AUTOMORPHIC EQUIVALENCE OF ONE-SORTED ALGEBRAS.

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Abstract

One of the central questions of universal algebraic geometry is: when two algebras have the same algebraic geometry? There are various interpretations of the sentence "Two algebras have the same algebraic geometry". One of these is automorphic equivalence of algebras, which is discussed in this paper, and the other interpretation is geometric equivalence of algebras. In this paper we consider very wide and natural class of algebras: one sorted algebras from IBN variety. The variety Θ is called an IBM variety if two free algebras $W(X), W(Y) \in \Theta$ are isomorphic if and only if the powers of sets $X$ and $Y$ coincide. In the researching of the automorphic equivalence of algebras we must study the group of automorphisms of the category $\Theta_0$ of the all finitely generated free algebras of $\Theta$ and the group of its automorphisms $\text{Aut}\Theta_0$. An automorphism $\Upsilon$ of the category $\mathfrak{A}$ is called inner if it is isomorphic to the identity automorphism or, in other words, if for every $A \in \text{Ob}\mathfrak{A}$ there exists $s_A^\Upsilon: A \to \Upsilon(A)$ isomorphism of these objects of the category $\mathfrak{A}$ and for every $\alpha \in \text{Mor}_\mathfrak{A}(A, B)$ the diagram

$$
\begin{array}{ccc}
A & \overset{s_A^\Upsilon}{\longrightarrow} & \Upsilon(A) \\
\downarrow \alpha & & \downarrow \Upsilon(\alpha) \\
B & \overset{s_B^\Upsilon}{\longrightarrow} & \Upsilon(B)
\end{array}
$$

is commutative. By [PZ, Theorem 2], if $\Theta$ is an IBN variety of one-sorted algebras, then every automorphism $\Psi \in \text{Aut}\Theta_0$ can be decomposed: $\Psi = \Upsilon \Phi$, where $\Upsilon, \Phi \in \text{Aut}\Theta_0$, $\Upsilon$ is an inner automorphism of and $\Phi$
is a strongly stable one (see Definition 3.1). In this situation every strongly stable automorphism defines the other algebraic structure on every algebra \( H \in \Theta \), such that the algebra \( H^* \) with this structure also belongs to our variety \( \Theta \) and (Theorem 4.1) even automorphically equivalent to the algebra \( H \), i.e., has the same algebraic geometry. From this we conclude the necessary and sufficient conditions for two algebras to be automorphically equivalent. We formulate these conditions by using the notion of geometric equivalence of algebras. It means that we reduce automorphic equivalence of algebras to the simpler notion of geometric equivalence. This paper is a continuation of the research which was started in [PZ].

1 Introduction.

We denote by \( \Omega \) the signature of algebras of the variety \( \Theta \). Let \( X_0 = \{x_1, x_2, \ldots, x_n, \ldots\} \) be a countable set of symbols, \( \mathcal{F}(X_0) \) - set of all finite subset of \( X_0 \). We will consider the category \( \Theta^0 \), which objects are all free algebras \( W(X) \) of the variety \( \Theta \) generated by the finite subsets \( X \in \mathcal{F}(X_0) \). Morphisms of the category \( \Theta^0 \) are homomorphisms of these algebras.

In universal algebraic geometry we consider the ”set of equations” \( T \subseteq B^2 \) for some \( B \in \text{Ob}\Theta^0 \) and we ”resolve” these equations in the \( \text{Hom}(B, H) \) - ”affine space over the algebra \( H \in \Theta^0 \). We denote \( T_H^* = \{ \mu \in \text{Hom}(B, H) \mid T \subseteq \ker \mu \} \). This is the set of all solutions of the set of equations \( T \). For every set of ”points” of affine space \( R \subseteq \text{Hom}(B, H) \) we can consider a congruence of equations defined by this set: \( R'_H = \bigcap_{\mu \in R} \ker \mu \). Also for every set \( T \subseteq B^2 \) we can consider its algebraic closer according the algebra \( H \): \( T_H' \). The set \( T \subseteq B^2 \) is called \( H \)-closed if \( T = T_H' \). \( H \)-closed set is always a congruence. The lattices of the \( H \)-closed congruences in the algebra \( B \in \text{Ob}\Theta^0 \) we denote \( \text{Cl}_H(B) \). We can consider the category of coordinate algebras connected with the algebra \( H \in \Theta \). This category we denote by \( \text{C}_\Theta(H) \). The objects of this category is quotients algebras \( W(X)/T \), where \( X \in \mathcal{F}(X_0) \), \( T \in \text{Cl}_H(W(X)) \). The morphisms of this category is the homomorphism of algebras. This category describes the algebraic geometry of the algebra \( H \). An answer to the our central question: are two algebras have the same algebraic geometry - we can obtain by various comparisons of these categories. All these definitions we can see, for example, in [P1], [P2] and [P3].

Definition 1.1. (P1) Let \( H_1 \) and \( H_2 \) be algebras in \( \Theta \). The algebras \( H_1 \) and \( H_2 \) are called geometrically equivalent if \( \text{Cl}_{H_1}(B) = \text{Cl}_{H_2}(B) \) for every \( B \in \text{Ob}\Theta^0 \).

Algebras \( H_1 \) and \( H_2 \) are geometrically equivalent if and only if the categories \( C_\Theta(H_1) \) and \( C_\Theta(H_2) \) coincide.

The notion of weak geometric equivalence of algebras defined in [P1] Section 4] together with the closely connected notion of similarity of algebras. In this
paper we will give to this notion more natural name: automorphic equivalence of algebras.

**Definition 1.2.** Let $H_1$ and $H_2$ be algebras in $\Theta$. The algebras $H_1$ and $H_2$ are called automorphically equivalent if there exists an automorphism $\Phi : \Theta^0 \to \Theta^0$ and for every $B \in \text{Ob} \Theta^0$ there exists a bijection

$$\alpha(\Phi)_B : \text{Cl}_{H_1}(B) \to \text{Cl}_{H_2}(\Phi(B))$$

such that these bijections are coordinated with automorphism $\Phi$ in this sense: if $B_1, B_2 \in \text{Ob} \Theta^0$, $\mu_1, \mu_2 \in \text{Hom}(B_1, B_2)$, $T \in \text{Cl}_{H_1}(B_2)$ and

$$\tau \mu_1 = \tau \mu_2$$

then

$$\tilde{\tau} \Phi(\mu_1) = \tilde{\tau} \Phi(\mu_2),$$

where $\tau : B_2 \to B_2/T$, $\tilde{\tau} : \Phi(B_2) \to \Phi(B_2)/\alpha(\Phi)_{B_2}(T)$ are natural epimorphisms.

In [2] proved that if bijections $\{\alpha(\Phi)_B \mid B \in \text{Ob} \Theta^0\}$ are coordinated with automorphism $\Phi$, then they defined uniquely by $\Phi$.

By the method of [2] it can be proved that algebras $H_1$ and $H_2$ are automorphically equivalent if and only if exists the pair $(\Phi, \Psi)$ where $\Phi : \Theta^0 \to \Theta^0$ is an automorphism, $\Psi : C_{\Theta}(H_1) \to C_{\Theta}(H_2)$ is an isomorphism and these three conditions

A. $\Psi(W(X)/\text{Id}(H_1, X)) = W(Y)/\text{Id}(H_2, Y)$, where $\Phi(W(X)) = W(Y)$,

B. $\Psi(W(X)/T) = W(Y)/\tilde{T}$, where $T \in \text{Cl}_{H_1}(W(X))$, $\tilde{T} \in \text{Cl}_{H_2}(W(Y))$,

C. natural epimorphism $\varpi : W(X)/\text{Id}(H_1, X) \to W(X)/T$ is transformed to the natural epimorphism $\Psi(\varpi) : W(Y)/\text{Id}(H_2, Y) \to W(Y)/\tilde{T}$

are fulfilled $(\text{Id}(H, X) = \bigcap_{\varphi \in \text{Hom}(W(X), H)} \ker \varphi$ is a minimal $H$-closed congruence in the $(W(X))^2$). It should be remarked that if the pair $(\Phi, \Psi)$, which fulfills condition A. - C. exists, then the isomorphism $\Psi$ is defined uniquely by $\Phi$.

The basic facts about automorphic equivalence are these:

1. If algebras $H_1$ and $H_2$ are geometrically equivalent then they are automorphically equivalent with the $\Phi = \text{id}_{\Theta^0}$ and $\alpha(\Phi)_B$ is the identity mapping on $\text{Cl}_{H_1}(B) = \text{Cl}_{H_2}(B)$.

2. If an automorphism $\Phi : \Theta^0 \to \Theta^0$ provides the automorphic equivalence of algebras $H_1$ and $H_2$, then the automorphism $\Phi^{-1} : \Theta^0 \to \Theta^0$ provides the automorphic equivalence of algebras $H_2$ and $H_1$ with the $\alpha(\Phi^{-1})_B = \left(\alpha(\Phi)_{\Phi^{-1}(B)}\right)^{-1}$.  

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3. If an automorphism $\Phi_1 : \Theta^0 \to \Theta^0$ provides the automorphic equivalence of algebras $H_1$ and $H_2$, and an automorphism $\Phi_2 : \Theta^0 \to \Theta^0$ provides the automorphic equivalence of algebras $H_2$ and $H_3$, then $\Phi_2 \Phi_1$ provides the automorphic equivalence of algebras $H_1$ and $H_3$ with the $\alpha(\Phi_2 \Phi_1)_B = \alpha(\Phi_2)_{\Phi_1(B)} \alpha(\Phi_1)_B$.

4. If an automorphism $\Phi : \Theta^0 \to \Theta^0$ which provides the automorphic equivalence of algebras $H_1$ and $H_2$ is an inner automorphism, then $H_1$ and $H_2$ are geometrically equivalent.

Facts 1., 2., 3. and 4. were formulated in [Pl2] for the similarity of algebras, but they can be very easily established for the automorphic equivalence. From 1., 2. and 3. we conclude that the automorphic equivalence is a reflexive, symmetric and transitive relation.

We will use this well known fact that conjugation of the inner automorphism of some category $\mathfrak{R}$ by the arbitrary automorphism of this category is also an inner automorphism.

2 Derived and verbal operations.

2.1 Verbal (polynomial) operations.

Before the explanation of the notion of the verbal operation we will introduce the short notation, which will be widely used in this paper. In this notation $k$-tuple $(c_1, \ldots, c_k) \in C^k$ ($C$ is an arbitrary set) we denote by single letter $c$ and we will even allow ourself to write $c \in C$ instead $c \in C^k$ and to write "homomorphism $\alpha : A \ni a \to b \in B"$ instead "homomorphism $\alpha : A \to B$", which transforms $a_i$ to the $b_i$, where $1 \leq i \leq k$.

For every word $w(x) \in W(X)$, where $X = \{x_1, \ldots, x_k\} \subset X_0$ and every $H \in \Theta$ we can define a $k$-ary operation $w_H^*(h) = w(h)$ (in full notation $h = (h_1, \ldots, h_k) \in H^k$, $x = (x_1, \ldots, x_k) \in (W(X))^k$) or, more formal, $w_H^*(h) = \gamma_h(w)$, where $\gamma_h$ is a well defined homomorphism $\gamma_h : W(X) \ni x \to h \in H$ (in full notation: homomorphism $\gamma_h : W(X) \to H$, which transforms $x_i$ to the $h_i$ for $1 \leq i \leq k$). This operation we call the verbal operation induced on the algebra $H$ by the word $w(x) \in W(X)$. If we will be very precise, we must say that "word $w(x) \in W(X)$" is actually a class $[w(x)]_{\Lambda_X}$ of words in the absolutely free algebra of our signature $F(X)$ generated by set of symbols $X$, which are congruent to the word $w(x)$ according the congruence $\Lambda_X$ of all identities of the variety $\Theta$ in $(F(X))^2$. We define an operation $w_H^*$ on the algebra $H$ of our variety $\Theta$. So the result of substitution $w(h)$, or, by other words, the image $\gamma_h(w)$ does not depend on what word from the class $[w(x)]_{\Lambda_X}$ we take. Because of that, we use the expression "word $w(x) \in W(X)$" and will avoid the redundant punctuality, which will only impede the explanation. Also we must remark that if the word $w(x)$ is generated by the set $X' \subset X$ such as $X' \neq X$, then some variables in the operation $w_H^*$ are fictive (the results of
this operation do not depend on them), but all our consideration is valid in this case.

**Remark 2.1.** By [Gr 1.8, Lemma 8], if $H_1, H_2$ are algebras of the variety $\Theta$ and $\varphi$ a homomorphism from $H_1$ to $H_2$, then $\varphi(w^h_{H_1}(h)) = w^h_{H_2}(\varphi(h))$ for every $h \in H_1$, i. e., $\varphi$ will be a homomorphism from $H_1$ to $H_2$ as algebras with signature $\Omega \cup \{w^h_{H_1}\}$ and $\Omega \cup \{w^h_{H_2}\}$ respectively.

### 2.2 Derived operations.

We have another way to define additional algebraic operations on the arbitrary algebra. Let us have an algebra $C$ with the signature $\Omega$ and a bijection $s : C \to \omega$. For every $\omega \in \Omega$ we can define the derived operation $\bar{\omega}_C$ by this way: $\bar{\omega}_C(c) = s(\omega_C(s^{-1}(c)))$ for every $c \in C$ ($\omega_C$ is a realization of the operation $\omega$ in the algebra $C$). By this definition $s$ will be an isomorphism from the algebra $C$ with operations $\omega_C$ to the algebra $\bar{C}$, which has same domain and operations $\bar{\omega}_C$ ($\omega \in \Omega$). Operations $\bar{\omega}_C$ ($\omega \in \Omega$) we call "derived operations induced on the $C$ by the bijection $s$".

Now we will interweave the notions of derived and verbal operations. Let us have a system of bijections $\{s_B : B \to B \mid B \in \text{Ob}\Theta^0\}$ which fulfills these two conditions:

**B1)** for every homomorphism $\alpha : A \to B$ ($A, B \in \text{Ob}\Theta^0$) the mappings $s_B \circ s_A^{-1}$ and $s_B^{-1} \circ s_A$ is also homomorphisms from $A$ to $B$ and

**B2)** $s_B|_X = id_X$ for every $B = W(X) \in \text{Ob}\Theta^0$.

If arity of $\omega \in \Omega$ is $k$, we take $X_\omega = \{x_1, \ldots, x_k\} \subset X_0$. $A_\omega = W(X_\omega)$ - free algebra in $\Theta$. We have that $\omega(x) \in A_\omega (x = (x_1, \ldots, x_k))$ so there exists $w_\omega(x) \in A_\omega$ such that

$$w_\omega(x) = s_{A_\omega}(\omega(x)). \quad (2.1)$$

For every $H \in \Theta$ we denote $\omega^*_H$ the verbal operation induced on the algebra $H$ by $w_\omega(x)$. $H^*$ will be the algebra, which has the same domain as the algebra $H$ and its operations are $\{\omega^*_H \mid \omega \in \Omega\}$. As it was proved in [DZ Theorem 3]

$$\omega^*_H = \bar{\omega}_B \quad (2.2)$$

for every $B \in \text{Ob}\Theta^0$, where $\bar{\omega}_B$ is a derived operations induced on the $B$ by the bijection $s_B$. But $\bar{\omega}_H$ is not defined for $H \in \Theta \setminus \text{Ob}\Theta^0$.

By (2.2) for every $B = W(X) \in \text{Ob}\Theta^0$ we have that $s_B : B \to B^*$ is an isomorphism. So the system of words $\{w_\omega(x) \in A_\omega = W(X_\omega) \mid \omega \in \Omega\}$ fulfills these two conditions:

**Op1)** $X_\omega = \{x_1, \ldots, x_k\}$, where $k$ is an arity of $\omega$, for every $\omega \in \Omega$;

**Op2)** for every $B = W(X) \in \text{Ob}\Theta^0$ there exists an isomorphism $\sigma_B : B \to B^*$ (algebra $B^*$ has same domain as the algebra $B$ and its operations $\omega^*_B$ are induced by $w_\omega(x)$ for every $\omega \in \Omega$) such as $\sigma_B|_X = id_X$ with $\sigma_B = s_B$. This system of words we denote $\mathfrak{W}(S)$.
3 Systems of bijections, systems of words and strongly stable automorphisms.

Now let us have a system of words \( W = \{ w_\omega(x) \in A_\omega = W(X_\omega) \ | \ \omega \in \Omega \} \) which fulfills conditions Op1) and Op2), then we can take a system of bijections \( \mathfrak{S}(W) = \{ \sigma_B : B \to B \ | \ B \in \text{Ob}\Theta^0 \} \).

By Op2) \( B^* \) is a free algebra in the \( \Theta \) with generators \( \sigma_B(x) = x \ (B = W(X), \ x \in X) \).

By Section 2 we can induce the operation \( \omega^*_B \) by \( w_\omega(x) \ (\omega \in \Omega) \) on every algebra \( H \in \Theta \). As above \( H^* \) the algebra which has the same domain as the algebra \( H \) and the operations \( \{ \omega^*_H \ | \ \omega \in \Omega \} \). By Remark 2.1 if \( \varphi : H_1 \to H_2 \) \((H_1, H_2 \in \Theta)\) is a homomorphism then \( \varphi : H_1^* \to H_2^* \) is also a homomorphism.

**Proposition 3.1.** \( H^* \in \Theta \) for every \( H \in \Theta \).

**Proof.** There exists \( B \in \text{Ob}\Theta^0 \), such that \( H \) is an epimorphic image of \( B \). So, \( H^* \) is an epimorphic image of \( B^* \). But \( B^* \in \Theta \), hence \( H^* \in \Theta \). ■

We can consider the signature \( \Omega^* = \{ \omega^* \ | \ \omega \in \Omega \} \). Between signatures \( \Omega^* \) and \( \Omega \) there is a symmetry:

**Proposition 3.2.** In every algebra \( H \in \Theta \) operations \( \omega \in \Omega \) are verbal operations defined by the system of words \( U = \{ u^*_\omega(x) \ | \ \omega \in \Omega \} \) written by the signature \( \Omega^* \). The system of words \( U \) fulfills conditions Op1) and Op2) with the system of isomorphism \( \sigma_B^{-1} : B^* \to B \).

**Proof.** \( \omega(x) \in A_\omega = W(X_\omega) \), where \( X_\omega = \{ x_1, \ldots, x_k \} \) and \( k \) is the arity of \( \omega \), so there exists \( u(x) \in A_\omega \) such that \( \omega(x) = \sigma_A(u_\omega(x)) \). We denote by \( u^*_\omega(x) \) the word, which we receive from the word \( u_\omega(x) \) by replacement of all operations \( \omega \in \Omega \) by operations \( \omega^* \in \Omega^* \) respectively. \( \sigma_{A_\omega} \) is an isomorphism from the algebra \( A_\omega \) with operations \( \omega \in \Omega \) to the algebra \( A^*_\omega \) with operations \( \omega^* \in \Omega^* \) and \( \sigma_{A_\omega}(x) = x \) for every \( x \in X_\omega \), so \( \sigma_{A_\omega}(u_\omega(x)) = u^*_\omega(x) \). \( \omega(h) = \gamma_h(\omega(x)) = \gamma_h(u^*_\omega(x)) \), where \( \gamma_h : A_\omega \ni x \to h \in H \) is a homomorphism from \( A_\omega \) to \( H \) and from \( A^*_\omega \) to \( H^* \). \( A^*_\omega \) with operations \( \omega^* \in \Omega^* \) is a free algebra in the variety \( \Theta \) with the set of generators \( X_\omega \). So the operations \( \omega \in \Omega \) are verbal operations. Condition Op1) is fulfilled by constructions of the words \( u^*_\omega(x) \). Condition Op2) is obvious. ■

And by Remark 2.1 in which we change \( H_1, H_2 \) to \( H_1^*, H_2^* \) correspondingly and vice versa we have

**Corollary 1.** If \( H_1, H_2 \in \Theta \) and \( \varphi : H_1^* \to H_2^* \) is a homomorphism, then \( \varphi : H_1 \to H_2 \) is also a homomorphism.

And now we can prove

**Proposition 3.3.** The system of bijections \( \mathfrak{S}(W) = \{ \sigma_B : B \to B \ | \ B \in \text{Ob}\Theta^0 \} \) fulfills conditions B1) and B2).
Proof. Actually, we must only prove condition B1). If \( \alpha : A \to B \) \((A, B \in \text{Ob}\Theta^0)\), then, because the operations in \( A^* \) and \( B^* \) are induced by the same words, by Remark 2.1 \( \alpha \) is a homomorphism from \( A^* \) to \( B^* \), so \( \sigma_{B^*}^{-1} \alpha \sigma_{A^*} \) will be a homomorphism from \( A \) to \( B \). \( \sigma_{B^*}^{-1} \alpha \sigma_{A^*} \) is a homomorphism from \( A^* \) to \( B^* \).

So, by a Corollary from Proposition 3.2 it is a homomorphism from \( A \) to \( B \).

**Proposition 3.4.** If \( S = \{ s_B : B \to B \mid B \in \text{Ob}\Theta^0 \} \) is a system of bijections, which fulfills conditions B1) and B2), then \( \Theta (\mathfrak{W}(S)) = S \). If \( W = \{ w_\omega (x) \mid \omega \in \Omega \} \) is a system of words, which fulfills conditions Op1) and Op2), then \( \mathfrak{W}(\Theta (W)) = W \).

**Proof.** Let \( S = \{ s_B : B \to B \mid B \in \text{Ob}\Theta^0 \} \). \( \mathfrak{W}(S) \) be a system of words defined by the formula (3.1). By Section 2.2 this system fulfills conditions Op1) and Op2) with \( \sigma_B = s_B \) for every \( B \in \text{Ob}\Theta^0 \). So \( \Theta (\mathfrak{W}(S)) = S \).

Let \( W = \{ w_\omega (x) \mid \omega \in \Omega \} \) be a system of words, which fulfills conditions Op1) and Op2). \( \Theta (W) = \{ \sigma_B : B \to B \mid B \in \text{Ob}\Theta^0 \} \), where isomorphisms \( \sigma_B \) are remembered in condition Op2). Let \( \mathfrak{W}(\Theta (W)) = \{ w'_{\omega} (x) \mid \omega \in \Omega \} \). \( \mathfrak{W}(\Theta (W)) \) is defined by formula (2.1), so \( \sigma_{A^*} (\omega (x)) = w'_{\omega} (x) \). Also, by Op2), \( \sigma_{A^*} (\omega (x)) = \omega_{A^*} (\sigma_A (x)) = \omega_{A^*} (w_{\omega} (x)) = \omega (x) \). Therefore, \( w_{\omega} (x) = w'_{\omega} (x) \) for every \( \omega \in \Omega \) and \( \mathfrak{W}(\Theta (W)) = W \).

Now we introduce one of the central notion of our paper: strongly stable automorphism of the category \( \Theta^0 \). Automorphisms of this kind are closely connected with the system of words, which fulfills conditions Op1) and Op2) via system of bijections, which fulfills conditions B1) and B2).

**Definition 3.1.** Automorphism of the category \( \Theta^0 \) are called **strongly stable** if it fulfills these three conditions

A1) \( \Phi \) preserves all objects of \( \Theta^0 \),

A2) there exists a system of bijections \( \{ s_B : B \to B \mid B \in \text{Ob}\Theta^0 \} \) such that \( \Phi \) acts on the morphisms of \( \Theta^0 \) by this system, i. e.,

\[
\Phi (\alpha) = s_B^{s_B} \alpha (s_A^{s_A})^{-1}
\]

for every \( \alpha \in \text{Hom} (A, B) \) \((A, B \in \text{Ob}\Theta^0)\) and

A3) \( s_B^{s_B} |_X = id_X \) for every \( B = W (X) \in \text{Ob}\Theta^0 \).

Obviously, if we have a strongly stable automorphisms \( \Phi \), then the system of bijections \( \{ s_B : B \to B \mid B \in \text{Ob}\Theta^0 \} \) fulfills conditions B1) and B2). We must remark, that if a strongly stable automorphisms \( \Phi \) of \( \Theta^0 \) fulfills conditions A2) and A3) with the system of bijections \( \{ s_B : B \to B \mid B \in \text{Ob}\Theta^0 \} \), it can fulfills these conditions with many other systems of bijections which fulfill conditions B1) and B2).

Contrariwise, if we have a system of bijections \( S = \{ s_B : B \to B \mid B \in \text{Ob}\Theta^0 \} \) which fulfills conditions B1) and B2) then we can define a strongly stable automorphism \( \Phi (S) = \Phi \) of the category \( \Theta^0 \) by this way: \( \Phi \) preserves all objects
of the category \( \Theta^0 \) and acts on its morphisms according formula (3.1) with \( s_B^\phi = s_B \). Of course, two different system of bijections \( S_1 = \{ s_{1,B}^\phi : B \to B \mid B \in \mathrm{Ob}\Theta^0 \} \) and \( S_2 = \{ s_{2,B}^\phi : B \to B \mid B \in \mathrm{Ob}\Theta^0 \} \) which fulfill conditions B1) and B2) can provide by formula (3.1) the same action on homomorphisms and, so, they will provide the same strongly stable automorphism of the category \( \Theta^0 \).

4 Automorphic equivalence of one-sorted algebras.

In this Section we assume that there is a strongly stable automorphisms \( \Phi \) of the category \( \Theta^0 \). It fulfills condition A2) and A3) with the system of bijections \( S = \{ s_B^\phi : B \to B \mid B \in \mathrm{Ob}\Theta^0 \} \), which fulfills conditions B1) and B2). Then we have a system of words \( \mathfrak{W}(S) = \{ w_\omega(x) \mid \omega \in \Omega \} \) which fulfills conditions Op1) and Op2) with \( \sigma_B = s_B^\phi \). As above, \( \omega_H \) is an operation induced on \( H \) by the word \( w_\omega(x) \) \((H \in \Theta)\) and \( H^* \) is the algebra which has same domain as the algebra \( H \) and the operations \( \{ \omega_H \mid \omega \in \Omega \} \). By Proposition 3.1 \( H^* \in \Theta \).

**Proposition 4.1.** Let \( B \in \mathrm{Ob}\Theta^0 \), \( H \in \Theta \) and \( T \subset B^2 \). If \( T \) is an \( H \)-closed congruence, then \( \sigma_B^{-1}T \) is an \( H^* \)-closed congruence.

**Proof.** We shall consider the diagram

\[
\begin{array}{ccc}
B & \xrightarrow{\sigma_B} & B \\
\downarrow \psi & & \downarrow \varphi \\
H^* & \xrightarrow{\varphi} & H
\end{array}
\]

If \( \varphi \in \mathrm{Hom}(B, H) \) then, \( \varphi \in \mathrm{Hom}(B^*, H^*) \), \( \sigma_B \in \mathrm{Hom}(B, B^*) \), so \( \varphi \sigma_B \in \mathrm{Hom}(B, H^*) \). If \( \psi \in \mathrm{Hom}(B, H^*) \) then \( \psi \sigma_B^{-1} \in \mathrm{Hom}(B^*, H^*) \). \( \mathfrak{W}(S) \) fulfills conditions Op1) and Op2), so we can use Corollary 1 from Proposition 3.2 and conclude that, \( \psi \sigma_B^{-1} \in \mathrm{Hom}(B, H) \).

If \( T \) is a \( H \)-closed congruence, then \( \bigcap_{\varphi \in T_H} \ker \varphi = T \). If \( \varphi \in \mathrm{Hom}(B, H) \) and \( \ker \varphi \supseteq T \), then \( \varphi \sigma_B \in \mathrm{Hom}(B, H^*) \) and \( \ker \varphi \sigma_B = \sigma_B^{-1} \ker \varphi \supseteq \sigma_B^{-1}T \). So

\[
\bigcap_{\varphi \in \sigma_B^{-1}T_H} \ker \psi \subseteq \bigcap_{\psi \in \sigma_B^{-1}T_H} \ker \varphi \sigma_B = \sigma_B^{-1}T.
\]

**Remark 4.1.** Similar we can prove that if \( T \subset B^2 \) and \( T \) is a \( H^* \)-closed congruence, then \( \sigma_B T \) is a \( H \)-closed congruence.

**Theorem 4.1.** Automorphism \( \Phi^{-1} \) provides the automorphic equivalence of algebras \( H \) and \( H^* \) in the variety \( \Theta \).

**Proof.** For every \( B \in \mathrm{Ob}\Theta^0 \) we take the monotone bijections \( Cl_H(B) \ni T \to \sigma_B^{-1}T \in Cl_H^*(B) \). And now we need to prove that if \( B_1, B_2 \in \mathrm{Ob}\Theta^0 \), \( \mu_1, \mu_2 \in \mathrm{Hom}(B_1, B_2) \), \( T \in Cl_H(B_2) \), \( \tau : B_2 \to B_2/T \) and \( \tilde{\tau} : B_2 \to B_2/\sigma_B^{-1}T \) are natural homomorphisms and

\[
\tau \mu_1 = \tau \mu_2,
\]

(4.1)
We denote \( \sigma_i = \sigma_{B_i} \) (\( i = 1, 2 \)). From \( \Phi \) we have \( (\mu_1 \sigma_1(b), \mu_2 \sigma_1(b)) \in T \) for every \( b \in B \), \( (\sigma_1^{-1} \mu_1 \sigma_1(b), \sigma_2^{-1} \mu_2 \sigma_1(b)) = (\Phi^{-1}(\mu_1)(b), \Phi^{-1}(\mu_2)(b)) \in \sigma_2^{-1}T \). Therefore \( \Phi \) is fulfilled. \( \blacksquare \)

5 Automorphic equivalence and geometric equivalence.

**Theorem 5.1.** Let the algebras \( H_1 \) and \( H_2 \) belongs to the variety \( \Theta \). They are automorphically equivalent in \( \Theta \) if and only if the algebra \( H_1 \) geometrically equivalent to the algebra \( H_2^* \), where \( H_2^* \) is an algebra which has the same domain as the algebra \( H_2 \) and its operations are induced by some system of words \( \{ w_\omega(x) \mid \omega \in \Omega \} \), which fulfills conditions Op1) and Op2).

**Proof.** Let an automorphism \( \Psi : \Theta^0 \to \Theta^0 \) provide the automorphic equivalence of the algebras \( H_1 \) and \( H_2 \). By [PZ, Theorem 2] \( \Psi \) can be decomposed: \( \Psi = \gamma \Phi \), where \( \gamma \) is an inner automorphism of \( \Theta^0 \) and \( \Phi \) is a strongly stable one. \( S = \{ s_B^B : B \to B \mid B \in \text{Ob}\Theta^0 \} \) is a system of bijections described in conditions A2) and A3). So the system \( S \) fulfills the conditions B1) and B2). We have the system of words \( \mathcal{M}(S) = \{ w_\omega(x) \mid \omega \in \Omega \} \). Let \( H_2^* \) be an algebra which has the same domain as the algebra \( H_2 \) and its operations are induced by the system \( \mathcal{M}(S) \). By **Theorem 4.1** \( \Phi^{-1} \) provides the automorphic equivalence of the algebras \( H_2 \) and \( H_2^* \). Hence (see Introduction), an automorphism \( \Lambda = \Phi^{-1}T \Phi = \Phi^{-1} \Psi \) provides the automorphic equivalence of algebras \( H_1 \) and \( H_2^* \). But, \( \Lambda \) is an inner automorphism (see Introduction), hence \( H_1 \) and \( H_2^* \) are geometrically equivalent.

Let us have a system of words \( W = \{ w_\omega(x) \mid \omega \in \Omega \} \), which fulfills conditions Op1) and Op2) and \( H_1 \) is geometrically equivalent to \( H_2^* \) (\( H_2^* \) is the algebra with the same domain as the algebra \( H_2 \) and its operations are induced by \( \{ w_\omega(x) \mid \omega \in \Omega \} \)). By **Proposition 3.3** the system of bijections \( \mathcal{S}(W) = \{ \sigma_B : B \to B \mid B \in \text{Ob}\Theta^0 \} \) fulfills conditions B1) and B2), where \( \sigma_B \) is an isomorphism which is described in condition Op2). We can consider the system of words \( \mathcal{M}(\mathcal{S}(W)) \) and the automorphism \( \Phi(\mathcal{S}(W)) = \Phi \) which acts on morphisms of the category \( \Theta^0 \) by bijections \( \{ \sigma_B : B \to B \mid B \in \text{Ob}\Theta^0 \} \). By **Proposition 3.3** \( \mathcal{M}(\mathcal{S}(W)) = W \), so, by **Theorem 4.1** the automorphism \( \Phi^{-1} \) provides the automorphic equivalence of the algebras \( H_2 \) and \( H_2^* \). But automorphic equivalence is a symmetric and transitive relation. \( \blacksquare \)

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