Duality and Representations for New Exotic Bialgebras

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Abstract

We find the exotic matrix bialgebras which correspond to the two nontriangular nonsingular $4 \times 4$ $R$-matrices in the classification of Hietarinta, namely, $R_{S0,3}$ and $R_{S1,4}$. We find two new exotic bialgebras $S03$ and $S14$ which are not deformations of the of the classical algebras of functions on $GL(2)$ or $GL(1|1)$. With this we finalize the classification of the matrix bialgebras which unital associative algebras generated by four elements. We also find the corresponding dual bialgebras of these new exotic bialgebras and study their representation theory in detail. We also discuss in detail a special case of $R_{S1,4}$ in which the corresponding algebra turns out to be a special case of the two-parameter quantum group deformation $GL_{p,q}(2)$.

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1 Introduction

Until now there was no complete list of the matrix bialgebras which are unital associative algebras generated by four elements. Naturally, since the co-product relations are the classical ones we first mention the two related to $GL(2)$, namely, the standard $GL_{pq}(2)$ [1] and nonstandard (Jordanian) $GL_{gh}(2)$ [2] two-parameter deformations. For the supergroup $GL(1|1)$ there are also two: the standard $GL_{pq}(1|1)$ [3–5] and the hybrid (standard-nonstandard) $GL_{gh}(1|1)$ [6] two-parameter deformations. Recently, in [7] it was shown that there are no more deformations of $GL(2)$ or $GL(1|1)$. In particular, it was shown that these four deformations match the distinct triangular $4 \times 4$ $R$-matrices from the classification of [8] which are deformations of the trivial $R$-matrix (corresponding to undeformed $GL(2)$).

Naturally, there are matrix bialgebras generated by four elements, which are not deformations of the classical algebra of functions over the group $GL(2)$ or the supergroup $GL(1|1)$. Those should correspond to $4 \times 4$ $R$-matrices which are not deformations of the trivial $R$-matrix. Studying the classification of [8] we noticed altogether five nonsingular such $R$-matrices. The triangular ones were introduced in [7] and their duals were found and studied in detail in [9]. In the latter paper we called these bialgebras exotic.

In the present paper we finalize the explicit classification of the matrix bialgebras generated by four elements, by studying those that correspond to the two non-triangular nonsingular $4 \times 4$ $R$-matrices of [8], namely, $R_{S0,3}$ and $R_{S1,4}$ which also are not deformations of the trivial $R$-matrix.

The paper is organized as follows. Section 2 just introduces general notation. In Section 3 we study the matrix bialgebra $S_{03}$ which corresponds to $R_{S0,3}$. We find the dual bialgebra $s_{03}$ and study the representation theory of $s_{03}$ in detail. In Sections 4 and 5 we study the matrix bialgebras $S_{14}$ and $S_{14o}$ which correspond to $R_{S1,4}$ for two distinctive regions of the deformation parameter $q$: $q^2 \neq 1$ and $q^2 = 1$, respectively. In both cases we find the corresponding dual bialgebras and their representation theory. In Section 6 we present our conclusions and outlook.

2 Generalities

In this paper we consider matrix bialgebras which are unital associative algebras generated by four elements $a, b, c, d$. The co-product and co-unit relations are the classical ones:

$$
\delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \otimes a + b \otimes c & a \otimes b + b \otimes d \\ c \otimes a + d \otimes c & c \otimes b + d \otimes d \end{pmatrix} \tag{2.1a}
$$

$$
\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.1b}
$$
However, the bialgebras under consideration are not Hopf algebras, except one. This shall be discussed separately in each case.

It shall be convenient to make the following change of generators:

\[ \tilde{a} = \frac{1}{2}(a + d), \quad \tilde{d} = \frac{1}{2}(a - d), \quad \tilde{b} = \frac{1}{2}(b + c), \quad \tilde{c} = \frac{1}{2}(b - c). \] (2.2)

With the new generators we have:

\[
\begin{align*}
\delta \left( \begin{array}{c} \tilde{a} \\ \tilde{b} \\ \tilde{c} \\ \tilde{d} \end{array} \right) &= \left( \begin{array}{cccc} \tilde{a} \otimes \tilde{a} + \tilde{b} \otimes \tilde{b} - \tilde{c} \otimes \tilde{c} + \tilde{d} \otimes \tilde{d} & \tilde{a} \otimes \tilde{b} + \tilde{b} \otimes \tilde{a} - \tilde{c} \otimes \tilde{d} + \tilde{d} \otimes \tilde{c} \\ \tilde{a} \otimes \tilde{c} + \tilde{c} \otimes \tilde{a} - \tilde{b} \otimes \tilde{d} + \tilde{d} \otimes \tilde{b} & \tilde{a} \otimes \tilde{d} + \tilde{d} \otimes \tilde{a} - \tilde{b} \otimes \tilde{c} + \tilde{c} \otimes \tilde{b} \end{array} \right) \\
\varepsilon \left( \begin{array}{c} \tilde{a} \\ \tilde{b} \\ \tilde{c} \\ \tilde{d} \end{array} \right) &= \left( \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \end{array} \right)
\end{align*}
\] (2.3)

3 Algebra S03

3.1 Bialgebra relations

In this Section we consider the matrix bialgebra S03. We obtain it by applying the RTT relations of [10]:

\[ R T_1 T_2 = T_2 T_1 R , \] (3.1)

where \( T_1 = T \otimes 1_2 \), \( T_2 = 1_2 \otimes T \), for the case when \( R = R_{S03} \), where:

\[ R_{S03} \equiv \left( \begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{array} \right) \] (3.2)

This \( R \)-matrix is given in [8].

The relations which follow from (3.1) and (3.2) are:

\[ b^2 + c^2 = 0 , \quad a^2 - d^2 = 0 , \] (3.3)

\[ cd = ba , \quad dc = -ab , \]
\[ bd = ca , \quad db = -ac , \]
\[ da = ad , \quad cb = -bc . \]

In terms of the generators \( \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \) we have:

\[ \tilde{b}^2 = \tilde{c}^2 = 0 , \quad \tilde{a}\tilde{d} = \tilde{d}\tilde{a} = 0 , \] (3.4)
\[ \tilde{a}\tilde{b} = 0 , \quad \tilde{b}\tilde{d} = 0 , \]
\[ \tilde{d}\tilde{c} = 0 , \quad \tilde{c}\tilde{a} = 0 . \]
In view of the above relations we conclude that this bialgebra has no PBW basis. Indeed, the ordering following from (3.4) is cyclic:
\[ \tilde{a} > \tilde{c} > \tilde{d} > \tilde{b} > \tilde{a} \] (3.5)

Thus, the basis consists of building blocks like \( \tilde{a}^k \tilde{c} \tilde{d}^\ell \tilde{b} \) and cyclic. Explicitly the basis can be described by the following monomials:

\( \tilde{a}^{k_1} \tilde{c} \tilde{d}^{\ell_1} \tilde{b} \cdots \tilde{a}^{k_n} \tilde{c} \tilde{d}^{\ell_n} \tilde{b} \tilde{a}^{k_{n+1}} \), \( n, k_i, \ell_i \in \mathbb{Z}_+ \) (3.6a)

\( \tilde{d}^{\ell_1} \tilde{b} \tilde{a}^{k_1} \tilde{c} \cdots \tilde{d}^{\ell_n} \tilde{b} \tilde{a}^{k_n} \), \( n, k_i, \ell_i \in \mathbb{Z}_+ \) (3.6b)

\( \tilde{a}^{k_1} \tilde{c} \tilde{d}^{\ell_1} \tilde{b} \cdots \tilde{a}^{k_n} \tilde{c} \tilde{d}^{\ell_n}, \) \( n, k_i, \ell_i \in \mathbb{Z}_+ \) (3.6c)

\( \tilde{d}^{\ell_1} \tilde{b} \tilde{a}^{k_1} \tilde{c} \cdots \tilde{d}^{\ell_n} \tilde{b} \tilde{a}^{k_n} \tilde{c} \tilde{d}^{\ell_{n+1}} \), \( n, k_i, \ell_i \in \mathbb{Z}_+ \) (3.6d)

We shall call the elements of the basis 'words'. The one-letter words are the generators \( \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \); they are obtained from (3.6a), (3.6b), (3.6c), (3.6d), resp., for \( n = 0, k_1 = 1, n = 1, k_1 = \ell_1 = 0, n = 1, k_1 = \ell_1 = 0, n = 0, \ell_1 = 1, \) resp. The unit element \( 1_A \) is obtained from (3.6b) or (3.6c) for \( n = 0 \).

### 3.2 Dual algebra

Two bialgebras \( \mathcal{U}, \mathcal{A} \) are said to be in duality \cite{11} if there exists a doubly nondegenerate bilinear form

\[ \langle \ , \ \rangle : \mathcal{U} \times \mathcal{A} \longrightarrow \mathbb{C}, \quad \langle \ , \ \rangle : (u, a) \mapsto \langle u, a \rangle, \quad u \in \mathcal{U}, \ a \in \mathcal{A} \] (3.7)

such that, for \( u, v, a, b \in \mathcal{U}, \mathcal{A} \):

\[ \langle u, ab \rangle = \langle \delta_{\mathcal{U}}(u), a \otimes b \rangle, \quad \langle uv, a \rangle = \langle u \otimes v, \delta_{\mathcal{A}}(a) \rangle \] (3.8a)

\[ \langle 1_{\mathcal{U}}, a \rangle = \varepsilon_{\mathcal{A}}(a), \quad \langle u, 1_{\mathcal{A}} \rangle = \varepsilon_{\mathcal{U}}(u) \] (3.8b)

Two Hopf algebras \( \mathcal{U}, \mathcal{A} \) are said to be in duality \cite{11} if they are in duality as bialgebras and if

\[ \langle \gamma_{\mathcal{U}}(u), a \rangle = \langle u, \gamma_{\mathcal{A}}(a) \rangle \] (3.9)

It is enough to define the pairing (3.7) between the generating elements of the two algebras. The pairing between any other elements of \( \mathcal{U}, \mathcal{A} \) follows then from relations (3.8) and the standard bilinear form inherited by the tensor product.

The duality between two bialgebras or Hopf algebras may be used also to obtain the unknown dual of a known algebra. For that it is enough to give the pairing between the generating elements of the unknown algebra with arbitrary elements of the basis of the known algebra. Using these initial pairings and the duality properties one may find the unknown algebra. One such possibility is given in \cite{10}. However, their approach is not universal. In particular, it is not enough for the algebras considered here, (as will become clear) and will be used only as consistency check.
Another approach was initiated by Sudbery [12]. He obtained $U_q(sl(2)) \otimes U(u(1))$ as the algebra of tangent vectors at the identity of $GL_q(2)$. The initial pairings were defined through the tangent vectors at the identity. However, such calculations become very difficult for more complicated algebras. Thus, in [13] a generalization was proposed in which the initial pairings are postulated to be equal to the classical undeformed results. This generalized method was applied in [13] to the standard two-parameter deformation $GL_{p,q}(2)$, (where also Sudbery’s method was used), then in [14] to the multiparameter deformation of $GL(n)$, in [15] to the matrix quantum Lorentz group of [16], in [17] to the Jordanian two-parameter deformation $GL_{g,h}(2)$, in [6] to the hybrid two-parameter deformation of the superalgebra $GL_{q,h}(1|1)$, in [18] to the multiparameter deformation of the superalgebra $GL(m|n)$, in [9] to the firstly discussed three exotic bialgebras. (We note that the dual of $GL_{p,q}(2)$ was obtained also in [19] by methods of $q$-differential calculus.)

Let us denote by $s_{03}$ the unknown yet dual algebra of $S_{03}$, and by $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ the four generators of $s_{03}$. Like in [13] we define the pairing $\langle Z, f \rangle$, $Z = \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, $f$ is from (3.6), as the classical tangent vector at the identity:

$$\langle Z, f \rangle \equiv \varepsilon \left( \frac{\partial f}{\partial y} \right),$$

(3.10)

where $(Z, y) = (\tilde{A}, \tilde{a}), (\tilde{B}, \tilde{b}), (\tilde{C}, \tilde{c}), (\tilde{D}, \tilde{d})$. Explicitly, we get:

$$\langle \tilde{A}, f \rangle = \varepsilon \left( \frac{\partial f}{\partial \tilde{a}} \right) = \begin{cases} k & \text{for } f = \tilde{a}^k \\ 0 & \text{otherwise} \end{cases}$$

(3.11a)

$$\langle \tilde{B}, f \rangle = \varepsilon \left( \frac{\partial f}{\partial \tilde{b}} \right) = \begin{cases} 1 & \text{for } f = \tilde{b}\tilde{a}^k \\ 0 & \text{otherwise} \end{cases}$$

(3.11b)

$$\langle \tilde{C}, f \rangle = \varepsilon \left( \frac{\partial f}{\partial \tilde{c}} \right) = \begin{cases} 1 & \text{for } f = \tilde{a}^k\tilde{c} \\ 0 & \text{otherwise} \end{cases}$$

(3.11c)

$$\langle \tilde{D}, f \rangle = \varepsilon \left( \frac{\partial f}{\partial \tilde{d}} \right) = \begin{cases} 1 & \text{for } f = \tilde{d} \\ 0 & \text{otherwise} \end{cases}$$

(3.11d)

Using the above we obtain:

**Proposition 1:** The generators $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ introduced above obey the following
relations:

\[ [\tilde{A}, Z] = 0, \quad Z = \tilde{B}, \tilde{C}, \quad (3.12a) \]
\[ \tilde{A}D = \tilde{D}A = \tilde{D}^3 = \tilde{B}^2 \tilde{D} = \tilde{D} \tilde{B}^2 = \tilde{D}, \quad (3.12b) \]
\[ [	ilde{B}, \tilde{C}] = -2 \tilde{D}, \quad (3.12c) \]
\[ \tilde{D} \tilde{B} = -\tilde{B} \tilde{D} = \tilde{C} \tilde{D}^2 = \tilde{D}^2 \tilde{C}, \quad (3.12d) \]
\[ \{\tilde{C}, \tilde{D}\} = 0, \quad (3.12e) \]
\[ \tilde{B}^2 + \tilde{C}^2 = 0, \quad (3.12f) \]
\[ \tilde{B}^3 = \tilde{B}, \quad (3.12g) \]
\[ \tilde{C}^3 = -\tilde{C}, \quad (3.12h) \]
\[ \tilde{B}^2 \tilde{A} = \tilde{A}. \quad (3.12i) \]

\[ \delta_U(\tilde{A}) = \tilde{A} \otimes 1_U + 1_U \otimes \tilde{A} \quad (3.13a) \]
\[ \delta_U(\tilde{B}) = \tilde{B} \otimes 1_U + (1_U - \tilde{B}^2) \otimes \tilde{B} \quad (3.13b) \]
\[ \delta_U(\tilde{C}) = \tilde{C} \otimes (1_U - \tilde{B}^2) + 1_U \otimes \tilde{C} \quad (3.13c) \]
\[ \delta_U(\tilde{D}) = \tilde{D} \otimes (1_U - \tilde{B}^2) + (1_U - \tilde{B}^2) \otimes \tilde{D} \quad (3.13d) \]
\[ \varepsilon_U(Z) = 0, \quad Z = \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}. \quad (3.13e) \]

\( \tilde{A}, \tilde{B}^2 = -\tilde{C}^2 \) and \( \tilde{D}^2 \) are Casimir operators. The bialgebra s03 is not a Hopf algebra.

**Proof:** Using the assumed duality the algebraic relations (3.12) are shown by calculating their pairings with the basis monomials \( f \) from (3.6). In particular, we have
(giving only the nonzero pairings):

\[
\langle \tilde{A}\tilde{B} , f \rangle = \langle \tilde{B}\tilde{A} , f \rangle = k + 1 , \quad \text{for } f = \tilde{b}\tilde{a}^k \quad (3.14a)
\]

\[
\langle \tilde{A}\tilde{C} , f \rangle = \langle \tilde{C}\tilde{A} , f \rangle = k + 1 , \quad \text{for } f = \tilde{a}^k\tilde{c} \quad (3.14b)
\]

\[
\langle \tilde{A}\tilde{D} , f \rangle = \langle \tilde{D}\tilde{A} , f \rangle = \langle \tilde{D}^3 , f \rangle = \langle \tilde{B}^2\tilde{D} , f \rangle = \langle \tilde{D}\tilde{B}^2 , f \rangle = - \langle \tilde{B}\tilde{C} , f \rangle = \langle \tilde{C}\tilde{B} , f \rangle = \langle \tilde{D} , f \rangle = 1 , \quad \text{for } f = \tilde{d} \quad (3.14c)
\]

\[
\langle \tilde{B}\tilde{C} , f \rangle = \langle \tilde{C}\tilde{B} , f \rangle = 1 , \quad \text{for } f = \tilde{b}\tilde{a}^k\tilde{c} \quad (3.14d)
\]

\[
\langle \tilde{D}\tilde{B} , f \rangle = - \langle \tilde{B}\tilde{D} , f \rangle = \langle \tilde{C}\tilde{D}^2 , f \rangle = \langle \tilde{D}^2\tilde{C} , f \rangle = 1 , \quad \text{for } f = \tilde{c} \quad (3.14e)
\]

\[
\langle \tilde{D}\tilde{C} , f \rangle = - \langle \tilde{C}\tilde{D} , f \rangle = 1 , \quad \text{for } f = \tilde{b} \quad (3.14f)
\]

\[
\langle \tilde{B}^2 , f \rangle = - \langle \tilde{C}^2 , f \rangle = 1 , \quad \text{for } f = \tilde{a}^k \quad (3.14g)
\]

\[
\langle \tilde{B}^3 , f \rangle = \langle \tilde{B} , f \rangle = 1 , \quad \text{for } f = \tilde{b}\tilde{a}^k \quad (3.14h)
\]

\[
\langle -\tilde{C}^3 , f \rangle = \langle \tilde{C} , f \rangle = 1 , \quad \text{for } f = \tilde{a}^k\tilde{c} \quad (3.14i)
\]

\[
\langle \tilde{B}^2\tilde{A} , f \rangle = \langle \tilde{A} , f \rangle = k , \quad \text{for } f = \tilde{a}^k \quad (3.14j)
\]

For the proof of (3.14h,i) is used (3.14g). The facts that \( \tilde{A}, \tilde{B}^2 = -\tilde{C}^2 \) and \( \tilde{D}^2 \) are Casimir operators follow easily from (3.12) having in mind also (3.14g,j) and that \( \langle \tilde{D}^2, \tilde{a} \rangle = 1 \) is the only nonzero pairing of \( \tilde{D}^2 \). For the proof of (3.13a-d) we use the duality property (3.8a)

\[
\langle Z , f_1 f_2 \rangle = \langle \delta_U(Z) , f_1 \otimes f_2 \rangle
\]

for every generator \( Z \) of \( s_{03} \) and for every \( f_1, f_2 \in S_{03} \), then we calculate separately the LHS and RHS and compare the results. We also use that

\[
\tilde{B}^2 Z = Z , \quad \text{for } Z = \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \quad (3.15)
\]

- this is contained in (3.12) except for \( Z = \tilde{C} \) for which we use that \( \langle \tilde{B}^2\tilde{C}, \tilde{a}^k\tilde{c} \rangle = 1 \) is the only nonzero pairing of \( \tilde{B}^2\tilde{C} \). Formulae (3.13e) follow from \( \varepsilon_U(Z) = \langle Z, 1_A \rangle \), cf. (3.8b), and the defining relations (3.11). To show that \( s_{03} \) is not a Hopf algebra we suppose that it is, i.e., there should exist an antipode \( \gamma \). Then then we use one of the Hopf algebra axioms:

\[
m \circ (\text{id} \otimes \gamma) \circ \delta = i \circ \varepsilon \quad (3.16)
\]

as maps \( s_{03} \rightarrow s_{03} \), where \( m \) maps to the usual product in the algebra: \( m(Y \otimes Z) = YZ \), \( Y, Z \in s_{03} \) and \( i \) is the natural embedding of the number field \( \mathbb{C} \) into \( s_{03} \):
\( i(\mu) = \mu 1_U, \ \mu \in \mathbb{C}. \) Applying this to the generator \( \tilde{B} \) we would get:

\[
\tilde{B} + \left( 1_U - \tilde{B}^2 \right) \gamma(\tilde{B}) = 0
\]

which is a contradiction, since \( 1_U - \tilde{B}^2 \) is zero when multiplied by anything except by \( 1_U \), and in the latter case the product is not equal to \( -\tilde{B} \). From this we see that \( \gamma \) can not be defined on \( \tilde{C} \) and \( \tilde{D} \) since their coproducts involve \( \tilde{B} \). The antipode may be introduced only if we restrict to the subalgebra generated by \( \tilde{A} \), but the bialgebra \( s03 \) as a whole is not a Hopf algebra.

\[\text{Corollary 1:} \] The algebra generated by the generator \( \tilde{A} \) is a sub-bialgebra of \( s03 \). The algebra \( s03' \) generated by the generators \( \tilde{B}, \tilde{C}, \tilde{D} \) is a nine-dimensional sub-bialgebra of \( s03 \) with PBW basis:

\[
1_U, \ \tilde{B}, \ \tilde{C}, \ \tilde{D}, \ \tilde{B}\tilde{C}, \ \tilde{B}\tilde{D}, \ \tilde{D}\tilde{C}, \ \tilde{B}^2, \ \tilde{D}^2
\]

\[\text{Proof:} \] The statement follows immediately from relations (3.12,3.13). We comment only the PBW basis of the subalgebra \( s03' \). Indeed, a priori it has a PBW basis:

\[
\tilde{B}^k \tilde{D}^\ell \tilde{C}^m, \quad k, \ell \leq 2, \ m \leq 1
\]

the restrictions following from (3.12b,f,g). Furthermore it is easy to see that there are no cubic (and consequently higher order) elements of the basis. For some of the cubic elements this is clear from (3.12). For the rest we have:

\[
\begin{align*}
\tilde{B}\tilde{D}\tilde{C} &= -\tilde{D}^2\tilde{C}^2 = \tilde{D}^2\tilde{B}^2 = \tilde{D}^2 \\
\tilde{B}^2\tilde{C} &= -\tilde{C}^3 = \tilde{C} \\
\tilde{B}\tilde{D}^2 &= -\tilde{C}\tilde{D}^3 = \tilde{D}\tilde{C}
\end{align*}
\]

also using (3.12). Thus, the basis is given by (3.17) the algebra is indeed nine-dimensional.

\[\text{Remark 1:} \] The algebra \( s03 \) is not the direct sum of the two subalgebras described in the preceding Corollary since both subalgebras have nontrivial action on each other, e.g., \( \tilde{B}^2\tilde{A} = \tilde{A}, \ \tilde{A}\tilde{D} = \tilde{D} \). The algebra \( s03 \) is a nine-dimensional associative algebra over the central algebra generated by \( \tilde{A} \).

### 3.3 Regular representation

We start with the study of the left regular representation (LRR) of the subalgebra \( s03' \). For this we need the left multiplication table:

|    | 1  | \( \tilde{B} \) | \( \tilde{C} \) | \( \tilde{B}^2 \) | \( \tilde{B}\tilde{C} \) | ... |
|----|----|----------------|----------------|----------------|----------------|----|
| \( \tilde{B} \) | \( \tilde{B} \) | \( \tilde{B}^2 \) | \( \tilde{B}\tilde{C} \) | \( \tilde{B} \) | \( \tilde{C} \) | ... |
| \( \tilde{C} \) | \( \tilde{C} \) | \( \tilde{B}\tilde{C} + 2\tilde{D} \) | \( -\tilde{B}^2 \) | \( \tilde{C} \) | \( -\tilde{B} + 2\tilde{D}\tilde{C} \) | ... |
| \( \tilde{D} \) | \( \tilde{D} \) | \( -\tilde{B}\tilde{D} \) | \( \tilde{D}\tilde{C} \) | \( \tilde{D} \) | \( -\tilde{D}^2 \) | ... |
The LRR hence contains the subrepresentation generated as a vector space by 
\{\tilde{D}, \tilde{D}^2, \tilde{B}\tilde{D}, \tilde{D}\tilde{C}\}, which decomposes into two two-dimensional irreps
\begin{align}
v_1^0 &= \tilde{D} + \tilde{D}^2, \quad v_1^1 = \tilde{B}\tilde{D} + \tilde{D}\tilde{C}, \\
\tilde{B}\begin{pmatrix} v_0^1 \\ v_1^1 \end{pmatrix} &= \begin{pmatrix} v_0^1 \\ v_1^1 \end{pmatrix}, \\
\tilde{C}\begin{pmatrix} v_0^1 \\ v_1^1 \end{pmatrix} &= \begin{pmatrix} -v_1^1 \\ v_0^1 \end{pmatrix}, \\
\tilde{D}\begin{pmatrix} v_0^1 \\ v_1^1 \end{pmatrix} &= \begin{pmatrix} v_0^1 \\ -v_1^1 \end{pmatrix}
\end{align}
(3.20)
and
\begin{align}
v_0^2 &= \tilde{B}\tilde{D} - \tilde{D}\tilde{C}, \quad v_1^2 = \tilde{D} - \tilde{D}^2, \\
\tilde{B}\begin{pmatrix} v_0^2 \\ v_1^2 \end{pmatrix} &= \begin{pmatrix} v_0^2 \\ v_1^2 \end{pmatrix}, \\
\tilde{C}\begin{pmatrix} v_0^2 \\ v_1^2 \end{pmatrix} &= \begin{pmatrix} -v_1^2 \\ v_0^2 \end{pmatrix}, \\
\tilde{D}\begin{pmatrix} v_0^2 \\ v_1^2 \end{pmatrix} &= \begin{pmatrix} v_0^2 \\ -v_1^2 \end{pmatrix}
\end{align}
(3.21)
These two irreps are isomorphic by the map \((v_0^1, v_1^1) \rightarrow (v_0^2, v_1^2)\). On both of them the Casimirs \(\tilde{B}^2, \tilde{D}^2\) take the value 1. (Also the Casimir \(\tilde{A}\) of s03 has the value 1.)

The LRR contains also the trivial one-dimensional representation generated by the vector \(v = \tilde{B}^2 - 1_U\). On this vector all Casimirs and moreover all generators of s03 take the value 0.

The quotient of the LRR by the above three sub-modules has the following multiplication table:

\[
\begin{array}{cccc}
1 & \tilde{B} & \tilde{C} & \tilde{B}\tilde{C} \\
\tilde{B} & \tilde{B} & \tilde{B}\tilde{C} & \tilde{C} \\
\tilde{C} & \tilde{C} & \tilde{B}\tilde{C} & -\tilde{B}^2 \\
\tilde{D} & 0 & 0 & 0 & 0 \\
\end{array}
\]

Thus the quotient decomposes into a direct sum of four one-dimensional representations, generated as vector spaces by
\[v_{\epsilon,\epsilon'} = \tilde{B} + \epsilon 1_U - i\epsilon\epsilon'\tilde{C} - i\epsilon\tilde{B}\tilde{C}, \quad \epsilon, \epsilon' = \pm\]
(3.24)
On the latter vectors we have the following action:
\[
\tilde{B}v_{\epsilon,\epsilon'} = \epsilon v_{\epsilon,\epsilon'}, \quad \tilde{C}v_{\epsilon,\epsilon'} = i\epsilon' v_{\epsilon,\epsilon'}, \quad \tilde{D}v_{\epsilon,\epsilon'} = 0.
\]
(3.25)
Obviously, on all of them the Casimirs \(\tilde{B}^2, \tilde{D}^2\) take the values 1, 0, respectively. However, these four representations are not isomorphic to each other.

To summarize, there are seven irreps of s03' which are obtained from the LRR:
• one-dimensional trivial (all generators act by zero)
• two-dimensional with both Casimirs $\tilde{B}^2, \tilde{D}^2$ having value 1.
• four one-dimensional with Casimir values $1, 0$ for $\tilde{B}^2, \tilde{D}^2$, respectively.

Turning to the algebra $s_{03}$ we note that it inherits the representation structure of its subalgebra $s_{03}'$. On the representations (3.20,3.22) the Casimir $\tilde{A}$ has the value 1, while on the trivial irrep generated by $v = \tilde{B}^2 - 1_{1d}$ the Casimir $\tilde{A}$ has the value 0. However, on the one-dimensional irreps generated by (3.24) the Casimir $\tilde{A}$ has no fixed value. Thus, the list of the irreps of $s_{03}$ arising from the LRR is:
• one-dimensional trivial
• two-dimensional with all Casimirs $\tilde{A}, \tilde{B}^2, \tilde{D}^2$ having value 1.
• four one-dimensional with Casimir values $\mu, 1, 0$ for $\tilde{A}, \tilde{B}^2, \tilde{D}^2$, respectively, $\mu \in \mathbb{C}$.

Finally, we note that we could have studied also the right regular representation of $s_{03}$. The list of irreps would be the same as the one obtained above.

3.4 Weight representations

Here we consider weight representations. These are representations which are built from the action of the algebra on a weight vector with respect to one of the generators. We start with a weight vector $v_0$ such that:

$$\tilde{D} v_0 = \lambda v_0$$

(3.26)

where $\lambda \in \mathbb{C}$ is the weight. As we shall see the cases $\lambda \neq 0$ and $\lambda = 0$ are very different.

We start with $\lambda \neq 0$. In that case from $\tilde{D}^3 = \tilde{D}$ follows that $\lambda^2 = 1$, while from $\tilde{B}^2 \tilde{D} = \tilde{D}$ follows that $\tilde{B}^2 v_0 = v_0$. Further, from (3.12d) follows that $\tilde{C} v_0 = -\lambda \tilde{B} v_0$. Thus, acting with the elements of $s_{03}$ on $v_0$ we obtain a two-dimensional representation, e.g.:

$$v_0, \tilde{B} v_0$$

(3.27)

(and we could have chosen $v_0, \tilde{C} v_0$ as its basis). This representation is irreducible. The action is given as follows:

|   | $v_0$ | $\tilde{B} v_0$ |
|---|---|---|
| $\tilde{B}$ | $\tilde{B} v_0$ | $v_0$ |
| $\tilde{C}$ | $-\lambda \tilde{B} v_0$ | $\lambda v_0$ |
| $\tilde{D}$ | $\lambda v_0$ | $-\lambda \tilde{B} v_0$ |
Both Casimirs $\tilde{B}^2, \tilde{D}^2$ take the value 1.

Let now $\lambda = 0$. In this case acting with the elements of $s03$ on $v_0$ we obtain a five-dimensional representation:

$$v_0, \tilde{B} v_0, \tilde{C} v_0, \tilde{B} \tilde{C} v_0, \tilde{B}^2 v_0.$$ (3.28)

This representation is reducible. It has a one-dimensional subrepresentation spanned by the vector $w = v v_0 = (\tilde{B}^2 - 1_U)v_0$. This is the trivial representation since all generators act by zero on it. After we factor out this representation the factor-representation splits into four one-dimensional representations spanned by the following vectors $w_{\epsilon, \epsilon'} = v_{\epsilon, \epsilon'} v_0$, where $v_{\epsilon, \epsilon'}$ are from (3.24) and the action of the generators is as given in (3.25). Thus, these irreps are as those obtained from the LRR.

To summarize, there are six irreps of $s03'$ which are obtained as weight irreps of the generator $\tilde{D}$:

- one-dimensional trivial
- one two-dimensional with both Casimirs $\tilde{B}^2, \tilde{D}^2$ having value 1.
- four one-dimensional with Casimir values 1, 0 for $\tilde{B}^2, \tilde{D}^2$, respectively.

Turning to the algebra $s03$ we note that it inherits the representation structure of its subalgebra $s03'$, however, the value of the Casimir $\tilde{A}$ is not fixed except on the trivial irrep. Thus, the list of the irreps of $s03$ which are obtained as weight irreps of the generator $\tilde{D}$ is:

- one-dimensional trivial
- one two-dimensional with Casimir values $\mu, 1, 1$ for $\tilde{A}, \tilde{B}^2, \tilde{D}^2$, respectively, $\mu \in \mathbb{C}$.
- four one-dimensional with Casimir values $\mu, 1, 0$ for $\tilde{A}, \tilde{B}^2, \tilde{D}^2$, respectively, $\mu \in \mathbb{C}$.

Finally, we note that it is not possible to construct weight representations w.r.t. generator $\tilde{B}$ (or $\tilde{C}$).

### 3.5 Representations of $s03$ on $S03$

Here we shall study the representations of $s03$ obtained by the use of its right regular action (RRA) on the dual bialgebra $S03$. The RRA is defined as follows:

$$\pi_R(Z)f \equiv f_{(1)} \langle Z, f_{(2)} \rangle, \quad Z \in s03, \quad Z \neq 1_U, \quad f \in S03,$$ (3.29a)

$$\pi_R(1_U)f \equiv f, \quad f \in S03.$$ (3.29b)
where we use Sweedler’s notation for the co-product: \( \delta(f) = f_{(1)} \otimes f_{(2)} \). (Note that we can not use the left regular action since that would be given by the formula: \( \pi_L(Z)f = \langle \gamma_U(Z), f_{(1)} \rangle f_{(2)} \) and we do not have an antipode.) More explicitly, for the generators of \( s03 \) we have:

\[
\begin{align*}
\pi_R(\tilde{A})(\tilde{a} \quad \tilde{b} \quad \tilde{c} \quad \tilde{d}) &= (\tilde{a} \quad \tilde{b}) \\
\pi_R(\tilde{B})(\tilde{a} \quad \tilde{b} \quad \tilde{c} \quad \tilde{d}) &= (\tilde{b} \quad \tilde{a}) \\
\pi_R(\tilde{C})(\tilde{a} \quad \tilde{b} \quad \tilde{c} \quad \tilde{d}) &= (-\tilde{c} \quad \tilde{d}) \\
\pi_R(\tilde{D})(\tilde{a} \quad \tilde{b} \quad \tilde{c} \quad \tilde{d}) &= (\tilde{d} \quad -\tilde{c})
\end{align*}
\]

(3.30a–d)

\[\pi_R(Z) 1_A = 1_A \langle Z, 1_A \rangle = 1_A \varepsilon_U(Z) = 0, \quad Z = \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}
\]

(3.30e)

For the action on the elements (words) of \( S03 \) we use a Corollary of (3.29):

\[\pi_R(Z)fg = \pi_R(\delta_U(Z))(f \otimes g)
\]

(3.31)

where \( f, g \) are arbitrary words from (3.6). Further we shall need the notion of the 'length' \( \ell(f) \) of the word \( f \). It is defined naturally as the number of the letters of \( f \); in addition we set \( \ell(1_A) = 0 \). Now we obtain from (3.31):

\[
\begin{align*}
\pi_R(\tilde{A})f &= \ell(f)f \\
\pi_R(\tilde{B})f \cdot g &= (\pi_R(\tilde{B})f) \cdot g \\
\pi_R(\tilde{C})f \cdot g &= f \cdot (\pi_R(\tilde{C})g) \\
\pi_R(\tilde{D})f &= 0, \quad \text{if } \ell(f) > 1
\end{align*}
\]

(3.32a–d)

From (3.32b,c) it is obvious that the only nonzero action of \( \tilde{B}, \tilde{C} \) actually is:

\[
\begin{align*}
\pi_R(\tilde{B})(\tilde{a} \quad \tilde{b} \quad \tilde{c} \quad \tilde{d}) \cdot f &= (\tilde{b} \quad \tilde{a}) \cdot f \\
\pi_R(\tilde{C})f \cdot (\tilde{a} \quad \tilde{b} \quad \tilde{c} \quad \tilde{d}) &= f \cdot (\tilde{c} \quad \tilde{d})
\end{align*}
\]

(3.33a,b)

From (3.32a) it is obvious that we can classify the irreps by the value \( \mu_A \) of the Casimir \( \tilde{A} \) which runs over the nonnegative integers. For fixed \( \mu_A \) the basis of the corresponding representations is spanned by the words \( f \) such that \( \ell(f) = \mu_A \). Thus, we have:

- \( \mu_A = 0 \)

This is the one-dimensional trivial representation spanned by the unit element \( 1_A \) on which all generators of \( s03 \) have zero action.
\[ \mu_A = 1 \]

This representation is four-dimensional spanned by the four generators \( \tilde{a}, \tilde{b}, \tilde{c}, \tilde{d} \) of \( S_03 \). It is reducible and decomposes in two two-dimensional irreps with basis vectors:

\[
\begin{align*}
 v_0^1 &= \tilde{a} + \tilde{d} = a , \quad v_1^1 = \tilde{b} + \tilde{c} = b , \\
 v_2^1 &= \tilde{b} - \tilde{c} = c , \quad v_1^2 = \tilde{a} - \tilde{d} = d .
\end{align*}
\] (3.34)

The RRA of \( \tilde{B}, \tilde{C}, \tilde{D} \) on these vectors is as (3.21,3.23):

\[
\begin{align*}
 \pi_R(\tilde{B}) \begin{pmatrix} v_0^k \\ v_1^k \end{pmatrix} &= \begin{pmatrix} v_1^k \\ -v_0^k \end{pmatrix} , \\
 \pi_R(\tilde{C}) \begin{pmatrix} v_0^k \\ v_1^k \end{pmatrix} &= \begin{pmatrix} -v_1^k \\ v_0^k \end{pmatrix} , \\
 \pi_R(\tilde{D}) \begin{pmatrix} v_0^k \\ v_1^k \end{pmatrix} &= \begin{pmatrix} v_0^k \\ -v_1^k \end{pmatrix} .
\end{align*}
\] (3.35)

These two irreps are isomorphic by the map \((v_0^1, v_1^1) \rightarrow (v_0^2, v_1^2)\). On both of them the Casimirs \( \tilde{B}^2, \tilde{D}^2 \) take the value 1.

\[ \mu_A = 2 \]

This representation is eight-dimensional spanned by \( \tilde{a}, \tilde{a}'\tilde{c}, \tilde{b}, \tilde{b}'\tilde{c}, \tilde{c}'\tilde{b}, \tilde{c}'\tilde{d}, \tilde{d}'\tilde{b} \). It is reducible and decomposes in eight one-dimensional irreps with basis vectors:

\[
\begin{align*}
 v_{1,\epsilon,\epsilon'} &= (\tilde{a} + \epsilon \tilde{b})(\tilde{a} + i\epsilon' \tilde{c}) \\
 v_{2,\epsilon,\epsilon'} &= (\tilde{d} + \epsilon \tilde{c})(\tilde{d} + i\epsilon' \tilde{b}) \\
 \epsilon, \epsilon' &= \pm
\end{align*}
\] (3.36)

The RRA of \( \tilde{B}, \tilde{C}, \tilde{D} \) on these vectors is as (3.25):

\[
\begin{align*}
 \pi_R(\tilde{B}) v_{1,\epsilon,\epsilon'} &= \epsilon v_{1,\epsilon,\epsilon'} , \\
 \pi_R(\tilde{C}) v_{1,\epsilon,\epsilon'} &= i\epsilon' v_{1,\epsilon,\epsilon'} , \\
 \pi_R(\tilde{D}) v_{1,\epsilon,\epsilon'} &= 0 .
\end{align*}
\] (3.37)

The irrep with vector \( v_{1,\epsilon,\epsilon'} \) is isomorphic to the irrep with vector \( v_{2,\epsilon,\epsilon'} \). Thus, there are only four distinct irreps parametrized by \( \epsilon, \epsilon' \). On all of them the Casimirs \( \tilde{B}^2, \tilde{D}^2 \) take the value 1,0, respectively.

\[ \mu_A = N > 2 \]

These representations are reducible and decompose in one-dimensional irreps with basis vectors:

\[
\begin{align*}
 v_{1,\epsilon,\epsilon'} &= (\tilde{a} + \epsilon \tilde{b}) \cdot f_1 \cdot (\tilde{a} + i\epsilon' \tilde{c}) \\
 v_{2,\epsilon,\epsilon'} &= (\tilde{d} + \epsilon \tilde{c}) \cdot f_2 \cdot (\tilde{d} + i\epsilon' \tilde{b}) \\
 v_{3,\epsilon,\epsilon'} &= (\tilde{a} + \epsilon \tilde{b}) \cdot f_3 \cdot (\tilde{a} + i\epsilon' \tilde{c}) \\
 v_{4,\epsilon,\epsilon'} &= (\tilde{d} + \epsilon \tilde{c}) \cdot f_4 \cdot (\tilde{a} + i\epsilon' \tilde{c}) \\
 \epsilon, \epsilon' &= \pm , \\
 \ell(f_k) &= N - 2
\end{align*}
\] (3.38)
The RRA of $\tilde{B}, \tilde{C}, \tilde{D}$ on these vectors is as exactly as (3.25). The irrep with vector $v^{k}_{\epsilon, \epsilon'}$ is isomorphic to the irrep with vector $v^{n}_{\epsilon, \epsilon'}$. Thus, there are only four distinct irreps as in the case above. On all of them the Casimirs $\tilde{B}^2, \tilde{D}^2$ take the value 1,0, respectively.

To summarize the list of irreps of $s03'$ is the same as given in subsection 3.3. The list of irreps of $s03$ here is smaller since the Casimir $\tilde{A}$ can take only nonnegative integer values. Thus, the list of the irreps of $s03$ using the dual bialgebra $S03$ as carrier space is:

- one-dimensional trivial
- two-dimensional with all Casimirs $\tilde{A}, \tilde{B}^2, \tilde{D}^2$ having value 1.
- four one-dimensional with Casimir values $\mu, 1, 0$ for $\tilde{A}, \tilde{B}^2, \tilde{D}^2$, respectively, $\mu \in \mathbb{N} + 1$.

The difference in the two lists is natural since here more structure (the co-product) is involved. Speaking more loosely the irreps here may be looked upon as 'integrals' of the irreps obtained in subsection 3.3.

4 Algebra $S14$

4.1 Bialgebra relations

In this Section we consider the matrix bialgebra $S14$. We obtain it by applying the RTT relations (3.1) for the case $R = R_{S1,4}$, when $q^2 \neq 1$ where:

$$R_{S1,4} \equiv \begin{pmatrix} 0 & 0 & 0 & q \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ q & 0 & 0 & 0 \end{pmatrix}$$  \hspace{1cm} (4.1)$$

This $R$-matrix is given in [8].

The relations which follow from (3.1) and (4.1) when $q^2 \neq 1$ are:

$$b^2 - c^2 = 0 , \quad a^2 - d^2 = 0 \quad (4.2)$$
$$ab = ba = 0 , \quad ac = ca = 0$$
$$bd = db = 0 , \quad cd = dc = 0$$

In terms of the generators $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$

$$\tilde{b}\tilde{c} + \tilde{c}\tilde{b} = 0 \quad \tilde{a}\tilde{d} + \tilde{d}\tilde{a} = 0 \quad (4.3)$$
$$\tilde{a}\tilde{b} = \tilde{b}\tilde{a} = 0 \quad \tilde{a}\tilde{c} = \tilde{c}\tilde{a} = 0$$
$$\tilde{b}\tilde{d} = \tilde{d}\tilde{b} = 0 \quad \tilde{c}\tilde{d} = \tilde{d}\tilde{c} = 0$$
From the above relations it is clear that the PBW basis of $S_{14}$ is:

\begin{equation}
\tilde{a}^k \tilde{d}^\ell, \quad \tilde{b}^k \tilde{c}^\ell
\end{equation}

### 4.2 Dual algebra

Let us denote by $s_{14}$ the unknown yet dual algebra of $S_{14}$, and by $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ the four generators of $s_{14}$. We define the pairing $\langle Z, f \rangle$, $Z = \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, $f$ is from (4.4), as (3.10). Explicitly, we obtain:

\begin{align*}
\langle \tilde{A}, f \rangle &= \varepsilon \left( \frac{\partial f}{\partial a} \right) = \begin{cases} k\delta_{00} & f = \tilde{a}^k \tilde{d}^\ell \\ 0 & f = \tilde{b}^k \tilde{c}^\ell \end{cases} \\
\langle \tilde{B}, f \rangle &= \varepsilon \left( \frac{\partial f}{\partial b} \right) = \begin{cases} 0 & f = \tilde{a}^k \tilde{d}^\ell \\ \delta_{k1}\delta_{00} & f = \tilde{b}^k \tilde{c}^\ell \end{cases} \\
\langle \tilde{C}, f \rangle &= \varepsilon \left( \frac{\partial f}{\partial c} \right) = \begin{cases} 0 & f = \tilde{a}^k \tilde{d}^\ell \\ \delta_{k0}\delta_{01} & f = \tilde{b}^k \tilde{c}^\ell \end{cases} \\
\langle \tilde{D}, f \rangle &= \varepsilon \left( \frac{\partial f}{\partial d} \right) = \begin{cases} \delta_{01} & f = \tilde{a}^k \tilde{d}^\ell \\ 0 & f = \tilde{b}^k \tilde{c}^\ell \end{cases}
\end{align*}

We shall need (as in [9]) the auxiliary operator $E$ such that

\begin{equation}
\langle E, f \rangle = \begin{cases} 1 & \text{for } f = 1_A \\ 0 & \text{otherwise} \end{cases}
\end{equation}

Using the above we obtain:

**Proposition 2:** The generators $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ introduced above obey the following relations:

\begin{align*}
\tilde{C} &= \tilde{D}\tilde{B} = -\tilde{B}\tilde{D} \\
[\tilde{A}, \tilde{D}] &= 0 \\
\tilde{A}\tilde{B} &= \tilde{B}\tilde{A} = \tilde{D}^2\tilde{B} = \tilde{B}^3 = \tilde{B} \\
EZ &= ZE = 0, \quad Z = \tilde{A}, \tilde{B}, \tilde{D}.
\end{align*}

\begin{align*}
\delta_U(\tilde{A}) &= \tilde{A} \otimes 1_U + 1_U \otimes \tilde{A} \\
\delta_U(\tilde{B}) &= \tilde{B} \otimes E + E \otimes \tilde{B} \\
\delta_U(\tilde{D}) &= \tilde{D} \otimes K + 1_U \otimes \tilde{D}, \quad K \equiv (-1)^{\tilde{A}} \\
\delta(E) &= E \otimes E
\end{align*}
\[ \varepsilon_U(Z) = 0 \quad \text{and} \quad Z = \tilde{A}, \tilde{B}, \tilde{D} \quad (4.9a) \]
\[ \varepsilon_U(E) = 1 \quad (4.9b) \]

\( \tilde{A}, \tilde{B}^2 \) and \( \tilde{D}^2 \) are Casimir operators. The bialgebra \( s_{14} \) is not a Hopf algebra.

**Proof:** Using the assumed duality the algebraic relations (4.7) are shown by calculating their pairings with the basis monomials \( f \) from (4.4). In particular, we have (giving only the nonzero pairings):

\[ \langle \tilde{D} \tilde{B}, f \rangle = -\langle \tilde{B} \tilde{D}, f \rangle = \langle \tilde{C}, f \rangle = 1 \quad \text{for} \quad f = \tilde{c} \quad (4.10a) \]
\[ \langle \tilde{A} \tilde{D}, f \rangle = \langle \tilde{D} \tilde{A}, f \rangle = k + 1 \quad \text{for} \quad f = \tilde{a}^k \tilde{d} \quad (4.10b) \]
\[ \langle \tilde{A} \tilde{B}, f \rangle = \langle \tilde{B} \tilde{A}, f \rangle = \langle \tilde{D}^2 \tilde{B}, f \rangle = \]
\[ = \langle \tilde{D}^3, f \rangle = \langle \tilde{B}, f \rangle = 1 \quad \text{for} \quad f = \tilde{b} \quad (4.10c) \]

The facts that \( \tilde{A}, \tilde{B}^2 \) and \( \tilde{D}^2 \) are Casimir operators follow easily from (4.7). The proof of (4.8) is done as the proof of (3.13). We also use that

\[ \langle \tilde{A}^n, \tilde{a}^k \tilde{d}^\ell \rangle = k^n \delta_{\ell 0} \quad (4.11) \]

and hence:

\[ \langle K, \tilde{a}^k \tilde{d}^\ell \rangle = (-1)^k \delta_{\ell 0} \quad (4.12) \]

There is no antipode for the bialgebra \( s_{14} \). Indeed, suppose that there was such. Then by applying the Hopf algebra axiom (3.16) to the operator \( E \) we would get:

\[ E \gamma(E) = 1_U \]

which would lead to contradiction after multiplication from the left with \( Z = \tilde{A}, \tilde{B}, \tilde{D} \) (we would get \( 0 = Z \)). From this follows also that the generator \( \tilde{B} \) does not have an antipode, since from (3.16) to the \( \tilde{B} \) we would get:

\[ \tilde{B} \gamma(E) + E \gamma(\tilde{B}) = 0 \]

Thus, the bialgebra \( s_{14} \) is not a Hopf algebra. ♠

**Corollary 2:** The algebra generated by the generator \( \tilde{A} \) is a sub-bialgebra of \( s_{14} \). The algebra \( s_{14}' \) generated by \( \tilde{B}, \tilde{D} \) is a subalgebra of \( s_{14} \), but is not a sub-bialgebra (cf. (4.8b,c)). It has the following PBW basis:

\[ \tilde{B}, \tilde{B}^2, \tilde{D} \tilde{B}, \tilde{D}^2 \tilde{B}, \tilde{D}^\ell, \ell = 0, 1, 2, ... \quad (4.13) \]

where we use the convention \( \tilde{D}^0 = 1_U \). ♠
4.3 Regular representation

We start with the study of the right regular representation of the subalgebra $s_{14}'$. For this we use the right multiplication table:

| $\tilde{B}$ | $\tilde{B}^2$ | $\tilde{D}\tilde{B}$ | $\tilde{D}\tilde{B}^2$ | $\tilde{D}^{2k}$ | $\tilde{D}^{2k+1}$ |
|-------------|--------------|----------------------|------------------------|----------------|-------------------|
| $\tilde{B}$ | $\tilde{B}^2$ | $\tilde{B}$ | $\tilde{D}\tilde{B}^2$ | $\tilde{D}\tilde{B}$ | $\tilde{B}$ | $\tilde{D}^{2k+1}$ |
| $\tilde{D}$ | $-\tilde{D}\tilde{B}$ | $\tilde{D}\tilde{B}^2$ | $-\tilde{B}$ | $\tilde{B}^2$ | $\tilde{D}^{2k+1}$ | $\tilde{D}^{2k+2}$ |

From the above table follows that there is a four-dimensional subspace spanned by $\tilde{B}, \tilde{B}^2, \tilde{D}\tilde{B}, \tilde{D}\tilde{B}^2$. It is reducible and decomposes into four one-dimensional representations spanned by:

$$v_{\epsilon,\epsilon'} = \tilde{B} + \epsilon\tilde{B}^2 - \epsilon'\tilde{D}\tilde{B} + \epsilon\epsilon'\tilde{D}\tilde{B}^2$$  \hspace{1cm} (4.14)

The action of $\tilde{B}, \tilde{D}$ on these vectors is:

$$\tilde{B}v_{\epsilon,\epsilon'} = \epsilon v_{\epsilon,\epsilon'}, \hspace{0.5cm} \tilde{D}v_{\epsilon,\epsilon'} = \epsilon' v_{\epsilon,\epsilon'}$$  \hspace{1cm} (4.15)

The value of the Casimirs $\tilde{B}^2, \tilde{D}^2$ on these vectors is 1.

The quotient of the RRR by the above submodules has the following multiplication table:

| $\tilde{D}^{2k}$ | $\tilde{D}^{2k+1}$ |
|------------------|-------------------|
| $\tilde{B}$ | 0 | 0 |
| $\tilde{D}$ | $\tilde{D}^{2k+1}$ | $\tilde{D}^{2k+2}$ |

This representation is reducible. It contains an infinite set of nested submodules $V^n \supset V^{n+1}$, $n = 0, 1, ..., $ where $V^n$ is spanned by $\tilde{D}^{n+\ell}$, $\ell = 0, 1, ....$ Correspondingly there is an infinite set of one-dimensional irreducible factor-modules $F^n \equiv V^n/V^{n+1}$, (generated by $\tilde{D}^n$) which are all isomorphic to the trivial representation since the generators $\tilde{B}, \tilde{D}$ act as zero on them. Thus there are five irreps arising from the RRR of $s_{14}'$:

- one-dimensional trivial
- four one-dimensional with both Casimirs $\tilde{B}^2, \tilde{D}^2$ having value 1.

Turning to the algebra $s_{14}$ we note that it inherits the representation structure of its subalgebra $s_{14}'$. On the representations (4.14) the Casimir $\tilde{A}$ has the value 1. However, on the one-dimensional irreps $F^n$ the Casimir $\tilde{A}$ has no fixed value. Thus, the list of the irreps arising from the RRR of $s_{14}$ is:

- one-dimensional with Casimir values $\mu, 0, 0$ for $\tilde{A}, \tilde{B}^2, \tilde{D}^2$, respectively, $\mu \in \mathbb{C}$.
- four one-dimensional with all Casimirs $\tilde{A}, \tilde{B}^2, \tilde{D}^2$ having value 1.
4.4 Weight representations

Here we study weight representations, first w.r.t. \( \tilde{D} \), as in (3.26). The resulting representation of \( s14' \) is three-dimensional:

\[
v_0, \; \tilde{B}v_0, \; \tilde{B}^2v_0.
\]

(4.16)

It is reducible and contains one one-dimensional and one two-dimensional irrep:

- **one-dimensional**

  \[
  w_0 = (\tilde{B}^2 - 1_{d})v_0, \\
  \tilde{B}w_0 = 0, \\
  \tilde{D}w_0 = \lambda w_0,
  \]

  (4.17)

  \( \lambda \in \mathbb{C} \).

- **two-dimensional**

  \[
  \{v_0, v_1 = \tilde{B}v_0\}, \\
  \tilde{B}\begin{pmatrix}v_0 \\ v_1\end{pmatrix} = \begin{pmatrix}v_1 \\ v_0\end{pmatrix}, \\
  \tilde{D}\begin{pmatrix}v_0 \\ v_1\end{pmatrix} = \lambda\begin{pmatrix}v_0 \\ -v_1\end{pmatrix}
  \]

  (4.19)

  with \( \lambda = \pm 1 \).

Turning to the algebra \( s14 \) we note that it inherits the representation structure of its subalgebra \( s14' \). On the one-dimensional irrep (4.17) the Casimir \( \tilde{A} \) has no fixed value since \( \tilde{B} \) is trivial, and [\( \tilde{A}, \tilde{D} \)] = 0. On the two-dimensional irrep (4.19) the Casimir \( \tilde{A} \) has the value 1 since \( \tilde{A}\tilde{B} = \tilde{B} \).

Thus, there are the following irreps of \( s14 \) which are obtained as weight irreps of the generator \( \tilde{D} \):

- **one-dimensional** with Casimir values \( \mu, 0, \lambda^2 \) for \( \tilde{A}, \tilde{B}^2, \tilde{D}^2 \), respectively, \( \mu, \lambda \in \mathbb{C} \).

- **two two-dimensional** with all Casimirs \( \tilde{A}, \tilde{B}^2, \tilde{D}^2 \) having the value 1.

Next we consider weight representations w.r.t. \( \tilde{B} \):

\[
\tilde{B}v_0 = \nu v_0,
\]

(4.21)

with \( \nu \in \mathbb{C} \). From \( \tilde{B}^3 = \tilde{B} \) follows that \( \nu = 0, \pm 1 \). Acting with the generators we obtain the following representation vectors: \( v_\ell = \tilde{D}^\ell v_0 \). We have that \( \tilde{D}v_\ell = v_{\ell+1} \).

Further we consider first the case \( \nu^2 = 1 \). Then we apply the relation \( \tilde{D}^2\tilde{B} = \tilde{B} \) to \( v_\ell \) and we get:

\[
\tilde{D}^2\tilde{B}v_\ell = (-1)^\ell \nu v_{\ell+2} = \tilde{B}v_\ell = (-1)^\ell \nu v_\ell
\]

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from which follows that we have to identify \( v_{\ell+2} \) with \( v_\ell \). Thus the representation is given as follows:

\[
\{v_0, v_1 = \tilde{D} v_0\}
\]

\[
\tilde{B}\left( \frac{v_0}{v_1} \right) = \nu\left( -\frac{v_0}{v_1} \right), \quad \tilde{D}\left( \frac{v_0}{v_1} \right) = \left( \frac{v_1}{v_0} \right)
\]

(4.22)

(4.23)

On this irrep both Casimirs \( \tilde{B}^2, \tilde{D}^2 \) have value 1, \( (\nu^2 = 1) \).

Further we consider the case \( \nu = 0 \). This representation is reducible. It contains an infinite set of nested submodules \( V^n \supset V^{n+1}, n = 0, 1, ..., \) where \( V^n \) is spanned by \( \tilde{D}^{n+\ell} v_0, \ell = 0, 1, ... \). Correspondingly there is an infinite set of one-dimensional irreducible factor-modules \( F^n \equiv V^n/V^{n+1}, \) (generated by \( \tilde{D}^n v_0 \)) which are all isomorphic to the trivial representation since the generators \( \tilde{B}, \tilde{D} \) act as zero on them.

Turning to the algebra \( s_{14} \) we note that it inherits the representation structure of its subalgebra \( s_{14}' \), with the value of the Casimir \( \tilde{A} \) being not fixed if \( \tilde{B} \) acts trivially, and being 1, if \( \tilde{B} \) acts non trivially.

Thus, there are the following irreps of \( s_{14} \) which are obtained as weight irreps of the generator \( \tilde{B} \):

- one-dimensional with Casimir values \( \mu, 0, 0 \) for \( \tilde{A}, \tilde{B}^2, \tilde{D}^2 \), respectively, \( \mu \in \mathbb{C} \).
- two two-dimensional with all Casimirs \( \tilde{A}, \tilde{B}^2, \tilde{D}^2 \) having the value 1.

### 4.5 Representations of \( s_{14} \) on \( S_{14} \)

Here we shall study the representations of \( s_{14} \) obtained by the use of its right regular action (RRA) on the dual bialgebra \( S_{14} \). The RRA is defined as in (3.29). For the generators of \( s_{14} \) we have:

\[
\pi_R(\tilde{A}) \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}
\]

(4.24a)

\[
\pi_R(\tilde{B}) \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} \tilde{b} & \tilde{a} \\ \tilde{d} & \tilde{c} \end{pmatrix}
\]

(4.24b)

\[
\pi_R(\tilde{D}) \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} \tilde{d} & -\tilde{c} \\ -\tilde{b} & \tilde{a} \end{pmatrix}
\]

(4.24c)

\[
\pi_R(E) \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]

(4.24d)

\[
\pi_R(Z) 1_A = 1_A \langle Z, 1_A \rangle = 1_A \varepsilon_U(Z) = \begin{cases} 0, & Z = \tilde{A}, \tilde{B}, \tilde{D} \\ 1, & Z = E \end{cases}
\]

(4.24e)
For the action on the basis of $S14$ we use formula (3.31). We obtain:

\[
\pi_R(A) \tilde{a}^n \tilde{d}^k = (n+k)\tilde{a}^n \tilde{d}^k, \quad \pi_R(A) \tilde{b}^n \tilde{c}^k = (n+k)\tilde{b}^n \tilde{c}^k \quad (4.25a)
\]

\[
\pi_R(B) \tilde{a}^n \tilde{d}^k = \delta_{k0}\delta_{n1} \tilde{b} + \delta_{n0}\delta_{k1} \tilde{c}, \quad \pi_R(B) \tilde{b}^n \tilde{c}^k = \delta_{k0}\delta_{n1} \tilde{a} + \delta_{n0}\delta_{k1} \tilde{d} \quad (4.25b)
\]

\[
\pi_R(D) \tilde{a}^k \tilde{d}^\ell = (-1)^{\ell+1} \tilde{a}^{k+1} \tilde{d}^{\ell-1} + (-1)^{\ell} k \tilde{a}^{k-1} \tilde{d}^{\ell+1} \quad (4.25c)
\]

\[
\pi_R(D) \tilde{b}^k \tilde{c}^\ell = (-1)^{\ell+1} \tilde{b}^{k+1} \tilde{c}^{\ell-1} + (-1)^{\ell+1} k \tilde{b}^{k-1} \tilde{c}^{\ell+1} \quad (4.25d)
\]

We see that similarly to subsection 3.5 the Casimir $\tilde{A}$ acts as the length of the elements of $S14$, i.e., (3.30) holds. Thus, also here we classify the irreps by the value $\mu_A$ of the Casimir $\tilde{A}$ which runs over the nonnegative integers. For fixed $\mu_A$ the basis of the corresponding representations is spanned by the elements $f$ such that $\ell(f) = \mu_A$.

The dimension of each such representation is:

\[
\dim(\mu_A) = \begin{cases} 
2(\mu_A + 1) & \text{for } \mu_A \geq 1 \\
1 & \text{for } \mu_A = 0 
\end{cases} 
\quad (4.26)
\]

The classification goes as follows:

- $\mu_A = 0$
  This is the one-dimensional trivial representation spanned by $1_A$.

- $\mu_A = 1$
  This representation is four-dimensional spanned by the four generators $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ of $S14$. It decomposes in two two-dimensional isomorphic to each other irreps with basis vectors as in (3.34) - this is due to the fact that the action (4.24b,c) is the same as the action (3.30). The value of the Casimirs $\tilde{B}^2, \tilde{D}^2$ is 1.

- Each representation for fixed $\mu_A \geq 2$ is reducible and decomposes in two isomorphic representations: one built on the basis $\tilde{a}^k \tilde{d}^\ell$, and the other built on the basis $\tilde{b}^k \tilde{c}^\ell$, each of dimension $\mu_A + 1$. Thus, for $\mu_A \geq 2$ we shall consider only the representations built on the basis $\tilde{a}^k \tilde{d}^\ell$. These representations are also reducible and they all decompose in one-dimensional irreps. Further, the action of $\tilde{B}$ is zero, thus, we speak only about the action of $\tilde{D}$.

- $\mu_A = 2n$, $n = 1, 2, ...$
  For fixed $n$ the representation decomposes into $2n + 1$ one-dimensional irreps. On one of these, which is spanned by the element:

\[
w_0 = \sum_{k=0}^{n} \binom{n}{k} \tilde{a}^{2n-2k} \tilde{d}^{2k} ,
\quad (4.27)
\]

the generator $\tilde{D}$ acts by zero. The rest of the irreps are enumerated by the parameters: $\pm, \tau$, where $\tau = 2, 4, ..., 2n = \mu_A$, and are spanned by the
vectors:

\[ u^\pm = u_0 \pm \tau u_1 , \]  
\[ u_0 = \sum_{k=0}^{n} \alpha_k \bar{a}^{2n-2k} \bar{d}^{2k} , \quad \alpha_0 = 1 , \]  
\[ u_1 = \sum_{k=0}^{n-1} \beta_k \bar{a}^{2n-2k-1} \bar{d}^{2k+1} , \quad \beta_0 = 1 , \]

where \( \bar{D} \) acts by:

\[ \pi_R(\bar{D}) u^\pm = \pm \tau u^\pm \]  
which follows from:

\[ \pi_R(\bar{D}) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} \tau^2 u_1 \\ u_0 \end{pmatrix} \]  
Note that the value of the Casimir \( \bar{D}^2 \) is equal to \( \tau^2 \). The coefficients \( \alpha_k, \beta_k \) depend on \( \tau \) and are fixed from the two recursive equations which follow from (4.30):

\[ \tau^2 \beta_k = (2n-2k+1)\alpha_k - 2(k+1)\alpha_{k+1} , \quad k = 0, ..., n-1; \quad \text{(4.31a)} \]  
\[ \alpha_k = (2k+1)\beta_k - (2n-2k+1)\beta_{k-1} , \quad k = 0, ..., n , \quad \text{(4.31b)} \]

where we set \( \beta_{-1} \equiv 0, \beta_n \equiv 0 \).

- \( \mu_A = 2n + 1, n = 1,2,... \)

For fixed \( n \) the representation is \((2n+2)\)-dimensional and decomposes into \(2n+2\) irreps which are enumerated by two parameters: \( \pm, \tau \), where \( \tau = 1,3,5,...,2n+1 = \mu_A \), and are spanned by the vectors:

\[ w^\pm = w_0 \pm \tau w_1 , \]  
\[ w_0 = \sum_{k=0}^{n} \alpha'_k \bar{a}^{2n-2k+1} \bar{d}^{2k} , \quad \alpha'_0 = 1 , \]  
\[ w_1 = \sum_{k=0}^{n} \beta'_k \bar{a}^{2n-2k} \bar{d}^{2k+1} , \quad \beta'_0 = 1 , \]

on which \( \bar{D} \) acts by (4.29). Note that the value of the Casimir \( \bar{D}^2 \) is equal to \( \tau^2 \). The coefficients \( \alpha'_k, \beta'_k \) are fixed from the two recursive equations which follow from (4.29):

\[ \tau^2 \beta'_k = (2n-2k+1)\alpha'_k - 2(k+1)\alpha'_{k+1} , \quad k = 0, ..., n; \quad \text{(4.33)} \]  
\[ \alpha'_k = (2k+1)\beta'_k - 2(n-k+1)\beta'_{k-1} , \quad k = 0, ..., n , \quad \text{(4.34)} \]

where we set \( \alpha'_{n+1} \equiv 0, \beta'_{-1} \equiv 0 \).
To summarize the list of irreps of $s_{14}$ on $S_{14}$ is:

- one-dimensional trivial
- two two-dimensional with all Casimirs $\tilde{A}, \tilde{B}^2, \tilde{D}^2$ having the value 1.
- one-dimensional enumerated by $n = 1, 2, ...$ which for fixed $n$ have Casimir values $2n, 0, 0$ for $\tilde{A}, \tilde{B}^2, \tilde{D}^2$, respectively.
- pairs of one-dimensional irreps enumerated by $n = 1, 2, ..., \tau = 2, 4, ..., 2n$, which have Casimir values $2n, 0, \tau^2$ for $\tilde{A}, \tilde{B}^2, \tilde{D}^2$, respectively.
- pairs of one-dimensional irreps enumerated by $n = 1, 2, ..., \tau = 1, 3, ..., (2n + 1)$, which have Casimir values $2n + 1, 0, \tau^2$ for $\tilde{A}, \tilde{B}^2, \tilde{D}^2$, respectively.

Finally, we note in the irreps of $s_{14}$ on $S_{14}$ all Casimirs can take only nonnegative integer values.

5 Algebra $S_{14o}$

5.1 Bialgebra relations

In this Section we consider the matrix bialgebra $S_{14o}$. We obtain it by applying the RTT relations (3.1) for the case $R = R_{S_{14}}$, cf. (4.1), when $q^2 = 1$. We shall consider the case $q = 1$ (the case $q = -1$ is equivalent, cf. below). For $q = 1$ the relations following from (3.1) and (4.1) are:

$$a^2 = d^2, \quad b^2 = c^2 = 0, \quad ab = ba = ac = ca = bd = db = cd = dc = 0$$  \hspace{1cm} (5.1)

or in terms of the generators $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$:

$$\tilde{b}\tilde{a} = \tilde{a}\tilde{b}, \quad \tilde{c}\tilde{a} = -\tilde{a}\tilde{c}, \quad \tilde{d}\tilde{a} = -\tilde{a}\tilde{d}, \quad \tilde{c}\tilde{b} = -\tilde{b}\tilde{c}, \quad \tilde{d}\tilde{b} = -\tilde{b}\tilde{d}, \quad \tilde{d}\tilde{c} = \tilde{c}\tilde{d}$$  \hspace{1cm} (5.2)

(The case $q = -1$ is obtained from the above through the exchange $\tilde{b} \leftrightarrow \tilde{c}$.)

From the above relations it is clear that we can choose any ordering of the PBW basis. For definiteness we choose for the PBW basis of $S_{14o}$:

$$\tilde{a}^k \tilde{b}^\ell \tilde{c}^m \tilde{d}^n$$  \hspace{1cm} (5.3)

5.2 Dual algebra

Let us denote by $s_{14o}$ the unknown yet dual algebra of $S_{14o}$, and by $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$ the four generators of $s_{14o}$. We define the pairing $\langle Z, f \rangle$, $Z = \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, $f$ is from
\[ (\tilde{A}, f) = \varepsilon \left( \frac{\partial f}{\partial \tilde{a}} \right) = \begin{cases} k & \text{for } f = \tilde{a}^k \\ 0 & \text{otherwise} \end{cases} \quad (5.4a) \]
\[ (\tilde{B}, f) = \varepsilon \left( \frac{\partial f}{\partial \tilde{b}} \right) = \begin{cases} 1 & \text{for } f = \tilde{a}^k \tilde{b} \\ 0 & \text{otherwise} \end{cases} \quad (5.4b) \]
\[ (\tilde{C}, f) = \varepsilon \left( \frac{\partial f}{\partial \tilde{c}} \right) = \begin{cases} 1 & \text{for } f = \tilde{a}^k \tilde{c} \\ 0 & \text{otherwise} \end{cases} \quad (5.4c) \]
\[ (\tilde{D}, f) = \varepsilon \left( \frac{\partial f}{\partial \tilde{d}} \right) = \begin{cases} 1 & \text{for } f = \tilde{a}^k \tilde{d} \\ 0 & \text{otherwise} \end{cases} \quad (5.4d) \]

Using the above we obtain:

**Proposition 3:** The generators \( \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \) introduced above obey the following relations:

\[ [\tilde{A}, Z] = 0 , \quad Z = \tilde{B}, \tilde{C}, \tilde{D} \quad (5.5a) \]
\[ [\tilde{B}, \tilde{C}] = -2\tilde{D}, \quad [\tilde{B}, \tilde{D}] = -2\tilde{C}, \quad [\tilde{C}, \tilde{D}] = -2\tilde{B} \quad (5.5b) \]

\[ \delta_U(\tilde{A}) = \tilde{A} \otimes 1_U + 1_U \otimes \tilde{A} \quad (5.6a) \]
\[ \delta_U(\tilde{B}) = \tilde{B} \otimes 1_U + 1_U \otimes \tilde{B} \quad (5.6b) \]
\[ \delta_U(\tilde{C}) = \tilde{C} \otimes K + 1_U \otimes \tilde{C} , \quad K = (-1)^{\tilde{A}} \quad (5.6c) \]
\[ \delta_U(\tilde{D}) = \tilde{D} \otimes K + 1_U \otimes \tilde{D} \quad (5.6d) \]
\[ \varepsilon_U(Z) = 0 , \quad Z = \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \quad (5.7) \]
\[ \gamma_U(\tilde{A}) = -\tilde{A} , \quad \gamma_U(\tilde{B}) = -\tilde{B} , \quad \gamma_U(\tilde{C}) = -\tilde{C} K , \quad \gamma_U(\tilde{D}) = -\tilde{D} K \quad (5.8) \]

**Proof:** The proof of (5.5) goes as the standard duality between the classical \( U(gl(2)) \) and \( GL(2) \), cf. e.g., [13]. The proof of (5.6,5.7,5.8) is also standard, except the factor \( K \) which appears while calculating:

\[ \langle \tilde{C}, \tilde{a}^k \rangle = (-1)^k \langle \tilde{C}, \tilde{a}^k \tilde{c} \rangle = (-1)^k \]

which on the other hand is equal to (supposing an unknown yet \( K \)):

\[ \langle \delta_U(\tilde{C}), \tilde{c} \otimes \tilde{a}^k \rangle = \langle \tilde{C} \otimes K + 1_U \otimes \tilde{C} , \tilde{c} \otimes \tilde{a}^k \rangle = \langle K, \tilde{a}^k \rangle \]

Comparing the two RHSs we conclude that \( K = (-1)^{\tilde{A}} \). The same follows from calculating \( \langle \tilde{D}, \tilde{a}^k \rangle \). \( \blacklozenge \)

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Corollary 3: The auxiliary generator $K = (-1)^\frac{3}{2}$ is central and $K^{-1} = K$. Its co-algebra relations are:

\[
\delta_u(K) = K \otimes K, \quad \varepsilon_u(K) = 1, \quad \gamma_u(K) = K \quad \blacklozenge
\] (5.9)

Corollary 4: The algebra generated by the generator $\tilde{A}$ is a Hopf subalgebra of $s14o$. The algebra $s14o'$ generated by $\tilde{B}, \tilde{C}, \tilde{D}$ is a subalgebra of $s14o$, but is not a Hopf subalgebra because of the operator $K$ in the co-algebra structure. The algebras $s14o, s14o'$ are isomorphic to $U(gl(2)), U(sl(2))$, respectively. The latter is seen from the following:

\[
X^\pm \equiv \frac{1}{2}(\tilde{D} \mp \tilde{C})
\]

\[
[\tilde{B}, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = \tilde{B}.
\] (5.10b)

Indeed the last line presents the standard $sl(2)$ commutation relations. However, the generators $X^\pm$ inherit the $K$ dependence in the coalgebra operations:

\[
\delta_u(X^\pm) = X^\pm \otimes K + 1_u \otimes X^\pm
\]

\[
\varepsilon_u(X^\pm) = 0
\]

\[
\gamma_u(X^\pm) = -X^\pm K
\] (5.11c)

The algebra $s14o$ is graded:

\[
\text{deg } X^\pm = \pm 1, \quad \text{deg } \tilde{A} = \text{deg } \tilde{B} = 0, \quad (\implies \text{deg } K = 0) \quad \blacklozenge
\] (5.12)

Based on the above Corollary we are able to make the following important observation. The algebra $s14o$ may be identified with a special case of the Hopf algebra $U_{p,q}$, which was found in [13] as the dual of $GL_{p,q}(2)$. To make direct contact with [13] we need to replace there $(p^{1/2}, q^{1/2}) \rightarrow (p, q)$, then to set $q = p^{-1}$, and at the end to set $p = -1$. (The necessity to set values in such order is clear from, e.g., the formula for the co-product in (5.21) of [13].) The generators from [13] $K, p^K, H, X^\pm$ correspond to $\tilde{A}, \tilde{K}, \tilde{B}, X^\pm$ in our notation.

More than this. It turns out that the corresponding algebras in duality, namely, $S14o$ and $GL_{p,q}(2)$ may be identified setting $q, p$ as above. To make this evident we make the following change of generators:

\[
\hat{a} = \tilde{a} + \tilde{b}, \quad \hat{b} = \tilde{d} - \tilde{c}, \quad \hat{c} = \tilde{c} + \tilde{d}, \quad \hat{d} = \tilde{a} - \tilde{b}.
\] (5.13)

For these generators the commutation relations are:

\[
\hat{a} \hat{b} = -\hat{a} \hat{b}, \quad \hat{c} \hat{a} = -\hat{a} \hat{c}, \quad \hat{d} \hat{a} = \hat{a} \hat{d}, \quad \hat{c} \hat{b} = \hat{b} \hat{c}, \quad \hat{b} \hat{d} = -\hat{d} \hat{b}, \quad \hat{c} \hat{d} = -\hat{d} \hat{c}
\] (5.14)

i.e., exactly those of $GL_{p,q}(2)$ (cf. [1]) for $p = q = -1$. Furthermore the co-product and co-unit are as for $GL_{p,q}(2)$ or $GL(2)$, i.e., as in (2.1). For the antipode we
have to suppose that the determinant $ad - p^{-1}bc$ from [1], which here becomes (cf. $p = -1$):

$$\omega = \hat{a}\hat{d} + \hat{b}\hat{c},$$  \hspace{1cm} (5.15)

is invertible, or, that $\omega \neq 0$ and we extend the algebra by an element $\omega^{-1}$ so that:

$$\omega\omega^{-1} = \omega^{-1}\omega = 1_{A}, \quad \delta_{\mathcal{U}}(\omega^{\pm 1}) = \omega^{\pm 1} \otimes \omega^{\pm 1}, \quad \varepsilon_{\mathcal{U}}(\omega^{\pm 1}) = 1, \quad \gamma_{\mathcal{U}}(\omega^{\pm 1}) = \omega^{\mp 1}$$  \hspace{1cm} (5.16)

Then the antipode is given by:

$$\gamma_{\mathcal{U}} \left( \begin{array}{cc} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{array} \right) = \omega^{-1} \left( \begin{array}{cc} \hat{d} & \hat{b} \\ \hat{c} & \hat{a} \end{array} \right)$$  \hspace{1cm} (5.17)

or in a more compact notation:

$$\gamma_{\mathcal{U}} (M) = M^{-1}$$  \hspace{1cm} (5.18)

Indeed, we have:

$$\left( \begin{array}{cc} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{array} \right) \left( \begin{array}{cc} \hat{d} & \hat{b} \\ \hat{c} & \hat{a} \end{array} \right) = \omega \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right)$$  \hspace{1cm} (5.19)

This relation between $s_{140}, S_{140}$ and $\mathcal{U}_{p,q}, GL_{p,q}(2)$ was not anticipated since the corresponding $R$-matrices $R_{S_{1,4}}$ and $R_{S_{2,1}}$ are listed in [8] as different and furthermore non-equivalent. It turns out that this is indeed the case, except in the case we have stumbled upon. To show this we first recall:

$$R_{S_{2,1}} = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & p & 1 - pq & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$  \hspace{1cm} (5.20)

which for $q = p^{-1} = -1$ becomes:

$$R_0 \equiv (R_{S_{2,1}})_{q = p^{-1} = -1} = \left( \begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$  \hspace{1cm} (5.21)

Further, we need:

$$R_{\pm} \equiv (R_{S_{1,4}})_{q = \pm 1} = \left( \begin{array}{cccc} 0 & 0 & 0 & \pm 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \pm 1 & 0 & 0 & 0 \end{array} \right)$$  \hspace{1cm} (5.22)
Now we can show that $R_\pm$ can be transformed by "gauge transformations" to $R_0$, namely, we have:

$$R_0 = (U_\pm \otimes U_\pm) R_\pm (U_\pm \otimes U_\pm)^{-1}$$  \hspace{1cm} (5.23a)$$
$$U_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad U_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$  \hspace{1cm} (5.23b)

In accord with this we have:

$$\hat{T} \equiv \left( \begin{array}{cc} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{array} \right), \quad T \equiv \left( \begin{array}{cc} a & b \\ c & d \end{array} \right), \quad \hat{T} = U_+ T (U_+)^{-1} \Rightarrow$$  \hspace{1cm} (5.24)

$$\hat{a} = \frac{1}{2}(a + b + c + d), \quad \hat{b} = \frac{1}{2}(a - b + c - d),$$
$$\hat{c} = \frac{1}{2}(a + b - c - d), \quad \hat{d} = \frac{1}{2}(a - b - c + d)$$  \hspace{1cm} (5.25)

which is equivalent to substituting (2.2) in (5.13).

The use of $U_-$ would lead to different relations between hatted and unhatted generators, which, however, would not affect the algebra relations. Indeed:

$$\hat{T}' \equiv \left( \begin{array}{cc} \hat{a}' & \hat{b}' \\ \hat{c}' & \hat{d}' \end{array} \right), \quad T' \equiv \left( \begin{array}{cc} a' & b' \\ c' & d' \end{array} \right), \quad \hat{T}' = U_- T' (U_-)^{-1} \Rightarrow$$  \hspace{1cm} (5.26)

$$\hat{a}' = \frac{1}{2}(a' - ib' + ic' + d'), \quad \hat{b}' = \frac{1}{2}(-ia' + b' + c' + id'),$$
$$\hat{c}' = \frac{1}{2}(ia' + b' + c' - id'), \quad \hat{d}' = \frac{1}{2}(a' + ib' - ic' + d')$$  \hspace{1cm} (5.27)

But this becomes equivalent to (5.25) with the changes:

$$(\hat{a}', \hat{b}', -i\hat{c}', \hat{d}') \mapsto (\hat{a}, \hat{b}, \hat{c}, \hat{d}), \quad (a', -ib', ic', d') \mapsto (a, b, c, d)$$  \hspace{1cm} (5.28)

while the (inverse) changes (5.28) do not affect (5.14), (5.1).

### 5.3 Representations of $s_{14o}$ on $S_{14o}$

The regular representation of $s_{14o}$, $(s_{14o}')$ on itself and its weight representations are the same as those of $U(gl(2))$, $(U(sl(2)))$ due to (5.10). The situation is different for the representations of $s_{14o}$ on $S_{14o}$ since these involve the coalgebra structure. However, the deviation from the trivial coalgebra structure is only via the sign operator $K$, and as we shall see in some representations there remains no trace of this.

In treating the representations of $s_{14o}$ on $S_{14o}$ we shall use the known construction for the induced representations of $U_{p,q}$ on $GL_{p,q}(2)$ from [20] and the relation between $s_{14o}, S_{14o}$ and $U_{p,q}, GL_{p,q}(2)$ that we established in the previous subsection. For the comparison with [20] we should note the parametrization used there: $p = ts^{1/2}$, $q = ts^{-1/2}$. Thus, using $q = p^{-1}, p = -1$ we need to substitute: $t \to 1$, $\sqrt{s} \to -1$. Further, one should substitute the operator $h$ from [20] with $i\hat{B}/2$, and expand in
order to get the action of $\tilde{B}$. Finally, one should substitute the operator $r$ from [20]
with $\sqrt{s^A} = (-1)^A = K$. In fact it is easier to derive the necessary formulae directly,
which we shall proceed to do in a compact way.

Here we shall employ both the left action and the right action. We start by calculating the left action using:

$$
\pi_L(Z)f \equiv \langle \gamma_U(Z), f_{(1)} \rangle f_{(2)}, \quad Z \in s_{14}, \ Z \neq 1_U, \ f \in S_{14}, \quad (5.29a)
$$

$$
\pi_L(1_U)f \equiv f, \quad f \in S_{14}. \quad (5.29b)
$$

using for the PBW basis:

$$
\hat{a}^j \hat{d}^k \hat{b}^\ell \hat{c}^n. \quad (5.30)
$$

For the left action on the elements of $S_{14}$ we use a Corollary of (5.29):

$$
\pi_L(Z)fg = \pi_L(\delta'_U(Z))(f \otimes g) \quad (5.31)
$$

where is used the opposite comultiplication $\delta'_U \equiv \sigma \circ \delta_U$, where $\sigma$ is the permutation in $U \otimes U$. We find:

$$
\pi_L(A) \begin{pmatrix} \hat{a}^k & \hat{b}^k \\ \hat{c}^k & \hat{d}^k \end{pmatrix} = -k \begin{pmatrix} \hat{a}^k & \hat{b}^k \\ \hat{c}^k & \hat{d}^k \end{pmatrix} \quad (5.32a)
$$

$$
\pi_L(K) \begin{pmatrix} \hat{a}^k & \hat{b}^k \\ \hat{c}^k & \hat{d}^k \end{pmatrix} = (-1)^k \begin{pmatrix} \hat{a}^k & \hat{b}^k \\ \hat{c}^k & \hat{d}^k \end{pmatrix} \quad (5.32b)
$$

$$
\pi_L(B) \begin{pmatrix} \hat{a}^k & \hat{b}^k \\ \hat{c}^k & \hat{d}^k \end{pmatrix} = k \begin{pmatrix} -\hat{a}^k & -\hat{b}^k \\ \hat{c}^k & \hat{d}^k \end{pmatrix} \quad (5.32c)
$$

$$
\pi_L(X^+) \begin{pmatrix} \hat{a}^k & \hat{b}^k \\ \hat{c}^k & \hat{d}^k \end{pmatrix} = k \begin{pmatrix} (-1)^{k-1} \hat{a}^{k-1} \hat{c} & \hat{d}^{k-1} \\ 0 & 0 \end{pmatrix} \quad (5.32d)
$$

$$
\pi_L(X^-) \begin{pmatrix} \hat{a}^k & \hat{b}^k \\ \hat{c}^k & \hat{d}^k \end{pmatrix} = k \begin{pmatrix} 0 & 0 \\ \hat{a}^{k-1} & (-1)^{k-1} \hat{d}^{k-1} \hat{b} \end{pmatrix} \quad (5.32e)
$$

$$
\pi_L(K) 1_A = 1, \quad \pi_L(Z) 1_A = 0, \quad Z = \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \quad (5.32f)
$$

(We give also the action of $K$ though it follows from that of $A$.)
Next we calculate the right action as in (3.29) to find:

\[
\pi_R(A) \left( \begin{array}{c} \hat{a}^k \\ \hat{b}^k \\ \hat{c}^k \\ \hat{d}^k \end{array} \right) = k \left( \begin{array}{c} \hat{a}^k \\ \hat{b}^k \\ \hat{c}^k \\ \hat{d}^k \end{array} \right) \quad (5.33a)
\]

\[
\pi_R(K) \left( \begin{array}{c} \hat{a}^k \\ \hat{b}^k \\ \hat{c}^k \\ \hat{d}^k \end{array} \right) = (-1)^k \left( \begin{array}{c} \hat{a}^k \\ \hat{b}^k \\ \hat{c}^k \\ \hat{d}^k \end{array} \right) \quad (5.33b)
\]

\[
\pi_R(B) \left( \begin{array}{c} \hat{a}^k \\ \hat{b}^k \\ \hat{c}^k \\ \hat{d}^k \end{array} \right) = k \left( \begin{array}{c} \hat{a}^k \\ \hat{b}^k \\ \hat{c}^k \\ \hat{d}^k \end{array} \right) \quad (5.33c)
\]

\[
\pi_R(X^+) \left( \begin{array}{c} \hat{a}^k \\ \hat{b}^k \\ \hat{c}^k \\ \hat{d}^k \end{array} \right) = k \left( \begin{array}{c} 0 \\ (-1)^{k-1} \hat{a}^k \hat{b}^{k-1} \\ 0 \\ \hat{d}^{k-1} \hat{c} \end{array} \right) \quad (5.33d)
\]

\[
\pi_R(X^-) \left( \begin{array}{c} \hat{a}^k \\ \hat{b}^k \\ \hat{c}^k \\ \hat{d}^k \end{array} \right) = k \left( \begin{array}{c} \hat{a}^{k-1} \hat{b} \\ 0 \\ (-1)^{k-1} \hat{d}^{k-1} \hat{c} \end{array} \right) \quad (5.33e)
\]

\[
\pi_R(K) 1_A = 1 \quad \pi_R(Z) 1_A = 0, \quad Z = \tilde{A}, \tilde{B}, \tilde{C}, \tilde{D} \quad (5.33f)
\]

By (5.29) and (5.33) we have defined left and right action of \( s_{14o} \) on \( S_{14o} \). As in the classical case the left and right actions commute, and we shall use the right action to reduce the left regular representation. Following [21] we would like the right action to mimic some properties of a lowest weight module, i.e., annihilation by the lowering (negative grade) generator \( X^- \) and scalar action by the (exponent of the) Cartan (zero grade) generators \( \tilde{A} \) (or \( K \)) and \( \tilde{B} \). Such action is the reason we call these representations induced. We start with functions which are formal power series in the PBW basis:

\[
f = \sum_{j,k,\ell,m \in \mathbb{Z}_+} \mu_{j,k,\ell,m} \hat{a}^j \hat{b}^k \hat{c}^\ell \hat{d}^m \quad (5.34)
\]

The right-action conditions we mentioned are:

\[
\pi_R(X^-) f = 0 \quad (5.35a)
\]

\[
\pi_R(\tilde{A}) f = \rho f, \quad \pi_R(\tilde{B}) f = -\nu f \quad (5.35b)
\]

From (5.35a) follows that our functions would not depend on \( \hat{a} \) and \( \hat{c} \), except through the determinant \( \omega \), since \( \pi_R(X^-) \omega = 0 \). We also have:

\[
\pi_R(\tilde{A}) \omega^n = 2n \omega^n, \quad \pi_R(K) \omega^n = \omega^n, \quad \pi_R(\tilde{B}) \omega^n = 0, \quad \pi_R(X^\pm) \omega^n = 0. \quad (5.36)
\]

Thus, we continue with:

\[
f = \sum_{k,\ell,\mu, \nu \in \mathbb{Z}_+, n \in \mathbb{Z}} \mu_{k,\ell,\mu, \nu} \hat{a}^k \hat{b}^\ell \omega^n \quad (5.37)
\]

on which conditions (5.35b) lead to: \( k+\ell+2n = \rho \in \mathbb{Z}, \quad k+\ell = \nu \in \mathbb{Z}_+, \quad \rho-\nu \in 2\mathbb{Z} \), and the summation becomes single:

\[
f = \sum_{\ell \in \mathbb{Z}_+} \mu_{\ell, \ell} u_\ell, \quad u_\ell \equiv \hat{b}^\ell \hat{a}^{\nu-\ell} \omega^{(\rho-\nu)/2} \quad (5.38)
\]
(where we changed the ordering since this would give simpler formulae for the action). Now if neither \( \hat{b} \) or \( \hat{d} \) has an inverse the representations will be finite-dimensional, in contrast to the classical case. However, these finite-dimensional representations we shall obtain also if we suppose that either \( \hat{b} \) or \( \hat{d} \) has an inverse (see below), and in the same time we shall have infinite-dimensional representations. Thus, further we shall suppose that \( \hat{d} \) has an inverse. This means that we can allow \( k \) in (5.38) to take any integer values, and then the same is true for \( \nu \).

Now we shall work out the representation (left) action for the basis \( u_\ell \) for which we need first the left action on \( \omega \):

\[
\pi_L(A)\omega^n = -2n\omega^n, \quad \pi_L(K)\omega^n = \omega^n, \quad \pi_L(B)\omega^n = 0, \quad \pi_L(X^\pm)\omega^n = 0. \tag{5.39}
\]

We also remark that the action on \( \hat{d}^k, \omega^n \) for negative \( k, n \) is given again by (5.32), (5.39). (This can be checked, e.g., by calculating \( \pi_L(Z)\hat{d}^k \hat{d}^{-k} = \pi_L(Z)1_A \) in two different ways: (5.31) for the LHS, and (5.32) for the RHS.) Then the rules are:

\[
\begin{align*}
\pi_{\nu,\rho}(A)u_\ell &= -\rho u_\ell & \text{(5.40a)} \\
\pi_{\nu,\rho}(B)u_\ell &= (\nu - 2\ell)u_\ell & \text{(5.40b)} \\
\pi_{\nu,\rho}(X^+)u_\ell &= (-1)^{\ell-1}\ell u_{\ell+1} & \text{(5.40c)} \\
\pi_{\nu,\rho}(X^-)u_\ell &= (-1)^{\ell+1}(\ell - \nu)u_{\ell+1} & \text{(5.40d)}
\end{align*}
\]

where \( \pi_{\nu,\rho} \) denotes \( \pi_L \) with the parameter dependence made explicit.

Thus, we have obtained infinite-dimensional representations of \( S_{140} \) parametrized by two integers \( \nu, \rho \). We shall denote by \( C_{\nu,\rho} \) the corresponding representation space. Note that these representations are highest weight representations since we have:

\[
\pi_L(X^+)u_0 = 0. \quad \text{If the parameter } \nu \text{ is nonnegative, } \nu \in \mathbb{Z}^+, \text{ then the corresponding representation is reducible, due to the fact that } \pi_L(X^-)u_\nu = 0. \quad \text{Thus, the vectors } u_0, ..., u_\nu \text{ form an invariant subspace, of dimension } \nu+1, \text{ which shall denote by } E_{\nu,\rho}, \nu \in \mathbb{Z}^+. \quad \text{Thus, if } \nu \in \mathbb{Z}^+ \text{ we have two irreducible representations with representation spaces isomorphic to } E_{\nu,\rho} \text{ and to } C_{\nu,\rho}/E_{\nu,\rho} \text{ (the latter is infinite-dimensional). If } \nu \notin \mathbb{Z}^+ \text{ the representation } C_{\nu,\rho} \text{ is irreducible.}
\]

From the above we are prompted to use the variable \( \eta \equiv \hat{b}\hat{d}^{-1} \). This is also related to the following Gauss decomposition of \( S_{140} \):

\[
\begin{pmatrix}
\hat{a} & \hat{b} \\
\hat{c} & \hat{d}
\end{pmatrix} =
\begin{pmatrix}
1 & \hat{b}\hat{d}^{-1} \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\omega\hat{d}^{-1} & 0 \\
0 & \hat{d}
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
\hat{d}^{-1}\hat{c} & 1
\end{pmatrix}
\tag{5.41}
\]

i.e., from here the natural variables are \( \eta, \hat{d}, \omega \). Thus, we use also the functions:

\[
\varphi = \sum_{\ell \in \mathbb{Z}^+} \alpha_\ell v_\ell, \quad v_\ell \equiv \eta^\ell \hat{d}^\nu \omega^{(\rho-\nu)/2} \tag{5.42}
\]

The action of the generators on the variable \( \eta \) is:

\[
\begin{pmatrix}
\hat{A} & K \\
\hat{B} & X^\pm
\end{pmatrix} \eta^\ell =
\begin{pmatrix}
0 & \eta^\ell \\
-2\ell \eta^\ell & \pm\ell \eta^{\ell+1}
\end{pmatrix} \tag{5.43}
\]
and the action on the basis is as for $u_\ell$ except for $X^\pm$:

\begin{align}
\pi_{\nu,\ell}(X^+) v_\ell &= \ell v_{\ell-1} \\
\pi_{\nu,\ell}(X^-) v_\ell &= (\nu - \ell) v_{\ell+1}
\end{align}

(5.44a) (5.44b)

Thus, in this basis there is no trace of the non-triviality of the co-product of $X^\pm$.

More than this we can reduce the representations directly to the classical \( U(gl(2)) \) if we introduce the restricted functions $\hat{\varphi}(\eta)$ by the operators:

\begin{align}
\hat{A}_{\nu,\rho} : C_{\nu,\rho} &\rightarrow \hat{C}_{\nu,\rho} , \\
\hat{A}^{-1}_{\nu,\rho} : \hat{C}_{\nu,\rho} &\rightarrow C_{\nu,\rho} , \\
\varphi(\eta, \dot{d}, \omega) &= (\hat{A}^{-1}_{\nu,\rho} \hat{\varphi})(\eta, \dot{d}, \omega) \equiv \hat{\varphi}(\eta) \dot{d}^\nu \omega^{(\rho-\nu)/2}
\end{align}

We denote the representation space of $\hat{\varphi}(\eta)$ by $\hat{C}_{\nu,\rho}$ and the representation acting in $\hat{C}_{\nu,\rho}$ by $\hat{\pi}_{\nu,\rho}$. The properties of $\hat{C}_{\nu,\rho}$ follow from the intertwining requirements [21]:

\begin{align}
\hat{\pi}_{\nu,\rho} \circ \hat{A}_{\nu,\rho} &= \hat{A}_{\nu,\rho} \circ \pi_{\nu,\rho} , \\
\pi_{\nu,\rho} \circ \hat{A}^{-1}_{\nu,\rho} &= \hat{A}^{-1}_{\nu,\rho} \circ \hat{\pi}_{\nu,\rho}
\end{align}

In particular, the representation action of $\hat{\pi}_{\nu,\rho}$ on $\eta^\ell$ is given by the same formulae as the action of $\pi_{\nu,\rho}$ on $v_\ell$.

At this moment, we should note that since we have functions of one variable $\eta$ we can treat it as complex variable $z$. In these terms we recover from the action of $\hat{\pi}_{\nu,\rho}$ the classical vector-field representation of $gl(2)$ (with $\partial_z \equiv d/dz$):

\begin{align}
\hat{A} \hat{\varphi} &= -\rho \hat{\varphi} , \\
\hat{B} \hat{\varphi} &= (\nu - 2z\partial_z) \hat{\varphi} , \\
X^+ \hat{\varphi} &= \partial_z \hat{\varphi} , \\
X^- \hat{\varphi} &= (\nu z - z^2 \partial_z) \hat{\varphi}
\end{align}

(5.47)

Of course, the importance of the non-trivial co-product for $X^\pm$ will be felt in the construction of the tensor products of the representations.

### 6 Conclusions and outlook

In this paper we have found the exotic matrix bialgebras which correspond to the two non-triangular nonsingular $4 \times 4$ $R$-matrices of [8], namely, $R_{S03}$ and $R_{S14}$ which are not deformations of the trivial $R$-matrix. We study three bialgebras denoted by: $S03$, $S14$, $S14o$, the latter two cases corresponding to $R_{S14}$ for deformation parameter $q^2 \neq 1$ and $q^2 = 1$, respectively. We have found the corresponding dual bialgebras $s03$, $s14$, $s14o$, and studied their representation theory.

For the bialgebras $s03$ and $s14$ we have studied the regular representation (the algebra acting on itself), the weight representations, and the representations in which the algebra acts on the dual matrix bialgebra. The representation theory is degenerate: the irreps are finite-dimensional of maximal dimension 4 and 2 for $s03$ and $s14$, respectively. For future use we shall say that the bialgebras $s03, S03$ and $s14, S14$ are exotic (adding to the list of exotic bialgebras termed so in [9]).
The algebras $s_{14o}, S_{14o}$ turned out to be Hopf algebras and to be special cases of the two-parameter deformations $U_{p,q}, GL_{p,q}(2)$, namely, they would be obtained from the latter by setting $q = p^{-1}$ and then $p = -1$. This was not anticipated since the corresponding $R$-matrices were different and seemingly nonequivalent cases of the classification of [8]. Thus, this became a case study important methodologically, and so we have made the exposition according to the way we proceeded. In fact, the algebra $s_{14o}$ is equivalent even to $U(gl(2))$, and the only nontriviality is in the Hopf algebra structure. Thus, the regular and weight representations are as those of $U(gl(2))$. The induced representations of $s_{14o}$ on $S_{14o}$ could also be extracted from the equivalence with the two-parameter $p, q$ deformations but their consideration is also important methodologically.

To conclude, we should stress that with this paper we finalize the explicit classification of the matrix bialgebras generated by four elements. There are altogether nine such bialgebras, four of which are quantum groups and are deformations of the classical algebras of functions on $GL(2)$ and $GL(1|1)$ (two in each case), and the other five bialgebras, which we call exotic, are not such deformations.

Further, we would like to study the spectral decomposition and Baxterisation of these exotic algebras and associated noncommutative geometries, cf. [22].

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