RESTRICTED DIOPHANTINE APPROXIMATION IN REAL QUADRATIC NUMBER FIELDS

STEPHAN BAIER AND DWAI PAYAN MAZUMDER

Abstract. The distribution of $\alpha p$ modulo one, where $p$ runs over the rational primes and $\alpha$ is a fixed irrational real, has received a lot of attention. It is natural to ask for which exponents $\nu > 0$ one can establish the infinitude of primes $p$ satisfying $||\alpha p|| \leq p^{-\nu}$. The latest record in this regard is Kaisa Matomäki’s landmark result $\nu = 1/3 - \varepsilon$ which presents the limit of currently known technology. Recently, Glyn Harman, and, jointly, Marc Technau and the first-named author, investigated the same problem in the context of imaginary quadratic fields. Glyn Harman obtained an analog for $\mathbb{Q}(i)$ of his result in the context of $\mathbb{Q}$, which yields an exponent of $\nu = 7/22$. Marc Technau and the first-named author produced an analogue of Bob Vaughan’s result $\nu = 1/4 - \varepsilon$ for all imaginary quadratic number fields of class number 1. In the present article, we establish an analog of the last-mentioned result for real-quadratic fields of class number 1. This setting involves the additional complication of an infinite group of units in the ring of integers. Moreover, although the basic sieve approach remains the same (we use an ideal version of Harman’s sieve), the problem takes a different flavor since it becomes truly 2-dimensional. We reduce it eventually to a counting problem which is, interestingly, related to roots of quadratic congruences. To approximate them, we use an approach by Christopher Hooley based on the theory of binary quadratic forms.

Contents

1. Introduction and main results 2
1.1. History 2
1.2. Notations 3
1.3. Dirichlet approximation in real-quadratic number fields 3
1.4. Good and bad $(x_1, x_2)$ 4
1.5. Main result 4
2. Smoothed setup 5
3. Bounds for $\Psi(q)$ and $F(q)$ 7
4. Poisson summation 8
5. Harman’s Sieve 10
5.1. Notations and a basic lemma 10
5.2. The sieve result 10
5.3. Applying Harman’s sieve to real-quadratic fields 15
6. Treatment of the type II sum 16
6.1. Initial transformations 16
6.2. Cutting off summations 16
6.3. Removing the exponential weights 18
6.4. Applying Cauchy-Schwarz 19
6.5. Applying Poisson summation 20
7. Counting problem 20
7.1. Approximating $(x_1, x_2)$ 21
7.2. A congruence relation between $a$ and $b$ 22
7.3. Counting solutions of systems of linear congruences 22
7.4. Detecting congruences using additive characters 23
7.5. Estimating the type II sum 24
8. Roots of quadratic congruences 25

Date: May 28, 2020.
2010 Mathematics Subject Classification. Primary 11J71; Secondary 11J17, 11K60, 11J25, 11R11, 11R44, 11L20, 11N35, 11N36, 11L07, 11D79, 11H55, 11J70.

Key words and phrases. distribution modulo one, Diophantine approximation, real quadratic fields, sieves, distribution of prime ideals, smoothed sums, Poisson summation, roots of quadratic congruences, binary quadratic forms, continued fractions, Lebesgue measure.
1. Introduction and main results

1.1. History. Dirichlet’s approximation theorem for the rationals implies that for every \( x_0 \in \mathbb{R} \setminus \mathbb{Q} \), there are infinitely many pairs \((p, q)\) of coprime integers \( p \) and \( q \) such that

\[
\left| x_0 - \frac{p}{q} \right| \leq q^{-2}.
\]

Whereas the above statement is easy to prove just using pigeonhole principle or the continued fraction expansion of \( x_0 \), the problem of Diophantine approximation by fractions with denominator restricted to primes is hard. This problem has a long history and triggered the development of important tools in analytic number theory. It is not difficult to prove using the generalized Riemann Hypothesis for Dirichlet \( L \)-functions that for every \( \varepsilon > 0 \) and \( \nu = 1/3 \), there are infinitely many pairs \((p, \pi)\) with \( p \) an integer and \( \pi \) a prime such that

\[
\left| x_0 - \frac{p}{\pi} \right| \leq \pi^{-1-\nu+\varepsilon},
\]

but an improvement beyond \( \nu = 1/3 \) seems difficult even conditionally. One might expect that (1) holds for infinitely many \((p, \pi)\) if \( \nu = 1 \). Important unconditional results started with Vinogradov [16] who showed that \( \nu = 1/5 \) is admissible using his intricate non-trivial treatment of trigonometrical sums over primes. Vaughan [15] simplified this treatment introducing his famous identity for sums over primes and improved the exponent to \( \nu = 1/4 \) by refining Fourier-analytic arguments in Vinogradov’s method. Harman [6] developed a new sieve method which enabled him to establish the exponent \( \nu = 3/10 \). These results were subsequently improved by Jia and Harman in several papers. In particular, Harman got the exponent \( \nu = 7/22 \) in [7]. Heath-Brown and Jia [10] brought Kloosterman sums into the picture and made a number of further innovations to reach \( \nu = 16/49 \) which falls short of the exponent \( 1/3 \). Finally, in a landmark paper, Matomäki [14] managed to reach \( \nu = 1/3 \) by using bounds for averages of Kloosterman sums. This is considered the limit of the current technology.

It is interesting to put the said problem on restricted Diophantine approximation in the framework of number fields. Novel ideas are required to make the classical methods work in this context, and so far there are only a few recent results in this regard. The first-named author [1] extended the classical problem, in slightly generalized form, to \( \mathbb{Q}(i) \) and his method led to an exponent of \( \nu = 1/12 \). Harman [7] established the full analog of his above-mentioned result with \( \nu = 7/22 \) for \( \mathbb{Q}(i) \) by introducing a number of novelties into his method, in particular, a clever estimation of trigonometrical sums over regions of \( \mathbb{C} \) with sharp cutoff, which are more difficult to handle than trigonometrical sums over intervals in \( \mathbb{R} \) since the geometry of the regions comes into play. The first-named author and Technau [2] considered the problem for all imaginary quadratic number fields of class number 1 and obtained an exponent corresponding to \( \nu = 1/4 \). They avoided the said trigonometrical sums with sharp cutoff by using a smoothed version of Harman’s sieve for imaginary quadratic fields and Poisson summation to transform smooth trigonometrical sums over all of \( \mathbb{C} \). Their result can be put in the following form.
Theorem 1. Assume that \( \mathbb{Q}(\sqrt{-d}) \) with \( d > 1 \) a square-free positive integer has class number 1. Let \( x_0 \in \mathbb{C} \setminus \mathbb{Q}(\sqrt{-d}) \). Then there exist infinitely many non-zero prime ideals \( p \) in \( \mathcal{O} \), the ring of integers of \( \mathbb{Q}(\sqrt{-d}) \), such that
\[
|x_0 - \frac{p}{q}| \leq N(p)^{-1/2-1/8+\varepsilon}
\]
for some generator \( q \) of \( p \) and \( p \in \mathcal{O} \).

In simpler words, under the assumptions of the above theorem, there exist infinitely many prime elements \( \pi \in \mathcal{O} \) such that
\[
|x_0 - \frac{p}{\pi}| \leq |\pi|^{-1/4+\varepsilon}
\]
for a suitable \( p \in \mathcal{O} \). This corresponds to Vaughan’s exponent \( \nu = 1/4 \) in the classical setting. In this article, we prove an analog of Theorem 1 for real-quadratic number fields of class number 1. There will be a certain Diophantine restriction to the pairs \((x_1, x_2)\) in \( \mathbb{R}^2 \) replacing \( x_0 \) in this context, but we will see that, in the sense of the Lebesgue measure, almost all of these \((x_1, x_2)\) will satisfy this restriction.

The real-quadratic setting is more difficult and of different flavor because the problem becomes truly two-dimensional and we have to handle the infinite unit group. We will be led to an interesting counting problem for solutions of two-dimensional systems of linear congruences whose resolution depends on information on the Diophantine properties of roots of quadratic congruences, as established by Hooley [11] making use of the theory of quadratic forms. This is a new feature, as compared to the imaginary-quadratic case. Our starting point will again be a smoothed version of Harman’s sieve for quadratic number fields, where we here use a formulation with ideals instead of algebraic integers. Before we state our main theorem, we introduce some notations which will be used throughout this article and review Dirichlet approximation in real-quadratic number fields.

1.2. Notations.
- We assume that \( d \) is a positive square-free integer satisfying \( d \equiv 3 \mod 4 \) in which case the ring of integers of \( \mathbb{Q}(\sqrt{d}) \) equals \( \mathbb{Z}(\sqrt{d}) \).
- We denote the ring of integers of \( \mathbb{Q}(\sqrt{d}) \) by \( \mathcal{O} \) and write \( \mathbb{K} := \mathbb{Q}(\sqrt{d}) \).
- We denote the set of ideals in \( \mathcal{O} \) by \( \mathcal{I} \).
- We denote the two embeddings of \( \mathbb{Q}(\sqrt{d}) \), given by the identity and conjugation, by
\[
\sigma_1(\alpha + \beta\sqrt{d}) := \alpha + \beta\sqrt{d}
\]
and
\[
\sigma_2(\alpha + \beta\sqrt{d}) := \alpha - \beta\sqrt{d}.
\]
- We write \( \sigma(\mathbb{K}) := \{(\sigma_1(\gamma), \sigma_2(\gamma)) : \gamma \in K\} \).
- We assume that \( \mathbb{Q}(\sqrt{d}) \) has class number 1 so that all ideals in \( \mathcal{O} \) are principal.
- The norm of an ideal \( q \in \mathcal{O} \) will be denoted by \( N(q) \).
- We write \( N(q) \) for the modulus of the norm over \( \mathbb{Q} \) of an algebraic integer \( q \in \mathcal{O} \), i.e. \( N(q) = N((q)) \).
- If \( p, q \in \mathcal{O} \), we write \( \text{gcd}(p, q) \approx t \) to mean that \( t \) is a greatest common divisor of \( p \) and \( q \) in \( \mathcal{O} \). We note that \( t \) is unique up to units in \( \mathcal{O} \).
- \((x_1, x_2)\) is a pair of real numbers which does not belong to \( \sigma(\mathbb{K}) \).
- \( N, x \) and \( \delta \) will be variables, where \( N > 1 \) is a natural number, \( x > 1 \) is a real number and \( N^{-1} \leq \delta \leq 1 \).
- \( \varepsilon \) is an arbitrary but fixed positive number.
- As usual, we write \( f = O(g) \) or \( f \ll g \) if the functions \( f \) and \( g \) satisfy \( |f| \leq c|g| \) for some positive constant \( c \), and we write \( f \asymp g \) if \( f \ll g \) and \( g \ll f \).
- We allow all \( O \)-constants to depend on \( \varepsilon, d \) and \((x_1, x_2)\).

1.3. Dirichlet approximation in real-quadratic number fields. A general version of Dirichlet’s approximation theorem for number fields given in [13, Theorem 1.8], which is based on a result of Burger, [5, Lemma 5.1], implies the following version for the case of real-quadratic number fields \( \mathbb{Q}(\sqrt{d}) \), where no congruence conditions on \( d \) and no conditions on the class number are required here.
Theorem 2. There exists a constant $C > 0$ with the following property. If $(x_1, x_2) \in \mathbb{R}^2 \setminus \sigma(\mathbb{K})$, then there are infinitely many pairs $(p, q) \in \mathcal{O} \times (\mathcal{O} \setminus \{0\})$ such that

$$|x_i - \frac{\sigma_i(p)}{\sigma_i(q)}| \leq \frac{C}{|\sigma_i(q)|\sqrt{N(q)}} \quad \text{for } i = 1, 2,$$

where $N(q) := \sigma_1(q)^2 + \sigma_2(q)^2$.

Since $|\sigma_i(q)|\sqrt{N(q)} \geq |\sigma_1(q)\sigma_2(q)| = N(q)$ for $i = 1, 2$ and $N(q) \geq N(q)$ whenever $\tilde{p}/\tilde{q} = p/q$ and $p$ and $q$ are coprime in $\mathcal{O}$, the following is an immediate Corollary.

Corollary 3. There exists a constant $C > 0$ with the following property. If $(x_1, x_2) \in \mathbb{R}^2 \setminus \sigma(\mathbb{K})$, then there are infinitely many principal ideals $q \in \mathcal{I} \setminus \{0\}$ such that

$$|x - \frac{\sigma_i(p)}{\sigma_i(q)}| \leq \frac{C}{N(q)} \quad \text{for } i = 1, 2$$

for some generator $q$ of $\mathcal{O}$ and $p \in \mathcal{O}$ coprime to $q$.

Our main result on restricted Diophantine approximation in $\mathbb{Q}(\sqrt{d})$ will depend on a certain Diophantine property of $(x_1, x_2)$ which we introduce next.

1.4. Good and bad $(x_1, x_2)$. By Corollary 3, there are infinitely many natural numbers $W$ such that

$$|x_i - \frac{\sigma_i(u + v\sqrt{d})}{\sigma_i(f + g\sqrt{d})}| \leq \frac{1}{N(f + g\sqrt{d})} \quad \text{for } i = 1, 2$$

for suitable $u, v, f, g \in \mathbb{Z}$ with $u + v\sqrt{d}$ and $f + g\sqrt{d}$ coprime in $\mathcal{O}$ and $N(f + g\sqrt{d}) = W$. We shall require the following notion.

Definition 4. For $\eta > 0$, we call $(x_1, x_2)$ good if there is an infinite sequence of natural numbers $W$ such that (2) holds with $N(f + g\sqrt{d}) = W$, $\gcd(u + v\sqrt{d}, f + g\sqrt{d}) = 1$ and $Z = \gcd(f, g) = O(W^\nu)$, where $\gcd(f, g)$ is meant to be the largest natural number dividing both $f$ and $g$. We call $(x_1, x_2)$ bad if it is not good.

We shall show in subsection 9.3 that, in the sense of the Lebesgue measure, almost all $(x_1, x_2) \in \mathbb{R}^2 \setminus \sigma(\mathbb{K})$ are good. Moreover, we shall supply an explicit construction of particular good $(x_1, x_2)$ in subsection 9.4.

1.5. Main result. The following is our main result.

Theorem 5. Assume that $\mathbb{Q}(\sqrt{d})$ has class number 1, where $d$ is a positive square-free integer satisfying $d \equiv 3 \mod 4$. Let $\varepsilon$ be any positive real number. Suppose further that $(x_1, x_2) \in \mathbb{R}^2 \setminus \sigma(\mathbb{K})$ is $\eta$-good in the above sense. Set

$$\nu := \frac{1}{8} - \frac{\eta}{1 + 2\eta},$$

Then there exist infinitely many non-zero prime ideals $\mathfrak{p}$ in the ring $\mathcal{O}$ of integers of $\mathbb{Q}(\sqrt{d})$ such that

$$|x - \frac{\sigma_i(p)}{\sigma_i(q)}| \leq N(\mathfrak{p})^{-1/2 - \nu + \varepsilon} \quad \text{for } i = 1, 2$$

for some generator $q$ of $\mathfrak{p}$ and $p \in \mathcal{O}$. If $(x_1, x_2)$ is good, then the above holds with $\nu = 1/8$.

In particular, for a concrete subset of $\mathbb{R}^2$ of full Lebesgue measure, the good $(x_1, x_2)$, we have the real-quadratic analog to Theorem 1. We note that Theorem 5 gives a nontrivial estimate except when $(x_1, x_2)$ is not $\eta$-good for any $\eta < 1/8$. The set of these $(x_1, x_2)$ has Lebesgue measure 0 since it is a subset of the set of bad $(x_1, x_2)$.

The restriction to $d \equiv 3 \mod 4$ is non-essential and just made for convenience because under this condition we can write all elements of $\mathcal{O}$ in the form $a + b\sqrt{d}$ with $a, b$ integers, and we have that $d$ is odd, which will turn out convenient in subsection 9.2 but is not needed anywhere else. It should not cause much trouble to establish Theorem 5 also for the case when $d \equiv 1, 2 \mod 4$ along the same lines. The requirement of $\mathbb{Q}(\sqrt{d})$ having class number 1 is of significance, though. It is
not the sieve which restricts us to class number 1, but rather we make frequent use of the interplay between ideals and elements of \( \mathcal{O} \), which is not possible in the same way if the class number is greater than 1. However, in contrast to the imaginary-quadratic case in which one has only finitely many fields of class number 1 by the celebrated resolution of the Gauss class number problem by Baker-Heegner-Stark, it is conjectured that there exist finitely many real-quadratic number fields \( \mathbb{Q}(\sqrt{d}) \) of class number 1, and this should remain true when \( d \) is restricted to integers congruent to 3 modulo 4.

The authors aim to improve the exponent to \( \nu = \frac{7}{44} \) in future work. This would correspond to Harman’s above-mentioned \( \frac{7}{22} \)-results in the cases of \( \mathbb{Q} \) and \( \mathbb{Q}(i) \). To achieve a result of this strength, no major changes in the method are required except that the asymptotic Harman sieve for quadratic number fields needs to be replaced by a lower bound sieve analogous to the one used by Harman in the cases of \( \mathbb{Q} \) and \( \mathbb{Q}(i) \). This is ongoing work.

Acknowledgements. We would like to thank the Ramakrishna Mission Vivekananda Educational and Research Institute for an excellent work environment. The second-named author’s research has been supported by a UGC Net fellowship.

2. Smoothed setup

We begin with smoothing our Diophantine problem. Throughout the sequel, we let \( \mathcal{C} \) be a natural number which will be fixed in the course of this article depending on \( \varepsilon \) and no other parameter. We shall make use of the function

\[
 f(x) = (\exp(-\pi x^2) - \exp(-2\pi x^2))^C, 
\]

which has, on the one hand, exponential decay as \( |x| \to \infty \), and satisfies, on the other hand, the bound

\[
 f(x) \ll_C |x|^{2C}
\]

for all real \( x \), which is strong if \( x \) is small. We also write

\[
 \Omega_\Delta(x) := \exp\left(-\pi \cdot \frac{x^2}{\Delta^2}\right). 
\]

We define two functions \( \omega, \tilde{\omega} : \mathcal{I} \to \mathbb{R}^+ \) as

\[
 \omega(q) := \frac{\delta^2}{2\sqrt{d}} \cdot \Psi(q)
\]

and

\[
 \tilde{\omega}(q) := \Psi(q) \cdot F(q)
\]

with

\[
 \Psi(q) := \sum_{\substack{k \in \mathcal{O} \\ (k) = q}} f\left(\frac{\sigma_1(k)}{\sqrt{N}}\right) f\left(\frac{\sigma_2(k)}{\sqrt{N}}\right)
\]

and

\[
 F(q) := \sum_{p \in \mathcal{O}} \Omega_{\sqrt{N}}\left(x_1 - \frac{\sigma_1(p)}{\sigma_1(q)}\right) \Omega_{\sqrt{N}}\left(x_2 - \frac{\sigma_2(p)}{\sigma_2(q)}\right),
\]

where \( q \) in (9) is any generator of \( \mathfrak{q} \), i.e.

\[
 q = (q).
\]

Below we will see that \( \omega(q) \) and \( \tilde{\omega}(q) \) are well-defined.

It is easy to see that the sum over \( p \) in (9) converges. This is because

\[
 \Lambda(q; x_1, x_2) = \left\{ \left(x_1 - \frac{\sigma_1(p)}{\sigma_1(q)}, x_2 - \frac{\sigma_2(p)}{\sigma_2(q)}\right) \in \mathbb{R}^2 : p \in \mathcal{O}\right\}
\]

is a shifted lattice in \( \mathbb{R}^2 \), and \( \Omega_\Delta(x) \) is exponentially decreasing for \( |x| \to \infty \). Moreover, we have the upper bound

\[
 F(q) \ll 1 + \frac{\delta^2}{N} \cdot \mathcal{N}(q),
\]

which will be provided in section 3.
The convergence of the sum over \( q \) in (8) is easy to see as well, and moreover we have the upper bound
\[
\Psi(q) \ll \exp \left( -\pi DC \cdot \frac{\mathcal{N}(q)}{N} \right) \cdot \log N
\]
for every ideal \( q \in \mathcal{I} \setminus \{0\} \) and a suitable constant \( D > 0 \), which will be proved in section 3, where we shall also provide the lower bound
\[
\Psi(q) \gg 1 \quad \text{if } N \leq \mathcal{N}(q) \leq 2N.
\]

We still need to show that \( F(q) \) is independent of the choice of the generator \( q \), i.e., the sum on the right-hand side of (9) is invariant under a change of variables \( q \to uq \), where \( u \) is a unit in \( \mathcal{O} \). Putting \( uq \) in place of \( q \), we get
\[
\sum_{p \in \mathcal{O}} \Omega_{d/\sqrt{N}} \left( x_1 - \frac{\sigma_1(p)}{\sigma_1(qu)} \right) \Omega_{d/\sqrt{N}} \left( x_2 - \frac{\sigma_2(p)}{\sigma_2(qu)} \right)
\]
which equals \( F(q) \) upon making the change of variables \( pu^{-1} \to p \).

Thus, we have seen that \( \Psi(q) \) and \( F(q) \) are well-defined, and the definitions of \( \omega(q) \) and \( \tilde{\omega}(q) \) in (6) and (7) together with the bounds (12) and (11) and \( \delta \leq 1 \) give
\[
\omega(q) \ll \delta^2 \exp \left( -\pi DC \cdot \frac{\mathcal{N}(q)}{N} \right) \cdot \log N \quad \text{and} \quad \tilde{\omega}(q) \ll \exp \left( -\frac{\pi}{2} \cdot DC \cdot \frac{\mathcal{N}(q)}{N} \right) \cdot \log N.
\]

Our goal is to derive a lower bound for the quantity
\[
\tilde{T}(N) := \sum_{p} \tilde{\omega}(p),
\]
where the sum on the right-hand side runs over all prime ideals \( p \in \mathcal{I} \setminus \{0\} \). The convergence of this sum is again ensured due to the bound for \( \tilde{\omega}(q) \) in (14). The above quantity \( \tilde{T}(N) \) measures, up to units, the number of prime elements \( q \in \mathcal{O} \) with norm of size about \( N \) such that \( |x_i - \sigma_i(p)/\sigma_i(q)| \) is not much larger than \( \delta/\sqrt{N} \) for \( i = 1, 2 \) and a suitable \( p \in \mathcal{O} \). Our approach is, following Harman’s philosophy, to compare \( \tilde{T}(N) \) with the quantity
\[
T(N) := \sum_{p} \omega(p),
\]
where the sum on the right-hand side again runs over all prime ideals \( p \in \mathcal{I} \setminus \{0\} \). The quantity \( T(N) \) measures, up to units, the number of prime elements \( q \in \mathcal{O} \) with norm of size about \( N \), scaled by a factor of \( \delta^2/(2\sqrt{d}) \). Using Landau’s prime ideal theorem together with and (6) and (13), we obtain
\[
T(N) \gg \delta^2 \cdot \frac{N}{\log N}.
\]
The main task of this article is to show that, under suitable conditions on \( \delta \), the difference
\[
\tilde{T}(N) - T(N)
\]
is small compared to \( \delta^2 N/\log N \) so that the above lower bound for \( T(N) \) yields one for \( \tilde{T}(N) \). We shall establish the following.

**Theorem 6.** Assume that \( (x_1, x_2) \in \mathbb{R}^2 \setminus \sigma(\mathbb{K}) \) is \( \eta \)-good, where \( \eta > 0 \). Suppose that \( \varepsilon \leq 1/14 \). Then there exist infinitely many natural numbers \( N \) such that
\[
\tilde{T}(N) - T(N) \ll \delta^2 N^{1-\varepsilon}
\]
and hence
\[
\tilde{T}(N) \gg \delta^2 \cdot \frac{N}{\log N},
\]
provided that
\[
1 \geq \delta \geq N^{-\nu+15\varepsilon},
\]
where
\[
\nu := \frac{1/8 - \eta}{1 + 2\eta}.
\]
If \((x_1, x_2)\) is good, we can choose \(\eta\) as small as we wish and therefore get the following as an immediate Corollary.

**Corollary 7.** Assume that \((x_1, x_2) \in \mathbb{R}^2 \setminus \sigma(\mathbb{K})\) is good. Suppose that \(\varepsilon \leq 1/14\). Then there exist infinitely many natural numbers \(N\) such that

\[
\tilde{T}(N) - T(N) \ll \delta^2 N^{1-\varepsilon}
\]

and hence

\[
\tilde{T}(N) \gg \delta^2 \frac{N}{\log N},
\]

provided that

\[
1 \geq \delta \geq N^{-1/8+16\varepsilon}.
\]

In the last section, we shall derive our main result, Theorem 5 from Theorem 6 and Corollary 7.

3. Bounds for \(\Psi(q)\) and \(F(q)\)

In this section, we shall show that the sum on the right-hand side of (8) converges and that \(\Psi(q)\) satisfies (12) and (13). We shall also provide the bound (11) for \(F(q)\). To this end, we first prove the following lemma, which will also be needed in later parts of this paper.

**Lemma 8.** There exist constants \(c_1, c_2 > 0\) depending only on \(\mathbb{K}\) with the following property. For every \(m_0 \in \mathcal{O} \setminus \{0\}\), there exists a unit \(u\) in \(\mathcal{O}\) such that

\[
c_1 \sqrt{N(m_0)} \leq |\sigma_i(u m_0)| \leq c_2 \sqrt{N(m_0)} \quad \text{for} \quad i = 1, 2.
\]

**Proof.** Let \(\epsilon\) be the fundamental unit, i.e. the smallest unit in \(\mathcal{O}\) exceeding 1. Then for any \(k \in \mathbb{Z}\), we have

\[
\sigma_1(\epsilon^k m_0) = \epsilon^k \sigma_1(m_0)
\]

and

\[
\sigma_2(\epsilon^k m_0) = \epsilon^{-k} \sigma_2(m_0).
\]

If \(\rho\) is a real number satisfying

\[
|\epsilon^\rho \sigma_1(m_0)| = |\epsilon^{-\rho} \sigma_2(m_0)|,
\]

then since

\[
|\epsilon^\rho \sigma_1(m_0)| \cdot |\epsilon^{-\rho} \sigma_2(m_0)| = N(m_0),
\]

it follows that

\[
|\epsilon^\rho \sigma_1(m_0)| = \sqrt{N(m_0)} = |\epsilon^{-\rho} \sigma_2(m_0)|.
\]

This real number \(\rho\) is given by

\[
\rho = \frac{\log |\sigma_2(m_0)| - \log |\sigma_1(m_0)|}{2 \log \epsilon}.
\]

Take \(k := |\rho|\). Then it follows that

\[
\epsilon^{-1} \sqrt{N(m_0)} \leq |\sigma_1(\epsilon^k m_0)| \leq \epsilon \sqrt{N(m_0)} \quad \text{for} \quad i = 1, 2.
\]

Now the claim follows with \(c_1 = \epsilon^{-1}, c_2 = \epsilon\) and \(u = \epsilon^k\). □

Using the above lemma and noting that all generators of \(q\) are of the form \(\pm m e^n\), where \(m\) is any fixed generator, \(\epsilon\) is the fundamental unit and \(n\) runs over the integers, we may write \(\Psi(q)\) in the form

\[
\Psi(q) = 2 \sum_{n \in \mathbb{Z}} f \left( \frac{\sigma_1(m e^n)}{\sqrt{N}} \right) f \left( \frac{\sigma_2(m e^n)}{\sqrt{N}} \right) = 2 \sum_{n \in \mathbb{Z}} f \left( \frac{\sigma_1(m)}{\sqrt{N}} \right) f \left( \frac{\sigma_2(m)}{\sqrt{N}} \right),
\]

where \((m) = q\) and \(\sigma_{1,2}(m) \simeq \sqrt{N(q)}\). This gives immediately the lower bound in (13) since

\[
\Psi(q) \geq f \left( \frac{\sigma_1(m)}{\sqrt{N}} \right) f \left( \frac{\sigma_2(m)}{\sqrt{N}} \right) \gg 1 \quad \text{if} \quad N \leq N(q) \leq 2N.
\]

The upper bound follows from

\[
\sum_{n \in \mathbb{Z}} f \left( \frac{\sigma_1(m)}{\sqrt{N}} \right) f \left( \frac{\sigma_2(m)}{\sqrt{N}} \right) \leq 2 \sum_{n=0}^{\infty} \exp \left( -\pi D C \cdot \frac{N(q)}{N} \cdot 2^n \right) \ll \exp \left( -\pi D C \cdot \frac{N(q)}{N} \right) \cdot \log N
\]

for a suitable constant \(D > 0\), where we use \(N(q) \geq 1\).
At this point, we apply Lemma 8 to establish a related result which will be needed only in subsection 9.3.

**Lemma 9.** For \( q \in \mathcal{O} \), let

\[
\hat{N}(q) = \sigma_1(q)^2 + \sigma_2(q)^2.
\]

Then we have

\[
\sum_{q \in \mathcal{O}} \frac{1}{\hat{N}(q)} \ll \frac{1}{N(a)}
\]

for any non-zero (principal) ideal \( a \in \mathcal{I} \).

**Proof.** As above, we may write

\[
\sum_{q \in \mathcal{O}} \frac{1}{\hat{N}(q)} = \sum_{n \in \mathbb{Z}} \frac{1}{\sigma_1(a \epsilon^n)^2 + \sigma_2(a \epsilon^n)^2},
\]

where \((a) = a\) with \(\sigma_1(a) \approx \sqrt{N(a)}\) and \(\epsilon\) is the fundamental unit. It follows that

\[
\sum_{q \in \mathcal{O}} \frac{1}{\hat{N}(q)} = \sum_{n \in \mathbb{Z}} \frac{1}{\sigma_1(a)^2 \epsilon^{2n} + \sigma_2(a)^2 \epsilon^{-2n}} \ll \sum_{n=0}^{\infty} \frac{1}{N(a) \epsilon^{2n}} \ll \frac{1}{N(a)},
\]

which completes the proof. \(\square\)

Our bound (11) follows from the exponential decay of the Gaussian and the fact that \(\Lambda(q; x_1, x_2)\), defined in (10), is a shifted lattice in \(\mathbb{R}^2\) with a fundamental parallelogram of area equal to \(1/\hat{N}(q)\) and dimensions \(\ll 1/\sqrt{\hat{N}(q)}\). To see the latter, we employ again Lemma 8: The set \(\Lambda(q; x_1, x_2)\) does only depend on \(q = (q)\) but not on the generator \(q\) of this ideal. Using Lemma 8, we may choose \(q\) in such a way that \(\sigma_1(q) \approx \sqrt{\hat{N}(q)}\). Then

\[
\Lambda(q; x_1, x_2) = (x_1, x_2) - \{u a + v b : (u, v) \in \mathbb{Z}^2\}
\]

with

\[
a := \left(\frac{1}{\sigma_1(q)}, \frac{\sqrt{d}}{\sigma_2(q)}\right) \quad \text{and} \quad b := \left(\frac{1}{\sigma_1(q)}, -\frac{\sqrt{d}}{\sigma_2(q)}\right),
\]

which are vectors of lengths \(\ll 1/\sqrt{\hat{N}(q)}\) which span a parallelogram of area equal to \(1/\hat{N}(q)\), as claimed.

### 4. Poisson summation

In this section, we shall transform \(F(q)\), defined in (9), using the 2-dimensional Poisson summation formula, given below.

**Lemma 10.** Suppose that \(f \in L^1(\mathbb{R}^n)\). Let \(\hat{f}\) be the Fourier transform of this function, defined as

\[
\hat{f}(y) := \int_{\mathbb{R}^n} f(x) e(-x \cdot y) dx
\]

for \(y \in \mathbb{R}^n\). Suppose that \(\hat{f} \in L^1(\mathbb{R}^n)\) and

\[
|f(x)| + |\hat{f}(x)| \ll (1 + |x|)^{-(n+\varepsilon)}
\]

for some \(\varepsilon > 0\) and all \(x \in \mathbb{R}^n\), where \(|x|\) is the Euclidean norm of \(x\). Then for all \(z \in \mathbb{R}^n\) we have

\[
\sum_{m \in \mathbb{Z}^n} \hat{f}(m) e(m \cdot z) = \sum_{m \in \mathbb{Z}^n} f(m + z),
\]

where the series on the left-hand and right-hand sides are absolutely convergent, respectively. In particular,

\[
\sum_{m \in \mathbb{Z}^n} \hat{f}(m) = \sum_{m \in \mathbb{Z}^n} f(m).
\]

**Proof.** See, for example, [4]. \(\square\)

Using Lemma 10, we shall establish the following.
Lemma 11. We have
\begin{equation}
F(q) = \frac{\delta^2}{2\sqrt{d}} \sum_{p \in \mathcal{O}} e \left( \frac{\sigma_2(pq) x_2 - \sigma_1(pq) x_1}{2\sqrt{d}} \right) \cdot \exp \left( -\pi \cdot \frac{\sigma_1(pq)^2 + \sigma_2(pq)^2}{4d} \cdot \frac{\delta^2}{N} \right).
\end{equation}

Proof. Let \( f : \mathbb{R}^2 \to \mathbb{C} \) be defined as
\[
f(x, y) = \Omega_{\delta/\sqrt{N}} \left( x_1 - \frac{x + y\sqrt{d}}{\sigma_1(q)} \right) \cdot \Omega_{\delta/\sqrt{N}} \left( x_2 - \frac{x - y\sqrt{d}}{\sigma_2(q)} \right)
\]
so that
\[
\Omega_{\delta/\sqrt{N}} \left( x_1 - \frac{\sigma_1(p)}{\sigma_1(q)} \right) \cdot \Omega_{\delta/\sqrt{N}} \left( x_2 - \frac{\sigma_2(p)}{\sigma_2(q)} \right) = f(x + y\sqrt{d})
\]
if \( p = x + y\sqrt{d} \). The exponential decay of the functions \( \Omega_{\delta/\sqrt{N}} \) ensures that \( f \in L^1(\mathbb{R}^2) \). Now we calculate the Fourier transform of \( f \). Making the linear change of variables
\[
u = \frac{\sqrt{N}}{\delta} \cdot \left( x_1 - \frac{x + y\sqrt{d}}{\sigma_1(q)} \right), \quad \omega = \frac{\sqrt{N}}{\delta} \cdot \left( x_2 - \frac{x - y\sqrt{d}}{\sigma_2(q)} \right),
\]
and using the definition of \( \Omega_\Delta \) in (5), we obtain
\begin{equation}
\hat{f}(\alpha, \beta) = \int_{\mathbb{R}^2} f(x, y)e(-x\alpha - y\beta) dydx
\end{equation}
\begin{equation}
= \frac{\delta^2}{2\sqrt{d}} \cdot e(-A\alpha - B\beta) \cdot \int_{\mathbb{R}^2} \exp \left( -\pi(u^2 + v^2) \right) \cdot e(C(\alpha, \beta)u - D(\alpha, \beta)v) \cdot du \cdot dv,
\end{equation}
where
\[
A := \frac{\sigma_1(q)x_1 + \sigma_2(q)x_2}{2},
B := \frac{\sigma_1(q)x_1 - \sigma_2(q)x_2}{2\sqrt{d}},
C(\alpha, \beta) := \frac{(\beta + \sqrt{d}\alpha)\sigma_1(q)}{2\sqrt{d}} \cdot \frac{\delta}{\sqrt{N}},
D(\alpha, \beta) := \frac{(\beta - \sqrt{d}\alpha)\sigma_2(q)}{2\sqrt{d}} \cdot \frac{\delta}{\sqrt{N}}.
\]
Calculating the double integral in the second line of (18), we get
\[
\hat{f}(\alpha, \beta) = \frac{\delta^2}{2\sqrt{d}} \cdot e \left( \frac{(\beta - \sqrt{d}\alpha)\sigma_2(q)x_2 - (\beta + \sqrt{d}\alpha)\sigma_1(q)x_1}{2\sqrt{d}} \right) \times
\exp \left( -\pi \cdot \frac{\beta^2\sigma_1(q)^2 + \beta\alpha\sigma_2(q)^2 + \alpha^2\sigma_2(q)^2}{4d} \cdot \frac{\delta^2}{N} \right).
\]
Clearly, \( \hat{f} \in L^1(\mathbb{R}^2) \). Now the Poisson summation formula implies the result of Lemma (11) after recalling the definition of \( F(q) \) in (9). \( \square \)

Plugging (17) into (7), we obtain
\begin{equation}
\tilde{\omega}(q) = \Psi(q) \cdot \frac{\delta^2}{2\sqrt{d}} \cdot \sum_{p \in \mathcal{O}} e \left( \frac{\sigma_2(pq)x_2 - \sigma_1(pq)x_1}{2\sqrt{d}} \right) \cdot \exp \left( -\pi \cdot \frac{\sigma_1(pq)^2 + \sigma_2(pq)^2}{4d} \cdot \frac{\delta^2}{N} \right).
\end{equation}
We note that the right-hand side is still independent of the choice of the generator \( q \) of \( q \). Further, we observe that \( \omega(q) \), defined in (6), equals the contribution of \( p = 0 \) on the right-hand side of (19) so that
\begin{equation}
\tilde{\omega}(q) - \omega(q) = \Psi(q) \cdot \frac{\delta^2}{2\sqrt{d}} \cdot \sum_{p \in \mathcal{O} \setminus \{0\}} e \left( \frac{\sigma_2(pq)x_2 - \sigma_1(pq)x_1}{2\sqrt{d}} \right) \times
\exp \left( -\pi \cdot \frac{\sigma_1(pq)^2 + \sigma_2(pq)^2}{4d} \cdot \frac{\delta^2}{N} \right).
\end{equation}
This will be essential in establishing a non-trivial bound for the difference $\tilde{T}(N) - T(N)$. Our key tool will be a version of Harman’s sieve for real-quadratic fields, to be worked out in the next section.

5. Harmann’s Sieve

In this section, we prove a weighted version of Harman’s asymptotic sieve for ideals in the ring of integers of a real quadratic field (not necessarily of class number 1). We follow closely the proof of a weighted integer version of Harman’s sieve for imaginary quadratic number fields in [2, section 7].

5.1. Notations and a basic lemma.

- Throughout, we suppose that $K \subseteq \mathbb{R}$ is some real quadratic field with class number 1 and ring of integers $O$.
- We denote by $I$ the set of all ideals of $O$.
- We denote by $\mathcal{P}$ the set of all non-zero prime ideals of $O$.
- We set $\mathcal{P}(z) = \{ p \in \mathcal{P} \mid N(p) < z \}$.
- We set
  \[ \mathcal{P}(z) = \prod_{p \in \mathcal{P}(z)} p. \]

- We denote by $d_k(a)$ the number of ways to write an ideal $a \subseteq O$ as a product of $k$ ideals. In particular, $d_2(a) = d(a)$ is the number of ideal divisors of $a$.
- We write $(a, b) = 1$ if the ideals $a$ and $b$ are coprime, i.e., $O$ is the only divisor of both $a$ and $b$.
- For a general condition $(C)$, we write
  \[ 1_{(C)} := \begin{cases} 1 & \text{if } (C) \text{ is satisfied}, \\ 0 & \text{if } (C) \text{ is not satisfied}. \end{cases} \]

If $M$ is a set, we write
\[ 1_M(x) := \begin{cases} 1 & \text{if } x \in M, \\ 0 & \text{otherwise}. \end{cases} \]

The following lemma, known as "cosmetic surgery" will be used for the separation of variables.

**Lemma 12.** For any two distinct real numbers $\rho, \gamma > 0$ and $T \geq 1$ one has
\[ \left| 1_{\gamma < \rho} - \frac{1}{\pi} \int_{-T}^T e^{i\gamma t} \sin(\rho t) \frac{dt}{t} \right| \ll \frac{1}{T|\gamma - \rho|}, \]
where the implied constant is absolute.

**Proof.** See, for instance, [9, Lemma 2.2]. \qed

5.2. The sieve result.

**Theorem 13** (Weighted version of Harmann’s asymptotic sieve for quadratic fields). Let $x \geq 3$ be real and let $\omega, \tilde{\omega} : I \to \mathbb{R}^+$ be two bounded functions such that, for both $w = \omega$ and $w = \tilde{\omega}$,
\[ \lim_{R \to \infty} \sum_{a \in I \atop N(a) < R} d_4(a) w(a) \leq X \]
for $X \geq 1$. Suppose further one has $Y > 1$, $0 < \mu < 1$, $0 < \kappa < 1/2$ and $M \in (x^\mu, x)$ with the following property:
For any sequences $(a_a)_{a \in I}, (b_b)_{b \in I}$ of complex numbers with $|a_a| \leq 1$ and $|b_b| \leq d(b)$, one has,
\[ T_I := \sum_{a, b \in I \atop N(a) < M} a_a \omega(ab) - \tilde{\omega}(ab) \leq Y \]
and
\[ T_{II} := \sum_{a, b \in I \atop x^\mu < N(a) < x^{\mu + \kappa}} a_a b_b \omega(ab) - \tilde{\omega}(ab) \leq Y. \]
Then
\begin{equation}
|S(\omega, x^a) - S(\tilde{\omega}, x^a)| \ll Y(\log(xX))^3,
\end{equation}
where
\[ S(w, z) = \sum_{\substack{a \in \mathcal{I} \\
(a, P(z)) = 1}} w(a). \]

Proof. First we define the Möbius function \( \mu \) for ideals. Assume that \( a \) is an ideal in \( \mathcal{O} \) with prime ideal factorization
\[ a = \prod_{j=1}^{k} p_j^{\alpha_j}. \]
Then we set
\[ \mu(a) := \begin{cases} 
(-1)^k & \text{if } \alpha_j = 1 \text{ for } j = 1, 2, \ldots, k, \\
0 & \text{otherwise.} 
\end{cases} \]
Since
\[ \sum_{d \mid a} \mu(d) = \begin{cases} 
1 & \text{if } a = \mathcal{O} \\
0 & \text{otherwise,}
\end{cases} \]
we have
\begin{equation}
S(w, z) = \sum_{b \in \mathcal{I} \setminus 0} w(b) \sum_{d \mid P(z)} \mu(d) = \sum_{d \mid P(z)} \mu(d) \sum_{a \in \mathcal{I} \setminus 0} w(ad).
\end{equation}
Let
\begin{equation}
\Delta(d) = \sum_{a \in \mathcal{I} \setminus 0} (w(ad) - \tilde{\omega}(ad)).
\end{equation}
Applying (25) for \( w = \omega \) and \( w = \tilde{\omega} \) yields
\begin{equation}
S(\omega, z) - S(\tilde{\omega}, z) = \sum_{d \mid P(z)} \mu(d) \Delta(d) = \sum_{d \mid P(z)} \mu(d) \sum_{N(d) < M} + \sum_{d \mid P(z)} \mu(d) \sum_{N(d) \geq M} \Delta(d) = S^a + S^b, \text{ say.}
\end{equation}
Using (22) with \( a_0 = \mu(d)1_{d \mid P(z)} \), we deduce that \( |S^a| \leq Y \). Therefore, to prove the theorem, it suffices to show that
\begin{equation}
|S^b| \ll Y(\log(xX))^3.
\end{equation}

The next step is to arrange \( S^b \) into subsums according to the "sizes" of the prime factors in \( d \) (where \( d \) is the summation variable in (27)). To have some notion of size, fix some total order \( \prec \) on \( P(z) \) such that if \( N(p) < N(p') \), then \( p \prec p' \) (many such orders exist, and all will do equally well). Moreover, for \( p \in P(z) \) let
\[ \Pi(p) = \prod_{q \prec p} q. \]
Now take \( g : \mathcal{I} \longrightarrow C \) to be any function. We may group the terms of the sum
\[ S = \sum_{d \mid P(z)} \mu(d)g(d) \]
according to the largest factor \( p_1 \) of \( d \) with respect to \( \prec \), getting the Buchstamb-type identity
\begin{equation}
S = g(\mathcal{O}) - \sum_{p_1 \in P(z)} \sum_{d \mid \Pi(p_1)} \mu(d)g(p_1d).
\end{equation}
Similarly, for the part \( \sum_{d \mid \Pi(p_1)} \mu(d)g(p_1d) \), we have
\begin{equation}
\sum_{d \mid \Pi(p_1)} \mu(d)g(p_1d) = g(p_1) - \sum_{p_2 \prec p_1} \sum_{d \mid \Pi(p_1)} \mu(d)g(p_1, p_2d).
\end{equation}
Minding the innermost sum on the right-hand side above, it is obvious that the above identity can be iterated if so desired. To describe for which sub-sums iteration is beneficial, we decompose \( P(z) \) into

\[
P(z) = \{ p \in \mathbb{P}(z) \mid \mathcal{N}(p) > x^\mu \} \cup \{ p \in \mathbb{P}(z) \mid \mathcal{N}(p) \leq x^\mu \}
\]

where

\[
P_1 \cup Q_1, \text{ say,}
\]

and inductively for \( s = 2, 3, \ldots \),

\[
Q_s = \{ (p_1, \ldots, p_{s-1}, p_s) \in \mathbb{P}(z)^s \mid p_s \prec p_{s-1}, (p_1, \ldots, p_{s-1}) \in Q_{s-1} \}
\]

where

\[
P_s = \{ (p_1, \ldots, p_{s-1}, p_s) \mid \mathcal{N}(p_1, p_2, \ldots, p_s) > x^\mu \}
\]

and

\[
Q_s = \{ (p_1, \ldots, p_{s-1}, p_s) \mid \mathcal{N}(p_1, p_2, \ldots, p_s) \leq x^\mu \}.
\]

Assuming that \( g \) vanishes on arguments \( a \) with \( \mathcal{N}(a) \leq x^\mu \), and on applying (29) and (30), we have

\[
S = - \left( \sum_{p_i \in P_1} + \sum_{p_i \in Q_1} \right) \sum_{\delta \mid \Pi(p_1)} \mu(\delta) g(p_1, \delta)
\]

\[
= - \sum_{p_i \in P_1} \sum_{\delta \mid \Pi(p_1)} \mu(\delta) g(p_1, \delta) + \sum_{(p_i, p_j) \in P_0 \circ \Pi(p_1)} \mu(\delta) g(p_1, p_2, \delta)
\]

\[
+ \sum_{(p_i, p_j) \in Q_0 \circ \Pi(p_1)} \mu(\delta) g(p_1, p_2, \delta).
\]

On iterating this process - always applying (30) to the \( Q \)-part - it transpires that

\[
S = \sum_{s \leq t} (-1)^s \sum_{(p_1, \ldots, p_s) \in P_0 \circ \Pi(p_1)} \mu(\delta) g(p_1, p_2, \ldots, p_s, \delta)
\]

\[
+ (-1)^t \sum_{(p_1, \ldots, p_t) \in Q_0 \circ \Pi(p_1)} \mu(\delta) g(p_1, p_2, \ldots, p_t, \delta)
\]

for any \( t \in \mathbb{N} \). Since the product of \( t \) prime ideals has norm greater than or equal to \( 2^t \), we have

\[
Q_t = \emptyset \text{ for } t > \frac{\mu}{\log 2} \log x.
\]

Hence,

\[
S = \sum_{s \leq t} (-1)^s \sum_{(p_1, \ldots, p_s) \in P_0 \circ \Pi(p_1)} \mu(\delta) g(p_1, p_2, \ldots, p_s, \delta)
\]

for

\[
t := \left\lfloor \frac{\log x}{\log 2} \right\rfloor + 1 \ll \log x.
\]

We apply this to \( S^0 \) with \( g(a) = \Delta(a)1_{\mathcal{N}(a) \geq M} \). Note that since \( M > x^\mu \), we have \( g(a) = 0 \) for all \( \mathcal{N}(a) \leq x^\mu \), as was assumed in the above arguments. Thus,

\[
S^0 = \sum_{s \leq t} (-1)^s S^0(s),
\]

where

\[
S^0(s) = \sum_{\delta = p_1 \cdots p_s} \mu(\delta) \Delta(a, \delta).
\]

Another application of (30) gives

\[
S^0(s) = \sum_{\delta = p_1 \cdots p_s} \Delta(a) - \sum_{\delta = p_1 \cdots p_s} \sum_{\mathcal{N}(a) \geq M} \mu(\delta) \Delta(a, \delta)
\]

\[
= S^0_1(s) - S^0_2(s), \text{ say.}
\]
Given \( a = p_1 \cdots p_{s-1} p_s \) with 
\[
(p_1, \ldots, p_{s-1}, p_s) \in \mathcal{P}_s \quad \text{and} \quad (p_1, \ldots, p_{s-1}) \in \mathcal{Q}_{s-1},
\]
and noting that \( \mathcal{N}(p_s) \leq \mathcal{N}(p_1) < z = x^\mu \), we have 
\[
x^\mu < \mathcal{N}(a) = \mathcal{N}(p_1 \cdots p_{s-1} p_s) < x^{\mu+\kappa}.
\]
Using this, we find that \( S_1^a(s) \) can be expressed as 
\[
\sum_{a,b \in \mathcal{Z}} a_0(\omega(ab) - \tilde{\omega}(ab)),
\]
where the coefficients 
\[
a_0 = 1_{\{\mathcal{N}(a) \geq M\}} 1_{(p_1, \ldots, p_s) \in \mathcal{P}_s} (a)
\]
are only supported on \( a \) with \( x^\mu < \mathcal{N}(a) < x^{\mu+\kappa} \). Hence by \( (34) \), 
\[
|S_1^a(s)| \leq Y.
\]

Moving on to \( S_2^a(s) \), we expand the definition \( (26) \) of \( \Delta \), getting 
\[
S_2^a(s) = S_2^a(s, \omega) - S_2^a(s, \tilde{\omega}),
\]
where
\[
S_2^a(s, w) := \sum_{(p_1, \ldots, p_s) \in \mathcal{P}_s} \sum_{d \in \mathcal{I}(p)} \mu(d) \sum_{b \in \mathcal{Z}} w(ab dp^d)
\]
\[
= \sum_{(p_1, \ldots, p_s) \in \mathcal{P}_s} \sum_{n \in \mathcal{Z}} \sum_{(d \in \mathcal{I}(p), d | n)} \mu(d) w(an).
\]
In order to apply \( (23) \), we must disentangle the variables \( a \) and \( n \) in the above summation. To this end, we split 
\[
\sum_{p < p_s} = \sum_{p < p_s} 1_{\mathcal{N}(p) = \mathcal{N}(p_s)} + \sum_{p < p_s} 1_{\mathcal{N}(p) < \mathcal{N}(p_s)}
\]
to obtain a decomposition 
\[
S_2^a(s, w) = S_2^{b=\infty}(s, w) + S_2^{b<\infty}(s, w), \quad \text{say.}
\]
For \( S_2^{b<\infty}(s, w) \) we have 
\[
S_2^{b<\infty}(s, w) = \sum_{(p_1, \ldots, p_s) \in \mathcal{P}_s} \sum_{n \in \mathcal{Z}} \sum_{(d \in \mathcal{I}(p), d | n)} \mu(d) \chi(a, d, p, p_s) w(an),
\]
where 
\[
\chi(a, d, p, p_s) = 1_{\{\mathcal{N}(a) \geq M\}} 1_{\{\mathcal{N}(p) < \mathcal{N}(p_s)\}}.
\]
and the sum \( S_2^{b=\infty}(s, w) \) can be expressed similarly but needs a little more care.

The first summation on the right-hand side of \( (35) \) contains at most one term because \( K \) is a quadratic extension of \( \mathbb{Q} \) and hence for each \( l \) there are at most two prime ideals with norm \( l \). We will write \( \mathcal{P}_s' \) for the set of \( (p_1, \ldots, p_s) \in \mathcal{P}_s \) for which there is such a term, that is, some \( p < p_s \) with \( \mathcal{N}(p) = \mathcal{N}(p_s) \). Furthermore, let 
\[
\mathcal{P}'(z) := \{ p \in \mathcal{P}(z) : \text{there exists } p_s \text{ such that } p < p_s \text{ and } \mathcal{N}(p) = \mathcal{N}(p_s) \}.
\]
Then 
\[
S_2^{b=\infty}(s, w) := \sum_{(p_1, \ldots, p_s) \in \mathcal{P}_s' \cap \mathcal{P}'(z)} \sum_{n \in \mathcal{Z}} \sum_{(d \in \mathcal{I}(p), d | n)} \mu(d) \tilde{\chi}(a, d, p, p_s) w(an),
\]
where
\[
\tilde{\chi}(a, d, p, p_s) = 1_{\{\mathcal{N}(a) \geq M\}} 1_{\{\mathcal{N}(p) = \mathcal{N}(p_s)\}}
\]
\[
= 1_{\{\mathcal{N}(a) \geq M\}} 1_{\{\mathcal{N}(p) \leq \mathcal{N}(p_s)\}} - \chi(a, d, p, p_s).
\]
We choose some real number $\rho$ with $|\rho| \leq 1/2$ and $\{M + \rho\} = 1/2$, where $\{.\}$ denotes the fractional part. Then the condition $N(\alpha \theta) \geq M$ is equivalent to $\log N(\alpha \theta) \geq \log (M + \rho)$ and

$$\left| \log N(\alpha \theta) - \log (M + \rho) \right| \geq \log \frac{x + 1}{x + 1/2} \geq \frac{1}{3x}.$$ 

Therefore, Lemma 12 shows that

$$1_{\{N(\alpha \theta) \geq M\}} = 1 - \frac{1}{\pi} \int_{-T}^{T} N(\alpha \theta)^i t \sin(t \log(M + \rho)) \frac{dt}{t} + O\left(\frac{x}{T}\right)$$

for every $T \geq 1$. Similarly,

$$1_{\{N(p) < N(p_\ast)\}} = \frac{1}{\pi} \int_{-T}^{T} e^{it/2} e^{itN(p)} \sin(tN(p_\ast)) \frac{dt}{t} + O\left(\frac{1}{T}\right)$$

and

$$1_{\{N(p) \leq N(p_\ast)\}} = \frac{1}{\pi} \int_{-T}^{T} e^{-it/2} e^{itN(p)} \sin(tN(p_\ast)) \frac{dt}{t} + O\left(\frac{1}{T}\right).$$

Thus,

$$S^{\rho < N_\ast}_{2}(s, \omega) = \frac{1}{\pi} \int_{-T}^{T} \sum_{a, n \in I} a(a(t)b_n(t)\omega(a)\frac{dt}{t})$$

$$- \frac{1}{\pi^2} \int_{-T}^{T} \int_{-T}^{T} \sum_{a, n \in I} a(a(t, \tau)b_n(t, \tau)\omega(a)\frac{dt}{t})$$

$$+ O\left(\frac{x}{T^2} + \frac{1}{T} \int_{-T}^{T} |\sin(\tau \log(M + \rho))| \frac{dt}{\tau} \right) \times$$

$$\left(\sum_{(p_\ast, \ldots, p_\ast) \in \mathcal{P}_s, n \in I} \sum_{b, \omega} \sum_{\rho \in \mathcal{P}_0} \sum_{b \rho} w(\omega)\right)$$

with coefficients

$$a(a(t)) = \begin{cases} \sin(tN(p_\ast)) & \text{if there exists } (p_\ast, \ldots, p_\ast) \in \mathcal{P}_s \text{ such that } a = p_1 \cdots p_s, \\ 0 & \text{otherwise,} \end{cases}$$

$$b_n(t) = \sum_{p \in \mathcal{P}(x)} \sum_{b, \omega} e^{it/2} e^{itN(p)} \mu(\omega),$$

$$a(a(t, \tau)) = a(a(t))N(p_\ast)^{it} \sin(t \log(M + \rho)),$$

$$b_n(t, \tau) = \sum_{p \in \mathcal{P}(x)} \sum_{b, \omega} e^{it/2} e^{itN(p)} \mu(\omega)(Nn)^{it}. $$

We proceed by gathering some intermediate information before applying (23). Clearly,

$$|b_n(t)|, |b_n(t, \tau)| \leq d(n).$$

For the other coefficients we always have

$$|a(a(t))|, |a(a(t, \tau))| \leq 1,$$

yet if $t$ and $\tau$ are small, we can do better: if $|t| \leq x^{-1/2}$ and $|\tau| \leq (\log(x + 1/2))^{-1}$, then

$$|a(a(t))| \leq \sqrt{x}|t|, \quad |a(a(t, \tau))| \leq \sqrt{x}|t||\tau| |\log\left(x + \frac{1}{2}\right)|.$$ 

In view of this, we must deal with functions $f : \mathbb{R} \times (1, \infty) \rightarrow \mathbb{R}$ of the shape

$$f(t, \eta) = \begin{cases} \eta t & \text{if } |t| \leq \eta^{-1}, \\ 1 & \text{otherwise} \end{cases}$$
and their integrals

\[
\int_{-T}^{T} f(t, \eta) \frac{dt}{|t|} \ll \frac{1}{T} \left( \int_{0}^{\eta^{-1}} dt + \int_{\eta^{-1}}^{T} \frac{dt}{t} \right) \ll 1 + \log(T\eta).
\]

Lastly, we note that by (21),

\[
\sum_{p_1, \ldots, p_k \in \mathcal{P}_s} \sum_{\eta \in \mathcal{N}(0, p_1 \cdots p_k)} \sum_{\nu, \rho \in \mathcal{V}} \sum_{\eta \in \mathcal{N}(0, p_1 \cdots p_k)} w(\eta) \ll \sum_{a \in \mathcal{I} \setminus 0} d_4(a)w(a) \ll X.
\]

Gathering all information we got so far, we may derive a bound for

\[
\mathcal{E} = |S_{2, <}^b(s, \omega) - S_{2, <}^b(s, \tilde{\omega})|
\]
as follows: after applying (38) with \(w = \omega\) and \(w = \tilde{\omega}\), the O-terms are treated directly with (41) and (42), whereas for the rest one may apply (23). Here it is important to use (40) for small \(|\tau|\) and \(|\tau|\) first - prior to applying (23) - and (41) then bounds the integrals. Therefore, after some computations, we infer

\[
\mathcal{E} \ll Y \log(Tx) (1 + \log(T \log(x + 1/2))) + XT^{-1} (x + \log(T \log(x + 1/2))).
\]

Of course, the same arguments also apply to \(S_{2, =}^{b, \pm}(s, \omega)\): In view of (37), we have to apply them twice, but in both cases the coefficients corresponding to (39) obey the same bounds we used to derive (43). Consequently, (43) also holds with \(S_{2, =}^{b, \pm}\) in place of \(S_{2, <}^b\). In total, recalling (33), (34) and (36), we have

\[
|S_{2}^b(s)| \ll Y + \text{the bound from } (43)
\]
and it transpires that choosing \(T = xX\) suffices to yield a bound of \(\ll Y (\log(xX))^2\). On plugging this into (32) and recalling (31), we infer (28). Hence, the theorem is proved.

\[
5.3. \text{Applying Harman’s sieve to real-quadratic fields.} \text{ Now we apply Harman’s sieve, Theorem 13, with}
\]

\[
\mathbb{K} := \mathbb{Q}(\sqrt{d}), \quad \mathcal{O} = \mathbb{Z}[\sqrt{d}], \quad x^{-1/2} < \delta \leq 1/2, \quad N := \left\lfloor x^{1-\varepsilon} \right\rfloor, \quad X := C(\varepsilon)\delta^2x, \quad \kappa = 1/2,
\]

and \(\omega\) and \(\tilde{\omega}\) as defined in (6) and (7), the constant \(C(\varepsilon) > 0\) above only depending on \(\varepsilon\). Here and in the following, we take into account that for every given \(k \in \mathbb{N}\) and \(\varepsilon > 0\), \(d_k(q) \leq N(q)^{2\varepsilon}\) if \(N(q)\) is large enough. We shall obtain non-trivial estimates for the bilinear sums in (22) and (23) for specific choices of the variables \(x\), \(M\) and \(\mu\). According to the usual terminology, the sum in (22) is called type I and that in (23) type II sum. We observe that under the above conditions, the sums \(\mathcal{T}(N)\) and \(\mathcal{T}(N)\) defined in (15) and (16) satisfy

\[
\mathcal{T}(N) = S(\omega, x^{\kappa}) + O \left( x^{1/2} \log x \right)
\]

and

\[
\tilde{\mathcal{T}}(N) = S(\tilde{\omega}, x^{\kappa}) + O \left( x^{1/2} \log x \right).
\]

This is because

\[
S(w, x^{\kappa}) = S(w, x^{1/2}) = \sum_{\substack{a \in \mathcal{I} \setminus 0 \\text{new}}} w(a) + O \left( x^{-100} \right)
\]

\[
= w(\mathcal{O}) + \sum_{p \in \mathcal{P}(\varepsilon)} w(p) + O \left( x^{-100} \right)
\]

\[
\sum_{p \in \mathcal{P}(\varepsilon)} w(p) + O \left( x^{1/2} \log x \right)
\]

\[
\sum_{p \in \mathcal{P}(\varepsilon)} w(p) + O \left( x^{1/2} \log x \right)
\]

for \(w = \omega, \tilde{\omega}\), where we use (14) and \(\delta \leq 1\). In the next sections, we will deal with the Type I and Type II sums.
6. Treatment of the Type II Sum

6.1. Initial Transformations. Plugging (20) into (23), we obtain

\[ T_{II} = \frac{\delta^2}{2\sqrt{d}} \left| \sum_{m,n \in \mathbb{Z} \setminus (0)} a_m b_n (\omega(mn) - \tilde{\omega}(mn)) \right| \]

= \frac{\delta^2}{2\sqrt{d}} \left| \sum_{m,n \in \mathbb{Z} \setminus (0)} a_m b_n \Psi(mn) \cdot \sum_{p \in \mathcal{O} \setminus \{0\}} E(pk) E(pk) \right|

where \( k \) is any generator of the ideal \( mn \), and we set

\[ E(l) := \exp \left( -\pi \frac{\sigma_1(l)^2 + \sigma_2(l)^2}{4d} \frac{\delta^2}{N} \right) \]

and

\[ E(l) := e \left( \frac{\sigma_2(l)x_2 - \sigma_1(l)x_1}{2\sqrt{d}} \right) \]

We also use the notation

\[ G(k) := f \left( \frac{\sigma_1(k)}{\sqrt{N}} \right) f \left( \frac{\sigma_2(k)}{\sqrt{N}} \right) \]

below, where \( f \) is defined as in (3).

We shall choose a generator \( m \) for each of the ideals \( m \) in such a way that \( |\sigma_i(m)| \approx \sqrt{N(m)} \) for \( i = 1, 2 \), which is possible due to Lemma 8. Moreover, using the definition of \( \Psi(mn) \) and the independence of the sum over \( p \) from the choice of \( k \) as generator of \( mn \), we write, for \( m, n \) and \( m \) fixed,

\[ \Psi(mn) \cdot \sum_{p \in \mathcal{O} \setminus \{0\}} E(pk) E(pk) = \sum_{k \in \mathcal{O}} G(k) \sum_{\substack{p \in \mathcal{O} \setminus \{0\}} \ \ (k) = mn} E(pk) E(pk) \]

\[ \text{(46)} \]

on setting \( n := k/m \). Unfolding the sum

\[ \sum_{n \in \mathcal{O} \setminus \{0\}} \sum_{(n) = n} \]

we therefore obtain

\[ T_{II} = \frac{\delta^2}{2\sqrt{d}} \left| \sum_{m \in \mathcal{R}} \sum_{n \in \mathcal{O} \setminus \{0\}} a_m b_n \sum_{p \in \mathcal{O} \setminus \{0\}} E(p, mn) E(p, mn) \right| \]

where \( \mathcal{R} \) is a maximal system of mutually non-associate elements of \( \mathcal{O} \) satisfying \( |\sigma_i(m)| \approx \sqrt{N(m)} \) for \( i = 1, 2 \), and

\[ a_m := a_{(m)}, \quad b_n := b_{(n)} \]

and

\[ E(p, k) := f \left( \frac{\sigma_1(k)}{\sqrt{N}} \right) f \left( \frac{\sigma_2(k)}{\sqrt{N}} \right) \exp \left( -\pi \cdot \frac{\sigma_1(pk)^2 + \sigma_2(pk)^2}{4d} \right) \]

\[ \text{(47)} \]

6.2. Cutting off summations. For convenience, we would like to cut off the summation over \( n \) and \( p \) at appropriate points so that we are left with finite sums only. Taking \( |\sigma_i(m)| \approx \sqrt{N(m)} \) for \( i = 1, 2 \) and \( N := \left\lfloor x^{1-\varepsilon} \right\rfloor \) into account, the weight function

\[ G(mn) = f \left( \frac{\sigma_1(mn)}{\sqrt{N}} \right) f \left( \frac{\sigma_2(mn)}{\sqrt{N}} \right) \]

becomes negligible if

\[ \max\{|\sigma_1(n)|, |\sigma_2(n)|\} > \sqrt{\frac{x}{N(m)}}. \]
Therefore, it suffices to take only those \(n\)'s into consideration for which
\[
\max\{|\sigma_1(n)|, |\sigma_2(n)|\} \leq \sqrt{\frac{x}{N(m)}}.
\]
Moreover, we may discard all \(p\)'s for which
\[
\frac{\sigma_1(pmn)^2 + \sigma_2(pmn)^2}{4d} \cdot \frac{\delta^2}{N} \gg x^\varepsilon.
\]
Now comes the point where we make use of the exponent \(C\). If
\[
\sigma_1(mn)^2 \leq x^{-\varepsilon}N \quad \text{or} \quad \sigma_2(mn)^2 \leq x^{-\varepsilon}N,
\]
then
\[
G(mn) \ll x^{-2C\varepsilon}
\]
using (4). Now we may choose
\[
C := \left\lfloor \frac{100}{\varepsilon} \right\rfloor
\]
so that
\[
E(mn) \ll x^{-200}
\]
if \(mn\) satisfies (50). The contribution of these \(mn\) becomes negligibly small so that we can assume that
\[
\sigma_1(mn)^2 > x^{-\varepsilon}N \quad \text{and} \quad \sigma_2(mn)^2 > x^{-\varepsilon}N
\]
in which case inequality (49) holds if
\[
|\sigma_1(p)| > x^\varepsilon \delta^{-1} \quad \text{or} \quad |\sigma_2(p)| > x^\varepsilon \delta^{-1}.
\]
Hence, it suffices to consider \(p\)'s such that
\[
|\sigma_{1,2}(p)| \leq x^\varepsilon \delta^{-1}.
\]
We deduce that
\[
T_{II} = \frac{\delta^2}{2\sqrt{d}} \sum_{m \in \mathcal{R}} \sum_{n \in \mathcal{O}(0)} \sum_{x^\mu < N(m) < x^{\mu+\varepsilon}} \sum_{|\sigma_{1,2}(n)| \leq (x/N(m))^{1/2}} \sum_{p \in \mathcal{O}(0)} \sum_{|\sigma_{1,2}(p)| \leq x^\varepsilon \delta^{-1}} E(p, mn) E(pmn) + O(x^{-100}).
\]
Moreover, we divide the \(m\)-sum into \(O(\log x)\) subsums
\[
\sum_{\kappa \leq N(m) \leq 2K} \sum_{x^\mu < K \leq x^{\mu+\varepsilon}} |\Sigma_K| + O(x^{-100}),
\]
over dyadic intervals, getting
\[
T_{II} \ll (\log x)^2 \sup_{x^\mu < K \leq x^{\mu+\varepsilon}} |\Sigma_K| + O(x^{-100}),
\]
where
\[
\Sigma_K := \sum_{m \in \mathcal{R}} \sum_{n \in \mathcal{O}(0)} \sum_{x^\mu < N(m) \leq 2K} \sum_{|\sigma_{1,2}(n)| \leq (x/K)^{1/2}} \sum_{p \in \mathcal{O}(0)} \sum_{|\sigma_{1,2}(p)| \leq x^\varepsilon \delta^{-1}} E(p, mn) E(pmn)
\]
with \(a_m := 0\) if \(N(m) > x^{\mu+\varepsilon}\). Here we note, again looking at \(G(mn)\), that the contribution of \(n\)'s with \((x/N(m))^{1/2} < |\sigma_1(n)| \leq (x/K)^{1/2}\) or \((x/N(m))^{1/2} < |\sigma_2(n)| \leq (x/K)^{1/2}\) is negligible.

Now we use the definition of \(f\) in (3) and expand the \(C\)-th powers implicit in the definition of \(E(p, k)\) in (47). We are led to a linear combination of sums of the form
\[
\Sigma_{K,j} := \sum_{m \in \mathcal{R}} \sum_{n \in \mathcal{O}(0)} \sum_{x^\mu < N(m) \leq 2K} \sum_{|\sigma_{1,2}(n)| \leq (x/K)^{1/2}} \sum_{p \in \mathcal{O}(0)} \sum_{|\sigma_{1,2}(p)| \leq x^\varepsilon \delta^{-1}} E_j(p, mn) E(pmn),
\]
where \(j := (j_1, j_2) \in \mathbb{N}^2\) and
\[
E_j(p, k) := \exp\left(-\pi \cdot \frac{\sigma_1(pk)^2 + \sigma_2(pk)^2}{4d} \cdot \frac{\delta^2}{N}\right) \cdot \exp\left(-\pi \cdot \frac{j_1 \sigma_1(k)^2 + j_2 \sigma_2(k)^2}{N}\right)
\]
with \( j_1 \) and \( j_2 \) bounded by \( 2\mathcal{C} \). Hence, (51) turns into
\[
T_{11} \ll (\log x) \delta^2 \sup_{x^\varepsilon \leq K \leq x^{1+\varepsilon}} \sum_{1 \leq j \leq 2\mathcal{C}} |\Sigma_{K,j}| + O \left( x^{-100} \right),
\]
where the summation condition on \( j \) means that \( 1 \leq j_1, j_2 \leq 2\mathcal{C} \). In the following, we bound \( |\Sigma_{K,j}| \).

### 6.3. Removing the exponential weights
Whereas in the later treatment of the type I sums, the weight \( E_j(p, mn) \) will be essential for performing Poisson summation in the smooth sum over \( n \), it is convenient to remove it when dealing with the type II sums. This will be done using inverse Mellin transform. Smooth weights of a more suitable shape will be re-introduced after applying Cauchy-Schwarz.

We begin by writing the Gaussian as an inverse Mellin transform in the form
\[
\exp \left( -y^2 \right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} |y|^{-s} \frac{\Gamma(s/2)}{2} ds
\]
for all \( y \in \mathbb{R} \), where \( c \) is any positive number. This implies
\[
\begin{align*}
\exp \left( -\pi \cdot \frac{\sigma_1(pmn)^2}{4d} \right) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s/2)}{\sqrt{4dN}} ds_1, \\
\exp \left( -\pi \cdot \frac{\sigma_2(pmn)^2}{4d} \right) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s/2)}{\sqrt{4dN}} ds_2, \\
\exp \left( -\pi \cdot \frac{j_1\sigma_1(mn)^2}{N} \right) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s/2)}{\sqrt{N}} ds_3, \\
\exp \left( -\pi \cdot \frac{j_2\sigma_2(mn)^2}{N} \right) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s/2)}{\sqrt{N}} ds_4.
\end{align*}
\]
Write \( s := (s_1, s_2, s_3, s_4) \) and \( ds := ds_1ds_2ds_3ds_4 \). Then it follows that
\[
\Sigma_{K,j} = \frac{1}{(4\pi i)^4} \int_{c-i\infty}^{c+i\infty} \cdots \int_{c-i\infty}^{c+i\infty} j_1^{-s_1/2} j_2^{-s_2/2} \Phi(s) \prod_{i=1}^4 \Gamma \left( \frac{s_i}{2} \right) \Sigma_{K}(s) ds,
\]
where
\[
\Phi(s) := \left( \frac{N}{\pi} \right)^{(s_1+s_2+s_3+s_4)/2} (4d)^{(s_1+s_2)/2} \delta^{-(s_1+s_2)}
\]
and
\[
\Sigma_K(s) := \sum_{m \in \mathcal{R}} \sum_{n \in \mathcal{O}(\{0\}) \leq (x/K)^{1/2}} \sum_{p \in \mathcal{O}(\{0\}) \leq x^{1/3}(x/K)^{1/2}} \sum_{\omega \in \{\pm 1\}} c_p(s_1, s_2) \mathcal{E}(pmn)
\]
with
\[
\begin{align*}
a_m(s) := & a_m|\sigma_1(m)|^{-s_1-s_3}|\sigma_2(m)|^{-s_2-s_4}, \\
b_n(s) := & b_n|\sigma_1(n)|^{-s_1-s_3}|\sigma_2(n)|^{-s_2-s_4}, \\
c_p(s_1, s_2) := & |\sigma_1(p)|^{-s_1}|\sigma_2(p)|^{-s_2}.
\end{align*}
\]
We set
\[
c := \frac{1}{\log x},
\]
Then, if
\[
N^{-1/2} \leq \delta \leq 1,
\]
we have
\[
\Phi(s) = O(1)
\]
and
\[
a_m(s) \ll |a_m| \leq 1, \quad b_n(s) \ll |b_n| \leq d(n), \quad c_p(s_1, s_2) = O(1)
\]
for all \( s \) with \( \Re(s_i) = c \) for \( i = 1, \ldots, 4 \) and \( m, n, p \) in the relevant summation ranges.

### 6.4 Applying Cauchy-Schwarz

Next we bound \( \Sigma_K(s) \). We first re-arrange summations and use the triangle inequality and the bounds \( c_p(s_1, s_2) = O(1) \) and \( a_m = O(1) \) to get

\[
\Sigma_K(s) \ll \sum_{p \in \mathbb{O}\setminus\{0\}} \sum_{n \in \mathcal{R}} \sum_{\sigma_1, \sigma_2 \leq N/m \leq 2K} b_n(s)\mathcal{E}(pmn) .
\]

Now we apply the Cauchy-Schwarz inequality and the definition of \( \mathcal{R} \) to get

\[
|\Sigma_K(s)|^2 \ll x^{2\varepsilon}\delta^{-2}K \quad \sum_{p \in \mathbb{O}\setminus\{0\}} \sum_{n \in \mathcal{R}} \sum_{\sigma_1, \sigma_2 \leq N/m \leq 2K} b_n(s)\mathcal{E}(pmn) .
\]

Writing \( k = pm \) and using the definition of \( \mathcal{R} \), we deduce that

\[
|\Sigma_K(s)|^2 \ll x^{2\varepsilon}\delta^{-2}K \quad \sum_{k \in \mathbb{O}\setminus\{0\}} \left( \sum_{p \in \mathbb{O}} \sum_{m \in \mathcal{R}} \frac{1}{pm=k} \right) \sum_{n \in \mathcal{R}} b_n(s)\mathcal{E}(kn) .
\]

for some constant \( c > 0 \). Since the number of ideal divisors of \( k \) is bounded by \( O(N(k)^c) \) and \( m \) runs over mutually non-associate elements of \( \mathcal{O} \), we have

\[
\sum_{p, m \in \mathcal{O}} \frac{1}{pm=k} \ll x^{\varepsilon}.
\]

Furthermore, we introduce a smooth weight to extend the summation over \( k \) to all integers, obtaining

\[
|\Sigma_K(s)|^2 \ll x^{3\varepsilon}\delta^{-2}K \sum_{k \in \mathcal{O}} \exp \left( -\pi \cdot \frac{\sigma_1(k)^2 + \sigma_2(k)^2}{x^{2\varepsilon}\delta^{-2}K} \right) \sum_{n \in \mathcal{O}\setminus\{0\}} b_n(s)\mathcal{E}(kn) .
\]

We expand the square, use the fact that \( \mathcal{E}(l) = \mathcal{E}(-l) \) and move in the summation over \( k \) to deduce that

\[
|\Sigma_K(s)|^2 \ll x^{3\varepsilon}\delta^{-2}K \sum_{k \in \mathcal{O}} b_n(s)b_{n_2}(s) \times \sum_{n_1, n_2 \in \mathbb{O}\setminus\{0\}} \sum_{\sigma_1, \sigma_2 \leq N/m \leq 2K} \exp \left( -\pi \cdot \frac{\sigma_1(k)^2 + \sigma_2(k)^2}{x^{2\varepsilon}\delta^{-2}K} \right) \mathcal{E}(k(n_1 - n_2)) .
\]

Using \( b_n(s) \ll d((n_1)) \ll x^{\varepsilon} \), and writing \( n = n_1 - n_2 \), it follows that

\[
|\Sigma_K(s)|^2 \ll x^{5\varepsilon}\delta^{-2}K \sum_{n \in \mathcal{O}} \sum_{\sigma_1, \sigma_2 \leq N/m \leq 2K} \left( \sum_{n_1 - n_2 = n} 1 \right) \times \sum_{k \in \mathcal{O}} \exp \left( -\pi \cdot \frac{\sigma_1(k)^2 + \sigma_2(k)^2}{x^{2\varepsilon}\delta^{-2}K} \right) \mathcal{E}(kn) .
\]

Clearly,

\[
\sum_{n_1 - n_2 = n} 1 \ll xK^{-1},
\]

and the bound follows.
Our approach is similar to that in the settings of rational or Gaussian primes, namely to use

$$|\Sigma_K(s)|^2 \ll x^{1+\varepsilon} \delta^{-2} \sum_{n \in \mathcal{O}} \left| \sum_{k \in \mathcal{O}} \exp \left( -\pi \cdot \frac{\sigma_1(k)^2 + \sigma_2(k)^2}{x^{2\varepsilon} \delta^{-2} K} \right) \mathcal{E}(kn) \right|.$$  

6.5. Applying Poisson summation. Now we use again the Poisson summation formula to transform the sum over $k$ above. For $(u, v) \in \mathbb{R}^2$ set

$$f(u, v) := \exp \left( -\pi \cdot \frac{2(u^2 + v^2 \delta^2)}{x^{2\varepsilon} \delta^{-2} K} \right) \cdot e \left( \frac{(\sigma_2(n)x_2 - \sigma_1(n)x_1)u - (\sigma_2(n)x_2 + \sigma_1(n)x_1)v\sqrt{\delta}}{2\sqrt{\delta}} \right)$$

so that

$$\exp \left( -\pi \cdot \frac{\sigma_1(k)^2 + \sigma_2(k)^2}{x^{2\varepsilon} \delta^{-2} K} \right) \mathcal{E}(kn) = f(u, v)$$

if $k = u + v\sqrt{\delta}$ with $(u, v) \in \mathbb{Z}^2$. Clearly, $f \in L^1(\mathbb{R}^2)$. We calculate the Fourier transform of $f$ to be

$$\hat{f}(\alpha, \beta) = \frac{1}{2\sqrt{\delta}} \cdot x^{2\varepsilon} \delta^{-2} K \times \exp \left( \frac{-\pi}{2} \cdot \frac{x^{2\varepsilon} \delta^{-2} K}{2\sqrt{\delta}} \left( \beta - \sigma_2(n)x_2 - \sigma_1(n)x_1 \right)^2 + \left( \alpha - \frac{(\sigma_2(n)x_2 + \sigma_1(n)x_1)}{2} \right)^2 \right).$$

Obviously, this function is also in $L^1(\mathbb{R}^2)$. Hence, by Lemma 10, we get

$$\sum_{k \in \mathcal{O}} \exp \left( -\pi \cdot \frac{\sigma_1(k)^2 + \sigma_2(k)^2}{x^{2\varepsilon} \delta^{-2} K} \right) \mathcal{E}(kn) = \frac{1}{2\sqrt{\delta}} \cdot x^{2\varepsilon} \delta^{-2} K \times \sum_{(\alpha, \beta) \in \mathbb{R}^2} \exp \left( \frac{-\pi}{2} \cdot \frac{x^{2\varepsilon} \delta^{-2} K}{2\sqrt{\delta}} \left( \beta - \sigma_2(n)x_2 - \sigma_1(n)x_1 \right)^2 + \left( \alpha - \frac{(\sigma_2(n)x_2 + \sigma_1(n)x_1)}{2} \right)^2 \right).$$

which is negligibly small if

$$\left| \frac{\sigma_2(n)x_2 - \sigma_1(n)x_1}{2\sqrt{\delta}} \right| > \delta K^{-1/2}$$

or

$$\left| \frac{\sigma_2(n)x_2 + \sigma_1(n)x_1}{2} \right| > \delta K^{-1/2}$$

and bounded by $O \left( x^{2\varepsilon} \delta^{-2} K \right)$ otherwise. Hence, we obtain

$$|\Sigma_K(s)|^2 \ll x^{1+7\varepsilon} \delta^{-4} K \times \sum_{n \in \mathcal{O}} \chi_J \left( \left| \frac{\sigma_2(n)x_2 - \sigma_1(n)x_1}{2\sqrt{\delta}} \right| \right) : \chi_J \left( \left| \frac{\sigma_2(n)x_2 + \sigma_1(n)x_1}{2} \right| \right),$$

where $\chi_J$ is the characteristic function of the interval

$$J := \left[ -\delta K^{-1/2}, \delta K^{-1/2} \right].$$

7. Counting problem

We are now down to a counting problem reminiscent of that appearing in the treatment of the same problem in the settings of rational or Gaussian integers. However, the present counting problem has a different flavor since it is a truly 2-dimensional problem. Indeed, we will see that new ingredients are required such as results about the approximation of roots of quadratic congruences. Our approach is similar to that in the settings of rational or Gaussian primes, namely to use Diophantine approximation to replace $x_1$ and $x_2$ by elements of $\mathbb{Q}(\sqrt{\delta})$. 

7.1. **Approximating** \((x_1, x_2)\). Dirichlet’s approximation theorem for \(\mathbb{Q}(\sqrt{d})\) gives us, in a precise sense, a set of \(\theta \in \mathbb{Q}(\sqrt{d})\) such that \(x_1\) and \(x_2\) have a good simultaneous approximation by \(\sigma_1(\theta)\) and \(\sigma_2(\theta)\), respectively. We will work out the details in section 9. Assume for now we have an approximation satisfying

\[
|x_i - \sigma_i(\theta)| \leq \Delta \quad \text{for } i = 1, 2,
\]

where \(\theta \in \mathbb{Q}(\sqrt{d})\). Then it follows from (56) that

\[
|\Sigma_K(s)|^2 \ll x^{1 + 7\varepsilon} \delta^{-4} K \times \sum_{n \in \mathbb{Q}} \chi_J \left( \left\| \frac{\sigma_2(n\theta) - \sigma_1(n\theta)}{2\sqrt{d}} \right\| \right) \cdot \chi_J \left( \left\| \frac{\sigma_2(n\theta) + \sigma_1(n\theta)}{2} \right\| \right),
\]

where

\[
\mathcal{J} := [-\tilde{\Delta}, \tilde{\Delta}]
\]

with

\[
\tilde{\Delta} := \delta K^{-1/2} + 2x^{1/2}K^{-1/2} \Delta.
\]

Writing

\[
n = \lambda + \mu \sqrt{d}, \quad \theta = \sigma + \tau \sqrt{d}
\]

with \(\lambda, \mu \in \mathbb{Z}\) and \(\sigma, \tau \in \mathbb{Q}\), the above turns into

\[
|\Sigma_K(s)|^2 \ll x^{1 + 7\varepsilon} \delta^{-4} K \sum_{(\lambda, \mu) \in \mathbb{Z}^2} \chi_{\mathcal{J} \times \mathcal{J}} \left( \left\| \left( \begin{array}{c} \sigma \\ \tau \\
\end{array} \right) \right\| \right) \cdot \chi_J \left( \left\| \lambda \sigma + \mu \tau d \right\| \right),
\]

which may be rewritten as

\[
|\Sigma_K(s)|^2 \ll x^{1 + 7\varepsilon} \delta^{-4} K \sum_{(\lambda, \mu) \in \mathbb{Z}^2} \chi_{\mathcal{J} \times \mathcal{J}} \left( \left\| \left( \begin{array}{c} \sigma \\ \tau \\
\end{array} \right) \right\| \right) \cdot \chi_J \left( \left\| \lambda \sigma + \mu \tau d \right\| \right),
\]

where we set

\[
\left\| \left( \begin{array}{c} x \\ y \\
\end{array} \right) \right\| := \left( \left\| x \right\| \right).
\]

Hence, if

\[
\theta = \frac{a}{W} + \frac{b}{W} \sqrt{d} \quad \text{with } a, b \in \mathbb{Z}, \quad W \in \mathbb{N},
\]

then

\[
|\Sigma_K(s)|^2 \ll x^{1 + 7\varepsilon} \delta^{-4} K \sum_{(\lambda, \mu) \in \mathbb{Z}^2} \chi_{\mathcal{J} \times \mathcal{J}} \left( \left\| \left( \begin{array}{c} a/W \\ b/W \\
\end{array} \right) \right\| \right) \cdot \chi_J \left( \left\| \left( \begin{array}{c} \lambda \\ \mu \\
\end{array} \right) \right\| \right).
\]

Finally, we may express the above using congruences modulo \(W\) as

\[
|\Sigma_K(s)|^2 \ll x^{1 + 7\varepsilon} \delta^{-4} K \sum_{(\lambda, \mu) \in \mathbb{Z}^2} \sum_{(\alpha, \beta) \in \mathbb{Z}^2} 1,
\]

where “\(C\)” stands for the congruence condition

\[
\left( \begin{array}{cc} a & bd \\ b & a \\
\end{array} \right) \left( \begin{array}{c} \lambda \\ \mu \\
\end{array} \right) \equiv \left( \begin{array}{c} \alpha \\ \beta \\
\end{array} \right) \mod W
\]

and

\[
U := 2x^{1/2}K^{-1/2} \quad \text{and} \quad V := W \tilde{\Delta}.
\]
7.2. A congruence relation between $a$ and $b$. In our application of the Dirichlet approximation theorem for $\mathbb{Q}(\sqrt{d})$ in section 9, we shall write $\theta$ in the form

\begin{equation}
\theta = \frac{u + v\sqrt{d}}{f + g\sqrt{d}}
\end{equation}

where $u, v, f, g \in \mathbb{Z}$ and

\begin{equation}
gcd(u + v\sqrt{d}, f + g\sqrt{d}) \approx 1 \text{ in } \mathcal{O}.
\end{equation}

Further, we shall make the natural choice

\[ W := N(f + g\sqrt{d}) = |f^2 - g^2d| \]

so that

\[ |a + b\sqrt{d}| = |(u + v\sqrt{d})(f - g\sqrt{d})|. \]

It follows that

\[ |a^2 - b^2d| = N(a + b\sqrt{d}) = N(u + v\sqrt{d})W \]

and hence

\begin{equation}
a^2 - b^2d \equiv 0 \mod W.
\end{equation}

This congruence relation will be crucial in what follows.

7.3. Counting solutions of systems of linear congruences. Recalling (61), we need to detect $\alpha, \beta, \lambda, \mu$ satisfying the system of congruences

\begin{equation}
\begin{cases}
a\lambda + bd\mu & \equiv \alpha \mod W, \\
b\lambda + a\mu & \equiv \beta \mod W.
\end{cases}
\end{equation}

Multiplying the first congruence with $b$ and the second with $a$ and then subtracting and using (65), we get

\begin{equation}
ba \equiv a\beta \mod W.
\end{equation}

We first count pairs $(\alpha, \beta)$ lying the relevant range for which (67) is satisfied. Then we fix $(\alpha, \beta)$ and count how many pairs $(\lambda, \mu)$ in the relevant range satisfy (66). After fixing $(\alpha, \beta)$, assuming that $(\lambda_0, \mu_0)$ is one particular solution, any other solution of (66) is of the form $(\lambda_0 + \tilde{\lambda}, \mu_0 + \tilde{\mu})$, where

\begin{equation}
\begin{cases}
a\tilde{\lambda} + bd\tilde{\mu} & \equiv 0 \mod W, \\
b\tilde{\lambda} + a\mu & \equiv 0 \mod W.
\end{cases}
\end{equation}

We throw away the first congruence and just count the number of pairs $(\tilde{\lambda}, \tilde{\mu})$ satisfying

\begin{equation}
b\tilde{\lambda} + a\tilde{\mu} \equiv 0 \mod W,
\end{equation}

where $(\lambda_0 + \tilde{\lambda}, \mu_0 + \tilde{\mu})$ lies in the relevant range, i.e.

\[ |\lambda_0 + \tilde{\lambda}|, |\mu_0 + \tilde{\mu}| \leq U. \]

Since

\[ |\lambda_0|, |\mu_0| \leq U, \]

it suffices to count $(\tilde{\lambda}, \tilde{\mu})$ satisfying (69) and

\[ |\tilde{\lambda}|, |\tilde{\mu}| \leq 2U. \]

So relabeling $\tilde{\lambda}$ and $\tilde{\mu}$ as $\lambda$ and $-\mu$, respectively, we are led to counting solutions $(\alpha, \beta, \lambda, \mu)$ of the system of independent congruences

\begin{equation}
\begin{cases}
ba & \equiv a\beta \mod W, \\
b\lambda & \equiv a\mu \mod W,
\end{cases}
\end{equation}

subject to the conditions

\[ |\lambda|, |\mu| \leq 2U \quad \text{and} \quad |\alpha|, |\beta| \leq V. \]

These congruences are of the same shape, and therefore it suffices to count the number $\Theta(X; a, b; W)$ of solutions $(A, B)$ with

\[ |A|, |B| \leq X \]

of the single congruence

\begin{equation}
bB \equiv aA \mod W.
\end{equation}
In summary, we conclude from (72) that

\[ |\sum_{s} K(t)|^2 \ll x^{1+7\varepsilon \delta-4K} \cdot \Theta(2U; a, b; W) \cdot \Theta(V; a, b; W). \]

7.4. Detecting congruences using additive characters. Next we bound \( \Theta(X; a, b; W) \) under the condition that

\[ a = Za', \quad b = Zb', \quad W = Z^2W', \quad \gcd(b', W') = 1 \]

or

\[ a = Za', \quad b = Zb', \quad W = Z^2W', \quad \gcd(a', W') = d \]

for suitable \( a', b', W', Z \in \mathbb{Z} \) with \( W', Z > 0 \). In section 9, we will prove that (73) or (74) holds.

We observe that under both (73) and (74), (65) is equivalent to

\[ (a')^2 \equiv (b')^2d \mod W' \]

on taking out a factor of \( Z^2 \), and (71) is equivalent to

\[ b'B \equiv a'A \mod W'Z \]

on taking out a factor of \( Z \). Moreover, (76) implies the congruence

\[ b'B \equiv a'A \mod W' \]

which we will use in place of (76) because we are not able to utilize the factor \( Z \) in (76). Further, under (73), if \( b' \) is a multiplicative inverse of \( b' \mod W' \) (i.e., \( b'b' \equiv 1 \mod W' \)), then (77) is equivalent to

\[ B \equiv a'B \mod W'. \]

We will confine ourselves to treating the case when (73) is satisfied because the other case is very similar and gives the same results. Indeed, in the case of (74), we write \( a'' = a'/d \) and \( W'' = W'/d \) and reduce (75) further to

\[ (a'')^2d \equiv (b')^2 \mod W''. \]

Moreover, we deduce from (77) that

\[ b'B \equiv a''Ad \mod W'' \]

and, noting \( \gcd(a'', W'') = 1 \), transform this into

\[ Ad \equiv b'a''B \mod W''. \]

where \( a'' \) is a multiplicative inverse modulo \( W'' \). The counting problem now takes the same shape as above, with the replacements

\[ W' \rightarrow W'' \]
\[ a' \rightarrow b' \]
\[ b' \rightarrow a'' \]
\[ A \rightarrow B \]
\[ B \rightarrow Ad. \]

An inspection of the method below shows that the occurrence of an extra factor of \( d \) in the last line presents no problems. Thus, we assume (73) to hold throughout the following.
We detect the congruence (78) using additive characters, writing
\[ \Theta(X; a, b; W) \leq \sum_{|A| |B| \leq X \atop B \equiv a \overline{f} A \mod W'} \frac{1}{W'} \sum_{h=0}^{W'-1} e \left( \frac{h(B - a \overline{f} A)}{W'} \right) \]
\[ = \frac{1}{W'} \sum_{|A| |B| \leq X} \sum_{h=0}^{W'-1} e \left( \frac{hB}{W'} \right) \sum_{|A| \leq X} e \left( \frac{ha \overline{f} A}{W'} \right) \]
\[ \ll \frac{1}{W'} \sum_{0 \leq h \leq W'/2} \min \left\{ X, \left\| \frac{h}{W'} \right\|^{-1} \right\} + \sum_{W'/X < h \leq W'/2} \frac{1}{h} \min \left\{ X, \left\| \frac{ha \overline{f}}{W'} \right\|^{-1} \right\} \]
\[ \ll (\log 2W') \sup_{H \leq X \atop W'/X < H < W'} \frac{1}{H} \sum_{0 \leq h \leq H} \min \left\{ X, \left\| \frac{ha \overline{f}}{W'} \right\|^{-1} \right\} \]
if \( X \geq 1 \). Now we use the following lemma, which is a standard tool in this circle of problems.

**Lemma 14.** Suppose that \( X, H \geq 1 \) and \( \gamma \in \mathbb{R} \) satisfies
\[ |\gamma - \frac{u}{r}| \ll r^{-2} \]
for some \( u \in \mathbb{Z} \) and \( r \in \mathbb{N} \) with \( \gcd(u, r) = 1 \), then
\[ \sum_{0 \leq h \leq H} \min \left\{ X, \left\| \frac{h\gamma}{r} \right\|^{-1} \right\} \ll \left( 1 + \frac{H}{r} \right) (X + r) \log 2r, \]
where the implied constant in (81) depends only on that in (80).

**Proof.** This can be proved in a similar way as [3, Lemma 6.4.4] by dividing the summation range into intervals of length \( r \).

Applying Lemma 14 to the last line of (79), we deduce that
\[ \Theta(X; a, b; W) \ll \left( \frac{X^2}{W'} + \frac{rX}{W'} + \frac{X}{r} + 1 \right) (\log 2r)(\log 2W'), \]
provided we have a Diophantine approximation of the form
\[ \left| \frac{a \overline{f}}{W'} - \frac{u}{r} \right| \ll r^{-2} \text{ with } \gcd(u, r) = 1. \]

We shall prove in section 8 that this is the case for some \( r \gg \sqrt{W'} \), which is a crucial point in this article. Therefore, (82) turns into
\[ \Theta(X; a, b; W) \ll \left( \frac{X^2}{W'} + 1 \right) \log^2 2W' \ll \left( \frac{X^2 Z^2}{W} + 1 \right) \log^2 W. \]

7.5. **Estimating the type II sum.** Plugging (84) into (72) and recalling the definitions of \( U \) and \( V \) in (62) and \( \Delta \) in (59), we obtain the estimate
\[ |\Sigma_K(8)|^2 \ll x^{1+\delta - 4} K \left( xK^{-1} W^{-1} Z^2 + 1 \right) \left( \delta^2 + xK \right) \]
\[ = x^{\delta - 4} \left( (\delta^2 + xK) \left( x^2 K^{-1} Z^2 + xW Z^2 \right) + x^2 W^{-1} Z^2 + xK \right), \]
provided that
\[ W \ll x^{100}. \]
Taking square-root on both sides of (85), plugging the resulting estimate into (53), recalling (54) and (55), applying Stirling’s formula to bound the Gamma factors, and integrating, we obtain

\[ |\Sigma_{K,j}| \ll x^{5\varepsilon} \delta^{-2} \left( \left( \delta + x^{1/2} \Delta \right) \left( xK^{-1/2}Z^2 + x^{1/2}W^{1/2}Z \right) + xW^{-1/2}Z + x^{1/2}K^{1/2} \right). \]

Reversing the roles of \( m \) and \( n \) in the whole process, we get the same estimate with \( x/K \) in place of \( K \), i.e.,

\[ |\Sigma_{K,j}| \ll x^{5\varepsilon} \delta^{-2} \left( \left( \delta + x^{1/2} \Delta \right) \left( x^{1/2}K^{1/2}Z^2 + x^{1/2}W^{1/2}Z \right) + xW^{-1/2}Z + x^{1/2}K^{-1/2} \right). \]

Taking the first estimate when \( K \leq x^{1/2} \) and the second one if \( K \geq x^{1/2} \) and employing (52), we obtain

\[ T_{11} \ll x^{6\varepsilon} \left( \left( \delta + x^{1/2} \Delta \right) \left( x^{1-\nu/2}Z^2 + x^{(1+\nu+\kappa)/2}Z^2 + x^{1/2}W^{1/2}Z \right) + xW^{-1/2}Z + x^3/4 \right) \]

if

\[ \mu \leq 1/2 \leq \mu + \kappa. \]

8. Roots of quadratic congruences

Now we want to establish the important Diophantine approximation (83). We recall that we have the congruence (75) which is equivalent to

\[ (a\overline{m})^2 \equiv d \mod W'. \]

Hence, it suffices to prove the following.

**Lemma 15.** Let \( d > 1 \) be an integer which is not a perfect square. Then there exist positive constants \( c_1, c_2, c_3 \) only depending on \( d \) such that the following holds. If \( Q \) is a positive integer and \( x \) is a solution to the quadratic congruence

\[ x^2 \equiv d \mod Q, \]

then there exist \( u, r \in \mathbb{Z} \) with \( \gcd(u, r) = 1 \) such that

\[ c_1 \sqrt{Q} \leq |r| \leq c_2 \sqrt{Q} \]

and

\[ \left| \frac{x}{Q} - \frac{u}{r} \right| \leq \frac{c_3}{r^2}. \]

8.1. Relation between quadratic congruences and quadratic forms. Our approach follows Hooley’s treatment of roots of quadratic congruences in [11, section] and makes use of the theory of binary quadratic forms. Recall that the discriminant of the quadratic form

\[ ax^2 + 2hxy + by^2 \]

is the quantity

\[ D = h^2 - ab. \]

If the congruence

\[ x^2 \equiv d \mod Q \]

has a solution \( \omega \), then the quadratic form

\[ Qx^2 + 2\omega xy + \frac{\omega^2 - d}{Q}y^2 \]

represents \( Q \) for \( (x, y) = (1, 0) \) and has discriminant \( d \). We shall turn it into a reduced form and then express \( \omega/Q \) in a way which is sufficient to prove Lemma 15.

Any form equivalent to (89) is obtained via a change of variables

\[ \begin{pmatrix} r & \beta \\ s & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}, \]

where the matrix above is in \( \text{SL}_2(\mathbb{Z}) \). In particular, \( (x, y) = (1, 0) \) is taken to \( (x', y') = (r, s) \) via this change of variables. Therefore, any form

\[ ax^2 + 2hxy + by^2 \]

equivalent to (89) represents \( Q \) in the form

\[ ar^2 + 2hrs + bs^2 = Q, \]
where \( r \) and \( s \) are relatively prime integers such that for suitable relatively prime integers \( \alpha \) and \( \beta \), one has
\[
(92) \quad r\alpha - s\beta = 1
\]
and
\[
(93) \quad Qx^2 + 2\omega xy + \frac{\omega^2 - d}{Q} y^2 = a(rx + \beta y)^2 + 2hr(rx + \beta y)(sx + \alpha y) + b(sx + \alpha y)^2.
\]
Comparing the coefficients of \( xy \) on both sides, we calculate that
\[
\omega = \frac{ar\beta + hrs\beta + brs\alpha}{r^2 + 2hrs + bs^2},
\]
and hence, using (92) and (91),
\[
\frac{\omega}{Q} = \frac{ar\beta + h(r\alpha + s\beta) + bs\alpha}{ar^2 + 2hrs + bs^2} = \frac{ar^2\beta + hr\alpha + hrs\beta + brs\alpha}{r(ar^2 + 2hrs + bs^2)} = \frac{ar^2\beta + hr(s\beta + 1) + hrs\beta + bs(s\beta + 1)}{r(ar^2 + 2hrs + bs^2)} = \frac{\beta(ar^2 + 2hrs + bs^2) + hr + bs}{r(ar^2 + 2hrs + bs^2)} = \frac{\beta + \frac{hr + bs}{rQ}}{r},
\]
where \( \gcd(\beta, r) = 1 \).

8.2. Evaluating the approximation. Thus we have established the approximation (94) for the ratio of a root \( \omega \) of the quadratic congruence \( x^2 \equiv d \mod Q \) and its modulus \( Q \). To complete the proof of Lemma 15, we now assume without loss of generality that (90) is reduced and establish that for suitable solutions \((r, s)\) of (91), we have
\[
|r| \asymp \sqrt{Q} \quad \text{and} \quad \frac{hr + bs}{rQ} \ll \frac{1}{r^2},
\]
where the implied constants are only allowed to depend on \( d \). Since there are only finitely many reduced forms of discriminant \( d \), we may treat \( a, h, b \) like constants, and it suffices to establish that given \( a, h, b \), we have
\[
(95) \quad |r| \asymp \sqrt{Q} \quad \text{and} \quad |s| \ll |r|
\]
for suitable integers \( r, s \) satisfying (91), where the implied constants in (95) depend only on \( a, b, h \).

We use the following information given in [11, page 109]. Without loss of generality, we may take \( a \) to be positive and \( b \) to be negative. Let
\[
m := \gcd(a, h, b)
\]
and \((T, U)\) the least solution of the Pellian equation
\[
T^2 - dU^2 = m.
\]
Then there exists precisely one solution \((r, s)\) of (91) such that
\[
r > 0 \quad \text{and} \quad 0 < s \leq \frac{aU}{T - hU} \cdot r.
\]
Moreover, on the set
\[
S := \left\{(x, y) \in \mathbb{R} : x > 0, \ 0 < y \leq \frac{aU}{T - hU} \cdot x\right\},
\]
the form
\[
a x^2 + 2hxy + by^2
\]
takes positive values only. With this information, we are able to finish off our proof easily. Let
\[
\eta_1 := \min_{y \in [0, aU/(T - hU)]} (a + 2hy + by^2)
\]
and
\[ \eta_2 := \max_{y \in [0, u t/(f^2 - u t^2)]} (a + 2by + by^2). \]

From the above, \( 0 < \eta_1 < \eta_2 \), and
\[ \eta_1 x^2 \leq ax^2 + 2hxy + by^2 \leq \eta_2 x^2 \]
whenever \((x, y) \in S\). Hence, \( r \approx \sqrt{1} \), and the claim (95) is established. This completes the proof of Lemma 15.

9. Dirichlet approximation in \( \mathbb{Q}(\sqrt{d}) \)

9.1. Diophantine approximation. Now we use Dirichlet’s approximation theorem in \( \mathbb{Q}(\sqrt{d}) \) to approximate \( x_1 \) and \( x_2 \) simultaneously. By Corollary 3, there is an infinite increasing sequence of natural numbers \( q \) such that
\[
\begin{align*}
|x_1 - \frac{u + v \sqrt{d}}{f + g \sqrt{d}}| &\leq \frac{C}{W}, \\
|x_2 - \frac{u - v \sqrt{d}}{f - g \sqrt{d}}| &\leq \frac{C}{W},
\end{align*}
\]
(96)
where \( u, v, f, g \) are rational integers, \( u + v \sqrt{d} \) and \( f + g \sqrt{d} \) are relatively prime in \( \mathcal{O} \),
\[ W := N(f + g \sqrt{d}), \]
and \( C \) is a positive constant only depending on \( d \). We further write
\[ \frac{u + v \sqrt{d}}{f + g \sqrt{d}} = \frac{a}{W} + \frac{b}{W} \sqrt{d} \]
with \( a, b \in \mathbb{Z} \).

9.2. Reducing \( a, b \) and \( W \). Now we want to show that \( a, b \) and \( W \) allow for a reduction in the form given in (73) or (74). We have
\[
\begin{align*}
a + b \sqrt{d} &= (u + \sqrt{d}v)(f - \sqrt{d}g), \\
W &= |(f + g \sqrt{d})(f - g \sqrt{d})|
\end{align*}
\]
and
\[ \gcd(u + v \sqrt{d}, f + g \sqrt{d}) \approx 1 \approx \gcd(u - v \sqrt{d}, f - g \sqrt{d}). \]

Let
\[ Z := \gcd(f, g) \]
and
\[ f' := \frac{f}{Z}, \quad g' := \frac{g}{Z}, \quad W' := \frac{W}{Z^2} \]
so that \( \gcd(f', g') = 1 \). Then by (97), we also have \( Z | a \) and \( Z | b \). Let
\[ a' := \frac{a}{Z}, \quad b' := \frac{b}{Z}. \]

It follows that
\[
\begin{align*}
a' + b' \sqrt{d} &= (u + v \sqrt{d})(f' - g' \sqrt{d}), \\
W' &= |(f' + g' \sqrt{d})(f' - g' \sqrt{d})|
\end{align*}
\]
and
\[ \gcd(u + v \sqrt{d}, f' + g' \sqrt{d}) \approx 1 \approx \gcd(u - v \sqrt{d}, f' - g' \sqrt{d}). \]

Assume that
\[ \gcd\left(f' + g' \sqrt{d}, f' - g' \sqrt{d}\right) \approx t. \]
(101)
Then, since
\[ (f' + g' \sqrt{d}) + (f' - g' \sqrt{d}) = 2f' \quad \text{and} \quad (f' + g' \sqrt{d}) - (f' - g' \sqrt{d}) = 2g' \sqrt{d}, \]
it follows that \( t | 2f' \) and \( t | 2g' \sqrt{d} \). Since \( \gcd(f', g') = 1 \) (and hence \( \gcd(f', g') \approx 1 \) in \( \mathcal{O} \)), this implies \( t | 2\sqrt{d} \). But then \( t | \sqrt{d} \) because otherwise \( 2 | t \) and hence \( 2 | f' \) and \( 2 | g' \) by (101) which contradicts...
the coprimality of $f'$ and $g'$. Now the only possibilities are $t = \pm \sqrt{d}$ and $t = \pm 1$. In the first case, $d|f'$ and necessarily $\gcd(g', d) = 1$ because otherwise $\gcd(f', g') \neq 1$. Using the equations
\begin{equation}
2a' = (u + v\sqrt{d})(f' - g'\sqrt{d}) + (u - v\sqrt{d})(f' + g'\sqrt{d})
\end{equation}
and
\begin{equation}
2b'\sqrt{d} = (u + v\sqrt{d})(f' - g'\sqrt{d}) - (u - v\sqrt{d})(f' + g'\sqrt{d})
\end{equation}
(which follow from (98)) together with (100), we deduce that
\[ \gcd(2a', f' \pm g'\sqrt{d}) \approx t \approx \gcd\left(2b'\sqrt{d}, f' \pm g'\sqrt{d}\right). \]
If $t = \pm 1$, then using (99), it follows that $\gcd(b', W') = 1$. If $t = \pm \sqrt{d}$, then $\sqrt{d}|2a'$ and hence $d|2a'$, from which we deduce that $\gcd(2a', W') = d$. Since we assumed that $d \equiv 3 \mod 4$, it follows that $\gcd(a', W') = d$. In summary, we have either (73) or (74).

Now we have filled all gaps in section 7. Further, in view of (96), we may take
\[ \theta = \frac{u + v\sqrt{d}}{f + g\sqrt{d}} \]
and
\begin{equation}
\Delta = \frac{C}{W}
\end{equation}
in (57). Hence, our estimate (87) for the type II sum turns into
\begin{equation}
T_{II} \ll x^{\delta c} \left( \left( \delta + x^{1/2}w^{-1} \right) \left( x^{-\mu/2}Z^2 + x^{(1+\mu+\nu)/2}Z^2 + x^{1/2}W^{1/2}Z \right) + xW^{-1/2}Z + x^{3/4} \right).
\end{equation}
The appearance of powers of $Z$ above presents a serious problem. To overcome this, we introduced the notions of good and bad $(x_1, x_2)$ in subsection (1.4). Recalling definition 4, if $(x_1, x_2)$ is $\eta$-good, then (105) yields
\begin{equation}
T_{II} \ll x^{\delta c} \left( \left( \delta + x^{1/2}w^{-1} \right) \left( x^{-\mu/2}W^{2\eta} + x^{(1+\mu+\nu)/2}W^{2\eta} + x^{1/2}W^{1/2+\eta} \right) + xW^{-\eta/2} + x^{3/4} \right).
\end{equation}
We note that we may assume, without loss of generality, that $\eta \leq 1/2$ since $Z^2 \leq W$. In the next subsection we shall show that, in the sense of the Lebesgue measure, for every $\eta \in (0, 1/2]$, almost all $(x_1, x_2)$ are $\eta$-good, which implies that almost all $(x_1, x_2)$ are good.

9.3. Almost all $(x_1, x_2)$ are good. Now we provide a measure theoretical proof of the following result. Here we point out that $\sigma(K)$ has Lebesgue measure 0 in $\mathbb{R}^2$.

**Theorem 16.** Given $\eta > 0$, almost all $(x_1, x_2) \in \mathbb{R}^2 \setminus \sigma(K)$ are $\eta$-good.

**Proof.** Throughout this proof, we set $\mathcal{X} := \mathbb{R}^2 \setminus \sigma(K)$, $\alpha := (x_1, x_2)$ and $\pi := \sigma(n)$ if $n \in \mathcal{O}$ for convenience. If $q \in \mathcal{O} \setminus \{0\}$, we set
\[ B(q) := \bigcup_{p \in \mathcal{O}} \left[ \left\{ \frac{p}{q} - \frac{1}{F(q)} \right\} \frac{p}{q} + \frac{1}{F(q)} \right] \times \left[ \left\{ \frac{\pi}{q} - \frac{1}{F(q)} \right\} \frac{\pi}{q} + \frac{1}{F(q)} \right] \cap \mathcal{X}, \]
where
\begin{equation}
F(q) := |q| \sqrt{\tilde{N}(q)/C}
\end{equation}
with $C$ being the positive constant from Theorem 2 and
\[ \tilde{N}(q) := \sigma_1(q)^2 + \sigma_2(q)^2. \]
Further, we set
\[ Z(q) := \gcd(f, g). \]
Noting that $F(q) \geq \tilde{N}(q)/C$, we observe that the set of $\eta$-good $\alpha$’s contains the set
\[ G = \bigcap_{\tilde{N} \in \mathbb{N}} \bigcup_{q \in \mathcal{O}(\{0\})} B(q), \]
where $M$ is any positive number. Now it suffices to prove that the complement $G^c$ of $G$ in $\mathcal{X}$ has Lebesgue measure 0.
We further note that it is sufficient to restrict ourselves to \( \alpha \)'s contained in the parallelogram \( P \) with sides given by the vectors \((1, 1)\) and \((\sqrt{d}, -\sqrt{d})\). This is for the following reason: If \( \alpha = (x_1, x_2) \in B(q) \), then also \( \tilde{\alpha} = (x_1 + a + b\sqrt{d}, x_2 + a - b\sqrt{d}) \in B(q) \) for any \( a, b \in \mathbb{Z} \). The set
\[
\{(a + b\sqrt{d}, a - b\sqrt{d}) : (a, b) \in \mathbb{Z}^2\}
\]
may be written in the form
\[
\{(a(1, 1) + b(\sqrt{d}, -\sqrt{d}) : (a, b) \in \mathbb{Z}^2\} = \Lambda,
\]
which forms a lattice in \( \mathbb{R}^2 \), generated by the vectors \((1, 1)\) and \((\sqrt{d}, -\sqrt{d})\). Hence, it suffices to show that \( P \cap G^c \) has Lebesgue measure 0.

We may write
\[
P \cap G^c = P \setminus \left( \bigcap_{N \in \mathbb{N}} \bigcup_{N \geq M} \bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q) \right),
\]
where
\[
\tilde{B}(q) := P \cap B(q).
\]
Next, we write
\[
\bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q) \supseteq \left( \bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q) \right) \setminus \left( \bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q) \right) = \left( \bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q) \right) \cap \left( \bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q) \right)^c,
\]
where the complement in the last line is taken inside \( P \cap X \). It follows that
\[
\bigcap_{N \in \mathbb{N}} \bigcup_{N \geq M} \bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q) \supseteq \left( \bigcap_{N \in \mathbb{N}} \bigcup_{N \geq M} \bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q) \right) \cap \left( \bigcap_{N \in \mathbb{N}} \bigcup_{N \geq M} \bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q) \right)^c.
\]
Using Dirichlet's approximation theorem for \( Q(\sqrt{d}) \), Theorem 2, together with the above restriction-to-\( P \) argument, the set
\[
\bigcap_{N \in \mathbb{N}} \bigcup_{N \geq M} \bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q)
\]
agrees with \( P \) up to a set of measure 0. It follows that
\[
\mu \left( P \setminus \left( \bigcap_{N \in \mathbb{N}} \bigcup_{N \geq M} \bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q) \right) \right) \leq \mu \left( \bigcup_{q \in \mathcal{O} \setminus \{0\}} \bigcup_{Z(q) > N(q)^{\gamma}} \tilde{B}(q) \right) = \mu \left( \bigcup_{Z(q) > N(q)^{\gamma}} \tilde{B}(q) \right),
\]
where \( \mu \) is the Lebesgue measure. Further,
\[
\bigcup_{q \in \mathcal{O} \setminus \{0\}} \tilde{B}(q) = \bigcup_{Z(q) > N(q)^{\gamma}} \bigcup_{Z(q) > M} \tilde{B}(q)
\]
and, setting \( q' = q/Z \) and \( q' = f' + g' \sqrt{d} \),

\[
\bigcup_{q \in \mathcal{O} \setminus \{0\}, \ M/N(q) < Z^{1/n}, \ Z(q) = Z} \hat{\mathcal{B}}(q) = \bigcup_{q' \in \mathcal{O} \setminus \{0\}, \ M/Z^2 < N(q') < Z^{1/n}, \ (f', g') = 1} \bigcup_{p \in \mathcal{O}} P \cap \left( \left[ \frac{p}{Zq'} - \frac{1}{Z^2 F(q')}, \frac{p}{Zq'} + \frac{1}{Z^2 F(q')} \right] \right) \times \left[ \frac{p}{Zq'} - \frac{1}{Z^2 F(q')}, \frac{p}{Zq'} + \frac{1}{Z^2 F(q')} \right] \subset \frac{1}{Z} \bigcup_{q \in \mathcal{O} \setminus \{0\}, \ N(q) < Z^{1/n}} \bigcup_{p \in \mathcal{O}} (\mathcal{P}) \cap \left( \left[ \frac{p}{q} - \frac{1}{ZF(q)}, \frac{p}{q} + \frac{1}{ZF(q)} \right] \right) \times \left[ \frac{p}{q} - \frac{1}{ZF(q)}, \frac{p}{q} + \frac{1}{ZF(q)} \right] \right),
\]

where in the last line, we have replaced \( q' \) by \( q \). Given \( q \), the number of \( p \)'s such that \( Z\mathcal{P} \) has non-empty intersection with the rectangle

\[
R_Z(p, q) = \left[ \frac{p}{q} - \frac{1}{ZF(q)}, \frac{p}{q} + \frac{1}{ZF(q)} \right] \times \left[ \frac{p}{q} - \frac{1}{ZF(q)}, \frac{p}{q} + \frac{1}{ZF(q)} \right]
\]

is bounded by \( O(Z^2 N(q)) \). To see this, note that

\[
F(q), F(\overline{q}) \geq N(q)
\]

for all \( q \), and for fixed \( q \), the points

\[
\left( \frac{p}{q}, \frac{p}{\overline{q}} \right), \ p \in \mathcal{O}
\]

form a lattice \( \Lambda(q) \) in \( \mathbb{R}^2 \) which contains \( \Lambda = \Lambda(1) \) as a sublattice of index \( N(q) \) in \( \Lambda(q) \). Moreover,

\[
\mu(R_Z(p, q)) = \frac{1}{Z^2 F(q) F(q)} = \frac{C^2}{Z^2 N(q) N(q)}.
\]

Combining everything and using countable subadditivity, we arrive at

\[
\mu(P \cap G^c) \ll \sum_{Z > M^n} \sum_{q \in \mathcal{O} \setminus \{0\}, \ N(q) < Z^{1/n}} \frac{1}{Z^2 N(q)}.
\]

By Lemma 9, for any given non-zero (principal) ideal \( a \), we have

\[
\sum_{q \in \mathcal{O}} \frac{1}{N(q)} = a \ll \frac{1}{N(a)}
\]

Hence,

\[
\mu(P \cap G^c) \ll \sum_{Z > M^n} \frac{1}{Z^2} \sum_{a \in \mathcal{O} \setminus \{0\}, \ N(q) < Z^{1/n}} \frac{1}{N(a)} \ll_{\eta} \sum_{Z > M^n} \frac{\log Z}{Z^2} \ll M^{-\eta/2}.
\]

Since this holds for all positive \( M \), it follows that

\[
m(P \cap G^c) = 0.
\]

This completes the proof. \( \square \)

We deduce the following.

**Corollary 17.** Almost all \((x_1, x_2) \in \mathbb{R}^2\) are good.

**Proof.** Let \( \mathcal{G}_q \) be the set of \( \eta \)-good \((x_1, x_2)\) and \( \mathcal{G} \) be the set of good \((x_1, x_2)\). Then

\[
\mathcal{G} = \bigcap_{n=1}^{\infty} \mathcal{G}_{1/n}.
\]
Hence,
\[ G^c = \bigcup_{n=1}^{\infty} G^c_{1/n} \]
and therefore
\[ \mu(G^c) \leq \sum_{n=1}^{\infty} \mu(G^c_{1/n}) = 0, \]
which completes the proof. \(\square\)

9.4. Constructing good \((x_1, x_2)\). After we have seen that almost all \((x_1, x_2)\) are good, we supplement an explicit construction of good \((x_1, x_2)\) using continued fractions. As in the previous subsection, if \(a \in \mathcal{O}\), let \(\overline{a} \in \mathcal{O}\) be the conjugate of \(a\), i.e., \(\overline{a} := \sigma_2(a)\). We pick a sequence \((a_k)_{k \in \mathbb{N} \cup \{0\}}\) in \(\mathcal{O}\) satisfying the condition
\[ \inf_{k \in \mathbb{N}} a_k > 0 \quad \text{and} \quad \inf_{k \in \mathbb{N}} \overline{a_k} > 0 \]
which we assume to hold throughout this subsection. In particular, \(a_k, \overline{a_k} > 0\) for all \(k \geq 0\). Under this condition, the continued fractions
\[ x_1 = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} \quad \text{and} \quad x_2 = \overline{a_0} + \frac{1}{\overline{a_1} + \frac{1}{\overline{a_2} + \cdots}} \]
both converge. Our idea is to choose \((a_k)_{k \in \mathbb{N} \cup \{0\}}\) in such a way that the convergents of these continued fractions produce sequences
\[ \left( \frac{a_k + v_k \sqrt{d}}{f_k + g_k \sqrt{d}}, \frac{u_k - v_k \sqrt{d}}{f_k - g_k \sqrt{d}} \right) \]
of good simultaneous approximations of \((x_1, x_2)\) with \(\gcd(f_k, g_k) = 1\), thus establishing that \((x_1, x_2)\) is good.

Similarly as in the context of continued fractions for integers, we write
\[ a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} =: [a_0 : a_1, a_2, \ldots] \]
and its convergents as
\[ a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_k}}} =: [a_0 : a_1, a_2, \ldots, a_k] = \frac{p_k}{q_k} = \frac{u_k + v_k \sqrt{d}}{f_k + g_k \sqrt{d}} \quad \text{for} \quad k \geq 0, \]
where \(p_k, q_k \in \mathcal{O}\) are coprime and \(u_k, v_k, f_k, g_k \in \mathbb{Z}\). Here, \(p_k\) and \(q_k\) are unique only up to units. Clearly, we have
\[ \frac{p_k}{q_k} = [a_0, a_1, a_2, \ldots, a_k]. \]

The results in the following lemma are standard in the context of continued fractions for rational integers and can be proved in our setting of algebraic integers in \(\mathcal{O}\) in an analogous way. For proofs in the classical setup, we refer the reader to [12].

**Lemma 18.** (i) For \(k \geq 2\), assume that
\[ \frac{p_l}{q_l} = [a_0 : a_1, a_2, \ldots, a_l] \quad \text{if} \quad l = k - 2, k - 1. \]
Then
\[ \frac{p_k}{q_k} = [a_0 : a_1, a_2, \ldots, a_k] \]
for
\[ p_k = a_k p_{k-1} + p_{k-2}, \]
\[ q_k = a_k q_{k-1} + q_{k-2}. \]
(ii) If the sequences \((p_k)_{k \in \mathbb{N}}\) and \((q_k)_{k \in \mathbb{N}}\) satisfy the recursive relation (109) for all \(k \geq 2\) and, without loss of generality, 
\[ p_0 = a_0, \quad q_0 = 1, \quad p_1 = a_0 a_1 + 1, \quad q_1 = a_1, \]
then we have \(q_k > 0\) for all \(k \geq 0\),
\[ q_k p_{k-1} - p_k q_{k-1} = (-1)^k \]
for all \(k \geq 1\) and, consequently,
\[ \left| x_1 - \frac{p_{k-1}}{q_{k-1}} \right| \leq \frac{1}{q_k q_{k-1}} \quad \text{and} \quad \left| x_2 - \frac{p_{k-1}}{q_{k-1}} \right| \leq \frac{1}{q_k q_{k-1}} \]
for all \(k \geq 1\).

Now we begin with our recursive construction. In each step, we have to ensure that \(\gcd(f_k, g_k) = 1\), and we also need a strong enough growth of \(q_k\) and \(\overline{q}_k\) to get sufficiently strong approximations. For the start, we take
\[ a_0 := 1 \quad \text{and} \quad a_1 := f_1 + g_1 \sqrt{d} \quad \text{with} \quad f_1 > d g_1 > 0 \quad \text{and} \quad \gcd(f_1, g_1) = 1. \]
Then, as in Lemma 18(i), we may take
\[ p_0 = 1, \quad q_0 = f_0 + g_0 \sqrt{d} = 1, \quad p_1 = f_1 + 1 + g_1 \sqrt{d}, \quad q_1 = f_1 + g_1 \sqrt{d}. \]
We check that for \(k = 2\), the following conditions hold.
\[ \gcd(f_{k-2}, g_{k-2}) = 1, \]
\[ \gcd(f_{k-1}, g_{k-1}) = 1, \]
\[ \gcd(f_{k-1}, f_{k-2}) = 1, \]
\[ f_{k-2} > d g_{k-2} > 0, \]
\[ f_{k-1} > d g_{k-1} > 0, \]
\[ f_{k-1} > f_{k-2}^2. \]
Assume this holds for \(k \geq 2\). We shall construct \(a_k, p_k, q_k\) in such a way that the conditions in (110) hold with \(k + 1\) in place of \(k\), i.e.,
\[ \gcd(f_{k-1}, g_{k-1}) = 1, \]
\[ \gcd(f_k, g_k) = 1, \]
\[ \gcd(f_k, f_{k-1}) = 1, \]
\[ f_{k-1} > d g_{k-1} > 0, \]
\[ f_k > d g_k > 0, \]
\[ f_k > f_{k-1}^2. \]
In the following, we write
\[ a_k = s_k + t_k \sqrt{d}, \quad \text{where} \quad s_k, t_k \in \mathbb{Z}. \]
According to Lemma 18(i), we may then take
\[ q_k = a_k q_{k-1} + q_{k-2} \]
\[ = (s_k + t_k \sqrt{d})(f_{k-1} + g_{k-1} \sqrt{d}) + (f_{k-2} + g_{k-2} \sqrt{d}) \]
\[ = (s_k f_{k-1} + t_k g_{k-1} + f_{k-2}) + (t_k f_{k-1} + s_k g_{k-1} + g_{k-2}) \sqrt{d} \]
and hence
\[ f_k = s_k f_{k-1} + t_k g_{k-1} + f_{k-2}, \]
\[ g_k = t_k f_{k-1} + s_k g_{k-1} + g_{k-2}. \]
If \(s_k, t_k > 0\), which we want to assume from now on, then it follows from \(f_{k-2}, g_{k-1} > 0\) that \(f_k > f_{k-1}\). So if \(f_k\) is prime, then \(\gcd(f_k, f_{k-1}) = 1\). If, in addition, \(f_k > d g_k\), which is the case if
\[ s_k (f_{k-1} - d g_{k-1}) + (f_{k-2} - d g_{k-2}) > d (f_{k-1} - g_{k-1}) t_k, \]
then \(\gcd(f_k, g_k) = 1\). So to establish (111), all we need is to find \(s_k\) and \(t_k\) such that (112) holds and \(f_k\) is a prime satisfying \(f_k > f_{k-1}^2\).
Since \(\gcd(f_{k-1}, f_{k-2}) = 1\), Dirichlet’s theorem on the infinitude of primes in arithmetic progressions ensures the existence of \(s_k > 0\) such that \(s_k f_{k-1} + f_{k-2}\) is prime and \(s_k f_{k-1} + f_{k-2} > d g_{k-1}\).
So \( \gcd(\tilde{s}_k f_{k-1} + f_{k-2}, d q_{k-1}) = 1 \), and by the same theorem we can choose \( t_k \) in such a way that \( \tilde{s}_k f_{k-1} + t_k g_{k-1} d + f_{k-2} \) is prime. The pair \( (s_k, t_k) = (\tilde{s}_k, t_k) \) may not satisfy condition (112). To enforce this situation, we observe that since \( \tilde{s}_k f_{k-1} + t_k g_{k-1} d + f_{k-2} \) is prime, we have \( \gcd(\tilde{s}_k f_{k-1}, t_k g_{k-1} d + f_{k-2}) = 1 \), and so again by Dirichlet’s above-mentioned theorem, we can find \( \lambda > 0 \) such that \( \lambda \tilde{s}_k f_{k-1} + t_k g_{k-1} d + f_{k-2} \) is a prime greater than \( f_{k-1}^2 \) and

\[
\lambda \tilde{s}_k f_{k-1} - d g_{k-1} + (f_{k-2} - d g_{k-2}) > d(f_{k-1} - g_{k-1}) t_k.
\]

Here we recall that \( f_{k-1} > d g_{k-1} \) and \( f_{k-2} > d g_{k-2} \) by (110). Now choosing \( s_k := \lambda \tilde{s}_k \), the condition (112) is satisfied and \( f_k \) is a prime greater than \( f_{k-1}^2 \). Hence, the conditions in (111) are established.

We aim to show that under this construction, the pair \( (x_1, x_2) \) given by (108) is good. Since the construction gives \( (f_k, g_k) = 1 \) for all \( k \), it suffices to show that it satisfies (96) for all \( (p, q) = (p_k, q_k) \) with \( k \) large enough, and that \( \mathcal{N}(q_k) \to \infty \) as \( n \to \infty \). To this end, we observe that the condition \( f_k \geq \max \{d g_k, f_{k-1}^2 \} \) for all \( k \geq 1 \) implies

\[
q_k^2 \asymp \mathcal{N}(q_k) \asymp \frac{q_k^2}{\nu_k}
\]

and

\[
q_k \gg q_{k-1}^2 \quad \text{and} \quad \nu_k \gg q_{k-1}^{-2}
\]

for all \( k \geq 1 \). The above claim follows from Lemma 18(ii), which establishes that \( (x_1, x_2) \) is good.

10. Bounding the type I sum

10.1. Initial transformations. We are left with bounding the type I sum \( T_I \) arising from our application of Harman’s sieve in section 5. Proceeding precisely as in the initial treatment of the type II sum \( T_{II} \) in subsections 6.1 and 6.2, but leaving the \( n \)-summation uncut, we arrive at

\[
T_I \ll (\log x)^2 \sup_{1/2 \leq K \leq M} \sum_{1 \leq j \leq 2C} |\Sigma'_{K,j}| + O(x^{-100}),
\]

where

\[
\Sigma'_{K,j} := \sum_{K \leq N(m) \leq 2K} \sum_{n \in \mathcal{O}\setminus\{0\}} a_m \sum_{p \in \mathcal{O}\setminus\{0\}} E_j(p, mn) \mathcal{E}(pmn)
\]

with

\[
\mathcal{E}(l) := e \left( \frac{\sigma_2(l)x_2 - \sigma_1(l)x_1}{2\sqrt{d}} \right)
\]

and

\[
E_j(p, k) := \exp \left( -\pi \cdot \frac{\sigma_1(pk)^2 + \sigma_2(pk)^2}{4d} \cdot \frac{\delta_k^2}{N} \right) \cdot \exp \left( -\pi \cdot \frac{j_1 \sigma_1(k)^2 + j_2 \sigma_2(k)^2}{N} \right).
\]

In the following, we bound \( \Sigma'_{K,j} \). Unlike in subsection 6.3, we here don’t remove the weight functions but rather make use of them in directly applying the Poisson summation formula to the smooth sum over \( n \), which will be carried out in the next subsection. We still pull in the sum over \( n \) and use \( |a_m| \leq 1 \) to obtain

\[
\Sigma'_{K,j} \leq \sum_{K \leq N(m) \leq 2K} \sum_{p \in \mathcal{O}\setminus\{0\}} \sum_{n \in \mathcal{O}\setminus\{0\}} E_j(p, mn) \mathcal{E}(pmn).
\]

10.2. Applying Poisson summation. Now we use Poisson summation formula to transform the sum over \( n \) above. For \((u, v) \in \mathbb{R}^2\) set

\[
f(u, v) := e \left( -\frac{\sigma_1(pm)x_1}{2\sqrt{d}} \cdot (u + v \sqrt{d}) + \frac{\sigma_2(pm)x_2}{2\sqrt{d}} \cdot (u - v \sqrt{d}) \right) \times
\]

\[
\exp \left( -\pi \cdot \left( \frac{\sigma_1(pm)^2 \delta_k^2}{4dN} + \frac{j_1 \sigma_1(m)^2}{N} \right) (u + v \sqrt{d})^2 + \left( \frac{\sigma_2(pm)^2 \delta_k^2}{4dN} + \frac{j_2 \sigma_2(m)^2}{N} \right) (u - v \sqrt{d})^2 \right)
\]

so that

\[
E_j(p, mn) \mathcal{E}(pmn) = f(u, v)
\]
if \( n = u + v\sqrt{a} \) with \((u, v) \in \mathbb{Z}^2\). Clearly, \( f \in L^1(\mathbb{R}^2) \). Making the linear change of variables
\[
u + v\sqrt{a} = \sigma, \quad u - v\sqrt{a} = \tau,
\]
we calculate the Fourier transform of \( f \) to be
\[
\hat{f}(\alpha, \beta) = 2\sqrt{d} \left( (\sigma_1(pm)^2\delta^2/N + 4dj_1\sigma_1(m)^2/N)^{-1/2} \right. \end{align*}
\[
\left. \left( (\sigma_2(pm)^2\delta^2/N + 4dj_2\sigma_2(m)^2/N)^{-1/2} \right) \right.
\]
\[
\exp \left( -\pi \cdot \left( \frac{(\beta + \alpha\sqrt{d} + \sigma_1(pm)x_1)^2}{(\sigma_1(pm)^2\delta^2/N + 4dj_1\sigma_1(m)^2/N)} + \frac{(-\beta + \alpha\sqrt{d} - \sigma_2(pm)x_2)^2}{(\sigma_2(pm)^2\delta^2/N + 4dj_2\sigma_2(m)^2/N)} \right) \right)
\]
\[
\ll \frac{N_1}{K} \cdot \exp \left( -\pi \cdot \frac{N_1}{K} \cdot \left( (\beta + \alpha\sqrt{d} + \sigma_1(pm)x_1)^2 + (-\beta + \alpha\sqrt{d} - \sigma_2(pm)x_2)^2 \right) \right).
\]
if \( m \) and \( p \) satisfy the relevant summation conditions in (114). Obviously, the above function \( \hat{f} \) is also in \( L^1(\mathbb{R}^2) \) and hence, we may use Lemma 10 to deduce that
\[
\sum_{n \in \mathcal{O}} E_\ell(p, mn)E(pm) = \sum_{(\alpha, \beta) \in \mathbb{Z}^2} \hat{f}(\alpha, \beta)
\]
\[
\ll \frac{x}{K} \cdot \sum_{(\alpha, \beta) \in \mathbb{Z}^2} \exp \left( -\pi \cdot \frac{N_1}{K} \cdot \left( (\beta + \alpha\sqrt{d} + \sigma_1(pm)x_1)^2 + (-\beta + \alpha\sqrt{d} - \sigma_2(pm)x_2)^2 \right) \right).
\]
The right-hand side is negligible unless
\[
|\beta + \alpha\sqrt{d} + \sigma_1(pm)x_1| \leq x^{2\varepsilon - 1/2} K^{1/2}
\]
and
\[
|\beta - \alpha\sqrt{d} - \sigma_2(pm)x_2| \leq x^{2\varepsilon - 1/2} K^{1/2}.
\]
The above inequalities imply
\[
|\beta + \sigma_2(pm)x_2 + \sigma_1(pm)x_1| \leq x^{2\varepsilon - 1/2} K^{1/2}
\]
and
\[
|\alpha - \sigma_2(pm)x_2 - \sigma_1(pm)x_1| \leq x^{2\varepsilon - 1/2} K^{1/2}
\]
on upon subtracting and adding and dividing by 2 and \( 2\sqrt{d} \), respectively. Hence, the right-hand side of (115) is negligible unless
\[
\left| \frac{\sigma_2(pm)x_2 + \sigma_1(pm)x_1}{2} \right| \leq x^{2\varepsilon - 1/2} K^{1/2} \quad \text{and} \quad \left| \frac{\sigma_2(pm)x_2 - \sigma_1(pm)x_1}{2\sqrt{d}} \right| \leq x^{2\varepsilon - 1/2} K^{1/2},
\]
in which case it is bounded by \( O(x/K) \). Hence, we deduce from (114) that
\[
\Sigma_{K,j} \ll x^{2\varepsilon} K \delta^{-2} + xK^{-1} \times \sum_{m \in \mathcal{R}} \sum_{p \in \mathcal{O}\setminus\{0\}} \chi_{\ell} \left( \left| \frac{\sigma_2(pm)x_2 + \sigma_1(pm)x_1}{2} \right| \right) \chi_{\ell} \left( \left| \frac{\sigma_2(pm)x_2 - \sigma_1(pm)x_1}{2\sqrt{d}} \right| \right)
\]
where the first summand is on the right-hand side comes from the contribution of \( n = 0 \), and \( J' \) is the interval
\[
J' := [-x^{2\varepsilon - 1/2} K^{1/2}, x^{2\varepsilon - 1/2} K^{1/2}].
\]
Writing \( k = pm \) similarly as in subsection 6.4, we deduce that
\[
\Sigma_{K,j} \ll x^{2\varepsilon} K \delta^{-2} + x^{1+\varepsilon} K^{-1} \times \sum_{k \in \mathcal{O}\setminus\{0\}} \chi_{\ell} \left( \left| \frac{\sigma_2(k)x_2 + \sigma_1(k)x_1}{2} \right| \right) \chi_{\ell} \left( \left| \frac{\sigma_2(k)x_2 - \sigma_1(k)x_1}{2\sqrt{d}} \right| \right)
\]
for some constant \( c > 0 \).
10.3. Estimating the type I sum. The bound (117) has the same shape as (56), and we can therefore directly apply our method in section 7 to bound this sum. In place of (72), we now obtain
\[ \Sigma'_{K,j} \leq x^{2\varepsilon} K \delta^{-2} + x^{1+\varepsilon} K^{-1} \cdot \Theta(2U'; a, b; W) \cdot \Theta(V'; a, b; W), \]
where
\[ U' := cx^2 \delta^{-1} K^{1/2} \quad \text{and} \quad V' = W(x^{2\varepsilon-1/2} K^{1/2} + cx^2 \delta^{-1} K^{1/2} \Delta). \]
The choices of \( U' \) and \( V' \) above are due to the fact that \( k \) is of the form
\[ k = \lambda + \mu \sqrt{d} \]
with
\[ \lambda, \mu \leq U'. \]
We recall (84) which states that
\[ \Theta(X; a, b; W) \ll \left( \frac{X^2 Z^2}{W} + 1 \right) \log^2 W, \]
and \( \Delta = C/W \) (see (104)). Hence, we have
\[ \Sigma'_{K,j} \ll x^{2\varepsilon} K \delta^{-2} + x^{1+\varepsilon} K^{-1} (\delta^{-2} K W^{-1} Z^2 + 1) \]
\[ \ll x^{2\varepsilon} K \delta^{-2} + x^{8\varepsilon} (K \delta^{-2} Z^4 + K \delta^{-4} x W^{-2} Z^4 + \delta^{-2} x W^{-1} Z^2 + W Z^2 + x K^{-1}) \]
under the condition (86) which states that \( W \ll x^{100}. \)
The bound for \( \Sigma'_{K,j} \) which we need to beat is \( \ll x \). Therefore, if \( K \) is small, (119) will not suffice. However, we can exclude small \( K \), as we shall see in the next subsection.

10.4. Excluding small \( K \). Repeating our method in section 7, we are led to counting solutions \((\alpha, \beta, \lambda, \mu)\) of the system of congruences
\[ \begin{cases} a \lambda + b d \mu & \equiv \alpha \mod W, \\ b \lambda + a \mu & \equiv \beta \mod W, \end{cases} \]
which is (66) in subsection 7.3, subject to the conditions
\[ |\lambda|, |\mu| \leq U' \quad \text{and} \quad |\alpha|, |\beta| \leq V'. \]
Using the reduction in subsection 7.4, (120) implies
\[ \begin{cases} a' \lambda + b' d \mu & \equiv \alpha \mod W', \\ b' \lambda + a' \mu & \equiv \beta \mod W', \end{cases} \]
where \( a' = a/Z, b' = b/Z, W' = W/Z^2 \) and \( \gcd(b', W') = 1 \). Moreover, we deduced the congruence (67), which states that
\[ b a \equiv b \alpha \mod W. \]
By the same reduction, the above congruence implies
\[ \alpha \equiv a' b' \beta \mod W', \]
where \( b' \) is a multiplicative inverse modulo \( W' \). We recall that, according to (83),
\[ \frac{|a' b' \beta - u|}{r} \ll r^{-2} \quad \text{with} \quad \gcd(u, r) = 1 \]
for some integer \( r \) with \( r \approx \sqrt{W'} \). Therefore, there is a constant \( c_1 > 0 \) such that (122) has no solutions \((\alpha, \beta)\) with \( |\alpha|, |\beta| \leq c_1 \sqrt{W'} \) except for the trivial one, which is \((\alpha, \beta) = (0, 0)\). Hence, if \( V' \leq c_1 \sqrt{W'} \), then (121) becomes a homogeneous system
\[ \begin{cases} a' \lambda + b' d \mu & \equiv 0 \mod W', \\ b' \lambda + a' \mu & \equiv 0 \mod W'. \end{cases} \]
From the second congruence, it follows that
\[ a' b' \mu \equiv -\lambda \mod W'. \]
Again using (123) and \( |\mu|, |\lambda| \leq U' \) by (118), this congruence has only the solution \((\lambda, \mu) = (0, 0)\) in the above range if \( U' \leq c_1 \sqrt{W'} \). But since \( k \neq 0 \), this solution is excluded. In summary, the system (68) has no solutions if
\[ U' = cx^2 \delta^{-1} K^{1/2} \leq c_1 \sqrt{W'} = c_1 \sqrt{W}/Z. \]
and
\[ V' = W(x^{2\epsilon-1/2}K^{1/2} + C\epsilon x^\delta K^{1/2}W^{-1}) \leq c_1 \sqrt{W}/ = c_1 \sqrt{W}/Z. \]
Therefore, the sum over \( k \) in (117) is empty and thus
\[ (124) \quad \Sigma_{K,j} \ll x^{2\epsilon-\delta^2} K \quad \text{if} \quad K \leq c_2 \min \left\{ x^{-2\epsilon-\delta^2} WZ^{-2}, x^{-1-4\epsilon-1} Z^{-2} \right\} \]
for some constant \( c_2 > 0 \). Combining (119) and (124), we get the estimate
\[ (125) \quad \Sigma_{K,j} \ll x^{13\epsilon} \left( K^{\delta-2} Z^4 + K^{\delta-4} xW^{-2} Z^4 + \delta^{-2} xW^{-1} Z^2 + WZ^2 \right) \]
Plugging this into (113), we obtain
\[ T_1 \ll x^{13\epsilon} \left( MW^{4\eta} + M^{\delta-2} xW^{4\eta-2} + xW^{2\eta-1} + \delta^2 W^{2\eta+1} \right) . \]
If \((x_1, x_2)\) is \( \eta \)-good, then this gives the final bound
\[ (125) \quad T_1 \ll x^{13\epsilon} \left( MW^{4\eta} + M^{\delta-2} xW^{4\eta-2} + xW^{2\eta-1} + \delta^2 W^{2\eta+1} \right) . \]

11. Conclusion

Now we apply Theorem (13). Recalling the definitions of \( \omega(q) \) and \( \tilde{\omega}(q) \) in (6) and (7) and the bound (14), we have
\[ \lim_{R \to \infty} \sum_{\mathbb{N}(a) < R} d_4(a)w(a) \ll N^{1-\epsilon} \ll x \]
if \( w = \omega \) or \( w = \tilde{\omega} \). Now combining (24), (44), (45),(106) and (125) and choosing
\[ \mu = \frac{1}{4}, \quad \kappa = \frac{1}{2}, \quad M = 2x^{1/4}, \]
we obtain
\[ (126) \quad x^{-13\epsilon}(T(N) - \tilde{T}(N)) \ll x^{1/4}W^{4\eta} + \delta^{-2} x^{5/4} W^{4\eta-2} + xW^{2\eta-1} + \delta^2 W^{2\eta+1} + \left( \delta + x^{1/2} W^{-1} \right) \left( x^{7/8} W^{\eta} + x^{1/2} W^{1/2+\eta} \right) + xW^{\eta-1/2} + x^{3/4}. \]
Now we optimize the parameters. First, we choose \( x \) depending on \( W \) in such a way \( \delta = x^{1/2} W^{-1} \), i.e.
\[ x := (\delta W)^2 \]
and hence
\[ W \asymp x^{1/2} \delta^{-1}. \]
Then (126) turns into
\[ x^{-13\epsilon}(T(N) - \tilde{T}(N)) \ll x^{1/4+2\eta} \delta^{-4\eta} + x^{7/8+\eta} \delta^{1-2\eta} + x^{3/4+\eta/2} \delta^{1/2-\eta} + x^{3/4}. \]
Recalling \( N := \lfloor x^{1-\epsilon} \rfloor \), the estimate
\[ T(N) - \tilde{T}(N) \ll \delta^2 N^{1-\epsilon} \]
in Theorem 6 holds if \( \epsilon \leq 1/14 \) and
\[ \delta \geq N^{-\nu+15\epsilon} \]
with
\[ \nu := \min \left\{ \frac{3/4 - 2\eta}{2 + 4\eta}, \frac{1/8 - \eta}{1 + 2\eta}, \frac{1/4 - \eta/2}{3/2 + \eta}, \frac{1}{8} \right\} = \frac{1/8 - \eta}{1 + 2\eta} \]
This establishes Theorem 6.
12. Unsmoothing

Now we want to use Theorem 6 and Corollary 7 to establish our main result, Theorem 5. We look first at the definition of $F(q)$ is (9). The set

$$\Lambda(q) := \left\{ \left( \frac{\sigma_1(p)}{\sigma_2(q)}, \frac{\sigma_2(p)}{\sigma_2(q)} \right) : p \in \mathcal{O} \right\}$$

is a lattice in $\mathbb{R}^2$ which has a fundamental parallelogram whose area equals $1/\mathcal{N}(q)$ and whose dimensions are $\approx 1/\sqrt{\mathcal{N}(q)}$, by the same arguments as at the end of section 3. If

$$\mathcal{N}(q) \leq N^{1+\varepsilon}$$

and $N$ is large enough, then taking into account the exponential decay of $\Omega_{y/\sqrt{N}}(y)$, we have

$$F(q) \ll 1$$

if there exists $p \in \mathcal{O}$ such that

$$|x - \frac{\sigma_1(p)}{\sigma_2(q)}| \leq \frac{\delta}{N^{1/2-\varepsilon}}$$

for $i = 1, 2,$ and $F(q)$ is negligible otherwise. Using this together with (12) and the definition of $\tilde{\omega}(q)$ in (6), it follows that $\mathcal{F}(N)/\log N$ is dominated by the number of prime ideals $p \in \mathcal{I} \setminus 0$ satisfying

$$\mathcal{N}(p) \leq N^{1+\varepsilon}$$

and (127) for some $q$ generating $p$ and $p \in \mathcal{O}$, provided $\mathcal{F}(N) \gg 1$ and $\delta \leq N^{-2\varepsilon}$ which latter we may assume without loss of generality. This together with Theorem 6 and Corollary 7 implies the main result upon re-defining $\varepsilon$.

References

[1] S. Baier; *Diophantine approximation on lines in $C^2$ with Gaussian prime constraints*. Eur. J. Math. 3(3):614-649 (2017).
[2] S. Baier and M. Technau; *On the distribution of $ap$ modulo one in imaginary quadratic fields with class number one*. preprint, arXiv:1905.07623, 2019.
[3] J. Brückner; *Einführung in die analytische Zahlentheorie*. Berlin-Heidelberg: Springer-Verlag, 1995.
[4] D. Bump; *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge: Cambridge University Press, 1997.
[5] E. Burger; *Homogeneous Diophantine approximation in $S$-integers*. Pacific J. Math. 152(2):211-253, 1992.
[6] G. Harman; *On the distribution of $ap$ modulo one*. J. London Math. Soc. (2), 27(1):9-18, 1983.
[7] G. Harman; *On the distribution of $ap$ modulo one. II*. Proc. London Math. Soc. (3), 72(2):241-260, 1996.
[8] G. Harman; *Diophantine approximation with Gaussian primes*, preprint, to appear in Quart. J. Math., 2019.
[9] G. Harman; *Prime detecting sieves*. Princeton, NJ: Princeton University Press, 2007.
[10] D.R. Heath-Brown and C. Jia; *The distribution of $ap$ modulo one*. Proc. London Math. Soc. (3), 84(1):79-104, 2002.
[11] C. Hooley; *On the number of divisors of quadratic polynomials*. Acta Math. 110, 97–114, 1963.
[12] A.Y. Khinchin; *Continued Fractions*. Chicago and London: The University of Chicago Press, Dover Publication, INC, 1964.
[13] T.N. Ly; *Diophantine Approximation in Algebraic Number Fields and Flows on Homogeneous Spaces*, PhD Thesis at Brandeis University, Waltham, 2016.
[14] K. Matomäki; *The distribution of $ap$ modulo one*. Math. Proc. Camb. Philos. Soc. 147(2):267–283, 2009.
[15] R.C. Vaughan; *On the distribution of $ap$ modulo 1*. Mathematika, 24:135–141, 1978.
[16] I.M. Vinogradov; *The method of trigonometrical sums in the theory of numbers*. Translated from the Russian, revised and annotated by K.F. Roth and Anne Davenport. Reprint of the 1954 translation. New York: Dover Publications, 2004.

Stephan Baier, Ramakrishna Mission Vivekananda Educational Research Institute, Department of Mathematics, G. T. Road, PO Belur Math, Howrah, West Bengal 711202, India
E-mail address: stephanbaier2017@gmail.com

Dwaiyan Mazumder, Ramakrishna Mission Vivekananda Educational Research Institute, Department of Mathematics, G. T. Road, PO Belur Math, Howrah, West Bengal 711202, India
E-mail address: dvmaz.1993@gmail.com