A presentation of PGL(2,Q)

A. Muhammed Uludağ*

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Abstract

We give a conjectural presentation of the infinitely generated group PGL(2,Q) with an infinite list of relators.

1 Introduction

The group PGL₂(Q) is defined as the projectivization of the group GL₂(Q). Our aim is to give a conjectural presentation of it.

This group contains the subgroup PSL₂(Q) of 2×2 rational projective matrices of determinant 1. One has the inclusions

\[
PSL₂(\mathbb{Z}) < PGL₂(\mathbb{Z}) < PSL₂(\mathbb{Q})^\pm < PGL₂(\mathbb{Q}) < PGL₂(\mathbb{R}),
\]

\[
PSL₂(\mathbb{Z}) < PSL₂(\mathbb{Q}) < PGL₂(\mathbb{Q})^+ < PSL₂(\mathbb{R}),
\]

where

PSL₂(\mathbb{Q})^\pm is the group of 2×2 rational projective matrices of determinant ±1.
PGL₂(\mathbb{Z}) is the group of 2×2 integral projective matrices of determinant 1.
PSL₂(\mathbb{Z}) is the group of 2×2 integral projective matrices of determinant 1.
PGL₂(\mathbb{Q})^+ is the group of 2×2 rational matrices of determinant > 0.

Every inclusion in the above list is of infinite index, except the first one. The modular group PSL₂(\mathbb{Z}) and its \( \mathbb{Z}/2\mathbb{Z} \)-extension PGL₂(\mathbb{Z}) in this hierarchy are finitely generated and their presentations are known. Other groups are not finitely generated and to our knowledge their presentations are not known.

*Department of Mathematics, Galatasaray University Çırağan Cad. No. 36, 34349 Beşiktaş İstanbul, Turkey
Dresden classifies the finite subgroups and finite-order elements (see Section below) of this group. Besides a few papers about some $\text{PGL}_2(\mathbb{Q})$-cocycles related to some partial zeta values and to generalized Dedekind sums, (see [8], [9]) we were unable to spot any works about this group.

An infinitely presented group is usually not a very friendly object, nevertheless, due to its connection to the lattice $\mathbb{Z}^2$, the group $\text{PGL}_2(\mathbb{Q})$ merits a better treatment. It turns out that its presentation is not so complicated, and one has the following parallelism between $\text{PGL}_2(\mathbb{Z})$ and $\text{PGL}_2(\mathbb{Q})$. The Borel subgroup $B(\mathbb{Z})$ of $\text{PGL}_2(\mathbb{Z})$, which by definition is the set of upper triangular elements, is generated by the translation $T_z := 1 + z$ and the reflection $V_z := -z$ (where we take the liberty to consider the elements of $\text{PGL}_2(\mathbb{Q})$ as elements of the Möbius group of $\mathbb{P}^1(\mathbb{R})$). Since $T = KV$, the Borel subgroup is also generated by the involutions $V$ and $K := 1 - z$, showing that $B(\mathbb{Z})$ is the infinite dihedral group. The group $\text{PGL}_2(\mathbb{Z})$ itself is generated by its Borel subgroup $B(\mathbb{Z})$ and the involution $U_z := 1/z$. Note that the derived subgroup of $B(\mathbb{Z})$ is $\mathbb{Z}$.

Likewise, $\text{PGL}_2(\mathbb{Q})$ is generated by its Borel subgroup $B(\mathbb{Q})$ and $U$. Here $B(\mathbb{Q})$ is infinitely generated but nevertheless is quite similar to the infinite dihedral group in that its derived subgroup is $\mathbb{Q}$. It admits the presentation

$$B(\mathbb{Q}) \simeq \langle K, H_p \mid H_p^{-1}TPH_p = T, [H_p, H_q] = 1, p, q = -1, 2, 3, 5, 7, \ldots \rangle$$

where the elements $H_nz := nz$ are the homotheties (note that $T = KV$ and $V = H_{-1}$). Its abelianization is $\mathbb{Q}^\times$. These claims are easily verified by using the matrix description of $B(\mathbb{Q})$.

The subgroup $\langle T, H_p \rangle < B(\mathbb{Q})$ admits the presentation

$$\langle T, H_p \mid H_p^{-1}TPH_p = T \rangle,$$

i.e. it is the Baumslag-Solitar group $BS(p, 1)$. Its abelianization is $\mathbb{Z} \times \mathbb{Z}/(p - 1)\mathbb{Z}$. See [7], [3] for more about $BS(p, 1)$.

The derived subgroup of $\text{PGL}_2(\mathbb{Q})$ is $\text{PSL}_2(\mathbb{Q})$. It is simple [4]. The abelianization of $\text{PGL}_2(\mathbb{Q})$ is the multiplicative group of integers modulo square integers. This latter is an infinitely generated 2-torsion group. It follows that any quotient of $\text{PGL}_2(\mathbb{Q})$ is an abelian 2-torsion group.

2 The monoid of integral matrices

Let $\Lambda$ be the free abelian group of rank 2 and consider the monoid of all $\mathbb{Z}$-module morphisms $\Lambda \to \Lambda$. By using the standard generators for $\Lambda$, we may identify this
monoid with the monoid of $2 \times 2$ matrices over $\mathbb{Z}$, denoted by $\mathbb{M}_2(\mathbb{Z})$. Thus we consider the monoid

$$\mathbb{M}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} p & q \\ r & s \end{pmatrix} \mid p, q, r, s \in \mathbb{Z} \right\},$$

under the usual matrix product. Set-wise, $\mathbb{M}_2(\mathbb{Z})$ is just $\mathbb{Z}^4$. It admits the set of scalar matrices $\langle nI \mid n \in \mathbb{Z} \rangle$ as a submonoid. The quotient of $\mathbb{M}_2(\mathbb{Z})$ by this submonoid will be referred to as its projectivization and denoted by $\mathbb{P}\mathbb{M}_2(\mathbb{Z})$. Denote by $\mathbb{M}_2^{\text{n.s.}}(\mathbb{Z})$ the set of non-singular matrices and by $\mathbb{M}_2^{\text{\ast}}(\mathbb{Z})$ those with a square determinant.

The set of invertible elements inside $\mathbb{P}\mathbb{M}_2(\mathbb{Z})$ consists of projective two-by-two non-singular matrices with rational (or equivalently integral) entries and is denoted by $\text{PGL}_2(\mathbb{Q})$. To be more precise, denote by $[M]$ the projectivization of $M$. Then

$$\text{PGL}_2(\mathbb{Q}) := \{ [M] : M \in \mathbb{M}_2(\mathbb{Z}), \quad \det(M) \neq 0 \} = \mathbb{P}\mathbb{M}_2^{\text{n.s.}}(\mathbb{Z}).$$

The group $\text{PGL}_2(\mathbb{Q})$ is isomorphic to the group of integral Möbius transformations of $\mathbb{P}^1(\mathbb{R})$. It contains the projectivization of the set of matrices with rational entries and of determinant 1 as a subgroup, the group $\text{PSL}_2(\mathbb{Q})$. Hence,

$$\text{PSL}_2(\mathbb{Q}) := \{ [M] : M \in \mathbb{M}_2(\mathbb{Z}), \quad \det(M) \text{ is a square} \} = \mathbb{P}\mathbb{M}_2^{\text{\ast}}(\mathbb{Z}).$$

The multiplicative group $\mathbb{Z}^\times$ admits a subgroup which consists of square rationals which we denote as $\mathbb{Z}^{2\times}$. The group $\mathbb{Z}^\times / \mathbb{Z}^{2\times}$ of square-free rationals is the torsion abelian group generated by $\{ x_{-1} \} \cup \{ x_p : p \text{ prime} \}$ subject to the relations $x_i^2 = 1$ for each $i = -1, 2, 3, \ldots$. It is isomorphic to $\mathbb{Q}^\times / \mathbb{Q}^{2\times}$.

The map $\det : \mathbb{M}_2(\mathbb{Z}) \to \mathbb{Z}$ is equivariant under the action of $a \in \mathbb{Z}$ by

$$\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}) \to \begin{pmatrix} ap & aq \\ ar & as \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z})$$
on the left and by $t \to a^2t$ on the right. Hence we may projectivize it as follows:

$$\begin{array}{cccccc}
1 & \to & \langle \pm I \rangle & \to & \mathbb{I} \mathbb{Z}^\times & \to & \mathbb{Z}^{2\times} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \mathbb{M}_2(\mathbb{Z}) & \to & \mathbb{M}_2^{\text{n.s.}}(\mathbb{Z}) & \overset{\det}{\to} & \mathbb{Z}^\times \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \to & \text{PSL}_2(\mathbb{Q}) & \to & \text{PGL}_2(\mathbb{Q}) & \overset{\text{pdet}}{\to} & \mathbb{Z}^\times / \mathbb{Z}^{2\times} \\
\end{array}$$
3 Generators

The modular group. It is well known (see [3]) that the extended modular group \( \text{PGL}_2(\mathbb{Z}) \) admits the presentation
\[
\langle V, U, K \mid V^2 = U^2 = K^2 = (VU)^2 = (KU)^3 = 1 \rangle.
\]
The modular group is the subgroup \( \text{PSL}_2(\mathbb{Z}) \) is generated by \( S(z) = -1/z \) and \( L(z) = (z - 1)/z \) and admits the presentation \( \text{PSL}_2(\mathbb{Z}) = \langle S, L \mid S^2 = L^3 = 1 \rangle \).

Homotheties. For \( r = m/n \in \mathbb{Q} \), consider the homothety \( H_r : z \mapsto rz \). In matrix form
\[
H_r = \begin{bmatrix} r & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} m & 0 \\ 0 & n \end{bmatrix} \implies H_r^{-1} = H_{1/r}, \quad H_rH_s = H_sH_r = H_{rs}.
\]
Hence every homothety is a product of “prime” homotheties. \( H_{-1} \) is involutive. The remaining homotheties forms a free abelian group of infinite rank.

Involutions. Let \( r = m/n \in \mathbb{Q} \) and consider the involutions \( I_r(z) := r/z = m/nz \in \text{PGL}_2(\mathbb{Q}) \), which is represented by the traceless matrix
\[
I_r = \begin{bmatrix} 0 & r \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & m \\ n & 0 \end{bmatrix}.
\]
Note that \( \text{pdet}(I_p) = x_p \). One has
\[
I_rI_s = H_{r/s}, \quad I_sI_r = H_{s/r}, \quad [I_r, I_s] = I_rI_sI_r^{-1}I_s^{-1} = H_{r^2s^2} \in \text{PSL}_2(\mathbb{Q})
\]
In particular, \( I_r \) and \( I_s \) do not commute unless \( r = \pm s \).

The set of involutions \( \{I_r : r \in \mathbb{Q} \} \) generate a group containing the homotheties and which consists of two lines inside \( \text{PM}_2(\mathbb{R}) \):
\[
\langle I_r \mid r \in \mathbb{Q}^\times \rangle = \left\{ \begin{bmatrix} 0 & m \\ m & 0 \end{bmatrix}, \begin{bmatrix} 0 & n \\ n & 0 \end{bmatrix} \mid m, n \in \mathbb{Z}, mn \neq 0 \right\}
\]
Every involution \( I_r \) can be expressed in terms of the following list of involutions:
\[
V(z) = -z, \quad S(z) = I_{-1}(z) = -\frac{1}{z}, \quad I_2(z) = \frac{2}{z}, \quad I_3(z) = \frac{3}{z}, \ldots, I_p, \ldots \text{ (p prime)}.
\]
For example, \( I_{pq} = I_pI_qI_q = I_pUI_q \) where \( U(z) = I_1 = SV \). The involution \( V = SU = I_{-1}I_1 \) commutes with all involutions \( I_n \) for \( n \in \mathbb{Z} \). Also, \( S = I_{-1} \) and \( U = I_1 \) commute, otherwise \( I_p \) and \( I_q \) do not commute.
Proposition 1 The set \( \{ K, S, V \} \cup \{ I_p : p : \text{prime} \} \) generates \( \text{PGL}_2(\mathbb{Q}) \).

Proof. Denote the translation \( T_{r/s}(z) := z + r/s \). Then \( T_{r/s} = H_{r/s} K V H_{s/r} \), and unless \( p = 0 \), we have

\[
\frac{p z + q}{r z + s} = \frac{p}{r} \left\{ 1 - \frac{ps-qr}{z + \frac{s}{r}} \right\} = H_{p/r} K H_{(ps-qr)/pr} T_{s/r}(z).
\]

If \( p = 0 \), then

\[
\frac{p z + q}{r z + s} = \frac{q}{r z + s} = \frac{1}{r} \left( \frac{z + \frac{s}{r}}{r} \right) = H_{q/r} U T_{s/r}(z).
\]

Now \( U = K I_p(KV)^p V I_p K \) for any \( p \), as one may easily check. Finally, the result follows since we can express the homotheties in terms of involutions. \( \blacksquare \)

Corollary 1 The set \( \{ T, U, V \} \cup \{ H_p : p : \text{is prime } \} \) generates \( \text{PGL}_2(\mathbb{Q}) \).

When \( p = 2 \) the relation \( U = K I_p(KV)^p V I_p K \) implies \( U = K I_2 K V K I_2 K \), i.e. either one of the generators \( V \) and \( U \) can be eliminated from a generating set. The case \( p = 2 \) also implies that \( U \) and \( V \) are conjugate elements in \( \text{PGL}_2(\mathbb{Q}) \). They are not conjugates inside \( \text{PGL}_2(\mathbb{Z}) \).

4 Elements of finite order

\( \text{PGL}_2(\mathbb{R}) \) contains elements of any order. This is not true for \( \text{PGL}_2(\mathbb{Q}) \). Finite order matrices are elliptic so have \( \text{tr}^2(M)/\det(M) < 4 \). It is known [4] that the order of \( M \) can be 2, 3, 4 or 6 and \( M \) is \( \text{PGL}_2(\mathbb{Q}) \)-conjugate to

\[
M_3 := \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{(with fixed points} \quad \frac{-1 \pm i \sqrt{3}}{2} \text{)} \quad \text{if the order is 3} \quad (1)
\]

\[
M_4 := \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{(with fixed points} \quad \pm i \text{)} \quad \text{if the order is 4} \quad (2)
\]

\[
M_6 := \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{(with fixed points} \quad \frac{1 \pm i \sqrt{3}}{2} \text{)} \quad \text{if the order is 6} \quad (3)
\]

Elements of order two.

\[
M = \begin{bmatrix} p & q \\ r & s \end{bmatrix} \quad \Rightarrow \quad M^2 = \begin{bmatrix} p^2 + qr & pq + qs \\ rp + rs & rq + s^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Rightarrow
\]
\[ p^2 + qr = s^2 + rq \text{ and } q(p + s) = r(p + s) = 0. \]

Hence, \( p^2 = s^2 \) and there are two possibilities: If \( p = s \neq 0 \) yields the identity. Otherwise \( p = -s \), and \( q, r \) are free parameters, yielding the matrices of type

\[
M = \begin{bmatrix} p & q \\ r & -p \end{bmatrix}, \quad \text{with } \det(M) = -p^2 - qr \neq 0
\]

These account for the set of all non-singular traceless matrices.

**Elements of order three.** Routine calculations shows that if \( M^3 = I \) then

\[
M = \begin{bmatrix} p/(p + s) & q/(p + s) \\ -(p^2 + sp + s^2)/q(p + s) & s/(p + s) \end{bmatrix} \in \text{PSL}_2(\mathbb{Q}) \quad \text{if } p, q, s \in \mathbb{Q}
\]

We observe that these matrices are of trace one.

**Elements of order four.** Routine calculations shows that if \( M^4 = I \) then

\[
M = \begin{bmatrix} p/(p + s) & q/(p + s) \\ -(p^2 + sp + s^2)/2q & s/(p + s) \end{bmatrix} \quad q(p + s) \neq 0 \quad \text{with } \det = \frac{(p + s)^2}{2}
\]

These are not in \( \text{PSL}_2(\mathbb{Q}) \). Their normalized trace is \( \sqrt{2} \).

**Elements of order six.** Routine calculations shows that if \( M^6 = I \) then

\[
M = \begin{bmatrix} p/(p + s) & q/(p + s) \\ -(p^2 + ps + s^2)/3q & s/(p + s) \end{bmatrix} \quad q(p + s) \neq 0 \quad \text{with } \det = \frac{(p + s)^2}{3}
\]

These are not in \( \text{PSL}_2(\mathbb{Q}) \). Their normalized trace is \( \sqrt{3} \).

**Remark:** One may ask the question, over which fields \( K \), the group \( \text{PGL}(2, K) \) admits elements of finite orders other then 2, 3, 4 and 6? To handle this general case we introduce a new coordinate system as follows:

\[
\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} x + y & z - t \\ z + t & x - y \end{bmatrix}, \quad \begin{cases} t := \sqrt{y^2 + z^2 - \delta^2} \\ x^2/\delta^2 := \xi \end{cases} \implies \begin{bmatrix} x + y & z - t \\ z + t & x - y \end{bmatrix}^n = I \iff\begin{bmatrix} \delta \sqrt{\xi} + y & z - \sqrt{y^2 + z^2 - \delta^2} \\ z + \sqrt{y^2 + z^2 - \delta^2} & \delta \sqrt{\xi} - y \end{bmatrix}^n = I
\]

For \( n \) odd, this reduces to just one equation of degree \((n - 1)/2\) in \( \xi \) which reads

\[
\begin{align*}
n &= 3 & \implies \quad 3\xi + 1 \\
n &= 5 & \implies \quad 5\xi^2 + 10\xi + 1 \\
n &= 7 & \implies \quad 7\xi^3 + 35\xi^2 + 21\xi + 1, \text{ etc.}
\end{align*}
\]
It can be shown that elements of higher orders appear in cyclotomic fields. The group $\text{PGL}(2, K)$ contains elements of any finite order, if $K$ is the maximal abelian extension of $\mathbb{Q}$. We believe that for $K$ a number field, it should be possible to give a presentation of this group, in a way similar to what we do here.

5 Relators

Here is our conjectural presentation for $\text{PGL}_2(\mathbb{Q})$:

| Generators: $\{T, U, V = H_{-1}, H_2, \ldots H_p, \ldots \mid p : \text{prime}\}$. |
| Relators: |
| (I) $U^2 = V^2 = (UV)^2 = 1$ |
| (II) $[H_p, H_q] = 1 \forall p, q$ |
| (III) $(UH_p)^2 = 1 \forall p$ |
| (IV) $H_p^{-1}T^pH_p = T^q \forall p$ |
| (V) $V = T^{-1}UTUT^{-1}U$ |
| (VI) $(TUT^{-1}U)^3 = 1$ (redundant) |
| (VII) $(H_2UTUT^{-1}U)^4 = 1$ |
| (VIII) $(H_3UTUTUT^{-2})^6 = 1$ |

| Dictionary: |
| $T(z) := z + 1$: Translation |
| $V(z) := H_{-1}(z) = -z$: Reflection |
| $H_p(z) = pz$: Homothety |
| $TUT^{-1}U(z) = R^2(z) = 1/(1 - z)$: 3-rotation |
| $H_2UTUT^{-1}U(z) = -2(z + 1)/z$: 4-rotation |
| $H_3UTUTUT^{-2}(z) = (3z - 3)/(2z - 3)$: 6-rotation |

It is straightforward to check the validity of these relators. The difficulty lies in proving that there are no other relators independent from the above ones. This list have been found by a non-systematic search. The last three equations originates from the finite order elements. The redundancy of the relator (VI) will become evident in the alternative presentation we give below.

5.1 Another presentation of $\text{PGL}_2(\mathbb{Q})$

Here we rewrite the above presentation in terms of involutions, by expressing the homotheties in terms of $I_p$’s. The aim is to provide a framework for future research.
**Generators:** \( \{K, V, U = I_1, I_2, \ldots I_p, \ldots \mid p : \text{prime}\} \).

**Relators:**

| (I)  | \( U^2 = V^2 = (UV)^2 = 1 \) |
|------|-------------------------------|
| (II) | \( (I_p U I_q)^2 = 1 \forall p, q \) |
| (III)| \( I_p^2 = 1 \forall p \)       |
| (IV) | \( K I_p (KV)^p I_p V K = U \forall p \) |
| (V)  | \( (KU)^3 = 1 \)                 |
| (VI) | \( (KU)^6 = 1 \) (redundant)     |
| (VII)| \( (I_2 V KUV)^4 = 1 \)        |
| (VIII)| \( (I_3 U K V K U V K)^6 = 1 \) |

**Dictionary:**

- \( I_{-1} = S = UV = -1/z \)
- \( L = KU = 1 - 1/z \)
- \( I_1 I_{-1} = H_{-1} = V = -z \)
- \( T = LS = KV = K I_1 I_{-1} = z + 1 \)
- \( I_1 = U = SV = 1/z \)
- \( K = 1 - z \)

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