DYNAMICS OF 2-INTERVAL PIECEWISE AFFINE MAPS AND HECKE-MAHLER SERIES

MICHEL LAURENT AND ARNALDO NOGUEIRA
(Communicated by Raphäel Krikorian)

ABSTRACT. Let \( f : [0,1) \to [0,1) \) be a 2-interval piecewise affine increasing map which is injective but not surjective. Such a map \( f \) has a rotation number and can be parametrized by three real numbers. We make fully explicit the dynamics of \( f \) thanks to two specific functions \( \delta \) and \( \phi \) depending on these parameters whose definitions involve Hecke-Mahler series. As an application, we show that the rotation number of \( f \) is rational, whenever the three parameters are all algebraic numbers, extending thus the main result of [16] dealing with the particular case of 2-interval piecewise affine contractions with constant slope.

1. INTRODUCTION

DEFINITION 1. Let \( I = [0,1) \) be the unit interval. Let \( \lambda, \mu, \delta \) be three real numbers. Assume

\[
0 < \lambda < 1, \mu > 0, 1 - \lambda < \delta < d_{\lambda, \mu} := \begin{cases} 
1 & \text{if } \lambda \mu < 1, \\
\frac{\mu - \lambda \mu}{\mu - 1} & \text{if } \lambda \mu \geq 1.
\end{cases}
\]

Set \( \eta = \frac{1 - \delta}{\lambda} \) and define a map \( f = f_{\lambda, \mu, \delta} : I \to I \) by the formula

\[
f(x) = \begin{cases} 
\lambda x + \delta & \text{if } 0 \leq x < \eta, \\
\mu(\lambda x + \delta - 1) & \text{if } \eta \leq x < 1.
\end{cases}
\]

The restrictions of \( f \) to the intervals \([0, \eta)\) and \([\eta, 1)\) are increasing affine functions with slopes \( \lambda \) and \( \lambda \mu \), respectively. The symmetry \((x, y) \mapsto (1 - x, 1 - y)\) on \([0,1]^2\) exchanges the two slopes. Our assumption that the first segment of the graph has a slope \( \lambda \) less than 1 is thus unrestricted. Observe that \( 1 - \lambda < d_{\lambda, \mu} \leq 1 \) and that the bound \( \delta < d_{\lambda, \mu} \) yields the injectivity of \( f \). Indeed, the inequality

\[
\delta > \mu(\lambda + \delta - 1)
\]

holds true for any \( \delta \) in the interval \( 1 - \lambda < \delta < 1 \) when \( \lambda \mu < 1 \) (\( f \) is then a piecewise affine contracting map) and is equivalent to

\[
\delta < \frac{\mu - \lambda \mu}{\mu - 1} = d_{\lambda, \mu}.
\]
when $\lambda \mu \geq 1$ (see Figure 1). Notice that in the limit case $\delta = d_{\lambda, \mu}$ with $\lambda \mu > 1$, the map $f = f_{\lambda, \mu, \delta}$ becomes a bijection and we deduce by a continuity argument that its rotation number is $-\log \lambda / \log \mu$. We recall that this result had already been obtained in [3].

We are concerned with the dynamics of the family of interval maps $f = f_{\lambda, \mu, \delta}$. We plan to relate their dynamics to the so-called Hecke-Mahler series in two variables $\lambda$ and $\mu$ (see Section 2 for definitions). The paper [16] deals with the case $\mu = 1$ where the slope is constant. Although the map $f = f_{\lambda, \mu, \delta}$ is not necessarily a piecewise contraction, we extend part of the results established in [16] to a 2-slope setting.

The dynamics of interval piecewise affine contractions has been studied by many authors, amongst others [6, 7, 8, 9, 11, 12, 13, 19, 22, 23]. According to [24], every map $f = f_{\lambda, \mu, \delta}$ has a rotation number $\rho = \rho_{\lambda, \mu, \delta}$, $0 < \rho < 1$. Although $f$ is not necessarily a piecewise contraction, we will prove that if $\rho$ takes an irrational value, then the closure $\overline{C}$ of the limit set $C := \bigcap_{k \geq 1} f^k(I)$ of $f$ is a Cantor set and $f$ is topologically conjugated to the rotation map $x \in I \mapsto x + \rho \mod 1$ on $C$. When the rotation number is rational, the map $f$ has at most one periodic orbit (exactly one in most cases) and the limit set $C$ equals the periodic orbit when it does exist. More precisely, either $f$ or $f^{-1}$ (a slight modification of the map $f$ whose definition is postponed to Section 8) has a periodic cycle.

We make the above mentioned qualitative results fully explicit thanks to formulae involving Hecke-Mahler series. Our approach is based on the study of a conjugacy function $\phi$ which may be written down in terms of Hecke-Mahler series. The method was already performed in the special case $\mu = 1$ where the two slopes are equal. It is motivated by Coutinho’s thesis [9] and has been recently reworked in [4, 14, 16]. The general case involving two different slopes is quite similar. Theorem 3 gives the value of the rotation number $\rho_{\lambda, \mu, \delta}$ in terms of the values of the three parameters $\lambda, \mu, \delta$, while Theorem 6 describes the behaviour of the orbits of $f$ and their relations with the conjugacy $\phi$. 
Next we introduce some standard notations. For any real function \( f(x) \) of the real variable \( x \), we denote by

\[
    f(x^-) = \lim_{y \to x^-} f(y) \quad \text{and} \quad f(x^+) = \lim_{y \to x^+} f(y),
\]

respectively, the left limit and the right limit of \( f \) at \( x \), whenever these limits do exist. As usual \([x]\) and \(\lceil x \rceil\) stand, respectively, for the integer floor and the integer ceiling of the real number \( x \). In particular, we have \([x] = [x] + 1\) for any real number \( x \not\in \mathbb{Z} \) and \([x] = [x] \) when \( x \in \mathbb{Z} \). We denote by \(\{x\} = x - [x]\) the fractional part of \( x \). The length of an interval \( J \subset \mathbb{R} \) is denoted by \(|J|\).

We first define a real function \( \delta(\lambda, \mu, \rho) \) as follows.

**Definition 2.** For positive real numbers \( \lambda, \mu \) and \( \rho \) such that \( \lambda \mu \rho < 1 \), set

\[
    \sigma = \sigma(\lambda, \mu, \rho) := \sum_{k \geq 1} \left( \left\lfloor (k + 1) \rho \right\rfloor - \lfloor k \rho \rfloor \right) \lambda^k \mu^{\lfloor k \rho \rfloor},
\]

and

\[
    \delta(\lambda, \mu, \rho) = \frac{(1 - \lambda)(1 + \mu \sigma)}{1 + (\mu - 1) \sigma}.
\]

For real numbers \( \lambda \) and \( \mu \) with \( 0 < \lambda < 1 \) and \( \mu > 0 \), set

\[
    r_{\lambda, \mu} = \begin{cases} 
        1 & \text{if } \lambda \mu < 1, \\
        -\frac{\log \lambda}{\log \mu} & \text{if } \lambda \mu \geq 1.
    \end{cases}
\]

![Figure 2](image_url)

**Figure 2.** Plot of the map \( \rho \to \delta(0.9, 0.8, \rho) \)

The series \( \sigma(\lambda, \mu, \rho) \) converges when \( 0 \leq \rho < r_{\lambda, \mu} \). For fixed \( \lambda \) and \( \mu \) with \( 0 < \lambda < 1, \mu > 0 \), the map \( \rho \to \delta(\lambda, \mu, \rho) \) is increasing in the interval \( 0 \leq \rho < r_{\lambda, \mu} \) and it has a left discontinuity at each rational value (see Figure 2). It is continuous.
for any irrational $\rho$ and right continuous everywhere. The function $\delta$ enables us to compute the rotation number of $f_{\lambda,\mu,\delta}$ thanks to the following:

**Theorem 3.** Let $\lambda$ and $\mu$ be real numbers with $0 < \lambda < 1$ and $\mu > 0$. Then the application $\delta \to \rho_{\lambda,\mu,\delta}$ is a continuous non decreasing function sending the interval $(1 - \lambda, d_{\lambda,\mu})$ onto the interval $(0, r_{\lambda,\mu})$ and satisfying the following properties:

(i) Let $p/q$ be a rational number with $0 < p/q < r_{\lambda,\mu}$, where $p$ and $q$ are relatively prime integers. Then $\rho_{\lambda,\mu}$ takes the value $p/q$ if, and only if, $\delta$ is located in the interval

$$\delta(\lambda,\mu,(p/q)^{-}) \leq \delta \leq \delta(\lambda,\mu,p/q)$$

with the explicit formulae

$$\delta(\lambda,\mu,(p/q)^{-}) = \frac{(1 - \lambda)(1 + \mu S + \lambda^{q^{-1}}\mu^{p}(1 - \lambda))}{1 + (\mu - 1)S + \lambda^{q^{-1}}\mu^{p-1}(\mu - \lambda\mu - 1)}$$

$$\delta(\lambda,\mu,(p/q)^{+}) = \frac{(1 - \lambda)(1 + \mu S)}{1 + (\mu - 1)S - \lambda^{q^{-1}}\mu^{p-1}},$$

where

$$S = S(\lambda,\mu,p/q) := \sum_{k=1}^{q-2} \left( \lceil (k+1)p/q \rceil - \lfloor kp/q \rfloor \right) \lambda^{k}\mu^{\lfloor kp/q \rfloor}$$

and the sum $S$ equals 0 when $q = 2$.

(ii) For every irrational number $\rho$ with $0 < \rho < r_{\lambda,\mu}$, the real number $\delta(\lambda,\mu,\rho)$ is the only value of $\delta$ with $1 - \lambda < \delta < d_{\lambda,\mu}$ for which $\rho_{\lambda,\mu,\delta} = \rho$.

Roughly speaking, the two maps $\rho \to \delta(\lambda,\mu,\rho)$ and $\delta \to \rho_{\lambda,\mu,\delta}$ are “inverse” from each other, meaning that their graphs are symmetric with respect to the main diagonal. In the special case $\mu = 1$, we recover the formulae obtained in [16] for the map $f_{\lambda,1,\delta}$, which coincides with the contracted rotation $x \to \{\lambda x + \delta\}$. Notice that the formulae of our Theorem 3 are consistent with those of [6, Theorem 4.15], dealing with the subfamily of contractions $f_{\lambda,\mu,\delta}$ with $\lambda\mu < 1$, although the formulations greatly differ.

Our approach gives an alternative proof of a result in [3] about the rotation number of the piecewise affine circle homeomorphism $f$ with slopes $0 < \lambda < 1$ and $\lambda\mu > 1$ obtained for the limit value $\delta = d_{\lambda,\mu}$ in Definition 1.

**Corollary.** Let $0 < \lambda < 1$ and $\mu > 1/\lambda$ be real numbers. Put $d_{\lambda,\mu} = \frac{\mu - \lambda\mu}{\mu - 1}$ and $\eta = \frac{1 - d_{\lambda,\mu}}{\lambda} = \frac{\lambda\mu - 1}{\mu - \lambda}$. Then the circle homeomorphism $f : I \to I$ given by

$$f(x) = f_{\lambda,\mu,d_{\lambda,\mu}}(x) = \begin{cases} \lambda x + d_{\lambda,\mu} & \text{if } 0 \leq x < \eta, \\ \mu(\lambda x + d_{\lambda,\mu} - 1) & \text{if } \eta \leq x < 1, \end{cases}$$

has rotation number equal to $r_{\lambda,\mu} = -\log \lambda / \log \mu$.

Applying now a classical transcendence result, which is stated as Theorem 10 below, to the number $\delta(\lambda,\mu,\rho)$, we deduce from the assertion (ii) of Theorem 3 the following:
THEOREM 4. Let $\lambda, \mu, \delta$ be algebraic real numbers with $0 < \lambda < 1$, $\mu > 0$ and $1 - \lambda < \delta < d_{\lambda, \mu}$. Then, the rotation number $\rho_{\lambda, \mu, \delta}$ takes a rational value.

Notice that Theorem 4 no longer holds for the value $\delta = d_{\lambda, \mu}$ when $\lambda \mu > 1$. Indeed, $d_{\lambda, \mu} = \frac{\mu - \lambda \mu}{\mu - 1}$ is an algebraic number when $\lambda$ and $\mu$ are algebraic, while $f$ has rotation number $-\log \lambda / \log \mu$ by the above corollary. This ratio is a transcendental number when $\lambda$ and $\mu$ are non-zero algebraic numbers, unless $\lambda$ and $\mu$ are multiplicatively dependent.

We now investigate the behaviour of the iterates of $f = f_{\lambda, \mu, \delta}$ thanks to an explicit conjugacy map $\phi$.

DEFINITION 5. Let $\lambda, \mu, \rho$ be three positive real numbers such that $\lambda \mu \rho < 1$, and let $\delta$ be an arbitrary real number. Let $\phi_{\lambda, \mu, \delta, \rho} : \mathbb{R} \to \mathbb{R}$ be the real function defined by the convergent series

$$
\phi_{\lambda, \mu, \delta, \rho}(y) = \lfloor y \rfloor + \frac{1 - \delta}{\lambda} + \sum_{k=0}^{+\infty} \lambda^k \mu^{\lfloor y - k \rho \rfloor} \left( \frac{\lambda + \delta - 1}{\lambda} + |y - (k+1)\rho| - |y - k\rho| \right).
$$

THEOREM 6. Let $\lambda, \mu, \delta$ be three real numbers with $0 < \lambda < 1$, $\mu > 0$ and $1 - \lambda < \delta < d_{\lambda, \mu}$. Set $\rho = \rho_{\lambda, \mu, \delta}$ and $\phi = \phi_{\lambda, \mu, \delta, \rho}$.

(i) Assume that $\rho$ is irrational. Then $C = \phi(I)$ and the restriction of $f = f_{\lambda, \mu, \delta}$ to the invariant set $C$ is conjugate by $\phi$ to the rotation $R_{\rho} : y \mapsto y + \rho \mod 1$. In other words, we have the commutative diagram:

$$
\begin{array}{ccc}
I & \xrightarrow{R_{\rho}} & I \\
\phi \downarrow & & \downarrow \phi \\
C & \xrightarrow{f} & C.
\end{array}
$$

\[\text{Figure 3. A plot of } f_{\lambda, \mu, d_{\lambda, \mu}} \text{ for } \lambda = 1/2, \mu = 3\]
Moreover $\phi(I)$ is a Cantor set and for every $x \in I$, the $\omega$-limit set

$$\omega(x) := \bigcap_{n \to +\infty} \bigcup_{k \geq n} f^k(x)$$

equals $\overline{C}$.

(ii) Assume that $\rho = p/q$ is rational, where $p$ and $q$ are relatively prime, and that

$$\delta(\lambda, \mu, (p/q)^-) \leq \delta < \delta(\lambda, \mu, p/q).$$

Then

$$C = \phi(I) = \{\phi(m/q); 0 \leq m \leq q-1\}$$

is a cycle of order $q$ and we have the commutative diagram:

$$\begin{array}{ccc}
\{m/q; 0 \leq m \leq q-1\} & \xrightarrow{R_{p/q}} & \{m/q; 0 \leq m \leq q-1\} \\
\phi \downarrow & & \phi \downarrow \\
C & \xrightarrow{f} & C
\end{array}$$

where $R_{p/q}$ denotes the rotation $y \mapsto y + \frac{p}{q} \mod 1$. Moreover, for every $x \in I$, the $\omega$-limit set $\omega(x)$ equals $C$.

(iii) When $\delta = \delta(\lambda, \mu, \frac{p}{q})$, the limit set $C$ is empty and $\phi(I)$ is a finite set with $q$ elements containing $1$. For every $x \in I$, the $\omega$-limit set $\omega(x)$ coincides with $\phi(I)$.

**Figure 4.** Plot of the function $\phi_{0.95,0.9,\delta,(\sqrt{5}-1)/2}$ in the range $0 \leq y \leq 1$, where $\delta = \delta(0.95,0.9,(\sqrt{5}-1)/2) = 0.6617\ldots$
The paper is organized as follows. In Section 2, we introduce Hecke-Mahler series and relate them to our functions $\delta$ and $\phi$. Then, Theorem 4 easily follows from Theorem 3. The purpose of Sections 3 and 4 is to establish the basic conjugacy equations (1) and (3). This goal is achieved thanks to Lemma 14, where some relations connecting the parameter $\delta$ with values of the function $\delta(\lambda, \mu, p)$ are needed, as for instance the inequalities (2) in the case (3). It turns out that these constraints characterize the rotation number $\rho_{\lambda, \mu, \delta}$. As a consequence of the method, we establish Theorem 3 in Section 5. The next two sections provide additional information on the dynamics of $f$ in the case of an irrational rotation number (Proposition 15 in Section 6), or a rational one (Proposition 16 of Section 7). In both cases, we explicitly describe the iterated images $f^n(I)$, $n \geq 1$. Finally Section 8 deals with the exceptional values of the form $\delta = \delta(\lambda, \mu, p/q)$ for which no periodic cycle exists.

2. HECKE-MAHLER SERIES AND TRANSCENDENTAL NUMBERS

2.1. On Hecke-Mahler series. We introduce the following sums:

**Definition 7.** Let $\lambda$, $\mu$, and $\rho$ be positive real numbers such that $0 < \lambda < 1, 0 < \lambda \mu^h < 1$. We set, for every real number $x$,

$$
\Psi_\rho(\lambda, \mu) = \sum_{k \geq 0} \sum_{h \leq k \rho} \lambda^k \mu^h,
$$

$$
\Phi_\rho(\lambda, \mu, x) = \sum_{k \geq 0} \sum_{0 \leq i < k \rho + x} \lambda^k \mu^i,
$$

with the convention that a sum indexed by an empty set equals zero.

Notice that $\Psi_\rho(\lambda, \mu)$ is a right continuous function in the variable $\rho$, while the function $\Phi_\rho(\lambda, \mu, x)$ is left continuous in both variables $\rho$ and $x$. We have the relation

$$
\Phi_\rho(\lambda, \mu, 0^+) = \frac{1}{1 - \lambda} + \Psi_\rho(\lambda, \mu).
$$

Viewed as power series in the two variables $\lambda$ and $\mu$, these two functions are called Hecke-Mahler series which have been studied especially from a diophantine point of view [1, 2, 5, 10, 15, 17, 18, 20, 21]. We relate our functions $\delta(\lambda, \mu, p)$ and $\phi_{\lambda, \mu, \delta}(x)$ respectively to $\Psi_\rho(\lambda, \mu)$ and $\Phi_\rho(\lambda, \mu, x)$.

**Lemma 8.** Let $\lambda, \mu, \rho$ be real numbers with $0 < \lambda < 1, \mu > 0$ and $0 < \rho < r_{\lambda, \mu}$. Then the following equality holds

$$
\sigma(\lambda, \mu, \rho) = \sum_{k \geq 1} \left( \lfloor (k + 1) \rho \rfloor - \lfloor k \rho \rfloor \right) \lambda^k \mu^{\lfloor k \rho \rfloor} = \frac{1 - \lambda}{\lambda \mu} \Psi_\rho(\lambda, \mu).
$$

**Proof.** Reverting the summation order for the indices $h, k$ involved in $\Psi_\rho(\lambda, \mu)$, we obtain

$$
\Psi_\rho(\lambda, \mu) = \sum_{h \geq 1} \sum_{k \geq \frac{h}{\rho}} \lambda^k \mu^h = \sum_{h \geq 1} \sum_{k \geq \frac{h}{\rho}} \lambda^k \mu^h = \frac{1}{1 - \lambda} \sum_{h \geq 1} \lambda^\left\lfloor \frac{h}{\rho} \right\rfloor \mu^h.
$$
A positive integer \( k \) is of the form \( \left\lfloor \frac{h}{p} \right\rfloor \) for some some positive integer \( h \) if and only if \( \frac{h}{p} \leq k < \frac{h}{p} + 1 \), or equivalently \( (k-1)p < h \leq kp \). There exists at most one integer \( h \) in the interval \( ((k-1)p, kp) \) whose length is \( \rho < r_{\lambda,\mu} \leq 1 \). The integer \( h \) does exist exactly when \([((k-1)p) = [kp] − 1 \) and then \( h = [kp] = [(k-1)p] + 1 \). Otherwise, \([((k-1)p) = [kp] \).

Moreover, it is right continuous everywhere and continuous at any irrational point \( \rho \).

Proof. Using Lemma 8, we can rewrite \( \delta(\lambda, \mu, \rho) \) in the form

\[
\delta(\lambda, \mu, \rho) = \frac{(1 - \lambda)(1 + \mu \sigma(\lambda, \mu, \rho))}{1 + (\mu - 1)\sigma(\lambda, \mu, \rho)} = \mu(1 - \lambda) \frac{\lambda + (1 - \lambda)\Psi(\lambda, \mu)}{\lambda \mu + (1 - \lambda)(\mu - 1)\Psi(\lambda, \mu)}.
\]

We distinguish two cases whether \( \lambda \mu < 1 \) or not.

When \( \lambda \mu < 1 \), we have \( d_{\lambda,\mu} = r_{\lambda,\mu} = 1 \). The series \( \Psi(\lambda, \mu) \) converges for any \( \rho \in (0,1) \) and the map \( \rho \to \Psi(\lambda, \mu) \) is obviously increasing, since \( \Psi(\lambda, \mu) \) is a sum of powers of \( \lambda \) and \( \mu \) and the set of summation indices \( (h, k) \) enlarges when \( \rho \) grows. We easily compute

\[
\lim_{\rho \searrow 0} \Psi(\lambda, \mu) = 0 \quad \text{and} \quad \lim_{\rho \nearrow 1} \Psi(\lambda, \mu) = \frac{\lambda^2 \mu}{(1 - \lambda)(1 - \lambda \mu)}.
\]

It follows that

\[
0 < \Psi(\lambda, \mu) < \frac{\lambda^2 \mu}{(1 - \lambda)(1 - \lambda \mu)}
\]

for any \( \rho \in (0,1) \). Since \( \lambda \) differs from 0 and 1, the homographic function

\[
x \to \mu(1 - \lambda) \frac{\lambda + (1 - \lambda)x}{\lambda \mu + (1 - \lambda)(\mu - 1)x}
\]

We deduce

**Corollary.** Let \( 0 < \lambda < 1 \) and \( \mu > 0 \). Then the map \( \rho \to \delta(\lambda, \mu, \rho) \) is strictly increasing on the interval \( 0 < \rho < r_{\lambda,\mu} \) and sends the interval \( (0, r_{\lambda,\mu}) \) into the interval \( (1 - \lambda, d_{\lambda,\mu}) \) with limit values

\[
\lim_{\rho \searrow 0} \delta(\lambda, \mu, \rho) = 1 - \lambda \quad \text{and} \quad \lim_{\rho \nearrow r_{\lambda,\mu}} \delta(\lambda, \mu, \rho) = d_{\lambda,\mu}.
\]

Moreover, it is right continuous everywhere and continuous at any irrational point \( \rho \).
is increasing on the interval $0 < x < \frac{\lambda^2 \mu}{(1-\lambda)(1-\lambda\mu)}$, and sends this interval onto $(1-\lambda, 1)$. By composition, we obtain that the image of $(0, 1)$ by the map $\rho \mapsto \Phi(\lambda, \mu, \rho)$ is contained in the interval $(1-\lambda, 1)$.

When $\lambda \mu \geq 1$, we have $r_{\lambda, \mu} = -\log \lambda / \log \mu$ and the series $\sum_{\rho} (\Phi(\lambda, \mu, \rho))$ tends to $+\infty$ when $\rho$ tends to $r_{\lambda, \mu}$ from below. Thus, we obtain in this case,

$$0 < \sum_{\rho} \Phi(\lambda, \mu) < +\infty \quad \text{and} \quad 1-\lambda < \delta(\lambda, \mu, \rho) < \frac{\mu(1-\lambda)}{\mu-1} = d_{\lambda, \mu}.$$

For the continuity property, observe that the floor function $x \mapsto \lfloor x \rfloor$ is right continuous on $\mathbb{R}$ and continuous on $\mathbb{R} \sim \mathbb{Z}$.

We now give an alternative formula for the function $\phi_{\lambda, \mu, \rho}$ in terms of the Hecke-Mahler series $\Phi_{\rho}$.

**Lemma 9.** Let $0 < \lambda < 1$, $\mu > 0$ and $0 < \rho < r_{\lambda, \mu}$. Then, for any real number $x > -1$, we have the equalities

$$\left(\Phi(\lambda, \mu, x) = \sum_{k \geq 0} \frac{\lambda^k \mu^{\lfloor k \rho x \rfloor}}{1-\lambda} \right) \text{ and } \left(\Phi(\lambda, \mu, x) = \sum_{k \geq 1} \frac{\lambda^k \mu^{\lfloor k \rho x \rfloor}}{1-\lambda} \right),$$

where the indeterminate ratio $(1-\lambda)/(1-\lambda)$ equals $[x]$ when $\mu = 1$. Moreover, the formula

$$\phi_{\lambda, \mu, \rho}(y) = [y] + \frac{\delta}{1-\lambda} - \frac{\delta - \mu(\lambda + \delta - 1)}{\lambda} \Phi_{\rho}(\lambda, \mu, -[y])$$

holds for any real number $y$.

**Proof.** From Definition 7, we can write

$$\Phi(\lambda, \mu, x) = \sum_{k \geq 0} \lambda^k \sum_{0 \leq l \leq \lfloor k \rho x \rfloor} \mu^l = \sum_{k \geq 0} \lambda^k \frac{1-\mu^{\lfloor k \rho x \rfloor}}{1-\mu}.$$

which implies (5). For equation (6), multiplying (5) by $1-\lambda$, we find

$$\left(1-\lambda\right) \sum_{k \geq 0} \lambda^k \mu^{\lfloor k \rho x \rfloor} = \sum_{k \geq 0} \lambda^k \mu^{\lfloor k \rho x \rfloor} - \sum_{k \geq 1} \lambda^k \mu^{\lfloor (k-1) \rho x \rfloor} = \mu^{\lfloor x \rfloor} + \lambda \sum_{k \geq 0} \lambda^k \left(\mu^{\lfloor (k+1) \rho x \rfloor} - \mu^{\lfloor k \rho x \rfloor}\right).$$

Observe that, for any integer $k \geq 0$, $\lfloor (k+1) \rho x \rfloor - \lfloor k \rho x \rfloor$ takes only the value 0 or 1. Therefore

$$\mu^{\lfloor (k+1) \rho x \rfloor} - \mu^{\lfloor k \rho x \rfloor} = (\mu - 1)(\lfloor (k+1) \rho x \rfloor - \lfloor k \rho x \rfloor) \mu^{\lfloor k \rho x \rfloor}.$$
We obtain the equality
\[(1 - \lambda) \left( \frac{1}{1 - \lambda} - (1 - \mu) \Phi_\rho(\lambda, \mu, x) \right)\]
\[= \mu^{[x]} + \lambda (\mu - 1) \sum_{k\geq 0} \left( (k + 1) \rho + x \right) - (k \rho + x) ) \lambda^k \mu^{[k+1]} \]
from which formula (6) follows.

The map \( y \mapsto [y] - |y - \rho| \) has period 1 for any integer \( k \). We can thus replace \( y \) by its fractional part \([y]\) in the sum over \( k \) occurring in Definition 5 giving \( \phi_{\lambda, \mu, \delta, \rho} \). Observe also that \([x] = -[-x]\) for any real number \( x \). We can therefore rewrite \( \phi_{\lambda, \mu, \delta, \rho}(y) \) in the form
\[\phi_{\lambda, \mu, \delta, \rho}(y) = [y] + \frac{1 - \delta}{\lambda} + \sum_{k=0}^{\infty} \lambda^k \mu^{[\rho(y)]} \left( \frac{\lambda + \delta - 1}{\lambda} - (k + 1) \rho - [y] + [k \rho - [y]] \right) \]

Using (5) and (6) for \( x = -[y] \) and noting that \([-y] = 0\), we obtain
\[\phi_{\lambda, \mu, \delta, \rho}(y) = [y] + \frac{1 - \delta}{\lambda} + \lambda + \delta - 1 \left( \frac{1}{1 - \lambda} - (1 - \mu) \Phi_\rho(\lambda, \mu, -[y]) \right) - \frac{1 - \lambda}{\lambda} \Phi_\rho(\lambda, \mu, -[y]) \]
\[= [y] + \frac{\delta}{1 - \lambda} - \frac{\delta - \mu (\lambda + \delta - 1)}{\lambda} \Phi_\rho(\lambda, \mu, -[y]) \]

**Corollary.** Let \( 0 < \lambda < 1, \mu > 0, 0 < \rho < r_{\lambda, \mu} \) and \( 1 - \lambda < \delta < d_{\lambda, \mu} \). Then the function \( \phi_{\lambda, \mu, \delta, \rho} \) is right continuous and non-decreasing on the interval \( I = [0, 1) \).

Moreover,

(i) \( \phi_{\lambda, \mu, \delta, \rho} \) is strictly increasing on \( I \), if \( \rho \) is irrational.

(ii) If \( \rho = \frac{p}{q} \) is rational, the function \( y \mapsto \phi_{\lambda, \mu, \delta, \rho/q}(y) \) is constant on each interval \([\frac{p}{q}, \frac{p+1}{q})\), \( n \in \mathbb{Z} \).

(iii) In any case, the relation \( \phi_{\lambda, \mu, \delta, \rho}(y + 1) = \phi_{\lambda, \mu, \delta, \rho}(y) + 1 \) holds for any real number \( y \).

**Proof.** The function \( x \mapsto \Phi_\rho(\lambda, \mu, x) \) is clearly non-decreasing and strictly increasing when \( \rho \) is irrational. By Lemma 9, we have
\[\phi_{\lambda, \mu, \delta, \rho}(y) = \frac{\delta}{1 - \lambda} - \frac{\delta - \mu (\lambda + \delta - 1)}{\lambda} \Phi_\rho(\lambda, \mu, -[y]) \]
when \( 0 \leq y < 1 \). Notice that the coefficient \(-\frac{\delta - \mu (\lambda + \delta - 1)}{\lambda}\) is negative by the assumption \( \delta < d_{\lambda, \mu} \) (see Figure 1). This yields that \( \phi_{\lambda, \mu, \delta, \rho} \) is non-decreasing. Also the right continuity of \( \phi_{\lambda, \mu, \delta, \rho} \) follows from the left continuity of the function \( x \mapsto \Phi_\rho(\lambda, \mu, x) \). The other assertions are straightforward.

2.2. **Proof of Theorem 4.** Let us begin with the following result on the transcendency of values of the Hecke-Mahler function, due to Loxton and Van der Poorten [17]. See also the survey article [18]. Further information on the proof of Theorem 10 will be given in the Appendix.
**Theorem 10.** Let \( \lambda \) and \( \mu \) be non-zero algebraic numbers and let \( \rho \) be an irrational real number with \( 0 < \rho < 1 \). Assume that \( 0 < |\lambda| < 1 \) and \( |\lambda| |\mu|^p < 1 \). Then \( \Psi_\rho(\lambda, \mu) \) is a transcendental number.

Using the homographic relations (4), both numbers \( \delta(\lambda, \mu, \rho) \) and \( \Psi_\rho(\lambda, \mu) \) are simultaneously either algebraic or transcendental. Then, it follows from Theorem 10 that \( \delta(\lambda, \mu, \rho) \) is a transcendental number for any irrational real number \( 0 < \rho < r_{\lambda, \mu} \). As a consequence of the assertion (ii) of Theorem 3, the rotation number \( \rho_{\lambda, \mu, \delta} \) cannot be an irrational number \( \rho \) when \( \lambda, \mu, \delta \) are algebraic numbers. It is therefore a rational number. Theorem 4 is established.

3. Properties of the function \( \phi \)

Let \( \lambda, \mu, \delta, \rho \) be four real numbers satisfying the inequalities

\[
0 < \lambda < 1, \mu > 0, 1 - \lambda < \delta < d_{\lambda, \mu}, 0 < \rho < r_{\lambda, \mu}.
\]

We estimate in this technical section the value of the function \( \phi_{\lambda, \mu, \delta, \rho} \) at the points 0 and \( 1 - \rho \) according to the values of \( \delta \). We stress that \( \rho \) is not assumed here to be the rotation number of the map \( f_{\lambda, \mu, \delta} \). On the opposite, we shall make use of our results to identify this rotation number \( \rho_{\lambda, \mu, \delta} \) in the subsequent Section 5, and thus proving Theorem 3. Our estimates are based on numerical relations between some special values of the Hecke-Mahler series \( \Phi_\rho \) and the function \( \sigma \), as for instance the formulæ (9) to (12) below.

**Lemma 11.** Assume that \( 0 < \rho < r_{\lambda, \mu} \) is irrational. Let \( \delta = \delta(\lambda, \mu, \rho) \) and \( \phi = \phi_{\lambda, \mu, \delta, \rho} \). Then the following equalities hold

\[
\phi(0) = 0 \quad \text{and} \quad \phi(1 - \rho) = \frac{1 - \delta}{\lambda} = \eta.
\]

**Proof.** Recall the formula

\[
\delta = \frac{(1 - \lambda)(1 + \mu \sigma)}{1 + (\mu - 1)\sigma}, \quad \text{where} \quad \sigma = \sum_{k \geq 1} \left( [k + 1] \rho - [k \rho] \right) \lambda^k \mu^{|kp|}.
\]

Notice first that we have the equalities

\[
\Phi_\rho(\lambda, \mu, 0) = \sum_{k \geq 10} \sum_{l < k \rho} \lambda^k \mu^l = \sum_{k \geq 1} \lambda^k + \sum_{k \geq 11} \sum_{l < k \rho} \lambda^k \mu^l = \frac{\lambda}{1 - \lambda} + \Psi_\rho(\lambda, \mu) = \frac{\lambda}{1 - \lambda} (1 + \mu \sigma),
\]

the last one coming from Lemma 8 and noting that the strict inequality \( l < k \rho \) is equivalent to \( l \leq k \rho \), when \( \rho \) is irrational. It follows from Lemma 9 and (7) that

\[
\phi(0) = \frac{\delta}{1 - \lambda} - \frac{\delta - \mu(\lambda + \delta - 1)}{\lambda} \Phi_\rho(\lambda, \mu, 0) = \frac{\delta}{1 - \lambda} - \frac{\delta - \mu(\lambda + \delta - 1)}{1 - \lambda} (1 + \mu \sigma)
\]

\[
= \frac{\mu \left( \delta (1 + (\mu - 1)\sigma) - (1 - \lambda)(1 + \mu \sigma) \right)}{1 - \lambda} = 0.
\]
For the value $\phi(1 - \rho)$, we compute $\Phi_{\rho}(\lambda, \mu, \rho - 1)$ using (6). Noting that $[\rho - 1] = 0$, we find

$$
\frac{1 - \lambda}{\lambda} \Phi_{\rho}(\lambda, \mu, \rho - 1) = \sum_{k \geq 0} \frac{1}{\lambda \mu} \sum_{k \geq 1} \{(k + 1)\rho + \rho - 1 - [k \rho + \rho - 1]\} \lambda^k \mu^{[k \rho + \rho - 1]}
$$

$$
= \frac{1}{\lambda \mu} \sum_{k \geq 1} \{(k + 1)\rho - [k \rho]\} \lambda^k \mu^{[k \rho]}
$$

$$
= \frac{1}{\lambda} \sum_{k \geq 1} \{(k + 1)\rho - [k \rho]\} \lambda^k \mu^{[k \rho]} = \frac{\sigma}{\lambda},
$$

since $[k \rho] = [k \rho] + 1$ for any integer $k \geq 1$. Therefore

$$
\phi(1 - \rho) = \frac{\delta}{1 - \lambda} - \frac{\delta - \mu(\lambda + \delta - 1)}{\lambda} \Phi_{\rho}(\lambda, \mu, \rho - 1)
$$

$$
= \frac{\delta}{1 - \lambda} - \frac{\delta - \mu(\lambda + \delta - 1)}{\lambda} \frac{\sigma}{1 - \lambda}
$$

$$
= \frac{\delta(\lambda + (\mu - 1)\sigma) - (1 - \lambda)\mu\sigma}{\lambda(1 - \lambda)}
$$

$$
= \frac{(1 - \lambda)(1 - \delta) + \delta(1 + (\mu - 1)\sigma) - (1 - \lambda)(1 + \mu\sigma)}{\lambda(1 - \lambda)}
$$

$$
= \frac{1 - \delta}{\lambda},
$$

since $\delta = (1 - \lambda)(1 + \mu\sigma)/(1 + (\mu - 1)\sigma)$.

When $\rho$ is a rational number $p/q$, the function $\phi_{\lambda, \mu, \delta, p/q}$ is constant on any interval of the form $[n/q, n+1/q)$, $n \in \mathbb{Z}$, and has a positive jump at the endpoints $\mathbb{Z}/q$. In this case, we have the analogous

**Lemma 12.** Assume that $\rho = p/q < r_{\lambda, \mu}$ and $\delta(\lambda, \mu, (p/q)^-) \leq \delta < \delta(\lambda, \mu, p/q)$. Put $\phi = \phi_{\lambda, \mu, \delta, p/q}$. Then

$$
\phi\left(-\frac{1}{q}\right) = \phi(0^-) < 0 \leq \phi(0)
$$

and

$$
\phi\left(\frac{q - p - 1}{q}\right) = \phi\left(\left(1 - \frac{p}{q}\right)^-\right) < \frac{1 - \delta}{\lambda} \leq \phi\left(\frac{1 - p}{q}\right).
$$

**Proof.** Set $\sigma = \sigma(\lambda, \mu, \frac{p}{q})$ and $\sigma^- = \sigma(\lambda, \mu, (\frac{p}{q})^-)$. We first show that

$$
(8) \quad \sigma = \frac{S + \lambda^{q-1} \mu^{p-1}}{1 - \lambda q \mu^p} \quad \text{and} \quad \sigma^- = \frac{S + \lambda q \mu^{p-1}}{1 - \lambda q \mu^p},
$$

where we recall the notation

$$
S = \sum_{k=1}^{q-2} \left\{(k + 1) \frac{p}{q} - \left\lfloor k \frac{p}{q} \right\rfloor\right\} \lambda^k \mu^{[k \frac{p}{q}]}
$$

from Theorem 3. By Definition 2, we have

$$
\sigma = \sum_{k \geq 1} \left\{[(k + 1) \frac{p}{q} - \left\lfloor k \frac{p}{q} \right\rfloor]^+\right\} \lambda^k \mu^{[k \frac{p}{q}]}.\]
Observe that \( [(k + q) \frac{p}{q}] = [k \frac{p}{q}] + p \). Splitting the above sum over \( k \) according to the various classes of \( k \) modulo \( q \) and noting that \( [(q - 1) \frac{p}{q}] = p - 1 \), we obtain the first formula

\[
\sigma = \frac{1}{1 - \lambda^q \mu^p} \sum_{k=1}^{q} \left( [(k + 1) \frac{p}{q}] - [k \frac{p}{q}] \right) \lambda^k \mu^{[k \frac{p}{q}]} = \frac{S + \lambda q^{-1} \mu^{p-1}}{1 - \lambda^q \mu^p}.
\]

Similarly, we have

\[
\sigma^- = \sum_{k \geq 1} \left( [(k + 1) p/q] - [(kp/q)] \right) \lambda^k \mu^{[(kp/q) -]} = \frac{1}{1 - \lambda^q \mu^p} \sum_{k=1}^{q} \left( [(k + 1) \frac{p}{q}] - [k \frac{p}{q}] \right) \lambda^k \mu^{[(kp/q) -]} = \frac{S + \lambda q \mu^{p-1}}{1 - \lambda^q \mu^p}.
\]

Now, we establish the formulae

\[
(9) \quad \Phi_{p/q}(\lambda, \mu, 0) = \frac{\lambda(1 + \mu\sigma^-)}{1 - \lambda},
\]

\[
(10) \quad \Phi_{p/q}(\lambda, \mu, (-1)^+) = \Phi_{p/q}(\lambda, \mu, -1 + \frac{1}{q}) = \frac{\lambda \sigma^-}{1 - \lambda},
\]

\[
(11) \quad \Phi_{p/q}(\lambda, \mu, p q^{-1}) = \frac{\sigma^-}{1 - \lambda},
\]

\[
(12) \quad \Phi_{p/q}(\lambda, \mu, (p q^{-1})^+) = \Phi_{p/q}(\lambda, \mu, p q^{-1} + \frac{1}{q}) = \frac{\sigma^-}{1 - \lambda}.
\]

To that purpose, we observe that the function \( x \mapsto \Phi_{p/q}(\lambda, \mu, x) \) is constant on each interval \((\frac{n}{q}, \frac{n+1}{q})\), \( n \in \mathbb{Z} \), and we use formula (6). Gathering as above the various classes of \( k \) modulo \( q \), we obtain the sums

\[
\frac{1 - \lambda}{\lambda} \Phi_{p/q}(\lambda, \mu, 0) = \sum_{k \geq 0} \left( [(k + 1) \frac{p}{q}] - [k \frac{p}{q}] \right) \lambda^k \mu^{[k \frac{p}{q}]} = \frac{S}{1 - \lambda^q \mu^p} = \frac{1 + \mu S}{1 - \lambda^q \mu^p} = 1 + \mu \sigma^-,
\]

since \( [k \frac{p}{q}] = [k \frac{p}{q}] + 1 \) for \( 1 \leq k \leq q - 1 \), \( [(q - 1) \frac{p}{q}] = p \) and by using (8).
Similarly, we have the equalities
\[
\frac{1 - \lambda}{\lambda} \Phi_{p/q}(\lambda, \mu, \frac{1}{q} - 1) = \sum_{k=0}^{\lfloor \frac{kp+1}{q} \rfloor - 1} \left( \left( \frac{(k+1)p+1}{q} \right) - \left( \frac{kp+1}{q} \right) \right) \lambda^k \mu^{\lfloor \frac{kp+1}{q} \rfloor - 1}
\]

\[
= \frac{\sum_{k=0}^{q-1} \left( \left( \frac{(k+1)p+1}{q} \right) - \left( \frac{kp+1}{q} \right) \right) \lambda^k \mu^{\lfloor \frac{kp+1}{q} \rfloor - 1}}{1 - \lambda^q \mu^p}
\]

\[
= \frac{\lambda^{q-1} \mu^{p-1} + \sum_{k=1}^{q-2} \left( \left( \frac{(k+1)p}{q} \right) - \left( \frac{kp}{q} \right) \right) \lambda^k \mu^{\lfloor \frac{kp}{q} \rfloor}}{1 - \lambda^q \mu^p}
\]

\[
= \frac{\lambda^{q-1} \mu^{p-1}}{1 - \lambda^q \mu^p} + \sigma,
\]

since \( \left\lfloor \frac{kp+1}{q} \right\rfloor = \left\lfloor \frac{kp}{q} \right\rfloor + 1 \). For the value \( x = \frac{p}{q} - 1 \), we find
\[
\frac{1 - \lambda}{\lambda} \Phi_{p/q}(\lambda, \mu, \frac{p}{q} - 1) = \sum_{k=0}^{\lfloor \frac{kp+1}{q} \rfloor - 1} \left( \left( \frac{(k+1)p}{q} \right) - \left( \frac{kp}{q} \right) \right) \lambda^k \mu^{\lfloor \frac{kp+1}{q} \rfloor - 1}
\]

\[
= \frac{1}{\lambda \mu} \sum_{k=1}^{q-1} \left( \left( \frac{(k+1)p}{q} \right) - \left( \frac{kp}{q} \right) \right) \lambda^k \mu^{\lfloor \frac{kp}{q} \rfloor}
\]

\[
= (1 + \mu \sigma^-) - \frac{\sigma^-}{\lambda},
\]

taking again the computations used for \( \Phi_{p/q}(\lambda, \mu, 0) \). Finally, we get
\[
\frac{1 - \lambda}{\lambda} \Phi_{p/q}(\lambda, \mu, \frac{p+1}{q} - 1) = \sum_{k=0}^{\lfloor \frac{(k+2)p+1}{q} \rfloor - 1} \left( \left( \frac{(k+1)p+1}{q} \right) - \left( \frac{(k+1)p+1}{q} \right) \right) \lambda^k \mu^{\lfloor \frac{(k+1)p+1}{q} \rfloor - 1}
\]

\[
= \frac{1}{\lambda} \sum_{k=1}^{\lfloor \frac{(k+1)p+1}{q} \rfloor - 1} \left( \left( \frac{(k+1)p+1}{q} \right) - \left( \frac{kp+1}{q} \right) \right) \lambda^k \mu^{\lfloor \frac{kp+1}{q} \rfloor - 1} = \frac{\sigma}{\lambda},
\]

by the above computation of \( \Phi_{p/q}(\lambda, \mu, \frac{1}{q} - 1) \). The formulae (9) to (12) are established.

We now use Lemma 9 in order to estimate values of \( \phi \). We have
\[
\phi(0) = \frac{\delta}{1 - \lambda} - \frac{\delta - \mu(\lambda + \delta - 1)}{\lambda} \Phi_{p/q}(\lambda, \mu, 0).
\]

Then (9) yields
\[
(1 - \lambda)\phi(0) = \delta - (\delta - \mu(\lambda + \delta - 1))(1 + \mu \sigma^-)
\]

\[
= \mu \left( \delta(1 + (\mu - 1)\sigma^-) - (1 - \lambda)(1 + \mu \sigma^-) \right)
\]

\[
= \mu(1 + (\mu - 1)\sigma^-)(\delta - \delta(\lambda, \mu, (p/q)^-))
\]

since \( \delta(\lambda, \mu, (p/q)^-) = (1 - \lambda)(1 + \mu \sigma^-)/(1 + (\mu - 1)\sigma^-) \). Observe now that the factor \( 1 + (\mu - 1)\sigma^- \) is always positive. Indeed, we deduce from Lemma 8 and its corollary that
\[
0 \leq \sigma^- < \frac{1 - \lambda}{\lambda^\mu} \lim_{\rho \to \lambda, \mu} \Psi_p(\lambda, \mu) = \begin{cases} \frac{\lambda}{1 - \mu} & \text{if } \lambda \mu < 1, \\ +\infty & \text{if } \lambda \mu \geq 1. \end{cases}
\]
Therefore $1 + (\mu - 1)\sigma^-$ is bounded from below by 1 when $\mu \geq 1$ and by $(1 - \lambda)/(1 - \lambda \mu)$ when $\mu < 1$. It follows that $\phi(0)$ is $\geq 0$ if and only if $\delta \geq \delta(\lambda, \mu, (p/q)^-)$.

For the value $\phi(0^-)$, observe that $-\{y\} = -1 - y$ tends to $-1$ from above when $y < 0$ tends to $0$. Lemma 9 provides now the formula

$$\phi(0^-) = -1 + \frac{\delta}{1 - \lambda} - \frac{\delta - \mu(\lambda + \delta - 1)}{\lambda} \Phi_{\rho, q}(\lambda, \mu, (1^-)).$$

Then (10) yields

$$(1 - \lambda)\phi(0^-) = (\lambda + \delta - 1)(1 + \mu \sigma) - \delta \sigma$$

(14)

$$= \delta(1 + (\mu - 1)\sigma) - (1 - \lambda)(1 + \mu \sigma)$$

$$= (1 + (\mu - 1)\sigma)(\delta - \delta(\lambda, \mu, p/q)).$$

It follows that $\phi(0^-)$ is negative if and only if $\delta < \delta(\lambda, \mu, p/q)$.

We now deal with the lower bound at the point $1 - p/q$. Using Lemma 9 and (11), we find

$$\phi\left(1 - \frac{p}{q}\right) = \frac{\delta}{1 - \lambda} - \frac{\delta - \mu(\lambda + \delta - 1)}{\lambda} \Phi_{\rho, q}(\lambda, \mu, p/q - 1)$$

$$= \frac{\delta}{1 - \lambda} - \frac{\delta - \mu(\lambda + \delta - 1)}{\lambda} \frac{\sigma^-}{1 - \lambda}$$

$$= \frac{\delta(\lambda + (\mu - 1)\sigma^-) - (1 - \lambda)\mu \sigma^-}{\lambda(1 - \lambda)}$$

$$\geq \frac{1 - \delta}{\lambda},$$

since the expression

$$\delta(1 + (\mu - 1)\sigma^-) - (1 - \lambda)(1 + \mu \sigma^-) = (1 + (\mu - 1)\sigma^-)(\delta - \delta(\lambda, \mu, (p/q)^-))$$

appearing above in the numerator is $\geq 0$.

The computations are similar for the left limit at the point $1 - p/q$. Using (12), we find

$$\phi\left(1 - \frac{p}{q}\right)^- = \frac{\delta}{1 - \lambda} - \frac{\delta - \mu(\lambda + \delta - 1)}{\lambda} \Phi_{\rho, q}\left(\lambda, \mu, \left(\frac{p}{q} - 1\right)^-\right)$$

$$= \frac{\delta}{1 - \lambda} - \frac{\delta - \mu(\lambda + \delta - 1)}{\lambda} \frac{\sigma^-}{1 - \lambda} = \frac{\delta(\lambda + (\mu - 1)\sigma) - (1 - \lambda)\mu \sigma}{\lambda(1 - \lambda)}$$

$$= \frac{(1 - \lambda)(1 - \delta) + \delta(1 + (\mu - 1)\sigma^-) - (1 - \lambda)(1 + \mu \sigma^-)}{\lambda(1 - \lambda)}$$

$$< \frac{1 - \delta}{\lambda},$$

since

$$\delta(1 + (\mu - 1)\sigma) - (1 - \lambda)(1 + \mu \sigma) = (1 + (\mu - 1)\sigma)(\delta - \delta(\lambda, \mu, p/q)) < 0.$$
4. The Lift $F$

Let $F : \mathbb{R} \to \mathbb{R}$ be the real function defined by

$$F(x) = F_{\lambda, \mu, \delta}(x) = \begin{cases} \lambda x + \delta + (1 - \lambda) \lfloor x \rfloor & \text{if } 0 \leq \{x\} < \eta, \\ \mu(\lambda x + \delta - 1) + 1 + (1 - \lambda \mu) \lfloor x \rfloor & \text{if } \eta \leq \{x\} < 1. \end{cases}$$

Then $F$ is a lift of $f$, meaning that $F$ satisfies the following properties:

(i) For every $x \in \mathbb{R}$, we have $\{F(x)\} = f(\{x\})$.

(ii) $F(x + 1) = F(x) + 1$, for every $x \in \mathbb{R}$.

(iii) $F$ is an increasing function on $\mathbb{R}$ which is continuous on each interval of $\mathbb{R} \sim \mathbb{Z}$ and right continuous everywhere.

![Figure 5. Plot of $F_{1/2,1/2,3/4}(x)$ in the interval $-1 \leq x < 1$](image)

Let $x \in \mathbb{R}$ and let $(x_k)_{k \geq 0}$ be the forward orbit of $x$ by $F$, where $x_k = F^k(x)$ stands for the $k$-th iterate of the function $F$. When $\{x\}$ belongs to $C = \bigcap_{k \geq 1} f^k(\mathbb{I})$, we denote, moreover, by $(x_{-k})_{k \geq 0}$ the backward orbit of $x$ by $F$, where $x_{-k} = F^{-k}(x)$ is the $k$-th preimage of $x$ by $F$. This makes sense since $\{x_{-k}\} = f^{-k}(\{x\})$ is the $k$-th preimage of $\{x\}$ by $f$ and $x_{-k-1} = F^{-1}(x_{-k})$ is the inverse image of $x_{-k}$ by the injective map $F$, noting that $x_{-k} \in F(\mathbb{R}) = f(\mathbb{I}) + \mathbb{Z}$ for all $k \geq 0$.

A fundamental property is that any forward orbit $(x_k)_{k \geq 0}$ can be computed explicitly in terms of its initial point $x$ and of the associated coding sequence

$$[x_{k+1}] - [x_k] \in [0, 1].$$

We have $[x_{k+1}] = [x_k]$ iff $\{x_k\} \in [0, \eta)$ while $[x_{k+1}] = [x_k] + 1$ iff $\{x_k\} \in [\eta, 1)$. It turns out that, for any orbit, this coding sequence is either periodic when the rotation number $\rho$ is rational, or a Sturmian sequence of slope $\rho$ in the irrational case. We have the following explicit recursion formulae which motivate our definition of the conjugacy $\phi$:
Lemma 13. Let $x \in \mathbb{R}$ and let $(x_k)_{k \geq 0}$ be the forward orbit of $x$ by $F$. For any non-negative integers $m$ and $n$, we have the relation

$$x_{m+n} = [x_{m+n}] + \lambda^n \mu^{[x_{m+n}] - [x_m]} \left( [x_m] - \frac{1 - \delta}{\lambda} \right) + \frac{1 - \delta}{\lambda} + \sum_{k=0}^{n-1} \lambda^k \mu^{[x_{m+n}-k]} \left( \frac{\lambda + \delta - 1}{\lambda} + [x_{m+n}-(k+1)] - [x_{m+n}-k] \right).$$

Moreover, assume that $[x] \in C$. Let $(x_{-k})_{k \geq 0}$ be the backward orbit of $x$ by $F$ and assume that there exist two real numbers $y$ and $\rho$ with $0 < \rho < r_{\lambda, \mu}$ such that $[x_k] = [y + k\rho]$ for all integer $k \leq 0$. Then, we have the series expansion

$$x = [x] + \frac{1 - \delta}{\lambda} + \sum_{k=0}^{+\infty} \lambda^k \mu^{[y] - [y - k\rho]} \left( \frac{\lambda + \delta - 1}{\lambda} + [y - (k+1)\rho] - [y - k\rho] \right) = \phi_{\lambda, \mu, \delta, \rho}(y).$$

Proof. Notice that $[x_{k+1}] - [x_k] \in \{0, 1\}$ for any integer $k$. Put

$$a_k := \lambda \mu^{[x_{k+1}] - [x_k]} = \begin{cases} \lambda & \text{if } [x_k] = [x_{k+1}], \\ \lambda \mu & \text{if } [x_{k+1}] = [x_k] + 1 \end{cases}$$

$$b_k := 1 + \mu^{[x_{k+1}] - [x_k]} (\delta - 1) + (1 - \lambda \mu^{[x_{k+1}] - [x_k]} ) [x_k]$$

$$= \begin{cases} \delta + (1 - \lambda) [x_k] & \text{if } [x_k] = [x_{k+1}], \\ \mu (\delta - 1) + 1 + (1 - \lambda \mu) [x_k] & \text{if } [x_{k+1}] = [x_k] + 1, \end{cases}$$

so that $F(x_k) = a_k x_k + b_k$. Thus $x_{k+1} = a_k x_k + b_k$. Let $n$ be a positive integer. Composing these affine relations for $k = m, \ldots, m+n-1$, we obtain by induction on $n$ the formula

$$x_{m+n} = a_m \cdots a_{m+n-1} x_m + \sum_{k=0}^{n-1} a_{m+k+1} \cdots a_{m+n-1} b_{m+k} = v_0 x_m + \sum_{k=0}^{n-1} v_{k+1} b_{m+k},$$

where we have set

$$v_k := \prod_{j=k}^{n-1} a_{m+j} = \lambda^{n-k} \mu^{[x_{m+n}] - [x_{m+k}]} \quad \text{for } 0 \leq k \leq n,$$

noting that $v_{k+1} a_{m+k} = v_k$. We display the terms

$$v_{k+1} b_{m+k} = v_{k+1} + \frac{\delta - 1}{\lambda} v_k + (v_{k+1} - v_k) [x_{m+k}].$$
appearing in the above sum. Note that \( v_n = 1 \). Thus, by Abel’s summation, we find

\[
x_{m+n} = v_0 x_m + \sum_{k=0}^{n-1} v_{k+1} + \frac{\delta - 1}{\lambda} v_k + (v_{k+1} - v_k) [x_{m+k}]
\]

\[
= v_0 \left( x_m - [x_m] + \frac{\delta - 1}{\lambda} \right) + [x_{m+n-1}] + 1 + \frac{\lambda + \delta - 1}{\lambda} \sum_{k=1}^{n-1} v_k
\]

\[
+ \sum_{k=1}^{n-1} v_k ( [x_{m+k-1}] - [x_{m+k}] )
\]

\[
= v_0 \left( (x_m) + \frac{\delta - 1}{\lambda} \right) + [x_{m+n}] + \frac{1 - \delta}{\lambda}
\]

\[
+ \sum_{k=1}^{n} v_k \left( \frac{\lambda + \delta - 1}{\lambda} + [x_{m+k-1}] - [x_{m+k}] \right).
\]

Replacing the index of summation \( k \) by \( n - k \) in the last sum, we find

\[
\sum_{k=1}^{n} v_k \left( \frac{\lambda + \delta - 1}{\lambda} + [x_{m+k-1}] - [x_{m+k}] \right)
\]

\[
= \sum_{k=0}^{n-1} v_{n-k} \left( \frac{\lambda + \delta - 1}{\lambda} + [x_{m+n-(k+1)}] - [x_{m+n-k}] \right)
\]

\[
= \sum_{k=0}^{n-1} \lambda^k \mu^{[x_{m+n}] - [x_{m+n-k}]} \left( \frac{\lambda + \delta - 1}{\lambda} + [x_{m+n-(k+1)}] - [x_{m+n-k}] \right).
\]

The first assertion is established.

For the second one, observe that the first assertion remains valid for \( m \) negative when \( \{x\} \) belongs to \( C \), since then \( x \) has a preimage \( F^m(x) \) and we apply the formula at this point. Choosing \( m = -n \) and letting \( n \) tend to infinity, we get the stated series expansion, noting that \( v_0 = \lambda^n \mu^{[x]} - [x_n] = \lambda^n \mu^{[n\rho-y]} \) tends to 0 as \( n \) tends to infinity, since \( \rho < r_{\lambda,\mu} \). \( \square \)

The next result is crucial in our approach. It shows that \( \phi \) satisfies a functional equation as in Theorem 6.

**Lemma 14.** Let \( \lambda, \mu, \delta \) and \( \rho \) be real numbers such that \( 0 < \lambda < 1, \mu > 0, 0 < \rho < r_{\lambda,\mu} \). We assume that \( \delta = \delta(\lambda, \mu, \rho) \) when \( \rho \) is irrational, or that \( \delta \) belongs to the interval \( (2) \) when \( \rho = p/q \) is rational. Put \( \phi = \phi_{\lambda,\mu,\delta,\rho} \) and \( F = F_{\lambda,\mu,\delta} \). Then, the relations

\[
[\phi(y)] = [y] \quad \text{and} \quad F(\phi(y)) = \phi(y + \rho)
\]

hold for any real number \( y \). Thus, the \( F \)-orbit \( (x_k)_{k \in \mathbb{Z}} \) of \( x = \phi(y) \) is given by the sequence

\[
x_k = \phi(y + k\rho), \ \forall k \in \mathbb{Z}.
\]

**Proof:** We first show that \( 0 \leq \phi(y) < 1 \) when \( 0 \leq y < 1 \). In the case \( \rho \) irrational, Lemma 11 gives \( \phi(0) = 0 \). Thus \( \phi(1) = \phi(0) + 1 = 1 \) and the corollary of Lemma 9
asserts that the function $\phi$ is strictly increasing. Therefore

$$0 = \phi(0) \leq \phi(y) < \phi(1) = 1.$$ 

In the rational case, the function $\phi$ is non-increasing and constant on each interval $\left(\frac{n}{n+1}, \frac{n+1}{n}\right)$, $n \in \mathbb{Z}$. Now, we know that $\phi(0) \geq 0$ and $\phi(0^-) < 0$ by Lemma 12. Therefore

$$0 \leq \phi(0) \leq \phi(y) \leq \phi(1^-) = \phi(0^-) + 1 < 1.$$ 

For any $y \in \mathbb{R}$, we can write

$$\phi(y) = \phi([y] + \{y\}) = [y] + \phi((y)).$$ 

We have thus proved that $\{\phi(y)\} = \{y\}$ and $\{\phi(y)\} = \phi((y))$ for all real number $y$.

We now prove the relation $F(\phi(y)) = \phi(y + \rho)$. By definition of $F$, we have to deal with two expressions for the value of $F(\phi(y))$ depending whether the fractional part $\{\phi(y)\}$ is smaller than $\eta = (1 - \delta)/\lambda$ or not. But $\{\phi(y)\} = \phi((y))$ and Lemmata 11 and 12 yield that $\phi((y))$ belongs the interval $[0, \eta)$ when $\{y\} < 1 - \rho$, and to the other interval $[\eta, 1)$ when $\{y\} \geq 1 - \rho$, since $\phi$ is non-decreasing. The computation splits into two cases.

Suppose first that $\{y\} < 1 - \rho$. Then $\{y + \rho\} = \{y\}$. Moreover, $\{\phi(y)\} = \phi((y)) < \eta$ by Lemmata 11 and 12 and the increasing monotonicity of the function $\phi$. Using the expression of $F$ in the intervals $[n, n + \eta), n \in \mathbb{Z}$, we obtain the equalities:

$$F(\phi(y)) = \lambda \phi(y) + \delta + (1 - \lambda)\{\phi(y)\}$$

$$= \delta + (1 - \lambda)\{y\}$$

$$+ \lambda \left(\{y\} + 1 - \frac{\delta}{\lambda} + \sum_{k=0}^{+\infty} \lambda^k \mu_{\{y\} - \{y - k\rho\}} \left(\frac{\lambda + \delta - 1}{\lambda} + \{y - (k + 1)\rho\} - \{y - k\rho\}\right)\right)$$

$$= \{y\} + 1 + \sum_{k=0}^{+\infty} \lambda^k \mu_{\{y\} - \{y - k\rho\}} \left(\frac{\lambda + \delta - 1}{\lambda} + \{y - (k + 1)\rho\} - \{y - k\rho\}\right)$$

$$= \{y\} + 1 + \sum_{k=1}^{+\infty} \lambda^k \mu_{\{y\} - \{y - (k-1)\rho\}} \left(\frac{\lambda + \delta - 1}{\lambda} + \{y - k\rho\} - \{y - (k-1)\rho\}\right)$$

$$= \{y\} + \{y + \rho\} + 1$$

$$+ \sum_{k=1}^{+\infty} \lambda^k \mu_{\{y + \rho\} - \{y + \rho - k\rho\}} \left(\frac{\lambda + \delta - 1}{\lambda} + \{y + \rho - (k + 1)\rho\} - \{y + \rho - k\rho\}\right)$$

$$= \{y + \rho\} + 1 - \frac{\lambda + \delta - 1}{\lambda}$$

$$+ \sum_{k=0}^{+\infty} \lambda^k \mu_{\{y + \rho\} - \{y + \rho - k\rho\}} \left(\frac{\lambda + \delta - 1}{\lambda} + \{y + \rho - (k + 1)\rho\} - \{y + \rho - k\rho\}\right)$$

$$= \phi(y + \rho).$$

The case $\{y\} \geq 1 - \rho$ is similar. Then $\{y + \rho\} = \{y\} + 1$ and $\{\phi(y)\} \geq \eta$ by Lemmata 11 and 12. We now use the expression of $F$ in the intervals $[n + \eta, n + 1), n \in \mathbb{Z}$. 
We then obtain the equalities:
\[ F(\phi(y)) = \lambda \mu \phi(y) + \mu \delta + 1 - \mu + (1 - \lambda \mu) [\phi(y)] \]
\[ = \mu \delta + 1 - \mu + (1 - \lambda \mu) [y] \]
\[ + \lambda \mu \left( [y] + \frac{1 - \delta}{\lambda} + \sum_{k=0}^{+\infty} \lambda^k \mu [y]^{1 - [y - \rho]} \left( \frac{\lambda + \delta - 1}{\lambda} + [y - (k + 1) \rho] - [y - k \rho] \right) \right) \]
\[ = [y] + 1 + \sum_{k=0}^{+\infty} \lambda^{k+1} \mu [y]^{1 - [y - \rho]} \left( \frac{\lambda + \delta - 1}{\lambda} + [y - (k + 1) \rho] - [y - k \rho] \right) \]
\[ = [y] + 1 + \sum_{k=1}^{+\infty} \lambda^k \mu [y]^{1 - [y - \rho]} \left( \frac{\lambda + \delta - 1}{\lambda} + [y - (k - 1) \rho] - [y - (k - 1) \rho] \right) \]
\[ = [y + \rho] - \left( \frac{\lambda + \delta - 1}{\lambda} - 1 \right) \]
\[ + \sum_{k=0}^{+\infty} \lambda^k \mu [y + \rho] - [y + \rho - \rho] \left( \frac{\lambda + \delta - 1}{\lambda} + [y + \rho - (k + 1) \rho] - [y + \rho - k \rho] \right) \]
\[ = \phi([y + \rho]). \]

5. Proof of Theorem 3 and its Corollary

Let \( x \in \mathbb{R} \) and let \( (x_k)_{k \geq 0} \) be the forward orbit of \( x \) by \( F \). It is known (see [24]) that the limit
\[ \rho_{\lambda, \mu, \delta} = \lim_{k \to \infty} \frac{x_k}{k} \]
exists and does not depend on the initial point \( x \). The number \( \rho_{\lambda, \mu, \delta} \) is called the rotation number of the map \( f = f_{\lambda, \mu, \delta} \).

Fix \( \lambda \) and \( \mu \) with \( 0 < \lambda < 1, \mu > 0 \). Let \( \delta \) be a real number in the interval \( 1 - \lambda < \delta < d_{\lambda, \mu} \). By the corollary of Lemma 8, the following alternative holds. Either \( \delta \) belongs to the image of the interval \( 0 < \rho < r_{\lambda, \mu} \) by the function \( \rho \mapsto \delta(\lambda, \mu, \rho) \), or
\[ \delta(\lambda, \mu, (p/q)^-) \leq \delta < \delta(\lambda, \mu, p/q) \]
for some rational number \( p/q \) with \( 0 < p/q < 1 \) (these intervals are the jumps of the increasing function \( \rho \mapsto \delta(\lambda, \mu, \rho) \)). In the latter case, Lemma 14 yields that \( \rho_{\lambda, \mu, \delta} = p/q \). Indeed, we select an initial point \( x \) of the form \( x = \phi_{\lambda, \mu, \delta, p/q}(y) \) for an arbitrary \( y \in \mathbb{R} \), so that \( x_k = \phi_{\lambda, \mu, \delta, p/q}(y + kp/q) \) for every integer \( k \geq 0 \). Then,
\[
\rho_{\lambda, \mu, \delta} = \lim_{k \to \infty} \frac{x_k}{k} = \lim_{k \to \infty} \frac{\lfloor x_k \rfloor}{k} = \lim_{k \to \infty} \frac{\lfloor y + k \frac{p}{q} \rfloor}{k} = \frac{p}{q}.
\]

It remains to deal with parameters \( \delta \) in the image, in other words \( \delta = \delta(\lambda, \mu, \rho) \) for some \( 0 < \rho < r_{\lambda, \mu} \). When \( \rho \) is irrational, Lemma 14 yields as well that \( \rho_{\lambda, \mu, \delta} = \rho \). When \( \rho = p/q \) is rational, we may use a general argument of continuity. Since \( F \) is strictly increasing, [25, Theorem 5.8] tells us that the rotation number \( \rho_{\lambda, \mu, \delta} \) is a continuous function of the parameter \( \delta \). Thus
\[ \rho_{\lambda, \mu, \delta} = \lim_{\delta' \to \delta} \rho_{\lambda, \mu, \delta'} = \frac{p}{q}. \]
Since we have already proved that $\rho_{\lambda, \mu, \delta'}$ is constant and equal to $p/q$ when $\delta'$ is located in the right open interval

$$ \delta(\lambda, \mu, (p/q)^-) \leq \delta' < \delta(\lambda, \mu, p/q). $$

We express now $\delta(\lambda, \mu, p/q)$ and $\delta(\lambda, \mu, (p/q)^-)$ in term of the finite sum $S$. Recalling formula (8), we obtain

$$ \delta(\lambda, \mu, p/q) = \frac{(1 - \lambda)(1 + \mu \sigma)}{1 + (\mu - 1)\sigma} = \frac{(1 - \lambda)(1 + \mu S + \lambda^{q-1} \mu^p (1 - \lambda))}{1 + (\mu - 1) S + \lambda^{q-1} \mu^p (\mu - \lambda \mu - 1)} $$

and

$$ \delta(\lambda, \mu, (p/q)^-) = \frac{(1 - \lambda)(1 + \mu \sigma^-)}{1 + (\mu - 1)\sigma^-} = \frac{(1 - \lambda)(1 + \mu S)}{1 + (\mu - 1) S - \lambda^{q-1} \mu^p (\mu - \lambda \mu - 1)}. $$

It remains to prove the corollary. By [25, Theorem 5.8], we know that the rotation number of the homeomorphism $f_{\lambda, \mu, \delta}$ equals the left limit of $\rho_{\lambda, \mu, \delta}$ as $\delta$ tends to $d_{\lambda, \mu}$ from below. Now, Theorem 3 tells us that this left limit equals $r_{\lambda, \mu}$.

6. Irrational Rotation Number

We prove part (i) of Theorem 6 and we give furthermore a description of the iterated images $f^n(I)$ when the rotation number $\rho$ is irrational.

In this case, the function $\phi$ is strictly increasing on $\mathbb{R}$ and has jumps at the points $l \rho + \mathbb{Z}$, $l \geq 1$. Put

$$ \xi_l = \phi(l \rho), \quad \text{and} \quad \xi_l^- = \phi((l \rho)^-), \quad l \geq 1. $$

All the intervals $[\xi_l^-, \xi_l], l \geq 1$, are pairwise disjoint and contained in $I = (0, 1)$.

**Proposition 15.** For any integer $n \geq 1$, we have the decomposition into disjoint intervals

$$ f^n(I) = I \sim \bigcup_{l=1}^n [\xi_l^-, \xi_l], $$

and the formulae

$$ f^{l}(0) = \frac{1 - \delta}{\lambda} + \sum_{k=0}^l \lambda^{l-k} \mu^{\lfloor kp \rfloor - \lfloor kp \rfloor} \left( \frac{\lambda + \delta - 1}{\lambda} + \lfloor (k - 1) \rho \rfloor - \lfloor k \rho \rfloor \right), $$

$$ f^{l}(1^-) = f^{l}(0) - \lambda^{l-1} \mu^{\lfloor l \rho \rfloor} (\delta - \mu (\lambda + \delta - 1)). $$

Moreover, the set equalities

$$ C := \bigcap_{n=1} f^n(I) = \phi(I) = I \sim \bigcup_{l \geq 1} [\xi_l^-, \xi_l] \quad \text{and} \quad \overline{C} = \overline{\phi(I)} = I \sim \bigcup_{l \geq 1} (\xi_l^-, \xi_l) $$

hold. The set $\overline{C}$ is topologically homeomorphic to a Cantor set.

**Proof.** Lemma 14 shows that for any $k \geq 0$ and any $l \geq 1$, we have the equalities

$$ F^k(\xi_l) = \phi((l \rho) + k \rho) = \phi((k + l) \rho - (l \rho)) $$

$$ = \phi((k + l) \rho) + \lfloor (k + l) \rho \rfloor - \lfloor l \rho \rfloor $$

$$ = \xi_{k+l} + \lfloor (k + l) \rho \rfloor - \lfloor l \rho \rfloor = \xi_{k+l} + \lfloor l \rho \rfloor + k \rho. $$
and, since $|(k+l)\rho| \neq 0$,

$$F^k(\xi^-) = \phi((l\rho)^- + k\rho) = \phi(((k+l)\rho)^- - |l\rho|)$$

$$= \phi(((k+l)\rho)^- + [(k+l)\rho] - |l\rho|)$$

$$= \xi^-_{k+l} + [(k+l)\rho] - |l\rho| = \xi^-_{k+l} + |l\rho| + k\rho,$$

where $F^k$ stands for the $k$-th iterate of $F$. Since $F$ is increasing and continuous on $\mathbb{R} \sim \mathbb{Z}$, it follows that

$$F^k((\xi^-_l, \xi_l)) = [\xi^-_{k+l}, \xi_{k+l}] + |l\rho| + kp,$$

so that any number $z \in F^k((\xi^-_l, \xi_l))$ has integer part $|z| = |(l\rho) + kp|$.

We first show that

$$\phi(I) = I \sim \bigcup_{l \geq 1} [\xi^-_l, \xi_l].$$

Since $\phi$ is right continuous and increasing, no point of $\phi(I)$ is located in an interval of the form $[\xi^-_l, \xi_l)$, $l \geq 1$. Thus, we have the inclusion $\phi(I) \subseteq I \sim \bigcup_{l \geq 1} [\xi^-_l, \xi_l]$. The reversed inclusion $\phi(I) \supseteq I \sim \bigcup_{l \geq 1} [\xi^-_l, \xi_l]$ follows straightforwardly from the right continuity of $\phi$. Indeed, let $x \in I$ which is located outside the intervals $[\xi^-_l, \xi_l)$, $l \geq 1$. For every $n \geq 1$, define an index $l_n$ among the integers $1 \leq l \leq n$ for which $x \leq \xi_l$ and $\xi_{l_n}$ is the closest to $x$. It is readily seen that the decreasing sequence $(l_n\rho)$ converges to a number $y$ and that $x = \phi(y)$ by right continuity of $\phi$.

We know by Lemma 11 that the critical point $\eta = \phi(1 - \rho)$ is located in the image $\phi(I)$. In particular, this critical point $\eta$ does not belong to any interval $[\xi^-_l, \xi_l)$, $l \geq 1$. The function $f$ is thus continuous on each interval $[\xi^-_l, \xi_l)$, so that we deduce from (15) that

$$f((\xi^-_l, \xi_l)) = [\xi^-_{l+1}, \xi_{l+1})$$

by reducing modulo 1. Now, Lemmata 11 and 14 yield the equalities

$$\xi_1 = \phi(\rho) = F(\phi(0)) = F(0) = \delta \quad \text{and} \quad \xi^-_1 = \phi(\rho^-) = F(0^-) = \mu(\lambda + \delta - 1).$$

Looking at Figure 1, we immediately observe that $f(I) = I \sim [\xi^-_1, \xi_1)$, as announced. Taking now the image by $f$ and using (16), since $f$ is injective, we find

$$f^2(I) = f((\xi^-_1, \xi_1)) = I \sim ((\xi^-_1, \xi_1) \cup [\xi^-_2, \xi_2)).$$

Arguing by induction on $n \geq 1$, we thus deduce from (16) the required equality

$$f^n(I) = I \sim \bigcup_{l=1}^n [\xi^-_l, \xi_l].$$

Letting $n$ tend to infinity, we finally obtain that

$$\bigcap_{n \geq 1} f^n(I) = I \sim \bigcup_{l=1} \{\xi^-_l, \xi_l\} = \phi(I).$$
It remains to establish the explicit formulae giving $\xi_l$ and $\xi_l^-$. Lemma 13 and Lemma 14 deliver the expression

$$\phi(l\rho) = [l\rho] + \lambda^l \mu^{[l\rho]} \left( [0] - \frac{1-\delta}{\lambda} \right) + \frac{1-\delta}{\lambda}$$

$$+ \sum_{k=0}^{l-1} \lambda^k \mu^{[l\rho]-[(l-k)\rho]} \left( \frac{\lambda + \delta - 1}{\lambda} + [(l-k)\rho] - [(l-k)\rho] \right)$$

$$= [l\rho] + \frac{1-\delta}{\lambda} (1-\lambda^l \mu^{[l\rho]}) + \sum_{k=1}^{l} \lambda^{l-k} \mu^{[(l-k)\rho]} \left( \frac{\lambda + \delta - 1}{\lambda} + [(k-1)\rho] - [k\rho] \right)$$

$$= [l\rho] + \frac{1-\delta}{\lambda} + \sum_{k=0}^{l-1} \lambda^{l-k} \mu^{[l\rho]-[(l-k)\rho]} \left( \frac{\lambda + \delta - 1}{\lambda} + [(k-1)\rho] - [k\rho] \right),$$

since $[\phi(k\rho)] = [k\rho]$ for every integer $k$. Taking the fractional part, we obtain the formula

$$\xi_l = \phi([l\rho]) = \phi([l\rho]) = \frac{1-\delta}{\lambda} + \sum_{k=0}^{l-1} \lambda^{l-k} \mu^{[l\rho]-[(l-k)\rho]} \left( \frac{\lambda + \delta - 1}{\lambda} + [(k-1)\rho] - [k\rho] \right).$$

The equality $\xi_l = f^l(0)$ immediately follows from Lemmata 11 and 14.

Similarly $f^l(1^-) = \xi_l^- = \phi([l\rho^-])$. We have $[0^-] = 1$, $[0^-] = -1$ and $[k\rho^-] = [k\rho]$ for any integer $k \geq 1$. Then, Lemma 13 gives

$$\phi([l\rho^-]) = [l\rho] + \lambda^l \mu^{[l\rho]} + \left( 1 - \frac{1-\delta}{\lambda} \right) + \frac{1-\delta}{\lambda} + \lambda^{l-1} \mu^{[l\rho]} \left( \frac{\lambda + \delta - 1}{\lambda} - 1 \right)$$

$$+ \sum_{k=0}^{l-2} \lambda^k \mu^{[l\rho]-[(l-k)\rho]} \left( \frac{\lambda + \delta - 1}{\lambda} + [(l-k)\rho] - [(l-k)\rho] \right)$$

$$= \phi([l\rho]) - \lambda^{l-1} \mu^{[l\rho]} (\delta - \mu(\lambda + \delta - 1)).$$

As a corollary of the above formulae, let us briefly prove that $\overline{C}$ is a Cantor set. The Lebesgue measure of $\overline{C} = I = \bigcup_{l \geq 1} (\xi_l, \xi_l^-)$ is equal to

$$1 - \sum_{l \geq 1} \xi_l - \xi_l^- = 1 - (\delta - \mu(\lambda + \delta - 1)) \sum_{l \geq 1} \lambda^{l-1} \mu^{[l\rho]}.$$

Using (5) and (7), we easily compute the sum

$$\sum_{l \geq 1} \lambda^{l-1} \mu^{[l\rho]} = \frac{1 + (\mu - 1)\sigma}{1 - \lambda},$$

where $\sigma = \sigma(\lambda, \mu, \rho)$. On the other hand, we have

$$\delta = \frac{(1-\lambda)(1+\mu\sigma)}{1+(\mu-1)\sigma} \quad \text{so that} \quad \delta - \mu(\lambda + \delta - 1) = \frac{1-\lambda}{1+(\mu-1)\sigma}.$$

Therefore $\overline{C}$ is a null set. Consequently, it has no inner point. Moreover, $\overline{C} = \overline{\phi(I)}$ has no isolated point, since the function $\phi$ is strictly increasing and right continuous. It follows that the compact set $\overline{C}$ is homeomorphic to a Cantor set. \qed
In order to complete the proof of the assertion (i) of Theorem 6, we now show that for any point \( x \in I \), the \( \omega \)-limit set \( \omega(x) \) coincides with \( \overline{\mathcal{W}(f)} \). To that purpose, we consider the \( F \)-orbit \( x_k := F^k(x) \), \( k \geq 0 \), of \( x \). When \( x = \phi(y) \) belongs to \( \phi(I) \), Lemma 14 shows that \( x_k = \phi(y + kp) \) so that \( F^k(x) = \{ x_k \} = \phi((y + kp)) \). The sequence of fractional parts \((y + kp)! \) is dense in \( I \), since \( \rho \) is an irrational number. Thus the set \( \omega(x) \) of accumulation points of the orbit \( (F^k(x))_{k \geq 0} \) is equal to \( \overline{\mathcal{W}(f)} \). It remains to deal with points \( x \in I \) not belonging to \( \phi(I) \), that is to say \( x \in [\xi^-_l, \xi^+_l) \) for some \( l \geq 1 \). By (13), we know that \( F^k(x) \in [\xi^-_{l+k}, \xi^+_{l+k}) \). Since \( \xi_{l+k} - \xi^-_{l+k} \) tends to 0 as \( k \) tends to infinity, it follows that \( \omega(x) = \omega(\xi_l) = \overline{\mathcal{W}(f)} \).

7. Rational rotation number

We prove the statement (ii) of Theorem 6 and add a dynamical description of the iterated images of \( f = f_\lambda, \mu, \delta \). We assume throughout this section that the inequalities (2) are fulfilled for some rational number \( p/q \). Then, Theorem 3 asserts that the rotation number of \( f \) equals \( p/q \). Put \( \phi = \phi_\lambda, \mu, \xi, p/q \) and set

\[
\zeta_m = \phi \left( \frac{m}{q} \right) \quad \text{for} \quad 0 \leq m \leq q - 1.
\]

It is convenient to extend the sequence \((\zeta_m)_{0 \leq m \leq q - 1}\) by \( q \)-periodicity setting \( \zeta_m = \zeta_r \) for any integer \( m \in \mathbb{Z} \), where \( r \) is the remainder in the euclidean division of \( m \) by \( q \). Then, Lemma 14 yields the formula

\[
f(\zeta_m) = \zeta_{m + p}, \quad \forall m \in \mathbb{Z}.
\]

As \( \phi \) is non-decreasing \( \zeta_0 \leq \cdots \leq \zeta_{q-1} \), we claim that these numbers are distinct. If not, there exists \( m \) such that \( \zeta_m = \zeta_{m+1} \). Iterating \( f \) thus \( \zeta_{m+k} = \zeta_{m+k+1} \), \( \forall k \in \mathbb{Z} \), obtaining that the function \( \phi \) is constant, in contradiction, for instance, with Lemma 12. Moreover, \( \zeta_{q-1} = \phi(1^-) = \phi(0^-) + 1 < 1 \), again by Lemma 12. So, we have an increasing sequence

\[
0 \leq \xi_0 < \cdots < \xi_{q-1} < 1
\]

in \( I \). It follows that the set \( \phi(I) = \{ \xi_0, \ldots, \xi_{q-1} \} \) is an \( f \)-cycle of order \( q \), on which \( f \) acts by the substitution \( m \rightarrow m + p \) modulo \( q \).

Recall that \( f(\eta) = 0 \) and \( f(\eta^-) = 1 \), where \( \eta = (1 - \delta)/\lambda \) is the critical point of the map \( f \). Moreover, Lemma 12 shows that \( \xi_{q-p-1} < \eta < \xi_{q-p} \). If \( \eta \) does not belong to \( \phi(I) \), we have strict inequalities \( \xi_{q-p-1} < \eta < \xi_{q-p} \) and \( f(\xi_{q-p}) = \xi_0 > 0 \). Otherwise \( \eta = \xi_{q-p} \) and \( \xi_0 = f(\eta) = 0 \). The latter case occurs only when \( \delta \) coincides with the left end point of the interval (2). Indeed (13) shows that \( \phi(0) \) vanishes if and only if \( \delta = \delta(\lambda, \mu, (p/q)^-) \).

\[
\begin{array}{cccccccccccc}
0 & \xi_0 & \xi_1 & \xi_{q-p-1} & \xi_{q-p} & \xi_{q-1} & 1 & 0 & \xi_0 & \xi_1 & \xi_2 & \cdots & \xi_{q-1} & 1 \\
& & & & & & & & & & & & \\
& & & \eta & & & & & & & & &
\end{array}
\]

Figure 6. Case \( \zeta_0 > 0 \) and Case \( \zeta_0 = 0 \)
The next proposition provides a partition of the image \( f^n(I) \), \( n \geq 1 \), into disjoint intervals. It is convenient to consider circular intervals (or circle arcs identifying \( I \) with \( \mathbb{R}/\mathbb{Z} \)). For any \( a, b \) both belonging to \( I \), we set

\[
[a, b) = \begin{cases} 
\text{the usual interval } [a, b) & \text{if } a < b \\
[0, b) \cup (a, 1) & \text{if } a > b.
\end{cases}
\]

We write for instance \([\zeta_{q-1}, \zeta_0) = [0, \zeta_0) \cup [\zeta_{q-1}, 1)\).]

**Proposition 16.** For any integer \( l \geq 1 \), the circular interval \([f^l(1^-), f^l(0)]\) is contained in \([\zeta_{lp-1}, \zeta_{lp}) = [f^l(\zeta_{q-1}), f^l(\zeta_0)]\). We have \( f^l(1^-) < f^l(0) \) when \( l \) is not divisible by \( q \) and \( f^l(1^-) > f^l(0) \) when \( l \) is a multiple of \( q \). The decomposition into disjoint intervals

\[
f^n(I) = I \sim \bigcup_{l=1}^n [f^l(1^-), f^l(0)], \quad \text{when } 0 \leq n \leq q-1,
\]

and

\[
f^n(I) = I \sim \bigcup_{l=-q+1}^n [f^l(1^-), f^l(0)], \quad \text{when } n \geq q,
\]

holds true. Moreover \( f^n(I) \) has Lebesgue measure \( \leq (\lambda^q \mu^p)^{\lfloor n/q \rfloor} \).

**Proof.** Let us consider the partition of \( I \)

\[
I = [\zeta_0, \zeta_1) \cup [\zeta_1, \zeta_2) \cup \ldots \cup [\zeta_{q-2}, \zeta_{q-1}) \cup [\zeta_{q-1}, \zeta_0)
\]

into disjoint circular intervals. The action of \( f \) on these intervals is drawn in Figure 7. The inclusions

\[
f([\zeta_{lp-1}, \zeta_{lp}) \subseteq [\zeta_{(l+1)p-1}, \zeta_{(l+1)p}]
\]

**Figure 7.** Dynamics of the map \( f \) with \( \zeta_0 > 0 \) on the left and \( \zeta_0 = 0 \) on the right. The arrows indicate the action of \( f \) on the intervals.
hold for $1 \leq l \leq q$ and we have equality
\[ f([\zeta_{lp-1}, \zeta_{lp}) = [\zeta_{(l+1)p-1}, \zeta_{(l+1)p}) \]
for $1 \leq l \leq q - 1$. However, for $l = q$, we have a strict inclusion:
\[ f([\zeta_{q-1}, \zeta_{0})) = [\zeta_{p-1}, f(1^-)) \cup [f(0), \zeta_{p}) = [\zeta_{p-1}, \zeta_{p}) \sim [f(1^-), f(0)). \]
Thus (17) holds for $1$. Applying $f$ to this decomposition, we observe the appearance of a second “hole” $[f^2(1^-), f^2(0))$ contained in the interval $[\zeta_{2p-1}, \zeta_{2p})$, just over the first hole $[f(1^-), f(0))$ dotted on Figure 7. Iterating $f$ again, we obtain (17) for $1 \leq n \leq q$ (note that the formulæ (17) and (18) coincide for $n = q$). At the $q$-th iteration, each interval $[\zeta_{lp-1}, \zeta_{lp})$, $1 \leq l \leq q$, has been holed and $\eta$ does not belong to $f^q(I)$. Then, $f$ exchanges the intervals and contracts them. We thus obtain formula (18) for $n \geq q$.

We finally prove the bound $|f^n(I)| \leq (\lambda q^p)^{|n/q|}$ for the Lebesgue measure of the iterated images $f^n(I)$. We claim that
\[ |f^{n+q}(I)| = \lambda q^p |f^n(I)| \]
for every integer $n \geq 0$. Suppose for instance $n \geq q$ and rewrite (18) in the form
\[ f^n(I) = \bigcup_{l=n-q+1}^{n} [\zeta_{lp-1}, f^l(1^-)) \cup [f^l(0), \zeta_{lp}] . \]
Each interval $H = [\zeta_{lp-1}, f^l(1^-)$ or $H = [f^l(0), \zeta_{lp})$ involved in the disjoint union (20) is contained either in $[0, \eta)$ or in $[\eta, 1)$. Keeping track of the iterated images
\[ f^m(H) = [\zeta_{(l+m)p-1}, f^{l+m}(1^-)] \quad \text{or} \quad f^m(H) = [f^{l+m}(0), \zeta_{(l+m)p}] , \]
for $1 \leq m \leq q$ in the decomposition (20) at level $n+m$ (instead of $n$), we observe that $f^m(H)$ is an interval contained in $[0, \eta)$ for $q-p$ values of $m$ and in $[\eta, 1)$ for the $p$ remaining values of $m$ (see Figure 6). Notice that the image $f(J)$ of any interval $J$ contained in $[0, \eta)$ (resp. $[\eta, 1)$) is an interval whose length equals $\lambda |J|$ (resp. $\lambda \mu |J|$). The length of $f^{m+1}(H)$ equals the length of $f^m(H)$ multiplied either by $\lambda$ or by $\lambda \mu$, according whether $f^m(H)$ is contained in $[0, \eta)$ or in $[\eta, 1)$. It follows that
\[ |f^q(H)| = \lambda q^{-p}(\lambda \mu)^p |H| = \lambda q^p |H| . \]
Summing over the disjoint intervals $H$ occurring in (20), we obtain (19) for $n \geq q$. The proof for $n \leq q-1$ is similar, now based on the decomposition (17). Using euclidean division, write $n = q\lfloor n/q \rfloor + r$, with $0 \leq r < q$. Equation (19) yields the required bound
\[ |f^n(I)| = (\lambda q^p)^{\lfloor n/q \rfloor} |f^r(I)| \leq (\lambda q^p)^{\lfloor n/q \rfloor} . \]

We deduce from Proposition 16 the following explicit decomposition of the images $f^n(I)$.

**Corollary.** Let $n \geq q$. Denote by $\tilde{p}$ the multiplicative inverse of $p$ modulo $q$. For every integer $l$ with $n-q+1 \leq l \leq n$, let $m$ be the unique integer in the interval
n - q + 1 ≤ m ≤ n which is congruent to l + p modulo q. Then, the decomposition into disjoint intervals

\[ f^n(I) = \bigcup_{l=n-q+1}^{n} [f^l(0), f^m(1^-)] \]

holds true. For every integer l with \( n - q + 1 \leq l \leq n \), the interval \([f^l(0), f^m(1^-)]\) contains the point \( \zeta_{lp} \).

**Proof.** Recall the decomposition (20) and observe that \( \zeta_{mp} = \zeta_{lp} \) and that \( l \mapsto m \) is a bijection of the set \{\( n - q + 1, \ldots, n \)\}. Collecting the intervals involved in (20) by pairs \((l, m)\), we find

\[ f^n(I) = \bigcup_{l=n-q+1}^{n} [\zeta_{lp}, f^m(1^-)] \cup [f^l(0), \zeta_{lp}] = \bigcup_{l=n-q+1}^{n} [f^l(0), f^m(1^-)]. \]

It follows from the corollary that

\[ C := \bigcap_{n \geq 0} f^n(I) = \{\zeta_0, \ldots, \zeta_{q-1}\} = \phi(I). \]

Indeed, \( \{\zeta_0, \ldots, \zeta_{q-1}\} \) is contained in \( f^n(I) \), for every \( n \geq 1 \). The image \( f^n(I) \), for \( n \geq q \), equals the union of \( q \) intervals whose lengths shrink to 0, as \( n \to \infty \), noting that Proposition 16 delivers the bound

\[ |f^n(I)| \leq (\lambda \mu)^{p/q} q^{n/q}, \]

where \( \lambda \mu^{p/q} \) is less than 1, since \( p/q < r_{\lambda, \mu} \). This proves (21). Since these \( q \) disjoint intervals rotate under the action of \( f \), we obtain as well that the \( \omega \)-limit set \( \omega(x) \) equals \( C \), for any \( x \in I \). The proof of the assertion (ii) of Theorem 6 is complete.

### 8. The Right Endpoint

We deal here with the exceptional value \( \delta = \delta(\lambda, \mu, p/q) \). Let us first explain the reasons why the arguments expanded in Section 7 do not apply to this value. Indeed, (14) shows that \( \phi(0^-) = 0 \) in this case. Then

\[ \zeta_{q-1} = \phi(1^-) = \phi(0^-) + 1 = 1, \]

and the set \( \{\zeta_0, \ldots, \zeta_{q-1}\} \) cannot be of course an \( f \)-cycle, since it contains the point 1 which is outside the set of definition \( I \) of the map \( f \). We slightly modify the map \( f \) in order that the obstruction no longer holds.

Put \( J = (0, 1] \). As usual, let \( \lambda, \mu, \delta \) be three real numbers with

\[ 0 < \lambda < 1, \mu > 0, 1 - \lambda < \delta < d_{\lambda, \mu} \]

and let \( \rho = \rho_{\lambda, \mu, \delta} \) be the rotation number of \( f = f_{\lambda, \mu, \delta} \). Recall the associated lift \( F = F_{\lambda, \mu, \delta} \) and the conjugacy \( \phi = \phi_{\lambda, \mu, \delta, \rho} \). We introduce three functions \( f^- \), \( F^- \) and \( \phi^- \) which are the left limit of \( f, F \) and \( \phi \) respectively.
**Definition 17.** (i) Let $f^- : J \to I$ be the map defined by

$$f^{-}(x) = f(x^-) = \begin{cases} \lambda x + \delta & \text{if } 0 < x \leq \eta, \\ \mu(\lambda x + 1 - 1) & \text{if } \eta < x \leq 1. \end{cases}$$

(ii) Let $F^- : \mathbb{R} \to \mathbb{R}$ be the map defined by

$$F^{-}(x) = F(x^-) = \begin{cases} F(x) & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ F(x) - \delta + \mu(\lambda + \delta - 1) & \text{if } x \in \mathbb{Z}, \end{cases}$$

(iii) Let $\phi^- : \mathbb{R} \to \mathbb{R}$ be the map defined by

$$\phi^{-}(y) = \phi(y^-) = \left[ y \right] - \frac{\lambda + \delta - 1}{\lambda} + \sum_{k=0}^{\infty} \lambda^k \mu^{\left[ y - \left( k + 1 \right) \rho \right]} \left( \frac{\lambda + \delta - 1}{\lambda} + \left[ y - 1 + \rho \right] - \left[ y - k \rho \right] \right).$$

The maps $f$ and $f^-$ share almost the same dynamical behaviour and we present the analogies, omitting the proofs which follow the lines of Sections 3 and 4. The two maps $f$ and $f^-$ coincide on $(0, 1) \sim \{\eta\}$ and differ at the critical point $\eta = (1 - \delta)/\lambda$ where $f(\eta) = 0$ and $f^-(\eta) = 1$. Thus, any $f$-orbit contained in $(0, 1)$ does not contain the point $\eta$ and is also an $f^-$-orbit. The function $F^-$ turns out to be a lift for the circle map $f^-$, identifying now the circle $\mathbb{R}/\mathbb{Z}$ with the interval $J$. It follows that both maps $f$ and $f^-$ have the same rotation number $\rho = \rho_{\lambda, \mu, \delta}$. One can show that when $\rho$ is irrational, or when $\rho = p/q$ is rational and

$$\delta \left( \lambda, \mu, (p/q)^- \right) < \delta \leq \delta \left( \lambda, \mu, p/q \right),$$

the functional equation $F^-(\phi^-(y)) = \phi^-(y + \rho)$ holds for any $y \in \mathbb{R}$. Moreover, when $\rho$ is irrational, the closure $\phi^-(J) = \phi(I)$ is the Cantor set considered in Theorem 6 (i) and any $f^-$-orbit approaches this Cantor set. When (22) holds, every $f^-$-orbit approaches cyclically the $f^-$ periodic cycle

$$\phi^-(I) = \{ \phi^-(1/q), \ldots, \phi^-(1) \} = \{ \zeta_0, \ldots, \zeta_{q-1} \} = \phi(I)$$

of order $q$.

From now, let us fix $\delta = \delta(\lambda, \mu, p/q)$ and put $f = f_{\lambda, \mu, \delta}$. Since $\delta$ belongs to the interval (22), $\phi(I)$ is an $f^-$-cycle of order $q$ containing the point 1. Let $\partial$ be an $f^-$-orbit. If $0 \notin \partial$, then $\partial$ is also an $f^-$-orbit which converges to $\phi(I)$. It follows in particular that there exists no finite $f$-cycle. Indeed, arguing as in Section 7, this finite cycle should be an attractor for any $f$-orbit $\partial$, and it would be equal to $\phi(I)$, which is impossible since $1 \in \phi(I)$. Suppose now that $0 \in \partial$. Then $0$ appears only once in $\partial$. If not, $\partial$ would contain a finite $f$-cycle. Hence, some tail of $\partial$ does not contain 0 and $\partial$ converges as well to $\phi(I)$. Part (iii) of Theorem 6 is established.
APPENDIX. ON THE TRANSCENDENCY OF THE NUMBER $\Psi_\rho(\lambda, \mu)$

We give some hints on the proof of Theorem 10. Let $(p_n/q_n)_{n \geq 0}$ be the sequence of convergents of the irrational number $\rho$. Loxton and Van der Poorten have established in [17, Theorem 8] that the series $\Psi_\rho(\lambda, \mu)$ takes a transcendental value for any non-zero algebraic numbers $\lambda$ and $\mu$ within the domain of convergence

$$|\lambda| < 1, \quad |\lambda| |\mu|^{\rho} < 1,$$

of the power series $\Psi_\rho$, assuming moreover that $\lambda q_n \mu p_n \neq 1$ for every $n \geq 0$. We show that this last assumption is unnecessary. See also Sections 2.9 and 2.10 of the monograph [20].

Let $\rho = [0; a_1, a_2, \ldots]$ be the continued fraction expansion of the irrational number $\rho$. The proof of Theorem 10 splits into two parts, according whether the sequence $(a_k)_{k \geq 1}$ of partial quotients is bounded or not. For $n \geq 0$, we set $\rho_n = [0; a_{n+1}, a_{n+2}, \ldots]$, so that $\rho_0 = \rho$ and $\rho_{n+1} = \left\{ \frac{1}{\rho_n} \right\}$.

When the sequence $(a_k)_{k \geq 1}$ is bounded, Theorem 10 follows from the Theory of Mahler’s functions, see [20, Theorem 2.10.3]. One uses the chain of functional equations

$$\Psi_\rho(\lambda, \mu) = \sum_{l=0}^{n-1} (-1)^l \frac{\lambda^{q_l+q_{l+1}} \mu^{p_{l+1}+p_{l+1}}}{(1 - \lambda q_l \mu^{p_l})(1 - \lambda q_{l+1} \mu^{p_{l+1}})} + (-1)^n \Psi_{\rho_n}(\lambda q_n \mu^{p_n}, \lambda q_{n-1} \mu^{p_{n-1}}),$$

which links the Hecke-Mahler power series $\Psi_\rho$ and $\Psi_{\rho_n}$ for any $n \geq 1$. However, we have the identity

$$\sum_{l=0}^{n-1} (-1)^l \frac{\lambda^{q_l+q_{l+1}} \mu^{p_{l+1}+p_{l+1}}}{(1 - \lambda q_l \mu^{p_l})(1 - \lambda q_{l+1} \mu^{p_{l+1}})} = \frac{\lambda \mu \sum_{k=1}^{q_n} ((k+1)\rho - [k\rho]) \lambda^k \mu^{[k\rho]}}{(1 - \lambda)(1 - \lambda q_n \mu^{p_n})}$$

of rational functions in $\lambda$ and $\mu$. The equalities (24) can be proved by induction on $n$, using the classical recursion formulae defining the characteristic Sturmian sequence $((k+1)\rho - [k\rho])_{k \geq 1}$. Thus, the equalities (23) make sense for $n$ large enough, even if some denominator factor $1 - \lambda q_l \mu^{p_l}$ vanishes. See also [20, Lemma 2.3.3] for an alternative argument. Now, the machinery of Mahler’s functions, applied to the chain of relations (23) at the algebraic point $(\lambda, \mu)$, delivers the transcendency of the number $\Psi_\rho(\lambda, \mu)$.

When the sequence $(a_k)_{k \geq 1}$ is unbounded, we let $n$ tend to infinity in (24). Using furthermore Lemma 8, we obtain the formula

$$\Psi_\rho(\lambda, \mu) = \frac{\lambda \mu}{1 - \lambda} \left( \sum_{k=1}^{\infty} ((k+1)\rho - [k\rho]) \lambda^k \mu^{[k\rho]} \right) = \sum_{l=0}^{\infty} (-1)^l \frac{\lambda^{q_l+q_{l+1}} \mu^{p_{l+1}+p_{l+1}}}{(1 - \lambda q_l \mu^{p_l})(1 - \lambda q_{l+1} \mu^{p_{l+1}})},$$

The series on the right hand side above converges very quickly. Since the ratios $q_{l+1}/q_l$ and $p_{l+1}/p_l$ are unbounded, Liouville’s inequality applied to truncated sums of this series, shows that $\Psi_\rho(\lambda, \mu)$ is a transcendent number, and even that it is a Liouville’s number when $\lambda$ and $\mu$ are rational numbers.
Acknowledgments. We would like to thank the anonymous referee whose remarks and comments have greatly improved our text. We graciously acknowledge the support of Région Provence-Alpes-Côte d’Azur through the project APEX Systèmes dynamiques: Probabilités et Approximation Diophantienne PAD, CEFIPRA through the project No. 5801-B and the program MATHAMSUD projet No. 38889TM DCS: Dynamics of Cantor systems.

REFERENCES

[1] W. W. Adams and J. L. Davison, A remarkable class of continued fractions, Proc. Amer. Math. Soc., 65 (1977), 194–198.
[2] P. E. Böhmer, Über die Transzendenz gewisser dyadischer Brüche, Math. Ann., 96 (1927), 367–377.
[3] M. D. Boshernitzan, Dense orbits of rationals, Proc. Amer. Math. Soc., 117 (1993), 1201–1203.
[4] J. P. Bowman and S. Sanderson, Angels’ staircases, Sturmian sequences, and trajectories on homothety surfaces, J. Mod. Dyn., 16 (2020), 109–153.
[5] J. M. Borwein and P. B. Borwein, On the generating function of the integer part: \(|na + \gamma|\), J. Number Theory, 43 (1993), 293–318.
[6] J. Brémond, Dynamics of injective quasi-contractions, Ergodic Theory Dynam. Systems, 26 (2006), 19–44.
[7] Y. Bugeaud, Dynamique de certaines applications contractantes, linéaires par morceaux, sur \([0,1)\), C. R. Acad. Sci. Paris Sér I Math., 317 (1993), 575–578.
[8] Y. Bugeaud and J.-P. Conze, Calcul de la dynamique de transformations linéaires contractantes mod 1 et arbre de Farey, Acta Arith., 88 (1999), 201–218.
[9] R. Coutinho, Dinâmica simbólica linear, Ph.D Thesis, Instituto Superior Técnico, Universidade Técnica de Lisboa, 1999.
[10] L. V. Danilov, Certain classes of transcendental numbers, Math. Zametki, 12 (1972), 149–154.
[11] E. J. Ding and P. C. Hemmer, Exact treatment of mode locking for a piecewise linear map, J. Statist. Phys., 46 (1987), 99–110.
[12] O. Feely and L. O. Chua, The effect of integrator leak in \(\Sigma – \Delta\) modulation, IEEE Transactions on Circuits and Systems, 38 (1991), 1293–1305.
[13] M. Hata, Neurons–A Mathematical Ignition, Series on Number Theory and its Applications, 9, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2015.
[14] S. Janson and A. Öberg, A piecewise contractive dynamical system and Phragmén’s election method, Bull. Soc. Math. France, 147 (2019), 395–441.
[15] T. Komatsu, A certain power series and the inhomogeneous continued fraction expansions, J. Number Theory, 59 (1996), 291–312.
[16] M. Laurent and A. Nogueira, Rotation number of contracted rotations, J. Mod. Dyn., 12 (2018), 175–191.
[17] J. H. Loxton and A. J. van der Poorten, Arithmetic properties of certain functions in several variables. III, Bull. Austral. Math. Soc., 16 (1977), 15–47.
[18] J. H. Loxton and A. J. van der Poorten, Transcendence and algebraic independence by a method of Mahler, in Transcendence Theory: Advances and Applications (Proc. Conf., Univ. Cambridge, Cambridge, 1976), Academic Press, London, 1977, 211–226.
[19] J. Nagumo and S. Sato, On a response characteristic of a mathematical neuron model, Kybernetik, 10 (1972), 155–164.
[20] K. Nishioka, Mahler Functions and Transcendence, Springer Lecture Notes in Mathematics, 1631, Springer-Verlag, Berlin, 1996.
[21] K. Nishioka, I. Shiokawa and J. Tamura, Arithmetical properties of a certain power series, J. Number Theory, 42 (1992), 61–87.
[22] A. Nogueira and B. Pires, Dynamics of piecewise contractions of the interval, *Ergodic Theory Dynam. Systems*, **35** (2015), 2198–2215.

[23] A. Nogueira, B. Pires and R. A. Rosales, Topological dynamics of piecewise \( \lambda \)-affine maps, *Ergodic Theory Dynam. Systems*, **38** (2018), 1876–1893.

[24] F. Rhodes and C. L. Thompson, Rotation numbers for monotone functions on the circle, *J. London Math. Soc. (2)*, **34** (1986), 360–368.

[25] F. Rhodes and C. L. Thompson, Topologies and rotation numbers for families of monotone functions on the circle, *J. London Math. Soc. (2)*, **43** (1991), 156–170.

MICHEL LAURENT <michel-julien.laurent@univ-amu.fr>: Aix-Marseille Université, CNRS, Centrale Marseille, Institut de Mathématiques de Marseille, 163 Avenue de Luminy, Case 907, 13288, Marseille Cédex 9, France

ARNALDO NOGUEIRA <arnaldo.nogueira@univ-amu.fr>: Aix-Marseille Université, CNRS, Centrale Marseille, Institut de Mathématiques de Marseille, 163 Avenue de Luminy, Case 907, 13288, Marseille Cédex 9, France