A short note on extension theorems and their connection to universal consistency in machine learning†

Andreas Christmann¹, Florian Dumpert¹, Dao-Hong Xiang²,¹

¹ Department of Mathematics, University of Bayreuth, Germany
² Department of Mathematics, Zhejiang Normal University, Jinhua, Zhejiang 321004, China

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Abstract

Statistical machine learning plays an important role in modern statistics and computer science. One main goal of statistical machine learning is to provide universally consistent algorithms, i.e., the estimator converges in probability or in some stronger sense to the Bayes risk or to the Bayes decision function. Kernel methods based on minimizing the regularized risk over a reproducing kernel Hilbert space (RKHS) belong to these statistical machine learning methods. It is in general unknown which kernel yields optimal results for a particular data set or for the unknown probability measure. Hence various kernel learning methods were proposed to choose the kernel and therefore also its RKHS in a data adaptive manner. Nevertheless, many practitioners often use the classical Gaussian RBF kernel or certain Sobolev kernels with good success. The goal of this short note is to offer one possible theoretical explanation for this empirical fact.

Key words and phrases. Machine learning; kernel learning; universal consistency; Dugundji extension theorem; Lusin theorem; dense; reproducing kernel Hilbert space.

1 Introduction

Regularized empirical risk minimization over large classes \( \mathcal{F} \) of functions \( f : \mathcal{X} \to \mathcal{Y} \) have attracted a lot of interest during the last decades in statistical machine learning. Here \( \mathcal{X} \) and \( \mathcal{Y} \) denote the so-called input space and output space, respectively. Of particular importance is the case that \( \mathcal{F} \) equals a reproducing kernel Hilbert space \( H \) specified by its corresponding

†Corresponding author: A. Christmann, Email: andreas.christmann@uni-bayreuth.de

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kernel \( k \). Fields of applications range from classification, regression, and quantile regression to ranking, similarity learning and minimum entropy learning. Probably the most important goal of such machine learning methods is universal consistency, i.e., convergence in probability to the Bayes risk defined as the infimum of the risk over all measurable functions or to the Bayes decision function, if it exists. To achieve this goal, one typically splits up the total error into a stochastic error and into a non-stochastic approximation error. Concentration inequalities are then often used to upper bound the stochastic error. Denseness arguments are typically used to show that the infimum of the risk when minimizing over \( F \) equals the Bayes risk, i.e.,

\[
\inf_{f \in F} \mathcal{R}_{L,P}(f) = \inf_{f \text{ measurable}} \mathcal{R}_{L,P}(f),
\]

where \( L \) denotes a loss function and \( P \) denotes a probability measure. Two of the most successful special cases in statistical machine learning are the following ones. Let the input space \( \mathcal{X} \) be a compact metric space. Then a continuous kernel is called universal, if its RKHS \( H \) is dense with respect to the supremum norm in \( C(\mathcal{X}) \), see Definition \ref{def:universality}. For more general input spaces, one often assumes that the RKHS is dense in some \( L^p(\mu) \) for all probability measures \( \mu \) on \( \mathcal{X} \), where \( p \geq 1 \) is some constant, see e.g., Steinwart and Christmann (2008, Lem. 4.59, Thm. 4.63).

The main goal of this paper is to address the question whether denseness of the RKHS with respect to the supremum norm in \( C(\mathcal{X}) \) can sometimes be weakened.

To fix ideas, let \( n \) be a positive integer and \( D = ((x_1, y_1), \ldots, (x_n, y_n)) \) be a given data set, where the value \( x_i \in \mathcal{X} \) denotes the input value and \( y_i \in \mathcal{Y} \) denotes the output value of the \( i \)-th data point. Let

\[
L : \mathcal{X} \times \mathcal{Y} \times \mathbb{R} \to [0, \infty)
\]

be a loss function of the form \( L(x, y, f(x)) \), where \( f(x) \) denotes the predicted value for \( y \), if \( x \) is observed, and \( f : \mathcal{X} \to \mathbb{R} \) is a real-valued function. Most often \( L \) is assumed to be a convex loss function, i.e., \( L(x, y, \cdot) \) is convex for any fixed pair \((x, y) \in \mathcal{X} \times \mathcal{Y}\). Many regularized learning methods are then defined as minimizers of the optimization problem

\[
\inf_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} L(x_i, y_i, f(x_i)) + \text{pen}(\lambda_n, f),
\]

where the set \( \mathcal{F} \) consists of functions \( f : \mathcal{X} \to \mathbb{R} \), \( \lambda_n > 0 \) is a regularization constant, and \( \text{pen}(\lambda_n, f) \geq 0 \) is some regularization term to avoid overfitting for the case, that \( \mathcal{F} \) is rich. One example is that \( \mathcal{F} \) is a reproducing kernel Hilbert space \( H \) and \( \text{pen}(\lambda_n, f) = \lambda_n \|f\|_H^2 \), i.e.,

\[
\inf_{f \in H} \frac{1}{n} \sum_{i=1}^{n} L(x_i, y_i, f(x_i)) + \lambda_n \|f\|_H^2,
\]

see e.g., Vapnik (1995, 1998), Poggio and Girosi (1998), Wahba (1999), Schölkopf and Smola (2002), Cucker and Zhou (2007), Smale and Zhou (2007), Steinwart and Christmann (2008) and the references cited therein. If the output space \( \mathcal{Y} \) is a general Hilbert space, regularized learning with kernels have been investigated, e.g., by Micchelli and Pontil (2005b) and Caponnetto and De Vito (2007). We also like to mention two other regularization terms: \( \text{pen}(\lambda_n, f) := \lambda_n \|f\|_H^p \) for some \( p \geq 1 \) and elastic nets, see e.g., De Mol et al. (2009).
In recent years there is increasing interest in related pairwise learning methods where a pairwise loss function

\[ L : \mathcal{X} \times \mathcal{Y} \times \mathcal{X} \times \mathcal{Y} \times \mathbb{R} \times \mathbb{R} \to [0, \infty) \]

is used and optimization problems like

\[
\inf_{f \in H} \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} L(x_i, y_i, x_j, y_j, f(x_i), f(x_j)) + \lambda_n \|f\|_H^2
\]

have to be solved. An example of this class of learning methods occurs when one is interested in minimizing Renyi’s entropy of order 2, see e.g., Hu et al. (2013), Fan et al. (2014), and Ying and Zhou (2015) for consistency and fast learning rates. Another example arises from ranking algorithms, see e.g., Clémencen et al. (2008) and Agarwal and Niyogi (2009). Other examples include gradient learning, and metric and similarity learning, see e.g., Mukherjee and Zhou (2006), Xing et al. (2003), and Cao et al. (2016). We refer to Christmann and Zhou (2015) for robustness aspects of pairwise learning algorithms.

In practice, the loss function is usually determined by the concrete application. However, it is not always clear how to choose a kernel and therefore its RKHS in a reasonable manner. There exist so many papers on learning the kernel from the data – often called kernel learning or multiple kernel learning – that it is impossible to cite all them here, but we like to mention a few. One popular approach is to consider a linear or a convex combination of several fixed kernels or of their corresponding reproducing kernel Hilbert spaces. Lanckriet et al. (2004) proposed to learn the kernel matrix with semidefinite programming and Micchelli and Pontil (2005a) proposed to learn the kernel function via regularization. Ying and Zhou (2007, Thm. 3) proposed learning with Gaussian RBF-kernels with flexible bandwidth parameters. A direct method for building sparse kernel learning algorithms was proposed by Wu et al. (2006). Large scale multiple kernel learning was investigated by Sonnenburg et al. (2006). Bach (2008) considered consistency of the group lasso and multiple kernel learning. We also refer to Rakotomamonjy et al. (2008) and Gönen and Alpaydın (2011) for multiple kernel learning algorithms and to Koltchinskii and Yuan (2010) for sparsity considerations of such algorithms. Learning rates of multiple kernel learning with L1 and elastic-net regularizations and a trade-off between sparsity and smoothness were considered by Suzuki and Sugiyama (2013). A different approach was given by Ong et al. (2005), who proposed learning the kernel via hyperkernels. The idea behind hyperkernels is to consider the kernel \( k : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \) as an unknown function from \( \mathcal{X}^2 \) to \( \mathbb{R} \). To estimate \( k \) from the data set, a second kernel \( \tilde{k} : \mathcal{X}^2 \times \mathcal{X}^2 \to \mathbb{R} \) is constructed and optimization over the RKHS corresponding to \( k \) is done.

The rest of the paper has the following structure. To improve the readability of this paper, we list some well-known results on kernels and reproducing kernel Hilbert spaces in Section 2 and on extension theorems and Lusin’s theorem in Section 3. Section 4 contains with Theorem 4.2 our result.

## 2 Kernels

Kernels and reproducing kernel Hilbert spaces (RKHSs) play a central role in modern non-parametric statistics and machine learning. We refer to Berg et al. (1984). Benyamini and Lindenstrauss
Let $K \in \{\mathbb{R}, \mathbb{C}\}$ and $X$ be a non-empty set. A function $k : X \times X \rightarrow K$ is called a kernel on $X$ if there exists a $K$-Hilbert space $H$ and a map $\Phi : X \rightarrow H$ such that for all $x, x' \in X$ we have

$$k(x, x') = \langle \Phi(x'), \Phi(x) \rangle_H.$$  

We call $\Phi$ a feature map and $H$ a feature space of $k$. A $K$-Hilbert function space $H$ consists of functions mapping from $X$ into $K$.

**Definition 2.1.** Let $X \neq \emptyset$ and $H$ be a $K$-Hilbert function space over $X$.

(i) A function $k : X \times X \rightarrow K$ is called a reproducing kernel of $H$ if we have $k(\cdot, x) \in H$ for all $x \in X$ and the reproducing property

$$f(x) = \langle f, k(\cdot, x) \rangle_H$$

holds for all $f \in H$ and all $x \in X$. The function $\Phi : X \rightarrow H$, $\Phi(x) := k(\cdot, x)$ is called canonical feature map of $k$.

(ii) The space $H$ is called a reproducing kernel Hilbert space (RKHS) over $X$ if for all $x \in X$ the Dirac functional $\delta_x : H \rightarrow K$ defined by $\delta_x(f) := f(x)$, $f \in H$, is continuous.

It is well-known that every Hilbert function space with a reproducing kernel is an RKHS and that, conversely, every RKHS has a unique reproducing kernel which can be determined by the Dirac functionals. We will consider in the following only $K = \mathbb{R}$, because $\mathbb{R}$-valued kernels are most often used in practice.

A kernel $k$ is bounded if and only if $\|k\|_\infty := \sup_{x \in X} \sqrt{k(x, x)} < \infty$.

The Gaussian RBF kernel $k_\gamma$ defined on $X \subset \mathbb{R}^d$, where $d \in \mathbb{N}$ and the width $\gamma > 0$, is given by

$$k_\gamma(x, x') = \exp \left( -\frac{\|x - x'\|^2}{\gamma^2} \right), \quad x, x' \in X.$$  

It is well-known that $k_\gamma$ is bounded and continuous and that hence all functions $f$ in its RKHS $H_\gamma$ are bounded and continuous, too.

The following notion of universal kernels was introduced by Steinwart (2001, Def. 4). Please note the combination of a compact metric input space and a continuous kernel.

**Definition 2.2.** A continuous kernel $k$ on a compact metric space $(X, d_X)$ is called universal if the RKHS $H$ of $k$ is dense in $C(X)$ with respect to the supremum norm, i.e., for every function $f \in C(X)$ and every $\varepsilon > 0$ there exists a function $g \in H$ with

$$\|f - g\|_\infty \leq \varepsilon.$$  

Please note that the rather strong supremum norm is used in Definition 2.2. This is to some extent surprising, because the universal consistency of learning algorithms is often defined by a weaker mode of convergence, e.g., convergence in probability, to the Bayes risk or to the Bayes decision function. For kernel based regression, we refer e.g., to Gyorfi et al. (2002)
for the least squares loss function and to Christmann and Steinwart (2007, Thm. 12) for general convex loss functions of growth type \( p \geq 1 \). We refer to Christmann and Steinwart (2008, Thm. 5, Thm. 6) for kernel based quantile regression.

Universal kernels can separate compact and disjoint subsets of a compact metric space as the following results shows.

**Proposition 2.3** (Steinwart (2001, Prop. 5)). Let \((\mathcal{X}, d_\mathcal{X})\) be a compact metric space and \( k \) be a universal kernel on \( \mathcal{X} \) with RKHS \( H \). Then for all compact and mutually disjoint subsets \( K_1, \ldots, K_n \subset \mathcal{X} \), all \( \alpha_1, \ldots, \alpha_n \in \mathbb{R} \) and all \( \varepsilon > 0 \) there exists a function \( g \) induced by \( k \), i.e., there exists \( w \in H \) such that \( g(x) = \langle w, k(\cdot, x) \rangle_H \) for all \( x \in \mathcal{X} \), with \( \|g\|_\infty \leq \max_i |\alpha_i| + \varepsilon \) such that

\[
\left\| g_{|K_i} - \sum_{i=1}^n \alpha_i 1_{K_i} \right\|_\infty \leq \varepsilon, \tag{2.4}
\]

where \( K := \bigcup_{i=1}^n K_i \) and \( g_{|K} \) denotes the restriction of \( g \) to \( K \) and \( 1_{K_i} \) denotes the indicator function on \( K_i \).

We refer to Micchelli et al. (2006) and the references given therein for additional results on universal kernels and relationships between their RKHSs and \( C(\mathcal{X}) \). Special emphasis is given in that paper to translation invariant kernels having the form \( k(x, x') = h(x - y) \) for continuous functions \( h : \mathbb{R}^d \to \mathbb{R} \) and to radial kernels \( k(x, x') = \phi(\|x - x'\|^2) \) on \( \mathcal{X} \subset \mathbb{R}^d \) for appropriate functions \( \phi : [0, \infty) \to \mathbb{R} \). Such kernels were already investigated by Schoenberg (1938). We refer to Wu (1995) and Wendland (1995) for radial kernels with compact support. Many Wendland kernels have a Sobolev space as RKHS, for details we refer to Wendland (2003, Thm. 10.35).

The next result on the denseness of RKHSs in some \( L_p(\mu) \) spaces is also useful to prove universal consistency results of kernel based methods, we refer e.g., to Steinwart and Christmann (2008, Thm. 4.26, Lem. 4.59).

**Theorem 2.4.** Let \( \mathcal{X} \) be a measurable space, \( \mu \) be a \( \sigma \)-finite measure on \( \mathcal{X} \), and \( H \) be a separable RKHS over \( \mathcal{X} \) with measurable kernel \( k : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \). Assume that there exists a \( p \in [1, \infty) \) such that \( \|k\|_{L_p(\mu)} := (\int_{\mathcal{X}} k^{p/2}(x, x)d\mu(x))^1/p < \infty \). Then

(i) \( H \) consists of \( p \)-integrable functions and the inclusion \( \text{id} : H \to L_p(\mu) \) is continuous with \( \| \text{id} : H \to L_p(\mu) \| \leq \| k \|_{L_p(\mu)} \).

(ii) The adjoint of this inclusion is the operator \( S_k : L_{p'}(\mu) \to H \) defined by

\[
S_k g(x) := \int_{\mathcal{X}} k(x, x')g(x')d\mu(x'), \quad g \in L_{p'}(\mu), \ x \in \mathcal{X}, \tag{2.5}
\]

where \( p' \) is defined by \( \frac{1}{p} + \frac{1}{p'} = 1 \).

(iii) \( H \) is dense in \( L_p(\mu) \) if and only if \( S_k : L_{p'}(\mu) \to H \) is injective.

(iv) If the operator \( S_k \) defined in (2.5) is injective, then \( H \) is dense in \( L_q(h\mu) \) for all \( q \in [1, p] \) and all measurable \( h : \mathcal{X} \to [0, \infty) \) with \( h \in L_s(\mu) \), where \( s := \frac{p}{p - q} \).
One can show that the operator \( S_k \) is injective for any real-valued Gaussian RBF kernel \( k \), given by (2.2), which yields the following result, see e.g., Steinwart and Christmann (2008, Thm. 4.63).

**Theorem 2.5.** Let \( \gamma > 0 \), \( p \in [1, \infty) \), and \( \mu \) be a finite measure on \( \mathbb{R}^d \). Then the RKHS \( H_\gamma(\mathbb{R}^d) \) of the real-valued Gaussian RBF kernel \( k_\gamma \) is dense in \( L_p(\mu) \).

Scovel et al. (2010, Cor. 4.9) proved the following more general result. Let \( X \subset \mathbb{R}^d \), where \( X \) is not necessarily compact, and \( k : X \times X \to \mathbb{R} \) be a non-constant radial kernel \( k \). Then the RKHS of \( k \) is dense in \( L_p(\mu) \) for all \( p \in [1, \infty) \) and all finite measures \( \mu \) on \( \mathbb{R}^d \). Furthermore, if \( X \subset \mathbb{R}^d \) is compact, then \( k \) is universal.

3 Extension theorems and Lusin’s theorem

To improve the readability of the paper, we now cite some facts from topology, see e.g., Dugundji (1966) or Dudley (2002). A topological space \((X, \tau)\) is called normal if for each pair of disjoint closed sets \( E_1 \subset X \) and \( E_2 \subset X \) there are disjoint open sets \( O_i \) with \( E_i \subset O_i \), \( i \in \{1, 2\} \). Every metric space and every compact Hausdorff space are normal. Recall, that a subspace of a normal space need not be normal. However, a closed subspace of a normal space is normal.

Let \((X, \tau_X)\) and \((W, \tau_W)\) be two topological spaces, \( A \subset X \) closed, and \( f : A \to W \) a continuous function. A continuous function \( F : X \to W \) such that \( F(a) = f(a) \) for all \( a \in A \) is called an extension of \( f \) (over \( X \) relative to \( W \)). The classical Tietze (or Tietze-Urysohn) extension theorem shows that an extension of a real-valued function \( f \) is possible for normal spaces, see Dudley (2002, Thm. 2.6.4, p. 65).

**Theorem 3.1** (Tietze-Urysohn extension theorem). Let \((X, \tau_X)\) be a normal topological space and \( A \) be a closed subset of \( X \). Then for any \( c \geq 0 \) and each of the following subsets \( W \) of \( \mathbb{R} \) with usual topology, every continuous function \( f : A \to W \) can be extended to a continuous function \( F : X \to W \):

\[
\begin{align*}
(i) & \quad W = [-c, +c]. \\
(ii) & \quad W = (-c, +c). \\
(iii) & \quad W = \mathbb{R}.
\end{align*}
\]

The following extension theorem was proven by Dugundji (1951, Thm.4.1). This theorem makes a stronger assumption on \( X \), but a weaker assumption on \( W \). Recall that a linear topological space is a vector space \( W \) equipped with a Hausdorff topology such that the two maps \( \alpha : W \times W \to W \) and \( m : \mathbb{R} \times W \to W \) (Euclidean topology on \( \mathbb{R} \)) are continuous, see Dugundji (1966, p. 413). A linear topological space \( W \) is locally convex if for each \( w \in W \) and neighborhood \( U(w) \) there is a convex neighborhood \( V \) such that \( w \in V \subset U(w) \), see Dugundji (1966, p. 414).

**Theorem 3.2** (Dugundji extension theorem). Let \((X, d_X)\) be a metric space, \( A \) be a closed subset of \( X \), \( W \) be a locally convex linear topological space, and \( f : A \to W \) a continuous
map. Then there exists an extension $F : X \to W$ of $f$, i.e., $F : X \to W$ is a continuous function with $F(a) = f(a)$ for every $a \in A$. Furthermore, $F(X)$ is a subset of the convex hull of $f(A)$.

Of course, there is a close relationship between Borel measurable functions and continuous functions: if $f$ is a continuous map between metric spaces, then $f$ is Borel-measurable. However, it is well-known that there is a much deeper relationship between continuity and Borel measurability. The following theorem is a generalisation of the classical Lusin theorem for real-valued functions to more general domain and range spaces, see Dudley (2002, Thm. 7.5.2, p. 244).

**Theorem 3.3** (Lusin’s theorem I). Let $(X, \tau)$ be any topological space and $\mu$ be a finite, closed regular Borel measure on $X$. Let $(W, d_W)$ be a separable metric space and let $f : X \to W$ be a Borel measurable function. Then for any $\varepsilon > 0$ there is a closed set $X_\varepsilon \subset X$ such that $\mu(X \setminus X_\varepsilon) < \varepsilon$ and the restriction of $f$ to $X_\varepsilon$ is continuous.

There exist other versions of Lusin’s theorem for $X$ a Polish space or a locally compact space and compact sets $X_\varepsilon \subset X$. Here, we only cite the following result taken from Denkowski et al. (2003, Thm. 2.5.15, p. 187). Recall, that a topological space $(Y, \tau_Y)$ is called a Polish space, if the topology $\tau_Y$ is metrizable by some metric $d_Y$ such that $(Y, d_Y)$ is a complete separable metric space.

**Theorem 3.4** (Lusin’s theorem II). Let $(X, \tau)$ be a Polish space, $(W, d_W)$ be a separable metric space, $f : X \to W$ be a Borel measurable function, and $\mu$ be a finite Borel measure on $(X, B_X)$. Then for any $\varepsilon > 0$ there is a compact set $X_\varepsilon \subset X$ such that $\mu(X \setminus X_\varepsilon) < \varepsilon$ and the restriction of $f$ to $X_\varepsilon$ is continuous.

One reason why Polish spaces are interesting in probability theory and statistics is the fact, than regular conditional probabilities are uniquely defined, see Dudley (2002, Thm. 10.2.2, p. 345). Furthermore, disintegration allows then to split a probability measure $P$ defined on $(X \times Y, \mathcal{A} \otimes \mathcal{B}_Y)$ into the marginal distribution $P_X$ on $(X, \mathcal{A})$ and the conditional distribution $P(\cdot | x)$ of a random variable $Y$ given $X = x$, see e.g., Dudley (2002, Thm. 10.2.1, p. 343f).

### 4 Result

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $(W, \tau_W)$ be a separable metric space equipped with the Borel $\sigma$-algebra $B_W$. Denote the set of all $(\mathcal{A}, B_W)$-measurable functions by $L_0(\Omega, W)$ and the corresponding factor space of equivalence classes of functions, which are $P$-almost everywhere identical, by $L_0(\Omega, W)$. Then the Ky Fan metric defined on $L_0(\Omega, W) \times L_0(\Omega, W)$ is given by

$$d_{KyFan}(f_1, f_2) := \inf \{ \varepsilon \geq 0; P(d_W(f_1, f_2) > \varepsilon) \leq \varepsilon \}$$

for any $f_1, f_2 \in L_0(\Omega, W)$. This metric metrizes convergence in probability, i.e., if $f, f_n \in L_0(\Omega, W)$, $n \in \mathbb{N}$, then $f_n \to f$ in probability if and only if

$$\lim_{n \to \infty} d_{KyFan}(f_n, f) = 0,$$  \hspace{1cm} (4.1)
see e.g., Dudley (2002, Thm. 9.2.2). Furthermore, \( L_0(\Omega, \mathcal{W}) \) is even complete for the Ky Fan metric, if \((\Omega, \mathcal{A}, \mathbb{P})\) is a probability space and if \((\mathcal{W}, d_\mathcal{W})\) a complete separable metric space, see Dudley (2002, Thm. 9.2.3, p. 290). For our purpose it is more comfortable to consider equivalent metrics and we will prove the next simple result to improve the readability of this note.

**Lemma 4.1.** Let \((\Omega, \mathcal{A}, \mathbb{P})\) be a probability space, \((\mathcal{W}, d_\mathcal{W})\) be a separable metric space, and \(f, f_n : (\Omega, \mathcal{A}) \rightarrow (\mathcal{W}, \mathcal{B}_\mathcal{W})\) be measurable functions, \(n \in \mathbb{N}\). Let \(\psi : [0, \infty) \rightarrow [0, 1]\) be a continuous, subadditive, and monotone increasing function with \(\psi(0) = 0\) and \(\psi(x) > 0\), if \(x > 0\), i.e., \(\psi(x_1 + x_2) \leq \psi(x_1) + \psi(x_2)\), and \(\psi(x_1) \leq \psi(x_2)\) for all \(x_1, x_2 \in [0, \infty)\) with \(x_1 \leq x_2\). Then:

\(\psi\)

(i) The function \(d_\psi : L_0(\Omega, \mathcal{W}) \times L_0(\Omega, \mathcal{W}) \rightarrow [0, \infty)\) defined by

\[
d_\psi(f_1, f_2) := \int \psi(d_\mathcal{W}(f_1, f_2)) \, d\mathbb{P}, \quad f_1, f_2 \in L_0(\Omega, \mathcal{W}),
\]

is a metric on \(L_0(\Omega, \mathcal{W})\).

(ii) We have \(f_n \rightarrow f\) in probability if and only if

\[
\lim_{n \rightarrow \infty} d_\psi(f_n, f) = 0.
\]

**Proof of Lemma 4.1.** Part (i). Obviously, for any \(f_1, f_2 \in L_0(\Omega, \mathcal{W})\) we have \(d_\psi(f_1, f_2) = d_\psi(f_2, f_1) \geq 0\) and \(d_\psi(f_1, f_2) = 0\) if and only if \(f_1 = f_2\), because \(d_\mathcal{W}\) is a metric and \(\psi(x) > 0\) if \(x > 0\). The triangle inequality for \(d_\psi\) follows from the triangle inequality for \(d_\mathcal{W}\) and the subadditivity of \(\psi\). Hence \(d_\psi\) is a metric.

Part (ii). Let us assume that \(f_n \rightarrow f\) in probability. Then we have, for all \(\varepsilon > 0\), that \(P(d_\mathcal{W}(f_n, f) > \varepsilon) \rightarrow 0\), if \(n \rightarrow \infty\). Because \(\psi\) maps into the interval \([0, 1]\) and \(\psi\) is monotone increasing, it follows

\[
\psi(d_\mathcal{W}(f_n, f)) = \psi(d_\mathcal{W}(f_n, f)) \cdot 1_{\{d_\mathcal{W}(f_n, f) > \varepsilon\}} + \psi(d_\mathcal{W}(f_n, f)) \cdot 1_{\{d_\mathcal{W}(f_n, f) \leq \varepsilon\}} \leq 1 \cdot 1_{\{d_\mathcal{W}(f_n, f) > \varepsilon\}} + \psi(\varepsilon) \cdot 1_{\{d_\mathcal{W}(f_n, f) \leq \varepsilon\}}.
\]

Therefore,

\[
\int \psi(d_\mathcal{W}(f_n, f)) \, d\mathbb{P} \leq \int (1_{\{d_\mathcal{W}(f_n, f) > \varepsilon\}} + \psi(\varepsilon) \cdot 1_{\{d_\mathcal{W}(f_n, f) \leq \varepsilon\}}) \, d\mathbb{P} = P(d_\mathcal{W}(f_n, f) > \varepsilon) + \psi(\varepsilon) \cdot P(d_\mathcal{W}(f_n, f) \leq \varepsilon).
\]

Taking limits yields

\[
0 \leq \lim_{n \rightarrow \infty} \int \psi(d_\mathcal{W}(f_n, f)) \, d\mathbb{P} \leq \psi(\varepsilon), \quad \forall \varepsilon > 0,
\]
which proves one direction, because $\psi$ is continuous and $\psi(0) = 0$.

Let us now assume that $\int \psi(d_W(f_n, f)) \, dP \rightarrow 0$, if $n \rightarrow \infty$. The function $\psi$ is non-negative and monotone increasing by assumption. Hence

$$0 \leq \psi(\varepsilon) \cdot 1_{\{d_W(f_n, f) > \varepsilon\}} \leq \psi(d_W(f_n, f)) \cdot 1_{\{d_W(f_n, f) > \varepsilon\}} \leq \psi(d_W(f_n, f)).$$

Integrating with respect to $P$ and then taking limits yields

$$0 \leq \lim_{n \rightarrow \infty} \int \psi(\varepsilon) \cdot 1_{\{d_W(f_n, f) > \varepsilon\}} \, dP \leq \lim_{n \rightarrow \infty} \int \psi(d_W(f_n, f)) \, dP = 0, \quad \forall \varepsilon > 0.$$

Since $\psi(\varepsilon) > 0$ for all $\varepsilon > 0$, we conclude $\int 1_{\{d_W(f_n, f) > \varepsilon\}} \, dP = P(d_W(f_n, f) > \varepsilon) \rightarrow 0$, if $n \rightarrow \infty$. \hfill \Box

Special cases are $\psi_1(x) = x/(1 + x)$, see Jacod and Protter [2004, Thm. 17.1], and $\psi_2(x) = \min\{1, x\}, \ x \geq 0$, see e.g. Steinwart and Christmann [2008, Problem 9.2, p. 353], respectively. The metric $d_{\psi_2}$ was used to derive consistency in probability of support vector machines for kernel based quantile regression, see Steinwart and Christmann [2008, Thm. 9.7, p. 343].

We can now give our result which can be interesting for statistical machine learning if the input set is $\mathcal{X}$, the output space is $\mathcal{Y}$ and a function class $\mathcal{F}$ containing functions $f : \mathcal{X} \rightarrow \mathcal{H}$ are considered, where $\mathcal{H} \subset \mathcal{Y}$. Special cases are $\mathcal{Y} = \mathcal{H}$ and $\mathcal{Y} = \mathcal{H} = \mathbb{R}$.

**Theorem 4.2.** Let $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$ be complete separable metric spaces and $(\mathcal{H}, \| \cdot \|_{\mathcal{H}})$ be a separable Hilbert space with metric $d_{\mathcal{H}} := \| \cdot - \cdot \|_{\mathcal{H}}$. Equip these spaces with their Borel $\sigma$-algebras $\mathcal{B}_{\mathcal{X}}, \mathcal{B}_{\mathcal{Y}},$ and $\mathcal{B}_{\mathcal{H}}$, respectively. Let $P$ be a probability measure on $(\mathcal{X} \times \mathcal{Y}, \mathcal{B}_{\mathcal{X} \times \mathcal{Y}})$. Denote the set of all continuous functions $f : (\mathcal{X}, d_{\mathcal{X}}) \rightarrow (\mathcal{H}, d_{\mathcal{H}})$ by $C(\mathcal{X}, \mathcal{H})$. Let $\mathcal{F}$ be a subset of $L_0(\mathcal{X}, \mathcal{H})$, where $\mathcal{F}$ is either a dense subset of $C(\mathcal{X}, \mathcal{H})$ or $\mathcal{F}$ contains a dense subset of $C(\mathcal{X}, \mathcal{H})$, where denseness is with respect to the metric $d_\psi$ defined in (4.2).

Then $\mathcal{F}$ is dense in $L_0(\mathcal{X}, \mathcal{H})$ with respect to the metric $d_\psi$, i.e., for all $\varepsilon > 0$ and for all $f \in L_0(\mathcal{X}, \mathcal{H})$ there exists $g_{\varepsilon, f} \in \mathcal{F}$ such that

$$d_\psi(f, g_{\varepsilon, f}) < \varepsilon. \tag{4.4}$$

Please note, that the denseness notions in (4.4) and in (2.3) differ: $d_\psi$ used in (4.4) metrizes the convergence in probability for $\mathcal{H}$-valued random quantities $f_n$, see Lemma 1.1 whereas the much stronger supremum norm is used in (2.3).

**Proof of Theorem 4.2.** Because $(\mathcal{Y}, d_{\mathcal{Y}})$ is a complete separable metric space and hence a Polish space, we can split the probability measure $P$ into its marginal distribution $P_X$ and its conditional distribution $P(\cdot | x), \ x \in \mathcal{X}$. Fix $\varepsilon > 0$ and $f \in L_0(\mathcal{X}, \mathcal{H})$. Lusin’s theorem, see Theorem 3.3 gives the existence of a closed set $X_{\varepsilon, f} \in \mathcal{B}(\mathcal{X})$ such that

$$P((\mathcal{X} \setminus X_{\varepsilon, f}) \times \mathcal{Y}) = P_X(\mathcal{X} \setminus X_{\varepsilon, f}) < \frac{\varepsilon}{2}$$

and the existence of a continuous function

$$h_{\varepsilon, f} : (X_{\varepsilon, f}, d_{\mathcal{X}|X_{\varepsilon, f}}) \rightarrow (\mathcal{H}, d_{\mathcal{H}})$$

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such that
\[ h_{\varepsilon,f}(x) = f(x), \quad x \in X. \] (4.5)
Because \( h_{\varepsilon,f} \) is continuous, it is of course \((\mathcal{B}_X \cap X_{\varepsilon,f}, \mathcal{B}_H)\)-measurable. Recall that every normed space and in particular every Hilbert space is a Hausdorff locally convex space. Hence we can apply Dugundji’s extension theorem, see Theorem 3.2 which guarantees the existence of a continuous – and therefore \((\mathcal{B}_X, \mathcal{B}_H)\)-measurable – function
\[ F_{\varepsilon,f} : (X, d_X) \to (H, d_H), \]
such that
\[ F_{\varepsilon,f}(x) = h_{\varepsilon,f}(x) \quad \forall x \in X. \] (4.6)
Obviously, the continuous function \( F_{\varepsilon,f} \) will in general not be identical to the measurable function \( f \). Denote the indicator function of some set \( A \) by \( 1_A \). Because \( \psi \) maps into the interval \([0, 1] \), it follows
\[
d_\psi(f, F_{\varepsilon,f}) = \int \psi(d_H(f, F_{\varepsilon,f})) \, dP = \int \psi(d_H(f, F_{\varepsilon,f})) 1_{X_{\varepsilon,f}} \, dP + \int \psi(d_H(f, F_{\varepsilon,f})) 1_{X \setminus X_{\varepsilon,f}} \, dP
\]
\[
\leq \int 0 \cdot 1_{X_{\varepsilon,f}} \, dP + \int 1 \cdot 1_{X \setminus X_{\varepsilon,f}} \, dP = \mathbb{P}(X \setminus X_{\varepsilon,f}) < \frac{\varepsilon}{2}.
\]
If the continuous function \( F_{\varepsilon,f} \) is an element of \( F \), then we can choose \( g_{\varepsilon,f} = F_{\varepsilon,f} \) and we have \( d_\psi(f, g_{\varepsilon,f}) < \frac{\varepsilon}{2} \). If the continuous function \( F_{\varepsilon,f} \) is not an element of \( F \), then the denseness assumption of \( F \) guarantees the existence of a continuous function \( g_{\varepsilon,f} \in F \) such that \( d_\psi(F_{\varepsilon,f}, g_{\varepsilon,f}) < \frac{\varepsilon}{2} \). We then obtain \( d_\psi(f, g_{\varepsilon,f}) < \varepsilon \) by the triangle inequality, which completes the proof.

\( \square \)

**Example 4.3.** Let \( k : X \times X \to \mathbb{R} \) be a universal kernel with RKHS \( H \), where \((X, d_X)\) is a compact metric space. Then, for all \( f \in C(X, \mathbb{R}) \) and for all \( \varepsilon > 0 \), there exists \( g_{\varepsilon,f} \in H \) such that \(||f - g_{\varepsilon,f}\|_{\infty} < \varepsilon \). As convergence with respect to the supremum norm implies convergence in probability, we immediately obtain that for all \( f \in C(X, \mathbb{R}) \) and for all \( \varepsilon > 0 \) there exists \( g_{\varepsilon,f} \in H \) such that \( d_\psi(f, g_{\varepsilon,f}) < \varepsilon \). One special case is the Gaussian RBF-kernel \( k_\gamma \) with bandwidth \( \gamma > 0 \) defined on some compact set \( X \subset \mathbb{R}^d \). This kernel is well-known to be universal, see e.g., \textbf{Steinwart and Christmann (2008, Cor. 4.58)}. Another special case is the universal kernel \( k_\sigma \) defined on the set of all Borel probability measures, see \textbf{Christmann and Steinwart (2011, Example 1)} for details. Let \( X = M_1(\Omega, \mathcal{B}(\Omega)) \), where \((\Omega, d_\Omega) \) is some compact metric space and \( k_\Omega \) is a continuous kernel on \( \Omega \) with canonical feature map \( \Phi_\Omega \) and RKHS \( H_\Omega \). Assume that \( k_\Omega \) is a so-called characteristic kernel in the sense that the function \( \rho : X \to H_\Omega \) defined by \( \rho(P) = \mathbb{E}_P \Phi_\Omega \) is injective. Then the Gaussian-type RBF-kernel
\[
k_\sigma(P, P') := \exp\left(-\frac{1}{\gamma^2} \|\Phi_P - \Phi_{P'}\|_{H_\Omega}^2\right), \quad P, P' \in M_1(\Omega, \mathcal{B}(\Omega)),
\]
is a universal kernel on \( X \) and obviously even bounded.
Example 4.4. Let $X$ be a complete separable metric space and $Y = [-M, +M]$ for some fixed constant $M \in (0, \infty)$. Let $L$ be a convex and Lipschitz continuous loss function with Lipschitz constant $|L|_1 > 0$. Consider the minimizer $f_{L,D,\lambda_n}$ defined by minimizing (1.3).

If $f_0 \in \text{arg min} \{R_{L,P}(f) \mid f \in L_0(X,Y)\}$ exists, it follows directly that $f_0(x) \in [-M, +M]$ for all $x \in X$. Therefore it is natural to project $f_{L,D,\lambda_n}$ onto $[-M, +M]$ obtaining

$$\hat{f}_{L,D,\lambda_n} := \max \{-M, \min \{+M, f_{L,D,\lambda_n}\}\},$$

see e.g., Cucker and Zhou (2007, Section 10.2). Hence, let us define

$$\mathcal{F} := \{g = \max \{-M, \min \{+M, f\} \mid f \in H\},$$

where $H$ is the RKHS corresponding to the chosen kernel $k$.

If $\mathcal{F}$ is dense in $L_0(X,Y)$ with respect to $d_\psi$, then, for $f_0 \in \mathcal{F}$ and for all $\varepsilon > 0$, there exists a sequence $(g_n)_{n \in \mathbb{N}} \subset \mathcal{F}$ and a positive integer $n_0$ such that for all $n \geq n_0$ it holds $d_\psi(g_n, f_0) < \varepsilon$. Hence Lemma 4.1 implies that $g_n \to f_0$ in probability for $n \to \infty$.

Under the conditions that $(g_n)_{n \in \mathbb{N}} \subset \mathcal{F}$ and therefore $|g_n(x)| \leq M$ for all $n \in \mathbb{N}$ and all $x \in X$, it holds true that

$$\lim_{n \to \infty} \|g_n - f_0\|_{L_1(P_X)} = 0 \quad (4.7)$$

for all marginal distributions $P_X$ on $X$. Recalling the Lipschitz continuity of the loss function $L$ we have, see e.g., Steinwart and Christmann (2008, Lem. 2.19),

$$|R_{L,P}(f) - R_{L,P}(f_0)| \leq |L|_1 \|f - f_0\|_{L_1(P_X)} \quad \text{for all } f \in \mathcal{F}. \quad (4.8)$$

Combining (4.7) and (4.8) we obtain

$$\lim_{n \to \infty} R_{L,P}(g_n) = R_{L,P}(f_0),$$

where $g_n \in \mathcal{F}$ for all $n \in \mathbb{N}$.

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