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A Semi-group Expansion for Pricing Barrier Options

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Abstract

This paper presents a new asymptotic expansion method for pricing continuously monitoring barrier options. In particular, we develop a semi-group expansion scheme for the Cauchy-Dirichlet problem in the second-order parabolic partial differential equations (PDEs) arising in barrier option pricing. As an application, we propose a concrete approximation formula under a stochastic volatility model and demonstrate its validity by some numerical experiments.

Keywords: Barrier options, Asymptotic expansion, Stochastic volatility model, Semi-group representation, Cauchy-Dirichlet problem.

1 Introduction

Since the Merton’s seminal work ([15]) barrier options have been quite popular and important products in both academics and financial business for the last four decades. In particular, fast and accurate computation of their prices and Greeks is highly desirable in the risk management, which is a tough task under the finance models commonly used in practice. Thus, it has been one of the central issues in the mathematical finance community. Among various approaches to attacking the problem, this paper proposes a new semi-group expansion scheme under general diffusion setting.

Firstly, let us note that the value of a continuously monitoring down-and-out barrier option is expressed as the following form:

\[ C \text{Barrier}(T, x) = E[f(X_T^x)1_{\{\tau > T\}}] = E[f(X_T^x)1_{\{\min_{t \in [0, T]} X_t > L\}}]. \] (1.1)

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Here, $T > 0$ is a maturity of the option, and $(X_t^x)_t$ denotes a vector process starting from $x$ including a price process of the underlying asset (usually given as the solution of a certain stochastic differential equation (SDE)). Also, $L$ stands for a constant lower barrier, that is $L < x$, and $\tau$ is the hitting time to $L$:

$$\tau = \inf\{t \in [0, T] : X_t^x \leq L\}. \hspace{1cm} (1.2)$$

It is well-known that a possible approach in computation of $C_{\text{Barrier}}(T, x)$ is the Euler-Maruyama scheme, which stores the sample paths of the process $(X_t^x)_t$ through an $n$-time discretization with the step size $T/n$. When applying this scheme to pricing a continuously monitoring barrier option, one kills the simulated process, say $(\bar{X}_t^x)_i$ if $\bar{X}_t^x$ exits from the domain $(L, \infty)$ until the maturity $T$. The usual Euler-Maruyama scheme is suboptimal since it does not control the diffusion paths between two successive dates $t_i$ and $t_{i+1}$: the diffusion paths could have crossed the barriers and come back to the domain without being detected. It is also known that the error between $C_{\text{Barrier}}(T, x)$ and $\bar{C}_{\text{Barrier}}(T, x)$, the barrier option price obtained by the Euler-Maruyama scheme is of order $\sqrt{T/n}$, as opposed to the order $T/n$ for standard plain-vanilla options. (See [7]) Thus, various Monte-Carlo schemes have been proposed for improving the order of the error. (See [17] for instance.)

One of the other tractable approaches for calculating $C_{\text{Barrier}}(T, x)$ is to derive an analytical approximation. If we obtain an accurate approximation formula, it is a powerful tool for pricing continuously monitoring barrier options because we need not rely on Monte-Carlo simulations anymore. However, from a mathematical viewpoint, deriving an approximation formula by applying stochastic analysis is not an easy task since the Malliavin calculus cannot be directly applied. It is due to the non-existence of the Malliavin derivative $D_t \tau$ (see [4]) and to the fact that the minimum (maximum) process of the Brownian motion has only the first-order differentiability in the Malliavin sense. Thus, neither approach in [12] nor in [20] can be applied directly to valuation of continuously monitoring barrier options, while they are applicable to pricing discrete barrier options. (See [19] for the detail.)

This paper proposes a new general method for the approximation of barrier option prices. Particularly, our objective is to pricing barrier options when the dynamics of the underlying asset price is described by the following perturbed SDE:

$$\begin{align*}
\frac{dX_t^{\varepsilon, x}}{dt} &= b(X_t^{\varepsilon, x}, \varepsilon)dt + \sigma(X_t^{\varepsilon, x}, \varepsilon)dB_t, \\
X_0^{\varepsilon, x} &= x,
\end{align*} \hspace{1cm} (1.3)$$

where $\varepsilon$ is a small parameter, which will be defined precisely in the next section. In this case, the barrier option price (1.1) is characterized as a solution of the Cauchy-Dirichlet problem:

$$\begin{align*}
\frac{\partial}{\partial t} u^\varepsilon(t, x) + \mathcal{L}^\varepsilon u^\varepsilon(t, x) &= 0, \quad (t, x) \in [0, T) \times (L, \infty), \\
u^\varepsilon(T, x) &= f(x), \quad x > L, \\
u^\varepsilon(t, L) &= 0, \quad t \in [0, T],
\end{align*} \hspace{1cm} (1.4)$$

where the differential operator $\mathcal{L}^\varepsilon$ is determined by the diffusion coefficients $b$ and $\sigma$. Next, we introduce an asymptotic expansion formula:

$$u^\varepsilon(t, x) = u^0(t, x) + \varepsilon v_1^0(t, x) + \cdots + \varepsilon^{n-1} v_{n-1}^0(t, x) + O(\varepsilon^n), \hspace{1cm} (1.5)$$
where \( O \) denotes the Landau symbol. The function \( u^0(t, x) \) is the solution of (1.4) with \( \varepsilon = 0 \): if \( b(x, 0) \) and \( \sigma(x, 0) \) have some simple forms such as constants (as in the Black-Scholes model), we already know the closed form of \( u^0(t, x) \) and hence obtain the price. Then, we are able to get the approximate value for \( u^\varepsilon(t, x) \) through evaluation of the coefficient functions

\[ v_k^0(t, x), \ldots, v_{n-1}^0(t, x). \]

In fact, they are also characterized as the solution of a certain PDE with the Dirichlet condition. By formal asymptotic expansions, (1.5) above and (1.6) below,

\[
L^\varepsilon = L^0 + \varepsilon L_1^0 + \cdots + \varepsilon^{n-1} L_{n-1}^0 + \cdots,
\]

we can derive the following PDE which \( v_k^0(t, x) \) satisfies:

\[
\begin{align*}
\frac{\partial}{\partial t} v_k^0(t, x) + L^0 v_k^0(t, x) + g_k^0(t, x) &= 0, \quad (t, x) \in [0, T) \times (L, \infty), \\
v_k^0(T, x) &= 0, \quad x > L, \\
v_k^0(t, L) &= 0, \quad t \in [0, T],
\end{align*}
\]

(1.7)

where \( g_k^0(t, x) \) will be given explicitly in Section 3. Moreover, by applying the Feynman-Kac approach to the PDE (1.7), we obtain a semi-group representation of \( v_k^0 \). That is, for each

\[ k = 1, \ldots, n - 1, \]

\[
v_k^0(T - t, x) = \sum_{l=1}^{k} \sum_{(\beta^l)_1 \in \mathbb{N}^l, \sum_i \beta^i = k} \int_0^t \int_0^{t_1} \cdots \int_0^{t_{l-1}} P_{t-t_1}^D \tilde{L}^0_{\beta^1} P_{t_1-t_2}^D \tilde{L}^0_{\beta^2} \cdots P_{t_{l-1}-t}^D \tilde{L}^0_{\beta^l} P_{t}^D f(x) dt_1 \cdots dt_l,
\]

(1.8)

where \((P_t^D)\) is a semi-group defined in Section 3. We will justify the above argument in a mathematically rigorous manner in the sections 2 and 3.

The theory of the Cauchy-Dirichlet problem for this kind of the second order parabolic PDE is well understood for the case of bounded domains (see [5], [6] and [14] for instance). As for an unbounded domain case such as (1.4), [18] provides the existence and uniqueness results for a solution of the PDE and the Feynman-Kac type formula, the part of which will be cited as Theorem 1 in Section 2. However, some mathematical difficulty exists for applying the results of [18] to the PDE (1.7). More precisely, the function \( g_k^0(t, x) \) may be divergent at \( t = T \). Hence, in order to obtain an asymptotic expansion (1.5), we generalize the result of [18] and the argument of the Feynman–Kac representation. Furthermore, we derive a new representation (1.8) for \( v_k^0(t, x) \) by using the semi-group \((P_t^D)\). We notice that such a form is convenient for evaluation of \( v_k^0(t, x) \) in concrete examples.

We also apply our method to pricing a barrier option in a stochastic volatility model. Then, as an example of (1.8) we obtain a new approximation formula of the barrier option price \( C_{\text{Barrier}}^{SV, \varepsilon} \) under a stochastic volatility model as follows: for the initial value of the logarithmic underlying price \( x \), the maturity \( T \) and the lower barrier \( L \),

\[
C_{\text{Barrier}}^{SV, \varepsilon}(T, e^x) = \mathbb{E}[f(S_T^x)1_{\{\min_{0\leq t\leq T} S_t > L}\}]
\]

\[
\simeq P_T^D \hat{f}(x) + \varepsilon \int_0^T P_{T-r}^D \hat{L}_1^0 P_r^D f(x) dr,
\]

(1.9)
where \( (S_t^{ε,x})_t \) is the underlying asset price process, \( f \) is a payoff function and \( \tilde{f}(x) = f(e^x) \). Here, \( P^D_T \tilde{f}(x) \) is regarded as the down-and-out barrier option price in the Black-Scholes model. Moreover, we confirm practical validity of our method through a numerical example given in Section 4.

Finally, we remark that there exist the previous works on barrier option pricing such as [2], [3], [8], [9], which start with some specific models (e.g. Black-Scholes model or some type of fast mean-reversion model), and derive approximation formulas for discretely or continuously monitoring barrier option prices. Our approach is to firstly develop a general semi-group expansion scheme for the Cauchy-Dirichlet problem under multi-dimensional diffusion setting; then as an application, we provide a new approximation formula under a certain class of stochastic volatility model.

The organization of this paper is as follows: the next section prepares the existence and uniqueness result for the Cauchy-Dirichlet problem in the second-order parabolic PDE associated with barrier option pricing. Section 3 presents our main result for an asymptotic expansion of barrier option prices. Section 4 shows numerical examples under a stochastic volatility model. Section 5 concludes. Finally, Appendix A provides the proofs of the results in the main text, and Appendix B shows some generalization of Section 2 and Section 3.

2 Preparation

This section shows the existence and uniqueness result for the Cauchy-Dirichlet problem in the second-order parabolic PDE associated with the valuation of barrier options.

Suppose first that the underlying asset price is described by the following perturbed SDE:

\[
\begin{cases}
    dX_t^{ε,x} = b(X_t^{ε,x}, ε)dt + σ(X_t^{ε,x}, ε)dB_t, \\
    X_0^{ε,x} = x,
\end{cases}
\tag{2.1}
\]

where \( ε \) is a small parameter. Let \( b : \mathbb{R}^d \times I \rightarrow \mathbb{R}^d \) and \( σ : \mathbb{R}^d \times I \rightarrow \mathbb{R}^{d \otimes R^m} \) be Borel measurable functions \((d, m \in \mathbb{N})\) where \( I \) is an interval on \( \mathbb{R} \) including the origin 0 (for instance \( I = (−1, 1) \)). We consider the SDE (2.1) for any \( x \in \mathbb{R}^d \) and \( ε \in I \); in the condition [A] below, we will introduce the assumptions for existence and uniqueness of a solution of (2.1).

We are interested in evaluation of the following barrier option price: for a small \( ε \),

\[
w^ε(t, x) = E \left[ \exp \left( -\int_0^{T-t} c(X_r^{ε,x}, ε)dr \right) f(X_{T-t}^{ε,x})1_{\{τ_D(X^{ε,x}) ≥ T-t\}} \right], \quad (t, x) ∈ [0, T] \times \bar{D}
\tag{2.2}
\]

for Borel measurable functions \( f : \mathbb{R}^d \rightarrow \mathbb{R} \) and \( c : \mathbb{R}^d \times I \rightarrow \mathbb{R} \), a positive real number \( T > 0 \) and a domain \( D \subset \mathbb{R}^d \); \( \bar{D} \subset \mathbb{R}^d \) is the closure of \( D \) and \( τ_D(w), \ w ∈ C([0, T]; \mathbb{R}^d) \), stands for the first exit time from \( D \), that is

\[ τ_D(w) = \inf\{t ∈ [0, T]; w(t) ∉ D\}. \]

Let us define a second order differential operator \( \mathcal{L}^ε \) by

\[
\mathcal{L}^ε = \frac{1}{2} \sum_{i,j=1}^d a^{ij}(x, ε) \frac{∂^2}{∂x^i∂x^j} + \sum_{i=1}^d b^i(x, ε) \frac{∂}{∂x^i} - c(x, ε),
\]

where \( a^{ij} \) are Borel measurable functions from \( \mathbb{R}^d \times I \) to \( \mathbb{R}^{d \times d} \), \( b^i : \mathbb{R}^d \times I \rightarrow \mathbb{R}^d \), \( c : \mathbb{R}^d \times I \rightarrow \mathbb{R} \).
where \( a_{ij} = \sum_{k=1}^{d} \sigma^{ik} \sigma^{jk} \). We consider the following Cauchy-Dirichlet problem for a PDE of parabolic type:

\[
\begin{align*}
\frac{\partial}{\partial t} u^\varepsilon(t, x) + \mathcal{L}^\varepsilon u^\varepsilon(t, x) &= 0, \quad (t, x) \in (0, T) \times D, \\
u^\varepsilon(T, x) &= f(x), \quad x \in D, \\
u^\varepsilon(t, x) &= 0, \quad (t, x) \in [0, T] \times \partial D.
\end{align*}
\]

(2.3)

Now we introduce a series of the assumptions necessary for the existence and the uniqueness of the classical solution of (2.3).

[A] There is a positive constant \( A_1 \) such that

\[
|\sigma^{ij}(x, \varepsilon)|^2 + |b^i(x, \varepsilon)|^2 \leq A_1(1 + |x|^2), \quad x \in \mathbb{R}^d, \quad \varepsilon \in I, \quad i, j = 1, \ldots, d.
\]

Moreover, for each \( \varepsilon \in I \) it holds that \( \sigma^{ij}(\cdot, \varepsilon), b^i(\cdot, \varepsilon) \in \mathcal{L} \) for \( i, j = 1, \ldots, d \), where \( \mathcal{L} \) is the set of locally Lipschitz continuous functions defined on \( \mathbb{R}^d \):

\[
\mathcal{L} = \{ f \in C(\mathbb{R}^d; \mathbb{R}); \text{ for any compact set } K \subset \mathbb{R}^d, \exists C_K > 0 \text{ such that } |f(x) - f(y)| \leq C_K|x - y|, x, y \in K \}
\]

Remark 1. Note that under [A], the existence and uniqueness of a solution of (2.1) are guaranteed on any filtered probability space equipped with a standard \( d \)-dimensional Brownian motion, and Corollary 2.5.12 in [11] and Lemma 3.2.6 in [16] imply

\[
E[\sup_{0 \leq t \leq \tau_D} |X^\varepsilon_{\tau_D} - x|^2] \leq C_l t^{l-1}(1 + |x|^{2m}), \quad (t, x) \in [0, T] \times \mathbb{R}^d, \quad l \in \mathbb{N}
\]

(2.4)

for some \( C_l > 0 \) which depends only on \( l \) and \( A_1 \). Moreover, \( (X^\varepsilon_{\tau_D})_r \) has the strong Markov property.

[B] The function \( f(x) \) is continuous on \( \bar{D} \) and there are \( C_f > 0 \) and \( m \in \mathbb{N} \) such that

\[
|f(x)| \leq C_f(1 + |x|^{2m}), \quad x \in \mathbb{R}^d.
\]

Moreover, \( f(x) = 0 \) on \( \mathbb{R}^d \setminus D \).

Remark 2. The assumption [B] guarantees the continuity of a solution of (2.3) on the so called parabolic boundary \( \Sigma = \partial D \times [0, T) \cup \bar{D} \times \{T\} \), in addition to the continuity and polynomial growth of \( f \).

[C] \( c(x, \varepsilon) \) is non-negative (i.e. \( c(x, \varepsilon) \geq 0 \)). Moreover, for each \( \varepsilon \in I \), it holds that \( c(\cdot, \varepsilon) \in \mathcal{L} \).

[D] The boundary \( \partial D \) has the outside strong sphere property, that is, for each \( x \in \partial D \) there is a closed ball \( E \) such that \( E \cap D = \phi \) and \( E \cap \bar{D} = \{x\} \).

Remark 3. The assumption [D] provides the regularity of each point in \( \partial D \). (c.f.[5]) Also, [18] points out that [D] with the ellipticity of the matrix \( (a^{ij}(x, \varepsilon))_{ij} \) in \( [E] \) below gives

\[
P(\tau_D(X^\varepsilon, x) = \tau_D(X^\varepsilon, x)) = 1.
\]
The matrix \((a_{ij}(x, \varepsilon))_{ij}\) is locally elliptic in the sense that for each \(\varepsilon \in I\) and compact set \(K \subset \mathbb{R}^d\) there is a positive number \(\mu_{\varepsilon,K}\) such that
\[
\sum_{i,j=1}^{d} a_{ij}(x, \varepsilon) \xi_i \xi_j \geq \mu_{\varepsilon,K} |\xi|^2
\]
for any \(x \in K\) and \(\xi \in \mathbb{R}^d\).

**Remark 4.** Note that although the condition [E] (local ellipticity) is necessary for the existence of classical solution of our PDE (See Remark 2.2 in [18]), the assumption can be removed through consideration of viscosity solutions rather than classical solutions by applying Theorem 8.2 in [1] and Theorems 4.4.3 and 7.7.2 in [16]. Note that we need the additional assumption such that \(I \subset [0, \infty)\) by technical reason in this case.

Under the assumptions [A]–[E] above, we have the following existence and uniqueness result due to Theorem 3.1 in [18].

**Theorem 1.** Assume [A]–[E]. For each \(\varepsilon \in I\), \(u^\varepsilon(t, x)\) is a (classical) solution of (2.3) and
\[
\sup_{(t,y) \in [0,T] \times D} |u^\varepsilon(t, x)|/(1 + |x|^{2m}) < \infty. \tag{2.5}
\]
Moreover, if \(w^\varepsilon(t, x)\) is also a solution of (2.3) satisfying the growth condition
\[
\sup_{(t,y) \in [0,T] \times D} |w^\varepsilon(t, x)|/(1 + |x|^{2m'}) < \infty
\]
for some \(m' \in \mathbb{N}\), then \(u^\varepsilon = w^\varepsilon\).

### 3 Asymptotic Expansion of Barrier Option Price

Our purpose is to present an asymptotic expansion of the barrier option price \(u^\varepsilon(t, x)\):
\[
u^\varepsilon(t, x) = u_0^0(t, x) + \varepsilon v_1^0(t, x) + \cdots + \varepsilon^{n-1} v_{n-1}^0(t, x) + O(\varepsilon^n), \quad \varepsilon \to 0. \tag{3.1}
\]
Here, the coefficient function \(v_k^0(t, x), k = 1, \ldots, n - 1\) are (formally) given as the solution of
\[
\begin{align*}
\frac{\partial}{\partial t} v_k^0(t, x) + L_0 v_k^0(t, x) + g_k^0(t, x) &= 0, \quad (t, x) \in [0, T) \times D, \\
v_k^0(T, x) &= 0, \quad x \in D, \\
v_k^0(t, x) &= 0, \quad (t, x) \in [0, T] \times \partial D, \tag{3.2}
\end{align*}
\]
where \(g_k^0(t, x)\) is given inductively by
\[
g_k^0(t, x) = \mathcal{L}_k^0 u_0^0(t, x) + \sum_{l=1}^{k-1} \mathcal{L}_{k-l}^0 v_l^0(t, x), \tag{3.3}
\]
where \(\mathcal{L}_k^0\) is defined as follows:
\[
\mathcal{L}_k^0 = \frac{1}{k!} \left\{ \frac{d}{dx} \sum_{i,j=1}^{d} \frac{\partial^2 a_{ij}}{\partial x^i \partial x^j}(x, 0) \right\} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left( x, 0, \frac{\partial}{\partial x^i} \right) - \frac{\partial}{\partial x^k} \left( x, 0, \frac{\partial}{\partial x^k} \right). \tag{3.4}
\]
To study the asymptotic expansion, we put the following assumptions in addition to [A]–[E]. Firstly, by the next condition we can properly define \(\mathcal{L}_k^0, k \in \mathbb{N}\) in (3.4) above.
Let \( n \in \mathbb{N} \). The functions \( a^{ij}(x, \varepsilon), b^i(x, \varepsilon) \) and \( c(x, \varepsilon) \) are \( n \)-times continuously differentiable in \( \varepsilon \). Furthermore, each of derivatives \( \partial^k a^{ij}/\partial \varepsilon^k, \partial^k b^i/\partial \varepsilon^k, \partial^k c/\partial \varepsilon^k, k = 1, \ldots, n - 1, \) has a polynomial growth rate in \( x \in \mathbb{R}^d \) uniformly in \( \varepsilon \in I \).

To state the existence of the functions \( v_0^i(t, x) \) in the asymptotic expansion (3.1), we first prepare the following set.

**Definition 1.** The set \( \mathcal{H}^{m,p} \) of \( g \in C([0, T] \times \bar{D}) \) is defined to satisfy the following condition: There is some \( M^g \in C([0, T]) \cap L^p([0, T], dt) \) such that

\[
|g(t, x)| \leq M^g(t)(1 + |x|^{2m}), \quad t \in [0, T), \quad x, y \in \bar{D}.
\]

Given this definition of the set \( \mathcal{H}^{m,p} \), we put the next condition on \( u^0 \).

**[G]** \( u^0 \in \mathcal{G}^m \), where

\[
\mathcal{G}^m = \left\{ g \in C^{1,2}([0, T] \times D) \cap C([0, T] \times \bar{D}) ; \quad \frac{\partial g}{\partial x^i} \in \mathcal{H}^{m,2}, \quad \frac{\partial^2 g}{\partial x^i \partial x^j} \in \mathcal{H}^{m,1}, \quad i, j = 1, \ldots, d \right\}.
\]

Now we examine the conditions necessary for the classical solution to the PDE (3.2). Let us start with the case of \( k = 1 \). By the assumption \([G]\), we have \( g_0^i \in \mathcal{H}^{m,1} \) for some \( m \in \mathbb{N} \) by the definition of \( g_k^0 \) with \( k = 1 \) in (3.3). Thus we can define

\[
v_1^0(t, x) = E \left[ \int_0^{(T-t)\wedge \tau_D} \exp \left( - \int_0^r c(X^0_v, 0) dv \right) g_1^0(t + r, X^0_r) dr \right].
\]

Therefore, if we assume that \( v_1^0 \in C^{1,2}([0, T] \times D) \) we can show that \( v_1^0 \) is the solution of (3.2) with \( k = 1 \), that is, we can confirm that

\[
\frac{\partial}{\partial t} v_1^0(t, x) + \mathcal{L}^0 v_1^0(t, x) + g_1^0(t, x) = 0.
\]

Note that the relations \( v_1^0(T, \cdot) = 0 \) and \( v_1^0 = 0 \) on \([0, T] \times \partial D\) are obvious.

Next, let us give some comments on the smoothness of \( v_1^0 \). In many cases as in the Black–Scholes model (see (4.10) in Section 4), we can rewrite (3.6) as

\[
v_1^0(t, x) = \int_0^{T-t} \int_D g_1^0(t + r, y)p(r, x, y)dydr
\]

for some \( p(r, x, y) \). Thus, if \( p \) has a “good” smoothness property, the smoothness of \( v_1^0 \) also holds such as

\[
\frac{\partial}{\partial t} v_1^0(t, x) = - \lim_{s \to T} \int_D g_1^0(s, y)p(s - t, x, y)dy + \int_0^t \int_D \frac{\partial}{\partial t} g_1^0(t + r, y)p(r, x, y)dr,
\]

\[
\frac{\partial}{\partial x^i} v_1^0(t, x) = \int_0^t \int_D g_1^0(t + r, y) \frac{\partial}{\partial x^i} p(r, x, y)dr,
\]

for some \( p(r, x, y) \).
\[
\frac{\partial^2}{\partial x^2 \partial x^3} v_0(t, x) = \int_0^t \int_D g_1^0(t + r, y) \frac{\partial^2}{\partial x^2 \partial x^3} p(r, x, y) dr.
\]

Moreover, if \( v_0^0 \) is in \( G^{m_1} \) for some \( m_1 \in \mathbb{N} \), we also have \( g^0_k \in \mathcal{H}^{\tilde{m}_1, 1} \) for some \( \tilde{m}_1 \in \mathbb{N} \) by the definition of \( g^0_k \) with \( k = 2 \) in (3.3). Then, we can define \( v_0^0 \) similarly as \( v_1^0 \). Furthermore, under some suitable smoothness conditions for \( v_0^0 \), which may be given by the smoothness property of \( p(r, x, y) \), we are able to show that \( v_0^0 \) is the classical solution of (3.2) with \( k = 2 \).

Thus, the observation above leads us to our final assumption.

\[ [H] \quad \text{It holds that } v^0_k \in G^{m_n}, \ k = 1, \ldots, n - 1 \text{ for some } m_n \in \mathbb{N}, \text{ where} \]

\[
v^0_k(t, x) = E \left[ \int_0^{(T-t)\wedge T_{\partial D}(X_0^0,x)} \exp \left( - \int_0^r c(X_v^0, x, 0) dv \right) g^0_k(t + r, X_v^0, x) dr \right]. \tag{3.7}
\]

Then, we can show the next result. The proof is given in Section 6.1 of Appendix.

**Theorem 2.** Assume \([A]−[H]\). Then, for each \( k = 1, \ldots, n - 1 \), \( v^0_k \) is the classical solution of (3.2) and satisfies

\[
|v^0_k(t, x)| \leq C_k(1 + |x|^{2m_k}), \quad (t, x) \in [0, T] \times \mathbb{R}^d \tag{3.8}
\]

for some \( C_k, m_k > 0 \).

Note that the uniqueness of the solutions of (3.2) follows from the same arguments as in the proof of Theorem 5.7.6 in [10]. That is, we obtain the next proposition.

**Proposition 1.** For any function \( g \) which has a polynomial growth rate in \( x \) uniformly in \( t \), a classical solution of (3.2) is unique in the following sense: if \( v \) and \( w \) are classical solutions of (3.2) and \( |v(t, x)| + |w(t, x)| \leq C(1 + |x|^{2m}) \) for some \( C, m > 0 \), then \( v = w \).

Now, we are able to state our first main result on the asymptotic expansion. The proof is given in Section 6.2 of Appendix.

**Theorem 3.** Assume \([A]−[H]\). There are positive constants \( C_n \) and \( \tilde{m}_n \) which are independent of \( \varepsilon \) such that

\[
\left| u^\varepsilon(t, x) - (u^0(t, x) + \sum_{k=1}^{n-1} \varepsilon^k v^0_k(t, x)) \right| \leq C_n(1 + |x|^{2\tilde{m}_n})\varepsilon^n, \quad (t, x) \in [0, T] \times \tilde{D}.
\]

Next, we construct a semi-group corresponding to \( (X^0_{\varepsilon, t}) \) (that is, \( (X^\varepsilon_{t, x}) \), with \( \varepsilon = 0 \)) and \( D \). Then, based on this semi-group we can obtain more explicit representation for the coefficient function \( v^0_k(t, x) \) than the right hand side of (3.7).

Let \( C_0^D(D) \) be the set of bounded continuous functions \( f : \tilde{D} \rightarrow \mathbb{R} \) such that \( f(x) = 0 \) on \( \partial D \). Obviously, \( C_0^D(D) \) equipped with the sup-norm becomes a Banach space.

For \( t \in [0, T] \) and \( f \in C_0^D(D) \), we define \( P^D_t f : \tilde{D} \rightarrow \mathbb{R} \) by

\[
P^D_t f(x) = E \left[ \exp \left( - \int_0^t c(X_v^0, x, 0) dv \right) f(X^0_{t, x}) 1_{\{r_D(X^0_{t, x}) \geq t\}} \right], \tag{3.9}
\]

where \( c(x, 0) \) is non-negative. We notice that \( P^D_t f(x) \) is equal to \( u^0(T - t, x) \) with the payoff function \( f \). Then, we have the following result:
Proposition 2. Under the assumptions [A]–[E], the mapping \( P^D_t : C^0_b(\bar{D}) \rightarrow C^0_b(\bar{D}) \) is well-defined and \( (P^D_t)_{0 \leq t \leq T} \) is a contraction semi-group.

Proof. Let \( f \in C^0_b(\bar{D}) \). The relations \( P^D_0 f = f \), \( P^D_t f|_{\partial D} = 0 \) and \( \sup_{\bar{D}} |P^D_t f| \leq \sup_{\bar{D}} |f| \) are obvious. The continuity of \( P^D_t f \) is by Lemma 4.3 in [18]. The semi-group property is verified by a straightforward calculation. ■

Remark 5. Note that \( (P^D_t) \) also has the semi-group property on the set \( C^0_p(\bar{D}) \) of continuous functions \( f \), each of which has a polynomial growth rate and satisfies \( f(x) = 0 \) on \( \partial D \).

Finally, we show our second main result on the semi-group representation of the coefficient function \( v^0_k \) in the expansion, whose proof is given in Section 6.3 in Appendix.

Theorem 4. Under Assumptions [A]–[H], for each \( k = 1, \ldots, n-1 \)
\[
v^0_k(T - t, x) = \sum_{l=1}^{k} \sum_{(\beta')_{i=1}^{l} \in \mathbb{N}} \int_0^t \cdots \int_0^{t_{l-1}} P^D_{t-l} \cdots P^D_{t-1} \cdots P^D_{t} f(x) dt_l \cdots dt_t.
\]

(3.10)

4 Application to Barrier Option Pricing in Stochastic Volatility Environment

This section demonstrates the effectiveness of our method in stochastic volatility environment: Section 4.1 derives concrete approximation formulas, and Section 4.2 shows numerical examples.

4.1 Approximation of Barrier Option Prices in a Stochastic Volatility Model

We consider the following stochastic volatility model.
\[
\begin{align*}
dS^\varepsilon_t &= (c - q)S^\varepsilon_t dt + \sigma^\varepsilon_t S^\varepsilon_t dB^1_t, \quad S^\varepsilon_0 = S, \\
d\sigma^\varepsilon_t &= \varepsilon(\theta - \sigma^\varepsilon_t) dt + \varepsilon\nu \sigma^\varepsilon_t (\rho dB^1_t + \sqrt{1 - \rho^2} dB^2_t), \quad \sigma^\varepsilon_0 = \sigma,
\end{align*}
\]

where \( c, q > 0, \varepsilon \in [0,1), \lambda, \theta, \nu > 0, \rho \in [-1,1] \) and \( B = (B^1, B^2) \) is a two dimensional Brownian motion. Here \( c \) and \( q \) represent a domestic interest rate and a foreign interest rate, respectively when we consider the currency options. Clearly, applying Itô’s formula, we have its logarithmic process:
\[
\begin{align*}
dx^\varepsilon_t &= (c - q - \frac{1}{2}(\sigma^\varepsilon_t)^2) dt + \sigma^\varepsilon_t dB^1_t, \quad x^\varepsilon_0 = x = \log S, \\
d\sigma^\varepsilon_t &= \varepsilon(\theta - \sigma^\varepsilon_t) dt + \varepsilon\nu \sigma^\varepsilon_t (\rho dB^1_t + \sqrt{1 - \rho^2} dB^2_t), \quad \sigma^\varepsilon_0 = \sigma.
\end{align*}
\]

(4.1)
Also, its generator is expressed as

$$\mathcal{L}^\varepsilon = \left( c - q - \frac{1}{2} \sigma^2 \right) \frac{\partial}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2}{\partial x^2} + \varepsilon \rho \nu \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} + \varepsilon \lambda(\theta - \sigma) \frac{\partial}{\partial \sigma} + \varepsilon^2 \frac{1}{2} \nu^2 \sigma^2 \frac{\partial^2}{\partial \sigma^2}. \quad (4.3)$$

In this case, $\mathcal{L}_1^0$ which is defined by (3.4) with $k = 1$ is given as

$$\hat{\mathcal{L}}_1^0 = \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} + \lambda(\theta - \sigma) \frac{\partial}{\partial \sigma}. \quad (4.4)$$

We will apply the asymptotic expansion in the previous section to (4.2) and give an approximation formula for a barrier option price, which is given under a risk-neutral probability measure as

$$C_{\text{Barrier}}(T - t, e^x) = E \left[ e^{-c(T-t)} f(S_{t}^{x,e^x}) 1_{\{T - t\}} \right],$$

where $f$ stands for a payoff function and $L(< S)$ is a barrier price.

Then, $u^\varepsilon(t, x) = C_{\text{Barrier}}^{SV,\varepsilon}(T - t, e^x)$ satisfies the following PDE:

$$\begin{cases} \left( \frac{\partial}{\partial t} + \mathcal{L}^\varepsilon - c \right) u^\varepsilon(t, x) = 0, \quad (t, x) \in (0, T] \times D, \\ u^\varepsilon(T, x) = \bar{f}(x), \quad x \in \bar{D}, \\ u^\varepsilon(t, l) = 0, \quad t \in [0, T]. \end{cases} \quad (4.5)$$

where $\bar{f}(x) = \max\{e^x - K, 0\}$, $D = (l, \infty)$ and $l = \log L$. We obtain the 0-th order $u^0$ as

$$u^0(t, x) = P_{T-t}^D \bar{f}(x) = E\left[ e^{-c(T-t)} \bar{f}(X_{T-t}^x) 1_{\{\tau_{D} < \infty\}} \right]. \quad (4.6)$$

Remark 6. $u^0$ satisfies the PDE (4.5) with $\varepsilon = 0$. Although the condition $[E]$ in Section 2 does not seem to be satisfied in this case, the volatility process $(\sigma^2_t)$ becomes a constant $\sigma > 0$, and so (4.2) is reduced to a one-dimensional SDE. Then, (4.5) with $\varepsilon = 0$ becomes a non-degenerating PDE with fixed $\sigma$. Therefore, we need not take care of the lack of the condition $[E]$ in this example.

Setting $\alpha = c - q$, we note that $P_{T-t}^D \bar{f}(x) = C_{\text{Barrier}}^{BS}(T - t, e^x, \alpha, \sigma, L)$ is the price of the down-and-out barrier call option under the Black-Scholes model:

$$C_{\text{Barrier}}^{BS}(T - t, e^x, \alpha, \sigma, L) = C_{\text{Barrier}}^{BS}(T - t, e^x, \alpha, \sigma) - \left( e^x \right)^{1 - \frac{2\alpha}{L}} C_{\text{Barrier}}^{BS} \left( T - t, \frac{L^2}{e^x}, \alpha, \sigma \right). \quad (4.7)$$

Here, we recall that the price of the plain vanilla option under the Black-Scholes model is given as

$$C_{\text{BS}}(T - t, e^x, \alpha, \sigma) = e^{-q(T-t)} e^x N(d_1(T - t, x, \alpha)) - e^{-c(T-t)} KN(d_2(T - t, x, \alpha)), \quad (4.8)$$

where

$$d_1(t, x, \alpha) = \frac{x - \log K + \alpha t}{\sigma \sqrt{t}} + \frac{1}{2} \sigma \sqrt{t},$$

$$d_2(t, x, \alpha) = \frac{x - \log K - \alpha t}{\sigma \sqrt{t}} - \frac{1}{2} \sigma \sqrt{t}.$$
\[d_2(t, x, \alpha) = d_1(t, x, \alpha) - \sigma \sqrt{t}\]

\[N(x) = \int_{-\infty}^{x} n(y) dy,\]

\[n(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}.\]

Note also that

\[P(\tau_D(X_0^t) \geq t| X_0^t) = 1 - \exp \left(-\frac{2(x - l)(X_0^t - l)}{\sigma^2 t}\right)\] on \(\{X_0^t > l\}\).

Therefore, for \(g \in C^0_p(\bar{D})\) we have

\[P_D^D g(x) = E[P(\tau_D(X_0^t) \geq t| X_0^t)e^{-ct}g(X_0^t)1\{X_0^t \geq l\}] = \int_{l}^{\infty} e^{-ct}g(y)p(t, x, y)dy, \quad (4.9)\]

where

\[p(t, x, y) = \frac{1}{\sqrt{2\pi}\sigma^2 t} e^{-\frac{(x-l)^2}{\sigma^2 t}} e^{-\frac{(y-x-l)^2}{2\sigma^2 t}},\]

\[\mu = \alpha - \sigma^2/2 = (c - q - \sigma^2/2).\]

Then, we show the following main result in this section.

**Theorem 5.** We obtain an approximation formula for the down-and-out barrier call option under the stochastic volatility model (4.1):

\[C_{SV,\varepsilon}^{\text{Barrier}}(T, e^x) = C_{BS}^{\text{Barrier}}(T, e^x, \alpha, \sigma, L) + \varepsilon v_1^0(0, x) + O(\varepsilon^2), \quad (4.11)\]

where

\[v_1^0(0, x) = e^{-cT} \int_{0}^{T} \int_{l}^{\infty} 1 \left(\frac{1}{\sqrt{2\pi}\sigma^2 s} e^{-\frac{(x-l)^2}{\sigma^2 s}} e^{-\frac{(y-x-l)^2}{2\sigma^2 s}} \vartheta(s, y) dy ds, \quad (4.12)\]

\[\vartheta(t, x) = e^{\alpha(T-t)} \rho \nu \sigma e^x n(d_1(T-t, x, \alpha)) (-d_2(T-t, x, \alpha)) + 2e^{\alpha(T-t)} \rho \nu \alpha \left(\frac{e^x}{L}\right)^{\frac{-2\alpha}{\sigma^2}} \ln(c_1(T-t, x, \alpha)) \sqrt{T-t}\]

\[= e^{\alpha(T-t)} \rho \nu \sigma e^x n(d_1(T-t, x, \alpha)) (-d_2(T-t, x, \alpha)) + 2e^{\alpha(T-t)} \rho \nu \alpha \left(\frac{e^x}{L}\right)^{\frac{-2\alpha}{\sigma^2}} \ln(c_1(T-t, x, \alpha)) \sqrt{T-t}\]

\[= e^{\alpha(T-t)} \rho \nu \sigma e^x \left(\frac{e^x}{L}\right)^{\frac{-2\alpha}{\sigma^2}} \ln(c_1(T-t, x, \alpha)) c_1(T-t, x, \alpha)\]

\[\times \left\{ C_{BS} \left(T-t, \frac{L^2}{e^x}, \alpha, \sigma \right) \left[1 + (x - \log L) \left(1 - \frac{2\alpha}{\sigma^2}\right)\right] - (x - \log L) e^{-q(T-t)} \frac{L^2}{e^x} N(c_1(T-t, x, \alpha))\right\}\]
+ \lambda(\theta - \sigma)e^{\alpha(T-t)}e^x_n(d_1(T - t, x, \alpha))\sqrt{T-t} \\
- \lambda(\theta - \sigma) \left( \frac{e^x}{L} \right)^{-\frac{2x}{\sigma^2}} e^{\alpha(T-t)\text{L}(c_1(T - t, x, \alpha))\sqrt{T-t}} \\
- e^{c(T-t)}\lambda(\theta - \sigma) \frac{4\alpha}{\sigma^3} \left( \log \frac{e^x}{L} \right) \left( \frac{e^x}{L} \right)^{1-\frac{2x}{\sigma^2}} C^{\text{BS}} \left( T - t, \frac{L^2}{e^x}, \alpha, \sigma \right), \quad (4.13)

and

\begin{align*}
c_1(t, x, \alpha) = \frac{2t - x - \log K + \alpha t}{\sigma \sqrt{t}} + \frac{1}{2} \sigma \sqrt{t}.
\end{align*}

**Proof.** Firstly, note that when \( k = 1 \) in Theorem 4, we have

\begin{align*}
v_1^0(T - t, x) = \int_0^t P^D_{1-s} \tilde{\mathcal{L}}_1^0 P^D_r f(x)dr.
\end{align*}

Thus, we see the expansion

\begin{align*}
C^{\text{SV,}\varepsilon}_{\text{Barrier}}(T - t, e^x) = C^{\text{BS}}_{\text{Barrier}}(T - t, e^x, \alpha, \sigma, L) + \varepsilon \int_0^{T-t} P^D_s \tilde{\mathcal{L}}_1^0 P^D_{T-s} \bar{f}(x)ds + O(\varepsilon^2). \quad (4.14)
\end{align*}

The first-order approximation term \( v_1^0(t, x) = \int_0^{T-t} P^D_s \tilde{\mathcal{L}}_1^0 P^D_{T-s} \bar{f}(x)ds \) is given by

\begin{align*}
v_1^0(t, x) = \int_0^{T-t} e^{-c_s} P^D_s \tilde{\mathcal{L}}_1^0 e^{-c(T-t-s)} P^D_{T-s} \bar{f}(x)ds \\
= e^{-c(T-t)} \int_0^{T-t} P^D_s \tilde{\mathcal{L}}_1^0 P^D_{T-s} \bar{f}(x)ds,
\end{align*}

where \( \tilde{P}^D_t \) is defined by

\begin{align*}
\tilde{P}^D_t \bar{f}(x) = \int_1^\infty \frac{1}{\sqrt{2\pi\sigma^2s}} \left( 1 - e^{-\frac{2(x - \bar{y})\sqrt{\sigma^2s}}{\sigma^2}} \right) e^{-\frac{(y-x-(\sigma\sqrt{\sigma^2s})^2)}{2\sigma^2s}} \bar{f}(y)dy.
\end{align*}

Define \( \vartheta(t, x) \) as

\begin{align*}
\vartheta(t, x) = \tilde{\mathcal{L}}_1^0 \tilde{P}^D_{T-t} f(e^x) \\
= e^{c(T-t)} \rho \nu \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} C^{\text{BS}}_{\text{Barrier}}(T - t, e^x, \alpha, \sigma, L) + e^{c(T-t)} \lambda(\theta - \sigma) \frac{\partial}{\partial \sigma} C^{\text{BS}}_{\text{Barrier}}(T - t, e^x, \alpha, \sigma, L).
\end{align*}

A straightforward calculation shows that the above function agrees with the right-hand side of (4.13). Then we get the assertion. \( \blacksquare \)

Remark that through numerical integrations with respect to time \( s \) and space \( y \) in (4.12), we easily obtain the first order approximation of the down-and-out option prices.

Next, as a special case of (4.1) we consider the following stochastic volatility model with no drifts:

\begin{align*}
dS^\varnothing_t = \sigma^\varnothing_t S^\varnothing_t dB^1_t, \quad S^\varnothing_0 = S > 0, \quad (4.15)
\end{align*}
\[
d\sigma_t^\varepsilon = \varepsilon \nu \sigma_t^\varepsilon (\rho dB_1^t + \sqrt{1-\rho^2} dB_2^t), \quad \sigma_0^\varepsilon = \sigma > 0.
\]

where \(\varepsilon \in [0, 1]\), \(\rho \in [-1, 1]\) and \(B = (B^1, B^2)\) is a two dimensional Brownian motion. In this case, we can provide a simpler approximation formula than in Theorem 5.

By Itô’s formula, the following logarithmic model is obtained.

\[
dX_t^\varepsilon = -\frac{1}{2} (\sigma_t^\varepsilon)^2 dt + \sigma_t^\varepsilon dB_1^t, \quad X_0^\varepsilon = x = \log S,
\]

\[
d\sigma_t^\varepsilon = \varepsilon \nu \sigma_t^\varepsilon (\rho dB_1^t + \sqrt{1-\rho^2} dB_2^t), \quad \sigma_0^\varepsilon = \sigma.
\]

Again, the barrier option price is given by

\[
C_{SV,\varepsilon}^{\text{Barrier}}(T, e^x) = E \left[ f(S_T^{\varepsilon})1_{\{\min_{0 \leq u \leq T} S_u > L\}} \right],
\]

where \(f\) stands for a payoff function and \(L(< S)\) is a barrier price.

The differentiation operators \(L^\varepsilon, L_1^0\) and the PDE are same as (4.3)–(4.5) with \(c = q = 0\) and \(\lambda = 0\). Also, the barrier option price in the Black-Scholes model coincides with (4.7) with no drift, that is,

\[
C_{BS}^{\text{Barrier}}(T, S) = C_{BS}(T, S) - \left( \frac{S}{L} \right) C_{BS} \left( T, \frac{L^2}{S} \right),
\]

where \(C_{BS}(T, S)\) is the driftless Black-Scholes formula of the European call option given by

\[
C_{BS}(T, S) = SN(d_1(T, \log S)) - KN(d_2(T, \log S))
\]

with

\[
d_1(t, x) = \frac{x - \log K + \sigma^2 t/2}{\sigma \sqrt{t}}, \quad d_2(t, x) = d_1(t, x) - \sigma \sqrt{t}.
\]

Then, we reach the following expansion formula which only needs 1-dimensional numerical integration.

**Theorem 6.** \(C_{SV,\varepsilon}^{\text{Barrier}}(T, e^x) = C_{BS}^{\text{Barrier}}(T, e^x) + \varepsilon v_0^1(0, x) + O(\varepsilon^2)\), where

\[
v_0^1(0, x) = -\frac{1}{2} T \nu \rho \sigma \left\{ e^x n(d_1(T, x))d_2(T, x) + Ln(c_1(T, x))c_1(T, x) \right\} + \frac{\nu \rho L(x-t) \log(L/K)}{2\pi \sigma} \int_0^T \frac{(T-s)^{1/2}}{s^{3/2}} \exp \left( -\frac{c_2(T-s, L/K) + c_2(s, L/e^x)}{2} \right) ds, \]

\[
c_1(t, x) = \frac{\log(L^2/e^x K) + \sigma^2 t/2}{\sigma \sqrt{t}}, \quad c_2(t, y) = \left( \frac{\log y + \sigma^2 t/2}{\sigma \sqrt{t}} \right)^2.
\]

**Proof.** See Appendix 6.4. \(\blacksquare\)
4.2 Numerical Example

Finally, applying the our approximation formulas in Theorem 5 and Theorem 6, we present numerical experiments for European down-and-out barrier call prices. First, let us denote $u_0 = C_{\text{Barrier}}^{BS}(T, S)$ and $v_0^1 = v_0^1(0, \log S)$. Then, we see

$$C_{\text{Barrier}}^{SV,\varepsilon}(T, S) \simeq u_0 + \varepsilon v_0^1.$$

In the following we report the results of the numerical experiments, where the numbers in the parentheses show the error rates (%) relative to the benchmark prices of $C_{\text{Barrier}}^{SV,\varepsilon}(T, S)$; they are computed by Monte-Carlo simulations with 100,000 time steps and 1,000,000 trials. We check the accuracy of our approximations by changing the model parameters. Case 1–6 show the results for the stochastic volatility model with drifts of the underlying price process or/and the volatility process (4.1), while Case 7 shows the result for the stochastic volatility model with no drifts (4.15). There, we apply the formula in Theorem 5 to Case 1–6 and the formula in Theorem 6 to Case 7, respectively.

Apparently, our approximation formula $u_0 + \varepsilon v_0^1$ improves the accuracy against $C_{\text{Barrier}}^{SV,\varepsilon}(T, S)$, and it is observed that $\varepsilon v_0^1$ accurately compensates for the difference between $C_{\text{Barrier}}^{SV,\varepsilon}(T, S)$ and $C_{\text{Barrier}}^{BS}(T, S)$, which confirms the validity of our method.

1. 

$$S = 100, \quad \sigma = 0.15, \quad c = 0.01, \quad q = 0.0, \quad \varepsilon \nu = 0.2, \quad \rho = -0.5,$$

$$\varepsilon \lambda = 0.00, \quad \theta = 0.00, \quad L = 95, \quad T = 0.5, \quad K = 100, \quad 102, \quad 105.$$

| Strike | Benchmark $u_0$ | Our Approximation $(u_0^u + \varepsilon v_0^1)$ | Barrier Black-Scholes $u_0^u$ |
|--------|----------------|--------------------------------|----------------------------|
| 100    | 3.468          | 3.466 (-0.05%)                 | 3.495 (0.80%)              |
| 102    | 2.822          | 2.822 (0.00%)                 | 2.866 (1.57%)              |
| 105    | 1.986          | 1.986 (0.01%)                 | 2.052 (3.36%)              |

2. 

$$S = 100, \quad \sigma = 0.15, \quad c = 0.05, \quad q = 0.0, \quad \varepsilon \nu = 0.35, \quad \rho = -0.7,$$

$$\varepsilon \lambda = 0.00, \quad \theta = 0.00, \quad L = 95, \quad T = 0.5, \quad K = 100, \quad 102, \quad 105.$$

3. 

$$S = 100, \quad \sigma = 0.15, \quad c = 0.05, \quad q = 0.0, \quad \varepsilon \nu = 0.35, \quad \rho = -0.7,$$

$$\varepsilon \lambda = 0.00, \quad \theta = 0.00, \quad L = 95, \quad T = 0.5, \quad K = 100, \quad 102, \quad 105.$$
Table 2: Down-and-Out Barrier Option

| Strike | Benchmark | Our Approximation ($u^0 + \varepsilon v^1_0$) | Barrier Black-Scholes ($u^0$) |
|--------|-----------|---------------------------------|-------------------------------|
| 100    | 3.421     | 3.423 (0.07%)                   | 3.495 (2.18%)                |
| 102    | 2.753     | 2.757 (0.18%)                   | 2.866 (4.13%)                |
| 105    | 1.885     | 1.890 (0.23%)                   | 2.052 (8.88%)                |

Table 3: Down-and-Out Barrier Option

| Strike | Benchmark | Our Approximation ($u^0 + \varepsilon v^1_0$) | Barrier Black-Scholes ($u^0$) |
|--------|-----------|---------------------------------|-------------------------------|
| 100    | 4.352     | 4.349 (-0.07%)                   | 4.399 (1.06%)                |
| 102    | 3.585     | 3.586 (0.02%)                   | 3.665 (2.24%)                |
| 105    | 2.560     | 2.563 (0.11%)                   | 2.696 (5.31%)                |

4.

\[ S = 100, \sigma = 0.15, c = 0.05, q = 0.1, \varepsilon = 0.2, \rho = -0.5, \varepsilon \lambda = 0.00, \theta = 0.00, L = 95, T = 0.5, K = 100, 102, 105. \]

Table 4: Down-and-Out Barrier Option

| Strike | Benchmark | Our Approximation ($u^0 + \varepsilon v^1_0$) | Barrier Black-Scholes ($u^0$) |
|--------|-----------|---------------------------------|-------------------------------|
| 100    | 2.231     | 2.224 (-0.31%)                   | 2.268 (1.64%)                |
| 102    | 1.758     | 1.754 (-0.27%)                   | 1.812 (3.02%)                |
| 105    | 1.172     | 1.168 (-0.31%)                   | 1.243 (6.05%)                |

5.

\[ S = 100, \sigma = 0.15, c = 0.01, q = 0.0, \varepsilon = 0.2, \rho = -0.5, \varepsilon \lambda = 0.2, \theta = 0.25, L = 95, T = 0.5, K = 100, 102, 105. \]

6.

\[ S = 100, \sigma = 0.15, c = 0.01, q = 0.0, \varepsilon = 0.2, \rho = -0.5, \varepsilon \lambda = 0.5, \theta = 0.25, L = 95, T = 0.5, K = 100, 102, 105. \]

7.

\[ S = 100, \sigma = 0.15, c = 0.0, q = 0.0, \varepsilon = 0.2, \rho = -0.5, \varepsilon \lambda = 0.0, \theta = 0.0, L = 95, T = 0.5, K = 100, 102, 105. \]
### Table 5: Down-and-Out Barrier Option

| Strike | Benchmark | Our Approximation ($u^0 + \varepsilon v^0_1$) | Barrier Black-Scholes ($u^0$) |
|--------|-----------|---------------------------------------------|-------------------------------|
| 100    | 3.523     | 3.517 (-0.16%)                             | 3.495 (-0.77%)                |
| 102    | 2.891     | 2.888 (-0.09%)                             | 2.866 (-0.85%)                |
| 105    | 2.066     | 2.065 (-0.06%)                             | 2.052 (-0.64%)                |

### Table 6: Down-and-Out Barrier Option

| Strike | Benchmark | Our Approximation ($u^0 + \varepsilon v^0_1$) | Barrier Black-Scholes ($u^0$) |
|--------|-----------|---------------------------------------------|-------------------------------|
| 100    | 3.587     | 3.594 (0.20%)                               | 3.495 (-2.55%)                |
| 102    | 2.976     | 2.987 (0.39%)                               | 2.866 (-3.68%)                |
| 105    | 2.170     | 2.183 (0.59%)                               | 2.052 (-5.41%)                |

### 5 Conclusion

This paper has proposed an approximation scheme for barrier option prices by applying a new semi-group expansion to the Cauchy-Dirichlet problem in the second order parabolic partial differential equations (PDEs). As an application, we have derived a semi-group expansion formula under a certain type of stochastic volatility model and confirmed the validity of our method through numerical examples. Developing concrete computational schemes under various models is our next research topic.
Table 7: Down-and-Out Barrier Option

| Strike | Benchmark | Our Approximation ($u_0^v + \varepsilon v_0^v$) | Barrier Black-Scholes ($u_0^v$) |
|--------|-----------|---------------------------------|---------------------------------|
| 100    | 3.261     | 3.258 (-0.09%)                  | 3.290 (0.90%)                   |
| 102    | 2.640     | 2.639 (-0.02%)                  | 2.686 (1.78%)                   |
| 105    | 1.841     | 1.841 (0.01%)                   | 1.911 (3.77%)                   |

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6 Appendix A: Proof of Theorem 2, 3, 4, 6

6.1 Proof of Theorem 2

First, by the definition of $v_0^k$, we easily get $v_0^k(T, x) = 0$ for $x \in D$ and $v_0^k(t, x) = 0$ for $(t, x) \in [0, T] \times \partial D$.

Next, fix any $x \in D$. By the Markov property, we have

$$J(t \wedge \tau_D(X_0^0, x))v_0^k(t \wedge \tau_D(X_0^0, x), X_{t \wedge \tau_D(X_0^0, x)}^0) = J(t)v_0^k(t, X_{t \wedge \tau_D(X_0^0, x)}^0)1_{\{\tau_D(X_0^0, x) \geq t\}}$$

for each $t \in [0, T]$, where $J(r) = \exp \left(-\int_0^r c(X_v^0, 0)dv\right)$ and $(F_r)_r$ is the Brownian filtration. This implies that

$$M_t := J(t \wedge \tau_D(X_0^0, x))v_0^k(t \wedge \tau_D(X_0^0, x), X_{t \wedge \tau_D(X_0^0, x)}^0) + \int_0^{t \wedge \tau_D(X_0^0, x)} J(r)g_k^0(r, X_r^{0, x})dr$$

is a local martingale. On the other hand, applying Ito’s formula, we have that

$$M_t = M_0 + \int_0^t \left\{ \left( \frac{\partial}{\partial t} + \mathcal{L}^0 \right) v_k^0(r, X_r^{0, x}) + g_k^0(r, X_r^{0, x}) \right\} 1_{\{\tau_D(X_0^0, x) \geq r\}}dr$$
\[ + \sum_{i,j=1}^{d} \int_{0}^{t} J(r) \sigma_{ij}^{*}(X^{0,x}_{r}, 0) \frac{\partial}{\partial x^{i}} v_{k}^{0}(r, X^{0,x}_{r}) 1_{\{\tau_{D}(X^{0,x}) \geq r\}} dB_{r}^{i} \]

for each \( t \in [0, T] \). Thus, the uniqueness of decompositions of semimartingales gives us

\[ \int_{0}^{t} \left\{ \left( \frac{\partial}{\partial t} + \mathcal{L}^{0} \right) v_{k}^{0}(r, X^{0,x}_{r}) + g_{k}^{0}(r, X^{0,x}_{r}) \right\} 1_{\{\tau_{D}(X^{0,x}) \geq r\}} dr = 0, \quad t \in [0, T]. \]

Therefore, for each fixed \( t \in (0, T) \),

\[ \frac{1}{h} \int_{t}^{t+h} \left\{ \left( \frac{\partial}{\partial t} + \mathcal{L}^{0} \right) v_{k}^{0}(r, X^{0,x}_{r}) + g_{k}^{0}(r, X^{0,x}_{r}) \right\} 1_{\{\tau_{D}(X^{0,x}) \geq r\}} dr = 0 \]

holds for any small enough \( h > 0 \). Since \( x \in D \), by letting \( h \to 0 \), we obtain

\[ \left( \frac{\partial}{\partial t} + \mathcal{L}^{0} \right) v_{k}^{0}(t, x) + g_{k}^{0}(t, x) = 0. \]

Finally we prove (3.8) by mathematical induction. When \( k = 0 \), the assertion is easily obtained by (2.4), (3.4), [F] and [G]. Now we assume that (3.8) holds for \( 1, 2, \ldots, k-1 \). Then, by (3.4), (3.3) and [F], we have

\[ |g_{k}^{0}(t, x)| \leq C(1 + |x|^{2m}) \sum_{|\alpha| \leq 2} \left( |D_{\alpha} v^{0}(t, x)| + \sum_{i=1}^{k-1} |D_{\alpha} v_{i}^{0}(t, x)| \right) \]

for some \( C, m > 0 \), where \( \alpha = (i_{1}, \ldots, i_{d}) \in \{0, 1, 2, \ldots\}^{d} \) is a multi-index, \( |\alpha| = i_{1} + \cdots + i_{d} \) and \( D_{\alpha} = \partial^{\alpha}/(\partial x^{1})^{i_{1}} \cdots (\partial x^{d})^{i_{d}} \). By the induction hypothesis and [G]–[H], we see that

\[ \sum_{|\alpha| \leq 2} \left( |D_{\alpha} v^{0}(t, x)| + \sum_{i=1}^{k-1} |D_{\alpha} v_{i}^{0}(t, x)| \right) \leq C' M(t)(1 + |x|^{2m'}) \]

for some \( C', m' > 0 \) and \( M \in L^{1}([0, T), dt) \). Therefore, we get

\[ |g_{k}^{0}(t, x)| \leq C'' M(t)(1 + |x|^{2m''}) \]

for some \( C'', m'' > 0 \). Then we obtain

\[ |v_{k}^{0}(t, x)| \leq C'' E \left[ \int_{0}^{(T-t)} M(t + r)(1 + |X^{0,x}_{r}|^{2m''}) dr \right] \]

\[ \leq C' \left( 1 + E \sup_{0 \leq r \leq T} |X^{0,x}_{r}|^{2m''} \right) \int_{0}^{T} M(r) dr \]

\[ \leq C'' C_m T^{m'-1} \left( \int_{0}^{T} M(r) dr \right) (1 + |x|^{2m''}) \]

by virtue of (2.4). Thus (3.8) also holds for \( k \). Now we complete the proof of Theorem 2.
6.2 Proof of Theorem 3

First, we generalize the definitions of $\mathcal{L}^0_k$, $g^0_k$ and $v^0_k$. For $k, n \geq 1$, We define

$$\mathcal{L}^\varepsilon_k = \frac{1}{(k-1)!} \left\{ \frac{1}{2} \sum_{i,j=1}^d \int_0^1 (1-r)^{k-1} \frac{\partial^k u^{ij}}{\partial x^i \partial x^j}(x,r\varepsilon)dr \frac{\partial^2}{\partial x^i \partial x^j} + \sum_{i=1}^d \int_0^1 (1-r)^{k-1} \frac{\partial^k b^i}{\partial x^i}(x,r\varepsilon)dr \frac{\partial}{\partial x^i} - \int_0^1 (1-r)^{k-1} \frac{\partial^k c}{\partial x^i}(x,r\varepsilon)dr \right\},$$

$$g^\varepsilon_n(t,x) = \mathcal{L}^\varepsilon_n u^0(t,x) + \sum_{k=1}^{n-1} \mathcal{L}_n \mathcal{L}^0_{n-k} v^0_k(t,x) + \sum_{k=1}^{n-2} \varepsilon^k \left\{ \mathcal{L}^\varepsilon_n v^0_k(t,x) + \sum_{l=k+1}^{n-1} \mathcal{L}^0_{n-l} v^0_l(t,x) \right\} + \varepsilon^{n-1} \mathcal{L}^\varepsilon_n v^0_{n-1}(t,x),$$

where $g^\varepsilon_1(t,x)$ and $g^\varepsilon_2(t,x)$ are understood as $g^\varepsilon_1(t,x) = \mathcal{L}^\varepsilon_1 u^0(t,x)$ and $g^\varepsilon_2(t,x) = \mathcal{L}^\varepsilon_2 u^0(t,x) + \mathcal{L}^0_1 v^0_1(t,x) + \varepsilon \mathcal{L}^\varepsilon_1 v^0_1(t,x)$, respectively.

We consider the following Cauchy-Dirichlet problem:

$$\begin{cases}
-\frac{\partial}{\partial t} v(t,x) - \mathcal{L}^\varepsilon v(t,x) - g^\varepsilon_n(t,x) = 0, \quad (t,x) \in [0,T) \times D, \\
v(T,x) = 0, \quad x \in D, \\
v(t,x) = 0, \quad (t,x) \in [0,T] \times \partial D.
\end{cases} \tag{6.1}$$

For $\varepsilon \neq 0$, we define $v^\varepsilon_n = [u^\varepsilon - \{u^0 + \sum_{k=1}^{n-1} \varepsilon^k v^0_k(t,x)\}] / \varepsilon^n$. Obviously, we see

$$u^\varepsilon(t,x) = u^0(t,x) + \sum_{k=1}^{n-1} \varepsilon^k v^0_k(t,x) + \varepsilon^n v^\varepsilon_n(t,x). \tag{6.2}$$

Proposition 3. The function $v^\varepsilon_n$ is a solution of (6.1).

Proof. It is obvious that $v^\varepsilon_n(T,x) = 0$ for $x \in D$ and $v^\varepsilon_n(t,x) = 0$ for $(t,x) \in [0,T] \times \partial D$. Apply Taylor’s theorem to $\mathcal{L}^\varepsilon$ in (2.3) to observe

$$\mathcal{L}^\varepsilon u^\varepsilon(t,x) = \left\{ \mathcal{L}^0 + \sum_{k=1}^{n-1} \varepsilon^k \mathcal{L}^0_k + \varepsilon^n \mathcal{L}^\varepsilon_n \right\} u^\varepsilon(t,x). \tag{6.3}$$

Since $u^0$ is the solution of (2.3) with $\varepsilon = 0$, we get

$$\frac{\partial}{\partial t} u^0(t,x) + \mathcal{L}^0 u^0(t,x) = 0. \tag{6.4}$$

Similarly, since $v^0_k$ is a solution of (3.2), we have

$$\frac{\partial}{\partial t} v^0_k(t,x) + \mathcal{L}^0 v^0_k(t,x) + \mathcal{L}^0_k u^0(t,x) + \sum_{l=1}^{k-1} \mathcal{L}^0_{k-l} v^0_l(t,x) = 0. \tag{6.5}$$
Combining (6.2)–(6.5) and Theorem 1, we obtain
\[
\varepsilon^n \left\{ \frac{\partial}{\partial t} v_n^\varepsilon(t, x) + \mathcal{L}_0 v_n^\varepsilon(t, x) + \sum_{l=1}^{n-1} \mathcal{L}_{l-1}^0 v_l^0(t, x) \right\} \\
+ \sum_{k=n+1}^{2n-2} \varepsilon^k \left\{ \mathcal{L}_{k-n}^0 v_n^\varepsilon(t, x) + \mathcal{L}_{k-n}^0 v_k^0(t, x) + \sum_{l=k-n+1}^{n-1} \mathcal{L}_{l-k}^0 v_l^0(t, x) \right\} \\
+ \varepsilon^{2n-1} \left\{ \mathcal{L}_{n-1}^0 v_n^\varepsilon(t, x) + \mathcal{L}_{n-1}^0 v_{n-1}^0(t, x) \right\} \\
+ \varepsilon^{2n-2} \sum_{k=n+1}^{n-2} \varepsilon^k \left\{ \mathcal{L}_{k-n}^0 v_n^\varepsilon(t, x) + \mathcal{L}_{k-n}^0 v_k^0(t, x) + \sum_{l=k-n+1}^{n-1} \mathcal{L}_{l-k}^0 v_l^0(t, x) \right\} \\
+ \varepsilon^2 n \left\{ \mathcal{L}_{n-1}^0 v_n^\varepsilon(t, x) + \mathcal{L}_{n-1}^0 v_{n-1}^0(t, x) \right\} = 0,
\]
and thus,
\[
\frac{\partial}{\partial t} v_n^\varepsilon(t, x) + \mathcal{L}_0 v_n^\varepsilon(t, x) + g_n^\varepsilon(t, x) = 0.
\]
This implies the assertion. \(\blacksquare\)

Set
\[
\tilde{v}_n^\varepsilon(t, x) = E \left[ \int_0^{\tau_\mathcal{D}(X_0, x) \wedge (T-t)} \exp \left( - \int_0^r c(X_0, x, v, \varepsilon) dv \right) g_n^\varepsilon(t + r, X_0, x) dr \right].
\]
By [G]–[H], we find that there are \(C_n > 0, \tilde{m}_n \in \mathbb{N}\) which are independent of \(\varepsilon\) and the function \(M_n \in C([0, T]) \cap L^1([0, T], dt)\) determined by \(u_0^0, v_1^0, \ldots, v_{n-1}^0\) such that
\[
|g_n^\varepsilon(t, x)| \leq C_n M_n(t) (1 + |x|^{2\tilde{m}_n}). \tag{6.6}
\]
The inequalities (2.4) and (6.6) imply
\[
|\tilde{v}_n^\varepsilon(t, x)| \leq C'_n \int_t^T M_n(r) dr (1 + |x|^{2\tilde{m}_n}) \tag{6.7}
\]
for some \(C'_n > 0\) which is also independent of \(\varepsilon\).

**Proposition 4.** \(v_n^\varepsilon = \tilde{v}_n^\varepsilon\).

**Proof.** The assertion is easily obtained by the similar argument to the one in Theorem 5.1.9 in [13]. \(\blacksquare\)

**Proof of Theorem 3.** By (6.2) and Proposition 4, we have
\[
u_n^\varepsilon(t, x) - (u_0^0(t, x) + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t, x)) = \varepsilon^n \tilde{v}_n^\varepsilon(t, x).
\]
Our assertion is now immediately obtained by the inequality (6.7). \(\blacksquare\)

### 6.3 Proof of Theorem 4

1. Firstly, let us consider the case for \(k = 1\). Let \(g \in \mathcal{H}^{m,1}\). Observe that
\[
\int_0^{(T-t) \wedge \tau_\mathcal{D}(X_0, x)} \exp \left( - \int_0^r c(X_0, x, 0) dv \right) g(t + r, X_0, x) dr
\]
21
If the assertion holds for 1, . . . , k, . . . , k

Thus, under the assumption [H], we see

\[ v_1^0(T - t, x) = \mathbb{E} \left[ \int_0^t \exp \left( - \int_0^r c(X_{v_0}^{0,x}, 0)dr \right) g(t + r, X_{r}^{0,x})1_{\{\tau_D(X_{0,x}^0) \geq r\}}dr \right] \]

= \int_0^T P_r^D g(t + r, x)dr.

Thus, under the assumption [H], we see

\[ v_k^0(T - t, x) = \mathbb{E} \left[ \int_0^t \exp \left( - \int_0^r c(X_{v_0}^{0,x}, 0)dr \right) g(t + r, X_{r}^{0,x})1_{\{\tau_D(X_{0,x}^0) \geq r\}}dr \right] \]

= \int_0^T P_r^D \mathcal{L}_k^0 u_0(T - t + r, x)dr

= \int_0^T P_r^D \mathcal{L}_k^0 P_L^D f(x)dr. \tag{6.8}

Thus, we have the assertion for \( k = 1 \).

2. If the assertion holds for 1, . . . , k − 1, then

\[ v_k^0(T - t, x) = \int_0^t P_{t_0}^D \{ \mathcal{L}_k^0 u_0^0 + \sum_{l=1}^{k-1} \mathcal{L}_{k-1}^0 v_l^0 \}(T - t + t_0, \cdot)(x)dt_0 \]

= \int_0^t P_{t_0}^D \mathcal{L}_k^0 P_L^D f(x)dt_0

+ \sum_{l=1}^{k-1} \left( \sum_{m=1}^{l} \sum_{\beta_m=1}^{m} \int_0^t \int_0^{t_1} \ldots \int_0^{t_{l-1}} P_{t_0}^D \mathcal{L}_k^0 \mathcal{L}_{k-1}^0 P_{t_0}^D \mathcal{L}_{k-2}^0 P_{t_1}^D \mathcal{L}_{k-3}^0 P_{t_2}^D \ldots P_{t_{l-2}}^D \mathcal{L}_{k-l}^0 P_{t_{l-1}}^D f(x)dt_{l-1} \ldots dt_1 dt_0 \right)

= \sum_{l=1}^{k} \left( \sum_{m=1}^{l} \sum_{\beta_m=1}^{m} \int_0^t \int_0^{t_1} \ldots \int_0^{t_{l-1}} P_{t_0}^D \mathcal{L}_k^0 \mathcal{L}_{k-1}^0 P_{t_0}^D \mathcal{L}_{k-2}^0 P_{t_1}^D \mathcal{L}_{k-3}^0 P_{t_2}^D \ldots P_{t_{l-2}}^D \mathcal{L}_{k-l}^0 P_{t_{l-1}}^D f(x)dt_{l-1} \ldots dt_1 \right)

Thus, our assertion is also true for \( k \). Then we complete the proof of Proposition 4 by mathematical induction.
6.4 Proof of Theorem 6

By the asymptotic expansion in Section 3 and Theorem 4 with $k = 1$, we see that the expansion

$$C_{SV,\varepsilon}^{\text{Barrier}}(T, e^x) = C_{BS,\varepsilon}^{\text{Barrier}}(T, e^x) + \varepsilon v_1^0(0, x) + O(\varepsilon^2)$$

holds with

$$v_1^0(t, x) = \int_0^{T-1} P_{T-t-r}^D \mathcal{Q}^0 P_r^D f(x)dr. \quad (6.9)$$

Then, we have the following proposition for an expression of $v_1^0(0, x)$. The proof is given in Section 6.4.1.

**Proposition 5.**

$$v_1^0(0, x) = \frac{T}{2} \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_T^D f(x) - \frac{1}{2} \mathbb{E}[(T - \tau_D(X_{0,x}))\nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-\tau_D(X_{0,x})}^D h(l)1_{\{\tau_D(X_{0,x}) < T}\}].$$

We remark that the expectation in the above equality can be represented as

$$\frac{1}{2} \mathbb{E}[(T - \tau_D(X_{0,x}))\nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-\tau_D(X_{0,x})}^D h(l)1_{\{\tau_D(X_{0,x}) < T\}] = \int_0^T \frac{(T - s)}{2} \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-s}^D h(l) h(s, x - l) ds, \quad (6.10)$$

where $h(s, x - l)$ is the density function of the first hitting time to $l$ defined by

$$h(s, x - l) = \frac{-(l - x)}{\sqrt{2\pi \sigma^2 s}} \exp \left( -\frac{(l - x + \sigma^2 s/2)^2}{2\sigma^2 s} \right). \quad (6.11)$$

Now we evaluate

$$\nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} P_T^D f(x) = \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} C_{BS}(t, e^x) - \nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} \left\{ \left( \frac{e^x}{L} \right) C_{BS} \left( t, \frac{L^2}{e^x} \right) \right\}. \quad (6.12)$$

Note that

$$\frac{\partial}{\partial \sigma} C_{BS} \left( t, \frac{e^x}{L} \right) = e^x n(d_1(t, x)) \sqrt{t}, \quad (6.12)$$

and

$$\frac{\partial}{\partial \sigma} \left\{ \left( \frac{e^x}{L} \right) C_{BS} \left( t, \frac{L^2}{e^x} \right) \right\} = L n(c_1(t, x)) \sqrt{t}. \quad (6.13)$$

Then we have

$$\nu \rho \sigma^2 \frac{\partial^2}{\partial x \partial \sigma} C_{BS} \left( t, \frac{e^x}{L} \right) = \nu \rho \sigma^2 e^x n(d_1(t, x)) \sqrt{t} \left\{ 1 - \frac{d_1(t, x)}{\sigma \sqrt{t}} \right\} = -\nu \rho \sigma e^x n(d_1(t, x)) d_2(t, x) \quad (6.14)$$
and
\[ \nu \rho^2 \frac{\partial^2}{\partial x \partial \sigma} \left\{ \left( \frac{e^x}{L} \right) C^{BS} \left( l, \frac{L^2}{e^x} \right) \right\} = \nu \rho \sigma \ln(c_1(t, x))c_1(t, x). \] (6.15)

Combining (6.12), (6.14) and (6.15), we get
\[ \nu \rho^2 \frac{\partial^2}{\partial x \partial \sigma} P_t^D \tilde{f}(x) = \nu \rho \sigma \left\{ e^n(d_1(t, x))(-d_2(t, x)) - \ln(c_1(t, x))c_1(t, x) \right\}. \] (6.16)

Substituting (6.16) into (6.10), we have
\[ \nu \rho^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-t} f(l) = \nu \rho \sigma \ln(d_1(t, l))(-d_2(t, l)) - \rho \sigma \ln(c_1(t, l))c_1(t, l) \]
\[ = \nu \rho \sigma \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(l - \log K + \frac{1}{2} \sigma^2 t)^2}{2\sigma^2 t} \right) \left( -2(l - \log K) \right). \] (6.17)

By Proposition 5, (6.10), (6.16) and (6.17), we reach the assertion.

6.4.1 Proof of Proposition 5
First, we notice the following relation:
\[ \frac{1}{2} \mathbb{E}[(T - \tau_D(X^{0,x}))\nu \rho^2 \frac{\partial^2}{\partial x \partial \sigma} P_{T-\tau_D(X^{0,x})} \tilde{f}(l)1_{\{\tau_D(X^{0,x})<T\}}] = -\int_0^T \frac{1}{2} \nu \rho L \frac{e^{-\frac{(l-x)^2}{2\sigma^2(t-s)}}}{\sqrt{2\pi}\sigma^2} ds. \]
(6.19)
Also, we have
\[
\left( \frac{\partial}{\partial t} + \mathcal{L}^0 \right) \int_0^{T-t} P^D_{T-t-r} \left( \nu \rho^2 \frac{\partial^2}{\partial x \partial \sigma} P^D_r \right) \bar{f}(x) \, dr = - \mathcal{L}^0_1 P^D_{T-t} \bar{f}(x), \quad x \in (l, \infty). \tag{6.20}
\]
Therefore, the function
\[
\eta(t, x) = \int_0^{T-t} P^D_{T-t-r} \left( \nu \rho^2 \frac{\partial^2}{\partial x \partial \sigma} P^D_r \right) \bar{f}(x) \, dr - \frac{T-t}{2} \mathcal{L}^0_1 P^D_{T-t} \bar{f}(l), \quad t \in [0, T).
\tag{6.21}
\]
satisfies the following PDE
\[
\left\{ \begin{aligned}
\left( \frac{\partial}{\partial t} + \mathcal{L}^0 \right) \eta(t, x) &= 0, \quad (t, x) \in [0, T) \times (l, \infty), \\
\eta(T, x) &= 0, \quad x \in [l, \infty), \\
\eta(t, l) &= -\mathcal{L}^0_1 P^D_{T-t} \bar{f}(l), \quad t \in [0, T).
\end{aligned} \right.
\]
Then Theorem 6.5.2 in [6] implies
\[
\eta(0, x) = -\frac{1}{2} \mathbb{E}[(T - \tau_D(X^{0,x}) \nu \rho^2 \frac{\partial^2}{\partial x \partial \sigma} P^D_{T-\tau_D(X^{0,x})} \bar{f}(l) 1_{\{\tau_D(X^{0,x}) < T\}}]. \tag{6.22}
\]
By (6.21) and (6.22), we get the assertion. □

7 Appendix B: Generalization

This section generalizes the results given in Section 2 and Section 3 to treat more general cases covered in the main text. Particularly, let \(d' \in \{1, \ldots, d\}\), and we regard \(X^{\varepsilon,x,i}_t\) as logarithm of the underlying asset prices for \(i \leq d'\), and as parameter processes (e.g. a stochastic volatility and a stochastic interest rate) for \(i > d'\). Also, we assume \(I \subset [0, \infty)\) in this section for a technical reason introduced later.

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_t, P)\) be a filtered space equipped with a standard Brownian motion \((B_t)_t\).

Set
\[
\begin{align*}
\hat{b}^i(y, \varepsilon) &= \begin{cases} 
y^i & i \leq d', \\
\frac{1}{2} \sum_{j=1}^d (\sigma^{ij}(\pi(y), \varepsilon))^2 & i > d',
\end{cases} \\
\hat{\sigma}^{ij}(y, \varepsilon) &= \begin{cases} 
y^i \sigma^{ij}(\pi(y), \varepsilon) & i \leq d', \\
\sigma^{ij}(\pi(y), \varepsilon) & i > d',
\end{cases}
\end{align*}
\tag{7.1}
\]
where \(\pi(y) = (\log y^1, \ldots, \log y^{d'}, y^{d'+1}, \ldots, y^d) \in \mathbb{R}^d\).

Next, we introduce new assumptions for the generalization: \([A']-[C']\) and \([G']-[H']\) below with \([D], [F]\) in Section 2 and Section 3, respectively are necessary for the generalization (Theorem 9 below) of the asymptotic expansion.
[A'] For each $\epsilon \in I$ it holds that $\sigma^{ij}(\cdot, \epsilon), b^i(\cdot, \epsilon) \in \mathcal{L}$, and that $\hat{\sigma}^{ij}(\cdot, \epsilon), \hat{b}^i(\cdot, \epsilon)$ and $c(\pi(\cdot), \epsilon)$ are also in $\mathcal{L}$. Here, $\mathcal{L}$ is defined in the assumption [A] of Section 2, that is the set of locally Lipschitz continuous functions defined on $\mathbb{R}^d$.

Moreover, there exists a solution $(X_t^\epsilon)^t$ of SDE (2.1) and for any $m > 0$ there are $m', C > 0$ such that

$$\sup_{0 \leq \tau \leq t} E[|Y_{t, \epsilon}^\epsilon|^2] \leq Ct^{m-1}(1 + |y|^{2m'})$$

where

$$Y_{t, \epsilon}^\epsilon = \lambda(X_{t, \epsilon, \pi(y)}),$$

$$\lambda(x) = (e^{x^1}, \ldots, e^{x^d}, x^{d+1}, \ldots, x^d) \in \mathbb{R}^d.$$

**Remark 7.** Note that Ito’s formula implies that $(Y_{t, \epsilon}^\epsilon)_t$ is a solution of

$$\begin{cases}
dY_{t, \epsilon}^\epsilon = \hat{b}(Y_{t, \epsilon}^\epsilon, \epsilon)dt + \hat{\sigma}(Y_{t, \epsilon}^\epsilon, \epsilon)dB_t, \\
Y_{0, \epsilon}^\epsilon = y.
\end{cases}$$

[B'] The function $f(x)$ is represented by the continuous function $\hat{f} : \mathbb{R}^d \rightarrow \mathbb{R}$ as $f(x) = \hat{f}(\lambda(x))$. There exists $C_f > 0$ such that $|\hat{f}(y)|^2 \leq C_f(1 + |y|^{2m})$, $y \in \mathbb{R}^d$. Moreover, $f(x) = 0$ on $\mathbb{R}^d \setminus D$.

[C'] In addition to the condition [C] ($c(x, \epsilon) \geq 0$ and $c(\cdot, \epsilon) \in \mathcal{L}$), there is a constant $A_2^\epsilon > 0$ such that

$$|\hat{\sigma}^{ij}(y, \epsilon)|^2 + |\hat{b}^i(y, \epsilon)|^2 \leq A_2^\epsilon(1 + |y|^2), \quad i, j = 1, \ldots, d,$$

$$c(x, \epsilon)^2 \leq A_2^\epsilon(1 + |\lambda(x)|^{2m}).$$

**Remark 8.** We remark that [C'] implies

$$|\sigma^{ij}(x, \epsilon)|^2 + |b^i(x, \epsilon)|^2 \leq A_3^\epsilon(1 + |y|^2), \quad i, j = 1, \ldots, d$$

for some $A_3^\epsilon > 0$.

We note that Theorem 3.1 in [18] no longer works for the PDE (2.3):

$$\begin{cases}
\frac{\partial}{\partial t} u^\epsilon(t, x) + \mathcal{L}^\epsilon u^\epsilon(t, x) = 0, \quad (t, x) \in [0, T) \times D, \\
u^\epsilon(T, x) = f(x), \quad x \in D, \\
u^\epsilon(t, x) = 0, \quad (t, x) \in (0, T] \times \partial D
\end{cases}$$

under [A’]–[B’]. Now, we focus on the generator

$$\mathcal{L}^\epsilon = \frac{1}{2} \sum_{i, j=1}^d \hat{a}^{ij}(y, \epsilon) \frac{\partial^2}{\partial y^i \partial y^j} + \sum_{i=1}^d \hat{b}^i(y, \epsilon) \frac{\partial}{\partial y^i} - c(\pi(y), \epsilon),$$
rather than $L^\varepsilon$, where $\tilde{\alpha}^{ij} = \sum_{k=1}^{d} \sigma^{ik} \sigma^{jk}$. Moreover, define

$$\hat{D} = \{ y \in \mathbb{R}^d ; y^i > 0, \ i = 1, \ldots, d' \ \text{and} \ \pi(y) \in D \}$$

and $\hat{u}^\varepsilon(t, y) = u^\varepsilon(t, \pi(y)) \ ((t, y) \in [0, T] \times \hat{D}), \ 0 ((t, y) \in [0, T] \times \partial \hat{D})$. Then the function $\hat{u}^\varepsilon$ is expected to be the solution of

$$\begin{cases}
-\frac{\partial}{\partial t} \hat{u}^\varepsilon(t, y) - \mathcal{L}^\varepsilon \hat{u}^\varepsilon(t, y) = 0, \ (t, y) \in [0, T] \times \hat{D}, \\
\hat{u}^\varepsilon(T, y) = \hat{f}(y), \quad x \in \hat{D}, \\
\hat{u}^\varepsilon(t, y) = 0, \quad (t, y) \in [0, T] \times \partial \hat{D},
\end{cases} \tag{7.6}$$

and we obtain the following existence result.

**Theorem 7.** Assume $[A']$--$[C']$ and $[D]$. Then, $u^\varepsilon(t, x)$ is a continuous viscosity solution of (2.3). Moreover, $\hat{u}^\varepsilon(t, y)$ is a continuous viscosity solution of satisfying

$$\sup_{(t, y) \in [0, T] \times \hat{D}} \frac{|\hat{u}^\varepsilon(t, y)|}{(1 + |y|^{2m'})} < \infty. \tag{7.7}$$

**Proof.** The latter assertion is by the similar argument to the proof of Proposition 6 in Appendix 7.1. Then, the simple calculation gives the former assertion. \hfill \blacksquare

**Remark 9.** Here we no longer require a local ellipticity condition $[E]$, because we consider viscosity solutions of (2.3) and (7.6) rather than classical solutions: we can directly show that the function $u^\varepsilon(t, x)$ (which is given in the form of a stochastic representation) becomes the viscosity solution of the corresponding PDE.

If we further assume a local ellipticity condition such as $[E]$, we may show the existence of classical solutions which is characterized as

$$\hat{u}^\varepsilon(t, y) = E \left[ \exp \left( -\int_0^{T-t} c(\pi(Y_{r}^{\varepsilon,y}), \varepsilon) dr \right) \hat{f}(Y_{T-t}^{\varepsilon,y})1_{\{\tau_D(Y_{T-t}^{\varepsilon,y}) \geq T-t \}} \right].$$

Moreover, applying Theorem 8.2 in [1] and Theorem 7.7.2 in [16] to (7.6), we have the following uniqueness theorem.

**Theorem 8.** Assume $[A']$--$[C']$ and $[D]$. If $\hat{u}^\varepsilon(t, y)$ is a continuous viscosity solution of (7.6) satisfying the growth condition (7.7), then $\hat{u}^\varepsilon = \hat{u}^\varepsilon$.

For our generalization of the asymptotic expansion stated as Theorem 9 below, we need to modify the assumptions $[G]$ and $[H]$ in the previous sections.

In order to state the existence of a function $v_k^0(t, x)$, we prepare the following set which slightly modifies $\mathcal{H}^{m,p}$ in Definition 1. Moreover, we define $\hat{G}^m$ similarly to $G^m$, replacing $\mathcal{H}^{m,1}$ and $\mathcal{H}^{m,2}$ in the definition with $\hat{\mathcal{H}}^{m,1}$ and $\hat{\mathcal{H}}^{m,2}$, respectively.

**[G']** The condition $[G]$ holds replacing $G^m$ with $\hat{G}^m$. That is, $u^0 \in \hat{G}^m$, where

$$\hat{G}^m = \left\{ g \in C^{1,2}([0, T] \times \hat{D}) \cap C([0, T] \times \hat{D}) \mid \begin{array}{l}
\frac{\partial g}{\partial x_i} \in \hat{\mathcal{H}}^{m,2}, \\
\frac{\partial^2 g}{\partial x_i \partial x_j} \in \hat{\mathcal{H}}^{m,1}, \ i, j = 1, \ldots, d\end{array} \right\},$$

and the set $\hat{\mathcal{H}}^{m,p}$ of $g \in C([0, T] \times \hat{D})$ is given by the following:
Definition 2. The set $\hat{H}^{m,n}$ of $g \in C([0,T] \times \bar{D})$ is defined to satisfy the following condition: There is some $M_g \in C([0,T]) \cap L^{\infty}([0,T], dt)$ such that

$$|g(t,x)| \leq M_g(t)(1 + |\iota(x)|^{2m}), \quad t \in [0,T], \ x, y \in \bar{D}. \quad (7.8)$$

Accordingly, the condition $[H]$ is replaced by the following:

$[H']$ The condition $[H]$ holds replacing $G^m$ with $\hat{G}^m$: It holds that $\hat{v}_k^0 \in \hat{G}^m_n$, $k = 1, \ldots, n - 1$ for some $m_n \in \mathbb{N}$.

Then, we obtain the generalization of Theorem 3 whose proof is given in Appendix 7.1.

Theorem 9. Assume $[A']-[C']$, $[D]$, $[F]$ and $[G']-[H']$. Then, there are positive constants $C_n$ and $m_n$ which are independent of $\varepsilon$ such that

$$\left| u^\varepsilon(t,x) - (u^0(t,x) + \sum_{k=1}^{n-1} \varepsilon^k \hat{v}_k^0(t,x)) \right| \leq C_n(1 + |\iota(x)|^{2\tilde{m}_n})\varepsilon^n, \quad (t,x) \in [0,T] \times \bar{D}. \quad (7.9)$$

7.1 Proof of Theorem 9

Let $v_n^\varepsilon$ and $\hat{v}_n^\varepsilon$ be as in Section 6.2. Thanks to the assumption $I \subset [0,\infty)$ and $[G']-[H']$, we can apply similar argument to the proof of Proposition 3, which tells us that $v_n^\varepsilon$ is a viscosity solution of (6.1). That is, we obtain the next proposition.

Proposition 6. The function $\hat{v}_n^\varepsilon$ is a continuous viscosity solution of (6.1).

Proof. Until the end of the proof we suppress $\varepsilon$ in the notation. First, we check the continuity. By the similar argument to the proof of Lemma 4.2 in [18], we see that $v_n^\varepsilon$ is continuous on $[0,T] \times \bar{D}$. Moreover, similarly to (6.7), we see that there are a function $M_n \in C([0,T]) \cap L^1([0,T], dt)$ and constants $\tilde{C}_n, \tilde{m}_n > 0$ such that

$$|\hat{v}_n(t,x)| \leq \tilde{C}_n \int_0^T M_n(r)dr(1 + |\iota(x)|^{2\tilde{m}_n}). \quad (7.9)$$

Thus we get

$$\sup_{x \in K \cap D} |\hat{v}_n(t,x)| \leq C_n'(1 + \sup_{x \in K} |\iota(x)|^{2m}) \left\{ \int_0^T M_n(r)dr - \int_0^t M_n(r)dr \right\} \rightarrow 0, \quad t \rightarrow T$$

for any compact set $K \subset \mathbb{R}^d$. Thus, $v_n^\varepsilon$ is continuous on $[0,T] \times \bar{D}$.

Next, we show that $v_n^\varepsilon$ is a viscosity subsolution of (6.1). By the definition of $\hat{v}_n$, we easily get $\hat{v}_n(T,x) = 0$ for $x \in D$ and $\hat{v}_n(t,x) = 0$ for $(t,x) \in [0,T] \times \partial D$. Now take any $(t,x) \in [0,T] \times \bar{D}$ and let $\varphi$ be $C^{1,2}$-function such that $v_n^\varepsilon - \varphi$ has a maximum 0 at $(t,x)$. We may assume that $\varphi$ and its derivatives have polynomial growth rates in $x$ uniformly in $t$. By the Markov property, we have

$$E\left[ J(h \wedge \tau_D(X^x))\hat{v}_n(t + h \wedge \tau_D(X^x), X^x_{t+h \wedge \tau_D(X^x)}) \right] = E\left[ J(h)\bar{v}_n(t + h, X^x_h)1_{\{\tau_D(X^x) \geq h\}} \right]$$

$$= E\left[ \int_{h}^{(T-t) \wedge \tau_D(X^x_{t+h})+h} J(r)g_n(t + r, X^x_r)dr1_{\{\tau_D(X^x) \geq h\}} \right]$$
for \( h \in (0, T-t) \), where \( J(r) = \exp \left( - \int_{t}^{r} c(X_u^x, \varepsilon) dv \right) \). Since \( \tau_D(X_{\tau+h}^x) = \tau_D(X^x) - h \) on \( \{ \tau_D(X^x) \geq h \} \), we obtain

\[
E \left[ J(h \land \tau_D(X^x)) \tilde{v}_n \left( t + h \land \tau_D(X^x), X_{h\land\tau_D(X^x)}^x \right) \right] = \tilde{v}_n(t, x) - E \left[ \int_0^{h \land \tau_D(X^x)} J(r) g_n(t + r, X^x_r) dr \right].
\]

Therefore,

\[
\varphi(t, x) = \tilde{v}_n(t, x)
\]

\[
= E \left[ J(h \land \tau_D(X^x)) \tilde{v}_n \left( t + h \land \tau_D(X^x), X_{h\land\tau_D(X^x)}^x \right) \right] + E \left[ \int_0^{h \land \tau_D(X^x)} g_n(t + r, X^x_r) dr \right]
\]

\[
\leq E \left[ J(h \land \tau_D(X^x)) \varphi \left( t + h \land \tau_D(X^x), X_{h\land\tau_D(X^x)}^x \right) \right] + E \left[ \int_0^{h \land \tau_D(X^x)} g_n(t + r, X^x_r) dr \right].
\]

Note that \( [A'], [C'] \) and (7.5) imply that

\[
\int_0^h J(r) \sigma^{ij}(X^x_r, \varepsilon) \frac{\partial}{\partial x^j} \varphi(t + r, X^x_r) dB^i_r
\]

is a martingale. Thus, applying Ito’s formula, we get

\[
- \frac{1}{h} \int_0^h E \left[ \left( \frac{\partial}{\partial t} + \mathcal{L} \right) \varphi(t + r, X^x_r) + g_n(t + r, X^x_r) \right] \mathbb{1}_{\{\tau_D(X^x) \geq h\}} dr \leq 0.
\]

Letting \( h \to 0 \), we see that

\[
- \frac{\partial}{\partial t} \varphi(t, x) - \mathcal{L} \varphi(t, x) - g_n(t, x) \leq 0.
\]

Hence, \( \tilde{v}_n \) is a viscosity subsolution of (6.1). By the similar argument, we also find that \( \tilde{v}_n \) is a viscosity supersolution.

To see the equivalence \( v_n^\varepsilon = \tilde{v}_n^\varepsilon \), we need to give a new proof of Proposition 4 under the assumptions of Theorem 9.

**Proof of Proposition 4.** Set \( \bar{u}_n^\varepsilon(t, x) = u^0(t, x) + \sum_{k=1}^{n-1} \varepsilon^k v_k^0(t, x) + \varepsilon^n \tilde{v}_n^\varepsilon(t, x) \). The analogous argument of the proof of Proposition 3 implies that \( \bar{u}_n^\varepsilon \) is the continuous viscosity solution of (2.3). Thus \( \bar{u}_n^\varepsilon(t, \pi(y)) \) is the continuous viscosity solution of (7.6). By (7.9), we see that \( \bar{u}_n^\varepsilon(t, \pi(y)) \) has a polynomial growth rate in \( y \) uniformly in \( t \). Then, Theorem 8 leads us to \( \bar{u}_n^\varepsilon(t, \pi(y)) = \tilde{u}^\varepsilon(t, y) \), which implies \( \bar{u}_n^\varepsilon(t, x) = u^\varepsilon(t, x) \). This equality, (6.2) and (7.9) imply the assertion.

Now, we obtain the assertion of Theorem 9 by the same way as that of Theorem 3.