A mixed least-squares finite element formulation with explicit consideration of the balance of moment of momentum, a numerical study

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Abstract

Important conditions in structural analysis are the fulfillment of the balance of linear momentum (vanishing resultant forces) and the balance of angular momentum (vanishing resultant moment), which is not a priori satisfied for arbitrary element formulations. In this contribution, we analyze a mixed least-squares (LS) finite element formulation for linear elasticity with explicit consideration of the balance of angular momentum. The considered stress-displacement \( \boldsymbol{\sigma} - \mathbf{u} \) formulation is based on the squared \( L^2(\mathcal{M}) \)-norm minimization of the residuals of a first-order system of differential equations. The formulation is constructed by means of two residuals, that is, the balance of linear momentum and the constitutive equation. Motivated by the crucial point of weighting factors within LS formulations, a scale independent formulation is constructed. The displacement approximation is performed by standard Lagrange polynomials and the stress approximation with Raviart-Thomas functions. The latter ansatz functions do not a priori fulfill the symmetry of the Cauchy stress tensor. Therefore, a redundant residual, the balance of angular momentum \( \left( (\mathbf{x} - \mathbf{x}_0) \times (\text{div} \boldsymbol{\sigma} + \mathbf{f}) + \mathbf{axl}[\boldsymbol{\sigma} - \boldsymbol{\sigma}^T] \right) \), is introduced and the results are discussed from the engineering point of view, especially for coarse mesh discretizations. However, this formulation shows an improvement compared to standard LS \( \boldsymbol{\sigma} - \mathbf{u} \) formulations, which is considered here in a numerical study.

Keywords

balance of moment of momentum, linear elasticity, mixed least-squares finite element method

1 | INTRODUCTION

The consideration of finite element formulations using standard displacement methods are well established in structural analysis of engineering problems within the frameworks of small and finite deformations. Unfortunately, the applicability of standard displacement formulations is limited by certain constraints, which can lead to locking behavior and unreliable results in the displacement and stress fields, see for example, Babuška and Suri. An alternative to the primal finite element method are mixed variational principles where for example, displacements and stresses are approximated directly. One major advantage of mixed methods is the robustness in the presence of certain limiting situations, for example, incompressible or nearly...
incompressible materials. Here, the mainly used approaches are the Hellinger-Reissner type principles in terms of displacement and stresses and furthermore the Hu-Washizu type principles in terms of displacement, strains and stresses.\cite{2–4} The stability of this mixed variational principles are ensured for the framework of linear elasticity with combinations of polynomial degrees that provide stable and unique finite element (FE)-approximations, by the fulfillment of the Ladyzhenskaya-Babuška-Brezzi (LBB-) condition.\cite{5–7} However, the saddle-point structure and the associated stability issues can be avoided by use of the mixed least-squares (LS) finite element formulations, which result in a minimization problem.

The proposition of this work is the consideration of the mixed least square finite element method (LSFEM) which has already been successfully applied to several problems in computational mechanics, see for example, the textbooks by Jiang\cite{8} and Bochev and Gunzburger.\cite{9} In contrast to other mixed formulations the LSFEM is not restricted by the LBB-condition and therefore the combination of arbitrary polynomial orders for the approximation of the displacements \( u \) and stresses \( \sigma \) is in principle possible without losing stability properties. In addition, the resulting functional can be used as a posteriori error estimator, see Bochev and Gunzburger,\cite{9} and Cai and Starke\cite{10} and leads to positive definite system matrices. Nevertheless, disadvantages of the LSFEM are a weak performance for low order elements and the crucial impact of the residuals and their associated weightings on the accuracy of the solutions.\cite{11–13}

In the field of solid mechanics first formulations based on a div-curl-grad system of first-order in terms of velocity, vorticity and pressure for the Stokes equation with applications to linear elasticity are given for example, in Cai et al.\cite{14} First-order system LS formulations based on \( H(\text{div}) \)-conforming stress elements in combination with \( H(\text{div}) \)-conforming displacements were studied by Cai et al.\cite{10,20,21} where vector-valued functions of Raviart-Thomas type in \( H(\text{div}) \) space are considered.\cite{22} In Schwarz et al.\cite{13,23} the performance for nearly incompressible materials and the not a priori fulfillment of the stress symmetry, due to the approximation with Raviart-Thomas functions, is discussed. Furthermore, a modified first variation of the LS functional is introduced, which leads to an improvement of the performance of low order elements. An extension of the functional by adding an additional redundant residual for the fulfillment of stress symmetry is done in Schwarz et al.\cite{24,25} and Igelbücher et al.\cite{26}

The main aspect of this contribution is the investigation of the fulfillment of the balance of angular momentum, especially for coarse and moderate mesh densities. However, after some algebraic manipulations the balance of angular momentum is represented by two terms, the symmetry of the stress tensor \( \sigma = \sigma^T \) and the cross product of the balance of linear momentum with the related distance to a fixed reference point \((x - x_0) \times (\text{div} \sigma + f)\), where \( f \) denotes the body force. From the continuum mechanical point of view the balance of angular momentum is ensured by the symmetry of the Cauchy stress tensor, if in addition the balance of momentum is exactly fulfilled, that is, \( \text{div} \sigma + f = 0 \). As mentioned before, the discrete counterpart of the LS functional, based on the RT\textsubscript{m} approximation of the stresses does not a priori fulfill the stress symmetry condition, which is fulfilled in the limit case \( h \to 0 \), where \( h \) denotes the characteristic finite element size (diameter). However, the symmetry of the stress tensor is controlled by the right side of the inequality \( \|\sigma - \sigma^T\|_{L^2(B)}^2 \leq c\|\sigma - \sigma + \mu \nabla \cdot \psi(\epsilon)\|_{L^2(B)}^2 \), see Cai and Starke.\cite{10}

The stress-displacement based mixed LS formulations as presented in the functional below, consisting of the balance of linear momentum and the constitutive equation is given by

\[
F(\sigma, u) = \frac{1}{2} \left\| \frac{\omega_m}{\mu} (\text{div} \sigma + f) \right\|_{L^2(B)}^2 + \frac{1}{2} \left\| \frac{\omega_c}{\mu} (\sigma - \mathbb{C} : \nabla u) \right\|_{L^2(B)}^2 ,
\]

with the shear modulus \( \mu \) and the fourth order elasticity tensor \( \mathbb{C} \). A drawback of this formulation is the unsatisfying fulfillment of the equilibrium of forces \( R = \sum F_i = 0 \) and equilibrium of moments \( M_R(x_0) = 0 \) w.r.t. a fixed reference point \( x_0 \).

An illustration of the equilibrium of forces and moments is demonstrated by consideration of a cantilever beam of dimensions 5 × 1. The left side of the cantilever beam is clamped and a boundary traction of \( \sigma \cdot n = 0 \) w.r.t. a fixed reference point \( x_0 \).

An example is investigated by use of the underlying standard stress-displacement LS formulation (1). For the investigation of this boundary value problem we calculate the resulting support reactions \((A_H, A_V, M_A)\) based on the equilibrium of forces and moments. A comparison is performed based on the analytical solution \((A_H = 0, A_V = 0.1, M_A = 0.5)\) and a linear standard displacement element \( P_1 \). The support reactions are determined based on nodal reaction forces for the displacement element \( P_1 \) and based on the resulting tractions for the mixed LS method with an RT\textsubscript{1}/P\textsubscript{2} element type evaluated at the clamped left side (\( \partial B_a \)). The displacement element yields to an explicit fulfillment of the support reactions for forces and moments (see Figure 1). In contrast to the displacement element the LS formulation leads to an unsatisfying enforcement of the support reactions \( A_V \) and \( M_A \). The result for the horizontal force \( A_H \) at \( \partial B_a \) show for all weightings \( \alpha = \{ \omega_m, \omega_c \} \) a very good agreement to the analytical solution of \( A_H = 0 \). For the vertical force \( A_V = 0.1 \) only for the weighted formulation with \( \omega_c \leq 0.1 \) a satisfying solution is obtained. The calculations using \( \alpha_1 = \{ 1, 1 \} \) and \( \alpha_2 = \{ 1, 0.5 \} \) yield no adequate results. A motivation
FIGURE 1  Clamped cantilever: Setup, exemplary coarse mesh density and results for the fulfillment of equilibrium of forces and moments plotted vs the number of equations $neq$ ($E = 70$ kN/mm², $\mu = 26.12$ kN/mm²)

for the applied weighting parameters is given later on in the numerical study. However, the resulting moment $M_A$ with respect to the point (0, 0) ($M_A = 0.5$) is not fulfilled for any setup of weighting parameters for the standard stress-displacement LS formulation (1), as depicted at the top right in Figure 1. In comparison with the satisfying solution of the standard displacement formulation all weighted LS formulations show an insufficient approximation of the support reaction for the moment. Based on this an improvement of the formulation will be introduced in the following, especially for the correct representation of support reactions of moment $M_A$ and vertical force $A_V$.

In this contribution, we investigate a LS formulation additionally based on the balance of angular momentum, in order to present an improvement of the LS approximation quality w.r.t. the support reaction for the moment and vertical force. The paper is outlined as follows, at first the underlying fundamentals of the method, as for example, the construction of the functional and the derivative of the residual forms are briefly introduced. Furthermore, an extensive numerical investigation of the proposed formulation is performed by means of convergence studies with respect to displacement and stresses as well as convergence order of the LS functional. Additionally, the crucial impact of weighting factors on the approximation qualities of the LS functional is presented for problems in solid mechanics.

2  THE MIXED LS FINITE ELEMENT METHOD

2.1  Construction and fundamentals

In the following we introduce the mixed LS finite element method based on minimizing the $L^2(B)$-norm of the residuals in the first-order system of differential equations, cf. for example, Jiang[8] and Bochev and Gunzburger.[9] Introducing the body of interest $B$, parameterized in $x \in \mathbb{R}^3$, with the boundary $\partial B$ consisting of two subsets, the Dirichlet boundary $\partial B_D$ and the Neumann boundary $\partial B_N$, satisfying

$$\partial B = \partial B_D \cup \partial B_N \quad \text{and} \quad \partial B_D \cap \partial B_N = \emptyset.$$  (2)

The construction of the first-order system LS in terms of displacements $u$ and stress field $\sigma$ is described for the theory of linear elasticity for example, in Cai and Starke.[10] As a construction rule for the minimization problem by application of the squared
$L^2(B)$-norm to a first-order system of $n$ (differential) equations given in residual forms $R_i$, we define

$$P = \sum_{i=1}^{n} \frac{1}{2} \| \omega_i R_i \|_{L^2(B)}^2 = \sum_{i=1}^{n} \int_B \frac{1}{2} \omega_i^2 |R_i| \cdot R_i \, dV \rightarrow \text{min.}$$

(3)

Here, the functional is depending on the weighting factors denoted with $\omega_i$ and the $L^2(B)$-norm is given by

$$\| \cdot \|_{L^2(B)} = \left( \int_B | \cdot |^2 \, dV \right)^{1/2},$$

(4)

compare, for example, Jiang[8] and Bochev and Gunzburger.[9] For the proposed formulation the governing system of differential equations in residual form is defined in terms of the balance of momentum $R_m$ and the constitutive equation as Hooke’s law $R_c$, that is,

$$R_m = \text{div}\sigma + f \quad \text{and} \quad R_c = \sigma - C : \varepsilon,$$

(5)

where the symmetric strain tensor is defined as $\varepsilon = \nabla \cdot u = \frac{1}{2} (\nabla u + \nabla^T u)$. $f$ denotes the body force and the fourth-order elasticity tensor is introduced as $C = A \otimes I + 2\mu I$ or in index notation given by $C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + 2\mu \delta_{ik} \delta_{jl}$, in terms of the fourth-order identity tensor $I$, the second-order identity tensor $I$ and the Lamé constants $\lambda$ and $\mu$. As mentioned before the stress symmetry condition of the Cauchy stress tensor $\sigma$ is not fulfilled a priori based on the utilization of $RT_m$ functions for the stress approximation and only enforced in a weak sense, see for example, Boffi et al.[27] and Cockburn et al.[28] Nevertheless, it is controlled through the right side of the inequality

$$\| \sigma - \sigma^T \|_{L^2(B)}^2 \leq c \| \sigma - \partial_t \psi(\varepsilon) \|_{L^2(B)}^2 = c \| \sigma - C : \varepsilon \|_{L^2(B)}^2,$$

(6)

where $\psi(\varepsilon) = \frac{1}{2} \varepsilon : C : \varepsilon$ denotes the free energy function and $c$ is a positive constant.[10] However, an additional control of this symmetry constraint, as suggested by Schwarz et al.[24] is performed with the introduction of the balance of moment of momentum. The balance of moment of momentum states that the material time derivative of the moment of momentum, $\frac{d}{dt} \int_B (x - x_0) \times \rho \hat{x} \, dV$, with respect to a fixed reference point $x_0$, is equal to the resultant moment of all forces acting on $B$:

$$\int_B (x - x_0) \times \rho \hat{x} \, dV = \int_B (x - x_0) \times f \, dV + \int_{\partial B} (x - x_0) \times t \, dA.$$  

(7)

Here, the traction vector is denoted by $t$, the density by $\rho$, the velocity by $x$ and the acceleration by $\hat{x}$. Neglecting accelerations and using Cauchy’s theorem ($t = \sigma \cdot n$) yields

$$\int_B (x - x_0) \times f \, dV + \int_{\partial B} (x - x_0) \times \sigma \cdot n \, dA = 0.$$  

(8)

Applying Gauss theorem and exploiting the identity $\nabla(x-x_0) = I$ leads to

$$0 = \int_B ((x-x_0) \times (\text{div}\sigma + f) + axl[\sigma - \sigma^T]) \, dV \quad \text{with} \quad axl[\sigma - \sigma^T] := \begin{pmatrix} \sigma_{32} - \sigma_{23} \\ \sigma_{13} - \sigma_{31} \\ \sigma_{21} - \sigma_{12} \end{pmatrix}.$$  

(9)

Therefore, the third residual of the proposed LS functional is defined by

$$R_a = (x - x_0) \times (\text{div}\sigma + f) + axl[\sigma - \sigma^T].$$  

(10)

The construction of the single parts of the functional by applying the squared $L^2(B)$-norm and introducing weighting factors $\omega_i$ with $i = m, c, a$ yields:

$$F_m = \frac{1}{2} \| \omega_m (\text{div}\sigma + f) \|_{L^2(B)}^2,$$
\[ F_c = \frac{1}{2} \| \omega_c (\sigma - C : \nabla \mathbf{u}) \|_{L^2(B)}^2, \]
\[ F_a = \frac{1}{2} \| \omega_a ((x-x_0) \times (\text{div} \sigma + f) + \text{axl}[\sigma - \sigma^T]) \|_{L^2(B)}^2, \]

The complete LS functional for linear elasticity depending on the balance of momentum, the constitutive equation and the balance of angular momentum is given with the abbreviation \( r = x - x_0 \) by

\[ F = \frac{1}{2} \int_B \omega_m^2 (\text{div} \sigma + f) \cdot (\text{div} \sigma + f) \, dV + \frac{1}{2} \int_B \omega_c^2 (\sigma - C : \epsilon) : (\sigma - C : \epsilon) \, dV \]
\[ + \frac{1}{2} \int_B \omega_a^2 \{ r \times (\text{div} \sigma + f) + \text{axl}[\sigma - \sigma^T] \} \cdot \{ r \times (\text{div} \sigma + f) + \text{axl}[\sigma - \sigma^T] \} \, dV. \]  
(12)

For the minimization we require that the first variation of the functional with respect to displacements and stresses have to be equal to zero, \( \delta_u F \) and \( \delta_\sigma F \). The construction of the first variation of the formulation based on \( \mathbf{u} \) and \( \sigma \) with \( \delta \mathbf{r} = 0 \), yields

\[ \delta F = \int_B \omega_m^2 \text{div} \delta \sigma \cdot (\text{div} \sigma + f) \, dV + \int_B \omega_c^2 (\delta \sigma - C : \nabla^i (\delta \mathbf{u})) : (\sigma - C : \nabla^i \mathbf{u}) \, dV \]
\[ + \int_B \omega_a^2 \{ r \times \text{div} \delta \sigma + \text{axl}[\delta \sigma - \delta \sigma^T] \} \cdot \{ r \times (\text{div} \sigma + f) + \text{axl}[\sigma - \sigma^T] \} \, dV, \]  
(13)

and the second variations reads

\[ \Delta \delta F = \int_B \omega_m^2 \text{div} \delta \sigma \cdot \text{div} \Delta \sigma \, dV + \int_B \omega_c^2 (\delta \sigma - C : \nabla^i (\Delta \mathbf{u})) : (\Delta \sigma - C : \nabla^i \Delta \mathbf{u}) \, dV \]
\[ + \int_B \omega_a^2 \{ r \times \text{div} \delta \sigma + \text{axl}[\delta \sigma - \delta \sigma^T] \} \cdot \{ r \times \text{div} \Delta \sigma + \text{axl}[\Delta \sigma - \Delta \sigma^T] \} \, dV. \]  
(14)

Following the approach of considering a formulation which is scale independent, similar to Bell and Surana,[29] which used dimensionless variables, a physical weighting of the residuals, is performed by means of the Lamé constant \( \mu \) and a characteristic length \( \tilde{l} \). Therefore, the weighting factors \( \omega_i \) are chosen to be of the form

\[ \omega_m = \frac{\alpha_m \tilde{l}}{\mu}, \quad \omega_c = \frac{\alpha_c}{\mu}, \quad \text{and} \quad \omega_a = \frac{\alpha_a}{\mu}, \]  
(15)

where \( \alpha_i \) denotes the new dimensionless (ie, unit 1) weighting factors. This weighting parameters fulfill the condition of a scale independent functional, that is, we consider a dimensionless formulation of the functional. The parameter \( \tilde{l} \) have to be chosen reasonably, possible measures are for example, the length, height or the square root of the area of the boundary value problem. A discussion of the characteristic length and the influence of the weighting parameters will be investigated in more detail as a part of the numerical examples.

### 2.2 Discretization

The linear problem is solved by minimizing the functional with respect to appropriate finite-dimensional spaces. Therefore, let \((u, \sigma) \in V \times X\) be the solution spaces for the displacement and stress field and their discrete counterparts \( V_h \) and \( X_h \) based on a triangulation \( T_h \) of \( B \). Here, we choose for a conforming discretization the following function spaces:

\[ V = H^1(B) = \{ u \in L^2(B)^d : \nabla u \in L^2(B)^d \}, \]  
(16)

and

\[ X = H(\text{div}, B) = \{ \sigma \in L^2(B)^{d \times d} : \text{div} \sigma \in L^2(B)^d \}, \]  
(17)

respectively, with the dimension \( d \). We introduce the corresponding finite element subspaces \( V_h^k \) and \( X_h^m \), ie, \( V_h^k \) is the space of continuous piecewise polynomial functions of order \( k \geq 1 \), chosen as Lagrange functions \( (P_k) \), and \( X_h^m \) is the space of continuous piecewise polynomial functions of order \( m \), given by vector-valued Raviart-Thomas functions \( (RT_m) \), see for example, Schröder et al.[30]
2.3 Rate of convergence

A detailed description of the determination of optimal convergence rates is discussed, for example, in Brenner and Scott, [31] Boffi et al., [32] and Bathe, [33] For the determination of the optimal rate of convergence in the theory of elastic problems, we apply

\[ \|u - u_h\|_{H^1(B)} \leq c h^{m-1} |u|_{H^m(B)} \]

with the constant \(c\) independent of \(h\) and \(m = k + 1\) depending on \(k\) as the degree of complete polynomial. In general, \(h\) is chosen as the length of a side of an element or as the diameter of a circle circumscribing the element. Following Brenner and Scott, [31] we define 0 \(\leq s \leq \min\{m, r + 1\}\) with the continuity of the finite element solution \(C^r\) for \(r \geq 0\). For considering complete polynomials in \(x\) and \(y\) for the two dimensional case, for triangular elements all possible terms of the form \(x^\alpha y^\beta\) are present, with \(\alpha + \beta = k\) and \(k\) denotes the degree to which the polynomial is complete, compare chapter 4.3.5 in Bathe, [33] The degree of a complete polynomial for the displacement approximation is based on \(\mathbf{P}_2\) given with \(k = 2\). Furthermore, here \(s = 1\) due to \(u \in H^1(B)\) and therefore we obtain

\[ \|u - u_h\|_{H^1(B)} \leq c h^2 |u|_{H^1(B)} \rightarrow \mathcal{O}(h^2). \]

Furthermore, for the stress approximation based on quadratic Raviart-Thomas functions \(RT_1\) we have, \(k = 1\) and with \(q \in H^m(B)\) and \(\text{div} q \in H^r(B)\)

\[ \|q - q_h\|_{H^1(B)} \leq c h^n |q|_{H^m(B)}, \]

\[ \|\text{div}(q - q_h)\|_{H^0(B)} \leq c h^n |\text{div} q|_{H^r(B)}, \]

with \(n \leq k + 1\), which yields

\[ \|q - q_h\|_{H^1(B)} \leq c h^2 |q|_{H^1(B)} \rightarrow \mathcal{O}(h^2), \]

\[ \|\text{div}(q - q_h)\|_{H^0(B)} \leq c h^2 |\text{div} q|_{H^1(B)} \rightarrow \mathcal{O}(h^2). \]

For combining these estimates for the displacements and stresses, we expect, under the assumption of a \(RT_1P_2\) finite element type, an optimal rate of convergence for the theory of elastic problems of two.

The obtained order of convergence for an underlying boundary value problem is crucially influenced for example, by the regularity of the boundary value problem, the optimality of the mesh regularity and the used refinement strategy, that is, regular or adaptive refinement strategies. It must be noted, that the simple application of \(L^2\)-norms to residuals stemming from different continuum equations does not a priori lead to norm-equivalence. But, in the framework of this work, norm-equivalence is not part of the investigated issues. For further information on the method, the reader is referred to for example, Bochev and Gunzburger, [19] In the investigation of the formulation based on the following numerical examples we discuss a boundary value problem with a so-called corner singularity given in Cook’s membrane problem (boundary conditions change from clamped \(u = 0\) to stress-free conditions \(\sigma \cdot n = 0\)) and the perforated plate with an expected stress concentration at the circular hole of the plate.

3 NUMERICAL INVESTIGATION

For the numerical validation of the proposed plane strain formulation, we investigate first the fulfillment of the equilibrium of forces and resulting moments. Additionally the reliability of the displacement approximation of the mixed LS formulation and as a second part the stress distribution is analyzed with respect to different residual weightings. For further validations we consider the convergence within the squared \(L^2(B)\)-norm of the LS functional as well as the single functional parts, that is, the momentum balance, the constitutive equation and the balance of angular momentum.

For all analyzed numerical examples we chose the reference point \(x_0 = (0, 0)^T\). The numerical results could be sensitive to the choice of the reference point. In the considered boundary value problems a variation of \(x_0 \in B\) leads to deviations for the stresses and displacements that are negligible small (\(\ll 1\%)\). In general, a careful consideration of the point is necessary.

In our analysis we choose \(\tilde{t} = \sqrt{A}\), where \(A\) is the area of the boundary value problem. Especially for thin bar-like structures, where the height of the BVP tends to zero and therefore the area of the BVP tends to zero, this approach will not yield to
satisfying results. In such a case the length of the bar would be a suitable characteristic length. However, for the here discussed numerical examples, where the sizes of height and length of the BVPs have a comparable range, the approach is valid.

### 3.1 Cantilever beam

As a first brief example, for the performance of the LS formulation with explicit consideration of the balance of angular momentum, we discuss again the cantilever beam as presented in the introduction. The boundary conditions are described by a clamped left side $u=0$, an applied traction on the right side with $\sigma \cdot n = (0, 0.1)^T$ and stress-free boundary conditions on the top and bottom side of the cantilever $\sigma \cdot n = 0$ (see Figure 2). The material data is given with $E = 70 \text{ kN/mm}^2$ and $\mu = 26.12 \text{ kN/mm}^2$. As in the introductory calculations we determine the reaction forces and moment, which have to be $A_H = 0$, $A_V = 0.1$ and $M_A = 0.5$, depicted in Figure 2. As a discretization pattern we consider five times the number of elements per side in $x_1$-direction as in $x_2$-direction. The weighting parameter setups are chosen as $\alpha_i = \{ \alpha_m, \alpha_c, \alpha_a \}$ with $\alpha_1 = \{1, 1, 1\}$, $\alpha_2 = \{1, 0.5, 1\}$, $\alpha_3 = \{1, 0.1, 1\}$, $\alpha_4 = \{1, 0.05, 1\}$ and $\alpha_5 = \{1, 0.01, 1\}$.

The results for the proposed formulation show a strong improvement for the determination of the support reaction of the moment $M_A$ for all weighting setups as well as an improvement for the vertical support reaction $A_V$ compared to the results obtained based on formulation (1). For the weights $\alpha_{3,4,5}$ we obtain for all support reactions satisfying results for moderate mesh densities. The horizontal support reaction $A_H$ shows a deviation in comparison with the results shown in Figure 1, but at least for the choice of weightings $\alpha_{3,4,5}$ we obtain the expected solution. These outcomes are investigated further for two additional numerical examples, that is, the Cook’s membrane and the perforated plate problem.

### 3.2 Cook’s membrane

The well-known Cook’s membrane problem is depicted in Figure 3. The boundary value problem is described by a clamped displacement boundary at the left side $u = 0$. All other edges are assumed to be stress boundaries with stress-free boundary...
FIGURE 3  Cook’s membrane: Material parameters, boundary conditions, geometry and exemplary mesh density conditions at the upper and lower edge and \( \sigma \cdot n = (0, 1)^T \) on the right edge. Furthermore, we choose as reference position vector \( x_0 \) the origin \((0,0)\). Material parameters, boundary conditions and geometry are depicted in Figure 3. The boundary value problem is discretized by the same number of elements in \( x_1 \)- and \( x_2 \)-direction. For comparison of the obtained results a quadratic standard Galerkin element \( P_2 \) is used. In the underlying example we will restrict ourselves to the weighting parameter setups with \( \alpha_i = \{ \alpha_m, \alpha_c, \alpha_a \} \) with \( \alpha_1 = \{ 1, 1, 1 \} \), \( \alpha_2 = \{ 1, 0.1, 1 \} \), \( \alpha_3 = \{ 1, 5, 1 \} \) and the formulation (1) denoted by \( \alpha_4 = \{ 1, 0.1, 0 \} \).

3.2.1  Evaluation of reaction forces

The resulting reaction forces in horizontal and vertical direction and additionally the resulting moment with respect to the origin at \((0, 0)\) are investigated at the clamped edge of the Cook's membrane problem. A comparison of the obtained results for the LS formulation with different weighting factors is performed using the standard displacement element with \( P_2 \). In the case of the standard displacement formulation the reaction forces are calculated based on the sum of nodal forces \( F^I \), taken from the right-hand side vector, and the resulting moment is evaluated based on the nodal forces on \( \partial B_u \) in horizontal direction multiplied by the nodal distance in vertical direction to the origin \((0,0)\). For determining the support reactions based on the LS element the normal component of the Cauchy stresses \( \sigma \cdot n \) are considered and evaluated at the boundary \( \partial B_u \). The moments are determined by means of the resulting horizontal force \( A_H \) at the boundary \( \partial B_u \). The calculation rules for the support reactions of the LS and standard displacement formulations are given in Figure 4.

As a solution the sum of all forces \( F^I \) in horizontal \((x_1)\) and vertical \((x_2)\) direction and the moment can be analytically calculated as

\[
\sum_{I \in \partial B_u} F^I_{x_1} = 0, \quad \sum_{I \in \partial B_u} F^I_{x_2} = 16, \quad \text{and} \quad \sum_{I \in \partial B_u} F^I_{x_1} \cdot x_2 = 768, \quad (22)
\]

where \( I \in \partial B_u \) denotes all nodes at the clamped edge. In Figure 5, the difference in the convergence of the formulations with the weighting combinations are visualized. The LS element with \( \alpha_1 \) and \( \alpha_3 \), lead for the determination of horizontal and vertical reaction forces to unsatisfying results, which can be seen especially for the detailed view at the bottom of Figure 5. The standard displacement element \( P_2 \) is satisfying the expected solution for all illustrated discretizations. The weighted LS functional with \( \alpha_2 \) leads to a significant increase in the accuracy of the reaction forces in horizontal and vertical direction, especially compared to the other formulations.
to the formulation using $\alpha_3$. Furthermore, the convergence of the resulting moment at the clamped boundary leads for the weighting combination $\alpha_2$ to a good agreement with the analytical solution. In contrast to that the formulations considering $\alpha_1$ and $\alpha_3$ are not satisfying for the shown mesh densities. Additionally, $\alpha_4$ denotes the LS formulation (1) depending on the balance of linear momentum and the constitutive relation is evaluated (see Figure 5). For this formulation the reaction forces $A_H, A_V$ as well as the support reaction for the moment $M_A$ do not yield satisfying results. This observation illustrates the improvement of the results by considering the balance of angular momentum within the LS formulation. The deviations to the analytical solutions of the LS formulation are exemplary shown in Table 1. The performance of the functionals weighted with $\alpha_2$ and $\alpha_3$ regarding the vertical and horizontal reaction forces can be additionally illustrated by the course of reaction forces over the boundary $\partial B_h$ (see Figure 6). Therein, the normal stress component $\sigma_{11}$ and vertical stress component $\sigma_{21}$ show convergence to the expected distribution of normal and vertical stresses.

### 3.2.2 Displacement and stress approximation

The results for the displacement convergence of the upper right point $(48, 60)$, of the problem shown in Figure 3, are depicted in Figure 7, where we can observe a good convergence behavior for the displacements for the formulation with $\alpha_1$ and $\alpha_2$. For the setup $\alpha_3$ the results, compared to the reference solution of the $P_2$ displacement element, are not satisfying for the here presented mesh densities.

![Figure 5](image)

**Figure 5** Convergence of reaction forces over number of equations $\text{neq}$ (for all formulations [top], zoom of $A_H, A_V$, and $M_A$ [bottom])

(Analytical solutions: $A_H = 0, A_V = 16$, and $M_A = 768$)

| $\# \text{neq}$ | $\alpha_i$ | $|u_2 - u_{2h}|$ | $|A_H - A_{Hh}|$ | $|A_V - A_{Vh}|$ | $|M_A - M_{Ah}|$ |
|----------------|------------|----------------|----------------|----------------|----------------|
| 2800           | $\alpha_1$ | $3.4186 \times 10^{-1}$ | $4.3376 \times 10^{-1}$ | $4.9188 \times 10^{-1}$ | $4.1636 \times 10^1$ |
| 44 800         | $\alpha_1$ | $5.9259 \times 10^{-2}$ | $7.0459 \times 10^{-2}$ | $9.4787 \times 10^{-2}$ | $7.4581 \times 10^0$ |
| 2800           | $\alpha_2$ | $1.2690 \times 10^{-1}$ | $2.5256 \times 10^{-2}$ | $4.2087 \times 10^{-3}$ | $1.3234 \times 10^0$ |
| 44 800         | $\alpha_2$ | $2.2467 \times 10^{-2}$ | $3.3809 \times 10^{-3}$ | $3.6539 \times 10^{-4}$ | $2.1109 \times 10^{-1}$ |
| 2800           | $\alpha_3$ | $1.2182 \times 10^{0}$ | $3.3995 \times 10^{-1}$ | $9.1717 \times 10^{0}$ | $2.9409 \times 10^2$ |
| 44 800         | $\alpha_3$ | $2.9794 \times 10^{-1}$ | $1.3917 \times 10^{-1}$ | $2.4079 \times 10^{0}$ | $7.6732 \times 10^1$ |
The distribution of the von Mises stress $\sigma_{vM}$ is illustrated in Figures 8 and 9. As in the previous investigations, the weaker performance of the formulation with $\alpha_3$ can be observed. However, the choice of $\alpha_2$ result in a similar stress distribution as presented by the quadratic displacement element $P_2$. For a finer mesh density all formulations show the same distribution (see Figure 9). In Figure 10A, a convergence study for the von Mises stress $\sigma_{vM}$ at the mid-point of the lower edge (24, 22) and the fulfillment of the Cauchy stress symmetry condition are presented. Therein, the crucial influence of weighting factors is once more illustrated. The fulfillment of the stress symmetry condition for the Cauchy stress tensor is depicted in Figure 10B, where all formulations yield to a sufficient fulfillment of the condition.

All numerical evaluations show that the performance of the proposed mixed LS finite element formulation is crucially depending on the chosen setup of weighting parameters, which can be seen for the convergence studies for the displacements and von Mises stress, Figures 7 and 10A as well as for the von Mises stress distribution in Figure 8 and the fulfillment of the stress symmetry condition in Figure 10B. The formulation with $\alpha_2$ leads for all analyzed examples to satisfying results for support reactions as well as displacement and stress convergence, which are in line with the standard displacement element $P_2$. In contrast to this, $\alpha_1$ and $\alpha_3$, are showing weaker convergence behavior, especially for the investigation of support reactions. The evaluation of reaction forces show that an improvement can be achieved, by additionally introducing the balance of angular momentum in the LS functional, especially in the enforcement of the equilibrium of moments.
3.2.3 Convergence behavior and order of the LS formulation

A further investigation of the $L^2(B)$-norm convergence behavior of the functional, by analyzing the separate functionals $F_m$, $F_c$, $F_a$, see Equation (11), and the convergence of the single residuals $||R_i||^2$, for $i = m, c, a$ without consideration of the weightings $\omega_i$, are depicted in Figure 11. The comparison for the setup of $\alpha_2$ and the one with $\alpha_3$ show for both formulations a similar convergence of the single residuals and the functional parts. Here, a rate of convergence for the formulation with $\alpha_2$ of approximately 0.64 and 0.61 using $\alpha_3$ is obtained for the last regular mesh refinement step, compare Table 2. The underlying Cook’s membrane example is not sufficiently regular based on a singularity at the top left corner and by applying an uniform mesh refinement the optimal rate of convergence of $O(h^2)$ can not be reached. In order to show the influence of the balance of
FIGURE 11  Convergence of the residuals $\|R\|_2^2$ (top) and the functionals $F_1$ and the functional $F$ (bottom) with $i = m, c, a$ over the number of equations using a $RT_1P_2$ element with $\alpha_2$ (left) and $\alpha_3$ (right).

TABLE 2  Results for the order of convergence for the Cook’s membrane problem using a $RT_1P_2$ element

| Level $l$ | $\# nel$ | $\# neq$ | $F | \alpha_2$ (order) | $F | \alpha_3$ (order) |
|-----------|-----------|-----------|------------------------|------------------------|
| 0         | 32        | 448       | $1.8973 \times 10^{-6}$ | $6.5112 \times 10^{-4}$ |
| 1         | 128       | 1792      | $6.3751 \times 10^{-7}$ | $3.3479 \times 10^{-4}$ (0.47985) |
| 2         | 512       | 7168      | $2.3510 \times 10^{-7}$ | $1.7310 \times 10^{-4}$ (0.47581) |
| 3         | 2048      | 28 672    | $9.0806 \times 10^{-8}$ | $8.2852 \times 10^{-5}$ (0.53151) |
| 4         | 8192      | 114 688   | $3.6239 \times 10^{-8}$ | $3.7400 \times 10^{-5}$ (0.57375) |
| 5         | 32 768    | 458 752   | $1.4751 \times 10^{-8}$ | $1.6319 \times 10^{-5}$ (0.59824) |
| 6         | 131 072   | 1 835 008 | $6.0845 \times 10^{-9}$ | $7.0029 \times 10^{-6}$ (0.61026) |

angular momentum we divide it into two parts namely

$$ R_{a1} = r \times (\text{div}\sigma + f) \quad \text{and} \quad R_{a2} = a x l[\sigma - \sigma^T]. $$

(23)

The evaluation of these parts of the balance of angular momentum is additionally depicted in Figure 11, which shows that both parts have similar values and therefore a similar influence on the functional. This visualization is from a mathematical point of view unnecessary. However, since the third residual is mathematically redundant, due to the fact that $R_{a1}$ is fulfilled based on the balance of linear momentum and $R_{a2}$ is controlled by the constitutive equation, the illustration can be performed to show the influence of these two parts. This illustrates, especially for coarse and moderate mesh densities, that the balance of linear momentum itself is not sufficient for the fulfillment of reaction forces and that the balance of angular momentum have to be considered separately in the functional.

3.3  Perforated plate

As a third numerical example the perforated plate problem which exhibits a stress concentration at the left and right point next to the circular hole is analyzed. The displacement of the plate is fixed in both directions at the left side of the circle and in
\( x_2 \)-direction on the right side of the circular hole. Furthermore, the inner circle, as well as the right and left edge of the plate are assumed to be stress boundaries with stress-free boundary conditions. The upper and lower edge of the plate is loaded by \( \sigma \cdot n = (0, 10)^T \). Furthermore, we consider for the reference position vector \( x_0 \) the origin \((0,0)\). Material parameters, boundary conditions and the geometry can be found in Figure 12.

For the perforated plate example similar effects concerning the reliability of the solution fields, as in the previous Cook’s membrane example, are determined. Here, depicted by means of a displacement convergence of the top right corner node \( A(b, b) \) of the plate and a von Mises stress convergence study at node \( B(2r, 2r) \) of the underlying formulation using different weighting parameters, compare Figure 13. Analogously to the investigation of the Cook’s membrane problem we consider \( \alpha_i = \{\alpha_m, \alpha_c, \alpha_a\} \) with \( \alpha_1 = \{1, 1, 1\} \), \( \alpha_2 = \{1, 0.1, 1\} \), and \( \alpha_3 = \{1, 5, 1\} \). Both convergence studies show for the investigated perforated plate problem the satisfying performance of all three setups \( \alpha_i \) for the approximation of displacements and the stresses. As in the

**System setup:**

| Section               | Equation                           | Units       |
|-----------------------|------------------------------------|-------------|
| Upper side            | \( \sigma \cdot n = (0, 10)^T \) kN/mm\(^2\) |             |
| Lower side            | \( \sigma \cdot n = (0, -10)^T \) kN/mm\(^2\) |             |
| Left circle side      | \( u(-r, 0) = (0, 0)^T \) mm         |             |
| Right circle side     | \( u_r(r, 0) = 0 \) mm               |             |
| Young’s modulus       | \( E = 70 \) kN/mm\(^2\)             |             |
| Poisson’s ratio       | \( \nu = 0.34 \)                     |             |
| Shear modulus         | \( \mu = 26.12 \) kN/mm\(^2\)        |             |
| Length and height     | \( b = 100 \) mm                    |             |
| Radius of the hole    | \( r = 10 \) mm                     |             |

**Figure 12** Material parameters, boundary conditions and geometry of perforated plate problem

**Figure 13** Displacement convergence at node A in \( x_2 \)-direction (left) and convergence of the von Mises stress \( \sigma_{vM} \) at node B for the perforated plate problem

**Figure 14** Convergence of the residuals \( \|R_i\|^2 \) (top) and functionals \( F_i \) and the functional \( F \) (bottom) using a \( RT_0P_2 \) element with \( \alpha_2 \) (left) and \( \alpha_3 \) (right)
previous discussion of the Cook’s membrane problem the weighting setups are crucial for the performance of the formulation. In addition to the comparison of the stress and displacement convergence of the formulations further insight into the importance of the choice of the weighting factors are given by the convergence behavior of the functional $\mathcal{F}$, the single functional parts $\mathcal{F}_i$ and the corresponding residual parts $\|R_i\|^2$ with $i = m, c, a$ are compared exemplarily for $\alpha_2$ and $\alpha_3$ respectively, see Figure 14 for a $RT_0P_2$ and Figure 15 for a $RT_1P_2$ element type. The comparison of the applied element types ($RT_0P_2, RT_1P_2$) illustrates the impact of the chosen interpolation order for the rate of convergence. Furthermore, the resulting rate of convergence for the LS functional is determined based on an uniform mesh refinement, compare Tables 3 and 4. An increase of the rate of convergence is observed for the $RT_1P_2$ element type in comparison with the $RT_0P_2$ element formulation, which is illustrated in Tables 3 and 4. However, both formulations will not yield the best possible order of convergence for the perforated plate problem, with order $O(h)$ for $RT_0P_2$ and respectively $O(h^2)$ for the $RT_1P_2$ element type, due to the uniform mesh refinement. Further results for the von Mises stress distribution on the deformed body of the perforated plate problem considering different element combinations with $RT_0P_2$ and $RT_1P_2$ show the satisfying results compared to the linear and quadratic standard displacement element $P_1$ and $P_2$, see Figure 16. A satisfying accordance of the von Mises stress distribution $\sigma_{vM}$ is reached at least for an uniform mesh refinement including 4096 elements.

**Figure 15** Convergence of the residuals $\|R_i\|^2$ (top) and functionals $\mathcal{F}_i$ and the functional $\mathcal{F}$ (bottom) using a $RT_1P_2$ element with $\alpha_2$ (left) and $\alpha_3$ (right)

**Table 3** Results for the convergence order for the perforated plate problem using a $RT_0P_2$ element

| Level $l$ | # nel | # neq | $\mathcal{F}$ | $|\alpha_2|$ (order) | $|\alpha_3|$ (order) |
|-----------|-------|-------|---------------|----------------------|----------------------|
| 0         | 256   | 1853  | $1.4309 \times 10^{-4}$ | $1.4712 \times 10^{-2}$ |
| 1         | 1024  | 7293  | $5.3783 \times 10^{-5}$ | (0.7142) | $6.7544 \times 10^{-3}$ | (0.5682) |
| 2         | 4096  | 28925 | $2.2516 \times 10^{-5}$ | (0.6319) | $2.7838 \times 10^{-3}$ | (0.6433) |
| 3         | 16384 | 115197| $9.2461 \times 10^{-6}$ | (0.6441) | $9.5501 \times 10^{-4}$ | (0.7742) |
| 4         | 65536 | 459773| $3.4743 \times 10^{-6}$ | (0.7072) | $2.7041 \times 10^{-4}$ | (0.9116) |
| 5         | 147456| 1033725| $1.8214 \times 10^{-6}$ | (0.7970) | $1.2355 \times 10^{-4}$ | (0.9669) |

**Table 4** Results for the convergence order for the perforated plate problem using a $RT_1P_2$ element

| Level $l$ | # nel | # neq | $\mathcal{F}$ | $|\alpha_2|$ (order) | $|\alpha_3|$ (order) |
|-----------|-------|-------|---------------|----------------------|----------------------|
| 0         | 256   | 3581  | $1.9240 \times 10^{-5}$ | $7.0073 \times 10^{-3}$ |
| 1         | 1024  | 14333 | $6.5664 \times 10^{-6}$ | (0.7751) | $2.7940 \times 10^{-3}$ | (0.6629) |
| 2         | 4096  | 57341 | $1.8411 \times 10^{-6}$ | (0.9172) | $8.0739 \times 10^{-4}$ | (0.8954) |
| 3         | 16384 | 229373| $3.5885 \times 10^{-7}$ | (1.1795) | $1.2856 \times 10^{-4}$ | (1.3254) |
| 4         | 65536 | 917501| $4.6162 \times 10^{-8}$ | (1.4793) | $1.1965 \times 10^{-5}$ | (1.7128) |
| 5         | 147456| 2064381| $1.1704 \times 10^{-8}$ | (1.6921) | $2.6240 \times 10^{-6}$ | (1.8711) |
\textbf{FIGURE 16} \(\sigma_{vM}\) on FE meshes A, \(\text{nel} = 256\) (left), \(\text{nel} = 4096\) (mid) and \(\text{nel} = 65536\) (right) for \(RT_0P_2; B, RT_1P_2; C, \) with \(\alpha_2; D, P_1; \) and E, \(P_2\) element
CONCLUDING REMARKS AND OUTLOOK

The proposed linear elastic mixed LS finite element formulation with explicit consideration of the balance of angular momentum describes the improvement concerning the fulfillment of the support reactions compared to the standard LSFEM, see Equation (1). Especially the reaction moment, which exact calculation is extremely important in engineering applications, and the crucial impact of weighting factors within the LS methodology has been investigated in detail. Furthermore, the proposed formulation enforces the fulfillment of the (global) balance of angular momentum including the symmetry of the Cauchy stress tensor, the latter condition is not a priori fulfilled by the use of vector-valued Raviart-Thomas functions for the stress approximation. The numerical investigation of the formulations with explicit consideration of the balance of angular momentum show a significant impact on the approximate fulfillment of support reactions compared to a standard LS approach.

Further analysis of the here presented mixed LS finite element formulation for hyperelastic material models and for a consideration of an approximation of the stress field directly in the deformed configuration should be performed based on the gained insights in the field of linear elasticity.

ACKNOWLEDGEMENTS

The authors gratefully acknowledge the support by the Deutsche Forschungsgemeinschaft in the Priority Program 1748 “Reliable simulation techniques in solid mechanics. Development of nonstandard discretization methods, mechanical and mathematical analysis” under the project “Approximation and Reconstruction of Stresses in the Deformed Configuration for Hyperelastic Material Models”—project number 392587488 (SCHR 570/34-1).

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How to cite this article: Igelbüscher M, Schröder J, Schwarz A. A mixed least-squares finite element formulation with explicit consideration of the balance of moment of momentum, a numerical study. *GAMM-Mitteilungen*. 2020;43:e202000009. https://doi.org/10.1002/gamm.202000009