FROM OHKAWA TO STRONG GENERATION VIA APPROXIMABLE TRIANGULATED CATEGORIES
- A VARIATION ON THE THEME OF AMNÓN NEEMAN’S NAGOYA LECTURE SERIES

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Abstract. This survey stems from Amnon Neeman’s lecture series at Ohakawa’s memorial workshop. Starting with Ohakawa’s theorem, this survey intends to supply enough motivation, background and technical details to read Neeman’s recent papers on his “approximable triangulated categories” and his $\mathcal{D}_\text{coh}^b(X)$ strong generation sufficient criterion via de Jong’s regular alteration, even for non-experts.

1. Introduction

This survey stems from Amnon Neeman’s lecture series at Ohakawa’s memorial workshop. The original lecture series started and ended with Ohkawa’s theorem on the stable homotopy category. In the beginning Ohkawa’s theorem was presented in its lovely, original form. The lecture series then meandered through some—definitely not all—of the developments and generalizations made by others in the years following Ohkawa’s paper. And at the end came what was then a recent result of Amnon Neeman’s—and the relevance was that the Ohkawa set and its properties, as developed in the years following Ohkawa, turned out to be key to the proof of the recent theorem.

Here, our presentation significantly modifies Neeman’s original presentation, partially fueled by other distinguished submissions to this proceedings, mostly to motivate topologists to get interested in this rich subject. For this purpose, we have reorganized and expanded the original framework of Amnon Neeman’s lecture series.

Still, the underlying philosophy of Neeman’s presentation to start with Ohkawa’s theorem remains kept in this survey. And most significantly, following a strong request of Professor Neeman, we reviewed Neeman’s recent proof of:

$\mathcal{D}_\text{coh}^b(X)$ strong generation sufficient criterion via de Jong’s regular alteration with enough background and technical details, expanding and sometimes even modifying parts of the original proof so as to make this review beginner-friendly from

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a homotopy theorist’s point of view. Actually, this proof of Neeman also makes critical use of, in addition to de Jong’s regular alteration, a couple of Thomason’s theorems:

- First, the fundamental theorem of Hopkins, Neeman, Thomason and others on the classification of thick tensor ideals of $\text{D}^{\text{perf}}(X)$, the $\text{D}^{\text{perf}}(X) = \text{D}_{\text{qc}}(X)^c$ analogue of the Hopkins-Smith thick subcategory theorem of $\mathcal{SH}^{\text{fin}} = \mathcal{SH}^c$ whose proof heavily depends upon the (Devinatz-)Hopkins-Smith nilpotency theorem.
- Second, Thomason’s localization theorem on $\text{D}^{\text{perf}}(X \setminus Z)$, for which Neeman found a homotopy theoretical proof in the framework of Miller’s finite localization.

Considering these circumstance, we have also explained the role of (Devinatz-)Hopkins-Smith nilpotency theorem in the proof of Hopkins-Smith thick subcategory theorem, as well as essentially all the details of Neeman’s proof of Thomason’s localization theorem.

Now the rest of this survey is organized as follows:

§2: The first goal of this section is to recall Ohkawa’s theorem in stable homotopy theory. Ohkawa’s theorem claims the Bousfield classes in the stable homotopy category $\mathcal{SH}$ form a set which is very mysterious and beyond our imagination. Then the second goal of this section is the fundamental theorem of Hopkins, Neeman, Thomason and others, which roughly states the analogue of the Bousfield classes in $\text{D}_{\text{qc}}(X)$, in contrast to the Ohakawa’s case of $\mathcal{SH}$, form a set with a clear algebro-geometric description. For these purposes, standard facts about the Bousfield localization and triangulated categories are reviewed, including the existence of Bousfield localization for perfectly generated triangulated subcategories, Miller’s finite localization for triangulated subcategories generated a set of compact objects, and the telescope conjecture.

§3: In reality, Hopkins was not motivated by Ohkawa’s Theorem 2.25 for his influential paper in algebraic geometry [Hop85] (Theorem 2.37). Instead, Hopkins was motivated by his own theorem with Smith [HS98] in the triangulated subcategory $\mathcal{SH}^c$ consisting of compact objects, whose validity was already known to them back around the time Hopkins wrote [Hop85]. In this section, we review this theorem of Hopkins-Smith, emphasizing the way how (Devinatz-)Hopkins-Smith nilpotency theorem is used in its proof. In Theorem 3.7 we summarize the main stories in $\mathcal{SH}^c_{(p)} \subset \mathcal{SH}_{(p)}$ (the Ohkawa theorem, the Hopkins-Smith theorem, Miller’s version of the Ravenel telescope conjecture $(C \circ I \cong \text{Id}_{\mathcal{SH}^c_{(p)}})$, and the conjectures of Hovey and Hovey-Palmieri) in the following succinct commutative diagram:
We then review analogues of the Hopkins-Smith theorem in the motivic setting by Joachimi and Kelly. Also, inspired by this influence of Hopkins-Smith theorem to algebra and algebraic geometry, we briefly reviewed the couple of most prominent conjectures in homotopy theory, the telescope conjecture and the chromatic splitting conjecture, following a suggestion of Professor Morava.

§4: From the previous two sections, we are naturally led to investigate $D_{qc}(X)^c$. However, the story is not so simple. Whereas there is a conceptually simple algebro-geometrical interpretation $D_{qc}(X)^c = D_{perf}(X)$, it is its close relative (actually equivalent if $X$ is smooth over a field) $D_{coh}^b(X)$ which traditionally has been intensively studied because of its rich geometric and physical information. So, we wish to understand both $D_{coh}^b(X)$ and $D_{perf}(X)$. In this section, we start with brief, and so inevitably incomplete, summaries of $D_{coh}^b(X)$ and $D_{perf}(X)$, focusing on their usages. Still, we hope this would convince non-experts that $D_{coh}^b(X)$ and $D_{perf}(X)$ are very important objects to study. Amongst of all, we shall recall the fundamental theorem of Hopkins, Neeman, Thomason and others on the classification of thick tensor ideals of $D_{perf}(X)$ and the Thomason’s localization theorem on $D_{perf}(X \setminus Z)$, both of which play critical roles in Neeman’s proof of the strong generation of $D_{coh}^b(X)$ reviewed in §5. For the classification of thick tensor ideals of $D_{perf}(X)$, we shall establish the following commutative diagram [(39)] in Theorem [1.15] which is the $D_{qc}(X) = D_{perf}(X)$ analogue of the Hopkins-Smith theorem, coupled with the fundamental theorem of Hopkins, Neeman, Thomason, and others, reviewed in §2, which is the $D_{qc}(X)$ analogue of he Ohkawa theorem:

This commutative diagram is very important because it encapsulates the story (of not only this article, but also of this proceedings!). In fact, this commutative diagram in $D_{qc}^c \subset D_{qc}$, which is the analogue of the commutative diagram in $SH^c \subset SH$ (introduced in §3), leads us to extend these commutative diagrams to other triangulated categories. Furthermore, the mutually inverse arrows at the bottom right of the diagram yield a positive
solution to the telescope conjecture (see Theorem 4.15 and Remark 4.16 for more detail), unlike the original problematic telescope conjecture in $\mathcal{SH}_{(p)}$ which shows up in the commutative diagram (1.1) (see the paragraph after Theorem 3.3). Finally, to close this section, we shall review Neeman’s recent result, which claims two close relatives $D^b_{\text{coh}}(X)$ and $D^\text{perf}(X)$ actually determine each other, and its main technical tool: approximable triangulated category whose principal example is $D_{\text{qc}}(X)$, as well as $\mathcal{SH}$.

§5:: Having been convinced that $D^b_{\text{coh}}(X)$ and $D^\text{perf}(X)$ carry rich information and are intimately related to each other in the previous section, we review here Neeman’s recent investigations of the important “strong generation” property, in the sense of Bondal and Van den Bergh [BVdB03], for $D^b_{\text{coh}}(X)$ and $D^\text{perf}(X)$. The focus here (and in this paper) is Neeman’s $D^b_{\text{coh}}(X)$ strong generation sufficient criterion via de Jong’s regular alteration, for which we give a substantial part of its proof, including some modifications.

- Start with the $D_{\text{qc}}(X)$ strong compact generation sufficient criterion Theorem 5.12, and give an outline of its proof, emphasizing where the approximability of $D_{\text{qc}}(X)$ is used
- Applying both the fundamental theorem of Hopkins, Neeman, Thomason and others on the classification of thick tensor ideals of $D^\text{perf}(X)$ and the Thomason localization theorem on $D^\text{perf}(X \setminus Z)$, both of which were reviewed in §4, we shall show how the $D_{\text{qc}}(X)$ strong compact generation sufficient criterion Theorem 5.12 reviewed above, implies the $D_{\text{qc}}(X)$ strong bounded generation sufficient criterion via de Jong’s regular alteration Theorem 5.12. Here, we extend and partially modify Neeman’s proof in order to make this review beginner-friendly.
- Having the $D_{\text{qc}}(X)$ strong compact generation sufficient criterion available, we can prove our desired $D^b_{\text{coh}}(X)$ strong generation sufficient criterion via de Jong’s regular alteration Theorem 5.6. However, this proof is rather involved, and requires, in addition to Christensen’s theory of phantom masp, some algebrao-geometric result which we had to put in a black box. We have located this black box in Lemma 5.7 (ii).

Neeman’s own results presented in this survey are not exactly what he talked about at the workshop. For instance, although the “strong generation” of $D^b_{\text{coh}}(X)$ and $D^\text{perf}(X)$ was still a major issue in Neeman’s lecture series, Neeman’s theory of approximable triangulated category, which first appeared in Neeman’s series of arxiv preprints in 2017, was not touched upon during 2015 lectures. Likewise, nothing was mentioned from §3 and §4 in this survey during 2015 lectures. In contrast, Neeman actually talked about other results of his own, but they have been omitted in this survey. All of these decisions were made in order to make this proceedings a “coherent story,” with this survey at its philosophical core. In fact, the author, who happened to be both an organizer of the workshop and an editor of this follow-up proceedings, became confident that the mathematics presented by Neeman at the
workshop vividly interacts with lots of other talks at the workshop and articles submitted to this proceedings. So, the author repeatedly mentioned such interactions whenever appropriate.

In spite of such an excitement, the first version of this paper was just a twenty page short list of results with no proof, but it was the requests and the suggestions by Professor Neeman and Professor Morava, which prompted the author to revise this article repeatedly to contain lots of useful results, including many proofs!

The author would like to express his hearty thanks to Professor Amnon Neeman for his beautiful lecture series, his encouragement to write up his lecture series from the author’s perspective as a non-expert, and his request to write a beginner-friendly survey of his proof of the $D^b_{\text{coh}}(X)$ strong generation sufficient criterion, in such a way that the roles of the homotopical ideas of Bousfield, Ohkawa, Hopkins-Smith and others in its proof become transparent. Not only that, Professor Neeman kindly read a preliminary version of this survey and offered the author many useful suggestions including locating author’s confusions.

The author’s thanks also goes to Professor Jack Morava for his suggestion to emphasize the telescope conjecture and the chromatic splitting in this article, as well as many inspiring and useful comments, some of which emerged as footnotes of this paper.

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Still, the author is solely responsible for any left over mistakes and confusions, as a matter of course.

Professor Haynes Miller informed the author of interesting works of Ruth Joachimi and Tobias Barthel, both of which have been incorporated in this survey and our proceedings. As an editor of this proceedings, the author would like to thank Professor Miller for these information and other valuable information, all of which were so crucial in organizing this proceedings.

To conclude the introduction, the author dedicates this survey to Professor Tetsusuke Ohkawa, the author’s former colleague at Hiroshima University. Probably the author should express his heartfelt gratitude to Professor Tetsusuke Ohkawa with rhetorical flourish... However, the author does not have such an ability, and, what is probably even more importantly, the author knows very well that Professor Ohkawa prefers interesting mathematics much more than such rhetorical flourish! So, the author would like to close this section with a homework on behalf of Professor Tetsusuke Ohkawa to be submitted to Professor Tetsusuke Ohkawa:

**Homework 1.1.** Extend the commutative diagrams below to other triangulated categories:

\[\text{Diagram}\]

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2 Actually, the author thought even such a short list is exciting.
The first goal of this section is to recall Ohkawa’s theorem in stable homotopy theory. Ohkawa’s theorem claims the Bousfield classes in the stable homotopy category $\text{SH}$ form a set which is very mysterious and beyond our imagination. Then the second goal of this section is the fundamental theorem of Hopkins, Neeman, Thomason and others, which roughly states the analogue of the Bousfield classes in $\text{D}_{qc}(X)$, in contrast to the Ohkawa’s case of $\text{SH}$, form a set with a clear algebro-geometric description.

Since both $\text{SH}$ and $\text{D}_{qc}(X)$ are triangulated categories, we start with recalling some basic terminologies of triangulated categories.

2.1. Bousfield localizations. Let $\mathcal{T}$ be a triangulated category. The suspension functor is denoted by $\Sigma$. In this article all triangulated categories are assumed to have small Hom-sets, except Verdier quotients to be defined now. In fact, to study highly rich objects like triangulated categories, we should “localize” at various stages. This is exactly what Verdier [Ver77] did in the context of derived categories.

Concerning this sentence, Professor Morava communicated the following thoughts to the author: “When I read it I was reminded of a quotation from the English writer Sir Thomas Browne (from ‘Urn Burial’, in 1658):

What song the Sirens sang, or what name Achilles assumed when he hid himself among women, though puzzling questions, are not beyond all conjecture...

I believe understanding the structure of Ohkawa’s set (perhaps by defining something like a topology on it) is very important, not just for homotopy theory but for mathematics in general. An analogy occurs to me, to other very complicated objects (like the Stone–Čech compactification of the rationals or the reals, or maybe the Mandelbrot set) which are very mysterious but can approached as limits of more comprehensible objects. Indeed I wonder if this is what Neeman’s theory of approximable triangulated categories points toward.”

Let us briefly recall the localization in the abelian category setting: [Gab62, III,1] [GM03, p.122, Exer.9]. Just as we may start with thick triangulated categories for Verdier quotients, which we will see in Remark 2.3 (iii), to localize an abelian category $\mathcal{A}$ by its full subcategory $\mathcal{B}$, we start
**Definition 2.1** (Verdier quotient (a.k.a. Verdier localization)). (see also \[Ver77\] and \[Nee01, Chapter 2\]) For a triangulated category $\mathcal{T}$ and its triangulated subcategory $\mathcal{S}$, the Verdier quotient (a.k.a. Verdier localization) $\mathcal{T}/\mathcal{S}$ is a “triangulated category”, which are characterized by the following properties:

- $\text{Ob}(\mathcal{T}/\mathcal{S}) = \text{Ob}(\mathcal{T})$. For $X, Y \in \text{Ob}(\mathcal{T}/\mathcal{S}) = \text{Ob}(\mathcal{T})$, the class of morphisms is given by

$$
\text{Hom}_{\mathcal{T}/\mathcal{S}}(X, Y) = \frac{\text{diagrams of the form } (X \stackrel{l}{\leftarrow} Z \stackrel{f}{\rightarrow} Y) \text{ with } l, f \in \text{Hom}_\mathcal{T}, \text{Cone}(l) \in \text{Ob}(\mathcal{S})}{(X \stackrel{l_1}{\leftarrow} Z_1 \stackrel{f_1}{\rightarrow} Y_1) \simeq (X \stackrel{l_2}{\leftarrow} Z_2 \stackrel{f_2}{\rightarrow} Y_2) \iff \exists \text{ commutative diagram:}}
$$

with assuming $\mathcal{S}$ is a Serre subcategory, i.e. for any exact sequence $0 \to B' \to B \to B'' \to 0$ in $\mathcal{A}$, $(B \in \mathcal{B} \iff (B' \in \mathcal{B} \text{ and } B'' \in \mathcal{B})))$ Then the quotient category $\mathcal{A}/\mathcal{B}$, in the sense of Gabriel, Grothendieck, Serre, is of the following form:

$$
\text{Ob } \mathcal{A}/\mathcal{B} := \text{Ob } \mathcal{A}; \quad \text{\textquotedblright Hom\textquotedblright}_\mathcal{A}/\mathcal{B}(A, A') := \lim_{\Delta \subseteq B, \Delta' \subseteq B} \text{Hom}_\mathcal{A}(A/A', A'/\Delta')
$$

Thus, an element of $\text{Hom}_{\mathcal{A}/\mathcal{B}}(A, A')$ is of the following form:

$\begin{tikzpicture}
\node (A) at (0,0) {$A$};
\node (A') at (2,0) {$A'$};
\node (B) at (0,-2) {$B \ni A/A$};
\node (A'/A) at (2,-2) {$A'/\Delta'$};
\draw[->] (A) to (A');
\draw[->] (B) to (A); \node at (1,-1) {$\Delta$};
\draw[->] (B) to (A'/A); \node at (1,-3) {$\Delta'$};
\end{tikzpicture}$

However, if we consider a similar diagram in the setting of derived categories, we may take the homotopy pullback $\tilde{A}$ as in the following diagram:

$\begin{tikzpicture}
\node (A) at (0,0) {$A$};
\node (A') at (2,0) {$A'$};
\node (A/\Delta) at (0,-2) {$\tilde{A}$};
\node (A'/\Delta') at (2,-2) {$A'/\Delta'$};
\draw[->] (A) to (A');
\draw[->] (A/) to (A'); \node at (1,-1) {$\tilde{A}$};
\draw[->] (A/) to (A'/\Delta'); \node at (1,-3) {$\tilde{A}$};
\end{tikzpicture}$

Here, arrows with $\star$ are local maps, and so, this gives a pair of maps $(A \leftarrow \tilde{A} \rightarrow A')$, which is a typical element in the “Hom” class in the Verdier quotient.

**WARNING!**: In this article, we follow the convention of \[Nee01, Def.1.5.1\] \[Kra10, 4.5\] for a triangulated subcategory, which is automatically full by this convention. On the other hand, it is not so in the convention of \[Tho97\] p.3.1.1.

6 Verdier quotient does not necessarily have small Hom-sets.
• The Verdier localization functor

\[ F_{\text{univ}} : \mathcal{T} \to \mathcal{T}/S \]

\[ X \mapsto X \]

\[ (X \xrightarrow{f} Y) \mapsto (X \xrightarrow{id X} X \xrightarrow{f} Y) \]

(3)

is universal for all triangulated functors \( F : \mathcal{T} \to \mathcal{T} \) which sends all morphisms \( (Z \xrightarrow{l} X) \) with \( \text{Cone}(l) \in \text{Ob}(S) \) to invertible morphisms.

• The triangulated structure of \( \mathcal{T}/S \) is induced from that of \( \mathcal{T} \) via the Verdier localization functor \( F_{\text{univ}} \):

– The suspension \( \Sigma_{\mathcal{T}/S} \) of \( \mathcal{T}/S \) is induced from the suspension \( \Sigma_{\mathcal{T}} \) of \( \mathcal{T} \):

\[ \Sigma_{\mathcal{T}/S} : \mathcal{T}/S \to \mathcal{T}/S \]

\[ X \mapsto \Sigma_{\mathcal{T}} X \]

\[ (X \xleftarrow{Z} Z \xrightarrow{f} Y) \mapsto (\Sigma_{\mathcal{T}} X \xrightarrow{\Sigma_{\mathcal{T}} l} \Sigma_{\mathcal{T}} Z \xrightarrow{\Sigma_{\mathcal{T}} f} \Sigma_{\mathcal{T}} Y) \]

– A distinguished triangle in \( \mathcal{T}/S \) is isomorphic to the Verdier localization functor \( F_{\text{univ}} \) image of a distinguished triangle in \( \mathcal{T} \).

As is always the case with such a localization procedure, the Verdier localization does not necessarily have small Hom-sets. It was Neeman’s insight \cite{Nee92b, Nee96, Nee01} to make use of the Bousfield localization \cite{Bou79}, which was introduced in the context of stable homotopy theory, to take care of this problem in general triangulated category theory.

To explain this theory of Neeman, we now prepare some definitions.

**Definition 2.2.** (WARNING! A **triangulated subcategory** is by definition \cite[Def.1.5.1]{Nee01} \cite[4.5]{Kra10} automatically full.)

1. A triangulated subcategory \( S \) of a triangulated category \( \mathcal{T} \) with small coproducts is called localizing, if it is closed under coproducts in \( \mathcal{T} \).
2. A triangulated subcategory \( S \) of \( \mathcal{T} \) is called thick, if it closed under direct summands in \( \mathcal{T} \).
3. \cite[p99,Rem.2.1.39]{Nee01} The thick closure \( \hat{S} \) of a triangulated subcategory \( S \) of a triangulated category \( \mathcal{T} \) is the triangulated subcategory of \( \mathcal{T} \) consisting of direct summands in \( \mathcal{T} \) of objects in \( S \).
4. \cite[1.4]{Tho97} A triangulated subcategory \( S \) of a triangulated category \( \mathcal{T} \) is called dense, if \( \hat{S} = \mathcal{T} \).

**Remark 2.3.** (i) Every localizing triangulated subcategory is thick, for any direct summand decomposition in \( \mathcal{T} \):

\[ \exists x \in \bigoplus (1 - e)x \]

(can be realized using the cones in \( S \)):

\[
\begin{cases}
  ex = \text{Cone} \left( \bigoplus N x \to \bigoplus N x : (\xi_n)_{n \in \mathbb{N}} \mapsto (\xi_n - e\xi_{n-1})_{n \in \mathbb{N}} \right) \\
  (1 - e)x = \text{Cone} \left( \bigoplus N x \to \bigoplus N x : (\xi_n)_{n \in \mathbb{N}} \mapsto (\xi_n - (1 - e)\xi_{n-1})_{n \in \mathbb{N}} \right)
\end{cases}
\]
(ii) A triangulated subcategory $S$ of a triangulated category $T$ is thick if and only if $S = \hat{S}$.

(iii) [Nee01, p99,Rem.2.1.39] The thick closure is nothing but the kernel of the Verdier localization functor: For a triangulated subcategory $S$ of a triangulated category $T$, $\hat{S} = \text{Ker}(F_{\text{univ}} : T \to T/S)$.

(iv) [Nee01, p.148,Cor.4.5.12] If $S$ is a dense triangulated subcategory of a triangulated category $T$, then,

$$\forall x \in T, \quad x \oplus \Sigma x \in S.$$ (4)

To see this, since $\exists y \in T$ s.t. $x \oplus y \in S$, form a triangle:

$$x \oplus 0 \oplus y \xrightarrow{0 \oplus 0 \oplus \text{id}_Y} 0 \oplus x \oplus y \xrightarrow{0 \oplus \text{id}_x \oplus \text{id}_Y} \Sigma x \oplus x \oplus 0,$$

where the first and the second terms are contained in $S$: $x \oplus 0 \oplus y \cong 0 \oplus x \oplus y \cong x \oplus y \in S$, and so is the third term: $\Sigma x \oplus x \cong \Sigma x \oplus x \oplus 0 \in S$, as desired.

From Remark 2.3 (iii), to search for criteria which guarantee the Verdier quotient to have small Hom-sets, we may start with a thick triangulated subcategory $S$ of $T$. Also, while the original Bousfield localization [Bou79] require $T$ to have small coproducts, there are many cases where we wish Verdier quotients $T/S$ to have small Hom-sets, even when $T$ does not have small coproducts. Now, Neeman [Nee01] proposed the following general definition for Bousfield localization:

**Definition 2.4.** [Nee01, Def.9.1.1,Def.9.1.3,Def.9.1.4,Def.9.1.10, Kra10] (i) Let $S$ be a thick subcategory of a triangulated category $T$. Then the pair $S \subset T$ is said to possess a Bousfield localization functor when the Verdier localization functor $F_{\text{univ}} : T \to T/S$ has a right adjoint $G : T/S \to T$, which is called the Bousfield localization functor. The resulting composite

$$L := G \circ F_{\text{univ}} : T \xrightarrow{F_{\text{univ}}} T/S \xrightarrow{G} T$$

is also called the Bousfield localization functor by an abuse of terminology.

(ii) $S \subset T$ is, by definition, the full subcategory of $S$-colocal objects.

(iii) $S^\perp \subset T$ is, by definition, the full subcategory of $L$-local objects or $S$-local objects.

An adjoint functor between triangulated categories showed up in the above definition, but such an adjoint functor actually becomes a triangulated functor:

**Lemma 2.5.** [Nee01, Lem.5.3.6] Suppose a pair of adjoint functors between triangulated categories are given:

$$S \xleftarrow{F} T \xrightarrow{G} S$$

If either one of $F$ or $G$ is a triangulated functor, then so is the other.

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7 If $T$ is essentially small, this result also follows immediately from a general result reviewed later in Proposition 4.3.

8 We do not require $T$ to have small coproducts in this definition.
We shall freely use this useful fact for the rest of this article.

Still, readers might worry that the more existence of a right adjoint $G : \mathcal{T}/S \to \mathcal{T}$ in the definition of the above Bousfield localization too weak. However, in this particular case, we have a very special property that the natural map from the category of fractions $\mathcal{T} \left[ \Sigma(F_{\text{univ}})^{-1} \right]$ to the Verdier quotient $\mathcal{T}/S$ becomes an equivalence:

$$\mathcal{T} \left[ \Sigma(F_{\text{univ}})^{-1} \right] \xrightarrow{\cong} \mathcal{T}/S,$$

where $\Sigma(F_{\text{univ}})$ is the collection of morphisms in $\mathcal{T}$ whose image in $\mathcal{T}/S$ is invertible, i.e. those maps in $\mathcal{T}$ whose mapping cone is in $S$. And, using this useful fact, we can see any right adjoint $G : \mathcal{T}S \to \mathcal{T}$ is fully faithful by applying the following useful fact:

**Lemma 2.6.** (see [GZ67, I,Prop.1.3] [Kra10, Prop.2.3.1]). For an adjoint pair: $\xymatrix{ \mathcal{C} \ar[r]^-F & \mathcal{D} & G \ar[l]_-G}$, the following conditions are equivalent:

- The right adjoint $G$ is fully faithful.
- The adjunction $F \circ G \to \text{Id}_\mathcal{D}$ is an isomorphism.
- The functor $\overline{F} : \mathcal{C} \left[ \Sigma(F)^{-1} \right] \to \mathcal{D}$ satisfying $F = \overline{F} \circ Q_{\Sigma(F)}$ is an equivalence, where $\Sigma(F)$ is the collection of morphisms in $\mathcal{T}$ whose images in $\mathcal{T}'$ by $F$ becomes invertible, and $Q_{\Sigma(F)} : \mathcal{C} \to \mathcal{C} \left[ \Sigma(F)^{-1} \right]$ is the canonical quotient functor to the category of fractions.

Thus, from Lemma 2.6 and Lemma 2.5 we obtain the following:

**Proposition 2.7.** Any right adjoint $G : \mathcal{T}/S \to \mathcal{T}$ in Neeman’s definition of the Bousfield localization Definition 2.4 is automatically a fully faithful triangulated functor.

In fact, as is well known, if a triangulated functor $F : \mathcal{T} \to \mathcal{T}'$ enjoys good properties listed in Lemma 2.6 then we have the following very useful result:

**Proposition 2.8.** (see e.g. [Rou10, Lem.3.4]) If a triangulated functor $F : \mathcal{T} \to \mathcal{T}'$ has a fully faithful right adjoint $G$ or a right adjoint $G$ with its adjunction an isomorphism $F \circ G \xrightarrow{\cong} \text{Id}_\mathcal{D}$, then $\text{Ker} F$ becomes a thick triangulated subcategory of $\mathcal{T}$, and $F$ induces the following equivalence of triangulated categories:

$$\mathcal{T}/\text{Ker} F \xrightarrow{\cong} \mathcal{T}'.$$

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9 This is an adjoint pair of functors between ordinary categories, and we are not considering any triangulated structure.

10 Goes back at least to Verdier.

11 Let us recall the following precursor of this result in the setting of abelian categories, which goes back at least to Gabriel (see also [Rou10, Lem.3.2]): If an exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories has a fully faithful right adjoint $G$ (i.e. the adjunction $F \circ G \to \text{Id}_\mathcal{B}$ is an isomorphism, then $\text{Ker} F$ is Serre subcategory of $\mathcal{A}$, and $F$ induces the following equivalence of abelian categories: $\mathcal{A}/\text{Ker} F \xrightarrow{\cong} \mathcal{B}$, where the left hand side is the abelian quotient category in the sense of Gabriel, Grothendieck, Serre.
Going back to Bousfield localization, we prepare some more definitions to state its basic properties.

**Definition 2.9.**

1. (WARNING! These conventions are those of [Kra10, 4.8], which are the opposite of [Nee01, Def.9.1.10; Def.9.1.11]!) For a full subcategory $A$ of $T$, define the full subcategory $A^\perp$ of $T$ by

   $$A^\perp = \{ t \in T \mid \text{Hom}_T(A, t) = 0 \}. $$

   Dually, $^\perp A$ is defined by

   $$^\perp A = \{ t \in T \mid \text{Hom}_T(t, A) = 0 \}. $$

2. For full subcategories $A$ and $B$ of $T$, denote by $A*B$ the full subcategory of $T$ consisting of all objects $y \in T$ for which there exists a triangle $x \to y \to z \to \Sigma x$ with $x \in A$ and $z \in B$.

**Proposition 2.10.** [Nee01, Prop.9.1.18; Th.9.1.16; Th.9.1.13; Cor.9.1.14] [Kra10, Prop.4.9.1]

Let $S$ be a thick subcategory of a triangulated category $T$. Then the following assertions are equivalent.

1. The inclusion functor $I : S \hookrightarrow T$ has a right adjoint $\tilde{\Gamma} : T \to S$.
2. $T = S*S^\perp$.
3. $S \subset T$ possesses a **Bousfield localization functor**, i.e. the Verdier localization functor $F_{\text{univ}} : T \to T/S$ has a right adjoint $G : T/S \to T$.
4. The composite $E : S^\perp \hookrightarrow T \to T/S$ is an equivalence.
5. The inclusion $J : S^\perp \hookrightarrow T$ has a left adjoint $T \to S^\perp$ and $^\perp(S^\perp) = S$.

These equivalent conditions can be succinctly expressed, via the standard adjoint functor notation, as follows:

$$S \xrightarrow{I} T \xrightarrow{F_{\text{univ}}} T/S$$

**Remark 2.11.** Assume that the inclusion $I : S \hookrightarrow T$ has a right adjoint $\tilde{\Gamma}$ as in Proposition 2.10.1. Then, for each $t \in T$, embed the counit of adjunction $\Gamma(t) = \tilde{\Gamma}(t) \to t$, where $\Gamma : T \to T$ is called the Bousfield colocalization functor for the pair $T \to T/S$, into a triangle

$$\Gamma(t) \to t \to L(t) \to \Sigma \Gamma(t),$$

which yields a functor $L : T \to T$. Then we see $L(t) \in S^\perp$, which

- implies $T = S*S^\perp$ in Proposition 2.10.2;

---

12 We do not require $T$ to have small coproducts.
13 An arrow above is left adjoint to the arrow below.
14 A Bousfield colocalization functor means its opposite functor is a Bousfield localization functor [HPS97, Def.3.1.1] [Kra10, 2.8]. **WARNING:** This terminology is not consistent with that of Bousfield [Bou79] (see [HPS97, Rem.3.1.4] ).
• yields a left adjoint \( \tilde{L} : T \to S^\perp \) to the inclusion \( J : S^\perp \to T \), stated in Proposition 2.10.5, and \( \tilde{L} \) yields the Bousfield localization functor, recovering the above functor \( L \) by the composition

\[
L = J \circ \tilde{L} : T \to S^\perp \to T. 
\]

(6)

• yields a left adjoint \( G : T/S \to T \) to the Verdier localization functor \( F_{\text{univ}} : T \to T/S \) as the composition \( G : T/S \xrightarrow{\tilde{L}} S^\perp \to T \) stated in Proposition 2.10.3, and,

• assuming Proposition 2.10.4, \( \tilde{L} \) is equivalent to \( E^{-1} \circ F_{\text{univ}} : T \to T/S \to S^\perp \).

Remark 2.12. Actually, the property in Proposition 2.10.2 is exactly what Bondal-Orlov [BO02, Def.3.1] call semiorthogonal decomposition and denote by

\[
T = \langle S^\perp, S \rangle. 
\]

(7)

Of course, the fundamental question is when Bounsfield location exists. Now, Neeman’s insight [Nee01, Th.8.4.4] is to apply Brown representability to construct Bousfield localization. We now review this development following mostly Krause [Kra02, Kra10].

Definition 2.13. Let \( T \) be a triangulated category with small coproducts.

(i) [Nee01, Def.6.2.8] A set \( G \) of objects in \( T \) is said to generate \( T \), if \((\bigcup_{n \in \mathbb{Z}} \Sigma^n G)^\perp = 0\), i.e., given \( t \in T \),

\[
\forall g \in G, \forall n \in \mathbb{Z}, \quad \text{Hom}_T(\Sigma^n g, t) = 0 \quad \implies \quad t = 0.
\]

(ii) An element \( t \in T \) is called compact if, for every set of objects \( \{t_\lambda\}_{\lambda \in \Lambda} \) in \( T \), the natural map

\[
\bigoplus_{\lambda \in \Lambda} \text{Hom}_T(t_\lambda, t) \to \text{Hom}_T(t, \bigoplus_{\lambda \in \Lambda} t_\lambda)
\]

is an isomorphism.

(iii) \( T \) is called compactly generated, if \( T \) is generated by a set of compact objects in \( T \).

(iii) (c.f. [Kra02, Def.1] [Kra10, 5.1] (see also [Nee01, Def.8.1.2])) A set of objects \( P \) in \( T \) is said to perfectly generate \( T \), if,

(1) \( P \) generates \( T \),

(2) for every countable set of morphisms \( x_i \to y_i \) in \( T \) such that \( T(p, x_i) \to T(p, y_i) \) is surjective for all \( p \in P \) and \( i \), the induced map

\[
T(p, \prod_i x_i) \to T(p, \prod_i y_i)
\]

is surjective.

\( T \) is called perfectly generated, if \( T \) is perfectly generated by a set \( P \) of objects in \( T \).

\footnote{Strictly speaking, the definition here is slightly differently from Krause’s, but essentially the same.}
Remark 2.14. Any compactly generated triangulated category is perfectly generated.

Theorem 2.15 (Brown representability). ([Kra02] Th.A) Suppose a triangulated category $\mathcal{T}$ is perfectly generated.

1. A functor $F : \mathcal{T}^{op} \to Ab$, the category of abelian groups, is cohomological and sends coproducts in $\mathcal{T}$ to products in $Ab$ if and only if $F \cong \mathcal{T}(\cdot, t)$ for some object $t$ in $\mathcal{T}$.

2. A triangulated functor $\mathcal{T} \to \mathcal{U}$ preserves small coproducts if and only if it has a right adjoint.

From the second part of this theorem and the second characterization of Bousfield localization in Proposition 2.10, we immediately obtain the following:

Corollary 2.16 (Existence of Bousfield localization). ([Kra10, Prop.5.2.1] [Nee01, Prop.9.1.19] Bousfield localization exists for any perfectly generated triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$, a triangulated category with small coproducts.

Corollary 2.17. Bousfield localization exists for any compactly generated triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$, a triangulated category with small coproducts.

To be precise, the “compactly generated” assumption adapted in [Nee92b, Lem.1.7] meant the smallest localizing triangulated subcategory containing the generating set is the entire triangulated category. But this can be reconciled by the following corollary of Corollary 2.16:

Corollary 2.18. ([Nee01, Th.8.3.3;Prop.8.4.1] Suppose $\mathcal{T}$ is perfectly generated by a set $P$ of objects in $\mathcal{T}$, then $\mathcal{T} =$ the smallest localizing triangulated subcategory containing $P$.

For a special case of Corollary 2.17, Neeman and Miller gave a simple explicit homotopy theoretical construction of Bousfield localization with a nice property:

Theorem 2.19. ([Mil92] [Nee92b, Lem.1.7] For any localizing triangulated subcategory $\mathcal{R}$ of a compactly generated triangulated category with small coproducts $\mathcal{T}$ such that $\mathcal{R}$ is the smallest coidealizing triangulated subcategory containing a set $R$ consisting of compact objects in $\mathcal{T}$,

1. Bousfield localization exists, given explicitly by Miller’s finite localization.

This claim itself is a special case of Corollary 2.17.
• Then Miller’s finite localization of \( x \in \mathcal{T} \) is simply given by the mapping telescope:
\[
x \rightarrow Lx := \text{hocolim}(x_n).
\]

(2) Miller’s finite localization is smashing, i.e. \( L \) preserves arbitrary coproducts.

Let us record the above definition of “smashing”, because this definition of “smashing” without smash (tensor) product is not the traditional Ravenel’s definition [Rav84]:

**Definition 2.20.** [Kra10, 5.5] A Bousfield localization \( L : \mathcal{T} \to \mathcal{T} \) is smashing if \( L \) preserves arbitrary coproducts in \( L \). Then, \( S = \text{Ker } L \) is also called smashing.

We have the following equivalent characterizations of smashing Bousfield localization without smash (tensor) product:

**Proposition 2.21.** [Kra10, Prop.5.5.1] For a thick subcategory \( S \) of a triangulated category with small coproducts, suppose there is a Bousfield localization \( L = G \circ F_{\text{univ}} : \mathcal{T} \to \mathcal{T} \) for the pair \( S \to \mathcal{T} \) in the following set-up: (see (8)):
\[
S \xrightarrow{I} \mathcal{T} \xrightarrow{F_{\text{univ}}} \mathcal{T}/S
\]

Then the following conditions are equivalent:

1. Bousfield localization \( L = G \circ F_{\text{univ}} \) is smashing, i.e. \( L = G \circ F_{\text{univ}} : \mathcal{T} \to \mathcal{T} \) preserves coproducts (see Definition 2.21).
2. Bousfield colocalization \( \Gamma = I \circ \tilde{\Gamma} : \mathcal{T} \to \mathcal{T} \) preserves coproducts.
3. The right adjoint \( G : \mathcal{T}/S \to \mathcal{T} \) of the Verdier quotient \( F_{\text{univ}} : \mathcal{T} \to \mathcal{T}/S \) preserves coproducts.
4. The right adjoint \( \tilde{\Gamma} : \mathcal{T} \to S \) of the canonical inclusion \( I : S \to \mathcal{T} \) preserves coproducts.
5. The full subcategory \( S^\perp \) of all \( L \)-local \( (S\)-local) objects is localizing.

If \( \mathcal{T} \) is perfectly generated, in addition the following is equivalent.

6. In the set-up (8), both \( \tilde{\Gamma} \) and \( G \) have right adjoints and (3) is amplified to a recollement [18] of the following form:
\[
S \xrightarrow{I} \mathcal{T} \xrightarrow{F_{\text{univ}}} \mathcal{T}/S
\]

Later in Proposition [2.28], all of these conditions are shown to be equivalent to Ravenel’s [Rav84], when \( \mathcal{T} \) is a rigidly compactly generated tensor triangulated category. Smashing localization is frequently referred in the context of the telescope conjecture, which asks whether the converse of the second claim in Theorem 2.19 holds or not [19]:

17 This “perfectly generated” condition is used to apply Brown representability (Theorem 2.15) to construct two right adjoints in the recollement.
18 For the precise definition of recollement, consult [BBD82, 1.4].
19 Strictly speaking, this is the telescope conjecture without smash (tensor) product, but coincides with the original Ravenel’s telescope conjecture for \( \mathcal{T} = \mathcal{SH} \), and more generally for rigidly
Conjecture 2.22 (Telescope conjecture without smash (tensor) product).

[HPS97 Def.3.3.2,Def.3.3.8] (see also Proposition 2.28) In a rigidly compactly
generated tensor triangulated category $T$, a smashing localization $L: T \to T$ is a
finite localization, i.e. $\text{Ker } L$ is generated by a set of compact objects in $T$.

After we take into account the tensor product structure, we shall revisit the finite
localization and the telescope conjecture in Theorem 3.3 For now, we record
another easy consequence of Miller’s finite localization construction presented in The-
orem 2.19.

Proposition 2.23. (See [Nee92b p.556, from 7th to 10th lines])

Let $R$ be a set of compact objects in a triangulated category with small coproducts
$\mathcal{T}$, and $\mathcal{R}$ be the smallest localizing triangulated subcategory contain-
ing $R$.

Then, every element in $\mathcal{R}^c$ is isomorphic in $\mathcal{R}^c$ to a direct summand of a finite
extensions of finite coproducts of elements in $R$. In particular, $\mathcal{R}^c$ is essentially
small.

In fact, for any $X \in \mathcal{R}$, the Bousfield localization with respect to the pair $\langle R \rangle =
\mathcal{R} \subset \mathcal{R}$, is trivial for any $x \in \mathcal{R}$:

$$x \to Lx := \text{hocolim}(x_n) \simeq 0.$$ 

Then, if $x \in \mathcal{R}^c$, this map becomes trivial at some “finite” stage, which implies $x$
is a direct summand of a finite extensions of finite coproducts of elements in $R$, as
claimed.

2.2. Bousfield classes and Ohkawa’s theorem. Now we focus on a special case:
let $\mathcal{T} = SH$ be the homotopy category of spectra. Then $\mathcal{T}$ is a triangulated
category with coproducts. It has the smash product $\wedge: \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ and the unit object
$S^0 \in \mathcal{T}$ which make $\mathcal{T}$ a tensor triangulated category.

The smash product preserves coproducts in each variable. $\mathcal{T}$ is generated by $\{S^0\}$,
and $\mathcal{T}$ satisfies Brown representability.

For each $H \in \mathcal{T}$, put $H_* = H \wedge (-)$. We consider the localizing triangulated
subcategory

$$\text{Ker } H_* = \{ t \in \mathcal{T} \mid H \wedge t = 0 \},$$

which is called the Bousfield class of $H$.

Theorem 2.24 (Bousfield [Bou79]). Let $\mathcal{T} = SH$ be the homotopy category of
spectra.

1. If $S \subset SH$ is a localizing triangulated subcategory which is generated by a
set of objects, then a Bousfield localization exists for $S$.
2. For every $H \in SH$, there exists a set of objects which generates $\text{Ker } H_*$. 

Therefore a Bousfield localization exists for $\text{Ker } H_*$. 

compactly generated tensor triangulated categories [HPS97 Def.3.3.2,Def.3.3.8] (see also Proposition 2.28).

For a serious treatment of the definition of “tensor triangulated category,” consult [May01].
Now, we can state the truly surprising theorem of Ohkawa:  

**Theorem 2.25** (Ohkawa [Ohk89]). \{ Ker $H_*$ | $H \in \mathcal{SH}$ \} is a set.  

We note that no explicit structure of this set is known.  
For more details, including a proof, of the Ohakawa theorem, see the survey [CasPr] in this proceedings.  

2.3. **Casacuberta-Gutiérrez-Rosický theorem, motivic analogue of Ohkawa’s theorem.** Ohkawa’s theorem is a statement in the stable homotopy category $\mathcal{SH}$, which is “a part” of the Morel-Voevodsky stable homotopy category $\mathcal{SH}(k)$ when $k \subseteq \mathbb{C}$, via the retraction of the following form:  

$$  
\begin{array}{ccccc}
\mathcal{SH} & \longrightarrow & \mathcal{SH}(k) & R_\kappa & \mathcal{SH} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{SH} & \longrightarrow & \mathcal{SH}(k) & id & \mathcal{SH} 
\end{array}  
$$  

(10)  

So, a natural question here is whether there is a shadow of Ohkawa’s theorem in this algebro-geometrical setting, i.e. whether there is a motivic analogue of Ohkawa’s theorem or not.  
Now, Casacuberta-Gutiérrez-Rosický [CGR14] answered this question affirmatively under some very mild assumption.  

**Theorem 2.26.** [CGR14 Cor.3.6] For each Noetherian scheme $S$ of finite Krull dimension, there is only a set of distinct Bousfield classes in the stable motivic homotopy category $\mathcal{SH}(S)$ with base scheme $S$.  

Once again, no explicit structure of this set is known.  
For various generalizations of Ohkawa’s theorem, see afore-quoted [CGR14], also [KOSPr] and the review [CRPr], both in this proceedings.  

2.4. **Localizing tensor ideals of derived categories and the fundamental theorem of Hopkins, Neeman, Thomason and others.** In both Ohkawa’s Theorem 2.25 and its algebro-geometric shadow Theorem 2.26, the resulting sets are completely mysterious and beyond our imagination. However, if we take a look at the algebro-geometrical shadow of Ohkawa’s theorem from a different angle, i.e. by considering $D_{qc}(X)$ for a fixed Noetherian scheme instead of $\mathcal{SH}(k)$, then we see an explicit set representing clear algebro-geometric information. This is the fundamental theorem of Hopkins, Neeman, Thomason, and others, which has been the guiding principle of the area.  
Now, the tensor structure is essential for this fundamental theorem, and we must start with some review of fundamental facts about general tensor triangulated categories and Bousfield localization from the tensor triangulated category point of view.  

---  

21 Somewhat surprisingly, Ohkawa’s theorem had been elusive from researchers’ attention for more than a decade. It was the paper of Dwyer and Palmieri [DP01] which drew researchers’ attention to Ohkawa’s surprising theorem.  
22 For a concise summary of the academic life of Professor Tetsusuke Ohkawa, see [MatPr] in this proceedings.
Definition 2.27. Let $\mathcal{T}$ be a tensor triangulated category.

1. A triangulated subcategory $\mathcal{I}$ of $\mathcal{T}$ is called a tensor ideal prime if
\[
\mathcal{T} \otimes \mathcal{I} \subset \mathcal{I};
\]
and it is a tensor ideal and $(\mathcal{T} \setminus \mathcal{I}) \otimes (\mathcal{T} \setminus \mathcal{I}) \subset (\mathcal{T} \setminus \mathcal{I}) \neq \emptyset$.

2. [LMS86, Chapter III] (see also [HPS97, App.A], [BF11, p.1163]) An element $x$ in a closed symmetric monoidal triangulated category $(\mathcal{T}, \otimes, \text{Hom})$ is called strongly dualizable or simply rigid, if the natural map $Dx \otimes y \to \text{Hom}(x, y)$, where $Dx := \text{Hom}(x, 1)$, is an isomorphism for all $y \in \mathcal{T}$.

3. [HPS97, Def.1.1.4] (see also [BF11, Hyp.1.1]) A closed symmetric monoidal triangulated category $(\mathcal{T} = \langle G \rangle, \otimes, \text{Hom})$ is called a unital algebraic stable homotopy category or a rigidly compactly generated tensor triangulated category, if $1$ is compact and $\mathcal{T} = \langle G \rangle$ for a set $G$ of rigid and compact objects.

Now, we are ready to reconcile our previous definition (Definition 2.20) of smashing localization with Ravel’s original definition in [Rav84] for rigidly compactly generated tensor triangulated categories:

Proposition 2.28. [HPS97, Def.3.3.2] For a thick subcategory $\mathcal{S}$ of a closed symmetric monoidal triangulated category with small coproducts $(\mathcal{T} = \langle G \rangle, \otimes, \text{Hom})$, suppose there is a Bousfield localization $L : \mathcal{T} \to \mathcal{T}$ for the pair $\mathcal{S} \to \mathcal{T}$. Consider the following “smishing” conditions:

(S): (Ravenel’s original definition of smashing localization [Rav84]):
\[
L \cong L(1) \otimes -, \quad \text{where } 1 \text{ is the unit object of } (\mathcal{T}, \otimes).
\]

(C): (The definition of smashing localization in Definition 2.20):
\[
L \text{ preserves arbitrary coproducts.}
\]

Then, the implication (S) $\implies$ (C) always holds. If $\mathcal{T}$ is also a rigidly compactly generated tensor triangulated category, the converse (C) $\implies$ (S) also holds, and so, (C) and (S) become equivalent.

Proof. The implication (S) $\implies$ (C) is easy:
\[
L\left(\bigoplus_{\lambda} x_{\lambda}\right)^{(S)} \cong L(1) \otimes \left(\bigoplus_{\lambda} x_{\lambda}\right) \cong \bigoplus_{\lambda} \left(L(1) \otimes x_{\lambda}\right)^{(S)} \cong \bigoplus_{\lambda} Lx_{\lambda}.
\]

23 If $x \in \mathcal{T}$ is strongly dualizable, i.e., rigid, the natural map $x \to D^2x$ is an isomorphism [LMS86, Chapter III] [HPS97, Th.2.5.(b)].

24 In a rigidly compactly generated tensor triangulated category, any compact object is rigid, for, by Proposition 2.23, any compact object is seen to be isomorphic to a direct summand of a finite extensions of finite coproducts of rigid elements. In particular, in a rigidly compactly generated tensor triangulated category, $1$ is both rigid and compact.

25 Recall in this case $\mathcal{T}$ becomes distributive, because for any objects $x_{\lambda}$ ($\lambda \in \Lambda$), $y, z$ in $\mathcal{T}$,
\[
\text{Hom}\left(\bigoplus_{\lambda} x_{\lambda} \otimes y, z\right) \cong \bigoplus_{\lambda} \text{Hom}(x_{\lambda}, \text{Hom}(y, z)) \cong \bigoplus_{\lambda} \text{Hom}(x_{\lambda} \otimes y, z) \cong \bigoplus_{\lambda} \text{Hom}(x_{\lambda} \otimes y, z) \cong \text{Hom}\left(\bigoplus_{\lambda} x_{\lambda} \otimes y, z\right).
\]
For the converse \((C) \implies (S)\), first note that \((C)\) implies those \(x \in \mathcal{T}\) which satisfies \(L(1) \otimes x \cong Lx\) form a localizing triangulated subcategory of \(\mathcal{T}\), even without the rigidly compactly generated assumption. For instance, if \(L(1) \otimes x_\lambda \cong Lx_\lambda \ \forall \lambda \in \Lambda\), then

\[
L(1) \otimes (\oplus x_\lambda) \cong \oplus (L(1) \otimes x_\lambda) \cong \oplus Lx_\lambda \ \overset{(C)}{=} L(\oplus x_\lambda).
\]

Now, we are reduced to showing \(L(1) \otimes g \cong Lg\) for any rigid element \(g\). For this, we start with the tensor product of the localization distinguished sequence for \(1\) with \(g\):

\[
\Gamma(1) \otimes g \rightarrow (g \cong 1 \otimes g) \rightarrow L(1) \otimes g,
\]

and apply the Bousfield localization \(L\) to drive the equivalence \(L(1) \otimes g \cong Lg\) as follows:

\[
\begin{align*}
\left(\ast \ \overset{(TI)}{\cong} L(\Gamma(1) \otimes g) \right) & \rightarrow (Lg \cong L(1 \otimes g)) \\
\overset{\sim}{\Rightarrow} \left( L(L(1) \otimes g) \overset{(R)}{=} L\text{Hom}(Dg, L(1)) \overset{(L)}{=} \text{Hom}(Dg, L(1)) \overset{(R)}{=} L(1) \otimes g \right),
\end{align*}
\]

where \((TI)\) holds because \(\text{Ker} \ L\) is a tensor ideal, \((R)\) holds because \(g\) is rigid, and \((L)\) holds because \(\text{Hom}(Dg, L(1))\) is \(L\)-local.

\[
\square
\]

In general, when we talk about smashing Bousfield localization in tensor triangulated setting, we adopt the following equivalent conditions, where the localizing tensor ideal \(\mathcal{I}\) is called a \textit{smashing ideal} \cite[Def.2.15]{BF11}:

\textbf{Proposition 2.29 (See \cite[Th.2.13]{BF11}).} Let \(\mathcal{T}\) be a tensor triangulated category with coproducts, and let \(\mathcal{I}\) be a localizing tensor ideal of \(\mathcal{T}\) for which a Bousfield localization exists. Define the Bousfield localization functor \(L\colon \mathcal{T} \rightarrow \mathcal{I}^\perp\) as in Remark \ref{rem:smashingideal}. Then the following assertions are equivalent.

\begin{enumerate}
\item \textbf{(TI)} \(\mathcal{I}^\perp\) is a tensor ideal. That is, \(\mathcal{I} \otimes \mathcal{I}^\perp \subset \mathcal{I}^\perp\).
\item \textbf{(S)} \(L\) is smashing in Ravenel’s sense: \(L \cong L(1) \otimes -\).
\end{enumerate}

\textbf{Remark 2.30.} \((TI)\) is a tensor triangulated analogue of Proposition \ref{prop:smashingideal}.

\textbf{Proof of Proposition 2.29.} Now, for the implication \((TI) \implies (S)\), consider the tensor product of the localization distinguished sequence for \(1\) with \(x \in \mathcal{T}\):

\[
\Gamma(1) \otimes x \rightarrow (x \cong 1 \otimes x) \rightarrow L(1) \otimes x,
\]

where \(\Gamma(1) \otimes x \in \mathcal{I}\) because \(\mathcal{I}\) is a tensor ideal by assumption, and \(L(1) \otimes x \in \mathcal{I}^\perp\) because \(\mathcal{I}^\perp\) is also a tensor ideal by \((TI)\). Then, from the uniqueness of the localization distinguished sequence for \(x \in \mathcal{T}\), we find \(Lx \cong L(1) \otimes x\), which implies \((S)\).

The converse \((S) \implies (TI)\) is easy; for, if \(l = L(l) \in \mathcal{I}^\perp\) be a \(\mathcal{I}\)-local object and \(x \in \mathcal{T}\), then

\[
l \otimes x = L(l) \otimes x \overset{(S)}{=} (L(1) \otimes l) \otimes x = L(1) \otimes (l \otimes x) \overset{(S)}{=} L(l \otimes x) \in \mathcal{I}^\perp.
\]

\[
\square
\]
In the above proposition, we started with a localizing tensor ideal for which a Bousfield localization exists. However, we have the following example of a localizing tensor ideal for which an existence of the Bousfield localization is problematic:

**Example 2.31.** Let $\mathcal{T} = \mathcal{SH}$ be the homotopy category of spectra. For every $H \in \mathcal{T}$, its Bousfield class $\text{Ker\,}H^*$ is a localizing tensor ideal. The subcategory

$$\text{Ker\,}H^* = \{ \, t \in \mathcal{T} \mid \text{Hom}(t, \Sigma^i H) = 0 \text{ for all } i \in \mathbb{Z} \}$$

called the cohomological Bousfield class of $H$, is also a localizing tensor ideal. Actually, as was noticed by Hovey [Hov95b, Prop. 1.1], any Bousfield class is a cohomological Bousfield class:

$$\text{Ker\,}H^* = \text{Ker\,(IH)^*},$$

where $IH$ is the Brown-Comenetz dual of $H$, characterized by: $(IH)^*(t) = \text{Hom\,(}H_*(t), \mathbb{Q}/\mathbb{Z})$, $\forall t \in \mathcal{T}$.

Here, Hovey [Hov95b] and Hovver-Palmieri [HP99] proposed the following conjectures, any one of which implies that an arbitrary localizing tensor ideal $\text{Ker\,}H^*$ admits a Bousfield localization:

**Conjecture 2.32.**

(i) [Hov95b, Conj. 1.2] Every cohomological Bousfield class is a Bousfield class.

(ii) Every localizing tensor ideal is a Bousfield class.

(iii) [HP99, Conj. 9.1] Every localizing triangulated subcategory is a Bousfield class.

Of course, $(iii) \implies (ii) \implies (i)$, for we have an obvious inclusions of classes:

- Bousfield-Ohkawa set := The class of Bousfield classes
- $\subseteq$ The class of cohomological Bousfield classes
- $\subseteq$ The class of localizing tensor ideals
- $\subseteq$ The class of localizing triangulated subcategories,

where all the inclusions become $=$ if the above conjecture $(iii)$ holds. However, even $(i)$ is still open, and so it is still unknown even whether the the class of cohomological Bousfield classes becomes a set or not. Similarly, it is still unknown even whether any cohomological Bousfield class admits a Bousfield localization or not. Here, we shall show an analogue of $(ii)$ holds holds with an explicit geometric description of its set structure for $D_{qc}(X)$.

For a scheme $X$, $D_{qc}(X)$ is the derived category of complexes of arbitrary modules on $X$ whose cohomologies are quasi-coherent. If $X$ is quasi-compact and separated, then $D_{qc}(X)$ is equivalent to $D(\mathcal{QCoh}
\, X)$, where $D(\mathcal{QCoh}\, X)$ is the derived category of complexes of quasi-coherent sheaves on $X$ ([BN93, Corollary 5.5]). Here we have the nice theorem of Gabriel [Gab62] and Rosenberg [Ros04]:

**Theorem 2.33.** Any quasi-compact and separated scheme $X$ can be reconstructed from $\mathcal{QCoh}\, X$.

---

26 Conjecture 2.32 should be taken more seriously. In fact, Professor Peter May is very glad to see Conjecture 2.32 is advertised here.
Glancing at this theorem of Gabriel and Rosenberg, we naturally hope $D_{qc}(X) \cong D(QCoh X)$ would carry rich information of $X$.

Now $D_{qc}(X) \cong D(QCoh X)$ is a tensor triangulated category with coproducts, with respect to the derived tensor product $- \otimes^L_X -$, which is defined using flat resolutions (see e.g. [Lip09 (2.5.7)]), and the unit object given by the structure sheaf $O_X$. Let us also recall the following standard facts about derived functors:

**Proposition 2.34.**

(i) (see e.g. [Lip09 (2.1.1)(2.7.2)(3.1.3)(3.9.1)(3.6.4)] ) For any map of schemes $f : X \to Y$, we can define the derived pullback triangulated functor

$$L f^* : D_{qc}(Y) \to D_{qc}(X),$$

via flat resolutions.

Furthermore, we have a natural functorial isomorphism

$$L f^* L g^* \sim L (g f)^*$$

(ii) (see e.g. [Lip09 (2.1.1)(2.3.7)(3.1.2)(3.9.2)(3.6.4)]) For any quasi-compact and quasi-separated map of schemes $f : X \to Y$, we can define the derived direct image (a.k.a. derived pushforward) triangulated functor

$$R f_* : D_{qc}(X) \to D_{qc}(Y),$$

via injective resolutions. Furthermore, we have a natural functorial isomorphism

$$R (g j)_* \sim R g_* R f_*,$$

when both $f$ and $g$ and quasi-compact and quasi-separated maps.

(iii) (see e.g. [Lip09 (3.6.10)]) For any quasi-compact and quasi-separated map of schemes $f : X \to Y$, $(L f^*, R f_*)$ gives an afjunction pair:

$$D_{qc}(Y) \xrightarrow{L f^*} D_{qc}(X) \xleftarrow{R f_*}$$

(iv) (see e.g. [Lip09 (3.2.1)(3.9.4)]) For any quasi-compact and quasi-separated map of schemes $f : X \to Y$, the projection formula holds, i.e. we have natural isomorphisms for any $F \in D_{qc}(X), G \in D_{qc}(Y)$:

$$(R f_* F) \otimes^L G \xrightarrow{\cong} R f_* (F \otimes^L L f^* G), \quad G \otimes^L R f_* F \xrightarrow{\cong} R f_* (L f^* G \otimes^L F)$$

To investigate an analogue of Ohkawa’s theorem for $D_{qc}(X)$, we must consider localizing tensor ideals of $D_{qc}(X)$. However, those smashing (localizing tensor ideals) are sometimes, more important. To study such (smashing) localizing tensor ideals of $D_{qc}(X)$, an appropriate concept of “stalk” becomes crucial:

**Definition 2.35.** (compare with [ATJLSS04 Proof of Th.4.12] [ILN15 App.A])

Let $x \in X$ be a point in a scheme. Then we have the following canonical maps

\[ \text{[27]} \] Our presentation of “supports” in this definition and next proposition is somewhat different from those given in [ATJLSS04 Proof of Th.4.12] [ILN15 App.A], but the author hopes this would be more transparent to the reader.
Then, for \( E \in \mathcal{D}_{qc}(X) \), we have four notions of “supports”:

\[
\text{supp}(E) := \{ x \in X \mid L(i_x)^*E \neq 0 \in \mathcal{D}_{qc}(\text{Spec } k) \}
\]

\[
\subseteq \text{Supp}(E) := \{ x \in X \mid (i_x)^*E \neq 0 \in \mathcal{D}_{qc}(\text{Spec } \mathcal{O}_{X,x}) \};
\]

\[
\text{supph}(E) := \{ x \in X \mid L(i_x)^*(\oplus_{\bullet \in \mathbb{Z}} \mathcal{H}^\bullet E) \neq 0 \in \mathcal{D}_{qc}(\text{Spec } k) \}
\]

\[
\subseteq \text{Supph}(E) := \{ x \in X \mid \oplus_{\bullet \in \mathbb{Z}} (\mathcal{H}^\bullet E)_x = (l_x)^*(\oplus_{\bullet \in \mathbb{Z}} \mathcal{H}^\bullet E) \neq 0 \in \text{QCoh}(\text{Spec } \mathcal{O}_{X,x}) \}
\]

where:

- \( \mathcal{H}^\bullet E \) is the associated homology sheaves, regarded as a chain complex with trivial boundaries, of \( E \).

- The inclusive relations follow from \( L(l_x)^*(l_x)^* = L(l_x)^*L(l_x)^* \cong L(l_x r_x) = L(i_x)^* \), where the former equality follows from \( L(l_x)^* = (l_x)^* \), a consequence of the flatness of \( l_x \), and the latter isomorphism is a direct consequence of Proposition 2.34(i).

- These inclusions relations become equalities when \( E \in \mathcal{D}^b_{\text{coh}}(X) \) because of Nakayama’s lemma.

- If it becomes necessary to distinguish these four concepts, we call \( \text{supp}(E) \) the \underline{small} support of \( E \), \( \text{Supp}(E) \) the \underline{large} support of \( E \), \( \text{supph}(E) \) the \underline{large homology} support of \( E \), and \( \text{Supph}(E) \) the \underline{homology support} of \( E \), because this is a tractable ordinary sheaf theoretical support for the associated homology sheaves \( \oplus_{\bullet \in \mathbb{Z}} \mathcal{H}^\bullet E \).

Then the following useful fact will be used later:

**Proposition 2.36.** (i) Given \( E \in \mathcal{D}_{qc}(X) \), we have for any \( x \in X \) and \( \bullet \in \mathbb{Z} \),

\[
\mathcal{H}^\bullet((l_x)^*E) \cong (l_x)^*(\mathcal{H}^\bullet E) \in \text{QCoh}(\text{Spec } \mathcal{O}_{X,x}).
\]

Consequently, for any \( E \in \mathcal{D}_{qc}(X) \),

\[
\text{Supp } E = \text{Supph } E.
\]

(ii) The commutative diagram of quasi-coherent sheaves in (12) restricts to coherent sheaves, and for any \( E \in \mathcal{D}^b_{\text{coh}}(X) \), all the four concepts of supports in Definition 2.35 coincide:

\[
\text{supp } E = \text{Supp } E = \text{Supph } E = \text{supph } E.
\]
Proof. In view of Definition 2.35, we only have to verify the first claim in (i): \( H^\bullet ((l_x)^* E) \cong (l_x)^* (H^\bullet E) \in \text{QCoh} (\text{Spec} \mathcal{O}_{X,x}) \). However, this follows immediately from the flatness of \( l_x \) which implies \( (l_x)^\ast \) preserves exactness at the cochain level. \( \square \)

Now, the fundamental theorem of Hopkins, Neeman, Thomason and others classify (smashing) localizing tensor ideals of \( D_{qc}(X) \) under a mild assumption of \( X \):

**Theorem 2.37.** ([Hop85] [Nee92a, Th.2.8, Th.3.3] [Tho97], [ATJLS04, Cor.4.6; Cor.4.13; Th.5.6] [BF11, Cor.6.8] [DS13, Cor.6.8; Ex.6.9] [HR17, Th.B]) Let \( X \) be a Noetherian scheme. Then every localizing tensor ideal of \( D_{qc}(X) \) is of the form

\[
\text{Ker } H_* = \{ Q \in D_{qc}(X) \mid \text{supp } Q \subseteq S \},
\]

for some \( S \subset X \).

The subcategory \( \text{Ker } H_* \) is smashing if and only if the corresponding \( S \subset X \) is closed under specialization.

Note those \( S \)'s with \( S \subset X \) clearly form a set. So, we see an analogue of Ohkawa’s theorem, however with a clear algebro-geometrical interpretation of “the Bousfield-Ohkawa set” in contrast to the case of Ohkawa’s theorem. Furthermore, Theorem 2.37 solves Conjecture 2.32 (ii) affirmatively for the case \( D_{qc}(X) \).

Also note that, in the special case when \( S \) in Theorem 2.37 is \( Z = X \setminus U \subset X \), the complement of a quasi-compact Zariski open immersion \( j : U \hookrightarrow X \), we have the following equivalence for not only noetherian, but also more general quasicompact, quasiseparated schemes (in which case, as \( L_j^* \) has a right adjoint \( R_j^* \) with \( \epsilon : L_j^* R_j \to \text{id} \) an isomorphism, we may apply Proposition 2.28): \[28\]

\[
D_{qc}(X) / (D_{qc})_Z (X) \xrightarrow{\cong} D_{qc}(U),
\]

(14)

where \( (D_{qc})_Z (X) := \{ Y \in D_{qc}(X) \mid \text{Supp } Y \subseteq Z \} = \text{Ker } L_j^* \). In this generality of quasicompact, separated schemes, Bousfield localization \( L \) is smashing (see Proposition 2.29), given explicitly as follows:

\[
L = R_j^* L_j^* = (R_j^* \mathcal{O}_U) \otimes_{\mathcal{O}_X} - : D_{qc}(X) \to D_{qc}(X) / (D_{qc})_Z (X) \xrightarrow{\text{Lj\#}} D_{qc}(U) \xrightarrow{R_j} D_{qc}(X).
\]

(15)

\[28\] So, should had been known to Verdier.

\[29\] Let us recall the following precursor of this result in the setting of abelian category of quasi-coherent sheaves, which should go back at least to Gabriel (see e.g. [Ron10]). In the proof of Prop.3.1]: \( \text{QCoh}(X) / \text{QCoh}_Z(X) \xrightarrow{\cong} \text{QCoh}(U) \), where the left hand side is the abelian quotient category in the sense of Gabriel, Grothendieck, Serre.

\[30\] Unlike Theorem 2.37 stated under the noetherian assumption, (14) is stated under more general quasicompact, quasiseparated assumption. Therefore, in this equality \( (D_{qc})_Z (X) := \{ Y \in D_{qc}(X) \mid \text{Supp } Y \subseteq Z \} = \text{Ker } L_j^* \), we may not replace \( \text{Supp} \) with \( \text{supp} \). In fact, without the noetherian hypothesis, Theorem 2.37 becomes very bad as was shown in [Nec00]. The author is grateful to Professor Neeman for this reference.
3. Hopkins-Smith theorem and its motivic analogue

In reality, Hopkins was not motivated by Ohkawa’s Theorem 2.25 for his influential paper in algebraic geometry [Hop85] (Theorem 2.37). Instead, Hopkins was motivated by his own theorem with Smith [HS98] in the sub stable homotopy category \( \mathcal{SH}^c \), consisting of compact objects, whose validity was already known to them back around the time Hopkins wrote [Hop85].

**Theorem 3.1.** [HS98] For any prime \( p \), any thick (épaisse) subcategories of the subtriangulated category \( \mathcal{SH}^c_{(p)} \) consisting of compact objects

\[
\mathcal{SH}^c_{(p)} = \mathcal{SH}^{fin}_{(p)} = \text{the homotopy category of } p\text{-local finite spectra}
\]

is of the form

\[
C_n := \ker E(n-1)_*|_{\mathcal{SH}^{fin}_{(p)}} = \left\{ X \in \mathcal{SH}^{fin}_{(p)} \mid E(n-1) \wedge X = 0 \right\}
\]

\[
= \ker K(n-1)_*|_{\mathcal{SH}^{fin}_{(p)}} = \left\{ X \in \mathcal{SH}^{fin}_{(p)} \mid K(n-1) \wedge X = 0 \right\}.
\]  

Furthermore, these form a decreasing filtration of \( \mathcal{F}_{(p)} \):

\[
\{ * \} \subsetneq \cdots \subsetneq C_{n+1} \subsetneq C_n \subsetneq C_{n-1} \subsetneq \cdots \subsetneq C_1 \subsetneq C_0 = \mathcal{SH}^{fin}_{(p)}.
\]  

In this Hopkins-Smith classification of thick triangulated subcategories of \( \mathcal{SH}^c \), the first step is an easy observation that any thick triangulated subcategory of \( \mathcal{SH}^c \) is a thick (tensor) ideal. Furthermore, \( E(n-1) \) and \( K(n-1) \) are the \( (n-1) \)-st Johnson-Wilson spectrum and Morava K-theory, respectively, and the equality \( \ker E(n-1)_*|_{\mathcal{SH}^{fin}_{(p)}} = \ker K(n-1)_*|_{\mathcal{SH}^{fin}_{(p)}} \) in (16) and the inclusions (17) are consequences of the following results found in Ravenel’s paper [Rav84]:

**Theorem 3.2.** (i) [Rav84, Th.2.1(d)] \( \ker E(n-1)_* = \ker (\lor_{0 \leq i \leq n-1} K(i))_* \)

(ii) [Rav84, Th.2.11] For \( X \in \mathcal{SH}^{fin}_{(p)} \), if \( K(i)_*X = 0 \), then \( K(i-1)_*X = 0 \).

By the Hopkins-Smith work [HS98], the smashing conjecture for \( E(n) \) [Rav84] also holds [Rav92], and so, \( \ker E(n-1)_* \) in (16) is a smashing tensor ideal. Actually, the first equality in (16) is a part of the following elegant reformulation of the telescope conjecture [Rav84, HPS97, Def.3.3.8] (see also Conjecture 2.22) by Miller [Mil92], [HPS97, Th.3.3.3] (here we follow more recent formulations of [BF11, Th.4.1;Def.4.2] [HR17, Cor.2.1;Def.3.1]).

**Theorem 3.3.** (Miller’s finite localization and the Ravenel telescope conjecture) Let \( \mathcal{T} \) be a rigidly compactly generated tensor triangulated category. Let \( \mathcal{S}(\mathcal{T}) \) denote the collection of all smashing localizing tensor ideals of \( \mathcal{T} \), and let \( \mathcal{T}(\mathcal{T}^c) \) denote the collection of all thick tensor ideals of \( \mathcal{T}^c \).

(i) [Mil92, Cor.6;Prop.9] [HR17, Th.1.7] For any \( C \in \mathcal{T}(\mathcal{T}^c) \), the smallest localizing

\[\text{ideal of } \mathcal{T}^c \]
triangulated subcategory $⟨C⟩$ containing $C$ in $T$ is smashing, i.e. $C ∈ S(T)$.

Thus, we obtain the inflation map:

$$I : T(T^c) → S(T).$$

(ii) [HPS97, Th.3.3.3] There is also the contraction map:

$$C : S(T) → T(T^c); \quad S ↦ S \cap T^c,$$

which enjoys:

$$C \circ I = id_{T(T^c)} : T(T^c) \xrightarrow{I} S(T)$$

(iii) [Mil92, Cor.6;Prop.9;Cor.10] The telescope conjecture for $S ∈ S(T)$ holds if and only if, in addition to $C \circ I = id_{T(T^c)}$ stated in (ii), the following also holds:

$$I \circ C(S) = S ∈ S(T)$$

(iv) [Mil92, Cor.6;Prop.9;Cor.10] The telescope conjecture for $T$ holds if and only if $I$ and $C$ give mutually inverse equivalence:

$$C \circ I = id_{T(T^c)} : T(T^c) \xrightarrow{I} S(T) : id_{S(T)} = I \circ C.$$

However, the telescope conjecture of this generality has been shown to be false [Kel94], and even the original telescope conjecture for $SH$ is now believed to be false by many experts, including Ravenel himself [MRS01]. Still, algebraicists have shown the validity of its various algebraic analogues (e.g. [BF11], [KS10], [BIKP18]) as we shall review an algebraic analogue of the Hopkins-Smith theorem, in conjunction with the above telescope conjecture, later in Theorem 4.15. Furthermore, Krause [Kra00] showed the underlying philosophical message of the telescope conjecture that smashing tensor ideals are completely characterized by their restrictions to compact objects. In fact, whereas the original telescope conjecture only concerns local compact objects, Krause proves his characterization of smashing tensor ideals via “local maps” between compacts objects. For more details, consult Krause’s own paper [Kra00].

Going back to the Hopkins-Smith theorem, a major part of its proof was to show:

**Theorem 3.4.** [HS98, Th.7] Any thick subcategory of $SH^\text{fin}_{(p)}$ is of the form $C_n$ for some $n ∈ \mathbb{Z}_{≥0}$.

To show this, Hopkins-Smith prepared the following version of the nilpotency theorem [HS98, Cor.2.5], building upon their earlier collaboration work with Devinatz [DHS88]:

**Theorem 3.5.** [HS98, Cor.2.5.ii)] For a map $f : F → A$ between finite $p$-local spectra and another finite $P$-local spectra $Y$, the following conditions are equivalent:

- ∃m ≫ 0 such that $f^\wedge m \wedge I_Y : F^{\wedge m} \wedge Y → A^{\wedge m} \wedge Y$ is null.
- $0 ≤ ∀ n < ∞, \quad K(n)_*(f \wedge I_Y) = 0$.

Now, to prove Theorem 3.4 it suffices to prove the following:

---

\[32\] This telescope conjecture is equivalent to the telescope conjecture without product Conjecture 2.22 via Proposition 2.28 and Proposition 2.29.
Lemma 3.6. [HS98, (2.9)] Let \( C \) be a thick subcategory of \( \mathcal{SH}(p) \) and \( X, Y \) be \( p \)-local finite spectra. Then, if \( X \in C \) and \( \{ n \in \mathbb{Z}_{\geq 0} \mid K(n)_* Y \neq 0 \} \subseteq \{ n \in \mathbb{Z}_{\geq 0} \mid K(n)_* X \neq 0 \} \), then \( Y \in C \).

Actually, if Lemma 3.6 is shown to be correct, together with Ravenel’s Theorem 3.2(ii), it would imply \( C = C_m \), where \( m = \min \{ n \in \mathbb{Z}_{\geq 0} \mid C_n \subseteq C \} \).

Then, the proof of Lemma 3.6 in [HS98] proceeds as follows (see also [Rav92]:

- Starting with \( X \), let \( e : S^0 \to X \wedge DX \) be the \( S \)-dual of the identity map \( : I_X : X \to X \), and extend it to a triangle with a map between \( p \)-local finite spectra \( f : F \to S^0 \) as the fiber as follows:

\[
F \xrightarrow{f} S^0 \xrightarrow{e} X \wedge DX \simeq C_f, \quad \text{the cofiber of } f. \tag{18}
\]

- Applying the smash product with \( Y \) to (18), we obtain:

\[
F \wedge Y \xrightarrow{f \wedge I_Y} S^0 \wedge Y \simeq Y \xrightarrow{e \wedge I_Y} X \wedge DX \wedge Y \simeq C_f \wedge Y, \tag{19}
\]

for which, we claim

\[
0 \leq \forall n < \infty, \quad K(n)_*(f \wedge I_Y) = 0. \tag{20}
\]

- If \( K(n)_* Y = 0 \) then \( K(n)_*(I_Y) = 0 \), which implies the triviality of (20), by the Kunneth theorem for Morava \( K \)-theories:

\[
K(n)_*(X \wedge Y) \cong K(n)_* X \otimes_{K(n)_*} K(n)_* Y \quad \text{for any } p \text{-local spectra } X, Y \tag{21}
\]

- If \( K(n)_* Y \neq 0 \) then \( K(n)_* X \neq 0 \) by the assumption of Lemma 3.6.

Then, by the duality isomorphism for Morava \( K \)-theories:

\[
\text{Hom}_{K(n)_*}(K(n)_* X, K(n)_* Y) = \text{Hom}_{K(n)_*}(K(n)_* (Y \wedge DX)) = K(n)_*(Y \wedge DX)
\]

for any \( p \)-local spectra \( X, Y \),

we also find the non-triviality: \( K(n)_*(e \wedge I_Y) \neq 0 \). But, this in turn implies the triviality: \( K(n)_*(f \wedge I_Y) = 0 \) from the Morava \( K \)-theory exact sequence associated to (19), making use of the Morava Kunneth isomorphism (21) again.

- Since (20), we may apply the Hopkins-Smith nilpotency Theorem 3.5 to \( f \wedge I_Y \) in (19) to find \( m \gg 0 \) such that \( f^\wedge m \wedge I_Y : F^\wedge m \wedge Y \to (S^0)^\wedge m \wedge Y \cong Y \) is null. This implies:

\[
Y \text{ is a direct summand of } C_{f^\wedge m \wedge I_Y} \cong C_{f^\wedge m} \wedge Y. \tag{23}
\]

- By the assumption, \( X \in C \), but as the thick subcategory \( C \) of \( \mathcal{SH}(p) \) is also a thick ideal, this implies \( C_f \cong X \wedge DX \in C \).
For any \( n \in \mathbb{N} \), consider the commutative diagram:

\[
\begin{array}{ccc}
F \wedge F \rightarrow (S^0)^n \wedge F & \cong & F \\
\downarrow & & \downarrow \\
F \wedge (n+1) \rightarrow (S^0)^{(n+1)} & \cong & S^0 \\
\downarrow & & \downarrow \\
C_f & & C_f
\end{array}
\]

From this, we obtain a triangle

\[ C_f \wedge F \rightarrow C_{f^{n+1}} \rightarrow C_f \]

Since \( C_f \in \mathcal{C} \) and \( \mathcal{C} \) is a tensor ideal, we see inductively from this triangle that

\[ C_{f^m} \in \mathcal{C} \quad (\forall m \in \mathbb{N}) \quad (24) \]

Since \( \mathcal{C} \) is a thick ideal, we conclude from (23) and (24) that \( Y \in \mathcal{C} \). This complete the proof of Lemma 3.6.

Now, the basic philosophy underlying the above picture of Hopkins-Smith was already perceived by Morava much earlier (see the ‘exercises’ in §0.5 of [Mor85], whose preprint version was circulated nearly a decade ago before its publication). For a modern development of Morava \( K \)-theory, consult Morava’s own paper [MorPr] in this proceedings.

The author believes the Hopkins-Smith theorem (Theorem 3.1) and the Ohkawa theorem (Theorem 2.25) are best understood, when they are appreciated simultaneously in a single commutative diagram. Since this commutative diagram can be drawn for more general rigidly compactly generated tensor triangulated category \( \mathcal{T} \), let us first set up our notations of our interests in this generality:

- \( \mathbb{L}(\mathcal{T}) \): the collection of localizing tensor ideals of \( \mathcal{T} \).
- \( \mathbb{S}(\mathcal{T}) \): the collection of smashing localizing tensor ideals of \( \mathcal{T} \).
- \( \mathbb{T}(\mathcal{T}^e) \): the collection of thick tensor ideals of \( \mathcal{T}^e \).
- \( \mathbb{B}(\mathcal{T}) \): the collection of Bousfield classes, i.e. those of the form \( \text{Ker}(h \otimes -) \subseteq \mathbb{L}(\mathcal{T}) \) \( (h \in \mathcal{T}) \).

Now let us specialize to the case \( \mathcal{T} = \mathcal{SH}(p) \):

**Theorem 3.7.** In \( \mathcal{SH}(p) \), the Ohkawa theorem, the Hopkins-Smith theorem, Miller’s version of the Ravenel telescope conjecture \( (C \circ I \cong \text{Id}_{\mathcal{T}(\mathcal{SH}(p))}) \), and the conjectures of Hovey and Hovey-Palmieri can be simultaneously expressed in the following
succinct commutative diagram:

\[
\begin{array}{ccc}
\text{mysterious set} & \xrightarrow{\text{Ohkawa Th.}} & \mathcal{B}(\mathcal{SH}(p)) \\
\downarrow & & \uparrow \\
\text{chromatic hierarchy} & \xrightarrow{\text{Hopkins-Smith Th.}} & \mathcal{F}(\mathcal{SH}^{\text{fin}}(p))
\end{array}
\]

\[
\xrightarrow{\approx} \downarrow \quad \xrightarrow{I \text{ (split inj.)}} \quad \xrightarrow{\mathcal{C} \text{ (split surj.)}} \mathcal{L}(\mathcal{SH}(p)) = \mathcal{S}(\mathcal{SH}(p))
\]

For more on the Hopkins-Smith theorem and related “chromatic mathematics,” see [Rav92] and, for some of the latest developments, see [BarPr] [BRPr] [TorPr] in this proceedings. Actually, Bartel’s survey [BarPr] focuses upon the telescope [Rav81] [Rav92] and chromatic splitting conjectures [Hov95a], which are major directions of research, not only in chromatic homotopy theory, but also in stable homotopy theory as a whole. Considering the traditional influence of stable homotopy theory, initiated by Hopkind, Rickard, Neeman, Thomason and others, to the representation theory of finite dimensional algebras and the derived category theory in algebraic, as is highlighted by Brown representability, Bousfield localization, Hopkins-Smith theorem, researchers in these areas might better to keep this fact in mind.

Comparing with the telescope conjecture, the chromatic splitting conjecture appears to be elusive for them. In short, the chromatic splitting conjecture predicts, for a \( p \)-completed finite spectrum \( F \), the first map in the canonical cofiber sequence

\[
\text{Hom} \left( L_{E(n-1)}S^0, L_{E(n)}F \right) \to L_{E(n-1)}F \to L_{E(n-1)}L_{K(n)}F
\]

is trivial; stated differently, the second map in (26) is split injective. \(^{34}\)

In fact, Hopkins [Hov95a, Conj.4.2(iv)] further predicted, presumably hoping to provide a program to prove the triviality of the first map in (26), an explicit decomposition of \( \text{Hom} \left( L_{E(n-1)}S^0, L_{E(n)}F \right) \), inspired by Morava’s old observation [Mor85, Rem.2.2.5]. The structure of \( \text{Hom} \left( L_{E(n-1)}S^0, L_{E(n)}F \right) \) is highly reflected by its divisible homotopy group elements. In general, divisible homotopy group elements of a spectrum \( X \) can be isolated in the spectrum \( \text{Hom}(L_0S^0, X) \), which is in the current case:

\[
\text{Hom} \left( L_0S^0, \text{Hom} \left( L_{E(n-1)}S^0, L_{E(n)}F \right) \right) \cong \text{Hom} \left( L_0S^0 \wedge L_{E(n-1)}S^0, L_{E(n)}F \right) \\
\cong \text{Hom} \left( L_0S^0, L_{E(n)}F \right)
\]

To understand this, Morava [Mor14] suggested to consider the following cohomology theory \( L_n^* \):

\[
X \mapsto L_n^* (X) := \text{Hom} \left( \pi_-, \text{Hom} \left( L_0S^0, L_{E(n)}X \right), \mathbb{Q} \right)
\]

\(^{33}\) A trend here is to apply the higher algebra technique of Lurie [Lur09] [Lur16] to understand chromatic phenomena [BRPr] and [TorPr], where the latter contains a concise review of higher algebra technology. Different kinds of applications of Lurie’s higher algebra technique can be seen in [M1Pr] [M2Pr].

\(^{34}\) This splitting conjecture implies, for any \( p \)-completed finite spectrum \( F \) and any infinite subset \( \{ n_i \}_{i=1}^\infty \subseteq \mathbb{N} \) such that \( \mathbb{N} \setminus \{ n_i \}_{i=1}^\infty \subseteq \mathbb{N} \), the natural map \( F \to \prod_{i=1}^\infty L_{K(n_i)}F \) is split injective. For this and much more, consult [Hov95a] [BarPr].
Actually, Morava [Mor14] noticed the validity of the Hopkins’ prediction on the explicit structure of $\text{Hom}(L_{E(n-1)} S^0, L_{E(n)} F)$ would imply the cohomology theory $L^a_n$ is represented by the $p$-adic rationalization of the spectrum:

$$
\Sigma^{2n} \left( \bigvee \left\{ n_i \in \mathbb{Z}_{>0} \right\}^\infty \right) \left( \prod_{i=1}^\infty (n_i!) \right) \left( \prod_{i=1}^\infty U(i - 1)^{n_i} \right) \right) \quad (27)
$$

While Hopkins’ prediction [Hov95a] above of the explicit decomposition of $\text{Hom}(L_{E(n-1)} S^0, L_{E(n)} F)$, which the above work of Morava [Mor14] is based upon, is known to hold for $n = 1$ or $n = 2$ and $p \geq 3$, Beaudry [Bea17] has recently shown it to fail for the case $n = 2$ and $p = 2$. Still, as was pointed out to the author by Tobias Barthel, the above formula (27), which was derived from Morava’s calculation, still holds even for this troublesome case of $n = 2$ and $p = 2$, because the discrepancy found by Beaudry [Bea17] is $p$-torson and so vanishes rationally. Thus, it could well be the case (27) holds for any pair of a prime $p$ and a natural number $n$.

Furthermore, it could be the case that Hopkins’ prediction of the explicit decomposition of $\text{Hom}(L_{E(n-1)} S^0, L_{E(n)} F)$ still holds, consequently so does Morava’s deduction (27) above, when the base prime $p$ is sufficiently large comparing with the height $n$.

It would be fantastic, if, as Professor Morava dreams of, there hold formulae analogous to the predicted Hopkins’ and Morava’s, in algebraic examples like $D_{\text{qc}}(X)$, where the fundamental theorem of Hopkins, Neeman, Thomason and others gave us an explicit “Bousfield-Okawa set”, not only for Bousfield classes, but also for localized tensor ideals, whereas the original Okawa’s set for $\mathcal{SH}$ only takes into account Bousfield classes and is not explicit at all. Furthermore, as we mentioned before, while the telescope conjecture is now believed to be false by many experts, algebraicists have shown the validity of its various algebraic analogues. So, why not for the chromatic splitting conjecture, as Professor Morava dreams of!

Actually, restricting to the conjectured splitting of the second map in (26), recent effort of Beaudry-Goerss-Henn [BGH17] has shown its validity even for the case $n = p = 2$, which is the case [Bea17] showed Hopkins’ conjectural decomposition of $\text{Hom}(L_{E(n-1)} S^0, L_{E(n)} F)$ is false. Furthermore, Barthel-Heard-Valenzuela [BHV18] has recently proved an algebraic analogue of the conjectural splitting of the second map in (26). For this and much more, consult Bartel’s survey [BarPr].

Going back to the Hopkins-Smith theorem, it is natural to look after its motivic analogue [10] (This means efforts to classify thick (tensor) ideals of $\mathcal{SH}(k)^c.$).

In this regard, Ruth Joachimi [JoaPr] constructed some motivic thick ideals in $\mathcal{SH}(k)^c$ for $k \subseteq \mathbb{C}$:

**Theorem 3.8.** [JoaPr Th.13]

\[35\text{It appears that [Mor14 p.4 Corollary] should be modified as in [27]}\]
(1) If $k \subseteq \mathbb{C}$, then $(\mathcal{SH}(k)^c)_{(p)}$ contains at least an infinite chain of different thick ideals, given by $R_k^{-1}(\mathcal{C}_n), 0 \leq n \leq \infty$, where $R_k$ denotes the $p$-localisation of the restriction of $R_k$ to $\mathcal{SH}(k)^c$:

![Diagram](image)

Here,
- $c_k$ is induced from the constant presheaf functor [JoapPr, Th.10], which restricts to the compact objects [JoapPr, Rem.53, Prop.58, Prop.61].
- The existence of $R_k$ follows since $R_k$ preserves compactness [JoapPr, Prop.61].

(2) If $k \subseteq \mathbb{R}$, then $(\mathcal{SH}(k)^c)_{(p)}$ contains at least a two-dimensional lattice of different thick ideals, given by $(R_k')^{-1}(\mathcal{C}_{m,n})$, for all $(m,n) \in \Gamma_p$ (see [JoapPr, Def.35] for the definition of $\Gamma_p$ and more detail):

![Diagram](image)

Here,
- [JoapPr Th.11] $c'_k : (\mathcal{SH}(\mathbb{Z}/2)^c)_{(p)} \to (\mathcal{SH}(k))_{(p)}$ is induced by $c' : sSet(\mathbb{Z}/2) \to sPre(Sm/\mathbb{R})$
  
  $$M \mapsto \left( \coprod_{M^{\mathbb{Z}/2}} \coprod_{(M\setminus M^{\mathbb{Z}/2})/(\mathbb{Z}/2)} \text{Spec } \mathbb{C} \right),$$

  which restricts to the compact objects [JoapPr Rem.53, Prop.58, Prop.61].
- (Strickland’s theorem [JoapPr Cor.34]) Any thick ideal in the category $(\mathcal{SH}(\mathbb{Z}/2)^c)_{(p)}$ is of the form
  
  $$\mathcal{C}_{m,n} = \{ X \mid \phi^{(1)}(X) \in \mathcal{C}_m \text{ and } \phi^{\mathbb{Z}/2}(X) \in \mathcal{C}_n \},$$

  where $m,n \in [0, \infty]$.

36 Strickland’s theorem for $G = \mathbb{Z}/2$ has recently been generalized to arbitrary finite group $G$ by Balmer-Sanders [BS17].
Just like the nilpotency theorem Theorem 3.5 was crucial in the proof of Hopkins-Smith theorem Theorem 3.1, the above theorem of Strickland is shown by first proving an appropriate nilpotency theorem [JoaPr, Th.3]. At the same time, Joachimi [JoaPr] explains various difficulties in proving an appropriate nilpotency theorem in the motivic setting. Furthermore, the above Joachimi’s construction of motivic thick ideals in $\text{SH}(k)^c$ for $k \subseteq \mathbb{C}$ is so far limited to importing the Hopkins-Smith stable homotopy thick ideals in $\text{SH}^c$. Thus, constructions of motivic thick ideals of truly algebro-geometric origin is highly desired. For details and much more of Joachimi’s work, construct her own exposition [JoaPr] in this proceeding.

For a case of $k \not\subseteq \mathbb{C}$, Kelly [KelPr] obtained the following surprisingly simple description of the set of prime thick tensor ideals $\text{Spc}((\text{SH}(\mathbb{F}_q)^c_{\mathbb{Q}}))$ up to a couple of widely believed conjectures:

**Theorem 3.9.** [KelPr, Th.1.1] Let $\mathbb{F}_q$ be a field with a prime power, $q$, number of elements. Suppose that for all connected smooth projective varieties $X$ we have:

\[ CH^i(X; j)_{\mathbb{Q}} = 0; \quad \forall j \neq 0; i \in \mathbb{Z} \quad (\text{Beilinson-Parshin conjecture}), \]

\[ CH^i(X)_{\mathbb{Q}} \otimes CH_i(X)_{\mathbb{Q}} \to CH_0(X)_{\mathbb{Q}} \text{ is non-degenerate}. \quad (\text{Rat. and num. equiv. agree}) \]

Then

\[ \text{Spc}((\text{SH}(\mathbb{F}_q)^c_{\mathbb{Q}})) \cong \text{Spec}(\mathbb{Q}). \]

For details, consult Kelly’s own exposition [KelPr] in this proceeding.

4. $D^b_{\text{coh}}(X)$ AND $D^{\text{perf}}(X)$

In the last two sections, we reviewed:

- Ohkawa’s theorem in $\text{SH}$, which states the Bousfield classes form a somewhat mysterious set.
- Its analogue in $D_{\text{qc}}(X)$ is explicitly computable: the fundamental theorem of Hopkins, Neeman,..., identifies the set of Bousfield classes with the set of of localizing tensor ideals, which turns out to have a concrete and algebro-geometric description.
- Hopkins’ motivation of his fundamental theorem in $D_{\text{qc}}(X)$ was his own theorem with Smith in $\text{SH}^c$.

Thus, we are naturally led to investigate $D_{\text{qc}}(X)^c$. However, the story is not so simple. Whereas there is a conceptually simple categorical interpretation $D_{\text{qc}}(X)^c = D^{\text{perf}}(X)$, it is its close relative (actually equivalent if $X$ is smooth over a field) $D^b_{\text{coh}}(X)$ which traditionally has been intensively studied because of its rich geometric and physical information. 38

So, we wish to understand both $D^b_{\text{coh}}(X)$ and $D^{\text{perf}}(X)$.

\[ \text{See Definition 4.22 for this concept.} \]

38 Or, researchers might prefer “♥-felt” $D^b_{\text{coh}}(X) \cong D^b(\text{Coh}(X))$ (although separated, not mere quasi-separated, assumption is needed for this equivalence) over simply formal $D^{\text{perf}}(X) \cong D_{\text{qc}}(X)^c$. ...
In this section, we start with brief, and so inevitably incomplete, summaries of $D^b_{\text{coh}}(X)$ and $D^\text{perf}(X)$, focusing on their usages. Still, we hope this would convince non-experts that $D^b_{\text{coh}}(X)$ and $D^\text{perf}(X)$ are very important objects to study.

Then, we shall review Neeman’s recent result, which claims these two close relatives $D^b_{\text{coh}}(X)$ and $D^\text{perf}(X)$ actually determine each other, and its main technical tool: approximable triangulated category.

4.1. $D^b_{\text{coh}}(X)$.

- There is a classical functoriality result of Grothendieck:

\textbf{Theorem 4.1.} [Gro61, Th.3.2.1] Let $f: X \to Y$ be a proper morphism with $Y$ locally noetherian. Then

$$Rf_* D^b_{\text{coh}}(X) \subset D^b_{\text{coh}}(Y).$$

Actually, there is a sharp converse (i.e. we do not have to check $Rf_* D^b_{\text{coh}}(X) \subset D^b_{\text{coh}}(Y)$) to Theorem 4.1 [LN07, Cor.4.3.2] [Nee17, Lem.0.20]:

\textbf{Theorem 4.2.} [Nee17, Lem.0.20] Let $f: X \to Y$ be a separated, finite-type morphism of noetherian schemes such that

$$Rf_* D^\text{perf}(X) \subset D^b_{\text{coh}}(Y).$$

Then $f$ is proper.

- For an essentially small triangulated category $\mathcal{T}$, its Grothendieck $K_0$-group $K_0(\mathcal{T})$ is defined by generators and relations as follows [Nee01, Def.4.5.8] [Nee05, Def.1]:

$$K_0(\mathcal{T}) := \frac{\mathbb{Z}\{[X]|\text{[X] is an isomorphism class of } X \in \mathcal{T}\}}{\mathbb{Z}\{[X] - [Y] + [Z]|\text{there is a distinguished triangle } X \to Y \to Z \to \Sigma X\}} (28)$$

- Having defined $K_0(\mathcal{T})$, we should not be too optimistic to hope $K_0(\mathcal{T})$ always carries a rich information of $\mathcal{T}$. In fact, if $\mathcal{T}$ contains an arbitrary countable direct sum (coproduct) \textsuperscript{39}, then, for any $X \in \mathcal{T}$, we have a distinguished triangle of the following form:

$$\bigoplus_{n \in \mathbb{N}} X \xrightarrow{\text{index shift}} \bigoplus_{n \in \mathbb{N}} X \to X \to \Sigma \big(\bigoplus_{n \in \mathbb{N}} X\big)$$

From the defining relation of $K_0(\mathcal{T})$ \textsuperscript{28}, this implies $[X] = 0 \in K_0(\mathcal{T})$ for any $X \in \mathcal{T}$. By the definition \textsuperscript{28}, this means $K_0(\mathcal{T}) = 0$ whenever $\mathcal{T}$ contains an arbitrary countable direct sum (coproduct). As a very important special case, we emphasize:

$$K_0(D_{\text{qc}}(X)) = 0.$$  

- Grothendieck $K_0$-group is useful to classify dense subcategories of an essentially small triangulated subcategory.

\textsuperscript{39} Having arbitrary small coproducts was an indispensqble assumption for Brown representability and Bousfield localization (Theorem 2.15 Corollary 2.16).
Proposition 4.3. [Tho97, p.5,Lem.2.2,p.6,Cor.2.3] [Nee01, Prop.4.5.11] Suppose a triangulated subcategory $S$ of an essentially small triangulated category $T$ is dense, i.e. $\hat{S} = T$. Then,

1. The induced map $K_0(S) \to K_0(T)$ is a monomorphism.
2. For any $X \in T$,
   
   $X \in S \iff [X] \in \text{Im}(K_0(S) \to K_0(T))$.

Theorem 4.4. [Tho97, p.5,Th.2.1] For an essentially small triangulated category $T$, there is a one-to-one correspondence between the dense triangulated subcategories of $T$ and the subgroups of $K_0(T)$:

\[
\{\text{dense triangulated subcategories of } T\} \cong \{ \text{subgroups of } K_0(T) \} \quad S \mapsto \text{Im } (K_0(S) \to K_0(T))
\]

\[\triangle\] subcategory consisting of $X \in T$ with $[X] \in H \subseteq K_0(T) \iff H$

For any small abelian category $\mathcal{A}$, the functor $\mathcal{D}^b$ comes with the canonical embedding $\mathcal{A} \to \mathcal{D}^b(\mathcal{A})$, which induces an equivalence of Grothendieck $K$-groups of an abelian category $\mathcal{T}$ and a triangulated category $\mathcal{D}^b(\mathcal{A})$:

\[
K_0(\mathcal{A}) \cong K_0(\mathcal{D}^b(\mathcal{A})),
\] (29)

Whenever a bounded $t$-structure is given on $\mathcal{T}$, if we denote by $\mathcal{T}^\bigtriangleup$ its heart, then we have another isomorphism of $K_0$-groups of an abelian category and a triangulated category:

\[
K_0(\mathcal{T}^\bigtriangleup) \cong K_0(\mathcal{T}).
\] (30)

Applying (30) to $\mathcal{T} = \mathcal{D}_{\text{coh}}^b(X)$, $\mathcal{T}^\bigtriangleup = \text{Coh}(X)$, we find the canonical isomorphism:

\[
K(\text{Coh}(X)) \cong K(\mathcal{D}_{\text{coh}}^b(X)).
\] (32)

The sheaf theory has its origin in Oka-Cartan theory of complex functions of several variables (see e.g. [Ohs15] for a general picture, and [OhsPr] for a review of the $L^2$-technique in complex geometry, both by Ohsawa [41]). The pivotal achievement at the time was Oka’s Coherence Theorem, which states that the structure sheaf $\mathcal{O}_M$ of a complex manifold $M$ is coherent (for a

\[\footnote{If we apply (29) in order to obtain the isomorphism (32), we must require the extra “separated” assumption, for then we should also use the isomorphism:

\[\mathcal{D}_{\text{coh}}^b(X) = \mathcal{D}^b(\text{Coh}(X)),\] (31)

which requires the “separated” assumption of $X$. This fact, and the above approach to use was communicated to the author by Professor Neeman.}]

\[\footnote{Professor Takeo Ohsawa is the AMS Stefan Bergman Prize 2014 recipient. His survey paper [OhsPr] in this proceedings is a concise summary of his work for which this prize was awarded. It was his Bergman Prize money which enabled us to invite distinguished lecturers to Ohkawa’s memorial conference at Nagoya University in the summer of 2015. Takeo Ohsawa was also Tetsuske Ohkawa’s highschool classmate at Kanazawa University High School in Kanazawa, Japan.}]

\[\footnote{40} \]
From the viewpoint of algebraic geometry, interest of complex manifolds emerge through the GAGA theorem of Serre \cite{Ser55}, which, for a proper scheme $X$ over $\text{Spec}\mathbb{C}$, can be stated as an equivalence of abelian categories of coherent modules \cite[XII, Th.4.4]{GR63}:

$$\phi^* : \text{Coh}(X) \xrightarrow{\sim} \text{Coh}(X^{an}),$$

where $\phi : X^{an} \to X$ is the canonical morphism from the associated analytic space $X^{an}$ of $X$ \cite[XII,1.1]{GR63}, and $\phi^*$ consequently induces isomorphisms of resulting derived categories:

$$\mathcal{D}_{\text{Coh}}(X) \xrightarrow{\sim} \mathcal{D}_{\text{Coh}}(X^{an}); \quad \mathcal{D}_{\text{coh}}^b(X) \xrightarrow{\sim} \mathcal{D}_{\text{coh}}^b(X^{an}); \cdots$$

Recently, Jack Hall \cite{Hal18} proposed a unified treatment of “GAGA type theorems,” in which, a prominent role of Oka’s coherence theorem became transparent in his deduction of the classical GAGA theorem \cite[Example 7.5]{Hal18} (also consult the updated version of \cite[Remark 1.7 and Appendix A]{Nee18a} to appreciate how short and simple the Jack Hall’s new proof is).

- Derived categories in the complex analytic setting shows up in the Kontsevich homological mirror symmetry \cite{Kon95} which in the Calabi-Yau setting is of the following form:

$$\mathcal{D}_{\text{coh}}^b(X) \cong \mathcal{D}_{\text{Fuk}}^b(X^\vee), \quad (33)$$

where $X$ is expected to be a mirror of $X^\vee$, given by a sigma model:

$$(M, I, \omega, B),$$

where we only note $I$ is the complex structure of $M$, and that whose category of $D$-branes of type $B$ (B-model) is the left side of (33):

$$DB(M, I, \omega, B) \cong \mathcal{D}_{\text{coh}}^b(M, I) \cong \mathcal{D}_{\text{coh}}^b(X).$$

On the other hand, $\mathcal{D}_{\text{Fuk}}^b(X^\vee)$, the derived Fukaya category consisting of Lagrangian submanifolds of the mirror $X^\vee$, is not a derived category of an abelian category (but of an $A_\infty$ category; see \cite{FOOO1, FOOO2} for more details).

- Recall that $\mathcal{D}_{\text{coh}}^b(X)$ is given by the composite of functors:

$$\mathcal{D}_{\text{coh}}^b : X \xrightarrow{\text{Coh}} \text{Coh}(X) \xrightarrow{\mathcal{D}_{\text{coh}}^b} \mathcal{D}_{\text{coh}}^b(\text{Coh}(X)) = \mathcal{D}_{\text{coh}}^b(X). \quad (34)$$

It is instructive to keep reconstruction problems arising from these functors in mind. For instance, Theorem 2.33 of Gabriel-Rosenberg can be specialized

---

42 $X$ being proper over $\text{Spec}(\mathbb{C})$ implies (as part of the definition of properness) that it is separated, hence $\mathcal{D}_{\text{coh}}^b(\text{Coh}(X)) = \mathcal{D}_{\text{coh}}^b(X)$. Hence, these two isomorphisms are trivial consequences of the isomorphism $\phi^* : \text{Coh}(X) \xrightarrow{\sim} \text{Coh}(X^{an})$. These two isomorphisms are supplied just for reader’s information.

43 Of course, there are many other mathematical approaches to physics. For instance, some of Costello’s approach to quantum field theory via Lurie’s higher algebra \cite{Lur09, Lur16} point of view are touched upon in Matsuoka’s surveys \cite{M1Pr, M2Pr} in this proceedings.
to the following (which is essentially the original theorem of Gabriel [Gab62])
reconstruction theorem with respect to Coh:

**Theorem 4.5.** Any Noetherian and separated scheme $X$ can be reconstructed from $\text{Coh}(X)$.

- Glancing at this theorem of Gabriel, we naturally hope $D^b_{\text{coh}}(X)$ would carry rich information of $X$. Concerning the reconstruction problem associated with (34), any smooth connected projective variety with either $K_X$ ample or $-K_X$ ample can be reconstructed from $D^b_{\text{coh}}(X)$ (the Bondal-Orlov reconstruction theorem [BO01]).

- On the other hand, among those $X$ with trivial $K_X$ like an abelian variety or Calabi-Yau, many examples of so-called Fourier-Mukai partners, i.e. non-isomorphic smooth projective varieties with equivalent $D^b_{\text{coh}}$, have been produced, starting with Mukai [Muk82]. Thus, the restriction for the composite $D^b_{\text{coh}}: X \mapsto D^b_{\text{coh}}(X)$ in (34) does not hold in general. Considering the Gabriel reconstruction Theorem 4.5, we find this failure results from that of the reconstruction of $D^b_{\text{coh}}$ among those $X$ with trivial $K_X$. This suggests an existence of a moduli of hearts of $D^b_{\text{coh}}(X)$ for these $X$.

- If $X$ is affine locally regular and finite-dimensional, then we have the following canonical equivalence (which is a local assertion):

$$D^b_{\text{coh}}(X) \cong D^\text{perf}(X)$$

This, in turn, suggests the Verdier quotient

$$D^b_{\text{Sg}}(X) := D^b_{\text{coh}}(X)/D^\text{perf}(X)$$

reflects singular information of $X$, and is consequently called the derived category of singularities [Orl04 Def.1.8].

In the Kontsevich homological mirror symmetry, a mirror of varieties other than Calabi-Yau is not expected to be given by a sigma model. For a variety with either $K_X$ ample or $-K_X$ ample, its mirror is expected to be given by a Landau-Ginzburg model

$$(Y, I, \omega, B, W),$$

where $W : Y \to \mathcal{A}^1$ is a regular function called the superpotential. In this case, the category of $D$-branes of type $B$ is, via its identification with the category of matrix factorizations, shown to be of the following form [KL03, Orl04, Orl12]:

$$DB(Y, I, \omega, B, W) \cong \prod_{\lambda \in \mathcal{A}^1} D^b_{\text{Sg}} \left(W^{-1}(\lambda)\right).$$

44 Theorem 4.5 is reduced to Theorem 2.33 for QCoh($X) \cong \text{Ind Coh}(X)$ under the Noetherian hypothesis [Lur04 Lem.3.9]. See also [CG15 p.2] [Per09].

45 As we shall briefly review later, Bridgeland’s space of stability conditions is a kind of moduli space of “enriched hearts” of a triangulated category.
This oracle of physics [35], which highlights essentially only the singular part, might appear surprising for mathematicians. However, in the development of the minimal model program in birational geometry, it has become clear that we should take into account singular information even if we are only interested in smooth ones [MP97][KM98][Mat02].

Now, close relationship between $D^b_{\text{coh}}$ and birational geometry have been observed [BO02, Kaw02]. A central problem here is the Kawamata DK-hypothesis:

**Conjecture 4.6.** [Kaw17 Conj.1.2] For birationally equivalent smooth projective varieties $X, Y$, suppose there exists a smooth projective variety $Z$ with birational morphisms $f : Z \to X, g : Z \to Y$.

\[ K\text{-equivalence} \implies D\text{-equivalence}: \]

\[ \text{(i.e. } f^*K_X \sim g^*K_Y \text{ (linearly equivalent))} \]

\[ \text{implies} \]

\[ D\text{-equivalence} \quad \text{(i.e. } D^b_{\text{coh}}(X) \cong D^b_{\text{coh}}(Y)) \]

\[ K\text{-inequality} \implies \text{fully faithful triangulated functor:} \]

\[ \text{(i.e. there exists an effective divisor } E \text{ on } Z \text{ s.t.} f^*K_X + E \sim g^*K_Y \text{ (linearly equivalent))} \]

\[ \text{implies} \quad \left( \right. \text{there is a fully faithful functor of triangulated categories} \]

\[ D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(Y). \]

While the converse ($D\text{-equivalence} \implies K\text{-equivalence}$) does not hold in general [Ueh04], if there is a fully faithful functor $\Psi : D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(Y)$, then we obtain a semi-orthogonal decomposition [7] [BvdB03]:

\[ D^b_{\text{coh}}(Y) = \langle \Psi(D^b_{\text{coh}}(X)) \perp, \Psi(D^b_{\text{coh}}(X)) \rangle \] (36)

Motivated by the Kontsevich homological mirror symmetry, some previously unexpected structures of $D^b_{\text{coh}}(X)$ have been discovered:

- Motivated by the generalized Dehn twist associated with the Lagrangian spheres of the (hypothetical) mirror $X^\vee$, Seidel-Thomas [ST01] constructed a braid group $B_{m+1}$ action under the presence of the spherical $A_m$-configuration, i.e. there are $E_i \in D^b_{\text{coh}}(X) \ (1 \leq i \leq m)$ such that the following two conditions are satisfied:

  (sphericality):: For $1 \leq i \leq m$, $E_i \otimes \omega_X \cong E_i$ and

  \[ \text{Hom}_{D^b_{\text{coh}}(X)}(E_i, E_i[r]) = \begin{cases} 
  \mathbb{C} & \text{if } r = 0, \dim X \\
  0 & \text{if } r \neq 0, \dim X
  \end{cases} \]

  (A_m-configuration)::

  \[ \text{dim}_\mathbb{C} \oplus_r \text{Hom}_{D^b_{\text{coh}}(X)}(E_i, E_j[r]) = \begin{cases} 
  1 & |i - j| = 1 \\
  0 & |i - j| \geq 2
  \end{cases} \]
Going back to the reconstruction problem of $D^b$ in (34), existence of Fourier-Mukai partners suggests an existence of a moduli of hearts of $D^b_{\text{coh}}(X) = D^b(\text{Coh}(X))$.

To begin with, we recall a related toy model for $\text{Coh}(X)$, where we can construct moduli spaces, $M_{\mathcal{O}_X}(P)$ for a fixed Hilbert polynomial, by restricting to (Gieseker-Maruyama-Simpson) (semi)-stable sheaves [HL10, Th.4.3.4].

Thus, it’s not surprising that some kind of stability condition is needed to construct a moduli in of hearts of $D^b_{\text{coh}}(X) = D^b(\text{Coh}(X))$.

In fact, axiomatizing Douglas’ study [Dou02] of the Π-stability of D-branes, Bridgeland [Bri07] proposed a way of constructing a moduli space of “enriched hearts,” space of stability conditions, out of certain triangulated categories. Bridgeland [Bri07] defined a stability condition on a triangulated category $D$ to be a data $(Z, A)$ such that:

* $A \subset D$ is the heart of a bounded t-structure on $D$.
* $Z : K(A) \to \mathbb{C}$ is a stability function, i.e.
  * $Z : K(A) \to \mathbb{C}$ is a group homomorphism.
  * For any $E \in A \setminus \{0\}$,
    $$Z(E) := r(E) \exp(i\pi \phi(E)) \quad (r(E) > 0, 0 < \phi(E) \leq 1)$$
    \[ \in \mathbb{H} := \{ r \exp(i\pi \phi) \mid r > 0, 0 < \phi \leq 1 \}. \]
  * This stability function $Z : K(A) \to \mathbb{C}$ is furthermore a stability condition, i.e. any $E \in A$ admits a Harder-Narasimhan filtration:
    $$0 = E_0 \subset E_1 \subset \cdots \subset E_n = E,$$
    such that
    * each $F_i = E_i/E_{i-1}$ is $Z$-semistable, i.e. for all nonzero subobjects $F'_i \subset F_i$ we have
      $$\phi(F'_i) \leq \phi(F_i).$$
    * $\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n).$
    Since $Z$ is a homomorphism, we can easily verify:
    $$E, F : Z\text{-semistable s.t. } \phi(E) > \phi(F) \implies \text{Hom}_A(E, F) = 0.$$

Thus, topologists should recognize a similarity between the Harder-Narasimhan filtration and the (finite) Postnikov tower with the following analogy

$$K(\pi_1, n_1), K(\pi_2, n_2) : \text{Eilenberg-MacLane spectra s.t. } n_1 > n_2 \implies \text{Hom}_{\mathcal{SH}}(K(\pi_1, n_1), K(\pi_2, n_2)) = H^{n_2}(K(\pi_1, n_1), \pi_2) = 0.$$ Here, we wish to vary the heart $A = D^\triangledown$ while fixing the amibient triangulated category $D$. For this purpose, in view of [29], we impose
an extra structure on the stability function, i.e.

\[ K(\mathcal{D}^\vee) \xrightarrow{\cong} K(\mathcal{D}) \xrightarrow{\mathbb{Z}} \mathbb{C}, \]

where

\[ \Gamma \text{ is a finitely generated free abelian group,} \]

\[ \text{s.t. } \Gamma \otimes_{\mathbb{Z}} \mathbb{R} \text{ is equipped with a norm} \]

\[ \text{(which allows us to define } \| \text{cl}(E) \| \text{ for } E \in K(\mathcal{D}) \text{).} \]

\[ \text{cl} : \Gamma \to \mathbb{C} \text{ is a homomorphism} \]

We further impose the support property [KS08]:

\[ \left\{ \frac{|Z(E)|}{\| \text{cl}(E) \|} \right\} \quad E \in \left( \bigcup_{i \in \mathbb{Z}} \mathcal{D}^\vee[i] \right) \setminus \{0\} \text{ is bounded.} \]

When we fix \( \mathcal{D} \) with such a homomorphism \( K(\mathcal{D}) \to \Gamma \), Bridgeland [Bri07] showed the set of such stability conditions can be topologized and becomes a complex manifold \( \text{Stab}_\Gamma(\mathcal{D}) \).

However, for the case of our interest \( \mathcal{D} = \mathcal{D}^b_{\text{coh}}(X) \), as soon as \( \dim X \geq 3 \), there is no stability condition on \( \mathcal{D} = \mathcal{D}^b_{\text{coh}}(X) \) with \( \mathcal{D}^\vee = \text{Coh}(X) \) [Tod09, Lem.2.7], and even the existence of such a stability condition is problematic, i.e. the possibility of \( \text{Stab}_\Gamma(\mathcal{D}) = \emptyset \) is yet to be excluded.

4.2. \( \mathcal{D}^\text{perf}(X) \).

- The functoriality results for \( \mathcal{D}^b_{\text{coh}} \) reviewed in Theorem 4.1 and Theorem 4.2 have the following analogue for \( \mathcal{D}^\text{perf} \):

**Theorem 4.7.** [LN07, Th.1.2] [Nee17, Ill.0.19] For a separated, finite-type morphism of noetherian schemes \( f : X \to Y \),

\[ RF_* \mathcal{D}^\text{perf}(X) \subset \mathcal{D}^\text{perf}(Y) \quad \text{(i.e. perfect)} \quad \iff \quad f \text{ is proper and of finite Tor-dimension} \]

- \( \mathcal{D}^\text{perf}(X) \) can be directly recovered from \( \mathcal{D}^\text{qc}(X) \):

**Theorem 4.8.** (Nee96, BVdB03) The canonical functor

\[ \mathcal{D}^\text{perf}(X) \to \mathcal{D}^\text{qc}(X) \]

identifies \( \mathcal{D}^\text{perf}(X) \) as the full triangulated subcategory \( \mathcal{D}^\text{qc}(X)^c \) of compact objects in \( \mathcal{D}^\text{qc}(X) \):

\[ \mathcal{D}^\text{perf}(X) = \mathcal{D}^\text{qc}(X)^c \]

- Thomason-Trobaugh [TT90, App.F] proved \( \mathcal{D}^\text{perf}(X) = \mathcal{D}^\text{qc}(X)^c \) is essentially small (i.e. equivalent to a small category) for any quasi-compact and quasiseparated scheme \( X \) (e.g. for any noetherian scheme). Starting with this, Thomason [Tho97, Th.3.15] classified thick tensor triangulated ideals of \( \mathcal{D}^\text{perf}(X) = \mathcal{D}^\text{qc}(X)^c \) for any quasi-compact and quasiseparated scheme \( X \). Here, we review Paul Balmer’s generalization [Bal05] of such a classification to certain essentially small tensor triangulated categories.

**Definition 4.9.** For a tensor triangulated category \( \mathcal{K} \),
A thick tensor ideal $\mathcal{I} \subset \mathcal{K}$ is called _radical_ if

$$\mathcal{I} = \sqrt{\mathcal{I}} := \{ a \in \mathcal{K} \mid \exists n \geq 1 \text{ such that } a^\otimes n \in \mathcal{I} \}.$$  

The collection of radical thick tensor ideals of $\mathcal{K}$ is denoted by $\mathbb{R}(\mathcal{K})$. 

A proper thick tensor ideal $\mathcal{P} \subset \mathcal{K}$ is called _prime_, if

$$a \otimes b \in \mathcal{P} \Longrightarrow a \in \mathcal{P} \text{ or } b \in \mathcal{P}.$$  

If $\mathcal{K}$ is further essentially small, its _spectrum_ $\text{Spc}(\mathcal{K})$ is given by the following (set, by the “essentially small” assumption):

$$\text{Spc}(\mathcal{K}) = \{ \mathcal{P} \subset \mathcal{K} \mid \mathcal{P} \text{ is a proper prime thick tensor ideal of } \mathcal{K} \},$$

which is endowed with the topology whose open subsets are of the form

$$U(\mathcal{E}) := \{ \mathcal{P} \in \text{Spc}(\mathcal{K}) \mid \mathcal{E} \cap \mathcal{P} \neq \emptyset \} \quad (\mathcal{E} \subset \mathcal{K});$$

in other words, given by the closed basis $\{ \text{supp}(a) \}_{a \in \mathcal{K}}$, where

$$\text{supp}(a) = \{ \mathcal{P} \in \text{Spc}(\mathcal{K}) \mid a \notin \mathcal{P} \}$$

is the support of $a \in \mathcal{K}$.  

For a general topological space $T$ (we are particularly interested in the case $T = \text{Spc}(\mathcal{K})$), a subset $Y \subset T$ of the form

$$Y = \bigcup_{i \in I} Y_i \quad \text{with each complement } X \setminus Y_i \text{ open and quasi-compact}$$

is called a _Thomason subset_ of $T$. The set of Thomason subsets of $T$ is denoted by $\text{Tho}(T)$.

**Theorem 4.10.** (i) [Bal05, Th.4.10] [Bal10, Th.14] [BF11, Th.5.9] For an essentially small tensor triangulated category $\mathcal{K}$, there are mutually inverse isomorphisms between radical thick tensor ideals of $\mathcal{K}$ and Thomason subsets of $\text{Spc}(\mathcal{K})$:

$$\mathcal{K}_- : \text{Tho}(\text{Spc}(\mathcal{K})) \cong \mathbb{R}(\mathcal{K}) : \text{supp}$$

$$Y \mapsto \mathcal{K}_Y := \{ a \in \mathcal{K} \mid \text{supp}(a) \subset Y \}$$

$$(37) \quad \text{supp}(\mathcal{R}) := \bigcup_{a \in \mathcal{R}} \text{supp}(a) \leftarrow | \mathcal{R}$$

(ii) [Bal07, Prop.2.4] Suppose further $\mathcal{K}$ is rigid, then every thick tensor ideal is radical, and so, $\mathbb{R}(\mathcal{K}) = \mathcal{T}(\mathcal{K})$. Consequently, the mutually inverse isomorphisms in (i) becomes the following:

$$\mathcal{K}_- : \text{Tho}(\text{Spc}(\mathcal{K})) \cong \mathcal{T}(\mathcal{K}) : \text{supp}$$
Theorem 4.11. [Tho97] [Bal05, Cor. 5.6] [BKS07, Cor. 5.2] [Bal10, Th. 16]

For a quasi-compact and quasi-separated scheme $X$, its underlying topological space $|X|$ is homeomorphic to the spectrum $\text{Spec}(D_{\text{perf}}(X))$ via

$$|X| \xrightarrow{\cong} \text{Spec}(D_{\text{perf}}(X))$$

$$x \mapsto \mathcal{P}(x) := \left\{ P \in D_{\text{perf}}(X) \mid P_x \cong 0 \right\}.$$

For any $P \in D_{\text{perf}}(X)$, this homeomorphism restricts to the homeomorphism

$$\text{Supph}(P) \xrightarrow{\cong} \text{supp}(P),$$

where $\text{Supph}(P) \subseteq X$ is the homological support of $P \in D_{\text{perf}}(X)$, i.e. the usual sheaf theoretical support of the total homology of $P$ given in Definition 2.36 Proposition 2.37.

From Theorem 4.11 Theorem 4.10 (ii) yields the following theorem of Thomason, which is a $D_{qc}(X)$ analogue of the Hopkins-Smith Theorem 3.1:

Theorem 4.12. [Tho97, Th. 3.15] For a quasi-compact and quasi-separated scheme $X$, there are mutually inverse isomorphisms between thick tensor ideals of $D_{\text{perf}}(X)$ and Thomason subsets of $|X|:

$$D_{\text{perf}}^-(X) : \text{Tho}(|X|) \xrightarrow{\cong} \mathcal{T}(D_{\text{perf}}(X)) : \text{supp}$$

$$Y \mapsto D_{\text{perf}}^+(X) := \left\{ P \in D_{\text{perf}}(X) \mid \text{Supph}(P) \subset Y \right\} \quad (38)$$

$$\text{supp}(\mathcal{R}) := \bigcup_{a \in \mathcal{R}} \text{supp}(a) \longleftrightarrow \mathcal{R}.$$

Remark 4.13. [Nee17, Lem. 3.1]

For an object $H$ of a tensor triangulated category $\mathcal{T}$, denote by $\langle H \rangle_\otimes$ the thick tensor ideal (tensor) generated by $H$. Then we easily see:

$$\langle H \rangle_\otimes = \bigcup_{t \in \mathbb{N}, C \in \mathcal{T}} \langle C \otimes H \rangle_N,$$

where the notation $\langle - \rangle_N$ is recalled in Definition 4.28.

Many tensor triangulated categories $\mathcal{T}$ are (tensor) generated by a single element.

It should be mentioned that, just like the nilpotency theorem Theorem 3.3 was crucial in the proof of Hopkins-Smith theorem Theorem 3.1 some algebro-geometric analogue of (Devinatz-)Hopkins-Smith nilpotency is crucial to prove these algebro-geometric analogues of the Hopkins-Smith theorem (see e.g. [Nee92a, Th. 1.1] [Tho97 Th. 3.6, Th. 3.8] ). In this direction, Hovey-Palmieri-Strickland [HPS97, 5] developed a general theory how nilpotence implies classifications of thick subcategories.

Now, the following simple consequence of the above theorem of Thomason will be used later:

Corollary 4.14. For a quasi-compact and quasi-separated scheme $X$, any thick tensor ideal generated by a single $H \in D_{\text{perf}}(X)$ with $\text{Supph}(H) = |X|$ is all of $D_{\text{perf}}(X)$. 
In terms of $D_{\text{perf}}(X) = D_{\text{qc}}(X)^c$, we may refine the smashing part of the fundamental theorem of Hopkins, Neeman, Thomason and others (Theorem 2.37) to become an algebraic analogue of the Hopkins-Smith theorem (Theorem 3.1), with an extra bonus of the validity of an algebraic analogue of the telescope conjecture. We shall review it now, together with (a restatement of) Theorem 2.37. For the notations below, consult the list just before Theorem 3.7.

Theorem 4.15. ([Hop85] [Nee92a, Th.2.8,Th.3.3] [Tho97], [ATJLSS04, Cor.4.6;Cor.4.13;Th.5.6] [BF11, Cor.6.8] [DS13, Cor.6.8;Ex.6.9] [HR17 Th.B])

For a Noetherian scheme $X$, we have a commutative diagram consisting of mutually inverse horizontal arrows:

\[
\begin{array}{c}
2^{|X|} \quad \{Q \in D_{\text{qc}}(X) \mid \text{supp}(Q) \subseteq -\} \\
\text{Tho}(|X|) \quad D_{\text{perf}}(X) \\
\text{supp} \quad \text{supp} \quad \text{supp} \\
\downarrow \quad \downarrow \quad \downarrow \\
\bigcup \quad \bigcup \quad \bigcup \\
\{Q \in D_{\text{qc}}(X) \mid \text{supp}(Q) \subseteq -\} \\
\text{L}(D_{\text{qc}}(X)) \\
\end{array}
\]

(39)

Here,

- The upper side mutually inverse arrows are those in Theorem 2.37, which is the analogue of the Ohkawa theorem and an affirmative solution of the Hovey Conjecture 2.32 (ii) for $D_{\text{qc}}(X)$.

- The lower left side mutually inverse arrows are those in Thomason’s Theorem 4.12, which is a $D_{\text{qc}}(X)$ analogue of the Hopkins-Smith Theorem 3.1.

Remark 4.16. The above commutative diagram (39) encapsulates our story; starting with Ohkawa’s theorem in $\mathcal{SH}$, we then move on to the $D_{\text{qc}}$ analogue, encountering the fundamental theorem of Hopkins, Neeman, Thomason and others; then going back to $\mathcal{SH}^c$ to appreciate the Hopkins-Smith thick category theorem, and then, moving back again to the $D_{\text{qc}}^c$ analogue, we discover the above fantastic Theorem 4.15.

In fact, the commutative diagram (39) is a $D_{\text{qc}}^c \subset D_{\text{qc}}$ analogue of the commutative diagram (25) for $\mathcal{SH}_{(p)}^c \subset \mathcal{SH}_{(p)}$. Thus the underlying message here is to extend the commutative diagrams of (39) and (25) to other triangulated categories. There is a paper of Iyenger-Krause [IK12] in this direction, and this is exactly the theme of our Homework in the introduction.

- The mutually inverse arrows at the bottom right of the diagram yield a positive solution of the telescope conjecture (Theorem 3.3(iv)) by [BF11 Cor.6.8] [HR17 Th.B]).

However, the analogue of (14) for $D_{\text{perf}}$ does not hold in general, for $L_j^* : D_{\text{perf}}(X) \to D_{\text{perf}}(U)$ is not surjective in general. Still, as was noticed by
Thomason-Trobaugh [TT90], there is a similar equivalence as soon as we apply the thick closure $(-)^{\wedge}$:

**Theorem 4.17** (Thomason’s localization theorem). Under the situation of (14), i.e. let $X$ be a quasicompact and quasiseparated scheme, $Z = X \setminus U \subseteq X$, the complement of a quasi-compact Zariski open immersion $j : U \hookrightarrow X$, we have a triangulated embedding

$$D_{\text{perf}}(X)/ (D_{\text{perf}})_Z(X) \subset D_{\text{perf}}(U),$$

which yields an equivalence upon applying the thick closure:

$$\left(D_{\text{perf}}(X)/ (D_{\text{perf}})_Z(X)\right)^{Lj^*_\perp} \xrightarrow{\sim} D_{\text{perf}}(U). \quad (40)$$

In applications, we sometime have to take care of elements in $(D_{\text{perf}})_Z(X)$. Then we wonder if they are in the image of $R_i_* D_{\text{perf}}(Z)$ or not. Now, Rouquier [Rou08] gave an affirmative answer for a weaker question in the coherent setting:

Let us recall the following related result in the setting of abelian category of quasi-coherent sheaves, which should go back at least to Gabriel (see e.g. [Rou10, Prop.3.1]):

$$\text{Coh}(X)/\text{Coh}_Z(X) \xrightarrow{\sim} \text{Coh}(U),$$

where the left hand side is the abelian quotient category in the sense of Gabriel, Grothendieck, Serre.

The following interesting historical account on the difficulty of generalizing statements in $D_{\text{qc}}$

$$D_{\text{qc}}(X)/ (D_{\text{qc}})_Z(X) \xrightarrow{Lj^*_\perp} D_{\text{qc}}(U)$$

and the precursor in the setting of abelian categories reviewed in footnote 27:

$$\text{QCoh}(X)/\text{QCoh}_Z(X) \xrightarrow{\sim} \text{QCoh}(U)$$

to the setting of $D_{\text{perf}}$, has been communicated to the author by Professor Neeman:

... But the right adjoints $j_* : \text{QCoh}(U) \to \text{QCoh}(X)$ and $Rj_* : D_{\text{qc}}(U) \to D_{\text{qc}}(X)$ fail to preserve the finite subcategories $\text{Coh}(-)$ and $D_{\text{perf}}(-)$. For these categories some work is needed. Especially in the case of $D_{\text{perf}}(-); for a long time all that was known was that $Lj^* : D_{\text{perf}}(X) \to D_{\text{perf}}(U)$ isn’t surjective on objects, hence the natural map

$$\frac{D_{\text{perf}}(X)}{\text{Ker}(Lj^*)} \to D_{\text{perf}}(U)$$

couldn’t be an equivalence. So the assumption was that this map had to be worthless. Thomason’s ingenious insight was that the old counterexamples were a red herring. Up to idempotent completion this map is an equivalence, and in particular induces an isomorphism in higher $K$-theory. This of course required proof. Thomason gave a rather involved proof, following SGA6, and I noticed that the proof simplifies and generalizes when one uses the methods of homotopy theory.

It was an amusing role reversal: Thomason, the homotopy theorist, had the brilliant idea but gave a clumsy proof using the techniques of algebraic geometry, while I, the algebraic geometer, simplified the argument with the techniques of homotopy theory.
Theorem 4.18. [Rou08, Lem.7.40] Let \( X \) be a separated noetherian scheme and \( Z \) be its closed subscheme given by the ideal sheaf \( \mathcal{I} \) of \( \mathcal{O}_X \). For \( n \in \mathbb{N} \), let \( Z_n \) be the closed subscheme of \( X \) with ideal sheaf \( \mathcal{I}_n \) and \( i_n : Z_n \to X \) the corresponding immersion. Then,

\[
\forall Q \in \left( \mathcal{D}^b_{\text{coh}} \right)_Z(X), \quad \exists n \in \mathbb{N}, \exists P_n \in \mathcal{D}^b_{\text{coh}}(Z_n) \text{ s.t. } Q = R_{i_n}^*P_n.
\]

While the original proof of Theorem 4.17 given in [TT90] is purely algebro geometric in the spirit of SGA6, Neeman [Nee92b, Th.2.1] gave a proof from a general triangulated category theoretical point of view, in the homotopy theoretical spirit of Bousfield, Ohakawa, and others, building upon Corollary 2.19 [Nee92b, Lem.1.7]:

Theorem 4.19 (Neeman’s generalization of Thomason’s localization theorem). Let \( T \) be a compactly generated triangulated category, generated by a set \( K \) consisting of compact objects in \( T \). For a subset \( S \subseteq K \), set \( S \) be the smallest localizing triangulated subcategory containing \( S \). Then, the canonical sequence of triangulated categories

\[
S \to T \to T/S
\]

induces another sequence of triangulated categories of compact objects

\[
S^c \to T^c \to (T/S)^c,
\]

which induces an equivalence

\[
S^c = S \cap T^c,
\]

a fully faithful embedding

\[
T^c/S^c \to (T/S)^c,
\]

and, although it may fail to induce an equivalence \( T^c/S^c \to (T/S)^c \), it does induce an equivalence upon applying the thick closure:

\[
(T^c/S^c)^{\wedge} \to (T/S)^c.
\]

Proof. (i) The first triangulated functor in (42) is an easy consequence of Proposition 2.23. The second triangulated functor in (42) is induced by the smashing Bousfield localization functor \( T \to T/S \), which preserves arbitrary coproducts Theorem 2.19. Then for \( c \in T^c, t_\lambda \in T \ (\lambda \in \Lambda) \), regarding \( T/S \) as the full subcategory of \( L \)-local objects, we evaluate as follows:

\[
\text{Hom}_{T/S} (L c, \oplus_{\lambda \in \Lambda} L t_\lambda) = \text{Hom}_T (L c, \oplus_{\lambda \in \Lambda} L t_\lambda) \overset{L : \text{smashing}}{=} \text{Hom}_T (L c, L (\oplus_{\lambda \in \Lambda} t_\lambda)) = \text{Hom}_T (L c, L (\oplus_{\lambda \in \Lambda} t_\lambda)) \overset{L : \text{compact}}{=} \oplus_{\lambda \in \Lambda} \text{Hom}_T (c, L t_\lambda) = \oplus_{\lambda \in \Lambda} \text{Hom}_{T/S} (L c, L t_\lambda),
\]

which implies \( L c \) is also compact.

On the other hand, Krause [Kra10] gave a conceptually simple, though more involved, proof of the existence of (42), applying the following easy observation [Kra10, Lem.5.4.1.(1)], which goes back at least to [Nee96, Th.5.1]
where the converse, i.e. compactness preservation of \( F \implies \) small coproducts preservation of \( G \), is also shown under the additional compact generation assumption of \( \mathcal{T} \):

For any pair of adjoint triangulated functors \( \mathcal{T} \xrightarrow{F} \mathcal{U} \) such that

\( G \) preserves small coproducts,
\( F \) preserves compactness.

\[ \vdash \) In fact, for any \( c \in \mathcal{T}^c, \ u_\lambda \in \mathcal{U} \ (\lambda \in \Lambda) \),
\[ \text{Hom}_\mathcal{U}(Fc, \oplus_\lambda u_\lambda) = \text{Hom}_\mathcal{T}(c, G(\oplus_\lambda u_\lambda)) = \oplus_\lambda \text{Hom}_\mathcal{T}(c, G(u_\lambda)) = \oplus_\lambda \text{Hom}_\mathcal{U}(Fc, u_\lambda) \] 

Now, (42) is induced from (41) by applying this easy observation to the recollement given by Proposition 2.21.6.

(ii) To see (43), first note \( S^c \supset S \cap \mathcal{T}^c \) is trivial from the definition. Then (43) follows since converse \( S^c \subset S \cap \mathcal{T}^c \) also follows from (42).

(iii) For (44), suffices to show the composite
\[ \text{Hom}_{\mathcal{T}/S^c}(c, c') \to \text{Hom}_{\mathcal{T}/S}(c, c') \cong \text{Hom}_{\mathcal{T}/S}(c, \text{hocolim}(x_n)) \] 

is an isomorphism.

For the surjectivity, take \( (f : c \to \text{hocolim}(x_n)) \in \text{Hom}_{\mathcal{T}}(c, \text{hocolim}(x_n)) \), then we can find its preimage \( (c \xleftarrow{\star} c \times_{c^0} c' \xrightarrow{\exists f_n} c') \in \text{Hom}_{\mathcal{T}/S^c}(c, c') \) by a straightforward contemplation summarized in the following commutative diagram:

\[ \begin{array}{ccc}
  & & x_0 \\
  & c' \xrightarrow{\exists f_n} \exists c_n & \xrightarrow{\exists f_n} x_n \\
  \xleftarrow{\star} \ c \times_{c^0} c' \xrightarrow{\exists f_n} \exists c_n & \xrightarrow{\exists f_n} x_n & \text{hocolim}(x_n) \\
\end{array} \]

Here, \( c_n \) is some compact object so that arrows with \( \star \) have cones of the form finite extension of finite coproducts of elements in \( S \), and \( c \times_{c^0} c' \) is the homotopy pullback (see e.g. [TT90, p.252,(1.1.2.5)]).

For the injectivity, suppose \( (c \xleftarrow{\star} \ c \xrightarrow{\exists f_n} c') \in \text{Hom}_{\mathcal{T}/S^c}(c, c') \) is sent to \( (c \xleftarrow{\star} \ c \xrightarrow{\exists f_n} \text{hocolim}(x_n)) = 0 \in \text{Hom}_{\mathcal{T}}(c, \text{hocolim}(x_n)) \). Then we can see \( (c \xleftarrow{\star} \ c \xrightarrow{\exists f_n} c') = (x \xleftarrow{\star} c \times_{c^0} c' \xrightarrow{\exists f_n} c') = 0 \in \text{Hom}_{\mathcal{T}/S^c}(c, c') \) by a straightforward

\[ \text{This is the involved part of this proof, for the existence of recollement there requires Brown representability.} \]
contemplation summarized in the following commutative diagram:

\[
\begin{array}{c}
\tilde{c} \times_{c_m} c' \ar[r]^-{0} \ar[d]^-{r} & c' \ar[d]^-{0} \ar[r] & x_0 \\
\star \ar[r] \ar[d] & \exists c'_m \ar[r] \ar[d] & x_m \\
\star \ar[u] \ar[r] & \star \ar[u] \\
\star \ar[u] \ar[r] & \star \ar[u] \\
\end{array}
\]

Here, \(c'_m\) is some compact object so that arrows with \(\star\) have cones of the form finite extension of finite coproducts of elements in \(S\), and \(c \times_{c_m} c'\) is the homotopy pullback \([TT90, \text{p.252}, (1.1.2.5)]\).

(iv) To see \((15)\), write \(T = \langle K \rangle\), and observe from the construction of the Verdier quotient \(T \xrightarrow{F_{\text{univ}}} T/S\) that \(T/S = \langle F_{\text{univ}}(K) \rangle\), where \(F_{\text{univ}}(K) \subseteq T^c/S^c \subseteq (T/S)^c\) by \((12)\) and \((14)\). Now apply Proposition \(2.23\) to conclude any object \(y\) of \((T/S)^c\) is a direct summand of a finite extension (in \((T/S)^c\)) of finite direct sums of objects in \(F_{\text{univ}}(K) \subseteq T^c/S^c\), which is a full triangulated subcategory by \((14)\). This implies the desired equivalence upon thick closure \((15)\): \((T^c/S^c) \xrightarrow{\approx} (T/S)^c\).

\[\square\]

The following consequence of Theorem 4.17 and Remark 2.3 (iv) will be used later:

**Corollary 4.20.** Let \(X\) be a Noetherian scheme, and \(Z = X \setminus U \subset X\), the complement of a quasi-compact Zariski open immersion \(j : U \hookrightarrow X\). Then, for any \(P \in D^\text{perf}(U)\), there exists \(H \in D^\text{perf}(X)\) such that

\[L_j^* H \cong P \oplus \Sigma P \in D^\text{perf}(U)\.

Now, to motivate Balmer’s construction reviewed next, let us single out the following slight strengthening of Theorem 4.19 (and so also of Theorem 4.17):

**Theorem 4.21.** Under the same assumption of Theorem 4.19, the extrinsic thick closure equivalence \((15)\) can be upgraded to the intrinsic idempotent completion equivalence:

\[(T^c/S^c)^\dagger \xrightarrow{\approx} (T/S)^c.\]  

(46)

In particular, under the same assumption of Theorem 4.17, we have an equivalence upon applying the idempotent completion:

\[(D^\text{perf}(X)/ (D^\text{perf})^\ast_Z(X))^\dagger \xrightarrow{L_j^*} D^\text{perf}(U).\]  

(47)

To show \((46)\), it suffices to show \((T^c/S^c)^\dagger \cong (T^c/S^c)^\dagger\) thanks to \((15)\). For this, note from \((14)\) a fully faithful embedding \(T^c/S^c \to T/S\). Here, \(T/S\) is idempotent complete, because \(T/S\) is first seen to be equipped with arbitrary

---

\[50\] For the fact that the idempotent completion of a triangulated category has a natural structure of a triangulated category, there is a proof in Balmer-Schlichting \([BS01]\).
small coproducts by Theorem 2.19.2, Proposition 2.21.5, Proposition 2.10.5, and then we may apply Remark 2.3.(i) to find \( T^c/S^c \) is idempotent complete. Thus, any added idempotent object of \( (T^e/S^e)^c \) shows up in \( T/S \), but, because of \( T^c/S^c \subseteq (T/S)^c \) and any direct summand of a compact object is still compact, these added idempotent objects actually show up in \( (T/S)^c \). This implies the desired (46).

- In view of Theorem 4.11, we wonder whether the spectrum \( X \) is reconstructed from \((D^{\text{perf}}(X), \otimes^L)\). But, this is nothing but the theorem of Paul Balmer [Bal05]:

**Definition 4.22.** For an essentially small tensor triangulated category \( K \), we defined in Definition 4.9 the spectrum (topological space) \( \text{Spc}(K) \).

- Here, motivated by (47), we can construct a presheaf of tensor triangulated categories by

\[
U \mapsto K(U) := (K/K_Z)^c,
\]

where \( K_Z := \{a \in K \mid \text{supp}(a) \subseteq Z\} \) with \( Z := X \setminus U \) and \( \text{supp}(a) := \text{Spc}(K) \setminus U(a) = \{P \in \text{Spc}(K) \mid a \notin P\} \).

- Finally, we obtain the ringed space

\[
\text{Spec}(K) = (\text{Spc}(K), \mathcal{O}_K),
\]

as the sheafification of the presheaf of commutative rings

\[
U \mapsto \text{End}_{K(U)}(1),
\]

where \( 1 \) is the unit object of the tensor triangulated category \( K(U) \).

Now Balmer’s reconstruction theorem [Bal05] states:

**Theorem 4.23.** For a quasi-compact and quasi-separated scheme \( X \), we have an isomorphism of ringed spaces \(51\)

\[
\text{Spec}(D^{\text{perf}}(X), \otimes^L) \cong X.
\]

4.3. \( D^b_{\text{coh}}(X) \) and \( D^{\text{perf}}(X) \) determine each other. With the concepts “approximable”, “noetherian approximable”, “metric”, “preferred \( t \)-structure”, and “Cauchy sequence” in a black box, Amnon Neeman’s strategy to prove this may be summarized as follows:

- [Nee18c, Ex.8.4]: Out of an approximable triangulated category \( T \) with a preferred \( t \)-structure \( (T^{\leq 0}, T^{\geq 0}) \), we can construct a couple of triangulated categories \( S \) with metrics:

1. \( S = T^c \subset T \), and \( M_i = T^c \cap T^{\leq -i} \).
2. \( S = [T^b_c]^{\text{op}} \), and \( M_i^{\text{op}} = T^b_c \cap T^{\leq -i} \).

---

\(51\) The weaker reconstruction just as a topological space was already shown by Thomason (see Theorem 4.11) in the course of his establishing a \( D_{\text{qc}}(X) \) analogue of the Hopkins-Smith theorem (see Theorem 4.12 Theorem 4.10).
For an essentially small triangulated category $\mathcal{S}$ with a metric $\{M_i\}$, we define three full subcategories $\mathcal{L}(\mathcal{S}), \mathcal{C}(\mathcal{S}), \mathcal{G}(\mathcal{S})$ of the category
$$\text{Mod} - \mathcal{S} := \text{additive functors } \mathcal{S}^{\text{op}} \to \mathbb{Z} - \text{Mod}.$$ 
With $Y : \mathcal{S} \to \text{Mod} - \mathcal{S}; A \mapsto Y(A) := \text{Hom}(\cdot, A)$ the Yoneda functor, we set
$$\mathcal{L}(\mathcal{S}) := \left\{ \text{colim} Y(E_i) \in \text{Mod} - \mathcal{S} \mid E_i, \text{ is a Cauchy sequence in } \mathcal{S} \right\}$$
$$\mathcal{C}(\mathcal{S}) := \{ A \in \text{Mod} - \mathcal{S} \mid \text{For every } j \in \mathbb{Z} \text{ there exists } i \in \mathbb{Z} \text{ with } \text{Hom}(Y(M_i), \Sigma^{-j} A) = 0 \}$$
$$\mathcal{G}(\mathcal{S}) := \mathcal{L}(\mathcal{S}) \cap \mathcal{C}(\mathcal{S}).$$
By construction, we see [Nee18b, Obs.2.3]
$$\mathcal{G}(\mathcal{S}) = \bigcap_{j \in \mathbb{Z}} \bigcup_{i \in \mathbb{N}} [Y(\Sigma^j E_i)]^\perp$$
Intuitively, $\mathcal{G}(\mathcal{S})$ consists of compactly supported objects (for contained in $\mathcal{C}(\mathcal{S})$) of the Cauchy completion with respect to the given metric inside the Ind-completion given by the Yoneda embedding (for contained in $\mathcal{L}(\mathcal{S})$).
Appriori, it is not clear whether $\mathcal{G}(\mathcal{S})$ is triangulated or not. However, Neeman proves:

**Theorem 4.24.** [Nee18b, Def.2.10, Th.2.11] $\mathcal{G}(\mathcal{S})$ becomes a triangulated category with the distinguished triangles of the form $\text{colim} Y(A_i) \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \xrightarrow{h_i} \Sigma A_i$), where $(A_i \xrightarrow{f_i} B_i \xrightarrow{g_i} C_i \xrightarrow{h_i} \Sigma A_i)$ is a Cauchy sequence of triangles in $\mathcal{S}$.

[Nee18c, Th.8.8] With the metrics as above, we have triangulated equivalences
(1) $\mathcal{G}([\mathcal{T}^c]) = [\mathcal{T}^b_c]$.
(2) If $\mathcal{T}$ is noetherian then $\mathcal{G}([\mathcal{T}^b_c]^{\text{op}}) = [\mathcal{T}^c]^{\text{op}}$.

[Nee18a] Ex.3.6] The above theory works when $X$ is separated and quasi-compact: If $X$ is separated and quasi-compact, $\mathcal{T} = D_{qc}(X)$ is approximable with the standard $t$-structure in the preferred equivalence class.

Consequently, we obtain our desired result:

When $X$ is separated and quasi-compact, we have the following:
(1) $\mathcal{G}(D^{\text{perf}}(X)) = D^{\text{coh}}(X)$.
(2) If $X$ is further noetherian, $\mathcal{G}(D^{\text{coh}}(X)]^{\text{op}}) = [D^{\text{perf}}(X)]^{\text{op}}.$

For the rest of this section, we explain the concepts of “approximable”, “noetherian approximable”, “metric”, “preferred $t$-structure”, and “Cauchy sequence”, which were put in a black box in the above summary. We urge readers to consult Neeman’s own survey [Nee18c] for more details about the approximable triangulated categories.

Now, it is rather straightforward to define “metric” and “Cauchy sequence”.

Definition 4.25. [Nee18b, Def.1.2] [Nee18c, Def.8.3] A metric on a triangulated category $S$ is a sequence of additive subcategories $\{M_i, i \in \mathbb{N}\}$, satisfying:

1. $M_{i+1} \subseteq M_i$ for every $i \in \mathbb{N}$.
2. Any $b \in S$, with a distinguished triangle $a \to b \to c$ s.t. $a, c \in M_i$, belongs to $M_i$.

Definition 4.26. [Nee18b, Def.1.6] [Nee18c, Def.8.5] A Cauchy sequence in $S$, a triangulated category with a metric $\{M_i\}$, is a sequence $E_1 \to E_2 \to E_3 \to \cdots$ such that, for any $i \in \mathbb{N}, j \in \mathbb{Z}$, there exists $M \in \mathbb{N}$ such that, $\text{Cof}(E_m \to E_{m'}) \in \Sigma^{-j}M_i$ for any $m' > m \geq M$.

Next, we aim at “preferred $t$-structure”, but we shall make a little detour for some later purpose.

Definition 4.27. [Nee18c, Rem.3.1] Let $A$ be a full subcategory of a category $T$. Define the full subcategories $\text{add}A$, $\text{Add}A$, and $\text{smd}A$ as follows.

1. Assume $T$ has finite coproducts. $\text{add}A$ consists of all finite coproducts of objects in $A$.
2. Assume $T$ has coproducts. Add$A$ consists of all coproducts of objects in $A$.
3. $\text{smd}A$ consists of all direct summands in $T$ of objects in $A$.

The following construction will play major roles:

Definition 4.28. [Nee18c, Def.3.3] [Nee17, Rem.0.1] Given $A \subset T$, a full subcategory of a triangulated category, and possibly infinite integers $m \leq n$, define the full subcategories:

1. $A[m, n] = \bigcup_{i=m}^n A[-i]$.
2. For $l \in \mathbb{N}$, define inductively the full subcategory $\langle A \rangle_l^{[m,n]}$ (resp. $\overline{\langle A \rangle}_l^{[m,n]}$ if $T$ has coproducts) as follows.
   
   a. $\langle A \rangle_l^{[m,n]} = \text{smd}((\text{add}A)[m, n])$ (resp. $\overline{\langle A \rangle}_l^{[m,n]} = \text{smd}(\text{Add}A[m, n])$)
   
   b. $\langle A \rangle_l^{[m,n]} = \text{smd}(\langle A \rangle_l^{[m,n]} * \langle A \rangle_l^{[m,n]})$ (resp. $\overline{\langle A \rangle}_l^{[m,n]} = \text{smd}(\overline{\langle A \rangle}_l^{[m,n]} * \overline{\langle A \rangle}_l^{[m,n]})$).
3. For the case $m = -\infty, n = \infty$ and $l \in \mathbb{N}$, following Bondal-Van den Bergh [BVdB03], we shall simply denote as follows:

$$\langle A \rangle_l := \langle A \rangle_l^{[-\infty, \infty]} \quad (\text{resp.} \overline{\langle A \rangle}_l := \overline{\langle A \rangle}_l^{[-\infty, \infty]})$$

Whereas the above definition might look complicated, its major part is reflected in the following simpler definition:

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52 It was Neeman’s insight to notice surprising usefulness of introducing related categories $\langle A \rangle_l^{[m,n]}$ and $\overline{\langle A \rangle}_l^{[m,n]}$ as well.
Definition 4.29. [Nee17 Def.1.3] Given $\mathcal{A} \subset \mathcal{T}$, a full subcategory of a triangulated category, and $l \in \mathbb{N}$, define inductively the full subcategory $\text{coprod}_l(\mathcal{A})$ (resp. $\text{Coprod}_l(\mathcal{A})$) if $\mathcal{T}$ has coproducts as follows:

1. $\text{coprod}_1(\mathcal{A}) = \text{add}(\mathcal{A})$ (resp. $\text{Coprod}_1(\mathcal{A}) = \text{Add}(\mathcal{A})$),
2. $\text{coprod}_{l+1}(\mathcal{A}) = \text{coprod}_1(\mathcal{A}) \ast \text{coprod}_l(\mathcal{A})$ (resp. $\text{Coprod}_{l+1}(\mathcal{A}) = \text{Coprod}_1(\mathcal{A}) \ast \text{Coprod}_l(\mathcal{A})$).

The key for Definition 4.29 to reflect a major part of Definition 4.28 is the following elementary observation of Bondal-Van den Bergh [BVdB03]:

Lemma 4.30. [BVdB03 Lem.2.2.1] Let $\mathcal{A}$ and $\mathcal{B}$ be full subcategories of a triangulated category with small coproducts. Then:

1. $\text{smd}(\mathcal{A} \ast \mathcal{B}) \subset \text{smd}(\mathcal{A} \ast \mathcal{B})$, $\mathcal{A} \ast \text{smd}(\mathcal{B}) \subset \text{smd}(\mathcal{A} \ast \mathcal{B})$;
2. $\text{smd}(\text{smd}(\mathcal{A} \ast \mathcal{B})) = \text{smd}(\mathcal{A} \ast \text{smd}(\mathcal{B})) = \text{smd}(\mathcal{A} \ast \mathcal{B})$.

To show the first inclusion of (1): $\text{smd}(\mathcal{A} \ast \mathcal{B}) \subset \text{smd}(\mathcal{A} \ast \mathcal{B})$, pick $x \in \text{smd}(\mathcal{A} \ast \mathcal{B}$) fitting in a triangle:

$$s \to x \to b \quad (s \in \text{smd}(\mathcal{A}), b \in \mathcal{B}),$$

for which we pick $s' \in \mathcal{T}$ with $s \oplus s' \in \mathcal{A}$ and form a new triangle:

$$s \oplus s' \to x \oplus s' \to b.$$ 

This shows the desired $x \in \text{smd}(\mathcal{A} \ast \mathcal{B})$. The second inclusion of (1): $\mathcal{A} \ast \text{smd}(\mathcal{B}) \subset \text{smd}(\mathcal{A} \ast \mathcal{B})$ is shown similarly. Then (2) follows immediately from (1).

Using Lemma 4.30, we can easily prove, by induction on $l$, the following transparent expression relating Definition 4.28 with Definition 4.29.

Corollary 4.31. (c.f. [Nee17 Cor.1.11]) Given $\mathcal{A} \subset \mathcal{T}$, a full subcategory of a triangulated category, a natural number $l \in \mathbb{N}$, and possibly infinite integers $m \leq n$,

$$\langle \mathcal{A} \rangle_{m,n}^{[m,n]} = \text{smd}(\text{coprod}_l \mathcal{A}[m,n]), \quad \langle \mathcal{A} \rangle_{l}^{m,n} = \text{smd}(\text{Coprod}_l \mathcal{A}[m,n]).$$

The following Proposition 4.32 follows immediately by combining the second equality of Corollary 4.31 and Lemma 4.33 below. Philosophically Proposition 4.31 may be viewed as saying that $\langle - \rangle_l$ and $\text{Coprod}_l(-)$ are interchangeable.

Proposition 4.32. (c.f. [Nee17 Cor.1.11]) Given $\mathcal{A} \subset \mathcal{T}$, a full subcategory of a triangulated category, a natural number $l \in \mathbb{N}$, and possibly infinite integers $m \leq n$,

$$\text{Coprod}_l(\mathcal{A}[m,n]) \subseteq \langle \mathcal{A} \rangle_{l}^{m,n} \subseteq \text{Coprod}_{2l}(\mathcal{A}[m-1,n]).$$

We include a proof of the following Lemma 4.33 to highlight the point at which infinite coproducts are used. Just in case the reader is wondering: the finite analogue of Proposition 4.32 is false. While the inclusion $\text{coprod}_l(\mathcal{A}[m,n]) \subseteq \langle \mathcal{A} \rangle_{l}^{m,n}$ is true and easy, it isn’t in general true that $\langle \mathcal{A} \rangle_{l}^{m,n} \subseteq \text{coprod}_{2l}(\mathcal{A}[m-1,n]).$

Lemma 4.33. (c.f. [Nee17 Lem.1.9]) Let $\mathcal{B}$ a subcategory of $\mathcal{T}$, a triangulated category with coproducts, and $l \in \mathbb{N}$. Then

$$\text{Coprod}_l(\mathcal{B}) \subseteq \text{smd}(\text{Coprod}_l(\mathcal{B})) \subseteq \text{Coprod}_{2l}(\mathcal{B}[-1,0]).$$
Proof. The first inclusion is obvious. For the second inclusion, recall from Remark 2.3(i) that
\[ \forall x \in \text{smd} \left( \text{Coprod}_l(B) \right), \exists b \in \text{Coprod}_l(B) \text{ and an idempotent } e : b \rightarrow b, \text{ s.t. } x = eb = \text{Cone} \left( \oplus_b \rightarrow \oplus_b \right). \]

From this, we obtain the following triangle:
\[ \oplus_N b \rightarrow \oplus_N b \rightarrow x \rightarrow \Sigma \left( \oplus_N b \right), \]
where \( \oplus_N b \in \text{Add} \left( \text{Coprod}_l(B) \right) = \text{Coprod}_l(B) \) and so \( \Sigma \left( \oplus_N b \right) \in \Sigma \text{Coprod}_l(B) = \text{Coprod}_l(\Sigma B) \).

Thus, \( x \in \text{Coprod}_l(B) \ast \text{Coprod}_l(\Sigma B) \subseteq \text{Coprod}_l(B \cup \Sigma B) \ast \text{Coprod}_l(B \cup \Sigma B) \subseteq \text{Coprod}_2(B \Sigma B). \)

The constructions \( \langle - \rangle_l \) and \( \langle - \rangle_l \) are older than \( \text{coprod}_l(-) \) and \( \text{Coprod}_l(-) \), and for most purposes they work just fine. But there are results which become much easier to prove by working with \( \text{coprod}_l(-) \) and \( \text{Coprod}_l(-) \); for example the reader can look at the proof of [BNP18, Lem.4.4].

The constructions \( \langle - \rangle_l \) and \( \langle - \rangle_l \) are older than \( \text{coprod}_l(-) \) and \( \text{Coprod}_l(-) \), and for most purposes they work just fine. But there are results which become much easier to prove by working with \( \text{coprod}_l(-) \) and \( \text{Coprod}_l(-) \); for example the reader can look at the proof of [BNP18, Lem.4.4].

Thus one way to view the difference is to regard \( \text{coprod}_l(-) \) and \( \text{Coprod}_l(-) \) as technically more powerful than the older \( \langle - \rangle_l \) and \( \langle - \rangle_l \).

Now, in practice, as their constructions suggest, \( \text{coprod}_l(-) \) (resp. \( \text{Coprod}_l(-) \)) are more tractible than \( \langle A \rangle^{[m,n]}_l \) (resp. \( \langle A \rangle^{[m,n]}_l \)). However, \( \langle A \rangle^{[m,n]}_l \) (resp. \( \langle A \rangle^{[m,n]}_l \)) occurs more frequently, for instance,

**Theorem 4.34.** [ATJLS03 Th.A] (See also [Nee18a Ex.0.13]) For a triangulated category \( T \) with coproducts and a compact generator \( G \in T \), there is a unique \( t \)-structure of the following form:

\[ (T^{\leq 0}_G, T^{\geq 0}_G) := \left( \langle G \rangle^{[-\infty,0]}, \langle G \rangle^{[-\infty,0]} \right)^{\perp} [1]. \]

**Definition 4.35.** [Nee18c Def.7.3, Rem.7.4]

1. Two \( t \)-structures \( (T^{\leq 0}_1, T^{\geq 0}_1) \) and \( (T^{\leq 0}_2, T^{\geq 0}_2) \) are called equivalent, if there exists \( A \in \mathbb{N} \) with
\[ T^{\leq -A}_1 \subset T^{\leq 0}_2 \subset T^{\leq A}_1. \]

2. For a triangulated category \( T \) with coproducts and a compact generator, a \( t \)-structure \( (T^{\leq 0}, T^{\geq 0}) \) is in the preferred equivalence class if it is equivalent to \( (T^{\leq 0}_G, T^{\geq 0}_G) \) for some compact generator \( G \) (in fact, for every compact generator).

The importance of “preferred equivalence class” is that \( T^- \), \( T^+ \), and \( T^b \), recalled in the next definition, are independent of the particular representative \( (T^{\leq 0}, T^{\geq 0}) \) in the preferred equivalence class [Nee18c Fact.0.5.(iii)].

**Definition 4.36.** [Nee18c Def.7.5, Def.7.6]

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The author is grateful to Professor Neeman for this reference.
(1) Given a $t$-structure $(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0})$, we have the usual subcategories: 
\[ \mathcal{T}^- = \bigcup_n \mathcal{T}_{\leq n}, \quad \mathcal{T}^+ = \bigcup_n \mathcal{T}_{\geq n}, \quad \mathcal{T}^b = \mathcal{T}^- \cap \mathcal{T}^+. \]

(2) For a triangulated category $\mathcal{T}$ with coproducts and a compact generator, choose a $t$-structure $(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0})$ in the preferred equivalence class, define the full subcategories $\mathcal{T}_c^-$ and $\mathcal{T}_c^b$ as follows:
\[ \mathcal{T}_c^- := \left\{ F \in \mathcal{T} \mid \text{For any } n \in \mathbb{N} \text{ there exists a triangle } E \to F \to D \to E[1] \text{ with } E \text{ compact and } D \in \mathcal{T}_{\leq n-1} \right\}, \quad \mathcal{T}_c^b := \mathcal{T}^b \cap \mathcal{T}_c^-. \]

Intuitively, $\mathcal{T}_c^-$ is the closure, with respect to the metric $\mathcal{M}_i = \mathcal{T}^{\leq -i}$, of $\mathcal{T}^c$. $\mathcal{T}_c^-$ and $\mathcal{T}_c^b$ in the above definition do not depend on the choice of compact generator $G$ and are both intrinsic [Nee18c, Rem.7.7, Fact.0.5.(iv)].

Now we are ready state the fundamental concepts of “approximable” and “noetherian (approximable)” triangulated categories:

**Definition 4.37.** [Nee18a Def.0.21, Nee18c Def.4.1] A triangulated category $\mathcal{T}$ with coproducts is called approximable if there exists a compact generator $G \in \mathcal{T}$, a $t$-structure $(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0})$, and $A \in \mathbb{N}$ such that

1. $G[A] \in \mathcal{T}_{\leq 0}$ and $\text{Hom}(G[-A], \mathcal{T}_{\leq 0}) = 0$.
2. For every object $F \in \mathcal{T}_{\leq 0}$, there exists a triangle
\[ E \to F \to D \to E[1], \]
with $D \in \mathcal{T}_{\leq -1}$ and $E \in (G)_{\leq A}$.

From the definition, we find for any approximable triangulated category $\mathcal{T}$, the closure, with respect to the metric $\mathcal{M}_i = \mathcal{T}^{\leq -i}$, of $\bigcup_n (G)^{[-n,n]}$ is nothing but $\mathcal{T}^-$. Thus we may intuitively say every object in $\mathcal{T}^-$ may be “Taylor approximable” regarding $(G)^{[-n,n]}$ as consisting of “Taylor polynomials of $G$ of degree $\leq n$.” [Nee18c Dis.0.1, Rem.02].

**Definition 4.38.** [Nee18b Def.5.1, Nee18c Not.8.9] Suppose $\mathcal{T}$ is a triangulated category with coproducts, and assume it has a compact generator $G$ with $\text{Hom}(G, \Sigma^i G) = 0$ for $i \gg 0$. We declare $\mathcal{T}$ to be noetherian if there exists $N \in \mathbb{N}$ and a $t$-structure $(\mathcal{T}_{\leq 0}, \mathcal{T}_{\geq 0})$ in the preferred equivalence class, s.t.
\[ \forall X \in \mathcal{T}_c^-, \exists \text{ triangle } A \to X \to B \text{ s.t. } A \in \mathcal{T}_c^- \cap \mathcal{T}_{\leq 0}, \quad B \in \mathcal{T}_c^- \cap \mathcal{T}_{\geq N} = \mathcal{T}_c^b \cap \mathcal{T}_{\geq N}. \]

**Remark 4.39.** (i) The noetherian hypothesis is somewhat weaker than the assumption that there exists a $t$-structure in the preferred equivalence class which restricts to a $t$-structure on $\mathcal{T}_c^-$. (ii) [Nee18a Fac.0.23, Exa.3.6] For a quasicompact and separated scheme $X$, the standard $t$-structure on $\mathcal{T} = D_{\text{qc}}(X)$ is in the preferred equivalence class. Suppose further that $X$ is noetherian, then $\mathcal{T}_c^- = D_{\text{coh}}^-$, the category of bounded-above complexes of coherent sheaves, and so, the standard $t$-structure, which is in the preferred equivalence class, on $\mathcal{T} = D_{\text{qc}}(X)$ restricts to a $t$-structure on $\mathcal{T} = D_{\text{qc}}(X)$. This implies $D_{\text{qc}}(X)$ becomes noetherian in the sense of Definition [4.38 provided $X$ is
noetherian and separated. This is the origin of the terminology “noetherian” of Definition 4.38.

(iii) **WARNING!** The “noetherian” triangulated category of Definition 4.38 is nothing to do with the “Noetherian” stable homotopy category of [HPS97, Def.6.0.1].

For instance, for the case of $\mathcal{T} = \mathcal{SH}$, the stable homotopy category of spectra, it is easy to see $\mathcal{T}^-_c$ consists of those spectra $X$ whose homotopy group $\pi_i(X)$ is a finitely generated abelian groups for each $i$ and vanishes for $i \ll 0$. Thus, the standard $t$-structure, which is obviously in the preferred equivalence class, restricts to a $t$-structure on $\mathcal{T}^-_c$. This implies $\mathcal{SH}$ is noetherian in the sense of Definition 4.38 [Nee18a, Fac.0.23].

On the other hand, $\mathcal{SH}$ is clearly NOT a Noetherian stable homotopy category in the sense of [HPS97, Def.6.0.1], for the graded ring of the stable homotopy category of spheres $\pi_*S^0$ is not a Noetherian graded commutative ring, which can be easily seen by applying the Nishida nilpotency, the precursor of (Devinatz-)Hopkins-Smith nilpotency.

Then we have the following somewhat straightforward result to produce examples of approximable triangulated categories:

**Proposition 4.40.** [Nee18a, Ex.3.3]

If $\mathcal{T}$ has a compact generator $G$, such that $\text{Hom}(G, \Sigma^i G) = 0$ for all $i > 0$, then $\mathcal{T}$ is approximable. Just take the $t$-structure $(\mathcal{T}^-_G, \mathcal{T}^+_G)$ of Theorem 4.34 with $A = 1$.

From this, we immediately see the stable homotopy category $\mathcal{SH}$ is approximable. (actually noetherian, as was remarked in Remark 4.39(iii)).

Our principal example of approximable triangulated categories is supplied by the following theorem:

**Theorem 4.41.** [Nee18a, Ex.3.6]

Let $X$ be a quasicompact, separated scheme. Then the category $\mathcal{D}_{qc}(X)$ is approximable. (actually noetherian if $X$ is further noetherian, as was remarked in Remark 4.39(ii)).

The proof is very involved and we urge readers to consult Neeman’s original paper [Nee18a].

For now, we shall record the following application of approximability:

**Corollary 4.42.** [Nee18c, Lem.6.5] [Nee19a, Th.0.18] Let $X$ be a quasicompact, separated scheme, let $G \in \mathcal{D}_{qc}(X)$ be a compact generator, and let $u : U \to X$ be an open immersion with $U$ quasicompact. Then

$$\exists n \in \mathbb{N} \text{ s.t. } R\!u_*\mathcal{O}_U \in \langle G \rangle_{n}^{[-n,n]} \subset \mathcal{D}_{qc}(X).$$

Outline of the proof of Corollary 4.42 using approximability presented in [Nee18c]:

**Step 1:** $\exists l \in \mathbb{N} \text{ s.t. } \text{Hom}\left(R\!u_*\mathcal{O}_U, \mathcal{D}_{qc}(X)^{\leq -l}\right) = 0$.  

54 Unlike (14) and Theorem 4.17, the general case (where $X$ is quasicompact and quasiseparated) is still open – see [Nee18a, Just above Lem.3.5].
Step 2 (This is where the approximability of $D\text{qc}(X)$ is used!): By the approximability of $D\text{qc}(X)$, for some $n \in \mathbb{N}$ and a triangle:

$$E \to R_u\mathcal{O}_U \to D$$

with $D \in D\text{qc}(X)_{ \leq -l}$ and $E \in \ell \left\langle G \right\rangle_n^{[-n,n]}$.

Step 3: From Step 1 and Step 2, the map $R_u\mathcal{O}_U \to D$ in Step 2 is 0, which implies $R_u\mathcal{O}_U$ is a direct summand of $E \in \ell \left\langle G \right\rangle_n^{[-n,n]}$, as desired.

For details about the approximable triangulated categories, consult Neeman’s own survey [Nee18c].

5. Strong generation in derived categories of schemes

In the previous section, we saw $D\text{perf}(X)$ and $D\text{coh}(X)$ carry rich information and are intimately related to each other. In this section, we would like to investigate the important “strong generation” property, in the sense of Bondal and Van den Bergh [BVdB03], for $D\text{perf}(X)$ and $D\text{coh}(X)$, via approximable triangulated category techniques.

For this purpose, we have to start with what we mean by a “generator” of $D\text{perf}(X)$ and $D\text{coh}(X)$, because our previous definition of a generator in Definition 2.13 only works for triangulated categories with small coproducts, which $D\text{perf}(X)$ and $D\text{coh}(X)$ are not.

**Definition 5.1.** [Nee18c, Expl.5.4] Let $G$ be an element of a triangulated category $S$. Then, in the notation of Definition 4.28,

1. $G$ is called a classical generator if $S = \bigcup_n \ell \left\langle G \right\rangle_n^{[-n,n]}$.
2. $G$ is called a strong generator if there exists an integer $l > 0$ with $S = \bigcup_n \ell \left\langle G \right\rangle_l^{[-n,n]}$. In this case, $S$ is called strongly generated.

With this opportunity, let us record the following important concept intimately related to the above definition:

**Definition 5.2.** [Rou08, Def.3.2] The Rouquier dimension of a triangulated category $S$, denoted by $\dim S$, is the smallest $d$ for which there exists $G \in S$ with $S = \bigcup_n \ell \left\langle G \right\rangle_{d+1}^{[-n,n]}$.

**Remark 5.3.** (i) Rouquier [Rou08] proved the following properties of the Rouquier dimension of $D\text{coh}(X)$:

- [Rou08, Prop.7.9] For a smooth quasiprojective scheme $X$ over a field, we have $\dim D\text{coh}(X) \leq 2 \dim X$.
- [Rou08, Prop.7.16] For a reduced separated scheme $X$ of finite type over a field, $\dim D\text{coh}(X) \geq \dim X$.
- [Rou08, Th.7.17] For a smooth affine scheme $X$ of finite type over a field, $\dim D\text{coh}(X) = \dim X$.

There is some subtlety here. See e.g. [Nee18c] footnote 4 in Proof of Lem.5; Sketch 7.19.(i)]
(ii) For a sample of examples of Rouquier dimension in affine case, see [IT14, DT15a, DT15b] for instance.

On the other hand, Neeman deduces strong generation of $\mathcal{D}_{\text{perf}}(X)$ and $\mathcal{D}_{\text{coh}}^b(X)$ from some properties of $\mathcal{D}_{\text{qc}}(X)$:

**Definition 5.4.** Let $X$ be a separated scheme.

1. $\mathcal{D}_{\text{qc}}(X)$ is called strongly compactly generated if there exists $G \in \mathcal{D}_{\text{perf}}(X)$ and an integer $l > 0$ with $\mathcal{D}_{\text{qc}}(X) = \langle G \rangle_{l,\infty}$.
2. $\mathcal{D}_{\text{qc}}(X)$ is called strongly boundedly generated if there exists $G \in \mathcal{D}_{\text{coh}}^b(X)$ and an integer $l > 0$ with $\mathcal{D}_{\text{qc}}(X) = \langle G \rangle_{l,\infty}$.

**Remark 5.5.** From Proposition 4.32, we may replace the required equality $\mathcal{D}_{\text{qc}}(X) = \langle G \rangle_{l,\infty}$ showing up twice in Definition 5.4 with more tractible $\mathcal{D}_{\text{qc}}(X) = \text{COPROD}_l(G_{(-\infty,\infty)})$ (of course, $l$ here is a doubling of old $l$).

**Theorem 5.6.** Let $X$ be a separated scheme.

1. [Nee17, Proof of Lem.2.2] If $\mathcal{D}_{\text{qc}}(X)$ is strongly compactly generated, then $\mathcal{D}_{\text{perf}}(X)$ is strongly generated.
2. [Nee17, Proof of Lem.2.7] Suppose $X$ is noetherian. If $\mathcal{D}_{\text{qc}}(X)$ is strongly boundedly generated, then $\mathcal{D}_{\text{coh}}^b(X)$ is strongly generated.

To prove these claims, the following observation is crucial:

**Lemma 5.7.**

1. [Nee17, Prop.1.8.(i)] Let $\mathcal{T}$ be a triangulated category with coproducts, and let $\mathcal{B}$ be a subcategory of $\mathcal{T}^c$. Then, for any $l \in \mathbb{N}$,
   $$\mathcal{T}^c \cap \text{COPROD}_l(\mathcal{B}) \subseteq \text{smd}(\text{COPROD}_l(\mathcal{B})).$$

2. [Nee17, Lem.2.6] Let $X$ be a noetherian scheme, and let $G$ be an object in $\mathcal{D}_{\text{coh}}^b(X)$. Then, for any $l \in \mathbb{N}$,
   $$\mathcal{D}_{\text{coh}}^b(X) \cap \text{COPROD}_l(G_{(-\infty,\infty)}) \subseteq \text{smd}(\text{COPROD}_l(G_{(-\infty,\infty)})).$$

Of course, we are going to apply (i) with

$$\mathcal{T} = \mathcal{D}_{\text{qc}}(X), \quad \mathcal{B} = G_{(-\infty,\infty)} \subseteq \mathcal{T}^c = \mathcal{D}_{\text{perf}}(X).$$

Then (i) becomes

$$\mathcal{D}_{\text{perf}}(X) \cap \text{COPROD}_l(G_{(-\infty,\infty)}) \subseteq \text{smd}(\text{COPROD}_l(G_{(-\infty,\infty)})),$n

a clear analogue of (ii).

However, the point is that we can not prove (ii) with a generality like (i). In fact, while the proof of (i) is somewhat straightforward, the proof of (ii) is more involved. For instance (see [Nee17, Proof of Lem.2.4]), the “phantom ideal” $\mathcal{I}$, consisting of those maps $f : x \to y$ such that any composite $\Sigma^i G \to x \overset{f}{\to} y$ vanishes for any $i \in \mathbb{Z}$ and any map $\Sigma^i G \to x$ is studied carefully, resorting Christensen’s phantom map theory:
Theorem 5.8. [Chr98, Th.1.1] Suppose \((\mathcal{P}, \mathcal{I})\) is a projective class of a triangulated category \(\mathcal{T}\), i.e. \(\mathcal{P}\) is a collection of objects in \(\mathcal{T}\), \(\mathcal{I}\) is a collection of maps in \(\mathcal{T}\), such that

- \(\mathcal{P} - \text{null} = \mathcal{I}\), where \(\mathcal{P} - \text{null}\) is the collection of \(\mathcal{P}\)-phantom maps”, i.e. those maps \(x \to y\) such that the composite \(p \to x \to y\) is zero for all objects \(p \in \mathcal{P}\) and all maps \(p \to x\). (This condition makes \(\mathcal{I}\) an ideal.)

- \(\mathcal{I} - \text{proj} = \mathcal{P}\), where \(\mathcal{I} - \text{proj}\) is the collection of all objects \(p\) such that the composite \(p \to x \to y\) is zero for all maps \(x \to y\) in \(\mathcal{I}\) and all maps \(p \to x\).

- For any object \(x \in \mathcal{T}\), there exists a triangle \(p \to x \to y\) with \(p \in \mathcal{P}\) and \(x \to y\) in \(\mathcal{I}\).

Then, for any \(n \in \mathbb{N}\), \((\mathcal{P}_n, \mathcal{I}_n)\) is also a projective class, where \(\mathcal{I}_n\) is the \(n\)-th power of the “phantom ideal” \(\mathcal{I}\), and \(\mathcal{P}_n = \langle \mathcal{P} \rangle^n\), which is by defined inductively analogous to Definition 4.28:

\[
\langle \mathcal{P} \rangle_1 = \mathcal{P}, \quad \langle \mathcal{P} \rangle_{i+1} = \text{smd} (\langle \mathcal{P} \rangle_1 \ast \langle \mathcal{P} \rangle_i).
\]

But, we also need some algebro-geometric input also to prove (ii) (see [Nee17, Lem.2.5] [LN07, Th.4.1]).

Anyway, assuming Lemma 5.7 the proof of Theorem 5.6 becomes straightforward:

Proof of Theorem 5.6 assuming Lemma 5.7. In both cases, assuming the respective assumption on \(D_{\text{qc}}(X)\), together with Remark 5.5, the claims follow as follows:

\[
D_{\text{perf}}(X) = D_{\text{perf}}(X) \cap D_{\text{qc}}(X) = D_{\text{perf}}(X) \cap \text{Coprod}_l(G(\mathbb{R}, \mathbb{R})) \\
\subseteq \text{smd} (\text{coprod}_l(G(\mathbb{R}, \mathbb{R}))) \subseteq \cup_n (G)^{[-n,n]}.
\]

\[
D^b_{\text{coh}}(X) = D^b_{\text{coh}}(X) \cap D_{\text{qc}}(X) = D^b_{\text{coh}}(X) \cap \text{Coprod}_l(G(\mathbb{R}, \mathbb{R})) \\
\subseteq \text{smd} (\text{coprod}_2(G(\mathbb{R}, \mathbb{R}))) \subseteq \cup_n (G)^{[-n,n]}.
\]

5.1. Strong generation of \(D_{\text{perf}}(X)\). From Theorem 5.6.1, we search for situations when \(D_{\text{qc}}(X)\) becomes strongly compactly generated:

Theorem 5.9 (Max Kelly [Kel65]). Suppose \(X = \text{Spec } R\) is affine. Then \(D_{\text{qc}}(X)\) is strongly compactly generated if and only if \(R\) is of finite global dimension.

Theorem 5.10 (Bondal–Van den Bergh [BVdB03]). Let \(X\) be smooth scheme of finite type over a field \(k\). Then \(D_{\text{qc}}(X)\) is strongly compactly generated.

Theorem 5.10 has recently been improved by Orlov as a characterization of the strong generation of \(D_{\text{perf}}(X)\):

Theorem 5.11. (Orlov [Orl16, Th.3.27]) Let \(X\) be a separated noetherian scheme of finite K"{u}rull dimension over an arbitrary field \(k\). Assume that the square \(X \times X\) is noetherian too. Then the following conditions are equivalent:

1. \(X\) is regular;
2. \(D_{\text{perf}}(X)\) is strongly generated.
It was this paper of Orlov [Orl16] which motivated Neeman to develop his theory of approximable triangulated category (see e.g. [Nee17, p.6, the paragraph before Rem.0.10]).

In fact, the approximability of $D_{qc}(X)$ allowed Neeman to prove the following statement by reducing to the Kelly’s old theorem in a straightforward way, i.e. by induction on the number of open affines covering $X$:

**Theorem 5.12.** (Neeman [Nee17, Th.2.1]) Let $X$ be a quasi-compact separated scheme. If $X$ can be covered by open affines Spec $R_i$ with $R_i$ of finite global dimension, then $D_{qc}(X)$ is strongly compactly generated.

**Proof.** (Outline of a proof of Theorem 5.12 following [Nee18c, Sketch.6.6]) Proceed as follows:

- Write $X = \bigcup_{1 \leq i \leq r} U_i$ with $u_i : U_i = \text{Spec}(R_i)$, by assumption.
- By induction on $r$ using the Mayer-Vietoris sequence [Rou08, Prop.5.10] (as in the proof given in [Nee17, Proof of Theorem 2.1]), we find $D_{qc}(X) = \left( \text{add} \left[ \bigcup_{i=1}^{r} Ru_i \ast D_{qc}(U_i) \right] \right) \ast \left( \text{add} \left[ \bigcup_{i=1}^{r} Ru_i \ast D_{qc}(U_i) \right] \right) \ast \cdots \ast \left( \text{add} \left[ \bigcup_{i=1}^{r} Ru_i \ast D_{qc}(U_i) \right] \right)$.

By a minor variant of Max Kelly’s Theorem 5.9,

$$\exists l \in \mathbb{N}, \text{ s.t. } 1 \leq i \leq r, \quad D_{qc}(U_i) = \langle O_{U_i} \rangle_{\langle -\infty, \infty \rangle}^{-l(n,n)}.$$  \hfill (52)

From Corollary 4.42 (recall this is where the approximability of $D_{qc}(X)$ was exploited),

$$\exists n \in \mathbb{N} \text{ s.t. } 1 \leq i \leq r, \quad Ru_i \ast O_{U_i} \in \langle G \rangle_{\langle -\infty, \infty \rangle}^{-l(n,n)} \subset D_{qc}(X).$$  \hfill (53)

From (52) and (53),

$$Ru_i \ast D_{qc}(U_i) = Ru_i \left[ \langle O_{U_i} \rangle_{\langle -\infty, \infty \rangle}^{-l(n,n)} \right] \subset \langle Ru_i \ast O_{U_i} \rangle_{\langle -\infty, \infty \rangle}^{-l(n,n)} \subset \langle G \rangle_{\langle -\infty, \infty \rangle}^{-l(n,n)},$$

and so

$$\text{add} \left[ \bigcup_{i=1}^{r} Ru_i \ast D_{qc}(U_i) \right] \subset \langle G \rangle_{\langle -\infty, \infty \rangle}^{-l(n)}.$$  \hfill (54)

From (51) and (54), we obtain the desired strong compact generation of $D_{qc}(X)$:

$$D_{qc}(X) = \langle G \rangle_{\langle -\infty, \infty \rangle}^{-l(n)}, \quad \square$$

Now, Neeman proves his main theorem on strong generation of $D_{perf}(X)$:

**Theorem 5.13.** (Neeman [Nee17, Th.0.5] [Nee18c, Th.6.1]) Let $X$ be a quasi-compact separated scheme. Then $D_{perf}(X)$ is strongly generated if and only if $X$ can be covered by open affines Spec $R_i$ with $R_i$ of finite global dimension.

**Proof.** “if” part: This is immediate from Theorem 5.12 and Theorem 5.6(1).
“only if” part: [Nee17, Rem.0.10] By Thomason-Trobaugh [TT90] recalled in Theorem [L17] and [L17], we have an equivalence upon idempotent completion:

\[(\mathcal{D}^{\text{perf}}(X)/\mathcal{D}^{\text{perf}}_Z(X))^t \xrightarrow{L_j^*} \mathcal{D}^{\text{perf}}(U).\]

Thus, if \(G \in \mathcal{D}^{\text{perf}}(X)\) is a strong generator, then so is \(L_j^*G \in \mathcal{D}^{\text{perf}}(U)\).

Now the strong generation of an affine \(U = \text{Spec}(R)\) forces \(R\) to be of finite global dimension, as is shown in [Rou08, Prop.7.25].

5.2. Strong generation of \(\mathcal{D}^{b}_{\text{coh}}(X)\). Here, we start with a nice theorem of Rouquier:

**Theorem 5.14.** (Rouquier [Rou08, Th.7.39]) Let \(X\) be a scheme of finite type over a perfect field \(k\). Then \(\mathcal{D}_{\text{qc}}(X)\) is strongly boundedly generated, and \(\mathcal{D}^{b}_{\text{coh}}(X)\) is strongly generated.

To go further, let us recall:

- the canonical map \(\mathcal{D}^{\text{perf}}(X) \rightarrow \mathcal{D}^{b}_{\text{coh}}(X)\) is an isomorphism when \(X\) is smooth over a field, and in this case, the strong generation of \(\mathcal{D}^{b}_{\text{coh}}(X) \cong \mathcal{D}^{\text{perf}}(X)\) is already discussed in the previous subsection.

- the Verdier quotient \(\mathcal{D}^{sg}(X) = \mathcal{D}^{b}_{\text{coh}}(X)/\mathcal{D}^{\text{perf}}(X)\) reflects singular information of \(X\).

Thus, we must take care of singular property of \(X\). However, while Theorem 5.13 is easy and classical in the case where \(X\) is affine, this problem is *neither easy nor classical for affine \(X\).* See [Nee18c, H.S..6.12] for more on this point. 56

Now, for this purpose, Neeman turned his attention to de Jong’s alteration: 57

**Definition 5.15.** [deJ96, deJ97, Oor98, Nee17, Remi.0.13] Let \(X\) be a noetherian scheme. A regular alteration of \(X\) is a proper, surjective morphism \(f: Y \rightarrow X\), so that

1. \(Y\) is regular and finite dimensional.
2. There is a dense open set \(U \subset X\) over which \(f\) is finite.

Now, Neeman proves:

**Theorem 5.16.** (Neeman [Nee17, Th.2.3])

Let \(X\) be a noetherian scheme, and assume every closed subscheme \(Z \subset X\) admits a regular alteration. Then \(\mathcal{D}_{\text{qc}}(X)\) is strongly boundedly generated.

---

56 In fact, when \(X\) is affine, strong generation of \(\mathcal{D}_{\text{qc}}(X)\) has been proved by Iyengar and Takahashi [IT16] under different hypotheses, and using quite different techniques, from Neeman’s Theorem 5.16. And they give examples where strong generation fails; see [IT16] and references therein.

57 (Gabber’s strengthening [Gab05] of) de Jong’s alteration is now widely used in the Morel-Voevodsky motivic stable homotopy theory. see e.g. [K13, HKO17]. For an introductory review of de Jong’s alteration, consult Oort’s [Oor18] for instance.
Proof. (Outline of a proof of Theorem 5.16 following [Nee17] Proof that Theorem 2.3 follows from Theorem 2.1.) This is proved in the following order:

- Suppose there is a counterexample $X$ to Theorem 5.16 (SBG criterion). Since $X$ is noetherian, we may choose a minimal closed subscheme $Z \subset X$ which does not satisfy Theorem 5.16 (SBG criterion).
- Replacing $X$ by $Z$, may assume all proper closed subschemes $Z \subset X$ satisfy Theorem 5.16 (SBG criterion).
- To prove Theorem 5.16 (SBG criterion) for $X$, we may assume it is reduced: for, let $j : X_{\text{red}} \to X$ be the inclusion of the reduced part of $X$, and let $\mathcal{J}$ be the corresponding ideal sheaf with $\mathcal{J}^n = 0$. Then, expressing any $C \in D_{qc}(X)$ by a complex of quasi-coherent sheaves, we obtain a filtration
  \[ 0 = \mathcal{J}^n C \subset \mathcal{J}^{n-1} C \subset \cdots \subset \mathcal{J} C \subset C, \]
  with $\mathcal{J}^j C/\mathcal{J}^{j+1} \in R_{j*} D_{qc}(X_{\text{red}})$ ($0 \leq j \leq n-1$). Then, as in [Rou08, 7.3], we find:
  \[ C \in [R_{j*} D_{qc}(X_{\text{red}})]^n = [R_{j*} D_{qc}(X_{\text{red}})] \ast [R_{j*} D_{qc}(X_{\text{red}})] \ast \cdots \ast [R_{j*} D_{qc}(X_{\text{red}})]. \]
  So, it suffices to prove the strong bounded generation
  \[ D_{qc}(X_{\text{red}}) = \text{Coprod}_{\tilde{N}} \left( \tilde{G}(\infty, \infty) \right) \]
  for some $\tilde{N} \in \mathbb{N}$ and some $\tilde{G} \in D^b_{\text{coh}}(X_{\text{red}})$, for then we would get:
  \[ D^b_{\text{coh}}(X) \subseteq [R_{j*} D_{qc}(X_{\text{red}})]^n = [R_{j*} \text{Coprod}_{\tilde{N}} \left( \tilde{G}(\infty, \infty) \right)]^n \]
  \[ \subseteq \left[ \text{Coprod}_{\tilde{N}} \left( (R_{j*}\tilde{G})(\infty, \infty) \right) \right]^n = \text{Coprod}_{\tilde{N}^n} \left( (R_{j*}\tilde{G})(\infty, \infty) \right), \]
  where $R_{j*}\tilde{G} \in D^b_{\text{coh}}(X)$ by Theorem 4.1. So, the strong bounded generation of $D^b_{\text{coh}}(X)$ would follow.
- Now that we may assume $X$ is reduced, we may apply de Jong’s regular alteration to $X$:
  \[
  \begin{array}{ccc}
  Y & \xrightarrow{f} & X \\
  \text{proper & surjective} & & \\
  f^{-1}(U) & \xrightarrow{f|_{f^{-1}(U)}} & U
  \end{array}
  \]
  where we may apply Theorem 5.12 (SCG criterion) to $Y$, because $Y$ is finite-dimensional, separated and regular: Here, let us consider $Rf_* (\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y) \in D^b_{\text{coh}}(X)$ (see Theorem 4.1). Then,

---

58 This proof does not directly use the of approximability of $D_{qc}(X)$, the approximability enters only indirectly, when we appeal to Theorem 5.10. What we want to highlight here, following a strong suggestion of Professor Neeman, is the pivotal role that the homotopy-theoretical ideas of Bousfield, Ohkawa, Hopkins-Smith and many others play in the reduction.
In [Nee17, Proof that Theorem 2.3 follows from Theorem 2.4], Neeman concluded that Theorem 2.3 follows from Theorem 2.4. It quite problematic, and usually, such an honest map $H$ (or Miller’s finite) localization instantaneously at the top of his or her head. Thus, homotopy reader is required to come up with this kind of patch spelled out in terms of elementary Bousfield our “honest map” $H$, and pretend the map $\tilde{H}$ and $H$. Professor Neeman, and the author replaced his own patch, which concentrates on some sort of patch is needed. The “patch” presented above was communicated to the author by with Professor Neeman’s “patch”, which concentrates on $\tilde{H}$ (see (63)), because Professor Neeman’s “patch” which concentrates on $\tilde{H}$ does not exist. Thus, this is trivial for the canonical map (which is the Bousfield localization) and the claim for $\psi$ follows from the local isomorphism (59).

• To the local isomorphism (56), applying the adjoint isomorphism $\mathrm{Hom}_{D_{qc}(U)}(L_j^*H, L_j^*R_j s(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y)) \cong \mathrm{Hom}_{D_{qc}(X)}(H, R_j sL_j^*R_j s(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y))$, we obtain a map $\psi$:

$$\psi : H \rightarrow R_j sL_j^*R_j s(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y).$$

(57)

Recall, since $(D_{qc})_Z(X)$ is compactly generated ([Rou08, Th.6.8]), we can apply Miller’s finite localization Theorem 2.19 to form the Verdier quotient with the equivalence (12):

$$D_{qc}(X)/ (D_{qc})_Z(X) \cong D_{qc}(U),$$

(58)

and that $R_j sL_j^*$ which shows up in the target of the $\psi$ map (57) can be interpreted as the Bousfield localization, as in (13), which is consequently expressed by a mapping telescope $\text{hocolim}$ as Miller’s finite localization (Theorem 2.19). Then, consider the following pair of maps:

$$H \xrightarrow{\psi} R_j sL_j^*R_j s(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y) = \text{hocolim}(R_0) \xleftarrow{\text{canonical map}} R_0 = R_j s(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y).$$

(59)

The both maps in (59) are local isomorphism, i.e. isomorphisms when restricted $U$. This is trivial for the canonical map (which is the Bousfield localization) and the claim for $\psi$ follows from the local isomorphism (56).

59 **WARNING!** In [Nee17] Proof that Theorem 2.3 follows from Theorem 2.4, Neeman concluded the existence of an honest map $H \rightarrow R_j s(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y)$ corresponding to (60). However, this is quite problematic, and usually, such an honest map $H \rightarrow R_j s, \mathcal{O}_Y \oplus \Sigma R_j s, \mathcal{O}_Y$ does not exist. Thus, some sort of patch is needed. The “patch” presented above was communicated to the author by Professor Neeman, and the author replaced his own patch, which concentrates on $\tilde{H}$ (see (63)), with Professor Neeman’s “patch”, which concentrates on $\tilde{H}$ (see (63)), because Professor Neeman’s patch delivers a simple message how to read [Nee17] Proof that Theorem 2.3 follows from Theorem 2.4: just replace $H$ with $\tilde{H}$ and pretend the map $\psi' : \tilde{H} \rightarrow R_j s(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y)$ obtained in (63) as our “honest map” $H \rightarrow R_j s(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y)$, and then, just proceed as is written in [Nee17] Proof that Theorem 2.3 follows from Theorem 2.4.

According to Professor Neeman, this leap and omission of justification is standard. So, the reader is required to come up with this kind of patch spelled out in terms of elementary Bousfield (or Miller’s finite) localization instantaneously at the top of his or her head. Thus, homotopy theoretical insight is prerequisite to read Professor Neeman’s papers!
– Since $H \in D^{\text{perf}}(X) = D_{qc}(X)^{c}$ is compact, arguing as in Proposition 2.23 and its comments below, we may factorize the pair of maps (59) as follows:

$$H \xrightarrow{\psi} \text{hocolim}(R_n) \xrightarrow{c} R_f \ast (\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y),$$  

(60)

where:
- $\tilde{R}$ is obtained from $R_f \ast (\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y) \in D_{coh}^b(X)$ via $\tilde{c}$ by a finite step extensions of finite coproducts of elements in $D^{\text{perf}}(X)$. Thus, we have a triangle of the following form:

$$R_f \ast (\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y) \xrightarrow{\tilde{c}} \tilde{R} \to Q' \ (Q' \in (D^{\text{perf}})_Z(X), \tilde{R} \in D_{coh}^b(X))$$  

(61)

* From (61), we see $\tilde{c}$ is a local isomorphism, then, since $c$ is also a local isomorphism, $\iota$ is a local isomorphism as well from the right hand side commutative diagram of (60). Then, since $\phi$ is also a local isomorphism, from the left hand side commutative diagram of (60), we find $\tilde{\psi}$ is also a local isomorphism. Thus, we have a triangle of the following form:

$$Q'' \to H \xrightarrow{\tilde{\psi}} \tilde{R} \to Q' \ (Q'' \in (D_{coh}^b)_Z(X))$$  

(62)

– Take the homotopy pullback $\tilde{\mathcal{H}}$ of the pair of maps $H \xrightarrow{\psi} \tilde{R} \xleftarrow{\tilde{c}} R_f \ast (\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y)$ obtained in (60):

$$\tilde{\mathcal{H}} := H \times^R_{\tilde{R}} R_f \ast (\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y)$$  

(63)

where:
- From (61), the homotopy pullback diagram (63) and $H \in D^{\text{perf}}(X)$, we have a triangle of the following form:

$$\tilde{\mathcal{H}} \xrightarrow{\tilde{\psi}} H \to Q' \ (Q' \in (D^{\text{perf}})_Z(X), H, \tilde{\mathcal{H}} \in D^{\text{perf}}(X))$$  

(64)

* From (62) and the homotopy pullback diagram (63), we have a triangle of the following form:

$$Q'' \to \tilde{\mathcal{H}} \xrightarrow{\tilde{\phi}} R_f \ast (\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y) \ (Q'' \in (D_{coh}^b)_Z(X))$$  

(65)
Concerning the homological support \( \text{Supph}(\bar{H}) \) of \( \bar{H} \in D^{\text{perf}}(X) \), we see:
- \( \text{Supph}(\bar{H}) \) is closed, because \( \bar{H} \in D^{\text{perf}}(X) \) implies \( H^* \bar{H} \) is of finite type as an \( \mathcal{O}_X \)-module, and so we may apply [Stack Lem.17.9.6] for instance.

\[
\text{Supph}(\bar{H}) \bigcap U \subseteq \text{Supph}(H) \bigcap U \subseteq \text{Supph}(Rf_*(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y)) \bigcap U
\]

Thus the homological support \( \text{Supph}(\bar{H}) \) is whole \( X \). Then, we can apply Corollary [4.14] of Thomason’s theorem of Thomason sets (Theorem [4.12]) to conclude that, \( \langle \bar{H} \rangle \), the tensor ideal generated by \( \bar{H} \), is the whole \( D^{\text{perf}}(X) \), which obviously contains \( \mathcal{O}_X \). Then, applying Remark [4.13] and Proposition [4.32], we may pick some \( C \in D^{\text{perf}}(X) \) and \( L \in \mathbb{N} \) such that

\[
\mathcal{O}_X \in \langle C \otimes \bar{H} \rangle_L \subseteq \text{Coprod}_L \left( \left( C \otimes \bar{H} \right) (-\infty, \infty) \right).
\]

Consequently, for any \( D \in D_{\text{qc}}(X) \),

\[
D = D \otimes \mathcal{O}_X \in \langle D \otimes C \otimes \bar{H} \rangle_L \subseteq \text{Coprod}_L \left( \left( D \otimes C \otimes \bar{H} \right) (-\infty, \infty) \right).
\]

- Having (67) in mind, we apply \( D \otimes C \otimes - \) to (65) to obtain the following triangles:

\[
D \otimes C \otimes Q'' \rightarrow D \otimes C \otimes \bar{H} \rightarrow D \otimes C \otimes Rf_*(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y)
\]

where \( Rf_*(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y) \in D_{\text{coh}}^b(X) \), \( Q'' \in (D_{\text{coh}}^b)^Z(X) \).

- For \( Y \), obtained by de Jong’s regular alteration, we may apply Theorem [5.12] to conclude its strong compact generation. Thus, \( \exists G \in D^{\text{perf}}(X) \), \( \exists N \in \mathbb{N} \), s.t. \( D_{\text{qc}}(Y) = \text{Coprod}_N (G(-\infty, \infty)) \). Hence,

\[
Lf^*(D \otimes C) \otimes (\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y) \in D_{\text{qc}}(Y) = \text{Coprod}_N (G(-\infty, \infty)) (G \in D^{\text{perf}}(X))
\]

Consequently, by the projection formula,

\[
D \otimes C \otimes Rf_*(\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y) = \text{Rf}_* \left( Lf^*(D \otimes C) \otimes (\mathcal{O}_Y \oplus \Sigma \mathcal{O}_Y) \right)
\]

where \( \text{Rf}_*G \in D_{\text{coh}}^b(X) \) by Theorem [4.11].

- For \( Q'' \in (D_{\text{coh}}^b)^Z(X) \) in (68), we may apply Rouquier’s Theorem [4.18] to find \( n \in \mathbb{N}, P_n \in D_{\text{coh}}^b(Z_n) \) s.t.

\[
Q'' = \text{Rf}_{n*}P_n \quad (P_n \in D_{\text{coh}}^b(Z_n)).
\]

- For \( Z_n \), whose underlying space is equal to that of the proper closed subscheme \( Z \) of \( X \) from their constructions in Theorem [4.18], we may apply Theorem [5.16] by inductive assumption to conclude their strong
bounded generations. Thus, \( \exists G'' \in D^{\text{coh}}_{\text{qc}}(Z_n), \exists M \in \mathbb{N} \) s.t. \( D_{\text{qc}}(Z_n) = \text{Coprod}_M(G''(-\infty, \infty))) \). Hence,
\[
\text{Li}_n^*(D \otimes C) \otimes P_n \in D_{\text{qc}}(Z_n) = \text{Coprod}_M(G''(-\infty, \infty)))) \quad (G'' \in D^{\text{coh}}_{\text{qc}}(Z_n)) \tag{71}
\]
Consequently, by the projection formula,
\[
D \otimes C \otimes Q'' = D \otimes C \otimes R\text{i}_{i_n*}P_n = R\text{i}_{i_n*} \left( \text{Li}_n^*(D \otimes C) \otimes P_n \right)
\in R\text{i}_{i_n*} \text{Coprod}_M(G''(-\infty, \infty))) \subseteq \text{Coprod}_M((R\text{i}_{i_n*}G'')(-\infty, \infty)))
\]
where \( R\text{i}_{i_n*}G'' \in D^{\text{coh}}_{\text{qc}}(X) \) by Theorem 4.1

- From (65, 69, 72), we find
\[
D \otimes C \otimes \tilde{H} \in \text{Coprod}_M((R\text{i}_{i_n*}G'')(-\infty, \infty))) \ast \text{Coprod}_N((Rf_*G)(-\infty, \infty)))
\]
\[
\subseteq \text{Coprod}_M((Rf_*G \oplus R\text{i}_{i_n*}G'')(-\infty, \infty))) \ast \text{Coprod}_N((Rf_*G \oplus R\text{i}_{i_n*}G'')(\infty, \infty)))
\]
\[
\subseteq \text{Coprod}_{M+N}((Rf_*G \oplus R\text{i}_{i_n*}G'')(\infty, \infty)))
\]
where \( Rf_*G \oplus R\text{i}_{i_n*}G'' \in D^{\text{coh}}_{\text{qc}}(X) \).

- Finally, from (67, 73) we see for any \( D \in D_{\text{qc}}(X) \),
\[
\begin{align*}
D \in & \text{Coprod}_{2L} \left( (D \otimes C \otimes \tilde{H})(-\infty, \infty) \right) \\
\subseteq & \text{Coprod}_{2L} \left( (\text{Coprod}_{M+N}((Rf_*G \oplus R\text{i}_{i_n*}G'')(\infty, \infty)))\right)\otimes (D \otimes C \otimes \tilde{H})(-\infty, \infty)) \right)
\end{align*}
\tag{74}
\]
\[
\subseteq \text{Coprod}_{2L(M+N)}((Rf_*G \oplus R\text{i}_{i_n*}G'')(-\infty, \infty)))
\]
where \( Rf_*G \oplus R\text{i}_{i_n*}G'' \in D^{\text{coh}}_{\text{qc}}(X) \). Thus, we have obtained the desired
\[
D_{\text{qc}}(X) = \text{Coprod}_{2L(M+N)}((Rf_*G \oplus R\text{i}_{i_n*}G'')(\infty, \infty)))
\]
which shows the strong bounded generation of \( D_{\text{qc}}(X) \) for \( Rf_*G \oplus R\text{i}_{i_n*}G'' \in D^{\text{coh}}_{\text{qc}}(X) \).

From Theorem 5.16 and Theorem 5.6 (2), we obtain Neeman’s main theorem on strong generation of \( D^{\text{coh}}_{\text{qc}}(X) \):

**Theorem 5.17.** (Neeman [Nee17 Th.0.15] [Nee18 Th.6.11]) Let \( X \) be a noetherian scheme, and assume every closed subscheme \( Z \subseteq X \) admits a regular alteration. Then \( D^{\text{coh}}_{\text{qc}}(X) \) is strongly generated.

From [loc.196, loc.197] and [Nay09], we see any \( X \), which is separated and essentially of finite type over a separated excellent scheme \( S \) of dimension \( \leq 2 \), satisfies the assumptions of Theorem 5.16 and Theorem 5.17. Thus, Theorem 5.16 and Theorem 5.17 generalize Rouquier’s Theorem 5.14

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60 In Neeman’s corresponding calculation [Nee17 1st paragraph in p.24], the extension length of Coprod was doubled to be \( 2(M + N) \) rather than \( M + N \) given in (73). However, the author does not see such a need, and so, the author opted to present as in (73).
For more details about strong generations of $D^{\text{perf}}(X)$ and $D^{b}_{\text{coh}}(X)$, consult Nee- man’s original article [Nee17] and the survey [Nee18c].

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