Derivation of Jacobian formula with Dirac delta function

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Abstract
We demonstrate how to make the coordinate transformation or change of variables from Cartesian coordinates to curvilinear coordinates by making use of a convolution of a function with Dirac delta functions whose arguments are determined by the transformation functions between the two coordinate systems. By integrating out an original coordinate with a Dirac delta function, we replace the original coordinate with a new coordinate in a systematic way. A recursive use of Dirac delta functions allows the coordinate transformation successively. After replacing every original coordinate into a new curvilinear coordinate, we find that the resultant Jacobian of the corresponding coordinate transformation is automatically obtained in a completely algebraic way. In order to provide insights on this method, we present a few examples of evaluating the Jacobian explicitly without resort to the known general formula.

Keywords: Jacobian, Dirac delta function, coordinate transformation, chain rule of partial derivatives

1. Introduction

A coordinate transformation or change of variables from a coordinate system to another in multi-dimensional integrals has widely been applied to a variety of fields in mathematics and physics. This transformation always involves a factor called the Jacobian, which is the determinant of the Jacobian matrix. The matrix elements of the Jacobian matrix are the first-order partial derivatives of the new coordinates with respect to the original coordinates. The formula

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for the change of variables from $n$-dimensional variables $x_1, x_2, \ldots, x_n$ to $q_1, q_2, \ldots, q_n$ is expressed in terms of the Jacobian $\mathcal{J}$:

$$\int dx_1 dx_2 \ldots dx_n f(x_1, \ldots, x_n) = \int dq_1 dq_2 \ldots dq_n \mathcal{J} F(q_1, \ldots, q_n),$$

(1)

where the integrand $f(x_1, \ldots, x_n)$ is a function of the independent variables $x_1, \ldots, x_n$ and $F(q_1, \ldots, q_n) = f[x_1(q_1, \ldots, q_n), \ldots, x_n(q_1, \ldots, q_n)]$. In general, the variables $x_1, \ldots, x_n$ can be treated as the Cartesian coordinates of an $n$-dimensional Euclidean space and the variables $q_1, q_2, \ldots, q_n$ form a set of curvilinear coordinates representing the same Euclidean space. We assume that each curvilinear coordinate $q_i$ is uniquely defined by the Cartesian coordinates: $q_i = q_i(x_1, \ldots, x_n)$. We also assume that the inverse transformation is uniquely defined as $x_i = x_i(q_1, \ldots, q_n)$. Then the Jacobian $\mathcal{J}$ can be expressed as

$$\mathcal{J} = \left| \frac{\partial(x_1, x_2, \ldots, x_n)}{\partial(q_1, q_2, \ldots, q_n)} \right| = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \ldots & \frac{\partial x_1}{\partial q_n} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \ldots & \frac{\partial x_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial q_1} & \frac{\partial x_n}{\partial q_2} & \ldots & \frac{\partial x_n}{\partial q_n} \end{vmatrix} = |\mathcal{D}et|,$$

(2)

where $\mathcal{D}et$ stands for the determinant.

In physics, the Jacobian appears frequently when one makes the change of variables between Cartesian and curvilinear coordinates in various physical quantities involving surface or volume integrals. However, in most physics textbook including classical mechanics and electromagnetism usually abstract descriptions are provided. In many textbooks of calculus or mathematical physics [1], the Jacobian formula is derived in the following way: first, one forms the multi-variable differential volume by applying the change of variables. Next, one imposes a geometrical argument that the infinitesimal volume is invariant under the transformation [2]. The invariance of the volume can also be confirmed by applying Green’s theorem to show that $\int_S dx_1 dx_2 = \int_T \mathcal{J} dq_1 dq_2$, where $S$ is a rectangular region in the $x_1, x_2$ plane and $T$ is the corresponding region [3]. Although experienced teachers or researchers may follow the abstract logic in this kind of Jacobian derivation without difficulties, the concept is rather unclear or less intuitive to undergraduate physics-major students who are not familiar with advanced mathematical concepts of multi-variable calculus.

The Dirac delta function $\delta(x)$ is not a well-defined function but a distribution defined only through integration: $\int_{-\infty}^{\infty} dx \delta(x) = 1$. For any smooth function $f(x)$, $\int_{-\infty}^{\infty} dx \delta(x - y)f(x) = f(y)$. This property can be applied to changing the integration variable. Recently, some of us introduced an alternative proof of Cramer’s rule by making use of Dirac delta functions [4]. It turns out that the method with a convolution of a coordinate vector with Dirac delta functions provides a systematic way to change integration variables from original coordinates to new coordinates. This change of variables enables us to reproduce Cramer’s rule.

The Dirac delta function technique exploited in the derivation of Cramer’s rule in reference [4] can be immediately applicable to the evaluation of the Jacobian for the coordinate transformation or change of variables after replacing the coordinate vector with an arbitrary function. However, the direct application of the approach in reference [4] is limited to a linear transformation between two coordinate systems because Cramer’s rule applies only to the system of linear equations. Thus the method is not applicable to the transformation involving a set of curvilinear coordinates which are frequently used in physics.
The main goal of this work is to present a more intuitive derivation of the Jacobian involving any coordinate transformation and to demonstrate how it works with heuristic examples. We develop an alternative derivation of the Jacobian formula as well as the coordinate transformation by convolving a function with Dirac delta functions. Our derivation relies only on the direct integration of Dirac delta functions whose arguments involve the coordinate transformation rules. Hence, we expect that students who are familiar with the Dirac delta function can compute the Jacobian formula in any coordinate transformations by themselves without referring to a reference. The approach that we present in this paper is quite straightforward and requires mostly algebraic computation skills.

In this paper, we derive the Jacobian for a coordinate transformation or change of variables in the case of a non-linear transformation by convolving an arbitrary function with Dirac delta functions. While the basic strategy to perform the coordinate transformation is similar to that employed in reference [4], the integration of the original coordinates is rather involved because of the non-linear property of the transformation functions between the two coordinate systems. The integration of the original coordinates can be carried out by making use of Dirac delta functions. An extra factor containing partial derivatives of corresponding coordinate variables appears in front of the original integrand after that multiple integration. It turns out that the extra factor can be evaluated by making use of the chain rule of partial derivatives. A recursive use of Dirac delta functions enables us to achieve the coordinate transformation successively. After replacing every coordinate with new coordinates, we identify the resultant extra factor in the integrand with the Jacobian for the coordinate transformation or change of variables.

The derivation of the Jacobian formula presented in this paper is new to our best knowledge. It is remarkable that our derivation is free of borrowing abstract and advanced mathematical concepts unlike the other derivations available. Instead, we exploit a simple concept of integration of the one-dimensional Dirac delta function repeatedly in combination with a purely algebraic manipulation in reorganizing the extra factor by applying chain rules. This intuitive and systematic approach is expected to be pedagogically useful in upper-level mathematics or physics courses in practice of the recursive use of both the Dirac delta function and the chain rule of partial derivatives.

This paper is organized as follows: in section 2, we introduce notations that are frequently used in the rest of the paper, make a rough sketch of our strategy and present a formal derivation of the Jacobian factor for the coordinate transformation from the \( n \)-dimensional Cartesian coordinates to a set of curvilinear coordinates. Section 3 is devoted to the explicit evaluation of the Jacobians for a few examples. Conclusions are given in section 4 and a rigorous derivation of the chain rule for partial derivatives is given in appendices.

2. Derivation of the Jacobian

2.1. Strategy and notation

In this subsection, we present our strategy to derive the Jacobian for a coordinate transformation or change of variables from the Cartesian coordinates \( x_i \) to the curvilinear coordinates \( q_i \) with transformation functions

\[
q_i = q_i(x_1, \ldots, x_n)
\]

for \( i = 1 \) through \( n \), where \( n \) is a positive integer. We assume that the two sets of coordinates describe a single point uniquely and, therefore, the two sets of coordinates have a one-to-one correspondence although the transformation is in general non-linear. Thus the transformation
in equation (3) is invertible: the inverse transformation from the curvilinear coordinates to the Cartesian coordinates exists. If the curvilinear coordinates are a linear combination of the Cartesian coordinates, then the linear transformation is invertible if the transformation matrix is non-singular: the determinant of the matrix is not vanishing. Then, the inverse transformation can be written as

$$x_i = x_i(q_1, \ldots, q_n).$$

(4)

The basic strategy to derive the Jacobian with Dirac delta functions is the same as that for the derivation of Cramer’s rule for a partial set of a coordinate transformation or change of variables given in reference [4]. One could immediately apply the approach in reference [4] to find the Jacobian as long as the transformation (3) is linear. In general, the transformation functions (3) are non-linear. Here we develop a generalized version of the approach in reference [4] in order to consider arbitrary curvilinear coordinates.

We define an $n$-dimensional integral $I_n$,

$$I_n = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} d^n x \ f(x_1, \ldots, x_n),$$

(5)

where $d^n x \equiv dx_1 \ldots dx_n$ is the $n$-dimensional differential volume element and the integrand $f(x_1, \ldots, x_n)$ is an arbitrary function of the Cartesian coordinates. We assume that every Cartesian coordinate is integrated over the region $(-\infty, \infty)$. We define the unity

$$\mathbb{1}_i \equiv \int dq_i \delta[q_i - q_i(x_1, \ldots, x_n)] = 1$$

(6)

for $i = 1$ through $n$. Multiplying the unity $\mathbb{1}_i$ to the integral $I_n$, one can integrate out the integration variable $x_i$ by making use of the Dirac delta function keeping the $q_i$ integral unevaluated. By applying this process to $I_n$ recursively from $i = 1$ through $n$, we complete the change of the integration variables from the Cartesian coordinates to the curvilinear coordinates. While we have suppressed the bounds of the integration for the curvilinear coordinate $q_i$’s in equation (6), the curvilinear coordinates are assumed to be integrated over the entire region to cover the whole Euclidean space represented by the Cartesian coordinates by a single time.

We first compute $\mathbb{1}_1 \times I_n$:

$$I_n = \mathbb{1}_1 \times I_n = \int dq_1 \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} d^n x \ f(x_1, \ldots, x_n) \delta[q_1 - q_1(x_1, \ldots, x_n)].$$

(7)

By definition, we integrate over $x_1$ by making use of the delta function $\delta[q_1 - q_1(x_1, \ldots, x_n)]$ keeping the $q_1$ integral unevaluated:

$$I_n = \int dq_1 \int_{-\infty}^{\infty} d^n x \ f(q_1, x_2, \ldots, x_n),$$

(8)

where $x_1$ is expressed in terms of $q_1$ and $x_j$’s for $j = 2$ through $n$ satisfying the condition that the argument of the delta function vanishes, $q_1 - q_1(x_1, \ldots, x_n) = 0$. Because the explicit forms of $f(x_1)$ and $x_j$ vary depending on the integration step, we adopt the notation $f(q_1, x_2, \ldots, x_n)$ after the $x_1$ integration. We will present more detailed explanations for this notation in the later part of this subsection. The extra factor $\mathcal{G}_1$ is the remnant of the integration of the delta function and its explicit form will be given later in this paper.

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Next, we multiply \( I_2 \) to \( I_n \) in equation (8) to find that

\[
I_n = I_2 \times I_n = \int dq_1 dq_2 \int_{-\infty}^{\infty} dx_2 \ldots dx_n \frac{1}{\mathcal{G}_2} f(q_1, \ldots, x_n) \delta \{ q_2 - q_2(q_1, x_2, \ldots, x_n) \},
\]

where every \( x_1 \) in the argument of the delta function as well as the integrand function is replaced with the expression in terms of \( q_1 \) and \( x_j \)'s for \( j = 2 \) through \( n \). Performing the integration over \( x_2 \) by making use of the delta function, we find

\[
I_n = \int dq_1 dq_2 \int_{-\infty}^{\infty} dx_3 \ldots dx_n \frac{1}{\mathcal{G}_2} f(q_1, q_2, x_3, \ldots, x_n).
\]

After the \( x_2 \) integration every \( x_2 \) in the integrand is expressed in terms of \( q_1, q_2 \) and \( x_j \)'s for \( j = 3 \) through \( n \). Again, \( \mathcal{G}_2 \) is the remnant of the integration of the delta functions.

In this way, we integrate over \( x_k \) for \( k = 1 \) through \( n \) successively. Finally, after the integration over \( x_n \) we find that \( I_n \) reduces into the \( n \)-dimensional multiple integral over the curvilinear coordinates \( q_i \) for \( i = 1 \) through \( n \) only:

\[
I_n = \int dq_1 \ldots dq_n \frac{1}{\mathcal{G}_n} f(q_1, \ldots, q_n),
\]

where \( \mathcal{G}_n \) is the remnant of the integration of the delta functions. At this stage, the integrand acquires an additional factor \( 1/\mathcal{G}_n \) in front of the original integrand \( f \) in equation (5). This extra factor is identified with the Jacobian.

In an intermediate step, for example, where the integration over \( x_j \) (\( 1 \leq j \leq n \)) is carried out, \( x_1, \ldots, x_j \) in the integrand must be replaced with the expressions in terms of \( q_1, \ldots, q_j \) and \( x_{j+1}, \ldots, x_n \) as

\[
x_k = x_k(q_1, \ldots, q_j, x_{j+1}, \ldots, x_n)
\]

for \( k = 1 \) through \( j \). Each \( x_k \) in equation (12) is determined by the condition that the argument of the corresponding Dirac delta function vanishes:

\[
q_k - q_k \left[ x_1(q_1, \ldots, q_j, x_{j+1}, \ldots, x_n), \ldots, x_j(q_1, \ldots, q_j, x_{j+1}, \ldots, x_n), x_{j+1}, \ldots, x_n \right] = 0.
\]

One must keep in mind that the explicit form of each \( x_k \) varies depending on the integration step as is displayed in equation (12). Thus, one must distinguish, for example, \( x_k(q_1, \ldots, q_{j-1}, x_j, \ldots, x_n) \) from \( x_k(q_1, \ldots, q_{j-1}, x_j, \ldots, x_n) \), where the former is the expression after the integration over \( x_j \) and the latter is that after the integration over \( x_{j-1} \). This notation is also applied to the original integrand function \( f \) and the extra factor \( \mathcal{G}_n \).

As an explicit example, we consider the coordinate transformation between the two-dimensional Cartesian coordinates and the polar coordinates with the transformation functions

\[
r = \sqrt{x^2 + y^2} \quad \text{and} \quad \theta = \arctan \frac{y}{x}
\]

(14)

and the inverse transformation functions

\[
x = r \cos \theta \quad \text{and} \quad y = r \sin \theta.
\]

(15)

In an intermediate step, \( y \) can be expressed in terms of \( \theta \) and \( x \) as \( y(\theta, x) = x \tan \theta \), which must be distinguished from \( y(\theta, r) = r \sin \theta \).
Since the dependence of a coordinate variable varies according to the integration step, one must take special care of dealing with their partial derivatives. In order to avoid such an ambiguity, we introduce a notation for the partial derivative with subscripts of variables that are held constant. In the above two-dimensional transformation, the partial derivative of $\theta$ with respect to $y$ holding $x$ fixed is denoted by

$$\left( \frac{\partial \theta}{\partial y} \right)_x = \frac{x}{x^2 + y^2}, \quad (16)$$

while the partial derivative of $\theta$ with respect to $y$ holding $r$ fixed is represented by

$$\left( \frac{\partial \theta}{\partial y} \right)_r = \frac{1}{\sqrt{r^2 - y^2}}. \quad (17)$$

It is apparent from equations (16) and (17) that

$$\left( \frac{\partial \theta}{\partial y} \right)_x \neq \left( \frac{\partial \theta}{\partial y} \right)_r. \quad (18)$$

In a general case, for a variable $q_j(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n)$, we denote the partial derivative of $q_j$ with respect to $x_a$ holding $q_1, \ldots, q_i, x_{i+1}, \ldots, x_n$ fixed with the subscript $(i)$ as

$$\left( \frac{\partial q_j}{\partial x_a} \right)_{(i)} = \frac{\partial q_j(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n)}{\partial x_a}, \quad (19)$$

where $i + 1 \leq j \leq n$ and $i + 1 \leq a \leq n$. Similarly, for a variable $x_k(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n)$, the partial derivative of $x_k$ with respect to $q_b$ holding $q_1, \ldots, q_i, x_{i+1}, \ldots, x_n$ fixed is denoted by

$$\left( \frac{\partial x_k}{\partial q_b} \right)_{(i)} = \frac{\partial x_k(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n)}{\partial q_b}, \quad (20)$$

where $1 \leq k \leq i$ and $1 \leq b \leq i$. We denote the partial derivative of $x_k$ with respect to $x_c$ holding $q_1, \ldots, q_i, x_{i+1}, \ldots, x_n$ fixed by

$$\left( \frac{\partial x_k}{\partial x_c} \right)_{(i)} = \frac{\partial x_k(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n)}{\partial x_c}, \quad (21)$$

where $1 \leq \ell \leq i$ and $i + 1 \leq c \leq n$. Finally, we express the partial derivative of $q_j$ with respect to $x_c$ holding $x_1, \ldots, x_n$ without subscript:

$$\frac{\partial q_j}{\partial x_c} = \frac{\partial q_j(x_1, \ldots, x_n)}{\partial x_c}, \quad (22)$$

where $1 \leq \ell \leq n$ and $1 \leq c \leq n$.

In the coordinate transformation from $(x_1, \ldots, x_n)$ to $(q_1, \ldots, q_n)$, we define the function $G_k$, which is relevant for a partial set of integral variables corresponding to a transformation from $(x_1, \ldots, x_k)$ to $(q_1, \ldots, q_k)$, as
\[ G_k = \begin{vmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_2}{\partial x_1} & \cdots & \frac{\partial q_k}{\partial x_1} \\ \frac{\partial q_1}{\partial x_2} & \frac{\partial q_2}{\partial x_2} & \cdots & \frac{\partial q_k}{\partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial q_1}{\partial x_k} & \frac{\partial q_2}{\partial x_k} & \cdots & \frac{\partial q_k}{\partial x_k} \end{vmatrix} \tag{23} \]

for \( k = 1 \) through \( n \). Note that \( G_n \) is the inverse of the Jacobian \( J_n \) which is defined by

\[ J_n = \begin{vmatrix} \frac{\partial x_1}{\partial q_1} & \frac{\partial x_1}{\partial q_2} & \cdots & \frac{\partial x_1}{\partial q_k} \\ \frac{\partial x_2}{\partial q_1} & \frac{\partial x_2}{\partial q_2} & \cdots & \frac{\partial x_2}{\partial q_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_k}{\partial q_1} & \frac{\partial x_k}{\partial q_2} & \cdots & \frac{\partial x_k}{\partial q_k} \end{vmatrix}. \tag{24} \]

### 2.2. 2-dimensional case

In this subsection, we consider a two-dimensional integral \( I_2 \) for the integration of an arbitrary function \( f(x_1, x_2) \):

\[ I_2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_1 dx_2 f(x_1, x_2). \tag{25} \]

The Cartesian coordinates \( x_1 \) and \( x_2 \) are transformed into curvilinear coordinates \( q_1 \) and \( q_2 \) with the transformation relations

\[ q_1 = q_1(x_1, x_2), \quad q_2 = q_2(x_1, x_2). \tag{26} \]

which are assumed to be invertible and non-singular. Thus, \( x_1 \) and \( x_2 \) can be expressed in terms of \( q_1 \) and \( q_2 \):

\[ x_1 = x_1(q_1, q_2), \quad x_2 = x_2(q_1, q_2). \tag{27} \]

We multiply the unities

\[ 1_i = \int dq_i \delta[q_i - q_i(x_1, x_2)] = 1 \tag{28} \]

to \( I_2 \) for \( i = 1, 2 \), sequentially. First, we multiply \( 1_1 \) in equation (28) to \( I_2 \). Then, \( I_2 \) can be expressed as

\[ I_2 = \int dq_1 \int_{-\infty}^{\infty} dx_1 dx_2 \delta[q_1 - q_1(x_1, x_2)] f(x_1, x_2). \tag{29} \]

The integration over \( x_1 \) can be carried out by making use of the delta function for \( q_1 \) as
\[ \int_{-\infty}^{\infty} dx_1 \, \delta[q_1 - q_1(x_1, x_2)] = \frac{1}{\mathcal{G}_1(q_1, x_2)} \]

which leads to

\[ I_2 = \int dq_1 \int_{-\infty}^{\infty} dx_2 \, \frac{1}{\mathcal{G}_1(q_1, x_2)} f(q_1, x_2). \] (31)

After the integration over \( x_1 \), every \( x_1 \) on the right-hand side of equation (30) and \( \delta[q_2 - q_2(x_1, x_2)] f(x_1, x_2) \) in equation (28) must be replaced with \( x_1(q_1, x_2) \) satisfying the condition that the argument of the delta function vanishes, \( q_1 - q_1(x_1, x_2) = 0 \). Then the delta function for \( q_1 \) and \( f(x_1, x_2) \) can be expressed as \( \delta[q_2 - q_2(q_1, x_2)] \) and \( f(q_1, x_2) \), respectively.

After multiplying \( I_1 \) in equation (28) to \( I_2 \) in equation (31), the integration over \( x_2 \) can be performed by making use of the remaining delta function for \( q_2 \) as

\[ \int_{-\infty}^{\infty} dx_2 \, \delta[q_2 - q_2(q_1, x_2)] = \frac{1}{\left( \frac{\partial}{\partial q_2} \right)_{x_2=q_2(q_1,q_2)}} \mathcal{G}_1(q_1, q_2) \mathcal{G}_2(q_1, q_2), \] (32)

After the integration over \( x_2 \), every \( x_2 \) on the right-hand sides of equations (30) and (32) and in \( f(q_1, x_2) \) is replaced with \( x_2(q_1, q_2) \), which can be obtained from the condition that the argument of the delta function vanishes, \( q_2 - q_2(q_1, x_2) = 0 \). Then, we can express \( x_1 \) in terms of \( q_1 \) and \( q_2 \) by replacing \( x_2 \) with \( x_2(q_1, q_2) \) in \( x_1(q_1, x_2) \). We have obtained the last equality of equation (32) by making use of the chain rule for the partial derivatives. A rigorous proof of this formula is given in equation (A5) of appendix A.

After both \( x_1 \) and \( x_2 \) are integrated out, the integral \( I_2 \) reduces into

\[ I_2 = \int dq_1 \, dq_2 \, \frac{1}{\mathcal{G}_1} \times \frac{\mathcal{G}_1}{\mathcal{G}_2} f[x_1(q_1, q_2), x_2(q_1, q_2)], \]

where the Jacobian for the change of variables is identified as \( \mathcal{J} = \mathcal{J}_2 = 1/\mathcal{G}_2 \). This completes the proof of the Jacobian for a two-dimensional coordinate transformation or change of variables.

### 2.3. 3-dimensional case

In this subsection, we extend the results in the previous subsection to the three-dimensional case. This is a special case of the \( n \)-dimensional coordinate transformation or change of variables, which we will prove in the next subsection. However, it is worthwhile to prove the three-dimensional case in detail for a pedagogical purpose.

We consider a three-dimensional integral \( I_3 \) for an arbitrary function \( f(x_1, x_2, x_3) \):

\[ I_3 = \int_{-\infty}^{\infty} dx_1 \, dx_2 \, dx_3 \, f(x_1, x_2, x_3). \] (34)
The Cartesian coordinates $x_i$ for $i = 1$ through 3 are transformed into the curvilinear coordinates $q_i$’s as

$$q_1 = q_1(x_1, x_2, x_3), \quad q_2 = q_2(x_1, x_2, x_3), \quad q_3 = q_3(x_1, x_2, x_3). \quad (35)$$

The inverse transformation can be expressed as

$$x_1 = x_1(q_1, q_2, q_3), \quad x_2 = x_2(q_1, q_2, q_3), \quad x_3 = x_3(q_1, q_2, q_3). \quad (36)$$

We carry out the change of variables by multiplying the unites $1$ for $i = 1$ through 3 to $I_3$, sequentially. First, after multiplying $1_1$ in equation (37) to $I_3$, we find that $I_3$ can be expressed as

$$I_3 = \int dq_1 \int_{-\infty}^{\infty} dx_1 dx_2 dx_3 \frac{1}{g_1(q_1, x_2, x_3)} f(q_1, x_2, x_3). \quad (38)$$

Similarly to the two-dimensional case, we perform the integration over $x_1$ by making use of the Dirac delta function for $q_1$ as

$$I_3 = \int dq_1 \int_{-\infty}^{\infty} dx_1 \delta[q_1 - q_1(x_1, x_2, x_3)] f(x_1, x_2, x_3). \quad (39)$$

which leads to

$$I_3 = \int dq_1 \int_{-\infty}^{\infty} dx_2 dx_3 \frac{1}{g_1(q_1, x_2, x_3)} f(q_1, x_2, x_3). \quad (40)$$

After the integration over $x_1$, every $x_1$ in the integrand and remaining delta functions is replaced with $x_1(q_1, x_2, x_3)$ satisfying the condition that the argument of the Dirac delta function vanishes, $q_1 - q_1(x_1, x_2, x_3) = 0$. Then, the delta function for $q_2$ in equation (37) can be expressed as $\delta[q_2 - q_2(q_1, x_2, x_3)]$.

After multiplying $1_2$ in equation (37) to $I_3$ in equation (40), the integration over $x_2$ can be carried out by making use of the Dirac delta function for $q_2$ as

$$I_3 = \int dq_1 dq_2 \int_{-\infty}^{\infty} dx_3 \frac{1}{g_1(q_1, q_2, x_3)} \frac{g_2(q_1, q_2, x_3)}{g_2(q_1, q_2, x_3)} f(q_1, q_2, x_3). \quad (41)$$

where the last equality comes from equation (B5). Then, equation (41) is expressed as

$$I_3 = \int dq_1 dq_2 \int_{-\infty}^{\infty} dx_3 \frac{1}{g_1(q_1, q_2, x_3)} \frac{g_2(q_1, q_2, x_3)}{g_2(q_1, q_2, x_3)} f(q_1, q_2, x_3). \quad (42)$$

$x_2(q_1, q_2, x_3)$ is determined from the condition that the argument of the Dirac delta function vanishes, $q_2 - q_2(q_1, x_2, x_3) = 0$. Substituting $x_2(q_1, q_2, x_3)$ into $x_1(q_1, x_2, x_3)$, we obtain $x_1 =$
In this subsection, we consider the two-dimensional case and the argument of the Dirac delta function for \( q_3 \) in equation (37) is expressed as \( \delta[q_3 - q_3(q_1, q_2, x_3)] \).

Finally, after multiplying \( I_3 \) in equation (37) to \( I_3 \) in equation (42), we integrate over \( x_3 \) by taking into account the delta function for \( q_3 \) as

\[
\int_{-\infty}^{\infty} dx_3 \frac{1}{G_1G_2G_3} f[x_1(q_1, q_2, q_3), x_2(q_1, q_2, q_3), x_3(q_1, q_2, q_3)]
\]

which leadsto

\[
I_3 = \int dq_1 dq_2 dq_3 \frac{1}{G_1G_2G_3} f[x_1(q_1, q_2, q_3), x_2(q_1, q_2, q_3), x_3(q_1, q_2, q_3)]
\]

where the last equality comes from equation (B11). After the integration over \( x_3 \), every \( x_3 \) in the integrand and the right-hand sides of equations (39), (41) and (43) is replaced with \( x_3(q_1, q_2, q_3) \) which is determined from the condition that the argument of the delta function vanishes, \( q_3 - q_3(q_1, q_2, x_3) = 0 \). Substituting \( x_3(q_1, q_2, q_3) \) into \( x_1(q_1, q_2, x_3) \) and \( x_2(q_1, q_2, x_3) \), we can obtain the expressions for \( x_1 = x_1(q_1, q_2, q_3) \) and \( x_2 = x_2(q_1, q_2, q_3) \), respectively. Then, we can express the integrand \( f(x_1, x_2, x_3) \) in terms of \( q_1, q_2 \) and \( q_3 \) and the integral \( I_3 \) is expressed as

\[
I_3 = \int dq_1 dq_2 dq_3 \frac{1}{G_1G_2G_3} f[x_1(q_1, q_2, q_3), x_2(q_1, q_2, q_3), x_3(q_1, q_2, q_3)]
\]

where the Jacobian for the change of variables is identified as \( J = J_3 = 1/G_3 \). This completes the proof of the Jacobian for a three-dimensional coordinate transformation or change of variables.

2.4. \( n \)-dimensional case

In this subsection, we consider the \( n \)-dimensional integral \( I_n \) for a function \( f(x_1, \ldots, x_n) \) defined in equation (5) by multiplying the unities \( 1_i \) in equation (6) to \( I_n \) in equation (5) sequentially. Then, we integrate \( I_n \), which is multiplied by the unity, over \( x_i \) for \( i = 1 \) through \( n \) successively by making use of the Dirac delta function

\[
\int_{-\infty}^{\infty} dx_i \delta[q_i - q_i(x_1, \ldots, x_n)].
\]

After the integration over all \( x_i \) variables, the corresponding Jacobian formula is obtained by employing mathematical induction.

First, the integration over \( x_1 \) can be carried out from equation (7) and the result for the integration over \( x_1 \) can easily be generalized from the two-dimensional version in equation (30) as

\[
\int_{-\infty}^{\infty} dx_1 \frac{1}{G_1} f(q_1, x_2, \ldots, x_n)
\]

which leads to

\[
I_n = \int dq_1 \int_{-\infty}^{\infty} dx_2 \ldots dx_n \frac{1}{G_1(q_1, x_2, \ldots, x_n)} f(q_1, x_2, \ldots, x_n).
\]
After the $x_1$ integration, every $x_1$ in the integrand of equation (7) and $\mathcal{G}_1$ in equation (46) is replaced with

$$x_1 = x_1(q_1, x_2, \ldots, x_n).$$

(48)

The constraint equation coming from the convolution with the Dirac delta function in equation (46) is

$$q_1 - q_1[x_1(q_1, x_2, \ldots, x_n), x_2, \ldots, x_n] = 0.$$

(49)

Then, we carry out the integration over $x_i$ for $i = 1$ through $n - 1$ by multiplying $\mathbf{1}_i$ to $I_n$ in equation (47) sequentially. We assume that, after the integration over $x_i$ for $i = 1$ through $n - 1$, the result of the integration of Dirac delta functions is

$$\int_{-\infty}^{\infty} dx_1 \cdots dx_i \prod_{j=1}^i \delta[q_j - q_j(x_1, \ldots, x_n)] = \frac{1}{\mathcal{G}_1 \mathcal{G}_2 \cdots \mathcal{G}_{i-1}} \mathcal{G}_i(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n).$$

(50)

which leads to

$$I_n = \int dq_1 \cdots dq_i \int_{-\infty}^{\infty} dx_{i+1} \cdots dx_n
\times \frac{1}{\mathcal{G}_i(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n)} f(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n).$$

(51)

For any $j \leq i$ every $x_j$ in the integrand of equation (5) and $\mathcal{G}_j$’s in equation (50) is replaced with

$$x_j = x_j(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n)$$

(52)

after the integrations over $x_1$ through $x_i$. There are $i$ constraint equations coming from the convolution with Dirac delta functions in equation (50):

$$q_j - q_j[x_1(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n), \ldots, x_{i+1}, \ldots, x_n] = 0,$$

(53)

where $j$ runs from 1 through $i$.

After multiplying $1_{i+1}$ to $I_n$ in equation (51), we integrate out one more Cartesian coordinate $x_{i+1}$ to find that

$$\int_{-\infty}^{\infty} dx_{i+1} \delta[q_{i+1} - q_{i+1}(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n)] x_j = x_j(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n)$$

$$= \frac{1}{\mathcal{G}_{i+1}(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n)} x_j = x_j(q_1, \ldots, q_i, x_{i+1}, \ldots, x_n)$$

(54)

where $1 \leq j \leq i$ and $1 \leq k \leq i + 1$. The proof of the last equality can be found in equation (C8) in appendix C. In combination with equations (50) and (54), we find that
\[
\int_{-\infty}^{\infty} \, \cdots \, \int_{-\infty}^{\infty} dx_1 \cdots dx_{i+1} \prod_{j=1}^{i+1} \delta[q_j - q_j(x_1, \ldots, x_n)] \\
= \frac{1}{G_1} \times \frac{G_1}{G_2} \times \cdots \times \frac{G_{i-1}}{G_i} \times \frac{G_i}{G_{i+1}} = \frac{1}{G_{i+1}(q_1, \ldots, q_{i+1}, x_{i+2}, \ldots, x_n)}. \quad (55)
\]

For any \( j \leq i + 1 \) every \( x_j \) in the integrand of equation (7) and \( G_j \)'s in equation (55) is replaced with

\[
x_j = x_j(q_1, \ldots, q_{i+1}, x_{i+2}, \ldots, x_n) \quad (56)
\]
after the integrations over \( x_1 \) through \( x_{i+1} \). There are \( i + 1 \) constraint equations coming from the convolution with Dirac delta functions in equation (55):

\[
q_j - q_j \left[ x_1(q_1, \ldots, q_{i+1}, x_{i+2}, \ldots, x_n), \ldots, x_i \left( q_1, \ldots, q_{i+1}, x_{i+2}, \ldots, x_n \right), x_{i+1}, \ldots, x_n \right] = 0, \quad (57)
\]

where \( j \) runs from 1 through \( i + 1 \). According to mathematical induction, this proves that the assumption in equation (50) with the constraints (52) and (53) is true for all \( i = 1 \) through \( n \).

Finally, the integral \( I_n \) can be expressed in terms of \( q_1, \ldots, q_n \) as

\[
I_n = \int d^n q \frac{1}{G_n} f(x_1, \ldots, x_n) |_{x_1 = x_2 = \cdots = q_n} = \int d^n q J_n f(x_1, \ldots, x_n) |_{x_1 = x_2 = \cdots = q_n}, \quad (58)
\]

where \( 1 \leq k \leq n \). This completes the proof of the Jacobian \( J = J_n = 1/G_n \) for an \( n \)-dimensional coordinate transformation or change of variables.

### 3. Application

Since the proof of the Jacobian formula in the previous section is rather abstract, readers who are not familiar with the notation might be confused. In this section, we present a few explicit examples of deriving the Jacobian without resort to the general formula derived in the previous section. We expect that the explicit examples will help readers to understand the method presented in the previous section more intuitively and to apply it to a specific change of variables.

#### 3.1. Spherical coordinates

In this subsection, we consider the change of variables from the three-dimensional Cartesian coordinates \((x, y, z)\) to the spherical coordinates \((r, \theta, \phi)\). The Cartesian coordinates can be expressed in terms of the radius \( r \), the polar angle \( \theta \) and the azimuthal angle \( \phi \) as

\[
x = r \sin \theta \cos \phi, \quad y = r \sin \theta \sin \phi, \quad z = r \cos \theta. \quad (59)
\]

We use \( \cos \theta \) instead of \( \theta \) as the integration variable and reorganize the order of multiple integrations in order to simplify the computation. That is, \((x_1, x_2, x_3)\) in section 2.3 corresponds to \((z, y, x)\) while \((q_1, q_2, q_3)\) corresponds to \((\cos \theta, \phi, r)\), respectively. However, it turns out that the integral is invariant under this reordering. We also note that one can use, for instance, \( \sin \theta \)
instead of \( \cos \theta \) as an integration variable and it will not alter the result of the integration. Conventionally, the arctangent function is defined in the region \([ -\frac{\pi}{2}, \frac{\pi}{2} ]\). The period of the tangent function is \( \pi \), while the azimuthal angle ranges from 0 to \( 2\pi \). Thus the angle \( \phi \) for \( x > 0 \) is set to be arctan \( \frac{y}{x} \in [ -\frac{\pi}{2}, \frac{\pi}{2} ] \) while that for \( x < 0 \) is set to be \( \pi + \arctan \frac{y}{x} \in [ \frac{\pi}{2}, \frac{3\pi}{2} ] \) in order to make the transformation function invertible in the entire range. Then the inverse transformation of equation (59) is expressed as

\[
\cos \theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \quad (60a)
\]

\[
\phi = \begin{cases} 
\arctan \frac{y}{x} \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right), & \text{for } x > 0, \\
\pi + \arctan \frac{y}{x} \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right), & \text{for } x < 0, \\
\frac{\pi}{2}, & \text{for } x = 0 \text{ and } y > 0, \\
-\frac{\pi}{2}, & \text{for } x = 0 \text{ and } y < 0, \\
0, & \text{for } x = y = 0,
\end{cases} \quad (60b)
\]

\[
r = \sqrt{x^2 + y^2 + z^2}. \quad (60c)
\]

We consider a three-dimensional integral \( J_3 \) with an arbitrary integrand \( f(x, y, z) \)

\[
J_3 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx \, dy \, dz \, f(x, y, z). \quad (61)
\]

We multiply the unities

\[
\Psi_1 = \int_{-1}^{1} d \cos \theta \, \delta \left( \cos \theta - \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) = 1, \quad (62a)
\]

\[
\Psi_2 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi \left[ \delta \left( \phi - \arctan \frac{y}{x} \right) \Theta(x) + \delta \left( \phi - \arctan \frac{y}{x} - \pi \right) \Theta(-x) \right] = 1, \quad (62b)
\]

\[
\Psi_3 = \int_{0}^{\infty} dr \, \delta \left( r - \sqrt{x^2 + y^2 + z^2} \right) = 1 \quad (62c)
\]

to \( J_3 \) without changing the value of the integral sequentially.

\[
\Theta(x) = \begin{cases} 
1, & \text{for } x > 0, \\
\frac{1}{2}, & \text{for } x = 0, \\
0, & \text{for } x < 0.
\end{cases} \quad (63)
\]

Here, the Heaviside step function \( \Theta(x) \) is defined by

First, we integrate out the \( x_1 = z \) coordinate by multiplying \( \Psi_1 \) in equation (62a) to \( J_3 \) in equation (61). The integration over \( z \) can be performed as

\[
\int_{-\infty}^{\infty} dz \, \delta \left( \cos \theta - \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right) = \frac{\sqrt{x^2 + y^2}}{\sin^2 \theta}. \quad (64)
\]

After the integration over \( z \), every \( z \) in the integrand of \( J_3 \) is replaced with the expression in terms of \( \cos \theta \), \( y \) and \( x \): \( z = \sqrt{x^2 + y^2} / \tan \theta \), where we omit the simple conversion between
trigonometric functions here and after. The integrand of the remaining double integral over $x$ and $y$ is a function of $\theta$, $x$ and $y$:

$$J_3 = \int_{-1}^{1} d\cos \theta \int_{-\infty}^{\infty} dx f(x, y, \sqrt{x^2 + y^2}/\tan \theta) \frac{\sqrt{x^2 + y^2}}{\sin^3 \theta}. \quad (65)$$

This can also be obtained by substituting $\cos \theta$, $y$ and $x$ into equation (39).

Next, we integrate out the $x_3 = y$ coordinate by multiplying $1_2$ in equation (62b) into $J_3$ in equation (65). The integration over $y$ can be performed as

$$\int_{-\infty}^{\infty} dy \delta \left( \phi - \arctan \frac{y}{x} \right) \Theta(x) + \delta \left( \phi - \arctan \frac{y}{x} - \pi \right) \Theta(-x) \right)$$

$$= \frac{1}{\left| \arctan \frac{x}{y} \right|} \left[ \Theta(x) + \Theta(-x) \right] \bigg|_{y = x \tan \phi} = \frac{|x|}{\cos^2 \phi}. \quad (66)$$

where we have used the identity

$$\Theta(x) + \Theta(-x) = 1. \quad (67)$$

Equation (66) can also be obtained by substituting $\cos \theta$, $\phi$ and $x$ into equation (41). After the integration over $y$, every $y$ in the integrand of $J_3$ is replaced with the expression in terms of $\cos \theta$, $\phi$ and $x$. The integrand of the remaining integral over $x$ is a function of $\cos \theta$, $\phi$ and $x$:

$$J_3 = \int_{-1}^{1} d\cos \theta \int_{0}^{2\pi} d\phi \int_{-\infty}^{\infty} dx f(x, y, \sqrt{x^2 + y^2}/\tan \theta) \frac{\sqrt{x^2 + y^2}}{\sin^3 \theta} \frac{|x|}{\cos^2 \phi} \bigg|_{y = x \tan \phi}$$

$$= \int_{-1}^{1} d\cos \theta \int_{0}^{2\pi} d\phi \int_{-\infty}^{\infty} dx f[x, x \tan \phi, |x|/(\cos \phi \tan \theta)] \frac{x^2}{\sin^3 \theta} \frac{1}{\cos^2 \phi}. \quad (68)$$

where $y = x \tan \phi$. After the integrations over both $z$ and $y$, $y = x \tan \phi$ and $z = x/(\cos \phi \tan \theta)$.

Finally, after multiplying $1_2$ in equation (62c) into $J_3$ in equation (68), the integration over $x_3 = x$ can be performed as

$$\int_{-\infty}^{\infty} dx \delta(r - \sqrt{x^2 + y^2 + z^2}) \bigg|_{y = x \tan \phi, z = \frac{x}{\cos \phi \sin \theta}}$$

$$= \int_{-\infty}^{\infty} dx \delta \left[ r - \frac{x}{\cos \phi \sin \theta} \right] = |\cos \phi| \sin \theta. \quad (69)$$

This can also be obtained by substituting $\cos \theta$, $\phi$ and $r$ into equation (43). Here, the radius $r$ is non-negative because the Dirac delta function requires that $r = \sqrt{x^2 + y^2 + z^2}$ and $x^2 + y^2 + z^2$ is non-negative. After the integration over all of the Cartesian coordinates, $x$, $y$ and $z$ are expressed as in equation (59).

Substituting equation (59) into (64), (66), (69), and $f(x, y, z)$, we find that $J_3$ can be expressed in terms of the spherical coordinates as

$$J_3 = \int_{0}^{\infty} dr \int_{-1}^{1} d\cos \theta \int_{0}^{2\pi} d\phi \frac{r^2 \sin \theta \sin \phi \cos \phi}{\sin^3 \theta} \cos \phi \sin \theta f(r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$$

$$x \frac{r \sin \theta \sin \phi \cos \phi}{\sin^3 \theta} \cos \phi \sin \theta f(r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$$

$$y \frac{r \sin \theta \sin \phi \cos \phi}{\sin^3 \theta} \cos \phi \sin \theta f(r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$$

$$z \frac{r \sin \theta \sin \phi \cos \phi}{\sin^3 \theta} \cos \phi \sin \theta f(r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$$
This exactly reproduces the result which can be obtained by applying the formula (44) [1].

In this subsection, we extend the three-dimensional case to the $n$-dimensional coordinate transformation from the $n$-dimensional Cartesian coordinates $(x_1, \ldots, x_n)$ to the $n$-dimensional polar coordinates $(r, \theta_1, \theta_2, \ldots, \theta_{n-2}, \phi)$. Here, $r$ is the radius and there are $n-2$ polar angles $\theta_i$’s and a single azimuthal angle $\phi$. The corresponding transformation functions are expressed as

$$
x_1 = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-3} \sin \theta_{n-2} \cos \phi,
$$
$$
x_2 = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-3} \sin \theta_{n-2} \sin \phi,
$$
$$
x_3 = r \sin \theta_1 \sin \theta_2 \ldots \sin \theta_{n-3} \cos \theta_{n-2},
$$
$$
\vdots
$$
$$
x_{n-1} = r \sin \theta_1 \cos \theta_2,
$$
$$
x_n = r \cos \theta_1. \tag{72}
$$

We reorganize the order of integrations of the Cartesian coordinates as $(x_n, x_{n-1}, \ldots, x_1)$ for simplicity and the corresponding curvilinear coordinates are reorganized as $(r, \theta_1, \theta_2, \ldots, \theta_{n-2}, \phi, r)$. We consider an $n$-dimensional integral $J_n$ with an arbitrary integrand $f(x_1, x_2, \ldots, x_n)$:

$$
J_n = \int_{-\infty}^{\infty} d^n x \ f(x_1, x_2, \ldots, x_n), \tag{73}
$$

where $d^n x = dx_1 dx_2 \ldots dx_n$. We multiply the unities

$$
\mathbb{1}_i = \int_{-1}^{1} \cos \theta_i \delta \left( \cos \theta_i - \frac{x_{n-i+1}}{\sqrt{x_1^2 + \cdots + x_{n-i+1}^2}} \right) = 1, \quad i = 1, \ldots, n-2, \tag{74a}
$$
$$
\mathbb{1}_{n-1} = \int_{0}^{2\pi} d\phi \left[ \delta \left( \phi - \arctan \frac{x_2}{x_1} \right) \Theta(x_1) + \delta \left( \phi - \arctan \frac{x_2}{x_1} - \pi \right) \Theta(-x_1) \right] = 1, \tag{74b}
$$
$$
\mathbb{1}_n = \int_{0}^{\infty} dr \delta \left( r - \sqrt{x_1^2 + \cdots + x_n^2} \right) = 1 \tag{74c}
$$

for $i = 1$ through $n$ to $J_n$, sequentially, keeping the integral invariant. $\Theta(x)$ in equation (74b) is the Heaviside step function defined in equation (60b). There are numerous ways to perform the multiple integrations over the $n$ Cartesian coordinates. Our strategy to integrate out $x_i$’s is...
as follows: according to the integrand of the right-hand side of equation (74a), the integration over \( x_{n-i+1} \) in the integral \( J_n \) provides the constraint to the polar angle \( \theta_i \). Thus we choose to integrate over from \( x_0 \) to \( x_1 \) to express them in terms of the polar angles from \( \theta_1 \) through \( \theta_{n-2} \). Then we integrate out \( x_2 \) to express \( x_n \) through \( x_2 \) in terms of the \( n-2 \) polar angles and the azimuthal angle \( \phi \) by making use of equation (74b). As the last step, we integrate out \( x_1 \) to determine all of the Cartesian coordinates in terms of the spherical polar coordinates by making use of (74c).

First, after multiplying \( \frac{1}{i} \) in equation (74a) into \( J_n \), the integration over \( x_{n-i+1} \) for \( i = 1 \) through \( n-2 \) can be performed as

\[
\int_{-\infty}^{\infty} dx_{n-i+1} \delta \left( \cos \theta_i - \frac{x_{n-i+1}^{2}}{\sqrt{x_1^2 + \cdots + x_{n-i}^2}} \right) = \sqrt{x_1^2 + \cdots + x_{n-i}^2} \sin^3 \theta_i, \tag{75}
\]

where one can obtain the same results from equations (46) and (54) taking care of the order of integration. After the integration over \( x_{n-i+1} \), we make the replacement

\[
x_{n-i+1} = \sqrt{x_1^2 + \cdots + x_{n-i}^2} \tan \theta_i. \tag{76}
\]

Then, \( J_n \) is expressed as

\[
J_n = \int_{-1}^{1} d \cos \theta_1 \cdots \int_{-1}^{1} d \cos \theta_{n-2} \times \int_{-\infty}^{\infty} dx_1 dx_2 f(x_1, x_2, \cos \theta_1, \ldots, \cos \theta_{n-2}) \prod_{i=1}^{n-2} \frac{\sqrt{x_1^2 + \cdots + x_{n-i}^2}}{\sin^3 \theta_i}, \tag{77}
\]

where every \( x_i \) in the last factor for \( i = 3 \) through \( n \) is replaced by that in equation (76).

After multiplying \( \frac{1}{n-1} \) in equation (74b) into \( J_n \) in equation (77), the integration over \( x_2 \) can be carried out in a similar manner as is done in equation (66). The result is

\[
\int_{-\infty}^{\infty} dx_2 \left[ \delta \left( \phi - \arctan \frac{x_2}{x_1} \right) \Theta(x_1) + \delta \left( \phi - \arctan \frac{x_2}{x_1} - \pi \right) \Theta(-x_1) \right] = \frac{|x_1|}{\cos^2 \phi}, \tag{78}
\]

where \( x_2 = x_1 \tan \phi \) after the integration. This can also be obtained from equation (54) while keeping the results in equations (75) and (76). Then, \( x_{n-i+1} \) for \( i = 1 \) through \( n-1 \) can be expressed as

\[
x_{n-i+1} = \frac{|x_1|}{\cos \phi} \sin \theta_{n-2} \cdots \sin \theta_{i+1} \tan \theta_i. \tag{79}
\]

Then, we find that

\[
J_n = \int_{-1}^{1} d \cos \theta_1 \cdots \int_{-1}^{1} d \cos \theta_{n-2} \int_{0}^{2\pi} d\phi \times \int_{-\infty}^{\infty} dx_1 f(x_1, \phi, \cos \theta_1, \ldots, \cos \theta_{n-2}) \frac{|x_1|}{\cos^2 \phi} \prod_{i=1}^{n-2} \frac{\sqrt{x_1^2 + \cdots + x_{n-i}^2}}{\sin^3 \theta_i}, \tag{80}
\]

16
where every $x_i$ in the last factor for $i = 2$ through $n$ is replaced by that in equation (79).

Finally, after multiplying $I_{2n}$ in equation (74c) to $J_n$ in equation (80), the integration over $x_1$ can be performed like equation (69) and we obtain

$$
\int_{-\infty}^{\infty} dx_1 \delta \left( r - \sqrt{x_1^2 + \cdots + x_n^2} \right) = \int_{-\infty}^{\infty} dx_1 \delta \left[ r - \frac{x_1}{\cos \phi \sin \theta_{n-2} \cdots \sin \theta_1} \right]
$$

where we have omitted the replacements of $x_2 = x_1 \tan \phi$ and $x_3$ through $x_n$ that can be obtained from equation (79) on the left-hand side. This result can also be obtained from equation (54) while keeping the results in equations (75), (76), (78) and (79). After integrating out all of the Cartesian coordinates, we reproduce the expression for every $x_i$ that is given in equation (72).

Combining all of the results listed above, we find that the $n$-dimensional coordinate transformation or change of variables is carried out as

$$
J_n = \int_0^\infty dr \int_0^\pi d\theta_1 \cdots \int_0^{\pi} d\theta_{n-2} \int_0^{2\pi} d\phi \ 
\frac{r \sin \theta_1 \cdots \sin \theta_{n-2} \cos \phi}{\cos^2 \phi} \sin \phi \sin \theta_1 \cdots \sin \theta_{n-2} \times f (r \sin \theta_1 \cdots \sin \theta_{n-2} \cos \phi, r \sin \theta_1 \cdots \sin \theta_{n-2} \sin \phi, r \sin \theta_1 \cdots \sin \theta_{n-2} \cos \theta_2, r \cos \theta_1) \n$$

\[ \times \frac{r \sin \theta_1 \cdots \sin \theta_{n-3} \cos \theta_{n-2}, \cdots, r \sin \theta_1 \cos \theta_2, r \cos \theta_1}{\sin^2 \theta_1 \sin \theta_{n-2}} \times f (r \sin \theta_1 \cdots \sin \theta_{n-2} \cos \phi, r \sin \theta_1 \cdots \sin \theta_{n-2} \sin \phi, r \sin \theta_1 \cdots \sin \theta_{n-2} \cos \theta_2, r \cos \theta_1) . \]

(82)

The extra factor $r^{n-1} \sin^{n-2} \theta_1 \cdots \sin^2 \theta_{n-3} \sin \theta_{n-2}$ in front of the original integrand $f$ is identified with the Jacobian

$$
J = r^{n-1} \sin^{n-2} \theta_1 \cdots \sin^2 \theta_{n-3} \sin \theta_{n-2}.
$$

(83)

This exactly reproduces the result in references [5, 6], which can be obtained by applying the general formula (58).

4. Conclusions

We have derived the general formula for the Jacobian of the transformation from the $n$-dimensional Cartesian coordinates to arbitrary curvilinear coordinates by making use of Dirac delta functions, whose arguments correspond to the transformation functions between the two coordinate systems. The multiplication of the trivial identities (6) to the original integral enables us to integrate out the original integration variables corresponding to the Cartesian coordinates systematically. By making use of the chain rule for the partial derivatives, we can carry out the integration over the Cartesian coordinates successively and end up with the integral expressed in terms of the curvilinear coordinates. Then, the Jacobian can be read off by comparing the integrands of the resultant integral with the original one. It turns out that the formula derived in this paper exactly reproduces the Jacobian for the coordinate transformation or change of variables.
We have presented a few examples, where we have integrated out the Cartesian coordinates by making use of Dirac delta functions explicitly without resort to the general formula for the Jacobian derived in this paper. We find that the formulas obtained in these explicit examples are exactly the same as those in the general formula (58). Since the derivation of the Jacobian in the general case that makes use of the chain rule of the partial derivatives is rather abstract, we expect that these examples will give insights on understanding the derivation concretely.

To our best knowledge, this derivation of the Jacobian factor by making use of Dirac delta functions for the coordinate transformation or change of variables from the \( n \)-dimensional Cartesian coordinates to the curvilinear coordinates is new. Although there are several ways to derive the Jacobian available in textbooks [1–3], our derivation could be pedagogically useful in upper-level mathematics or physics courses in practice using Dirac delta functions successively. Compared with the methods popular in the textbook level, our method is more intuitive because we have employed only the explicit calculation of elementary single-dimensional integrals without relying on abstract geometrical interpretations or more abstract Green’s theorem with which undergraduate physics-major students are not usually familiar. Furthermore, a detailed derivation of the chain rule for the partial derivatives, which is employed to prove the Jacobian formula, should be a nice working example with which one can understand a rigorous usage of the partial derivatives with multi-dimensional variables without ambiguity.

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Appendix A. 2-dimensional case

In this section, we consider a two-dimensional coordinate transformation from the Cartesian coordinates \((x_1, x_2)\) to the curvilinear coordinates \((q_1, q_2)\). The total differentials of \(q_1\) and \(x_1\) can be expressed as

\[
dq_1 = \frac{\partial q_1}{\partial x_1} dx_1 + \frac{\partial q_1}{\partial x_2} dx_2, \tag{A1a}
\]

\[
dx_1 = \left(\frac{\partial x_1}{\partial q_1}\right)_{(i)} dq_1 + \left(\frac{\partial x_1}{\partial x_2}\right)_{(i)} dx_2, \tag{A1b}
\]

where \(q_1 = q_1(x_1, x_2)\) in equation (A1a) and \(x_1 = x_1(q_1, x_2)\) in equation (A1b), respectively. Note that the definition of the partial derivative with a subscript are given in section 2.1: \((\partial x_1/\partial q_1)_{(i)}, (\partial x_1/\partial x_2)_{(i)}\), and \((\partial q_1/\partial x_1)_{(2)}\) are defined in equations (20)–(22), respectively.
Replacing $dx_1$ in equation (A1a) with the right-hand side of equation (A1b), we obtain

$$dq_1 = \frac{\partial q_1}{\partial x_1} \frac{\partial x_1}{\partial q_1} dq_1 + \left[ \frac{\partial q_1}{\partial x_1} \frac{\partial x_1}{\partial x_2} + \frac{\partial q_1}{\partial x_2} \right] dx_2. \quad (A2)$$

Comparing the coefficients of the differentials on both sides, we find that

$$\frac{\partial q_1}{\partial x_1} \left( \frac{\partial x_1}{\partial q_1} \right)_{(1)} = 1, \quad (A3a)$$

$$\frac{\partial q_1}{\partial x_1} \left( \frac{\partial x_1}{\partial x_2} \right)_{(1)} + \frac{\partial q_1}{\partial x_2} = 0. \quad (A3b)$$

We note that equation (A3a) is trivial. By making use of equation (A3b), we find that the factor in the denominator of equation (32) \( (\partial q_2/\partial x_2)_{(1)} \) can be expressed as

$$\left( \frac{\partial q_2}{\partial x_2} \right)_{(1)} = \frac{\partial q_2}{\partial q_1} \left( \frac{\partial x_1}{\partial x_2} \right)_{(1)} + \frac{\partial q_2}{\partial x_2} = \frac{\det \left( \frac{\partial q_1}{\partial x_1} \frac{\partial q_2}{\partial x_2} \right)}{\det \left( \frac{\partial q_1}{\partial x_1} \right)} . \quad (A4)$$

which leads to

$$\left( \frac{\partial q_2}{\partial x_2} \right)_{(1)} = \frac{q_2}{q_1} . \quad (A5)$$

**Appendix B. 3-dimensional case**

In this section, we consider a three-dimensional coordinate transformation from the Cartesian coordinates \((x_1, x_2, x_3)\) to the curvilinear coordinates \((q_1, q_2, q_3)\). First, we consider the total differentials of \(q_1 \) and \(x_1\) which can be expressed as

$$dq_1 = \frac{\partial q_1}{\partial x_1} dx_1 + \frac{\partial q_1}{\partial x_2} dx_2 + \frac{\partial q_1}{\partial x_3} dx_3, \quad (B1a)$$

$$dx_1 = \left( \frac{\partial x_1}{\partial q_1} \right)_{(1)} dq_1 + \left( \frac{\partial x_1}{\partial x_2} \right)_{(1)} dx_2 + \left( \frac{\partial x_1}{\partial x_3} \right)_{(1)} dx_3, \quad (B1b)$$

where \(q_1 = q_1(x_1, x_2, x_3)\) in equation (B1a) and \(x_1 = x_1(q_1, x_2, x_3)\) in equation (B1b), respectively. Replacing \(dx_1\) in equation (B1a) with the right-hand side of equation (B1b), we obtain

$$dq_1 = \frac{\partial q_1}{\partial x_1} \left( \frac{\partial x_1}{\partial q_1} \right)_{(1)} dq_1 + \left[ \frac{\partial q_1}{\partial x_1} \left( \frac{\partial x_1}{\partial x_2} \right)_{(1)} + \frac{\partial q_1}{\partial x_2} \right] dx_2$$

$$+ \left[ \frac{\partial q_1}{\partial x_1} \left( \frac{\partial x_1}{\partial x_3} \right)_{(1)} + \frac{\partial q_1}{\partial x_3} \right] dx_3. \quad (B2)$$

Comparing the coefficients of the differentials on both sides, we find that
\[
\frac{\partial q_1}{\partial x_1} \left( \frac{\partial x_1}{\partial q_1} \right)_{(1)} = 1, \quad \text{(B3a)}
\]
\[
\frac{\partial q_1}{\partial x_1} \left( \frac{\partial x_1}{\partial q_1} \right)_{(1)} + \frac{\partial q_1}{\partial x_2} = 0, \quad \text{(B3b)}
\]
\[
\frac{\partial q_1}{\partial x_1} \left( \frac{\partial x_1}{\partial q_1} \right)_{(1)} + \frac{\partial q_1}{\partial x_3} = 0, \quad \text{(B3c)}
\]

where equation \(\text{(B3a)}\) is trivial.

By making use of equation \(\text{(B3b)}\), we find that the factor in the denominator of equation \(\text{(41)}\) \((\partial q_2/\partial x_2)_{(1)}\) can be expressed as

\[
\left( \frac{\partial q_2}{\partial x_2} \right)_{(1)} = \frac{\partial q_2}{\partial x_2} - \frac{\partial q_2}{\partial x_1} \left( \frac{\partial x_1}{\partial q_1} \right)_{(1)}
\]

which leads to

\[
\left| \left( \frac{\partial q_2}{\partial x_2} \right)_{(1)} \right| = \frac{q_2}{q_1} \quad \text{(B5)}
\]

In order to prove equation \(\text{(B11)}\), we take into account the total differentials of \(q_1, x_1, q_2\) and \(x_2\). The total derivatives can be expressed as

\[
dq_1 = \frac{\partial q_1}{\partial x_1} dx_1 + \frac{\partial q_1}{\partial x_2} dx_2 + \frac{\partial q_1}{\partial x_3} dx_3, \quad \text{(B6a)}
\]
\[
dq_2 = \frac{\partial q_2}{\partial x_1} dx_1 + \frac{\partial q_2}{\partial x_2} dx_2 + \frac{\partial q_2}{\partial x_3} dx_3, \quad \text{(B6b)}
\]
\[
dx_1 = \left( \frac{\partial x_1}{\partial q_1} \right)_{(2)} dq_1 + \left( \frac{\partial x_1}{\partial q_2} \right)_{(2)} dq_2 + \left( \frac{\partial x_1}{\partial x_3} \right)_{(2)} dx_3, \quad \text{(B6c)}
\]
\[
dx_2 = \left( \frac{\partial x_2}{\partial q_1} \right)_{(2)} dq_1 + \left( \frac{\partial x_2}{\partial q_2} \right)_{(2)} dq_2 + \left( \frac{\partial x_2}{\partial x_3} \right)_{(2)} dx_3, \quad \text{(B6d)}
\]

where \(q_1 = q_1(x_1, x_2, x_3)\) in equation \(\text{(B6a)}\), \(q_2 = q_2(x_1, x_2, x_3)\) in equation \(\text{(B6b)}\), \(x_1 = x_1(q_1, q_2, x_3)\) in equation \(\text{(B6c)}\), and \(x_2 = x_2(q_1, q_2, x_3)\) in equation \(\text{(B6d)}\), respectively. Substituting equations \(\text{(B6c)}\) and \(\text{(B6d)}\) into equations \(\text{(B6a)}\) and \(\text{(B6b)}\), we obtain

\[
dq_1 = \left[ \left( \frac{\partial q_1}{\partial x_1} \right)_{(2)} \left( \frac{\partial x_1}{\partial q_1} \right)_{(2)} \right] dq_1 + \left[ \left( \frac{\partial q_1}{\partial x_2} \right)_{(2)} \left( \frac{\partial x_2}{\partial q_1} \right)_{(2)} \right] dq_2 + \left[ \left( \frac{\partial q_1}{\partial x_3} \right)_{(2)} \left( \frac{\partial x_3}{\partial q_1} \right) \right] dx_3,
\]
\[
dq_2 = \left[ \left( \frac{\partial q_1}{\partial x_1} \right)_{(2)} \left( \frac{\partial x_1}{\partial q_2} \right)_{(2)} \right] dq_1 + \left[ \left( \frac{\partial q_1}{\partial x_2} \right)_{(2)} \left( \frac{\partial x_2}{\partial q_2} \right)_{(2)} \right] dq_2 + \left[ \left( \frac{\partial q_1}{\partial x_3} \right)_{(2)} \left( \frac{\partial x_3}{\partial q_2} \right) \right] dx_3.
\]
By making use of Cramer’s rule, we find that

\[
\frac{\partial q_2}{\partial x_1} \left( \frac{\partial x_1}{\partial q_1} \right)_{(2)} + \frac{\partial q_2}{\partial x_2} \left( \frac{\partial x_2}{\partial q_1} \right)_{(2)} + \frac{\partial q_2}{\partial x_3} \left( \frac{\partial x_3}{\partial q_1} \right)_{(2)} \frac{dq_1}{dq_1}
\]

\[
+ \left[ \frac{\partial q_2}{\partial x_1} \left( \frac{\partial x_1}{\partial q_2} \right)_{(2)} + \frac{\partial q_2}{\partial x_2} \left( \frac{\partial x_2}{\partial q_2} \right)_{(2)} + \frac{\partial q_2}{\partial x_3} \left( \frac{\partial x_3}{\partial q_2} \right)_{(2)} \right] \frac{dq_2}{dq_2}
\]

\[
+ \left[ \frac{\partial q_2}{\partial x_1} \left( \frac{\partial x_1}{\partial x_3} \right)_{(2)} + \frac{\partial q_2}{\partial x_2} \left( \frac{\partial x_2}{\partial x_3} \right)_{(2)} + \frac{\partial q_2}{\partial x_3} \left( \frac{\partial x_3}{\partial x_3} \right) \right] \frac{dx_3}{dx_3}.
\]  

(B7)

Comparing both sides of equation (B7), we find two relevant non-trivial equations:

\[
\frac{\partial q_1}{\partial x_1} \left( \frac{\partial x_1}{\partial x_3} \right)_{(2)} + \frac{\partial q_1}{\partial x_2} \left( \frac{\partial x_2}{\partial x_3} \right)_{(2)} = \frac{\partial q_1}{\partial x_3}.
\]

(B8)

By making use of Cramer’s rule, we find that

\[
\left( \frac{\partial x_1}{\partial x_3} \right)_{(2)} = \frac{\operatorname{det} \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} \end{pmatrix}}{\operatorname{det} \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} \end{pmatrix}}, \quad \left( \frac{\partial x_2}{\partial x_3} \right)_{(2)} = -\frac{\operatorname{det} \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} \end{pmatrix}}{\operatorname{det} \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} \end{pmatrix}}.
\]

(B9)

Then, by making use of equation (B9), we find that the factor \( \left( \frac{\partial q_3}{\partial x_3} \right)_{(2)} \) in the denominator of equation (43) can be expressed as

\[
\left( \frac{\partial q_3}{\partial x_3} \right)_{(2)} = \frac{\partial q_3}{\partial x_1} \left( \frac{\partial x_1}{\partial x_3} \right)_{(2)} + \frac{\partial q_3}{\partial x_2} \left( \frac{\partial x_2}{\partial x_3} \right)_{(2)} + \frac{\partial q_3}{\partial x_3} \left( \frac{\partial x_3}{\partial x_3} \right)_{(2)}
\]

\[
= \frac{\operatorname{det} \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} & \frac{\partial q_1}{\partial x_3} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} & \frac{\partial q_2}{\partial x_3} \\ \frac{\partial q_3}{\partial x_1} & \frac{\partial q_3}{\partial x_2} & \frac{\partial q_3}{\partial x_3} \end{pmatrix}}{\operatorname{det} \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \frac{\partial q_1}{\partial x_2} \\ \frac{\partial q_2}{\partial x_1} & \frac{\partial q_2}{\partial x_2} \end{pmatrix}}.
\]

(B10)

which leads to

\[
\left| \frac{\partial q_3}{\partial x_3} \right| = \frac{\partial q_3}{\partial x_3}.
\]

(B11)
Appendix C. $n$-dimensional case

The total differentials of $q_1,\ldots, q_i, x_1,\ldots, x_i$ can be expressed as

$$dq_1 = \sum_{a=1}^{n} \frac{\partial q_1}{\partial x_a} dx_a,$$

$$\vdots$$

$$dq_i = \sum_{a=1}^{n} \frac{\partial q_i}{\partial x_a} dx_a,$$

$$dx_1 = \sum_{a=1}^{i} \left[ \left( \frac{\partial x_1}{\partial q_a} \right)_{(i)} dq_a \right] + \sum_{a=i+1}^{n} \left[ \left( \frac{\partial x_1}{\partial x_a} \right)_{(i)} dx_a \right],$$

$$\vdots$$

$$dx_i = \sum_{a=1}^{i} \left[ \left( \frac{\partial x_i}{\partial q_a} \right)_{(i)} dq_a \right] + \sum_{a=i+1}^{n} \left[ \left( \frac{\partial x_i}{\partial x_a} \right)_{(i)} dx_a \right].$$  \hspace{1cm} (C1)

The partial derivative of $q_{i+1}$ with respect to $x_{i+1}$ holding $q_1,\ldots, q_i, x_{i+2},\ldots, x_{n}$ fixed is

$$\left( \frac{\partial q_{i+1}}{\partial x_{i+1}} \right)_{(i)} = \sum_{a=1}^{i} \left[ \frac{\partial q_{i+1}}{\partial x_a} \left( \frac{\partial x_a}{\partial q_{i+1}} \right) \right] + \frac{\partial q_{i+1}}{\partial x_{i+1}}.$$  \hspace{1cm} (C2)

Substituting $dx_1,\ldots, dx_i$ into $dq_j$, we obtain

$$dq_j = \sum_{a=1}^{i} \frac{\partial q_j}{\partial x_a} \left[ \sum_{b=1}^{i} \left( \frac{\partial x_a}{\partial q_b} \right)_{(i)} dq_b + \sum_{b=i+1}^{n} \left( \frac{\partial x_a}{\partial x_b} \right)_{(i)} dx_b \right] + \sum_{a=i+1}^{n} \frac{\partial q_j}{\partial x_a} dx_a,$$  \hspace{1cm} (C3)

where $1 \leq j \leq i$. Because the coefficient of $dx_{i+1}$ in (C3) should be 0, we obtain the following equation:

$$\begin{pmatrix}
\frac{\partial q_1}{\partial x_1} & \ldots & \frac{\partial q_1}{\partial x_i} \\
\vdots & \ddots & \vdots \\
\frac{\partial q_i}{\partial x_1} & \ldots & \frac{\partial q_i}{\partial x_i}
\end{pmatrix}
\begin{pmatrix}
\left( \frac{\partial x_1}{\partial q_{i+1}} \right)_{(i)} \\
\vdots \\
\left( \frac{\partial x_i}{\partial q_{i+1}} \right)_{(i)}
\end{pmatrix}
= - \begin{pmatrix}
\frac{\partial q_1}{\partial x_{i+1}} \\
\vdots \\
\frac{\partial q_i}{\partial x_{i+1}}
\end{pmatrix}.$$  \hspace{1cm} (C4)

By making use of Cramer’s rule, we find that

$$\left( \frac{\partial x_j}{\partial x_{i+1}} \right)_{(i)} = \frac{\mathcal{D} \mathcal{E} \mathcal{I} \left[ \left( \frac{1}{i} \right)^{i-j} \left( \frac{\partial q_j}{\partial q_k} \right) \right]}{\mathcal{D} \mathcal{E} \mathcal{L} \left[ \left( \frac{1}{i} \right)^{i} \right]}.$$  \hspace{1cm} (C5)

where
\[ J_{i\times i}^{-1} = \begin{pmatrix} \frac{\partial q_1}{\partial x_1} & \cdots & \frac{\partial q_1}{\partial x_i} \\ \vdots & \ddots & \vdots \\ \frac{\partial q_i}{\partial x_1} & \cdots & \frac{\partial q_i}{\partial x_i} \end{pmatrix}, \]

\[ \partial q_i = \begin{pmatrix} \frac{\partial q_1}{\partial x_1} \\ \vdots \\ \frac{\partial q_i}{\partial x_1} \end{pmatrix}, \quad (C6) \]

and \((J_{i\times i}^{-1})^j(\partial q_i)\) is identical to \(J_{i\times i}^{-1}\) except that the \(j\)th column is replaced with \(\partial q_i\). Substituting (C5) into (C2), we obtain

\[ \left( \frac{\partial q_{i+1}}{\partial x_{i+1}} \right)_{(i)} = \frac{\partial q_{i+1}}{\partial x_{i+1}} - \sum_{j=1}^{i} \frac{\text{Det} \left[ (J_{i\times i}^{-1})^j(\partial q_i) \right]}{\text{Det} \left[ (J_{i\times i}^{-1}) \right]} \frac{\partial q_{i+1}}{\partial x_j} \]

\[ = \frac{1}{\text{Det} \left[ (J_{i\times i}^{-1}) \right]} \sum_{j=1}^{i+1} (-1)^{i+1-j} \mathcal{M}_{j+1}(J_{(i+1)\times (i+1)}) \frac{\text{Det} \left[ (J_{i\times i}^{-1})_{(i+1)} \right]}{\text{Det} \left[ (J_{i\times i}^{-1}) \right]} \quad \text{(C7)} \]

Here, the \(ij\) minor \(\mathcal{M}_{j}(\mathcal{A})\) of an \(n \times n\) square matrix \(\mathcal{A}\) is the determinant of a matrix whose \(i\)th row and \(j\)th column are removed from \(\mathcal{A}\). Hence, equation (C7) leads to

\[ \left| \left( \frac{\partial q_{i+1}}{\partial x_{i+1}} \right)_{(i)} \right| = \left| \frac{\partial q_{i+1}}{\partial x_i} \right| \quad \text{(C8)} \]

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