Extensions of the Schur majorisation inequalities

Rajendra Bhatia

Ashoka University, Sonepat, Haryana, 131029, India

Rajesh Sharma

Department of Mathematics and Statistics, H.P. University, Shimla-5, India

Abstract

Let $\lambda_j$ and $a_{jj}$, $1 \leq j \leq n$, be the eigenvalues and the diagonal entries of a Hermitian matrix $A$, both enumerated in the increasing order. We prove some inequalities that are stronger than the Schur majorisation inequalities $\sum_{j=1}^{r} \lambda_j \leq \sum_{j=1}^{r} a_{jj}$, $1 \leq r \leq n$.

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1. Introduction

Let $A$ be an $n \times n$ complex Hermitian matrix. Let the eigenvalues and the diagonal entries of $A$ both be enumerated in increasing order as

$$\lambda_1(A) \leq \lambda_2(A) \leq \cdots \leq \lambda_n(A),$$

and

$$a_{11} \leq a_{22} \leq \cdots \leq a_{nn},$$

respectively. We then have

$$\lambda_1(A) \leq a_{11} \quad \text{and} \quad \lambda_n(A) \geq a_{nn}.$$
These inequalities are included in the Schur majorisation inequalities that say: for every \( 1 \leq r \leq n \)

\[
\sum_{j=1}^{r} \lambda_j(A) \leq \sum_{j=1}^{r} a_{jj},
\]

with equality in the case \( r = n \). These inequalities are of fundamental importance in matrix analysis and have been the subject of intensive work. See, e.g. Bhatia [1], Horn and Johnson [4] and Marshall and Olkin [5].

In this note we obtain some inequalities that are stronger than (1.3) and (1.4). These give estimates of eigenvalues in terms of quantities easily computable from the entries of \( A \).

Given the \( n \times n \) Hermitian matrix \( A = [a_{ij}] \), let

\[
r_i = \sum_{j \neq i} |a_{ij}|, \quad 1 \leq i \leq n \tag{1.5}
\]

and

\[
q_i = \sum_{j \neq i} |a_{ij}|^2, \quad 1 \leq i \leq n. \tag{1.6}
\]

A permutation similarity does not change either the eigenvalues or the diagonal entries of \( A \). Nor does it change the quantities \( r_i \) and \( q_i \). We assume that such a permutation similarity has been performed and the ordering (1.2) for diagonal entries has been achieved.

To rule out trivial cases, we assume that \( A \) is not a diagonal matrix.

Our first theorem is a strengthening of the inequalities (1.3).

**Theorem 1** For every \( n \times n \) Hermitian matrix \( A \), we have

\[
\lambda_1(A) \leq a_{11} - \frac{q_1}{\max_i (a_{ii} + r_i) - a_{11}},
\]

\[
\lambda_n(A) \geq a_{nn} + \frac{q_n}{a_{nn} - \min_i (a_{ii} - r_i)}. \tag{1.7}
\]

The next two theorems give inequalities stronger than (1.4).
Theorem 2. Let $A$ be an $n \times n$ Hermitian matrix. Then for $1 \leq r \leq n - 1$ and $r < t \leq n$, we have
\[
\sum_{i=1}^{r} \lambda_i(A) \leq \sum_{i=1}^{r} a_{ii} - \frac{\sum_{i=1}^{r} |a_{is}|^2}{a_{tt} - \min_{i=1, \ldots, r} \left( a_{ii} - \sum_{s=1 \atop s \neq i}^{r+1} |a_{is}| \right)}.
\] (1.9)

Theorem 3. Let $A$ be an $n \times n$ Hermitian matrix. Then for $1 \leq r \leq n - 1$, $1 \leq k \leq r$, and $r < t \leq n$, we have
\[
\sum_{i=1}^{r} \lambda_i(A) \leq \sum_{i=1}^{r} a_{ii} - \sqrt{(a_{tt} - a_{kk})^2 + 4|a_{tk}|^2} - (a_{tt} - a_{kk}).
\] (1.10)

2. Proofs

Our proofs rely upon two basic theorems of matrix analysis. Let $\mathbb{M}(n)$ be the algebra of all $n \times n$ complex matrices and let $\Phi : \mathbb{M}(n) \to \mathbb{M}(k)$ be a positive unital linear map, [3]. Then the Bhatia-Davis inequality [2] says that for every Hermitian matrix $A$ whose spectrum is contained in the interval $[m, M]$, we have
\[
\Phi(A^2) - \Phi(A)^2 \leq (MI - \Phi(A))(\Phi(A) - mI) \leq \left( \frac{M - m}{2} \right)^2 I.
\] (2.1)

Cauchy’s interlacing principle says that if $A_r$ is an $r \times r$ principal submatrix of $A$, then
\[
\lambda_j(A) \leq \lambda_j(A_r), \quad 1 \leq j \leq r.
\] (2.2)

See Chapter III of [1] for this and other facts used here.

2.1. Proof of Theorem 1

Let $\varphi : \mathbb{M}(n) \to \mathbb{C}$ be a positive unital linear functional and let the eigenvalues of Hermitian element $A \in \mathbb{M}(n)$ be arranged as in (1.1). From the first inequality (2.1), we have
\[
\varphi(A^2) - \varphi(A)^2 \leq (\lambda_n(A) - \varphi(A))(\varphi(A) - \lambda_1(A)).
\] (2.3)
Suppose $\lambda_n(A) \neq \varphi(A)$. Then, from (2.3), we have

$$\lambda_1(A) \leq \frac{\varphi(A^2) - \varphi(A)}{\lambda_n(A) - \varphi(A)}.$$  \hspace{1cm} (2.4)

Further, by the Gersgorin disk theorem, we have

$$\lambda_n(A) \leq \max_i (a_{ii} + r_i).$$  \hspace{1cm} (2.5)

Combining (2.4) and (2.5), we get

$$\lambda_1(A) \leq \frac{\varphi(A^2) - \varphi(A)}{\max_i (a_{ii} + r_i) - \varphi(A)}.$$  \hspace{1cm} (2.6)

Choose $\varphi(A) = a_{11}$. Then, $\varphi$ is a positive unital linear functional and $\varphi(A^2) - \varphi(A)^2 = q_1$. So, (2.6) yields (1.7).

Suppose $\lambda_n(A) = \varphi(A) = a_{11}$. Then, from (1.2) and (1.3), we have $a_{11} = a_{22} = \cdots = a_{nn}$ and from (2.3), $\varphi(A^2) - \varphi(A)^2 = 0$. Therefore, $q_i = 0$ for all $i = 1, 2, \ldots, n$. But then $A$ is a scalar matrix.

The inequality (1.8) follows on using similar arguments. The derivation requires lower bound of $\lambda_n(A)$ from (2.3) which is analogous to (2.4), $\lambda_1(A) \geq \min_i (a_{ii} - r_i)$ and $\varphi(A) = a_{nn}$.

\section*{2.2. Proof of Theorem 2}

The trace of $A$ is the sum of the eigenvalues of $A$. Therefore,

$$\lambda_n(A) = \text{tr} A - \sum_{i=1}^{n-1} \lambda_i(A).$$  \hspace{1cm} (2.7)

Combining (1.8) and (2.7), we find that

$$\sum_{i=1}^{n-1} \lambda_i(A) \leq \sum_{i=1}^{n-1} a_{ii} - \frac{q_n}{a_{nn} - \min_i (a_{ii} - r_i)}.$$  \hspace{1cm} (2.8)

Apply (2.8) to the principal submatrix $P$ of $A$ containing diagonal entries $a_{11}, a_{22}, \ldots, a_{rr}, a_{tt}$, we get that

$$\sum_{i=1}^{r} \lambda_i(P) \leq \sum_{i=1}^{r} a_{ii} - \frac{\sum_{s=1}^{r} |a_{ts}|^2}{a_{tt} - \min_{i=1,\ldots,r} \left(a_{ii} - \sum_{s=1, s \neq i}^{r+1} |a_{is}| \right)}.$$  \hspace{1cm} (2.9)
By the interlacing inequalities (2.2), \( \sum_{i=1}^{r} \lambda_i(A) \leq \sum_{i=1}^{r} \lambda_i(P) \). So, (2.9) gives (1.9).

\[ \blacksquare \]

2.3. Proof of Theorem 3

By the Cauchy interlacing principle (2.2), the largest eigenvalue of \( A \) is greater than or equal to the largest eigenvalue of any \( 2 \times 2 \) principal submatrix of \( A \). Further, the eigenvalues of
\[
\begin{bmatrix}
a_{rr} & a_{rs} \\
a_{rs} & a_{ss}
\end{bmatrix}
\]
are
\[
\frac{1}{2} \left( a_{rr} + a_{ss} \pm \sqrt{(a_{rr} - a_{ss})^2 + 4|a_{rs}|^2} \right).
\]
On using these two facts, we see that
\[
\lambda_n(A) \geq a_{nn} + \frac{\sqrt{(a_{nn} - a_{kk})^2 + 4|a_{kn}|^2} - (a_{nn} - a_{kk})}{2} \tag{2.10}
\]
for all \( k = 1, 2, \ldots, n - 1 \). Combining (2.7) and (2.10), we find that
\[
\sum_{i=1}^{n-1} \lambda_i(A) \leq \sum_{i=1}^{n-1} a_{ii} - \frac{\sqrt{(a_{nn} - a_{kk})^2 + 4|a_{kn}|^2} - (a_{nn} - a_{kk})}{2} \tag{2.11}
\]
Apply (2.11) to the principal submatrix \( Q \) of \( A \) containing \( a_{11}, a_{22}, \ldots, a_{rr}, a_{tt} \), we find that for \( k = 1, 2, \ldots, r \), we have
\[
\sum_{i=1}^{r} \lambda_i(Q) \leq \sum_{i=1}^{r} a_{ii} - \frac{\sqrt{(a_{tt} - a_{kk})^2 + 4|a_{tk}|^2} - (a_{tt} - a_{kk})}{2} \tag{2.12}
\]
The inequality (2.12) yields (1.10), on using the interlacing inequalities (2.2).

\[ \blacksquare \]

We show by means of an example that (1.9) and (1.10) are independent.

**Example 1** Let
\[
A = \begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 3
\end{bmatrix}, \quad B = \begin{bmatrix}
1 & 2 & 3 \\
2 & 1 & 4 \\
3 & 4 & 1
\end{bmatrix}.
\]

Then (1.9) gives the estimate \( \lambda_1(A) + \lambda_2(A) < \frac{10}{3} \), while (1.10) gives the weaker estimate \( \frac{2 - \sqrt{5}}{2} \) for the same quantity. On the other hand from (1.9) we get that \( \lambda_1(B) + \lambda_2(B) < -\frac{11}{7} \), while from (1.10) we see that the same quantity is not bigger than \(-2\).
References

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