WEIGHTED PBW DEGENERATIONS AND TROPICAL FLAG VARIETIES

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ABSTRACT. We study algebraic, combinatorial and geometric aspects of weighted PBW-type degenerations of (partial) flag varieties in type $A$. These degenerations are labeled by degree functions lying in an explicitly defined polyhedral cone, which can be identified with a maximal cone in the tropical flag variety. Varying the degree function in the cone, we recover, for example, the classical flag variety, its abelian PBW degeneration, some of its linear degenerations and a particular toric degeneration.

INTRODUCTION

PBW degenerations of modules and projective varieties gained a lot of attention in the past decade, this fast growing subject provides new links between combinatorics, geometric representation theory, toric geometry and quiver Grassmannians to name but a few. The origin is a simple observation, namely let $n^-$ be the complex Lie algebra of strictly lower triangular $n \times n$-matrices and $U(n^-)$ be its universal enveloping algebra. By setting to one the degree of $f_{i,j}$ in the basis of elementary matrices, one obtains a filtration on $U(n^-)$ and every cyclically generated $n^-$-module ([FFL1]). The associated graded structure is then abelian. This construction can be further transferred to (partial) flag varieties $F$, identified with highest weight orbits of an algebraic group, and provides the PBW degenerate flag variety [Fe1]. This machinery was generalized in various directions, for an overview one may refer to [CFFFR] or [FFoL2].

In this paper, we take a different approach. Instead of giving each $f_{i,j}$ degree 1, we consider a weight system $A$, which is defined as a collection of integers $a_{i,j}$. Such a weight system induces a filtration of $n^-$ and $U(n^-)$. One could then ask for conditions on the weight system such that the associated graded vector spaces admit reasonable algebraic structures, e.g. the associated graded space of $n^-$ inherits a natural non-trivial graded Lie algebra structure.

In fact we provide an explicit description of a polyhedral cone $K$, such that for any weight system $A$ in the cone one can use the machinery of PBW degenerations to construct degenerate cyclic modules and degenerate flag varieties $F^A$.

We study their combinatorial, algebraic and geometric properties, depending on the relative position of $A$ in $K$. We point out that for certain weight systems discussions of the varieties $F^A$ can be found in the literature. For example, if $a_{i,j} = 0$ for all $i,j$, then we obtain just the classical flag variety; if $a_{i,j} = 1$ we obtain the PBW degenerate flag variety $F^a$ ([Fe1]); if $a_{i,j} = (j-i+1)(n-j)$ we obtain the degeneration into a toric variety discussed in ([FFR], [FPL3]). In fact, we prove that the latter degeneration is obtained for any weight system $A$ in the relative interior of $K$ (Theorem 5.1). We also obtain (Remark 6.3) some of the linear degenerate flag varieties (PBW locus) discussed in ([CFFFR]).

From now on, let $A$ be a weight system in $K$, we show that one can embed (similarly to the classical case) the degenerate flag variety $F^A$ into a product of projective spaces of (degenerate) representation spaces. Classically, the ideal $I^0$ of Plücker relations on the Plücker coordinates describes the image of the flag variety in this product. Using our weight system, we naturally
attach a degree $s^A_i$ to each Plücker coordinate. Let $I^A$ be the initial ideal of $I^0$ with respect to these degrees. Then the first theorem is

**Theorem** (Theorem 3.3 and Proposition 3.5). The ideal $I^A$ is the defining ideal of $F^A$ with respect to the embedding above. Moreover, $I^A$ is generated by its quadratic part.

In [Fe1] a monomial basis in the homogeneous coordinate ring of the abelian degenerate flag variety $F^n$ has been constructed using PBW semi-standard Young tableaux. It turns out that these monomials form a basis in the homogeneous coordinate ring of $F^A$ for all $A \in \mathcal{K}$ (again generalizing results from [FFL3]).

As we mentioned above, the degenerate flag varieties $F^A$ are labeled by weight systems belonging to a certain explicitly described cone $\mathcal{K}$. $F^A$ only depends on the relative position in the cone and the flag varieties degenerate along the face lattice of $\mathcal{K}$.

**Theorem** (Proposition 6.1 and Proposition 6.2). Let $H_A$ (resp. $H_B$) be the minimal face of $\mathcal{K}$ that contains a weight system $A$ (resp. $B$). If $H_A = H_B$, then $F^A \simeq F^B$ as projective varieties. Moreover, if $H_B \subseteq H_A$, then $F^A$ is a degeneration of $F^B$.

Recall the degrees $s^A_i$ attached to the Plücker coordinates. Let $\mathcal{C} = \{(s^A_i) \mid A \in \mathcal{K}\}$ be the set of all collections of degrees obtained from all weight systems $A \in \mathcal{K}$. By construction, $F^A$ is irreducible and thus the initial ideal $I^A$ does not contain any monomials. This implies that $\mathcal{C}$ is contained in the tropical flag variety $[\text{BLMM}]$. We prove the following theorem.

**Theorem** (Theorem 7.3). $\mathcal{C}$ is a maximal cone in the tropical flag variety.

We derive explicit inequalities providing a non-redundant description of the facets of the cone $\mathcal{C}$. To the best of our knowledge, this is the first appearance of a precise description of a maximal cone for each $n > 1$ (see [SS, MaS, BLMM] for partial results in this direction).

The paper is organized as follows. In Section 1 we recall definitions and results concerning the classical and PBW degenerate representations and flag varieties. In Section 2 we define the weighted PBW degenerations and derive their first properties. In Section 3 we write down the quadratic Plücker relations for the flag varieties $F^A$. Section 4 contains the proof of Theorem 3.3 describing the ideals of the degenerate flag varieties. The toric degenerations corresponding to the weight systems in the interior of the cone of weight systems are considered in Section 5. Section 6 discusses the cone $\mathcal{K}$. The link to the theory of tropical flag varieties is described in Section 7. Finally, in Section 8 we prove a Borel-Weil-type theorem for the degenerate flag varieties $F^A$.

1. Preliminaries

1.1. The classical theory. For a Lie algebra $\mathfrak{g}$, let $\mathcal{U}(\mathfrak{g})$ denote the enveloping algebra associated with it. Fix an integer $n \geq 2$ and consider the Lie algebra $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ of complex $n \times n$-matrices with trace 0. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, the summands being, respectively, the subalgebra of upper triangular nilpotent matrices, the subalgebra of diagonal matrices and the subalgebra of lower triangular nilpotent matrices. Denote the corresponding set of simple roots $\alpha_1, \ldots, \alpha_{n-1} \in \mathfrak{h}^*$ and denote the set of positive roots $\Phi^+$. Then

$$\Phi^+ = \{\alpha_{i,j} = \alpha_i + \ldots + \alpha_{j-1} \mid 1 \leq i < j \leq n\}.$$ 

For every positive root $\alpha_{i,j}$ we fix a non-zero element $f_{i,j} \in \mathfrak{n}_-$ in the weight space of weight $-\alpha_{i,j}$ in such a way that the following relation holds whenever $i \leq k$:

$$[f_{i,j}, f_{k,l}] = \begin{cases} f_{i,l}, & \text{if } j = k, \\ 0, & \text{otherwise}. \end{cases}$$ (1)
Let $\omega_1, \ldots, \omega_{n-1}$ be the fundamental weights of $\mathfrak{g}$. Let $\lambda = a_1\omega_1 + \ldots + a_{n-1}\omega_{n-1} \in \mathfrak{h}^*$ be an integral dominant weight for some $a_i \in \mathbb{Z}_{\geq 0}$. We denote $d$ the tuple $(d_1, \ldots, d_s)$ where $\{d_1 < \ldots < d_s\}$ is the set of all $i$ for which $a_i \neq 0$.

Let $L_\lambda$ denote the finite-dimensional irreducible $\mathfrak{g}$-representation with highest weight $\lambda$. We fix a non-zero highest weight vector $v_\lambda \in L_\lambda$, then $L_\lambda = \mathcal{U}(\mathfrak{n}_-)v_\lambda$. Recall that for $h \in \mathfrak{h}$, $hv_\lambda = \lambda(h)v_\lambda$ and $n_+v_\lambda = 0$.

Let $N$ be the Lie group of lower unitriangular complex $n \times n$-matrices, $N$ is diffeomorphic to $\mathbb{C}^{(2)}$ and thus connected and simply connected. The Lie algebra $\text{Lie}N$ can be naturally identified with $\mathfrak{n}_-$ which provides an action of $N$ on $L_\lambda$ via the exponential map. We then have $CNv_\lambda = L_\lambda$. Furthermore, we also obtain an action of $N$ on the projectivization $\mathbb{P}(L_\lambda)$. Let $u_\lambda \in \mathbb{P}(L_\lambda)$ be the point corresponding to the line $\mathbb{C}v_\lambda$. The closure $\overline{Nv_\lambda} \subset \mathbb{P}(L_\lambda)$ is known as the (partial) flag variety which we will denote $F_\lambda$.

Let $V = \mathbb{C}^n$ be the vector representation of $\mathfrak{g}$ with basis $e_1, \ldots, e_n$ and $f_{i,j}$ mapping $e_i$ to $e_j$ while mapping $e_l$ with $l \neq i$ to 0. Then for $1 \leq k \leq n-1$ we have $L_{\omega_k} = \wedge^k V$. For $1 \leq i_1, \ldots, i_k \leq n$, we denote

$$e_{i_1, \ldots, i_k} := e_{i_1} \wedge \ldots \wedge e_{i_k}. $$

Then for any $\sigma \in S_k$,

$$e_{i_1, \ldots, i_k} = (-1)^{\ell(\sigma)}e_{i_{\sigma(1)}, \ldots, i_{\sigma(k)}}$$

where $\ell(\sigma)$ is the inversion number of $\sigma$. The vector space $L_{\omega_k}$ admits the basis

$$\{e_{i_1, \ldots, i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n\}. $$

The vector $e_{1, \ldots, k}$ is a highest weight vector and we assume that $v_{\omega_k} = e_{1, \ldots, k}$. Note that $f_{i,j}$ maps $e_{i_1, \ldots, i_k}$ to 0 whenever $i \notin \{i_1, \ldots, i_k\}$ or $j \in \{i_1, \ldots, i_k\}$ and otherwise to $e_{i'_1, \ldots, i'_k}$ where $i'_i = i$ when $i \neq i$ and $i'_j = j$ when $i = i$.

We define the $\mathfrak{g}$-representation

$$U_\lambda = L_{\omega_1}^{\otimes a_1} \otimes \ldots \otimes L_{\omega_{n-1}}^{\otimes a_{n-1}},$$

and denote

$$w_\lambda = v_{\omega_1}^{\otimes a_1} \otimes \ldots \otimes v_{\omega_{n-1}}^{\otimes a_{n-1}} \in U_\lambda.$$ 

The subrepresentation $\mathcal{U}(\mathfrak{g})w_\lambda \subset U_\lambda$ is isomorphic to $L_\lambda$ by identifying $w_\lambda$ with $v_\lambda$. We thus obtain the embedding $F_\lambda \subset \mathbb{P}(L_\lambda) \subset \mathbb{P}(U_\lambda)$.

We consider the Segre embedding

$$\mathbb{P}(L_{\omega_1})^{a_1} \times \ldots \times \mathbb{P}(L_{\omega_{n-1}})^{a_{n-1}} \subset \mathbb{P}(U_\lambda)$$

and the embedding

$$\mathbb{P}_d = \mathbb{P}(L_{\omega_{d_1}}) \times \ldots \times \mathbb{P}(L_{\omega_{d_s}}) \subset \mathbb{P}(L_{\omega_1})^{a_1} \times \ldots \times \mathbb{P}(L_{\omega_{n-1}})^{a_{n-1}}$$

where $\mathbb{P}(L_{\omega_{d_i}})$ is embedded diagonally into $\mathbb{P}(L_{\omega_{d_i}})^{a_{d_i}}$. Let $y_\lambda$ be the point in $\mathbb{P}(U_\lambda)$ corresponding to $\mathbb{C}w_\lambda$. The definition of the Segre embedding implies that $Ny_\lambda \subset \mathbb{P}_d$ which gives us an embedding $F_\lambda \subset \mathbb{P}_d$.

The polynomial ring

$$R_d = \mathbb{C}[X_{i_1, \ldots, i_{d_j}} \mid 1 \leq j \leq s, 1 \leq i_1 < \ldots < i_{d_j}]$$

is the homogeneous coordinate ring of $\mathbb{P}_d$ with homogeneous coordinate $X_{i_1, \ldots, i_{d_j}}$ dual to the basis vector $e_{i_1, \ldots, i_{d_j}} \in L_{\omega_{d_j}}$. In particular, the ring $R_d$ is graded by $\mathbb{Z}_{\geq 0}$ with the generator $X_{i_1, \ldots, i_{d_j}}$ having homogeneity degree $(0, \ldots, 1, \ldots, 0)$ with the 1 being the $j$th coordinate. We embed $\mathbb{Z}_{\geq 0}$ into $\mathfrak{h}^*$ and view these homogeneity degrees as integral dominant weights by setting $\text{deg} \ X_{i_1, \ldots, i_{d_j}} = \omega_{d_j}$. 

Let us denote \( X_{i_1,\ldots,i_p} = (-1)^{l(\sigma)} X_{i_{\sigma(1)},\ldots,i_{\sigma(p)}} \) for any \( p \in \{d_1,\ldots,d_s\}, \{i_1,\ldots,i_p\} \subset \{1,\ldots,n\} \) and \( \sigma \in S_p \). We consider the ideal \( I_d \) of \( R_d \) generated by the following quadratic elements (known as Plücker relations). For \( 1 \leq k \leq q \) and a pair of collections of pairwise distinct elements \( 1 \leq i_1,\ldots,i_p \leq n \) and \( 1 \leq j_1,\ldots,j_q \leq n \) where \( p, q \in \{d_1,\ldots,d_s\} \) with \( p \geq q \), the corresponding Plücker relation is:

\[
X_{i_1,\ldots,i_p}X_{j_1,\ldots,j_q} - \sum_{\{r_1,\ldots,r_k\} \subset \{i_1,\ldots,i_p\}} X_{i_1',\ldots,i_p'}X_{r_1,\ldots,r_k,j_{k+1},\ldots,j_q} \tag{2}
\]

where \( i_i' = j_m \) if \( i_i = r_m \) for some \( 1 \leq m \leq k \) and \( i_i' = i_i \) otherwise. Note that the Plücker relations \((2)\) are all homogeneous elements of the ring.

**Theorem 1.1.** The ideal of the subvariety \( F_\lambda \subset \mathbb{P}_d \) is precisely \( I_d \).

In particular, \( F_\lambda \) only depends on the set \( \{d_1,\ldots,d_s\} \). We also mention that the dimension of the component of \( R_d/I_d \) of homogeneity degree \( \lambda \) is equal to \( \dim L_\lambda \).

The theory presented in this subsection can, for instance, be found in [C] [Ful1].

1.2. **Abelian PBW degenerations.** We give a brief overview of the theory of abelian PBW (Poincaré–Birkhoff–Witt) degenerations following [FPL1] and [FG1].

The universal enveloping algebra \( \mathcal{U} = \mathcal{U}(\mathfrak{n}_-) \) is equipped with a \( \mathbb{Z}_{\geq 0} \)-filtration known as the PBW filtration. The \( m \)th component of the filtration is defined as

\[
(\mathcal{U})^a = \text{span}\{(f_{i_1,j_1}\ldots f_{i_k,j_k} \mid k \leq m)\},
\]

the linear span of all PBW monomials of PBW degree no greater than \( m \). By the PBW theorem, the associated graded algebra \( \text{gr} \mathcal{U} \) is isomorphic to the symmetric algebra \( \mathcal{S}^* (\mathfrak{n}_-) \).

The \( \mathbb{Z}_{\geq 0} \)-filtration on \( \mathcal{U} \) induces a \( \mathbb{Z}_{\geq 0} \)-filtration on \( L_\lambda \) via \( (L_\lambda)^a_m = (\mathcal{U})^a_m v_\lambda \). The associated graded space \( \text{gr} L_\lambda \) is naturally a module over the associated graded algebra \( \text{gr} \mathcal{U} = \mathcal{S}^* (\mathfrak{n}_-) \). This \( \mathcal{S}^*(\mathfrak{n}_-) \)-module is most commonly known as the **PBW degeneration of** \( L_\lambda \) and is denoted by \( L_\lambda^a \) with the “\( a \)” standing for “abelian” in reference to the commutativity of \( \mathcal{S}^* (\mathfrak{n}_-) \). We will refer to this object as the **abelian PBW degeneration** or simply the **abelian degeneration** to distinguish it among the more general objects studied in this paper. Note that the component \((L_\lambda)^a_0\) is precisely \( \mathbb{C} v_\lambda \) and the 0th homogeneous component of \( L_\lambda^a \) is one-dimensional, let \( v_\lambda^a \) be the image of \( v_\lambda \) therein. It can be easily seen that \( L_\lambda^a = \mathcal{S}^* (\mathfrak{n}_-) v_\lambda^a \).

Now observe that \( \mathcal{S}^* (\mathfrak{n}_-) \) is the universal enveloping algebra \( \mathcal{U}(\mathfrak{n}_a) \) of the abelian Lie algebra \( \mathfrak{n}_a \) on the vector space \( \mathfrak{n}_- \). We denote \( f_{i,j}^a \in \mathfrak{n}_a \) the image of \( f_{i,j} \) under the canonical linear isomorphism \( \mathfrak{n}_- \to \mathfrak{n}_a \).

\( L_\lambda^a \) is a representation of \( \mathfrak{n}_a \) and, consequently, of the corresponding connected simply connected Lie group \( N^a \). The group \( N^a \) is just \( \mathbb{C}^{(2)} \) with \( \mathbb{C} \) viewed as Lie group under addition. Further, we have an action of \( N^a \) on \( \mathbb{P}(L_\lambda^a) \). We denote \( u_\lambda^a \) the point in \( \mathbb{P}(L_\lambda^a) \) corresponding to \( \mathbb{C} v_\lambda^a \) and consider the closure \( N^a u_\lambda^a \subset \mathbb{P}(L_\lambda^a) \). This subvariety is known as the **PBW degenerate flag variety** or the **abelian degeneration** of \( F_\lambda \) and is denoted \( F^a_\lambda \).

Similarly to the above classical situation we define the \( \mathfrak{n}_a \)-representation

\[
U_\lambda^a = (L_\lambda^a)^{\otimes a_1} \otimes \ldots \otimes (L_\lambda^a)^{\otimes a_{n-1}}
\]

and the vector

\[
w_\lambda^a = (u_\lambda^a)^{\otimes a_1} \otimes \ldots \otimes (u_\lambda^a)^{\otimes a_{n-1}} \in U_\lambda^a.
\]

The subrepresentation \( \mathcal{U}(\mathfrak{n}_a) u_\lambda^a \subset U_\lambda^a \) is isomorphic to \( L_\lambda^a \) via identifying \( w_\lambda^a \) with \( v_\lambda^a \).

We define the subvariety

\[
\mathbb{P}_d^a = \mathbb{P}(L_\lambda^a)^{d_1} \times \ldots \times \mathbb{P}(L_\lambda^a)^{d_s} \subset \mathbb{P}(U_\lambda^a)
\]
via the Segre embedding and obtain the embedding $F^a_\lambda \subset \mathbb{P}^a_d$.

For every $1 \leq k \leq n$ and $i_1 < \ldots < i_k$ we may consider the least $m$ such that $e_{i_1, \ldots, i_k} \in (L_{\omega_k})_m$ and denote $e_{i_1, \ldots, i_k}^a \in L_{\omega_k}^a$ the image of $e_{i_1, \ldots, i_k} \in (L_{\omega_k})_m/(L_{\omega_k})_{m-1}$ (with $(L_{\omega_k})_{-1} = 0$). We will also use the notation $s_{i_1, \ldots, i_k}^a = m$.

The vectors $e_{i_1, \ldots, i_k}^a$ comprise a basis in $L_{\omega_k}^a$. This allows us to view the ring

$$R_d^a = \mathbb{C}[\{X_{i_1, \ldots, i_d}^a \mid 1 \leq j \leq s, 1 \leq i_1 < \ldots < i_d\}]$$

as the homogeneous coordinate ring of $\mathbb{P}^a_d$ where the homogeneous coordinate $X_{i_1, \ldots, i_d}^a$ is the dual basis element to $e_{i_1, \ldots, i_d}^a$. We introduce an additional $\mathbb{Z}_{\geq 0}$-grading on $R_d^a$ by setting $\text{grad}^a X_{i_1, \ldots, i_d}^a = s_{i_1, \ldots, i_d}^a$. Now consider the isomorphism

$$\varphi : R_d \to R_d^a, \quad X_{i_1, \ldots, i_d} \mapsto X_{i_1, \ldots, i_d}^a.$$

Let $I_d^a$ denote the initial ideal in $\text{grad}^a(\varphi(I_d))$, i.e. the ideal spanned by the elements obtained by considering an $X \in \varphi(I_d)$ and then taking the sum of monomials in $X$ with lowest grading $\text{grad}^a$ (the initial part in $\text{grad}^a$ of $X$).

**Theorem 1.2 (Fe1).** The ideal of the subvariety $F^a_\lambda \subset \mathbb{P}^a_d$ is precisely $I_d^a$.

The following more explicit characterization of $I_d^a$ is also given in [Fe1].

**Proposition 1.3 (Fe1).** $I_d^a$ is generated by its quadratic part, i.e. it is generated by the initial parts (with respect to $\text{grad}^a$) of the Plücker relations (2).

### 1.3. FFLV bases and FFLV polytopes.

In [FFL1] certain combinatorial monomial bases in $L^a_\lambda$ and $L^a_\lambda$ are constructed. We briefly recall the definitions of these bases.

First we define the set $\Pi_\lambda$ that parametrizes the elements in each basis. This set is comprised of certain arrays of integers each containing $\binom{\lambda}{2}$ elements. Each array $T$ consists of elements $T_{i,j}$ with $1 \leq i < j \leq n$. We visualize $T$ as a number triangle in the following way:

$$
\begin{array}{cccc}
T_{1,2} & T_{2,3} & \cdots & T_{n-1,n} \\
T_{1,3} & & \cdots & T_{n-2,n} \\
& \cdots & \cdots \\
& & T_{1,n}
\end{array}
$$

Thus a horizontal row contains all $T_{i,j}$ with a given difference $j - i$.

To specify when $T \in \Pi_\lambda$ the notion of a Dyck path is used. We understand a Dyck path to be a sequence of pairs of integers $((i_1, j_1), \ldots, (i_N, j_N))$ with $1 \leq i < j \leq n$ such that $j_1 - i_1 = j_N - i_N = 1$ (both lie in the top row) and $(i_{k+1}, j_{k+1})$ is either $(i_k + 1, j_k)$ or $(i_k, j_k + 1)$ for any $1 \leq k \leq N-1$ (either the upper-right or the bottom-right neighbor of $(i_k, j_k)$). The set $\Pi_\lambda$ consists of all arrays $T$ whose elements are nonnegative integers such that for any Dyck path $d = ((i_1, j_1), \ldots, (i_N, j_N))$ one has

$$T_{i_1,j_1} + \ldots + T_{i_N,j_N} \leq a_{i_1} + a_{i_1+1} + \ldots + a_{i_N}.$$

We will denote the right hand side by $M(\lambda, d)$. For a number triangle $T$ and a Dyck path $d$ we will denote the left-hand side above via $S(T, d)$. We will refer to $T \in \Pi_\lambda$ as FFLV patterns.

**Theorem 1.4 (FFL1).** The set

$$\left\{ \left( \prod_{i,j} (f_{i,j}^a)^{T_{i,j}} \right) e_{\lambda}^a \mid T \in \Pi_\lambda \right\}$$
Corollary 1.5. Each set of the form
\[ \left\{ \left( \prod_{i,j} l_{i,j}^{-r_{i,j}} \right) v_\lambda \mid T \in \Pi_\lambda \right\}, \]
where the order of the factors is chosen arbitrarily for each \( T \), constitutes a basis in \( L_\lambda^2 \).

Note that the order of the factors in the above product does not matter in view of the abelianity of \( n^a \). The following is easily seen to follow.

**Corollary 1.5.** Each set of the form
\[ \left\{ \left( \prod_{i,j} l_{i,j}^{-r_{i,j}} \right) v_\lambda \mid T \in \Pi_\lambda \right\}, \]
where the order of the factors is chosen arbitrarily for each \( T \), constitutes a basis in \( L_\lambda \).

Next we define the **FFLV polytope** \( Q_\lambda \). The polytope is contained in \( \mathbb{R}^{G(\lambda)} \) with coordinates enumerated by pairs \( 1 \leq i < j \leq n \). A point \( x = (x_{i,j}) \) in this space is visualized as a number triangle in the same exact fashion as the arrays comprising \( \Pi_\lambda \). We have \( x \in Q_\lambda \) if and only if all \( x_{i,j} \geq 0 \) and for any Dyck path \( d \) one has
\[ S(x, d) \leq M(\lambda, d). \]
We see that \( Q_\lambda \) is indeed a convex polytope and \( \Pi_\lambda \) is precisely the set of integer points therein.

An obvious but important property of the FFLV polytopes is \( Q_\lambda + Q_\mu = Q_{\lambda + \mu} \) (Minkowski sum) for any pair of integral dominant weights. A much less obvious property proved in [FFL1] is the following **Minkowski property**.

**Lemma 1.6.** For any integral dominant weights \( \lambda \) and \( \mu \), one has \( \Pi_\lambda + \Pi_\mu = \Pi_{\lambda + \mu} \), where + is the Minkowski sum of sets.

**1.4. PBW Young tableaux.** With the weight \( \lambda \) let us associate the integers \( \lambda_i = a_i + \ldots + a_{n-1} \) for \( 1 \leq i \leq n \). These \( \lambda_i \) comprise a non-increasing sequence of nonnegative integers and thus define a Young diagram (always in English notation) which we will also denote \( \lambda \). Note that \( a_i \) is precisely the number of columns of height \( i \) in \( \lambda \).

Consider \( Y \), a filling of \( \lambda \) (Young tableau of shape \( \lambda \)), we will denote \( Y_{i,j} \) its element in the \( i \)th row and \( j \)th column. We say that \( Y \) is a **PBW Young tableau** (or “PBW tableau” for short) if all of its elements are integers from \( [1,n] \) and for every \( 1 \leq j \leq \lambda_1 \) the following hold (here \( \lambda'_j \) denotes the height of the \( j \)th column in \( \lambda \)).

1. \( Y_{i,j} \neq Y_{k,j} \) for any \( 1 \leq i < k \leq \lambda'_j \).
2. For any \( 1 \leq i \leq \lambda'_j \) if \( Y_{i,j} \leq \lambda'_j \), then \( Y_{i,j} = i \).
3. For any \( 1 \leq i \neq k \leq \lambda'_j \) if \( Y_{i,j} > Y_{k,j} > \lambda'_j \), then \( i < k \).

We say that \( Y \) is a **PBW semistandard** Young tableau (or “PBW SSYT” for short) if, apart from the above three conditions, whenever \( j > 1 \) we also have

4. For every \( 1 \leq i \leq \lambda'_j \) there exists a \( 1 \leq k \leq \lambda'_{j-1} \) such that \( Y_{k,j-1} \geq Y_{i,j} \).

We denote \( \mathcal{Y}_\lambda \) the set of all PBW SSYTs of shape \( \lambda \).

Let \( Z \) be a PBW tableau of shape \( \omega_{d_1} \) for some \( 1 \leq j \leq s \), i.e. consisting of one column of height \( d_j \). Let \( i_1 < \ldots < i_{d_j} \) be the elements of \( Z \) reordered increasingly. We denote \( X(Z) = X_{i_1, \ldots, i_{d_j}} \). Next, for a PBW tableau \( Y \) of shape \( \lambda \) we denote
\[ X(Y) = \prod_{Z: \text{column of } Y} X(Z) \in R_d. \]
We will denote \( X^a(Y) \) the image of \( X(Y) \) under the isomorphism \( \varphi \).

If \( \mu \) is a \( \mathbb{Z}_{\geq 0} \)-linear combination of \( \omega_{d_1}, \ldots, \omega_{d_s} \), then the monomials \( X(Y) \) with \( Y \) ranging over all PBW tableaux of shape \( \mu \) comprise a basis in the component of \( R_d \) of homogeneity
degree $\mu$. We consider the ring $R_d^\mu$ to also be $\mathfrak{h}^*$-graded (the grading being induced by $\varphi$) and the previous statement, also holds with $\mathfrak{a}$ appended where necessary.

**Theorem 1.7** ([Fe1]). For $\mu$ as above, the images of the monomials $X^\mu(Y)$ with $Y$ ranging over $\mathcal{Y}_\mu$ comprise a basis in the component of $R_d^\mu/I_d^\mu$ of homogeneity degree $\mu$.

In particular, since the corresponding homogeneous components of a homogeneous ideal and of its initial part with respect to some grading have the same dimension, we see that $|\mathcal{Y}_\lambda| = \dim L_\lambda$.

We point out that the analogous statement for the non-degenerate case also holds. This is not found in [Fe1] but will follow from the more general Corollary 1.4.

**Theorem 1.8.** For $\mu$ as above, the images of the monomials $X(Y)$ with $Y$ ranging over $\mathcal{Y}_\mu$ comprise a basis in the component of $R_d/I_d$ of homogeneity degree $\mu$.

**Remark 1.9.** One may also give the definition of PBW SSYTs in terms of a certain partial order. Consider the poset $P$ the elements of which are nonempty proper subsets of $[1, n]$ and the order relation is defined as follows. For $x, y \in P$ we set $x \leq y$ whenever $|x| \geq |y|$ and the (unique) PBW tableau with two columns such that the $x$ is the set of elements in its first column and $y$ is the set of elements in its second column is PBW semistandard, i.e. satisfies conditions (2)-(4) above. (One may easily verify that this is indeed an order relation.) The above theorem then asserts that the monomials which are standard with respect to this partial order map to a basis in the homogeneous coordinate ring of $F^\mu$.

The theory in this subsection should be reminiscent of the classical description of the coordinate ring $R_d/I_d$ in terms of semistandard Young tableaux which also provide bases in the homogeneous components and correspond to standard monomials with respect to a certain partial order (see [Ful1] for details).

To complete this section we prove a combinatorial fact that is not found in [Fe1]. First we introduce the partial order $\leq$ on the set of all pairs $(i, j)$ with $1 \leq i < j \leq n$ with $(i, j) \leq (i', j')$ whenever $i \leq i'$ and $j \leq j'$. For $1 \leq k \leq n - 1$ we consider the set $\Pi_{\omega_k}$; it is comprised of all $T$ such that for $1 \leq i < j \leq n$, (i) $T_{i,j} \in \{0, 1\}$; (ii) if $T_{i,j} = 1$ then $i \leq k$ and $j \geq k + 1$; (iii) the set of pairs $(i, j)$ with $T_{i,j} = 1$ forms an antichain with respect to $\leq$.

For a PBW tableau $Z$ of shape $\omega_k$ let us denote $\tau(Z) \in \Pi_{\omega_k}$ the FFLV pattern with

$$\tau(Z)_{i,j} = \begin{cases} 1, & \text{if } Z_{i,1} = j \text{ and } j > i; \\ 0, & \text{otherwise.} \end{cases}$$

The above description of $\Pi_{\omega_k}$ shows that this gives us a bijection between $\mathcal{Y}_{\omega_k}$ and $\Pi_{\omega_k}$. In fact, one sees that the element of the FFLV basis corresponding to $T$ is $e_{Z_{1,1}} \wedge \ldots \wedge e_{Z_{k,1}}$.

We now define $\tau$ on any PBW tableau $Y$ of shape $\lambda$ by denoting $Z_i$ the PBW tableau of shape $\omega_{\lambda_i}$ found in the $i$th column of $Y$ and setting $\tau(Y) = \sum_{i=1}^{\lambda_1} \tau(Z_i)$.

**Lemma 1.10.** When restricted to the set $\mathcal{Y}_\lambda$ the map $\tau$ provides a bijection onto $\Pi_{\lambda}$.

**Proof.** The fact that for $Y \in \mathcal{Y}_\lambda$ one has $\tau(Y) \in \Pi_{\lambda}$ is immediate from Lemma 1.6.

Let us describe the inverse of $\tau$. Consider some $T \in \Pi_{\lambda}$ and consider $\lambda'_1 = d_k$, i.e. the largest $i$ with $a_i > 0$. Let $T^1 \in \Pi_{\omega_{\lambda'_1}}$ be such that $T^1_{i,j} = 1$ whenever $i \leq \lambda'_1$, $j \geq \lambda'_1 + 1$ and $(i, j)$ is maximal among all $(i', j')$ with $T_{i',j'} > 0$ with respect to $\leq$. We then observe that $T - T^1 \in \Pi_{\lambda - \omega_{\lambda'_1}}$ and define $T^2$ for the pair $\lambda - \omega_{\lambda'_1}, T - T^1$ in the same way as $T^1$ was defined for the pair $\lambda, T$. By iterating this procedure we obtain a decomposition of $T$ into the sum $T^1 + \ldots + T^{\lambda_1}$ with $T^i \in \Pi_{\omega_{\lambda'_i}}$. 

Denote $Z_t$ the unique element of $Y_{\omega_{\lambda_t}}$ with $\tau(Z_t) = T^t$ and denote $\zeta(T)$ the PBW tableau of shape $\lambda$ having $Z_t$ as its $i$th column. By definition, for any $1 \leq l < \lambda_1$ and for any $T_{i,j}^{l+1} = 1$ one either has $j < \lambda'_1 + 1$ or one has $T_{i,j}^{l+1} = 1$ for some $(i', j') \geq (i, j)$. One easily checks that this is equivalent to the two-column tableau with first column $Z_t$ and second column $Z_{t+1}$ being PBW semistandard which shows that $\zeta(T)$ is PBW semistandard. It is now straightforward to verify that $\tau$ and $\zeta$ are mutually inverse. 

2. **Weighted PBW degenerations**

A weight system $A = (a_{i,j})_{1 \leq i < j \leq n}$ is a collection of integers $a_{i,j}$ such that

(a) $a_{i,i+1} + a_{i+1,i+2} \geq a_{i,i+2}$ for $1 \leq i \leq n - 2$ and 
(b) $a_{i,j} + a_{i+1,j+1} \geq a_{i,j+1} + a_{i+1,j}$ for $1 \leq i < j - 1 \leq n - 2$.

The reasons for these requirements should become evident below.

We immediately derive a larger set of inequalities.

**Proposition 2.1.** We have

\[ \text{(A)} \quad a_{i,j} + a_{j,k} \geq a_{i,k} \text{ for } 1 \leq i < j < k \leq n \text{ and} \]

\[ \text{(B)} \quad a_{i,j} + a_{k,l} \geq a_{i,l} + a_{k,j} \text{ for } 1 \leq i < k < j < l \leq n. \]

**Proof.** In (a) and (b) replace $i$ with $i'$ and $j$ with $j'$. Then the inequality in (A) can be obtained as the sum of the inequality in (a) for $i' = j - 1$ and the inequalities in (b) for all $i \leq i' \leq j - 1$ and $j \leq j' \leq k - 1$ other than $i' = j - 1, j' = j$.

The inequality in (B) can be obtained as the sum of the inequalities in (b) for all $i \leq i' \leq k - 1$ and $j \leq j' \leq l - 1$. 

The weight system allows us to define a $\mathbb{Z}$-filtration of the Lie algebra $\mathfrak{n}_-$ by setting for $m \in \mathbb{Z}$

$$ (\mathfrak{n}_-)_m = \text{span} (\{f_{i,j} | a_{i,j} \leq m\}). $$

Condition (A) above together with the commutation relations ensure that we indeed have a filtered Lie algebra, i.e. $[(\mathfrak{n}_-)_t, (\mathfrak{n}_-)_m] \subset (\mathfrak{n}_-)_{t+m}$. Let $\mathfrak{n}_A^t$ be the associated graded Lie algebra

$$ \mathfrak{n}_A^t = \bigoplus_{m \in \mathbb{Z}} (\mathfrak{n}_-)_{t+m} = \bigoplus_{m \in \mathbb{Z}} (\mathfrak{n}_A^t)_m. $$

The Lie algebra $\mathfrak{n}_A^t$ is spanned by elements $f_{i,j}^A$ with $f_{i,j}^A$ being the image of $f_{i,j} \in (\mathfrak{n}_-)_{a_{i,j}}$ in $(\mathfrak{n}_A^t)_{a_{i,j}}$. The commutation relations in $\mathfrak{n}_A^t$ are then given by: for $i \leq k$,

$$ [f_{i,j}^A, f_{k,l}^A] = \begin{cases} f_{i,l}^A, & \text{if } j = k \text{ and } a_{i,j} + a_{k,l} = a_{i,l}; \\ 0, & \text{otherwise.} \end{cases} \quad (3) $$

For a PBW monomial $M = f_{i_1,j_1} \cdots f_{i_N,j_N} \in \mathcal{U}$ define its $A$-degree to be $\deg^A M = a_{i_1,j_1} + \cdots + a_{i_N,j_N}$. We obtain a $\mathbb{Z}$-filtration on $\mathcal{U}$ with the $m$th component being

$$ \mathcal{U}_m = \text{span}(\{M = f_{i_1,j_1} \cdots f_{i_N,j_N} | \deg^A M \leq m\}). $$

We obtain a filtered algebra structure on $\mathcal{U}$ and denote the associated graded algebra $\mathcal{U}_A = \text{gr} \mathcal{U}$. We will denote the $m$th homogeneous component $\mathcal{U}_m/\mathcal{U}_{m-1} = \mathcal{U}_m^A$.

For every pair $1 \leq i < j \leq n$ we may consider the element $f_{i,j}^A \in \mathcal{U}_A^t$ which is the image of $f_{i,j} \in \mathcal{U}_{a_{i,j}}$ (the reason for the apparent conflict of notations will become clear in a moment). It is evident that these $f_{i,j}^A \in \mathcal{U}_A^t$ satisfy the commutation relations (3) with the commutator being induced by the multiplication in $\mathcal{U}_A^t$. Furthermore, the PBW theorem implies that these
elements generate the algebra $\mathcal{U}^A$. The universal property of $\mathcal{U}(n^A)$ now shows that we have a surjective homomorphism from $\mathcal{U}(n^A)$ to $\mathcal{U}^A$ mapping $f_{i,j}^A \in n^A \subset \mathcal{U}(n^A)\to f_{i,j}^A \in \mathcal{U}^A$.

**Proposition 2.2.** The above homomorphism is an isomorphism.

*Proof.* Consider an arbitrary linear ordering “<” of the $f_{i,j} \in \mathcal{U}$. Let us show that for $m \in \mathbb{Z}$ the monomials $M = f_{i_1,j_1} \ldots f_{i_N,j_N}$ with $f_{i_1,j_1} \leq \ldots \leq f_{i_N,j_N}$ and $\deg^A M \leq m$ span $\mathcal{U}_m$. Indeed, consider an arbitrary monomial $L \in \mathcal{U}$ with $\deg^A L \leq m$. Now apply the commutation relations (1) to reorder the $f_{i,j}$ in $L$ and all the appearing summands to express $L$ as a linear combination of $M = f_{i_1,j_1} \ldots f_{i_N,j_N}$ with $f_{i_1,j_1} \leq \ldots \leq f_{i_N,j_N}$ (this is the standard PBW procedure). The inequalities in (A) ensure that all the monomials in the resulting expression are of degree no greater than $m$.

Next, consider the induced ordering of the $f_{i,j}^A \in \mathcal{U}^A$ which we also denote “<”. To prove the proposition it suffices to show that all products $f_{i_1,j_1}^A \ldots f_{i_N,j_N}^A \in \mathcal{U}^A$ for which $f_{i_1,j_1}^A \leq \ldots \leq f_{i_N,j_N}^A$ are linearly independent in $\mathcal{U}^A$. Indeed, consider $k$ such products $M_1^A, \ldots, M_k^A$ and suppose that they are linearly dependent. We may assume that all $M_i^A \in \mathcal{U}_m^A$ for some $m$. Now consider the corresponding monomials $M_1, \ldots, M_k \in \mathcal{U}$ where $M_i$ is obtained from $M_i^A$ by simply removing the $^A$ superscript from each $f_{i,j}^A$. We see that a linear combination of these $M_1, \ldots, M_k \in \mathcal{U}$ lies in $\mathcal{U}_{m-1}$ which contradicts the conclusion from the previous paragraph.

Now let us consider the $\mathbb{Z}$-filtration of $L_\lambda$ given by $(L_\lambda)_m = \mathcal{U}_m v_\lambda$. Denote the associated graded space $L_\lambda^A$ and its homogeneous components $(L_\lambda^A)_m$. Since for $l, m \in \mathbb{Z}$ we have $\mathcal{U}_l(L_\lambda)_m \subset (L_\lambda)_l + m$, the space $L_\lambda^A$ is naturally a graded $\mathcal{U}_\lambda^A$-representation. This representation is the *weighted PBW degeneration* of $L_\lambda$ and will be often referred to as “degenerate representation” for brevity.

Note that $v_\lambda \in (L_\lambda)_0$ and that $v_\lambda \notin (L_\lambda)_m$ for all $m < 0$. Therefore, we may consider the “highest weight vector” $v_\lambda^A \in L_\lambda^A$ which is the image of $v_\lambda$ in $(L_\lambda^A)_0$. We then have $L_\lambda^A = \mathcal{U}^A v_\lambda^A$ and $(L_\lambda^A)_m = \mathcal{U}_m^A v_\lambda^A$.

Since $L_\lambda^A$ is a representation of $\mathcal{U}^A$ and, consequently, of $n^A$, it is also a representation of the corresponding connected simply connected Lie group which we denote $N^A$. Then $N^A$ also acts on the projectivization $\mathbb{P}(L_\lambda^A)$. In this projectivization we may consider the point $u_\lambda^A$ corresponding to $\mathbb{C} v_\lambda$ and the closure $F_\lambda^A = N^A u_\lambda^A$. This subvariety will be referred to as the *weighted PBW degeneration* of the partial flag variety associated with $\lambda$ or, more compactly, the *degenerate flag variety*.

**Remark 2.3.** If $a_{i,j} = 0$ for all $1 \leq i < j \leq n$, we obtain the non-degenerate objects $n^A = n_{-}$, $L_\lambda^A = L_\lambda$ and $F_\lambda^A = F_\lambda$. If $a_{i,j} = 1$ for all $1 \leq i < j \leq n$, we obtain the abelian degenerations $n^a$, $L_\lambda^a$ and $F_\lambda^a$. These observations will be generalized via Proposition 6.1 (see the paragraph following the Proposition).

**Remark 2.4.** We see that in order to define $n^A$, $L_\lambda^A$ and $F_\lambda^A$ we only make use of the inequalities in (A). However, the inequalities in (B) become crucial in the subsequent sections and, in particular, in the proofs of Lemma 6.2 and Theorem 3.4.

3. **Degenerate Plücker relations**

For $1 \leq k \leq n - 1$ consider the fundamental representation $L_{\omega_k}$ (and recall its properties specified in subsection 1.1). We have $(L_{\omega_k})_m = 0$ for $m < 0$, hence for each tuple $1 \leq i_1 < \ldots < i_k \leq n$ we may choose the least $m$ such that $e_{i_1, \ldots, i_k} \in (L_{\omega_k})_m$. Let $e_{i_1, \ldots, i_k}^A \in L_{\omega_k}$ be the image of $e_{i_1, \ldots, i_k}$ in $(L_\lambda^A)_m$.
Proposition 3.1. The vectors $e_{i_1,\ldots,i_k}^A$ comprise a basis in $L_{\omega_k}^A$.

Proof. This follows directly from the fact that the image of $v_{\omega_k}$ under the action of any monomial in the $f_{i,j}$ is equal to $\pm e_{i_1,\ldots,i_k}$ for some $1 \leq i_1 < \ldots < i_k \leq n$. \hfill \Box

For each tuple $1 \leq i_1 < \ldots < i_k \leq n$ let $s_{i_1,\ldots,i_k}^A$ be the integer such that $e_{i_1,\ldots,i_k}^A \in (L_{\omega_k}^A)_{s_{i_1,\ldots,i_k}^A}$.

We now proceed to generalize the constructions in subsections [1.1] and [1.2] to give an explicit description of the homogeneous coordinate ring of $F_\lambda^A$ (with respect to a certain projective embedding) in terms of generators and relations.

We define the $U^A$-representation

$$U_\lambda^A = (L_{\omega_1}^A)^{\otimes \alpha_1} \otimes \ldots \otimes (L_{\omega_{n-1}}^A)^{\otimes \alpha_{n-1}}.$$ 

In $U_\lambda^A$ we distinguish the vector

$$w_\lambda^A = (v_{\omega_1}^A)^{\otimes \alpha_1} \otimes \ldots \otimes (v_{\omega_{n-1}}^A)^{\otimes \alpha_{n-1}}.$$ 

The following key fact holds.

Lemma 3.2. There is an isomorphism between $L_\lambda^A$ and the cyclic subrepresentation $U^A w_\lambda^A \subset U_\lambda^A$ mapping $v_\lambda^A$ to $w_\lambda^A$.

The proof of this lemma will be given in the next section.

Now let us consider the Segre embedding

$$\mathbb{P}(L_{\omega_1}^A)^{\alpha_1} \times \ldots \times \mathbb{P}(L_{\omega_{n-1}}^A)^{\alpha_{n-1}} \subset \mathbb{P}(U_\lambda^A)$$ 

and the embedding

$$\mathbb{P}_d^A = \mathbb{P}(L_{\omega_{d_{i_1}}}^A) \times \ldots \times \mathbb{P}(L_{\omega_{d_{i_t}}}^A) \subset \mathbb{P}(L_{\omega_1}^A)^{\alpha_1} \times \ldots \times \mathbb{P}(L_{\omega_{n-1}}^A)^{\alpha_{n-1}}$$

where $\mathbb{P}(L_{\omega_{d_{i_j}}}^A)$ is embedded diagonally into $\mathbb{P}(L_{\omega_{d_{i_j}}}^A)^{d_{i_j}}$.

Denote $y_{\lambda}^A$ the point in $\mathbb{P}(U_\lambda^A)$ corresponding to $C w_\lambda^A$ and recall the Lie group $N^A$.

Proposition 3.3. We have $N^A y_{\lambda}^A \subset \mathbb{P}_d^A \subset \mathbb{P}(U_\lambda^A)$.

Proof. This follows immediately from the definition of the Segre embedding. \hfill \Box

In view of Lemma [3.2] the closure $\overline{N^A y_{\lambda}^A}$ is precisely $F_\lambda^A$ and we have, therefore, embedded $F_\lambda^A$ into $\mathbb{P}_d^A$. Let us consider homogeneous coordinates $X_{i_1,\ldots,i_{d_j}}^A$ on $\mathbb{P}_d^A$ for all $1 \leq j \leq s$ and all $1 \leq i_1 < \ldots < i_{d_j} \leq n$ with $X_{i_1,\ldots,i_{d_j}}^A$ being dual to $e_{i_1,\ldots,i_k}^A$.

We introduce a grading on the ring $R_d^A = \mathbb{C}[\{X_{i_1,\ldots,i_{d_j}}^A\}]$ by setting $\text{grad}^A X_{i_1,\ldots,i_{d_j}}^A = s_{i_1,\ldots,i_{d_j}}^A$.

Let $\varphi^A : R_d \to R_d^A$ be the isomorphism sending $X_{i_1,\ldots,i_{d_j}}$ to $X_{i_1,\ldots,i_{d_j}}^A$, and $I_d^A$ be the initial ideal $I_{\text{grad}^A(\varphi^A(I_d))}$.

Theorem 3.4. The ideal of the subvariety $F_\lambda^A \subset P_d^A$ is precisely $I_d^A$.

We postpone the proof of this theorem till the next section. Also in Section [5] we will prove the following

Proposition 3.5. The ideal $I_d^A$ is generated by its quadratic part.
This means that $I_d^A$ is generated by the initial parts of the relations (2).

An important property of $F^A_\lambda$ is that it provides a flat degeneration of $F_\lambda$ in the following sense. Consider the ring $R_d = R_d^A \otimes \mathbb{C}[t]$, it is the coordinate ring of the variety $F^A_d = R_d^A \times \mathbb{A}^1$ (the affine line). Now consider some $\delta \in I_d$. Via $(\varphi^A)^{-1}$ we may view $\text{grad}^A$ as a grading on $R_d$. Let $M$ be the minimal degree with respect to $\text{grad}^A$ among the monomials appearing in $\delta$ with a nonzero coefficient. Let $\theta^A$ be the homomorphism from $R_d$ to $R_d$ sending $X_{i_1} \ldots X_{i_k}$ to $t^{\delta_{i_1} \ldots \delta_{i_k}} X_{i_1} \ldots X_{i_k}$. Let $I_d^A \subset R_d^A$ be the ideal generated by the expressions $t^{-M}\theta^A(\delta)$ ranging over all $\delta$ (in fact, one sees that $I_d^A$ is generated by the expressions $t^{-M}\theta^A(\delta)$ already as a $\mathbb{C}[t]$-module).

Denote $\mathfrak{S}^A_d \subset \mathfrak{P}^A_d$ the subvariety given by the ideal $I_d^A$. Let $\pi_2 : \mathfrak{P}^A_d \to \mathbb{A}^1$ be the projection onto the second component. One sees that $(R_d/I_d^A)/(t)$ is isomorphic to $R_d^A/I_d^A$, i.e. the fiber $\pi_2^{-1}(0) \cap \mathfrak{S}^A_d$ is equal to $F^A_\lambda$, while $(R_d/I_d^A)/(t-c)$ for $c \neq 0$ is isomorphic to $R_d/I_d$, i.e. any fiber $\mathfrak{S}^A_d \cap \pi_2^{-1}(c)$ with $c \neq 0$ is isomorphic to $F_\lambda$.

The flatness of the degeneration will follow from the somewhat stronger

**Proposition 3.6.** The ring $R_d/I_d^A$ is free over $\mathbb{C}[t]$.

**Proof.** The ring $R_d$ is also $\mathfrak{h}^*$-graded by $\text{deg} X_{i_1} \ldots X_{i_k} = \omega_k$ and $\text{deg} t = 0$. The ideal $I_d^A$ is deg-homogeneous. We choose a weight $\mu \in \mathbb{Z}\{\omega_1, \ldots, \omega_d\}$ and prove that $R_d/\mu/I_d^A$ is free over $\mathbb{C}[t]$ where $R_d/\mu$ and $I_d^A/\mu$ are the components of homogeneity degree $\mu$ in the respective spaces.

Denote $I_{d,\mu}$ the component of homogeneity degree $\mu$ in $I_d$. We may choose a basis $\delta_1, \ldots, \delta_{D_{\mu}}$ in $I_{d,\mu}$ such that the initial parts in $\text{grad}^A(\varphi^A(\delta_i))$ comprise a basis in the homogeneous component $I_{d,\mu}^A$. Indeed, consider the subspace of $I_{d,\mu}$ of such $\delta$ that $\text{grad}^A(\text{in}_{\text{grad}^A}(\varphi^A(\delta)))$ is as large as possible and choose a basis in this subspace. Then consider the subspace of $I_{d,\mu}$ of such $\delta$ that $\text{grad}^A(\text{in}_{\text{grad}^A}(\varphi^A(\delta)))$ takes one of the two largest possible values and extended the previously chosen basis to a basis in this space. Increasing the subspace by adding one value of $\text{grad}^A(\text{in}_{\text{grad}^A}(\varphi^A(\delta)))$ at a time will result in a basis with the desired property.

We may extended the basis $\{\delta_i\}$ to a basis in the homogeneous component $R_{d,\mu}$ by a set of monomials $M_1, \ldots, M_{\dim L_{\mu}}$ in such a way that the set $\{\text{in}_{\text{grad}^A}(\varphi^A(\delta_i))\} \cup \{\varphi^A(M_i)\}$ is a basis in $R_{d,\mu}$. Let $\mathcal{O}$ be the subset in $R_{d,\mu}$ comprised of the expressions $t^{-\text{grad}^A(\text{in}_{\text{grad}^A}(\varphi^A(\delta)))} \varphi^A(\delta_i)$ and $\mathfrak{M}$ be the subset in $R_{d,\mu}$ comprised of the monomials $\varphi^A(M_i)$. It is straightforward to check that $\mathcal{O}$ generates $\mathfrak{S}^A_{d,\mu}$ as a $\mathbb{C}[t]$-module and $\mathcal{O} \cup \mathfrak{M}$ is a $\mathbb{C}[t]$-basis in $R_{d,\mu}$. The proposition follows. \hfill \Box

One sees that the above proof applies in the general case: one could take any homogeneous ideal instead of $I_d$ and any grading instead of $\text{grad}^A$.

4. **Proof of Theorem 3.4**

In this section we will prove Lemma 3.2 and then derive Theorem 3.4.

We start off by giving the following explicit description of the integers $s_{i_1, \ldots, i_k}^A$. Choose $1 \leq i_1 < \ldots < i_k \leq n$ and let the integers $p_1 < \ldots < p_l$ comprise the difference $\{1, \ldots, k\} \backslash \{i_1, \ldots, i_k\}$ while the integers $q_1 > \ldots > q_l$ comprise the difference $\{i_1, \ldots, i_k\} \backslash \{1, \ldots, k\}$.

**Proposition 4.1.** In the above notations

$$s_{i_1, \ldots, i_k}^A = a_{p_1, q_1} + \ldots + a_{p_l, q_l}.$$
Proof. Consider a monomial $M = f_{x_1,y_1} \cdots f_{x_N,y_N} \in \mathcal{U}$ such that $Mv_{\omega_k} = \pm e_{i_1,\ldots,i_k}$ of minimal possible degree $\deg^A$. Then $s^A_{i_1,\ldots,i_k} = \deg^A M$.

Let us also assume that among all possible choices, $M$ is comprised of the least possible number of $f_{x,y}$, i.e. $N$ is as small as possible with the above properties holding.

First, let us show that we have $x_j \leq k$ and $y_j \geq k + 1$ for all $1 \leq j \leq N$. Indeed, consider the largest $j$ such that either $x_j > k$ or $y_j < k + 1$. Suppose first that $x_j > k$. Note that $f_{x,y}$ acts nontrivially only on those $e^A_{i_1,\ldots,i_k}$ for which $x_j \in \{i_1,\ldots,i_k\}$. Since $v_{\omega_k} = e_{1,\ldots,k}$, this implies that there exists some $j' > j$ such that $y_j' = x_j$. Since all $f_{x,y}$ with $x \leq k$ and $y \geq k + 1$ commute pairwisely, we may assume that $j' = j + 1$. Now, the image of $f_{x_{j+1},y_{j+2}} \cdots f_{x_N,y_N} v_0$ under the action of $f_{x_j,y_j} f_{x_{j+1},x_j}$ coincides with its image under the action of $f_{x_{j+1},y_j}$, i.e. we may replace $f_{x_j,y_j} f_{x_{j+1},y_{j+1}}$ with $f_{x_{j+1},y_j}$ and obtain a monomial $M'$ of no greater degree $\deg^A$ (due to inequality (A)) such that $M'v_0 = Mv_0$.

If $y_j < k + 1$, then we may assume that $x_{j+1} = y_j$ and define $M'$ of no greater degree by replacing $f_{x_j,y_j} f_{x_{j+1},y_{j+1}}$ with $f_{x_j,y_j} + 1$.

Now observe that a product of the form $f_{x,y} f_{x,y'}$ or $f_{x,y} f_{x',y}$ annihilates $L_{\omega_k}$. Therefore, with the mentioned commutativity taken into account, all $x_j$ as well as all $y_j$ are pairwise distinct. This means that $N = l$ and the set $\{x_1,\ldots,x_N\}$ is precisely $\{p_1,\ldots,p_l\}$ while $\{y_1,\ldots,y_N\}$ is precisely $\{q_1,\ldots,q_l\}$. Now suppose that we have $x_j \leq x_{j'}$ and $y_j \leq y_{j'}$ for some $j \neq j'$. Then we may replace $f_{x_j,y_j} f_{x_{j'},y_{j'}}$ with $f_{x_j,y_j} f_{x_{j'},y_{j'}}$ and obtain a monomial that is of no greater degree (due to inequality (B)) and once again sends $v_{\omega_k}$ to $\pm e_{i_1,\ldots,i_k}$. Repeating this procedure we will eventually obtain a monomial in which either $x_j > x_{j'}$ or $y_j > y_{j'}$ for any $j \neq j'$, i.e. precisely the monomial $f_{p_1,q_1} \cdots f_{p_l,q_l}$. \(\square\)

The above proposition is directly related to FFLV bases and PBW tableaux. Indeed, the discussion preceding Lemma 1.10 implies that if $Z$ is the unique one-column PBW tableau with content $\{i_1,\ldots,i_k\}$, then $\tau(Z)_{p_j,q_j} = 1$ for all $1 \leq j \leq l$ while all other $\tau(Z)_{i,j} = 0$. Therefore, Proposition 4.1 tells us that the basis $\{e^A_{i_1,\ldots,i_k}\}$ coincides (up to sign change) with

$$\left\{ \left\langle \prod (f_{i,j}^A)^{T_{i,j}} \right\rangle v^A_{\omega_k} \mid T \in \Pi_{\omega_k} \right\}.$$

We will denote the FFLV pattern $\tau(Z)$ corresponding to the basis vector $e^A_{i_1,\ldots,i_k}$ via $T(e^A_{i_1,\ldots,i_k})$.

We next prove

**Proposition 4.2.** For every $T \in \Pi_\lambda$ choose a monomial $M_T \in \mathcal{U}^A$ of the form $\prod (f_{i,j}^A)^{T_{i,j}}$ where the factors are ordered arbitrarily. The set $\{M_T w^A_\lambda \mid T \in \Pi_\lambda\}$ is linearly independent.

Proof. The space $U^A_\lambda$ has a basis comprised of all vectors of the form $e = e_1 \otimes \cdots \otimes e_{a_1+\cdots+a_n-1}$ where the first $a_1$ factors are of the form $e^A_1 \in L^A_{\omega_1}$, the next $a_2$ are of the form $e^A_{i_1,i_2} \in L^A_{\omega_2}$ and so on. With each such basis vector $e$ we may associate $T(e) \in \Pi_\lambda$. Indeed, set $T(e) = T(e^1) + \ldots + T(e_{a_1+\cdots+a_n-1})$ (a sum of points in $\mathbb{R}(\lambda)$). We have $T(e) \in \Pi_\lambda$ in view of the Minkowski property Lemma 1.6. We have decomposed $U^A_\lambda$ into the direct sum of spaces

$$(U^A_\lambda)_T = \bigoplus_{T(e)=T} C e$$

with $T$ ranging over $\Pi_\lambda$.

Next, let us define a partial order on the set of all number triangles $T = \{T_{i,j} \mid 1 \leq i < j \leq n\}$ and, in particular, on $\Pi_\lambda$. For such a triangle $T$ let $\gamma(T)$ be the sequence of all elements $T_{i,j}$ ordered first by $i + j$ increasing and then by $i$ increasing, i.e., from left to right and within one vertical column from the bottom up. We then write $T_1 \preceq T_2$ whenever $\gamma(T_1)$ is no greater
than $\gamma(T_2)$ lexicographically, that is $T_1 = T_2$ or for the least $i$ such that $\gamma(T_1)_i \neq \gamma(T_2)_i$ one has $\gamma(T_1)_i < \gamma(T_2)_i$. Note that the order $\preceq$ is additive: $T_1 \preceq T_2$ and $T_3 \preceq T_4$ imply $T_1 + T_3 \preceq T_2 + T_4$.

This partial order induces a partial order on monomials in $U^A$ by setting

$$\prod (f_{i,j}^A)^{T_{i,j}} \preceq \prod (f_{i,j}^A)^{T_{i,j}}$$

whenever $T_1 \preceq T_2$ for arbitrary orderings of the factors. This order is multiplicative.

Consider some $1 \leq i_1 < \ldots < i_k \leq n$ and a monomial $M$ with $Mv_{a_1} = \pm e_{i_1,\ldots,i_k}$. First, we must have $\deg A M = s_{i_1,\ldots,i_k}$. Second, the proof of Proposition 3.1 shows that the monomial $\prod (f_{i,j}^A)^{T(e_{i_1,\ldots,i_k})_{i,j}}$ can be obtained from $M$ by replacing $f_{i,j}^A f_{i,l}^A$ with $f_{i,j}^A$ for $i < j < l$, replacing $f_{i,j}^A f_{i,m}^A$ with $f_{i,m}^A f_{i,j}^A$ for $i < l < j < m$ and commuting the factors. None of these operations increase the monomial with respect to $\preceq$, i.e. $M \preceq \prod (f_{i,j}^A)^{T(e_{i_1,\ldots,i_k})_{i,j}}$.

Now consider $T \in \Pi_\lambda$ and the vector $M_T w^A_{\lambda}$. This vector decomposes as

$$M_T w^A_{\lambda} = \sum_{T_1^{i_1,1} + \ldots + T_1^{a_1} + \ldots + T_1^{n-1}} \left( \prod_{i,j} (f_{i,j}^A)^{T_{i,j}} v_{\omega_1}^A \right) \otimes \ldots \otimes \left( \prod_{i,j} (f_{i,j}^A)^{T_{i,j}} v_{\omega_{n-1}}^A \right).$$

(4)

Here the sum ranges over all decompositions of $T$ into a sum of number triangles $T^{k,l}$ with non-negative integer elements over all $1 \leq k \leq n - 1$ and $1 \leq l \leq a_k$. The set of all factors $f_{i,j}$ over all the products $\prod (f_{i,j}^A)^{T^{k,l}}$ is in one-to-one correspondence with the factors in $M_T$ and in each such product they are ordered in the same way as in $M_T$. Suppose $\prod (f_{i,j}^A)^{T^{k,l}} v_{\omega_1}^A$ is nonzero and equal to $\pm e_{i_1,\ldots,i_k}$. This implies that $\deg A (\prod (f_{i,j}^A)^{T^{k,l}}) = s_{i_1,\ldots,i_k}$ and, in view of the discussion above, that $T^{k,l} \succeq (e_{i_1,\ldots,i_k})$. This means that every summand in the right-hand side of (4) is contained in some $(U^A_{\lambda})_T$ with $T' \preceq T$. Moreover, the Minkowski property implies that for at least one summand we have $T^{k,l} \in \Pi_{\omega_k}$ for all $k$ and $l$, i.e. at least one of the summands is contained in $(U^A_{\lambda})_T$. The linear independence follows.

We have thus shown that $\dim (U^A w^A_{\lambda}) \geq |\Pi_\lambda| = \dim L_\lambda = \dim L^A_\lambda$. To prove Lemma 3.2 it now suffices to show the existence of a surjective homomorphism from $L^A_{\lambda}$ to $U^A w^A_{\lambda}$ taking $v^A_{\lambda}$ to $w^A_{\lambda}$.

**Proof of Lemma 3.2** We are to show that for every $S \in U^A$ we have $Sw^A_{\lambda} = 0$ whenever $Sv^A_{\lambda} = 0$. Indeed, consider some $S$ with $Sw^A_{\lambda} = 0$, we may assume that $S \in U^A_m$ for some integer $m$.

The relation $Sv^A_{\lambda} = 0$ means that there exists some $S' \in U$ such that $S'v_{\lambda} = 0$ and $S' = S_0 + S_1$ with $S_0$ being obtained from $S$ by simply removing all the $A$ superscripts and $S_1 \in U_{m-1}$. We then have $S'w_{\lambda} = 0 \in U_{\lambda}$.

$U_{\lambda}$ has a basis comprised of all vectors of the form $e = e^1 \otimes \ldots \otimes e^{a_1 + \ldots + a_{n-1}}$ where the first $a_1$ factors are of the form $e_i \in L_{\omega_1}$, the next $a_2$ are of the form $e_{i_1,i_2} \in L_{\omega_2}$ and so on. For such an $e$ suppose that some $e^1$ is equal to $e_{i_1,\ldots,i_k}$ and let $m_i$ be the least integer such that $e_{i_1,\ldots,i_k} \in (L_{\omega_k})_{m_i}$, i.e. $m_i = s_{i_1,\ldots,i_k}$. We then set $m^A(e) = m_1 + \ldots + m_{a_1 + \ldots + a_{n-1}}$ and obtain
the decomposition of $U_\lambda$ into a direct sum of the spaces
\[ (U_\lambda)_N = \bigoplus_{m^A(e) = N} \mathbb{C}e. \]

Now consider the map $\Psi$ from $U_\lambda$ to $U^A_\lambda$ obtained as the tensor product of maps from $L_{\omega_k}$ to $L^A_{\omega_k}$ taking $e_{i_1, \ldots, i_k}$ to $e^A_{i_1, \ldots, i_k}$. Let $M \in \mathcal{U}$ be a monomial with $\deg^A M = m$ and $M^A \in \mathcal{U}^A$ be the monomial obtained from $M$ by adding $^A$ superscripts to all the factors. By considering the decompositions of $M_w_\lambda$ and $M^A_w_\lambda$ in the respective bases, one sees that $M_w_\lambda \in \bigoplus_{m' \leq m}(U_\lambda)_{m'}$ and that $M^A_w_\lambda = \Psi((M_w_\lambda)_m)$ with $(.)_m$ denoting the projection onto $(U_\lambda)_m$. In particular, $(S_1 w_\lambda)_m = 0$ and, therefore,

\[ S^A_w_\lambda = \Psi((S_0 w_\lambda)_m) = \Psi((S' w_\lambda)_m) = 0. \]

Together Proposition 4.2 and Lemma 3.2 have the following implication, that connects FFLV patterns with bases of representations.

**Corollary 4.3.** For every $T \in \Pi_\lambda$ choose a monomial $M_T \in \mathcal{U}^A$ of the form $\prod (f_{i,j})^{T_{i,j}}$ where the factors are ordered arbitrarily. The set $\{M_T v^A_\lambda \mid T \in \Pi_\lambda\}$ constitutes a basis in $L^A_\lambda$.

We are now ready to prove Theorem 3.4.

**Proof of Theorem 3.4** Recall the $\mathfrak{h}^*$-grading deg of $R^A_d$ given by $\deg X_{i_1, \ldots, i_k} = \omega_k$. We are to show that the deg-homogeneous ideal $J$ spanned by all deg-homogeneous polynomials $X \in R^A_d$ vanishing on $N^A y^A_\lambda$ is precisely $I^A_d$. However, the commutation relations \[\{\]\ easily imply that the Lie algebra $n^A$ is nilpotent which means that its exponential map to $N^A$ is surjective (in view of the simply connectedness, it is bijective). Therefore, $N^A y^A_\lambda = \exp(n^A) y^A_\lambda$.

For any $f \in n^A$ the point $\exp(f)y^A_\lambda \in \mathbb{P}(U^A_\lambda)$ corresponds to the line

\[ \mathbb{C}(\exp(f)v^A_{\omega_{i_1}}^\lambda) \otimes \cdots \otimes (\exp(f)v^A_{\omega_{n-1}}^\lambda) \]

and, by the definition of the Segre embedding, coincides with the point
\[ (\exp(f)u^A_{i_1})^{a_1} \times \cdots \times (\exp(f)u^A_{\omega_{n-1}})^{a_{n-1}} \in \mathbb{P}(L^A_{\omega_{i_1}}) \times \cdots \times \mathbb{P}(L^A_{\omega_{n-1}})^{a_{n-1}} \]

and with the point
\[ (\exp(f)u^A_{\omega_{i_1}}) \times \cdots \times (\exp(f)u^A_{\omega_{n-1}}) \in \mathbb{P}^A_d. \]

Let us denote $C^A_{i_1, \ldots, i_k}((\{c_{i,j}\})')$ the coordinate of $\exp(f)v^A_{\omega_k}$ corresponding to the basis vector $e^A_{i_1, \ldots, i_k}$ where $f = \sum c_{i,j} f_{i,j}$. Via the standard expansion of the action of $\exp(f)$ on $L_{\omega_k}$ as a power series in the action of $f$ one sees that each $C^A_{i_1, \ldots, i_k}((\{c_{i,j}\}))$ depends polynomially on the $c_{i,j}$. We may, therefore, view $C^A_{i_1, \ldots, i_k}$ as an element of $\mathbb{C}[[z_{i,j}]]$ with $1 \leq i < j \leq n$. Let us introduce $n - 1$ additional indeterminates $z_1, \ldots, z_{n-1}$. Then $J$ is precisely the kernel of the homomorphism $\psi^A : R^A_d \rightarrow \mathbb{C}[[z_{i,j}, z_{i,j}]]$ mapping $X^A_{i_1, \ldots, i_k}$ to $z_k C^A_{i_1, \ldots, i_k}$.

Next, for $f' = \sum_{i,j} c_{i,j} f_{i,j} \in n_-$ denote $C^A_{i_1, \ldots, i_k}((\{c_{i,j}\})')$ the coordinate of $\exp(f')v^A_{\omega_k}$ (for $\exp : n_- \rightarrow N$) corresponding to the basis vector $e_{i_1, \ldots, i_k}$. The same reasoning shows that $C^A_{i_1, \ldots, i_k}$ also depends polynomially on the $c_{i,j}$ and we view $C^A_{i_1, \ldots, i_k}$ as an element of $\mathbb{C}[[z_{i,j}]]$. One may then observe that $C^A_{i_1, \ldots, i_k}$ is the initial part of $C^A_{i_1, \ldots, i_k}$ with respect to the grading $\text{grad}^A$ defined by $\text{grad}^A(z_{i,j}) = a_{i,j}$. Note that this initial part is the component of grading $s^A_{i_1, \ldots, i_k}$ in $C^A_{i_1, \ldots, i_k}$.

For any $\text{grad}^A$-homogeneous $X \in I^A_d$ we have a relation $X' \in I_d$ such that $\text{inv}_{\text{grad}^A}(\phi(X')) = X$. On one hand, in view of Theorem 1.1, $\psi(X') = 0$ where $\psi$ is the homomorphism from
$R_d$ to $\mathbb{C}\{z_{i,j},z_l\}$ mapping $X_{i_1,\ldots,i_k}$ to $z_k C_{i_1,\ldots,i_k}$. On the other hand, $\psi^A(X)$ is the initial part (the component of grading $\text{grad}^4(X)$) of $\psi(X')$ with respect to $\text{grad}^4$ where $\text{grad}^4$ is the extension of $\text{grad}^4$ by $\text{grad}^4_z z_l = 0$, i.e. $\psi^A(X) = 0$. This shows that $I_d^A \subset J$.

The ring $\mathbb{C}\{z_{i,j},z_l\}$ is $\mathbb{N}$-graded by $\text{deg} z_k = \omega_k$ and $\text{deg} z_{i,j} = 0$; the homomorphisms $\psi$ and $\psi^A$ are deg-homogeneous. To prove the reverse inclusion $J \subset I_d^A$ we show that the dimensions of the homogeneous components of the image of $\psi^A$ are no less than the corresponding dimensions for the ring $R_d^A/I_d^A$.

For a weight $\mu \in \mathbb{Z}_{\geq 0}\{\omega_{d_1},\ldots,\omega_{d_k}\}$ the component of degree $\mu$ has dimension $\dim L_\mu$ which is simply due to $I_d^A$ being an initial ideal of $I_d$. We extend the notations from subsection 1.4 by denoting $X^A(Z)$ the indeterminate $X_{i_1,\ldots,i_k}$ for the unique PBW tableau $Z$ of shape $\omega_k$ and content $\{i_1,\ldots,i_k\}$. For a PBW tableau $Y$ of shape $\mu$ with columns $Z_{\mu_1}$, $Z_{\mu_2}$ we denote $X^A(Y) = X^A(Z_{\mu_1}) \ldots X^A(Z_{\mu_2})$. We prove the announced inequality by showing that the images under $\psi^A$ of the monomials $X^A(Y)$ with $Y \in \mathcal{Y}_\mu$ are linearly independent.

Let us take a closer look at the polynomial $C_{i_1,\ldots,i_k}^A$: it is the sum of products $\pm z_{i_1,j_1} \ldots z_{i_N,j_N}/N!$ over all sequences $(i_1,j_1),\ldots,(i_N,j_N)$ such that $\deg^A(f_{i_1,j_1}^A \ldots f_{i_N,j_N}^A) = s_{i_1,\ldots,i_k}$ and $
abla_{i_1,\ldots,i_k}^A (f_{i_1,j_1}^A \ldots f_{i_N,j_N}^A) v_{\omega_k} = \pm e_{i_1,\ldots,i_k}^A$.

In particular, we see that the monomial $\prod_{i,j} T(e_{i_1,\ldots,i_k}^A)$ appears with coefficient $(-1)^{\sigma_{i_1,\ldots,i_k}}$ for $\sigma_{i_1,\ldots,i_k} \in S_k$ such that $(i_{\sigma_{i_1,\ldots,i_k}(1)},\ldots,i_{\sigma_{i_1,\ldots,i_k}(k)})$ is a PBW tableau. The partial order $\preceq$ from the proof of Proposition 4.1 implies a monomial order on $\mathbb{C}\{z_{i,j}\}$ by setting $z_{i_1,j_1} \ldots z_{i_N,j_N} \preceq z_{i_1',j_1'} \ldots z_{i_N',j_N'}$, whenever $f_{i_1,j_1}^A \ldots f_{i_N,j_N}^A \preceq f_{i_1',j_1'}^A \ldots f_{i_N',j_N'}^A$. The discussion in the proof of Proposition 4.1 implies that $(-1)^{\sigma_{i_1,\ldots,i_k}} \prod_{i,j} T(e_{i_1,\ldots,i_k}^A) z_{i,j}$ is the initial term in $C_{i_1,\ldots,i_k}^A$ with respect to monomial order $\preceq$.

We extend $\preceq$ to a multiplicative partial order (also $\preceq$) on the set of monomials in $\mathbb{C}\{z_{i,j},z_l\}$ by comparing the images of these monomials in $\mathbb{C}\{z_{i,j}\}$ under the map sending every $z_l$ to 1 and every $z_{i,j}$ to itself. If we denote $b_1,\ldots,b_{n-1}$ the coordinates of $\mu$ in the basis of fundamental weights, then we see that $\pm \prod z_l^{b_l} \prod z_{i,j}^{\tau(Y)_{i,j}}$ is the unique $\preceq$-minimal term in $\psi^A(X^A(Y))$. Since these monomials are distinct for distinct $Y$, the linear independence of the polynomials $X^A(Y)$ with $Y \in \mathcal{Y}_\mu$ follows.

The above proof has the following implication.

**Corollary 4.4.** For $\mu \in \mathbb{Z}_{\geq 0}\{\omega_{d_1},\ldots,\omega_{d_k}\}$, the images of the monomials $X^A(Y)$ with $Y$ ranging over $\mathcal{Y}_\mu$ comprise a basis in the component of $R_d^A/I_d^A$ of homogeneity degree $\mu$.

5. **Toric degenerations and monomial annihilating ideals**

The following fact may easily be deduced from the above results.

**Theorem 5.1.** If the weight system $A$ is such that all inequalities of types (a) and (b) (or, equivalently, all inequalities of types (A) and (B)) are strict, then the degenerate flag variety $F_\lambda^A$ is the toric variety corresponding to the polytope $Q_\lambda$. 

Proof. The proof of Proposition 3.1 shows that the only monomial taking \( v_{w_k}^A \) to some \( \pm e_{i_1, \ldots, i_k}^A \) is \( \prod (f_{i,j}^A)^{T(e_{i_1, \ldots, i_k})} \) (up to a permutation of the commuting factors). The proof of Theorem 3.4 then shows that

\[
\psi^A(X_{i_1, \ldots, i_k}) = (-1)^{a_{i_1, \ldots, i_k}} \prod z_{i,j}^{T(e_{i_1, \ldots, i_k})}.
\]  

(5)

When \( k = d_j \), the monomials (5) generate the coordinate ring of the toric variety given by the polytope \( Q_{\omega_k} \). However, the image of \( \psi^A \) (which is isomorphic to \( R_d^A/I_d^A \)) is generated by the monomials (3) with \( k \) ranging over \( \{d_1, \ldots, d_s\} \) and all possible \( 1 \leq i_1 < \ldots < i_k \leq n \). The theorem now follows from Lemma 1.6.

Now note that the commutation relations (3) show that the Lie algebra \( \mathfrak{n}^A \) and the algebra \( U^A \) are commutative whenever all the inequalities of type (A) are strict. In other words, we have \( U^A = \mathbb{C}[\{f_{i,j}^A\}] \) which allows us to speak of monomial ideals in \( U^A \).

**Theorem 5.2.** If the weight system \( A \) is such that all inequalities of types (a) and (b) are strict, then the annihilating ideal of \( L^A_\lambda \) in \( U^A \) is the monomial ideal spanned by the monomials \( \prod (f_{i,j}^A)^{S_{i,j}} \) with \( S \) ranging over the set \( \{1 \leq i < j \leq n\} \setminus \Pi_\lambda \).

**Proof.** We are to show that every monomial \( \prod (f_{i,j}^A)^{S_{i,j}} \) with \( S \in \mathbb{Z}_{\geq 0}^{\{1 \leq i < j \leq n\}} \setminus \Pi_\lambda \) acts trivially on \( L^A_\lambda \). We do so by invoking Lemma 3.2 and showing that it acts trivially on \( w_\lambda^A \in U^A_\lambda \).

Indeed, \( \prod (f_{i,j}^A)^{S_{i,j}} (w_\lambda^A) \) is equal to a sum of expressions of the form

\[
\left( \prod (f_{i,j}^A)^{s_{i,j}^{1,1}} (v_{\omega_1}^A) \right) \otimes \ldots \otimes \left( \prod (f_{i,j}^A)^{s_{i,j}^{1,n}} (v_{\omega_n}^A) \right) \otimes \ldots \otimes \left( \prod (f_{i,j}^A)^{s_{i,j}^{n-1,1}} (v_{\omega_1}^A) \right) \otimes \ldots \otimes \left( \prod (f_{i,j}^A)^{s_{i,j}^{n-1,n-1}} (v_{\omega_{n-1}}^A) \right)
\]  

(6)

where the sum of all \( S_{i,j}^{k,l} \) is equal to \( S_{i,j} \) for all \( i, j \). However, every monomial in the \( f_{i,j}^A \) takes \( v_{w_k}^A \) to a vector of the form \( \pm e_{i_1, \ldots, i_k}^A \) and the only monomial taking \( v_{w_k}^A \) to \( \pm e_{i_1, \ldots, i_k}^A \) is \( \prod (f_{i,j}^A)^{T(e_{i_1, \ldots, i_k})} \). Hence, every monomial \( \prod (f_{i,j}^A)^{T} \) with \( T \notin \Pi_{\omega_k} \) annihilates \( v_{w_k}^A \). Therefore, (3) is nonzero only if every \( \{s_{i,j}^{k,l}\}_{1 \leq i < j \leq n} \in \Pi_{\omega_k} \) which is impossible in view of \( S \notin \Pi_\lambda \).

We have shown that the same combinatorial conditions (which hold for a “generic” weight system) are sufficient for both \( F_\lambda^A \) being the toric variety given by \( Q_\lambda \) and for the annihilating ideal of \( L^A_\lambda \) being spanned by the monomials with exponent vectors outside of \( Q_\lambda \). This is not a coincidence, a slightly closer look at the proof of Theorem 3.4 shows that whenever the annihilating ideal of \( L^A_\lambda \) is as above, \( F_\lambda^A \) is the toric variety in question.

**Remark 5.3.** Recall that in [FFR] the authors construct a weight system \( A \) such that the annihilating ideal of a cyclic vector of \( L^A_\lambda \) is monomial. This weight system satisfies all the inequalities of types (A) and (B).

### 6. The Cone of Degenerations

Conditions (a) and (b) define a polyhedral cone \( K \) in \( \mathbb{R}^{\{1 \leq i < j \leq n\}} \). One sees that \( K \) is the product of the linear subspace of dimension \( n - 1 \) given by \( a_{i,j} = a_{i,i+1} + \ldots + a_{j-1,j} \) for all \( 1 \leq i < j \leq n \) and of the simplicial cone of dimension \( \binom{n-1}{2} \) given by all \( a_{i,i+1} = 0 \) and all inequalities of types (a) and (b). We show that, in a sense, all the degenerations that are obtained in this paper are parametrized by the faces of \( K \).
Proposition 6.1. If the minimal face of $\mathcal{K}$ containing weight system $A$ coincides with the minimal face of $\mathcal{K}$ containing some weight system $B = \{b_{i,j}\}$, then the following hold.

i) The map $\Theta : \mathfrak{n}^A \rightarrow \mathfrak{n}^B$ taking $f_{i,j}^A$ to $f_{i,j}^B$ is an isomorphism of Lie algebras.

ii) The representations $L_{\lambda}^A$ and $L_{\lambda}^B$ are isomorphic (where $\mathfrak{n}^A$ and $\mathfrak{n}^B$ are identified by $\Theta$).

iii) The varieties $F_X^A$ and $F_X^B$ coincide as subvarieties in $\mathbb{P}^A_d$ (where $\mathbb{P}^B_d$ is identified with $\mathbb{P}^A_d$ via the isomorphisms between $L^A_{\omega_k}$ and $L^B_{\omega_k}$).

Proof. 

i) The commutation relations \((3)\) are determined by which inequalities of type (A) are strict and which are not. As seen in the proof of Proposition \(2.1\) every inequality of type (A) decomposes into the sum of inequalities of types (a) and (b). Hence, the commutation relations in $\mathfrak{n}^A$ are determined by which inequalities of types (a) and (b) are strict and which are not, i.e. by the minimal face of $\mathcal{K}$.

ii) First let us consider the case of a fundamental weight $\lambda = \omega_k$. We show that the map sending $e_{i_1,\ldots,i_k}^A$ to $e_{i_1,\ldots,i_k}^B$ provides the desired isomorphism. For some $e_{i_1,\ldots,i_k}^A$, suppose that a monomial $M$ in the $f_{i,j}^A$ is such that $M e_{i_1,\ldots,i_k}^A = \pm e_{i_1,\ldots,i_k}^A$. Then, as seen in the proof of Proposition \(4.1\), the monomial $\Pi(f_{i,j}^A)^T(e_{i_1,\ldots,i_k})_{i,j}$ can be obtained from $M$ by replacing $f_{i,j}^A f_{i,j}^A$ with $f_{i,j}^A$ for $i < j < l$, replacing $f_{i,j}^A f_{i,m}^A$ with $f_{i,m}^A f_{i,j}^A$ for $i < l < j < m$ and commuting the factors. However, $\deg^A M = s_{i_1,\ldots,i_k}$ and none of these operations increase the degree $\deg^A$, therefore, each one of them must preserve $\deg^A$.

In other words, whether a monomial takes $v_{\omega_k}$ to $\pm e_{i_1,\ldots,i_k}^A$ depends on whether one may obtain $\Pi(f_{i,j}^A)^T(e_{i_1,\ldots,i_k})_{i,j}$ from $M$ by a series of the above operations preserving $\deg^A$, i.e. on which inequalities of types (A) and (B) are equalities. The same holds for weight system $B$ and our assertion follows.

The general case now follows directly from Lemma \(3.2\).

iii) The polynomial $C_{i_1,\ldots,i_k}^A$ introduced in the proof of Theorem \(3.4\) is determined by the set of monomials in $U^A$ taking $v_{\omega_k}$ to $\pm e_{i_1,\ldots,i_k}^A$. This set, as observed in the proof of part ii), is determined by the minimal face of $\mathcal{K}$ containing $A$. Therefore, the map $\psi^A$ and its kernel $I^A_d$ are also determined by the minimal face of $\mathcal{K}$ containing $A$ and our assertion follows. \(\square\)

In particular, we see that $\mathfrak{n}^A$, $L_{\lambda}^A$ and $F_{\lambda}^A$ coincide with the non-degenerate objects $\mathfrak{n}_-$, $L_{\lambda}$ and $F_{\lambda}$ whenever $A$ is contained in the minimal face of $\mathcal{K}$, i.e. the $(n-1)$-dimensional linear space given by the equations $a_{i,j} = a_{i,i+1} + \ldots + a_{j-1,j}$. When all inequalities of type (a) (or (A)) are strict but all the inequalities of type (b) (or (B)) are equalities, $\mathfrak{n}^A$, $L_{\lambda}^A$ and $F_{\lambda}^A$ coincide with the abelian degenerations $\mathfrak{n}^A$, $L_{\lambda}^A$ and $F_{\lambda}^A$. When the minimal face containing $A$ is maximal, i.e. $A$ lies in the interior of $\mathcal{K}$, we are in the toric situation discussed in Section \(5\).

We have shown that the ideal of the variety $F_{\lambda}^A$ is an initial ideal of the ideal of the non-degenerate flag variety $F_{\lambda}$. We will now generalize this fact.

Proposition 6.2. Suppose that the weight system $B$ is such that the minimal face of $\mathcal{K}$ containing $B$ coincides with the minimal face of $\mathcal{K}$ containing $A$. Then the ideal $(\varphi^B)^{-1}(I^A_d)$ is the initial ideal of $(\varphi^B)^{-1}(I^A_d)$ with respect to grading $\text{grad}^B$ on $R_d$.

Proof. Since the dimensions of the homogeneous components coincide, it suffices to consider some $\delta \in I^A_d$ and show that $\text{in}_{\text{grad}^B}((\varphi^A)^{-1}(\delta)) \in (\varphi^B)^{-1}(I^B_d)$. In other words, we are to show that

$$
\psi^B \left( \varphi^B \left( \text{in}_{\text{grad}^B}((\varphi^A)^{-1}(\delta)) \right) \right) = 0.
$$
Now, if a monomial $M$ in the $I_{d}^{B}$ takes $v_{w_{k}}^{B}$ to $\pm e_{i_{1},...,i_{k}}^{B}$, then the monomial obtained from $M$ by replacing the $B$ superscripts with $A$ superscripts takes $v_{w_{k}}^{A}$ to $\pm e_{i_{1},...,i_{k}}^{A}$. This follows from the set of inequalities of types (a) and (b) that are equalities being larger for $A$ than for $B$. This implies that $C_{i_{1},...,i_{k}}^{B}$ is the initial part of $C_{i_{1},...,i_{k}}^{A}$ with respect to grad$^{B}$. Therefore,

$$
\psi^{B} \left( \varphi^{B} \left( \text{in}_{\text{grad}^{A}}((\varphi^{A})^{-1}(\delta)) \right) \right)
$$

is the initial part of $\psi^{A}(\delta) = 0$ with respect to grad$^{B}$. \hfill $\square$

We may now replicate Proposition 3.6 and the preceding discussion to show that in the assumptions of Proposition 6.2 the variety $F_{d}^{B}$ provides a flat degeneration of $F_{d}^{A}$.

To complete this section we derive Proposition 3.5.

**Proof of Proposition 3.5.** First we consider the case when $A$ lies in the interior of $K$, i.e. the case discussed in Section 5. If we denote $J_{d}^{A}$ the ideal in $I_{d}^{A}$ generated by the quadratic part of $I_{d}^{A}$, then, in view of Corollary 4.4, it suffices to show that for every monomial $M \in R_{d}^{A}$ there exists some $\delta \in J_{d}^{A}$ such that $M + \delta$ is a linear combination of monomials of the form $X^{A}(Y)$ with $Y$ PBW semistandard.

Recall the partial order $\preceq$ from Remark 1.9. Consider a product $X_{i_{1},...,i_{k}}^{A}X_{i'_{1},...,i'_{k'}}^{A} \in R_{d}^{A}$ such that $k \geq k'$ and $\{i_{1},...,i_{k}\} \not\subseteq \{i'_{1},...,i'_{k'}\}$. Let $Z$ denote the PBW tableau with elements $i_{1},...,i_{k}$ in its first column and $i'_{1},...,i'_{k'}$ in its second column. By our assumption, $Z$ is not PBW semistandard, i.e. we have a maximal $i_{0}$ such that for all $i_{0} \leq i \leq k$ we have $Z_{i,1} < Z_{i,2}$. Let us denote $W$ the tableau obtained from $Z$ by exchanging $Z_{i,1}$ with $Z_{i,2}$ for all $i \leq i_{0}$. Note that $W$ is also a PBW tableau and that $W_{i_{0},2} < Z_{i_{0},2}$. Let $j_{1} < ... < j_{k}$ be the elements in the first column of $W$ and $j'_{1} < ... < j'_{k'}$ be the elements in its second column. Then we have

$$
T(e_{i_{1},...,i_{k}}^{A}) + T(e_{i'_{1},...,i'_{k'}}^{A}) = T(e_{j_{1},...,j_{k}}^{A}) + T(e_{j'_{1},...,j'_{k'}}^{A})
$$

which, in view of the discussion in Section 5, means that we have a relation of the form

$$
X_{i_{1},...,i_{k}}^{A}X_{i'_{1},...,i'_{k'}}^{A} \pm X_{j_{1},...,j_{k}}^{A}X_{j'_{1},...,j'_{k'}}^{A} \in I_{d}^{A}.
$$

Now consider any monomial $X^{A}(Y) \in R_{d}^{A}$ with $Y$ a PBW tableau that is not PBW semistandard. For some column $j$ we have $\{Y_{j}\} \not\subseteq \{Y_{j+1}\}$ and we may apply the procedure from the previous paragraph to the $j$th and $j + 1$st column of $Y$ to obtain a new PBW tableau $Y'$ with $X^{A}(Y') \pm X^{A}(Y'') \in J_{d}^{A}$. Furthermore, the lowest differing element in the rightmost differing column of $Y$ and $Y'$ will be smaller in $Y'$ than in $Y$, i.e. $Y'$ will be smaller with respect to the corresponding lexicographic order. Consequently, iteration of this procedure will provide the desired relation for the monomial $X^{A}(Y)$.

We derive the general case from the toric case. Consider a relation $\delta \in I_{d}^{A}$ homogeneous with respect to both deg and grad$^{A}$. Consider any weight system $B$ in the interior of $K$. By Proposition 6.2 we have an $\delta' \in I_{d}^{B}$ such that

$$
\text{in}_{\text{grad}^{B}}((\varphi^{A})^{-1}(\delta)) = (\varphi^{B})^{-1}(\delta').
$$

Let $\delta' = K_{1}P_{1}^{B} + \ldots + K_{n}P_{n}^{B}$ for deg-homogeneous $K_{i} \in R_{d}^{B}$ and quadratic deg-homogeneous and grad$^{B}$-homogeneous $P_{i}^{B} \in I_{d}^{B}$. We have quadratic $P_{i} \in I_{d}^{A}$ with

$$
\varphi^{B} \left( \text{in}_{\text{grad}^{B}}((\varphi^{A})^{-1}(P_{i})) \right) = P_{i}^{B}
$$
and, consequently,
\[ \text{in}_{\text{grad}^{B}((\varphi^{B})^{-1}(K_{1})(\varphi^{A})^{-1}(P_{1}) + \ldots + (\varphi^{B})^{-1}(K_{n})(\varphi^{A})^{-1}(P_{n}))} = \text{in}_{\text{grad}^{B}((\varphi^{B})^{-1}(\delta'))}. \]

Therefore, if we introduce the grading \( \text{grad}^{B} \) on \( R^{A}_{d} \) via \( \varphi^{A} \), the \( \text{grad}^{B} \) grading of the \( \text{grad}^{B} \)-initial part of the difference
\[ \delta - \varphi^{A}((\varphi^{B})^{-1}(K_{1}))P_{1} - \ldots - \varphi^{A}((\varphi^{B})^{-1}(K_{n}))P_{n} \]
is greater than \( \text{grad}^{B}(\delta') \). However, within a given deg-homogeneous component the grading \( \text{grad}^{B} \) takes only a finite number of values. Therefore, passing from \( \delta \) to the above difference and then iterating the procedure will express \( \delta \) as a \( R^{A}_{d} \)-linear combination of polynomials in the quadratic part of \( I^{A}_{d} \).

**Remark 6.3.** Let us consider the face \( F \) of \( K \) defined by the condition that all inequalities in (B) are equalities. \( F \) has \( 2^{n-2} \) subfaces determined by the set of inequalities in (A) turning into equalities. There is a one-to-one correspondence between the subfaces of \( F \) and the weight systems given in Proposition 7 of [CFFFR]. More precisely, each subface contains exactly one weight system from Proposition 7 of [CFFFR] (the number of these weight systems is \( 2^{n-2} \) – note the notation shift \( n + 1 \to n \) from [CFFFR] to this paper).

### 7. A MAXIMAL CONE IN THE TROPICAL FLAG VARIETY

We briefly recall some basic facts on the tropical flag varieties (see [MaS], [SS]). Let us consider a \((2^{n} - 2)\)-dimensional real vector space. A point \( s \in \mathbb{R}^{2^{n}-2} \) has coordinates \( s_{I} = s_{i_{1}, \ldots, i_{k}} \) labeled by collections \( 1 \leq i_{1} < \ldots < i_{k} \leq n \) with \( 1 \leq k < n \). A point \( s \) defines a grading on the polynomial ring in Plücker variables \( X_{I} \) attaching degree \( s_{I} \) to \( X_{I} \).

We consider the Plücker embedding of the flag variety \( F_{l_{n}} := SL_{n}/B \) into the product of projective spaces. For the ideal \( J \) consisting of all multi-homogeneous polynomials in \( X_{I} \) vanishing on the image of \( SL_{n}/B \) and a point \( s \in \mathbb{R}^{2^{n}-2} \) we consider the initial ideal \( J_{s} \) (with respect to the grading defined by \( s \)). The tropical flag variety \( \text{trop}(F_{l_{n}}) \) consists of points \( s \) such that \( J_{s} \) does not contain monomials.

**Remark 7.1.** From the tropical point of view, it is natural to quotient the space \( \mathbb{R}^{2^{n}-2} \) by \( \mathbb{R}^{n-1} \), since a point \( s = (s_{I}) \) belongs to the tropical flag variety if and only if so all the points of the form \( \{s_{I} + b_{I}l\} \) for any collection of numbers \( (b_{I})_{k=1}^{n-1} \in \mathbb{R}^{n-1} \). In what follows we choose a normalization assuming that \( s_{1, \ldots, k} = 0 \) for all \( k = 1, \ldots, n - 1 \).

We consider the map from the cone \( K \) of weight systems to \( \mathbb{R}^{2^{n}-2} \) that maps \( A = \{a_{i,j}\} \) to \( \{s_{A}^{I}\} \), \( I = \{i_{1}, \ldots, i_{k}\} \) defined in Proposition 4.1. Recall that if \( p_{1} < \ldots < p_{l} \) comprise the difference \( \{1, \ldots, k\} \setminus \{i_{1}, \ldots, i_{k}\} \) and the integers \( q_{1} > \ldots > q_{l} \) comprise the difference \( \{i_{1}, \ldots, i_{k}\} \setminus \{1, \ldots, k\} \), then
\[ s_{i_{1}, \ldots, i_{k}}^{A} = a_{p_{1}, q_{1}} + \ldots + a_{p_{l}, q_{l}}. \]

Let \( h: \mathbb{R}^{n(n-1)/2} \to \mathbb{R}^{2^{n}-2} \) be the linear map defined by formula (7). We denote the image \( h(K) \) by \( C \). In particular, \( C \) is contained in the \((2^{n} - 2)\)-dimensional vector space, whose coordinates will be denoted by \( s_{I} \), \( I \) being a proper subset of \( \{1, \ldots, n\} \). We note that for any point \( s \) in the image of \( h \) one has \( s_{1, \ldots, k} = 0 \) for \( 1 \leq k \leq n - 1 \). The following Lemma describes \( C \) explicitly.

**Lemma 7.2.** The cone \( C \) is cut out by the following set of equalities and inequalities:
Proof. The first and third conditions are obvious from the definition. The second condition comes from the fact that for any \( i < k < j \) one has \( s_{1,\ldots,i-1,i+1,\ldots,k,j} = s_{1,\ldots,i-1,i+1,\ldots,l,j} \), where \( i < j \leq n \) and any \( i \leq k < l < j \) one has \( s_{1,\ldots,i-1,i+1,\ldots,k,j} = s_{1,\ldots,i-1,i+1,\ldots,l,j} \).

Theorem 7.3. \( C \) is a maximal cone in the tropical flag variety for any \( n > 1 \).

Proof. Since all the degenerate flag varieties are irreducible, the cone \( C \) is contained in the tropical flag variety \( \text{trop}(\mathcal{F}l_n) \). Now assume that there exists a larger cone in \( \text{trop}(\mathcal{F}l_n) \) containing \( C \). Since the dimension of a maximal cone in \( \text{trop}(\mathcal{F}l_n) \) is equal to \( n(n-1)/2 = \dim SL_n/B \) (see e.g. [SS]) and \( \dim C = (n-1)/2 \) we conclude that if a maximal cone \( C' \subset \text{trop}(\mathcal{F}l_n) \) contains \( C \), then \( C' \) is contained in \( h(\mathbb{R}^{n(n-1)/2}) \) (if a point of \( C' \) is not contained in \( h(\mathbb{R}^{n(n-1)/2}) \), then the dimension of the convex hull of this point and \( C \) is greater than \( \dim C \)).

This implies that if a point \( s \) is contained in the maximal cone \( C' \subset \text{trop}(\mathcal{F}l_n) \), then \( s \) satisfies conditions \([ii], [iii], [iv]\). So we are left to show that a point \( s \in C' \) satisfies conditions \([iv]\) and \([v]\).

For \( 1 \leq i \leq n-2 \) we consider Plücker relation

\[
X_{1,\ldots,i,i+2}X_{1,\ldots,i+1,i+1}X_{1,\ldots,i+1,i+2} - X_{1,\ldots,i-1,i+1,i+2}X_{1,\ldots,i+1,i+2}.
\]

If \( s_{1,\ldots,i-1,i+1} + s_{1,\ldots,i+2} < s_{1,\ldots,i-1,i+1}, \) then the initial part of this relation is a monomial \( X_{1,\ldots,i+2}X_{1,\ldots,i-1,i+1}, \) which implies that \( s \notin \text{trop}(\mathcal{F}l_n) \). We conclude that inequality \([iv]\) holds true for all \( i \).

Now for \( 1 \leq i < j - 1 \leq n-2 \) we consider the Plücker relation

\[
X_{1,\ldots,i-1,i+1,j}X_{1,\ldots,i,j+1}X_{1,\ldots,i-1,i+1,j+1}X_{1,\ldots,i,j} - X_{1,\ldots,i-1,j,j+1}X_{1,\ldots,i-1,i+1,j+1}X_{1,\ldots,i,j+1}.
\]

If \( s_{1,\ldots,i-1,i} + s_{1,\ldots,i,j+1} < s_{1,\ldots,i-1,i+1,j} + s_{1,\ldots,i,j} \), then the initial term of this relation is the monomial \( X_{1,\ldots,i-1,i+1,j}X_{1,\ldots,i,j+1}, \) which implies that \( s \notin \text{trop}(\mathcal{F}l_n) \). We conclude that \([v]\) holds true.

We have shown that for any \( s \in C' \) one has \( s \in C \). Hence \( C = C' \).

8. Line bundles and BW theorem

Consider the embedding \( \iota_\lambda : F^A_\lambda \hookrightarrow \mathbb{P}(L^A_\lambda) \) and the line bundle \( \mathcal{O}_{\mathbb{P}(L^A_\lambda)}(1) \). We denote \( L^A_\mu \) the pullback \( \iota_\lambda^*(\mathcal{O}_{\mathbb{P}(L^A_\lambda)}(1)) \). Note that, since \( F^A_\lambda \) depends only on \( d \), the sheaf \( L^A_\mu \) on \( F^A_\lambda \) is now defined for all \( \mu \in \mathbb{Z}_{\geq 0}\{\omega_d, \ldots, \omega_d\} \).

These line bundles can also be obtained in a different way. The \( h^* \)-grading on \( R^A_d \), the coordinate ring of \( \mathbb{P}^A_d \), induces an \( h^* \)-grading on its structure sheaf. Denote \( \mathcal{O}_d(\lambda) \) the component of homogeneity degree \( \lambda \). This sheaf is seen to be the restriction of \( \mathcal{O}_{\mathbb{P}(L^A_\lambda)}(1) \). Therefore, the restriction of \( \mathcal{O}_d(\lambda) \) to \( F^A_\lambda \) coincides with the restriction of \( \mathcal{O}_{\mathbb{P}(L^A_\lambda)}(1) \) to \( F^A_\lambda \), the latter being \( L^A_\lambda \).
In view of the previous paragraph, for $\mu \in \mathbb{Z}_{\geq 0} \{\omega_{d_1}, \ldots, \omega_{d_6}\}$ the space of global sections $H^0(F^A_{\lambda}, L^A_{\mu})$ is the homogeneous component of degree $\mu$ in the coordinate ring $R^A_d/I^A_d$. Furthermore, $F^A_{\lambda}$ is obviously acted on by $N^A$ and $L^A_{\lambda}$ is $N^A$-equivariant. This leads us to an analogue of the Borel-Weil theorem.

**Theorem 8.1.** As a representation of $N^A$ the space $H^0(F^A_{\lambda}, L^A_{\mu})$ is isomorphic to the dual $(L^A_{\mu})^*$.  

**Proof.** Since $\mathcal{L}^A_{\mu}$ is defined as the restriction of $\mathcal{O}_{F^A_{\lambda}}(1)$ every functional on $L^A_{\mu}$ provides a global section of $\mathcal{L}^A_{\mu}$ and we have a map from $(L^A_{\mu})^*$ to $H^0(F^A_{\lambda}, \mathcal{L}^A_{\mu})$. This map is injective since the linear hull $\mathbb{C}N^A(v^A_{\mu}) = \mathcal{U}^A v^A_{\mu}$ is all of $L^A_{\mu}$, i.e. $N^A(v^A_{\mu})$ is not contained in any proper linear subspace, hence, no global section of $\mathcal{O}_{F^A_{\lambda}}(1)$ vanishes on $F^A_{\lambda}$. However, we have identified $H^0(F^A_{\lambda}, \mathcal{L}^A_{\mu})$ with the component of homogeneity degree $\mu$ in $R^A_d/I^A_d$ and Corollary 4.3 shows that the dimension of the latter component is precisely $\dim L^A_{\mu}$. The theorem follows. \(\square\)

To extend this fact to an analogue of the Borel-Weil theorem (or, rather, its restriction to the case of integral dominant weights) we prove the acyclicity of $L^A_{\mu}$.

**Theorem 8.2.** One has $H^m(F^A_{\lambda}, \mathcal{L}^A_{\mu}) = 0$ for all $\mu \in \mathbb{Z}_{\geq 0} \{\omega_{d_1}, \ldots, \omega_{d_6}\}$ and all $m > 0$.  

**Proof.** We first note that for a weight system $A$ lying in the interior of $\mathcal{K}$ the claim of the theorem holds (since the line bundle $\mathcal{L}^A_{\mu}$ is generated by its sections, see [Ful2], Section 3.5). Now, given a weight system $B \in \mathcal{K}$, Proposition 6.2 gives a flat family over $\mathbb{A}^1$ with the generic fiber $F^B_{\lambda}$ and $F^A_{\lambda}$ as the special fiber. Then the upper semi-continuity theorem for the cohomology groups [III], chapter III, Theorem 12.8 implies the claim of our theorem. \(\square\)

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