COMPATIBLE STRUCTURES ON UNARY BINARY NONSYMMETRIC OPERADS WITH QUADRATIC AND CUBIC RELATIONS

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Abstract. Various compatibility conditions among replicated copies of operations in a given algebraic structure have appeared in many contexts in recent years. Taking an uniform approach, this paper gives an operadic study of compatibility conditions for nonsymmetric operads with unary and binary operations, and homogeneous quadratic and cubic relations. This generalizes the previous studies for binary quadratic operads. We consider three compatibility conditions, namely the linear compatibility, matching compatibility and total compatibility, with increasingly strict restraints among the replicated copies. The linear compatibility is in Koszul dual to the total compatibility, while the matching compatibility is self dual. Further, each compatibility can be expressed in terms of either one or both of the two Manin square products.

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1. Introduction

This paper studies nonsymmetric operads encoding algebraic structures with replicated copies of operations satisfying various compatibility conditions among these copies. The relations of the compatibility conditions with Koszul duality and Manin products are established.

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1.1. Algebraic structures with replicated operations.

1.1.1. Linear compatibility of operations. An important property of derivations and its multi-dimensional generalizations such as tangent vectors is their closure under taking linear combinations, allowing them to form a vector space and further a Lie algebra. Such properties cannot be expected for other operators or operations. Thus it is natural to investigate the conditions under which linear combinations of multiple copies of a given operation still have the same properties of this operation. Such a condition is called a linear compatibility condition.

The notion of linear compatibility first appeared for the Lie bracket, arising from the pioneering work [30] of Magri in the study of the integrable Hamiltonian equation. There a bi-Hamiltonian system could be defined by a Poisson algebra with two linearly compatible Poisson (Lie) brackets. Such a structure was abstracted to the notion of a bihamiltonian algebra and studied in the context of operads and Koszul duality [12], and further applied to quadratic algebras in the sense of Manin and a conjecture of Feigin [3]. The Koszul property of the linearly compatible Lie operad was verified in [13] applying posets of weighted partitions. This approach was further extended in [9, 10] to the study of free linearly compatible Lie algebras with multiple brackets.

Linear compatibility is naturally related to linear or infinitesimal deformations, studying when a linear turbulence $\mu + a\nu$ of an operation $\mu$ is still the same kind of operation. In this direction, deformations of bihamiltonian algebras were developed in [27] and continued in [7] where a vanishing conjecture raised in [27] was proved applying spectral sequences.

1.1.2. Compatibilities of binary quadratic operads and nonsymmetric operads. Linear compatibility of the Lie operad was generalized to binary quadratic operads by Strohmayer [37] where it was also shown that linear compatibility is in Koszul dual to another naturally defined total compatibility, in the sense that the linear compatibility of an operad has its Koszul dual as the total compatibility of the dual operad.

In the remarkable work of Bruned, Hairer and Zambotti [5] on algebraic renormalization of regularity structures, another compatible condition for pre-Lie algebra with multiple operations emerged [15], called multiple pre-Lie algebras. As it turns out, its associative analogue with two multiplications was introduced in [45] as $A_{s}^{(2)}$ and further studied in [42] under the name of matching associative dialgebras. In [44], such matching conditions were systematically studied for Rota-Baxter algebras, dendriform algebras and pre-Lie algebras.

Linearly compatible associative algebras were studied in [34] for matrix algebras and especially for linear deformations. It was further showed in [35] that a pair of linear compatible associative products gives rise to a hierarchy of integrable systems of ODEs via the Lenard-Magri scheme [30]. Double constructions of linearly compatible associative algebras have been studied further in the direction of compatible associative bialgebras, associative Yang-Baxter equations and Goncharov’s path Hopf algebras [21, 33, 40].

In [6], quantum bi-Hamiltonian systems were built on linearly compatible associative algebras. In [11], linear compatible associative algebras were studied as $S_{n}$-modules and free objects were constructed in terms of rooted trees and grafting, and further related to the Hopf algebras of Connes-Kreimer, Grossman-Larson and Loday-Ronco. Homotopy linear compatible algebras were introduced and the homotopy transfer theorem were proved in [41].

Totally compatible associative algebras and Lie algebra with two multiplications were further studied in [43] in connection with tridendriform algebras and PostLie algebras.

1.1.3. Compatibilities of Rota-Baxter operators. More recently, compatible unary operations, that is, linear operators, have also been studied [44] under the name of matching Rota-Baxter operators. Such studies were motivated on the one hand by imposing to Rota-Baxter operators...
the linear closeness of the derivations noted at the very beginning and, on the other hand, by
the matching pre-Lie algebras arising from the work [5] on algebraic renormalization of regu-

larity structures. Furthermore, such structure underlies the algebraic study of Volterra integral
equations [18, 24].

From the viewpoint of deformation theory, while deformation theory for algebraic structures
with binary operations is quite general and complete, as deformation of binary (quadratic) oper-

ads [29], its study when the algebraic structure has nontrivial unary operations is experiencing
rapid developments quite recently. See [8, 26, 38] and the references therein.

1.2. Compatibility conditions of unary binary operads. Motivated by these recent develop-

ments of algebraic structures with compatible unary and binary operations that satisfy quadratic
or cubic relations, we give a systematic study of operads with compatible operations, generalizing
the existing treatments of binary quadratic operads in several directions. Thus our approach

(a) includes unary as well as binary operations,
(b) examines cubic as well as quadratic relations, and
(c) relates several compatibility conditions by Koszul duality and Manin products.

In order to give a systematic treatment, we will focus on nonsymmetric (ns) operads in this
paper and leave a detailed discussion of the other cases to a later study.

The main part of the paper is organized in three sections for three compatibility conditions.

In Section 2, we set up the stage of our study, on an unary binary quadratic/cubic ns operad
\( P \), and provide many examples. We then introduce the general structure of linearly compatible
algebras over such an operad, for a parameter set \( \Omega \). We then define the linearly compatible
operad \( P_{\Omega}^{\text{LC}} \) of \( P \) that encodes the linearly compatible algebras over \( P \) (Theorem 2.12). It is
shown that \( P_{\Omega}^{\text{LC}} \) is the Manin black square product of \( P \) with the linearly compatible operad
\( A_{\Omega}^{\text{LC}} \) of the associative algebra (Proposition 2.14).

In Section 3, the matching compatible operad \( P_{\Omega}^{\text{MT}} \) of \( P \) is introduced and its self duality for
the Koszul dual is proved (Theorem 3.8). Examples of self-dual operads with nontrivial unary
operations are provided. Further, \( P_{\Omega}^{\text{MT}} \) can be obtained from \( P \) by taking its Manin black square
product and white square product with \( A_{\Omega}^{\text{MT}} \), the matching associative operad (Proposition 3.11).

The notion of totally compatible operad \( P_{\Omega}^{\text{TC}} \) is introduced in Section 4 and its duality with
\( (P_{\Omega}^{\text{TC}})_d \) is proved (Theorem 4.5). It is further shown that \( P_{\Omega}^{\text{TC}} \) can be obtained from the Manin
white square product of \( P \) with the totally compatible associative operad \( A_{\Omega}^{\text{TC}} \) (Corollary 4.6).

See [16] for an operadic study of a related structure, called family algebraic structures.

Notation. Throughout this paper, we will work over a fixed field \( k \) of characteristic zero. It is
the base field for all vector spaces, algebras, tensor products, as well as linear maps.

2. Linear compatibility and the Manin black square product

In this section, we introduce the notion of unary binary quadratic/cubic ns operads and then
study for such operads the first of our compatibility conditions, that is, the linear compatibility
condition.

2.1. Unary binary quadratic/cubic ns operads. We give the notion and examples of unary
binary quadratic/cubic ns operads and refer the reader to [4, 29, 32] for further details on operads.

Definition 2.1. A nonsymmetric (ns) operad is an arity graded vector space \( P = \{ P_0, P_1, \ldots \} \)
equipped with an element id \( \in P_1 \) and composition maps:

\[
\gamma := \gamma_k^{r_1, \ldots, r_k} : P_k \otimes P_{r_1} \otimes \cdots \otimes P_{r_k} \longrightarrow P_{r_1 + \cdots + r_k}, \quad (\mu, v_1, \ldots, v_k) \mapsto \gamma(\mu; v_1, \ldots, v_k)
\]
which satisfy the following properties:

(a) (associativity)
\[
\gamma(\mu; \gamma(v_1; \omega_{1,1}, \ldots, \omega_{1,f_1}), \ldots, \gamma(v_k; \omega_{k,1}, \ldots, \omega_{k,f_k})) = \gamma(\gamma(\mu; v_1, \ldots, v_k); \omega_{1,1}, \ldots, \omega_{1,f_1}, \omega_{2,1}, \ldots, \omega_{k,1}, \ldots, \omega_{k,f_k}).
\]

(b) (unitality) \(\gamma(\text{id}; \mu) = \mu, \quad \gamma(\mu; \text{id}, \ldots, \text{id}) = \mu\).

An ns operad \(\mathcal{P}\) is called reduced if \(\mathcal{P}_0 = 0\). All operads are assumed to be reduced in this paper. A morphism of ns operads is a morphism of the arity graded vector spaces that is compatible with the compositions.

Recall that, for any vector space \(V\), the pair \(\text{End}_V := (\{\text{Hom}(V^\otimes n, V)\}_{n \geq 0}, \gamma)\) is an ns operad, with \(\gamma(f; g_1, \ldots, g_k)\) the usual composition of multivariate functions.

**Definition 2.2.** Let \(\mathcal{P}\) be an ns operad.

(a) A \(\mathcal{P}\)-algebra is a vector space \(V\) with a morphism of ns operads \(\mathcal{P} \xrightarrow{\rho} \text{End}_V\). We say that the operad \(\mathcal{P}\) encodes the \(\mathcal{P}\)-algebras.

(b) Let \((V, \rho_V)\) and \((W, \rho_W)\) be \(\mathcal{P}\)-algebras. A morphism of \(\mathcal{P}\)-algebras is a linear map \(f : V \to W\) such that
\[
\rho_W(\mu)(f(v_1), \ldots, f(v_n)) = \rho_W(\mu(f(v_1), \ldots, f(v_n)), \text{ for all } n \geq 0, \mu \in \mathcal{P}_n, v_i \in V.
\]

Let \(E = \{E_0, E_1, \ldots\}\) be an arity graded vector space. The free ns operad \(\mathcal{T}(E)\) on \(E\) can be constructed as follows. Let \(\mathcal{T}(E)_n\) be the vector space spanned by all decorated planar rooted trees with \(n\) leaves whose each internal (non-leaf) vertex \(v\) is decorated by an element of \(E_{|\text{in}(v)|}\), where \(\text{in}(v)\) is the set of inputs of the vertex \(v\) in the planar rooted tree. Consider the arity graded vector space
\[
\mathcal{T}(E) := \{\mathcal{T}(E)_0, \mathcal{T}(E)_1, \ldots\}.
\]
Define a composition product \(\gamma(t; t_1, \ldots, t_n)\) by grafting the root of \(t_i \in \mathcal{T}(E)_n\) to the \(i\)-th leaf of \(t \in \mathcal{T}(E)_k\) for \(1 \leq i \leq k\). This composition product \(\gamma\) makes \(\mathcal{T}(E)\) into an ns operad. Let \(i : E \to \mathcal{T}(E)\) denote the embedding map identifying the operation \(\mu \in E(n)\) with the \(n\)-th corolla decorated by \(\mu\).

**Lemma 2.3.** [29] Let \(E = \{E_0, E_1, \ldots\}\) be an arity graded vector space. Then \((\mathcal{T}(E), \gamma)\) together with the natural embedding \(i : E \to \mathcal{T}(E)\) is the free ns operad on \(E\).

Any ns operad \(\mathcal{P}\) can be presented as the quotient of a free ns operad modulo an operadic ideal:
\[
\mathcal{P} = \mathcal{P}(E, R) := \mathcal{T}(E)/\langle R \rangle,
\]
where \(E\) and \(R\) are called the generators and relations of \(\mathcal{P}\), respectively. \(\mathcal{P}\) is called finitely generated if \(E_n\) is finite dimensional for all \(n \geq 0\).

For any element \(t\) of \(\mathcal{T}(E)\), its weight is defined to the number of internal vertices of \(t\) as a decorated planar rooted tree. For example, for the following planar decorated rooted trees in \(\mathcal{T}(E)\),
\[
\begin{align*}
\begin{array}{c}
\bullet \\
\end{array},
\begin{array}{c}
\bullet \\
\bullet
\end{array},
\begin{array}{c}
\bullet \\
\bullet_d
\end{array},
\begin{array}{c}
\bullet_p \\
\bullet_p
\end{array},
\begin{array}{c}
\bullet_p \\
\bullet_p
\end{array},
\begin{array}{c}
\bullet_p \\
\bullet_p
\end{array},
\begin{array}{c}
\bullet_p \\
\bullet_p
\end{array},
\begin{array}{c}
\bullet_p \\
\bullet_p
\end{array},
\end{align*}
\]
the first one has weight 1, the second and third ones have weight 2 and the last three ones have weight 3. Denote by \(\mathcal{T}(E)^{(n)}\) the subset of elements of \(\mathcal{T}(E)\) of weight \(n\). In particular,
\[
\mathcal{T}(E)^{(0)} = \text{kid} \text{ and } \mathcal{T}(E)^{(1)} = E.
\]
In this paper, we focus on unary binary quadratic/cubic operads defined as follows.

**Definition 2.4.** Let $\mathcal{P} = \mathcal{T}(E)/(R)$ be an ns operad.

(a) We call $\mathcal{P}$ **unary binary** if $E = \{0, E_1, E_2, 0, \ldots, 0, \ldots\}$. If $E_1 = \text{kid}$, then the operad is called **binary**.

(b) A relation in $R$ is said to be **homogeneous** if it is in $\mathcal{T}(E)^{(k)}$ for some $k \geq 0$. In particular, it is called **quadratic** (resp. **cubic**) if it is in $\mathcal{T}(E)^{(2)}$ (resp. $\mathcal{T}(E)^{(3)}$). We call $\mathcal{P}$ **quadratic/cubic** if each relation in $R$ is either quadratic or cubic.

Note that a quadratic/cubic operad can have quadratic relations and cubic relations at the same time, but cannot have a quadratic term and a cubic term in the same relation. For example, the relation for a Rota-Baxter algebra (see Example 2.5.(c) and (d)) is quadratic/cubic when the weight is zero, but is not quadratic/cubic when the weight is nonzero.

For later applications, we give an explicit description of unary binary quadratic/cubic ns operads. For distinction, we will use suffix notion for arity of relations. More precisely, an $n$-ary relation will be called a relation in arity $n$.

A unary binary quadratic/cubic nc operad $\mathcal{P}$ can be presented by

(1) $\mathcal{P} = \mathcal{P}(E, R) = \mathcal{T}(E)/(R),$

for which

(a) $E = \{0, E_1, E_2, 0, \ldots, 0, \ldots\}$, where

(i) $E_1$ is spanned by

\[
\left\{ p_1, p_2, \ldots, p_t \right\},
\]

(ii) $E_2$ is spanned by

\[
\left\{ \ldots, \ldots \right\}.
\]

(b) $R := R_1 \sqcup R_2 \sqcup R_3 \sqcup R_4 := (R_{1,2} \sqcup R_{1,3}) \sqcup (R_{2,2} \sqcup R_{2,3}) \sqcup (R_{3,2} \sqcup R_{3,3}) \sqcup R_{4}$, where

(i) $R_{1,2}$ is the set of quadratic relations in arity one:

\[
R_{1,2} := \left\{ r_{1,2}^m(P_k, P_t) := \sum_{1 \leq k, t \leq i} \alpha_{k, t}^m p_k p_t \mid 1 \leq n \leq n_{1,2} \right\},
\]

(ii) $R_{1,3}$ is the set of cubic relations in arity one:

\[
R_{1,3} := \left\{ r_{1,3}^m(P_k, P_t, P_m) := \sum_{1 \leq k, t, m \leq i} \alpha_{k, t, m} p_k p_t p_m \mid 1 \leq n \leq n_{1,3} \right\},
\]

(iii) $R_{2,2}$ is the set of quadratic relations in arity two:

\[
R_{2,2} := \left\{ r_{2,2}^m(P_k, i) := \sum_{1 \leq k, i \leq j} \beta_{k, i}^m p_k i p_i + \gamma_{k, i}^m p_k i p_i + \delta_{k, i}^m i p_k p_i \mid 1 \leq n \leq n_{2,2} \right\},
\]

(iv) $R_{2,3}$ is the set of cubic relations in arity two:

\[
R_{2,3} := \left\{ r_{2,3}^m(P_k, P_t, i) := \sum_{1 \leq k, t, i \leq j} \beta_{k, t, i}^m p_k p_t i p_i + \gamma_{k, t, i}^m p_k p_t i p_i + \delta_{k, t, i}^m i p_k p_t p_i \mid 1 \leq n \leq n_{2,3} \right\},
\]
(v) \( R_{3,2} \) is the set of quadratic relations in arity three:

\[
R_{3,2} := \left\{ r_{3,2}^n(i, j) := \sum_{1 \leq i, j, s} \gamma_{i,j}^{(1)} + \gamma_{i,j}^{(2)} \mid 1 \leq n \leq n_{3,2} \right\}.
\]

(vi) \( R_{3,3} \) is the set of cubic relations in arity three:

\[
R_{3,3} := \left\{ r_{3,3}^n(P_k, i, j) := \sum_{1 \leq k \leq l \leq j \leq s} \gamma_{k,i,j}^{(1)} + \gamma_{k,i,j}^{(2)} + \gamma_{k,i,j}^{(3)} + \gamma_{k,i,j}^{(4)} \gamma_{k,i,j}^{(5)} \mid 1 \leq n \leq n_{3,3} \right\}.
\]

(vii) \( R_4 \) is the set of cubic relations in arity four:

\[
R_4 := \left\{ r_4^n(i, j, p) := \sum_{1 \leq i, j, p, s} \gamma_{i,j,p}^{(1)} + \gamma_{i,j,p}^{(2)} + \gamma_{i,j,p}^{(3)} \mid 1 \leq n \leq n_4 \right\}.
\]

Here the Greek letters are coefficients in \( k \), and \( n_{1,2}, n_{1,3}, n_{2,2}, n_{2,3}, n_{3,2}, n_{3,3} \) and \( n_4 \) are nonnegative integers, with the convention that when any of the integers is zero, then the corresponding set is empty. Note that if \( \mathcal{P} \) is a quadratic operad, then the above relations are reduced to

\[
R = R_1 \sqcup R_2 \sqcup R_3 = R_{1,2} \sqcup R_{2,2} \sqcup R_{3,2}.
\]

We give the following examples to fix notations for later applications and to demonstrate the wide range of structures covered by the notion of unary binary quadratic/cubic ns operads.

**Example 2.5.**

(a) The **associative algebra** is encoded by the associative operad \( \mathcal{A} \), which is binary quadratic and defined by

\[
E = E_2 = k \begin{array}{c} \cdot \end{array} \quad \text{and} \quad R = R_{3,2} = \left\{ \begin{array}{c} + \end{array} - \right\}.
\]

(b) The **differential associative algebra** [25] is encoded by the differential associative operad \( \mathcal{D}er.\mathcal{A} \) [28], defined by the unary and binary generators

\[
E_1 = k \begin{array}{c} d \end{array}, \quad E_2 = k \begin{array}{c} \cdot \end{array}
\]

and quadratic relations

\[
R_2 = R_{2,2} = \left\{ \begin{array}{c} + \end{array} \begin{array}{c} - \end{array} - \begin{array}{c} \cdot \end{array} \begin{array}{c} \cdot \end{array} \right\}, \quad R_3 = R_{3,2} = \left\{ \begin{array}{c} + \end{array} - \begin{array}{c} - \end{array} \right\}
\]

corresponding to the Leibniz rule and the associativity law.

(c) The **Rota-Baxter associative algebra** (of weight zero) [2, 23] is encoded by the Rota-Baxter associative operad \( \mathcal{RBA} \), defined by the unary binary generators

\[
E_1 = k \begin{array}{c} p \end{array}, \quad E_2 = k \begin{array}{c} \cdot \end{array}
\]

and the quadratic and cubic relations

\[
R_2 = R_{2,3} = \left\{ \begin{array}{c} p \end{array} \begin{array}{c} p \end{array} - \begin{array}{c} - \end{array} \begin{array}{c} p \end{array} \begin{array}{c} p \end{array} \right\}, \quad R_3 = R_{3,2} = \left\{ \begin{array}{c} + \end{array} - \begin{array}{c} - \end{array} \right\}
\]
(d) The Rota-Baxter algebra with nonzero weight $\lambda$ does not give a unary binary quadratic/cubic ns operad since the Rota-Baxter relation

$$P(x)P(y) - P(P(x)y) - P(xP(y)) - \lambda P(xy) = 0$$

contains both quadratic and cubic terms. As its homogeneous analog, the Nijenhuis associative algebra is encoded by the unary binary quadratic/cubic ns operad $\mathcal{N\mathcal{A}}$, with generators

$$E_1 = k \delta_1, \quad E_2 = k \delta_2$$

and relations

$$R_2 = R_{2,3} = \left\{ \begin{array}{c} - \frac{P}{P} \delta_1 \delta_2 \delta_3, \quad \frac{P}{P} \delta_1 \delta_2 \delta_3 \\ - \frac{P}{P} \delta_1 \delta_2 \delta_3, \quad \frac{P}{P} \delta_1 \delta_2 \delta_3 \end{array} \right\}, \quad R_3 = R_{3,2} = \left\{ \begin{array}{c} \frac{1}{1} \delta_1 \delta_2 \delta_3 \delta_4, \quad \frac{1}{1} \delta_1 \delta_2 \delta_3 \delta_4 \end{array} \right\}.$$

(e) Let $\Delta$ be a nonempty set. A $\Delta$-differential algebra [25] is an algebra with multiple differential operators $\delta \in \Delta$ that commute with each other. This algebraic structure is encoded by the unary binary quadratic ns operad, with generators

$$E_1 = k \delta \in \Delta, \quad E_2 = k \delta \delta$$

and relations

$$R_1 = R_{1,2} = \left\{ \begin{array}{c} \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6, \quad \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \\ \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6, \quad \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \end{array} \right\}, \quad R_2 = R_{2,2} = \left\{ \begin{array}{c} \frac{1}{1} \delta_1 \delta_2 \delta_3 \delta_4, \quad \frac{1}{1} \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \delta_9 \end{array} \right\},$$

$$R_3 = R_{3,2} = \left\{ \begin{array}{c} \frac{1}{1} \delta_1 \delta_2 \delta_3 \delta_4, \quad \frac{1}{1} \delta_1 \delta_2 \delta_3 \delta_4 \delta_5 \delta_6 \delta_7 \delta_8 \delta_9 \end{array} \right\}.$$

(f) A Hom-associative algebra [31] is encoded by the Hom-associative operad $\mathcal{H}om\mathcal{A}$ defined by the unary and binary generators

$$E_1 = k \alpha, \quad E_2 = k \alpha \alpha$$

and cubic relation

$$R_3 = R_{3,3} = \left\{ \begin{array}{c} \frac{1}{1} \alpha_1 \alpha_2 \alpha_3 \alpha_4, \quad \frac{1}{1} \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7 \delta_8 \delta_9 \end{array} \right\}.$$

(g) A cubic associative algebra [4] is encoded by the cubic associative operad $\mathcal{C\mathcal{A}}$ defined by the binary generator

$$E = E_2 = k \alpha \alpha$$

and the cubic relations in arity four

$$R = R_4 = \left\{ \begin{array}{c} \frac{1}{1} \alpha_1 \alpha_2 \alpha_3 \alpha_4, \quad \frac{1}{1} \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7 \delta_8 \delta_9 \end{array} \right\}.$$
2.2. **Linearly compatible operads.** The linear compatibility of a binary quadratic operad with two duplicated copies of operations has been studied in [37].

**Definition 2.6.** [37, Definition A] Let $\mathcal{P}$ be a binary quadratic operad and $V$ a vector space. Let $A = (V, \mu_1, \ldots, \mu_k)$ and $B = (V, \nu_1, \ldots, \nu_k)$ be two $\mathcal{P}$-algebras such that $\mu_i$ and $\nu_i$ are the corresponding binary operations for $1 \leq i \leq k$. Denote $\eta_i := \alpha \mu_i + \beta \nu_i$ for some $\alpha, \beta \in k$. The pair $A, B$ are called **linearly compatible** if $C = (V, \eta_1, \ldots, \eta_k)$ is a $\mathcal{P}$-algebra for any choice of $\alpha$ and $\beta$. This is equivalent to requiring that $C$ is a $\mathcal{P}$-algebra for $\alpha = \beta = 1$.

We will study the more general case of a unary binary operad with quadratic/cubic relations, with any number of replicated copies of the unary and binary operations. To motivate the general notion, we first discuss an example.

**Example 2.7.** Let $V$ be a vector space. Let $A = (V, \bullet_1, P_1)$ and $B = (V, \bullet_2, P_2)$ be Rota-Baxter associative algebras of weight zero on the same underlying vector space. Define

$$\ast := \alpha \bullet_1 + \beta \bullet_2$$

and $R := \alpha P_1 + \beta P_2$ for $\alpha, \beta \in k$.

The two Rota-Baxter algebras are called **linearly compatible** if the triple $(V, \ast, R)$ is still a Rota-Baxter algebra of weight zero.

Imposing the associativity for $\ast$ and the Rota-Baxter relation for $R$, we find that $(V, \ast, R)$ is a Rota-Baxter associative algebra for all choices of $\alpha$ and $\beta$ if and only if

$$(\bullet_1 y) \bullet_2 z + (\bullet_2 y) \bullet_1 z = x \bullet_1 (y \bullet_2 z) + x \bullet_2 (y \bullet_1 z),$$

and

$$P_1(x) \bullet_2 P_1(y) + P_1(x) \bullet_1 P_2(y) + P_2(x) \bullet_1 P_1(y)$$

$$= P_1(P_1(x) \bullet_2 y) + P_1(x \bullet_2 P_1(y)) + P_2(P_1(x) \bullet_1 y) + P_1(x \bullet_1 P_2(y))$$

$$+ P_1(P_2(x) \bullet_1 y) + P_2(x \bullet_1 P_1(y)),$$

$$P_1(x) \bullet_2 P_2(y) + P_2(x) \bullet_2 P_1(y) + P_2(x) \bullet_1 P_2(y)$$

$$= P_2(P_1(x) \bullet_2 y) + P_1(x \bullet_2 P_2(y)) + P_1(P_2(x) \bullet_2 y) + P_2(x \bullet_2 P_1(y))$$

$$+ P_2(P_2(x) \bullet_1 y) + P_2(x \bullet_1 P_2(y)).$$

These should be the relations for the operad encoding linearly compatible Rota-Baxter algebras with two replicated copies.

Generalizing this to algebras on an ns operad, we give

**Definition 2.8.** Let $\Omega$ be a nonempty set. Let $\mathcal{P} = \mathcal{P}(E)/\langle R \rangle$ be an operad with generators $\mu_1, \ldots, \mu_k$ and such that each relation in $R$ is homogeneous. For each $\omega$ in a nonempty set $\Omega$, let $\mu_{\omega,1}, \ldots, \mu_{\omega,k}$ be a replicate of $\mu_1, \ldots, \mu_k$. Let $V$ be a vector space such that, for each $\omega \in \Omega$, the tuple

$$A_{\omega} := (V, \mu_{\omega,1}, \ldots, \mu_{\omega,k})$$

is a $\mathcal{P}$-algebra. Then $V$ is called an **($\Omega$-)linearly compatible $\mathcal{P}$-algebra** if, for any $c := (c_\omega)_{\omega \in \Omega}$ with $c_\omega \in k$ and the linear combination $v_i := \nu_{c,i} := \sum_{\omega \in \Omega} c_\omega \mu_{\omega,i}, i = 1, \ldots, k$, the tuple

$$A := A_c := (V, v_1, \ldots, v_k)$$

is still a $\mathcal{P}$-algebra.

Since $\mu_{\omega,i}, \omega \in \Omega$, are spacial cases of $v_i, i = 1, \ldots, k$, we have
Proposition 2.9. Let $\Omega$ be a nonempty set and let $P$ be an operad with generators $\mu_1, \ldots, \mu_k$ and such that each relation in $R$ is homogeneous. A vector space $V$ is an $(\Omega)$-linearly compatible $P$-algebra if and only if $(V, v_1, \ldots, v_k)$ is a $P$-algebra for any linear combination $v_i := \sum_{\omega \in \Omega} c_{\omega} \mu_i, c_\omega \in k, \omega \in \Omega$.

Now we determine the operad that encodes linearly compatible $P$-algebras for any unary binary quadratic/cubic operad $\mathcal{P}$. Let $\mathcal{P} = \mathcal{T}(E)/\langle R \rangle$ be such an operad, with its operations and relations shown in (2) – (10). Let $\Omega$ be a nonempty set. We consider a family of ns operads
\[
\mathcal{P}_\omega = \mathcal{T}(E_\omega)/\langle R_\omega \rangle, \quad \text{for } \omega \in \Omega,
\]
that are just copies of $\mathcal{P}$. As for $\mathcal{P}$, we describe $\mathcal{P}_\omega$ by the generators
\[
E_{\omega,1} := \{P_{\omega,1}, P_{\omega,2}, \ldots, P_{\omega,\ell} \}, \quad E_{\omega,2} := \{\emptyset, \bullet, \cdots, \emptyset \}, \quad E_{\omega,n} := 0, \quad n \neq 1, 2, \omega \in \Omega.
\]
The relations
\[
R_\omega := R_{\omega,1,2} \cup R_{\omega,1,3} \cup R_{\omega,2,2} \cup R_{\omega,2,3} \cup R_{\omega,3,2} \cup R_{\omega,3,3} \cup R_{\omega,4}, \quad \text{for } \omega \in \Omega
\]
are given by the same coefficients as the corresponding ones of $R$. For example,
\[
R_{\omega,1,2} = \left\{ r_{1,2}^n(P_{\omega,k}, P_{\omega,\ell}) := \sum_{1 \leq k, \ell \leq l} a_{k,\ell}^{\omega} \left[ P_{\omega,k} \right]^{P_{\omega,\ell}} \left| 1 \leq n \leq n_{1,2} \right. \right\}.
\]
Define the arity graded space
\[
\bigoplus_{\omega \in \Omega} E_{\omega} := \left\{ 0, \bigoplus_{\omega \in \Omega} E_{\omega,1}, \bigoplus_{\omega \in \Omega} E_{\omega,2}, 0 \ldots \right\}.
\]
For each $\omega \in \Omega$, by embedding $\mathcal{P}_\omega$ into $\mathcal{T}(\bigoplus_{\omega \in \Omega} E_{\omega})/\langle \cup R_\omega \rangle$, a $\mathcal{T}(\bigoplus_{\omega \in \Omega} E_{\omega})/\langle \cup R_\omega \rangle$-algebra is a family of $\mathcal{P}$-algebras which are not necessarily compatible with one another in any way.

To describe the compatible conditions among the copies $\mathcal{P}_\omega$, we build on the notations in Eqs. (4) – (10) and define, for $\mu, \nu, \omega \in \Omega$,
\[
R_{1,2}^{1,\nu} := \left\{ r_{1,2}^n(P_{\mu,k}, P_{\nu,\ell}) \left| 1 \leq n \leq n_{1,2} \right. \right\},
\]
\[
R_{1,3}^{1,\nu} := \left\{ r_{1,3}^n(P_{\mu,k}, P_{\nu,\ell}, P_{\omega,\mu}) \left| 1 \leq n \leq n_{1,3} \right. \right\},
\]
\[
R_{2,2}^{2,\nu} := \left\{ r_{2,2}^n(P_{\mu,k}, i_\nu) \left| 1 \leq n \leq n_{2,2} \right. \right\},
\]
\[
R_{2,3}^{2,\nu} := \left\{ r_{2,3}^n(P_{\mu,k}, P_{\omega,\nu}, i_\omega) \left| 1 \leq n \leq n_{2,3} \right. \right\},
\]
\[
R_{3,2}^{3,\nu} := \left\{ r_{3,2}^n(i_\mu, j_\nu) \left| 1 \leq n \leq n_{3,2} \right. \right\},
\]
\[
R_{3,3}^{3,\nu} := \left\{ r_{3,3}^n(P_{\mu,k}, i_\nu, j_\omega) \left| 1 \leq n \leq n_{3,3} \right. \right\},
\]
\[
R_{4}^{4,\nu} := \left\{ r_{4}^n(i_\mu, j_\nu, P_\omega) \left| 1 \leq n \leq n_4 \right. \right\}.
\]
For example,
\[
R_{1,2}^{1,\nu} = \left\{ r_{1,2}^n(P_{\mu,k}, P_{\nu,\ell}) := \sum_{1 \leq k, \ell \leq l} a_{k,\ell}^{\mu,\nu} \left[ P_{\mu,k} \right]^{P_{\nu,\ell}} \left| 1 \leq n \leq n_{1,2} \right. \right\},
\]
\[
R_{2,2}^{2,\nu} = \left\{ r_{2,2}^n(P_{\mu,k}, i_\nu) := \sum_{1 \leq k, \ell \leq l} \beta_{k,\ell}^{\mu,\nu} \left[ P_{\mu,k} \right]^{P_{\nu,\ell}} + \beta_{k,\ell}^{2,\nu} P_{\mu,k} \left| 1 \leq n \leq n_{2,2} \right. \right\}.
\]
Note that, comparing the above notions with Eq. (12), we have

\[ R_{i,2}^{\omega,\nu,\omega} = R_{i,2}^{\omega,\nu,\omega} \]

and \( R_{i,3}^{\omega,\nu,\omega} = R_{i,3}^{\omega,\nu,\omega} \)

for \( i = 1, 2, 3 \) and \( \omega \in \Omega \).

We will use the following general notation for our discussion.

**Notation 2.10.** Let \( X := \{ x_i \mid i \in I \} \) and \( Y := \{ y_j \mid j \in I \} \) be two sets of elements in a vector space \( V \), parameterized by the same set \( I \). Denote \( X \uplus Y := \{ x_i + y_j \mid i \in I \} \).

In particular, when \( I = [n] := \{ 1, \ldots, n \} \), then the above \( X \) and \( Y \) can be regarded as vectors \( X = (x_1, \ldots, x_n) \) and \( Y = (y_1, \ldots, y_n) \). Then \( X \uplus Y \) is simply the addition of the two vectors.

Then together with the notations in Eqs. (13) – (19), we have

\[
R_{1,2}^{\mu,\nu,\omega} \uplus R_{1,2}^{\omega,\nu,\omega} = \left\{ r_{1,2}^{\mu}(P_{\mu,k}, P_{\nu,\ell}, P_{\omega,\ell}) \mid 1 \leq n \leq n_{1,2} \right\}
\]

\[
R_{1,3}^{\mu,\nu,\omega} \uplus R_{1,3}^{\omega,\nu,\omega} = \left\{ r_{1,3}^{\mu}(P_{\mu,k}, P_{\nu,\ell}, P_{\omega,m}) \mid 1 \leq n \leq n_{1,3} \right\}
\]

\[
R_{2,2}^{\mu,\nu,\omega} \uplus R_{2,2}^{\omega,\nu,\omega} = \left\{ r_{2,2}^{\mu}(P_{\mu,k}, P_{\nu,\ell}, P_{\omega,m}) + r_{2,2}^{\nu}(P_{\omega,\ell}, P_{\omega,m}) \mid 1 \leq n \leq n_{2,2} \right\}
\]

\[
R_{2,3}^{\mu,\nu,\omega} \uplus R_{2,3}^{\omega,\nu,\omega} = \left\{ r_{2,3}^{\mu}(P_{\mu,k}, P_{\nu,\ell}, i_{\omega}) \mid 1 \leq n \leq n_{2,3} \right\}
\]

\[
R_{3,2}^{\mu,\nu,\omega} \uplus R_{3,2}^{\omega,\nu,\omega} = \left\{ r_{3,2}^{\mu}(P_{\mu,k}, i_{\omega}, P_{\omega,\ell}) \mid 1 \leq n \leq n_{3,2} \right\}
\]

Now we are ready for our main concept in this section. Set

\[
R_L := \bigcup_{i=1,2,3} \left( \bigcup_{\mu \neq \nu \in \Omega} (R_{i,2}^{\mu,\nu,\omega} \uplus R_{i,2}^{\nu,\omega,\omega}) \bigcup_{\mu \neq \nu \in \Omega} (R_{i,3}^{\mu,\nu,\omega} \uplus R_{i,3}^{\nu,\omega,\omega}) \bigcup_{\mu,\nu,\omega \in \Omega} (R_{i,3}^{\mu,\nu,\omega}) \right)
\]

\[
= \bigcup_{i=1,2,3} \left( \bigcup_{\mu \neq \nu \in \Omega} (R_{i,2}^{\mu,\nu,\omega}) \bigcup_{\mu \neq \nu \in \Omega} (R_{i,3}^{\mu,\nu,\omega}) \right)
\]

**Definition 2.11.** Let \( \Omega \) be a nonempty set. Let \( \mathcal{P} \) be a unary binary quadratic/cubic ns operad given in Eq. (1). We call the ns operad

\[
\mathcal{P}_{\Omega}^{LC} := \mathcal{T} \left( \bigoplus_{\omega \in \Omega} E_{\omega} \right) \bigcup \bigcup_{\omega \in \Omega} \left( \bigcup_{\mu \neq \nu \in \Omega} (R_{i,2}^{\mu,\nu,\omega}) \right)
\]

the **linearly compatible operad** of \( \mathcal{P} \) with parameter \( \Omega \).

Encoding linearly compatible algebras, we give
Theorem 2.12. Let $\Omega$ be a nonempty set. Let $\mathcal{P} = \mathcal{P}(E, R)$ be a unary binary quadratic/cubic ns operad. A vector space $V$ is a $\mathcal{P}^{LC}$-algebra if and only if it is an $\Omega$-linearly compatible $\mathcal{P}$-algebra.

Proof. Let $\mathcal{P}$ be a unary binary quadratic/cubic ns operad given in Eq. (1). Let $V$ be a vector space. On the one hand, by Definition 2.11, $V$ is a $\mathcal{P}^{LC}$-algebra if and only if elements of $V$ satisfy the relations in $\bigcup_{\omega \in \Omega} R_\omega \cup R_L$.

On the other hand, if $V$ is an $\Omega$-linearly compatible $\mathcal{P}$-algebra. Then by Proposition 2.9, $V$ with operations

$$\sum_{\omega \in \Omega} c_\omega P_{\omega,1}, \ldots, \sum_{\omega \in \Omega} c_\omega P_{\omega,t}, \sum_{\omega \in \Omega} c_\omega S_\omega$$

is a $\mathcal{P}$-algebra for any choice of $c := (c_\omega)_{\omega \in \Omega}$ with $c_\omega \in k$, that is, elements of $V$ satisfy the relations in

$$R_c = R_{c,1,2} \sqcup R_{c,1,3} \sqcup R_{c,2,2} \sqcup R_{c,2,3} \sqcup R_{c,3,2} \sqcup R_{c,3,3} \sqcup R_{c,4}.$$

Explicitly,

$$R_{c,1,2} = \left\{ r_{1,2}^n \left( \sum_{\omega \in \Omega} c_\omega P_{\omega,k}, \sum_{\omega \in \Omega} c_\omega P_{\omega,f} \right) \bigg| 1 \leq n \leq n_{1,2} \right\},$$

$$R_{c,1,3} = \left\{ r_{1,3}^n \left( \sum_{\omega \in \Omega} c_\omega P_{\omega,k}, \sum_{\omega \in \Omega} c_\omega P_{\omega,f}, \sum_{\omega \in \Omega} c_\omega P_{\omega,m} \right) \bigg| 1 \leq n \leq n_{1,3} \right\},$$

$$R_{c,2,2} = \left\{ r_{2,2}^n \left( \sum_{\omega \in \Omega} c_\omega P_{\omega,k}, \sum_{\omega \in \Omega} c_\omega i_\omega \right) \bigg| 1 \leq n \leq n_{2,2} \right\},$$

$$R_{c,2,3} = \left\{ r_{2,3}^n \left( \sum_{\omega \in \Omega} c_\omega P_{\omega,k}, \sum_{\omega \in \Omega} c_\omega P_{\omega,f}, \sum_{\omega \in \Omega} c_\omega i_\omega \right) \bigg| 1 \leq n \leq n_{2,3} \right\},$$

$$R_{c,3,2} = \left\{ r_{3,2}^n \left( \sum_{\omega \in \Omega} c_\omega i_\omega, \sum_{\omega \in \Omega} c_\omega j_\omega \right) \bigg| 1 \leq n \leq n_{3,2} \right\},$$

$$R_{c,3,3} = \left\{ r_{3,3}^n \left( \sum_{\omega \in \Omega} c_\omega P_{\omega,k}, \sum_{\omega \in \Omega} c_\omega i_\omega, \sum_{\omega \in \Omega} c_\omega j_\omega \right) \bigg| 1 \leq n \leq n_{3,3} \right\},$$

$$R_{c,4} = \left\{ r_4^n \left( \sum_{\omega \in \Omega} c_\omega i_\omega, \sum_{\omega \in \Omega} c_\omega j_\omega, \sum_{\omega \in \Omega} c_\omega P_\omega \right) \bigg| 1 \leq n \leq n_4 \right\}.$$

Thus we are left to verify

$$\langle R_c \rangle = \left\{ \bigcup_{\omega \in \Omega} R_\omega \cup R_L \right\},$$

which, by comparing the arities and Eq. (21), is equivalent to the following equations, for $i = 1, 2, 3$:

$$kR_{c,i,2} = k \bigg\{ \bigcup_{\omega \in \Omega} R_{\omega,i,2} \cup \bigcup_{\mu \neq \nu \in \Omega} R_{i,2}^{\mu,\nu} \cup R_{i,2}^{\nu,\mu} \bigg\},$$

$$kR_{c,i,3} = k \bigg\{ \bigcup_{\omega \in \Omega} R_{\omega,i,3} \cup \bigcup_{\mu \neq \nu \in \Omega} \left( R_{i,3}^{\mu,\nu} \cup R_{i,3}^{\nu,\mu} \cup R_{i,3}^{\nu,\mu} \right) \cup \bigcup_{\mu \neq \nu \in \Omega} R_{i,3}^{\mu,\nu} \bigg\},$$

$$kR_{c,4} = k \bigg\{ \bigcup_{\omega \in \Omega} R_{\omega,4} \cup \bigcup_{\mu \neq \nu \in \Omega} \left( R_{4}^{\mu,\nu} \cup R_{4}^{\nu,\mu} \cup R_{4}^{\nu,\mu} \right) \cup \bigcup_{\mu \neq \nu \in \Omega} R_{4}^{\mu,\nu} \bigg\}.$$
Now expanding the set of quadratic relations \( R_{c,1} \) by linearity, we obtain

\[
k R_{c,1,2} = k \left\{ r^n_{1,2} \left( \sum_{\omega \in \Omega} c_{\omega}P_{\omega,k}, \sum_{\omega \in \Omega} c_{\omega}P_{\omega,\ell} \right) \bigg| 1 \leq n \leq n_{1,2} \right\}
\]

\[
= k \left\{ \sum_{\mu \in \Omega} c_{\mu} c_{\nu} r^n_{1,2}(P_{\mu,k}, P_{\nu,\ell}) \bigg| 1 \leq n \leq n_{1,2} \right\}
\]

\[
= k \left( \sum_{\omega \in \Omega} c_{\omega}^2 r^n_{1,2}(P_{\omega,k}, P_{\omega,\ell}) + \sum_{\mu \neq \omega \in \Omega} c_{\mu} c_{\omega} r^n_{1,2}(P_{\mu,k}, P_{\omega,\ell}) \bigg| 1 \leq n \leq n_{1,2} \right) \}
\]

\[
= k \left( r^n_{1,2}(P_{\omega,k}, P_{\omega,\ell}) + \sum_{\omega \in \Omega} c_{\omega}^2 r^n_{1,2}(P_{\mu,k}, P_{\omega,\ell}) \bigg| 1 \leq n \leq n_{1,2} \right) \}
\]

Thus Eq. (22) holds for \( i = 1 \). Similarly, expanding \( R_{c,2,2} \) and \( R_{c,3,2} \) by linearity, we obtain Eq. (22) for \( i = 2, 3 \).

For the cubic relations \( R_{c,2,3} \), expanding \( R_{c,2,3} \) by linearity, we have

\[
k R_{c,2,3} = k \left\{ r^n_{2,3} \left( \sum_{\omega \in \Omega} c_{\omega}P_{\omega,k}, \sum_{\omega \in \Omega} c_{\omega}P_{\omega,\ell}, \sum_{\omega \in \Omega} c_{\omega}i_{\omega} \right) \bigg| 1 \leq n \leq n_{2,3} \right\}
\]

\[
= k \left\{ \sum_{\mu,\nu,\omega \in \Omega} c_{\mu} c_{\nu} c_{\omega} r^n_{2,3}(P_{\mu,k}, P_{\nu,\ell}, i_{\omega}) \bigg| 1 \leq n \leq n_{2,3} \right\}
\]

\[
= k \left( \sum_{\omega \in \Omega} c_{\omega}^3 r^n_{2,3}(P_{\omega,k}, P_{\omega,\ell}, i_{\omega}) + \sum_{\mu \neq \nu \neq \omega \in \Omega} c_{\mu} c_{\nu} c_{\omega} r^n_{2,3}(P_{\mu,k}, P_{\nu,\ell}, i_{\omega}) \bigg| 1 \leq n \leq n_{2,3} \right) \}
\]

\[
= k \left( r^n_{2,3}(P_{\omega,k}, P_{\omega,\ell}, i_{\omega}) \right| \mu, \nu, \omega \in \Omega \bigg| \text{distinct} \right) \}
\]

(by the arbitrariness of \( c_{\mu}, c_{\nu}, c_{\omega} \in k \))

Therefore Eq. (23) also holds, for \( i = 2 \). Similarly, expanding the cubic relations \( R_{c,2,3}, R_{c,3,3} \) and \( R_{c,4} \) by linearity, we obtain Eq. (22) for \( i = 2, 3 \), and Eq. (24). This completes the proof.

\[\square\]

### 2.3. Linear compatibility by the Manin black square product

We now establish the relationship between the linear compatibility and the Manin black square products of ns operads.
Proposition 2.14. Let $\mathcal{P} = \mathcal{I}(E)/\langle R \rangle$ and $\mathcal{Q} = \mathcal{I}(F)/\langle S \rangle$ be two finite generated binary quadratic ns operads with relations
\[ R = \left\{ \sum_{i,j \in E} k_{i,j}^{m} \, i \otimes j \mid 1 \leq m \leq r \right\} \]
and
\[ S = \left\{ \sum_{k,\ell \in F} k_{k,\ell}^{n} \, i \otimes \ell \mid 1 \leq n \leq s \right\}. \]

(a) The Manin black product of $\mathcal{P}$ and $\mathcal{Q}$ is the operad $\mathcal{P} \blacktriangleright \mathcal{Q} := \mathcal{I}(E \otimes F)/\langle R \blacktriangleright S \rangle$, where
\[ R \blacktriangleright S := \left\{ \sum_{i,j \in E, k,\ell \in F} k_{i,j}^{m} \, i \otimes (k \otimes \ell) - k_{i,j}^{n} \, i \otimes \ell \mid 1 \leq m \leq r, 1 \leq n \leq s \right\}. \]

(b) The Manin white product of $\mathcal{P}$ and $\mathcal{Q}$ is the operad $\mathcal{P} \blacklozenge \mathcal{Q} := \mathcal{I}(E \otimes F)/\langle R \blacklozenge S \rangle$, where
\[ R \blacklozenge S := \left\{ \sum_{i,j \in E, k,\ell \in F} k_{i,j}^{m} \, i \otimes \ell - k_{i,j}^{n} \, i \otimes \ell \mid 1 \leq m \leq r \right\} \cup \left\{ \sum_{i,j \in E, k,\ell \in F} k_{i,j}^{n} \, i \otimes \ell \mid 1 \leq n \leq s \right\}. \]

The following result shows that linear compatibility can be achieved by taking a black square product.

Proposition 2.14. Let $\Omega$ be a nonempty finite set. Let $\mathcal{P}$ be a finitely generated binary quadratic ns operad. Then
\[ \mathcal{P}^{\text{LC}}_{\Omega} \cong \mathcal{A}^{\text{LC}}_{\Omega} \blacktriangleright \mathcal{P}. \]

Proof. By Definition 2.11, for the associative operad $\mathcal{A}s = \mathcal{I}(E)/\langle R \rangle$ in Example 2.5 (a), we have
\[ \mathcal{A}s^{\text{LC}}_{\Omega} = \mathcal{I}\left( \bigoplus_{\omega \in \Omega} E_{\omega} \Big/ \bigcup_{\omega \in \Omega} R_{\omega} \cup R_{K} \right) = \mathcal{I}\left( \bigoplus_{\omega \in \Omega} E_{\omega} \Big/ \bigcup_{\omega \in \Omega} R_{\omega} \cup \left( \bigcup_{\mu \neq \nu \in \Omega} R^{\mu,\nu} \cup R^{\nu,\mu} \right) \right), \]
where
\[ \bigoplus_{\omega \in \Omega} E_{\omega} = \bigoplus_{\omega \in \Omega} k^{\omega} \]
and
\[ \bigcup_{\omega \in \Omega} R_{\omega} \cup \left( \bigcup_{\mu \neq \nu \in \Omega} R^{\mu,\nu} \cup R^{\nu,\mu} \right) = \left\{ Y^{\omega} - Y^{\mu} \right\} \cup \left\{ Y^{\omega} + Y^{\mu} - Y^{\nu} - Y^{\nu'} \right\}. \]

Let $\mathcal{P} = \mathcal{I}(F)/\langle S \rangle$ with
\[ F := F_{2} := k\left\{ Y^{\omega}, Y^{\nu}, \ldots, Y^{\nu'} \right\}, \quad S := S_{3,2} := \left\{ \sum_{1 \leq i,j \leq s} \gamma_{i,j}^{1} \, i \otimes j + \gamma_{i,j}^{2} \, i \otimes j \mid 1 \leq d \leq s \right\}. \]

By Eq. (11), we obtain
\[ \left( \bigoplus_{\omega \in \Omega} E_{\omega} \right) \otimes F = \left( \bigoplus_{\omega \in \Omega} k^{\omega} \right) \otimes \left( \bigoplus_{1 \leq i \leq s} k^{i} \right) = \bigoplus_{\omega \in \Omega} k^{\omega} \otimes i \cong \bigoplus_{\omega \in \Omega} k^{\omega} \otimes F_{\omega} \]
and by Definition 2.13 (a),
\[
\bigcup_{\omega \in \Omega} \left( \bigcup_{\mu \neq \nu} R_{\omega} \cup \bigcup_{\mu \neq \nu} R_{\omega}^{\nu, \mu} \right) S
\]
\[
= \bigcup_{\omega \in \Omega} \left( R_{\omega} S \cup \bigcup_{\mu \neq \nu} \left( R_{\omega}^{\nu, \mu} S \right) \right)
\]
\[
= \bigcup_{\omega \in \Omega} \bigg\{ \sum_{1 \leq i, j \leq s} \gamma_{i,j}^{1,n} Y_{i,j}^{1,n} + \gamma_{i,j}^{2,n} Y_{i,j}^{2,n} \bigg\} \bigg\{ \sum_{1 \leq i, j \leq s} \gamma_{i,j}^{1,n} Y_{i,j}^{1,n} + \gamma_{i,j}^{2,n} Y_{i,j}^{2,n} \bigg\} \bigg| 1 \leq d \leq s \bigg\} \text{ (by Eq. (26))}
\]
\[
= \bigcup_{\omega \in \Omega} \bigg\{ \sum_{1 \leq i, j \leq s} \gamma_{i,j}^{1,n} Y_{i,j}^{1,n} + \gamma_{i,j}^{2,n} Y_{i,j}^{2,n} \bigg\} \bigg\{ \sum_{1 \leq i, j \leq s} \gamma_{i,j}^{1,n} Y_{i,j}^{1,n} + \gamma_{i,j}^{2,n} Y_{i,j}^{2,n} \bigg\} \bigg| 1 \leq d \leq s \bigg\}
\]
\[
= \bigcup_{\omega \in \Omega} S_{\omega, 3, 2} \cup \bigcup_{\mu \neq \nu} S_{\omega, 3, 2}^{\nu, \mu} \text{ (by Eqs. (17) and (20)).}
\]

Hence
\[
\mathcal{P}_{\Omega}^{\text{LC}} \mathcal{O} = \mathcal{T} \left( \bigoplus_{\omega \in \Omega} E_{\omega} \otimes F \bigg/ \left( \bigcup_{\omega \in \Omega} R_{\omega} \cup \bigcup_{\mu \neq \nu} R_{\omega}^{\nu, \mu} \right) S \right)
\]
\[
\cong \mathcal{T} \left( \bigoplus_{\omega \in \Omega} F_{\omega} \bigg/ \left( \bigcup_{\omega \in \Omega} S_{\omega, 3, 2} \cup \bigcup_{\mu \neq \nu} S_{\omega, 3, 2}^{\nu, \mu} \right) \right)
\]
\[
= \mathcal{P}_{\Omega}^{\text{LC}}
\]

3. The matching compatibility and Koszul self duality

This section is devoted to the operadic study of another compatibility condition of algebraic structures carrying multiple copies of the same operations. It will be called the matching compatibility, which has stronger conditions than the linear compatibility and has a self dual property.

3.1. Matching compatibility. By splitting Eq. (21) according to arities and degrees, we define
\[
R_{M} := \bigcup_{i \in \{1, 2, 3\}} \left( \bigcup_{\mu \neq \nu} R_{1, 2}^{\mu, \nu} \cup \bigcup_{\mu, \nu, \omega \neq \emptyset \text{ not all equal}} R_{1, 3}^{\mu, \nu, \omega} \right) \cup \bigcup_{\mu, \nu, \omega, \omega \neq \emptyset \text{ not all equal}} R_{4}^{\mu, \nu, \omega}
\]

Definition 3.1. Let \( \Omega \) be a nonempty set. Let \( \mathcal{P} = \mathcal{T}(E) / \langle R \rangle \) be a unary binary quadratic/cubic ns operad given in Eq. (1). We call the ns operad
\[
\mathcal{P}_{\Omega}^{\text{MT}} := \mathcal{T} \left( \bigoplus_{\omega \in \Omega} E_{\omega} \bigg/ \left( \bigcup_{\omega \in \Omega} R_{\omega} \cup R_{M} \right) \right)
\]
the matching operad of \( \mathcal{P} \) with parameter set \( \Omega \).
By Eqs. (12) and (20), we have
\[ \bigcup_{\omega \in \Omega} R_\omega \cup \mathbb{R}_M = \bigcup_{\mu, \nu, \omega \in \Omega \atop \in \{1, 2, 3\}} \left( R_{\mu, \nu}^{\mu, \nu \omega} \cup R_{\mu, \nu}^{\mu \nu \omega} \cup R_{\mu, \nu}^{\mu \nu \omega} \right). \]

Thus
\[ \mathfrak{P}_{\Omega}^{\text{MT}} = \mathcal{P} \left( \bigoplus_{\omega \in \Omega} E_\omega \bigg/ \bigcup_{\mu, \nu, \omega \in \Omega \atop \in \{1, 2, 3\}} \left( R_{\mu, \nu}^{\mu, \nu \omega} \cup R_{\mu, \nu}^{\mu \nu \omega} \cup R_{\mu, \nu}^{\mu \nu \omega} \right) \right). \]

Since \( \langle R_L \rangle \subseteq \langle R_M \rangle \), we have

**Proposition 3.2.** Let \( \Omega \) be a nonempty set. Let \( \mathfrak{P} \) be a unary binary quadratic/cubic ns operad given in Eq. (1). Then there is an epimorphism of ns operads
\[ \mathfrak{P}_{\Omega}^{\text{LC}} \longrightarrow \mathfrak{P}_{\Omega}^{\text{MT}}. \]

In other words, any \( \mathfrak{P}_{\Omega}^{\text{MT}} \)-algebra is a \( \mathfrak{P}_{\Omega}^{\text{LC}} \)-algebra.

We give some examples on the level of algebras.

**Example 3.3.** Let \( \Omega \) be a finite set.

(a) When \( \mathfrak{P} \) is the operad of associative algebras, \( \mathfrak{P}_{\Omega}^{\text{MT}} \) gives the matching associative algebra [42, 45], defined to be a vector space \( A \) equipped with a family of binary operations \( \cdot_\omega : A \otimes A \to A \), satisfying the matching associativity
\[ (x \cdot_\alpha y) \cdot_\beta z = x \cdot_{\alpha \cdot_\beta} (y \cdot_\beta z), \text{ for all } x, y, z \in A, \alpha, \beta \in \Omega. \] (29)

(b) When \( \mathfrak{P} \) is the operad of Rota-Baxter algebras of weight zero, \( \mathfrak{P}_{\Omega}^{\text{MT}} \) gives the following algebraic structure with multiple copies \( \cdot_\alpha \) of the multiplication and multiple copies \( P_\alpha \) of the Rota-Baxter operator, satisfying Eq. (29) and
\[ P_\alpha(x) \cdot_\gamma P_\beta(y) = P_\alpha(x \cdot_\gamma P_\beta(y)) + P_\beta(P_\alpha(x) \cdot_\gamma y) \text{ for all } x, y \in A, \]
where \( \alpha, \beta, \gamma \in \Omega \) with exactly two of which to be the same.

(c) In the above example, if the multiplications \( \cdot_\alpha \) are taken to be the same, we recover the notion of matching Rota-Baxter algebras introduced in [44].

(d) Taking \( \mathfrak{P} \) to be the operad \( \mathsf{Dend} \) of dendriform algebras. A \( \mathsf{Dend}_{\Omega}^{\text{MT}} \)-algebra is a matching dendriform algebra \( (D, \{<_\alpha, >_\alpha \mid \alpha, \beta \in \Omega\}) \), characterized by the relations
\[ (x <_\alpha y) <_\beta z = x <_\alpha (y <_\beta z) + x <_\alpha (y >_\beta z), \]
\[ (x >_\alpha y) <_\beta z = x >_\alpha (y <_\beta z), \]
\[ (x <_\alpha y) >_\beta z + (x >_\alpha y) >_\beta z = x >_\alpha (y >_\beta z), \text{ for all } x, y, z \in D. \]

Then it is immediately checked that \( (D, \{*_\alpha :=<_\alpha + >_\alpha \mid \alpha \in \Omega\}) \) is a matching associative algebra in Item (a). This relationship of matching compatibility with the splitting of operads [1, 22, 36] holds for any binary quadratic ns operad. We note that this notion of matching dendriform algebra is different from the one defined in [44]. For example, instead of the first equation, the notion in [44] has the equation
\[ (x <_\alpha y) <_\beta z = x <_\alpha (y <_\beta z) + x <_\beta (y >_\alpha z). \]
3.2. The Koszul dual of unary binary quadratic operads. The Koszul duality for binary operads was given by Ginzburg and Kapranov [20] for binary quadratic operads and by Getzler [19] for binary operads. See also [17, 29].

**Definition 3.4.** (a) An quadratic cooperad \( C(E, R) \) associated to the cogenerators \( E \) and the corelations \( R \) is a sub-cooperad of the cofree cooperad \( \mathcal{T}^n(E) \) such that the composite

\[
C(E, R) \hookrightarrow \mathcal{T}^n(E) \twoheadrightarrow \mathcal{T}^n(E)^{(2)}/R
\]

is 0.

(b) Let \( \mathcal{P} = \mathcal{T}(E)/\langle R \rangle \) be a quadratic operad. Let \( s \) be the suspension. Define the Koszul dual cooperad of \( \mathcal{P} \) to be the quadratic cooperad

\[
\mathcal{P}^\perp := C(sE, s^2R).
\]

(c) Let \( \mathcal{P} = \mathcal{T}(E)/\langle R \rangle \) be a quadratic operad. Let \( \mathcal{I}^n \) be the cooperad defined by \( \mathcal{I}^n_c := \text{Hom}((sk)^n, sk) \). Define the Koszul dual operad of \( \mathcal{P} \) to be the linear dual

\[
\mathcal{P}^\perp := \left(\mathcal{P}^c \otimes \mathcal{P}^\perp\right)^*.
\]

For the operad of differential algebras, the Koszul dual was explicitly computed by Loday in [28]. Generalizing the approach, we obtain

**Theorem 3.5.** For any finitely generated unary binary quadratic ns operad \( \mathcal{P} = \mathcal{T}(E)/\langle R \rangle \) with

\[
R = R_1 \sqcup R_2 \sqcup R_3 := R_{1,2} \sqcup R_{2,2} \sqcup R_{3,2},
\]

its Koszul dual operad is given by

\[
\mathcal{P}^\perp = \mathcal{T}(E^*)/\langle R^\perp \rangle,
\]

where \( R^\perp := \{R_1^\perp, R_2^\perp, R_3^\perp\} \), while \( R_1^\perp, R_2^\perp \) and \( R_3^\perp \) are respectively the orthogonal subspaces of \( R_1, R_2 \) and \( R_3 \) in \( E_1 \otimes E_1, E_1 \otimes E_2 \oplus E_2 \otimes E_1 \oplus E_1 \otimes E_2 \otimes E_1 \) and \( E_2 \otimes E_2 \otimes E_2 \otimes E_2 \).

We give some notations before the proof. Since \( \mathcal{P} \) is a unary binary quadratic ns operad, the generator is \( E = \{0, E_1, E_2, \ldots \} \) and

\[
\mathcal{T}(E)^{(2)} = \left\{ \mathcal{T}(E)^{(2)}_1, \mathcal{T}(E)^{(2)}_2, \mathcal{T}(E)^{(2)}_3 \right\}
\]

\[
= \left\{ k[\alpha \circ_1 \beta | \alpha, \beta \in E_1], k[\alpha \circ_1 \mu, \mu \circ_1 \alpha, \mu \circ_2 \alpha | \alpha \in E_1, \mu \in E_2], k[\mu \circ_1 \nu, \mu \circ_2 \nu | \mu, \nu \in E_2] \right\}
\]

\[
= \left\{ E_1 \otimes E_1, E_1 \otimes E_2 \oplus E_2 \otimes E_1 \oplus E_2 \otimes E_1 \otimes E_2 \oplus E_2 \otimes E_2 \otimes E_2 \right\}.
\]

Thus

\[
\left( \mathcal{T}(E)^{(2)} \right)^* = \left\{ \left( \mathcal{T}(E)^{(2)}_1 \right)^*, \left( \mathcal{T}(E)^{(2)}_2 \right)^*, \left( \mathcal{T}(E)^{(2)}_3 \right)^* \right\}.
\]

We identify \( \mathcal{T}(E^*)^{(2)} \) with the dual of \( \mathcal{T}(E)^{(2)} \) by means of the nondegenerate bilinear form

\[
\langle \cdot, \cdot \rangle_i : \mathcal{T}(E^*)^{(2)} \otimes \mathcal{T}(E)^{(2)} \rightarrow k, \text{ for } i = 1, 2, 3,
\]

\[
\langle \alpha^* \circ_1 \beta^*, \alpha \circ_1 \beta \rangle_1 := \alpha^*(\alpha) \beta^*(\beta) \in k, \quad \langle \alpha^* \circ_1 \mu^*, \alpha \circ_1 \mu \rangle_2 := -\alpha^*(\alpha) \mu^*(\mu) \in k,
\]

\[
\langle \mu^* \circ_1 \alpha^*, \mu \circ_1 \alpha \rangle_2 := \mu^*(\mu) \alpha^*(\alpha) \in k, \quad \langle \mu^* \circ_2 \alpha^*, \mu \circ_2 \alpha \rangle_2 := \mu^*(\mu) \alpha^*(\alpha) \in k,
\]

\[
\langle \mu^* \circ_1 \nu^*, \mu \circ_1 \nu \rangle_3 := \mu^*(\mu) \nu^*(\nu) \in k, \quad \langle \mu^* \circ_2 \nu^*, \mu \circ_2 \nu \rangle_3 := -\mu^*(\mu) \nu^*(\nu) \in k,
\]

\[
\langle \cdot, \cdot \rangle_i := 0, \text{ otherwise , where } \alpha, \beta \in E_1, \mu, \nu \in E_2.
\]
Then, for \( i = 1, 2, 3 \), the orthogonal subspace \( R_i^\perp \subseteq \mathcal{I}(E^*_i)^{(2)} \) is

\[
R_i^\perp = \left\{ x \in \mathcal{I}(E^*_i)^{(2)} \mid \langle x, R_i \rangle = 0 \right\}.
\]

**Proof. (of Theorem 3.5)** Since \( \mathcal{I}_i^c = \text{Hom}((sk)^n, sk) \), the cogenerators of \( \mathcal{I}_i^c \) in arity 1 and 2 are \( \text{id} \) and \( s^{-1} \), respectively. Thus the cogenerators of the cooperad \( \mathcal{I}_i^c \otimes \mathcal{P}_i \) is \( E \) with degree 0.

Notice that \( R \) is an arity graded subspace of \( \mathcal{I}(E)^{(2)} \) with

\[
\begin{align*}
\mathcal{I}(E)_1^{(2)} &= E_1 \otimes E_1, \\
\mathcal{I}(E)_2^{(2)} &= E_1 \otimes E_2 \oplus E_2 \otimes E_1, \\
\mathcal{I}(E)_3^{(2)} &= E_2 \otimes E_2 \oplus E_2 \otimes E_2.
\end{align*}
\]

From Definition 3.4, we have

\[
\mathcal{P}_i = \mathcal{C}(sE, s^2R) \quad \text{and} \quad \mathcal{I}_i^c \otimes \mathcal{P}_i = \mathcal{C}([E_1, E_2], R).
\]

For the cooperad \( \mathcal{C}([E_1, E_2], R) \), the associated linear dual operad is the quadratic operad \( \mathcal{P}(E^*, R^*) \), where \( R^\perp \) is obtained as follows:

\[
\begin{array}{ccc}
\text{Ker}(\pi) & \overset{i}{\longrightarrow} & (\mathcal{I}(E)^{(2)})^* \\
\downarrow & & \downarrow \phi = \pi \\
R^\perp & \overset{\pi}{\longrightarrow} & \mathcal{I}(E^*)^{(2)}.
\end{array}
\]

Here the isomorphism \( \phi \) is induced by the scalar product in Eq. (30).

As an example, we compute the Koszul dual of the operad of \( \Delta \)-differential algebras in Example 2.5 (e). Taking \( \Delta \) to be a singleton gives [28, Proposition 7.2].

**Proposition 3.6.** Let \( \Delta \) be a nonempty finite set. Then the Koszul dual operad of the operad of \( \Delta \)-differential algebra is presented by generators

\[
E_1^* = \left\{ k \mid \delta^* \mid \delta \in \Delta \right\}, \quad E_2^* = k \mathcal{V},
\]

and relations

\[
R_1^\perp = \left\{ \frac{\delta^*}{\delta_1^*} + \frac{\delta^*}{\delta_2^*} \mid \delta_1, \delta_2 \in \Delta \right\}, \quad R_2^\perp = \left\{ \frac{\mu^*}{\delta^* - \delta^*}, \frac{\mu^*}{\delta^* - \mu^*} \mid \delta \in \Delta \right\}, \quad R_3^\perp = \left\{ \frac{\mu^*}{\mu^* - \mu^*} \right\}.
\]

**Proof.** By Example 2.5 (e), Theorem 3.5 and Eq. (31), we have

\[
\begin{align*}
\langle \frac{\delta^*}{\delta_1^*} + \frac{\delta^*}{\delta_2^*}, \frac{\delta^*}{\delta_1} \rangle = \langle \delta_1^* \circ_1 \delta_2^*, \delta_1 \circ_1 \delta_2 \rangle_1 - \langle \delta_2^* \circ_1 \delta_1^*, \delta_2 \circ_1 \delta_1 \rangle_1 = \delta_1^* (\delta_1) \delta_2^* (\delta_2) - \delta_2^* (\delta_2) \delta_1^* (\delta_1) = 0,
\end{align*}
\]

and

\[
\begin{align*}
\left\langle \frac{\mu^*}{\delta^* - \delta^*}, \frac{\mu^*}{\delta^* - \mu^*} - \frac{\mu^*}{\delta^*} \right\rangle_2 &= \langle \delta^* \circ_1 \mu^*, \delta \circ_1 \mu \rangle_2 + \langle \mu^* \circ_1 \delta^*, \mu \circ_1 \delta \rangle_2 \\
&= - \delta^* (\delta) \mu^* (\mu) + \mu^* (\mu) \delta^* (\delta) = 0
\end{align*}
\]
for any \( \delta, \delta_1, \delta_2 \in \Delta \). Similarly,

\[
\begin{pmatrix}
\mu_\delta - \mu_\delta - \mu_\delta - \mu_\delta \\
\mu_\mu - \mu_\mu - \mu_\mu - \mu_\mu
\end{pmatrix}_2 = 0,
\]

and

\[
\begin{pmatrix}
\mu_\delta - \mu_\delta - \mu_\delta - \mu_\delta \\
\mu_\mu - \mu_\mu - \mu_\mu - \mu_\mu
\end{pmatrix}_3 = (\mu^* \circ_1 \mu^* \circ \mu \circ_1 \mu)_3 + (\mu^* \circ_2 \mu^* \circ \mu \circ_2 \mu)_3 = \mu^*(\mu)\mu^*(\mu)^* - \mu^*(\mu)\mu^*(\mu) = 0.
\]

We now compute the dimension of \( \mathcal{T}(E)_2^{(2)} \):

\[
\dim(\mathcal{T}(E)_2^{(2)}) = |\Delta|^2, \quad \dim(\mathcal{T}(E)_2^{(2)}) = 3|\Delta|, \quad \dim(\mathcal{T}(E)_3^{(2)}) = 2,
\]

the dimension of the relation \( kR \):

\[
\dim(kR_1) = \frac{|\Delta|(|\Delta| - 1)}{2}, \quad \dim(kR_2) = |\Delta|, \quad \dim(kR_3) = 1,
\]

and the dimension of the orthogonal space \( R^\perp \):

\[
\dim(R_1^\perp) = \frac{|\Delta|(|\Delta| + 1)}{2}, \quad \dim(R_2^\perp) = 2|\Delta|, \quad \dim(R_3^\perp) = 1.
\]

Then the conclusion follows from the equality

\[
\dim(R_1^\perp) + \dim(kR_i) = \dim(\mathcal{T}(E)_i^{(2)}) \quad \text{for } i = 1, 2, 3.
\]

\section{3.3. Self duality of the matching compatibility}

A remarkable property of the matching compatibility is its self duality. Let us first record an easy fact for later use.

\textbf{Lemma 3.7.} Let \( U, V \) be vector spaces, \( W = U \oplus V \) and \( \langle -, - \rangle : W^* \otimes W \to k \) the natural pairing \( u^* \otimes v \mapsto u^*(v) \). Suppose \( X \subseteq U \) and \( Y \subseteq V \). Then

\[
(X \cup Y)^\perp = k(X^\perp|_U \cup Y^\perp|_V).
\]

\textbf{Theorem 3.8.} Let \( \Omega \) be a nonempty finite set. Let \( \mathcal{P} = \mathcal{T}(E)/\langle R \rangle \) be a finitely generated unary binary quadratic ns operad. Then \( (\mathcal{P}_\Omega^{MT})^! = (\mathcal{P}_\Omega^1)^{MT} \).

\textbf{Proof.} Since the operad \( \mathcal{P} = \mathcal{T}(E, R) \) is quadratic, it follows from Eqs. (4) – (10) that

\[
R_{1,3} = R_{2,3} = R_{3,3} = R_4 = \emptyset.
\]

Abbreviate

\[
R_1 := R_{1,2} = \left\{ r^n_{1,2}(P_k, P_r) \mid 1 \leq n_1 \leq n_1 \right\},
\]

\[
R_2 := R_{2,2} = \left\{ r^n_{2,2}(P_k, i) \mid 1 \leq n_2 \leq n_2 \right\},
\]

\[
R_3 := R_{3,2} = \left\{ r^n_{2,2}(i, j) \mid 1 \leq n_3 \leq n_3 \right\},
\]

and

\[
R_i^{\mu, \nu} := R_i^{\mu, \nu}, \quad \text{for } i = 1, 2, 3.
\]

Eq. (28) gives

\[
(R_1^\perp)_M = \bigcup_{\mu \neq \nu \in \Omega} \left( (R_1^\perp)_1^{\mu, \nu} \cup (R_1^\perp)_2^{\mu, \nu} \cup (R_1^\perp)_3^{\mu, \nu} \right).
\]

By Definition 3.1 and Theorem 3.5,

\[
(\mathcal{P}_\Omega^{MT})^! = \mathcal{T}\left( \bigoplus_{\omega \in \Omega} E_\omega \right) / \left( \bigcup_{\omega \in \Omega} R_\omega \cup R_M \right)^!.
\]
\[
(\mathcal{P}^1)^{\text{MT}}_{\Omega} = \mathcal{T}\left( \bigoplus_{\omega \in \Omega} E^*_\omega \right) / \left( \bigcup_{\omega \in \Omega} (\mathbb{R}^1)_\omega \cup (\mathbb{R}^1)_M \right).
\]

We identify \( \mathcal{T}\left( \bigoplus_{\omega \in \Omega} E^*_\omega \right) \) with \( \mathcal{T}\left( (\bigoplus_{\omega \in \Omega} E^*_\omega)^* \right) \) by \( \bigoplus_{\omega \in \Omega} E^*_\omega \cong (\bigoplus_{\omega \in \Omega} E^*_\omega)^* \). Denote by \( \mathcal{T}_\Omega \) (resp. \( \mathcal{T}_\mathbb{F} \)) the subspace of \( \mathcal{T}\left( \bigoplus_{\omega \in \Omega} E^*_\omega \right) \) generated by trees with vertices decorated by \( E^*_\omega \) (resp. \( E^*_\omega, E^*_\omega^\prime, \ldots, E^*_\omega^r \), for \( 1 \leq n \leq |\Omega| \)), for some \( \omega \in \Omega \) (resp. for \( \omega_i \in \Omega \), not all identical). Then

\[
(33) \quad \mathcal{T}\left( \bigoplus_{\omega \in \Omega} E^*_\omega \right) = \mathcal{T}_\Omega \oplus \mathcal{T}_\mathbb{F}.
\]

Denote by \( \mathcal{T}_{\mathbb{F}, n} \) the component of \( \mathcal{T}_\mathbb{F} \) in arity \( n \). We only need to verify \( \left( \bigcup_{\omega \in \Omega} (\mathbb{R}^1)_\omega \cup (\mathbb{R}^1)_M \right) \) amounts to the three equations:

\[
(34) \quad k\left( \bigcup_{\omega \in \Omega} (\mathbb{R}^1)_\omega \cup (\mathbb{R}^1)_M \right) = k\left( \bigcup_{\omega \in \Omega} (\mathbb{R}^1)_\omega \cup (\mathbb{R}^1)_M \right) = k\left( \bigcup_{\omega \in \Omega} (\mathbb{R}^1)_\omega \cup (\mathbb{R}^1)_M \right).
\]

Now the left hand side of the equation is

\[
k\left( \bigcup_{\omega \in \Omega} (\mathbb{R}^1)_\omega \cup (\mathbb{R}^1)_M \right) = k\left( \bigcup_{\omega \in \Omega} (\mathbb{R}^1)_\omega \cup (\mathbb{R}^1)_M \right) = k\left( \bigcup_{\omega \in \Omega} (\mathbb{R}^1)_\omega \cup (\mathbb{R}^1)_M \right) = k\left( \bigcup_{\omega \in \Omega} (\mathbb{R}^1)_\omega \cup (\mathbb{R}^1)_M \right).
\]

Applying the arity grading and Eq. (32), then Eq. (34) amounts to the three equations:

\[
(35) \quad k\left( \bigcup_{\mu \neq \in \Omega} (R^1)_i \right) = k\left( \bigcup_{\mu \neq \in \Omega} (R^1)_i \right), \quad i = 1, 2, 3.
\]

Denote

\[
(36) \quad (R^1)_i := \left\{ r^n_{k_1}(P_{k_1}^*, P_{k_2}^*) : 1 \leq n \leq n_1 \right\}.
\]

Then Eq. (35) for \( i = 1 \) follows from

\[
k\left( \bigcup_{\mu \neq \in \Omega} (R^1)_i \right) = k\left( \bigcup_{\mu \neq \in \Omega} (R^1)_i \right) = k\left( \bigcup_{\mu \neq \in \Omega} (R^1)_i \right).
\]
Corollary 3.9. (a) ([42]) The operad of matching (associative) algebras is self dual.
(b) The operad of (multiple) matching Poisson algebras is self dual.

We also give some examples of self dual operads which have nontrivial unary operations.

Corollary 3.10. (a) Let $\mathcal{P}$ be the operad with generators 

$$
E_1 = \mathbb{k}\{d_1, d_2\}, \quad E_2 = \mathbb{k} \gamma,
$$

and relations

$$
R_1 = \{d_1, d_2\}, \quad R_3 = \{\gamma - \gamma\}, \\
R_2 = \{d_1 - d_1, d_2 - d_2\}.
$$

Then $\mathcal{P}$ is self dual.
(b) For the operad $\mathcal{P}$ above and any nonempty finite set $\Omega$, the operad $\mathcal{P}_\Omega$ is self dual.

Proof. Item (a) can be verified by the same computation as for Proposition 3.6. Then Item (b) follows from Theorem 3.8.

We next show that taking a matching family of an operad can be obtained by taking the black square product or the white square product.

Proposition 3.11. Let $\Omega$ be a nonempty finite set. Let $\mathcal{P}$ be a finitely generated binary quadratic ns operad. Then

$$
\mathcal{P}_\Omega \cong \mathcal{A}_\Omega \boxtimes \mathcal{P} \quad \text{and} \quad \mathcal{P}_\Omega \cong \mathcal{A}_\Omega \square \mathcal{P}.
$$

Proof. For the associative operad $\mathcal{A} = \mathcal{T}(E)/\langle R \rangle$ in Example 2.5 (a), by Definition 2.11, we have

$$
\mathcal{A}_\Omega^{MT} = \mathcal{T}\left( \bigoplus_{\omega \in \Omega} E_\omega \right)/\langle \bigcup_{\omega \in \Omega} R_\omega \cup R_M \rangle = \mathcal{T}\left( \bigoplus_{\omega \in \Omega} E_\omega \right)/\langle \bigcup_{\omega \in \Omega} R_\omega \cup \left( \bigcup_{\mu \neq \nu \in \Omega} R_{\mu, \nu} \right) \rangle,
$$

where the generators

$$
\bigoplus_{\omega \in \Omega} E_\omega = \bigoplus_{\omega \in \Omega} k^{\omega}
$$

and relations are given in Eq. (26). Let $\mathcal{P} = \mathcal{T}(F)/\langle S \rangle$ with

$$
F := F_2 := \mathbb{k}\{\gamma, \gamma', ..., \gamma\}
$$

and

$$
S := S_{3,2} := \left\{ \sum_{1 \leq i,j \leq s} \gamma_{i,j}^{1,2} \gamma_{i,j} + \gamma_{i,j}^{3,2} \gamma_{i,j} \bigg| 1 \leq d \leq s \right\}.
$$

Recall from Eq. (27) that

$$
\left( \bigoplus_{\omega \in \Omega} E_\omega \right) \otimes F \cong \bigoplus_{\omega \in \Omega} F_\omega.
$$

By Definitions 2.13 (a), we obtain

$$
\left( \bigcup_{\omega \in \Omega} R_\omega \cup \bigcup_{\mu \neq \nu \in \Omega} R_{\mu, \nu} \right) \boxtimes S
$$
Therefore

\[ \mathcal{A}^{\text{MT}}_{\Omega} \mathcal{P} = \mathcal{F}(\bigoplus_{\omega \in \Omega} E_\omega \otimes F) / \langle \bigcup_{\omega \in \Omega} R_\omega \cup \bigcup_{\mu \neq \nu \in \Omega} R^{\mu, \nu} \rangle S \]

\[ \cong \mathcal{F}(\bigoplus_{\omega \in \Omega} F_\omega) / \langle \bigcup_{\omega \in \Omega} S_{\omega, 3, 2} \cup \bigcup_{\mu \neq \nu \in \Omega} S^{\mu, \nu}_{3, 2} \rangle \]

\[ = \mathcal{A}^{\text{MT}}_{\Omega} \mathcal{P}. \]

To prove the second isomorphism, we recall the duality between the black square products and white square products for any finitely generated binary ns operad \( \mathcal{P} \) and \( \mathcal{Q} \) [39]:

\[ (\mathcal{P} \boxdot \mathcal{Q})^! = \mathcal{P}^! \boxdot \mathcal{Q}^! \]

Applying it to the first isomorphism and utilizing Theorem 3.8, we obtain

\[ (\mathcal{P}^{\text{MT}})^!_\Omega \cong (\mathcal{A}^{\text{MT}}_{\Omega} \mathcal{P})^! \cong (\mathcal{A}^{\text{MT}}_{\Omega} \mathcal{P})^! \cong \mathcal{A}^{\text{MT}}_{\Omega} \mathcal{P}, \]

hence the result after replacing \( \mathcal{P}^! \) by \( \mathcal{P} \).

\[ \square \]

4. Total compatibility and Koszul duality

In this section, we study the total compatibility. It is stronger than the matching compatibility and is in duality with the linear compatibility.

4.1. Totally compatible operads. Let \( \mathcal{P} \) be a unary binary quadratic/cubic ns operad. Recall from Eqs. (4) – (10) that the relations \( R \) of \( \mathcal{P} \). For \( r = \sum_{t} \alpha_t t \in R \) with \( \alpha_t \in k \), the support \( \text{Supp}(r) \) of \( r \) is defined to be the set of decorated tree \( t \) with \( \alpha_t \neq 0 \). For \( S \subseteq R \), denote

\[ \text{Supp}(S) := \bigcup_{t \in S} \text{Supp}(r). \]

Let \( \Omega \) be a nonempty set. For \( \mu, \nu \in \Omega \), we define

\[ R^{\mu, \nu}_{T, 1, 2} := \{ t_{1, 2}(P_{\mu, k}, P_{\nu, t}) - t_{1, 2}(P_{\nu, k}, P_{\mu, t}) \mid t_{1, 2}(P_{k}, P_{t}) \in \text{Supp}(R_{1, 2}), \} \]

\[ R^{\mu, \nu}_{T, 1, 3} := \{ t_{1, 3}(P_{\mu, k}, P_{\nu, t}, P_{\mu, m}) - t_{1, 3}(P_{\nu, k}, P_{\mu, t}, P_{\mu, m}), \} \]
\[ t_{1,3}(P_{\mu,k}, P_{\nu,\ell}, P_{\mu,m}) - t_{1,3}(P_{\mu,k}, P_{\mu,\ell}, P_{\nu,m}) \mid t_{1,3}(P_k, P_\ell, P_m) \in \text{Supp}(R_{1,3}) \],

\[ R_{T,2,2}^{\mu,v} := \{ t_{2,2}(P_{\mu,k}, i_\nu) - t_{2,2}(P_{\nu,k}, i_\mu) \mid t_{2,2}(P_k, i) \in \text{Supp}(R_{2,2}) \}, \]

\[ R_{T,2,3}^{\mu,v} := \{ t_{2,3}(P_{\mu,k}, P_{\nu,\ell}, i_\mu) - t_{2,3}(P_{\nu,k}, P_{\mu,\ell}, i_\nu) \mid t_{2,3}(P_k, P_\ell, i) \in \text{Supp}(R_{2,3}) \}, \]

Set

\begin{equation}
R_T := \bigcup_{\mu, v \in \Omega} (R_{T,1,2}^{\mu,v} \cup R_{T,2,2}^{\mu,v} \cup R_{T,2,3}^{\mu,v}).
\end{equation}

Roughly speaking, these relations means that the decorated trees appearing in the relations of \( \mathcal{P} \) can carry arbitrary decorations in \( \Omega \).

**Example 4.1.** Let \( \Omega = \{1, 2\} \). For the associative operad \( \mathcal{AS} \), we have

\[ \text{Supp}(R_{3,2}) = \{\quad \}
\]

\[ R_{T,3,2}^{1,2} = \{\quad - \quad\}
\]

Now we give our last compatibility condition for operads. Recall that \( R_M \) is given in Eq. (28).

**Definition 4.2.** Let \( \Omega \) be a nonempty set. Let \( \mathcal{P} = (\mathcal{T}(\mathcal{E})/(R) \) be a unary binary quadratic/cubic ns operad. We call

\[ \mathcal{P}_\Omega^{TC} := \mathcal{T}\left( \bigoplus_{\omega \in \Omega} E_\omega \right) / \left( \bigcup_{\omega \in \Omega} R_\omega \cup R_M \cup R_T \right) \]

the **totally compatible operad** of \( \mathcal{P} \) with parameter \( \Omega \).

**Example 4.3.** Let \( \Omega = \{1, 2\} \) and \( \mathcal{P} \) be the operad of Rota-Baxter algebras with the unary operation \( P \) and binary operation \( \bullet \). Then a \( \mathcal{P}_\Omega^{TC} \)-algebra is a vector space \( A \) with associative multiplications \( \bullet_1, \bullet_2 \) and Rota-Baxter operators \( P_1, P_2 \) satisfying the additional conditions

\[(x \bullet_1 y) \bullet_2 z = (x \bullet_2 y) \bullet_1 z = x \bullet_1 (y \bullet_2 z) = x \bullet_2 (y \bullet_1 z) \]

and

\[ P_i(x) \bullet_j P_j(y) = P_i(P_i(x) \bullet_j y) + P_j(x \bullet_j P_j(y)), \]

\[ P_i(x) \bullet_i P_j(y) = P_j(x) \bullet_i P_j(y), \]

\[ P_j(P_i(x) \bullet_i y) = P_i(P_j(x) \bullet_i y) = P_i(P_i(x) \bullet_j y), \]

\[ P_i(x \bullet_i P_j(y)) = P_j(x \bullet_j P_i(y)) = P_j(x \bullet_j P_i(y)), \]

for \( i \neq j \in \{1, 2\} \). Here the last three lines are relations \( R_{T,2,3}^{\mu,v} \), for \( \mu, v \in \Omega \).
Since $\mathcal{P}^{TC}$ has the extra conditions $R_T$ beyond $\mathcal{P}^{MT}$, we have

**Proposition 4.4.** Let $\mathcal{P}$ be a unary binary quadratic/cubic ns operad. Then there is an epimorphism of ns operads

$$\mathcal{P}^{MT}_\Omega \longrightarrow \mathcal{P}^{TC}_\Omega.$$ 

In other words, a $\mathcal{P}^{TC}_\Omega$-algebra is a $\mathcal{P}^{MT}_\Omega$-algebra.

4.2. **Total compatibility, Koszul duality and Manin white square products.** We first show that the total compatibility is in Koszul dual to the linear compatibility.

**Theorem 4.5.** Let $\Omega$ be a nonempty finite set. Let $\mathcal{P} = \mathcal{F}(E)/\langle R \rangle$ be a finitely generated unary binary quadratic ns operad. Then $(\mathcal{P}^{LC})^! = (\mathcal{P}^{TC})^!$ and $(\mathcal{P}^{TC})^! = (\mathcal{P}^{LC})^!$.

**Proof.** We first prove $(\mathcal{P}^{LC})^! = (\mathcal{P}^{TC})^!$. Since the operad $\mathcal{P} = \mathcal{P}(E,R)$ is quadratic, it follows from Eqs. (4) – (10) that 

$$R_{1,3} = R_{2,3} = R_{3,3} = R_4 = 0.$$ 

Denote

- $R_1 := R_{1,2} = \left\{ r^i_1(P_k, P_L) := r^i(P_k, P_L) \mid 1 \leq n \leq n_1 \right\},$
- $R_2 := R_{2,2} = \left\{ r^i_2(P_k, i) := r^i(P_k, i) \mid 1 \leq n \leq n_2 \right\},$
- $R_3 := R_{3,2} = \left\{ r^i_3(i, j) := r^i(i, j) \mid 1 \leq n \leq n_3 \right\},$

and

$$R_{i,2}^{\mu,\nu} := r_{i,2}^{\mu,\nu}, \quad R_{T,i,2}^{\mu,\nu} := r_{T,i,2}^{\mu,\nu} \text{ for } i = 1, 2, 3.$$ 

By Eq. (28), we have

$$\quad (R^M)_{\mu} = \bigcup_{\mu \neq v \in \Omega} (\mathcal{P}^{\mu,\nu})_{\mu}.$$ 

By Definitions 2.11 and 4.2, and Theorem 3.5, we obtain

$$\quad (\mathcal{P}^{LC})^! = \mathcal{T}\left( \bigoplus_{\omega \in \Omega} E_{\omega}^* \right) / \left( \bigcup_{\omega \in \Omega} R_{\omega} \cup R_L \right)^1,$$

and

$$\quad (\mathcal{P}^{TC})^! = \mathcal{T}\left( \bigoplus_{\omega \in \Omega} E_{\omega}^* \right) / \left( \bigcup_{\omega \in \Omega} (R^M_\omega) \cup (R^T_\omega) \right).$$

Identify $\mathcal{T}\left( \bigoplus_{\omega \in \Omega} E_{\omega}^* \right)$ with $\mathcal{T}\left( \bigoplus_{\omega \in \Omega} E_{\omega}^* \right)$ by $\bigoplus_{\omega \in \Omega} E_{\omega}^* \cong \left( \bigoplus_{\omega \in \Omega} E_{\omega}^* \right)$. Denote by $\mathcal{T}_\Omega$ (resp. $\mathcal{T}_\Omega^*$) the subspace of $\mathcal{T}\left( \bigoplus_{\omega \in \Omega} E_{\omega}^* \right)$, spanned by trees with vertices decorated by $E_{\omega}^*$ (resp. by $E_{\omega_1}^*, E_{\omega_2}^*, \ldots, E_{\omega_n}^*$, for $1 \leq n \leq |\Omega|$), for some $\omega \in \Omega$ (resp. for $\omega \in \Omega$ not all identical). Then

$$\quad \mathcal{T}\left( \bigoplus_{\omega \in \Omega} E_{\omega}^* \right) = \mathcal{T}_\Omega \oplus \mathcal{T}_{\Omega^*}.$$ 

Denote by $\mathcal{T}_{\Omega^*,n}$ the component of $\mathcal{T}_{\Omega^*}$ in arity $n$. We only need to show

$$\quad \langle \bigcup_{\omega \in \Omega} R_{\omega} \cup R_L \rangle_1 = \langle \bigcup_{\omega \in \Omega} (R^M_\omega) \cup (R^T_\omega) \rangle_1,$$

which follows from the equality

$$\quad (R^M_\omega) = \bigcup_{\omega \in \Omega} (R^M_\omega) \cup (R^T_\omega).$$
The left hand side of the equality is
\[
\kappa\left(\bigcup_{\omega \in \Omega} R_{\omega} \cup R_{L}\right) = \kappa\left(\bigcup_{\omega \in \Omega} R_{\omega} \cup R_{L}\bigg|_{\beta_0} \bigcup \left(\bigcup_{\omega \in \Omega} R_{\omega} \cup R_{L}\bigg|_{\beta_F}\right)^{-1}\right)
\]
(by Eq. (39))
\[
= \kappa\left(\bigcup_{\omega \in \Omega} R_{\omega} \cup R_{L}\bigg|_{\beta_0} \bigcup \left(\bigcup_{\mu \in \Omega} R_{\mu} \cup R_{L}\bigg|_{\beta_F}\right)^{-1}\right)
\]
(by Lemma 3.7)
\[
= \kappa\left(\bigcup_{\omega \in \Omega} \left(\bigcup_{\omega \in \Omega} \left(\bigcup_{\mu \in \Omega} R_{\mu} \cup R_{L}\bigg|_{\beta_F}\right)^{-1}\right)\bigg|_{\beta_0}\right)
\]
(by Eq. (21) and Lemma 3.7).

Applying the arity grading, and Eqs. (37) and (38), one finds that Eq. (40) is equivalent to the equations:
\[
\kappa\left(\bigcup_{\mu \neq \nu \in \Omega} \left(\bigcup_{\omega \in \Omega} \left(\bigcup_{\mu \in \Omega} R_{\mu} \cup R_{L}\bigg|_{\beta_F}\right)^{-1}\right)\bigg|_{\beta_0}\right) = \kappa\left(\bigcup_{\mu \neq \nu \in \Omega} \left(\bigcup_{\omega \in \Omega} \left(\bigcup_{\mu \in \Omega} R_{\mu} \cup R_{L}\bigg|_{\beta_F}\right)^{-1}\right)\bigg|_{\beta_0}\right), \quad i = 1, 2, 3.
\]

First denote
\[
(R^+)_1 = \left\{ \sum_{1 \leq k,l \leq t} \alpha_{k,l}^n \left| P_{k,l}^n \right| 1 \leq n \leq n_1' \right\}.
\]
Then Eq. (41) for \(i = 1\) follows from
\[
\kappa\left(\bigcup_{\mu \neq \nu \in \Omega} \left(\bigcup_{\omega \in \Omega} \left(\bigcup_{\mu \in \Omega} R_{\mu} \cup R_{L}\bigg|_{\beta_F}\right)^{-1}\right)\bigg|_{\beta_0}\right)
\]
\[
= \kappa\left\{ \sum_{1 \leq k,l \leq t} \alpha_{k,l}^n \left( \left| P_{k,l}^n \right| + \left| P_{k,l}^{\mu} \right| \right) \bigg| 1 \leq n \leq n_1' \right\} \bigg|_{\beta_0}\n\]
\[
= \kappa\left\{ \sum_{1 \leq k,l \leq t} \alpha_{k,l}^n \left( \left| P_{k,l}^n \right| - \left| P_{k,l}^{\mu} \right| \right) \bigg| 1 \leq n \leq n_1' \right\} \bigg|_{\beta_0}\n\]
\[
= \kappa\left( \left\{ \sum_{1 \leq k,l \leq t} \alpha_{k,l}^n \left| P_{k,l}^n \right| \bigg| \mu \neq \nu \in \Omega, 1 \leq n \leq n_1' \right\} \bigcup \left\{ \left| P_{k,l}^{\mu} \right| \bigg| \mu \neq \nu \in \Omega \right\} \right)\bigg|_{\beta_0}\n\]
\[
= \kappa\left( \bigcup_{\mu \neq \nu \in \Omega} \left(\bigcup_{\omega \in \Omega} \left(\bigcup_{\mu \in \Omega} R_{\mu} \cup R_{L}\bigg|_{\beta_F}\right)^{-1}\right)\bigg|_{\beta_0}\right).
\]

We similarly verify Eq. (41) for \(i = 2, 3\). This completes the proof of \((\mathcal{P}^{\text{LC}})^! = (\mathcal{P})^{\text{TC}}^!\). Since \((\mathcal{P})^! = \mathcal{P}\) for any quadratic operad \(\mathcal{P}\), we have \(\mathcal{P}^{\text{LC}} = ((\mathcal{P}^{\text{LC}})^!)^! = ((\mathcal{P})^{\text{TC}})^!\) and so \((\mathcal{P})^{\text{LC}} = (\mathcal{P}^{\text{TC}})^!\).

We finally show that taking the total compatibility of an operad amounts to taking the Manin white square product with the operad of totally compatible associative algebras.
Corollary 4.6. Let $\mathcal{P}$ be a unary binary quadratic/cubic ns operad and $\Omega$ a nonempty set. Then $\mathcal{P}_{\Omega}^{TC} \cong \mathcal{A}_{\Omega}^{TC} \Box \mathcal{P}$.

Proof. The isomorphism follows from taking the Koszul dual of the isomorphism in Proposition 2.14 and applying Theorem 4.5:

\[
\mathcal{P}_{\Omega}^{TC} = (\mathcal{P}^{!})^{LC}_{\Omega} = (\mathcal{P}^{!})^{LC}_{\Omega} \equiv (\mathcal{A}_{\Omega}^{LC})^{!}(\mathcal{P}^{!})^{!} = (\mathcal{A}_{\Omega}^{TC})^{!} \Box \mathcal{P} = \mathcal{A}_{\Omega}^{TC} \Box \mathcal{P}.
\]

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