Efficient Bayesian inference for multivariate factor stochastic volatility models with leverage*

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Abstract

This paper discusses the efficient Bayesian estimation of a multivariate factor stochastic volatility (Factor MSV) model with leverage. We propose a novel approach to construct the sampling schemes that converges to the posterior distribution of the latent volatilities and the parameters of interest of the Factor MSV model based on recent advances in Particle Markov chain Monte Carlo (PMCMC). As opposed to the approach of Chib et al. (2006) and Omori et al. (2007), our approach does not require approximating the joint distribution of outcome and volatility innovations by a mixture of bivariate normal distributions. To sample the free elements of the loading matrix we employ the interweaving method used in Kastner et al. (2017) in the Particle Metropolis within Gibbs (PMwG) step. The proposed method is illustrated empirically using a simulated dataset and a sample of daily US stock returns.

Keywords: Interweaving method; Particle Metropolis within Gibbs; Pseudo marginal Metropolis Hastings;

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1 Introduction

The analysis of financial time series has become an important research area over the last two decades, with both methodological and computational developments making it possible to estimate more complex models. Two well-known classes of models, the GARCH and stochastic volatility (SV), have been proposed to model financial time series volatility (see Bollerslev et al. (1994) and Ghysels et al. (1996)). However, current real world financial applications call for jointly modeling many simultaneous and co-varying observations over time. Recently, the literature has dealt with the development of multivariate models and estimation of such models. Factor multivariate stochastic volatility (factor MSV) models are increasingly used because they are able to model the volatility dynamics of a large system of financial or economic time series when the common features in these series can be captured by a small number of latent factors. Our article focuses on the model formulated by Chib et al. (2006) and extends it to include leverage.

A computationally efficient method of estimating a high dimensional factor MSV model is necessary if such models are to be applied to real world financial applications. Bayesian MCMC methods have been proposed to estimate the parameters of the factor MSV model (see for example, Chib et al. (2006); Han (2006); Aguilar and West (2000)). Based on results reported in the literature, such as Chib et al. (2006), estimating a factor MSV using current Bayesian simulation methods is neither exact nor flexible for two reasons. The first is related to sampling the latent volatilities. Chib et al. (2006) use the approach proposed by Kim et al. (1998) to approximate the joint distribution of outcome innovations by a suitably constructed seven-component
mixture of normal distributions. The second is related to sampling the latent factors and the associated free parameters in the loading matrix. Our aim is to outline a reliable and efficient method for exact Bayesian inference that performs well and is easy to implement and extend.

We develop a general approach to constructing sampling schemes that converge to the correct posterior distribution of the latent volatilities and the parameters of interest of the Factor MSV based on recent advances in Particle Markov chain Monte Carlo (PMCMC). The sampling schemes generate particles as auxiliary variables. Andrieu et al. (2010) proposed two particle MCMC samplers. The first is Pseudo Marginal Metropolis-Hastings (PMMH), where the parameters are generated with the latent states integrated out. The second is a Particle Gibbs (PG) algorithm. PG is a Monte carlo approximation of the standard Gibbs sampling procedure which uses sequential Monte carlo (SMC) to update the states given the parameters. Andrieu et al. (2010) shows that the augmented target density of these two algorithms has the joint posterior density of the parameters and states as a marginal density. Furthermore, Mendes et al. (2016) proposed a general PMCMC sampler which combine the PG and PMMH. This mixed sampler is highly efficient when there is a set of parameters that is not highly correlated with the latent states which can be generated using PG, and another set of parameters that is highly correlated with the latent states and is generated using the PMMH sampler.

In this paper, we develop a version of PG of Andrieu et al. (2010) and mixed sampler of Mendes et al. (2016) to sample both the latent volatilities and the parameters of Factor MSV. Note that in this case, our approach also does not require to approximate the joint distribution of outcome and
volatility innovations by a ten-component mixture of bivariate normal distributions \cite{omori2007}. To sample the free elements of the loading matrix we employ interweaving method as in \cite{kastner2017} in the Particle Metropolis within Gibbs (PMwG) step. The proposed method is illustrated empirically using simulated dataset and a sample of daily US stock returns.

The remainder of this paper is organized as follows. Section 2 describes the Factor MSV model in detail. Section 3 gives an in-depth discussion of the estimation algorithm and its implementation. Section 4 presents measures of sampling efficiency for a simulated dataset. Section 5 discusses an empirical application to US stock returns. Section 6 concludes.

2 Factor SV Model with leverage in the Idiosyncratic Error

2.1 Model

Let $y_t = (y_{1t}, ..., y_{pt})'$ denote the $p$ observations at time $t$ and suppose that conditional on $k$ unobserved factors $f_t = (f_{1t}, ..., f_{kt})'$, we have

$$y_t = B f_t + u_t,$$

where $B$ is an unknown $p \times k$ factor loadings matrix of unknown parameters.

$$\begin{pmatrix} u_t \\ f_t \end{pmatrix} \sim N_{p+k} \left( \begin{pmatrix} 0 \\ V_t \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & D_t \end{pmatrix} \right)$$
are conditionally independent Gaussian random vectors. The time varying variance matrices \( V_t \) and \( D_t \) are taken to depend upon unobserved random variables \( h_{1t} = (h_{11,t}, ..., h_{1p,t}) \) and \( h_{2t} = (h_{21,t}, ..., h_{2k,t}) \) in the form

\[
V_t = V_t(h_{1t}) = \text{diag}\{\exp(h_{11,t}), ..., \exp(h_{1p,t})\}: p \times p
\]

\[
D_t = D_t(h_{2t}) = \text{diag}\{\exp(h_{21,t}), ..., \exp(h_{2k,t})\}: k \times k
\]

where each \( h_{1i,t} \) and \( h_{2j,t} \) follows an independent three parameter \((\mu_{1i}, \phi_{1i}, \tau^2_{1i}, \mu_{2j}, \phi_{2j}, \tau^2_{2j})\) stochastic volatility process

\[
h_{1st} - \mu_{1s} = \phi_{1s}(h_{1st-1} - \mu_{1s}) + \eta_{1st}, \eta_{1st} \sim N(0, \tau^2_{1s}), s = 1, ..., p \tag{2}
\]

and

\[
h_{2jt} - \mu_{2j} = \phi_{2j}(h_{2jt-1} - \mu_{2j}) + \eta_{2jt}, \eta_{2jt} \sim N(0, \tau^2_{2j}), j = 1, ..., k. \tag{3}
\]

We model the joint distributions of outcome innovations and volatilities as follows

\[
\begin{pmatrix}
    u_{st} \\
    \eta_{1st}
\end{pmatrix} \sim N_p \left(0, \begin{pmatrix}
    \exp(h_{1st}) & \rho_{1s}\tau_{1s}\exp(h_{1st}/2) \\
    \rho_{1s}\tau_{1s}\exp(h_{1st}/2) & \tau^2_{1s}
\end{pmatrix}\right), s = 1, ..., p,
\]

where \( \rho_s \) is the correlation coefficient between \( u_{st} \) and \( \eta_{1st} \) and it is used to measure the leverage effect. [Harvey and Shephard (1996)] were the first to proposed the univariate SV model with leverage effects in discrete time.
First, to prevent factor rotation and column switching, we follow the usual convention and set the upper triangular part of \(B\) to zero and \(\text{diag}(B)\) non-zero (e.g. Geweke and Zhou, 1996). This parameterisation imposes an order dependence. Secondly, the model is also not identified without identifying the scaling of either the \(k\)th column of \(B\) or the the variance of \(f_{kt}\). The usual solution is to set the diagonal elements of the factor loading matrix \(B_{jj}\) to one, for \(j = 1, \ldots, r\), while the level \(\mu_{2j,t}\) of the factor volatilities \(h_{2j,t}\) is modeled to be unknown. As noted by Kastner et al. (2017), this approach imposes that the first \(k\) variables are leading the factors, and making the variable ordering dependence stronger. We follow Kastner et al. (2017) and leave the diagonal elements \(B_{jj}\) unrestricted and set the level \(\mu_{2j}\) of the factor volatilities \(h_{2j,t}\) to zero for \(j = 1, \ldots, k\). An intuitive explanation is that the “leadership” of a factor is shared by several series. Each column of \(B\) is only identified up to a possible sign switch, we solve this problem a posteriori, by running our PMCMC sampler and identify the factor loading signs afterwards.

3 Proposed PMCMC algorithm

3.1 Preliminaries

If we let \(\mathcal{F}_{t-1}\) denote the history of the \(\{y_t\}\) process up to time \(t - 1\), and \(p(h_{1t}, h_{2t}|\mathcal{F}_{t-1}, \Theta)\) the density of latent variables \(h_{1t}\) and \(h_{2t}\) conditioned on
\((F_{t-1}, \Theta)\), then the likelihood function of \(\Theta\) given the data \(y = (y_1, ..., y_T)\) is

\[
p(y|\Theta) = \prod_{t=1}^{T} \left( \int p(y_t|h_{1t}, h_{2t}, \Theta) p(h_{1t}, h_{2t}|F_{t-1}, \Theta) \, dh_{1t} \, dh_{2t} \right)
\]

\[
= \prod_{t=1}^{T} \int N_p\left(y_t|RV_t^{1/2}T_1^{-1/2}\eta_{1t}, \Omega_t\right) p(h_{1t}, h_{2t}|F_{t-1}, \psi) \, dh_{1t} \, dh_{2t}\]

where \(R = diag\{\rho_1, ..., \rho_p\}\), \(T_1 = diag\{\tau_{11}, ..., \tau_{1p}\}\), \(N_p\) is the multivariate normal density function with the mean \(RV_t^{1/2}T_1^{-1/2}\eta_{1t}\) marginalised over \(f_t\), with the variance given by

\[
\Omega_t = BD_tB' + V_t - R^2V_t.
\]

It is clear to see that neither \(p(h_{1t}, h_{2t}|F_{t-1}, \psi)\) nor the integral of \(N_p\left(y_t|RV_t^{1/2}T_1^{-1/2}\eta_{1t}, \Omega_t\right) \) over \((h_{1t}, h_{2t})\) are available in closed form. We utilise PMCMC algorithm to develop a novel Bayesian estimation approach for this model. Firstly, we discuss sampling the factor loading matrix \(B\) and the latent factors \(f = \{f_j\}, j = 1, ..., k\), where \(f_{j,.} = (f_{j1}, ..., f_{jT})'\). Conditional on knowing \(h_1, h_2,\) and \(f\), the \(B\) can be sampled conditionally on each other from the multivariate normal distribution similar to a standard factor model [Lopes and West, 2004].

Sampling the factor loadings \(B'_{s,.}\) for \(s = 1, ..., p\), conditionally on \(f\) from the conditional posterior density \(\tilde{\pi}(B'_{s,.}|f, y_{s,.}, h_{1s,.})\) can be done independently for each \(s\), by performing a Gibbs-update from

\[
B'_{s,.}|f, y_{s,.}, h_{1s,.} \sim N_{k_s}(a_{sT}, b_{sT}),
\]
where $b_{sT} = \left(F_s'V_s^{-1}F_s + B_0^{-1}I_{k_s}\right)^{-1}$ and $a_{sT} = b_{sT}F_s'\left(V_s^{-1}y_s - V_s^{-1} \frac{\rho_t}{\eta_t} \exp(h_{1s.}/2) \eta_t \right)$, 

$$F_s = \begin{bmatrix} f_{11} & \cdots & f_{k,1} \\ \vdots & \ddots & \vdots \\ f_{1T} & \cdots & f_{kT} \end{bmatrix}$$

and

$$V_s = \begin{bmatrix} \exp(h_{1s,1}) (1 - \rho^2_s) & \cdots & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & \exp(h_{1s,T}) (1 - \rho^2_s) \end{bmatrix}.$$ 

Sampling of $\{f_t\}$: The sampling of the factors are completed by sampling $\{f_t\}$ from the distribution $\{f_t\} | y, \{h_{1t}\}, \{h_{2t}\}, B$. After completing some algebra, we can show that $\{f_t\}$ can be sampled from Gaussian with variance $b_t = \left(B' \left(V_t - R^2V_t\right)^{-1} B + D_t^{-1}\right)^{-1}$ and mean $a_t = b_tB' \left(\left(V_t - R^2V_t\right)^{-1} y_t - \left(V_t - R^2V_t\right)^{-1} RV_t^{1/2}T_1^{-1/2} \eta_t \right)$.

Sampling $\rho, \mu, \phi, \tau^2, \{h_{1t}\}$, and $\{h_{2t}\}$: in the next step of the algorithm, given $(y, B, f_t)$, and the conditional independence of the errors, we exploit the fact that this models separates into $p$ univariate SV models with leverage, and $k$ univariate SV models. This shows that the latent idiosyncratic and factor volatilities and SV specific parameters can be sampled series-by-series. This is one of the reason that our approach is scalable in both $p$ and $k$.

### 3.2 State Space Models and Sequential Monte Carlo Methods

In this section, we first briefly describe state space model in general. Let $\{h_t\}_{t \in N}$ is a latent Markov process with initial density $p_\theta(h_1)$ and state transition density $p_\theta(h_t|h_{t-1})$ for $t = 1, ..., T$. The latent process $\{h_t\}_{t \in N}$
is observed only through \( \{ y_t \}_{t \in N} \), whose value at time \( t \) depends on the value of hidden state at time \( t \). This \( p_\theta (y_t|h_t) \) is often called observation/measurement density. The joint probability density function of \((h_{1:T},y_{1:T})\) is

\[
p(h_{1:T},y_{1:T}|\theta) = p_\theta (h_1) \prod_{t=2}^{T} p_\theta (h_t|h_{t-1}) p_\theta (y_t|h_t).
\]

We also define the likelihood as \( Z_{1:T}(\theta) = \prod_{t=1}^{T} Z_t(\theta) \), where \( Z_1(\theta) = p(y_1|\theta) \) and \( Z_t(\theta) = p(y_t|y_{1:t-1},\theta) \). The joint filtering density of \( h_{1:t} \) can be written as

\[
\pi_t(h_{1:t}) = p(h_{1:t}|y_{1:t},\theta) = \frac{p(h_{1:t},y_{1:t}|\theta)}{Z_{1:t}(\theta)}
\]

and the joint posterior density of \( \theta \) and \( h_{1:T} \) is given by

\[
p(\theta,h_{1:T}|y_{1:T}) = \frac{p(h_{1:T},y_{1:T}|\theta)p(\theta)}{Z_{1:T}},
\]

where \( Z_{1:T} = \int_\Theta Z_{1:T}(\theta)p(\theta)\,d\theta \) is the marginal likelihood.

For a Bayesian analysis in a non-linear, non-Gaussian state space model, such as SV model with or without leverage, the “ideal” Gibbs sampler targeting the joint posterior density \( p(\theta,h_{1:T}|y_{1:T}) \) consists of sampling alternately from the full conditional posteriors \( p_\theta (h_{1:T}|y_{1:T}) \) and \( p(\theta|h_{1:T},y_{1:T}) \). This is typically infeasible since exact sampling from \( p_\theta (h_{1:T}|y_{1:T}) \) is impossible. The particle Gibbs approach of [Andrieu et al., 2010] and the mixed sampler of [Mendes et al., 2016] use a sequential Monte Carlo (SMC) algorithm to obtain approximate samples from \( p_\theta (h_{1:T}|y_{1:T}) \).

Sequential Monte Carlo (SMC) algorithms consist of recursively producing a weighted particles \( \{ h_{1:t}^i, w_t^i \}_{i=1}^{N} \) such that the intermediate target density
\( p(h_{1:t}|y_{1:t}, \theta) \) can be approximated by
\[
\pi_t(dh_{1:t}) = \sum_{i=1}^{N} W_i^t \delta_{h_{1:t}^i}(dh_{1:t}), \quad W_t^i = \frac{w_t^i}{\sum_{i=1}^{N} w_t^i},
\]
where \( \delta \) denote the Dirac delta mass located at \( h \). Suppose that at the end of period \( (t-1) \), we have a set of particle \( \{h_{1:t-1}^i, W_{t-1}^i\}_{i=1}^N \). Once we have a new observation \( y_t \), we propagate the particles \( h_{1:t-1}^i \) to \( h_{1:t}^i = (h_t^i, h_{1:t-1}^i) \) using the importance sampling (IS) density \( m(h_t^i|h_{1:t-1}^i) \) and updating the corresponding importance sampling weights according to
\[
w_t^i = W_{t-1}^i \frac{p(y_t|h_t^i) p(h_t^i|h_{1:t-1}^i)}{m(h_t^i|h_{1:t-1}^i)},
\]
with the corresponding normalised weights calculated as \( W_t^i = w_t^i / \sum_{i=1}^{N} w_t^i \).

The variance of IS weights \( w_t^i \) in (5) increases exponentially with the time period \( t \) and hence reducing the effective sample size in the particle filter. This is known as 'weight degeneracy' problem. To avoid this problem, SMC algorithm needs to include a resampling step before propagating the particles \( h_{1:t-1}^i \) to \( h_{1:t}^i = (h_t^i, h_{1:t-1}^i) \). The \( N \) 'ancestor particles' from \( \{h_{1:t-1}^i\}_{i=1}^N \) is sampled according to their normalised IS weights \( \{W_{t-1}^i\} \) and then set the IS weights \( W_{t-1}^i \) all equal to \( 1/N \). Popular resampling schemes include multinomial, residual, stratified, and systematic resampling.

Another issue in implementing SMC efficiently is the choice of the IS densities \( q_t(h_t|h_{1:t-1}^i) \). In general, this requires to select \( m_t(h_t|h_{1:t-1}^i) \) as a close approximation to the period-\( t \) conditional density \( \pi_t(h_t|h_{1:t-1}^i) \). The most popular selection for importance sampling densities are the transition densities \( p_\theta(h_t^i|h_{t-1}^i) \) used by Bootstrap Particle Filter (Gordon et al. 1993). In
the case of the measurement density $p_\theta (y_t | h_t)$ is quite flat in $h_t$, this selection typically sufficient. There are more advances particle filter algorithms developed in the literature, such as Particle efficient importance sampling of Scharth and Kohn (2016) and Auxiliary Particle filter of Pitt and Shephard (1999) that are more efficient than standard bootstrap particle filter.

The SV model with leverage can be expressed in the form of state space model consisting of the measurement density

$$p_\theta (y_{st} | h_{1st}, h_{1st-1}) \sim N \left( B_s f_t + \frac{\rho_{1s}}{\tau_{1s}} \exp \left( h_{1st}/2 \right) ( h_{1st} - \mu_{1s} - \phi_{1s} (h_{1st-1} - \mu_{1s})) , (1 - \rho_{1s}^2) \exp \left( h_{1st} \right) \right)$$

and this following state transition density

$$p_\theta (h_{1st} | h_{1st-1}) \sim N \left( \mu_{1s} + \phi_{1s} (h_{1st-1} - \mu_{1s}), \tau_{1s}^2 \right), \text{ for } s = 1, \ldots, p,$$

where, $p_\theta (h_{1i1}) \sim N \left( \mu_{1i}, \frac{\tau_{1i}^2}{1 - \phi_{1i}^2} \right)$. For each $j = 1, \ldots, k$, the SV model can be expressed in the form of state space model consisting of the measurement density

$$p (f_{jt} | h_{2jt}) \sim N (0, \exp (h_{2jt}))$$

and this following state transition density

$$p (h_{2jt} | h_{2jt-1}) \sim N \left( \phi_{2j} h_{2jt-1}, \tau_{2j}^2 \right),$$

where $p_\theta (h_{2j1}) \sim N \left( \mu_{2j}, \frac{\tau_{2j}^2}{1 - \phi_{2j}^2} \right)$. 


3.3 Target Distribution of Particle Markov chain Monte Carlo (PMCMC)

In this section, we first define the appropriate target density for factor MSV that include all the random variables which are produces by SMC to generate $h_{1s,1:T}$ for $s = 1, ..., p$ and $h_{2j,1:T}$ for $j = 1, ..., r$. We first approximate the joint filtering densities $\{p(h_{1st}|y_{s,1:t}, \Theta) : t = 1, ..., T\}$ for $s = 1, ..., p$ and $\{p(h_{2jt}|f_{j,1:t}, \Theta) : t = 1, ..., T\}$ sequentially, using particles, i.e. weighted samples $(h_{1s}^{1:N}, w_{1s}^{1:N})$ and $(h_{2j}^{1:N}, w_{2j}^{1:N})$, drawn from some important densities $m_{1s}^{\Theta}$ and $m_{2j}^{\Theta}$ for $t = 1, ..., T$, respectively. A valid resampling scheme $M_1 (a_{1s-1}^{1:N})$, where each $a_{1s-1}^{i} = k$ indexed a particle in $(h_{1s}^{1:N}, w_{1s}^{1:N})$ and is chosen with probability $w_{1s}^{i}$, $M_2 (a_{2jt-1}^{1:N})$ is defined similarly. The Sequential Monte Carlo algorithm used in this paper is the same as in Andrieu et al. (2010) and it is given in the Appendix A. We denote the vector of particles by

$$U_{1s,1:T} := (h_{1s}^{1:N}, ..., h_{1s,T}, C_{1s,1}, ..., C_{1s,T-1})$$

and

$$U_{2j,1:T} := (h_{2j}^{1:N}, ..., h_{2j,T}, C_{2j,1}, ..., C_{2j,T-1})$$

This SMC algorithm also provides an estimate of the likelihood $Z_{1s} (U_{1s,1:T}, \Theta) = \prod_{t=1}^{T} \left( \frac{1}{N} \sum_{i=1}^{N} w_{1s,t}^{i} \right)$, for $s = 1, ..., p$ and $Z_{2j} (U_{2j,1:T}, \Theta) = \prod_{t=1}^{T} \left( \frac{1}{N} \sum_{i=1}^{N} w_{2j,t}^{i} \right)$ for $j = 1, ..., k$. The joint distribution of the particles given the parameters
are

$$\psi_{1s}(U_{1s,1:T}^{1:N}|\Theta) := \prod_{i=1}^{N} m_{1s1}^{\Theta}(h_{1s1}^i) \prod_{t=2}^{T} \left\{ M(a_{1s1t-1}^{1:N}|w_{1st-1}^{1:N}) \prod_{i=1}^{N} m_{1st}^{\Theta}(x_{1st1t}^i|n_{1st1t-1}^i) \right\}$$

for \(s = 1, \ldots, p\) and

$$\psi_{2j}(U_{2j,1:T}^{1:N}|\Theta) := \prod_{i=1}^{N} m_{2j1}^{\Theta}(h_{2j1}^i) \prod_{t=2}^{T} \left\{ M(a_{2j1t-1}^{1:N}|w_{2j1t-1}^{1:N}) \prod_{i=1}^{N} m_{2jt}^{\Theta}(x_{2jt1t}^i|n_{2jt1t-1}^i) \right\}$$

for \(j = 1, \ldots, k\). We then construct a target distribution on an augmented space that includes the particles \(U_{1s,1:T}^{1:N}\), for \(s = 1, \ldots, p\) and \(U_{2j,1:T}^{1:N}\) for \(j = 1, \ldots, k\).

In this paper, we use simple ancestral tracing method of [Kitagawa (1996)](#) to sample one particle from the final particle filter. The method is equivalent to sampling index \(J_{1s} = j_{1s}\), for \(s = 1, \ldots, P\), with probability \(w_{J_{1s}}\), tracing back its ancestral lineage \(C_{1s1:T-1}^{J_{1s}}\) and choosing the particle \(h_{1s1:T}^{J_{1s}} = (h_{1s1}^{C_{1s1}^{J_{1s}}}, \ldots, h_{1sT}^{C_{1sT}^{J_{1s}}})\), and \(h_{2j1:T}^{J_{2j}}\) can be obtained similarly. Further, let us denote

$$u_{1s1:T}^{(-J_{1s})} = \left\{ h_{1s1}^{(-C_{1s1}^{J_{1s}})}, \ldots, h_{1sT-1}^{(-C_{1sT-1}^{J_{1s}})}, h_{1sT}^{(-J_{1s})}, a_{1s1}^{(-C_{1s1}^{J_{1s}})}, \ldots, a_{1sT-1}^{(-C_{1sT-1}^{J_{1s}})} \right\}.$$
Then, the target distribution is given by

\[
\hat{\pi}^N \left( h_{1,1:T}, h_{2,1:T}, C_{1,1:T-1}, C_{2,1:T-1}, J_1, J_2, U_{1,1:T}^{(-J_1)}, U_{2,1:T}^{(-J_2)}, \Theta, f \right) :=
\]

\[
p \left( h_{1,1:T}, h_{2,1:T}, \Theta, f | y_{1:T} \right) \prod_{s=1}^{P} \frac{\psi_{1s} \left( U_{1s,1:T} | \Theta \right)}{m_{1s}^\Theta \left( h_{1s1}^{C_{1s1}1} \right) \prod_{t=2}^{T} \bar{w}_{1s,t-1} m_{1st}^\theta \left( h_{1st} | h_{1st-1}^{C_{1st1}} \right)} \prod_{j=1}^{k} \frac{\psi_{2j} \left( U_{2j,1:T} | \Theta \right)}{m_{2j1}^\Theta \left( h_{2j1}^{C_{2j1}1} \right) \prod_{t=2}^{T} \bar{w}_{2j,t-1} m_{2jt}^\theta \left( h_{2jt} | h_{2jt-1}^{C_{2jt1}} \right)}.
\]

Equation (6) has the following marginal distribution

\[
\hat{\pi}^N \left( h_{1,1:T}, h_{2,1:T}, C_{1,1:T-1}, C_{2,1:T-1}, J_1, J_2, \Theta, f \right)
\]

\[
:= \int \int \hat{\pi}^N \left( h_{1,1:T}, h_{2,1:T}, C_{1,1:T-1}, C_{2,1:T-1}, J_1, J_2, U_{1,1:T}^{(-J_1)}, U_{2,1:T}^{(-J_2)}, \Theta, f \right) dU_{1,1:T}^{(-J_1)} dU_{2,1:T}^{(-J_2)}
\]

\[
:= p \left( h_{1,1:T}, h_{2,1:T}, \Theta, f | y_{1:T} \right).
\]

This is defined to be the target density of interest up to the factor $1/N^T$ representing a discrete uniform density over the index variables in $(C_{1,1:T-1}, C_{2,1:T-1})$ and hence

\[
\hat{\pi}^N \left( h_{1,1:T}, h_{2,1:T}, \Theta, f \right) := p \left( h_{1,1:T}, h_{2,1:T}, \Theta, f | y_{1:T} \right).
\]

It follows that Pseudo Marginal, particle Gibbs, and mixed samplers leaves
the target density \( p(h_{1,1:T}, h_{2,1:T}, \Theta, f| y_{1:T}) \) invariant and delivers under weak regularity conditions a sequence of draws \( \{h_{1,1:T}^{(l)}, h_{2,1:T}^{(l)}, \Theta^{(l)}, f^{(l)}\} \) whose marginal distributions converge for any \( N > 1 \) to \( p(h_{1,1:T}, h_{2,1:T}, \Theta, f| y_{1:T}) \) as \( l \to \infty \) (Andrieu et al., 2010).

3.4 PMCMC (Particle Gibbs and Mixed) Sampling Schemes

This section describes a version of Particle Gibbs (PG) of Andrieu et al. (2010) and mixed sampling schemes of Mendes et al. (2016) for Factor MSV using the target distributions given in Section 3.3.

3.4.1 Particle Gibbs (PG) and Particle Metropolis within Gibbs (PMwG)

The Particle Gibbs is a standard Gibbs sampler for the augmented target distribution in equation (6). The Gibbs sampler for this augmented density requires a different type of SMC algorithm, referred to as conditional SMC, where one of the particles is specified a priori. This reference particle denoted by \( (h_{1,1:T}^0, h_{2,1:T}^0) \) is then retained throughout the entire SMC sampling process (Andrieu et al., 2010). To accomplish this, we also need special resampling schemes. We use conditional systematic resampling in Chopin and Singh (2013). The conditional SMC algorithm is given in Appendix B.

The PG or PMwG sampling schemes for Factor MSV with leverage in Section 2 can be proceed as follows:

1. Loop over \( \theta_{1s} : (\mu_{1s}, \phi_{1s}, \rho_{1s}, \tau_{1s}^2) \), for each \( s = 1, \ldots, p \)
(a) Sample from conditional distribution $p \left( \theta_{1s} | h_{11,1:T}^{j_s}, C_{11,1:T}^{j_s}, J_1, \theta_{-1s}, \theta_{1s}, f, B \right)$ if it is available.

(b) Or, sample $\theta_{1s}^* \sim q_{1s} \left( . | h_{11,1:T}^{j_s}, C_{11,1:T}^{j_s}, J_1, \theta_{-1s}, \theta_{1s}, f, B \right)$

(c) Accept with probability

$$
\alpha \left( \theta_{1s}^*, \theta_{1s} | h_{11,1:T}^{j_s}, C_{11,1:T}^{j_s}, J_1, \theta_{-1s}, f, B \right) = 1 \wedge \frac{\pi \left( \theta_{1s}^* | h_{11,1:T}^{j_s}, C_{11,1:T}^{j_s}, J_1, \theta_{-1s}, f, B \right)}{\pi \left( \theta_{1s} | h_{11,1:T}^{j_s}, C_{11,1:T}^{j_s}, J_1, \theta_{-1s}, f, B \right)} \times \frac{q_{1s} \left( \theta_{1s}^* | h_{11,1:T}^{j_s}, C_{11,1:T}^{j_s}, J_1, \theta_{-1s}, \theta_{1s}^*, f, B \right)}{q_{1s} \left( \theta_{1s} | h_{11,1:T}^{j_s}, C_{11,1:T}^{j_s}, J_1, \theta_{-1s}, \theta_{1s}, f, B \right)}.
$$

2. Loop over $\theta_{2j} : (\phi_{2j}, \tau_{2j}^2)$, for each $j = 1, ..., k$

(a) Sample from conditional distribution $p \left( \theta_{2j} | h_{21,1:T}^{j_2}, C_{21,1:T}^{j_2}, J_2, \theta_{-2j}, \theta_{2j}, f, B \right)$ if it is available.

(b) Sample $\theta_{2j}^* \sim q_{2j} \left( . | h_{21,1:T}^{j_2}, C_{21,1:T}^{j_2}, J_2, \theta_{-2j}, \theta_{2j}, f, B \right)$

(c) Accept with probability

$$
\alpha \left( \theta_{2j}^*, \theta_{2j} | h_{21,1:T}^{j_2}, C_{21,1:T}^{j_2}, J_2, \theta_{-2j}, f, B \right) = 1 \wedge \frac{\pi \left( \theta_{2j}^* | h_{21,1:T}^{j_2}, C_{21,1:T}^{j_2}, J_2, \theta_{-2j}, f, B \right)}{\pi \left( \theta_{2j} | h_{21,1:T}^{j_2}, C_{21,1:T}^{j_2}, J_2, \theta_{-2j}, f, B \right)} \times \frac{q_{2j} \left( \theta_{2j}^* | h_{21,1:T}^{j_2}, C_{21,1:T}^{j_2}, J_2, \theta_{-2j}, \theta_{2j}^*, f, B \right)}{q_{2j} \left( \theta_{2j} | h_{21,1:T}^{j_2}, C_{21,1:T}^{j_2}, J_2, \theta_{-2j}, \theta_{2j}, f, B \right)}.
$$

3. Generate $B$ from $\pi \left( B | h_{11,1:T}^{j_1}, C_{11,1:T}^{j_1}, J_1, h_{21,1:T}^{j_2}, C_{21,1:T}^{j_2}, J_2, \Theta, \beta, f \right)$.

4. Generate $f_t$ for $t = 1, ..., T$ from $\pi \left( f_t | \{ h_{1t} \}, \{ h_{2t} \}, B, \Theta \right)$.

5. For $s = 1, ..., p$, sample $U_{1s,1:T}^{(-J_{1s})} \sim \pi \left( U_{1s,1:T}^{(-J_{1s})} | h_{1s,1:T}^{J_{1s}}, C_{1s,1:T}^{J_{1s}}, J_{1s}, \Theta \right)$, this is the conditional sequential Monte Carlo step, in which a particle
and the associated sequence of ancestral indices $C_{1:s,1:T−1}^{J_{1:s}}$ are kept unchanged. The conditional sequential Monte Carlo is a procedure that resamples all the particles and indices except for $U_{1:s,1:T}^{J_{1:s}}$.

6. For $s = 1, \ldots, p$, sample $J_{1:s} \sim \pi(J_{1:s}|U_{1:s,1:T}, \Theta)$

7. For $j = 1, \ldots, k$, sample $U_{2:j,1:T}^{(−J_{2:j})} \sim \pi(U_{2:j,1:T}^{−J_{2:j}}|h_{2:j,1:T}, C_{2:j,1:T−1}, J_{2:j}, \Theta)$, this is conditional sequential Monte Carlo step and is given in Appendix B.

8. For $j = 1, \ldots, k$, sample $J_{2:j} \sim \pi(J_{2:j}|U_{2:j,1:T}, \Theta)$.

As is known in the literature that this PG or PMwG implemented using bootstrap particle filter with resampling steps at every period of $t$, have a very poor mixing, especially when the time period $T$ is large. This is due to path degeneracy problem (Lindsten et al., 2014). The consequence of this path degeneracy problem is that at iteration step $l$ the new path trajectory $h_{1:T}^{(l)}$ tend to coalesce with the previous one $h_{1:T}^{(l−1)}$ which is retained as the reference particle trajectory in conditional sequential Monte Carlo sampling. The resulting particle degenerate toward this reference trajectory, and leads to poor mixing Markov chain.

In order to address the mixing problem of the PG caused by path degeneracy, we add additional Ancestor Sampling steps to conditional SMC (PGAS), which assign at each time period $t$ a new artificial $h_{1:t−1}$ history to the reference path $h_{1:T}$. The PGAS augments each period-$t$ conditional SMC resampling step by randomly selecting from the set $\{h_{1:t−1}^i\}_{i=1}^N$ (including the reference trajectory) one ancestor particle which is used as a new history to the partially reference trajectory $h_{1:T}^{(l)}$. In Lindsten et al. (2014), the PGAS
is implemented using bootstrap particle filter, the ancestor sampling weights are given by
\[ \bar{w}_{t-1|T} \propto w_{t-1|T} p_\theta (h^j_t | h^i_{t-1}) . \]

Lindsten et al. (2014) shows that the invariance property of PG is not violated by this additional ancestor sampling step. Because this ancestor sampling step assign a new ancestor to \( h^j_{t:T} \) in each period, then it will produce new trajectory \( h'_{t:T} \) that tends to be different from the reference trajectory \( h^j_{t:T} \).

3.4.2 Mixed Sampling Schemes

Mendes et al. (2016) proposed a mixed PMCMC sampler which combine the PG and pseudo marginal method. This mixed sampler is highly efficient when there is a set of parameters that is not highly correlated with the latent states which can be generated using PG, and another set of parameters that is highly correlated with the latent states and is generated using the PMMH sampler. After some experimentation with univariate SV, we found that \( PMMH (\tau^2) + PG (\mu, \phi) \) is the most efficient sampler. by using their notation, for Factor MSV model, we follow

\[ PMMH (\tau^2_{11}, \ldots, \tau^2_{1s}, \tau^2_{21}, \ldots, \tau^2_{2k}) + PG (\mu_1, \phi_1, \rho_1, s = 1, \ldots, p; \mu_j, \phi_j, j = 1, \ldots, k; B, f) . \]

The sampling scheme is given by:

1. Pseudo Marginal step for \( \tau^2_{1s} \), For \( s = 1, \ldots, P \)
   
   (a) Sample \( \tau^*_{1s, \tau^2} \sim q_{1s,\tau^2} \left( \cdot | U_{1,1:T}, U_{2,1:T}, J_1, \theta_{-\tau^2_{1s}}; f, B \right) \)
   
   (b) Sample \( U^*_{1s,1:T} \sim \psi \left( \cdot | U_{1-s,1:T}, U_{2,1:T}, \theta_{-\tau^2_{1s}}, \tau^2_{1s}; f, B \right) \)
(c) Sample $J_{1s}^*$ from $\pi \left( \cdot \bigm| U_{1s,1:T}^*, U_{1-\tau_{1s}^2,1:T}, U_{2,1:T}, \theta_{-\tau_{1s}^2}, \tau_{1s}^{2*}, f, B \right)$

(d) Accept with probability:

$$
\alpha \left( U_{1s,1:T}, J_{1s}, \tau_{1s}^2; U_{1s,1:T}^*, J_{1s}, \tau_{1s}^{2*}; f, \theta_{-\tau_{1s}^2}, B \right) = \frac{Z \left( \theta_{-\tau_{1s}^2}, \tau_{1s}^{2*}, U_{1s,1:T}^* \right)}{Z \left( \theta_{-\tau_{1s}^2}, \tau_{1s}^2, U_{1s,1:T} \right)} \times \frac{q_{1s,\tau^2} \left( \tau_{1s}^2 \bigm| U_{1s,1:T}^*, J_{1s}, \theta_{-\tau_{1s}^2}, f, \tau_{1s}^{2*} \right)}{q_{1s,\tau^2} \left( \tau_{1s}^2 \bigm| U_{1s,1:T}, J_{1s}, \theta_{-\tau_{1s}^2}, f, \tau_{1s}^2 \right)} \times \frac{p \left( \tau_{1s}^{2*} \bigm| \theta_{-\tau_{1s}^2} \right)}{p \left( \tau_{1s}^2 \bigm| \theta_{-\tau_{1s}^2} \right)}
$$

(7)

2. Pseudo Marginal step for $\tau_{2j}^2$, for $j = 1, \ldots, k$

(a) Sample $\tau_{2j}^{2*} \sim q_{2j,\tau^2} \left( \cdot \bigm| U_{1,1:T}, U_{2,1:T}, J_2, \theta_{-\tau_{2j}^2}, f, B \right)$

(b) Sample $U_{2j,1:T}^* \sim \psi \left( \cdot \bigm| U_{1,1:T}, U_{2-\tau_{2j}^2,1:T}, \theta_{-\tau_{2j}^2}, \tau_{2j}^{2*}, f, B \right)$

(c) Sample $J_{2j}^*$ from $\pi \left( \cdot \bigm| U_{1,1:T}, U_{2-\tau_{2j}^2,1:T}, U_{2j,1:T}^*, \theta_{-\tau_{2j}^2}, \tau_{2j}^{2*}, f, B \right)$

(d) Accept with probability:

$$
\alpha \left( U_{2j,1:T}, J_{2j}, \tau_{2j}^2; U_{2j,1:T}^*, J_{2j}, \tau_{2j}^{2*}; f, \theta_{-\tau_{2j}^2}, B \right) = \frac{Z \left( \theta_{-\tau_{2j}^2}, \tau_{2j}^{2*}, U_{2j,1:T}^* \right)}{Z \left( \theta_{-\tau_{2j}^2}, \tau_{2j}^2, U_{2j,1:T} \right)} \times \frac{q_{2j,\tau^2} \left( \tau_{2j}^2 \bigm| U_{2j,1:T}^*, J_{2j}, \theta_{-\tau_{2j}^2}, f, \tau_{2j}^{2*} \right)}{q_{2j,\tau^2} \left( \tau_{2j}^2 \bigm| U_{2j,1:T}, J_{2j}, \theta_{-\tau_{2j}^2}, f, \tau_{2j}^2 \right)} \times \frac{p \left( \tau_{2j}^{2*} \bigm| \theta_{-\tau_{2j}^2} \right)}{p \left( \tau_{2j}^2 \bigm| \theta_{-\tau_{2j}^2} \right)}
$$

(8)

3. Followed by step 1 to 8 of PG algorithm, except that the $\tau_{1s}^2$, $s = 1, \ldots, p$
and $\tau_{2j}^2$, $j = 1, \ldots, k$ are not generated in step 1 and 2 of PG algorithm.

Note that part 3 is the same as PG or Particle Metropolis within Gibbs algorithm described in Section 3.4.1. Part 1 and 2 also generates the variable
$J_1$ and $J_2$ which select the trajectory for each series and factors. This is necessary since $J_1$ and $J_2$ are used in the PG/PMwG step.

3.5 Prior Distributions

To perform Bayesian inference, the prior distributions for the parameters need to be specified. Independently, for each $s = 1, \ldots, p$, priors for the idiosynchratic SV parameters $p(\mu_{1s}, \phi_{1s}, \tau_{1s}^2) = p(\mu_{1s}) p(\phi_{1s}) p(\tau_{1s}^2)$, where the prior for $p(\mu_{1s}) \propto 1$, the prior for the persistence parameter $\phi_s \in (-1, 1)$ follows $U(-1, 1)$, and the prior for $\tau$ follow half-cauchy distribution such that the prior for $\tau^2$ is given by

$$p(\tau^2) = \frac{I(\tau > 0)}{\pi (1 + \tau^2) \sqrt{\tau^2}}.$$ 

The prior for $\rho_s$ for $s = 1, \ldots, p$ is $U(-1, 1)$. Same prior is used for factor SV parameters $(\mu_{2j}, \phi_{2j}, \tau_{2j}^2)$ for $j = 1, \ldots, k$. The initial state $h_{1s,1}$ and $h_{2j,1}$ are distributed according to the stationary distribution of the AR(1) process, i.e. $h_{1s,1}|\mu_{1s}, \phi_{1s}, \tau_{1s}^2 \sim N(\mu_{1s}, \tau_{1s}^2/(1 - \phi_{1s}^2))$ and $h_{2j,1}|\mu_{2j}, \phi_{2j}, \tau_{2j}^2 \sim N(\mu_{2j}, \tau_{2j}^2/(1 - \phi_{2j}^2))$. For every unrestricted element of the factor loadings matrix $B$, we choose independent Gaussian distributions, i.e. $p(B_{sj}) \sim N(0, 1)$.

3.6 Sampling Factor Loading $B$ using Interweaving Method

It is well-known that sampling factor loading $B$ conditioned on $\{f_t\}$ and then sampling $\{f_t\}$ conditioned on $B$ is very inefficient and leads to extremely slow convergence and poor mixing. To overcome this problem, Chib et al. (2006) sample the factor loading matrix $B$ from $p(B|h_1, h_2, \Theta_{-B})$ without condi-
tioning on the factor $f$. However, without conditioning on the factor $f$, the full conditional distribution is not available in closed form and to sample from it requires Metropolis-Hastings update with high dimensional and complex proposal that is based on numerically maximising the conditional posterior and approximate the hessian of log-posterior at MCMC iteration. In this paper we employ simpler approach based on an ancillarity-sufficiency interweaving strategy (ASIS), in particular deep interweaving strategy, introduced by Kastner et al. (2017). We briefly describe the deep interweaving strategy.

The parameterisation underlying deep interweaving is given by

$$y_t = B^* f_t^* + u_t, \quad f_t^* | h^*_{2j,t} \sim N_k \left(0, diag \left(e^{h^*_{21,t}}, ..., e^{h^*_{2k,t}} \right) \right), \quad (9)$$

with a lower triangular factor loading matrix $B^*$ where $B_{11}^* = 1, ..., B_{kk}^* = 1$. The factor model in equation (1) can be reparameterised into factor model in equation (9) using a simple linear transformation

$$f_t^* = D f_t, B^* = BD^{-1}$$

for $t = 1, ..., T$. The $k$ latent factor volatilities $h_{2j,t}^*$ follow alternative univariate SV models with the level $\mu_{2j} = \log B_{jj}^2$ rather than zero as in factor SV model in Section 2. The transformation of the factor volatilities is given by

$$h_{2j,t}^* = h_{2j,t} + \log B_{jj}^2, t = 0, ..., T, j = 1, ..., k$$

In between step 3 and 4 of the PG and step 5 and 6 of the mixed sampler, we add this following deep interweaving algorithm and perform these steps
independently for each \( j = 1, \ldots, k \)

- Determine the vector \( B^*_j \), where \( B^*_j = B^\text{old}_{s,j} / B^\text{old}_{jj} \) in the \( j \)th column of the transformed factor loading matrix \( B^* \).

- Define \( h^*_2 = h^\text{old}_2 + 2 \log |B^\text{old}_{jj}| \) and sample \( B^\text{new}_{jj} \) from \( p \left( B_{jj} | B^*_j, h^*_2, \phi_{2j}, \tau_{2j}^2 \right) \), see [E] and [Kastner et al. (2017)] for details.

- Update \( B_{..j} = \frac{B^\text{new}_{jj}}{B^\text{old}_{jj}} B^\text{old}_{..j}, f_{..j} = \frac{B^\text{old}_{jj}}{B^\text{new}_{jj}} f^\text{old}_{..j}, \) and \( h^\text{new}_2 = h^\text{old}_2 + 2 \log |B^\text{old}_{jj}|. \)

### 3.7 Normal-Gamma prior distribution for factor loading matrix \( B \)

The standard prior for each element of the factor loading matrix \( B \) is an independent zero-mean normal distribution, \( N(0, \sigma^2 = 1) \) for each \( s = 1, \ldots, p \) and \( j = 1, \ldots, k \). Following [Griffin and Brown (2010)], we model the variance each variance \( \sigma^2_{sj} \) as a random variable and placing hyperprior on \( \sigma^2_{sj} \) as follows

\[
B_{sj} | \sigma^2_{sj} \sim N(0, \sigma^2_{sj}), \quad \sigma^2_{sj} | \lambda_s^2 \sim G(a_s, \lambda_s^2 / 2).
\]

We let \( \lambda_s^2 \sim G(c_s, d_s) \), where \( c_s \) and \( d_s \) are fixed hyperparameters. The choice of \( a \) and \( \lambda \) plays an important role in the estimation. As the shape parameter \( a_s \) decreases these include distributions that place a lot of mass close to zero, and at the same time heavy tails. This implies that choosing small \( a_s \) imposes strong shrinkage towards zero, and choosing large \( a_s \) imposes a little shrinkage towards zero. The Bayesian Lasso prior of [Park and Casella (2008)] is a special case when \( a_s = 1 \).
After completing some algebra, for \( s = 1, \ldots, p \), sample the full conditional distribution of \( \lambda_s^2 | \sigma_s^2 \) from

\[
\lambda_s^2 | \sigma_s^2 \sim G \left( c_s + a_s k_s, d_s + \frac{1}{2} \sum_{s=1}^{k_s} \sigma_{s j}^2 \right)
\]

where \( k_s = \min(s, k) \), and then sample the full conditional distribution of \( \sigma_{s j}^2 | \lambda_s^2, B_{s j} \) from

\[
\sigma_{s j}^2 | \lambda_s^2, B_{s j} \sim GIG \left( a_s - \frac{1}{2}, \lambda_s^2, B_{s j}^2 \right),
\]

where the generalised inverse Gaussian \( GIG (m, k, l) \) distribution has a density proportional to

\[
x^{m-1} \exp \left\{ -\frac{1}{2} (kx + l/x) \right\}.
\]

Let \( \Psi_s = \text{diag} (\sigma_{i1}^{-2}, \sigma_{i2}^{-2}, \ldots, \sigma_{ik_s}^{-2}) \), we can draw

\[
B_{s,.} | f, y_{s,.}, h_{1s,.} \sim N_{k_s} (a_{sT}, b_{sT}),
\]

where \( b_{sT} = (F_s' V_s^{-1} F_s + \Psi_s)^{-1} \) and \( a_{sT} = b_{sT} (V_s^{-1} y_{s,.} - V_s^{-1} \rho_s \tau_{1s} \exp (h_{1s,.} / 2) \eta_{1s,.}) \).

### 3.8 Sampling Idiosyncratic and Factor SV parameters

In the Particle Gibbs (PG) algorithm, each individual SV parameters is drawn from the full conditional distribution \( \mu_{1s} | h_{1s}, \Theta_{-\mu_{1s}}, \phi_{1s} | h_{1s}, \Theta_{-\phi_{1s}}, \rho_s | h_{1s}, \Theta_{-\rho_s}, \) and \( \tau_{1s}^2 | h_{1s}, \Theta_{-\tau_{1s}^2}, \) respectively. For sampling \( \tau_{1s}^2 \), we obtain pro-
posal from inverse gamma distribution with \( scale = (T - 1) / 2 \) and \( shape = M / 2 \), where

\[
M = \left( (1 - \phi_{1s}^2) (h_{1s,1} - \mu_{1s})^2 + \sum_{t=2}^{T} (h_{1s,t} - \mu_{1s} - \phi_{1s} (h_{1s,t-1} - \mu_{1s}))^2 \right).
\]

Then, the acceptance probability is equal to \( \min(1, R) \) with

\[
R = \frac{\prod_{t=1}^{T} P(y_{st} \mid h_{1st}, h_{1st-1}, \theta_{-\phi_{1s}}, \tau_{1s}^2) (1 + \tau_{1s}^2)}{\prod_{t=1}^{T} P(y_{st} \mid h_{1st}, h_{1st-1}, \theta_{-\phi_{1s}}, \tau_{1s}^2) (1 + \tau_{1s}^2)}.
\]

For sampling \( \phi_{1s} \), we obtain proposal from \( q(\phi_{1s} \mid h_{1s}, \Theta_{-\phi_{1s}}) \sim N(c_{\phi}, d_{\phi}) \), where

\[
d_{\phi} = \frac{\tau_{1s}^2}{\sum_{t=2}^{T} (h_{1s,t-1} - \mu_{1s})^2 - (h_{1s,1} - \mu_{1s})^2}
\]

and

\[
c_{\phi} = d_{\phi} \frac{\sum_{t=2}^{T} (h_{1s,t} - \mu_{1s}) (h_{1s,t-1} - \mu_{1s})}{\tau_{1s}^2}.
\]

The acceptance probability is equal to \( \min(1, R) \) with

\[
R = \frac{\prod_{t=1}^{T} P(y_{st} \mid h_{1st}, h_{1st-1}, \theta_{-\phi_{1s}}, \phi_{1s}^*) (1 + \phi_{1s}^2)}{\prod_{t=1}^{T} P(y_{st} \mid h_{1st}, h_{1st-1}, \theta_{-\phi_{1s}}, \phi_{1s}) \sqrt{1 + \phi_{1s}^2}}.
\]

For sampling \( \mu_{1s} \), we obtain proposal from normal distribution \( q(\mu_{1s} \mid h_{1s}, \Theta_{-\mu_{1s}}) \sim N(c_{\mu}, d_{\mu}) \), where

\[
d_{\mu} = \frac{\tau_{1s}^2}{1 - \phi_{1s}^2 + (T - 1) (1 - \phi_{1s})^2}.
\]
and
\[ c_\mu = d_\mu \frac{h_{1s,1} (1 - \phi_{1s}^2) + \sum_{t=2}^{T} h_{1s,t} - \phi_{1s} h_{1s,t-1} + \phi_{1s}^2 h_{1s,t-1} - \phi_{1s} h_{1s,t-1}}{\tau_{1s}^2}. \]

The acceptance probability is equal to \( \min (1, R) \) with
\[ R = \frac{\prod_{t=1}^{T} p(y_{st}|h_{1st}, h_{1st-1}, \theta_{-\mu_{1s}}, \mu_{1s}^*)}{\prod_{t=1}^{T} p(y_{st}|h_{1st}, h_{1st-1}, \theta_{-\mu_{1s}}, \mu_{1s})}. \]

The full conditional distribution can also be derived for \( \mu_{2j}|h_{2j}, \Theta_{-\mu_{2j}}, \phi_{2j}|h_{2j}, \theta_{-\phi_{2j}}, \) and \( \tau_{2j}^2|h_{2j}, \theta_{-\tau_{2j}}. \) For sampling \( \tau_{2s}^2 \), we obtain proposal from inverse gamma distribution with \( \text{scale} = (T - 1)/2 \) and \( \text{shape} = M/2 \), where
\[ M = \left( (1 - \phi_{2s}^2) (h_{2s,1} - \mu_{2s})^2 + \sum_{t=2}^{T} (h_{2s,t} - \mu_{2s} - \phi_{2s} (h_{2s,t-1} - \mu_{2s}))^2 \right). \]

Then, the acceptance probability is equal to \( \min (1, R) \) with
\[ R = \frac{(1 + \tau_{2s}^2)}{(1 + \tau_{2s}^*)}. \]

For sampling \( \phi_{2s} \), we obtain proposal from \( q(\phi_{2s}|h_{2s}, \Theta_{-\phi_{2s}}) \sim N(c_\phi, d_\phi) \), where
\[ d_\phi = \frac{\tau_{2s}^2}{\sum_{t=2}^{T} (h_{2s,t-1} - \mu_{2s})^2 - (h_{2s,1} - \mu_{2s})^2} \]
and
\[ c_\phi = d_\phi \frac{\sum_{t=2}^{T} (h_{2s,t} - \mu_{2s}) (h_{2s,t-1} - \mu_{2s})}{\tau_{2s}^2}. \]
The acceptance probability is equal to \( \min (1, R) \) with

\[
R = \sqrt{\frac{1 + \phi_{1s}^2}{1 + \phi_{1s}^2}}.
\]

For sampling \( \mu_{2s} \), we obtain from normal distribution \( p (\mu_{2s} | h_{2s}, \Theta_{-\mu_{2s}}) \sim N (c_\mu, d_\mu) \), where

\[
d_\mu = \frac{\tau_{2s}^2}{1 - \phi_{2s}^2 + (T - 1)(1 - \phi_{2s})^2}
\]

and

\[
c_\mu = d_\mu h_{2s,1} (1 - \phi_{2s}^2) + \sum_{t=2}^T h_{2s,t} - \phi_{2s} h_{2s,t} + \phi_{2s}^2 h_{2s,t-1} - \phi_{2s} h_{2s,t-1} \frac{\tau_{2s}^2}{r_{2s}^2}.
\]

Next, we discuss about the Hamiltonian Monte Carlo proposal to sample the parameter \( \rho_s \) for \( s = 1, ..., p \) from conditional posterior density \( \tilde{\pi} (\rho_s | h_{1t}, h_{2t}, \Theta_{-\rho_s}) \). It can be used to generate distant proposals for the Particle Metropolis within Gibbs algorithm to avoid the slow exploration behaviour that results from simple random walk proposals. Suppose we want to sample from a distribution with pdf proportional to \( \exp (L (\rho_s)) \), where \( L (\rho_s) = \log \tilde{\pi} (\rho_s | h_{1t}, h_{2t}, \Theta_{-\rho_s}) \) is the logarithm of the conditional posterior density of \( \rho_s \) (up to a normalising constant). In Hamiltonian Monte Carlo (Neal, 2011), we augment an auxiliary momentum variable \( r_s \) for each parameter \( \rho_s \) with density \( p (r_s) = N (r_s | 0, 1) \). The joint density follows in factorised form as

\[
p (\rho_s, r_s | h_{1t}, h_{2t}, \Theta_{-\rho_s}, y) \propto \exp \left( L (\rho_s) - \frac{1}{2} r_s^2 \right) \propto \exp (-H (\rho_s, r_s)).
\]
This augmented model can be interpreted as Hamiltonian system where \( \rho_s \) denotes a parameter's position, \( r_s \) denotes the momentum, \( \mathcal{L}(\rho_s) \) is a negative potential energy function of the parameters \( \rho_s \), and \( \frac{1}{2} r_s^2 \) is the kinetic energy function of the parameters, and \( -H(\rho_s, r_s) \) is the total negative energy of the parameters and momentum variables and the function \( H(\rho_s, r_s) \) is often called Hamiltonian. At the end of this algorithm, we will discard the momentum variable \( r_s \), obtaining a new \( \rho_s \) that is still distributed as \( \exp(\mathcal{L}(\rho_s)) \). Equation (10) is factorisable because the conditional distribution of momentum does not depend on the parameter values.

In the Hamiltonian dynamics, the parameters \( \rho_j \) and the momentum variables \( r_j \) are moved along a continuous time \( t \) according to the following differential equations

\[
\frac{d\rho_s}{dt} = \frac{\partial H}{\partial r_s} = r_s \\
\frac{dr_s}{dt} = -\frac{\partial H}{\partial \rho_s} = \nabla_{\rho_s} \mathcal{L}(\rho_s),
\]

where \( \nabla_{\rho_s} \) denotes the gradient with respect to the parameter \( \rho_s \). In implementation, this Hamiltonian dynamics needs to be approximated by discretised time, using small step size \( \epsilon \). We can simulate the evolution over time of \( (\rho_s, r_s) \) via “leapfrog” integrator. The one step Leapfrog update is given as

\[
\begin{align*}
    r_s \left( t + \frac{\epsilon}{2} \right) &= r_s (t) + \epsilon \nabla_{\rho_s} \mathcal{L}(\rho_s (t)) / 2 \\
    \rho_s (t + \epsilon) &= \rho_s (t) + \epsilon r_s \left( t + \frac{\epsilon}{2} \right) \\
    r_s (t + \epsilon) &= r_s (t + \epsilon/2) + \epsilon \nabla_{\rho_s} \mathcal{L}(\rho_s (t + \epsilon)) / 2
\end{align*}
\]
Each leapfrog step is time reversible by negation of the step size, $\epsilon$. Since leapfrog integrator provides mapping $(\rho_s, r_s) \rightarrow (\rho^*_s, r^*_s)$ that are both time-reversible and volume preserving (Neal, 2011), then the Metropolis-Hastings algorithm with acceptance probability given by $\min \left( 1, \frac{\exp(L(\rho^*_s) - \frac{1}{2}r^*_s^2)}{\exp(L(\rho_s) - \frac{1}{2}r_s^2)} \right)$ produces an ergodic, time reversible Markov chain that satisfies detailed balance and whose stationary density is $p(\rho_s, r_s|h_{1t}, h_{2t}, \Theta_{-\rho_s}, y)$ (Liu, 2001; Neal, 1996). A summary of the Hamiltonian Monte Carlo algorithm is given in Algorithm 1.

### Algorithm 1 Hamiltonian Monte Carlo

Given $\rho^0_s$, $\epsilon$, Leap, $S$, where Leap is the number of Leapfrog updates.

- For $l = 1$ to $L$
  - Sample $r^0_s \sim N(0, 1)$.
  - Set $\rho^l_s \leftarrow \rho^{l-1}_s$, $\rho^*_s \leftarrow \rho^{l-1}_s$, and $r^*_s \leftarrow r^0_s$.
    - For $i = 1$ to Leap
      - Set $(\rho^*_s, r^*_s) \leftarrow \text{Leapfrog}(\rho^*_s, r^*_s, \epsilon)$
    end for
  end for

With probability $\alpha = \min \left( 1, \frac{\exp(L(\rho^*_s) - \frac{1}{2}r^*_s^2)}{\exp(L(\rho_s) - \frac{1}{2}r_s^2)} \right)$, set $\rho^l_s = \rho^*_s$, $r^l_s = -r^*_s$.

The performance of HMC depends strongly on choosing suitable values for $\epsilon$ and $L$. The step size $\epsilon$ determines how well the leapfrog integration can approximate Hamiltonian dynamics. If we set $\epsilon$ too large, then the simulation error is large and yield low acceptance rate. However, if we set $\epsilon$ too small, then the computational burden is high to obtain distant proposals. In the same way, if we set $L$ too small, the proposal will be close to the current value of parameters, resulting in undesirable random walk behaviour and
slow mixing. If $L$ is too large, HMC will generate trajectories that retrace back their steps. In this paper, we use No-U-Turn sampler (NUTS) with dual averaging algorithm developed by Hoffman and Gelman (2014) and Nesterov (2009), respectively, that still leaves the target density invariant and satisfies time reversibility to adaptively select $L$ and $\epsilon$, respectively.

In the Mixed sampler, for sampling $\tau_{1s}^2$ for $s = 1, ..., p$ and $\tau_{2j}^2$ for $j = 1, ..., k$ are done in Pseudo Marginal (PM) step. In PM step, the gradient of log-posterior cannot be computed exactly and need to be estimated. The efficiency of PMMH will then depend crucially on how accurately we can estimate the gradient of log-posterior. If the error in the estimate of the gradient is too large, then there will be no advantage in using proposals with derivatives information over a random walk proposal (Nemeth et al., 2016). In this paper, we employ a single step of the leapfrog algorithm that has an update of the form

$$
\tau_j^2(t + \epsilon) = \tau_j^2(t) + \frac{\epsilon^2}{2} \nabla_{\tau_j^2} \mathcal{L}_{\tau_j^2} \left( \tau_j^2(t) \right) + \epsilon r_j(t)
$$

$$
r_j(t + \epsilon) = r_j(t) + \frac{\epsilon}{2} \nabla_{\tau_j^2} \mathcal{L}_{\tau_j^2} \left( \tau_j^2(t) \right) + \frac{\epsilon}{2} \nabla_{\tau_j^2} \mathcal{L}_{\tau_j^2} \left( \tau_j^2(t + \epsilon) \right)
$$

This update is a discrete pre-conditioned Langevin diffusion as employed in Metropolis Adjusted Langevin Algorithm (MALA) (Roberts and Stramer, 2003). The algorithm to estimate the gradient of log-posterior is given in Appendix 2.
4 Simulation Study

In order to compare different sampling schemes in terms of sampling efficiency, a simple simulation study is conducted. We use $T = 1000$ periods of data using a model with $k = 2$ factors and $p = 10$ dimensions. We set $\phi_{1s} = 0.98$, $\rho_{1s} = -0.1$, $\mu_{1s} = 0.01$, and $\tau_{1s}^2 = 0.05$, for all $s = 1, \ldots, p$, and also $\phi_{2j} = 0.98$, and $\tau_{2j}^2 = 0.05$ for $j = 1, \ldots, k$, and

$$B' = \begin{bmatrix} 1 & 0.9 & 0.8 & 0.7 & 0.6 & 0.5 & 0.4 & 0.3 & 0.2 & 0.1 \\ 0 & 1 & 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 & 0.7 & 0.8 \end{bmatrix}$$

In this simulation study, the total number of MCMC iterations is 15000, with the first 5000 discarded as burn in replications. The number of particles is 500. We conduct a simulation study in order to compare three different approaches to estimation: PG, particle Gibbs with additional ancestor sampling step (PGAS), and Mixed samplers. To define our measure of the inefficiency of different sampling schemes that takes computing time into account, we first define the Integrated Autocorrelation Time ($IACT_\theta$). For a univariate parameter $\theta$, $IACT$ is estimated by

$$IACT(\theta_{1:M}) := 1 + 2 \sum_{t=1}^{L} \hat{\rho}_t(\theta_{1:M}),$$

where $\hat{\rho}_t(\theta_{1:M})$ denotes the empirical autocorrelation at lag $t$ of $\theta_{1:M}$ (after the burnin periods have been discarded). A lower value of $IACT$ indicates that the chain mixed well. Here, $L$ is chosen as the first index for which the empirical autocorrelation satisfies $|\hat{\rho}_t(\theta_{1:M})| < 2/\sqrt{M}$, i.e. when the empirical autocorrelation coefficient is statistically insignificant. Our measure of
inefficiency of sampling scheme is the time normalised variance

\[ TNV := IACT_{\text{mean}} \times CT, \quad (11) \]

where \( CT \) is the computing time and \( IACT_{\text{mean}} \) be the mean of IACT’s over all parameters.

Tables 7, 8, and 9 in the Appendix show the inefficiency factors for all the parameters of factor MSV with leverage for the mixed, PG, and PGAS samplers, respectively. Table 1 summarises the simulation results and shows that the mixed sampler is more than 4 times more efficient than the PG sampler and more than 2 times more efficient than PGAS sampler. PGAS sampler is 2 times more efficient than PG sampler. Figures 1 and 2 show the estimated trajectory of idiosyncratic log-variances \( h_{1s,t} \) for \( s = 1, \ldots, p \) and factor log variances \( h_{2j,t} \) for \( j = 1, \ldots, k \) from mixed sampler. They estimate the true trajectory of idiosyncratic and factor log variances well.

Table 1: Comparison of Different Sampling Schemes. Time in seconds (\( N = 500 \)) PG: Particle Gibbs, PGAS: Particle Gibbs with additional ancestor sampling step, Mix.: \( PMMH \left( \tau_{11}, \ldots, \tau_{1p}, \tau_{21}, \ldots, \tau_{2k} \right) + PG (\mu_{1s}, \phi_{1s}, \rho_{1s}, s = 1, \ldots, p; \mu_{1j}, \phi_{1j}, j = 1, \ldots, k; B, f) \).

|            | PG   | PGAS  | Mix. |
|------------|------|-------|------|
| Time       | 1.07 | 1.18  | 1.75 |
| \( IACT_{\text{Mean}} \) | 70.45 | 31.05 | 10.42 |
| TNV        | 75.38 | 36.64 | 18.23 |
| Rel. TNV   | 4.13 | 2.01  | 1    |
Figure 1: Plots of estimated trajectories against the true trajectories of idiosyncratic log-variances $h_{1s,t}$ for $s = 1, \ldots, 10$
5 Empirical Application to US stock returns

We applied the estimation described above to a sample of daily US stock returns. The data, provided by Kenneth French, consisted of the daily returns for 16 industry portfolios and it is given in Table 2. We used a sample running from 13rd August 1993 to 17th July 2001, a total of 2000 observations. We consider models with $1 - 4$ factors. The total number of MCMC iterations is 15000, with the first 5000 discarded as burn in replications. The number of particles is 1000. Table 3 summarises the estimation results for $1 - 4$
factors with standard normal prior $N(0, 1)$ for each element of factor loading matrix. It is clear that mixed sampler is always better than the PGAS sampler for all cases. Given the outcome of these comparisons, the remaining analysis are based on the results from the mixed sampler.

Table 2: List of Industry Portfolios

| Industry  | Industry  | Industry  | Industry |
|-----------|-----------|-----------|----------|
| 1         | Automobiles | 5         | Fabricated |
| 2         | Chemicals  | 6         | Banks     |
| 3         | Textiles   | 7         | Food      |
| 4         | Drugs, Soap, etc. | 8        | Machinery |
| 9         | Mining     | 10        | Oil       |
| 11        | Other      | 12        | Retail    |
| 13        | Steel      | 14        | Transportation |
| 15        | Consumer Durables | 16       | Utilities |

Table 3: Comparison of Different Sampling Schemes for Factor MSV with leverage. Time in seconds ($N = 1000$)

|          | 1 Factor | 2 Factor | 3 factor | 4 factor |
|----------|----------|----------|----------|----------|
| PGAS     | Time     | 2.63     | 2.65     | 2.70     | 2.75     |
| Mixed    | 4.35     | 4.40     | 4.45     | 4.50     |
| IACT     | Mean     | 132.04   | 122.04   | 111.42   | 101.80   |
|          | 15.96    | 16.69    | 29.76    | 117.05   |
| TNV      | 347.27   | 323.41   | 300.83   | 401.80   |
|          | 69.43    | 73.44    | 132.43   | 117.05   |
| Rel. TNV | 5.00     | 4.40     | 2.27     | 3.43     |
|          | 1        | 1        | 1        | 1        |

In this paper, we select the number of factor using deviance information criterion (DIC). In the seminal paper, Spiegelhalter et al. (2002) proposed a concept of deviance information criterion (DIC) for model comparison. The model selection is based on the deviance, which is given by

$$D(\theta) = -2 \log p(y|\Theta) + 2 \log h(y),$$

where $p(y|\Theta)$ is the likelihood function of the parametric model and $h(y)$ is some fully specified standardising term that is only a function of the data.
For model comparison purposes, we set $h(y) = 1$ for all models. The effective number of parameters $p_D$ is defined as

$$p_D = \overline{D(\theta)} - D(\hat{\theta}),$$

where

$$\overline{D(\theta)} = -2E_\theta [\log p(y|\Theta) | y] + 2 \log h(y)$$

is the posterior mean deviance and $\hat{\theta}$ is an estimate of $\theta$, which is usually set to be posterior mode or mean. Thus, the deviance information criterion is defined as

$$DIC = \overline{D(\theta)} + p_D$$

$$= -4E_\theta [\log p(y|\Theta) | y] + 2 \log p(y|\hat{\theta}).$$

Given a set of models for the given data, the preferred model is the one with the minimum DIC value. Celeux et al. (2006) pointed that there are a number of alternative definitions of the DIC in the latent variable models. In this paper, we follow the definitions of DIC that are based on conditional likelihood, which is given by:

$$DIC_7 = -4E_{\theta,Z} [\log p(y|\Theta, Z) | y] + 2 \log p(y|\hat{Z}, \hat{\theta}),$$

where $\left(\hat{Z}, \hat{\theta}\right)$ is the joint maximum a posterior (MAP) estimate, and $Z$ consists of all the latent volatilities in the model. The first term on the right hand side can be estimated by averaging the log-conditional likelihoods $\log p(y|\Theta, Z)$ over the posterior draws of $(Z, \theta)$.

Table 4 shows the DIC values for 1-4 factors with different priors for ele-
ments of factor loading matrix. We compare standard normal prior $N(0, 1)$, normal-gamma prior with $a_s = 1$ and $a_s = 0.5$ for all $s = 1, ..., p$. We set the hyper-hyperparameters $c_s = d_s = 2$ for all $s = 1, ..., p$. The best model is the four-factor model with normal-gamma prior ($a_s = 0.5$). We found no evidence for the leverage effects in the dataset, with posterior credible intervals of each $\rho_s$ including zero, except the mining industry. We begin by discussing the log-variances of the latent factors, visualised in Figure 3 and the corresponding posterior means of the factor loadings given in Table 6. The first factor can clearly be interpreted as the mining industry driven one. The automobiles, transportation and steel industries also load very highly on this factor. Factor 1’s log-variance appears quite volatile through out the sample period. Factor 2’s log-variances appears slightly less volatile than the first factor, and generally very smooth and more persistent. It is also driven by mining industry. Retail and steel industries also loads very highly on this factor. The third factor volatility shows a similar overall pattern as the second. The fourth factor volatility shows a similar pattern as the first, but slightly more volatile. Figure 4 shows the marginal posterior means of univariate volatilities for all 16 US stock returns from 13rd August 1993 to 17th July 2001. In general, the log-volatilities are generally smooth and less volatile, except, the fabricated, utilities, and bank industries.

Table 4: Selecting Number of Factors based on mixed sampler using DIC criterion

| Number of Factors | N (0, 1) prior | N-G prior ($a_s = 1$) | N-G prior ($a_s = 0.5$) |
|-------------------|---------------|-----------------------|------------------------|
| 1                 | 73893.38      | 73965.72              | NA                     |
| 2                 | 72607.25      | 72466.73              | NA                     |
| 3                 | 71594.45      | 71626.78              | 71490.18               |
| 4                 | 71288.95      | 71334.78              | 71285.17               |
Table 5: The Inefficiency of factor loading matrix for different priors

| Number of Factors | N (0, 1) prior | N-G prior ($a_s = 1$) | N-G prior ($a_s = 0.5$) |
|-------------------|----------------|-----------------------|-------------------------|
| 3                 | 18.65          | 29.56                 | 22.46                   |
| 4                 | 26.77          | 27.01                 | 28.95                   |

Table 6: Posterior Means of factor loading matrix of four factor models with N-G prior ($a_s = 0.5$)

| Category                | Value 1  | Value 2  | Value 3  | Value 4  |
|-------------------------|----------|----------|----------|----------|
| Automobiles             | 0.73     | 0        | 0        | 0        |
| Chemicals               | 0.68     | -0.59    | 0        | 0        |
| Textiles                | 0.58     | 0.05     | 0.21     | 0        |
| Drugs, Soap, etc.       | 0.58     | 0.48     | -0.26    | 0.42     |
| Fabricated              | 0.62     | -0.01    | 0.21     | 0.03     |
| Banks                   | 0.56     | -0.50    | 0.24     | 0.02     |
| Food                    | 0.71     | 0.21     | 0.17     | 0.19     |
| Machinery               | 0.51     | 0.10     | -0.32    | 0.38     |
| Mining                  | 0.83     | 0.71     | 1.61     | 0.11     |
| Oil                     | 0.39     | -0.05    | 0.11     | -0.11    |
| Other                   | 0.48     | -0.59    | -0.12    | 0.17     |
| Retail                  | 0.70     | 0.57     | 0.94     | 0.17     |
| Steel                   | 0.72     | 0.61     | 0.01     | 0.09     |
| Transportation          | 0.72     | -0.67    | 0.85     | -0.10    |
| Consumer Durables       | 0.70     | -0.25    | 0.11     | 0.03     |
| Utilities               | 0.33     | -0.15    | -0.16    | 0.21     |
Figure 3: Marginal Posteriors of the factor log-variances $(mean \pm 2sd)$, $h_{2j,t}$ for $j = 1, 2, 3, 4$
6 Conclusions

Estimating time-varying covariance matrices of financial times series is an active area of research. In this paper, we employ factor multivariate stochastic volatility (factor MSV) models with leverage because they are able to model the volatility dynamics of a large system of financial or economic time series when the common features in these series can be captured by a small number of latent factors. To conduct efficient and reliable statistical inference, we propose a sampler based on recent developments in PMCMC methods. Our
article demonstrates that a version of general PMCMC sampler of Mendes et al. (2016) provides a flexible and efficient framework to carry out inference on factor MSV models with leverage. The resulting parameter estimates mix well. The proposed method is illustrated using simulated and real datasets.

A Generic Sequential Monte Carlo (SMC) Algorithm

1. For $t = 1$
   
   (a) Sample $h_1^i$ from $m_1^\theta (h)$, for $i = 1, ..., N$
   
   (b) Calculate the importance weights

   $$w_1^i = \frac{p_\theta (y_1 | h_1^i) p_\theta (h_1^i)}{m_1^\theta (h_1^i)}, i = 1, ..., N.$$ 

   and normalised them to obtain $\bar{w}_1^{1:N}$.

2. For $t > 1$
   
   (a) Sample the ancestral indices $C_{t-1}^{1:N} \sim M (a_{t-1}^{1:N} | \bar{w}_{t-1}^{1:N})$.
   
   (b) Sample $h_t^i$ from $m_t^\theta (h_t | h_t^{a_{t-1}}, h_t^{(i)})$, $i = 1, ..., N$.
   
   (c) Set $h_{1:t}^{(i)} = \left( h_{1:t-1}^{a_{t-1}}, h_t^{(i)} \right)$.
   
   (d) Calculate the importance weights

   $$w_t^i = \frac{p_\theta (y_1 | h_t^i) p_\theta (h_t^i | h_t^{a_{t-1}}, h_t^{(i)})}{m_t^\theta (h_t^i | h_t^{a_{t-1}})}, i = 1, ..., N.$$
and normalised them to obtain $\overline{w}_t^{1:N}$.

**B Conditional Sequential Monte Carlo**

In the following, we describe the steps of conditional particle filter to draw $(h_{1:T})$ [Andrieu et al. 2010].

1. Fix $h_{1:T}^j$ and $C_{1:T-1}^j$.

2. For $t = 1$

   (a) Sample $h_1^i$ from $m_\theta^q(h_1)$, for $i \in \{1, ..., N\} \setminus \{b_1^j\}$.

   (b) Calculate the importance weights

   $w_1^i = \frac{p_\theta(y_1|h_i^1) p_\theta(h_i^1)}{m_\theta^q(h_i^1)}, i = 1, ..., N.$

   and normalised them to obtain $\overline{w}_1^{1:N}$.

3. For $t > 1$

   (a) Sample the ancestral indices $C_{t-1}^{(-b_t^j)} \sim M \left( a_{t-1}^{(-b_t^j)} | \overline{w}_{t-1}^{1:N} \right)$.

   (b) Sample $h_t^i$ from $m_\theta^q\left(h_t|h_{t-1}^{a_t^i-1}\right)$, $i = 1, ..., N \setminus \{b_t^j\}$.

   (c) Set $h_{1:t}^{(i)} = \left(h_{1:t-1}^{(a_t^i-1)}, h_t^{(i)}\right)$, $i = 1, ..., N$.

   (d) Calculate the importance weights

   $w_t^i = \frac{p_\theta(y_t|h_t^i) p_\theta(h_t^i|h_{t-1}^{a_t^i-1})}{m_\theta^q\left(h_t^i|h_{t-1}^{a_t^i-1}\right)}, i = 1, ..., N.$

   and normalised them to obtain $\overline{w}_t^{1:N}$. 
C Conditional Sequential Monte Carlo for Ancestral Sampling Algorithm

In the following, we describe the steps of conditional particle filter with ancestor sampling to draw \( (h_{1:T}) \) (Lindsten et al., 2014).

1. Fix \( h_{1:T}^j \) and \( C_{1:T-1}^j \).

2. For \( t = 1 \)

   (a) Sample \( h_i^1 \) from \( m_1^\theta(h_1) \), for \( i \in \{1, ..., N\} \setminus \{b_1^j\} \).

   (b) Calculate the importance weights

   \[
   w_i^1 = \frac{p_\theta(y_1|h_i^1) p_\theta(h_i^1)}{m_1^\theta(h_i^1)}, \ i = 1, ..., N. 
   \]

   and normalise them to obtain \( \overline{w}_{1:N}^1 \).

3. For \( t > 1 \)

   (a) Sample the ancestral indices \( C_{t-1}^{-b_t^j} \sim M \left( a(-b_t^j) | \overline{w}_{t-1}^1 \right) \).

   (b) Draw \( b_{t-1}^{(j)} \) from \( p \left( b_{t-1}^{(j)} = k \right) \propto w_{t-1}^{(k)} p_\theta \left( h_{t-1}^{b_{t-1}^{(j)}} | h_{t-1}^{(k)} \right) \).

   (c) Sample \( h_i^t \) from \( m_t^\theta \left( h_t | h_{t-1}^{a_{t-1}^i} \right), \ i = 1, ..., N \setminus \{b_t^j\} \).

   (d) Calculate the importance weights

   \[
   w_i^t = \frac{p_\theta(y_t|h_i^t) p_\theta \left( h_i^t | h_{t-1}^{a_{t-1}^i} \right)}{m_t^\theta \left( h_i^t | h_{t-1}^{a_{t-1}^i} \right)}, \ i = 1, ..., N. 
   \]

   and normalised them to obtain \( \overline{w}_{1:N}^t \).
D Estimating Gradients of Log-Posterior using Particle Filter

This section presents the construction of the proposal density in Pseudo Marginal (PM) step that makes use of the derivatives of the log likelihood. Poyiadjis et al. (2011) were the first to show how the particle filter methods can be used to estimate the derivatives of the log likelihood for state space models. Their methods might suffer from a computational cost that is quadratic in the number of particles. Nemeth et al. (2016) proposed an alternative method whose computational cost is linear in the number of particles. They use a combination of kernel density estimation and Rao-Blackwellisation to reduce the Monte Carlo error of the estimates.

For non-linear and non-Gaussian state space models it is not possible to obtain the score and observed information matrix exactly. If it is possible to obtain a particle approximation of \( p(h_{1:T}|y_{1:T}, \theta) \), then this approximation can be used to estimate the score vector \( \nabla \log p(y_{1:T}|\theta) \) using Fisher’s identity (Cappe et al., 2005)

\[
\nabla \log p(y_{1:T}|\theta) = \int \nabla \log p(h_{1:T}, y_{1:T}|\theta) p(h_{1:T}|y_{1:T}, \theta) dh_{1:T}.
\]

where

\[
\nabla \log p(h_{1:T}, y_{1:T}|\theta) = \sum_{t=1}^{T} \{ \nabla \log g_\theta(y_t|h_t) + \nabla \log p_\theta(h_t|h_{t-1}) \}.
\]

The algorithm to estimate Gradient is given in [2].
Algorithm 2 Algorithm to estimate Gradient and Hessian Matrix [Nemeth et al. (2016)]

- Initialise: set $m_0^{(i)} = 0$ and $n_0^{(i)} = 0$ for $i = 1, \ldots, N$, where $N$ is the number of particles, and $S_0 = 0$ and $B_0 = 0$.

- At iteration $t = 1, \ldots, T$
  
  - Run the Particle Filter to obtain $\{h_t^{(i)}\}_{i=1}^N$, $\{a_i\}_{i=1}^N$, and $\{w_t^{(i)}\}_{i=1}^N$, where $w_t^{(i)}$ is the weight of particle $i$ at time $t$. $a_i$ is the ancestor index of particle $i$ at time $t - 1$.
  
  - Normalised the weights $W_t^{(i)} = \frac{w_t^{(i)}}{\sum w_t^{(i)}}$.

- Update the $m_t^{(i)}$ and $n_t^{(i)}$ as follows
  
  $$m_t^{(i)} = \lambda m_{t-1}^{(k_i)} + (1 - \lambda) S_{t-1} + \nabla \log g_\theta(y_t|h_t^{(i)}) + \nabla \log p_\theta(h_t^{(i)}|h_{t-1}^{(k_i)})$$

- Update the score vector
  
  $$S_t = \sum_{i=1}^N W_t^{(i)} m_t^{(i)}$$

Setting $\lambda = 1$ gives the [Poyiadjis et al. (2011)] algorithm. [Nemeth et al. (2016)] shows that bias and variance of both score estimate vary according to $\lambda$. Reducing the value of $\lambda$ has the effect of increasing the bias, but it reduces the Monte Carlo variance of estimates. They also show that by setting $\lambda \approx 0.95$ will produce an estimate for the score with linearly increasing variance and minimal bias. We use $\lambda = 0.95$ in all our application.
E Sampling the scaling parameters in the deep interweaving representations

In the deep interweaving representation, we sample the scaling parameter $B_{jj}$ indirectly through $\mu_{2j}$, $j = 1, \ldots, k$. The implied prior $p(\mu_{2j}) \propto \exp(\mu_{2j}/2 - \exp(\mu_{2j})/2)$ and the density $p(B_{*j}^j|\mu_{2j}) \sim N_{k_j}(0, \exp(-\mu_{2j}) I_{k_j})$ and the likelihood given by Equation (3) yields the posterior

$$p(\mu_{2j}|B_{*j}^j, h_{2j}, \phi_{2j}, \tau_{2j}^2) \propto p(h_{2j},|\mu_{2j}, \phi_{2j}, \tau_{2j}^2) p(B_{*j}^j|\mu_{2j}) p(\mu_{2j}),$$

which is not in recognisable form. As in Kastner et al. (2017), we draw a proposal for $\mu_{2j}^{prop}$ from $N(A,B)$ where

$$A = \frac{\sum_{t=2}^{T-1} h_{2j,t}^* + (h_{2j,1}^* - \phi_{2j} h_{2j,1}) / (1 - \phi_{2j})}{T + 1/B_0}, B = \frac{\tau_{2j}^2 / (1 - \phi_{2j})^2}{T + 1/B_0}.$$

Denoting the current value $\mu_{2j}$ by $\mu_{2j}^{old}$, the new value $\mu_{2j}^{prop}$ gets accepted with probability $\min(1, R)$, where

$$R = \frac{p(\mu_{2j}^{prop}) p(h_{2j,1}^*, \phi_{2j}, \tau_{2j}^2) p(B_{*j}^j|\mu_{2j}^{prop})}{p(\mu_{2j}^{old}) p(h_{2j,1}^*, \phi_{2j}, \tau_{2j}^2) p(B_{*j}^j|\mu_{2j}^{old})} \times \frac{p_{aux}(\mu_{2j}^{old} | \phi_{2j}, \tau_{2j}^2)}{p_{aux}(\mu_{2j}^{prop} | \phi_{2j}, \tau_{2j}^2)},$$

where

$$p_{aux}(\mu_{2j}^{old} | \phi_{2j}, \tau_{2j}^2) \sim N(0, B_0 \tau_{2j}^2 / (1 - \phi_{2j})^2).$$

When normal-gamma prior is used, the density of $B_{*j}^j|\mu_{2j}$ is given by

$$p(B_{*j}^j|\mu_{2j}) \sim N_{k_j}(0, \Psi_j \exp(-\mu_{2j})).$$
where $\Psi_j = \text{diag}(\sigma^2_{1j}, ..., \sigma^2_{pj})$.

## F Simulation Results

Table 7: Multivariate Factor SV model with leverage Simulation Results:
Mixed Sampler with $N = 500$

|     | IACT | IACT | IACT | IACT | IACT | IACT |
|-----|------|------|------|------|------|------|
| $\phi_1$ | 12.16 | 17.92 | $\mu_1$ | 1.89 | $\rho_1$ | 12.31 | $\beta_{11}$ | 6.33 | $\beta_{22}$ | 5.86 |
| $\phi_2$ | 8.85 | 16.43 | $\mu_2$ | 1.14 | $\rho_2$ | 20.25 | $\beta_{21}$ | 6.32 | $\beta_{23}$ | 5.41 |
| $\phi_3$ | 11.51 | 14.59 | $\mu_3$ | 1.36 | $\rho_3$ | 23.23 | $\beta_{31}$ | 6.46 | $\beta_{24}$ | 6.24 |
| $\phi_4$ | 5.33 | 15.88 | $\mu_4$ | 1.05 | $\rho_4$ | 15.85 | $\beta_{41}$ | 6.37 | $\beta_{25}$ | 5.13 |
| $\phi_5$ | 4.70 | 8.83 | $\mu_5$ | 1.02 | $\rho_5$ | 30.09 | $\beta_{51}$ | 6.44 | $\beta_{26}$ | 5.12 |
| $\phi_6$ | 14.37 | 19.56 | $\mu_6$ | 1.31 | $\rho_6$ | 24.93 | $\beta_{61}$ | 6.30 | $\beta_{27}$ | 5.52 |
| $\phi_7$ | 6.78 | 13.70 | $\mu_7$ | 1.10 | $\rho_7$ | 37.98 | $\beta_{71}$ | 6.32 | $\beta_{28}$ | 5.84 |
| $\phi_8$ | 4.66 | 9.25 | $\mu_8$ | 1.04 | $\rho_8$ | 30.94 | $\beta_{81}$ | 6.22 | $\beta_{29}$ | 6.11 |
| $\phi_9$ | 6.17 | 13.30 | $\mu_9$ | 1.04 | $\rho_9$ | 30.43 | $\beta_{91}$ | 5.72 | $\beta_{2,10}$ | 6.27 |
| $\phi_{10}$ | 8.12 | 16.65 | $\mu_{10}$ | 1.45 | $\rho_{10}$ | 19.90 | $\beta_{10,1}$ | 6.26 |
| $\phi_{f1}$ | 10.36 | 12.64 |  |  |  |  |  |  |
| $\phi_{f2}$ | 13.96 | 18.24 |  |  |  |  |  |  |
Table 8: Multivariate Factor SV model with leverage Simulation Results:
PGAS Sampler with \( N = 500 \)

| \( \phi \) | IACT | IACT | IACT | IACT | IACT |
|---|---|---|---|---|---|
| \( \phi_1 \) | 87.72 | \( \tau_1^2 \) | 229.96 | \( \mu_1 \) | 3.07 | \( \rho_1 \) | 17.16 | \( \beta_{11} \) | 5.31 | \( \beta_{22} \) | 7.17 |
| \( \phi_2 \) | 128.27 | \( \tau_2^2 \) | 273.72 | \( \mu_2 \) | 1.04 | \( \rho_2 \) | 23.46 | \( \beta_{21} \) | 5.47 | \( \beta_{23} \) | 6.71 |
| \( \phi_3 \) | 96.86 | \( \tau_3^2 \) | 188.35 | \( \mu_3 \) | 1.64 | \( \rho_3 \) | 32.85 | \( \beta_{31} \) | 5.43 | \( \beta_{24} \) | 6.05 |
| \( \phi_4 \) | 83.42 | \( \tau_4^2 \) | 336.61 | \( \mu_4 \) | 1.02 | \( \rho_4 \) | 24.44 | \( \beta_{41} \) | 5.39 | \( \beta_{25} \) | 4.94 |
| \( \phi_5 \) | 30.09 | \( \tau_5^2 \) | 214.09 | \( \mu_5 \) | 1.08 | \( \rho_5 \) | 40.87 | \( \beta_{51} \) | 5.39 | \( \beta_{26} \) | 6.26 |
| \( \phi_6 \) | 271.72 | \( \tau_6^2 \) | 510.85 | \( \mu_6 \) | 1.23 | \( \rho_6 \) | 32.51 | \( \beta_{61} \) | 5.44 | \( \beta_{27} \) | 7.24 |
| \( \phi_7 \) | 53.51 | \( \tau_7^2 \) | 234.28 | \( \mu_7 \) | 1.12 | \( \rho_7 \) | 45.32 | \( \beta_{71} \) | 5.45 | \( \beta_{28} \) | 7.64 |
| \( \phi_8 \) | 27.35 | \( \tau_8^2 \) | 174.22 | \( \mu_8 \) | 1.02 | \( \rho_8 \) | 30.07 | \( \beta_{81} \) | 5.48 | \( \beta_{29} \) | 7.52 |
| \( \phi_9 \) | 49.86 | \( \tau_9^2 \) | 188.67 | \( \mu_9 \) | 1.02 | \( \rho_9 \) | 42.76 | \( \beta_{91} \) | 5.28 | \( \beta_{2,10} \) | 7.92 |
| \( \phi_{10} \) | 95.05 | \( \tau_{10}^2 \) | 259.60 | \( \mu_{10} \) | 1.17 | \( \rho_{10} \) | 19.33 | \( \beta_{10,1} \) | 6.34 |
| \( \phi_{f1} \) | 42.32 | \( \tau_{f1}^2 \) | 121.84 |
| \( \phi_{f2} \) | 99.06 | \( \tau_{f2}^2 \) | 202.43 |

Table 9: Multivariate Factor SV model with leverage Simulation Results:
PGAS Sampler with \( N = 500 \)

| \( \phi \) | IACT | IACT | IACT | IACT | IACT | IACT |
|---|---|---|---|---|---|---|
| \( \phi_1 \) | 44.55 | \( \tau_1^2 \) | 100.13 | \( \mu_1 \) | 1.26 | \( \rho_1 \) | 9.56 | \( \beta_{11} \) | 5.53 | \( \beta_{22} \) | 7.20 |
| \( \phi_2 \) | 22.34 | \( \tau_2^2 \) | 81.51 | \( \mu_2 \) | 1.18 | \( \rho_2 \) | 14.33 | \( \beta_{21} \) | 5.54 | \( \beta_{23} \) | 4.81 |
| \( \phi_3 \) | 57.44 | \( \tau_3^2 \) | 100.61 | \( \mu_3 \) | 1.36 | \( \rho_3 \) | 18.13 | \( \beta_{31} \) | 5.67 | \( \beta_{24} \) | 5.79 |
| \( \phi_4 \) | 34.07 | \( \tau_4^2 \) | 126.11 | \( \mu_4 \) | 1.02 | \( \rho_4 \) | 11.96 | \( \beta_{41} \) | 5.56 | \( \beta_{25} \) | 5.30 |
| \( \phi_5 \) | 24.27 | \( \tau_5^2 \) | 72.48 | \( \mu_5 \) | 1.11 | \( \rho_5 \) | 24.22 | \( \beta_{51} \) | 5.61 | \( \beta_{26} \) | 6.92 |
| \( \phi_6 \) | 49.25 | \( \tau_6^2 \) | 103.15 | \( \mu_6 \) | 1.22 | \( \rho_6 \) | 21.72 | \( \beta_{61} \) | 5.50 | \( \beta_{27} \) | 7.49 |
| \( \phi_7 \) | 20.92 | \( \tau_7^2 \) | 61.51 | \( \mu_7 \) | 1.04 | \( \rho_7 \) | 20.34 | \( \beta_{71} \) | 5.53 | \( \beta_{28} \) | 7.94 |
| \( \phi_8 \) | 17.76 | \( \tau_8^2 \) | 83.77 | \( \mu_8 \) | 1.04 | \( \rho_8 \) | 22.47 | \( \beta_{81} \) | 5.46 | \( \beta_{29} \) | 8.58 |
| \( \phi_9 \) | 24.50 | \( \tau_9^2 \) | 72.55 | \( \mu_9 \) | 1.04 | \( \rho_9 \) | 17.18 | \( \beta_{91} \) | 5.02 | \( \beta_{2,10} \) | 10.17 |
| \( \phi_{10} \) | 69.57 | \( \tau_{10}^2 \) | 184.45 | \( \mu_{10} \) | 1.19 | \( \rho_{10} \) | 12.84 | \( \beta_{10,1} \) | 5.74 |
| \( \phi_{f1} \) | 21.62 | \( \tau_{f1}^2 \) | 68.15 |
| \( \phi_{f2} \) | 65.94 | \( \tau_{f2}^2 \) | 146.21 |
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