CANAL HYPERSURFACES GENERATED BY NON-NUL L CURVES IN LORENTZ-MINKOWSKI 4-SPACE

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Abstract. In the present paper, firstly we obtain the general expression of the canal hypersurfaces which are formed as the envelope of a family of pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones in $E_4^1$ and give their some geometric invariants such as unit normal vector fields, Gaussian curvatures, mean curvatures and principal curvatures. Also we give some results about their flatness and minimality conditions and Weingarten canal hypersurfaces. Also, we obtain these characterizations for tubular hypersurfaces in $E_4^1$ by taking constant radius function and finally we construct some examples and visualize them with the aid of Mathematica.

1. GENERAL INFORMATION AND BASIC CONCEPTS

Canal surfaces, which are the class of surfaces formed by sweeping a sphere, have been investigated by Monge in 1850. Thus, a canal surface can be seen as the envelope of a moving sphere with varying radius, defined by the trajectory $\beta(s)$ of its centers and a radius function $r(s)$. If the radius function is constant, then the canal surface is called a tubular surface or pipe surface.

Canal surfaces (especially tubular surfaces) have been applied to many fields, such as the solid and the surface modeling for CAD/CAM, construction of blending surfaces, shape re-construction and they are useful to represent various objects such as pipe, hose, rope or intestine of a body [16], [27].

In this context, canal and tubular (hyper)surfaces have been studied by many mathematicians in different three, four or higher dimensional spaces (see [1], [5], [7], [8], [9], [11], [12], [13], [14], [15], [18], [19], [21], [24], [25], [26], [27], [29], [30] and etc.).

Since the extrinsic differential geometry of submanifolds in Lorentz-Minkowski 4-space $E_4^1$ is of special interest in Relativity Theory, lots of studies about curves and (hyper)surfaces have been done in this space and this motivated us to construct the canal hypersurfaces in $E_4^1$. Now, let us recall some basic concepts for curves and hypersurfaces in $E_4^1$.

If $\mathbf{x} = (x_1, x_2, x_3, x_4)$, $\mathbf{y} = (y_1, y_2, y_3, y_4)$ and $\mathbf{z} = (z_1, z_2, z_3, z_4)$ are three vectors in $E_4^1$, then the inner product and vector product are defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 \quad (1.1)$$

and

$$\mathbf{x} \times \mathbf{y} \times \mathbf{z} = \det \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}, \quad (1.2)$$

respectively.

A vector $\mathbf{x} \in E_4^1 - \{0\}$ is called spacelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$; timelike if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ and lightlike (null) if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$. In particular, the vector $\mathbf{x} = 0$ is spacelike. Also, the norm of the vector $\mathbf{x}$ is $\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$. A curve $\beta(s)$ in $E_4^1$ is spacelike, timelike or lightlike (null), if all its

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velocity vectors $\beta'(s)$ are spacelike, timelike or lightlike, respectively and a non-null (i.e. timelike or spacelike) curve has unit speed if $\langle \beta', \beta' \rangle = \pm 1$ [22].

If $\{F_1, F_2, F_3, F_4\}$ is the moving Frenet frame along the timelike or spacelike curve $\beta(s)$ in $E_4^1$, where we’ll call $F_1, F_2, F_3$ and $F_4$ are unit tangent vector field, principal normal vector field, binormal vector field and trinormal vector field, respectively, then the Frenet equations can be given according to the causal characters of non-null Frenet vector fields $F_1, F_2, F_3$ and $F_4$ as follows [28]:

If the curve $\beta(s)$ is timelike, i.e. $\langle F_1, F_1 \rangle = -1, \langle F_2, F_2 \rangle = \langle F_3, F_3 \rangle = \langle F_4, F_4 \rangle = 1$, then

$$\begin{bmatrix} F'_1 \\ F'_2 \\ F'_3 \\ F'_4 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}; \quad (1.3)$$

if the curve $\beta(s)$ is spacelike with timelike principal normal vector field $F_2$, i.e. $\langle F_2, F_2 \rangle = -1, \langle F_1, F_1 \rangle = \langle F_3, F_3 \rangle = \langle F_4, F_4 \rangle = 1$, then

$$\begin{bmatrix} F'_1 \\ F'_2 \\ F'_3 \\ F'_4 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}; \quad (1.4)$$

if the curve $\beta(s)$ is spacelike with timelike binormal vector field $F_3$, i.e. $\langle F_3, F_3 \rangle = -1, \langle F_1, F_1 \rangle = \langle F_2, F_2 \rangle = \langle F_4, F_4 \rangle = 1$, then

$$\begin{bmatrix} F'_1 \\ F'_2 \\ F'_3 \\ F'_4 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}; \quad (1.5)$$

if the curve $\beta(s)$ is spacelike with timelike trinormal vector field $F_4$, i.e. $\langle F_4, F_4 \rangle = -1, \langle F_1, F_1 \rangle = \langle F_2, F_2 \rangle = \langle F_3, F_3 \rangle = 1$, then

$$\begin{bmatrix} F'_1 \\ F'_2 \\ F'_3 \\ F'_4 \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & k_3 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}; \quad (1.6)$$

where $k_1, k_2, k_3$ are the first, second and third curvatures of the non-null curve $\beta(s)$.

Furthermore, if $p$ is a fixed point in $E_4^1$ and $r$ is a positive constant, then the pseudo-Riemannian hypersphere is defined by

$$S^3_1(p, r) = \{x \in E_4^1 : \langle x - p, x - p \rangle = r^2 \},$$

the pseudo-Riemannian hyperbolic space is defined by

$$H^3_0(p, r) = \{x \in E_4^1 : \langle x - p, x - p \rangle = -r^2 \},$$

the pseudo-Riemannian null hypercone is defined by

$$Q^3_1 = \{x \in E_4^1 : \langle x - p, x - p \rangle = 0 \}.$$

In the present study, we construct the canal hypersurfaces in $E_4^1$ as the envelope of a family of pseudo hyperspheres, pseudo hyperbolic hyperspheres or null hypercones whose centers lie on a non-null space curve.
On the other hand, the differential geometry of different types of (hyper)surfaces in 4-dimensional spaces has been a popular topic for geometers, recently ([2], [3], [4], [6], [17], [20] and etc). In this context, let \( \Gamma \) be a hypersurface in \( E^4_1 \) given by
\[
\Gamma : U \subset E^3 \rightarrow E^4_1
\]
\[
(u_1, u_2, u_3) \rightarrow \Gamma(u_1, u_2, u_3) = (\Gamma_1(u_1, u_2, u_3), \Gamma_2(u_1, u_2, u_3), \Gamma_3(u_1, u_2, u_3), \Gamma_4(u_1, u_2, u_3)).
\]

Then the Gauss map (i.e., the unit normal vector field), the matrix forms of the first and second fundamental forms are
\[
N_{\Gamma} = \frac{\Gamma_{u_1} \times \Gamma_{u_2} \times \Gamma_{u_3}}{\| \Gamma_{u_1} \times \Gamma_{u_2} \times \Gamma_{u_3} \|},
\]
\[
[g_{ij}] = \begin{bmatrix}
g_{11} & g_{12} & g_{13} 
g_{21} & g_{22} & g_{23} 
g_{31} & g_{32} & g_{33}
\end{bmatrix},
\]
and
\[
[h_{ij}] = \begin{bmatrix}
h_{11} & h_{12} & h_{13} 
h_{21} & h_{22} & h_{23} 
h_{31} & h_{32} & h_{33}
\end{bmatrix},
\]
respectively. Here \( g_{ij} = \langle \Gamma_{u_i}, \Gamma_{u_j} \rangle \), \( h_{ij} = \langle \Gamma_{u_i u_j}, N_{\Gamma} \rangle \), \( \Gamma_{u_i} = \frac{\partial \Gamma}{\partial u_i} \), \( \Gamma_{u_i u_j} = \frac{\partial^2 \Gamma}{\partial u_i \partial u_j} \), \( i, j \in \{1, 2, 3\} \).

Also, the matrix of shape operator of the hypersurface (1.7) is
\[
S = [a_{ij}] = [g^{ij}].[h_{ij}],
\]
where \([g^{ij}]\) is the inverse matrix of \([g_{ij}]\).

With the aid of (1.9)-(1.11), the Gaussian curvature and mean curvature of a hypersurface in \( E^4_1 \) are given by
\[
K = \varepsilon \frac{\det[h_{ij}]}{\det[g_{ij}]},
\]
and
\[
3\varepsilon H = tr(S),
\]
respectively. Here, \( \varepsilon = \langle N_{\Gamma}, N_{\Gamma} \rangle \). We say that a hypersurface is flat or minimal, if it has zero Gaussian or zero mean curvature, respectively. For more details about hypersurfaces in \( E^4_1 \), we refer to [10], [23] and etc.

In the second section of this paper, we obtain the general expression of the canal hypersurfaces which are formed as the envelope of a family of pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones in \( E^4_1 \) and give some characterizations for them. In the third section, we obtain these results for tubular hypersurfaces by taking constant radius function and in the last section, we construct some examples and visualize them with the aid of Mathematica.

2. CANAL HYPERSURFACES GENERATED BY NON-NUL L CURVES IN \( E^4_1 \)

In this section, firstly we construct the canal hypersurfaces which are formed as the envelope of a family of pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones in \( E^4_1 \). After that, we obtain some important geometric invariants such as unit normal vector fields, Gaussian curvatures, mean curvatures and principal curvatures of these canal hypersurfaces in general form. Also, we give some results for flat, minimal and Weingarten canal hypersurfaces in \( E^4_1 \).
2.1. CONSTRUCTION OF CANAL HYPERSURFACES.

Here, we prove a theorem which gives us the general parametric expressions of 11 different types of canal hypersurfaces obtained by pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones in $E_4^4$. Also, we write the parametric expressions of these canal hypersurfaces explicitly.

**Theorem 1.** The canal hypersurfaces which are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres in $E_4^4$ generated by spacelike or timelike center curves with non-null Frenet vector fields can be parametrized by

$$
\mathbf{C}^{(j;\lambda)}(s, t, w) = \beta(s) - \lambda \varepsilon_1 r(s)r'(s)F_1(s) \mp r(s)\sqrt{r'(s)^2 - \lambda \varepsilon_1 \left( \sum_{i=2}^{4} a_i(s, t, w)F_i(s) \right)},
$$

(2.1)

where

$$
\begin{align*}
    i) & \quad g(F_1, F_j) = -1 = \varepsilon_j \quad \text{and} \quad \text{for } i \neq j, \varepsilon_i = 1, \quad i, j \in \{1, 2, 3, 4\}; \\
    ii) & \quad \begin{cases} 
        j = 1 \Rightarrow a_2(s, t, w) = \cos t \cos w, \quad a_3(s, t, w) = \sin t \cos w, \quad a_4(s, t, w) = \sin w, \\
        j = 2, 3, 4 \Rightarrow a_j(s, t, w) = \cosh t \cosh w, \quad a_{j+1}(s, t, w) = \sinh w, \quad a_{j+2}(s, t, w) = \sinh t \cosh w, \\
        a_5(s, t, w) = a_2(s, t, w), \quad a_6(s, t, w) = a_3(s, t, w); 
    \end{cases}
\end{align*}
$$

(2.2)

also, we suppose $r'(s)^2 > \lambda \varepsilon_1$ and if the canal hypersurface is foliated by pseudo hyperspheres or pseudo hyperbolic hyperspheres, then $\lambda = 1$ or $\lambda = -1$, respectively.

Furthermore, the canal hypersurfaces which are formed as the envelope of a family of null hypercones can be parametrized by

$$
\mathbf{C}^{(j;0)}(s, t, w) = \beta(s) \mp \left( \sum_{i=2}^{4} a_i(s, t, w)F_i(s) \right), \quad j = 2, 3, 4
$$

(2.3)

where $a_i(s, t, w)$ are arbitrary functions satisfying

$$
\sum_{i=2}^{4} \varepsilon_i a_i^2(s, t, w) = 0.
$$

(2.4)

It is obvious from this expression that, the canal hypersurface $\mathbf{C}^{(1;0)}$ cannot be defined.

Here, the center curve which generates the canal hypersurface is timelike or spacelike if $j = 1$ or $j = 2, 3, 4$, respectively.

**Proof.** Let $\beta : I \subset \mathbb{R} \rightarrow E_4^4$ be a spacelike or timelike center curve parametrized by arc-length with non-zero curvature and $g(F_1, F_j) = \varepsilon_j = -1$, where $F_1(s), F_2(s), F_3(s), F_4(s)$ are unit tangent, principal normal, binormal and trinormal vectors of $\beta(s)$, respectively. Then, the parametrization of the envelope of pseudo hyperspheres ($\lambda = 1$) (resp. pseudo hyperbolic hyperspheres ($\lambda = -1$) or null hypercones ($\lambda = 0$)) defining the canal hypersurfaces $\mathbf{C}^{(j;\lambda)}$ in $E_4^4$ can be given by

$$
\mathbf{C}^{(j;\lambda)}(s, t, w) - \beta(s) = a_1(s, t, w)F_1(s) + a_2(s, t, w)F_2(s) + a_3(s, t, w)F_3(s) + a_4(s, t, w)F_4(s),
$$

(2.5)

where $a_i(s, t, w)$ are differentiable functions of $s, t, w$ on the interval $I$. Furthermore, since $\mathbf{C}^{(j;\lambda)}(s, t, w)$ lies on the pseudo hyperspheres (resp. pseudo hyperbolic hyperspheres or null hypercones), we have

$$
g(\mathbf{C}^{(j;\lambda)}(s, t, w) - \beta(s), \mathbf{C}^{(j;\lambda)}(s, t, w) - \beta(s)) = \lambda r^2(s)
$$

(2.6)

which leads to from (2.3) that

$$
\varepsilon_1 a_1^2 + \varepsilon_2 a_2^2 + \varepsilon_3 a_3^2 + \varepsilon_4 a_4^2 = \lambda r^2
$$

(2.7)

and

$$
\varepsilon_1 a_1 a_1 + \varepsilon_2 a_2 a_2 + \varepsilon_3 a_3 a_3 + \varepsilon_4 a_4 a_4 = \lambda r r_s,
$$

(2.8)
where \( r(s) \) is the radius function; \( r = r(s), \ r_s = \frac{dr(s)}{ds}, \ a_i = a_i(s, t, w), \ a_{is} = \frac{\partial a_i(s, t, w)}{\partial s}. \)

Also from (1.3)-(1.6), the non-null Frenet vectors \( F_i(s), \ i \in \{1, 2, 3, 4\}, \) of the spacelike or timelike curve \( \beta(s) \) satisfy the relations

\[
\begin{align*}
F'_1(s) &= k_1(s)F_2(s), \\
F'_2(s) &= \varepsilon_3\varepsilon_4k_1(s)F_1(s) + k_2(s)F_3(s), \\
F'_3(s) &= \varepsilon_1\varepsilon_4k_2(s)F_2(s) + k_3(s)F_4(s), \\
F'_4(s) &= \varepsilon_1\varepsilon_3k_3(s)F'_3(s).
\end{align*}
\]

Here, \( \beta(s) \) is a timelike curve if \( \varepsilon_1 = -1. \) Also, \( \beta(s) \) is a spacelike curve with timelike principal normal \( F_2 \) or timelike binormal \( F_3 \) or timelike trinormal \( F_4 \) if \( \varepsilon_2 = -1 \) or \( \varepsilon_3 = -1 \) or \( \varepsilon_4 = -1, \)

respectively, where \( g(F_i, F_i) = \varepsilon_i, \ i \in \{1, 2, 3, 4\}. \)

So, differentiating (2.3) with respect to \( s \) and using the Frenet formula (2.7), we get

\[
\mathbf{c}^{(j; \lambda)}(s, t, w) - \beta(s) = (1 + \varepsilon_3\varepsilon_4a_2k_1+a_1s) F_1 + (a_1k_1+\varepsilon_1\varepsilon_4a_3k_2+a_2s) F_2 + (a_2k_2+\varepsilon_1\varepsilon_4a_4k_3+a_3s) F_3 + (a_3k_3+a_4s) F_4.
\]

where \( \mathbf{c}^{(j; \lambda)}(s, t, w) = \frac{\partial \mathbf{c}^{(j; \lambda)}(s, t, w)}{\partial s}. \)

Furthermore, \( \mathbf{c}^{(j; \lambda)}(s, t, w) - \beta(s) \) is a normal vector to the canal hypersurfaces, which implies that

\[
g(\mathbf{c}^{(j; \lambda)}(s, t, w) - \beta(s), \mathbf{c}^{(j; \lambda)}(s, t, w)) = 0
\]

and so, from (2.3), (2.8) and (2.9) we have

\[
\varepsilon_1a_1(1 + \varepsilon_3\varepsilon_4a_2k_1+a_1s) + \varepsilon_2a_2(a_1k_1+\varepsilon_1\varepsilon_4a_3k_2+a_2s) + \varepsilon_3a_3(a_2k_2+\varepsilon_1\varepsilon_4a_4k_3+a_3s) + \varepsilon_4a_4(a_3k_3+a_4s) = 0.
\]

Using (2.6) in (2.10), we get

\[
\varepsilon_1a_1 + \varepsilon_1a_1a_1s + \varepsilon_2a_2a_2s + \varepsilon_3a_3a_3s + \varepsilon_4a_4a_4s = 0
\]

and thus, from (2.6) and the definition of \( \varepsilon_i, \) we obtain

\[
a_1 = -\varepsilon_1\lambda rr_s.
\]

Hence, using (2.12) in (2.5), we reach that

\[
\varepsilon_1(\varepsilon_2a_2^2+\varepsilon_3a_3^2+\varepsilon_4a_4^2) = -r^2(-\varepsilon_1\lambda + \lambda^2r_s^2).
\]

Here from (2.13);

\textbf{i}: if \( \lambda = 1 \) or \( \lambda = -1, \) then the canal hypersurfaces \( \mathbf{c}^{(j; \lambda)} \) which is formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres in \( E_1^4 \) generated by spacelike or timelike center curves can be parametrized by (2.11);

\textbf{ii}: if \( \lambda = 0, \) then the canal hypersurfaces \( \mathbf{c}^{(j; 0)} \) which is formed as the envelope of a family of null hypercones in \( E_1^4 \) generated by spacelike or timelike center curves can be parametrized by (2.2)

and this completes the proof. \( \square \)

Thus, the explicit parametric expressions of the canal hypersurfaces \( \mathbf{c}^{(j; \lambda)}(s, t, w) \) and \( \mathbf{c}^{(j; 0)}(s, t, w) \) from Theorem [1] are as follows:
\[ \mathcal{C}^{(1;1)} = \beta + r r' F_1 \mp r \sqrt{r'^2 + 1} (\cos t \cos w F_2 + \sin t \cos w F_3 + \sin w F_4), \]
\[ \mathcal{C}^{(1;1)} = \beta - r r' F_1 \mp r \sqrt{r'^2 - 1} (\cos t \cos w F_2 + \sin t \cos w F_3 + \sin w F_4), \]
\[ \mathcal{C}^{(2;1)} = \beta - r r' F_1 \mp r \sqrt{r'^2 - 1} (\cosh t \cosh w F_2 + \sinh w F_3 + \sinh t \cosh w F_4), \]
\[ \mathcal{C}^{(2;1)} = \beta + r r' F_1 \mp r \sqrt{r'^2 + 1} (\cosh t \cosh w F_2 + \sinh w F_3 + \sinh t \cosh w F_4), \]
\[ \mathcal{C}^{(3;1)} = \beta - r r' F_1 \mp r \sqrt{r'^2 - 1} (\sinh t \cosh w F_2 + \cosh t \cosh w F_3 + \sinh w F_4), \]
\[ \mathcal{C}^{(3;1)} = \beta + r r' F_1 \mp r \sqrt{r'^2 + 1} (\cosh t \cosh w F_2 + \cosh t \cosh w F_3 + \sinh w F_4), \]
\[ \mathcal{C}^{(4;1)} = \beta - r r' F_1 \mp r \sqrt{r'^2 - 1} (\cosh w F_2 + \sinh t \cosh w F_3 + \cosh t \cosh w F_4), \]
\[ \mathcal{C}^{(4;1)} = \beta + r r' F_1 \mp r \sqrt{r'^2 + 1} (\sinh w F_2 + \sinh t \cosh w F_3 + \cosh t \cosh w F_4) \] (2.14)

and
\[ \mathcal{C}^{(2;0)} = \beta \mp \sqrt{a_3^2 + a_4^2} F_2 + a_3 F_3 + a_4 F_4, \]
\[ \mathcal{C}^{(3;0)} = \beta + a_2 F_2 \mp \sqrt{a_2^2 + a_4^2} F_3 + a_4 F_4, \]
\[ \mathcal{C}^{(4;0)} = \beta + a_2 F_2 + a_3 F_3 \mp \sqrt{a_2^2 + a_3^2} F_4. \] (2.15)

Remark 1. As one can see in Theorem 1, we deal with the canal hypersurfaces satisfying \( r'(s)^2 > \lambda \varepsilon_1 \). If we assume that \( r'(s)^2 < \lambda \varepsilon_1 \), then the canal hypersurfaces \( \mathcal{C}^{(2;1)}, \mathcal{C}^{(3;1)} \) and \( \mathcal{C}^{(4;1)} \) can be rewritten as
\[ \mathcal{C}^{(2;1)} = \beta - r r' F_1 \mp r \sqrt{1 - r'^2} (\cos t \sinh w F_2 + \cosh w F_3 + \sinh t \sinh w F_4), \]
\[ \mathcal{C}^{(3;1)} = \beta - r r' F_1 \mp r \sqrt{1 - r'^2} (\sinh t \cosh w F_2 + \cosh t \sinh w F_3 + \cosh w F_4), \]
\[ \mathcal{C}^{(4;1)} = \beta - r r' F_1 \mp r \sqrt{1 - r'^2} (\cosh w F_2 + \sinh t \sinh w F_3 + \cosh t \sinh w F_4). \] (2.16)

2.2. CURVATURES OF CANAL HYPER SURFACES.

At the beginning of this subsection, we must note that we’ll obtain the following results by taking \( r'(s)^2 > \lambda \varepsilon_1 \) and ”\( \mp \)” which is in (2.11) as ”\( + \)”. Similar characterizations can be obtained by taking \( r'(s)^2 < \lambda \varepsilon_1 \) and ”\( \mp \)” as ”\( - \)”, (or \( r'(s)^2 < \lambda \varepsilon_1 \) and ”\( + \)” or ”\( - \)”).

Here, we’ll obtain some important geometric invariants of these canal hypersurfaces by proving the following theorem:

**Theorem 2.** The unit normal vector fields \( N^{(j;\lambda)} \), Gaussian curvatures \( K^{(j;\lambda)} \), mean curvatures \( H^{(j;\lambda)} \) and principal curvatures \( \mu_i^{(j;\lambda)} \), \( i \in \{1, 2, 3\} \), of the canal hypersurfaces \( \mathcal{C}^{(j;\lambda)}(s, t, w) \), given
by (2.1) in $E_4^1$, are

\begin{align}
N^{(j;\lambda)} &= -\varepsilon_3 \varepsilon_4 \lambda^j \left( -\lambda \varepsilon_1 r' F_1(s) + \sqrt{r'^2 - \lambda \varepsilon_1} \sum_{i=2}^{4} a_i(s, t, w) F_i(s) \right), \\
K^{(j;\lambda)} &= \frac{\varepsilon_3 \varepsilon_4 \lambda^j}{3} \left( \frac{2}{r} + \frac{r k_1^2 f_j (r'^2 - \lambda \varepsilon_1) + r'' (r'^2 - \lambda \varepsilon_1 + r r'') + \varepsilon_2 \lambda k_1 f_j \sqrt{r'^2 - \lambda \varepsilon_1} (r'^2 - \lambda \varepsilon_1 + 2 r r'')} {r'^2 - \lambda \varepsilon_1 + \varepsilon_2 \lambda r k_1 f_j \sqrt{r'^2 - \lambda \varepsilon_1 + r r''}} \right), \\
H^{(j;\lambda)} &= \frac{\varepsilon_3 \varepsilon_4 \lambda^j}{3} \left( \frac{2}{r} + \frac{r k_1^2 f_j (r'^2 - \lambda \varepsilon_1) + r'' (r'^2 - \lambda \varepsilon_1 + r r'') + \varepsilon_2 \lambda k_1 f_j \sqrt{r'^2 - \lambda \varepsilon_1} (r'^2 - \lambda \varepsilon_1 + 2 r r'')} {r'^2 - \lambda \varepsilon_1 + \varepsilon_2 \lambda r k_1 f_j \sqrt{r'^2 - \lambda \varepsilon_1 + r r''}} \right),
\end{align}

where

\begin{align}
\mu_1^{(j;\lambda)} &= \mu_2^{(j;\lambda)} = \varepsilon_3 \varepsilon_4 \lambda^j, \\
\mu_3^{(j;\lambda)} &= \varepsilon_3 \varepsilon_4 \lambda^j \left( \frac{r k_1^2 f_j (r'^2 - \lambda \varepsilon_1) + r'' (r'^2 - \lambda \varepsilon_1 + r r'') + \varepsilon_2 \lambda k_1 f_j \sqrt{r'^2 - \lambda \varepsilon_1} (r'^2 - \lambda \varepsilon_1 + 2 r r'')} {r'^2 - \lambda \varepsilon_1 + \varepsilon_2 \lambda r k_1 f_j \sqrt{r'^2 - \lambda \varepsilon_1 + r r''}} \right),
\end{align}

Proof. Firstly, with the aid of the first derivatives of (2.14) according to $s$, $t$ and $w$, we obtain the normals of the canal hypersurfaces $C^{(j;\lambda)}(s, t, w)$ from (1.8) as

\begin{align}
N^{(1;1)} &= -r'(s) F_1(s) - \sqrt{r'(s)^2 + 1} \left( \cos t \cos w F_2 + \sin t \cos w F_3 + \sin w F_4 \right), \\
N^{(1;-1)} &= -r'(s) F_1(s) + \sqrt{r'(s)^2 - 1} \left( \cos t \cos w F_2 + \sin t \cos w F_3 + \sin w F_4 \right), \\
N^{(2;1)} &= r'(s) F_1(s) - \sqrt{r'(s)^2 - 1} \left( \cos t \cos w F_2 + \sin w F_3 + \sin t \cos w F_4 \right), \\
N^{(2;-1)} &= -r'(s) F_1(s) + \sqrt{r'(s)^2 + 1} \left( \cos t \cos w F_2 + \sin w F_3 + \sin t \cos w F_4 \right), \\
N^{(3;1)} &= -r'(s) F_1(s) + \sqrt{r'(s)^2 - 1} \left( \sin t \cos w F_2 + \cosh t \cos w F_3 + \sinh w F_4 \right), \\
N^{(3;-1)} &= -r'(s) F_1(s) - \sqrt{r'(s)^2 + 1} \left( \sin t \cos w F_2 + \cosh t \cos w F_3 + \sinh w F_4 \right), \\
N^{(4;1)} &= -r'(s) F_1(s) + \sqrt{r'(s)^2 - 1} \left( \sinh w F_2 + \sinh t \cos w F_3 + \cosh t \cos w F_4 \right), \\
N^{(4;-1)} &= r'(s) F_1(s) + \sqrt{r'(s)^2 + 1} \left( \sinh w F_2 + \sinh t \cos w F_3 + \cosh t \cos w F_4 \right)
\end{align}

and so, we can write (2.17). Now, the coefficients of the first and second fundamental forms of the canal hypersurfaces $C^{(1;\lambda)}$ which are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic
hyperspheres in $E^4_1$ generated by the timelike center curve are

$$
g_{11}^{(1;\lambda)} = \frac{1}{4} \left( r^2 \left( 4k^2_3 \cos^2 w - 4k^2_2 \cos t \sin 2w \right) - 2k^2_2 k^2_3 \cos t \sin 2w - 3k^2_3 \right) \left( 1 + \frac{1}{4} \right) - 4\lambda \right) - \lambda \frac{r^2 k^2_1}{k^2_1} \left( \cos^2 t \cos^2 w + (\lambda \cos^2 t \cos^2 w - \lambda) r^2 \right) - 2\lambda r + \lambda \frac{r^2}{\lambda+r^2} \right)
$$

$$
g_{12}^{(1)} = g_{21}^{(1)} = r^2 \left( k^2_1 (r^2 + \lambda) \cos w - \lambda k^2_1 r^2 \cos t \sin t - k^2_3 (r^2 + \lambda) \cos t \sin w \right) \cos w,
$$

$$
g_{13}^{(1)} = g_{31}^{(1)} = r^2 \left( k^2_1 (r^2 + \lambda) \sin t - \lambda k^2_1 r^2 \cos t \sin w \right) \cos w,
$$

$$
g_{22}^{(1)} = (r^2 + \lambda) r^2 \cos^2 w,
$$

$$
g_{23}^{(1)} = g_{32}^{(1)} = 0,
$$

$$
g_{33}^{(1)} = (r^2 + \lambda) r^2;
$$

$$\{2.23\}

the coefficients of the first and second fundamental forms of the canal hypersurfaces $\mathbf{C}^{(2;\lambda)}$ which are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres in $E^4_1$ generated by the spacelike center curve with the timelike principal normal vector are

$$
g_{11}^{(2;\lambda)} = \frac{1}{4} \left( 4 - 4\lambda r^2 + r^2 (r^2 - \lambda) \right) \left( \cos t \left( 2t + 2 \cos^2 t \cos 2w - 3k^2_3 \right) \right) \left( 1 + \frac{1}{4} \right) - 4\lambda k^2_3 \left( 4 - 4\lambda r^2 + r^2 \right) \left( r^2 - \lambda k^2_3 \cos^2 w \sinh 2t \right)
$$

$$+ \left( \cos^2 t \cos^2 w - 1 \right) r^2 - \lambda \cos^2 t \cos^2 w - 2\lambda r + \lambda \frac{r^2}{\lambda+r^2} \right)
$$

$$+ \left( \cos t \left( 2t + 2 \cos^2 t \cos 2w - 3k^2_3 \right) \right) \left( 1 + \frac{1}{4} \right) - 4\lambda k^2_3 \left( 4 - 4\lambda r^2 + r^2 \right) \left( r^2 - \lambda k^2_3 \cos^2 w \sinh 2t \right)
$$

$$+ \left( \cos^2 t \cos^2 w - 1 \right) r^2 - \lambda \cos^2 t \cos^2 w - 2\lambda r + \lambda \frac{r^2}{\lambda+r^2} \right)
$$

$$g_{12}^{(2;\lambda)} = g_{21}^{(2;\lambda)} = r^2 \left( k^2_1 (r^2 + \lambda) \cos w - \lambda k^2_1 r^2 \cos t \sin t - k^2_3 (r^2 + \lambda) \cos t \sin w \right) \cos w,
$$

$$
g_{13}^{(2;\lambda)} = g_{31}^{(2;\lambda)} = r^2 \left( k^2_1 (r^2 + \lambda) \sin t - \lambda k^2_1 r^2 \cos t \sin w \right) \cos w,
$$

$$
g_{22}^{(2;\lambda)} = (r^2 - \lambda) r^2 \cos^2 w,
$$

$$
g_{23}^{(2;\lambda)} = g_{32}^{(2;\lambda)} = 0,
$$

$$
g_{33}^{(2;\lambda)} = r^2 (r^2 - \lambda);
$$

$$\{2.25\}

the coefficients of the first and second fundamental forms of the canal hypersurfaces $\mathbf{C}^{(3;\lambda)}$ which are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres
in $E^4_1$ generated by the spacelike center curve with the timelike binormal vector are

\[
\begin{aligned}
g_{11}^{(3;\lambda)} &= \frac{r^2 - \lambda}{4} \left( r^2 \left( k_2^2 \left( 3 + \cosh 2t + 2 \cosh 2w \sinh^2 t \right) + 4k_3^2 \cosh^2 w - 4k_2k_3 \sinh t \sinh 2w \right) - 4\lambda \right) \\
&\quad + \frac{r^2k_1^2}{4} \left( r^2 \cosh^2 w - \lambda \sinh^2 w - 2\lambda \sinh 2w \right) \left( 2\lambda r'' - \frac{\lambda r'''}{\sqrt{r^2 - \lambda}} \right) \\
g_{12}^{(3;\lambda)} &= g_{21}^{(3;\lambda)} = r^2 \left( k_2 \left( r^2 - \lambda \right) \cosh w - \lambda k_1 r' \sqrt{r^2 - \lambda} \cosh t \sinh t + r'' \cosh t \right) \cosh^2 w, \\
g_{13}^{(3;\lambda)} &= g_{31}^{(3;\lambda)} = r^2 \left( k_3 \left( r^2 - \lambda \right) \cosh t - \lambda k_1 r' \sqrt{r^2 - \lambda} \cosh t \sinh w \right), \\
g_{22}^{(3;\lambda)} &= (r^2 - \lambda) r^2 \cosh^2 w, \quad g_{32}^{(3;\lambda)} = g_{33}^{(3;\lambda)} = 0, \quad g_{33}^{(3;\lambda)} = (r^2 - \lambda) r^2,
\end{aligned}
\]

\[\text{(2.27)}\]

and the coefficients of the first and second fundamental forms of the canal hypersurfaces $\mathcal{C}^{(4;\lambda)}$ which are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres in $E^4_1$ generated by the spacelike center curve with the timelike trinormal vector are

\[
\begin{aligned}
g_{11}^{(4;\lambda)} &= -\frac{\lambda + r''}{4} \left( r^2 \left( k_2^2 \left( 2 \cosh 2t \cosh^2 w + \cosh 2w - 3 \right) + 4k_3^2 \cosh^2 w + 4k_2k_3 \cosh t \sinh 2w \right) - 4\lambda \right) \\
&\quad + \frac{r^2k_1^2}{4} \left( r^2 \cosh^2 w - \lambda \sinh^2 w - 2\lambda r'' - \frac{2\lambda r'''}{\lambda + r''} \right) \cosh w \sinh t - \lambda \left( r^2 - \lambda + r'' \right) \sinh w \left( r^2 - \lambda + r'' \right) \cosh w \sinh t, \\
g_{12}^{(4;\lambda)} &= g_{21}^{(4;\lambda)} = r^2 \left( r^2 - \lambda \right) \left( k_2 \cosh w + k_3 \cosh w \right) \cosh w, \\
g_{13}^{(4;\lambda)} &= g_{31}^{(4;\lambda)} = -r^2 \left( \lambda k_1 r' \sqrt{r^2 - \lambda} \cosh w + k_2 \left( r^2 - \lambda \right) \cosh t \right), \\
g_{22}^{(4;\lambda)} &= r^2 \left( r^2 - \lambda \right) \cosh^2 w, \quad g_{32}^{(4;\lambda)} = g_{33}^{(4;\lambda)} = 0, \quad g_{33}^{(4;\lambda)} = r^2 \left( r^2 - \lambda \right)
\end{aligned}
\]

\[\text{(2.29)}\]

and these imply that

\[
\begin{aligned}
\det g_{ij}^{(j;\lambda)} &= -\lambda A^2 r^4 \left( r^2 - \lambda \varepsilon_1 \right) \left( r^2 - \lambda \varepsilon_1 + \varepsilon_2 \lambda k_1 f_j r \sqrt{r^2 - \lambda} \varepsilon_1 + r'' \right)^2, \\
\det h_{ij}^{(j;\lambda)} &= -\lambda A^2 r^2 \left( r^2 - \lambda \varepsilon_1 \right) \left( \varepsilon_3 \varepsilon_4 \lambda^j f_j^2 k_2^2 r \left( r^2 - \lambda \varepsilon_1 \right) + \varepsilon_3 \varepsilon_4 \lambda^j r'' \left( r^2 - \lambda \varepsilon_1 + r'' \right) \right) + \varepsilon_2 \varepsilon_3 \varepsilon_4 \lambda^j f_j k_1 r \sqrt{r^2 - \lambda} \varepsilon_1 \left( r^2 - \lambda \varepsilon_1 + 2r'' \right),
\end{aligned}
\]

\[\text{(2.31)}\]

\[\text{(2.32)}\]

where $A = \cos w$, for $j = 1$ and $A = \cos w$, for $j = 2, 3, 4$. Thus, from (1.12), (2.31) and (2.32), the Gaussian curvatures are obtained as (2.18).
Here, obtaining the first of the first fundamental forms and using these and second fundamental forms in (1.11), we obtain the components of the shape operators of canal hypersurfaces $T_{\gamma}^{(j;\lambda)}$ as

$$
S_{11}^{[1;\lambda]} = \frac{\lambda k_1^2 r (r^2-\lambda) \cos^2 t \cos^2 w + \lambda \nu^2 (r^2 - \lambda + 2r'') k_1 \sqrt{r^2-\lambda} (r^2 - \lambda + 2r'') \cos t \cos w}{\left(r^2 - \lambda + \nu (\lambda k_1 \sqrt{r^2-\lambda} \cos t \cos w + \nu') \right)^2},
$$

$$
S_{21}^{[1;\lambda]} = -\lambda \sqrt{r^2 + \lambda} \left(k_1 r' \sin t \sec w - \lambda k_2 \sqrt{r^2 + \lambda} + \nu (\lambda k_3 \sqrt{r^2 + \lambda} \cos t \tan w) \sec w \right),
$$

$$
S_{31}^{[1;\lambda]} = \frac{\lambda (k_3 r^2 + \lambda) \sin t - k_1 r' \sin w \lambda k_2 \sqrt{r^2 + \lambda} \cos t \cos w}{r^2 - \lambda + \nu (\lambda k_1 \sqrt{r^2 + \lambda} \cos t \cos w + \nu')},
$$

$$
S_{22}^{[1;\lambda]} = S_{33}^{[1;\lambda]} = -\frac{\lambda}{r},
$$

$$
S_{12}^{[1;\lambda]} = S_{13}^{[1;\lambda]} = S_{23}^{[1;\lambda]} = S_{32}^{[1;\lambda]} = 0;
$$

$$
S_{11}^{[2;\lambda]} = k_1^2 r (r^2 - \lambda) \cosh^2 t \cos^2 w + \lambda \nu^2 (r^2 - \lambda + 2r'') k_1 \sqrt{r^2 - \lambda} \cos t \cos w,
$$

$$
S_{21}^{[2;\lambda]} = \frac{(\lambda k_1 r^2 + \lambda) \sin t \sec hw + (k_3 \cosh t - k_2 \sinh t) \sqrt{r^2 + \lambda} \tan w \sec hw}{r (r^2 - \lambda + \nu (\lambda k_3 \sqrt{r^2 + \lambda} \cos t \cosh w)}.
$$

$$
S_{31}^{[2;\lambda]} = -k_1 r' \sqrt{r^2 - \lambda} \cosh t \sin w - \lambda k_2 \sqrt{r^2 - \lambda} \cosh t + \lambda \lambda k_3 (r^2 - \lambda) \sinh t,
$$

$$
S_{22}^{[2;\lambda]} = S_{33}^{[2;\lambda]} = -\frac{\lambda}{r},
$$

$$
S_{12}^{[2;\lambda]} = S_{13}^{[2;\lambda]} = S_{23}^{[2;\lambda]} = S_{32}^{[2;\lambda]} = 0;
$$

$$
S_{11}^{[3;\lambda]} = -\lambda k_1^2 r (r^2 - \lambda) \sinh^2 t \cos^2 w + \nu^2 (r^2 - \lambda + 2r'') k_1 \sqrt{r^2 - \lambda} \sinh t \cos w,
$$

$$
S_{21}^{[3;\lambda]} = \frac{\sec hw (k_1 r^2 + \lambda) \sinh t \sec hw - \lambda (k_2 - k_3 \sinh t \tan w) (r^2 - \lambda)}{r (r^2 - \lambda + \nu (\lambda k_3 \sqrt{r^2 + \lambda} \cosh t \cosh w))},
$$

$$
S_{31}^{[3;\lambda]} = -k_1 r' \sqrt{r^2 - \lambda} \cosh t \sin w - \lambda k_2 \sqrt{r^2 + \lambda} \cosh t \cosh w \cosh w + \nu (\lambda k_3 \sqrt{r^2 + \lambda} \cosh t \cosh w) \sin w + \lambda \lambda k_3 (r^2 - \lambda) \sinh t,
$$

$$
S_{22}^{[3;\lambda]} = S_{33}^{[3;\lambda]} = -\frac{\lambda}{r},
$$

$$
S_{12}^{[3;\lambda]} = S_{13}^{[3;\lambda]} = S_{23}^{[3;\lambda]} = S_{32}^{[3;\lambda]} = 0;
$$

$$
S_{11}^{[4;\lambda]} = \frac{-k_1^2 r (r^2 - \lambda) \sinh^2 t \cos^2 w - \nu^2 (r^2 - \lambda + 2r'') k_1 \sqrt{r^2 - \lambda} \sinh t \sin w}{\left(r^2 - \lambda + \nu (\lambda k_3 \sqrt{r^2 - \lambda} \sinh w + \nu') \right)^2},
$$

$$
S_{21}^{[4;\lambda]} = \frac{-\lambda (k_3 + k_2 \tan w) \cosh t (r^2 - \lambda)}{r \sqrt{r^2 - 1 + r (k_1 \sqrt{r^2 - \lambda} \sinh w + \lambda \nu')}},
$$

$$
S_{31}^{[4;\lambda]} = \frac{-k_1 r' \sqrt{r^2 - \lambda} \cosh w + \lambda k_2 (r^2 - \lambda) \sinh t}{r (r^2 - \lambda + \nu (\lambda k_3 \sqrt{r^2 + \lambda} \sinh w + \nu') \sinh w + \lambda \lambda k_3 (r^2 - \lambda) \sinh t),}
$$

$$
S_{22}^{[4;\lambda]} = S_{33}^{[4;\lambda]} = -\frac{1}{r},
$$

$$
S_{12}^{[4;\lambda]} = S_{13}^{[4;\lambda]} = S_{23}^{[4;\lambda]} = S_{32}^{[4;\lambda]} = 0.
$$

From (1.13) and (2.33)-(2.36), we obtain the mean curvatures of the canal hypersurfaces $T_{\gamma}^{(j;\lambda)}$ as (2.19).

Finally, from (2.33)-(2.36), we get

$$
\text{det}(S - \mu I) = \left(\mu - \frac{\varepsilon_3 \varepsilon_4 \lambda}{r}\right)^2 \left(\varepsilon_3 \varepsilon_4 \lambda^2 \left(\frac{\lambda k_1^2 r^2 + \lambda k_2 \sqrt{r^2 - \lambda} \sinh t + \lambda \lambda k_3 \sqrt{r^2 - \lambda} \sinh w}{\left(r^2 - \lambda \varepsilon_1 + \lambda \varepsilon_1 + \nu'\right)} \right) - \mu \right)\left(\frac{\lambda k_1^2 r^2 + \lambda k_2 \sqrt{r^2 - \lambda} \sinh t + \lambda \lambda k_3 \sqrt{r^2 - \lambda} \sinh w}{\left(r^2 - \lambda \varepsilon_1 + \lambda \varepsilon_1 + \nu'\right)} \right)^2
$$

and solving the equation of det$(S - \mu I) = 0$ we have (2.20). So, the proof is completed. \qed
Here, from (2.22) or (2.31), we can state the following proposition:

**Proposition 1.** The canal hypersurfaces $C^{(j;λ)}(s,t,w)$ in $E^4_1$ are spacelike or timelike if $λ = -1$ or $λ = 1$, respectively.

### 2.3. SOME GEOMETRIC CHARACTERIZATIONS FOR CANAL HYPERSURFACES.

From (2.18) and (2.19), we can obtain the following important relation between the Gaussian and mean curvatures of the canal hypersurfaces $C^{(j;λ)}$:

**Theorem 3.** The Gaussian and mean curvatures of the canal hypersurfaces $C^{(j;λ)}(s,t,w)$, given by (2.7) in $E^4_1$, satisfy

$$3H^{(j;λ)}r - K^{(j;λ)}r^3 - 2ε_3ε_4λ^3 = 0.\tag{2.38}$$

**Theorem 4.** The canal hypersurfaces $C^{(j;λ)}(s,t,w)$, given by (2.7) in $E^4_1$, are flat if and only if $k_1 = 0$ and $r(s) = as + b$, $a, b \in \mathbb{R}$, $a \neq ±1$.

**Proof.** Firstly, let we suppose that the canal hypersurfaces $C^{(j;λ)}(s,t,w)$ in $E^4_1$ are flat. Then from (2.18) we get

$$rk^2_1(r'^2 - λε_1) f_j^2 + ε_2λk_1 r^{r'^2 - λε_1}(r'^2 - λε_1 + 2rr'' f_j + r''(r'^2 - λε_1 + rr'') = 0.\tag{2.39}$$

Since $\{f_j, f_j^2, 1\}$ are linearly independent, we have

$$rk^2_1(r'^2 - λε_1) = ε_2λk_1 r^{r'^2 - λε_1}(r'^2 - λε_1 + 2rr'') = r''(r'^2 - λε_1 + rr'') = 0.\tag{2.40}$$

From the (2.40), we get $k_1 = 0$ and $r''(r'^2 - λε_1 + rr'') = 0$. When $k_1 = 0$, from (2.18) we have $r'^2 - λε_1 + rr'' \neq 0$ and so it must be $r'' = 0$.

Conversely, if $k_1 = 0$ and $r(s) = as + b$, $a, b \in \mathbb{R}$, $a \neq ±1$, from (2.18) we get $K = 0$ and this completes the proof. □

**Theorem 5.** The canal hypersurfaces $C^{(j;λ)}(s,t,w)$, given by (2.7) in $E^4_1$, are minimal if and only if $k_1 = 0$ and the radius $r(s)$ satisfies

$$\int \frac{dr}{\sqrt{ε_1λ + (\frac{4}{3})}} = ±s + c_2, c_1, c_2 \in \mathbb{R}.\tag{2.41}$$

**Proof.** Firstly, let we suppose that the canal hypersurfaces $C^{(j;λ)}(s,t,w)$ in $E^4_1$ are minimal. Then from (2.19) we get

$$-3r^2k^2_1(ε_1λ - r'^2) f_j^2 - ε_2λk_1 r^{r'^2 - λε_1}(5ε_1λ - 5r'^2 - 6rr'') f_j + 2 + 2r'^4 - 5ε_1λrr'' + 3r^2r'' + r^2(-4ε_1λ + 5rr'') = 0.\tag{2.42}$$

Since $\{f_j, f_j^2, 1\}$ are linearly independent, we have

$$\left\{\begin{array}{l}
-3r^2k^2_1(ε_1λ - r'^2) = 0, \\
ε_2λk_1 r^{r'^2 - λε_1}(5ε_1λ - 5r'^2 - 6rr'') = 0, \\
2 + 2r'^4 - 5ε_1λrr'' + 3r^2r'' + r^2(-4ε_1λ + 5rr'') = 0.
\end{array}\right.\tag{2.42}$$

From (2.42), we get $k_1 = 0$ and $2 + 2r'^4 - 5ε_1λrr'' + 3r^2r'' + r^2(-4ε_1λ + 5rr'') = 0$; i.e.

$$(2ε_1λ - 2r'^2 - 3rr'')(ε_1λ - r'^2 - rr'') = 0.\tag{2.43}$$

When $k_1 = 0$, from (2.19) we have $r'^2 - λε_1 + rr'' \neq 0$ and so it must be

$$- 2(r'^2 - ε_1λ) - 3rr'' = 0.\tag{2.44}$$

Now, let us solve the equation (2.44).
If we take \( r'(s) = h(s) \), we get
\[
    r'' = h' = \frac{dh}{dr} \frac{dr}{ds} = \frac{dh}{dr} h. \tag{2.45}
\]
Using (2.45) in (2.44), we have
\[
    3r \frac{dh}{dr} h + 2h^2 - 2\epsilon_1 \lambda = 0. \tag{2.46}
\]
From (2.44), \( r'(s) = h(s) \neq 0 \) and so we reach that
\[
    \frac{3h}{2(\epsilon_1 \lambda - h^2)} dh = \frac{dr}{r}. \tag{2.47}
\]
By integrating (2.47), we have
\[
    h = \pm \sqrt{\epsilon_1 \lambda + \left( \frac{c_1}{r} \right)^\frac{4}{3}}, \tag{2.48}
\]
where \( c_1 \) is constant. Since \( r' = \frac{dr}{ds} = h \), from (2.48) we get
\[
    \int \frac{dr}{\sqrt{\epsilon_1 \lambda + \left( \frac{c_1}{r} \right)^\frac{4}{3}}} = \pm \int ds. \tag{2.49}
\]
Conversely, if \( k_1 = 0 \) and \( r(s) \) satisfies
\[
    \int \frac{dr}{\sqrt{\epsilon_1 \lambda + \left( \frac{c_1}{r} \right)^\frac{4}{3}}} = \pm s + c_2, \quad c_1, c_2 \in \mathbb{R},
\]
then we have \( H = 0 \) and this completes the proof. \( \square \)

Now, if
\[
    H_sK_t - H_tK_s = 0, \quad H_sK_w - H_wK_s = 0, \quad H_tK_w - H_wK_t = 0, \tag{2.50}
\]
hold on a hypersurface, then we call the hypersurface as \((H, K)_{st}\)-Weingarten, \((H, K)_{sw}\)-Weingarten, \((H, K)_{tw}\)-Weingarten hypersurface, respectively, where \( H_s = \frac{\partial H}{\partial s} \) and so on. So, from (2.18) and (2.19) we have

**Theorem 6.** The canal hypersurfaces \( E^{(j;\lambda)}(s, t, w) \), given by (2.7) in \( E_1^4 \), are \((H, K)_{tw}\)-Weingarten.

**Proof.** From (2.18) and (2.19), we get \( H_t^{(j;\lambda)} K_s^{(j;\lambda)} - H_s^{(j;\lambda)} K_t^{(j;\lambda)} = 0 \) and this completes the proof. \( \square \)

**Theorem 7.** The canal hypersurfaces \( E^{(j;\lambda)}(s, t, w) \), given by (2.7) in \( E_1^4 \), are \((H, K)_{sw}\)-Weingarten if and only if \( k_1 = 0 \) or \( r(s) \) is constant.

**Proof.** From (2.18) and (2.19), we have
\[
    H_s^{(j;\lambda)} K_w^{(j;\lambda)} - H_w^{(j;\lambda)} K_s^{(j;\lambda)} = 0. \tag{2.51}
\]
If \( H_s^{(j;\lambda)} K_w^{(j;\lambda)} - H_w^{(j;\lambda)} K_s^{(j;\lambda)} = 0 \), then from (2.51) we have
\[
    2k_1 r' \left( r'^2 + \epsilon_1 \right)^2 \left( r^2 - \epsilon_1 + \frac{r''}{\epsilon_1} \right) f_{jw} = 0. \tag{2.52}
\]
and so,
\[ k_1 r' \left( r'^2 - \lambda \epsilon_1 + r r'' \right) \left( \varepsilon_2 \lambda \sqrt{r'^2 - \lambda \epsilon_1} (r'^2 - \lambda \epsilon_1 + r r'') + k_1 r (r'^2 - \lambda \epsilon_1) f_j \right) = 0. \] (2.53)

From (2.51), the second and third component of (2.53) cannot be zero and so it must be
\[ k_1 r' = 0. \] (2.54)

This completes the proof. \qed

**Theorem 8.** The canal hypersurfaces \( \mathcal{C}^{(j;\lambda)}(s, t, w) \), given by (2.7) in \( E^4_1 \), are \((H, K)_st\)-Weingarten and the canal hypersurfaces \( \mathcal{C}^{(1;\lambda)}(s, t, w) \), \( \mathcal{C}^{(2;\lambda)}(s, t, w) \), \( \mathcal{C}^{(3;\lambda)}(s, t, w) \) are \((H, K)_st\)-Weingarten if and only if \( k_1 = 0 \) or \( r(s) = \text{constant} \).

**Proof.** From (2.18) and (2.19), we have
\[
H_s^{(j;\lambda)} K_t^{(j;\lambda)} - H_t^{(j;\lambda)} K_s^{(j;\lambda)} = \left( \frac{2 \varepsilon_2 \lambda^2}{4 \lambda^2 - 1} k_1 r' \left( r'^2 - \lambda \epsilon_1 \right)^2 - \left( \frac{\varepsilon_2 \lambda^2}{4 \lambda^2 - 1} k_1 r' f_j' + 2 r'^2 - \lambda \epsilon_1 + \varepsilon_2 \lambda k_1 r \sqrt{r'^2 - \lambda \epsilon_1} f_j + r r'' \right) \right) f_j
\]
\[
= - \frac{3 r^4 \left( r'^2 - \lambda \epsilon_1 + \varepsilon_2 \lambda k_1 r \sqrt{r'^2 - \lambda \epsilon_1} f_j + r r'' \right)}{3 r^4 \left( r'^2 - \lambda \epsilon_1 + \varepsilon_2 \lambda k_1 r \sqrt{r'^2 - \lambda \epsilon_1} f_j + r r'' \right)^3}.
\] (2.55)

Using (2.21), for \( j = 1, 2, 3 \) the proof can be seen with similar method in Theorem 7. Also, for \( j = 4 \) we have \( f_4 = 0 \) and so \( H_s^{(4;\lambda)} K_t^{(4;\lambda)} - H_t^{(4;\lambda)} K_s^{(4;\lambda)} = 0 \) and this completes the proof. \qed

3. **TUBULAR HYPERSURFACES GENERATED BY NON-NULL CURVES IN \( E^4_1 \)**

In this section, we give the above results, which are given for the canal hypersurfaces in \( E^4_1 \), for tubular hypersurfaces.

**Theorem 9.** The tubular hypersurfaces \( \mathcal{T}^{(j;\lambda)}(s, t, w) \) which are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres in \( E^4_1 \) generated by spacelike or timelike center curves can be parametrized by
\[
\begin{align*}
\mathcal{T}^{(1;1)}(s, t, w) &= \beta(s) \mp r(s) \left( \cos t \cos w F_2(s) + \sin t \cos w F_3 + \sin w F_4 \right), \\
\mathcal{T}^{(2;1)}(s, t, w) &= \beta(s) \mp r(s) \left( \cosh t \sinh w F_2(s) + \cosh w F_3 + \sinh t \sinh w F_4 \right), \\
\mathcal{T}^{(2;1)}(s, t, w) &= \beta(s) \mp r(s) \left( \cosh t \cosh w F_2(s) + \sinh w F_3 + \sinh t \cosh w F_4 \right), \\
\mathcal{T}^{(3;1)}(s, t, w) &= \beta(s) \mp r(s) \left( \sinh t \sinh w F_2(s) + \cosh t \sinh w F_3 + \cosh t \cosh w F_4 \right), \\
\mathcal{T}^{(3;1)}(s, t, w) &= \beta(s) \mp r(s) \left( \sinh t \cosh w F_2(s) + \cosh t \sinh w F_3 + \cosh t \cosh w F_4 \right), \\
\mathcal{T}^{(4;1)}(s, t, w) &= \beta(s) \mp r(s) \left( \cosh w F_2(s) + \sinh t \sinh w F_3 + \sinh t \cosh w F_4 \right), \\
\mathcal{T}^{(4;1)}(s, t, w) &= \beta(s) \mp r(s) \left( \cosh t \cosh w F_3 + \sinh t \coth w F_3 + \cosh t \sinh w F_4 \right).
\end{align*}
\] (3.1)

Also, there is no tubular hypersurface \( \mathcal{T}^{(1;1)}(s, t, w) \) which is formed as the envelope of a family of pseudo hyperbolic hyperspheres in \( E^4_1 \) generated by timelike center curves.

Furthermore the tubular hypersurfaces \( \mathcal{T}^{(j;0)}(s, t, w) \) which are formed as the envelope of a family of null hypercones in \( E^4_1 \) generated by spacelike or timelike center curves can be parametrized by (2.15).

**Proof.** Let the center curve \( \beta : I \subseteq \mathbb{R} \to E^4_1 \) be a arc-length parametrized timelike or spacelike curve with non-zero curvature. Then, the parametrization of the envelope of pseudo hyperspheres
Hence, using (3.6) in (3.4), we have
\[
\text{Theorem 10.}
\]
\[
\begin{align*}
\mathcal{T}^{(j;\lambda)}(s, t, w) - \beta(s) &= a_1(s, t, w)F_1(s) + a_2(s, t, w)F_2(s) + a_3(s, t, w)F_3(s) + a_4(s, t, w)F_4(s). \\
\end{align*}
\]
(3.2)
Furthermore, since \( \mathcal{T}^{(j;\lambda)}(s, t, w) \) lies on the pseudo hyperspheres (resp. pseudo hyperbolic hyperspheres or null hypercones), we have
\[
g(\mathcal{T}^{(j;\lambda)}(s, t, w) - \beta(s), \mathcal{T}^{(j;\lambda)}(s, t, w) - \beta(s)) = \lambda r^2
\]
(3.3)
which leads to from (3.2) that
\[
\varepsilon_1a_1^2 + \varepsilon_2a_2^2 + \varepsilon_3a_3^2 + \varepsilon_4a_4^2 = \lambda r^2
\]
(3.4)
and
\[
\varepsilon_1a_1a_{1s} + \varepsilon_2a_2a_{2s} + \varepsilon_3a_3a_{3s} + \varepsilon_4a_4a_{4s} = 0,
\]
(3.5)
where \( \lambda = 1 \), \( \lambda = -1 \) and \( \lambda = 0 \) if the tubular hypersurfaces are obtained by pseudo hyperspheres, pseudo hyperbolic hyperspheres and null hypercones, respectively; \( r \) is constant radius \( a_i = a_i(s, t, w), a_{is} = \frac{\partial a_i(s, t, w)}{\partial s}, \) and so on.

With similar procedure in the proof of the Theorem 1, we reach that
\[
a = 0.
\]
(3.6)
Hence, using (3.6) in (3.4), we have
\[
\varepsilon_2a_2^2 + \varepsilon_3a_3^2 + \varepsilon_4a_4^2 = \lambda r^2.
\]
(3.7)
From (3.2) and (3.7), we obtain the parametric expressions of the tubular hypersurfaces \( \mathcal{T}^{(j;\lambda)}(s, t, w) \) as (3.1) and (2.15).

Now, let we give the Gaussian and mean curvatures of tubular hypersurfaces \( \mathcal{T}^{(j;\lambda)}(s, t, w) \) given by (3.1) with similar method used for canal hypersurfaces.

**Theorem 10.** The Gaussian and mean curvatures of tubular hypersurfaces \( \mathcal{T}^{(j;\lambda)}(s, t, w) \), given by (3.1) in \( E^4 \), are
\[
\begin{align*}
K^{1;1} &= \frac{k_1\cos t \cos w}{r^2(1+r k_1 \cos t \cos w)}, & H^{1;1} &= \frac{2+3r k_1 \cos t \cos w}{3r(1+r k_1 \cos t \cos w)}, \\
K^{2;1} &= \frac{k_1 \cosh t \sin w}{r^2(1+r k_1 \cosh t \sin w)}, & H^{2;1} &= \frac{2+3r k_1 \cosh t \sin w}{3r(1+r k_1 \cosh t \sin w)}, \\
K^{2;-1} &= \frac{k_1 \cosh t \cosh w}{r^2(1+r k_1 \cosh t \cosh w)}, & H^{2;-1} &= \frac{2+3r k_1 \cosh t \cosh w}{3r(1+r k_1 \cosh t \cosh w)}, \\
K^{3;1} &= \frac{k_1 \sinh t \sin w}{r^2(1+r k_1 \sinh t \sin w)}, & H^{3;1} &= \frac{2+3r k_1 \sinh t \sin w}{3r(1+r k_1 \sinh t \sin w)}, \\
K^{3;-1} &= \frac{k_1 \sinh t \cosh w}{r^2(1+r k_1 \sinh t \cosh w)}, & H^{3;-1} &= \frac{2+3r k_1 \sinh t \cosh w}{3r(1+r k_1 \sinh t \cosh w)}, \\
K^{4;1} &= \frac{k_1 \cosh w}{r^2(1+r k_1 \cosh w)}, & H^{4;1} &= \frac{2+3r k_1 \cosh w}{3r(1+r k_1 \cosh w)}, \\
K^{4;-1} &= \frac{k_1 \sinh w}{r^2(1+r k_1 \sinh w)}, & H^{4;-1} &= \frac{2+3r k_1 \sinh w}{3r(1+r k_1 \sinh w)}.
\end{align*}
\]
(3.8)
From (3.8), we can state the following theorem:

**Theorem 11.** The tubular hypersurfaces \( \mathcal{T}^{(j;\lambda)}(s, t, w) \), given by (3.1) in \( E^4 \), are \((H, K)_{tw-}\) Weingarten, \((H, K)_{ts-}\) Weingarten and \((H, K)_{sw-}\) Weingarten hypersurfaces.
4. VISUALIZATION

In this section, we construct examples for canal hypersurfaces \( C^{(1;\lambda)}(s, t, w) \) and \( C^{(3;\lambda)}(s, t, w) \) which are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres in \( E_1^4 \) with the aid of a timelike curve and a spacelike curve with timelike binormal vector, separately.

Firstly, let we take the unit speed timelike curve

\[
\beta_1(s) = \left(2 \sinh s, 2 \cosh s, \sqrt{3} \cos s, \sqrt{3} \sin s\right) \tag{4.1}
\]

in \( E_1^4 \). The Frenet vectors and curvatures of the curve \( \beta_1 \) are

\[
\begin{align*}
F_1^\beta &= \left(2 \cosh s, 2 \sinh s, -\sqrt{3} \sin s, \sqrt{3} \cos s\right), \\
F_2^\beta &= \left(\frac{2}{\sqrt{7}} \sinh s, \frac{2}{\sqrt{7}} \cosh s, -\sqrt{\frac{2}{7}} \cos s, -\sqrt{\frac{2}{7}} \sin s\right), \\
F_3^\beta &= \left(-\sqrt{3} \cosh s, -\sqrt{3} \sinh s, 2 \sin s, -2 \cos s\right), \\
F_4^\beta &= \left(\frac{\sqrt{3}}{2} \sinh s, \frac{\sqrt{3}}{2} \cosh s, \frac{2}{\sqrt{7}} \cos s, \frac{2}{\sqrt{7}} \sin s\right),
\end{align*}
\]

\[
k_1^\beta = \sqrt{7}, \quad k_2^\beta = 4 \sqrt{\frac{3}{7}}, \quad k_3^\beta = \frac{1}{\sqrt{7}}.
\]  

From (2.14), the canal hypersurfaces \( C^{(1;\lambda)}(s, t, w) \) which are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres in \( E_1^4 \) generated by the curve \( \beta_1 \) are

\[
C^{(1;1)}(s, t, w) = \begin{pmatrix}
-2s \cosh s (4 + \sqrt{15} \cos w \sin t) + \frac{2}{7} \sinh s \left(7 + \sqrt{35} s \left(2 \cos t \cos w + \sqrt{3} \sin w\right)\right), \\
-2s \sinh s (4 + \sqrt{15} \cos w \sin t) + \frac{2}{7} \cosh s \left(7 + \sqrt{35} s \left(2 \cos t \cos w + \sqrt{3} \sin w\right)\right), \\
-4s \sin s \left(\sqrt{3} - \sqrt{5} \cos w \sin t\right) + \cos s \left(\sqrt{3} - 2 \sqrt{\frac{2}{7}} s \left(\sqrt{3} \cos t \cos w - 2 \sin w\right)\right) + 4s \cos s \left(\sqrt{3} - \sqrt{5} \cos w \sin t\right)
\end{pmatrix},
\]

and

\[
C^{(1;\lambda)}(s, t, w) = \begin{pmatrix}
-2s (4 + 3 \cos w \sin t) \cosh s + \frac{2}{7} \left(7 + 2 \sqrt{21} s \cos t \cos w + 3 \sqrt{7} s \sin w\right) \sinh s, \\
-2s (4 + 3 \cos w \sin t) \sinh s + \frac{2}{7} \left(7 + 2 \sqrt{21} s \left(2 \cos t \cos w + \sqrt{3} \sin w\right)\right) \cosh s, \\
4 \sqrt{3} s \left(1 + \cos w \sin t\right) \sin s + \left(\sqrt{3} - \frac{2}{\sqrt{7}} s \left(3 \cos t \cos w - 2 \sqrt{3} \sin w\right)\right) \cos s, \\
\sin s \left(\sqrt{3} - \frac{2}{\sqrt{7}} s \left(3 \cos t \cos w - 2 \sqrt{3} \sin w\right)\right) - 4 \sqrt{3} s \left(1 + \cos w \sin t\right) \cos s
\end{pmatrix},
\]

where the radius function has been taken as \( r(s) = 2s \). From (2.18), (2.19) and (2.20), the Gaussian, mean and principal curvatures of the canal hypersurfaces \( C^{(1;1)} \) and \( C^{(1;\lambda)} \) are obtained as

\[
\begin{align*}
K^{(1;1)} &= \frac{5 \left(\sqrt{35} + 14 s \cos t \cos w\right) \cos t \cos w}{4 s^2 (5 + 2 \sqrt{35} s \cos t \cos w)^2}, \\
H^{(1;1)} &= \frac{1}{3} \left[\frac{s + \frac{5 \left(\sqrt{35} + 14 s \cos t \cos w\right) \cos t \cos w}{\left(5 + 2 \sqrt{35} s \cos t \cos w\right)^2}}\right], \\
\mu_1^{(1;1)} &= \frac{1}{2s}, \quad \mu_2^{(1;1)} = \frac{5 \left(\sqrt{35} + 14 s \cos t \cos w\right) \cos t \cos w}{(5 + 2 \sqrt{35} s \cos t \cos w)^2}, \quad \mu_3^{(1;1)} = \frac{5 \left(\sqrt{35} + 14 s \cos t \cos w\right) \cos t \cos w}{(5 + 2 \sqrt{35} s \cos t \cos w)^2}.
\end{align*}
\]
and

\begin{align*}
K^{1;1;1} &= \frac{3(\sqrt{21} - 14 s \cos t \cos w) \cos t \cos w}{4s^2 \left(3 - 2\sqrt{21} \cos t \cos w \right)^2}, \\
H^{1;1;1} &= \frac{-3\sqrt{5} + 5 \sqrt{21} \cos t \cos w - 42 s^2 \cos^2 t \cos w}{s \left(3 - 2\sqrt{21} \cos t \cos w \right)^2}, \\
\mu_1^{1;1;1} &= \frac{1}{2s}, \quad \mu_2^{1;1;1} = \frac{1}{2s}, \quad \mu_3^{1;1;1} = \frac{3(\sqrt{21} - 14 s \cos t \cos w) \cos t \cos w}{(3 - 2\sqrt{21} \cos t \cos w)^2}, \\
\end{align*}

(4.6)

respectively. In the following figures, one can see the projections of the canal hypersurfaces (4.3) and (4.4) for \( w = 2 \) and \( r(s) = 2s \) into \( x_2x_3x_4 \)-spaces in (A) and (B), respectively.

![Figure 1](image)

Figure 1

Secondly, let we take the unit speed spacelike curve with timelike binormal vector

\[ \beta_2(s) = \left( \sqrt{3} \sinh s, \sqrt{3} \cosh s, 2 \cos s, 2 \sin s \right) \]

(4.7)

in \( E^4_1 \). The Frenet vectors and curvatures of the curve (4.7) are

\begin{align*}
F_1^{\beta_2} &= \left( \sqrt{3} \cosh s, \sqrt{3} \sinh s, -2 \sin s, 2 \cos s \right), \\
F_2^{\beta_2} &= \left( \sqrt{\frac{3}{7}} \sinh s, \sqrt{\frac{3}{7}} \cosh s, -2 \sin s, \frac{2}{\sqrt{7}} \cos s \right), \\
F_3^{\beta_2} &= \left( 2 \cosh s, 2 \sinh s, -\sqrt{3} \sin s, \sqrt{3} \cos s \right), \\
F_4^{\beta_2} &= \left( \frac{2}{\sqrt{7}} \sinh s, \frac{2}{\sqrt{7}} \cosh s, \sqrt{\frac{3}{7}} \cos s, \sqrt{\frac{3}{7}} \sin s \right), \\
\end{align*}

(4.8)

\[ k_1^{\beta_2} = \sqrt{7}, \quad k_2^{\beta_2} = 4 \sqrt{\frac{3}{7}}, \quad k_3^{\beta_2} = \frac{1}{\sqrt{7}}. \]

From (2.14), the canal hypersurfaces \( \mathcal{C}^{(3;1)}(s, t, w) \) which are formed as the envelope of a family of pseudo hyperspheres or pseudo hyperbolic hyperspheres in \( E^4_1 \) generated by the curve (4.7) are

\[ \mathcal{C}^{(3;1)}(s, t, w) = \]

(4.9)

\[ \left( \sqrt{\frac{3}{7}} \left( 28s (-1 + \cosh t \cosh w) \cosh s + (7 + 2\sqrt{21}) \cosh w \sinh t + 4\sqrt{7} \sinh w \right) \right) \sinh s, \]

\[ \sqrt{\frac{3}{7}} \left( 28s (-1 + \cosh t \cosh w) \sinh s + (7 + 2\sqrt{21}) \cosh w \sinh t + 4\sqrt{7} \sinh w \right) \cosh s, \]

\[ -6s \sin s \cosh t \cosh w + 2 (\cos s + 4s \sin s) + \frac{2}{\sqrt{7}} (-2\sqrt{3} \cosh w \sinh t + 3 \sinh w) \cos s, \]

\[ 2s (-4 + 3 \cosh t \cosh w) \cos s + \frac{2}{\sqrt{7}} (7 - 2\sqrt{21} \cosh w \sinh t + 3\sqrt{7} \sinh w) \sin s \]
and

\[ c^{(3; -1)}(s, t, w) = \begin{cases} 
4s \left( \sqrt{3} + \sqrt{5} \cosh t \cosh w \right) \cosh s + \left( \sqrt{3} + 2 \sqrt{\frac{2}{3}} s \left( \sqrt{3} \cosh w \cosh t + 2 \sinh w \right) \right) \sinh s, \\
4s \left( \sqrt{3} + \sqrt{5} \cosh t \cosh w \right) \sinh s + \left( \sqrt{3} + 2 \sqrt{\frac{2}{3}} s \left( \sqrt{3} \cosh w \cosh t + 2 \sinh w \right) \right) \cosh s, \\
-2s \left( 4 + \sqrt{15} \cosh t \cosh w \right) \sin s + \frac{2}{7} \left( 7 + \sqrt{3} \sinh s \right) \cos s, \\
2s \left( 4 + \sqrt{15} \cosh t \cosh w \right) \cos s + \frac{2}{7} \left( 7 + \sqrt{3} \sinh s \right) \cosh s 
\end{cases} \]

(4.10)

where the radius function has been taken as \( r(s) = 2s \). From (2.18), (2.19) and (2.20), the Gaussian, mean and principal curvatures of the canal hypersurfaces \( c^{(3; 1)} \) and \( c^{(3; -1)} \) are obtained as

\[
\begin{align*}
K^{(3; 1)} &= - \frac{4s^2 \left( \sqrt{3} + 2 \sqrt{\frac{2}{3}} s \left( \sqrt{3} \cosh w \cosh t + 2 \sinh w \right) \right) \cosh t \cosh w \sinh s}{4s^2 \left( \sqrt{3} + 2 \sqrt{\frac{2}{3}} s \left( \sqrt{3} \cosh w \cosh t + 2 \sinh w \right) \right)^2}, \\
H^{(3; 1)} &= \frac{3 + 5 \sqrt{21} s \cosh w \sinh t + 42 s^2 \cosh^2 w \sinh^2 t}{3 + 5 \sqrt{21} s \cosh w \sinh t}, \\
\mu_1^{(3; 1)} &= \mu_2^{(3; 1)} = \frac{1}{2s}, \quad \mu_3^{(3; 1)} = - \frac{3 \left( \sqrt{21} + 14 s \cosh w \cosh t \right) c\cosh w \cosh t}{3 + 5 \sqrt{21} s \cosh w \sinh t} 
\end{align*}
\]

(4.11)

and

\[
\begin{align*}
K^{(3; -1)} &= - \frac{5 \left( \sqrt{3} + 2 \sqrt{\frac{2}{3}} s \left( \sqrt{3} \cosh w \cosh t + 2 \sinh w \right) \right) \cosh t \cosh w \sinh t}{4s^2 \left( 5 - 2 \sqrt{3} s \cosh w \sinh t \right)^3}, \\
H^{(3; -1)} &= \frac{1}{8} \left( \frac{1}{s} + \frac{5 \left( \sqrt{3} + 2 \sqrt{\frac{2}{3}} s \left( \sqrt{3} \cosh w \cosh t + 2 \sinh w \right) \right) \cosh t \cosh w \sinh t}{\left( 5 - 2 \sqrt{3} s \cosh w \sinh t \right)^2} \right), \\
\mu_1^{(3; -1)} &= \mu_2^{(3; -1)} = \frac{1}{2s}, \quad \mu_3^{(3; -1)} = \frac{5 \left( \sqrt{3} + 2 \sqrt{\frac{2}{3}} s \left( \sqrt{3} \cosh w \cosh t + 2 \sinh w \right) \right) \cosh t \cosh w \sinh t}{\left( 5 - 2 \sqrt{3} s \cosh w \sinh t \right)^2} 
\end{align*}
\]

(4.12)

respectively. In the following figures, one can see the projections of the canal hypersurfaces (4.9) and (4.10) for \( w = 2 \) and \( r(s) = 2s \) into \( x_2x_3x_4 \)-spaces in (A) and (B), respectively.

**Figure 2**
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