Ergodic quasi-invariant measures on topologically mixing subshifts are isomorphic to Bernoulli shifts

Doureid Hamdan

Abstract

We prove that a shift ergodic measure on a topologically mixing sub-shift is isomorphic to a Bernoulli shift whenever it is quasi invariant under permutations of finite number of coordinates. We prove also that Gibbs measures on topologically mixing subshift of finite type are quasi invariant.

1 Introduction, Notation

Usually a sequence \((X_n)_{n \in \mathbb{Z}}\) of random variables is said exchangeable if the law \(P_\sigma\) of the process \((X_{\sigma(n)})_{n \in \mathbb{Z}}\) is equal the law \(P\) of the process \((X_n)_{n \in \mathbb{Z}}\), for every permutation \(\sigma\) of the set \(\mathbb{Z}\) of integers, which leaves fixed all but a finite number of integers. Naturally, \((X_n)_{n \in \mathbb{Z}}\) will then be said quasi exchangeable if \(P_\sigma\) is equivalent to \(P\), for any such \(\sigma\).

In [6] Hewitt and Savage enlarge the category of the state space and obtain a generalization of De Finetti’s Theorem which says that an an exchangeable sequence of random variables is a mixture of i.i.d. sequences. We note that exchangeability implies stationarity. In [5] it is proved that, if the dynamical system generated by a stationary quasi exchangeable process is ergodic, then it is isomorphic to a Bernoulli process, and in the particular case where the family of all Radon-Nikodym derivatives is uniformly integrable, then the process is exchangeable. Also an application of one result from [5] yields that the translation invariant determinantal processes considered in [2] are isomorphic to Bernoulli processes.

In this paper we shall consider only finite state space stationary processes \((X_n)_{n \in \mathbb{Z}}\) whose underlying dynamical system is a topologically mixing subshift. We prove that they are isomorphic to Bernoulli shifts, whenever they are quasi exchangeable and ergodic, a result which generalizes Theorem 1 of [5].

Keywords: Ergodic processes, quasi exchangeable sequence, Subshift of finite type, Gibbs measure, translation-invariant determinantal process, Bernoulli system, De Finetti’s Theorem.

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First we establish the terminology and notation which will be needed later.

We suppose that, for every \( n \in \mathbb{Z} \), \( X_n \) is the \( n^{th} \) coordinate on the product space \( \Omega := K^\mathbb{Z} \), where \( K \) is the finite state space, and also that the law \( P \) is concentrated on a subshift in \( \Omega \).

Then \( F := \text{the sigma algebra of all subsets of } K, \)
\[ \Omega := K^\mathbb{Z}, \]
\( \) is endowed with the product sigma algebra \( E := F^{\otimes \mathbb{Z}} \), and
\[ X_n(\omega) = \omega_n, \text{ for all integer } n \in \mathbb{Z} \text{ and all } \omega \in \Omega. \]

The shift transformation is denoted \( S : (S\omega)_n = \omega_{n+1}, n \in \mathbb{Z}, \omega \in \Omega. \)

Let \( G \) be the group of all permutations of \( \mathbb{Z} \) and \( H \subset G \), be the subgroup of all permutations with finite support:
\[ \sigma \in H \iff \sigma \in G, \text{ and } \exists N, \sigma(n) = n, \forall n, |n| \geq N. \]

For any \( \tau \in G \), let the transformation \( T_\tau : \Omega \to \Omega \), be defined for all \( \omega \in \Omega \), by
\[ (T_\tau(\omega))_n = \omega_{\tau(n)}, \forall n \in \mathbb{Z}. \]

Then \( T_\tau \) is \( E \)-measurable, and, when \( K \) is a topological space, \( T_\tau \) is continuous for the product topology on \( \Omega \). Also for all \( \sigma \) and \( \tau \) in \( H \),
\[ T_{\tau \circ \sigma} = T_\sigma \circ T_\tau, \]
from which follows that
\[ T_{\sigma^{-1}} = T_{\sigma -1}. \]

Recall that then the quasi exchangeability of the sequence \((X_n)_{n \in \mathbb{Z}}\) is equivalent to the following

**Definition 1**

We say that a sequence \((X_n)_{n \in \mathbb{Z}}\) of random variables, with law \( \mu \), is quasi exchangeable if \( \mu \circ T_\sigma^{-1} \) is equivalent to \( \mu \), for all permutation \( \sigma \in H \).

In this case we denote the Radon-Nikodym derivative of \( \mu \circ T_\sigma^{-1} \) with respect to \( \mu \), by \( \phi_\sigma \)
\[ \phi_\sigma := \frac{d\mu \circ T_\sigma^{-1}}{d\mu}. \]

Before setting the precise statement, we establish some more notation.

We endow \( \Omega \) with the product topology of the discrete topologies on \( K \). Let \( \Sigma \) be a closed shift-invariant subset of \( \Omega \). We assume that \( \forall j \in K \), there exists \( x \in \Sigma \) such that \( x_0 = j \).

We assume also that the system \((\Sigma, S)\) is topologically mixing, which means that for non empty open sets \( U, V \) of \( \Sigma \), there is an \( N \) such that \( U \cap S^m V \neq \emptyset \), for all \( m \geq N \).

We shall also use the following notations.
If \( L \) and \( s \) are integers with \( L \geq 0 \) and \( s \geq 1 \), and if \( A_{-L}, \ldots, A_s \) are measurable subsets of \( K \), we set

\[
\Pi(A_{-L}, \ldots, A_0) := \{ \omega \in \Sigma : \omega_{-L} \in A_{-L}, \ldots, \omega_0 \in A_0 \} \tag{5}
\]
\[
F(A_1, \ldots, A_s) := \{ \omega \in \Sigma : \omega_1 \in A_1, \ldots, \omega_s \in A_s \}, \tag{6}
\]

and for all \( I \subset \mathbb{Z} \),

\[
\mathcal{A}_I := \text{the smallest algebra containing the sets } \{ \omega \in \Sigma : \omega_j \in A_j \}, j \in I, A \in \mathcal{F}, \]
and

\[
\mathcal{B}_I := \text{the sigma algebra generated by } \mathcal{A}_I. \]

In the following particular cases:

\[
I = \{ n \in \mathbb{Z} : n \leq 0 \} \quad \mathcal{A}_I \text{ is denoted } \mathcal{A}_{\leq 0}
\]
\[
I = \{ n \in \mathbb{Z} : n \geq p \} \quad \mathcal{A}_I \text{ is denoted } \mathcal{A}_{\geq p}
\]
\[
I = \mathbb{Z} \quad \mathcal{A}_I \text{ is denoted } \mathcal{A}.
\]

The same notation will be used for \( \mathcal{B}_I \), in particular \( \mathcal{B}_\mathbb{Z} = \mathcal{B} \).

Also, the smallest sigma-algebra which contains two sigma-algebras \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) is denoted by \( \mathcal{F}_1 \vee \mathcal{F}_2 \), and the smallest one containing a family \( \{ \mathcal{F}_j : j \in J \} \), of sigma-algebras is denoted \( \bigvee_{j \in J} \mathcal{F}_j \).

For any \( \sigma \in H \), let us consider the subset

\[
\Gamma_\sigma := \Sigma \cap T_{\sigma^{-1}} \Sigma. \tag{7}
\]

Then \( \Gamma_\sigma \) is a clopen set in \( \Sigma \) and the following implication holds

\[
\sigma \text{ is an involution } \Rightarrow T_\sigma \Gamma_\sigma = \Gamma_\sigma = T_{\sigma^{-1}} \Gamma_\sigma.
\]

Let

\[
H_+ := \{ \sigma \in H : \sigma \text{ is an involution, } \Gamma_\sigma \neq \emptyset \}, \tag{8}
\]

and, for every \( a \in K \),

\[
L(a) = \{ j \in K : \{ x \in \Sigma : x_0 = j, x_1 = a \} \neq \emptyset \}
\]

and

\[
R(a) = \{ j \in K : \{ x \in \Sigma : x_0 = a, x_1 = j \} \neq \emptyset \}. \tag{9}
\]

[ so that \( j \in L(a) \iff a \in R(j) \).]

Let

\[
Q := \{ (a, b) \in K \times K : a \in L(b) \}
\]

Let us also use the notation

\[
(x_j \in A_j, j \in J) := \{ x \in \Sigma : x_j \in A_j, j \in J \}, \quad J \subset \mathbb{Z}, \tag{10}
\]
where, for every \( j \), \( A_j \) is a subset of \( K \), and in particular, we set

\[
C_k(a, b, c, d) := (x_0 = a, x_1 = b, x_k = c, x_{k+1} = d)
\]

so that, with

\[
\tilde{A}(a, b) := (x_0 \in Lb, x_1 \in Ra),
\]

it is clear that for \((a, b), (c, d) \in \mathcal{Q}\), we have the inclusions

\[
A(a, b) \subset \tilde{A}(a, b),
\]

and also the following equalities

\[
C_k = C_k(a, b, c, d) = A \cap S^{-k}B, \quad D_k = D_k(a, b, c, d) = \tilde{A} \cap S^{-k}\tilde{B}.
\]

Hence

\[
C_k(a, b, c, d) \subset D_k(a, b, c, d).
\]

By topological mixing and because \( \mathcal{Q} \) is finite, there exists \( k_0 \) such that

\[
C_k(a, b, c, d) \neq \emptyset, \quad \forall k \geq k_0, \quad \forall (a, b), (c, d) \in \mathcal{Q}.
\]

Hence, also, there exists \( P_0(k) \) such that

\[
C_k(a, b, c, d) \cap S^{-P}D_k(a, b, c, d) \neq \emptyset, \quad \forall k \geq k_0, \quad \forall (a, b), (c, d) \in \mathcal{Q}, \quad \forall P \geq P_0(k).
\]

For all natural numbers \( k \) and \( P \), such that \( 1 \leq k < P \), let us consider the permutation (involution) \( \sigma := \sigma_{P,k} \in H \) which translates the "interval" \( I := \mathbb{N} \cap [1, ..., k] \) by \( P \), translates the "interval" \( P + I = \mathbb{N} \cap [P+1, ..., P+k] \) by \(-P\), and leaves fixed all \( n \in \mathbb{Z} \), which are not in the disjoint union \( I \cup (P + I) \), that is, which is defined by

\[
\sigma(j) = P + j, \quad \text{and} \quad \sigma(P + j) = j, \quad \text{if} \quad 1 \leq j \leq k
\]

so that

\[
T_\sigma(\omega)_n = \omega_n, \quad \text{if} \quad n \notin \{1, ..., k\} \cup \{P + j : j = 1, ..., k\}
\]

\[
T_\sigma(\omega)_j = \omega_{P+j} \quad \text{and} \quad T_\sigma(\omega)_{P+j} = \omega_j \quad \text{for} \quad 1 \leq j \leq k.
\]

In this particular case, where \( \sigma = \sigma_{P,k} \), which will be used in the proof of Theorem 1, we shall denote \( \Gamma_\sigma \) by \( \Gamma_{P,k} \), or, when \( k \) is fixed, simply by \( \Gamma_P \). We have then the following equality

\[
\Gamma_P = \bigcup_{(a, b), (c, d) \in \mathcal{Q}} C'_k(a, b, c, d) \cap S^{-P}D_k(a, b, c, d).
\]
We note that, by (19) and (22),
\[
\forall k > k_0, \forall P > k \vee P_0(k), \text{ the involution } \sigma_{P,k} \text{ belongs to } H_+.
\] (23)

To prove the equality (*), let \( x \in \Sigma \) and consider \( T_\sigma x \), where \( \sigma = \sigma_{P,k} \). Write them as an infinite strings
\[
\cdots x_{-1}x_0x_1x_2\cdots x_kx_{k+1}\cdots x_{P-1}x_Px_{P+1}\cdots x_{P+k-1}x_Px_{P+k}\cdots \\
\cdots x_{-1}x_0x_{P+1}x_{P+2}\cdots x_{P+k}x_{k+1}\cdots x_{P-1}x_Px_{P+1}\cdots x_{k-1}x_kx_{P+k}\cdots
\]
the first line representing \( x \), and the second corresponds to \( T_\sigma x \). Then we have the following implications
\[
x \in \Gamma_P \iff x \in \Sigma, T_\sigma x \in \Sigma \\
x \in \Sigma, x_{P+1} \in R(x_0), x_{P+k} \in L(x_{k+1}), x_P \in L(x_1), x_{P+k+1} \in R(x_k) \iff \exists a, b, c, d, x_0 = a, x_1 = b, x_k = c, x_{k+1} = d,
\]
\[
(S^P x)_1 \in Ra, (S^P x)_k \in Ld, (S^P x)_0 \in Lb, (S^P x)_{k+1} \in Rc \\
\iff \exists a, b, c, d, x \in C_k(a, b, c, d), S^P x \in D_k(a, b, c, d)
\]
which prove the equality (*).

Let \( T \) be the two-sided tail sigma field
\[
T := \bigcap_{n \geq 1} (B_{\leq -n} \vee B_{\geq n}).
\]

## 2 The main result

**Definition 2**
A probability measure \( \mu \) on \( \Sigma \) is said quasi invariant if for any involution \( \sigma \), the restrictions to \( \Gamma_{\sigma} := \Sigma \cap T_{\sigma}^{-1}\Sigma \), of the two measures \( \mu \) and \( \mu \circ T_{\sigma}^{-1} \), are equivalent.

In this case, we still, as in the full shift case, let \( \phi_{\sigma} \) denote the Radon Nikodym derivative of \( \mu \circ T_{\sigma}^{-1} \) restricted to \( \Gamma_{\sigma} \) with respect to the restriction of \( \mu \) to the same set.
In the particular case where \( \sigma = \sigma_{P,k} \), \( \phi_{\sigma} \) will be denoted \( \phi_{\sigma_{P,k}} \).

**Theorem 1**
Let \((\Sigma, S)\) be topologically mixing subshift. Let \( \mu \) be a shift invariant probability measure on \( \Sigma \), such that the system \((\Sigma, S, \mu)\) is ergodic. Suppose that \( \mu \) is quasi invariant. Then \((\Sigma, S, \mu)\) is isomorphic to a Bernoulli shift.

**Proof:** The natural numbers \( P \) and \( k \) in this proof are as in (23). Let \( R \in \mathbb{N} \) be such that \( R + k < P \). For any positive \( n \in \mathbb{N} \), such that \( n \leq R \), let \( V_1, \ldots, V_n \) be subsets of \( K \). Using
the notation as in (5) and (6), and setting
\[ E = \Pi(A_{-L}, ..., A_0) := \{ \omega \in \Sigma : \omega_{-L} \in A_{-L}, ..., \omega_0 \in A_0 \}, \tag{24} \]
then from the equality \( T^{-1}_{\sigma} \Gamma_P = \Gamma_P \), from the definition of \( \sigma = \sigma_{P,k} \) given in (20) and (21), and from the definition of \( T_{\sigma} \), we obtain the following equality for all \( T \in T \), and \( E \) as in (24),
\[
\mu(T^{-1}_{\sigma}(\Gamma_P \cap E \cap F(A_1, ..., A_k) \cap S^{-k}F(V_1, ..., V_n) \cap S^{-P}F(B_1, ..., B_k) \cap T) = \\
\mu(\Gamma_P \cap E \cap F(B_1, ..., B_k) \cap S^{-k}F(V_1, ..., V_n) \cap S^{-P}F(A_1, ..., A_k))
\]
because of the following two equalities
\[
T^{-1}_{\sigma}(E \cap F(A_1, ..., A_k) \cap S^{-k}F(C_1, ..., C_M) \cap S^{-P}F(B_1, ..., B_k) = \\
E \cap F(B_1, ..., B_k) \cap S^{-k}F(C_1, ..., C_M) \cap S^{-P}F(A_1, ..., A_k)
\]
and
\[
T^{-1}_{\sigma} T = T.
\]
Recall that
\[
\Gamma_P = \bigcup_{(a,b),(c,d) \in \Omega} C_k(a,b,c,d) \cap S^{-P}D_k(a,b,c,d), \tag{25}
\]
and let us find, modulo \( \mu \) zero set, the following subset
\[
R = R_{a,b,c,d} := T^{-1}_{\sigma}(C_k(a,b,c,d) \cap S^{-P}D_k(a,b,c,d)).
\]
First
\[
T_{\sigma}x \in C_k(a,b,c,d) \iff x_0 = a, x_{P+1} = b, x_{P+k} = c, x_{k+1} = d \\
\iff x_0 = a, x_{k+1} = d, x_{P+1} = b, x_{P+k} = c
\]
and, if we set \( y = T_{\sigma}x \),
\[
T_{\sigma}x \in S^{-P}D_k(a,b,c,d) \iff S^P T_{\sigma}x \in D_k(a,b,c,d) \iff \\
(S^P y)_0 \in Lb, (S^P y)_1 \in Ra, (S^P y)_k \in Ld, (S^P y)_{k+1} \in Rc \\
\iff y_P \in Lb, y_{P+1} \in Ra, y_{P+k} \in Ld, y_{P+k+1} \in Rc \\
\iff \ x_P \in Lb, x_1 \in Ra, x_k \in Ld, x_{P+k+1} \in Rc \\
\iff \ x_1 \in Ra, x_k \in Ld, x_P \in Lb, x_{P+k+1} \in Rc
\]
So that
\[
x \in R \iff x_0 = a, x_1 \in Ra, x_k \in Ld, x_{k+1} = d, x_P \in Lb, x_{P+1} = b, \\
x_{P+k} = c, x_{P+k+1} \in Rc
\]
so that modulo \( \mu \) null set
\[
x \in R \iff x_0 = a, x_{k+1} = d, x_{P+1} = b, \\
x_{P+k} = c
\]
that is
\[
R = (x_0 = a, x_{k+1} = d) \cap S^{-P}(x_1 = b, x_k = c)
\]
It follows
\[
R \cap T_{\sigma}^{-1}(E \cap F(A_1, ..., A_k) \cap S^{-k}F(V_1, ..., V_n) \cap S^{-P}F(B_1, ..., B_k) = \\
R \cap E \cap F(B_1, ..., B_k) \cap S^{-k}F(V_1, ..., V_n) \cap S^{-P}F(A_1, ..., A_k)
\]
so that
\[
\int_{T \cap C_k(a,b,c,d) \cap S^{-P}D_k(a,b,c,d) \cap E \cap F(A_1, ..., A_k) \cap S^{-k}F(V_1, ..., V_n) \cap S^{-P}F(B_1, ..., B_k)} \phi_\sigma d\mu = \\
\int_{T \cap E \cap F(B_1, ..., B_k) \cap S^{-k}F(V_1, ..., V_n) \cap S^{-P}F(A_1, ..., A_k)} \mu(T \cap R \cap E \cap F(B_1, ..., B_k) \cap S^{-k}F(V_1, ..., V_n) \cap S^{-P}F(A_1, ..., A_k))
\]
which we write as
\[
\int_{T \cap E \cap F(A_1, ..., A_k) \cap S^{-k}F(V_1, ..., V_n)} 1_{C_k(a,b,c,d)}[1_{D_k(a,b,c,d)}1_{F(B_1, ..., B_k)}] \circ S^P \phi_\sigma d\mu = \\
\int_{T \cap E \cap F(B_1, ..., B_k) \cap S^{-k}F(V_1, ..., V_n)} 1_{(x_0=a, x_{k+1}=d)}[1_{(x_1=b, x_k=c)}1_{F(A_1, ..., A_k)}] \circ S^P d\mu. \quad (26)
\]
Define two sequences \( \delta^Q_{(A_1, ..., A_k)}(b, c) \) and \( \xi^Q_{B_1, ..., B_k}(a, b, c, d), Q \geq R + k + 1, \) by
\[
\delta^Q_{(A_1, ..., A_k)}(b, c) := \frac{1}{Q} \sum_{P=R+k+1}^{Q} [1_{(x_1=b, x_k=c)}1_{F(A_1, ..., A_k)}] \circ S^P
\]
and, recalling that \( \sigma = \sigma_{P,k}, \)
\[
\xi^Q_{B_1, ..., B_k}(a, b, c, d) := \frac{1}{Q} \sum_{P=R+k+1}^{Q} [1_{D_k(a,b,c,d)}1_{F(B_1, ..., B_k)}] \circ S^P \phi_\sigma
\]
so that, by (26), for any \( Q \geq R + k + 1, \)
\[
\int_{T \cap E \cap F(A_1, ..., A_k) \cap S^{-k}F(V_1, ..., V_n)} 1_{C_k(a,b,c,d)}\xi^Q_{B_1, ..., B_k}(a, b, c, d) d\mu = \\
\int_{T \cap E \cap F(B_1, ..., B_k) \cap S^{-k}F(V_1, ..., V_n)} 1_{(x_0=a, x_{k+1}=d)}\delta^Q_{(A_1, ..., A_k)}(b, c) d\mu. \quad (27)
\]
The sequence \( (\xi^Q_{B_1, ..., B_k}(a, b, c, d))_{Q \geq R+k+1} \) is bounded in \( L^\infty(\mu)^*. \) By Alaoglu-Bourbaki theorem ([3] Theorem 2., p. 424), it then admits at least one weak-star cluster point. Because,
by the mean ergodic Theorem, the sequence \( \delta_{(A_1,\ldots,A_k)}^q(b,c) \) converges in \( L^2(\mu) \) norm, the\eqref{eq:mean_ergodic} implies that for any weak star cluster point \( \eta_{B_1,\ldots,B_k}(a,b,c,d) \) of the sequence \((\xi_{B_1,\ldots,B_k}^q(a,b,c,d))_{Q \geq R+k+1} \), we have
\[
\eta_{B_1,\ldots,B_k}(a,b,c,d)(C_k(a,b,c,d) \cap T \cap E \cap F(A_1,\ldots,A_k) \cap S^{-k}F(V_1,\ldots,V_n)) = \\
\mu((x_1 = b, x_k = c) \cap F(A_1,\ldots,A_k)) \times \mu((x_0 = a, x_{k+1} = d) \cap T \cap E \cap F(B_1,\ldots,B_k) \cap S^{-k}F(V_1,\ldots,V_n))
\]
In the particular case where \( B_1 = \ldots = B_k = K \), we denote \( \eta_{B_1,\ldots,B_k}(a,b,c,d) \) by \( \eta(a,b,c,d) \). Also the dependence on \( k \) is specified by
\[
\eta^k = \eta(a,b,c,d) = \eta(a,b,c,d).
\]
Then, for \( E = E_+ \cap (x_0 = a) \), where \( E_+ \) is of the form
\[
E_+ = \Pi(A_{-L},\ldots,A_{-1},K) := \{ \omega \in \Sigma : \omega_{-L} \in A_{-L},\ldots,\omega_{-1} \in A_{-1} \}
\]
we have
\[
\eta^k((x_0 = a, x_1 = b, x_k = c, x_{k+1} = d) \cap T \cap E_+ \cap (x_1 \in A_1,\ldots,x_k \in A_k, x_{k+1} \in V_1,\ldots,x_{k+n} \in V_n)) = \\
\mu((x_1 = b, x_k = c, x_1 \in A_1,\ldots,x_k \in A_k)) \times \\
\mu((x_0 = a, x_{k+1} = d) \cap T \cap E_+ \cap (x_0 = a) \cap (x_{k+1} \in V_1,\ldots,x_{k+n} \in V_n))
\]
which, when \( b \in A_1, c \in A_k \) and \( d \in V_1 \), we rewrite as
\[
\eta^k(T \cap E_+ \cap (x_0 = a, x_1 = b, x_2 \in A_2,\ldots,x_{k-1} \in A_{k-1}, x_k = c, x_{k+1} = d, x_{k+2} \in V_2,\ldots,x_{k+n} \in V_n)) = \\
\mu(x_1 = b, x_2 \in A_2,\ldots,x_{k-1} \in A_{k-1}, x_k = c) \times \mu(T \cap E_+ \cap (x_0 = a, x_{k+1} = d, x_{k+2} \in V_2,\ldots,x_{k+n} \in V_n))
\]
In particular, for any \( l \) with \( 2 \leq l \leq k-1 \),
\[
\eta^k(T \cap E_+ \cap (x_0 = a, x_1 = b, x_2 \in A_2,\ldots,x_l \in A_l, x_k = c, x_{k+1} = d, x_{k+2} \in V_2,\ldots,x_{k+n} \in V_n)) = \\
\mu(x_1 = b, x_2 \in A_2,\ldots,x_l \in A_l, x_k = c) \times \mu(T \cap E_+ \cap (x_0 = a, x_{k+1} = d, x_{k+2} \in V_2,\ldots,x_{k+n} \in V_n))
\]
that is
\[
\eta^k(T \cap E_+ \cap (x_0 = a, x_1 = b, x_2 \in A_2,\ldots,x_l \in A_l) \cap S^{-k}(x_0 = c, x_1 = d, x_2 \in V_2,\ldots,x_n \in V_n)) = \\
\mu((x_1 = b, x_2 \in A_2,\ldots,x_l \in A_l) \cap S^{-k}(x_0 = c)) \times \mu(T \cap E_+ \cap (x_0 = a) \cap S^{-k}(x_1 = d, x_2 \in V_2,\ldots,x_n \in V_n))
\]
Let
\[
\alpha^k_{a,b} = \alpha^k := \sum_{c,d} \eta^k(a,b,c,d).
\]
Then, in particular for \( V_2 = \ldots = V_n = K \), we obtain
\[
\alpha^k_{a,b}(T \cap E_+ \cap (x_0 = a, x_1 = b, x_2 \in A_2,\ldots,x_l \in A_l) = \\
\mu(x_1 = b, x_2 \in A_2,\ldots,x_l \in A_l) \times \mu(T \cap E_+ \cap (x_0 = a)).
\]
Clearly the sequence $(\eta^k(a, b, c, d))_{k \geq 3}$ is bounded in $L^\infty(\mu)$. The same holds then for the sequence $(\alpha_{a,b}^k)_{k \geq 3}$. It follows that every weak star cluster point $\alpha_{a,b} \in L^\infty(\mu)$ of the sequence $(\alpha_{a,b}^k)_{k \geq 3}$, satisfies: for any $l \geq 2$,

$$\alpha_{a,b}(T \cap E_- \cap (x_0 = a, x_1 = b, x_2 \in A_2, \ldots, x_l \in A_l)) = \mu(T \cap E_- \cap (x_0 = a)) \times \mu(x_1 = b, x_2 \in A_2, \ldots, x_l \in A_l)$$

so that $\alpha_{a,b}$ extends uniquely to a countably additive $\tilde{\alpha}_{a,b}$ to the algebra of the sets of the form $E \cap (x_0 = a, x_1 = b) \cap F$, where $E \in \bigvee_{j \leq 1} B_j$, $F \in \bigvee_{j \geq 2} B_j$, and

$$\tilde{\alpha}_{a,b}(T \cap E \cap (x_0 = a, x_1 = b) \cap F) = \mu(T \cap E \cap (x_0 = a)) \times \mu((x_1 = b) \cap F). \tag{28}$$

Also this last equality still holds for $E$ in the $\mu$ completion of $\bigvee_{j \leq 1} B_j$ and $F$ in the $\mu$ completion of $\bigvee_{j \geq 2} B_j$. [In particular it still holds for all $E, F \in \Pi_S$, where $\Pi_S$ is the Pinsker sigma algebra of the system $(\Omega, S, \mu)$, so that we have

**Proposition 1**

*The Pinsker sigma algebra of the system $(\Omega, S, \mu)$ is trivial.*

**Proof**

In fact if $R \in \Pi_S$ with $\mu(R) > 0$, then

$$\tilde{\alpha}(R \cap (x_0 = a, x_1 = b) \cap R) = \mu(R \cap (x_0 = a)) \times \mu((x_1 = b) \cap R)$$

and also $\tilde{\alpha}(R \cap (x_0 = a, x_1 = b) \cap R) = \mu(R \cap (x_0 = a)) \times \mu((x_1 = b) \cap R)$ imply $\mu(R \cap (x_0 = a)) \times \mu((x_1 = b) \cap R) = \mu(R \cap (x_0 = a)) \times \mu((x_1 = b))$.

Then, for $a$ with $\mu(R \cap (x_0 = a)) \neq 0$, we have

$$\mu((x_1 = b) \cap R) = \mu((x_1 = b)), \forall b$$

thus $\mu(R) = 1$.

] Now, we continue the proof of Theorem 1. Let

$$\nu := \sum_{a,b} \tilde{\alpha}_{a,b}.$$

Then from (28) we get

$$\nu(T \cap E \cap F) = \mu(T \cap E) \times \mu(F), \; \forall T \in \mathcal{T}, \; E \in \mathcal{B}_{\leq 0}, \; F \in \mathcal{B}_{\geq 1}. \tag{29}$$

Let $\mathcal{G} := \mathcal{B}_{\leq 0} \times \mathcal{B}_{\geq 1}$ and $\mathcal{L} := (\mathcal{T} \vee \mathcal{B}_{\leq 0}) \times \mathcal{B}_{\geq 1}$. Then (29) shows that $\nu$ is countably additive on $\mathcal{L}$. Thus $\nu$ extends uniquely to a countably additive measure $\nu_1$ to the sigma algebra $\sigma(\mathcal{L})$ generated by $\mathcal{L}$. Since $\mathcal{G} \subset \mathcal{L}$, $\nu$ extends uniquely to a countably additive measure $\nu_2$ on $\sigma(\mathcal{G})$ also. Because $\sigma(\mathcal{G}) = \sigma(\mathcal{L}) = \mathcal{B}$, and $\nu_1$ is a countably additive extension of
ν from $G$ to $\sigma(G)$, we have $\nu_1 = \nu_2 =: \tilde{\nu}$. Then in particular $\tilde{\nu}$ is countably additive on $\mathcal{B}$ and satisfies the two following equalities

$$\tilde{\nu}(E \cap F) = \mu(E) \times \mu(F), \quad \forall E \in \mathcal{B}_{\leq 0}, \ F \in \mathcal{B}_{\geq 1}. \quad (30)$$

and

$$\tilde{\nu}(T) = \mu(T), \quad \forall T \in \mathcal{T} \quad (31)$$

which mean that $\mu$ is "faiblement de Bernoulli". Since "faiblement de Bernoulli" is equivalent to weak Bernoulli ([7], Proposition 2) and also, a system which is weak Bernoulli is isomorphic to a Bernoulli system [4], the proof is complete.
3 Application

3.1 Quasi-invariance of Gibbs measures

If $\Lambda$ is an $n \times n$ matrix of zeros and ones, let $K := \{0, 1, ..., n - 1\}$ and

$$\Sigma_\Lambda := \{x \in K^\mathbb{Z} : \Lambda_{x_i, x_{i+1}} = 1, \forall i \in \mathbb{Z}\}.$$

We assume that $\forall j \in K$, there exists $x \in \Sigma_\Lambda$ such that $x_0 = j$.

**Definition 3**

Let $\phi : \Sigma_\Lambda \to \mathbb{R}$ be continuous. A Gibbs measure for $\phi$ is a shift invariant probability measure $\mu_\phi$ on $\Sigma_\Lambda$ for which one can find constants $c_1 > 0, c_2 > 0$ and $p$ such that

$$c_1 \leq \frac{\mu_\phi(\{y \in \Sigma_\Lambda : y_j = x_j, j = 0, ..., m\})}{\exp(-pm + \sum_{j=0}^{m-1} \phi(S^j x))} \leq c_2,$$

(32)

for every $x \in \Sigma_\Lambda$ and $m \geq 0$.

Recall that the system $(\Sigma_\Lambda, S)$ is topologically mixing if for some $M, \Lambda^M_{i,j} > 0$, for all $i, j$.

Let, as in [1], $\mathcal{F}_\Lambda$ be the set of $\phi$ which satisfies

$$\text{var}_k \phi \leq b\alpha^k, \forall k \geq 0,$$

for some $\alpha \in [0, 1]$ and $b$, where $\text{var}_k \phi$ is defined by

$$\text{var}_k \phi := \sup \{|\phi(x) - \phi(y)| : x, y \in \Sigma_\Lambda, x_j = y_j, \forall |j| \leq k\}.$$

Clearly, $\mathcal{F}_\Lambda$ contains every $\phi$ which depends only on a finite number of coordinates.

**Definition 4**:

Two functions $\phi, \psi \in C(\Sigma_\Lambda)$ are homologous with respect to the shift $S$, if there is a $u \in C(\Sigma_\Lambda)$ such that

$$\psi(x) = \phi(x) + u(x) - u \circ S(x), \quad \forall x \in \Sigma_\Lambda.$$

We recall the following results from [1]:

**Theorem A**:

Suppose $(\Sigma_\Lambda, S)$ topologically mixing and let $\phi \in \mathcal{F}_\Lambda$. Then

(i) there exists a unique Gibbs measure $\mu_\phi$ for $\phi$.
(ii) If $\psi$ is cohomologous to $\phi$ then $\mu_\psi = \mu_\phi$.
(iii) $\phi$ is cohomologous to some $\psi \in \mathcal{F}_\Lambda$ with $\psi(x) = \psi(y)$ whenever $x_j = y_j$ for all $j \geq 0$. 
(iv) \((\Sigma_A, S, \mu_\phi)\) is isomorphic to a Bernoulli system.

We prove the following proposition, from which the statement (iv) in Theorem A, follows as a corollary of Theorem 1:

**Proposition 2**

For every potential \(\phi \in \mathcal{F}_A\) the Gibbs measure \(\mu_\phi\) is quasi-invariant by any involution which moves only a finite number of coordinates.

**Proof** Denote \(\mu_\phi\) by \(\mu\). Observe first that the inequalities (32) in the definition can be written as

\[
\mu(\{y \in \Sigma_A : y_j = x_j, j = 0, ..., m\}) = a_m(x) \exp(-pm + \sum_{j=0}^{m-1} \phi(S^j x)),
\]

with

\[c_1 \leq a_m(x) \leq c_2;\]

and, in particular, impilee that the measure of any cylinder is non null.

Let

\[H_A := \{\sigma \in H : \sigma \text{ is an involution, } T_\sigma \Sigma_A \cap \Sigma_A \neq \emptyset\}\]

. Then

\[\sigma \in H_A \iff \sigma^{-1} \in H_A,\]

because of the equality

\[T_{\sigma^{-1}}(\Sigma_A \cap T_\sigma \Sigma_A) = T_{\sigma^{-1}} \Sigma_A \cap \Sigma_A.\]

Let \(\sigma \in H_A, \ \tau = \sigma^{-1}\) and \(M(\tau)\) be the smallest natural number such that

\[|j| \geq M(\tau) \Rightarrow \tau(j) = j.\]

We note then that, for any \(N \geq M(\tau)\), the restriction of \(\tau\) to the set \(\{-N, ..., N\}\) is a permutation of this set. Let \(x \in \Sigma_A \cap T_{\sigma^{-1}} \Sigma_A\), and for all \(N \geq M(\tau)\),

\[C_N(x) = C := \{y \in \Sigma_A : y_j = x_j, j = -N, ..., N\}.\]

Then, by shift-invariance of \(\mu\),

\[\mu(C) = \mu(\{y \in \Sigma_A : y_0 = x_{-N}, y_1 = x_{-N+1}, ..., y_{2N} = x_N\},\]

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hence by (33), we obtain

\[ \mu(C_N(x)) = \mu(C) = a_{2N}(S^{-N}x) \exp(-p \times 2N + \sum_{j=0}^{2N-1} \phi(S^jS^{-N}x)) \]

that is

\[ \mu(C_N(x)) = a_{2N}(S^{-N}x) \exp(-2Np + \sum_{q=-N}^{N-1} \phi(S^q x)) \]  

(34)

and also, since, as noted before, \( \sigma \) restricted to \( \{ -N, \ldots, N \} \) is a permutation of \( \{ -N, \ldots, N \} \), we have

\[ T_\sigma^{-1}C = \{ y \in \Sigma \Lambda : y_{\sigma(j)} = x_j, j = -N, \ldots, j = N \} \]

\[ = \{ y \in \Sigma \Lambda : y_j = x_{\tau(j)}, j = -N, \ldots, j = N \}, \]

where \( \tau := \sigma^{-1} \), and because \( x_{\tau(j)} = (T_\tau x)(j) \), for all \( j \), we obtain

\[ T_\sigma^{-1}C = C_N(T_\tau x), \]

so by (34),

\[ \mu(T_\sigma^{-1}C) = a_{2N}(S^{-N}T_\tau x) \exp(-2Np + \sum_{q=-N}^{N-1} \phi(S^qT_\tau x)). \]  

It follows that

\[ \frac{\mu(T_\sigma^{-1}C)}{\mu(C)} = \frac{a_{2N}(S^{-N}T_\tau x)}{a_{2N}(S^{-N}x)} \times \exp\left( \sum_{q=-N}^{N-1} (\phi(S^qT_\tau x) - \phi(S^q x)) \right). \]  

(35)

Set, \( \tau \) being fixed,

\[ G_{N,\tau}(x) = G_N(x) := \sum_{q=-N}^{N-1} (\phi(S^qT_\tau x) - \phi(S^q x)), \]

so that (35) becomes

\[ \frac{\mu(T_\sigma^{-1}C)}{\mu(C)} = \frac{a_{2N}(S^{-N}T_\tau x)}{a_{2N}(S^{-N}x)} \times \exp(G_N(x)). \]  

(36)

But, according to Theorem A, we can suppose that \( \phi \) depends only on the non negative coordinates \( x_0, x_1, \ldots \)

\[ \phi(x) = \phi_1(x) = \phi_1(x_0, x_1, \ldots). \]  

(37)
Let $b > 0$ and $\alpha \in ]0, 1[$ such that $\text{var}_k \phi_1 \leq b \alpha^k$, $\forall k \geq 0$. Then

$$G_N(x) = \sum_{q=-N}^{N-1} (\phi_1(S^qT_\tau x) - \phi_1(S^q x)) = \sum_{q=-N}^{N-1} (\phi_1(x_{\tau(q)}, x_{\tau(q)+1}, \ldots) - \phi_1(x_q, x_{q+1}, \ldots)),$$

so that if $N \geq M(\tau) + 1$, we can write

$$G_N(x) = H_N(x) + F_N(x),$$

where

$$H_N(x) = \sum_{q=-N}^{-M(\tau)} (\phi_1(x_{\tau(q)}, x_{\tau(q)+1}, \ldots) - \phi_1(x_q, x_{q+1}, \ldots)),$$

$$F_N(x) = \sum_{q=-M(\tau)+1}^{N-1} (\phi_1(x_{\tau(q)}, x_{\tau(q)+1}, \ldots) - \phi_1(x_q, x_{q+1}, \ldots)).$$

Clearly, for any $N \geq M(\tau)$, we have

$$F_N(x) = F_{M(\tau)}(x).$$

Also, since for any $q \leq -M(\tau)$,

$$(S^qT_\tau x)_j = (S^q x)_j, \forall j, 0 \leq j \leq -M(\tau) - q,$$

we get

$$| \phi_1(S^qT_\tau x) - \phi_1(S^q x) | \leq \text{var}_{-M(\tau)-q} \phi_1 \leq b \alpha^{-M(\tau)-q},$$

and then

$$\sum_{q=-N}^{-M(\tau)} | (\phi_1(x_{\tau(q)}, x_{\tau(q)+1}, \ldots) - \phi_1(x_q, x_{q+1}, \ldots)) | \leq \sum_{q=-M(\tau)-q}^{N-M(\tau)} \text{var}_{-M(\tau)-q} \phi_1 = \sum_{k=0}^{N-M(\tau)} \text{var}_k \phi_1 \leq \sum_{k=0}^{\infty} \text{var}_k \phi_1 \leq \frac{b}{1 - \alpha}.$$

It follows that, the sequence $H_N(x)$ converges and thus $G_N(x)$ converges also to

$$\lim_{N \to \infty} G_N(x) = \sum_{q \in \mathbb{Z}} (\phi(S^qT_\tau x) - \phi(S^q x)),$$

and

$$| G_N(x) | \leq | F_{M(\tau)}(x) | + \frac{b}{1 - \alpha},$$

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from which and because $F_{M(\tau)}$ is continuous on the compact space $\Sigma_\Lambda$, we conclude that for some constant $C$,

$$\sup_{N \geq M(\tau)} \sup_x |G_N(x)| \leq C,$$

which, in view of (36), gives

$$\frac{c_1}{c_2} \times \exp(-C) \leq \frac{\mu(T_{\sigma}^{-1}C)}{\mu(C)} \leq \frac{c_2}{c_1} \times \exp(C), \quad (39)$$

because of the two following inequalities

$$\frac{c_1}{c_2} \leq \frac{a_{2N}(S^{-N}T_{\tau}x)}{a_{2N}(S^{-N}x)} \leq \frac{c_2}{c_1}.$$

Let $\alpha := \frac{c_1}{c_2} \exp(-C)$ and $\beta := \frac{c_2}{c_1} \exp(C)$, so that (39) reads as

$$\alpha \mu(C) \leq \mu(T_{\sigma}^{-1}C) \leq \beta \mu(C),$$

for all cylinder $C$ in the algebra generated by the coordinates in $\{-N, ..., N\}$, and thus, by finite additivity, these inequalities still hold for any set $E$ in that algebra. Since $N \geq M(\tau)$ is arbitrary, the equivalence of $\mu$ and $\mu \circ T_{\sigma}^{-1}$ follows, and this proves the quasi-invariance of $\mu = \mu_\phi$.□

**Remark**

In [10] the authors define a Gibbs measure $\mu$ of a map $\phi : \Sigma_\Lambda \to \mathbb{R}$, with summable variation, to be a probability measure $\mu$ on $\Sigma_\Lambda$ such, in our setting, that for any $\sigma \in H_\Lambda$, the Radon-Nikodym derivative $\frac{d\mu \circ T_{\sigma}}{d\mu}$ satisfies

$$\log\left(\frac{d\mu \circ T_{\sigma}}{d\mu}(x)\right) = \sum_{k \in \mathbb{Z}} (\phi(S^k T_{\sigma} x) - \phi(S^k x)),$$

for $\mu$ almost all $x$. Then, according to this definition, a Gibbs measure is quasi-invariant. They then give a proof of the fact that, for every $\phi$, with summable variation, there exists a unique Gibbs measure $\mu_\phi$, such that the dynamical system $(\Sigma_\Lambda, S, \mu_\phi)$ is a $K$-system.

By (36) and (38), recalling that $\tau = \sigma^{-1}$, we have

$$\log\left(\frac{d\mu \circ T_{\tau}}{d\mu}(x)\right) = \lim_{N} \log\left(\frac{a_{2N}(S^{-N}T_{\tau}x)}{a_{2N}(S^{-N}x)}\right) + \sum_{q \in \mathbb{Z}} (\phi(S^q T_{\tau} x) - \phi(S^q x))$$

or equivalently

$$\log\left(\frac{d\mu \circ T_{\tau}}{d\mu}(x)\right) = \log(g_{\tau}(x)) + \sum_{q \in \mathbb{Z}} (\phi(S^q T_{\tau} x) - \phi(S^q x))$$

where

$$\frac{c_1}{c_2} \leq g_{\tau}(x) \leq \frac{c_2}{c_1}.$$
so that the Gibbs measures as in [10] are Gibbs measures according to the given Definition 3 which is quoted from [1].

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Sorbonne Université, UMR 8001, Laboratoire de Probabilités, Statistique et Modélisation, Boîte courrier 158, 4 Place Jussieu, F-75752 Paris Cedex 05, France.
E-mail: doureid.hamdan@upmc.fr