Web ontology representation and reasoning via fragments of set theory

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Abstract. In this paper we use results from Computable Set Theory as a means to represent and reason about description logics and rule languages for the semantic web. Specifically, we introduce the description logic $\mathcal{DL}^4LQS^R(D)$–admitting features such as min/max cardinality constructs on the left-hand/right-hand side of inclusion axioms, role chain axioms, and datatypes–which turns out to be quite expressive if compared with $\mathcal{SROIQ}(D)$, the description logic underpinning the Web Ontology Language OWL. Then we show that the consistency problem for $\mathcal{DL}^4LQS^R(D)$-knowledge bases is decidable by reducing it, through a suitable translation process, to the satisfiability problem of the stratified fragment $4LQS^R$ of set theory, involving variables of four sorts and a restricted form of quantification. We prove also that, under suitable not very restrictive constraints, the consistency problem for $\mathcal{DL}^4LQS^R(D)$-knowledge bases is $\text{NP}$-complete. Finally, we provide a $4LQS^R$-translation of rules belonging to the Semantic Web Rule Language (SWRL).

1 Introduction

Computable Set Theory is a research field started in the late seventies with the purpose of studying the decidability of the satisfiability problem for fragments of set theory. The most efficient decision procedures designed in this area have been implemented within the reasoner $\texttt{Etnanova/Referee}$ [1] and constitute its inferential core. A wide collection of decidability results obtained up to 2001 can be found in the monographs [2, 3].

Most of the decidability results and applications in computable set theory concern one-sorted multi-level syllogistics, namely collections of formulae admitting variables of one sort only, which range over the Von Neumann universe of sets. Only a few stratified syllogistics, where variables of multiple sorts are allowed, have been investigated, despite the fact that in many fields of computer science and mathematics one often has to deal with multi-sorted languages. For instance, in Description Logics one has to consider entities of different types, namely individual elements, concepts, namely sets of individuals, and roles, namely binary relations over elements.
Recently, one-sorted multi-level fragments of set theory allowing one to express constructs related to multi-valued maps have been studied (see [4–6]) and applied in the realm of knowledge representation. In [7], for instance, an expressive description logic, called $\mathcal{DL}(\text{M} \text{LSS}_{2,m},)$, has been introduced and the consistency problem for $\mathcal{DL}(\text{M} \text{LSS}_{2,m})$-knowledge bases has been proved NP-complete. $\mathcal{DL}(\text{M} \text{LSS}_{2,m})$ has been extended with additional description logic constructs and SWRL rules in [5], proving that the decision problem for the resulting description logic, called $\mathcal{DL}(\forall \pi_0,)$, is still NP-complete under some conditions. Finally, in [6] $\mathcal{DL}(\forall \pi_0)$ has been extended with some metamodelling features. However, none of the above-mentioned description logics provides any functionality to deal with datatypes, a simple form of concrete domains that are relevant in real-world applications.

In this paper we introduce an expressive description logic, $\mathcal{DL}(4LQS^R)(D)$ (more simply referred to as $\mathcal{DL}_4^D$ in the rest of the paper), that can be represented in the decidable four-level stratified fragment of set theory $4LQS^R$. The logic $\mathcal{DL}_4^D$ supports datatypes, and admits concept constructs such as full negation, union and intersection of concepts, concept domain and range, existential quantification and min cardinality on the left-hand side of inclusion axioms, universal quantification and max cardinality on the right-hand side of inclusion axioms. It also supports role constructs such as role chains on the left hand side of inclusion axioms, union, intersection, and complement of roles, and properties on roles such as transitivity, symmetry, reflexivity, and irreflexivity.

We shall prove that the consistency problem for $\mathcal{DL}_4^D$-knowledge bases is decidable via a reduction to the satisfiability problem for formulae of $4LQS^R$. The latter problem was proved decidable in [8]. We shall also show that the consistency problem for $\mathcal{DL}_4^D$-knowledge bases involving only suitably constrained $\mathcal{DL}_4^D$-formulae is NP-complete. Such restrictions are not very limiting: in fact, it turns out that the constrained logic allows one to represent real world ontologies such as Ontoceramic, designed for ancient ceramic cataloguing in collaboration with archaeological experts (see [9, 10]).

The logic $\mathcal{DL}_4^D$ is not an extension of $\text{SROIQ}(D)$, the description logic upon which the W3C standard OWL 2 DL is based, as it admits existential (resp., universal) quantification only on the left-hand (resp., right-hand) side of inclusion axioms. However, $\mathcal{DL}_4^D$ supports chain axioms that are more liberal than the ones supported by $\text{SROIQ}(D)$, as they can involve roles that are not subject to any regularity restriction. Moreover, Boolean combination of roles is supported even on the right-hand side of chain axioms. The latter fact is particularly relevant to the problem of expressing rules in OWL. We will briefly illustrate how $4LQS^R$ can be used to express SWRL rules in Section 3.1.

The paper is organized as follows. In Section 2 we review the syntax and semantics of the set-theoretic fragment $4LQS^R$ and of the logic $\text{SROIQ}(D)$. Then, in Section 3, we present the description logic $\mathcal{DL}_4^D$ and prove that the decidability of the consistency problem for $\mathcal{DL}_4^D$-knowledge bases can be reduced to the satisfiability problem for $4LQS^R$-formulae. In particular, in Section 3.1 we
show that SWRL rules can be represented within the $4LQS^R$ fragment. Finally, in Section 4 we draw our conclusions and give some hints to future work.

2 Preliminaries

In this section we introduce concepts and notions that will be used in the paper.

2.1 The set-theoretic fragment $4LQS^R$

In order to define the fragment $4LQS^R$, it is convenient to first introduce the syntax and semantics of a more general four-level quantified language, denoted $4LQS$. Then we provide some restrictions on quantified formulae of $4LQS$ that characterize $4LQS^R$. We recall that the satisfiability problem for $4LQS^R$ has been proved decidable in [8].

$4LQS$ involves the four collections of variables $\mathcal{V}_0$, $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$, where:
- $\mathcal{V}_0$ contains variables of sort 0, denoted by $x, y, z, ...$;
- $\mathcal{V}_1$ contains variables of sort 1, denoted by $X^1_1, Y^1_1, Z^1_1, ...$;
- $\mathcal{V}_2$ contains variables of sort 2, denoted by $X^2_1, Y^2_1, Z^2_1, ...$;
- $\mathcal{V}_3$ contains variables of sort 3, denoted by $X^3_1, Y^3_1, Z^3_1, ...$.

In addition to variables, $4LQS$ involves also pair terms of the form $\langle x, y \rangle$, for $x, y \in \mathcal{V}_0$. $4LQS$-quantifier-free atomic formulae are classified as:
- level 0: $x = y$, $x \in X^1_1$, $\langle x, y \rangle \in X^2_1$, $\langle x, y \rangle \in X^3_1$, where $x, y \in \mathcal{V}_0$, $\langle x, y \rangle$ is a pair term, $X^1_1 \in \mathcal{V}_1$, $X^2_1 \in \mathcal{V}_2$, $X^3_1 \in \mathcal{V}_3$;
- level 1: $X^1_1 = Y^1_1$, $X^1_1 \in X^2_1$, with $X^1_1, Y^1_1 \in \mathcal{V}_1$, $X^2_1 \in \mathcal{V}_2$;
- level 2: $X^2_1 = Y^2_1$, $X^2_1 \in X^3_1$, with $X^2_1, Y^2_1 \in \mathcal{V}_2$, $X^3_1 \in \mathcal{V}_3$.

$4LQS$ purely universal formulae are classified as:
- level 1: $(\forall z_1) ... (\forall z_n) \varphi_0$, where $z_1, ..., z_n \in \mathcal{V}_0$ and $\varphi_0$ is any propositional combination of quantifier-free atomic formulae of level 0;
- level 2: $(\forall Z_1^1) ... (\forall Z_m^1) \varphi_1$, where $Z_1^1, ..., Z_m^1 \in \mathcal{V}_1$ and $\varphi_1$ is any propositional combination of quantifier-free atomic formulae of levels 0 and 1 and of purely universal formulae of level 1;
- level 3: $(\forall Z_1^2) ... (\forall Z_m^2) \varphi_2$, where $Z_1^2, ..., Z_m^2 \in \mathcal{V}_2$ and $\varphi_2$ is any propositional combination of quantifier-free atomic formulae and of purely universal formulae of levels 1 and 2.

$4LQS$-formulae are all the propositional combinations of quantifier-free atomic formulae of levels 0, 1, 2 and of purely universal formulae of levels 1, 2, 3.

Let $\varphi$ be a $4LQS$-formula. Without loss of generality, we can assume that $\varphi$ contains only $\neg$, $\land$, $\lor$ as propositional connectives. Further, let $S_\varphi$ be the syntax tree for a $4LQS$-formula $\varphi$,\(^1\) and let $\nu$ be a node of $S_\varphi$. We say that a

\(^1\) The notion of syntax tree for $4LQS$-formulae is similar to the notion of syntax tree for formulae of first-order logic. A precise definition of the latter can be found in [11].
A 4LQS-formula \( \psi \) occurs within \( \varphi \) at position \( \nu \) if the subtree of \( S_\varphi \) rooted at \( \nu \) is identical to \( S_\psi \). In this case we refer to \( \nu \) as an occurrence of \( \psi \) in \( \varphi \) and to the path from the root of \( S_\varphi \) to \( \nu \) as its occurrence path. An occurrence of \( \psi \) within \( \varphi \) is positive if its occurrence path deprived by its last node contains an even number of nodes labelled by a 4LQS-formula of type \( \neg \chi \). Otherwise, the occurrence is said to be negative.

A 4LQS-interpretation is a pair \( \mathcal{M} = (D, M) \) where \( D \) is any non-empty collection of objects (called domain or universe of \( \mathcal{M} \)) and \( M \) is an assignment over variables in \( V_0, V_1, V_2, V_3 \) such that

- \( Mx = D \), for each \( x \in V_0 \);
- \( MX^1 \in pow(D) \), for each \( X^1 \in V_1 \);
- \( MX^2 \in pow(pow(D)) \), for each \( X^2 \in V_2 \);
- \( MX^3 \in pow(pow(pow(D))) \), for each \( X^3 \in V_3 \);

(we recall that \( pow(s) \) denotes the powerset of \( s \)).

We assume that pair terms are interpreted \( \text{à la} \) Kuratowski, and therefore we put \( M(x, y) =_{DF} \{(Mx), \{Mx, My\}\} \). The presence of a pairing operator in the language is very useful for the set theoretic representation of the logic \( DL_{FB} \) and of SWRL rules introduced in Sections 3 and 3.1, respectively. Moreover, even though several pairing operators are available (see [12]), encoding ordered pairs \( \text{à la} \) Kuratowski turns out to be quite straightforward, at least for our purposes.

Next, let

- \( \mathcal{M} = (D, M) \) be a 4LQS-interpretation,
- \( x_1, \ldots, x_n \in V_0 \), \( X^1_1, \ldots, X^m_1 \in V_1 \), \( X^2_1, \ldots, X^2_p \in V_2 \),
- \( u_1, \ldots, u_n \in D \), \( U^1_1, \ldots, U^1_m \in pow(D) \), \( U^2_1, \ldots, U^2_p \in pow(pow(D)) \).

By \( \mathcal{M}[x_1/u_1, \ldots, x_n/u_n, X^1_1/U^1_1, \ldots, X^1_m/U^1_m, X^2_1/U^2_1, \ldots, X^2_p/U^2_p] \), we denote the interpretation \( \mathcal{M}' = (D, M') \) such that \( M'x_i = u_i \), for \( i = 1, \ldots, n \), \( M'X^k_j = U^k_j \), for \( j = 1, \ldots, m \), \( M'X^2_k = U^2_k \), for \( k = 1, \ldots, p \), and which otherwise coincides with \( M \) on all remaining variables. Let \( \varphi \) be a 4LQS-formula and let \( \mathcal{M} = (D, M) \) be a 4LQS-interpretation. The notion of satisfiability of \( \varphi \) by \( \mathcal{M} \) (denoted by \( \mathcal{M} \models \varphi \)) is defined inductively over the structure of \( \varphi \). Quantifier-free atomic formulae are evaluated in a standard way according to the usual meaning of the predicates ‘\( \varepsilon \)’ and ‘\( = \)’, and purely universal formulae are evaluated as follows:

- \( \mathcal{M} \models (\forall z_1) \ldots (\forall z_n) \varphi_0 \) iff \( \mathcal{M}[z_1/u_1, \ldots, z_n/u_n] \models \varphi_0 \), for all \( u_1, \ldots, u_n \in D \);
- \( \mathcal{M} \models (\forall Z^1_1) \ldots (\forall Z^1_m) \varphi_1 \) iff \( \mathcal{M}[Z^1_1/U^1_1, \ldots, Z^1_n/U^1_n] \models \varphi_1 \), for all \( U^1_1, \ldots, U^1_n \in pow(D) \);
- \( \mathcal{M} \models (\forall Z^2_1) \ldots (\forall Z^2_m) \varphi_2 \) iff \( \mathcal{M}[Z^2_1/U^2_1, \ldots, Z^2_n/U^2_n] \models \varphi_2 \), for all \( U^2_1, \ldots, U^2_n \in pow(pow(D)) \).

Finally, compound formulae are interpreted according to the standard rules of propositional logic. If \( \mathcal{M} \models \varphi \), then \( \mathcal{M} \) is said to be a 4LQS-model for \( \varphi \). A 4LQS-formula is said to have a 4LQS-model if it has a 4LQS-model. A 4LQS-formula is valid if it is satisfied by all 4LQS-interpretations.

Next we present the fragment \( 4LQS^R \) of 4LQS of our interest, namely the collection of the formulae \( \psi \) of 4LQS fulfilling the restrictions:
1. for every purely universal formula \((\forall Z^1_1), \ldots, (\forall Z^1_m_\phi_1)\) of level 2 occurring in 
\(\psi\) and every purely universal formula \((\forall z_1), \ldots, (\forall z_n_\phi_0)\) of level 1 occurring
negatively in \(\phi_1\), the condition

\[ \neg \phi_0 \rightarrow \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{m} z_i \in Z^1_j \]

is a valid 4LQS-formula (in this case we say that \((\forall z_1), \ldots, (\forall z_n)_\phi_0\) is linked
to the variables \(Z^1_1, \ldots, Z^1_m\));

2. for every purely universal formula \((\forall Z^2_1), \ldots, (\forall Z^2_n_\phi_2)\) of level 3 in 
\(\psi\):
   - every purely universal formula of level 1 occurring negatively in \(\phi_2\) and
     not occurring in a purely universal formula of level 2 is only allowed to
     be of the form

\[ (\forall z_1), \ldots, (\forall z_n) \neg \bigwedge_{i=1}^{n} \bigwedge_{j=1}^{n} (z_i, z_j) = Y^2_{ij}, \]

with \(Y^2_{ij} \in V^2\), for \(i, j = 1, \ldots, n\);
   - purely universal formulae \((\forall Z^1_1), \ldots, (\forall Z^1_m_\phi_1)\) of level 2 may occur only
     positively in \(\phi_2\).

Restriction 1 has been introduced for technical reasons concerning the decidability of the satisfiability problem for the fragment. In fact it guarantees that satisfiability is preserved in a suitable finite submodel of \(\psi\). Restriction 2 allows one to express binary relations and several operations on them while keeping simple, at the same time, the decision procedure (for space reasons details are not included here but can be found in [8]).

We observe that the semantics of 4LQS\(^R\) plainly coincides with that of 4LQS.

In the 4LQS\(^R\)-fragment one can express several set-theoretic constructs such as a restricted variant of the set former, which in turn allows one to express other significant set operators such as binary union, intersection, set difference, the singleton operator, the powerset operator, etc. Within the fragment 4LQS\(^R\),
it is also possible to define binary relations over elements of a domain together
with conditions on them (i.e., reflexivity, transitivity, weak connectedness, ir
reflexivity, intransitivity) which characterize accessibility relations of well-known modal logics. In particular, the normal modal logic K\(45\) can be translated in the 
4LQS\(^R\)-fragment. Again, the interested reader is referred to [8] for details.

2.2 Description Logics

Description Logics (DL) are a family of formalisms widely used in the field
of Knowledge Representation to model application domains and to reason on
them [13]. DL knowledge bases describe models that are based on individual
elements (or, more simply, individuals), classes whose elements are individual
names, and binary relationships between individuals. These three types of se
mantic entities are syntactically denoted by means of individual names, concept
names, and role names. In addition, DL provide operators for combining concept and role names into complex concept and role expressions. One of the leading application domains for DL is the semantic web. In fact, the most recently developed semantic web language, namely OWL 2, is based on a very expressive description logic with datatypes \( D \), called \( \mathcal{SROIQ}(D) \). Extensions of DL with datatypes have been studied and analyzed in [14, 15].

The logic \( \mathcal{SROIQ}(D) \) is briefly introduced in the next section (the interested reader is referred to [16] for details).

2.2.1 The description logic \( \mathcal{SROIQ}(D) \). Let \( D = (N_D, N_C, N_F, {D}) \) be a datatype map in the sense of [15], where \( N_D \) is a finite set of datatypes, \( N_C \) is a function assigning a set of constants \( N_C(d) \) to each datatype \( d \in N_D \), \( N_F \) is a function assigning a set of facets \( N_F(d) \) to each \( d \in N_D \), and \( D \) is a function assigning a datatype interpretation \( d^D \) to each datatype \( d \in N_D \), a facet interpretation \( f^D \subseteq d^D \) to each facet \( f \in N_F(d) \), and a data value \( e^D_d \in d^D \) to every constant \( e_d \in N_C(d) \). We shall assume that the interpretations of the datatypes in \( N_D \) are nonempty pairwise disjoint sets.

A facet expression for a datatype \( d \in N_D \) is a formula \( \psi_d \) constructed from the elements of \( N_F(d) \cup \{ \top, \bot \} \) by applying a finite number of times the connectives \( \neg, \land, \land \). The function \( D \) is extended to facet expressions for \( d \in N_D \) by putting \( \top^D = d^D, \bot^D = \emptyset, (\neg f)^D = d^D \setminus f^D, (f_1 \land f_2)^D = f_1^D \cap f_2^D, \) and \( (f_1 \lor f_2)^D = f_1^D \cup f_2^D \), for \( f, f_1, f_2 \in N_F(d) \).

A data range \( dr \) for \( D \) is either a datatype \( d \in N_D \), or a finite enumeration of datatype constants \( \{ e_{d_1}, \ldots, e_{d_n} \} \), with \( e_{d_i} \in N_C(d_i) \) and \( d_i \in N_D \), or a facet expression \( \psi_d \), for \( d \in N_D \), or their negation.

Let \( R_A, R_D, C, I \) be denumerable pairwise disjoint sets of abstract role names, concrete role names, concept names, and individual names, respectively. The set of abstract roles is defined as \( R_A \cup \{ R^- \mid R \in R_A \} \cup U \), where \( U \) is the universal role and \( R^- \) is the inverse role of \( R \).

A role inclusion axiom (RIA) is an expression of the form \( w \sqsubseteq R \), where \( w \) is a finite string of roles not including the universal role \( U \) and \( R \) is an abstract role name distinct from the universal role \( U \).

An abstract role hierarchy \( R^H_a \) is a finite collection of RIAs.

A concrete role hierarchy \( R^H_c \) is a finite collection of concrete role inclusion axioms \( T_i \sqsubseteq T_j \), where \( T_i, T_j \in R_D \).

A role assertion is an expression of one of the types: Ref\( (R) \), Irref\( (R) \), Sym\( (R) \), Asym\( (R) \), Tra\( (R) \), and Dis\( (R, S) \), where \( R, S \in R_A \cup \{ R^- \mid R \in R_A \} \).

Given an abstract role hierarchy \( R^H_a \) and a set of role assertions \( R^A \) without transitivity or symmetry assertions (Sym\( (R) \) can be represented by a RIA of type \( R^- \sqsubseteq R \)), the set of roles that are simple in \( R^H_a \cup R^A \) is inductively defined as follows: (a) a role name is simple if it does not occur on the right hand side of a RIA in \( R^H_a \), (b) an inverse role \( R^- \) is simple if \( R \) is, and (c) if \( R \) occurs on the right hand of a RIA in \( R^H_a \), then \( R \) is simple if, for each \( w \sqsubseteq R \in R^H_a, w = S \), for a simple role \( S \).
A set of role assertions $R^A$ is called simple if all roles $R, S$ appearing in role assertions of the form $\text{Irref}(R)$, $\text{Asym}(R)$, or $\text{Dis}(R, S)$ are simple in $R^A$.

An SROIQ(D)-TBox is a set $R = R_H^D \cup R_D^H \cup R^A$ such that $R_H^D$ is a regular abstract role hierarchy, $R_D^H$ is a concrete role hierarchy, and $R^A$ is a finite simple set of role assertions. A formal definition of regular abstract role hierarchy can be found in [16].

Before introducing the formal definitions of TBox and of ABox, we define the set of SROIQ(D)-concepts as the smallest set such that:

- every concept name and the constants $\top, \bot$ are concepts,
- if $C, D$ are concepts, $R$ is an abstract role (possibly inverse), $S$ is a simple role (possibly inverse), $T$ is a concrete role, $dr$ is a data range for $D$, $a$ is an individual, and $n$ is a non-negative integer, then $C \sqcap D, C \sqcup D, \neg C, \{a\}, \forall R.C, \exists R.C, \exists S.Self, \forall T.dr, \exists T.dr, \geq n.S.C, \leq n.S.C$ are also concepts.

A general concept inclusion axiom (GCI) is an expression $C \sqsubseteq D$, where $C, D$ are SROIQ(D)-concepts. An SROIQ(D)-TBox $T$ is a finite set of CGIs.

Any expression of one of the following forms: $a : C, (a, b) : R, (a, e_d) : T, (a, b) : \neg R, (a, e_d) : \neg T, a = b, a \neq b$, where $a, b$ are individuals, $e_d$ is a constant in $N_C(d)$, $R$ is a (possibly inverse) abstract role, $P$ is a concrete role, and $C$ is a concept, is called an individual assertion. An SROIQ(D)-ABox $\mathcal{A}$ is a finite set of individual assertions.

An SROIQ(D)-knowledge base is a triple $\mathcal{K} = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ such that $\mathcal{R}$ is an SROIQ(D)-RBox, $\mathcal{T}$ an SROIQ(D)-TBox, and $\mathcal{A}$ an SROIQ(D)-ABox. The semantics of SROIQ(D) is given by means of an interpretation $I = (\Delta^T, \Delta_D, ^I)$, where $\Delta^T$ and $\Delta_D$ are non-empty disjoint domains such that $d^D \subseteq \Delta_D$, for every $d \in N_D$, and $^I$ is an interpretation function. The interpretation of concepts and roles, axioms and assertions is defined in Table 1.

| Name                  | Syntax  | Semantics                                                                 |
|-----------------------|---------|---------------------------------------------------------------------------|
| concept               | $A$     | $A^I \subseteq \Delta^I$                                                 |
| ab. (resp., cn.) rl.  | $R$     | $R^I \subseteq \Delta^I \times \Delta^I$                               |
| ind. (resp., d. cs.) | $a$     | $(a)^I = \{a\}$                                                         |
| nominal               | $d$     | $d^D \subseteq \Delta_D$                                                |
| dtype (resp., ng.)    | $e_d$   | $\psi_d = \{e_{d_1}, \ldots, e_{d_n}\} \cap \Delta_D \setminus d^D$    |
| data range            | $\top$  | $\Delta^I$                                                               |
| top (resp., bot.)     | $\neg C$| $(\neg C)^I = \Delta^I \setminus C$                                    |
| conj. (resp., disj.)  | $C \sqcap D$| $(C \sqcap D)^I = C^I \cap D^I$                                        |
|                       | $\forall R.C$| $(\forall R.C)^I = \{x \in \Delta^1 : \forall y \in \Delta^1, (x,y) \in R^I \rightarrow y \in C^1\}$ |
|                       | $\exists R.C$| $(\exists R.C)^I = \{x \in \Delta^1 : \exists y \in C^1, (x,y) \in R^I\}$ |
|                       | $\exists S.Self$| $(\exists S.Self)^I = \{x \in \Delta^1 : (x,x) \in R^I\}$              |
| exist. restriction    | $\exists T.dr$| $(\exists T.dr)^I = \{x \in \Delta^1 : \exists y \in d^D_D, (x,y) \in R^I\}$ |
| self concept          | $\forall T.dr$| $(\forall T.dr)^I = \{x \in \Delta^1 : \forall y \in \Delta_D, (x,y) \in R^I \rightarrow y \in d^D\}$ |
| datatype exists       | $\leq n.R.C$| $(\leq n.R.C)^I = \{x \in \Delta^1 : \{y \in C^1 : (x,y) \in R^I\} \leq n\}$ |
| datatype value        | $\geq n.R.C$| $(\geq n.R.C)^I = \{x \in \Delta^1 : \{y \in C^1 : (x,y) \in R^I\} \geq n\}$ |
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\textbf{Table 1: Semantics of } \textit{SROIQ(D)}.  

\begin{tabular}{|c|c|c|}
\hline
qual. datatype & \( \leq_n T.dr \) & \( \{ \leq_n T.dr \}^1 = \{ x \in \Delta^1 : |\{ y \in dDR : (x,y) \in T^1 \}| \leq n \} \) \\
number restr. nominals & \( \geq_n T.dr \) & \( \{ \geq_n T.dr \}^1 = \{ x \in \Delta^1 : |\{ y \in dDR : (x,y) \in T^1 \}| \geq n \} \) \\
universal role & \( U \) & \( (U)^1 = \Delta^1 \times \Delta^1 \) \\
imverse role & \( R^\sim \) & \( (R^\sim)^1 = \{ (x,y) \mid (x,y) \in R^1 \} \) \\
concept subsum. & \( C_1 \sqsubseteq C_2 \) & \( I \models_D C_1 \sqsubseteq C_2 \iff C_1^1 \subseteq C_2^1 \) \\
ab. role subsum. & \( R_1 \sqsubseteq R_2 \) & \( I \models_D R_1 \sqsubseteq R_2 \iff R_1^1 \subseteq R_2^1 \) \\
role incl. axiom & \( S_1 \ldots S_n \subseteq R \) & \( I \models_D S_1 \ldots S_n \sqsubseteq R \iff S_1^1 \circ \ldots \circ S_n^1 \subseteq R^1 \) \\
con. role subsum. & \( T_1 \sqsubseteq T_2 \) & \( I \models_D T_1 \sqsubseteq T_2 \iff T_1^1 \subseteq T_2^1 \) \\
symmetric role & Sym(R) & \( I \models_D \text{Sym}(R) \iff (R^{-1})^1 \subseteq R^1 \) \\
asymmetric role & Asym(R) & \( I \models_D \text{Asym}(R) \iff R^1 \cap (R^{-1})^1 = \emptyset \) \\
transitive role & Tra(R) & \( I \models_D \text{Tra}(R) \iff R^1 \circ R^1 \subseteq R^1 \) \\
disjoint role & Dis(R,S) & \( I \models_D \text{Dis}(R,S) \iff R^1 \cap S^1 = \emptyset \) \\
reflexive role & Ref(R) & \( I \models_D \text{Ref}(R) \iff \{ (x,x) \mid x \in \Delta^1 \} \subseteq R^1 \) \\
irreflexive role & Iref(R) & \( I \models_D \text{Iref}(R) \iff R^1 \setminus \{ (x,x) \mid x \in \Delta^1 \} = \emptyset \) \\
func. ab. role & Fun(R) & \( I \models_D \text{Fun}(R) \iff (R^{-1})^1 \circ R^1 \subseteq \{ (x,x) \mid x \in \Delta^1 \} \) \\
func. cn. role & Fun(T) & \( I \models_D \text{Fun}(T) \iff \{ (x,y) \in T^1 \text{ and } (x,z) \in T^1 \} \) \\
concept assertion & \( a : C_1 \) & \( I \models_D a : C_1 \iff \{ a^1 \} \subseteq C_1^1 \) \\
agreement & \( a = b \) & \( I \models_D a = b \iff a^1 = b^1 \) \\
disagreement & \( a \neq b \) & \( I \models_D a \neq b \iff \neg (a^1 = b^1) \) \\
ab. role asser. & \( (a,b) : R \) & \( I \models_D (a,b) : R \iff \{ a^1, b^1 \} \in R^1 \) \\
ab. role asser. & \( (a,c,d) : T \) & \( I \models_D (a,c,d) : T \iff \{ a^1, c^1, d^1 \} \in T^1 \) \\
ng. ab. role asser. & \( (a,b) : \neg R \) & \( I \models_D (a,b) : \neg R \iff \neg \{ a^1, b^1 \} \in R^1 \) \\
ng. cn. role asser. & \( (a,c,d) : \neg T \) & \( I \models_D (a,c,d) : \neg T \iff \neg \{ a^1, c^1, d^1 \} \in T^1 \) \\
\hline
\end{tabular}

\textit{Legenda.} ab: abstract, cn.: concrete, rl.: role, ind.: individual, d. cs.: datatype constant, dtype: datatype, ng.: negated, bot.: bottom, incl.: inclusion, asser.: assertion.

Let } A, R, T \text{ be, respectively, an } \textit{SROIQ(D)-ABox}, an \textit{SROIQ(D)-RBox}, and an } \textit{SROIQ(D)-TBox}. An interpretation } I = (\Delta^1, \Delta_D^1, \ldots, T^1) \text{ is a } D\text{-model of } R \text{ (resp., } T), \text{ and we write } I \models_D R \text{ (resp., } I \models_D T), \text{ if } I \text{ satisfies each axiom in } R \text{ (resp., } T) \text{ according to the semantic rules in Table 1. Analogously, } I = (\Delta^1, \Delta_D^1, \ldots, T^1) \text{ is a } D\text{-model of } A, \text{ and we write } I \models_D A, \text{ if } I \text{ satisfies each assertion in } A \text{, according to the semantic rules in Table 1.}

An } \textit{SROIQ(D)} \text{-knowledge base } K = (A, T, R) \text{ is consistent if there is an interpretation } I = (\Delta^1, \Delta_D^1, \ldots) \text{ that is a } D\text{-model of } A, \text{ and } T, \text{ and } R.\text{ Decidability of the consistency problem for } \textit{SROIQ(D)} \text{-knowledge bases was proved in [16] by means of a tableau-based decision procedure and its computational complexity was shown to be } \text{N2EXPTIME-complete in [17].}

\section{The logic } \mathcal{DL}(4LQS^R)(D)\text{.}

In this section we introduce the description logic } \mathcal{DL}(4LQS^R)(D) \text{ (shortly referred to as } \mathcal{DL}_D^4) \text{ and prove that the consistency problem for } \mathcal{DL}_D^4 \text{-knowledge}
bases is decidable by reducing it to the satisfiability problem for 4LQS$^R$-formulae. Then we show that under certain restrictions the consistency problem for $\mathcal{DL}_D^4$-knowledge bases is NP-complete. Finally we briefly illustrate how SWRL-rules can be translated into the language of $4LQS^R$.

Let $\mathbf{D}, \mathbf{R}, \mathbf{R_D}, \mathbf{I}, \mathbf{C}$ be as in Section 2.2.1.

(a) $\mathcal{DL}_D^4$-datatype, (b) $\mathcal{DL}_D^4$-concept, (c) $\mathcal{DL}_D^4$-abstract role, and (d) $\mathcal{DL}_D^4$-concrete role terms are constructed according to the following syntax rules:

(a) $t_1, t_2 \rightarrow dr \mid \neg t_1 \mid t_1 \cap t_2 \mid t_1 \sqcup t_2 \mid \{e_d\}$,
(b) $C_1, C_2 \rightarrow A \mid B \mid \neg C_1 \mid C_1 \cup C_2 \mid C_1 \cap C_2 \mid \{a\} \mid \exists R.\text{Self} \mid \exists R_1.\{a\} \mid \exists P.e_d$,
(c) $R_1, R_2 \rightarrow S \mid U \mid R_1^\exists \mid \neg R_1 \mid R_1 \sqcup R_2 \mid R_1 \cap R_2 \mid R_{C_1} \mid R_{C_1} \cap c_2 \mid id(C)$,
(d) $P \rightarrow T \mid \neg P \mid P_{C_1} \mid P_{t_1} \mid P_{C_1,t_1}$,

where $dr$ is a data range for $\mathbf{D}$, $t_1, t_2$ are datatype terms, $e_d$ is a constant in $N_C(d)$, $a$ is an individual name, $A$ is a concept name, $C_1, C_2$ are $\mathcal{DL}_D^4$-concept terms, $S$ is an abstract role name, $R, R_1, R_2$ are $\mathcal{DL}_D^4$-abstract role terms, $T$ a concrete role name, and $P$ a $\mathcal{DL}_D^4$-concrete role term.

A $\mathcal{DL}_D^4$-knowledge base is a triple $K = (\mathcal{R}, \mathcal{T}, \mathcal{A})$ such that $\mathcal{R}$ is a $\mathcal{DL}_D^4$-RBox, $\mathcal{T}$ is a $\mathcal{DL}_D^4$-TBox, and $\mathcal{A}$ a $\mathcal{DL}_D^4$-ABox. A $\mathcal{DL}_D^4$-RBox is a collection of statements of the following forms: $R_1 \equiv R_2$, $R_1 \subseteq R_2$, $R_1 \ldots R_n \subseteq R_{n+1}$, $\text{Sym}(R_1)$, $\text{Sym}(R_1)$, $\text{Asym}(R_1)$, $\text{Ref}(R_1)$, $\text{Irref}(R_1)$, $\text{Dis}(R_1, R_2)$, $\text{Tra}(R_1)$, $\text{Fun}(R_1)$, $P_1 \equiv P_2$, $P_1 \subseteq P_2$, $\text{Fun}(P_1)$, where $R_1, R_2$ are $\mathcal{DL}_D^4$-abstract role terms and $P_1, P_2$ are $\mathcal{DL}_D^4$-concrete role terms. A $\mathcal{DL}_D^4$-TBox is a set of statements of the types:

- $C_1 \equiv C_2$, $C_1 \subseteq C_2$, $C_1 \sqcup \forall R.\{a\} \subseteq C_2$, $\exists R_1.\{a\} \subseteq C_2$, $\forall R_1.\{a\} \subseteq C_2$, $\exists R_1.\{a\} \subseteq C_2$,
- $t_1 \equiv t_2$, $t_1 \sqcup t_2$, $C_1 \sqcup \forall P_1.t_1 \sqcup \exists P_1.t_1 \sqcup C_1$, $\forall P_1.t_1 \sqcup C_1$, $\forall P_1.t_1 \sqcup C_1$,

where $C_1, C_2$ are $\mathcal{DL}_D^4$-concept terms, $t_1, t_2$ datatype terms, $R_1$ a $\mathcal{DL}_D^4$-abstract role term, $P_1$ a $\mathcal{DL}_D^4$-concrete role term.

A $\mathcal{DL}_D^4$-ABox is a set of assertions of the forms: $a : C_1$, $(a, b) : R_1$, $(a, b) : \neg R_1, a = b, a \neq b, e_d : t_1, (a, e_d) : P_1, (a, e_d) : \neg P_1, (a, e_d) : \exists P_1, (a, e_d) : \forall P_1, (a, e_d) : \neg R_1$, where $C_1$ is a $\mathcal{DL}_D^4$-concept term, $t_1$ is a datatype term, $R_1$ is a $\mathcal{DL}_D^4$-abstract role term, $P_1$ is a $\mathcal{DL}_D^4$-concrete role term, $a, b$ are individual names, and $e_d$ is a constant in $N_C(d)$.

The semantics of $\mathcal{DL}_D^4$ is similar to that of $SROIQ(\mathbf{D})$ (cf. Section 2.2.1). The interpretation of terms, axioms, and assertions of $\mathcal{DL}_D^4$ shared with $SROIQ(\mathbf{D})$ is illustrated in Table 1 while the semantics of terms and statements specific to $\mathcal{DL}_D^4$ is described in Table 2. The notions of $\mathbf{D}$-model of a $\mathcal{DL}_D^4$-RBox, $\mathcal{DL}_D^4$-TBox, $\mathcal{DL}_D^4$-ABox, and the notion of consistency of a $\mathcal{DL}_D^4$-knowledge base are similar to the ones described in Section 2.2.1 for $SROIQ(\mathbf{D})$.

| Name                  | Syntax | Semantics |
|-----------------------|--------|-----------|
| data range $dr$       | $dr$   | $dr^D \subseteq \Delta_D^D$ |
| negative datatype term| $\neg t_1$ | $(-t_1)^D = \Delta_D^D \setminus t_1^D$ |
| datatype terms intersection | $t_1 \cap t_2$ | $(t_1 \cap t_2)^D = t_1^D \cap t_2^D$ |
| datatype terms union  | $t_1 \sqcup t_2$ | $(t_1 \sqcup t_2)^D = t_1^D \sqcup t_2^D$ |
| constant in $N_C(d)$  | $e_d$  | $e_d^D \subseteq d^D$ |
In the following theorem we prove the decidability of the consistency problem for $DL^4$-knowledge bases.

**Theorem 1.** Let $\mathcal{K}$ be a $DL^4$-knowledge base. Then, one can construct a $4LQS^R$-formula $\psi_\mathcal{K}$ s.t. $\psi_\mathcal{K}$ is satisfiable if and only if $\mathcal{K}$ is consistent.

**Proof.** As a preliminary step, observe that the statements of the $DL^4$-knowledge base $\mathcal{K}$ that need to be considered are those of the following types:

- $C_1 \equiv \top$, $C_1 \equiv \neg C_2$, $C_1 \equiv C_2 \cup C_3$, $C_1 \equiv \{a\}$, $C_1 \equiv R_1 \cap C_2$, $\exists R_1 C_1 \subseteq C_2$,
  $\geq_n R_1 . C_1 \subseteq C_2$, $C_1 \subseteq \forall P_1 . t_1$, $\forall P_1 . t_1 \subseteq C_1$, $\geq_n P_1 . t_1 \subseteq C_1$,
  $C_1 \subseteq \leq_n P_1 . t_1$,
- $R_1 \equiv U$, $R_1 \equiv \neg R_2$, $R_1 \equiv R_2 \cup R_3$, $R_1 \equiv R_2 \cap R_3$, $R_1 \equiv \text{id}(C_1)$, $R_1 \equiv R_2 \cup C_1$,
  $R_1 \equiv R_2 \cap C_1$, $R_1 \equiv R_2 \cap C_1$, $R_1 \equiv \text{Ref}(R_1)$, $\text{Irref}(R_1)$, $\text{Dis}(R_1, R_2)$, $\text{Fun}(R_1)$,
- $P_1 \equiv p_2$, $P_1 \equiv \neg P_2$, $P_1 \equiv P_2 \cup P_2$, $P_1 \equiv P_2 \cap P_2$, $P_1 \equiv P_2 \cup C_1$, $P_1 \equiv P_2 \cap C_1$,
  $P_1 \equiv t_1 \equiv \neg t_2$, $t_1 \equiv t_2 \equiv t_3$, $t_1 \equiv \{e_d\}$,
- $a : C_1$, $\langle a, b \rangle : R_1$, $(a, b) \equiv \neg R_1$, $a = b$, $a \neq b$, $e_d : t_1$, $(a, e_d) : P_1$, $(a, e_d) : \neg P_1$.

In order to define the $4LQS^R$-formula $\psi_\mathcal{K}$, we shall make use of a mapping $\tau$ from the $DL^4$-statements (and their conjunctions) listed above into $4LQS^R$-formulas. To prepare for the definition of $\tau$, we map injectively individuals $a$ and constants $e_d \in N_C(d)$ into level 0 variables $x_a$ and $x_{e_d}$, the constant concepts $\top$ and $\bot$, datatype terms $t$, and concept terms $C$ into level 1 variables $X^1_U$, $X^1_L$, $X^1_T$, $X^1_C$, respectively, and the universal relation on individuals $U$, abstract role terms $R$, and concrete role terms $P$ into level 3 variables $X^3_U$, $X^3_R$, and $X^3_P$, respectively.

Then the mapping $\tau$ is defined as follows:

---

2 The use of level 3 variables to model abstract and concrete role terms is motivated by the fact that their elements, that is ordered pairs $(x, y)$ are encoded in Kuratowski’s style as $\{(x), \{x, y\}\}$, namely as collections of sets of objects. Variables of level 2 are used in the formulae $\psi_\mathcal{K}$ and $\psi_\mathcal{K}$ of the construction to model the fact that level 3 variables representing role terms are binary relations.
\[ \tau(C_1 \equiv \top) = \text{def} (\forall z)(z \in X_{\text{def}}^1 \leftrightarrow z \in X^1) \),
\[ \tau(C_1 \equiv \neg C_2) = \text{def} (\forall z)(z \in X_{\text{def}}^1 \leftrightarrow \neg(z \in X_{\text{def}}^2)) \),
\[ \tau(C_1 \equiv C_2 \sqcup C_3) = \text{def} (\forall z)(z \in X_{\text{def}}^1 \leftrightarrow (z \in X_{\text{def}}^2 \lor z \in X_{\text{def}}^3)) \),
\[ \tau(C_1 \equiv \{a\}) = \text{def} (\forall z)(z \in X_{\text{def}}^1 \leftrightarrow z = x_a) \),
\[ \tau(C_1 \sqsubseteq \forall \text{Ref}(C_1, C_2) = \text{def} (\forall z_1)(\forall z_2)(z_1 \in X_{\text{def}}^1 \rightarrow ((z_1, z_2) \in X^3_{\text{def}} \rightarrow z_2 \in X_{\text{def}}^2)) \),
\[ \tau(\exists \text{Ref}(C_1, C_2) = \text{def} (\forall z_1)(\forall z_2)(((z_1, z_2) \in X^3_{\text{def}} \land z_2 \in X_{\text{def}}^1) \rightarrow z_1 \in X_{\text{def}}^2) \),
\[ \tau(C_1 \equiv \exists \text{Ref}(a)) = \text{def} (\forall z)(z \in X_{\text{def}}^1 \leftrightarrow (z, x_a) \in X^3_{\text{def}}) \),
\[ \tau(C_1 \sqsubseteq \forall \text{Ref}(\forall \text{Ref}(C_1, C_2)) = \text{def} (\forall z)(\forall z_1) \ldots (\forall z_{n+1})(z \in X_{\text{def}}^1 \rightarrow )
\]
\[ (\bigwedge_{i=1}^{n+1}(z_i \in X_{\text{def}}^2 \land (z, z_i) \in X^3_{\text{def}}) \rightarrow \bigvee z_i = z_j) \),
\[ \tau(\forall \text{Ref}(C_1, C_2) = \text{def} (\forall z)(\forall z_1) \ldots (\forall z_n)(\bigwedge_{i=1}^{n}(z_i \in X_{\text{def}}^1 \land (z, z_i) \in X^3_{\text{def}}) \rightarrow \bigwedge_{i<j}z_i \neq z_j) \rightarrow z \in X_{\text{def}}^2) \),
\[ \tau(C_1 \equiv \forall \text{Ref}(P_1, t_1) = \text{def} (\forall z_1)(\forall z_2)(z_1 \in X_{\text{def}}^1 \rightarrow ((z_1, z_2) \in X^3_{\text{def}} \rightarrow z_2 \in X_{\text{def}}^1)) \),
\[ \tau(\exists \text{Ref}(P_1, t_1) = \text{def} (\forall z_1)(\forall z_2)(((z_1, z_2) \in X^3_{\text{def}} \land z_2 \in X_{\text{def}}^1) \rightarrow z_1 \in X_{\text{def}}^2) \),
\[ \tau(C_1 \equiv \exists \text{Ref}(t_1)) = \text{def} (\forall z)(z \in X_{\text{def}}^1 \leftrightarrow (z, x_{t_1}) \in X^3_{\text{def}}) \),
\[ \tau(C_1 \sqsubseteq \forall \text{Ref}(\exists \text{Ref}(P_1, t_1)) = \text{def} (\forall z)(\forall z_1) \ldots (\forall z_{n+1})(z \in X_{\text{def}}^1 \rightarrow )
\]
\[ (\bigwedge_{i=1}^{n+1}(z_i \in X_{\text{def}}^1 \land (z, z_i) \in X^3_{\text{def}}) \rightarrow \bigvee z_i = z_j) \),
\[ \tau(\forall \text{Ref}(t_1)) = \text{def} (\forall z)(\forall z_1)(\forall z_2)((z_1, z_2) \in X^3_{\text{def}} \land z_2 \in X_{\text{def}}^1 \land z_1 = z_2) \),
\[ \tau(t_1 \equiv U) = \text{def} (\forall z)(Z^2 \in X_{\text{def}}^1 \leftrightarrow Z^2 \in X_{\text{def}}^1) \),
\[ \tau(t_1 \equiv t_2) = \text{def} (\forall z)(\forall z_1)(\forall z_2)((z_1, z_2) \in X^3_{\text{def}} \leftrightarrow ((z_1, z_2) \in X_{\text{def}}^1 \land z_2 \in X_{\text{def}}^2)) \),
\[ \tau(t_1 \equiv R_{t_1} \sqcup t_2) = \text{def} (\forall z)(\forall z_1)(\forall z_2)((z_1, z_2) \in X^3_{\text{def}} \leftrightarrow (Z^2 \in X_{\text{def}}^1 \lor Z^2 \in X_{\text{def}}^2)) \),
\[ \tau(t_1 \equiv R_{t_1} \sqcap t_2) = \text{def} (\forall z)(\forall z_1)(\forall z_2)((z_1, z_2) \in X^3_{\text{def}} \leftrightarrow (Z^2 \in X_{\text{def}}^1 \land Z^2 \in X_{\text{def}}^2)) \),
\[ \tau(t_1 \equiv \neg t_2) = \text{def} (\forall z)(\forall z_1)(\forall z_2)((z_1, z_2) \in X^3_{\text{def}} \leftrightarrow (z_2 \in X_{\text{def}}^1)) \),
\[ \tau(t_1 \equiv t_2 \lor t_3) = \text{def} (\forall z)(\forall z_1)(\forall z_2)((z_1, z_2) \in X^3_{\text{def}} \leftrightarrow (z_1 \in X_{\text{def}}^1 \lor z \in X_{\text{def}}^3)) \),
\[ \tau(t_1 \equiv t_2 \cap t_3) = \text{def} (\forall z)(\forall z_1)(\forall z_2)((z_1, z_2) \in X^3_{\text{def}} \leftrightarrow (z_1 \in X_{\text{def}}^1 \land z \in X_{\text{def}}^3)) \),
\[ \tau(t_1 \equiv \{e_d\}) = \text{def} (\forall z)(z \in X_{\text{def}}^1 \leftrightarrow z = x_{e_d}) \),
\[ \tau(a : C_1) = \text{def} x_a \in X_{\text{def}}^1 \),
\[ \tau((a, b) : R_1) = \text{def} (x_a, x_b) \in X^3_{\text{def}} \).
\[ \tau((a, b) : \neg R_1) = \text{Def} \neg(\langle x_a, x_b \rangle \in X^3_{R_1}), \]
\[ \tau(a = b) = \text{Def} \langle x_a, x_b \rangle, \]
\[ \tau(a \neq b) = \text{Def} \neg(\langle x_a, x_b \rangle), \]
\[ \tau(e_d : t_1) = \text{Def} \langle x_{e_d}, x_{t_1} \rangle \in X^1_{t_1}, \]
\[ \tau((a, e_d) : P_1) = \text{Def} \langle x_a, x_{e_d} \rangle \in X^3_{P_1}, \]
\[ \tau((a, e_d) : \neg P_1) = \text{Def} \neg(\langle x_a, x_{e_d} \rangle \in X^3_{P_1}), \]
\[ \tau(a \land \beta) = \text{Def} \tau(a) \land \tau(\beta). \]

Let \( K \) be our \( D\mathcal{L}^D_{\mathbb{P}}\)-knowledge base, and let \( \text{cpt}_K, \text{ari}_K, \text{crl}_K, \) and \( \text{ind}_K \) be, respectively, the sets of concept, of abstract role, of concrete role, and of individual names in \( K \). Moreover, let \( N^K_D \subseteq N_D \) be the set of datatypes in \( K \), \( N^K_F \) a restriction of \( N_F \) assigning to every \( d \in N^K_D \) the set \( N^K_F(d) \) of facets in \( N_F(d) \) and in \( K \). Analogously, let \( N^K_C \) be a restriction of the function \( N_C \) associating to every \( d \in N^K_D \) the set \( N^K_C(d) \) of constants contained in \( N_C(d) \) and in \( K \). Finally, for every datatype \( d \in N^K_D \), let \( \mathbf{bf}^D_{\mathbb{P}}(d) \) be the set of facet expressions for \( d \) occurring in \( K \) and not in \( N_F(d) \cup \{ \top^d, \bot^d \} \). We define the 4LQSR-formula \( \varphi_K \) expressing the consistency of \( K \) as follows:

\[
\varphi_K = \text{Def} \bigwedge_{i=1}^{12} \psi_i \land \bigwedge_{H \in K} \tau(H),
\]

where

- \( \psi_1 = \text{Def} \left( \forall z (z \in X^1 \leftrightarrow \neg(z \in X^1_D)) \land \left( \forall z (z \in X^1 \lor z \in X^1_D) \land \neg(\forall z) \land \neg(\forall z) \land \neg(z \in X^1_D) \right) \right) \)
- \( \psi_2 = \text{Def} \left( \left( \forall z \right)(z \in X^1 \leftrightarrow z \in X^1_D) \land \left( \forall z \right)\neg(z \in X^1_D) \right) \)
- \( \psi_3 = \text{Def} \bigwedge_{A \in \text{cpt}_K} \left( \forall z \right)(z \in X^1 \rightarrow z \in X^1_D) \)
- \( \psi_4 = \text{Def} \left( \bigwedge_{d \in N^K_D} \left( \forall z \right)(z \in X^1 \rightarrow z \in X^1_D) \land \neg(z \in X^1_D) \right) \)
- \( \psi_5 = \text{Def} \bigwedge_{d \in N^K_D} \left( \left( \forall z \right)(z \in X^1 \leftrightarrow z \in X^1_D) \land \neg(z \in X^1_D) \right) \)
- \( \psi_6 = \text{Def} \bigwedge_{d \in N^K_D} \bigwedge_{a \in \text{ind}_K} \left( \forall z \right)(z \in X^1_D \rightarrow z \in X^1_D) \)
- \( \psi_7 = \text{Def} \left( \left( \forall z \right)(z \in X^1_D \leftrightarrow z \in X^1_D) \right) \)
- \( \psi_8 = \text{Def} \left( \forall z \right)(Z^2 \in X^2_D \rightarrow \neg(\forall z)(Z^2) \land (\forall z)(Z^2) \land (\forall z)(Z^2) \) \)
- \( \psi_9 = \text{Def} \left( \forall z \right)(Z^2 \in X^2_D \rightarrow \neg(\forall z)(Z^2) \land (\forall z)(Z^2) \land (\forall z)(Z^2) \) \)
- \( \psi_{10} = \text{Def} \left( \forall z \right)(Z^2 \in X^2_D \rightarrow \neg(\forall z)(Z^2) \land (\forall z)(Z^2) \land (\forall z)(Z^2) \) \)
- \( \psi_{11} = \text{Def} \left( \forall z \right)(z \in X^1 \leftrightarrow z \in X^1_D) \)
- \( \psi_{12} = \text{Def} \left( \forall z \right)(z \in X^1 \leftrightarrow z \in X^1_D) \)

\[ \land \bigwedge_{a \in \text{ind}_K} \left( \forall z \right)(z \in X^1 \leftrightarrow z \in X^1_D) \]
- $\psi_{12} \overset{\text{def}}{=} \bigwedge_{d \in N_D^K} \bigwedge_{\psi_d \in \mathcal{B}_D(d)} (\forall z)(z \in X^1_{\psi_d} \leftrightarrow z \in \sigma(X^1_{\psi_d}))$,

with $\sigma$ the transformation function from 4LQS$^R$-variables of level 1 to 4LQS$^R$-formulae recursively defined, for $d \in N_D^K$, by

$$\sigma(X^1_{\psi_d}) = \begin{cases} X^1_{\psi_d} & \text{if } \psi_d \in N_D^K(d) \cup \{\top, \bot\} \\ \neg \sigma(X^1_{\psi_d}) & \text{if } \psi_d = \neg \chi_d \\ \sigma(X^1_{\psi_d}) \wedge \sigma(X^1_{\psi_d}) & \text{if } \psi_d = \chi_d \wedge \varphi_d \\ \sigma(X^1_{\psi_d}) \vee \sigma(X^1_{\psi_d}) & \text{if } \psi_d = \chi_d \vee \varphi_d. \end{cases}$$

In the above formulae, the variable $X^1_i$ denotes the set of individuals $I$, $X^1_d$ a datatype $d \in N_D^K$, $X^1_D$ a superset of the union of datatypes in $N_D^K$, $X^1_{\top_d}$ and $X^1_{\bot_d}$ the constants $\top_d$ and $\bot_d$, and $X^1_{f_d}$, $X^1_{\psi_d}$ a facet $f_d$ and a facet expression $\psi_d$, for $d \in N_D^K$, respectively. In addition, $X^1_A$, $X^3_R$, $X^3_T$ denote a concept name $A$, an abstract role name $R$, and a concrete role name $T$ occurring in $\mathcal{K}$, respectively. Finally, $X^1_{\{e_1, \ldots, e_n\}}$ denotes a data range $\{e_1, \ldots, e_n\}$ occurring in $\mathcal{K}$, and $X^1_{\{a_1, \ldots, a_n\}}$ a finite set $\{a_1, \ldots, a_n\}$ of nominals in $\mathcal{K}$.

Clearly, the constraints $\psi_1, \psi_2$ have been introduced to guarantee that each model of $\varphi_{\mathcal{K}}$ can be easily transformed into a $\mathcal{DL}^4_{D}$-interpretation.

Next we show that the consistency problem for $\mathcal{K}$ is equivalent to the satisfiability problem for $\varphi_{\mathcal{K}}$.

Let us first assume that $\varphi_{\mathcal{K}}$ is satisfiable. It is not hard to see that $\varphi_{\mathcal{K}}$ is satisfied by a 4LQS$^R$-model of the form $\mathcal{M} = (D_1 \cup D_2, M)$, where:

- $D_1$ and $D_2$ are disjoint nonempty sets and $\bigcup_{d \in N_D^K} d \subseteq D_2$,
- $M X^1_I = \overset{\text{def}}{=} D_1$, $M X^1_D = \overset{\text{def}}{=} D_2$,
- $M X^1_d = \overset{\text{def}}{=} d\mathcal{D}$, for every $d \in N_D^K$,
- $M X^1_{f_d} = \overset{\text{def}}{=} f_d\mathcal{D}$, for every $f_d \in N_D^K(d)$, with $d \in N_D^K$.

Exploiting the fact that $\mathcal{M}$ satisfies the constraints $\psi_1, \psi_2$, it is then possible to define a $\mathcal{DL}^4_{D}$-interpretation $\mathcal{I}_\mathcal{M} = (\Delta^4, \Delta_D, \mathcal{I})$, by putting $\mathcal{I}^4 = \overset{\text{def}}{=} M X^4_I$, $\Delta_D = \overset{\text{def}}{=} M X^4_D$, $\mathcal{I}^4 = \overset{\text{def}}{=} M X^4_A$, for every concept name $A \in \text{cpt}_{\mathcal{K}}$, $S^4 = \overset{\text{def}}{=} M X^4_S$, for every abstract role name $S \in \text{ar}_{\mathcal{K}}$, $T^4 = \overset{\text{def}}{=} M X^4_T$, for every concrete role name $T \in \text{cr}_{\mathcal{K}}$, and $a^4 = \overset{\text{def}}{=} M x_a$, for every individual $a \in \text{ind}_{\mathcal{K}}$.

Since $\mathcal{M} \models \Delta^4$, $\mathcal{I}_\mathcal{M}$ satisfies $\tau(H)$ and, as can be easily checked, $\mathcal{I}_\mathcal{M} \models D H$ if and only if $\mathcal{M} \models \tau(H)$, for every statement $H \in \mathcal{K}$, we plainly have $\mathcal{I}_\mathcal{M} \models D H$, namely $\mathcal{K}$ is consistent, as we wished to prove.

Conversely, let $\mathcal{K}$ be a consistent $\mathcal{DL}^4_{D}$-knowledge base. Then, there is a $\mathcal{DL}^4_{D}$-interpretation $\mathcal{I} = (\Delta^4, \Delta_D, \mathcal{I})$ such that $\mathcal{I} \models D \mathcal{K}$. We show how to construct, out of the datatypemap $\mathcal{D}$ and the $\mathcal{DL}^4_{D}$-interpretation $\mathcal{I}$, a 4LQS$^R$-interpretation $\mathcal{M}_{\mathcal{I}, D} = (D_{\mathcal{I}, D}, M_{\mathcal{I}, D})$ which satisfies $\varphi_{\mathcal{K}}$. Let us put $D_{\mathcal{I}, D} = \overset{\text{def}}{=} \Delta^4 \cup \Delta_D$ and define $M_{\mathcal{I}, D}$ by putting $M_{\mathcal{I}, D} X^1_I = \overset{\text{def}}{=} \Delta^4$, $M_{\mathcal{I}, D} X^1_D = \overset{\text{def}}{=} \Delta_D$, $M_{\mathcal{I}, D} X^3_I = \overset{\text{def}}{=} U^4$, $M_{\mathcal{I}, D} X^3_R = \overset{\text{def}}{=} d r^D$, for every variable $X^1_{f_d}$ in $\varphi$ denoting a data range $dr$ occurring in $\mathcal{K}$, $M_{\mathcal{I}, D} X^3_T = \overset{\text{def}}{=} A^4$, for every $X^3_T$ in $\varphi$ denoting a concept name in $\mathcal{K}$, and $M_{\mathcal{I}, D} X^3_S = \overset{\text{def}}{=} S^4$, for every $X^3_S$ in $\varphi$ denoting an abstract role name in
\( K \). Variables \( X \), denoting concrete role names, and variables \( x, x_e \), denoting individuals and datatype constants, respectively, are interpreted in a similar way. From the definitions of \( D \) and \( I \), it follows easily that \( M_{I,D} \) satisfies the formulae \( \psi_1-\psi_{12} \) and \( \tau(H) \), for every statement \( H \in K \), and, therefore, that \( M_{I,D} \) is a model for \( \varphi_K \).

Some considerations on the expressive power of the logic \( DL^4_{D} \) are in order. Despite \( DL^4_{D} \) allows one to express existential quantification and at-least number restriction (resp., universal quantification and at-most number restriction) only on the left- (resp., right-) hand side of inclusion axioms, it is more liberal than \( SROIQ(D) \) in the construction of role inclusion axioms since the roles involved are not required to be subject to any ordering relationship. For example, the role hierarchy \( \{RS \sqsubseteq S, RT \sqsubseteq R, VT \sqsubseteq T, VS \sqsubseteq V\} \) presented in [16] and not expressible in \( SROIQ(D) \) is admitted by the language of \( DL^4_{D} \). Moreover, the notion of simple role is not needed in the definition of role inclusion axioms and of axioms involving number restrictions. In addition, Boolean operators on roles are admitted and can be introduced in inclusion axioms such as, for instance, \( R_1 \sqsubseteq R_2 \cap R_3 \) and \( R_1 \sqsubseteq \neg R_2 \sqcup R_3 \). Finally, \( DL^4_{D} \) treats derived datatypes by admitting datatype terms constructed from data ranges by means of a finite number of applications of the Boolean operators. Basic and derived datatypes can be used inside inclusion axioms involving concrete roles.

Remark 1. For a fixed positive integer \( h \), a \( DL^4_{D} \)-knowledge base \( K \) is said to be \( h \)-restricted if an atom of any of the forms \( R_1 \ldots R_n \sqsubseteq R, R, C_1 \sqsubseteq C_2, \geq n_2 R, C_1 \sqsubseteq C_2, \geq n_3 P.t_1 \sqsubseteq t_2, C_1 \sqsubseteq \leq n_4 R, C_2, t_1 \sqsubseteq \leq n_5 P.t_2 \) occurs in \( K \) only if \( n_1, n_2, n_3, n_4, n_5 \leq h \).

It turns out that by using the same function \( \tau \) introduced in the proof of Theorem 1 and some additional constraints, the consistency problem for a \( h \)-restricted \( DL^4_{D} \)-knowledge base \( K \) can be expressed by a formula \( \varphi'_K \) such that

(i) \( \varphi'_K \) belongs to the sublanguage \( (4LQS^R)^h \) of \( 4LQS^R \), whose satisfiability problem is \( NP \)-complete (see [8] for details), and

(ii) the size of \( \varphi'_K \) is polynomially related to that of \( K \).

From (i) and (ii) above, and from \( NP \)-completeness of the satisfiability problem for propositional logic, it follows immediately that the consistency problem for \( h \)-restricted \( DL^4_{D} \)-knowledge bases is \( NP \)-complete.

In practice, \( h \)-restricted \( DL^4_{D} \)-knowledge bases are quite expressive: for instance, in [10] we have shown that the ontology \( Ontoceramic \), for ceramics classification, is representable in \( (4LQS^R)^3 \) and, much in the same way, it can be shown that it is representable as a 3-restricted \( DL^4_{D} \)-knowledge base.

3.1 Translating SWRL-rules into \( 4LQS^R \)-formulae

The possibility of extending ontologies with rules has become a fundamental requirement to increase the expressiveness and the reasoning power of OWL knowledge bases. In a general sense, a rule is any sentence stating that if a set of premises is satisfied in a given model, then a certain conclusion must be satisfied
in the same model. Although OWL is provided with several sorts of conditionals, these are, however, very constrained. Moreover, it is not possible to mix directly classes (concepts) and properties (roles) and include non-monotonic reasoning such as negation as failure.\(^{3}\) Such considerations led to the definition of SWRL [18], a rule language combining OWL with the Unary/Binary Datalog fragment of the Rule Markup Language. SWRL allows users to write rules containing OWL constructs providing more reasoning capabilities than OWL alone.

An SWRL-rule \(r\) has the form \((\forall x_1, \ldots, x_n)(B \Rightarrow H)\), where:

- \(B\) (the body of \(r\)) and \(H\) (the head of \(r\)) are conjunctions of atoms of the following types: \(x \in C\), \(y \in t\), \(\langle x, y \rangle \in R\), \(\langle x, y \rangle \in T\), \(x = y\), \(x \neq y\), with \(C\) a concept name, \(t\) a datatype, \(R\) an abstract role name, \(T\) a concrete role name, and \(x, y\) either individuals or variables (in the specific cases of atoms of the forms \(y \in t\) and \(\langle x, y \rangle \in T\), \(y\) can be either a datatype constant or a variable), and
- \(\text{Var}(H) \subseteq \text{Var}(B) = \{x_1, \ldots, x_n\}\), where \(\text{Var}(H)\) and \(\text{Var}(B)\) are the sets of variables occurring in \(H\) and in \(B\), respectively.

In Table 3 we give some examples showing how SWRL-rules can be expressed by \(4LQS\)\(_{\text{R}}\)-formulae. For space reasons we do not provide here a formal translation function. However, it is not hard to see that it could be constructed by modifying the map \(\tau\) introduced in the proof of Theorem 1.

| Type of Rule | Rule |
|--------------|------|
| SWRL-rule | \(\text{hasParent}(X, Y), \text{hasBrother}(Y, Z) : \neg \text{hasUncle}(X, Z)\). |
| \(4LQS\)\(_{\text{R}}\)-rule | \((\forall x)(\forall y)(\forall z)(\langle x, y \rangle \in X^3_{\text{hasParent}} \wedge \langle y, z \rangle \in X^3_{\text{hasBrother}} \rightarrow \langle x, z \rangle \in X^3_{\text{hasUncle}})\). |
| SWRL-rule | \(\text{Location}(X), \text{Trauma}(Y), \text{isLocationOf}(X, Y), \text{isPartOf}(X, Z)\). |
| \(4LQS\)\(_{\text{R}}\)-rule | \((\forall x)(\forall y)(\forall z)(x \in X^1_{\text{Location}} \wedge y \in X^1_{\text{Trauma}} \wedge \langle x, z \rangle \in X^3_{\text{isPartOf}} \rightarrow \langle z, y \rangle \in X^3_{\text{isLocationOf}})\). |
| SWRL-rule | \(\text{Person}(X), \text{hasAge}(X, Y), (Y \geq 18) : \neg \text{Adult}(X)\). |
| \(4LQS\)\(_{\text{R}}\)-rule | \((\forall x)(\forall y)(x \in X^1_{\text{Person}} \wedge \langle x, y \rangle \in X^3_{\text{hasAge}} \wedge y \in X^3_{\geq 18} \rightarrow x \in X^1_{\text{Adult}})\). |
| SWRL-rule | \(\text{Region}(Y), \text{hasLocation}(X, Y) : \neg \text{hasRegion}(X, Y)\). |
| \(4LQS\)\(_{\text{R}}\)-rule | \((\forall x)(\forall y)(y \in X^3_{\text{Region}} \wedge \langle x, y \rangle \in X^3_{\text{hasLocation}} \rightarrow \langle x, y \rangle \in X^3_{\text{hasRegion}})\). |

Table 3: Examples of rule translation.

4 Conclusions and Future Work

We have introduced the description logic \(DL^4_{D}\) which admits, among other features, datatype reasoning, role chain axioms without regularity conditions on roles, \(\text{min}\) (resp., \(\text{max}\)) cardinality construct on the left-hand (resp., right-hand) side of inclusion axioms extended to non-simple roles, constructs of full negation,

\(^{3}\) We recall that a logic is non-monotonic if some conclusions can be invalidated when more knowledge is added.
union, and intersection for abstract roles. As discussed at the end of Section 3, the logic $\mathcal{DL}^4_D$ turns out to be quite expressive, if compared with $\mathcal{SROIQ(D)}$, the logic underpinning the Web Ontology Language OWL. However, although $\mathcal{DL}^4_D$ is endowed with features not supported by $\mathcal{SROIQ(D)}$, it is not a proper extension of it, as $\mathcal{DL}^4_D$ admits existential (resp., universal) quantification only on the left-hand (resp., right-hand) side of inclusion axioms.

Through a suitable translation process, we have then shown that the consistency problem for $\mathcal{DL}^4_D$-knowledge bases can be effectively reduced to the satisfiability problem for the decidable fragment of set theory $4LQS^R$. Moreover, in the restricted case in which a $\mathcal{DL}^4_D$-knowledge base $\mathcal{K}$ can involve only role chain axioms $R_1 \ldots R_m \sqsubseteq R$ and inclusion axioms $\geq_n R.C_1 \subseteq C_2$, $C_1 \subseteq \leq_p R.C_2$ such that $m$, $n$, and $p$ do not exceed a fixed constant (hence independent of the size of $\mathcal{K}$), we have shown that the consistency problem is NP-complete, as it can be polynomially reduced to the satisfiability problem for a subfragment of $4LQS^R$ which has an NP-complete decision problem. Finally, we have also translated SWRL-rules into the $4LQS^R$ language.

We plan to introduce the constructs of union and intersection of concrete roles and to extend our results to include also datatype groups (here we have considered only a simple form of datatypes) and to admit Boolean operators on concrete roles by defining a suitable strategy of datatype checking. Moreover, we intend to extend the fragment $4LQS^R$ with metamodelling capabilities [19–21], so as to make it possible to define concepts containing other concepts and roles (i.e., meta-concepts) and relationships between concepts or between roles (i.e., meta-roles). Finally, we intend to implement efficient reasoners for suitable fragments of $4LQS^R$.

References

1. J. T. Schwartz, D. Cantone, and E. G. Omodeo, Computational Logic and Set Theory: Applying Formalized Logic to Analysis. Texts in Computer Science, Springer-Verlag New York, Inc., 2011.
2. D. Cantone, A. Ferro, and E. G. Omodeo, Computable set theory. No. 6 in International Series of Monographs on Computer Science, Oxford Science Publications, Oxford, UK: Clarendon Press, 1989.
3. D. Cantone, E. Omodeo, and A. Policriti, Set theory for computing: from decision procedures to declarative programming with sets. Monographs in Computer Science, New York, NY, USA: Springer-Verlag, 2001.
4. D. Cantone, C. Longo, and M. Nicolosi Asmundo, “A decision procedure for a two-sorted extension of multi-level syllogistic with the Cartesian product and some map constructs,” in Proceedings of the 25th Italian Conference on Computational Logic (CILC 2010), Rende, Italy, July 7-9, 2010 (W. Faber and N. Leone, eds.), vol. 598, pp. 1–18 (paper 11), CEUR Workshop Proceedings, ISSN 1613-0073, June 2010.
5. D. Cantone, C. Longo, and M. Nicolosi Asmundo, “A decidable quantified fragment of set theory involving ordered pairs with applications to description logics,” in Computer Science Logic, 20th Annual Conference of the EACSL, CSL 2011, September 12-15, 2011, Bergen, Norway, Proceedings, pp. 129–143, 2011.
6. D. Cantone and C. Longo, “A decidable two-sorted quantified fragment of set theory with ordered pairs and some undecidable extensions,” Theor. Comput. Sci., vol. 560, pp. 307–325, 2014.
7. D. Cantone, C. Longo, and A. Pisasale, “Comparing description logics with multilevel syllogistics: the description logic $DL(\text{MLSS}_{\Delta,m})$,” in 6th Workshop on Semantic Web Applications and Perspectives (Bressanone, Italy, Sep. 21-22, 2010) (P. Traverso, ed.), pp. 1–13, 2010.
8. D. Cantone and M. Nicolosi Asmundo, “On the satisfiability problem for a 4-level quantified syllogistic and some applications to modal logic,” Fundamenta Informaticae, vol. 124, no. 4, pp. 427–448, 2013.
9. D. Cantone, M. Nicolosi-Asmundo, D. F. Santamaria, and F. Trapani, “An ontology for ceramics cataloguing,” in Computer Applications and Quantitative Methods in Archaeology (CAA), 2015.
10. D. F. Santamaria, A Set-Theoretical Representation for OWL 2 Profiles. LAP Lambert Academic Publishing, ISBN 978-3-659-68797-6, 2015.
11. N. Dershowitz and J.-P. Jouannaud, “Rewrite systems,” in Handbook of Theoretical Computer Science (Vol. B) (J. van Leeuwen, ed.), pp. 243–320, Cambridge, MA, USA: MIT Press, 1990.
12. A. Formisano, E. G. Omodeo, and A. Policriti, “Three-variable statements of set-pairing,” Theor. Comput. Sci., vol. 322, no. 1, pp. 147–173, 2004.
13. F. Baader, I. Horrocks, and U. Sattler, “Description logics as ontology languages for the semantic web,” in Festschrift in honor of Jörg Siekmann, Lecture Notes in Artificial Intelligence, pp. 228–248, Springer-Verlag, 2003.
14. I. Horrocks and U. Sattler, “Ontology reasoning in the SHOQ(D) description logic,” in Proc. of IJCAI 2001, pp. 199–204, 2001.
15. B. Motik and I. Horrocks, “Owl datatypes: Design and implementation,” in Proc. of the 7th Int. Semantic Web Conference (ISWC 2008), vol. 5318 of LNCS, pp. 307–322, Springer, October 26–30 2008.
16. I. Horrocks, O. Kutz, and U. Sattler, “The even more irresistible SROIQ,” in Proc. of the 10th Int. Conf. on Principles of Knowledge Representation and Reasoning (P. Doherty, J. Mylopoulos, and C. A. Welty, eds.), pp. 57–67, AAAI Press, 2006.
17. Y. Kazakov, “RIQ and SROIQ are harder than SHOIQ,” in Proc. of the 11th International Conference, KR 2008, Sydney, Australia, Sept. 16-19, 2008 (G. Brewka and J. Lang, eds.), pp. 274–284, 2008.
18. W. W. W. C. (W3C), “SWRL: A semantic web rule language.” http://www.w3.org/Submission/SWRL/.
19. B. Motik, “On the properties of metamodeling in owl,” in In 4th Int. Semantic Web Conf. (ISWC 2005), pp. 548–562, 2005.
20. B. Glimm, S. Rudolph, and V. J., “Integrated metamodeling and diagnosis in OWL 2,” in The Semantic Web - ISWC 2010 - 9th Int. Semantic Web Conf., ISWC 2010, Shanghai, China, November 7-11, 2010, Revised Selected Papers, Part I, pp. 257–272, 2010.
21. M. Homola, J. Kluka, V. Svátek, and M. Vacura, “Typed higher-order variant of SROIQ - why not?,” in Proc. of the 27th Int. Workshop on Description Logics, Vienna, Austria, July 17-20, 2014., vol. 1193, pp. 567–578, CEUR Workshop Proceedings, ISSN 1613-0073, 2014.