On the notion of a differential operator in noncommutative geometry

G. Sardanashvily

Department of Theoretical Physics, Moscow State University, 117234 Moscow, Russia
E-mail: sard@grav.phys.msu.su
URL: http://webcenter.ru/~sardan/

Abstract. The algebraic notion of a differential operator on a module over a commutative ring is not extended to a module over a noncommutative ring.

Let $K$ be a commutative ring and $A$ a commutative $K$-ring. By a module $P$ over a commutative ring $A$ throughout is meant a central bimodule, i.e., $ap = pa$ for all $p \in P$ and $a \in A$. Let $Q$ be another $A$-module. There are three equivalent definitions of a $Q$-valued differential operator on an $A$-module $P$ [1, 2, 3] (see items (i) – (iii)). None of them is extended to modules over a noncommutative ring. Therefore, a different definition is suggested.

Note that the definition (iii) is based on the notion of jet modules of a commutative ring. These jet modules provide the standard formulation of the Lagrangian formalism on modules over commutative and graded commutative rings [2, 4, 5]. However, the notion of a jet module fails to be extended to a noncommutative ring.

(i) Let $P$ and $Q$ be modules over a commutative $K$-ring $A$. The $K$-module $\text{Hom}_K(P, Q)$ of $K$-module homomorphisms $\phi : P \rightarrow Q$ can be endowed with the two $A$-module structures

$$ (a \phi)(p) := a \phi(p), \quad (a \star \phi)(p) := \phi(ap), \quad a \in A, \quad p \in P. \quad (1) $$

For the sake of convenience, we will refer to the second one as the $A^\star$-module structure. Let us put

$$ \delta_c \phi = c \phi - c \star \phi, \quad c \in A. \quad (2) $$

An element $\Delta \in \text{Hom}_K(P, Q)$ is called a $Q$-valued $s$-order differential operator on $P$ if

$$ \delta_{c_0} \circ \ldots \circ \delta_{c_s} \Delta = 0 $$

for any tuple of $s + 1$ elements $c_0, \ldots, c_s$ of $A$. The set $\text{Diff}_s(P, Q)$ of these operators inherits the $A$-module structures (1), and $\text{Diff}_{s-1}(P, Q) \subset \text{Diff}_s(P, Q)$. For instance, zero
order differential operators are $\mathcal{A}$-module homomorphisms of $P$ to $Q$, i.e., $\text{Diff}_0(P, Q) = \text{Hom}_{\mathcal{A}}(P, Q)$. For our purpose, it suffices to consider first order differential operators. A first order differential operator $\Delta$ satisfies the condition

$$\delta_a \circ \delta_b \Delta(p) = \Delta(abp) - a\Delta(bp) - b\Delta(ap) + ab\Delta(p) = 0, \quad b, a \in \mathcal{A}. \quad (3)$$

(ii) One can think of a ring $\mathcal{A}$ as being an $\mathcal{A}$-module generated by its unit element $1$. A $Q$-valued first order differential operator $\Delta$ on $\mathcal{A}$ fulfils the relation

$$\Delta(ab) = a\Delta(b) + b\Delta(a) - ab\Delta(1), \quad \forall a, b \in \mathcal{A}. \quad (4)$$

It is a $Q$-valued derivation of $\mathcal{A}$ if $\Delta(1) = 0$, i.e., the Leibniz rule

$$\Delta(ab) = a\Delta(b) + b\Delta(a), \quad \forall a, b \in \mathcal{A}, \quad (5)$$

holds. Hence, any first order differential operator on $\mathcal{A}$ falls into the sum

$$\Delta(a) = a\Delta(1) + [\Delta(a) - a\Delta(1)]$$

of a zero order differential operator $a\Delta(1)$ and a derivation $\Delta(a) - a\Delta(1)$. Note that any zero order differential operator $\Delta$ on $\mathcal{A}$ is uniquely given by its value $\Delta(1)$. Then, there is the $\mathcal{A}$-module isomorphism $\text{Diff}_0(\mathcal{A}, Q) = Q$ via the association

$$Q \ni q \mapsto \Delta_q \in \text{Diff}_0(\mathcal{A}, Q),$$

where $\Delta_q$ is defined by the equality $\Delta_q(1) = q$. Let us consider the $\mathcal{A}$-module morphism

$$h : \text{Diff}_1(\mathcal{A}, Q) \to Q, \quad h(\Delta) = \Delta(1). \quad (6)$$

One can show that any $Q$-valued first order differential operator $\Delta \in \text{Diff}_1(P, Q)$ on $P$ uniquely factorizes

$$\Delta : P \overset{i_\Delta}{\to} \text{Diff}_1(\mathcal{A}, Q) \overset{h}{\to} Q \quad (7)$$

through the morphism $h$ (6) and some homomorphism

$$f_\Delta : P \to \text{Diff}_1(\mathcal{A}, Q), \quad (f_\Delta p)(a) = \Delta(ap), \quad a \in \mathcal{A}, \quad (8)$$

of the $\mathcal{A}$-module $P$ to the $\mathcal{A}^\ast$-module $\text{Diff}_1(\mathcal{A}, Q)$. Hence, the assignment $\Delta \mapsto f_\Delta$ defines the isomorphism

$$\text{Diff}_1(P, Q) = \text{Hom}_{\mathcal{A}}(P, \text{Diff}_1(\mathcal{A}, Q)). \quad (9)$$
(iii) Given an \( \mathcal{A} \)-module \( P \), let us consider the tensor product \( \mathcal{A} \otimes_{K} P \) over \( K \). We put
\[
\delta^b(a \otimes p) = (ba) \otimes p - a \otimes (bp), \quad p \in P, \quad a, b \in \mathcal{A}.
\]
(10)
Let us denote by \( \mu^{k+1} \) the submodule of \( \mathcal{A} \otimes_{K} P \) generated by elements of the type
\[
\delta^b \circ \cdots \circ \delta^k (a \otimes p).
\]
The \( k \)-order jet module \( \mathcal{J}^k(P) \) of the module \( P \) is defined as the quotient of the \( K \)-module \( \mathcal{A} \otimes_{K} P \) by \( \mu^{k+1} \). We denote its elements \( a \otimes_k p \). In particular, the first order jet module \( \mathcal{J}^1(P) \) consists of elements \( a \otimes p \) modulo the relations
\[
\delta^a \circ \delta^b (1 \otimes p) = 1 \otimes (abp) - a \otimes (bp) - b \otimes (ap) + ab \otimes p = 0.
\]
(11)
The \( K \)-module \( \mathcal{J}^1(P) \) is endowed with the following two \( \mathcal{A} \)-module structures:
\[
b(a \otimes_k p) := ba \otimes_k p, \quad b \ast (a \otimes_k p) := a \otimes_k (bp).
\]
(12)
There exists the module morphism
\[
J^1 : P \ni p \rightarrow 1 \otimes_1 p \in \mathcal{J}^1(P)
\]
(13)
of the \( \mathcal{A} \)-module \( P \) to the \( \mathcal{A}^* \)-module \( \mathcal{J}^1(P) \) such that \( \mathcal{J}^1(P) \) seen as an \( \mathcal{A} \)-module is generated by elements \( J^1 p, p \in P \). Conversely, there is the epimorphism
\[
\pi_0^1 : \mathcal{J}^1(P) \ni a \otimes_1 p \rightarrow ap \in P.
\]
(14)
A glance at \( \delta_b (2) \) and \( \delta^b (10) \) shows that the morphism \( J^1 \) (13) is a first order differential operator on \( P \). As a consequence, any \( Q \)-valued first order differential operator \( \Delta \) on \( P \) uniquely factorizes
\[
\Delta : P \xrightarrow{J^1} \mathcal{J}^1(P) \xrightarrow{f^A} Q
\]
(15)
through the morphism \( J^1 \) (13) and some \( \mathcal{A} \)-module homomorphism \( f^A : \mathcal{J}^1(P) \rightarrow Q \). The assignment \( \Delta \mapsto f^A \) defines the isomorphism
\[
\text{Diff}_1(P, Q) = \text{Hom}_{\mathcal{A}}(\mathcal{J}^1(P), Q).
\]
(16)

Now, let \( \mathcal{A} \) be a noncommutative \( K \)-ring and \( P \) a \( \mathcal{A} \)-bimodule. Let \( \mathcal{Z}_A \) be the center of \( \mathcal{A} \) and \( \mathcal{Z}_P \) the center of \( P \) (i.e., \( \mathcal{Z}_P \) consists of elements \( p \in P \) such that \( ap = pa \) for all \( a \in \mathcal{A} \)). Let \( Q \) be another \( \mathcal{A} \)-bimodule. The \( K \)-module \( \text{Hom}_K(P, Q) \) can be provided
with the left $\mathcal{A}$-module structures (1) and the similar right ones. The left $\mathcal{A}$-module homomorphisms $\Delta : P \to Q$ obey the conditions $\delta_c \Delta = 0$ and, therefore, can be regarded as left $Q$-valued zero order differential operators on $P$. One can also write the condition (3). However, a problem is that, if $\mathcal{A}$ is noncommutative, zero order differential operators (e.g., $Q = P$ and $\Delta = \text{Id}_P$) fail to satisfy this condition.

If $P = \mathcal{A}$, a $Q$-valued zero order differential operator $\Delta$ on $\mathcal{A}$ takes its value $\Delta(1)$ only in the center $Z_Q$ of $Q$. Therefore, one can rewrite the condition (4) as

$$\Delta(ab) = a\Delta(b) + \Delta(a)b - ab\Delta(1), \quad \forall a, b \in \mathcal{A}.$$  

This provides the definition of a $Q$-valued first order differential operator $\Delta$ on $\mathcal{A}$. It is the sum of a $Q$-valued derivation $\partial$ of $\mathcal{A}$ which obeys the Leibniz rule

$$\partial(ab) = (\partial a)b + a\partial b, \quad a, b \in \mathcal{A}, \quad (17)$$

and some zero order differential operator. Therefore, we have $\Delta(1) \in Z_Q$. The $Q$-valued first order differential operators on $\mathcal{A}$ make up a left $Z\mathcal{A}$-module $\text{Diff}_1(\mathcal{A}, Q)$. Then, one may try to define a $Q$-valued first order differential operator on an $\mathcal{A}$-bimodule $P$ as the composition $\Delta = h \circ f$ (7). However, such an operator takes its values only in the center $Z_Q$ of $Q$ since $\Delta(p) = (fp)(1) \in Z_Q$.

The composition (15) fails to provide the definition of a differential operator on a module over a noncommutative ring either. The key point is that the jet module of a noncommutative ring is ill defined. Namely, the projection $\pi_0^1$ (14) of the zero element (11) in $J^1(P)$ fails to be zero in $P$.

Using the fact that derivations of a noncommutative $K$-ring $\mathcal{A}$ with values in an $\mathcal{A}$-bimodule are well defined, one can suggest the following definition of first order differential operators on modules over a noncommutative rings. Let $P$ and $Q$ be bimodules over a noncommutative $K$-ring $\mathcal{A}$. A $K$-module homomorphism $\Delta \in \text{Hom}_K(P, Q)$ of $P$ to $Q$ is said to be a $Q$-valued first order differential operator on $P$ if it obeys the condition

$$\Delta(ab) = (\overrightarrow{\partial} a)(pb) + a\Delta(p)b + a(\overleftarrow{\partial} b)(p), \quad (18)$$

where $\overrightarrow{\partial}$ and $\overleftarrow{\partial}$ are derivations of $\mathcal{A}$ which take their values in the modules $\text{Hom}^r_\mathcal{A}(P, Q)$ and $\text{Hom}^l_\mathcal{A}(P, Q)$ of right $\mathcal{A}$-module homomorphisms and left $\mathcal{A}$-module homomorphisms of $P$ to $Q$, respectively. Namely, $(\overrightarrow{\partial} a)(pb) = (\overrightarrow{\partial} a)(p)b$ and $(\overleftarrow{\partial} b)(ap) = a(\overleftarrow{\partial} b)(p)$. Note that $\text{Hom}^r_\mathcal{A}(P, Q)$ and $\text{Hom}^l_\mathcal{A}(P, Q)$ are $\mathcal{A}$-bimodules such that

$$(a\phi)(p) := a\phi(p), \quad (\phi a)(p) := \phi(ap), \quad \phi \in \text{Hom}^r_\mathcal{A}(P, Q),$$

$$(a\varphi)(p) := \varphi(pa), \quad (\varphi a)(p) := \varphi(p)a, \quad \varphi \in \text{Hom}^l_\mathcal{A}(P, Q),$$
for all $a \in A$.

For instance, let $P = P^*$ be a differential calculus over a $K$-ring $A$ provided with an associative multiplication $\circ$ and a coboundary operator $d$. Then, $d$ is a $P$-valued first order differential operator on $P$. It obeys the condition (18) which reads

$$d(apb) = (da) \circ pb + a(dp)b + ap \circ db.$$ 

Another important example is a Dubois–Violette connection $\nabla$ on an $A$-bimodule $P$ [2, 3, 6]. It associates to every $A$-valued derivation $u$ of $A$ a $P$-valued first order differential operator $\nabla_u$ which obeys the Leibniz rule

$$\nabla_u(apb) = u(a)pb + a\nabla_u(p)b + apu(b).$$

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