Homeomorphisms of $\overline{U} \times R$ and rotation number

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Abstract

Suppose $U \subset \mathbb{R}^2$ is bounded open and contractible and $H : \overline{U} \times R \rightarrow \overline{U} \times R$ is a homeomorphism leaving invariant $U \times R$. If $\partial U$ is locally connected and not a simple closed curve $H$ induces a homeomorphism of the solid cylinder leaving invariant sufficiently many vertical lines to determine a rotation number. If $\partial U$ is not locally connected $H$ admits a natural notion of rotation number despite a general absence of an induced homeomorphism of the solid cylinder.

1 Introduction

The following theorem, credited to Ursell and Young [11] is central to applications ([1], [3], [2], [4], [5], [9]) of prime end theory to dynamical systems:

Theorem 1 Suppose $U \subset \mathbb{R}^2$ is bounded open and contractible, $\psi : U \rightarrow int(D^2)$ is conformal, and $h : \overline{U} \rightarrow \overline{U}$ is a homeomorphism such that $h(U) = U$. Then $\psi h \psi^{-1}$ can be extended to a homeomorphism of $D^2$.

In particular if $h$ is orientation preserving then $\psi h \psi^{-1}_{s1} : \partial D^2 \rightarrow \partial D^2$ determines a rotation number, measuring in a sense the average rotation by $h$ of $\partial U$ about $U$.

Specific examples of higher dimensional versions of Theorem 1 are in short supply despite a general criteria established in [4].

For example if $\partial U$ is not locally connected, there is no corresponding version of Theorem 1 for domains $U \times R \subset \mathbb{R}^3$ with $\Psi : U \times R \rightarrow int(D^2) \times R$ defined via $\psi(u,t) = (\psi(u), t))$. Example 4 exhibits a homeomorphism $H : \overline{U} \times R \rightarrow \overline{U} \times R$ such that $H(U \times R) = U \times R$ but $\Psi H \Psi^{-1}$ cannot be extended to a homeomorphism of $D^2 \times R$.
The main result of this paper (Theorem 18) salvages a notion of rotation number for such homeomorphisms $H : \overline{U} \times R \to \overline{U} \times R$. The basic idea is that $H$ preserves the circular order of a certain collection of sets, each of which can be understood as the product of $R$ with an interval of accessible prime ends of $U$. If $\partial U$ is not locally connected this provides enough information to determine a homeomorphism $g : S^1 \to S^1$ whose rotation number we declare to be that of $H$.

On the other hand a useful ‘3 dimensional prime end theory’ exists if $\partial U$ is locally connected. Theorem 13 shows $\Psi$ induces a homeomorphism of the (3 cell) two point compactification of $D^2 \times R$. Moreover if $\partial U$ is not a simple closed curve of the cutpoints of $\partial U$ help to determine a discrete collection of invariant boundary lines which in turn determine a rotation number.

Both notions of rotation number are invariant under topological conjugacy, and agree with the usual rotation number of $h$ in the special case $H(u,t) = (h(u),t)$ where $h : \overline{U} \to \overline{U}$ is a homeomorphism such that $h(U) = U$.

2 Preliminaries

Suppose throughout this paper $U \subset R^2$ is bounded open and contractible, $\partial U$ denotes $\overline{U}\setminus U$, $D^2 \subset R^2$ denotes the closed unit disk and $\psi : U \to \text{int}(D^2)$ is conformal. All function spaces will have the compact open topology.

Define $\Psi : U \times R \to \text{int}(D^2) \times R$ via $\Psi(u,t) = (\psi(u),t)$.

Let $D^3 = \{(x,y,z) \in R^3 | |\sqrt{x^2 + y^2 + z^2} - 1| \leq 1\}$, the standard 3 cell.

If $J \subset S^1$ is connected let $\text{int}(J) = J$ if $J = \{x\}$. Otherwise let $\text{int}(J)$ denote the union of all open intervals contained in $J$.

If $Y$ is a topological space attach two points $\{\infty\}$ and $\{-\infty\}$ to $Y \times R$ creating $(Y \times J) \cup \{\infty, -\infty\}$ topologized such that $(y_n,t_n) \to \{\infty\}$ iff $t_n \to \infty$ and $(y_n,t_n) \to \{-\infty\}$ iff $t_n \to -\infty$.

In similar fashion we attach two points $\{\infty\}$ and $\{-\infty\}$ to $Y \times (-1,1)$ and create a new space $\overline{Y \times (-1,1)}$ topologized such that $(y_n,t_n) \to \{-\infty\}$ iff $t_n \to -1$ and $(y_n,t_n) \to \{\infty\}$ iff $t_n \to 1$.

3 The map $\phi : X^* \to D^2$

The following procedure creates a complete metric space $X^*$ whose underlying set can be seen as the union of $U$ and the accessible prime ends of $U$.

Define a metric $d^* : U \times U \to R$ such that $d^*(x,y) < \varepsilon$ iff there exists a map $f : [0,1] \to U$ such that $f(0) = x, f(1) = y$ and $\forall s,t \ |f(t) - f(s)| < \varepsilon$. Let $(X^*, d^*)$ denote the metric completion of $(U, d^*)$.
Let $\partial X^* = X^* \setminus U^*$.

**Lemma 2** There exists a map $\phi : X^* \to D^2$ such that $\phi$ is uniformly continuous, one to one and $\phi(U^*) = \text{int}(D^2)$.

**Proof.** Let $\overline{id} : X^* \to \overline{U}$ denote the unique extension of the uniformly continuous identity map $id : U^* \to U$. Uniform continuity of $\psi(id)$ is essentially a consequence of Theorem 1 and is proved in Theorem 3.1 of [7]. Define $\phi : X^* \to D^2$ to be the unique continuous extension of $\psi(id)$. Suppose $x \neq y$ and $\{x, y\} \subset X^*$. If $\overline{id}(x) \neq \overline{id}(y)$ then $\{\overline{id}(x), \overline{id}(y)\} \subset \partial U$ and $\{x, y\} \subset \partial X^*$. Construct a closed topological disk $E \subset \overline{U}$ such that $\overline{id}(x) \in \partial E$ and $E \setminus \{\overline{id}(x)\} \subset U$. If $d^*(x, y) \neq 0$ then $\text{int}(E) \cap \partial U \neq \emptyset$ and $\overline{id}(x)$ and $\overline{id}(y)$ determine distinct prime ends. Let $z \in \partial E \setminus \{\overline{id}(x)\}$. By Theorem 2.15 [7] $\psi$ determines a bijection between the prime ends of $U$ and $\partial D^2$. In particular $\partial E \setminus \{z\}$ determines distinct endcuts which map under $\psi$ to distinct points of $\partial D^2$. Hence $\phi$ is one to one. ■

**Remark 3** The injective map $\phi : X^* \to D^2$ need not be an embedding. For example if $U \subset R^2$ is the region bounded by a ‘Warsaw circle’. The canonical map from $\partial X^* \to \partial D^2$ is a continuous bijection but not a homeomorphism.

4 Failure of homeomorphism extension

**Example 4** Suppose $U \subset R^2$ is the interior of the standard Warsaw circle. There there exists $H \in G(U \times R)$ such that $\Psi H \Psi^{-1} : \text{int}(D^2) \times R \to \text{int}(D^2) \times R$ is not extendable to a homeomorphism of $D^2 \times R$.

**Proof.** Let $U \subset R^2$ be the interior of a standardly embedded Warsaw circle such that the impression of the bad prime end (the closed interval ‘limit bar’) is precisely $\partial U \cap (\{0\} \times R)$. Define a homeomorphism $H : U \times R \to U \times R$ via $H(x, y, t) = (x, y, y + t)$. Note $\Psi H \Psi^{-1}$ is not continuously extendable near the bad prime end since $H$ is not level preserving on the impression of the bad prime end. ■

5 Homeomorphisms of $S^1$ and rotation number

The notion of rotation number of a homeomorphism of the unit circle dates back to Poincare. Its properties are derived in Devaney’s book [8] formally for diffeomorphisms but the proofs are valid for homeomorphisms. See also [8] for a helpful survey.
Theorem 5 Suppose \( g : S^1 \to S^1 \) is an orientation preserving homeomorphism, \( \Pi : R \to S^1 \) is the covering map \( \Pi(\theta) = e^{2\pi i \theta}, x \in R \) and \( G_1, G_2 : R \to R \) are homeomorphisms such that \( \Pi(G_1) = g\Pi \). Then the following limits exist and differ by an integer: \( \lim_{n \to \infty} \frac{G_n(x)}{n} \) and \( \lim_{n \to \infty} \frac{G_n(x)}{n} \). This number \((\mod 1)\) is the rotation number of \( g \) and is invariant under the choice of \( x \). If \( g \) reverses orientation, then \( g \) has two fixed points and we declare \( g \) to have rotation number 0. The rotation number of \( g \) is invariant under topological conjugacy (if \( h : S^1 \to S^1 \) is a homeomorphism and \( \hat{g}^* = hgh^{-1} \) then \( \text{rot}(\hat{g}^*) = \text{rot}(g) \)). Finally, \( \text{rot} \) is a continuous function on the space of homeomorphisms of \( S^1 \).

6 Sets with circular order and rotation number

Given 4 distinct points \( \{x_1, x_2, x_3, x_4\} \subset S^1 \) declare \( x_1 < x_2 < x_3 < x_4 \) if there exists a homeomorphism \( g : S^1 \to S^1 \) such that \( g(i^n) = n \).

Note it is allowed that \( g \) reverses orientation. Suppose \( \{J_1, J_2, J_3, J_4\} \) is a collection of distinct pairwise disjoint nonempty subsets of \( S^1 \). Declare \( J_1 < J_2 < J_3 < J_4 \) if \( x_1 < x_2 < x_3 < x_4 \) whenever \( x_i \in J_i \).

Suppose \( J \) is a proper connected subset of \( S^1 \). Define \( \text{int}(J) = J \) if \( J = \{x\} \) and define \( \text{int}(J) \) to be the largest open interval contained in \( J \) otherwise.

Definition 6 Suppose \( A^* \) is a collection pairwise disjoint subsets of \( S^1 \) and \( h : A^* \to A^* \) is a bijection. Then \( h \) is order preserving if there exists a homeomorphism \( g : S^1 \to S^1 \) such that \( g(J) = h(J) \) for each element \( J \in A^* \). The homeomorphism \( g \) is said to be compatible with \( h \). If \( \left|A^*\right| \geq 3 \) then declare \( h \) orientation preserving/reversing iff \( g \) is orientation preserving/reversing.

Lemma 7 Suppose \( A^* \) is a collection pairwise disjoint subsets of \( S^1 \) such that \( \left|A^*\right| \geq 4 \) and \( \bigcup_{J \in A^*} J \) is dense in \( S^1 \). Suppose \( J = \text{int}(J) \) for each \( J \in A^* \). Suppose \( h : A^* \to A^* \) is a bijection leaving invariant the set of nontrivial elements of \( A^* \). Then \( h \) is order preserving if and only if \( h(J_3) \) and \( h(J_4) \) belong to the same component of \( S^1 \{h(J_1) \cup h(J_2)\} \) whenever \( J_1 < J_2 < J_3 < J_4 \).

Proof. Suppose \( h \) is order preserving. Let \( g : S^1 \to S^1 \) be a homeomorphism such that \( g(J) = h(J) \) for each \( J \in A^* \). Suppose \( J_1 < J_2 < J_3 < J_4 \). Then \( J_3 \) and \( J_4 \) belong to the same component of \( S^1 \{J_1 \cup J_2\} \). Since \( g \) is a homeomorphism \( g(J_3) \) and \( g(J_4) \) belong to the same component of \( S^1 \{g(J_1) \cup g(J_2)\} \). The conclusion follows since \( g(J_3) = h(J_3) \). Conversely suppose \( h(J_3) \) and \( h(J_4) \) belong to the
same component of $S^1 \setminus \{h(J_1) \cup h(J_2)\}$ whenever $J_1 < J_2 < J_3 < J_4$.

Suppose $J_1 < J_2 < J_3 < J_4$. Then either $h(J_1) < h(J_2) < h(J_3) < h(J_4)$ or $h(J_1) < h(J_2) < h(J_4) < h(J_3)$. Suppose in order to obtain a contradiction that $h(J_1) < h(J_2) < h(J_4) < h(J_3)$. Then $h(J_2)$ and $h(J_3)$ lie in opposite components of $S^1 \setminus \{h(J_1) \cup h(J_4)\}$. On the other hand by hypothesis $J_4 < J_1 < J_2 < J_3$ and hence $h(J_2)$ and $h(J_3)$ lie in the same component of $S^1 \setminus \{h(J_1) \cup h(J_4)\}$ and we have a contradiction. Thus $h(J_1) < h(J_2) < h(J_3) < h(J_4)$ and globally the bijection $h$ either preserves or reverses orientation. Let $g : S^1 \to S^1$ be the unique homeomorphism mapping $J$ linearly onto $h(J)$ whenever $J$ is a nontrivial component of $A^\ast$.

**Lemma 8** Suppose $A^\ast$ is a collection of pairwise disjoint connected subsets of $S^1$ such that $|A^\ast| \geq 3$ and $\cup_{J \in A^\ast} J$ is dense in $S^1$. Suppose $J = \text{int}(J)$ whenever $J \in A^\ast$ and suppose $h : A^\ast \to A^\ast$ is an order preserving bijection as demonstrated by the compatible orientation preserving homeomorphisms $g_1 : S^1 \to S^1$ and $g_2 : S^1 \to S^1$. Then $g_1$ and $g_2$ have the same rotation number.

**Proof.** Let $B \subset S^1$ denote the complement of the union of the open intervals of $A^\ast$. Then $B \neq \emptyset$, $g_1(B) = B$ and $g_{2|B} = g_{1|B}$. Thus there exists $b^\ast \in R$ and lifts $G_1 : R \to R$ and $G_2 : R \to R$ respectively of $g_1$ and $g_2$ such that $\pi(b^\ast) \in B$ and $G^n_1(b^\ast) = G^n_2(b^\ast) \forall n$. Hence by Theorem 5 $g_1$ and $g_2$ have the same rotation number.

7 **The order preserving bijection $h : A^\ast \to A^\ast$**

Define a surjection $p : D^2 \times [-1, 1] \to D^3$ via $p(x, y, t) = ((1 - |t|)x, (1 - |t|)y, t)$. Note $p$ collapses $D^2 \times \{-1\}$ and $D^2 \times \{1\}$ to points and is otherwise injective.

Note the map $\Phi_{X^\ast \times (-1, 1)} : X^\ast \times (-1, 1) \to D^2 \times (-1, 1)$ defined such that $\Phi(x, t) = (\phi(x), t)$ can be continuously extended to an injective map $\Phi : X^\ast \times (-1, 1) \hookrightarrow D^2 \times (-1, 1)$ such that $\Phi(\infty) = \infty$ and $\Phi(-\infty) = -\infty$.

Define a metric on $X^\ast \times R$ via $d((x, t), (y, t)) = \max(d^*(x, y), |s - t|)$.

Define a set $A^\ast \subset 2^{|D^2|}$ such that $\beta \in A^\ast$ iff $\beta = \text{int}(\phi(X))$ for some path component $X \subset \partial X^\ast$.

**Theorem 9** The homeomorphism $H : \overline{U} \times R \to \overline{U} \times R$ induces a canonical homeomorphism $H^* : X^\ast \times R \to X^\ast \times R$ such that $H^*(\partial X^\ast \times R) = \partial X^\ast \times R$ and such that $H^*$ is extendable to a homeomorphism of $(X^\ast \times R) \cup \{\infty, -\infty\}$.
Proof. We first obtain, as follows, an induced homeomorphism $H^*: X^* \times R \to X^* \times R$ such that $H^*(U^* \times R) = U^* \times R$ and for each $S$ and $T$ there exists $S'$ and $T'$ such $H^*(X^* \times [S, T]) \cup H^{-1}(X^* \times [S, T]) \subset X^* \times [S', T']$. Suppose $\{(z_n, t_n)\}$ is Cauchy in $U^* \times R$. Since $\{t_n\}$ is Cauchy choose $S$ and $T$ such that $\forall n (z_n, t_n) \in X^* \times [S, T]$. Moreover for each $n, m$ we may choose a path $\gamma_{nm} \subset U^* \times [S, T]$ such that $\gamma_{nm}$ connects $(z_n, t_n)$ to $(z_m, t_m)$ and $\text{diam}(\gamma_{nm}) < 2d((z_n, t_n), (z_m, t_m))$. Hence $\lim_{n,m \to \infty} \text{diam}(\gamma_{nm}) = 0$. However, since $U \times [S, T]$ is compact, both $H_{\tau \times [S, T]}$ and $H_{\tau \times [S, T]}^{-1}$ are uniformly continuous. Thus $\lim_{n,m \to \infty} \text{diam}(H(\gamma_{nm})) = 0 = \lim_{n,m \to \infty} \text{diam}(H^{-1}(\gamma_{nm}))$. Hence $H$ and $H^{-1}$ preserve Cauchy sequences in $U^* \times R$ and thus are extendable to maps $H^*: X^* \times R \to X^* \times R$ and $(H^{-1})^*: X^* \times R \to X^* \times R$ such that $H^*((H^{-1})^*) = (H^{-1})^*H = id$. Since $id$ is a bijection it follows that $H^*$ and $(H^{-1})^*$ are bijections and hence homeomorphisms. By definition $H(U \times R) = H^{-1}(U \times R) = U \times R$. Since compact subsets of $\overline{U} \times R$ are bounded we may choose $S'$ and $T'$ such that the compactum $H(\overline{U} \times [S, T]) \cup H^{-1}(\overline{U} \times [S, T]) \subset \overline{U} \times [S', T']$. Note $\overline{U} \times [S, T]$ and hence $H(\overline{U} \times [S, T])$ separate $\overline{U} \times R$ into two components. Since $\overline{U} \times (T, \infty)$ is connected $H(\overline{U} \times (T, \infty))$ intersects and contains exactly one component of $(\overline{U} \times R)(\overline{U} \times [S', T'])$. Thus if $(z_n, t_n) \to \infty$ then $H(z_n, t_n)$ converges either to $\infty$ to $-\infty$. Thus both $H^*$ and $(H^*)^{-1}$ are extendable to maps of $(X^* \times R) \cup \{-\infty, \infty\}$ and both must be homeomorphisms since $H^*(H^{-1})$ is fixes pointwise a dense set.

Corollary 10 The induced homeomorphism $H^*: X^* \times R \to X^* \times R$ induces a canonical homeomorphism $H^{**}: X^* \times (-1, 1) \to X^* \times (-1, 1)$ and a canonical bijection $h: A^* \to A^*$.

Proof. Let $H^*: X^* \times R \to X^* \times R$ be induced from $H: \overline{U} \times R \to \overline{U} \times R$ as in Theorem 9. Let $T: R \to (-1, 1)$ be any homeomorphism. Define a homeomorphism $H^{**}: X^* \times (-1, 1) \to X^* \times (-1, 1)$ such that $H^{**}(x, T(t)) = (y, T(s))$ if $H^*(x, t) = (y, s)$. Since $H^{**}(\partial X^* \times (-1, 1)) = \partial X^* \times (-1, 1)$ the homeomorphism $H^{**}$ permutes the path components of $\partial X^* \times (-1, 1)$. Each path component of $\partial X^* \times (-1, 1)$ is of the form $X \times (-1, 1)$ where $X$ is a path component of $\partial X^*$. By definition $\beta \in A^*$ iff there exists a path component $X \subset \partial X^*$ such that $\beta = int\phi(X)$. Thus we define a bijection $h: A^* \to A^*$ satisfying $h(\beta) = \gamma$ if $\beta = int\phi(X)$ and $H^{**}(X \times (-1, 1)) = Y \times (-1, 1)$ and $\gamma = int\phi(Y)$.

Lemma 11 Suppose $D \subset D^3$ is a topological disk such that $int(D) \subset int(D^3)$, and $\partial D \subset \partial D^3$. Suppose $\{(0, 0, 1), (0, 0, -1)\} \subset \partial D$ and suppose $\alpha: [0, 1] \to D^3 \setminus D$ satisfies $\alpha(t) \in int(D^3)$ iff $0 < t < 1$. Sup-
pose $\beta_1$ and $\beta_2$ are disjoint nonempty connected subsets of $S^1$. Suppose $\partial D \subset p((\beta_1 \cup \beta_2) \times [-1,1])$. Suppose $\{\alpha(0), \alpha(1)\} \cap p((\beta_1 \cup \beta_2) \times [-1,1]) = \emptyset$. Then $\pi_1(p^{-1}(\alpha(0), \alpha(1)))$ is contained in a single component of $S^1 \setminus \{\beta_1 \cup \beta_2\}$.

**Proof.** The topological disk $D$ separates $D^3$ into two components and $\partial D$ separates $\partial D^3$ into two components. Moreover the two components of $\partial D^3 \setminus \partial D$ are contained in distinct components of $D^3 \setminus D$. Hence, since $\text{int}(\alpha)$ is connected and $\text{int}(\alpha) \cap D = \emptyset$, $\partial \alpha$ belongs to a single component of $\partial D^3 \setminus \partial D$. Moreover $\partial D^3 \setminus p((\beta_1 \cup \beta_2) \times [-1,1])$ contains at most two components. No two of these components are contained in the same component of $\partial D^3 \setminus \partial D$. Thus $\{\alpha(0), \alpha(1)\}$ is contained in a single component of $\partial D^3 \setminus p((\beta_1 \cup \beta_2) \times [-1,1])$. Hence $\pi_1(p^{-1}(\alpha(0), \alpha(1)))$ is contained in a single component of $S^1 \setminus \{\beta_1 \cup \beta_2\}$. ■

**Corollary 12** If $|A^-| \geq 4$ the bijection $h : A^- \to A^-$ is order preserving.

**Proof.** Suppose $\gamma_1 < \gamma_2 < \gamma_3 < \gamma_4$ and $\gamma_i \in A^-$. Choose 4 points $x_i \in X_i$. Let $\text{int} \phi X_i = \gamma_i$. Let $\lambda \subset X^*$ be an arc connecting $x_1$ to $x_2$ such that $\text{int}(\lambda) \subset U$. Let $A \subset X^*$ be an arc connecting $x_3$ to $x_4$ such that $\text{int}(A) \subset U$ and such that $A \cap \lambda = \emptyset$. Let $\Lambda = \{\infty, -\infty\} \cup (\lambda \times (-1,1)) \subset X^* \times (-1,1)$. Observe $\Lambda$ is compact and homeomorphic to $D^2$. Let $D = p \Phi(H^* \Lambda)$. Let $\alpha = p \Phi(H^* (A \times \{0\}))$. Let $\beta_i = h(\gamma_i)$. Apply Lemma [14] to conclude $\beta_3$ and $\beta_4$ belong to the same component of $S^1 \setminus \{\beta_1 \cup \beta_2\}$. Thus by Lemma [4] $h$ is order preserving. ■

8 Main results

Define $G(U) = \{H : \overline{U} \to \overline{U} | H \text{ is a homeomorphism and } H(U) = U\}$. Define $G(U \times R) = \{H : \overline{U \times R} \to \overline{U \times R} | H \text{ is a homeomorphism and } H(U \times R) = U \times R\}$.

Recalling Theorems [11] and [5] define $\text{rot}(h) : G(U) \to R$ such that $\text{rot}(h)$ is the rotation number of $\psi h \psi^{-1}$.s1.

8.1 The case $\partial U$ is not locally connected

Suppose $\partial U$ is not locally connected.

Recall the bijection $h : A^- \to A^-$ from Corollary [10]. Define a function $\text{Rot} : G(U \times R) \to R$ as follows.

If $|A^-| \geq 3$ recall Corollary [12] note $h$ is order preserving, and define $\text{Rot}(H) = \text{Rot}(g)$ where $g : S^1 \to S^1$ is any homeomorphism compatible with $h$.

If $|A^-| \leq 2$ define $\text{Rot}(H) = 0$ if $H$ is orientation reversing and fixes $\{\infty, -\infty\}$ pointwise or if $H$ is orientation preserving and swaps
\{\infty, -\infty\}. Otherwise define Rot(H) to be 0 or \(\frac{1}{2}\) determined, respectively, by whether \(h = ID_A\) or not.

### 8.2 The case \(\partial U\) is locally connected

The case \(\partial U\) is locally connected deserves special treatment since \(U \times R\) admits a homeomorphism extension theorem analogous to Theorem 9 and seems to provide a useful model of ‘3 dimensional prime end theory’ in the sense of [11].

**Theorem 13** Suppose \(\partial U\) is locally connected and \(H \in G(U \times R)\). Then \(\Psi H \Psi^{-1} : int(D^2) \times R \to int(D^2) \times R\) can be extended to a canonical homeomorphism of the 3 cell \((D^2 \times R) \cup \{\infty, -\infty\}\).

**Proof.** Since \(\partial U\) is locally connected the conformal map \(\psi^{-1} : int(D^2) \to U\) can be continuously extended to a surjective map \(\bar{\psi}^{-1} : D^2 \to \overline{U}\) (Theorem 2.1 p20 [10]). To show \(X^*\) is compact, it suffices, since \(X^*\) is complete, to show that each sequence in \(X^*\) has a Cauchy subsequence. Suppose \(y_n\) is a sequence in \(X^*\). Let \(x_n = \phi(y_n)\). Let \(\{x_{nk}\} \subset D^2\) be a Cauchy subsequence of \(\{x_n\}\). Consider the chords \([x_{nk}, x_{nk}]\] \(\subset D^2\). Since \(\bar{\psi}^{-1}\) is uniformly continuous, \(\text{diam} \ (\bar{\psi}^{-1}[x_{nk}, x_{nk}]) \to 0\). Hence \(d^*(y_{nk}, y_{nk}) \to 0\). Consequently \(X^*\) is compact and the injective map from Lemma 2 \(\phi : X^* \to D^2\) is a homeomorphism. Thus the homeomorphism from Theorem 9 \(H^* : X^* \times R \to X^* \times R\) induces a homeomorphism of \(D^2 \times R\) leaving invariant the ends \(\{\infty\}\) and \(\{-\infty\}\).

To see that \(H\) is canonical let \(j : U^* \times R \to U \times R\) denote identity. Suppose \(H_N \to H\) in \(G(U \times R)\). Note \(\{j^{-1}H_Nj\}\) is uniformly equicontinuous and converges pointwise to \(j^{-1}Hj\). Hence \(H_N^* \to H^*\) uniformly. Thus \(\Psi H_N^* (x) = \Psi H^* (x)\) uniformly.

To define rotation number when \(\partial U\) is locally connected we seek a nonempty proper totally disconnected set \(B^* \subset S^1\) such that \(B^* \times R\) is invariant under the induced homeomorphism \(\overline{\psi H \Psi^{-1}} : D^2 \times R \to D^2 \times R\).

The point \(x \in \partial U\) is a **cutpoint** if \(\partial U \setminus \{x\}\) not connected. Let \(f : D^2 \to \overline{U}\) denote the continuous extension of \(\psi^{-1}\). The map \(f\) is light since the prime ends of \(U\) are in bijective correspondence with points of \(S^1\) (Theorem 2.15 [10]). Let \(B = \{x \in S^1 | f(x)\) is a cutpoint of \(\partial U\}\).

**Lemma 14** \(z \in B\) if and only if \(|f^{-1}(z)| \geq 2\).

**Proof.** If \(z \in \partial U\) and \(|f^{-1}(z)| = 1\) then \(z \notin B\) is since \(S^1 \setminus f^{-1}(z)\) is connected and hence \(f(S^1 \setminus f^{-1}(z)) = \partial U \setminus z\) is connected.

Suppose \(|f^{-1}(z)| \geq 2\) choose \(x \neq y\) such that \(f(y) = f(x) = z\). Let \([x, y] \subset D^2\) denote the chord from \(x\) to \(y\). If \(V\) is a component
Lemma 15 Suppose \( x \neq y, \{x, y\} \subset S^1 \) and \( z = f(x) = f(y) \). Suppose \( a \) and \( b \) lie in distinct components of \( S^1 \setminus \{x, y\} \). Suppose \( \{a, b\} \cap f^{-1}(z) = \emptyset \). Then \( f(a) \neq f(b) \).

Proof. The points \( a \) and \( b \) belong to the boundaries of distinct components of \( int(D^2) \setminus \{x, y\} \). These components in turn map into distinct complementary domains of the simple closed curve \( f([x, y]) \), since points of \( int(D^2) \) can approach the chord \([x, y]\) from distinct sides. By hypothesis \( \{f(a), f(b)\} \cap f([x, y]) = \emptyset \). Thus \( f(a) \) and \( f(b) \) belong to distinct components of \( f([x, y]) \) and in particular \( f(a) \neq f(b) \).

Lemma 16 If \( \partial U \) is not a simple closed curve then \( B \neq \emptyset \) and \( B \neq S^1 \).

Proof. If \( \partial U \) is not a simple closed curve then by Lemma 14 \( f \) is not one to one and hence \( B \neq \emptyset \). Let \( a_0 \subset S^1 \) be any closed interval with distinct endpoints \( x_0 \) and \( y_0 \) such that \( z_0 = f(x_0) = f(y_0) \). Proceeding recursively, if possible let \( \alpha_{n+1} \subset \alpha_n \setminus f^{-1}(z_n) \) be a closed interval with distinct endpoints \( x_{n+1} \) and \( y_{n+1} \) such that \( f(x_{n+1}) = f(y_{n+1}) \) and \( |x_{n+1} - y_{n+1}| < \frac{1}{n+1} \). Suppose the process never terminates. Let \( a = \lim x_n = \lim y_n \). Suppose \( f(b) = f(a) \). By Lemma 15 \( b \in \cap \alpha_n \). Thus \( b = a \) and by Lemma 12 \( a \notin B \). If the process terminates then there must exist a nontrivial open interval \( (x, y) \subset \alpha_0 \) such that \( f_j \) is one to one and \( f(x) = f(y) \). Let \( a \in (x, y) \) and apply Lemma 15 to conclude \( a \notin B \).

Lemma 17 The following are equivalent. 1) \( b \notin B \). 2) Suppose \( g : S^1 \to U \times R \) satisfies \( g(1) = (f(b), t') \) and \( g(\theta) \in U \times R \) if \( \theta \neq 1 \). Then there exists a map \( G : D^2 \to U \times R \) such that \( G(z) \in U \times R \) whenever \( z \in int(D^2) \) and \( G_{S^1} = g \).

Proof. Let \( \pi_1 : U \times R \to U \) and \( \pi_2 : U \times R \to R \) denote the projection maps. Suppose \( b \notin B \) and \( g : S^1 \to U \times R \) satisfies \( \pi_1 g(1) = f(b) \) and \( g(\theta) \in U \times R \) if \( \theta \neq 1 \). By Lemma 12 \( |f^{-1}(f(b))| = 1 \) and hence the formula \( \alpha = f^{-1}(\pi_1 g) : S^1 \to D^2 \) determines a map. Let \( \beta : D^2 \to D^2 \) be a continuous extension of \( \alpha \) such that \( \beta(z) \in int(D^2) \) if \( z \in int(D^2) \). Define \( G' : D^2 \to U \) via \( G' = f(\beta) \). Since \( R \) is contractible extend \( \pi_2(g) : S^1 \to R \) to a map \( h : D^2 \to R \). Define \( G : D^2 \to U \times R \) via \( G(z) = (G'(z), h(z)) \).

Conversely suppose \( b \in B \). By Lemma 14 choose \( b' \in S^1 \) such that \( b' \neq b \) and \( f(b') = f(b) \). Let \( [b, b'] \subset D^2 \) denote the chord. Let \( g' \)
map $S^1$ homeomorphically onto the simple closed curve $f([b,b']) \subset \overline{U}$ such that $g^*(1) = f(b) = f(b')$. Both components of $R^2 \backslash \text{im}(g^*)$ contain points of $\partial U$. Hence $g^*$ is essential in $\{f(b)\} \cup (R^2 \backslash \partial U)$. In particular $G$ cannot exist since $\pi_1(G)$ would show that $g^*$ is inessential.

Since condition 2) is a topological property $B \times R$ maps onto itself under the induced homeomorphism $\Psi H^{-1} : D^2 \times R \rightarrow D^2 \times R$. It follows that if $B' = \{x \in S^1\} \{x\}$ is a component of $B$ or $x$ is an endpoint of a nontrivial interval of $B$ then $B' \neq \emptyset$, $B'$ is totally disconnected, and $\Psi H^{-1}(B' \times R) = B' \times R$. If $|B'| \neq 2$ let $g : S^1 \rightarrow S^1$ be any homeomorphism such that $g(x) = y$ whenever $x$ and $y$ are components of $B'$ such that $\Psi H^{-1}(\{x\} \times R) = \{y\} \times R$. Define $\text{Rot}(H) = \text{rot}(g)$. If $|B'| = 2$ define $\text{Rot}(H) = 0$ if $\Psi H^{-1}$ is orientation reversing and fixes $\{\infty, -\infty\}$ pointwise or if $\Psi H^{-1}$ is orientation preserving and swaps $\{\infty, -\infty\}$. Otherwise define $\text{Rot}(H)$ to be 0 or $\frac{1}{2}$ determined, respectively, by whether $\Psi H^{-1}$ interchanges the lines of $B' \times R$ or not.

8.3 Main Theorem on rotation number over $U \times R$

**Theorem 18** Suppose $\partial U$ is not a simple closed curve. The function $\text{Rot} : G(U \times R) \rightarrow R$ is continuous. If $F, H \in G(\overline{U} \times R)$ then $\text{Rot}(H) = \text{Rot}(FHF^{-1})$. If there exists $h \in G(\overline{U})$ such that $\forall z, t H(z, t) = (h(z), t)$ then $\text{Rot}(H) = \text{rot}(h)$.

**Proof.** If $|A^-| \geq 3$ then by Corollary 12 $\text{Rot}$ is well defined. Suppose $g, f : S^1 \rightarrow S^1$ are homeomorphisms compatible, respectively with $H$ and $F$. Then $fgf^{-1}$ is compatible with $FGF^{-1}$. Hence $\text{Rot}(H) = \text{rot}(g) = \text{rot}(fgf^{-1}) = \text{Rot}(FGF^{-1})$. For continuity of $\text{Rot}$ suppose $H_n \rightarrow H$. Let $h_n : A^- \rightarrow A^-$ and $h : A^- \rightarrow A^-$ denote the corresponding order preserving bijections. If $J$ is a nontrivial component of $A^-$ then $h_n(J)$ is eventually constant. Construct compatible homeomorphisms $g_n : S^1 \rightarrow S^1$ and $g : S^1 \rightarrow S^1$ as in Lemma 7. Note eventually $g_n|J = g|J$ and hence $\text{rot}(g_n) = \text{rot}(g)$ eventually. If each component $J$ of $A^-$ is trivial then $g_n \rightarrow g$ pointwise on a dense set $A^- \subset S^1$ and hence $g_n \rightarrow g$ uniformly. Thus by Theorem 5 $\text{rot}(g_n) \rightarrow \text{rot}(g)$.

If $\partial U$ is not locally connected and $|A^-| \leq 2$ the theorem follows from the definition of $\text{Rot}$.

If $\partial U$ is locally connected the homeomorphism $\Psi H^{-1}$ varies continuously with $H$. The theorem follows from the definition of $\text{Rot}$ and from Theorem 5. ■
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