Caculus of Variation and the $L^2$-Bergman Metric on Teichmüller Space

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Abstract

The canonical metric on a surface is of nonpositive curvature, so it is natural to study harmonic maps between canonical metrics on a surface in a fixed homotopy class. Through this approach, we establish the $L^2$-Bergman metric on Teichmüller space as the second variation of energy functionals of chosen families of harmonic maps.

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1 Introduction

We present a geometric analytic approach to the $L^2$-Bergman metric on Teichmüller space of Riemann surfaces in this paper.

Teichmüller space $\mathcal{T}_g$ is the space of conformal structures on a compact, smooth, oriented, closed Riemann surface $\Sigma$ of genus $g \geq 1$, where two conformal structures $\sigma$ and $\rho$ are equivalent if there is a biholomorphic map between $(\Sigma, \sigma)$ and $(\Sigma, \rho)$ in the homotopy class of the identity map. When $g \geq 2$, $\sigma$ and $\rho$ are equivalent if there is a biholomorphic map between $(\Sigma, \sigma)$ and $(\Sigma, \rho)$ in the homotopy class of the identity map.

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Teichmüller space $T_g$ is naturally a complex manifold of complex dimension $3g - 3 > 1$, and the cotangent space at $\Sigma$ is identified with $QD(\Sigma)$, the space of holomorphic quadratic differentials. (1).

Since $T_g$ is a complex manifold, it is natural to study its metric geometry. There are several interesting metrics defined on Teichmüller space, all have advantages and disadvantages. These metrics reflect different perspectives of Teichmüller space. Among those metrics, two are named after S. Bergman. One of which, we still call it the Bergman metric, comes from the Bergman kernel function of a complex manifold: as a bounded complex domain $T_g$ carries an invariant Kählerian Bergman metric, defined by the line element

$$ds^2 = \sum \frac{\partial^2 \log K(z, z)}{\partial z \partial \bar{z}} dz_i \wedge d\bar{z}_j,$$

where $K(z, \zeta)$ is the Bergman kernel. This Bergman metric is complete (8).

The main object of this paper is the other metric which sometimes bears the name of Bergman. It is a Weil-Petersson type metric on Teichmüller space, i.e., it is obtained from duality by a $L^2$ inner product. In order to distinguish from the first Bergman metric, it may be appropriate to call this metric the $L^2$-Bergman metric.

From the classical Riemann surface theory, the period map $p: \Sigma \to J_{\Sigma}$ embeds the surface $\Sigma$ to its Jacobian $J_{\Sigma}$. The pullback metric of the flat metric on $J_{\Sigma}$ via this period map thus defines the so-called canonical metric or Bergman metric on $\Sigma$, denoted by $\rho_B$. This metric $\rho_B$ is of nonpositive Gaussian curvature, and when $g \geq 2$, the curvature vanishes if and only if the surface is hyperelliptic and only at $2g + 2$ Weierstrass points (9 12), in other words, the Gaussian curvatures characterize hyperelliptic surfaces. There is a unique canonical metric in every conformal structure.

The induced $L^2$-Bergman cometric is defined on $QD(\Sigma)$ by $L^2$-norm

$$\|\phi\|^2_B = \int_{\Sigma} \frac{\left|\phi\right|^2}{\rho_B},$$

thus we obtain a metric on Teichmüller space by duality. This is a Riemannian, Hermitian metric, invariant under the mapping class group.

This metric has been studied by Haberman and Jost who showed that it is incomplete (17). Roughly speaking, with respect to the $L^2$-Bergman metric, boundary points of the moduli space $M_g$ corresponding to pinching a nonseparating curve on the surface are at infinite distance from the interior, while boundary points of $M_g$ corresponding to pinching a separating curve on the surface are at finite distance from the interior. In a sense, the $L^2$-Bergman metric detects topology of the surface.

One of the motivations of this study is to compare the $L^2$-Bergman metric with more intensively studied Weil-Petersson metric. These two metrics are
both defined from duality from $L^2$ inner products, and they are both incomplete. However, the $L^2$-Bergman metric does not depend on the uniformization theorem. The difference between hyperbolic metric (constant curvature $-1$) and the canonical metric on the surface results in different behavior of the induced $L^2$ metrics on Teichmüller space. The Weil-Petersson metric is of negative curvature ([15], [18]), we are yet to understand the curvature properties of the $L^2$-Bergman metric.

In this paper, we take a variational approach to the study of the $L^2$-Bergman metric. To do so, we fix a conformal structure $(\sigma, z)$ with conformal coordinates $z$. For each canonical metric $\rho$ on the surface $\Sigma$, one obtains a quadratic differential $\phi(z)dz^2$ which is the Hopf differential of the unique harmonic map from $\sigma$ to $\rho$. This quadratic differential is holomorphic with respect to the conformal structure $(\sigma, z)$, therefore an element of the space $QD(\Sigma)$. We thus obtain a map $\phi$ between Teichmüller space $\mathcal{T}_g$ and $QD(\Sigma)$, sending $\rho$ to $\phi(z)dz^2$. We show that this map is a global homeomorphism, hence it provides global coordinates to $\mathcal{T}_g$. The following theorem is an analog to Wolf’s theorem in the case of hyperbolic metrics ([17]).

**Theorem 1.1.** The map $\phi : \mathcal{T}_g \rightarrow QD(\Sigma)$ is a homemorphism.

We note that, in the case of hyperbolic metrics, the injectivity of the map $\phi$ is a direct application of Bochner’s identities and maximum principle, as seen in [13], relying on the fact that hyperbolic metric is of constant curvature $-1$. In the case of canonical metrics, this is rather difficult since canonical metric has varied curvatures.

With the homeomorphism theorem in mind, we then consider a family of harmonic maps between canonical metrics on a surface and show that the second variation of an energy functional is the $L^2$-Bergman metric of two infinitesimal cotangent vectors on Teichmüller space. In the case of varying target metrics, we find:

**Theorem 1.2.** Let $w(t) : (\Sigma, \sigma(z)|dz|^2) \rightarrow (\Sigma, \rho(t)|dw|^2)$ be a family of harmonic maps between canonical metrics on surface $\Sigma$, where $\rho(0) = \sigma$, for $|t| < \epsilon$ small. Then the second variation of the energy functional of $w(t)$, at $t = 0$, is given by the $L^2$-Bergman metric of infinitesimal holomorphic quadratic differentials (up to a constant).

Similar result holds in the case of varying domain metrics:

**Theorem 1.3.** Let $w(s) : (\Sigma, \sigma(s)) \rightarrow (\Sigma, \rho)$ be a family of harmonic maps between canonical metrics on surface $\Sigma$, where $\sigma(0) = \rho$, for $|s| < \epsilon$ small. Then the second variation of the energy functional of $w(t)$, at $t = 0$, is given
by the $L^2$-Bergman metric of infinitesimal holomorphic quadratic differentials (up to a constant).

This paper is organized as follows. We introduce the preliminaries in section 2, then prove the homeomorphism theorem 1.1 in section 3. Section Four is devoted to a variational approach to the study of the $L^2$-Bergman metric, where we prove theorem 1.2 (varying the target metric) in §4.1 and theorem 1.3 (varying the domain metric) in §4.2.

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2 Preliminaries

On a compact Riemann surface $\Sigma$ of genus $g > 1$, the dimension of the space of Abelian differentials of the first kind, or holomorphic one forms, is $g$. There is a natural pairing of Abelian differentials defined on this space:

$$<\mu, \nu> = \frac{\sqrt{-1}}{2} \int_{\Sigma} \mu \wedge \bar{\nu}$$

Let $\{\omega_1, \omega_2, \cdots, \omega_g\}$ be a basis of Abelian differentials, normalized with respect to the $A$-cycles of some symplectic homology basis $\{A_i, B_i\}_{1 \leq i \leq g}$, i.e., $\int_{A_i} \omega_j = \delta_{ij}$. Thus the period matrix $\Omega_{ij} = \int_{B_i} \omega_j$. One finds that, since not all Abelian differentials vanish at the same point according to Riemann-Roch, the period matrix is then symmetric with positive definite imaginary part: $\text{Im}\Omega_{ij} = <\omega_i, \omega_j>$ ([5]).

The canonical metric $\rho_B$ on surface $\Sigma$ is the metric associated to the $(1,1)$ form given by

$$\frac{\sqrt{-1}}{2} \sum_{i,j=1}^{g} (\text{Im}\Omega)_{ij}^{-1} \omega_i(z) \overline{\omega_j(\bar{z})}.$$

It is not hard to see that this metric is the pull-back of the Euclidean metric from the Jacobian variety $J(\Sigma)$ via the period map ([5]).

Remark 2.1. It is easy to see that the area of the surface $\Sigma$ with respect to the canonical metric is a constant, i.e., $\int_{\Sigma} \rho_B = g$. Sometimes the canonical metric is also referred to $\text{area}$ to unify the surface area.

It is known that, when $g \geq 2$, the Gaussian curvature $K_c$ satisfies $K_c \leq 0$ ([6], [12]), and $K_c(p) = 0$ for some $p \in \Sigma$ if and only if $\Sigma$ is hyperelliptic and $p$ is one of the $2g + 2$ classical Weierstrass points of $\Sigma$ ([12]).
The Weil-Petersson cometric on Teichmüller space is defined on the space of holomorphic quadratic differentials \( QD(\Sigma) \) by the \( L^2 \)-norm:

\[
\|\phi\|_{WP}^2 = \int_{\Sigma} \frac{|\phi|^2}{\sigma} \, dz d\bar{z}
\]  

(1)

where \( \sigma |dz|^2 \) is the hyperbolic metric on \( \Sigma \). By duality, we obtain a Riemannian metric on the tangent space of \( Tg \).

The \( L^2 \)-Bergman metric on \( Tg \) is similarly defined by duality from the \( L^2 \)-norm

\[
\|\phi\|_{B}^2 = \int_{\Sigma} \frac{|\phi|^2}{\rho_B} .
\]

We now introduce harmonic maps between canonical metrics on a surface as much of our analysis will focus on the techniques of harmonic maps.

For a Lipschitz map \( w : (\Sigma, \sigma|dz|^2) \rightarrow (\Sigma, \rho|dw|^2) \), where \( \sigma|dz|^2 \) and \( \rho|dw|^2 \) are metrics on \( \Sigma \), and \( z \) and \( w \) are conformal coordinates on \( \Sigma \), one follows some notations of Sampson ([13]) to define

\[
\mathcal{H}(z) = \frac{\rho(w(z))}{\sigma(z)} |w_z|^2, \quad \mathcal{L}(z) = \frac{\rho(w(z))}{\sigma(z)} |w_{\bar{z}}|^2.
\]

Then the energy density of \( w \) is simply \( e(w) = \mathcal{H} + \mathcal{L} \), and the total energy is then given by

\[
E(w, \sigma, \rho) = \int_{\Sigma} e \sigma |dz|^2,
\]

which depends on the target metric and conformal structure of the domain. The map \( w \) is called harmonic if it is a critical point of this energy functional, i.e., it satisfies Euler-Lagrange equation:

\[
w_{zz} + \frac{\rho w}{\rho} w_{\bar{z}} w_{z\bar{z}} = 0.
\]

The \( (2,0) \) part of the pullback \( w^* \rho \) is the so-called Hopf differential:

\[
\phi(z) dz^2 = (w^* \rho)^{(2,0)} = \rho w_z \bar{w}_z dz^2.
\]

It is routine to check that \( w \) is harmonic if and only if \( \phi dz^2 \in QD(\Sigma) \), and \( w \) is conformal if and only if \( \phi = 0 \).

One also finds that

\[
\mathcal{H}(z) \mathcal{L}(z) = \frac{\phi \bar{\phi}}{\sigma^2} = \frac{|\phi|^2}{\sigma^2}
\]

(2)

and the Jacobian functional is \( J(z) = \mathcal{H}(z) - \mathcal{L}(z) \).
Now assume both $\sigma$ and $\rho$ are canonical metrics on surface $\Sigma$ (then they represent two different conformal structures unless they are biholomorphic). Since the target surface $(\Sigma, \rho)$ has negative Gaussian curvatures almost everywhere, with possibly finitely many flat points, the classical theory of harmonic maps guarantees that there is a unique harmonic map $w : (\Sigma, \sigma) \to (\Sigma, \rho)$ in the homotopy class of the identity, moreover, this map $w$ is a diffeomorphism with $J > 0$ and $\mathcal{H} > 0$ (4, 9, 13, 14, 3, 11).

3 A Homeomorphism

The method of harmonic maps has been a great computational tool in Teichmüller theory (see 10). In the case of hyperbolic metrics on a compact Riemann surface, the second variation of the energy of the harmonic map $w = w(\sigma, \rho)$, with respect to the domain metric $\sigma$ (or target metric $\rho$) at $\sigma = \rho$, yields the Weil-Petersson metric on $T_g$ (16, 17). In our case of canonical metrics on a Riemann surface, we prove a homeomorphism theorem, the theorem 1.1, to link Teichmüller space of canonical metrics to the space $QD(\Sigma)$.

To define this map, we fix a point $\sigma$ in Teichmüller space, with conformal coordinates $z$. Thus $\sigma$ is a conformal structure on surface $\Sigma$. For each canonical metric $\rho$ on $\Sigma$, we obtain the unique harmonic map $w(\sigma, \rho)$ in the homotopy class of the identity map, since $\rho$ is of nonpositive curvature, with only possibly finitely many flat points on $\Sigma$. The associated Hopf differential of the harmonic map $w(\sigma, \rho)$ is then given by $\phi(z)dz^2 = \rho(w(z))w_\bar{z}w_zdz^2$. Therefore the map $\phi : T_g \to QD(\Sigma)$ which sends $\rho$ to $\phi(z)dz^2$ is well defined.

Remark 3.1. Sampson considered this map in the case that of hyperbolic metrics (13), and showed that it is continuous and one-to-one. Later Wolf showed the map is actually a homeomorphism (17). The condition of constant Gaussian curvature of hyperbolic metric is essential in the argument of proving this map is injective.

We start with a technical lemma, which is only slightly different than the case of hyperbolic metrics, as shown in 17. For Hopf differential $\phi$ corresponding to metric $\rho$, we define $\|\phi\| = \int_{\Sigma} |\phi|dzd\bar{z}$. We need to show $\|\phi\|$ is approximately the total energy of the harmonic map $w$ in a large scale, i.e.,

Lemma 3.2. $\|\phi\| \to \infty$ if and only if $E(\rho) \to \infty$

Proof. This harmonic map $w(z)$ is naturally quasiconformal, and we write its Beltrami differential as $\nu = \frac{w_\bar{z}}{w_z}$, and $|\nu| < 1$. 

6
We abuse our notation to write $\sigma$ as the domain canonical metric, and $dA = \sigma(z)dzd\bar{z}$ is the area element of the domain surface. Recall from section two, we have density functions $H(z) = \frac{\rho(w(z))}{\sigma(z)}|w_z|^2$ and $L(z) = \frac{\rho(w(z))}{\sigma(z)}|w_{\bar{z}}|^2$.

The total energy is $E(\rho) = \int_{\Sigma}(H(z) + L(z))dA$, and the Jacobian determinant of the map is $J(z) = H(z) - L(z)$. Note that $H(z) > L(z) \geq 0$.

It is not hard to see that $|\phi|^2 = \sigma^2 H L$, and $|\nu|^2 = \frac{\epsilon}{H} < 1$.

Therefore, we now have

$$
\|\phi\| = \int_{\Sigma} |\phi|dzd\bar{z} = \int_{\Sigma} H|\nu|\sigma dzd\bar{z}
= \int_{\Sigma} H|\nu|dA < \int_{\Sigma} HdA
\leq \int_{\Sigma} (H + L)dA = E(\rho).
$$

For the opposite direction, we find $L \leq \sqrt{H \lambda}$ since $L \leq H$, and therefore,

$$
E(\rho) = \int_{\Sigma}(H + L)dA = \int_{\Sigma} J(z)dA + 2 \int_{\Sigma} LdA
\leq Area(\Sigma, \sigma) + 2 \int_{\Sigma} \sqrt{H \lambda}dA
= g + 2 \int_{\Sigma} \frac{|\phi|}{\sigma}dA = g + 2\|\phi\|.
$$

Here we used the fact that $Area(\Sigma, \sigma) = g$, as pointed out in remark 2.1. This completes the proof of the lemma.

We now start to prove the homeomorphism theorem.

**Proof.** (Proof of theorem 1.1): It is clear that this map is continuous because of the uniqueness of harmonic map $w$ in the homotopy class of the identity map. We want to show this map is a local diffeomorphism and proper.

We firstly notice that the map $\phi$ is a local diffeomorphism. To see this, we consider a sufficiently small neighborhood of $\sigma$ and a family of harmonic maps $w(t) : (\Sigma, \sigma) \rightarrow (\Sigma, \rho(t))$ between canonical metrics near $t = 0$, where $\rho(t)$ is a family of canonical metrics with $\rho(0) = \sigma$. It is easy to see that $w(0) = z$, the identity map. Associated Hopf differentials of this family are given by $\phi(t)dz^2 = \rho(t)w_z(t)\bar{w}_z(t)dz^2$ with $\phi(0) = 0$. we take $t$-derivative on $\phi(t)$ at $t = 0$ to find that
\[
\frac{d\phi(t)}{dt}|_{t=0} = \rho(0) w_z(0) \frac{d\bar{w}_z(t)}{dt}|_{t=0} = \sigma \frac{d\bar{w}_z(t)}{dt}|_{t=0}.
\]
This shows that \( \frac{dw(t)}{dt}|_{t=0} \) is conformal, provided that \( \frac{d\phi(t)}{dt}|_{t=0} = 0 \). So the map \( d\phi \) is nonsingular, and \( \phi \) is a local diffeomorphism by applying inverse function theorem.

We then apply a slightly rearranged argument of Wolf (\[17\]) (on hyperbolic metrics) to show map \( \phi \) is proper.

Given that \( T_g \) and \( QD(\Sigma) \) are finite dimensional spaces, and from lemma 3.2, it suffices to show the energy function \( E(\rho) \) is a proper map from Teichmüller space to \( \mathbb{R} \) (see theorem 2.7.1, \[2\]). In other words, we need to show the set \( B = \{ \rho \in T_g : E(\rho) \leq K \} \) is a compact subset of \( T_g \). Without loss of generality, we assume \( id : (\Sigma, \sigma) \rightarrow (\Sigma, \rho) \) is harmonic, or we can choose \( w^* \rho \) to represent the equivalency class \([\rho]\).

Consider a geodesic ball \( B(x_0, \delta) \) for some \( x_0 \) in domain surface \( (\Sigma, \sigma) \), where positive constant \( \delta < \min\{1, inj_\sigma(\Sigma)^2\} \), where \( inj_\sigma(\Sigma) \) is the injectivity radius of \( \Sigma \) with respect to the metric \( \sigma \). Notice that a harmonic map between surfaces does not depend on the choice of metrics on the domain, but on the choice of conformal structures of the domain surface. Therefore, we can choose \( \sigma \) to be hyperbolic in this argument, and then introduce polar coordinates \((r, \theta)\) in the hyperbolic disk \( B(x_0, \delta) \) so that \( \sigma = dr^2 + sinh^2(r) d\theta^2 \).

For \( r < \sqrt{\delta} < 1 \), we have \( sinh(r) < 2r \) and then
\[
\int_0^{\sqrt{\delta}} \frac{dr}{sinh(r)} > \frac{1}{2} \int_0^{\sqrt{\delta}} \frac{dr}{r} = \frac{1}{4} |log\delta|.
\]
Now considering the annulus \( A(x_0) = A(x_0, \delta, \sqrt{\delta}) \) centered at \( x_0 \) of inner and outer radii \( \delta \) and \( \sqrt{\delta} \), respectively, in domain metric \( \sigma \), we apply the upper bound of the energy to find
\[
\int_A \| \frac{\partial}{\partial \theta} \|^2 \frac{dr}{sinh(r)} d\theta \leq \int_A \| \frac{\partial}{\partial r} \|^2 + \frac{1}{sinh^2(r)} \| \frac{\partial}{\partial \theta} \|^2 sinh(r) dr d\theta \leq 2 \int_{\Sigma} e(z) dA = 2E(\rho) \leq 2K.
\]
Thus there exists \( \delta < r < \sqrt{\delta} \) such that
\[
\int_0^{2\pi} \| \frac{\partial}{\partial \theta} \|^2 d\theta \leq \frac{8K}{|log\delta|}.
\]
For this \( r \) and two points \( x_3 \) and \( x_4 \) on the boundary of the disk \( B_\sigma(x_0, r) \),
and two points $x_1$ and $x_2$ in $B_\sigma(x_0, \delta)$, we now have,

$$d_\rho(w(x_1), w(x_2)) = d_\rho(x_1, x_2) \leq d_\rho(x_3, x_4) \leq \int_0^{2\pi} \| \frac{\partial}{\partial \theta} \|_\rho^2 d\theta \leq 4\sqrt{2K\pi |\log\delta|}.$$  

Applying the Courant-Lebesgue lemma, we conclude that the energy $E(\rho)$ is proper, hence so is the map $\phi$.

We have showed that the map $\phi$ is a proper local diffeomorphism between Teichmüller space and $QD(\Sigma)$. It is clear that $\mathcal{T}_g$ is path-connected and $QD(\Sigma)$ is a simply connected Hausdorff space. Standard theory of covering map between manifolds implies such a local homeomorphism is actually a global homeomorphism. This completes the proof of theorem 1.1.

4 Variations of the Energy

In the previous section, we showed that the map $\phi : \mathcal{T}_g \to QD(\Sigma)$ is a homeomorphism, thus its inverse map $\phi^{-1}$ provides coordinates for any canonical metric $\rho \in \mathcal{T}_g$. We will study these coordinates in this section, i.e., we apply variational approach to derive the infinitesimal $L^2$-Bergman norm on Teichmüller space. We will separate the cases where either target canonical metrics are varying or domain canonical metrics are varying, in subsections 4.1 and 4.2, respectively.

4.1 Varying the Target

To prove theorem 1.2, we need to develop some infinitesimal calculations for the variation of a harmonic map between canonical metrics. This technique is an analog to that of Wolf’s on the case of hyperbolic metrics, which plays an important role in studying the Weil-Petersson geometry of Teichmüller space.

Now we consider a family of harmonic maps $w(t) : (\Sigma, \sigma) \to (\Sigma, \rho(t))$ between canonical metrics, where $w(t)$ varies real analytically in $t$ for $|t| < \epsilon$, and $\rho(t)$ is a family of canonical metrics with $\rho(0) = \sigma$, therefore $w(0) = z$. Associated Hopf differentials are given by $\phi(t) dz^2 = \rho(t) w_z(t) \bar{w}_z(t) dz^2$ with $\phi(0) = 0$.

For $t = (t^\alpha, t^\beta)$, denote $\phi_\alpha = \frac{d\phi(t)}{dt^\alpha}|_{t=0}$ and $\phi_\beta = \frac{d\phi(t)}{dt^\beta}|_{t=0}$ as infinitesimal holomorphic quadratic differentials.
Recall that the holomorphic and antiholomorphic functions of this family of harmonic maps are

\[ H(t) = \frac{\rho(w(t))}{\sigma(z)} |w_z(t)|^2, \quad L(t) = \frac{\rho(w(t))}{\sigma(z)} |w_{\bar{z}}(t)|^2. \]

We denote

\[ H_\alpha = \frac{dH(t)}{dt} |_{t=0}, \quad L_\alpha = \frac{dL(t)}{dt} |_{t=0}, \quad L_{\alpha\bar{\beta}} = \frac{d^2}{dt^2} |_{t=0} L(t), \]

and we assign similar meaning for \( H_{\alpha\bar{\beta}} \) and \( E_{\alpha\bar{\beta}} \).

We also write

\[ K(t) = K(\rho(t)) = -\frac{1}{2} \Delta \rho \log \rho \]

as the Gaussian curvature of the metric \( \rho(t) \), and assign obvious meaning to \( K_\alpha \) and \( K_{\alpha\bar{\beta}} \). Since \( K(\sigma) \leq 0 \) and is negative everywhere except possibly finitely many points, it is not hard to see that the operator \( \Delta \sigma + 2K(\sigma) \) is invertible on \( (\Sigma, \sigma) \), and we denote \( D_B = -2(\sigma + 2K(\sigma))^{-1} \).

**Lemma 4.1.** For this family of harmonic maps \( w(t) \), the following holds:

(i) \( H(0) = 1 \) and \( L(0) = 0 \);

(ii) \( L_\alpha \equiv 0 \), \( H_\alpha = D_B(K_\alpha) \), and \( \int_{\Sigma} H_\alpha \sigma = 0 \);

(iii) \( L_{\alpha\bar{\beta}} = \frac{\phi_\alpha \phi_{\bar{\beta}}}{\sigma^2} \);

(iv) \( H_{\alpha\bar{\beta}} = D_B(K_{\alpha\bar{\beta}}) + D_B(K_{\bar{\beta}} D_B(K_\alpha)) + D_B(K_\alpha D_B(K_{\bar{\beta}})) - D_B(K(\sigma) \frac{\phi_\alpha \phi_{\bar{\beta}}}{\sigma^2}) - \frac{1}{2} D_B(\Delta \sigma (D_B(K_{\alpha}) D_B(K_{\bar{\beta}}))). \)

**Proof.**

(i) This is true since the map \( w(t) \) is the identity map at time \( t = 0 \).

(ii) Recalling formula (2):

\[ H(t)L(t) = \frac{\phi(t)\tilde{\phi}(t)}{\sigma^2}, \]

we take \( t \)-derivative at \( t = 0 \), to find that

\[ H_\alpha L(0) + H(0) L_\alpha = \frac{\phi_\alpha \tilde{\phi}(0) + \phi(0) \tilde{\phi}_{\alpha}}{\sigma^2}. \]

The righthand side is zero as \( \phi(0) = 0 \). Therefore \( L_\alpha \equiv 0 \).

We notice that \( \int_{\Sigma} (H(t) - L(t)) \sigma = \int_{\Sigma} J(t) \sigma = g \) is independent of the parameter \( t \). Therefore

\[ \int_{\Sigma} H_\alpha \sigma = \int_{\Sigma} L_\alpha \sigma = 0. \]
From standard Bochner identities, we have
\[
\Delta \sigma \log H = 2K(\sigma) - 2K(\rho)(H - L).
\] (3)

Therefore \(\Delta \sigma \log H(t) = 2K(\sigma) - 2K(\rho(t))(H(t) - L(t))\) and

\[
\begin{align*}
\Delta \sigma H_\alpha &= \Delta \sigma \frac{H_\alpha}{H(0)} \\
&= -2K_\alpha(H(0) - L(0)) - 2K(\rho(0))(H_\alpha - L_\alpha) \\
&= -2K_\alpha - 2K(\sigma)H_\alpha.
\end{align*}
\]

We now obtain \((\Delta \sigma + 2K(\sigma))H_\alpha = -2K_\alpha\) and then

\[
H_\alpha = -2(\Delta \sigma + 2K(\sigma))^{-1}(K_\alpha) = D_B(K_\alpha).
\]

(iii) To calculate next variation, we consider formula
\[
H(t)L_\beta = \phi(t)\bar{\phi}(t)\sigma^2
\]
again. We find
\[
H_\alpha L_\beta(0) + H_\beta L_\alpha + H_\alpha L_\beta + H(0)L_\alpha L_\beta = \frac{\phi_\alpha \bar{\phi}_\beta}{\sigma^2}.
\]

Therefore \(L_\alpha L_\beta = \frac{\phi_\alpha \bar{\phi}_\beta}{\sigma^2}\).

(iv) We take second \(t\)-derivative from (3) to find
\[
\Delta \sigma(H_{\alpha\beta} - H_\alpha H_\beta) = -2K_{\alpha\beta} - 2K_\alpha H_\beta - 2K_\beta H_\alpha \\
- 2K(\sigma)(H_{\alpha\beta} - L_{\alpha\beta}),
\]
then we obtain that
\[
(\Delta \sigma + 2K(\sigma))(H_{\alpha\beta}) = \Delta(H_\alpha H_\beta) - 2K_{\alpha\beta} - 2K_\alpha H_\beta - 2K_\beta H_\alpha \\
- 2K(\sigma)(H_{\alpha\beta} - L_{\alpha\beta}).
\]

Now we apply formulas \(H_\alpha = D_B(K_\alpha)\), and \(H_\beta = D_B(K_\beta)\), and \(L_{\alpha\beta} = \frac{\phi_\alpha \bar{\phi}_\beta}{\sigma^2}\) to above equation to complete the proof of this lemma.

**Remark 4.2.** It is very interesting to compare our situation with the case of variations of a harmonic map between hypers on the surface. If we assume all metrics on the surface are hyperbolic with constant Gaussian curvature, under the same notations, then we have the following comparison: (i) and (iii) in lemma 4.1 hold; (ii) also holds except furthermore, \(H_\alpha \equiv 0\), i.e., the
holomorphic energy reaches its minimum at time zero; (iv) of the lemma takes the form of $H_{\alpha\bar{\beta}} = D\left(\frac{\phi_\alpha\bar{\phi}_\beta}{\sigma^2}\right)$, where $D = -2(\Delta_\sigma - 2)^{-1}$ is a compact, self-adjoint operator. Operator $D_B$ in (iv) of lemma 4.1 is not self-adjoint for $L^2$ functions, while $-K(\sigma)D_B$ is, and coincides with operator $D$ when $K \equiv -1$.

We now consider the variations of the corresponding total energy $E(t)$ of the family $w(t)$ near $t = 0$, i.e., we show theorem 1.2 in following equivalent form:

**Theorem 4.3.** $\frac{d^2}{dt^2}\big|_{t=0}E(t) = 2 < \phi_\alpha, \phi_\beta >_B$.

*Proof.* The total energy is

$$E(t) = \int_\Sigma (H(t) + L(t))\sigma$$

$$= \int_\Sigma (H(t) - L(t))\sigma + 2 \int_\Sigma L(t)\sigma$$

$$= \int_\Sigma J(t)\sigma + 2 \int_\Sigma L(t)\sigma$$

$$= g + 2 \int_\Sigma L(t)\sigma \geq g,$$

since $E(0) = g$ is equal to the area of the surface. Thus $E(t)$ reaches its global minimum $g$ at $t = 0$ from (i) of lemma 4.1.

From (ii) of lemma 4.1, it is easy to see that $t = 0$ is a critical point of $E(t)$ as

$$E_\alpha = \int_\Sigma (H_\alpha + L_\alpha)\sigma = 0.$$

and

$$E_{\alpha\bar{\beta}} = \int_\Sigma (H_{\alpha\bar{\beta}} + L_{\alpha\bar{\beta}})\sigma$$

$$= 2 \int_\Sigma L_{\alpha\bar{\beta}}\sigma = 2 \int_\Sigma \frac{\phi_\alpha\bar{\phi}_\beta}{\sigma^2}\sigma$$

$$= 2 < \phi_\alpha, \phi_\beta >_B.$$

\[\square\]

**4.2 Varying the Domain**

In this subsection, we consider a family of harmonic maps between fixed target metric and varying domain metrics.
Again, since the target metric is negatively curved (except possibly finitely many flat points), we have the existence and uniqueness of a harmonic map in the homotopy class of the identity, and this map is a diffeomorphism. In other words, let \( w(s) : (\Sigma, \sigma(s)) \rightarrow (\Sigma, \rho|dw|^2) \) be this family of harmonic maps near the identity map, where \( w(s) \) varies real analytically in \( s \) for \( |s| < \epsilon \), and \( \sigma(s) \) is a family of canonical metrics with \( \sigma(0) = \rho \), therefore \( w(0) = z \).

Associated Hopf differentials are given by \( \phi(s)dz(s)^2 = \rho w_z(s)\overline{w_z(s)}dz(s)^2 \) with \( \phi(0) = 0 \).

For \( s = (s^a, s^b) \), similar to last subsection, we denote \( \phi_a = \frac{\partial\phi(s)}{\partial s^a}|_{s=0} \) and assign similar meanings to \( H_a \), and \( \mathcal{L}_{ab} \), etc.

Let \( K(s) = -\frac{1}{2}\Delta_{\sigma(s)}\log\sigma(s) \) be the Gaussian curvature of the surface \( (\Sigma, \sigma(s)) \) and denote \( K_a = \frac{\partial K(s)}{\partial s^a}|_{s=0} \). Again, since \( K(\rho) \leq 0 \) and is negative everywhere except possibly finitely many points, the operator \( \Delta_\rho + 2K(\rho) \) is invertible on \( (\Sigma, \rho) \), and we denote \( D'_B = -2(\Delta_\rho + 2K(\rho))^{-1} \). This operator \( D'_B \) is not self-adjoint for \( L^2 \) functions.

We firstly calculate the variations of these two density functions \( \mathcal{H}(s) \) and \( \mathcal{L}(s) \). It is interesting to notice the difference with the case of varying the target showed in lemma 4.1.

**Lemma 4.4.** For this family of harmonic maps \( w(s) \), the following holds:

(i) \( \mathcal{H}(0) = 1 \) and \( \mathcal{L}(0) = 0 \);

(ii) \( \mathcal{L}_a \equiv 0 \), \( \mathcal{H}_a = -D'_B(K_a) \), and \( \int_\Sigma \mathcal{H}_a\sigma = 0 \);

(iii) \( \mathcal{L}_{ab} = \frac{\phi_a\phi_b}{\sigma^2} \).

**Proof.**

(i) It is true since the map \( w(0) \) is the identity map.

(ii) We take \( s^a \)-derivative of \( \mathcal{H}(s)\mathcal{L}(s) = \frac{\phi(s)\phi(s)}{\sigma^2(s)} \) to find

\[
\mathcal{H}_a\mathcal{L}(0) + \mathcal{H}(0)\mathcal{L}_a = \frac{\phi_a\phi(0) + \phi(0)\phi_a}{\sigma^2(0)} + \left[ \frac{\partial}{\partial s^a} \frac{1}{\sigma^2(s)} \right]|_{s=0}|\phi(0)|^2,
\]

and this implies \( \mathcal{L}_a = 0 \), for \( \phi(0) = 0 \).

Therefore \( \int_\Sigma \mathcal{H}_a\sigma = \int_\Sigma \mathcal{L}_a\sigma = 0 \).

To calculate \( \mathcal{H}_a \), recalling formula (3):

\[
\Delta_{\sigma(s)}\log\mathcal{H}(s) = 2K(\sigma(s)) - 2K(\rho)(\mathcal{H}(s) - \mathcal{L}(s)),
\]

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we find that
\[
\frac{\partial}{\partial s} \sigma |_{s=0} (\Delta_{\sigma(s)}) \log \mathcal{H}(0) + \Delta_{\rho}(\mathcal{H}_a) = 2K_a - 2K(\rho)(\mathcal{H}_a - \mathcal{L}_a) \\
= 2K_a - 2K(\rho)(\mathcal{H}_a).
\]
Therefore \((\Delta_{\rho} + 2K(\rho))(\mathcal{H}_a) = 2K_a\), and so \(\mathcal{H}_a = -D'B(K_a)\).

(iii) For the second variation of \(\mathcal{L}(s)\), we have
\[
\mathcal{L}_{\bar{a}\bar{b}} = \mathcal{H}_{\bar{a}\bar{b}} \mathcal{L}(0) + \mathcal{H}_\bar{b} \mathcal{L}_a + \mathcal{H}_a \mathcal{L}_\bar{b} + \mathcal{H}(0) \mathcal{L}_{\bar{a}\bar{b}} \\
= \frac{\phi_a \bar{\phi}_b}{\rho^2} + \left[ \frac{\partial (\frac{1}{2} \sigma^2(s))}{\partial \bar{s}_b} \right] |_{s=0} [\phi_a \bar{\phi}(0) + \phi(0) \bar{\phi}_a] \\
+ \left[ \frac{\partial (\frac{1}{2} \sigma^2(s))}{\partial s_a} \right] |_{s=0} [\phi_b \bar{\phi}(0) + \phi(0) \bar{\phi}_b] + \left[ \frac{\partial^2 (\frac{1}{2} \sigma^2(s))}{\partial s_a \partial \bar{s}_b} \right] |_{s=0} [\phi(0) \bar{\phi}(0)] \\
= \frac{\phi_a \bar{\phi}_b}{\rho^2}.
\]

Now we show the equivalent form of theorem 1.3:

**Theorem 4.5.** \(\frac{\partial^2}{\partial s^a \partial s^b} |_{s=0} E(s) = 2 < \phi_a, \phi_b >_B\).

**Proof.** The total energy is now
\[
E(s) = \int \Sigma (\mathcal{H}(s) + \mathcal{L}(s)) \sigma(s) \\
= \int \Sigma J(s) \sigma(s) + 2 \int \Sigma \mathcal{L}(s) \sigma(s) \\
= g + 2 \int \Sigma \mathcal{L}(s) \sigma(s) \geq g,
\]
where \(E(0) = g\) reaches the global minimum.

Together with \(\mathcal{L}(0) = 0\), we then find
\[
E_a = 2 \frac{\partial^2}{\partial s^a \partial s^b} |_{s=0} \int \Sigma \mathcal{L}(s) \sigma(s) = 0,
\]
and then \(s = 0\) is also a critical point of \(E(s)\).

Now we consider \(\frac{\partial^2}{\partial s^a \partial s^b} |_{s=0} E(s)\) from lemma 4.4. We apply \(\mathcal{L}(0) = \mathcal{L}_a = \mathcal{L}_\bar{b} = 0\) to find
\[
\frac{\partial^2}{\partial s^a \partial s^b} |_{s=0} E(s) = 2 \frac{\partial^2}{\partial s^a \partial \bar{s}_b} |_{s=0} \left\{ \int \Sigma \mathcal{L}(s) \sigma(s) \right\} \\
= 2 \int \Sigma \frac{\phi_a \bar{\phi}_b}{\rho^2} \rho \\
= 2 < \phi_a, \phi_b >_B.
\]
Remark 4.6. Teichmüller space is a complex manifold (when \( g \geq 2 \)), so it has its own complex structure. For Riemannian metrics on this complex manifold, it is ideal that metrics are compatible with the complex structure. Ahlfors (\[1\]) showed that the Weil-Petersson metric is Kählerian. From the definition, we know that the \( L^2 \)-Bergman metric is an Hermitian metric, yet it is unknown if the \( L^2 \)-Bergman metric is actually Kählerian.

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