CONJUGACY DEPTH FUNCTIONS OF WREATH PRODUCTS OF ABELIAN GROUPS

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Abstract. In this note, we complete the study of asymptotic behaviour of conjugacy separability of the general case of wreath products of finitely abelian groups where the base group is possibly infinite. In particular, we provide super-exponential upper and lower bounds for conjugacy separability of wreath products where the base group contains $\mathbb{Z}$ and, combining with previous work of the authors, we provide asymptotic bounds for conjugacy separability depth functions of all wreath products of finitely generated abelian groups. As an application, we give exponential lower bounds for infinitely many wreath products where the acting group is not necessarily abelian.

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1. Introduction

Studying infinite, finitely generated groups through their finite quotients is a common method in group theory. Groups in which one can distinguish elements using their finite quotients are called residually finite. Formally speaking, a group $G$ is said to be residually finite if for every pair of distinct elements $f, g \in G$ there exists a finite group $Q$ and a surjective homomorphism $\pi: G \to Q$ such that $\pi(f) \neq \pi(g)$ in $Q$. Group properties of this type are called separability properties.

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and are usually defined by what types of subsets we want to distinguish. In this article, we study quantitative aspects of **conjugacy separability**, meaning that we will study groups in which one can distinguish conjugacy classes using finite quotients. To be more specific, a group \( G \) is said to be conjugacy separable if for every pair of nonconjugate elements \( f, g \in G \) there exists a finite group \( Q \) and a surjective homomorphism \( \pi: G \to Q \) such that \( \pi(f) \) is not conjugate to \( \pi(g) \) in \( Q \).

1.1. **Motivation.** One of the original reasons for studying separability properties in groups is that they provide an algebraic analogue to decision problems in finitely presented groups. To be more specific, if \( S \subseteq G \) is a separable subset such that \( S \) is recursively enumerable and where one can always effectively construct the image of \( S \) under the canonical projection onto a finite quotient of \( G \), then one can then decide whether a word in the generators of \( G \) represents an element belonging to \( S \) simply by checking finite quotients. Indeed, it was proved by Mal’tsev \[16\], adapting the result of McKinsey \[18\] to the setting of finitely presented groups, that the word problem is solvable for finitely presented, residually finite groups in the following way. Given a finite presentation \( \langle X \mid R \rangle \) and a word \( w \in F(X) \) where \( F(X) \) is the free group with the generating set \( X \), one runs two algorithms in parallel. The first algorithm enumerates all the products of conjugates of the relators and their inverses and checks whether \( w \) appears on the list whereas the second algorithm enumerates all finite quotients of \( G \) and checks whether the image of the element of \( G \) represented by \( w \) is nontrivial. In other words, the first algorithm is looking for a witness of the triviality of \( w \) whereas the second algorithm is looking for a witness of the nontriviality of \( w \). Using an analogous approach, Mostowski \[20\] showed that the conjugacy problem is solvable for finitely presented, conjugacy separable groups. In a similar fashion, finitely presented, LERF groups have solvable generalised word problem meaning that the membership problem is uniformly solvable for every finitely generated subgroup. In general, algorithms that involve enumerating finite quotients of an algebraic structure are sometimes called algorithms of **Mal’tsev-Mostowski type** or **McKinsey’s algorithms**.

Given an algorithm, it is natural to ask how much computing power is necessary to produce an answer. In the case of algorithms of Mal’tsev-Mostowski type, one can measure their space complexity by the associated depth functions. Given a residually finite group \( G \) with a finite generating set \( S \), its residual finiteness depth function \( RF_{G,S}: \mathbb{N} \to \mathbb{N} \) quantifies how deep within the lattice of normal subgroups of finite index of \( G \) one needs to look to be able to decide whether or not a word of length at most \( n \) with respect to \( S \) represents a nontrivial element. In particular, if \( w \) is a word in the alphabet \( S \) of length at most \( n \), then either \( G \) has a finite quotient of size at most \( RF_{G,S}(n) \) in which the image of the element represented by \( w \) is nontrivial, and if there is no finite quotient of size less than or equal to \( RF_{G,S}(n) \) in which the image of \( w \) is nontrivial, then \( w \) must represent the trivial word. In particular, we see that the the residual finiteness depth function of \( G \) with respect to the generating set \( S \) fully determines the size of finite quotients McKinsey’s needs to generate in order to give produce an answer. Since every finite group can be fully described by its Cayley table, we see that the space complexity of the word problem of \( G \) with respect to the generating set \( S \) can be bounded from above by \( (RF_{G,S}(n))^2 \). Moreover, the notion of depth function can be generalised to different separability properties. In this note, we study conjugacy separability depth functions which denote as \( \text{Conj}_{G,S}(n) \) which is a function that measures how
deep within the lattice of normal subgroups of finite index one needs to go in order to be able to distinguish distinct conjugacy classes of elements of word length at most \( n \) with respect to the finite generating subset \( S \). Just like in computational complexity, we study these functions up to asymptotic equivalence. See subsection 2.1 for the precise definitions of depth functions and the corresponding asymptotic notions.

1.2. Statement of the results. Not much is known about the asymptotic behaviour of conjugacy depth functions. Lawton, Louder, and McReynolds [15] showed that if \( G \) is a nonabelian free group or the fundamental group of a closed oriented surface of genus \( g \geq 2 \), then \( \text{Conj}_G(n) \leq n^{n^2} \). For the class of finitely generated nilpotent groups, the second named author and Deré [7] showed that if \( G \) is a finite extension of a finitely generated abelian group, then \( \text{Conj}_G(n) \leq (\log(n))^d \) for some natural number \( d \), and when \( G \) is a virtually nilpotent group that is not virtually abelian, then there exist natural numbers \( d_1 \) and \( d_2 \) such that \( n^{d_1} \leq \text{Conj}_G(n) \leq n^{d_2} \). In [10], the authors of this note gave a general formula in terms of depth functions of the factors for upper bounds for conjugacy depth functions of wreath products of conjugacy separable groups, generalising Remeslennikov’s classification of conjugacy separable wreath products [21]. However, when applied directly to wreath products of abelian groups, the formulas given in [10] produce rather rough upper bounds. Applying [10, Theorem C] to the lamplighter group \( \mathbb{F}_2 \wr \mathbb{Z} \), one gets that its conjugacy depth function can be bounded from above by the function \( 2^{n^{n^2}} \), and by an application of [10, Theorem C] to the group \( \mathbb{Z} \wr \mathbb{Z} \), one gets that its conjugacy depth function can be bounded from above by the function \( n^{n^{n^2}} \). The former of these bounds was significantly improved in [11, Theorem 1.1], where the authors showed that the conjugacy function of a group \( A \wr B \), where \( A \) is finite abelian and \( B \) is finitely generated abelian of torsion free rank \( k \), can be bounded from above by \( 2^{n^{2^k}} \) and that it can be bounded from below by \( 2^n \). Moreover, they were able to precisely compute the asymptotic behaviour of conjugacy separability for groups of the form \( A \wr B \) where \( A \) is finite and \( B \) is virtually cyclic which includes the lamplighter group. In particular, they demonstrated that \( \text{Conj}_{A\wr B}(n) \approx 2^n \).

In this note, we extend the methods introduced in [11, Theorem 1.1] to the setting of all the groups of the form \( A \wr B \), where both \( A \) and \( B \) are finitely generated abelian groups. This allows us to use more effective methods to obtain much better upper bounds than those presented in [10]. Additionally, we are able to use methods from commutative algebra to produce lower bounds.

Letting \( f, g : \mathbb{N} \to \mathbb{N} \) be nondecreasing functions, we write \( f \preceq g \) if there is a constant \( C \in \mathbb{N} \) such that \( f(n) \leq Cg(Cn) \) for all \( n \in \mathbb{N} \). If \( f \preceq g \) and \( g \preceq f \), we then write \( f \approx g \).

Together with [11, Theorem 1.1], Theorem 1.1 provides the best known result for the asymptotic behaviour of conjugacy separability of wreath products of finitely generated abelian groups with infinite acting group.

**Theorem 1.1.** Let \( A \) be an infinite, finitely generated abelian group, and suppose that \( B \) is an infinite finitely generated abelian group. If \( B \) has torsion free rank 1, then

\[
(\log(n))^n \leq \text{Conj}_{A\wr B}(n) \leq (\log(n))^{n^2}.
\]
If $B$ has torsion free rank $k > 1$, then

$$(\log(n))^n \leq \text{Conj}_{A \wr B}(n) \leq (\log(n))^{n^{2k+2}}.$$  

Combining Theorem 1.1 and [11, Theorem 1.1], we have the following corollary, which gives the best known result for asymptotic behaviour of conjugacy separability of wreath products of finitely generated abelian groups with infinite acting group.

**Corollary 1.2.** Let $A$ be a finitely generated abelian group, and suppose that $B$ is an infinite, finitely generated group.

Suppose that $A$ is finite. If $B$ has torsion free rank 1, then

$$(\log(n))^n \leq \text{Conj}_{A \wr B}(n) \leq (\log(n))^{n^2}.$$  

If $B$ has torsion free rank $k > 1$, then

$$(\log(n))^n \leq \text{Conj}_{A \wr B}(n) \leq (\log(n))^{n^{2k+2}}.$$  

Suppose that $A$ is finite. If $B$ has torsion free rank 1, then

$$\text{Conj}_{A \wr B}(n) \approx 2^n$$  

If $B$ has torsion free rank $k > 1$, then

$$2^n \leq \text{Conj}_{A \wr B}(n) \leq 2^{n^{2k}}.$$  

1.3. **Outline of the paper.** In Section 2 we recall standard mathematical notions and concepts that will be used throughout the paper. In Subsection 2.1 we recall the notions of word length, depth functions and the associated asymptotic notation. In Subsection 2.2 we recall the basic terminology of wreath products of groups. Finally, in Subsubsection 2.3 we recall the notion of Laurent polynomial rings and show that wreath products with infinite cyclic acting group can be seen as subgroup of $GL(2, R)$, where $R$ is a Laurent polynomial ring, and finish by giving a conjugacy criterion for such groups purely in terms of commutative algebra.

In Section 3 we use methods from commutative algebra to produce lower bounds for the conjugacy depth functions by constructing infinite sequences of pairs of non-conjugate elements that require large quotients in order to remain non-conjugate.

In Section 4 we use combinatorial methods together with the conjugacy criterion for wreath products of abelian groups to construct upper bounds for such wreath products.

Finally, in Section 5 we combine the lower bound obtained in Section 3 together with the upper bounds constructed in Section 4 to prove Theorem 1.1. We then proceed to apply our methods to give lower bounds on the conjugacy depth function for wreath products where the acting group may not necessarily be abelian.

2. **Preliminaries**

We will use additive notation for abelian groups. We denote $\mathbb{F}_p$ as the finite field of $p$ elements where $p$ is prime. We denote $\text{Sym}(n)$ as the symmetric group on $n$ letters. For $x, y \in G$, we write that $x \sim_G y$ if there exists an element $z \in G$ such that $z x z^{-1} = y$ and suppress the subscript when $G$ is clear from context.

We say that a subgroup is $H \leq G$ is **conjugacy embedded** in $G$ if for every $f, g \in H$ we have that $f \sim_H g$ if and only if $f \sim_G g$. Following this definition, one can easily check that the relation of being conjugacy embedded is transitive: if
Remark 2.1. Let $G$ be a group, and let $R \leq G$ be a subgroup. If $R$ is a retract of $G$, then $R$ is conjugacy embedded in $G$.

The next lemma allows us to the reduced study of conjugacy in a semidirect product of abelian groups $A \rtimes B$ to conjugacy in $A \rtimes (B/K)$ where $K$ is the kernel of the action of $B$ on $A$.

Lemma 2.2. Suppose that $A$ and $B$ are finitely generated abelian groups. We claim that if $(a_1, b) \simeq (a_2, b)$ in $A \rtimes B$ if and only if $(a_1, b) \sim (a_2, b) \mod (0, K)$ where $K$ is the kernel of the action of $B$ on $A$.

Proof. Since the backwards direction is clear, we may assume that $(a_1, b_1) \sim (a_2, b_2)$ in $A \rtimes B$. Suppose for a contradiction that there exists $(x, y) \in A \rtimes B$ and $k \in K$ such that $(x, y) \cdot (a_1, b) \cdot (x, y)^{-1} = (a_2, b) \cdot (0, k)$. Thus, we have

\[
(x, y) \cdot (a_1, b) \cdot (x, y)^{-1} = (a_2, b) \cdot (0, k) \\
(x + y \cdot a_1, yb) \cdot (-y^{-1} \cdot x, y^{-1}) = (a_2, bk) \\
(x + y \cdot a_1 - b \cdot x, b) = (a_2, bk).
\]

Hence, we must have that $k = 0$. Therefore, we have $(x, y)(a_1, b) \cdot (x, y)^{-1} = (a_2, b)$ which is a contradiction. Hence, we have our claim.

2.1. Asymptotic notions and depth functions. Given a finitely generated group $G$ with finite generating subset $S$, one can define the word length function $\| \cdot \|_S : G \to \mathbb{N} \cup \{0\}$ as

\[\|g\|_S = \min\{\|w\| \mid w \in F(S) \text{ and } w =_{G} g\}.\]

Word length is a standard tool in geometric group theory that is used to equip $G$ with a left invariant metric $d_S : G \times G \to \mathbb{N}$ given by $d_S(g_1, g_2) = \|g_1^{-1}g_2\|_S$. We will use $B_{G, S}(n)$ to denote the ball of radius $n$ centred around the identity, i.e. $B_{G, S}(n) = \{g \in B \mid \|g\|_S \leq n\}$. When the generating subset $S$ is clear from context, we suppress the subscript and write $B_G(n)$ instead.

The conjugacy separability depth function of $G$ is defined in the following way. Suppose that $G$ is a conjugacy separable group, and let $f, g \in G$ be a pair of elements such that $f \not\sim_G g$. We then let

\[\text{CD}_G(f, g) = \min\{|G/N| \mid N \trianglelefteq_{f.i.} G \text{ and } fN \not\sim_{G/N} gN \},\]

and we say that $\text{CD}_G(f, g)$ is the conjugacy separability depth (conjugacy depth for short) of the pair $(f, g)$ in $G$. Similar to the definition of the residual finiteness depth function, given a finite generating subset $S \subseteq G$, the conjugacy separability depth function $\text{Con}_{f}^S : \mathbb{N} \to \mathbb{N}$ is defined as

\[\text{Con}_{f}^S(n) = \max\{\text{CD}_G(f, g) \mid f, g \in B_{G, S}(n) \text{ and } f \not\sim_G g\}.\]

Note that both the functions $RF_{G, S}(n)$ and $\text{Con}_{f}^S(n)$ depend on the choice of the generating subset $S$. However, one can easily check that the asymptotic behaviour does not. It is well known that a change of a generating subset is an
quasi-isometry: if \( S_1, S_2 \subset G \) are two finite generating subsets of a group \( G \), then \( \| \cdot \|_{S_1} \approx \| \cdot \|_{S_2} \). The same holds for depth functions. Recall that if \( f, g: \mathbb{N} \to \mathbb{N} \) are nondecreasing functions, we write \( f \leq g \) if there is a constant \( C \in \mathbb{N} \) such that \( f(n) \leq C g(n) \) for all \( n \in \mathbb{N} \), and if \( f \preceq g \) and \( g \preceq f \), then we write \( f \approx g \). When \( G \) is a finitely generated, conjugacy separable group with two finite generating subsets \( S_1 \) and \( S_2 \), we have \( \text{Conj}_{G,S_1}(n) \approx \text{Conj}_{G,S_2}(n) \), see [15] for more details.

As we are only interested in the asymptotic behaviour of the above defined functions, we will suppress the choice of generating subset whenever we reference the depth functions or the word-length.

We conclude this subsection with the following easy lemma.

**Lemma 2.3.** Suppose that \( G \) is a finitely generated conjugacy separable group with a finitely generated subgroup \( R \leq G \), and suppose that \( R \leq G \) is a retract. Then

\[
\text{Conj}_R(n) \preceq \text{Conj}_G(n).
\]

**Proof.** Let \( \rho: G \to R \) be the corresponding retraction. First we show that there is a finite generating set \( X \subseteq G \) such that \( X = X_R \cup X_K \), \( R = \langle X_R \rangle \) and \( \langle X_K \rangle \cap R = \{1\} \). Suppose that \( G = \langle X' \rangle \), where \( X = \{x_1, \ldots, x_m\} \). We set \( X_R = \{\rho(x_1), \ldots, \rho(x_m)\} \) and \( X_K = \{\rho(x_1)^{-1}x_1, \ldots, \rho(x_m)^{-1}x_m\} \). It then follows that \( R = \langle X_R \rangle \) and \( \langle X_K \rangle \leq \ker(\rho) \), so \( \langle X_K \rangle \cap R = \{1\} \). Therefore, we have \( |r|_{x_R} = \|r\|_X \) for every \( r \).

Now suppose that \( r_1, r_2 \in \text{B}_R(n) \) are given such that \( r_1 \not\sim_R r_2 \). Following the previous paragraph together with Remark 2.1, we see that \( r_1, r_2 \in \text{B}_G(n) \) and that \( r_1 \not\sim_G r_2 \). Thus, we finish by showing that \( \text{CD}_R(r_1, r_2) \leq \text{CD}_G(r_1, r_2) \). Suppose that \( N_R \) is a normal finite index subgroup of \( R \) that realises \( \text{CD}_R(r_1, r_2) \) and that there is a finite index normal subgroup \( N_G \) of \( G \) such that \( r_1 N_G \not\sim r_2 N_G \) in \( G/N_G \) and \( |G/N_G| < |R/N_R| \). Since \( R/(R \cap N_G) \leq G/N_G \), we see that \( r_1 (R \cap N_G) \not\sim r_2 (R \cap N_G) \) in \( R/(R \cap N_G) \). It follows that

\[
|R/(R \cap N_G)| \leq |G/N_G| < |R/N_R|,
\]

which is a contradiction with the assumption that \( N_R \) realises \( \text{CD}_G(r_1, r_2) \). Therefore, we see that \( \text{Conj}_R(n) \preceq \text{Conj}_G(n) \). \( \square \)

### 2.2. Wreath products

For groups \( A \) and \( B \), we denote the restricted wreath product of \( A \) and \( B \), written as \( A \wr B \), by

\[
A \wr B = \left( \bigoplus_{b \in B} A \right) \rtimes B.
\]

where \( B \) acts on \( \bigoplus_{b \in B} A \) via left multiplication on the coordinates. An element \( f \in \bigoplus_{b \in B} A \) is understood as a function \( f: B \to A \) such that \( f(b) \neq 1 \) for only finitely many \( b \in B \). With a slight abuse of notation, we will use \( A^B \) to denote \( \bigoplus_{b \in B} A \). The left action of \( B \) on \( A^B \) is then realised as \( b \cdot f(x) = f(bx) \).

The set of elements on which \( f \) does not vanish is called the **support** and will be denoted as

\[
\text{supp}(f) = \{ b \in B \mid f(b) \neq 1 \}.
\]

The **range** of \( f \) will be denoted as

\[
\text{rng}(f) = \{ f(x) \mid x \in B \}.
\]
Following the given notation, if $H \leq A$ and $K \leq B$, we will use $H^K$ to denote the subset

$$H^K = \{ f \in A^B \mid \text{supp}(f) \subseteq K \text{ and } \text{rng}(f) \subseteq H \}.$$

Keeping this in mind, the wreath product $H \wr K$ can be identified in an obvious way with the subgroup $H^K \rtimes K \leq A \wr B$. The following statement was proved in [11, Lemma 2.2].

**Lemma 2.4.** Let $A, B$ be finitely generated abelian groups and suppose that $R_A \leq A$ and $R_B \leq B$ are retracts. Then the group $R_A \wr R_B$ is a retract of $A \wr B$. In particular, $R_A \wr R_B$ is conjugacy embedded in $A \wr B$ and $\text{Conj}_R(n) \leq \text{Conj}_G(n)$.

The following can be derived from [10, Lemma 5.13].

**Lemma 2.5.** Let $A, B$ be finitely generated abelian groups. Then there exists a constant $C > 0$ such that for every $f \in A^B$ and $b \in B$ there is $f' \in A^B$ such that $f' b \sim_{A \wr B} f b$ and where the elements of supp($f'$) lie in distinct cosets of $\langle b \rangle$ in $B$. Finally, we have that $f' b \in B_{A \wr B}(C \|f b\|)$.

Suppose that $b \in B$ and $f \in A^B$ is a function with a finite support. We say that $f$ is **minimal** with respect to $b$ if all elements of supp($f$) lie in distinct cosets of $\langle b \rangle$ in $B$. We will say that an element $f b \in A \wr B$ is **reduced** if $f$ is minimal with respect to $b$.

The following lemma is a special case of [10, Lemma 5.13].

**Lemma 2.6.** Let $A, B$ be finitely generated groups, and suppose that $b \in B$ and $f : B \to A$ are given such that $f b \in B_{A \wr B}(n)$. Then there exists a constant $C$ independent of $b$ and $f$ and $f' \in A^B$ such that $f' b \sim f b$, $f' b$ is reduced, and $\|f' b\| \leq C \|f b\|$.

The following statement, which provides a conjugacy criterion for wreath products of abelian groups, follows from [10, Lemma 5.14].

**Lemma 2.7.** Let $A, B$ be abelian groups, and let $G = A \wr B$ be their wreath product. Let $f_1, f_2 \in A^B$, $b_1, b_2 \in B$ be such that the elements $f_1 b_1$ and $f_2 b_2$ are reduced. Then $f_1 b_1 \sim_{A \wr B} f_2 b_2$ if and only if $b_1 = b_2$ and $f_1 b \in f_2 b^B$, i.e., there exists $c \in B$ such that $c \text{supp}(f_1) = \text{supp}(f_2)$ and $f_1(c x) = f_2(x)$ for all $x \in B$.

One interpretation of Lemma 2.7 is that by ensuring that we are only working with reduced elements of $A \wr B$, we only need to worry about them being conjugate by an element from $B$.

The next lemma relates word length of an element of a wreath product to the size of the support of its function part and the size of the elements in the range of the function part.

**Lemma 2.8.** Let $A, B$ be finitely generated groups and let $G = A \wr B$ be their wreath product. Then there exists a constant $C > 0$ such that whenever $g = f b$, where $f \in A^B$ and $b \in B$, then

1. $\text{supp}(f) \subseteq B_B(C \|g\|)$,
2. $\text{rng}(f) \subseteq B_A(C \|g\|)$.

**Proof.** Use [6, Theorem 3.4].

Given a wreath product $A \wr B$ with a surjective homomorphism $\pi : B \to \overline{B}$, we denote $\bar{\pi} : A \wr B \to A \wr \overline{B}$ as the canonical extension of $\pi$ to all of $A \wr B$. Similarly,
if \( \pi: A \to \overline{A} \) is a surjective homomorphism, we denote \( \tilde{\pi}: A \wr B \to \overline{A} \wr B \) as the canonical extension of \( \pi \) to all of \( A \wr B \). We refer readers unfamiliar with the concept of a canonical extension to [10] Subsection 4.1 for the formal definition and more details.

2.3. Linear groups over Laurent polynomial rings. Much of the following discussion, which includes undefined notation and terms, can be found in [2] 8 14. We will write \( \mathbb{Z}[x] \) to denote the ring of polynomials in the variable \( x \) with integer coefficients. Furthermore, we will use \( \mathbb{Z}[x, x^{-1}] \) to denote the ring of Laurent polynomials with integer coefficients.

We first note that \( \mathbb{Z}[x, x^{-1}] \) is the localisation of the ring \( \mathbb{Z}[x] \) on the set \( S = \{ x^m \mid m \in \mathbb{N} \} \). In particular, proper ideals of \( \mathbb{Z}[x, x^{-1}] \) are in one-to-one correspondence with ideals of \( \mathbb{Z}[x] \) that don’t intersect the set \( S \). Therefore, for any ideal \( I \subset \mathbb{Z}[x] \) where \( I \cap S = \emptyset \), we have that \( \mathbb{Z}[x, x^{-1}]/(S^{-1} I) = S^{-1}(\mathbb{Z}[x, y]/I) \). If \( m \) is a maximal ideal of \( \mathbb{Z}[x, x^{-1}] \), we then have that \( m = (p, f) \) where \( p \) is prime number and \( f \) is an irreducible polynomial modulo \( p \). We also have that \( |\mathbb{Z}[x, x^{-1}]/m| = p^k \) where \( k = \text{deg}(f) \).

We focus on the following representation of \( \mathbb{Z} \wr \mathbb{Z} \) as a group of matrices with coefficients in the ring \( \mathbb{Z}[x, x^{-1}] \). First, let us define a function \( P: \mathbb{Z}^2 \to \mathbb{Z}[x, x^{-1}] \) defined by

\[
P(f) = \sum_{m \in \mathbb{Z}^k} f(m)x^m.
\]

One can easily check that, in the context of finitely supported functions, \( P \) is a bijection and that for any \( r \in \mathbb{Z} \), \( f, g \in \mathbb{Z}^2 \), and \( m \in \mathbb{Z} \) the following hold:

\begin{enumerate}
  \item \( P(rf) = rP(f) \),
  \item \( P(f + g) = P(f) + P(g) \),
  \item \( P(m \cdot f) = x^mP(f) \).
\end{enumerate}

We will use these three equalities without mention.

**Lemma 2.9.** The group \( \mathbb{Z} \wr \mathbb{Z} \) is isomorphic to the subgroup of \( \text{GL}(2, \mathbb{Z}[x, x^{-1}]) \) given by

\[
L_R = \left\{ \begin{pmatrix} x^m & P \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z}, P \in \mathbb{Z}[x, x^{-1}] \right\}.
\]

**Proof.** Let \( \varphi: \mathbb{Z} \wr \mathbb{Z} \to L_R \) be the map given by

\[
\varphi(fm) = \begin{pmatrix} x^m & P(f) \\ 0 & 1 \end{pmatrix}.
\]

It is easy to see that this map is bijective. Thus, we need to show that it is a homomorphism which we do directly by computation. Let \( f_1, f_1 \in \mathbb{Z}^2 \) and \( m_1, m_2 \in \mathbb{Z} \) be arbitrary. We can then write:

\[
\varphi(f_1m_2) \cdot \varphi(f_2m_2) = \begin{pmatrix} x^{m_1} & P(f_1) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x^{m_2} & P(f_2) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{m_1} \cdot x^{m_1} & P(f_1) + x^{m_1}P(f_2) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{m_1 + m_2} & P(f_1 + m_1 \cdot f_2) \\ 0 & 1 \end{pmatrix} = \varphi((m_1 + m_2, f_1 + m_1 \cdot f_2)) = \varphi((f_1m_1) \cdot (f_2m_2)).
\]

\( \square \)
For the rest of this article, we denote the element

\[ \begin{bmatrix} x^m & P \\ 0 & 1 \end{bmatrix} \]

as \((P, m)\).

The following lemma allows us to understand finite quotients of \(\mathbb{Z}/\mathbb{Z}\) in terms of cofinite ideals of \(\mathbb{Z}[x, x^{-1}]\). For the following, we identify \(\mathbb{Z}[x, x^{-1}]\) with the normal subgroup of \(\mathbb{Z}/\mathbb{Z}\) given by elements of the form \((P, 0)\), where \(P \in \mathbb{Z}[x, x^{-1}]\).

**Lemma 2.10.** Let \(N \leq \mathbb{Z}/\mathbb{Z}\). Then \(N \cap \mathbb{Z}[x, x^{-1}]\) is an ideal in \(\mathbb{Z}[x, x^{-1}]\). In particular, if \(N\) is finite index, then \(N \cap \mathbb{Z}[x, x^{-1}]\) is a cofinite ideal of \(\mathbb{Z}[x, x^{-1}]\).

**Proof.** Let \(M = N \cap \mathbb{Z}[x, x^{-1}]\). We note for \((P, 0) \in M\) that

\[ (0, m)(P, 0)(0, -m) = (x^m P, 0) \in M \]

since \(M\) is normal. In particular, we have that \(M\) is closed under multiplication by monomials in \(\mathbb{Z}[x, x^{-1}]\). Additionally, for \((P_1, 0), (P_2, 0) \in M\) we have that

\[ (P_1, 0)(P_2, 0) = (P_1 + P_2, 0). \]

That implies \(M\) is closed under addition. It is clear for \(r \in \mathbb{Z}\) and \(P \in \mathbb{Z}[x, x^{-1}]\) that \((rP, 0) \in M\) since multiplying \(P\) by \(r\) is the same as adding \(r\) copies of \(P\). Thus, for \((P, 0) \in M\) and a general element \(\sum_{m \in \mathbb{Z}} a_m x^m\) of \(\mathbb{Z}[x, x^{-1}]\), we may write

\[ P \cdot \sum_{m \in \mathbb{Z}} a_m x^m = \sum_{m \in \mathbb{Z}} a_m x^m P \in M. \]

Thus, \(M\) is an ideal in \(\mathbb{Z}[x, x^{-1}]\). Moreover, the second part of the statement immediately follows. \(\square\)

The following lemma gives the explicit expression for the conjugacy class of an arbitrary element of \(\mathbb{Z}/\mathbb{Z}\).

**Lemma 2.11.** For \((P, m) \in \mathbb{Z}/\mathbb{Z}\), its conjugacy class is given by

\[ \{ (x^\ell P + (x^m - 1)Q, m) \mid \ell \in \mathbb{Z}, Q \in \mathbb{Z}[x, x^{-1}] \}. \]

**Proof.** Let \(Q \in \mathbb{Z}[x, x^{-1}]\) and \(\ell \in \mathbb{Z}\) be arbitrary. We can then write

\[ (Q, \ell)(P, m)(Q, \ell)^{-1} = (Q + x^\ell P, \ell + m) \begin{pmatrix} -x^{\ell} Q, -\ell \\ Q + x^\ell P - x^{\ell+m} x^{-\ell} Q, \ell + m - \ell \end{pmatrix} = (Q + x^\ell P - x^{\ell+m} Q, \ell + m - \ell) = (Q + x^\ell P - x^m Q, m) = (x^\ell P + (1 - x^m)Q, m). \]

Since \(Q\) was arbitrary, we may replace it by \(-Q\) and thus getting

\[ (Q, \ell)(P, m)(Q, \ell)^{-1} = (x^\ell P + (x^m - 1)Q, m). \]

From here, our statement is clear. \(\square\)

Let \(S\) be a finite generating subset for \(\mathbb{Z}/\mathbb{Z}\). For a Laurent polynomial \(P \in \mathbb{Z}[x, x^{-1}]\), we finish this section with the following lemma which bounds the coefficients and the degree of \(P\) in terms of the word length of \(P\) with respect to \(S\). Its proof is a straightforward consequence of Lemma 2.11.

**Lemma 2.12.** Let \(S\) be a finite generating subset for \(\mathbb{Z}/\mathbb{Z}\). There exists a constant \(C > 0\) satisfying the following. Let \(P \in \mathbb{Z}[x, x^{-1}]\), and let \(v \in \mathbb{Z}\). Finally, let \(x = (P, v)\). If \(a_\ell x^\ell\) is a monomial in \(P\), then \(|\ell| \leq C\|x\|_S\) and \(|a_\ell| \leq C\|x\|_S\).
3. Lower Bounds

In this section, we provide asymptotic lower bounds for \( \text{Conj}_{A,B}(n) \) where \( A \) and \( B \) are infinite, finitely generated abelian groups. We start with the group \( \mathbb{Z} \times \mathbb{Z} \) as seen in Proposition 3.2. The general case is done by Proposition 3.3. Before we start, we have the following lemma.

**Lemma 3.1.** Let \( m, n \in \mathbb{N} \) and \( d \in \mathbb{Z} \). Then
\[
(x^m - 1) \equiv (x^{\gcd(m,n)} - 1) \pmod{(x^n - 1, d)}.
\]

**Proof.** We note that \( m = \ell \gcd(m,n) \) for some integer \( \ell \). Therefore,
\[
x^m \equiv x^{\ell \gcd(m,n)} \equiv 1 \pmod{(x^{\gcd(m,n)} - 1)}.
\]
Hence, \( x^{\gcd(m,n)} - 1 \mid x^m - 1 \), and thus,
\[
(x^m - 1) \in (x^{\gcd(m,n)} - 1) \pmod{(x^n - 1, d)}.
\]

For the other inclusion, we note that there exist integers \( t, s \) such that \( \gcd(m,n) = tm + sn \). Hence, we may write
\[
(x^m - 1)(x^n - 1) = x^{tm+sn} - x^m - x^n + 1 \pmod{(x^n - 1, d)}
\]
\[
= x^{\gcd(m,n)} - x^n - (x^m - 1) \pmod{(x^n - 1, d)}
\]
\[
\equiv x^{\gcd(m,n)} - 1 - (x^m - 1) \pmod{(x^n - 1, d)}.
\]
Since \( (x^m - 1)(x^n - 1) \in (x^n - 1, d) \), we have
\[
x^{\gcd(m,n)} - 1 \equiv x^m - 1 \pmod{(x^n - 1, d)}.
\]
Hence,
\[
(x^m - 1) \equiv (x^{\gcd(m,n)} - 1) \pmod{(x^n - 1, d)}. \quad \square
\]

**Proposition 3.2.** \( (\log n)^n \preceq \text{Conj}_{\mathbb{Z}\times\mathbb{Z}}(n) \).

**Proof.** We need to find an infinite sequence of pairs of nonconjugate elements \( \{f_i, g_i\} \) such that \( \log(C \max(||f_i||,||g_i||)) \) \( \leq \) \( C \text{CD}_{\mathbb{Z}\times\mathbb{Z}}(f_i, g_i) \) for some \( C > 0 \). For ease of writing, we denote
\[
G = \mathbb{Z} \times \mathbb{Z} \cong \mathbb{Z}[x, x^{-1}] \times \mathbb{Z}
\]
where \( \mathbb{Z} \) acts by multiplication on \( \mathbb{Z}[x, x^{-1}] \) by \( x \).

Let \( \{q_i\} \) be an enumeration of the primes, and let \( \alpha(i) = \ell \text{lcm}(1, \ldots, q_i - 1) \). Finally, let \( k_i \) be the smallest integer such that \( \alpha(i) \leq 2^{k_i} \). We define the elements \( f_i, g_i \in \mathbb{Z} \times \mathbb{Z} \) as
\[
f_i = (\alpha(i)(x^{2^{k_i}} - 1), 2^{k_i}) \quad \text{and} \quad g_i = (\alpha(i)(x^{2^{k_i}} - 1 + x^{2^{k_i-1}} - 1), 2^{k_i}).
\]

To see that \( f_i \) is not conjugate to \( g_i \), we set \( t_i = (2^{k_i}, x^{2^{k_i}} - 1) \subseteq \mathbb{Z}[x, x^{-1}] \) to be an ideal and \( H = t_i \times q_i \mathbb{Z} \leq \mathbb{Z} \times \mathbb{Z} \). We also see that \( |(\mathbb{Z} \times \mathbb{Z})/H| = 2^{k_i} q_i^{2^{k_i}} \). We have that
\[
\pi_H(f_i) = (0, 0) \quad \text{and} \quad \pi_H(g_i) = (\alpha(i)(x^{2^{k_i-1}} - 1), 0) \neq 0
\]
where \( \pi_H : G \to G/H \) is the natural projection. Therefore, \( f_i \not\sim g_i \) in \( G \).

Now suppose that \( N \leq \mathbb{Z} \times \mathbb{Z} \) is a finite index subgroup where \( |\mathbb{Z} \times \mathbb{Z}/N| = q_i^{2^{k_i}} \). We will show that \( f_i N \sim g_i N \) in \( G/N \).

We note by Lemma 2.10 that \( \mathcal{J}_N = N \cap \mathbb{Z}[x, x^{-1}] \) is an ideal in \( \mathbb{Z}[x, x^{-1}] \). In particular, \( \mathcal{J} \times N \cap \mathbb{Z} \) is a normal subgroup in \( G \). Similarly, \( N \cap \mathbb{Z} = b\mathbb{Z} \) for some \( b \in \mathbb{Z} \). Therefore, we denote \( N' = \mathcal{J}_N \times b\mathbb{Z} \). We see that \( N \) is a finite index normal
subgroup where $N' \leq N$. In particular, if $f_i N \not\sim g_i N$ in $G/N$, then $f_i N' \not\sim g_i N'$ in $G/N'$. Therefore, we may assume that $G/N$ takes the form $\mathbb{Z}[x, x^{-1}]/\mathcal{J} \rtimes (\mathbb{Z}/b\mathbb{Z})$ where $\mathcal{J}$ is a cofinite ideal and $b$ is an integer.

Following Lemma 2.2 we see that $f_i N'' \sim g_i N''$ for $N'' = \mathcal{J}_N \rtimes (b\mathbb{Z} + K)$ where $K \leq \mathbb{Z}$ is the preimage under the projection modulo $b$ of the kernel of the action of $\mathbb{Z}/b\mathbb{Z}$ on $\mathbb{Z}[x, x^{-1}]/\mathcal{J}_N$. Letting $b_0 \geq 0$ be such that $b_0 \mathbb{Z} = b\mathbb{Z} + K$, we note that $N''$ is a finite index normal subgroup where

$$G/N'' = (\mathbb{Z}[x, x^{-1}] \rtimes \mathbb{Z}) / (\mathcal{J}_N \rtimes b\mathbb{Z}) \simeq \mathbb{Z}[x, x^{-1}]/\mathcal{J}_N \rtimes (\mathbb{Z}/b\mathbb{Z}).$$

Therefore, by the above discussion, we may assume that $G/N \cong \mathbb{Z}[x, x^{-1}]/\mathcal{J} \rtimes (\mathbb{Z}/b\mathbb{Z})$ where $\mathbb{Z}/b\mathbb{Z}$ acts faithfully on $\mathbb{Z}[x, x^{-1}]/\mathcal{J}$.

We now show we may assume that $\mathbb{Z}/b\mathbb{Z}$ acts freely on $\mathbb{Z}[x, x^{-1}]/\mathcal{J}$. Suppose that there are polynomials $\rho(x), \lambda(x) \in \mathbb{Z}[x, x^{-1}]/\mathcal{J}$ such that $x^m \rho(x) + \mathcal{J}_N = x^m \lambda(x) + \mathcal{J}_N$ for some $0 \leq m < b$. Since $x^m$ is a unit, we may cancel and write $\rho(x) + \mathcal{J}_N = \lambda(x) + \mathcal{J}_N$ which gives our claim.

Let $\ell$ be the multiplicative order of $x + \mathcal{J}_N$ in $\mathbb{Z}[x, x^{-1}]/\mathcal{J}$. We claim that $\ell = b$. By definition, we have that $b$ is the smallest integer such that $x^b \rho(x) + \mathcal{J} = \rho(x) + \mathcal{J}$ for all $\rho(x) \in \mathbb{Z}[x, x^{-1}]$. In particular, we have that $x^b \cdot 1 = 1 \mod \mathcal{J}$. Thus, we have that $\ell \mid b$. If $\ell \leq b$, then we have that $x^\ell \rho(x) = \rho(x) \mod \mathcal{J}$ for all $\rho(x) \in \mathbb{Z}[x, x^{-1}]$. However, that implies $\mathbb{Z}/b\mathbb{Z}$ doesn’t act faithfully on $\mathbb{Z}[x, x^{-1}]/\mathcal{J}$ which is a contradiction. Therefore, we have that $\ell = b$.

Since $\mathbb{Z}/b\mathbb{Z}$ acts freely and transitively on the set of powers of $x \mod \mathcal{J}$ in $\mathbb{Z}[x, x^{-1}]/\mathcal{J}$, we have that $[\mathbb{Z}[x, x^{-1}]/\mathcal{J}] = d^b$ where $d$ is the characteristic of the finite ring $\mathbb{Z}[x, x^{-1}]/\mathcal{J}_N$. We note that the ideal $(d, x^b - 1)$ is contained in the ideal $\mathcal{J}$ and that $[\mathbb{Z}[x, x^{-1}]/(d, x^b - 1)] = d^b$. It follows that $\mathcal{J}_N = (d, x^b - 1)$.

If $d < q_i$, then $d \mid \alpha(i)$, and subsequently,

$$\alpha(i)(x^{k_i} - 1), \alpha(i)(x^{k_i} - 1 + x^{2k_i-1} - 1) \in (d, x^{b_0} - 1).$$

Hence, $f_i = g_i \mod N$. Therefore, we may assume that $d \geq q_i$.

By Lemma 2.11 we may write the conjugacy class of $f_i$ as

$$\{(x^\ell (x^{2k_i} - 1) + (x^{2k_i} - 1)Q, 2^{k_i}) \mid \ell \in \mathbb{Z}, Q \in \mathbb{Z}[x, x^{-1}]\}.$$ 

Thus, we have that $f_i \sim g_i$ if and only if

$$x^{2k_i} - 1 + x^{2k_i-1} - 1 \in \{x^\ell (x^{2k_i} - 1) + (x^{2k_i} - 1)Q, \mid \ell \in \mathbb{Z}, Q \in \mathbb{Z}[x, x^{-1}]\}$$

which is equivalent to

$$x^{2k_i-1} - 1 \in \{x^\ell (1 + Q)(x^{2k_i} - 1) \mid \ell \in \mathbb{Z}, Q \in \mathbb{Z}[x, x^{-1}]\}.$$ 

Since $x^\ell + 1 + Q$ can be any Laurent polynomial, we have that $f_i \sim g_i$ if and only if

$$x^{2k_i-1} - 1 \in \{Q(x^{2k_i} - 1) \mid Q \in \mathbb{Z}[x, x^{-1}]\}.$$ 

By Lemma 3.1 we have that

$$x^{2k_i} - 1 \equiv x^{\gcd(2k_i,b)} - 1 \mod (d, x^b - 1).$$

Therefore, we may write the conjugacy class of $f_i$ in $(\mathbb{Z} \rtimes \mathbb{Z})/N$ as

$$\{\{Q(x^{2t} - 1) \mid Q \in \mathbb{Z}[x, x^{-1}]\} \mod (d, x^b - 1),$$

where $0 \leq t < k_i$. Therefore, $f_i N \sim g_i N$ if and only if

$$x^{2k_i-1} - 1 \in \{\{Q(x^{2t} - 1) \mid Q \in \mathbb{Z}[x, x^{-1}]\} \mod (d, x^b - 1).$$
Since $2^i \mid 2^{k_i-1}$, it is well known that $x^{2^i} - 1 \mid x^{2^{k_i-1}} - 1$. Therefore,

$$x^{2^{k_i-1}} - 1 \in \{(Q(x^{2^i} - 1) \mid Q \in \mathbb{Z}[x, x^{-1}] \mod (d, x^b - 1)$$

when $\gcd(2^{k_i-1}, b) \leq 2^{k_i-1}$. Hence, if $b \leq 2^{k_i-1}$, then $\gcd(2^{k_i-1}, b) \leq 2^{k_i-1}$, and subsequently, $f_i N \sim g_i N$. We see that

$$q_i^{2^{k_i}} \leq \text{CD}_{\mathbb{Z}/\mathbb{Z}}(f_i, g_i) < 2^{k_i} q_i^{2^{k_i}}.$$

Recall that $\alpha(i) = \exp\{v(q_i - 1)\}$, where $v: \mathbb{N} \to \mathbb{N}$ is the second Chebyshev’s function. The Prime Number Theorem [25, 1.2] then implies that there are constants $C_0^0, C_0^1 > 0$ such that $2^{C_0^0 q_i} \leq \alpha(i) \leq 2^{C_0^1 q_i}$. Following the definition of $k_i$, we see that there are constants $C_1^-, C_1^+ > 0$ such that $2^{C_1^- q_i} \leq 2^{k_i} \leq 2^{C_1^+ q_i}$. From the construction of the elements $f_i, g_i$, it can be easily seen that there is a constant $C'$ such that

$$\alpha(i)2^{k_i} \leq n_i \leq C'\alpha(i)2^{k_i},$$

where $n_i = \max\{|f_i||, |g_i||)$. Following the previous discussion, we see that there are constants $C_2^-, C_2^+ > 0$ such that $2^{C_2^- q_i} \leq n_i \leq 2^{C_2^+ q_i}$. In particular, we see that $q_i \leq \log(Cn_i)$ for some $C > 0$. Therefore, $q_i^{2^{k_i}} \geq \log(C^{-n_i})^{C^{-n_i}}$. To sum up, we constructed an infinite sequence of non-conjugate elements $f_i, g_i \in \mathbb{Z} \wr \mathbb{Z}$, where $n_i = \max\{|f_i|, |g_i|\}$, that are conjugate in every finite quotient of $\mathbb{Z} \wr \mathbb{Z}$ of size smaller than $\log(C^{-n_i})^{C^{-n_i}}$, where $C^{-} > 0$ is some constant. We see that

$$\log(C^{-n_i})^{C^{-n_i}} < \text{CD}_{\mathbb{Z}/\mathbb{Z}}(f_i, g_i),$$

meaning that

$$(\log(n))^n \leq \text{Conj}_{\mathbb{Z}/\mathbb{Z}}(n),$$

which concludes the proof. \qed

We finish this section by giving asymptotic lower bounds for all wreath products of finitely generated abelian groups where the acting group is infinite.

**Proposition 3.3.** Let $A$ and $B$ be finitely generated abelian groups where $B$ is infinite. If $A$ is infinite, then

$$(\log(n))^n \leq \text{Conj}_{A \wr B}(n).$$

Proof. Let us first note that we may decompose $A$ and $B$ as direct sums $A \simeq T_A \mathbin{\oplus} \mathbb{Z}^k$ and $B \simeq T_B \mathbin{\oplus} \mathbb{Z}^d$ where $T_A$ and $T_B$ are the torsion subgroups of $A$ and $B$, respectively. Since we assumed that both $A$, $B$ are infinite, we see that $d, k > 0$. In particular, $A$ contains an element $a$ of an infinite order such that $\langle a \rangle$ is a retract of $A$ and $B$ contains an element $b$ of an infinite order such that $\langle b \rangle$ is a retract of $B$.

By Lemma [24] we see that the subgroup $\mathbb{Z} \wr \mathbb{Z} \simeq \langle a \rangle \wr \langle b \rangle$ is a retract of $A \wr B$ and $\text{Conj}_{\langle a \rangle \wr \langle b \rangle}(n) \leq \text{Conj}_{A \wr B}(n)$. Therefore, Proposition 3.2 implies that

$$\log(n)^n \leq \text{Conj}_{\mathbb{Z}/\mathbb{Z}}(n) \leq \text{Conj}_{A \wr B}(n).$$

\qed
4. Upper bounds for wreath products of abelian groups

The following statement was proved in [11, Proposition 4.9].

**Proposition 4.1.** Let $A$ be an abelian group and $B$ be an infinite, finitely generated abelian group. Let $f, g: B \to A$ be finitely supported functions and $b \in B$ an element such that $fb, gb \in B_{\text{AIB}}(n)$, the elements $fb$ and $gb$ are reduced, and $fb \not\sim_{AIB} gb$. Then there exists a surjective homomorphism $\pi: B \to B$ to a finite group such that the elements $\tilde{\pi}(fb), \tilde{\pi}(gb)$ are reduced and $\tilde{\pi}(fb) \not\sim \tilde{\pi}(gb)$ in $A_1B$, where $\tilde{\pi}: A_1B \to A_1B$ is the canonical extension of $\pi$ to the whole of $A_1B$.

Moreover, there exists a constant $C > 0$ independent of $n$ such that if $B$ has torsion-free rank $1$, then we have $|B| \leq Cn$, and if $B$ is of torsion-free rank $k > 1$, we then have $|B| \leq Cn^{2k}$.

When given a wreath product of finitely generated abelian groups $A \wr B$, the following lemma will allow us to construct an upper bound for size of the quotient of the base group given two elements of $fb$ and $gb$ where $f, g: B \to A$ are finitely supported functions.

**Lemma 4.2.** Let $A$ and $B$ be finitely generated abelian groups where $A$ is infinite and $B$ is finite. Let $f, g: B \to A$ be finitely supported functions with $b \in B$ such that $fb, gb \in B_0(n)$, the elements $fb$ and $gb$ are reduced, and $fb \not\sim_{AIB} gb$. There exists a surjective homomorphism $\pi: A \to \overline{A}$ to a finite group $\overline{A}$ such that

$$|\overline{A}| \leq \min \left\{ \log(Cn)^2|B|, \log(Cn)^{Cn^2} \right\}$$

and where $\tilde{\pi}(fb) \not\sim \tilde{\pi}(gb)$ in $\overline{A} \wr B$ for some constant $C > 0$ independent of $f, g, b, n$.

**Proof.** Since $fb \not\sim_{AIB} gb$ and the elements $fb, gb$ are reduced, Lemma 2.7 implies that one of the following must true:

1. $\text{supp}(f)$ is not a translate of $\text{supp}(g)$ in $B$,
2. for every $a \in B$ such that $a + \text{supp}(f) = \text{supp}(g)$ there exists some $x \in \text{supp}(g)$ such that $f(x + a) \neq g(x)$.

Lemma 2.8 implies there exists a constant $C_1 > 0$ such that

$$|\text{supp}(f)|, |\text{supp}(g)| \leq C_1n$$

and that $\text{rng}(f),\text{rng}(g) \subseteq B_A(C_1n)$. Denote $R = \text{rng}(f) \cup \text{rng}(g)$. Clearly $|R| \leq 2C_1n$ and $R \subseteq B_A(C_1n)$.

First, we will show that there is a finite group $\overline{A}$ satisfying the requirements on conjugacy such that $|\overline{A}| \leq \log(Cn)^2|B|$. Suppose that $\text{supp}(f)$ is not a translate of $\text{supp}(g)$ in $B$. Since $A$ is a finitely generated abelian group, its residual finiteness depth function is equivalent to $\log(n)$. It then follows that for every $r \in R \subseteq B$ there is a normal finite index subgroup $K_r$ of $A$ such that $r \notin K_r$ and $|A : K_r| \leq \log(C_1n)$. Set $K = \cap_{r \in R}K_r$ with natural projection given by $\pi: A \to A/K$. As none of the elements in $R$ get mapped to the identity, we see that $\text{supp}(\tilde{\pi}(f)) = \text{supp}(f)$ and $\text{supp}(\tilde{\pi}(g)) = \text{supp}(g)$. In particular, we see that the elements $\tilde{\pi}(fb), \tilde{\pi}(gb)$ are reduced. It follows that $\text{supp}(\tilde{\pi}(f))$ is not a translate of $\text{supp}(\tilde{\pi}(g))$. Therefore, we have that $\tilde{\pi}(fb)$ is not conjugate to $\tilde{\pi}(gb)$ in $(A/K) \wr B$ by Lemma 2.7. Clearly,

$$|A/K| \leq \log(C_1n)^2|R| \leq \log(C_1n)^2|B|$$

Suppose that $\text{supp}(f)$ is a translate of $\text{supp}(g)$ and let $T \subseteq B$ be the set of all elements of $B$ that translate $\text{supp}(f)$ onto $\text{supp}(g)$. By assumption, for every $t \in T$
there is \( x \in B \) such that \( f(t + x) \neq g(x) \). For every such \( t \), there exists a normal finite index subgroup \( K_t \) of \( A \) such that \( f(t + x_t)K_t \neq g(x_t)K_t \) for some \( x_t \in B \) and \( |A : K_t| \leq \log(C_1n) \). Denote

\[
K = \bigcap_{r \in R} K_r \cap \bigcap_{t \in T} K_t,
\]

where \( K_r \) is defined as in the previous paragraph, and let \( \pi : A \to A/K \) be the natural projection. Clearly, \( \text{supp}(\tilde{\pi}(f)) = \text{supp}(f) \) and \( \text{supp}(\tilde{\pi}(g)) = \text{supp}(g) \) and, again, we see that the elements \( \tilde{\pi}(fb), \tilde{\pi}(gb) \) are reduced. From the construction of the \( K \) we see that for every \( t \in T \) there is \( x_t \in B \) such that

\[
\tilde{\pi}(f)(x_t + t) = \pi(f(x_t + t)) \neq \pi(g(x_t)) = \tilde{\pi}(g)(x_t),
\]

meaning that \( \tilde{\pi}(fb) \) is not conjugate to \( \pi(gb) \) by Lemma 2.7. To bound the index of \( K \), we can write

\[
|A/K| \leq \log(C_1n)^{|R|} \log(C_1n)^{|T|} \leq \log(C_1n)^{2|R|}.
\]

Now we show that there is a finite group \( \overline{A} \) satisfying the requirements on conjugacy such that \( \overline{A} \lneq \log(C_n)^{C_1n^2} \). Suppose that \( \text{supp}(f) \) is not a translate of \( \text{supp}(g) \) in \( B \). Since \( A \) is a finitely generated abelian group, its residual finiteness depth function is equivalent to \( \log(n) \). That means that for every \( r \in R \) there is \( K_r \subseteq A \) such that \( r \notin K_r \) and \( |A/K_r| \leq \log(C_2n) \) for some constant \( C_2 > 0 \). Set \( K = \bigcap_{r \in R} K_r \) with natural projection given by \( \pi : A \to A/K \). As none of the elements in \( R \) get mapped to the identity, we see that \( \text{supp}(\tilde{\pi}(f)) = \text{supp}(f) \) and \( \text{supp}(\tilde{\pi}(g)) = \text{supp}(g) \) and that the elements \( \tilde{\pi}(fb), \tilde{\pi}(gb) \) are reduced. It follows that \( \text{supp}(\tilde{f}) \) is not a translate of \( \text{supp}(\tilde{g}) \). Therefore, we have that \( \tilde{\pi}(fb) \) is not conjugate to \( \tilde{\pi}(gb) \) in \( (A/K)/B \) by Lemma 2.7. Clearly,

\[
|A/K| \leq \log(C_2n)^{|R|} \leq \log(C_2n)^{2C_1n} \leq \log(2C_2n)^{2C_1n}.
\]

Now suppose for every \( a \in B \) such that \( a + \text{supp}(f) = \text{supp}(g) \) there exists some \( x \in \text{supp}(g) \) such that \( f(x + a) \neq g(x) \). For every \( r \in R \), let \( K_r \) be defined as in the previous paragraph recalling that \( |A/K_r| \leq \log(C_0n) \). For every \( \{r, s\} \in R \) there is \( K_{r,s} \subseteq A \) such that \( r - s \notin K_{r,s} \) and \( |A/K_{r,s}| \leq \log(C_2n) \). Denote

\[
K = \bigcap_{r \in R} K_r \cap \bigcap_{\{r,s\} \in R} K_{r,s}
\]

with associated natural projection given by \( \pi : A \to A/K \). Following the same argument as in the previous case, we see that \( \text{supp}(\tilde{\pi}(f)) = \text{supp}(f), \text{supp}(\tilde{\pi}(g)) = \text{supp}(g) \), and the elements \( \tilde{\pi}(fb), \tilde{\pi}(gb) \) are reduced. Now suppose that \( a \in B \) is given such that \( a + \text{supp}(\tilde{\pi}(f)) = \text{supp}(\tilde{\pi}(g)) \). That is equivalent to \( a + \text{supp}(f) = \text{supp}(g) \). Thus, there is some \( x_a \in B \) such that \( f(x_a + a) \neq g(x_a) \). However, from the construction of \( K \) we see that

\[
\tilde{\pi}(f)(x_a + a) = \pi(f(x_a + a)) \neq \pi(g(x_a)) = \tilde{\pi}(g)(x_a).
\]

In particular, Lemma 2.7 implies that \( \tilde{\pi}(fb) \) is not conjugate to \( \tilde{\pi}(g) \) in \( (A/K)/B \). Finally, we finish with

\[
|A/K| \leq \log(C_2n)^{|R|} \cdot \log(2C_2n)^{|\{r,s\}|} \leq \log(2C_2n)^{2C_1n} \leq \log(4C_2n)^{4C_1n^2}.
\]

Combining \( 1 \) and \( 2 \) immediately yields the result. \( \square \)
Combining Proposition 4.1 together with Lemma 4.2 gives the following upper bound for wreath products of infinite, finitely generated abelian groups.

**Proposition 4.3.** Suppose that $A, B$ are infinite finitely generated abelian groups. If $B$ is virtually cyclic then

$$\text{Conj}_{A \wr B}(n) \leq (\log(n))^{n^2}.$$  

Otherwise,

$$\text{Conj}_{A \wr B}(n) \leq (\log(n))^{n^{2k+2}}$$

where $k$ is the torsion-free rank of $B$.

**Proof.** Let $k$ denote the torsion-free rank of $B$, and suppose that $n \in \mathbb{N}, f, g \in A^B$, and $b, c \in B$ are given such that $fb, gc \in B_{A \wr B}(n)$ and $fb \not\sim gc$.

Suppose first that $b \neq c$. Since $b - c \in B_B(2n)$, [4] Corollary 2.3 implies there exists a constant $C_1 > 0$ and a surjective homomorphism $\varphi: B \rightarrow Q$ such that $\varphi(b) \neq \varphi(c)$ and where $|Q| \leq C_1 \log(C_1 n)$. Since $A$ is abelian, we have that $\varphi(b)$ and $\varphi(c)$ are non-conjugate. By composing $\varphi$ with the projection of $A \wr B$ onto $B$ which we also denote $\varphi$, we have a surjective homomorphism $\varphi: A \wr B \rightarrow Q$ such that $\varphi(fb) \not\sim \varphi(gc)$ and where $|Q| \leq C_1 \log(C_1 n)$. Thus, we may assume that $b = c$.

Following Lemma 2.6 there exist a constant $C_0$ and functions $f', g' \in A^B$ such that $f'b \sim fb$ and $g'b \sim gb$, the elements $f'b, g'b$ are reduced, and $fb, gb \in B_{A \wr B}(C_0 n)$.

Following Proposition 4.1, there is a constant $C_2$ (independent of $n, f, g, b, c$) and a finite abelian group $\bar{B}$ together with a surjective homomorphism $\tilde{\pi}: B \rightarrow \bar{B}$ such that the elements $\tilde{\pi}_B(f'b)$ and $\tilde{\pi}_B(g'b)$ are reduced, $\tilde{\pi}_B(fb)$ is not conjugate to $\tilde{\pi}_B(gc)$ in $A \wr B$, where $|\bar{B}| \leq C_0 C_B n$ if $B$ has torsion free rank 1 and where $|\bar{B}| \leq C_0 C_B n^{2k}$ if $B$ has torsion free rank $k > 1$.

Set $\bar{G} = A \wr \bar{B}$. One can easily check that $\tilde{\pi}_B(B^G(C_0 n)) \subseteq B_{\bar{G}}(C_0 n)$. In particular, $\tilde{\pi}_B(f'b), \tilde{\pi}_B(g'b) \subseteq B_{\bar{G}}(C_0 n)$. Lemma 4.2 implies that there is a constant $C_A$ (also independent of $n, f, g, b, c$) and a finite abelian group $\bar{A}$ together with a surjective homomorphism $\tilde{\pi}_A: A \rightarrow \bar{A}$ such that

$$|\bar{A}| \leq \min \left\{ \log(C_A n)^{2|B|}, \log(C_A n)^{C_A n^2} \right\}$$

and where $\tilde{\pi}_A(\tilde{\pi}_B(fb))$ is not conjugate to $\tilde{\pi}_A(\tilde{\pi}_B(gc))$ in $\bar{A} \wr \bar{B}$.

Now, $\bar{A} \wr \bar{B}$ is a finite group. If $B$ has torsion free rank 1, then $|B| \leq C_0 C_B n$. Therefore, if $B$ is virtually cyclic, we set $C = 2C_0 C_B^2$, and we compute

$$|\bar{A} \wr \bar{B}| = |\bar{B}| \cdot |\bar{A}|^{2|\bar{B}|} \leq C_0 C_B n \cdot (\log(C_A n)^{2C_0 C_B n})^{C_0 C_B n^2} = C_0 C_B n \cdot (\log(C_A n))^{2C_0 C_B^2 n^2} \leq C n (\log(C n))^{C_A n^2}.$$  

Interpreting $|\bar{A} \wr \bar{B}|$ as a function of $n$ immediately yields

$$|\bar{A} \wr \bar{B}| \leq C n (\log(C n))^{C_A n^2} \approx n \log(n)^n \approx \log(n)^n.$$  

Therefore, we have that

$$\text{Conj}_{C}(n) \leq \log(n)^n.$$
Now suppose that \( k > 1 \). Set \( C = C_0 \mathcal{C}_A \mathcal{C}_B \). We then compute
\[
|A \wr B| = |B| \cdot |A|^{\mathcal{I}} \leq C_0 \mathcal{C}_B n^{2k} \cdot \left( \log(C_A n)^{C_A n^2} \right)^{C_0 \mathcal{C}_B n^{2k}} \\
= C_0 \mathcal{C}_B n^{2k} \cdot (\log(C_A n))^{C_0 \mathcal{C}_A \mathcal{C}_B n^{2k+2}} \\
\leq Cn^{2k} (\log(Cn))^{(Cn)^{2k+2}}.
\]
Interpreting \( |A \wr B| \) as a function of \( n \) immediately yields
\[
|A \wr B| \leq Cn^{2k} (\log(Cn))^{Cn^{2k+2}} \approx n^{2k} \log(n)^{n^{2k+2}} \approx \log(n)^{n^{2k+2}},
\]
and therefore,
\[
\text{Conj}_G(n) \leq \log(n)^{n^{2k+2}},
\]
which concludes our proof. \( \square \)

5. Proof of the main theorem and applications

This section is devoted to the proofs of the main theorems. We start with Theorem 1.1 which we restate for the reader’s convenience.

**Theorem 1.1** Let \( A \) be an infinite, finitely generated abelian group, and suppose that \( B \) is an infinite finitely generated abelian group. If \( B \) has torsion free rank 1, then
\[
(\log(n))^n \leq \text{Conj}_{AIB}(n) \leq (\log(n))^2.
\]
If \( B \) has torsion free rank \( k > 1 \), then
\[
(\log(n))^n \leq \text{Conj}_{AIB}(n) \leq (\log(n))^{n^{2k+2}}.
\]

**Proof.** For the upper bounds, we appeal to Proposition 4.3 and for the lower bound, we appeal to Proposition 3.3. \( \square \)

In the rest of this section, we use Corollary 1.2 to derive lower bounds for the conjugacy separability depth function where the acting group is not necessarily abelian. We start with the case when the acting group is nilpotent. For the statement of the theorem, we denote the center of a group \( G \) as \( Z(G) \).

**Theorem 5.1.** Let \( A \) be a nontrivial finitely generated abelian group, and suppose that \( G \) is a conjugacy separable finitely generated group with separable cyclic subgroups. Suppose that \( Z \leq Z(G) \). If \( A \) is finite, then
\[
2^n \leq \text{Conj}_{AIG}(n).
\]
Otherwise,
\[
(\log(n))^n \leq \text{Conj}_{AIG}(n).
\]

**Proof.** Let \( B = Z^k \leq Z(G) \). We claim that if \( (f, b), (g, b) \in A \wr B \) then \( x \sim_{AIG} y \) if and only if there exists \( z \in A \wr B \) such that \( z x z^{-1} = y \). Suppose first that \( x \sim_{AIG} y \). [10] Lemma 5.13 implies we may assume that elements of \( \text{supp}(f) \) (respectively \( \text{supp}(g) \)) lie in different right cosets of \( \langle b \rangle \). [10] Lemma 5.14 implies that \( x \sim_{AIG} y \) if and only if there exists a \( c \in C_G(b) \) such that \( c f c^{-1} = g \). That implies for all \( x \in G \), we have that \( f(cx) = g(x) \). We note that \( g(x) \neq 0 \) only if \( x \in B \). That implies \( f(cx) \neq 0 \) only if \( cx \in B \). Hence, \( c \in B \). In particular, if
\[(h, c) \in A \wr B\text{ where } c \notin B\text{, then } (h, c) \cdot x \cdot (h, c)^{-1} \neq y.\text{ Thus, if } (h, c) \cdot x \cdot (h, c)^{-1} = y,\text{ then } c \in B.\text{ Hence, we may write}
\[
(h, c)(f, b)(h, c)^{-1} = (h, c)(f, b)((c^{-1} \cdot h, c^{-1})
= (h + c \cdot f, cb)(c^{-1} \cdot h, c^{-1})
= (h + c \cdot f + (cb) \cdot c^{-1} \cdot h, g)
= (h + c \cdot f + b \cdot h, g).
\]
Thus, if \((h, c) \cdot x \cdot (h, c) = y\), we must have that
\[
supp(h + g' \cdot f + g \cdot h) \subseteq B.
\]
Suppose \(supp(h) \not\subseteq B\). In this case, we note that \(supp(h + g \cdot h) \not\subseteq B\) and that \(supp(c \cdot f) \subseteq B\). Therefore, \(supp(h + g' \cdot f + g \cdot h) \not\subseteq B\). Hence, we have that \((h, c) \cdot x \cdot (h, c)^{-1} \neq y\) which is a contradiction. Therefore, \(supp(f) \subseteq B\) which implies \((h, c) \in A \wr B\). Since the other direction is clear, we have our claim.

Subsequently, we have \(Conj_{A \wr B}(n) \leq Conj_{A \wr G}(n)\). Our theorem then follows from Corollary 1.2. □

Since the rank of the center of an infinite, finitely generated nilpotent group is always positive, we have the following corollary.

**Corollary 5.2.** Let \(A\) be a finitely generated abelian group, and let \(N\) be an infinite, finitely generated nilpotent group. If \(A\) is finite, then
\[
2^n \leq Conj_{A \wr N}(n).
\]
Otherwise,
\[
(log(n))^n \leq Conj_{A \wr N}(n).
\]
In both cases, we have that \(Conj_{A \wr N}(n)\) has at least exponential growth.

Finally, we consider the case when the acting group contains an isomorphic copy of the integers as a retract.

**Theorem 5.3.** Let \(A\) be a nontrivial finitely generated abelian group, and suppose that \(G\) is a conjugacy separable finitely generated group with separable cyclic subgroups that contains an infinite cyclic group as a retract. If \(A\) is finite, then
\[
2^n \leq Conj_{A \wr G}(n).
\]
Otherwise,
\[
(log(n))^n \leq Conj_{A \wr G}(n).
\]

**Proof.** Let \(g \in G\) be an element of infinite order such that \(\langle g \rangle\) is a retract in \(G\). Then by Lemma 2.4, we see that \(A \wr \langle g \rangle \cong A \wr \mathbb{Z}\) is a retract in \(A \wr G\) and \(Conj_{A \wr \langle g \rangle}(n) \leq Conj_{A \wr G}(n)\). The rest then follows by Corollary 1.2. □

**Corollary 5.4.** Let \(A\) be a finitely generated abelian group. And suppose that \(G\) belongs to one of the following classes of groups:

(i) right-angled Artin groups,
(ii) infinite finitely generated nilpotent and polycyclic groups,
(iii) limit groups,
(iv) fundamental groups of hyperbolic fibered 3-manifolds,
(v) graph products of any of the above.
If $A$ is finite, then
\[ 2^n \preceq \text{Conj}_{A \rtimes G}(n). \]

Otherwise,
\[ (\log(n))^n \preceq \text{Conj}_{A \rtimes G}(n). \]

Proof. Right-angled Artin groups were shown to be conjugacy separable by Minasyan in [19, Theorem 1.1], cyclic subgroup separable by Green in [12, Theorem 2.16], and checking that a right-right angled Artin group admits an infinite cyclic retract is easy - just consider an endomorphism that maps all but one generator to the identity.

For fundamental groups of closed orientable surfaces, they are known to be subgroup separable due to Scott [22, 23] and basic structure theory of abelian groups. The fact that they are conjugacy separable follows from Martino [17]. Lastly, it is clear that they have infinite cyclic retracts due to the fact that they have infinite abelianizations.

Infinite polycyclic and finitely generated nilpotent groups are well known to conjugacy separable and subgroup separable. Proofs of these results can be found in [24, Theorem 3, pg59] and [16]. The fact that these classes of groups follows from the fact that infinite polycyclic groups and infinite finitely generated abelian groups always have infinite abelianization by basic structure results found in [24].

For limit groups, it was shown by Wilton that they are subgroup separable in [5, Theorem A], and Chagas and Zalesskii demonstrated that they are conjugacy separable in [5, Theorem 1.1]. The fact that they have infinite cyclic retracts follows from the fact that limit groups are fully residually free.

The proof that fundamental groups of hyperbolic 3-manifold groups are conjugacy separable follows from Hamilton, Wilton, and Zalesskii [13, Theorem 1.3] and subgroup separability follows from [11, Corollary 4.2.3]. The fact that they have infinite cyclic retracts follows from these fact that these groups have the form $\pi_1(\Sigma_g) \rtimes \mathbb{Z}$ where $\Sigma_g$ is a closed orientable genus $g \geq 2$ surface.

The class of conjugacy separable groups is closed under forming graph products by [9, Theorem 1.1]. The class of cyclic subgroup separable groups is closed under forming graph products by [3, Theorem A]. Finally, to see that a graph product of groups that admits an infinite cyclic retract again admits an infinite cyclic retracts easily follows from the fact that all vertex groups are themselves retracts. 

The vast range of examples we were able to construct in this paper either with Theorem 5.1 and Theorem 5.3 lead us to believe that the lower bounds we produced cannot be relaxed and therefore we state the following conjecture.

**Conjecture 5.5.** Let $A$ be a finitely generated abelian group and let $G$ be a conjugacy separable group with separable cyclic subgroups. If $A$ is finite, then
\[ 2^n \preceq \text{Conj}_{A \rtimes G}(n). \]

Otherwise,
\[ (\log(n))^n \preceq \text{Conj}_{A \rtimes G}(n). \]

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