Abstract

Equip each point $x$ of a homogeneous Poisson process $P$ on $\mathbb{R}$ with $D_x$ edge stubs, where the $D_x$ are i.i.d. positive integer-valued random variables with distribution given by $\mu$. Following the stable multi-matching scheme introduced by Deijfen, Häggström and Holroyd [1], we pair off edge stubs in a series of rounds to form the edge set of an infinite component $G$ on the vertex set $P$. In this note, we answer questions of Deijfen, Holroyd and Peres [2] and Deijfen, Häggström and Holroyd [1] on percolation (the existence of an infinite connected component) in $G$. We prove that percolation may occur a.s. even if $\mu$ has support over odd integers. Furthermore, we show that for any $\varepsilon > 0$ there exists a distribution $\mu$ such that $\mu({1}) > 1 - \varepsilon$ such that percolation still occurs a.s..

Keywords: Poisson process, random graph, matching, percolation.

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1 Introduction

In this paper, we study certain matching processes on the real line. Let $D$ be a random variable with distribution $\mu$ supported on the positive integers. Generate a set of vertices $P$ by a Poisson point process of intensity 1 on $\mathbb{R}$. Equip each vertex $x \in P$ with a random number $D_x$ of edge stubs, where the $(D_x)_{x \in P}$ are i.i.d. random variables with distribution given by $D$. Now form edges in rounds by matching edge stubs in the following manner. In each round, say that two vertices $x, y$ are compatible if they are not already joined by an edge and both $x$ and $y$ still possess some unmatched edge stubs.
Two such vertices form a \textit{mutually closest compatible pair} if \( x \) is the nearest \( y \)-compatible vertex to \( y \) in the usual Euclidean distance and vice-versa. For each such mutually closest compatible pair \((x, y)\), remove an edge stub from each of \( x \) and \( y \) to form the edge \( xy \). Repeat the procedure indefinitely.

This matching scheme, known as \textit{stable multi-matching}, was introduced by Deijfen, Häggström and Holroyd \cite{Deijfen2007}, who showed that it almost surely exhausts the set of edge stubs, yielding an infinite graph \( G = G(\mu) \) with degree distribution given by \( \mu \).

A natural question to ask is which degree distributions \( \mu \) (if any) yield an infinite connected component in \( G \). For example if \( \mu(\{1\}) = 1 \), then no such component exists, while if \( \mu(\{2\}) = 1 \), Deijfen, Holroyd and Peres \cite{Deijfen2008} suggest that percolation (the existence of an infinite component) occurs almost surely. Note that by (a version of) Kolmogorov’s zero–one law, the probability of percolation occurring is zero or one. Also, as shown by Deijfen, Holroyd and Peres (see \cite{Deijfen2008}, Proposition 1.1), an infinite component in \( G \), if it exists, is almost surely unique.

Taking the Poisson point process in \( \mathbb{R}^d \) for some \( d \geq 1 \) and applying the stable multi-matching scheme mutatis mutandis, we obtain the \( d \)-dimensional Poisson graph \( G_d \). Deijfen, Häggström and Holroyd proved the following result on percolation in \( G_d \):

\begin{theorem} \textbf{(Deijfen, Häggström and Holroyd \cite{Deijfen2007} Theorem 1.2)} \end{theorem}

\( (i) \) For all \( d \geq 2 \) there exists \( k = k(d) \) such that if \( \mu(\{n \in \mathbb{N} : n \geq k\}) = 1 \), then almost surely \( G_d \) percolates.

\( (ii) \) For all \( d \geq 1 \), if \( \mu(\{1, 2\}) = 1 \) and \( \mu(\{1\}) > 0 \), then almost surely \( G_d \) does not percolate.

Their proof of part \( (i) \) of Theorem 1.1 relies on a comparison of the \( d \)-dimensional stable multi-matching process with dependent site percolation on \( \mathbb{Z}^d \). In particular, since the threshold for percolation in \( \mathbb{Z} \) is trivial, their argument cannot say anything about percolation in the 1-dimensional Poisson graph \( G = G_1 \).

Related to part \( (ii) \) of Theorem 1.1, Deijfen, Häggström and Holroyd asked the following question.

\begin{question} \textbf{(Deijfen, Häggström and Holroyd).} Does there exist some \( \varepsilon > 0 \) such that if \( \mu(\{1\}) > 1 - \varepsilon \), then almost surely \( G_d \) contains no infinite component? \end{question}

In subsequent work on \( G = G_1 \), Deijfen, Holroyd and Peres \cite{Deijfen2008} observed that simulations suggested percolation might not occur when \( \mu(\{3\}) = 1 \), and asked whether the presence of odd degrees kills off infinite components in general.

\begin{question} \textbf{(Deijfen, Holroyd and Peres).} Is it true that percolation in \( G = G_1 \) occurs almost surely, if and only if, \( \mu \) has support only on the even integers? \end{question}
In this paper we prove the following theorem, answering Question 2 negatively:

**Theorem 1.2.** There exist degree distributions $\mu$ with support on the odd integers, such that the stable multi-matching process a.s. yields an infinite component.

Furthermore, we show in Remark 3.1 that for any $\varepsilon > 0$ we can construct such a degree distribution $\mu$ with $\mu(\{1\}) > 1 - \varepsilon$, thus also answering Question 1 negatively. We note however that the distribution $\mu$ we construct has unbounded support; it would be interesting to find an example with bounded support only.

### 2 Proof of Theorem 1.2

The idea behind the proof of Theorem 1.2 is to set $\mu(\{d_i\}) = 1/2^i$ for a sharply increasing sequence of integers $(d_i)_{i \in \mathbb{N}}$. Suppose we are given a vertex $x_i$ with degree $D_{x_i} = d_i$. By choosing $d_i$ large enough we can ensure that with probability close to 1, there exists some vertex $x_{i+1}$ with $D_{x_{i+1}} = d_{i+1}$ that is connected to $x_i$ by an edge of $G$. Let $E_i, i \geq 1$, be the event that a given vertex $x_i$ of degree $d_i$ is connected to some vertex $x_{i+1}$ of degree $d_{i+1}$.

Starting from a vertex $x_1$ of degree $d_1$, we see that if $\bigcap_{i=1}^{\infty} E_i$ occurs, then there is an infinite path $x_1x_2x_3\ldots$ in $G$. If the events $(E_i)_{i \in \mathbb{N}}$ were independent of each other, then $\mathbb{P}(\bigcap_{i=1}^{\infty} E_i) = \prod_{i \in \mathbb{N}} \mathbb{P}(E_i)$, which we could make strictly positive by letting the sequence $(d_i)_{i \in \mathbb{N}}$ grow sufficiently quickly, ensuring in turn that percolation occurs a.s.. Of course the events $(E_i)_{i \in \mathbb{N}}$ as we have loosely defined them above are highly dependent. We circumvent this problem by working with a sequence of slightly more restricted events, for which we do have full independence. As we have no restrictions on the $d_i$ other than their growth rate, each of them can be chosen to be odd or even as we please.

Before we begin the proof, let us introduce the following notation. Given $x \in \mathcal{P}$, let $B(x,r)$ be the collection of all vertices in $\mathcal{P}$ within distance at most $r$ of $x$. We say that a pair of vertices $(x,y)$ with degrees $(D_x, D_y)$ is **strongly connected** if $|B(x,|y-x|)| \leq D_x$ and $|B(y,|y-x|)| \leq D_y$. Observe that if a pair of vertices $(x,y)$ is strongly connected, then there will a.s. be an edge between $x$ and $y$ in the stable multi-matching scheme.

**Proof of Theorem 1.2.** Set $d_i = 10 \cdot 4^i$ and $\mu(\{d_i\}) = \frac{1}{2^i}$ for each $i \in \mathbb{N}$. Suppose that we condition on a particular vertex $x_i$ of degree $d_i$ belonging to the point process $\mathcal{P}$. (Formally, we consider a Palm version of $\mathcal{P}$, i.e. a version of $\mathcal{P}$ conditioned on $x_1 \in \mathcal{P}$, with the rest of the process taken as a stationary background. Note that the Palm version of a homogeneous Poisson process has the same distribution as the unconditioned process itself with an added point; see [4, Chapter 11].) We condition further on there
Selecting $i_F$ together with $B$ yields that $P(C(x))$ is at distance at least 0.

Let $A_i(x_i)$ be the event that there is a vertex $x_{i+1} \in P$ with degree $d_{i+1}$ such that $0.1d_i < |x_{i+1} - x_i| < 0.2d_i$. Viewing $P$ as the union of two thinned Poisson point processes, one of intensity $2^{-(i+1)}$ giving us the vertices of degree $d_{i+1}$ and another of intensity $1 - 2^{-(i+1)}$ giving us the rest of the vertices, we see that $P((A_i(x_i))^c) = e^{-\frac{0.2d_i}{2^{i+1}}} = e^{-2^i}$. If $A_i(x_i)$ occurs, let $x_{i+1}$ denote the a.s. unique vertex of degree $d_{i+1}$ which is nearest to $x_i$ among those degree $d_{i+1}$ vertices lying at distance at least 0.1$d_i$ from $x_i$.

Let $B_i(x_i)$ be the event that there are at most 0.3$d_i$ vertices $x \in P$ with $0.1d_i < |x - x_i| < 0.2d_i$. Furthermore, given a point $y$ (not necessarily belonging to our process $P$) with $0.1d_i < |y - x_i| < 0.2d_i$, we let $C_i(x_i, y)$ be the event that there are at most 0.6$d_i$ vertices $x$ lying at distance at least 0.2$d_i$ from $x_i$ and at most 0.4$d_i$ from $y$. A quick calculation (using the Chernoff bound) yields that $P(B_i(x_i))^c = e^{-3(\log \frac{2}{5} - \frac{1}{2})4^i + O(i)}$ and $P(C_i(x_i, y))^c \leq e^{-6(\log \frac{2}{5} - \frac{1}{2})4^i + O(i)}$ respectively.

Finally let $D_i(x_i) = A_i(x_i) \cap B_i(x_i) \cap C_i(x_i, x_{i+1})$. If $D_i(x_i)$ occurs, then $x_i$ and $x_{i+1}$ are strongly connected, since our initial assumption $F_i(x_i)$ together with $B_i(x_i)$ tells us that $|B(x_i, |x_i - x_{i+1}|) \leq 0.6d_i$, while $F_i(x_i)$ together with $B_i(x_i) \cap C_i(x_i, x_{i+1})$ yield that $|B(x_{i+1}, 0.4d_i)| \leq 1.2d_i = 0.3d_{i+1}$ (see Figure 1). This last statement is exactly our initial conditioning $F_i(x_i)$ with $i$ replaced by $i + 1$; hence $D_i(x_i) \cap F_i(x_i) \subseteq F_{i+1}(x_{i+1})$.

By the union bound, we have

$$P(D_i(x_i) | F_i(x_i)) \geq 1 - P((A_i(x_i))^c | F_i(x_i)) - P((B_i(x_i))^c | F_i(x_i))$$

$$- \sup_{y : |y - x_i| \in (0.1d_i, 0.2d_i)} P((C_i(x_i, y))^c | F_i(x_i))$$

$$> 1 - e^{-2^i(1 + o(1))}.$$

Selecting $i_0$ sufficiently large and some arbitrary vertex $x_{i_0}$ of degree $d_{i_0}$ as a
starting point, we may define events $D_{i_0}(x_{i_0}), D_{i_0+1}(x_{i_0+1}), \ldots$ inductively, each conditional on its predecessors, with
\[
\mathbb{P}\left(\bigcap_{i \geq i_0} D_i(x_i) | F_{i_0}(x_{i_0})\right) = \prod_{i \geq i_0} \mathbb{P}(D_i(x_i) | \bigcap_{j<i} D_j(x_j) \cap F_{i_0}(x_{i_0})) > 1 - 2 \sum_{i \geq i_0} e^{-2^i} > 0.
\]
From any vertex $x_{i_0} \in \mathcal{P}$ there is, with strictly positive probability, an infinite path in $G$, $x_{i_0}, x_{i_0+1}, \ldots$ through vertices of increasing degrees $d_{i_0}, d_{i_0+1}, \ldots$. It follows that $G$ a.s. contains a strongly connected infinite component.

Remark 2.1. The pairs $(x_{i_0}, x_{i_0+1}), (x_{i_0+1}, x_{i_0+2}), \ldots$ remain strongly connected if we increase the degrees. Thus, if a given degree distribution $\mu$ a.s. results in a strongly connected infinite component in $G(\mu)$, then any degree distribution $\mu'$ that stochastically dominates $\mu$ will also a.s. yield a strongly connected infinite component in $G(\mu')$.

Thus, if we set $d'_i = d_i + 1$ and $\mu'(d'_i) = 2^{-i}$, then the associated Poisson graph $G = G(\mu')$ a.s. percolates, though all vertices have odd degrees.

3 Concluding remarks

Remark 3.1. Note that our proof of Theorem 1.2 does not use any information about $d_i$ for $i < i_0$. In particular, we could set $\mu(\{1\}) = 1 - 2^{-i_0+1}$ and $\mu(\{d_i\}) = 2^{-i}$ for $i \geq i_0$ and still have a distribution for which $G$ percolates a.s.. Choosing $i_0$ sufficiently large, this implies a negative answer to Question 1.

Remark 3.2. The existence of degree distributions that a.s. result in an infinite component in dimensions $d ≥ 2$ was established in [1, Theorem 1.2 a)]. Our proof of Theorem 1.2 for $G = G_1(\mu)$ easily adapts to higher dimensions $d ≥ 2$ (with $d$-dimensional balls and annuli replacing intervals and punctured intervals, and the sequence $(d_i)_{i \in \mathbb{N}}$ being scaled accordingly), giving a different approach to the construction of examples in that setting.

The distribution $\mu$ we construct in Theorem 1.2 has unbounded support, and the expected degree of a vertex in $G(\mu)$ is infinite. We believe however that the answer to Questions 1 and 2 should still remain negative if $\mu$ is required to have bounded support. Indeed we conjecture the following:

Conjecture 3.3. For every $\varepsilon > 0$, there exists $k = k(\varepsilon)$ such that if $\mu(\{n \in \mathbb{N} : n \geq k\}) > \varepsilon$, then percolation occurs a.s. in $G = G_1(\mu)$.

One might expect that there is a critical value $d_*$ of the expected degree for percolation. We believe however that no such critical value exists:
**Conjecture 3.4.** There is no critical value $d_\star$ such that if $\mathbb{E}(D) < d_\star$ then a.s. percolation does not occur, while if $\mathbb{E}(D) > d_\star$, then a.s. percolation occurs in the stable multi-matching scheme.

Let us give some motivation for this conjecture. By [1, Theorem 1.2 b]), for any $\mu$ with support on $\{1, 2\}$ and $\mu(\{1\}) > 0$, $G_1(\mu)$ a.s. does not percolate. So any putative critical value must satisfy $d_\star \geq 2$. Now, pick $\varepsilon > 0$ and choose $\delta \gg d_\star$. Let $\mu$ be a degree distribution with support on $\{1, \delta\}$, such that the expected degree satisfies $\mathbb{E}(D) < d_\star - \varepsilon$. By definition of $d_\star$ this would imply $G(\mu)$ a.s. does not percolate. Assign degrees independently at random to the vertices of $G(\mu)$. Perform the first $\delta/2$ stages of the stable multi-matching process. By then most degree 1 vertices have been matched (and in fact matched to other degree 1 vertices). Now force the remaining degree 1 vertices to match to their future partners. Consider the vertices that had originally been assigned $\delta$ edge stubs. A number of these edge stubs will have been used up by the process so far, and the number of edge stubs left at each vertex is not independent; nevertheless we expect most degree $\delta$ vertices will have at least $\delta/4$ edge stubs left, and that the number of stubs left will be almost independently distributed. Thus, we believe the stable multi-matching scheme on the remaining edge stubs of the degree $\delta$ vertices will contain as a subgraph the edges of a stable multi-matching scheme on a thinned Poisson point process on $\mathbb{R}$ corresponding to the degree $\delta$ vertices, and with degrees given by some random variable $D'$ with $\mathbb{E}(D') > \delta/4 \gg d_\star$. Since rescaling does not affect the stable multi-matching process, this would imply $G(\mu)$ a.s. percolates (by definition of $d_\star$), a contradiction.

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