Abstract

A four point function of basic Neveu-Schwarz exponential fields is constructed in the $N = 1$ supersymmetric Liouville field theory. Although the basic NS structure constants were known previously, we present a new derivation, based on a singular vector decoupling in the NS sector. This allows to stay completely inside the NS sector of the space of states, without referencing to the Ramond fields. The four-point construction involves also the NS blocks, for which we suggest a new recursion representation, the so-called elliptic one. The bootstrap conditions for this four point correlation function are verified numerically for different values of the parameters.
1. The $N = 1$ super Liouville

Construction of the super Liouville field theory (SLFT) is motivated by the following action, which appeared in the non-critical superstring theory in 1981 [1]

\[
\mathcal{L}_{\text{SLFT}} = \frac{1}{8\pi} (\partial_a \phi)^2 + \frac{1}{2\pi} (\psi \bar{\partial} \psi + \bar{\psi} \partial \bar{\psi}) + 2i\mu b^2 \bar{\psi} \psi e^{b \phi} + 2\pi b^2 \mu^2 e^{2b \phi}
\]

where $b$ is the standard “quantum” parameter related through the “background charge” $Q = b^{-1} + b$ to the central charge

\[
\hat{c} = 1 + 2Q^2
\]

of the superconformal algebra generated by the supercurrent $S(z)$, $\bar{S}(\bar{z})$ and the stress tensor $T(z)$, $\bar{T}(\bar{z})$. Traditionally the scale parameter $\mu$ is called the (super) cosmological constant.

Let us recall some details about SLFT necessary for the forthcoming discussion (see e.g., [2, 3, 4, 5, 6]). The space of fields splits into the so called Neveu-Schwarz [7] (NS) and Ramond [8] (R) sectors, the supercurrent components $(S, \bar{S})$ being respectively single or double valued near the point where the corresponding operator is located. Apparently the first, NS sector, is closed under the operator product expansions (OPE). It is completely consistent to consider it separately. This is what we’re going to do in the present publication, meaning to include the R sector in the future.

Respectively, the NS fields belong to highest weight representation of the NS superconformal algebra

\[
\{G_k, G_l\} = 2L_{k+l} + \frac{\hat{c}}{2} \left( k^2 - \frac{1}{4} \right) \delta_{k+l}
\]

\[
[L_n, G_k] = \left( \frac{n}{2} - k \right) G_{n+k}
\]

\[
[L_m, L_n] = (m - n)L_{m+n} + \frac{\hat{c}}{8}(m^3 - m)\delta_{m+n}
\]

where the subscripts $m, n$ are integer and $k, l$ are half-integer. In fact there are two copies of algebra (1.3), the “right” one $SVir$ generated by $S(z)$ and $T(z)$, and the “left” $\overline{SVir}$ constructed from the left-moving components $\bar{S}(\bar{z})$ and $\bar{T}(\bar{z})$. The space is classified in the highest weight representations of $SVir \otimes \overline{SVir}$. The basic NS fields are the scalar primary fields $V_a(x)$ corresponding to the highest weight vectors

\[
L_n V_a = 0; \quad \bar{L}_n V_a = 0; \quad \text{for } n > 0
\]

\[
G_k V_a = 0; \quad \bar{G}_k V_a = 0; \quad \text{for } k > 0
\]

\[
L_0 V_a = \bar{L}_0 V_a = \Delta_a V_a
\]

where

\[
\Delta_a = \frac{a(Q - a)}{2}
\]

and $a$ is a (complex) continuous parameter. It is sometimes instructive to think of these basic operators as of the properly regularized exponentials $V_a = \exp(a\phi)$ of the fundamental
bosonic field entering the Lagrangian (1.1). This is particularly useful in the region of the configuration space where \( \phi \to -\infty \). Here one can neglect the interaction terms in the action (1.1), the fields \( \phi \) and \( (\psi, \bar{\psi}) \) behave as a free boson and a free Majorana fermion and the exponential expression can be given an exact sense.

All other NS fields are the \( SVir \otimes SVir \) descendents of these basic ones. It will prove convenient to distinguish the descendents of integer and half-integer level, for which we reserve (somewhat loosely) the terms “even” descendents, often marked by the index “e”, and the “odd” ones, referred to as “o”. It will be also useful to introduce special notations for the components of the multiplets under the standard super Poincare algebra, a subalgebra of \( SVir \otimes SVir \) generated by \( G^{-1/2} \), \( \bar{G}^{-1/2} \) and \( L_0 - \bar{L}_0 \). From this point of view \( V_a \) is the “bottom” component of the supermultiplet, which includes also

\[
\begin{align*}
\Lambda_a &= G^{-1/2}V_a = -ia\psi e^a \phi \\
\bar{\Lambda}_a &= \bar{G}^{-1/2}V_a = -ia\bar{\psi} e^a \bar{\phi} \\
W_a &= G^{-1/2}\bar{G}^{-1/2}V_a = a^2\bar{\psi} \psi e^a \phi - 2i\pi \mu ab e^{(a+b)\phi}
\end{align*}
\]

Here we partially borrow apt notations from ref. [9] and also give a “free field” interpretations of the corresponding components. The basic Ward identities are

\[
\begin{align*}
T(z)V_a(0) &= \frac{\Delta_a}{z^2}V_a(0) + \frac{1}{z}\partial V_a(0) + \text{reg} \\
T(z)\Lambda_a(0) &= \frac{\Lambda_a}{z^2} + \frac{1}{2}\Lambda_a(0) + \frac{1}{z}\partial \Lambda_a(0) + \text{reg} \\
S(z)V_a(0) &= \frac{1}{z}\Lambda_a(0) + \text{reg} \\
S(z)\Lambda_a(0) &= \frac{2\Delta_a}{z^2}V_a(0) + \frac{1}{z}\partial V_a(0) + \text{reg}
\end{align*}
\]

We explicitly present the holomorphic relations and quote them for \( V_a \) and \( \Lambda_a \) only. The “right” superconformal properties of the doublet \( \bar{\Lambda}_a, W_a \) are the same as of \( V_a, \Lambda_a \) and the “left” ones of \( V_a, \bar{\Lambda}_a \) and \( \Lambda_a, W_a \) are similar to (1.7) with obvious modifications caused by the anticommutativity of the right and left “fermionic” generators \( G_k \) and \( \bar{G}_k \).

Local properties of SLFT in the NS sector are encoded in the basic operator product expansion (here and below for the sake of brevity we denote \( \Delta = \Delta_{Q/2+iP} \) and \( \Delta_i = \Delta_{a_i} \), wherever it cannot cause any misunderstanding)

\[
V_{a_1}(x)V_{a_2}(0) = \int \frac{dP}{4\pi} (x\bar{x})^{\Delta - \Delta_1 - \Delta_2} \left( \mathbb{C}^{Q/2+iP} [V_{Q/2+iP}(0)]_{ee} + \tilde{\mathbb{C}}^{Q/2+iP} [V_{Q/2+iP}(0)]_{oo} \right)
\]

This OPE is continuous and involves integration over the “momentum” \( P \). Precisely as in the bosonic Liouville field theory [10] the integration contour is basically along the real axis, but should be deformed sometimes under analytic continuation in the parameters \( a_1 \) and \( a_2 \). It is a good idea to make such deformations explicit collecting the result in the form
of the so-called discrete terms \[10\]. In \((1.8)\) \([V_p]\) denotes the contribution of the primary field \(V_p\) and its superconformal descendents to the operator product expansion. Unlike the standard conformal symmetry, not all these contributions are prescribed unambiguously by the superconformal invariance, the even and the odd ones entering independently. From here come two different structure constants \(C_{\alpha_1 \alpha_2}^{a_1}\) and \(\bar{C}_{\alpha_1 \alpha_2}^{a_1}\) in the OPE \((1.8)\), while \([V_{\alpha}]_{ee}\) and \([\bar{V}_{\alpha}]_{oo}\) denote respectively the collections of “even-even” and “odd-odd” descendents. As usual \[11\] both towers of descendents enjoy the factorization in the product of holomorphic and antiholomorphic “chains”

\[
[V_{\alpha}(0)]_{ee} = C^{\Delta_1, \Delta_2}_e (\Delta_{\alpha}, x) \bar{C}^{\Delta_1, \Delta_2}_e (\Delta_{\alpha}, \bar{x}) \ V_{\alpha}(0) \\
[V_{\alpha}(0)]_{oo} = C^{\Delta_1, \Delta_2}_o (\Delta_{\alpha}, x) \bar{C}^{\Delta_1, \Delta_2}_o (\Delta_{\alpha}, \bar{x}) \ V_{\alpha}(0)
\]

where each of the “chain operators”

\[
C^{\Delta_1, \Delta_2}_e (\Delta, x) = 1 + x \frac{\Delta + \Delta_1 - \Delta_2}{2\Delta} \ L_{-1} + O(x^2) \\
C^{\Delta_1, \Delta_2}_o (\Delta, x) = \frac{x^{1/2}}{2\Delta} \ G_{-1/2} + O(x^{3/2})
\]

(and the same for \(\bar{C}_e\) and \(\bar{C}_o\) with the “right” \(SVir\) operators \(G_k\) and \(L_n\) replaced by the “left” ones \(\bar{G}_k\) and \(\bar{L}_n\)) is determined uniquely by superconformal symmetry once the normalization of the first term is fixed, e.g., as in \((1.10)\), essentially in the same way as it occurs in the usual CFT case.

The basic NS structure constants \(C^{a_1/2-iP}_{a_1} a_2\) and \(\bar{C}^{a_1/2-iP}_{a_1} a_2\) in \((1.8)\) have been evaluated through the bootstrap technique quite a while ago in refs. \([6,12]\). Here we quote their result in terms of the three-point functions

\[
\langle V_{a_1}(x_1) V_{a_2}(x_2) V_{a_3}(x_3) \rangle = C_{a_1,a_2,a_3} \left( x_{12} x_{12} x_{23} x_{23} x_{31} x_{31} x_{31} x_{31} x_{31} \right)^{\Delta_3+1-2} (1.11) \\
\langle W_{a_1}(x_1) W_{a_2}(x_2) W_{a_3}(x_3) \rangle = (1/2 - \Delta_1 - \Delta_2 - \Delta_3)^2 \bar{C}_{a_1,a_2,a_3} \left( x_{12} x_{12} x_{23} x_{23} x_{31} x_{31} x_{31} x_{31} x_{31} \right)^{\Delta_3+1-2+1/2} (1.11)
\]

Here and henceforth we denote \(x_{ij} = x_i - x_j\) and also use the abbreviations like \(\Delta_{1+2-3} = \Delta_1 + \Delta_2 - \Delta_3\) etc. All the other three point functions of different supermultiplet components are expressed through these via the superprojective invariance \(SL(2|1) \otimes SL(2|1)\). For example

\[
\langle W_{a_1}(x_1) W_{a_2}(x_2) V_{a_3}(x_3) \rangle = (\Delta_1 + \Delta_2 - \Delta_3)^2 C_{a_1,a_2,a_3} \left( x_{12} x_{12} x_{23} x_{23} x_{31} x_{31} x_{31} x_{31} x_{31} \right)^{\Delta_3+1-2} (1.12) \\
\langle W_{a_1}(x_1) V_{a_2}(x_2) V_{a_3}(x_3) \rangle = \bar{C}_{a_1,a_2,a_3} \left( x_{12} x_{12} x_{23} x_{23} x_{31} x_{31} x_{31} x_{31} x_{31} \right)^{\Delta_3+1-2+1/2}
\]

\(^3\)This means even in the left and even in the right sector. Terms “odd-odd”, “even-odd” etc. have similar sense.
The superconformal symmetry allows to express all three point functions with different components (1.6) of the supermultiplets through the two constants $C_{a_1a_2a_3}$ and $\tilde{C}_{a_1a_2a_3}$. Both constants $C_{a_1a_2a_3}$ and $\tilde{C}_{a_1a_2a_3}$ are symmetric in all their arguments and related to the structure constants as (normalizations chosen in (1.10) are important in these relations)

$$C_{a_1a_2a_3} = C_{a_1a_2a_3} ; \quad \tilde{C}_{a_1a_2a_3} = -\tilde{C}_{a_1a_2a_3}$$

(1.13)

Following [6, 12] they have the following explicit form (a stands here for $a_1 + a_2 + a_3$)

$$C_{a_1a_2a_3} = \left( \pi \mu \gamma \left( \frac{Qb}{2} \right) b^{1-b^2} \right)^{(Q-a)/b} \frac{\Upsilon'_{NS}(0) \Upsilon_{NS}(2a_1) \Upsilon_{NS}(2a_2) \Upsilon_{NS}(2a_3)}{\Upsilon_{NS}(a-Q) \Upsilon_{NS}(a_1+3-1) \Upsilon_{NS}(a_2+3-1) \Upsilon_{NS}(a_3+3-1)}$$

(1.14)

$$\tilde{C}_{a_1a_2a_3} = \left( \pi \mu \gamma \left( \frac{Qb}{2} \right) b^{1-b^2} \right)^{(Q-a)/b} \frac{2i \Upsilon'_{NS}(0) \Upsilon_{NS}(2a_1) \Upsilon_{NS}(2a_2) \Upsilon_{NS}(2a_3)}{\Upsilon_{R}(a-Q) \Upsilon_{R}(a_1+3-1) \Upsilon_{R}(a_2+3-1) \Upsilon_{R}(a_3+3-1)}$$

where we make use of convenient notations from ref. [9] for the special functions

$$\Upsilon_{NS}(x) = \Upsilon_b \left( \frac{x}{2} \right) \Upsilon_b \left( \frac{x + Q}{2} \right)$$

$$\Upsilon_{R}(x) = \Upsilon_b \left( \frac{x + b}{2} \right) \Upsilon_b \left( \frac{x + b^{-1}}{2} \right)$$

(1.15)

expressed in terms of the standard in the Liouville field theory “upsilon” function $\Upsilon_b$ (see [13, 10]). For us the following functional relations are important

$$\Upsilon_{NS}(x + b) = b^{-bx} \gamma \left( \frac{1}{2} + \frac{bx}{2} \right) \Upsilon_{R}(x)$$

$$\Upsilon_{R}(x + b) = b^{1-bx} \gamma \left( \frac{bx}{2} \right) \Upsilon_{NS}(x)$$

(1.16)

and the same with b replaced by $b^{-1}$. Finally

$$\Upsilon'_{NS}(0) = \frac{1}{2} \Upsilon_b \left( \frac{b}{2} \right) \Upsilon_b \left( \frac{1}{2b} \right)$$

(1.17)

All these expressions correspond to the “natural” normalization of the exponential fields, where the two-point function reads

$$\langle V_a(x) V_a(0) \rangle = \frac{D_{NS}(a)}{(x, \bar{x})^{2\Delta_a}}$$

(1.18)

4 A derivation based on the superprojective Ward identities, which does not exploit a superfield formalism, is presented in Appendix A.
with
\[ D_{NS}(a) = \left( \pi \mu \gamma \left( \frac{bQ}{2} \right) \right)^{(Q-2a)b^{-1}} \frac{b^2 \gamma \left( ba - \frac{1}{2} - \frac{b^2}{2} \right)}{\gamma \left( \frac{1}{2} + \frac{b^2}{2} - ab^{-1} \right)} \] 

As usual, in the natural normalization the fields \( V_a \) and \( V_{Q-a} \) are identified through the following “reflection relations”
\[ V_a = D_{NS}(a)V_{Q-a} \] 

In refs. [6,12] (see also [9]) expressions (1.14) were derived on the basis of the singular vector decoupling in the singular Ramond representation. This approach has many advantages, being the simplest known and giving simultaneously the OPE structure constants in the whole space of fields, Neveu-Schwarz and Ramond. In this note we find it instructive, however, to rederive (1.14) with the help of pure NS bootstrap, requiring the NS null-vector decoupling. Albeit technically more complicated, this derivation allows to stay completely inside the NS sector of SLFT and provides a good exercise in classical analysis. This program is described in the following section.

Once the structure constants are determined, expression (1.8) gives directly the integral representation for the four point function of four “bottom” NS primary fields
\[ \langle V_{a_1}(x_1)V_{a_2}(x_2)V_{a_3}(x_3)V_{a_4}(x_4) \rangle = \] 
\[ (x_{41} \bar{x}_{41})^{-2\Delta_1} (x_{24} \bar{x}_{24})^{\Delta_1+3-2-4} (x_{34} \bar{x}_{34})^{\Delta_1+2-3-4} (x_{23} \bar{x}_{23})^{\Delta_4-1-2-3} G \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \bigg| x, \bar{x} \right) \]

where
\[ x = \frac{x_{12}x_{34}}{x_{23}x_{41}} \]

and the “reduced” four point function admits the following “s-channel” representation
\[ G \left( \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \end{array} \bigg| x, \bar{x} \right) = \int \frac{dP}{4\pi} C_{a_1 a_2}^{Q/2-iP} C_{a_3 a_4}^{Q/2+iP} F_e \left( \begin{array}{c} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{array} \bigg| |x| \right) F_e \left( \begin{array}{c} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{array} \bigg| |\bar{x}| \right) \]
\[ - \int \frac{dP}{4\pi} \tilde{C}_{a_1 a_2}^{Q/2-iP} \tilde{C}_{a_3 a_4}^{Q/2+iP} F_o \left( \begin{array}{c} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{array} \bigg| |x| \right) F_o \left( \begin{array}{c} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \end{array} \bigg| |\bar{x}| \right) \]

For simplicity we omit possible discrete terms. Superconformal NS blocks \( F_e \) and \( F_o \) effectively sum up respectively “even” and “odd” descendants of the primary NS field of dimension \( \Delta = Q^2/8 + P^2/2 \). Notice the minus sign in front of the second term in (1.23). It is due to the anticommutativity of the “right” and “left” odd chain operators in (1.9). The blocks are constructed unambiguously on the basis of superconformal invariance. The problem of their evaluation has been addressed recently in refs. [14,15] and will be reconsidered below.

\[ ^5 \text{This convention is consistent with the minus sign in the second relation in eq.(1.13). The pure imaginary structure constant } C_{a_1 a_2}^{P} \text{ in (1.14) assures positive sign before the net contribution of the odd levels (see below).} \]
in sections 3, 4 and 5. First we recapitulate the so-called recursive “c-representation” [16], which has been developed in [14,15] and allows to effectively reconstruct the blocks iteratively as a series in the parameter $x$. Better convergent is the so-called “delta”, or “elliptic” representation, whose recursions give a much faster convergent series in the elliptic parameter $q$. Such representation has been constructed in [17] for the ordinary conformal algebra. Presently in section 5 we suggest a generalization of the elliptic representation for the $N = 1$ superconformal case. In this paper we restrict ourselves only to the blocks with all four external fields “bottom” supermultiplet components (and those related to this case by the superprojective symmetry).

An important property of a consistent euclidean quantum field theory is the associativity of the algebra of operator product expansions. This property ensures the correlation functions to be single-valued over the euclidean slice of the complex space-time. It is commonly believed to give an euclidean interpretation of the standard Minkowskian locality. In particular, the four-point function is single-valued (or, enjoys the crossing symmetry) if the following two identities hold

$$G \left( a_1 a_3 \bigg| x, \bar{x} \right) = \left( 1 - x \right) \left( 1 - \bar{x} \right)^{-2\Delta_1} G \left( a_1 a_4 \bigg| \frac{x}{x - 1}, \frac{\bar{x}}{\bar{x} - 1} \right)$$

$$G \left( a_1 a_3 \bigg| x, \bar{x} \right) = G \left( a_4 a_2 \bigg| a_3 a_4 \bigg| 1 - x, 1 - \bar{x} \right)$$

In the form (1.23) the first relation is a trivial consequence of the transformation properties of the blocks. The second, however, becomes a non-trivial relation for the structure constants. Its closed formulation in terms of the structure constants requires explicit knowledge of the fusion (or crossing) matrix for a general superconformal block, which is not currently available (see [18] for an explicit construction in the bosonic case). Under these circumstances, we find it of value to verify this relation numerically, taking the explicit form of the structure constants and using the fast convergent elliptic representation for the superconformal blocks. This we perform in section 6 and find a reasonable numerical support for the relations (1.24) for certain randomly chosen values of the parameters.

Section 7 is devoted to the discussion and outlook.

2. Singular vector bootstrap in NS sector

At certain values of the parameter $a$ the SLFT operator $V_a$ is a highest vector of a singular representation of $SVir$. In the NS sector, which we only consider in this paper, the simplest singular representation has dimension $\Delta_{1,3} = -1/2 - b^2$ with the primary field $V_{-b}$ as the highest weight vector. The singular vector appearing at the level $3/2$ reads

$$(G_{-1/2}^3 + b^2 G_{-3/2}) V_{-b}$$

There is also a singular vector of the same form in the left $SVir$ sector. Vanishing of all singular vectors in the singular representations can be taken as the basic dynamical principle
of SLFT, precisely like in the ordinary bosonic LFT. In particular, setting $\text{SLFT}$ and its “left” counterpart to zero leads to non-trivial equations for the correlation functions. It is easy to see, e.g., considering the three point function with the field $V_{-b}$, that such “decoupling equation” restricts the form of the operator product expansion of a product of this degenerate field and arbitrary primary $V_a$ to the following “discrete” form

$$V_{-b}V_a = (x\bar{x})^{ab}C_-(a) [V_{-b}]_{ee} + (x\bar{x})^{1/2} \tilde{C}_0(a) [V_a]_{oo} + C_+(a)(x\bar{x})^{1-\bar{b}a+b^2} [V_{a+b}]_{ee}$$

where $C_-(a)$, $\tilde{C}_0(a)$ and $C_+(a)$ are “special” (unlike the “generic” ones in the OPE (1.8)), or “discrete” structure constants. It is instructive to understand how the general OPE (1.8) with the structure constants (1.14) results in the discrete expansion (2.2) if one of the parameters $a_1$ or $a_2$ is set to the singular value $-b$. This calculation is performed in the Appendix B.

Below in this section we will use the bootstrap conditions together with the “decoupling principle” to derive unambiguously the generic NS structure constants (1.14). In particular, the values of the special structure constants in (2.2) are recovered uniquely up to an overall scale. It is instructive, however, to give a “perturbative” derivation, similar to that presented e.g., in [19], because firstly it is simpler and more transparent, and secondly it gives a heuristic link with the SLFT Lagrangian (1.1), in particular relates the scale parameter to the cosmological constant $\mu$. The idea is simply a version of the “screening” calculus, invented by B. Feigin and D. Fuchs (and further developed by V. Fateev and V. Dotsenko [20]) where the role of the screening operator is played by the interaction term $2i\mu W_b$ in (1.1). E.g., neglecting formally this term, one considers $\phi, \psi$ as free fields, so that the exponentials $V_{-b}$ and $V_a$ enjoy the free field fusion to $V_{a-b}$ with

$$C_-(a) = 1$$

The next term with $[V_a]_{oo} = (2\Delta_a)^{-2}(x\bar{x})^{1/2}W_a + \ldots$ requires one insertion of the perturbation $2i\mu b^2 \int \bar{\psi}\psi e^{b\phi} d^2x.$ Thus

$$\tilde{C}_0(a) = -2i\mu b^2(Q - a)^2 \int \langle \bar{\psi}\psi e^{b\phi} (y)V_a(0)V_{-b}(1)\bar{\psi}\psi e^{(Q-a)\phi}(\infty) \rangle$$

$$= 2\pi i\mu \gamma(ab - b^2)$$

Finally, we need to make two perturbative insertions in order to create the last term $[V_{a+b}]_{ee} = V_{a+b} + \ldots$ in the OPE (2.2). This results in

$$C_+(a) = \frac{(-2i\mu b^2)^2}{2} \int \langle \bar{\psi}\psi e^{b\phi}(y_1)\bar{\psi}\psi e^{b\phi}(y_2)V_a(0)V_{-b}(1)V_{Q-a-b}(\infty) \rangle$$

$$= 2\mu b^4 \int [(y_1 - y_2)(\bar{y}_1 - \bar{y}_2)]^{-b^2-1} \prod_{i=1}^{2} (y_i\bar{y}_i)^{-ab} [(1 - y_i)(1 - \bar{y}_i)]^{b^2} d^2 y_i$$

$$= \pi^2 \mu b^4 \gamma^2 \left( \frac{1}{2} + \frac{b^2}{2} \right) \gamma \left( -\frac{1}{2} - \frac{b^2}{2} + ab \right) \gamma \left( \frac{1}{2} - \frac{b^2}{2} - ab \right)$$

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In the last calculation we have used the following integration formula

\[
\frac{1}{n!} \int \prod_{i=1}^{n} (z_i \bar{z}_i)^{n-1} [1 - z_i] [1 - \bar{z}_i]^{n-1} d^2 z_i \prod_{i>j} [(z_i - z_j)(\bar{z}_i - \bar{z}_j)]^{2g}
\]

where \( \mu + \nu + \lambda = 1 - 2(n - 1)g \). This formula belongs to Dotsenko and Fateev [21].

It is a simple exercise in operator product expansion (see e.g. [19]) to show that the two-point function (1.18) satisfies the functional relation

\[
\frac{D_{NS}(a)}{D_{NS}(a + b)} = C_+(a)
\]

This is indeed the case for (1.19) together with (2.5). The “dual” functional relation

\[
\frac{D_{NS}(a)}{D_{NS}(a + b^{-1})} = \pi^2 \tilde{\mu}^2 b^4 \gamma^2 \left( \frac{1}{2} + \frac{b^2}{2} \right) \gamma \left( \frac{-1}{2} - \frac{b^2}{2} + ab^{-1} \right) \gamma \left( \frac{1}{2} - \frac{b^2}{2} - ab^{-1} \right)
\]

is also satisfied if the “dual cosmological constant” \( \tilde{\mu} \) is related to \( \mu \) as

\[
\pi \tilde{\mu} \gamma \left( \frac{1}{2} + \frac{b^2}{2} \right) = \left( \pi \mu \gamma \left( \frac{1}{2} + \frac{b^2}{2} \right) \right)^{b^2}
\]

Now, let us turn to the four point function with one singular primary field \( V_{-b} \) and three arbitrary ones. It is natural to renumber the operators, setting in (1.21) \( V_{a_1} \) to be \( V_{-b} \) and denoting \( V_{a_2}, V_{a_3} \) and \( V_{a_4} \) as \( V_1, V_2 \) and \( V_3 \), their dimensions being \( \Delta_{a_1}, \Delta_2 \) and \( \Delta_3 \) respectively. Thus, in the notations of (1.21)

\[
g(x, \bar{x}) = G\left( \frac{-b}{a_1} \left| \begin{array}{c} a_2 \\ a_3 \end{array} \right| \ x, \bar{x} \right)
\]

It has been shown in [14] that this function satisfies the following third order linear differential equation

\[
\frac{1}{b^2} g''' + \frac{1 - 2b^2}{b^2} \frac{1 - 2x}{x(1-x)} g'' + \left( \frac{b^2 + 2\Delta_1}{(1-x)^2} + \frac{b^2 + 2\Delta_2}{x(1-x)} + \frac{2 - 3b^2 + 2\Delta_{1+2-3}}{x(1-x)} \right) g' + \left( \frac{2\Delta_2(1+b^2)}{(1-x)^3} - \frac{2\Delta_1(1+b^2)}{x^3} + \frac{\Delta_{2-1} + (1 - 2x)(b^4 + b^2(1/2 - \Delta_{1+2-3} - \Delta_{1+2}))}{x^2(1-x)^2} \right) g = 0
\]

The same equation holds with respect to \( \bar{x} \). Near \( x \to 0 \) it has three exponents \( ba_1, 1 + b^2 \) and \( 1 - ba_1 + b^2 \), which correspond, respectively, to the dimensions \( \Delta_{a_1-b}, \Delta_{a_1} + 1/2 \) and
\( \Delta_{a_1+b} \). They give rise to three s-channel blocks

\[
\mathcal{F}_e^{(-)}(x) = x^{a_1b} (1 + \ldots) \\
\mathcal{F}_o^{(0)}(x) = x^{1+b^2} \left( \frac{1}{2\Delta_{a_1}} + \ldots \right) \\
\mathcal{F}_e^{(+))(x) = x^{1-ba_1+b^2} (1 + \ldots) \tag{2.12}
\]

where in the brackets stand regular series in \( x \) and the normalization correspond to the general convention of (1.10). The correlation function is combined as

\[
g(x, \bar{x}) = -\tilde{\mathcal{C}}_0(a_1)\tilde{\mathcal{C}}_{a_1,a_2,a_3}\mathcal{F}_o^{(0)}(x)\mathcal{F}_o^{(0)}(\bar{x}) + C_-(a_1)\mathcal{F}_e^{(-)}(x)\mathcal{F}_e^{(-)}(\bar{x}) + C_+(a_1)\mathcal{F}_e^{(+))(x)\mathcal{F}_e^{(+))(\bar{x}) \tag{2.13}
\]

It turns out that (2.11) is of the type considered by Dotsenko and Fateev in [20] and can be solved in terms of two-fold contour integrals. Explicitly

\[
\mathcal{F}_e^{(-)}(x) = \frac{x^{a_1b}(1-x)^{a_2b} \Gamma \left( \frac{1}{2} - \frac{b^2}{2} \right) \Gamma \left( \frac{1}{2} - \frac{b^2}{2} + ba_1 \right) \Gamma(ba_1 - b^2) F_-(A, B, C, g; x)}{\Gamma(-1 - b^2) \Gamma \left( \frac{1}{2} + \frac{ba_{1+2-3}}{2} \right) \Gamma \left( \frac{ba_{1+2-3}}{2} - \frac{b^2}{2} \right) \Gamma \left( \frac{ba_{1+2-3}}{2} - \frac{b^2}{2} \right) } \\
\mathcal{F}_o^{(0)}(x) = \frac{x^{1+b^2} (1-x)^{a_2b} \Gamma (-ba_1 + b^2) \Gamma(ba_1 - b^2) F_0(A, B, C, g; x)}{\Gamma \left( \frac{1}{2} + \frac{ba_{2+3-1}}{2} \right) \Gamma \left( \frac{ba_{2+3-1}}{2} - \frac{b^2}{2} \right) \Gamma \left( \frac{ba_{2+3-1}}{2} - \frac{b^2}{2} \right) \Gamma \left( \frac{b^2}{2} - \frac{ba_{1+2+3}}{2} + b^2 \right) } \tag{2.14}
\]

and

\[
\mathcal{F}_e^{(+))(x) = \frac{x^{1-ba_1+b^2} (1-x)^{a_2b} \Gamma \left( \frac{1}{2} - \frac{b^2}{2} \right) \Gamma \left( \frac{3}{2} - \frac{ba_1 + b^2}{2} \right) \Gamma \left( \frac{3}{2} - \frac{ba_1 + b^2}{2} \right) \Gamma \left( 1 - \frac{ba_{1+2+3}}{2} + \frac{b^2}{2} \right) }{\Gamma(-1 - b^2) \Gamma \left( \frac{1}{2} + \frac{ba_{2+3-1}}{2} \right) \Gamma \left( \frac{ba_{2+3-1}}{2} - \frac{b^2}{2} \right) \Gamma \left( \frac{ba_{2+3-1}}{2} - \frac{b^2}{2} \right) \Gamma \left( 1 - \frac{ba_{1+2+3}}{2} + \frac{b^2}{2} \right) } \tag{2.15}
\]

Functions \( F_-(A, B, C, g; x) \) and \( F_0(A, B, C, g; x) \) are regular series in \( x \) and admit the following integral representations

\[
F_-(x) = \int_0^1 dt_1 \int_0^{t_1} dt_2 (t_1 - t_2)^{2g} (t_1t_2)^{-A-B-C-2} [(1 - t_1)(1 - t_2)]^B [(1 - xt_1)(1 - xt_2)]^C \tag{2.16}
\]

\[
F_+(x) = \int_0^1 dt_1 \int_0^{t_1} dt_2 (t_1 - t_2)^{2g} (t_1t_2)^A [(1 - t_1)(1 - t_2)]^C [(1 - xt_1)(1 - xt_2)]^B \tag{2.16}
\]

\[
F_0(x) = \int_0^{t_1} 2^{-A-B-C-2g} (1 - t_1)^B (1 - xt_1)^C dt_1 \int_0^{t_1} dt_2 t_2^A [1 - xt_2]^B [(1 - t_2)^C (1 - xt_1t_2)^{2g}
\]
Parameters $A, B, C$ and $g$ are related to the super Liouville ones $a_1, a_2, a_3$ and $b$ as

$$A = -\frac{1}{2} + \frac{ba_{a_2+3-1}}{2}; \quad B = -\frac{1}{2} + \frac{ba_{a_2+3}}{2}$$

$$C = \frac{1}{2} - \frac{ba_{a_2+3}}{2} + b^2; \quad g = -\frac{1}{2} - \frac{b^2}{2}$$

(2.17)

For the sake of completeness in Appendix C we recapitulate from [20] the third order differential equation of Dotsenko-Fateev type, relevant integral representations of the solutions, as well as their monodromy properties.

In particular, the combination (2.13) is a single-valued function of the two-dimensional coordinate $(x, \bar{x})$ if

$$\frac{C_+(a_1)C_{a_1+b,a_2,a_3}}{C_-^{(a_1)}C_{a_1-b,a_2,a_3}} = -\gamma (ba_1)(ba_1-b^2)\gamma^2\left(\frac{1}{2}-\frac{b^2}{2}+ba_1\right) \times$$

$$\gamma\left(\frac{1}{2}+\frac{ba_{a_2+3-1}}{2}\right)\gamma\left(\frac{ba_{a_2+3-1}}{2} - \frac{b^2}{2}\right)\gamma\left(\frac{3}{2} - \frac{ba_{a_2+3}}{2} + b^2\right) \gamma\left(1 - \frac{ba_{a_2+3}}{2} + \frac{b^2}{2}\right)$$

$$\gamma\left(\frac{1}{2} + \frac{ba_{a_2+3} - \frac{b^2}{2}}{2}\right)\gamma\left(\frac{3}{2} - \frac{ba_{a_2+3}}{2} + b^2\right) \gamma\left(\frac{ba_{a_2+3}}{2} - \frac{b^2}{2}\right)$$

(2.18)

and

$$\frac{\tilde{C}_0(a_1)\tilde{C}_{a_1+a_2,a_3}}{C_-^{(a_1)}C_{a_1-b,a_2,a_3}} = -\gamma \left(\frac{1}{2} - \frac{b^2}{2}\right)\gamma \left(\frac{1}{2} - \frac{b^2}{2} + ba_1\right) \gamma^2(ba_1 - b^2) \times$$

$$\gamma\left(\frac{1}{2} + \frac{ba_{a_2+3-1}}{2}\right)\gamma\left(\frac{3}{2} - \frac{ba_{a_2+3}}{2} + b^2\right)$$

$$\gamma\left(\frac{ba_{a_2+3} - \frac{b^2}{2}}{2}\right)\gamma\left(\frac{ba_{a_2+3}}{2} - \frac{b^2}{2}\right)$$

(2.19)

These formulas, together with the explicit expressions (2.3), (2.4) and (2.5) for the special structure constants, result in the following functional relations for the three-point functions $C_{a_1,a_2,a_3}$ and $\tilde{C}_{a_1,a_2,a_3}$

$$\frac{C_{a_1+b,a_2,a_3}}{C_{a_1-b,a_2,a_3}} = \gamma\left(\frac{1}{2} + \frac{b^2}{2} + a_1 b\right)\gamma\left(\frac{1}{2} - \frac{b^2}{2} + a_1 b\right)$$

$$\pi^2 b^4 \gamma^2\left(\frac{1}{2} + \frac{b^2}{2}\right)\gamma\left(\frac{1}{2} - \frac{b^2}{2} + a_1 b\right) \gamma\left(\frac{1}{2} - \frac{b^2}{2} + a_1 b\right) \times$$

$$\gamma\left(1 - \frac{1}{2}ba_{a_2+3} + \frac{1}{2}b^2\right)\gamma\left(\frac{1}{2} - \frac{ba_{a_2+3}}{2} + b^2\right) \gamma\left(\frac{1}{2} + \frac{1}{2}ba_{a_2+3} - \frac{1}{2}b^2\right) \gamma\left(\frac{1}{2} + \frac{1}{2}ba_{a_2+3} - \frac{1}{2}b^2\right)$$

(2.20)

and

$$\frac{\tilde{C}_{a_1,a_2,a_3}}{\tilde{C}_{a_1-b,a_2,a_3}} =$$

$$2i\gamma\left(\frac{1}{2} - \frac{b^2}{2} + ba_1\right)\gamma(ba_1 - b^2)\gamma\left(\frac{1}{2} + \frac{ba_{a_2+3-1}}{2}\right)\gamma\left(\frac{3}{2} - \frac{ba_{a_2+3}}{2} + b^2\right) \times$$

$$\pi b^4 \gamma\left(\frac{1}{2} + \frac{b^2}{2}\right)\gamma\left(\frac{ba_{a_2+3}}{2} - \frac{1}{2}b^2\right) \gamma\left(\frac{ba_{a_2+3}}{2} - \frac{1}{2}b^2\right)$$

(2.21)
It is verified directly through the shift relations (1.16) that the structure constants (1.14) satisfy these functional relations. A standard consideration [22] with the “dual” functional relation (i.e., the relation with $b \to b^{-1}$ and $\mu \to \bar{\mu}$) now applies and allows to argue that, at least at real irrational values of $b^2$, expressions (1.14) give the unique solution to the singular vector decoupling equations.

3. Analytic structure of NS block

Here we analyze in more details the properties of the superconformal blocks $F_e$ and $F_o$ which enter the integral representation (1.23) of the four-point function. The “chain operators” introduced in (1.9) are mostly important in this analysis. The “right-left” factorization allows to concentrate on the holomorphic part only and then combine it with the (mostly identical) antiholomorphic one. As it has been found in [15] and [14], apart from the OPE (1.8) one has to consider the similar OPE $\Lambda^a_1(x)V^{\Delta}(0)$. Commutation relations

\[
\{G_k, \Lambda_a(x)\} = x^{k+1/2} \left( \frac{2\Delta_a(k+1/2)}{x}V_a(x) + \frac{\partial}{\partial x}V_a(x) \right)
\]

follow directly from (1.7). Acting by any generator $G_k$ with $k > 0$ on the left hand side of (1.8) one finds

\[
\Lambda^a_1(x)V^{\Delta}(0) = x^{-1/2} \int \frac{dP}{4\pi} (x\bar{x})^{\Delta - \Delta_1 - \Delta_2} \left( C^{Q/2+ip}_{a_1,a_2} \tilde{\mathcal{C}}^e(\Delta_a, x) \overline{\mathcal{C}}^o(\Delta_a, \bar{x})V^{Q/2+ip}(0) + \tilde{\mathcal{C}}^{Q/2+ip}_{a_1,a_2} \tilde{\mathcal{C}}^e(\Delta_a, x) \overline{\mathcal{C}}^o(\Delta_a, \bar{x})V^{Q/2+ip}(0) \right)
\]

where $\tilde{\mathcal{C}}^e(\Delta_a, x)$ and $\tilde{\mathcal{C}}^o(\Delta_a, x)$ are new chain operators. It is convenient to unify the even and odd ones as the joint series in integer and half-integer powers of $x$

\[
\mathcal{C}(\Delta, x) = C^e(\Delta, x) + C^o(\Delta, x)
\]

\[
\tilde{\mathcal{C}}(\Delta, x) = \tilde{C}^e(\Delta, x) + \tilde{C}^o(\Delta, x)
\]

From (3.1) one finds

\[
\tilde{\mathcal{C}}(\Delta, x)V_{\Delta} = x^k G_k \mathcal{C}(\Delta, x)V_{\Delta}
\]

\[
\mathcal{C}(\Delta, x)V_{\Delta} = x^k \left( \Delta + 2\Delta_1 k - \Delta_2 - k + x \frac{\partial}{\partial x} \right) G_k \tilde{\mathcal{C}}(\Delta, x)V_{\Delta}
\]

For the coefficients in the level expansion

\[
\mathcal{C}(\Delta, x) = \sum_N x^N C_N(\Delta) ; \quad \tilde{\mathcal{C}}(\Delta, x) = \sum_N x^N \tilde{C}_N(\Delta)
\]

\[\text{These operators depend also on the dimensions } \Delta_1 = \Delta_{a_1} \text{ and } \Delta_2 = \Delta_{a_2}. \text{ For not to overload the notations, this dependence is not indicated explicitly.}\]
where \( N \) runs over non-negative integer and half-integer numbers (levels), these relations read \((k \leq N)\)

\[
G_k \mathcal{C}_N(\Delta)V_\Delta = \tilde{\mathcal{C}}_{N-k}(\Delta)V_\Delta \\
G_k \tilde{\mathcal{C}}_N(\Delta)V_\Delta = (\Delta + 2\Delta_1 k - \Delta_2 + N - k) \mathcal{C}_{N-k}(\Delta)V_\Delta
\]

This system turns out to be enough to reconstruct the chain operators up to two overall constants. The latter can be fixed by the conditions \( C_0 = 1 \) and \( \tilde{C}_0 = 1 \). Explicitly one finds, level by level

\[
\mathcal{C}_{1/2}(\Delta) = \frac{1}{2\Delta}G_{-1/2} \\
\mathcal{C}_1(\Delta) = \frac{\Delta + \Delta_1 - \Delta_2}{2\Delta}L_{-1} \\
\mathcal{C}_{3/2}(\Delta) = \frac{\Delta + \Delta_1 - \Delta_2 + 1/2}{2\Delta(2\Delta + 1)}G_{-1/2}^3 + \frac{2(\Delta_1 - \Delta_2)}{2\Delta^2 + 2\tilde{c}\Delta - 6\Delta + \tilde{c}}O_{-3/2} \\
\mathcal{C}_2(\Delta) = \left(\frac{\Delta + \Delta_1 - \Delta_2}{2\Delta} + \frac{1}{2}\right)(\Delta + \Delta_1 - \Delta_2 + 1)G_{-1/2}^3 \\
\quad + \frac{2(\Delta_1 - \Delta_2)^2 + \Delta - (2\Delta + 1)(\Delta_1 + \Delta_2)}{(2\Delta + 3)(4\Delta^2 + 2\tilde{c}\Delta - 6\Delta + \tilde{c})}G_{-1/2}O_{-3/2} \\
\quad + \frac{3(\Delta_1 - \Delta_2)^2 - 2\Delta(\Delta_1 + \Delta_2) - \Delta^2}{2\Delta(2\Delta + 3)(16\Delta + 3\tilde{c} - 3)}O_{-2} (3.7)
\]

and

\[
\tilde{\mathcal{C}}_{1/2}(\Delta) = \frac{\Delta + \Delta_1 - \Delta_2}{2\Delta}G_{-1/2} \\
\tilde{\mathcal{C}}_1(\Delta) = \frac{\Delta + \Delta_1 - \Delta_2 + 1/2}{2\Delta}L_{-1} \\
\tilde{\mathcal{C}}_{3/2}(\Delta) = \frac{\Delta + \Delta_1 - \Delta_2 + 1/2}{2\Delta(2\Delta + 1)}G_{-1/2}^3 \\
\quad + \frac{2(\Delta_1 - \Delta_2)^2 + \Delta - (2\Delta + 1)(\Delta_1 + \Delta_2)}{4\Delta^2 + 2\tilde{c}\Delta - 6\Delta + \tilde{c}}O_{-3/2} \\
\tilde{\mathcal{C}}_2(\Delta) = \left(\frac{\Delta + \Delta_1 - \Delta_2}{2\Delta} + \frac{1}{2}\right)(\Delta + \Delta_1 - \Delta_2 + 3/2)G_{-1/2}^3 \\
\quad + \frac{2(\Delta_1 - \Delta_2)(\Delta + \Delta_2 - 3/2)}{(2\Delta + 3)(4\Delta^2 + 2\tilde{c}\Delta - 6\Delta + \tilde{c})}G_{-1/2}O_{-3/2} \\
\quad + \frac{12(\Delta_1 - \Delta_2)^2 - (2\Delta + 3)(2\Delta - 1 + 4(\Delta_1 + \Delta_2))}{8\Delta(2\Delta + 3)(16\Delta + 3\tilde{c} - 3)}O_{-2} (3.8)
\]

where we have used the abbreviations

\[
O_{-3/2} = \frac{2}{2\Delta + 1}L_{-1}G_{-1/2} - G_{-3/2} (3.9) \\
O_{-2} = 3L_{-1}^2 - 4\Delta L_{-2} - 3G_{-3/2}G_{-1/2}
\]
Further coefficients are systematically evaluated level by level as a solution of (3.4). At each level (3.6) is a finite dimensional linear problem, the coefficients being polynomials in \( \hat{c}, \Delta \) and the “external dimensions” \( \Delta_1 \) and \( \Delta_2 \). The determinant of the system is the Kac determinant [23] of the corresponding level in the representation of the superconformal algebra. Therefore, the singularities of \( \mathcal{C}(\Delta) \) are (simple in general) poles located at the singular dimensions \( \Delta = \Delta_{m,n} \), where \( (m, n) \) is a pair of positive integers,

\[
\Delta_{m,n} = \frac{Q^2}{8} - \frac{\lambda_{m,n}^2}{2} \tag{3.10}
\]

and we introduced a notation

\[
\lambda_{m,n} = \frac{mb^{-1} + nb}{2} \tag{3.11}
\]

In the NS sector, which we only consider here, \( m \) and \( n \) are either both even or both odd. At this values in the \( SVir \) module over \( V_\Delta \) appears a singular vector \( D_{m,n}V_{m,n} \), a primary field of dimension \( \Delta_{m,n} + mn/2 = \Delta_{m,n} \). Here, as in ref. [24], we denoted \( V_{m,n} = V_{\Delta_{m,n}} \) and introduced a set of “singular vector creation” operators \( D_{m,n} \), which are graded polynomials in the generators \( G_{-k} \) and \( L_{-n} \) (with the coefficients depending in \( \hat{c} \) (or \( b \) only) such that \( G_kD_{m,n}V_{m,n} = 0 \) for every half integer \( k > 0 \). These operators are supposed to be normalized as in [24]

\[
D_{m,n} = G_{mn}^{1/2} + \ldots \tag{3.12}
\]

Below we will need also the “conjugate” operator \( D_{m,n}^\dagger \) defined, as in ref. [24] through the following conjugation rules \( L_n^\dagger = L_{-n} \) and \( G_k^\dagger = G_{-k} \). Apparently

\[
D_{m,n}^\dagger D_{m,n}V_\Delta(0) = r_{m,n}'(\Delta - \Delta_{m,n})V_\Delta(0) + O((\Delta - \Delta_{m,n})^2) \tag{3.13}
\]

The coefficient \( r_{m,n}' \) (the “logarithmic norm” of the singular vector) has been explicitly evaluated in [24]

\[
r_{m,n}' = 2^{mn-1} \prod_{k=1-m, \ell=1-n}^{m,n,k+\ell \in \mathbb{Z}} \lambda_{k,l} \tag{3.14}
\]

The singular vector \( D_{m,n}V_{m,n} \), once appeared in the chain vectors (3.3), gives rise to its own chains \( \mathcal{C}(\Delta_{m,-n}, x)D_{m,n}V_{m,n} \) and \( \mathcal{C}(\Delta_{m,-n}, x)D_{m,n}V_{m,n} \) which by themselves satisfy the chain equations (3.6) with \( \Delta = \Delta_{m,n} \). Apparently

\[
\text{res}_{\Delta = \Delta_{m,n}} \mathcal{C}(\Delta, x) = x^{mn/2}X_{m,n}\mathcal{C}(\Delta_{m,-n}, x)D_{m,n} \tag{3.15}
\]

where \( X_{m,n} \) are certain coefficients. Since \( \mathcal{C}_e(\Delta, x) \) with \( \tilde{\mathcal{C}}_o(\Delta, x) \) and \( \mathcal{C}_o(\Delta, x) \) with \( \tilde{\mathcal{C}}_e(\Delta, x) \) form two independent systems, we need to treat separately the integer and half-integer
chains. Denote

\[
\text{res}_{\Delta=\Delta_{m,n}} C_{e}(\Delta, x) = x^{mn/2} \left\{ \begin{array}{ll}
X^{(e)}_{m,n} C_{e}(\Delta_{m,-n}, x) D_{m,n} & m, n \text{ even} \\
X^{(e)}_{m,n} C_{o}(\Delta_{m,-n}, x) D_{m,n} & m, n \text{ odd} 
\end{array} \right.
\]

(3.16)

\[
\text{res}_{\Delta=\Delta_{m,n}} C_{o}(\Delta, x) = x^{mn/2} \left\{ \begin{array}{ll}
X^{(o)}_{m,n} C_{e}(\Delta_{m,-n}, x) D_{m,n} & m, n \text{ even} \\
X^{(o)}_{m,n} C_{o}(\Delta_{m,-n}, x) D_{m,n} & m, n \text{ odd} 
\end{array} \right.
\]

and let \( C(\Delta, x) \) with \( \tilde{C}(\Delta, x) \) be normalized as above. The coefficients \( X^{(e)}_{m,n} \) and \( X^{(o)}_{m,n} \) are then uniquely defined. By construction they are polynomials in the external dimensions \( \Delta_{1} \) and \( \Delta_{2} \). To describe these coefficients it is convenient to define the following “fusion polynomials” \[14, 15\]

\[
P^{(e)}_{m,n}(x) = \prod_{k \in \{1-m,2,m-1\}, l \in \{1-n,2,n-1\}} (x - \lambda_{k,l}) \quad (x - \lambda_{k,l}) \mod 4 = 0
\]

(3.17)

\[
P^{(o)}_{m,n}(x) = \prod_{k \in \{1-m,2,m-1\}, l \in \{1-n,2,n-1\}} (x - \lambda_{k,l}) \quad (x - \lambda_{k,l}) \mod 4 = 2
\]

Here e.g. \( \{1-m, 2, m-1\} \) means “from \( 1-m \) to \( m-1 \) with step 2”, i.e., \( 1-m, 3-m, \ldots, m-1 \).

The degree of these polynomials

\[
p_{e,o}(m, n) = \text{deg} P^{(e,o)}_{m,n}(x)
\]

(3.18)

coincides with the number of multipliers in the products (3.17)

\[
p_{e,o}(m, n) = mn/2 \quad \text{at } m, n \text{ even}
\]

\[
p_{e}(m, n) = mn/2 - 1/2 \quad \text{at } m, n \text{ odd}
\]

\[
p_{o}(m, n) = mn/2 + 1/2
\]

(3.19)

In particular, the parity of \( P^{(e,o)}_{m,n}(x) \) is that of the integer \( p_{e,o}(m, n) \). In the current context it turns out convenient to parameterize the external dimensions in terms of new variables \( \lambda_{i} \) as

\[
\Delta_{i} = \frac{Q^2}{8} - \frac{\lambda_{i}^2}{2}
\]

(3.20)

We will need some more notations. Consider a three point function of formal chiral fields, like \( \langle V_{m,n}(\infty)D_{m,n}^{\dagger}V_{\Delta_{1}(1)}V_{\Delta_{2}(0)} \rangle \) and \( \langle V_{m,n}(\infty)D_{m,n}^{\dagger}V_{\Delta_{1}(1)}V_{\Delta_{2}(0)} \rangle \). Application of the Ward identities, which read in this context simply as

\[
\{G_{k}, V_{\Delta_{1}(1)}\} = \Lambda_{\Delta_{1}(1)}
\]

(3.21)

\[
\{G_{k}, \Lambda_{\Delta_{1}(1)}\} = (2k\Delta_{1} - \Delta_{2} + \Delta_{m,n} + N)V_{\Delta_{1}(1)}
\]
\( N \) is some integer or half integer, corresponding to the level of the descendent of \( V_{m,n}(\infty) \) is a purely algebraic procedure and leads to the relations

\[
\langle V_{m,n}(\infty) D_{m,n}^\dagger V(1) V_{\Delta}(0) \rangle = Y_{m,n}(\Delta_1, \Delta_2) \left\{ \begin{array}{ll}
\langle V_{m,n}(\infty) V_{\Delta}(1) V_{\Delta}(0) \rangle & \text{at } m, n \text{ even} \\
\langle V_{m,n}(\infty) A_{\Delta}(1) V_{\Delta}(0) \rangle & \text{at } m, n \text{ odd} 
\end{array} \right.
\] (3.22)

\[
\langle V_{m,n}(\infty) D_{m,n}^\dagger A_{\Delta}(1) V_{\Delta}(0) \rangle = Y_{m,n}(\Delta_1, \Delta_2) \left\{ \begin{array}{ll}
\langle V_{m,n}(\infty) A_{\Delta}(1) V_{\Delta}(0) \rangle & \text{at } m, n \text{ even} \\
\langle V_{m,n}(\infty) V_{\Delta}(1) V_{\Delta}(0) \rangle & \text{at } m, n \text{ odd} 
\end{array} \right.
\]

Apparantly the \( Y_{m,n}(\Delta_1, \Delta_2) \) are polynomials in \( \Delta_1 \) and \( \Delta_2 \), the leading order term being generated by the term in \( D_{m,n} \) with maximum number of generators \( G_k \), i.e., the one quoted in (3.12). Thus the degree of \( Y_{m,n}(\Delta_1, \Delta_2) \) is \( p_{e,o}(m, n) \) with leading term

\[
Y_{m,n}(\Delta_1, \Delta_2) = (\Delta_1 - \Delta_2)^{p_{e,o}(m, n)} + \text{lower order terms}
\] (3.23)

On the other hand, the standard decoupling consideration requires this polynomials to be proportional to the product \( P_{m,n}^{(e,o)}(\lambda_1 + \lambda_2) P_{m,n}^{(e,o)}(\lambda_2 - \lambda_1) \). The last is apparently a polynomial in \( \Delta_1 \) and \( \Delta_2 \) with leading order term \( (2\Delta_1 - 2\Delta_2)^{p_{e,o}(m, n)} \) and therefore

\[
Y_{m,n}(\Delta_1, \Delta_2) = 2^{-p_{e,o}(m, n)} P_{m,n}^{(e,o)}(\lambda_1 + \lambda_2) P_{m,n}^{(e,o)}(\lambda_2 - \lambda_1)
\] (3.24)

Consider first the case \( m, n \) even and the following products

\[
\langle V_\Delta(\infty) D_{m,n}^\dagger C_\Delta(\Delta, x) V_\Delta(0) \rangle
\]

\[
= \frac{x^{m/2} X_{m,n}^{(e)}}{\Delta - \Delta_{m,n}} \langle V_\Delta(\infty) D_{m,n}^\dagger C_\Delta(\Delta_{m,n}, x) D_{m,n} V_\Delta(0) \rangle + O(\Delta - \Delta_{m,n})
\] (3.25)

\[
\langle V_\Delta(\infty) D_{m,n}^\dagger G_{1/2} C_\Delta(\Delta, x) V_\Delta(0) \rangle
\]

\[
= \frac{x^{m/2} X_{m,n}^{(e)}}{\Delta - \Delta_{m,n}} \langle V_\Delta(\infty) D_{m,n}^\dagger G_{1/2} C_\Delta(\Delta_{m,n}, x) D_{m,n} V_\Delta(0) \rangle + O(\Delta - \Delta_{m,n})
\]

The estimates \( O(\Delta - \Delta_{m,n}) \) follows from the observation, that all non-polar terms in \( C_\Delta(\Delta, x) \) or \( G_{1/2} C_\Delta(\Delta, x) \) are orthogonal to \( V_\Delta(\infty) D_{m,n}^\dagger \) at \( \Delta = \Delta_{m,n} \). Similarly, at \( m, n \) odd we have

\[
\langle V_\Delta(\infty) D_{m,n}^\dagger C_\Delta(\Delta, x) V_\Delta(0) \rangle
\]

\[
= \frac{x^{m/2} X_{m,n}^{(e)}}{\Delta - \Delta_{m,n}} \langle V_\Delta(\infty) D_{m,n}^\dagger C_\Delta(\Delta_{m,n}, x) D_{m,n} V_\Delta(0) \rangle + O(\Delta - \Delta_{m,n})
\] (3.26)

\[
\langle V_\Delta(\infty) D_{m,n}^\dagger G_{1/2} C_\Delta(\Delta, x) V_\Delta(0) \rangle
\]

\[
= \frac{x^{m/2} X_{m,n}^{(e)}}{\Delta - \Delta_{m,n}} \langle V_\Delta(\infty) D_{m,n}^\dagger G_{1/2} C_\Delta(\Delta_{m,n}, x) D_{m,n} V_\Delta(0) \rangle + O(\Delta - \Delta_{m,n})
\]

Then the right hand sides are evaluated using (3.13) and compared with the left hand sides coming from (3.22). This results in

\[
X_{m,n}^{(e,o)} = \frac{Y_{m,n}^{(e,o)}(\Delta_1, \Delta_2)}{r_{m,n}}
\] (3.27)

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Notice, that a simple change of the notations turns the above consideration to a proof of the residue formula for the complementary chain \( \tilde{C}(\Delta, x) \)

\[
\text{res}_{\Delta=\Delta_{m,n}} \tilde{C}_e(\Delta, x) = x^{mn/2} \left\{ \begin{array}{ll}
X^{(o)}_{m,n} \tilde{C}_e(\Delta_{m,n}, x) D_{m,n} & \text{even} \\
X^{(e)}_{m,n} \tilde{C}_o(\Delta_{m,n}, x) D_{m,n} & \text{odd}
\end{array} \right.
\]

(3.28)

\[
\text{res}_{\Delta=\Delta_{m,n}} \tilde{C}_o(\Delta, x) = x^{mn/2} \left\{ \begin{array}{ll}
X^{(o)}_{m,n} \tilde{C}_o(\Delta_{m,n}, x) D_{m,n} & \text{even} \\
X^{(e)}_{m,n} \tilde{C}_o(\Delta_{m,n}, x) D_{m,n} & \text{odd}
\end{array} \right.
\]

(3.29)

These simple analytic properties are inherited by the corresponding superconformal blocks. Define the odd and even blocks as

\[
\mathcal{F}_{e,o} \left( \begin{array}{c}
\Delta_1 \\
\Delta_2 \\
\Delta_4
\end{array} \bigg| |\Delta| \bigg| x \right) = x^{\Delta_1-\Delta_2} \left\langle V_{\Delta_4}(\infty)V_{\Delta_3}(1) \mathcal{C}^{\Delta_1,\Delta_2}_{e,o}(\Delta, x) \right\rangle V_\Delta(0)
\]

(3.30)

where again \( V_\Delta \) are formal (chiral) primary fields and we have restored the implicit dependence of the chain operator on the external dimensions. Normalization is fixed by

\[
\left\langle V_{\Delta_4}(\infty)V_{\Delta_3}(1)G_{-1/2}V_\Delta(0) \right\rangle = 1
\]

(3.31)

The poles of \( \mathcal{C}^{\Delta_1,\Delta_2}_{e,o}(\Delta) \) turn to the poles of the blocks, the residues being evaluated similarly (to be more compact we suppress the external dimensions in the arguments of the blocks)

\[
\text{res}_{\Delta=\Delta_{m,n}} \mathcal{F}_e(\Delta, x) = B^{(e)}_{m,n} \left( \begin{array}{c}
\Delta_1 \\
\Delta_2 \\
\Delta_4
\end{array} \bigg| \Delta \bigg| x \right) \mathcal{F}_e(\Delta_{m,n}, x) \quad \text{at } m, n \text{ even}
\]

(3.32)

\[
\text{res}_{\Delta=\Delta_{m,n}} \mathcal{F}_o(\Delta, x) = B^{(o)}_{m,n} \left( \begin{array}{c}
\Delta_1 \\
\Delta_2 \\
\Delta_4
\end{array} \bigg| \Delta \bigg| x \right) \mathcal{F}_o(\Delta_{m,n}, x) \quad \text{at } m, n \text{ odd}
\]

Matrix elements \( \left\langle V_{\Delta_4}(\infty)V_{\Delta_3}(1)D_{m,n}V_{m,n}(0) \right\rangle \) are evaluated by the same algebraic procedure

\[
\left\langle V_{\Delta_4}(\infty)V_{\Delta_3}(1)D_{m,n}V_{m,n}(0) \right\rangle = Y^{(e)}_{m,n}(\Delta_3, \Delta_4) \left\langle V_{\Delta_4}(\infty)V_{\Delta_3}(1)V_{m,n}(0) \right\rangle \quad \text{at } m, n \text{ even}
\]

(3.33)

where \( \hat{D}_{m,n} = G_{-1/2}D_{m,n} \) and \( \hat{V}_{m,n}(0) = G_{-1/2}V_{m,n} \). Combining all together we find [14,15]

\[
B^{(e,o)}_{m,n} \left( \begin{array}{c}
\Delta_1 \\
\Delta_2 \\
\Delta_4
\end{array} \right) = Y^{(e,o)}_{m,n}(\Delta_1, \Delta_2)Y^{(e,o)}_{m,n}(\Delta_3, \Delta_4)
\]

(3.34)

Important symmetries

\[
B^{(e,o)}_{m,n} \left( \begin{array}{c}
\Delta_1 \\
\Delta_2 \\
\Delta_4
\end{array} \right) = B^{(e,o)}_{m,n} \left( \begin{array}{c}
\Delta_3 \\
\Delta_4 \\
\Delta_2
\end{array} \right) = B^{(e,o)}_{m,n} \left( \begin{array}{c}
\Delta_2 \\
\Delta_4 \\
\Delta_1
\end{array} \right)
\]

(3.35)

\[
B^{(e,o)}_{m,n} \left( \begin{array}{c}
\Delta_1 \\
\Delta_2 \\
\Delta_4
\end{array} \right) = (-)^{\rho_{e,o}(m,n)} B^{(e,o)}_{m,n} \left( \begin{array}{c}
\Delta_2 \\
\Delta_3 \\
\Delta_4
\end{array} \right)
\]

(3.36)

follow directly from the symmetry properties of the fusion polynomials.
4. $\hat{c}$-recursion

Analytic properties observed in the previous section give rise to convenient relations for the superconformal blocks $F_e(\Delta, x)$ and $F_o(\Delta, x)$ (we suppress sometimes the explicit dependence on the external dimensions) which allow their simple recursive evaluation, e.g. as a power series in $x$. The first way is to consider analytic properties in the central charge $\hat{c}$ of the superconformal algebra \cite{14,15}. In this case the Kac dimensions (3.10) appear as (again in general simple) poles in $\hat{c}$ at

$$\hat{c} = \hat{c}_{m,n}(\Delta) = 5 + 2T_{m,n}(\Delta) + 2T_{m,n}^{-1}(\Delta) \tag{4.1}$$

where again $(m,n)$ is a pair of natural numbers, both even or both odd, while $T_{m,n}(\Delta)$ is a root of the quadric (3.10) in $b^2$

$$T_{m,n}(\Delta) = \frac{1 - 4\Delta - mn + \sqrt{16\Delta^2 + 8(mn - 1)\Delta + (m - n)^2}}{n^2 - 1} \tag{4.2}$$

$$T_{m,n}^{-1}(\Delta) = \frac{1 - 4\Delta - mn - \sqrt{16\Delta^2 + 8(mn - 1)\Delta + (m - n)^2}}{m^2 - 1}$$

For this particular root all singularities corresponding to $m = 1$ are sent to infinity and only the pairs with $m > 1$ count. Notice that the root chosen is non-singular at $n = 1$ so that (4.2) is understood as

$$T_{m,1}(\Delta) = \frac{m^2 - 1}{2(1 - 4\Delta - m)} \tag{4.3}$$

Corresponding residues are read off from those in (3.33) when being expressed in terms of $\Delta$. In the present context it is convenient to use the symmetry of the polynomials (3.17) to make the multipliers in the residues (3.24) explicit polynomials in $\Delta_1$ and $\Delta_2$. At $m,n$ even the multipliers in $P_{m,n}^{(e,o)}(x)$ always enter in pairs $(x - \lambda_k, i)(x - \lambda_{-k}, -i)$ and one can “fold” the product $2^{-p_{e,o}(m,n)}P_{m,n}^{(e,o)}(\lambda_1 - \lambda_2)P_{m,n}^{(e,o)}(\lambda_1 + \lambda_2)$ as follows

$$Y_{m,n}^{(e,o)}(\Delta_1, \Delta_2) = \prod_{k \in \{1,2,m-1\}, \ell \in \{1-n,2,n-1\}} Y_{k,\ell}^{(m,n)}(\Delta_1, \Delta_2, \Delta) \tag{4.4}$$

where

$$Y_{k,\ell}^{(m,n)}(\Delta_1, \Delta_2, \Delta) = (\Delta_1 - \Delta_2)^2 + \Lambda_{k,\ell}^{(m,n)}(\Delta)(\Delta_1 + \Delta_2 - \Lambda_{k,\ell}^{(m,n)}(\Delta)) + \frac{1}{4} \left( \Lambda_{k,\ell}^{(m,n)}(\Delta) \right)^2 \tag{4.5}$$

and

$$\Lambda_{k,\ell}^{(m,n)}(\Delta) = \frac{k^2T_{m,n}^{-1}(\Delta) + 2kl + l^2T_{m,n}(\Delta)}{4} \tag{4.6}$$

Similar folding is possible also at $m, n$ odd if the degree of $P_{m,n}$ is even. If it is odd, the term with $(k, l) = (0, 0)$ leads to an extra multiplier $\Delta_1 - \Delta_2$.,
Once the multipliers in (3.33) are expressed in terms of $\Delta$, the corresponding residues in $\hat{c}$ read

$$\text{res}_{\hat{c} = \hat{c}_{m,n}} F_e(\hat{c}, \Delta, x) = \hat{B}_{m,n}^{(e)}(\Delta) \begin{cases} F_e(\hat{c}_{m,n}, \Delta + mn/2, x) & \text{at } m, n \text{ even} \\ F_o(\hat{c}_{m,n}, \Delta + mn/2, x) & \text{at } m, n \text{ odd} \end{cases}$$

$$\text{res}_{\hat{c} = \hat{c}_{m,n}} F_o(\hat{c}, \Delta, x) = \hat{B}_{m,n}^{(o)}(\Delta) \begin{cases} F_o(\hat{c}_{m,n}, \Delta + mn/2, x) & \text{at } m, n \text{ even} \\ F_e(\hat{c}_{m,n}, \Delta + mn/2, x) & \text{at } m, n \text{ odd} \end{cases}$$

where

$$\hat{B}_{m,n}^{(e,o)}(\Delta) = B_{m,n}^{(e,o)}(\Delta) \frac{16(T_{m,n}(\Delta) - T_{m,n}^{-1}(\Delta))}{(n^2 - 1)T_{m,n}(\Delta) - (m^2 - 1)T_{m,n}^{-1}(\Delta)}$$

(4.8)

the last fraction corresponding to $-\partial \hat{c}/\partial \Delta$ along the $(m, n)$ Kac quadric (3.10).

The asymptotic of $F_{e,o}(\hat{c}, \Delta, x)$ at $\hat{c} \to \infty$ has been recovered in [14, 15]. Analytic properties in $\hat{c}$ sum up to the following relations

$$F_{e,o}(\hat{c}, \Delta, x) = f_{e,o}(\Delta, x) + \sum_{m,n \text{ even}} \frac{\hat{B}_{m,n}^{(e,o)}(\Delta)}{\hat{c} - \hat{c}_{m,n}(\Delta)} F_{e,o}(\hat{c}_{m,n}, \Delta + mn/2, x)$$

$$+ \sum_{m,n \text{ odd} \atop m > 1} \frac{\hat{B}_{m,n}^{(e,o)}(\Delta)}{\hat{c} - \hat{c}_{m,n}(\Delta)} F_{o,e}(\hat{c}_{m,n}, \Delta + mn/2, x)$$

(4.9)

where

$$f_e(\Delta, x) = x^{\Delta - \Delta_1 - \Delta_2} F_1(\Delta + \Delta_1 - \Delta_2, \Delta + \Delta_3 - \Delta_4, 2\Delta, x)$$

$$f_o(\Delta, x) = \frac{1}{2\Delta} x^{\Delta - \Delta_1 - \Delta_2 + 1/2} F_1(\Delta + \Delta_1 - \Delta_2 + \frac{1}{2}, \Delta + \Delta_3 - \Delta_4 + \frac{1}{2}, 2\Delta + 1, x)$$

(4.10)

Equations (4.9) apparently can be used for recursive evaluation of the coefficients in the series expansions

$$F_{e,o}(\hat{c}, \Delta, x) = x^{\Delta - \Delta_1 - \Delta_2} \sum_k x^k F_{e,o}^{(N)}(\hat{c}, \Delta)$$

(4.11)

where the sum is over non-negative integer (for $F_e$) or half-integer (for $F_o$) numbers

$$F_{e,o}^{(N)}(\hat{c}, \Delta) = f_{e,o}^{(N)}(\Delta) + \sum_{m,n \text{ even} \atop m,n/2 \leq N} \frac{\hat{B}_{m,n}^{(e,o)}(\Delta)}{\hat{c} - \hat{c}_{m,n}(\Delta)} F_{e,o}^{(N-mn/2)}(\hat{c}_{m,n}, \Delta + mn/2)$$

$$+ \sum_{m,n \text{ odd} \atop m,n/2 \leq N \atop m > 1} \frac{\hat{B}_{m,n}^{(e,o)}(\Delta)}{\hat{c} - \hat{c}_{m,n}(\Delta)} F_{o,e}^{(N-mn/2)}(\hat{c}_{m,n}, \Delta + mn/2)$$

(4.12)
In (4.12)
\[ f^{(N)}_{o}(\Delta) = \frac{(\Delta + \Delta_1 - \Delta_2)_N (\Delta + \Delta_3 - \Delta_4)_N}{N!(2\Delta)_N} \]
\[ f^{(N)}_{e}(\Delta) = \frac{(\Delta + \Delta_1 - \Delta_2 + 1/2)_{N-1/2} (\Delta + \Delta_3 - \Delta_4 + 1/2)_{N-1/2}}{(N - 1/2)!(2\Delta)_{N+1/2}} \]

5. Elliptic recursion

For practical calculations another relation turns out to be much more convenient. This relation is called the elliptic recursion (since it requires a parametrization in terms of the elliptic functions) or, sometimes, the $\Delta$-recursion, because it is based on the analytic properties of the blocks in $\Delta$ instead of $\hat{c}$. As in ref. [17] we understand the variable $x$ as the modulus of the elliptic curve $y^2 = t(1 - t)(1 - xt)$ and introduce the ratio of its periods
\[ \tau = i \frac{K(1 - x)}{K(x)} \]
where
\[ K(x) = \int_0^1 \frac{dt}{2y(t)} \]
is the complete elliptic integral of the first kind. Let also $q = \exp(i\pi\tau)$ and denote in the standard way
\[ \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}; \quad \theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} \]
so that
\[ x = \frac{\theta_2^4(q)}{\theta_4^4(q)} \]
inverts (5.1). This elliptic parametrization has important advantages. Eq.(5.4) maps the half plane Im $\tau > 0$ to the universal covering of the $x$-plane with punctures at 0, 1 and $\infty$. The power expansions (4.11) of the blocks in $x$ converge inside the disk $|x| < 1$. Once reexpanded in $q$ it converges inside $|q| < 1$, i.e., on the whole covering and therefore gives there a uniform approximation. Naturally, even in the region $|x| < 1$ it is expected to converge faster. The elliptic recursion, which we describe now, gives the blocks directly in terms of the elliptic variable and allows to generate the $q$-series in a simple way.

Define the “elliptic” blocks $H_{e,o}(\Delta, q)$ through the relations
\[ F_e\left( \begin{array}{ccc} \Delta_1 & \Delta_2 & |\Delta| \\ \Delta_3 & \Delta_4 & |x| \end{array} \right) = (16q)^{\Delta - Q/2} \frac{\theta_3^{Q_2/8 - \Delta_1 - \Delta_2} \theta_4^{Q_2/8 - \Delta_1 - \Delta_3}}{\theta_3^{\sum_{i=1}^{4} \Delta_i - 3Q^2/2}(q)} H_e\left( \begin{array}{ccc} \lambda_1 & \lambda_3 & |\Delta| \\ \lambda_2 & \lambda_4 & q \end{array} \right) \]
\[ 2F_o\left( \begin{array}{ccc} \Delta_1 & \Delta_2 & |\Delta| \\ \Delta_3 & \Delta_4 & |x| \end{array} \right) = (16q)^{\Delta - Q/2} \frac{\theta_3^{Q_2/8 - \Delta_1 - \Delta_2} \theta_4^{Q_2/8 - \Delta_1 - \Delta_3}}{\theta_3^{\sum_{i=1}^{4} \Delta_i - 3Q^2/2}(q)} H_o\left( \begin{array}{ccc} \lambda_1 & \lambda_3 & |\Delta| \\ \lambda_2 & \lambda_4 & q \end{array} \right) \]
where the parametrization (3.20) of the external dimensions $\Delta_i$ in terms of $\lambda_i$ is implied. In order, the elliptic blocks satisfy the following relations (the elliptic recursion)

$$H_e(\Delta, q) = \theta_3(q^2) + \sum_{m,n \text{ even}} \frac{q^{mn/2} R_{m,n}^{(e)}}{\Delta - \Delta_{m,n}} H_e(\Delta_{m,-n}, q) + \sum_{m,n \text{ odd}} \frac{q^{mn/2} R_{m,n}^{(e)}}{\Delta - \Delta_{m,n}} H_o(\Delta_{m,-n}, q)$$

$$H_o(\Delta, q) = \sum_{m,n \text{ even}} \frac{q^{mn/2} R_{m,n}^{(o)}}{\Delta - \Delta_{m,n}} H_o(\Delta_{m,-n}, q) + \sum_{m,n \text{ odd}} \frac{q^{mn/2} R_{m,n}^{(o)}}{\Delta - \Delta_{m,n}} H_e(\Delta_{m,-n}, q)$$

the residues reading simply

$$R_{m,n}^{(e,o)} \approx \frac{P_{m,n}^{(e,o)} (\lambda_1 + \lambda_2) P_{m,n}^{(e,o)} (\lambda_1 - \lambda_2) P_{m,n}^{(e,o)} (\lambda_3 + \lambda_4) P_{m,n}^{(e,o)} (\lambda_3 - \lambda_4)}{r_{m,n}^{(e,o)}}$$

These relations take into account the analytic properties of the superconformal blocks in $\Delta$ described in section 3. In addition they imply that in the limit $\Delta \to \infty$

$$H_e(\Delta, q) = \theta_3(q^2) + O(\Delta^{-1})$$

$$H_o(\Delta, q) = \Delta^{-1} \theta_2(q^2) + O(\Delta^{-2})$$

The first asymptotic is plugged explicitly into the relations, while the second is automatically generated by the recursion. To derive the relations (5.6) we need to justify the $\Delta \to \infty$ asymptotic of the blocks, summed up in (5.5) and (5.8). The arguments will be reported elsewhere.

Like the $\hat{c}$-recursion of section 4, relations (5.6) allow to evaluate recursively the series expansions of the elliptic blocks in powers of $q$

$$H_{e,o}(\Delta, q) = \sum_N q^N h_{e,o}^{(N)}(\Delta)$$

Again $H_e(\Delta, q)$ expands in non-negative integer powers of $q$ while $H_o(\Delta, q)$ is a series in positive half-integer ones. Relations (5.6) give

$$h_{e,o}^{(N)}(\Delta) = \eta_{e,o}^{(N)} + \sum_{m,n \text{ even}} \frac{P_{m,n}^{(e,o)} h_{e,o}^{(N-mn/2)}(\Delta_{m,-n})}{\Delta - \Delta_{m,n}} + \sum_{m,n \text{ odd}} \frac{P_{m,n}^{(e,o)} h_{e,o}^{(N-mn/2)}(\Delta_{m,-n})}{\Delta - \Delta_{m,n}}$$

where $\eta_{e}^{(N)} = 0$ and $\eta_{o}^{(N)}$ are coefficients in the $q$-expansion of $\theta_3(q^2)$. In practice this relation allows a much better algorithm as compared to the $\hat{c}$-recursion (4.12), mostly because $\hat{c}$ is fixed and the values $R_{m,n}^{(e,o)}$, $\Delta_{m,n}$ remain the same at all iteration steps (unlike (4.12), where at each level they have to be recomputed for the “shifted” values of $\Delta$). Once a necessary number of residues and dimensions is available, the remaining recursive procedure runs very fast.
6. Superconformal bootstrap

With the structure constants (1.14) and superconformal blocks known one is in the position to evaluate the four-point function (1.23) of basic NS fields in SLFT. The goal of this section is to verify numerically the crossing symmetry relations (1.24).

For the reasons discussed above, we will use the elliptic representation (5.5) of the blocks. The four-point function (1.23) acquires the form

\[ G \left( \frac{a_1}{a_2}, \frac{a_3}{a_4} \mid x, \bar{x} \right) = \frac{(x\bar{x})^{Q^2/8-\Delta_1-\Delta_2}(1-x)(1-\bar{x})^{Q^2/8-\Delta_1-\Delta_3}}{[\theta_3(q)\theta_3(\bar{q})]^4\sum_1 \Delta_1-3g^2/2} g \left( \frac{a_1}{a_2}, \frac{a_3}{a_4} \mid \tau, \bar{\tau} \right) \]  

where

\[ g \left( \frac{a_1}{a_2}, \frac{a_3}{a_4} \mid \tau, \bar{\tau} \right) = \int \frac{dP}{4\pi} |16q|^P \left[ C_{a_1,a_2}^{Q/2+iP} C_{a_3,a_4}^{Q/2-iP} H_e (\Delta, q) H_e (\Delta, \bar{q}) - \tilde{C}_{a_1,a_2}^{Q/2+iP} \tilde{C}_{a_3,a_4}^{Q/2-iP} H_o (\Delta, q) H_o (\Delta, \bar{q}) \right] \]

and \( \Delta = Q^2/8 + P^2/2 \). The first of the relations (1.24) in terms of the function \( g \) is verified analytically. Indeed, the identities

\[ H_e \left( \frac{\lambda_1}{\lambda_2}, \frac{\lambda_3}{\lambda_4} \mid \Delta \right) - q = H_o \left( \frac{\lambda_1}{\lambda_2}, \frac{\lambda_3}{\lambda_4} \mid \Delta \right) q \]  

for the elliptic blocks (the latter are easily derived from (5.6) and the symmetries (3.34) of the residues) directly result in

\[ g \left( \frac{a_1}{a_2}, \frac{a_3}{a_4} \mid \tau, \bar{\tau} \right) = g \left( \frac{a_1}{a_2}, \frac{a_3}{a_4} \mid \tau + 1, \bar{\tau} + 1 \right) \]  

The second relation in eq. (1.24) in terms of the function \( g \) reads

\[ g \left( \frac{a_1}{a_2}, \frac{a_3}{a_4} \mid \tau, \bar{\tau} \right) = (\tau \bar{\tau})^{3Q^2/4-2\sum_1 \Delta} g \left( \frac{a_1}{a_2}, \frac{a_3}{a_4} \mid \frac{1}{\tau}, \frac{1}{\bar{\tau}} \right) \]

It is a difficult mathematical problem to recover this property from the representation (2.10). However, with the fast algorithms based on the elliptic representation of the blocks, it is an affordable problem for numerical analysis.

As a simplest numerical test we have chosen the external parameters \( a_1 = a_2 = a_3 = a_4 = Q/2 \). The structure constants (1.14) vanish for these values of the external parameters, so that we take a first derivative in all four of them. This means that we consider the four point function of the primary fields \( V_{Q/2}^\prime = \phi \exp(Q\phi/2) \). Denoting

\[ \frac{\partial^4}{\partial a_1 \partial a_2 \partial a_3 \partial a_4} g \left( \frac{a_1}{a_2}, \frac{a_3}{a_4} \mid \tau, \bar{\tau} \right) \bigg|_{a_i = Q/2} = 4Y_{NS}^4(0)Y_R^2(b) f(\tau, \bar{\tau}) \]  

(6.6)
we find
\[ f(\tau, \bar{\tau}) = \int \frac{dP}{4\pi} |16q|^P \left[ r_e(P) \left| H_e \left( \begin{array}{ccc} 0 & 0 & |\Delta| q \\ 0 & 0 & |\Delta| q \end{array} \right) \right|^2 + r_o(P) \left| H_o \left( \begin{array}{ccc} 0 & 0 & |\Delta| q \\ 0 & 0 & |\Delta| q \end{array} \right) \right|^2 \right] \] (6.7)

The auxiliary functions \( r_e(P) \) and \( r_o(P) \) read in terms of standard upsilon functions \( \Upsilon_b(x) \)
\[ r_e(P) = \frac{\Upsilon_b(iP)\Upsilon_b(-iP)\Upsilon_b^2(iP + Q/2)}{\Upsilon_b^2(Q/4 + iP/2)\Upsilon_b^2(Q/4 - iP/2)} \] (6.8)

and allow the following integral representations
\[ r_e(P) = P^2 \exp \left\{ \int_0^\infty \frac{dt}{t} \left[ -\frac{(1 + 6b^2 + b^4)e^{-t}}{2b^2} + \frac{8\cos(Pt/2)\cosh[(b + 1/b)t/4] - 2\cos(Pt)\cosh^2[(b - 1/b)t/4] - 6}{\sinh(t/2b)\sinh(bt/2)} \right] \right\} \] (6.9)

\[ r_o(P) = P^2 \exp \left\{ \int_0^\infty \frac{dt}{t} \left[ -\frac{(1 - 2b^2 + b^4)e^{-t}}{2b^2} + \frac{8\cos(Pt/2)\cosh[(b - 1/b)t/4] - 2\cos(Pt)\cosh^2[(b - 1/b)t/4] - 6}{\sinh(t/2b)\sinh(bt/2)} \right] \right\} \] (6.10)

In Figure 1 the values of \( f(\tau, \bar{\tau}) \) (solid curves) are compared with that of \( (\tau \bar{\tau})^{-2Q^2/4} \times f(-1/\tau, -1/\bar{\tau}) \) (symbols) for pure imaginary \( \tau = it \) and certain values of the parameter \( b \). To give an idea about the accuracy, some numbers are shown in the Table 1. These numbers correspond to the approximation of the elliptic blocks \( H_e(q) \) and \( q^{1/2}H_o(q) \) as power series up to the order \( q^6 \). It seems like the main source of the discrepancy is in this approximation. For example at \( t = 0.5 \) \((q^2 \approx 0.0432)\) the numbers differ in the fifth decimal digit (e.g., the numbers 7.12534 and 7.12512 in the Table), while the approximation improved up to the order \( q^{10} \) gives seven correct decimal digits (respectively 7.12511575 and 7.12511599).

We have performed a similar comparison for some different values of \( a_1, a_2, a_3, a_4 \) (chosen at random, but close enough to \( Q/2 \) to preserve the convergence of the representations (6.9) and (6.10)). E.g., at
\[ a_1 = \frac{9}{14}Q; \quad a_3 = \frac{2Q}{3}; \quad a_2 = \frac{13Q}{30}; \quad a_4 = \frac{7Q}{10} \] (6.11)
we have for \( b = 2.8 \) and \( t = 0.3 \)
\[ f_{a_1,a_2,a_3,a_4}(t) = 2141.5325 \] (6.12)
\[ t^{3Q^2/2-4} \sum_i \Delta_i f_{a_1,a_2,a_3,a_4}(1/t) = 2141.5101 \]
7. Discussion

In the present paper a preliminary analysis of the bootstrap properties has been performed for the four point functions in the $N = 1$ supersymmetric Liouville field theory. Some, mostly numerical, arguments are presented that the NS operator algebra, based on the structure constants (1.14), satisfies the locality property. Of course, a separate analysis is needed to include the Ramond sector. This problem remains for future work.

Even in the purely NS sector the program is not yet completely finished. We considered only the four point functions of four “bottom” components $V$ of any supermultiplet. Correlation functions involving other components, like $\langle VVVW \rangle$ or $\langle \Lambda \Lambda VV \rangle$ etc., remain to be studied. Obviously they can be expressed in terms of the structure constants (1.14), like in eq. (1.23) for $\langle VVVV \rangle$, with different superconformal blocks. We arrive at the problem to describe completely the set of 32 different blocks like (in obvious notations)

\[
\mathcal{F}_{\epsilon,o} \left( \Delta_1, \Delta_2, \Delta_3, \Delta_4 | \Delta | x \right) = \langle V_4(\infty)\Lambda_3(1) | \Delta, \hat{\Delta} | V_1(x)V_2(0) \rangle \tag{7.1}
\]

\[
\mathcal{F}_{\epsilon,o} \left( \hat{\Delta}_1, \Delta_2, \Delta_3, \Delta_4 | \Delta | x \right) = \langle V_4(\infty)\Lambda_3(1) | \Delta, \hat{\Delta} | \Lambda_1(x)\Lambda_2(0) \rangle
\]

etc.

where in the present context $V$ and $\Lambda$ are formal chiral components of the “right” supermul-
\[
\begin{array}{|c|c|c|c|c|}
\hline
  t & f(t) & t^{-Q^2/2}f(1/t) & f(t) & t^{-Q^2/2}f(1/t) \\
  \text{for } b = 0.8 & \text{for } b = 0.8 & \text{for } b = i\pi/4 & \text{for } b = i\pi/4 \\
\hline
0.1 & 25.900821 & 25.456246 & 1.5305811 & 1.5232342 \\
0.2 & 15.372272 & 15.365205 & 1.5267458 & 1.5264759 \\
0.3 & 11.123984 & 11.123971 & 1.4398383 & 1.4396855 \\
0.4 & 8.7059097 & 8.7060123 & 1.3522805 & 1.3521968 \\
0.5 & 7.1251188 & 7.125178 & 1.2708572 & 1.2708109 \\
0.6 & 6.0067008 & 6.0067341 & 1.1971168 & 1.1970912 \\
0.7 & 5.1733043 & 5.1733228 & 1.1308767 & 1.1308632 \\
0.8 & 4.528805 & 4.5288147 & 1.0713992 & 1.0713936 \\
0.9 & 4.0160968 & 4.0161008 & 1.016392 & 1.0163936 \\
\hline
\end{array}
\]

Table 1: Numerical data for \( f(t) \) for \( b = 0.8 \) and \( b = i\pi/4 \).

tiplet, and, by definition

\[
\langle V_4(\infty)\Lambda_3(1) | \Delta, \tilde{\Delta} | V_1(x)V_2(0) \rangle = \langle V_4(\infty)\Lambda_3(1)C_{e,o}^{\Delta_1,\Delta_2}(\Delta, x)V_\Delta(0) \rangle
\]

(7.2)

(similarly for other components).

The problem is slightly less involved than it seems at first glance. A study of the superprojective Ward identities, similar to that in Appendix A, allows to reduce the set to 8 independent functions. Moreover, their analytic properties follow directly from those of the basic chains \( C_{e,o}(\Delta, x) \) and \( \tilde{C}_{e,o}(\Delta, x) \). In particular, the residues at the singular dimensions \( \Delta = \Delta_{m,n} \) are expressed similarly to (3.33) in terms of the fusion polynomials. Construction of the recursive representations, analogous to (4.9) or (5.6), presents a separate problem, which we hope to analyze in the future.

Another topic of a future report is the justifications of the asymptotic (5.5) and (5.8) of the block at \( \Delta \to \infty \), which has been simply conjectured in section 5 in order to write down the elliptic recursion relations. This analysis requires several essential steps and presently we prefer to skip it.

The superconformal blocks, considered above, are expected to satisfy certain “crossing”, or “fusion” relations, the set with different values of the intermediate dimension forming an infinite dimensional representation of the monodromy group. In the present case the group is equivalent to the modular group generated by the maps \( x \to x/(x-1) \) and \( x \to 1-x \). In terms of the elliptic parameter \( \tau \) these are \( \tau \to \tau + 1 \) and \( \tau \to -1/\tau \). The representation is completely determined by the relations

\[
\mathcal{F}_e \left( \begin{array}{cc}
\Delta_1 & \Delta_3 \\
\Delta_2 & \Delta_4 \\
\end{array} \right) |\Delta| x = e^{-i\pi(\Delta-\Delta_1-\Delta_2)(1-x)}/2\Delta_1 \mathcal{F}_e \left( \begin{array}{cc}
\Delta_1 & \Delta_3 \\
\Delta_2 & \Delta_4 \\
\end{array} \right) |\Delta| \frac{xe^{i\pi}}{1-x}
\]

\[
\mathcal{F}_o \left( \begin{array}{cc}
\Delta_1 & \Delta_3 \\
\Delta_2 & \Delta_4 \\
\end{array} \right) |\Delta| x = e^{-i\pi(\Delta-\Delta_1-\Delta_2+1/2)(1-x)}/2\Delta_1 \mathcal{F}_o \left( \begin{array}{cc}
\Delta_1 & \Delta_3 \\
\Delta_2 & \Delta_4 \\
\end{array} \right) |\Delta| \frac{xe^{i\pi}}{1-x}
\]

(7.3)

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and

\[ F_\varepsilon \left( \frac{\Delta_1}{\Delta_2}, \frac{\Delta_3}{\Delta_4} \mid \Delta_P \mid x \right) = \sum_{\varepsilon' = e,o} \int \frac{dP'}{4\pi} K_{\varepsilon'} \left( \frac{\Delta_1}{\Delta_2}, \frac{\Delta_3}{\Delta_4} \mid P, P' \right) F_{\varepsilon'} \left( \frac{\Delta_1}{\Delta_2}, \frac{\Delta_3}{\Delta_4} \mid \Delta_{P'} \mid x \right) \]  

(7.4)

Here we introduced an index \( \varepsilon = e,o \) to unify the notations, and \( K_{\varepsilon'} (P, P') \) is the kernel of a certain integral operator, called the crossing matrix (or, sometimes, the fusion matrix). In our construction the first relation follow directly from the symmetries (6.3) of the elliptic blocks. The second one is very non-trivial; the crossing matrix \( K_{\varepsilon'} (P, P') \) remaining to be found. In the case of the ordinary (non-supersymmetric) conformal block an explicit expression for the crossing matrix has been conjectured in ref. [18] on the basis of an appropriate quantum group analysis. Similar analysis seems to be feasible in the present superconformal situation.

Finally, we hope that the construction of the SLFT four point function, started in the present paper, will turn useful for the applications in the Liouville supergravity as well as in non-critical superstring theory.

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A. Superprojective invariance. Three point function

In this Appendix we consider only Ward identities related to the “right” superconformal algebra, formed by the holomorphic components \( S(z) \) and \( T(z) \). Respectively, supermultiplets consist of the highest weight vectors \( V_i \) or \( \bar{\Lambda}_i \) defined as in (1.6) (in this section we omit the parameter \( a \) near the primary field to give place for the identification number 1, 2 etc.) and the “top components” \( \Lambda_i \) or \( W_i \). To be definite, we will talk about the multiplet \( (V_i, \Lambda_i) \). It seems to require no comments how to combine the results below to the complete holomorphic-antiholomorphic combinations.
To describe all $2^3$ possible three point functions denote

\[ C_{123} = \langle V_1 V_2 V_3 \rangle \]
\[ C_{1\bar{2}\bar{3}} = \langle \Lambda_1 V_2 V_3 \rangle \]

\[
\text{etc.}
\]

From the operator product expansions (1.7) we have the following supercurrent Ward identities

\[
\langle S(z) \Lambda_1 V_2 V_3 \rangle = \left( \frac{2\Delta_1}{(z-x_1)^2} + \frac{1}{z-x_1} \frac{\partial}{\partial x_1} \right) C_{123} - \frac{1}{z-x_2} C_{1\bar{2}\bar{3}} - \frac{1}{z-x_3} C_{1\bar{2}\bar{3}}
\]

\[
\langle S(z) V_1 \Lambda_2 V_3 \rangle = \left( \frac{2\Delta_2}{(z-x_2)^2} + \frac{1}{z-x_2} \frac{\partial}{\partial x_2} \right) C_{123} + \frac{1}{z-x_1} C_{1\bar{2}\bar{3}} - \frac{1}{z-x_3} C_{1\bar{2}\bar{3}}
\]

\[
\langle S(z) V_1 V_2 \Lambda_3 \rangle = \left( \frac{2\Delta_3}{(z-x_3)^2} + \frac{1}{z-x_3} \frac{\partial}{\partial x_3} \right) C_{123} + \frac{1}{z-x_1} C_{1\bar{2}\bar{3}} + \frac{1}{z-x_2} C_{1\bar{2}\bar{3}}
\]

where $\Delta_1$, $\Delta_2$ and $\Delta_3$ are respectively the dimensions of $V_1$, $V_2$ and $V_3$. As $S(z) = O(z^{-3})$ at $z \to \infty$, the following super projective identities hold

\[
\frac{\partial}{\partial x_1} C_{123} = C_{1\bar{2}\bar{3}} + C_{1\bar{2}\bar{3}}
\]

\[
\frac{\partial}{\partial x_2} C_{123} = -C_{1\bar{2}\bar{3}} + C_{1\bar{2}\bar{3}}
\]

\[
\frac{\partial}{\partial x_3} C_{123} = -C_{1\bar{2}\bar{3}} - C_{1\bar{2}\bar{3}}
\]

\[
\left( x_1 \frac{\partial}{\partial x_1} + 2\Delta_1 \right) C_{123} = x_2 C_{1\bar{2}\bar{3}} + x_3 C_{1\bar{2}\bar{3}}
\]

\[
\left( x_2 \frac{\partial}{\partial x_2} + 2\Delta_2 \right) C_{123} = -x_1 C_{1\bar{2}\bar{3}} + x_3 C_{1\bar{2}\bar{3}}
\]

\[
\left( x_3 \frac{\partial}{\partial x_3} + 2\Delta_3 \right) C_{123} = -x_1 C_{1\bar{2}\bar{3}} - x_2 C_{1\bar{2}\bar{3}}
\]

These identities involve only the correlation functions with even number of “fermions” $\Lambda_i$. Eliminating the derivatives one finds

\[
2\Delta_1 C_{123} = -x_{12} C_{1\bar{2}\bar{3}} + x_{31} C_{1\bar{2}\bar{3}}
\]

\[
2\Delta_2 C_{123} = -x_{12} C_{1\bar{2}\bar{3}} - x_{23} C_{1\bar{2}\bar{3}}
\]

\[
2\Delta_3 C_{123} = x_{31} C_{1\bar{2}\bar{3}} - x_{23} C_{1\bar{2}\bar{3}}
\]

(A.4)
and thus

\[ C_{123} = -\frac{\Delta_2 + \Delta_3 - \Delta_1}{x_{23}} C_{123} \]
\[ C_{1\overline{2}3} = \frac{\Delta_1 + \Delta_3 - \Delta_2}{x_{31}} C_{123} \]
\[ C_{\overline{1}23} = -\frac{\Delta_1 + \Delta_2 - \Delta_3}{x_{12}} C_{123} \]  

(A.5)

Being substituted to differential equations this sums up to

\[ \frac{\partial}{\partial x_1} C_{123} = -\frac{\Delta_1 + \Delta_2 - \Delta_3}{x_{12}} C_{123} - \frac{\Delta_1 + \Delta_3 - \Delta_2}{x_{13}} C_{123} \]
\[ \frac{\partial}{\partial x_2} C_{123} = -\frac{\Delta_1 + \Delta_2 - \Delta_3}{x_{21}} C_{123} - \frac{\Delta_2 + \Delta_3 - \Delta_1}{x_{23}} C_{123} \]  

(A.6)

\[ \frac{\partial}{\partial x_3} C_{123} = -\frac{\Delta_1 + \Delta_3 - \Delta_2}{x_{31}} C_{123} - \frac{\Delta_2 + \Delta_3 - \Delta_1}{x_{32}} C_{123} \]

and gives finally

\[ C_{123} = \frac{C}{x_{12} + \Delta_1 + \Delta_2 - \Delta_3 x_{23} + \Delta_1 x_{31} + \Delta_3 - \Delta_2} \]  

(A.7)

where \( C \) is an integration constant, independent on \( x_1, x_2 \) and \( x_3 \).

Another Ward identity, relevant for the even in fermions functions, is

\[ \langle S(z) \Lambda_1 \Lambda_2 \Lambda_3 \rangle = \left( \frac{2\Delta_1}{(z - x_1)^2} + \frac{1}{z - x_1} \frac{\partial}{\partial x_1} \right) C_{1\overline{2}3} \]
\[ - \left( \frac{2\Delta_2}{(z - x_2)^2} + \frac{1}{z - x_2} \frac{\partial}{\partial x_2} \right) C_{\overline{1}23} + \left( \frac{2\Delta_3}{(z - x_3)^2} + \frac{1}{z - x_3} \frac{\partial}{\partial x_3} \right) C_{12\overline{3}} \]  

(A.8)

In the same manner it gives

\[ \frac{\partial}{\partial x_1} C_{1\overline{2}3} - \frac{\partial}{\partial x_2} C_{\overline{1}23} + \frac{\partial}{\partial x_3} C_{12\overline{3}} = 0 \]  

(A.9)

\[ \left( 2\Delta_1 + x_1 \frac{\partial}{\partial x_1} \right) C_{1\overline{2}3} - \left( 2\Delta_2 + x_2 \frac{\partial}{\partial x_2} \right) C_{\overline{1}23} + \left( 2\Delta_3 + x_3 \frac{\partial}{\partial x_3} \right) C_{12\overline{3}} = 0 \]

It is straightforward to verify that these relations are satisfied identically with the explicit expressions (A.5).

For the odd fermion number functions consider the Ward identity

\[ \langle S(z) V_1 V_2 V_3 \rangle = \frac{1}{z - x_1} C_{1\overline{2}3} + \frac{1}{z - x_2} C_{\overline{1}23} + \frac{1}{z - x_3} C_{12\overline{3}} \]  

(A.10)

It follows that

\[ C_{123} + C_{1\overline{2}3} + C_{12\overline{3}} = 0 \]  

(A.11)
\[ x_1 C_{1\overline{2}3} + x_2 C_{\overline{1}23} + x_3 C_{12\overline{3}} = 0 \]

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This system is solved in terms of a single function $\tilde{C}_{123}$
\[
\begin{align*}
C_{123} &= x_{23}\tilde{C}_{123} \\
C_{132} &= x_{31}\tilde{C}_{123} \\
C_{123} &= x_{12}\tilde{C}_{123}
\end{align*}
\]
Next, we need the identities
\[
\begin{align*}
\langle S(z)V_1\Lambda_2\Lambda_3 \rangle &= \frac{1}{z-x_1}C_{123} \\
&\quad + \left(\frac{2\Delta_2}{(z-x_2)^2} + \frac{1}{z-x_2}\frac{\partial}{\partial x_2}\right)C_{123} - \left(\frac{2\Delta_3}{(z-x_3)^2} + \frac{1}{z-x_3}\frac{\partial}{\partial x_3}\right)C_{123} \\
\langle S(z)\Lambda_1V_2\Lambda_3 \rangle &= -\frac{1}{z-x_2}C_{123} \\
&\quad + \left(\frac{2\Delta_1}{(z-x_1)^2} + \frac{1}{z-x_1}\frac{\partial}{\partial x_1}\right)C_{123} - \left(\frac{2\Delta_3}{(z-x_3)^2} + \frac{1}{z-x_3}\frac{\partial}{\partial x_3}\right)C_{123} \\
\langle S(z)\Lambda_1\Lambda_2V_3 \rangle &= \frac{1}{z-x_3}C_{123} \\
&\quad + \left(\frac{2\Delta_1}{(z-x_1)^2} + \frac{1}{z-x_1}\frac{\partial}{\partial x_1}\right)C_{123} - \left(\frac{2\Delta_2}{(z-x_2)^2} + \frac{1}{z-x_2}\frac{\partial}{\partial x_2}\right)C_{123}
\end{align*}
\]
They result in the relations
\[
\begin{align*}
\frac{\partial}{\partial x_2}C_{123} - \frac{\partial}{\partial x_3}C_{123} + C_{123} &= 0 \\
\frac{\partial}{\partial x_3}C_{123} - \frac{\partial}{\partial x_1}C_{123} - C_{123} &= 0 \\
\frac{\partial}{\partial x_1}C_{123} - \frac{\partial}{\partial x_2}C_{123} + C_{123} &= 0
\end{align*}
\]
(A.13)
\[
\begin{align*}
\left(2\Delta_2 + x_2\frac{\partial}{\partial x_3}\right)C_{123} - \left(2\Delta_3 + x_3\frac{\partial}{\partial x_3}\right)C_{123} + x_1C_{123} &= 0 \\
\left(2\Delta_1 + x_1\frac{\partial}{\partial x_1}\right)C_{123} - \left(2\Delta_3 + x_3\frac{\partial}{\partial x_3}\right)C_{123} - x_2C_{123} &= 0 \\
\left(2\Delta_1 + x_1\frac{\partial}{\partial x_1}\right)C_{123} - \left(2\Delta_2 + x_2\frac{\partial}{\partial x_2}\right)C_{123} + x_3C_{123} &= 0
\end{align*}
\]
They result in the relations
\[
\begin{align*}
\left(2\Delta_2 + x_2\frac{\partial}{\partial x_3}\right)C_{123} - \left(2\Delta_3 + x_3\frac{\partial}{\partial x_3}\right)C_{123} + x_1C_{123} &= 0 \\
\left(2\Delta_1 + x_1\frac{\partial}{\partial x_1}\right)C_{123} - \left(2\Delta_3 + x_3\frac{\partial}{\partial x_3}\right)C_{123} - x_2C_{123} &= 0 \\
\left(2\Delta_1 + x_1\frac{\partial}{\partial x_1}\right)C_{123} - \left(2\Delta_2 + x_2\frac{\partial}{\partial x_2}\right)C_{123} + x_3C_{123} &= 0
\end{align*}
\]
All of them are satisfied by
\[
\tilde{C}_{123} = \frac{\tilde{C}}{x_2^{\Delta_1+\Delta_2-\Delta_3+1/2}x_3^{\Delta_2+\Delta_3-\Delta_1+1/2}x_1^{\Delta_1+\Delta_3-\Delta_2+1/2}}
\]
with a new integration constant $\tilde{C}$, and
\[
C_{123} = (1/2 - \Delta_1 - \Delta_2 - \Delta_3)\tilde{C}_{123}
\]
B. General OPE and special structure constants

It is instructive to show how the general continuous OPE \( (1.8) \) turns to the discrete one \( (2.2) \) if one of the parameters \( a_1 \) or \( a_2 \) is set to the degenerate value \(-b\). Let us take \( a_2 = -b + \epsilon \) and consider the first term in \( (1.8) \) with \( C_{a_1,a_2}^p \) given by \( (1.14) \)

\[
C_{a_1,a_2}^p = \left( \pi \mu \gamma \left( \frac{Qb}{2} \right) b^{1-b^2} \right)^{(p-a_1-a_2)/b} \times \frac{\Upsilon_{NS}(0) \Upsilon_{NS}(2a_1) \Upsilon_{NS}(2a_2) \Upsilon_{NS}(2Q - 2p)}{\Upsilon_{NS}(Q + p - a_1 - a_2) \Upsilon_{NS}(a_1 + p - a_1) \Upsilon_{NS}(a_1 + a_2 + p - Q)}
\]

At \( a_2 \to -b \) this expression vanishes due to the zero of the multiplier \( \Upsilon_{NS}(2a_2) \). This means that the integral term in \( (1.8) \) disappears and only the discrete terms contribute. The latter are due to the singularities of the integral, which come from the pole structure of the integrand. Expression \( (B.1) \) has poles in \( p \) at (the four lines of singularities here correspond respectively to the four multipliers in the denominator of \( (B.1) \))

\[
a_1 + a_2 - Q - mb^{-1} - nb \quad \text{and} \quad a_1 + a_2 + mb^{-1} + nb \\
a_1 - a_2 - mb^{-1} - nb \quad \text{and} \quad Q + a_1 - a_2 + mb^{-1} + nb \\
a_2 - a_1 - mb^{-1} - nb \quad \text{and} \quad Q + a_2 - a_1 + mb^{-1} + nb \\
Q - a_1 - a_2 - mb^{-1} - nb \quad \text{and} \quad 2Q - a_1 - a_2 + mb^{-1} + nb
\]

where \((m, n)\) – any pair of non-negative integers of the same parity. At \( a_2 = -b + \epsilon \) the poles at \( p = a_1 - b + \epsilon \) and \( p = a_1 + b + \epsilon \) of the first multiplier in the denominator of \( (B.1) \) come across the poles at \( p = a_1 - b - \epsilon \) and \( p = a_1 + b - \epsilon \) of the second multiplier producing two singular terms. The same singularity appears from the two “reflection symmetric” pinches at \( p = Q - a_1 + b \) and \( p = Q - a_1 - b \). Due to the symmetry properties of the integrand and the reflection relation \( (1.20) \) the “reflected” terms give the same contributions and don’t need separate consideration.

First, let us pick up the pole at \( p = a_1 - b - \epsilon \)

\[
\text{res}_{p=a_1-b-\epsilon} C_{a_1,a_2}^p = \frac{\Upsilon_{NS}(0) \Upsilon_{NS}(2a_1) \Upsilon_{NS}(-2b + 2\epsilon) \Upsilon_{NS}(2Q - 2a_1 + 2b)}{\Upsilon_{NS}(Q - 2\epsilon) \Upsilon_{NS}(-2b) \Upsilon_{NS}(2a_1) \Upsilon_{NS}(2a_1 - 2b - Q)} = 1 \quad (B.3)
\]

Similar calculation for the residue at \( p = a_1 + b - \epsilon \) results in

\[
\text{res}_{p=a_1+b-\epsilon} C_{a_1,a_2}^p = \left( \pi \mu \gamma \left( \frac{Qb}{2} \right) b^{1-b^2} \right)^2 \frac{\Upsilon_{NS}(2a_1) \Upsilon_{NS}(2a_1 + 2b - Q)}{\Upsilon_{NS}(2a_1 + 2b) \Upsilon_{NS}(2a_1 - Q)} \quad (B.4)
\]
Second, let’s work out the contribution of the second term with

\[
\hat{\mathcal{C}}_{a_1,a_2}^p = \left( \pi \mu \gamma \left( \frac{Qb}{2} \right) \right)^{(p-a_1-a_2)/b} \times \frac{2i\mathcal{T}'_{\text{NS}}(0)\mathcal{T}_{\text{NS}}(2a_1)\mathcal{T}_{\text{NS}}(2a_2)\mathcal{T}_{\text{NS}}(2Q-2p)}{\mathcal{T}_R(Q+p-a_1-a_2)\mathcal{T}_R(a_2+p-a_1)\mathcal{T}_R(a_1+p-a_2)\mathcal{T}_R(a_1+a_2+p-Q)} (B.5)
\]

The pole structure is given by the same formula \[B.2\] where now \((m,n)\) is a pair of non-negative integers of opposite parity. At \(a_2 = -b + \epsilon\) we have to pick up a singular term at \(p = a_1 - \epsilon\)

\[
\text{res }_{p=a_1-\epsilon} \hat{\mathcal{C}}_{a_1,-b+\epsilon}^p = \left( \pi \mu \gamma \left( \frac{Qb}{2} \right) \right) \frac{2i\mathcal{T}'_{\text{NS}}(0)\mathcal{T}_{\text{NS}}(2a_1)\mathcal{T}_{\text{NS}}(-2b+2\epsilon)\mathcal{T}_{\text{NS}}(2Q-2p)}{\mathcal{T}_R(Q+b-2\epsilon)\mathcal{T}'_R(-b)\mathcal{T}_R(2a_1+b)\mathcal{T}_R(2a_1-b-Q)} (B.6)
\]

Residues \[B.3\], \[B.4\] and \[B.6\] can be compared with the special stricture constants, derived in section 2 in terms of the “screening” integrals.

C. Dotsenko-Fateev type equation

Substitution

\[ g = x^{a_{1}b}(1 - x)^{a_{2}b} F \]  

renders eq.\((2.11)\) to the form

\[
x^2(1 - x)^2 F''' - x(1 - x)(K_1 x - K_2(1 - x)) F'' + (L_1 x^2 + L_2(1 - x)^2 - L_3 x(1 - x)) F' + (M_1 x - M_2(1 - x)) F = 0
\]

where

\[
K_1 = -2g - 3B - 3C; \quad K_2 = -2g - 3A - 3C \\
L_1 = (B + C)(2B + 2C + 2g + 1); \quad L_2 = (A + C)(2A + 2C + 2g + 1) \\
L_3 = 4AB + 4(2A + 2B + 2C + 1)C + 4(A + B + 3C)g + 4g^2 + 2g \\
M_1 = -2C(A + B + C + g + 1)(2B + 2C + 2g + 1) \\
M_2 = -2C(A + B + C + g + 1)(2A + 2C + 2g + 1)
\]

while the parameters \(A, B, C\) and \(g\) are related to \(a_1, a_2, a_3\) and \(b\) as in eq.\((2.17)\). Consider the two-fold contour integrals

\[
I_{\alpha\beta}(x) = \int_{C_\alpha} \int_{C_\beta} dt_1 dt_2 \left| t_1 t_2 \right|^A \left| (1 - t_1)(1 - t_2) \right|^B \left| (x - t_1)(x - t_2) \right|^C \left| t_1 - t_2 \right|^{2g} (C.4)
\]
at $\alpha \neq \beta$ and

\[
I_{\alpha\alpha}(x) = \frac{1}{2} \int dt_1 dt_2 |t_1 t_2|^A |(1 - t_1)(1 - t_2)|^B |(x - t_1)(x - t_2)|^C |t_1 - t_2|^{2g} \quad \text{(C.5)}
\]

where the contours $C_{\alpha}$, $\alpha = 1, 2, 3, 4$ are numbered as follows

\[
C_1 = (-\infty, 0]; \quad C_2 = [0, x]; \quad C_3 = [x, 1]; \quad C_4 = [1, \infty) \quad \text{(C.6)}
\]

It is verified directly that all these integrals are solutions to eq. (C.2).

Of all the nine integrals only three are independent. As a base one can choose the set with a diagonal monodromy around the point $x = 0$

\[
\mathcal{I}_1(x) = I_{44}(x) \sim \mathcal{I}_1^{(0)}(1 + \ldots) \\
\mathcal{I}_2(x) = I_{24}(x) \sim x^{1 + A + C} \mathcal{I}_2^{(0)}(1 + \ldots) \quad \text{(C.7)} \\
\mathcal{I}_3(x) = I_{22}(x) \sim x^{2 + 2A + 2C + 2g} \mathcal{I}_3^{(0)}(1 + \ldots)
\]

where $\ldots$ stands for a regular series in $x$. The constants

\[
\mathcal{I}_1^{(0)} = \frac{\Gamma(2g)\Gamma(1 + B)\Gamma(1 + B + g)\Gamma(-1 - 2g - A - B - C)\Gamma(-1 - g - A - B - C)}{\Gamma(g)\Gamma(-g - A - C)\Gamma(-A - C)} \\
\mathcal{I}_2^{(0)} = \frac{\Gamma(1 + A)\Gamma(1 + B)\Gamma(1 + C)\Gamma(-1 - 2g - A - B - C)}{\Gamma(2 + A + C)\Gamma(-2g - A - C)} \\
\mathcal{I}_3^{(0)} = \frac{\Gamma(2g)\Gamma(1 + A)\Gamma(1 + A + g)\Gamma(1 + C)\Gamma(1 + C + g)}{\Gamma(g)\Gamma(2 + A + C + g)\Gamma(2 + A + C + 2g)} \quad \text{(C.8)}
\]

are calculated using the Selberg integral [25]

\[
\frac{1}{n!} \int_0^1 \prod_{i=1}^n dt_i t_i^{n-1} (1 - t_i)^{\nu-1} \prod_{i>j} |t_i - t_j|^{2g} = \prod_{k=0}^{n-1} \frac{\Gamma(g + kg)\Gamma(\mu + kg)\Gamma(\nu + kg)}{\Gamma(g)\Gamma(\mu + \nu + (n - 1 + k)g)} \quad \text{(C.9)}
\]

Another base

\[
\mathcal{J}_1(x) = I_{11}(x) \sim \mathcal{J}_1^{(0)}(1 + \ldots) \\
\mathcal{J}_2(x) = I_{13}(x) \sim (1 - x)^{1 + B + C} \mathcal{J}_2^{(0)}(1 + \ldots) \quad \text{(C.10)} \\
\mathcal{J}_3(x) = I_{33}(x) \sim (1 - x)^{2 + 2B + 2C + 2g} \mathcal{J}_3^{(0)}(1 + \ldots)
\]

where the dots now replace a regular series in $1 - x$, enjoys diagonal monodromy around $x = 1$. Apparently

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Footnote: It is implied that the parameters are chosen in a way to ensure convergence of all these integrals. Otherwise, standard regularization is in order.
\[ \mathcal{I}_\alpha = \sum_\beta \mathcal{M}_{\alpha \beta} \mathcal{J}_\beta \] (C.11)

where the “fusion matrix” \( \mathcal{M}_{\alpha \beta} \) is evaluated by manipulating the contours of integration. It has the following entries [21]

\[
\begin{align*}
\mathcal{M}_{11} &= \frac{\sin \pi A \sin \pi (g + A)}{\sin \pi (C + B) \sin \pi (g + C + B)} \\
\mathcal{M}_{12} &= -\frac{\sin \pi A \sin \pi C}{\sin \pi (C + B) \sin \pi (2g + C + B)} \\
\mathcal{M}_{13} &= \frac{\sin \pi C \sin \pi (g + C)}{\sin \pi (g + C + B) \sin \pi (2g + C + B)} \\
\mathcal{M}_{21} &= -\frac{2 \cos \pi g \sin \pi (g + A) \sin \pi (g + A + B + C)}{\sin \pi (C + B) \sin \pi (g + C + B)} \\
\mathcal{M}_{22} &= \frac{\sin \pi C \sin \pi (g + A + B + C) - \sin \pi (g + B) \sin \pi A}{\sin \pi (C + B) \sin \pi (g + C + B) \sin \pi (2g + C + B)} \\
\mathcal{M}_{23} &= \frac{\sin \pi g \sin \pi (g + C) \sin \pi (g + B)}{\sin \pi (g + C + B) \sin \pi (2g + C + B)} \\
\mathcal{M}_{31} &= \frac{\sin \pi (g + A + B + C) \sin \pi (2g + A + B + C)}{\sin \pi (C + B) \sin \pi (g + C + B)} \\
\mathcal{M}_{32} &= \frac{\sin \pi B \sin \pi (2g + A + B + C)}{\sin \pi (C + B) \sin \pi (2g + C + B)} \\
\mathcal{M}_{33} &= \frac{\sin \pi B \sin \pi (g + B)}{\sin \pi (g + C + B) \sin \pi (2g + C + B)}
\end{align*}
\] (C.12)

Next, it is verified [21] that the combination

\[ X_1 \mathcal{I}_1(x) \mathcal{I}_1(\bar{x}) + X_2 \mathcal{I}_2(x) \mathcal{I}_2(\bar{x}) + X_3 \mathcal{I}_3(x) \mathcal{I}_3(\bar{x}) \] (C.13)

is a single-valued function of \((x, \bar{x})\) if

\[
\begin{align*}
X_3 &= \frac{\sin \pi A \sin \pi C \sin \pi (A + C) \sin \pi (A + g) \sin \pi (C + g)}{\sin \pi B \sin \pi (B + g) \sin \pi (A + B + C + g) \sin \pi (A + C + 2g) \sin \pi (A + B + C + 2g)} \\
X_1 &= \frac{\sin \pi A \sin \pi B \sin \pi (A + B + C + g) \sin \pi (A + C + 2g)}{\sin \pi (A + C + g) \sin \pi A \sin \pi C} \\
X_2 &= \frac{\sin \pi \pi (A + g)}{\sin \pi (A + B + C + g) \sin \pi (A + C + 2g) \sin \pi (A + C + 2g)} \] (C.14)

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