Some New Simpson’s and Newton’s Formulas Type Inequalities for Convex Functions in Quantum Calculus

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Abstract: In this paper, using the notions of $q^2$-quantum integral and $q^2$-quantum derivative, we present some new identities that enable us to obtain new quantum Simpson’s and quantum Newton’s type inequalities for quantum differentiable convex functions. This paper, in particular, generalizes and expands previous findings in the field of quantum and classical integral inequalities obtained by various authors.

Keywords: Simpson’s inequalities; Newton’s inequalities; quantum calculus; convex functions

1. Introduction

Thomas Simpson developed crucial techniques for numerical integration and estimation of definite integrals, which became known as Simpson’s law during his lifetime (1710–1761). However, J. Kepler used a similar approximation nearly a century before, so it is also known as Kepler’s law. Since Simpson’s rule includes the three-point Newton–Cotes quadrature rule, estimations based entirely on a three-step quadratic kernel are often referred to as Newton-type results.

(1) Simpson’s quadrature formula (Simpson’s 1/3 rule):
\[
\int_{k_1}^{k_2} F(x) \, dx \approx \frac{k_2 - k_1}{6} \left[ F(k_1) + 4F \left(\frac{k_1 + k_2}{2} \right) + F(k_2) \right].
\]

(2) Simpson’s second formula or Newton–Cotes quadrature formula (Simpson’s 3/8 rule):
\[
\int_{k_1}^{k_2} F(x) \, dx \approx \frac{k_2 - k_1}{8} \left[ F(k_1) + 3F \left(\frac{2k_1 + k_2}{3} \right) + 3F \left(\frac{k_1 + 2k_2}{3} \right) + F(k_2) \right].
\]

Within the literature, there are a plethora of estimations correlated with certain quadrature laws, one of which is the following estimation known as Simpson’s inequality:
Theorem 1. Let a mapping $\mathcal{F}: [\kappa_1, \kappa_2] \to \mathbb{R}$ be four times continuously differentiable on $(\kappa_1, \kappa_2)$, and let $\|\mathcal{F}^{(4)}\|_\infty = \sup_{x \in (\kappa_1, \kappa_2)} |\mathcal{F}^{(4)}(x)| < \infty$. Then, one has the inequality

$$\left| \frac{1}{3} \left[ \frac{\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2} + 2\mathcal{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) \right] - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(x)\,dx \right| \leq \frac{1}{2880} \|\mathcal{F}^{(4)}\|_\infty (\kappa_2 - \kappa_1)^4.$$ 

Many authors have concentrated on the Simpson’s type inequality in different categories of mappings in recent years. Since convexity theory is an easy and efficient way to solve a large number of problems from various branches of pure and applied mathematics; some mathematicians have worked on the results of Simpson’s and Newton’s type in obtaining a convex map. Dragomir et al. [1], for example, introduced the recent Simpson’s inequalities and their applications in numerical integration quadrature formulas. In addition, Alomari et al. in [2] determined several inequalities of Simpson’s type for $s$-convex functions. The variance of Simpson’s type inequality dependent on convexity was then noted by Sarikaya et al. in [3]. The authors presented Newton’s inequality for harmonic convex and $p$-harmonic convex mappings in [4,5]. Iftekhar et al., in [6], described a new Newton-type inequality for functions with the local fractional derivative, which is generalized and convex.

On the other hand, several works in the field of $q$-analysis are being carried out, beginning with Euler, in order to achieve mastery in the mathematics that drives quantum computing. The link between physics and mathematics is referred to as $q$-calculus. It has a wide range of applications in mathematics, including number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other disciplines, as well as mechanics, relativity theory, and quantum theory [7,8]. Quantum calculus is also useful in quantum information theory, which is an interdisciplinary field that includes computer science, information theory, philosophy, and cryptography, among other things [9,10]. This important branch of mathematics is thought to have been invented by Euler. The $q$-parameter was used by Newton in his work on infinite series. Jackson [11] was the first to develop the $q$-calculus, which knew no limit calculus, in a systematic way. Al-Salam [12] introduced a $q$-analogue of the $q$-fractional integral and a $q$-Riemann–Liouville fractional in 1966. Since then, the amount of research in this area has steadily increased. Specifically, Tariboon and Ntouyas [13] demonstrated the $\kappa_1 D_q$-difference operator and $q^{\kappa_1}$-integral in 2013. The notions of the $e^2 D_q$ difference operator and $q^{e^2}$-integral were given by Bermudo et al. [14] in 2020. In [15], Sadjaj generalized to quantum calculus and introduced the notions of post-quantum calculus or $\mathcal{Q}$-calculus. Soontharanon et al. [16] introduced the concepts of fractional $(p,q)$-calculus later on. In [17], Tunç and Göv gave the post-quantum variant of the $\kappa_1 D_q$-difference operator and $q^{\kappa_1}$-integral. Recently, in [18], Chu et al. introduced the notions of the $Q^2 D_{p,q}$ derivative and $(p,q)^{Q^2}$-integral in [18].

For various types of functions, quantum and post-quantum integrals have been used to investigate many integral inequalities. The authors used $\kappa_1 D_q$, $Q^2 D_q$-derivatives and $q^{\kappa_1}$, $Q^2$-integrals to prove HH integral inequalities and their left–right estimates for convex and coordinated convex functions in [19–28]. Noor et al. presented a generalized version of quantum integral inequalities in [29]. Nwaeeze et al. proved certain parameterized quantum integral inequalities for generalized quasi-convex functions in [30]. In [31], Khan et al. proved quantum HH inequality using the green function. For convex and coordinated convex functions, Budak et al. [32], Ali et al. [33,34], and Vivas-Cortez et al. [35] developed new quantum Simpson’s and quantum Newton’s type inequalities. For quantum Ostrowski’s inequalities for convex and coordinated convex functions one can consult [36–38]. Using the $\kappa_1 D_{p,q}$-difference operator and $(p,q)_{\kappa_1}$-integral, Kurt et al. [39] generalized the results of [21] and proved HH type inequalities and their left estimates. Recently, Latif et al. [40] discovered the right estimates of HH type inequalities, which had previously been proven by Kurt et al. [39]. Chu et al. [18] used the concepts of the $Q^2 D_{p,q}$-derivative and $(p,q)^{Q^2}$-integral to prove Ostrowski’s inequalities. Recently, Vivas-Cortez et al. [41]
generalized the results of [14] and proved HH type inequalities and their left estimates using the $p,q^3$-difference operator and $(p,q)^{3/2}$-integral.

Inspired by the ongoing studies, we use the $q$-integral to develop some new quantum Simpson’s and quantum Newton’s formulas type inequalities for $q$-differentiable convex functions, and these inequalities can be helpful for finding the bounds of Simpson’s and Newton’s formulas for numerical integration. We also show that the newly developed inequalities are extensions of some previously known inequalities.

The following is the structure of this paper: Section 2 provides a brief overview of the fundamentals of $q$-calculus as well as other related studies in this field. In Section 3, we establish two crucial identities that play an essential role in developing the main results of this paper. The Simpson’s and Newton’s type inequalities for $q$-differentiable functions via $q$-integrals are described in Section 4. The relationship between the findings reported here and similar findings in the literature are also taken into account. Section 5 concludes with some recommendations for future research.

2. Preliminaries of $q$-Calculus and Some Inequalities

In this section, we first present the definitions and some properties of quantum derivatives and quantum integrals. We also mention some well-known inequalities for quantum integrals. The following notation will be used frequently in this work (see [8]):

$[n]_q = \frac{q^n - 1}{q - 1}$.

Jackson defined the $q$-Jackson integral from 0 to $\kappa_2$ for $0 < q < 1$ as follows:

$$\int_0^{\kappa_2} F(x) \, dq_\kappa x = (1 - q)\kappa_2 \sum_{k=0}^{\infty} q^k F(\kappa_2 q^k)$$  (1)

provided the sum converges absolutely [11].

In [11], Jackson also defined the $q$-Jackson integral on any closed interval $[\kappa_1, \kappa_2]$:

$$\int_{\kappa_1}^{\kappa_2} F(x) \, dq_\kappa x = \int_{\kappa_1}^{\kappa_2} F(x) \, dq_\kappa x - \int_{\kappa_1}^{0} F(x) \, dq_\kappa x.$$  (2)

Tariboon and Ntouyas defined the following $q_{\kappa_1}$-derivative and $q_{\kappa_1}$-integral:

**Definition 1** ([13]). The $q_{\kappa_1}$-derivative of the mapping $F : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ is defined as:

$$\kappa_1 D_q F(x) = \frac{F(x) - F(qx + (1 - q)\kappa_1)}{(1 - q)(x - \kappa_1)}, \quad x \neq \kappa_1.$$  (3)

If $x = \kappa_1$, we write $\kappa_1 D_q f(\kappa_1) = \lim_{x \to \kappa_1} \kappa_1 D_q f(x)$ if it exists and it is finite.

**Definition 2** ([13]). The $q_{\kappa_1}$-definite integral of the mapping $F : [\kappa_1, \kappa_2] \rightarrow \mathbb{R}$ on $[\kappa_1, \kappa_2]$ is defined as:

$$\int_{\kappa_1}^{\kappa_2} F(x) \, dq_\kappa x = (1 - q)(\kappa_2 - \kappa_1) \sum_{k=0}^{\infty} q^k F(q^k \kappa_2 + (1 - q^k)\kappa_1).$$

Alp et al. proved quantum Hermite–Hadamard inequalities for $q_{\kappa_1}$-integrals by utilizing the convex functions, as follows:
Theorem 2 ([21]). For a convex mapping \( \mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R} \) that is differentiable on \([\kappa_1, \kappa_2]\), the following inequality holds:

\[
\mathcal{F}\left(\frac{q\kappa_1 + \kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(x) \ k_1 \ d_q x \leq \frac{q\mathcal{F}(\kappa_1) + \mathcal{F}(\kappa_2)}{2},
\]

where \( q \in (0, 1) \).

On the other hand, Bermudo et al. defined a new quantum derivative and a quantum integral, which are called the \( q^2 \)-derivative and \( q^2 \)-integral:

**Definition 3** ([14]). The \( q^2 \)-derivative of a mapping \( \mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R} \) is defined as:

\[
k_1 D_q \mathcal{F}(x) = \frac{\mathcal{F}(qx + (1 - q)\kappa_2) - \mathcal{F}(x)}{(1 - q)(\kappa_2 - x)}, \ x \neq \kappa_2.
\]

If \( x = \kappa_2 \), we write \( k_1 D_q f(\kappa_2) = \lim_{x \to \kappa_2} k_1 D_q f(x) \) if it exists and it is finite.

**Definition 4** ([14]). The \( q^2 \)-definite integral of a mapping \( \mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R} \) on \([\kappa_1, \kappa_2]\) is defined as:

\[
\int_{\kappa_1}^{\kappa_2} \mathcal{F}(x) \ k_1 d_q x = (1 - q)(\kappa_2 - \kappa_1) \sum_{k=0}^{\infty} q^k \mathcal{F}
\]

Bermudo et al. also proved the corresponding quantum Hermite–Hadamard inequalities for the \( q^2 \)-integral:

**Theorem 3** ([14]). For a convex mapping \( \mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R} \) that is differentiable on \([\kappa_1, \kappa_2]\), the following inequality holds:

\[
\mathcal{F}\left(\frac{\kappa_1 + q\kappa_2}{2}\right) \leq \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(x) \ k_2 d_q x \leq \frac{\mathcal{F}(\kappa_1) + q\mathcal{F}(\kappa_2)}{2},
\]

where \( q \in (0, 1) \).

Moreover, we will need to use the subsequent Lemma in our key results:

**Lemma 1** ([8]). We have the equality

\[
\int_{\kappa_1}^{\kappa_2} (x - \kappa_1)^a \ k_1 d_q x = \frac{(\kappa_2 - \kappa_1)^{a+1}}{[a + 1]_q}
\]

for \( a \in \mathbb{R} \setminus \{-1\} \).

### 3. Identities

We deal with identities, which is necessary to attain our main estimations in this section. We first establish an identity based on a two steps kernel in the following Lemma.

**Lemma 2.** Let \( \mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R} \) be a \( q^2 \)-differentiable function on \((\kappa_1, \kappa_2)\) and \( 0 < q < 1 \). If \( k_2 D_q \mathcal{F} \) is continuous and integrable on \([\kappa_1, \kappa_2]\), then one has the identity

\[
\frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} (\mathcal{F}(s) k_2 d_q s) - \frac{1}{[6]_q} \left[ \mathcal{F}(\kappa_1) + q^2 \mathcal{F}(\kappa_2) \right]
\]

(6)
\[
\begin{align*}
= \varphi(\kappa_2 - \kappa_1) \int_0^1 \Lambda(s) \, \kappa_2 D_q \mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \, d_q s,
\end{align*}
\]

where:
\[
\Lambda(s) = \begin{cases} 
    s - \frac{1}{[6]_q}, & s \in \left[0, \frac{1}{[6]_q}\right) \\
    s - \frac{[5]_q}{[6]_q}, & s \in \left[\frac{1}{[6]_q}, 1\right]. 
\end{cases}
\]

**Proof.** Using Formula (2), from the definition of the function \(\Lambda(s)\), we find that
\[
\int_0^1 \Lambda(s) \, \kappa_2 D_q \mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \, d_q s 
= [5]_q - 1 \int_0^{[5]_q} \kappa_2 D_q \mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \, d_q s 
+ \int_0^{[5]_q} \left(s - \frac{[5]_q}{[6]_q}\right) \kappa_2 D_q \mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \, d_q s.
\]

By Definition 3, one also has
\[
\kappa_2 D_q \mathcal{F}(s\kappa_1 + (1-s)\kappa_2) = \frac{\mathcal{F}(qs\kappa_1 + (1-qs)\kappa_2) - \mathcal{F}(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)}. 
\]

Now, if we substitute the above Equation into (7), we obtain:
\[
\int_0^1 \Lambda(s) \, \kappa_2 D_q \mathcal{F}(s\kappa_1 + (1-s)\kappa_2) \, d_q s 
= [5]_q - 1 \int_0^{[5]_q} \frac{\mathcal{F}(qs\kappa_1 + (1-qs)\kappa_2) - \mathcal{F}(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)} \, d_q s 
+ \int_0^{[5]_q} \frac{\mathcal{F}(qs\kappa_1 + (1-qs)\kappa_2) - \mathcal{F}(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)} \, d_q s 
- [5]_q \int_0^{[5]_q} \frac{\mathcal{F}(qs\kappa_1 + (1-qs)\kappa_2) - \mathcal{F}(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)} \, d_q s.
\]

Calculating the first integral in the right-hand side of (8) by taking into account the case when \(\kappa_1 = 0\) of Definition 2, it is found that
\[
\int_0^{[5]_q} \frac{\mathcal{F}(qs\kappa_1 + (1-qs)\kappa_2) - \mathcal{F}(s\kappa_1 + (1-s)\kappa_2)}{(1-q)(\kappa_2 - \kappa_1)} \, d_q s 
= \frac{1}{(\kappa_2 - \kappa_1)} \left[2\right]_q \left( \sum_{k=0}^{\infty} q^{k+1} \mathcal{F}\left(\frac{q^{k+1}}{[2]_q} - \kappa_1 + \left(1 - \frac{q^{k+1}}{[2]_q}\right) \kappa_2 \right) \right).
\]
Proof. Using Formula (2), from the definition of the function \( \Delta(s) \), we find that

\[
\int_0^1 \Delta(s) \, \kappa^2 D_q F(s \kappa_1 + (1 - s) \kappa_2) \, d_q s
\]

If we similarly observe the other integrals in the right-hand side of (8), from Definition 4, then we obtain

\[
\int_0^1 \frac{F(q s \kappa_1 + (1 - q s) \kappa_2) - F(s \kappa_1 + (1 - s) \kappa_2)}{(1 - q)(\kappa_2 - \kappa_1)} \, d_q s
\]  
(10)

and

\[
\int_0^1 \frac{\varphi(q s \kappa_1 + (1 - q s) \kappa_2) - \varphi(s \kappa_1 + (1 - s) \kappa_2)}{(1 - q)(\kappa_2 - \kappa_1)s} \, d_q s
\]  
(11)

Substituting Expressions (9)–(11) into (8), and later multiplying both sides of the resulting identity by \( q(\kappa_2 - \kappa_1) \), Equality (6) can be captured. \( \square \)

We now observe how an equality comes out when we use the kernel mapping with three sections.

Lemma 3. Let \( F : [\kappa_1, \kappa_2] \rightarrow \mathbb{R} \) be a \( \kappa^2 \)-differentiable function on \( (\kappa_1, \kappa_2) \) and \( 0 < q < 1 \). If \( \kappa^2 D_q F \) is continuous and integrable on \( [\kappa_1, \kappa_2] \), then one has the identity

\[
\frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F(s) \, \kappa^2 d_q s
\]

\[
= \frac{1}{8} \left[ F(\kappa_1) + \frac{q^3}{2} F \left( \kappa_1 + \frac{q}{2} \kappa_2 \right) \right] + \frac{q^2}{2} D_q F \left( \frac{[2q^2_1 + q^2_2]}{[3q]} \right)
\]

where

\[
\Delta(s) = \begin{cases} 
  s - \frac{[1]}{[3]}, & s \in \left( 0, \frac{1}{[3]} \right) \\
  s - \frac{[1]}{[2]}, & s \in \left( \frac{1}{[2]}, \frac{[2]}{[3]} \right) \\
  s - \frac{[1]}{[3]}, & s \in \left( \frac{[2]}{[3]}, 1 \right)
\end{cases}
\]

Proof. Using Formula (2), from the definition of the function \( \Delta(s) \), we find that

\[
\int_0^1 \Delta(s) \, \kappa^2 D_q F(s \kappa_1 + (1 - s) \kappa_2) \, d_q s
\]
The desired result can be obtained. If the same steps in the proof of Lemma 2 are applied for the rest of this proof, the desired result can be obtained.

4. Main Results

For brevity, we start this section with the following notations, which will be used in the new results:

\[ A_1(q) = \frac{2q^2[2]_q^2 + [6]_q^2([6]_q - [3]_q)}{[2]_q^3[3]_q[6]_q}, \]  
\[ B_1(q) = \frac{2q[3]_q[6]_q - q^2}{[2]_q[3]_q[6]_q} + \frac{1}{[2]_q^2} \left( \frac{q + q^2}{[3]_q} - \frac{q^2 + 2q}{[6]_q} \right), \]  
\[ A_2(q) = \frac{2q^2[5]_q^3}{[2]_q^3[3]_q[6]_q} + \frac{[6]_q \left( 1 + [2]_q^3 \right) - [3]_q[5]_q \left( 1 + [2]_q^2 \right)}{[2]_q^3[3]_q[6]_q}, \]  
\[ B_2(q) = \frac{2q^2[5]_q^3[6]_q - q^2[5]_q^3}{[2]_q^3[3]_q[6]_q^3} + \frac{q^2[5]_q}{[2]_q[3]_q[6]_q} - \frac{q[5]_q}{[2]_q[6]_q} - \frac{1}{[2]_q^3} \left( \frac{[5]_q \left( 2q + q^2 \right)}{[6]_q} - \frac{q + q^2}{[3]_q} \right), \]  
\[ A_3(q) = \frac{2q^2[3]_q^3 + [8]_q^2 \left( [8]_q[2]_q - [3]_q^2 \right)}{[8]_q^3[3]_q[2]_q}, \]  
\[ B_3(q) = \frac{2q[8]_q[3]_q - q^2}{[8]_q^3[2]_q[3]_q} + \frac{[3]_q^2 - [2]_q}{[3]_q^4[2]_q} \left( 1 - [3]_q[2]_q \right) + \frac{[3]_q^2[3]_q^2}{[8]_q^3[2]_q}, \]  
\[ A_4(q) = \frac{2q^2[2]_q^2}{[2]_q^3[3]_q} + \frac{[2]_q \left( 1 + [2]_q^3 \right) - [3]_q^2 \left( 1 + [2]_q^2 \right)}{[3]_q^4[2]_q}, \]  
\[ B_4(q) = \frac{2q[2]_q^2 - q}{[3]_q^3} - \frac{q^2}{[3]_q^2} - A_4(q), \]  
\[ A_5(q) = \frac{2q^2[7]_q^3}{[8]_q^3[2]_q[3]_q} + \frac{[2]_q[8]_q \left( [2]_q^2 + [3]_q^3 \right) - [7]_q[3]_q^2 \left( [2]_q^2 + [3]_q^2 \right)}{[3]_q^4[8]_q[2]_q}. \]
and
\[ B_5(q) = 2 \left( \frac{q[7]_q^2 [8]_q [3]_q}{[2]_q [3]_q} - \frac{q^2 [7]_q^3}{[2]_q [3]_q} - \frac{q[7]_q}{[2]_q [8]_q} \right) + \frac{[2]_q ([3]_q - [2]_q^2)}{[3]_q^4} - \left( q + \frac{q^2}{2} \right) \frac{[7]_q}{[2]_q [8]_q}. \] (21)

4.1. Simpson’s Inequalities for $q^{k_2}$-Quantum Integral

In this subsection, we prove some Simpson’s type inequalities by using Lemma 2. Let us start to find some new quantum estimates by using Lemma 2. We first examine a new result for functions whose $q^{k_2}$-derivatives in modulus are convex in the following theorem.

**Theorem 4.** Let $\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a $q^{k_2}$-differentiable function on $(\kappa_1, \kappa_2)$ such that $q^{k_2} D_q \mathcal{F}$ is continuous and integrable on $[\kappa_1, \kappa_2]$. If $|q^{k_2} D_q \mathcal{F}|$ is convex on $[\kappa_1, \kappa_2]$, then we have following inequality for $q^{k_2}$-integrals:

\[
\left| \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(s) \, q^{k_2} d_q s - \frac{1}{[6]_q} \left[ \mathcal{F}(\kappa_1) + q^{2} [4]_q \mathcal{F}\left( \frac{\kappa_1 + q \kappa_2}{2} \right) + q \mathcal{F}(\kappa_2) \right] \right| \leq q(\kappa_2 - \kappa_1) \left\{ \left| q^{k_2} D_q \mathcal{F}(\kappa_1) \right| \left| A_1(q) + A_2(q) \right| + \left| q^{k_2} D_q \mathcal{F}(\kappa_2) \right| \left| B_1(q) + B_2(q) \right| \right\},
\]

where $0 < q < 1$, and $A_1(q)$, $A_2(q)$, $B_1(q)$, $B_2(q)$ are given as in (12)–(15), respectively.

**Proof.** On taking the modulus in Lemma 2, because of the properties of modulus, we find that

\[
\left| \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(s) \, q^{k_2} d_q s - \frac{1}{[6]_q} \left[ \mathcal{F}(\kappa_1) + q^{2} [4]_q \mathcal{F}\left( \frac{\kappa_1 + q \kappa_2}{2} \right) + q \mathcal{F}(\kappa_2) \right] \right| \leq q(\kappa_2 - \kappa_1) \int_{0}^{\frac{1}{[6]_q}} s - \frac{1}{[6]_q} \left| q^{k_2} D_q \mathcal{F}(s) (\kappa_1 + (1 - s) \kappa_2) \right| d_q s
\]

\[ + q(\kappa_2 - \kappa_1) \int_{\frac{1}{[6]_q}}^{1} s - \frac{[5]_q}{[6]_q} \left| q^{k_2} D_q \mathcal{F}(s) (\kappa_1 + (1 - s) \kappa_2) \right| d_q s. \]

To calculate integrals in the right-hand side of (23), using the convexity of $|q^{k_2} D_q \mathcal{F}|$, it follows that

\[
\int_{0}^{\frac{1}{[6]_q}} s - \frac{1}{[6]_q} \left| q^{k_2} D_q \mathcal{F}(s) (\kappa_1 + (1 - s) \kappa_2) \right| d_q s
\]

\[ \leq \left| q^{k_2} D_q \mathcal{F}(\kappa_1) \right| \left| \int_{0}^{\frac{1}{[6]_q}} s - \frac{1}{[6]_q} \right| d_q s + \left| q^{k_2} D_q \mathcal{F}(\kappa_2) \right| \left| \int_{\frac{1}{[6]_q}}^{1} (1 - s) \right| d_q s - \frac{1}{[6]_q} \left| d_q s. \right|
\]

Now, if we apply the concept of Lemma 1 for $\kappa_1 = 0$ to the above quantum integrals, we attain

\[
\int_{0}^{\frac{1}{[6]_q}} s - \frac{1}{[6]_q} \right| d_q s = \int_{0}^{\frac{1}{[6]_q}} s \left( \frac{1}{[6]_q} - s \right) d_q s + \int_{\frac{1}{[6]_q}}^{1} s \left( s - \frac{1}{[6]_q} \right) d_q s
\]

\[ = 2 \int_{0}^{\frac{1}{[6]_q}} s \left( \frac{1}{[6]_q} - s \right) d_q s + \int_{\frac{1}{[6]_q}}^{1} s \left( s - \frac{1}{[6]_q} \right) d_q s
\]
we have

\[ \frac{2q^2[2]^2 + [6]^2([6] - [3])}{[2]^{3}[3][6]} \]

and

\[
\int_0^{\frac{1}{[6]}} (1 - s) \left| s - \frac{1}{[6]} \right| d_q s
\]

\[ = \int_0^{\frac{1}{[6]}} (1 - s) \left( \frac{1}{[6]} - s \right) d_q s + \int_0^{\frac{1}{[6]}} (1 - s) \left( s - \frac{1}{[6]} \right) d_q s
\]

\[ = 2 \int_0^{\frac{1}{[6]}} (1 - s) \left( \frac{1}{[6]} - s \right) d_q s + \int_0^{\frac{1}{[6]}} (1 - s) \left( s - \frac{1}{[6]} \right) d_q s
\]

\[ = 2 \frac{q[3][6] - q^2}{[2][3][6]} + \frac{1}{[2]} \left( q + q^2 - q^2 + 2q \right). \]

Thus, we obtain

\[
\int_0^{\frac{1}{[6]}} \left| s - \frac{1}{[6]} \right| \left| \kappa_2 D_q F(s \kappa_1 + (1 - s) \kappa_2) \right| d_q s
\]

\[ \leq \left| \kappa_2 D_q F(\kappa_1) \right| \left( \frac{2q^2[2]^2 + [6]^2([6] - [3])}{[2]^{3}[3][6]} \right) \]

\[ + \left| \kappa_2 D_q F(\kappa_2) \right| \left( 2 \frac{q[3][6] - q^2}{[2][3][6]} + \frac{1}{[2]} \left( q + q^2 - q^2 + 2q \right) \right). \]

Similarly, using Equation (2) in addition to the convexity of \( |\kappa_2 D_q F | \) and Lemma 1, we have

\[
\int_0^{1} \left| s - \frac{[5]}{[6]} \right| \left| \kappa_2 D_q F(s \kappa_1 + (1 - s) \kappa_2) \right| d_q s
\]

\[ \leq \left| \kappa_2 D_q F(\kappa_1) \right| \left( \frac{2q^2[5]^3 + [6]^3\left( 1 + [2]^2 \right) - [3][6][5]_q \left( 1 + [2]^3 \right)}{[2]^{3}[3][6]} \right) \]

\[ + \left| \kappa_2 D_q F(\kappa_2) \right| \left( 2 \frac{q[5][6][3] - q^2[5]^3}{[2][3][6]} + \frac{q^2}{[2][3]} - \frac{q[5]}{[2][6]} \right) \]

\[ - \frac{1}{[2]} \left[ \frac{[5][2q + q^2]}{[6]} - q + q^2 \right]. \]

By putting (24) and (25) into (23), we attain Inequality (22), which finishes the proof. \( \square \)

**Corollary 1.** Under the assumptions of Theorem 4 with \( q \to 1^- \), we have the following inequality of Simpson’s type for the function whose modulus values of first derivative are convex (see [2]):

\[ \frac{1}{[5]} \left[ F(\kappa_1) + 4F \left( \frac{\kappa_1 + \kappa_2}{2} \right) + F(\kappa_2) \right] - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(s)ds \]

\[ \leq \frac{5(\kappa_2 - \kappa_1)}{72} \left[ \left| F'(\kappa_1) \right| + \left| F'(\kappa_2) \right| \right]. \]
Now, we observe how the inequalities come out when we use the mappings whose powers of \(q\)-\(r\)-derivatives in absolute value are convex.

**Theorem 5.** Let \(\mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R}\) be a \(q\)-\(r\)-differentiable function on \(\kappa_1, \kappa_2\) such that \(q\)-\(r\)-Dq\(\mathcal{F}\) is continuous and integrable on \([\kappa_1, \kappa_2]\). If \(q\)-\(r\)-Dq\(\mathcal{F}\)^\(p_1\) is convex on \([\kappa_1, \kappa_2]\) for some \(p_1 > 1\), then we have following inequality for \(q\)-\(r\)-integrals:

\[
\left| \frac{1}{(\kappa_2 - \kappa_1) q^{2}} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(s) \, q^2 \, ds \right| - \frac{1}{6 q} \left[ \mathcal{F}(\kappa_1) + q^2 \mathcal{F} \left( \frac{\kappa_1 + q \kappa_2}{2 q} \right) + q \mathcal{F}(\kappa_2) \right] \leq q(\kappa_2 - \kappa_1) \left( \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \right) \times \\
\times \left( \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \right)
\]

where \(0 < q < 1\) and \(\frac{1}{p_1} + \frac{1}{r_1} = 1\).

**Proof.** Applying the well-known H"{o}lder's inequality for quantum integrals to the integrals in the right-hand side of (23), it is found that

\[
\left| \frac{1}{(\kappa_2 - \kappa_1) q^{2}} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(s) \, q^2 \, ds \right| - \frac{1}{6 q} \left[ \mathcal{F}(\kappa_1) + q^2 \mathcal{F} \left( \frac{\kappa_1 + q \kappa_2}{2 q} \right) + q \mathcal{F}(\kappa_2) \right] \leq q(\kappa_2 - \kappa_1) \left( \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \right) \times \\
\times \left( \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \right)
\]

By using the convexity of \(q\)-\(r\)-Dq\(\mathcal{F}\)^\(p_1\), we obtain

\[
\left| \frac{1}{(\kappa_2 - \kappa_1) q^{2}} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(s) \, q^2 \, ds \right| - \frac{1}{6 q} \left[ \mathcal{F}(\kappa_1) + q^2 \mathcal{F} \left( \frac{\kappa_1 + q \kappa_2}{2 q} \right) + q \mathcal{F}(\kappa_2) \right] \leq q(\kappa_2 - \kappa_1) \left( \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \right) \times \\
\times \left( \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \left( \int_{0}^{1} q^2 \, ds \right)^{\frac{1}{p_1}} \right)
\]
\[
q(k_2 - k_1) \left[ \left( \int_0^{1/2} s - \frac{1}{[6]_q} \right)^{r_1} d_q s \right]
\]

\[
\left( |\kappa_2 D_q F(k_1)|^{r_1} \int_0^{1/2} s d_q s + |\kappa_2 D_q F(k_2)|^{r_1} \int_0^{1/2} (1 - s) d_q s \right)^{\frac{1}{r_1}}
\]

\[
+ q(k_2 - k_1) \left[ \left( \int_0^{1/2} s - \frac{5}{[6]_q} \right)^{r_1} d_q s \right]
\]

\[
\left( |\kappa_2 D_q F(k_1)|^{r_1} \int_0^{1/2} s d_q s + |\kappa_2 D_q F(k_2)|^{r_1} \int_0^{1/2} (1 - s) d_q s \right)^{\frac{1}{r_1}}
\].

To calculate the integrals in the right-hand side of (27), if we first use Rule (1), then we obtain

\[
\int_0^{1/2} s - \frac{1}{[6]_q} \right)^{r_1} d_q s = \frac{1 - q}{[2]_q} \sum_{k_2=0}^{\infty} q^{k_2} \left| \frac{q^{k_2}}{2^{r_1}} - \frac{1}{[6]_q} \right|^{r_1}
\]

\[
\leq \frac{1 - q}{[2]_q} \sum_{k_2=0}^{\infty} q^{k_2} \left| \frac{1}{2^{r_1}} - \frac{1}{[6]_q} \right|^{r_1}
\]

\[
= \left( \frac{1}{[2]_q} - \frac{1}{[6]_q} \right)^{r_1} \frac{1}{[2]_q}
\]

\[
= q^{r_1} \frac{[4]_q^{r_1}}{[2]_q^{r_1+1} [6]_q^{r_1}}
\]

Similarly, we have

\[
\int_0^{1/2} s - \frac{5}{[6]_q} \right)^{r_1} d_q s \leq \frac{[2]_q^{r_1+1} [5]_q^{r_1} - q^{r_1} [4]_q^{r_1}}{[2]_q^{r_1+1} [6]_q^{r_1}}
\]

For the other integrals in the right-hand side of (27), using the case when \(k_1 = 0\) of Lemma 1, we find that

\[
\int_0^{1/2} s d_q s = \frac{1}{[2]_q^3}
\]

\[
\int_0^{1/2} (1 - s) d_q s = \frac{q^2 + 2q}{[2]_q^3}
\]

Similarly, we get

\[
\int_0^{1/2} s d_q s = \frac{q^2 + 2q}{[2]_q^3}
\]

\[
\int_0^{1/2} (1 - s) d_q s = \frac{q^3 + q^2 - q}{[2]_q^3}
\]

By substituting (28)–(33) into (27), we obtain the desired Inequality (26), which completes the proof. \(\square\)
Theorem 6. Let $F : [\kappa_1, \kappa_2] \to \mathbb{R}$ be a $q^{\kappa_2}$-differentiable function on $(\kappa_1, \kappa_2)$ such that $\kappa_2 D_q F$ is continuous and integrable on $[\kappa_1, \kappa_2]$. If $\kappa_2 D_q F |^{p_1}$ is convex on $[\kappa_1, \kappa_2]$ for some $p_1 \geq 1$, then we have following inequality for $q^{\kappa_2}$-integrals:

$$\left| \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F(s) \, q^{\kappa_2} d_q s - \frac{1}{[6]_q} \left[ F(\kappa_1) + q^2 [4]_q F \left( \frac{\kappa_1 + q \kappa_2}{[2]_q} \right) + q F(\kappa_2) \right] \right|$$

$$\leq q(\kappa_2 - \kappa_1) \left[ \left( \frac{2q}{[2]_q [6]_q^2} + \frac{q^2 [3]_q - q}{[6]_q [2]_q} \right)^{1 - \frac{1}{p_1}} \right. \times \left(A_1(q) |^{\kappa_2} D_q F(\kappa_1) |^{p_1} + B_1(q) |^{\kappa_2} D_q F(\kappa_2) |^{p_1} \right) \right.$$}

$$+ \left. \left( \frac{2q [5]_q^2}{[2]_q [6]_q^2} + \frac{1}{[2]_q} - \frac{[5]_q}{[6]_q} - \frac{[5]_q [2]_q^2 - [6]_q}{[6]_q [2]_q^2} \right)^{1 - \frac{1}{p_1}} \times \left(A_2(q) |^{\kappa_2} D_q F(\kappa_1) |^{p_1} + B_2(q) |^{\kappa_2} D_q F(\kappa_2) |^{p_1} \right) \right] \right)^{\frac{1}{p_1}},$$

where $0 < q < 1$, and $A_1(q), A_2(q), B_1(q), B_2(q)$ are given as in (12)–(15), respectively.

Proof. Utilizing from the results in the proof of Theorem 4 after applying the well-known Power mean inequality to the integrals in the right-hand side of (23), owing to the convexity of $|^{\kappa_2} D_q F |^{p_1}$, we find that

$$\left| \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F(s) \, q^{\kappa_2} d_q s - \frac{1}{[6]_q} \left[ F(\kappa_1) + q^2 [4]_q F \left( \frac{\kappa_1 + q \kappa_2}{[2]_q} \right) + q F(\kappa_2) \right] \right|$$

$$\leq q(\kappa_2 - \kappa_1) \left[ \left( \int_{0}^{\frac{1}{q^2}} \left| s - \frac{1}{[6]_q} \right| d_q s \right)^{1 - \frac{1}{p_1}} \times \left( |^{\kappa_2} D_q F(\kappa_1) |^{p_1} \int_{0}^{\frac{1}{q^2}} \left| s - \frac{1}{[6]_q} \right| d_q s \right) \right.$$}

$$+ \left. |^{\kappa_2} D_q F(\kappa_2) |^{p_1} \int_{0}^{\frac{1}{q^2}} \left| s - \frac{1}{[6]_q} \right| d_q s \right)^{\frac{1}{p_1}} \times \left(A_1(q) |^{\kappa_2} D_q F(\kappa_1) |^{p_1} + B_1(q) |^{\kappa_2} D_q F(\kappa_2) |^{p_1} \right) \right.$$}

$$\leq q(\kappa_2 - \kappa_1) \left[ \left( \int_{0}^{\frac{1}{q^2}} \left| s - \frac{1}{[6]_q} \right| d_q s \right)^{1 - \frac{1}{p_1}} \times \left(A_1(q) |^{\kappa_2} D_q F(\kappa_1) |^{p_1} + B_1(q) |^{\kappa_2} D_q F(\kappa_2) |^{p_1} \right) \right.$$}
\[ q (\kappa_2 - \kappa_1) \left( \int_0^1 \left| s - \left[ \frac{5}{6} \right] \right| dq_s \right)^{1 - \frac{1}{p_1}} \times \left( A_2(q) \left| \kappa_2 D_q F(\kappa_1) \right|^{p_1} + B_2(q) \left| \kappa_2 D_q F(\kappa_2) \right|^{p_1} \right)^{\frac{1}{p_1}}. \] \]

We also observe that
\[ \int_0^1 \left| s - \frac{1}{[6]} \right| dq_s = 2 \int_0^{\frac{1}{[6]}} \left( \frac{1}{[6]} - s \right) dq_s + \int_0^{\frac{1}{[6]}} \left( s - \frac{1}{[6]} \right) dq_s = \frac{2q}{[2][6]^2} + q^3[3] - q, \]
and by using similar operations, we have
\[ \int_0^1 \left| s - \frac{[5]}{[6]} \right| dq_s = 2q \frac{[5]^2}{[2][6]^2} + \frac{1}{[2]} - \frac{[5]}{[6]} - \frac{[5][2]^2}{[6][2]^3}. \]

By substituting (36) and (37) into (35), we attain the required inequality (34). Hence, the proof is completed. \( \square \)

**Corollary 2.** Under the given conditions in Theorem 6 with \( q \to 1^- \), we have the following inequality given by Alomari et al. (see [2]):
\[ \left| \frac{1}{6} \left[ F(\kappa_1) + 4F \left( \frac{\kappa_1 + \kappa_2}{2} \right) + F(\kappa_2) \right] - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(s) ds \right| \leq \frac{1}{1296} \left( \frac{5}{72} \right)^{1 - \frac{1}{p_1}} \left( \kappa_2 - \kappa_1 \right) \times \left[ 61 \left| F'(\kappa_1) \right|^{p_1} + 29 \left| F'(\kappa_2) \right|^{p_1} \right]^{\frac{1}{p_1}} + \left[ 29 \left| F'(\kappa_1) \right|^{p_1} + 61 \left| F'(\kappa_2) \right|^{p_1} \right]^{\frac{1}{p_1}}. \]

4.2. Newton’s Inequalities for \( q^2 \)-Quantum Integral

In this subsection, we establish some new Newton’s type inequalities by using Lemma 3.

In the next theorems, we will present new quantum estimates by using the convexity of \( q^2 \)-derivatives in modulus and Lemma 3.

**Theorem 7.** Let \( F : [\kappa_1, \kappa_2] \to \mathbb{R} \) be a \( q^2 \)-differentiable function on \( (\kappa_1, \kappa_2) \) such that \( \kappa_2 D_q F \) is continuous and integrable on \([\kappa_1, \kappa_2]\). If \( \left| \kappa_2 D_q F \right| \) is convex on \([\kappa_1, \kappa_2]\), then we have the following inequality for \( q^2 \)-integrals:
\[ \left| \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F(s) dq_s - \frac{1}{[8]} \left[ F(\kappa_1) + \frac{q^3[6]}{[2]} F \left( \kappa_1 + \frac{q^2[2]}{[3]} \kappa_2 \right) \right] \right| \]
\[ + \left| \frac{q^2[6]}{[2]} F \left( \frac{[2] \kappa_1 + q^2[3] \kappa_2}{[3]} \right) + qF(\kappa_2) \right| \]
\[ \leq q(\kappa_2 - \kappa_1) \left\{ \left| \kappa_2 D_q F(\kappa_1) \right| \left[ A_3(q) + A_4(q) + A_5(q) \right] + \left| \kappa_2 D_q F(\kappa_2) \right| \left[ B_3(q) + B_4(q) + B_5(q) \right] \right\}, \]

where \( 0 < q < 1 \), and \( A_3(q), A_4(q), A_5(q), B_3(q), B_4(q), B_5(q) \) are given as in (16)-(21), respectively.
Proof. On taking the modulus in Lemma 3, we gain

\[
\left| \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(s) \ k_2 s d s - \frac{1}{[8]_q} \left[ \mathcal{F}(\kappa_1) + \frac{q^3[6]_q}{[2]_q} \mathcal{F}\left(\frac{\kappa_1 + q[2]_q \kappa_2}{[3]_q}\right)\right] \right| \\
+ \frac{q^2[6]_q}{[2]_q} \mathcal{F}\left(\frac{[2]_q \kappa_1 + q^2 \kappa_2}{[3]_q}\right) + q \mathcal{F}(\kappa_2) \leq \left| q(\kappa_2 - \kappa_1) \int_{\kappa_1}^{\kappa_2} \left| \mathcal{F}'(s) \right| ds \right|
\]

\[
\leq q(\kappa_2 - \kappa_1) \left[ \mathcal{F}(\kappa_1) + \left| \mathcal{F}(\kappa_2) \right| \right],
\]

which is given by Noor et al. in [5].

Remark 1. Under the given conditions of Theorem 7 with the limit \( q \to 1^- \), we have following inequality:

\[
\frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(s) ds - \frac{1}{8} \left[ \mathcal{F}(\kappa_1) + 3 \mathcal{F}\left(\frac{2 \kappa_1 + \kappa_2}{3}\right) + 3 \mathcal{F}\left(\frac{\kappa_1 + 2 \kappa_2}{3}\right) + \mathcal{F}(\kappa_2) \right] \leq \frac{25}{576} (\kappa_2 - \kappa_1) \left[ \mathcal{F}'(\kappa_1) + \left| \mathcal{F}'(\kappa_2) \right| \right],
\]

Theorem 8. Let \( \mathcal{F} : [\kappa_1, \kappa_2] \to \mathbb{R} \) be a \( q^2 \)-differentiable function on \( [\kappa_1, \kappa_2] \) such that \( q^2 D_q \mathcal{F} \) is continuous and integrable on \( [\kappa_1, \kappa_2] \). If \( \left| q^2 D_q \mathcal{F} \right|^{p_1} \) is convex on \( [\kappa_1, \kappa_2] \), for some \( p_1 > 1 \), then we have the following inequality for \( q^2 \)-integrals:

\[
\left| \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} \mathcal{F}(s) \ k_2 s d s - \frac{1}{[8]_q} \left[ \mathcal{F}(\kappa_1) + \frac{q^3[6]_q}{[2]_q} \mathcal{F}\left(\frac{\kappa_1 + q[2]_q \kappa_2}{[3]_q}\right)\right] \right| \\
+ \frac{q^2[6]_q}{[2]_q} \mathcal{F}\left(\frac{[2]_q \kappa_1 + q^2 \kappa_2}{[3]_q}\right) + q \mathcal{F}(\kappa_2) \leq q(\kappa_2 - \kappa_1) \left[ \mathcal{F}(\kappa_1) + \left| \mathcal{F}(\kappa_2) \right| \right] \\
\times \left( \frac{1}{[3]_q[2]_q} \left| k_2 D_q \mathcal{F}(\kappa_1) \right|^{p_1} + \frac{[3]_q[2]_q - 1}{[3]_q[2]_q} \left| \mathcal{F}(\kappa_2) \right|^{p_1} \right)^{\frac{1}{p_1}} \\
+ \left( \frac{q^{1[2]_q} - q^{2[2]_q}}{[3]_q[2]_q} \right)^{\frac{1}{p_1}} \\
\times \left( q^2 + \frac{2}{[3]_q[2]_q} \left| k_2 D_q \mathcal{F}(\kappa_1) \right|^{p_1} + \frac{q[3]_q[2]_q - (q^2 + 2q)}{[3]_q[2]_q} \left| \mathcal{F}(\kappa_2) \right|^{p_1} \right)^{\frac{1}{p_1}}
\]
Theorem 9. Let \( F : [\kappa_1, \kappa_2] \to \mathbb{R} \) be a \( q^{2s} \)-differentiable function on \([\kappa_1, \kappa_2]\) such that \( s^2D_q F \) is continuous and integrable on \([\kappa_1, \kappa_2]\). If \( |s^2D_q F|^{p_1} \) is convex on \([\kappa_1, \kappa_2]\) for some \( p_1 \geq 1 \), then we have the following inequality for \( q^{2s}\)-integrals:

\[
\left| \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F(s) \, s^2 \, dq_s - \frac{1}{[q]_{q^2}} \left[ F(\kappa_1) + q^2[6]_{q^2} F \left( \frac{\kappa_1 + q[2]_{q^2}}{[3]_{q^2}} \right) \right] \right| \leq q(\kappa_2 - \kappa_1) \left( \int_{[q]_{q^2}}^{[2]_{q^2}} s - \frac{1}{[q]_{q^2}} \, dq_s \right)^{\frac{1}{p_1}} \left( \int_{[q]_{q^2}}^{[2]_{q^2}} |s^2D_q F(\text{sk}_1 + (1-s)\kappa_2)|^{p_1} \, dq_s \right)^{\frac{1}{p_1}} + q(\kappa_2 - \kappa_1) \left( \int_{[q]_{q^2}}^{[2]_{q^2}} s - \frac{1}{[q]_{q^2}} \, dq_s \right)^{\frac{1}{p_1}} \left( \int_{[q]_{q^2}}^{[2]_{q^2}} |s^2D_q F(\text{sk}_1 + (1-s)\kappa_2)|^{p_1} \, dq_s \right)^{\frac{1}{p_1}}.
\]

Proof. If we apply Hölder’s inequality to the expressions in the right-hand side of (39), then we obtain

\[
\left| \frac{1}{(\kappa_2 - \kappa_1)} \int_{\kappa_1}^{\kappa_2} F(s) \, s^2 \, dq_s - \frac{1}{[q]_{q^2}} \left[ F(\kappa_1) + q^2[6]_{q^2} F \left( \frac{\kappa_1 + q[2]_{q^2}}{[3]_{q^2}} \right) \right] \right| \leq q(\kappa_2 - \kappa_1) \left( \int_{[q]_{q^2}}^{[2]_{q^2}} s - \frac{1}{[q]_{q^2}} \, dq_s \right)^{\frac{1}{p_1}} \left( \int_{[q]_{q^2}}^{[2]_{q^2}} |s^2D_q F(\text{sk}_1 + (1-s)\kappa_2)|^{p_1} \, dq_s \right)^{\frac{1}{p_1}} + q(\kappa_2 - \kappa_1) \left( \int_{[q]_{q^2}}^{[2]_{q^2}} s - \frac{1}{[q]_{q^2}} \, dq_s \right)^{\frac{1}{p_1}} \left( \int_{[q]_{q^2}}^{[2]_{q^2}} |s^2D_q F(\text{sk}_1 + (1-s)\kappa_2)|^{p_1} \, dq_s \right)^{\frac{1}{p_1}}.
\]

For the rest of this proof, if the same procedure that was used in the proof of Theorem 5 is applied to the above inequality, then Result (40) can be captured. \( \square \)

Theorem 9.
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\[
\begin{align*}
&+ \left(2 \frac{q [7]_q^2}{[8]_q [2]_q} + \frac{[3]_q^2 + [2]_q^2}{[8]_q [3]_q} - \frac{[7]_q (3 + 2)_{[q]}}{[8]_q [3]_q} \right)^{1 - \frac{1}{p_1}} \\
&\times \left( A_5(q) \left| \frac{2}{k_2} D_{k_1} \mathcal{F}(k_1) \right|^{p_1} + B_5(q) \left| \frac{2}{k_2} D_{k_1} \mathcal{F}(k_2) \right|^{p_1} \right)^{\frac{1}{p_1}},
\end{align*}
\]

where \( 0 < q < 1 \), and \( A_5(q), A_4(q), A_5(q), B_3(q), B_4(q), B_5(q) \) are given as in (16)–(21), respectively.

**Proof.** If the strategy that was used in the proof of Theorem 6 is applied by taking into account Lemma 3, the desired Inequality (41) can be attained. \( \square \)

**Remark 2.** Under the conditions of Theorem 9 with the limit \( q \to 1^- \), we obtain the following inequality:

\[
\left| \frac{1}{k_2 - k_1} \int_{k_1}^{k_2} \mathcal{F}(s) ds \right| \leq \frac{k_2 - k_1}{36} \left\{ \left( \frac{17}{16} \right)^{1 - \frac{1}{2}} \left( \frac{251}{1152} \left| \mathcal{F}'(k_2) \right|^{p_1} + \frac{973}{1152} \left| \mathcal{F}'(k_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \\
+ \left( \frac{1}{2} \left| \mathcal{F}'(k_2) \right|^{p_1} + \frac{1}{2} \left| \mathcal{F}'(k_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \\
+ \left( \frac{17}{16} \right)^{1 - \frac{1}{2}} \left( \frac{251}{1152} \left| \mathcal{F}'(k_2) \right|^{p_1} + \frac{973}{1152} \left| \mathcal{F}'(k_1) \right|^{p_1} \right)^{\frac{1}{p_1}} \right\},
\]

which can be found in [5].

5. Conclusions

We conclude our work by stating that, using the concepts of quantum derivative and quantum integral, we proved some new Simpson’s and Newton’s type quantum integral inequalities for quantum differentiable convex functions, and these inequalities can be helpful for finding the bounds of Simpson’s and Newton’s formulas for numerical integration. It is important to note that by considering the limit \( q \to 1^- \) in our key results, our results were transformed into some new and well-known results. We believe it is an interesting and creative problem for upcoming researchers who will be able to obtain similar inequalities for various types of convexity and co-ordinated convexity in their future work.
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