A high-order and fast scheme with variable time steps for the time-fractional Black-Scholes equation

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1 INTRODUCTION

Recently, the option theory has been widely used in financial and economic fields, so the study of option pricing becomes more important in both theoretical significance and practical application. The classical Black-Scholes model, which is a second-order parabolic partial differential equation related to stock price and time, is used for pricing European or American put and call options on stock. However, the assumptions of the classical Black-Scholes model are so idealistic that many characteristic properties of markets cannot be captured. Therefore, several modified models have been proposed, such as Lévy models, stochastic volatility models, and fractional Black-Scholes models.

For the past few years, fractional differential equations have been widely used in mathematics, physics, biology, geology, chemical systems, and other scientific fields (see previous studies). This is because the realistic modeling of some phenomena depends on time or historical time can be successfully realized by using fractional differential equations. Due to the non-locality of fractional derivatives, many processes in the study of some materials with memory process, genetic properties, and heterogeneity can be modeled more accurately by fractional derivatives than integer order derivatives. With the proposals of the fractional partial differential equation about stochastic model and financial theory, a growing number of scholars began to study fractional option pricing model and made significant progress. Wyss considered the pricing of option derivatives under a time-fractional Black-Scholes equation preliminarily by replacing the time first-order derivative with a fractional derivative of order \( \alpha (0 < \alpha \leq 1) \) and derived a closed-form solution for European vanilla options. Cartea and del-Castillo-Negrete displayed that some particular Lévy processes satisfy a fractional partial differential equation and employed numerical methods to solve the related fractional models in order to price exotic options, in particular barrier options. Jumarie applied the fractional Taylor formula to remove effects

In this paper, a high-order and fast numerical method is investigated for the time-fractional Black-Scholes equation. In order to deal with the typical weak initial singularity of the solution, we construct a finite difference scheme with variable time steps, where the fractional derivative is approximated by the nonuniform Alikhanov formula and the sum-of-exponentials (SOE) technique. In the spatial direction, an average approximation with fourth-order accuracy is employed. The stability and the convergence with fourth-order accuracy in space of the proposed scheme are religiously derived by the energy method. Numerical examples are given to demonstrate the theoretical statement.

KEYWORDS
fast algorithm, high-order method, time-fractional Black-Scholes equation, variable time steps

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of the non-zero initial value of the function. Under the Itô lemma of fractional order illustrated in the special case of a fractional growth with white noise, they derived the time and space fractional Black-Scholes equations. By assuming that the stock price dynamics follows a fractional Itô process, Liang et al.\cite{10,11} proposed a bi-fractional Black-Merton-Scholes model of option pricing. A slightly simplified version based on Liang et al.’s model\cite{11} was then discussed in Chen et al.\cite{7}; they assumed that the underlying asset price is consistent with the classical Brownian motion and the spatial-fractional derivative in the governing equation disappears, but the time-fractional derivative remains.

Since the analytical solution of fractional differential equations is always hard to find, it is necessary to study efficient numerical methods for related problems. In this paper, we will discuss a high-order finite difference method for the time-fractional Black-Scholes equation\cite{7}:

\[
\frac{\partial^\alpha C}{\partial \tau^\alpha} + \frac{1}{2} s^2 \frac{\partial^2 C}{\partial S^2} + (r - D) s \frac{\partial C}{\partial S} - r C = 0, \quad (S, \tau) \in \Omega \times [0, T),
\]

\[
C(S, T) = R(S), \quad S_l < S < S_r,
\]

\[
C(S, \tau) = P(\tau), \quad C(S_r, \tau) = Q(\tau), \quad \tau \in [0, T),
\]

where \( C(S, \tau) \) is the time-\( \tau \) price of an European-style double barrier option with the underlying \( S \), \( \tilde{\Omega} = (S_l, S_r) \subset \mathbb{R}^+, \) \( \tau \) is the current time, \( T \) is the expiry, \( r \) is the risk-free interest rate, \( D \) is the dividend yield, and \( \phi \) is the volatility of the returns. The functions \( P \) and \( Q \) are the rebates paid when the corresponding barrier is hit, and \( R \) is the payoff of the option. The time derivative in (1.1) is defined as

\[
\frac{\partial^\alpha C}{\partial \tau^\alpha} = \int_{\xi}^{T} \omega_{\tau - \alpha}(\tau - \eta) \frac{\partial \eta}{\partial \tau} C(S, \eta) d\eta \quad 0 < \alpha < 1,
\]

where the kernel \( \omega_{\tau}(\tau - \eta) := t^{\tau-1}/1(\tau), \) \( t > 0. \)

We notice that (see also Staelen & Hendy\cite{22}) by taking the auxiliary variables: \( x = \ln S, \) \( t = T - \tau \) and the function \( w(x, t) = C(e^x, T - t), \) one has

\[
-\frac{\partial^\alpha C}{\partial \tau^\alpha} = \int_{0}^{t} \omega_{\tau - \alpha}(t - s) \frac{\partial w(x, s)}{\partial s} ds = D^\alpha_t w,
\]

where \( D^\alpha_t \) represents the Caputo derivative of order \( \alpha \in (0, 1) \). Then, the problem (1.1) can be transformed into the following equation with constant coefficients:

\[
D^\alpha_t w - a \frac{\partial^2 w}{\partial x^2} - b \frac{\partial w}{\partial x} + c w = 0, \quad (x, t) \in \Omega \times (0, T],
\]

\[
w(x, 0) = r(x), \quad x \in \Omega,
\]

\[
w(x_l, t) = p(t), \quad w(x_r, t) = q(t), \quad t \in (0, T],
\]

where \( a = \frac{1}{2} \phi^2, \) \( b = r - a - D, \) \( c = r \) and \( \Omega = (x_l, x_r). \)

Moreover, denote \( u(x, t) := w(x, t) - z(x, t), \) where

\[
z(x, t) := \frac{q(t) - p(t)}{x_r - x_l} (x - x_l) + p(t).
\]

It is easy to see that the problem (1.3) is equivalent to the next equations with homogeneous boundary conditions:

\[
D^\alpha_t u = a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial x} - c u + f(x, t), \quad (x, t) \in \Omega \times (0, T],
\]

\[
u(x, 0) = \phi(x), \quad x \in \Omega,
\]

\[
u(x_l, t) = 0, \quad (x, t) \in \partial \Omega x \in (0, T],
\]

\[
u(x_r, t) = 0, \quad (x, t) \in \partial \Omega x \in (0, T],
\]

\[
u(x_r, t) = 0, \quad (x, t) \in \partial \Omega x \in (0, T],
\]
where
\[
\begin{align*}
  f(x, t) &= b \frac{q(t) - p(t)}{x_i - x_j} - cz - D_t^\alpha z, \\
  \varphi(x) &= r(x) - \frac{q(0) - p(0)}{x_i - x_j}(x - x_i) - p(0).
\end{align*}
\]

In recent years, several numerical methods for solving fractional Black-Scholes model have been developed. In Cen et al.\textsuperscript{23} a difference scheme on nonuniform time grids is proposed for an equivalent integral-differential equation of the problem (1.1), but it is only first-order convergent in time. For the problem (1.4), Zhang et al.\textsuperscript{13} discussed a discrete implicit numerical scheme that has temporal \((2 - \alpha)\)-order and spatial second-order convergence accuracies. De Staelen and Hendy\textsuperscript{22} investigated an implicit numerical scheme with a temporal accuracy of \((2 - \alpha)\)-order and spatial accuracy of fourth-order by using the Fourier analysis method. Roul\textsuperscript{24} and Roul and Goura\textsuperscript{25} studied finite difference methods with variable time step sizes are found to be very efficient and fairly popular in recent years to solve the weak points to catch the rapid variation of the solution and using large steps while the solution changes slowly. Numerical methods with variable time step sizes (the mesh will be nonuniform), that is, concentrating more mesh points around the (weak) singular and the restrictions of some classical approximations based on sufficient smooth solutions.

To deal with the weak singularity of the solutions, a natural and efficient way is to implement numerical methods with variable step sizes (the mesh will be nonuniform), that is, concentrating more mesh points around the (weak) singular and the restrictions of some classical approximations based on sufficient smooth solutions. It should be noticed that the numerical methods proposed in papers\textsuperscript{13,22,24,25} are based on the regularity assumption that the analytical solution is smooth enough in the time direction. However, the solution of time-fractional differential equations generally exhibits weak singularity near the initial time, which makes most of the classical numerical methods based on smooth assumptions are difficult to achieve the high-order convergence in the general situations. One may refer to Jin et al.\textsuperscript{26} and Stynes et al.\textsuperscript{27} for the discussion on the regularity of the solution of time-fractional diffusion equations and the restrictions of some classical approximations based on sufficient smooth solutions.

To deal with the weak singularity of the solutions, a natural and efficient way is to implement numerical methods with variable step sizes (the mesh will be nonuniform), that is, concentrating more mesh points around the (weak) singular points to catch the rapid variation of the solution and using large steps while the solution changes slowly. Numerical methods with variable time step sizes are found to be very efficient and fairly popular in recent years to solve the weak initial singularity of time-fractional partial differential equations.\textsuperscript{27-36} In view of the practical advantage of the nonuniform mesh technique, we will discuss the Alikhanov formula with variable steps to develop an efficient finite difference scheme with second-order temporal accuracy for the time-fractional Black-Scholes equation with weak singular solutions. The sum-of-exponentials (SOE) technique\textsuperscript{37} will also be utilized at the same time to rewrite the discrete Caputo derivative to save the computation costs. Meanwhile, a high-order average approximation will be employed to approximate the space derivatives and such that the proposed fully discrete scheme is fourth-order accuracy in the spatial direction. The stability of the proposed scheme will be established according to the analysis framework developed in Liao et al.\textsuperscript{31,38} and some matrix analysis techniques. Based on the following regularity assumptions on the exact solution \(u\) (for \(0 < t \leq T\)):

\[
\left\| \frac{\partial^{k+1} u}{\partial t^k \partial x^l} \right\|_{L^n} \leq C(1 + t^{\sigma-k}), \text{ for } k = 0, 1, 2, 3, \ l = 0, 1, 2; \tag{1.5}
\]

\[
\left\| \frac{\partial^m u}{\partial x^m} \right\|_{L^n} \leq C, \text{ for } m = 3, 4, 5, 6, \tag{1.6}
\]

where \(\sigma \in (0, 1) \cup (1, 2)\) is a regularity parameter, and under weak mesh restrictions, we can show that the proposed nonuniform scheme is unconditionally convergent with second-order accuracy in time and fourth-order accuracy in space.

We list the convergence of previous numerical methods for solving the time-fractional Black-Scholes equation in Table 1, where \(N\) is the total time node and \(\tau\) and \(h\) are the time and space step sizes, respectively. As can be seen from Table 1, the convergence accuracy of the numerical methods proposed in this paper will be higher than any of the previous relevant algorithms, even though most of them are based on sufficient smoothness assumptions of the analytical solution in time. It is the main advantage of this paper.

| Method                | Regularity assumption in time | Rate     | Mesh type      |
|-----------------------|------------------------------|----------|----------------|
| Cen et al.\textsuperscript{23} | \(\| u^{(1+k)} \|_{L^n} \leq C(1 + |t|^{-k})\) | \(\mathcal{O}(N^{-1} + h^2)\) | Nonuniform mesh |
| Zhang et al.\textsuperscript{13} | \(u(t, \cdot) \in C^2[0, T]\) | \(\mathcal{O}(r^{2-\alpha} + h^2)\) | Uniform mesh |
| De Staelen and Hendy\textsuperscript{22} | \(u(t, \cdot) \in C^3[0, T]\) | \(\mathcal{O}(r^{2-\alpha} + h^4)\) | Uniform mesh |
| Roul\textsuperscript{24} | \(u(t, \cdot) \in C^4[0, T]\) | \(\mathcal{O}(r^{2-\alpha} + h^4)\) | Uniform mesh |
| Roul and Goura\textsuperscript{25} | \(u(t, \cdot) \in C^5[0, T]\) | \(\mathcal{O}(r^{2-\alpha} + h^4)\) | Uniform mesh |

**TABLE 1** Convergence of previous relevant numerical methods
Our main contribution is twofold:

- For the time-fractional Black-Scholes model with weak singular solutions, we employ the Alikhanov formula with variable steps and the high-order average approximation to develop an efficient finite difference scheme. At the same time, the SOE technique is utilized for the discrete Caputo derivative to reduce computational costs.
- Taking the weak initial singularity of the solution into account, and based on weak mesh restrictions, we drive the unconditional stability and convergence of $O(\tau^2 + h^4)$, where $\tau$ is the maximal time step size, for the proposed scheme. To the best of our knowledge, the convergence accuracy of the proposed method is the highest one of the current numerical methods for the governing problem.

The rest of the paper is organized as follows. In Section 2, we introduce a spatial fourth-order approximation for the governing problem and derive some necessary properties of the discrete coefficients of the fast nonuniform Alikhanov formula. In Section 3, based on the fast nonuniform Alikhanov formula and the spatial fourth-order approximation, we construct an efficient nonuniform finite difference scheme for the time-fractional Black-Scholes equation. The unconditional stability and the convergence of second-order in time and fourth-order in space for the proposed scheme are well displayed by the energy method. Numerical examples are provided in Section 4 to demonstrate the theoretical statement. A brief conclusion is followed in Section 5.

## 2 | THE HIGH-ORDER AND NONUNIFORM APPROXIMATIONS

### 2.1 | Spatial high-order approximation

Some notations are needed. For a positive integer $M$, the spatial step size $h = (x_i - x_i)/M$, the discrete grid $\Omega_h := \{x_i + ih \mid 1 \leq i \leq M-1\}$, and $\Omega_h := \Omega_h \cup \partial \Omega$. Denote the space of grid functions $V_h := \{v_i \mid v_i \text{ vanishes on } \partial \Omega_h, 0 \leq i \leq M\}$. For two grid functions $v_i, w_i \in V_h$, the inner product is denoted as $\langle v, w \rangle := h \sum_{i=1}^{M} v_i w_i$, and the discrete $L^2$ norm is $\|v\| := \sqrt{\langle v, v \rangle}$. Define spatial central difference operators $\delta^2 v_i := (v_{i+1} - 2v_i + v_{i-1})/h^2$ and $\delta^2 v_i := (v_{i+1} - v_{i-1})/(2h)$.

In order to obtain a spatial high accuracy numerical scheme, we will utilize a fourth-order approximation that is derived in de Staelen and Hendy\textsuperscript{22} to discretize the space derivatives of time-fractional Black-Scholes equations (1.4). Let us briefly review it in the following.

Applying the Taylor formula at the grid points $x_i (1 \leq i \leq M - 1)$, and based on the assumption (1.6), we have

$$
\frac{\partial u(x_i, t)}{\partial x} = \delta^2 u(x_i, t) - \frac{h^2}{6} \frac{\partial^3 u(x_i, t)}{\partial x^3} + O(h^4),
$$

(2.1)

$$
\frac{\partial^2 u(x_i, t)}{\partial x^2} = \delta^4 u(x_i, t) - \frac{h^2}{12} \frac{\partial^4 u(x_i, t)}{\partial x^4} + O(h^4).
$$

(2.2)

Denote $g(x, t) := D^\alpha_t u(x, t) + cu(x, t) - f(x, t)$. It follows from the first equation in (1.4) and (2.1)–(2.2) that

$$
a \delta^2 u(x_i, t) + b \delta^4 u(x_i, t) - \frac{h^2}{12} \left( \frac{\partial^4 u(x_i, t)}{\partial x^4} + 2b \frac{\partial^3 u(x_i, t)}{\partial x^3} \right) + O(h^4) = g(x_i, t).
$$

(2.3)

On the other hand, supposing $g(x, \cdot) \in C^2(\Omega)$, the Taylor formula shows that

$$
\frac{\partial^3 u(x_i, t)}{\partial x^3} = \frac{1}{a} \left( \delta^2 g(x_i, t) - b \delta^4 u(x_i, t) \right) + O(h^2),
$$

(2.4)

$$
\frac{\partial^4 u(x_i, t)}{\partial x^4} = \frac{1}{a} \left( \delta^2 g(x_i, t) - b \frac{\partial^3 u(x_i, t)}{\partial x^3} \right) + O(h^2).
$$

(2.5)
Substituting (2.4)–(2.5) in (2.3), one has
\[
\frac{h^2}{12} \left( \delta_{,x}^2 g(x, t) + \frac{b}{a} \delta_{,x} g(x, t) \right) + g(x, t) = \left( a + \frac{h^2 b^2}{12a} \right) \delta_{,x}^2 u(x, t) + b \delta_{,x} u(x, t) + O(h^4). \tag{2.6}
\]

Thus, from (2.6), we have a high-order operator \( H := \frac{h^2}{12} \left( \delta_{,x}^2 + \frac{b}{a} \delta_{,x} \right) + 1 \) to implement a spatial fourth-order accurate approximation.

### 2.2 Fast nonuniform Alikhanov formula

Our numerical method will be implemented on possible nonuniform time partitions: \( 0 = t_0 < t_1 < t_2 < \cdots < t_N = T \), where \( N \) is a positive integer. Denote a fractional time level \( t_{n-\theta} := \theta t_{n-1} + (1-\theta) t_n \) for an offset parameter \( \theta = a/2 \), and take \( \tau_k := t_k - t_{k-1} (1 \leq k \leq N) \) as the \( k \)th time step size, and \( r := \max_{1 \leq k \leq N} \tau_k \) being the maximum step size. Besides, the local step-size ratios are defined as
\[
\rho_k := \frac{\tau_k}{\tau_{k+1}} \quad \text{for} \quad 1 \leq k \leq N-1, \quad \text{and} \quad \rho := \max_{1 \leq k \leq N-1} \rho_k.
\]

The numerical analysis of our proposed scheme will be based on the following weak assumptions on the temporal mesh:

**M1.** The maximum time-step ratio is \( \rho = 7/4 \).

**M2.** There is a constant \( C_r > 0 \) such that \( \tau_k \leq C_r r \min \left\{ 1, t_k^{1/r} \right\} \) for \( 1 \leq k \leq N \), with \( t_k \leq C_r t_{k-1} \) and \( \tau_k/\tau_k \leq C_r r_{k-1}/\tau_{k-1} \) for \( 2 \leq k \leq N \).

We next introduce the time approximation for the Caputo derivative. For any time sequence \( (\nu^k)_{k=0}^N \), define the backward difference \( \nabla_r \nu^k := \nu^k - \nu^{k-1} \) and the interpolated value \( \nu^{\theta} := \theta \nu^{\theta-1} + (1-\theta) \nu^{\theta} \). Denoting \( \Pi_{\nu,k} \nu \) the linear interpolation of a function \( \nu \) with respect to the nodes \( t_{k-1} \) and \( t_k \), and \( \Pi_{\nu,k} \nu \) the quadratic interpolation of a function \( \nu \) with respect to the nodes \( t_{k-1}, t_k, \) and \( t_{k+1} \). The corresponding interpolation errors are denoted by
\[
\left( \Pi_{\nu,k} \nu \right)(t) := \nu(t) - (\Pi_{\nu,k} \nu)(t) \quad \text{for} \quad p \in \{1, 2\}.
\]

To obtain a second-order scheme, we apply the Alikhanov formula on possible nonuniform meshes \(31\) to approximate the Caputo derivative. Meanwhile, the SOE technique is employed to result a nonuniform and fast Alikhanov formula in order to reduce the computational costs.

First of all, we review the SOE approximation (see also Jiang et al.\(^37\), Theorem 2.5 or Liao et al.\(^32\), Lemma 5.1), which is designed for the kernel function \( \omega_{a-\theta}(t) \) on the interval \([\Delta t, T]\):

**Lemma 2.1.** For the given \( a \in (0, 1) \), an absolute tolerance error \( \epsilon \ll 1 \), a cut-off time \( \Delta t > 0 \) and a final time \( T \), there exists a positive integer \( N_\delta \), positive quadrature nodes \( s^i \) and corresponding positive weights \( \sigma^i (1 \leq i \leq N_\delta) \) such that
\[
\left| \omega_{a-\theta}(t) - \sum_{i=1}^{N_\delta} \sigma^i e^{-\beta t} \right| \leq \epsilon, \quad \forall t \in [\Delta t, T],
\]
where the number \( N_\delta \) satisfies \( N_\delta = \Theta(\log^2 N) \) for \( T \approx 1 \).

Next, the Caputo fractional derivative at the time point \( t_{n-\theta} \) will be divided into two parts: an integral over \([0, t_{n-1}] \) (the historical part) and an integral over \([t_{n-1}, t_{n-\theta}] \) (the local part). The local part will be approximated directly via a linear interpolation, and the historical part will be evaluated by the SOE approximation given in Lemma 2.1, that is,
\[
D^\theta_a u(t_{n-\theta}) \approx \int_{t_{n-1}}^{t_{n-\theta}} \omega_{a-\theta}(t_{n-\theta} - s)(\Pi_{a} u)^{(1)}(s) ds + \int_{t_{n-1}}^{t_{n-\theta}} \sum_{i=1}^{N_\delta} \sigma^i e^{-\beta(t_{n-\theta} - s)} u'(s) ds \tag{2.7}
\]
\[
= a^{(n)} u^{(n)} + \sum_{i=1}^{N_\delta} \sigma^i Q^i(t_{n-1}), \quad n \geq 1,
\]
where \(1 \leq k \leq n\)
\[
a_{n-k}^{(n)} := \frac{1}{\tau_k} \int_{t_{k-1}}^{t_{\min(t_k,t_{k+1})}} \omega_{1-a}(t_{n-\theta} - s) \, ds, \quad (2.8)
\]
\[
Q^l(t_0) := 0, \quad Q^l(t_k) := \int_0^{t_k} e^{-\varphi(t_{k+1}-s)} u'(s) \, ds.
\]

The quantity \(Q^l(t_k)\) can be approximated by using the quadratic interpolation and a recursive formula, i.e.,
\[
Q^l(t_k) \approx \int_0^{t_k} e^{-\varphi(t_{k+1}-s)} u'(s) \, ds + \int_{t_{k-1}}^{t_k} e^{-\varphi(t_{k+1}-s)} (\Pi_{1,k} u)'(s) \, ds
\]
\[
= e^{-\varphi(t_{k+1}+(1-\theta)t_{k+1})} Q^l(t_{k-1}) + a^{(k,l)} \nabla_x u^k + b^{(k,l)} (\rho_k \nabla_x u^{k+1} - \nabla_x u^k),
\]
in which the positive coefficients \(a^{(k,l)}\) and \(b^{(k,l)}\) are, respectively, determined by
\[
a^{(k,l)} := \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} e^{-\varphi(t_{k+1}-s)} \, ds, \quad b^{(k,l)} := \int_{t_{k-1}}^{t_k} e^{-\varphi(t_{k+1}-s)} \frac{2(s-t_{k-1/2})}{\tau_k (\tau_k + \tau_{k+1})} \, ds.
\]

Thus, from (2.7)–(2.9), the fast Alikhanov formula is presented as
\[
(D^\varphi_t u)^{n-\theta} = a_0^{(n)} \nabla_x u^u + \sum_{l=1}^{N_q} \sigma_l Q^l(t_{n-1}), \quad n \geq 1.
\]

It can be observed that the average storage of the approximation (2.10) is \(O(N_q)\) instead of \(O(N)\), where the latter one is generated from classical Alikhanov approximation, while computing the discrete Caputo derivative at the terminal point \(t_N\). Thus, the total computational cost of the corresponding numerical scheme with the SOE approximation will be far less than that of the standard schemes with classical Alikhanov approximation while \(N\) is large.

One may notice that the discrete formula (2.10) has the following alternative form:
\[
(D^\varphi_t u)^{n-\theta} = \int_{t_{n-1}}^{t_n} \omega_{1-a}(t_{n-\theta} - s)(\Pi_{1,n} u)'(s) \, ds + \sum_{k=1}^{n-1} \int_{t_{k-1}}^{t_k} \sum_{l=1}^{N_q} \sigma_l e^{-\varphi(t_{k+1}-s)} (\Pi_{1,k} u)'(s) \, ds
\]
\[
= a_0^{(n)} \nabla_x u^u + \sum_{k=1}^{n-1} \sum_{l=1}^{N_q} \sigma_l \left( c^{(k,l)} \nabla_x u^k + d^{(k,l)} (\rho_k \nabla_x u^{k+1} - \nabla_x u^k) \right),
\]
where the discrete coefficients \(c^{(k,l)}\) and \(d^{(k,l)}\) are defined by
\[
c^{(k,l)} := \frac{1}{\tau_k} \int_{t_{k-1}}^{t_k} e^{-\varphi(t_{k+1}-s)} \, ds, \quad (2.12)
\]
\[
d^{(k,l)} := \int_{t_{k-1}}^{t_k} e^{-\varphi(t_{k+1}-s)} \frac{2(s-t_{k-1/2})}{\tau_k (\tau_k + \tau_{k+1})} \, ds. \quad (2.13)
\]

Rearranging the terms in (2.11), we obtain the compact form of (2.11):
\[
(D^\varphi_t u)^{n-\theta} = \sum_{k=1}^{n} A_{n-k}^{(n)} \nabla_x u^k, \quad n \geq 1,
\]
where the discrete convolution kernel \( A_{n-k}^{(n)} \) is defined as follows: \( A_{0}^{(1)} := a_{0}^{(1)} \) if \( n = 1 \) and, for \( n \geq 2 \),
\[
A_{n-k}^{(n)} := \begin{cases} 
    a_{0}^{(n)} + \sum_{l=1}^{N_{t}} \rho_{n-l} d^{(n-l,k)}, & k = n, \\
    \sum_{l=1}^{N_{t}} \rho_{l} \left( \rho_{k-l} d^{(k-l-1,l)} + c^{(k-l,l)} - d^{(l,k)} \right), & 2 \leq k \leq n-1, \\
    \sum_{l=1}^{N_{t}} \rho_{l} (c^{(l,l)} - d^{(l,l)}), & k = 1.
\end{cases}
\] (2.14)

To analyze the proposed numerical scheme later, we need to show that above discrete convolution kernel \( A_{n-k}^{(n)} \) fulfill two basic properties,\(^{38}\) i.e.,

A1. there is a constant \( \pi_{A} > 0 \) such that
\[
A_{n-k}^{(n)} \geq \frac{1}{\pi_{A} t_{k}} \int_{t_{k-1}}^{t_{k}} \omega_{1-n}(t_{n} - s) ds \quad \text{for} \quad 1 \leq k \leq n \leq N;
\]

A2. the discrete kernels are positive and monotone, that is,
\[
A_{0}^{(n)} \geq A_{1}^{(n)} \geq A_{2}^{(n)} \geq \ldots \geq A_{n-1}^{(n)} > 0 \quad \text{for} \quad 1 \leq k \leq n \leq N.
\]

We first derive some properties of the discrete coefficients \( c^{(k,l)} \) and \( d^{(k,l)} \).

**Lemma 2.2** (Liao et al.\(^{31}\), Lemma 2.1). For any function \( q \in C^{2}([t_{k-1}, t_{k}])\),
\[
\int_{t_{k-1}}^{t_{k}} (s - t_{k-1/2}) q(s) ds = -\int_{t_{k-1}}^{t_{k}} \left( \Pi_{1,k} q \right) (s) ds \\
= \frac{1}{2} \int_{t_{k-1}}^{t_{k}} (s - t_{k-1})(t_{k} - s) q''(s) ds.
\]
Applying Lemma 2.2, the definition (2.13) of \( d^{(k,l)} \) gives
\[
d^{(k,l)} = \int_{t_{k-1}}^{t_{k}} e^{-\rho_{l}(s-t_{k-1})} \frac{2(s-t_{k-1/2})}{t_{k}(t_{k} + t_{k+1})} ds \\
= s' \left( t_{k} - t_{k-1} \right) \int_{t_{k-1}}^{t_{k}} e^{-\rho_{l}(s-t_{k-1})} \frac{(s-t_{k-1})(t_{k} - s)}{t_{k}(t_{k} + t_{k+1})} ds, \quad 1 \leq k \leq n - 1.
\] (2.15)

Since \( 0 < (s-t_{k-1})(t_{k} - s) < \frac{\tau_{k}^{2}}{4} \) for \( t_{k-1} < s < t_{k} \), we have
\[
d^{(k,l)} \leq s' \int_{t_{k-1}}^{t_{k}} e^{-\rho_{l}(s-t_{k-1})} \frac{\tau_{k}^{2}}{4t_{k}(t_{k} + t_{k+1})} ds \\
= \frac{s' \rho_{k}}{4(1 + \rho_{k})} \int_{t_{k-1}}^{t_{k}} e^{-\rho_{l}(s-t_{k-1})} ds, \quad 1 \leq k \leq n - 1.
\] (2.16)

For simplicity of presentation, we let
\[
J^{(k,l)} := s' \int_{t_{k-1}}^{t_{k}} \frac{s - t_{k-1}}{t_{k}} e^{-\rho_{l}(s-t_{k-1})} ds, \quad 1 \leq k \leq n - 1.
\] (2.17)
Lemma 2.3. For $1 \leq k \leq n - 1$, the positive coefficients $d^{(k, l)}$ in (2.13) satisfy (i) $I^{(k, l)} \geq \frac{1+\rho_k}{\rho_k} d^{(k, l)}$; (ii) $J^{(k, l)} \geq \frac{2(1+\rho_k)}{\rho_k} d^{(k, l)}$; (iii) $J^{(k+1, l)} \geq \frac{1}{\rho_k} J^{(k, l)}$.

Proof. The alternative definition (2.15) of $d^{(k, l)}$ gives the result (i) directly since $0 < s - t_{k-1} < \tau_k$ for $s \in (t_{k-1}, t_k)$. Since $e^{-\phi(t_{k-1}, s)} > 0$ for $0 < s < t_{n-1}$, we apply Lemma 2.2 to find

$$s^l \int_{t_{k-1}}^{t_k} \left( s - t_{k-1} \right) e^{-\phi(t_{k-1}, s)} ds = \frac{\left( s^l \right)^2}{2 \tau_k} \int_{t_{k-1}}^{t_k} \left( s - t_{k-1} \right) (t_k - s) e^{-\phi(t_{k-1}, s)} ds > 0,$$

and then, $J^{(k, l)} > \frac{s^l}{\tau_k} J^{(k, l)} e^{-\phi(t_{k-1}, s)} ds$ for $1 \leq k \leq n - 1$. So the inequality (ii) follows immediately from (2.16). We now introduce an auxiliary function

$$G_k(z) := \frac{s^l}{\tau_k} \int_{t_{k-1}}^{t_k} (s - t_{k-1}) e^{-\phi(t_{k-1}, s)} ds, \quad 1 \leq k \leq n-1, \quad z \in [0, 1],$$

with its first-order derivative $G'_k(z) = \frac{z^l \tau_k}{\tau_k} e^{-\phi(t_{k-1}, t_k + z \tau_k)}$ for $1 \leq k \leq n - 1$. By using the Cauchy differential mean-value theorem, there exist $\xi \in (0, 1)$ such that

$$\frac{J^{(k+1, l)}}{J^{(k, l)}} = \frac{G_{k+1}(1)}{G_k(1)} = \frac{G_{k+1}(1) - G_{k+1}(0)}{G_k(1) - G_k(0)} = \frac{G'_{k+1}(\xi)}{G_k(\xi)} \geq \frac{\tau_k e^{-\phi(t_{k-1}, t_k + \xi \tau_k)}}{\tau_k e^{-\phi(t_{k-1}, t_k)}} = \frac{1}{\rho_k},$$

because $e^{-\phi(t_{k-1}, s)} > 0$ is increasing and $t_k > t_{k-1} + \xi \tau_k$. The inequality (iii) follows. \qed

Lemma 2.4. The positive coefficients $c^{(k, l)}$ in (2.12) satisfy

$$c^{(k+1, l)} - c^{(k, l)} = I^{(k+1, l)} + J^{(k, l)}, \quad 1 \leq k \leq n-1 (2 \leq n \leq N).$$

Proof. For fixed $n (2 \leq n \leq N)$, from the definition (2.12), we exchange the order of integration to find (for $1 \leq k \leq n - 1$)

$$c^{(k+1, l)} - c^{(k, l)} = \int_{t_{k-1}}^{t_k} e^{-\phi(t_{k-1}, s)} \frac{d^{(k, l)}}{\tau_k} ds = \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} e^{-\phi(t_{k-1}, y)} \frac{d^2}{\tau_k} dy ds = I^{(k+1, l)}. \quad (2.18)$$

Similarly, for $1 \leq k \leq n - 1 (2 \leq n \leq N)$,

$$c^{(k, l)} - c^{(k-1, l)} = \int_{t_{k-1}}^{t_k} e^{-\phi(t_{k-1}, s)} \frac{d^{(k-1, l)}}{\tau_k} ds = -J^{(k, l)}. \quad (2.19)$$

The proof is complete. \qed

Lemma 2.5. If $M1$ holds, for $1 \leq k \leq n-1 (2 \leq n \leq N)$, the positive coefficients $c^{(k, l)}$ in (2.12) satisfy

$$c^{(k+1, l)} - c^{(k, l)} = \begin{cases} d^{(2, l)}, & k = 1, \\ d^{(k+1, l)} + \rho_{k-1} d^{(k-1, l)} & 2 \leq k \leq n - 1. \end{cases}$$
Proof. By using Lemma 2.3 (ii) and (iii),

\[ \frac{\rho_{k-1}^2}{2(1 + \rho_{k-1})} f^{(k, l)} \geq \frac{\rho_{k-1}^2}{2(1 + \rho_{k-1})} f^{(k-1, l)} \geq \rho_{k-1} d^{(k-1, l)}, \quad 2 \leq k \leq n - 1. \]

Since \( 2 + 2y - y^3 \geq 9/64 \) for \( y \in [0, 7/4] \), the mesh assumption M1 leads to

\[ j^{(k, l)} = \frac{\rho_{k-1}^2}{2(1 + \rho_{k-1})} f^{(k, l)} + 2 + 2\rho_{k-1} - \rho_{k-1}^3 f^{(k, l)} > \rho_{k-1} d^{(k-1, l)}, \quad 2 \leq k \leq n - 1. \]  \hspace{1cm} (2.20)

Hence, it follows from Lemma 2.3 (i) and Lemma 2.4 that

\[ c^{(k+1, l)} - c^{(k, l)} = f^{(k+1, l)} + f^{(k, l)} \begin{cases} \frac{d^{(2, l)}}{k = 1}, & 2 \leq k \leq n - 1, \end{cases} \]

The proof is complete. \( \square \)

**Lemma 2.6.** If the tolerance error \( \epsilon \) of the SOE approximation satisfies \( \epsilon \leq \frac{\theta}{1-\alpha} \omega_{1-a}(T) \), then the discrete coefficients \( a_{n-k}^{(n)} \) of (2.8) satisfy

(i) \( a_0^{(2)} - \sum_{l=1}^{N_q} \omega_l c^{(l, l)} \geq 0 \); (ii) \( a_0^{(n)} - \sum_{l=1}^{N_q} \omega_l c^{(n-1, l)} - \sum_{l=1}^{N_q} \omega_l \rho_{n-2} d^{(n-2, l)} > 0 \).

Proof. Noticing \( a_0^{(n)} = \frac{1-\alpha}{1-\alpha^2} \omega_{1-a}(t_n-a - t_{n-1}) \) from (2.8), then we have

\[ a_0^{(n)} - a_1^{(n)} \geq a_0^{(n)} - \omega_{1-a}(t_n-a - t_{n-1}) = \frac{\theta}{1-\alpha} \omega_{1-a}(t_n-a - t_{n-1}) \geq \frac{\theta}{1-\alpha} \omega_{1-a}(T) \geq \epsilon. \]

Therefore, Lemma 2.1 gives the result (i) directly since \( a_0^{(2)} - \sum_{l=1}^{N_q} \omega_l c^{(l, l)} \geq a_0^{(2)} - a_1^{(2)} - \epsilon \geq 0 \). By (2.19) and (2.20), we have

\[ \sum_{l=1}^{N_q} \omega_l (c^{(l, l-1)} + \rho_{n-2} d^{(n-2, l)}) < \sum_{l=1}^{N_q} \omega_l (c^{(l, l-1)} + f^{(l-1, l)}) = \sum_{l=1}^{N_q} \omega_l e^{-\int (t_{n, 1} - t_{n-1})}. \]

Then, we apply Lemma (2.1) to arrive that

\[ a_0^{(n)} - \sum_{l=1}^{N_q} \omega_l c^{(l, l-1)} - \sum_{l=1}^{N_q} \omega_l \rho_{n-2} d^{(n-2, l)} > \epsilon + \omega_{1-a}(t_n-a - t_{n-1}) - \sum_{l=1}^{N_q} \omega_l e^{-\int (t_{n, 1} - t_{n-1})} \geq 0. \]

So the inequality (ii) is proved. \( \square \)

We now verify that the coefficients \( A_{n-k}^{(n)} \) satisfy A1 and A2. Part (I) in the next lemma ensures that assumption A2 is valid, while part (II) implies that assumption A1 holds true with \( \pi_A = 11/4 \).

**Lemma 2.7.** If the tolerance error \( \epsilon \) of SOE approximation satisfies \( \epsilon \leq \min \left\{ \frac{7}{11} \omega_{1-a}(T), \frac{\theta}{1-\alpha} \omega_{1-a}(T) \right\} \), then the discrete convolutional kernel \( A_{n-k}^{(n)} \) in (2.14) satisfies

(I) \( A_{n-k-1}^{(n)} > A_{n-k}^{(n)} > 0, \quad 1 \leq k \leq n - 1, \)

(II) \( A_{n-k}^{(n)} \geq \frac{\pi_A}{11/4} \int_{t_n}^{t_1} \omega_{1-a}(t_n-s) ds, \quad 1 \leq k \leq n. \)
Proof. Recalling the definition (2.14), it is not difficult to verify that

1. if \( k = 1 \) for \( n = 2 \),

\[
A_{0}^{(2)} - A_{1}^{(2)} = a_{0}^{(2)} + \sum_{i=1}^{N_{q}} \sigma^{i} \rho_{1} d_{i}^{(1,1)} - \sum_{i=1}^{N_{q}} \sigma^{i} (c_{i}^{(1,1)} - d_{i}^{(1,1)})
\]

\[
= \sum_{i=1}^{N_{q}} \sigma^{i} (\rho_{1} + 1) d_{i}^{(1,1)} + a_{0}^{(2)} - \sum_{i=1}^{N_{q}} \sigma^{i} c_{i}^{(1,1)};
\]

2. if \( k = n - 1 \) for \( n \geq 3 \),

\[
A_{n-2}^{(n)} - A_{n-1}^{(n)} = \sum_{i=1}^{N_{q}} \sigma^{i} (\rho_{1} d_{i}^{(1,1)} + c_{i}^{(2,1)} - d_{i}^{(2,1)}) - \sum_{i=1}^{N_{q}} \sigma^{i} (c_{i}^{(1,1)} - d_{i}^{(1,1)})
\]

\[
= \sum_{i=1}^{N_{q}} \sigma^{i} (\rho_{1} + 1) d_{i}^{(1,1)} + \sum_{i=1}^{N_{q}} \sigma^{i} (c_{i}^{(2,1)} - c_{i}^{(1,1)} - d_{i}^{(2,1)})
\]

3. if \( k = 1 \) for \( n \geq 3 \),

\[
A_{n-k-1}^{(n)} - A_{n-k}^{(n)} = \sum_{i=1}^{N_{q}} \sigma^{i} (\rho_{k} d_{i}^{(k,1)} + c_{i}^{(k+1,1)} - d_{i}^{(k+1,1)}) - \sum_{i=1}^{N_{q}} \sigma^{i} (\rho_{k-1} d_{i}^{(k-1,1)} + c_{i}^{(k,1)} - d_{i}^{(k,1)})
\]

\[
= \sum_{i=1}^{N_{q}} \sigma^{i} (\rho_{k} + 1) d_{i}^{(k,1)} + \sum_{i=1}^{N_{q}} \sigma^{i} (c_{i}^{(k+1,1)} - c_{i}^{(k,1)} - d_{i}^{(k+1,1)} - \rho_{k-1} d_{i}^{(k-1,1)}).
\]

Hence, the claimed inequality in part (I) follows from Lemmas 2.5 and 2.6 directly.

According to the definitions (2.14) and (2.8), the inequality in part (II) holds obviously, while \( k = n \). Under the assumption \textbf{M1}, and by using Lemma 2.3 (i) and (2.18), one has

\[
\frac{\rho_{k}}{\rho_{k} + 1} \frac{c_{i}^{(k,1)} - e^{-q_{n-t_{t_{k-1}}}}}{c_{i}^{(k,1)}} \leq \frac{7}{11} \left( c_{i}^{(k,1)} - e^{-q_{n-t_{t_{k-1}}}} \right).
\]

Moreover, from (2.8) and (2.12), and using Lemma 2.1, we have

\[
\sum_{i=1}^{N_{q}} \sigma^{i} c_{i}^{(k,1)} \geq a_{n-k}^{(n)} - \epsilon, \text{ for } 1 \leq k \leq n - 1.
\]

Thus, the lower bounds of \( A_{n-k}^{(n)} \) for \( 1 \leq k \leq n - 1 \) follow from Lemma 2.1 because the definition (2.14) implies that

\[
A_{n-k}^{(n)} \geq \sum_{i=1}^{N_{q}} \sigma^{i} c_{i}^{(k,1)} - q_{n-t_{t_{k-1}}} \geq \frac{4}{11} \sum_{i=1}^{N_{q}} \sigma^{i} c_{i}^{(k,1)} + \frac{7}{11} \sum_{i=1}^{N_{q}} \sigma^{i} e^{-q_{n-t_{t_{k-1}}}}
\]

\[
\geq \frac{4}{11} a_{n-k}^{(n)} + \frac{7}{11} \omega_{1-a}(t_{n-t_{t_{k-1}}}) - \epsilon \geq \frac{4}{11} a_{n-k}^{(n)} + \frac{7}{11} \omega_{1-a}(T) - \epsilon.
\]

The proof of part (II) is complete. \( \square \)
### 3.1 The numerical scheme

Let \( u^n \) be the discrete approximation of solution \( u(x_i, t_n) \) for \( x_i \in \bar{\Omega}_h, 0 \leq n \leq N \). Considering the first equation in (1.4) at the grid points \((x_i, t_{n-	heta})\), utilizing the fast nonuniform Alikhanov formula (2.10) and the spatial high-order approximation (2.6), we can obtain

\[
\mathcal{H}g_i^{n-	heta} = \left( a + \frac{h^2b^2}{12a} \right) \delta^2_i u_i^{n-	heta} + b\delta_i u_i^{n-	heta} + R_i^{n-	heta} \text{ for } x_i \in \Omega_h, 1 \leq n \leq N, \tag{3.1}
\]

where

\[
g_i^{n-	heta} := (D_i^a u_i)^{n-	heta} + cu_i^{n-	heta} - f(x_i, t_{n-	heta}),
\]

and \( R_i^{n-	heta} = \mathcal{H}(R_{1i})^{n-	heta} + \mathcal{H}(R_{2i})^{n-	heta} + (R_{3,i})^{n-	heta} + (R_{4,i})^{n-	heta} \), in which \( (R_{3,i})^{n-	heta} = \mathcal{O}(h^4) \) (according to (2.6)), and

\[
\begin{align*}
(R_{1i})^{n-	heta} &= (D_i^a u_i)^{n-	heta} - D_i^a u(x_i, t_{n-	heta}), \\
(R_{2i})^{n-	heta} &= -c(R_{i})^{n-	heta}, \\
(R_{3,i})^{n-	heta} &= \left( a + \frac{h^2b^2}{12a} \right) \delta^2_x(R_{i})^{n-	heta} + b\delta_x(R_{i})^{n-	heta},
\end{align*}
\]

with the error of the weighted time approximation at \( t_{n-	heta} \) is given as

\[
(R_{i})^{n-	heta} = u(x_i, t_0) - \left[ \theta u(x_i, t_{n-1}) + (1 - \theta)u(x_i, t_n) \right].
\]

Based on the regularity assumption (1.5) and the mesh condition \( \textbf{M1} \), and referring to Liao et al.,\textsuperscript{31} Lemma 3.6 and Lemma 3.8 we can obtain that

\[
\begin{align*}
\sum_{k=1}^{n} p_{n-k}^{(n)} |(R_{1})_{i}^{k-	heta}| &\leq C \left( \tau_1^\sigma / \sigma + \tau_1^{\sigma-3} \tau_3^3 + \frac{1}{1 - \alpha} \max_{2 \leq k \leq n} t_i^\sigma \tau_k^{\sigma-3} / \tau_{k-1} \right), \\
\sum_{k=1}^{n} p_{n-k}^{(n)} |(R_{2i})^{k-	heta}| &\leq C \left( \tau_1^{\sigma+\theta} / \sigma + \tau_1^{\sigma-2} \max_{2 \leq k \leq n} t_i^{\sigma-2} / \tau_{k-1} \right),
\end{align*}
\]

where \( p_{n-k}^{(n)} \) is called the discrete complementary convolution kernels, satisfying the basic rule: \( \sum_{j=k}^{n} p_{n-j}^{(n)} A_{j-k}^{(j)} \equiv 1 \) for \( 1 \leq k \leq n \leq N \). Moreover, the complementary kernels are nonnegative and satisfy\textsuperscript{38}, Lemma 2.1

\[
\sum_{j=1}^{n} p_{n-j}^{(n)} \tau_1^{\alpha+\beta m-1} (t_j) \leq \frac{11}{4} \tau_1^{\alpha+\beta m} (t_n) \text{ for } m = 0, 1, \text{ and } 1 \leq n \leq N. \tag{3.4}
\]

One may refer to Liao et al.\textsuperscript{31,38} for more details about the \( p_{n-k}^{(n)} \), which is a crucial tool in the numerical analysis. With (3.4) (for \( m = 1 \)), it is easy to show that

\[
\max_{1 \leq k \leq n} \sum_{j=1}^{k} p_{k-j}^{(k)} |(R_{3,i})^{j-	heta}| \leq Ch^4. \tag{3.5}
\]

Thus, combining (3.2)–(3.3) and (3.5), it holds

\[
\max_{1 \leq k \leq n} \sum_{j=1}^{k} p_{n-j}^{(n)} \|R^{k-	heta}\| \leq C \left( \tau_1^\sigma / \sigma + \max_{2 \leq k \leq n} t_i^\sigma \tau_k^{\sigma-3} / \tau_{k-1} + \tau_1^\sigma \max_{2 \leq k \leq n} t_i^{\sigma-2} \tau_k^2 + h^4 \right). \tag{3.6}
\]
Omitting the truncation errors in (3.1), we get a fast and high-order nonuniform scheme for the problem (1.4):

\[ H(D_t^n u^{n-\theta} + cu^{n-\theta} - f^{n-\theta}) = \left( a + \frac{b^2}{2h} \right) \delta^2 u^{n-\theta}_i + b \delta^3 u^{n-\theta}_i, \quad x_i \in \Omega_n, \ 1 \leq n \leq N, \]  

(3.7)
equipped with the initial condition \( u_0^n = \varphi_i \) for \( x_i \in \Omega_n \), and the boundary conditions \( u^n_M = 0 \).

Denote \( u^n := (u^n_1, u^n_2, \ldots, u^n_{M-1})^T \), \( f^{n-\theta} = (f^{n-\theta}_1, f^{n-\theta}_2, \ldots, f^{n-\theta}_{M-1})^T \), and the matrices associated to spatial central differences

\[ A := \begin{bmatrix} -2 & 1 & \cdots & 1 \\ 1 & -2 & \cdots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & 1 & -2 \end{bmatrix}, \quad S := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ -1 & \cdots & -1 & 0 \end{bmatrix}. \]

We can rewrite the scheme (3.7) into the following equivalent matrix-vector equation:

\[ HD_t^n u^{n-\theta} + cH u^{n-\theta} - Hf^{n-\theta} - f^{n-\theta} = \left( \frac{a}{n^2} + \frac{b^2}{24a} \right) Au^{n-\theta} + \frac{b}{2h} Su^{n-\theta}, \]

where

\[ f^{n-\theta} = \begin{bmatrix} \left( \frac{1}{12} - \frac{hb}{24a} \right) f^{n-\theta}_0, 0, \ldots, 0, \left( \frac{1}{12} + \frac{hb}{24a} \right) f^{n-\theta}_M \end{bmatrix}^T, \]

and \( H = \frac{1}{12}A + \frac{hb}{24a}S + I \) with \( I \) being the \((M-1)\)-dimensional unit matrix.

### 3.2 Stability and convergence

**Lemma 3.1** (Liao et al. 31, Corollary 2.3). Under the condition **M1**, the discrete Caputo formula satisfies

\[ \langle (D_t^n v)^{n-\theta}, v^{n-\theta} \rangle \geq \frac{1}{2} \sum_{k=1}^{n} A_{n-k}^{(n)} \sqrt{\langle v^k \rangle^2}, \quad \text{for} \ 1 \leq n \leq N. \]

According to Liao et al., 38, Theorem 3.1 and Remark 1 we have the following lemma.

**Lemma 3.2.** Let the assumptions **A1** and **A2** hold, and let \((\lambda^i)^{N-1}_{i=0}\) be a given non-positive sequence. If the non-negative time sequences \((v^k)^N_{k=1}\) and \((\nu^k)^N_{k=0}\) satisfy

\[ \sum_{k=1}^{n} A_{n-k}^{(n)} \sqrt{\langle v^k \rangle^2} \leq \sum_{k=1}^{n} \lambda_{n-k} (\nu^{k-\theta})^2 + v^{n-\theta} \nu^n, \quad \text{for} \ 1 \leq n \leq N, \]

then the \((v^k)^N_{k=0}\) satisfies

\[ v^n \leq v^0 + \max_{1 \leq k \leq n} \sum_{i=1}^{k} F^{(k)}_{k-j} \nu^j \leq v^0 + \frac{11}{4} \Gamma(1 - \alpha) \max_{1 \leq j \leq n} \{ t^\alpha \nu^j \}, \quad \text{for} \ 1 \leq n \leq N. \]

To show the stability and convergence of the proposed scheme, we first discuss some main properties of the matrices in (3.8).

**Lemma 3.3** (Laub 39). Let symmetric matrix \( M \in \mathbb{R}^{m \times m} \) with eigenvalues \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m \). Then, for all \( w \in \mathbb{R}^{m \times 1}, \)

\[ \lambda_m w^T w \leq w^T M w \leq \lambda_1 w^T w. \]

**Lemma 3.4.** The matrix \( H^T H \) satisfies \( \frac{5}{12} w^T w \leq w^T H^T H w \leq w^T w \) for any real vector \( w \).
Proof. Denote

\[ B := \begin{bmatrix} 10 & 1 & 1 \\ 1 & 10 & 1 \\ \vdots & \vdots & \vdots \\ 1 & 10 & 1 \\ 1 & 10 \end{bmatrix}. \]

Then, \( H = \frac{1}{12}A + \frac{b}{24a}hS + I = \frac{1}{12}B + \frac{b}{24a}hS, \) and

\[
H^T H = \left( \frac{1}{12}B + \frac{b}{24a}hS^T \right) \left( \frac{1}{12}B + \frac{b}{24a}hS \right)
\]

\[
= \frac{1}{144}B^2 + \frac{b}{288a}hBS + \frac{b}{288a}hS^TB + \frac{b^2}{576a^2}h^2S^TS
\]

\[
= \frac{1}{144} \left[ B^2 + \frac{b}{2a}h(BS + S^TB) + \frac{b^2}{4a^2}h^2S^TS \right].
\]

Noticing that

\[
BS + S^TB = \begin{bmatrix} -2 & 0 & \cdots & 0 \\ 0 & -2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & \cdots & 2 \end{bmatrix}, \quad S^TS = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}, \quad (3.9)
\]

therefore,

\[
H^T H = \frac{1}{144} \begin{bmatrix} c_2 & 20 & 1 - c_1 h^2 \\ 20 & 102 + 2c_1 h^2 & 20 \\ 1 - c_1 h^2 & 20 & 102 + 2c_1 h^2 \\ \vdots & \vdots & \vdots \\ 1 - c_1 h^2 & 20 & 102 + 2c_1 h^2 \\ 20 & 1 - c_1 h^2 & 20 \\ \vdots & \vdots & \vdots \\ 1 - c_1 h^2 & 20 & 102 + 2c_1 h^2 \\ 20 & 1 - c_1 h^2 & 20 \\ c_3 \end{bmatrix}
\]

where \( c_1 = \frac{b^2}{2a^2}; \ c_2 = 101 - \frac{b}{a}h + c_1 h^2; \ c_3 = 101 + \frac{b}{a}h + c_1 h^2. \) It is easy to check that \( H^T H \) is diagonally dominant.

Next, we divide \( H^T H \) into two parts to study its numerical range:

\[
H^T H = \frac{1}{144} \begin{bmatrix} 21 - c_1 h^2 & 20 & 1 - c_1 h^2 \\ 20 & 41 - c_1 h^2 & 20 \\ 1 - c_1 h^2 & 20 & 42 - 2c_1 h^2 \\ \vdots & \vdots & \vdots \\ 1 - c_1 h^2 & 20 & 41 - c_1 h^2 \\ 20 & 1 - c_1 h^2 & 20 \\ \vdots & \vdots & \vdots \\ 1 - c_1 h^2 & 20 & 41 - c_1 h^2 \\ 20 & 1 - c_1 h^2 & 20 \\ 80 - \frac{b}{a}h + 2c_1 h^2 \\ 61 + 3c_1 h^2 \\ 60 + 4c_1 h^2 \\ \vdots \\ 60 + 4c_1 h^2 \\ 61 + 3c_1 h^2 \\ 80 + \frac{b}{a}h + 2c_1 h^2 \end{bmatrix}
\]

\[
:= H_1 + H_2.
\]

For small \( h, \) by using the Gersgorin's circle theorem, it is easy to know that \( \lambda_{\text{min}}(H_1) \geq 0 \) and \( \lambda_{\text{min}}(H_2) \geq 60/144 = 5/12. \) By Lemma 3.3, for any real vector \( w, \) it holds

\[
w^T H^T H w = w^T H_1 w + w^T H_2 w \geq \frac{5}{12}w^T w.
\]
Similarly, we have the decomposition

\[
H^T H = \frac{1}{144} \begin{bmatrix}
-21 + c_1 h^2 & 20 & 1 - c_1 h^2 \\
20 & -41 + c_1 h^2 & 20 & 1 - c_1 h^2 \\
1 - c_1 h^2 & 20 & -42 + 2c_1 h^2 & 20 & 1 - c_1 h^2 \\
& & & & \ddots \ddots \ddots
\end{bmatrix} + \frac{1}{144} \begin{bmatrix}
122 - \frac{b h}{a} & 143 + c_1 h^2 \\
144 & -41 + c_1 h^2 \\
& & & & \ddots \ddots \ddots
\end{bmatrix} + \frac{1}{144} \begin{bmatrix}
122 + \frac{b h}{a}
\end{bmatrix}
\]

\[\vdots = H_3 + H_4.\]

The Gershgorin's circle theorem gives \(\lambda_{\text{max}}(H_3) \leq 0\) and \(\lambda_{\text{max}}(H_4) \leq 144/144 = 1\) for small \(h\), which leads to

\[w^T H^T H w = w^T H_3 w + w^T H_4 w \leq w^T w.\]

\[\square\]

**Lemma 3.5.** The matrices \(H^T A + AH\) and \(\frac{a}{h^2}(H^T A + AH) + \frac{b}{2h}(H^T S + S^T H)\) are negative semi-definite.

**Proof.** Straightforward computations show that

\[
H^T A + AH = \left(\frac{1}{12}B + \frac{b}{24a}hS\right) A + A \left(\frac{1}{12}B + \frac{b}{24a}hS\right) = \frac{1}{6} \begin{bmatrix}
-19 - \frac{b h}{2a} & 8 & 1 \\
8 & -18 & 8 & 1 \\
1 & 8 & -18 & 8 & 1 \\
& & & & \ddots \ddots \ddots \\
1 & 8 & -18 & 8 & 1 \\
& & & & \vdots
\end{bmatrix}
\]

So \(H^T A + AH\) is diagonally dominant with small \(h\). By similar arguments to the proof of Lemma 3.4, we can check that \(\lambda_{\text{max}}(H^T A + AH) \leq 0\), which leads to first part of the desired result.

For the second part, we notice that

\[
\frac{a}{h^2}(H^T A + AH) + \frac{b}{2h}(H^T S + S^T H)
\]

\[
= \frac{1}{6h^2} \begin{bmatrix}
c_4 & 8a & a - ac_1 h^2 \\
8a & -18a + 2ac_1 h^2 & 8a & a - ac_1 h^2 \\
a - ac_1 h^2 & 8a & -18a + 2ac_1 h^2 & 8a & a - ac_1 h^2 \\
& & & & \ddots \ddots \ddots \\
8a & a - ac_1 h^2 & 8a & -18a + 2ac_1 h^2 & 8a & c_5
\end{bmatrix}
\]

where \(c_4 = -19a - bh + ac_1 h^2\), \(c_5 = -19a + bh + ac_1 h^2\). One may easy to find that the above matrix is also diagonally dominant. Similarly, it is negative semi-definite. \(\square\)

We are now ready to display the stability and convergence of our proposed scheme.
Theorem 3.6 (Stability). If the assumptions A1–A2 hold, then the numerical scheme (3.7) is stable and satisfies

\[
\|u^n\| \leq \sqrt{12} \left( \|u^0\| + 2 \max_{1 \leq k \leq n} \sum_{j=1}^{k} F^{(k)}_{k-j}(\|f_{n-j}^{\theta}\| + \|\hat{f}_{n-j}^{\theta}\|) \right) \\
\leq \sqrt{12} \left( \|u^0\| + \frac{11}{2} (1 - \alpha) \max_{1 \leq i \leq n} \{ \tau_i^n(\|f_{n-\theta}^{\theta}\| + \|\hat{f}_{n-\theta}^{\theta}\|) \} \right), \quad 1 \leq n \leq N.
\]

Proof. Multiplying both sides of (3.8) with \(h(u^{a-\theta})^TH^T\), one has

\[
h(u^{a-\theta})^TH^THD^n\theta u^{a-\theta} + ch(u^{a-\theta})^TH^THu^{a-\theta} - h(u^{a-\theta})^TH^THf^{a-\theta} - h(u^{a-\theta})^TH^TH\hat{f}^{a-\theta} \\
= \left( \frac{a}{h^2} + \frac{b^2}{12a} \right) h(u^{a-\theta})^TH^T A u^{a-\theta} + \frac{b}{2h} h(u^{a-\theta})^TH^T S u^{a-\theta}.
\]

According to Lemmas 3.3 and 3.5, the right-hand side of (3.10) satisfies

\[
\begin{aligned}
&\left( \frac{a}{h^2} + \frac{b^2}{12a} \right) h(u^{a-\theta})^TH^T A u^{a-\theta} + \frac{b}{2h} h(u^{a-\theta})^TH^T S u^{a-\theta} \\
&= \frac{h}{2} \left( \frac{a}{h^2} + \frac{b^2}{12a} \right) (u^{a-\theta})^T (H^T A + AH) u^{a-\theta} + \frac{b}{4} (u^{a-\theta})^T (H^T S + S^T H) u^{a-\theta} \leq 0,
\end{aligned}
\]

where the identity \(u^TWu = \frac{1}{2}u^T(W + W^T)u\) has been applied.

From (3.10)–(3.11), and utilizing Lemma 3.4 and the Cauchy-Schwarz inequality, we get

\[
\begin{aligned}
h(u^{a-\theta})^TH^T HD^n\theta u^{a-\theta} \\
&\leq h(u^{a-\theta})^TH^T Hf^{a-\theta} + h(u^{a-\theta})^TH^T \hat{f}^{a-\theta} \\
&\leq h(v^{a-\theta})^T (Hf^{a-\theta}) + h(v^{a-\theta})^T \hat{f}^{a-\theta} \\
&\leq \sqrt{h(v^{a-\theta})^T (v^{a-\theta})} \sqrt{h(Hf^{a-\theta})^T (Hf^{a-\theta})} + \sqrt{h(v^{a-\theta})^T (v^{a-\theta})} \sqrt{h(\hat{f}^{a-\theta})^T (\hat{f}^{a-\theta})} \\
&\leq \sqrt{h(v^{a-\theta})^T (v^{a-\theta})} \sqrt{h(f^{a-\theta})^T (f^{a-\theta})} + \sqrt{h(v^{a-\theta})^T (v^{a-\theta})} \sqrt{h(\hat{f}^{a-\theta})^T (\hat{f}^{a-\theta})},
\end{aligned}
\]

where \(v^{a-\theta} = Hu^{a-\theta}\), and it leads to (by taking \(v^n_i = Hu^n_i\))

\[
\langle (D^n_i v^{a-\theta}, v^{a-\theta}) \leq \|v^{a-\theta}\| (\|f^{a-\theta}\| + \|\hat{f}^{a-\theta}\|) \leq (\theta\|v^{a-\theta}\| + (1 - \theta)\|\hat{v}^{a-\theta}\|) (\|f^{a-\theta}\| + \|\hat{f}^{a-\theta}\|).
\]

(3.12)

Hence, it follows from Lemmas 3.1 and 3.2, and (3.12) that

\[
\|v^n\| \leq \|v^0\| + 2 \max_{1 \leq k \leq n} \sum_{j=1}^{k} F^{(k)}_{k-j}(\|f^{a-\theta}\| + \|\hat{f}^{a-\theta}\|) \\
\leq \|v^0\| + \frac{11}{2} (1 - \alpha) \max_{1 \leq i \leq n} \{ \tau_i^n(\|f^{a-\theta}\| + \|\hat{f}^{a-\theta}\|) \}. \quad \text{for } 1 \leq n \leq N.
\]

Finally, the claimed result can be achieved provided the following bound arising from Lemma 3.4:

\[
\frac{5}{12} \|u^n\|^2 \leq \|v^n\|^2 \leq \|u^0\|^2.
\]

□
Theorem 3.7 (Convergence). Let \( e^n = u(x_i, t_n) - u_0^n; \) if M1, M2, and the regularity assumptions (1.5)–(1.6) hold, then the proposed scheme (3.7) is convergent with

\[
\|e^n\| \leq C(\tau^{\min\{\sigma, 2\}} + h^4), \quad 1 \leq n \leq N.
\]

Proof. We have the following error equation:

\[
H D_t^n e_i^n = \left( a + \frac{h^2 b^2}{12a} \right) \delta_t^2 e_i^n + b \delta_t e_i^n - c H e_i^n + R_i^n, \quad x_i \in \Omega_h, \quad 1 \leq n \leq N. \tag{3.13}
\]

By using analogous derivations to the proof of Theorem 3.6, and notice (3.6), we can get

\[
\|e^n\| \leq \sqrt{\frac{12}{5}} \left( \|e^0\| + 2 \max_{1 \leq k \leq n} \sum_{j=1}^{k} P^{(k)}_{k-j} \|R^n\| \right) \leq C \left( \frac{\tau_1^{\sigma} + \max_{2 \leq k \leq n} \frac{\tau_1^{\sigma}-3}{\tau_1^{\sigma}-1} \frac{\tau_1^{\sigma}}{\tau_1^{\sigma}-1} + \max_{2 \leq k \leq n} \frac{\tau_1^{\sigma}}{\tau_1^{\sigma}} + h^4 \right).
\]

If the mesh assumption M2 holds, we have \( \tau_1 \leq C \tau_\gamma \) and (for \( 2 \leq k \leq n \))

\[
t_1^{\sigma} \frac{\tau_1^{\sigma}-3}{\tau_1^{\sigma}-1} \frac{\tau_1^{\sigma}}{\tau_1^{\sigma}} \leq C t_k^{\max(0,\sigma-(3-\sigma)/\gamma)} \tau_{\min\{2,\sigma\}} ,
\]

\[
t_1^{\sigma}-2 \frac{\tau_1^{\sigma}}{\tau_1^{\sigma}} \leq C t_k^{\max(0,\sigma-2/\gamma)} \tau_{\min\{2,\sigma\}} ;
\]

see also Liao et al.\(^{31}\), equations (3.11) and (3.12) Thus, the claimed result follows immediately. \( \square \)

4 | NUMERICAL IMPLEMENTATIONS

In this section, we carry out numerical experiments to illustrate our theoretical statements. In our computations, the spatial domain is divided uniformly into \( M \) subintervals, and the time interval is divided by a general nonuniform grid with \( N \) parts. To test the sharpness of our error estimate, we choose the graded time mesh \( t_k = T(k/N)^\gamma \); one may notice that \( \sigma = \alpha \) for our examples. Meanwhile, the error parameter in SOE algorithm is set as \( \epsilon = 10^{-12} \). To understand the proposed numerical scheme (3.7) more clearly, we display its framework in Algorithm 1.

Algorithm 1 The algorithm framework of the proposed scheme (3.7)

Given: \( a, b, c, T \), the initial condition \( u^0 \), the source term \( f \) and \( \tilde{Q} = 0 \);

1: Construct matrices \( A, S, H \);
2: Calculate coefficient matrix \( Y = cH - (a/h^2 + b^2/12a)A - b/2h * S \);
3: for \( k = 1 : N \) do
4: Calculate \( X = d^{(n)}_0 + \sum_{l} (w^i * b^{(k-1,l)} \rho_{k-1}) \);
5: Calculate \( Q = e^{-\gamma (\theta \tau_1 + (1-\theta)\tau_1)} \star \tilde{Q} + (a^{(k-1,b)} - b^{(k-1,b)}) (u^{k-1} - u^{k-2}) \);
6: Calculate \( L^k = XH + (1-\theta)Y \);
7: Calculate \( R^k = XH * u^{k-1} + \theta Y * u^{k-1} - H \sum_{l} (w^i * Q) + H f^k + \tilde{f}^k \);
8: Calculate \( u^k = (L^k)^{-1} \star R^k \);
9: Calculate \( \tilde{Q} = Q + b^{(k-1,b)} \rho_{k-1} (u^k - u^{k-1}) \);
10: end for
Example 4.1. We consider the problem (1.4) with \( \Omega = (0, 1), T = 1 \), the initial condition \( u(x, 0) = x^3(1 - x)^3 \), and the source term

\[
f(x, t) = x^3(1 - x)^3 \left( \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \Gamma(\alpha + 1) \right) - \left( a \frac{\partial^2 u}{\partial x^2} + b \frac{\partial u}{\partial x} - cu \right) (t^\alpha + t + 1)
\]

is chosen to such that the exact solution is \( u(x, t) = x^3(1 - x)^3(t^\alpha + t + 1) \). Here, we take \( \alpha = 0.5 \), \( b = -0.45 \), and \( c = 0.05 \).

The numerical results of the proposed scheme (3.7) in solving Example 4.1 are recorded in Tables 2–4. In each run, the discrete \( L^2 \)-norm solution error

\[
E_2(M, N) = \max_{1 \leq n \leq N} \| u(\cdot, t_n) - u^n \|
\]

The temporal and spatial rate of convergence is estimated, respectively, by

\[
Rate_t = \log_2 \left[ \frac{E_2(M, N)}{E_2(M, 2N)} \right] \quad \text{and} \quad Rate_h = \log_2 \left[ \frac{E_2(M, N)}{E_2(2M, N)} \right].
\]

To show the experimental merit of the SOE approximation, we make the following denotations:

- Fast scheme represents the proposed scheme (3.7).
- Direct scheme represents the proposed scheme (3.7) without the SOE approximation.

Table 2 shows the numerical results of fast and direct schemes for different \( \alpha \) with the grading parameter \( \gamma = 2/\alpha \). It is seen that the errors and the rates of the two methods are almost the same; hence, the SOE approximation does not lose accuracy with fitted tolerance \( \epsilon \), and it is clear that the proposed numerical method is convergent with second-order accuracy in time, which agrees well with the theoretical statement. In addition, the curves of the CPU times of fast and direct schemes when solving Example 4.1 are displayed in Figure 1. Obviously, the fast scheme saves many computational costs compared with the direct scheme. By comparing with the lines having slopes equal to 1 and 2, it indicates that the fast scheme works with almost linear complexity of \( O(N) \) and is much faster than the direct scheme whose complexity is \( O(N^2) \). One may also refer to Liao et al.,33 Jiang et al.,37 Lyu et al.,40 and Yan et al41 for more similar comparisons on this kind of fast approximation with the classical approximation while solving the time-fractional partial differential equations.

To further explore the numerical behavior, the numerical accuracy of the proposed scheme (3.7) for smaller values of \( \alpha \) is recorded in Table 3. For the theoretical optimal mesh grading \( \gamma = 2/\alpha \), the proposed scheme fails to converge with desired orders, whereas by taking a minor mesh grading, i.e., \( \gamma = 2 \), the scheme works well with the second-order temporal accuracy. The reason should be that the optimal grading \( \gamma = 2/\alpha \) takes big values, while \( \alpha \) is close to zero, which makes the graded time mesh become very dense near the initial point and the coefficients (2.14) of the discrete Caputo derivative will also be very large. Thus, the numerical accuracy should be very sensitive to the relatively large rounding error as \( \alpha \) is small. It indicates that the error estimates (3.2) and (3.3) may not be very sharp; we would like to go into further investigation on this point in the future study. Nevertheless, we can take small grading values, e.g., \( \gamma = 2 \), while \( \alpha \) is small in the practical situation. To confirm the spatial fourth-order accuracy, we list the numerical results of fast scheme with the mesh grading \( \gamma = 2 \) in solving the example in Table 4.
FIGURE 1  Comparisons in CPU times of fast and direct schemes, where $M = 10$, $a = 0.5$, and $r = 2/a$ [Colour figure can be viewed at wileyonlinelibrary.com]

TABLE 3  Numerical temporal accuracy of Fast-Scheme for fixed $M = 100$ and small $a$

| $N$ | $E_2(M,N)$ | Rate | $E_2(M,N)$ | Rate |
|-----|-------------|-------|-------------|-------|
| 4   | 7.6536e+02 | *     | 1.8916e-04 | *     |
| 8   | 2.2200e+08 | -18.1460 | 5.8516e-05 | 1.6927 |
| 16  | 1.4241e+14 | -19.2911 | 1.1534e-03 | -4.3010 |
| 32  | 3.0622e+19 | -17.7141 | 4.5004e-01 | -8.6080 |

Note: Asterisk (*) stands for null data.

TABLE 4  Numerical spatial accuracy of fast scheme for fixed $N = 500$ and different $a$

| $M$ | $E_2(M,N)$ | Rate | $E_2(M,N)$ | Rate | $E_2(M,N)$ | Rate |
|-----|-------------|-------|-------------|-------|-------------|-------|
| 4   | 3.7442e-02 | *     | 3.4913e-02 | *     | 3.2215e-02 | *     |
| 8   | 2.3697e-03 | 3.9819 | 2.2105e-03 | 3.9813 | 2.0398e-03 | 3.9812 |
| 16  | 1.4780e-04 | 4.0030 | 1.3857e-04 | 3.9957 | 1.2784e-04 | 3.9960 |
| 32  | 1.0824e-05 | 3.7714 | 8.7548e-06 | 3.9844 | 8.0032e-06 | 3.9976 |

Note: Asterisk (*) stands for null data.

Example 4.2. We consider the original model (1.1) with nonhomogeneous boundary conditions

$$\frac{\partial^4 C}{\partial x^4} + \frac{1}{2} \frac{\partial^2 S}{\partial x^2} \frac{\partial^2 C}{\partial x^2} + (r - D) S \frac{\partial C}{\partial x} - r C = 0,$$
$$C(S, T) = (\ln S)^2 + (\ln S)^2 + 1,$$
$$C(0, \zeta) = (T - \zeta + 1)^2, \ C(1, \zeta) = 3(T - \zeta + 1)^2.$$

Here, the parameters are set as $T = 1$, $(S_1, S_r) = (0, 1)$, $\rho = 1$, $r = 1$, and $D = 0$.

To solve Example 4.2, we rewrite it into the form of (1.4) in order to apply the proposed scheme (3.7). It can be known by calculation that $a = 0.5$, $b = 0.5$, $c = 0.05$, $f(x, t) = (2b - c - 2cx)(t + 1)^2 - (4x + 2) \big( \frac{t^2}{16} + 2 \frac{x^2}{16} \big)$, and $\varphi(x) = x^3 + x^2 - 2x$.

The numerical results of Example 4.2 are listed in Tables 5 and 6. Since there is no exact solution for this example, we take the approximate errors $\tilde{E}_2(M, N) = \| u_M^N - u_M^N \|$ and $\tilde{E}_2(M, N) = \| u_M^N - u_M^N \|$, where $u_M^N$ is the numerical solution with mesh nodes $N, M$, and $u_M^N$ and $u_M^N$ are the numerical solutions with relative dense meshes ($N = 1,024$ and $M = 1,024$). The
temporal and spatial convergence rates are calculated respectively by

\[ Rate_t = \log_2 \left( \frac{\tilde{E}_2(M, N)}{\tilde{E}_2(2M, N)} \right) \quad \text{and} \quad Rate_h = \log_2 \left( \frac{\tilde{E}_2(M, N)}{\tilde{E}_2(2M, N)} \right). \]

By choosing appropriate mesh gradings, the numerical results of the proposed scheme for solving the Example 4.2 are displayed in Tables 5 and 6, which demonstrate that the proposed method works very well with the temporal second-order and spatial fourth-order convergence accuracies for the general time-fractional Black-Scholes equation.

**Example 4.3.** We then consider the time-fractional Black-Scholes model describing a double barrier knock-out option

\[
\frac{\partial^\alpha C}{\partial \zeta^\alpha} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S \frac{\partial C}{\partial S} - rC = 0, \quad (S, \zeta) \in (S_L, S_r) \times [0, T),
\]

\[ C(S, T) = \max\{S - K, 0\}, \]

\[ C(S_L, \zeta) = 0, \quad C(S_r, \zeta) = 0. \]
Here, the parameters are set as \( r = 0.03, \ K = 10, \ \varphi = 0.45, \ T = 1\) year, \( D = 0.01, \) and \((S_l, S_r) = (3, 15).\)

Figure 2 displays the double barrier option price of Example 4.3 solved by the proposed numerical scheme (3.7) for different \( \alpha \) and fixed \( M = N = 100, \) where the grading parameter \( \gamma = 2 \) while \( \alpha = 0.3 \) and \( \gamma = 2/\alpha \) otherwise. It is observed that the characteristics of Figure 2 are completely consistent with fig. 3 in Chen et al.\(^7\) Moreover, we can find from the figure that the double barrier option price will be influenced by the order of time-fractional derivative \( \alpha. \) Compared with the time-fractional Black-Scholes model, the classical Black-Scholes model tends to overprice the double barrier knock-out option when the underlying is close to the lower barrier. From a certain underlying value and onwards (close to the strike price \( K \)), the classical Black-Scholes model tends to underprice the price of options. It can be noticed that the smaller \( \alpha \) is, the larger pricing bias becomes. This suggests that the time-fractional Black-Scholes model may capture the characteristics of the significant movements more accurately.

**Example 4.4.** Consider the time-fractional Black-Scholes model governing European options\(^{13}\):

\[
\frac{\partial^\alpha C}{\partial \zeta^\alpha} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + (r - D)S \frac{\partial C}{\partial S} - rC = 0, \quad (S, \zeta) \in (S_l, S_r) \times (0, T).
\]

For the European call option, the initial and boundary conditions are

\[
C(S, T) = \max\{S - K, 0\}, \quad C(S_l, \zeta) = 0, \quad C(S_r, \zeta) = S_r - Ke^{-(1-\zeta)}.
\]
For the European put option, the initial and boundary conditions are

\[ C(S, T) = \max\{K - S, 0\}, \quad C(S_l, \zeta) = Ke^{-r(T-\zeta)}, \quad C(S_r, \zeta) = 0. \]

Here, the parameters are set as \( r = 0.05, K = 50, \varphi = 0.25, D = 0, T = 1 \) (year), and \((S_l, S_r) = (0,100)\).

We now apply the proposed numerical scheme (3.7) to solve Example 4.4 for different values of \( \alpha \). The curves of the call option price and put option price are plotted in Figures 3 and 4, respectively, where the grading parameter \( \gamma = 2 \), while \( \alpha = 0.3 \) and \( \gamma = 2/\alpha \) otherwise. As can be seen from them, the order of fractional derivative \( \alpha \) has an effect on the prices of European options.

5 | CONCLUSION

We proposed a high-order and nonuniform finite difference method for solving the time-fractional Black-Scholes equation. The numerical method is constructed by combining the fast nonuniform Alikhanov formula and a spatial fourth-order average approximation. The unconditional stability and convergence of second order in time and fourth order in space are rigorously derived by the energy method. Numerical examples are included, and the results indicated that the proposed numerical method works very accurately.

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CONFLICT OF INTEREST

The authors state that this work does not have any conflict of interest.

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