HARNACK INEQUALITY FOR A SUBELLiptic PDE IN NONdivergence Form

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Abstract. We consider subelliptic equations in non divergence form of the type

$$Lu = \sum_{i \leq j} a_{ij} X_j X_i u = 0$$

where $X_j$ are the Grushin vector fields, and the matrix coefficient is uniformly elliptic. We obtain a scale invariant Harnack’s inequality on the $X_j$’s CC balls for nonnegative solutions under the only assumption that the ratio between the maximum and minimum eigenvalues of the coefficient matrix is bounded. In the paper we first prove a weighted Aleksandrov Bakelman Pucci estimate, and then we show a critical density estimate, the double ball property and the power decay property. Once this is established, Harnack’s inequality follows directly from the axiomatic theory developed by Di Fazio, Gutierrez and Lanconelli in [6].

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1. Introduction

Let $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the vector fields

$$X_1 = \partial_{x_1}, \quad X_2 = x_1 \partial_{x_2}.$$ (1)

We define the second order partial differential operator

$$L = a_{11} X_1^2 + 2a_{12} X_2 X_1 + a_{22} X_2^2$$ (2)

and we assume there are $\lambda, \Lambda > 0$ such that for all $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$

$$\lambda (\xi_1^2 + \xi_2^2) \leq a_{11} \xi_1^2 + 2a_{12} \xi_1 \xi_2 + a_{22} \xi_2^2 \leq \Lambda (\xi_1^2 + \xi_2^2).$$ (3)
The positive constants $\lambda, \Lambda$ are called ellipticity constants with respect to the couple $(X_1, X_2)$. The second order operator $L$ is degenerate elliptic and in non divergence form with bounded coefficients and it is a prototype of subelliptic pdo’s, because $[X_1, X_2] = \partial_{x_2}$.

Our motivation to study the operator $L$ in (2) comes from the geometric theory of several complex variable, where nonlinear second order Partial Differential Equations of “degenerate elliptic”- type appear. In particular, in looking for a characterization property of domains of holomorphy in term of a differential property of the boundary (pseudoconvexity), one has to handle the Levi curvatures equations, which are fully nonlinear equations in non-divergence form (see e.g. [17], [18]). The existence theory for viscosity solutions to such equations is quite well settled down, mainly thanks to the papers [4], [21]. On the contrary, the problem of the regularity is well understood only in $\mathbb{R}^3$ (see [3]) and it is still widely open in higher dimension. This is mainly due to the lack of pointwise estimates for solutions to linear sub-elliptic equations with rough coefficients. Very recently, in a joint work with Cristian Gutierrez and Ermanno Lanconelli [12], we recognize that these equations in cylindrical coordinates are non divergence pde’s $Lu = f$, with $L$ structured as in (2).

The purpose of this paper is to establish a scale invariant Harnack’s inequality on balls $B$ of the Carnot Carathéodory (in brief CC) distance given by the vector fields in (1). Precisely, we will show that for all nonnegative solutions $u$ to $Lu = 0$

$$\sup_B u \leq C \inf_B u,$$

with a positive constant $C$ depending only on $\lambda$ and $\Lambda$.

When $L$ is a standard uniformly elliptic operator, this is the celebrated Harnack’s inequality of Krylov and Safonov and its proof depends in a crucial way upon the maximum principle of Aleksandrov Bakelman Pucci (in brief ABP), see [11, Section 9.8], [13, Theorem 2.1.1] and [2, Section 3].

It is not known if such a principle holds true in a general subelliptic context in a form such as [13, Theorem 2.1.1]. In [14] we proved a maximum principles of these type on the Heisenberg group, but with a reminder. Moreover, in [5] it is proved that the ABP maximum principle fails in the space of functions with second order horizontal derivatives in $L^p$ for subcritical $p$, i.e. $0 < p < Q$, where $Q$ the homogeneous dimension. Roughly speaking, a key problem in the subelliptic case is that the homogeneous dimension is always strictly greater than the number of the vector fields generating $\mathbb{R}^n$. Our main idea to overcome this obstacle is to introduce a weight in the Lebesgue measure, which allows us to handle subcritical $L^p$ norms.

In this paper we prove that a weighted ABP maximum principle holds in our context (see Theorem 2.5). This is mainly due to the particular structure of the symmetric matrix

$$(4) \begin{pmatrix} X_1^2 u & X_2 X_1 u \\ X_2 X_1 u & X_2^2 u \end{pmatrix},$$

whose determinant is the determinant of the real Hessian matrix of $u$ times a weight (see (6)). We then follow the classical proof of the ABP in [2], but by measuring the right hand side with a weighted measure.

Structural theorems for the CC balls then allow us to build ad hoc barriers for the geometry of the problem to get a critical density estimate. Moreover, the double ball property is obtained by performing an idea of Giulio Tralli in [22].
Definition 1.1 (Double Ball Property). Let $r > 0$ and $y \in \mathbb{R}^2$ and define the set of functions

$$K = \{ u \in C^2(B(y,3r)) \cap C(B(y,3r)) : u \geq 0 \text{ and } Lu \leq 0 \text{ in } B(y,3r), u \geq 1 \text{ on } B(y,r) \}.$$

We say that $L$ satisfies the Double Ball Property in $B(y,3r)$ if there exists a constant $\gamma > 0$, only depending on the ellipticity constants $\lambda, \Lambda$, such that $u \geq \gamma$ in $B(y,2r)$ for all $u \in K$.

In [15] Gutierrez and Tournier proved this property for elliptic equations on the Heisenberg group $\mathbb{H}^1$. In [22] Tralli proved that it holds true for a general Carnot group of step two. He first recognized that, via the weak Maximum Principle, the double ball property is a consequence of a kind of solvability of the Dirichlet problem in the exterior of any homogeneous ball. His main tools are the structure of two-dimensional non-abelian nilpotent Lie groups and the existence of suitable local barrier functions in the interior of the Gauge ball at any boundary point.

Unfortunately, in our situation there exist no two-dimensional non-abelian nilpotent Lie groups associated to the vector fields in (1) (see for instance [20]). However, in Section 6 we perform Tralli’s idea and we prove the Double Ball property for the operator $L$ as a consequence of a very general Ring Theorem (see Theorem 6.2).

Once the critical density, the double ball property and the power decay property are proved, Harnack’s inequality follows directly from the theory developed by Di Fazio, Gutierrez and Lanconelli in [6]. Indeed, they proved an axiomatic theory to establish the scale invariant Harnack inequality in very general settings. In particular, their procedure applies in Carnot Carathéodory metric spaces.

Recently, Gutierrez and Tournier in [15] and Tralli in [23] provided direct proofs (in the Heisenberg group $\mathbb{H}^1$ and in $H$-type groups, respectively) of the critical density estimates for super solutions using barriers, but under the restrictive assumption that the matrix of the coefficients $a_{ij}$ is a small perturbation of the Identity matrix.

The paper is organized as follows. Section 2 contains a few preliminaries and the proof of the ABP maximum principle. In Section 3 we prove two structure theorems of the CC ball which, together with the results proved by Franchi and Lanconelli in [10], play a central role in our study of barriers. The construction of the barrier and the critical density estimate are established in Section 4 and Section 5, respectively. In Section 6, we prove the Ring theorem and the existence of suitable uniform barrier functions for the operator $L$ in (2). Finally, we prove the double ball property and we indicate how to obtain the power decay property and the invariant Harnack’s inequality from the results in [6].

2. The ABP with a weight

We first introduce some standard notations and well known facts.

Definition 2.1. Let $w : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}^2$. We say that an affine function $\ell$ is a supporting hyperplane for $w$ at $x_0 \in \Omega$ in $\Omega$ if $\ell$ touches $w$ by below at $x_0$ in $\Omega$, i.e. $\ell(x_0) = w(x_0)$ and $\ell(x) \leq w(x)$ for any $x \in \Omega$.

Let $u$ be a continuous function in a open convex set $\Omega$. The convex envelope of $u$ in $\Omega$ is
defined by
\[ \Gamma(u)(x) = \sup_w \{ w(x) : w \leq u \text{ in } \Omega, w \text{ convex in } \Omega \} \]
\[ = \sup_\ell \{ \ell(x) : \ell \leq u \text{ in } \Omega, \ell \text{ is affine} \} \]
for \( x \in \Omega. \)

Obviously, \( \Gamma(u) \) is a convex function in \( \Omega \) and the set \( \{ u = \Gamma(u) \} \) is called the contact set.

**Definition 2.2.** The normal mapping of \( u \in C(\Omega) \), or sub-differential of \( u \), is the set valued function \( Du \) defined by
\[ Du(x_0) = \{ p \in \mathbb{R}^2 : u(x) \geq u(x_0) + p \cdot (x - x_0) \text{ for all } x \in \Omega \} \]

Given \( E \subset \Omega \), we define \( Du(E) = \bigcup_{x \in E} Du(x) \).

When \( u \) is differentiable \( Du \) is basically the gradient of \( u \).

**Theorem 2.3.** If \( \Omega \) is open and \( u \in C(\Omega) \) then the class
\[ S = \{ E \subset \Omega : Du(\Omega) \text{ is Lebesgue measurable} \} \]
is a Borel \( \sigma \) algebra. The set function \( \mu_u : S \to \mathbb{R} \) defined by
\[ \mu_u(E) = |Du(E)| \]
is a Borel measure and it is finite on compact sets. The measure \( \mu_u \) is called the Monge Ampère measure associated with the function \( u \).

Moreover, if \( u \in C^2(\Omega) \) is a convex function \( \Omega \), then
\[ \mu_u(E) = \int_E \det(D^2u)(x)dx \]
for any Borel set \( E \subset \Omega \).

We suggest the reference [13, Theorem 1.1.13 and Example 1.1.14] for the proof.

One can prove the following classical ABP estimate.

**Theorem 2.4.** Let \( \Omega \subset \mathbb{R}^2 \) be a bounded open set and assume \( u \in C(\bar{\Omega}) \), \( u \geq 0 \) on \( \partial \Omega \). Define \( u^-(x) = \max\{-u(x), 0\} \) and let \( \Gamma_u \) be the convex envelope of \( -u^- \) in a Euclidean ball \( B_\varepsilon(2d) \) of radius \( 2d \) such that \( \Omega \subset B_\varepsilon(2d) \) and extend \( u \equiv 0 \) outside \( \Omega \)
\[ \sup_\Omega u^- \leq \frac{d}{c} (\mu_{\Gamma_u}(\{u = \Gamma_u\} \cap \Omega))^{1/2} \]
where \( d = \text{diam } \Omega \) is the Euclidean diameter of \( \Omega \), \( c \) is a positive universal constant.

We suggest the reference [13, Theorem 1.4.5] for a detailed proof of Theorem 2.4.

The main result of this section is the following.

**Theorem 2.5 (Weighted ABP Maximum Principle).** Let \( \Omega \) be a bounded domain of \( \mathbb{R}^2 \) and let \( u \in C(\bar{\Omega}) \) and \( u \geq 0 \) on \( \partial \Omega \). Define \( u^-(x) = \max\{-u(x), 0\} \), \( f^+ = \max\{f(x), 0\} \). Moreover, \( \Gamma_u \) is the convex envelope of \( -u^- \) in a ball \( B_\varepsilon(2d) \) of radius \( 2d \) such that \( \Omega \subset B_\varepsilon(2d) \) and we have extended \( u \equiv 0 \) outside \( \Omega \). Assume \( u \in C^2(\Omega) \) is a classical solution of \( Lu(x) \leq f(x)x^2 \) in \( \Omega \), with \( f \) bounded. Then,
\[ \sup_\Omega u^- \leq C \text{diam}(\Omega) \left( \int_{\Omega \cap \{u = \Gamma_u\}} (f^+(x))^2 x^2_1 dx \right)^{\frac{1}{2}} \]
Here \( C \) is a positive universal constant only depending on \( \Lambda, \lambda \). Moreover, \( \text{diam}(\Omega) \) is the Euclidean diameter of \( \Omega \).

**Proof.** We first assume that \( u \) is strictly convex in the contact set \( \{ u = \Gamma_u \} \). Then \( Du \) is a one-to-one map. Moreover, the contact set \( \{ u = \Gamma_u \} \) has empty intersection with the line \( \{ x_1 = 0 \} \), because \( (a_{11}u_{11})(0, x_2) = Lu(0, x_2) \leq 0 \), and since \( a_{11} \) is positive then \( u_{11}(0, x_2) \leq 0 \), while \( u \) is strictly convex in \( \{ u = \Gamma_u \} \). Remark that if \( u \in C^2(\Omega) \) is convex then the symmetric matrix

\[
X^2u = \begin{pmatrix}
X_1^2u & X_2X_1u \\
X_2X_1u & X_2^2u
\end{pmatrix}
= \begin{pmatrix}
u_{11} & x_{1}u_{12} \\
x_{1}u_{12} & x_{1}^2u_{22}
\end{pmatrix}
\]

is nonnegative definite. Here we have denoted by \( u_{ij} = \partial_x \partial_{x_j} u \).

Moreover, for \( u \) convex we have

\[
det(D^2u) = (u_{11}u_{22} - u_{12}^2) = \frac{(u_{11}x_1^2u_{22} - (x_1u_{12})^2)}{x_1^2} = \frac{(X_1^2uX_2^2u - (X_2X_1u)^2)}{x_1^2} = \frac{\det(X^2u)}{x_1^2} \leq \frac{(\text{trace}(AX^2u))^2}{4x_1^2 \text{det } A}
\]

for every \( A > 0 \) and \( A \) symmetric.

Let \( \overline{B}_\varepsilon(2d) \) be a Euclidean ball containing \( \Omega \). Since \( \Gamma_u \) is convex, it follows that \( \Gamma_u \) has a supporting hyperplane at \( x_0 \), for \( x_0 \in \overline{B}_\varepsilon(2d) \cap \{ u = \Gamma_u \} \subset \Omega \). Since in addition \( \Gamma_u(x_0) = u(x_0) \), this hyperplane is also a supporting hyperplane to \( u \) at the same point. That is \( D\Gamma_u(x_0) \subset Du(x_0) \) for \( x_0 \in \overline{B}_\varepsilon(2d) \cap \{ u = \Gamma_u \} \subset \Omega \), and by recalling Theorem 2.3 we have

\[
|D\Gamma_u(\{ u = \Gamma_u \} \cap \Omega)| \leq |Du(\{ u = \Gamma_u \} \cap \Omega)| = \int_{\{ u = \Gamma_u \} \cap \Omega} (det D^2u)dx.
\]

By recalling that \( u \) is convex in \( \{ u = \Gamma_u \} \cap \Omega \) we can apply the matrix inequality (7) in the right hand side of (8) and we have

\[
\int_{\{ u = \Gamma_u \} \cap \Omega} (det D^2u)dx \leq C \int_{\{ u = \Gamma_u \} \cap \Omega} \frac{(Lu)^2}{x_1^2}dx.
\]

where \( C \) is a positive constant depending on \( \Lambda, \lambda \). By applying Theorem 2.4 and (8), (9) and recalling that \( Lu(x) \leq f(x) x_1^2 = f^+(x) x_1^2 \) on the contact set, we get the desired ABP estimate (5).

For the general case, \( u \) is only convex in the contact set. Let \( S_0 = \{ x \in \Omega : \det D^2u = 0 \} \). By Sard’s Theorem (see [7] or [8]) we have \( |Du(S_0)| = 0 \). Since \( E = \{ u = \Gamma_u \} \cap \Omega \) is a Borel set, \( E \cap S_0 \) and \( E \setminus S_0 \) are also Borel sets. Hence

\[
|Du(E)| = |Du(E \cap S_0)| + |Du(E \setminus S_0)| = |Du(E \setminus S_0)|
\]

and by (8) and (9) we have

\[
|D\Gamma_u(E)| \leq |Du(E)| = |Du(E \setminus S_0)| = \int_{E \setminus S_0} (det D^2u)dx \leq C \int_{E \setminus S_0} \frac{(Lu)^2}{x_1^2}dx \leq C \int_{E \setminus S_0} f^2 x_1^2 dx \leq C \int_{E} f^2 x_1^2 dx.
\]
As a corollary, we get the weak maximum principle for the operator $L$ in (2).

**Theorem 2.6** (Weak Maximum Principle). Let $\Omega$ be a bounded open set in $\mathbb{R}^2$ and let $u, v \in C^2(\Omega) \cap C(\Omega)$ such that $u \leq v$ on $\partial \Omega$ and $Lv \leq Lu$ in $\Omega$. Then, $u \leq v$ in $\Omega$.

**Proof.** Apply Theorem 2.5 to $w = v - u$. □

An alternative proof of the Weak Maximum Principle, without using Theorem 2.5, can be found in [16, Corollary 1.3].

3. GRUSHIN METRIC AND SUBLEVEL SETS

In this section we recall definition and basic properties of the Grushin metric.

The vector fields defined in (1) induce on $\mathbb{R}^2$ a metric $d_{CC}$ in the following way (see [10], [9] and [19], [1]).

**Definition 3.1** (Carnot Carathéodory metric). A Lipschitz continuous curve $\gamma : [0, T] \to \mathbb{R}^2$, $T \geq 0$, is subunit if there exists a vector of measurable functions $h = (h_1, h_2) : [0, T] \to \mathbb{R}^2$ such that $\gamma'(t) = \sum_{j=1}^2 h_j(t)X_j(\gamma(t))$ and $\sum_{j=1}^2 h_j^2(t) \leq 1$ for a.e. $t \in [0, T]$. Define the Carnot Carathéodory distance $d_{CC} : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, +\infty)$ by setting

$$d_{CC}(x, y) = \inf\{T \geq 0 : \text{there exists a subunit curve } \gamma : [0, T] \to \mathbb{R}^2$$

such that $\gamma(0) = x$ and $\gamma(T) = y\}.$

It is well-known that $d_{CC}(x, y)$ is finite for all $x, y$, because the vector fields are smooth and satisfy Hörmander condition (see [1], [19]).

The first structure Theorem below, which is a special case of the results proved by Franchi and Lanconelli in [10], plays a central role in our study of barriers.

We denote by $B_{CC}(x, r) = \{y \in \mathbb{R}^2 : d_{CC}(x, y) < r\}$ the balls in $\mathbb{R}^2$ defined by the metric $d_{CC}$.

For $j = 1, 2$ define the functions $F_j : \mathbb{R}^2 \times [0, +\infty) \to [0, +\infty)$ by

$$F_1(x, r) = r, \quad F_2(x, r) = r(|x_1| + F_1(x, r)) = r(|x_1| + r).$$

Note that $r \to F_j(x, r)$ is increasing and it satisfies the following doubling property

$$F_j(x, 2r) \leq CF_j(x, r), x \in \mathbb{R}^2, 0 < r < \infty$$

for all $j = 1, 2$.

The structure of the balls $B_{CC}(x, r)$ can be described by means of the boxes

$$Box(x, r) := \{x + h : |h| < F_j(x, r), j = 1, 2\}$$

$$= \{x + h : |h_1| < r, |h_2| < r|x_1| + r^2\}.$$  

For any fixed $x \in \mathbb{R}^2$ the function $F_j(x, \cdot)$ is strictly increasing and maps $]0, +\infty[$ onto itself. We denote its inverse by $G_j(x, \cdot) = F_j(x, \cdot)^{-1}.$ Precisely, $G_1(x, r) = r$ and $G_2(x, r) = \frac{|x_1| + \sqrt{x_1^2 + 4r^2}}{2}.$ The following structure theorem is proved in [10]

**Theorem 3.2.** There exists a constant $C > 0$ such that:

$$Box(x, C^{-1}r) \subset B_{CC}(x, r) \subset Box(x, Cr), x \in \mathbb{R}^2, r \in ]0, +\infty[.$$
and
\[ C^{-1}d_{CC}(x, y) \leq \sum_{j=1}^{2} G_j(x, |y_j - x_j|) \leq C d_{CC}(x, y), \quad x, y \in \mathbb{R}^2. \]

Here
\[ (10) \quad \sum_{j=1}^{2} G_j(x, |y_j - x_j|) = |y_1 - x_1| + \sqrt{x_1^2 + 4|y_2 - x_2|} - |x_1|. \]

For \( j = 1, 2 \) define the real numbers \( d_j \) by \( d_1 = 1, d_2 = 2 \). We say that \( d_j \) is the degree of the variable \( x_j \). Note that \( F_j(0, r) = r^{d_j} \).

Moreover, there exists a group of dilations \((\delta_t)_{t>0}\),
\[ (11) \quad \delta_t : \mathbb{R}^2 \to \mathbb{R}^2, \quad \delta_t(x) = (tx_1, t^2x_2) \]
such that the vector fields \( X_j \) are \( \delta_t \)-homogeneous of degree one, i.e. for every \( u \in C^1(\mathbb{R}^2) \),
\[ (12) \quad X_j(u \circ \delta_t)(x) = t(X_j u)(\delta_t(x)), \]
for all \( x \in \mathbb{R}^2 \) and \( t > 0 \).

The number \( Q = d_1 + d_2 = 3 \) is the homogeneous dimension of \( \mathbb{R}^2 \) with respect to \( \delta_t \).
The size of balls in the metric \( d_{CC} \) has been described by Franchi and Lanconelli \[10\] by means of the boxes \( Box(x, r) \). Precisely, there exists a positive constant \( C > 0 \) such that
\[ (13) \quad C^{-1}F(x, r) \leq |B_{CC}(x, r)| \leq CF(x, r) \]
for every \( x \in \mathbb{R}^2 \) and \( r > 0 \). Here \( |B| \) is the Lebesgue measure of \( B \) and \( F(x, r) = F_1(x, r) \cdot F_2(x, r) = r^2(|x_1| + r) \).

This means that the measure of balls with radius \( r \) and center at \( x \) with \( r \leq |x_1| \) is of Euclidean type \( |x_1| r^2 \), whereas the Lebesgue measure of \( B_{CC}(x, r) \) with \( |x_1| < r \) is comparable to \( r^3 \). Unfortunately, Theorem 3.2 and (13) are not enough to conclude that the ring condition in [6, Definition 2.6] holds true in \( B_{CC} \).

Inspired by (10), we then define a new quasi distance in \( \mathbb{R}^2 \) as
\[ \tilde{d}(x, y) = |x_1 - y_1| + \sqrt{x_1^2 + y_1^2 + 4|x_2 - y_2|} - \sqrt{x_1^2 + y_1^2}, \]
where \( x = (x_1, x_2), y = (y_1, y_2) \). Obviously, \( \tilde{d} \) is \( 1/2 \)-Hölder continuous.

We denote by
\[ (14) \quad B(x, r) = \{ y \in \mathbb{R}^2 : \tilde{d}(x, y) < r \} \]
the balls in \( \mathbb{R}^2 \) defined by the quasi metric \( \tilde{d} \).

**Theorem 3.3** (I Structure Theorem). There exists a constant \( C > 0 \) such that:
\[ Box(y, C^{-1}r) \subset B(y, r) \subset Box(y, Cr), y \in \mathbb{R}^2, r \in ]0, +\infty[ \]

**Proof.** If \( x \in B(y, r) \) then \( |x_1 - y_1| < r \) and \( \sqrt{x_1^2 + y_1^2 + 4|x_2 - y_2|} < r + \sqrt{x_1^2 + y_1^2} \). In particular,
\[ 4|x_2 - y_2| < (r + \sqrt{x_1^2 + y_1^2})^2 - (x_1^2 + y_1^2) = r^2 + 2r\sqrt{x_1^2 + y_1^2} \]
\[ \leq r^2 + 2r\sqrt{(|y_1| + r)^2 + y_1^2} \leq r^2 + 2r(2|y_1| + r) = 3r^2 + 4r|y_1|. \]
Hence, \( x \in Box(y, r) \).
Vice versa, if \( x \in Box(y, r) \), then \( |x_1 - y_1| < r \) and \( |x_2 - y_2| < r(r + |y_1|) \). We have

\[
|x_1 - y_1| + \sqrt{x_1^2 + y_1^2 + 4|x_2 - y_2|} - \sqrt{x_1^2 + y_1^2} < r + \sqrt{x_1^2 + y_1^2 + 4r(r + |y_1|)} - \sqrt{x_1^2 + y_1^2} = r + \sqrt{x_1^2 + (|y_1| + 2r)^2} - \sqrt{x_1^2 + y_1^2} \leq r + 2r.
\]

Hence, \( x \in B(y, 3r) \).

By Theorem 3.3 it is easy to recognize that there exists a positive constant \( C > 0 \) such that

\[
C^{-1}r^2(r + |y_1|) \leq |B(y, r)| \leq C r^2(r + |y_1|)
\]

for every \( y \in \mathbb{R}^2 \) and \( r > 0 \).

Moreover, there exists a positive constant \( C_D \) such that the following Doubling Property holds true

\[
0 < |B(x, 2r)| \leq C_D |B(x, r)|, \quad \forall x \in \mathbb{R}^2, \forall r > 0.
\]

The quasi metric balls \( B \) satisfy the following Ring Condition, which will be crucial in Section 6.

**Theorem 3.4 (Ring Condition).** There exists a nonnegative function \( \omega \), such that

- \( |B(x, r) \setminus B(x, (1 - \varepsilon)r)| \leq \omega(\varepsilon)|B(x, r)| \), for every ball \( B(x, r) \) and for all small \( \varepsilon > 0 \).
- \( \omega(\varepsilon) = O(\varepsilon) \) as \( \varepsilon \to 0^+ \).

**Proof.** For every fixed \( y \in \mathbb{R}^2 \) and for every \( r > 0 \), by Fubini’s Theorem we have

\[
f(r) := \int_{B(y, r)} dx = \int_{y_1 - r}^{y_1 + r} \left( \int_{\mathcal{Q}} dx_2 \right) dx_1 = \frac{1}{2} \int_{-r}^{r} \left( (r - |t|)^2 + 2(r - |t|)\sqrt{(y_1 + t)^2 + y_1^2} \right) dt.
\]

Remark that the function \( f \) is differentiable at any point and

\[
f'(r) = \int_{-r}^{r} \left( (r - |t|) + \sqrt{(y_1 + t)^2 + y_1^2} \right) dt = r^2 + \int_{-r}^{r} \sqrt{(y_1 + t)^2 + y_1^2} dt,
\]

\[
0 \leq f'(r) \leq r^2 + 4|y_1|r + r^2 \leq 4r(r + |y_1|).
\]

In particular, by Lagrange mean value Theorem, we have that there exists \( \theta \in (1 - \varepsilon, 1) \) such that

\[
|B(y, r) \setminus B(y, (1 - \varepsilon)r)| = f(r) - f((1 - \varepsilon)r) = f'(\theta r)\varepsilon r \leq 4\varepsilon r^2(r + |y_1|) \leq c\varepsilon|B(y, r)|,
\]

where \( c \) is a positive universal constant because of (15).

In the sequel we will also need the following characterization of \( Box(y, r) \) and consequently of \( B(y, r) \) in term of the function

\[
\rho(x, y) = (x_1^2 - y_1^2)^2 + 4(x_2 - y_2)^2)^{1/4}.
\]

It is well known that the function \( \Gamma(x, 0) := \rho^{2-Q}(x, 0) \) is the fundamental solution with pole at the origin of the subelliptic Laplacian \( X_1^2 + X_2^2 \), where \( Q = 3 \) is the homogeneous dimension.
For every fixed \( r > 0 \) and \( y = (y_1, y_2) \in \mathbb{R}^2 \) define
\[
\tilde{g}_r(x, y) = \begin{cases} 
\rho(x, y), & \text{if } |y_1| < r \\
\frac{1}{|y_1|}\rho^2(x, y), & \text{if } |y_1| \geq r
\end{cases}
\]

Remark 3.5. The function \( \tilde{g}_r(x, y) \) has two zeros at \( (y_1, y_2) \) and at \( (-y_1, y_2) \) and it is an even function with respect to the first variable \( x_1 \).

In the sequel we will study the sublevel sets of the function \( \tilde{g}_r(\cdot, y) \)
\[
\hat{G}(y, r) := \{ x \in \mathbb{R}^2 : \tilde{g}_r(x, y) < r \}.
\]

In order to avoid two zeros, for every fixed \( r > 0 \) we also define
\[
g_r(x, y) = \begin{cases} 
\rho(x, y), & \text{if } |y_1| < r \\
\frac{1}{|y_1|}\rho^2(x, y), & \text{if } |y_1| \geq r \text{ and } x_1y_1 \geq 0 \\
+\infty, & \text{if } |y_1| \geq r \text{ and } x_1y_1 < 0
\end{cases}
\]

In the following Theorem, which is the main result of this section, we compare the sublevel sets of the function \( g_r(\cdot, y) \)
\[
G(y, r) := \{ x \in \mathbb{R}^2 : g_r(x, y) < r \}
\]
with the Box\( (y, r) \).

Theorem 3.6 (II Structure Theorem). There exists a constant \( C > 0 \) such that:
\[
\text{Box}(y, C^{-1}r) \subset G(y, r) \subset \text{Box}(y, Cr), y \in \mathbb{R}^2, r \in ]0, +\infty[.
\]

Proof. To prove the first inclusion in (20), take \( x \in \text{Box}(y, r/5) \) and assume \( |y_1| < r \), then
\[
|x_1 - y_1| < r/5, \quad |x_2 - y_2| < r/5(r/5 + |y_1|) < 1/5(1/5 + 1)r^2
\]
and we get
\[
g_r^4(x, y) = \rho^4(x, y) = (x_1 - y_1)^2(x_1 + y_1)^2 + 4(x_2 - y_2)^2
\]
\[
< (r/5)^2(|x_1| + |y_1|)^2 + 4r^4(1/5)^2(1/5 + 1)^2
\]
\[
\leq (r/5)^2(|x_1 - y_1| + 2|y_1|)^2 + 4r^4(1/5)^2(1/5 + 1)^2
\]
\[
< r^4(1/5)^2(1/5 + 2)^2 + 4r^4(1/5)^2(1/5 + 1)^2
\]
\[
< r^4(1/5)(1/5 + 2)^2 = r^4(121/125) < r^4
\]
If \( x \in \text{Box}(y, r/5) \) and \( |y_1| \geq r \), then
\[
|x_1 - y_1| < r/5, \quad |x_2 - y_2| < r/5(r/5 + |y_1|) \leq 1/5(1/5 + 1)r|y_1|
\]
and we get \( x_1y_1 > 0 \) and
\[
g_r^2(x, y)|y_1|^2 = \rho^2(x, y) = (x_1 - y_1)^2(x_1 + y_1)^2 + 4(x_2 - y_2)^2
\]
\[
< (r/5)^2(|x_1| + |y_1|)^2 + 4r^2|y_1|^2(1/5)^2(1/5 + 1)^2
\]
\[
\leq (r/5)^2(|x_1 - y_1| + 2|y_1|)^2 + 4r^2|y_1|^2(1/5)^2(1/5 + 1)^2
\]
\[
< r^2|y_1|^2(1/5)^2(1/5 + 2)^2 + 4r^2|y_1|^2(1/5)^2(1/5 + 1)^2
\]
\[
< r^2|y_1|^2(1/5)(1/5 + 2)^2 = r^2|y_1|^2(121/125) < r^2|y_1|^2
\]
In particular, we have proved that
\[
\text{Box}(y, r/5) \subset G(y, r).
\]
To prove the second inclusion in (20) take \( x \in G(y, r) \) and assume \( |y_1| < r \), then
\[
|x_1^2 - y_1^2| < r^2, \quad 4|x_2 - y_2|^2 < r^2.
\]
In particular
\[
|x_2 - y_2| < \frac{r^2}{2} < r(r + |y_1|),
\]
and
\[
(21) \quad r^2 > |x_1 - y_1||x_1 + y_1| \geq |x_1 - y_1||x_1 - y_1| - 2|y_1|.
\]
We have two cases
- If \( |x_1 - y_1| \leq 2|y_1| \), then \( x \in Box(y, 2r) \).
- If \( |x_1 - y_1| > 2|y_1| \), then by (21)
\[
|x_1 - y_1| < \sqrt{|y_1|^2 + r^2} + |y_1| < 3r
\]
and \( x \in Box(y, 3r) \).

If \( x \in G(y, r) \) and \( |y_1| \geq r \) then \( x_1y_1 \geq 0 \)
\[
|x_1^2 - y_1^2| < r|y_1|, \quad 4|x_2 - y_2|^2 < r^2|y_1|^2.
\]
Thus, we have
\[
|x_2 - y_2| < \frac{r|y_1|}{2} < r(r + |y_1|),
\]
and, since \( x_1y_1 \geq 0 \)
\[
r|y_1| > |x_1 - y_1||x_1 + y_1| \geq |x_1 - y_1||y_1|.
\]
Hence \( x \in Box(y, r) \).
In particular, we have proved that
\[
G(y, r) \subset Box(y, 3r).
\]

\[ \square \]

4. Barriers

Let \( \rho \) be the function in (17).

**Lemma 4.1.** Let \( y = (y_1, 0) \in \mathbb{R}^2 \) be fixed and consider the function \( \phi(x) = \rho^\alpha(x, y) \). For \( \alpha \leq 2 - 3\Lambda/\lambda \), \( \phi \) is a classical solution of \( L\phi \geq 0 \) in the set \( \{ \rho > 0 \} \).

**Proof.** By a simple calculation we get
\[
L\phi = a_{11}\partial_{x_1} (\alpha \rho^{\alpha - 1}\partial_{x_1}) + 2a_{12}x_1\partial_{x_2}(\alpha \rho^{\alpha - 1}\partial_{x_1}) + a_{22}x_1^2\partial_{x_2}(\alpha \rho^{\alpha - 1}\partial_{x_2})
\]
\[
= a_{11}(\alpha(\alpha - 1)\rho^{\alpha - 2}\rho_{x_1}^2 + \alpha\rho^{\alpha - 1}\rho_{x_1x_1}) + 2a_{12}x_1(\alpha(\alpha - 1)\rho^{\alpha - 2}\rho_{x_1}\rho_{x_2} + \alpha\rho^{\alpha - 1}\rho_{x_1x_2})
\]
\[
+ a_{22}x_1^2(\alpha(\alpha - 1)\rho^{\alpha - 2}\rho_{x_2}^2 + \alpha\rho^{\alpha - 1}\rho_{x_2x_2})
\]
and
\[
\rho_{x_1} = \rho^{-3}(x_1^2 - y_1^2)x_1,
\]
\[
\rho_{x_2} = 2\rho^{-3}x_2,
\]
\[
(22) \quad \rho_{x_1x_1} = (3x_1^2 - y_1^2)\rho^{-3} - 3\rho^{-7}x_1^2(x_1^2 - y_1^2)^2 = \rho^{-7}(12x_2^2x_1^2 - y_1^2\rho^4)
\]
\[
\rho_{x_1x_2} = \rho^{-7}(-6x_2x_1(x_1^2 - y_1^2))
\]
\[
\rho_{x_2x_2} = 2\rho^{-3} - 12\rho^{-7}x_2^2 = \rho^{-7}(2(x_1^2 - y_1^2)^2 - 4x_2^2) = \rho^{-7}(3(x_1^2 - y_1^2)^2 - \rho^4)
\]

By substituting (23) in (22) and by taking into account (3) and that \( \alpha \leq -1 \) is negative and \( a_{11}, a_{22} \) are positive, we get

\[
L\phi = \alpha (\alpha - 1)x_1^2 \rho^{\alpha - 8} (a_{11}(x_1^2 - y_1^2)^2 + 4a_{12}x_2(x_1^2 - y_1^2) + 4a_{22}x_2^2)
\]
\[
+ \alpha \rho^{\alpha - 8} (a_{11}(12x_1^2y_2^2 - y_1^2)^2 - 12a_{12}x_2x_1^2(x_1^2 - y_1^2) + a_{22}x_1^2(3(x_1^2 - y_1^2)^2 - \rho^4))
\]
\[
\geq \alpha (\alpha - 1)x_1^2 \rho^{\alpha - 8} \left( \lambda \rho^4 - \alpha a_{11} \rho^{\alpha - 4} y_1^2 \right)
\]
\[
+ \alpha \rho^{\alpha - 8} (a_{11}(12x_1^2y_2^2 - y_1^2)^2 - 12a_{12}x_2x_1^2(x_1^2 - y_1^2) + a_{22}x_1^2(3(x_1^2 - y_1^2)^2 - \rho^4))
\]
\[
\geq - \alpha a_{11} \rho^{\alpha - 4} y_1^2 + \alpha x_1^2 \rho^{\alpha - 4} (\alpha - 2) \lambda + 3 \lambda \geq - \alpha a_{11} \rho^{\alpha - 4} y_1^2 \geq 0.
\]

\[\square\]

Theorem 4.2. There exists positive universal constants \( C > 0 \) and \( M > 1 \) such that for every \( y = (y_1, 0) \in \mathbb{R}^2 \) and \( r \in \mathbb{R}^+ \) there is a \( C^2 \) function \( \tilde{\varphi} : \mathbb{R}^2 \to \mathbb{R} \) such that

\[
\tilde{\varphi} \geq 0, \text{ on } \mathbb{R}^2 \setminus \tilde{G}(y, 2r),
\]

\[
\tilde{\varphi} \leq -2, \text{ in } \tilde{G}(y, r)
\]

\[
\tilde{\varphi} \geq -M, \quad L\tilde{\varphi}(x) \leq C \frac{x_1^2}{r^2(r + |y_1|)^2} \zeta(x), \text{ in } \mathbb{R}^2
\]

where \( 0 \leq \zeta \leq 1 \) is a continuous function in \( \mathbb{R}^2 \) with \( \text{supp} \zeta \subset \tilde{G}(y, r) \).

Proof. Consider the constant \( \alpha = 2 \cdot 3 \lambda / \lambda \) and recall that \( \alpha \leq -1 \). We define

\[
\varphi(x) = M_1 - M_2 \rho^\alpha(x, y), \text{ in } \mathbb{R}^2 \setminus \{ x : \rho(x, y) = 0 \}.
\]

We choose \( M_1, M_2 \) such that

\[
\varphi|_{\partial \tilde{G}(y, 2r)} = 0, \quad \varphi|_{\partial \tilde{G}(y, r)} = -2
\]

Then take \( -m = \varphi|_{\partial \tilde{G}(y, r/2)} \). We now show that \( M_1, M_2, m \) are positive. Indeed, if \( |y_1| \geq 2r \) then

\[
M_1 = \frac{2 \cdot 2^{\alpha/2}}{1 - 2^{\alpha/2}}, \quad M_2 = \frac{2}{(r|y_1|)^{\alpha/2}(1 - 2^{\alpha/2})}
\]

and

\[
-m = \frac{2}{1 - 2^{\alpha/2}} \left( 2^{\alpha/2} - \frac{1}{2^{\alpha/2}} \right) .
\]

If \( |y_1| < r/2 \) then

\[
M_1 = \frac{2 \cdot 2^\alpha}{1 - 2^\alpha}, \quad M_2 = \frac{2}{r^\alpha(1 - 2^\alpha)}
\]

and

\[
-m = \frac{2}{1 - 2^\alpha} \left( 2^\alpha - \frac{1}{2^\alpha} \right) .
\]

If \( r/2 \leq |y_1| < r \) then

\[
M_1 = \frac{2 \cdot 2^\alpha}{1 - 2^\alpha}, \quad M_2 = \frac{2}{r^\alpha(1 - 2^\alpha)}
\]
and

\[-m = \frac{2}{1 - 2^\alpha} \left( 2^\alpha - \left( \frac{|y_1|}{2r} \right)^{\alpha/2} \right) > \frac{2}{1 - 2^\alpha} \left( 2^\alpha - \frac{1}{2^{\alpha/2}} \right).\]

If \( r \leq |y_1| < 2r \) then

\[
(31) \quad M_1 = \frac{2 \cdot 2^\alpha r^{\alpha/2}}{|y_1|^{\alpha/2} - (4r)^{\alpha/2}} < \frac{2 \cdot 2^\alpha}{2^{\alpha/2} - 4^{\alpha/2}}, \quad M_2 = \frac{2}{r^{\alpha/2}(|y_1|^{\alpha/2} - (4r)^{\alpha/2})}
\]

and

\[-m = -2 \left( \frac{(|y_1|/2)^{\alpha/2} - (4r)^{\alpha/2}}{(|y_1|^{\alpha/2} - (4r)^{\alpha/2})} \right) > -2 \left( \frac{(1/2)^{\alpha/2} - 4^{\alpha/2}}{2^{\alpha/2} - 4^{\alpha/2}} \right).
\]

Therefore \( M_1 \) and \( m \) are uniformly bounded, while by (28), (29), (30), (31) we have

\[
(32) \quad \frac{\hat{C}}{r^{\alpha/2}(r + |y_1|)^{\alpha/2}} \leq M_1 \leq \frac{C}{r^{\alpha/2}(r + |y_1|)^{\alpha/2}}.
\]

Define

\[h(t) = \begin{cases} t, & \text{if } -m \leq t \\ -\int_{t-m}^0 \frac{1}{1 + x^{\alpha\beta}} ds - m, & \text{if } t < -m\end{cases}\]

where \( \beta \in \mathbb{N} \) will be chosen in a moment. For \( t < -m \) we have

\[h'(t) = \frac{1}{1 + (t + m)^{2\beta}} > 0, \quad h''(t) = -\frac{2\beta(t + m)^{2\beta - 1}}{(1 + (t + m)^{2\beta})^2},\]

therefore \( h \in C^2(\mathbb{R}) \).

Define

\[\tilde{\varphi}(x) = \begin{cases} h(\varphi(x)), & \text{if } \rho(x, y) > 0 \\ -\int_{-\infty}^0 \frac{1}{1 + x^{2\beta}} ds - m, & \text{if } \rho(x, y) = 0\end{cases}\]

Remark that \( \tilde{\varphi} \) satisfies (25), (26). By recalling (23), (24), (32) and by choosing \( \beta \in \mathbb{N} \) and \( 2\beta > \max\{1, 1 - 4/\alpha\} \), in \( \tilde{G}(y, r/2) \) we have

\[
L\tilde{\varphi} = h''(\varphi) \left( a_{11} \varphi_x^2 + 2a_{12} x_1 \varphi_x \varphi_x^2 + a_{22} x_1^2 \varphi_{x_2}^2 \right) + h'(\varphi) L\varphi
\leq |h''(\varphi)| \Lambda((X_1 \varphi)^2 + (X_2 \varphi)^2)
\leq |h''(\varphi)| \Lambda \hat{C} M_2 x_1^2 r^{\alpha - 4} \leq C M_2^{1-2\beta} \rho^{\alpha-4-2\alpha^2} x_1^2
\leq \frac{C}{r^2 (r + |y_1|)^{\alpha/2}} x_1^2
\]

because \( \rho < Cr^{1/2}(r + |y_1|)^{1/2} \) in \( \tilde{G}(y, r/2) \).

Finally, by Lemma 4.1, \( L\tilde{\varphi} = L\varphi \leq 0 \) in \( \mathbb{R}^2 \setminus \tilde{G}(y, r/2) \), and \( L\tilde{\varphi} \leq C \frac{x_1^2}{r^2 (r + |y_1|)^{\alpha/2}} \) in \( \tilde{G}(y, r/2) \). We now take a continuous function \( \zeta \) such that \( \zeta \equiv 1 \) in \( \tilde{G}(y, r/2) \), \( \zeta \equiv 0 \) outside \( \tilde{G}(y, 2r/3) \), \( 0 \leq \zeta \leq 1 \), \( \zeta \) is even in the \( x_1 \) variable and such that (27) holds true. \( \square \)
5. Critical Density

By combining Theorem 2.5 with the geometry of the sets $G$ and the barrier in Lemma 4.2 we get a first rough critical density estimate.

Recall that by definition $G(y, r) = \tilde{G}(y, r) \cap \{(x_1, x_2) : x_1 y_1 \geq 0\}$ for $|y_1| \geq r$, while $G(y, r) = \tilde{G}(y, r)$ for $|y_1| < r$ and that by Theorem 3.6 and Theorem 3.3 $B(y, r/c) \subset G(y, r) \subset B(y, cr)$ for a universal constant $c > 0$.

**Theorem 5.1.** There exist universal constants $0 < \nu < 1$ and $M > 1$ such that, for every $y = (y_1, 0) \in \mathbb{R}^2$ and $r > 0$, if $u \in C^2(G(y, 2r)) \cap C\{G(y, 2r)\}$ is a nonnegative solution of $Lu \leq 0$ in $G(y, 2r)$ and $\inf_{G(y, r)} w \leq 1$ then

$$
\{u \leq M\} \cap G(y, 3r/2) \geq \frac{\nu}{\max \{r + |y_1|, \frac{1}{r + |y_1|}\}|G(y, 3r/2)|}.
$$

Here $| \cdot |$ is the Lebesgue measure.

**Proof.** Take $\tilde{\varphi}$ as in Lemma 4.2 and $w = u + \tilde{\varphi}$. We have $Lw \leq C\frac{\xi^2}{(r + |y_1|)^4}$ in $G(y, 2r)$ and $w \geq 0$ on $\partial G(y, 2r)$ by (25). Moreover, $\inf_{G(y, r)} w \leq -1$ by (26).

We next apply Theorem 2.5 to $w \in G(y, 2r)$. By Theorem 3.3 and Theorem 3.6 we have $\text{diam}(G(y, 2r)) \leq C \max\{r, r + |y_1|\}$, $c^{-1}r^2(r + |y_1|) \leq |G(y, r)| \leq cr^2(r + |y_1|)$ and $\sup_{G(y, 2r)} |x_1| \leq \tilde{c}(r + |y_1|)$, for some positive universal constants $C, c, \tilde{c}$. By recalling that $0 \leq \zeta \leq 1$ and $\text{supp} \zeta \subset G(y, r)$ we get

\[
1 \leq C \text{diam}(G(y, 2r)) \left( \int_{\{w = \Gamma_y \} \cap G(y, 2r)} \frac{(x_1 C \zeta)^2}{r^4(r + |y_1|)^4} \, dx \right)^{1/2}
\leq C \max\{r, r + |y_1|\} \frac{\zeta^2}{r^2(r + |y_1|)} \left( \int_{\{w = \Gamma_y \} \cap G(y, 2r) \cap \tilde{G}(y, r)} \zeta^2 \, dx \right)^{1/2}
\leq \tilde{C} \max\{r, r + |y_1|\} \left( \int_{\{w = \Gamma_y \} \cap G(y, 2r) \cap \tilde{G}(y, r)} \zeta^2 \, dx \right)^{1/2}
\leq \tilde{C} e^{1/2} \max\{r, r + |y_1|\} \frac{(r^2 + |y_1|)^{1/2}}{|G(y, r)|^{1/2}} \left( \int_{\{w = \Gamma_y \} \cap G(y, 2r) \cap \tilde{G}(y, r)} \zeta^2 \, dx \right)^{1/2}
\leq \tilde{C} e^{1/2} \max\{r, r + |y_1|\} \frac{(r^2 + |y_1|)^{1/2}}{|G(y, r)|^{1/2}} \left( \int_{\{w = \Gamma_y \} \cap G(y, 2r) \cap \tilde{G}(y, r)} \zeta^2 \, dx \right)^{1/2}
\]

because $w = \Gamma_y$ implies $w(x) \leq 0$ and thus $u(x) \leq -\varphi(x) \leq M$.

Moreover,

\[
1 \leq C \max\{r, r + |y_1|\} \frac{|x_1 C \zeta|^2}{r^4(r + |y_1|)^4} \left( \int_{\{w = \Gamma_y \} \cap G(y, 2r) \cap \tilde{G}(y, r)} \zeta^2 \, dx \right)^{1/2}
\]

We now remark that, for $|y_1| < 3r/2$ or for $|y_1| \geq 2r$, we have $G(y, 2r) \cap \tilde{G}(y, 3r/2) = G(y, 3r/2)$ and the estimate (33) follows.
On the contrary, for $(3/2)r \leq |y_1| < 2r$ we have $G(y, 2r) \cap \tilde{G}(y, 3r/2) = G(y, 3r/2) \cup G(-y, 3r/2)$ with $G(y, 3r/2) \cap G(-y, 3r/2)$ empty. The idea is to apply estimate (34) in a smaller ball. Let us call $r = (3/2)R$. We obviously have $R = (2/3)r$ and $2R = (4/3)r < 2r$. Hence, $u \in C^2(G(y, 2R)) \cap C(\overline{G}(y, 2R))$ is a nonnegative solution of $Lu \leq 0$ in $G(y, 2R)$. If $\inf_{G(y, r)} u = \inf_{G(y, (3/2)R)} u \leq 1$ then by arguing as in estimate (34) we have

$$1 \leq \frac{\max\{1, (3/2)R + |y_1|\}}{(3/2)R + |y_1|^{1/2}} \frac{|\{u \leq M\} \cap G(y, 2R) \cap \tilde{G}(y, (3/2)R)|^{1/2}}{|G(y, (3/2)R)|^{1/2}}$$

and $|y_1| \geq (3/2)r = (9/4)R > 2R$. By recalling that $r = (3/2)R$, we have

$$1 \leq \frac{\max\{1, (3/2)R + |y_1|\}}{(3/2)R + |y_1|^{1/2}} \frac{|\{u \leq M\} \cap G(y, r)|^{1/2}}{|G(y, r)|^{1/2}} \leq \frac{\max\{1, (3/2)R + |y_1|\}}{(3/2)R + |y_1|^{1/2}} \frac{|\{u \leq M\} \cap G(y, 3r/2)|^{1/2}}{|G(y, 3r/2)|^{1/2}}.$$

We now perform the constant in Theorem 5.1 by taking into account the geometry of the problem.

**Theorem 5.2** (Critical Density). There exist universal constants $0 < \nu < 1$ and $M > 1$ such that, for every $y = (y_1, y_2) \in \mathbb{R}^2$ and $r > 0$, if $u \in C^2(G(y, 2r)) \cap C(\overline{G}(y, 2r))$ is a nonnegative solution of $Lu \leq 0$ in $G(y, 2r)$ and $\inf_{G(y, r)} u \leq 1$ then

$$|\{u \leq M\} \cap G(y, 3r/2)| \geq \nu |G(y, 3r/2)|.$$

Here $|\cdot|$ is the Lebesgue measure.

**Proof.** The strategy is to combine Theorem 5.1 with dilations and translations and it has been inspired us by Ermanno Lanconelli during a private conversation.

I STEP. Apply Theorem 5.1 for $y_1 \in [-1, 1]$ and $r = 1$. We then have that (33) holds true with a universal positive constant and precisely

$$|\{u \leq M\} \cap G((y_1, 0), 3/2)| \geq \nu |G((y_1, 0), 3/2)|.$$  \hfill (35)

II STEP. Assume $|y_1| \leq r$. We change variables and we recall (12) and (13), together with Theorem 3.3 and Theorem 3.6. We introduce a change of variable that preserves the equation: fix $y_2 \in \mathbb{R}$ and $r > 0$ and let

$$T(x) = T(x_1, x_2) = (rx_1, y_2 + r^2x_2).$$

Remark that $T(x) \in G(y, r)$ iff $x \in G((y_1/r, 0), 1)$. We define $\tilde{u}(x) = u(T(x))$. We have, for all $i \leq j \in \{1, 2\}$

$$X_i \tilde{u}(x) = rX_i u(T(x)),
\quad X_j X_i \tilde{u}(x) = r^2 X_j X_i u(T(x)).$$

Set $\tilde{a}_{11}(x) = a_{11}(T(x)), \tilde{a}_{12}(x) = a_{12}(T(x)), \tilde{a}_{22}(x) = a_{22}(T(x)), \text{ and } \tilde{L} = \tilde{a}_{11} X_1^2 + 2\tilde{a}_{12} X_2 X_1 + \tilde{a}_{22} X_2^2$, then $\tilde{L} \tilde{u}(x) = r^2 Lu(T(x))$. We have that $\tilde{u}$ satisfies the hypothesis of
Theorem 5.3. There exist universal constants \( \eta \) with respect to the quasi metric balls \( B \) and \( \nu \) such that, for every \( r > 0 \) and every \( y \), we have

\[
|\{u \leq M\} \cap G((y, r), 3/2)| \geq \nu |G((y, r), 3/2)|
\]

and by (13), together with Theorem 3.3 and Theorem 3.6, we conclude that for \( |y| \leq r \) we have

\[
|\{u \leq M\} \cap G(y, 3r/2)| = |T (\{u \leq M\} \cap G((y, r), 3/2))| = C r^3 |\{u \leq M\} \cap G((y, r), 3/2)| \geq \nu C r^3 G((y, r), 3/2)| = \tilde{\nu} |G(y, 3r/2)|.
\]

In particular, (37) holds true at \( y = (0, y_2) \) for every \( r > 0 \) and for every \( y_2 \).

**Remark 5.5.** Consider the symmetry \( (x_1, x_2) \mapsto (-x_1, x_2) \).

**Proof.** By Theorem 3.3 and Theorem 3.6 we can substitute the sets \( G \) in Theorem 5.2 with the quasi metric balls \( B \), after rescaling. \qed

**Theorem 5.3.** There exist universal constants \( \eta > 2, 0 < \nu < 1 \) and \( M > 1 \) such that, for all \( y \in \mathbb{R}^2 \) and \( r > 0 \), if \( u \in C^2(B(y, \eta r)) \cap C(\overline{B(y, \eta r)}) \) is a nonnegative solution of \( Lu \leq 0 \) in \( B(y, \eta r) \) and \( \inf_{B(y, \eta r)} u \leq 1 \) then

\[
|\{u \leq M\} \cap B(y, r)| \geq \nu |B(y, r)|
\]

**Proof.** By Theorem 3.3 and Theorem 3.6 we can substitute the sets \( G \) in Theorem 5.2 with the quasi metric balls \( B \), after rescaling. \qed

By arguing as in Theorem 5.2 and by taking into account (34) we get

**Corollary 5.4.** There exist universal constants \( 0 < \nu < 1 \) and \( M > 1 \) such that, for all \( y \in \mathbb{R}^2 \) and \( r > 0 \), if \( u \in C^2(G(y, 2r)) \cap C(G(y, 2r)) \) is a nonnegative solution of \( Lu \leq 0 \) in \( G(y, 2r) \) and \( \inf_{G(y, 2r)} u \leq 1 \) then

\[
|\{u \leq M\} \cap G(y, r)| \geq \nu |G(y, r)|.
\]

**Remark 5.5.** Consider the symmetry \( S \) with respect to the \( x_2 \) axis

\[
S(x_1, x_2) = (-x_1, x_2).
\]

We first show that \( S \) preserves the equation. Let us call \( u_S(x) = u(S(x)) \) and \( L_S = a_{11}(S(x))X_1^2 + 2a_{12}(S(x))X_2X_1 + a_{22}(S(x))X_2^2 \). We have that the coefficients of \( L_S \) satisfy (3) and \( L_S u_S(x) = (Lu)(S(x)) \).

By Corollary 5.4, there exist universal constants \( 0 < \nu < 1 \) and \( M > 1 \) such that, for all \( y \in \mathbb{R}^2 \) and \( r > 0 \), if \( u \in C^2(G(y, 2r)) \cap C(G(y, 2r)) \) is a nonnegative solution of \( Lu \leq 0 \) in \( G(y, 2r) \) we have:

\[
|\{u \leq M\} \cap G(y, r)| \geq \nu |G(y, r)|.
\]
• For $|y_1| < r$ or $|y_1| \geq 2r$, if $\inf_{G(y,r)} u \leq 1$ then
  \[ |\{u \leq M\} \cap G(y,r)| > \nu |G(y,r)|, \]
i.e.
  \[ |\{u > M\} \cap G(y,r)| < (1 - \nu) |G(y,r)|. \]

• For $r \leq |y_1| < 2r$, if $\inf_{G(y,r)} u \leq 1$ or $\inf_{G(y,r)} u_S \leq 1$ then
  \[ |\{u \leq M\} \cap G(y,r)| + |\{u_S \leq M\} \cap G(y,r)| > 2\nu |G(y,r)|, \]
i.e.
  \[ |\{u > M\} \cap G(y,r)| + |\{u_S > M\} \cap G(y,r)| < 2(1 - \nu) |G(y,r)|. \]

By taking the negation of the previous implications we have, respectively

• For $|y_1| < r$ or $|y_1| \geq 2r$, if
  \[ |\{u > M\} \cap G(y,r)| \geq (1 - \nu) |G(y,r)| \]
then $\inf_{G(y,r)} u > 1$.

• For $r \leq |y_1| < 2r$, if
  \[ |\{u > M\} \cap G(y,r)| + |\{u_S > M\} \cap G(y,r)| \geq 2(1 - \nu) |G(y,r)|, \]
then $\inf_{G(y,r)} u > 1$ and $\inf_{G(y,r)} u_S > 1$.

Let us define $\tilde{B}(y,r) = B(y,r) \cup B(S(y),r)$, with $S$ as in (38) and $B$ as in (14). By recalling the structure Theorem 3.6 and Theorem 3.3 and by rescaling we have

**Corollary 5.6.** There exist universal constants $0 < \epsilon, \theta < 1$ and $M > 1, \eta > 2$ such that, for all $y \in \mathbb{R}^2$ and $r > 0$, if $u \in C^2(B(y,\eta r)) \cap C(\tilde{B}(y,\eta r))$ is a nonnegative solution of $Lu \leq 0$ in $B(y,\eta r)$ and

\[ |\{u \geq M\} \cap \tilde{B}(y,r) \cap B(y,\eta r)| \geq \epsilon |\tilde{B}(y,r) \cap B(y,\eta r)|, \]
then $\inf_{\tilde{B}(y,\theta r) \cap B(y,\eta r)} u > 1$.

### 6. Double Ball Property, Power Decay Property and Harnack’s Inequality

We start with the definition of a uniform lower barrier function for a ring and for the operator $L$.

**Definition 6.1.** Let $0 \leq \gamma < 1$. A function $\Phi$ is a $\gamma$-lower barrier function for the ring $R(y,r,3r) := \tilde{G}(y,3r) \setminus G(y,3r)$ for every $y \in \mathbb{R}^2$ and $r > 0$ and for the operator $L$ if

- $\Phi \in C^2(R(y,r,3r)) \cap C(\overline{R(y,r,3r)})$
- $L\Phi \geq 0$ on $R(y,r,3r)$,
- $|\Phi|_{\partial \tilde{G}(y,3r)} \leq 0$
- $|\Phi|_{\partial G(y,r)} \leq 1$.
- $\inf_{\partial \tilde{G}(y,2r)} \Phi \geq \gamma$.

The main tool of this section is the following Ring Theorem, which has independent interest because you can reproduce it whenever you have the weak maximum principle.
Theorem 6.2 (Ring Theorem). Suppose there exists $\Phi$ a $\gamma$-barrier function for the ring $R(y,r,3r) = \tilde{G}(y,3r) \setminus \bar{G}(y,r)$ for every $y \in \mathbb{R}^2$ and $r > 0$ and for the operator $L$. Then the Double Ball Property (see Definition 1.1) holds true in $\tilde{G}(y,3r)$ with constant $\gamma$.

Proof. The main ingredient of the proof is the weak maximum principle, Theorem 2.6. Let $u \in C^2(\tilde{G}(y,3r)) \cap C(\bar{G}(y,3r))$ be a nonnegative classical solution of $Lu \leq 0$ in $\tilde{G}(y,3r)$ and assume $u \geq 1$ in $\tilde{G}(y,r)$. Let $\Phi$ be a $\gamma$-lower barrier function for the ring $R(y,r,3r) = \tilde{G}(y,3r) \setminus \bar{G}(y,r)$. By the weak maximum principle we have $u \geq \Phi$ in the ring $R(y,r,3r)$. In particular, $u \geq \inf_{\partial \tilde{G}(y,2r)} \Phi$ on $\partial \tilde{G}(y,2r)$. Now consider the function $u$ in $\tilde{G}(y,2r)$. We have $Lu \leq 0$ in $\tilde{G}(y,2r)$ and $u \geq \inf_{\partial \tilde{G}(y,2r)} \Phi \geq \gamma$ on $\partial \tilde{G}(y,2r)$. Then by the weak maximum principle we have $u \geq \gamma$ in $\tilde{G}(y,2r)$. \qed

In the following proposition we prove the existence of a $\gamma$-lower barrier function for a ring and for the operator $L$.

Proposition 6.3. There exists $\gamma > 0$ such that, for every $y \in \mathbb{R}^2$ and $r > 0$, there exists $\Phi$ a $\gamma$-lower barrier function for the ring $\tilde{G}(y,3r) \setminus \bar{G}(y,r)$ and for the operator $L$ in (2). Moreover, $\Phi$ is an even function with respect to $x_1$.

Proof. Let $\rho$ be the function in (17) and let $\phi = \rho^\alpha$ be the function in Lemma 4.1. Recall that $\alpha \leq 2 - 3\Lambda/\lambda$ and therefore $\alpha \leq -1$. Now take $\Phi = M_2 \phi - M_1$, and choose $M_1, M_2$ such that $\Phi_{|_{\partial \tilde{G}(y,3r)}} = 0$ and $\Phi_{|_{\partial \tilde{G}(y,r)}} = 1$. Obviously, $\Phi$ is an even function with respect to $x_1$. We will show that $M_1, M_2$ are positive. We distinguish four cases.

I CASE. If $|y_1| < r$ we have

$$M_1 = \frac{3^\alpha}{1 - 3^\alpha} > 0, \quad M_2 = \frac{1}{r^{\alpha}(1 - 3^\alpha)} > 0.$$ 

Let us put

$$M_3 = \Phi_{|_{\partial \tilde{G}(y,2r)}} = \frac{2^\alpha - 3^\alpha}{1 - 3^\alpha} > 0.$$

II CASE. If $3r \leq |y_1|$, we have

$$M_1 = \frac{3^{\alpha/2}}{1 - 3^{\alpha/2}} > 0, \quad M_2 = \frac{1}{(r|y_1|)^{\alpha/2}(1 - 3^{\alpha/2})} > 0.$$ 

Let us put

$$M_3 = \Phi_{|_{\partial \tilde{G}(y,2r)}} = \frac{2^{\alpha/2} - 3^{\alpha/2}}{1 - 3^{\alpha/2}} > 0.$$

III CASE. If $r \leq |y_1| < 2r$, we have

$$M_1 = \frac{3^\alpha}{(|y_1|/r)^{\alpha/2} - 3^\alpha} > 0, \quad M_2 = \frac{1}{(r|y_1|)^{\alpha/2} - (3r)^{\alpha}} > 0.$$ 

Let us put

$$M_3 = \Phi_{|_{\partial \tilde{G}(y,2r)}} = \frac{2^\alpha - 3^\alpha}{(|y_1|/r)^{\alpha/2} - 3^\alpha} \geq \frac{2^\alpha - 3^\alpha}{1 - 3^\alpha} > 0.$$

IV CASE. If $2r \leq |y_1| < 3r$, we have

$$M_1 = \frac{3^\alpha}{(|y_1|/r)^{\alpha/2} - 3^\alpha} > 0, \quad M_2 = \frac{1}{(r|y_1|)^{\alpha/2} - (3r)^{\alpha}} > 0.$$
Let us put
\[
M_3 = \Phi|_{\partial \tilde{G}(y,2r)} = \frac{(2|y_1|/r)^{\alpha/2} - 3^\alpha}{(|y_1|/r)^{\alpha/2} - 3^\alpha} \geq \frac{6^{\alpha/2} - 3^\alpha}{2^{\alpha/2} - 3^\alpha} > 0.
\]
Now choose
\[
\gamma = \min \left\{ \frac{2^\alpha - 3^\alpha}{1 - 3^\alpha}, \frac{2^{\alpha/2} - 3^{\alpha/2}}{1 - 3^{\alpha/2}}, \frac{6^{\alpha/2} - 3^\alpha}{2^{\alpha/2} - 3^\alpha} \right\}.
\]
By recalling Lemma 4.1 and (40), (41),(42),(43) we have that \(\Phi\) is a \(\gamma\)-lower barrier function for the ring \(\tilde{G}(y,3r) \setminus \tilde{G}(y,r)\) for every \(y \in \mathbb{R}^2\) and \(r > 0\) and for the operator \(L\) in (2).

As a consequence of Theorem 6.2 and of Proposition 6.3 we get

**Corollary 6.4.** The Double Ball Property for \(L\) holds true in the following families of sets

i) \(\tilde{G}(y,3r)\), for every \(y \in \mathbb{R}^2\) and \(r > 0\).

ii) \(\tilde{G}(y,3r)\), for \(|y_1| < r\) or for \(|y_1| \geq 3r\).

iii) \(\tilde{B}(y,\eta r) = B(y,\eta r) \cup B(S(y),\eta r)\) for every \(y \in \mathbb{R}^2\) and \(r > 0\). Here \(\eta > 2\) is a universal constant and \(S\) is the reflexion in (38).

iv) \(\tilde{B}(y,\eta r)\), for \(|y_1| < r\) or for \(|y_1| \geq \eta r\) and with \(\eta > 2\) as in iii).

**Proof.** By Proposition 6.3 there exists \(\Phi\) a \(\gamma\)-lower barrier function for the ring
\[
R(y,r,3r) = \tilde{G}(y,3r) \setminus \tilde{G}(y,r)
\]
for every \(y \in \mathbb{R}^2\) and \(r > 0\) and for the operator \(L\). By Theorem 6.2 we get i).

To prove ii) we distinguish two cases.

If \(|y_1| < r\) we have \(\tilde{G}(y,r) = G(y,r), \tilde{G}(y,2r) = G(y,2r), \tilde{G}(y,3r) = G(y,3r)\), and we apply Theorem 6.2.

If \(3r \geq |y_1|\), we have \(\tilde{G}(y,r) \cap \{x_1y_1 \geq 0\} = G(y,r), \tilde{G}(y,2r) \cap \{x_1y_1 \geq 0\} = G(y,2r), \tilde{G}(y,3r) \cap \{x_1y_1 \geq 0\} = G(y,3r)\), and we apply Theorem 6.2 in the halfplane \(\{x_1y_1 \geq 0\}\).

Statement iii) follows from i), by Theorem 3.6 and Theorem 3.3 and by rescaling.

Statement iv) follows from iii) by taking into account the geometry of the sets \(\tilde{B}\).

Remark that if \(r \leq |y_1| < 2r\), we have
\[
\tilde{G}(y,r) \cap \{x_1y_1 \geq 0\} = G(y,r), \tilde{G}(y,2r) = G(y,2r), \tilde{G}(y,3r) = G(y,3r).
\]
If \(2r \leq |y_1| < 3r\), we have
\[
\tilde{G}(y,r) \cap \{x_1y_1 \geq 0\} = G(y,r), \tilde{G}(y,2r) \cap \{x_1y_1 \geq 0\} = G(y,2r), \tilde{G}(y,3r) = G(y,3r).
\]

In both cases the geometry of the level sets \(G\) changes in passing trough the \(x_2\) axis. Our strategy to overcome this technical problem is to combine Corollary 6.4 with Corollary 5.6 to directly prove the following power decay property.

**Theorem 6.5** (Power Decay Property). There exist universal constants \(\eta > 2, 0 < \epsilon < 1\) and \(M > 1\) such that, for all \(y \in \mathbb{R}^2\) and \(r > 0\), if \(u \in C^2(B(y,\eta r)) \cap C(\overline{B(y,\eta r)})\) is a nonnegative solution of \(Lu \leq 0\) in \(B(y,\eta r)\) and \(\inf_{\tilde{B}(y,r) \cap B(y,\eta r)} u \leq 1\), then for every \(k \in \mathbb{N}\) we have
\[
\left| \left\{ u \geq M^k \right\} \cap \tilde{B}(y,r/2) \cap B(y,\eta r) \right| \leq \epsilon^k \left| \tilde{B}(y,r/2) \cap B(y,\eta r) \right|.
\]
The following Lemma is a crucial tool in the proof of Theorem 6.5.

Lemma 6.6. There exist \( \eta > 2, 0 < \epsilon < 1, M_1 > 1 \) such that for all \( M > 0 \), for all \( y \in \mathbb{R}^2 \) and \( r > 0 \), if \( u \in C^2(B(y, \eta r)) \cap C(\overline{B(y, \eta r)}) \) is a nonnegative solution of \( Lu \leq 0 \) in \( B(y, \eta r) \) with

\[
\left| \{ u \geq M \} \cap \tilde{B}(y, r) \cap B(y, \eta r) \right| \geq \epsilon |\tilde{B}(y, r) \cap B(y, \eta r)|
\]

then \( \inf_{\tilde{B}(y, r) \cap B(y, \eta r)} u > M/M_1 \).

Proof. Let \( M_0 > 1, \eta > 2, 0 < \epsilon, \theta < 1 \) be the constants in Corollary 6.4. Since

\[
\left| \{ M_0 u/M \geq M_0 \} \cap \tilde{B}(y, r) \cap B(y, \eta r) \right| \geq \epsilon |\tilde{B}(y, r) \cap B(y, \eta r)|
\]

then \( \inf_{\tilde{B}(y, r) \cap B(y, \eta r)} u > M/M_0 \).

By eventually enlarging \( \eta \), assume that Corollary 6.4 iii) holds true and that we can choose \( k \in \mathbb{N} \) such that \( 1 \leq 2^k \theta < \eta \). Let us call \( \gamma \) the constant in the Double Ball Property. By iterating \( k \) times Corollary 6.4 iii) we get \( u > M \gamma^k/M_0 \) in \( \tilde{B}(y, 2^k \theta r) \cap B(y, \eta r) \). In particular, \( u > M \gamma^k/M_0 \) in \( B(y, r) \cap B(y, \eta r) \). Now take \( M_1 = M_0/\gamma^k \).

Proof of Theorem 6.5. Let \( \eta > 2, M > 1 \) be universal constants such that Lemma 6.6 and Corollary 6.4 iii) hold true. Define \( E_k = \{ u \geq M^k \} \cap B(y, \eta r) \). By following the proof of [6, Theorem 4.7, conditions A1 and A2] and by recalling Theorem 3.4 and Lemma 6.6, construct a family of quasi metric balls \( B_k = B(y, t_k) \) with \( t_0 = r > t_1 > t_2 > \cdots > r/2 \) and choose \( 0 < \epsilon < 1 \) such that

\[
\left| \tilde{B}_{k+1} \cap E_{k+2} \right| \leq \epsilon \left| \tilde{B}_k \cap E_{k+1} \right|, \quad \forall k = 0, 1, \ldots.
\]

We have

\[
\left| \{ u \geq M^{k+2} \} \cap \tilde{B}(y, r/2) \cap B(y, \eta r) \right| \leq \epsilon^{k+1} |\tilde{B}(y, r) \cap B(y, \eta r)| \leq C_D \epsilon^{k+1} |\tilde{B}(y, r/2) \cap B(y, \eta r)|,
\]

where \( C_D \) is the doubling constant in (16). Now choose a positive integer \( k_0 \) such that \( \epsilon^{k_0} C_D < 1 \), and replace \( M \) with \( M^{2+k_0} \) to get the thesis.

By Theorem 6.5 we immediately get the following corollaries.

Corollary 6.7. There is a positive constant \( \eta > 2 \) such that the Double Ball Property for \( L \) holds true in \( B(y, \eta r) \) for all \( y \in \mathbb{R}^2 \) and \( r > 0 \).

Proof. Let \( \eta > 2 \) be such that Theorem 6.5 and Corollary 6.4 iii) hold true. Let \( u \in C^2(B(y, \eta r)) \cap C(\overline{B(y, \eta r)}) \) be a nonnegative classical solution of \( Lu \leq 0 \) in \( B(y, \eta r) \) and assume \( u \geq 1 \) in \( B(y, r) \). If \(|y_1| < r \) or \(|y_1| \geq \eta r \) we apply Corollary 6.4 vi).

If \( r \leq |y_1| < \eta r \), let \( M > 1 \) and \( 0 < \epsilon < 1 \) be the universal constants in Theorem 6.5. Now choose \( k \in \mathbb{N} \) such that \( \epsilon^k < 1/4 \). If \( u \geq 1/M^k \) in \( \tilde{B}(y, r) \) then by Corollary 6.4 iii) \( u \geq \gamma/M^k \) in \( \tilde{B}(y, 2r) \). In particular, \( u \geq \gamma/M^k \) in \( B(y, 2r) \) and the thesis follows.

On the contrary, if there is a point \( x_0 \in \tilde{B}(y, r) \) such that \( M^k u(x_0) < 1 \) then by considering the symmetry \( S \) in (38) and by Theorem 6.5 and (39) we get

\[
|\{ u \geq 1 \} \cap B(y, r/2)| + |\{ u_S \geq 1 \} \cap B(y, r/2)| \leq 2\epsilon^k |B(y, r/2)| < (1/2) |B(y, r/2)|
\]
and by recalling that $u \geq 1$ in $B(y, r)$, in particular $u \geq 1$ in $B(y, r/2)$, and
\[ |B(y, r/2)| \leq |\{u \geq 1\} \cap B(y, r/2)| + |\{u_S \geq 1\} \cap B(y, r/2)| < (1/2)|B(y, r/2)| \]
we get a contradiction. □

**Corollary 6.8.** There exist universal constants $\eta > 2$, $0 < \epsilon < 1$ and $M > 1$ such that, for all $y \in \mathbb{R}^2$ and $r > 0$, if $u \in C^2(B(y, \eta r)) \cap C(B(y, \eta r))$ is a nonnegative solution of $Lu \leq 0$ in $B(y, \eta r)$ and $\inf_{B(y,r) \cap B(y,\eta r)} u \leq 1$, then for every $k \in \mathbb{N}$ we have
\[ |\left\{ u \geq M^k \right\} \cap B(y, r/2) \cap B(y, \eta r)| \leq \epsilon^k |B(y, r/2) \cap B(y, \eta r)|. \]

**Proof.** In particular, $\inf_{B(y,r) \cap B(y,\eta r)} u \leq 1$, and by Theorem 6.5 we get
\[ |\left\{ u \geq M^k \right\} \cap B(y, r/2) \cap B(y, \eta r)| \leq 2\epsilon^k |B(y, r/2) \cap B(y, \eta r)|. \]
Now choose a positive integer $k_0$ such that $\epsilon^{k_0} < 1/2$, and replace $M$ with $M^{1+k_0}$ to get the thesis. □

An alternative proof of Corollary 6.8 can be obtained by applying Corollary 6.7, the critical density estimate in Theorem 5.3 and the results of Di Fazio et al. [6, Theorem 4.7].

By Corollary 6.8 and the results of Di Fazio et al. [6, Theorem 5.1] applied to
\[ K = \{u \in C^2(B(y, \eta r)) \cap C(B(y, \eta r)) : u \geq 0 \text{ and } Lu \leq 0 \text{ in } B(y, \eta r), \ u \geq 1 \text{ on } B(y, r) \} \]
we obtain the following invariant Harnack inequality.

**Theorem 6.9** (Harnack inequality). There exist constants $C$ and $\eta$, both bigger than 1 and depending only on the ellipticity constants, such that for every $y \in \mathbb{R}^2$ and $r > 0$, if $Lu = 0$ and $u \geq 0$ in $B(y, \eta r)$, $u \geq 1$ on $B(y, r)$, then
\[ \sup_{B(y,r)} u \leq C \inf_{B(y,r)} u. \]

The scale invariant Harnack’s inequality on balls $B_{CC}$ easily follows from Theorem 6.9 and from Theorem 3.2 and Theorem 3.3.

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