Kirszbraun extension on connected finite graph

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Abstract: We prove that the tight function introduced Sheffield and Smart (2012) \cite{10} is a Kirszbraun extension. In the real-valued case we prove that Kirszbraun extension is unique. Moreover, we produce a simple algorithm which calculates efficiently the value of Kirszbraun extension in polynomial time.

Key words: Minimal, Lipschitz, extension, Kirszbraun, harmonious.

1 Introduction

Let $A$ be a compact subset of $\mathbb{R}^n$. The best Lipschitz constant of a Lipschitz function $g : A \to \mathbb{R}^m$ is

$$\text{Lip}(g, A) := \sup_{x \neq y \in A} \frac{\|g(x) - g(y)\|}{\|x - y\|}$$

where $\|\cdot\|$ is Euclidean norm.

When $m = 1$, Aronsson in 1967 \cite{11} proved the existence of absolutely minimizing Lipschitz extension (AMLE), i.e., a extension $u$ of $g$ satisfying

$$\text{Lip}(u; V) = \text{Lip}(u, \partial V), \quad \text{for all } V \subset \subset \mathbb{R}^n \setminus A.$$  \hspace{1cm} (2)

Jensen in 1993 \cite{5} proved the uniqueness of AMLE under certain conditions.

In this chapter we begin by studying the discrete version of the existence and uniqueness of AMLE for case $m \geq 2$.

We define the function

$$\lambda(g, A)(x) := \inf_{y \in \mathbb{R}^m} \sup_{a \in A} \frac{\|g(a) - y\|}{\|a - x\|} \quad \text{if } x \in \mathbb{R}^n \setminus A.$$  \hspace{1cm} (3)

From Kirszbraun theorem (see \cite{3, 7}) the function $\lambda(g, A)$ is well-defined and

$$\lambda(g, A)(x) \leq \text{Lip}(g, A).$$

Moreover, (see \cite{3, Lemma 2.10.40}) for any $x \in \mathbb{R}^n \setminus A$ there exists a unique $y(x) \in \mathbb{R}^m$ such that

$$\lambda(g, A)(x) = \sup_{a \in A} \frac{\|g(a) - y(x)\|}{\|a - x\|},$$

\hspace{1cm} (4)
and \( y(x) \) belongs to the convex hull of the set
\[
B = \{ g(z) : z \in A \text{ and } \frac{\|g(z) - y(x)\|}{\|z - x\|} = \lambda(g, A)(x) \}.
\]
Thus we can define
\[
K(g, A)(x) := \begin{cases} 
  g(x) & \text{if } x \in A; \\
  \arg \min_{y \in \mathbb{R}^m} \sup_{a \in A} \frac{\|g(a) - y\|}{\|a - x\|} & \text{if } x \in \mathbb{R}^n \setminus A.
\end{cases}
\tag{5}
\]
We say that \( K(g, A)(x) \) is the Kirszbraun value of \( g \) restricted on \( A \) at point \( x \). The function \( K(g, A)(x) \) is the best extension at point \( x \) such that the Lipschitz constant is minimal. We produce a method to compute \( \lambda(g, A)(x) \) and \( K(g, A)(x) \) in section 4.

Let \( G = (V, E, \Omega) \) be a connected finite graph with vertices set \( V \subset \mathbb{R}^n \), edges set \( E \) and a non-empty set \( \Omega \subset V \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graph.png}
\caption{A simple picture of graph \( G \)}
\end{figure}

For \( x \in V \), we define
\[
S(x) := \{ y \in V : (x, y) \in E \}
\tag{6}
\]
to be the neighborhood of \( x \) on \( G \).

**Example 1.** In Figure 1 we have \( V = \{v_1, \ldots, v_6\}, E = \{e_1, \ldots, e_{10}\}, S(v_3) = \{v_1, v_2, v_4, v_5\} \).

Let \( f : \Omega \to \mathbb{R}^m \). We consider the following functional equation with Dirichlet’s condition:
\[
u(x) = \begin{cases} 
  K(u, S(x))(x) & \forall x \in V \setminus \Omega; \\
  f(x) & \forall x \in \Omega.
\end{cases}
\tag{7}
\]
We say that a function \( u \) satisfying (7) is a Kirszbraun extension of \( f \) on graph \( G \). This extension is the optimal Lipschitz extension of \( f \) on graph \( G \) since for any \( x \in V \setminus \Omega \), there is no way to decrease \( \text{Lip}(u, S(x)) \) by changing the value of \( u \) at \( x \).
In real valued case $m = 1$, the function $K(u, S(x))(x)$ was considered by Oberman [9] and he used this function to obtain a convergent difference scheme for the AMLE. Le Gruyer [6] showed the explicit formula for $K(u, S(x))(x)$ as follows

$$K(u, S(x))(x) = \inf_{z \in S(x)} \sup_{q \in S(x)} M(u, z, q)(x),$$

where

$$M(u, z, q)(x) := \frac{\|x - z\|u(q) + \|x - q\|u(z)}{\|x - z\| + \|x - q\|}.$$  

Le Gruyer studied the solution of (7) on a network where $K(u, S(x))(x)$ satisfying (8). This solution plays an important role in approximation arguments for AMLE in Le Gruyer [6].

The Kirszbraun extension $u$ is a generalization of the solution in the previous works of Le Gruyer for vector valued cases ($m \geq 2$). We prove that the tight function introduced by Sheffield and Smart (2012) [10] is a Kirszbraun extension. Therefore, we have the existence of a Kirszbraun extension, but in general Kirszbraun extension maybe not unique.

In the scalar case $m = 1$, Le Gruyer [6] defined a network on a metric space $(X, d)$ as follows

**Definition 2.** A network on a metric space $(X, d)$ is a couple $(N, U)$ where $N \subset X$ denotes a finite non-empty subset of $\mathbb{R}^n$ and $U$ a mapping $x \in N \rightarrow U(x) \subset N$ which satisfies

(P1) For any $x \in N$, $x \in U(x)$.

(P2) For any $x, y \in N$, $x \in U(y)$ iff $y \in U(x)$.

(P3) For any $x, y \in N$, there exists $x_1, ..., x_{n-1} \in G$ such that $x_1 = x$, $x_n = y$ and $x_i \in U(x_{i+1})$ for $i = 1, ..., n - 1$.

(P4) For any $x \in N$, any $y \in N \setminus U(x)$ there exists $z \in U(x)$ such that $d(z, y) \leq d(x, y)$.

In the above definition, $U(x)$ is called the neighborhood of $x$ on network $(N, U)$. Let $g : A \subset X \rightarrow \mathbb{R}$. In [6] Le Gruyer defined the Kirszbraun extension of $g$ with respect to the network (see [6] page 30) and he proved the existence and uniqueness of the Kirszbraun extension of $g$ on the network. In particular, when $X = \mathbb{R}^n$ equipped with the Euclidean norm, Le Gruyer obtained the approximation for AMLE by a sequence Kirszbraun extensions $(u_n)$ of networks $(N_n, U_n)$ ($n \in \mathbb{N}$) having some good properties.

Similarly to Le Gruyer’s result about the uniqueness of the Kirszbraun extension on a network, in this chapter we prove the uniqueness of the Kirszbraun extension $u$ of $f$ on graph $G$ when $m = 1$. The graph is more general than the network in some sense since there are many graphs that do not satisfy (P4). Moreover, in the scalar case $m = 1$, we produce a simple algorithm which calculates efficiently the value of Kirszbraun extension $u$ in polynomial time. This algorithm is similar to the algorithm produced by Lazarus el al. (1999) [8] when they calculate the Richman cost function. Assuming Jensen’s hypotheses [5], since this algorithm computes exactly solution of (7) and by using the argument of Le Gruyer [6] (the approximation for AMLE by a sequence Kirszbraun extensions $(u_n)$ of networks $(N_n, U_n)$ ($n \in \mathbb{N}$)), we obtain a new method to approximate the viscosity solution of Equation $\Delta_\infty u = 0$ under Dirichler’s condition $f$.

In the above algorithm, the explicit formula of $K(u, S(x))$ in (8) and the order structure of real number set play important role. The generalization of the algorithm to vector valued
case \((m \geq 2)\) is difficult since we do not know the explicit formula of \(K(u, S(x))\) when \(m \geq 2\) and \(\mathbb{R}^2\) does not have any useful order structure. Extending the results of the approximation of AMLE to vector valued cases \((m \geq 2)\) presents many difficulties which have limited the number of results in this direction, see [4] and the references therein.

2 The existence of Kirszbraun extension

In this section, we prove the existence of Kirszbraun extension satisfying Equation (7). Let \(G = (V, E, \Omega)\) be a connected finite graph with vertices set \(V \subset \mathbb{R}^n\), edges set \(E\) and a non-empty set \(\Omega \subset V\) and let \(f : \Omega \to \mathbb{R}^m\).

We denote \(E(f)\) to be the set of all extensions of \(f\) on \(G\). Let \(v \in E(f)\). The local Lipschitz constant of \(v\) at vertex \(x \in V\setminus \Omega\) is given by

\[
L_v(x) := \sup_{y \in S(x)} \frac{\|v(y) - v(x)\|}{\|y - x\|},
\]

where \(S(x)\) is neighborhood of \(x\) on \(G\).

**Definition 3.** [4] If \(u, v \in E(f)\) satisfy

\[
\max \{L_u(x) : L_u(x) > L_v(x), x \in V\setminus \Omega\} > \max \{L_v(x) : L_v(x) > L_u(x), x \in V\setminus \Omega\},
\]

then we say that \(v\) is tighter than \(u\) on \(G\). We say that \(u\) is a tight extension of \(f\) on \(G\) if there is no \(v\) tighter than \(u\).

**Theorem 4.** [10, Theorem 1.2] There exists a unique extension \(u\) that is tight of \(f\) on \(G\). Moreover, \(u\) is tighter than every other extension \(v\) of \(f\).

**Proposition 5.** Let \(u \in E(f)\). Let \(x \in V\setminus \Omega\), we define

\[
v(y) = \begin{cases} u(y), & \text{if } y \in V\setminus \{x\}, \\ K(u, S(x))(x), & \text{if } y = x. \end{cases}
\]

If \(K(u, S(x))(x) \neq u(x)\) then \(v\) is tighter than \(u\).

**Proof.**

*Step 1:* In this step we prove that for any \(y \in V\setminus \Omega\), we obtain

\[
L_v(y) \leq \max \{L_u(x), L_y(y)\}. \tag{9}
\]

Indeed,

*If \(y \notin S(x) \cup \{x\}\). Since \(v(y) = u(y)\) and \(v(z) = u(z)\) for all \(z \in S(y)\), we obtain

\[
L_v(y) = L_u(y).
\]

*If \(y = x\). Since \(v(x) \neq u(x)\) and \(v(x)\) is the Kirszbraun value of \(u\) restricted on \(S(x)\) at point \(x\), we have

\[
L_v(y) < L_u(y).
\]

\[1\] By convention, if \(C = \emptyset\) then \(\max C = 0\).
*If \( y \in S(x) \) we have

\[
Lv(y) = \max_{z \in S(y)} \frac{\|v(z) - v(y)\|}{\|z - y\|}
= \max_{z \in S(y) \setminus \{x\}} \left\{ \frac{\|v(x) - v(y)\|}{\|x - y\|}, \frac{\|u(z) - u(y)\|}{\|z - y\|} \right\}
\leq \max\{Lv(x), Lu(y)\}.
\]

Therefore, for any \( y \in V \setminus \Omega \) we have

\[
Lv(y) \leq \max\{Lv(x), Lu(y)\}.
\]

**Step 2:** In this step we prove that \( v \) is tighter than \( u \). It means that we need to show that

\[
\max\{Lv(y) : Lv(y) > Lu(y), y \in V \setminus \Omega\} < \max\{Lu(y) : Lu(y) > Lv(y), y \in V \setminus \Omega\}
\]

Indeed, if \( Lv(y) > Lu(y) \) then from (9) we have \( Lv(y) \leq Lv(x) \). Thus

\[
\max\{Lv(y) : Lv(y) > Lu(y), y \in V \setminus \Omega\} \leq Lv(x) \tag{10}
\]

Since \( v(x) \neq u(x) \) and \( v(x) \) is the Kirszbraun value of \( u \) restricted on \( S(x) \) at point \( x \), we have

\[
Lv(x) < Lu(x). \tag{11}
\]

From (10) and (11) we obtain

\[
\max\{Lv(y) : Lv(y) > Lu(y), y \in V \setminus \Omega\} \leq \max\{Lu(y) : Lu(y) > Lv(y), y \in V \setminus \Omega\}.
\]

We obtain the existence of a Kirszbraun extension satisfying Equation (7) as a consequence of the following theorem.

**Theorem 6.** If \( u \) is a tight extension of \( f \), then \( u \) is a Kirszbraun extension of \( f \).

**Proof.** Let \( u \) be a tight extension of \( f \). Suppose, by contradiction, that there are some \( x \in V \setminus \Omega \) such that

\[
K(u, S(x))(x) \neq u(x). \tag{12}
\]

we define

\[
v(y) = \begin{cases} u(y), & \text{if } y \in V \setminus \{x\}, \\ K(u, S(x)), & \text{if } y = x. \end{cases}
\]

By applying Proposition 5 we have \( v \) tighter than \( u \). This is impossible since \( u \) is tight of \( f \). \qed
In this section, let $G = (V, E, \Omega)$ be a connected finite graph, with vertices set $V \subset \mathbb{R}^n$, edges set $E$ and a non-empty set $\Omega \subset V$. Let $f : \Omega \to \mathbb{R}$.

We recall some properties of Kirszbraun function introduced in [5] which are useful in the proof of Theorem 8.

**Theorem 7.** Let $S = \{x_1, \ldots, x_n\} \subset \mathbb{R}^n$ and $u : S \to \mathbb{R}$. For each $x \in \mathbb{R}^n \setminus S$, we use the notation $d_i = \|x_i - x\|$, $i = 1, \ldots, n$.

(a) (see [9, Theorem 5]) We have

$$K(u, S)(x) = \frac{d_i u(x_i) + d_j u(x_j)}{d_i + d_j},$$

where $i, j$ are the indexes which satisfy

$$\frac{|u(x_i) - u(x_j)|}{d_i + d_j} = \max_{k,l=1} \left\{ \frac{|u(x_k) - u(x_l)|}{d_k + d_l} \right\}.$$

(b) (see [3, Lemma 2.10.40]) Let

$$\lambda(u, S)(x) := \inf_{y \in \mathbb{R}^m} \sup_{a \in S} \frac{|u(a) - y|}{\|a - x\|} \text{ if } x \in \mathbb{R}^n \setminus S. \quad (13)$$

then the set

$$B = \left\{ u(z) : z \in S \text{ and } \frac{\|u(z) - K(u, S)(x)\|}{\|z - x\|} = \lambda(u, S)(x) \right\},$$

is not empty, and $K(u, S)(x)$ belongs to the convex hull of $B$.

**Theorem 8.** There is a unique Kirszbraun extension $u$ of $f$ on the graph $G$. Moreover, the Kirszbraun extension $u$ of $f$ can be calculated in polynomial time.

Before proving Theorem 8, we need the following definition

**Definition 9.** Let $G' = (V', E', \Omega)$ be a subgraph of $G$, i.e. $\Omega \subset V' \subset V$ and $E' \subset E$. Let $u'$ be a Kirszbraun extension of $f$ on $G'$, a connecting path on $G'$ with respect to $u'$ is a sequence

$$v_0, e_1, v_1, \ldots, e_n, v_n \ (n \geq 1)$$

of distinct vertices and edges in $G$ such that

* each $e_i$ is an edge joining $v_{i-1}$ and $v_i$,
* $v_0$ and $v_n$ are in $V'$,
* for $1 \leq i < n$, $v_i$ is in $V \setminus V'$, and
* for $1 \leq i \leq n$, $e_i$ is in $E \setminus E'$

We define

$$c := \frac{|u'(v_n) - u'(v_0)|}{\sum_{i=1}^n \|v_i - v_{i-1}\|}.$$

We say that $c$ is the slope of the connecting path $v_0, e_1, v_1, \ldots, e_n, v_n$. 


Proof of Theorem 8. We construct an increasing sequence of subgraph $G_n = (V_n, E_n, \Omega)$ of $G$ and $u_n$ which is a Kirszbraun extension of $f$ on $G_n$. We finish the algorithm with a Kirszbraun extension $u$ on $G$.

**Step 1: Construct an increasing sequence of subgraph**

We begin with the trivial subgraph $G_1 = (V_1, E_1, \Omega)$ where $V_1 = \Omega$, $E_1 = \emptyset$ and let $u_1 = f$ on $\Omega$. It is clear that $u_1$ is a Kirszbraun extension of $f$ on $G_1$. The algorithm then proceeds in stages.

Suppose that after $n$ stages we have an increasing sequence of subgraph $G_l = (V_l, E_l, \Omega)$ of $G$ and $u_l$ is a Kirszbraun extension of $f$ on $G_l$ for $l = 1, ..., n$.

If there are no connecting paths on $G_n$ with respect to $u_n$, we go to step 2.

If there are some connecting paths on $G_n$ with respect to $u_n$. We construct $G_{n+1}$ subgraph of $G$ and $u_{n+1}$ Kirszbraun extension of $f$ on $G_{n+1}$ as follows:

Find a connecting path $v_0, e_1, v_1, ..., e_k, v_k$ ($k \geq 1$) on $G_n$ with respect to $u_n$ with largest possible slope $c_n$.

Without loss of generality, we label the vertices of the path so that $u_n(v_k) \geq u_n(v_0)$. We define

$$u_{n+1}(x) := \begin{cases} u_n(x), & \forall x \in G_n \\ u_n(v_0) + c_n \sum_{j=1}^{i} \|v_j - v_{j-1}\|, & \text{if } x = v_i \text{ for } i = 1, ..., k - 1. \end{cases}$$ (14)

$$V_{n+1} := V_n \cup \{v_1, ..., v_{k-1}\}$$

$$E_{n+1} := E_n \cup \{e_1, ..., e_k\}$$

We will show that $u_{n+1}$ is a Kirszbraun extension of $f$ on graph $G_{n+1} = (V_{n+1}, E_{n+1}, \Omega)$.

For $x \in V_{n+1}$, let

$$S_i(x) := \{y \in V_i : (x, y) \in E_i\} \text{ for } i \in \{1, ..., n+1\}.$$  

be the neighborhood of $x$ with respect to $G_i$.

**Case 1:** $x \in V_n \backslash \{v_0, v_k\}$.

We have $S_{n+1}(x) = S_n(x)$, $u_{n+1}(z) = u_n(z)$ for all $z \in S_{n+1}(x) \cup \{x\}$ and $u_n(x) = K(u_n, S_n(x))(x)$ since $u_n$ is Kirszbraun of $G_n$. Thus

$$u_{n+1}(x) = K(u_{n+1}, S_{n+1}(x))(x), \text{ for } x \in V_n \backslash \{v_0, v_k\}.$$ (14)

**Case 2:** $x \in \{v_1, ..., v_{k-1}\}$.

Noting that $S_{n+1}(v_i) = \{v_{i-1}, v_i\}$ for all $i = 1, ..., k - 1$. Moreover, from (14), we have

$$\frac{u_{n+1}(v_i) - u_{n+1}(v_{i-1})}{\|v_i - v_{i-1}\|} = c_n , \forall i : 1 \leq i \leq k.$$  

Hence

$$u_{n+1}(x) = K(u_{n+1}, S_{n+1}(x))(x) \text{ for } x \in \{v_1, ..., v_{n-1}\}.$$  

**Case 3:** $x \in \{v_0, v_k\}$.


We need to prove that

\[ u_{n+1}(v_0) = K(u_{n+1}, S_{n+1}(v_0))(v_0). \]  

(Proving \( u_{n+1}(v_k) = K(u_{n+1}, S_{n+1}(v_k))(v_k) \) is similar.)

To see (15), we must show that

\[
\sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - u_{n+1}(v_0)|}{\|x - v_0\|} \leq \inf_{y \in \mathbb{R}^m} \sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - y|}{\|x - v_0\|}.
\]

Noting that \( u_{n+1}(x) = u_n(x) \) for all \( x \in S_n(v_0) \cup \{v_0\} \), \( S_{n+1}(v_0) = S_n(v_0) \cup \{v_1\} \) and \( c_n = \frac{|u_{n+1}(v_1) - u_{n+1}(v_0)|}{\|v_1 - v_0\|} \). Moreover, since \( u_n \) is a Kirszbraun extension of \( f \) on \( G_n \), we have

\[
\sup_{x \in S_n(v_0)} \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|} \leq \inf_{y \in \mathbb{R}^m} \sup_{x \in S_n(v_0)} \frac{|u_n(x) - y|}{\|x - v_0\|}.
\]

Thus

\[
\sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - u_{n+1}(v_0)|}{\|x - v_0\|} = \sup_{x \in S_n(v_0) \cup \{v_1\}} \frac{|u_{n+1}(x) - u_{n+1}(v_0)|}{\|x - v_0\|} = \max_{x \in S_n(v_0)} \left\{ \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|}, \frac{|u_{n+1}(v_1) - u_{n+1}(v_0)|}{\|v_1 - v_0\|}, c_n \right\},
\]

and

\[
\max_{x \in S_n(v_0)} \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|} \leq \inf_{y \in \mathbb{R}^m} \sup_{x \in S_n(v_0)} \frac{|u_n(x) - y|}{\|x - v_0\|} \leq \inf_{y \in \mathbb{R}^m} \sup_{x \in S_{n+1}(v_0)} \frac{|u_{n+1}(x) - y|}{\|x - v_0\|}.
\]

Therefore, to obtain Equation (16), we need to prove that

\[ c_n \leq \frac{|u_n(x) - u_n(v_0)|}{\|x - v_0\|}, \]

for some \( x \in S_n(v_0) \).

Let \( F \) be the set of slope of connecting paths occurring in the algorithm. Remark that each edges and each vertices entered in our algorithm relate with a slope in \( F \). So that, for any \( y \in V_n \), there exist some \( x \in S_n(x) \) and \( c \in F \) such that

\[ c = \frac{|u_n(x) - u_n(y)|}{\|x - y\|}. \]

From above remark, to see (17), we need to show that the sequence of slope of connecting paths occurring in the algorithm is non-increasing. We show this in our present notation. Suppose that

\[ w_0, f_1, w_1, ..., f_m, w_m \ (m \geq 1) \]
is a connecting path on $G_{n+1}$ with respect to $u_{n+1}$ with slope $c_{n+1}$. We need to prove that $c_n \geq c_{n+1}$. We assume without loss of generality that $u_{n+1}(w_0) \leq u_{n+1}(w_m)$.

- If $w_0$ and $w_m$ are both in $V_n$ then the connecting path on $G_{n+1}$ with respect to $u_{n+1}$ is actually the connecting path on $G_n$ with respect to $u_n$. Therefore, since $c_n$ is the largest slope of connecting paths on $G_n$ with respect to $u_n$, we have $c_n \geq c_{n+1}$.

- If $w_0 = v_i$ and $w_m = v_j$ for some $0 \leq i < j \leq k$. We consider the path through the vertices

$$v_0, \ldots, v_{i-1}, w_0, \ldots, w_m, v_{j+1}, \ldots, v_k.$$ 

The slope of above path is

$$c = \frac{|u_n(v_k) - u_n(v_0)|}{\sum_{l=1}^{i} \|v_l - v_{l-1}\| + \sum_{l=1}^{m} \|w_l - w_{l-1}\| + \sum_{l=j+1}^{k} \|v_l - v_{l-1}\|}.$$ 

Since $c_n$ is the largest slope of connecting paths on $G_n$ with respect to $u_n$, we have $c_n \geq c$. Moreover,

$$c_n = \frac{|u_n(v_k) - u_n(v_0)|}{\sum_{l=1}^{k} \|v_l - v_{l-1}\|},$$ 

thus we obtain

$$\sum_{l=1}^{m} \|w_l - w_{l-1}\| \geq \sum_{l=i+1}^{j} \|v_l - v_{l-1}\|.$$ 

Hence

$$c_{n+1} = \frac{|u_{n+1}(w_m) - u_{n+1}(w_0)|}{\sum_{k=1}^{m} \|w_k - w_{k-1}\|} = \frac{|u_{n+1}(v_j) - u_{n+1}(v_i)|}{\sum_{k=1}^{m} \|w_k - w_{k-1}\|} \leq \frac{|u_{n+1}(v_j) - u_{n+1}(v_i)|}{\sum_{l=i+1}^{j} \|v_l - v_{l-1}\|} = c_n.$$ 

**Step 2: Completing the algorithm**

If there are no connecting paths on $G_n = (V_n, E_n, \Omega)$ with respect to $u_n$. Then each unlabeled vertex $v$ is connected via edges not in $E_n$ to exactly one vertex $w$ of $V_n$. We extend $u_n$ to the point $w$ by putting $u_n(w) := u_n(v)$. This completes the algorithm, and we obtains a Kirszbraun extension of $f$.

Since each stage adds at least one edge, and each stage can be accomplished by one shortest-path search for each pair of labeled vertices, this algorithm is calculated in polynomial time.

**Uniqueness**

Let $u$ be the Kirszbraun extension of $f$ defined by the algorithm above and $h$ be another Kirszbraun extension of $f$. Let $v$ be the first vertex added by algorithm such that $u(v) \neq h(v)$. 


• If \( v \) is added to a subgraph \( G' = (V', E', \Omega) \) as part of a connecting path through the vertices

\[ v_0, \ldots, v_k, \ldots, v_n \]

with slope \( c \) and \( v = v_k \).

We can assume without loss of generality that \( u(v_0) \leq u(v_n) \). Let

\[ \mathcal{L} = \{v_i : 0 \leq i \leq n, h(v_i) \geq u(v_i), h(v_i) - h(v_{i-1}) > u(v_i) - u(v_{i-1})\} \]

We prove that \( \mathcal{L} \neq \emptyset \). Indeed, by contradiction, suppose that \( \mathcal{L} = \emptyset \). Since \( u(v_0) = h(v_0) \) and \( \mathcal{L} = \emptyset \) we must have

\[ h(v_1) \leq u(v_1). \]

If \( h(v_2) > u(v_2) \) then

\[ h(v_2) - h(v_1) > u(v_2) - u(v_1). \]

Hence \( v_2 \in \mathcal{L} \). This contradicts with \( \mathcal{L} = \emptyset \). Thus we must have

\[ h(v_2) \leq u(v_2). \]

By induction, we have

\[ h(v_i) \leq u(v_i) \quad \forall i : 0 \leq i \leq k. \tag{19} \]

Since \( v = v_k, h(v) \neq u(v) \) and (19), we have \( h(v_k) < u(v_k) \). Thus if \( h(v_{k+1}) \geq u(v_{k+1}) \) then

\[ h(v_{k+1}) - h(v_k) > u(v_{k+1}) - u(v_k). \]

Hence \( v_{k+1} \in \mathcal{L} \). This contradicts with \( \mathcal{L} = \emptyset \). Thus we must have

\[ h(v_{k+1}) < u(v_{k+1}). \]

By induction, we have

\[ h(v_i) < u(v_i), \quad \forall k \leq i \leq n. \]

But we know that \( h(v_n) = u(v_n) \), thus we have a contradiction. Therefore \( \mathcal{L} \neq \emptyset \).

Let \( v_l \in \mathcal{L} \). We have

\[ \begin{cases} h(v_l) \geq u(v_l); \\ h(v_l) - h(v_{l-1}) > u(v_l) - u(v_{l-1}). \end{cases} \tag{20} \]

Hence

\[ \Delta := \frac{h(v_l) - h(v_{l-1})}{\|v_l - v_{l-1}\|} > \frac{u(v_l) - u(v_{l-1})}{\|v_l - v_{l-1}\|} = c \geq 0. \tag{21} \]

Set

\[ S(x) := \{y \in V, (x, y) \in E\} \quad \text{for } x \in V. \]
Since \( K(h, S(v))(v_i) = h(v_i) \), by applying Theorem [4] there exists \( z_1 \in S(v_i) \) such that
\[
\frac{h(z_1) - h(v_i)}{\|z_1 - v_i\|} = \max \{ \frac{h(y) - h(v_i)}{\|y - v_i\|} : y \in S(v_i) \}.
\]
Thus
\[
\frac{h(z_1) - h(v_i)}{\|z_1 - v_i\|} \geq \frac{h(v_i) - h(v_{i-1})}{\|v_i - v_{i-1}\|} = \Delta.
\]
We extend path of greatest \( z_1, z_2, \ldots \) such that \( z_{j+1} \in S(z_j) \) and
\[
\frac{h(z_{j+1}) - h(z_j)}{\|z_{j+1} - z_j\|} = \max \{ \frac{h(y) - h(z_j)}{\|y - z_j\|} : y \in S(z_j) \} \geq \Delta.
\]
This path must terminate with a \( z_m \in V' \).
Since \( \Delta > 0 \), we have
\[
h(z_m) > \ldots > h(v_i) \geq u(v_i) \geq u(v_0).
\]
Thus \( z_m \neq v_0 \).
Finally, consider the path through the vertices
\[
v_0, v_1, \ldots, v_i, z_1, \ldots, z_m.
\]
Set \( z_0 := v_i \). The above path is the connecting path on \( G' \) with respect to \( u \).
Moreover, \( c \) is the largest slope of connecting paths on \( V' \) with respect to \( u \), and
\[
u(v_0) = h(z_0), \quad u(z_m) = h(z_m), \quad h(z_0) = h(v_i) \geq u(v_i),
\]
\[
\frac{h(z_{i+1}) - h(z_i)}{\|z_{i+1} - z_i\|} \geq \Delta, \quad u(v_i) - u(v_0) \geq c, \quad \text{and} \quad \Delta > c.
\]
Thus we have
\[
c \geq \frac{u(z_m) - u(v_0)}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|} + \frac{\sum_{i=0}^{m-1} \|z_{i+1} - z_i\|}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|} \geq \frac{h(z_m) - h(z_0) + u(v_i) - u(v_0)}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|} + \frac{\sum_{i=0}^{m-1} \|z_{i+1} - z_i\|}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|} \geq
\]
\[
\frac{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|} + \frac{\sum_{i=0}^{m-1} \|z_{i+1} - z_i\|}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|} \geq \Delta \frac{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|} + \frac{\sum_{i=0}^{m-1} \|z_{i+1} - z_i\|}{\sum_{i=0}^{l-1} \|v_{i+1} - v_i\|} \geq c.
\]
The last inequality is obtained by $\Delta > c$. Thus we have a contradiction.

- If $v$ is added during the final step of the algorithm. We call $G' = (V', E', \Omega)$ to be the subgraph of $G = (V, E, \Omega)$ when we finish step 1 in the algorithm. Thus there are no connecting paths on $G'$ with respect to $u$. Therefore, $v$ is connected via edges not in $E'$ to exactly one vertex $w$ of $V'$.

We can find the largest connected subgraph $G'' = (V'', E'', \Omega)$ satisfying

$$v, w \in V'', V'' \cap V' = \{w\}, \text{ and } E'' \cap E' = \emptyset.$$  

From the definition of $u$, we have

$$h(w) = u(w) = u(x), \quad \forall x \in V''.$$  

Since $u(v) \neq h(v)$ and $h(w) = u(w) = u(v)$, we have $h(w) \neq h(v)$. Therefore, we must have

$$\sup_{z \in V''} h(z) \neq h(w) \text{ or } \inf_{z \in V''} h(z) \neq h(w).$$

Suppose $\sup_{z \in V''} h(z) \neq h(w)$ (we prove similar for the case $\inf_{z \in V''} h(z) \neq h(w)$). Let $v_0 \in V''$ such that

$$h(v_0) = \sup_{z \in V''} h(z) \neq h(w).$$

Set

$$S''(x) := \{y \in V'': (x, y) \in E''\}, \text{ for } x \in V'' \setminus \{w\},$$  

and

$$S(x) := \{y \in V: (x, y) \in E\}, \text{ for } x \in V \setminus \Omega.$$  

Noting that

$$S(x) = S''(x), \quad \forall x \in V'' \setminus \{w\}. \quad (22)$$

Since $G''$ is a connected graph, there exists a path through the vertices

$$v_0, v_1, ..., v_n, w$$

such that $v_i \in S''(v_{i-1}), \forall i \in \{1, ..., n\}$ and $w \in S''(v_n)$.

On the other hand, from (22) and since $h$ is Kirszbraun extension, we have

$$h(v_0) = \sup_{z \in V''} h(z) \geq \sup_{z \in S''(v_0)} h(z) = \sup_{z \in S(v_0)} h(z).$$

Thus applying Theorem 7 we have

$$h(v_0) = h(s), \quad \forall s \in S(v_0).$$

In particular, we have $h(v_0) = h(v_1)$. By induction, we obtain

$$h(v_0) = h(v_1) = ... = h(v_n) = h(w).$$

This contradicts with $h(w) \neq h(v_0)$.  

\[ \square \]
Remark 10. Assuming Jensen’s hypotheses [5], since this algorithm computes exactly solution of (7) and by using the argument of Le Gruyer [6] (the approximation for AMLE by a sequence Kirszbraun extensions \((u_n)\) of networks \((N_n, U_n)\) \((n \in \mathbb{N})\)), we obtain a new method to approximate the viscosity solution of Equation \(\Delta_{\infty}u = 0\) under Dirichler’s condition \(f\).

Definition 11. For any \(x, y \in V\). There exists a chain \(x_1, ..., x_n \in V\) such that \(x_1 = x, x_n = y\) and \(x_i \in S(x_{i+1})\) for \(i = 1, ..., n - 1\). To any chain we associate its length \(\sum_{i=1}^{n-1} \|x_i - x_j\|\). We define the geodesis metric \(d_g\) of Graph \(G\) by letting \(d_g(x, y)\) be the infimum of the length of chains connecting \(x\) and \(y\).

By using induction respect to increasing sequence of subgraph in the algorithm, we obtain the following theorem.

**Theorem 12.** Let \(u\) be the Kirszbraun extension of \(f\). We have

\[
\sup_{x, y \in V} \frac{\|u(x) - u(y)\|}{d_g(x, y)} \leq \sup_{x, y \in \Omega} \frac{\|f(x) - f(y)\|}{d_g(x, y)},
\]

and

\[
\inf_{z \in \Omega} f(z) \leq u(x) \leq \sup_{z \in \Omega} f(z), \quad \forall x \in V.
\]

4 Method to find \(K(f, S)(x)\) in general case for any \(m \geq 1\)

We fix \(S = \{p_1, ..., p_N\} \subset \mathbb{R}^n\) and \(f : S \to \mathbb{R}^m\) to be a Lipschitz function. Let \(x \in \mathbb{R}^n \setminus S\). We denote

\[
\lambda(f, S)(x) := \inf_{y \in \mathbb{R}^m} \sup_{i \in \{1, ..., N\}} \frac{\|y - f(p_i)\|}{\|x - p_i\|}.
\]

By applying Kirszbraun’s theorem (see [3] [7]) we have \(\lambda \leq \text{Lip}(f, S)\).

In this section, we show a method to compute \(\lambda(f, S)(x)\) and \(K(f, S)(x)\) given by (5).

We recall some results that will be useful in this section.

**Lemma 13.** ([3 Lemma 2.10.40]) There exists a unique \(y(x) \in \mathbb{R}^m\) such that

\[
\lambda(f, S)(x) = \sup_{a \in S} \frac{\|f(a) - y(x)\|}{\|a - x\|}, \tag{23}
\]

and \(y(x)\) belongs to the convex hull of the set

\[
B = \{f(z) : z \in S \text{ and } \frac{\|f(z) - y(x)\|}{\|z - x\|} = \lambda(f, S)(x)\}.
\]

Moreover, from the definition of \(K(f, S)(x)\), we have \(K(f, S)(x) = y(x)\).
To compute the value of $K(f, S)(x)$ we need some properties of Cayley-Menger determinant. We recall some definitions and basic results.

Let $x_1, ..., x_k \in \mathbb{R}^n$. We define the Cayley-Menger determinant of $(x_i)_{i=1}^k$ as

$$\Gamma(x_1, ..., x_k) := \det \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & \|x_1 - x_2\|^2 & \cdots & \|x_1 - x_k\|^2 \\ 1 & \|x_2 - x_1\|^2 & 0 & \cdots & \|x_2 - x_k\|^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \|x_k - x_1\|^2 & \|x_k - x_2\|^2 & \cdots & 0 \end{pmatrix}.$$ 

**Definition 14.** A $k$-simplex is a $k$-dimensional polytope which is the convex hull of its $k+1$ vertices. More formally, suppose the $k+1$ points $u_0, ..., u_k \in \mathbb{R}^n$ are affinely independent, which means $u_1 - u_0, ..., u_k - u_0$ are linearly independent. Then the $k$-simplex determined by them is the set of points

$$C = \{t_0u_0 + \cdots + t_ku_k : t_i \geq 0, 0 \leq i \leq k, \sum_{i=0}^k t_i = 1\}.$$ 

**Example 15.** A 2-simplex is a triangle, a 3-simplex is a tetrahedron.

The $k$-simplex and the Cayley-Menger determinant have beautiful relations by following theorem:

**Theorem 16.** [2, Lemma 9.7.3.4] Let $(x_i)_{i=1,...,k+2} \in \mathbb{R}^n$ be arbitrary points in $k$-dimensional Euclidean affine space $X$. Then $\Gamma(x_1, ..., x_{k+2}) = 0$. A necessary and sufficient condition for $(x_i)_{i=1,...,k+1}$ to be a $k$-simplex of $X$ is that $\Gamma(x_1, ..., x_{k+1}) \neq 0$.

**Lemma 17.** Let the point $u$ lie in the convex hull of the points $q_0, q_1, ..., q_s$ of $\mathbb{R}^m$. If $u'$ distinct from $u$, then for some $i$:

$$\|u - q_i\| \leq \|u' - q_i\|.$$ 

**Proof.** Choose $H$ to be the $(m-1)$-dimension (or hyperplane) through $u$ which is perpendicular to the segment $[u, u']$. Then for at least one value for $i$, $q_i$ must lie in the halfspace of $H$ which does not contain $u'$. Thus we have

$$\|u - q_i\| \leq \|u' - q_i\|.$$

**Proposition 18.** Suppose there exist $J \subset \{1, 2, ..., N\}$, $f_0$ inside convex hull of $\{f(p_j)\}_{j \in J}$ and $\lambda_0 > 0$ such that

$$\|f_0 - f(p_j)\| = \lambda_0 \|x - p_j\|, \quad \forall j \in J$$

and

$$\|f_0 - f(p_i)\| \leq \lambda_0 \|x - p_i\|, \quad \forall i \in \{1, ..., N\},$$

then $\lambda_0 = \lambda(f, S)(x)$ and $f_0 = K(f, S)(x)$. 

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Proof. We have

\[ \lambda_0 = \sup_{i \in \{1, \ldots, N\}} \frac{\|f_0 - f(p_i)\|}{\|x - p_i\|} \geq \inf_{y \in \mathbb{R}^m} \sup_{i \in \{1, \ldots, N\}} \frac{\|y - f(p_i)\|}{\|x - p_i\|} = \lambda(f, S)(x). \]

On the other hand, for any \( y \in \mathbb{R}^m \), by applying Lemma 17 there exists \( i \in J \) such that

\[ \|y - f(p_i)\| \geq \|f_0 - f(p_i)\| = \lambda_0 \|x - p_i\|. \]

Hence

\[ \sup_{i \in \{1, \ldots, N\}} \frac{\|y - f(p_i)\|}{\|x - p_i\|} \geq \lambda_0. \quad (24) \]

Since Inequality (24) is true for any \( y \in \mathbb{R}^m \), we have \( \lambda(f, S)(x) \geq \lambda_0 \). Thus \( \lambda(f, S)(x) = \lambda_0 \).

Therefore, we have

\[ \lambda(f, S)(x) = \sup_{i \in \{1, \ldots, N\}} \frac{\|f_0 - f(p_i)\|}{\|x - p_i\|}. \]

From Lemma 13 we have \( f_0 = K(f, S)(x) \).

A method to compute \( K(f, S)(x) \)

Recall that \( f : S \to \mathbb{R}^m \). By applying Lemma 13 we have

\[ \|f(a) - K(f, S)(x)\| \leq \lambda(f, S)(x)\|a - x\|, \quad \forall a \in S. \]

Moreover,

\[ B = \left\{ f(a) : a \in S \text{ and } \frac{\|f(a) - K(f, S)(x)\|}{\|a - x\|} = \lambda(u, S)(x) \right\}, \]

is not empty, and \( K(f, S)(x) \) belongs to the convex hull of \( B \).

Therefore, there exist \( \{f(p_{i_k})\}_{k=1,\ldots,l+1} \subset f(S) \) such that

(I) \( l \leq m \), where \( m \) is dimension of \( \mathbb{R}^m \);

(II) \( \{f(p_{i_k})\}_{k=1,\ldots,l+1} \) is a \( l \)-simplex. From Theorem 16 \( \{f(p_{i_k})\}_{k=1,\ldots,l+1} \) is a \( l \)-simplex to be equivalent to

\[ \Gamma(K(f, S)(x), f(p_{i_1}), \ldots, f(p_{i_{l+1}})) \neq 0; \quad (25) \]

(III) \( K(f, S)(x) \) belongs convex hull of \( \{f(p_{i_k})\}_{k=1,\ldots,l+1} \);

(IV)

\[ \|K(f, S)(x) - f(p_{i_k})\| = \lambda(f, S)(x)\|x - p_{i_k}\|, \quad \forall k = 1, \ldots, l + 1. \quad (26) \]

(V)

\[ \|f(a) - K(f, S)(x)\| \leq \lambda(f, S)(x)\|a - x\|, \quad \forall a \in S. \]

From the above observations, we obtain
Theorem 19. There exist \( \{f(p_i)\}_{k=1,...,l+1} \subset f(S) \) (1 \( \leq l \leq m \), where \( m \) is dimension of \( \mathbb{R}^m \)), \( f_{ij} \in \mathbb{R}^m \) and \( \lambda_{ij} \in \mathbb{R} \) satisfying some following properties

(a) \( f_{ij} \) inside convex hull of \( \{f(p_i)\}_{k=1,...,l+1} \).

(b) \( \Gamma(f(p_1), f(p_2), ..., f(p_{l+1})) \neq 0 \).

(c) \( \|f_{ij} - f(p_k)\| = \lambda_{ij} \|x-p_k\|, \forall k \in \{i_1, i_2, ..., i_{l+1}\} \).

(d) \( \|f_{ij} - f(p_k)\| \leq \lambda_{ij} \|x-p_k\|, \forall k \in \{1, ..., N\} \).

Moreover, from Proposition [18] we have \( f_{ij} = K(f,S)(x) \) and \( \lambda_{ij} = \lambda(f,S)(x) \).

Therefore, to compute the value of \( \lambda(u,S)(x) \) and \( K(u,S)(x) \), we need to find \( \{f(p_i)\}_{k=1,...,l+1} \subset f(S) \), \( f_{ij} \in \mathbb{R}^m \) and \( \lambda_{ij} \in \mathbb{R} \) satisfying the conditions (a),(b),(c),(d). We can do that step by step as follows

*Step 1:* For all \( i, j \in \{1, ..., N\}, (i \neq j) \). Let

\[
\begin{align*}
\lambda_{ij} & \equiv \frac{\|f(p_i) - f(p_j)\|}{\|x-p_i\| + \|x-p_j\|}, \\
\end{align*}
\]

We have \( f_{ij} \) inside convex hull of \( \{f(p_i), f(p_j)\} \) and

\[
\|f_{ij} - f(p_k)\| = \lambda_{ij} \|x-p_k\|, \quad \forall k \in \{i, j\}.
\]

Test the following condition

\[
\|f_{ij} - f(p_k)\| \leq \lambda_{ij} \|x-p_k\|, \quad \forall k \in \{1, ..., N\} 
\]

(27)

If \( (i, j) \) satisfies the above condition, then from Proposition [18] we have \( f_{ij} = K(f,S)(x) \) and \( \lambda_{ij} = \lambda(f,S)(x) \). We finish. If there is no \( (i, j) \in \{1, ..., N\}, (i \neq j) \) that satisfies the above condition, then we go to step 2.

*Step 2:* For all \( (i, j, k) \in \{1, ..., N\} \times \{1, ..., N\} \times \{1, ..., N\} \). Test the following condition

\[
\Gamma(f(p_i), f(p_j), f(p_k)) = 0.
\]

(28)

Let \( A \) is the set of all \( (i, j, k) \) that satisfies (28). We consider a \( (i, j, k) \in A \). Thus from Theorem [10] we have

- \( \{f(p_i), f(p_j), f(p_k)\} \) is 2-simplex.
- For any \( f_{ijk} \) inside convex hull of \( \{f(p_i), f(p_j), f(p_k)\} \) we have

\[
\Gamma(f_{ijk}, f(p_i), f(p_j), f(p_k)) = 0.
\]

We consider the following equations

\[
\begin{cases}
\Gamma(f_{ijk}, f(p_i), f(p_j), f(p_k)) = 0; \\
\|f_{ijk} - f(p_l)\| = \lambda_{ijk} \|x-p_l\|, \quad \forall l \in \{i, j, k\}; \\
\end{cases}
\]

We replace \( \|f_{ijk} - f(p_l)\| \) by \( \lambda_{ijk} \|x-p_l\| \) into the equation

\[
\Gamma(f_{ijk}, f(p_i), f(p_j), f(p_k)) = 0.
\]

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We obtain that

\[
0 = \Gamma(f_{i,j,k}, f(p_i), f(p_j), f(p_k))
\]

\[
= \det \begin{pmatrix}
0 & 1 & 1 & 1 \\
1 & 0 & \|f_{ijk} - f_i\|^2 & \|f_{ijk} - f_j\|^2 & \|f - f_k\|^2 \\
1 & \|f_i - f_j\|^2 & 0 & \|f_i - f_k\|^2 & \|f - f_k\|^2 \\
1 & \|f_j - f_k\|^2 & \|f_k - f_i\|^2 & 0 & \|f - f_k\|^2 \\
1 & \|f_k - f_{ijk}\|^2 & \|f_k - f_j\|^2 & \|f_k - f_j\|^2 & 0
\end{pmatrix}
\]

\[
= a(x)\lambda^4 + b(x)\lambda^2 + c(x),
\]

where \(a(x), b(x), c(x)\) are function only depending on \(x\) and initial data \(x_l, f(p_l)\) for \(l \in \{i, j, k\}\).

By solving the equation

\[
a(x)\lambda^4 + b(x)\lambda^2 + c(x) = 0,
\]

we obtain that \(\lambda_{ijk}\) is a positive real root of the above polynomial. It maybe that Equation (29) have no any positive real root. In this case, we consider another \((i', j', k') \in A\) until Equation (29) with respect to \((i', j', k')\) have a positive real root. We call \(L\) is the set of all positive real root of equation (29).

Let \(\lambda_{ijk} \in L\). We find \(f_{ijk}\) by solving the equations

\[
\|f_{ijk} - f(p_i)\| = \lambda_{ijk}\|x - p_i\|, \quad \forall l \in \{i, j, k\}. \tag{30}
\]

After that, we test the condition \(f_{ijk}\) in convex hull of \(f(p_i)\) for \(l \in \{i, j, k\}\), and test the following condition

\[
\|f_{ijk} - f(p_l)\| \leq \lambda_{ijk}\|x - p_l\|, \quad \forall l \in \{1, ..., N\}. \tag{31}
\]

If we has a \(\lambda_{ijk} \in L\) such that \(f_{ijk}\) in convex hull of \(f(p_i)\) satisfying Equations (30) and Inequalities (31) then from Proposition 18 we have \(f_{ijk} = K(f, S)(x)\) and \(\lambda_{ijk} = \lambda(f, S)(x)\). We finish. If there is no \((i, j, k) \in A\) that satisfies the above conditions, then we go to step 3.

**Step 3:** By the similar way as step 2 for \((i, j, k, l), (i, j, k, l, h), ...\) until we can find a \((i_1, ..., i_k) \subset \{1, ..., N\}\) such that \(f_{i_1i_2...i_k}\) and \(\lambda_{i_1i_2...i_k}\) satisfying some following properties

(a) \(f_{i_1i_2...i_k}\) inside convex hull of \(f(p_{i_1}), ..., f(p_{i_k})\)
(b) \(\Gamma(f(p_{i_1}), f(p_{i_2}), ..., f(p_{i_k})) \neq 0\).
(c) \(\|f_{i_1i_2...i_k} - f(p_l)\| = \lambda_{i_1i_2...i_k}\|x - p_l\|, \quad \forall l \in \{i_1, i_2, ..., i_k\}\).
(d) \(\|f_{i_1i_2...i_k} - f(p_l)\| \leq \lambda_{i_1i_2...i_k}\|x - p_l\|, \quad \forall l \in \{1, ..., N\}\).

By applying Proposition 18 we obtain \(f_{i_1i_2...i_k} = K(f, S)(x)\) and \(\lambda_{i_1i_2...i_k} = \lambda(f, S)(x)\).

**Remark 20.** By applying theorem 19, this method terminates when \(k = l + 1 \leq m + 1\), where \(m\) is dimension of \(\mathbb{R}^m\).
Remark 21. In step 3, when we solve $f_{i_1,i_2...i_k}$ by considering the equation

$$\Gamma(f_{i_1,i_2...i_k}, f(p_{i_1}), f(p_{i_2}), ..., f(p_{i_k})) = 0,$$

by replacing $\|f_{i_1,i_2...i_k} - f(p_l)\|$ by $\lambda_{i_1,i_2...i_k} \|x - p_l\|$, for $l \in \{i_1, i_2, ..., i_k\}$, this equation is equivalent to

$$a(x)\lambda_{i_1,i_2...i_k}^4 + b(x)\lambda_{i_1,i_2...i_k}^2 + c(x) = 0,$$

(32)

where $a(x), b(x), c(x)$ are function only depending on $x$ and initial data $x_l$, $f(p_l)$ for $l \in \{i_1, ..., i_k\}$. The polynomial $a(x)\lambda_{i_1,i_2...i_k}^4 + b(x)\lambda_{i_1,i_2...i_k}^2 + c(x)$, in fact, is 2-degree polynomial with variable $\lambda = \lambda_{i_1,i_2...i_k}$. Therefore, we can solve Equation (32) very fast to obtain exactly the value of $\lambda_{i_1,i_2...i_k}$.

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