Self-focusing, self modulation and stability properties of laser beam propagating in plasma: A variational approach

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Abstract. Laboratory as well as Particle in cell (PIC) simulation experiments reveal the strong flow of energetic electrons co-moving with laser beam in laser plasma interaction. Equation governing the evolution of complex envelope in slowly varying envelope approximation is nonlinear parabolic equation. A Lagrangian for the problem is set up and assuming a trial Gaussian profile, we solve the reduced Lagrangian problem for beam width and curvature. Besides self-focusing and self-modulation of laser beam, we observe that stability properties of such plasma system are studied about equilibrium values using this variational approach. We obtained an eigen value equation, which is cubic in nature and investigated the criterion for stability using Hurwitz conditions for laser beam plasma system.

1. Introduction
The development of new technology to generate very high power laser pulses has opened new vistas of novel applications in not only other fields but also in plasmas such as plasma based accelerators [1,2,3], advanced laser fusion schemes and new radiation sources [4,5,6,7]. This has given a boost to study propagation characteristics of high power lasers in plasma medium. Thus for the success of above mentioned applications it is only desirable that laser beam propagate over sufficient number of Rayleigh lengths ($R_d$). But in vacuum, laser propagation is limited by the diffraction process, the characteristic distance of which is Rayleigh length $Z_d = \frac{\pi r_0^2}{\lambda}$, where $\lambda$ is laser wavelength and $r_0$ is spot size. However, such situation is prone to large number of instabilities and other undesirable effects [5]. Self-focusing and filamentation are genuinely nonlinear basic physical mechanisms and plays crucial role in propagation of lasers in underdense plasma [8,9,10]. Self-focusing is due to increase of dielectric constant on axis relative to the edge of the beam, resulting from averaged quiver motion of electrons with their subsequent expulsion from the region of high intensity. This mechanism is of ponderomotive type and operates on time scale of the order of $r_0/\nu_s$, where $r_0$ is the dimension of the beam and $\nu$ is the ion acoustic speed [11]. As high power lasers are used, the quiver motion also reduces the local plasma frequency and can lead to relativistic self-focusing. Relativistic and ponderomotive self-focusing have been investigated by a number of researchers [12,13].

In order that laser beam propagate several Rayleigh lengths, the guiding of laser pulse becomes essential. Particularly in the presence of self-focusing, stable guiding over large underdense...
plasma is needed. Since self-focusing is sensitive to spatiotemporal profile of the laser beam intensity, external guiding systems seems to be favourable. Plasma channels [14,15,16,17,18,19] have been proposed as the means of guiding laser pulses over long distances. Several experimental methods have been proposed to create plasma channel [20]. Durfee and Milchberg [21,22] prepared channel of 1 cm length in high Z- gas but when the main pulse is injected the channel was easily destroyed by the production of additional free electrons due to further ionisation. Gual et al. [23] succeeded in the propagation of channels containing fully ionised helium ions and demonstrated the distortion free guiding of a high power laser pulse over a distance of 1.5 cm. Fauser and Langhoff [30] achieved a guiding over 180 times the Rayleigh lengths. If we create the channel by focusing an intense prepulse, then channeling pulse must propagate through a large plasma region without being absorbed or beam break up due to filamentation. This prepulse with Gaussian radial distribution will modify plasma density profile. Index of refraction of plasma thus created will have minimum on the axis and increases radially outwards. Such a medium will behave as a defocusing lens. The particle in cell (PIC) code developed by Durfee and Milchberg [21] predicted the formation of plasma channel with density minimum on the axis, a few second after the ionisation pulse is gone. The dynamics of second pulse sent through such a channel will be governed by competing processes of diffraction and refraction.

Based on some earlier observations, Liu and Tripathi [24] developed a theoretical model for the propagation of delayed pulse in the preformed channel. They used paraxial ray (PR) and Wentzel-Krammers-Brillouin (WKB) approximations to study the competing physical mechanisms of self-focusing and diffraction. The main drawback of this theory is that it overemphasizes the importance of filed close to beam axis and lacks global pulse dynamics. Their prediction showed that propagation is only limited to approximately six Rayleigh lengths. Later on Gill [25] used the variational approach to study the optical guiding of laser beam in non uniform plasma. The guided pulse was observed to propagate 12.5 Z_d before the diffraction process dominates.

Situation become piquant when laser beam along with electron beam propagates. Such observations is supported by the experiment and PIC simulations. In a high power short pulse laser plasma interactions, it is revealed that there is strong flow of energetic electrons co-moving with the laser beam. This would definitely play crucial role in ambipolar diffusion process and lead to significant change in dielectric constant. In the present research work, we are going to examine the role of energetic electrons on laser plasma channeling.

2. Dielectric constant
We consider two types of electrons, viz, plasma electrons and beam electrons. Let the density and velocity of beam electron be \( n_{eb} \) and \( n_b Z \), plasma electrons have no equilibrium drift and their density is \( n_{p0} - n_{eb} \). If we consider the ponderomotive force on electrons and beam taking into account the relative nature of the beam velocity, then following the procedure of Tiwari and Tripathi [26], we can show that the nonlinear permittivity of the medium can be written as:

\[
\varepsilon_{nl} = (1 - \frac{n_{eb}^0}{n_{p0}^0}) \omega^2 \left(1 - \exp(-\alpha |E|^2)\right) + \frac{\omega_{pb}^2}{\omega^2} \left(1 - \exp(-\beta |E|^2)\right)
\]

where,

\[
\alpha = \frac{e\alpha}{2m\omega^2 T_e}, \quad \beta = \frac{e\beta}{2m\omega^2 T_{eb}}, \quad \omega_{p0} = \left(\frac{4\pi n_{eb} e^2}{m}\right)^{\frac{1}{2}} \text{ and } \omega_{pb} = \left(\frac{4\pi n_{p0} e^2}{m}\right)^{\frac{1}{2}}
\]
Here \( \omega_{po} \) and \( \omega_{pb} \) is the frequency of the laser beam and electron beam, respectively, \( \gamma_0 \) is the relativistic factor. The value of \( \alpha \) and \( \beta \) in the above is 0.85 and 0.35.

3. Basic Formulation

The fundamental equation which governs the evolution of field in plasma medium is nonlinear wave equation. In slowly varying envelope approximation, and with the nonlinear dielectric constant given by [26], the parabolic equation governing the complex amplitude of laser beam is given by:

\[
2k_0 \frac{\partial A_0}{\partial z} + \nabla^2 A_0 + \frac{\omega^2}{c^2}(1 - \frac{n_{po}^2}{n_{no}^2}) \frac{\partial^2 A_0}{\partial z^2}(1 - \exp(-\alpha |E|^2)) + \frac{\omega^2_{pb}}{\omega^2_{no}}(1 - \exp(-\beta |E|^2))|A_0|^2 = 0
\]

(1)

The exact solution of equation (1) is not available and, we therefore seek either numerical and analytical approximate method. Although several approximate method are available. We have used a powerful variational method which have been used in several nonlinear wave problem in many physical systems [27]. In this approach, we can reformulate equation (1) into a variational problem corresponding to a Lagrangian \( L \) so as to make \( \frac{\partial L}{\partial A} = 0 \), is equivalent to equation (1).

Following [14,28,15,29], Lagrangian \( L \) is given by:

\[
L = -\frac{1}{2}\int |\frac{\partial A_0}{\partial r}|^2 + i k_0 r (A_0^* \frac{\partial A_0}{\partial z} - A_0 \frac{\partial A_0^*}{\partial z}) + \frac{\alpha}{2}(1 - \frac{n_{po}^2}{n_{no}^2}) \frac{\omega^2_{po}}{\omega^2_{no}} |A_0|^4 + \frac{1}{\alpha}(\exp(-\alpha |A_0|^2)) - \frac{1}{\alpha^2} + \frac{\omega^2_{pb}}{\omega^2_{no}} \left( |A_0|^2 + \frac{1}{\beta}(\exp(-\beta |A_0|^2)) \right) \left( |A_0|^2 - \frac{1}{\beta^2} \right)
\]

(2)

Thus the solution to the variational problem:

\[
\delta \iint Ldxdydz = 0
\]

(3)

also solves the nonlinear Schrodinger equation (1). The simplest choice of trial function, we assume Gaussian laser beam of the following form:

\[
A_0(r,z) = A_0(z) \exp[-\frac{r^2}{2a^2(z)} + ib(z)r^2]
\]

(4)

Using \( A_0 \) given by equation (4) in to \( L \), we integrate \( L \) to obtain:

\[
< L > \equiv \iint Ldxdydz
\]

(5)

Thus we have arrived at reduced variational problem. We solve the above integral to give:

\[
< L > = < L_0 > + < L_1 >
\]

(6)

where

\[
< L_0 > = -\int |A_0|^2 \left[ a^4 \left( \frac{1}{2a^2} + 2b^2 + k_0 \frac{db}{dz} \right) + \frac{ik_0 a^2}{2} \right] \left[ \frac{dA_0}{dz} - A_0 \frac{dA_0^*}{dz} \right]
\]

(7)

\[
< L_1 > = \frac{\omega^2 a^2}{2c^2} \left[ \omega_{po} \left| A_0 \right|^2 - \frac{1}{\alpha} E(\alpha \left| A_0 \right|^2) \right] + \omega_{no} \left| A_0 \right|^2 - \frac{1}{\beta} E(\beta \left| A_0 \right|^2)
\]
$$\omega_{sp} = \left(1 - \frac{n_{o b}^0}{n_{o e}^0}\right) \frac{\omega_p^2}{\omega^2}$$

$$\omega_{np} = \frac{\omega_{pb}^2}{\omega^2 T_0}$$

with exponential integral

$$E(\alpha |A_o|^2) = \int_0^1 \frac{1 - \exp(-t)}{t} dt$$

We obtain Euler-Lagrange equations using

$$\frac{\partial}{\partial \xi} \frac{\delta}{\partial \delta} = 0$$

where \( S \) denotes \( A'_0, A^*_0, a, b \) etc. and following procedure of [14], we get:

$$\frac{d^2 a}{d \xi^2} = \frac{\omega^2 a^3}{c^2}$$

$$\frac{\omega_m}{a_s a_0} - \frac{2 \omega_m}{a_s a_0} \left( \frac{E(\alpha |A_o|^2)}{2 \alpha |A_o|^2} \right) - \frac{2 \omega_m}{a_s a_0} \left( \frac{\exp(-\alpha |A_o|^2)}{\alpha |A_o|^2} \right)$$

Equation (8) and (9) are nonlinear ordinary differential equations governing the evolution of normalized beam width of the laser beam and phase developed during the propagation in our laser electron beam plasma system. It may further be mentioned that right hand side of equation (8) contains several terms each representing the physical mechanisms responsible for the evolution of beam during the propagation in plasma. For example, first term on right hand side is diffraction term which leads to divergence of beam in the absence of other terms. Second, fourth, fifth and seventh terms are due to ponderomotive force resulting from Gaussian nature of laser and electron beam. However, third and sixth term, which are the respectively the contributions of laser and electron beam, appear here due to averaging process considered in Lagrangian formulation. It may be mentioned that these terms do not appear in paraxial ray approximation.
theory. Further, both these terms (third and sixth) counteract the diffraction phenomenon and thus contribute to self-focusing process. It is the relative competition of the various terms which ultimately determines the fate of the beam width. It is worth mentioning that equation (8) and equation (9) cannot be solved analytically. As such, we have to use numerical methods for tractable solution. We have used the following set of parameters for the numerical computation:

\[
\omega_o = 1.778 \times 10^{14} \text{ rad/sec}, \quad a_o = 0.002 \text{ cm}, \quad \gamma_o = 2, \quad \omega_{pe} = 0.1 \omega_o, \quad a = 0.4, \quad \beta'/\alpha' = 0.2, \quad n_{0b} = 0.2, \quad k_o = 0.53 \times 10^4 \text{ cm}^{-1}.
\]

The results are shown in the form of graphs plotted in figure 1 exhibit normalized beam width function as a function of dimensionless distance of propagation \( \zeta \) for these values of laser beam intensity and electron beam parameters. It is observed that for lower values \( \alpha' |A_0|^2 \) and \( \beta' |A_0|^2 \) beam oscillations are slow. However, increase of these parameters leads to faster self-focusing oscillatory behaviour. The role of the electron beam is seen to guide the laser beam to a very long propagation distances. Since free streaming electrons which are modeled as an electron beam, are the results of intense laser beam propagation, it apparently leads to long distance of propagation both by contributing fifth and seventh, which helps the self-focusing and consequently counteract the diffraction process. This becomes more obvious on numerical computation and results are shown in figure 2. We have considered only a few Rayleigh lengths \( R_d \). However, the nature of propagation (figure 2) shows that beam propagation extends to hundreds of \( R_d \). These observations confirms the experimental observations [21] where guided wave propagates over hundreds of Rayleigh lengths. Figure 3. displays the phase \( \Phi \) as a function of \( \zeta \) for these values of intensity parameters. It is observed that phase is negative with distance of propagation. However, \( \Phi \) regularised is always negative (not shown). In the following section we discuss the stability criterion associated with nonlinear dynamics of the beam.

5. Stability Criterion

The stability properties of the system of ordinary differential equations is determined by solving them for equilibrium points [32]. The steady state solutions can be obtained from equations (10)-(12) for vanishing derivatives of amplitude, width and curvature. Variationally obtained Euler-Lagrange equations are the starting point in order to establish a stability criterion using the method of Lyapunov’s exponents [33].

To study the stability properties of the system [22], the following Jacoby determinant is constructed from derivatives with respect to amplitude, width and curvature in terms of \( S, F \) and \( G \):

\[
S = \frac{dA_0}{dz} = \frac{2b|A_0|}{k_0}, \quad F = \frac{da}{dz} = \frac{4ab}{k_0}, \quad G = \frac{d^2}{dz^2} \frac{A_0}{k_0}
\]

where
This leads to the following characteristic equation cubic in $\lambda$:

$$\lambda^3 + \alpha_1 \lambda^2 + \alpha_2 \lambda + \alpha_3 = 0$$

where

$$\alpha_1 = \frac{-2b}{k_0}$$

$$\alpha_2 = \frac{4}{k_0^2 a^2} + \frac{4 \omega^2 \omega_n}{k_0^2 a^2 c^2} \left( \exp(-\alpha |A_0|^2) \right) + \frac{4 \omega^2 \omega_{np}}{k_0^2 a^2 c^2} \left( \exp(-\beta |A_0|^2) \right) - \frac{16b^2}{k_0^2} - \frac{2 \omega^2 \omega_n}{k_0^2 a^2 c^2}$$

$$\left[1 - \exp(-\alpha |A_0|^2)\right] - \frac{2 \omega^2 \omega_{np}}{k_0^2 a^2 c^2} \beta |A_0|$$

$$\left[1 - \exp(-\beta |A_0|^2)\right] - \frac{2 \omega^2 \omega_{np}}{k_0^2 a^2 c^2}$$

$$\alpha_3 = \frac{-8b}{k_0^2 a^2} - \frac{8 \omega^2 \omega_n b}{k_0^2 a^2 c^2} \left( \exp(-\alpha |A_0|^2) \right) - \frac{8 \omega^2 \omega_{np} b}{k_0^2 a^2 c^2} \left( \exp(-\beta |A_0|^2) \right) + \frac{32b^3}{k_0^3} + \frac{4 \omega^2 \omega_n b}{k_0^2 a^2 c^2}$$

$$\left[1 - \exp(-\alpha |A_0|^2)\right] + \frac{4 \omega^2 \omega_{np} b}{k_0^2 a^2 c^2} \beta |A_0|$$

$$\left[1 - \exp(-\beta |A_0|^2)\right] + \frac{4 \omega^2 \omega_{np} b}{k_0^2 a^2 c^2} + \frac{4 \omega^2 \omega_{np} b}{k_0^2 a^2 c^2}$$

In order to have Lyapunov's stability, Hurwitz conditions must be fulfilled, i.e. $\alpha_2 - \alpha_1$ must be positive. According to the Routh-Hurwitz criterion, a necessary and sufficient condition for the stationary solutions to be stable is:
\[ \alpha_1 \alpha_2 - \alpha_3 > 0 \]

Equation (14) has a pair of purely imaginary roots at a critical point:

\[ \lambda = \pm iv, \quad v > 0 \quad (18) \]

We may substitute (18) into (14) and we get:

\[ v^2 - \alpha_2 = 0 \quad (19) \]

and

\[ \alpha_1 v^2 - \alpha_3 = 0 \quad (20) \]

The critical condition of the Hopf bifurcation is:

\[ f = \alpha_1 \alpha_2 - \alpha_3 = 0 \quad (21) \]

\( f > 0 \) is a necessary condition for the stationary solution to be stable, \( f < 0 \) is a necessary condition for the Hopf bifurcation to emerge. It is observed that the condition \( f > 0 \) is satisfied for the chosen set of parameters and therefore Hopf bifurcation, resulting from the unstable fixed point, does not come into play, leading to overall stability of the beam dynamics.

6. Conclusions

In this paper, we have studied the self-focusing and self phase modulation of laser electron beam plasma system. Increase in intensity leads to decrease in nonlinear term with dominance of self-focusing over the spatial dispersion. Contribution of the electron beam leads to guiding of the laser beam over very large number of Rayleigh lengths. The longitudinal phase may be positive or negative depending on the value of intensity parameter. In addition to this, the condition of Hopf bifurcation for determining the overall stability of beam dynamics, is satisfied in the present investigation.

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Figure 1. Variation of normalized beam width $a_n$ with dimensionless distance of propagation $\xi$ for different values of intensity parameters $\alpha' A_o^{-2}$ and $\beta A_o^{-2}$ with $a_o = 0.002 \text{ cm}, a = 0.4, k = 0.53 \times 10^4 \text{ cm}^{-1}, \text{ wrp} = 0.008, \text{ wnp} = 9.4 \times 10^{-4}, \omega_o = 1.778 \times 10^{14} \text{ rad/sec}$. Curve 1 corresponds to $\alpha'A_o^{-2}=1$ and $\beta A_o^{-2}=0.2$, Curve 2 corresponds to $\alpha'A_o^{-2}=2$ and $\beta A_o^{-2}=0.4$, Curve 3 corresponds to $\alpha'A_o^{-2}=3$ and $\beta A_o^{-2}=0.6$. 
Figure 2. Variation of normalized beam width \( a_n \) with dimensionless distance of propagation \( \xi \) for same set of parameters as in curve 1 of figure 1.

Figure 3. Plot of the longitudinal phase delay \( \phi (\xi) \) for different values of intensity parameters \( \alpha A_0^2 \) and \( \beta A_0^2 \) and with the other parameters the same as mentioned in the caption of figure 1. Curve 1 corresponds to \( \alpha A_0^2 = 1 \) and \( \beta A_0^2 = 0.2 \). Curve 2 corresponds to \( \alpha A_0^2 = 2 \) and \( \beta A_0^2 = 0.4 \).