Measure of Complexity in Self-Gravitating Systems using Structure Scalars

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Abstract

The aim of this paper is to present the definition of complexity for static self-gravitating anisotropic matter proposed in \(f(G, T)\) theory, where \(G\) is the Gauss-Bonnet term and \(T\) is the trace of energy momentum tensor. We evaluate field equations, Tolman-Oppenheimer-Volkoff equation, mass functions and structure scalars. Among the calculated modified scalar variables that are obtained from the orthogonal splitting of Riemann tensor, a single scalar function has been identified as the complexity factor. After exploring the corresponding Tolmann mass function, it is seen that the complexity factor along with the \(f(G, T)\) terms have greatly influenced its formulation and its role in the subsequent radial phases of the spherical system. We have also used couple of ansatz in order to discuss possible solutions of equations of motion in the study of the structure of compact object.

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I. Introduction

A system is a collection of ordered and interrelated elements. A minor/major disturbances among the elements of such a system may cause complexity. Many researchers put forward different definitions for complexity. Initially, entropy and information were considered a major criteria to check complexity. To scrutinize the complexity in mathematical physics, perfect crystal and ideal gas were considered. Atoms in perfect crystal are completely organized and have symmetry throughout the structure. A small portion completely describes the whole notion of the system, in that criterium it provide less information. While in ideal gas, all particles are randomly distributed and give maximum information by considering a small section. This analysis disclosed the fact that order and structure manifest no complexity. There must be some other factors.

López-Ruiz et al. [1] put forward a probabilistic and disequilibrium approach to describe the complexity of a physical system. It has many applications regarding different physical systems. Further, for different physical situations manipulation of complexity becomes easier to work with. This could not help in case of ideal gas and perfect crystal because complexity vanishes. Calbet and López-Ruiz [2] formulated the time evolution equations of “tetrahedral gas”. Under some constraints, the gas grows and maximum complexity occurs in phase space analysis at equilibrium. Expansion of statistical measure of complexity towards continuous system was examined by Catalan et al. [3] and it needs certain requirements.

Astrophysical objects have components like energy density, pressure, luminosity which can be utilized to measure complexity. The $C_{\text{LMC}}$ complexity is being calculated by the density of an astrophysical object, like white dwarf [4] and Chatzisavvas et. al [5] then applied this concept to neutron stars. To study evolution of the dynamical object, Herrera [6] introduced the condition of minimal complexity in addition to complexity factor. Furthermore, the same author [7] put forward the concept of complexity factor to axially symmetric static sources. Herrera et al. [8] obtained that five different scalar quantities from the orthogonal splitting of Riemann tensor exist in order to study the structure and evolution of self-gravitating spherical fluids. Yousaf et al. [9][11] also found same number variables in the realm of modified gravity. They have also checked their role in the modeling Raychaudhuri equations. Herrera et al. [12][15] studied the role of these scalar variables in the emergence of energy density inhomogeneity on the surface of anisotropic self-gravitating matter distributions. Bhatti et al. [16][17] extended these results and examined that under some constraints the Weyl scalar is also responsible for producing inhomogeneities over the regular distribution of fluid configurations.

Herrera et al. presented a new definition of complexity factor the static [18] and non-static [6] spherically symmetric spacetimes. Sharif and Butt [19] extended these results for the static cylindrical spacetimes. Herrera et al. [7] modified their own results for those
relativistic systems that has axially symmetric geometric distributions. Yousaf [20] also calculated such factor by taking corrections of Palatini \( f(R) \) gravity and found that one can study the complexity of the spherical structure with the help of structure scalars. recently, Herrera et al. [21] performed this analysis by taking Bondi metric and calculated a particular relation between the measure of complexity and vorticity of the system. They also formulated a way to differentiate natural and non-natural non-dissipative systems. Yousaf et al. [22,23] modified the definition of complexity factor in \( f(R, T, R_{\mu\nu}T^{\mu\nu}) \) gravity and calculated \( Y_{TF} \) as the corresponding complexity factor for the static relativistic systems.

Anisotropy and inhomogeneous energy density could play an important role in the study of the complexity factor. Bayin [24] presented the analytical solutions to study effects of anisotropy on the structure of compact objects, he also studied the rotating and radiating anisotropic spheres. When the system departs from the equilibrium, cracking appeared in self-gravitating spheres due to the inclusion of anisotropy [25]. For self gravitating anisotropic fluid, Patel and Vaidya [26] proposed four exact analytical solutions. Mak and Harko [27] proposed the exact analytical solution of static anisotropic quark, he deduced that due to the involvement of anisotropic pressure, mass and radius of quark star increase. Pinheiro and Chan [28] presented a new model with inhomogeneous energy density to a collapsing star and compared the behavior of physical characteristics, i.e., pressure, luminosity, energy, adiabatic index with homogeneous energy model.

General relativity (GR) gave the new directions to explore the universe by opening new ways of research area. No doubt, in the last century GR has obtained a great achievement in the study of relativistic structures and universe evolution. Baryon Acoustic oscillations, clusters of galaxies and Type Ia supernovae are some of the most important confirmations for the accelerated expansion of universe [29,31]. Recently, Baryon Oscillation Spectroscopic Survey (BOSS) measured the power spectrum and angle-averaged galaxy correlation function of galaxy cluster under the reconstruction of baryon acoustic oscillation (BAO) feature and found the most accurate distance constraint which are in agreement with the present supernova measurements [32]. From all these evidences, we have come to the point that there is some kind of enigmatic energy known as dark energy which could be the reason for this cosmic expansion. There exist several reviews on the issue of dark energy [33,34] and modified gravity theories [35–43] as an explanation for the cosmic acceleration.

To study the latest cosmological model and the role of dark energy/matter in the evolution of our cosmos, researchers used some other theories which could be called as modified theories. In these theories, the generic functions of scalar invariants are being added or replaced in place of Ricci scalar \( R \) in the Einstein-Hilbert action. First modification in GR was done by replacing \( R \) with generic function \( f(R) \). Lovelock gravity is generalization of GR in n-dimensional space [44]. \( \mathcal{R} \) is first lovelock scalar and Gauss-Bonnet (GB) invariant is the second lovelock scalar. The term GB consists of Riemann tensor, Ricci tensor and
Ricci scalar embroidered in a special way as $G = R^{\mu \nu \lambda \mu} R_{\mu \nu \lambda \mu} - 4 R^{\mu \nu} R_{\mu \nu} + R^2$.

Nojiri and Odintsov [45] used $f(G)$ gravity in order to study late-time cosmological aspects. Late-time era, coincidence problem and presence of dark matter with inhomogeneous equation of state are being discussed with $f(G)$, $f(R)$ and $f(R, G)$ models by the same authors [46]. The weak energy condition is established to analyze some pragmatic $f(G)$ models by utilizing values of hubble, jerk and snap parameters [47]. To study late-time cosmic acceleration, Felice and Tsujikawa [48] considered $f(G)$ gravity model as one of the best candidate because it satisfies the solar system constraints. Bamba et al. [49] explored some viability bounds on some specific $f(G)$ gravity models through numerical technique with the help of Hubble and snap parameters.

Recently, a new modified theory $f(G, T)$ was introduced by adding the trace of energy momentum tensor in the action. Sharif and Ikram [50] used reconstruction process with the help of power-law model in $f(G, T)$ theory and presented few energy conditions. Some of the differences in $f(G, T)$ theory in comparison to $GR$ are: the existence of an extra force, due to which particles pursue non-geodesic path and force is orthogonal to four velocity of the fluid. Also, there in non-minimal coupling between matter and $G$ terms, due to which we analyze the non-conserved energy-momentum tensor. Cosmologically possible $f(G, T)$ gravity models were being created by Noether symmetry approach [51]. Moreover, for spherically symmetric metrice, gravastars solutions are listed by Shamir and Ahmad [52]. Non-static curvature-matter coupling is assumed to analyze the shear, Raychaudhuri, and Weyl scalar equations. Also, Newman-Penrose formalism and Penrose-Hawking singularity theorems are studied best by $f(G, T)$ scalar functions [53]. The same author worked on the scalar structure in $f(G, T)$ gravity [54]. Recently, Yousaf et al. [55] calculated the complexity factor for the charged relativistic spherical systems in modified gravity.

In this paper, we study the role of complexity factor in the modeling of stellar structure in the background of $f(G, T)$ gravity. The outline of this paper is as follows. In Sec. II, we set up physical variables and field equations in the presence of $f(G, T)$ gravity. We have also expressed our field equations with the help of Tolman mass function. Sec. III is devoted to the orthogonal splitting of Riemann curvature tensor. This gives rise to set of four structure variables, among them one has been identified as a complexity factor. In Sec. IV, two proposed solutions of field equations are examined by vanishing complexity factor. Lastly, we conclude the results in Sec. V.

II. The Physical Variables And the Field Equations

In this section, we will describe the physical variables and its corresponding modified field equations to study the anisotropic self-gravitating fluid.
A. Field Equations

The action integral to formulate the field equation in $f(G, T)$ gravity is given by

$$S_{f(G, T)} = \frac{1}{2k^2} \int [f(G, T) + R] \sqrt{-g} d^4x + \int L_m \sqrt{-g} d^4x,$$  \hspace{1cm} (1)

Here, $L_m$ and $g$ serve as the lagrangian density and determinant of the metric tensor, respectively. The energy momentum tensor is described as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} L_m}{\delta g^{\mu\nu}}$$ \hspace{1cm} (2)

The following field equations for $f(G, T)$ are obtained by varying Eq.(1) with respect to metric tensor as

$$G^\mu_\nu = 8\pi T^\mu_\nu - (\Theta^\mu_\nu + T^\mu_\nu) f_T(G, T) + \frac{1}{2} \delta^\mu_\nu f(G, T)$$

$$+ (4R^\mu_\nu + 4R^l_\nu T^\mu_l - 2RR^\mu - 2R^{lmn} R_{\nu lmn}) f_G(G, T)$$

$$+ (4R^\mu_\nu \nabla^2 + 4\delta^\mu_\nu R^lm \nabla_l \nabla_m + 2R\nabla^\mu \nabla_\nu - 2\delta^\mu_\nu R\nabla^2 - 4R^l \nabla_l \nabla_\nu$$

$$- 4R^l_\nu \nabla^\mu \nabla_l - 4R^\mu_\nu \nabla^l \nabla^m) f_G(G, T),$$ \hspace{1cm} (3)

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ represents the Einstein tensor and $\nabla^2 = \Box = \nabla^l \nabla_l$ describe the d’Alembert operator and $\Theta_{\mu\nu} = -2T_{\mu\nu} - Pg_{\mu\nu}$. Also, the partial derivatives of $f(G, T)$ w.r.t $G$ and $T$ are represented by $f_G(G, T)$ and $f_T(G, T)$, respectively. It is worthy to stress that like in $f(R, T)$ theory [56], we have used $L_m = -P$ in the derivation of our field equations. The detailed derivation of these equations are described by Yousaf [53]. When $f(G, T) = f(G)$, the field equations reduce to $f(G)$ gravity.

One can write Eq. (3) in an alternative form as follows

$$G^\mu_\nu = 8\pi T^\mu_\nu^{(eff)} = 8\pi (T^\mu_\nu^{(M)} + T^\mu_\nu^{(GT)}).$$ \hspace{1cm} (4)

The aim of this paper is to explore the effects of $f(G, T)$ corrections on the definition of complexity factor for the static anisotropic spherically symmetric geometric distribution. Therefore, the source of the gravitation ($T^\mu_\nu^{(M)}$ mentioned in Eq.(1)) is assumed to be anisotropic matter given as follows

$$T^\mu_\nu^{(M)} = \mu u^\mu u_\nu - Ph^\mu_\nu + \Pi^\mu_\nu,$$ \hspace{1cm} (5)

where

$$\Pi^\mu_\nu = \Pi \left( s^\mu s_\nu + \frac{1}{3} h^\mu_\nu \right); \hspace{0.5cm} P = \frac{P_r + 2P_\perp}{3}$$

$$\Pi = P_r - P_\perp; \hspace{0.5cm} h^\mu_\nu = \delta^\mu_\nu - u^\mu u_\nu.$$ \hspace{1cm} (6)

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The line element for spherically symmetric metric is as follows
\[ ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2 [d\theta^2 + \sin^2 \theta d\phi^2] , \] (7)
and \( s^\mu \) has the representation
\[ s^\mu = \left( 0, e^{\frac{\lambda}{r}}, 0, 0 \right) , \] (8)
satisfying the properties \( s^\mu u_\mu = 0, s^\mu s_\mu = -1 \). In the energy-momentum tensor, four velocity vector \( u^\mu \) is written as
\[ u^\mu = \left( e^{-\frac{\nu}{r}}, 0, 0, 0 \right) . \] (9)
We can calculate the four acceleration from it as \( a^\alpha = u_\beta u^\beta \) and it is found that it has only one non-vanishing component which is
\[ a_1 = -\frac{\nu'}{2} , \] (10)
and the modified terms of \( f(G, T) \) are defined as
\[ T^\mu_{\nu(GT)} = \frac{1}{8 \pi} \left( (\mu + P)u^\mu u_\nu + \Pi^\mu_{\nu} \right) f_T(G, T) + \frac{1}{2} \delta^\mu_{\nu} f(G, T) \]
\[ + \left( 4R_{\mu\nu\rho\sigma} u^\mu u^\nu + 2RR_{\nu} - 2R_{\nu\mu\rho\sigma} u^\mu u^\nu \right) f_T(G, T) \]
\[ + \left( 4R_{\mu\nu} \nabla^2 + 4\delta^\mu_{\nu} R_{\mu\nu} - 2R_{\mu\nu} - 2R_{\mu\nu} \nabla^2 \nabla_{\mu\nu} \right) f_T(G, T) \]
\[ - \left( 4R_{\mu\nu} \nabla^2 + 4R_{\mu\nu} \nabla^2 + 2R_{\mu\nu} \nabla^2 \nabla_{\mu\nu} - 2\delta^\mu_{\nu} R^{\nu\mu} - 4R_{\nu\mu} \nabla^2 \nabla_{\mu\nu} \right) f_T(G, T) \] (11)
The corresponding \( f(G, T) \) field equations are
\[ \mu^{eff} = -\frac{1}{8 \pi} \left[ -\frac{1}{r^2} + e^{-\lambda} \left( \frac{1}{r^2} - \frac{\nu'}{r} \right) \right] , \] (12)
\[ P^{eff}_{r} = -\frac{1}{8 \pi} \left[ -\frac{1}{r^2} + e^{-\lambda} \left( \frac{1}{r^2} + \frac{\nu'}{r} \right) \right] , \] (13)
\[ P^{eff}_{\perp} = \frac{1}{32 \pi} \left( 2\nu'' + \nu'^2 - \lambda' \nu' + 2\frac{\nu' - \lambda'}{r} \right) , \] (14)
where prime denotes the derivatives with respect to \( r \). The values of \( \mu^{eff}, P^{eff}_{r} \) and \( P^{eff}_{\perp} \) are given in the Appendix of [55].

The covariant divergence of Eq.(3) is non-zero and is found to be
\[ \nabla^\mu T_{\mu\nu} = \frac{f_T(G, T)}{k^2 - f_T(G, T)} \left( (T_{\mu\nu} + \Theta_{\mu\nu}) \nabla^\mu (ln f_T(G, T)) \right) \]
\[ \nabla^\mu \Theta_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \nabla^\mu T \]. \quad (15)

The non-conserved hydrostatic equilibrium equation in \( f(G, T) \) gravity can be derived by using the Eqs. (12)-(14) as

\[ P_r^{\text{eff}} = -\nu' \left( \frac{1}{2}(P_r^{\text{eff}} + \mu^{\text{eff}}) + 2\left(\frac{P_{\perp}^{\text{eff}} - P_r^{\text{eff}}}{r}\right) + e^\lambda Z \right), \quad (16) \]

where value of \( Z \) is given in Appendix of [55]. Equation (16) is known as Tolman-Oppenheimer-Volkoff equation for anisotropic fluid.

From Eq. (13), one can get the value of the metric coefficient as follows

\[ \nu' = 2 \frac{m + 4\pi r^3 P_r^{\text{eff}}}{r(r - 2m)}, \quad (17) \]

where \( m \) is the mass function that can be expressed as

\[ 1 - e^{-\lambda} = \frac{2m}{r}, \quad (18) \]

One can write Eq. (16) after using Eq. (17) as follows

\[ P_r^{\text{eff}} = -\frac{m + 4\pi r^3 P_r^{\text{eff}}}{r(r - 2m)} \left( \mu^{\text{eff}} + P_r^{\text{eff}} \right) + 2\left(\frac{P_{\perp}^{\text{eff}} - P_r^{\text{eff}}}{r}\right) + e^\lambda Z, \quad (19) \]

With the help of Eqs. (12)-(14), Eq. (18) can also be written as

\[ m = 4\pi \int_0^r \tilde{r}^2 \mu^{\text{eff}} d\tilde{r}. \quad (20) \]

Now we assume that a 3-dimensional hypersurface \( \Sigma \) has differentiated our manifolds into two different portions. The one is known as an interior one whose manifold has been described already with the help of static spherically symmetric spacetime given in Eq. (7). The other portion also known as the exterior geometry, can be defined with the help of the following Schwarzschild metric as

\[ ds^2 = \left( 1 - \frac{2M}{r} \right) dt^2 - \frac{dr^2}{\left( 1 - \frac{2M}{r} \right)} - r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right). \quad (21) \]

Following Darmois [57], we need first and second conditions of continuity to match the two metrics (7) and (21) smoothly and gently at boundary surface \( \Sigma \). At the boundary surface, these give the following constraints

\[ e^{\nu_{\Sigma}} = \left( 1 - \frac{2M}{r_{\Sigma}} \right), \quad (22) \]
\[ e^{-\lambda e} = \left( 1 - \frac{2M}{r} \right), \tag{23} \]
\[ \left[ P_{eff}^r \right]_{\Sigma} = \chi_1, \tag{24} \]

where subscript \( \Sigma \) demonstrates that the subsequent values are calculated at \( \Sigma \), while the expression for \( \chi_1 \) is given in Appendix of \[55\]. It is worthy to stress that we have taken \( f_G = \tilde{f}_G, f_T = \tilde{f}_T, f = \tilde{f} \) while calculating the above mentioned constraints at the hypersurface. Such constraints need to be valid at the boundary for the smooth matching of manifolds. Such kind of condition with their detailed proof have been describe by Yousaf et al. \[58, 59\] for the \( f(R, T, R_{\mu\nu}T^{\mu\nu}) \) theory of gravity. The manifolds \([7] \) and \([21] \) are smoothly matched at boundary if and only if above three conditions \([22], [23] \) and \([24] \) are satisfied.

**B. The Riemann Curvature and the Weyl tensor**

The Weyl tensor has zero magnetic part for the spherical symmetric case. So it is expressed only in electric part \( (E_{\alpha\beta} = C_{\alpha\gamma\beta\delta}u^\gamma u^\delta) \) with

\[ C_{\mu\nu\kappa\lambda} = (g_{\mu\nu\alpha\beta}g_{\kappa\lambda\delta} - \eta_{\mu\nu\alpha\beta}\eta_{\kappa\lambda\delta}) u^\alpha u^\gamma E^{\beta\delta}, \tag{25} \]

where \( g_{\mu\nu\alpha\beta} = g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha}, \) and \( \eta_{\mu\nu\alpha\beta} \) denotes the Levi-Civita tensor. \( E_{\alpha\beta} \) can also be written as

\[ E_{\alpha\beta} = E \left( s_\alpha s_\beta + \frac{1}{3} h_{\alpha\beta} \right), \tag{26} \]

with

\[ E = -\frac{e^{-\lambda}}{4} \left[ \nu'' + \frac{\nu'^2 - \lambda'\nu'}{2} - \frac{\nu' - \lambda'}{r} + \frac{2(1 - e^\lambda)}{r^2} \right], \tag{27} \]

has the properties \( E_\alpha^\alpha = 0, \ E_{\alpha\gamma} = E_{(\alpha\gamma)} \) and \( E_{\alpha\gamma}u^\gamma = 0. \)

**C. The Mass Function And the Tolman Mass**

Here we are going to discuss the mass of an interior sphere at the surface \( \Sigma \) in two different forms and their relation with the Weyl tensor.

**The Mass Function**

From Eqs.\([4], [18] \) and \([26] \), we can write the mass function as

\[ m = \frac{4\pi}{3}r^3 \left( \mu_{eff} + P_{\perp}^{eff} - P_{r}^{eff} \right) + \frac{r^3E}{3}, \tag{28} \]
from which $E$ can be evaluated as

$$E = -\frac{4\pi}{r^3} \int_0^r \tilde{r}^3 \mu^\text{eff} d\tilde{r} + 4\pi \left( P^\text{eff}_r - P^\text{eff}_\perp \right).$$

(29)

After inserting Eq. (29) in Eq. (28), the mass function can be rewritten as

$$m(r) = \frac{4\pi}{3} r^3 \mu^\text{eff} - \frac{4\pi}{3} \int_0^r \tilde{r}^3 \mu^\text{eff} d\tilde{r}.$$  

(30)

Equation (29) shows the relation of the Weyl tensor with two physical properties of the fluid, i.e. density inhomogeneity and local anisotropic pressure, whereas in Eq. (30) mass function has been expressed in terms of homogeneous energy density and radial variations caused by density inhomogeneity in the presence of $f(G,T)$ dark source terms.

**Tolman Mass in $f(G,T)$ gravity**

The study to understand the effects of matter variables on the formulation of Tolman mass has gained utmost importance. Herrera et al. [60, 61] expressed the Tolman mass with the help of fluid matter variables. They also described the role of this mass function on the pace of spherical collapse. Bhatti et al. [62] modified their results and defined the same function in $f(R)$ and Maxwell-$f(R)$ theory of gravity for the relativistic spherical structures Here, we can write it as [63]

$$m_T = 4\pi \int_0^\Sigma \tilde{r}^2 e^{\frac{\nu - \lambda}{2}} \left( T^0_0(eff) - T^1_1(eff) - 2T^2_2(eff) \right) d\tilde{r}.$$  

(31)

As total energy of the fluid is measured in the form of Tolman formula, so the mass inside boundary $\Sigma$ of radius $r$ can be followed as

$$m_T = 4\pi \int_0^r \tilde{r}^2 \left( T^0_0(eff) - T^1_1(eff) - 2T^2_2(eff) \right) d\tilde{r}.$$  

(32)

The “active gravitational mass” plays an important role from the global concept of energy to the local level, which can seen from below as

$$m_T = \left[ m(r) + 4\pi r^3 P^\text{eff}_r \right].$$  

(33)

Alternatively, by using field equations (12), (13) and (14), we have

$$m_T = e^{\frac{\nu - \lambda}{2}} \nu r^2 \frac{\nu}{2}.$$  

(34)
The gravitational acceleration \( a = -s^x a_v \) of a test particle is obtained through Eq. (22) as
\[
a = e^{-\lambda} \nu' = e^{-\lambda} m_T, \tag{35}
\]
Another way to write \( m_T \) is
\[
m_T = (m_T)_{\Sigma} \left( \frac{r}{r_{\Sigma}} \right)^3 - r^3 \int_{r_{\Sigma}}^{r} \frac{8 \pi}{r} (P_\perp^{\text{eff}} - P_r^{\text{eff}}) + \frac{1}{r} \int_{0}^{r} 4 \pi r^3 \mu_{\text{eff}}^\gamma d\tilde{r} \tag{36}
\]
By making use of Eq. (29), we may write above equation as
\[
m_T = (m_T)_{\Sigma} \left( \frac{r}{r_{\Sigma}} \right)^3 - r^3 \int_{r_{\Sigma}}^{r} \frac{8 \pi}{r} (P_\perp^{\text{eff}} - P_r^{\text{eff}}) - E \] \tag{37}
In Eq. (36), the 2\text{nd} integral express \( m_T \) in the form of anisotropic pressure and inhomogeneous energy density in the presence of \( f(G, T) \) dark source terms. Now we move forward to the orthogonal splitting of Riemann tensor.

**III. The Orthogonal Splitting of the Riemann Tensor in \( f(G, T) \) Garvity**

For the first time Bel [64] introduced the orthogonal splitting of Riemann tensor. Herrera et al. [61,65] described the orthogonal splitting of Riemann tensor and elaborated this technique in the modeling of radiating and non-radiating stellar structure in GR. Now, after some alteration, one can write the following tensors as
\[
Y_{\alpha\beta} = R_{\alpha\gamma\beta\delta} u^\gamma u^\delta, \tag{38}
\]
\[
Z_{\alpha\beta} = *R_{\alpha\gamma\beta\delta} u^\gamma u^\delta = \frac{1}{2} \eta_{\alpha\gamma\mu} R_{\beta\delta}^{\mu} u^\gamma u^\delta, \tag{39}
\]
\[
X_{\alpha\beta} = *R_{\alpha\gamma\beta\delta} u^\gamma u^\delta = \frac{1}{2} \eta_{\alpha\delta} R_{\epsilon\mu\beta\gamma}^{\mu} u^\gamma u^\delta, \tag{40}
\]
Here the dual tensor is represented by *, i.e., \( R_{\alpha\beta\gamma\delta} = \frac{1}{2} \eta_{\epsilon\mu\gamma\delta} R_{\alpha\beta}^{\epsilon\mu} \). Now we orthogonally split the Riemann tensor. For that purpose, we can write the formula of Riemann tensor with the help of field equations as
\[
R_{\beta\delta}^{\alpha\gamma} = C_{\beta\delta}^{\alpha\gamma} + 16 \pi T^{\text{eff}}_{[\beta} [\alpha} [\delta] + 8 \pi T^{(\text{eff})} \left( \frac{1}{3} \delta_{[\beta}^{\alpha} \delta_{\delta]} - \delta_{[\beta}^{\alpha} \right), \tag{41}
\]
When we insert Eqs. (5) and (11) in Eq. (41), one can get the following tensorial quantities

\[ R_{(I)\beta\delta}^{\alpha\gamma} = 16\pi \mu \varepsilon_\alpha^{\alpha} \mu_{[\beta} \delta_{\gamma]} + 2\mu \varepsilon_{\alpha}^{\alpha} \mu_{[\beta} \delta_{\gamma]} - 16\pi P \varepsilon_{\alpha}^{\alpha} \delta_{\delta} + 8\pi (\mu - 3P) \left( \frac{1}{3} \delta_{[\beta} \delta_{\gamma]} - \delta_{\beta} \delta_{\gamma} \right) \]  

(42)

\[ R_{(II)\beta\delta}^{\alpha\gamma} = 16\pi \Pi_{[\beta}^{\alpha} \delta_{\gamma]} + 2\Pi_{\beta}^{\alpha} \delta_{\gamma]} + \delta_{[\beta} \delta_{\gamma]} f \]  

(43)

\[ R_{(III)\beta\delta}^{\alpha\gamma} = 4\mu \varepsilon_{\alpha}^{\alpha} E_{\beta}^{\gamma} - \epsilon_{\alpha}^{\alpha} \epsilon_{\beta\delta} E^{\mu\nu} \]  

(44)

\[ R_{(IV)\beta\delta}^{\alpha\gamma} = 2(\Pi_{\beta}^{\alpha} \delta_{\delta} - \Pi_{\delta}^{\alpha} \delta_{\beta} + \Pi_{\delta}^{\alpha} \delta_{\beta} + \Pi_{\delta}^{\alpha} \delta_{\beta}) f \]  

(45)

with the properties,

\[ \epsilon_{\alpha\gamma\beta} = \delta_{\alpha}^{\mu} \eta_{\mu\gamma\beta}, \quad \epsilon_{\alpha\gamma\beta} u_\beta = 0, \]  

(46)

and

\[ \epsilon^{\alpha\gamma\nu} \epsilon_{\alpha\nu\beta} = \delta_{\alpha}^{\mu} h_{\beta}^{\mu} - \delta_{\alpha}^{\mu} h_{\gamma}^{\mu} + u_\alpha (\delta_{\alpha}^{\mu} \delta_{\beta}^{\gamma} - \delta_{\beta}^{\mu} \delta_{\gamma}) \]  

(47)

Now we evaluate three tensors \( Y_{\alpha\beta} \), \( Z_{\alpha\beta} \) and \( X_{\alpha\beta} \) explicitly in form of physical variables as

\[ Y_{\alpha\beta} = \frac{4\pi}{3} (\mu + 3P) h_{\alpha\beta} + \frac{1}{6} (\mu + P) h_{\alpha\beta} f_T + E_{\alpha\beta} \]  

+ 4\pi \Pi_{\alpha\beta} + \frac{\Pi_{\alpha\beta}}{2} f_T + D_{\alpha\beta}^{(D)}, \]  

(48)

\[ X_{\alpha\beta} = \frac{8\pi}{3} \mu h_{\alpha\beta} - E_{\alpha\beta} + 4\pi \Pi_{\alpha\beta} + \frac{\Pi_{\alpha\beta}}{2} f_T + O_{\alpha\beta}^{(D)}, \]  

(49)
and

$$Z_{\alpha\beta} = I_{\alpha\beta}^{(D)}.$$  \hfill (50)

The values of dark source terms $D_{\alpha\beta}^{(D)}, O_{\alpha\beta}^{(D)}$ and $I_{\alpha\beta}^{(D)}$ are expressed in Appendix of [55].

The above tensors can also be written in form of some scalar functions, referred to as structure scalars. We can define four scalars functions from the tensors $X_{\alpha\beta}$ and $Y_{\alpha\beta}$, these are denoted by $X_T$, $X_{TF}$, $Y_T$, $Y_{TF}$. A fifth scalar is related to the tensor $Z_{\alpha\beta}$ which disappear here because of the the static spherical case. Energy density inhomogeneity, anisotropic pressure and Tolman mass are some of the elementary characteristics which are described through these scalars [66]. These scalars are found for $f(G,T)$ theory as follows

$$X_T = 8\pi\mu + Q^{(D)},$$  \hfill (51)

$$X_{TF} = 4\pi\Pi - E + \frac{\Pi}{2} f_T,$$  \hfill (52)

where the expression for $Q^{(D)}$ is given in Appendix of [55]. By using Eq.(29)

$$X_{TF} = \frac{4\pi}{r^3} \int_0^r \tilde{r}^3 \mu'^{\prime f} d\tilde{r} + \frac{\Pi}{2} f_T - 4\pi\Pi^{(D)},$$  \hfill (53)

$$Y_T = 4\pi (\mu + 3P_r - 2\Pi) + \frac{1}{2} (\mu + P) f_T + M^{(D)},$$  \hfill (54)

$$Y_{TF} = 4\pi\Pi + E + \frac{\Pi}{2} f_T + L^{(D)},$$  \hfill (55)

where $L^{(D)} = \frac{J_{\alpha\beta}^{(D)}}{s_{\alpha\beta} + h_{\alpha\beta}}$. The values of $M^{(D)}$ and $L^{(D)}$ are described in Appendix of [55]. By making use of $E$ from Eq.(29), we obtain

$$Y_{TF} = 8\pi\Pi - \frac{4\pi}{r^3} \int_0^r \tilde{r}^3 \mu'^{\prime f} d\tilde{r} + \frac{\Pi}{2} f_T + 4\pi\Pi^{(D)} + L^{(D)}. \hfill (56)$$

After considering Eqs.(53) and (56), we deduce that $X_{TF}$ is responsible for controlling the inhomogeneous energy in the presence of $f(G,T)$ corrections whereas $Y_{TF}$ is involved in the contribution of the anisotropic pressure, energy density inhomogeneity with extra curvature terms. This indicates that $Y_{TF}$ is serving as a complexity factor for our relativistic diagonally symmetric spherical geometry. From these equations, we also deduce that $X_{TF}$ and $Y_{TF}$ determine the local anisotropy of pressure with some dark source terms.

$$8\pi\Pi + \Pi f_T + L^{(D)} = X_{TF} + Y_{TF}.$$  \hfill (57)
To understand $Y_T$ and $Y_{TF}$ physically, we go back to Eqs. (36) and (37). From Eqs. (55) and (56), we obtain

$$m_T = (m_T)_{\Sigma} \left( \frac{r}{r_{\Sigma}} \right)^3 + r^3 \int_{r_{\Sigma}}^{r} \frac{e^{\frac{\nu + \lambda}{2}}}{r} [Y_{TF} - \frac{\Pi}{2} f_T - L^{(D)} + 4\pi \Pi^{(D)}] d\bar{r},$$

(58)

By comparing the above equation with Eq. (36), it can be seen that Tolman mass in term of anisotropic pressure and inhomogeneous energy density along with dark source terms in $f(G,T)$ gravity is represented by $Y_{TF}$. Alternatively, $Y_{TF}$ explains the modification of Tolman mass as compared to homogeneous matter. Lastly, Tolman mass can also be deduced in the form of

$$m_T = \int_{r_{\Sigma}}^{r} \bar{r}^2 e^{\frac{\nu + \lambda}{2}} \left( Y_T - \frac{1}{2}(\mu + P) f_T + 4\pi(\mu^{(D)} + 3P^{(D)}) - M^{(D)} \right) d\bar{r}.$$ (59)

It can be seen from the literature [53, 61, 65] that the structure scalar, $Y_T$ describes the effective form of mass density. In the above expression, it is noticed that $m_T$ is expressed with the help of $Y_T$. Thus, $m_T$ could be considered as a possible mathematical vehicle to understand the information as well as the subsequent changes in the matter terms along with the correction terms of $f(G,T)$ gravity.

### IV. Fluid Distributions With Zero Complexity Factor

The modified equations of motion make a system of three partial differential equations with five unknown functions ($\mu, \nu, \lambda, P^{eff}_T, P^{eff}_\perp$). In this direction, the vanishing complexity factor condition after applying $Y_{TF} = 0$ gives

$$\Pi = \frac{1}{8\pi + \frac{d}{2}} \left[ 4\pi \int_{0}^{r} \bar{r}^3 \mu^{eff} d\bar{r} - L^{(D)} - 4\pi \Pi^{(D)} \right].$$ (60)

It can be seen that the dark source terms reduces the complexity in the vanishing complexity factor condition. After applying the condition $Y_{TF} = 0$ one can solve these equations. However, we further need one more condition to solve this system. For this purpose, we proceed our results as follows.

### A. The Gokhroo and Mehra Ansätz

Elizalde et al. [67] claimed that such extra degrees of freedom mediated by exponential $f(R)$ terms may provide a unified picture of our accelerating universe at both late and early time
epochs. To study the behavior of compact objects, Gokhroo and Mehra [68] put forward solution of the field equations for anisotropic sphere with variable energy density. As we have merged all the modified geometric quantities in $\mu^{eff}$, so there would be no change in the formula arrangement except that now we are dealing the modified matter. The starting point is a supposition in the form of metric function $\lambda$ which is

$$e^{-\lambda} = 1 - \alpha r^2 + \frac{3K\alpha r^4}{5r_\Sigma^2},$$  \hspace{1cm} (61)$$

where $K$ is constant within the range of $(0, 1)$ and $\alpha = \frac{8\pi\mu_o}{3}$. Equations (12) and (13) give

$$\mu^{eff} = \mu_o \left(1 - \frac{Kr^2}{r_\Sigma^2}\right),$$  \hspace{1cm} (62)$$

and

$$m(r) = \frac{4\pi\mu_o r^3}{3} \left(1 - \frac{3Kr^2}{5r_\Sigma^2}\right).$$  \hspace{1cm} (63)$$

Further, we can write from Eqs.(13) and (14) as

$$8\pi(P_{eff} - P_{\perp}) = e^{-\lambda} \left[\frac{1}{r^2} + \frac{\nu'}{2} - \frac{\nu''}{2} - \left(\frac{\nu'}{2}\right)^2 + \frac{\lambda'}{2} \left(\frac{\nu'}{2} + \frac{1}{r}\right)\right] - \frac{1}{r^2}.$$  \hspace{1cm} (64)$$

To rewrite the line element, we introduce the new variables

$$e^{\nu(r)} = e^{\int(2z(r)-2/r)dr},$$  \hspace{1cm} (65)$$

and

$$e^{-\lambda} = y(r),$$  \hspace{1cm} (66)$$

and by inserting the Eqs. (65) and (66) in Eq (64), we obtain

$$y' + y \left[2z + \frac{2z'}{r} - \frac{4}{zr^2} - \frac{6}{r}\right] = -\frac{2}{z} \left(\frac{1}{r^2} + 8\pi\Pi\right).$$  \hspace{1cm} (67)$$

Then, the interior spherically symmetric line element becomes

$$ds^2 = -e^{\int(2z(r)-2/r)dr}dt^2 + \frac{z^2 e^{\int(2z(r)+4/rz(r))}}{r^6 \left(-2 \int \frac{z(r)(1+8\pi\Pi r^2)e^{\int(2z(r)+4/rz(r))dr}}{r^8}dr + C\right)^2}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2,$$  \hspace{1cm} (68)$$

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where $C$ is constant of integration. Now, the values of matter variables can be written with the help of the above mathematical relations as

\begin{align}
4\pi P^\text{eff}_r &= \frac{z(r-2m) + \frac{m}{r} - 1}{r^2}, \\
4\pi \mu^\text{eff} &= \frac{m'}{r^2}, \\
8\pi P^\text{eff}_\perp &= \left(1 - \frac{2m}{r}\right) \left(z' + z - \frac{z}{r} + \frac{1}{r^2}\right) + z \left(\frac{m}{r^2} - \frac{m'}{r}\right).
\end{align}

(69) \quad (70) \quad (71)

It can be noticed that the above mentioned results are regular at the origin $r = 0$ obeying $\mu^\text{eff} > 0, \mu^\text{eff} > P^\text{eff}_r, P^\text{eff}_\perp$ relations. The solution must fulfill the junction conditions to avert the singularity behavior of matter variables.

**B. The Polytrope with Disappearing Complexity Factor**

Self-gravitating fluid consists of different physical variables such as pressure, density, temperature etc. All these factors have different effects on the internal structure. The polytropic equation helps to study the compact objects in a better way. Here we would use the relation of effective energy density and pressure as we are going to deal the TOV equation, mass function, fluid with zero complexity which we have been derived in terms of effective matter in the scenario of $f(G, T)$ theory. The polytropic equation for anisotropic matter along with the vanishing complexity condition can be described as

\begin{align}
P^\text{eff}_r &= K \mu^\text{eff}(\gamma) = K \mu^\text{eff}(1+1/n), \\
Y_{TF} &= 0,
\end{align}

(72)

where constant $K$, $\gamma$ and $n$ are generally called as polytropic constant, polytropic exponent and polytropic index, respectively. We get two equations from polytropic equation of state, one of which is obtained from Tolman-Oppenheimer-Volkoff equation

\begin{align}
\xi^2 \frac{d\Psi}{d\xi} \left[1 - 2\nu\alpha(n+1)/\xi\right] + \nu + \alpha \frac{\xi^3 \Psi^{n+1} + 2\Pi \xi \Psi^{-n}}{P^\text{eff}_r (n+1)} \\
\times \left[1 - 2\nu\alpha(n+1)/\xi\right] = e^{\lambda Z} \left[\xi^2 (1 - 2\nu\alpha(n+1)/\xi) \right] \left[P^\text{eff}_r A \Psi^n (1 + \alpha \Psi)(1 + n)\right],
\end{align}

(73)

and other is

\begin{align}
d\nu = \xi^2 \Psi^n,
\end{align}

(74)
It could be beneficial to define few dimensionless variables as follows

\[
\alpha = \frac{P_{\text{eff}}}{\mu_{\text{eff}}}, \quad r = \frac{\xi}{A}, \quad A^2 = \frac{4\pi \mu_{\text{eff}} c}{\alpha(1 + n)}, \quad (75)
\]

\[
\Psi^n = \frac{\mu_{\text{eff}}}{\mu_c}, \quad \upsilon(\xi) = m(r)A^3/4\pi \mu_{\text{eff}},
\]

the subscript \(c\) describes that the corresponding quantities are calculated at the center of the sphere. We obtain two equations Eqs.(73) and (74) of first order with three unknown functions \(\Psi^n\), \(\upsilon\) and \(\Pi\) which depend on two parameters \(n\) and \(\alpha\). One extra condition is required to model the system, which is established from vanishing complexity factor condition as

\[
\frac{d\Pi}{d\xi} + 3\frac{\Pi}{\xi} = \frac{1}{(8\pi + \frac{f_T}{2})} \left[ 4\pi \mu_{\text{eff}} n\Psi^n-1 \frac{d\Psi}{d\xi} - \frac{\Pi f_{T,\xi}}{2} \right]. \quad (76)
\]

It gives us three ordinary differential Eqs.(73), (74) and (76) with three unknown functions \(\Psi^n\), \(\upsilon\) and \(\Pi\).

From Newtonian to general relativistic case we have two cases for polytropic equation of state, one of them is discussed in Eq.(72) and the other is given below

\[
P_{\text{eff}} = K \mu_{b_{\text{eff}}^{(\gamma)}} = K \mu_{b_{\text{eff}}^{(1+1/n)}}, \quad (77)
\]

where \(\mu_{b_{\text{eff}}}^{(\gamma)}\) indicates the baryonic mass density. For this case, the equations identical to Eqs. (73) and (76) are

\[
\xi^2 \frac{d\Psi_b}{d\xi} \left[ \frac{1 - 2\nu\alpha(n + 1)/\xi}{1 + \alpha \Psi_b} \right] + v + \alpha \xi^3 \Psi_b^{n + 1} + \frac{2\xi \Psi_b^{-n}}{P_{\text{eff}}(n + 1)} = e^\lambda \left[ \xi^2(1 - 2\nu\alpha(n + 1)/\xi) \right] \left[ \frac{\Psi_b^{n+1}}{P_{\text{eff}}(1 + \alpha \Psi_b)(1 + n)} \right], \quad (78)
\]

\[
\frac{d\Pi}{d\xi} + 3\frac{\Pi}{\xi} = \frac{1}{(8\pi + \frac{f_T}{2})} \left[ 4\pi \left( \mu_{b_{\text{eff}}}^{(\gamma)} n \Psi_b^{-1} \frac{d\Psi_b}{d\xi} \left( 1 + K(n + 1)\mu_{b_{\text{eff}}}^{(\gamma)} \frac{\Psi_b}{\xi} \right) \right) - \frac{3}{\xi} \left( (L^{(D)} + 4\pi^{(D)}) - L_{b_{\xi}}^{(D)} - 4\pi \frac{d\Pi^{(D)}}{d\xi} \right) - \frac{\Pi f_{T,\xi}}{2} \right], \quad (79)
\]

with \(\Psi_b^n = \frac{\mu_{b_{\text{eff}}}}{\mu_{b_{\text{eff}}}^{(\gamma)}}\). Equations (78) and (79) explain the TOV equation and vanishing complexity condition with dimensionless variables in the \(f(G, T)\) gravity, respectively in the second case.
The above equation describes the structure of spherical relativistic system with extra degrees of freedom mediated by $f(G, T)$ with different specific era of the cosmos controlled by an EoS parameter $K$. For instance,

1. The choice $K = 0$ in Eq.(78) describes TOV equation for non-relativistic matter configurations. The condition $K = 0$ in Eq.(72) corresponds that $p^{eff}_r = 0$, which means that dark source terms vanish out and we obtain a pressureless matter. Its so happened that one can analyze the hydrostatic equilibrium of a spherical object with the help of TOV equation. Furthermore, the input of $K = 0$ in Eq.(79) describes the equation of non-relativistic cloud having zero measure of complexity in it. However, the simultaneous solutions of Eqs.(78) and (79) could provide fruitful information for those non-relativistic spherical system that evolves non-adiabatically under the $f(G, T)$ gravity.

2. One can analyze the role of dark energy as well $f(G, T)$ corrections on the existence of the homogeneous relativistic spheres with the help of Eqs.(78) and (79) after considering $K = -1$. Thus one can investigate the structural evolution of the regular distribution of matter content with the effects of cosmic inflation and accelerated expansion from the above two equations.

3. Its could be possible that the spherical system could enter into the ultra-relativistic radiation phase. Thus, such a condition of the static irrotational spherical structure can be analyzed by keeping $K = \frac{1}{3}$ in Eqs.(78) and (79).

4. While the selection of $K \neq -1$ describes the set of equations of motion for a relativistic stellar structure in the phantom era of the universe with $f(G, T)$ corrections.

Another interesting feature of the above equations is that one can properties of analyze various regular objects at different phases of the universe by keeping specific values to the polytropic index $n$. For example, the choice $n = 1.5$ explains the polytropes for fully convective star cores [69,70], and $n = 3$ demonstrates the core of massive white dwarfs [71] and $n = \infty$ corresponds to isothermal sphere [72].

V. Conclusion

This paper is devoted to comprehend the consequences of $f(G, T)$ gravity on the dynamics of self-gravitating spherical objects. Self gravitating fluids in the field of astrophysics have such compelling characteristics which create curiosity among researchers to analyze their physical properties, like pressure, density, temperature, stability etc. For this reason,
we have analyzed static spherical symmetric-metric associated with anisotropic matter. In the formalism of $f(G, T)$ gravity, modified field and hydrostatic equilibrium equations are formulated. Specific relations between $m$ and $m_T$ are developed by using the formalism given by Misner-Sharp and Tolman, respectively. We have also performed complete orthogonal splitting of Riemann curvature tensor in $f(G, T)$ theory. Such a splitting lead us to get set of modified versions of scalar variables that woulds have a direct relations with the basic structural properties of the relativistic irrotational spherical object. It is seen here that like GR, the orthogonal breaking of Riemann tensor give rise to five scalar functions $X_{TF}$, $X_T$, $Y_{TF}$, $Y_T$ and $Z_T$ in $f(G, T)$ gravity. Herrera [18] introduced the idea of complexity for self-gravitating anisotropic fluids. The elementary supposition is that the system with isotropic pressure and homogenous energy density is less complex. Among the derived scalars variables, $Y_{TF}$ has been identified to be the complexity factor. We found some results that can be expressed as under.

(i) The scalar $Y_{TF}$ covers the energy density inhomogeneity and pressure anisotropy under the effect of extra curvature terms of $f(G, T)$ gravity, in a defined way.

(ii) Tolman mass in the presence of anisotropic pressure and inhomogeneous fluid is expressed in term of this scalar along with dark source term under $f(G, T)$ gravity.

(iii) For the non-static dissipative case, $Y_{TF}$ contributes the dissipative fluxes in addition to inhomogeneous energy density and local anisotropic pressure in the presence of modified terms.

(iv) The key reason for such a hypothesis lies in the fact that the scalar function $Y_{TF}$ includes influences from the inhomogeneity of energy density and the combined local pressure anisotropy in a very unique way, which disappears completely for the regular and locally isotropic distribution of fluids. It is important to note that the so-called complexity factor not only dissolves for the aforementioned simple configuration, but can also disappear when the two terms appearing in its definition, and featuring inhomogeneity of density and anisotropic pressure, nullify each other. Thus complexity can lead to some various varieties of systems, as mentioned in [1].

After developing modified field equations and matter distribution, we calculated the complexity of the system through one of the scalar variables $Y_{TF}$ given in Eq.(56) taken as complexity factor which includes the contribution of energy density inhomogeneity, anisotropic pressure and extra curvature terms. Moreover, we present two applications of field equations by taking the supposition of disappearing complexity factor condition i.e. $Y_{TF} = 0$. In the
first example, we have examined the effects of dark source terms by assuming specific energy density described by Gookhro and Mehra ansatz. In the second case, we have discussed the polytropic equation of state and adopted new dimensionless variables in order to write TOV equation, mass function and complexity factor condition. This work would enable us to understand the impact of complexity factor for self-gravitating objects in this modified theory. All the results of this modified theories are minimized to GR by using the limit $f(G, T) = R$.

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