BOOSTING AN ANALOGUE OF JORDAN’S
THEOREM FOR FINITE GROUPS

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Abstract. Let $C$ be a set of finite groups which is closed under
taking subgroups and let $d$ and $M$ be positive integers. Suppose
that for any $G \in C$ whose order is divisible by at most two distinct
primes there exists an abelian subgroup $A \subseteq G$ such that $A$
is generated by at most $d$ elements and $[G : A] \leq M$. We prove
that there exists a positive constant $C_0$ such that any $G \in C$
has an abelian subgroup $A$ satisfying $[G : A] \leq C_0$, and $A$
can be generated by at most $d$ elements. We also prove some related
results. Our proofs use the Classification of Finite Simple Groups.

1. Introduction

A celebrated theorem of C. Jordan [15] states the following.

Theorem (C. Jordan). For any $d$ there exists a constant $C$
such that any finite subgroup $G$ of $\text{GL}(d, \mathbb{C})$ has an abelian subgroup $A \subseteq G$ of
index at most $C$, and $A$ can be generated by at most $d$ elements.

The claim on the number of generators of $A$ is not usually part of the
statement of Jordan’s theorem, but from the perspective of this paper
it is a natural complement to the usual statement. To prove it, note
that if $A$ is a finite abelian subgroup of $\text{GL}(d, \mathbb{C})$ then, by simultaneous
diagonalization, $A$ is isomorphic to a subgroup of the group $D$
of $d \times d$ diagonal matrices whose diagonal entries are $|A|$-th roots unity; the
group $D$ is abelian and can be generated by at most $d$ elements, so any
subgroup of $D$ can be generated by at most $d$ elements.

In an alternative version of Jordan’s theorem, it is claimed that fi-
nite subgroups of $\text{GL}(d, \mathbb{C})$ have a normal abelian subgroup of index
bounded above by a constant depending only on $d$. However, since
$[G : A] = n$ implies $[G : \cap_{g \in G} g^{-1}Ag] \leq n!$, this version and the one we
stated are equivalent, up to using a different bound $C$. We note that,

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using the Classification of Finite Simple Groups, M. Collins [7] gives, for all $d$, the best possible bound for this alternative version.

Our aim in this paper is to provide a tool to prove the conclusion of Jordan’s Theorem for certain classes of finite groups. To put our result in context, we first review some extensions and analogues of Jordan’s theorem.

Jordan’s theorem can be generalized taking instead of $\text{GL}(d, \mathbb{C})$ any (finite dimensional) Lie group $R$ with a finite number of connected components: for any such $R$ there exist constants $C(R)$ and $d(R)$ with the property that any finite subgroup of $R$ has an abelian subgroup of index at most $C(R)$ which can be generated by at most $d(R)$ elements. This follows from the existence and uniqueness up to conjugation of maximal compact subgroups (see [13, Theorem 14.1.3]), Peter–Weyl’s theorem (see [4, Chap. III, §4, Theorem 4.1]), and the above version of Jordan’s theorem (see also [2]).

Similar to Jordan’s theorem, a result of Brauer and Feit [3] states that if $K$ is a field of characteristic $p > 0$ then any finite subgroup $G$ of $\text{GL}(d, K)$ has an abelian subgroup whose index is bounded above by a constant depending only on $d$ and on the size of the $p$-Sylow subgroups of $G$. Work of M. Collins [8] provides an interesting different modular analogue of Jordan’s Theorem.

Remarkably, the conclusion of Jordan’s theorem is also known or expected to be true when $\text{GL}(d, \mathbb{C})$ is replaced by some particular much bigger groups. Two notable examples are Serre’s conjecture on the Cremona group (see [26, §6.1]; some partial results appear in [26, Theorem 5.3], [25, 27]; see also [30]), and Ghys’s conjecture on the diffeomorphism group of smooth compact manifolds (see [11, Question 13.1] and, for some partial results, [31, §5] and [21]). In both cases, the big group on whose finite subgroups one is interested can be suitably understood as an infinite dimensional Lie group (see [9] for the case Cremona groups and [20] for the case of diffeomorphism groups). See [24] for a nice survey on these questions, mostly centered on Serre’s conjecture and some natural extensions of it.

Most of these statements and conjectures share the following pattern: one is given a set of finite groups $C$, closed under taking subgroups, and one proves (or wants to prove) that there exist constants $C_0$ and $d$ such

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1The statement of Ghys’s conjecture as written in [11] does not mention any bound on the number of generators of the abelian subgroup; the existence of such a bound (once the abelian subgroup has been proved to exist) follows from [19].
that any $G \in \mathcal{C}$ has an abelian subgroup of index at most $C_0$ which can be generated by at most $d$ elements. It will be useful for us to encode this property in a definition.

**Definition.** Let $\mathcal{C}$ be a set of finite groups, and let $C_0$ and $d$ be positive integers. We say that $\mathcal{C}$ satisfies the Jordan property $\mathcal{J}(C_0, d)$ if for every element $G \in \mathcal{C}$ there exists some abelian subgroup $A$ of $G$ such that $[G : A] \leq C_0$ and $A$ can be generated by at most $d$ elements.

A similar notion has been introduced and studied by V.L. Popov [23, 24], according to which a group $G$ is a Jordan group if there exists a constant $C$ such that any finite subgroup of $G$ has an abelian subgroup of index at most $C$. In [23] a group $G$ is said to have Jordan property if it is a Jordan group in this sense (this terminology is also used by Y. Prokhorov and C. Shramov in [25]).

More generally, we say that a set of finite groups $\mathcal{C}$ satisfies the Jordan property if there exist numbers $C_0$ and $d$ such that $\mathcal{C}$ satisfies $\mathcal{J}(C_0, d)$. The main result of this paper implies that if a set of finite groups $\mathcal{C}$ is closed under taking subgroups then, in order to check whether $\mathcal{C}$ satisfies the Jordan property, it suffices to consider the subset of $\mathcal{C}$ consisting of groups whose cardinal has at most two different prime divisors: if this subset has the Jordan property, then so does $\mathcal{C}$, although possibly with different constants. (Our main result is in fact slightly stronger than this.)

In order to give a precise statement, we introduce the following notation. If $\mathcal{C}$ is a set of finite groups, we denote by

$$\mathcal{T}(\mathcal{C}) \subset \mathcal{C}$$

the set of all $T \in \mathcal{C}$ such that there exist primes $p$ and $q$, a Sylow $p$-subgroup $P$ of $T$, and a normal Sylow $q$-subgroup $Q$ of $T$, such that $T = PQ$. (In particular, $T \in \mathcal{T}(\mathcal{C})$ implies $|T| = p^\alpha q^\beta$ for some primes $p$ and $q$ and some nonnegative integers $\alpha, \beta$).

Our main result is the following theorem (Theorem 3.8).

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*Note that in the literature on permutation groups there exists a concept called *Jordan group*, which was introduced by W.M. Kantor [16] (see also [5, Chap. 6, §6.8]) and which is different from the concept of Jordan group introduced by Popov. The only thing in common between the two notions seems to be that they are both inspired on theorems of C. Jordan — but not on the same one! (see [5, Chap. 6, Theorem 6.15] for the theorem of Jordan on which Kantor’s terminology is based).*
**Theorem.** Let $d$ and $M$ be positive integers. Let $\mathcal{C}$ be a set of finite groups which is closed under taking subgroups and such that $\mathcal{T}(\mathcal{C})$ satisfies the Jordan property $\mathcal{J}(M, d)$. Then there exists a positive integer $C_0$ such that $\mathcal{C}$ satisfies the Jordan property $\mathcal{J}(C_0, d)$.

This theorem is used in [22] to prove Ghys’s conjecture for manifolds without odd cohomology.

Our proof makes use of the Classification of Finite Simple Groups. While our proof would provide a specific value for the constant $C_0$ in terms of $M$ and $d$, we make no attempt to make it explicit or to find the best possible value.

The following corollary implies that the number $C_0$ in our main theorem can be chosen to depend only on $M$ and $d$, but not on the particular set $\mathcal{C}$. We use the following notation. If $G$ is a finite group, then $\mathcal{S}(G)$ denotes the set of all subgroups of $G$, and we define $\mathcal{T}(G) = \mathcal{T}(\mathcal{S}(G))$. We also denote by $\mathcal{A}_d(G)$ the set of abelian subgroups of $G$ which can be generated by at most $d$ elements.

**Corollary 1.1.** Given positive integers $d$ and $M$ there exists an integer $C_0$ with the following property. Let $G$ be any finite group. Suppose that for any $T \in \mathcal{T}(G)$ there exists some $A \in \mathcal{A}_d(T)$ satisfying $[T : A] \leq M$. Then there exists some $B \in \mathcal{A}_d(G)$ satisfying $[G : B] \leq C_0$.

**Proof.** Suppose the corollary is false. Then, there exist some integers $d$ and $M$ such that, for them, there does not exist any integer $C_0$ with the property specified in the statement. This implies that there exists a sequence of finite groups $G_1, G_2, G_3, \ldots$ such that for any $i$ and any $T \in \mathcal{T}(G_i)$ there exists some $A \in \mathcal{A}_d(T)$ satisfying $[T : A] \leq M$ and, if we define

$$C(G_i) := \inf \{[G_i : B] \mid B \in \mathcal{A}_d(G_i)\},$$

then $C(G_i) \to \infty$. Let $\mathcal{C} = \bigcup_i \mathcal{S}(G_i)$. Then $\mathcal{T}(\mathcal{C})$ satisfies $\mathcal{J}(M, d)$ but $\mathcal{C}$ does not satisfy $\mathcal{J}(C_0, d)$ for any value of $C_0$. This contradicts our main theorem, so the corollary is proved.

One can obtain variations on the theme of our main theorem and the previous corollary replacing $\mathcal{T}(\mathcal{C})$ by any subset of $\mathcal{C}$ containing $\mathcal{T}(\mathcal{C})$. Two natural choices are the following ones:

$$\mathcal{T}_0(\mathcal{C}) = \{T_0 \in \mathcal{C} : T_0 \text{ is a } \{p, q\}-\text{group for some primes } p \text{ and } q\},$$

$$\mathcal{S}ol(\mathcal{C}) = \{S \in \mathcal{C} : S \text{ is solvable}\}.$$
We have inclusions $\mathcal{T}(\mathcal{C}) \subset \mathcal{T}_0(\mathcal{C})$ and $\mathcal{T}(\mathcal{C}) \subset \text{Sol}(\mathcal{C})$: the first one is obvious and the second one is an easy exercise (in fact, Burnside’s $p^n q^\beta$-theorem, see e.g. [14, Theorem 7.8], implies that we also have an inclusion $\mathcal{T}_0(\mathcal{C}) \subset \text{Sol}(\mathcal{C})$). Hence, combining our main theorem and Corollary 1.1 we obtain immediately the following.

**Corollary 1.2.** Given positive integers $d$ and $M$ there exists an integer $C_0$ with the following property. Let $\mathcal{C}$ be a set of finite groups which is closed under taking subgroups. If $\text{Sol}(\mathcal{C})$ satisfies the Jordan property $J(M, d)$, then $\mathcal{C}$ satisfies the Jordan property $J(C_0, d)$.

The same corollary holds true replacing $\text{Sol}(\mathcal{C})$ by $\mathcal{T}_0(\mathcal{C})$.

A natural question is whether one could strengthen our main theorem replacing $\mathcal{T}(\mathcal{C})$ by some (in general) smaller subset. Although we can not answer completely this question at present, we can at least prove that it is not possible to replace $\mathcal{T}(\mathcal{C})$ by

$\mathcal{P}(\mathcal{C}) = \{P \in \mathcal{C} : P \text{ is a } p\text{-group for some prime } p\}$.

To justify this claim, let us denote by $G_p$ the group of affine transformations of the affine line over the finite field $F_p$, where $p$ is any prime. (In particular, $|G_p| = p(p-1)$.) Let

$\mathcal{L} = \{G \mid G \text{ is a subgroup of } G_p \text{ for some prime } p\}$.

The set $\mathcal{L}$ does not satisfy the Jordan property $J(C, d)$ for any $C$ and any $d$ (this follows from Lemma 2.8 below), and yet all elements of $\mathcal{P}(\mathcal{L})$ are abelian and cyclic. Indeed, for each $p$ we have an exact sequence $0 \to F_p^* \to G_p \to F_p^* \to 1$, where $F_p^*$ is the multiplicative group of units and $F_p$ is the additive group; both $F_p$ and $F_p^*$ are cyclic and their orders are coprime, so all Sylow subgroups of $G_p$ are abelian and cyclic. Hence, $\mathcal{P}(\mathcal{L})$ satisfies the Jordan property $J(1, 1)$, but $\mathcal{L}$ does not satisfy the Jordan property $J(C, d)$ for any $C$ and any $d$.

We close this introduction with a remark on style. This is a paper on finite groups which, we hope, will also be of interest to mathematicians whose main expertise is outside finite group theory. With these readers in mind and to make the paper generally easier to read, we give detailed and complete references to more of the results we use than we normally would if we were writing for an audience of only finite group theorists. An excellent reference for the basic notions and results on finite groups which we use is [14].

After completing this paper, we were informed by László Pyber that he had independently obtained some results related to the ones in this paper, but that his results have not yet appeared in print.
2. Preliminary lemmas

Recall that a quasisimple group is a finite group $S$ such that $S/Z(S)$ is a non-abelian simple group and $S$ is perfect. Here a finite group $S$ is perfect if $S' = S$, that is, $S$ has no nontrivial abelian homomorphic images. Following standard conventions as, for example, in [14], we define the layer $E(G)$ of a finite group $G$ to be the product of all the quasisimple subnormal subgroups of $G$ (the latter are called the components of $G$). For two subgroups $A$ and $B$ of a finite group $G$, we denote by $[A, B]$ the commutator subgroup of $A$ and $B$. We recall [14] Lemma 4.3 that $A$ normalizes $B$ if and only if $[A, B] \subseteq B$.

**Lemma 2.1.** Let $G$ be a finite group, let $G_1 = C_G(E(G))$, let $p$ be a prime, and let $P$ be an abelian Sylow $p$-subgroup $G_1$. Assume that $P$ is not a normal subgroup of $G_1$. Then there exists a prime $q$ and a nontrivial $q$-subgroup $Q$ of $G_1$ such that $p \neq q$ and $[P, Q] = Q$. (In particular, $P$ normalizes $Q$.)

**Proof.** Notice that $F(G)$, the Fitting subgroup of $G$, commutes with $E(G)$ (for example [14 Theorem 9.7]) so that $F(G) \subseteq G_1$, and more precisely $F(G) \subseteq F(G_1)$. Since $G_1$ is a normal subgroup of $G$ and $F(G_1) \subseteq G_1$ is characteristic, we also know that $F(G_1) \subseteq F(G)$, and therefore we have $F(G) = F(G_1)$. Denote as usual by $O_p(G)$ the largest normal $p$-subgroup of $G$. Notice that $O_p(G)$ is the Sylow $p$-subgroup of $F(G)$, and it is a normal subgroup of $G_1$. Since $P$ is a Sylow $p$-subgroup of $G_1$, we have $O_p(G) \subseteq P$. In particular, since $P$ is abelian, $P$ centralizes $O_p(G) = O_p(G_1)$. Suppose that $P \not\subseteq F(G)$. Then $P$ is a Sylow $p$-subgroup of $F(G)$, and this implies that $P = O_p(G)$, contradicting the fact that $P$ is not a normal subgroup of $G_1$. It follows that $P \not\subseteq F(G) = F(G_1)$. We claim that $E(G_1)$ is trivial. Otherwise, $G_1$ contains at least one component. Any quasisimple subnormal subgroup of $G_1$ is contained in $E(G)$. Since $G_1$ is also in the centralizer of $E(G)$ in $G$, this implies that any such subnormal subgroup is in $Z(E(G))$, which implies that it is solvable, a contradiction. Therefore, $E(G_1) = 1$ as claimed. In particular, the generalized Fitting subgroup of $G_1$ is $F(G)$. By for example [14 Corollary 9.9], this implies that there exists some prime $q$ such that $[P, O_q(G)] \neq 1$. By the remark above, $p \neq q$. Set $Q = [P, O_q(G)]$. It follows by, for example, [14 Lemma 4.29] that $Q = [P, Q]$. The lemma follows. 

**Lemma 2.2.** Let $H = PQ$ be a finite group which is the product of a $p$-subgroup $P$ and a normal subgroup $Q$, where $p$ is a prime, and $[P, Q] = Q$. Let $J$ be any proper subgroup of $H$. Then $[H : J] \geq p$. 

Proof. Let \( \Omega \) be the set of left cosets of \( J \) in \( H \). Then \(|\Omega| > 1\), and the action by left multiplication provides a group homomorphism

\[ \rho : H \to S_\Omega \]

from \( H \) to the symmetric group on \( \Omega \). Since the image of \( \rho \) is transitive, this image is not trivial. Now suppose \(|\Omega| < p\). Then \( \rho(P) \) is trivial since the Sylow \( p \)-subgroup of \( S_\Omega \) is trivial, and it follows from \( H = P[P, Q] \) that \( \rho(H) \) is trivial. This is a contradiction. Hence, the lemma holds. \( \square \)

Recall that the Frattini subgroup \( \Phi(G) \) of a finite group \( G \) is the intersection of all the maximal subgroups of \( G \).

Lemma 2.3. Let \( G \) be a finite group. Suppose that \( E(G) = 1 \). Then \( F(G)/\Phi(F(G)) \) is an abelian group of square-free exponent and

\[ C_G(F(G)/\Phi(F(G))) = F(G). \]

Proof. \( F(G) \) is the direct product of all the \( O_p(G) \) for all primes \( p \). It follows that \( \Phi(F(G)) \) is the direct product of all the \( \Phi(O_p(G)) \) for all primes \( p \). Therefore \( F(G)/\Phi(F(G)) \) is isomorphic to the direct product of the \( O_p(G)/\Phi(O_p(G)) \) for all primes \( p \). By, for example, [14, 1D.8], we have that \( O_p(G)/\Phi(O_p(G)) \) is elementary abelian, and it follows that \( F(G)/\Phi(F(G)) \) is abelian of square-free exponent.

Set

\[ C_1 = C_G(F(G)/\Phi(F(G))). \]

It is clear that \( C_1 \) is a normal subgroup of \( G \) such that \( F(G) \subseteq C_1 \). Let us assume that \( F(G) \neq C_1 \). Then the set \( \mathcal{S} \) of subnormal subgroups of \( G \) contained in \( C_1 \) and not contained in \( F(G) \) is nonempty, because \( C_1 \in \mathcal{S} \). Let \( C_2 \) be minimal among the elements of \( \mathcal{S} \).

Suppose that \( C_2F(G)/F(G) \) is not abelian. Then \( C_2F(G)/F(G) \) is a non-abelian simple group. Let \( \Gamma = C_2F(G)/F(G) \). We have inclusions \( \Gamma' \subset C_2'F(G)/F(G) \subseteq C_2F(G)/F(G) = \Gamma \) and, since \( \Gamma' = \Gamma \), we have \( C_2'F(G)/F(G) = C_2F(G)/F(G) \). This implies that \( C_2' \nsubseteq F(G) \), because \( C_2' \nsubseteq F(G) \). Hence, \( C_2' \in \mathcal{S} \), so, by the minimality of \( C_2 \), we have that \( C_2 = C_2' \) and \( C_2 \) is perfect. Let \( p \) be any prime. Then \( C_2 \) acts trivially on \( O_p(G)/\Phi(O_p(G)) \). Let \( q \) be a prime divisor of \( |C_2F(G)/F(G)| \) with \( p \neq q \), and let \( Q \) be a Sylow \( q \)-subgroup of \( C_2 \). Then by, for example, [14, 3D.4], we have that, since \( Q \) centralizes \( O_p(G)/\Phi(O_p(G)) \), we also have that \( Q \) centralizes \( O_p(G) \). This implies

\[ \text{By convention, maximal subgroup means a subgroup which is maximal among the proper subgroups.} \]
that $C_{C_2}(O_p(G))$ is a subnormal subgroup of $G$ contained in $C_2$, and, since $Q \subseteq C_{C_2}(O_p(G))$, $C_{C_2}(O_p(G))$ is not contained in $F(G)$. Again by the minimality, we obtain that $C_2 = C_{C_2}(O_p(G))$. Since this happens for all primes $p$, it follows that $C_2$ centralizes $F(G)$. Since $E(G) = 1$, by [14, Theorem 9.8] we deduce that $C_2 \subseteq F(G)$, which contradicts the definition of $C_2$.

Hence, $C_2F(G)/F(G)$ is abelian. By the minimality of $C_2$, we have that $C_2F(G)/F(G)$ has prime order, and we set $p = |C_2F(G)/F(G)|$. Let $P$ be a Sylow $p$-subgroup of $C_2$. Then, by the argument above, for every prime $q$ with $p \neq q$, we have that $P$ centralizes $O_q(G)$. We also have that $PO_p(G)$ centralizes $O_q(G)$. Since $PO_p(G)$ is a Sylow $p$-subgroup of $C_2F(G)$, it follows that $PO_p(G)$ is normalized by a Sylow $r$-subgroup of $C_2F(G)$ for every prime $r$, so that $PO_p(G)$ is a normal $p$-subgroup of $PF(G)$. This implies that $PO_p(G)$ is a subnormal $p$-subgroup of $G$ and this implies, by for example [14, Theorem 2.2], that $P \subseteq F(G)$. This final contradiction completes the proof of the lemma.

The following lemmas will be used later (in Lemma 3.5) to prove that if a set of groups $C$ satisfies the hypothesis of our main theorem, then the isomorphism classes of groups appearing as components of elements of $C$ form a finite collection. The crucial ingredient is to control the possible non-abelian simple groups, and for that we will use the Classification of the Finite Simple Groups [12, 29].

The classification provides an infinite list of finite simple groups such that every finite simple group is isomorphic to a member of this list. There are unfortunately discrepancies with the notation for finite simple groups in the literature. In this paper, we follow the notation in [12] for the list of finite simple groups. Table I in p. 8 of [op.cit.] also provides a list of alternative notations for these finite simple groups. While simple groups on this list can be isomorphic to other groups on the list in some cases, these few isomorphisms are all listed in [12, Table II, p. 10]. The table begins with the familiar abelian simple groups of prime order $\mathbb{Z}_p$, and the alternating groups $A_n$. This is followed by 16 families of groups which all depend on a parameter $q$ (and some also on another parameter $n$): these are the finite simple groups of Lie type. The table is then followed by the 26 sporadic simple groups. These cover, up to isomorphism, all the finite simple groups except the Tits group $^2F_4(2)'$ which appears in the table in footnote 2.
A description of the finite simple groups of Lie type appears in [6]. As we will see below, those finite simple groups of Lie type which depend on two parameters \( q \) and \( n \) are isomorphic to classical groups. Of course, many classical groups are not simple groups. We use the notation of [29] for the classical groups.

The following result refers to finite groups of Lie type and is probably well known (see for example [30]; in fact, there are much stronger results in the literature, e.g. [18]). We include a short proof with references for completeness.

**Lemma 2.4.** Let \( S \) be a finite simple group of Lie type, not isomorphic to a Suzuki group \( 2B_2(2^{2n+1}) \). Let \( q \) be the parameter corresponding to \( S \) according to [12, Table I, p. 8]. Assume that \( q \geq 4 \). Then \( S \) contains a subgroup isomorphic to some (possibly trivial) central extension of \( \text{PSL}(2, q) \).

**Remark 2.5.** The Suzuki groups \( 2B_2(2^{2n+1}) \) don’t contain subgroups isomorphic to a central extension of \( \text{PSL}(2, 2^{2n+1}) \) because their order \(|2B_2(q)| = q^2(q - 1)(q^2 + 1)| \) is not divisible by 3, whereas the order of \( \text{PSL}(2, 2^{2n+1}) \) is divisible by 3.

**Proof.** Suppose that \( S \) is isomorphic to an untwisted group of Lie type. By [6, Theorem 6.3.1] there exists a homomorphism from \( \text{SL}(2, q) \) to \( S \) with nontrivial image (see the formulas for \( x_r(t) \) in [6, p. 64]). Since we assume that \( q \geq 4 \), the group \( \text{PSL}(2, q) \) is simple, so the image of this homomorphism is isomorphic to either \( \text{SL}(2, q) \) or \( \text{PSL}(2, q) \). For any \( n \geq 2 \) the simple group \( 2A_n(q) \simeq \text{PSU}_{n+1}(q) \) contains a subgroup isomorphic to a central extension of \( \text{PSU}_2(q) \), and by [29, §2.6.1] there is an isomorphism \( \text{PSU}_2(q) \simeq \text{PSL}(2, q) \), so the result holds for \( 2A_n(q) \).

The simple group \( 2D_n(q) \) (for \( n \geq 2 \)) contains a subgroup isomorphic to \( 2D_2(q) \) [6, Theorem 14.5.2] and by [12, Table II, p. 10] \( 2D_2(q) \) is isomorphic to \( A_1(q^2) \). Since \( A_1(q^2) \simeq \text{PSL}_2(q^2) \), the result also holds for \( 2D_n(q) \). We have the following containments: \( 3D_4(q) > G_2(q) \) [29, §4.6.5, Theorem 4.3], \( 2G_2(3^{2m+1}) > \text{PSL}(2, 3^{2m+1}) \) [29, §4.5.3, Theorem 4.2], \( 2F_4(2^{2m+1}) > \text{SU}_3(2^{2m+1}) \) [29, §4.9.3, Theorem 4.5], and \( 2E_6(q) > F_4(q) \) [29, p. 173]. Since we have already proved the result for the smaller of these groups in each of the four cases (or for its quotient by a central subgroup, e.g. \( 2A_2(2^{2m+1}) \simeq \text{PSU}_3(2^{2m+1}) \) such a quotient of \( \text{SU}_3(2^{2m+1}) \)) the conclusion of the lemma holds for each of the larger groups. Hence, the proof of the lemma is now complete. \( \square \)

Recall that a finite permutation group \( G \) on a set \( \Omega \) is called a Frobenius group if \( G \) is transitive, no element of \( G \) except the identity fixes
more than one element of $\Omega$, and some non-identity element of $G$ fixes some element of $\Omega$ \cite[p. 191]{I}. (For example, the group of affine transformations of the affine line over a finite field is a Frobenius group.) If $\omega \in \Omega$, the stabilizer $G_\omega$ of $\omega$ in $G$ is called a Frobenius complement. Since $G$ acts transitively on $\Omega$, all Frobenius complements are conjugate, and hence they all have the same order. More generally, a finite group is called a Frobenius group if it is isomorphic to some permutation group which is a Frobenius group.

**Lemma 2.6.** Let $S$ be a finite simple group of Lie type. Let $q$ be the parameter corresponding to $S$ according to \cite[Table I, p. 8]{I}. Suppose that $q \geq 4$. Then $S$ contains subgroups $M$ and $N$, such that $M$ is solvable, $N$ is a normal subgroup of $M$, and $M/N$ is a Frobenius group with complement of order at least $(q - 1)/2$.

**Proof.** Suppose first that $S$ is not a Suzuki group. Then, by Lemma 2.4, $S$ has a subgroup which is a central extension of PSL$(2,q)$. Now PSL$(2,q)$ acts doubly transitively on the projective line of $\mathbb{F}_q$, in such a way that no non-identity element fixes more than two points. Let $B$ be the stabilizer of a point on the projective line. Then $B$ is a solvable group of order $q(q - 1)$ if $q$ is even, and $q(q - 1)/2$ if $q$ is odd. It follows immediately from the definition of Frobenius group, that $B$ is a Frobenius group in its action on the affine line. Since its Frobenius complement has order either $q - 1$ or $(q - 1)/2$ the lemma holds in this case.

Now suppose that $S$ is a Suzuki group. By Suzuki’s original article \cite{Suz}, $S$ has order $q^2(q - 1)(q^2 + 1)$, and acts faithfully and doubly transitively on a set $\Delta$ in such a way that $|\Delta| = q^2 + 1$ and no non-identity element of $S$ fixes more than two elements of $\Delta$. Set $B$ to be the stabilizer of a point in $\Delta$. Then $B$ is a Frobenius group with Frobenius complement of order $q - 1$. The group $S$ can be identified with a subgroup of GL$_4(2^{2m+1})$ \cite[§4.2.1]{St}, and with respect to this identification $B$ corresponds to a subgroup consisting of lower triangular matrices \cite[§4.2.2]{St}. Hence $B$ is solvable, so the lemma holds for Suzuki groups as well. 

Our next lemma depends on the Classification of the Finite Simple Groups \cite{CC, St}.

**Lemma 2.7.** Let $\Sigma$ be an infinite set of non-isomorphic finite non-abelian simple groups, and let $K$ be any positive constant. Then there exists some $S_0 \in \Sigma$, and some subgroups $M$ and $N$ of $S_0$ such that $M$
is solvable, $N$ is a normal subgroup of $M$, and $M/N$ is Frobenius group with Frobenius complement of order larger than $K$.

**Proof.** Assume the lemma is false. Suppose that $\Sigma$ and $K \geq 3$ provide a counterexample. Let $t$ be a power of a prime satisfying $t - 1 > K$. (This number $t$ will be fixed throughout the proof.) The group $F$ of affine transformations of the affine line over $\mathbb{F}_t$ is a solvable Frobenius subgroup of the group of permutations of the affine line, so $F$ is isomorphic to a subgroup of $S_t$. The Frobenius complements of $F$ have order $|F^*| = t - 1 > K$. Since $S_t$ is isomorphic to a subgroup of the alternating group $A_{t+2}$, in addition, $F$ is isomorphic to some subgroup of the simple alternating group $A_{t+2}$. It follows that no element of $\Sigma$ contains any subgroup isomorphic to $A_{t+2}$. In particular, if $A_n$ is isomorphic to some element of $\Sigma$ then $n \leq t + 1$. It follows that there are only a finite number of groups in $\Sigma$ which are isomorphic to alternating groups.

Suppose that $S$ is a finite simple group of Lie type from table [12. Table I in p. 8] and $S$ is isomorphic to some element of $\Sigma$. The table characterizes $S$ by one of 16 types and numbers $n$ and $q$. If the number $n$ is not explicitly given by the table, we take it to be the subindex, so, for example, if $S = ^3D_4(q)$ we set $n = 4$. By Lemma 2.6, we have $(q - 1)/2 \leq K$, so that $q \leq 2K + 1$ is bounded above. We next show that $n$ is also bounded above. 4 For this, we assume without loss of generality that $n \geq 9$.

Before proving that $n$ is bounded, we pause to prove an auxiliary result. Suppose that $G$ is a finite group, $G_0$ is a normal subgroup of $G$ such that $G/G_0$ is solvable, and $\phi : G_0 \rightarrow S$ is a surjective group homomorphism whose kernel $\ker(\phi)$ is solvable, where $S$ is isomorphic to an element of $\Sigma$. We claim that $G$ cannot contain subgroups $N$ and $T$ such that $T \unlhd N$, $T$ is solvable and $N/T \simeq A_{t+2}$. To prove the claim, assume that such groups $N$ and $T$ exist, and set $N_0 = N \cap G_0$. Then the composition factors of $NG_0/G_0 \simeq N/N_0 \cap G_0 = N_0$ are all solvable, so that $N_0$ contains exactly one non-solvable composition factor and this factor is isomorphic to $A_{t+2}$. The same condition holds for the image $\phi(N_0)$, since $\ker(\phi)$ is solvable. Since $A_{t+1}$ contains a subgroup isomorphic to a Frobenius group with complement larger than $K$, this contradicts our hypothesis. Therefore the claim is proved.

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4Our proof uses implicitly the Weyl groups of the classical finite groups of Lie type, but we avoid the explicit use of their theory for the benefit of a potential reader unfamiliar with it.

5This forces $S$ to be isomorphic to some classical group.
Now suppose $S = A_n(q) \simeq \text{PSL}_{n+1}(q)$. Then, by [29 §3.3.3], there exist groups $G = \text{GL}_{n+1}(q)$, $G_0 = \text{SL}_{n+1}(q)$ and a homomorphism $\phi$ as in the previous paragraph. Furthermore there exist subgroups $N$ and $T$ of $G$ such that $T \leq N$, $T$ is solvable and $N/T \simeq S_{n+1}$. If $n + 1 \geq t + 2$ then this contradicts what we know from the previous paragraph so that $n \leq t$ in this case. Suppose $S = 2A_n(q) \simeq \text{PSU}_{n+1}(q)$. Then, by [29 §3.3.3], there exist the groups $G = G_0 = \text{SU}_{n+1}(q)$ and a homomorphism $\phi$ as in the previous paragraph. Furthermore there exist subgroups $N$ and $T$ of $G$ such that $T \leq N$, $T$ is solvable and $N/T$ is isomorphic to the wreath product $Z_2 \wr S_m$, where $m$ is the integer part of $(n + 1)/2$. If $m \geq t + 2$ this contradicts what we know from the previous paragraph, so that $n \leq t + 1$ in this case. Essentially the same argument shows that if $S = D_n(q) \simeq \Omega^+_{2n}(q)$ then $n \leq t + 1$, and that if $S = 2D_n(q) \simeq \Omega^-_{2n}(q)$ then $n \leq t + 1$. Suppose $S = C_n(q) \simeq \text{PSp}_{2n}(q)$. Then, by [29 §3.7.4], there exist the groups $G = G_0 = \text{Sp}_{2n}(q)$ and a homomorphism $\phi$ as in the previous paragraph. Furthermore there exist subgroups $N$ and $T$ of $G$ such that $T \leq N$, $T$ is solvable and $N/T \simeq Z_2 \wr S_n$. If $n \geq t + 2$ this contradicts what we know from the previous paragraph, so that $n \leq t + 1$ in this case. Notice that, since $n \geq 9$, $S$ can not be any other finite simple group of Lie type. Hence, in all cases $n$ is bounded above, and there exist only a finite number of simple groups of Lie type on the list which can be isomorphic to groups in $\Sigma$.

Aside from alternating groups and groups of Lie type, [12 Table II, p. 10] contains only the 26 sporadic simple groups and the Tits group $2F_4(2)'$, and the abelian simple groups. Since the elements of $\Sigma$ are all non isomorphic to each other, they are non-abelian, and they can only be isomorphic to a finite number of alternating groups, and a finite number of finite simple groups of Lie type, we conclude that $\Sigma$ is finite. This contradicts our assumption, and completes the proof of the lemma. □

\textbf{Lemma 2.8.} Let $G$ be a Frobenius group of permutations of a finite set $\Omega$, and let $G_\omega$ be one of its Frobenius complements. Let $A$ be an abelian subgroup of $F$. Then $[F : A] \geq |G_\omega|$. 
Proof. It follows directly from the definition of a Frobenius group that $|Ω| > 1$. Let $N^* = G \setminus \bigcup_{ω ∈ Ω} G_ω$ and $N = \{1\} ∪ N^*$. A theorem of Frobenius \([11\ (35.24)]\) states that $N$ is a normal subgroup of $G$. Fix some $ω ∈ Ω$. Since all subgroups $\{G_{ω'} \mid ω' ∈ Ω\}$ are conjugate, we have $|G| = (|G_ω| - 1)|Ω| + |N|$. Since $N$ acts freely on $Ω$, the map $N ∋ γ \mapsto γ(ω) ∈ Ω$ is injective, so $|N| ≤ |Ω|$. Applying the same argument to the action of $G_ω$ on $Ω \setminus \{ω\}$ we get $|G_ω| ≤ |Ω| − 1$. We thus have

$$[G : G_ω] = \frac{|G|}{|G_ω|} = \frac{|G_ω| - 1)|Ω| + |N|}{|G_ω|} ≥ \frac{|G_ω| - 1)|Ω|}{|G_ω|} > |G_ω| - 1.$$

Since $[G : G_ω]$ is an integer, this gives $[G : G_ω] ≥ |G_ω|$. Similarly

$$[G : N] = \frac{|G|}{|N|} = \frac{|G_ω| - 1)|Ω| + |N|}{|N|} = \frac{|G_ω| - 1)|Ω|}{|N|} + 1 ≥ |G_ω|.$$

If $g ∈ G_ω$ and $h ∈ G$ are nontrivial commuting elements, we have $h ∈ G_ω$, because the action of $g$ on $Ω$ only fixes $ω$. Hence, if $A ⊆ G$ is abelian, then either $A ⊆ G_ω$ for some $ω ∈ Ω$ or $A ⊆ N$, so the lemma follows from the previous bounds on $[G : G_ω]$ and $[G : N]$. \(□\)

3. Results

For convenience, we now name the hypotheses which we will use throughout the rest of the paper. (Recall from the introduction that if $C$ is a set of finite groups then $T(C)$ denotes the set of all $T ∈ C$ such that there exist primes $p$ and $q$, a Sylow $p$-subgroup $P$ of $T$, and a normal Sylow $q$-subgroup $Q$ of $T$, such that $T = PQ$.)

**Hypotheses A.** $M$ and $d$ are positive integers, $C$ is a set of finite groups which is closed under taking subgroups, and $T = T(C)$ satisfies the Jordan property $J(M, d)$.

**Lemma 3.1.** Assume Hypotheses A. Let $p$ be any prime larger than $M$. Let $G ∈ C$, let $G_1 = C_G(E(G))$, and let $P$ be a Sylow $p$-subgroup $G_1$. Then $P$ is an abelian normal subgroup of $G_1$ and $P$ can be generated by at most $d$ elements.

**Proof.** Assume this is not the case, and that $P$ is a Sylow $p$-subgroup which contradicts the statement of the lemma. Since $P ∈ T$, by hypotheses there exists an abelian subgroup of $P$ of index at most $M < p$ and such that the subgroup can be generated by at most $d$ elements, so that $P$ is abelian and $P$ may be generated by at most $d$ elements. Since we assume that $P$ is not simultaneously abelian and normal in
$G_1$, we deduce that $P$ is not a normal subgroup of $G_1$. By Lemma 2.1 there exist a prime $q$ and a nontrivial $q$-subgroup $Q$ of $G_1$ such that $q \neq p$ and $[P, Q] = Q$. Furthermore, $P$ normalizes $Q$, so $PQ \subseteq G_1$ is a subgroup. Since $PQ \in \mathcal{T}$, there exists some abelian subgroup $J$ of $PQ$ such that $[PQ : J] \leq M$. Since $PQ$ is not abelian, $J$ is a proper subgroup of $PQ$. By Lemma 2.2, $[PQ : J] \geq p$. This contradicts the fact that $p > M$. This contradiction completes the proof of the lemma. □

**Lemma 3.2.** Assume Hypotheses A. Then there exists a positive integer $C_1$ with the following property. Let $G \in \mathcal{C}$. Set $G_1 = C_G(E(G))$, and let $R$ be the product of all the Sylow $p$-subgroups of $G_1$ for all primes $p > M$. Let $G_2 = C_{G_1}(R)$. Then $[G_1 : G_2] \leq C_1$.

**Proof.** Set $C_1 = M^{(d+1)M}$ so that $C_1 \geq 1$. If $G_1 = G_2$ the results holds, so we assume $G_1 \neq G_2$. In particular, $R$ is not trivial. Let $q$ be any prime dividing $[G_1 : G_2]$, and let $Q$ be a Sylow $q$-subgroup of $G_1$. By Lemma 3.1, $R$ is an abelian Hall subgroup of $G_1$, and it follows that $q \leq M$. Fix $p_0$ to be some prime divisor of $|R|$, and let $P_0$ be the Sylow $p_0$-subgroup of $R$. Since $Q P_0 \in \mathcal{T}$, by hypotheses, there exists an abelian subgroup $B$ of $QP_0$ such that $[QP_0 : B] \leq M$, and $B$ can be generated by at most $d$ elements. Since $p_0 > M$, it follows that $B$ contains a Sylow $p_0$-subgroup of $QP_0$, and therefore $B \supseteq P_0$. We set $B_0 = Q \cap B$. Then $[QP_0 : B] = [Q : B_0] \leq M$. Notice that $B_0$ is an abelian $q$-group generated by at most $d$ elements, and it acts trivially on $P_0$. Let $p$ be any prime divisor of $|R|$, and let $P$ be the Sylow $p$-subgroup of $R$. A similar argument shows that $[B_0 : C_{B_0}(P)] \leq M$. Let $B_1 = C_{B_0}(R)$. Then $B_1$ is the intersection of the $C_{B_0}(P)$ as we run through all the prime divisors $p$ of $|R|$. Then $B_0/B_1$ is an abelian $q$-group of exponent at most $M$ generated by at most $d$ elements, so that $|B_0/B_1| \leq M^d$. It follows that $[Q : B_1] \leq M^{d+1}$. Therefore, the $q$-part of $[G_1 : G_2]$ is at most $M^{d+1}$. Therefore

$$[G_1 : G_2] \leq M^{(d+1)M} = C_1,$$

as desired. □

**Lemma 3.3.** Assume Hypotheses A. Then there exists a positive constant $C_2$ such that whenever $G \in \mathcal{C}$ and $G_1 = C_G(E(G))$, then there exists an abelian subgroup $A$ of $G_1$ such that $[G_1 : A] \leq C_2$ and such that $A$ can be generated by at most $d$ elements.

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$^6$R is abelian because of the following general fact [13, Theorem 1.26]: if $\Gamma$ is a finite group all of whose Sylow subgroups are normal then $\Gamma$ is the direct product of its Sylow subgroups.
Proof. Assume the notation of Lemma 3.2. Set 
\[ C_2 = \left( (M!)^{M + d} \right)! M^M C_1. \]

Let \( G \in \mathcal{C} \). Set \( G_1 = C_G(E(G)) \), and let \( R \) be the product of all the Sylow \( p \)-subgroups of \( G_1 \) for all primes \( p > M \). Let \( G_2 = C_{G_1}(R) \). By Lemma 3.1, we know that \( R \) is an abelian normal Hall subgroup of \( G_2 \) and \( R \) can be generated by at most \( d \) elements. By the Schur-Zassenhaus Theorem, there exists a complement \( H \) to \( R \) in \( G_2 \). Then \( H \) is a Hall subgroup of \( G_2 \) for the set of primes smaller than or equal to \( M \). Since \( H \subseteq C_{G_1}(R) \), the elements of \( R \) commute with those of \( H \), and hence \( G_2 = H \times R \). This implies that \( H \) is the only Hall subgroup of \( G_2 \) for the set of primes not bigger than \( M \). Hence, \( H \) is characteristic in \( G_2 \), so \( H \) is normal in \( G \) (because \( G_2 \subseteq G \) : indeed, since by Lemma 3.1 \( R \subseteq G_1 \) is a normal Hall subgroup, \( R \) is characteristic in \( G_1 \), so \( G_1 \subseteq G \) implies \( G_2 \subseteq G \)). It then follows from the definition that \( E(H) \subseteq E(G) \), so that, since \( E(G) \) centralizes \( E(H) \subseteq G_1 \), we know that \( E(H) \) is abelian. This implies, again from the definition, that \( E(H) = 1 \). For each prime \( p \) with \( p \leq M \), since \( O_p(H) \in \mathcal{T} \), by hypotheses, there exists an abelian subgroup \( B_p \) of \( O_p(H) \) such that \( |O_p(H) : B_p| \leq M \) and such that \( B_p \) can be generated by at most \( d \) elements. Set \( B \) to be the product of all the \( B_p \) for \( p \leq M \). Then \( B \) is an abelian subgroup of \( F(H) \) which can be generated with at most \( d \) elements, and \( |F(H) : B| \leq M^M \) (\( B \) is abelian because if \( p \neq q \) then the elements of \( O_p(H) \) commute with those of \( O_q(H) \)). It follows that \( F(H) \) can be generated with at most \( M^M + d \) elements. Now \( F(H)/\Phi(F(H)) \) is abelian and of square-free exponent by Lemma 2.3. Since \( F(H)/\Phi(F(H)) \) can be generated by at most \( M^M + d \) elements and all prime divisors of \( |F(H)/\Phi(F(H))| \) are smaller than or equal to \( M \), this implies that
\[ |F(H)/\Phi(F(H))| \leq (M!)^{M + d}. \]
Again by Lemma 2.3, this implies that
\[ [H : F(H)] \leq \left( (M!)^{M + d} \right)! . \]
Now set \( A = BR \). Then \( A \) is an abelian subgroup of \( G_2 \) such that
\[ [G_2 : A] = [H : B] \leq \left( (M!)^{M + d} \right)! M^M \]
and such that \( A \) can be generated by at most \( d \) elements. It follows that
\[ [G_1 : A] \leq \left( (M!)^{M + d} \right)! M^M C_1 = C_2. \]
Hence the lemma holds. \( \square \)
Corollary 3.4. Assume Hypotheses A. Assume the notation of Lemma 3.3. Let $S \in \mathcal{C}$ be solvable. Then, there exists an abelian subgroup $A$ of $S$ such that $[S : A] \leq C_2$ and such that $A$ can be generated by at most $d$ elements.

Proof. This follows immediately from Lemma 3.3 because if $G = S$ then $E(G) = 1$ and $G = G_1 = S$. □

Lemma 3.5. Assume Hypotheses A. Then there exists a positive constant $C_3$ such that whenever $G \in \mathcal{C}$ is quasisimple then $|G| \leq C_3$.

Proof. Let $\Sigma$ be a full set of representatives of the isomorphism classes of simple groups $S$ of the form $S = G/Z(G)$ for some quasisimple $G \in \mathcal{C}$.

Suppose that $S \in \Sigma$ contains a solvable subgroup $H$ and a normal subgroup $H_0$ of $H$ such that $H/H_0$ is a Frobenius group with Frobenius complement of order $k$. Then there exists some $J \in \mathcal{C}$ such that $J$ is solvable and $H/H_0$ is a homomorphic image of $J$. By Corollary 3.4 there exists some abelian subgroup of $J$ of index at most $C_2$ in $J$. It follows that there exists an abelian subgroup of $H/H_0$ of index at most $C_2$. By Lemma 2.8 this tells us that $k \leq C_2$. By Lemma 2.7, this implies that $\Sigma$ is a finite set. □

Lemma 3.6. Assume Hypotheses A. Then there exists a positive constant $C_4$ such that whenever $G \in \mathcal{C}$ then the number of composition factors of $E(G)/Z(E(G))$ is at most $C_4$.

Proof. We set $C_4 = d + M$. Let $G \in \mathcal{C}$. By, for example, [14, Theorem 9.7], $E(G)/Z(E(G))$ is a direct product of simple groups. Say

$$E(G)/Z(E(G)) = S_1 \times \cdots \times S_n$$

\footnote{Also called Schur covering group. See [14, §5A] and [17]. The existence of the Schur representation group is proved in [17] Theorem 2.10.3 (for notation see [17, §2.7]). Note that the center $Z(R)$ is isomorphic to $H^2(S; \mathbb{C}^\times)$ and $R/Z(R) \simeq S$. Since $S$ is finite, we also have $H^2(S; \mathbb{C}^\times) \simeq H_2(S; \mathbb{Z})$, see [17] §2.7. In some respects, the Schur representation group is an analogue for finite groups of the universal covering group of a Lie group.}
where $n$ is the composition length and the $S_1, \ldots, S_n$ are non-abelian simple groups. For each $i = 1, \ldots, n$, pick a subgroup $T_i$ of order two of $S_i$ (we know that $T_i$ exists by the celebrated Feit–Thompson theorem \cite{10}, using the fact that $S_i$ is simple and non-abelian, hence non-solvable), and let $t_i$ be a 2-element of $E(G)$ which projects onto a generator for $T_i$. Each $t_i$ belongs to a different component times the center of $E(G)$, so that they all commute with each other, see for example \cite[Theorem 9.4]{14}. Let $B$ be the subgroup of $G$ generated by the $t_1, \ldots, t_n$. Now $B$ is an abelian 2-subgroup of $G$, and it can not be generated by fewer than $n$ elements. By our hypotheses, since $B \in T$, there exists an abelian subgroup $C$ of $B$ such that $C$ can be generated by at most $d$ elements and $[B : C] \leq M$. Hence, $B$ can be generated by at most $d + M = C_4$ elements. It follows that $n \leq C_4$, and the lemma holds.

\[ \square \]

**Corollary 3.7.** Assume Hypotheses A. Then there exists a positive constant $C_5$ such that whenever $G \in \mathcal{C}$ then $|E(G)| \leq C_5$.

**Proof.** Set $C_5 = C_3^{C_4}$. The results follows immediately from Lemma \ref{lem:3.3} and Lemma \ref{lem:3.6} since $E(G)$ is the product of as many components as the composition length of $E(G)/Z(E)$.

\[ \square \]

**Theorem 3.8.** Assume Hypotheses A. Then, there exists some positive integer $C_0$ such that $\mathcal{C}$ satisfies the Jordan property $J(C_0, d)$.

**Proof.** We use the notation of Lemma \ref{lem:3.3} and Corollary \ref{cor:3.7}. Set $C_0 = C_5! C_2$. Let $G \in \mathcal{C}$. Set $G_1 = C_G(E(G))$. By Lemma \ref{lem:3.3} there is an abelian subgroup $A$ of $G_1$ such that $[G_1 : A] \leq C_2$ and such that $A$ can be generated by at most $d$ elements. By Corollary \ref{cor:3.7} we have $|E(G)| \leq C_5$, and this implies that $[G : G_1] \leq C_5!$. This implies that $[G : A] \leq C_0$ and completes the proof of the theorem.

\[ \square \]

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