Eigenfunctions of the Multidimensional Linear Noise Fokker-Planck Operator via Ladder Operators

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Abstract

The eigenfunctions and eigenvalues of the Fokker-Planck operator with linear drift and constant diffusion are required for expanding time-dependent solutions and for evaluating our recent perturbation expansion for probability densities governed by a nonlinear master equation. Although well-known in one dimension, for multiple dimensions the eigenfunctions are not explicitly given in the literature. We develop raising and lowering operators for the Fokker-Planck (FP) operator and its adjoint, and use them to obtain expressions for the corresponding eigenvalues and eigenfunctions. We show that the eigenfunctions for the forward and adjoint FP operators form a bi-orthogonal set, and that the eigenfunctions reduce to sums of products of Hermite functions in a particular coordinate system.

Keywords: Ornstein-Uhlenbeck operator, Fokker-Planck eigenfunctions, Hermite functions, ladder operators.

1 Introduction

The Fokker-Planck equation (FPE) with linear drift and constant diffusion describes an Ornstein-Uhlenbeck process. In one dimension, the eigenfunctions are the well-known Hermite functions (Risken, 1989; Gardiner, 2000). The eigenfunctions enable expansion of time-dependent solutions to the FPE, and are required to evaluate our recent perturbation expansion for the probability densities arising from a nonlinear master equation (Leen and Friel, 2011, 2012; Leen et al., 2012). (Thomas and Grima have recently derived a similar approximation and applied it to one-dimensional chemical and gene expression systems (2015).) The multi-dimensional eigenfunctions are not given in convenient form in the literature. (Liberzon and Brockett (2000) discuss the eigenvalue spectrum, but do not give an explicit form of the complete set of eigenfunctions, nor discuss the eigenfunctions of the adjunct operator.) In the restricted case that the drift and diffusion are simultaneously diagonalizable, the equations separate and the solution reduces to products of the one-dimensional eigenfunctions. However for general drift and diffusion, the eigenfunctions are not simply obtained from the differential operators.

This note gives raising and lowering operators for the forward and adjoint Fokker-Planck operators that develop the corresponding complete sets of eigenfunctions. We give bi-orthogonality relations and show that in a special coordinate system, the eigenfunctions are sums of products of Hermite functions.

2 Operators and Eigenfunctions

Let \( x \in \mathbb{R}^N, \ \partial_i \equiv \partial/\partial x_i, \ i = 1, \ldots, N \). The forward and adjoint (backward) Fokker-Planck operators for an \( N \)-dimensional Ornstein-Uhlenbeck process are

\[
L q(x) \equiv - \partial_i (A^i_j \ x^j \ q) + \frac{1}{2}B^{ij} \partial_i \partial_j \ q
\]  

(1)

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and
\[ L^\dagger q(x) = A^i_j x^j \partial_i q + \frac{1}{2} B^{ij} \partial_i \partial_j q \]  
where repeated upper and lower indices in lower case Latin are summed over. Thus \((Ax)^i = A^i_j x^j \equiv \sum_j A^i_j x^j\) is the \(i^{th}\) component of the (linear) drift vector. We assume the diffusion matrix, whose elements are \(B^{ij}\), is positive definite.

Throughout we assume that the left \(w_I\) and right \(e_I\) eigenvectors of \(A\) form a complete set for \(\mathbb{R}^n\), and that their corresponding eigenvalues (perhaps complex) have negative real part\(^1\). These eigenvectors are normalized and bi-orthogonal
\[ w_I \cdot e_J = w_I^* \cdot e_J^* = \delta_{IJ} \]  

### 2.1 Stationary States

The stationary state satisfies
\[ L f_0(x) = 0 \]
and is given by
\[ f_0(x) = \frac{1}{\sqrt{2\pi}^N \det \Sigma} \exp \left( -\frac{1}{2} x^T \Sigma^{-1} x \right) \]  
The covariance matrix \(\Sigma\) is a solution to the Liapunov equation
\[ A \Sigma + \Sigma A^T = -B \]  
The corresponding stationary state of \(L^\dagger\) satisfies
\[ L^\dagger g_0 = 0 \]
and is given by
\[ g_0(x) = 1 \]  
As we will show, the eigenfunctions of \(L\) and \(L^\dagger\) form a biorthogonal set, and for the stationary (ground) states we have
\[ \langle g_0, f_0 \rangle = \int g_0^*(x) f_0(x) \, d^N x = 1 \]
as follows from the normalization of the multidimensional Gaussian \(f_0\).

### 2.2 Raising Operators

We generate the complete set of eigenfunctions of \(L\) and \(L^\dagger\) by application of raising operators to \(f_0\) and \(g_0\). The next two subsections address the two sets of eigenfunctions in turn.

#### 2.2.1 Forward Eigenfunctions

Let \(e_I, I = 1,\ldots,N\) be the right eigenvectors of \(A\) with corresponding eigenvalues \(\lambda_I\). (Recall we assume that all the eigenvalues of \(A\) have negative real part, corresponding to an asymptotically-stable fixed point at \(x = 0\).) Let \(e_I^i\) denote the \(i^{th}\) component of the \(I^{th}\) eigenvector. The operators
\[ V_I = -e_I^i \partial_i \equiv -e_I \cdot \nabla, \quad I = 1, 2, \ldots N \]  
satisfy the commutation relation
\[ [L, V_I] = \lambda_I V_I \]  
(There is no sum on \(I\).) The commutator establishes \(V_I\) as a raising operator for eigenfunctions of \(L\): If \(f_\lambda\) is an eigenfunction of \(L\) with eigenvalue \(\lambda\), then \(V_I f_\lambda\) is also an eigenfunction of \(L\) with eigenvalue \(\lambda + \lambda_I\).

\(^1\)Since \(A\) is real, any complex eigenvectors occur as conjugate pairs.
The application of products of the $V_i$ to $f_0$ generate new eigenfunctions of $L$ which we denote by subscripts indicating the number of applications of each operator. For example
\[ f_{n_1,n_2,0,0,...} = (V_1)^{n_1} (V_2)^{n_2} f_0 \quad \text{with} \quad L f_{n_1,n_2,0,0,...} = (n_1 \lambda_1 + n_2 \lambda_2) f_{n_1,n_2,0,0,...} \]
Since $[V_i,V_j] = 0$, the subscript labels on $f$ uniquely determine the eigenfunctions. We build a general eigenfunction of $L$ as
\[ f_{n_1,n_2,...,n_N} = \left( \prod_{I=1}^{N} (V_I)^{n_I} \right) f_0 \] (9)
which satisfies
\[ L f_{n_1,n_2,...,n_N} = \left( \sum_{I=1}^{N} n_I \lambda_I \right) f_{n_1,n_2,...,n_N} . \] (10)
Clearly from Eqn. (9)
\[ (V_I)^k f_{n_1,...,n_I,...,n_N} = f_{n_1,...,(n_I+k),...,n_N} . \] (11)

2.2.2 Adjoint Eigenfunctions

Let $w_I, I = 1, \ldots, N$ be the left eigenvectors of $A$ with eigenvalues $\lambda_I$, and let $w_I^i$ denote its $i^{th}$ component. (The subscript indicating the component is appropriate since we regard $w_I$ as a co-vector.) Define the operators
\[ \bar{V}_I = 2(w_I^i x^i) + 2 \left[ (A + \lambda_I)^{-1} B w_I^i \right] \partial_i = 2 \left[ w_I^i x^i - (\Sigma w_I^j) \partial_i \right] , \] (12)
where the second equality follows from the Liapunov equation (5). This operator satisfies the commutation relation
\[ [L^+, \bar{V}_I] = \lambda_I^* \bar{V}_I , \] (13)
which establishes it as a raising operator for eigenfunctions of $L^+$: If $g_\lambda$ is an eigenfunction of $L^+$ with eigenvalue $\lambda$, then $\bar{V}_I g_\lambda$ is an eigenfunction with eigenvalue $\lambda + \lambda_I^*$. Analogously to the forward eigenfunctions, acting with $\bar{V}_I$ on $g_0$ generates a new eigenfunction which we denote by subscripts indicating the number of applications of each raising operator. This subscript notation is free of ambiguity since the raising operators corresponding to different eigenvectors of $A$ commute
\[ [\bar{V}_I, \bar{V}_J] = -2 (w_I^j)^T \Sigma w_J^i + 2 (w_I^i)^T \Sigma w_J^i = 0 . \]
We construct a general eigenfunction of $L^+$ by repeated application of raising operators. Thus
\[ g_{n_1,n_2,...,n_N} = \prod_{I=1}^{N} (\bar{V}_I)^{n_I} g_0 , \] (14)
which satisfies
\[ L^+ g_{n_1,n_2,...,n_N} = \left( \sum_{I=1}^{N} n_I \lambda_I^* \right) g_{n_1,n_2,...,n_N} . \] (15)
Clearly, from Eqn. (14)
\[ (\bar{V}_I)^k g_{n_1,...,n_I,...,n_N} = g_{n_1,...,(n_I+k),...,n_N} . \] (16)

2.3 Lowering Operators

The two sets of raising operators are complemented by lowering operators. Their effect on the states, derived here, makes proving the bi-orthogonality property trivial.
2.3.1 Adjoint Eigenfunctions

Taking the adjoint of Eqn. (8) yields

\[
\left[ L^\dagger, V^\dagger \right] = -\lambda^* V^\dagger
\] (17)

which establishes \( V^\dagger_I = e_I^* \cdot \nabla \) as a lowering operator for eigenfunctions of \( L^\dagger \): If \( g_\lambda \) is an eigenfunction of \( L^\dagger \) with eigenvalue \( \lambda_I \), then \( V^\dagger_I g_\lambda \) is an eigenfunction of \( L^\dagger \) with eigenvalue \( \lambda - \lambda^*_I \). In particular, \( V^\dagger_I \) kills the stationary state

\[
V^\dagger_I g_0 = e_I^* \cdot \nabla g_0 = 0
\] .

Using the commutators

\[
\left[ V^\dagger_I, \bar{V}_J \right] = 2 e_I^* \cdot w^*_J = 2 \delta_{IJ}
\] (18)

and \( [\bar{V}_I, \bar{V}_J] = 0 \), and the expression for the adjoint eigenfunctions Eqn. (14), the action of \( V^\dagger_I \) on eigenfunctions of \( L^\dagger \) follows as

\[
V^\dagger_J g_{n_1,n_2,\ldots,n_N} = V^\dagger_J \left( \prod_{I=1}^N (\bar{V}_I)^{n_I} \right) g_0 = \left( \prod_{I \neq J} (\bar{V}_I)^{n_I} \right) V^\dagger_J (\bar{V}_J)^{n_J} g_0
\]

\[
= \left( \prod_{I \neq J} (\bar{V}_I)^{n_I} \right) \left[ (\bar{V}_J)^{n_J} V^\dagger_J + (2n_J)(\bar{V}_J)^{n_J-1} \right] g_0
\]

\[
= (2n_J) g_{n_1,n_2,\ldots,n_J-1,\ldots,n_N}
\] (19)

where the second line follows from the first by commuting \( V^\dagger_J \) past all \( n_J \) factors of \( \bar{V}_J \), and the third line follows since \( V^\dagger_J g_0 = e_J^* \cdot \nabla g_0 = 0 \). Multiple applications yield

\[
\left( V^\dagger_J \right)^k g_{n_1,n_2,\ldots,n_N} = \left\{ \begin{array}{ll}
2^k \frac{n_J!}{(n_J-k)!} g_{n_1,n_2,\ldots,n_J-k,\ldots,n_N}, & k \leq n_J \\
0, & k > n_J
\end{array} \right.
\] (20)

2.3.2 Forward Eigenfunctions

Taking the adjoint of Eqn. (13) we recover

\[
\left[ L, \bar{V}_J \right] = -\lambda_I \bar{V}_J
\] (21)

which establishes \( \bar{V}_J^\dagger \) as a lowering operator for eigenfunctions of \( L \): If \( f_\lambda \) is an eigenfunction of \( L \) with eigenvalue \( \lambda \), then \( \bar{V}_J^\dagger f_\lambda \) is an eigenfunction with eigenvalue \( \lambda - \lambda_I \). In particular, it kills the stationary state

\[
\bar{V}_J^\dagger f_0 = 2 (w_I x^i + (\Sigma w_I) \partial_i) f_0 = 0
\] (22)

having used the definition of \( f_0 \) in Eqn. (4). Similarly to the case for the adjoint lowering operators, it is straightforward to show that

\[
\bar{V}_J^\dagger f_{n_1,n_2,\ldots,n_N} = (2n_J) f_{n_1,\ldots,n_J-1,\ldots,n_N}
\] .

Multiple applications yield

\[
\left( \bar{V}_J^\dagger \right)^k f_{n_1,\ldots,n_N} = \left\{ \begin{array}{ll}
2^k \frac{n_J!}{(n_J-k)!} f_{n_1,\ldots,n_J-k,\ldots,n_N}, & k \leq n_J \\
0, & k > n_J
\end{array} \right.
\] (24)
3 Bi-Orthogonality and Normalization

The two sets of functions form a bi-orthogonal set. The usual result, that eigenfunctions corresponding to different eigenvalues are orthogonal can be strengthened. We will show that

\[ \langle g_{m_1, \ldots, m_N}, f_{n_1, \ldots, n_N} \rangle = \int g_{m_1, \ldots, m_N}^* (x) f_{n_1, \ldots, n_N} (x) \, d^N x \]

\[ = \left( \prod_{i=1}^{N} \delta_{m_i, n_i} 2^{n_i} (n_i!) \right) < g_0, f_0 > \]

\[ = \left( \prod_{i=1}^{N} \delta_{m_i, n_i} 2^{n_i} (n_i!) \right). \quad (25) \]

We consider three cases:

Case I Suppose \( n_J > m_J \) for some \( J \). Write

\[ \langle g_{m_1, \ldots, m_J, \ldots, m_N}, f_{n_1, \ldots, n_J, \ldots, n_N} \rangle = \langle g_{m_1, \ldots, m_J, \ldots, m_N}, V_J^{n_J} f_{n_1, \ldots, n_J=0, \ldots, n_N} \rangle = \left( (V_J^{n_J})^{n_J} g_{m_1, \ldots, m_J, \ldots, m_N}, f_{n_1, \ldots, n_J=0, \ldots, n_N} \right) = 0 \quad (26) \]

having used Eqn. (20) to arrive at the last equality.

Case II Now suppose instead that \( m_J > n_J \) for some \( J \). Write

\[ \langle g_{m_1, \ldots, m_J, \ldots, m_N}, f_{n_1, \ldots, n_J, \ldots, n_N} \rangle = \left( V_J^{m_J} g_{m_1, \ldots, m_J=0, \ldots, m_N}, f_{n_1, \ldots, n_J=0, \ldots, n_N} \right) = 0 \quad (27) \]

having used Eqn. (24) to arrive at the last equality.

Case III Suppose \( n_J = m_J \) for all \( J \). Write

\[ \langle g_{n_1, \ldots, n_N}, f_{n_1, \ldots, n_N} \rangle = \langle g_{n_1, \ldots, n_N}, V_1^{n_1} f_{0,n_2, \ldots, n_N} \rangle = \langle (V_1^{n_1})^{n_1} g_{n_1,n_2, \ldots, m_N}, f_{0,n_2, \ldots, n_N} \rangle = 2^{n_1} n_1! \langle g_{0,n_2, \ldots, n_N}, f_{0,n_2, \ldots, n_N} \rangle \]

\[ = 2^{n_1} n_1! \langle (V_2^{n_2})^{n_2} g_{0,n_3, \ldots, n_N}, f_{0,0,n_3, \ldots, n_N} \rangle = 2^{n_1} 2^{n_2} (n_1!) (n_2!) \langle g_{0,0,n_3, \ldots, n_N}, f_{0,0,n_3, \ldots, n_N} \rangle \]

\[ = \left( \prod_{J=1}^{N} 2^{n_J} (n_J!) \right) \langle g_0, f_0 \rangle \]

\[ = \left( \prod_{J=1}^{N} 2^{n_J} (n_J!) \right) \quad (28) \]

having used Eqn. (20) repeatedly. The three cases together prove the desired result Eqn. (25).

4 Analytic Form of the Eigenfunctions

In general coordinates, the eigenfunctions \( f_{n_1,n_2, \ldots, n_N} \) and \( g_{n_1, \ldots, n_N} \) do not assume a familiar analytic form. However they do if we transform to coordinates in which \( \Sigma = \frac{1}{2} \mathbb{I} \). (Since \( \Sigma \) is a real, symmetric, positive-
definite matrix, this is always possible.) In these coordinates,

\[ f_0(x) = \frac{1}{\sqrt{\pi^N}} e^{-x^T x} \quad (29) \]

\[ g_0(x) = 1. \quad (30) \]

In these coordinates the forward eigenfunctions are sums of products of Hermite polynomials times \( f_0 \) and the backward eigenfunctions are sums of products of Hermite polynomials.

### 4.1 Forward Eigenfunctions

To start, note that the usual generating expression for the Hermite polynomials (Abramowitz and Stegun, 1972) can be rearranged to read

\[ H_n(x_i) e^{-x^T x} = (-\partial_i)^n e^{-x^T x}. \quad (31) \]

Hence, application of \( V_I \) to \( f_0 \) yields

\[ V_I f_0(x) = -e_i^I \partial_i f_0(x) = \sum_{i=1}^{N} e_i^I H_1(x_i) f_0(x) \]

a linear combination of Hermite functions in each of the variables \( x_1, \ldots, x_N \). (When a coordinate index falls inside a function argument — as in the last expression — we will write the summation explicitly to avoid confusion.) Applying \( V_I \) to \( f_0 \), \( n_I \) times results in

\[ V_I^{n_I} f_0 = (-e_i^I \partial_i)^{n_I} f_0 \]

which can be evaluated using the multinomial theorem. Explicitly

\[ V_I^{n_I} f_0 = \sum_{\{i_1, \ldots, i_N = n_I\}} \frac{n_I!}{i_1! i_2! \ldots i_N!} \prod_{1 \leq k \leq N} (-e_i^k \partial_k)^{i_k} f_0 \]

\[ = \sum_{\{i_1, \ldots, i_N = n_I\}} \frac{n_I!}{i_1! i_2! \ldots i_N!} \prod_{1 \leq k \leq N} (e_i^k)^{i_k} H_{i_k}(x_k) f_0(x) \]

where the summation is over all values of indices satisfying the constraint \( i_1 + i_2 + \cdots + i_N = n_I \).

The action of two distinct ladder operators multiple times is

\[ V_I^{n_I} V_J^{n_J} f_0 = \sum_{\{i_1 + \ldots + i_N = n_I\}} \frac{n_I!}{i_1! i_2! \ldots i_N!} \frac{n_J!}{j_1! j_2! \ldots j_N!} \prod_{1 \leq k \leq N} (e_i^k)^{i_k} (e_j^k)^{j_k} (-\partial_k)^{i_k+j_k} f_0 \]

\[ = \sum_{\{i_1 + \ldots + i_N = n_I\}} \frac{n_I!}{i_1! i_2! \ldots i_N!} \frac{n_J!}{j_1! j_2! \ldots j_N!} \prod_{1 \leq k \leq N} (e_i^k)^{i_k} (e_j^k)^{j_k} H_{i_k+j_k}(x_k) f_0(x) \]

\[ = \sum_{\{i_1 + \ldots + i_N = n\}} \frac{n!}{i_1! i_2! \ldots i_N!} \frac{n!}{j_1! j_2! \ldots j_N!} \prod_{1 \leq k \leq N} (e_i^k)^{i_k} (e_j^k)^{j_k} H_{i_k+j_k}(x_k) f_0(x) \]

\[ = V_I^{n_I} V_J^{n_J} f_0 \quad (32) \]

This generalizes in the obvious way to products of the form

\[ f_{n_1, n_2, \ldots, n_N}(x) = V_I^{n_1} V_J^{n_2} \cdots V_K^{n_N} f_0(x) \quad (33) \]

So in our special coordinates the eigenfunctions of \( L \) are **sums of products of Hermite polynomials** times \( f_0 \).
4.2 Adjoint Eigenfunctions

In our special coordinates $\Sigma^{ij} = \frac{1}{2} \delta^{ij}$ so the backward raising operator \( \bar{V} \) simplifies to

$$
\bar{V}_I = 2 (w_I^* x^i) - (w_I^*)^i \partial_k ,
$$

(34)

where \((w_I^*)^i \equiv \delta^{ij} w_I^* j\). From the standard recursion relations for the Hermite polynomials one has

$$
\left(2x - \frac{d}{dx}\right) H_n(x) = H_{n+1}(x) .
$$

(35)

Thus using the form of $\bar{V}_I$ in our special coordinates \([34]\), its action on a product of Hermite polynomials is

$$
\bar{V}_I H_{i_1}(x_1) H_{i_2}(x_2) \cdots H_{i_N}(x_N) = \sum_{k=1}^N H_{i_1}(x_1) \cdots w_{I_k}^* H_{i_k+1}(x_k) \cdots H_{i_N}(x_N) .
$$

It is convenient to define the operator \( r^k \) which raises the order of \( H_m(x_k) \) by unity,

$$
r^k H_{i_1}(x_1) \cdots H_{i_k}(x_k) \cdots H_{i_N}(x_N) = H_{i_1}(x_1) \cdots H_{i_k+1}(x_k) \cdots H_{i_N}(x_N) .
$$

Then the action of $\bar{V}_I$ on a product of Hermite polynomials is

$$
\bar{V}_I H_{i_1}(x_1) \cdots H_{i_N}(x_N) = \sum_{k=1}^N w_{I_k}^* r^k H_{i_1}(x_1) \cdots H_{i_N}(x_N)
$$

$$
= \sum_{k=1}^N H_{i_1}(x_1) \cdots w_{I_k}^* H_{i_k+1}(x_k) \cdots H_{i_N}(x_N) .
$$

(36)

Writing

$$
g_0 = 1 = \bar{V}_I H_0(x_1) H_0(x_2) \cdots H_0(x_N)
$$

and using Eqn. \([36]\) gives

$$
\bar{V}_I g_0 = \sum_{k=1}^N w_{I_k}^* H_1(x_k)
$$

(37)

The action of $\bar{V}_I^{n_1}$ on $g_0$ is evaluated using the multinomial theorem

$$
\bar{V}_I^{n_1} g_0 = \sum_{(i_1 + \cdots + i_n = n_1)} \left( \frac{n_1!}{i_1! \cdots i_n!} \right) \prod_{1 \leq k \leq N} (w_{I_k}^* r^k)^{i_k} g_0
$$

$$
= \sum_{(i_1 + \cdots + i_n = n_1)} \left( \frac{n_1!}{i_1! \cdots i_n!} \right) \prod_{1 \leq k \leq N} (w_{I_k}^*)^{i_k} H_{i_k}(x_k)
$$

The action of two such operators on $g_0$ is

$$
\bar{V}_I^{n_1} \bar{V}_I^{n_2} g_0 = \sum_{(i_1 + \cdots + i_N = n_1) \atop (j_1 + \cdots + j_N = n_2)} \left( \frac{n_1!}{j_1! \cdots j_N!} \right) \left( \frac{n_2!}{i_1! \cdots i_N!} \right) \prod_{1 \leq k \leq N} (w_{I_k}^*)^{i_k} (w_{J_k}^*)^{j_k} H_{i_k+j_k}(x_k) .
$$

(38)

This generalizes in the obvious way to evaluate the general backward eigenfunction in Eqn. \([14]\).
5 Examples

Example 1 — Derivative and Multiplication Operators  
Recall that the left \((w_I)\) and right \((e_I)\) eigenvectors of \(A\) form a complete, bi-orthogonal set for \(\mathbb{R}^N\) (see Eqn. (3)). Then, from the definition of the raising operator \((7)\) for eigenfunctions of \(L\), we have

\[
\nabla = \sum_{I=1}^{N} w_I^* V_I^\dagger .
\]  

(39)

Similarly, from the definition of the raising operator \((12)\) for eigenfunctions of \(L^\dagger\) and this last result \((39)\), we have

\[
x = \frac{1}{2} \sum_{I=1}^{N} e_I^* \left[ \tilde{V}_I + 2 \sum_{J=1}^{N} (w_I^* \cdot \Sigma w_J^*) V_J^\dagger \right] .
\]  

(40)

Example 2 — Forward and Adjoint FPE Operators  
Using the results in Example 1, we can rewrite \(L\) as

\[
L = \frac{1}{2} \sum_{I} \lambda_I V_I \tilde{V}_I^\dagger
\]  

(41)

which can be verified by its action on the eigenfunctions

\[
L f_{n_1,\ldots,n_N} = \frac{1}{2} \sum_{I} \lambda_I V_I \tilde{V}_I^\dagger f_{n_1,\ldots,n_N} = \sum_{I} n_I \lambda_I V_I f_{n_1,\ldots,n_{I-1},\ldots,n_N} = \sum_{I} n_I \lambda_I f_{n_1,\ldots,n_N} .
\]  

(42)

Similarly, we recover \(L^\dagger\) as

\[
L^\dagger = \frac{1}{2} \sum_{I} \lambda_I^* \tilde{V}_I V_I^\dagger
\]  

(43)

which can be verified by its action on the eigenfunctions \(g_{n_1,\ldots,n_N}\). (Both Eqns. (41) and (43) have the flavor of quantum oscillator Hamiltonians involving products of lowering and raising operators.  The difference here is that \(L\) and \(L^\dagger\) are not Hermitian. The correspondence with quantum mechanics (in the 1-D case) is discussed by Gardiner (2009).)

Example 3 — Fourier Series  
Time-dependent solutions can be expanded in terms of the forward eigenfunctions exactly as in the one-dimensional case. Let \(K \equiv \{k_1, k_2, \ldots, k_N\}\) denote an index set for the eigenfunctions and eigenvalues. Then

\[
F(x,t) = \sum_{K} \alpha_K f_K(x) \exp(\lambda_K t)
\]  

(44)

clearly satisfies

\[
\partial_t F(x,t) = L F(x,t)
\]

where the coefficients are given by \(\alpha_K = \langle g_K, F(x,0) \rangle\). (Since the eigenvalues of \(A\) have negative real part, all the exponentials in (44) are decaying.) In multiple dimensions, the drift Jacobian \(A^\dagger\) can have one or more pairs of complex-conjugate eigenvalues and Eqn. (44) can represent damped oscillating solutions.

\[\text{Hence, } \frac{1}{2} V_I V_I^\dagger \text{ and } \frac{1}{2} \tilde{V}_I V_I^\dagger \text{ are occupation number operators for the } I^{th} \text{ eigenstate in } f_K \text{ and } g_K \text{ respectively.}\]
Example 4 — Solutions to Inhomogeneous Equations. In our perturbation solution for densities satisfying a nonlinear master equation (Leen and Friel, 2011, 2012; Leen et al., 2012), the \( i \)th order corrections to the equilibrium density is denoted \( P^{(i)}(x) \) and is given by inhomogeneous equations of the form

\[
L P^{(i)}(x) = q^{(i)}(x) ,
\]

where the \( q^{(i)} \) are known. Let \( K \equiv \{k_1, k_2, \ldots, k_N\} \) denote an index set for the eigenfunctions and eigenvalues. Next, expand \( P^{(i)}(x) \) in a linear combination of the eigenfunctions \( f_K(x) \), substitute that into (45), take the inner product with \( g_J(x) \), and use bi-orthogonality relations (25) to obtain

\[
P^{(i)}(x) = \sum_{K \neq 0} \frac{\langle g_K, q^{(i)} \rangle}{\lambda_K} \langle g_K, f_K \rangle f_K(x) \tag{46}
\]

where the summation excludes \( f_0 \).

6 Conclusion

We have provided raising and lowering operators to develop the eigenfunctions (and their corresponding eigenvalues) of the forward and adjoint multidimensional Fokker-Planck operators for the Ornstein-Uhlenbeck process. The eigenfunctions form a basis for expanding solutions to the time-dependent Fokker-Planck equation, and for a perturbation expansion of the densities arising from a nonlinear master equation (Leen and Friel, 2011, 2012; Leen et al., 2012; Thomas and Grima, 2015).

We gave bi-orthogonality and normalization results. We showed that in coordinates for which the covariance of the stationary state is spherically symmetric with variance one-half, the eigenfunctions reduce to sums of products of Hermite polynomials times \( f_0 \). In applications to time-dependent solutions of the Fokker-Planck equation and to inhomogeneous equations (see Example 3 and Example 4 in Section 5) one assumes the eigenfunctions \( f_K \) form a complete set on \( L^2 \). The proof of completeness is similar to that used to show completeness of the one-dimensional Hermite functions.

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