SimplicialDecomposability: a package for Macaulay 2

DAVID COOK II

Abstract. We introduce a new Macaulay 2 package, SimplicialDecomposability, which works in conjunction with the extant package SimplicialComplexes in order to compute a shelling order, if one exists, of a specified simplicial complex. Further, methods for determining vertex-decomposability are implemented, along with methods for determining k-decomposability.

Introduction. A simplicial complex ∆ on a finite vertex set V is a set of subsets of V closed under inclusion. Elements σ ∈ ∆ are called faces and the dimension of σ is #σ − 1. The dimension of ∆ is max dim σ. The f-vector of a ∆, where d = dim ∆ + 1, is the (d + 1)-tuple (f−1, . . . , fd−1), where fi is the number of faces of dimension i in ∆. Using this, the h-vector of ∆ is the d + 1-tuple (h0, . . . , hd) given by hj = ∑ j i=0 (−1)j−i(d−i)fi−1 for 0 ≤ j ≤ d.

The Stanley-Reisner ideal is the ideal I(∆) generated by the minimal non-faces of ∆ and the Stanley-Reisner ring is the ring K[∆] = K[V]/I(∆), for a given field K. Thus the Stanley-Reisner ideals of complexes on a given vertex set V are exactly the squarefree monomial ideals in K[V]. Using relations between the complex and the ideal, one can use tools from both algebra and combinatorics to study properties of both. For example, the h-vector of a complex ∆ is the coefficient-vector of the numerator of the Hilbert series of K[∆].

The package SimplicialComplexes by Sorin Popescu, Gregory G. Smith, and Mike Stillman already implements many methods for simplicial complexes in Macaulay 2 [M2], a software system designed to aid in research of commutative algebra and algebraic geometry. We introduce a new package, SimplicialDecomposability, for Macaulay 2 which provides several new methods for testing various forms of decomposability for simplicial complexes. Particularly, the package implements methods for testing shellability and vertex-decomposability.

Shellability. If a simplicial complex has one facet, say σ, then it is a simplex and is denoted 2σ. Let ∆ be a simplicial complex which has equi-dimensional facets, i.e., is pure. Then by [St, Definition III.2.1], ∆ is shellable if its facets can be ordered σ1,...,σn such that

\[ \bigcup_{j=1}^{i} 2\sigma_j \setminus \bigcup_{j=1}^{i-1} 2\sigma_j \]

has a unique minimal element for 2 ≤ i ≤ n, such an ordering is called a shelling order.

See [BW-1, Definition 2.1] for the definition of non-pure shellability, which is implemented in the package for non-pure complexes.
Shellability is of interest because it implies a number of nice properties. In particular, if a pure simplicial complex is shellable, then its Stanley-Reisner ring is Cohen-Macaulay over every field \( \text{[St, Theorem III.2.5]} \). Hence, its \( h \)-vector is non-negative and can be read off from any shelling order \( \text{[St, Theorem III.2.3]} \). Further still, the \( h \)-vectors of pure shellable complexes are numerically classified \( \text{[St, Theorems II.2.2 and II.3.3]} \).

We recall that the Alexander dual of a simplicial complex \( \Delta \) on vertex set \( V \) is the simplicial complex \( \Delta^\vee := \{ V \setminus F \mid F \not\in \Delta \} \). Further, we say an ideal \( I = (f_1, \ldots, f_n) \) has linear quotients if for \( 1 < i \leq n \), the quotient ideal \( (f_1, \ldots, f_{i-1}) : (f_i) \) is generated by linear forms.

In the following example we demonstrate [HHZ, Theorem 1.4(c)] which shows that a pure simplicial complex is shellable if and only if the Stanley Reisner ideal of the Alexander dual has linear quotients. We begin by constructing the polynomial ring \( R = \mathbb{Q}[a, b, c, d, e, f, g] \) and a simplicial complex \( D \), which we verify is pure. Note that loading the package SimplicialDecomposability automatically loads the package SimplicialComplexes.

```plaintext
i1 : needsPackage "SimplicialDecomposability";

i2 : R = QQ[a..g];

i3 : D = simplicialComplex monomialIdeal {a*b,a*c,b*c,c*d,d*e,d*f,f*g};

i4 : isPure D
  o4 = true
```

We can recover the sequence of linear quotients directly from a shelling order. We recall that a pure simplicial complex \( \Delta \) is shellable if there is an order of the facets \( F_1, \ldots, F_n \) such that for \( 0 < j < i \) there exists an \( x \in F_i \setminus F_j \) and a \( 0 < k < i \) such that \( F_i \setminus F_k = \{x\} \). The set of vertices associated to each \( i \) in the preceding statement generate the linear quotient order of \( I(\Delta^\vee) \) with respect to the given shelling order (see the proof of \( \text{[HHZ, Theorem 1.4(c)]} \)).

```plaintext
i5 : -- find the linear quotients from a shelling order
  linearQuotients = O \to (for i from 1 to #O-1 list
    unique flatten for j from 0 to i-1 list (ImJ = set support O_i - set support O_j;
      for k from 0 to i-1 list (ImK = set support O_i - set support O_k;
        if #ImK == 1 and isSubset(ImK, ImJ) then
          first toList ImK else continue)));

i6 : O1 = shellingOrder D
  o6 = {c*e*g, b*e*g, a*e*g, b*d*g, a*d*g, c*e*f, b*e*f, a*e*f}

i7 : linearQuotients O1
  o7 = {{b}, {a}, {d}, {d, a}, {f}, {f, b}, {f, a}}
```

We generate a shelling order \( O_1 \) of \( D \) with the method shellingOrder. This method attempts to build up a shelling order of \( D \) recursively using a depth-first search, adding one facet at a time. We note that in the non-pure case, the method only searches the remaining facets of largest dimension.

```plaintext
i8 : O1 = shellingOrder D
  o6 = {c*e*g, b*e*g, a*e*g, b*d*g, a*d*g, c*e*f, b*e*f, a*e*f}

i7 : linearQuotients O1
  o7 = {{b}, {a}, {d}, {d, a}, {f}, {f, b}, {f, a}}
```

It is sometimes beneficial to have more than one shelling order for a given simplicial complex. We can use the option Random with the method shellingOrder to first apply a random permutation to the facets before preceding with the recursion.
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i8 : O2 = shellingOrder(D, Random => true)
o8 = {b*e*g, a*e*g, a*d*g, b*e*f, c*e*f, a*e*f, c*e*g, b*d*g}
o8 : List
i9 : linearQuotients O2
o9 = {{a}, {d}, {f}, {c}, {f, a}, {c, g}, {d, b}}
o9 : List

Alternately, we may use the option Permutation with the method shellingOrder to force a given permutation on the facets before preceding with the recursion.

i10 : O3 = shellingOrder(D, Permutation => {3,2,1,0,4,5,6,7})
o10 = {b*d*g, b*e*g, a*e*g, c*e*g, a*d*g, c*e*f, b*e*f, a*e*f}
o10 : List
i11 : linearQuotients O3
o11 = {{e}, {a}, {c}, {a, d}, {f}, {f, b}, {f, a}}
o11 : List

Thus we now have multiple distinct linear quotient orders associated to the ideal $I(D^\vee)$, each coming from a distinct shelling order of $D$.

**Vertex-decomposability.** Let $\Delta$ be a pure simplicial complex and $\sigma$ a face of $\Delta$. Then the link and face deletion of $\Delta$ by $\sigma$ are the simplicial complexes

$$\text{link}_{\Delta} \sigma := \{ \tau \in \Delta \mid \sigma \cap \tau = \emptyset, \sigma \cup \tau \in \Delta \} \text{ and } \text{del}_{\Delta} \sigma := \{ \tau \in \Delta \mid \sigma \not\subseteq \tau \}.$$  

Then [PB, Definition 2.1] defines $\Delta$ to be *vertex-decomposable* if either $\Delta$ is a simplex or there exists a vertex $x \in \Delta$, called a *shedding vertex*, such that $\text{link}_{\Delta} x$ and $\text{del}_{\Delta} x$ are vertex-decomposable.

See [BW-2, Definition 11.1] for the definition of non-pure vertex-decomposability, which is implemented in the package for non-pure complexes. Also, see [Wo, Definitions 3.1 and 3.6] for the generalisation of vertex-decomposability, called $k$-decomposability. It is implemented in the package with the methods iskDecomposable and isSheddingFace.

Being vertex-decomposable is a strong property which implies many things. A pure vertex-decomposable simplicial complex is shellable [PB, Theorem 2.8] and hence has non-negative $h$-vector [St, Theorem III.2.3] and its Stanley-Reisner ring is Cohen-Macaulay [St, Theorem III.2.5]. Furthermore, the $h$-vectors are numerically classified for vertex-decomposable simplicial complexes [Lee, Theorem 3.5]. Moreover, the Stanley-Reisner ring of a pure vertex-decomposable complex is squarefree glicci, that is, in the Gorenstein liaison class of a complete intersection such that the even links are squarefree monomials [NR, Theorem 3.3].

In the following example we demonstrate that the simplicial complex $D$ from the previous example is indeed squarefree glicci. We use [NR, Remark 2.4] to find a basic double link of $I(D)$ to $I(\text{link}_{D} v)$, both in $R$, for some shedding vertex $v$ of $D$.

First, we verify that $D$ is vertex-decomposable. We then find all of its shedding vertices.

i12 : isVertexDecomposable D
o12 = true
i13 : select(allFaces(D, 0), v -> isSheddingVertex(D, v))
o13 = {a, b, c, d, f}
o13 : List
We choose the shedding vertex \( f \) of \( D \) and generate \( E = \text{link}_D f \). Moreover, we in turn find its shedding vertices.

\[
\text{i14 : } E = \text{link}(D, f); \text{ideal } E \\
o15 = \text{ideal } (a*b, a*c, b*c, d, f, g) \\
o15 : \text{Ideal of } R \\
i16 : \text{select(allFaces}(E, 0), v \rightarrow \text{isSheddingVertex}(E, v)) \\
o16 = \{a, b, c\} \\
o16 : \text{List}
\]

We now choose the shedding vertex \( c \) of \( D \) and generate \( F = \text{link}_E c \). Notice then that the Stanley Reisner ideal of \( F \) is a complete intersection.

\[
i17 : F = \text{link}(E, c); \text{ideal } F \\
o18 = \text{ideal } (a, b, c, d, f, g) \\
o18 : \text{Ideal of } R
\]

Hence, we now have the following sequence of basic double links in \( R \) which has squarefree terms on the even steps (the odd steps are omitted):

\[
\mathbb{Q}[D] = (ab, ac, bc, cd, de, df, fg) \sim \mathbb{Q}[E] = (ab, ac, bc, d, f, g) \sim \mathbb{Q}[F] = (a, b, c, d, f, g).
\]

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**Department of Mathematics, University of Kentucky, 715 Patterson Office Tower, Lexington, KY 40506-0027, USA**

**E-mail address:** dcook@ms.uky.edu