Homogenization of Bingham Flow in a thin domain with an oscillating boundary

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Abstract

In this paper we analyze the steady flow of an incompressible Bingham flow in a thin domain with rough boundary under the action of given external forces and with no-slip boundary condition on the whole boundary of the domain. Denoted by $\varepsilon$ the thickness of the domain and the roughness periodicity, this problem is described by non linear variational inequalities. We are interested in studying how the geometry of such a domain affects the asymptotic behaviour of the fluid when $\varepsilon$ tends to zero. The main mathematical tool is the unfolding method, introduced for the first time in [13] and recently adapted to two-dimensional thin domains with oscillating boundaries in [3]. Thanks to this method, we are able to obtain some a priori estimates both for the velocity and for the pressure without using any extension operators, which is the most important novelty of our paper with respect to previous works on the same subject. Hence we obtain the homogenized limit problem, which preserves the nonlinear character of the flow, and identify the effects of the microstructure in the corresponding effective equations. We conclude with the interpretation of the limit problem in terms of a non linear Darcy law.

Keywords: Non-Newtonian fluids, thin domain, oscillating boundary, unfolding operators.

AMS subject classifications: 76A05, 76A20, 76D08, 76M50, 74K10, 35B27

1 Introduction

In this paper we study the steady flow of an incompressible Bingham fluid in a thin domain with a rough boundary. Mathematical models involving thin domains are widely used to describe situations appearing naturally in numerous industrial and engineering applications. A relevant example is the classical lubrication problem describing the relative motion of two adjacent surfaces separated by a thin film of fluid acting as a lubricant. In the incompressible case, the main unknown is the pressure of the fluid. Once resolved the pressure, it is possible to compute other fundamental quantities such as the velocity field and the forces on the bounding surfaces.

On the other hand, to increase the hydrodynamic performance in various lubricated machine elements, for example journal bearings and thrust bearings, engineers also point out the importance of analyzing how the surface irregularities affects the thin film flow. From a mathematical point of view...
view, a thin domain with rough boundary is usually described by two parameters $\epsilon$ and $\eta$, different in general, which tend to zero. The first one, $\eta_\epsilon$, is the characteristic wavelength of the periodic roughness, and $\epsilon$ is the thickness of the domain, i.e. the distance between the surfaces. There are several papers studying the asymptotic behavior of fluids in thin domains with rough boundary in the case of Newtonian fluids, see for instance [4, 17, 18] and the references therein. However, for the non-Newtonian fluids the situation is completely different. The main reason is that the viscosity is a nonlinear function of the symmetrized gradient of the velocity (see [2]).

The Bingham fluid is a non-Newtonian fluid which behaves as a rigid body at low stresses but flows as a viscous fluid at high stress. This type of non-Newtonian fluid behavior is characterized by the existence of a threshold stress, called yield stress, which must be exceeded for the fluid to deform or flow. Once the externally applied stress is greater than the yield stress, the fluid exhibits Newtonian behavior. Typical examples of such fluids are some paints, toothpaste, the mud which can be used for the oil extraction, the volcanic lava or even the blood.

In this paper, denoted by $\epsilon$ the thickness of the domain and the roughness periodicity, we are interested in studying how the geometry of the thin domain with rough boundary affects the asymptotic behaviour of an incompressible Bingham fluid when $\epsilon$ tends to zero. We refer the reader to the very recent paper [25] and the references therein for the application of our study to problems issued from the real life applications. Indeed predicting lava flow pathways is important for understanding effusive eruptions and for volcanic hazard assessment. One particular challenge is understanding the interplay between flow pathways and substrate topography that is often rough on a variety of scales (< 1 m to 10 s km).

The physical description of the Bingham fluid was introduced in [5] while the mathematical model of the Bingham flow in a bounded domain was performed by G. Duvaut and J.L. Lions in [14]. The existence of the velocity and the pressure for such a flow was proved in the case of a bi-dimensional and of a three-dimensional domain.

There are several papers studying the asymptotic behavior of Bingham fluids in thin domains. In particular we can mention [11, 12] where the asymptotic behavior of a Bingham fluid in a thin layer of thickness $\epsilon$ is studied. In [10] the authors obtain and analyze the limit problem for a steady incompressible flow of a Bingham fluid in a thin T-like shape structure. Finally, in the recent paper [1] a dimension reduction and the unfolding operator method was used to describe the asymptotic behavior the flow of a Bingham fluid in thin porous media. We also refer the reader to [6, 7, 8, 9, 21] where the asymptotic behavior of a Bingham fluid in porous media is performed using different techniques in homogenization.

Our paper is based on the recent periodic unfolding method, see [13] for the first descriptions of the method and [3] for an adaptation of this method to two-dimensional thin domains with oscillating boundaries. Thanks to this method, we are able to obtain the homogenized limit problem as the thickness of the domain tends to zero and to identify the effects of the microstructure in the corresponding effective equations. The unfolding method is a very efficient tool to study periodic homogenization problems where the size of the periodic cell tends to zero. The idea is to introduce suitable changes of variables which transform every periodic cell into a simpler reference set by using a supplementary variable (microscopic variable).

The most important novelty of our paper, if compared with the works mentioned above, is to use directly the unfolding operator in thin domains with rough boundary, to obtain some a priori estimates both for the velocity and for the pressure. Acting in this way, we avoid to use Tartar’s argument, see [27], based on suitable extension operators. This approach, since no extension operators are required, allows to assume milder hypothesis on the regularity of the thin domain, necessary for the existence of such extensions. Moreover we underline that, following an approach similar to the one used to get our limit problem, we can recover the convergence results given in
[4], in the case of Newtonian fluids, see also [17, 18] for a generalization to the unstationary case, without using extension operators.

The paper is organized as follows.

In Section 2, we introduce our thin domain with rough top boundary $\Omega_\epsilon$, where the parameter $\epsilon$ represents either the thickness of the domain or the rough periodicity. Then we formulate the problem which models the flow in $\Omega_\epsilon$ of a viscoplastic incompressible Bingham fluid with velocity $u_\epsilon$ and pressure $p_\epsilon$ verifying the nonlinear variational inequality (2.4). Finally we give some notations useful in the sequel. In Section 3, we give some a priori estimates for both the velocity and the pressure. In Section 4 we introduce definition and properties of the unfolding operator adapted to thin domains with oscillating boundary introduced in [3] for the bidimensional case. Section 5 is devoted to state some convergences results for the unfolded velocity field taking into account the a priori estimates proved in Section 3, a suitable "rescaled" velocity field, which is typical for this kind of problem in thin domains, and the unfolding operator defined in Section 4.

Section 6 contains the most relevant result of our paper, Proposition 14 concerning the convergence of the unfolded pressure without the use of any extension operators. Moreover, if we assume that the thin domain is given by an exact number of basic cells, we can establish an interesting relationship between the limit of the unfolded pressure and the pressure $p_\epsilon$ itself (see Proposition 15).

In Section 7 we state and prove the main result of our paper, Theorem 16, which allow us to identify the limit problem. Finally in Section 8 we conclude with the interpretation of this limit problem, which preserves the nonlinear character of the flow. Indeed, in the case of forces independent of the vertical variable, both a nonlinear Darcy equation and a lower dimensional Bingham-like law arise (see Proposition 17).

2 The setting of the problem

Throughout the paper we will consider three-dimensional thin domains with an oscillatory behavior in its top boundary which are defined as follows

$$\Omega_\epsilon = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1, x_2) \in \omega, \ 0 < x_3 < \epsilon G(x_1/\epsilon, x_2/\epsilon)\},$$  

(2.1)

where $\omega$ denotes the unitary cell in $\mathbb{R}^2$, $\omega = (0, 1)^2$, the parameter $\epsilon$ is greater than zero and $G : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a smooth periodic function in the cell $Y = (0, L_1) \times (0, L_2)$ such that there exist two positive constants $G_0, G_1$ with $0 < G_0 < G(x, y) < G_1$, $\forall (x, y) \in \mathbb{R}^2$.

In order to simplify the notation we decompose each point $x \in \mathbb{R}^3$ according to

$$x = (\hat{x}, x_3), \text{ with } \hat{x} = (x_1, x_2) \in \mathbb{R}^2 \text{ and } x_3 \in \mathbb{R}.$$

Therefore, our thin domain is defined as follows

$$\Omega_\epsilon = \{(\hat{x}, x_3) \in \mathbb{R}^3 \mid \hat{x} \in \omega, \ 0 < x_3 < \epsilon G(\hat{x}/\epsilon)\}.$$

In $\Omega_\epsilon$ we consider the incompressible flow of a Bingham fluid, see [5], with viscosity and yield stress given by $\mu_\epsilon^2$ and $g_\epsilon$, respectively, where $\mu$ and $g$ are positive constants independent of $\epsilon$. The fluid velocity is denoted by $u_\epsilon$ while the pressure of the fluid is denoted by $p_\epsilon$. Then the stress tensor is defined by

$$\sigma_{ij} = -p_\epsilon \delta_{ij} + g_\epsilon \frac{D_{ij}(u_\epsilon)}{(D_{II}(u_\epsilon))^{1/2}} + 2\mu_\epsilon^2 D_{ij}(u_\epsilon),$$  

(2.2)
where \( \delta_{ij} \) is the Kronecker symbol and \( D_{ij} \) and \( D_{II} \) are defined by

\[
D_{ij}(u_\varepsilon) = \frac{1}{2} \left( \frac{\partial u_{i,1}}{\partial x_j} + \frac{\partial u_{j,1}}{\partial x_i} \right), 1 \leq i, j \leq 3,
\]

\[
D_{II}(u_\varepsilon) = \frac{1}{2} \sum_{i,j=1}^{n} D_{ij}(u_\varepsilon) D_{ij}(u_\varepsilon).
\]

**Remark 1** Notice that we will denote vector fields in three dimensions using bold face, \( u_\varepsilon = (u_{\varepsilon,1}, u_{\varepsilon,2}, u_{\varepsilon,3}) \). Moreover, the euclidean norm in \( \mathbb{R}^n \) is denoted by \( | \cdot | \).

Relation (2.2) represents the constitutive law of the Bingham fluid. In [14] it is shown that this constitutive law is equivalent to the following one:

\[
\begin{cases}
(\sigma_{II})^{\frac{1}{2}} < g_\varepsilon \iff D_{ij}(u_\varepsilon) = 0 \\
(\sigma_{II})^{\frac{1}{2}} \geq g_\varepsilon \iff D_{ij}(u_\varepsilon) = \frac{1}{2\mu_\varepsilon^2} \left( 1 - \frac{g_\varepsilon}{(\sigma_{II})^{\frac{1}{2}}} \right) \sigma_{ij}^e.
\end{cases}
\]

where \( \sigma_{II} \) and \( \sigma_{ij}^e \) are defined by

\[
\sigma_{II} = \frac{1}{2} \sum_{i,j=1}^{3} \sigma_{ij}^e \sigma_{ij}^e,
\]

\[
\sigma_{ij}^e = g \frac{D_{ij}}{(D_{II})^{\frac{1}{2}}} + 2\mu_\varepsilon^2 D_{ij}.
\]

We assume that the fluid is incompressible, i.e. the velocity field is divergence free, and we impose the no-slip condition on the boundary of the domain, \( u_\varepsilon = 0 \) on \( \partial \Omega_\varepsilon \). Therefore, the space of admissible velocity fields is given by

\[
V_\varepsilon = \left\{ v \in \left( H^1_0(\Omega_\varepsilon) \right)^3 : \text{div}(v) = 0 \right\}.
\]

Let us apply to the fluid an external body force \( f_\varepsilon \in (L^2(\Omega_\varepsilon))^3 \) defined by

\[
f_\varepsilon = f \left( \frac{x}{\varepsilon}, \frac{x_3}{\varepsilon} \right) \text{ a. e. } x \in \Omega_\varepsilon,
\]

(2.3)

where \( f \in (L^2(\Omega))^3 \), \( \Omega = \omega \times (0, G_1) \).

According to [14], there exists a unique velocity \( u_\varepsilon \in V_\varepsilon \) which satisfies the following inequality

\[
\mu_\varepsilon^2 \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla (v - u_\varepsilon) \, dx + g_\varepsilon \int_{\Omega_\varepsilon} |\nabla v| \, dx - g_\varepsilon \int_{\Omega_\varepsilon} |\nabla u_\varepsilon| \, dx
\]

\[
\geq \int_{\Omega_\varepsilon} f_\varepsilon \cdot (v - u_\varepsilon) \, dx, \forall v \in V_\varepsilon.
\]

(2.4)

Equivalently, see [6] [14], there exists \( p_\varepsilon \in L^2(\Omega_\varepsilon)/\mathbb{R} \) such that the couple \( (u_\varepsilon, p_\varepsilon) \) solves the following variational inequality

\[
\mu_\varepsilon^2 \int_{\Omega_\varepsilon} \nabla u_\varepsilon \cdot \nabla (v - u_\varepsilon) \, dx + g_\varepsilon \int_{\Omega_\varepsilon} |\nabla v| \, dx - g_\varepsilon \int_{\Omega_\varepsilon} |\nabla u_\varepsilon| \, dx
\]

\[
\geq \int_{\Omega_\varepsilon} f_\varepsilon (v - u_\varepsilon) \, dx + \int_{\Omega_\varepsilon} p_\varepsilon \text{div}(v - u_\varepsilon) \, dx, \forall v \in \left( H^1_0(\Omega_\varepsilon) \right)^3.
\]

(2.5)
Note that, in the pressure-velocity formulation $u_\epsilon$ is uniquely determined and it coincides with the solution of problem (2.4) but $p_\epsilon$ is not unique.

From the assumption (2.3) we have

$$\epsilon^{-1/2}||f_\epsilon||_{L^2(\Omega_\epsilon)}^3 \leq C$$

for some positive constant $C$ independent of the parameter $\epsilon > 0$. This assumption on the applied forces is usual in order to obtain appropriate estimates. In fact, the common choice of the applied forces $f_\epsilon$ in thin domains where the force does not depend on the vertical variable and the vertical component of the force is neglected satisfies this assumption, see Remark 3.

**Remark 2** Due to the order of the height of the thin domain it makes sense to consider the following rescaled Lebesgue measure

$$\rho_\epsilon(O) = \frac{1}{\epsilon} \mu(O), \forall O \subset \mathbb{R}^d,$$

which is widely considered in works involving thin domains, see e.g. [19, 22, 23, 24].

As a matter of fact, from now on, we use the following rescaled norms in the thin open sets

$$|||\varphi|||_{L^p(\Omega_\epsilon)} = \epsilon^{-1/p}|||\varphi|||_{L^p(\Omega)}, \forall \varphi \in L^p(\Omega_\epsilon), \ 1 \leq p < \infty,$$

$$|||\varphi|||_{W^{1,p}(\Omega_\epsilon)} = \epsilon^{-1/p}|||\varphi|||_{W^{1,p}(\Omega), \ 1 \leq p < \infty.}$$

For completeness we may denote $|||\varphi|||_{L^\infty(\Omega_\epsilon)} = |||\varphi|||_{L^\infty(\Omega)}$ and the norm for the dual space $H^{-1}(\Omega_\epsilon)$ is given by

$$|||\varphi|||_{H^{-1}(\Omega_\epsilon)} = \sup_{\psi \in H^1_0(\Omega_\epsilon)} \frac{1}{\epsilon} \left| \int_{\Omega_\epsilon} \varphi \psi dx \right|,$$

(2.7)

Therefore, the body forces satisfy

$$|||f_\epsilon|||_{L^2(\Omega_\epsilon)}^3 \leq C,$$

(2.8)

for some positive constant $C$ independent of the parameter $\epsilon > 0$.

**Remark 3** Since the thin domain shrinks in the vertical direction as $\epsilon$ tends to zero, it is usual to assume that the applied forces do not depend on $\epsilon$ and they are of the form

$$f(x) = (\hat{f}(\hat{x}), 0), \ a.e. \ x \in \Omega_\epsilon.$$ 

Notice that the third component is neglected and the force is independent of the vertical direction. Moreover, this particular $f$ satisfies

$$\epsilon^{-1/2}||f||_{L^2(\Omega_\epsilon)}^3 \leq C||\hat{f}||_{L^2(\omega)}^3 \leq C.$$

## 3 A priori estimates

In this section we follow the standard procedure to get the $a$ priori estimates for the velocity $u_\epsilon$ and the pressure $p_\epsilon$.

First, notice that the Poincaré inequality in the thin domain (2.1) can be written as
By using Hölder's inequality on the right hand side and the assumption (2.8) we have

\[ \epsilon\|\nabla \varphi\|_{L^2(\Omega)^3} \leq C \|\nabla \varphi\|_{L^2(\Omega)^3}, \]

where C is independent of \( \varphi \) and \( \epsilon \).

**Lemma 5** Let \( (u_\epsilon, p_\epsilon) \) be a solution of (2.5). Under the assumption (2.8), the following estimates hold

\[ \epsilon\|\nabla u_\epsilon\|_{L^2(\Omega)^3} \leq C, \]
\[ \|u_\epsilon\|_{L^2(\Omega)^3} \leq C, \]
\[ \|\nabla p_\epsilon\|_{H^{-1}(\Omega)} \leq \epsilon C, \]

with \( C \) a positive constant independent of \( \epsilon \).

**Proof.** Taking \( v = 0 \) and \( v = 2u_\epsilon \) as a test function in (2.5) we get

\[ -\mu \epsilon^2 \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla u_\epsilon \, dx - g \epsilon \int_{\Omega_\epsilon} |\nabla u_\epsilon| \, dx \geq - \int_{\Omega_\epsilon} f \cdot u_\epsilon, \, dx, \]
\[ \mu \epsilon^2 \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla u_\epsilon \, dx + g \epsilon \int_{\Omega_\epsilon} |\nabla u_\epsilon| \, dx \geq \int_{\Omega_\epsilon} f \cdot u_\epsilon, \, dx. \]

Consequently we obtain

\[ \mu \epsilon^2 \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla u_\epsilon \, dx + g \epsilon \int_{\Omega_\epsilon} |\nabla u_\epsilon| \, dx = \int_{\Omega_\epsilon} f \cdot u_\epsilon, \, dx. \]

By using Hölder's inequality on the right hand side and the assumption (2.8) we have

\[ \mu \epsilon^2 \|\nabla u_\epsilon\|_{L^2(\Omega)^3} \leq \|f\|_{L^2(\Omega)^3} \|u_\epsilon\|_{L^2(\Omega)^3} \leq C \|u_\epsilon\|_{L^2(\Omega)^3}. \]

Then, applying the classical Poincaré inequality, see Definition 4 we obtain

\[ \mu \epsilon^2 \|\nabla u_\epsilon\|_{L^2(\Omega)^3}^2 \leq \epsilon C \|\nabla u_\epsilon\|_{L^2(\Omega)^3}. \]

Therefore, from this last inequality and the Poincaré inequality we get estimates (3.1) and (3.2).

Finally, we are going to obtain the a priori estimate for the pressure. Let \( v_\epsilon \in (H^1_0(\Omega_\epsilon))^3 \). Then, taking \( v = v_\epsilon + u_\epsilon \) as a test function in (2.5) we get

\[ \mu \epsilon^2 \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla v_\epsilon \, dx + g \epsilon \int_{\Omega_\epsilon} |\nabla v_\epsilon + \nabla u_\epsilon| \, dx - g \epsilon \int_{\Omega_\epsilon} |\nabla u_\epsilon| \, dx \]
\[ \geq \int_{\Omega_\epsilon} f_\epsilon \cdot v_\epsilon \, dx + \int_{\Omega_\epsilon} p_\epsilon \text{div}(v_\epsilon) \, dx, \forall v_\epsilon \in (H^1_0(\Omega_\epsilon))^3. \]

Hence, by Holder’s inequality it follows that

\[ \frac{1}{\epsilon} \int_{\Omega_\epsilon} p_\epsilon \text{div}(v_\epsilon) \, dx \leq \mu \epsilon^2 \|\nabla u_\epsilon\|_{L^2(\Omega)^3} \|\nabla v_\epsilon\|_{L^2(\Omega)^3} + g \epsilon \|\nabla v_\epsilon\|_{L^2(\Omega)^3} + \|f\|_{L^2(\Omega)^3} \|v_\epsilon\|_{L^2(\Omega)^3}. \]

Consequently, by using estimates (2.8), (3.1) and Poincaré’s inequality we get

\[ \frac{1}{\epsilon} \int_{\Omega_\epsilon} p_\epsilon \text{div}(v_\epsilon) \, dx \leq C \epsilon \|\nabla v_\epsilon\|_{L^2(\Omega)^3}, \forall v_\epsilon \in (H^1_0(\Omega_\epsilon))^3, \]

which, in view of (2.7) in Remark 2 provides estimate (3.3).
4 The unfolding operator

In this section we extend the definition of the unfolding operator which was introduced in [3] to two dimensional thin domains with an oscillatory boundary. Moreover, we present some of the main properties of the unfolding operator that we will use to obtain the homogenized limit problem.

We will use similar notation as in [3]:

- \( N_\epsilon \) denotes the largest integer such that \( \epsilon L_1(N_\epsilon + 1) \leq 1 \).
- \( M_\epsilon \) denotes the largest integer such that \( \epsilon L_2(M_\epsilon + 1) \leq 1 \).
- \( \omega_{ij} = (i\epsilon L_1, (i + 1)\epsilon L_1) \times (j\epsilon L_2, (j + 1)\epsilon L_2) \) with \( i = 0, 1, \ldots, N_\epsilon, j = 0, 1, \ldots, M_\epsilon \).
- \( \omega^c = \text{Int}\left( \bigcup_{i=0}^{N_\epsilon} \bigcup_{j=0}^{M_\epsilon} \omega_{ij} \right) \), \( \overline{\omega_{ij}} \) denotes the closure of the open set \( \omega_{ij} \).
- \( \Lambda^c = \omega \setminus \omega^c \). Equivalently, \( \Lambda^c = ([\epsilon L_1(N_\epsilon + 1), 1) \times (0, 1)) \cup ((0, 1) \times [\epsilon L_2(M_\epsilon + 1), 1)) \).
- The representative cell which describes the thin structure is given by
  \[
  Y^* = \{ y \equiv (\hat{y}, y_3) \in \mathbb{R}^3 | \hat{y} \in Y, 0 < y_3 < G(\hat{y}) \}.
  \]
  Recall that \( Y = (0, L_1) \times (0, L_2) \).
- \( \Omega_0^0 \) denotes the set which contains all the cells totally included in \( \Omega_\epsilon \)
  \[
  \Omega_\epsilon^0 = \left\{ (\hat{x}, x_3) \in \mathbb{R}^3 | \hat{x} \in \omega^c, 0 < x_3 < \epsilon G(\hat{x}/\epsilon) \right\}.
  \]
- \( \Omega_1^1 = \Omega_\epsilon \setminus \Omega_\epsilon^0 \).
- By analogy with the definition of the integer and fractional part of a real number, for \( \hat{x} \in \mathbb{R}^2 \), \( \lfloor \hat{x} \rfloor_L \) denotes the unique pair of integers, \( \lfloor \hat{x} \rfloor_L = (k_1, k_2) \in \mathbb{Z}^2 \), such that \( \hat{x} \in [k_1 L_1, (k_1 + 1)L_1) \times [k_2 L_2, (k_2 + 1)L_2) \) and \( \lfloor \hat{x} \rfloor_L \in [0, L_1) \times [0, L_2) \) is such that \( \hat{x} = \lfloor \hat{x} \rfloor_L + \{ \hat{x} \}_L \). Then, if \( L \) denotes the pair \( (L_1, L_2) \), for each \( \epsilon > 0 \) and for every \( \hat{x} \in \mathbb{R}^2 \) there exists a unique pair of integers, \( \lfloor \hat{x} \rfloor_\epsilon_L \), such that
  \[
  \hat{x} = \epsilon \left\lfloor \frac{\hat{x}}{\epsilon} \right\rfloor_L L + \epsilon \left\{ \frac{\hat{x}}{\epsilon} \right\}_L, \quad \left\{ \frac{\hat{x}}{\epsilon} \right\}_L \in [0, L_1) \times [0, L_2).
  \] (4.1)

We are now in position to define the unfolding operator in our setting.

Definition 6 Let \( \varphi \) be a Lebesgue-measurable function in \( \Omega_\epsilon \). The unfolding operator \( \mathcal{T}_\epsilon \), acting on \( \varphi \), is defined as the following function in \( \omega \times Y^* \)

\[
\mathcal{T}_\epsilon(\varphi)(\hat{x}, y) = \begin{cases} 
\varphi\left(\epsilon \left\lfloor \frac{\hat{x}}{\epsilon} \right\rfloor_L L + \epsilon \hat{y}, \epsilon y_3 \right) & \text{for } (\hat{x}, \hat{y}, y_3) \in \omega^c \times Y^*, \\
0 & \text{for } (\hat{x}, \hat{y}, y_3) \in \Lambda^c \times Y^*.
\end{cases}
\]

In the following proposition we list the main properties of the unfolding operator previously defined.

Proposition 7 The unfolding operator \( \mathcal{T}_\epsilon \) has the following properties:
i) $\mathcal{T}_\varepsilon$ is a linear operator.

ii) $\mathcal{T}_\varepsilon(\varphi \psi) = \mathcal{T}_\varepsilon(\varphi) \mathcal{T}_\varepsilon(\psi) \quad \forall \varphi, \psi$ Lebesgue-measurable functions in $\Omega_\varepsilon$.

iii) Let $\varphi \in L^1(\Omega_\varepsilon)$. The following integral equality holds

$$\frac{1}{L_1 L_2} \int_{\omega \times Y^*} \mathcal{T}_\varepsilon(\varphi)(\hat{x}, y) d\hat{x} dy = \frac{1}{\varepsilon} \int_{\Omega^0} \varphi(x) dx$$

$$= \frac{1}{\varepsilon} \int_{\Omega^1} \varphi(x) dx - \frac{1}{\varepsilon} \int_{\Omega^2} \varphi(x) dx.$$

iv) For every $\varphi \in L^p(\Omega_\varepsilon)$ we have $\mathcal{T}_\varepsilon(\varphi) \in L^p(\omega \times Y^*)$, with $1 \leq p < \infty$. In addition, the following relationship exists between their norms:

$$\|\mathcal{T}_\varepsilon(\varphi)\|_{L^p(\omega \times Y^*)} = (L_1 L_2)^{\frac{1}{p}} \|\varphi\|_{L^p(\Omega^0)} \leq (L_1 L_2)^{\frac{1}{p}} \|\varphi\|_{L^p(\Omega^1)}.$$

v) For every $\varphi \in W^{1,p}(\Omega_\varepsilon)$, $1 \leq p \leq \infty$, one has

$$\frac{\partial}{\partial y_i} \mathcal{T}_\varepsilon(\varphi) = \varepsilon \mathcal{T}_\varepsilon \left( \frac{\partial \varphi}{\partial x_i} \right), \quad \text{for } i = 1, 2, 3. \quad (4.2)$$

vi) Let $\varphi$ be a measurable function on $Y^*$ extended by $Y-$periodicity in the first two variables. Then $\varphi^\varepsilon(x) = \varphi(\hat{x})$ is a measurable function on $\Omega^* \varepsilon$ such that

$$\mathcal{T}_\varepsilon(\varphi^\varepsilon)(\hat{x}, y) = \varphi(y), \quad \forall (\hat{x}, y) \in \omega^\varepsilon \times Y^*.$$

Furthermore, if $\varphi \in L^p(Y^*)$, with $1 \leq p \leq \infty$ then $\varphi^\varepsilon \in L^p(\Omega^*)$.

vii) Let $\{\varphi^\varepsilon\}$ be a sequence of functions in $L^p(\omega)$, $1 \leq p < \infty$, such that

$$\varphi^\varepsilon \xrightarrow{\varepsilon \to 0} \varphi \quad \text{strongly in } L^p(\omega).$$

Then

$$\mathcal{T}_\varepsilon(\varphi^\varepsilon) \xrightarrow{\varepsilon \to 0} \varphi \quad \text{strongly in } L^p(\omega \times Y^*).$$

Remark 8 The proofs of these properties are omitted since they follow directly from the properties proved in [3] for the unfolding operator defined in two-dimensional thin domains. Notice that, in view of property iii), we may say that the unfolding operator “almost preserves” the integral of the functions since the “integration defect” arises only from the cells which are not completely included in $\Omega_\varepsilon$ and it is controlled by the integral on $\Omega^1_\varepsilon$.

For every vector field $v \in H^1(\Omega_\varepsilon)^3$ the unfolding operator is naturally defined as follows:

$$\mathcal{T}_\varepsilon(v) = (\mathcal{T}_\varepsilon(v_1), \mathcal{T}_\varepsilon(v_2), \mathcal{T}_\varepsilon(v_3)).$$

Therefore, using basic properties of the unfolding operator we prove the following proposition.

Proposition 9 For every $v \in H^1(\Omega_\varepsilon)^3$ we have

$$\varepsilon \mathcal{T}_\varepsilon(\|\nabla v\|) = |\nabla_y \mathcal{T}_\varepsilon(v)|. \quad (4.3)$$
Proof.

\[ [T_\varepsilon(|\nabla \psi|)]^2 = T_\varepsilon(|\nabla \psi|^2) = T_\varepsilon \left( \frac{1}{\varepsilon} \left( \sum_{i,j=1}^n \left( \frac{\partial \psi_i}{\partial x_j} \right)^2 \right) \right) = \sum_{i,j=1}^3 \left( \frac{1}{\varepsilon} \frac{\partial}{\partial y_j} T_\varepsilon(\psi_i) \right)^2 = \frac{1}{\varepsilon^2} |\nabla_y T_\varepsilon(\psi)|^2. \]

To conclude this section we introduce the averaging operator \( U_\varepsilon \) which is the formal adjoint of the unfolding operator. We will use it to obtain some crucial convergences for the pressure.

**Definition 10** Let \( \varphi \) be a function in \( L^p(\omega \times Y^*) \), \( p \in [1, \infty] \), then we set

\[
U_\varepsilon(\varphi)(x) = \begin{cases} 
\frac{1}{L_1L_2} \int_Y \varphi \left( \left[ \frac{x}{\varepsilon} \right]_L + \epsilon \hat{y}, \left[ \frac{x}{\varepsilon} \right]_L, \frac{x_3}{\epsilon} \right) d\hat{y}, & \text{for } x \in \Omega^0, \\
0 & \text{for } x \in \Omega^1.
\end{cases}
\]

The following proposition provides the main properties of \( U_\varepsilon \).

**Proposition 11** The averaging operator satisfies the following properties.

i) \( U_\varepsilon \) is the formal adjoint of the unfolding operator \( T_\varepsilon \), in the sense that

\[
\frac{1}{L_1L_2} \int_{\omega \times Y^*} T_\varepsilon(\varphi) \psi \, d\hat{x}d\hat{y} = \frac{1}{\varepsilon} \int_{\Omega^\varepsilon} \varphi U_\varepsilon(\psi) \, dx,
\]

for \( \varphi \in L^q(\Omega^\varepsilon) \) and \( \psi \in L^p(\omega \times Y^*) \) with \( 1 \leq p, q \leq \infty \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

ii) The averaging operator \( U_\varepsilon \) is linear and continuous from \( L^p(\omega \times Y^*) \) to \( L^p(\Omega^\varepsilon) \), \( 1 \leq p \leq \infty \), and for every \( \varphi \in L^p(\omega \times Y^*) \) with \( p \in [1, \infty) \) one has

\[
|||U_\varepsilon(\varphi)|||_{L^p(\Omega^\varepsilon)} \leq \left( \frac{1}{L_1L_2} \right)^{1/p} ||\varphi||_{L^p(\omega \times Y^*)}.
\]

iii) For every \( \varphi \in D(\omega \times Y^*) \) which is \( Y \) – periodic in the variables \( y_1 \) and \( y_2 \), one has

\[
\frac{\partial}{\partial x_i} U_\varepsilon(\varphi) = \frac{1}{\varepsilon} U_\varepsilon \left( \frac{\partial \varphi}{\partial y_i} \right), \quad \text{for } i = 1, 2, 3.
\]

**Proof.** The proof of statements i) and ii) follows directly from the lines of the corresponding one in the case of two dimensional thin domains, see [3]. We prove here the last assertion. Let \( \psi \in D(\Omega^\varepsilon) \). Then, by property i) above and (4.2) we have

\[
\int_{\Omega^\varepsilon} U_\varepsilon(\varphi) \frac{\partial \psi}{\partial x_i} \, dx = \frac{\varepsilon}{L_1L_2} \int_{\omega \times Y^*} \varphi T_\varepsilon \left( \frac{\partial \psi}{\partial x_i} \right) \, d\hat{x}d\hat{y} = \frac{1}{L_1L_2} \int_{\omega \times Y^*} \varphi \frac{\partial}{\partial y_i} T_\varepsilon(\psi) \, d\hat{x}d\hat{y} = -\frac{1}{L_1L_2} \int_{\omega \times Y^*} \frac{\partial \varphi}{\partial y_i} T_\varepsilon(\psi) \, d\hat{x}d\hat{y} = -\frac{1}{\varepsilon} \int_{\Omega^\varepsilon} U_\varepsilon \left( \frac{\partial \varphi}{\partial y_i} \right) \psi \, dx. \]

9
5 Some convergence results for the velocity

In this section we state some weak convergences for the velocity field taking into account the a priori estimates (3.1) and (3.2). First, we define the rescaled velocity field $U_\epsilon$ and we show some of the properties of its limit.

In order to analyze the asymptotic behavior of the velocity field, we first perform a simple and typical change of variables in thin domains which consists in stretching in the $x_3$-direction by a factor $1/\epsilon$, $y_3 = x_3/\epsilon$. Then, the thin domain $\Omega_\epsilon$ is transformed into the domain

$$\Omega_\epsilon = \left\{ (\hat{x}, y_3) \in \mathbb{R}^3 \mid \hat{x} \in \omega, 0 < y_3 < G(\hat{x}/\epsilon) \right\}.$$  

Notice that the rescaled domain $\tilde{\Omega}_\epsilon$ is not thin anymore although it still presents an oscillatory behavior on the upper boundary.

Then, through the rescaling which transforms $\Omega_\epsilon$ to $\tilde{\Omega}_\epsilon$ we introduce the following notation:

$$U_\epsilon(\hat{x}, y_3) = u_\epsilon(\hat{x}, \epsilon y_3), \text{ a.e. } (\hat{x}, y_3) \in \tilde{\Omega}_\epsilon,$$

$$(\nabla, U_\epsilon)_{i,j} = \partial_{x_j} U_i^\epsilon, \quad (\nabla, U_\epsilon)_{i,3} = \frac{1}{\epsilon} \partial_{y_3} U_i^\epsilon, \text{ for } i = 1, 2, 3, \quad j = 1, 2,$$

$$\text{div}_\epsilon U_\epsilon = \partial_{x_1} U_1^\epsilon + \partial_{x_2} U_2^\epsilon + \frac{1}{\epsilon} \partial_{y_3} U_3^\epsilon.$$

Since the domain $\tilde{\Omega}_\epsilon$ “converges” in some sense to the rectangular parallelepiped $\Omega = \omega \times (0, G_1)$, as is usual in classical homogenization, extension of $U$ to the whole $\Omega$ can be used to obtain suitable estimates in the fixed domain $\Omega$ and to pass to the limit.

**Proposition 12** Let $\tilde{U}_\epsilon \in H_0^1(\Omega)$ be the extension by zero of $U_\epsilon$ to the rectangle $\Omega = \omega \times (0, G_1)$. Then, for a subsequence of $\epsilon$, still denoted by $\epsilon$, there exists $U \in H^1((0, G_1); L^2(\omega)^3)$ such that

$$\tilde{U}_\epsilon \xrightarrow{\epsilon \to 0} U \quad \text{in } H^1((0, G_1); L^2(\omega)^3).$$

Moreover, $U = (\hat{U}, 0)$ satisfies

$$\begin{cases}
\text{div}_\omega \left( \int_0^{G_1} \hat{U}(\hat{x}, y_3) dy_3 \right) = 0 \text{ in } \omega, \\
\left( \int_0^{G_1} \hat{U}(\hat{x}, y_3) dy_3 \right) \cdot nds = 0 \text{ on } \partial \omega.
\end{cases}$$

where $n$ is the outward normal to $\omega$.

**Proof.** From the a priori estimates (3.1) and (3.2) we deduce

$$\|\tilde{U}_\epsilon\|_{L^2(\Omega)^3} \leq C, \quad \|\nabla \tilde{U}_\epsilon\|_{L^2(\Omega)^3} \leq C, \quad \epsilon \|\partial_{x_i} \tilde{U}_\epsilon\|_{L^2(\Omega)^3} \leq C, \quad i = 1, 2.$$

Therefore, there exists $U \in H^1((0, G_1); L^2(\omega)^3)$ such that, for a subsequence, we have

$$\tilde{U}_\epsilon \xrightarrow{\epsilon \to 0} U \quad \text{in } L^2(\Omega)^3,$$

$$\frac{\partial \tilde{U}_\epsilon}{\partial y_3} \xrightarrow{\epsilon \to 0} \frac{\partial U}{\partial y_3} \quad \text{in } L^2(\Omega)^3,$$

$$\epsilon \frac{\partial \tilde{U}_\epsilon}{\partial x_i} \xrightarrow{\epsilon \to 0} z_i \quad \text{in } L^2(\Omega)^3, \quad i = 1, 2.$$
Moreover, taking into account that $\frac{\partial \hat{U}_i}{\partial x_i}$ is bounded in $H^{-1}(\Omega)^3$ we get $z_i = 0$, $i = 1, 2$.

Now we are going to prove that $U_3 = 0$. The incompressibility condition implies that

$$\epsilon \frac{\partial \hat{U}_1}{\partial x_1} + \epsilon \frac{\partial \hat{U}_2}{\partial x_2} + \frac{\partial \hat{U}_3}{\partial y_3} = 0. \quad (5.3)$$

Consequently, we have

$$\int_{\Omega} (\epsilon \frac{\partial \hat{U}_1}{\partial x_1} + \epsilon \frac{\partial \hat{U}_2}{\partial x_2} + \frac{\partial \hat{U}_3}{\partial y_3}) \varphi \, dx = 0, \forall \varphi \in D(\Omega).$$

Passing to the limit we get

$$\int_{\Omega} \frac{\partial U_3}{\partial y_3} \varphi \, dx = 0, \forall \varphi \in D(\Omega), \quad (5.4)$$

which implies that $U_3$ does not depend on $y_3$.

On the other side, the continuity of the trace operator from the space of functions $v$ such that $\|\tilde{v}\|_{L^2(\Omega)}$ and $\|\partial y_3 \tilde{v}\|_{L^2(\Omega)}$ is bounded to $L^2(\omega \times \{G_1\})$ and to $L^2(\omega \times \{0\})$ implies

$$U(\hat{x}, 0) = U(\hat{x}, G_1) = 0. \quad (5.5)$$

Hence, combining (5.4) and (5.5) we prove that $U_3 = 0$.

Finally we prove (5.2). Let $\varphi \in D(\omega)$. Multiplying (5.3) by $\frac{1}{\epsilon} \varphi$ and integrating by parts we get

$$\int_{\Omega} (\hat{U}_1 \frac{\partial \varphi}{\partial x_1} + \hat{U}_2 \frac{\partial \varphi}{\partial x_2}) \, d\hat{x} \, dy = 0. \quad (5.6)$$

Passing to the limit, thanks to convergence (5.1) we get the result.

Now, we have to take into account that the extension by zero of the velocity does not capture the effects of the rough boundary. Therefore, in the next proposition we get the limit for the unfolded velocity field $\mathcal{T}_\epsilon(u_\epsilon)$ which helps us to understand how the microscopic geometry of the domain affects the behavior of the fluid. Moreover, we show the relationship between this limit and $U$.

**Proposition 13** Let $u_\epsilon$ be the solution of (2.4). Then, for a subsequence of $\epsilon$ still denoted by $\epsilon$, there exists $u \in L^2(\omega; H^1(Y^*)^3)$ such that,

$$\mathcal{T}_\epsilon(u_\epsilon) \xrightarrow{\mathcal{C}} 0 u \quad w - L^2(\omega; H^1(Y^*)^3), \quad (5.6)$$

$$\epsilon \mathcal{T}_\epsilon \left( \frac{\partial u_\epsilon}{\partial x_i} \right) \xrightarrow{\mathcal{C}} \frac{\partial u}{\partial y_i}, \quad w - L^2(\omega \times Y^*)^3, \quad i = 1, 2, 3, \quad (5.7)$$

$$\text{div}_Y u = 0 \quad \text{in} \quad \omega \times Y^*, \quad (5.8)$$

$$u = 0 \quad \text{on} \quad \omega \times \{y_3 = 0\} \cup \omega \times \{y_3 = G(\hat{y})\}. \quad (5.9)$$

Moreover, since the function $u$ satisfies the following conditions

$$\int_0^{G_1} U \, dy_3 = \frac{1}{L_1 L_2} \int_{Y^*} u(\hat{x}, y) \, dy, \quad (5.10)$$

$$\int_{Y^*} u_3 \, dy = 0, \quad (5.11)$$

$$\text{div}_Y \left( \int_{Y^*} \hat{u} \, dy \right) = 0 \quad \text{in} \quad \omega, \quad (5.12)$$

$$\left( \int_{Y^*} \hat{u} \, dy \right) \cdot nds = 0 \quad \text{on} \quad \partial \omega. \quad (5.13)$$
Proof. From the a priori estimates (3.1), (3.2) and taking into account property iv) in Proposition 7 we have

\[ \|T_\epsilon(u_\epsilon)\|_{L^2(\omega \times Y^*)} \leq C, \quad \|\epsilon T_\epsilon \left( \frac{\partial u_\epsilon}{\partial x_i} \right)\|_{L^2(\omega \times Y^*)} \leq C, \quad i = 1, 2, 3. \]

Therefore, in view of property iv) in Proposition 7 we can ensure that there exists \( u \in L^2(\omega, H^1(Y^*))^3 \), such that, up to subsequences, convergences (5.6) and (5.7) hold.

Moreover, since \( \mathbf{u}_\epsilon \in V_\epsilon \) we have

\[ \sum_{i=1}^{3} T_\epsilon \left( \frac{\partial u_\epsilon}{\partial x_i} \right) = 0. \]

Then, multiplying the equality above by \( \epsilon \), using (4.2) and passing to the limit we easily obtain (5.8).

Finally, by using the \( Y \)-periodicity of the function \( G \) and taking into account that \( \mathbf{u}_\epsilon \) is zero on the boundary of \( \Omega_\epsilon \) we get

\[ T_\epsilon(u_\epsilon)|_{\omega \times \{y_3=0\}} = T_\epsilon(u_\epsilon)|_{y_3=0} = T_\epsilon(u_\epsilon)|_{x_3=0} = 0, \]

\[ T_\epsilon(u_\epsilon)|_{\omega \times \{y_3=G(y)\}} = \mathbf{u}_\epsilon \left( \epsilon \left[ \frac{\hat{x}}{\epsilon} \right] L + \epsilon \hat{y}, \epsilon G(\hat{y}) \right) = \mathbf{u}_\epsilon \left( \epsilon \left[ \frac{\hat{x}}{\epsilon} \right] L + \epsilon \hat{y}, \epsilon G(\hat{y}) \right) = 0, \]

which implies condition (5.9) on the trace of \( \mathbf{u} \).

Now we establish the relation between \( \mathbf{u} \) and the limit of the rescaled velocity \( \mathbf{U} \).

To do this, we consider \( \varphi \in D(\omega)^3 \). Then, using the definition of the rescaled operator and the unfolding operator we have

\[ \int_{\Omega} \tilde{U}_\epsilon \varphi \ d\hat{x} dy_3 = \int_{\Omega_\epsilon} U_\epsilon \varphi \ d\hat{x} dy_3 = \frac{1}{\epsilon} \int_{\Omega_\epsilon} u_\epsilon(x_1, x_2, x_3) \varphi(x_1, x_2) \ dx \]

\[ = \frac{1}{L_1L_2} \int_{\omega \times Y^*} T_\epsilon(u_\epsilon)T_\epsilon(\varphi)d\hat{x}dy. \]

Taking into account convergences (5.1) and (5.6) we can pass to the limit on the left and right hand side. Thus, we obtain

\[ \int_{\Omega} \mathbf{U} \varphi \ d\hat{x} dy_3 = \frac{1}{L_1L_2} \int_{\omega \times Y^*} \mathbf{u} \varphi(\hat{x}) \ d\hat{x} dy. \]

Consequently, we have

\[ \int_{\omega} \left( \int_0^{\tilde{G}_1} U(\hat{x}, y_3) dy_3 \right) \varphi \ d\hat{x} = \frac{1}{L_1L_2} \int_{\omega} \left( \int_{Y^*} u(\hat{x}, \hat{y}, y_3) dy_3 \right) \varphi d\hat{x} \quad \forall \varphi \in D(\omega)^3, \]

which is (5.10).

Moreover, since \( U^3 = 0 \) we have \( \int_{Y^*} u_3 d\hat{y} = 0 \). Finally, (5.2) and (5.10) immediately imply (5.12)

and (5.13). \( \blacksquare \)

6 Convergence results for the pressure

Obtaining appropriate convergences for the pressure is not immediate. Notice that, from the a priori estimate (3.3) and by using the Nečas inequality we have

\[ \|p_\epsilon\|_{L^2(\Omega_\epsilon)} \leq C(\Omega_\epsilon) \|\nabla p_\epsilon\|_{L^2(\Omega_\epsilon)} \leq \epsilon C(\Omega_\epsilon). \]
Therefore, it is no obvious how to obtain an estimate of the pressure in order to get a convergence result. To overcome this difficulty in previous papers the extension operator introduced by Tartar [27] was used. In [4, 20] an adaptation for thin domains was performed. However, in contrast to previous papers, we are going to obtain a convergence result for the unfolded pressure without extension operators. Notice that this approach allows to assume milder hypothesis on the regularity of the thin domain. In fact, instead of consider a uniformly bounded extension operator constructed on the unit cell we will prove that the sequence $T_\varepsilon(p_\varepsilon)$ is bounded in a fixed space $L^2(\omega \times Y^*)$.

Observe that the convergence for $T_\varepsilon(p_\varepsilon)$ gives us information on the original sequence, see Proposition [13].

**Proposition 14** Let $(u_\varepsilon, p_\varepsilon)$ be the solution of (2.3). Then, for a subsequence of $\varepsilon$ still denoted by $\varepsilon$, there exists $p \in L^2(\omega \times Y^*)$, independent of $y$, such that,

$$T_\varepsilon(p_\varepsilon) \xrightarrow{\varepsilon} p \quad w - L^2(\omega \times Y^*).$$

**Proof.** First, we define a family of operators which allows us to work in a domain independent of $\varepsilon$ where we will apply the Nečas inequality

$$(R_\varepsilon(p_\varepsilon))_{ij}(z_1, z_2, y_1, y_2, y_3) = T_\varepsilon(p_\varepsilon)(\varepsilon L_1(z_1+i), \varepsilon L_2(z_2+j), y_1, y_2, y_3),$$

for a.e. $(z_1, z_2, y_1, y_2, y_3) \in \omega \times Y^*$, with $i = 0, 1, \cdots, N_\varepsilon$ and $j = 0, 1, \cdots, M_\varepsilon$.

Taking into account that the unfolding operator is constant with respect to the first two variables in every $\omega_\varepsilon^i \times Y^*$ and performing a simple change of variables, for any $\Psi \in D(\omega \times Y^*)^5$ we have

$$\int_{\omega \times Y^*} (R_\varepsilon(p_\varepsilon))_{ij} \nabla \Psi d\hat{x}dy = \int_{\omega \times Y^*} (R_\varepsilon(p_\varepsilon))_{ij} \nabla \Psi d\hat{x}dy = \frac{1}{\varepsilon^2 L_1 L_2} \int_{\omega_\varepsilon^i \times Y^*} T_\varepsilon(p_\varepsilon) \nabla \Psi_{ij} d\hat{x}dy = \frac{1}{\varepsilon^2 L_1 L_2} \int_{\omega \times Y^*} T_\varepsilon(p_\varepsilon) \nabla \Psi_{ij} d\hat{x}dy,$$

where $\Psi_{ij}(\hat{x}, y) = \Psi(\frac{z_1 - \varepsilon L_1}{\varepsilon L_1}, \frac{z_2 - \varepsilon L_2}{\varepsilon L_2}, y)$ and $\tilde{\Psi}_{ij}$ denotes the extension by zero of $\Psi_{ij}$ to the whole $\omega \times Y^*$.

Now, by using basic properties of the adjoint operator $\mathcal{U}_\varepsilon$, see i) and iii) in Proposition [11] we get the following equality

$$\frac{1}{\varepsilon^2 L_1 L_2} \int_{\omega \times Y^*} T_\varepsilon(p_\varepsilon) \nabla \Psi_{ij} d\hat{x}dy = \frac{1}{\varepsilon^3} \int_{\Omega_\varepsilon} p_\varepsilon \mathcal{U}_\varepsilon(\nabla \tilde{\Psi}_{ij}) dx = \frac{1}{\varepsilon^2} \int_{\Omega_\varepsilon} p_\varepsilon \mathcal{U}_\varepsilon(\tilde{\Psi}_{ij}) dx.$$

Therefore, combining (6.2) and (6.3) and taking into account the a priori estimate (3.3) and Proposition [11] we have

$$\int_{\omega \times Y^*} (R_\varepsilon(p_\varepsilon))_{ij} \nabla \Psi d\hat{x}dy \leq C |||\nabla_x(\mathcal{U}_\varepsilon(\tilde{\Psi}_{ij}))|||_{L^2(\Omega_\varepsilon)} = \frac{C}{\varepsilon} \||\mathcal{U}_\varepsilon(\nabla_y \tilde{\Psi}_{ij})|||_{L^2(\Omega_\varepsilon)}$$

$$\leq \frac{C}{\varepsilon} ||\nabla_y \tilde{\Psi}_{ij}|||_{L^2(\omega \times Y^*)}.$$
Then, from (6.4) we obtain
\[ \int_{\omega \times Y^*} (\mathcal{R}_\epsilon(p_\epsilon))_{ij} \text{div} \Psi \, d\tilde{z} \, dy \leq C \| \nabla_y \Psi \|_{L^2(\omega \times Y^*}, \quad \forall \Psi \in D(\omega \times Y^*). \]
Consequently,
\[ \| \nabla (\mathcal{R}_\epsilon(p_\epsilon))_{ij} \|_{(H^{-1}(\omega \times Y^*))} \leq C, \quad \text{for } i = 0, 1, \cdots, \epsilon, \quad j = 0, 1, \cdots, M_\epsilon. \]
Consequently, by using the Nečas inequality we have
\[ \| (\mathcal{R}_\epsilon(p_\epsilon))_{ij} \|_{L^2(\omega \times Y^*)} \leq C \| \nabla (\mathcal{R}_\epsilon(p_\epsilon))_{ij} \|_{H^{-1}(\omega \times Y^*)} \leq C. \]
In view of the definition of \((\mathcal{R}_\epsilon(p_\epsilon))_{ij}\) we have
\[ \| \mathcal{T}_\epsilon(p_\epsilon) \|_{L^2(\omega \times Y^*)}^2 = c^2 L_1 L_2 \| (\mathcal{R}_\epsilon(p_\epsilon))_{ij} \|_{L^2(\omega \times Y^*)}^2 \leq C \epsilon^2 \text{ for } i = 0, 1, \cdots, \epsilon, \quad j = 0, 1, \cdots, M_\epsilon. \]
Finally, using the inequality above we have
\[ \| \mathcal{T}_\epsilon(p_\epsilon) \|_{L^2(\omega \times Y^*)}^2 = \sum_{i=1}^{\epsilon} \sum_{j=1}^{M_\epsilon} \| \mathcal{T}_\epsilon(p_\epsilon) \|^2_{L^2(\omega \times Y^*)} \leq C N_\epsilon M_\epsilon \epsilon^2 \leq C, \]
which implies, by weak compactness, the following convergence
\[ \mathcal{T}_\epsilon(p_\epsilon) \xrightarrow{\epsilon \to 0} p \quad \text{in } L^2(\omega \times Y^*). \]
Finally we shall prove that \(p\) does not depend on \(y\). Let us consider \(v^\epsilon(x) = \phi(x) \psi(x/\epsilon)\) where \(\phi \in D(\omega)\) and \(\psi \in H^1(\epsilon)\) such that \(\psi = 0\) on \(\{y_3 = 0\} \cup \{y_3 = G(\tilde{y})\}\). In view of \(vi\) in Proposition \(\mathcal{P}\), \(v^\epsilon \in H^1_0(\Omega^\epsilon)\) and satisfies
\[ \frac{\partial v^\epsilon}{\partial x_i} = \frac{\partial \phi}{\partial x_i} (\tilde{x}) \psi \left( \frac{x}{\epsilon} \right) + \frac{1}{\epsilon} \phi (\tilde{x}) \frac{\partial \psi}{\partial y_i} \left( \frac{x}{\epsilon} \right), \quad i = 1, 2, \quad \frac{\partial v^\epsilon}{\partial x_3} = \frac{1}{\epsilon} \phi (\tilde{x}) \frac{\partial \psi}{\partial y_3} \left( \frac{x}{\epsilon} \right), \]
\[ \text{div}_x (v^\epsilon) = \hat{\psi} \left( \frac{x}{\epsilon} \right) \text{div}_x \phi (\tilde{x}) + \frac{1}{\epsilon} \text{div}_y \psi \left( \frac{x}{\epsilon} \right). \]
Hence, by using properties \(v), vi\) and \(vii\) in Proposition \(\mathcal{P}\) we get
\[ \mathcal{T}_\epsilon (v^\epsilon) \xrightarrow{\epsilon \to 0} \phi \psi \quad \text{s-} L^2(\omega \times \epsilon)^3, \]
\[ \epsilon \mathcal{T}_\epsilon \left( \frac{\partial v^\epsilon}{\partial x_i} \right) \xrightarrow{\epsilon \to 0} \phi \frac{\partial \psi}{\partial y_i} \quad \text{s-} L^2(\omega \times \epsilon)^3, \quad i = 1, 2, \]
\[ \epsilon \mathcal{T}_\epsilon \left( \frac{\partial v^\epsilon}{\partial x_3} \right) \xrightarrow{\epsilon \to 0} \phi \frac{\partial \psi}{\partial y_3} \quad \text{s-} L^2(\omega \times \epsilon)^3, \]
\[ \epsilon \mathcal{T}_\epsilon \left( \text{div}_x v^\epsilon \right) \xrightarrow{\epsilon \to 0} \text{div}_y \psi \quad \text{s-} L^2(\omega \times \epsilon)^3. \]
Now let us take \(u_\epsilon + \epsilon v_\epsilon\) as test function in (2.5). We have
\[ \epsilon \int_{\Omega^\epsilon} p_\epsilon \text{div}_x v_\epsilon \, dx \leq \mu \epsilon^3 \int_{\Omega^\epsilon} \nabla u_\epsilon \cdot \nabla v_\epsilon \, dx + g \epsilon^3 \int_{\Omega^\epsilon} |\nabla v_\epsilon| \, dx - \epsilon \int_{\Omega^\epsilon} f_\epsilon v_\epsilon \, dx. \]
Then, we apply the unfolding operator to the previous variational inequality. By property \(iii\) in Proposition 7, we have
\[
\epsilon \int_{\omega \times Y^*} T_\epsilon(p_\epsilon) T_\epsilon(\text{div}_X v_\epsilon) \, d\hat{x}dy \leq \mu^2 \int_{\omega \times Y^*} T_\epsilon(\nabla u_\epsilon) \cdot T_\epsilon(\nabla v_\epsilon) \, d\hat{x}dy + g \int_{\omega \times Y^*} T_\epsilon(|\nabla v_\epsilon|) \, d\hat{x}dy
- \int_{\omega \times Y^*} T_\epsilon(f_\epsilon) T_\epsilon(v_\epsilon) \, d\hat{x}dy.
\]
(6.10)

According to convergences (5.7), (6.7), (6.8), we get
\[
\mu^2 \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla v_\epsilon \, dx \rightarrow \mu \int_{\omega \times Y^*} \nabla_y u \cdot (\phi \nabla_y \psi) \, d\hat{x}dy,
\]
(6.11)
hence the first integral in the right-hand side of (6.10) satisfies
\[
\mu^3 \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla v_\epsilon \, dx \rightarrow 0.
\]
(6.12)

By Proposition 9 one has the following equality
\[
ge \epsilon \int_{\omega \times Y^*} T_\epsilon(|\nabla v'|) \, d\hat{x}dy = g \int_{\omega \times Y^*} |\nabla_y T_\epsilon(v')| \, d\hat{x}dy.
\]
Notice that
\[
\left| \int_{\omega \times Y^*} |\nabla_y T_\epsilon(v')| \, d\hat{x}dy - \int_{\omega \times Y^*} |\phi \nabla_y \psi| \, d\hat{x}dy \right| \leq \int_{\omega \times Y^*} |\nabla_y T_\epsilon(v') - \phi \nabla_y \psi| \, d\hat{x}dy
= \int_{\omega \times Y^*} |\epsilon T_\epsilon(\nabla \phi) + T_\epsilon(\phi) \nabla_y \psi - \phi \nabla_y \psi| \, d\hat{x}dy.
\]

By using properties \(vii\) in Proposition 7 we have
\[
\epsilon T_\epsilon(\nabla \phi) \xrightarrow{\epsilon \to 0} 0 \quad \text{s.-L}^2(\omega \times Y^*), \quad T_\epsilon(\phi) \xrightarrow{\epsilon \to 0} \phi \quad \text{s.-L}^2(\omega \times Y^*),
\]
then we get the following convergence
\[
ge \epsilon \int_{\omega \times Y^*} T_\epsilon(|\nabla v'|) \, d\hat{x}dy \rightarrow g \int_{\omega \times Y^*} |\phi \nabla_y \psi| \, d\hat{x}dy.
\]
(6.13)

Hence the second integral in the right-hand side of (6.10) satisfies
\[
ge^2 \int_{\omega \times Y^*} T_\epsilon(|\nabla v'|) \, d\hat{x}dy \rightarrow 0.
\]
(6.14)

By \(vi\) in Proposition 7 and (6.6) we get
\[
\int_{\omega \times Y^*} T_\epsilon(f_\epsilon) T_\epsilon(v_\epsilon) \, d\hat{x}dy \rightarrow \int_{\omega \times Y^*} f(\phi \psi) \, d\hat{x}dy.
\]
(6.15)

Finally, by (7.2) and (6.9) we obtain
\[
\epsilon \int_{\omega \times Y^*} T_\epsilon(p_\epsilon) T_\epsilon(\text{div}_X v_\epsilon) \, d\hat{x}dy \rightarrow \int_{\omega \times Y^*} p \phi \nabla_y \psi.
\]
(6.16)
Then, by (6.12), (6.14), (6.15) and (6.16), we can pass to the limit when ε goes to zero in (6.10) and get
\[ \int_{\omega \times Y^*} p \phi \text{div}_Y \psi \leq 0 \quad \forall \phi \in \mathcal{D}(\omega) \text{ and } \psi \in H^1(Y^*)^3, \]
which by density of the tensor product \( \mathcal{D}(\omega) \otimes H^1(Y^*)^3 \) means
\[ \int_{\omega \times Y^*} p \text{div}_Y \Psi = 0 \quad \forall \Psi \in L^2(\omega; H^1(Y^*))^3, \]
i.e. the pressure \( p \) doesn’t depend on \( y \). ■

If we assume that the thin domain is given by an exact number of basic cells we can establish an interesting relationship between the limit of the unfolded pressure, \( \mathcal{T}_e(p_\epsilon) \), and the pressure \( p_\epsilon \). To do that we introduce the following function
\[ P_\epsilon = \frac{1}{\epsilon G_0} \int_0^{\epsilon G_0} p_\epsilon(\hat{x}, s) ds. \]  

**Proposition 15** Let \( P_\epsilon \) be the function defined in (6.17). There exists a constant \( C \) independent of \( \epsilon \) such that \( \| P_\epsilon \|_{L^2(\omega)} \leq C \). Moreover, for a subsequence of \( \epsilon \) still denoted by \( \epsilon \), \( P_\epsilon \) satisfies the following convergence
\[ P_\epsilon \overset{\epsilon \to 0}{\longrightarrow} p \quad w - L^2(\omega). \]

**Proof.** Notice that if we assume that the thin domain is given by a exact numbers of basic cells we have \( \| \mathcal{T}_\epsilon(\varphi) \|_{L^p(\omega \times Y^*)} = (L_1 L_2)^{\frac{3}{p}} \| \varphi \|_{L^p(\Omega_\epsilon)} \). Therefore, using Holder’s inequality we get
\[
\| P_\epsilon \|_{L^2(\omega)} = \left( \int_\omega \frac{1}{\epsilon G_0} \int_0^{\epsilon G_0} |p_\epsilon(\hat{x}, s)|^2 ds d\hat{x} \right)^{\frac{1}{2}} 
\leq \left( \int_\omega \frac{1}{\epsilon G_0} \int_0^{\epsilon G_0} |p_\epsilon(\hat{x}, s)|^2 ds d\hat{x} \right)^{\frac{1}{2}} 
\leq C \| p_\epsilon \|_{L^2(\Omega_\epsilon)} \leq C. 
\]
Moreover, for any function \( \varphi \in L^2(\omega) \) we have
\[ \int_\omega (P_\epsilon - p) \varphi d\hat{x} = \frac{1}{\epsilon G_0} \int_{\Omega_\epsilon} (p_\epsilon - p) \chi \varphi dx, \]
where \( \chi \) is the characteristic function of \( \Omega_\epsilon = \{ (\hat{x}, y_3) \in \mathbb{R}^3 \mid \hat{x} \in \omega, 0 < y_3 < \epsilon G_0 \} \).

Then, applying the unfolding operator we get
\[ \int_\omega (P_\epsilon - p) \varphi d\hat{x} = \frac{1}{G_0 L_1 L_2} \int_{\omega \times Y^*} (\mathcal{T}_\epsilon(p_\epsilon) - \mathcal{T}_\epsilon(p)\mathcal{T}_\epsilon(\chi)\mathcal{T}_\epsilon(\varphi))d\hat{x}dy, \]
where \( \mathcal{T}_\epsilon(\chi) \) is the characteristic function of \( \omega \times \{ (\hat{y}, y_3) \in \mathbb{R}^3 \mid \hat{y} \in (0, L_1) \times (0, L_2), 0 < y_3 < G_0 \} \).
Thus, taking into account property viii) in Proposition 4 and convergence (6.11) we have the desired result
\[ \int_\omega (P_\epsilon - p) \varphi d\hat{x} = \frac{1}{G_0} \int_{\omega \times Y^*} (\mathcal{T}_\epsilon(p_\epsilon) - \mathcal{T}_\epsilon(p)\mathcal{T}_\epsilon(\chi)\mathcal{T}_\epsilon(\varphi))d\hat{x}dy \overset{\epsilon \to 0}{\longrightarrow} 0. \]
7 The limit problem

We can now state the main result of our paper.

**Theorem 16** Let \( (u_\epsilon, p_\epsilon) \) be the solution of (2.3) with \( f_\epsilon \in L^2(\Omega_\epsilon)^3 \) satisfying (2.3) and (2.8). Then, there exist \( u \in L^2(\omega, H^1(Y^*)^3) \) and \( p \in L^2(\omega) \) such that

\[
\mathcal{T}_\epsilon(u_\epsilon) \xrightarrow{\epsilon \to 0} u \quad w - L^2(\omega; H^1(Y^*)^3) \tag{7.1}
\]

and

\[
\mathcal{T}_\epsilon(p_\epsilon) \xrightarrow{\epsilon \to 0} p \quad w - L^2(\omega \times Y^*) \tag{7.2}
\]

where the couple \((u, p)\) satisfies the following limit problem

\[
\frac{\mu}{\omega \times Y^*} \nabla_y u \cdot \nabla_y (\Psi - u) \, d\tilde{x} \, dy + g \int_{\omega \times Y^*} |\nabla_y \Psi| \, d\tilde{x} \, dy - g \int_{\omega \times Y^*} |\nabla_y u| \, d\tilde{x} \, dy
\geq \int_{\omega \times Y^*} f(\Psi - u) \, d\tilde{x} \, dy - \int_{\omega \times Y^*} \nabla \hat{\phi} \left( \hat{\Psi} - \hat{u} \right) \, d\tilde{x} \, dy \quad \forall \, \Psi \in \mathcal{V},
\tag{7.3}
\]

where

\[
\mathcal{V} = \left\{ \Psi \in L^2(\omega; H^1(Y^*))^3 : \Psi = 0 \mbox{ on } \omega \times \{y_3 = 0\} \cup \omega \times \{y_3 = G(\hat{y})\}, \mbox{ div}_y \Psi = 0, \right. \\
\left. \mbox{div}_{\hat{\omega}} \left( \int_{Y^*} \hat{\Psi} \, dy \right) = 0 \mbox{ in } \omega, \left( \int_{Y^*} \hat{\Psi} \, dy \right) \cdot nds = 0 \mbox{ on } \partial \omega \right\}.
\]

**Proof.** Notice that in view of Proposition 13 and Proposition 14 there exist \( u \in L^2(\omega, H^1(Y^*)^3) \) and \( p \in L^2(\omega \times Y^*) \) such that (7.1) and (7.2) are satisfied.

Now, we want to prove that the couple \((u, p)\) satisfies the limit problem (7.3). To this aim we consider \( \psi(x) = \phi(x) \varphi(x/\epsilon) \) where \( \phi \in \mathcal{D}(\omega) \) and \( \psi \in H^1(Y^*)^3 \) such that \( \psi = 0 \) on \( \{y_3 = 0\} \cup \{y_3 = G(\hat{y})\} \) and \( \mbox{div}_y \psi = 0 \). As previously it is easy to show that \( \psi^\epsilon \) satisfies (6.5). Moreover it holds

\[
\mbox{div}_x(\psi^\epsilon) = \hat{\psi}(\frac{x}{\epsilon}) \mbox{div}_\hat{x} \phi(\hat{x}). \tag{7.4}
\]

Hence we get (6.6), (6.7), (6.8) and

\[
\mathcal{T}_\epsilon \left( \mbox{div}_x \psi^\epsilon \right) \xrightarrow{\epsilon \to 0} \hat{\psi} \mbox{div}_\hat{x} \phi \quad w-L^2(\omega \times Y^*)^3. \tag{7.5}
\]

Let us take \( \psi = \psi^\epsilon \) as test function in (2.3) and apply the unfolding operator. Then we get

\[
\mu^2 \int_{\omega \times Y^*} \mathcal{T}_\epsilon(\nabla u_\epsilon) \cdot \mathcal{T}_\epsilon(\nabla (v^\epsilon - u_\epsilon)) \, d\tilde{x} \, dy + \mu \int_{\omega \times Y^*} |\nabla (\psi^\epsilon)| \, d\tilde{x} \, dy - \mu \int_{\omega \times Y^*} |\nabla u_\epsilon| \, d\tilde{x} \, dy
\geq \int_{\omega \times Y^*} \mathcal{T}_\epsilon(f_\epsilon) \mathcal{T}_\epsilon(v^\epsilon - u_\epsilon) \, d\tilde{x} \, dy + \int_{\omega \times Y^*} \mathcal{T}_\epsilon(p_\epsilon) \mathcal{T}_\epsilon \left( \mbox{div}_x (v^\epsilon - u_\epsilon) \right) \, d\tilde{x} \, dy. \tag{7.6}
\]

The first integral on the left-hand side of (7.6) can be written as

\[
\mu^2 \int_{\omega \times Y^*} \mathcal{T}_\epsilon(\nabla u_\epsilon) \cdot \mathcal{T}_\epsilon(\nabla (v^\epsilon - u_\epsilon)) \, d\tilde{x} \, dy = \mu^2 \int_{\omega \times Y^*} \mathcal{T}_\epsilon(\nabla u_\epsilon) \cdot \mathcal{T}_\epsilon(\nabla (v^\epsilon)) \, d\tilde{x} \, dy - \mu^2 \int_{\omega \times Y^*} \mathcal{T}_\epsilon(\nabla u_\epsilon) \cdot \mathcal{T}_\epsilon(\nabla u_\epsilon) \, d\tilde{x} \, dy.
\]
According to convergences (5.7), (6.7), (6.8), for the first term we have (6.11). Moreover, by standard weak lower-semicontinuity argument we have
\[
\liminf_{\epsilon \to 0} \varepsilon^2 \mu \int_{\omega \times Y^*} |\nabla (\nabla u_\epsilon)|^2 \, d\tilde{x} \, d\tilde{y} \geq \mu \int_{\omega \times Y^*} |\nabla_y u|^2 \, d\tilde{x} \, d\tilde{y}. \tag{7.7}
\]
By following the same argument as before, the second integral on the left-hand side of (7.3) satisfies (6.13). Moreover, Proposition 9 (5.7) and the standard weak lower-semicontinuity argument gives
\[
\liminf_{\epsilon \to 0} g\mu \int_{\omega \times Y^*} \mathcal{T}_0(|\nabla u_\epsilon|) \, d\tilde{x} \, d\tilde{y} \geq g \int_{\omega \times Y^*} |\nabla_y u| \, d\tilde{x} \, d\tilde{y}. \tag{7.8}
\]
By (vi) in Proposition 7 and from convergences (5.6) and (6.6) we get
\[
\int_{\omega \times Y^*} \mathcal{T}_0(f_\epsilon) \mathcal{T}_0(v_\epsilon - u_\epsilon) \, d\tilde{x} \, d\tilde{y} \to \int_{\omega \times Y^*} f(\phi_\epsilon - u) \, d\tilde{x} \, d\tilde{y}. \tag{7.9}
\]
Taking into account (7.4) and that \( \text{div}_x u_\epsilon = 0 \) the second integral in the right-hand side of (7.6) satisfies
\[
\int_{\omega \times Y^*} \mathcal{T}_0(p_\epsilon) \mathcal{T}_0(\text{div}_x (v_\epsilon - u_\epsilon)) \, d\tilde{x} \, d\tilde{y} = \int_{\omega \times Y^*} \mathcal{T}_0(p_\epsilon) \mathcal{T}_0(\text{div}_x (v_\epsilon)) \, d\tilde{x} \, d\tilde{y} = \int_{\omega \times Y^*} \mathcal{T}_0(p_\epsilon) \mathcal{T}_0(\text{div}_x \phi) \psi \, d\tilde{x} \, d\tilde{y}. \tag{7.10}
\]
Then, by convergences (7.2) and (7.5), we obtain
\[
\int_{\omega \times Y^*} \mathcal{T}_0(p_\epsilon) \mathcal{T}_0(\text{div}_x (v_\epsilon - u_\epsilon)) \, d\tilde{x} \, d\tilde{y} \to \int_{\omega \times Y^*} p \text{div}_x \phi \psi \, d\tilde{x} \, d\tilde{y}. \tag{7.10}
\]
Therefore, by collecting together convergences (6.11), (6.13), (7.7), (7.10), we obtain the following variational inequality
\[
\mu \int_{\omega \times Y^*} \nabla_y u \cdot (\phi \nabla_y \psi - \nabla_y u) \, d\tilde{x} \, d\tilde{y} + g \int_{\omega \times Y^*} |\phi \nabla_y \psi| \, d\tilde{x} \, d\tilde{y} - g \int_{\omega \times Y^*} |\nabla_y u| \, d\tilde{x} \, d\tilde{y}
\geq \int_{\omega \times Y^*} f(\phi_\epsilon - u) \, d\tilde{x} \, d\tilde{y} + \int_{\omega \times Y^*} p \text{div}_x \phi_\epsilon \psi \, d\tilde{x} \, d\tilde{y} \tag{7.11}
\]
\[
\forall \phi \in \mathcal{D}(\omega) \text{ and } \psi \in H^1(Y^*)^3 \text{ with } \text{div}_y (\psi) = 0 \text{ and } \psi = 0 \text{ on } \{y_3 = 0\} \cup \{y_3 = G(\tilde{y})\}
\]
which by density implies
\[
\mu \int_{\omega \times Y^*} \nabla_y u \cdot (\nabla_y \Psi - \nabla_y u) \, d\tilde{x} \, d\tilde{y} + g \int_{\omega \times Y^*} |\nabla_y \Psi| \, d\tilde{x} \, d\tilde{y} - g \int_{\omega \times Y^*} |\nabla_y u| \, d\tilde{x} \, d\tilde{y}
\geq \int_{\omega \times Y^*} f(\Psi - u) \, d\tilde{x} \, d\tilde{y} + \int_{\omega \times Y^*} p \text{div}_x \Psi \, d\tilde{x} \, d\tilde{y}, \tag{7.12}
\]
\[
\forall \Psi \in L^2(\omega; H^1(Y^*))^3, \text{ with } \text{div}_y \Psi = 0 \text{ and } \Psi = 0 \text{ on } \omega \times \{y_3 = 0\} \cup \omega \times \{y_3 = G(\tilde{y})\}.
\]
Since \( p \) does not depend on \( y \), by (5.12) and (5.13), we get
\[
\int_{\omega \times Y^*} p \text{div}_x \Psi \, d\tilde{x} \, d\tilde{y} = \int_{\omega \times Y^*} p \text{div}_x \left( \int_{Y^*} \tilde{u} \, d\tilde{y} \right) \, d\tilde{x} \, d\tilde{y} = \int_{\omega \times Y^*} p \text{div}_x \left( \hat{\Psi} - \hat{u} \right) \, d\tilde{x} \, d\tilde{y}
= \int_{\omega} p \text{div}_x \left( \int_{Y^*} (\hat{\Psi} - \hat{u}) \, d\tilde{y} \right) \, d\tilde{x} = -\int_{\omega \times Y^*} \nabla_x p (\hat{\Psi} - \hat{u}) \, d\tilde{x} \, d\tilde{y} + \int_{\partial \omega} p \left( \int_{Y^*} \hat{\Psi} \, d\tilde{y} \right) \cdot nds. \tag{7.13}
\]
Therefore (7.12) and (7.13) imply (7.3). 

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8 Conclusions

In this section we are interested on the solution of the limit problem \( (7.3) \) in the case of forces independent of the vertical variable, see Remark 3. As usual in the asymptotic study of fluids in thin domains and in classical porous media, we want to describe the limit problem introducing an auxiliary problem on the basic cell. More in particular, following the ideas of Lions and Sanchez-Palencia in [21] for the study of the Bingham flow in a classical porous medium, we want to show that the limit problem \( (7.3) \) in Theorem 16 can be interpreted as a non linear Darcy law. Therefore, if we assume the domain \( \Omega \) is given by an exact union of basic cells, we obtain the following proposition.

**Proposition 17** Let \((u_\epsilon, p_\epsilon)\) be the solution of \((2.5)\) with \(f_\epsilon(x) = (\hat{f}(x), 0)\) with \(\hat{f} \in L^2(\omega)^2\). Then, there exist \(U = (\hat{U}, 0) \in H^1((0, G_1); L^2(\omega)^3)\) and \(p \in L^2(\omega)\) such that

\[
\begin{align*}
\tilde{U}_\epsilon & \rightharpoonup U & \text{in } w - H^1((0, G_1); L^2(\omega)^3), \\
P_\epsilon & \rightharpoonup p & \text{in } w - L^2(\omega).
\end{align*}
\]

where \(\tilde{U}_\epsilon\) is the extension by zero of \(U_\epsilon(\hat{x}, y_3) = u_\epsilon(\hat{x}, \epsilon y_3)\) to the rectangle \(\Omega\). Moreover, \(\hat{V} = \int_0^{y_3} \hat{U} \, dy_3\) is the unique solution of

\[
\begin{align*}
\hat{V}(\hat{x}) & = A(\hat{f}(\hat{x}) - \nabla \hat{p}) & \text{in } \omega, \\
\text{div}_2 \hat{V} & = 0 & \text{in } \omega, \\
\hat{V}(\hat{x}) \cdot nds & = 0 & \text{on } \partial \omega,
\end{align*}
\]

where the nonlinear operator \(A(\cdot) : \mathbb{R}^2 \to \mathbb{R}^2\) is defined by

\[
A(\hat{\xi}) = \frac{1}{L_1 L_2} \int_{Y^*} \chi(\hat{\xi}) \, dy
\]

where \(\chi(\hat{\xi})\) is the unique solution of the following Bingham local problem on the basic cell

\[
\begin{align*}
\text{Find } \chi(\hat{\xi}) & \in \mathcal{V} \text{ such that}
\end{align*}
\]

\[
\begin{align*}
\mu \int_{Y^*} \nabla_y \chi(\hat{\xi}) \cdot \nabla_y (\Psi - \chi(\hat{\xi})) \, dy + g \int_{Y^*} |\nabla_y \Psi| \, dy - g \int_{Y^*} |\nabla_y \chi(\hat{\xi})| \, dy & \geq \int_{Y^*} \hat{\xi}(\Psi - \chi(\hat{\xi})) \, dy \\
\forall \Psi & \in \mathcal{V}.
\end{align*}
\]

**Proof.** Let us observe that under the hypotheses on the body force \(f_\epsilon\), the limit problem \((7.3)\) can be rewritten as

\[
\begin{align*}
\mu \int_{\omega \times Y^*} \nabla_y u \cdot \nabla_y (\Psi - u) \, dx dy & + g \int_{\omega \times Y^*} |\nabla_y \Psi| \, dx dy - g \int_{\omega \times Y^*} |\nabla_y u| \, dx dy \\
& \geq \int_{\omega \times Y^*} (\tilde{f} - \nabla \hat{p}) (\hat{\Psi} - \hat{u}) \, dx dy \quad \forall \Psi \in \mathcal{V}.
\end{align*}
\]

For every \(\hat{\xi} \in \mathbb{R}^2\) let \(\chi(\hat{\xi})\) be the unique solution of problem \((8.2)\).

From \((8.3)\) and \((8.2)\) we get

\[
u(\hat{x}, y) = \chi(y; \hat{f} - \nabla \hat{p}).
\]
By (5.12) and (5.13) we get
\[ \left( \int_{Y^*} \hat{u}(\hat{x}, y) d\nabla q \right)_{\omega} = 0 \quad \forall q \in H^1(\omega), \quad (8.4) \]
hence the pressure \( \hat{p} \) verifies
\[ \left( \int_{Y^*} \chi(y; \hat{f} - \nabla \hat{p}) d\nabla q \right)_{\omega} = 0 \quad \forall q \in H^1(\omega). \]
Defining the nonlinear operator \( A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) by
\[ A(\hat{\xi}) = \frac{1}{L_1 L_2} \int_{Y^*} \chi(y; \hat{\xi}) d\gamma, \]
the previous relation reads as
\[ \left( A(\hat{f} - \nabla \hat{p}), \nabla q \right)_{\omega} = 0 \quad \forall q \in H^1(\omega). \]
If we define the velocity of filtration as
\[ \hat{V}(x) = \frac{1}{L_1 L_2} \int_{Y^*} \hat{u}(\hat{x}, y) d\gamma = \int_{G_1} \hat{U}(\hat{x}) dy_3, \]
by taking into account (5.12), (5.13) and (8.4), we get the following nonlinear Darcy’s law
\[ \begin{cases} \hat{V} = A(\hat{f} - \nabla \hat{p}) & \text{in } \omega, \\ \hat{V} \cdot n = 0 & \text{on } \partial \omega, \\ \text{div}_x \hat{V} = 0 & \text{in } \omega. \end{cases} \]

Finally, we would like to point out that the newtonian fluid can be seen as a particular case of the Bingham fluid. Thus, taking \( g = 0 \) the initial problems corresponds to the following Stokes system:
\[ \mu \epsilon^2 \int_{\Omega_\epsilon} \nabla u_\epsilon \cdot \nabla v \, dx = \int_{\Omega_\epsilon} f_\epsilon v \, dx + \int_{\Omega_\epsilon} p_\epsilon \text{div}_x v \, dx, \quad \forall v \in (H^1_0(\Omega_\epsilon))^3, \]
where \( f_\epsilon(x) = (f(\hat{x}), 0) \) with \( \|f\|_{L^2(\omega)^2} \leq C. \)
Therefore, following a similar approach as the one used to get (7.3) we obtain to the limit the following problem
\[ \mu \int_{\omega \times Y^*} \nabla_y u \cdot \nabla_y \psi \, d\hat{x} dy = \int_{\omega \times Y^*} (f - \nabla \hat{p}) \psi \, d\hat{x} dy \forall \psi \in (L^2(\omega; H^1_0(Y^*))^3), \text{with div}_y \psi = 0. \]
In fact, notice that we can recover the convergence results given in [4], see also [17, 18] for a generalization to the unstationary case, without using extension operators. For instance, Theorem 3.1 and Theorem 3.2 are equivalent to our Proposition 12 and Proposition 14 respectively.

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