Transverse force on a moving vortex with the acoustic geometry

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Abstract

We consider the transverse force on a moving vortex with the acoustic metric using the \(\phi\)-mapping topological current theory. In the frame of effective spacetime geometry the vortex appear naturally by virtue of the vortex tensor in the Lorentz spacetime and we show that it is just the vortex derived with the order parameter in the condensed matter. With the usual Lagrangian we obtain the equation of motion for the vortex. At last, we show that the transverse force on the moving vortex in our equation is just the usual Magnus force in a simple model.
I. INTRODUCTION

Recently the analog model is very popular for studying the condensed matter physics (see [1] for a review). Since direct experimental probes of many important aspects of general relativity (GR) are extremely difficult, the possibility of using condensed matter system, such as Bose-Einstein condensates (BEC), to mimic certain aspects of GR could prove to be very important [2, 3, 4]. These analog models provide a bridge for interchanging conceptualizations of phenomena between various condensed matter systems and relativistic physics [1, 2, 3, 4, 5, 6, 7, 8, 9], sometimes they illuminate aspects of general relativity and sometimes the machinery of differential geometry can be used to illuminate aspects of the analog model. In this letter, we use mathematical methods developed in the framework of differential geometry to study the transverse force on a moving vortex in Bose-Einstein condensates. On the other hand, we have to mention that these condensed matter systems can also be used to simulate topological defects characteristic of gauge theories and which are considered to have played a cosmological role in the early stages of the evolution of the universe such as monopoles and cosmic strings.

In the analog model, what we are concerned is the propagation of small collective perturbations of the condensate around a background stationary state instead of solving the Gross-Pitaevskii (GP) equation with some given external potential. By virtue of the idea that an effective Lorentzian metric governs perturbative fluctuations, analog models can be given based on acoustic propagation in an irrotational vortex. Thus we can use the method of general relativity to investigate the ”effective space-time geometry” in a constant-speed-of-sound (iso-tachic) and almost incompressible (iso-pycnal) hydrodynamical flows [5]. With the so-called ”effective acoustic metric” the condensates can be regards as Lorentz spacetime [2, 3, 4]. It is natural that the cosmic string in the Lorentz spacetime will appear in such effective spacetime geometry as the vortex. Furthermore, in terms of the energy-momentum tensor we consider the equation of motion for the vortex and calculate the transverse force on a moving vortex in detail. We conclude that the Magnus force can be described with the effective acoustic metric in the frame of general relativity without any concrete model or hypothesis.
II. TOPOLOGICAL VORTEX IN THE BEC

Given the effective acoustic metric, the effective gravity arises in the BEC system [2,3,4], i.e. we can consider such system as the Lorentz spacetime as follows.

Bose–Einstein condensates are most usefully described by the nonlinear Schrödinger equation, also called the Gross–Pitaevskii equation:

$$i\hbar \frac{\partial}{\partial t} \psi(t, \vec{x}) = \left(-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}}(\vec{x}) + \lambda|\psi(t, \vec{x})|^2\right)\psi(t, \vec{x}).$$ (1)

Now use the Madelung representation [10] to put the Schrödinger equation in “hydrodynamic” form:

$$\psi = \sqrt{\rho} \exp(-i\theta m/\hbar).$$ (2)

Take real and imaginary parts: The imaginary part is a continuity equation for an irrotational fluid flow of velocity $\vec{v} \equiv \nabla \theta$ and density $\rho$; while the real part is a Hamilton–Jacobi equation (Bernoulli equation; its gradient leads to the Euler equation). Specifically:

$$\partial_t \rho + \nabla \cdot (\rho \nabla \theta) = 0. \quad (3)$$

$$\frac{\partial}{\partial t} \theta + \frac{1}{2}(\nabla \theta)^2 + \frac{\lambda \rho}{m} - \frac{\hbar^2}{2m^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} = 0. \quad (4)$$

That is, the nonlinear Schrödinger equation is completely equivalent to irrotational inviscid hydrodynamics with a particular form for the enthalpy

$$h = \int \frac{d\rho}{\rho} = \frac{\lambda \rho}{m}. \quad (5)$$

plus a peculiar derivative self-interaction:

$$V_Q = -\frac{\hbar^2}{2m^2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}. \quad (6)$$

The equation of state for this “quantum fluid” is calculated from the enthalpy

$$p = \frac{\lambda \rho^2}{2m}. \quad (7)$$

The corresponding speed of sound is

$$c_s^2 = \frac{dp}{d\rho} = \frac{\lambda \rho}{m}. \quad (8)$$
The disturbances propagate in an effective spacetime with metric $g_{\mu\nu}$, which was shown to be of the Painleve-Gullstrand form \cite{5, 6}

$$g_{00} = -\frac{\rho}{c}[c^2 - v^2], \quad g_{0i} = -\frac{\rho}{c}v_i, \quad g_{ij} = \frac{\rho}{c}\delta_{ij},$$

where the velocity $c$ plays the role of the speed of light and is equal to the sound speed for phonons. The metric has spacetime interval

$$ds^2 = \rho \left[ -c^2 dt^2 + \delta_{ij}(dx^i - v^i dt)(dx^j - v^j dt) \right],$$

where the indices on the background velocity $v^i$ are always raised and lowered using the flat 3-dimensional Cartesian metric, i.e. $v^i = v_i$ and $v^2 = v^i v_i$.

Consider the physical situation that the speed of sound is iso-tachic and independent of position and time, we can choose co-ordinates to set the speed $c$ of linear quasiparticle dispersion equal to unity. The (3+1)-dimensional Painleve-Gullstrand metric then reads \cite{5}

$$g_{\mu\nu} = \rho \begin{pmatrix} -1 + v^2 & -\vec{v} \\ -\vec{v} & 1 \end{pmatrix}$$

The inverse metric is

$$g^{\mu\nu} = \frac{1}{\rho} \begin{pmatrix} -1 & -v^j \\ -v^i & \delta^{ij} - v^i v^j \end{pmatrix}$$

Since the general relativity can be described with the tetrad field as the SO(3,1) gauge theory, the above BEC system should possess the similar effective tetrad field $e^a_\mu$ ($a$ and $\mu$ are SO(3,1) and space-time indices, respectively). As we known, $\omega_{\mu ab}$ is the connection of Lorentz group gauge theory

$$D_\mu \phi_a = \partial_\mu \phi_a - \omega_{\mu ab} \phi_b, \quad \mu = 0, 1, 2, 3, \quad a, b = 1, 2, 3, 4$$

and the corresponding pure connection is defined as

$$\omega_{abc} = e^A_\mu \omega_{\mu bc}.$$ 

With this connection the vortex tensor is proposed as

$$F_{\mu\nu} = e_{\nu a} D_\mu \omega_a - e_{\mu a} D_\nu \omega_a$$

where $\omega_a = \omega_{bab}$. For Lorentz spacetime, the torsion should be zero, i.e.

$$T_{\mu\nu a} = D_\mu e_{\nu a} - D_\nu e_{\mu a} = 0,$$
then, the vortex tensor

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{9} \]

where \( A_\mu = e_{\mu a} \omega_a \) can be looked as a U(1) connection.

In terms of Eqs. (9) the topological charge of vortex can be found

\[ q = \int_\Sigma F_{\mu\nu} dx^\mu \wedge dx^\nu. \tag{10} \]

As one has shown in [11], the U(1) gauge potential can be decomposed by the 2-dimensional unit vector fields \( n^A \) (\( n^A = \phi^A / \sqrt{\phi^B \phi^B} \)) as

\[ A_\mu = \frac{1}{2\pi} \varepsilon_{AB} n^A \partial_\mu n^B. \tag{11} \]

One can find that the charge of vortex can be expressed as

\[ q = \frac{1}{2\pi} \int_\Sigma \varepsilon_{AB} \partial_\mu n^A \partial_\nu n^B dx^\mu \wedge dx^\nu. \tag{12} \]

Following the \( \phi \)-mapping theory, it can be rigorously proved that

\[ q = \int_\Sigma \delta^2(\tilde{\phi}) D^\mu(\frac{\phi}{x}) dx^\mu \wedge dx^\nu, \tag{13} \]

where \( \Sigma \) is an arbitrary 2-dimensional surface,

\[ x^\mu = x^\mu(u^1, u^2), \tag{14} \]

and \( D^{\mu\nu}(\frac{\phi}{x}) \) is the tensor Jacobian which defined as

\[ \varepsilon^{AB} D^{\mu\nu} \left( \frac{\phi}{x} \right) = \varepsilon^{\mu\nu\lambda\sigma} \partial_\lambda \phi^A \partial_\sigma \phi^B. \tag{15} \]

The integral \( q \) can be rewritten with the usual Jacobian \( D(\phi/u) \)

\[ q = \int_\Sigma \delta^2(\tilde{\phi}) D(\frac{\phi}{u}) \sqrt{g_u} d^2 u. \tag{16} \]

We find that \( q \neq 0 \), only when

\[ \phi^A(x, t) = 0, \quad A = 1, 2. \tag{17} \]

Then solutions of above equations are vortices

\[ S_\alpha : \quad x^\mu = z^\mu_\alpha(\sigma, \tau), \quad \alpha = 1, 2, ..., l \tag{18} \]
which are the worldlines of vortices.

Using the $\phi$-mapping topological current theory we have

$$ q = \int_{\Sigma} \sum_{\alpha=1}^{l} W_{\alpha} \delta^2 (u^i - z^i_{\alpha}) d^2 u $$

(19)

where $z^i_{\alpha}(\alpha = 1, 2, ..., l)$ are the intersection points of vortices $S_{\alpha}$ with surface $\Sigma$ and $W_{\alpha}$ is the winding number. Then we find

$$ q = \sum_{\alpha=1}^{l} W_{\alpha}. $$

(20)

This is just our $\phi$-mapping topological current theory of vortices in the frame of effective acoustic geometry which shows that the vortices appear naturally in the hydrodynamical flows and the charge of vortices are topologically quantized by winding number. In the other word, the vortices will emerge in all those systems which can be described with the effective Lorentz geometry with the above discussion.

At the other hand, the vortex can be discussed with the order parameter $\psi(\vec{r}, t)$ directly. Starting from the Gross-Pitaevskii equation, one can construct the vortex current with the condensed wave function $\psi$

$$ \vec{j} = \frac{m}{\hbar} \nabla \times \vec{v}, $$

where the current velocity

$$ \vec{v} = -\frac{i\hbar}{2m}(\psi^* \nabla \psi - \psi \nabla \psi^*)/|\psi|^2, $$

is just the background velocity.

It is well known that the condensed wave function $\psi$ can be looked upon as a section of a complex line bundle with base manifold $M$ (in this paper $M = R^3 \otimes R$). Denoting the condensed wave function $\psi$ as

$$ \psi(\vec{x}, t) = \phi^1(\vec{x}, t) + i\phi^2(\vec{x}, t), $$

where $\phi^1(\vec{x})$ and $\phi^2(\vec{x})$ are two components of a two-dimensional vector field

$$ \vec{\phi} = (\phi^1, \phi^2) $$

in the (3+1)-dimensional space-time. The vortex current in the 3-dimensional space can be obtained

$$ j^i = \frac{1}{2\pi} \varepsilon^{ijk} \varepsilon_{AB} \partial_\nu n^A \partial_\lambda n^B, \quad i, j, k = 1, 2, 3 $$

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where $n^A$ is the two-dimensional unit vector field of the complex scalar field:

$$n^A = \frac{\phi^A}{||\phi||}, \quad ||\phi||^2 = \phi^A\phi^A, \quad A = 1, 2.$$ 

It is clear that the topological current is identically conserved \[12\], i.e.

$$\partial_i j^i = 0. \quad (21)$$

By making use of the $\phi$-mapping method, this topological current can be rewritten in a compact form \[13\],

$$j^i = D^i(\frac{\phi}{x})\delta(\vec{\phi}), \quad (22)$$

where $D^i(\frac{\phi}{x})$ is the vector Jacobians of $\phi(x):

$$D^i(\frac{\phi}{x}) = \frac{1}{2} \varepsilon^{ijk} \varepsilon_{AB} \partial_j \phi^A \partial_k \phi^B.$$ 

Thus we have the important relation between the topological current and the condensed wave function $\psi(\vec{x})$ in the Bose-Einstein condensation system. With this topological current the corresponding vorticity $\Gamma = \oint \vec{v} \cdot d\vec{l}$ can be given

$$\Gamma = \frac{h}{m} \sum_{\alpha=1}^{l} W_\alpha = q \frac{h}{m}.$$ 

One can find easily that the vortex in the frame of the effective acoustic geometry is the same vortex given with the order parameters. That means that we can discuss this kinds of topological defects in the condensed matter using the method of general relativity by virtue of the effective acoustic metric.

It is convenient to generalize the vortex current into the (3+1)-dimensional spacetime

$$j^{\mu\nu} = \varepsilon^{\mu\nu\lambda\rho} F_{\lambda\rho}. \quad (23)$$

It is easy to prove that

$$j^{\mu\nu} = \delta(\vec{\phi}) D^{\mu\nu}(\phi/x),$$

where $D^{\mu\nu}(\phi/x)$ is the tensor Jacobian

$$\varepsilon_{\mu\nu\lambda\rho} D^{\mu\nu}(\phi/x) = \varepsilon_{ab} \partial_\lambda \phi^a \partial_\rho \phi^b.$$ 

Based on this generalized vortex current we can consider the equation of emotion of vortex in the effective spacetime.
III. EQUATION OF EMOTION WITH THE ENERGY-MOMENTUM TENSOR

In the above section, we show the vortices exist in the effective acoustic spacetime geometry and their charges are quantized in the level of topology. With the vortex current $j^{\mu\nu}$, we can define the Lagrangian in the effective spacetime

$$L = T \sqrt{\frac{1}{2} g_{\mu\rho} j^{\mu\lambda} j^{\nu\rho}} = T \sqrt{\frac{1}{2} j^{\mu\nu} j^{\mu\nu}}$$

which is the generalization of Nielsen’s Lagrangian, where $T$ is a constant with dimension of $[\text{mass}]^2$. It is easy to obtain the energy-momentum tensor

$$T^{\mu\nu} = T \delta(\phi) D(\phi/x) g^{IJ} \frac{\partial x^\mu}{\partial u^I} \frac{\partial x^\nu}{\partial u^J}, \quad (23)$$

which shows that the tensor only appear in the zeroes of the order parameter field $\phi(x)$, i.e. the position of the vortices. Since the vortex is the only quasiparticles in the flowing fluids and offer the role of the matter term in the effective acoustic geometry, Eq. (23) is natural result for the energy-momentum tensor. From the principle of the least action or the formula $\nabla_{\mu} T^{\mu\nu} = 0$ we can obtain the equation of motion

$$\frac{1}{\sqrt{-g_u}} \frac{\partial}{\partial u^I} (\sqrt{-g_u} g^{IJ} \frac{\partial x^\lambda}{\partial u^J}) + \Gamma^\lambda_{\mu
u} g^{IJ} \frac{\partial x^\mu}{\partial u^I} \frac{\partial x^\nu}{\partial u^J} = 0,$$

which is the basic equations for us to discuss the transverse force on the moving vortex. If we choose the conformal gauge, the equation have the simple form

$$\partial_I \partial_J x^\lambda + \Gamma^\lambda_{\mu
u} \frac{\partial x^\mu}{\partial u^I} \frac{\partial x^\nu}{\partial u^J} = 0. \quad (24)$$

In the following we consider a simple model that the system only include one vortex and the background velocity is along with the $x$-direction, i.e. the velocity $\vec{v} = (v_1, 0, 0)$. With this 3-velocity the connection coefficients read

$$\Gamma^2_{00} = -\frac{1}{2} \partial_2 v_1^2, \quad \Gamma^2_{10} = \frac{1}{2} \partial_2 v_1. \quad (25)$$

Since we discuss the transverse force on the vortex, we choose an ideal model that the vortex coordinate $x^\mu$ satisfies

$$x^3 = \sigma, \ x^1 = x^1(\tau), \ x^2 = x^2(\tau),$$

and the vortex moves in the $x$-direction only

$$v_{\text{vortex}} = \dot{x}^1.$$
Then the Eq. (24) gives
\[ \partial_r \partial_r x^2 + (\Gamma^2_{00} \partial_r x^0 \partial_r x^0 + 2\Gamma^2_{10} \partial_r x^1 \partial_r x^0) = 0, \]
which can be calculate by virtue of the connection (25)
\[ x^2 - v_1 \partial_2 v_1 + \partial_2 v_1 \dot{x}^1 = 0, \]
i.e.
\[ \ddot{x}^2 = \partial_2 v_1 (v_1 - v_{vortex}). \]
Thus the transverse force \( F_t \) on the vortex is obtained with the effective acoustic geometry
\[ F_t = \rho_s \ddot{x}^2 = \rho_s \Omega (v_{vortex} - v_n), \]
where \( \rho_s \) is the density of fluids, \( \Omega = \partial_1 v_2 - \partial_2 v_1 = -\partial_2 v_1 \) is the vorticity of the fluid and \( v_n = v_1 \) means the background velocity of fluid. The last equation show that the vortex moving in the fluids will be exerted a transverse force, just the Magnus force.

IV. CONCLUSION

In this paper, we discuss the vortex in the BEC using the method of effective acoustic geometry. Instead of solving the concrete GP equation with some given external potential \( V_{ext}(x) \), we study the topological structure of vortices from the viewpoint of spacetime defects. We show that the vortex appears naturally in such a effective Lorentz spacetime and the charge of vortex is quantized in the topological level. Then, the energy-momentum tensor is given with the vortex current in the effective spacetime. The equation of vortex motion is derived in terms of the energy-momentum tensor in the frame of general relativity. Furthermore, we consider the transverse force on the vortex with this equation of motion. A simple model is calculated with the expression of transverse force by virtue of the effective acoustic metric. We find the transverse force in the frame of effective geometry is just the usual Magnus force upon a moving vortex.

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