Mathematical model of unemployment with a cyclical component

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Abstract
In this paper, we proposed and analyzed a new mathematical model of unemployment. Two types of unemployment are involved, structural and cyclical unemployment. The problem is modeled using a nonlinear of ordinary differential system. Three variables are considered, the structural unemployment (S), the employment (E) and the cyclical unemployment (C). Existence, positivity and boundedness of this model are proved. Local stability and global stability are established. The impact of different values of the parameters is analyzed by discussing their sensibility. Numerical simulations are given to confirm the main theoretical findings.

Keywords Unemployment · Mathematical model · Cyclical unemployment · Structural unemployment · Equilibrium · Stability

1 Introduction

Unemployment is a major problem for all countries. The political, economic and social efforts made by governments focus on economic and social development in order to implement programs and strategies to fight against unemployment phenomenon. The technological revolution that has been marked in recent years combined with inadequate training to the labor market, are among the main facts that cause structural and permanent unemployment. In addition, economic shocks, natural disasters and health crises are the main factors of cyclical unemployment.

According to the World Labor Organization in 2020 and 2021, Covid-19 has increased unemployment and affected 3.3 billion of population. Four out of five people are affected by the total or partial closure of workplaces. Due to this universal situation, the cyclical unemployment that still exists has known a great intensity. The efforts of governments to fight against structural unemployment are multiplying. The economies of the world are also called to invent measures to reduce cyclical unemployment and implement strategies and alternatives to overcome this crisis and recover the jobs.

Many researches using mathematical modeling of the unemployment problem have been the subject of several recent works [1–7]. To understand the dynamics of unemployment, we have followed up on the present work. In this context, it should be noted that the first investigation involving mathematical modeling that decree the unemployment problem was in 2011 by Misra and Singh [2]. In this work, the authors used the concepts introduced by Nikolopoulos and Tzanetis in 2003 [8] who presented a model of housing allowance for homeless families following a natural disaster. In 2013, Misra and Singh [9] developed a mathematical model with delay for control by creating new jobs. In 2015, Pathan and Bhathawala [10] analyzed the effect of self-employment on the unemployment rate and found that the problem of unemployment can be controlled in two manners. First, the government and the private sector must create more new vacancies. Second, it strongly stimulates the unemployed to think about the step of self-employment. In 2017, several researches interested in this problem are developed [4,6,7,11,12]. In [7], Pathan and Bhathawala extended their unemployment model to include four dynamic variables: the number of unemployed, the number of new migrant workers, the number of employed persons and the number of new jobs created by the state and the private sector. The authors suggested that if the territories allow new labor migrants, then this should create new vacancies in propor-
tion to the local unemployed, as well as labor migrants. In addition, they encouraged every possible attempt by the local unemployed and new migrant workers to become self-employed. Furthermore, Misra and Singh [11] retrieved the unemployment problem by examining the impact of the skills development pathways offered to the unemployed by some academic institutions as an important role for the control problem. They extended the previous nonlinear dynamic model to take into account four variables: unemployment, temporary/self-employment, permanent employment, and the avenues for the skill development. They found that as the effectiveness of the skills development pathway increased, the number of unemployed decreased while the number of casual/self-employed increased. In 2018, the authors in [13] found that to address the unemployment problem effectively, the government should create and provide more employment opportunities by increasing the rate of employment and decreasing the rate of diminution of available vacancies due to lack of government funds. In 2020, the authors in [14] studied the role of skills development in controlling unemployment. They used an extended mathematical model describing the importance of low-skilled and high-skilled persons in unemployment dynamics. They examined the role of highly skilled individuals endowed with entrepreneurial skills in job creation and explored the optimal strategies for minimizing the number of unemployed. They also studied the costs associated with the creation of qualified people and the job offers of highly qualified people. In 2021 [1], the authors analyzed how developing the skills of some unemployed people can reduce the unemployment problem and proposed how the unemployment problem can be controlled.

In this work, we investigate the problem of unemployment using a different modeling approach from those used in the previous researches. Then, we introduce the business cycle dimension by the subdivision of unemployment into structural and cyclical components. We are interested in the study of unemployment due to economic conditions resulting from shocks of lack of investments, decrease in demand and the effects of health crises that can affect the labor market. Recently, the world of employment was deeply affected by the global pandemic of Covid-19: unemployment increases and more precisely the cyclical unemployment. This work is devoted to the analysis of the unemployment problem caused by this kind of crisis. Motivated by the above researches [2,8] and the current economic situation, this paper considered a new model to analyze the problem of unemployment, where we subdivide the class of unemployment into two subclasses: structurally unemployed and cyclically unemployed people. This model contains three classes, structurally unemployed class \(S\), unemployed class \(E\) and cyclically unemployed class \(C\).

This paper is organized as follows: in Sect. 2, we present and explain the formulation of the problem; then, we prove the positivity and boundedness of the proposed model. In Sect. 3, we investigate the existence of the equilibrium point and its local stability. In Sect. 4, we present the global stability analysis of the equilibrium point using Lyapunov method. In Sect. 5, we present and study the impact of some key parameters on structurally unemployed and cyclically unemployed classes. Finally, we give some numerical simulations to support the theoretical analysis.

## 2 Model formulation positivity and boundedness

### 2.1 Model formulation

In the process of making a model, we assumed that all entrants to the structurally unemployed people are fully qualified and competent to occupy a job but that their skills lose relevance over time. The number of structurally unemployed people denoted by \(S(t)\) increases with constant rate \(\alpha\). Some of the structurally unemployed persons may become employed with a rate \(\alpha_1\). Some of employed persons \(E(t)\) may leave their jobs and join the structurally unemployed class with a rate \(\gamma\). On the other hand, some of the employers may fire or dismiss employees who go directly to cyclically unemployed class denoted by \(C(t)\) with the rate \(\alpha_2\). It is further assumed that some cyclically unemployed persons will try to return to their jobs in the employed class with a rate \(\alpha_3\) while the rest of persons who could not return to their jobs go directly to the structural unemployed class with a rate \(\alpha_3\). The diagram for this model is shown in Fig. 1. In this model, it is also assumed that the total numbers of available employment vacancies are limited and these are assumed to be constant.

The rate of variation in the number of structurally unemployed people who move into employment will be jointly proportional to \(S\) and to the number of available employment vacancies \((E_a - E)\), the rate of change of the cyclically unemployed people may become employed will be jointly proportional to \(C\) and to the number of available employment vacancies \((E_a - E)\), where \(E_a\) is the total number of available employment vacancies.

Hence, the dynamics of the model are governed by the following nonlinear differential system.

\[
\begin{align*}
\frac{dS(t)}{dt} &= -\alpha_1 S(t) (E_a - E) + \gamma E(t) + \alpha_2 C(t) - \mu_1 S(t), \\
\frac{dE(t)}{dt} &= \alpha_1 S(t) (E_a - E) + \alpha_2 C(t) (E_a - E) - \gamma E(t) \\
&\quad - \alpha E(t) - \mu_2 E(t), \\
\frac{dC(t)}{dt} &= -\alpha_2 C(t) (E_a - E) + \alpha E(t) - \alpha_3 C(t) - \mu_3 C(t).
\end{align*}
\]
The parameters of the model are positive constants defined as follows: 

- $\alpha_1$ Rate of increase in the number of structurally unemployed people.
- $\gamma$ Rate of employed who loose their jobs and join the structurally unemployed class.
- $\alpha$ Rate of employed who are dismissed from their jobs as a result of the crisis and join the cyclically unemployed class.
- $\alpha_2$ Rate of cyclically unemployed persons who return to their jobs in the employed class.
- $\alpha_3$ Rate of cyclically unemployed persons who unable to return to their jobs and move permanently to the structurally unemployed class.
- $\mu_1$ Rate of migration or death of structurally unemployed persons.
- $\mu_2$ Rate of migration, retirement or death of employed persons.
- $\mu_3$ Rate of migration or death of cyclically unemployed persons.

2.2 Positivity and boundedness

Theorem 1 If $(S(t), E(t), C(t)) \in \mathbb{R}_+^3$, then the set defined by

$$\Omega = \left\{(S, C, E) : 0 \leq S(t) + C(t) \leq \frac{A + (\gamma + \alpha)E_a}{\mu}, 0 \leq E \leq E_a \right\}$$

is positively invariant where $\mu = \min(\mu_1, \mu_3)$.

Proof We note that

$$\frac{dS(t)}{dt} \big|_{S=0} = A + \gamma E(t) + \alpha_3 C(t) \geq 0,$$
$$\frac{dE(t)}{dt} \big|_{E=0} = \alpha_1 S(t) E_a + \alpha_2 C(t) E_a \geq 0,$$
$$\frac{dC(t)}{dt} \big|_{C=0} = \alpha E(t) \geq 0.$$

This implies that for $t \geq 0$, all solutions remain nonnegative, under nonnegative initial data.

Now, we add the first and second equations of system (1), we get

$$\frac{dS(t)}{dt} + \frac{dC(t)}{dt} = A - \alpha_1 S(E_a - E) + \gamma E - \mu_1 S(t) - \alpha_2 C(E_a - E) - \alpha_3 C(t)$$
$$= A + \gamma E + \alpha E(t) - \mu_1 S(t) - \mu_3 C(t)$$
$$\leq A + \gamma E(t) + \alpha E(t) - \mu_1 S(t) - \mu_3 C(t)$$
$$\leq A + \gamma E(t) + \alpha E(t) - \min(\mu_1, \mu_3)(S(t) + C(t)),$$

with $E(t) \leq E_a$. Then, we obtain

$$\frac{dS(t)}{dt} + \frac{dC(t)}{dt} \leq A + \gamma E_a + \alpha E_a$$
$$\leq A + (\gamma + \alpha)E_a - \mu(S(t) + C(t)),$$

where $\mu = \min(\mu_1, \mu_3)$.

Using the supremum limit implies that

$$\lim_{t \to +\infty} \sup (S(t) + C(t)) \leq \frac{A + (\gamma + \alpha)E_a}{\mu}.$$

Then, we conclude that all solutions of system (1) are bounded and do not exit the region $\Omega$. therefore, $\Omega$ is positively invariant. □

3 Equilibrium analysis and local stability

3.1 Equilibrium analysis

The system (1) has only one nonnegative equilibrium, $Q^* (S^*, E^*, C^*)$, which may be obtained by solving the following set of algebraic equations:

$$A - \alpha_1 S(t)(E_a - E) + \gamma E(t) + \alpha_3 C(t) - \mu_1 S(t) = 0,$$

$$A - \gamma E(t) - \mu_2 E(t) - \mu_3 C(t) = 0,$$

$$A - \alpha_2 C(t)(E_a - E) - \mu_3 C(t) = 0.$$

(2)
\( \alpha_1 S(t) (E_a - E) + \alpha_2 C(t) (E_a - E) - \gamma E(t) \)
\[ -\alpha E(t) - \mu_2 E(t) = 0, \]
\[ -\alpha_2 C(t) (E_a - E) + \alpha E(t) - \alpha_3 C(t) - \mu_3 C(t) = 0. \]

From Eq. (3), we get
\[ E = \frac{(\alpha_1 S + \alpha_2 C) E_a}{\gamma + \alpha + \mu_2 + \alpha_1 S + \alpha_2 C}. \]

We substitute (5) in Eq. (2) and (4), we obtain the following equations in \( S \) and \( C \)
\[ A = \frac{\alpha_1 (\gamma + \alpha + \mu_2) E_a S}{\gamma + \alpha + \mu_2 + \alpha_1 S + \alpha_2 C} - \mu_1 S + \alpha_3 C \]
\[ + \frac{\alpha (\alpha_1 S + \alpha_2 C) E_a}{\gamma + \alpha + \mu_2 + \alpha_1 S + \alpha_2 C}, \]
\[ = 0, \]
\[ \gamma + \alpha + \mu_2 + \alpha_1 S + \alpha_2 C \]
\[ - \alpha_2 (\gamma + \alpha + \mu_2) E_a C, \]
\[ \frac{\alpha (\alpha_1 S + \alpha_2 C) E_a}{\gamma + \alpha + \mu_2 + \alpha_1 S + \alpha_2 C} \]
\[ + \frac{\alpha (\gamma + \alpha + \mu_2) E_a C}{\gamma + \alpha + \mu_2 + \alpha_1 S + \alpha_2 C}, \]
\[ - \mu_3 C - \alpha_3 C = 0. \]

Now to show the existence of equilibria, we plot the isoclines given by Eqs. (6) and (7) (see Fig. 2).

From Eq. (6), we may easily note the following facts
(i) For \( C = 0 \), Eq. (6) reduces to a quadratic equation in \( S \).

From this equation, we get one positive root, whereas the other root will be negative.

(ii) For \( S = 0 \), Eq. (6) reduces to a quadratic equation in \( C \); this equation has no solution in \( \mathbb{R}_+ \).

\[
M = \begin{bmatrix}
-\alpha_1 (E_a - E^*) - \mu_1 & 0 & \alpha_3 \\
\alpha_1 (E_a - E^*) - \mu_1 & -\alpha_1 S^* - \alpha_2 C^* - (\gamma + \alpha + \mu_2) & \alpha_2 (E_a - E^*) \\
0 & \alpha_2 C^* + \alpha & -\alpha_2 (E_a - E^*) - (\alpha_3 + \mu_1) 
\end{bmatrix}
\]

Similarly, for Eq. (7), we get the following facts
(i) For \( S = 0 \), \( C = 0 \) and \( C = -\frac{\alpha_2 (\gamma + \mu_2) E_a + (\alpha_3 + \mu_3)(\gamma + \mu_2 + \alpha)}{\alpha_2 (\alpha_3 + \mu_3)}, \)
which is negative.

(ii) \( C = \frac{-\alpha E_a}{\alpha_3 + \mu_3} \) as \( E \rightarrow \infty \).

(iii) \( \frac{dS}{dc} \) is positive for \( S > 0 \) and \( 0 \leq E \leq E_a \).

Hence, the two isoclines given by Eqs. (6) and (7) will intersect in the interior of the positive quadrant at \((S^*, C^*)\) (Fig. 2). Substituting \( S^* \) and \( C^* \) in Eq. (5) we get a positive value of \( E \), noted by \( E^* \). Hence, the model system (1) has only one nonnegative equilibrium \( Q^* (S^*, E^*, C^*) \).

\[ \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3 = 0, \]

where
\[ a_3 = a_1 \mu_1 (E_a - E^*)^2 + \mu_1 (E_a - E^*) (\alpha_3 S^* + \gamma + \alpha + \mu_2) \]
\[ + \mu_2 (\alpha_1 S^* + \alpha_2 C^* + \gamma + \alpha + \mu_2)(\alpha_3 + \mu_3) \]
\[ + \alpha_1 (E_a - E^*) \mu_3 \alpha_3 \]
\[ + \alpha_2 (E_a - E^*) (\alpha_2 C^* + \gamma + \mu_2), \]
\[ a_2 = a_1 (E_a - E^*) + (\alpha_2 C^* + \gamma + \alpha + \mu_2) + \mu_1 (\alpha_2 C^* + \gamma + \alpha + \mu_2) \]
\[ + (\alpha_3 + \mu_3)(\alpha_1 S^* + \alpha_2 C^* + \gamma + \alpha + \mu_2) \]
\[ + \alpha_2 (E_a - E^*) (\alpha_1 S^* + \gamma + \mu_2), \]
\[ a_1 = a_1 (E_a - E^*) + \alpha_1 S^* + \alpha_2 C^* + \alpha_3 + \gamma + \alpha + \mu_2. \]
Clearly \( a_1, a_2, a_3 \) are all positive and a little algebraic manipulation yields that \( a_1a_2 - a_3 > 0 \). Now by using the Routh–Hurwitz criterion, the local stability is concluded. \( \square \)

4 Global stability analysis

To investigate the global stability on the positive equilibrium \( Q^* (S^*, E^*, C^*) \) of (1), we state and prove the following theorem.

**Theorem 3** The equilibrium \( Q^* (S^*, E^*, C^*) \) is globally asymptotically stable in the region \( \Omega \) if the following conditions are satisfied

\[
\begin{align*}
(i) & \quad \alpha_1 (\gamma + \alpha S^*) E_a < \alpha(2\mu_1 - \alpha 3)) E^*, \\
(ii) & \quad \alpha_3 + \alpha_2 E_a (\alpha C^* + \alpha) < 2\alpha_2 (\alpha_2 + \mu_3) E^*, \\
(iii) & \quad \alpha_1 + \alpha_2 E_a < 2(\gamma + \alpha + \mu_2).
\end{align*}
\]

**Proof** To study the global stability behavior of the positive equilibrium \( Q^* (S^*, E^*, C^*) \) of (1), we use a suitable Lyapunov function.

Let us consider the following positive definite function on \( \Omega \).

\[
U(S, E, C) = \frac{1}{2} (S - S^*)^2 + \frac{1}{2} m_1 (C - C^*)^2 + \frac{1}{2} m_2 (E - E^*)^2.
\]

where the coefficients \( m_1 \) and \( m_2 \) are arbitrary constant to be chosen suitably later.

Then, we get

\[
\frac{dU}{dt} = -\mu_1 + \alpha_1 (E_a - E)(S - S^*)^2 - m_1 [\alpha_3 + \alpha_3
+ \alpha_2 (E_a - E)] (C - C^*)^2
- m_2 [\gamma + \alpha + \mu_2 + \alpha_1 S + \alpha_2 C] (E - E^*)^2
+ \alpha_3 (S - S^*)(C - C^*)
+ [\gamma + m_2 \alpha (E_a - E^*) + \alpha_1 S^*] (S - S^*)(E - E^*)
+ [m_2 \alpha_2 (E_a - E^*) + m_1] (\alpha + \alpha C^*) (C - C^*) (E - E^*).
\]

Choosing \( m_1 = \frac{\alpha_2 E^*}{\alpha C^* + \alpha} (S^* + \gamma / \alpha E^*) \), \( m_2 = \frac{\gamma}{\alpha_1 E^*} + S^*/E^* \), then \( \frac{dU}{dt} \) will be negative definite inside the region of attraction \( \Omega \) if the inequalities (i), (ii) and (iii) are satisfied. Hence, under these conditions, the equilibrium \( Q^* (S^*, E^*, C^*) \) is globally asymptotically stable. \( \square \)

5 Sensitivity analysis of the equilibrium values

In this section, we analyze the impact of different values of parameters on \( S^* \) and \( C^* \). The parameters considered here are the rate of structurally employed people \( \alpha_1 \), the rate of cyclically unemployed people who return to their jobs in the employed class \( \alpha_2 \) and the rate of cyclically unemployed people who are unable to return to their jobs and move permanently to the structurally employed people \( \alpha_3 \).

5.1 Variation of \( S^* \) with respect to \( \alpha_3 \)

Let

\[
\begin{align*}
f(S^*, C^*, \alpha_3) &= A - \frac{\alpha_1(\gamma + \alpha + \mu_2) E_a S^*}{\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*}
- \frac{\gamma (\alpha_1 S^* + \alpha_2 C^*) E_a}{\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*}, \\
g(S^*, C^*, \alpha_3) &= -\frac{\alpha_2(\gamma + \alpha + \mu_2) E_a C^*}{\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*}
+ \frac{\gamma (\alpha_1 S^* + \alpha_2 C^*) E_a}{\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*}
- (\alpha_3 + \mu_3) C^*.
\end{align*}
\]

Then,

\[
\frac{dS^*}{d\alpha_3} = \frac{\partial f}{\partial C^*} \frac{\partial g}{\partial \alpha_3} - \frac{\partial f}{\partial \alpha_3} \frac{\partial g}{\partial C^*}.
\]

From Eq. (8), we have

\[
\frac{df}{dC^*} = \frac{-\alpha_1(\gamma + \alpha + \mu_2) E_a (\alpha + \mu_2 + \alpha_2 C^*)^2}{(\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*)^2} - \mu_1,
\]

\[
\frac{df}{dC^*} = \frac{\alpha_2(\gamma + \alpha + \mu_2) E_a (\gamma + \alpha_3^*)}{(\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*)^2} + \alpha_3,
\]

\[
\frac{df}{d\alpha_3} = -C^*.
\]

and from Eq. (9) we get

\[
\frac{dg}{dC^*} = \frac{-\alpha_2(\gamma + \alpha + \mu_2) E_a (\gamma + \alpha + \mu_2 + \alpha_1 S^*)}{(\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*)^2} - (\mu_2 + \alpha_3),
\]

\[
\frac{dg}{d\alpha_3} = -C^*.
\]
\[
+ \frac{\alpha_3 (\gamma + \alpha + \mu_2) E_0 (\alpha + \mu_2 + \alpha_2 C^*)}{(\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*)^2} \\
+ \frac{k_2 (\gamma + \alpha + \mu_2) E_0 (\mu_2 + \alpha_1 S^*)}{(\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*)^2} + \mu_1 (\mu_3 + \alpha_3) > 0.
\]

(11)

Similarly, we have

\[
\frac{df}{dC^*} \cdot \frac{dg}{d\alpha_3} - \frac{df}{d\alpha_3} \cdot \frac{dg}{dC^*} = \frac{\mu_2 \alpha_2 (\gamma + \alpha + \mu_2) C^*}{(\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*)^2} + \mu_3 C^* > 0.
\]

(12)

From (11) and (12), we conclude that \( \frac{dS^*}{d\alpha_3} \) is positive; then, \( S^* \) increases when \( \alpha_3 \) increases.

5.2 Variation of \( C^* \) with respect to \( \alpha_2 \)

Let

\[
\begin{aligned}
f_1 (S^*, C^*, \alpha_2) &= \frac{\alpha_1 (\gamma + \alpha + \mu_2) E_0 S^*}{\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*} - \mu_1 S^* + \alpha_3 C^* \\
&\quad + \frac{\gamma (\alpha_1 S^* + \alpha_2 C^*) E_0}{(\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*)^2} \\
g_1 (S^*, C^*, \alpha_2) &= -\alpha_2 (\gamma + \alpha + \mu_2) E_0 C^* \\
&\quad + \frac{\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*}{\alpha (\alpha_1 S^* + \alpha_2 C^*) E_0} \\
&\quad + \frac{\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*}{(\alpha_1 S^* + \alpha_2 C^*) E_0}
\end{aligned}
\]

(13)

Then,

\[
\frac{dC^*}{d\alpha_2} = \frac{df_1}{dS^*} \frac{dg_1}{d\alpha_2} - \frac{df_1}{d\alpha_2} \frac{dg_1}{dS^*}
\]

(15)

where

\[
\begin{aligned}
\frac{df_1}{dS^*} \frac{dg_1}{d\alpha_2} - \frac{df_1}{d\alpha_2} \frac{dg_1}{dS^*} &= \frac{\alpha_1 (\gamma + \alpha + \mu_2)^2 E_0^2 C^* (\mu_2 (\alpha + \alpha_2 C^*) + \alpha (\gamma + \mu_2 + \alpha_1 S^*))}{(\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*)^3} \\
&\quad + \frac{\mu_1 (\gamma + \alpha + \mu_2) E_0 C^* (\gamma + \mu_2 + \alpha_1 S^*)}{(\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*)^4} > 0.
\end{aligned}
\]

(16)

\[
\begin{aligned}
\frac{df_1}{dC^*} \frac{dg_1}{d\alpha_2} - \frac{df_1}{d\alpha_2} \frac{dg_1}{dC^*} &= -\alpha_2 (\gamma + \alpha + \mu_2)^2 E_0^2 C^* (\mu_2 (\alpha + \alpha_2 C^*) + \alpha (\gamma + \mu_2 + \alpha_1 S^*)) \\
&\quad + \frac{\mu_1 (\gamma + \alpha + \mu_2) E_0 C^* (\gamma + \mu_2 + \alpha_1 S^*)}{(\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*)^4} \\
&\quad - \frac{\mu_1 (\gamma + \alpha + \mu_2) E_0 (\gamma + \mu_2 + \alpha_1 S^*)}{(\gamma + \alpha + \mu_2 + \alpha_1 S^* + \alpha_2 C^*)^4} \\
&\quad - \mu_1 (\mu_3 + \alpha_3) < 0.
\end{aligned}
\]

(17)

From (16) and (17), we deduce that \( \frac{dC^*}{d\alpha_2} < 0 \), then \( C^* \) decreases when \( \alpha_2 \) increases. In a similar way, we find that \( \frac{dS^*}{d\alpha_3} < 0 \). Hence, \( S^* \) decreases when \( \alpha_3 \) increases.

According to the above analysis, we conclude that the strategy to control the unemployment problem should focus on increasing the rate of structurally employed people \( \alpha_1 \) and the rate of cyclically unemployed people who return to their jobs in the employed class \( \alpha_2 \), while decreasing the rate of cyclically unemployed people who are unable to return to their jobs and move permanently to the structurally employed class \( \alpha_3 \).

6 Numerical simulations

In this section, we confirm the theoretical results by numerical experiments which are based on the simulation of the solutions of the mathematical model (1). Our tests are designed to verify the feasibility of our analysis regarding the stability conditions and also to show the effect of important factors on cyclical and structural unemployment according to their rate of change which are analyzed and discussed in Sect. 5. We have conducted some numerical calculations using MATLAB codes by choosing the following set of parameter values:

\[
A = 8500, \quad \alpha_1 = 0.004, \quad \alpha_2 = 0.009, \quad \alpha_3 = 0.008, \quad E_0 = 3500, \quad \mu_1 = 2, \quad \mu_2 = 0.9, \quad \mu_3 = 0.9, \quad \alpha = 0.9, \quad \gamma = 0.009.
\]

The equilibrium values for the model is \( Q^* = (2581, 3156, 594) \). Then, we find that all eigenvalues of the variational matrix corresponding to the equilibrium \( Q^*(S^*, C^*, E^*) \) for the model system 1 are negative, so the equilibrium \( Q^*(S^*, C^*, E^*) \) is locally asymptotically stable. Similarly we can note that for all of the above parameter values, the global stability conditions are also satisfied.

Different simulations are also performed for various initial values and the results are illustrated in Figs. 3, 4 and 5. These figures show, respectively, that the numerical solution of structural unemployment people \( S(t) \), employment people \( E(t) \) and cyclical unemployment \( C(t) \) for different initial values. We observe that all numerical solutions approach their equilibrium points for different choices of initial conditions. Hence, we infer that the system is globally asymptotically stable in the region of attraction around equilibrium point \( Q^*(S^*, C^*, E^*) \) for the above set of parameters.

Furthermore, we observe the impact of the different rate \( \alpha_2 \) on the cyclical unemployment, the impact of the different rate \( \alpha_3 \) on the structural unemployment. In Fig. 6, we remark that when \( \alpha_2 \) increases, the number of cyclical unemployment decreases. Unlike in Fig. 7, we remark that when \( \alpha_3 \) increases,
Fig. 3 The numerical simulation of the structural unemployment for various initial conditions $S(0)$

Fig. 4 The numerical simulation of the unemployment for various initial conditions $E(0)$

Fig. 5 The numerical simulation of the cyclical unemployment for various initial conditions $C(0)$

Fig. 6 Variation of the numerical simulation of the cyclical unemployment $C(t)$ for different values of rate $\alpha_2$

Fig. 7 Variation of the numerical simulation of structural unemployment $S(t)$ with time for different values of rate $\alpha_3$

the number of structural unemployment increases. this results confirm the analytical analysis findings in Sects. 5.1 and 5.2.

Figures 3, 4 and 5 show, respectively, that the numerical simulation of structural unemployment people $S(t)$, employment people $E(t)$ and cyclical unemployment $C(t)$ for the different initial values. Thus, we observe that the curves of solution converge to the unique equilibrium point. This result confirms the analysis presented in Sect. 4.
7 Conclusion

In this work we consider a new model of unemployment by introducing a cyclical compartment. After studied this new model we analyzed the impact of cyclical’s parameters on the dynamics of unemployment. First, we proved the well-posedness of the model. Then, we studied the existence of the problem steady state. Next, the local stability of the equilibrium point is given. Furthermore, we discussed the global stability of the model by constructing an appropriate Lyapunov function. In order to study the effect of some key parameters, a sensitivity analysis is presented and discussed. Finally, the theoretical findings are illustrated by numerical simulations. It is shown that our proposed modeling approach can help public authorities to simulate the effect of some economic policies which help to recover the cyclically unemployed people and prevent their fall into the structural unemployment.

For further researches, our study can be completed by an empirical analysis. In addition, the proposed model can be more accurate if we take into account the stochastic effects using a stochastic modeling approach.

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Declarations

Conflict of interest The authors declare the following financial interests/personal relationships which may be considered as potential competing interests.

References

1. Al-Maalwi R, Sarah Al-Sheikh HA, Ashi SA (2021) Mathematical modeling and parameter estimation of unemployment with the impact of training programs. Math Comput Simul 182:705–720
2. Misra AK, Singh Arvind K (2011) A mathematical model for unemployment. Nonlinear Anal Real World Appl 12(1):128–136
3. Al-Sheikh Sarah, Al-Maalwi Raneah, Ashi Hala (2021) A mathematical model of unemployment with the effect of limited jobs. Comptes Rendus. Math 359(04):283–290
4. Munoli SB, Gani S, Gani SR (2017) A mathematical approach to employment policies: an optimal control analysis. Int J Stat Syst 12(3):549–565
5. Daud A, Ghozali A (2015) Stability analysis of a simple mathematical. Casp. J. Appl. Sci. Res. 4(2):15–18
6. Galindo A, Torres DFM (2017) A simple mathematical model for unemployment: a case study in portugal with optimal control, stat. optim. Inf Comput 6:116–129
7. Pathan GPH, Bhatwa W (2017) Mathematical model for unemployment control-a numerical study. Int J Math Trends Technol 49(9):253–259
8. Nikolopulos CV, Tzanetis DE (2003) A model for housing allocation of a homeless population due to a natural disaster. Nonlinear Anal Real World Appl 4(4):561–579
9. Misra AK, Arvind S (2013) A delay mathematical model for the control of unemployment. Differ Equ Dyn Syst 21:07
10. Pathan G, Bhathawala PH (2015) A mathematical model for unemployment with effect of self employment. IOSR J Math 11:37–43
11. Misra AK, Singh AK, Singh PK (2017) Modeling the role of skill development to control unemployment. Differ Equ Dyn Syst 30:1–13
12. Pathan G, Bhathawala PH (2017) A mathematical model for unemployment-taking an action without delay. Int J Math Appl 12:41–48
13. Al-Maalwi RM, Ashi HA, Al-sheikh S (2018) Unemployment model. Appl Math Sci 12(21):989–1006
14. Singh AK, Singh PK, Misra AK (2020) Combating unemployment through skill development. Nonlinear Anal Model Control 25(11):919–937