Impact of a finite cut-off for the optical sum rule in the superconducting state

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A single band optical sum rule derived by Kubo can reveal a novel kind of superconducting state. It relies, however, on a knowledge of the single band contribution from zero to infinite frequency. A number of experiments over the past five years have used this sum rule; their data has been interpreted in support of 'kinetic energy-driven superconductivity'. However, because of the presence of unwanted interband optical spectral weight, they necessarily have to truncate their sum at a finite frequency. This work examines theoretical models where the impact of this truncation can be examined first in the normal state, and then in the superconducting state. The latter case is particularly important as previous considerations attributed the observed anomalous temperature dependence as an artifact of a non-infinite cutoff frequency. We find that this is in fact not the case, and that the sign of the corrections from the use of a non-infinite cutoff is such that the observed temperature dependence is even more anomalous when proper account is taken of the cutoff. On the other hand, in these same models, we find that the strong observed temperature dependence in the normal state can be attributed to the effect of a non-infinite cutoff frequency.

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I. INTRODUCTION

Kubo\textsuperscript{1} formulated two optical sum rules: the first involves all electrons in the system under study, and relates the total integrated area under the real part of the optical conductivity to basic parameters, the electron charge, the bare electron mass, and the electron density. The second sum rule focusses on a single band near the Fermi level, and relates the integrated area associated with intraband transitions to a single particle property:

\[
W(T) \equiv \int_0^{+\infty} d\nu \text{Re} \left[ \sigma_{xx}(\nu) \right] = \frac{\pi e^2}{4h^2} \left\{ \frac{1}{N} \sum_k \frac{\partial^2 \epsilon_k}{\partial k^2} n_k \right\}
\]

(1)

Here, \(n_k\) is the single electron occupation number, and \(\epsilon_k\) is the dispersion relation. Unlike the first Kubo sum rule, this relation depends on the system particulars (like the band structure), and external parameters, like the temperature (since \(n_k\) varies with temperature).

Over the last decade a number of optical measurements have been made in the high temperature superconductors\textsuperscript{2,3,4,5,6,7}, which indicate an anomalous temperature dependence of the optical sum rule. Two noteworthy observations have been made: first, in the normal state, as the temperature is decreased, the optical sum, \(W(T)\) increases. This is expected, even in a model without interactions, because of the thermal factor in the occupation (Sommerfeld’s expansion). However, the observed increase is an order of magnitude higher than that expected from a non-interacting model\textsuperscript{8}. Much of this discrepancy can be explained through interactions\textsuperscript{9,10} and/or phase fluctuations\textsuperscript{11}. However, it is possible that more mundane explanations exist, as will be discussed below.

The second observation is that the optical spectral weight increases below the superconducting transition temperature in a variety of optimally doped and underdoped high \(T_c\) samples. The 'standard' BCS result is that the optical spectral weight should decrease below the superconducting transition temperature\textsuperscript{12}. It is this second observation in particular that has captured much interest\textsuperscript{13}. At present, the notion of 'kinetic energy-driven' superconductivity\textsuperscript{14} is reasonably well supported by current optical sum rule measurements.

However, the optical sum requires integration over the entire spectral range. There are technical difficulties with this program, many of which have been overcome\textsuperscript{15}. In addition, there is a difficulty that the intraband contributions (required for the sum rule quoted in Eq. 1 above) may not be so readily separated from the interband contributions. Experimentalists generally impose a cutoff in the frequency integration, and then vary that cutoff to look for cutoff effects. If these are minimal, then one is satisfied that the sum represents the intraband contributions alone, and hence the measurement is examining the property described by Eq. 1. Almost immediately following the first ab-plane sum rule measurements, Karakozov et al.\textsuperscript{16} suggested that both observations could be explained by the use of a finite cutoff frequency. They made estimates of the changes in the normal state due to a temperature dependent scattering rate caused by the electron-phonon interaction, and in the superconducting state due to a temperature dependent superconducting order parameter. These estimates were generally ignored by other workers in the field, who instead relied on the sum rule itself, and used the right hand side of Eq. 1 to study trends and parameter dependencies. Recently Norman et al.\textsuperscript{17} have investigated the cutoff effect in the normal state in considerable detail, and have found that a cutoff frequency as used in the experiments can indeed lead to a substantial temperature dependence of the optical sum in the normal...
state, in agreement with Karakozov et al. We provide further support for this conclusion with calculations below. This means that the strong temperature dependence observed in the normal state can be quantitatively understood with a weak coupling picture.

The primary contribution of this work is a model calculation of the effect of a finite cutoff frequency in the superconducting state. We find that a finite cutoff leads to a decrease in the optical spectral weight in the superconducting state (in a model where no change is expected). This means that the observed increase in the superconducting state represents a lower bound on the expected). This means that the observed increase in the superconducting state (in a model where no change is expected). This means that the observed increase in the superconducting state represents a lower bound on the expected. This result is in the opposite direction of the estimate of Karakozov et al.\textsuperscript{17}, their result was made for the dirty limit (Mattis-Bardeen\textsuperscript{19}) result, which turns out to have high frequency properties that are very distinct from those with non-infinite scattering rate. The conclusion is that the increase in the optical spectral weight in the superconducting state does indeed suggest a decrease in kinetic energy, or, in more conventional language, a collapse of electronic scattering, in the superconducting state.\textsuperscript{13,20,21}

In this paper we first show results in the normal state, for a simple model of electrons coupled to an Einstein boson mode. This leads to a temperature dependent scattering rate, which, as first shown in Ref.\textsuperscript{17}, leads to significant temperature dependence in the normal state, in qualitative agreement with the observations. The same temperature dependence has minimal effect at temperatures where the superconducting state sets in; the high frequency scattering rate is essentially unaffected by the onset of superconductivity. However, the temperature dependence of the order parameter will give rise to adjustments in the spectral weight at high frequency (as well as low), resulting in a potentially significant temperature dependence in the optical spectral weight. This is the subject of the second part of the paper.

\section{II. NORMAL STATE}

For the conductivity in the normal state we use the expression\textsuperscript{22}

\[
\sigma(\nu + i\delta) = \frac{\omega_p^2}{4\pi \nu} \int_{-\infty}^{+\infty} f(\omega - \nu) - f(\omega) \frac{d\Omega}{\Omega} \left[ \Sigma(\omega + i\delta) + \Sigma(\nu - \omega + i\delta) \right]_0^{\Omega_0},
\]

where $\omega_p$ is the bare plasma frequency, $1/\tau_{\text{imp}}$ is the electron-impurity scattering rate (taken here to be independent of wave vector and frequency), $f(\omega) = 1/(\exp(\beta\omega) + 1)$ is the Fermi function ($\beta = 1/(k_BT)$), and $\Sigma(\omega + i\delta)$ is the self energy due to the electron-boson interaction\textsuperscript{23}. Note that we have not attempted vertex corrections; these have been discussed (see, for example, Ref.\textsuperscript{24}), and are suspected to be small. For definiteness, we use for the self energy the standard Migdal result, obtained for electron-phonon scattering\textsuperscript{25,26}

\[
\Sigma(z) = \int_0^{\infty} d\Omega \alpha^2 F(\Omega) \left[ -2\pi i(N(\Omega) + \frac{1}{2}) + \psi\left(1 + \frac{\Omega - z}{2\pi T}\right) - \psi\left(1 - \frac{\Omega - z}{2\pi T}\right) \right],
\]

where $N(\Omega) \equiv 1/(\exp(\beta\Omega) - 1)$ is the Bose function, and $\psi(x)$ is the digamma function, and the entire expression is required at $z = \omega + i\delta$.

The desired calculation is to use Eq.\textsuperscript{2} in the partial optical spectral weight,

\[
W(\nu_e) = \frac{2}{\pi} \int_0^{\nu_e} d\nu \frac{\sigma(\nu)}{\omega_p^2/(4\pi)},
\]

where we have included constants so that the integral is dimensionless, and, moreover, $W(\nu_e \to \infty) = 1$. It is important to note that with an infinite cutoff, i.e. in the spirit of the single band Kubo sum rule, Eq.\textsuperscript{1}, Eq.\textsuperscript{4} with the conductivity from Eq.\textsuperscript{2} inserted yields no temperature dependence whatsoever. This is because Eq.\textsuperscript{4} was derived with a band structure with quadratic dispersion, so Eq.\textsuperscript{1} clearly indicates (on the right-hand-side) that a constant is expected. This is a good model to use then, because any temperature dependence observed from Eq.\textsuperscript{4} can be definitely attributed to the cutoff. Since the cutoff $\nu_e$ is often taken to be high compared to other energy scales in the problem, it suffices to determine the optical conductivity accurately at high frequencies. Karakozov et al.\textsuperscript{17} noted (see also Norman et al.\textsuperscript{18}) that at high frequency the conductivity is, in fact, Drude-like. Indeed, one obtains

\[
\sigma(\nu + i\delta) \approx \frac{\omega_p^2}{1/\tau_\infty - i\nu},
\]

where

\[
1/\tau_\infty(T) \equiv 1/\tau_{\text{imp}} + 2\pi \int_0^{\infty} d\Omega \alpha^2 F(\Omega) \coth\left(\frac{\beta\Omega}{2}\right).
\]

Then the partial sum rule expression\textsuperscript{4} can be integrated analytically, and one obtains

\[
W_{\text{Dr}}(\nu_e) \approx \left(2/\pi\right) \tan^{-1} \left(\frac{\nu_e}{1/\tau_\infty(T)}\right).
\]

How accurate is the expression in Eq.\textsuperscript{7} compared with that obtained by integrating the full expression in Eq.\textsuperscript{2} ? To answer this question and examine trends we use an Einstein boson with frequency $\omega_E$ and electron-boson coupling strength $\lambda$. Then $\alpha^2 F(\Omega) = \frac{\omega_E}{2} \delta(\Omega - \omega_E)$, and the integral required in the self energy can be done analytically. The frequency scale $\omega_E$ represents the mean frequency of a broader $\alpha^2 F(\Omega)$ spectrum, such as those used by Norman et al.\textsuperscript{22}.\textsuperscript{22}
In Fig. 1 we show the real part of the conductivity vs. frequency for a system of electrons interacting with impurities (1/\(\tau_{\text{imp}}\) = 10 meV) and with Einstein bosons (\(\lambda = 1\), and \(\omega_E = 60\) meV). In the low frequency range (part (a)), the Drude approximation does not agree at all with the more precise Kubo result, particularly at low temperatures. However, in the high frequency range (part (b)), the Drude approximation agrees very well with the Kubo result, particularly beyond 2 eV. In Fig. 2 we show similar results for an example system with parameters identical to those in Fig. 1, except that \(\omega_E = 20\) meV. It is clear that for lower boson frequency (same \(\lambda\)), the Drude approximation works very well for even lower frequencies.

Note that here we have purposefully used a high frequency scattering rate to reproduce well the high frequency part of the conductivity. As we remarked above, this is required for the partial optical sum. When an understanding of the low frequency conductivity is required, a low frequency Drude form can be obtained through a different approximation. Details are given in Ref. 27.

To examine the temperature dependence in the normal state, we show in Fig. 3 the optical spectral weight up to some frequency \(\nu_c\) (= 1 eV in this case) vs. temperature for several model systems. Note that in all cases the integral shown approaches unity for \(\nu_c\to \infty\). Several key points are evident in this figure. First, in the absence of inelastic scattering, (\(\lambda = 0\), uppermost curve), there is no temperature dependence. The full sum rule (corresponding to \(W(\nu_c = \infty) = 1\) with our normalization) is not quite achieved, due to the non-zero elastic scattering rate (we use 1/\(\tau_{\text{imp}}\) = 10 meV for all these results). For a fixed boson frequency, \(\omega_E = 40\) meV, we increase the electron-boson coupling, \(\lambda\) (top three curves). It is clear that (a) the overall weight decreases with increasing coupling, and (b) the temperature dependence also increases. Moreover, the Drude approximation (shown with symbols) is less accurate as boson coupling increases. All of this can be understood through Eq. (6), as the effective scattering rate increases with increasing coupling.

For fixed coupling strength, the effect of boson frequency is also shown in Fig. 3. Clearly, as the boson frequency increases (lower three curves), the sum rule is less fulfilled, the temperature dependence diminishes, and the Drude formula becomes less accurate. The decreased temperature dependence is due to a scaling with temperature from Eq. (6): \(1/\tau_{\infty} = 1/\tau_{\text{imp}} + \pi \lambda \omega_E \coth(\beta \omega_E/2)\). Less accuracy is achieved with the Drude formula for higher boson frequency because the ‘high’ frequency limit for the Drude formula is no longer achieved at 1 eV when the boson frequency becomes higher than about 50 meV. The actual temperature dependence of the partial sum rule is exponential (for an Einstein boson). However, when Fig. 3 is re-plotted vs. \(T^2\), linear behaviour is observed for many of the parameters, over the temperature range relevant for the normal state (say, above 90 K). Norman et al. also found remarkable agreement with a \(T^2\) temperature dependence, though it was not explicit in their case either. In any event no further attempt is made here to optimize the agreement with experiment; we have no doubt that this could be done, but would require adjustment of unknown boson spectral functions and unknown band structure parameters.

The key point (made by Norman et al.) is that the magnitude of change in the normal state can be very large (of order 2% or more). The origin of this variation with temperature is the finite cutoff. Clearly this variation easily surpasses the amount seen experimentally.3,4,5,6,7.
and so the first observation alluded to in the introduction may merely be due to a finite cutoff in the optical spectral sum.

III. SUPERCONDUCTING STATE

The question then arises, can a similar cutoff effect, in the same direction (i.e. increase with decrease in temperature), occur in the superconducting state, in which case the observed anomalous behaviour could also be attributed to a finite cutoff? The answer is no, as we now explain.

First, note that the integrations must be done with care, as the effects we are trying to discern are as little as one tenth of a percent. Hence, we will focus on zero temperature, where some of the integrals involved can be done more accurately. We define

$$\Delta W(\nu_c) \equiv W_S(\nu_c) - W_N(\nu_c),$$  \hspace{1cm} (8)

where the subscripts refer to the superconducting and normal states, respectively. Again, for the models discussed here, this quantity is zero as $\nu_c \to \infty$, as the right hand side of Eq. (11) is a constant.

First, how does the effective infinite frequency scattering rate change in the superconducting state? The an-
swear is that it does not at all. As Kaplan et al.\textsuperscript{28} showed, the scattering rate in the superconducting state is significantly more complicated than in the normal state. However, as might be expected, in the high frequency limit, this expression reduces to that in the normal state, given by Eq. (9). Hence one can imagine that, just as in the normal state, the important frequency dependence in the superconducting state will be encapsulated in the infinite frequency limit of $1/\tau(\omega)$. For this reason, for the rest of this paper we focus on the BCS limit of Eliashberg theory, where the order parameter is not frequency-dependent. Another reason for doing this is that it more clearly separates the two questions; the first, the impact of the temperature dependence of the scattering on the sum rule, has already been addressed by normal state calculations in the first part of the paper. For the second question, the impact of the temperature dependence of the order parameter, a BCS calculation allows us to focus only on this aspect, and does not include any remnant temperature dependence of the scattering rate.\textsuperscript{29}

Nonetheless, some caution may be in order. Inspection of Fig. 5 (below) best illustrates the problem; on the scale of this figure, one cannot tell whether the superconducting and normal state conductivities actually cross. It may be that some subtle refinement of the theory may alter this state of affairs; for example, everything discussed here applies for an order parameter with s-wave symmetry. To our knowledge no one has performed these difficult calculations with a d-wave order parameter. This will be the subject of future work.

A. Mattis-Bardeen limit

Following Karakozov et al.\textsuperscript{17} we first make another simplification — we use the Mattis-Bardeen (MB), or dirty limit, where analytical expressions (at $T = 0$) are available. We have\textsuperscript{18},

\[
\frac{\sigma_S(\nu)}{\sigma_0} = \frac{\pi^2}{4} \delta(\nu) + \theta(\nu - 1) \left( \frac{1 + \nu}{\nu} E(m) - \frac{2}{\nu} K(m) \right) \quad (9)
\]

where $\nu \equiv W/W_0$, with $W_0$ the zero temperature energy gap, and $m \equiv [(\nu - 1)/(\nu + 1)]^2$, with the complete elliptic integral of the first and second kind defined\textsuperscript{19},

\[K(m) \equiv \int_0^{\pi/2} d\theta \frac{(1 - m \sin^2 \theta)^{1/2}}{\sin \theta}, \quad \text{and} \quad E(m) \equiv \int_0^{\pi/2} d\theta (1 - m \sin^2 \theta)^{1/2}, \]

respectively. Here $\sigma_0$ is the zero frequency normal state conductivity. This expression is easily integrated up to some finite cutoff; however, we require the difference in the optical spectral weight defined by Eq. (8), so the high frequency part is of most interest. Since $\Delta W(\infty) = 0$, we have

\[
\Delta W(\nu_c) = -\int_{\nu_c}^{\infty} d\nu \left( \sigma_S(\nu) - \sigma_N(\nu) \right), \quad (10)
\]

and only the high frequency part need be obtained accurately in Eq. (9).

Alternatively, for the Mattis-Bardeen result, we find, for high frequency,\textsuperscript{21}

\[
\frac{\sigma_S(\nu)}{\sigma_0} \approx 1 - \left( \frac{\Delta_0}{\nu} \right)^2 \left[ 1 + 2 \log \frac{2\nu}{\Delta_0} \right]. \quad (11)
\]

Then the integral in Eq. (10) can be done analytically, and the result is

\[
\frac{\Delta W(\nu_c)}{W_N(\nu_c)} \approx \left( \frac{\Delta_0}{\nu_c} \right)^2 \left[ 3 + 2 \log \frac{2\nu_c}{\Delta_0} \right]. \quad (12)
\]

Note that $W_N(\nu_c) = \sigma_0\nu_c$. Both the conductivity and the optical spectral weight difference as defined by Eq. (10) are plotted in Fig. 4 (the latter is normalized to the

![Fig. 4](color online) The Mattis-Bardeen result for the optical conductivity (left scale) and the optical spectral weight (right scale), as a function of frequency (note the bottom scale is for the conductivity and the top scale is for the spectral weight). In the Mattis-Bardeen limit the normal state is a constant; this is indicated by the horizontal (green) line at unity. The full numerical result for the conductivity is labelled and shown by the solid (red) curve, including a delta-function contribution at the origin. The analytical result, given by Eq. (11), is also shown, and is in remarkable agreement with the numerical result right down to $\nu = 2\Delta_0$. The full numerical result for the optical spectral weight integrated up to a frequency $\nu$ is shown by the solid (pink) curve. The approximate result, Eq. (12), is indiscernible from the numerical result. Note that as the frequency increases, the Mattis-Bardeen result always lies beneath the normal state.
normal state optical integral), along with their approximate counterparts, Eq. (11) and Eq. (12), respectively. Remarkably, the approximation for the conductivity is barely distinguishable from the full result, all the way down to $\nu \sim 2\Delta_0$, while the approximation for the sum rule difference cannot be seen on this plot (it is beneath the numerical result). Note the difference in both horizontal and vertical scales for the two quantities. The (normalized) sum rule difference is still more than 1% at $30\times$ the gap energy (top horizontal scale). This indicates that, because of the non-infinite frequency cut off, the optical spectral weight is expected to increase by a 'small' amount. The experiments reporting an anomalous increase in the superconducting state in the a-b axis conductivity typically report a change of less than 1/2%, so the 'small amount' referred to above is really not so small.

If this was the entire story, one could conclude that all 'anomalous' observations, i.e. the strong temperature dependence in the normal state, and the anomalous increase in the superconducting state, can be attributed to the use of a non-infinite frequency cutoff, as suggested by Karakozov and coworkers. Then these experimental results would not indicate any novel kind of physics. However, as we now illustrate, the Mattis-Bardeen limit is pathological in this matter, and the optical sum rule in the case of a non-infinite scattering rate behaves qualitatively very differently than the MB limit.

B. Non-infinite scattering rate

To our knowledge, Chubukov et al first noticed that, within a BCS formalism, the conductivity in the superconducting state actually exceeds that in the normal state. For large scattering rates they found that the difference occurred at a frequency comparable to the scattering rate, $1/\tau$. This then implies that the optical sum, obtained by integrating out to some high frequency, will become lower in the superconducting state than in the normal state.

Thus for any non-infinite scattering rate, a finite frequency cutoff (presumed to be higher than the scattering rate) will result in a lowering of the optical sum in the superconducting state. Recall, that in the models we use here, provided we integrate to infinite frequency, there should be no change in the optical sum rule as a function of temperature, even in the superconducting state.
Thus, the observed increase actually underestimates the "true" amount, since experiments are unable to integrate the optical spectral weight out to infinite frequency.

To illustrate this, we again confine ourselves to zero temperature, where integrals, etc. can be done more precisely. A somewhat compact expression for the real part of the conductivity, for any scattering rate is given by\textsuperscript{33,34,35}

\[
\sigma_1(\nu) = \frac{ne^2}{m} \frac{1}{2\nu} \text{Im} \int_{\Delta_0}^{\nu-\Delta_0} d\omega \left\{ \frac{1 + N(\omega)N(\nu - \omega) - P(\nu)P(\nu - \omega)}{\epsilon(\nu - \omega) + \epsilon(\omega) - i/\tau} - \frac{1 - N(\omega)N(\nu - \omega) + P(\nu)P(\nu - \omega)}{\epsilon(\nu - \omega) - \epsilon(\omega) - i/\tau} \right\}
\]

where we have used $\sigma_0 = \frac{ne^2}{m}$. Note that $\epsilon(\omega) \equiv \sqrt{\omega^2 - \Delta_0^2}$, and $N(\omega) = \omega/\epsilon(\omega)$ and $P(\omega) = \Delta_0/\epsilon(\omega)$, and all quantities in Eq. (13) are real except for the explicit imaginary $i/\tau$ in the denominators. No special definition for the square-roots is required since the frequency $\omega$ is always positive\textsuperscript{36}.

Results from Eq. (13) are obtained numerically. However, for large frequencies, an expansion is possible, and we obtain, to second order in $\Delta_0/\nu$, for any value of $1/\tau$\textsuperscript{37}

\[
\frac{\sigma_{1S}(\nu)}{\sigma_0} \approx \left( \frac{1/\tau}{\nu^2 + (1/\tau)^2} \right) \left( 1 - 2\left( \frac{\Delta_0}{\nu} \right)^2 \left[ 1 + \log\left( \frac{2\nu}{\Delta_0} \right) \right] - 2\left( \frac{\Delta_0}{\nu} \right)^2 \left[ 1 - 2\log\left( \frac{2\nu}{\Delta_0} \right) \right] \right).
\]

These results (dashed green curves) are plotted along with the numerical results (dotted blue curves) in Fig. 5 for a variety of scattering rates. The high frequency expansion given by Eq. (14) is remarkably accurate over the entire frequency range. Also shown is the Mattis-Bardeen limit (see Eq. (11) above) which was encountered already in Fig. 4. For the present discussion, the important characteristic of Eq. (14) is that it crosses the normal state result (solid red curves) at a frequency close to that given by the numerical result. Note that in Fig. 5 a possible crossing is not even apparent; hence in Fig. 6 we plot the difference in conductivities vs. frequency. Once again solid red (dashed green) curves refer to the full numerical (approximate analytical (Eq. (13)) result. Here we see that the frequency $\nu_x$, at which the difference in the conductivities $\sigma_{1S} - \sigma_{1N}$ crosses zero, is reasonably well described by the asymptotic formula above (dashed green curves) compared with the numerical results (solid red curves). From inspection of the figure, as the scattering rate increases, $\nu_x$ also increases, and of course the approximation for this improves, since we used a high frequency expansion. Note how well the difference in the Mattis-Bardeen limit is described by Eq. (14) (or Eq. (11)). Analytically, the crossover frequency is given (approximately) by the solution to $\nu_x = \frac{1}{2} \sqrt{\frac{(1+1/\log u_x)}{1-2/\log u_x}}$, where $u_x \equiv 2\nu_x/\Delta_0$. This crossover frequency approaches (very slowly) $1/\tau$ as $1/\tau \to \infty$.

The important point, however, is that the result as seen in Fig. 6 (and not really apparent in Fig. 5) shows a qualitatively different behaviour for the difference in conductivities when a non-infinite scattering rate is present. At frequencies somewhat higher than the scattering rate, the conductivity in the superconducting state always overshoots, by a small amount, the conductivity in the normal state\textsuperscript{32}. This feature is not present in the Mattis-Bardeen limit, and the latter is misleading on this point.

It should now be apparent that an integration of the conductivity in the superconducting state, say, up to a frequency of about $30\Delta_0$, will yield a small spectral weight compared to that in the normal state (since the curves in Fig. 6 are greater than zero above $30\Delta_0$, and the total spectral weight of the difference up to infinite frequency has to integrate to zero, this statement follows). In Fig. 7 we plot the normalized difference in spectral weights (superconducting minus normal) up to a cutoff frequency $\nu_c$. As anticipated, for finite scattering rate the expected optical sum difference up to some cutoff $\nu_c$, is always less than zero. This means that the 'error' incurred by integrating up to some non-infinite frequency will reinforce the conventional\textsuperscript{14} BCS result that leads to a prediction that the optical sum difference is negative. The data measured for four different doping levels
is indicated by the symbols. Our results indicate that
the observed positive change (in underdoped and opti-
mally doped samples) may in fact be even slightly larger
once the finite cutoff is accounted for. The degree of
this correction depends on the scattering rate, which is
not known with any certainty. For example, with an as-
sumed scattering rate of \(20\Delta_0\), and \(\Delta_0 \approx 15-25\) meV,
this yields a value of \(1/\tau \approx 300-500\) meV. This appears
to be quite high, but Eq. \(1\) indicates that for a boson
mode with coupling strength \(\lambda = 1\) at \(\Omega_F = 50\) meV,
one obtains \(1/\tau \approx 300\). Thus values in this range can be
expected.

IV. SUMMARY

The main conclusion of this work is that the sin-

cular weight in the superconducting state compared to the

normal state, whereas in the more realistic case of a non-
infinite frequency cutoff, the opposite is true.

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37. Note that the result in Ref. 32 (see their Eq. 9) agrees with ours to logarithmic accuracy; we include other terms of order ($\Delta_0/\nu$)$^2$ so that our agreement with the numerical result is considerably better.