COADJOINT ORBITOPE

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Abstract. We study coadjoint orbitopes, i.e. convex hulls of coadjoint orbits of a compact Lie group. We show that all the faces of such an orbitope are exposed. The face structure is studied by means of the momentum map and it is shown that every face is again a coadjoint orbitope. Up to conjugation the faces are completely determined by the momentum polytope and can be described in a simple way in terms of root data. Finally we consider the complex geometry of the coadjoint orbit and we prove that the submanifolds of the orbit that are extreme sets of a face are exactly the closed orbits of parabolic subgroups.

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Introduction

Let $K$ be a compact Lie group and let $K \to \text{GL}(V)$ be a finite-dimensional representation. An orbitope is by definition the convex envelope of an orbit of $K$ in $V$ (see [23]). An interesting class of orbitopes is given by the convex envelope of coadjoint orbits. We call these coadjoint orbitopes. The case of an integral orbit has been studied in [6], where it was realised that a remarkable construction introduced by Bourguignon, Li and Yau [8] in the

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case of complex projective space can be generalized to arbitrary flag manifolds. This allowed to show that the convex envelope of an integral coadjoint orbit is equivariantly homeomorphic to a Satake-Furstenberg compactification. This homeomorphism is constructed by integrating the momentum map, but unfortunately it is not explicit and its nature is not yet well-understood. On the other hand, the Satake-Furstenberg compactifications admit a very precise combinatorial description going back to Satake [24].

The first purpose of this paper is to give a precise description of the face structure of coadjoint orbitopes without the integrality assumption and without relying on the connection with Satake-Furstenberg compactifications. In the second place we establish the combinatorial description of the orbitope in terms of the notions put forward by Satake to describe his compactifications. This description is established directly, without need to mention the compactifications, and is valid for all orbits integral or not.

We describe the contents of the paper.

In §1 we recall some elementary facts from convex geometry. In §2 we fix the notation and recall some well-known results.

In §3 we study the face structure of coadjoint orbitopes using the symplectic properties of coadjoint orbits and the structure theory of compact Lie groups. Let \( O \) denote the coadjoint orbit and let \( \hat{O} \) be the convex hull of \( O \), i.e., the coadjoint orbitope. To describe a face \( F \) of \( \hat{O} \) it is enough to describe the set \( \text{ext} \ F \subset O \) that consists of the extreme points of \( F \). Therefore it is important to study the subsets \( O' \subset O \) of the form \( O' = \text{ext} \ F \), i.e., those subsets \( O' \subset O \) such that the convex hull of \( O' \) is a face of \( \hat{O} \). It turns out that for exposed faces \( \text{ext} \ F \) is a submanifold and it is best studied in terms of standard properties of the momentum map. The main results are the following. In the first place every face of a coadjoint orbitope is a coadjoint orbitope itself (Theorem 23). In particular faces of coadjoint orbitopes are orbitopes (see also Remark 26). In the second place, a coadjoint orbitope has only exposed faces (Theorem 36). This answers Question 1 of [23] for this particular class of orbitopes. These two properties are quite remarkable and show that coadjoint orbitopes are quite special among general orbitopes.

If we fix a maximal torus \( T \) of \( K \), then the orthogonal projection of \( \hat{O} \) onto \( t \) coincides with the intersection \( P = \hat{O} \cap t \) and is given as the convex hull of a Weyl group orbit or alternatively as the image of the momentum map with respect to the action of \( T \) on \( O \) (this is Kostant theorem). We show in §4 that the faces of the orbitope \( \hat{O} \) are completely determined by the faces of the momentum polytope \( P \). In fact up to conjugation the faces of \( \hat{O} \) are in one-to-one correspondence with the faces of the momentum polytope. In particular there are a finite number of faces up to conjugation.

This is applied in §5 where it is shown that the face structure yields a natural stratification of the boundary of a coadjoint orbitope. The strata are smooth fibre bundles over flag manifolds naturally associated with the faces.
In [6] we give a description of the faces of the coadjoint orbitope in terms of root data. We use the formalism of $x$-connected subset of simple roots developed by Satake [24]. This gives a combinatorial way to generate all the faces of the orbitope.

In §7 we completely characterize the submanifolds $O' \subset O$ of the form $O' = \text{ext } F$. The characterization is best expressed using the complex structure of the coadjoint orbit. The main result asserts that the submanifolds of the form $O' = \text{ext } F$ are exactly the closed orbits of parabolic subgroups of $G = K^\mathbb{C}$.

In the last section we prove that if $O$ is an integral orbit, the faces of $\hat{O}$ are also orbitopes of integral orbits. This is equivalent to saying that if $O$ comes from irreducible representation, also its faces are associated to irreducible representations.

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1. Preliminaries from convex geometry

It is useful to recall a few definitions and results regarding convex sets (see e.g. [25]). Let $V$ be a real vector space and $E \subset V$ a convex subset. The relative interior of $E$, denoted relint $E$, is the interior of $E$ in its affine hull. A face $F$ of $E$ is a convex subset $F \subset E$ with the following property: if $x, y \in E$ and relint$[x, y] \cap F \neq \emptyset$, then $[x, y] \subset F$. The extreme points of $E$ are the points $x \in E$ such that $\{x\}$ is a face. If $E$ is compact the faces are closed [25, p. 62]. If $F$ is a face of $E$ we say that relint $F$ is an open face of $E$. A face distinct from $E$ and $\emptyset$ will be called a proper face.

Assume for simplicity that a scalar product $\langle , \rangle$ is fixed on $V$ and that $E \subset V$ is a compact convex subset with nonempty interior.

**Definition 1.** The support function of $E$ is the function

$$h_E : V \to \mathbb{R} \quad h_E(u) = \max_{x \in E} \langle x, u \rangle.$$  \hspace{1cm} (2)

If $u \neq 0$, the hyperplane $H(E, u) := \{x \in E : \langle x, u \rangle = h_E(u)\}$ is called the supporting hyperplane of $E$ for $u$. The set

$$F_u(E) := E \cap H(E, u)$$

is a face and it is called the exposed face of $E$ defined by $u$ or also the support set of $E$ for $u$.

In using the notation $F_u(E)$ we will tacitly assume that the affine span of $E$ is $V$. Hence by definition an exposed face is proper. We notice that in general not all faces of a convex subsets are exposed. A simple example is given by the convex hull of a closed disc and a point outside the disc: the resulting convex set is the union of the disc and a triangle. The two vertices of the triangle that lie on the boundary of the disc are non-exposed 0-faces.
Lemma 3. If \( F \) is a face of a convex set \( E \), then \( \text{ext} \, F = F \cap \text{ext} \, E \).

Proof. It is immediate that \( F \cap \text{ext} \, E \subset \text{ext} \, F \). The converse follows from the definition of a face. \( \square \)

Lemma 4. If \( G \) is a compact group, \( V \) is a representation space of \( G \) and \( G \cdot x \) is an orbit of \( G \), then \( \text{conv}(G \cdot x) \) contains a fixed point of \( G \). Moreover any fixed point contained in \( \text{conv}(G \cdot x) \) lies in \( \text{relint} \, \text{conv}(G \cdot x) \).

Proof. Just pick \( x \in O \) and set

\[
\bar{x} := \int_G g \cdot x \, dg
\]

where \( dg \) is the Haar measure. Then \( \bar{x} \) is \( G \)-invariant and belongs \( \hat{O} \). Now let \( y \) be any fixed point of \( G \) that lies in \( \hat{O} \). By Theorem 6 there is a unique face \( F \subset \hat{O} \) such that \( y \) belongs to \( \text{relint} \, F \). Since \( \hat{O} \) is \( G \)-invariant and \( y \) is fixed by \( K \), it follows that \( a \cdot F = F \) for any \( a \in G \). So \( F \) is \( G \)-invariant, and hence also \( \text{ext} \, F \) is \( G \)-invariant. Since \( \text{ext} \, F \subset \text{ext} \, (\text{conv}(G \cdot x)) \subset G \cdot x \), it follows that \( \text{ext} \, F = G \cdot x \) and hence that \( F = G \cdot x \). \( \square \)

Proposition 5. If \( F \subset E \) is an exposed face, the set \( C_F := \{ u \in V : F = F_u(E) \} \) is a convex cone. If \( G \) is a compact subgroup of \( O(V) \) that preserves both \( E \) and \( F \), then \( C_F \) contains a fixed point of \( G \).

Proof. Let \( u_1, u_2 \in C_F \) and \( \lambda_1, \lambda_2 \geq 0 \) and set \( u = \lambda_1 u_1 + \lambda_2 u_2 \). We need to prove that if at least one of \( \lambda_1, \lambda_2 \) is strictly positive, then \( F = F_u(E) \). Assume for example that \( \lambda_1 > 0 \). It is clear that \( h_E(u) \leq \lambda_1 h_E(u_1) + \lambda_2 h_E(u_2) \). If \( x \in F \), then

\[
\langle x, u \rangle = \lambda_1 \langle x, u_1 \rangle + \lambda_2 \langle x, u_2 \rangle = \lambda_1 h_E(u_1) + \lambda_2 h_E(u_2).
\]

Hence \( h_E(u) = \lambda_1 h_E(u_1) + \lambda_2 h_E(u_2) \) and \( F \subset F_u(E) \). Conversely, if \( x \in F_u(E) \), then

\[
0 = h_E(u) - \langle x, u \rangle = \lambda_1 (h_E(u_1) - \langle x, u_1 \rangle) + \lambda_2 (h_E(u_2) - \langle x, u_2 \rangle).
\]

Since \( \lambda_1 > 0 \) we get \( h_E(u_1) - \langle x, u_1 \rangle = 0 \), so \( x \in F_{u_1}(E) = F \). Thus \( F = F_u(E) \). This proves the first fact. To prove the second, pick any vector \( u \in C_F \) and apply the previous lemma to the orbit \( G \cdot u \subset C_F \); this yields a \( G \)-invariant \( \bar{u} \in C_F \). \( \square \)

Theorem 6 ([25 p. 62]). If \( E \) is a compact convex set and \( F_1, F_2 \) are distinct faces of \( E \) then \( \text{relint} \, F_1 \cap \text{relint} \, F_2 = \emptyset \). If \( G \) is a nonempty convex subset of \( E \) which is open in its affine hull, then \( G \subset \text{relint} \, F \) for some face \( F \) of \( E \). Therefore \( E \) is the disjoint union of its open faces.

Lemma 7. If \( E \) is a compact convex set and \( F \subset E \) is a face, then \( \dim F < \dim E \).

Proof. If \( \dim F = \dim E \), then \( \text{relint} \, F \) is open in the affine span of \( E \), so \( \text{relint} \, F \subset \text{relint} \, E \). By the previous theorem this implies that \( F = E \). \( \square \)
Lemma 8. If \( E \) is a compact convex set and \( F \subset E \) is a face, then there is
a chain of faces
\[
F_0 = F \subset F_1 \subset \cdots \subset F_k = E
\]
which is maximal, in the sense that for any \( i \) there is no face of \( E \) strictly
contained between \( F_{i-1} \) and \( F_i \).

Proof. If \( F = E \) there is nothing to prove. Otherwise put \( F_0 := F \). If
there is no face strictly contained between \( F_0 \) and \( E \), just set \( F_1 = E \).
Otherwise we find a chain \( F_0 \subset F_1 \subset \cdots \subset F_k = E \). If this is not maximal, we
can refine it. Repeting this step we get a chain with \( k + 1 \) elements. Since
\( \dim F_{i-1} < \dim F_i \), \( k \leq n \). Therefore the chain gotten after at most \( n \) steps
is maximal. \( \Box \)

Lemma 9. If \( E \) is a convex subset of \( \mathbb{R}^n \), \( M \subset \mathbb{R}^n \) is an affine subspace
and \( F \subset E \) is a face, then \( F \cap M \) is a face of \( E \cap M \).

Proof. If \( x, y \in E \cap M \) and \( \text{relint}[x, y] \cap F \cap M \neq \emptyset \) then \( [x, y] \subset F \) since \( F \) is
a face, but \( [x, y] \) is also contained in \( M \) since \( M \) is affine. So \( [x, y] \subset F \cap M \) as desired.
\( \Box \)

2. COADJUNCT ORBITOPIES

Through the paper we will use the following notation. \( K \) denotes a
compact connected semisimple Lie group with Lie algebra \( \mathfrak{k} \). If \( T \subset K \) is a
maximal torus and \( \Pi \subset \Delta(\mathfrak{t}^\mathbb{C}, \mathfrak{t}^\mathbb{C}) \) is a set of simple roots, the Weyl chamber
of \( \mathfrak{t} \) corresponding to \( \Pi \) is defined by
\[
C^+ := \{ v \in \mathfrak{t} : -i\alpha(v) > 0 \text{ for any } \alpha \in \Delta^+ \}.
\]

\( B \) is the Killing form of \( \mathfrak{t}^\mathbb{C} \) and \( \langle , \rangle = -B|_{\mathfrak{k} \times \mathfrak{k}} \) is a scalar product on \( \mathfrak{k} \). By
means of \( \langle , \rangle \) we identify \( \mathfrak{k} \) with \( \mathfrak{k}^* \).

Lemma 10. Let \( T \subset K \) be a maximal torus, let \( \Delta \) be the root system of
\( (\mathfrak{t}^\mathbb{C}, \mathfrak{t}^\mathbb{C}) \) and let \( \Pi \subset \Delta \) be a base. Define \( H_{\alpha} \in \mathfrak{t}^\mathbb{C} \) by the formula
\( B(H_{\alpha}, \cdot) = \alpha(\cdot) \) and choose a nonzero vector \( X_{\alpha} \in \mathfrak{g}_{\alpha} \) for any \( \alpha \in \Delta \). For \( \alpha \in \Delta^+ \) set
\[
\begin{align*}
  u_{\alpha} := & \frac{1}{\sqrt{2}}(X_{\alpha} - X_{-\alpha}) \\
  v_{\alpha} := & \frac{i}{\sqrt{2}}(X_{\alpha} + X_{-\alpha}).
\end{align*}
\]
Then it is possible to choose the vectors \( X_{\alpha} \) in such a way that \( [X_{\alpha}, X_{-\alpha}] = H_{\alpha} \) and so that the set \( \{u_{\alpha}, v_{\alpha} | \alpha \in \Delta^+ \} \) be orthonormal
with respect to \( \langle , \rangle = -B \). Moreover for \( y \in \mathfrak{t} \)
\[
[y, u_{\alpha}] = -i\alpha(y){v_{\alpha}} \quad [y, v_{\alpha}] = i\alpha(y)u_{\alpha} \quad [u_{\alpha}, v_{\alpha}] = iH_{\alpha}.
\]

For a proof see e.g. [18, pp. 353-354]. Set
\[
\mathfrak{z}_{\alpha} = \mathbb{R}u_{\alpha} \oplus \mathbb{R}v_{\alpha}.
\]
(11)

Then
\[
\mathfrak{k} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{z}_{\alpha}.
\]
If $O$ is an adjoint orbit of $K$ and $x \in O$, then

$$T_xO = \text{Im} \, \text{ad} \ x = \bigoplus_{\alpha \in E} Z_{\alpha}$$

where $E := \{ \alpha \in \Delta_+ : \alpha(x) \neq 0 \}$. Denote by $v_O$ the vector field on $O$ defined by $v \in \mathfrak{k}$. Explicitly $v_O(x) = [u, x]$. Since we identify $\mathfrak{k} \cong \mathfrak{t}^*$ we may regard $O$ as a coadjoint orbit. As such it is equipped with a $K$-invariant symplectic form $\omega$, named after Kostant, Kirillov and Souriau, and defined by the following rule. For $u, v \in \mathfrak{k}$

$$\omega_x(u_O(x), v_O(x)) := \langle x, [u, v] \rangle.$$

See e.g. [17, p. 5]. $\omega$ is a $K$-invariant symplectic form on $O$ and the inclusion $O \hookrightarrow \mathfrak{k}$ is the moment map.

If $T \subset K$ is a maximal torus, we denote by $W(K, T)$ or simply by $W$ the Weyl group of $(K, T)$. We let $\pi : k \rightarrow t$ denote the orthogonal projection with respect to the scalar product $\langle , \rangle = -B$. Its restriction to $O$ is denoted by $\Phi_T : O \rightarrow t$; it is the momentum map for the $T$-action on $O$. $P := \Phi_T(O)$ is the momentum polytope. The following convexity theorem of Kostant [20] is the basic ingredient in the whole theory.

**Theorem 12 (Kostant).** Let $K$ be a compact connected Lie group, let $T \subset K$ be a maximal torus and let $O$ be a coadjoint orbit. Then $P$ is a convex polytope, ext $P = O \cap t$ and ext $P$ is a unique $W$-orbit.

There is a unique $K$-invariant complex structure $J$ on $O$ such that $\omega$ be a Kähler form. It can be described as follows (see [16, p. 113] for more information). Fix a maximal torus $T$ and a system of positive roots in such a way $x$ belongs to the closure of the positive Weyl chamber. Then the complex structure on $T_xO$ is given by the formula

$$Ju_\alpha = v_\alpha.$$ 

Set $G = K^C$. The action of $K$ on $O$ extends to an action $G \times O \rightarrow O$ which is holomorphic. If $v_O$ denotes the fundamental vector field induced by $v \in \mathfrak{g} = \mathfrak{k}^C$, this implies that

$$(iv)_O = Jv_O.$$ 

Let

$$b_- := \mathfrak{t}^C \oplus \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_{-\alpha}$$

denote the negative Borel subalgebra and let $B_-$ be the corresponding Borel subgroup. The following lemma is well-known.

**Lemma 13.** Let $T \subset K$ be a maximal torus and let $\Delta_+$ be a set of positive roots. If $x \in O \cap t$, then $x \in T^+$ if and only if $B_-$ is contained in the stabilizer $G_x$. 
3. Group theoretical description of the faces

Let $\mathcal{O} \subset \mathfrak{k}$ be a coadjoint orbit of $K$. The orbitope $\hat{\mathcal{O}}$ is by definition the convex hull of $\mathcal{O}$.

**Lemma 14.** $\text{ext } \hat{\mathcal{O}} = \mathcal{O}$. Moreover for any face $F \subset \hat{\mathcal{O}}$, $\text{ext } F = F \cap \mathcal{O}$.

**Proof.** This fact is common to all orbitopes [23, Prop. 2.2]. By construction $\text{ext } \hat{\mathcal{O}} \subset \mathcal{O}$. On the other hand $\mathcal{O}$ lies on a sphere, hence all points of $\mathcal{O}$ are exposed extreme points. This proves the first assertion. The second follows from the first and from Lemma 3. □

A submanifold $M \subset \mathbb{R}^n$ is called full if it is not contained in any proper affine subspace of $\mathbb{R}^n$.

**Lemma 15.** Let $K$ be a compact connected semisimple Lie group and let $\mathcal{O} \subset \mathfrak{k}$ be a coadjoint orbit. The orbit $\mathcal{O}$ is full if and only if every simple factor of $K$ acts nontrivially on $\mathcal{O}$.

**Proof.** Fix $x \in \mathcal{O}$. Let $M$ denote the affine hull of $\mathcal{O}$ in $\mathfrak{k}$ and let $V$ be the associated linear subspace, i.e. $M = x + V$. We claim that $M$ contains the origin. Since $\mathcal{O}$ is Ad-$K$-invariant, so are $M$ and $V$. Hence $V$ is an ideal and $V^\perp$ is an ideal as well. Write $x = x_0 + x_1$, with $x_0 \in V$ and $x_1 \in V^\perp$. For any $g \in K$, $gx - x \in V$, $g(x_0 - x_0) \in V$ and $gx_1 - x_1 \in V^\perp$. So $gx_1 - x_1 \in V \cap V^\perp$, i.e. $gx_1 = x_1$. This means that $x_1$ is a fixed point of the adjoint action. Since $K$ is semisimple, $x_1 = 0$, $x \in V$ and $M = V$ as desired. Let $K_i$, $i = 1, \ldots, r$ be the simple factors of $K$. Since $V$ is an ideal, $V = \bigoplus_{i \in I} \mathfrak{k}_i$ for some subset $I$ of $\{1, \ldots, r\}$. It is clear that $\mathfrak{k}_j \subseteq V$ if and only if $[\mathfrak{k}_j, V] = 0$ if and only if $K_j$ acts trivially on $\mathcal{O}$. This proves the first statement. □

Let $H$ be a compact connected Lie group (not necessarily semisimple) and let $\mathcal{O} \subset \mathfrak{h}$ be an orbit. There is a splitting of the algebra

$$\mathfrak{h} = \mathfrak{z} \oplus \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_r$$

(16)

where $\mathfrak{z}$ is the center of $\mathfrak{h}$ and $\mathfrak{k}_i$ are simple ideals. Let $K_i$ be the closed connected subgroups of $H$ with $\text{Lie } K_i = \mathfrak{k}_i$. So $H = Z \cdot K_1 \cdots K_r$, where $Z$ is the connected component of the identity in the center of $H$. Any two of these subgroups have finite intersection. We can reorder the factors in such a way that $K_i$ acts nontrivially on $\mathcal{O}$ if and only if $1 \leq i \leq q$ for some $q$ between 1 and $r$. Set

$$L := K_1 \cdots K_q \quad L' := K_{q+1} \cdots K_r.$$  

By construction there is a decomposition

$$H = Z \cdot L \cdot L'.$$  

(17)

Any two factors in this decomposition have finite intersection.
Lemma 18. For any \( x \in \mathcal{O} \), there is a unique decomposition \( x = x_0 + x_1 \) with \( x_0 \in \mathfrak{z} \) and \( x_1 \in \mathfrak{l} \). Moreover

\[
\mathcal{O} = H \cdot x = x_0 + L \cdot x_1,
\]

the affine span of \( \mathcal{O} \) is \( x_0 + \mathfrak{l} \) and \( x_0 \) belongs to \( \text{relint} \, \hat{\mathcal{O}} \).

Proof. Write \( x = x_0 + x_1 + x_2 \) with \( x_0 \in \mathfrak{z}, x_1 \in \mathfrak{l} \) and \( x_2 \in \mathfrak{l}' \). Since \( L' \cdot x = x \), the component \( x_2 \) is fixed by \( L' \). Since \( L' \) is semisimple, this forces \( x_2 = 0 \). It follows immediately that \( H \cdot x = x_0 + L \cdot x_1 \). By definition all simple factors of \( L \) act nontrivially on \( L \cdot x_1 \), hence the orbit \( L \cdot x_1 \) is full in \( \mathfrak{l} \) by Lemma 15. This proves that \( \text{aff}(\hat{\mathcal{O}}) = x_0 + \mathfrak{l} \). Since \( \mathcal{O} \subset x_0 + \mathfrak{l} \) and \( \mathfrak{l} \perp x_0 \), \( x_0 \) is the closest point to the origin. Such a point is unique because \( \hat{\mathcal{O}} \) is convex. Since \( \hat{\mathcal{O}} \) is \( H \)-invariant, \( x_0 \) is fixed by \( H \). The last statement follows from Lemma 4. \( \Box \)

The statement about the affine span is equivalent to \( L \cdot x_1 \) being full in \( \mathfrak{l} \). Therefore after possibly replacing \( K \) by \( L \) and translating by \( x_0 \) we can assume for most part of the paper that \( \mathcal{O} \) is full.

We are interested in the facial structure of \( \hat{\mathcal{O}} \) and we start by considering the structure of exposed faces.

Lemma 19. Assume that \( K \) is a compact connected Lie group, that \( H \subset K \) is a connected Lie subgroup of maximal rank and that \( \mathcal{O} \) is a coadjoint orbit of \( K \). Then

a) \( \mathcal{O} \cap \mathfrak{h} \) is a union of finitely many \( H \)-orbits;

b) if \( H \) is the centralizer of a torus, then \( \mathcal{O} \cap \mathfrak{h} \) is a symplectic submanifold of \( \mathcal{O} \).

Proof. Let \( T \) be a maximal torus of \( K \) contained in \( H \). Since \( \mathcal{O} \cap \mathfrak{h} \) is an \( H \)-invariant subset of \( H \) and \( T \) is a maximal torus of \( H \) we have \( \mathcal{O} \cap \mathfrak{h} = H \cdot (\mathcal{O} \cap \mathfrak{t}) \). But \( \mathcal{O} \cap \mathfrak{t} \) coincides with the an orbit of the Weyl group and is therefore finite. Hence \( \mathcal{O} \cap \mathfrak{h} \) is a finite union of \( H \)-orbits. This proves the first statement. For the second assume that \( H = Z_K(S) \) where \( S \subset K \) is a torus. Then \( \mathcal{O} \cap \mathfrak{h} = \mathcal{O}^S \) is the set of fixed points of \( S \), hence it is a symplectic submanifold of \( \mathcal{O} \). \( \Box \)

We start the analysis of the face structure of \( \hat{\mathcal{O}} \) by looking at the exposed faces. At the end of the section we will prove that all faces are exposed.

Let \( u \) be a nonzero vector in \( \mathfrak{k} \) and let \( \Phi_u : \mathcal{O} \rightarrow \mathbb{R} \) be the function \( \Phi_u(x) := \langle x, u \rangle \). Set

\[
\text{Max}(\Phi_u) := \{ x \in \mathcal{O} : \Phi_u(x) = \max_{\mathcal{O}} \Phi_u \}.
\]

\( \Phi_u \) is just the component of the moment map along \( u \). Then for \( x \in \mathcal{O} \) and \( u, v \in \mathfrak{k} \)

\[
d\Phi_u(x)(v_{\mathcal{O}}) = \omega_x(u_{\mathcal{O}}(x), v_{\mathcal{O}}(x)) = \langle x, [u, v] \rangle = \langle [x, u], v \rangle.
\]
This implies that \( x \in \mathcal{O} \) is a critical point of \( \Phi_u \) if and only if \( x \in \mathfrak{z}_t(u) \), i.e. \( \text{Crit}(\Phi_u) = \mathcal{O} \cap \mathfrak{z}_t(u) \).

**Lemma 20.** Let \( H = Z_K(u) \) be the centraliser of \( u \) in \( K \) and let \( F_u(\hat{O}) \) be the exposed face of \( \hat{O} \) defined by \( u \). Then

a) \( \max(\Phi_u) \) is an \( H \)-orbit;

b) \( \text{ext } F_u(\hat{O}) = \max(\Phi_u) \), so \( \text{ext } F_u(\hat{O}) \) is an \( H \)-orbit;

c) \( F_u(\hat{O}) \subset \mathfrak{z}_t(u) \).

**Proof.** By Atiyah theorem [2] the level sets of \( \Phi_u \) are connected. In particular \( \max(\Phi_u) \) is a connected component of \( \text{Crit}(\Phi_u) \). By the previous lemma it is an \( H \)-orbit. This proves (i). Let \( h_{\hat{O}} \) denote the support function of \( \hat{O} \), see [2]. Since \( \langle \cdot, u \rangle \) is a linear function, its maximum on \( \hat{O} \), that is \( h_{\hat{O}}(u) \), is attained at some extreme point, i.e. on \( \mathcal{O} \). Hence

\[
\max_{\mathcal{O}} \Phi_u = h_{\hat{O}}(u).
\]

By Lemma [14] \( \text{ext } F_u(\hat{O}) = F_u(\hat{O}) \cap \mathcal{O} = \{ x \in \mathcal{O} : \langle x, u \rangle = h_{\hat{O}}(u) \} = \max(\Phi_u) \). It follows immediately that \( F_u(\hat{O}) = \text{conv}(\max(\Phi_u)) \). Finally (iii) follows from the fact that \( \max(\Phi_u) \subset \text{Crit}(\Phi_u) = \mathcal{O} \cap \mathfrak{z}_t(u) \). \( \square \)

**Lemma 21.** Fix a maximal torus \( T \subset K \), a nonzero vector \( u \in \mathfrak{t} \) and a point \( x \in \mathcal{O} \cap \mathfrak{t} \). Then \( x \in \text{Crit}(\Phi_u) \) and \( x \) is a maximum point of \( \Phi_u \) if and only if there is a Weyl chamber in \( \mathfrak{t} \) whose closure contains both \( x \) and \( u \).

**Proof.** By assumption \( x \in \mathfrak{t} \subset \mathfrak{z}_t(u) \) and \( \mathfrak{z}_t(u) = \text{Crit}(\Phi_u) \). To check the second assertion recall that \( \Phi_u \) is a Morse-Bott function with critical points of even index (this is Frankel theorem, see e.g. [3], Thm. 2.3, p. 109 or [21, p. 186]) and any local maximum point is an absolute maximum point (see e.g. [3, p. 112]). Therefore \( x \) is a maximum point if and only if the Hessian \( D^2 \Phi_u(x) \) is negative semidefinite. Recall that \( T_x \mathcal{O} = \text{Im } ad_x \) and that

\[
f := ad_x |_{T_x \mathcal{O}} : T_x \mathcal{O} \rightarrow T_x \mathcal{O}
\]

is invertible. If \( w \in T_x \mathcal{O} \), then \( w = z_\mathcal{O}(x) = [z, x] \) for some \( z \in \mathfrak{k} \). The vector \( z \) can be chosen (uniquely) inside \( T_x \mathcal{O} \), i.e. \( z = -f^{-1}(w) \). Set \( \gamma(t) := \text{Ad}(\exp t z) \cdot x \). Then \( \gamma(0) = x, \dot{\gamma}(0) = [z, x] = w, \ddot{\gamma}(0) = [z, [z, x]] \), so

\[
D^2 \Phi_u(x)(w, w) = \frac{d^2}{dt^2} \bigg|_{t=0} h(\gamma(t)) = \langle \dot{\gamma}(0), u \rangle = \langle [z, x], [u, z] \rangle = \langle w, [u, z] \rangle = -\langle w, ad_u \circ f^{-1}(w) \rangle.
\]

(One can prove the same formula much more generally and by a more geometric argument, see [15], Prop. 2.5). Thus the quadratic form \( D^2 \Phi_u(x) \) is negative semidefinite if and only if the operator \( ad_u \circ f^{-1} \) is positive semidefinite. This operator preserves each \( Z_\alpha \) and its restriction to \( Z_\alpha \) is just multiplication by \( \alpha(u)/\alpha(x) \). Hence it is positive semidefinite iff and only
iff $\alpha(u)\alpha(x) \geq 0$ for any $\alpha \in \Delta$. This is equivalent to the condition that $x$ and $u$ lie in the closure of some Weyl chamber (see e.g. [14, p. 11]). □

The computation above goes back to [10] and to Heckman’s thesis [14].

**Lemma 22.** Let $F = F_u(\hat{O})$ be an exposed face of $\hat{O}$. Set $S := \exp(\mathbb{R}u)$ and $H = Z_K(S)$. Then (a) $S$ is a nontrivial torus and fixes $F$ pointwise, (b) ext $F$ is an adjoint orbit of $H := Z_K(S)$, (c) $F \subset \mathfrak{h}$.

**Proof.** Since $u \neq 0$ by the definition of exposed face, $S$ is a nontrivial torus. (b) follows from Lemma [20]. Moreover ext $F = \Max(\Phi_u) \subset \Crit(\Phi_u)$. Since $\Phi_u$ is the Hamiltonian function of the fundamental vector field on $\mathcal{O}$ associated to $u$, ext $F$ is fixed by $\exp(\mathbb{R}u)$ hence by $S$, thus proving (a). Finally ext $F \subset \mathfrak{z}_t(u) = \mathfrak{h}$ by Lemma [20]. □

**Theorem 23.** Let $F$ be a proper face of $\hat{O}$. Then there is a nontrivial torus $S \subset K$ with the following properties: (a) $S$ fixes $F$ pointwise, (b) ext $F$ is an adjoint orbit of $H := Z_K(S)$, (c) $F \subset \mathfrak{h}$, (d) $g \cdot F = F$ for any $g \in H$.

**Proof.** (d) is a direct consequence of (c). To prove (a)-(c) fix a chain of faces $F = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_k = \hat{O}$, such that for any $i$ there is no face strictly contained between $F_{i-1}$ and $F_i$. This is possible by Lemma [8]. We will prove (a)-(c) by induction on $k$. If $k = 1$, then $F$ is a maximal proper face. Since any face is contained in an exposed face, $F$ is necessarily exposed. Thus (a)-(c) follow from the previous lemma. We proceed with the induction. Let $k > 1$ and assume that the theorem is proved for faces contained in a maximal chain of length $k-1$. Fix $F$ with a maximal chain as above of length $k$. By the inductive hypothesis, the theorem holds for $F_1$, so that there is a nontrivial subtorus $S_1 \subset K$ which pointwise fixes $F_1$. Moreover, if we set $H_1 = Z_K(S_1)$ and $\mathfrak{h}_1 = \Lie H_1 = \mathfrak{z}_t(s_1)$, then $F_1 \subset \mathfrak{h}_1$ and ext $F_1$ is an orbit of $H_1$. In particular, if we choose a point $x \in \text{ext } F \subset \text{ext } F_1$, then $x \in F_1 = H_1 \cdot x$. Split $H_1 = Z \cdot L \cdot L'$ with $Z = Z(H_1)^0$ as in [17] and write $x = x_0 + x_1$ as in Lemma [18] with $x_0 \in \mathfrak{z} = \mathfrak{z}(\mathfrak{h}_1)$ and $x_1 \in \mathfrak{l}$, so that $x_1$ is a full orbit $\hat{O}' := L \cdot x_1$ is full in $\mathfrak{l}$ and $F' := F_0 - x_0 = F - x_0$ is the maximal face of $\hat{O}'$. Therefore $F'$ is an exposed face, i.e. there is some $u \in \mathfrak{l}$ such that $F' = F_u(\hat{O}')$. Set $S_2 := \exp(\mathbb{R}u)$. By the previous lemma ext $F'$ is an orbit of $Z_L(S_2)$. Moreover $x_1 \in \text{ext } F'$, because $x \in \text{ext } F$. Therefore ext $F' = Z_L(S_2) \cdot x_1$. Since $u \in \mathfrak{l}$ and $\mathfrak{l} \subset \mathfrak{h}_1$, $u$ commutes with $s_1$. So $S_1$ and $S_2$ commute and generate a torus $S$. Set $H := Z_K(S)$. If $g \in H$, then $g$ commutes with $S_1$, hence $g \in H_1$. It follows that $H \subset Z_{H_1}(S_2)$. Conversely, if $g \in Z_{H_1}(S_2)$, then $g$ also commutes with $S$, so $g \in H$. Thus we get $H = Z_{H_1}(S_2)$. Since $S_2 \subset L$, $Z \cdot L' \subset Z_{H_1}(S_2) = H$ and $H = Z \cdot Z_L(S_2) \cdot L'$. Since $Z \cdot L'$ fixes $x_1$ this implies that $H \cdot x_1 = Z_{H_1}(S_2) \cdot x_1 = Z_L(S_2) \cdot x_1 = \text{ext } F'$. Since $x_0 \in \mathfrak{z} = \mathfrak{z}(\mathfrak{h}_1)$, we conclude that ext $F = \text{ext } F' + x_0 = H \cdot x_1 + x_0 = H \cdot x$. This proves (b). Next observe that the previous lemma also ensures that $F' \subset \mathfrak{z}_t(u)$ and that $\mathfrak{z}_t(u) \subset \mathfrak{h}$. Since $x_0 \in \mathfrak{h}$ too, we conclude that $F = F' + x_0 \subset \mathfrak{h}$. This proves
(c). By definition \( h = z_k(s) \), so \( S \) fixes any point of \( h \) and in particular it fixes pointwise \( F \). Thus (a) is proved. □

We remark that the inductive argument used in the previous proof does not imply that all faces are exposed, since being an exposed face is not a transitive relation.

**Corollary 24.** If \( F \subset \hat{O} \) is a face, then \( \text{ext } F \) is a symplectic submanifold of \( O \).

**Proof.** Let \( S \) and \( H \) be as in Theorem 23. Then \( \text{ext } F \subset O \cap h \) is an \( H \)-orbit. The result follows directly from Lemma 19. □

**Corollary 25.** If \( F \subset \hat{O} \) is a face, there is a maximal torus \( T \subset K \) that preserves \( F \).

**Proof.** A maximal torus of \( H \) is also a maximal torus of \( K \). □

**Remark 26.** The above results shows that every face of \( \hat{O} \) is a coadjoint orbitope for some subgroup \( H \subset K \). One might wonder if a similar property holds for all orbitopes: if a group \( K \) acts linearly on \( V \) and \( O \) is an orbit, one might ask if every face of \( \hat{O} \) is an orbit of some subgroup of \( K \). The answer is negative in general. Counterexamples are provided e.g. by convex envelopes of orbits of \( S^1 \) acting linearly on \( \mathbb{R}^n \). These are called Carathéodory orbitopes, since their study goes back to [9]. In [26] there is a thorough study of the 4-dimensional case (see also [4]). It turns out (see Theorem 1 in [26]) that there are many 1-dimensional faces whose extreme sets are not orbits of any subgroup of \( K \). Therefore the fact that we just established, namely that the faces of a coadjoint orbitope are all orbitopes of the same kind, seems to be a rather remarkable property.

The subgroups \( S \) and \( H \) in Theorem 23 are not unique. Later in Theorem 36 (d) we will show that there is a canonical choice. Now we wish to show that one can always assume that \( S = Z(H)^0 \).

**Corollary 27.** In Theorem 23 we can assume that \( Z(H) \) acts trivially on \( F \) and that \( S = Z(H)^0 \).

**Proof.** Let \( p : k \to h \) denote the orthogonal projection. \( H \) acts on \( O \) in a Hamiltonian way with momentum map \( p|_O \). If \( x \in \text{ext } F \), then \( H \cdot x = \text{ext } F \) is a symplectic orbit by Corollary 24. Therefore \( H_x = H_{p(x)} \), see e.g. [13]. Thm. 26.8, p. 196]. Since \( p(x) \in h \), the stabilizer \( H_{p(x)} \) contains the center of \( H \). So \( Z(H) \subset H_x \). This proves the first statement. Next set \( S' = Z(H)^0 \). Then \( S' \) is a positive dimensional torus. To prove the second fact it is enough to show that changing \( S \) to \( S' \) does not change the centralizer, i.e. that \( H = Z_K(S') \). Since \( S' \subset Z(H) \), \( H \) and \( S' \) commute, so \( H \subset Z_K(S') \). On the other hand \( H \) is the centralizer of \( S \), so \( S \subset S' \), and \( Z_K(S') \subset Z_K(S) = H \). Therefore indeed \( H = Z_K(S') \). □

The following is an immediate consequence of Lemma 18.
Lemma 28. Let $F$ be a face of $\hat{O}$, $H \subset K$ a connected subgroup and assume that $\text{ext} \ F$ is an $H$-orbit and that $F \subset \mathfrak{h}$. Decompose $H$ as in (17), i.e. $L$ is the product of the simple factors of $(H, H)$ that act nontrivially on $F$, while $L'$ is the product of those factors that act trivially. If $x \in \text{ext} \ F$, then $x = x_0 + x_1$ with $x_0 \in \mathfrak{z}$ and $x_1 \in \mathfrak{l}$. Moreover
\[ \text{ext} \ F = H \cdot x = x_0 + L \cdot x_1 \]
and $L \cdot x_1 \subset \mathfrak{l}$ is full.

Now we fix a maximal torus $T$ and we use the notation of p. 6. We wish to show that the $T$-stable faces of $\hat{O}$ and the faces of the momentum polytope are in bijective correspondence. This will be used to prove that all faces of $\hat{O}$ are exposed. The relation between the $T$-invariant faces of $\hat{O}$ and the faces of $P$ will be studied further in the next section.

The following lemma is a consequence of Kostant convexity theorem. See [11, Lemma 7] for a proof in the context of polar representations.

Lemma 29. Let $K$ be a compact connected Lie group, $T \subset K$ be a maximal torus and let $\pi : \mathfrak{k} \to \mathfrak{t}$ be the orthogonal projection. Then (i) If $E \subset \mathfrak{k}$ is a $K$-invariant convex subset, then $E \cap \mathfrak{t} = \pi(E)$. (ii) If $A \subset \mathfrak{t}$ is a $W$-invariant convex subset, then $K \cdot A$ is convex and $\pi(K \cdot A) = A$.

Lemma 30. Let $T \subset K$ be a maximal torus and let $F \subset \hat{O}$ be a nonempty $T$-invariant face. Set $\sigma := \pi(\text{ext} \ F)$. Then $\sigma = \pi(F) = F \cap \mathfrak{t}$. Moreover $\sigma$ is a nonempty face of the momentum polytope $P$.

Proof. We prove this lemma in the same way as Kostant theorem is deduced from the Atiyah-Guillemin-Sternberg theorem. By Corollary 24 $\text{ext} \ F$ is a symplectic submanifold of $O$. $T$ acts on $\text{ext} \ F$ with momentum map given by the restriction of $\pi$ to $\text{ext} \ F$. By definition $\sigma = \pi(\text{ext} \ F)$ is the momentum polytope for this action. By the Atiyah-Guillemin-Sternberg theorem
\[ \sigma = \text{conv} \pi\left((\text{ext} \ F)^T\right) = \text{conv} \pi(\text{ext} \ F \cap \mathfrak{t}). \]
This means first of all that $\sigma$ is convex. Since $\pi$ is linear it follows that $\pi(F) = \text{conv} \pi(\text{ext} \ F) = \sigma$. On the other hand, since $\pi(\text{ext} F \cap \mathfrak{t}) = \text{ext} F \cap \mathfrak{t}$, we get
\[ \text{ext} \sigma \subset \text{ext} F \cap \mathfrak{t} \quad \sigma \subset F \cap \mathfrak{t}. \]
Conversely $F \cap \mathfrak{t} = \pi(F \cap \mathfrak{t}) \subset \pi(F)$. Since $\pi(F) = \sigma$ we get indeed $F \cap \mathfrak{t} = \sigma$. Thus the first part is proven. In particular we can apply this with $F = \hat{O}$, and we get that $P = \hat{O} \cap \mathfrak{t}$. That $F \cap \mathfrak{t}$ is a face of $P$ now follows directly from Lemma 9 without assuming that $F$ be $T$-invariant. To check that $\sigma \neq \emptyset$, recall that if a torus acts on a compact Kähler in a Hamiltonian way, then it has some fixed points. So $(\text{ext} \ F)^T = \text{ext} F \cap \mathfrak{t} \neq \emptyset$ and $\sigma \neq \emptyset$. □

Recall the following basic property of Hamiltonian actions (see e.g. [12, Thm. 3.6]).
**Lemma 32.** Let $M$ be a symplectic manifold and let $T$ be a torus that acts on $M$ in a Hamiltonian way with momentum map $\Phi : M \to \mathfrak{t}$. If $S \subset T$ is a subtorus that acts trivially on $M$, then $\Phi(M)$ is contained in a translate of $\mathfrak{s}^\perp$.

If $M \subset \mathbb{R}^n$ is an affine subspace, the linear subspace parallel to $M$ is called the direction of $M$ [5, p. 42]. Denote by $\mathfrak{s}^\perp$ the orthogonal space in $\mathfrak{t}$ to the direction of $\sigma$.

**Lemma 33.** Let $F$ be a proper face of $\hat{O}$, let $H$ be a subgroup as in Theorem 23 and let $T$ be a maximal torus of $H$. Then $\sigma := F \cap t$ is a proper face of $P$ and $\text{ext } F$ is a $Z_K(\sigma^\perp)$-orbit.

**Proof.** By assumption $\text{ext } F$ is an $H$-orbit. Hence it is a connected component of $\hat{O} \cap \mathfrak{h}$. In particular by Lemma 19 it is a symplectic submanifold of $\hat{O}$. By Corollary 27 $S := Z(H)^0$ is a nontrivial subtorus of $T$, which acts trivially on $\text{ext } F$. The momentum map for the $T$-action is the restriction of $\pi$. So by Lemma 32 $\sigma = \pi(\text{ext } F)$ is contained in a translate of $\mathfrak{s}^\perp$, i.e. $\mathfrak{s} \subset \sigma^\perp$. It follows that $Z_K(\sigma^\perp) \subset Z_K(\mathfrak{s}) = Z_K(S) = H$. Next consider the decomposition (17). We know that $L \cdot x_1 \subset \mathfrak{t}$ is a full orbit. Denoting by $\text{aff } (\cdot)$ the affine span

$$\text{aff } F = \text{aff } (\text{ext } F) = x_0 + \mathfrak{t}.$$ 

Since $x_0 \in \mathfrak{t}$, $(x_0 + \mathfrak{l}) \cap \mathfrak{t} = x_0 + (\mathfrak{l} \cap \mathfrak{t})$ and

$$\text{aff } \sigma = \text{aff } (F \cap \mathfrak{t}) \subset (\text{aff } F) \cap \mathfrak{t} = x_0 + (\mathfrak{l} \cap \mathfrak{t}).$$

Since $\mathfrak{l}$ is an ideal of $\mathfrak{t}$, it is the direct orthogonal sum of $\mathfrak{l} \cap \mathfrak{t}$ and some $Z_{\mathfrak{a}}$, see (11). Hence $\mathfrak{l} = (\mathfrak{l} \cap \mathfrak{t}) \perp (\mathfrak{l} \cap \mathfrak{t}^\perp)$. It follows that $\pi(\mathfrak{l}) = \mathfrak{l} \cap \mathfrak{t}$ and also, since $x_0 \in \mathfrak{t}$, that $\pi(x_0 + \mathfrak{l}) = x_0 + (\mathfrak{l} \cap \mathfrak{t})$. So

$$x_0 + (\mathfrak{l} \cap \mathfrak{t}) = \pi(x_0 + \mathfrak{l}) = \pi(\text{aff } F) \subset \text{aff } (\pi(F)) = \text{aff } \sigma.$$

From these two inclusions we get that $\text{aff } \sigma = x_0 + (\mathfrak{l} \cap \mathfrak{t})$. Therefore $\sigma^\perp$ is the orthogonal complement of $\mathfrak{l} \cap \mathfrak{t}$ in $\mathfrak{t}$. Since $\mathfrak{t} = \mathfrak{z} \perp (\mathfrak{l} \cap \mathfrak{t}) \perp (\mathfrak{l} \cap \mathfrak{t}^\perp)$, we get $\sigma^\perp = \mathfrak{z} \perp (\mathfrak{l} \cap \mathfrak{t}) \subset \mathfrak{z} \perp (\mathfrak{l} \cap \mathfrak{t}^\perp)$. So $[\mathfrak{l}, \sigma^\perp] \subset [\mathfrak{z}, \mathfrak{l} \cap \mathfrak{t}^\perp] = 0$ and $L \subset Z_K(\sigma^\perp)$. From the inclusions $L \subset Z_K(\sigma^\perp) \subset H$ and the fact that $L \cdot x = H \cdot x = \text{ext } F$ for any $x \in \text{ext } F$ we immediately get $Z_K(\sigma^\perp) \cdot x = \text{ext } F$. We already know (from Lemma 30) that $\sigma$ is a nonempty face of $P$. By Theorem 23 $\mathfrak{l} \neq \{0\}$, so $\sigma^\perp \neq \{0\}$, $\text{aff } \sigma \neq \mathfrak{t}$ and $\sigma \subsetneq P$. This shows that $\sigma$ is a proper face.

**Corollary 34.** Let $F_1, F_2$ be a proper faces of $\hat{O}$, let $H_1, H_2$ be corresponding subgroups as in Theorem 23 and let $T$ be a maximal torus of $K$ which is contained in both $H_1$ and $H_2$. If $F_1 \cap \mathfrak{t} = F_2 \cap \mathfrak{t}$, then $F_1 = F_2$.

**Proof.** Set $\sigma := F_1 \cap \mathfrak{t}$. Recall from (31) that $\text{ext } \sigma \subset \text{ext } F_1$ and pick $x \in \text{ext } \sigma$. Then we can apply the previous lemma to both faces and we get $\text{ext } F_1 = Z_K(\sigma^\perp) \cdot x = \text{ext } F_2$. The result follows.
If \( F \subset \hat{\mathcal{O}} \) is a face set
\[
H_F := \{ g \in K : gF = F \} \quad Z_F := Z(H_F)^0 \quad C_F := \{ u \in \mathfrak{t} : F = F_u(\hat{\mathcal{O}}) \}.
\]

The following is the main result of this section.

**Theorem 36.** All proper faces of \( \hat{\mathcal{O}} \) are exposed. More precisely, if \( F \) is a proper face \( F \subset \hat{\mathcal{O}} \), then

a) if \( T \subset H_F \) is a maximal torus, \( u \in \mathfrak{t} \) and \( F \cap \mathfrak{t} = F_u(P) \), then \( F = F_u(\hat{\mathcal{O}}) \);

b) there is a vector \( u \in \mathfrak{z}_F \) such that \( F = F_u(\hat{\mathcal{O}}) \);

c) if \( u \in C_F \cap \mathfrak{z}_F \), then \( H_F = Z_K(u) \) (in particular \( H_F \) is connected and \( Z_F \) has positive dimension);

d) the subgroup \( H_F \) satisfies (a)-(d) of Theorem \ref{Thm:Main}.\]

**Proof.** We start by proving (a) under the assumption that the maximal torus \( T \) is contained in some subgroup \( H \) that has the properties listed in Theorem \ref{Thm:Main}. By Lemma \ref{Lem:FaceExposure} \( \sigma := F \cap \mathfrak{t} = F \cap P \) is a proper face of \( P \). Since all faces of a polytope are exposed \cite[p. 95]{Ziegler}, there is a vector \( u \in \mathfrak{t} \) such that \( \sigma \) equals the exposed face of \( P \) defined by \( u \), i.e., \( \sigma = F_u(P) \). Since \( u \in \mathfrak{t} \) and \( P = \pi(\mathcal{O}) \), \( h_F(u) = \max_{\sigma \in O} \langle u, x \rangle = h_\mathcal{O}(u) \). Set \( F' := F_u(\hat{\mathcal{O}}) \). \( F' \) is a \( T \)-invariant face since \( u \) is fixed by \( T \). We wish to show that \( F = F' \).

The inclusion \( F \subset F' \) is immediate. Indeed if \( x \in F \), then \( \pi(x) \in \sigma \), so \( \langle x, u \rangle = h_F(u) = h_\mathcal{O}(u) \). It is also immediate that \( F' \cap \mathfrak{t} = \sigma \). So we have two faces \( F \) and \( F' \) with \( F \cap \mathfrak{t} = F' \cap \mathfrak{t} = \sigma \). Set \( H' := Z_K(u) \). By Lemma \ref{Lem:MaxFace} \( \operatorname{ext} F' = \operatorname{Max}(\Phi_u) \) is an \( H' \)-orbit and \( H' \) satisfies (a)-(d) of Theorem \ref{Thm:Main} for \( F' \). Clearly \( T \subset H' \) since \( u \in \mathfrak{t} \), and by hypothesis also \( T \subset H \). We can therefore apply Corollary \ref{Cor:FaceExposure} and we get \( F = F' \). In particular \( F = F_u(\hat{\mathcal{O}}) \) is an exposed face. We have thus proved (a) under the assumption that \( T \subset H \) for some \( H \) as in Theorem \ref{Thm:Main}. Next we show that the vector \( u \) can be chosen inside \( \mathfrak{z}_F \). The subgroup \( H_F \subset K \) is compact and preserves both \( \hat{\mathcal{O}} \) and \( F \). By Proposition \ref{Prop:FaceProperties} there is a vector \( u \in C_F \) that is fixed by \( H_F \). Note that \( H_F \) is of maximal rank since \( H \subset H_F \). If \( T \) is a maximal torus contained in \( H_F \), then \( u \) is is fixed by \( T \), so \( u \in \mathfrak{t} \subset \mathfrak{h}_F \). It follows that \( u \in \mathfrak{h}_F \) and since \( H_F \) fixes \( u \) it follows that \( u \in \mathfrak{z}_F \). Thus (b) is proved. To prove (c) assume that \( u \in \mathfrak{z}_F \) and that \( F = F_u(\hat{\mathcal{O}}) \). Then \( H_F \subset Z_K(u) \) since \( u \in \mathfrak{z}_F \). On the other hand \( \operatorname{ext} F = F_u(\hat{\mathcal{O}}) \cap \mathcal{O} = \operatorname{Max}(\Phi_u) = Z_K(u) \cdot \mathfrak{t} \) by Lemma \ref{Lem:MaxFace}. Therefore \( Z_K(u) \) preserves \( F \) and therefore \( Z_K(u) \subset H_F \) by definition. So \( H_F = Z_K(u) \) and (c) is proved. (d) follows from Lemma \ref{Lem:Main} and the fact that \( H_F = Z_K(u) \). Now we know that \( H_F \) itself has the properties of Theorem \ref{Thm:Main}. Hence (a) holds for any torus \( T \subset H_F \). \( \Box \)

**Remark 37.** In general the faces of an orbitope are not necessarily exposed. For example 4-dimensional Carathéodory orbitopes have non-exposed faces, see \cite[Thm. 1(5b)]{Caratheodory} (note that the author uses the word “facelet” for face
and “face” for exposed face). It is important to understand whether an orbitope has only exposed faces. Indeed this is Question 1 in [23]. The previous theorem shows that this is always the case for coadjoint orbitopes.

**Corollary 38.** If $O' \subset O$ is a smooth submanifold, then $\text{conv}(O')$ is a face of $\hat{O}$ if and only if there is a vector $u$ such that $O' = \text{Max}(\Phi_u)$.

**Proof.** Set $F = \text{conv}(O')$. From the fact that $O$ is contained in a sphere, it follows as in Lemma 14 that $\text{ext } F = O'$. Therefore the statement follows immediately from Lemma 20 and the fact that every face of $\hat{O}$ is exposed. □

This is a first characterization of the submanifolds that appear as $\text{ext } F$ for some face $F$. In §7 we will see that this characterization becomes much more transparent using the complex structure of $O$. An explicit characterization in terms of root data will be given in §6.

Various results about the faces have been established using some subgroup $H$ satisfying the properties stated in Theorem 23. Now we know that $H_F$ does satisfy these properties. Hence we can state those results more cleanly. This is done in Theorem 40 below. Next in Lemma 42 we will make precise the possible freedom in the choice of the group $H$. First of all decompose $H_F$ as in Lemma 28:

$$H_F = Z_F \cdot K_F \cdot K'_F. \quad (39)$$

$Z_F$ is defined in (35), $K_F$ is the product of the simple factors of $(H_F, H_F)$ that act nontrivially on $\text{ext } F$ and $K'_F$ is the product of the remaining factors.

**Theorem 40.** Let $T \subset K$ be a maximal torus.

a) If $F \subset \hat{O}$ is a proper $T$-invariant face, then $\sigma := F \cap t = \pi(F) = \pi(\text{ext } F)$ is a proper face of the momentum polytope $P$ and $\text{ext } F$ is a $Z_K(\sigma^\perp)$-orbit.

b) If $F_1$ and $F_2$ are $T$-invariant proper faces, then $F_1 \subset F_2$ if and only if $F_1 \cap t \subset F_2 \cap t$.

c) If $F_1$ and $F_2$ are $T$-invariant proper faces, then $F_1 = F_2$ if and only if $F_1 \cap t = F_2 \cap t$.

d) If $x \in \text{ext } F$, then $x = x_0 + x_1$ with $x_0 \in \mathfrak{z}_F$ and $x_1 \in \mathfrak{k}_F$. Moreover

$$\text{ext } F = x_0 + K_F \cdot x_1 \quad (41)$$

and $K_F \cdot x_1 \subset \mathfrak{t}_F$ is full.

**Proof.** If $F$ is $T$-stable, then $T \subset H_F$. So (a) follows from Lemma 33. (b) Set $\sigma_i := F_i \cap t$. If $F_1 \subset F_2$, then clearly $\sigma_1 \subset \sigma_2$. To prove the converse, assume that $\sigma_1 \subset \sigma_2$ and pick $x \in \sigma_1$. Then $Z_K(\sigma_1^\perp) \subset Z_K(\sigma_2^\perp)$ and $\text{ext } F_1 = Z_K(\sigma_1^\perp) \cdot x$. Thus $\text{ext } F_1 \subset \text{ext } F_2$. (c) follows immediately. (d) is just Lemma 28 stated for $H = H_F$. □

**Lemma 42.** If $F \subset \hat{O}$ is a face and $H \subset K$ is a connected subgroup, such that $F \subset \mathfrak{h}$ and $\text{ext } F$ is an $H$-orbit, then $K_F \subset H \subset H_F$ and $K_F = L$. 


Proof. Necessarily $F \neq \emptyset$. Since ext $F$ is an $H$-orbit, $H$ preserves ext $F$, hence $F$. So $H \subset H_F$ by definition \cite{23}. To prove the opposite inclusion, split as usual $H = Z \cdot L \cdot L'$ and write $x = x_0 + x_1$ as in Lemma \cite{23}. The orbit $L \cdot x_1 \subset \mathfrak{l}$ is full, so the affine span of $F$ is $x_0 + \mathfrak{l}$. Since also $H_F$ has the properties stated in Theorem \cite{23} we can repeat the same reasoning for $H_F$ instead of $H$. Thus we get that the affine span of $F$ is $x_0 + \mathfrak{t}_F$. Therefore $\mathfrak{l} = \mathfrak{t}_F$. So $L$ and $K_F$ are connected subgroups of $K$ with the same Lie algebra and therefore coincide. This implies $K_F = L \subset H$. \hfill $\square$

**Example 43.** Set $\mathfrak{f} = \mathfrak{su}(n+1) = \{ X \in \mathfrak{gl}(n+1, \mathbb{C}) : X + X^* = 0, \ Tr(X) = 0 \}$, $\mathcal{H} = \{ X \in \mathfrak{gl}(n+1) : X = X^* \}$ and $\mathcal{H}_1 = \{ X \in \mathcal{H} : Tr(X) = 1 \}$. We identify $\mathfrak{su}(n+1)$ with $\mathcal{H}_1$ using the map

$$\varphi : \mathfrak{su}(n+1) \to \mathcal{H}_1 \quad \varphi(X) = iX + \frac{\mathbf{1}_{n+1}}{n+1}.$$ 

The vector space of Hermitian matrices is endowed with an invariant scalar product, given by $\langle A, B \rangle = Tr(AB)$. Let $\mathcal{O} \subset \mathfrak{su}(n+1)$ be the coadjoint orbit corresponding to $\mathbb{P}^n(\mathbb{C})$ endowed with the Fubini-Study metric. Then $\mathcal{O}' = \varphi(\mathcal{O})$ is the set of orthogonal projectors onto lines, i.e.

$$\mathcal{O}' = \{ A \in \mathcal{H} : A^2 = A, \ rank(A) = 1 \}.$$ 

Using the spectral theorem it is easy to check that

$$\mathcal{O}' = \{ A \in \mathcal{H}_1 : A \geq 0, \ rank(A) = 1 \}$$

and

$$\mathcal{O}'' = \{ A \in \mathcal{H}_1 : A \geq 0 \}.$$ 

Given a Hermitian matrix $u \neq 0$ we wish to study the face

$$F := F_u(\mathcal{O}')$$

We can assume that $u$ be tangent to $\mathcal{H}_1$, i.e. $Tr\, u = 0$. Let

$$\mathbb{C}^{n+1} = V_1 \oplus \cdots \oplus V_s$$

be its eigenspace decomposition, i.e. $u|_{V_i} = \mu_i \mathbf{1}_{V_i}$. Since $u \neq 0$ and $Tr\, u = 0$ $s > 1$. We assume $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_s$. Let

$$\Phi : \mathcal{O}' \to \mathbb{R} \quad \Phi_u(x) = \langle u, x \rangle$$

be the height function with respect to $u$. The critical set of $\Phi_u$ is $\{ A \in \mathcal{O}' : [A, u] = 0 \}$. Since $[A, u] = 0$ if and only if $A(V_i) \subset V_i$, it follows that this is the set of projectors onto lines that are contained in some of the $V_i$’s, i.e. $Crit(\Phi_u) = \mathbb{P}(V_1) \sqcup \cdots \sqcup \mathbb{P}(V_s)$. For the same reason

$$Z_{SU(n+1)}(u) = S(U(V_1) \times \cdots \times U(V_s)).$$

Let $v_i$ be a non zero vector of $V_i$ and let $P_{v_i}$ denote the orthogonal projection onto the complex line $\mathbb{C} v_i$. Then

$$\mathbb{P}(V_i) = Z_{SU(n+1)}(u) \cdot P_{v_i}. $$
If \( A \in \text{Crit}(\Phi_u) \), then

\[
\Phi_u(A) = \mu_1 \text{Tr}(A|_{V_1}) + \cdots + \mu_s \text{Tr}(A|_{V_s}).
\]

Since \( \text{Tr}(A|_{V_i}) \geq 0 \) and \( s \sum_{i=1}^s \text{Tr}(A|_{V_i}) = \text{Tr}A = 1 \),
the maximum of \( \Phi_u \) is equal to \( \mu_s \) and it is attained exactly on \( \mathbb{P}(V_s) \). This means that
\[
\text{ext } F = \text{Max}(\Phi_u) = \mathbb{P}(V_s) \subset O'.
\]

So \( F \) consists of the operators in \( \widehat{O}' \) that are supported on \( V_s \). Notice that \( H_F = S(U(V_s) \times U(V_s^\perp)) \) and \( \mathcal{Z}_F = i\mathcal{R}v \) where \( v \) is the Hermitian operator such that \( v|_{V_s} = \text{Id}_{\dim V_s} \) \( v|_{V_s^\perp} = -\text{Id}_{\dim V_s^\perp} \).

In fact \( F = F_v(\widehat{O}') \). In particular in this example \( C_F \) is much larger than \( \mathcal{Z}_F \cap C_F \). The above computation shows that to each face corresponds a subspace, namely \( V_s \). Viceversa, given a subspace \( W \subset \mathbb{C}^{n+1} \), let \( w \) be the Hermitian operator such that \( w|_W = \text{Id}_{\dim W} \) \( w|_{W^\perp} = -\text{Id}_{\dim W^\perp} \).

Then
\[
F_w(\widehat{O}) = \{ A \in H_1 : A \geq 0, A|_{W^\perp} = 0 \} = \text{conv}(\mathbb{P}(W)).
\]

Therefore faces of \( \widehat{O}' \) are in one-to-one correspondence with the subspaces of \( \mathbb{C}^{n+1} \).

### 4. The role of the momentum polytope

Consider a full orbit \( O \subset \mathfrak{k} \), a maximal torus \( T \subset K \) and the momentum polytope \( P \). In this section we will study in detail the relation between the faces of \( \widehat{O} \) and those of \( P \). Denote by \( \mathcal{F}(\widehat{O}) \) the set of proper faces of \( O \) and by \( \mathcal{F}(P) \) the proper faces of the polytope \( P \). If \( F \) is a face of \( O \) and \( a \in K \), then \( a \cdot F \) is still a face, so \( K \) acts on \( \mathcal{F}(\widehat{O}) \). Similarly \( W = W(K,T) \) acts on \( \mathcal{F}(P) \). We wish to show that \( \mathcal{F}(\widehat{O})/K \cong \mathcal{F}(P)/W \).

**Lemma 44.** If \( F \) is a face of \( O \), there is a \( T \)-stable face \( F' \) which is conjugate to \( F \), i.e. \( F' = a \cdot F \) for some \( a \in K \). \( F' \) is unique up to conjugation by elements of \( N_K(T) \).

**Proof.** By Corollary 25 \( F \) is preserved by some maximal torus \( S \subset K \). There is \( a \in K \) such that \( S = a^{-1}Ta \). Hence \( F' = a \cdot F \) is preserved by \( T \). To prove uniqueness assume that \( F_1 \) and \( F_2 \) be \( T \)-stable faces of \( O \) and that \( F_2 = a \cdot F_1 \) for some \( a \in K \). Then \( H_{F_2} = aH_{F_1}a^{-1} \). In particular both \( T \) and
$aTa^{-1}$ are contained in $H_{F_2}$, so there is $b \in H_{F_2}$, such that $aTa^{-1} = bTb^{-1}$. Then $w = b^{-1}a \in N_K(T)$ and $w \cdot F_1 = b^{-1}a \cdot F_1 = b^{-1}F_2 = F_2$. □

Define a map

$$\varphi : \mathcal{F}(\hat{O})/K \to \mathcal{F}(P)/W$$

by the following rule: given $[F] \in \mathcal{F}(\hat{O})$ choose a $T$-invariant representative $F$ and set $\varphi([F]) := [F \cap t]$. By Lemma 30 $F \cap t$ is indeed a face of the polytope. By Lemma 44 if $F'$ is $T$-stable and $[F'] = [F]$ then $F' \cap t$ and $F \cap t$ are interchanged by some element of $W$. This shows that the map $\varphi$ is well-defined.

Now fix a face $F$ of $\hat{O}$ and a maximal torus $T \subset H_F$. Since $T \cap K_F$ is a maximal torus of $K_F$ and $T \cap K_F'$ is a maximal torus of $K_F'$, corresponding to the decomposition (39) there is a splitting

$$t = \mathfrak{z}_F \oplus (t \cap \mathfrak{t}_F) \oplus (t \cap \mathfrak{t}_F').$$

Denote by $W_F$ and $W_F'$ the Weyl groups of $(K_F, K_F \cap T)$ and $(K_F', K_F' \cap T)$ respectively. $W_F$ and $W_F'$ can be considered as subgroups of $W$. They commute and have the following sets of invariant vectors:

$$t^{W_F} = \mathfrak{z}_F \oplus \mathfrak{t}_F', \quad t^{W_F'} = \mathfrak{z}_F \oplus \mathfrak{t}_F, \quad t^{W_F \times W_F'} = \mathfrak{z}_F.$$

**Lemma 45.** Let $T \subset K$ be a maximal torus and let $F$ be a nonempty $T$-invariant face of $\mathcal{O}$. Set $\sigma := F \cap t$. Then (i) $W_F \times W_F'$ preserves $\sigma$; (ii) $F = H_F \cdot \sigma = K_F' \cdot \sigma$.

**Proof.** Recall that $\text{ext} F = x_0 + K_F \cdot x_1$. By Kostant theorem $\sigma = \pi(\text{ext} F) = \pi(x_0 + K_F \cdot x_1) = x_0 + \text{conv}(W_F \cdot x_1) = \text{conv}(W_F \cdot x)$. Hence $W_F$ preserves $\sigma$. Moreover $\sigma \subset \mathfrak{z}_F \oplus (t \cap \mathfrak{t}_F)$ hence $W_F'$ fixes $\sigma$ pointwise and (i) follows. Similarly, since $\sigma \subset \mathfrak{z}_F \oplus \mathfrak{t}_F$, $Z_F \cdot K_F'$ fixes $\sigma$ pointwise. Therefore $H_F \cdot \sigma = K_F \cdot \sigma$. By Lemma 29 $K_F \cdot (\sigma - x_0)$ is convex and the same is true of $x_0 + K_F \cdot (\sigma - x_0) = K_F' \cdot \sigma$. So $H_F \cdot \sigma = K_F' \cdot \sigma$ is convex. Since $\text{ext} F = H_F \cdot x \subset H_F \cdot \sigma$, it follows that $F \subset H_F \cdot \sigma$. On the other hand $\sigma \subset F$ and $F$ is $H_F$-invariant, so also $H_F \cdot \sigma \subset F$. This establishes (ii). □

If $\sigma$ is a face of $P$ set

$$G_\sigma := \{ g \in W : g(\sigma) = \sigma \}.$$ 

**Lemma 46.** If $\sigma \in \mathcal{F}(P)$ there is a vector $u \in t$ that is fixed by $G_\sigma$ and such that $\sigma = F_u(P)$. If $u$ is any such vector and $F := F_u(\mathcal{O})$, then $F \cap t = \sigma$, $G_\sigma = W_F \times W_F'$, $\mathfrak{z}_F = t^{G_\sigma}$ and $F$ does not depend on $u$ but only on $\sigma$.

**Proof.** The existence of $u$ follows directly from Lemma 45. By Lemma 29 (i) $W_F \times W_F' \subset G_\sigma$, so $u \in t^{W_F \times W_F'} = \mathfrak{z}_F$ and using Theorem 29 it follows that $H_F = C_K(u)$. Therefore the subgroup of $W$ that fixes $u$ is the Weyl group of $(H_F, T)$ i.e. $W_F \times W_F' = G_\sigma$. From this it follows that $\mathfrak{z}_F = t^{G_\sigma}$, that $H_F = C_K(\mathfrak{z}_F) = C_K(t^{G_\sigma})$ and in particular that $H_F$ and hence ext $F$ and $F$ only depend on $\sigma$. □
Define a map
\[ \psi: \mathcal{F}(P)/W \to \mathcal{F}(\hat{O})/K \]
by the following rule: given \( \sigma \), fix \( u \in t^G \sigma \) such that \( \sigma = F_u(P) \) and set
\[ \psi([\sigma]) := [F_u(\hat{O})]. \]
Thanks to the previous lemma \( F_u(\hat{O}) \) depends only on \( \sigma \), not on \( u \). It is clear that \( \psi \) is well-defined on equivalence classes.

**Proposition 47.** The maps \( \psi \) and \( \varphi \) are inverse to each other.

**Proof.** Let \( \sigma \) be a face of \( P \). Choose \( u \in t^G \sigma \) such that \( \sigma = F_u(P) \). Then \( F_u(\hat{O}) \) is \( T \)-stable, so \( \varphi \circ \psi([\sigma]) = \varphi([F_u(\hat{O})]) = [F_u(\hat{O}) \cap t] = [\sigma] \). So \( \varphi \circ \psi \) is the identity. It follows immediately from Theorem 40 (c) that \( \varphi \) is injective. Hence it is a bijection and \( \psi = \varphi^{-1} \). \( \square \)

5. **Smooth stratification**

As we saw in the previous section the group \( K \) acts on \( \mathcal{F}(\hat{O}) \), which is the set of faces of \( \hat{O} \) and this action has a finite number of orbits, which are in one-to-one correspondence with the orbits of the Weyl group on the finite set \( \mathcal{F}(P) \). Let \( B \) denote one of the orbits of \( K \) on \( \mathcal{F}(\hat{O}) \). We call \( B \) a face type. The set
\[ S_B := \bigcup_{F \in B} \text{relint } F. \]
is a subset of \( \partial \hat{O} \), because the faces \( F \in B \) are proper. Since every boundary point lies in exactly one open face (Theorem 4)
\[ \partial \hat{O} = \bigsqcup_{B \in \mathcal{F}(\hat{O})/K} S_B. \]
We call \( S_B \) the stratum corresponding to the face type \( B \). The purpose of this section is to show that the strata \( S_B \) yield a stratification of \( \hat{O} \) in the following sense.

**Theorem 48.** The strata are smooth embedded submanifolds of \( \mathfrak{k} \) and are locally closed in \( \partial \hat{O} \). For any stratum \( S_B \) the boundary \( \overline{S_B} - S_B \) is the disjoint union of strata of lower dimension.

There is an obvious map \( p: S_B \to B \) which maps a point \( x \in S_B \) to the unique face \( F \) such that \( x \in \text{relint } F \). To study \( S_B \) it is expedient to fix an element \( F \in B \). Thus \( B = \{ g \cdot F : g \in K \} \cong K/H_F \) and
\[ S_B = K \cdot \text{relint } F = \{ g \cdot x : g \in K, x \in \text{relint } F \}. \]
\( K \to K/H_F \) is a right principal bundle with structure group \( H_F \). Let
\[ \mathcal{E}_F = K \times^{H_F} \text{relint } F \]
be the associated bundle gotten from the action of $H_F$ on relint $F$. Note that $\mathcal{E}_F \to K/H_F$ is a homogeneous bundle in the sense that the left action of $K$ on $K/H_F$ lifts to an action of $K$ on $\mathcal{E}_F$ that is given by the following rule

$$a \cdot [g, x] := [ag, x] \quad a, g \in K, \ x \in \text{relint } F$$

(Here $[g, x]$ is the point in the associated bundle.)

**Proposition 49.** Let $B$ be a face type and let $F \in B$ be a representative. Define a map

$$f : \mathcal{E}_F \to \mathfrak{k} \quad f([g, x]) = g \cdot x.$$  

Then $f$ is a smooth $K$-equivariant embedding of $\mathcal{E}_F$ into $\mathfrak{k}$ with image $\mathcal{S}_B$. Therefore $\mathcal{S}_B$ is a smooth embedded submanifold of $\mathfrak{k}$. Moreover $p : \mathcal{S}_B \to B$ is a smooth fibre bundle.

**Proof.** It is straightforward to check that $f$ is well-defined, smooth and equivariant. It is also clear that $f(\mathcal{E}_F) = \mathcal{S}_B$. We proceed by showing that $f$ is injective. Recall from Theorem 6 that if $F_1$ and $F_2$ are different faces, then relint $F_1 \cap \text{relint } F_2 = \emptyset$. If $f([g, x]) = f([g_1, x_1])$ then $g_1^{-1} g \cdot x = x_1$. Since $x_1 \in \text{relint } F$ and $g_1^{-1} g \cdot x \in \text{relint}(g_1^{-1} g \cdot F)$ we get $g_1^{-1} g F = F$, so $[g, x] = [g_1, x_1]$ in $\mathcal{E}_F$. This shows that $f$ is injective. Next we show that $f$ is an immersion. Denote by $V$ the fibre of $\mathcal{E}_F$ over the origin of $K/H_F$. Since $\mathcal{E}_F$ is a homogeneous bundle and $f$ is equivariant, it is enough to show injectivity of $df_p$ at points $p \in V$, i.e. at points of the form $p = [e, x]$, $x \in \text{relint } F$. At such points

$$T_p \mathcal{E}_F = T_p V \oplus U$$

with

$$U = \left\{ \frac{d}{dt}_{|t=0} \exp(tv), x : v \in \mathfrak{h}_F^1 \right\}.$$  

Indeed $T_p V$ is the vertical space, while $U$ is the tangent space at $p$ of a local section of $K \to K/H_F$. The injectivity of $df_p$ will follow from the following three facts: a) $df_p|V$ is injective; b) $df_p|U$ is injective; c) $df_p(V) \cap df_p(U) = \{0\}$. (a) follows from the fact that $f|V$ is a diffeomorphism of $V$ onto relint $F$. To prove (b) observe first that if $x \in \text{relint } F$, then $\mathfrak{e}_x \subset \mathfrak{h}_F$. Indeed if $g \in K_x$ then $g \cdot x = x \in \text{relint}(g \cdot F) \cap \text{relint } F$, so $g \cdot F = F$ by Theorem 6 and $g \in H_F$. Therefore $K_x \subset H_F$ and $\mathfrak{e}_x \subset \mathfrak{h}_F$, as claimed. Now let $u$ be an element of $U$. By definition there is $v \in \mathfrak{k}$ such that

$$u := \left. \frac{d}{dt} \right|_{t=0} \exp(tv), x.$$  

Then

$$df_p(u) = \left. \frac{d}{dt} \right|_{t=0} f\left([\exp(tv), x]\right) = \left. \frac{d}{dt} \right|_{t=0} \exp(tv) \cdot x = [v, x].$$
(The bracket on right is the Lie bracket in \( \mathfrak{k} \)) If \( df_p(u) = 0 \), then \( [v, x] = 0 \) and \( v \in \mathfrak{k}_F \). Since \( v \in \mathfrak{h}_F^\perp \), this means that \( v = 0 \). Thus (b) is proved. Now observe that \( [\mathfrak{h}_F, \mathfrak{h}_F^\perp] \subset \mathfrak{h}_F^\perp \), since the adjoint action of \( H_F \) preserves \( \mathfrak{h}_F \) and \( \mathfrak{h}_F^\perp \). If \( v \in \mathfrak{h}_F^\perp \) and \( u \in U \) is given by \( (50) \), then \( df_p(u) = [v, x] \in \mathfrak{h}_F^\perp \) since \( x \in F \subset \mathfrak{h}_F \). So \( df_p(U) \subset \mathfrak{h}_F^\perp \). On the other hand \( df_p(T_pV) = T_{f(p)}(\text{relint} F) \subset \mathfrak{h}_F \). It follows that
\[
df_p(T_pV) \cap df_p(U) \subset \mathfrak{h}_F \cap \mathfrak{h}_F^\perp = \{0\}.
\]
Thus (c) is proved and \( f \) is an immersion. In order to prove that it is an embedding we shall prove that \( f \) is proper as a map \( f : \mathcal{E}_F \to \mathcal{S}_B = f(\mathcal{E}_F) \). Let \( \{y_n\} \) be a sequence in \( \mathcal{S}_B \) converging to some point \( y \in \mathcal{S}_B \). Set \( [g_n, x_n] := f^{-1}(y_n) \). We wish to show that \( \{[g_n, x_n]\} \) admits a convergent subsequence. Since \( K \) is compact by extracting a subsequence we can assume that \( g_n \to g \). Then \( y_n = f([g_n, x_n]) = g_n \cdot x_n \). Therefore \( x_n = g_n^{-1} \cdot y_n \to x := g^{-1} \cdot y \). Since \( y \in \mathcal{S}_B \), \( y \in \text{relint}(g^{-1} F) \) and \( x \in \text{relint} F \). Therefore \( [g_n, x_n] \to [g, x] \) as desired. \( \square \)

**Lemma 51.** If \( B \) is the face type of \( F \), then
\[
\dim \mathcal{S}_B = \dim K - \dim K'_F - \dim Z_F.
\]

**Proof.** \( \mathcal{S}_B \) is a fiber bundle over \( K/H_F \) with fibre \( \text{relint} F \). Since \( \dim F = \dim \mathfrak{t}_F \) we get the result. \( \square \)

We introduce a partial order on the face types, as follows: \( B_1 \preceq B_2 \) if for some (and hence for any) choice of representatives \( F_i \in B_i \) there is some \( g \in K \) such that \( gF_1 \subset F_2 \). This is a partial order. We write \( B_1 \prec B_2 \) if \( B_1 \preceq B_2 \) and \( B_1 \neq B_2 \).

**Proof of Theorem 48.** We already know that the strata are smooth embedded submanifold of \( \mathfrak{t} \). In particular they are locally closed subsets both of \( \mathfrak{t} \) and of \( \mathcal{O} \). By Prop. 49 \( \mathcal{S}_B = f(\mathcal{E}_F) = f(\mathcal{K} \times^{H_F} \text{relint} F) \). So
\[
\mathcal{S}_B = f(\mathcal{K} \times^{H_F} F) = \bigcup_{F \in B} F.
\]

Since any face \( F \) is the disjoint union of all proper faces contained in \( F \)
\[
\mathcal{S}_B = \bigcup_{F \in B} \text{relint} F \cup \bigcup_{C \prec B} \bigcup_{G \subset C} \text{relint} G = \mathcal{S}_B \cup \bigcup_{C \prec B} \mathcal{S}_C.
\]

To conclude we need to show that \( \dim \mathcal{S}_C < \dim \mathcal{S}_B \) if \( C \prec B \). Fix representatives \( F \in B \) and \( G \in C \) such that \( G \subseteq F \). By the previous lemma it is enough to show that \( \dim Z_F + \dim K'_F < \dim Z_G + \dim K'_G \). In fact \( Z_F \cdot K'_F \) fixes \( G \) pointwise since \( G \subset F \). Therefore \( Z_F \cdot K'_F \subset H_G \). On the other hand if \( x \in G \), then \( \text{aff}(G) = x + \mathfrak{t}_G \subset \text{aff}(F) = x + \mathfrak{t}_F \). Hence \( K_G \subset K_F \). It follows that \( \mathfrak{z}_F \oplus \mathfrak{t}_F^* \subset \mathfrak{t}_G \). Since \( \mathfrak{t}_G \) is semisimple, this shows that \( \mathfrak{z}_F \oplus \mathfrak{t}_F^* \perp \mathfrak{t}_G \). But \( \mathfrak{z}_F \oplus \mathfrak{t}_F^* \subset \mathfrak{h}_G \), so in fact \( \mathfrak{z}_F \oplus \mathfrak{t}_F^* \subset \mathfrak{z}_G \oplus \mathfrak{t}_G \). This proves the inequality \( \dim Z_F + \dim K'_F \leq \dim Z_G + \dim K'_G \). In the
case of equality, we would get \( Z_F \cdot K'_F = Z_G \cdot K'_G \), so \( Z_F = Z_G \), \( H_F = H_G \) and hence \( \text{ext } F = \text{ext } G \) and \( F = G \). □

Example 52. We shall describe the strata of the orbitope \( \mathring{O}' \) studied in Example 43. We saw there that the faces of \( \mathring{O}' \) are in one-to-one correspondence with subspaces of \( \mathbb{C}^{n+1} \). Two subspaces are interchanged by an element of \( \text{SU}(n + 1) \) if and only if they have the same dimension. So the orbit types are indexed by the dimension. Let \( W \subset \mathbb{C}^{n+1} \) be a subspace of dimension \( k \), let \( F = \text{conv}(\mathcal{P}(W)) \) be the corresponding face and let \( B \) be the orbit type of \( F \). Then

\[
B \cong K/H_F = \text{SU}(n + 1)/\text{S(U}(W) \times \text{U}(W^\perp)).
\]

Therefore \( B \) is simply the Grassmannian \( \mathbb{G}(k, n + 1) \). Since \( \text{relint } F = \{ A \in F : \text{rank } A = k \} \), it follows that

\[
\mathcal{S}_B = \{ A \in \mathcal{H}_1 : A \geq 0, \text{ rank } = k \}.
\]

In fact this is a bundle over the Grassmannian of \( k \)-planes. Finally, notice that \( H_F \) acts on \( \text{relint } F \) simply by the adjoint action of \( \text{SU}(W) \).

6. Satake combinatorics of the faces

In this section we describe the faces of \( \mathring{O} \) and the faces of the momentum polytope in terms of root data. The description uses the notion of \( x \)-connected subset of simple roots, which was introduced in [24]. In that paper Satake introduced certain compactifications of a symmetric space of noncompact type (the Satake-Furstenberg compactifications). The notion of \( x \)-connected subset was used in the study of the boundary components of these compactifications. It is no coincidence that faces of \( \mathring{O} \) and boundary components admit a description in terms of the same combinatorial data: in fact it was shown in [6] that the Satake compactifications of the symmetric space \( K^\mathbb{C}/K \) are homeomorphic to convex hulls of integral coadjoint orbit of \( K \). Here we do not use the link with the compactifications. Instead we show directly how to construct all the faces of \( \mathring{O} \) (up to conjugation) starting from the root data. This is accomplished for a general coadjoint orbit with no integrality assumption.

Fix a maximal torus \( T \) of \( K \) and a system of simple roots \( \Pi \subset \Delta = \Delta(\mathfrak{t}^\mathbb{C}, \mathfrak{k}^\mathbb{C}) \). As usual we identify \( \mathfrak{t}^\mathbb{C} \) with its dual using the Killing form \( B \). The roots get identified with elements of it.

**Definition 53.** A subset \( E \subset \mathfrak{t}^\mathbb{C} \) is connected if there is no pair of disjoint subsets \( D, C \subset E \) such that \( D \sqcup C = E \), and \( \langle x, y \rangle = 0 \) for any \( x \in D \) and for any \( y \in C \).

(A thorough discussion of connected subsets can be found in [22, §5].) Connected components are defined as usual. For example the connected components of \( \Pi \) are the subsets corresponding to the simple roots of the simple ideals in \( \mathfrak{k} \).
**Definition 54.** If $x$ is a nonzero vector of $t$, a subset $I \subset \Pi$ is called $x$-connected if $I \cup \{ix\}$ is connected.

Equivalently $I \subset \Pi$ is $x$-connected if and only if every connected component of $I$ contains at least one root $\alpha$ such that $\alpha(x) \neq 0$.

**Definition 55.** If $I \subset \Pi$ is $x$-connected, denote by $I'$ the collection of all simple roots orthogonal to $\{ix\} \cup I$. The set $J := I \cup I'$ is called the $x$-saturation of $I$.

The largest $x$-connected subset contained in $J$ is $I$. So $J$ is determined by $I$ and $I$ is determined by $J$. Given a subset $E \subset \Pi$ we will use the following notation:

$$t_{E} := t \cap \bigcap_{\alpha \in E} \ker \alpha$$
$$\Delta_{E} = \Delta \cap \text{span}_{\mathbb{R}}(E)$$
$$\Delta_{E,+} = \Delta_{E} \cap \Delta_{+}$$
$$t^{E} = \sum_{\alpha \in E} \mathbb{R}iH_{\alpha} = \text{orthogonal complement of } t_{E} \text{ in } t$$
$$\mathfrak{h}_{E} := t \oplus \bigoplus_{\alpha \in \Delta_{E,+}} Z_{\alpha}$$
$$t_{E} := t \oplus \bigoplus_{\alpha \in \Delta_{E,+}} Z_{\alpha}.$$ 

We denote by $T_{E}$, $H_{E}$, $K_{E}$ the corresponding connected subgroups. Note that $H_{E}$ is the subgroup associated to the subset $E \subset \Pi$, while $H_{F}$ is the subset associated to the face $F \subset \hat{O}$. This should cause no confusion.

**Lemma 56.** Let $O$ be a full coadjoint orbit and let $F \subset \hat{O}$ be a proper face. Assume that $u \in C_{F}$ and that $v \in C_{F} \cap \mathfrak{z}_{F}$. Let $\alpha \in \Delta$.

a) If $\alpha(u) = 0$, then $\alpha(v) = 0$.

b) If $-i\alpha(u) > 0$, then $-i\alpha(v) \geq 0$;

**Proof.** (a) $Z_{K}(u) \subset H_{F}$, since $F = F_{u}(\hat{O})$, and $H_{F} = Z_{K}(v)$ by Theorem 36. If $\alpha(u) = 0$, then $Z_{\alpha} \subset \mathfrak{z}_{t}(u) \subset \mathfrak{h}_{F} = \mathfrak{z}_{t}(v)$, hence $\alpha(v) = 0$. (b) Assume by contradiction that $-i\alpha(v) < 0$. Set $u_{t} = (1 - t)u + tv$. By Proposition 5 $C_{F}$ is convex, so $u_{t} \in C_{F}$ for any $t \in [0, 1]$. Since $-i\alpha(u) > 0$ and $-i\alpha(u_{t}) < 0$, there is some $s \in (0, 1)$ such that $\alpha(u_{s}) = 0$. Since $u_{s} \in C_{F}$ and $\alpha(v) \neq 0$, this would contradict (a). ∎

Denote by $C^{+}$ the positive Weyl chamber associated to $\Pi$. The following is immediate and well-known.

**Lemma 57.** If $v \in \overline{C^{+}}$, then $\mathfrak{z}_{t}(v) = \mathfrak{h}_{E}$ with $E = \{\alpha \in \Pi : \alpha(v) = 0\}$.

**Theorem 58.** Let $O$ be a full coadjoint orbit and let $x$ be the unique point in $O \cap \overline{C^{+}}$.

a) If $I \subset \Pi$ is $x$-connected and $J$ is its $x$-saturation, then

$$F := \text{conv}(H_{J} \cdot x)$$
is a face of $\hat{O}$. If $u \in t_J$ and $-i\alpha(u) > 0$ for any $\alpha \in \Pi - J$, then $F = F_u(\hat{O})$. Moreover

$$H_F = H_J \quad Z_F = T_J \quad K_F = K_I \quad K'_{F} = K'_I.$$  

(59)

b) Given an arbitrary subset $E \subset \Pi$, denote by $I$ the largest $x$-connected subset contained in $E$ and by $J$ the $x$-saturation of $I$. Then $H_E \cdot x = H_I \cdot x = H_J \cdot x$.

c) Any face of $\hat{O}$ is conjugate to one of the faces constructed in (a). More precisely, given a face $F$ and a maximal torus $T \subset H_F$ there are a base $\Pi \subset \Delta(t^E, t^C)$ and a subset $I \subset \Pi$ with the following properties: (i) if $C^+$ is the positive Weyl chamber corresponding to $\Pi$, then $\overline{C^+} \cap \text{ext} F \neq \emptyset$; (ii) if $x$ is the unique point in $\overline{C^+} \cap \text{ext} F$, then $I$ is $x$-connected and $F = \text{conv}(H_J \cdot x)$, where $J$ is the $x$-saturation of $I$.

Proof. (a) Since the set \{\alpha|_{t_J} : \alpha \in \Pi - J\} is a basis of $t^*_J$, we can pick $u \in t_J$ such that $\alpha(u) > 0$ for any $\alpha \in \Pi - J$. Then $Z_K(u) = H_J$. Set $F := F_u(\hat{O})$. We claim that $x \in F$. Indeed $x$ and $u$ belong to $\overline{C^+}$, so by Lemma 21 $x$ is a maximum point of $\Phi_u$, i.e. $x \in \text{ext} F$. By Lemma 20 $x \in F = Z_K(u) \cdot x$, so $F = \text{conv}(H_J \cdot x)$. This proves that $\text{conv}(H_J \cdot x)$ is indeed a face of $\hat{O}$.

By Lemma 12 $K_F \subset H_J = Z_K(u) \subset H_F$ and $K_F = K_I$. Pick $v \in C_F \cap t^F_J$ (this exists by Theorem 30). By Lemma 56 $-i\alpha(v) \geq 0$ for every $\alpha \in \Delta^+$, i.e. $v \in \overline{C^+}$. By Theorem 36 (c) and Lemma 57 $\Phi_F = \Phi_F(v) = \Phi_F$, where $E = \{\alpha \in \Pi : \alpha(v) = 0\}$. We claim that $E = J$. Indeed $\Phi_J \subset \Phi_F = \Phi_F$, so $J \subset E$. If we write $E = I \sqcup E'$, then $I' \subset E'$. Conversely, if $\alpha \in E'$, then $Z_{\alpha} \perp \mathfrak{t}_I = \mathfrak{t}_F$ (simply because the root space decomposition is orthogonal), so $Z_{\alpha} \subset t^F_J$. This entails on the one hand that $[Z_{\alpha}, t_I] = 0$, i.e. $\alpha \perp I$; on the other hand that $Z_{\alpha}$ fixes $x$, i.e. $\alpha(x) = 0$. This means in fact that $\alpha \in I'$. Hence $E = J$ as claimed and (59) follow.

(b) Split $E$ in connected components: $E = E_1 \sqcup \cdots \sqcup E_r$. We can assume that $E_j$ is $x$-connected iff $j \leq q$ for some $q$ between 1 and $r$. Then $I = E_1 \sqcup \cdots \sqcup E_q$. Set $E' := E - I = \sqcup_{j>q} E_j$. Then clearly $E' \subset I'$. So $E \subset J$. Let $F = \text{conv}(H_J \cdot x)$ be the face constructed from $J$ as in (a). Then $H_F = H_J$ and $K_F = K_I$. Since $I \subset E \subset J$, $K_I \subset H_E \subset H_J$. But $K_I \cdot x = K_F \cdot x = H_F \cdot x = H_J \cdot x$, so $H_E \cdot x = H_J \cdot x$ as desired.

(c) If $F = \hat{O}$, then $F = \text{conv}(H_J)$ with $I = J = \Pi$. Otherwise $F$ is a proper face. Fix a point $x \in \text{ext} F \cap \mathfrak{t}$. By Theorem 36 (b) there is a vector $u \in t_F$ such that $F = F_u(\hat{O})$. Then ext $F = \text{Max}(\Phi_u)$, so there is a Weyl chamber $\mathcal{C}$ such that $x, u \in \overline{\mathcal{C}}^+$. Let $\Pi$ be the base corresponding to $\mathcal{C}$. By Theorem 36 (c) $H_F = Z_K(u)$. Since $u \in \overline{\mathcal{C}}^+$, Lemma 57 says that $H_F = H_E$ with $E = \{\alpha \in \Pi : \alpha(u) = 0\}$. Let $I$ and $J$ be as in (b). Then $I$ is $x$-connected and using (b) we get ext $F = H_F \cdot x = H_E \cdot x = H_J \cdot x$. Thus $F = \text{conv}(H_J \cdot x)$ as desired. □
Remark 60. In the proof of (c) we have in fact that $E = J$. Indeed from (a) $H_F = H_J$, so $H_E = H_J$ i.e. $E = J$.

Example 61. Let $K = SU(n + 1)$, $n \geq 4$, and let $x \in \mathfrak{su}(n + 1)$ be the diagonal matrix $x = \text{diag}(i(n - 1), i(n - 1), -2i, \ldots, -2i)$. The coadjoint orbit through $x$ is the momentum image of the Grassmannian $G(2, n + 1)$. Let $t$ be the set of the diagonal matrices and denote by $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ the standard set of simple roots, i.e. $\alpha_i(\text{diag}(x_1, \ldots, x_{n+1})) = x_i - x_{i+1}$. The vector $x$ lies in the closure of the positive Weyl chamber containing $x$ and $\alpha_i(x) \neq 0$ if and only if $i = 2$. Therefore the $x$-connected subsets of $\Pi$ are the following:

a) $I^1_k = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}, 2 \leq k \leq n$;
b) $I^2_k = \{\alpha_2, \ldots, \alpha_k\}, 2 \leq k \leq n$.

For $i = 1, 2$ let $J^i_k$ be the $x$-saturation of $I^i_k$ and set $F^1_k = \text{conv}(H_{J^1_k} \cdot x)$. It is easy to check that $J^1_k = I^1_k \cup \{\alpha_{k+2}, \ldots, \alpha_n\}$. $K_{J^1_k}$ is the image of the embedding

$$\text{SU}(k + 1) \hookrightarrow \text{SU}(n + 1) \quad A \mapsto \begin{pmatrix} A & 0 \\ 0 & \text{Id} \end{pmatrix}$$

and $H_{J^1_k} = \text{S(U(k + 1) \times U(n - k))}$. Hence

$$\text{ext } F^1_k = K_{J^1_k} \cdot x = \text{SU}(k + 1)/\text{S(U(2) \times U(k - 1))},$$

is the complex Grassmannian $G(2, k + 1)$. The stratum corresponding to $F^1_k$ is a fiber bundle over $\text{SU}(n + 1)/\text{S(U(k + 1) \times U(n - k))} = G(k + 1, n + 1)$.

The $x$-saturation of $I^2_k$ is $J^2_k = I^2_k \cup \{\alpha_{k+2}, \ldots, \alpha_n\}$. $K_{J^2_k}$ is the image of the embedding

$$\text{SU}(k) \hookrightarrow \text{SU}(n + 1) \quad A \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \text{Id} \end{pmatrix}.$$

$H_{J^2_k} = \text{S(U(1) \times U(k) \times U(n - k))}$ and

$$\text{ext } F^2_k = K_{J^2_k} \cdot x = \text{SU}(k)/\text{S(U(1) \times U(k - 1))},$$

is a complex projective space $\mathbb{P}^{k-1}(\mathbb{C})$. The strata corresponding to $F^2_k$ is a fiber bundle over the flag manifold $\text{SU}(n + 1)/\text{S(U(1) \times U(k) \times U(n - k))}$.

7. Complex geometry of the faces

In the previous sections we have described the faces of $\hat{O}$ in terms of their extreme sets $\text{ext } F$ and have characterized the submanifolds $\text{ext } F \subset \mathcal{O}$ in various ways. Here we wish to add another characterization in terms of the complex structure of $\mathcal{O}$.

Theorem 62. Let $\mathcal{O}' \subset \mathcal{O}$ be a submanifold. The following conditions are equivalent.

a) $\mathcal{O}'$ is a compact orbit of a parabolic subgroup of $G$. 


b) There is a face $F$ of $\hat{O}$ such that $O' = \text{ext } F$.

c) $O'$ is compact and the subgroup

\[ P := \{ g \in G : g \cdot O' = O' \} \]

is a parabolic subgroup of $G$ that acts transitively on $O'$;

d) There are a maximal torus $T \subset K$, a Weyl chamber $C^+ \subset t$ and a subset $E$ of the corresponding set of simple roots $\Pi$ such that $O' \cap C^+ \neq \emptyset$ and $O'$ is an orbit of $H_E$.

**Proof.** That (d) is equivalent to (b) is the content of Theorem 58.

(a) $\Rightarrow$ (c) Since $O'$ is an orbit of some parabolic subgroup $Q$, the subgroup $P$ contains $Q$ so it is parabolic.

(c) $\Rightarrow$ (d) Since $P$ is parabolic we can find a maximal torus $T \subset K$ and a system of simple roots in $t$ in such a way that $B_- \subset P$. So $B_-$ acts on $O'$ and by the Borel fixed point theorem $B_-$ has some fixed point $x \in O'$. Since $x$ is fixed by $T \subset B_-$, $x \in t$ and it follows from Lemma 13 that $x \in C^+$. If $E \subset \Pi$ set

\[ u_E := \bigoplus_{\alpha \in \Delta^- - \Delta_E} g_\alpha \quad p_E := t^C \oplus \bigoplus_{\alpha \in \Delta^- - \Delta_E} g_\alpha. \]

Then $p_E = u_E^C \oplus u_E$ is a parabolic subalgebra. Denote by $U_E$ and $P_E$ the corresponding connected subgroups of $G$. Then $P_E$ is a parabolic subgroup, $U_E$ is its unipotent radical and $H_E^C$ is a Levi factor. In particular $P_E = H_E^C \cdot U_E$ and $U_E \triangleleft P_E$. Since $B_- \subset P$ there is some $E \subset \Pi$ such that $P = P_E$. Since $U_E \subset B_- \subset G_x$ we conclude that $O' = P_E \cdot x = H_E^C \cdot x$. As $O'$ is compact, the compact form $H_E$ must be transitive on $O'$. This concludes the proof.

(d) $\Rightarrow$ (a) First observe that $O' = H_E \cdot x$ is a complex submanifold since it is a connected component of the fixed point set of the torus $T_E$. Therefore $H_E^C$ preserves $O'$. By assumption there is $x \in C^+ \cap O'$. By Lemma 13 the stabilizer $G_x$ contains the negative Borel subgroup, so $U_E$ fixes $x$. If $x' \in O'$, there is $a \in H_E$ such that $x' = a \cdot x$. If $b \in U_E$ then $a^{-1}ba \in U_E$, so $a^{-1}ba \cdot x = x$ and $b \cdot x' = ba \cdot x = a \cdot x = x'$. Hence $U_E$ fixes pointwise $O'$. Therefore $P_E$ preserves $O'$ which is therefore a compact $P_E$-orbit. \qed

We notice that in condition (d) the set $E$ can be chosen to be the $x$-saturation of the maximal $x$-connected subset $I \subset E$ as shown in Theorem 58 (c).

The above result establishes a one-to-one correspondence between two rather distant classes of objects: on the one side the faces of the orbitope $\hat{O}$, on the other side the closed orbits of parabolic subgroups of $G$ inside $O$. To illustrate this correspondence recall the following fact.

**Lemma 64.** If $P \subset G$ is a parabolic subgroup, in $O$ there is only one orbit of $P$ which is closed.
Proof. Since the action is algebraic and $\mathcal{O}$ is a compact manifold, there is at least one orbit which is closed. Let $\mathcal{O}' \subset \mathcal{O}$ be a closed $P$-orbit and let $B \subset P$ be a Borel subgroup. Then $\mathcal{O}'$ is $B$-invariant, so it contains a closed $B$-orbit. But the $B$-orbits in $\mathcal{O}$ are just the Schubert cells and the only one which is closed is the fixed point of $B$. Hence any closed $P$-orbit contains this fixed point and this implies that the closed $P$-orbit is unique. □

The above uniqueness statement can also be considered from the point of view of the orbitope, as can be seen from the proof of the implication (c) ⇒ (d) in the previous theorem. Indeed, if $P$ is a parabolic subgroup, we write it as $P = P_E$ for some $E \subset \Pi$. Then there is a unique orbit of $H_E$ that is of the form $\text{ext} \mathcal{F}$, namely the orbit $H_E \cdot x$ for $x \in \mathcal{O} \cap \mathcal{F}$. Alternatively this orbit can be described as follows: choose $u \in t_E = z(H_E)$ such that $-i\alpha(u) > 0$ for $\alpha \in E$. Then the closed $P$-orbit is Max$(\Phi_u)$. In a sense to fix a parabolic subgroup $P_E$ is equivalent to fixing $H_E$ and the vector $u$. So once $P_E$ is fixed we know both $H_E$ and which component of $\mathcal{O} \cap h_E$ corresponds to the maximum of $\Phi_u$.

To conclude we wish to interpret geometrically condition (c) of Theorem 62. Let $\mathcal{O}' \subset \mathcal{O}$ be a complex submanifold of $\mathcal{O}$. Let $\mathcal{H}$ denote the Hilbert scheme of the projective manifold $\mathcal{O}$. If $Y \subset \mathcal{O}$ is a subscheme, let $[Y]$ be its Hilbert point. (See e.g. [1, Chapter IX].) The group $G$ acts on $\mathcal{H}$ by sending the Hilbert point $[Y]$ of a subscheme $Y \subset \mathcal{O}$ to $[g \cdot Y]$.

**Proposition 65.** Let $\mathcal{O}' \subset \mathcal{O}$ be a complex submanifold which is an orbit of some subgroup of $K$. Let $f : G \to \mathcal{H}$ be the map $f(g) := [g \cdot \mathcal{O}']$. Then the following conditions are all equivalent to condition (c) of Theorem 62:

i) $f(G)$ is compact;
ii) $f(K)$ is a subscheme of $\mathcal{H}$;
iii) $f(G) = f(K)$.

Proof. $f(G)$ is just the orbit of $G$ through the point $p = [\mathcal{O}'] \in \mathcal{H}$, while $f(K)$ is the orbit of $K$ through $p$. The subgroup $P$ defined in (63) is just the stabilizer $G_p$. Therefore $f(G) \cong G/P$. It follows immediately that the three conditions are equivalent to $P$ being parabolic, so they are implied by (c). Conversely, if they are satisfied, $P$ is parabolic. By assumption $\mathcal{O}'$ is an orbit of some subgroup $L \subset K$. Then $L \subset P$ and $\mathcal{O}'$ is a $P$-orbit, thus (c) holds. □

**Example 66.** Consider the orbitope of $\mathbb{P}^2(\mathbb{C})$ as described in Example 43. The complex lines satisfy the conditions in the proposition and in fact they do generate faces of $\hat{\mathcal{O}}$: if $\mathcal{O}' \subset \mathbb{P}^2(\mathbb{C})$ is a line the set $\text{conv}(\mathcal{O}')$ is a face of $\hat{\mathcal{O}}$. Also plane conics are complex submanifolds of $\mathbb{P}^2(\mathbb{C})$ that are homogeneous for a subgroup of $\text{SL}(3, \mathbb{C})$, namely $\text{SO}(3, \mathbb{C})$. Nevertheless the orbit of $\text{SL}(3, \mathbb{C})$ through a conic is not compact since smooth conics degenerate to singular ones. So conics do not satisfy the conditions above and in fact conics do not generate faces of $\hat{\mathcal{O}}$. 


Example 67. Let $L \subset K$ be the centralizer of a torus and let $O' \subset O$ be an orbit of $L$. As we have shown in general the set $F = \text{conv}(O')$ is not a face of $\hat{O}$. One condition is that $O' \subset l$. In fact if $L = Z_K(u)$, and $F = F_u(\hat{O})$, then $O' = \text{ext } F = \text{Max}(\Phi_u) \subset \text{Crit}(\Phi_u) = O \cap l$. This condition is not enough either. In fact $\text{Crit}(\Phi_u)$ will contain at least two orbits, one for the maximum and one for the minimum. These are "good" orbits, in the sense that they correspond to orbits, namely to $F_u(\hat{O})$ and $F_{-u}(\hat{O})$ respectively. The orbits in between in general do not generate faces. Consider the following example. Let $O \subset \mathfrak{su}(3)$ be the momentum image of the flag manifold of pairs $(L_1, L_2)$ where $L_1 \subset L_2 \subset \mathbb{C}^3$ and $\dim L_i = i$. Let $u = i\text{diag}(1, 1, -2)$. Set $V = \mathbb{C}^2 \times \{0\}$. Then $\text{Crit}(\Phi_u)$ has the following three connected components:

$$
C_1 = \{(L_1, L_2) \in O : L_1 \in \mathbb{P}(V), L_2 = L_1 \oplus \mathbb{C}e_3\}
$$
$$
C_2 = \{(L_1, L_2) \in O : L_1 \subset L_2 \subset V\}
$$
$$
C_3 = \{(L_1, L_2) \in O : L_1 = \mathbb{C}e_3\}.
$$

Each component is an orbit of $Z_K(u) = S(U(4) \times U(1))$. Let $P_i$ denote the stabilizer of $C_i$ for the action of $G = \text{SL}(3, \mathbb{C})$. Then $P_2 = \{g \in \text{SL}(5, \mathbb{C}) : g(V) = V\}$ and $P_3 = \{g : ge_3 = e_3\}$. These two subgroups are parabolic. So $C_2$ and $C_3$ correspond to faces, by Prop. [62]. On the other hand we claim that $P_1$ is the subgroup of $\text{SL}(5, \mathbb{C})$ of matrices of the form

$$
g = \begin{pmatrix} A & 0 \\ 0 & \lambda \end{pmatrix} \quad A \in \text{SL}(2, \mathbb{C}), \lambda \in \mathbb{C}^*. $$

It is clear that matrices of this form lie in $P_1$. Conversely assume $g \in P_1$. Then $g(V) = V$. Write $ge_3 = \lambda e_3 + w$ with $w \in V$. For any $v \in V - \{0\}$ the plane $g\text{span}(v, e_3) = \text{span}(gv, ge_3)$ contains $e_3$. Hence $w \in \text{span}(gv, e_3)$. Since $v \in V - \{0\}$ is arbitrary it follows that $w = 0$. The claim is proved, hence $P_1$ is not parabolic and $\text{conv}(C_1)$ is not a face of $\hat{O}$.

8. The case of an integral orbit

A coadjoint orbit $O \subset \mathfrak{k}$ is integral if $[\omega]/2\pi$ lies in the image of the natural morphism $H^2(O, \mathbb{Z}) \to H^2(O, \mathbb{R})$. (Here $\omega$ is the Kostant-Kirillov-Souriau form.) If $O$ is integral there is a complex line bundle $L \to O$ such that $[\omega] = 2\pi c_1(L)$. This line bundle can be made $K$-equivariant and holomorphic with respect to the structure $J$ on $O$ and it supports a unique $K$-invariant Hermitian bundle metric $h$ such that $\omega = iR(h)$. With this holomorphic structure the line bundle $L$ turns out to be very ample. Set $V := (H^0(O, L))^*$. Then $V$ inherits from $\omega$ and $h$ an $L^2$-scalar product. Moreover $V$ is an irreducible representation of $K$ and there is a unique orbit $M \subset \mathbb{P}(V)$ which is a complex submanifold of $\mathbb{P}(V)$. This orbit is simply connected. Fix on $M$ the restriction of the Fubini-Study form gotten from the $L^2$-scalar product on $V$. Since $K$ is semisimple there is a unique momentum map $\Phi : M \to \mathfrak{k}$ and $O = \Phi(M)$. Conversely, if there is an
irreducible $K$-representation $V$ such that $O = \Phi(M)$ for the unique complex orbit $M \subset \mathbb{P}(V)$, then $O$ is integral. This follows from the fact that \( \Phi \) is a symplectomorphism.

Another way to express integrality of $O$ is the following. Fix a maximal torus $T \subset K$ and choose a point $x \in O \cap t$. Recall that a linear functional $\lambda \in (it)^*$ is an algebraically integral weight if
\[
\frac{\langle \lambda, \alpha \rangle}{|\alpha|^2} = \frac{\lambda(H_{\alpha})}{|H_{\alpha}|^2} \in \mathbb{Z}
\]
for any root $\alpha \in \Delta(t, t)$, see e.g. [18, p. 265]. Then $O$ is integral if and only if $\lambda = \langle ix, \cdot \rangle$ is an algebraically integral weight. (For all this see [17, Chapter 1] or [19].)

**Theorem 68.** Let $O \subset k$ be an integral coadjoint orbit and let $F$ be a face of $\mathring{O}$. Write $\text{ext} \ F = x_0 + K_F \cdot x_1$ as in (41). Denote by $\langle \cdot, \cdot \rangle_F$ the scalar product on $k_F$ induced by the Killing form of $k_F$. Define $x_1' \in k_F$ by the following rule:
\[
\langle x_1', y \rangle_F = \langle x_1, y \rangle
\]
for all $y \in k_F$. Then $K_F \cdot x_1'$ is an integral coadjoint orbit in $k_F$.

**Proof.** This fact can be proved in a variety of ways using the various characterizations of integrality. One simple way is using the definition, i.e. the condition on the integrality of the Kostant-Kirillov-Souriau form. Let $\omega_F$ be the KSS form of $K_F \cdot x_1' \subset k_F$. Let $\mu \in k_F^*$ be the functional $\mu(y) = \langle x_1, y \rangle = \langle x_1', y \rangle_F$. The stabilizers (for the adjoint action) of $x_1$ and $x_1'$ are the same, because both coincide with the stabilizer of $\mu$ (for the coadjoint action). Moreover the stabilizers in $K_F$ of $x$ and of $x_1$ coincide since $x = x_0 + x_1$ and $x_0$ is fixed by $K_F$. Summing up we get that the stabilizers in $K_F$ of $x_1'$ and $x$ coincide. Hence the map
\[
j : K_F \cdot x_1' \hookrightarrow k \quad g \cdot x_1' \mapsto j(g \cdot x_1') := g \cdot x
\]
is an embedding of $K_F \cdot x_1'$ onto $\text{ext} \ F = K_F \cdot x \subset k$. We claim that $j^* \omega = \omega_F$. By equivariance it is enough to check that $j^* \omega = \omega_F$ at $x_1'$. Take $X, Y \in k_F$ and set $u = [X, x_1'], v = [Y, x_1']$. Then
\[
dj_{x_1'}(u) = \frac{d}{dt} \bigg|_{t=0} j(\text{Ad}(\exp tX) x_1') = \frac{d}{dt} \bigg|_{t=0} (\text{Ad}(\exp tX) x) = [X, x]
\]
and similarly $dj_{x_1'}(v) = [Y, x]$. Hence $j^* \omega(u, v) = \omega([X, x], [Y, x]) = \langle x, [X, Y] \rangle$. Since $[X, Y] \in k_F$ and $x_0 \in \mathfrak{z}_F$, $x_0 \perp [X, Y]$. Therefore $\langle x, [X, Y] \rangle = \langle x_1, [X, Y] \rangle = \langle x_1', [X, Y] \rangle_F = \omega_F(u, v)$. This proves that indeed $\omega_F = j^* \omega$ and thus $|\omega_f|/2\pi$ is integral if $|\omega|/2\pi$ is.

\[\square\]

**Remark 70.** Since the various definitions of integrality are equivalent, this theorem ensures that if $\langle ix, \cdot \rangle$ is an integral weight, then $\langle ix_1, \cdot \rangle_F$ is integral as well. Since integral weights give rise to representations, to each face $F$
of an integral coadjoint orbitope is attached an irreducible representation of $K_F$. If one fixes root data and $F$ is the face corresponding to an $x$-connected subset $I \subset \Pi$ as in \cite{bilotti2010satake}, then the representation corresponding to $F$ is the representation $V_I$ originally described by Satake \cite[p. 89]{satake1956compactifications} (see also \cite[p. 67]{borel2006compactifications}).

**Remark 71.** If $O$ is an integral orbit, then $O$ is the momentum image of a flag manifold $M$ provided with an invariant Hodge metric lying in a polarization $L \to M$. The space $H^0(M,L)$ is an irreducible representation $\tau$ of $K$. Out of these data one can construct a Satake-Furstenberg compactification $\overline{X}_\tau$ of the symmetric space $K^C/K$ and it is possible to define a homeomorphism (named after Bourguignon-Li-Yau) between this compactification and the orbitope $\hat{O}$. This was accomplished in \cite{bilotti2010satake}. Since this homeomorphism respects the boundary structure, some properties of the faces of $\hat{O}$ can be deduced in this way. The arguments in the present paper apply also to the non-integral case, give much more information and are more direct and geometric, since no use is made of the Bourguignon-Li-Yau map.

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