THE STRONG LEVINSON THEOREM FOR THE DIRAC EQUATION

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The strong Levinson theorem for the Dirac equation is a result in quantum mechanics that relates the scattering phase shifts at threshold to the number of bound states of a given parity. In this paper, the authors consider the Dirac equation in one space dimension in the presence of a symmetric potential well. They connect the scattering phase shifts at threshold $E = \pm m$ to the number of bound states via the relation

$$\Delta_{e,o}(+m) + \Delta_{e,o}(-m) = \pm \frac{\pi}{2} \{ \sin^2 \Delta_{e,o}(+m) - \sin^2 \Delta_{e,o}(-m) \} = n_{e,o} \pi$$

(1)

where the subscripts $e,o$ denote the parity of the state and $n_{e,o}$ stands for the number of even and odd bound states. Ma and Ni [3] obtain a similar result in three dimensions.

In contrast, however, to the Schrödinger case, knowledge of $\Delta_{e,o}(+m)$ is not sufficient to determine the number of bound states. As the potential deepens further a bound electron state can cross the value $E = -m$ and will give the strong Levinson theorem for positive energies. In this limit states that correspond (for $L \rightarrow \infty$) to a real wavevector $k$ with energy $|E| > m$ become continuum scattering states, whereas states that correspond to imaginary wavevectors become bound states. This technique of box normalization in the case of the Dirac equation has previously been employed by Bell and Rajam new [8]; more recently it was used in the context of the Schrödinger equation [9] to connect boundary conditions with phase shifts just as we will do here.

For definiteness we consider a potential $V_0(x)$ which satisfies $V_0(x) \leq 0$ everywhere with $V_0(x) = 0, |x| > a$ for some $a < L$. We follow Lin and assume that the potential is switched on adiabatically starting with $V = 0$ by considering the potential $\lambda V_0(x)$ and letting $\lambda$ vary smoothly from $\lambda = 0$ to $\lambda = 1$. The advantage of putting the system in a box and of the consequent discrete spectrum is that we can follow the energy eigenvalue of each state of a given $n$ as the potential deepens. This will allow us to show the behaviour of the even or odd phase shift $\Delta_{e,o}(+m)$ as a state crosses the value $E = +m$ and will give the strong Levinson theorem for positive energies. In contrast, however, to the Schrödinger case, knowledge of $\Delta_{e,o}(+m)$ is not sufficient to determine the number of bound states. As the potential deepens further a bound electron state can cross the value $E = 0$ and it eventually reaches $E = -m$ where it joins the negative energy

We consider the Dirac equation in one space dimension in the presence of a symmetric potential well. We connect the scattering phase shifts at $E = +m$ and $E = -m$ to the number of states that have left the positive energy continuum or joined the negative energy continuum respectively as the potential is turned on from zero.
continuum, since in one dimension the states at $|E| = m$ are not normalizable.

In addition to the scattering states $|E| > m$ and bound states $|E| < m$ for completeness we have to consider the special case of critical states that occur for certain values of the potential: these are the states called zero energy resonances or half-bound states [10] in the context of the Schrödinger equation, but in the Dirac equation these include states at both $E = -m$ as well as $E = m$ [11], [12]. We deal with critical potentials in the final section of the paper.

II. THE STRONG LEVINSON THEOREM IN THE ABSENCE OF CRITICAL STATES

We write the Dirac equation as a two-component spinor $u = \begin{pmatrix} f \\ g \end{pmatrix}$ where [11]

$$\frac{\partial f}{\partial x} + (E - V(x) + m)g = 0 \quad (2a)$$

$$\frac{\partial g}{\partial x} - (E - V(x) - m)f = 0 \quad (2b)$$

and then the even and odd free particle wavefunctions outside the well $x > a$ for positive energies are

$$u_e(k, x) = \frac{N_{e+}}{\sqrt{L}} \begin{pmatrix} \cos (kx + \Delta_e) \\ \frac{k}{\xi} \sin (kx + \Delta_e) \end{pmatrix} \quad (3a)$$

$$u_o(k, x) = \frac{N_{o+}}{\sqrt{L}} \begin{pmatrix} \sin (kx + \Delta_o) \\ -\frac{k}{\xi} \cos (kx + \Delta_o) \end{pmatrix} \quad (3b)$$

with corresponding expressions for $x < a$ where $e, o$ refer to even and odd states. The bound state wavefunction for $x > a$ is given by

$$u_b(x) = N_b e^{-\kappa x}, x > a \quad (4)$$

where $\kappa = \sqrt{m^2 - E_b^2}$ and for $x < -a$ according to its parity. The $N$ are normalisation factors and play no role in the discussion.

We now use Lin’s results for the behaviour of the phase shifts at threshold $k \to 0$ in the continuum case (i.e. without box normalisation) and in the absence of critical states. For a wavevector $k$ we define $\xi \equiv k a$. In the limit $\xi \to 0$ Lin finds that positive energies

$$\tan \Delta_e(+m) \sim \xi^{-(2\tilde{p}_{e,o} - 1)} \quad (5a)$$

$$\tan \Delta_o(+m) \sim \xi^{2\tilde{q}_{e,o} - 1} \quad (5b)$$

and for negative energies

$$\tan \Delta_e(-m) \sim \xi^{2\tilde{q}_{e,o} - 1} \quad (6a)$$

$$\tan \Delta_o(-m) \sim \xi^{-(2\tilde{p}_{e,o} - 1)} \quad (6b)$$

where $\tilde{p}_{e,o}, \tilde{q}_{e,o}$ are positive integers. This will allow us to determine whether a phase shift at threshold is 0 mod $\pi$ or $\pi/2$ mod $\pi$ and in particular shows that

$$\sin^2 \Delta_{e,o}(+m) - \sin^2 \Delta_{e,o}(-m) = \pm 1 \quad (7)$$

so the weak Levinson theorem in the absence of critical states is from Eq. (1)

$$\Delta_{e,o}(+m) + \Delta_{e,o}(-m) + \pi/2 = n_{e,o} \pi \quad (8)$$

We now turn to the counting argument. We impose the periodic boundary condition

$$\psi(-L) = \psi(L). \quad (9)$$

Since the potential is symmetric, parity conservation requires that the top component of an even spinor is even as is the bottom component of an odd spinor. Eq. (9) thus makes a non-trivial statement only about odd components and gives for both even and odd states

$$\sin (kL + \Delta_{e,o}) = \sin (-kL - \Delta_{e,o}) = 0 \quad (10)$$

and hence

$$k_n L + \Delta_{e,o}(k_n) = n\pi \quad (11)$$

where $n$ is an integer.

If we repeat this analysis for a free particle where $V(x) = 0$ for all $x$ we obtain

$$k_n L = n\pi \quad (12)$$

as expected. Since $k$ is nonnegative by definition, so is $n$ while Eqs. (3) shows that $k = n = 0$ is allowed. Thus for a free particle in a box states are labelled by integer values $n = 0, 1, 2, \ldots$ with corresponding values for the wavevector $k_n = 0, \pi/L, 2\pi/L, \ldots$

Assume a potential well of the form $V(x) = \lambda V_0(x)$, where we take $V_0(x) \leq 0$ in order to ensure that there is at least one bound state. $V_0(x)$ is symmetric and is less singular than 1 near threshold. Let us begin with positive energy even states. From Eq. (5a), $\tan \Delta_e(\xi) \to \infty$ as $\xi \to 0$ and so $\Delta_e(+m)$ must be a positive odd multiple of $\pi/2$ or

$$\Delta_e(+m) = \left(\mu_+ - \frac{1}{2}\right) \pi \quad (14)$$

where $\mu_+$ is a positive integer (the subscript refers to the sign of energy) whose physical meaning we find shortly. Substituting Eq. (14) in Eq. (11) gives for values of $k$ near threshold

$$k_n = \frac{\pi}{L} \left(n - \mu_+ + \frac{1}{2}\right) \quad (15)$$
Since $k_n$ is by definition non-negative

\[ n \geq \mu_+ \tag{16} \]

Note that when $V(x) = 0$, $n \geq 0$. To determine the meaning of $\mu_+$ compare $\lambda = 0$ to a potential well with a very small value $\lambda = \varepsilon > 0$. There is a one-to-one correspondence between states for different values of $\lambda$ and the energy levels $E_n$ for $n \geq 1$ are hardly changed by the potential. But the energy level $E_n$ for $n = 0$ becomes a bound state with energy $E_0 < m$ when $\lambda = \varepsilon$. So for this small value of $\lambda$, $\mu_+ = 1$ and one state has crossed $E = m$ from the continuum and become bound. Therefore from Eq. (14) the strong Levinson theorem is given by

\[ \Delta_e(+m) = \pi/2 \tag{17} \]

in agreement with the Schrödinger result for a weak potential well. Now increase $\lambda$ gradually. For some value of $\lambda$ another state becomes almost bound. We deal with critical states in the next section so increase $\lambda$ sufficiently in order that the new potential possesses 2 bound states. Now the continuum states would satisfy $n \geq 2$ and hence $\mu_+ = 2$. So the strong Levinson therem for this potential is $\Delta_e(+m) = 3\pi/2$. Thus we now can see that Eq. (14) gives the strong Levinson theorem where $\mu_+$ even states have crossed from the positive energy continuum and become bound as $\lambda$ has increased from 0 to 1.

Now consider odd states. A weak potential does not bind odd states and consequently $\Delta_o(+m) = 0$ for $\lambda = \varepsilon$. From Eq. (5b) we see that in general

\[ \Delta_o(+m) = \nu_+ \pi \tag{18} \]

where $\nu_+$ is a non-negative integer. Then since from Eq. (11) $k_n = \frac{\pi}{2} (n - \nu_+)$ we now have $n \geq \nu_+$ which implies that $\nu_+$ states have crossed $E = m$ and become bound as $\lambda$ increases from 0 to 1. Eq. (18) is the strong Levinson theorem for odd states. Similar expressions for the strong form of Levinson’s theorem for even and odd states at positive energies have been obtained by Farhi, Graham, Jaffe and Weigel [13] in the context of field theory.

If $\mu_+, \nu_+$ were the number of even and odd bound states of the potential these results would agree with Levinson’s theorem for the Schrödinger equation [14]. For strong potentials, however, this is not the case in the Dirac equation because some of these states will subsequently become supercritical as $\lambda$ increases, cross $E = -m$ and revert to the continuum. To illustrate this, consider the phase shifts for negative energy states $\Delta_{e,o}(-m)$. From the weak theorem Eq. (8) we have

\[ \Delta_e(-m) = n_e \pi - \pi/2 - \left( \mu_+ - \frac{1}{2} \right) \pi = (n_e - \mu_+)\pi \tag{19} \]

The difference $\mu_+ - n_e$ between the number of states $\mu_+$ which have crossed $E = n$ as $\lambda$ has increased from 0 to 1 and the number of bound states $n_e$ for a given potential with $\lambda = 1$ is just the number of states $\mu_-$ which have become supercritical and crossed $E = -m$. So the strong Levinson theorem for negative energy even states is

\[ \Delta_e(-m) = -\mu_- \pi \tag{20} \]

and for negative energy odd states

\[ \Delta_o(-m) = -\left( \nu_- + \frac{1}{2} \right) \pi \tag{21} \]

where $\nu_+ - n_o = \nu_-$ and $\nu_-$ is the number of odd states which have crossed $E = -m$ into the negative energy continuum.

It is perhaps worth commenting that we are using Lin’s definition [2] of the phase shift which differs from that normally used in non-relativistic scattering where the phase shift at finite energy $E$ is compared with that at $E = \infty$ where the phase shift is taken to be zero. This is done in order to remove ambiguities of multiples of $\pi$ in the value of the phase shift. But Parzen showed over 50 years ago [15] that the phase shift at infinite energy in the Dirac equation in three dimensions for any fixed angular momentum $j$ is given by

\[ \delta_j(\infty) = -\int_0^\infty V(r)dr \tag{22} \]

and so phase shifts cannot vanish at infinity in the Dirac equation. Lin defines his zero of phase shift to be at $V = 0$ and then turns on the potential. In one dimension he obtains the analogous relation to Eq. (22)

\[ \delta(\infty) = -\int_{-\infty}^\infty V(x)dx \tag{23} \]

Lin shows in detail that his method also removes any ambiguity in the definition of the phase shift since Eq. (23) needs to be satisfied.

### III. THE EFFECT OF CRITICAL STATES

For completeness we now must consider critical states. We need the analogues of Eqs. (5, 6) for the phase shifts of a critical state for $\xi \to 0$ For positive energy critical states Lin [2] shows

\[ \tan \Delta_e(+m) \sim \xi^{2p_e-1} \tag{24a} \]
\[ \tan \Delta_o(+m) \sim \xi^{-(2p_o-1)} \tag{24b} \]

and for negative (even or odd) energy states

\[ \tan \Delta_e(-m) \sim \xi^{-(2p_e-1)} \tag{25a} \]
\[ \tan \Delta_o(-m) \sim \xi^{2p_o-1} \tag{25b} \]

where the quantities $p_{e,o}$, $q_{e,o}$ are positive integers.

Consider $\Delta_{e,+}(m)$. According to Eq. (14) $\Delta_{e,+}(m) = \pi/2$ for small $\lambda$ and passes through the values $3\pi/2, 5\pi/2$.
as \( \lambda \) increases and states cross the \( E = +m \) level and become bound. During the process \( \Delta_{\nu, \pi}(m) \) goes through the values \( \pi, 2\pi, \ldots \). These are precisely the values at which (according to Eq.(24a)) the potential supports an even positive energy critical state. Thus the strong version of Levinson’s theorem reads as follows: if \( \mu_+ \) is the number of positive energy even states that have crossed \( E = +m \) and the next even state is critical with \( E = +m \) then

\[
\Delta_\nu(+m) = \mu_+ \pi \quad (26)
\]

Similarly for odd critical states. The phase shift \( \Delta_{\nu, \pi}(m) \) starts at zero and goes through the values \( \pi, 2\pi, \ldots \) as the potential deepens and positive energy odd states cross \( E = +m \) and become bound. Thus \( \Delta_\nu(m) \) goes through the values \( \pi/2, 3\pi/2, \ldots \) and according to Eqs. (24) these are the values where the potential supports an odd critical state at \( E = +m \). Thus the strong Levinson theorem states that if \( \nu_+ \) is the number of positive energy odd states that have crossed \( E = +m \) and the next odd state in line sits at \( E = +m \) then

\[
\Delta_\nu(+m) = \left( \nu_+ + \frac{1}{2} \right) \pi \quad (27)
\]

These results agree with those of Sassoli di Bianchi [14] for the Schrödinger equation with the replacement of \( \mu_+ \nu_+ \) by \( n_+, n_0 \).

For negative energies we find that if \( \mu_- \) is the number of even states that have crossed \( E = -m \) and there is a new even supercritical state at \( E = -m \) then

\[
\Delta_\nu(-m) = -\left( \mu_- + \frac{1}{2} \right) \pi \quad (28)
\]

and that if \( \nu_- \) is the number of odd states that have crossed \( E = -m \) and become supercritical and there is a new odd supercritical state at \( E = -m \) then

\[
\Delta_\nu(-m) = -\left( \nu_- + 1 \right) \pi \quad (29)
\]

IV. CONCLUSIONS AND ACKNOWLEDGEMENTS

We have shown that there is a strong Levinson theorem for the Dirac equation which allows the scattering phase shift at threshold to be determined for both positive and negative energies and both even and odd parities. The phase shift at threshold, however, does not determine the total number of bound states when the potential is strong. This is on account of what mathematicians call spectral flow: a strong attractive potential can become supercritical and thus states which had been bound can revert to the (negative energy) continuum.

We expect that our results can be extended to more general potentials without too much trouble. Nevertheless it seemed important to us to demonstrate for a simple class of potentials that the strong Levinson’s theorem does exist in the context of the Dirac equation given that previous attempts going back to the 1960s have been unsuccessful [16], [17], [4] and at best have shown the weak theorem. In addition we should emphasize that our results explain why a strong Levinson theorem for the Dirac equation in terms of the number of bound states is not always possible for potentials where \( |V| > 2m \).

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