POSITIVE SOLUTIONS OF SINGULAR MULTIPARAMETER p-LAPLACIAN ELLIPTIC SYSTEMS

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Abstract. In this paper, by using the eigenvalue theory, the sub-supersolution method and the fixed point theory, we prove the existence, multiplicity, uniqueness, asymptotic behavior and approximation of positive solutions for singular multiparameter p-Laplacian elliptic systems on nonlinearities with separate variables or without separate variables. Various nonexistence results of positive solutions are also studied.

1. Introduction. In this paper we first analyze the existence, uniqueness, asymptotic behavior and approximation of positive solutions to the p-Laplacian elliptic system

\[
\begin{cases}
-\Delta_p z_1 = \lambda_1 h_1(|x|)z_2^\alpha & \text{in } \Omega, \\
-\Delta_p z_2 = \lambda_2 h_2(|x|)z_1^\beta & \text{in } \Omega, \\
z_1 = z_2 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(1)

Here $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$, $1 < p < \infty$, $\lambda_1$, $\lambda_2 \not= 0$ are parameters, $\alpha$, $\beta > 0$, and $\Omega = \{x \in \mathbb{R}^N : |x| < 1\}$, $N \geq 2$. Moreover, $h_1$ and $h_2$ satisfy:

(H) $h_1(t)$, $h_2(t) \in L^1[0,1]$, and there exist $N_1$, $N_2 > 0$ such that $h_1(t) \geq N_1$, $h_2(t) \geq N_2$ a.e. on $[0,1]$.

Let $\varphi_p(s) = |s|^{p-2}s$, $(\varphi_p)^{-1} = \varphi_q$, $\frac{1}{p} + \frac{1}{q} = 1$. Then the solutions of

\[
\begin{cases}
-(r^{N-1}\varphi_p(u'_1))' = \lambda_1 r^{N-1}h_1(r)u_2^\alpha & \text{in } 0 < r < 1, \\
-(r^{N-1}\varphi_p(u'_2))' = \lambda_2 r^{N-1}h_2(r)u_1^\beta & \text{in } 0 < r < 1, \\
u'_1(0) = u'_2(0) = u_1(1) = u_2(1) = 0
\end{cases}
\]

(2)

are radial solutions to system (1). See Zhang and Feng [49] for full details of the transformation.

When $p = 2$, $\lambda_1 = \lambda_2 = 1$ and $h_1 = h_2 \equiv 1$ on $[0,1]$, we get the following system

\[
\begin{cases}
-\Delta z_1 = z_2^\alpha & \text{in } \Omega, \\
-\Delta z_2 = z_1^\beta & \text{in } \Omega, \\
z_1 = z_2 = 0 & \text{on } \partial \Omega.
\end{cases}
\]

(3)

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This problem can be thought of as the Lane-Emden system, which comes from modeling spatial phenomena in a variety of chemical and biological problems. Naturally positive solutions of (3) are of special interest, and the study of system (3) has attracted the attention of different researchers and it is a topic of current interest, see [6, 5, 9, 10, 15, 44, 47], and the references therein. In particular, Serrin and Zou [45] proved that the hyperbola

$$\frac{1}{\alpha + 1} + \frac{1}{\beta + 1} = \frac{N - 2}{N} \quad (4)$$

is the dividing curve of existence and nonexistence on the \(\alpha\beta\)-plane in several cases. At the same time, the authors obtained nonexistence results of positive solutions for (3) by applying the asymptotic estimates at infinity. When \(\lambda_1 = \lambda_2 = 1\), \(h_1\) and \(h_2\) are nonnegative continuous functions defined on \(\mathbb{R}^N\), system (3) has been studied by Lair and Wood in [37]. Under the case that \(h_1\) and \(h_2\) are radial, they proved the existence of entire large (i.e. blow up at infinity) positive solutions by using the maximum principle. On other related problems to (3) can be found in [2, 11, 14, 13, 17, 16, 22, 24, 26, 32, 37, 38, 40, 41, 51].

In this paper, we call a point \((\alpha, \beta)\) in the first quadrant of the \(\alpha\beta\)-plane critical, supercritical or subcritical if it is respectively on, above or below the hyperbola \(\alpha\beta = (p - 1)^2\), which is independent of dimension \(N\) and is simpler than (4). We will prove that (1) possesses a pair of continuous positive solutions above the hyperbola or below the hyperbola, does not admit a pair of continuous positive solutions on the hyperbola. The proofs depend heavily on the eigenvalue theory and Hölder’s inequality. Moreover, for any positive parameters \(\lambda_1\) and \(\lambda_2\), we prove the uniqueness and approximation of positive solutions for (2) by using the \(u_0\)-sublinear operator theory.

Similarly as Serrin and Zou [45] pointed out that there are various difficulties in dealing with various systems of elliptic equations, such as (1), partly due to the double parameters \(\lambda_1\) and \(\lambda_2\) and the imbalance between the two exponents \(\alpha\) and \(\beta\). This makes it generally impossible to apply techniques used in treating single equation to study system of elliptic equations, particularly for \(p\)-Laplacian systems. We hence introduce a composite operator, which is one of our main technologies to overcome these difficulties.

In contrast with the case: nonlinearities with separate variables, another case can be found in the following \(p\)-Laplacian system

$$\begin{cases}
-\Delta_p z_1 = g_1(|x|, z_1, z_2, a, b) \quad \text{in } \Omega, \\
-\Delta_p z_2 = g_2(|x|, z_1, z_2, a, b) \quad \text{in } \Omega, \\
(z_1, z_2) \to (0, 0) \quad \text{as } |x| \to \infty, \\
\frac{\partial z_1}{\partial n} = \frac{\partial z_2}{\partial n} = 0 \quad \text{on } |x| = r_0.
\end{cases} \quad (5)$$

where \(\Delta_p u := \text{div}(|\nabla u|^{p - 2} \nabla u), 1 < p < N\), \(a\) and \(b\) are parameters, \(g_1\) and \(g_2\) are continuous nonlinearities, \(\Omega = \{x \in \mathbb{R}^N : |x| > r_0 > 0\}\) and \(\frac{\partial w}{\partial n}\) is the outward normal derivative of \(w\) on \(\partial B_{r_0}\).

In an effort to look for positive radial solutions of (5), we apply the changes of variables \(r = |x|\) and \(t = r^{\frac{N - p}{p}} / r_0^{\frac{N - p}{p}}\) to transform system (5) to the following system

$$\begin{cases}
-(\varphi_p(u'))' = G_1(t, u, v, a, b), \quad t \in (0, 1), \\
-(\varphi_p(v'))' = G_2(t, u, v, a, b), \quad t \in (0, 1), \\
u(0) = v(0) = u'(1) = v'(1) = 0.
\end{cases} \quad (6)$$
where \( \varphi_p(s) = |s|^{p-2}s, \ 1 < p < N, \) and
\[
G_i(t, u, v, a, b) = \left( \frac{p-1}{N-p} \right)^{p(1-N)/N} g_i(r_0 t^{1-p/(N-p)}, u, v, a, b), \quad i \in \{1, 2\}.
\]

For full details of the transformation, see Appendix A.1 in Morris [42]. So, the study of positive radial solutions of system (5) is reduced to the study of positive solutions to system (6). To this goal, let
\[
\mathbb{H} = \left\{ h \in C((0, 1), (0, +\infty)) : \int_0^1 h(s) \, ds < +\infty \right\},
\]
and for \( i \in \{1, 2\} \), \( G_i \) satisfy
\[ G_i(t, v_1, v_2, a_1, b_1) \leq G_i(t, \tilde{v}_1, \tilde{v}_2, \tilde{a}_1, \tilde{b}_1) \]
whenever \((t, v_1, v_2, a_1, b_1) \leq (t, \tilde{v}_1, \tilde{v}_2, \tilde{a}_1, \tilde{b}_1)\), where the inequality is understood inside every component.

(C_1) Given \( a, b \geq 0 \), then for all \( d > 0 \) and \( i = 1, 2 \), there exist \( h_i \in \mathbb{H} \) such that
\[
0 \leq G_i(t, u, v, a, b) \leq h_i(t) \quad \text{for all} \quad (t, u, v) \in (0, 1) \times [0, d]^2.
\]

(C_2) There exist functions \( h_i \in \mathbb{H} \) and \( d_i > 0 \) such that
\[
\int_0^1 \varphi_p \left( \int_s^1 h(\tau) \, d\tau \right) \, ds < \frac{1}{2d_i}
\]
and
\[
\lim_{\| (u, v, a, b) \| \to 0^+} \frac{G_i(t, u, v, a, b)}{\varphi_p(u + v + a + b)} < \varphi_p(d_i) h_i(t).
\]

(C_3) There exist \( \alpha, \beta \in (0, \frac{1}{2}) \) such that
\[
\lim_{\| (u, v) \| \to +\infty} \frac{G_i(t, u, v, 0, 0)}{\varphi_p(u + v)} = +\infty
\]
uniformly for \( u, v \geq 0 \) with \( t \in [\alpha, 1 - \alpha] \), and
\[
\lim_{\| (a, b) \| \to +\infty} G_i(t, u, v, a, b) = +\infty
\]
uniformly for \( u, v \geq 0 \) with \( t \in [\beta, 1 - \beta] \).

We notice that many authors have paid more attention to system (5) when \( p = 2 \) and \( a = b = 0 \), for example, see [8, 3, 4, 7, 20, 29, 35, 39, 42, 43]; especially to the following elliptic differential systems:

\[
\begin{cases} 
\Delta u + g_1(|x|) f(u, v) = 0, \\
\Delta v + g_2(|x|) g(u, v) = 0
\end{cases}
\]

under different boundary conditions. Making use of the linking theorem and with the help of the Nehari manifold, Benrouhana [3] showed the multiplicity of solutions to system (7) in the whole space \( \mathbb{R}^N \). Kawano and Kusano [35] presented sufficient conditions to get the existence of entire solutions of (7) by applying the sub-supersolution method. In [43], Precup investigated the existence, localization and multiplicity of positive radial solutions of the elliptic differential system (7) in \( \Omega := \{ x \in \mathbb{R}^N : |x| > r_0 \} \) \((N \geq 3)\), under the conditions
\[
u_1 = u_2 = 0 \quad \text{for} \quad |x| = r_0 \quad \text{and} \quad u_1, u_2 \to 0 \quad \text{as} \quad |x| \to \infty.
\]
In [19], do Ó, Lorca, Sánchez and Ubilla studied an interesting problem on nonlinearities without separate variables

\[
\begin{align*}
-\Delta u &= g_1(|x|, u, v) \quad \text{for } |x| > 1 \text{ and } x \in \mathbb{R}^N, \\
-\Delta v &= g_2(|x|, u, v) \quad \text{for } |x| > 1 \text{ and } x \in \mathbb{R}^N, \\
(u, v) &= (a, b) \quad \text{for } |x| = 1, \\
(u, v) &\to (0, 0) \quad \text{as } |x| \to \infty.
\end{align*}
\]

They established essential results of existence of positive radial, decaying solutions for problem (8) with the help of the results for a second-order ordinary differential system. Recently, in [12], Chhetri, Sankar, Shivaji and Son analyzed the existence of positive radial solutions to the problem

\[
\begin{align*}
-\Delta_p u &= \lambda g_1(|x|) f(v) \quad \text{in } \Omega, \\
-\Delta_p v &= \lambda g_2(|x|) f(u) \quad \text{in } \Omega, \\
(u, v) &= (0, 0) \quad \text{if } |x| = r_0, \\
(u, v) &\to (0, 0) \quad \text{as } |x| \to \infty,
\end{align*}
\]

where \(\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u), \) \(1 < p < N, \) \(\lambda > 0\) is a parameter, and \(\Omega = \{ x \in \mathbb{R}^N : |x| > r_0 > 0 \}\). The authors considered the existence of a positive radial solution for small values of \(\lambda\) by applying degree theory and rescaling arguments.

Always for the system case, the interested readers may refer to Felmer, Manásevich and Thelin [21], Hai [30, 31], Hai and Shivaji [33], Son and Wang [46], Ahammou [1], Xiang, Zhang and Rădulescu [48].

On the basis of the above analysis, we extend the study of previous results in several ways. We consider system (2) when \(h_1, h_2 \in L^1[0, 1]\) (in particular, \(h_1, h_2\) are allowed to possess a finite number of singularities) and \(\lambda_1\) and \(\lambda_2\) are permitted to change signs. For any \(\lambda_1 \neq 0, \lambda_2 > 0 \) or for any \(\lambda_2 \neq 0, \lambda_1 > 0, \) we will verify that there exists a hyperbola \(\alpha \beta = (p - 1)^2\) such that system (2) admits a pair of positive solutions above the hyperbola or below the hyperbola, does not possess a pair of positive solutions on the hyperbola. Our results are often new even when \(h_1\) and \(h_2\) are continuous.

Moreover, we apply the sub-supersolution method and the fixed point index theory to show that there is a continuous curve \(\Gamma\) which divides the positive quadrant of the \((a, b)\)-plane into two disjoint sets \(S_1\) and \(S_2\) such that system (5) admits at least two positive solutions in \(S_1\), at least one positive solution on the boundary of \(S_1\), and no positive solution in \(S_2\). Our results extend the study in [19], [43] and [33] from Laplacian systems to \(p\)-Laplacian systems, and also improve that in [31] from nonsingular \(p\)-Laplacian systems to singular \(p\)-Laplacian systems. Moreover, it turns out that the case \(1 < p < N\) is more difficult to handle than the linear case: \(p = 2\). This needs some new ingredients in the arguments. One of the main results of this paper is the following theorem.

**Theorem 1.1.** Suppose that \((H)\) and \(\alpha \beta = (p - 1)^2\). Then there exists \(\lambda_* > 0\) such that system (2) admits no positive solution for \(0 < |\lambda_1| \lambda_*^\frac{2}{p} < \lambda_*\).

The rest of the paper is organized as follows. In Section 2, we analyze the existence and asymptotic behavior of positive solutions to (2) by using the eigenvalue theory due to Guo [27]. Section 3 is devoted to proving the uniqueness and approximation of positive solutions to (2) by using the \(u_0\)-sublinear operator theory introduced by Hu and Wang [34]. We present some necessary definitions and Lemmas that will be used to prove our main results of system (6) in Section...
4. Section 5 considers the existence of one nontrivial solution when the parameters \(a\) and \(b\) in (6) are sufficiently small. In Section 6, we introduce two theorems of the sub-supersolution method for singular systems. Section 7 gives a nonexistence result, as well as a priori estimate result used in Section 8 to prove the existence of two nontrivial solutions.

2. Existence and asymptotic behavior of positive solutions to (2). In this section, we apply the eigenvalue theory and the inequality technique to analyze the existence and asymptotic behavior of positive solutions to system (2). So we first collect some known results of the eigenvalue theory and some properties of \(\varphi_p\), which will be used in the subsequent proofs.

**Lemma 2.1.** (Corollary of Theorem 1, Guo [27]) Let \(A : E \rightarrow E\) be completely continuous. Suppose that \(A\theta = \theta\),
\[
\lim_{\|x\| \to 0} \frac{\|Ax\|}{\|x\|} = 0
\]
and
\[
\lim_{\|x\| \to +\infty} \frac{\|Ax\|}{\|x\|} = +\infty.
\]
Then the following two conclusions hold:

i) Every \(\mu \neq 0\) is an eigenvalue of \(A\), i.e., there exists \(x_\mu \in E, x_\mu \neq 0\) such that \(Ax_\mu = \mu x_\mu\);

ii) \(\lim_{\mu \to \infty} x_\mu = +\infty\).

**Lemma 2.2.** (Corollary of Theorem 2, Guo [27]) Let \(A : E \rightarrow E\) be completely continuous. Suppose that \(A\theta = \theta\),
\[
\lim_{\|x\| \to 0} \frac{\|Ax\|}{\|x\|} = +\infty
\]
and
\[
\lim_{\|x\| \to +\infty} \frac{\|Ax\|}{\|x\|} = 0.
\]
Then the following two conclusions hold:

i) Every \(\mu \neq 0\) is an eigenvalue of \(A\), i.e., there exists \(x_\mu \in E, x_\mu \neq 0\) such that \(Ax_\mu = \mu x_\mu\);

ii) \(\lim_{\mu \to \infty} x_\mu = 0\).

**Lemma 2.3.** (Lemma 2.1, Zhang and Feng [49]) Let \(p > 1, q > 1\) satisfy \(\frac{1}{p} + \frac{1}{q} = 1\). Then \(\varphi_p(s) = |s|^{p-2}s\) is odd, and
\[
s\varphi_p(s) > 0 \text{ if } s \neq 0, \quad \varphi_p(st) = \varphi_p(s)\varphi_p(t), \quad \varphi_p(0) = 0, \quad \varphi_p(1) = 1, \quad \varphi_p(-1) = -1,
\]
\[
\varphi_p(s + t) = \begin{cases} 2^{p-1}(\varphi_p(s) + \varphi_p(t)), & \text{if } p \geq 2, \ s, t > 0, \\ \varphi_p(s) + \varphi_p(t), & \text{if } 1 < p < 2, \ s, t > 0. \end{cases}
\]

On the other hand, \(\varphi_p(s)\) is increasing on \([0, \infty)\), and for \(a \geq 0, \varphi_p(s^a) = \varphi^a_p(s)\) on \([0, \infty)\).

A pair of functions \(u, v \in C[0, 1] \cap C^1(0, 1)\) with \(\varphi_p(u') = \varphi_p(v') \in C^1(0, 1)\) are called to be a positive solution of (2) if \(u(t), v(t) > 0\) for all \(t \in (0, 1)\), and \(u\) and \(v\) satisfy (2). Now let \(J = [0, 1]\) and take the Banach space to be \(E = C[0, 1]\) with supremum norm \(\|\cdot\|_C\).
Lemma 2.4. (Lemma 3.1, Feng, Du and Ge [23]) Suppose that (H) holds. If \( u, v \in C[0, 1] \cap C^1(0, 1) \) is a pair of solutions for system (2) if and only if \( u, v \in C[0, 1] \) is a pair of solutions of the following integral equations

\[
\begin{align*}
  u_1 &= \int_0^1 \varphi_\theta \left( \frac{1}{s^{N-1}} \int_0^s \lambda_1 h_1 \tau^{N-1} u_2^\alpha d\tau \right) ds, \\
  u_2 &= \int_0^1 \varphi_\theta \left( \frac{1}{s^{N-1}} \int_0^s \lambda_2 h_2 \tau^{N-1} u_1^\beta d\tau \right) ds,
\end{align*}
\]

and

\[
\min_{\frac{1}{4} \leq t \leq \frac{1}{2}} u_i(t) \geq \frac{1}{4} \| u_i \|_C,
\]

where \( i \in \{1, 2\} \).

Let \( K \subset E \) be

\[
K = \left\{ u \in E : u(t) \geq 0, t \in J, \min_{t \in [\frac{1}{4}, \frac{1}{2}]} u(t) \geq \frac{1}{4} \| u \|_C \right\},
\]

which is a norm cone in \( E \). We now give two solution operators. For any \( u \in K \), let \( T_i : K \to E \) be given by

\[
\begin{align*}
  T_1(u)(t) &= \int_0^1 \varphi_\theta \left( \frac{1}{s^{N-1}} \int_0^s h_1 \tau^{N-1} u^\alpha d\tau \right) ds, \\
  T_2(u)(t) &= \int_0^1 \varphi_\theta \left( \frac{1}{s^{N-1}} \int_0^s \lambda_2 h_2 \tau^{N-1} u^\beta d\tau \right) ds.
\end{align*}
\]

Notice that the image of each operator is a nonnegative continuous function on \( J \), so by Lemma 3.2 in [23], \( T_i (i \in \{1, 2\}) \) are completely continuous operators from \( K \) into \( K \).

Define a composite operator \( T = T_1 T_2 \). Naturally, \( T \) is also completely continuous from \( K \) to \( K \). Calculation shows that \((u_1, u_2) \in K \times K \) solves equations (13) and (14) if and only if \((u_1, u_2) \) satisfy \( u_1 = T_1 u_2, u_2 = T_2 u_1 \). Thus, if \( u_1 \in K \) is a fixed point of \( T \), define \( u_2 = T_2 u_1 \), then \( u_2 \in E \) such that \((u_1, u_2) \in K \times K \) solves equations (13) and (14); on the contrary, if \((u_1, u_2) \in K \times K \) solves equations (13) and (14), then \( u_1 \) is certainly a fixed point of \( T \).

Remark 1. It is not difficult to see that \( u_1 \) and \( u_2 \) do not solve system (2). It is natural to ask: how can we find the solution of system (2)? Next we will look for the solutions of system (2) by using the eigenvalue theory.

In fact, for any \( \lambda_1 \neq 0 \), in Theorem 2.5, we apply Lemmas 2.1-2.2 to find there exists \( u_{1\lambda_1} \neq \theta, u_{1\lambda_1} \in E \) such that \( T u_{1\lambda_1} = \frac{1}{\lambda_1} u_{1\lambda_1} \), i.e. \( \lambda_1 T u_{1\lambda_1} = u_{1\lambda_1} \), which shows that \( u_{1\lambda_1} \) is a fixed point of operator \( \lambda_1 T \). Then by \( \lambda_1 T = \lambda_1 T_1 T_2 \) we can define \( u_{2\lambda_1} = T_2 u_{1\lambda_1} \) and obtain \( u_{1\lambda_1} = \lambda_1 T_1 u_{2\lambda_1} \). Thus we get a pair of solutions \( u_{1\lambda_1}, u_{2\lambda_1} \) of system (2).

Theorem 2.5. Suppose that (H) holds. Then, for \( \alpha > 0, \beta > 0 \) and \( \alpha \beta \neq (p - 1)^2 \), we have the following conclusions:

(i) If \( \alpha \beta > (p - 1)^2 \), then for any \( \lambda_1 \neq 0 \) and \( \lambda_2 > 0 \), system (2) admits a pair of positive solutions \( u_{1\lambda_1}, u_{2\lambda_1} \) with \( u_{1\lambda_1} \neq 0 \), and

\[
\lim_{\lambda_1 \to 0} \| u_{1\lambda_1} \|_C = +\infty,
\]

\[
\lim_{\lambda_1 \to \infty} \| u_{1\lambda_1} \|_C = 0.
\]
(ii) If $\alpha \beta > (p - 1)^2$, then for any $\lambda_2 \neq 0$ and $\lambda_1 > 0$, system (2) admits a pair of positive solutions $u_{1\lambda_2}, u_{2\lambda_2}$ with $u_{2\lambda_2} \neq 0$, and
\[
\lim_{\lambda_2 \to 0} \|u_{2\lambda_2}\|_C = +\infty, \quad \lim_{\lambda_2 \to \infty} \|u_{2\lambda_2}\|_C = 0.
\]

(iii) If $0 < \alpha \beta < (p - 1)^2$, then for any $\lambda_1 \neq 0$ and $\lambda_2 > 0$, system (2) admits a pair of positive solutions $u_{1\lambda_1}, u_{2\lambda_1}$ with $u_{1\lambda_1} \neq 0$, and
\[
\lim_{\lambda_1 \to 0} \|u_{1\lambda_1}\|_C = 0,
\]

(iv) If $0 < \alpha \beta < (p - 1)^2$, then for any $\lambda_2 \neq 0$ and $\lambda_1 > 0$, system (2) admits a pair of positive solutions $u_{1\lambda_2}, u_{2\lambda_2}$ with $u_{2\lambda_2} \neq 0$, and \[
\lim_{\lambda_2 \to 0} \|u_{2\lambda_2}\|_C = 0.
\]

Proof. Here we only prove the conclusions (i) and (iii) hold since the proofs are similar when we verify (ii) and (iv). Next we verify that all the conditions of Lemmas 2.1-2.2 are satisfied.

On the one hand, for $u \in K$, we obtain by Lemma 2.3
\[
\|T_1 u\|_C = \max_{i,j \in J} \int_1^1 \varphi_q \left( \frac{1}{\tau} \int_0^\tau h_1(\tau)\tau^{N-1}u^\alpha(\tau)d\tau \right)d\tau
\]
\[
= \max_{i,j \in J} \int_1^1 \varphi_q \left( \int_0^\tau h_1(\tau)(\tau)^{N-1}u^\alpha(\tau)d\tau \right)d\tau
\]
\[
\leq \int_0^1 \varphi_q \left( \int_0^1 h_1(\tau)\|u\|_C^\alpha d\tau \right)d\tau
\]
\[
\leq \varphi_q(\|h_1\|_L^1)\varphi_q(\|u\|_C^\alpha)
\]
\[
\leq \varphi_q(\|h_1\|_L^1)\varphi_q^\beta(\|u\|_C).
\]

Similarly, we get
\[
\|T_2 u\|_C \leq \varphi_q(\lambda_2 \|h_2\|_L^1)\varphi_q^\beta(\|u\|_C).
\]

Thus, by Lemma 2.3 we get that
\[
\|T(u)\|_C = \|T_1 T_2(u)\|_C
\]
\[
\leq \varphi_q(\|h_1\|_L^1)\varphi_q^\alpha(\|T_2 u\|_C)
\]
\[
\leq \varphi_q(\|h_1\|_L^1)\varphi_q^\alpha(\|\varphi_q(\lambda_2 \|h_2\|_L^1)\varphi_q^\beta(\|u\|_C)\|_C)
\]
\[
= \varphi_q(\|h_1\|_L^1)\varphi_q^\alpha(\varphi_q(\lambda_2 \|h_2\|_L^1))\varphi_q^\beta(\|u\|_C)
\]
\[
= \varphi_q(\|h_1\|_L^1)\varphi_q^\alpha(\varphi_q(\lambda_2 \|h_2\|_L^1))\|u\|_C^{(q-1)(q-1)\alpha \beta}
\]
\[
= \varphi_q(\|h_1\|_L^1)\varphi_q^\alpha(\varphi_q(\lambda_2 \|h_2\|_L^1))\|u\|_C^{\frac{\alpha \beta}{q-1}}.
\]

We hence get from (18) the following two conclusions:
1) Supercritical case.
Since $\alpha \beta > (p - 1)^2$, we get
\[
\lim_{\|u\|_C \to 0} \frac{\|T u\|_C}{\|u\|_C} = 0.
\]

2) Subcritical case.
Since $0 < \alpha \beta < (p - 1)^2$, we get
\[
\lim_{\|u\|_C \to 0} \frac{\|T u\|_C}{\|u\|_C} = +\infty.
\]
On the other hand, for any \(u \in P\), we get
\[
\|T_1u\|_C = \max_{t \in J} \left( \int_t^1 \varphi_q \left( \int_s^t \frac{1}{\varphi_q} \int_0^\lambda h_1(\tau)\tau^{N-1}u^\alpha(\tau)d\tau \right) ds \right)
\geq \int_1^T \varphi_q \left( \int_\frac{1}{h_1(\frac{z}{s}) N-1} u^\alpha(\tau)d\tau \right) ds
\geq \int_1^T \varphi_q \left( \int_\frac{1}{h_1(\frac{z}{s}) N-1} u^{N-1} u^\alpha(\tau)d\tau \right) ds
\geq \int_1^T \varphi_q \left( \int_\frac{1}{N} N_1(\frac{z}{s}) N-1 \frac{1}{2} \|u\|_C^2 d\tau \right) ds
= \varphi_q \left( \frac{1}{4} N_1 \|u\|_C^2 \right) D,
\]
where
\[
D = \int_1^T \varphi_q \left( \int_\frac{1}{4} (\frac{z}{s})^{N-1} d\tau \right) ds.
\]
Similarly, for \(u \in P\), we can prove that
\[
\|T_2u\|_C \geq \varphi_q \left( \frac{1}{4} \lambda_2 N_2 \|u\|_C^\beta \right) D.
\]
We hence get from (21) and (22) that
\[
\|T(u)\|_C = \|T_1T_2(u)\|_C
\geq D \varphi_q \left( \frac{1}{4} N_1 \|T_2u\|_C^\alpha \right)
\geq D \varphi_q \left( \frac{1}{4} N_1 \|u\|_C^\alpha \right)
D \varphi_q \left( \frac{1}{4} \lambda_2 N_2 \|u\|_C^\beta \right)
= D \varphi_q \left( \frac{1}{4} N_1 D^\alpha \|u\|_C^\alpha \varphi_q \left( \frac{1}{4} \lambda_2 N_2 \|u\|_C^\beta \right) \right)
= D \varphi_q \left( \frac{1}{4} N_1 D^\alpha \varphi_q \left( \frac{1}{4} \lambda_2 N_2 \right) \varphi_q \left( \frac{1}{4} \lambda_2 N_2 \|u\|_C^\beta \right) \right)
= D \varphi_q \left( \frac{1}{4} N_1 D^\alpha \varphi_q \left( \frac{1}{4} \lambda_2 N_2 \right) \varphi_q \left( \frac{1}{4} \lambda_2 N_2 \|u\|_C^\beta \right) \right)
= D \varphi_q \left( \frac{1}{4} N_1 D^\alpha \varphi_q \left( \frac{1}{4} \lambda_2 N_2 \right) \varphi_q \left( \frac{1}{4} \lambda_2 N_2 \|u\|_C^\beta \right) \right)
\]
We so get the following two conclusions:
1) Supercritical case.
Since \(\alpha \beta > (p-1)^2\), we get
\[
\lim_{\|u\|_C \to +\infty} \frac{\|T_1u\|_C}{\|u\|_C} = +\infty.
\]
2) Subcritical case.
Since \(0 < \alpha \beta < (p-1)^2\), we get
\[
\lim_{\|u\|_C \to +\infty} \frac{\|T_1u\|_C}{\|u\|_C} = 0.
\]
Considering \(\alpha \beta > (p-1)^2\), from (19) and (24), together with Lemma 2.1, we find that: for any \(\lambda_1 \neq 0\), there exists \(u_{1\lambda_1} \in E\) with \(u_{1\lambda_1} \neq \theta\) such that \(T_{1\lambda_1} = \frac{1}{\lambda_1} u_{1\lambda_1}\); and
\[
\lim_{\lambda_1 \to 0} \|u_{1\lambda_1}\|_C = +\infty.
\]
Moreover, we obtain from (23) that
\[
\frac{1}{|\lambda_1|} \|u_{1\lambda_1}\|_C \leq \|Tu_{1\lambda_1}\|_C \geq D\varphi_q\left(\frac{1}{4} N_1 D^{\alpha} \varphi_q^\alpha \left(\frac{1}{4} \lambda_2 N_2\right)\right) \|u_{1\lambda_1}\|_C^{\frac{\alpha\beta}{(p-1)^2}}.
\]
which shows that
\[
\|u_{1\lambda_1}\|_C \leq \left|\lambda_1\right| D\varphi_q\left(\frac{1}{4} N_1 D^{\alpha} \varphi_q^\alpha \left(\frac{1}{4} \lambda_2 N_2\right)\right)^{\frac{(p-1)^2}{(p-1)^2-\alpha\beta}}. \tag{27}
\]
But
\[
\left|\frac{1}{\lambda_1} u_{1\lambda_1}\right| \leq \varphi_q(\|h_1\|_{L^1}) \varphi_q^\alpha \varphi_q(\|h_2\|_{L^1}) \|u\|_C^{\frac{\alpha\beta}{(p-1)^2}}.
\]
It hence follows from (27) that
\[
\|u_{1\lambda_1}\|_C \leq \left|\lambda_1\right| \varphi_q(\|h_1\|_{L^1}) \varphi_q^\alpha \varphi_q(\|h_2\|_{L^1}) \|u\|_C^{\frac{\alpha\beta}{(p-1)^2}} \leq \left|\lambda_1\right| \varphi_q(\|h_1\|_{L^1}) \varphi_q^\alpha \varphi_q(\|h_2\|_{L^1}) \lambda \neq 0. \tag{28}
\]
This gives the proof of (16) when \(\alpha \beta > (p-1)^2\).

Moreover, it follows from \(Tu_{1\lambda_1} = \frac{1}{\lambda_1} u_{1\lambda_1}\) that \(\lambda_1 Tu_{1\lambda_1} = u_{1\lambda_1}\), which shows that \(u_{1\lambda_1}\) is a fixed point of operator \(\lambda_1 T\). Then by \(\lambda_1 T = \lambda_1 T_1 T_2\) we can define \(u_{2\lambda_1} = T_2 u_{1\lambda_1}\) and obtain \(u_{1\lambda_1} = \lambda_1 T_1 u_{2\lambda_1}\). Thus we get a pair of solutions \(u_{1\lambda_1}, u_{2\lambda_1}\) of system (2) and \(u_{1\lambda_1}\) satisfies (15) and (16).

Next turning to \(0 < \alpha \beta < (p-1)^2\), then from (20) and (25), together with Lemma 2.2, we get that: for any \(\lambda_1 \neq 0\), there exists \(u_{1\lambda_1} \in E\) with \(u_{1\lambda_1} \neq \theta\) such that \(Tu_{1\lambda_1} = \frac{1}{\lambda_1} u_{1\lambda_1}\); and \(\lim_{\lambda_1 \to 0} \|u_{1\lambda_1}\|_C = 0\).

Moreover, we get from (28) that (17) holds when \(\lambda_1 \to 0\). The proof of Theorem 2.5 is completed.

**Remark 2.** In the proof of (ii) and (iv) in Theorem 2.5, we need define a new composite operator \(T^* = T_2^* T_1^*\), where
\[
T_1^*(u)(t) = \int_t^1 \varphi_q\left(\frac{1}{s^{N-1}} \int_0^s \lambda_1 h_1 \tau^{N-1} u^\alpha d\tau\right) ds,
\]
\[
T_2^*(u)(t) = \int_t^1 \varphi_q\left(\frac{1}{s^{N-1}} \int_0^s h_2 \tau^{N-1} u^\beta d\tau\right) ds.
\]
In Theorem 2.5 we consider the existence of positive solution in the supercritical case and subcritical case. Next we discuss what happen in the critical case: \(\alpha \beta = (p-1)^2\)? In fact, we will obtain the nonexistence result in Theorem 1.1 when \(\alpha \beta = (p-1)^2\).

**Proof of Theorem 1.1.** Assume \(u_1, u_2\) are a pair of positive solutions for system (2). We will prove that this leads to a contradiction for \(0 < |\lambda_1| \lambda^2 < \lambda^*_s\), where
\[
\lambda^*_s = \frac{1}{\varphi_q(\|h_1\|_{L^1}) \varphi_q^\alpha \varphi_q(\|h_2\|_C)}.
\]
Since \((T_1 T_2 u_1)(t) = (Tu_1)(t) = \frac{1}{\lambda_1} u_1(t)\) for \(t \in J\), it follows from (18) that
Remark 3. If we consider operator $T^*$, which is defined in Remark 2, then there exits $\lambda^* > 0$ such that (2) admits no positive solutions for $0 < |\lambda_2|\lambda_1^{\frac{\alpha}{\beta}} < \lambda^*$ when $\alpha \beta = (p-1)^2$.

Remark 4. The approaches to prove Theorem 1.1 and Theorem 2.5 can be applied to the following single elliptic equation

$$\|u_1\|_C \leq |\lambda_1|^{\frac{\alpha}{\beta}} (|h_1|_{L^1})^{\frac{\alpha}{\beta}} (|\varphi_q(\|h_2\|_{L^1})|)^{\frac{\alpha}{\beta}} u_1 = |\lambda_1|^{\frac{\alpha}{\beta}} (|h_1|_{L^1})^{\frac{\alpha}{\beta}} (|\varphi_q(\|h_2\|_{L^1})|)^{\frac{\alpha}{\beta}} u_1
$$

This is a contradiction, and our proof is finished.

Remark 5. We can generalize the approaches of Theorem 1.1 and Theorem 2.5 to study the more general $p$-Laplacian elliptic system

$$\begin{cases}
-\Delta_p z = \lambda h(|x|)z^\alpha & \text{in } \Omega, \\
z = 0 & \text{on } \partial\Omega,
\end{cases}$$

where $\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u)$, $1 < p < \infty$, $\lambda$ is a parameter, $\alpha > 0$, and $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$).

Remark 5. We can generalize the approaches of Theorem 1.1 and Theorem 2.5 to study the more general $p$-Laplacian elliptic system

$$\begin{cases}
-\Delta_p z_1 = \lambda_1 h_1(|x|)z_1^\alpha & \text{in } \Omega, \\
-\Delta_p z_2 = \lambda_2 h_2(|x|)z_2^\alpha & \text{in } \Omega, \\
\vdots \\
-\Delta_p z_n = \lambda_n h_n(|x|)z_n^\alpha & \text{in } \Omega, \\
z_1 = z_2 = \cdots = z_n = 0 & \text{on } \partial\Omega.
\end{cases}$$

Here, $\Delta_p u := \text{div}(|\nabla u|^{p-2} \nabla u)$, $1 < p < \infty$, for $i \in \{1, 2, \ldots, n\}$, $\lambda_i \neq 0$ are parameters, $\alpha_i > 0$, and $\Omega$ is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$). Thus, the results of Theorem 1.1 and Theorem 2.5 improve the study of Lan and Zhang [38] to some degree.

3. Uniqueness and approximation of positive solutions to (2). In this section, we intend to analyze a general result of uniqueness and approximation to system (2) by iterations of the solutions. Results to be verified below are true for any positive parameters $\lambda_1$ and $\lambda_2$. We hence may assume $\lambda_1 = \lambda_2 = 1$ for simplicity. Fix $\alpha > 0$ and $\beta > 0$ such that $\alpha \beta < (p-1)^2$ in (2).

It is well known that system (2) is equivalent to the following nonlinear integral equations

$$u_1 = \int_1^t \varphi_q \left( \frac{1}{s^{N-1}} \int_0^s h_1 \tau^{N-1} u_2^\alpha d\tau \right) ds, \quad (29)$$

$$u_2 = \int_1^t \varphi_q \left( \frac{1}{s^{N-1}} \int_0^s h_2 \tau^{N-1} u_1^\beta d\tau \right) ds. \quad (30)$$

Consider the operators

$$T_1 u(t) = \int_1^t \varphi_q \left( \frac{1}{s^{N-1}} \int_0^s h_1 \tau^{N-1} u_2^\alpha d\tau \right) ds, \quad (31)$$

$$T_2 u(t) = \int_1^t \varphi_q \left( \frac{1}{s^{N-1}} \int_0^s h_2 \tau^{N-1} u_1^\beta d\tau \right) ds. \quad (32)$$
Notice that the image of each operator is a nonnegative continuous function on $J$, so by Lemma 3.2 in [23], $T_i$ $(i \in \{1, 2\})$ are completely continuous operators from $P^*$ into $P^*$, where $P^*$ is given by

$$P^* := \{ u \in E : u(t) \geq 0, \ t \in J \}.$$

It is clear to see that $P^*$ is a normal cone in $E$.

Define a composite operator $T = T_1T_2$. Naturally, $T$ is also completely continuous from $E$ to $E$. Calculation shows that $(u_1, u_2) \in E \times E$ solves system (2) if and only if $(u_1, u_2)$ satisfy $u_1 = T_1u_2$, $u_2 = T_2u_1$. Thus, if $u_1 \in E$ is a fixed point of $T$, define $u_2 = T_2u_1$, then $u_2 \in E$ such that $(u_1, u_2) \in E \times E$ solves system (2); on the contrary, if $(u_1, u_2) \in E \times E$ solves system (2), then $u_1$ is certainly a fixed point of $T$.

**Remark 6.** Since $\lambda_1 = \lambda_2 = 1$, $(u_1, u_2) \in E \times E$ solves (31) and (32), and $(u_1, u_2) \in E \times E$ can solve system (2). So, we only need to verify that operator $T$ has at most one fixed point in the cone $P^*$.

Our results are proved via a sequence of lemmas. But we need to give a definition first.

**Definition 3.1.** (Definition 3.1, Hu and Wang [34]) Let $P$ be a cone from a real Banach space $E$. With some $u_0 \in P$ positive, $T : P \to P$ is called $u_0$-sublinear if:

(a) for any $x > 0$, there exist $\theta_1 > 0, \theta_2 > 0$ so that

$$\theta_1 u_0 \leq T x \leq \theta_2 u_0;$$

(b) for any $\theta_1 u_0 \leq x \leq \theta_2 u_0$ and $t \in (0, 1)$, there always exists some $\eta = \eta(x, t) > 0$ so that

$$T(tx) \geq (1 + \eta)Tx.$$

**Lemma 3.2.** (Lemma 3.3, Hu and Wang [34]) An increasing and $u_0$-sublinear operator $T$ can have at most one positive fixed point.

**Lemma 3.3.** Let $0 < \alpha\beta < (p - 1)^2$ and $u_0 = 1 - t$. Then the operator $T$ is $u_0$-sublinear.

**Proof.** First, we prove that for any $u > 0$ from $P^*$, there exist $\theta_1 = \theta_1(u) > 0$ and $\theta_2 = \theta_2(u) > 0$ such that $\theta_1 u_0 \leq T_1u \leq \theta_2 u_0$.

In fact, for any $u > 0$ from $P^*$, we have

$$T_1u(t) = \int_0^1 \varphi_q \left( \frac{1}{\varphi_q (\int_0^\tau h_1(\tau) t^{N - 1}u^\alpha(\tau)d\tau)} \right)ds$$

$$= \int_0^1 \varphi_q (\int_0^\tau h_1(\tau) t^{N - 1}u^\alpha(\tau)d\tau)ds$$

$$\leq \int_0^1 \varphi_q (\int_0^\tau h_1(\tau)||u||_C^\alpha d\tau)ds$$

$$\leq \varphi_q (||h_1||_{L_1}||u||_C^\alpha)(1 - t).$$

We so may take

$$\theta_2 = \varphi_q (||h_1||_{L_1}||u||_C^\alpha) = (||h_1||_{L_1}||u||_C^\alpha)^{p - 1} = (||h_1||_{L_1}||u||_C^\alpha)^{p - 1}.$$

On the other hand, from the proof of Lemma 3.1 in Feng, Du and Ge [23], we can choose $\theta_1 = ||T_1u||_C = (T_1u)(0)$ since $T_1(P) \subset T_1(P^*)$.

Next we give a direct proof. Considering that $(T_1u)(t)$ is strictly decreasing in $t$ and vanishes at 1, we can choose $c \in (0, 1)$ and put

$$\gamma = \left( \int_0^c N_1 t^{N - 1}u^\alpha(t)dt \right)^{\frac{1}{1 - p}}.$$
So, for \( t \in [0, c] \) we get
\[
(T_1 u)(t) \geq \int_t^1 \gamma dt \geq \gamma(1-t).
\]
Since \((T_1 u)(t) \geq (T_1 u)(c) = \gamma(1-c)\) for all \( t \in [0, c] \), we get
\[
(T_1 u)(t) \geq \gamma(1-c)(1-t)
\]
for all \( t \in J \).

Thus, we can take \( \theta_1 = \gamma(1-c) \) such that \( T_1 \) satisfies (a) of Definition 3.1.

Noticing that \( T_1, T_2 : P^* \to P^* \), so is \( T = T_1 T_2 \), and for \( u \in P^* \) with \( u > 0 \), we have \( T_2 u \in P^* \) and \( T_2 u > 0 \). We get that \( T = T_1 T_2 \) also satisfies (a) of Definition 3.1.

The proof is finished if \( A \) satisfies (b) of Definition 3.1.

In fact, letting \( \theta_1 u_0 \leq u \leq \theta_2 u_0 \), \( t \in (0, 1) \), then direct calculation shows \( T_1(tu) = t^{\frac{p}{p-1}} T_1(u), T_2(tu) = t^{\frac{p}{p-1}} T_2(u) \). Thus \( T(tu) = T_1(t^{\frac{p}{p-1}} T_2(u)) = t^{\frac{p}{p-1}} T_1 T_2(u) \geq (1+\eta)t Tu \) for some \( \eta > 0 \). This inequality holds because \( t \in (0, 1) \) and \( (\alpha \beta)^{\frac{p}{p-1}} < 1 \), and our proof is completed.

**Lemma 3.4.** If \( 0 < \alpha \beta < (p-1)^2 \), then system (2) admits a unique positive solution.

**Proof.** It is not difficult to see that \( T_1, T_2 \) are increasing operators on the partial order induced by \( P^* \). So is \( T = T_1 T_2 \). By Lemma 3.3, we know that \( T \) is \( u_0 \)-sublinear for \( u_0 = 1-t \). This together with Lemma 3.2 finishes the proof of Lemma 3.4. \( \square \)

**Lemma 3.5.** (Lemma 3.5, Hu and Wang [34]) If an increasing \( u_0 \)-sublinear operator \( T \) from a normal cone \( P \) into itself admits a positive fixed \( x^* \), then for any \( x_0 > 0 \) the iterations \( x_{n+1} = T x_n \) converge to \( x^* \). Namely \( \lim_{n \to \infty} \| x_n - x_0 \| = 0 \).

**Theorem 3.6.** Suppose that (H) and \( 0 < \alpha \beta < (p-1)^2 \). Then system (2) admits at most one positive solution. Moreover, if (2) possesses a positive solution \( u^* \), then for any \( u^0 \in P \) with \( u^0 > 0 \), the iteration \( u_n \), defined by
\[
u_{n+1} = T u_n
\]
converges to \( u^* \).

**Proof.** Now from the above lemmas, we can directly obtain Theorem 3.6. \( \square \)

**Remark 7.** Note that the procedure and method used in the proof of Lemma 3.3 are known in the literatures. See, for example, [28] and [34].

**Remark 8.** Comparing with system (2), we can handle more general ones, i.e.
\[
\begin{align*}
-(r N^{-1} \varphi_p(u_1'))' &= r N^{-1} h_1(r) f_1(u_2(t)) & \text{in} & & 0 < r < 1, \\
-(r N^{-1} \varphi_p(u_2'))' &= r N^{-1} h_2(r) f_2(u_1(t)) & \text{in} & & 0 < r < 1, \\
u_1'(0) &= u_2'(0) = u_1(1) = u_2(1) = 0
\end{align*}
\]

(35)

Similar arguments go through and we can get the following conclusion: for \( i \in \{1, 2\} \), let \( f_i : [0, \infty) \to [0, \infty) \) be continuous and nondecreasing such that for any \( u > 0 \) and \( t \in (0, 1) \) there always exists some \( \eta > 0 \) so that
\[
f_1(tu) \geq ((1+\eta)t)^{p-1}, \quad f_2(tu) \geq [(1+\eta)t]^{(p-1)^2}.
\]
Then system (35) admits at most one positive solution. Moreover, if (35) possesses a positive solution $u^*$, then for any $u^0 \in P$ with $u^0 > 0$, the iteration $u_n$, defined by
\[
 u_{n+1} = Tu_n
\]
converges to $u^*$.

Next we consider the nonexistence of positive solution for system (2) in the case $\lambda_1 = \lambda_2 = 1$ and $h_1(t) = h_2(t) \equiv 1$ on $J$.

**Theorem 3.7.** Suppose that (H) holds. Then system (2) admits no positive solution if $\alpha \neq 0$, $\beta \neq 0$ and $\alpha \beta = (p - 1)^2$.

**Proof.** As previous analysis, we only need to prove that operator $T$ does not have positive fixed point in $P$. For each $u \in P$, we get
\[
\|T_2u\|_C = \max_{t \in J} \int_1^t \varphi_q\left(\frac{1}{s-N} \int_0^s \tau^{N-1}u^\alpha(\tau)d\tau\right)ds \\
= \max_{t \in J} \int_1^t \varphi_q\left(\int_0^q (\tau^t)^{N-1}u^\alpha(\tau)d\tau\right)ds \\
\leq \int_0^1 \varphi_q\left(\int_0^1 (\tau^t)^{N-1}\|u\|^\alpha_2 d\tau\right)ds.
\]
Suppose that operator $T$ has a positive fixed point $u_0$ in $P$, then $u_0$ must be a concave function satisfying $u_0(1) = 0$ and $u_0(t) > 0$, $t \in [0,1)$. Thus if we choose $u = u_0$, then the last inequality in (36) must be strict. We hence get
\[
\|T_2u_0\|_C < \|u_0\|^{\frac{\alpha}{\alpha - 1}}_C.
\]
Similarly, we have $\|T_1u\|_C \leq \|u\|^{\frac{\alpha}{\alpha - 1}}_C$ for all $u \in P$. So we get
\[
\|Tu_0\|_C < \|u_0\|^{\frac{\alpha}{\alpha - 1}}_C = \|u_0\|_C.
\]
This is a contradiction with the assumption $Tu_0 = u_0$.

Finally, we consider the existence of positive for system (2) by using a well-known result of the fixed point index in a cone, which is used in [50].

**Lemma 3.8.** ([28]) Let $E$ be a real Banach space and $K$ be a cone in $E$. For $r > 0$, define $K_r = \{ x \in K : \|x\| < r \}$. Assume that $T : K_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{ x \in K : \|x\| = r \}$.

(i) If $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.

(ii) If $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

Let $E$ and $P$ be defined as in Section 2.

**Theorem 3.9.** Suppose that (H) holds. Then system (2) admits a positive solution if $\alpha \neq 0$, $\beta \neq 0$ and $\alpha \beta = (p - 1)^2$.

**Proof.** From (18) and (23) we respectively get
\[
\|T(u)\|_C \leq \varphi_q(\|h_1\|_L^1)\varphi_q^\alpha(\varphi_q(\lambda_2\|h_2\|_L^1))\|u\|^{\frac{\alpha}{\alpha - 1}}_C, \tag{37}
\]
\[
\|T(u)\|_C \geq D\varphi_q\left(\frac{1}{4}N_1D^\alpha\varphi_q(\lambda_2N_2)\right)\|u\|^{\frac{\alpha}{\alpha - 1}}_C. \tag{38}
\]

**Case 1.** $\alpha \beta > (p - 1)^2$.

Take $r_1$ small enough such that $0 < r_1 < 1$. For $u \in P$ with $\|u\|_C = r_1$, we get $\|Tu\|_C < \|u\|_C$ by (37). Similarly, by the estimate (38), we can choose $r_2$ large...
such that \( r_2 > r_1 \), and for each \( u \in P \) with \( \|Tu\|_C = r_2 \) it holds \( \|Tu\|_C > \|u\|_C \).

By Lemma 3.8

\[ i(T, P_{r_1}, P) = 1, \quad i(T, P_{r_2}, P) = 0. \]

We hence get \( i(T, P_{r_2} \setminus P_{r_1}, P) = -1 \) by the additivity of the fixed point index. Then obtain that \( T \) possesses a fixed point \( u_1 \) in \( P_{r_2} \setminus P_{r_1} \) by the existence property of the fixed point index. Define \( u_2 = T_2u_1 \). Then \((u_1, u_2)\) is the desired solution of (2).

**Case 2.** \( 0 < \alpha \beta < (p - 1)^2 \).

Take \( r_1 \) small enough such that \( r_3 > 0 \). For \( u \in P \) with \( \|u\|_C = r_3 \), we get \( \|Tu\|_C > \|u\|_C \) by (38). Similarly, by the estimate (37), we can choose \( r_4 \) large such that \( r_4 > r_3 \), and for each \( u \in P \) with \( \|Tu\|_C = r_4 \) it holds \( \|Tu\|_C < \|u\|_C \).

By Lemma 3.8

\[ i(T, P_{r_1}, P) = 1, \quad i(T, P_{r_2}, P) = 0. \]

The rest of the proof is similar to that of Case 1 and hence we omit it. \( \square \)

4. **Preliminaries and main results of system** \((6)\). In this section we mainly collect some known results to be used in the subsequent sections, and give the existence results of positive solutions for system \((6)\). A pair of functions \( u, v \in C([0, 1]) \) with \( \phi_p(u'), \phi_p(v') \in C^1(0, 1) \) is called to be a positive solution of \((6)\) if \( u(t), v(t) > 0 \) for all \( t \in (0, 1) \), and \( u \) and \( v \) satisfy \((6)\).

Let

\[ E = C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R}). \]

Then \( E \) is a Banach space with the norm \( \|(u, v)\| = \|u\|_\infty + \|v\|_\infty \), where \( \|u\|_\infty = \max_{t \in [0, 1]} |u(t)| \).

Define a cone \( P \) by

\[ P = \{(v_1, v_2) \in E : (v_1, v_2)(0) = (v_1, v_2)'(1) = (0, 0), \text{ and } v_1, v_2 \text{ are concave}\}. \]

Define an operator \( F : E \to E \) by

\[ F(\phi, \psi)(t) = (A(\phi, \psi)(t), B(\phi, \psi)(t)), \quad t \in [0, 1], \]

where

\[ A(\phi, \psi)(t) = \int_0^t \varphi_q \left( \int_s^1 G_1(\tau, \phi(\tau), \psi(\tau)) d\tau \right) d\sigma, \]

and

\[ B(\phi, \psi)(t) = \int_0^t \varphi_q \left( \int_s^1 G_2(\tau, \phi(\tau), \psi(\tau)) d\tau \right) d\sigma. \]

**Lemma 4.1.** Fix \( a, b \geq 0 \). Let conditions \((C_0)\) and \((C_1)\) hold. Then operator \( F : E \to E \) is well defined, \( F(P) \subset P \), and \( F \) is completely continuous.

**Proof.** Suppose that \((C_0)\) and \((C_1)\) hold. Then we get

\[ A(\phi, \psi)(t) \leq \int_0^1 \varphi_q \left( \int_0^1 h(\tau) d\tau \right) d\sigma < \infty, \quad \forall (\phi, \psi) \in E, \]

where \( d = \|(\phi, \psi)\| \). This shows that \( A \) is well defined for any \((\phi, \psi) \in E\).

Let \((\phi_n, \psi_n), (\phi, \psi) \in E, n \in \mathbb{N} \) be such that

\[ \|(\phi_n, \psi_n) - (\phi, \psi)\| \to 0 \quad \text{as } n \to \infty. \]

Define a function

\[ \xi_n(\tau) = |G_1(\tau, \phi_n, \psi_n, a, b) - G_1(\tau, \phi, \psi, a, b)|. \]
Then, for \( \tau \in (0, 1) \) and \( n \) sufficiently large, we get that \( \xi_n(\tau) \to 0 \) and \( 0 \leq \xi_n(\tau) \leq 2h_1(\tau) \).

We hence obtain that

\[
|A(\phi_n, \psi_n)(t) - A(\phi, \psi)(t)| = \left| \int_0^t \varphi_q \left( \int_s^1 G_1(\tau, \phi_n(\tau), \psi_n(\tau)) d\tau \right) ds \right|
\]

\[
- \int_0^t \varphi_q \left( \int_s^1 G_1(\tau, \phi_n(\tau), \psi_n(\tau)) d\tau \right) ds
\]

\[
= \int_0^t \left[ \varphi_q \left( \int_s^1 G_1(\tau, \phi_n(\tau), \psi_n(\tau)) d\tau \right) - \varphi_q \left( \int_s^1 G_1(\tau, \phi(\tau), \psi(\tau)) d\tau \right) \right] ds
\]

\[
= \int_0^t \left[ |G_n(s)|^{q-2}G_n(s) - |G(s)|^{q-2}G(s) \right] ds
\]

\[
= \int_0^t \left[ |G_n(s)|^{q-2}(G_n(s) - G(s)) + G(s)(|G_n(s)|^{q-2} - |G(s)|^{q-2}) \right] ds
\]

\[
= \int_0^t \left[ |G_n(s)|^{q-2}(G_n(s) - G(s)) + G(s)(|G_n(s)|^{q-2} - |G(s)|^{q-2}) \right] ds
\]

\[
+ |G_n(s)|^{q-4}|G(s)| + \cdots |G^{q-3})| ds
\]

\[
\leq \int_0^t \left[ |G_n(s)|^{q-2}(G_n(s) - G(s)) + G(s)(|G_n(s)|^{q-2} - |G(s)|^{q-2}) \right] ds
\]

\[
+ |G_n(s)|^{q-4}|G(s)| + \cdots |G^{q-3})| ds
\]

\[
\leq \int_0^t \left[ |G_n(s) - G(s)| |h_\tau^s(s)|^{q-2} + h_\tau^s(s)(|h_\tau^s(s)|^{q-3} + |h_\tau^s(s)|^{q-4}|h_\tau^s(s)| \right]
\]

\[
+ \cdots |h_\tau^s(s)|^{q-3}) | ds
\]

where

\[
G_n(s) = \int_s^1 G_1(\tau, \phi_n(\tau), \psi_n(\tau)) d\tau,
\]

\[
G(s) = \int_s^1 G_1(\tau, \phi(\tau), \psi(\tau)) d\tau,
\]

\[
h_\tau^s(s) = \int_s^1 h_1(\tau) d\tau.
\]

Thus, we obtain from Lebesgue Dominated Convergence Theorem that

\[
|A(\phi_n, \psi_n)(t) - A(\phi, \psi)(t)| \to 0,
\]

which implies that \( A \) is continuous. The proof of continuity for operator \( B \) is analogous.

On the other hand, a direct calculation implies that the operator \( A(\phi, \psi) \) is twice differentiable on \((0, 1)\) with the second derivative negative for \((\phi, \psi) \in P\). The same holds for \( B \). Thus, \( F(P) \subset P \). Let \((\phi_n, \psi_n) \subset E \) and \( \| (\phi_n, \psi_n) \| \leq L \).
Set
\[
\max_{i=1,2}\{A(\phi_n, \psi_n)(t), B(\phi_n, \psi_n)(t)\} \leq \max_{i=1,2}\left\{ \int_0^1 \varphi_q(\int_s^1 h_i(\tau)d\tau)ds < \infty \right\}.
\]
Therefore \( F(\phi_n, \psi_n) \) is uniformly bounded.

Letting \( t_1, t_2 \in J \) with \( t_2 < t_1 \), then
\[
|A(\phi_n, \psi_n)(t_1) - A(\phi_n, \psi_n)(t_2)| = |\int_0^{t_2} \varphi_q(\int_1^s G(t, \phi_n(\tau), \psi_n(\tau))d\tau)ds|
\]
\[
- |\int_{t_2}^{t_1} \varphi_q(\int_1^s G(t, \phi_n(\tau), \psi_n(\tau))d\tau)ds|
\]
\[
\leq |\int_{t_2}^{t_1} \varphi_q(\int_1^s h_1(\tau)d\tau)ds|
\]
\[
\to 0(t_2 \to t_1).
\]

This shows that \( A(\phi_n, \psi_n) \) is equicontinuous in \( J \). Similarly, we can prove that \( B(\phi_n, \psi_n) \) is also equicontinuous in \( J \). According to Arzelà-Ascoli Theorem, we can observe that the operator \( F \) is completely continuous.

The following well-known results are crucial in the proofs of our results.

Lemma 4.2. ([28, 36, 50]) Let \( X \) be a Banach space with norm \( \| \cdot \| \), and \( P \subset X \) be a cone in \( X \). For \( R > 0 \), define \( P_R = P \cap B[0, R] \), where \( B[0, R] \) denotes the closed ball of radius \( R \) centered at the origin of \( X \). Assume that \( F : P_R \to C \) is a completely continuous operator and that there exists \( 0 < r < R \) such that
\[
\|Fu\| < \|u\|, \text{ for all } u \in \partial P_r \text{ and } \|Fu\| > \|u\|, \text{ for all } u \in \partial P_R,
\]
or
\[
\|Fu\| > \|u\|, \text{ for all } u \in \partial P_r \text{ and } \|Fu\| < \|u\|, \text{ for all } u \in \partial P_R,
\]
where \( \partial P_R = \{u \in P : \|u\| = R\} \). Then \( F \) admits a fixed point \( u \in P \) with \( r < \|u\| < R \).

Next we recall the following theorems about fixed point index \( i(F, P_r, P) \) in a cone, where \( i(F, P_r, P) \) is called the fixed point index of \( F \) on \( P_r \) with respect to \( P \).

Lemma 4.3. ([28, 36, 50]) Let \( X \) be a Banach space with norm \( \| \cdot \| \) and \( P \) be a cone in \( X \); and let \( \Omega \) be a bounded open set in \( X \). Let \( 0 \in \Omega \), and let \( F : P \cap \overline{\Omega} \to P \) be compact. Suppose that \( Fu \neq \lambda u \) for all \( u \in P \cap \partial \Omega \) and all \( \lambda \geq 1 \). Then
\[
i(F, P \cap \Omega, P) = 1.
\]

In the rest of this section, we state an elementary result of concave functions that will be used in the subsequent sections.

Lemma 4.4. Suppose \( u(t) \) is a non-negative, concave, continuous function defined on \( J \). Then, for all \( \theta \in (0, \frac{1}{2}) \), we have
\[
\min_{t \in J_\theta = [\theta, 1-\theta]} u(t) \geq \theta \|u\|_\infty.
\]

Proof. The proof is similar to that of Lemma 3.1 in [23].

Our main results for system (6) are the following.
Theorem 4.5. Assume that (C_0) – (C_3) hold. Then there exist a constant \( \tilde{a} > 0 \) and a nonincreasing continuous function \( \Gamma : [0, \tilde{a}] \to \mathbb{R}_+ \) such that, for all \( a \in [0, \tilde{a}] \), system (6) admits

(i) at least one positive solution for \( 0 \leq b \leq \Gamma(a) \);
(ii) no positive solution for \( b > \Gamma(a) \);
(iii) at least two positive solutions for \( 0 < b < \Gamma(a) \), when \( G_1 \) and \( G_2 \) are increasing.

5. Existence of a positive solution for small parameters. In this section, we analyze the existence of a fixed point for operator \( F \) when \( a \) and \( b \) are sufficiently small.

Theorem 5.1. Suppose that (C_0) and (C_1) hold. Then there exist \( \tilde{R}_0 > 0 \) and \( \delta_0 \) so that, for all \( (\phi, \psi) \in P_{\tilde{R}_0} \) and all \( (a, b) \) with \( 0 < a + b < \delta_0 \), we get

\[
\| F(\phi, \psi) \| < \|(\phi, \psi)\|.
\]

Proof. From condition (C_2), we can take \( \sigma \in (0, 1) \) such that

\[
2d^* \int_0^1 \varphi_q \left( \int_0^1 h(\tau)d\tau \right) ds < 1 - \sigma,
\]

where \( d^* = \max\{d_1, d_2\} \), \( h(s) = \max\{h_1(s), h_2(s)\} \) for \( s \in (0, 1) \).

Also, there exists \( R > 0 \) so that, for \( 0 \leq u + v + a + b \leq R \) and \( t \in (0, 1) \), we obtain

\[
G_i(t, u, v, a, b) \leq \varphi_p(u + v + a + b) \varphi_p(d^*) h(t).
\]

Thus, we have

\[
A(\phi, \psi)(t) + B(\phi, \psi)(t') = \int_0^t \varphi_q \left( \int_0^1 G_1(\tau, \phi(t), \psi(t)) d\tau \right) ds + \int_0^t \varphi_q \left( \int_0^1 G_2(\tau, \phi(t), \psi(t)) d\tau \right) ds
\]
\[
\leq \int_0^1 \varphi_q \left( \int_0^1 \varphi_p(d^*) h(\tau) \varphi_p(\phi(\tau) + \psi(\tau) + a + b)d\tau \right) + \varphi_q \left( \int_0^1 \varphi_p(d^*) h(\tau) \varphi_p(\phi(\tau) + \psi(\tau) + a + b)d\tau \right) ds
\]
\[
\leq 2d^* \int_0^1 \varphi_q \left( \int_0^1 \varphi_p(d^*) h(\tau) \varphi_p(\|\phi\| + \|\psi\| + a + b)d\tau \right) + \varphi_q \left( \int_0^1 \varphi_p(d^*) h(\tau) \varphi_p(\|\phi\| + \|\psi\| + a + b)d\tau \right) ds
\]
\[
= 2d^* R \int_0^1 \varphi_q \left( \int_0^1 h(\tau)d\tau \right) ds
\]
\[
\leq \tilde{R}(1 - \sigma).
\]

Picking \( \tilde{R}_0 = \tilde{R}(1 - \sigma) \) and \( \delta_0 = \sigma \tilde{R} \), then for all \( (\phi, \psi) \in P_{\tilde{R}_0} \) and for \( 0 < a + b < \delta_0 \), we get

\[
\| F(\phi, \psi) \| = \| A(\phi, \psi) \|_{\infty} + \| B(\phi, \psi) \|_{\infty} < \tilde{R}_0 = \|(\phi, \psi)\|.
\]

\[\square\]

Theorem 5.2. Suppose that (C_0), (C_1) and (C_3) hold. Then there exists \( \tilde{R}_1 > 0 \) such that, for all \( (\phi, \psi) \in P_{\tilde{R}_1} \), we get

\[
\| F(\phi, \psi) \| > \|(\phi, \psi)\|.
\]

Proof. By contradiction, suppose that there exists an increasing sequence \( \tilde{R}_n \to +\infty \) and a sequence \( \{(\phi_n, \psi_n)\} \) in \( P \) such that the real sequence \( \tilde{R}_n \) defined by \( \|(\phi_n, \psi_n)\| = \tilde{R}_n \) would satisfy

\[
\| F(\phi_n, \psi_n) \| \leq \|(\phi_n, \psi_n)\|.
\]
We consider \((\|\phi_n\|_\infty + \|\psi_n\|_\infty)/\bar{R}_n \to 1\) as \(n \to +\infty\). Combining the monotonicity of the nonlinearities \(G_1\) and \(G_2\), the concavity of \(\phi_n\) and \(\psi_n\), and Lemma 4.3 we obtain

\[
\|F(\phi_n, \psi_n)\| = \|A(\phi_n, \psi_n)\|_\infty + \|B(\phi_n, \psi_n)\|_\infty \\
\geq \int_0^\alpha \varphi_q J_n^{1-\alpha} G_1(t, \phi_n(t), \psi_n(t), a, b) dt ds \\
+ \int_0^\alpha \varphi_q J_n^{1-\alpha} G_2(t, \phi_n(t), \psi_n(t), a, b) dt ds \\
\geq \int_0^\alpha \varphi_q J_n^{1-\alpha} G_1(t, \alpha \|\phi_n\|_\infty, \alpha \|\psi_n\|_\infty, 0, 0) dt ds \\
+ \int_0^\alpha \varphi_q J_n^{1-\alpha} G_2(t, \alpha \|\phi_n\|_\infty, \alpha \|\psi_n\|_\infty, 0, 0) dt ds \\
= \int_0^\alpha \varphi_q J_n^{1-\alpha} \frac{J_n(t)}{\varphi_p(\alpha \|\phi_n\|_\infty + \alpha \|\psi_n\|_\infty)} \varphi_p(\alpha \|\phi_n\|_\infty + \alpha \|\psi_n\|_\infty) dt ds \\
+ \int_0^\alpha \varphi_q J_n^{1-\alpha} \frac{J_n(t)}{\varphi_p(\alpha \|\phi_n\|_\infty + \alpha \|\psi_n\|_\infty)} \varphi_p(\alpha \|\phi_n\|_\infty + \alpha \|\psi_n\|_\infty) dt ds \\
= \alpha \left( \frac{\|\phi_n\|_\infty + \|\psi_n\|_\infty}{R_n} \right) M_n \varphi_q(y(1)).
\]

where
\[
J_1(t) = G_1(t, \alpha \|\phi_n\|_\infty, \alpha \|\psi_n\|_\infty, 0, 0), \\
J_2(t) = G_2(t, \alpha \|\phi_n\|_\infty, \alpha \|\psi_n\|_\infty, 0, 0)
\]

and
\[
M_n = \int_0^\alpha \varphi_q \left( \int_0^{J_1(t)} \frac{J_2(t)}{\varphi_p(\alpha \|\phi_n\|_\infty + \alpha \|\psi_n\|_\infty)} dt ds \\
+ \int_0^\alpha \varphi_q \left( \int_0^{J_2(t)} \frac{J_1(t)}{\varphi_p(\alpha \|\phi_n\|_\infty + \alpha \|\psi_n\|_\infty)} dt ds \right) \right) dt ds \\
\]

and \(M_n \to +\infty\) by assumption \((C_3)\), we would have
\[
1 \geq \alpha \left( \frac{\|\phi_n\|_\infty + \|\psi_n\|_\infty}{R_n} \right) M_n \varphi_q(y(1)).
\]

This contradiction completes the proof. \(\Box\)

According to Theorem 5.1 and Theorem 5.2, the following theorem is a direct result of Lemma 4.2.

**Theorem 5.3.** Let \((C_0), (C_1), (C_2)\) and \((C_3)\) hold. Then there exists \(\delta_0 > 0\) such that, for all \(a\) and \(b\) satisfying \(0 < a + b < \delta_0\), the operator \(F\) possesses a fixed point \((\phi, \psi) \in P\) verifying \(\bar{R}_0 < \|(\phi, \psi)\| < \bar{R}_1\).

6. The sub- and super-solutions method. In this section, we establish the classical sub-supersolution method for obtaining positive solutions of system (6). For this, we first study the system

\[
\begin{cases}
(\varphi_p(\psi(t)))' + G_1(t, u(t), v(t)) = 0, \\
(\varphi_p(\psi(t)))' + G_2(t, u(t), v(t)) = 0, \\
(u(0), v(0)) = (0, 0), \quad (u'(1), v'(1)) = (0, 0),
\end{cases}
\]

where \(G_1\) and \(G_2\) satisfy \((C_1)\).

Generally, we say that \((u, v)\) is an upper solution or a lower solution of system (39) if \((u, v)\) respectively satisfies the following inequalities

\[
\begin{cases}
(\varphi_p(\psi(t)))' + G_1(t, u(t), v(t)) \leq 0, \\
(\varphi_p(\psi(t)))' + G_2(t, u(t), v(t)) \leq 0, \\
(u(0), v(0)) \geq (0, 0), \quad (u'(1), v'(1)) \geq (0, 0),
\end{cases}
\]

or

\[
\begin{cases}
(\varphi_p(\psi(t)))' + G_1(t, u(t), v(t)) \geq 0, \\
(\varphi_p(\psi(t)))' + G_2(t, u(t), v(t)) \geq 0, \\
(u(0), v(0)) \leq (0, 0), \quad (u'(1), v'(1)) \leq (0, 0).
\end{cases}
\]
Theorem 6.1. Suppose that (C₀) and (C₁) hold. Let \((u, v)\) and \((\overline{u}, \overline{v})\) be a lower and upper solution of system (39). Moreover, we suppose that \((0, 0) \leq (u, v) \leq (\overline{u}, \overline{v})\). Then, system (39) admits a non-negative solution \((u, v)\) such that \((u, v) \leq (\overline{u}, \overline{v})\).

Proof. Let

\[
T_1(u, v)(t) = \int_0^t \varphi_q \left( \int_s^1 G_1(\tau, u(\tau), v(\tau)) \, d\tau \right) \, ds,
\]

\[
T_2(u, v)(t) = \int_0^t \varphi_q \left( \int_s^1 G_2(\tau, u(\tau), v(\tau)) \, d\tau \right) \, ds,
\]

and

\[
T(u, v)(t) = (T_1(u, v)(t), T_2(u, v)(t)).
\]

Then, the solutions of system (6.1) are equivalent to the fixed points of the operator \(T\) in the Banach space \(E = C([0, 1], R) \times C([0, 1], R)\) endowed with the norm \(||(u, v)|| = ||u||_{\infty} + ||v||_{\infty}||.

Now, we present an auxiliary operator \(\tilde{T}\) defined by

\[
\tilde{T}(u, v)(t) = (\tilde{T}_1(u, v)(t), \tilde{T}_2(u, v)(t)),
\]

where

\[
\tilde{T}_1(u, v)(t) = \int_0^t \varphi_q \left( \int_s^1 G_1(\tau, \eta_1(\tau, u), \eta_2(\tau, v)) \, d\tau \right) \, ds,
\]

\[
\tilde{T}_2(u, v)(t) = \int_0^t \varphi_q \left( \int_s^1 G_2(\tau, \eta_1(\tau, u), \eta_2(\tau, v)) \, d\tau \right) \, ds,
\]

and

\[
\eta_1(\tau, u) = \max\{q(\tau), \min\{u(\tau), \overline{u}(\tau)\}\},
\]

\[
\eta_2(\tau, v) = \max\{q(\tau), \min\{v(\tau), \overline{v}(\tau)\}\}.
\]

It is not difficult to verify that the operator \(\tilde{T}\) admits the following three properties:

1. \(\tilde{T}\) is a bounded and completely continuous operator.
2. If the pair \((u, v)\) \(\in X\) is a fixed point of \(\tilde{T}\), then \((u, v)\) is a fixed point of \(T\), with \((u, v) \leq (u, v) \leq (\overline{u}, \overline{v})\).
3. If \((u, v) = \mu \tilde{T}(u, v)\) with \(0 < \mu < 1\), then \(||(u, v)|| \leq l\) where \(l\) does not depend on \(\mu\) and \((u, v) \in X\).

Finally, using the topological degree of Leray-Schauder (see [18], Corollary 8.1, page 61), we get a fixed point of the operator \(T\). This finishes the proof of Theorem 6.1.

Theorem 6.2. Assume that the system (6) admits a non-negative solution. Then for \((0, 0) \leq (a_1, b_1) \leq (a_2, b_2)\), system (6) admits a non-negative solution.

Proof. Suppose that the system (6) admits a non-negative solution \((u^*, v^*)\). As \(G_1\) and \(G_2\) are non-decreasing functions in the last two variables, then for \(i = 1, 2\), we get

\[
G_i(t, u, v, a_i, b_i) \leq G_i(t, u^*, v^*, a_2, b_2).
\]

Thus, \((u^*, v^*)\) is an upper solution and we can find that \((0, 0)\) is a lower solution of system (6). The conclusion so follows from Theorem 6.1.
7. **A priori estimate and non-existence.** In the present section, we are devoted to a priori estimates of positive solutions for system (6). More precisely we obtain

**Theorem 7.1.** Let \((C_0), (C_1)\) and \((C_3)\) hold. Then there exists a constant \(L > 0\) independent of \(a\) and \(b\) such that, for all positive solutions \((u, v)\) of system (6), we have

\[
\| (u, v) \| \leq L.
\]

**Proof.** The proof is analogous to that of Theorem 5.2. In fact, we can suppose by contradiction that there exists a sequence of positive solution \((\| u_n \|_\infty, \| v_n \|_\infty) \in P\) of system (6) such that \(\| (u_n, v_n) \| \to \infty\). We can find

\[
\begin{align*}
\| (u_n, v_n) \| &= \| u_n \|_\infty + \| v_n \|_\infty \\
&\geq \int_0^\alpha \varphi_\tau \left( \int_\tau^{\infty} G_1(\tau, u_n(\tau), v_n(\tau), a, b) d\tau \right) ds \\
&\quad + \int_0^\alpha \varphi_\tau \left( \int_\tau^{\infty} G_2(\tau, v_n(\tau), a, b) d\tau \right) ds \\
&\geq \int_0^\alpha \varphi_\tau \left( \int_\tau^{\infty} G_1(\tau, u_n(\tau), a, b) d\tau \right) ds \\
&\quad + \int_0^\alpha \varphi_\tau \left( \int_\tau^{\infty} G_2(\tau, v_n(\tau), a, b) d\tau \right) ds \\
&= \int_0^\alpha \varphi_\tau \left( \int_\tau^{\infty} G_1(\tau, u_n(\tau), a, b) d\tau \right) ds \\
&\quad + \int_0^\alpha \varphi_\tau \left( \int_\tau^{\infty} G_2(\tau, v_n(\tau), a, b) d\tau \right) ds \\
&= \alpha (\| u_n \|_\infty + \| v_n \|_\infty) M_n,
\end{align*}
\]

where

\[
M_n' = \int_0^\alpha \varphi_\tau \left( \int_\tau^{\infty} G_1(\tau, u_n(\tau), a, b) d\tau \right) ds \\
+ \int_0^\alpha \varphi_\tau \left( \int_\tau^{\infty} G_2(\tau, v_n(\tau), a, b) d\tau \right) ds.
\]

It hence follows from (40) that

\[
1 \geq \alpha M_n',
\]

which is impossible because of \(M_n' \to \infty\). This completes the proof of Theorem 7.1. \(\square\)

**Theorem 7.2.** Let \((C_0), (C_1)\) and \((C_3)\) hold. Then there exists \(\rho > 0\) such that, for all \((a, b) \in (0, +\infty) \times (0, +\infty)\) with \(\| (a, b) \| > \rho\), system (6) admits no positive solution.

**Proof.** By contradiction, suppose that there exists a sequence \((a_n, b_n)\) satisfying \(|(a_n, b_n)| \to +\infty (n \to +\infty)\) such that, for each nature number \(n, a_n, b_n\) of system (6) has a pair of positive solutions \((u_n, v_n) \in P\). It follows from \((C_3)\) that, for any \(D > 0\), there exists a constant \(\rho > 0\) such that, for \(|(a_n, b_n)| > \rho\) and \(\beta \in (0, \frac{1}{2})\),

\[
G(t, 0, 0, a, b) \geq D, \forall t \in [\beta, 1 - \beta].
\]

Since \(|(a_n, b_n)| \to +\infty (n \to +\infty)\), for the above \(\rho > 0\), then there exists a natural number \(n_0\) such that, for \(n > n_0, \| (a_n, b_n) \| > \rho\). And then, for \(n > n_0\) and \(t \in [\beta, 1 - \beta]\), we get

\[
\| (u_n, v_n) \| = \| u_n \|_\infty + \| v_n \|_\infty \\
= \int_0^\alpha \varphi_\tau \left( \int_\tau^{\infty} G_1(\tau, u_n(\tau), v_n(\tau), a, b) d\tau \right) ds \\
+ \int_0^\alpha \varphi_\tau \left( \int_\tau^{\infty} G_2(\tau, v_n(\tau), a, b) d\tau \right) ds.
\]
\[ \geq \int_0^\beta \varphi_q \left( \int_0^{1-\beta} G_1(\tau, 0, 0, a, b) d\tau \right) d\tau + \int_0^\beta \varphi_q \left( \int_0^{1-\beta} G_2(\tau, 0, 0, a, b) d\tau \right) d\tau \]
\[ \geq 2\beta \varphi_q \left( \int_0^{1-\beta} D d\tau \right) \]
\[ = 2\beta \varphi_q (D(1-2\beta)). \]

Since \( D \) is arbitrarily large, we have \( \|(u_n, v_n)\| \to +\infty (n \to +\infty) \) in contradiction with Theorem 7.1. This gives the proof of Theorem 7.2. \( \square \)

Next, we define the set \( \mathbb{A} \) by
\[ \mathbb{A} = \{ a > 0 : \text{ system (6) admits a positive solution for some } b > 0 \} \]
By Theorem 5.1 and Theorem 7.2, we see that the set \( \mathbb{A} \) is non-empty and bounded.
This proves
\[ 0 < \bar{a} = \sup \mathbb{A} < \infty. \]
Applying the sub-supersolution method, we get that for all \( a \in (0, \bar{a}) \), there exists \( b > 0 \) such that system (6) admits a pair of positive solutions.
Define the function \( \Gamma : [0, \bar{a}] \to \mathbb{R}_+ \) by
\[ \Gamma(a) = \sup \{ b > 0 : \text{ system (6) possesses a positive solution} \}. \]

Theorem 6.2 yields that the function \( \Gamma \) is continuous and non-increasing. Moreover, it is easy to verify that \( \Gamma(0) > 0 \). Thus, we see that \( \Gamma(a) \) is attained. Actually, it suffices to apply Theorem 7.1 and the compactness of operator \( F \). Finally, by the definition of the function \( \Gamma \) we get that system (6) possesses at least one positive solution for \( 0 \leq b \leq \Gamma(a) \), and possesses no positive solutions for \( b > \Gamma(a) \), which verifies parts (i) and (ii) of Theorem 4.5, respectively.

8. \textbf{Existence of two positive solutions.} In this section, we will establish the existence of two positive solutions of system (6), which corresponds to proving part (iii) of Theorem 4.5. To this end, we will suppose that the nonlinearities \( G_1 \) and \( G_2 \) are increasing.

Fix \( a \in [0, \bar{a}] \) and suppose \( (\bar{\phi}, \bar{\psi}) \) is the solution of problem (6) which is gotten by applying Theorem 6.2. Our next theorem will show that system (6) has another solution in \( 0 < b < \Gamma(a) \).

\textbf{Theorem 8.1.} For every \( 0 < b < \Gamma(a) \), there exists \( \varepsilon_0 > 0 \) such that, for all \( 0 < \varepsilon \leq \varepsilon_0 \), and all \( t \in J \), we get
\[ \bar{\phi}_\varepsilon(t) > \int_0^t \varphi_q \left( \int_s^1 G_1(\tau, \bar{\phi}_\varepsilon(\tau), \bar{\psi}_\varepsilon(\tau), a, b) d\tau \right) d\tau \]
and
\[ \bar{\psi}_\varepsilon(t) > \int_0^t \varphi_q \left( \int_s^1 G_2(\tau, \bar{\phi}_\varepsilon(\tau), \bar{\psi}_\varepsilon(\tau), a, b) d\tau \right) d\tau, \]
where \( \bar{\phi}_\varepsilon(t) = \bar{\phi}(t) + \varepsilon \) and \( \bar{\psi}_\varepsilon(t) = \bar{\psi}(t) + \varepsilon \).

\textit{Proof.} If \( s = 1 \), then it is easy to see that Theorem 8.1 holds.
Next, we consider the case \( 0 \leq s < 1 \).
Since \( G_1 \) is increasing, for each \( 0 < b < \Gamma(a) \), we may seek out a positive constant \( I = I(b) \) such that, for any \( \tau \in [s, 1] \), we get
\[ G_1(\tau, \bar{\phi}(\tau), \bar{\psi}(\tau), a, \Gamma(a)) - G_1(\tau, \bar{\phi}(\tau), \bar{\psi}(\tau), a, b) \geq I > 0. \]
Applying that $G_1$ is uniform continuous, there exists $\varepsilon_0 > 0$ such that, for all $\tau \in [s, 1]$ and all $0 < \varepsilon \leq \varepsilon_0$, we get

$$|G_1(\tau, \bar{\phi}(\tau) + \varepsilon, \bar{\psi}(\tau) + \varepsilon, a, b) - G_1(\tau, \bar{\phi}(\tau), \bar{\psi}(\tau), a, b)| < \frac{I}{2}.$$  

Next, we study the functions $\zeta(\tau)$ and $\eta(\tau)$ given by

$$\zeta(\tau, s) = \varepsilon_q \left( \int_s^1 G_1(\tau, \bar{\phi}(\tau), \bar{\psi}(\tau), a, b)d\tau \right)$$

and

$$\eta(\tau, s) = \varepsilon_q \left( \int_s^1 G_1(\tau, \bar{\phi}(\tau), \bar{\psi}(\tau), a, b)d\tau \right).$$

Let $0 < \varepsilon \leq \varepsilon_0$. Then

$$\tilde{\phi}(t) > \int_0^t \varepsilon_q \left( \int_s^1 G_1(\tau, \bar{\phi}(\tau), \bar{\psi}(\tau), a, b)d\tau \right)ds$$

Substituting $\zeta(\tau, s)$ and $\eta(\tau, s)$ into (41), we get

$$\tilde{\phi}(t) > \int_0^t \varepsilon_q \left( \int_s^1 G_1(\tau, \bar{\phi}(\tau), \bar{\psi}(\tau), a, b)d\tau \right)ds$$
For simplicity, write $\bar{G}(s) = G$, $\bar{G}_\varepsilon(s) = \bar{G}_\varepsilon$. Then, we get

$$\varphi_q\left(\int_s^t G_1(\tau, \bar{\phi}(\tau), \bar{\psi}(\tau), a, \Gamma(a))d\tau\right) - \varphi_q\left(\int_s^t G_1(\tau, \bar{\phi}_\varepsilon(\tau), \bar{\psi}_\varepsilon(\tau), a, b)d\tau\right)$$

$$= \varphi_q(\bar{G}) - \varphi_q(\bar{G}_\varepsilon)$$

$$= |\bar{G}|^{-2}(\bar{G} - \bar{G}_\varepsilon) + \bar{G}_\varepsilon(\bar{G} - \bar{G}_\varepsilon)(|\bar{G}|^{-3} + |\bar{G}|^{-4}|G|_\varepsilon + \cdots + |\bar{G}|^{-3} G|_\varepsilon$$

$$\geq \frac{4}{3}(1-s)|\bar{G}|^{-2} + \bar{G}_\varepsilon(|G|^{-3} + |\bar{G}|^{-4}|G|_\varepsilon + \cdots + |\bar{G}|^{-3} G|_\varepsilon)$$

$$> 0.$$  

Thus, for $\varepsilon$ sufficiently small, from (42) and (43), we get

$$\tilde{\phi}_\varepsilon(t) > \int_0^t \varphi_q\left(\int_s^t G_1(\tau, \bar{\phi}(\tau), \bar{\psi}(\tau), a, b)d\tau\right)ds.$$  

Similarly, we can prove that

$$\tilde{\psi}_\varepsilon(t) > \int_0^t \varphi_q\left(\int_s^t G_2(\tau, \bar{\phi}(\tau), \bar{\psi}(\tau), a, b)d\tau\right)ds.$$  

\[\square\]

**Proof of part (iii) of Theorem 4.5.** Define a set $\Omega$ by

$$\Omega = \{(\phi, \psi) \in E : -\varepsilon < \phi(t) < \bar{\phi}_\varepsilon(t), -\varepsilon < \psi(t) < \bar{\psi}_\varepsilon(t), t \in J, \}$$

where $\tilde{\phi}_\varepsilon(t)$ and $\tilde{\psi}_\varepsilon(t)$ are defined in Theorem 8.1. It is easy to see that $\Omega$ is a bounded open set in $E$, and $(0,0) \in E$. Notice that a solution to system (6) belongs to $P \cap \tilde{\Omega}$, we also understand that $F : P \cap \tilde{\Omega} \rightarrow P$ is a compact operator.

Suppose that $(\phi, \psi) \in P \cap \partial\Omega$. Then there is a $t_0 \in (0,1)$ such that one of the following two cases hold: $\phi(t_0) = \bar{\phi}_\varepsilon(t_0)$ or $\psi(t_0) = \bar{\psi}_\varepsilon(t_0)$. On the case $\phi(t_0) = \bar{\phi}_\varepsilon(t_0)$, by Theorem 8.1, for all $\lambda \geq 1$, we get

$$A(\phi, \psi)(t_0) = \int_0^{t_0} \varphi_q\left(\int_s^t G_1(\tau, \phi(\tau), \psi(\tau), a, b)d\tau\right)ds$$

$$\leq \int_0^{t_0} \varphi_q\left(\int_s^t G_1(\tau, \bar{\phi}(\tau), \bar{\psi}(\tau), a, b)d\tau\right)ds$$

$$< \bar{\phi}_\varepsilon(t_0) = \phi(t_0) \leq \lambda\phi(t_0).$$

Similarly, we can show that $B(\phi, \psi)(t_0) < \lambda\psi(t_0)$ under the case $\psi(t_0) = \bar{\psi}_\varepsilon(t_0)$. Thus, for all $(\phi, \psi) \in P \cap \partial\Omega$ and all $\lambda \geq 1$, we have $F(\phi, \psi) \neq \lambda(\phi, \psi)$. It hence follows from Lemma 4.3 that

$$i(F, P \cap \Omega, P) = 1.$$

Moreover, according to Theorem 7.1, there is a $\tilde{r} > \tilde{R}_1$, where $\tilde{R}_1$ is as in Theorem 5.3, such that

$$\|F(\phi, \psi)\| > \|(\phi, \psi)\|$$

for each $\|(\phi, \psi)\| = \tilde{r}$ and each $(\phi, \psi) \in P$.

Let $\tilde{R} = \max\{L + 1, \tilde{r}, \|\tilde{\phi}_\varepsilon, \tilde{\psi}_\varepsilon\|\}$, where $L$ is defined in Theorem 5.1. Let

$$P_{\tilde{R}} = \{(\phi, \psi) \in P : \|\phi, \psi\| < \tilde{R}\}.$$  

Theorem 7.1 yields that $F(\phi, \psi) \neq (\phi, \psi)$ for $(\phi, \psi) \in \partial P_{\tilde{R}}$. Accordingly, by part (i) of Lemma 3.8, we get

$$i(F, P_{\tilde{R}}, P) = 0.$$  

Now by applying the additivity property of the fixed point index we get

$$i(F, P \cap \Omega, P) + i(F, P_{\tilde{R}} \setminus \overline{P \cap \Omega}, P) = i(F, P_{\tilde{R}}, P) = 0.$$
Because \( i(F, P \cap \Omega, P) = 1 \), we obtain \( i(F, P_R \setminus \overline{P \cap \Omega}, P) = -1 \). So there exists another fixed point in \( P_R \setminus \overline{\mathcal{U} \cap \Omega} \). This finishes the proof of Part (iii) of Theorem 4.5 \( \square \)

**Remark 9.** From the proof of part (iii) of Theorem 4.5, it is not difficult to see that Theorem 8.1 plays a key role in proving the existence of two positive solutions of (6). Here we extend the study in [19] from semilinear case to quasilinear case. From the proof of Theorem 8.1 that the case \( 1 < p < N \) is more difficult to handle than the linear case: \( p = 2 \), which needs some new ingredients in the arguments.

**Remark 10.** The approach to prove Theorem 4.5. can be applied to the single equation

\[
\begin{aligned}
&\begin{cases}
-\triangle_p z = g(|x|, z, a, b) \text{ in } \Omega, \\
z(x) \to 0 \text{ as } |x| \to \infty, \\
\frac{\partial z}{\partial n} = 0 \text{ on } |x| = r_0,
\end{cases}
\end{aligned}
\]  

(44)

where \( \triangle_p := \text{div}(|\nabla u|^{p-2} \nabla u) \), \( 1 < p < N \), \( a \) and \( b \) are parameters, \( g \) is continuous nonlinearities, and \( \Omega = \{ x \in \mathbb{R}^N : |x| > r_0 > 0 \} \).

Similar to deal with system (5), by applying the change of variables \( r = |x| \) and \( t = (\frac{|x|}{r_0})^{\frac{p-N}{p}} \), the study of radial solutions of problem (44) can be reduced to analyzing the following problem

\[
\begin{aligned}
&\begin{cases}
-(\varphi_p(u'))' = G(t, u, a, b), \quad t \in (0, 1), \\
u(0) = u'(1) = 0,
\end{cases}
\end{aligned}
\]  

(45)

where \( \varphi_p(s) = |s|^{p-2} s \), \( (\varphi_p)^{-1} = \varphi_{\frac{1}{p}} \), \( \frac{1}{p} + \frac{1}{q} = 1 \), and

\[
G(t, u, a, b) = (\frac{p-1}{N-p})^{p-1} t^{\frac{N(p-1-N)}{N-p}}\ g(r_0 t^{\frac{1}{N-p}}, u, a, b).
\]

We can get the following result, which we state without proof.

**Theorem 8.2.** Suppose that \((C_0) - (C_3)\) hold with \( h_i \) and \( G \), replaced by \( h \) and \( G \). Then, there exist a constant \( \bar{a} > 0 \) and a nonincreasing continuous function \( \Gamma : [0, \bar{a}] \to \mathbb{R}_+ \) such that, for all \( a \in [0, \bar{a}] \), problem (44) admits

(i) at least one positive solution for \( 0 \leq b \leq \Gamma(a) \);

(ii) no positive solution for \( b > \Gamma(a) \);

(iii) at least two positive solutions for \( 0 < b < \Gamma(a) \), when \( G \) is increasing.

**Remark 11.** The approach to prove Theorem 4.5 can be applied to the more general case

\[
\begin{aligned}
&\begin{cases}
-\triangle_p z_1 = g_1(|x|, z_1, z_2, a, b) \text{ in } \Omega, \\
-\triangle_p z_2 = g_2(|x|, z_1, z_2, a, b) \text{ in } \Omega, \\
(z_1, z_2) \to (0, 0) \text{ as } |x| \to \infty, \\
z_1 = z_2 = 0 \text{ on } |x| = r_0.
\end{cases}
\end{aligned}
\]  

(46)

Here \( \triangle_p := \text{div}(|\nabla u|^{p-2} \nabla u) \), \( 1 < p < N \), \( a \) and \( b \) are parameters, \( g_1 \) and \( g_2 \) are continuous nonlinearities, and \( \Omega = \{ x \in \mathbb{R}^N : |x| > r_0 > 0 \} \).

Similar to deal with system (5), by applying the change of variables \( r = |x| \) and \( t = (\frac{|x|}{r_0})^{\frac{p-N}{p}} \), the study of radial solutions of system (46) can be reduced to
analyzing the following system

\[
\begin{align*}
-(&\varphi_p(u'))' &= G_1(t, u, v, a, b), \quad t \in (0, 1), \\
-(&\varphi_p(v'))' &= G_2(t, u, v, a, b), \quad t \in (0, 1), \\
u(0) &= v(0) = v(1) = 0,
\end{align*}
\]

(47)

where \( \varphi_p(s) = |s|^{p-2}s, \quad (\varphi_p)^{-1} = \varphi_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \) and

\[
G_i(t, u, v, a, b) = \left( \frac{p-1}{N-p} \right)^{p} \left( \frac{1-N}{p-1} \right) g_i(r_0 t^{\frac{1}{N-p}}, u(t), v(t), a, b), \quad i \in \{1, 2\}.
\]

We can get the following theorem.

**Theorem 8.3.** Suppose that \((C_0) - (C_3)\) hold. Then, there exist a constant \( \tilde{a} > 0 \) and a nonincreasing continuous function \( \Gamma : [0, \tilde{a}] \to \mathbb{R}_+ \) such that, for all \( a \in [0, \tilde{a}] \), system (47) admits

(i) at least one positive solution for \( 0 \leq b \leq \Gamma(a) \);

(ii) no positive solution for \( b > \Gamma(a) \);

(iii) at least two positive solutions for \( 0 < b < \Gamma(a) \), when \( G_1 \) and \( G_2 \) are increasing.

**Proof.** The proof is similar to that of Theorem 4.5, so we omit it here. \( \square \)

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