The Groups and Nilpotent Lie Rings of Order $p^8$ with Maximal Class

Seungjai Lee$^a$ and Michael Vaughan-Lee$^b$

$^a$The Research Institute of Basic Sciences, Seoul National University, Seoul, South Korea; $^b$Christ Church, University of Oxford, Oxford, UK

**ABSTRACT**

We classify and count the nilpotent Lie rings of order $p^8$ with maximal class for $p \geq 5$. This also provides a classification of the groups of order $p^8$ with maximal class for $p \geq 11$ via the Lazard correspondence. We also record the number of nilpotent Lie rings/groups of order $p^8$ with maximal class for $n \leq 7$ from currently known data and discuss its asymptotic behavior as $n$ grows and its potential connection to Higman’s PORC conjecture.

**KEYWORDS**

$p$-group, maximal class, Lie $p$-ring, Higman’s PORC conjecture

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**1. Introduction**

For a prime $p$, the classification of groups of order $p^n$ for small $n$ has a long history dating back to the end of the 19th century; for a detailed account see [2] or [12]. Up to the present, the complete description of groups of order $p^n$ for any given $p$ is only available for $n \leq 7$, where the groups of order $p^7$ were classified by O’Brien and Vaughan-Lee [11] in 2005.

The complete classification of groups of order $p^8$ still looks out of reach, but here we manage to make some progress by giving a classification of the nilpotent Lie rings of order $p^8$ ($p \geq 5$) which are of maximal class (i.e., nilpotent of class 7). The classification gives us the following theorem.

**Theorem 1.** For $p \geq 5$ the number of nilpotent Lie rings of order $p^8$ which have maximal class is

$$4p^3 + 7p^2 + 9p + 6 + (6p + 11) \gcd(p - 1, 3) + 4 \gcd(p - 1, 5) + (p + 2) \gcd(p - 1, 7) + (p + 3) \gcd(p - 1, 8) + 2 \gcd(p - 1, 9) + \gcd(p - 1, 12).$$

By the Lazard correspondence between $p$-groups and nilpotent Lie rings, for $p \geq 11$ this formula also gives us the number of groups of order $p^8$ with maximal class.

Recall Higman’s long-standing PORC conjecture [8]: for a fixed $n$, the number $f(p, n)$ of isomorphism classes of groups of order $p^n$ is given by a polynomial in $p$ whose coefficients depend on the residue class of $p$ modulo some fixed integer $N(= N(n))$, so that $f(p, n)$ is Polynomial On Residue Classes (PORC). This conjecture is known to be true for $n \leq 7$ [11], but still open for $n \geq 8$.

Let $m(p, n)$ denote the number of isomorphism classes of groups of order $p^n$ with maximal class. Our result (Theorem 1) shows that $m(p, 8)$ is PORC. Furthermore, later in Section 5 we give a complete record of $m(p, n)$ for $3 \leq n \leq 8$ (c.f. Theorem 4), which confirms that $m(p, n)$ is PORC for $n \leq 8$. We also introduce some conjectures on the asymptotic behavior of $m(p, n)$ and $f(p, n)$ to initiate further investigations.

We have constructed a database of the nilpotent Lie rings of order $p^8$ with maximal class which will be included in the next release of the GAP package LiePRing [5]. The LiePRing package can then be used to provide complete lists of the nilpotent Lie rings of order $p^8$ with maximal class for any given $p \geq 5$, and can also be used to provide complete lists of the groups of order $p^n$ with maximal class for any given $p \geq 11$. Both GAP [7] and MAGMA [3] have databases of the groups of order 2$^8$ and 3$^8$, and it is easy to extract the groups of maximal class from these databases. For $p = 5, 7$ we can use the databases of groups of order $p^7$ to obtain a list of the groups of maximal class, and then use the Descendants function in GAP or MAGMA to compute the groups of order $p^8$ with maximal class. (If $P$ has order $p^8$ and maximal class then $P$ is a “descendant” of the quotient $P/Z$ where $Z$ is the center of $P$, and $P/Z$ is a group of order $p^7$ of maximal class.)

**2. Preliminaries**

Let $L$ be a nilpotent Lie ring of order $p^n$ ($n \geq 3$) and class $n - 1$. Here “class” means nilpotency class, and not $p$-class, which is the length of lower $p$-central series of $L$. So $L$ is a nilpotent Lie ring of maximal class. Let the lower central series of $L$ be

$$L > L^2 > L^3 > \cdots > L^{n-1} > L^n = \{0\},$$

The Research Institute of Basic Sciences, Seoul National University, Seoul, South Korea.
where $L^2 = \langle ab \mid a, b \in L \rangle$, and where for $i > 2$, $L^i = \langle ab \mid a \in L^{i-1}, b \in L \rangle$. (We denote the Lie product of $a$ and $b$ by $ab$, rather than by $[a, b]$.) Then $L/L^2$ is elementary Abelian of order $p^2$, and for $2 \leq i < n$ the quotient $L^i/L^{i+1}$ has order $p$. Note that if $n > 3$ then $L/L^{n-1}$ has maximal class $n - 2$.

**Lemma 2.** $pL \leq L^{n-1}$. 

**Proof.** The proof is by induction on $n$. Note that there is nothing to prove if $n = 3$. So assume that $n > 3$, and assume by induction that $pL \leq L^{n-2}$. Let $L$ be generated by $a, b$, so that we can assume that $pa, pb \in L^{n-2}$. If $pa \notin L^{n-1}$ then $a$ centralizes $L^{n-2}$. Similarly if $pb \notin L^{n-1}$ then $b$ centralizes $L^{n-2}$. If $pa + pb \notin L^{n-1}$ then $a + b$ centralizes $L^{n-2}$. Now if one or the other or both of $pa, pb$ lie outside $L^{n-1}$ then at least two of $pa, pb, pa + pb$ lie outside $L^{n-1}$, which implies that $L$ is generated by elements which centralize $L^{n-2}$. Clearly this is impossible, and so $pa, pb \in L^{n-1}$. 

Now let $L$ be a nilpotent Lie ring of order $p^8$ and maximal class $7$. Then $L/L^2$ is a nilpotent Lie ring of order $p^7$ with maximal class. By Lemma 2, $L/L^2$ has characteristic $p$. The GAP package LiePRing contains a database of the nilpotent Lie rings of order $p^7$ ($p \geq 5$). There are $p + 8$ Lie rings of maximal class and characteristic $p$ in the database. These are as follows. (Here $bab$ denotes $(ba)b$, $ba^3$ denotes $((ba)a)a$, $ba^4$ denotes $(((ba)a)a)a$, and so on.)

\[
\begin{align*}
(a, b \mid bab, ba^3b, pa, pb, class \, 6), & \quad (7.623) \\
(a, b \mid bab, ba^3b - ba^5, pa, pb, class \, 6), & \quad (7.627) \\
(a, b \mid bab - ba^5, ba^3b, pa, pb, class \, 6), & \quad (7.633) \\
(a, b \mid bab - ba^4, ba^3b, pa, pb, class \, 6), & \quad (7.641) \\
(a, b \mid bab - ba^4, ba^3b, pa, pb, class \, 6), & \quad (7.646) \\
(a, b \mid bab - ba^4, ba^3b - ba^5, pa, pb, class \, 6), & \quad (7.648) \\
(a, b \mid bab - ba^3, ba^3b - xba^5, pa, pb, class \, 6) & \quad (0 \leq x < p), (7.650) \\
(a, b \mid bab - ba^3 - ba^5, ba^3b - ba^5, pa, pb, class \, 6), & \quad (7.656) \\
(a, b \mid bab - ba^3 - wba^5, ba^3b - ba^5, pa, pb, class \, 6). & \quad (7.657)
\end{align*}
\]

The numbering $7.623, 7.627, ...$ gives the “LibraryName” of these Lie rings in the database. The parameter $w$ in $7.657$ is taken to be a (fixed) primitive element mod $p$. Note that $p$ in these presentations can be replaced by $5$ to give a complete list of the nilpotent Lie rings of order $5^7$ of maximal class and characteristic 5, replaced by $7$ to give a complete list of the nilpotent Lie rings of order $7^7$ of maximal class and characteristic 7, and so on.

### 3. Computing Descendants

We use the Lie ring generation algorithm as described in [9] and [11] to compute the descendants of order $p^8$ of the nilpotent Lie rings of order $p^7$ with maximal class. This algorithm is an analogue of the $p$-group generation algorithm described in [10]. The Lie ring generation algorithm makes use of the lower $p$-central series of a Lie ring $L$, which is defined in an analogous way to groups. We define the series

\[
L = L_1 \geq L_2 \geq L_3 \geq \cdots \geq L_c \geq \cdots
\]

by setting $L_1 = L$, $L_2 = L^2 + pL$, and for $c > 1$ we set $L_{c+1} = L_cL + pL_c$. (Here $L_cL$ is $\langle ab \mid a \in L_c, b \in L \rangle$.) Note that we use superscripts to denote terms of the lower central series, and subscripts to denote terms of the lower $p$-central series. In the case of a nilpotent Lie ring of order $p^n$ with maximal class the two series are identical. The ideal $L_c$ consists of all linear combinations of terms of the form

\[
a_1a_2 \ldots a_c, \; pa_1a_2 \ldots a_{c-1}, \; p^2a_1a_2 \ldots a_{c-2}, \ldots, \; p^{c-1}a_1.
\]

We say that $L$ has $p$-class $c$ if $L_{c+1} = \{0\}$, $L_c \neq \{0\}$.

If $L$ is a nilpotent Lie ring with finite order $p^n$ for some prime $p$, then $L_{c+1}$ will equal $\{0\}$ for some $c$. In fact if $L$ is nilpotent of class $k$, and if the exponent of $L$ as a finite Abelian group is $p^m$ then $L$ has $p$-class $c$ for some $c$ with $k \leq c < k + m$.

If $L$ and $M$ are two finite nilpotent Lie rings with prime-power order, then $L$ is a descendant of $M$ if $L/L_c \cong M$ for some $c \geq 2$. If $L/L_c \cong M$ and $L$ has $p$-class $c$ (so that $L_c \neq \{0\}$, $L_{c+1} = \{0\}$) then $L$ is an immediate descendant of $M$. Note that if $L$ is a descendant of $M$ then $L/L_2 \cong M/M_2$, so that $L$ and $M$ have the same generator number.

As described in Section 2, any nilpotent Lie rings of order $p^8$ with maximal class arise as descendants of nilpotent Lie rings of order $p^7$, of characteristic $p$ with maximal class. We refer the reader to [9] for a description of a method for computing the immediate descendants of a nilpotent Lie ring $M$. 

4. The Nilpotent Lie Rings of Order $p^8$ with Maximal Class

In this section we give a complete list of presentations for the nilpotent Lie rings of order $p^8$ with maximal class ($p \geq 5$). Many of the presentations involve parameters $w, x, y, z$. These parameters take integer values in the range $0, 1, \ldots, p - 1$. The parameter $w$ is always assumed to be a (fixed) primitive element mod $p$. Associated with the presentations are somewhat cryptic comments intended to describe when two sets of parameters give isomorphic Lie rings. For example, four of the descendants of 7.623 have a single parameter $x$, together with the comment "$x \neq 0, x \sim xd^6\). Here (and for all these comments) $d$ is assumed to range over all integers which are nonzero modulo $p$. So this comment is intended to mean that $x$ can take any value in the range $1, 2, \ldots, p - 1$ and that if $0 < x, y < p$ then $x$ and $y$ give isomorphic Lie rings if and only if $x = yd^6 \mod p$ for some integer $d$ which is not divisible by $p$. Actually there is no reason to restrict $x$ to the range $1, 2, \ldots, p - 1$ since $x$ is the coefficient of an element of order $p$ in the presentations of these Lie rings. So if $x = x' \mod p$ then $x$ and $x'$ give identical Lie rings.

The simplest way to "solve" these conditions is to treat them as defining equivalence relations over $GF(p)$. Two nonzero elements in $GF(p)$ give isomorphic algebras if and only if they lie in the same coset of the subgroup $\{d^6 | d \in GF(p)^*\}$ of the multiplicative group $GF(p)^*$ of nonzero elements in $GF(p)$. This subgroup has order $\frac{p - 1}{6}$ if $\gcd(p - 1, 3) = 3$ and order $\frac{p - 1}{2}$ if $\gcd(p - 1, 3) = 1$. If we let $u$ be a primitive element in $GF(p)$ then $1, u, u^2, u^3, u^4, u^5$ is a transversal for this subgroup when $\gcd(p - 1, 3) = 3$, and $1, u$ is a transversal for the subgroup if $\gcd(p - 1, 3) = 1$. So we obtain a complete and irredundant set of representatives for the isomorphism classes of these Lie rings as $x$ ranges over $1, 2, \ldots, p - 1$ by taking $x = 1, w, w^2 \mod p, w^3 \mod p, w^4 \mod p, w^5 \mod p$ when $\gcd(p - 1, 3) = 3$ and taking $x = 1, w$ when $\gcd(p - 1, 3) = 1$.

The comments associated with the other Lie rings with a single parameter $x$ are similar. One of the descendants of 7.627 has two parameters $x, y$ with the comment "$x, y \neq 0, [x, y] \sim [xd^6, yd^6]\)". We can solve this over $GF(p)$ by letting $y$ range over a transversal for the subgroup $\{d^6 | d \in GF(p)^*\}$ of the group $GF(p)^*$. For any given $y$ in this transversal the pair $[x, y]$ gives an isomorphic Lie ring to the pair $[x', y]$ if and only if $x' = xd$ for some $d \in GF(p)^*$ satisfying $d^6 = 1$. We can solve this equivalence relation on the values for $x$ over $GF(p)$, and lift the a set of representatives for the equivalence classes to integers in the range $1, 2, \ldots, p - 1$.

All the Lie rings in the list below have nilpotency class 7, but we leave the class unspecified, to save space.

4.1. The Descendants of 7.623

7.623 has $8 + 8 \gcd(p - 1, 3) + 2 \gcd(p - 1, 5) + \gcd(p - 1, 8)$ descendants of order $p^8$.

\[
\langle a, b | bab, ba^3b, pa, pb, ba^5b \rangle
\]

\[
\langle a, b | bab, ba^3b, pa - ba^6, pb, ba^5b \rangle
\]

\[
\langle a, b | bab, ba^3b, pa - xba^6, ba^5b \rangle \ (x \neq 0, x \sim xd^6)
\]

\[
\langle a, b | bab, ba^3b - ba^6, pa, pb, ba^5b \rangle
\]

\[
\langle a, b | bab, ba^3b - ba^6, pa - xba^6, pb, ba^5b \rangle \ (x \neq 0, x \sim xd^8)
\]

\[
\langle a, b | bab, ba^3b - ba^6, pa, pb - xba^6, ba^5b \rangle \ (x \neq 0, x \sim xa^6)
\]

\[
\langle a, b | bab - ba^6, ba^3b, pa, pb, ba^5b \rangle
\]

\[
\langle a, b | bab - ba^6, ba^3b, pa - xba^6, pb, ba^5b \rangle \ (x \neq 0, x \sim xd^{10})
\]

\[
\langle a, b | bab - ba^6, ba^3b, pa, pb - xba^6, ba^5b \rangle \ (x \neq 0, x \sim xa^6)
\]

\[
\langle a, b | bab, ba^3b, pa, pb, ba^6 \rangle
\]

\[
\langle a, b | bab, ba^3b, pa - ba^2b, pb, ba^6 \rangle
\]

\[
\langle a, b | bab, ba^3b, pa - wba^5b, pb, ba^6 \rangle
\]

\[
\langle a, b | bab, ba^3b, pa, pb - ba^2b, ba^6 \rangle
\]

\[
\langle a, b | bab, ba^3b, pa - xba^5b, pb - ba^5b, ba^6 \rangle \ (x \neq 0, x \sim xd^6)
\]
4.2. Descendants of 7.627

7.627 has \( p + 2 \gcd(p - 1, 3) + \gcd(p - 1, 7) \) descendants of order \( p^8 \).

\[
(a, b \mid bab, ba^2b - ba^5, pa, pb)
\]

\[
(a, b \mid bab, ba^3b - ba^5, pa - xba^6, pb) \quad (x \neq 0, x \sim xad^7)
\]

\[
(a, b \mid bab, ba^3b - ba^5, pa, pb - xba^6) \quad (x \neq 0, x \sim xa^6)
\]

\[
(a, b \mid bab, ba^3b - ba^5, pa - xba^6, pb - yba^6) \quad (x, y \neq 0, [x,y] \sim [xa^7, ya^6])
\]

4.3. Descendants of 7.633

7.633 has \( 4p + (p + 1) \gcd(p - 1, 3) + 2 \gcd(p - 1, 9) + \gcd(p - 1, 12) \) descendants of order \( p^8 \).

\[
(a, b \mid bab - ba^5, ba^3b, pa, pb, ba^3b)
\]

\[
(a, b \mid bab - ba^5, ba^3b, pa, pb - xba^6, ba^3b) \quad (x \neq 0, x \sim xa^6)
\]

\[
(a, b \mid bab - ba^5, ba^3b, pa - xba^6, pb, ba^5b) \quad (x \neq 0, x \sim xa^9)
\]

\[
(a, b \mid bab - ba^5, ba^3b, pa, pb, ba^5b)
\]

\[
(a, b \mid bab - ba^5, ba^3b - ba^6, pa - xba^6, pb, ba^5b) \quad (x \neq 0)
\]

\[
(a, b \mid bab - ba^5, ba^3b, pa, pb - xba^6, ba^5b) \quad (x \neq 0)
\]

\[
(a, b \mid bab - ba^5, ba^3b - ba^6, pa, pb - xba^6, ba^5b) \quad (x \neq 0)
\]

\[
(a, b \mid bab - ba^5, ba^3b, pa, pb, ba^6)
\]

\[
(a, b \mid bab - ba^5, ba^3b, pa - xba^5b, pb, ba^6) \quad (x \neq 0, x \sim xa^{12})
\]

\[
(a, b \mid bab - ba^5, ba^3b, pa, pb - xba^5b, ba^6) \quad (x \neq 0, x \sim xa^6)
\]

\[
(a, b \mid bab - ba^5, ba^3b, pa - xba^5b, pb - yba^5b, ba^6) \quad (x, y \neq 0, [x,y] \sim [xa^{12}, ya^6])
\]

4.4. Descendants of 7.641

7.641 has \( 5p - 3 + 2p \gcd(p - 1, 3) + 2 \gcd(p - 1, 5) + (p + 1) \gcd(p - 1, 8) \) descendants of order \( p^8 \).

\[
(a, b \mid bab - ba^4, ba^3b - xba^6, pa, pb, ba^5b) \quad (\text{all } x)
\]

\[
(a, b \mid bab - ba^4, ba^3b - xba^6, pa - yba^6, pb, ba^5b) \quad (y \neq 0, [x,y] \sim [x, ya^6])
\]

\[
(a, b \mid bab - ba^4, ba^3b - xba^6, pa, pb - yba^6, ba^5b) \quad (y \neq 0, [x,y] \sim [x, ya^6])
\]

\[
(a, b \mid bab - ba^4, ba^3b - ba^6, pa - xba^6, pb - yba^6, ba^5b) \quad (x, y \neq 0, [x,y] \sim [xa^8, ya^6])
\]

\[
(a, b \mid bab - ba^4, ba^3b, pa, pb, ba^6)
\]

\[
(a, b \mid bab - ba^4, ba^3b, pa - xba^5b, pb, ba^6) \quad (x \neq 0, x \sim xa^{10})
\]

\[
(a, b \mid bab - ba^4, ba^3b, pa, pb - xba^5b, ba^6) \quad (x \neq 0, x \sim xa^8)
\]

\[
(a, b \mid bab - ba^4, ba^3b, pa - xba^5b, pb - yba^5b, ba^6) \quad (x, y \neq 0, [x,y] \sim [xa^{10}, ya^6])
\]
4.5. Descendants of 7.646

7.646 has $p^2$ descendants of order $p^8$.

$$\langle a, b \mid bab - ba^4, ba^3b - ba^2, pa - xba^6, pb - yba^6 \rangle \quad (\text{all } x, y)$$

4.6. Descendants of 7.648

7.648 has $4p^2 - 3p + 1$ descendants of order $p^8$.

$$\langle a, b \mid bab - ba^4 - ba^5, ba^3b - xba^6, pa, pb, ba^5b \rangle \quad (x \neq 1)$$

$$\langle a, b \mid bab - ba^4 - ba^5, ba^3b - xba^6, pa - yba^6, pb, ba^5b \rangle \quad (x \neq 1, y \neq 0)$$

$$\langle a, b \mid bab - ba^4 - ba^5, ba^3b - xba^6, pa, pb - yba^6, ba^5b \rangle \quad (x \neq 1, y \neq 0)$$

$$\langle a, b \mid bab - ba^4 - ba^5, ba^3b - ba^6, pa - xba^6, pb - yba^6, ba^5b \rangle \quad (\text{all } x, y)$$

$$\langle a, b \mid bab - ba^4 - ba^5, ba^3b, pa - xba^5b, pb - yba^5b \rangle \quad (\text{all } x, y)$$

4.7. Descendants of 7.650

7.650 is a family of $p$ Lie rings, and between them they have

$$2p^3 + 3p^2 + p + 3p \gcd(p - 1, 3) + (p + 1) \gcd(p - 1, 7) + \gcd(p - 1, 8)$$

descendants of order $p^8$.

$$\langle a, b \mid bab - ba^3, ba^2b - ba^5, ba^5b, pb \rangle$$

$$\langle a, b \mid bab - ba^3, ba^2b - ba^5, ba^5b, pa - xba^6, pb \rangle \quad (x \neq 0, x \sim xd^7)$$

$$\langle a, b \mid bab - ba^3, ba^2b - ba^5, ba^5b, pa - xba^6, pb - yba^6 \rangle \quad (x \neq 0, x \sim xd^8)$$

$$\langle a, b \mid bab - ba^3, ba^2b - ba^5, xba^6, pa - xba^6, pb - yba^6 \rangle \quad (x, y \neq 0, [x, y, z] \sim [xd^7, yd^8])$$

$$\langle a, b \mid bab - ba^3 - xba^6, ba^2b - ba^5, ba^5b, pa - yba^6, pb - zba^6 \rangle \quad (x \neq 0, [x, y, z] \sim [xd^3, yd^7, zd^6])$$

$$\langle a, b \mid bab - ba^3, ba^2b - ba^5, pa, pb, ba^5b \rangle \quad (x \neq 1)$$

$$\langle a, b \mid bab - ba^3, ba^2b - xba^5, pa - yba^6, pb, ba^5b \rangle \quad (x \neq 1, y \neq 0, [x, y] \sim [x, yd^7])$$

$$\langle a, b \mid bab - ba^3, ba^2b - xba^5, pa, pb - yba^6, ba^5b \rangle \quad (x \neq 1, y \neq 0, [x, y] \sim [x, yd^8])$$

$$\langle a, b \mid bab - ba^3, ba^2b - xba^5, pa - yba^6, pb - zba^6, ba^5b \rangle \quad (x \neq 1, y, z \neq 0, [x, y, z] \sim [x, yd^7, zd^6])$$

$$\langle a, b \mid bab - ba^3, ba^2b - xba^5 - ba^6, pa - yba^6, pb - zba^6, ba^5b \rangle \quad (x \neq 1, \text{all } y, z)$$

$$\langle a, b \mid bab - ba^3, ba^2b - 3ba^5, pa, pb, ba^6 \rangle$$

$$\langle a, b \mid bab - ba^3, ba^2b - 3ba^5, pa - xba^5b, pb, ba^6 \rangle \quad (x \neq 0, x \sim xd^8)$$

$$\langle a, b \mid bab - ba^3, ba^2b - 3ba^5, pa, pb - xba^5b, ba^6 \rangle \quad (x \neq 0, x \sim xd^8)$$

$$\langle a, b \mid bab - ba^3, ba^2b - 3ba^5, pa - xba^5b, pb - yba^5b, ba^6 \rangle \quad (x, y \neq 0, [x, y] \sim [xd^8, yd^7])$$

$$\langle a, b \mid bab - ba^3, ba^2b - 3ba^5 - xba^6, pa - yba^6, pb - zba^6, ba^5b - ba^6 \rangle \quad (\text{all } x, y, z)$$
4.8. Descendants of 7.656

7.656 has \( p^3 - \frac{p^2-q}{2} \) descendants of order \( p^8 \).

\[
\langle a, b \mid bab - ba^3 - ba^5, ba^3b - ba^5, ba^5b, pa - xba^6, pb - yba^6 \rangle \ (\{x, y\} \sim \{-x, y\})
\]
\[
\langle a, b \mid bab - ba^3 - ba^5 - xba^6, ba^3b - ba^5, ba^5b, pa - yba^6, pb - zba^6 \rangle \ (x \neq 0, \{x, y, z\} \sim \{-x, y, z\})
\]
\[
\langle a, b \mid bab - ba^3 - ba^5 - xba^6, ba^3b - ba^5 - xba^6, ba^5b, pa - yba^6, pb - zba^6 \rangle \ (x \neq 0, \{x, y, z\} \sim \{-x, y, z\})
\]

4.9. Descendants of 7.657

7.657 has \( p^3 - \frac{p^2-q}{2} \) descendants of order \( p^8 \).

\[
\langle a, b \mid bab - ba^3 - wba^5, ba^3b - ba^5, ba^5b, pa - xba^6, pb - yba^6 \rangle \ (\{x, y\} \sim \{-x, y\})
\]
\[
\langle a, b \mid bab - ba^3 - wba^5 - xba^6, ba^3b - ba^5, ba^5b, pa - yba^6, pb - zba^6 \rangle \ (x \neq 0, \{x, y, z\} \sim \{-x, y, z\})
\]
\[
\langle a, b \mid bab - ba^3 - wba^5, ba^3b - ba^5 - xba^6, ba^5b, pa - yba^6, pb - zba^6 \rangle \ (x \neq 0, \{x, y, z\} \sim \{-x, y, z\})
\]

5. Higman's PORC Conjecture on \( p \)-Groups with Maximal Class

As mentioned in the introduction, let \( f(p, n) \) denote the number of isomorphism classes of groups of order \( p^n \). Higman's PORC conjecture can be rephrased as follows: for a fixed \( n \), there exists \( N(n) \in \mathbb{N} \) and finitely many polynomials \( W_i(X) \in \mathbb{Q}[X] \) for \( 0 \leq i \leq N(n) - 1 \) such that for almost all prime \( p \), if \( p \equiv i \mod N(n) \), then \( f(p, n) = W_i(p) \).

Theorem 3 ([1, 9, 11]). We have

\[
f(p, 3) = 5
\]
\[
f(p, 4) = 15 \ (\text{for } p > 2),
\]
\[
f(p, 5) = 2p + 61 + 2 \gcd(p - 1, 3) + \gcd(p - 1, 4) \ (\text{for } p > 3),
\]
\[
f(p, 6) = 3p^2 + 39p + 344 + 24 \gcd(p - 1, 3) + 11 \gcd(p - 1, 4) + 2 \gcd(p - 1, 5) \ (\text{for } p > 5),
\]
\[
f(p, 7) = 3p^3 + 12p^4 + 4p^5 - 70p^2 + 170p^3 + 707p + 2455 + 4p^2 + 4p + 291) \gcd(p - 1, 3)
\]
\[
\quad + (p^2 + 19p + 135) \gcd(p - 1, 4) + (3p + 31) \gcd(p - 1, 5) + 4 \gcd(p - 1, 7)
\]
\[
\quad + 5 \gcd(p - 1, 8) + \gcd(p - 1, 9) \ (\text{for } p > 5).
\]

Moreover,

\[
N(3) = 1, N(4) = 2, N(5) = 12, N(6) = 60, N(7) = 2520.
\]

Let \( m(p, n) \) denote the number of isomorphism classes of groups of order \( p^n \) with maximal class. Can we ask a weaker version of Higman's PORC conjecture on maximal class \( p \)-groups: that for a fixed \( n \), the number \( m(p, n) \) of isomorphism classes of groups of order \( p^n \) with maximal class is given by a polynomial in \( p \) whose coefficients depend on the residue class of \( p \) modulo some fixed integer \( M(= M(n)) \)? Theorem 1 shows that this is true for \( n = 8 \), since \( m(p, 8) \) is also PORC.

In fact, with Theorem 1 and from the available data we manage to calculate \( m(p, n) \) for \( n \leq 8 \).

Theorem 4. We have

\[
m(p, 3) = 2,
\]
\[
m(p, 4) = 4 \ (\text{for } p > 2),
\]
\[
m(p, 5) = 3 + 2 \gcd(p - 1, 3) + \gcd(p - 1, 4) \ (\text{for } p > 3),
\]
\[
m(p, 6) = 2p + 7 + 4 \gcd(p - 1, 3) + 4 \gcd(p - 1, 4) + 2 \gcd(p - 1, 5) \ (\text{for } p > 5),
\]
\[
m(p, 7) = 2p^2 + 5p + 3 + (2p + 2) \gcd(p - 1, 3) + (p - 2) \gcd(p - 1, 5)
\]
\[
\quad + \gcd(p - 1, 7) + \gcd(p - 1, 8) \ (\text{for } p > 5),
\]
\[
m(p, 8) = 4p^3 + 7p^2 + 9p + 6 + (6p + 11) \gcd(p - 1, 3) + 4 \gcd(p - 1, 5)
\]
\[
\quad + (p + 2) \gcd(p - 1, 7) + (p + 3) \gcd(p - 1, 8) + 2 \gcd(p - 1, 9)
\]
\[
\quad + \gcd(p - 1, 12) \ (\text{for } p > 7).
\]

Moreover,

\[
M(3) = 1, M(4) = 2, M(5) = 12, M(6) = 60, M(7) = 840, M(8) = 2520.
\]
From this result we can ask the following questions:

**Question 5.** [PORC conjecture on maximal class $p$-groups] Is $m(p, n)$ PORC for each $n \in \mathbb{N}$?

Higman [8] proved that the number of groups of order $p^n$ with $p$-class 2 is PORC for any arbitrary $n$. Evseev [6] extended this result by Higman and proved that for any $n$ the number of groups of order $p^n$ with central Frattini subgroup is PORC. Note that this is the only general result regarding Higman’s PORC conjecture that is currently known to be true for any arbitrary $n$.

Since counting the groups of order $p^n$ with maximal class is easier than counting all the groups of order $p^n$, and the family of maximal class $p$-groups is an interesting and important object of study in its own right, it is worth investigating the nature of $m(p, n)$. In particular, a positive answer to Question 5 would be a big breakthrough in Higman's PORC conjecture.

Furthermore, by looking at $N(n)$ and $M(n)$ for small $n$, one can observe for each fixed $n$ the prime factors of $N(n)$ and $M(n)$ are always less than or equal to $n$. For example, $N(6) = M(6) = 60 = 2^2 \cdot 3 \cdot 5$, where $2, 3, 5 \leq 6$, and $N(7) = 2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$, $m(7) = 840 = 2^3 \cdot 3 \cdot 5 \cdot 7$, where $2, 3, 5, 7 \leq 7$. Is this just a coincidence? Or are there some fundamental reasons for this phenomena?

**Conjecture 6.** For each $n$, if $m(p, n)$ is PORC, then $M(n)$ has a unique prime factorization $M(n) = p_1^{r_1} \cdots p_k^{r_k}$, where $p_1, \ldots, p_k \leq n$.

**Conjecture 7.** For each $n$, if $f(p, n)$ is PORC, then $N(n)$ has a unique prime factorization $N(n) = p_1^{r_1} \cdots p_l^{r_l}$, where $p_1, \ldots, p_l \leq n$.

Our work indeed supports Conjecture 6 for $n = 8$. Would Conjecture 7 be also true for $n = 8$? Since the essence of Higman’s PORC conjecture is the existence of $N(n)$ for each $n$, where the splitting behavior of $f(p, n)$ over $p$ is determined by congruence conditions of $p$ modulo $N(n)$, understanding this relation between $n$ and $N(n)$ (or $M(n)$) might provide important insight. In this light, more computations of $f(p, n)$ or $m(p, n)$ for larger $n$ would be useful for further investigations.

**Acknowledgments**

Our classification of the nilpotent Lie rings of order $p^n$ with maximal class ($p \geq 5$) is essentially a hand calculation, with some computer assistance with Magma [3]. To eliminate errors we used Eamonn O’Brien’s $p$-group generation algorithm [10] in Magma to compute the groups of order $p^n$ with maximal class for $11 \leq p \leq 43$, and confirmed that the number of groups in these cases agreed with the PORC formula given in Theorem 1. We also used Serena Cicalò and Willem de Graaf’s implementation of the Lazard correspondence in their GAP package LieRing [4] to obtain the groups corresponding to the nilpotent Lie rings in our database, so that we could compare them with the groups provided by the $p$-group generation algorithm. We used Eamonn O’Brien’s Standard Presentation function in Magma to prove that the two sets of groups are identical (up to isomorphism) for $11 \leq p \leq 43$.

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