On Gel’fand-Kolmogoroff type results

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August 4, 2020

Abstract

We prove that a vector bundle $E \to M$ is characterized by the associative structure of the symbols of the Lie algebra generated by all differential operators on $E$ which are eigenvectors of the Lie derivative in the direction of the Euler vector field.

We have also obtained similar result with the $\mathbb{R}$– algebra of smooth functions which are polynomial along the fibers of $E$.

Introduction

An associative algebra $\mathcal{A}(M)$ characterizes a topological space $M$ if, for any topological space $N$; the associative algebras $\mathcal{A}(M)$ and $\mathcal{A}(N)$ are isomorphic if, and only if, the topological spaces $M$ and $N$ are homeomorphic.

The algebraic characterization of topological spaces dates back to the end of the 30s of the last century with I. Gel’fand and A. Kolmogoroff in : [1]. It is established in this paper that a compact topological space $M$ is characterized by the associative algebra $C(M)$ of continuous functions on $M$ with values in $\mathbb{R}$ or $\mathbb{C}$.

The methods developed in [1] can be applied to the case of a differential manifold $M$, assumed of Hausdorff and with a countable basis, by considering the algebra $\mathcal{A}(M) = C^\infty(M)$ of smooth functions on $M$.

Thanks to a result due to Milnor, we know that any isomorphism of associative algebras $\Psi : \mathcal{A}(M) \to \mathcal{A}(N)$ is of the form

$$\Psi : f \mapsto f \circ \psi,$$

with $\psi : N \to M$ a diffeomorphism of differential manifolds.
The Lie-algebraic characterization of differential manifolds appeared for the first time in 1954 in a paper of P.E. Pursell and M.E. Shanks.

One way to generalize these results is to establish characterizations with Lie algebras larger than that of vector fields of a manifold. This is what J. Grabowski and N. Poncin proposed in [2,3,4]. We can also obtain a generalization of the result of Pursell and Shanks by characterizing vector bundles by Lie algebras. In this way, let us cite [5,9,10,11]. In [5] it is established that, under certain assumptions, a vector bundle can be characterized by the Lie algebra of its infinitesimal automorphisms.

1 From results due to Grabowski and Poncin

Let \( M \) be a manifold. We know that the algebras \( D(M) \) and \( S(M) \) characterize the manifold \( M \) by their respective structures of Lie algebras. This remains true if we consider their respective subalgebras \( D^1(M) \) and \( S^1(M) \). For their structures of \( \mathbb{R} \)-associative algebras, the following result provides a partial answer to the question.

**Theorem 1.1** Let \( M \) and \( N \) be differential manifolds. The \( \mathbb{R} \)-associative algebras \( D(M) \) and \( D(N) \) (respectively \( D^1(M) \) and \( D^1(N) \)) are isomorphic if, and only if, the differential manifolds \( M \) and \( N \) are diffeomorphic.

**Proof.** The assertion comes from the fact that any isomorphism of associative algebras between these algebras preserves their Lie bracket, the latter being nothing other than the bracket of commutators.

For the Poisson algebras \( S(M) \) and \( S^1(M) \), obtaining the characterization of the variety \( M \) by their associative structure requires another theoretical approach.

2 Some notions of algebra

In this part, we present some notions on the \( \mathbb{R} \)-commutative associative algebras which we use in the following, to obtain results of the Gel’fand-Kolmogoroff type. They are mainly taken from [13].
Let $\mathcal{F}$ be a $\mathbb{R}$–commutative associative algebra with unit. In the following lines, we will simply say that $\mathcal{F}$ is a $\mathbb{R}$–algebra. We denote by $|\mathcal{F}| := M$, the set of all homomorphisms of $\mathbb{R}$–algebras of $\mathcal{F}$ in $\mathbb{R}$:

$$M \ni x : \mathcal{F} \to \mathbb{R} : f \mapsto x(f).$$

The elements of $M$ are then called $\mathbb{R}$–points of the algebra $\mathcal{F}$ and the set $|\mathcal{F}|$, the dual space of $\mathbb{R}$–points.

A $\mathbb{R}$–algebra $\mathcal{F}$ is said to be geometric if the subset

$$\mathcal{I}(\mathcal{F}) = \bigcap_{x \in |\mathcal{F}|} \text{Ker} x$$

contains only the null element.

**Proposition 2.1** If $\mathcal{F}$ is a $\mathbb{R}$–algebra of functions over a given set $N$, then $\mathcal{F}$ is geometric.

**Proof.** Let us specify that by function on a set $N$ we mean a function $f : N \to \mathbb{R}$. Consider the application $\theta : N \to |\mathcal{F}|$ associating to any point $a \in N$, the homomorphism of $\mathbb{R}$–algebras $\mathcal{F} \to \mathbb{R} : f \mapsto f(a)$. In other words,

$$\theta(a)(f) = f(a), \quad \forall a \in N, \forall f \in \mathcal{F},$$

and we have that $\theta(a)$ is a homomorphism of $\mathbb{R}$–algebras. Observe that through the following definition

$$f(x) := x(f), x \in |\mathcal{F}|, f \in \mathcal{F},$$

any element of $\mathcal{F}$ can be seen as a function on the dual space $|\mathcal{F}|$. In particular, we have

$$f(\theta(a)) = \theta(a)(f) = f(a).$$

Therefore, if an element $f \in \mathcal{F}$ vanishes in $\theta(a)$, for all $a \in N$, then $f$ is the null element of the $\mathbb{R}$–algebra $\mathcal{F}$. From the inclusion

$$\bigcap_{x \in |\mathcal{F}|} \text{Ker} x \subset \bigcap_{a \in N} \text{Ker} \theta(a)$$

we then deduce that $\mathcal{F}$ is geometric. ■

A $\mathbb{R}$–geometric algebra $\mathcal{F}$ is called $C^\infty$–closed if for any finite collection of elements $f_1, \cdots, f_k \in \mathcal{F}$ and any function $g \in C^\infty(\mathbb{R}^k)$, there exists $f \in \mathcal{F}$ such that

$$f(a) = g(f_1(a), \cdots, f_k(a)), \quad \forall a \in |\mathcal{F}|.$$
Note that such an element \( f \in F \) is uniquely determined by \( f_i \) and \( g \) because \( F \) is geometric.

As an example, for any \( n \in \mathbb{N}_0 \), the \( \mathbb{R} \)-algebra \( C^\infty(\mathbb{R}^n) \) is \( C^\infty \)-closed.

Let \( F \) be a \( \mathbb{R} \)-geometric algebra that we identify with a subset of the algebra of functions over \( |F| \). Consider the set \( \overline{F} \) defined as the set of functions over \( |F| \) that can be written in the form

\[
g(f_1, \ldots, f_k), \quad k \in \mathbb{N}, \ f_i \in F, \ g \in C^\infty(\mathbb{R}^k).
\]

The set \( \overline{F} \) has an obvious structure of \( \mathbb{R} \)-algebra and \( F \) is indeed a subalgebra. By virtue of the previous proposition \( \overline{F} \) is geometric. This algebra is also \( C^\infty \)-closed.

By definition, the \( \mathbb{R} \)-algebra \( \overline{F} \) thus associated with the \( \mathbb{R} \)-geometric algebra \( F \) is called smooth envelope of \( F \). It has the following significant property.

**Proposition 2.2** Let \( F \) be a \( \mathbb{R} \)-geometric algebra and \( \overline{F} \) its smooth envelope. For any homomorphism \( \Psi : F \rightarrow F' \) of \( \mathbb{R} \)-algebras, with \( F' \) a \( \mathbb{R} \)-algebra \( C^\infty \)-closed, there exists a unique homomorphism of \( \mathbb{R} \)-algebras \( \overline{F} \rightarrow F' \) such that \( \Psi = \overline{\Psi} \circ i \), with \( i : F \rightarrow \overline{F} \) the canonical inclusion.

**Proof.** Consider the following relation

\[
\overline{\Psi}(g(f_1, \ldots, f_k)) = g(\Psi(f_1), \ldots, \Psi(f_k))
\]

defined from \( \overline{F} \) in \( F' \). This mapping is an application. This comes directly by using the fact that \( F' \) is geometric and by observing that for all \( a' \in |F'| \), we have \( a' \circ \overline{\Psi} \in |F| \).

For the uniqueness of this application, suppose there is an application \( \Psi : \overline{F} \rightarrow F' \) such that \( \Psi = \overline{\Psi} \circ i \). So for all \( a \in |F| \), we can write

\[
\Psi(g(f_1, \ldots, f_k)(a)) = g(f_1, \ldots, f_k)(a \circ \Psi) = g(i(f_1), \ldots, i(f_k))(a \circ \Psi) = g(f_1, \ldots, f_k)(a \circ \Psi \circ i) = g(f_1, \ldots, f_k)(a \circ \Psi) = g(\Psi(f_1), \ldots, \Psi(f_k))(a)
\]

We deduce the equality \( \overline{\Psi} = \overline{\Psi} \), since \( F' \) is geometric.

The application \( \overline{\Psi} \) thus defined is indeed a homomorphism of \( \mathbb{R} \)-algebras. Indeed, assume that

\[
g(f_1, \ldots, f_j) \cdot g_2(f'_1, \ldots, f'_k) = g''(f''_1, \ldots, f''_k).
\]
Then we have, for any $a \in |F'|$,
\[ \Psi(g''(f'_1, \ldots, f'_r))(a) = g''(\Psi(f'_1), \ldots, \Psi(f'_r))(a) \]
\[ = g''(f'_1, \ldots, f'_r)(a \circ \Psi) \]
\[ = (g(f_1, \ldots, f_j) \cdot g_2(f'_1, \ldots, f'_k))(a \circ \Psi) \]
\[ = (g(f_1, \ldots, f_j)(a \circ \Psi)) \cdot (g_2(f'_1, \ldots, f'_k)(a \circ \Psi)) \]
\[ = (\Psi(g(f_1, \ldots, f_j)) \cdot \Psi(g_2(f'_1, \ldots, f'_k)))(a); \]
which is enough, $F'$ being geometric. 

We then deduce the following result for which a proof can be found in [12].

**Proposition 2.3** Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two $\mathbb{R}$− geometric algebras. If $\Psi : \mathcal{F}_1 \to \mathcal{F}_2$ is an isomorphism of $\mathbb{R}$− algebras, then there exists a unique isomorphism of $\mathbb{R}$− algebras $\hat{\Psi} : \mathcal{F}_1 \to \mathcal{F}_2$ such that $\Psi = \hat{\Psi} \circ i_1$, with $i_1 : \mathcal{F}_1 \to \mathcal{F}_1$ the canonical inclusion.

**Proposition 2.4** Let $\mathcal{F}$ and $\mathcal{G}$ be two $\mathbb{R}$− geometric algebras. We then have the following implication
\[ \mathcal{F} \subset \mathcal{G} \implies \mathcal{F} \subset \mathcal{G}. \]
And the equality
\[ \mathcal{F} = \mathcal{F} \]
is a corollary.

**Proof.** As $\mathcal{F} \subset \mathcal{G}$, denote by $i : \mathcal{F} \to \mathcal{G}$ the canonical injection. According to Proposition 2.2, $i$ extends into a homomorphism of $\mathbb{R}$− algebras $\overline{i} : \mathcal{F} \to \mathcal{G}$. The formula defining $\overline{i}$ given at the beginning of the proof of the same Proposition 2.2 shows that $\overline{i}$ is nothing other than the canonical injection of $\mathcal{F}$ in $\mathcal{G}$.

From the inclusion $\mathcal{F} \subset \mathcal{F}$, we then deduce that $\mathcal{F} \subset \mathcal{F}$. Hence the conclusion.

Let us state a last result in this section and whose proof is given in [13].

**Proposition 2.5** Let $E \to M$ be a vector bundle. The smooth envelope of the $\mathbb{R}$− algebra $\mathcal{A}(E)$ of fiberwise polynomial functions of $E$ is the $\mathbb{R}$− algebra $C^\infty(E)$.

### 3 A Gel’fand-Kolmogoroff type result

For $E \to M$ and $F \to N$ two vector bundles, we establish in the following lines that any isomorphism between the associative algebras $\mathcal{A}(E)$ and $\mathcal{A}(F)$ induces an isomorphism of vector bundles.
Lemma 3.1 let π : E → M and η : F → N be two vector bundles. If Ψ : \(\mathcal{A}(E) \rightarrow \mathcal{A}(F)\) is an isomorphism of associative algebras, then

\[\Psi(A^0(E)) = A^0(F)\]

Proof. For any nonvanishing element \(u \in A^0(E)\), the function \(u^{-1} : e \mapsto \frac{1}{u(e)}\) is still an element of \(A^0(E)\). Since Ψ is an homomorphism, we have

\[A^0(F) \ni 1_F = \Psi(1_E) = \Psi(u.u^{-1}) = \Psi(u).\Psi(u^{-1})\]

This implies that \(Ψ(u)\) and \(Ψ(u^{-1})\) are two nonvanishing functions, fiberwise polynomial with zero degree because their product is fiberwise polynomial of zero degree.

In addition, for any element \(u \in A^0(E)\), the function \(u^2 + 1_E : e \mapsto u(e)^2 + 1\) is nonvanishing and belongs to \(A^0(E)\). It follows that

\[A^0(F) \ni \Psi(u^2 + 1_E) = \Psi(u).\Psi(u) + 1_F,\]

which shows that \(Ψ(u)\) is fiberwise polynomial and of zero degree. We just proved inclusion \(\Psi(A^0(E)) \subset A^0(F)\). The conclusion follows by applying the same reasoning to the inverse homomorphism \(Ψ^{-1}\).

Lemma 3.2 Any difféomorphism \(ϕ : \mathbb{R}^n \rightarrow \mathbb{R}^n\) such that

\[Pol(\mathbb{R}^n) = \{P \circ ϕ : P \in Pol(\mathbb{R}^n)\}\]

is polynomial and its ”linear part” coincides with the automorphism \(ϕ_∗\) of the \(\mathbb{R}\)–vector space \(\mathbb{R}^n\).

Proof. Let us first observe that \(ϕ\) is necessarily polynomial; in the sense that we can write

\[ϕ(y_1, \ldots, y_n) = (ϕ_1(y_1, \ldots, y_n), ϕ_2(y_1, \ldots, y_n), \ldots, ϕ_n(y_1, \ldots, y_n))\]

with, for any \(j \in \{1, \ldots, n\},\)

\[ϕ_j(y_1, \ldots, y_n) = λ_0^j + λ_1^j y_1 + \cdots + λ_n^j y_n + ϕ_{≥2}^j(y_1, \ldots, y_n)\]

where the expression \(ϕ_{≥2}^j(y_1, \ldots, y_n)\) is polynomial in \(y_1, \ldots, y_n\) of degree greater than or equal to 2. Indeed, this comes from the fact that, for any \(j \in \{1, \ldots, n\},\) the map

\[(y_1, \ldots, y_n) \in \mathbb{R}^n \mapsto y_j \in \mathbb{R}\]

is an element of \(Pol(\mathbb{R}^n)\).
We have that for any \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \),
\[
\varphi \ast_y = \begin{pmatrix}
\lambda_1 + \partial_1 \varphi_{\geq 2}(y) & \cdots & \lambda_1 + \partial_n \varphi_{\geq 2}(y) \\
\vdots & \ddots & \vdots \\
\lambda_n + \partial_1 \varphi_{\geq 2}(y) & \cdots & \lambda_n + \partial_n \varphi_{\geq 2}(y)
\end{pmatrix}
\]
is a linear automorphism of \( \mathbb{R}^n \). In particular, for \( y = 0 \), we have that the matrix
\[
A_\varphi = \begin{pmatrix}
\lambda_1 & \cdots & \lambda_1 \\
\vdots & \ddots & \vdots \\
\lambda_n & \cdots & \lambda_n
\end{pmatrix}
\]
is regular. Which completes the proof of the lemma. \( \blacksquare \)

**Theorem 3.3** Let \( E \to M \) and \( F \to N \) be two vector bundles of respective ranks \( n \) and \( n' \). The \( \mathbb{R} \)– algebras \( \mathcal{A}(E) \) and \( \mathcal{A}(F) \), of polynomial functions along the fibers, are isomorphic if and only if the vector bundles \( E \) and \( F \) are isomorphic.

**Proof.** Let \( \Psi : \mathcal{A}(F) \to \mathcal{A}(E) \) be an isomorphism of \( \mathbb{R} \)–algebras. Then, according to Proposition 2.3 and the Proposition 2.5, \( \Psi \) extends to an isomorphism of \( \mathbb{R} \)–algebras \( \overline{\Psi} : C^\infty(F) \to C^\infty(E) \). Therefore, according to Milnor result, there exists a diffeomorphism \( \varphi : E \to F \) such that,
\[
\overline{\Psi}(h) = h \circ \varphi, \quad \forall h \in C^\infty(F).
\]
In particular, since \( \overline{\Psi} \) extends \( \Psi \), we can write
\[
\mathcal{A}(E) = \{ u \circ \varphi : u \in \mathcal{A}(F) \} \tag{3.1}
\]
In view of the Lemma 3.1, we have
\[
\Psi \circ \pi_F^*(C^\infty(N)) = \pi_E^*(C^\infty(M)).
\]
We deduce the existence of a diffeomorphism \( \phi : M \to N \) with
\[
\Psi \circ \pi_F^*(g) = \pi_E^*(g \circ \phi), \quad \forall g \in C^\infty(N)
\]
Therefore, since \( \pi_F^*(g) \circ \varphi = \Psi(\pi_F^*(g)) \), we have, for any \( e \in E \),
\[
g(\pi_F(\varphi(e))) = g(\phi(\pi_E(e))).
\]
It becomes
\[
\pi_F \circ \varphi = \phi \circ \pi_E \tag{3.2}
\]
We have thus just shown that \((\phi, \varphi)\) is an isomorphism of \(E \to M\) on \(F \to N\), seen as differential fibrations, the local trivialization diffeomorphisms being the same as those which make vector bundles. We will therefore assume that the manifolds \(M\) and \(N\) coincide and consider \(\varphi\) as a \(M\)–isomorphism of fibrations between \(E \to M\) and \(F \to M\). Consider now an open cover \((U_{\alpha})\) of \(M\) by trivialization domains of both \(E\) and \(F\), where \(\sigma_{\alpha}\) and \(\rho_{\alpha}\) are the diffeomorphisms of local trivialization relating to \(E\) and \(F\) respectively. We then have, for any pair of indices \((\alpha, \beta)\), the relation
\[
\vartheta_{\beta \alpha} \circ \varphi_{\beta \alpha} = \varphi_{\alpha \beta} \circ \theta_{\beta \alpha}
\]
where \(\vartheta_{\beta \alpha}\) and \(\theta_{\alpha \beta}\) are transition diffeomorphisms associated respectively with \(E\) and \(F\), and,
\[
\varphi_{\beta \alpha} : U_{\alpha \beta} \times \mathbb{R}^n \to U_{\alpha \beta} \times \mathbb{R}^n : (x, y) \mapsto (x, \delta_{\beta \alpha}(x, y))
\]
is the restriction of \(\varphi_{\alpha} = \rho_{\alpha}^{-1} \circ (\varphi|_{\pi^{-1}(U_{\alpha})}) \circ \sigma_{\alpha}\).
We further observe that
\[
\varphi_{\alpha}^x : \mathbb{R}^n \to \mathbb{R}^n : y \mapsto \delta_{\alpha}(x, y)
\]
is a diffeomorphism, polynomial in \(y\), such that
\[
P \in Pol(\mathbb{R}^n) \mapsto P \circ \varphi_{\alpha}^x \in Pol(\mathbb{R}^n)
\]
is an isomorphism of \(\mathbb{R}\)–algebras. We write \(\varphi_{\alpha}^x(y)\) in the form
\[
(\lambda^0_{\alpha,1}(x) + \sum_{i=1}^{n} \lambda^i_{\alpha,1}(x)y^i + \varphi_{\alpha,1}^{\geq 2}(x, y), \ldots, \lambda^0_{\alpha,n}(x) + \sum_{i=1}^{n} \lambda^i_{\alpha,n}(x)y^i + \varphi_{\alpha,n}^{\geq 2}(x, y))
\]
where for any \(i \in [1, n] \cap \mathbb{N}\), \(\varphi_{\alpha,i}^{\geq 2}(x, y)\) is polynomial in \(y\) and each of whose terms is of degree greater than or equal to 2.
According to Lemma 3.2, we observe that \((\lambda_{\alpha,j}^i(x))\) is then a regular matrix. Hence, for any index \(\alpha\), we have a diffeomorphism
\[
\psi_{\alpha} : U_{\alpha} \times \mathbb{R}^n \to U_{\alpha} \times \mathbb{R}^n : (x, y) \mapsto (x, (\lambda_{\alpha,j}^i(x))(y))
\]
such that
\[
\psi_{\alpha}^x : y \mapsto (\sum_{i=1}^{n} \lambda_{\alpha,1}^i(x)y^i, \sum_{i=1}^{n} \lambda_{\alpha,2}^i(x)y^i, \ldots, \sum_{i=1}^{n} \lambda_{\alpha,n}^i(x)y^i)
\]
is an automorphism of the \(\mathbb{R}\)–vector space \(\mathbb{R}^n\).
We deduce from the relation (3.3) that
\[ \theta_{\beta\alpha} \circ \psi_{\beta\alpha} = \psi_{\alpha\beta} \circ \theta_{\beta\alpha}, \]
by setting for any pair of indices \((\alpha, \beta)\),
\[ \psi_{\beta\alpha} = \psi_\alpha |_{U_{\beta\alpha} \times \mathbb{R}^n}. \]
We deduce from the above that there exists a unique \(M\)–isomorphism of a vector bundle \(\psi : E \to F\) such that for any index \(\alpha\)
\[ \psi_\alpha = \rho_{\alpha}^{-1} \circ \psi \circ \sigma_\alpha. \]

**Remark 3.4** Observe that the vector bundles isomorphism \(\psi\), obtained through the previous Theorem 3.3 proof, induces, by the relation \(h \in C^\infty(F) \mapsto h \circ \psi \in C^\infty(E)\), an isomorphism of \(\mathbb{R}\)–algebras whose the restriction on fiberwise polynomial functions is a isomorphism of \(\mathbb{R}\)–algebras between \(A(F)\) and \(A(E)\) respecting the gradation.

## 4 Algebraic characterization of manifolds

The following result achieves what we start in section 1.

**Theorem 4.1** Let \(M\) be a manifold.

(a) Any isomorphism \(\Phi : S(M) \to S(N)\) of \(\mathbb{R}\)–associative algebras induces an isomorphism of \(\mathbb{R}\)–algebras that respects gradation.

(b) The associative \(\mathbb{R}\)–algebras \(S(M)\) and \(S(N)\) are isomorphic if and only if \(M\) and \(N\) are diffeomorphic manifolds.

**Proof.** The statement (a) is an immediate consequence of Remark 3.4. The point (b) can be obtained as corollary of the previous point (a) or by direct application of Lemme 3.1.

## 5 Algebraic characterization of vector bundles.

Let \(M\) be a manifold, \(E \to M\) be a vector bundle of rank \(n\) and \(T^*E \to E\) be the cotangent bundle of \(E\).

Consider the associative algebra \(S_E(E)\), sub-algebra of \(Pol(T^*E) := S(E)\), elements of the last space being functions that are polynomial along the fibers of \(T^*E\).

We can now state the following, the proof being the same as for point (a) of Theorem 4.1.
**Theorem 5.1** Let $E \to M$ and $F \to N$ be vector bundles. The associative algebras $\mathcal{D}_E(E)$ and $\mathcal{D}_F(F)$ are isomorphic if and only if the vector bundles $E$ and $F$ are isomorphic.

**Proposition 5.2** The smooth envelope of the geometric $\mathbb{R}$–algebra $\mathcal{S}_E(E)$ is given by

$$\mathcal{S}_E(E) = C^\infty(T^*E).$$

**Proof.** We observe that

$$\mathcal{S}_E(E) \supset \pi^*_T(E)(C^\infty(E)).$$

Indeed, in virtue of Proposition 2.5, we have on the one hand,

$$\pi^*_T(E)(C^\infty(E)) = \pi^*_T(E)(\mathcal{A}(E))$$

and on the other,

$$C^\infty(E) = \mathcal{A}(E) \cong \pi^*_T(E)(\mathcal{A}(E))$$

We thus obtain the identification

$$\pi^*_T(E)(\mathcal{A}(E)) \cong \pi^*_T(E)(\mathcal{A}(E));$$

which makes it possible to have the result announced by applying the previous Proposition 2.4 to the inclusion

$$\pi^*_T(E)(\mathcal{A}(E)) \subset \mathcal{S}_E(E).$$

Let consider $u \in Pol^k(T^*E)$. For any $e \in T^*E$, there is a canonical chart domain $U \ni e$ of $T^*E$ associated with a trivialisation $(V, \psi)$ of $E$ in which we can write

$$u(e) = \sum_{|r|+|s|=k} f_{rs}(x,y)\xi^r\eta^s,$$

where $(x,y)$ are local coordinates in $V$, $(x,y,\xi,\eta)$ being those corresponding in $U$ and $f_{rs} \in C^\infty(V)$.

Let us cover $M$ by such trivialization domains and extract a Palais cover from them. Denote the open cover of $T^*E$ associated with this Palais cover of $M$ by $O = O_1 \cup \cdots \cup O_N$ with $N \in \mathbb{N}$ and denote by $U_{i,\alpha}(\alpha \in J)$ the elements of $O_i$.

Consider a partition of the unit $\varphi_{i,\alpha}$ of $M$ subordinate to the cover $(U_{i,\alpha})$ of $M$ associated with $O$.

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\[1\]Indeed, it shown in [9] that the Lie-algebra $\mathcal{D}_E(E)$ characterizes the vector bundle $E$. 

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We know that each $\varphi_{i,\alpha}$ is compactly supported in $U_{i,\alpha}$. For each $U_{i,\alpha}$, consider a function $\phi_{i,\alpha}$ with compact support in $U_{i,\alpha}$ and which is equal to 1 where $\varphi_{i,\alpha}$ is nonvanishing.

We then have

$$\pi^*_{T^*E} \circ \pi^*_E(\varphi_{i,\alpha})u = \sum_{|r|+|s|=k} g_{i,\alpha}^{rs} \cdot v_{i,\alpha}^{rs},$$

where $g_{i,\alpha}^{rs} \in \pi^*_{T^*E}(C^\infty(E))$ and $v_{i,\alpha}^{rs} \in \mathcal{S}_E(E)$ are of support in $U_{i,\alpha}$ and are given by

$$g_{i,\alpha}^{rs} = \varphi_{i,\alpha} f_{i,\alpha}^{rs} \text{ and } v_{i,\alpha}^{rs} = \phi_{i,\alpha} \xi^r \eta^s.$$

Therefore, we can write

$$u = \sum_{|r|+|s|=k} \sum_{i,\alpha} g_{i,\alpha}^{rs} \cdot v_{i,\alpha}^{rs}$$

$$= \sum_{|r|+|s|=k} \left( \sum_{i} g_{i,\alpha}^{rs} \right) \cdot \left( \sum_{\beta} v_{i,\alpha}^{rs} \right)$$

$$= \sum_{|r|+|s|=k} \sum_{i} g_{i,\alpha}^{rs} \cdot v_{i,\alpha}^{rs}$$

since for $\alpha \neq \beta$, one has $g_{i,\alpha}^{rs} \cdot v_{i,\beta}^{rs} = 0$.

We then obtain $u \in \mathcal{S}_E(E)$, since we have just shown that $u$ decomposes into a sum whose terms are products having two factors, one in $\pi^*_{T^*E}(C^\infty(E))$ and the other in $\mathcal{S}_E(E)$. Therefore,

$$\text{Pol}(T^*E) \subset \mathcal{S}_E(E).$$

According to previous Proposition 2.4, we thus have the inclusion

$$\overline{\text{Pol}(T^*E)} \subset \mathcal{S}_E(E).$$

But we also have $\overline{\text{Pol}(T^*E)} = C^\infty(T^*E)$. It then comes

$$C^\infty(T^*E) \subset \mathcal{S}_E(E).$$
We then deduce the equality
\[ S^\varepsilon(E) = C^\infty(T^*E); \]
because the inclusion \( S^\varepsilon(E) \subset C^\infty(T^*E) \) comes from the Proposition 2.4, applied to the following inclusions
\[ S^\varepsilon(E) \subset Pol(T^*E) \subset Pol(T^*E). \]

We can now state the following result which will allow us to draw some conclusions on vector bundles, in relation to the structure of \( \mathbb{R}^\rightarrow \)-algebra of the space of symbols of homogeneous operators.

**Proposition 5.3** Let \( E \to M \) and \( F \to N \) be two vector bundles and \( T^*E \to E \) and \( T^*F \to F \) be the cotangent bundles associated with \( E \) and \( F \) respectively. Any isomorphism of \( \mathbb{R}^\rightarrow \)-algebras \( \Psi : S^\varepsilon(E) \to S^\varepsilon(F) \) extends to an unique isomorphism of \( \mathbb{R}^\rightarrow \)-algebras \( \Psi : C^\infty(T^*E) \to C^\infty(T^*F) \) such that
\[ \Psi(A^0((E))) = A^0((F)) \]
where \( A^0((E)) = \pi^*_T(E)(A^0(E)) \) and \( A^0((F)) \) is the \( \mathbb{R}^\rightarrow \)-algebra defined analogously.

**Proof.** If \( u \in A^0((E)) \subset Pol^0(T^*E) \), is nonvanishing on \( T^*E \), it is then the same for
\[ u^{-1} : T^*E \to \mathbb{R} : e \mapsto \frac{1}{u(e)} \]
which is an element of \( Pol^0(T^*E) \). But since \( u \) is a polynomial function of zero degree in \( y \), when we consider a system of local coordinates \((x, y, \xi, \eta)\), in a canonical chart of \( T^*E \), the same is true for \( u^{-1} \); and thus
\[ u^{-1} \in A^0((E)) \subset S^0_E(E) = \pi^*_T(E)(A(E)). \]
Therefore, the relation
\[ \overline{\Psi(u)} \cdot \overline{\Psi(u^{-1})} = 1_{T^*F} : e \mapsto 1 \]
becomes, because \( \overline{\Psi} \) extends \( \Psi \),
\[ \Psi(u) \cdot \Psi(u^{-1}) = 1_{T^*F}. \]
The above equality allows to conclude that \( \Psi(u) \) and \( \Psi(u^{-1}) \) are constant along the fibers of \( T^*F \). They thus are in
\[ S^0(F) = \pi^*_T(F)(A(F)). \]
But since invertible elements of $A(F)$ that have their inverses in $A(F)$ are in $A^0(F)$ and

$$\pi_{T^*F}|_{\mathcal{C}^\infty(F)} : \mathcal{C}^\infty(F) \to \mathcal{C}^\infty(F)$$

is an isomorphism of $\mathbb{R} -$algebras, we also conclude that $\Psi(u)$ is in $A^0((F))$. This prove the first equality. Since for any $u \in A^0((E))$, the element $u^2 + 1$ is a nonvanishing one, we have that $(\Psi(u))^2$ is an element of $A^0((F))$. We directly deduce that

$$\Psi(u) \in A_0((F)).$$

And the above relation achieves the proof of the proposition. \hfill $\blacksquare$

**Remark 5.4** Let $E \to M$ be a vector bundle. To any transition diffeomorphism of the vector bundle $E \to M$ of the form $(x, y) \mapsto (x, A(x)(y))$ corresponds a transition diffeomorphism of the tangent bundle $TE \to E$ of the form

$$(x, y, h, k) \mapsto (x, y, h, (A'(x) \cdot y)(h) + A(x)(k)),$$

where $A'(x) \cdot y$ is a $(n, m)$--type matrix whose element of row $l$ and column $k$ is given by

$$\sum_r \partial_k A_{l,r}(x)y^r$$

The above matrix is such that

$$(A'(x) \cdot y)(h) = (A_s \ h)(y).$$

The transition diffeomorphisms of $T^*E \to E$ are then given by

$$(x, y, \xi, \eta) \mapsto (x, y, \xi - t(A^{-1}(x) \circ (A'(x) \cdot y))(\eta), tA^{-1}(x)(\eta))$$

These diffeomorphisms thus define a differential fibration $T^*E \to M$ whose projection is given by

$$\pi : T^*E \to M : e \mapsto \pi_E \circ \pi_{T^*E}(e)$$

**Corollary 5.5** Let $E \to M$ and $F \to N$ be vector bundles. If the $\mathbb{R} -$algebras $S_E(E)$ and $S_E(F)$ are isomorphic, then the differential fibrations $T^*E \to M$ and $T^*F \to N$ are isomorphic.

**Proof.** Indeed, in virtue of Proposition 5.3, the $\mathbb{R} -$algebras $C^\infty(T^*E)$ and $C^\infty(T^*F)$ are isomorphic.
There then exists a diffeomorphism $\Phi : T^*E \rightarrow T^*F$ such that, in accordance with the proposition cited above, the $\mathbb{R}$--algebras isomorphism $\Psi$ between $S_E(F)$ and $S_F(E)$ is given by

$$\Psi : u \in S_E(F) \mapsto u \circ \Phi \in S_F(E).$$

The same Proposition 5.3 allows to write

$$\Psi(\pi^*_F \circ \pi^*_E(C^\infty(N))) = \pi^*_E \circ \pi^*_F(C^\infty(M)).$$

Therefore, the restriction of $\Psi$ to $\mathcal{A}^0(F)$ induces an isomorphism of $\mathbb{R}$-algebras $\Psi$ between $C^\infty(N)$ and $C^\infty(M)$. By Milnor result, there exists a diffeomorphism $\phi : M \rightarrow N$ such that

$$\Psi(g) = g \circ \phi.$$

We deduce from the above,

$$\Psi(\pi^*_F \circ \pi^*_E(g)) = \pi^*_E \circ \pi^*_F(g \circ \phi).$$

But we have the equality

$$(\pi^*_F \circ \pi^*_E(g)) \circ \Phi = \Psi(\pi^*_F \circ \pi^*_E(g)).$$

Hence, for any $e \in T^*E$, we have

$$(\pi^*_F \circ \pi^*_E(g))(\Phi(e)) = \pi^*_E \circ \pi^*_F(g \circ \phi)(e).$$

This is equivalent to

$$g(\pi_F \circ \pi_T(F)(\Phi(e))) = g(\phi(\pi_E \circ \pi_T(E)(e))), \forall e \in T^*E, \forall g \in C^\infty(N).$$

We deduce that

$$(\pi_F \circ \pi_T(F)) \circ \Phi = \phi \circ (\pi_E \circ \pi_T(E));$$

and the announced result is proved. ■

The above corollary also allows us to say that if the $\mathbb{R}$-algebras $S_E(E)$ and $S_F(F)$ are isomorphic, the basis $M$ and $N$ of the vector bundles $E$ and $F$ are diffeomorphic. It also implies that these bundles have the same rank.

To obtain a characterization of vector bundles, let us return to the previous $\Phi$ isomorphism and keep the notations of the proof of Theorem 3.3.

We can therefore consider $\Phi$ as an $M$-isomorphism of the fibrations $T^*E \rightarrow M$ and $T^*F \rightarrow M$.

Let now $(U_\alpha)$ be an open cover of $M$ by trivialization domains of $T^*E$ and $T^*F$. Let us denote by $\sigma_\alpha$ and $\rho_\alpha$ the local trivialization diffeomorphisms relating to these vector bundles.
We now can write
\[ \Phi^x : \mathbb{R}^{m+2n} \rightarrow \mathbb{R}^{m+2n} : (y, \xi, \eta) \mapsto \Delta_{\alpha}(x, y, \xi, \eta) \]
with \( \Phi_{\alpha} = \rho_{\alpha}^{-1} \circ (\Phi|_{\pi^{-1}(U_{\alpha})}) \circ \sigma_{\alpha} \).

In addition, observe that \( \Phi^x_{\alpha} \) is a diffeomorphism, polynomial on \( y, \xi, \eta \), such that
\[ P \in \text{Pol}(\mathbb{R}^{m+2n}) \mapsto P \circ \Phi^x_{\alpha} \in \text{Pol}(\mathbb{R}^{m+2n}) \]
is an automorphism of \( \mathbb{R} \)-algebra.

By Lemma 3.2 the linear part of \( \Phi^x_{\alpha} \) allows to define, for any index \( \alpha \), a diffeomorphism
\[ \Psi_{\alpha} : U_{\alpha} \times \mathbb{R}^{m+2n} \rightarrow U_{\alpha} \times \mathbb{R}^{m+2n} \]
We deduce, as before, that there exists a \( M \)-isomorphism of vector bundles and only one \( \Psi : T^* E \rightarrow T^* F \) such that for any index \( \alpha \)
\[ \Psi_{\alpha} = \rho_{\alpha}^{-1} \circ \Psi \circ \sigma_{\alpha} \].

We have thus just proved the following result.

**Proposition 5.6** Let \( E \rightarrow M \) be a vector bundle.
Then, the associative algebra \( S_E(E) \) characterizes the vector bundle \( T^* E \rightarrow M \).

If rather, we define
\[ \Phi^{x,y} : \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n} : (\xi, \eta) \mapsto \Delta_{\alpha}(x, y, \xi, \eta) \]
with \( \Phi_{\alpha} = \rho_{\alpha}^{-1} \circ (\Phi|_{\pi^{-1}(U_{\alpha})}) \circ \sigma_{\alpha} \), where this time, \( \rho \) and \( \sigma \) are seen as transition diffeomorphisms of \( T^* E \rightarrow E \) and \( T^* F \rightarrow F \), we obtain, in a similar way as in the previous lines, a \( M \)-isomorphism of vector bundles \( \Psi \) between \( T^* E \rightarrow E \) and \( T^* F \rightarrow F \).

We summarize this in the following statement.

**Theorem 5.1** Let \( E \rightarrow M \) be a vector bundle.
Then, the associative algebra \( S_E(E) \) characterizes the vector bundle \( E \rightarrow M \).

6 Classical limit of homogeneous operators of zero weight

Let \( E \rightarrow M \) be a vector bundle. We denote by
\[ D_0(E) = \{ T \in \mathcal{D}(E) : [\mathcal{E}, D] = 0 \} \]
the quantum Poisson algebra of homogeneous operators of zero weight.
Consider the classical Poisson algebra $S_0(E)$, classical limit of $D_0(E)$. We then have

$$S_0(E) = \bigoplus_{k \geq 0} S^k_0(E),$$

with $S^k_0(E) = \{ \sigma(T) : T \in D^k_0(E) \}$. Seen as associative subalgebra of $Pol(T^*E)$, we have

$$S^0_0(E) = \pi^*_{T^*E}(A^0(E)) = A^0((E)).$$

Locally, in a canonical chart of $T^*E$ associated with an adapted chart of $E$, an element $u$ of $S^k_0(E)$ is written in the form

$$u(e) = \sum_{|r| + |s| \leq k} f_{rs}(x)y^r\xi^s\eta^r,$$

where $e \in T^*E$ admits $(x, y, \xi, \eta)$ as local coordinates.

**Proposition 6.1** Let $E \to M$ and $F \to N$ be two vector bundles. If $\Psi : S_0(E) \to S_0(F)$ is an isomorphism of $\mathbb{R}$–algebras, then we have

$$\Psi(A^0((E))) = A^0((F)).$$

**Proof.** The proof is analogous to that of the Lemma 3.1. Indeed, let $u \in A^0((E))$ be nonvanishing on $T^*E$. Then $u = \pi^*_{T^*E}(u)$, with $u \in A^0(E)$ a nonvanishing function on $E$. Therefore, $u^{-1} \in A^0(E)$ and, using $u^{-1} = \pi^*_{T^*E}(u^{-1})$, we obtain the relation

$$u \cdot u^{-1} = 1_{T^*E} : e \mapsto 1.$$

Hence, we have

$$\Psi(u) \cdot \Psi(u^{-1}) = 1_{T^*F}.$$

We deduce that

$$\Psi(u) \in Pol^0(T^*F) \cap S_0(F) = A^0((F)).$$

Considering any element $u$ of $A^0((E))$, we have that $u^2 + 1$ is nonvanishing on $T^*E$, and it comes $\Psi(u) \cdot \Psi(u) \in A^0((F))$. We can then conclude because it implies

$$\Psi(u) \in A^0((F)).$$

We then deduce the following result.

**Corollary 6.2** Let $E \to M$ and $F \to N$ be two vector bundles. If the $\mathbb{R}$–algebras $S_0(E)$ and $S_0(F)$ are isomorphic then the differential manifolds $M$ and $N$ are diffeomorphic.

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2This comes from the local form of homogeneous operators established in [9]
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