{1 \leq n \leq x} \tilde{f}(n)e^{2\pi int} = \sum_{m \leq x^{1/k}} m^{k-p}f(m)e^{2\pi imkt},
\]
\[
\tilde{F}(t) = \sum_{n=1}^{\infty} \frac{\tilde{f}(n)e^{2\pi int}}{n} = \sum_{m=1}^{\infty} \frac{m^{k-p}f(m)e^{2\pi imkt}}{mk} = F_{k,p}(t).
\]

Applying Lemma 3.1 with $\phi(u) = u^{k-p}$ and $a(m) = f(m)e^{2\pi imkt}$ (so that $A(u) = \sum_{m \leq u} f(m)e^{2\pi imkt} = S_k(u, t)$), we obtain
\[
\left| \sum_{m \leq x^{1/k}} m^{k-p}f(m)e^{2\pi imkt} \right| = \left| S_k(x^{1/k}, t)x^{k-p}/k - \int_{1}^{x^{1/k}} S_k(u, t)(k-p)u^{k-p-1}du \right|
\leq Cx^\alpha/kx^{(k-p)/k} + C|k-p| \int_{1}^{x^{1/k}} u^\alpha u^{k-p-1}du.
\]

Note that, by (1.14), $0 < \alpha + k - p < k$. Hence,
\[
\left| \sum_{m \leq x^{1/k}} m^{k-p}f(m)e^{2\pi imkt} \right| \leq Cx^{(\alpha+k-p)/k} + C|k-p| \frac{x^{(\alpha+k-p)/k-1}}{\alpha + k - p}
\leq \tilde{C} x^{(\alpha+k-p)/k},
\]
where
\[
\tilde{C} = \left( 1 + \frac{|k-p|}{\alpha + k - p} \right) C.
\]
Combining (4.4) with (4.2) yields
\begin{equation}
\tilde{S}(x, t) \leq \tilde{C}x^{(\alpha+k-p)/k} = \tilde{C}x^{\tilde{\alpha}},
\end{equation}
where
\begin{equation}
\tilde{\alpha} = \frac{\alpha + k - p}{k}.
\end{equation}

Our assumption (1.14) ensures that the exponent \(\tilde{\alpha}\) defined in (4.7) satisfies \(0 < \tilde{\alpha} < 1\). We can therefore apply Theorem 1.1 to conclude that
\begin{equation}
|\tilde{F}(t + h) - \tilde{F}(t)| \leq \tilde{C}_1 h^{1-\tilde{\alpha}} \quad (t \in \mathbb{R}, h > 0),
\end{equation}
where
\begin{equation}
\tilde{C}_1 = C_1(\tilde{C}, \tilde{\alpha}) = C_1 \left( 1 + \frac{|k-p|}{\alpha + k - p} \right) C_2(k, p, C, \alpha)
\end{equation}
is the constant in Theorem 1.3. In view of (4.3) this proves the bound (1.16) of Theorem 1.3 for \(F_{k,p}(t)\), which in turn implies analogous bounds (with the same constant) for the functions \(|F_{k,p}(t)|\), \(\text{Re} F_{k,p}(t)\), and \(\text{Im} F_{k,p}(t)\).

This completes the proof of Theorem 1.3. \(\square\)

5. Applications

5.1. Weierstrass type functions. The classical Weierstrass functions are defined by
\begin{equation}
\sum_{k=1}^{\infty} a^k \cos(2\pi b^k t),
\end{equation}
where \(a\) and \(b\) are positive real numbers satisfying \(1/b < a < 1\). These functions have been introduced more than a century ago by Weierstrass and Hardy (see [Har16]) as examples of continuous, but nowhere differentiable functions, and they have been extensively studied in the literature. In particular, it is now known that the function (5.1) is Hölder continuous with exponent \(-\log_b a\), that this exponent is both globally and locally optimal [Bar15], and that the Hausdorff [She18] and box-counting dimensions [Har16] of the graph of this function over the interval \([0, 1]\) are both equal to \(2 + \log_b a\).

Here we consider generalizations of the Weierstrass function of the form
\begin{equation}
W_{a,b}(f; t) = \sum_{k=1}^{\infty} f(k) a^k e^{2\pi i b^k t},
\end{equation}
with arbitrary bounded coefficients \(f(k)\). From Theorem 1.1 we will derive the following result, which shows that, for integer values of \(b\), this much more general class of functions satisfies the same Hölder and dimension bounds as the classical Weierstrass function (5.1).

**Corollary 5.1.** Let \(a\) be a real number satisfying \(0 < a < 1\), let \(b\) be an integer satisfying \(b > 1/a\), and let \(f : \mathbb{N} \to \mathbb{C}\) be an arbitrary bounded function. Then we have:
\begin{itemize}
  \item[(i)] The function \(W_{a,b}(f; t)\) is Hölder continuous with exponent \(-\log_b a\), and the same holds for the real-valued functions \(|W_{a,b}(f; t)|\), \(\text{Re} W_{a,b}(f; t)\), and \(\text{Im} W_{a,b}(f; t)\).
  \item[(ii)] The upper box-counting dimensions of the graphs of the functions \(|W_{a,b}(f; t)|\), \(\text{Re} W_{a,b}(f; t)\), and \(\text{Im} W_{a,b}(f; t)\) over the interval \([0, 1]\) are bounded by \(\leq 2 + \log_b a\).
\end{itemize}

**Remark.** The restriction of the parameter \(b\) in the corollary to integer values is a purely technical one as it allows us to derive the result directly from Theorem 1.1. By adapting the proof of Theorem 1.1, one can show that the corollary remains valid without this restriction.
Proof. Let $f : \mathbb{N} \to \mathbb{C}$ be a bounded function. By rescaling $f$ if necessary we may assume, without loss of generality, that $|f(n)| \leq 1$ for all $n \in \mathbb{N}$. Define an arithmetic function $\tilde{f}$ by

$$\tilde{f}(n) = \begin{cases} (ab)^k f(k) & \text{if } n = b^k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{otherwise}, \end{cases}$$

and let $\tilde{S}(x, t)$ and $\tilde{F}(t)$ be defined as in (1.6) and (1.1), but with respect to the arithmetic function $\tilde{f}(n)$. Setting

$$k_x = \lfloor \log_b x \rfloor,$$

we then have

$$\tilde{S}(x, t) = \sum_{1 \leq n \leq x} \tilde{f}(n) e^{2\pi i n t} = \sum_{1 \leq k \leq k_x} (ab)^k f(k) e^{2\pi i b^k t},$$

$$\tilde{F}(t) = \frac{\sum_{n=1}^{\infty} \tilde{f}(n) e^{2\pi i n t}}{n} = \sum_{k=1}^{\infty} f(k) a^k e^{2\pi i b^k t} = W_{a,b}(f; t).$$

Since, by assumption, $b > 1/a$ and $|f| \leq 1$, we have

$$\left| \sum_{1 \leq k \leq k_x} (ab)^k f(k) e^{2\pi i b^k t} \right| \leq \sum_{1 \leq k \leq k_x} (ab)^k \leq \frac{(ab)^{k_x+1}}{ab - 1} \leq \frac{ab}{ab - 1} x^{1+\log_b a}.$$  

Combining (5.7) with (5.5), it follows that the function $\tilde{f}$ satisfies the condition (1.7) of Theorem 1.1 with exponent $\alpha = 1 + \log_b a = \log(ab)/\log b$ and constant $C = C_{a,b} = ab/(ab - 1)$. Moreover, our assumptions $0 < a < 1$ and $ab > 1$ ensure that $\alpha$ satisfies the condition $0 < \alpha < 1$ of Theorem 1.1. The theorem therefore implies that the Fourier series $\tilde{F}(t)$ is Hölder continuous with exponent $1 - \alpha = -\log_b a$, and that the same holds for the functions $|\tilde{F}(t)|$, $\Re \tilde{F}(t)$, and $\Im \tilde{F}(t)$. Since, by (5.6), $\tilde{F}(t) = W_{a,b}(t)$, this proves the claim of part (i) of the corollary.

Part (ii) follows from Corollary 1.2.

Remark. In the case when $f \equiv 1$, the function $\Re W_{a,b}(t)$ reduces to the classical Weierstrass function (5.1). In view of the remarks at the beginning of the section, the bounds $-\log_b a$ and $2 + \log_b a$ for the Hölder exponent and dimensions provided by the corollary are best-possible in this case. This example shows that the bounds provided by Theorem 1.1 on the Hölder exponent and dimensions are optimal.

5.2. Riemann type functions. The Riemann function

$$\sum_{n=1}^{\infty} \frac{\sin(\pi n^2 t)}{n^2}$$

is another classical example of a function that is continuous, but non-differentiable almost everywhere. In contrast to the Weierstrass function (5.1), which has the same pointwise Hölder exponent, $\eta(t) = -\log_b a$, at each point $t$ (see [Har17]), the pointwise Hölder exponent $\eta(t)$ of the Riemann function is strongly dependent on the diophantine approximation properties of $t$. More precisely, the set $\{\eta(t) : t \in [0,1]\}$ of pointwise Hölder exponents consists of the closed interval $[1/2, 3/4]$ along with the single point 1; see, for example, [Dui91] and [Ja96]. It follows that 1/2 is the best-possible uniform Hölder exponent for the Riemann function.
As an application of Theorem 1.3 we show in the following corollary that the same Hölder exponent, 1/2, holds for a class of generalized Riemann functions defined by

\[
R(f; t) = \sum_{n=1}^{\infty} \frac{f(n)e^{2\pi in^2t}}{n^2},
\]

where the coefficients \(f(n)\) are bounded, but otherwise arbitrary.

**Corollary 5.2.** Let \(f : \mathbb{N} \to \mathbb{C}\) be an arbitrary bounded arithmetic function. Then we have:

(i) The function \(R(f; t)\) is Hölder continuous with exponent 1/2, and the same holds for the real-valued functions \(|R(f; t)|\), \(\text{Re} R(f; t)\), and \(\text{Im} R(f; t)\).

(ii) The upper box-counting dimensions of the graphs of the functions \(|R(f; t)|\), \(\text{Re} R(f; t)\), and \(\text{Im} R(f; t)\) over the interval [0, 1] are bounded by \(\leq 3/2\).

**Proof.** As in the proof of Corollary 5.1, we may assume without loss of generality that \(f : \mathbb{N} \to \mathbb{C}\) is bounded by 1. Note that the series \(R(f; t)\) defined in (5.9) is exactly the series \(F_{k,p}(f; t)\) of Theorem 1.3 when \(p = k = 2\). Moreover, by our assumption that \(|f| \leq 1\) we have

\[
|S_2(x, t)| = \sum_{1 \leq n \leq x} f(n)e^{2\pi in^2t} \leq \sum_{1 \leq n \leq x} |f(n)| \leq x.
\]

Thus the assumption (1.15) of Theorem 1.3 holds with constants \(C = 1\) and \(\alpha = 1\). Moreover, the inequality (1.14) is satisfied when \(\alpha = 1\) and \(k = p = 2\). The theorem therefore implies that \(R(f; t)\) is Hölder continuous with exponent \((p - \alpha)/k = (2 - 1)/2 = 1/2\). This proves part (i) of the corollary. The assertions of part (ii) follow from Corollary 1.4.

5.3. **Fourier series associated with the Möbius and Liouville functions.** The Möbius function \(\mu\) is the arithmetic function defined by \(\mu(1) = 1\); \(\mu(n) = (-1)^r\) if \(n\) is the product of \(r\) distinct primes; and \(\mu(n) = 0\) otherwise. The closely related Liouville function \(\lambda\) is defined by \(\lambda(1) = 1\) and \(\lambda(n) = (-1)^r\) if \(n\) is the product of \(r\) not necessarily distinct primes.

Davenport [Dav37] showed that the exponential sums \(S(\mu; x, t)\) associated with the Möbius function satisfy

\[
S(\mu; x, t) = \sum_{1 \leq n \leq x} \mu(n)e^{2\pi int} \ll_A x(\log x)^{-A},
\]

uniformly in \(t\), for any fixed constant \(A\). Here the Vinogradov notation “\(f(x) \ll g(x)\)” means that there exist constants \(C\) and \(x_0\) such that \(|f(x)| \leq C|g(x)|\) holds for all \(x \geq x_0\) and the subscript \(A\) in \(\ll_A\) indicates that these constants may depend on \(A\).

Using (5.10) along with an elementary argument that relates estimates for \(S(\lambda; x, t)\) to estimates of the same quality for \(S(\mu; x, t)\), Bateman and Chowla [BC63] showed that the Fourier series associated with \(\mu\) and \(\lambda\), i.e., the functions

\[
F(\mu; t) = \sum_{n=1}^{\infty} \frac{\mu(n)e^{2\pi int}}{n}, \quad F(\lambda; t) = \sum_{n=1}^{\infty} \frac{\lambda(n)e^{2\pi int}}{n},
\]

converge uniformly and hence represent continuous functions of \(t\).

In light of the Bateman-Chowla result it is natural to ask whether these series are Hölder continuous. Theorem 1.1 yields such a result provided a stronger form of the estimate (5.10), with a saving of a power of \(x\), is available. Unconditionally, no such estimate is known to date. However, Baker and Harman [BH91] showed that under the assumption of the Generalized Riemann Hypothesis one has

\[
\sum_{1 \leq n \leq x} \mu(n)e^{2\pi int} \ll_\epsilon x^{3/4+\epsilon}, \quad \sum_{1 \leq n \leq x} \lambda(n)e^{2\pi int} \ll_\epsilon x^{3/4+\epsilon},
\]
for any fixed $\epsilon > 0$ and uniformly in $t$.

It is conjectured (see, e.g., (6) in [BP98]) that the exponent $3/4$ in these estimates can be replaced by $1/2$, i.e., that, for any fixed $\epsilon > 0$, we have uniformly in $t$

\begin{equation}
\sum_{1 \leq n \leq x} \mu(n) e^{2\pi i n t} \ll_{\epsilon} x^{1/2+\epsilon}, \quad \sum_{1 \leq n \leq x} \lambda(n) e^{2\pi i n t} \ll_{\epsilon} x^{1/2+\epsilon}.
\end{equation}

In view of Parseval's identity the exponent $1/2$ here cannot be further improved.

By Theorem [1.1] and Corollary [1.2] the estimates (5.12) and (5.13) translate into bounds for the Hölder exponent and dimensions of the Fourier series $F(\mu; t)$ and $F(\lambda; t)$:

**Corollary 5.3.** Assume the Generalized Riemann Hypothesis holds. Then we have:

(i) The functions $F(\mu; t)$ and $F(\lambda; t)$ are Hölder continuous with exponent $1/4 - \epsilon$, for any $\epsilon > 0$, and the same holds for the absolute values and the real and imaginary parts of these functions.

(ii) The upper box-counting dimensions of the graphs of the functions $|F(\mu; t)|$, $\Re F(\mu; t)$, and $\Im F(\mu; t)$ over the interval $[0, 1]$ are bounded by $\leq 7/4$, and the same holds for the dimensions of the graphs associated with $F(\lambda; t)$.

Moreover, under conjectures (5.13) the bounds $1/4 - \epsilon$ and $7/4$ in (i) and (ii) can be replaced by $1/2 - \epsilon$ and $3/2$, respectively.

The assertion about the Hölder continuity of the Möbius Fourier series $F(\mu; t)$ is stated (though in a rather different form) in a recent paper of Veech [VPPF18] (see Remark 5.4). The other assertions of the corollary do not seem to be in the literature.

Several authors considered the more general exponential sums

\begin{equation}
S_k(\mu; x, t) = \sum_{1 \leq n \leq x} \mu(n) e^{2\pi i n^k t},
\end{equation}

where $k$ is a positive integer. In particular, assuming the Generalized Riemann Hypothesis, Zhan and Liu [ZL96] showed that if $k \geq 2$, then for any fixed $\epsilon > 0$ and uniformly in $t$, one has

\begin{equation}
S_k(\mu; x, t) \ll_{\epsilon} x^{\alpha_k + \epsilon}
\end{equation}

with $\alpha_k = 1 - 2^{1-2k}/k$. Via Theorem [1.3] and Corollary [1.4] this translates into the following bounds for the Hölder exponent and dimensions of the Fourier series

\begin{equation}
F_{k,k}(\mu; t) = \sum_{n=1}^{\infty} \frac{\mu(n) e^{2\pi i n^k t}}{n^k}.
\end{equation}

**Corollary 5.4.** Let $k$ be an integer with $k \geq 2$. Assume that the Generalized Riemann Hypothesis holds. Then we have:

(i) The function $F_{k,k}(\mu; t)$ is Hölder continuous with exponent $1 - (1 - 2^{1-2k}/k) - \epsilon$ for any $\epsilon > 0$, and the same holds for the absolute values and the real and imaginary parts of $F_{k,k}(\mu; t)$.

(ii) The upper box-counting dimensions of the graphs of the functions $|F_{k,k}(\mu; t)|$, $\Re F_{k,k}(\mu; t)$, and $\Im F_{k,k}(\mu; t)$ over the interval $[0, 1]$ are bounded by $\leq 1 + (1 - 2^{1-2k}/k)$.

**Remarks.**

(1) Similar bounds on Hölder exponents and dimensions can be obtained from Theorem [1.3] for the more general functions $F_{k,p}(\mu; t) = \sum_{n=1}^{\infty} \mu(n) e^{2\pi i n^k t}/n^p$. For example, if $p$ satisfies $1 < p \leq k$, then, under the assumption of the Generalized Riemann Hypothesis, $F_{k,p}(\mu; t)$ is Hölder continuous with exponent $(p - 1 + 2^{1-2k}/k)$. 
(2) Since the function $\mu(n)$ is bounded by 1, we have the trivial bound
\begin{equation}
|S_k(\mu; x, t)| \leq x \quad (x \geq 1, \ t \in \mathbb{R}).
\end{equation}
Applying Theorem 1.3 with this bound instead of (5.15) yields, for $k \geq 2$, the conclusions of the corollary with H"older exponent $1 - 1/k$ in place of $1 - (1 - 2^{1-2k})/k - \epsilon$ and dimension bound $1 + 1/k$ in place of $1 + (1 - 2^{1-2k})/k$. These bounds can be regarded as “baseline” bounds for these quantities that do not depend on the oscillating nature of the M"obius function.

(3) In the other direction, the most optimistic estimate for $S_k(\mu; x, t)$ one can hope for is a squareroot bound analogous to (5.13), i.e., a bound of the form
\begin{equation}
\sum_{1 \leq n \leq x} \mu(n)e^{2\pi in^kt} \ll x^{1/2+\epsilon},
\end{equation}
for any fixed $\epsilon > 0$ and uniformly in $t$. Under this assumption, Theorem 1.3 yields the assertions of the corollary with H"older exponent $1 - 1/(2k) - \epsilon$ and dimension bound $1 + 1/(2k)$.

Table 1 summarizes the H"older exponents for the series $F_{k,k}(\mu; t)$, $k = 1, \ldots, 4$, obtained from the above corollaries and the remarks following Corollary 5.4.

| $k$ | 1     | 2     | 3     | 4    |
|-----|-------|-------|-------|------|
| Unconditional | —     | $\frac{1}{2} = 0.5000$ | $\frac{2}{3} = 0.6666 \ldots$ | $\frac{3}{4} = 0.7500$ |
| Assuming GRH | $\frac{1}{4} = 0.2500$ | $\frac{9}{16} = 0.5625$ | $\frac{65}{96} = 0.6770 \ldots$ | $\frac{385}{512} = 0.7519 \ldots$ |
| Assuming (5.18) | $\frac{1}{2} = 0.5000$ | $\frac{1}{2} = 0.5000$ | $\frac{3}{4} = 0.7500$ | $\frac{5}{6} = 0.8333 \ldots$ | $\frac{7}{8} = 0.8750$ |

**Table 1.** H"older exponents for the generalized M"obius series $F_{k,k}(\mu; t) = \sum_{n=1}^{\infty} \mu(n)e^{2\pi in^kt}/n^k$, under various assumptions. The exponents corresponding to the values in the last two rows are understood to be of the form $\eta - \epsilon$, where $\eta$ is the value given in the table and $\epsilon$ is an arbitrarily small positive number.

Figures 2–5 below show how the fractal nature of the M"obius series $F_{k,k}(\mu; t)$ becomes less and less prominent as $k$ gets larger. This is consistent with the dimension bounds in part (ii) of Corollary 5.4, which approach 1 as $k \to \infty$.

![Path](image1.png) ![Graph](image2.png)

(A) Path $F(\mu; t)$, $0 \leq t \leq 1$ (B) Graph of $|F(\mu; t)|$, $0 \leq t \leq 1$

**Figure 2.** The Fourier series $F(\mu; t) = \sum_{n=1}^{\infty} \mu(n)e^{2\pi in^t}/n$. 
6. CONCLUDING REMARKS

We conclude this paper by discussing some possible extensions of our results and some related questions and open problems.
6.1. **Localized versions of Theorems [1.1] and [1.3]** A key aspect of Theorems [1.1] and [1.3] is the uniformity in \( t \) of both the bound [1.7] for \( S(x, t) \) and the bound [1.8] for \( F(t + h) - F(t) \). This raises the question whether these theorems can be localized, in the sense that assuming a bound for the exponential sums \( S(x, t) \) at a particular point \( t \) yields a bound for the local (or pointwise) Hölder exponent of the associated Fourier series at that point (see Definition [2.1]). In other words, if we have a bound for \( S(x, t) \) of the form [1.7], but with the constant \( C = C(t) \) and the exponent \( \alpha = \alpha(t) \) being allowed to depend on \( t \), can we obtain non-trivial bounds for the pointwise Hölder exponents \( \eta(t) \) of the function \( F(t) \)?

A result of this type, if true, would have significant applications. For example, in the case of the Möbius function \( \mu(n) \), Murty and Sankaranarayanan [MS02] proved unconditionally (i.e., without assuming the Generalized Riemann Hypothesis) a bound of the form \( S(\mu; x, t) \ll_\epsilon x^{4/5+\epsilon} \) for a certain class of numbers \( t \) that includes all algebraic irrational numbers. If a localized version of Theorem [1.1] were available, such a bound would imply non-trivial bounds on the local Hölder exponent of the Möbius Fourier series \( F(\mu; t) \) for the same class of numbers \( t \).

Our method relies on the uniformity of the exponential sum bound [1.7] in an essential manner and does not seem to be capable of yielding a localized version of the type mentioned. The most we can prove in this direction is that an exponential sum bound that is uniform in some open neighborhood of \( t \) implies a corresponding bound for the Hölder exponent in that same neighborhood. This, however, would not be sufficient for the application to the Möbius Fourier series mentioned above.

The question whether there is a local analog of Theorem [1.1] that relates estimates for \( S(x, t) \) at a particular point \( t \) to estimates for the local or pointwise Hölder exponent of \( F(t) \) at the same point \( t \) remains open. We expect, however, that such a result, if true, would not hold in nearly the same generality as Theorem [1.1].

6.2. **Lower bounds on the Hölder exponent and bi-Hölder continuity.** Theorem [1.1] yields, under the assumption [1.7], a bound of the form

\[
|F(t + h) - F(t)| \leq C_1 h^n
\]

uniformly in \( t, h \in (0, 1) \). A natural question is whether a similar bound in the other direction holds, i.e., whether one has, with suitable constants \( C'_1 > 0 \) and \( \eta' > 0 \),

\[
|F(t + h) - F(t)| \geq C'_1 h^{\eta'}
\]

uniformly in \( t, h \in (0, 1) \). A function satisfying both (6.1) and (6.2) is called bi-Hölder continuous.

Lower bounds of the form [6.2] (or localized versions of such bounds) have been established for certain narrowly defined classes of functions such as the Weierstrass function and functions of Riemann type. However, for more general classes of functions such as the functions considered in Theorem [1.1] very little is known. In particular, whether the Möbius Fourier series \( F(\mu; t) \) satisfies a non-trivial lower bound of the form [6.2] is not known.

6.3. **The constants in Theorems [1.1] and [1.3]** As we have noted, the exponent \( 1 - \alpha \) in the Hölder estimate [1.8] of Theorem [1.1] is optimal. This raises the question to what extent the constant \( C_1 = C_1(C, \alpha) \) in this theorem is also optimal. A small improvement can be obtained by keeping the factor \( (2\pi + 4\pi/\alpha) \) in (3.14) instead of replacing it by the larger value \( 6\pi/\alpha \). This results in the value

\[
C_1(C, \alpha) = \frac{8\alpha + 2\pi(\alpha + 2)(1 - \alpha)}{\alpha(1 - \alpha)} C
\]

for the constant in Theorem [1.1] which is slightly smaller than the value \( [1.9] \) given in that theorem though has similar asymptotic behavior as \( \alpha \to 0^+ \) or \( \alpha \to 1^- \). Is this constant best-possible, at least as far as its dependence on \( C \) and \( \alpha \) is concerned?
Since scaling the coefficients $f(n)$ by a fixed constant scales the associated exponential sum $S(x,t)$ and Fourier series $F(t)$ by the same constant, it is obvious that $C_1$ must depend linearly on the constant $C$ in (1.7). On the other hand, the dependence on $\alpha$ is less clear. In particular, one can ask whether, as $\alpha$ approaches 0 or 1, the constant $C_1$ is necessarily unbounded. With our approach this seems unavoidable: Lemmas 3.2 and 3.3 introduce factors proportional to $1/(1 - \alpha)$ and $1/\alpha$, respectively, which in turn forces the constant $C_1$ in Theorem 1.1 to depend on $\alpha$ at a rate roughly proportional to $1/(\alpha(1 - \alpha))$.

Similar questions can be asked about the constant $C_2$ in Theorem 1.3 and its dependence on the parameters $k$, $p$, $C$, and $\alpha$.

6.4. Lower dimension bounds. Theorem 1.1 provides upper bounds on the box-counting dimensions of the path $F([0, 1])$ and the graph $G(F)$. In the case of the Weierstrass function, these bounds are known to be sharp, i.e., they also represent lower bounds on the dimension. One can ask whether non-trivial lower bounds on the dimensions of $F([0, 1])$ and $G(F)$ can be obtained in the setting of Theorem 1.1. More precisely, what non-trivial conditions can be assumed on $f$ or $S(f; x, t)$ in order to deduce a corresponding non-trivial lower bound on the dimensions of $F([0, 1])$ and $G(F)$? Note the connection of this question to the discussion in Section 6.2 in view of (3.11) in Fra21 and Corollary 11.2 in Fra14, which imply lower dimension bounds for $F([0, 1])$ and $G(F)$, respectively, if a bi-Hölder condition holds for $F$.

6.5. Bounds on Hausdorff and Assouad dimensions. Corollaries 1.2 and 1.4 give bounds on the box-counting dimension, $\dim_B$, of graphs associated with the functions $F(t)$ and $F_{k,p}(t)$. Other notions of dimension that have been studied in the literature include the Hausdorff dimension, denoted by $\dim_H$, and the Assouad dimension, denoted by $\dim_A$; see Fra21, pp. 6 and Fra21, p. 10 for precise definitions of these concepts.

It is known (see Fra21, Lemma 2.4.3) that, for any bounded set $E \subset \mathbb{R}^d$,

$$\dim_H E \leq \dim_B E \leq \dim_A E.$$  

Thus any upper bound on the box-counting dimension is also an upper bound on the Hausdorff dimension. It follows that the upper bounds for the box-counting dimension provided by Corollaries 1.2 and 1.4 also hold for the Hausdorff dimension.

On the other hand, (6.3) also shows that the Assouad dimension is bounded below by the (upper) box-counting dimension, so no similar conclusion can be drawn for the Assouad dimension. Moreover, there is no analog of the inequalities of Proposition 2.3 for the Assouad dimension. That is, there is no non-trivial bound for the Assouad dimension of the path $F([0, 1])$ and the graph $G(F)$ in terms of the Hölder exponent of the function $F(t)$. It would be interesting to know if one can obtain non-trivial bounds on the Assouad dimension under assumptions similar to those of Corollary 1.2. Even in the special case of the Weierstrass function, determining the exact value of the Assouad dimension of its graph is still an open problem (see Fra21, Question 17.11.1).

6.6. Random Fourier series. A natural model for Fourier series with pseudo-random coefficients such as the Möbius and Liouville functions $\mu(n)$ and $\lambda(n)$ is a random series of the form

$$F(X; t) = \sum_{n=1}^{\infty} \frac{X_n e^{2\pi int}}{n},$$

with coefficients given by a sequence of independent random variables $X_n$ that take on values $+1$ and $-1$ with probability $1/2$ each. The behavior (with probability 1) of such random series can serve as a source for conjectures for deterministic series such as those associated with the Möbius and Liouville functions.

Random Fourier series of this type have been investigated in the literature. In particular, a result of Kahane (see Theorem 3 in Chapter 7 of Kah85) shows that, with probability 1, the series (6.4)
is Hölder continuous with exponent $1/2$, and that this exponent is best-possible. Thus it seems reasonable to conjecture that the Möbius series $F(\mu;t)$ is Hölder continuous with exponent $1/2 - \epsilon$, for any fixed $\epsilon$, and that the constant $1/2$ here is best possible.

Recently, Kowalski and Sawin [KS16] considered series of the form (6.4), where the $X_n$ are independent random variables having the Sato-Tate distribution. They showed that this particular random series arises naturally in number theory as a limiting process of random functions associated with Kloosterman sums, and they established a variety of properties of this series.

**References**

[Apo76] Tom M. Apostol, *Introduction to analytic number theory*, Undergraduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1976. MR 0434929

[Bar15] Krzysztof Barański, *Dimension of the graphs of the Weierstrass-type functions*, Fractal geometry and stochastics V, Progr. Probab., vol. 70, Birkhäuser/Springer, Cham, 2015, pp. 77–91. MR 3558151

[BBR14] Krzysztof Barański, Balázs Bárány, and Julia Romanowska, *On the dimension of the graph of the classical Weierstrass function*, Adv. Math. 265 (2014), 32–59. MR 3255455

[BC63] P. T. Bateman and S. Chowla, *Some special trigonometrical series related to the distribution of prime numbers*, J. London Math. Soc. 38 (1963), 372–374. MR 153639

[BF95] J. Bohman and C.-E. Fröberg, *Heuristic investigation of chaotic mapping producing fractal objects*, BIT 35 (1995), no. 4, 669–615. MR 1431353

[BH91] R. C. Baker and G. Harman, *Exponential sums formed with the Möbius function*, J. London Math. Soc. (2) 43 (1991), no. 2, 193–198. MR 1111578

[BP98] A. Balog and A. Perelli, *On the $L^2$ mean of the exponential sum formed with the Möbius function*, J. London Math. Soc. (2) 57 (1998), no. 2, 275–288. MR 1644201

[CC99] Fernando Chamizo and Antonio Córdoba, *Differentiability and dimension of some fractal Fourier series*, Adv. Math. 142 (1999), no. 2, 335–354. MR 1680194

[CU07] Fernando Chamizo and Adrián Ubis, *Some Fourier series with gaps*, J. Anal. Math. 101 (2007), 179–197. MR 2346544

[Dav37] Harold Davenport, *On some infinite series involving arithmetical functions (ii)*, The Quarterly Journal of Mathematics (1937), no. 1, 313–320.

[Dui91] J. J. Duistermaat, *Self-similarity of “Riemann’s nondifferentiable function”*, Nieuw Arch. Wisk. (4) 9 (1991), no. 3, 303–337. MR 1166151

[Ece20] Daniel Eceizabarrena, *Geometric differentiability of Riemann’s non-differentiable function*, Adv. Math. 366 (2020), 107091, 39. MR 4072795

[Fal14] Kenneth Falconer, *Fractal geometry*, third ed., John Wiley & Sons, Ltd., Chichester, 2014, Mathematical foundations and applications. MR 3236784

[Fra21] Jonathan M. Fraser, *Assouad dimension and fractal geometry*, Cambridge Tracts in Mathematics, vol. 222, Cambridge University Press, Cambridge, 2021. MR 4411274

[Har16] G. H. Hardy, *Weierstrass’s non-differentiable function*, Trans. Amer. Math. Soc. 17 (1916), no. 3, 301–325. MR 1501044

[Hun98] Brian R. Hunt, *The Hausdorff dimension of graphs of Weierstrass functions*, Proc. Amer. Math. Soc. 126 (1998), no. 3, 791–800. MR 1452806

[Jaf96] Stéphane Jaffard, *The spectrum of singularities of Riemann’s function*, Rev. Mat. Iberoamericana 12 (1996), no. 2, 441–460. MR 1402673

[Kah85] Jean-Pierre Kahane, *Some random series of functions*, second ed., Cambridge Studies in Advanced Mathematics, vol. 5, Cambridge University Press, Cambridge, 1985. MR 833073

[KMPY84] James L. Kaplan, John Mallet-Paret, and James A. Yorke, *The Lyapunov dimension of a nowhere differentiable attracting torus*, Ergodic Theory Dynam. Systems 4 (1984), no. 2, 261–281. MR 766105

[KS16] Emmanuel Kowalski and William F. Sawin, *Kloosterman paths and the shape of exponential sums*, Compos. Math. 152 (2016), no. 7, 1489–1516. MR 3530449

[MS02] M. Ram Murty and A. Sankaranarayanan, *Averages of exponential twists of the Liouville function*, Forum Math. 14 (2002), no. 2, 273–291. MR 1880914

[She18] Weixiao Shen, *Hausdorff dimension of the graphs of the classical Weierstrass functions*, Math. Z. 289 (2018), no. 1-2, 223–266. MR 3803788

[VFF18] William A. Veech, Giovanni Forni, and Jon Fickenscher, *Riemann sums and Möbius*, J. Anal. Math. 135 (2018), no. 2, 413–436, Final revisions by Giovanni Forni and Jon Fickenscher. MR 3829605

[ZL96] T. Zhan and J.-Y. Liu, *Exponential sums involving the Möbius function*, Indag. Math. (N.S.) 7 (1996), no. 2, 271–278. MR 1621332
