Analytical solution for the structure of ADAFs

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ABSTRACT
The standard advection-dominated accretion flow (ADAF) is studied using a set of self-similar analytical solutions in spherical coordinates. Our new solutions are useful for studying ADAFs without dealing with the usual mathematical complexity. We assume that the $r\phi$ component of the stress tensor dominates and the latitudinal component of the velocity is negligible; moreover, the fluid is incompressible and the solutions are radially self-similar. We show that our analytical solutions display most of the important properties of ADAFs that have already been obtained through detailed numerical solutions. According to our solutions, the density and pressure of the flow decrease from the equator to the polar regions and this reduction depends on the amount of advected energy. We also show analytically that an ADAF tends to a quasi-spherical configuration as more energy is advected with the radial flow.

Key words: accretion, accretion discs – black hole physics – galaxies: active.

1 INTRODUCTION
Various theoretical models have been proposed to understand accreting systems over at least the last four decades (e.g. Shakura & Sunyaev 1973; Ichimaru 1977; Anderson 1987; Abramowicz et al. 1988; Narayan & Yi 1994; Chen, Abramowicz & Lasota 1997; Narayan, Kato & Honma 1997; Blandford & Begelman 1999; Igumenshchev, Abramowicz & Narayan 2000). In these models, the mechanisms of energy transport in an accreting system and its radiative efficiency are among the most important physical factors. The standard model of accretion discs (Shakura & Sunyaev 1973) is successful in explaining the spectrum of some accreting systems such as discs around young stars (e.g. Hartmann 2000). However, radiatively inefficient accretion flows have also been proposed (e.g. Ichimaru 1977; Narayan & Yi 1994) to explain other astronomical objects like discs in active galactic nuclei (AGNs). This type of accretion flow is generally hot, because the heat due to the turbulence is advected with the flow instead of radiating out of the system. Although the idea of such flows was originally proposed by Ichimaru (1977), the first analytical description of advection-dominated accretion flows (ADAFs) was presented by Narayan & Yi (1994) using a set of height-integrated similarity solutions. ADAFs are generally hot and geometrically thick and their rotational velocity is sub-Keplerian, due to the non-negligible effect of the gradient of pressure in the radial direction. On the other hand, when there is efficient cooling the rotational profile tends to the Keplerian profile, which is similar to the standard disc configuration.

The accretion-flow type can be classified by the ratio $M/M_E$, where $M$ is the mass-accretion rate and $M_E$ is the Eddington accretion rate. The structure of the disc is described by the standard accretion model (Shakura & Sunyaev 1973), if we have $M \lesssim M_E$. For an accretion rate much smaller than the Eddington accretion rate (i.e. $M \ll M_E$), on the other hand, optically thin ADAF solutions are appropriate for describing the flow (e.g. Ichimaru 1977; Narayan & Yi 1994). The disc will be an optically thick or slim disc (e.g. Abramowicz et al. 1988) if the accretion rate becomes larger than the Eddington rate ($M \gg M_E$). Hot accretion seems to be applicable in describing the properties of accretion flows around black holes in X-ray binaries and AGNs (e.g. Greene, Ho & Ulvestad 2006; Körding, Jester & Fender 2006; Ludwig et al. 2012). Also, one of the best astronomical objects for the study of hot accretion flows is Sgr A* (e.g. Falcke & Melia 1997; Falcke & Biermann 1999; Bower et al. 2004; Yusef-Zadeh et al. 2006). To analyse the steady-state structure of ADAFs, one can start with the standard hydrodynamical equations in cylindrical or spherical coordinates. The original study of Narayan & Yi (1994) uses a height-integrated version of the basic equations in cylindrical coordinates. Subsequent studies extended these solutions extensively by considering various physical ingredients like magnetic field, thermal conduction or even outflows (e.g. Zhang & Dai 2008; Bu, Yuan & Xie 2009). Although these solutions are fully analytical, one should note that the three-dimensional structure of ADAFs cannot be described properly using these vertically averaged solutions, because height integration is a poor approximation when the flow is geometrically thick. These points motivated Narayan & Yi (1995, hereafter NY95) to re-analyse the steady-state structure of ADAFs in spherical polar coordinates, where the central mass is located at the centre of the system. These solutions have also been extended...

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by many authors over the years (e.g. Tanaka & Menou 2006; Xue & Wang 2005; Bu et al. 2009; Jiao & Wu 2011).

None of the previous works has reported a fully analytical solution for the structure of ADAFs in spherical polar coordinates, to the best of our knowledge. In this article, we report a fully analytical solution in spherical coordinates. Our analytical solutions, despite their simplicity, display most of the properties of the previous numerical or semi-analytical steady solutions of ADAFs in a spherical system. In the next section, the basic equations of a standard ADAF model in spherical coordinates are presented. We obtain similarity solutions in Section 3 and their properties are explored. We conclude with a summary of our results in the final section.

2 GENERAL FORMULATION

Our basic equations are the standard hydrodynamic equations in spherical coordinates \((r, \theta, \varphi)\), where an object with mass \(M\) is at its centre. Temporal variation of the physical quantities is not considered, which means the flow is steady-state. The flow is assumed to be axisymmetric and radial and rotational components of the velocity are considered. The latitudinal component of the velocity is assumed to be zero, i.e. \(v_\theta = 0\). However, this assumption is relaxed by some authors who are interested in investigating the steady-state structure of ADAFs with outflows (e.g. Jiao & Wu 2011). We do not consider outflows and so the continuity equation for our incompressible flow becomes

\[
\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho v_r) = 0, \tag{1}
\]

where \(\rho(r, \theta)\) and \(v_r(r, \theta)\) are the density and the radial component of velocity.

Now, we can write three components of the momentum equation. Assuming that the \(\varphi\) component of the viscosity stress tensor is dominant is a key assumption, which greatly simplifies the equations by reducing the number of terms. Doing so, viscous terms do not appear in the radial and latitudinal components of the equation of motion, but the viscous term has a vital role in the azimuthal component of the momentum equation (see also Jiao & Wu 2011). This simplification not only reduces the order of the differential equations, but also brings down the number of necessary boundary conditions. Moreover, the \(\alpha\) viscosity prescription is used. Shearing-box magnetohydrodynamics simulations have also shown that the vertically averaged stress is proportional to the vertically averaged total thermal pressure (e.g. Hirose, Blaes & Krolik 2009). Thus, the components of the equation of motion become

\[
\frac{\partial v_r}{\partial r} = \frac{GM}{r^2} - \frac{1}{\rho} \frac{\partial p}{\partial r}, \tag{2}
\]

\[
\frac{1}{r \rho} \frac{\partial p}{\partial \theta} - \frac{v_\theta^2}{r} \cot \theta = 0, \tag{3}
\]

\[
\frac{\partial v_\phi}{\partial r} + \frac{v_\theta v_r}{r} + \frac{1}{\rho r^2} \frac{\partial}{\partial \varphi} (r^2 t_{\varphi \varphi}) = 0, \tag{4}
\]

where \(p(r, \theta)\) and \(v_r(r, \theta)\) are the pressure and rotational velocity of the flow, respectively. Here, \(t_{\varphi \varphi}\) is the \(\varphi\) component of the viscosity tensor and we prescribe it based on the \(\alpha\) prescription, i.e. \(t_{\varphi \varphi} \approx -\alpha p\).

The energy equation reduces to a simple form, if we assume the \(r\varphi\) component of the stress tensor to be dominant. The energy equation is written as (see also Jiao & Wu 2011)

\[
\rho \frac{\partial e}{\partial t} + \frac{p}{\rho} \frac{\partial v_r}{\partial r} = f t_{\varphi \varphi} \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right), \tag{5}
\]

where \(f\) is the advective factor and \(e\) is the internal energy of the gas,

\[
\rho e = \frac{p}{\gamma - 1}, \tag{6}
\]

where \(\gamma\) is the heat capacity ratio. We also assume that both the input parameters, \(f\) and \(\gamma\), are constant.

Thus, equations (1)–(6) constitute our basic equations of the model to be solved subject to the appropriate boundary conditions. Most previous semi-analytical studies generally construct radially self-similar solutions, where the structure equations are solved numerically by integrating over polar angle. We also assume radial self-similarity, but the polar angle parts are obtained analytically. Moreover, the fluid is incompressible, the \(r\varphi\) component of the stress tensor dominates and \(v_\theta = 0\). In the next section, we obtain our analytical solutions based on these basic assumptions.

3 ANALYSIS

We introduce the following self-similar solutions:

\[
\rho(r, \theta) = \rho(\theta) r^{-3/2}, \tag{7}
\]

\[
v_r(r, \theta) = v_\theta(\theta) \sqrt{GM/r}, \tag{8}
\]

\[
v_\phi(r, \theta) = v_\phi(\theta) \sqrt{GM/r}, \tag{9}
\]

\[
p(r, \theta) = p(\theta) G M r^{-5/2}. \tag{10}
\]

Now, we can substitute the self-similar solutions into the above basic equations. Thus,

\[
5p(\theta) + p(\theta) [v_r(\theta)^2 - 2 + 2v_\phi(\theta)^2] = 0, \tag{11}
\]

\[
\frac{dp(\theta)}{d\theta} - \rho(\theta) v_\phi(\theta)^2 \cot \theta = 0, \tag{12}
\]

\[
\alpha p(\theta) + p(\theta) v_r(\theta) v_\phi(\theta) = 0, \tag{13}
\]

\[
(3\gamma - 5)v_r(\theta) - 3\alpha f(\gamma - 1)v_\phi(\theta) = 0. \tag{14}
\]

As we show below, these differential and algebraic equations are integrable. As boundary conditions, we assume the flow is symmetric with respect to the equatorial plane \(\theta = \pi/2\) and it is sufficient to know one of the physical quantities, say density. The rest of the quantities at the equatorial plane are obtained from the equations. We obtain the density at the equatorial plane from the accretion rate (see below).

We can now obtain the angular part of the solution analytically. To our knowledge, all previous self-similar solutions for ADAFs in spherical coordinates are not fully analytical. However, our simple
analytical solutions represent some of the basic features of ADAFs clearly. From the energy equation (14), we have
\[ v_r(\theta) = -\frac{\alpha}{\epsilon} v_\phi(\theta), \] (15)
where \( \epsilon' = \epsilon / f \) and \( \epsilon = (5/3 - \gamma)/(\gamma - 1) \). Substituting the above equation for \( v_r(\theta) \) into equation (13), we obtain
\[ p(\theta) = \frac{1}{\epsilon} \frac{\rho(\theta) v_\phi(\theta)}{\epsilon}. \] (16)
Using the above equation and equation (15), the rotational velocity is obtained from (11), i.e.
\[ v_\phi(\theta) = \frac{\sqrt{2 \epsilon}}{g(\alpha, \epsilon'}), \] (17)
and equation (15) gives
\[ v_r(\theta) = -\frac{\sqrt{2 \epsilon}}{g(\alpha, \epsilon'}). \] (18)
where \( g(\alpha, \epsilon') = \sqrt{\alpha^2 + 5 \epsilon' + 2 \epsilon'^2} \). Thus, both radial and rotational velocities have no dependence on the polar angle \( \theta \).

Having equation (16), we can integrate equation (12) analytically, i.e.
\[ p(\theta) = p \left( \frac{\pi}{2} \right) (\sin \theta)^{\epsilon'}, \] (19)
and then
\[ p(\theta) = p \left( \frac{\pi}{2} \right) (\sin \theta)^{\epsilon'}, \] (20)
where \( p(\pi/2) = (2\epsilon'/g') p(\pi/2) \).

The mass accretion rate \( \dot{M} \) is written as
\[ \dot{M} = -\int 2\pi r^2 \sin \theta \rho(r, \theta) v(r, \theta) d\theta. \] (21)

We determine the density at the equatorial plane by assuming that the accretion rate is fixed. By substituting our solutions into the above equation, we obtain
\[ \rho \left( \frac{\pi}{2} \right) = \frac{\dot{m}}{J(\epsilon')} \] (22)
where the non-dimensional accretion rate is \( \dot{m} = \dot{M} / (2\pi \sqrt{GM}) \) and
\[ J(\epsilon') = \int_0^{\pi} (\sin \theta)^{1+\epsilon'} d\theta = \sqrt{\pi} \left( \frac{1 + \epsilon'}{2} \right) \frac{\Gamma \left( 1 + \frac{\epsilon'}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{\epsilon'}{2} \right)}, \] (23)
where \( \Gamma \) is the standard Gamma function. Therefore,
\[ \rho \left( \frac{\pi}{2} \right) = \frac{g(\alpha, \epsilon') \Gamma \left( \frac{1}{2} + \frac{\epsilon'}{2} \right)}{\sqrt{2\pi} \alpha} \frac{\Gamma \left( \frac{1}{2} + \frac{\epsilon'}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{\epsilon'}{2} \right)} \dot{m} \] (24)
and
\[ \rho \left( \frac{\pi}{2} \right) = \sqrt{\frac{\pi}{2}} \frac{\epsilon'}{\alpha} \frac{\Gamma \left( \frac{1}{2} + \frac{\epsilon'}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{\epsilon'}{2} \right)} \dot{m}. \] (25)

Fig. 1 shows the profile of the density versus the polar angle for \( \dot{m} = 1 \) and different values of \( \epsilon' \). Each curve is labelled by the corresponding value of \( \epsilon' \). For a fixed \( \gamma \), a larger \( \epsilon' \) implies less energy advected with the flow. For small values of \( \epsilon' \), which means the flow is fully advective, we see that the density has little variation from the pole to the equator. Thus, the solution corresponds to a nearly spherical accretion flow. As the value of \( \epsilon' \) increases, however, the mass distribution concentrates more around the equatorial regions, so that the density contrast between \( \rho(0) \) and \( \rho(\pi/2) \) is enhanced significantly. A case with a large value of \( \epsilon' \) resembles a standard thin disc configuration, where there is significant cooling. The behaviour of the density distribution based on our analytical solution is very similar to what was shown by NY95 in their fig. 1.

We also found that both \( v_r(\theta) \) and \( v_\phi(\theta) \) are independent of the polar angle. The \( \theta \)-independence of the components of the velocity arises almost by construction, i.e. from equations (17) and (18). This is a major qualitative difference in comparison with the NY95 solutions, where the radial velocity is zero at the poles and increases monotonically towards the equator. NY95 showed that variation of \( v_r(\theta) \) with polar angle becomes less significant as the value of \( \epsilon' \) increases. Some authors also found outflows at the poles (see e.g. Tanaka & Menou 2006; Jiao & Wu 2011). As for the rotational velocity and the temperature, there is no polar dependence according to our new solutions. In the NY95 solutions, the square of the sound speed can vary by almost an order of magnitude between the pole and the equator. In contrast to these qualitative differences, we think, our \( \theta \)-independent values for the radial and rotational velocities and the sound speed are consistent with the polar-angle average of the NY95 solutions. Although our new solutions do not contain polar-angle dependence, these solutions recover these physical quantities in an angle-averaged sense.

We now explore the variation of the radial and rotational velocities with the input parameters. In Fig. 2, variations of the radial velocity (top) and rotational velocity (bottom) versus \( \alpha \) are displayed for different values of \( \epsilon' \). For a fixed value of \( \alpha \), as more dissipated turbulent energy is advected with the flow, the deviation of the rotational velocity from the Keplerian profile becomes more significant. We also showed that the radial velocity decreases with increasing \( \epsilon' \). Moreover, rotational velocity has little dependence on the value of \( \alpha \) unless the flow becomes fully advective. However, the radial velocity increases significantly with \( \alpha \), though its dependence on the value of viscosity coefficient becomes less significant as more energy radiates out of the system. Therefore, our solutions display in an angle-average sense most of the properties of the components of velocity that have already been obtained by NY95.
We also showed analytically that the geometrical thickness of the flow depends sensitively on the amount of advected energy. Note that the density and pressure distributions are obtained without a height-integration procedure. As more energy is advected with the radial flow, not only does the temperature increase, but also the density distribution tends to a quasi-spherical configuration. Although this typical behaviour of ADAFs has already been discussed through detailed numerical approaches, our model confirms this behaviour through an analytical solution. Moreover, the advantage of having analytic solutions for the structure of ADAFs allows one to use these solutions for modelling astrophysical systems where the properties of ADAFs are among the input parameters.

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Figure 2. Variation of radial velocity (top) and rotational velocity (bottom) versus the coefficient of viscosity $\alpha$ for different values of $\epsilon$. Each curve is labelled by the corresponding value of $\epsilon$.

We can also calculate the Bernoulli parameter $Be$ corresponding to our solutions. If this parameter is positive, then it means we may have outflows. Based on this hypothesis, NY95 calculated the Bernoulli parameter for their solutions and found ranges of the input parameters within which parameter $Be$ is positive. Subsequent studies studied outflows from ADAFs extensively using similarity methods (e.g. Xue & Wang 2005; Tanaka & Menou 2006; Jiao & Wu 2011). However, Tanaka & Menou (2006) found no relation between outflows and the Bernoulli parameter in their solutions. We note that the outflow found in Tanaka & Menou (2006) is driven by thermal conduction (as opposed to the way the viscous flow equations are solved). Here, we calculate $Be$ for our solutions to see if the profile of this parameter is similar to what was obtained by NY95. The non-dimensional Bernoulli parameter $b$ is written as

$$ b = \frac{Be}{\Omega_k R^2} = \frac{1}{2} v_r^2 + \frac{1}{2} (v_\theta \sin \theta)^2 - 1 + \frac{\gamma}{\gamma - 1} c_s^2, $$

where $c_s$ is the sound speed and $\Omega_k$ is the Keplerian angular velocity. Upon substituting our solutions into the above equation, we obtain

$$ b(\theta) = \left( \frac{\epsilon}{g} \right)^2 (\sin^2 \theta + 3 f - 2). $$

Clearly, the parameter $b(\theta)$ is positive for all $\theta$ for $f > 2/3 \approx 0.66$. However, this critical value of the advection parameter has been found by NY95 as $f \approx 0.446$. Our Bernoulli parameter behaves qualitatively differently from the behaviour of the NY95 solutions. More specifically, the new solutions have $b(\theta)$ increasing monotonically towards the equator, which is exactly the opposite of the behaviour of the NY95 solutions. In the new solutions, the radial and rotational velocity and the sound speed are all constant with respect to the polar angle, because of our approximations. In the previous solutions, however, all of these depended on the polar angle. Since the new solutions give values for $v_r$, $v_\theta$ and $c_s$ that are consistent with the polar-angle-averaged values of the previous solutions, computation of the Bernoulli parameter seems only to be valid in angle-averaged magnitude.

4 CONCLUSIONS

We reported a set of self-similar analytical solutions for the structure of ADAFs in spherical polar coordinates. Although our solutions are obtained without solving partial differential equations numerically, their physical properties are similar to the previous known ADAF solutions. Most of the previous similarity solutions for the steady-state structure of ADAFs in spherical coordinates are obtained through solving a set of differential equations subject to suitable boundary conditions at the pole and the equator, which seems to be a challenging problem. However, our analytical solutions could be useful for those authors wishing to study the properties of ADAFs without struggling with those numerical difficulties.

Because of our simplifying assumptions, physical quantities such as the velocity components and the sound speed are independent of the polar angle, but in an angle-average sense these solutions are consistent with previous solutions despite missing their polar-angle dependence entirely. Our similarity solutions are based on a few assumptions, such as neglecting all components of the stress tensor except the $r\phi$ component. This key hypothesis greatly reduces the number of terms in the momentum and energy equations. The solutions are also parametrized by a fixed accretion rate. This enabled us to calculate the density and the pressure at the equator analytically in terms of the non-dimensional accretion rate, coefficient of viscosity and amount of advected energy.

We also showed analytically that the geometrical thickness of the flow depends sensitively on the amount of advected energy. Note that the density and pressure distributions are obtained without a height-integration procedure. As more energy is advected with the radial flow, not only does the temperature increase, but also the density distribution tends to a quasi-spherical configuration. Although this typical behaviour of ADAFs has already been discussed through detailed numerical approaches, our model confirms this behaviour through an analytical solution. Moreover, the advantage of having analytic solutions for the structure of ADAFs allows one to use these solutions for modelling astrophysical systems where the properties of the ADAFs are among the input parameters.
