CO-SEIFERT FIBRATIONS OF COMPACT FLAT 3-ORBIFOLDS

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Abstract. This paper is a continuation of our previous paper [9] in which we developed the theory for classifying geometric fibrations of compact, connected, flat n-orbifolds, over a 1-orbifold, up to affine equivalence. In this paper, we apply our theory to classify all the geometric fibrations of compact, connected, flat 3-orbifolds, over a 1-orbifold, up to affine equivalence.

1. Introduction

An n-dimensional crystallographic group (n-space group) is a discrete group Γ of isometries of Euclidean n-space $E^n$ whose orbit space $E^n/\Gamma$ is compact. If Γ is an n-space group, then $E^n/\Gamma$ is a compact, connected, flat n-orbifold, and conversely if M is a compact, connected, flat n-orbifold, then there is an n-space group Γ such that M is isometric to $E^n/\Gamma$.

In our paper [5], we proved that a geometric fibration of $E^n/\Gamma$ corresponds to a space group extension

$$1 \rightarrow N \hookrightarrow \Gamma \twoheadrightarrow \Gamma/N \rightarrow 1.$$

The corresponding geometric fibration of $E^n/\Gamma$ is said to be a co-Seifert fibration when the base space is a 1-orbifold, or equivalently, when $\Gamma/N$ is a 1-space group.

In our previous paper [9], we develop the theory for classifying all co-Seifert geometric fibrations of compact, connected, flat n-orbifolds up to affine equivalence. By Theorems 5 and 10 of [5], this problem is equivalent to classifying all pairs $(\Gamma, N)$ such that Γ is an n-space group and N is a normal subgroup of Γ, such that $\Gamma/N$ is infinite cyclic or infinite dihedral, up to isomorphism. In our paper [8], we proved that for each dimension n, there are only finitely many isomorphism classes of such pairs. In [9], we described the classification for $n = 2$. In this paper, we describe the classification for $n = 3$. In the process, we determine the group of affinities of every compact, connected, flat 2-orbifold. In particular, we show that the group of affinities of a flat torus has an interesting free product with amalgamation decomposition. Finally, as an application, we give an explanation for all but one of the enantiomorphic 3-space group pairs.

2. Organization and Description of the Classification

This paper is a continuation of [9], and we will refer to [9] for all definitions and basic results. As explained in §9 of [9], the classification follows from knowledge of the action of the structure group. In this paper, we describe the action of the structure group for the generalized Calabi constructions corresponding to the Seifert and dual co-Seifert geometric fibrations of compact, connected, flat 3-orbifolds given in Table 1 of [5]. We did not describe the co-Seifert fibration projections in Table 1 of [5] because we did not have an efficient way of giving a description. Our generalized Calabi construction gives us a simple way to simultaneously describe
both a Seifert fibration and its dual co-Seifert fibration of a compact, connected, flat 3-orbifold via Theorem 4 and Corollary 1 of [9] (see also [7]).

We will organize our description into 17 tables, one for each 2-space group type of the generic fiber of the co-Seifert fibration. We will go through the 2-space groups in reverse order because the complexity of the description generally increases in reverse order of the IT number of a 2-space group. That the computer generated affine classification of the co-Seifert geometric fibrations described in Table 1 of [5] is correct and complete follows from Theorems 7 – 12 of [9] and Lemmas 1 – 33 below.

(1) In each of these 17 tables, the first column lists the IT (international tables) number of the 3-space group \( \Gamma \).

(2) The second column lists the generic fibers \((V/N, V^\perp/K)\) of the co-Seifert and Seifert fibrations corresponding to a 2-dimensional, complete, normal subgroup \( N \) of \( \Gamma \) with \( V = \text{Span}(N) \) and \( K = N^\perp \). We will denote a circle by \( O \) and a closed interval by \( I \).

(3) The third column lists the isomorphism type of the structure group \( \Gamma/NK \), with \( C_n \) indicating a cyclic group of order \( n \), and \( D_n \) a dihedral group of order \( 2n \).

(4) The fourth column lists the quotients \((V/(\Gamma/K), V^\perp/(\Gamma/N))\) under the action of the structure group. Note that \((V/(\Gamma/K), V^\perp/(\Gamma/N))\) are the bases of the Seifert and co-Seifert fibrations.

(5) The fifth column indicates how generators of the structure group act diagonally on the Cartesian product \( V/N \times V^\perp/K \). The structure group action was derived from the standard affine representation of \( \Gamma \) in Table 1B of [1]. We denote a rotation of \( O \) of \((360/n)°\) by \( n\text{-rot.} \), and a reflection of \( O \) or \( I \) by \( \text{ref.} \). We denote the identity map by \( \text{idt.} \).

(6) The sixth column lists the classifying pairs for the co-Seifert fibrations. These are pairs \( \{ \alpha, \beta \} \) of affinities of \( V/N \) that classify co-Seifert fibrations via Theorems 7 – 10 of [9]. For \( C_n \) actions, \( \alpha \) and \( \beta \) are inverse affinities of order \( n \). For \( D_1 \) actions, \( \alpha = \text{idt.} \), and \( \beta \) has order 2. For \( D_n \) actions, with \( n > 1 \), \( \alpha \) and \( \beta \) are affinities of order 2 whose product has order \( n \). In particular, for \( D_n \) actions given by \( \{ \alpha, \text{ref.}, \gamma, n\text{-rot.} \} \), the classifying pair is \( \{ \alpha, \beta \} \) with \( \gamma = \alpha\beta \), except for the cases \( n = 3, 6 \) at the end of Tables 16 and 17, which require an affine deformation. We indicate a classifying pair that falls into the case \( E_1 \cap E_2 \neq \{0\} \) of Theorem 10 of [9] by an asterisk.

Tables 1 – 17 were first computed by hand, and then they were double checked by a computer calculation.

Let \( C_\infty \) be the standard infinite cyclic 1-space group \( \langle e_1 + I \rangle \), and let \( D_\infty \) be the standard infinite dihedral 1-space group \( \langle e_1 + I, -I \rangle \).

If \( M \) is a 2-space group, we define the symmetry group of the flat orbifold \( E^2/M \) by \( \text{Sym}(M) = \text{Isom}(E^2/M) \). We will identify \( V/N \) with \( E^2/M \) and the group of affinities \( \text{Aff}(V/N) \) of \( V/N \) with \( \text{Aff}(M) = \text{Aff}(E^2/M) \).

3. **Generic Fiber** *632* (30° – 60° right triangle) with IT number 17

The 2-space group \( M \) with IT number 17 is *632* in Conway’s notation [2] or \( p6m \) in IT notation. See space group 4/4/1/1 in Table 1A of [1] for the standard affine representation of \( M \). The flat orbifold \( E^2/M \) is a 30° – 60° right triangle.
Lemma 1. If $M$ is the 2-space group $632$, then $\text{Sym}(M) = \text{Aff}(M) = \{\text{idt.}\}$, and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism.

Proof. $\text{Sym}(M) = \{\text{idt.}\}$, since a symmetry of $E^2/M$ fixes each corner point. We have that $Z(M) = \{I\}$ by Lemma 5 of [6]. Hence $\Omega : \text{Sym}(M) \to \text{Out}_E(M)$ is an isomorphism by Theorems 1 and 2 of [6]. We have that $\text{Out}_E(M) = \text{Out}(M)$ by Lemma 9 of [6] and Table 5A of [1], and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism by Theorems 1 and 3 in [6]. Therefore $\text{Sym}(M) = \text{Aff}(M)$. □

We represent $\text{Out}(M)$ by $\text{Sym}(M)$ via the isomorphism $\Omega : \text{Sym}(M) \to \text{Out}(M)$. The set $\text{Isom}(C_\infty, M)$ consists of one element corresponding to the pair of inverse elements $\{\text{idt.}, \text{idt.}\}$ of $\text{Sym}(M)$ by Theorem 7 of [9]. The corresponding affine equivalence class of co-Seifert fibrations corresponds to row 1 of Table 1.

The set $\text{Isom}(D_\infty, M)$ consists of one element corresponding to the pair of identity elements $\{\text{idt.}, \text{idt.}\}$ of $\text{Sym}(M)$ by Theorem 8 of [9]. The corresponding affine equivalence class of co-Seifert fibrations corresponds to row 2 of Table 1.

Notice that the first row of Table 1 says that $E^3/\Gamma$ is the Cartesian product $632 \times O$, and the second row of Table 1 says that $E^3/\Gamma$ is the Cartesian product $632 \times I$. The corresponding co-Seifert geometric fibrations of $E^3/\Gamma$ are the Cartesian product fibrations of $632 \times O$ and $632 \times I$ with fiber $632$.

4. Generic Fiber 632 (turnover) with IT number 16

The 2-space group $M$ with IT number 16 is $632$ in Conway’s notation or $p6$ in IT notation. See space group 4/3/1/1 in Table 1A of [1] for the standard affine representation of $M$. The flat orbifold $E^2/M$ is a turnover with 3 cone points obtained by gluing together two congruent $30^\circ - 60^\circ$ right triangles along their boundaries. The 632 turnover is orientable. Let c-ref. denote the central reflection between the two triangles.

Lemma 2. If $M$ is the 2-space group 632, then $\text{Sym}(M) = \text{Aff}(M) = \{\text{idt.}, \text{c-ref.}\}$, and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism.

Proof. $\text{Sym}(M) = \{\text{idt.}, \text{c-ref.}\}$, since a symmetry of $E^2/M$ fixes each cone point. We have that $\text{Sym}(M) = \text{Aff}(M)$ and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism as in Lemma 1. □

We represent $\text{Out}(M)$ by $\text{Sym}(M)$ via the isomorphism $\Omega : \text{Sym}(M) \to \text{Out}(M)$. The set $\text{Isom}(C_\infty, M)$ consists of two elements corresponding to the pairs of inverse elements $\{\text{idt.}, \text{idt.}\}$ and $\{\text{c-ref.}, \text{c-ref.}\}$ of $\text{Sym}(M)$ by Theorem 7 of [9]. The corresponding affine equivalence classes of co-Seifert fibrations correspond to the two rows of Table 2 whose column 4 second quotient is $O$. 

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### Table 1. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type $632$ with IT number 17

| no. fibers | grp. quotients | group action | classifying pair |
|------------|----------------|--------------|------------------|
| 183        | ($632, O$)     | $C_1$        | ($632, O$) ($\text{idt.}, \text{idt.}$) |
| 191        | ($632, I$)     | $C_1$        | ($632, I$) ($\text{idt.}, \text{idt.}$) |
The set $\text{Isom}(C_{\infty}, M)$ consists of two elements corresponding to the pairs of inverse elements $\{\text{idt., idt.}\}$ and $\{\text{c-ref., c-ref.}\}$ of $\text{Sym}(M)$ by Theorem 7 of [9]. The corresponding affine equivalence classes of co-Seifert fibrations correspond to the two rows of Table 3 whose column 4 second quotient is O.

The set $\text{Isom}(D_{\infty}, M)$ consists of three elements corresponding to the pairs of elements $\{\text{idt., idt.}\}$, $\{\text{c-ref., c-ref.}\}$, $\{\text{idt., c-ref.}\}$ of order 1 or 2 of $\text{Sym}(M)$ by Theorem 8 of [9]. The corresponding affine equivalence classes of co-Seifert fibrations correspond to the three rows of Table 3 whose column 4 second quotient is I.

The set $\text{Isom}(D_{\infty}, M)$ consists of three elements corresponding to the pairs of elements $\{\text{idt., idt.}\}$, $\{\text{c-ref., c-ref.}\}$, $\{\text{idt., c-ref.}\}$ of order 1 or 2 of $\text{Sym}(M)$ by Theorem 8 of [9]. The corresponding affine equivalence classes of co-Seifert fibrations correspond to the three rows of Table 3 whose column 4 second quotient is I.
| no. | fibers | grp. quotients | structure group action | classifying pair |
|-----|--------|----------------|------------------------|------------------|
| 156 | (+333, O) | $C_1$ (+333, O) | (idt., idt.) | {idt., idt.} |
| 160 | (+333, O) | $C_3$ (3+3, O) | (3-rot., 3-rot.) | {3-rot., 3-rot.$^{-1}$} |
| 164 | (+333, O) | $C_2$ (632, I) | (t-ref., ref.) | {t-ref., t-ref.} |
| 166 | (+333, O) | $D_3$ (632, O) | (t-ref., ref.), (3-rot., 3-rot.) | {t-ref., t-ref.'} |
| 186 | (+333, I) | $C_1$ (+333, I) | (idt., idt.) | {idt., idt.} |
| 187 | (+333, I) | $D_1$ (632, I) | (t-ref., ref.) | {idt., t-ref.} |
| 194 | (+333, I) | $C_2$ (632, O) | (t-ref., 2-rot.) | {t-ref., t-ref.} |

Table 4. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type $*333$ with IT number 14

6. Generic Fiber $*333$ (equilateral triangle) with IT number 14

The 2-space group $M$ with IT number 14 is $*333$ in Conway’s notation or $p3m1$ in IT notation. See space group 4/2/1/1 in Table 1A of [1] for the standard affine representation of $M$. The flat orbifold $E^2/M$ is an equilateral triangle $\triangle$. The symmetry group of $\triangle$ is a dihedral group of order 6. There is one conjugacy class of symmetries of order 2 represented by a triangle reflection $t$-ref. in the perpendicular bisector of a side. There is one conjugacy class of symmetries of order 3 represented by a rotation 3-rot. of 120° about the center of the triangle. Define the triangle reflection $t$-ref.$' = (t$-ref.$)(3$-rot.$)$.

**Lemma 4.** If $M$ is the 2-space group $*333$, then $\text{Sym}(M)$ is the dihedral group $\langle t$-ref.$, 3$-rot.$ \rangle$ of order 6, and $\text{Sym}(M) = \text{Aff}(M)$, and $\Omega : \text{Aff}(M) \rightarrow \text{Out}(M)$ is an isomorphism.

**Proof.** $\text{Sym}(M) = \langle t$-ref.$, 3$-rot.$ \rangle$, since a symmetry permutes the corner points. We have that $\text{Sym}(M) = \text{Aff}(M)$ and $\Omega : \text{Aff}(M) \rightarrow \text{Out}(M)$ is an isomorphism as in Lemma 1. □

We represent $\text{Out}(M)$ by $\text{Sym}(M)$ via the isomorphism $\Omega : \text{Sym}(M) \rightarrow \text{Out}(M)$. The set $\text{Isom}(C_\infty, M)$ consists of three elements corresponding to the pairs of inverse elements $\{\text{idt.}, \text{idt.}\}, \{3$-rot.$, 3$-rot.$^{-1}\}, \{t$-ref.$, t$-ref.$\}$ of $\text{Sym}(M)$ by Theorem 7 of [9]. The corresponding affine equivalence classes of co-Seifert fibrations correspond to the three rows of Table 4 whose column 4 second quotient is O.

The set $\text{Isom}(D_\infty, M)$ consists of four elements corresponding to the pairs of elements $\{t$-ref.$, t$-ref.$\}, \{t$-ref.$, t$-ref.$'\}, \{\text{idt.}, \text{idt.}\}, \{\text{idt.}, t$-ref.$\}$ of order 1 or 2 of $\text{Sym}(M)$ by Theorem 8 of [9]. The corresponding affine equivalence classes of co-Seifert fibrations correspond to the four rows of Table 4 whose column 4 second quotient is I.

7. Generic Fiber $333$ (turnover) with IT number 13

The 2-space group $M$ with IT number 13 is $333$ in Conway’s notation or $p3$ in IT notation. See space group 4/1/1/1 in Table 1A of [1] for the standard affine representation of $M$. The flat orbifold $E^2/M$ is a turnover with three 120° line points obtained by gluing together two congruent equilateral triangles along their boundaries. The $333$ turnover is orientable. The symmetry group of this orbifold is the direct product of the subgroup of order 2, generated by the central reflection
The set \( \text{Isom}(C) \) consists of six elements corresponding to the pairs of inverse elements \( \{\text{idt.}, \text{idt.}\}, \{3\text{-rot.}, 3\text{-rot.}^{-1}\}, \{c\text{-ref.}, c\text{-ref.}\}, \{t\text{-ref.}, t\text{-ref.}\}, \{6\text{-sym.}, 6\text{-sym.}^{-1}\}, \{2\text{-rot.}, 2\text{-rot.}\} \) of \( \text{Sym}(M) \) by Theorem 7 of [9].

| no. fibers | grp. quotients | structure group action | classifying pair |
|------------|----------------|------------------------|------------------|
| 143 (333, O) | \( C_1 \) (333, O) | (idt., idt.) | {idt., idt.} |
| 146 (333, O) | \( C_3 \) (333, O) | (3-rot., 3-rot.) | \{3-rot., 3-rot.\} |
| 147 (333, O) | \( C_2 \) (632, I) | (2-rot., ref.) | \{2-rot., 2-rot.\} |
| 148 (333, O) | \( D_3 \) (632, I) | (2-rot., ref.), (3-rot., 3-rot.) | \{2-rot., 2-rot.\} |
| 149 (333, O) | \( C_2 \) (*333, I) | (c-ref., ref.) | \{c-ref., c-ref.\} |
| 150 (333, O) | \( C_2 \) (3*3, I) | (t-ref., ref.) | \{t-ref., t-ref.\} |
| 155 (333, O) | \( D_3 \) (*333, I) | (t-ref., ref.), (3-rot., 3-rot.) | \{t-ref., t-ref.\} |
| 158 (333, O) | \( C_2 \) (*333, O) | (c-ref., 2-rot.) | \{c-ref., c-ref.\} |
| 159 (333, O) | \( C_2 \) (3*3, O) | (t-ref., 2-rot.) | \{t-ref., t-ref.\} |
| 161 (333, O) | \( C_6 \) (3*3, O) | (6-sym., 6-rot.) | \{6-sym., 6-sym.\} |
| 163 (333, O) | \( D_2 \) (*632, I) | (c-ref., ref.), (t-ref., 2-rot.) | \{c-ref., 2-rot.\} |
| 165 (333, O) | \( D_2 \) (*632, I) | (t-ref., ref.), (c-ref., 2-rot.) | \{t-ref., 2-rot.\} |
| 167 (333, O) | \( D_6 \) (*632, I) | (t-ref., ref.), (6-sym., 6-rot.) | \{t-ref., 2-rot.\} |
| 173 (333, O) | \( C_2 \) (632, O) | (2-rot., 2-rot.) | \{2-rot., 2-rot.\} |
| 174 (333, I) | \( C_1 \) (333, I) | (idt., idt.) | {idt., idt.} |
| 176 (333, I) | \( D_1 \) (632, I) | (2-rot., ref.) | \{idt., 2-rot.\} |
| 182 (333, O) | \( D_2 \) (*632, I) | (c-ref., ref.), (2-rot., 2-rot.) | \{c-ref., t-ref.\} |
| 188 (333, I) | \( D_1 \) (*333, I) | (c-ref., ref.) | \{idt., c-ref.\} |
| 190 (333, I) | \( D_1 \) (3*3, I) | (t-ref., ref.) | \{idt., t-ref.\} |

Table 5. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type 333 with IT number 13

There are 3 conjugacy classes of symmetries of order 2, the class of the central reflection \( c\text{-ref.} \), the class of the triangle reflection \( t\text{-ref.} \), and the class of the half-turn around a cone point 2-rot., defined so that 2-rot. = (c-ref.)(t-ref.).

There is one conjugacy class of symmetries of order 3 represented by a rotation 3-rot. that cyclically permutes the cone points. There is one conjugacy class of dihedral subgroups of order 4, represented by the group \{idt., c-ref., t-ref., 2-rot.\}. There is one conjugacy class of symmetries of order 6 represented by 6-sym. = (c-ref.)(3-rot.). There are two conjugacy classes of dihedral subgroups of order 6, the class of the symmetry group of a triangular side of the turnover generated by \( t\text{-ref.} \) and 3-rot., and the class of the orientation-preserving subgroup generated by 2-rot. and 3-rot.

Define \( t\text{-ref.}' = (t\text{-ref.})(3\text{-rot.}) \) and \( 2\text{-rot.}' = (c\text{-ref.})(t\text{-ref.}) \).

Lemma 5. If \( M \) is the 2-space group 333, then \( \text{Sym}(M) \) is the dihedral group \( (c\text{-ref.}, t\text{-ref.}, 3\text{-rot.}) \) of order 12, and \( \text{Sym}(M) = \text{Aff}(M) \), and \( \Omega : \text{Aff}(M) \to \text{Out}(M) \) is an isomorphism.

Proof. \( \text{Sym}(M) = (c\text{-ref.}, t\text{-ref.}, 3\text{-rot.}) \), since a symmetry permutes the cone points. We have that \( \text{Sym}(M) = \text{Aff}(M) \) and \( \Omega : \text{Aff}(M) \to \text{Out}(M) \) is an isomorphism as in Lemma 1.

We represent \( \text{Out}(M) \) by \( \text{Sym}(M) \) via the isomorphism \( \Omega : \text{Sym}(M) \to \text{Out}(M) \).
The set $\text{Isom}(D_\infty, M)$ consists of thirteen elements corresponding to the pairs of elements \{2-rot., 2-rot.\}, \{2-rot., 2-rot.'\}, \{c-ref., c-ref.\}, \{t-ref., t-ref.\}, \{t-ref., t-ref.'\}, \{c-ref., 2-rot.\}, \{t-ref., 2-rot.\}, \{t-ref., 2-rot.'\}, \{idt., idt.\}, \{idt., 2-rot.\}, \{c-ref., t-ref.\}, \{idt., c-ref.\}, \{idt., t-ref.\} of order 1 or 2 of $\text{Sym}(M)$ by Theorem 8 of [9].

**Example 1.** Let $\Gamma$ be the affine 3-space group with IT number 163 in Table 1B of [1]. Then $\Gamma = \langle t_1, t_2, t_3, A, \beta, C \rangle$ where $t_i = e_i + I$ for $i = 1, 2, 3$ are the standard translations, and $\beta = \frac{1}{2}e_3 + B$, and

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$  

The group $N = \langle t_1, t_2, A \rangle$ is a complete normal subgroup of $\Gamma$ with $V = \text{Span}(N) = \text{Span}\{e_1, e_2\}$. The flat orbifold $V/N$ is a 333 turnover. Let $K = N^\perp$. Then $K = \langle t_3 \rangle$. The flat orbifold $V^\perp/K$ is a circle. The structure group $\Gamma/NK$ is a dihedral group of order 4 generated by $NK\beta$ and $NK\gamma$. The elements $NK\beta$ and $NK\gamma$ act on the circle $V^\perp/K$ as reflections. The elements $NK\beta$ and $NK\gamma$ both fix the cone point of $V/N$ represented by $(0, 0, 0)$. The other two cone points of $V/N$ are represented by $(2/3, 1/3, 0)$, which is the fixed point of $t_1A$, and by $(1/3, 2/3, 0)$, which is the fixed point of $t_1t_2A$. The element $NK\beta$ acts as the central reflection of $V/N$, since it fixes all three cone points. The element $NK\gamma$ acts as the halfturn around the cone point represented by $(0, 0, 0)$, since $NK\gamma$ preserves the orientation of $V/N$ because $C$ preserves the orientation of $V$. By Theorems 8 and 12 of [9], the classifying pair for the co-Seifert fibration determined by $(\Gamma, N)$ is \{c-ref., 2-rot.\}.

8. **Generic Fiber 4*2 (cone) with IT number 12**

The 2-space group $M$ with IT number 12 is 4*2 in Conway’s notation or $p4g$ in IT notation. See space group 3/2/1/2 in Table 1A of [1] for the standard affine representation of $M$. The flat orbifold $E^2/M$ is a cone with one 90° cone point and one 90° corner point obtained by gluing together two congruent 45° – 45° right triangles along two sides opposite 45° and 90° angles. Let c-ref. denote the central reflection between the triangles.

**Lemma 6.** If $M$ is the 2-space group 4*2, then $\text{Sym}(M) = \text{Aff}(M) = \{\text{idt.}, \text{c-ref.}\}$, and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism.

**Proof.** $\text{Sym}(M) = \{\text{idt.}, \text{c-ref.}\}$, since a symmetry of $E^2/M$ fixes the cone point and the corner point. We have that $\text{Sym}(M) = \text{Aff}(M)$ and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism as in Lemma 1. \hfill \Box

The derivation of Table 6 for fiber of type 4*2 is the same as the derivation of Table 3 for fiber of type 3*3.

9. **Generic Fiber +442 (45° – 45° right triangle) with IT number 11**

The 2-space group $M$ with IT number 11 is +442 in Conway’s notation or $p4m$ in IT notation. See space group 3/2/1/1 in Table 1A of [1] for the standard affine representation of $M$. The flat orbifold $E^2/M$ is a 45° – 45° right triangle. Let t-ref. denote the triangle reflection of $E^2/M$. 

no. fibers  grp. quotients  group action  classifying pair
100  (4∗2, O)  C₁  (4∗2, O)  (idt., idt.)  {idt., idt.}  
108  (4∗2, O)  C₂  (*442, O)  (c-ref., 2-rot.)  {c-ref., c-ref.}  
125  (4∗2, O)  C₂  (*442, I)  (c-ref., ref.)  {c-ref., c-ref.}  
127  (4∗2, I)  C₁  (4∗2, I)  (idt., idt.)  {idt., idt.}  
140  (4∗2, I)  D₁  (*442, I)  (c-ref., ref.)  {idt., c-ref.}  

Table 6. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type 4∗2 with IT number 12

no. fibers  grp. quotients  group action  classifying pair
99  (*442, O)  C₁  (*442, O)  (idt., idt.)  {idt., idt.}  
107  (*442, O)  C₂  (*442, O)  (t-ref., 2-rot.)  {t-ref., t-ref.}  
123  (*442, I)  C₁  (*442, I)  (idt., idt.)  {idt., idt.}  
129  (*442, O)  C₂  (*442, I)  (t-ref., ref.)  {t-ref., t-ref.}  
139  (*442, I)  D₁  (*442, I)  (t-ref., ref.)  {idt., t-ref.}  

Table 7. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type *442 with IT number 11

Lemma 7. If \( M \) is the 2-space group *442, then \( \text{Sym}(M) = \text{Aff}(M) = \{\text{idt., t-ref.}\} \), and \( \Omega : \text{Aff}(M) \to \text{Out}(M) \) is an isomorphism.

Proof. \( \text{Sym}(M) = \{\text{idt., t-ref.}\} \), since a symmetry of \( E^2/M \) fixes the 90° corner point and permutes the other two corner points. We have that \( \text{Sym}(M) = \text{Aff}(M) \) and \( \Omega : \text{Aff}(M) \to \text{Out}(M) \) is an isomorphism as in Lemma 1. \( \square \)

We represent \( \text{Out}(M) \) by \( \text{Sym}(M) \) via the isomorphism \( \Omega : \text{Sym}(M) \to \text{Out}(M) \). The set \( \text{Isom}(C_\infty, M) \) consists of two elements corresponding to the pairs of inverse elements \( \{\text{idt., idt.}\} \) and \( \{\text{t-ref., t-ref.}\} \) of \( \text{Sym}(M) \) by Theorem 7 of [9]. The corresponding affine equivalence classes of co-Seifert fibrations correspond to the first two rows of Table 7.

The set \( \text{Isom}(D_\infty, M) \) consists of three elements corresponding to the pairs of elements \( \{\text{idt., idt.}\}, \{\text{t-ref., t-ref.}\}, \{\text{idt., t-ref.}\} \) of order 1 or 2 of \( \text{Sym}(M) \) by Theorem 8 of [9]. The corresponding affine equivalence class of co-Seifert fibrations corresponds to the last three rows of Table 7.

10. Generic Fiber 442 (Turnover) with IT number 10

The 2-space group \( M \) with IT number 10 is 442 in Conway’s notation or \( p4 \) in IT notation. See space group 3/1/1/1 in Table 1A of [1] for the standard affine representation of \( M \). The flat orbifold \( E^2/M \) is a turnover with 3 cone points obtained by gluing together two congruent 45° – 45° right triangles along their boundaries. The 442 turnover is orientable. The symmetry group of this orbifold is a dihedral group of order 4 consisting of the identity symmetry, the halfturn 2-rot., the central reflection c-ref. between the two triangles, and the triangle reflection t-ref.
Lemma 8. If $M$ is the 2-space group 442, then $\text{Sym}(M)$ is the dihedral group $\{\text{idt., 2-rot, c-ref., t-ref.}\}$, and $\text{Sym}(M) = \text{Aff}(M)$, and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism.

Proof. $\text{Sym}(M) = \{\text{idt., 2-rot, c-ref., t-ref.}\}$, since a symmetry of $E^2/M$ fixes the 180° cone point and permutes the other two cone points. We have that $\text{Sym}(M) = \text{Aff}(M)$ and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism as in Lemma 1. □

We represent $\text{Out}(M)$ by $\text{Sym}(M)$ via the isomorphism $\Omega : \text{Sym}(M) \to \text{Out}(M)$. The set $\text{Isom}(C_\infty, M)$ consists of four elements corresponding to the pairs of inverse elements $\{\text{idt., idt.}\}, \{\text{2-rot., 2-rot.}\}, \{\text{c-ref., c-ref.}\}, \{\text{t-ref., t-ref.}\}$ of $\text{Sym}(M)$ by Theorem 7 of [9]. The set $\text{Isom}(D_\infty, M)$ consists of ten elements corresponding to the pairs of elements $\{\text{idt., idt.}\}, \{\text{2-rot., 2-rot.}\}, \{\text{idt., 2-rot.}\}, \{\text{c-ref., c-ref.}\}, \{\text{t-ref., t-ref.}\}, \{\text{c-ref., t-ref.}\}, \{\text{idt., c-ref.}\}, \{\text{c-ref., 2-rot.}\}, \{\text{idt., t-ref.}\}, \{\text{t-ref., 2-rot.}\}$ of order 1 or 2 of $\text{Sym}(M)$ by Theorem 8 of [9].

11. Generic Fiber 2*22 (bonnet) with IT number 9

The 2-space group $M$ with IT number 9 is 2*22 in Conway’s notation or cmm in IT notation. See space group 2/2/2/1 in Table 1A of [1] for the standard affine representation of $M$. The flat orbifold $E^2/M$ is a bonnet. The most symmetric bonnet is the square bonnet obtained by gluing together two congruent squares along the union of two adjacent sides. This orbifold has one 180° cone point and two 90° corner points. The symmetry group of this orbifold is a dihedral group of order 4 consisting of the identity symmetry, the central reflection c-ref. between the two squares, the diagonal reflection d-ref., and the halfturn 2-rot. Both c-ref. and 2-rot. transpose the two corner points of the square bonnet, whereas d-ref. fixes each of the corner points of the square bonnet.

| no. | fibers | grp. | quotients | structure group action | classifying pair |
|-----|--------|------|-----------|------------------------|------------------|
| 75  | (442, O) | $C_1$ | (442, O)  | (idt., idt.)           | {idt., idt.}     |
| 79  | (442, O) | $C_2$ | (442, O)  | (2-rot., 2-rot.)       | {2-rot., 2-rot.} |
| 83  | (442, I) | $C_1$ | (442, I)  | (idt., idt.)           | {idt., idt.}     |
| 85  | (442, O) | $C_2$ | (442, I)  | (2-rot., ref.)         | {2-rot., 2-rot.} |
| 87  | (442, I) | $D_1$ | (442, I)  | (2-rot., ref.)         | {idt., 2-rot.}   |
| 89  | (442, O) | $C_2$ | (+442, I) | (c-ref., ref.)         | {c-ref., c-ref.} |
| 90  | (442, O) | $C_2$ | (4*2, I)  | (t-ref., ref.)         | {t-ref., t-ref.} |
| 97  | (442, O) | $D_2$ | (+442, I) | (c-ref., ref.), (2-rot., 2-rot.) | {c-ref., t-ref.} |
| 103 | (442, O) | $C_2$ | (+442, O) | (c-ref., 2-rot.)       | {c-ref., c-ref.} |
| 104 | (442, O) | $C_2$ | (4*2, O)  | (t-ref., 2-rot.)       | {t-ref., t-ref.} |
| 124 | (442, I) | $D_1$ | (+442, I) | (c-ref., ref.)         | {idt., c-ref.}   |
| 126 | (442, O) | $D_2$ | (+442, I) | (c-ref., ref.), (t-ref., 2-rot.) | {c-ref., 2-rot.} |
| 128 | (442, I) | $D_1$ | (4*2, I)  | (t-ref., ref.)         | {idt., t-ref.}   |
| 130 | (442, O) | $D_2$ | (+442, I) | (t-ref., ref.), (c-ref., 2-rot.) | {t-ref., 2-rot.} |

Table 8. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type 442 with IT number 10.
Lemma 9. If $M$ is the 2-space group $2\times 22$ and $E^2/M$ is a square bonnet, then $\text{Sym}(M)$ is the dihedral group $\{\text{idt.}, 2\text{-rot}, \text{c-ref.}, \text{d-ref.}\}$, and $\text{Sym}(M) = \text{Aff}(M)$, and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism.

Proof. $\text{Sym}(M) = \{\text{idt.}, 2\text{-rot}, \text{c-ref.}, \text{d-ref.}\}$, since a symmetry of $E^2/M$ fixes the cone point and permutes the corner points. We have that $\text{Sym}(M) = \text{Aff}(M)$ and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism as in Lemma 1. □

The derivation of Table 9 for fiber of type $2\times 22$ is similar to the derivation of Table 8 for fiber of type $442$.

12. Generic Fiber $22\times$ (projective pillow) with IT number 8

The 2-space group $M$ with IT number 8 is $22\times$ in Conway’s notation or $pgg$ in IT notation. See space group 2/2/1/3 in Table 1A of [1] for the standard affine representation of $M$. The flat orbifold $E^2/M$ is a projective pillow. The most symmetric projective pillow is the square projective pillow obtained by gluing the opposite sides of a square $\square$ by glide reflections with axes the lines joining the midpoints of opposite sides of $\square$. This orbifold has two $180^\circ$ cone points represented by diagonally opposite vertices of $\square$, and a center point represented by the center of $\square$. The symmetry group of this orbifold is a dihedral group of order 8, represented by the symmetry group of $\square$, consisting of the identity symmetry, two midline reflections, m-ref. and m-ref', two diagonal reflections, d-ref. and d-ref', a half turn 2-rot., and two order 4 rotations, 4-rot. and 4-rot. $^{-1}$, with 4-rot. = (d-ref')(m-ref'). There are three conjugacy classes of order 2 symmetries, the classes represented by m-ref, 2-rot., and d-ref. There is one conjugacy class of order 4 symmetries. There are two conjugacy classes of dihedral subgroups of order 4, the group generated by the midline reflections, and the group generated by the diagonal reflections.

Lemma 10. If $M$ is the 2-space group $22\times$ and $E^2/M$ is a square projective pillow, then $\text{Sym}(M)$ is the dihedral group (m-ref, d-ref) of order 8, and $\text{Sym}(M) = \text{Aff}(M)$, and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism.

| no. fibers | grp. | quotients | structure group action | classifying pair |
|------------|------|-----------|------------------------|-----------------|
| 35 (2*22, O) | $C_1$ (2*22, O) | (idt., idt.) | (idt., idt.) |
| 42 (2*22, O) | $C_2$ (*2222, O) | (c-ref., 2-rot.) | (c-ref., c-ref.) |
| 65 (2*22, I) | $C_1$ (2*22, I) | (idt., idt.) | (idt., idt.) |
| 67 (2*22, O) | $C_2$ (*2222, I) | (c-ref., ref.) | (c-ref., c-ref.) |
| 69 (2*22, I) | $D_1$ (*2222, I) | (c-ref., ref.) | (idt., c-ref.) |
| 101 (2*22, O) | $C_2$ (*442, O) | (d-ref., 2-rot.) | (d-ref., d-ref.) |
| 102 (2*22, O) | $C_2$ (4*2, O) | (2-rot., 2-rot.) | (2-rot., 2-rot.) |
| 111 (2*22, O) | $C_2$ (*442, I) | (d-ref., ref.) | (d-ref., d-ref.) |
| 113 (2*22, O) | $C_2$ (4*2, I) | (2-rot., ref.) | (2-rot., 2-rot.) |
| 121 (2*22, O) | $D_2$ (*442, I) | (d-ref., ref.), (c-ref., 2-rot.) | (d-ref., 2-rot.) |
| 132 (2*22, I) | $D_1$ (*442, I) | (d-ref., ref.) | (idt., d-ref.) |
| 134 (2*22, O) | $D_2$ (*442, I) | (c-ref., ref.), (2-rot., 2-rot.) | (c-ref., d-ref.) |
| 136 (2*22, I) | $D_1$ (4*2, I) | (2-rot., ref.) | (idt., 2-rot.) |
| 138 (2*22, O) | $D_2$ (*442, I) | (c-ref., ref.), (d-ref., 2-rot.) | (c-ref., 2-rot.) |

Table 9. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type $2\times 22$ with IT number 9.
Lemma 11. Out(M) is an isomorphism as in Lemma 2-rot.

The 2-space group M with IT number 7 is 22/pillowcase with IT number 8.

Proof. Sym(M) = \langle \text{idt., idt.} \rangle, since a symmetry of \(E^2/M\) fixes the center point and permutes the cone points. We have that Sym(M) = Aff(M) and \(\Omega : \text{Aff}(M) \rightarrow \text{Out}(M)\) is an isomorphism as in Lemma 1.

We represent Out(M) by Sym(M) via the isomorphism \(\Omega : \text{Sym}(M) \rightarrow \text{Out}(M)\). The set Isom(\(C_\infty\), M) consists of five elements corresponding to the pairs of inverse elements \{\text{idt., idt.}\}, \{\text{m-ref., m-ref.}\}, \{\text{2-rot., 2-rot.}\}, \{\text{d-ref., d-ref.}\}, \{\text{4-rot., 4-rot.}^{-1}\} of Sym(M) by Theorem 7 of [9].

The set Isom(\(D_\infty\), M) consists of twelve elements corresponding to the pairs of elements \{\text{2-rot., 2-rot.}\}, \{\text{m-ref., m-ref.}\}, \{\text{idt., idt.}\}, \{\text{idt., m-ref.}\}, \{\text{m-ref., 2-rot.}\}, \{\text{idt., 2-rot.}\}, \{\text{m-ref., m-ref.'}\}, \{\text{d-ref., d-ref.}\}, \{\text{d-ref., d-ref.'}\}, \{\text{d-ref., 2-rot.}\}, \{\text{idt., d-ref.}\}, \{\text{d-ref., m-ref.}\} of order 1 or 2 of Sym(M) by Theorem 8 of [9].

13. Generic Fiber 22* (pillowcase) with IT number 7

The 2-space group M with IT number 7 is 22* in Conway's notation or \(\text{pmg}\) in IT notation. See space group 2/2/1/2 in Table 1A of [1] for the standard affine representation of M. The flat orbifold \(E^2/M\) is a pillowcase obtained by gluing together two congruent rectangles along the union of three of their sides. This orbifold has two 180° cone points. The symmetry group of this orbifold is a dihedral group of order 4 consisting of the identity symmetry, the central reflection c-ref. between the two rectangles, the halfturn 2-rot., and the midline reflection m-ref.

Lemma 11. If M is the 2-space group 22*, then Sym(M) is the dihedral group \{\text{idt., c-ref., m-ref., 2-rot.}\}, and Sym(M) = Aff(M), and \(\Omega : \text{Aff}(M) \rightarrow \text{Out}(M)\) is an isomorphism.
Table 11. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type $22\ast$ with IT number 7

| no. | fibers | grp. quotients | structure group action | classifying pair |
|-----|--------|---------------|-----------------------|-----------------|
| 28  | $(22\ast, O)$ | $C_1$ | $(22\ast, O)$ | (idt., idt.) |
| 39  | $(22\ast, O)$ | $C_2$ | $(2\ast 22, O)$ | (c-ref., 2-rot.) |
| 40  | $(22\ast, O)$ | $C_2$ | $(22\ast, O)$ | (2-rot., 2-rot.) |
| 46  | $(22\ast, O)$ | $C_2$ | $(2\ast 22, O)$ | (m-ref., 2-rot.) |
| 49  | $(22\ast, O)$ | $C_2$ | $(2\ast 22, 1)$ | (c-ref., ref.) |
| 51  | $(22\ast, I)$ | $C_1$ | $(22\ast, I)$ | (idt., idt.) |
| 53  | $(22\ast, O)$ | $C_2$ | $(2\ast 22, 1)$ | (m-ref., ref.) |
| 57  | $(22\ast, O)$ | $C_2$ | $(2\ast 22, I)$ | (2-rot., 2-rot.) |
| 63  | $(22\ast, I)$ | $D_1$ | $(2\ast, I)$ | (2-rot., ref.) |
| 64  | $(22\ast, O)$ | $D_2$ | $(2\ast 22, I)$ | (m-ref., ref.), (c-ref., 2-rot.) |
| 66  | $(22\ast, O)$ | $D_2$ | $(2\ast 22, I)$ | (c-ref., ref.), (2-rot., 2-rot.) |
| 67  | $(22\ast, I)$ | $D_1$ | $(2\ast 22, I)$ | (c-ref., ref.) |
| 72  | $(22\ast, O)$ | $D_2$ | $(2\ast 22, I)$ | (c-ref., ref.), (m-ref., 2-rot.) |
| 74  | $(22\ast, I)$ | $D_1$ | $(2\ast 22, I)$ | (m-ref., ref.) |

Proof. Sym(M) = \{idt., c-ref., m-ref., 2-rot\}, since a symmetry of $E^2/M$ permutes the cone points. We have that Sym(M) = Aff(M) and $\Omega : \text{Aff}(M) \rightarrow \text{Out}(M)$ is an isomorphism as in Lemma 1. □

The derivation of Table 11 for fiber of type $22\ast$ is similar to the derivation of Table 8 for fiber of type 442.

14. Generic Fiber $*2222$ (rectangle) with IT number 6

The 2-space group M with IT number 6 is $*2222$ in Conway’s notation or pmm in IT notation. See space group 2/2/1/1 in Table 1A of [1] for the standard affine representation of M. The flat orbifold $E^2/M$ is a rectangle. A rectangle has four 90° corner points. The most symmetric rectangle is a square $\Box$. The symmetry group of $\Box$ is a dihedral group of order 8 consisting of the identity symmetry, two midline reflections, m-ref. and m-ref.’, two diagonal reflections, d-ref. and d-ref.’, a halfturn 2-rot., and two order 4 rotations, 4-rot. and 4-rot.$^{-1}$, with 4-rot. = (d-ref.)(m-ref.). There are three conjugacy classes of order 2 symmetries, the classes represented by m-ref., 2-rot., and d-ref. There is one conjugacy class of order 4 symmetries. There are two conjugacy classes of dihedral subgroups of order 4, the group generated by the midline reflections, and the group generated by the diagonal reflections.

Lemma 12. If M is the 2-space group $*2222$ and $E^2/M$ is a square, then Sym(M) is the dihedral group \{m-ref., d-ref\} of order 8, and Sym(M) = Aff(M), and $\Omega : \text{Aff}(M) \rightarrow \text{Out}(M)$ is an isomorphism.

Proof. Sym(M) = \{m-ref., d-ref\}, since a symmetry of $E^2/M$ permutes the corner points. We have that Sym(M) = Aff(M) and $\Omega : \text{Aff}(M) \rightarrow \text{Out}(M)$ is an isomorphism as in Lemma 1. □

The derivation of Table 12 for fiber of type $*2222$ is similar to the derivation of Table 10 for fiber of type $22\times$. 
Observe that

Proof. $M$ is a Möbius band. We have that $A$ and $B$ are transposes $e_1$ and $e_2$. Then $M$ is a 2-space group with IT number 5. The Conway notation for $M$ is $\ast \times$ and the IT notation is $cn$. The flat orbifold $E^2/M$ is a Möbius band. We have that $Z(M) = \langle e_1 + e_2 + I \rangle$.

Lemma 13. If $M = \langle e_1 + I, e_2 + I, A \rangle$, with $A$ transposing $e_1$ and $e_2$, and $N_A(M)$ is the normalizer of $M$ in $\text{Aff}(E^2)$, then

$N_A(M) = \{ b + B : b_1 - b_2 \in \mathbb{Z} \text{ and } B \in \langle -I, A \rangle \}.$

Proof. Observe that $b + B \in N_A(M)$ if and only if $B \in N_A(\langle e_1 + I, e_2 + I \rangle)$ and $(b + B)A(b + B)^{-1} \in M$. Now $B \in N_A(\langle e_1 + I, e_2 + I \rangle)$ if and only if $B \in \text{GL}(2, \mathbb{Z})$. As $(b + B)A(b + B)^{-1} = b - BAB^{-1}b + BAB^{-1}$, we have that $(b + B)A(b + B)^{-1} \in M$ if and only if $BAB^{-1} = A$ and $b - Ab \in \mathbb{Z}^2$. Hence $b + B \in N_A(M)$ if and only if $B \in \langle -I, A \rangle$ and $b_1 - b_2 \in \mathbb{Z}$. □

A fundamental polygon for the action of $M$ on $E^2$ is the $45^\circ - 45^\circ$ right triangle with vertices $v_1 = (0,0), v_2 = (1,0), v_3 = (1,1)$. The reflection $A$ fixes the hypotenuse $[v_1, v_3]$ pointwise. The glide reflection $e_1 + A$ maps the side $[v_1, v_2]$ to the side $[v_2, v_3]$. The boundary of the Möbius band $E^2/M$ is represented by the hypotenuse $[v_1, v_3]$, and the central circle of $E^2/M$ is represented by the line segment $[(1/2, 0), (1, 1/2)]$ joining the midpoints of the two short sides of the triangle.

Restricting a symmetry of the Möbius band $E^2/M$ to a symmetry of its boundary circle gives an isomorphism from the symmetry group of the Möbius band to the symmetry group of its boundary circle. There are two conjugacy classes of elements of order 2, the class of the reflection $c$-ref. $= (e_1/2 + e_2/2 + I)$, in the central circle of the Möbius band, and the class of a halfturn 2-rot. $= (-I)$, about the

| no. | fibers | grp. | quotients | structure group action | classifying pair |
|-----|--------|------|-----------|-----------------------|------------------|
| 25  | (+2222, O) | $C_1$ | (+2222, O) | (idt., idt.) | {idt., idt.} |
| 38  | (+2222, O) | $C_2$ | (+2222, O) | (m-ref., 2-rot.) | {m-ref., m-ref.} |
| 44  | (+2222, O) | $C_2$ | (2*2*2, O) | (2-rot., 2-rot.) | {2-rot., 2-rot.} |
| 47  | (+2222, I) | $C_1$ | (+2222, I) | (idt., idt.) | {idt., idt.} |
| 51  | (+2222, O) | $C_2$ | (+2222, O) | (m-ref., ref.) | {m-ref., m-ref.} |
| 59  | (+2222, O) | $C_2$ | (2*2*2, I) | (2-rot., ref.) | {2-rot., 2-rot.} |
| 63  | (+2222, O) | $D_2$ | (+2222, I) | (m-ref., ref.), (m-ref., 2-rot.) | {m-ref., m-ref.'} |
| 65  | (+2222, I) | $D_1$ | (+2222, I) | (m-ref., ref.) | {idt., m-ref.} |
| 71  | (+2222, I) | $D_1$ | (2*2*2, I) | (2-rot., ref.) | {idt., 2-rot.} |
| 74  | (+2222, O) | $D_2$ | (+2222, I) | (m-ref., ref.), (2-rot., 2-rot.) | {m-ref., m-ref.'} |
| 105 | (+2222, O) | $C_2$ | (+4*4*2, O) | (d-ref., 2-rot.) | {d-ref., d-ref.} |
| 109 | (+2222, O) | $C_4$ | (+4*2, O) | (4-rot., 4-rot.) | {4-rot., 4-rot.} |
| 115 | (+2222, O) | $C_2$ | (+4*4*2, I) | (d-ref., ref.) | {d-ref., d-ref.} |
| 119 | (+2222, O) | $D_2$ | (+4*4*2, I) | (d-ref., ref.), (2-rot., 2-rot.) | {d-ref., d-ref.'} |
| 131 | (+2222, I) | $D_1$ | (+4*4*2, I) | (d-ref., ref.) | {idt., d-ref.} |
| 137 | (+2222, O) | $D_2$ | (+4*4*2, I) | (d-ref., ref.), (d-ref., 2-rot.) | {d-ref., 2-rot.} |
| 141 | (+2222, O) | $D_4$ | (+4*4*2, I) | (d-ref., ref.), (4-rot., 4-rot.) | {d-ref., m-ref.'} |

Table 12. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type $\ast 2222$ with IT number 6
Proof. The Lie group Sym(M) is isomorphic to O(2), since restricting a symmetry of the Möbius band M to a symmetry of its boundary circle O is an isomorphism from Sym(M) to Isom(O) by Lemma 1 of [E]. There is one conjugacy class of dihedral symmetry groups of order 4, represented by the group {idt., c-ref., 2-rot., 2-rot′}.

Lemma 14. If M = (e_1 + I, e_2 + I, A), with A transposing e_1 and e_2, then the Lie group Sym(M) is isomorphic to O(2), and Sym(M) = Aff(M), and Ω : Aff(M) → Out(M) maps the subgroup {idt., 2-rot.} isomorphically onto Out(M).

Proof. The Lie group Sym(M) is isomorphic to O(2), since restricting a symmetry of E^2/M to a symmetry of its boundary circle O is an isomorphism from Sym(M) to Isom(O) by Lemma 1 of [6] and Lemma 13. We have that Sym(M) = Aff(M) by Lemmas 1 and 7 of [6] and Lemma 13. The epimorphism Ω : Aff(M) → Out(M) maps the group {idt., 2-rot.} isomorphically onto Out(M) by Theorem 3 of [6], since 2-rot. acts as a reflection of O.

We represent Out(M) by the subgroup {idt., 2-rot.} of Sym(M) via the epimorphism Ω : Sym(M) → Out(M). The set Isom(C_∞, M) consists of two elements corresponding to the pairs of inverse elements {idt., idt.} and {2-rot., 2-rot.} of Sym(M) by Theorem 7 of [9].

The set Isom(D_∞, M) consists of six elements corresponding to the pairs of elements {2-rot., 2-rot.}, {idt., idt.}, {c-ref., c-ref.}, {idt., c-ref.}, {idt., 2-rot.}, {2-rot., c-ref.} of order 1 or 2 of Sym(M) by Lemma 8 of [9] and Theorems 9 and 10 of [9]. Notice that if 2-rot′ = (xe_1 + xe_2 - I), for some x ∈ R, then the pairs {2-rot., 2-rot′} and {2-rot., 2-rot′} determine the same element of Isom(D_∞, M) by Theorem 9 of [9]. The pairs {2-rot., c-ref.} and {2-rot′, c-ref.} determine the same element of Isom(D_∞, M), since they are conjugate.

16. Generic Fiber ×× (Klein bottle) with IT number 4

Let M = (e_1 + I, e_2 + I, e_1/2 + A) where A = diag(1, -1). Then M is a 2-space group with IT number 4. The Conway notation for M is ×× and the IT notation is pg. The flat orbifold E^2/M is a Klein bottle. We have that Z(M) = ⟨e_1 + I⟩.

Lemma 15. If M = (e_1 + I, e_2 + I, e_1/2 + A), with A = diag(1, -1), and N_A(M) is the normalizer of M in Aff(E^2), then

N_A(M) = {b + B : 2b_2 ∈ Z and B ∈ ⟨-I, A⟩}.
Proof. Observe that $b+B \in N_{A}(M)$ if and only if $B \in N_{A}((e_{1}+I,e_{2}+I))$ and $(b+B)(e_{1}/2+A)(b+B)^{-1} \in M$. Now $B \in N_{A}((e_{1}+I,e_{2}+I))$ if and only if $B \in GL(2,\mathbb{Z})$. As $(b+B)(e_{1}/2+A)(b+B)^{-1} = b+Be_{1}/2-BAB^{-1}b+BAB^{-1}$, we have that $(b+B)(e_{1}/2+A)(b+B)^{-1} \in M$ if and only if $BAB^{-1} = A$ and $b - Ab+Be_{1}/2-e_{1}/2 \in \mathbb{Z}^{2}$. Hence $b+B \in N_{A}(M)$ if and only if $B \in (-I,A)$ and $2b_{2} \in \mathbb{Z}$. □

A fundamental polygon for the action of $M$ on $E^{2}$ is the rectangle with vertices $v_{1} = (0,-1/2), v_{2} = (1/2,-1/2), v_{3} = (1/2,1/2)$ and $v_{4} = (0,1/2)$. The vertical translation $e_{2} + I$ maps the side $[v_{1},v_{2}]$ to the side $[v_{4},v_{3}]$. The horizontal glide reflection $e_{1}/2 + A$ maps the side $[v_{1},v_{4}]$ to the side $[v_{2},v_{3}]$. The Klein bottle $E^{2}/M$ has two short horizontal geodesics represented by the line segments $[v_{1},v_{2}]$ and $[(0,0),(1/2,0)]$. The union of these two short geodesics is invariant under any symmetry of the Klein bottle. Therefore the horizontal quarterline geodesic, which we call the central circle, represented by the union of the line segments $[(0,-1/4),(1/2,-1/4)]$ and $[(0,1/4),(1/2,1/4)]$ is invariant under any symmetry of the Klein Bottle. The symmetry group of the Klein bottle is the direct product of the subgroup of order 2, generated by the reflection in the central circle, and the subgroup consisting of the horizontal translations and the reflections in a vertical geodesic. The latter subgroup restricts to the symmetry group of the central circle.

There are five conjugacy classes of symmetries of order 2, the class of the reflection $m$-ref. $= A_{*} = (e_{1}/2+I)$, in the short horizontal geodesics, the class of the vertical halfturn $2$-sym. $= (e_{2}/2+I)$, the class of the reflection $c$-ref. $= (e_{2}/2+A)$, in the central circle, the class of the reflection $v$-ref. $= (-A)_{*} = (e_{1}/2-I)$, in the vertical geodesic represented by $[v_{1},v_{4}]$, and the class of the halfturn $2$-rot. $= (e_{2}/2-I)_{*}$ around a pair of antipodal points on the central circle. Define $v$-ref.’ $= (-I)_{*}$ and $2$-rot.’ $= (e_{1}/2+e_{2}/2-I)$.

There are five conjugacy classes of dihedral symmetry groups of order 4, the classes of the groups $\{\text{idt.}, m$-ref., $2$-sym., $c$-ref., $\}$, $\{\text{idt.}, m$-ref., $v$-ref., $v$-ref.’ $\}$, $\{\text{idt.}, m$-ref., $2$-rot., $2$-rot.’ $\}$, $\{\text{idt.}, 2$-sym., $v$-ref., $2$-rot.’ $\}$, and $\{\text{idt.}, c$-ref., $v$-ref., $2$-rot.’ $\}$.

Lemma 16. If $M = (e_{1}+I,e_{2}+I,e_{1}/2+A)$, with $A = \text{diag}(1,-1)$, then the Lie group $\text{Sym}(M)$ is isomorphic to $C_{2} \times O(2)$, and $\text{Sym}(M) = \text{Aff}(M)$, and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ maps the subgroup $\{\text{idt.}, c$-ref., $v$-ref., $2$-rot.$\}$ isomorphically onto $\text{Out}(M)$. Moreover $\Omega(2$-sym. $) = \Omega(c$-ref.$)$.

Proof. Let $O$ be the central circle of the Klein bottle, and let $\Lambda$ be the group of all symmetries of the Klein bottle that leave each horizontal geodesic invariant. Restriction induces an isomorphism from $\Lambda$ to $\text{Isom}(O)$ by Lemma 1 of [6] and Lemma 15. We have that $\text{Sym}(M) = \langle c$-ref.$\rangle \times \Lambda$, since every symmetry of the Klein bottle leaves $O$ invariant, and $c$-ref. commutes with every symmetry in $\Lambda$. Therefore $\text{Sym}(M)$ is isomorphic to $C_{2} \times O(2)$. We have that $\text{Sym}(M) = \text{Aff}(M)$ by Lemmas 1 and 7 of [6] and Lemma 15. The epimorphism $\Omega : \text{Aff}(M) \to \text{Out}(M)$ maps $\{\text{idt.}, c$-ref., $v$-ref., $2$-rot.$\}$ isomorphically onto $\text{Out}(M)$ by Theorem 3 of [6], since $v$-ref. acts as a reflection of O. Moreover $\Omega(2$-sym. $) = \Omega((c$-ref.$)(m$-ref.$)) = \Omega(c$-ref.$)$. □

We represent $\text{Out}(M)$ by the subgroup $\{\text{idt.}, c$-ref., $v$-ref., $2$-rot.$\}$ of $\text{Sym}(M)$ via the epimorphism $\Omega : \text{Sym}(M) \to \text{Out}(M)$. The set $\text{Isom}(C_{\infty},M)$ consists of four elements corresponding to the pairs of inverse elements $\{\text{idt., idt.}, \{c$-ref., $c$-ref.$\}$,
Remark 1. The Seifert fibrations \((\ast\times), \) and \((2\times 2, 2)\), with IT numbers 9 and 15, respectively, in Table 1 of [3] were replaced by two different affinely equivalent Seifert fibrations in Table 1 of [5]. The Seifert fibrations \((\ast\times)\) and \((2\times 2, 2)\) have

\[
\begin{array}{cccc}
\text{no.} & \text{fibers} & \text{grp.} & \text{quotients} & \text{structure group action} & \text{classifying pair} \\
7 & (\times \times, O) & C_1 & (\times \times, O) & (\text{idt.}, \text{idt.}) & (\text{idt.}, \text{idt.}) \\
9 & (\times \times, O) & C_2 & (\times \times, O) & (2\text{-sym.}, 2\text{-rot.}) & (2\text{-sym.}, 2\text{-sym.}) \\
13 & (\times \times, O) & C_2 & (22\times I) & (\text{v-ref.}, \text{ref.}) & (\text{v-ref.}, \text{v-ref.}) \\
14 & (\times \times, O) & C_2 & (22\times I) & (2\text{-rot.}, \text{ref.}) & (2\text{-rot.}, 2\text{-rot.}) \\
15 & (\times \times, O) & D_2 & (22\times I) & (\text{v-ref.}, \text{ref.}), (2\text{-sym.}, 2\text{-rot.}) & (\text{v-ref.}, 2\text{-rot.}) \\
26 & (\times \times, I) & C_1 & (\times \times, I) & (\text{idt.}, \text{idt.}) & (\text{idt.}, \text{idt.}) \\
27 & (\times \times, O) & C_2 & (**I, I) & (\text{m-ref.}, \text{ref.}) & (\text{m-ref.}, \text{m-ref.}) \\
29 & (\times \times, O) & C_2 & (\times \times, I) & (2\text{-sym.}, \text{ref.}) & (2\text{-sym.}, 2\text{-sym.}) \\
29 & (\times \times, O) & C_2 & (22\times O, O) & (\text{v-ref.}, \text{2-rot.}) & (\text{v-ref.}, \text{v-ref.}) \\
30 & (\times \times, O) & C_2 & (**I, I) & (\text{c-ref.}, \text{ref.}) & (\text{c-ref.}, \text{c-ref.}) \\
33 & (\times \times, O) & C_2 & (22\times O) & (2\text{-rot.}, 2\text{-rot.}) & (2\text{-rot.}, 2\text{-rot.}) \\
36 & (\times \times, I) & D_1 & (\times \times, I) & (2\text{-sym.}, \text{ref.}) & (\text{idt.}, 2\text{-sym.}) \\
37 & (\times \times, O) & D_2 & (**I, I) & (\text{m-ref.}, \text{ref.}), (2\text{-sym.}, 2\text{-rot.}) & (\text{m-ref.}, \text{c-ref.}) \\
39 & (\times \times, I) & D_1 & (**I, I) & (\text{m-ref.}, \text{ref.}) & (\text{idt.}, \text{m-ref.}) \\
41 & (\times \times, O) & D_2 & (**I, I) & (\text{c-ref.}, \text{ref.}), (\text{m-ref.}, \text{2-rot.}) & (\text{c-ref.}, 2\text{-sym.}) \\
45 & (\times \times, O) & D_2 & (**I, I) & (\text{m-ref.}, \text{ref.}), (\text{c-ref.}, \text{2-rot.}) & (\text{m-ref.}, 2\text{-sym.}) \\
46 & (\times \times, I) & D_1 & (**I, I) & (\text{c-ref.}, \text{ref.}) & (\text{idt.}, \text{c-ref.}) \\
52 & (\times \times, O) & D_2 & (2\times 2, 2) & (\text{c-ref.}, \text{ref.}), (2\text{-rot.}, 2\text{-rot.}) & (\text{c-ref.}, \text{v-ref.}) \\
54 & (\times \times, O) & D_2 & (\ast 2222, I) & (\text{m-ref.}, \text{ref.}), (\text{v-ref.'}, 2\text{-rot.}) & (\text{m-ref.}, \text{v-ref.}) \\
56 & (\times \times, O) & D_2 & (22\times I) & (\text{m-ref.}, \text{ref.}), (2\text{-rot.'}, 2\text{-rot.}) & (\text{m-ref.}, 2\text{-rot.}) \\
57 & (\times \times, I) & D_1 & (22\times I) & (\text{v-ref.}, \text{ref.}) & (\text{idt.}, \text{v-ref.}) \\
60 & (\times \times, O) & D_2 & (22\times I) & (2\text{-sym.}, \text{ref.}), (2\text{-rot.'}, 2\text{-rot.}) & (2\text{-sym.}, \text{v-ref.}) \\
60 & (\times \times, O) & D_2 & (2\times 2, 2) & (\text{c-ref.}, \text{ref.}), (2\text{-rot.}, 2\text{-rot.}) & (\text{c-ref.}, 2\text{-rot.}) \\
61 & (\times \times, O) & D_2 & (22\times I) & (\text{v-ref.}, \text{2-rot.}) & (2\text{-sym.}, 2\text{-rot.}) \\
62 & (\times \times, I) & D_1 & (22\times I) & (2\text{-rot.}, \text{ref.}) & (\text{idt.}, 2\text{-rot.}) \\
\end{array}
\]

Table 14. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type \(\times \times\) with IT number 4

\{v-ref., v-ref.\}, \{2-rot., 2-rot.\} of Sym(M) by Theorem 7 of [9]. Notice that the pairs \{2-sym., 2-sym.\} and \{c-ref., c-ref.\} determine the same element in Isom(C_\infty, M), since \(\Omega(2\text{-sym.}) = \Omega(c\text{-ref.})\).

The set Isom(D_\infty, M) consists of twenty one elements corresponding to the pairs of elements {v-ref., v-ref.}, {2-rot., 2-rot.'}, {idt., idt.}, {m-ref., m-ref.}, {2-sym., 2-sym.'}, {c-ref., c-ref.'}, {idt., 2-sym.}, {m-ref., c-ref.}, {idt., m-ref.}, {c-ref., 2-sym.}, {m-ref., 2-sym.}, {idt., c-ref.}, {c-ref., v-ref.}, {m-ref., v-ref.}, {m-ref., 2-rot.}, {idt., v-ref.}, {2-sym., v-ref.}, {c-ref., 2-rot.}, {2-sym., 2-rot.'}, {idt., 2-rot.} of order 1 or 2 of Sym(M) by Lemma 8 of [9] and Theorems 9 and 10 of [9]. Notice that the pairs \{v-ref., v-ref.\} and \{v-ref., v-ref.'\} determine the same element of Isom(D_\infty, M) by Theorem 9 of [9]. The same is true for \{2-rot., 2-rot.'\} and \{2-rot., 2-rot.'\}, \{v-ref., 2-rot.\} and \{v-ref., 2-rot.'\}, \{idt., v-ref.\} and \{idt., v-ref.'\}, \{c-ref., v-ref.\} and \{c-ref., v-ref.'\}, \{m-ref., v-ref.\} and \{m-ref., v-ref.'\}, \{m-ref., 2-rot.\} and \{m-ref., 2-rot.'\}. The pairs \{2-sym., v-ref.\}, \{2-sym., v-ref.'\}, \{c-ref., 2-rot.\} and \{c-ref., 2-rot.'\}, \{idt., 2-rot.\} and \{idt., 2-rot.'\} determine the same element of Isom(D_\infty, M), since they are conjugate.

Remark 1. The Seifert fibrations \((\ast\times)\) and \((2\times 2, 2)\), with IT numbers 9 and 15, respectively, in Table 1 of [3] were replaced by two different affinely equivalent Seifert fibrations in Table 1 of [5]. The Seifert fibrations \((\ast\times)\) and \((2\times 2, 2)\) have
Lemma 18. If \( \text{Sym}(M) \) is isomorphic to \( \text{Sym}(M) \) with IT number 3, the structure group actions for these fibrations are \( \langle \text{c-ref.}, \text{2-rot.} \rangle \) and \( \langle \text{v-ref.}, \text{ref.} \rangle \), \( \langle \text{c-ref.}, \text{2-rot.} \rangle \) respectively.

The action \( \langle \text{c-ref.}, \text{2-rot.} \rangle \) corresponds to the pair \( \{ \text{c-ref.}, \text{c-ref.} \} \), and the action \( \langle \text{v-ref.}, \text{ref.} \rangle \), \( \langle \text{c-ref.}, \text{2-rot.} \rangle \) corresponds to the pair \( \{ \text{v-ref.}, \text{2-rot.} \} \). Hence, these co-Seifert fibrations are affinely equivalent to the co-Seifert fibrations described in rows 2 and 5 of Table 14 respectively, by Theorems 7 and 9 of [9] respectively.

17. Generic Fiber ** (Annulus) with IT Number 3

Let \( M = \langle e_1 + I, e_2 + I, A \rangle \) where \( A = \text{diag}(1, -1) \). Then \( M \) is a 2-space group with IT number 3. The Conway notation for \( M \) is ** and the IT notation is \( pm \).

The flat orbifold \( E^2/M \) is an annulus. We have that \( Z(M) = \langle e_1 + I \rangle \).

Lemma 17. If \( M = \langle e_1 + I, e_2 + I, A \rangle \), with \( A = \text{diag}(1, -1) \), then

\[
N_A(M) = \{ b + B : 2b_2 \in \mathbb{Z} \text{ and } B \in \langle -I, A \rangle \}.
\]

Proof. Observe that \( b + B \in N_A(M) \) if and only if \( B \in N_A(\langle e_1 + I, e_2 + I \rangle) \) and \( (b + B)A(b + B)^{-1} \in M \). Now \( B \in N_A(\langle e_1 + I, e_2 + I \rangle) \) if and only if \( B \in \text{GL}(2, \mathbb{Z}) \).

As \( (b + B)A(b + B)^{-1} = b - BAB^{-1}b + BAB^{-1} \), we have that \( (b + B)A(b + B)^{-1} \in M \) if and only if \( BAB^{-1} = A \) and \( b - Ab \in \mathbb{Z}^2 \). Hence \( b + B \in N_A(M) \) if and only if \( B \in \langle -I, A \rangle \) and \( 2b_2 \in \mathbb{Z} \). \( \square \)

A fundamental polygon for the action of \( M \) on \( E^2 \) is the rectangle with vertices \( v_1 = (0, 0), v_2 = (1, 0), v_3 = (1, 1/2) \), and \( v_4 = (0, 1/2) \). The reflection \( A \) fixes the side \([v_1, v_2]\) pointwise, the reflection \( e_2 + A \) fixes the side \([v_3, v_2]\) pointwise, and the horizontal translation \( e_1 + I \) maps the side \([v_1, v_4]\) to the side \([v_2, v_3]\). The central geodesic of the annulus \( E^2/M \), which we call the central circle, is represented by the horizontal line segment \([0, 1/4], (1, 1/4)\] is invariant under any symmetry of the annulus. The symmetry group of \( E^2/M \) is the direct product of the subgroup of order 2, generated by the reflection in the central circle, and the subgroup consisting of the horizontal rotations and the reflections in a pair of antipodal vertical line segments. The latter subgroup restricts to the symmetry group of the central circle.

There are five conjugacy classes of symmetries of order 2, the class of the reflection \( \text{c-ref.} = (e_2/2 + I)_* \), in the central circle, the class of the horizontal halfturn \( \text{2-sym.} = (e_1+2I)_* \), the class of the halfturn glide-reflection \( \text{g-ref.} = (e_1+2+e_2+2I)_* \) in the central circle, the class of a reflection \( \text{v-ref.} = (-I)_* \) in a pair of antipodal vertical line segments of the annulus, and the class of the halfturn \( \text{2-rot.} = (e_2/2 - I)_* \) around a pair of antipodal points on the central circle. Define \( v-ref.' = (e_1/2 - I)_* \) and \( 2-rot.' = (e_1/2 + e_2/2 - I)_* \).

There are five conjugacy classes of dihedral symmetry groups of order 4, the classes of the groups \( \{ \text{idt.}, \text{c-ref.}, \text{2-sym.}, \text{g-ref.} \} \), \( \{ \text{idt.}, \text{c-ref.}, \text{v-ref.}, \text{2-rot.} \} \), \( \{ \text{idt.}, \text{2-sym.}, \text{v-ref.'} \} \), \( \{ \text{idt.}, \text{2-sym.}, \text{2-rot.'} \} \), and \( \{ \text{idt.}, \text{g-ref.} \} \).

Lemma 18. If \( M = \langle e_1 + I, e_2 + I, A \rangle \), with \( A = \text{diag}(1, -1) \), then the Lie group \( \text{Sym}(M) \) is isomorphic to \( C_2 \times O(2) \), and \( \text{Sym}(M) = \text{Aff}(M) \), and \( \Omega : \text{Aff}(M) \rightarrow \text{Out}(M) \) maps the subgroup \( \{ \text{idt.}, \text{c-ref.}, \text{v-ref.'}, \text{2-rot.'} \} \) isomorphically onto \( \text{Out}(M) \).

Proof. Let \( O \) be the central circle of the annulus, and let \( A \) be the group of all symmetries of the annulus that leave each horizontal geodesic invariant. Restriction induces an isomorphism from \( A \) to \( \text{Isom}(O) \) by Lemmas 1 of [6] and 17. We have that \( \text{Sym}(M) = \langle \text{c-ref.} \rangle \times \Lambda \), since every symmetry of the annulus leaves \( O \) invariant, and \( \text{c-ref.} \) commutes with every symmetry in \( \Lambda \). Therefore \( \text{Sym}(M) \) is isomorphic to
The derivation of Table 15 for fiber of type ** is similar to the derivation of Table 14 for fiber of type ××.

18. Generic Fiber 2222 (pillow) with IT number 2

Let $M = \langle e_1 + I, e_2 + I, -I \rangle$ where $e_1, e_2$ are the standard basis vectors of $E^2$. Then $M$ is a 2-space group with IT number 2. The Conway notation for $M$ is 2222 and the IT notation is $p2$. The flat $E^2/M$ is a square pillow. A flat 2-orbifold affinely equivalent to $\Box$ is called a pillow. A pillow is orientable and has four $180^\circ$ cone points.

Lemma 19. If $M = \langle e_1 + I, e_2 + I, -I \rangle$, then $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism.

Proof. We have that $Z(M) = \{I\}$ by Lemma 5 of [6]. Hence $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism by Theorems 1 and 3 of [6].
Lemma 20. If $M = \langle e_1 + I, e_2 + I, -I \rangle$, then

$$N_A(M) = \left\{ \frac{m}{2} e_1 + \frac{n}{2} e_2 + A : m, n \in \mathbb{Z} \text{ and } A \in \text{GL}(2, \mathbb{Z}) \right\}.$$

Proof.Observe that $a + A \in N_A(M)$ if and only if $A \in N_A(\langle e_1 + I, e_2 + I \rangle)$ and $(a + I)(-I)(a + I)^{-1} \in M$. Now $A \in N_A(\langle e_1 + I, e_2 + I \rangle)$ if and only if $A \in \text{GL}(2, \mathbb{Z})$. As $(a + I)(-I)(a + I)^{-1} = 2a - I$, we have that $(a + I)(-I)(a + I)^{-1} \in M$ if and only if $a = \frac{m}{2} e_1 + \frac{n}{2} e_2$ for some $m, n \in \mathbb{Z}$. \hfill \Box

Lemma 21. Let $M = \langle e_1 + I, e_2 + I, -I \rangle$. The group $\text{Aff}(M)$ has a normal dihedral subgroup $K$ of order 4 generated by $(e_1/2 + I)_*$ and $(e_2/2 + I)_*$. The map $\eta : \text{Aff}(M) \to \text{PGL}(2, \mathbb{Z})$, defined by $\eta((a + A)_*) = \pm A$ for each $A \in \text{GL}(2, \mathbb{Z})$, is an epimorphism with kernel $K$. The map $\sigma : \text{PGL}(2, \mathbb{Z}) \to \text{Aff}(M)$, defined by $\sigma(\pm A) = A_*$, is a monomorphism, and $\sigma$ is a right inverse of $\eta$.

Proof. We have that $Z(M) = \{ I \}$ by Lemma 5 of [6], and $\Omega : \text{Aff}(M) \to \text{Out}(M)$ is an isomorphism by Theorems 1 and 3 of [6]. Hence $K = \Omega^{-1}(\text{Out}_K^1)$ is a normal subgroup of $\text{Aff}(M)$ by Lemma 9 of [6]. Let $T$ be the translation subgroup of $N_A(M)$. Then $K = \Phi(\text{TM}/M)$ by Lemmas 7 and 8 of [6]. Now $\text{TM}/M \cong T/T \cap M$ is a dihedral group of order 4 generated by $(e_1/2 + I)_M$ and $(e_2/2 + I)_M$ by Lemma 20. Hence $K$ is a dihedral group generated by $(e_1/2 + I)_*$ and $(e_2/2 + I)_*$.

The point group $\Pi_A$ of $N_A(M)$ is $\text{GL}(2, \mathbb{Z})$ by Lemma 20. Hence, the map $\eta : \text{Aff}(M) \to \text{PGL}(2, \mathbb{Z})$, defined by $\eta((a + A)_*) = \pm A$ for each $A \in \text{GL}(2, \mathbb{Z})$, is an epimorphism with kernel $K$ by Lemma 9 of [6]. The map $\sigma : \text{PGL}(2, \mathbb{Z}) \to \text{Aff}(M)$ is a well-defined homomorphism by Lemma 7 of [6]. The map $\sigma$ is a monomorphism, since $\sigma$ is a right inverse of $\eta$. \hfill \Box

Lemma 22. Let

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}.$$

The group $\text{PGL}(2, \mathbb{Z})$ is the free product of the dihedral subgroup $\langle \pm A, \pm C \rangle$ of order 4 and the dihedral subgroup $\langle \pm B, \pm C \rangle$ of order 6 amalgamated along the subgroup $\langle \pm C \rangle$ of order 2. Every finite subgroup of $\text{PGL}(2, \mathbb{Z})$ is conjugate to a subgroup of either $\langle \pm A, \pm C \rangle$ or $\langle \pm B, \pm C \rangle$.

Proof. The matrices $A, B, C$ have order 2, and $(AC)^2 = -I$, and $(BC)^3 = -I$. Hence $\langle \pm A, \pm C \rangle$ is a dihedral group of order 4, and $\langle \pm B, \pm C \rangle$ is a dihedral group of order 6.

The group $\text{PGL}(2, \mathbb{Z})$ acts effectively by isometries on the upper complex half-space model of hyperbolic 2-space $H^2$ by

$$\pm \begin{pmatrix} a & b \\ c & c \end{pmatrix} z = \frac{az + b}{cz + d} \quad \text{if } \det \begin{pmatrix} a & b \\ c & c \end{pmatrix} = 1,$$

and

$$\pm \begin{pmatrix} a & b \\ c & c \end{pmatrix} z = \frac{az + b}{cz + d} \quad \text{if } \det \begin{pmatrix} a & b \\ c & c \end{pmatrix} = -1.$$
is the reflection in the $y$-axis, $R_2$ is the inversion in the circle $|z| = 1$, and $R_3$ is the reflection in the line $x = \frac{1}{2}$. Moreover

$$(\pm A)z = R_1(z), \quad (\pm C)z = R_2(z), \quad (\pm B)z = R_3(z).$$

By Poincaré's fundamental polygon theorem (Theorem 13.5.3 [4]), the group $\text{PGL}(2, \mathbb{Z})$ has the presentation

$$\langle R_1, R_2, R_3; R_1^2, R_2^2, R_3^2, (R_1R_2)^2, (R_2R_3)^3 \rangle,$$

which is equivalent to the presentation

$$\langle R_1, R_2, R_3, R_4; R_1^2, R_2^2, (R_1R_2)^2, R_2 = R_4, R_3^2, R_4^2, (R_4R_3)^3 \rangle.$$

Therefore $\text{PGL}(2, \mathbb{Z})$ is the free product with amalgamation of the dihedral group $\langle R_1, R_2 \rangle$ of order 4 and the dihedral group $\langle R_2, R_3 \rangle$ of order 6 amalgamated along the subgroup $\langle R_2 \rangle$ of order 2.

By the torsion theorem for amalgamated products of groups, every finite subgroup of $\text{GL}(2, \mathbb{Z})$ is conjugate to a subgroup of $\pm \langle A, \pm C \rangle$ or $\langle \pm B, \pm C \rangle$. \hfill \Box

**Lemma 23.** Let $M = \langle e_1 + i, e_2 + i, -1 \rangle$, let $K = \langle (e_1/2 + i)_*, (e_2/2 + i)_* \rangle$, and let $A, B, C$ be defined as in Lemma 22. The group $\text{Aff}(M)$ is the free product of the subgroup $\langle K, A_*, C_* \rangle$ of order 16 and the subgroup $\langle K, B_*, C_* \rangle$ of order 24 amalgamated along the subgroup $\langle K, C_* \rangle$ of order 8. Every finite subgroup of $\text{Aff}(M)$ is conjugate to a subgroup of either $\langle K, A_*, C_* \rangle$ or $\langle K, B_*, C_* \rangle$.

**Proof.** The map $\eta : \text{Aff}(M) \to \text{PGL}(2, \mathbb{Z})$, defined by $\eta((a + A)_*) = \pm A$, is an epimorphism with kernel $K$ by Lemma 21. The group $\text{Aff}(M)$ acts on the Bass-Serre tree of the amalgamated product decomposition of $\text{PGL}(2, \mathbb{Z})$ via $\eta$. Hence $\text{Aff}(M)$ is the amalgamated product of the groups $\eta^{-1}(\langle \pm A, \pm C \rangle) = \langle K, A_*, C_* \rangle$ and $\eta^{-1}(\langle \pm B, \pm C \rangle) = \langle K, B_*, C_* \rangle$ along the subgroup $\eta^{-1}(\langle \pm C \rangle) = \langle K, C_* \rangle$. \hfill \Box

Let $\Delta = \langle e_1 + i, e_1/2 + \sqrt{3}e_2/2 + i, -1 \rangle$. Then $\Delta$ is a 2-space group, and the flat orbifold $E^2/\Delta$ is a tetrahedral pillow $\Delta$.

**Lemma 24.** Let $M = \langle e_1 + i, e_2 + i, -1 \rangle$, and let $\Delta = \langle e_1 + i, e_1/2 + \sqrt{3}e_2/2 + i, -1 \rangle$. Let $K = \langle (e_1/2 + i)_*, (e_2/2 + i)_* \rangle$, and let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}.$$

Then $\text{Sym}(M) = \langle K, A_*, C_* \rangle$, and $DMD^{-1} = \Delta$, and $D_2 : \text{Aff}(M) \to \text{Aff}(\Delta)$ is an isomorphism, and $\text{Sym}(\Delta) = D_2(\langle K, B_*, C_* \rangle)$.

**Proof.** We have that $Z(M) = \{I\}$ by Lemma 5 of [6], and so $\text{Sym}(M)$ is finite by Corollary 2 of [6]. Now $\text{Sym}(M)$ is a finite subgroup of $\text{Aff}(M)$ that contains $\langle K, A_*, C_* \rangle$. Hence $\text{Sym}(M) = \langle K, A_*, C_* \rangle$, since $\langle K, A_*, C_* \rangle$ is a maximal finite subgroup of $\text{Aff}(M)$ by Lemma 23.

Clearly $DMD^{-1} = \Delta$, and so $D_2 : \text{Aff}(M) \to \text{Aff}(\Delta)$ is an isomorphism. Now $D_2(\langle K, B_*, C_* \rangle)$ is a subgroup of $\text{Sym}(\Delta)$, since $DMD^{-1}, DCD^{-1} \in O(2)$. Hence $D_2^{-1}(\text{Sym}(\Delta))$ is a finite subgroup of $\text{Aff}(M)$ that contains $\langle K, B_*, C_* \rangle$. Therefore $D_2^{-1}(\text{Sym}(\Delta)) = \langle K, B_*, C_* \rangle$, since $\langle K, B_*, C_* \rangle$ is a maximal finite subgroup of $\text{Aff}(M)$ by Lemma 23. Hence $\text{Sym}(\Delta) = D_2(\langle K, B_*, C_* \rangle)$. \hfill \Box
A square pillow is formed by identifying the boundaries of two congruent squares. The symmetry group of a square pillow is the direct product of the subgroup of order 2 generated by the central reflection between the two squares, and the subgroup of order 8 corresponding to the symmetry group of the two squares. A fundamental domain for the square pillow \( \square = E^2/M \) is the rectangle with vertices \((0, 0), (1/2, 0), (1/2, 1), (0, 1)\). This rectangle is subdivided into two congruent squares that correspond to the two sides of the pillow \( \square \).

There are seven conjugacy classes of symmetries of order 2 of \( \square \) represented by

1. the central halfturn \( c\text{-rot.} = (e_1/2 + e_2/2 + I) \) about the centers of the squares,
2. the midline halfturn \( m\text{-rot.} = (e_1/2 + I) \) about the midpoints of opposite sides of the squares,
3. the central reflection \( c\text{-ref.} = A \) between the two squares,
4. the midline reflection \( m\text{-ref.} = (e_1/2 + A) \),
5. the antipodal map \( 2\text{-sym.} = (e_1/2 + e_2/2 + A) \),
6. the diagonal reflection \( d\text{-ref.} = C \), and
7. the diagonal halfturn \( d\text{-rot.} = (AC) \).

There are two conjugacy classes of symmetries of order 4 of \( \square \), the class of the 90° rotation \( 4\text{-rot.} = (e_1/2 + AC) \) about the centers of the squares, and the class of \( 4\text{-sym.} = (e_1/2 + C) \).

There are nine conjugacy classes of dihedral symmetry groups of order 4 of \( \square \), the classes of the groups \( K = \{ \text{idt., c-rot., m-rot., m-rot.'} \}, \{ \text{idt., m-ref., 2-sym., m-rot.'} \}, \{ \text{idt., c-rot., d-ref., d-ref.'} \}, \{ \text{idt., c-ref., m-ref., m-rot.'} \}, \{ \text{idt., c-rot., m-ref., m-ref.'} \}, \{ \text{idt., c-ref., c-rot., 2-sym.} \}, \{ \text{idt., c-rot., d-rot., d-rot.'} \}, \{ \text{idt., c-ref., d-ref., d-rot.'} \}, \{ \text{idt., 2-sym., d-ref., d-rot.'} \} \).

There are four conjugacy classes of dihedral symmetry groups of order 8 of \( \square \), the classes of the groups \( \{ \text{m-rot., d-rot.}, \langle \text{m-ref., d-ref.}, \langle \text{m-rot., d-rot.}, \langle \text{m-ref., d-rot.} \} \). Moreover \( 4\text{-rot.} = (\text{m-rot.})(\text{d-rot.}) = (\text{m-ref.})(\text{d-ref.}) \) and \( 4\text{-sym.} = (\text{m-rot.})(\text{d-ref.}) = (\text{m-ref.})(\text{d-rot.}) \).

A tetrahedral pillow is realized by the boundary of a regular tetrahedron. The symmetry group of a tetrahedral pillow corresponds to the symmetric group on its four cone points (vertices). A fundamental domain for the tetrahedral pillow \( \Delta = E^3/\Lambda \) is the equilateral triangle with vertices \((0, 0), (1, 0), (1/2, \sqrt{3}/2)\). This triangle is subdivided into four congruent equilateral triangles that correspond to the four faces of a regular tetrahedron.

There are two conjugacy classes of symmetries of order 2 of \( \Delta \), the class of the halfturn \( 2\text{-rot.} = (e_1/2 + I) \), with axis joining the midpoints of a pair of opposite edges of the tetrahedron, corresponding to the product of two disjoint transpositions of vertices, and the class of the reflection \( \text{ref.} = (DBD^{-1}) = A \), corresponding to a transposition of vertices.

There is one conjugacy class of symmetries of order 3 of \( \Delta \), the class of the 120° rotation \( 3\text{-rot.} = (DBCD^{-1}) \), corresponding to a 3-cycle. There is one conjugacy class of elements of order 4 of \( \Delta \), the class of 4-cyc. \( = (e_1/2 + DCD^{-1}) \), corresponding to a 4-cycle.

There are two conjugacy classes of dihedral symmetry groups of order 4 of \( \Delta \), the class of the group of halfturns, and the class of the group generated by two perpendicular reflections.
There is one conjugacy class of dihedral symmetry groups of order 6 of △, the class of the stabilizer of a vertex (and the opposite face) of △. There is one conjugacy class of dihedral symmetry groups of order 8 of △, since all Sylow 2-subgroups of the symmetry group of △ are conjugate.

The square pillow □ is affinely equivalent to the tetrahedral pillow △ by Lemma 24. The symmetries m-rot. and 4-sym. of □ are conjugate by D* to the symmetries 2-rot. and 4-cyc. of △ respectively. The affinity 3-aff. = (BC)* of □ is conjugate by D* to the symmetry 3-rot. of △. The group \{idt., c-rot, m-rot., m-rot.’\} of symmetries of □ is conjugate by D* to the group of halfturns of △. The group \{idt., c-rot, d-ref., d-ref.’\} of symmetries of □ is conjugate by D* to a group of symmetries of △ generated by two perpendicular reflections.

Define the affinity 2-aff. of □ by 2-aff. = B*. Define the reflection ref.’ of △ by ref.’ = (DCD’)-1*. The group \langle 2-aff., d-ref. \rangle of affinities of □ is conjugate by D* to the dihedral group \langle ref., ref.’ \rangle of symmetries of △ of order 6. The group \langle m-rot., d-ref. \rangle of symmetries of □ is conjugate by D* to the dihedral group of symmetries \langle 2-aff., ref., ref.’ \rangle of △ of order 8.

Every affinity of □ of finite order has order 1, 2, 3, or 4 by Lemma 23. There are six conjugacy classes of affinities of □ of order 2 represented by m-rot., c-ref., m-ref., 2-sym., d-ref., and d-rot. Note that c-rot. and m-rot. are conjugate in Aff(M), and 2-aff. and d-ref. are conjugate in Aff(M), since B and C are conjugate in GL(2, Z).

There is one conjugacy class of affinities of □ of order 3 represented by 3-aff. There are two conjugacy classes of affinities of □ of order 4 represented by 4-rot. and 4-sym.

We represent Out(M) by Aff(M) via the isomorphism \( \Omega : \text{Aff}(M) \rightarrow \text{Out}(M) \). The set Isom\((C_\infty, M)\) consists of ten elements corresponding to the pairs of inverse elements \{idt., idt.\}, \{m-rot., m-rot\}, \{c-ref., c-ref\}, \{m-ref., m-ref\}, \{2-sym., 2-sym\}, \{d-ref., d-ref\}, \{d-rot., d-rot\}, \{3-aff., 3-aff., 3-aff., 3-aff,-1\}, \{4-sym., 4-sym., 4-sym,-1\}, \{4-rot., 4-rot., 4-rot,-1\} of Aff(M) by Theorem 7 of [9].

The set Isom\((D_\infty, M)\) consists of forty four elements corresponding to the remaining classifying pairs of elements of Aff(M) in Table 16 by Theorem 8 of [9].

19. Generic Fiber \( \circ \) (torus) with IT number 1

Let \( M = \langle e_1 + I, e_2 + I \rangle \) where \( e_1, e_2 \) are the standard basis vectors of \( E^2 \). Then M is a 2-space group with IT number 1. The Conway notation for M is \( \circ \) and the IT notation is \( p1 \). The flat orbifold \( E^2/M \) is a square torus □. A flat 2-orbifold affinely equivalent to □ is called a torus. A torus is orientable.

**Lemma 25.** If \( M = \langle e_1 + I, e_2 + I \rangle \), then Out(M) = Aut(M) = GL(2, Z), and

\[ N_A(M) = \{a + A : a \in E^2 \text{ and } A \in \text{GL}(2, Z)\} \]

**Proof.** We have that Out(M) = Aut(M), since M is abelian, and Aut(M) = GL(2, Z), since every automorphism of M extends to a unique linear automorphism of \( E^2 \) corresponding to an element of GL(2, Z).

Observe that \( a + A \in N_A(M) \) if and only if \( A \in N_A((e_1 + I, e_2 + I)) \). Hence \( a + A \in N_A(M) \) if and only if \( A \in \text{GL}(2, Z) \). □

**Lemma 26.** Let \( M = \langle e_1 + I, e_2 + I \rangle \). The group \text{Aff}(M)\) has a normal, infinite, abelian subgroup \( K = \{a + I, a \in E^2\} \). The map \( \eta : \text{Aff}(M) \rightarrow \text{GL}(2, Z) \), defined by \( \eta((a + A)_*) = A \) for each \( A \in \text{GL}(2, Z) \), is an epimorphism with kernel K. The
| no. fibers | grp. | quotients | structure group action | classifying pair |
|------------|------|-----------|------------------------|------------------|
| 3 | (2222, O) | C_1 | (2222, O) | (idt., idt.) | {idt., idt.} |
| 5 | (2222, O) | C_2 | (2222, O) | (m-rot., 2-rot.) | {m-rot., m-rot.} |
| 10 | (2222, I) | C_1 | (2222, I) | (idt., idt.) | {idt., idt.} |
| 12 | (2222, I) | D_1 | (2222, I) | (m-rot., ref.) | {m-rot., m-rot.} |
| 13 | (2222, O) | C_2 | (2222, I) | (m-rot., ref.) | {m-rot., m-rot.} |
| 15 | (2222, O) | D_2 | (2222, I) | (c-rot., ref.), (m-rot.’, 2-rot.) | {c-rot., m-rot.} |
| 16 | (2222, O) | C_2 | (2222, I) | (c-rot., ref.) | {c-rot., c-rot.} |
| 17 | (2222, O) | C_2 | (22+, I) | (m-rot., ref.) | {m-rot., m-rot.} |
| 18 | (2222, O) | C_2 | (22+, I) | (2-sym., ref.) | {2-sym., 2-sym.} |
| 20 | (2222, O) | C_2 | (22+, I) | (2-sym., ref.), (m-rot.’, 2-rot.) | {2-sym., m-rot.} |
| 21 | (2222, O) | C_2 | (22+, I) | (d-rot., ref.) | {d-rot., d-rot.} |
| 21 | (2222, O) | D_2 | (22+, I) | (d-rot., ref.), (m-rot., 2-rot.) | {d-rot., m-rot.} |
| 22 | (2222, O) | D_2 | (22+, I) | (d-rot., ref.), (c-rot., 2-rot.) | {d-rot., d-rot.} |
| 23 | (2222, O) | D_2 | (22+, I) | (c-rot., ref.), (c-rot., 2-rot.) | {c-rot., c-rot.} |
| 24 | (2222, O) | D_2 | (22+, I) | (m-rot., ref.), (c-rot., 2-rot.) | {m-rot., m-rot.} |
| 27 | (2222, O) | C_2 | (22+, O) | (c-ref., 2-rot.) | {c-ref., c-ref.} |
| 30 | (2222, O) | C_2 | (22+, O) | (m-rot., 2-rot.) | {m-rot., m-rot.} |
| 34 | (2222, O) | C_2 | (22+, O) | (2-sym., 2-rot.) | {2-sym., 2-sym.} |
| 37 | (2222, O) | C_2 | (22+, O) | (d-ref., 2-rot.) | {d-ref., d-ref.} |
| 43 | (2222, O) | C_2 | (22+, O) | (4-sym., 4-rot.) | {4-sym., 4-sym.} |
| 48 | (2222, O) | D_2 | (22+, I) | (c-ref., ref.), (2-sym., 2-rot.) | {c-ref., c-ref.} |
| 49 | (2222, O) | D_1 | (22+, I) | (c-ref., ref.) | {c-ref., c-ref.} |
| 50 | (2222, O) | D_2 | (22+, I) | (c-ref., ref.), (m-rot., 2-rot.) | {c-ref., m-rot.} |
| 52 | (2222, O) | D_2 | (22+, I) | (c-ref., ref.), (2-sym., 2-rot.) | {c-ref., m-rot.} |
| 52 | (2222, O) | D_2 | (22+, I) | (c-ref., ref.), (m-rot.’, 2-rot.) | {c-ref., m-rot.} |
| 53 | (2222, O) | D_1 | (22+, I) | (m-rot., ref.) | {m-rot., m-rot.} |
| 54 | (2222, O) | D_2 | (22+, I) | (m-rot., ref.), (c-ref., 2-rot.) | {m-rot., m-rot.} |
| 56 | (2222, O) | D_2 | (22+, I) | (c-ref., ref.), (c-ref., 2-rot.) | {c-ref., 2-sym.} |
| 58 | (2222, I) | D_1 | (22+, I) | (2-sym., ref.) | {2-sym., 2-sym.} |
| 60 | (2222, O) | D_2 | (22+, I) | (2-sym., ref.), (m-rot., 2-rot.) | {2-sym., m-rot.} |
| 66 | (2222, I) | D_1 | (22+, I) | (d-ref., ref.) | {d-ref., d-ref.} |
| 68 | (2222, O) | D_2 | (22+, I) | (c-ref., ref.), (d-ref.’, 2-rot.) | {c-ref., d-ref.’} |
| 70 | (2222, O) | D_2 | (22+, I) | (m-rot., ref.), (4-sym., 4-rot.) | {m-rot., d-ref.} |
| 77 | (2222, O) | C_2 | (442, O) | (d-rot., 2-rot.) | {d-rot., d-rot.} |
| 80 | (2222, O) | C_4 | (442, O) | (4-rot., 4-rot.) | {4-rot., 4-rot.} |
| 81 | (2222, O) | C_2 | (442, O) | (d-rot., ref.) | {d-rot., d-rot.} |
| 82 | (2222, O) | D_2 | (442, O) | (d-rot., ref.), (c-rot., 2-rot.) | {d-rot., d-rot.’} |
| 84 | (2222, I) | D_1 | (442, O) | (d-rot., ref.) | {d-rot., d-rot.} |
| 86 | (2222, O) | D_2 | (442, O) | (c-ref., ref.), (d-ref.’, 2-rot.) | {c-ref., d-ref.’} |
| 88 | (2222, O) | D_2 | (442, O) | (m-rot., ref.), (4-rot., 4-rot.) | {m-rot., d-ref.} |
| 93 | (2222, O) | D_2 | (442, O) | (c-ref., ref.), (d-rot., 2-rot.) | {c-ref., d-ref.} |
| 94 | (2222, O) | D_2 | (442, O) | (2-sym., ref.), (d-rot., 2-rot.) | {2-sym., d-ref.} |
| 98 | (2222, O) | D_2 | (442, O) | (m-rot., ref.), (4-rot., 4-rot.) | {m-rot., d-ref.} |
| 112 | (2222, O) | D_2 | (442, O) | (c-ref., ref.), (d-ref., 2-rot.) | {c-ref., d-ref.} |
| 114 | (2222, O) | D_2 | (442, O) | (2-sym., ref.), (d-ref., 2-rot.) | {2-sym., d-ref.’} |
| 116 | (2222, O) | D_2 | (442, O) | (d-rot., ref.), (c-ref., 2-rot.) | {d-rot., d-rot.} |
| 118 | (2222, O) | D_2 | (442, O) | (d-ref., ref.), (2-sym., 2-rot.) | {d-ref., d-ref.’} |
| 122 | (2222, O) | D_4 | (442, O) | (m-rot., ref.), (4-sym., 4-rot.) | {m-rot., d-ref.} |
| 171 | (2222, O) | C_3 | (632, O) | (3-rot., 3-rot.) | {3-aff., 3-aff.’} |
| 180 | (2222, O) | D_3 | (632, O) | (ref., ref.), (3-rot., 3-rot.) | {2-aff., d-ref.} |

Table 16. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type 2222 with IT number 2.
map \( \sigma : \text{GL}(2, \mathbb{Z}) \to \text{Aff}(M) \), defined by \( \sigma(A) = A_* \), is a monomorphism, and \( \sigma \) is a right inverse of \( \eta \).

**Proof.** The map \( \Phi : N_A(M) \to \text{Aff}(M) \), defined by \( \Phi(a + A) = (a + A)_* \), is an epimorphism with kernel \( M \) by Lemma 7 of [6]. Let \( T \) be the translation subgroup of \( N_A(M) \). Then \( \Phi \) maps the normal subgroup \( T/M \) of \( N_A(M)/M \) onto the normal subgroup \( K \) of \( \text{Aff}(M) \). We have that

\[
(N_A(M)/M)/(T/M) = N_A(M)/T = \text{GL}(2, \mathbb{Z}).
\]

Hence \( \eta : \text{Aff}(M) \to \text{GL}(2, \mathbb{Z}) \) is an epimorphism with kernel \( K \). The map \( \sigma : \text{GL}(2, \mathbb{Z}) \to \text{Aff}(M) \) is a well-defined homomorphism by Lemma 7 of [6]. The map \( \sigma \) is a monomorphism, since \( \sigma \) is a right inverse of \( \eta \). \( \square \)

**Lemma 27.** Let

\[
A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

The group \( \text{GL}(2, \mathbb{Z}) \) is the free product of the dihedral subgroup \( \langle A, C \rangle \) of order 8 and the dihedral subgroup \( \langle B, C \rangle \) of order 12 amalgamated along the dihedral subgroup \( \langle -I, C \rangle \) of order 4. Every finite subgroup of \( \text{GL}(2, \mathbb{Z}) \) is conjugate to a subgroup of either \( \langle A, C \rangle \) or \( \langle B, C \rangle \).

**Proof.** Let \( \pi : \text{GL}(2, \mathbb{Z}) \to \text{PGL}(2, \mathbb{Z}) \) be the natural projection defined by \( \pi(A) = \pm A \). Then \( \text{GL}(2, \mathbb{Z}) \) acts on the Bass-Serre tree of the amalgamated product decomposition of \( \text{PGL}(2, \mathbb{Z}) \), given in Lemma 22, via \( \pi \). Hence \( \text{GL}(2, \mathbb{Z}) \) is the amalgamated product of the groups \( \pi^{-1}(\langle \pm A, \pm C \rangle) = \langle A, C \rangle \) and \( \pi^{-1}(\langle \pm B, \pm C \rangle) = \langle B, C \rangle \) along the subgroup \( \pi^{-1}(\langle \pm C \rangle) = \langle -I, C \rangle \), since \( \ker(\pi) = \{ \pm I \} \), and \( (AC)^2 = -I \) and \( (BC)^3 = -I \).

By the torsion theorem for amalgamated products of groups, every finite subgroup of \( \text{GL}(2, \mathbb{Z}) \) is conjugate to a subgroup of either \( \langle A, C \rangle \) or \( \langle B, C \rangle \). \( \square \)

**Lemma 28.** The group \( \text{GL}(2, \mathbb{Z}) \) has seven conjugacy classes of elements of finite order, the class of \( I \), three conjugacy classes of elements of order 2, represented by \( -I, A, C \), one conjugacy class of elements of order 3 represented by \( (BC)^2 \), one conjugacy class of elements of order 4 represented by \( AC \), and one conjugacy class of element of order 6 represented by \( BC \).

**Proof.** By Lemma 27, an element of \( \text{GL}(2, \mathbb{Z}) \) of finite order is conjugate to either \( I, -I, A, C, AC, B, BC \) or \( (BC)^2 \). As \( -I \) commutes with every element of \( \text{GL}(2, \mathbb{Z}) \), we have that \( -I \) is conjugate to only itself. The element \( A \) is not conjugate to \( C \) by a simple proof by contradiction. The matrices \( B \) and \( C \) are conjugate in \( \text{GL}(2, \mathbb{Z}) \), since

\[
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.
\]

**Lemma 29.** Let \( M = \langle e_1 + I, e_2 + I \rangle \), let \( K = \{(a + I)_* : a \in E^2\} \), and let \( A, B, C \) be defined as in Lemma 27. The group \( \text{Aff}(M) \) is the free product of the subgroup \( \langle K, A_*, C_* \rangle \) and the subgroup \( \langle K, B_*, C_* \rangle \) amalgamated along the subgroup \( \langle K, (-I)_*, C_* \rangle \). Every finite subgroup of \( \text{Aff}(M) \) is conjugate to a subgroup of either \( \langle K, A_*, C_* \rangle \) or \( \langle K, B_*, C_* \rangle \).
Proof. The map $\eta : \text{Aff}(M) \to \text{GL}(2, \mathbb{Z})$, defined by $\eta((a + A)_*) = A$ for each $A \in \text{GL}(2, \mathbb{Z})$, is an epimorphism with kernel $K$ by Lemma 26. Hence $\text{Aff}(M)$ acts on the Bass-Serre tree of the amalgamated product decomposition of $\text{GL}(2, \mathbb{Z})$, given in Lemma 27, via $\eta$. Therefore $\text{Aff}(M)$ is the amalgamated product of the groups $\eta^{-1}((A, C)) = \langle K, A_*, C_* \rangle$ and $\eta^{-1}((B, C)) = \langle K, B_*, C_* \rangle$ along the subgroup $\eta^{-1}((-I, C)) = \langle K, (-I)_*, C_* \rangle$.

By the torsion theorem for amalgamated products of groups, every finite subgroup of $\text{Aff}(M)$ is conjugate to a subgroup of either $\langle K, A_*, C_* \rangle$ or $\langle K, B_*, C_* \rangle$. \hfill $\Box$

Let $\Lambda = \langle e_1 + I, e_1/2 + \sqrt{3}e_2/2 + I \rangle$. Then $\Lambda$ is a 2-space group, and the flat orbifold $E^2/\Lambda$ is a hexagonal torus $\diamondsuit$.

Lemma 30. Let $M = \langle e_1 + I, e_2 + I \rangle$, and let $\Lambda = \langle e_1 + I, e_1/2 + \sqrt{3}e_2/2 + I \rangle$. Let $K = \{(a + I)_* : a \in E^2\}$, and let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & -1/2 \\ 0 & \sqrt{3}/2 \end{pmatrix}. $$

Then $\text{Sym}(M) = \langle K, A_*, C_* \rangle$, and $\text{DMD}^{-1} = \Lambda$, and $D_I : \text{Aff}(M) \to \text{Aff}(\Lambda)$ is an isomorphism, and $\text{Sym}(\Lambda) = D_I(K, B_*, C_*)$.

Proof. Now $\text{Out}_E(M) = \text{Aut}_E(M)$ is a finite subgroup of $\text{Aut}(M) = GL(2, \mathbb{Z})$ that contains $\langle A, C \rangle$. Hence $\text{Aut}_E(M) = \langle A, C \rangle$, since $\langle A, C \rangle$ is a maximal finite subgroup of $\text{GL}(2, \mathbb{Z})$ by Lemma 27. The group $K$ is the connected component of $\text{Sym}(M)$ that contains $I_*$. Hence $\text{Sym}(M) = \langle K, A_*, C_* \rangle$ by Theorem 2 of [6].

Clearly $\text{DMD}^{-1} = \Lambda$, and so $D_\# : \text{Aut}(M) \to \text{Aut}(\Lambda)$ and $D_I : \text{Aff}(M) \to \text{Aff}(\Lambda)$ are isomorphisms. Now $D_\#(B, C)$ is a subgroup of $\text{Aut}_E(\Lambda)$, since $DCD^{-1} \in O(2)$. Hence $D_\#^{-1}(\text{Aut}_E(\Lambda))$ is a finite subgroup of $\text{Aff}(M)$ that contains $\langle B, C \rangle$. Therefore $D_\#^{-1}(\text{Aut}_E(\Lambda)) = \langle B, C \rangle$, since $\langle B, C \rangle$ is a maximal finite subgroup of $\text{GL}(2, \mathbb{Z})$ by Lemma 27. Hence $D_I^{-1}(\text{Sym}(\Lambda)) = \langle K, B_*, C_* \rangle$ by Theorem 2 of [6] and Lemma 10 of [6]. Therefore $\text{Sym}(\Lambda) = D_I(K, B_*, C_*)$. \hfill $\Box$

The square torus $\square = E^2/M$ is formed by identifying the opposite sides of the square fundamental domain for $M$, with vertices $(\pm 1/2, \pm 1/2)$, by translations. By Lemma 30, the group $\text{Isom}(\square) = \text{Sym}(M)$ is the semidirect product of the translation subgroup $K = \{(a + I)_* : a \in E^2\}$ and the dihedral subgroup $\langle A_*, C_* \rangle$ of order 8 induced by the symmetry group $\langle A, C \rangle$ of the square fundamental domain of $M$. The elements of $K$ of order at most 2 form the dihedral group of order 4,

$$K_2 = \{I_*, (e_1/2 + I)_*, (e_2/2 + I)_*, (e_1 + e_2 + I)_*\}. $$

By Lemma 31 below, the group $\text{Isom}(\square)$ has six conjugacy classes of elements of order 2 represented by

1. the horizontal halfturn h-rot. $= (e_1/2 + I)_*$ (or vertical halfturn v-rot. $= (e_2/2 + I)_*$),
2. the antipodal map 2-sym. $= (e_1/2 + e_2/2 + I)_*$,
3. the halfturn 2-rot. $= (-I)_*$,
4. the horizontal reflection h-ref. $= A_*$ (or vertical reflection v-ref. $= (-A)_*$),
5. the horizontal glide-reflection h-grf. $= (e_1/2 - A)_*$ (or vertical glide-reflection v-grf. $= (e_2/2 + A)_*$), and
6. the diagonal reflection d-ref. $= C_*$ (or perpendicular diagonal reflection e-ref. $= (-C)_*$).
Lemma 31. Let \( k \in E^2 \), and let \( K \in \text{GL}(2, \mathbb{Z}) \). Then there exists \( a \in E^2 \) such that \( (a + I)(k + K)(a + I)^{-1} = K \) if and only if \( k \in \text{Im}(K - I) \).

Proof. This follows from the fact that

\[
(a + I)(k + K)(-a + I) = a + k - Ka + K = (I - K)a + k + K.
\]

In the following discussion and Table 17, an apostrophe “′” on a symmetry means that the symmetry is multiplied on the left by \( h\text{-rot.} \), a prime symbol “′′” on a symmetry means that the symmetry is multiplied on the left by \( v\text{-rot.} \), and a double prime symbol “′′′” on a symmetry means that the symmetry is multiplied on the left by 2-sym. These alterations do not change the conjugacy class of the symmetry by Lemma 31. Note that conjugating by \( d\text{-ref.} = C \), transposes the “h-” and “v-” prefixes, and so also the “′” and “′′” superscripts.

By Lemma 33 below, the group \( \text{Isom}(\Box) \) has 12 conjugacy classes of dihedral subgroups of order 4, the classes of the groups \( K_2 = \{ \text{idt.}, h\text{-rot.}, v\text{-rot.}, 2\text{-sym.} \} \), \( \{ \text{idt.}, h\text{-rot.}, 2\text{-rot.}, 2\text{-rot.}' \} \), \( \{ \text{idt.}, 2\text{-sym.}, 2\text{-rot.}, 2\text{-rot.}'' \} \), \( \{ \text{idt.}, h\text{-rot.}, v\text{-ref.}, h\text{-grf.} \} \), \( \{ \text{idt.}, v\text{-rot.}, v\text{-ref.}, v\text{-grf.}' \} \), \( \{ \text{idt.}, 2\text{-sym.}, d\text{-ref.}, d\text{-ref.}'' \} \), \( \{ \text{idt.}, 2\text{-rot.}, h\text{-ref.}, h\text{-grf.} \} \), \( \{ \text{idt.}, 2\text{-rot.}, h\text{-grf.}' \} \), \( \{ \text{idt.}, 2\text{-rot.}, d\text{-ref.}, e\text{-ref.} \} \), and \( \{ \text{idt.}, 2\text{-rot.}, d\text{-ref.}, e\text{-ref.} \} \).

The isometries

- 4-rot. = \((AC)_\star\), and
- 4-sym. = \((e_2/2 + C)_\star\)

are symmetries of \( \Box \) of order 4, and the groups \( \langle h\text{-ref.}, d\text{-ref.} \rangle \), \( \langle v\text{-grf.}, d\text{-ref.} \rangle \), and \( \langle v\text{-rot.}, d\text{-ref.} \rangle \) are dihedral subgroups of \( \text{Isom}(\Box) \) of order 8. Moreover 4-rot. = \((h\text{-ref.})(d\text{-ref.})\) and 4-rot.' = \((v\text{-grf.})(d\text{-ref.})\) and 4-sym. = \((v\text{-rot.})(d\text{-ref.})\).

The hexagonal torus \( \mathcal{O} = E^2/\Lambda \) is formed by identifying the opposite sides of the regular hexagon fundamental domain, with vertices \((\pm 1/2, \pm \sqrt{3}/6), (0, \pm \sqrt{3}/3)\), by translations. By Lemma 30, the group \( \text{Isom}(\mathcal{O}) \) is the semidirect product of the subgroup \( K = \{ (a + I)_\star : a \in E^2 \} \) and the dihedral subgroup \( \langle DBD^{-1}_\star, DCD^{-1}_\star \rangle \) of order 12 induced by the symmetry group \( \langle DBD^{-1}, DCD^{-1} \rangle \) of the regular hexagon fundamental domain of \( \Lambda \).

We denote the symmetry of \( \mathcal{O} \) represented by the 180°, 120°, 60° rotation about the center of the regular hexagon fundamental domain of \( \Lambda \) by

- 2-rot. = \(-I)_\star\)
- 3-rot. = \((D(BC)^2D^{-1})_\star\) and
- 6-rot. = \((DBCD^{-1})_\star\)

respectively.

Define affinities of \( \Box \) by

- 3-aff. = \(((BC)^2)_\star\) and
- 6-aff. = \((BC)_\star\).

The affinities 3-aff. and 6-aff. of \( \Box \) are conjugate by \( D \), to the symmetries 3-rot. and 6-rot. of \( \mathcal{O} \) respectively.

Define the reflection l-ref. of \( \mathcal{O} \) by

- l-ref. = \((DCD^{-1})_\star\).

Then l-ref. is represented by the reflection in the line segment joining the midpoints \( \pm (1/4, \sqrt{3}/4) \) of two opposite sides of \( \mathcal{O} \).

Define the reflection m-ref. of \( \mathcal{O} \) by

- m-ref. = \((DBD^{-1})_\star\).
Then $m$-ref. is represented by the reflection in the line segment joining the opposite vertices $(0, \pm \sqrt{3}/3)$ of $\varhexagon$.

Define the reflections $n$-ref. and $o$-ref. of $\varhexagon$ by

- $n$-ref. $= (D(BC)^2CD^{-1})_*$ $= (DBCBD^{-1})_*$ and
- $o$-ref. $= (D(BC)^3CD^{-1})_*$ $= (DCD^{-1})_*$.

The groups $\langle l$-ref., $n$-ref. $\rangle$ and $\langle m$-ref., $o$-ref. $\rangle$ are dihedral groups of order 6 with $3$-rot. = $(n$-ref.$)(l$-ref.$) = (o$-ref.$)(m$-ref.$)$. The group $\langle l$-ref., $m$-ref.$\rangle$ is a dihedral group of order 12 with $6$-rot. = $(m$-ref.$)(l$-ref.$)$. The symmetries $2$-rot., $d$-ref., $e$-ref. of $\varhexagon$ are conjugate by $D_*$ to the symmetries $2$-rot., $l$-ref., $o$-ref. of $\varhexagon$ respectively.

Define affinities of $\varhexagon$ by

- $m$-aff. $= B_*$ and
- $n$-aff. $= (BC)_*$

The affinities $m$-aff. and $n$-aff. of $\varhexagon$ are conjugate by $D_*$ to the symmetries $m$-ref. and $n$-ref. of $\varhexagon$, respectively. The groups $\langle d$-ref., $n$-aff.$\rangle$, $\langle m$-aff., $e$-ref.$\rangle$, $\langle d$-ref., $m$-aff.$\rangle$ are conjugate by $D_*$ to the groups $\langle l$-ref., $n$-ref.$\rangle$, $\langle m$-ref., $o$-ref.$\rangle$, $\langle l$-ref., $m$-ref.$\rangle$ respectively.

**Lemma 32.** Let $M = \langle e_1 + I, e_2 + I \rangle$, and let $\kappa = (k + K)_*$ be an element of $\text{Aff}(M)$ such that $K$ has finite order. Then $K$ is conjugate to exactly one of $I, -I, A, C, (BC)^2, AC$ or $BC$.

1. If $K = I$ and $\kappa$ has order 2, then $\kappa \in K_2$ and $\kappa$ is conjugate to $h$-rot. $= (e_1/2 + I)_*$.
2. If $K = -I$, then $\kappa$ is conjugate to $2$-rot. $= (-I)_*$.
3. If $K$ is conjugate to $A$ and $\kappa$ has order 2, then $\kappa$ is conjugate to exactly one of $h$-ref. $= A_*$ or $v$-grf. $= (e_2/2 + A)_*$.
4. If $K$ is conjugate to $C$ and $\kappa$ has order 2, then $\kappa$ is conjugate to $d$-ref. $= C_*$.
5. If $K$ is conjugate to $(BC)^2$, then $\kappa$ is conjugate to $3$-aff. $= (BC)^2_*$.
6. If $K$ is conjugate to $AC$, then $\kappa$ is conjugate to $4$-rot. $= (AC)_*$.
7. If $K$ is conjugate to $BC$, then $\kappa$ is conjugate to $6$-aff. $= (BC)_*$.

**Proof.** The matrix $K$ is conjugate to exactly one of $I, -I, A, C, (BC)^2, AC$ or $BC$ by Lemma 28. Hence, we may assume that $K = I, -I, A, C, (BC)^2, AC$ or $BC$ by Lemma 28. Hence, we may assume that $K = I, -I, A, C, (BC)^2, AC$ or $BC$.

1. As $\kappa^2 = 2k + I$, we have that $2k \in \mathbb{Z}^2$. Hence $\kappa \in K_2$. We have that $C(e_1/2 + I)C^{-1} = e_2/2 + I$ and $B(e_2/2 + I)B^{-1} = e_1/2 + e_2/2$. Hence $\kappa$ is conjugate to $(e_1/2 + I)_*$.
2. We have that $(k/2 + I)(-I)(-k/2 + I) = k - I$.
3. Observe that $(k + A)^2 = 2k^2e_2 + I$, and so $2k \in \mathbb{Z}$. Hence $\kappa = (k_1e_1 + A)_*$ or $\kappa = (e_2/2 + k_1e_1 + A)_*$. We have that $(k_1e_1/2 + I)A(-k_1e_1/2 + I) = k_1e_1 + A$, and so $(k_1e_1/2 + I)(e_2/2 + A)(-k_1e_1/2 + I) = e_2/2 + k_1e_1 + A$. The isometries $h$-ref. and $v$-grf. are not conjugate in $\text{Aff}(M)$, since $h$-ref. fixes points of $\varhexagon$ whereas $v$-grf. does not.
4. We have that $(k + C)^2 = (k_1 + k_2)e_1 + (k_1 + k_2)e_2 + I$. Hence $k_1 + k_2 \in \mathbb{Z}$, and so $\kappa = (k_1e_1 - k_1e_2 + C)_*$. We have that $\kappa$ is conjugate to $C_*$ by Lemma 31.
5. We have that $\kappa$ is conjugate to $((BC)^2)_*$ by Lemma 31. (6) The proofs of cases (6) and (7) are similar to the proof of (5).

**Lemma 33.** Let $M = \langle e_1 + I, e_2 + I \rangle$. Let $\kappa = (k + K)_*$ and $\lambda = (\ell + L)_*$ be distinct elements of order 2 of $\text{Aff}(M)$ such that $\Omega(\kappa \lambda)$ has finite order in $\text{Out}(M)$. 

Let $H = (K, L)$. Then $K$ and $L$ have order 1 or 2 and either $H$ is a cyclic group of order 1 or 2 or $H$ is a dihedral group of order 4, 6, 8 or 12.

(1) If $H = \{I\}$, then $\langle \kappa, \lambda \rangle = K_2$ and $\{\kappa, \lambda \}$ is conjugate to $\{h\text{-rot., v-rot.}\}$.

(2) If $H$ has order 2, then $H$ is conjugate to exactly one of $\langle -I \rangle$, $\langle CA \rangle$ or $\langle C \rangle$.

(a) If $K = L = -I$, then $\langle \kappa, \lambda \rangle$ is conjugate to $\{h\text{-rot., 2-rot.}\}$.

(b) If $K = L = -I$, then $\langle \kappa, \lambda \rangle$ is conjugate to $\{2\text{-rot., } (v + I), 2\text{-rot.}\}$ with $Kv = -v$.

(c) If $K = L$ and $L$ is conjugate to $A$, then $\{\kappa, \lambda \}$ is conjugate to exactly one of $\{h\text{-rot., v-ref.}, \{v\text{-rot., v-ref.}, \{2\text{-sym., v-ref.}, \{h\text{-rot., h-grf.}, \{v\text{-rot., h-grf.} \text{ or } \{2\text{-sym., h-grf.}\}.$

(d) If $K = L$ and $L$ is conjugate to $A$, then $\{\kappa, \lambda \}$ is conjugate to exactly one of $\{h\text{-ref., } (v + I), \text{h-ref.}, \{h\text{-ref., } (v + I), \text{v-grf.}, \{v\text{-grf., v-grf.} \text{ or } \{v\text{-grf., h-grf.}\}.$

(e) If $K = L$ and $L$ is conjugate to $C$, then $\{\kappa, \lambda \}$ is conjugate to exactly one of $\{2\text{-sym., d-ref.} \text{ or } \{v\text{-rot., d-ref.}\}.$

(f) If $K = L$ and $L$ is conjugate to $C$, then $\{\kappa, \lambda \}$ is conjugate to $\{d\text{-ref., } (v + I), \text{d-ref.}\}$ with $Cv = -v$.

(3) If $H$ has order 4, then $H$ is conjugate to exactly one of $\langle -I, A \rangle$ or $\langle -I, C \rangle$.

(a) If $K = -I$ and $L$ is conjugate to $A$, then $\{\kappa, \lambda \}$ is conjugate to exactly one of $\{2\text{-rot., } (v + I), \text{h-ref.} \text{ or } \{2\text{-rot., } (v + I), \text{v-grf.}\}$ with $Av = -v$.

(b) If $K = -L$ and $K$ is conjugate to $A$, then $\{\kappa, \lambda \}$ is conjugate to exactly one of $\{h\text{-ref., v-ref.}, \{v\text{-ref., v-ref.} \text{ or } \{v\text{-ref., h-grf.}\}.$

(c) If $K = -L$ and $L$ is conjugate to $C$, then $\{\kappa, \lambda \}$ is conjugate to $\{2\text{-rot., } (v + I), \text{d-ref.}\}$ with $Cv = -v$.

(d) If $K = -L$ and $K$ is conjugate to $C$, then $\{\kappa, \lambda \}$ is conjugate to $\{d\text{-ref., } (v + I), \text{d-ref.}\}$.

(4) If $H$ has order 8, then $H$ is conjugate to $\langle A, C \rangle$ and $\{\kappa, \lambda \}$ is conjugate to exactly one of $\{h\text{-ref., d-ref.} \text{ or } \{v\text{-grf., d-ref.}\}.$

(5) If $H$ has order 6, then $H$ is conjugate to exactly one of $\langle BCB, C \rangle$ or $\langle -C, B \rangle$.

(a) In the former case, $\{\kappa, \lambda \}$ is conjugate to $\{n\text{-aff., d-ref.}\}.$

(b) In the latter case, $\{\kappa, \lambda \}$ is conjugate to $\{e\text{-ref., m-aff.}\}.$

(6) If $H$ has order 12, then $H$ is conjugate to $\langle B, C \rangle$ and $\{\kappa, \lambda \}$ is conjugate to $\{m\text{-aff., d-ref.}\}.$

Proof. As $\kappa^2 = (k + Kk + K^2)_*$, we have that $k + Kk \in \mathbb{Z}$ and $K^2 = I$. Hence $K$ has order 1 or 2. Likewise $L$ has order 1 or 2. Now $KL$ has finite order by Theorem 3 of [6] and Lemma 26, moreover $KL$ has order 1, 2, 3, 4 or 6 by Lemma 32. Hence, either $H$ is cyclic of order 1 or 2 or $H$ is dihedral of order 4, 6, 8 or 12.

(1) We have that $\langle \kappa, \lambda \rangle = K_2$ by Lemma 32(1), moreover $B_* \{h\text{-rot., v-rot.}\} B_*^{-1} = \{h\text{-rot., 2-sym.}\}$ and $C_* \{h\text{-rot., 2-sym.}\} C_*^{-1} = \{v\text{-rot., 2-sym.}\}.$

(2) If $H$ has order 2, then $H$ is conjugate to exactly one of $\langle -I \rangle$, $\langle A \rangle$ or $\langle C \rangle$ by Lemma 28; moreover if $H$ is conjugate to $\langle -I \rangle$, then $H = \langle -I \rangle$.

(a) By conjugating $\{\kappa, \lambda \}$, we may assume that $\lambda = 2\text{-rot.}$ by Lemma 32(2). Then $\{\kappa, \lambda \}$ is conjugate to $\{h\text{-rot., 2-rot.}\}$ as in the proof of Lemma 32(1).

(b) By Lemma 32(2), we may assume that $\kappa = K_2$ and $\{\kappa, \lambda \}$ is conjugate to $\{h\text{-rot., 2-rot.}\}$ with $K\ell = -\ell$.

(c) By conjugating $\{\kappa, \lambda \}$, we may assume that $\lambda = h\text{-ref.}$ or $v\text{-grf.}$ by Lemma 32(3). Then $\kappa \in K_2$ by Lemma 32(1), and so there are six possibilities. These six
pairs are nonconjugate since h-ref. is not conjugate to v-grf., the centralizer of \( A \) in \( \text{GL}(2, \mathbb{Z}) \) is \((-I, A)\), and \((-I, A)\) centralizes \( K_2 \).

(d) By Lemma 32(3), we may assume that \( K = A \) and \( \kappa = \text{h-ref.} \) or \( \text{v-grf.} \). Now \( \lambda = (v + A)_* \) or \( \lambda = (e_2/2 + v + A)_* \), with \( Av = -v \) by the proof of Lemma 32(3). Hence \( \lambda = (v + I)_* \), \text{h-ref.} or \( \lambda = (v + I)_* \), \text{v-grf.} with \( Av = -v \).

(e) By conjugating \( \{\kappa, \lambda\} \), we may assume that \( \lambda = \text{d-ref.} \) by Lemma 32(4). Then \( \kappa \in K_2 \) by Lemma 32(1), and so there are three possibilities. We have that \( C(e_1/2 + C)C^{-1} = e_2/2 + C \). The pair \{v-rot., d-ref.\} is not conjugate to \{2-sym, d-ref.\}, since \( (v-rot.) \) (d-ref.) has order 4 while \( (2-sym.) \) (d-ref.) has order 2.

(f) By Lemma 32(4), we may assume that \( K = C \) and \( \kappa = C_* = \text{d-ref.} \). Then \( \lambda = (\ell + C)_* = (\ell + I)_* \), d-ref. with \( C\ell = -\ell \) by the proof of Lemma 32(4).

(3) If \( H \) has order 4, then \( H \) is conjugate to exactly one of \((-I, A)\) or \((-I, C)\) by Lemma 27.

(a) By conjugating \( \{\kappa, \lambda\} \), we may assume that \( L = A \) and \( \kappa = \text{2-rot.} \) by Lemma 31. Now \( \lambda = (v + A)_* \), or \( \lambda = (e_2/2 + v + A)_* \), with \( Av = -v \) by the proof of Lemma 32(3). Hence \( \lambda = (v + I)_* \), \text{h-ref.} or \( \lambda = (v + I)_* \), \text{v-grf.} with \( Av = -v \).

(b) By conjugating \( \{\kappa, \lambda\} \), we may assume that \( K = A \) and \( \kappa \lambda = \text{2-rot.} \). By the proof of Lemma 32(3), we have that \( \kappa = (k_1 e_1 + A)_* \), or \( (e_2/2 + k_1 e_1 + A)_* \), and \( \lambda = (\ell e_2 - A)_* \), or \( (e_1/2 + \ell e_2 - A)_* \). As \( \kappa \lambda \) is 2-rot., we have that \( \{\kappa, \lambda\} \) is either \{h-ref., v-grf.\}, \{h-ref., v-grf.'\}, \{v-grf., h-grf.'\}, or \{v-grf., v-grf.'\}. Moreover \( C_* = \{\text{h-ref.'}, \text{h-grf.}\} C^{-1}_{-} = \{\text{v-ref.'}, \text{v-grf.}\} \).

(c) By conjugating \( \{\kappa, \lambda\} \), we may assume that \( L = C \) and \( \kappa = \text{2-rot.} \). By the proof of Lemma 32(4), we have that \( \lambda = (v + C)_* = (v + I)_* \), d-ref. with \( Cv = -v \).

(d) By conjugating \( \{\kappa, \lambda\} \), we may assume that \( K = C \) and \( \kappa \lambda = \text{2-rot.} \). By the proof of Lemma 32(4), we have that \( \kappa = (k_1 e_1 - k_1 e_2 + C)_* \). Now \( (\ell - C)^2 = (\ell_1 - \ell_2) e_1 + (\ell_2 - \ell_1) e_2 + I \). Hence \( \ell_1 - \ell_2 \in \mathbb{Z} \), and so \( \lambda = (\ell_1 e_1 + \ell_1 e_2 - C)_* \). Now \((k_1 e_1 - k_1 e_2 + C)(\ell_1 e_1 + \ell_1 e_2 - C) = (k_1 + \ell_1) e_1 + (\ell_1 - k_1) e_1 - I \). Hence \( k_1 + \ell_1, \ell_1 - k_1 \in \mathbb{Z} \). Thus we may take \( \ell_1 = k_1 \) and \( k_1 = 0 \) or 1/2. Then \( \{\kappa, \lambda\} \) is either \{d-ref., e-ref.\} or \{d-ref.'', e-ref.'\}. Moreover \((e_1/2 + I) C(-e_1/2 + I) = e_1/2 - e_2/2 + C \) and \((e_1/2 + I)(-C)(-e_1/2 + I) = e_1/2 + e_2/2 - C \).

(4) If \( H \) has order 8, then \( H \) is conjugate to \((A, C)\) by Lemma 27. By conjugating \( \{\kappa, \lambda\} \), we may assume that \( K = BC B \) and \( L = C \) and \( \kappa \lambda = \text{4-rot.} \). By Lemma 32(3), we have that \( \lambda = (k_1 e_1 + A)_* \), or \( (k_1 e_1 + e_2/2 + A)_* \), and by Lemma 32(4), we have that \( \lambda = (\ell_1 e_1 - \ell_1 e_2 + C)_* \). Now \((k_1 e_1 + A)(\ell_1 e_1 - \ell_1 e_2 + AC) = (k_1 - \ell_1) e_1 - \ell_1 e_2 + AC \), and we may take \( k_1 = \ell_1 = 0 \), moreover \((k_1 e_1 + e_2/2 + A)(\ell_1 e_1 - \ell_1 e_2 + C) = (k_1 - \ell_1) e_1 + (1/2 - \ell_1) e_2 + AC \), and we may take \( k_1 = \ell_1 = 1/2 \). Then we have that \( \{\kappa, \lambda\} = \{\text{h-ref.}, \text{d-ref.}\} \) or \{v-grf.'', d-ref.'\}''. Moreover, we have that \((e_1/4 - e_2/4 + I)(e_2/4 + A)(-e_1/4 + e_2/4 + I) = e_1/2 + e_2/2 + A \) and we have that \((e_1/4 - e_2/4 + I) C(-e_1/4 + e_2/4 + I) = e_1/2 - e_2/2 + C \).

(5) If \( H \) has order 6, then \( H \) is conjugate to exactly one of \((B^2, C)\) or \((B^2, CB)\) by Lemma 27.

(a) By conjugating \( \{\kappa, \lambda\} \), we may assume that \( K = BC B \) and \( L = C \) and \( \kappa \lambda = \text{3-aff.} \). Now \((k + BC B)^2 = (2k_2 - k_1) e_2 + I \). Hence \( \kappa = 2k_2 e_1 + k_2 e_2 + BC B \).

By Lemma 32(4), we have that \( \lambda = (\ell_1 e_1 - \ell_1 e_2 + C)_* \). Now, we have that \( \kappa \lambda = (2k_2 - \ell_1) e_1 + (k_2 - 2\ell_1) e_2 + (BC B)^2 \). Hence, we may take \( k_2 = \ell_1 = 0 \), and so \( \kappa = (BC B)_* = \text{n-aff.} \) and \( \lambda = C_* = \text{d-ref.} \).
(b) By conjugating \( \{ \kappa, \lambda \} \), we may assume that \( K = -C \) and \( L = B \) and \( \kappa \lambda = 3\text{-aff} \). Now \((k - C)^2 = (k_1 - k_2)e_1 + (k_2 - k_1)e_2 + I \). Hence \( \kappa = (k_1e_1 + k_2e_2 - C) \). Now \((\ell + B)^2 = \ell_2e_1 + 2\ell_1e_2 + I \). Hence \( \lambda = B \) = 3-aff. Now \((k_1e_1 + k_1e_2 - C)B = k_1e_1 + k_1e_2 + (BC)^2 \). Hence \( \kappa = (-C) \) = e-ref.

(6) If \( H \) has order 12, then \( H \) is conjugate to \((B, C)\) by Lemma 27. By conjugating \( \{ \kappa, \lambda \} \), we may assume that \( K = B \) and \( L = C \) and \( \kappa \lambda = 6\text{-aff} \). By the proof of 5(b), we have that \( \kappa = B_1 = m\text{-aff} \). By Lemma 32(4), we have that \( \lambda = (k_1e_1 - k_1e_2) \). Now \( B(k_1e_1 - k_1e_2 + C) = -2k_1e_1 - k_1e_2 + BC \). Hence \( \lambda = C_1 = d\text{-ref} \). □

We represent \( \text{Out}(M) \) by the image of the monomorphism \( \sigma : \text{GL}(2, \mathbb{Z}) \to \text{Aff}(M) \) (see Lemmas 25 and 26). The set \( \text{Isom}(C_\infty, M) \) consists of seven elements corresponding to the pairs of inverse elements \( \{ \text{idt, idt.} \}, \{2\text{-rot., 2\text{-rot.}} \}, \{\text{h-ref., h-ref.} \}, \{d\text{-ref., d-ref.} \}, \{4\text{-rot., 4\text{-rot.} }^{-1} \}, \{3\text{-aff., 3\text{-aff.} }^{-1} \}, \{6\text{-aff., 6\text{-aff.} }^{-1} \} \) of \( \sigma(\text{GL}(2, \mathbb{Z})) \) by Lemma 28 and Theorem 7 of [6]. The pairs \( \{\text{h-ref., h-ref.} \} \) and \( \{\text{v-ref., v-ref.} \} \) determine the same element of \( \text{Isom}(C_\infty, M) \), since they are conjugate.

The set \( \text{Isom}(D_\infty, M) \) consists of thirty-four elements corresponding to the remaining classifying pairs of elements of \( \text{Aff}(M) \) in Table 17 by Lemma 33 and Theorems 9 and 10 of [6]. Note that one may have to conjugate a classifying pair in Table 17 to make it correspond to a pair in Lemma 33.

20. Enantiomorphic 3-space group pairs

In this section, we apply our theory to give an explanation for all but one of the enantiomorphic 3-space group pairs. An enantiomorphic 3-space group pair consists of a pair \((\Gamma_1, \Gamma_2)\) of isomorphic 3-space groups all of whose elements are orientation-preserving such that there is no orientation-preserving affinity of \( E^3 \) that conjugates one to the other. There are 11 enantiomorphic 3-space group pairs up to isomorphism, and 10 of these have 2-dimensional, complete, normal subgroups; moreover, these complete normal subgroups are unique.

The first enantiomorphic pair \((\Gamma_1, \Gamma_2)\) has IT numbers 76 and 78. The geometric co-Seifert fibration of \( E^3/\Gamma_1 \) is described in Table 17. The action of the cyclic structure group of order 4 on \( \phi \times O \) is given by \((4\text{-rot., 4\text{-rot.})} \). The action of the structure group for \( E^3/\Gamma_2 \) is given by \((4\text{-rot. }^{-1}, 4\text{-rot.}) \), since the classifying pair is \( \{4\text{-rot., 4\text{-rot. }^{-1}} \} \).

Let \( N_i \) be the unique 2-dimensional, complete, normal subgroup of \( \Gamma_i \) for \( i = 1, 2 \). We have that \( \text{Span}(N_i) = E^2 \). Let \( \phi \) be an affinity of \( E^3 \) such that \( \phi \Gamma_1 \phi^{-1} = \Gamma_2 \). Then \( \phi N_1 \phi^{-1} = N_2 \), since \( \phi N_1 \phi^{-1} \) is a 2-dimensional, complete, normal subgroup of \( \Gamma_2 \). Let \( \overline{\phi} : E^2 \to E^2 \) be the restriction of \( \phi \), and let \( \phi' : (E^2)^\perp \to (E^2)^\perp \) be the restriction of \( \phi \) followed by orthogonal projection. Then \( \phi \) is orientation-preserving if and only if either \( \overline{\phi} \) and \( \phi' \) are both orientation-preserving (orientation-preserving case) or \( \overline{\phi} \) and \( \phi' \) are both orientation-reversing (orientation-reversing case).

We now prove that the pair \((\Gamma_1, \Gamma_2)\) is enantiomorphic. Assume that \( \phi \) is orientation-preserving. By Theorem 3.3 of [8], we have the equation

\[
\Xi_2 P_2^{-1}(\phi')P_1 = (Dp_1)_*(\overline{\phi})^2 \Xi_1.
\]

The map \( P_2^{-1}(\phi')P_1 : \Gamma_1/N_1 \to \Gamma_2/N_2 \) is an isomorphism of infinite cyclic groups that maps the positively oriented generator of \( \Gamma_1/N_1 \) to the positively oriented generator of \( \Gamma_2/N_2 \) if and only if \( \phi' \) is orientation-preserving. The map \( \Xi_2 : \Gamma_1/N_1 \to \text{Isom}(E^2/N_i) \) is the homomorphism induced by the action of \( \Gamma_1/N_1 \) on \( E^2/N_i \), which factors through the action of the structure group on \( E^2/N_i \) for each \( i = 1, 2 \). The
| no. | fibers | grp. | quotients | structure group action | classifying pair |
|-----|--------|------|-----------|------------------------|-----------------|
| 1   | (α, O) | C₁   | (O, O)    | (idt., idt.)           | {idt., idt.}    |
| 2   | (α, O) | C₂   | (2222, I) | (2-rot., ref.)         | {2-rot., 2-rot.} |
| 3   | (α, O) | C₂   | (**)     | (v-ref., ref.)         | {v-ref., v-ref.} |
| 4   | (α, O) | C₂   | (××, I)  | (h-grf., ref.)         | {h-grf., h-grf.} |
| 4   | (α, O) | C₂   | (2222, O) | (2-rot., 2-rot.)       | {2-rot., 2-rot.} |
| 5   | (α, O) | C₂   | (**)     | (v-ref., ref.)         | {v-ref., h-grf.} |
| 6   | (α, I) | C₁   | (O, I)    | (idt., idt.)           | {idt., idt.}    |
| 7   | (α, O) | C₂   | (α, I)   | (v-rot., ref.)         | {v-rot., v-rot.} |
| 7   | (α, O) | C₂   | (**)     | (v-ref., v-ref.)       | {v-ref., v-ref.} |
| 8   | (α, I) | D₁   | (O, I)    | (h-rot., ref.)         | {h-rot., h-rot.} |
| 9   | (α, O) | D₂   | (O, I)   | (v-rot., ref.)         | {v-rot., 2-sym.} |
| 9   | (α, O) | C₂   | (××, O)  | (2-sym., 2-rot.)       | {d-ref., d-ref.} |
| 11  | (α, I) | D₁   | (2222, I)| (2-rot., ref.)         | {2-rot., 2-rot.} |
| 13  | (α, O) | D₂   | (2222, I)| (h-ref., ref.)         | {h-ref., 2-rot.} |
| 14  | (α, O) | D₂   | (2222, I)| (v-rot., ref.)         | {h-ref., h-ref.} |
| 14  | (α, O) | D₂   | (22+, I) | (h-ref., ref.)         | {h-ref., h-ref.} |
| 15  | (α, O) | D₂   | (2×2, I) | (e-ref., ref.)         | {e-ref., 2-rot.} |
| 17  | (α, O) | D₂   | (2222, I)| (v-ref., ref.)         | {v-ref., h-ref.} |
| 18  | (α, O) | D₂   | (2×2+, I)| (2-rot., ref.)         | {h-ref., h-ref.} |
| 19  | (α, O) | D₂   | (2×2, I) | (v-rot., ref.)         | {v-rot., v-rot.} |
| 20  | (α, O) | D₂   | (2×2+, I)| (d-ref., ref.)         | {d-ref., d-ref.} |
| 28  | (α, I) | D₁   | (**)     | (idt., ref.)           | {idt., h-ref.}  |
| 29  | (α, O) | D₂   | (**)     | (v-ref., ref.)         | {v-rot., v-grf.} |
| 30  | (α, O) | D₂   | (**)     | (v-ref., ref.)         | {v-rot., h-ref.} |
| 31  | (α, I) | D₁   | (××, I)  | (v-grf., ref.)         | {v-grf., v-grf.} |
| 32  | (α, O) | D₂   | (**)     | (h-ref., ref.)         | {h-ref., h-ref.} |
| 33  | (α, O) | D₂   | (**)     | (v-ref., ref.)         | {v-ref., h-ref.} |
| 33  | (α, O) | D₂   | (**)     | (v-ref., ref.)         | {v-ref., h-ref.} |
| 34  | (α, O) | D₂   | (**)     | (2-sym., ref.)         | {2-sym., v-grf.} |
| 34  | (α, O) | D₂   | (**)     | (2-sym., ref.)         | {2-sym., v-grf.} |
| 40  | (α, I) | D₁   | (**)     | (e-ref., ref.)         | {idt., e-ref.}  |
| 41  | (α, O) | D₂   | (**)     | (e-ref., ref.)         | {e-ref., 2-sym.} |
| 43  | (α, O) | D₄   | (**)     | (v-ref., ref.)         | {v-rot., d-ref.} |
| 76  | (α, O) | C₄   | (442, O)  | (4-rot., 4-rot.)       | {4-rot., 4-rot.} |
| 91  | (α, O) | D₄   | (442I, O)| (4-rot., 4-rot.)       | {4-rot., 4-rot.} |
| 92  | (α, O) | D₄   | (42, O)  | (4-rot., 4-rot.)       | {4-rot., 4-rot.} |
| 144 | (α, O) | C₃   | (333, O)  | (3-aff., 3-aff.)       | {3-aff., 3-aff.} |
| 151 | (α, O) | D₃   | (333, O)  | (3-aff., 3-aff.)       | {3-aff., 3-aff.} |
| 152 | (α, O) | D₃   | (3×3, I)  | (3-aff., 3-aff.)       | {3-aff., 3-aff.} |
| 169 | (α, O) | C₆   | (632, O)  | (6-aff., 6-aff.)       | {6-aff., 6-aff.} |
| 178 | (α, O) | D₆   | (632, I)  | (6-aff., 6-aff.)       | {6-aff., 6-aff.} |

Table 17. The classification of the co-Seifert fibrations of 3-space groups whose generic fiber is of type α with IT number 1.
image of the positively oriented generator of $\Gamma_1/N_1$ under $\Xi_2P_2^{-1}(\phi'), P_1$ is therefore 
$(4\text{-rot.})^{-1} = (AC)_p^{-1}$ in the orientation-preserving case and $4\text{-rot.} = (AC)_p$ in the orientation-reversing case.

The map $(Dp_1)_*(\overline{\phi})_2\Xi_1$ is harder to evaluate because $(Dp_1)_*$ is a crossed homomorphism from $\Gamma_1/N_1$ to the connected component $\mathcal{K}_2$ of the identity of the Lie group $\text{Isom}(E^2/N_2)$. To simplify, we apply the epimorphism $\Omega: \text{Aff}(E^2/N_2) \to \text{Out}(N_2)$ whose kernel is $\mathcal{K}_2$ by Theorem 3 of [6]. This cancels the action of $(Dp_1)_*$. To simply further, we identify $\text{Out}(N_2)$ with $\text{GL}(2,\mathbb{Z})$ as in Lemma 25. The map $(\overline{\phi})_2: \text{Aff}(E^2/N_1) \to \text{Aff}(E^2/N_2)$ is induced by conjugating by the affinity $(\overline{\phi})_*$.

It is now clear that the pair $(\Gamma_1, \Gamma_2)$ is enantiomorphic because in the orientation-preserving case, there is no element of $\text{SL}(2,\mathbb{Z})$ that conjugates $AC$ to $(AC)_p^{-1}$, and in the orientation-reversing case, there is no element of $\text{GL}(2,\mathbb{Z})$ of determinant $-1$ that conjugates $AC$ to $AC$. The enantiomorphic IT number pairs $(144, 145), (169, 170), and (171, 172)$ are enantiomorphic by a similar argument.

The second enantiomorphic pair $(\Gamma_1, \Gamma_2)$ has IT numbers 91 and 95. The co-Seifert fibration of $E^3/\Gamma_1$ is described in Table 17. The action of the dihedral structure group of order 8 on $\circ \times O$ is given by $\text{h-ref., ref.}$, $(4\text{-rot., 4-rot.})$. The action of the structure group for $E^3/\Gamma_2$ is given by $(\text{d-ref., ref.})$, $(4\text{-rot.})$, since the classifying pair is $\{(\text{h-ref., d-ref.})\}$. Recall that $\text{h-ref.} = A$ and $\text{d-ref.} = C$.

Let $N_i$ be the unique 2-dimensional, complete, normal subgroup of $\Gamma_i$ for $i = 1, 2$. Then $\Gamma_i/N_i$ is an infinite dihedral group for each $i = 1, 2$. By considering the image of a pair of Coxeter generators of $\Gamma_1/N_1$, the same argument as before shows that the pair $(\Gamma_1, \Gamma_2)$ is enantiomorphic because in the orientation-preserving case, there is no element of $\text{SL}(2,\mathbb{Z})$ that conjugates the pair $(A, C)$ to the pair $(C, A)$, and in the orientation-reversing case, there is no element of $\text{GL}(2,\mathbb{Z})$ of determinant $-1$ that conjugates the pair $(A, C)$ to itself. The enantiomorphic IT number pairs $(92, 96), (151, 153), (152, 154), (178, 179), and (180, 181)$ are enantiomorphic by a similar argument.

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