Compacton-like solutions of the hydrodynamic system describing relaxing media

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Abstract We show the existence of a compacton-like solutions within the relaxing hydrodynamic-type model and perform numerical study of attracting features of these solutions

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1 Introduction

In this paper there are studied solutions to evolutionary equations, describing wave patterns with compact support. Different kinds of wave patterns play key rules in natural processes. They occur in nonlinear transport phenomena [1], serve as a channels of information transfer in animate systems [2], and very often assure stability of some dynamical processes [3, 4].

One of the most advanced mathematical theory dealing with wave patterns’ formation and evolution is the soliton theory [5]. The origin of this theory goes back to Scott Russell’s description of the solitary wave movement in the surface of channel filled with water. It was the ability of the wave to move quite a long distance without any change of shape which stroke the imagination of the first chronicler of this phenomenon. In 1895 Korteweg and de Vries put forward their famous equation

\[ u_t + \beta u u_x + u_{xxx} = 0, \]

(1)
describing long waves’ evolution on the surface of a shallow water. They also obtained the analytical solution to this equation, corresponding to the solitary wave:

\[ u = \frac{12 a^2}{\beta} \text{sech}^2[a(x - 4a^2 t)]. \]

(2)

Both the already mentioned report by Scott Russell as well as the model suggested to explain his observation did not involve a proper impact till the middle of 60-th of the XX century when there have been established a number of outstanding
features of equation (1) finally becoming aware as the consequences of its complete integrability [5].

In recent years ago there have been discovered another type of solitary waves referred to as compactons [6]. These solutions inherit main features of solitons, but differ from them in one point: their supports are compact.

A big progress is actually observed in studying compactons and their properties, yet most papers dealing with this subject are concerned with the compactons being the solutions to either completely integrable equations, or those which produce a completely integrable ones when being reduced onto subset of a traveling wave (TW) solutions [7, 8, 9].

In this paper compacton-like solutions to the hydrodynamic-type model taking account of the effects of temporal non-locality are considered. Being of dissipative type, this model is obviously non-Hamiltonian. As a consequence, compactons are shown to exist merely for selected values of the parameters. In spite of such restriction, existence of this type of solution in significant for several reasons. Firstly, the mere existence of this solutions is connected with the presence of relaxing effects and rather cannot be manifested in any local hydrodynamic model. Secondly, solutions of this type manifest some attractive features and can be treated as some universal mechanism of the energy transfer in media with internal structure leading to the given type of the hydrodynamic-type modeling system.

The structure of the paper is following. In section 2 we give a geometric insight into the soliton and compacton TW solutions, revealing the mechanism of appearance of generalized solutions with compact supports. In section 3 we introduce the modeling system and show that compacton-like solutions do exist among the set of TW solutions. In section 4 we perform the numerical investigations of the modeling system based on the Godunov method and show that compacton-like solutions manifest attractive features.

2 Solitons and compactons from the geometric viewpoint

Let us discuss how the solitary wave solution to (1) can be obtained. Since the function \( u(\cdot) \) in the formula (2) depend on the specific combination of the independent variables, we can use for this purpose the ansatz \( u(t, x) = U(\xi) \), with \( \xi = x - V t \). Inserting this ansatz into equation (1) we get, after one integration, following system of ODEs:

\[
\begin{align*}
\dot{U}(\xi) &= -W(\xi), \\
\dot{W}(\xi) &= \frac{\beta}{2} U(\xi) \left( U(\xi) - \frac{2V}{\beta} \right).
\end{align*}
\]  

System (3) is a Hamiltonian system describing by the Hamilton function

\[
H = \frac{1}{2} \left( W^2 + \frac{\beta}{3} U^3 - V U^2 \right).
\]
Every solutions of (3) can be identified with some level curve $H = C$. Solution (2) corresponds to the value $C = 0$ and is represented by the homoclinic trajectory shown in Fig. 1 (the only trajectory in the right half-plane going through the origin). Since the origin is an equilibrium point of system (3) and penetration of the homoclinic loop takes an infinite “time” then the beginning of this trajectory corresponds to $\xi = -\infty$ while its end - to $\xi = +\infty$. This assertion is equivalent to the statement that solution (2) is nonzero for all finite values of the arguments.

Now let us discuss the geometric structure of compactons. For this purpose we restore to the original equation which is a nonlinear generalization to classical Korteweg - de Vries equation [6]:

$$u_t + \alpha u^m u_x + \beta (u^n)_{xxx} = 0.$$  \hspace{1cm} (4)

Like in the case of equation (1), we look for the TW solutions $u(t, x) = U(\xi)$, where $\xi = x - Vt$. Inserting this ansatz into (4) we obtain, after one integration, the following dynamical system:

$$\frac{dU}{dT} = -n \beta U^2 W,$$

$$\frac{dW}{dT} = U^{3-n} \left[-V U + \frac{\alpha}{m+1} U^{m+1} - n \beta U^{n-2} W \right],$$

where $\frac{d}{dT} = n \beta U^2 \frac{d}{d\xi}$. All the trajectories of this system are given by its first integral

$$\frac{\alpha}{(m+1)(5+m-n)} U^{5+m-n} - \frac{V}{5-n} U^{5-n} + \frac{\beta n}{2} (U W)^2 = H = \text{const.}$$

Phase portrait of system (5) shown in Fig. 2 are similar to some extent to that corresponding to system (3). Yet the critical point $U = W = 0$ of system (5) lies on the line of singular points $U = 0$. And this implies that modulus of the tangent vector field along the homoclinic trajectory is bounded from below by a
positive constant. Consequently the homoclinic trajectory is penetrated in a finite
time and the corresponding generalized solution to the initial system (4) is the
compound of a function corresponding to the homoclinic loop (which now has a
compact support) and zero solution corresponding to the rest point $U = W = 0$.
In case when $m = 1$, $\beta = 1/2$ and $n = 2$ such solution has the following analytical
representation [6]:

$$u = \begin{cases} \frac{8\sqrt{V}}{3\alpha} \cos^2 \sqrt{\frac{1}{4} \xi} \quad \text{when } |\xi| < \frac{\pi}{\sqrt{\alpha}}, \\ 0 \quad \text{when } |\xi| \geq \frac{\pi}{\sqrt{\alpha}}. \end{cases}$$

(6)

It is quite obvious that similar mechanism of creating the compacton-like solutions
can be realized in case of non-Hamiltonian system as well, but in contrast to the
Hamiltonian systems, the homoclinic solution is no more expected to form a one-
parameter family like this is the case with solution (6). In fact, in the modeling
system described in the following section, homoclinic solution appears as a result of
a bifurcation following the birth of the limit cycle and its further interaction with
the nearby saddle point.

Let us note in conclusion that we do not distinguish solutions having the compact
supports and those which can be made so by proper change of variables. In particular,
the solutions we deal with in the following sections, are realized as compact
perturbations evolving in a self-similar mode on the background of the stationary
inhomogeneous solution of a system of PDEs.

3 Relaxing hydrodynamic-type model and its qualitative investigations

We consider the following system [10]:

$$u_t + p_x = \gamma,$$
\[
V_t - u_x = 0, \\
\tau p_t + \frac{\chi}{V^2} u_x = \frac{\kappa}{V} - p.
\]

Here \(u\) is mass velocity, \(V\) is specific volume, \(p\) is pressure, \(\gamma\) is acceleration of the external force, \(\kappa\) and \(\chi/\tau\) are squares of the equilibrium and "frozen" sound velocities, respectively, \(t\) is time, \(x\) is mass (Lagrangean) coordinate. The first two equations are convenient balance equations for momentum and mass, while the last one is the dynamical equation of state, taking into account relaxing properties of the media.

We perform the factorization \([11]\) of system (7) (or, in other words, passage to an ODE system describing TW solutions), using its symmetry properties summarized in the following statement.

**Lemma 1.** System (7) is invariant with respect to one-parameter groups of transformations generated by the infinitesimal operators

\[
\begin{cases}
\hat{X}_0 = \frac{\partial}{\partial t}, & \hat{X}_1 = \frac{\partial}{\partial x}, \\
\hat{X}_2 = p \frac{\partial}{\partial p} + x \frac{\partial}{\partial x} - V \frac{\partial}{\partial V}.
\end{cases}
\]

**Proof.** Invariance with respect to one parameter groups generated by the operators \(\hat{X}_0, \hat{X}_1\) is a direct consequence of the fact that system (7) does not depend explicitly on \(t\) and \(x\). Operator \(\hat{X}_2\) is the generator of scaling transformation

\[
u' = u, \quad p' = e^\alpha p, \quad V' = e^{-\alpha} V, \quad t' = t \quad \text{and} \quad x' = e^\alpha x.
\]

Invariance of system (7) with respect to this transformation is easily verified by direct substitution.

The above symmetry generators are composed on the following combination:

\[
\hat{Z} = \frac{\partial}{\partial t} + \xi \left[ (x - x_0) \frac{\partial}{\partial x} + p \frac{\partial}{\partial p} - V \frac{\partial}{\partial V} \right].
\]

It is obvious that operator \(\hat{Z}\) belongs to the Lie algebra of the symmetry group of (7). Therefore expressing the old variables in terms of four independent solutions of equation \(\hat{Z} J(t, x) = 0\), we gain the reduction of the initial system \([11]\). Solving the equivalent system

\[
\frac{dt}{\Gamma} = \frac{d(x_0 - x)}{\xi(x_0 - x)} = \frac{dp}{\xi p} = \frac{dV}{-\xi V} = \frac{du}{0},
\]

we get the following ansatz, leading to reduction:

\[
u = U(\omega), \quad p = \Pi(\omega)(x_0 - x), \quad V = R(\omega)/(x_0 - x), \quad \omega = \xi t + \log \frac{x_0}{x_0 - x}.
\]

In fact, inserting this ansatz into the second equation of system (7), we get the quadrature

\[
U = \xi R + \text{const}.
\]
and the following dynamical system:

\[
\begin{align*}
\xi \Delta(R) R' &= -R \left[ \sigma R \Pi - \kappa + \tau \xi R \gamma \right] = F_1, \\
\xi \Delta(R) \Pi' &= \xi \left\{ \xi R (R \Pi - \kappa) + \chi (\Pi + \gamma) \right\} = F_2,
\end{align*}
\]  

(11)

where \((\cdot)' = d(\cdot)/d\omega\), \(\Delta(R) = \tau (\xi R)^2 - \chi\), \(\sigma = 1 + \tau \xi\).

In case when \(\gamma < 0\), system (11) has three stationary points in the right half-plane. One of them, having the coordinates \(R_0 = 0\), \(\Pi_0 = -\gamma\), lies in the vertical coordinate axis. Another one having the coordinates \(R_1 = -\kappa/\gamma\), \(\Pi_1 = -\gamma\) is the only stationary point lying in the physical parameters range. The last one having the coordinates

\[
R_2 = \sqrt{\frac{\chi}{\tau \xi^2}}, \quad \Pi_2 = \frac{\kappa - \tau \xi \gamma R_2}{\sigma R_2},
\]

lies on the line of singular points \(\tau (\xi R)^2 - \chi = 0\).

As was announced earlier, we are looking for the homoclinic trajectory arising as a result of a limit cycle destruction. So in the first step we should assure the fulfillment of the Andronov-Hopf theorem statements in some stationary point. Since the only good candidate for this purpose is the point \(A(R_1, \Pi_1)\), we put the origin into this point by making the following change of the coordinates \(X = R - R_1\), \(Y = \Pi - \Pi_1\) which gives us the system

\[
\begin{align*}
\xi \Delta(R) \left( \begin{array}{c} X \\ Y \end{array} \right)' &= \left[ \begin{array}{c} \kappa \\ -R_1^2 \sigma \\ \kappa \xi^2, \\ (\xi R_1)^2 + \chi \xi \end{array} \right] \left( \begin{array}{c} X \\ Y \end{array} \right) + \left( \begin{array}{c} H_1 \\ H_2 \end{array} \right),
\end{align*}
\]

(12)

where

\[
H_1 = - \left( \Pi_1 X^2 + 2\sigma R_1 XY + \sigma X^2 Y \right),
\]

\[
H_2 = \xi^2 \left( \Pi_1 X^2 + 2R_1 XY + X^2 Y \right).
\]

A necessary condition for the limit cycle appearance read as follows [12]:

\[
\text{sp} \hat{M} = 0 \iff (\xi R_1)^2 + \chi \xi = \kappa,
\]

\[
\text{det} \hat{M} > 0 \iff \Omega^2 = \kappa \xi \Delta(R_1) > 0.
\]

(13)

(14)

where \(\hat{M}\) is the linearization matrix of system (12). The inequality (14) will be fulfilled if \(\xi < 0\) and the coordinate \(R_1\) lies inside the set \((0, \sqrt{\chi/(\tau \xi^2)})\). Note that another option, i.e. when \(\xi > 0\) and \(\Delta > 0\) is forbidden from physical reason [13].

On view of that, the critical value of \(\xi\) is expressed by the formula

\[
\xi_{cr} = - \frac{\chi + \sqrt{\chi^2 + 4\kappa R_1^2}}{2R_1^2}.
\]

(15)

**Remark.** Note that as a by-product of inequalities inequalities (13), (14) we get the following relations:

\[-1 < \tau \xi < 0.\]

(16)
To accomplish the study of the Andronov-Hopf bifurcation, we are going to calculate the real part of the first Floquet index $C_1$ \[12\]. For this purpose we use the transformation

\[ y_1 = X, \quad y_2 = -\frac{\kappa}{\Omega} X - \frac{\sigma R_1^2}{\Omega} Y, \quad (17) \]

enabling to pass from system \[12\] to the canonical one having the following anti-diagonal linearization matrix

\[ \hat{M}_{ij} = \Omega (\delta_{2i} \delta_{1j} - \delta_{1i} \delta_{2j}). \]

For this representation formulae from \[12, 14\], are directly applied and, using them we obtain the expression:

\[ 16 R_1^2 \Omega^2 Re C_1 = -\kappa \left\{ 3 \kappa^2 + (\xi R_1)^2 (3 - \xi \tau) - \kappa (\xi R_1)^2 (6 + \tau \xi) \right\}. \]

Using \[13\], we get after some algebraic manipulation the following formula:

\[ Re C_1 = \frac{\kappa}{16 \Omega^2 R_1^2} \left\{ 2 \kappa \xi \tau (\xi R_1)^2 - \chi \tau (\xi^2 R_1)^2 - 3 (\chi \xi)^2 \right\}. \]

Since for $\xi = \xi_{cr} < 0$ expression in braces is negative, the following statement is true:

**Lemma 2.** If $R_1 < R_2$ then in vicinity of the critical value $\xi = \xi_{cr}$ given by the formula \[15\] a stable limit cycle appears in system \[11\].

We’ve formulated conditions assuring the appearance of periodic orbit in proximity of stationary point $A(R_1, \Pi_1)$ yet in order that the required homoclinic bifurcation would ever take place, another condition should be fulfilled, namely that on the same restrictions upon the parameters critical point $B(R_2, \Pi_2)$ is a saddle. Besides, it is necessary to pose the conditions on the parameters assuring that the stationary point $B(R_2, \Pi_2)$ lies in the first quadrant of the phase plane for otherwise corresponding stationary solution which is needed to to compose the compacton would not have the physical interpretation. Below we formulate the statement addressing both of these questions.

**Lemma 3.** Stationary point $B(R_2, \Pi_2)$ is a saddle lying in the first quadrant for any $\xi > \xi_{cr}$ if the following inequalities hold:

\[ -\tau \xi_{cr} R_2 < R_1 < R_2. \quad (18) \]

**Proof.** First we are going to show that the eigenvalues $\lambda_{1,2}$ of the system’s \[11\] Jacobi matrix

\[ \hat{M} = \frac{\partial (F_1, F_2)}{\partial (R, \Pi)} \bigg|_{R_2, \Pi_2} = \begin{bmatrix} \kappa, & \frac{\sigma R_1^2}{\tau \xi^2} \\ \frac{\xi^2 [\kappa (\sigma - 2) + 2 \gamma R_2 (\sigma - 1)]}{\sigma}, & -\frac{\kappa \xi}{\tau} \end{bmatrix} \quad (19) \]
are real and have different signs. Since the eigenvalues of $\hat{M}$ are expressed by the formula
\[
\lambda_{1,2} = \frac{\text{sp} \hat{M} \pm \sqrt{\left(\text{sp} \hat{M}\right)^2 - 4 \det \hat{M}}}{2},
\]
it is sufficient to show that
\[
\det \hat{M} < 0.
\]
(20)
In fact, we have
\[
\det \hat{M} = -\frac{X \sigma \kappa}{\tau} - \frac{X}{\tau} [\kappa (\sigma - 2) + 2 \gamma R_2 (\sigma - 1)] =
\]
\[
= -\frac{X 2 \gamma \tau}{\tau} \xi \left(\frac{\kappa}{\gamma} + R_2\right) = 2 \chi \xi \gamma (R_1 - R_2) < 0.
\]
To finish the proof, we must show that stationary point $B(R_2, \Pi_2)$ lies in the first quadrant. This is equivalent to the statement that
\[
\kappa - \tau \xi_{cr} \gamma R_2 > 0.
\]
Carrying the first term into the RHS and dividing the inequality obtained by $\gamma < 0$, we get the inequality $-\tau \xi_{cr} R_2 < R_1$. The latter implies inequalities $-\tau \xi R_2 < R_1 < R_2$ which are valid for any $\xi > \xi_{cr}$. And this ends the proof.

Numerical studies of system’s (11) behavior reveal the following changes of regimes (cf. Fig. 3). When $\xi < \xi_{cr}$, $A(R_1, \Pi_1)$ is a stable focus; above the critical value a stable limit cycle softly appears. Its radius grows with the growth of parameter $\xi$ until it gains the second critical value $\xi_{cr2} > \xi_{cr}$ for which the homoclinic loop appears in place of the periodic trajectory. The homoclinic trajectory is based upon the stationary point $B(R_2, \Pi_2)$ lying on the line of singular points $\Delta(R) = 0$ so it corresponds to the generalized compacton-like solution to system (7). We obtain this solution sewing up the TW solution corresponding to homoclinic loop with stationary inhomogeneous solution
\[
u(t, x) = p(t, x) - \Pi_2 (x_0 - x), \quad V = V(t, x) = \frac{R_2}{(x_0 - x)}.
\]
corresponding to critical point $B(R_2, \Pi_2)$. So, strictly speaking it is different from the ”true” compacton, which is defined as a solution with compact support. Note, that we can pass to the compactly supported function by the following change of variables:
\[
\pi(t, x) = p(t, x) - \Pi_2 (x_0 - x), \quad \nu(t, x) = V(t, x) - \frac{R_2}{(x_0 - x)}.
\]

4 Numerical investigations of system (7)

4.1 Construction and verification of the numerical scheme.

We construct the numerical scheme basing on the S.K.Godunov method [15]. Since the inhomogeneous terms appearing in (7) destroy the scaling invariance, we look,
Figure 3: Changes of phase portrait of system (11): (a) $A(R_1, \Pi_1)$ is the stable focus; (b) $A(R_1, \Pi_1)$ is surrounded by the stable limit cycle; (c) $A(R_1, \Pi_1)$ is surrounded by the homoclinic loop; (d) $A(R_1, \Pi_1)$ is the unstable focus;

in accordance with common practice [16], for the solution of the Riemann problem $(V_1u_1p_1)$ at $x < 0$ $(V_2, u_2, p_2)$ at $x > 0$ to corresponding homogeneous system

$$
egin{align*}
    u_t + p_x &= 0, \\
    V_t - u_x &= 0, \\
    p_t + \frac{\chi}{\tau V^2} V_t &= 0.
\end{align*}
$$

Using the acoustic approximation, we find the functions $U, P$ in the sector $-Ct < x < Ct, \quad C = \sqrt{\frac{\chi}{\tau V^2}}$:

$$
U = \frac{u_1 + u_2}{2} + \frac{p_1 - p_2}{2C},
$$

$$
P = \frac{p_1 + p_2}{2} + C \frac{u_1 - u_2}{2}.
$$

where $V_0 = \frac{V_1 + V_2}{2}$. Expression for the function $V$ is omitted since it does not take part in the construction of the scheme on this step.

At some additional assumption the Riemann problem can be solved without resorting to the acoustic approximation. Let us assume that

$$
p = \frac{\chi}{\tau V}.
$$

As it easily seen, this relation is the particular integral of the third equation of system (22). Employing this formula, we can write down the first two equations as the following closed system:

$$
\left( \frac{\partial}{\partial t} + \tilde{A} \frac{\partial}{\partial x} \right) \begin{pmatrix} u \\ V \end{pmatrix} = 0,
$$

where $\tilde{A} = \begin{pmatrix} 0 & -\chi/\tau V^2 \\ -1 & 0 \end{pmatrix}$.
Solving the eigenvalue problem \( \det | \tilde{A} - \lambda I | = 0 \), we find that the characteristic velocities satisfy the equation
\[ \lambda^2 = C_{L\infty}^2 = \chi/(\tau V^2). \]
Now we look for the Riemann invariants in the form of infinite series
\[ r_\pm = V \sum_{n=0}^{\infty} A_n^\pm u^n. \]
It is not difficult to verify by direct inspection that the following relations hold:
\[
D_\pm V = \left( \frac{\partial}{\partial t} \pm C_{L\infty} \frac{\partial}{\partial x} \right) V = u_x \pm C_{L\infty} V_x = Q_\pm, \quad (26)
\]
\[
D_\pm u = \pm C_{L\infty} Q_\pm. \quad (27)
\]
Using (26), we find the recurrence
\[ A_n^\pm = (\mp 1)^n \frac{A_0}{n!(\sqrt{\chi/\tau})^n}, \]
and finally obtain the expression for Riemann invariants:
\[ r_\pm = A_0 V \exp \left( \mp u/\sqrt{\chi/\tau} \right). \quad (28) \]
So under the assumption that \( p = \frac{\chi\tau V}{\sqrt{\chi/\tau}} \), system (22) can be rewritten in the following form:
\[ D_\pm r_\pm = 0. \quad (29) \]
Using (24) and (29) we get the solution of Riemann problem in the sector \( \sqrt{\chi/(\tau V_1^2)} t < x < \sqrt{\chi/(\tau V_2^2)} t \):
\[
U = \sqrt{\chi/\tau} \ln Z, \quad \quad P = p_2 + \sqrt{\chi/\tau} C_2 [Z \exp (-u_2/\sqrt{\chi/\tau}) - 1],
\]
where \( C_i = \sqrt{\chi/\tau}/V_i \equiv C_{L\infty}(V_i), \ i = 1, 2; \)
\[
Z = (E + \sqrt{Q})/(2C_2 \sqrt{\chi/\tau}), \quad E = \exp (u_2/\sqrt{\chi/\tau}) \{ p_1 - p_2 + \sqrt{\chi/\tau} (C_2 - C_1) \}, \quad Q = E^2 + 4\chi/\tau C_1 C_2 \exp [(u_1 + u_2)/\sqrt{\chi/\tau}].
\]
Note that (30) is reduced to (23) when \( |p_1 - p_2| << 1, \ |u_1 - u_2| << 1. \)
The difference scheme for (7) takes the following form:
\[
(u^n_i - u^{n+1}_i) \Delta x - (p^n_i + 1/2 - p^{n+1}_i + 1/2) \Delta t = -\gamma \Delta t \Delta x,
\]
\[
(V^n_i - V^{n+1}_i) \Delta x + (u^n_i + 1/2 - u^{n+1}_i + 1/2) \Delta t = 0,
\]
\[
\left( p^n_i - \frac{\chi}{\tau V^n_i} \right) \Delta x - \left( p^{n+1}_i + 1 - \frac{\chi}{\tau V^{n+1}_i} \right) \Delta x = -f \Delta t \Delta x,
\]
where \((u_{i-1/2}^n, p_{i-1/2}^n, p_{i+1/2}^n)\), \((u_{i+1/2}^n, p_{i+1/2}^n)\) are solutions of Riemann problems \((V_{i-1}^n, u_{i-1}^n, p_{i-1}^n), (V_{i}^n, u_{i}^n, p_{i}^n), (V_{i+1}^n, u_{i+1}^n, p_{i+1}^n)\), correspondingly, 
\[
f = f(p_i^k, V_i^k) = \frac{\kappa}{V_i^k} - p_i^k,
\]
k is equal to either \(n\) or \(n + 1\). The choice \(k = n\) leads to the explicit Godunov scheme
\[
\begin{aligned}
    u_i^{n+1} &= u_i^n + \frac{\Delta t}{\Delta x} (p_{i-1/2}^n - p_{i+1/2}^n) + \gamma \Delta t \\
    V_i^{n+1} &= V_i^n + \frac{\Delta t}{\Delta x} (u_{i+1/2}^n - u_{i-1/2}^n) \\
    p_i^{n+1} &= p_i^n + \frac{\kappa}{\gamma} \left( \frac{1}{V_i^n} - \frac{1}{V_i^{n+1}} \right) + f(p_i^n, V_i^n) \Delta t
\end{aligned}
\]  
(31)

The scheme (31) was tested on invariant TV solutions of the following form:
\[
u = U(\omega), \quad p = P(\omega), \quad V = V(\omega), \quad \omega = x - Dt.
\]  
(32)

Inserting (32) into first two equations of system (7), one can obtain the following first integrals:
\[
\begin{aligned}
    U &= u_1 + D(V_1 - V), \\
    P &= p_1 + D^2(V_1 - V),
\end{aligned}
\]  
(33)

where \(V_1 = \lim_{\omega \to \infty} V(\omega)\). Let us assume in addition that \(u_1 = 0\), while \(p_1 = \kappa/V_1\). With this assumption constants \((u_1, p_1, V_1)\) satisfy the initial system.

Inserting \(U\) and \(P\) into the third equation of system (7) we get:
\[
\frac{dV}{d\omega} = -V \frac{[D^2V^2 - SV + \kappa]}{\tau D[C_T^2 - (DV)^2]} = F(V),
\]  
(34)

where \(C_T = \sqrt{\chi/\tau}\), \(S = p_1 + D^2V_1\). Equation (34) has three critical points:
\[
V = V_0 = 0, \quad V = V_1, \quad V = V_2 = \kappa/(V_1 D^2).
\]

If the inequality \(V_2 = \kappa/(V_1 D^2) < V_1\) holds and the line \(\chi/\tau - (DV)^2 = 0\) is outside the interval then \((V_2, V_1)\) then constants
\[
u_{-\infty} = u_2 = \frac{(DV_1)^2 - \kappa}{DV_1} > 0, \quad p_{-\infty} = p_2 = \kappa/V_2, \quad V_{-\infty} = V_2;
\]
deliver the second stationary solution to the initial system and solution to (34) corresponds to a smooth compressive wave connecting these two stationary solutions.

Results of numerical solving the Cauchy problem based on the Godunov scheme (31) are shown in Fig. 4. As the Cauchy data we took the smooth self-similar solution obtained by numerical solving equation (34) and employment of the first integrals (33). So we see that the numerical scheme quite well describes the self-similar evolution of the initial data.
Figure 4: Temporal evolution of Cauchy data defined by solutions of equation (34) and the first integrals (33). Following values of the parameters were chosen during the numerical simulation: $\kappa = 0.5$, $\chi = 0.25$, $\tau = 0.1$, $V_1 = 0.5$, $D = 3.1$.

4.2 Numerical investigations of the temporal evolution and attractive features of compactons.

Below we present the results of numerical solving of the Cauchy problem for system (7). In numerical experiments we used the values of the parameters taken in accordance with the preliminary results of qualitative investigations and corresponding to the homoclinic loop appearance in system (11). As the Cauchy data we got the generalized solution describing the compacton and obtained by the preliminary solving of system (11) and employment of formulae (10), (21). Results of the numerical simulation are shown in Fig. 5. It is seen that compacton evolves for a long time in a stable self-similar mode.

Additionally the numerical experiments revealed that the wave packs being created by sufficiently wide family of initial data tend under certain conditions to the compacton solution. Following family of the initial perturbations have been considered in the numerical experiments:

\[
p = \begin{cases} 
  p_0(x_0 - x) & \text{when } x \in (0, a) \cup (a + l, x_0) \\
  (p_0 + p_1)(x_0 - x) + w(x - a) + h & \text{when } x \in (a, a + l),
\end{cases}
\]

\[u = 0, \quad V = \kappa/p.\]

Here $a$, $l$, $p_1$, $w$, $h$ are parameters of the perturbation defined on the background of the inhomogeneous stationary solution (21). Note that parameter $l$ defines the width of the initial perturbation. Varying broadly parameters of the initial perturbation, we observed in numerical experiments that, when fixing e.g. value of $l$, it is possible to fit in many ways the rest of parameters such that one of the wave packs created by the perturbation (namely that one which runs ”downwards” towards the direction of diminishing pressure) in the long run approaches compacton solution. Whether
Figure 5: Numerical solution of system (7) in case when the invariant homoclinic solution is taken as the Cauchy data

the wave pack would approach the compacton solution or not occurs to depend on that part of energy of the initial perturbation which is carried out "downwards". Assuming that the energy is divided between two wave packs created more or less in half, we can use for the rough estimation of convergency the total energy of the initial perturbation, consisting of the internal energy \( E_{int} \) and the potential energy \( E_{pot} \):

\[
E = E_{int} + E_{pot} = \int \left[ \varepsilon_{int} + \varepsilon_{pot} \right] dx,
\]

where \( \varepsilon_{int}, \) and \( \varepsilon_{pot} \) are local densities of the corresponding terms.

Function \( \varepsilon_{pot} \) is connected with forces acting in system by means of the evident relation

\[
\gamma - \frac{1}{\rho} \frac{\partial p}{\partial x_e} = -\frac{\partial \varepsilon_{pot}}{\partial x_e},
\]

where \( x_e \) is physical (Eulerian) coordinate connected with the mass Lagrangean coordinate \( x \) as follows:

\[
x = \int V^{-1} d x_e.
\]

From this we extract the expression

\[
E_{pot} = \int \varepsilon_{pot} dx = \int_{\Omega} \left[ \int_{x_1}^{x_e} \left( V \frac{\partial p}{\partial x_e} - \gamma \right) dx_e' \right] V^{-1} dx_e,
\]

where \( \Omega \) is the support of initial perturbation. Employing in the above integral the relation \( V \frac{\partial p}{\partial x_e} = \frac{\partial p}{\partial x} \), we obtain the following formula:

\[
E_{pot} = \frac{\kappa l}{\alpha + \beta} \left[ \left( \frac{1+k}{k} \right) \ln(1+k) - 1 \right],
\]

where \( k = [P(a+l) - P(a)]/P(a), \quad P(z) = \alpha z + \beta, \quad \alpha = w - (p_0 + p_1), \quad \beta = (p_1 + p_0)x_0 - aw. \)
Figure 6: Perturbations of stationary invariant solutions of system (7) (left) and TW solutions created by these perturbations (right) on the background of the invariant compacton-like solution (dashed)
Figure 7: Evolution of the wave patterns created by the local perturbations which do not satisfy the energy criterion.

For $\chi = 1.5, \kappa = 10, \gamma = -0.04, \tau = 0.07, x_0 = 120$ convergency was observed when $E_{tot}$ was close to 45 (see Figures below).

Function $\varepsilon_{int}$ is obtained from the second law of thermodynamics written for the adiabatic case: $(\partial \varepsilon_{int}/\partial V)_S = -p = -\kappa/V$. From this we get

$$\varepsilon_{int} = c - \kappa \ln V.$$  

To obtain the energy of perturbation itself, we should subtract from this value the energy density of stationary inhomogeneous solution $c - \kappa \ln V_0$, so finally we get

$$E_{int} = \int_{\Omega} (\varepsilon_{int} - \varepsilon_{int}^0) V^{-1} dx_e = \kappa \int_{\Omega} \ln V_0/V dx.$$  

Using the formula (3.7.20), we finally obtain

$$E_{int} = \kappa \left\{ \ln \frac{P(a + l)}{P_0(a + l)} + \frac{P(a)}{\alpha} \ln \left[ 1 + \frac{\alpha l}{P(a)} \right] + \frac{P_0(a)}{p_2} \ln \left[ 1 - \frac{p_2 l}{P_0(a)} \right] \right\},$$  

where $P_0(z) = p_0(x_0 - z)$.

Numerical experimenting shows that the energy norm serves as sufficiently good criterium of convergency. At $\chi = 1.5, \kappa = 10, \gamma = -0.04, \tau = 0.07, x_0 = 120$...
convergency was observed when \( E \in (43, 47) \). Patterns of evolution of the wave perturbations is shown in Figs 6. For comparison we also show the temporal evolution of the wave packs created by the perturbations for which \( E \not\in (43, 47) \) (Figs. 7).

Thus there is observed some correlation between the energy of initial perturbation and convergency of the wave packs created to the compacton solution.

5 Conclusions and discussion.

In this work we have discussed the origin of generalized TW solutions called compactons and have shown the existence of such solutions within the hydrodynamic-type model of relaxing media. The main results concerning this subject can be summarized as follows:

- A family of TW solutions to (7) includes a compacton in case when an external force is present (more precisely, when \( \gamma < 0 \)).
- Compacton solution to system (7) occurs merely at selected values of the parameters: for fixed \( \kappa, \gamma \) and \( \chi \) there is a unique compacton-like solution, corresponding to the value \( \xi = \xi_{cr} \).

Qualitative and numerical analysis of the corresponding ODE system describing the TW solutions to initial system served us as a starting point in numerical investigations of compactons, based on the Godunov method. Numerical investigations reveal that compacton encountering in this particular model form a stable wave pattern evolving in a self-similar mode. It is also obtained a numerical evidence of attracting features of this structure: a wide class of initial perturbations creates wave packs tending to compacton. Convergency only weakly depend on the shape of initial perturbation and is mainly caused by fulfillment of the energy criterion. This criterium is far from being perfect. In fact, it is not sensible on the form of initial perturbation, which, in turn, influences the part of the the total energy getting away by the wave pack moving "downwards". Besides, employment of the Godunov scheme does not enable to obtain more strict quantitative measure of convergency. But in spite of these discrepancies the effect of convergency is evidently observed and this will be the topic of our further study to develop more strict criteria of convergency as well as trying to realize whether the compacton solution serves as true or intermediate [17, 18] asymptotics.
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