Supersymmetry on Graphs and Networks

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Abstract

We show that graphs, networks and other related discrete model systems carry a natural supersymmetric structure, which, apart from its conceptual importance as to possible physical applications, allows to derive a series of spectral properties for a class of graph operators which typically encode relevant graph characteristics.
1 Introduction

In the following we show that graphs and networks support in a very natural way a supersymmetric structure. Our starting point is a class of geometrically relevant graph-operators acting on the direct sum, $\mathcal{H}$, of the vertex-Hilbert space, $\mathcal{H}_0$ and the edge-Hilbert space, $\mathcal{H}_1$. As to these technical prerequisites cf. the analysis being made in [1] and further references given there.

Of particular importance is a graph-Dirac operator, $D$, introduced there, which maps $\mathcal{H}_0$ into $\mathcal{H}_1$ and vice versa. Furthermore we have a natural Laplace operator on $\mathcal{H}$ which together with $D$ and another supercharge forms a closed $(N = 2)$-superalgebra. This abstract susy-structure, if concretely represented by our graph-operators, allows, among other things, to derive a series of useful spectral properties of these operators.

The natural existence of this susy structure on graphs (and related models) may also be of some relevance in a wider context. We have been promoting a discrete network approach to quantum space-time physics in recent years (see for example [15] or [16] and further references given there) which is assumed to underly our more macroscopic continuum physics on the Planck-scale. Our present analysis shows that these discrete model systems, perhaps contrary to naive wisdom, are in fact quite rich as to their structural properties.

We conclude this introduction with the remark that such technical properties of graphs have also been mentioned elsewhere in a however different context and using a different formalism (cf. [17]). So we should add the remark that there may exist papers in other fields of research being related to our work but which we presently are not aware of.

2 Some Concepts from Algebraic Graph Theory

To set the stage, we briefly introduce some concepts employed in algebraic graph theory but in a form particularly adapted to our own framework (see also [1]; as to the wider context cf. [2] to [6] and further references given there). The graph, $G$, consists of a countable set of vertices (or nodes), $V$, with $x_i$ the labelled vertices, and an edge set, $E$, the directed edges, pointing from $x_i$ to $x_j$, denoted by $d_{ij}$. We assume (for convenience) the graph to be simple, that is, an edge, $d_{ij}$, can be associated with the ordered pair of vertices, $(x_i, x_j)$ with $i \neq j$.

So-called unoriented graphs with the edges consisting of unordered pairs, $\{x_i, x_j\}$, can be subsumed in the above framework in two different ways, each of which having a certain advantage of its own. On the one hand, we can give the graph an arbitrary orientation, that is we associate to each undirected edge, $\{x_i, x_j\}$, one of the two possible choices, $(x_i, x_j)$ or $(x_j, x_i)$. It turns out that most of the concepts and calculations do not depend on the particular choice (see below). On the other hand, we can associate an unoriented but orientable graph
with a directed graph so that each edge occurs twice, that is if $d_{ij}$ belongs to $E$, $d_{ji}$ also belongs to $E$.

For reasons of simplicity we assume our graph to be *locally finite*, that is, each vertex is only incident with a finite number of edges. A slightly stronger assumption is to assume this *vertex degree* to be globally finite over the vertex set. For a directed graph we then have ingoing edges and outgoing edges relative to a given node with the respective vertex degrees, $v_i^{\text{in}}$ and $v_i^{\text{out}}$ and the total degree $v_i = v_i^{\text{in}} + v_i^{\text{out}}$.

For such a graph we can introduce two Hilbert spaces, a vertex Hilbert space, $\mathcal{H}_0$, and an edge Hilbert space, $\mathcal{H}_1$, with orthonormal bases the set of vertices, $x_i$, and the set of directed edges, $d_{ij}$. This means, we introduce a formal scalar product on $\mathcal{H}_0, \mathcal{H}_1$ respectively with

$$ (x_i, x_j) = \delta_{ij} \quad , \quad (d_{ij}, d_{lm}) = \delta_{il} \delta_{jm} $$

and with vectors being the formal sums

$$ f = \sum_{i=1}^{\infty} f_i x_i \quad , \quad g = \sum_{i,j=1}^{\infty} g_{ij} d_{ij} \quad \text{with} \quad f_i, g_{ij} \in \mathbb{C} $$

with $\sum |f_i|^2 < \infty$ and $\sum |g_{ik}|^2 < \infty$.

Remark: We treat the vertices and edges as abstract basis elements (in a way similar to the *group algebra* of a group). One can of course consider the abstract vectors equally well as discrete functions over the vertex- or edge-set, respectively and the basis vectors as elementary indicator functions.

If we deal with an undirected but orientable graph, employing the second variant introduced above, we found it convenient (cf. [1]) to introduce the superposition

$$ b_{ij} := d_{ij} - d_{ji} = -b_{ji} $$

and relate it to an undirected but orientable edge.

## 3 Operators on Graphs

In [1] we introduced two operators, interpolating between $\mathcal{H}_0$ and $\mathcal{H}_1$. We define them on the basis vectors:

$$ d(x_i) := \sum_k d_{ki} - \sum_{k'} d_{ik'} $$

with the first sum running over the ingoing edges relative to $x_i$, the second sum running over the outgoing edges. In the case of a symmetric (or undirected graph;
version two) we have

\[ d(x_i) := \sum_k (d_{ki} - d_{ik}) = \sum_k b_{ki} \]  

(5)

This operator is closely related to a sort of non-commutative discrete differential calculus on graphs as we have

\[ df = \sum_{i,k} (f_k - f_i) d_{ik} \]  

(6)

A simple calculation shows that the adjoint, \( d^* \), acts on the basis vectors of \( \mathcal{H}_1 \) as follows:

\[ d^*(d_{ik}) = x_k - x_i \]  

(7)

In algebraic graph theory (finite graphs) the so-called incidence matrix, \( B \), is introduced, having the entry 1 if vertex \( x_i \) is the positive end of a certain (ingoing) edge, and having a \(-1\) if it is the negative end (outgoing edge) (see for example [3]). This matrix corresponds to our operator \( d^* \).

Another important operator is the adjacency matrix, \( A \), being a map from \( \mathcal{H}_0 \) to \( \mathcal{H}_0 \) and having (in ordinary graph theory of (un)oriented graphs) a \(+1\) at entry \((i,j)\) if \( x_i \) and \( x_j \) are connected by an edge. This matrix is a symmetric operator, \( a_{ij} = a_{ji} \). In our more general context (which includes however the ordinary situation as a special case) of directed graphs one can introduce the in-adjacency matrix, \( A^\text{in} \), and the out-adjacency matrix, \( A^\text{out} \), with \( A = A^\text{in} + A^\text{out} \). In our (operator)-notation they are given by

\[ A x_i = \sum_{k \sim i} \epsilon_{ki} x_k, \quad A^\text{in} x_i = \sum_{k \to i} x_k, \quad A^\text{out} x_i = \sum_{i \to k'} x_{k'} \]  

(8)

with \( \sim \) designating the unordered pair \( \{x_i, x_k\} \), \( k \to i \) the ordered pair \((k,i)\) and \( \epsilon_{k,i} \) is either one or two depending on the two possible cases of one directed edge between node \( x_i \) and node \( x_k \) or two directed edges, pointing in opposite directions.

These operators can be built up from more elementary operators (cf. [1]):

\[ d_1 x_i = \sum_k d_{ki}, \quad d_2 x_i = \sum_{k'} d_{ik'} \]  

(9)

\[ d_1^* d_{ik} = x_k, \quad d_2^* d_{ik} = x_i \]  

(10)

so that

\[ d = d_1 - d_2, \quad d^* = d_1^* - d_2^* \]  

(11)

\[ d_1^* d_1 x_i = v_i^{\text{in}} \cdot x_i, \quad d_2^* d_2 x_i = v_i^{\text{out}} \cdot x_i \]  

(12)
\[ d_1^*d_2 x_i = \sum_{i \to k'} x_{k'} , \quad d_2^*d_1 x_i = \sum_{k \to i} x_k \]  

where \( v_i^{in} \), \( v_i^{out} \) is the in-, out degree of vertex \( x_i \) respectively. We hence have

**Lemma 3.1** The in-, out-vertex degree matrices read

\[ V^{in} = d_1^*d_1 , \quad V^{out} = d_2^*d_2 \]  

The in-, out-adjacency matrices read

\[ A^{in} = d_2^*d_1 , \quad A^{out} = d_1^*d_2 \]  

\[ A = A^{in} + A^{out} \text{ is symmetric.} \]

**Proposition 3.2** The so-called graph Laplacian is the following positive operator

\[ -\Delta := d^*d = \left( V^{in} + V^{out} \right) - \left( A^{in} + A^{out} \right) = V - A \]  

The reason to call this operator a Laplacian stems from the observation that it acts like a second order partial difference operator on functions of \( H_0 \).

\[ -\Delta f = \sum_i f_i \left( v_i^{in} x_i + v_i^{out} x_i - \sum_{k \to i} x_k - \sum_{i \to k} x_k \right) \]  

and after a simple relabelling of indices

\[ -\Delta f = -\sum_i \left( \sum_{k \to i} f_k + \sum_{i \to k} f_k - v_i^{in} f_i - v_i^{out} f_i \right) x_i \]

\[ = -\sum_i \left( \sum_{k \to i} (f_k - f_i) + \sum_{i \to k} (f_k - f_i) \right) x_i = -\sum_i \left( \sum_{k \sim i} \epsilon_{ki} (f_k - f_i) \right) x_i \]  

which reduces to the ordinary expression in the undirected case.

Forming now the direct sum \( \mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \), we can introduce yet another important graph operator which closely entangles geometric and functional analytic properties of graphs (and similar structures); see \[ \mathbb{I} \].

**Definition 3.3** We define the graph Dirac operator as follows

\[ D : \mathcal{H} \to \mathcal{H} \text{ with } D := \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} , \quad H = \begin{pmatrix} \mathcal{H}_0 \\ \mathcal{H}_1 \end{pmatrix} \]  

**Observation 3.4**

\[ D^2 = D D = \begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \]  

with \( d^*d = -\Delta \).
The action of $dd^*$ on a basis vector $d_{ik}$ reads

$$dd^* d_{ik} = d(x_k - x_i) = \sum_{k'} d_{k'k} - \sum_{k''} d_{kk''} - \sum_l d_{li} + \sum_{l'} d_{ll'}$$  \(21\)

which, after some relabelling and introduction of the Kronecker delta function can be written as

$$dd^* d_{ik} = \sum_{m,j} (d_{mj}\delta_{jk} - d_{jm}\delta_{jk} - d_{mj}\delta_{ij} + d_{jm}\delta_{ij})$$  \(22\)

For a function $g = \sum g_{ik} d_{ik}$ we hence get

$$dd^* g = \sum_{l,m} \left( \sum_i g_{im} - g_{mi} - g_{il} + g_{li} \right) d_{lm}$$  \(23\)

In the simple case of the one-dimensional directed lattice, $\mathbb{Z}_1$, with directed edges only pointing from $x_i \to x_{i+1}$, we get a transparent expression:

$$dd^* \left( \sum_i g_{i,i+1} d_{i,i+1} \right) = -\sum_i (g_{i+1,i+2} + g_{i-1,i} - 2g_{i,i+1}) d_{i,i+1}$$  \(24\)

i.e., it represents also a discrete second derivative operator on the level of edges. For the directed lattice, $\mathbb{Z}_n$, it can be related to what is called the vector Laplacian in the continuum (cf. [7], p.575).

There is a pendant in the calculus of differential forms on general Riemannian manifolds where, with the help of the Hodge-star operation, we can construct a dual, $\delta$, to the ordinary exterior derivative. The generalized Laplacian then reads

$$-\Delta = \delta d + d\delta$$  \(25\)

with $\delta$ (modulo certain combinatorial prefactors) corresponding to our $d^*$ (see for example [8] or [9]).

### 4 Susy on Graphs

We introduce the following simple model of a supersymmetric algebra, consisting of two hermitean charges, $Q_1$, $Q_2$, and a Hamiltonian

$$H_S = Q_1^2 = Q_2^2 \quad \text{hence} \quad [H_S, Q_{1,2}] = 0$$  \(26\)

$$\{Q_1, Q_2\} = 0 \quad \text{hence} \quad \{Q_i, Q_j\} = 2H_S \cdot \delta_{ij}$$  \(27\)

Defining $Q_+, Q_-$ as

$$Q_+ := 2^{-1}(Q_1 + iQ_2), \quad Q_- := 2^{-1}(Q_1 - iQ_2)$$  \(28\)
implying
\[ Q_1 = Q_+ + Q_- , \quad Q_2 = -i(Q_+ - Q_-) \] (29)
we get
\[ Q_+^2 = Q_-^2 = 0 , \quad H_S = \{Q_+, Q_-\} , \quad [H_S, Q_\pm] = 0 \] (30)
That is, these three generators create a closed (N=2) susy-algebra.

We now make the following correspondence with our graph operators:
\[ H_S = D^2 = D D = \begin{pmatrix} d^* d & 0 \\ 0 & dd^* \end{pmatrix} \] (31)
\[ Q_+ = \begin{pmatrix} 0 & 0 \\ d & 0 \end{pmatrix} , \quad Q_- = \begin{pmatrix} 0 & d^* \\ 0 & 0 \end{pmatrix} \] (32)
yielding
\[ Q_1 = D = \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} , \quad Q_2 = \begin{pmatrix} 0 & i d^* \\ -i d & 0 \end{pmatrix} \] (33)
In other words, the charge \( Q_1 \) is our original Dirac operator. We can now check all the above abstract (anti) commutation relations and find that they are fulfilled by our representation in form of graph operators. We see that \( Q_+ \) is essentially a map from the subspace \( \mathcal{H}_0 \) to the subspace \( \mathcal{H}_1 \) and \( Q_- \) from \( \mathcal{H}_1 \) to \( \mathcal{H}_0 \) while \( Q_+ \) vanishes on \( \mathcal{H}_1 \), \( Q_- \) on \( \mathcal{H}_0 \). Therefore we can tentatively associate \( \mathcal{H}_0 \) with the bosonic and \( \mathcal{H}_1 \) with the fermionic subspace. Furthermore there exists a natural grading operator on \( \mathcal{H} \), given by
\[ \chi := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (34)
We have \( Q_2 = i Q_1 \cdot \chi \). Both \( Q_1 \) and \( Q_2 \) anticommute with \( \chi \) and are therefore called supercharges of an abstract supersymmetric quantum mechanics (cf. [10], sect. 8.3 or [11], [12] respectively). We note that many of the susy properties follow already from the existence of this abstract structure.

**Remark 4.1** Note that \( \chi \) is both selfadjoint and unitary with
\[ \chi = \chi^* = \chi^{-1} , \quad \chi^2 = 1 \] (35)
The projectors on \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) are given by
\[ P_0 = 1/2(1 + \chi) , \quad P_1 = 1/2(1 - \chi) \] (36)
respectively, as for ordinary continuum Dirac operators.
5 Some Graph-Spectral Properties following from Susy

We indicated already in [1] (see also [2]) that a variety of spectral properties on graphs are encoded in our Laplace or Dirac operator. For a finite, connected (for reasons of simplicity only) graph we have for example

Observation 5.1 The following operator kernel, range properties hold.

\[ \dim (\text{Rg } d^*) = n - 1, \quad \dim (\text{Ker } d^*) = \dim (\text{Rg } d^\perp) = \sum v_i^\text{in} - (n - 1) \geq 0 \]

(37)

\[ \dim (\text{Rg } d) = \dim (\text{Ker } d^*) = n - 1 \quad (38) \]

that is, \( \text{Rg } d^* \) and \( \text{Rg } d \) have the same dimension, \((n - 1)\), and the dimension of \( \text{Ker } d \) is one (\( n \) designates the order of the graph i.e. the number of vertices).

Note that \( \sum v_i^\text{in} = \sum v_i^\text{out} = \sum v_i/2 = \#(\text{edges}) \) is the dimension of \( \mathcal{H}_1 \) as each (directed) edge occurs as an ingoing edge for exactly one node.

To give an idea how these results can be proved, take for example the first statement. \( \text{Ker } d \) is spanned by the vector \( \sum x_i \) as each edge, occurring as an in-edge for, say, \( x_i \) occurs as an out-edge for some other \( x_j \), hence \( d(\sum x_i) \) vanishes. With \( \text{Rg } A^* = (\text{Ker } A)^\perp \) the result follows and the other results follow from simple vector space mathematics and properties of the adjoint (note for example that for finite dimensions \( \dim (\text{Ker } d) + \dim (\text{Rg } d) = n \)).

For general infinite graphs our susy structure allows to infer more interesting spectral results. Note that, to keep matters simple, we restricted ourselves to graphs with globally bounded node degree. Therefore all our operators are bounded and there are hence no operator domain problems. In the following the polar decomposition of an operator turns out to be useful. With \( A \) a closed operator from a Hilbert space, \( X_1 \), to a Hilbert space, \( X_2 \) we have the following representation

\[ A = S \cdot |A| \quad (39) \]

with \( |A| = (A^* A)^{1/2} \) a positive operator from \( X_1 \) to \( X_1 \), \( S \) a partial isometry, mapping \( \text{Rg } |A| \) isometrically onto \( \text{Rg } A \) (see for example [13]).

We write \( d = S|d| \) with \( |d| = (d^* d)^{1/2} \). We hence have \( d^* = |d| S^* \) and with \( dd^* = |d|^2 = S|d|^2 S^* \) and consequently \( |d^*| = S|d| S^* \) (the square root):

\[ d = S|d| = |d^*| S, \quad d^* = |d| S^* = S^* |d^*| \quad (40) \]

For our \( Q_1 := D \) we have

Lemma 5.2 The polar decomposition of \( Q_1 = D \) is

\[ Q_1 = \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} = \begin{pmatrix} 0 & S^* |d^*| \\ S|d| & 0 \end{pmatrix} = \begin{pmatrix} 0 & S^* \\ S & 0 \end{pmatrix} \cdot \begin{pmatrix} |d| & 0 \\ 0 & |d^*| \end{pmatrix} \quad (41) \]
We furthermore have

**Lemma 5.3**

\[ \text{Ker } Q_i = \text{Ker } d \oplus \text{Ker } d^* = \text{Ker } Q_i^2 \]  

(42)

with corresponding formulas holding for the respective orthogonal complements. For a finite connected graph we hence have (by observation 5.1)

\[ \text{Dim} \left( \text{Ker } Q_1, Q_2 \right) = \sum v_i^\text{in} - (n - 1) + 1 = \#(\text{edges}) - (n - 2) \]  

(43)

For the proof note that for e.g. a bounded \( \text{Ker}(A^*A) = \text{Ker}(A) \) as \( \text{Ker}(A^*) = \text{Rg}(A)^\perp \).

**Proposition 5.4** On \( (\text{Ker } d)^\perp \oplus (\text{Ker } d^*)^\perp \) we have

\[ \begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} = \begin{pmatrix} S^*dd^*S & 0 \\ 0 & Sd^*dS \end{pmatrix} \]  

(44)

That is, \( d^*d \) on \( \text{Rg } d^* \) is unitarily equivalent to \( dd^* \) on \( \text{Rg } d \). This is a generalisation to infinite dimensional spaces of a previous result concerning the dimension of the respective subspaces. It follows that the spectra of \( d^*d \) and \( dd^* \) coincide away from zero! We have

\[ d^*d f = Ef \Rightarrow dd^* S f = Ef \]  

(45)

Therefore the eigenvalue spectrum of \( H_S \) away from zero is at least twofold degenerate.

\[ H_S (f, g)^T = E (f, g)^T \Rightarrow d^*d f = Ef \text{ and } dd^* g = E g \]  

(46)

hence, \( (f, 0)^T \) and \( (0, g)^T \) are eigenvectors of \( H_S \) to the same eigenvalue and are at the same time eigenvectors of \( \chi \) to the eigenvalues \( \pm 1 \).

Furthermore, as \( H_S = Q_1^2 = Q_2^2 \) and \( \{Q_1, Q_2\} = 0 \), certain combinations of the above eigenvectors yield common eigenvectors of the pairs \( H_S, Q_i \).

**Proposition 5.5** The eigenvalues away from zero of \( H_S = Q_i^2 \) are at least twofold degenerate. With \( E > 0 \) being an eigenvalue of \( H_S = Q_1^2 = Q_2^2 \) with \( (f, g)^T \) the corresponding eigenvector of \( Q_1 \), the eigenvalue of \( Q_1 \) is \( +E^{1/2} \) or \( -E^{1/2} \) and, due to the anticommutation relation between \( Q_1 \) and \( Q_2 \), \( (f, g)^T \) is another eigenvector to \( Q_1 \) with eigenvalue \( -E^{1/2} \) or \( +E^{1/2} \). We thus see that both \( Q_1 \) and \( Q_2 \) have a symmetric eigenvalue spectrum.

We can make the result a little bit more explicit by making canonical choices.
Observation 5.6 With \((f, g)^T\) an eigenvector of \(Q_1\) to eigenvalue \(\lambda\) (and hence \((f, 0)^T, (0, g)^T\) “pure” eigenvectors of \(H_S\) to eigenvalue \(\lambda^2\)), \((f, -g)^T\) is an eigenvector to eigenvalue \(-\lambda\). Correspondingly, a straightforward calculation shows that \((if, g)^T\) is an eigenvector to \(Q_2\) with eigenvalue \(\lambda\) and \((if, -g)^T\) the eigenvector to eigenvalue \(-\lambda\). All these vectors are eigenvectors of \(H_S\) to eigenvalue \(E = \lambda^2\). However, only the pairs belonging to \(Q_1\) or \(Q_2\), respectively, are linearly independent. We see that \(Q_{1,2}\) necessarily mix the pure bosonic and fermionic eigenstates of \(H_S\) (see also sect. 2.3.3 of [10]).

Boilt down to the two operators \(d^*d\) and \(dd^*\) we have that with \(f\) being an eigenvector of \(d^*d\), \(df\) is an eigenvector of \(dd^*\) to the same eigenvalue with a corresponding result for \(dd^*, g\) and \(d^*g\) \((dd^*df = dd^*df = Edf)\). Hence, all eigenvectors of \(dd^*\) are of the form \(g = df, f\) an eigenvector of \(d^*d\), both belonging to the same eigenvalue (see also [11] sect. 5.2.3). Furthermore, the eigenvectors, \((f, g)^T\) of, for example, \(Q_1\) are characterized by the following symmetry property:

\[
d^* g = \lambda f, \quad df = \lambda g
\]  

We conclude from our preceding findings that the susy structure relates the spectral properties of \(d, d^*, d^*d, dd^*\) to each other. On the other hand it seems to say (at least as far as we can see) not much about the spectrum of, for example, \(d^*d\) as such.

6 The Zero Eigenspace

In supersymmetric quantum mechanics the zero eigenspace of \(H_S\) is particularly interesting and is associated with the notion of supersymmetry breaking (see for example [10]). Our above lemma 5.3 makes an explicit statement about the dimension for a finite connected graph. We see that the eigenspace is in many cases highly degenerate and always has a dimension bigger than zero. This can be seen as follows. Each connected graph contains a spanning tree (see e.g. [14]). A finite tree has \((n - 1)\) edges (see below). Hence each connected graph contains at least \((n - 1)\) edges so that

\[
\text{Dim}(\text{Ker}H_S) \geq 1
\]  

In case of a finite tree we have the following.

A finite tree of order \(n\) has \((n - 1)\) edges. This can most easily be seen by choosing a base vertex, \(x_0\), and then starting from the outer vertices. Each edge corresponds to exactly one vertex, ending with the base vertex, which does not correspond to an edge.

Lemma 6.1 For a finite tree the preceding formula reduces to

\[
\text{Ker} \ d^* = (n - 1) - (n - 1) = 0
\]
We infer that \( \text{Dim}(\text{Ker } H_S) = 1 \).

The situation is more complicated for infinite connected graphs. In this case the kernel of \( d \) is zero. The formula
\[
0 = d f = \sum_{ik} (f_i - f_k)d_{ik}
\]
implies \( f_i = f_k \) for all pairs \((i, k)\) occurring in the sum. The only normalisable vector in the infinite case has \( f_i = 0 \) for all \( i \).

On the other hand, for an oriented or (more generally) directed graph each cycle lies in the zero eigenspace of \( d^* \). More specifically, denoting the cycle by the edge sequence \( x_{i_1}, \ldots, x_{i_l} \) with \( x_{i_j} \) linked to \( x_{i_1} \), we choose the following vector in \( H_1 \):
\[
g := \sum_{\nu} \left[ d_{i_\nu, i_{\nu+1}} \right]
\]
with \( \left[ d_{i_\nu, i_{\nu+1}} \right] \) denoting either \( d_{i_\nu, i_{\nu+1}} \) if the edge is pointing from \( x_{i_\nu} \) to \( x_{i_{\nu+1}} \) or \( -d_{i_{\nu+1}, i_\nu} \) if it points in the opposite direction. Applying \( d^* \) to this vector yields the nullvector in \( H_0 \). Furthermore we state without proof that (genuinely) different cycles are linearly independent and span the kernel of \( d^* \). This can be shown by exploiting the existence of a spanning tree (see, for example [14], p.53). We then have

**Observation 6.2** For an infinite connected directed graph the zero eigenspace of \( Q_1, Q_2 \) or \( H_S \) is purely fermionic and consists of the cycle space.

This implies that the susy-Hamiltonian for an infinite tree has no non-trivial zero eigenvectors. We provide a separate proof for this statement as it employs a possibly useful technical property. We again pick a base vertex, \( x_0 \), and construct the spheres
\[
\Gamma_i(x_0) := \{ x_i, d(x_0, x_i) = l \}
\]
with \( d \) denoting the canonical graph metric. We can infer that for a tree all the \((x_i - x_k)\) so that \( x_i, x_k \) are linked by an edge are linearly independent in \( H_0 \) and, together with \( x_0 \) span \( H_0 \). We then have that
\[
0 = d^* g = \sum g_{ik}(x_i - x_k)
\]
implies \( g_{ik} = 0 \) for all occurring pairs.

**Remark 6.3** This fact can also be exploited for general graphs by using a spanning tree.
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