The Algebra of the Pseudo-Observables III: Transformations and Time Evolution

Edoardo Piparo
Liceo Scientifico Statale “Archimede”, Viale Regina Margherita 3, I-98121 Messina, Italy
E-mail: edoardo.piparo@istruzione.it

Abstract. This paper is the third part concluding the introduction of the powerful algebra of the pseudo-observables. In this article will be dealt how to treat the time evolution, and, more in general, the transformations, in the framework of the new theory. It will be shown that this requires only minor changes with respect to the Dirac-Jordan transformation theory that can be found in almost any textbook, with some more care about the treatment of the continuous limit. A remarkable difference is also in the introduction of the time reversal, which gives the opportunity to a deeper insight about the Hermitian transposition. In the conclusions, we will examine better the relationship between time evolution and measurement, a very problematic aspect in the framework of the Copenhagen interpretation (and in many others ones). Finally we will present a list of compelling reasons for which the physics community would seriously have to evaluate a switching to the new formalism.

Keywords: Quantum dynamics, time evolution, interpretation of quantum mechanics

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1. Introduction

This third paper completes the reformulation of Quantum Mechanics in the framework of the algebra of the pseudo-observables. After the introduction of the algebra in [1], referred from now on as the first paper, in order to show that quantum mechanics is the unique minimal description of physical reality and the presentation of the hierarchical structure of the observer networks, we fronted the measurement problem in [2], referred from now on as the second paper, giving, in our opinion, a solution to it and bringing a deeper insight on what to mean for “physical reality”.

We will deal how to treat the time evolution, and, more in general, the transformations, in the framework of the algebra of the pseudo-observables.

From an historical point of view, the formulation of the quantum mechanical equations of motion wasn’t much linear. In 1925-26, on the one side, Werner Heisenberg[3], Max Born and Pascual Jordan[4,5] gave rise to matrix mechanics, in the framework of which time evolution is described by the Heisenberg equation; on the other side, Erwin Schrödinger[6], carrying forward the ideas of de Broglie, created wave mechanics, in which time evolution is described by the well-known Schrödinger equation. The two approaches found a reconciliation essentially in the 1930 classic The Principles of Quantum Mechanics[7], in which Paul Dirac showed that Schrödinger’s and Heisenberg’s approaches were two different representations of the same theory. The investigations in the field of the quantum statistical mechanics, led also to the introduction the density matrix[8], whose time evolution is described by means of the von Neumann equation.

In this paper we will show as all these equations can be found in the framework of the algebra of the pseudo-observables, by following procedures very similar to those found in almost any textbook, according the spirit of the Dirac-Jordan transformation theory[9]. The main difference is that we will work with finite Hilbert spaces and a discretized time evolution, recovering the continuity by finally assuming suitable limits.

2. Transformations of pseudo-observables

2.1. Transformations

The study of the physical phenomena is indeed the more important aspect of the physical analysis of reality. A physical phenomenon consists in an evolution process of the physical properties of a system. It is, therefore, characterized in a natural way through transformations. A transformation is a functional bound between an initial situation and a final situation. A first hypothesis that it is necessary to assume about such bound is that it is a biunivocal correspondence between observables - and more generally pseudo-observables - in the sense that it must be possible to trace the effects (final situation) to the causes (initial situation). By virtue of this reversibility principle, therefore, a transformation must be an invertible map among pseudo-observables. By indicating with $\tau$ such a map, it must transform observables into observables and so, if $O$ is a generic observable, it must result:

$$ O^\dagger = O \Rightarrow (\tau (O))^\dagger = \tau (O) \quad .$$

The map must also preserve the links between pseudo-observables, it has, therefore, to be a ring automorphism. This implies that, for two given pseudo-observables $A$ and $B$, $\tau$ must be additive:

$$ \tau (A + B) = \tau (A) + \tau (B)$$

and must preserve the multiplication relationship:

$$ \tau (AB) = \tau (A) \tau (B) \quad .$$
We will make the further assumption that constants (real or complex) are invariant under transformations, i.e. that, if $\gamma$ is a whatever complex constant, it results:

$$\tau(\gamma) = \gamma . \quad (4)$$

By these two fundamental requirements, the two important lemmas immediately derive:

(i) If $P$ is a whatever pseudo-observable, of real part $P_R$ and imaginary part $P_I$, one has:

$$\tau(P^\dagger) = \tau(P_R - i P_I) = (\tau(P_R) + i \tau(P_I))^\dagger = (\tau(P))^\dagger .$$

(ii) Transformations are linear mappings, so, if $\gamma_1$ and $\gamma_2$ are two complex constants and $P_1$ and $P_2$ are two pseudo-observables, one has:

$$\tau(\gamma_1 P_1 + \gamma_2 P_2) = \gamma_1 \tau(P_1) + \gamma_2 \tau(P_2) .$$

We will show now how these properties fully characterize the transformations.

We start analyzing what happens applying a transformation $\tau$ to the projectors. The following theorems hold:

(i) By applying a transformation to a projectors $I$, another projector is obtained. In fact, $\tau(I)$, by virtue of hypothesis, is an observable and, according to the (3), one has:

$$(\tau(I))^2 = \tau(I^2) = \tau(I)$$

having made use of the property (2) in the first paper, that justifies also the thesis.

(ii) By applying a transformation to two mutually exclusive projectors $I_1$ and $I_2$, one obtains two mutually exclusive projectors:

$$I_1 I_2 = 0 \Rightarrow \tau(I_1) \tau(I_2) = \tau(I_1 I_2) = \tau(0) = 0 .$$

(iii) By applying a transformation to a set $\{I_j\}$ of projectors for which it holds the closure relation (7) in the first paper, one obtains a set of projectors satisfying the same closure relation:

$$\sum_j I_j = 1 \Rightarrow \sum_j \tau(I_j) = \tau \left( \sum_j I_j \right) = \tau(1) = 1 .$$

One can, therefore, infer that by applying a transformation to a projector basis one obtains a new projector basis. We, finally, prove that by applying a transformation to an elementary projector, another elementary projector is obtained. Let, in fact, $I$ be an elementary projector and suppose by absurd $\tau(I)$ not to be elementary, that is there exist two non-null mutually exclusive projectors $J_1$ and $J_2$ such that:

$$\tau(I) = J_1 + J_2 . \quad (5)$$

By applying to both sides of the (5) the inverse transformation $\tau^{-1}$, then, one would have:

$$I = \tau^{-1}(J_1) + \tau^{-1}(J_2)$$

that it would be an absurd, since, for what previously proved, this would imply that $I$ is equal to the sum of two non-null mutually exclusive projectors, because transformations, according their fundamental properties, associate non-null pseudo-observables to non-null pseudo-observables.

We will, now, analyze the effects of a transformation upon a dyad basis $\{\Gamma_{jk}\}$, proving that the set $\{\tau(\Gamma_{jk})\}$ is a new dyad basis. To this end, we write every dyad $\Gamma_{jk}$ of the basis $\{\Gamma_{jk}\}$
as a dyadic form relative to the pair \((I_j, I_k)\) of elementary projectors, belonging to an elementary projector basis \(\{I_j\}\), having the core \(C_{jk}\): 
\[
\Gamma_{jk} = I_j C_{jk} I_k .
\]
(6)
By applying the transformation \(\tau\) to both sides of (6), one has:
\[
\tau(\Gamma_{jk}) = \tau(I_j) \tau(C_{jk}) \tau(I_k) .
\]
(7)
Remembering what has been above proved about the application of a transformation to an elementary projector basis, one concludes that also \(\tau(\Gamma_{jk})\) are dyadic forms, which forms a basis. Observing, besides, that it results:

(i) \(\tau(\Gamma_{jj}) = \tau(I_j)\)

(ii) \(\tau(\Gamma_{jk})^\dagger = \tau(\Gamma_{kj})\)

(iii) \(\tau(\Gamma_{jl}) \tau(\Gamma_{l'k}) = \delta_{l,l'} \tau(\Gamma_{jk})\)

it is concluded that also the set \(\{\tau(\Gamma_{jk})\}\) is a dyad basis, i.e. by applying a transformation to a dyad basis a new dyad basis is obtained. For what stated in subsection 4.3 in the first paper, it will therefore have to exist a change of basis unitary pseudo-observable \(W\) such that it results:

\[
\tau(\Gamma_{jk}) = W \Gamma_{jk} W^\dagger .
\]
(7)
We will say that the transformation is induced by the unitary pseudo-observable \(W\) and that \(W\) is the unitary pseudo-observable associated to the transformation.

Let, now, \(P\) be a whatever element of the space \(\mathcal{P}\) of the pseudo-observables. One can decompose \(P\) according the dyad basis \(\{\Gamma_{jk}\}\), as in equation (48) in the first paper, obtaining:
\[
P = \sum_{j,k} \varpi_{jk} \Gamma_{jk}
\]
where the components \(\varpi_{jk}\) are suitable complex constants. By applying to both sides of this relation the transformation \(\tau\) and exploiting its linearity, the \(\tau^2\) and the distributivity of the product of pseudo-observables over addition, one obtains the following expression for the transformation equation in terms of the associated unitary pseudo-observable:
\[
\tau(P) = \sum_{j,k} \varpi_{jk} \tau(\Gamma_{jk}) = \sum_{j,k} \varpi_{jk} W \Gamma_{jk} W^\dagger =
\]
\[
= W \left( \sum_{j,k} \varpi_{jk} \Gamma_{jk} \right) W^\dagger = WPW^\dagger .
\]
(8)
Since \(W\) is an unitary pseudo-observable, its real part \(W_R\) is compatible with its imaginary part \(W_I\). In fact, according to equations (21) and (66), both of them in the first paper, one has:
\[
[W_R, W_I] = \left[ \frac{W + W^\dagger}{2}, \frac{W - W^\dagger}{2i} \right] = \frac{W^\dagger W - WW^\dagger}{2i} = \frac{1 - 1}{2i} = 0 .
\]
The real and the imaginary part of \(W\) will be therefore allowed to be expressed as linear combinations of projectors of the same basis \(\{I_j\}\):
\[
W_R = \sum_j \alpha_j I_j \quad \text{and} \quad W_I = \sum_j \beta_j I_j
\]
(9)
where \( \alpha_j \) and \( \beta_j \) are suitable real coefficients. By making use of the (9), one finds, therefore, the following expression for the unitary pseudo-observable \( W \) associated to the transformation:

\[
W = \sum_j (\alpha_j + i\beta_j) I_j .
\]

(10)

By substitution of this expression in equation (66) in the first paper, one obtains the following relation among the coefficients:

\[
\alpha_j^2 + \beta_j^2 = 1
\]

(11)

valid for each index \( j \). The (11) implies that, for each index \( j \), it exists an angle \( \vartheta_j \), that can be supposed to be, for instance, between \(-\pi\) and \(\pi\), such that it results:

\[
\begin{align*}
\alpha_j &= \cos \vartheta_j , \\
\beta_j &= \sin \vartheta_j .
\end{align*}
\]

(12)

By substitution of these relations in the (10), finally one has:

\[
W = \sum_j (\cos \vartheta_j + i \sin \vartheta_j) I_j = \sum_j e^{i\vartheta_j} I_j
\]

(13)

having made use of the Euler’s formula.

By considering, now, the observable:

\[
G := \sum_j \vartheta_j I_j
\]

(14)

according to the definition of a function of an observable, equation (12) in the first paper, one has:

\[
W = \cos (G) + i \sin (G) = e^{iG} .
\]

(15)

We will call \( G \) the generatrix of the transformation. Note, however, that it is not uniquely identified by the transformation itself, since the elements of its spectrum are defined to the less of \( 2\pi \) multiples.

Consider, now, an observable \( A \), expressed as a linear combination of the projectors of the associated basis:

\[
A = \sum_j a_j I_{A=a_j} .
\]

By applying to both side the transformation \( \tau \), by virtue of the property of linearity, one obtains:

\[
\tau (A) = \sum_j a_j \tau (I_{A=a_j})
\]

(16)

where, according to what stated above, the set \( \{ \tau (I_{A=a_j}) \} \) of observables is a projector basis. Since the coefficients \( a_j \) are all distinguished among themselves, by comparison of the (10) with the equation (8) in the first paper, it is concluded that \( \tau (A) \) has the same spectrum of \( A \) and that besides it results:

\[
\tau (I_{A=a_j}) = I_{\tau (A)=a_j} .
\]

(17)

Consider, finally, a function \( f \), defined according the equation (14) in the first paper, of a complete set \( \{ O_r \} \) of compatible observables, which, decomposed according the elementary projector basis \( \{ I_j \} \) associated to the space of the observables compatible with them, result given by:

\[
O_r = \sum_j o_{r,j} I_j .
\]

(18)
If you indicate with $O$ the $n$-tuple formed by the elements of the complete set of compatible observables, by applying to the observable $f(O)$ the transformation $\tau$, by virtue of the linearity property ad of the (16), one has:

$$\tau (f(O)) = \sum_j f(o_j) \tau (I_j) = f(\tau (O))$$  \hspace{1cm} (19)$$

where it was put:

$$\tau (O) := (\tau (O_1), \ldots, \tau (O_r), \ldots)$$  \hspace{1cm} (20)$$
i.e. by applying a transformation to a function of compatible observables it is obtained as a result the function of the observables obtained by applying the transformation to the starting observables.

2.2. Transformation invariance

It is, now, interesting to characterize the main invariants under a transformation. In this context, an invariant is an observable, or more generally a pseudo-observable, or a value, real or complex, associated to one or more observables or pseudo-observables, that does not change by applying the transformation.

We will begin by demonstrating that the observables invariant under a transformation $\tau$, induced by the unitary pseudo-observable $W$, are all and only those compatible with the generatrix $G$ of the transformation. Let, in fact, $A$ be an observable compatible with $G$ and so commuting with it:

$$[A, G] = 0$$

Since if two observables commute also each function of the one commutes with every function of the other, the above relation implies:

$$[A, \cos (G)] = [A, \sin (G)] = 0$$

and therefore:

$$[A, W] = [A, \cos (G) + i \sin (G)] = [A, \cos (G)] + i [A, \sin (G)] = 0.$$  \hspace{1cm} (21)$$

By the last relation it follows:

$$AW = WA$$

by which, by multiplying both sides by $W^\dagger$ to the right and exploiting the unitarity of $W$, one obtains:

$$A = WAW^\dagger = \tau (A)$$

which proves that each observable compatible with $G$ is also an invariant under the transformation. If, conversely, an observable $A$ is an invariant under the transformation $\tau$, it will be:

$$A = \tau (A) = WAW^\dagger.$$  \hspace{1cm} (21)$$

By multiplying both sides of this relation to the right by $W$ and exploiting the unitarity of this last, one has:

$$AW = WA$$

that is the observable $A$ commutes with $W$. By transposition of both sides of the (21), then one has that $A$ also commutes with $W^\dagger$. It is so concluded that $A$ is compatible both with the real part
and the imaginary part of $W$. There will therefore be an elementary projector basis with respect to which it is possible express, as linear combinations of the elements of the basis, both the observable $A$ and $W$, and so, by virtue of the definition (14), also the generatrix $G$. The observables $A$ and $G$ will be, therefore, compatible.

We will, now, prove that the trace of an observable, as defined in subsection 2.2 in the second paper, is an invariant properties of the transformation. To this end it suffices to observe that a transformation, as showed at the end of subsection (2.1), does not alter nor the observable spectrum, nor the multiplicity of the relative terms and therefore, according to definition (9) in the second paper, does not change the trace. More in general, for a whatever pseudo-observable $P$, by virtue of the third property of the trace, one has:

$$
\text{tr} \left( \tau (P) \right) = \text{tr} \left( WPW^\dagger \right) = \text{tr} \left( PW^\dagger W \right) = \text{tr} \left( P \right)
$$

(22)

where it was also exploited the unitarity of the pseudo-observable $W$ associated to the transformation $\tau$.

A further important consequence of the (22), by virtue of the definition (23) in the second paper, of the relation (3) and of the first lemma about transformations, is that also inner products are invariant under transformations. By considering, in fact, two whatever pseudo-observables $X$ and $Y$, one has:

$$
\langle \tau (X) | \tau (Y) \rangle = \text{tr} \left( \tau (X)^\dagger \tau (Y) \right) = \text{tr} \left( \tau (X^\dagger Y) \right) = \\
= \text{tr} \left( X^\dagger Y \right) = \langle X | Y \rangle.
$$

(23)

It should be, however, pointed out that one is not allowed to infer by such property that the expectation value of an observable or of a pseudo-observable is invariant under a transformation, despite the fact that, according to the equation (75) in the second paper, this can be expressed as the inner product between the pseudo-observable and the density observable. The density observable is, in fact, somewhat peculiar, being built substantially according an a posteriori process, in order to summarize the statistical properties of the possible measurements relative to a given system in a certain state, in the spirit of the Bayesian inference [10, 11]. In introducing the concept of transformation, it was implicitly adopted the Heisenberg picture, i.e. it was assumed that the transformations change the observables while leaving unaltered the states. In such a picture one has, therefore, to assume that the density observable relative to the transformed observables is the same of that relative to the starting ones. The expectation value of a pseudo-observable $P$, to which is applied a transformation $\tau$, will therefore change, on the basis of the equation (75) in the second paper, according to the law:

$$
\langle P \rangle = \langle D | P \rangle \mapsto \langle \tau (P) \rangle = \langle D | \tau (P) \rangle
$$

(24)

According to the (24) and exploiting the invariance of the inner product, equation (23), besides, one has:

$$
\langle \tau (P) \rangle = \langle D | \tau (P) \rangle = \langle \tau (\tau^{-1} (D)) | \tau (P) \rangle = \langle \tau^{-1} (D) | P \rangle
$$

(25)

On the basis of such relation, it is therefore inferred that the expectation values of observables and pseudo-observables are invariant for those states for which the density observable is invariant under the transformation, i.e., according to what proved at the beginning of this subsection, for those ones for which $D$ is compatible with the generatrix $G$ of the transformation.

Equation (25), shows as the expectation value after applying a transformation can be calculated as the starting one by making use of a suitably transformed density observable, so justifying,
as an alternative to the Heisenberg picture, the well-known Schrödinger picture, in which the observables remain unaltered while to evolve are the states, described in terms of the density $D$. In such a picture, as a result of a transformation $\tau$, the system state changes, passing from the situation described by the density $D$ to the one described by the density $D' = \tau^{-1}(D)$, that is:

$$D \xrightarrow{\tau} D' = \tau^{-1}(D).$$  \hfill (26)

If $\Psi_j$ is the state vector associated to the pure state of index $j$, the expectation value a pseudo-observable $P$, according equation (86) in the second paper, is given by:

$$\langle P \rangle_j = \langle \Psi_j | P\Psi_j \rangle.$$  

After applying the transformation $\tau$, according to (8), this will change becoming:

$$\langle \tau(P) \rangle_j = \langle \Psi_j | \tau(P) \Psi_j \rangle = \langle W^\dagger \Psi_j | PW^\dagger \Psi_j \rangle.$$  \hfill (27)

It is easily verified that, due to unitarity of $W$, the set $\{W^\dagger \Psi_j\}$ has all the properties of a set of state vectors, as defined in section 2 of the second paper, relative to the anti-transformed dyad basis. This implies that one may consider $W^\dagger \Psi_j$ as the transformed state vector associated to the pure state of index $j$, that is:

$$\Psi_j \xrightarrow{\tau} \Psi'_j = W^\dagger \Psi_j$$  \hfill (28)

that is, therefore, the law expressing the transformation of the state vectors in the Schrödinger picture.

### 2.3. Translations

Consider an observable $A$ decomposed, according to the (8) in the first paper, as a linear combination of the projectors of the associated basis:

$$A = \sum_j a_j I_{A=a_j}$$

clearly independent on the order of the terms in the summation. We will agree in sorting the spectrum elements in ascending order, such that it results: $j > k \Rightarrow a_j > a_k$.

By indicating with $\epsilon$ an arbitrarily chosen positive real constant, the sequence $\{j\epsilon\}$ is monotonically increasing and therefore, by a suitable restriction of the number of its terms, can be put in a biunivocal correspondence with that of the spectrum elements of $A$, through the mapping:  

$$f : j\epsilon \rightarrow a_j$$  \hfill (29)

According to the definition given in the first paper, equation (11), we will be therefore allowed to think $A$ as a function of the observable $A_L$, defined by the equation:

$$A_L := \sum_j j\epsilon I_{A=a_j}$$  \hfill (30)

that is:

$$A = f(A_L)$$

where $f$ is the function defined in (29).
We will call an observable whose spectrum is of the form of that of \( A \) a **linear spectrum observable**. The constant \( \epsilon \) is the resolution of the observable, that is the minimal variation possible for the quantity. The arbitrarily in the choose of its value reflects the arbitrariness in choosing the units of measurement.

Let \( Q \) be a linear spectrum observable with resolution \( \epsilon \). We will suppose, besides, that the number of the elements of the spectrum of \( Q \) is very large, such that one is legitimated to assume it equal to \( 2n \approx 2n + 1 \), where \( n \) is a natural number much greater than 1. In other words the number of the elements of the spectrum is a potential infinite. This assumption is justified by mathematical convenience reasons, in order to avoid several troubles with actual infinities, and by the fact that the set of the effective outcomes obtained in all the measurement sessions is necessarily finite. According to the (30), we will then be able to decompose \( Q \) as:

\[
Q = \sum_j j \epsilon I_j
\]  

(31)

where \( \{ I_j \} \) is the projector basis associated to the observable and the index \( j \) assumes all the integer values between \(-n\) and \(n\).

We will assume that all the generalized coordinates, required to describe a physical system at any given time instant, are linear spectrum observables (**homogeneity hypothesis**).

By indicating with \( \delta \) a real constant, we will introduce, with the appropriate hypotheses, the translation of displacement \( \delta \) of the observable \( Q \), as that transformation \( \tau_\delta \) that applied to the observable makes it result:

\[
\tau_\delta (Q) = Q + \delta
\]  

(32)

in which the sum in the right-hand side is to be meant in a manner specified below.

To this end, we firstly decompose the displacement \( \delta \) in the form:

\[
\delta = \xi + s \epsilon
\]  

(33)

where \( s \) is an integer and \( \xi \) a real constant between 0 and \( \epsilon \):

\[
0 \leq \xi < \epsilon.
\]  

(34)

It should be observed that such decomposition is unique for each value of \( \delta \).

By substituting the (31) and the (33) into the (32) and exploiting the linearity of the transformation and the closure relation of the projectors, equation (7) in the first paper, therefore, it must result:

\[
\sum_j j \epsilon \tau_\delta (I_j) = \sum_j (j \epsilon) I_j + (\xi + r \epsilon) \sum_j I_j =
\]

\[
= \sum_j (\xi + (j + r) \epsilon) I_j = \sum_j (\xi + j \epsilon) I_{j-r}.
\]  

(35)

Since, as seen in subsection 2.3, the transformed observable has the same spectrum as the starting one, equation (35) can be satisfied only by admitting that \( \xi = 0 \), that the sum in the right-hand side of the (32) is modulo \( 2n \epsilon \) and that the difference \( j - r \) is modulo \( 2n \).
2.4. Conjugate momenta

If $S = e^{iG}$ is the unitary pseudo-observable associated to the minimal translation $\delta = \epsilon$ whose generatrix is $G$, it’s easy to verify that the translation of displacement $s\epsilon$ is induced by the unitary pseudo-observable $S^s = e^{isG}$, whose generatrix is $sG$. By virtue of this direct proportionality between the displacement and the generatrix, one it is allowed to assume that it results:

$$G := \frac{\epsilon}{\hbar} P$$

(36)

where the observable $P$ is the conjugate momentum of $Q$ and the reduced Planck constant $\hbar$ is introduced for reasons of choice of the measurement units. The unitary pseudo-observable associated to the minimal translation can so be written in the form:

$$S = e^{i\frac{\epsilon}{\hbar} P}$$

(37)

and therefore thought as a function of the conjugate momentum. Due to the periodicity of the complex exponential function, by adding to the momentum $P$ a whatever multiple of $2\pi\hbar/\epsilon$ the pseudo-observable $S$ does not change. The terms of the spectrum of the momentum $P$ will be, therefore, distributed in an interval of diameter of $2\pi\hbar/\epsilon$ and one will be allowed to assume that the generic eigenvalue $p_k$ of $P$ satisfies the limitations:

$$-\frac{\pi \hbar}{\epsilon} \leq p_k \leq \frac{\pi \hbar}{\epsilon}.$$  

(38)

The fact that the sum in the right-hand side of the (32) is modulo $2n\epsilon$, implies that it has to be:

$$\tau_{2n\epsilon} (Q) = Q$$

and therefore:

$$S^{2n} = 1$$  

(39)

that is:

$$2n\frac{\epsilon}{\hbar}p_k = 2k\pi \Rightarrow p_k = \frac{k\pi \hbar}{n\epsilon} \quad \text{with} \quad k = -n, \ldots, n$$

(40)

where it was made use of the limitations (38) and that $n$ is much greater than 1. From this follows that also the momentum $P$ is a linear spectrum observable, with $2n$ elements in its spectrum, and, therefore, can be written in the form:

$$P = \sum_k \frac{k\pi \hbar}{n\epsilon} \tilde{I}_k$$

(41)

where $\{\tilde{I}_k\}$ is the projector basis associated to $P$.

In order to better specify the relationship between the observables $Q$ and $P$, we substitute the (36) in the (38), obtaining:

$$\tau_{\epsilon} (Q) = e^{i\frac{\epsilon}{\hbar} P} Q e^{-i\frac{\epsilon}{\hbar} P} = Q + \epsilon$$

(42)

where, due to the hypothesis that is $n \gg 1$, one can neglect that the sum in the right-hand side would have to be modulo $2n\epsilon$. By (42), it immediately follows:

$$\frac{e^{i\frac{\epsilon}{\hbar} P} Q e^{-i\frac{\epsilon}{\hbar} P} - Q}{\epsilon} = 1$$
that in the limit of the resolution that tends to zero, $\epsilon \to 0^+$, gives the well-known **canonical commutation relation**:

$$\lim_{\epsilon \to 0^+} \frac{1}{\hbar} [Q, P] = 1$$

which has the structure of a **Poisson bracket** of canonical coordinates.

It is, however, to be observed that the (43) is exactly true only in the limits $n \to \infty$ and $\epsilon \to 0$. It is, instead, impossible to satisfy for finite value of $n$, as observed already by Weyl\(^1\). According to the (18) in the second paper, in fact, it would result: $\text{tr} ([Q, P]) = 0$; whereas, by indicating with $d$ the number of the elementary projectors in a basis $\{I_j\}$, that, by the assumptions, is much greater than 1, due to the linearity of the trace functional and according to the (21) in the second paper, for the right-hand side, one has: $\text{tr} (1) = \text{tr} (\sum_j I_j) = \sum_j \text{tr} (I_j) = d \gg 1$.

### 3. Time evolution

#### 3.1. Time evolution in the Heisenberg picture

The physical phenomena are describable as evolution processes of observables while passing of the time. The observation of a physical phenomenon thus requires the measurement of a certain set of observable at different times. Let $\tau$ be the **minimum possible temporal separation** between two subsequent measurements. We will assume the **hypothesis of the homogeneity of the time**, according to which this minimum time separation is always the same.

In this subsection we will analyze time evolution in the **Heisenberg picture**.

We start studying the time evolution of an observable characterizing the system that doesn’t explicitly depend on time. This is describable as a transformation in a new one, having the same spectrum but, usually, incompatible with the initial observable.

Let $Q$ be a tuple of generalized coordinates that gives a complete description of the system and $P$ the tuple of the corresponding conjugate momenta. The time evolution of the system is therefore described by means of a transformation $\theta$ that makes it pass from the canonical coordinates, $(Q(t), P(t))$, relative to the time $t$ to those corresponding to the time $t + \tau$, according to the relations:

$$\begin{align*}
\theta (Q(t)) &= Q(t + \tau) \\
\theta (P(t)) &= P(t + \tau)
\end{align*}$$

By indicating with $U$, **pseudo-observable of minimal evolution**, the unitary pseudo-observable associated to such transformation and calling $G$ its generatrix, similarly as seen for translations, we will put:

$$G = \frac{\tau}{\hbar} H$$

where $H$ is the **Hamiltonian observable**.

Consider, now, an observable $O(Q(t), P(t))$, given by a function of an observable obtained by summation and multiplication of the canonical coordinates or of functions of compatible groups of them. According to the properties of transformations, it will be:

$$O(Q(t + \tau), P(t + \tau)) = O(\theta (Q(t)), \theta (P(t))) = \theta (O(Q(t), P(t))) .$$

---

\(^1\) Weyl, Hermann. (1927). "Quantenmechanik und Gruppentheorie." Sitzungsberichte der Preußischen Akademie der Wissenschaften zu Berlin. Math.-physikalische Klasse, 1927, 163-176.
If, for brevity, one puts \( O(t) := O(Q(t), P(t)) \), according to the (8) and to (15), therefore, one has:

\[
O(t + \tau) = UO(t)U^\dagger = e^{i\tilde{\tau}H}O(t)e^{-i\tilde{\tau}H}. \tag{47}
\]

If the observable \( O \) depends also explicitly on time, i.e. it is of the form:

\[
O(Q(t), P(t), t),
\]

the evolution may be split into two (simultaneous) steps:

\[
O(Q(t), P(t), t) \mapsto O(Q(t), P(t), t + \tau) \tag{48}
\]

\[
O(Q(t), P(t), t + \tau) \mapsto O(Q(t + \tau), P(t + \tau), t + \tau) \tag{49}
\]

that is in the process (48), into which the observable is made evolve while keeping fixed the canonical coordinates and in the process (49), into which the canonical coordinates evolve while ignoring the explicit dependence on time of the observable. According to such splitting and by applying to the second step the (47), by putting, for brevity,

\[
O(t + \tau) := O(Q(t + \tau), P(t + \tau), t + \tau),
\]

one has:

\[
O(t + \tau) = e^{i\tilde{\tau}H(t)}O(t + \tau)P(t + \tau)e^{-i\tilde{\tau}H(t)} \tag{50}
\]

Treating \( \tau \) as an infinitesimal and by a Taylor expansion of both sides of the (50) to first order in it, one obtains, by omitting the arguments, the following relation:

\[
\frac{dO}{dt} = \frac{1}{i\hbar}[O, H] + \frac{\partial O}{\partial t} \tag{51}
\]

that is the general form of the well-known Heisenberg equation.

Consider now the **temporal abscissa** observable \( T \) of an event. By thinking \( T \) as a function of the canonical coordinate, by virtue of equation (47), one has:

\[
e^{i\tilde{\tau}H}T(t)e^{-i\tilde{\tau}H} = T(t + \tau) = T(t) + \tau. \tag{52}
\]

The time evolution transformation \( \theta \) then represents for the temporal abscissa \( T \) a translation of minimal (temporal) displacement \( \tau \).

### 3.2. Constants of motion and symmetries

The time evolution gives rise to several important **conservation laws**.

First of all, if two observables, that does not depend explicitly on time, are initially compatible they remain compatible afterward. In fact, by indicating with \( A(t) \) and \( B(t) \) the two observables at a time \( t \) and if it results \([A(t), B(t)] = 0\), for the properties of the transformations, one has:

\[
[A(t + \tau), B(t + \tau)] = [\theta(A(t)), \theta(B(t))] = \theta([A(t), B(t)]) = 0. \tag{53}
\]

Consider, now, a one-parameter group of transformations induced by unitary pseudo-observables \( R_\xi \) of the form:

\[
R_\xi = e^{iF}\tag{54}
\]

where \( \xi \) is a real parameter and \( F \) is an observable. Equation (54) characterizes a **Lie group**, in the parameter \( \xi \), whose generator is the observable \( F \). In the hypothesis that \( F \) does not explicitly depend on time, such group of transformation represents a **continuous symmetry** for the system if each transformation leaves unchanged the Hamiltonian observable:

\[
R_\xi HR_\xi^{-1} = H. \tag{55}
\]
By a Taylor expansion of the \([54]\) with respect to \(\xi\), treated as in infinitesimal, one obtains:

\[
H + \xi i [F, H] + \frac{\xi^2}{2} i [F, i [F, H]] + \cdots = H
\]

from which it follows:

\[
[F, H] = 0 .
\]

So we have retrieved, in the formalism of the pseudo-observables, the well-known correspondence between the continuous symmetries and the compatibility among their generators and the Hamiltonian observable. According to the Heisenberg equation \([47]\), one therefore has:

\[
F(t + \tau) = e^{i\frac{\tau}{\hbar}H} F(t) e^{-i\frac{\tau}{\hbar}H} = F(t)
\]

that implies the the generator \(F\) is a constant of motion, as stated, in a more general form, by the Noether’s theorem \([13]\).

As again well-known, equation \([57]\) implies that if the Hamiltonian observable does not depend explicitly on time it is a constant of motion and the terms of its spectrum \(\{\varepsilon_j\}\) give the possible outcomes of energy measurement of the system.

In such hypothesis, consider an observable \(O(Q(t), P(t))\) that also does not depend explicitly on time. The time evolution of \(O\) is given by the \([47]\), from which it follows:

\[
O(t) = U^\dagger O(t + \tau) U \Rightarrow O(t - \tau) = U^\dagger O(t) U
\]

that describes the time reversed evolution of the observable \(O\). By analyzing such relation, it is argued that, if \(Z\) is a generic pseudo-observable appearing in the \([58]\), the time reversal \((T)\) implies the substitutions:

\[
\begin{align*}
Z &\rightarrow Z^\dagger \\
\tau &\rightarrow -\tau .
\end{align*}
\]

Generalizing such result, it is assumed that time reversal implies the application of the substitutions \([59]\) to every pseudo-observable.

By applying the \([59]\) to the pseudo-observable of minimal evolution \(U = e^{i\frac{\tau}{\hbar}H}\), it is argued that the reversed time evolution is generated by the same Hamiltonian observable that generates the direct temporal evolution (reversibility principle), provided that this last does not explicitly depend on time, that is:

\[
T(H) = H .
\]

The application of time reversal to the temporal abscissa \(T\) of an event makes, according to the second substitution in the \([59]\), reverse the sign of each term in its spectrum and therefore:

\[
T(T) = -T .
\]

Since, besides, the substitutions implied by the \([59]\), does not affect any generalized coordinate \(Q\), one has:

\[
T(Q) = Q .
\]

Applying, finally, the time reversal to the unitary pseudo-observable \(S\) associated to the minimal translation of the coordinate \(Q\), since it must remain unaffected, it is argued that time reversal reverses the signs of the conjugate momenta:

\[
T(P) = -P .
\]
For what above stated, the effect of the time reversal on a generic observable will be given by:

\[ T(O(Q, P, t)) = O(Q, -P, -t) \]  

(64)

It is important, now, to specify the relationship, given by the first substitution in the (59), between time reversal and transposition. **Time reversal, indeed, implies the inversion of the observation order**, as it was already heuristically stated in subsection 3.2 in the first paper. This association, therefore, finds here a more rigorous justification.

### 3.3. Time evolution in the Schrödinger picture

The time evolution in the Schrödinger picture can be easily described by making use of what discussed at the end of subsection (2.2). First of all, it is worth remembering that in this picture the observables evolves only if they depend *explicitly* on time. Time evolution so affects mainly the quantum state, as determined through a Bayesian inference on the basis of the outcomes of an initial set of compatible measurements. If the initial state is reputed to be described as a pure state associated to a state vector \( \Psi \), according to the (28) and to the definition of the pseudo-observable of minimal evolution given in subsection (3.1), one has:

\[ \Psi(t + \tau) = U^\dagger \Psi(t) = e^{-i\tau \hat{H}(t)} \Psi(t) \]  

(65)

from which, by treating \( \tau \) as an infinitesimal and by a Taylor expansion of both sides of the (65) to first order in it, omitting, for brevity, the time argument, it follows:

\[ i\hbar \frac{d\Psi}{dt} = H \Psi \]  

(66)

that is the well-known *Schrödinger equation of motion*.

In the more general case in which the initial state of the physical system is described, as a mixed state, in terms of the density observable \( D \), according to the (26), one, instead, obtains the following time evolution equation:

\[ D(t + \tau) = U^\dagger D(t) U = e^{-i\tau \hat{H}(t)} D(t) e^{i\tau \hat{H}(t)} \]  

(67)

By treating again \( \tau \) as an infinitesimal and by a Taylor expansion of both sides of the (67) to first order in it, omitting, for brevity, the time argument, one finally obtains the well-known *von Neumann equation*:

\[ i\hbar \frac{dD}{dt} = [H, D] \]  

(68)

### 4. Conclusions

#### 4.1. The continuous limit

The way followed to derive the results in this article may appear dissatisfying the Mathematical readers, due to the somewhat coarse limit procedures used. In this subsection I will try to better justify the reasons for doing so.

In subsection 2.3 the generalized coordinates were initially introduced as linear spectrum observables, with a discrete and finite spectrum. We was forced to do this, because of the simplifying assumptions taken in the first paper. But there is more. A continuous spectrum implies the physical possibility of resolve two different eigenvalues, however close they can be. This corresponds to the possibility of performing an exact measurement of a generalized coordinate \( Q \), that is a
measurement whose standard deviation is 0, but, in force of the Heisenberg uncertainty relations and of the canonical commutation relation, this would involve an actual infinite deviation in the conjugate momentum, that would reflect itself in an actual infinite amount of energy to be used in the measurement process. Since this is impossible, this implies that it is also impossible to experimentally prove the continuity of the spectrum of a generalized coordinate. It is also clearly impossible to experimentally prove the infinity of the spectrum and also its boundlessness. This concepts exist only in the Mathematics Realm, since it is true what Kronecker, as quoted by Weber in 1893, said:

“Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk”

[“God made the integers, all else is the work of man”]

but also actual infinities, with all their well-known paradoxes, cannot find place in the Physics World.

This may appear again as an excess of “empiricism”, but there is other. The validity of the canonical commutation relation requires the boundless of at least one of a pair of canonical coordinates. This boundlessness cannot be approximate, not even as a result of a limit, since the argument of Weyl, readjusted for pseudo-observables in subsection 2.4, proves that it isn’t even approximately true for finite and bounded canonical coordinates. At this regard, it is not to be forgotten that the whole canonical formalism relies on the hypothesis of the continuity of the coordinates.

The above arguments apply also to the temporal abscissa, that, by virtue of the 52 may be considered as having as conjugate momentum the Hamiltonian observable.

Effectively, a discrete structure of space and time at a Planck scale may be considered not surprising. The point is so how to recover, in a coherent manner, the canonical formalism. I think that this might be done not in terms of relationship among observables but in terms of expectation value and wave functions, for which continuity and analyticity are recovered, even for finite Hilbert spaces.

This, however, will be the subject of future investigations.

4.2. Time evolution and observation

A delicate point in the quantum mechanical description of the reality is that of the relationship between time evolution and observation. To be more precise, as already discussed in the second paper, an observation, here meant as equivalent to a measurement process, breaks down the continuity of the time evolution of the quantum states (it is here convenient to assume the Schrödinger picture), causing a “collapse” of the density observable.

As widely discussed in the conclusions of the second paper, this way of seeing the things is misleading: as a matter of fact, the density observable is not a “normal” observable, but a property summarizing the description that an observer can give of the physical system when “adopting” a suitable measurement setup, giving birth to a new meta-observer. In this sense the density observable is relative to the meta-observer, so each changing of the latter alters also the first, through a logical - not physical - process of Bayesian inference.

But what does it happen if a second observer views “externally” the measurement act of the first? Here the word “externally” means that there is no communication between the two observers. The situation is well described by the relational vision of the Quantum Mechanics of Rovelli [14, 15].

To be clear, let’s indicate with A (“Alice”) the first observer, with B (“Bob”) the second one and with O the physical system “measured” by A.
From the point of view of Bob, the observer A is an object of observation, describable as a quantum system entangled with O in the measurement act. According to Bob the state of this entangled system (A+O) evolves in a continuous manner over time - before, during and after the measurement of A - according to the equations of motion seen above (possibly using the von Neumann one). The final state is, however, a mixed one, in which appears all the possible outcomes of the measurement performed by A.

From the point of view of Alice - that doesn’t know what B is seeing and inferring - the measurement changes the observational situation, due to the information acquired through the measurement setup, that changes the meta-observer of which she is part. This fact forces a change in the density observable adopted to describe O, that must now correspond to the pure state associated with the outcome of her measurement. But Alice cannot follow the time evolution of the density observable during the measurement process, because she cannot observe herself as an external object!

If, after the measurement, Alice and Bob communicate with each other, then they give rise to a new higher-level meta-observer, so Bob “observes” a “collapse” of the density observable describing the measurement process to agree with Alice for the outcome found. Again the “collapse” is due to the change of meta-observer, which requires a revision in the density observable.

4.3. Seven reasons to change

Most of the results of this paper may not seem “extraordinary”, since, after all - apart from the treatment of transformations, the continuity issues and the time reversal - they may be simply obtained by standard textbook Quantum Mechanics by substitution of the operators and the state vectors with the appropriate pseudo-observables - with the remarkable exception of the time reversal. Effectively the principal reason in doing so was just to show how easy is to switch from the Dirac-von Neumann formalism to the new one.

So why would one have to change, if, after all, equations are, formally, the same?

In this conclusive subsection I will try to give a list of compelling reasons to do this:

(i) The standard formalism is based on a somewhat “schizophrenic” choice on the fundamental entities of the theory: abstract state vectors, whose unclear and ambiguous ontology made flourish a plethora of different interpretations of Quantum Mechanics, and an algebra of operators, part of which corresponding to observable, whose definition is vague, with a a far from being clear connection with the outcomes of a measurement. The new formalism is characterized by an ontology clear and “parsimonious”, making reference to the classification criteria of Pykacz\[16\]; gives the observables a due central role, defining them as precise algebraic objects, and gives clear and physically transparent definitions of the only type of entities - the pseudo-observables - and of the operations and the functionals needed to describe a physical system and to define the measurement outcomes. The new theory is also “embedded” with a clear and unambiguous interpretation, based both on the “relational” view of Rovelli and on Quantum Bayesianism, sweeping away all the others from the field.

(ii) Passing to new formalism requires almost no efforts, since to each valid expression in the Dirac-von Neumann formalism corresponds a valid expression in the framework of the algebra of pseudo-observables, simply substituting the operators and the state vectors with the corresponding pseudo-observables. So, for instance, if $\Psi_i$ and $\Psi_j$ are the state vector pseudo-observables corresponding to the “kets” $|i\rangle$ and $|j\rangle$ and if $A$ is the
observable represented by the operator $\hat{A}$, the matrix element $\langle i | \hat{A} | j \rangle$ corresponds to the inner product $\langle \Psi_i | A \Psi_j \rangle$. Therefore for a Physicist who embraces a purely instrumentalist vision\cite{17}, that is one - as a matter of fact the majority... - that explicitly avoids any explanatory role of the theory, using it only as a mere instrument of calculation, the changing is really minimal!

(iii) Even if, however, each expression in the Dirac-von Neumann formalism has a corresponding one in the framework of the algebra of the pseudo-observables, the converse is not true! So the latter offers new possibilities to find more convenient, clear and compact demonstrations of relevant physical relations.

(iv) The whole formalism of the algebra of the pseudo-observables is simpler than the standard one, requiring only initial undergraduate level knowledge, giving the opportunity to a wider diffusion of a deeper understanding of Quantum Mechanics.

(v) The theory was constructed on the basis of very general logical properties, proving that quantum mechanics is the unique minimal description of physical reality\cite{1}. Unlike a diffuse practice, besides, in this theory Mathematics is constructed to fit well-defined physical ideas, not chosen among the existing models to fit experimental data and generic physical conceptions.

(vi) The conceptual bases of the theory are physically transparent and allows to easily overcome deep conceptual troubles in the Dirac-von Neumann axiomatic system, as, for instance, the measurement problem\cite{2} (see also subsection 4.2).

(vii) The algebra of the pseudo-observables entails the capability of being a descriptive instrument that goes beyond the simply reformulation of the Quantum Mechanics. I have already obtained some preliminary result which shows that this theory can give us a deeper insight in the more intimate properties of the space and of the time, till the Planck scale! This however will, hopefully, be the subject of future investigations.

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