Further boundary conditions in heat conduction problems in multilayer structures

K V Trubitsyn¹, G V Mikheeva¹, R M Klebleev¹, O Y Kurganova¹

¹Samara State Technical University, Molodogvardeiskaya street 244, Samara, Russia, 443100

Abstract. This paper presents an approximate analytical solution of the heat conduction problem for a two-layer plate under symmetric boundary conditions of the first kind. The solution was determined on the basis of the property of the parabolic heat transfer equation associated with an infinite velocity of heat propagation, by determining the accessory unknown function and accessory boundary conditions in the integral heat balance method. Local coordinate systems are given in order to obtain the simplest possible coordinate system satisfying the matching conditions and boundary conditions for each separate layer. An accessory unknown function is the temperature change over time in the center of symmetry. The use of this function in the heat balance integral method allows for reduction in the solution of the initial partial differential equation to the integration of an ordinary differential equation with respect to the additional unknown function. Further boundary conditions are defined in such a way that their satisfaction by an unknown solution is equivalent to the satisfaction of the equations at the boundary points. Studies have shown that the equation fulfillment at the boundaries leads to their fulfillment within the regions.

1. Introduction
Complications in solving heat conduction problems for multilayer bodies are to be satisfied by an unknown solution the matching conditions specified in the form of equal temperatures and heat flows at the contact of the layers. The use of classical analytical methods in this case leads to the solution of transcendental equation systems with respect to the eigenvalues of the boundary value problem, the solutions of which can be found only by numerical methods [1 – 3]. In [4 – 7], a multilayer construction is reduced to a single layer construction with numerous physical properties of the medium on a basis of the generalized functions theory. Obtaining a solution to the problem in this case is significantly simplified due to the absence of the need to satisfy the matching conditions that are included in the differential equation of the boundary value problem and are fulfilled in the process of obtaining its solution. However, these solutions are expressed by complex functional series, which are of little use for engineering purposes.

In [2, 3], as applied to heat conduction problems for multilayer bodies, a method based on the combined use of exact (Fourier, Laplace integral transforms, etc.) and approximate (Kantorovich, Bubnov-Galerkin, etc.) analytical methods is considered. This method allows to solve the systems of ordinary differential equations using coordinate functions that exactly satisfy the matching and boundary conditions. Obtaining a solution is significantly more complicated with a large amount of
approximations, and, moreover, these solutions are overloaded by huge mathematical expressions, and therefore, we have to limit the number of approximations that may result in low precision.

The main motivation for obtaining an approximate analytical solution is that the exact solutions for such problems have not yet been found. Known approximate analytical solutions are characterized by the complexity of their determination, uncomfortable analytical expressions, which complicates their employment in engineering practice and especially their application to solving inverse heat conduction problems. The solutions obtained in this work are written with maximum simplicity and precision, which is quite sufficient for engineering purposes. If necessary, the accuracy can be improved by increasing the number of approximations.

2. Mathematical statement of the problem
To simplify the process of obtaining solutions of heat conduction problem for multilayer structures and final expressions, the heat balance integral method with the determination of the accessory unknown function and accessory boundary conditions using local coordinates was applied. Let us consider the application of the method on the example of obtaining a solution to the following boundary value problem for a two-layer plate

\[
\begin{align*}
\frac{\partial T_1(x_1,t)}{\partial t} &= a_1 \frac{\partial^2 T_1(x_1,t)}{\partial x_1^2}; \\
\frac{\partial T_2(x_2,t)}{\partial t} &= a_2 \frac{\partial^2 T_2(x_2,t)}{\partial x_2^2};
\end{align*}
\]

\[
(t > 0; \quad 0 < x_i < \delta_i);
\]

\[
T_i(x_i,0) = T_{i0};
\]

\[
T_i(0,t) = T_{w};
\]

\[
T_i(\delta_i,t) = T_{r};
\]

\[
\lambda_i \frac{\partial T_i(\delta_i,t)}{\partial x_1} = \lambda_i \frac{\partial T_i(0,t)}{\partial x_2};
\]

\[
\frac{\partial T_2(\delta_2,t)}{\partial x_2} = 0,
\]

where \(T_i, x_i, (i = 1, 2)\) are temperature and coordinates of the first and second layers; \(t\) – time; \(\delta_i, a_i, \lambda_i, (i = 1, 2)\) – thicknesses, thermal diffusivity coefficients and thermal conductivity coefficients of the layers, respectively; \(T_w\) – initial temperature; \(T_r\) – wall temperature at \(x_1 = \delta_2\); \(\delta = \delta_1 + \delta_2\) – total thickness of the layers.

These are the following dimensionless variables and parameters:

\[
\Theta_i = \frac{T_r - T_{w}}{T_0 - T_{w}}, \quad (i = 1, 2);
\]
where \( \Theta_1, (i = 1, 2) \) is dimensionless temperature; \( \text{Fo} \) – Fourier number (dimensionless time); \( \xi_1, \xi_2 \) – dimensionless coordinates of the first and second layer; \( \Delta_1, \Delta_2 \) – dimensionless layer thicknesses.

Taking into account the notations (9), the problem (1) – (8) will be

\[
\frac{\partial \Theta_1(\xi_1, \text{Fo})}{\partial \text{Fo}} = \frac{\partial^2 \Theta_1(\xi_1, \text{Fo})}{\partial \xi_1^2}, \quad (\text{Fo} > 0; \ 0 < \xi_1 < \Delta_1); \tag{10}
\]

\[
\frac{\partial \Theta_2(\xi_2, \text{Fo})}{\partial \text{Fo}} = \frac{\partial^2 \Theta_2(\xi_2, \text{Fo})}{\partial \xi_2^2}, \quad (\text{Fo} > 0; \ 0 < \xi_2 < \Delta_2); \tag{11}
\]

Taking of the notations (9), the problem (1) – (8) will be

\[
\Theta_1(\xi_1, 0) = 1; \tag{12}
\]

\[
\Theta_2(\xi_2, 0) = 1; \tag{13}
\]

\[
\Theta_1(0, \text{Fo}) = 0; \tag{14}
\]

\[
\Theta_1(\Delta_1, \text{Fo}) = \Theta_2(0, \text{Fo}); \tag{15}
\]

\[
\lambda \frac{\partial \Theta_1(\Delta_1, \text{Fo})}{\partial \xi_1} = \lambda \frac{\partial \Theta_2(0, \text{Fo})}{\partial \xi_2}; \tag{16}
\]

\[
\frac{\partial \Theta_1(\Delta_1, \text{Fo})}{\partial \xi_1} + \frac{\partial \Theta_2(\Delta_2, \text{Fo})}{\partial \xi_2} = 0. \tag{17}
\]

3. Obtaining an approximate analytical solution

Accessory unknown function \([8 – 10]\)

\[
q(\text{Fo}) = \Theta_2(\Delta_2, \text{Fo}), \tag{18}
\]

that depicts the temperature change over time at a point \( \xi_2 = \Delta_2 \). Since the temperature at this point is an unknown value of problem (10) – (17), its separate consideration does not change this problem and is only an additional way to simplify the process of obtaining a solution. Due to the infinite velocity of the heat distribution described by the solutions of parabolic equations (10), (11), the temperature at the point \( \xi_2 = \Delta_2 \) will change immediately after the application of condition (14). Therefore, the function range will include the entire time range of the non-stationary process \((0 < \text{Fo} < \infty)\).

The solutions of problem (10) – (17) for each layer are as follows

\[
\Theta_1(\xi_1, \text{Fo}) = \sum_{n=1}^{s_1} b_1(q) \varphi_{1n}(\xi_1); \tag{19}
\]

\[
\Theta_2(\xi_2, \text{Fo}) = \sum_{n=1}^{s_2} b_2(q) \varphi_{2n}(\xi_2), \tag{20}
\]

where \( b_i(q) \) – unknown coefficients; \( \varphi_{1n}(\xi_1), \varphi_{2n}(\xi_2) \) – coordinate functions for the first and second layers, which are in such a form that the unknown solutions (19), (20) exactly satisfy the boundary and matching conditions in advance in any approximation.

It is clear that due to the adoption of the same system of unknown coefficients \( b_i(q) \) for each layer, the two-layer system is reduced to a single-layer system with variable physical properties of the medium, which are taken into account by different coordinate functions \( \varphi_{1n}(\xi_1) \) and \( \varphi_{2n}(\xi_2) \) for each of the layers.

Consider the construction of coordinate functions that exactly satisfy the boundary conditions (14), (17) and the matching conditions (15), (16). Formulas for them are as follows

\[
\varphi_{1n}(\xi_1) = \xi_1 - \xi_1^{2n}; \tag{21}
\]

\[
\varphi_{2n}(\xi_2) = B_{1n} \xi_2 - B_{2n} \xi_2^{2n}, \tag{22}
\]

where \( B_{1n}, B_{2n}, B_{3n} \) – unknown coefficients.

It is clear that the solution (19) with the coordinate function (21) exactly satisfies the boundary condition (14). Therefore, to determine the unknown coefficients \( B_{1n}, B_{2n}, B_{3n} \), conditions (15) –
(17) are used. Substituting (21), (22) into these conditions, we obtain

\[ B_{1k} = \Delta_1 - \Delta_1^{2k}, \]

\[ B_{2k} = N \left(1 - 2k \Delta_1^{2k-1}\right); \]

\[ B_{3k} = B_{2k} (2k \Delta_1^{2k-1}); \]

\[ N = \lambda_1 / \lambda_2. \]

Taking into account the found values of the coefficients \( B_{1k}, B_{2k}, B_{3k} \), formula (22) is as followed:

\[ \varphi_{2k}(\xi) = \Delta_1 - \Delta_1^{2k} + N \left(1 - 2k \Delta_1^{2k-1}\right) \left(\xi_2 - \frac{\xi_2^{2k}}{2k \Delta_1^{2k-1}}\right). \] (23)

Taking into account formulas (21), (23), the relations (19), (20) exactly satisfy the boundary conditions (14), (17) and the matching conditions (15), (16). Unknown coefficients \( b_k(q) \) are found from condition (18) and some additional boundary conditions. To determine the first of them, we shall differentiate condition (14) with respect to the variable \( F_0 \)

\[ \frac{\partial \Theta_1(0, F_0)}{\partial F_0} = 0. \] (24)

Relation (24), taking into account equation (10), is reduced to an accessory boundary condition

\[ \frac{\partial^2 \Theta_1(0, F_0)}{\partial \xi_1^2} = 0. \] (25)

Differentiate (25) with respect to the variable \( F_0 \)

\[ \frac{\partial^2}{\partial \xi_1^2} \left(\frac{\partial \Theta_1(0, F_0)}{\partial \xi_1} \right) = 0. \] (26)

Relation (26), taking into account equation (10), is reduced to the following accessory boundary condition

\[ \frac{\partial \Theta_1(0, F_0)}{\partial \xi_1^2} = 0. \] (27)

By differentiating each previous additional condition with respect to the variable \( F_0 \), taking into account equation (10), we can obtain any number of additional boundary conditions. The general formula for them will be

\[ \frac{\partial^2 \Theta_1(0, F_0)}{\partial \xi_1^2} = 0, \quad (i = 1, 2, 3, \ldots). \] (28)

At the point \( \xi_1 = 0 \), condition (18) is also given. Therefore, accessory boundary conditions corresponding to this relation should be found. Differentiating (18) with respect to the variable \( F_0 \), we obtain

\[ \frac{\partial q(F_0)}{\partial F_0} = \frac{\partial \Theta_1(\Delta_1, F_0)}{\partial F_0}. \] (29)

Relation (29), taking into account equation (11), is reduced to an additional condition

\[ \frac{a_2}{a_1} \frac{\partial^3 \Theta_1(\Delta_1, F_0)}{\partial \xi_1^3} = d q(F_0) / d F_0. \] (30)

Differentiating (30) with respect to the variable \( F_0 \) and comparing with equation (10), we find one more additional condition

\[ \left(\frac{a_2}{a_1}\right)^{i} \frac{\partial^i \Theta_1(\Delta_1, F_0)}{\partial \xi_1^i} = d^i q(F_0) / d F_0^i, \quad (i = 1, 2, 3, \ldots). \] (31)

The general formula for them, including the function \( q(F_0) \), is as followed

\[ \left(\frac{a_2}{a_1}\right)^{i} \frac{\partial^i \Theta_1(\Delta_1, F_0)}{\partial \xi_1^i} = d^i q(F_0) / d F_0^i, \quad (i = 1, 2, 3, \ldots). \] (32)

In addition to the above additional boundary conditions obtained, which should be fulfilled at points \( \xi_1 = 0 \) and \( \xi_2 = \Delta_2 \), it is also necessary to use additional boundary conditions at the point \( \xi_2 = \Delta_2 \), found on the basis of the boundary condition (17). To find them, we differentiate (17) with respect to \( F_0 \)
\[
\frac{\partial}{\partial \xi_2} \left( \frac{\partial \Theta_2(\Delta_2, F_0)}{\partial F_0} \right) = 0 .
\]  

(33)

Using equation (11), we determine an accessory boundary condition

\[
\frac{\partial^2 \Theta_2(\Delta_2, F_0)}{\partial \xi_2^2} = 0 .
\]  

(34)

Differentiating (34) with respect to the variable \( F_0 \) taking into account equation (11), we find one more additional boundary condition at the point \( \xi_2 = \Delta_2 \)

\[
\frac{\partial^3 \Theta_2(\Delta_2, F_0)}{\partial \xi_2^3} = 0 .
\]  

(35)

The general formula for them is as follows

\[
\frac{\partial^{2i+1} \Theta_i(\Delta_i, F_0)}{\partial \xi_2^{2i+1}} = 0 , \quad (i = 1, 2, 3, \ldots) .
\]  

(36)

Note that relation (20) when using coordinate functions of the form (22) in any approximation exactly satisfies the additional boundary conditions (36).

To determine the unknown coefficients \( b_i(q) \) of relations (19), (20) in the first approximation, we use condition (18) and additional boundary conditions obtained by the general formulas (28) and (32). Substituting (19), (20), (limited to three terms of the series) into (18), (28), (32) (at \( \eta = \lambda_2(28 \lambda_2 \Delta_1^4 + 120 \Delta_2 \lambda_1 \Delta_1^3 - 108 \Delta_2 \lambda_1 \Delta_1^2 - 30 \Delta_2 \lambda_1 \Delta_1^1 + 8 \lambda_2 \Delta_1 - 5 \Delta_2 \lambda_1 + 4 \lambda_2 \Delta_1^0) \))

\[
b_i(q) = \frac{b_i(q) - 2}{3} (q' \eta_1 + q \eta_2) / \eta ; \quad b_2(q) = (q' \eta_2 + q \eta_3) / \eta ,
\]  

(38)

where

\[
\eta = \lambda_2(28 \lambda_2 \Delta_1^4 + 72 \lambda_2 \Delta_1^3 + 120 \Delta_2 \lambda_1 \Delta_1^2 - 108 \Delta_2 \lambda_1 \Delta_1^1 - 30 \Delta_2 \lambda_1 \Delta_1^0 - 8 \lambda_2 \Delta_1 + 5 \Delta_2 \lambda_1 + 4 \lambda_2 \Delta_1^0); \quad \eta_1 = 2 \lambda_2(6 \lambda_2 \Delta_1^4 + 30 \Delta_2 \lambda_1 \Delta_1^3 - 5 \lambda_2 \lambda_1 - 6 \lambda_2 \Delta_1); \quad \eta_2 = 30 \lambda_1(6 \Delta_1^3 - 1); \quad \eta_3 = 12 \lambda_1(4 \Delta_1^3 - 1).
\]

Relation (19), (20) after determination of \( b_i(q) \), \( k = 1, 2, 3 \) will be

\[
\Theta_1(\xi_1, F_0) = -\frac{2}{3} (q' \eta_1 + q \eta_2) \varphi_{11} / \eta + (q' \eta_1 + q \eta_2) \varphi_{12} / \eta ;
\]  

(39)

\[
\Theta_2(\xi_2, F_0) = -\frac{2}{3} (q' \eta_2 + q \eta_3) \varphi_{21} / \eta + (q' \eta_2 + q \eta_3) \varphi_{22} / \eta .
\]  

(40)

Then we require the relations (19), (20) not to satisfy the initial equations (10), (11), but some equations averaged over the thickness of the corresponding layer to the equations (the heat balance integral)

\[
\int_0^{\lambda_1} \left( \frac{\partial \Theta_1(\xi_1, F_0)}{\partial F_0} - \frac{\partial^2 \Theta_1(\xi_1, F_0)}{\partial \xi_2^2} \right) d\xi_1 +
\int_0^{\lambda_1} \left( \frac{\partial \Theta_2(\xi_2, F_0)}{\partial F_0} - \frac{\partial^2 \Theta_2(\xi_2, F_0)}{\partial \xi_2^2} \right) d\xi_2 = 0.
\]  

(41)

Substituting (39), (40) into (41), we can find

\[
K \frac{d^2 q}{d F_0^2} + L \frac{dq}{d F_0} + M q = 0 .
\]  

(42)
where \( K = 0.58 \), \( L = 1.082 \), \( M = 0.0766 \) with the following initial values: 

- \( a_1 = 12.5 \cdot 10^{-6} \text{ m}^2/\text{s} \); 
- \( a_2 = 6 \cdot 10^{-6} \text{ m}^2/\text{s} \); 
- \( \lambda_1 = 45.24 \text{ W}/(\text{m} \cdot \text{K}) \); 
- \( \lambda_2 = 16.24 \text{ W}/(\text{m} \cdot \text{K}) \); 
- \( \delta_1 = 0.002 \text{ m} \); 
- \( \delta_2 = 0.004 \text{ m} \); 
- \( \Delta_1 = 0.33 \); 
- \( \Delta_2 = 0.67 \).

Integrating equation (42), we find

\[
q(Fo) = C_1 \exp \left( -\frac{(L^2 - 4KM)^{1/2} + L}{2K} \right) + C_2 \exp \left( -\frac{(L^2 - 4KM)^{1/2} - L}{2K} \right).
\]

where \( C_1, C_2 \) are integration constants, which are found from the initial conditions (12), (13).

Finding the odds of the initial conditions (12), (13), and requiring their orthogonality to the coordinate functions \( \phi_{11}(\xi_1), \phi_{21}(\xi_2), (k = 1, 2) \), we find

\[
\begin{align*}
\int_0^{\xi_1} \left[ \Theta_1(\xi_1, 0) - 1 \right] \phi_{11}(\xi_1) d\xi_1 + \int_0^{\xi_2} \left[ \Theta_2(\xi_2, 0) - 1 \right] \phi_{21}(\xi_2) d\xi_2 &= 0; \\
\int_0^{\xi_1} \left[ \Theta_1(\xi_1, 0) - 1 \right] \phi_{12}(\xi_1) d\xi_1 + \int_0^{\xi_2} \left[ \Theta_2(\xi_2, 0) - 1 \right] \phi_{22}(\xi_2) d\xi_2 &= 0.
\end{align*}
\]

Substituting (39), (40) (taking into account (43)) into (44) with respect to the integration constants \( C_1, C_2 \), we have a system of two algebraic equations. We can find \( C_1 = -1.307 \), \( C_2 = 1.608 \). The solution of problem (10) – (17) in the first approximation is found from (39), (40).

Figure 2. Temperature distribution in a two-layer plate with different properties of the layers. 
Calculations by formulas (19), (20) at \( n = 3 \).

Figure 3. Temperature distribution in a two-layer plate with similar layer properties. — by formulas (19), (20) at \( n = 3 \); ○ — exact solution [11].
Relations (39), (40) exactly satisfy the boundary conditions (14), (17) and the matching conditions (15), (16). Equations (10), (11) and initial conditions (12), (13) in this case are fulfilled only approximately.

Figure 2 shows the calculation results by formulas (19), (20) at \( n = 3 \). Figure 3 shows the calculation results in the case when a two-layer plate is reduced to a single-layer \( (n = 3) \), that is, at \( \lambda = \lambda_1, \ a = a_1 \). The analysis allows to conclude that in the range \( 0,4 < Fo < \infty \) the highest possible error value to the exact solution [11] does not exceed 6%.

4. Conclusion
An approximate analytical solution of the heat conduction problem for a two-layer plate under symmetric boundary conditions of the first kind was determined using the found coordinate system functions that precisely satisfy the matching conditions and boundary conditions by introducing the accessory unknown function and accessory boundary conditions.

In order to construct the simplest form of coordinate functions that exactly satisfy the matching and boundary conditions, local coordinate systems are used. Their use tremendously simplifies both the process of constructing coordinate functions and the final expressions for them.

Using the accessory unknown function that characterizes the temperature change over time at the center of symmetry allows one to reduce the solution of the partial differential equation to the solution of the ordinary differential equation. Additional boundary conditions are found so that their fulfillment by the unknown solutions is equivalent to the satisfaction of differential equations at the boundaries of the region. It is proven that the fulfillment of equations at the boundaries of the region leads to their fulfillment inside the region.

5. Acknowledgments.
The reported study was funded by RFBR, project number 20-38-70021.

6. References
[1] Belyaev N M and Ryadno A A 1978 Methods of Unsteady Thermal Conductivity (Moscow: Higher School) p 328.
[2] Kudinov V A 1999 Analytical solutions of boundary value problems for multilayer structures Izvestiya of the Russian Academy of Sciences. Power Engineering 5 85-92.
[3] Kudinov V A, Kartashov E M and Kalashnikov V V 2005 Analytical Solutions of Heat and Mass Transfer and Thermoelasticity Problems for Multilayer Structures (Moscow: Higher School) p 340.
[4] Kolyano Y M 1978 Application of generalized functions in thermomechanics of piecewise homogeneous bodies Mathematical Methods and Physical and Mechanical Fields 7 7-11.
[5] Kolyano Y M and Popovich V S 1975 Nonstationary Temperature Field in Docked Plates Phys. and Chem. of Material Proc. 5 16-23.
[6] Kudinov V A, Kudinov I V and Skvortsova M P 2015 Generalized functions and additional boundary conditions in heat conduction problems for multilayered bodies Computational Mathematics and Mathematical Phys. 55 666-676.
[7] Kudinov I V, Kudinov V A, Kotova E V and Kuznetsova A E 2013 Generalized functions in thermal conductivity problems for multilayered constructions High Temperature 51 830-840.
[8] Kudinov I V, Kudinov V A and Kotova E V 2017 Additional boundary conditions in unsteady-state heat conduction problems High Temperature 55 541-548.
[9] Kudinov I V, Kudinov V A and Kotova E V 2016 Analytic solutions to heat transfer problems on a basis of determination of a front of heat disturbance Russian Mathematics 60 22-34.
[10] Kudinov I V, Kotova E V and Kudinov V A 2019 A method for obtaining analytical solutions to boundary value problems by defining additional boundary conditions and additional sought-for functions Numerical Analysis and Applications 12 126-136.
[11] Lykov A V 1967 Theory of Heat Conductivity (Moscow: Higher School) p 600.