RESIDUE IN INTERSECTION HOMOLOGY
FOR QUASIHOMOGENEOUS SINGULARITIES

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1. Residues in homology and the problem of lift

Let \( M^{n+1} \) be a complex manifold and let \( K \) be a hypersurface in \( M \). Suppose \( K \) has isolated singularities. We think of \( K \) as obtained from a manifold-with-boundary \( K^\circ \) by shrinking the components of the boundary. For simplicity assume that \( n > 1 \). Then the intersection homology of \( K \) is isomorphic to \([7],[4]\):

\[
IH_n(K) = \text{im} ([K] \cap : H^n(K) \to H_n(K)) = \\
= \text{im} ([K^\circ] \cap : H^n(K^\circ, \partial K^\circ) \to H_n(K^\circ, \partial K^\circ)) = \\
= \text{im} (H^n(K^\circ, \partial K^\circ) \to H^n(K^\circ)) = \\
= \ker (H^n(K^\circ) \to H^n(\partial K^\circ)).
\]

In [13] we define a residue of a closed form \( \omega \in \Omega^{n+1}(M \setminus K) \) as the class of the Leray [9] residue form \( \text{res} \omega = [\text{Res} \omega] \in H^n(K \setminus \Sigma) \simeq H_n(K) \). All coefficients of homology and cohomology are in \( \mathbb{C} \). The following question arises: can one integrate the residue form over cycles intersecting singularity. The question comes from partial differential equations; [14]. It turns out, that certain integrals have a meaning, but it is not clear in which spaces cycles and integrals should be considered. The possibility to give a meaning to the symbol \( \int_\xi \text{Res} \omega \) which would not depend on the homology class of \( \xi \) is simply a lift of \([\text{Res} \omega]\) to cohomology:

\[
\begin{align*}
H^n(K \setminus \Sigma) & \simeq H_n(K) & \xrightarrow{PD} & H^n(K) \\
[\text{Res} \omega] & = \text{res} \omega & \longrightarrow & ?
\end{align*}
\]

The homological residue \( \text{res} \omega \) can be defined equivalently by the Alexander duality:

\[
\begin{align*}
H^{p+1}(M \setminus K) & \xrightarrow{\delta} H^{p+2}(M, M \setminus K) & \xrightarrow{[M] \cap} & H_{2n-p}(K) \\
\omega & \longmapsto \delta \omega & \longmapsto & \text{res} \omega
\end{align*}
\]

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The previous case was $p = n$. Now $p$ is arbitrary and $K$ is a variety with not necessary isolated singularities. The question is the same: does the homological residue $res \omega$ belong to the image of Poincaré duality map. The Poincaré duality map factors through the intersection homology:

$$H^{2n-p}(K) \xrightarrow{PD} H_p(K) \xrightarrow{} IH^{m}_{p}(X).$$

The question of lift to intersection homology also seems to be reasonable. In many cases algebraically defined elements in $H^{*}(K)$ lift to $IH^{m}_{*}(K)$, e.g. Chern-MacPherson classes or arbitrary algebraic cycles; [3].

Consider a case when $M$ is a projective manifold and $K = \bigcup_{i \in I} D_i$ is a sum of smooth divisors with normal crossings. Then

$$IH^{m}_{p}(K) = \bigoplus_{i \in I} H^{2n-p}(D_i).$$

One can easily show that:

**Proposition 1.** Homological residue of a class $c \in H^{p}(M \setminus K)$ can be lift to the intersection homology if and only if $c$ belongs to $W_{p+1}H^{p}(M \setminus K)$ – the $(p+1)$–st term of the Deligne weight filtration.

Belonging to $W_{p+1}H^{p}(M \setminus K)$ means that $c$ is represented by a smooth form

$$\omega \in \Omega^{p-1}(M) \wedge \Omega^{1}(\log \langle K \rangle),$$

see e.g. [8 §5]. The above proposition remains true for an arbitrary singular projective variety. To prove this one can use a resolution of $K$, then push the residue to the intersection homology of the resolution and pull down using the result of [3]. Unfortunately this procedure uses desingularization, weight filtration and functoriality of intersection homology. Each of these ingredients is rather mysterious and hard to compute. We will not follow this direction. We restrict our attention to the case of isolated quasihomogeneous singularities.

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**2. Valuation and quasihomogeneous functions**

Let $\mathbb{C}(z_0, \ldots, z_n)$ be the field of the rational functions on $n + 1$ variables and let

$$v : \mathbb{C}(z_0, \ldots, z_n) \to \mathbb{Q}$$

be a valuation satisfying

1) $v(z_i) = a_i > 0$;
2) $v(fg) = v(f)v(g)$;
3) if $f = \sum f_i$, $f_i$ monomial, then $v(f) = \max\{v(f_i)\}$

If $f$ is a sum of monomials of the same weight, then we say that $f$ is quasihomogeneous (with respect to the valuation $v$). We define the valuation on forms

$$v : \mathbb{C}(z_0, \ldots, z_n) \otimes \Lambda^* (\mathbb{C}^{n+1}^*) \to \mathbb{Q}$$
putting \( v(dz_i) = v(z_i) = a_i \).

Suppose that in each singular point the hypersurface \( K \) is given by an equation \( s = 0 \) with \( s \) quasihomogeneous (in some coordinates and valuation). We assume that \( v(s) = 1 \). We define a number

\[
\kappa = v(dz_0 \wedge \cdots \wedge dz_n) = \sum_{i=0}^{n} a_i.
\]

It coincides (after the change of the sign) with 'the complex oscillation indicator' defined in [1; 13.1 p.258] and it is an analytic invariant of a singularity type. We define a condition:

**Condition 2.** For any choice of \( k_i \in \mathbb{N} \cup \{0\} \), \( i = 0, \ldots, n \) we have \( \kappa + \sum k_i a_i \neq 1 \).

Of course the Condition 2 is satisfied if \( \kappa > 1 \).

**Example 3.** Let \( s = z_0^3 + z_1^3 + z_2^4 \). Then \( a_0 = a_1 = \frac{1}{3} \) and \( a_2 = \frac{1}{4} \), \( \kappa = \frac{11}{12} \). The Condition 2 is satisfied.

3. A simple criterion to lift

We investigate residues of meromorphic forms of the type \((n + 1, 0)\) with a first order pole on \( K \), i.e. the forms which can be locally written as \( \omega = g s \, dz_0 \wedge \cdots \wedge dz_n \) with \( g \) holomorphic.

**Theorem 4.** Suppose that \( K \) of dimension \( n \) has isolated singularities given by quasihomogeneous equations. Let \( \omega \in \Omega^{n+1,0}(M \setminus K) \) be a meromorphic form with a first order pole on \( K \). If the Condition 2 is fulfilled in each singular point then the residue class of \( \omega \) lifts to intersection homology of \( K \).

The Theorem 4 is true in a greater generality; we assume that \( K \) has isolated singularities and the Condition 2 should be substituted by the following:

**Condition 5.** The number 0 does not belong to the spectrum of the singularity.

The concept of the spectrum as defined in [1; §13.3 p. 270] comes from the theory of oscillating integrals. It follows from the definition that \( \int_X \text{Res} \omega = 0 \) for any cycle contained in the link of a singular point; see [13; 1.3]. In this case residue lifts to the cohomology. The point is that the spectrum is computable from the Newton diagram; [1; §13.3 p. 274]. For quasihomogeneous singularities we have

\[
\{\text{spectrum of the singularity}\} \cap (-\infty, 0] = \{v(\frac{g}{s} dz_0 \wedge \cdots \wedge dz_n)\} \cap (-\infty, 0] = \{\kappa + \sum k_i a_i - 1 : k_i \in \mathbb{N} \cup \{0\}, i = 0, \ldots, n\} \cap (-\infty, 0]
\]

The spectrum can also be defined in terms of eigenvalues of the monodromy acting on the vanishing cycles filtered by weights; [11].

Now we prove the Theorem 4 using few well known facts from the intersection homology theory.

**Proof.** One should show that \( [\text{Res} \omega] \in \ker (H^n(K^0) \rightarrow H^n(\partial K^0)) \) that is for each link \( L \) in \( K \) \( [\text{Res} \omega|_L] = 0 \in H^n(L) \). We take a neighbourhood of a singular point
in which $K$ is given by a quasihomogeneous equation $s = 0$. We give a formula for $\text{Res } \omega$ in the points where $s_0 = \frac{\partial s}{\partial z_0} \neq 0$:

$$ds = \sum_{i=0}^{n} s_i dz_i,$$

$$dz_0 = \frac{1}{s_0} ds - \sum_{i=1}^{n} \frac{s_i}{s_0} dz_i,$$

$$\omega = \frac{g ds}{s} \wedge dz_1 \wedge \cdots \wedge dz_n = \frac{ds}{s} \wedge g dz_1 \wedge \cdots \wedge dz_n.$$

Let $r = \frac{g}{s_0} dz_1 \wedge \cdots \wedge dz_n$. Then $\text{Res } \omega = r |_K$ in the points where $s_0 \neq 0$. Now suppose that $g$ is quasihomogeneous (or we decompose $g$ into a quasihomogeneous components). Let $v(g) = \alpha$. We have

$$v(\omega) = v(ds) - v(s) + v(r) = v(r).$$

Then

$$v(r) = v(g) - v(s) + v(dz_0) + \cdots + v(dz_n) = \alpha - 1 + a_0 + \cdots + a_n = \alpha - 1 + \kappa.$$

Let $l$ be a natural number such that $l a_i \in \mathbb{N}$ for $i = 0, \ldots, n$. We construct a branched covering of $\mathbb{C}^{n+1}$:

$$\Phi : \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}$$

$$\hat{z}_0, \ldots, \hat{z}_n \longmapsto \hat{z}_0^{l a_0}, \ldots, \hat{z}_n^{l a_n}.$$

Let $\hat{v}$ be a standard valuation: $\hat{v}(\hat{z}_i) = 1$. The map $\Phi$ has the property:

$$\hat{v}(\Phi^* \eta) = l v(\eta)$$

for any $\eta \in \mathbb{C}(\hat{z}_0, \ldots, \hat{z}_n) \otimes \Lambda^* (\mathbb{C}^{n+1})^*$. We have

$$v(\Phi^* r) = l(\alpha - 1 + \kappa).$$

If we write $\Phi^* r = q d\hat{z}_1 \wedge \cdots \wedge d\hat{z}_n$ then $q$ is homogeneous function of weight

$$v(q) = l(\alpha - 1 + \kappa) - n.$$

The mapping $\Phi$ induces the branched covering of the links:

$$S^{2n+1} \cap \Phi^{-1}(K) = \hat{L} \xrightarrow{\Phi} L = K \cap \{z_0, \ldots, z_n : |z_0|^{2a_0} + \cdots + |z_n|^{2a_n} = 1\}$$

The degree of this map is $l \kappa$. Unfortunately $\hat{L}$ may be singular; see Example 6. To show that $[\text{Res } \omega|_L] = 0$ we will prove that $[\Phi^* \text{Res } \omega|_L] = 0$ in $IH^m_{n-1}(\hat{L})$. It is enough since the map

$$H^n(L) \xrightarrow{\Phi^*} H^n(\hat{L}) \rightarrow IH^m_{n-1}(\hat{L})$$
We write $\Phi$ in the trivialization of the bundle has a form: 

The map $\hat{r}$ for suitably chosen metrics on is a monomorphism with a splitting

$$IH_{n-1}^m(\hat{L}) \to H_{n-1}^m(\hat{L}) \xrightarrow{\Phi^*} H_{n-1}(L) \to H^n(L).$$

The last map is the inverse to the Poincaré duality isomorphism multiplied by $(l\kappa)^{-1}$, the maps to and from intersection homology are the canonical ones. To show vanishing in intersection homology we use a Gysin sequence of the fibration

$$S^1 \to \hat{L} \xrightarrow{p} \hat{L}/S^1$$

coming from the action of $\mathbb{C}^*$ on $\Phi^{-1}(K)$:

$$\to IH_n^m(\hat{L}/S^1) \xrightarrow{\cap e} IH_{n-2}^m(\hat{L}/S^1) \xrightarrow{p^*} IH_{n-1}^m(\hat{L}) \xrightarrow{p_*} IH_{n-1}^m(\hat{L}/S^1) \to .$$

The map $\cap e$ is the multiplication by the Euler class of the fibration; it is an isomorphism by hard Lefschetz since dim$_\mathbb{C} \hat{L}/S^1 = n - 1$; [2]. We interpret $IH_{n-1}^m(\hat{L})$ as the $L_2$-cohomology of the nonsingular part of $\hat{L}$:

$$IH_{n-1}^m(\hat{L}) = H_{(2)}^n(\hat{L} \setminus \Sigma) =: H_{(2)}^n(L)$$

for suitably chosen metrics on $\hat{L} \setminus \Sigma$ and $(\hat{L} \setminus \Sigma)/S^1$; see [6], [12]. Then the sequence has a form:

$$\to H_{(2)}^{n-2}(\hat{L}/S^1) \xrightarrow{\cap e} H_{(2)}^n(\hat{L}/S^1) \xrightarrow{p^*} H_{(2)}^n(\hat{L}) \xrightarrow{p_*} H_{(2)}^{n-1}(\hat{L}/S^1) \to .$$

The map $p_*$ is just the integration along the fibers of $p$. Let us calculate the integral in the trivialization of the bundle $\mathbb{C}^{n+1} \setminus \{0\} \xrightarrow{\bar{p}} \mathbb{P}^n$ over $U_0 = \{\hat{z}_0 \neq 0\} \subset \mathbb{P}^n$:

$$\mathbb{C}^* \times U_0 \longrightarrow p^{-1}(U_0), \quad u_0, u_1, \ldots, u_n \mapsto u_0, u_0u_1, \ldots, u_0u_n.$$ 

We write $\Phi^* r$ in $u$-coordinates:

$$\Phi^* r = q(\hat{z}_0, \ldots, \hat{z}_n) d\hat{z}_1 \wedge \cdots \wedge d\hat{z}_n = u_0^{l(\alpha+\kappa-1)-n} \bar{q}(u_1, \ldots, u_n)(u_1 du_0 + u_0 du_1) \wedge \cdots \wedge (u_n du_0 + u_0 du_n) = u_0^{l(\alpha+\kappa-1)-1} \bar{q}(u_1, \ldots, u_n) du_0 \sum_{i=1}^{n} (-1)^{i+1} u_i du_1 \wedge \cdots \wedge du_i + u_0 du_1 \wedge \cdots \wedge du_n = u_0^{l(\alpha+\kappa-1)-1} du_0 \wedge r_2 + \Theta,$$

where $r_2$ and $\Theta$ do not contain $du_0$ and $r_2$ does not depend on $u_0$. Thus the integral can be nonzero only if $\alpha + \kappa = 1$. It is impossible by the Condition 2 since $\alpha$ is a combination of $a_i$’s. Thus $p_* \Phi^*(\text{Res } \omega|_L) = 0$, so the residue lifts to intersection homology.
Example 6. Consider the polynomial

\[ s(x, y, z) = (x + z^2)^2 + y^2 - z^4. \]

It has an isolated singularity of the type \( A_3 \). It is quasihomogeneous with weights \( v(x) = v(y) = \frac{1}{2} \) and \( v(z) = \frac{1}{4} \). The polynomial \( \Phi^*(s) \) is:

\[ \Phi^*(s) = (x^2 + z^2)^2 + y^4 - z^4 = x^4 + 2x^2z^2 + y^4. \]

Zero is not an isolated singularity since for \( z = c = \text{const} \) we obtain:

\[ x^4 + 2x^2c^2 + y^4 \sim x^2 + y^4 \]

which is a singularity of the type \( A_3 \). The example of a singularity with \( \hat{L} \) nonsingular is \( z_0^{k_0} + \cdots + z_n^{k_n} \) for any choice of \( k_i \in \mathbb{N} \).

The Example 6 shows, that in the proof of the Theorem 3 we have to use the hard Lefschetz theorem for intersection homology instead of the standard one.

Remark. Note that (in the case of isolated singularities) if the residue class lifts to intersection homology then it lifts to cohomology. Choose \( p > 1 \). For \( \kappa > 1 \) we show that \( \text{Res} \omega \) is a form with the \( L_p \)-integrable norm in suitably chosen metric; [13]. If \( p \) is large then the \( L_p \)-cohomology of \( K \setminus \Sigma \) is isomorphic to the cohomology \( K \); see [5],[12]. This way we find a particular lift to cohomology. This lift depends on coordinates, but can it be calculated in terms of integrals.

4. Weighted blow-up

There is another way of looking at the calculation presented in the proof of the Theorem 4. Let the group \( G = \mathbb{Z}/l_0 \times \cdots \times \mathbb{Z}/l_n \) acts on the coordinates of \( \mathbb{C}^{n+1} \) by the multiplication by the roots of unity. Then \( K = \hat{K}/G \). We blow up \( \hat{K} \subset \mathbb{C}^{n+1} \) in 0 and obtain:

\[
\begin{array}{c}
\mathbb{C}^{n+1} \supset \hat{Y} \cup \mathbb{P}^n \\
\mathbb{P}r \\
\mathbb{C}^{n+1} \supset \hat{K} \xrightarrow{\Phi} K \xrightarrow{\Phi^i} Y \cup \mathbb{P}(v) \xrightarrow{\Phi} \hat{Y} / G \cup \mathbb{P}^n / G \subset \hat{C}^{n+1} / G \\
pr \\
= \hat{K} / G \subset \mathbb{C}^{n+1} / G = \mathbb{C}^{n+1}.
\end{array}
\]

Here \( \mathbb{P}(v) = \mathbb{P}^n / G \) is weighted projective space. We have \( \hat{Y} \cap \mathbb{P}^n = \hat{L} / S^1 \) and \( Y \cap \mathbb{P}(v) = L / S^1 \). The spaces \( \mathbb{P}(v), \mathbb{C}^{n+1} / G, Y \) and \( L / S^1 \) are homology manifolds; locally they are quotients of smooth manifolds by a finite group i.e. they are \( V \)-manifolds as defined by Steenbrink; [10]. From the homology point of view they can be treated as ordinary (smooth) Kähler manifolds.

5. Nonvanishing of the second residue

The last lines of the proof of the Theorem 4 lead to a definition of an element

\[
\text{res}_{2}\omega = \left[ \frac{1}{2\pi i} \int_{p} R e s \omega_{|L} \right] \in IH_{n-1}^{m}(\hat{L} / S^1).
\]
This is an obstruction to lift the residue class to cohomology. We call it the second residue. The class \( \text{res}_2\omega \) is \( G \)-invariant, so it is in
\[
IH_{n-1}^m(\hat{L}/S^1)^G = IH_{n-1}^m(L/S^1) = H^{n-1}(L/S^1).
\]
The form \( r_2 \) represents \( \text{res}_2\omega \). Since \( L/S^1 \) is \( V \)-manifold then its cohomology admits Hodge decomposition \([10]\) and \( \text{res}_2\omega \) is of \((n-1,0)\) type. The form \( r_2 \) is harmonic outside the singularities of \( L/S^1 \), so to show that it does not vanish in cohomology it suffices to check that it is not tautologically zero. Suppose that \( g \) is quasihomogeneous of the weight
\[
v(g) = \alpha = 1 - \kappa,
\]
hence \( v(\omega) = 0 \). We will show that
\[
\text{res}_2\omega = \text{res}_2\left( \frac{d}{s} dz_0 \wedge \cdots \wedge dz_n \right) \neq 0 \in H^{n-1}(L/S^1).
\]

We blow–up \( \mathbb{C}^{n+1} \supset \tilde{K} \) in 0. We calculate the form \( \Phi^*\omega \) pulled up to \( \mathbb{C}^{n+1} \) in the canonical coordinates (in the 0–th chart).
\[
\tilde{p}^*\Phi^*\omega = C \frac{\Phi^*g}{\Phi^*s} \left( \prod_{i=0}^{n} \hat{z}_{i}^{l_{a_{i}}-1} \right) dz_0 \wedge \cdots \wedge dz_n =
\]
\[
= C \frac{u_0^{l_{a}^0} \hat{\Phi}^*g}{u_0^l \Phi^*s} u_0^{l_{\kappa-n-1}} \left( \prod_{i=1}^{n} u_i^{l_{a_{i}}-1} \right) u_0^n du_0 \wedge \cdots \wedge du_n =
\]
\[
= C \frac{du_0}{u_0} \wedge \hat{\Phi}^*g \left( \prod_{i=1}^{n} u_i^{l_{a_{i}}-1} \right) du_1 \wedge \cdots \wedge du_n.
\]
Here \( \tilde{p}(u_1, \ldots, u_n) \) denotes \( p(1, u_1, \ldots, u_n) \). We see that the form \( \tilde{p}^*\Phi^*\omega \) has the logarithmic pole on the exceptional divisor. The form \( r_2|\hat{Y}_{\cup\mathbb{P}^n} \) is the second Leray residue; \([8], \ [9]\). We can decompose the form \( \tilde{p}^*\Phi^*\omega \) in a way
\[
\tilde{p}^*\Phi^*\omega = \frac{du_0}{u_0} \wedge \hat{\Phi}^*s \wedge r_2',
\]
where \( r_2' \) do not contain \( u_0 \) nor \( du_0 \). Then \( r_2'|\hat{Y}_{\cup\mathbb{P}^n} = r_2|\hat{Y}_{\cup\mathbb{P}^n} \). The function \( \hat{\Phi}^*(s) \) describes \( \hat{Y} \cup \mathbb{P}^n \) in \( \mathbb{P}^n \) for \( u_0 \neq 0 \), so to show that \( r_2|\hat{Y}_{\cup\mathbb{P}^n} \neq 0 \) it suffices check that \( \hat{\Phi}^*(s) \wedge r_2' \neq 0 \) on \( \hat{Y} \cup \mathbb{P}^n \). By the decomposition:
\[
u_0 \hat{\Phi}^*(s) \tilde{p}^*\Phi^*\omega = du_0 \wedge d\hat{\Phi}^*s \wedge r_2' = C \ du_0 \wedge \hat{\Phi}^*g \left( \prod_{i=1}^{n} u_i^{l_{a_{i}}-1} \right) du_1 \wedge \cdots \wedge du_n.
\]
The polynomial \( g \) has the lower weight than \( s \), thus \( \hat{\Phi}^*g \) does not vanish on \( \hat{Y} \cup \mathbb{P}^n \). Moreover \( \hat{Y} \cup \mathbb{P}^n \) is not contained in any of hyperplane \( u_i = 0 \). Thus \( d\hat{\Phi}^*s \wedge r_2' \neq 0 \) on \( \hat{Y} \cup \mathbb{P}^n \) and hence \( r_2'|\hat{Y}_{\cup\mathbb{P}^n} \neq 0 \). This way we proved
Theorem 7. Suppose that $g$ has the nonzero quasihomogeneous component of the weight $\alpha = 1 - \kappa$. Then the second residue of $\omega = \frac{1}{s}dz_0 \wedge \cdots \wedge dz_n$ does not vanish in $H^{n-1}(L/S^1)$.

Example 8, [13]. Consider the singularity of the type $P_3$:

$$s(z_0, z_1, z_2) = z_0^3 + z_1^3 + z_2^3$$

and $\omega = \frac{1}{s}dz_0 \wedge dz_1 \wedge dz_2$. Then $\tilde{L}/S^1 = L/S^1 \subset \mathbb{P}^2$. The second residue (i.e. the obstruction to lift) is:

$$\text{res}_2 \omega = \left[ \frac{1}{2\pi i} \int_p \text{Res} \omega \right] = \frac{1}{3} u_1 du_2 - u_2 du_1$$

in the notation used above. As one can check by hand the integral

$$\int_{L/S^1 \cup \mathbb{P}^2} \text{res}_2 \omega \neq 0.$$