On the almost everywhere convergence of the eigenfunction expansions from Liouville classes \( L_1^\alpha(T^N) \)

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Abstract. The relevance of waves in quantum mechanics naturally implies that the decomposition of arbitrary wave packets in terms of monochromatic waves plays an important role in applications of the theory. When eigenfunction expansions does not converge, then the expansions of the functions with certain smoothness should be considered. Such functions gained prominence primarily through their application in quantum mechanics. In this work we study the almost everywhere convergence of the eigenfunction expansions from Liouville classes \( L_1^\alpha(T^N) \), related to the self-adjoint extension of the Laplace operator in torus \( T^N \). The sufficient conditions for summability is obtained using the modified Poisson formula. Isomorphism properties of the elliptic differential operators is applied in order to obtain estimation for the Fourier series of the functions from the classes of Liouville \( L_1^\alpha \).

1. Introduction

Let \( T^N \) denotes \( N \)-dimensional torus

\( T^N = \{ x = (x_1, x_2, ..., x_N) : -\pi < x_j < \pi, j = 1, 2, ..., N \} \).

To any function \( f(x) \in L_1(T^N) \) we can assign its Fourier series

\[
\sum_{n \in \mathbb{Z}^N} f_n e^{i(n,x)} = \sum_{n_1 = -\infty}^{+\infty} ... \sum_{n_N = -\infty}^{+\infty} f_{n_1...n_N} e^{i(n_1x_1 + ... + n_Nx_N)},
\]

where \( f_n \) are Fourier coefficients of \( f \):

\[
f_n = (2\pi)^{-N} \int_{T^N} f(x)e^{-i(n,x)} dx, \quad n \in \mathbb{Z}^N.
\]

Let \( \Delta \) denotes a Laplace operator: \( \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_N^2} \). The operator \( -\Delta \) is considered in Hilbert space \( L_2(T^N) \), symmetric and nonnegative, therefore by Friedrichs theorem (see \([1]\))
it has nonnegative self adjoint extension which coincides with the closure $-\Delta$ of the Laplace operator. Let $I$ denotes identity operator, then it is easy to see that

$$(T-\Delta)e^{i(n,x)} = (1 + |n|^2)e^{i(n,x)}.$$  

For any $\beta \in R$ we define a fractional power of the operator $T-\Delta$ as follows

$$(T-\Delta)^\beta f(x) = \sum_{n \in \mathbb{Z}^N} (1 + |n|^2)^\beta f_n e^{i(n,x)}.$$  

The main result of the paper is the following: a function $f \in L_2(T^N)$ belongs to the Liouville class $L_2^\alpha(T^N)$, if 

$$(T-\Delta)^{\alpha/2} f \in L_2(T^N).$$  

The latter is equivalent to the condition

$$\sum_{n \in \mathbb{Z}^N} (1 + |n|^2)^\alpha |f_n|^2 < \infty.$$  

where the expression on the left side is called a norm of $f \in L_2^\alpha(T^N)$ and denoted by $\|f\|_{L_2^\alpha(T^N)}$.

2. Main result

In this paper we deal with the problems on the convergence of the multiple Fourier series in the classes of Liouville $L_p^\alpha(T^N)$. From eigenvalue point of view, the natural question is in what conditions does the series

$$\sum_{\nu=0}^{\infty} \left( \sum_{|n|^2 = \nu} f_n e^{i(n,x)} \right)$$

approximate $f$? To answer this question we need to define a spherical partial sums of Fourier series

$$E_\lambda f(x) = \sum_{|n|^2 < \lambda} f_n e^{i(n,x)}.$$  

The main result of the paper is the following:

**Theorem 2.1.** If $f \in L_1^\alpha(T^N).$ Then for $\alpha > \frac{N+1}{2}$ one has

i) $\| \sup_{\lambda > 0} |E_\lambda f(x)| \|_{L_1(T^N)} \leq C(N, \alpha) \|f\|_{L_1^\alpha(T^N)}$;

ii) $\lim_{\lambda \to \infty} E_\lambda f(x) = f(x)$, almost everywhere in $T^N$.

The problems related to the almost everywhere convergence of the eigenfunction expansions from Liouville classes intensively investigated starting works of Bastis [2], [3], [4], where the eigenfunction expansions corresponding the Laplace operator are considered. The almost everywhere convergence of the partial sums of the Fourier integrals in the classes of Liouville $L_p^\alpha(R^N)$ are investigated in paper [5], where the condition $\alpha > (N-1)/2$ guarantees the almost everywhere convergence of the spectral expansions of the Laplace operator. So far no results on the almost everywhere convergence of multiple Fourier series in the classes of $L_p^\alpha(T^N)$. In such case the difficulty arises because of not applicability of Poisson formula. In the current work we found new approach for the almost everywhere convergence problem in the Liouville classes by using the estimation of Lebesgue constant and applying Jackson theorems. The statement of the Theorem in the case of unit sphere is obtained in [6].
3. Preliminaries

3.1. Lebesgue constant

The spherical partial sum $E_\lambda f(x)$ of the Fourier series of $f$ is the convolution $f * \Theta_\lambda$ with the corresponding Dirichlet kernel (spectral function)

$$\Theta_\lambda(x) = \sum_{|n|^2 < \lambda} e^{i(n,x)}$$

and the convergence of the partial sums $E_\lambda f$ depend upon estimations for the spherical Lebesgue constants:

$$\|\Theta_\lambda(x)\|_{L_1(T^N)} = \int_{T^N} \left| \sum_{|n|^2 < \lambda} e^{i(n,x)} \right| dx.$$

Lemma 3.1. There exists a constant $C$, which depends only on $N$, such that for any nonnegative integral $k$ we have

$$\int_{T^N} \left| \sum_{|n| < k} e^{i(n,x)} \right| dx \leq C k^{N-1/2}, \quad N \geq 1.$$

For the proof of the above result we refer readers to [7] and [8]. The exponent $\frac{N-1}{2}$ is sharp. This is a consequence of results in [9], where the authors do not restrict themselves to Fourier series, but study more general eigenfunction expansions.

Lemma 3.2. Let $f(x)$ be function from $L_1(T^N)$ and extended by periodicity to $\mathbb{R}^N$. Then there exist a sequence of trigonometric polynomials

$$S_k(x) = \sum_{|n|^2 \leq k} c_n e^{i(n,x)}, \quad k = 0, 1, 2, \ldots$$

such that

$$\lim_{k \to \infty} \int_{T^N} |S_k(x) - f(x)| dx = 0.$$

Furthermore if $\{\phi_n(x)\}$ approximate the identity in $L_1(T^N)$, then for any function $f \in L_1(T^N)$ the convolution $\phi_n * f$ will approximate the given function in the norm of $L_1$:

$$\lim_{n \to \infty} \|\phi_n * f - f\|_{L_1(T^N)} = 0.$$

The following is well-known fact from the theory of approximation by multiple Fourier series.

Lemma 3.3. If $g \in L_1(T^N)$, for sufficiently large values of $\lambda$

$$\|E_\lambda g\|_{L_1} = O(1)\lambda^{(N-1)/4}\|g\|_{L_1}.$$

Proof. Let $g \in L_1(T^N)$. As we defined before the spherical partial sum of the Fourier series of $g$ is $E_\lambda g(x)$ is the convolution $g * \Theta_\lambda$ with the corresponding Dirichlet kernel (spectral function)

$$E_\lambda g(x) = (g * \Theta_\lambda)(x).$$

Let consider a function $\phi \in C^\infty(\mathbb{R}^N)$, such that $\phi(x) = 1$ if $|x| \leq 1$, and $\phi(x) = 0$ if $|x| \geq 2$. Then we form the following harmonics

$$\phi_\lambda(x) = \sum_{n \in \mathbb{Z}^N} \phi \left( \frac{x}{\lambda} \right) e^{i(n,x)}.$$
For any $g \in L_1(T^N)$ the convolution $\Theta_\lambda * g$ can be estimated as follows

$$
\|\Theta_\lambda * g\|_{L_1} \leq \|\Theta_\lambda * \phi_\lambda * g\|_{L_1} + \|\Theta_\lambda\|_{L_1}\|\phi_\lambda * g - g\|_{L_1}.
$$

The estimations for Lebesgue constant above show that the $L_1$ norms of $\Theta_\lambda * \phi_\lambda$ are uniformly bounded with respect to $\lambda$. The second term can be estimated using the approximation by trigonometric polynomials. Let $k$ be an integer such that $k \leq \lambda < k + 1$. There exists a trigonometric polynomial

$$S_k(x) = \sum_{|n|^2 \leq k} c_n e^{i(n,x)}$$

which approximates a function $g$ in the norm of $L_1$ and $\phi_\lambda * S_k = S_k$, because $\phi(x) = 1$ if $|x| \leq 1$. Therefore one has

$$\|\phi_\lambda * g - g\|_{L_1} \leq \|\phi_\lambda * (S_k - g)\|_{L_1} + \|S_k - g\|_{L_1},$$

which shows that

$$\|\phi_\lambda * g - g\|_{L_1} = O(1)\|g\|_{L_1}, \quad \lambda \to \infty.$$

If take into account the Lebesgue estimates we have

$$\|\Theta_\lambda * g\|_{L_1} = O(1)\lambda^{(N-1)/4}\|g\|_{L_1},$$

for sufficiently large values of $\lambda$. \hfill \Box

### 3.2. The fractional powers of Laplace operator and the almost everywhere convergence of eigenfunction expansions in $L_1^\alpha(T^N)$

The operator of taking partial sums from Fourier series $E_\lambda$ and fractional powers of the operator $\sqrt{-\Delta}$ are commutative, which leads to the following representation for the partial sums:

$$E_\lambda f = (\sqrt{\Delta} - \Delta)^{\alpha} E_\lambda (\sqrt{\Delta} - \Delta)^{\alpha} f, \quad \alpha > 0.$$ 

By introducing the notations $\sigma_\lambda^{\alpha} = (\sqrt{\Delta} - \Delta)^{\alpha} E_\lambda$, and $g = (\sqrt{\Delta} - \Delta)^{\alpha} f$, we obtain

$$E_\lambda f(x) = \sigma_\lambda^{\alpha} g(x) = \int_{T^N} g(y) \tau_\lambda^{\alpha}(x,y) \, dy$$

where

$$\tau_\lambda^{\alpha}(x,y) = \sum_{|n|^2 < \lambda} (1 + |n|^2)^{-\alpha} e^{i(n,x-y)}.$$ 

We note that if $f \in L_0^2(T^N), p \geq 1$, then $g = (\sqrt{\Delta} - \Delta)^{\alpha} f \in L_0^p(T^N)$, it can be established using the isomorphism of the Liouville classes (see [10]).

The following Lemma helps to estimate $\sup_{\lambda > 0} |\sigma_\lambda^{\alpha} g(x)|$ in the norm of $L_1$.

**Lemma 3.4.** If $g \in L_1(T^N)$. Then for $\alpha > \frac{N-1}{2}$ such that one has

$$\sup_{\lambda > 0} |\sigma_\lambda^{\alpha} g(x)|_{L_1(T^N)} \leq \frac{C_\alpha}{\alpha - \frac{N-1}{2}} \|g\|_{L_1(T^N)},$$

where a constant $C_\alpha > 0$ does not have a singularity at $\alpha = \frac{N-1}{2}$. 

Proof. Let integer $k$ chosen as $k \leq \lambda < k + 1$. Then using the Abel transformation formula we obtain

$$
\sigma_\lambda^\alpha g(x) = \sum_{|n|^2 \leq k} (1 + |n|^2)^{-\frac{\alpha}{2}} g_n e^{inx} = \sum_{\nu=0}^{k} (1 + \nu)^{-\frac{\alpha}{2}} \sum_{|n|^2 = \nu} g_n e^{inx}
$$

$$
= (1 + k)^{-\frac{\alpha}{2}} \sum_{|n|^2 \leq k} g_n e^{inx} + \sum_{\nu=0}^{k-1} [(1 + \nu)^{-\frac{\alpha}{2}} - (2 + \nu)^{-\frac{\alpha}{2}}] \sum_{|n|^2 \leq \nu} g_n e^{inx}.
$$

Which leads to the following estimation

$$
|\sigma_\lambda^\alpha g(x)| \leq \sum_{\nu=0}^{\infty} [(1 + \nu)^{-\frac{\alpha}{2}} - (2 + \nu)^{-\frac{\alpha}{2}}] \sum_{|n|^2 \leq \nu} g_n e^{i(n,x)}.
$$

Shifting to the maximal operator and integrating over $T^N$ gives

$$
\int_{T^N} |\sigma_\lambda^\alpha g(x)| \, dx \leq \sum_{\nu=0}^{\infty} [(1 + \nu)^{-\frac{\alpha}{2}} - (2 + \nu)^{-\frac{\alpha}{2}}] \int_{T^N} \sum_{|n|^2 \leq \nu} g_n e^{i(n,x)} \, dx.
$$

The estimations for the partial sums of the Fourier series of $g \in L_1(T^N)$ gives

$$
\|\sigma_\lambda^\alpha g(x)\|_{L_1(T^N)} \leq C\|g\|_{L_1} \sum_{k=0}^{\infty} (1 + k)^{-\frac{\alpha}{2} - 1} k^{\frac{N-1}{4}}
$$

$$
\leq C\|g\|_{L_1} \left( 1 + \int_{1}^{\infty} t^{-\frac{\alpha}{2} + \frac{N-1}{4} - 1} dt \right) \leq \frac{C\alpha}{\alpha - \frac{N-1}{2}} \|g\|_{L_1(T^N)},
$$

here we used the condition $\alpha > (N - 1)/2$. This proves the statement of Lemma 3.4.

4. Proof of the main Theorem

We define a maximal operator $E_\alpha$ by the following

$$
E_\alpha f(x) = \sup_{\lambda > 0} |E_\lambda f(x)|,
$$

which is nonlinear operator. The well known fact about the maximal operator is that if it is bounded in $L_p(T^N)$, then the Fourier series of a function $f \in L_p(T^N)$ is almost everywhere convergent.

We notice that for all functions $f \in C^\infty(T^N)$ the following identity is true

$$
E_\alpha f(x) = \sup_{\lambda > 0} |\sigma_\lambda^\alpha (I - \Delta)^{\frac{\alpha}{2}} f|.
$$

By reference to the Lemma 3.4 we obtain

$$
\|E_\alpha f\|_{L_1} = \| \sup_{\lambda > 0} |\sigma_\lambda^\alpha (I - \Delta)^{\frac{\alpha}{2}} f| \|_{L_1} \leq \frac{C\alpha}{\alpha - \frac{N-1}{2}} \|(I - \Delta)^{\frac{\alpha}{2}} f\|_{L_1} = \frac{C\alpha}{\alpha - \frac{N-1}{2}} \|f\|_{L_1^\alpha}.
$$
Part one of Theorem 2.1 is established. We turn now to prove the almost everywhere convergence of the Fourier series as follows. Let us denote by $\Lambda f(x)$ the fluctuation of $E_\lambda f(x)$:

$$\Lambda f(x) = \left| \limsup_{\lambda \to \infty} E_\lambda f(x) - \liminf_{\lambda \to \infty} E_\lambda f(x) \right|.$$ 

It is obvious that

$$\Lambda f(x) \leq 2E_* f(x).$$

From the density of $C^\infty(T^N)$ in $L_1^\alpha(T^N)$ and for any $\varepsilon > 0$, the function $f \in L_1^\alpha(T^N)$ can be represented as the sum of two functions:

$$f(x) = f_1(x) + f_2(x),$$

where $f_1(x) \in C^\infty(T^N)$ and $\|f_2\|_{L_1^\alpha(T^N)} < \varepsilon$. Then we have

$$\Lambda f(x) = \Lambda \{f_1(x) + f_2(x)\} = \left| \limsup_{\lambda \to \infty} E_\lambda f_2(x) - \liminf_{\lambda \to \infty} E_\lambda f_2(x) \right| \leq C\|f_2\|_{L_1^\alpha(T^N)} \leq \varepsilon.$$ 

Since $\varepsilon$ is arbitrary, then $\Lambda f(x) = 0$ for almost all $x \in T^N$. Thus, we have the almost everywhere convergence on $T^N$ for partial sums of the function $f \in L_1^\alpha(T^N)$, $\alpha > (N - 1)/2$:

$$\lim_{\lambda \to \infty} E_\lambda f(x) = f(x).$$

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References
[1] Alimov Sh A and Ashurov R R and Pulatov A K 1991 *Multiple Fourier series and Fourier integrals*. In *Commutative Harmonic Analysis IV* (Berlin: Springer Berlin Heidelberg) pp 1-95
[2] Bastis A I 1983 *On some problems of convergence of spectral expansions of the elliptic differential operators* (Moscow: PhD thesis Moscow State University )
[3] Bastis A I 1983 *Mathematical Notes* 34(4) pp 782-789
[4] Bastis A I 1982 *Mathematical Notes* 32(3) pp 634-637
[5] Ashurov R R and Buvaev K T 1998 *Uzbek Mathematical Journal* 4 pp 14-23
[6] Ahmelev A A 1996 *Uzbek Mathematical Journal* 4 pp 17-25
[7] Babenko K I 1971 *On the mean convergence of multiple Fourier series and the asymptotic behaviour of the Dirichlet kernel of the spherical means* (Moscow: Preprint No 52 Institut Prikladnoy Matematiki Akademii Nauk SSSR )
[8] Shapiro H S 1975 *Journal of Approximation Theory*, 13, pp 40-44
[9] Alimov S A and Il’in V A 1971 *Differential Equations* 7 pp 516-543
[10] Nikol’ski S M 1969 *Approximation of Functions of Several Variables and Embedding Theorems* (Moscow: Nauka)