Topological states and braiding statistics using quantum circuits

J. Q. You,1,2 Xiao-Feng Shi,1,2 and Franco Nori2,3

1Department of Physics and Surface Physics Laboratory (National Key Laboratory), Fudan University, Shanghai 200433, China
2Advanced Study Institute, The Institute of Physical and Chemical Research (RIKEN), Wako-shi 351-0198, Japan
3Center for Theoretical Physics, Physics Department, Center for the Study of Complex Systems, University of Michigan, Ann Arbor, MI 48109-1040, USA

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Using superconducting quantum circuits, we propose an approach to construct a Kitaev lattice, i.e., an anisotropic spin model on a honeycomb lattice with three types of nearest-neighbor interactions. We study two particular cases to demonstrate topological states (i.e., the vortex and bond states) and show how the braiding statistics can be revealed. Our approach provides an experimentally realizable many-body system for demonstrating exotic properties of topological phases.

Topological quantum systems are currently attracting considerable interest because of their fundamental importance and potential applications in quantum computing. They exhibit topological phases of matter that are insensitive to local perturbations. This exotic property makes them appealing for fault-tolerant quantum computing.

Topological quantum computing can be implemented by storing and processing quantum information with anyons, which are neither bosons nor fermions, but obey anyonic braiding statistics. To achieve this, it is a centrally important issue to experimentally realize a topological quantum system. Recently, topological quantum computing using non-Abelian anyons in a fractional quantum Hall system was proposed, however, it has not been experimentally verified if the observed fractional quantum Hall states are the desired topological states for quantum computing. Indeed, up to now, anyons have never been directly observed in experiments. Thus, in order to guarantee anyonic topological phases in a many-body quantum system, one could design artificial, but exactly solvable lattice models that have such desired topological phases. The most important example is the Kitaev model on a honeycomb lattice, which is an anisotropic spin model with three types of nearest-neighbor interactions. Most strikingly, depending on the bond parameters, this model supports both Abelian and non-Abelian anyons. Therefore, if this topological model could be realized in experiments, it would provide exciting opportunities for experimentally demonstrating anyons and implementing topological quantum computing.

Proposals were made for implementing the Kitaev model using atomic optical lattices. Also, it was shown that optical lattices can be used to demonstrate the anyonic braiding statistics. However, its implementation using an optical lattice requires extremely low temperatures that are beyond current experimental capabilities. Moreover, to demonstrate anyonic braiding, the optical lattice needs to perform rotations on single atoms at selected sites using external laser beams. This is difficult because the atomic spacing is of the order of laser wavelength and the diffraction limit takes effect.

Here we propose an approach to construct a Kitaev lattice using superconducting quantum circuits. A charge qubit is placed at each site and the three types of nearest-neighbor interactions are achieved using different circuit elements. This circuit can demonstrate topological states (i.e., the vortex and bond states) and show how the braiding statistics can be revealed. Here, the quantum circuits behave like artificial spins and the Kitaev model can be constructed using the experimentally available technologies for superconducting qubits. Moreover, these charge qubits are tunable via external fields, making the required single-qubit rotations implementable for demonstrating the braiding statistics of the topological states. Our proposal provides an experimentally realizable many-body system for demonstrating exotic properties of the topological phases.

Kitaev model.—We use superconducting quantum circuits to construct an artificial Kitaev lattice (see Fig. 1). The building block of the lattice consists of four charge qubits [Fig. 1(a)], each placed at a lattice site of a hexagonal lattice [Fig. 1(b)]. Here, each charge qubit is a Cooper-pair box connected to a superconducting ring by a mutual inductance $M$. The voltage $V_i$ applied via the gate capacitance $C_g$ controls the magnetic field $\Phi_i$ piercing the SQUID loop. A circulating supercurrent $I_i$ in the $i$th charge qubit is

$$I_i = \frac{\cos(\phi_i)}{2\pi E_0} \sin(\pi \hat{\Phi}_i/\Phi_0),$$

where $\Phi_0 = h/2e$ is the flux quantum and $\phi_i$ is the phase drop across the $i$th Josephson junctions.
The mutual capacitance $C_m$ modifies the charging energy $2e^2/C_m$ of an isolated charge qubit to become $E_z = 2e^2C_m / \Lambda$.

When the electrostatic and the Josephson coupling energies of each charge qubit are included, the Hamiltonian of the lattice is $H = \sum_i [E_c(n_i - n_{gi})^2 - E_J(\Phi_i) \cos \varphi_i] + \sum_{x\text{-link}} K_x(j, k) + \sum_{y\text{-link}} K_y(j, k) + \sum_{z\text{-link}} K_z(j, k)$. For charge qubits, $E_c \gg E_J$. When the gate voltage $V_i$ is near the optimal point $e/C_q$, i.e., $n_{gi} \sim \frac{1}{2}$, only two charge states $|0\rangle_i$ and $|1\rangle$, corresponding to zero and one extra Cooper pairs in the box, are important for each qubit. In the spin-$\frac{1}{2}$ representation, based on the charge states $|0\rangle_i \equiv |\uparrow\rangle_i$ and $|1\rangle_i \equiv |\downarrow\rangle_i$, one has $n_i = \frac{1}{2}(1 - \sigma_i^z)$, $\cos \varphi_i = \frac{1}{2}\sigma_i^x$, and $\sin \varphi_i = \frac{1}{2}\sigma_i^y$. Here we consider the simple case with $n_{gi} = n_g$ (i.e., $V_i = V_g$) and $\Phi_i = \Phi_e$ for all qubits. The Hamiltonian of the system is then reduced to

$$H = J_z \sum_{x\text{-link}} \sigma_j^z \sigma_k^z + J_y \sum_{y\text{-link}} \sigma_j^y \sigma_k^y + J_z \sum_{z\text{-link}} \sigma_j^z \sigma_k^z + \sum_i (\hbar \sigma_i^x + h_x \sigma_i^y).$$

This is the Kitaev model on a honeycomb lattice, in the presence of a “magnetic field” with $z$- and $x$-components. Here $h_z = (E_c + \frac{1}{2}E_m)(n_g - \frac{1}{2})$, and $h_x = -\frac{1}{2}E_J(\Phi_e)$, with $E_J(\Phi_e) = 2E_J \cos(\pi \Phi_e / \Phi_0)$. The bond parameters are $J_z = \frac{1}{4}M \ell^2 \sin^2(\pi \Phi_0 / \Phi_0) \geq 0$, $J_y = -\xi (E_J(\Phi_e))^2 < 0$, and $J_z = \frac{1}{4}E_m > 0$. The magnetic field can be used to achieve single-qubit rotations for demonstrating anyonic braiding statistics.

**Topological excitations.**—Below we focus on two particular cases to show the properties of topological excitations in the Kitaev model (1):

(i) **Kitaev lattice in a weak magnetic field.** We first consider the case with $|h_z|, |h_x| \ll J_z, |J_y|, J_z$, and choose $V = \sum_i (h_x \sigma_i^x + h_x \sigma_i^x)$ as the perturbation. Using perturbation theory in the Green function formalism, one can construct an effective Hamiltonian $H_{\text{eff}}$ acting on the vortex-free sector: $H_{\text{eff}} = -(2h_z^2 / \Delta z) \sum_{x\text{-link}} \sigma_j^z \sigma_k^z - (2h_x^2 / \Delta z) \sum_{x\text{-link}} \sigma_j^x \sigma_k^x$, where $\delta z(x)$ is the excitation energy of the state $\sigma_j^z \sigma_k^z |\rangle$, and $|\rangle$ is the eigenstate of all plaquette operators $W_p$ [see Fig. [1]b] corresponding to the eigenvalue $w_p = 1$. Here the effective Hamiltonian $H_{\text{eff}}$ is only contributed by the second-order term because both the first- and third-order terms are zero. With the zeroth-order term (unperturbed Hamiltonian) included, the Hamiltonian of the system can be equivalently written as $H = J_z' \sum_{x\text{-link}} \sigma_j^z \sigma_k^z + J_y' \sum_{y\text{-link}} \sigma_j^y \sigma_k^y + J_z' \sum_{z\text{-link}} \sigma_j^z \sigma_k^z$, where the bond parameters $J_z'$ and $J_x'$ are renormalized to $J_z' = J_z - 2h_z^2 / \Delta z$, and $J_x' = J_x - 2h_x^2 / \Delta z$.

To show the topological excitations, we focus on the Abelian case [3] with $J_x' \gg J_z', J_y'$. The dominant part of $H$, i.e., $H_0 = J_z' \sum_{x\text{-link}} \sigma_j^z \sigma_k^z$, favors a highly degenerate
spin-pair operator on the ground state of neighboring plaquettes. This can be achieved by acting a pair of vortex excitations are Abelian anyons \([2, 4]\), which have an energy gap \(\Delta \varepsilon = 4J_{\text{eff}}\) above the ground state.

(ii) Kitaev lattice with dominant \(z\)-bonds. We choose \(h_z = 0\), which corresponds to the case with each charge qubit working at the optimal point \(n_g = \frac{1}{2}\). Moreover, we choose suitable circuit parameters to have \(J_z \gg |h_x|, J_z, |J_y|\), with \(|h_x|\) comparable to \(J_z, |J_y|\). Here we also use perturbation theory in the Green function formalism to derive the effective Hamiltonian. Up to the second-order term, the effective Hamiltonian can be written as

\[
H_{\text{eff}} = -K_{\text{eff}} \sum_{z-\text{link}} \sigma^z_i \sigma^z_k,
\]

with \(K_{\text{eff}} = h_z^2 / J_z\). The spin-pair operator \(\sigma^z_i \sigma^z_k\) at a \(z\)-bond is commutative with the unperturbed zero-order Hamiltonian \(H_0 = J_z \sum_{z-\text{link}} \sigma^x_i \sigma^x_k\), but anticommutative to the four plaquette operators \(W_p\) connected to this \(z\)-bond. Similar to \(W_p\), the pair operator \(\sigma^z_i \sigma^z_k\) also has two eigenvalues \(p_z = \pm 1\). Now the ground state \(|g\rangle\) of the system preserves \(p_z = 1\), i.e., \(\sigma^z_i \sigma^z_k \langle g| g\rangle\) for all \(z\)-bonds. When the pair operators \(\sigma^z_i\) and \(\sigma^z_k\) are separately applied to the ground state at the \(i\)th \(z\)-bond, the excited states \(|Z_i\rangle = \sigma^z_i |g\rangle\), and \(|Y_i\rangle = \sigma^z_k |g\rangle\) are two types of bond states; each corresponding to the change \(p_z = 1 \rightarrow -1\) at the \(i\)th \(z\)-bond and having an energy gap \(2K_{\text{eff}}\) above the ground state.

The braiding of topological excitations.—As shown in \([6]\), an \(e\)-type vortex looping around another \(e\) vortex does not produce a sign change to the wave function, while an \(e\) vortex looping around an \(m\)-type vortex does. This indicates anyonic statistics between the \(e\) and \(m\) vortex states. Here we show an alternative procedure for braiding a topological excitation with another, which applies to both the vortex and bond states.

First, successively apply spin-pair operations \(\tilde{\sigma}^z_{i} = \sigma^z_{i} \sigma^z_{k}\) to the ground state \(|g\rangle\) at three \(z\)-bonds in the vertical path \(P_v\) [see Fig. 2(a)]. In the case of Hamiltonian \([2]\), this creates a pair of \(e\) vortex states and move a vortex along the vertical path \(P_v\), while a bond state is generated in the case of Hamiltonian \([4]\). Then, we successively apply the operations \(\tilde{\sigma}^z_{i} = \sigma^z_{i} I_1\) at four \(z\)-bonds along the horizontal path \(P_h\) [see also Fig. 2(a)]. After these operations, the state of the system is \(U_h U_v |g\rangle\), where \(U_h = \tilde{\sigma}^z_1 \tilde{\sigma}^z_2 \tilde{\sigma}^z_3 \tilde{\sigma}^z_4\) and \(U_v = \tilde{\sigma}^z_5 \tilde{\sigma}^z_6 \tilde{\sigma}^z_7 \tilde{\sigma}^z_8\) correspond to the operations along the horizontal and vertical paths. Furthermore, successively apply \(U_v^{-1}\) and \(U_h^{-1}\) to the system, so as to fuse \([1, 3]\) the excitations to the vacuum (i.e., the ground state) [see Fig. 2(b)]. Now, the final state of the system is \(|\Psi_f\rangle = U_h^{-1} U_v^{-1} U_h U_v |g\rangle\). Because the paths \(P_v\) and \(P_h\) intersect at a lattice point, where \(\tilde{\sigma}^z_2\) and \(\tilde{\sigma}^z_5\) anticommute, one has \(U_h U_v = -U_v U_h\). Therefore, the final

\[
H_{\text{eff}} = -J_{\text{eff}} \sum_p W_p,
\]

where \(J_{\text{eff}} = J^2 x y / 16 J^3 z\). Clearly, the ground state of the system preserves \(w_p = 1\), i.e., \(W_p |g\rangle = |g\rangle\), for all plaquettes \(p\). A pair of vortex excitations are created when the system changes with \(w_p = 1 \rightarrow -1\) for two neighboring plaquettes. This can be achieved by acting a spin-pair operator on the ground state \(|g\rangle\): \(|Z_i\rangle = \tilde{\sigma}^z_i |g\rangle\), and \(|Y_i\rangle = \tilde{\sigma}^z_i |g\rangle\), with \(\tilde{\sigma}^z_i \equiv \sigma^z_i I_1\) and \(\tilde{\sigma}^z_i \equiv \sigma^z_i \sigma^z_i\). Here the two operators \(\tilde{\sigma}^z_i\) and \(I_1\) act on the ground state \(|g\rangle\) at the bottom and top sites of the \(i\)th \(z\)-link, respectively. This pair of vortex excitations can be braided using the following procedures.

FIG. 2: (Color online) Schematic diagram of the procedures for braiding topological excitations. (a) The operations \(U_h\) and \(U_v\) for creating different types of topological excitations, which are achieved by successively applying spin-pair operators at \(z\)-bonds along the horizontal (\(P_h\)) and vertical (\(P_v\)) paths. (b) A combined operation \(U_h^{-1} U_v^{-1} U_h U_v\) for braiding the topological excitations created in (a) and fusing them to the vacuum. (c) The operations \(U_h\) and \(U_v\) for creating the same types of topological excitations, which are achieved by successively applying spin-pair operators at \(z\)-bonds along \(P_h\) and \(P_v\).
state becomes $|\Psi_f\rangle = -|g\rangle$. In contrast, for two topological excitations of the same type, when similar operations are applied, the paths $P_1$ and $P_2$ do not intersect at a lattice point [see Fig. 2(c)]. Thus, $U_hU_v = U_vU_h$, and $|\Psi_f\rangle = |g\rangle$, yielding no sign change to the ground-state wave function.

The braiding statistics of topological excitations can be revealed by means of Ramsey-type interference \cite{1,12}. To achieve this, we keep the same $U_v$, as above, but use $U_h = (\tilde{\sigma}_x^0)^2(\tilde{\sigma}_x^0)^2(\tilde{\sigma}_x^0)^2(\tilde{\sigma}_x^0)^2$, where $(\tilde{\sigma}_x^0)^2 \equiv (\tilde{\sigma}_x^0)^2I$, i.e., each $\sigma_i^x$ is replaced by half of the rotation. In the braiding case shown in Fig. 2(a), $\tilde{\sigma}_x^2 = i(\tilde{\sigma}_x^0)^2i(\tilde{\sigma}_x^0)^2\tilde{\sigma}_x^0$ at the crossing point of paths $P_h$ and $P_v$. Thus, $|\Psi_f\rangle = U_v^{-1}U_hU_v|g\rangle = (\tilde{\sigma}_x^0)^2i(\tilde{\sigma}_x^0)^2|g\rangle = i(\tilde{Z}_2)$, similar to the case with an $e$ vortex looping around a superposition state of an $m$ vortex and the vacuum \cite{3}. However, in the case without braiding [see Fig. 2(b)], $|\Psi_f\rangle = U_v^{-1}U_hU_v|g\rangle = |g\rangle$. Therefore, the braiding of topological excitations can be distinguished by verifying if an excited state $|\tilde{Z}_2\rangle$ occurs at the crossing point of paths $P_h$ and $P_v$.

**Discussion and conclusion.**—When the magnetic flux in the SQUID loop of each charge qubit is set to $\Phi = \Phi_0/2$, then $h_z = 0$ and $J_y = 0$. Also, we assume that $E_c \gg J_x,J_z$. When $\Phi_e = \Phi_0/2$, one can shift the gate voltage, at the $i$th lattice point, far away from the usual working point $n_g \sim \frac{1}{2}$ of the Kitaev lattice for a period of time $\tau$. This yields a local $z$-type rotation on the $i$th qubit: $R^z_i(\theta) = \exp[-i(h_z\tau/h)|\sigma_i^z|] = \exp(-i\theta|\sigma_i^z|/2)$. When $\theta \equiv 2h_z\tau/h = \pi$ (where $n_g > \frac{1}{2}$), $R^z_i(\pi) = -i\sigma_i^z$, so the $\sigma_i^z$ operation on the $i$th qubit is given by $\sigma_i^z = e^{i\pi/2}R^z_i(\pi)$, while half of the rotation is $(\sigma_i^z)^2 = e^{i\pi/4}R^z_i(\pi/2)$. The corresponding inverse rotations can be achieved by shifting the gate voltage to $n_g < \frac{1}{2}$. Similarly, when $n_g = \frac{1}{2}$ and $\Phi_e = 0$, one has $h_z = 0$, $h_x = -E_J$, and $J_y = 0$. Here we now assume that $E_J \gg |J_{xy}|,|J_{xz}|$. When $n_g = \frac{1}{2}$, we also switch off the flux in the SQUID loop of the $i$th qubit for a time $\tau$ (the working point of this Kitaev lattice is usually at $0 < \Phi_e < \Phi_0/2$). This produces a local $x$-type rotation on the $i$th qubit: $R^x_i(\theta) = \exp[i(\delta E_J\tau/h)|\sigma_i^x|] = \exp(i\theta|\sigma_i^x|/2)$, where $\delta E_J = E_J - \frac{1}{2}E_J(\Phi_e)$. The $\sigma_i^x$ rotation on the $i$th qubit is $\sigma_i^x = e^{-i\pi/2}R^x_i(\pi)$, where $2\delta E_J\tau/h = \pi$. With both $\sigma_i^x$ and $\sigma_i^z$ rotations available for the $i$th qubit, the $\sigma_i^y$ rotation is given by $\sigma_i^y = e^{i\pi/2}\sigma_i^x\sigma_i^z$. Therefore, one can construct the operations $\tilde{\sigma}_i^x$ and $\tilde{\sigma}_i^z$ for generating topological excitations by using the single-qubit rotations $\sigma_i^x$ and $\sigma_i^z$.

In order to obtain accurate $z$- and $x$-type single-qubit rotations, we assume that $E_c$ and $E_J$ are much larger than the inter-qubit coupling. Actually, this strict condition can be loosened for realistic systems. As shown in \cite{12}, accurate effective single-qubit rotations can still be achieved using techniques from nuclear magnetic resonance when the inter-qubit coupling is small (instead of much smaller than $E_c$ and $E_J$). In the vortex- and bond-state cases studied here, to reveal the braiding statistics of topological excitations, one should verify the occurrence of the excited state $|\tilde{Z}_2\rangle$ at the crossing point of paths $P_h$ and $P_v$. This needs to distinguish $|\tilde{Z}_2\rangle$ from the ground state. Here the difference between $|\tilde{Z}_2\rangle$ and the ground state is the phase flip, induced by the single-qubit rotation $\tilde{\sigma}_i^x$, on the charge state of the $z$-bond at the crossing point. This phase flip of the charge state could be measured, e.g., using state tomography \cite{14}.

In the region $J_z' \geq J_x',|J_y'|$, it is shown \cite{15} that when acting on the ground state, the spin-pair operations $\tilde{\sigma}_i^x$ and $\tilde{\sigma}_i^y$ generally create both vortex states (Abelian anyons) and fermionic excitations. However, the weight of the anyons is dominant in the low-energy subspace if the number $m$ of the spin-pair operations for braiding anyons is not large and $J_z' \gg J_x',|J_y'|$. Let $J_z' = |J_y'| = J$. For example, when $m = 25$, $\gamma \equiv J/J_z' \sim 0.2$ \cite{15}.

In conclusion, we have proposed an approach to realize the Kitaev model on a honeycomb lattice using superconducting quantum circuits. Two particular cases are studied to demonstrate the topological states and the braiding statistics. Our approach provides an experimentally realizable many-body system for demonstrating exotic properties of these topological phases.

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