A New Condition for the Concavity Method of Blow-up Solutions to p-Laplacian Parabolic Equations

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Abstract

In this paper, we consider an initial-boundary value problem of the p-Laplacian parabolic equations

\[
\begin{cases}
  u_t(x,t) = \text{div}(|\nabla u(x,t)|^{p-2}\nabla u(x,t)) + f(u(x,t)), & (x,t) \in \Omega \times (0, +\infty), \\
  u(x,t) = 0, & (x,t) \in \partial \Omega \times [0, +\infty), \\
  u(x,0) = u_0 \geq 0, & x \in \Omega,
\end{cases}
\]

where $p \geq 2$ and $\Omega$ is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial \Omega$. The main contribution of this work is to introduce a new condition \( (C_p) \)

\[
\alpha \int_0^u f(s)ds \leq uf(u) + \beta u^p + \gamma, \quad u > 0
\]

for some $\alpha, \beta, \gamma > 0$ with $0 < \beta \leq \frac{(\alpha-p)\lambda_{1,p}}{p}$, where $\lambda_{1,p}$ is the first eigenvalue of $p$-Laplacian $\Delta_p$, and we use the concavity method to obtain the blow-up solutions to the above equations. In fact, it will be seen that the condition \( (C_p) \) improves the conditions ever known so far.

Keywords: Parabolic, p-Laplacian, Blow-up, Concavity method.

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1. Introduction

In this paper, we discuss the blow-up solutions for the following p-Laplacian parabolic equations

\[
\begin{cases}
  u_t(x,t) = \text{div}(|\nabla u(x,t)|^{p-2}\nabla u(x,t)) + f(u(x,t)), & (x,t) \in \Omega \times (0, +\infty), \\
  u(x,t) = 0, & (x,t) \in \partial \Omega \times [0, +\infty), \\
  u(x,0) = u_0 \geq 0, & x \in \Omega,
\end{cases}
\]

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where \( p \geq 2 \), \( \Omega \) is a bounded domain in \( \mathbb{R}^N (N \geq 1) \) with smooth boundary \( \partial \Omega \) and \( f \) is locally Lipschitz continuous on \( \mathbb{R} \), \( f(0) = 0 \) and \( f(u) > 0 \) for \( u > 0 \). Moreover, the initial data \( u_0 \) is assumed to be a non-negative and non-trivial function in \( C^1(\Omega) \) with \( u_0(x) = 0 \) on \( \partial \Omega \) for \( p = 2 \) and in \( L^\infty(\Omega) \cap W^{1,p}_0(\Omega) \) for \( p > 2 \), respectively.

There are many literatures dealing with a local existence of classical solutions (or weak solutions) to the equations (1). In general, it is well known that not all solutions of the equations (1) exist for all time. So, many authors have focused on the sufficient conditions for the local existence of solutions to the equations (1). In particular, for \( p = 2 \), Ball [1] derived sufficient conditions for the local existence solutions to the equations (1). On the other hand, for \( p > 2 \), Zhao [14] also derived sufficient conditions for the nonexistence of global solutions to the equations (1).

On the other hand, the blow-up solutions to the equations (1) have been studied by many authors. In particular, Levine and Payne [5] studied the abstract equation

\[
\begin{cases}
  P \frac{du}{dt} = -A(t)u + f(u(t)), & t \in [0, +\infty), \\
  u(0) = u_0,
\end{cases}
\]

where \( P \) and \( A \) are positive linear operators defined on a dense subdomain \( D \) of a real or complex Hilbert Space, in which they obtained the blow-up solutions, under abstract conditions

\[
2(\alpha + 1)F(x) \leq (x, f(x)), \quad F(u_0(x)) > \frac{1}{2} (u_0(x), Au_0(x))
\]

for every \( x \in D \), where \( F(x) = \int_0^1 (f(\rho x), x) d\rho \). This work has been recognized as a creative and elegant tool for giving criteria for the blow-up, which is called “the concavity method”. They also applied the method to some other equations or system of equations (See [6, 7]).

Afterwards, the method in the abstract form was changed into a concrete form by Philippin and Proytcheva [11] and applied to the same equation as (1) with \( p = 2 \). In fact, the condition (2) was changed into the form

\[
(A) \quad (2 + \epsilon)F(u) \leq uf(u), \quad u > 0,
\]

for some \( \epsilon > 0 \) and the initial data \( u_0 \) satisfies

\[
-\frac{1}{2} \int_\Omega |\nabla u_0(x)|^2 dx + \int_\Omega F(u_0(x)) dx > 0,
\]

where \( F(u) = \int_0^u f(s) ds \).

Since then, the concavity method has been used so far to derive the blow-up solutions the variants of the equations (1) or some other equations.

For example, Ding and Hu [2] adopted the condition (A) and

\[
-\int_\Omega \int_0^{u_0} \rho(y) dy dx + 2k(0) \int_\Omega \int_0^{u_0} f(s) ds dx > 0
\]
to get blow-up solutions to the equation
\[(g(u))_t = \nabla \cdot (\rho(|\nabla u|^2)\nabla u) + k(t)f(u),\]
assuming more that \(k(0) > 0, k'(t) \geq 0, \lim_{s \to 0^+} s^2 g'(s) = 0, g''(s) \leq 0,\) and
\[0 < s\rho(s) \leq (1 + \alpha) \int_0^s \rho(y)dy.\]

Another example is the work by Payne et al. in [12, 13] in which they obtained
the blow-up solutions to the equations
\[
\begin{cases}
    u_t = \Delta u - g(u), & \text{in } \Omega \times (0, +\infty), \\
    \frac{\partial u}{\partial n} = f(u), & \text{on } \partial \Omega \times (0, +\infty), \\
    u(x,0) = u_0(x) \geq 0,
\end{cases}
\]
when the Neumann boundary data \(f\) satisfies the condition (A).

On the other hands, the condition (A) for the nonlinear term \(f\) and the
condition (3) for the initial data \(u_0\) were relaxed by Bandle and Brunner [2] as
follows:
\[(B) \quad (2 + \epsilon)F(u) \leq uf(u) + \gamma, \quad u > 0\]
for some \(\epsilon > 0\) and the initial data \(u_0\) satisfying
\[-\frac{1}{2} \int_\Omega |\nabla u_0(x)|^2 dx + \int_\Omega [F(x,u_0) - \gamma]dx > 0, \tag{5}\]

Concerning the general case \(p > 2\), in 1993, Zhao [14] studied the following
equations
\[
\begin{cases}
    u_t = \text{div}(|\nabla u|^{p-2}\nabla u) + f(u), & \text{in } \Omega \times (0, T), \\
    u(x,t) = 0, & \text{on } \partial \Omega, \\
    u(x,0) = u_0, & \text{in } \Omega
\end{cases}
\]
and proved the blow-up solutions to the equations (6) under the condition
\[p F(u) \leq uf(u), \quad u > 0, \tag{7}\]
for some \(\epsilon > 0\) and the initial data \(u_0\) satisfying
\[-\frac{1}{p} \int_\Omega |\nabla u_0(x)|^p dx + \int_\Omega F(u_0(x))dx \geq \frac{4(p-1)}{T(p-2)^2} \int_\Omega u_0^2(x)dx \tag{8}\]

In 2002, Messaoudi [9] proved that the solutions to the same equations (11)
blow up under the condition
\[(A_p) \quad (p + \epsilon)F(u) \leq uf(u), \quad u > 0, \tag{11}\]
and the initial data \(u_0\) satisfying
\[-\frac{1}{p} \int_{\Omega} |\nabla u_0(x)|^p \, dx + \int_{\Omega} F(u_0(x)) \, dx > 0.\]

Looking into the above conditions (A), (A\(_p\)), (B) and so on more closely, we can see that they are independent of the eigenvalue of the Laplacian \(\Delta\) (or \(p\)-Laplacian \(\Delta_p\)), which is a constant depending on the domain \(\Omega\). From this point of view, there is, we think, a possibility that the above conditions can be improved and refined in a way, depending on the domain and the eigenvalue.

Being motivated by this point, we introduce a new condition as follows: for some \(\alpha, \beta, \gamma > 0\),

\[(C_p) \quad \alpha \int_0^u f(s) \, ds \leq uf(u) + \beta u^p + \gamma, \, u > 0,
\]

where \(0 < \beta \leq \frac{(\alpha-p)\lambda_{1,p}}{p}\) and \(\lambda_{1,p}\) is the first eigenvalue of the Laplacian \(\Delta_p\).

(For simplicity, the condition \((C_p)\) and \(\lambda_{1,p}\) are denoted by \((C)\) when \(p = 2\), respectively.)

The main theorems of this paper are as follows:

**Theorem** (Case 1 : \(p = 2\)). Let a function \(f\) satisfy the condition \((C)\). If the initial data \(u_0 \in C^1(\Omega)\) with \(u_0 = 0\) on \(\partial \Omega\) satisfies

\[-\frac{1}{2} \int_{\Omega} |\nabla u_0(x)|^2 \, dx + \int_{\Omega} \left[\int_0^{u_0(x)} f(s) \, ds - \gamma\right] \, dx > 0, \tag{9}\]

then the nonnegative classical solutions \(u\) to the equations \((1)\) blows up at finite time \(T^*\), in the sense of

\[\lim_{t \to T^*} \int_0^t \int_{\Omega} u^2(x, s) \, ds = +\infty,\]

where \(\gamma\) is the constant in the condition \((C)\).

**Theorem** (Case 2 : \(p > 2\)). Let a function \(f\) satisfy the condition \((C_p)\) and \(p > 2\). If the initial data \(u_0 \in L^\infty(\Omega) \cap W^{1,p}_0(\Omega)\) satisfies

\[-\frac{1}{p} \int_{\Omega} |\nabla u_0(x)|^p \, dx + \int_{\Omega} [F(u_0(x)) - \gamma] \, dx > 0, \tag{10}\]

then the nonnegative weak solutions \(u\) to the equations \((1)\) blows up at finite time \(T^*\), in the sense of

\[\lim_{t \to T^*} \int_0^t \int_{\Omega} u^2(x, s) \, ds = +\infty,\]

where \(\gamma\) is the constant in the condition \((C_p)\).
We organized this paper as follows: In Section 2, we discuss, when $p = 2$, the blow-up classical solutions using concavity method with the condition $(C_p)$ and in Section 3 when $p > 2$, we discuss the blow-up weak solutions using the same method with the condition $(C_p)$, which is the general case. Finally, in Section 4, we discuss the condition $(C_p)$, comparing with the conditions $(A_p)$, $(B)$, and (7) so on, together with the condition $J_p(0) > 0$ for the initial data.

2. Case 1 : $p = 2$ and the classical solutions

The local existence of the classical solutions to the equations (1) with the case $p = 2$ is well known (See Ball [1]). So, accepting the local existence, we focus ourselves on the discussion of the blow-up of the classical solutions to the equation (1) with $p = 2$.

The following lemma is going to be useful in the proof of Theorem 2.3.

**Lemma 2.1** ([4, 10]). There exist $\lambda_1 > 0$ and $\phi_1 \in H_0^1(\Omega)$ with $\phi_1 > 0$ in $\Omega$ such that

$$
\begin{align*}
-\Delta \phi_1 (x) &= \lambda_1 \phi_1 (x), \quad x \in \Omega, \\
\phi_1 (x) &= 0, \quad x \in \partial \Omega.
\end{align*}
$$

Moreover, $\lambda_1$ is given by

$$
\lambda_1 = \inf_{w \in H_0^1(\Omega)} \frac{\int_{\Omega} |\nabla w|^2 \, dx}{\int_{\Omega} |w|^2 \, dx} > 0.
$$

In the above, we recall that the number $\lambda_1$ is the first eigenvalue of $\Delta$ and $\phi_1$ is a corresponding eigenfunction.

Let us recall the condition $(C)$ : for some $\alpha, \beta, \gamma > 0$,

$$(C) \quad \alpha \int_0^u f(s) \, ds \leq uf(u) + \beta u^2 + \gamma, \quad u > 0,$$

where $0 < \beta \leq \frac{(\alpha - 2)\lambda_1}{2}$ and $\lambda_1$ is the first eigenvalue of the Laplacian $\Delta$ on $\Omega$.

**Remark 2.2.** We will discuss the condition $(C)$ in the section 3 comparing with the condition $(A)$ and $(B)$ introduced in the first section, together with the condition $J(0) > 0$ for the initial data.

Now, we state and prove our result for $p = 2$.

**Theorem 2.3.** Let a function $f$ satisfy the condition $(C)$. If the initial data $u_0 \in C^1(\overline{\Omega})$ with $u_0 = 0$ on $\partial \Omega$ satisfies

$$
-\frac{1}{2} \int_{\Omega} |\nabla u_0(x)|^2 \, dx + \int_{\Omega} \left[ \int_0^{u_0(x)} f(s) \, ds - \gamma \right] \, dx > 0, \tag{11}
$$

...
then the nonnegative classical solutions \( u \) to the equations \( \text{(1)} \) blows up at finite time \( T^* \), in the sense of

\[
\lim_{t \to T^*} \int_0^t \int_\Omega u^2(x,s) \, ds = +\infty,
\]

where \( \gamma \) is the constant in the condition \( \text{(C)} \).

**Proof.** We first define a functional \( J \) by

\[
J(t) := -\frac{1}{2} \int_\Omega |\nabla u(x,t)|^2 \, dx + \int_\Omega [F(u(x,t)) - \gamma] \, dx, \quad t \geq 0,
\]

where \( F(u) := \int_0^u f(s) \, ds \).

Then by \( \text{(11)} \),

\[
J(0) = -\frac{1}{2} \int_\Omega |\nabla u_0(x)|^2 \, dx + \int_\Omega [F(u_0(x)) - \gamma] \, dx > 0.
\]

and we can see that

\[
J(t) = J(0) + \int_0^t \frac{d}{dt} J(s) \, ds = J(0) + \int_0^t \int_\Omega u_t^2(x,s) \, dxds. \tag{12}
\]

Now, we introduce a new function

\[
I(t) = \int_0^t \int_\Omega u^2(x,s) \, dxds + M, \quad t \geq 0, \tag{13}
\]

where \( M > 0 \) is a constant to be determined later. Then it is easy to see that

\[
I'(t) = \int_\Omega u^2(x,t) \, dx
\]

\[
= \int_\Omega \int_0^t 2u(x,s) u_t(x,s) \, dsdx + \int_\Omega u_0^2(x) \, dx. \tag{14}
\]

Then we use integration by parts, the condition \( \text{(C)} \), Lemma \ref{2.1} and \ref{12}.
\[ I''(t) = \frac{d}{dt} \int_{\Omega} u^2(x, t) \, dx \]
\[ = \int_{\Omega} 2u(x, t) u_t(x, t) \, dx \]
\[ = \int_{\Omega} 2u(x, t) [\Delta u(x, t) + f(u(x, t))] \, dx \]
\[ = -2 \int_{\Omega} |\nabla u(x, t)|^2 \, dx + \int_{\Omega} 2u(x, t) f(u(x, t)) \, dx \]
\[ \geq -2 \int_{\Omega} |\nabla u(x, t)|^2 \, dx + \int_{\Omega} 2[\alpha F(u(x, t)) - \beta u^2(x, t) - \alpha \gamma] \, dx \quad (15) \]
\[ = 2\alpha \left[ -\frac{1}{2} \int_{\Omega} |\nabla u(x, t)|^2 \, dx + \int_{\Omega} [F(u(x, t)) - \gamma] \, dx \right] \]
\[ + (\alpha - 2) \int_{\Omega} |\nabla u(x, t)|^2 \, dx - 2\beta \int_{\Omega} u^2(x, t) \, dx \]
\[ \geq 2\alpha J(t) + [(\alpha - 2)\lambda_1 - 2\beta] \int_{\Omega} u^2(x, t) \, dx \]
\[ \geq 2\alpha \left[ J(0) + \int_0^t \int_{\Omega} u_t^2(x, s) \, dx \, ds \right]. \]

Using the Schwarz inequality, we obtain

\[ I'(t)^2 \]
\[ \leq 4 (1 + \delta) \left[ \int_{\Omega} \int_0^t u(x, s) u_t(x, s) \, ds \, dx \right]^2 \]
\[ + \left( 1 + \frac{1}{\delta} \right) \left[ \int_{\Omega} u_0^2(x) \, dx \right]^2 \]
\[ \leq 4 (1 + \delta) \left[ \int_{\Omega} \left( \int_0^t u^2(x, s) \, ds \right)^{\frac{1}{2}} \left( \int_0^t u_t^2(x, s) \, ds \right)^{\frac{1}{2}} \, dx \right]^2 \]
\[ + \left( 1 + \frac{1}{\delta} \right) \left[ \int_{\Omega} u_0^2(x) \, dx \right]^2 \]
\[ \leq 4 (1 + \delta) \left( \int_{\Omega} \int_0^t u^2(x, s) \, ds \, dx \right) \left( \int_{\Omega} \int_0^t u_t^2(x, s) \, ds \, dx \right) \]
\[ + \left( 1 + \frac{1}{\delta} \right) \left[ \int_{\Omega} u_0^2(x) \, dx \right]^2, \quad (16) \]

where \( \delta > 0 \) is arbitrary. Combining the above estimates (13), (15), and (16).
we obtain that for $\sigma = \delta = \sqrt{\frac{\alpha}{2}} - 1 > 0$,

\[
I''(t) I(t) - (1 + \sigma) I'(t)^2 \\
\geq 2\alpha \left[ J(0) + \int_0^t \int u_t^2 (x, s) dx ds \right] \left[ \int_0^t \int u^2 (x, s) dx ds + M \right] \\
- 4 (1 + \sigma) (1 + \delta) \left[ \int_{\Omega} \int_0^t u^2 (x, s) ds dx \right] \left[ \int_0^t \int_0^s u_t^2 (x, s) ds dx \right] \\
- (1 + \sigma) \left( 1 + \frac{1}{\delta} \right) \left[ \int_{\Omega} u_0^2 (x) dx \right]^2 \\
\geq 2\alpha M \cdot J(0) - (1 + \sigma) \left( 1 + \frac{1}{\delta} \right) \left[ \int_{\Omega} u_0^2 (x) dx \right]^2.
\]

Since $J(0) > 0$ by the assumption, we can choose $M > 0$ to be large enough so that

\[
I''(t) I(t) - (1 + \sigma) I'(t)^2 > 0. \tag{17}
\]

This inequality (17) implies that for $t \geq 0$,

\[
\frac{d}{dt} \left[ \frac{I'(t)}{I^{\sigma+1}(t)} \right] > 0 \text{ i.e. } I'(t) \geq \left[ \frac{I'(0)}{I^{\sigma+1}(0)} \right] I^{\sigma+1}(t).
\]

Therefore, it follows that $I(t)$ cannot remain finite for all $t > 0$. In other words, the solutions $u$ blow up in finite time $T^*$.

Remark 2.4. We estimate the blow-up time of the solutions to equation (1) roughly. Put

\[
M := \frac{\alpha - \frac{\alpha}{2} \left( 1 + \sqrt{\frac{\alpha}{2}} \right) \left[ \int_\Omega u_0^2 (x) dx \right]^2}{2\alpha \left[ \frac{1}{\delta} \int_\Omega |\nabla u_0 (x)|^2 dx + \int_\Omega [F(u_0 (x)) - \gamma] dx \right]}
\]

Then we obtain that

\[
\begin{cases}
I'(t) \geq \left[ \frac{K u_t^2 (x) dx}{M^{\sigma+1}} \right] I^{\sigma+1}(t), & t > 0, \\
I(0) = M,
\end{cases}
\]

which implies

\[
I(t) \geq \left[ \frac{1}{M^\sigma} - \frac{\sigma \int_\Omega u_0^2 (x) dx}{M^{\sigma+1}} \right]^{-\frac{1}{\sigma}}
\]

where $\sigma = \sqrt{\frac{\alpha}{2}} - 1 > 0$. Then the blow-up time $T^*$ satisfies

\[
0 < T^* \leq \frac{M}{\sigma \int_\Omega u_0^2 (x) dx}.
\]
3. Case 2: $p > 2$ and the weak solutions

In this section, we discuss the blow-up of solutions to the equations (1) for the case $p > 2$, which is the main part of our work. In order to make this section self-contained we state, without proof, a local existence result of [14].

**Theorem 3.1** (See [14]). Let $f$ be in $C(\mathbb{R})$ and there exists a function $g(u) \in C^1(\mathbb{R})$ such that

$$|f(u)| \leq g(u).$$

Then for any $u_0 \in L^\infty(\Omega) \cap W^{1,p}_0(\Omega)$, there exists $T > 0$ such that (1) has a solution $u \in L^\infty(\Omega \times (0,T)) \cap L^p(0,T; W^{1,p}_0(\Omega))$, $u_t \in L^2(\Omega \times (0,T))$.

The following lemmas are used when proving Theorem 3.4.

**Lemma 3.2** ([4, 10]). For $1 < p < \infty$, there exist $\lambda_{1,p} > 0$ and $\phi_{1,p} \in W^{1,p}_0(\Omega)$ with $\phi_{1,p} > 0$ in $\Omega$ such that

$$\begin{cases}
-\Delta_p \phi_{1,p}(x) = \lambda_{1,p}|\phi_{1,p}(x)|^{p-2}\phi_{1,p}(x), & x \in \Omega \\
\phi_{1,p}(x) = 0, & x \in \partial\Omega.
\end{cases}$$

Moreover, $\lambda_{1,p}$ is given by

$$\lambda_{1,p} = \inf_{w \in W^{1,p}_0(\Omega)} \frac{\int_\Omega |\nabla w|^p dx}{\int_\Omega |w|^p dx} > 0.$$ 

In the above, we recall that the number $\lambda_{1,p}$ is the first eigenvalue of the $p$-Laplacian $\Delta_p$ and $\phi_{1,p}$ is a corresponding eigenfunction.

Let us recall that for some $\alpha, \beta, \gamma > 0$,

$$(C_p) \quad \alpha \int_0^u f(s) ds \leq uf(u) + \beta u^p + \gamma, \quad u > 0,$$

where $0 < \beta \leq \frac{(\alpha - p)\lambda_{1,p}}{p}$ and $\lambda_{1,p}$ is the first eigenvalue of the $p$-Laplacian $\Delta_p$ on $\Omega$.

**Remark 3.3.** We will discuss the condition $(C_p)$ in the next section, comparing with the conditions $(A_p)$ and $(B_p)$ introduced in the first section, together with the initial data condition.

Now, we state and prove our main result.

**Theorem 3.4.** Let a function $f$ satisfy the condition $(C_p)$ and $p > 2$. If the initial data $u_0 \in L^\infty(\Omega) \cap W^{1,p}_0(\Omega)$ satisfies

$$-\frac{1}{p} \int_\Omega |\nabla u_0(x)|^p dx + \int_\Omega |F(u_0(x)) - \gamma| dx > 0,$$ (18)
then the nonnegative weak solutions $u$ to the equations \[ (1) \] blows up at finite time $T^*$, in the sense of

$$
\lim_{t \to T^*} \int_0^t \int_\Omega u^2(x, s) \, ds = +\infty,
$$

where $\gamma$ is the constant in the condition $(C_p)$.

The following lemma is essential in the proof of the above theorem.

**Lemma 3.5** (14). Let $u$ be the weak solutions to the equations \[ (1) \] with $|\nabla u_0| \in L^p(\Omega)$. Then

(i)

$$
\int_0^t \int_\Omega \frac{1}{2} (u^2(x, s))_t \, dx \, ds = \frac{1}{2} \int_\Omega [u^2(x, t) - u_0^2(x)] \, dx
= \int_0^t \int_\Omega [\frac{-|\nabla u(x, t)|^p}{t} + u(x, t) f(u(x, t))] \, dx \, ds
$$

(ii)

$$
\int_0^t \int_\Omega u_t^2(x, s) \, dx \, ds \leq -\frac{1}{p} \int_\Omega [|\nabla u(x, t)|^p - |\nabla u_0(x)|^p] \, dx
+ \int_\Omega [F(u(x, t)) - F(u_0(x))] \, dx,
$$

where $F(u) := \int_0^u f(s) \, ds$.

**Proof of Theorem 3.4.** We define a function $J_p$ by

$$
J_p(t) := -\frac{1}{p} \int_\Omega |\nabla u(x, t)|^p \, dx + \int_\Omega [F(u(x, t)) - \gamma] \, dx. \quad (19)
$$

Then it follows from \[ (18) \] and Lemma 3.5(ii) that,

$$
J_p(0) = -\frac{1}{p} \int_\Omega |\nabla u_0(x)|^p \, dx + \int_\Omega [F(u_0(x)) - \gamma] \, dx > 0
$$

and

$$
J_p(t) = -\frac{1}{p} \int_\Omega |\nabla u(x, t)|^p \, dx + \int_\Omega [F(u(x, t)) - \gamma] \, dx
\geq -\frac{1}{p} \int_\Omega |\nabla u_0(x)|^p \, dx + \int_\Omega [F(u_0(x)) - \gamma] \, dx + \int_0^t \int_\Omega u_t^2(x, s) \, dx \, ds,
= J_p(0) + \int_0^t \int_\Omega u_t^2(x, s) \, dx \, ds,
$$

(20)
On the other hand, we define a function $I_p$ by

$$I_p(t) := \int_0^t \int_{\Omega} u^2(x,s) \, dx \, ds + M, \ t \geq 0,$$  \hspace{1cm} (21)

where $M > 0$ is a constant to be determined later. Then it is easy to see that

$$I'_p(t) = \int_{\Omega} u^2(x,t) \, dx = \int_0^t \int_{\Omega} 2u(x,s) u_t(x,s) \, ds \, dx + \int_{\Omega} u_0^2(x) \, dx.$$ \hspace{1cm} (22)

Then by Lemma 3.5 (i), we can see that

$$I''_p(t) = \int_{\Omega} (u^2(x,t))_t \, dx = 2 \int_{\Omega} [-|\nabla u(x,t)|^p + u(x,t)f(u(x,t))] \, dx.$$ \hspace{1cm} (23)

By using the condition $(C_p)$, Lemma 3.2, and (20) in turn, we obtain that

$$I''_p(t) \geq -2 \int_{\Omega} |\nabla u(x,t)|^p \, dx + 2 \int_{\Omega} [\alpha F(u(x,t)) - \beta u^p(x,t) - \alpha \gamma] \, dx$$

$$= 2 \alpha J_p(t) + \frac{2(\alpha - p)}{p} \int_{\Omega} |\nabla u(x,t)|^p \, dx - 2 \beta \int_{\Omega} u^p(x,t) \, dx$$

$$\geq 2 \alpha J_p(t) + 2 \left[ \frac{(\alpha - p)\lambda_{1,p}}{p} - \beta \right] \int_{\Omega} u(x,t)^p \, dx$$

$$\geq 2 \alpha \left[ J_p(0) + \int_0^t \int_{\Omega} u_0^2(x,s) \, dx \, ds \right].$$ \hspace{1cm} (24)

Applying the Schwarz inequality, as done in Theorem 2.3, we obtain that

$$I_p''(t)^2 \leq 4(1 + \delta) \left( \int_{\Omega} \int_0^t u^2(x,s) \, dx \, ds \right) \left( \int_{\Omega} \int_0^t u_0^2(x,s) \, dx \, ds \right)$$

$$+ \left( 1 + \frac{1}{\delta} \right) \left[ \int_{\Omega} u_0^2(x) \, dx \right]^2,$$ \hspace{1cm} (25)

where $\delta > 0$ is arbitrary. Combining the above estimates (21), (24), and (25), we obtain that

$$I_p''(t)I_p(t) - (1 + \sigma)I'_p(t)^2 > 0,$$ \hspace{1cm} (26)

by choosing $\sigma = \delta = \sqrt{\frac{n}{1 - 2}} > 0$ and $M$ to be large enough. This means that the solutions $u$ blow up in finite time $T^*$. \hfill \Box
Remark 3.6. (i) In a same way as in Remark 2.4, the blow-up time $T^\ast$ can be estimated as follows

$$0 < T^\ast \leq \frac{\alpha \sigma}{\tilde{\alpha} - 2} \left(1 + \sqrt{\frac{\alpha}{\tilde{\alpha}}} \right) \left[ \int_{\Omega} u_0^2(x) \, dx \right]^2.$$ 

(ii) In fact, the above theorem and its proof can still works for the case $p = 2$.

4. Discussion on the Condition $(C_p)$ with the initial data conditions

In this section, we compare the conditions $(A_p)$ and $(C_p)$ each other and discuss the role of the condition $J_p(0) > 0$ for the initial data $u_0$.

As seen in the proof of Theorem 3.4, the concavity method is a tool for deriving the blow-up solution via the auxiliary function $J_p(t)$ under the condition $(A_p)$ or $(C_p)$, by imposing $J_p(0) > 0$, instead of the large initial data.

On the other hand, instead of the condition $(B_p)$ in Section 1, it is not difficult to consider $(B_p)$, in a similar form as in $(A_p)$ or $(C_p)$. In fact, to be strange, the condition $(B_p)$ is not seen in any literature, as far as the authors know.

Then for $p \geq 2$, let us recall the conditions as follows:

- $(A_p)$: $(p + \epsilon)F(u) \leq uf(u),$
- $(B_p)$: $(p + \epsilon)F(u) \leq uf(u) + \gamma,$
- $(C_p)$: $(p + \epsilon)F(u) \leq uf(u) + \beta u^p + \gamma,$

for every $u > 0$, where $0 < \beta \leq \frac{\epsilon \lambda_1}{\tilde{\alpha} - p}$ and $F(u) := \int_0^u f(s) \, ds$. Here, note that the constants $\epsilon, \beta$, and $\gamma > 0$ may be different in each case.

Then it is easy to see that $(A_p)$ implies $(B_p)$ and $(B_p)$ implies $(C_p)$, in turn. The difference between $(B_p)$ and $(C_p)$ is whether or not they depend on the domain. The condition $(B_p)$ is independent of the first eigenvalue $\lambda_{1,p}$ which depends on the domain $\Omega$. However, the condition $(C_p)$ depends on domain, due to the term $\beta u^p$ with $0 < \beta \leq \frac{\epsilon \lambda_1}{\tilde{\alpha} - p}$. From this point of view, the condition $(C_p)$ can be understood as a refinement of $(B_p)$, corresponding to the domain. On the contrary, if a function $f$ satisfies $(C_p)$ for every bounded domain $\Omega$ with smooth boundary $\partial \Omega$, then the first eigenvalue $\lambda_{1,p}$ can be arbitrary small so that the condition $(C_p)$ get closer to $(B_p)$ arbitrarily. Besides, as far as the authors know, there has not been any noteworthy condition for the concavity method other than $(A_p)$ or $(B_p)$.

On the other hand, using the fact that $(C_p)$ is equivalent to

$$\frac{d}{du} \left( \frac{F(u)}{u^{p+\epsilon}} - \frac{\gamma}{p+\epsilon} \cdot \frac{1}{u^{p+\epsilon}} - \frac{\beta}{\epsilon} \cdot \frac{1}{u^p} \right) \geq 0, \quad u > 0,$$
we can easily see that for every \( u > 0 \),

\[ (A_p) \text{ holds if and only if } F(u) = u^{p+\epsilon}h_1(u), \]

\[ (B_p) \text{ holds if and only if } F(u) = u^{p+\epsilon}h_2(u) + b, \]

\[ (C_p) \text{ holds if and only if } F(u) = u^{p+\epsilon}h_3(u) + au^p + b, \]

(27)

for some constants \( \epsilon > 0, a > 0, \) and \( b > 0 \) with \( 0 < a \leq \frac{\lambda_1p}{p} \), where \( h_1, h_2, \) and \( h_3 \) are nondecreasing functions on \( (0, +\infty) \). Here also, the constants \( \epsilon, a, \) and \( b \) may be different in each case. We note here that the nondecreasing functions \( h_1 \) is nonnegative on \((0, +\infty)\), but \( h_2 \) and \( h_3 \) may not be nonnegative, in general.

**Lemma 4.1.** Let \( f \) be a function satisfying \((C_p)\) and \( f(u) \geq \lambda u^{p-1}, u > 0, \) where \( \lambda > \lambda_{1,p} \). Then the condition \((C_p)\) implies that there exists \( m > 0 \) such that \( h_3(u) > 0 \) for \( u > m \). In this case, we can find \( \mu > 0 \) such that \( f(u) \geq \mu u^{p-1+\epsilon}, u \geq m \). Moreover, the conditions \((B_p)\) and \((C_p)\) are equivalent.

**Proof.** First, it follows from (27) and the fact \( \lambda > \lambda_{1,p} \) that \( F(u) \geq \frac{\lambda_1p}{p}u^p \geq \frac{\lambda_1p}{p}u^p \) and so that

\[ u^{p+\epsilon}h_3(u) = F(u) - au^p - b \geq \frac{\lambda - \lambda_1p}{p}u^p - b, \]

which goes to \(+\infty\), as \( u \to +\infty \). So, we can find \( m > 1 \) such that \( h_3(m) > 0 \), which implies that

\[ F(u) \geq u^{p+\epsilon}h_3(u), u \geq m. \]

Putting it into the condition \((C_p)\), we obtain

\[ u^{p+\epsilon}h_3(m) \leq uf(u) + \beta u^p + \gamma \]

or

\[ u^{p-1+\epsilon}h_3(m) \leq f(u) + \beta u + \gamma \leq (1 + \frac{\epsilon}{p})f(u) + \gamma, u \geq m > 1, \]

which gives

\[ f(u) \geq \mu u^{p-1+\epsilon}, u \geq m > 1 \]

for some \( \mu > 0 \) and another constant \( m \).

Now, assume that the condition \((C_p)\) is true. Since \( 0 < \beta \leq \frac{\epsilon\lambda_1p}{p} \) and \( f(u) \geq \lambda u^{p-1} > \lambda_{1,p}u^{p-1}, u > 0, \) it follows from \((C_p)\) that

\[ \epsilon_1 F(u) + (p + \epsilon_2) F(u) \leq uf(u) + \frac{\epsilon_1\lambda_1p}{p}u^p + \gamma, \]

where \( \epsilon_1 = \frac{\epsilon\lambda_1p}{\lambda} > 0 \) and \( \epsilon_2 = \epsilon - \epsilon_1 > 0 \). This implies that for every \( u > 0 \),

\[ uf(u) + \gamma \geq (p + \epsilon_2) F(u) + \epsilon_1 \int_0^u [f(s) - \lambda s^{p-1}] ds \]

\[ \geq (p + \epsilon_2) F(u), \]

which gives \((B_p)\).
Remark 4.2. In general, the constant $\alpha$ with $\alpha > p$ in $(C_p)$ cannot be replaced by $p$. But, assume $p > 2$ and $f$ satisfies a condition,

$$(C_p)' \quad pF(u) \leq uf(u) + \gamma, \ u > 0,$$

which comes from $(C_p)$ by replacing $\alpha$ by $p$ and taking $\beta = 0$. Then the inequalities (24) and (26) in the proof of Theorem 3.4 can be derived in an easy way as follows:

$$I_p''(t) \geq -2 \int_\Omega |\nabla u(x,t)|^p \, dx + 2 \int_\Omega [pF(u(x,t)) - p\gamma] \, dx$$
$$= 2pJ_p(t)$$

and

$$I_p''(t)I_p(t) - (1 + \sigma)I_p'(t)^2 > 0.$$

Therefore, we can prove that the weak solutions to the equations (11) for $p > 2$ blows up in a finite time, under the conditions $(C_p)'$ and $J_p(0) > 0$, which can be understood as an improvement of the result by Zhao [14].

Remark 4.3. It is well known that if $\int_0^+ \frac{du}{f(s)} = +\infty$ for some $m > 0$, the solutions to equations (11) is global. On the contrary, it has not been clear yet whether or not the condition $\int_0^+ \frac{du}{f(s)} < +\infty$ guarantees the blow-up solution. Instead, when $f(u) \geq \mu u^{(p-1)+\epsilon}, \ u \geq m$ for some $\epsilon > 0$ and $m > 0$, the solutions to the equations (11) blow up in a finite time, only if the initial data $u_0$ is sufficiently large (for more details, see [8]).

In general, the condition $(C_p)$ may not guarantee the blow-up solutions for any initial data $u_0$. In fact, we can easily see that a function $f(u) = au^{p-1}$ $(p > 2)$ satisfies $(C_p)$ if and only if $a \leq \lambda_{1,p}$. However, for any function $u_0$,

$$J(0) = -\frac{1}{p} \int_\Omega |\nabla u_0(x)|^p \, dx + \int_\Omega \left[ \frac{a}{p} u_0^p(x) - \gamma \right] \, dx$$
$$\leq \frac{a - \lambda_{1,p}}{p} \int_\Omega |u_0(x)|^p \, dx - \gamma|\Omega| < 0,$$

which means that there is no initial data $u_0$ satisfying $J(0) > 0$, when $f(u) = au^{p-1}, \ a \leq \lambda_{1,p}$. Of course, it is well known that the solutions to the equations (11) is global, in this case (see [8]).

So, we are here going to discuss when we can find initial data $u_0$ satisfies $J(0) > 0$. Consider a domain $\Omega$ with $\lambda_{1,p} > \frac{p}{p-1}$ and a nonnegative continuous function $f$ satisfying the condition $(C_p)$ with $\gamma = 1$ for simplicity and $f(s) \geq p\lambda_{1,p}s^{p-1}, \ s > 0$. Now, let us take $u_0(x) := \phi_{1,p}(x)$ where $\phi_{1,p}$ is an eigenfunction in Lemma 3.2 with $\int_\Omega [\phi_{1,p}(x)]^p \, dx = |\Omega|$. Then it follows that
\[ J(0) = -\frac{1}{p} \int_{\Omega} |\nabla \phi_{1,p}(x)|^p \, dx + \int_{0}^{\phi_{1,p}(x)} f(s) \, ds - |\Omega| \]
\[ \geq -\frac{\lambda_{1,p}}{p} \int_{\Omega} [\phi_{1,p}(x)]^p \, dx + \int_{0}^{\phi_{1,p}(x)} p\lambda_{1,p}s^{p-1} \, ds - |\Omega| \]
\[ = \lambda_{1,p} \left( 1 - \frac{1}{p} \right) \int_{\Omega} [\phi_{1,p}(x)]^p \, dx - |\Omega| \]
\[ = \left[ \lambda_{1,p} \left( 1 - \frac{1}{p} \right) - 1 \right] |\Omega| > 0. \]

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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