Effect of depth and width of periodic trapezoidal channels to water content in soil: a numerical study

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Abstract. Time-independent or steady infiltration problems from periodic furrows are studied. There are variations in the depth and width of the periodic furrows or channels. The problems are modelled using Richards equation. To solve the problems using the Dual Reciprocity Boundary Element Methods (DRBEM), the mathematical model must be changed into a modified Helmholtz equation. Solutions obtained are then compared and discussed to observe effects of the depth and width of the channels to soil’s moisture content.

1. Introduction
Studies of water flow in porous medium, especially in soils have been done by researchers. Some of such researchers or authors are Batu [1], Solekhudin [2], Clements and Lobo [3], Philip [4], and Raats [5]. For more specific studies or research, studies on infiltration from channels, the researches were conducted by, for instance, Lobo et al. [6], and Solekhudin [7, 8]. In these two researches, effect of width and depth of the channels has not been studied.

This paper reports some of results from study of a time independent infiltration problem from periodic channels or furrows into a homogenous soil. There are different geometries of trapezoidal furrows. To study the problem, the mathematical model of the problem is transformed into a modified Helmholtz equation using a set of transformations. A DRBEM is employed to obtain numerical solutions of the problems. From the solutions obtained, effect of depth and width of the channels on soil’s moisture content is discussed.

2. Methods
We refer a Cartesian frame $OXYZ$. It is considered a homogenous soil lying in the region $Z \geq 0$, where positive direction of $Z$-axis is pointing down. The soil surface includes periodic trapezoidal furrows with cross section does not alter in the $OY$ direction. The channel is sufficiently long and has surface area of 100 cm$^2$ per centimeter length in the $OY$ direction. The angle between horizontal lines and the hypotenuse of the channels is $30^\circ$. An illustration of the cross section of a channel is shown in Figure 1. Water infiltrates from the furrow or channel into the soil. Due to the symmetry of the infiltration problem described, the problem is solved over a semi-infinite region bounded by $Z \geq 0$ and $0 \leq X \leq 100$ cm. The notation of the region and its boundary are $R$ and $C$, respectively.
The boundary conditions in this study are similar to those used by Batu [9]. Specifically, we assume that flux on the surface of the channel is a constant. It is also assumed that there are no fluxes across the soil surface outside the channels, and \( \lim_{z \to -\infty} \frac{\partial \Theta}{\partial x} = 0 \) and \( \lim_{z \to -\infty} \frac{\partial \Theta}{\partial z} = 0 \) [9].

\[ \frac{\partial}{\partial X} \left( K \frac{\partial \psi}{\partial X} \right) + \frac{\partial}{\partial Z} \left( K \frac{\partial \psi}{\partial Z} \right) - \frac{\partial K}{\partial Z} = 0, \]  

(1)

Where \( K \) is the hydraulic conductivity and \( \psi \) is the suction potential. Equation (1) can be written as

\[ \frac{\partial^2 \Theta}{\partial X^2} + \frac{\partial^2 \Theta}{\partial Z^2} - \frac{\partial K}{\partial Z} = 0, \]  

(2)

where \( \Theta \) is the matric flux potential (MFP), formulated as

\[ \Theta = \int_{-\infty}^{\psi} K(s)ds = \frac{K(\psi)}{\alpha}. \]  

(3)

Here, \( K \) is as that in [10],

\[ K(\psi) = K_0 e^{\alpha \psi}, \]

where \( \alpha \) is a constant related to the fineness of the soil. The flux normal to the surface is given by

\[ F = U n_1 + V n_2, \]  

(4)

where \( n = (n_1, n_2) \) is normal pointing out the region,

\[ U = -\frac{\partial \Theta}{\partial X} \]

and

\[ V = \alpha \Theta - \frac{\partial \Theta}{\partial Z}. \]

The boundary conditions outlined in Section 2 may in terms of the MFP are as follows.

\[ \frac{\partial \Theta}{\partial n} = \alpha \theta n_2 + v_0, \]  

(5)

on the surface of the cross section of the channels.
\[
\frac{\partial \theta}{\partial n} = \alpha \theta n_2, \text{ on } Z = 0, \quad (6)
\]
\[
\frac{\partial \theta}{\partial n} = 0, \text{ on } X = 0 \text{ and } Z \geq 0, \quad (7)
\]
\[
\frac{\partial \theta}{\partial n} = 0, \text{ on } X = 100 \text{ cm and } Z \geq 0, \quad (8)
\]
\[
\frac{\partial \theta}{\partial x} \rightarrow 0 \text{ and } \frac{\partial \theta}{\partial z} \rightarrow 0 \text{ as } Z \rightarrow \infty. \quad (9)
\]

To transform Equation (2) into an equation containing dimensionless variables, we use the dimensionless variables

\[
\begin{align*}
\theta &= \frac{\pi}{50\alpha} \Theta, \\
x &= \frac{\alpha}{2} X, \\
z &= \frac{\alpha}{2} Z, \\
u &= \frac{\pi}{25v_0 \alpha} U, \\
v &= \frac{\pi}{25v_0 \alpha} V, \\
f &= \frac{\pi}{25v_0 \alpha} F,
\end{align*}
\]

(10)

where \(v_0\) is water flux at the surface of the channels. Substituting these variables into Equation (2) and (4), we have

\[
\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2} - 2 \frac{\partial \theta}{\partial z} = 0,
\]

(11)

and

\[
f = u n_1 + v n_2.
\]

(12)

where

\[
u = -\frac{\partial \theta}{\partial x},
\]

and

\[
v = 2\theta - \frac{\partial \theta}{\partial z}.
\]

Boundary conditions (5) – (9) become

\[
\begin{align*}
\frac{\partial \theta}{\partial n} &= \frac{\pi}{25\alpha} + 2\theta n_2, \text{ on the surface of the cross section of the channels,} \\
\frac{\partial \theta}{\partial n} &= 2\theta n_2, \text{ for } z = 0, \\
\frac{\partial \theta}{\partial n} &= 0, \text{ for } x = 0 \text{ and } z \geq 0, \\
\frac{\partial \theta}{\partial n} &= 0, \text{ for } x = 50\alpha \text{ and } z \geq 0, \\
\frac{\partial \theta}{\partial x} \rightarrow 0 \text{ and } \frac{\partial \theta}{\partial z} \rightarrow 0 \text{ as } Z \rightarrow \infty.
\end{align*}
\]

(13) - (17)

Employing transformation

\[
\theta = \varphi e^z,
\]

(18)

Equations (11) and (12) can be written as

\[
\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} - \varphi = 0,
\]

(19)

\[
u = -e^z \frac{\partial \varphi}{\partial x}, \quad v = e^z \left( \varphi - \frac{\partial \varphi}{\partial z} \right).
\]

(20)
and

\[ f = e^z \left[ -\frac{\partial \phi}{\partial x} n_1 + \left( \frac{\phi}{\partial z} \right) n_2 \right]. \]  

(21)

From Equation (21), we have

\[ \frac{\partial \phi}{\partial n} = \phi n_2 - e^{-z} f. \]  

(22)

Boundary conditions (13) – (17) can now be written as follows.

1. \( \frac{\partial \phi}{\partial n} = \phi n_2 + \frac{\pi}{25a} e^{-z} \), on the surface of the cross section of the channels,
2. \( \frac{\partial \phi}{\partial n} = -\phi \), on \( z = 0 \),
3. \( \frac{\partial \phi}{\partial n} = 0 \), on \( x = 0 \) and \( z \geq 0 \),
4. \( \frac{\partial \phi}{\partial n} = 0 \), on \( x = 50a \) and \( z \geq 0 \),
5. \( \frac{\partial \phi}{\partial n} = -\phi \), for \( 0 \leq x \leq 50a \) and \( z = \infty \).

Following the DRBEM procedure, the corresponding boundary integral equation for Equation (19)

\[ \lambda(\xi, \eta) \phi(\xi, \eta) = \int \int R \Psi(x, z; \xi, \eta) \phi(x, z) \, dx \, dz \]
\[ + \int \int C \left[ \phi(x, z) \frac{\partial}{\partial n} \left( \Psi(x, z; \varepsilon, \zeta) \right) - \Psi(x, z; \varepsilon, \zeta) \frac{\partial}{\partial n} \left( \phi(x, z) \right) \right] \, ds, \]

is

(28)

where \( \Psi(x, z; \xi, \eta) \) is the fundamental solution of Laplace’s equation. This integral equation

is then reduced into a system of linear algebraic equations (SLAE). After solving the SLAE,
solutions at any points in the region may be obtained.

3. Results and Discussion

Some of the numerical results obtained using the method described are presented. The periodic

trapezoidal channels considered in this study are in five different types, named by Furrow A, Furrow B, Furrow C, Furrow D, and Furrow E. The soil considered in this study is Pima Clay Loam, with \( \alpha = 0.014 \) cm [11]. The depth and the width of the channels for each type of furrows are shown in Table 1.

| Type of channel | Width  | Depth |
|-----------------|--------|-------|
| Furrow A        | 49.3301 cm | 2.5 cm |
| Furrow B        | 47.9904 cm | 7.5 cm |
| Furrow C        | 46.6506 cm | 12.5 cm |
| Furrow D        | 45.3109 cm | 17.5 cm |
| Furrow E        | 43.9711 cm | 22.5 cm |

To obtain values of \( \phi \) and \( \frac{\partial \phi}{\partial n} \) using the DRBEM, an imposed boundary to replace \( z = \infty \) is needed. After performing some computational trials, it is obtained that a suitable imposed boundary is
$z = 4$. The new boundary is then divided into segments or elements, and some interior points are selected to reduce Integrals Equation (28) into a SLAE. The number of elements, denoted by $N$, and interior collocation points, denoted by $M$, for the different types of furrows are summarized in Table 2. Using the quantity of elements and interior collocation points in Table 2, some numerical results are presented in Figure 2 and Table 3.

### Table 2. Number of segments and interior points for each channel’s type.

|        | Furrow A | Furrow B | Furrow C | Furrow D | Furrow E |
|--------|----------|----------|----------|----------|----------|
| $N$    | 210      | 209      | 208      | 207      | 206      |
| $M$    | 400      | 400      | 400      | 400      | 400      |

#### Figure 2. Graph of suction potentials for five different types of channels.

In Figure 2, amounts of suction potential from different types of furrows, $\psi$, at various values of $X$ for $0 \leq Z \leq 100$ cm are shown. From Figure 2, we observe that $\psi$ achieve its highest value at $x = 10$ cm. The lowest value of $\psi$ is achieved at $x = 90$ cm. This means that locations near the channels result in higher value of $\psi$ than those further. This implies that soils near the channels have higher water content than those further. It can be observed that graphs of $\psi$ at $X = 10$ cm and at $X = 30$ cm at shallow level of soil for Furrow A are closer than those for Furrow E. These results are probably some of the effects of the geometry of the furrows. For Furrow A, the length of horizontal part is 45 cm. Hence, the depth of the channels at $X = 10$ cm and at $X = 30$ cm are 2.5 cm. This results in closer values of $\psi$ for $X = 10$ cm and at $X = 30$ cm. For Furrow E, the length of horizontal part is 5 cm, and hence the depth of the channels for $X = 10$ cm is deeper than that for $X = 30$ cm. This results in further values of $\psi$ for $X = 10$ cm and at $X = 30$ cm.
Table 3. Values of suction potential at selected points.

| Point            | Values of suction potential (cm) |
|------------------|----------------------------------|
|                  | Furrow A | Furrow B | Furrow C | Furrow D | Furrow E |
| (10 cm, 25 cm)   | -37.3989 | -36.6320 | -35.8885 | -35.0549 | -33.9657 |
| (10 cm, 50 cm)   | -42.1606 | -41.7427 | -41.3647 | -40.9613 | -40.4971 |
| (10 cm, 75 cm)   | -44.6241 | -44.3882 | -44.1828 | -43.9674 | -43.7330 |
| (50 cm, 25 cm)   | -51.3384 | -51.7779 | -52.2310 | -52.7161 | -53.2869 |
| (50 cm, 50 cm)   | -49.8228 | -49.9141 | -50.0257 | -50.1499 | -50.3180 |
| (50 cm, 75 cm)   | -48.3525 | -48.3651 | -48.3912 | -48.4196 | -48.4703 |
| (90 cm, 25 cm)   | -70.8939 | -71.5401 | -72.1032 | -72.6655 | -73.2829 |
| (90 cm, 50 cm)   | -60.5700 | -60.9614 | -61.3140 | -61.6694 | -62.0717 |
| (90 cm, 75 cm)   | -54.8038 | -55.0169 | -55.2146 | -55.4150 | -55.6532 |

Values of suction potential, \( \psi \), for some points are shown in Table 3. At \( X = 10 \) cm, \( \psi \) reaches maximum when Furrow E created on the soil surface, and the minimum value occurs in the soil with Furrow A. This may because of the depth of the channels. Deeper furrows or channels, Furrow E, result in higher flux than those shallower, Furrow A, at any fixed level of soil less than 75 cm. In contrast, for \( X = 50 \) cm and \( X = 90 \) cm, \( \psi \) attain maximum values in the soil as Furrow A created, and the smallest values of \( \psi \) are in the soil with Furrow E. This is due to the width of the furrows, as \( X = 50 \) cm and \( X = 90 \) cm are outside the furrows or channels. Furrow A is the widest channel than other types of the channels.

4. Conclusion

Steady or time-independent infiltration problems are solved using a numerical method called the dual reciprocity boundary element method. The mathematical model of the problems is transformed into a modified Helmholtz equation and then employing the method to solve the modified Helmholtz equation to obtain suction potentials. The results presented illustrate the influence of the depth and the width of the channels. Deeper channels result in higher suction potential than those shallower, at locations under the channels. This result indicates that soils under deeper channels are moister than those under shallower channels. At location outside channels, wider channels result in higher suction potential than those narrower. Hence, moisture content in soils outside wider channels is higher than that in soils outside narrower channels.

5. References

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