THE ASYMPTOTIC DIMENSION OF QUOTIENTS BY FINITE GROUPS

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Abstract. Let $X$ be a proper metric space and let $F$ be a finite group acting on $X$ by isometries. We show that the asymptotic dimension of $F \setminus X$ is the same as the asymptotic dimension of $X$.

1. Introduction

If a metric space $X$ has asymptotic dimension $n$ and $F$ is a finite group acting isometrically on $X$, then it is easy to show that $F \setminus X$ has asymptotic dimension at most $|F|(n + 1) - 1$. We will show that the asymptotic dimension of the quotient is equal to the asymptotic dimension of $X$.

Theorem 1.1. Let $X$ be a proper metric space and let $F$ be a finite group acting isometrically on $X$. Then $F \setminus X$ has the same asymptotic dimension as $X$.

As a corollary we also obtain the following family version of the theorem.

Corollary 1.2. Let $\{X_i\}_{i \in I}$ be a set of proper metric spaces and let $\{F_i\}_{i \in I}$ be a set of finite groups such that $F_i$ acts isometrically on $X_i$. If $\{X_i\}_{i \in I}$ has asymptotic dimension $n$ uniformly and there exists $N \in \mathbb{N}$ with $|F_i| \leq N$ for all $i \in I$, then $\{F_i \setminus X_i\}$ has asymptotic dimension $n$ uniformly.

This article was motivated by the following.

In [Kas, Theorem A.1] the author proved that for a group $G$ with a finite dimensional classifying space for the family of finite subgroups $\mathcal{E}G$ the $K$-theoretic assembly map

$$H^{\infty}_n(\mathcal{E}G; \mathbb{K}_R) \to K_n(R[G])$$

is split injective for every ring $R$ if for any proper left-invariant metric on $G$ the family $\{F \setminus G \mid F \leq G \text{ finite}\}$ has finite asymptotic dimension uniformly (or more generally finite decomposition complexity).

While it is easy to show that if $G$ has finite asymptotic dimension, then for every $n \in \mathbb{N}$ the family $\mathcal{F}in(G)_n := \{F \setminus G \mid F \leq G, |F| \leq n\}$ has finite asymptotic dimension uniformly, the following question is open.

Question 1.3. Has $\mathcal{F}in(G) := \{F \setminus G \mid F \leq G \text{ finite}\}$ finite asymptotic dimension uniformly if $G$ has finite asymptotic dimension?
By Corollary 1.2 we have asdim $\mathcal{F}in(G)_n = \text{asdim } G$ and hence it is not possible to obtain a counterexample to Question 1.3 by finding a group $G$ for which the uniform asymptotic dimension of $\mathcal{F}in(G)_n$ goes to infinity with increasing $n \in \mathbb{N}$. Note that it is important that the metric is left-invariant and the quotient is taken from the left. For every finite subgroup $F \leq G$ the quotient $G/F$ is quasi-isometric to $G$, but this is in general not true for $F \setminus G$.

From Theorem 1.1 we also obtain the following corollary about the existence of equivariant covers.

**Corollary 1.4.** Let $X$ be a proper metric space with asymptotic dimension at most $n$ and let $F$ be a finite group acting isometrically on $X$. Then for every $R > 0$ there exists an $F$-equivariant, bounded cover of $X$ with Lebesgue number at least $R$ and dimension at most $n$.

The key ingredient of the proof of Theorem 1.1 is to use Dranishnikov’s result [Dra00, Theorem 6.2] comparing the asymptotic dimension with the topological dimension of the Higson corona. We recall the definition of the Higson corona and the comparison result in Section 3.

2. **Asymptotic dimension**

Asymptotic dimension is a coarse invariant of metric spaces introduced by Gromov [Gro93]. We begin by giving the definition and collect a few well-known facts.

**Definition 2.1.** Let $r > 0$. A metric space $X$ is the $r$-disjoint union of subspaces $X_i$, $i \in I$ if $X = \bigcup_{i \in I} X_i$ and for all $x \in X_i$, $y \in X_j$ with $i \neq j$ we have $d(x, y) > r$. In this case we write

$$X = \bigsqcup_{r\text{-disjoint}} \{X_i \mid i \in I\}.$$

**Definition 2.2.** A metric $X$ has asymptotic dimension at most $n$ if for each $r > 0$ there exist decompositions

$$X = \bigcup_{j=0}^{n} U_j, \quad U_i = \bigsqcup_{\lambda \in I_j} V_j^{\lambda}$$

such that $\sup\{\text{diam } V_j^{\lambda} \mid j \in \{0, \ldots, n\}, \lambda \in I_j\} < \infty$. We denote the asymptotic dimension of $X$ by asdim $X$.

A set $\{X_i\}_{i \in I}$ of metric spaces has asymptotic dimension at most $n$ if for each $r > 0$ there exist decompositions

$$X_i = \bigcup_{j=0}^{n} U_{i,j}, \quad U_{i,j} = \bigsqcup_{\lambda \in I_{i,j}} V_{i,j}^{\lambda}$$

such that $\sup\{\text{diam } V_{i,j}^{\lambda} \mid i \in I, j \in \{0, \ldots, n\}, \lambda \in I_{i,j}\} < \infty$.

**Remark 2.3.** Often a set $\{X_i\}_{i \in I}$ with the above property is said to have asymptotic dimension uniformly. We used this convention in the introduction but will omit the word uniformly from now on.

If a finite group $F$ acts isometrically on a metric space $X$, then we will use the following metric on $F \setminus X$:

$$d(Fx, Fx') = \min_{f \in F} d(x, fx')$$
We will also need an equivalent formulation of finite asymptotic dimension. For this recall the following definitions.

**Definition 2.4.** A cover $U$ of a metric space $X$

1. is *bounded* if $\sup_{U \in U} \text{diam } U < \infty$.
2. has *dimension at most* $n$ if every $x \in X$ is contained in at most $n + 1$ elements of $U$.
3. has *Lebesgue number* at least $R$ if for every $x \in X$ there exists $U \in U$ with $B_R(x) \subseteq U$.

**Proposition 2.5** ([Roe03, Theorem 9.9]). Let $X$ be a proper metric space. Then $\text{asdim } X \leq n$ if for each $R > 0$ there exists a bounded cover $U$ of $X$ such that no more than $n + 1$ members of $U$ meet any ball of radius $R$. Equivalently, for each $R > 0$ there exists a bounded cover $U$ of $X$ of dimension at most $n$ and Lebesgue number at least $R$.

**Lemma 2.6.** Let $X$ be a proper metric space with $\text{asdim } X = n$ and let $F$ be a finite group acting by isometries on $X$, then the quotient $F \backslash X$ has asymptotic dimension at most $(|F|)(n + 1) - 1$.

**Proof.** In this proof we will use the alternative description of asymptotic dimension from Proposition 2.5. Let $R > 0$ be given and choose a bounded cover $U$ of $X$ of dimension at most $n$ and Lebesgue number at least $R$.

For every $U \in U$ we have $\text{diam } F \backslash FU < \text{diam } U$ and thus $F \backslash U := \{ F \backslash FU | U \in U \}$ is a bounded cover of $F \backslash X$. The ball $B_R(x)$ of radius $R$ around $x \in X$ maps onto the ball $B_R(Fx) \subseteq F \backslash X$ and hence the Lebesgue number of $F \backslash U$ is bigger or equal to the Lebesgue number of $U$.

Furthermore, each of the preimages $fx \in X$ of $Fx \in F \backslash X$ is contained in at most $n + 1$ elements of $U$. Therefore, $Fx$ is contained in at most $|F|n(n + 1)$ elements of $F \backslash U$ and has dimension at most $|F|n(n + 1) - 1$. \hfill \square

For the proof of Corollary 1.2 we will need the following results relating the asymptotic dimension of a set of metric spaces to the asymptotic dimension of a single space.

**Lemma 2.7** ([FSW, Lemma 2.2]). Let $\{X_i\}_{i \in I}$ be a set of metric spaces. Then $\text{asdim } \{X_i\}_{i \in I} = \text{asdim } \{X_j\}_{j \in J}$ for any countable $J \subseteq I$.

**Definition 2.8.** For a set $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ of metric spaces with finite subspaces $Y_n \subseteq X_n$ and a sequence $\{f(n)\}_{n \in \mathbb{N}}$ of positive numbers with $f(n) \geq \text{diam } Y_n$ let $S(\{X_n\}, \{f_n\})$ denote the disjoint union $S(\{X_n\}, \{f_n\}) = \bigsqcup_{i \in I} X_i$ with the following metric. For $x \in X_n, y \in X_m$ we have

$$d(x, y) = \begin{cases} d_n(x, y) & \text{if } n = m \\ d_n(x, Y_n) + d_m(y, Y_m) + \max \{f(n), f(m)\} & \text{else} \end{cases}$$

This is indeed a metric since $f(n) \geq \text{diam } Y_n$.

**Proposition 2.9.** Let $\{(X_n, d_n)\}_{n \in \mathbb{N}}$ be as set of metric spaces with finite subspaces $Y_n \subseteq X_n$ and $\{f(n)\}_{n \in \mathbb{N}}$ a sequence of strictly increasing positive numbers with $f(n) \geq Y_n$. The metric space $S(\{X_n\}, \{f(n)\})$ has the same asymptotic dimension as $\{X_n\}_{n \in \mathbb{N}}$ and it is proper if and only if each $X_n$ is proper.
Proof. Let \( \text{asdim}\{X_n\}_{n \in \mathbb{N}} = d \).

By [Roe03, Proposition 9.13] for every \( N \in \mathbb{N} \) we have

\[
\text{asdim} \bigcup_{n=1}^{N} X_n = \max\{\text{asdim} X_n \mid n \leq N\} \leq \text{asdim}\{X_n\}_{n \in \mathbb{N}},
\]

where \( \bigcup_{n=1}^{N} X_n \) is considered as a subset of \( S(\{X_n\}, \{f_n\}) \). For \( r > 0 \) choose decompositions

\[
X_n = \bigcup_{i=0}^{d} X_{n,i}, \quad X_{n,i} = \bigcup_{\lambda \in J_{n,i}} Y_{n,i}^\lambda
\]

with \( \sup\{\text{diam} Y_{n,i}^\lambda \mid n \in \mathbb{N}, i \in \{0, \ldots, n\}, \lambda \in J_{n,i}\} < \infty \). Also choose \( N > R \) and a decomposition

\[
\bigcup_{n=1}^{N} X_n = \bigcup_{i=0}^{d} U_i, \quad U_i = \bigcup_{\lambda \in J_i} U_i^\lambda
\]

with \( \sup\{\text{diam} U_i^\lambda \mid i \in \{0, \ldots, n\}, \lambda \in J_i\} < \infty \). Then we can consider the decomposition

\[
S(\{X_n\}, \{f(n)\}) = \bigcup_{i=0}^{d} \left( U_i \cup \bigcup_{n=N+1}^{\infty} X_{n,i} \right)
\]

with

\[
U_i \cup \bigcup_{n=N+1}^{\infty} X_{n,i} = \bigcup_{\lambda \in J_i} U_i^\lambda \cup \bigcup_{n=N+1}^{\infty} Y_{n,i}^\lambda
\]

where the \( r \)-disjointness follows from \( f(n) \geq n \) and the definition of \( S(\{X_n\}, \{f(n)\}) \).

This shows that \( \text{asdim} S(\{X_n\}, \{f_n\}) \leq \text{asdim}\{X_n\}_{n \in \mathbb{N}} \). On the other hand any such decomposition for \( S(\{X_n\}, \{f(n)\}) \) can be restricted to the \( X_n \) to prove that \( \text{asdim}\{X_n\}_{n \in \mathbb{N}} \leq \text{asdim} S(\{X_n\}, \{f(n)\}) \).

If \( S(\{X_n\}, \{f(n)\}) \) is proper then all the subspaces \( X_n \) are proper as well. For \( N > r \) the ball \( B_r(x) \subseteq S(\{X_n\}, \{f(n)\}) \) for \( x \in X_n \) is contained in \( (B_r(x) \cap X_n) \cup \bigcup_{i=1}^{N} B_r(Y_i) \). This is a union of finitely many balls and thus compact if \( X_i \) is proper for all \( i \leq N \).

\( \square \)

Lemma 2.10. Let \( X \) be a metric space and let \( F \) be a finite group acting isometrically on \( X \). If \( \text{asdim} F\backslash X = n \), then for every \( R > 0 \) there exists an \( F \)-equivariant, bounded cover of \( X \) with Lebesgue number at least \( R \) and dimension at most \( n \). In particular, \( \text{asdim} X \leq \text{asdim} F\backslash X \).

Proof. By Proposition 2.5 there is a bounded cover \( \mathcal{U} \) of \( F\backslash X \) of dimension at most \( n \) and Lebesgue number at least \( R \). Let \( p \colon X \to F\backslash X \) be the projection. The cover \( \{p^{-1}(U) \mid U \in \mathcal{U}\} \) is an \( F \)-equivariant cover of dimension at most \( n \) and with Lebesgue number at least \( R \) but it might not be bounded.

Let \( s := \sup\{\text{diam} U \mid U \in \mathcal{U}\} \geq R \). Then for each \( x \in p^{-1}(U) \) we have \( p^{-1}(U) \subseteq \bigcup_{f \in F} B_s(fx) \). For \( x \in X \) let \( F_{x,s} \) be the subgroup of \( F \) generated by \( \{f \in F \mid d(x, fx) \leq 4s\} \). For \( x \in U \) define \( U_x := p^{-1}(U) \cap \bigcup_{f \in F_{x,s}} B_s(fx) \).

We will first show that \( \text{diam} U_x < 4s(|F| + 1) \). Given \( y, y' \in U_x \) there are \( f, f' \in F_{x,s} \) with \( \text{d}(y, fx) \leq s \) and \( \text{d}(y', fx) \leq s \). Therefore, \( \text{diam} U_x \leq \text{diam} F_{x,s} + 2s \).

We have \( \text{diam} F_{x,s,x} = \max_{f \in F_{x,s}} \text{d}(x, fx) \) and every \( f \in F_{x,s} \) can be written as \( f = f_1 \cdots f_k \) with \( \text{d}(x, f_kx) \leq 4s \) and \( k \leq |F| \). Hence \( \text{d}(x, fx) \leq \sum_{i=1}^{k} \text{d}(x, f_i x) \leq 4ks \).
For $U \in \mathcal{U}$ choose $x_U \in U$. Then $p^{-1}(U) = \bigcup_{f \in F} U_{fx_U}$. If $f^{-1}f' \in F_{x_U,s}$ then by definition we have $U_{fx_U} = U'_{fx_U}$. Now suppose $y \in B_s(U_{fx_U}) \cap B_s(U'_{fx_U})$, then there are $h \in F_{x_U,s}$ and $h' \in F_{x_U,s}$ with $d(y, hfx_U) \leq 2s$ and $d(y, h'f'x_U) \leq 2s$. Thus $d(hfx_U, h'f'x_U) \leq 4s$ and $f^{-1}h^{-1}h'f' \in F_{x_U,s}$. If $h \in F_{x_U,s}$ then $d(f^{-1}hfx_U, x_U) = d(hfx_U, f'x_U) \leq 4s$ and $f^{-1}h \in F_{x_U,s}$. It follows that

$$f^{-1}f' = f^{-1}h(f^{-1}h^{-1}h'f')(f'f^{-1}h^{-1}h') \in F_{x_U,s}$$

and thus $p^{-1}(U)$ is the $2s$ disjoint union of the $U_{fx_U}$ where one $f \in F$ per coset $F/F_{x_U,s}$ is chosen.

The cover

$$\{U_{fx_U} \mid U \in \mathcal{U}, f \in F\}$$

now is bounded, of dimension at most $n$ and has Lebesgue number at least $R$. □

3. The Higson corona

The following definitions can for example be found in [Dra00] and [Roe03].

Let $X$ be a proper metric space. Given a bounded and continuous function $f : X \to \mathbb{C}$ and $R > 0$, we define $\text{Var}_R f : X \to \mathbb{R}$ by

$$\text{Var}_R f(x) := \sup \{|f(x) - f(y)| \mid d(x, y) < R\}.$$ 

A function $f$ is a Higson function if $\text{Var}_R f \in C_0(X)$ for every $R > 0$. Furthermore, $C_h(X)$ is the $C^*$-algebra of all Higson functions. By the Gelfand-Naimark theorem $C_h(X) \cong C_0(X)$ is isomorphic to the $C^*$-algebra of continuous functions $C_0(hX) = C(hX)$ for some compactification $hX$ of $X$. The compact space $hX$ is unique up to homeomorphism and is called the Higson compactification of $X$. This yields a functor from the category of proper metric spaces and coarse maps to the category of compact Hausdorff spaces. The Higson corona is defined as $\nu X := hX \setminus X$. If a group $F$ acts on $X$ we denote the $C^*$-algebra of $F$-invariant Higson functions by $C_h(X)^F$.

**Proposition 3.1.** Let $F$ be a finite group acting isometrically on a proper metric space $X$. By functoriality this induces an action of $F$ on $hX$. We have the following isomorphisms

$$C(F \setminus hX) \cong C(hX)^F \cong C_h(X)^F \cong C_h(F \setminus X) \cong C(h(F \setminus X)).$$

Since $F$ is finite, the quotient $F \setminus hX$ is Hausdorff and thus $F \setminus \nu X$ and $\nu(F \setminus X)$ are homeomorphic.

**Proof.** All but the third isomorphism follow directly from the definition of $hX$ and the fact that a map $X \to \mathbb{C}$ is $F$-equivariant if and only if it factors through $F \setminus X \to \mathbb{C}$.

Since $F$ is finite, the pre-image of every compact subset $K \subseteq F \setminus X$ is again compact. Therefore, if $f : F \setminus X \to \mathbb{C}$ is a Higson function, then so is $X \to F \setminus X \xrightarrow{f} \mathbb{C}$. This implies the third isomorphism. □

The topological dimension $\dim X$ of a topological space $X$ is the smallest $n \in \mathbb{N}$ such that every open cover of $X$ has an open refinement of dimension at most $n$.

For the proof of the main theorem we need the following comparison.

**Theorem 3.2 ([Dra00, Theorem 6.2]).** If a proper metric space $X$ has finite asymptotic dimension, then $\dim(\nu X) = \operatorname{asdim} X$. 

4. Proof of the main theorem

The last ingredient we need is the following proposition.

**Proposition 4.1** ([Pea75, Proposition 9.2.16]). Let $X,Y$ be weakly paracompact, normal spaces. Let $f: X \to Y$ be a continuous, open surjection. If for every point $y \in Y$ the pre-image $f^{-1}(y)$ is finite, then $\dim(X) = \dim(Y)$.

**Proof of Theorem 1.1.** By Lemma 2.10 $\text{asdim } X = \infty$ implies $\text{asdim } F\setminus X = \infty$.

Hence let $X$ be a proper metric space with $\text{asdim } X = n < \infty$ and let $F$ be a finite group acting isometrically on $X$. By Lemma 2.6 the quotient $F\setminus X$ has again finite asymptotic dimension and thus by Theorem 3.2 its asymptotic dimension is the same as the dimension of $\nu(F\setminus X)$. By Proposition 3.1 we have a homeomorphism

$$F\setminus \nu(X) \cong \nu(F\setminus X).$$

The map $\nu X \to F\setminus \nu X$ is surjective and open, since it is the projection under a group action. The space $\nu X$ is a compact Hausdorff space and hence also $F\setminus \nu(X)$ is compact and Hausdorff. In particular both spaces are paracompact and normal.

Now using Proposition 4.1 and Theorem 3.2 together with the above we get

$$\text{asdim } F\setminus X = \dim \nu(F\setminus X) = \dim F\setminus \nu X = \dim \nu X = \text{asdim } X = n.$$  

□

**Proof of Corollary 1.2.** By Lemma 2.7 it suffices to consider the case where $I$ is countable and thus we will use $\mathbb{N}$ as index set instead. Since $|F_n| \leq N$ there exist finitely many finite groups $H_j$, $j = 1, \ldots, k$ such that each $F_n$ is isomorphic to some $H_j$. Now let $H := \bigoplus_{j=1}^k H_j$ act on $X_n$ as follows. Choose one $H_j$ isomorphic to $F_n$ and let $H_j$ act on $X_n$ using this isomorphism. All other components act trivially. Then $\{H\setminus X_n\}_{n \in \mathbb{N}}$ is isometric to $\{F_n\setminus X_n\}_{n \in \mathbb{N}}$. Choose any sequence of points $x_n \in X_n$ and let $Y_n := Hx_n$ and $f(n) := \text{diam } Y_n + n$.

Then $H$ acts componentwise on $S(\{X_n\}, \{f(n)\})$ and this action is isometric since $d(fx, Y_n) = d(x, Y_n)$ for all $x \in X_n$, $f \in H$. Furthermore, $H\setminus S(\{X_n\}, \{f(n)\})$ is isometric to $S(\{H\setminus X_n\}, \{f(n)\})$ where for the definition of the later we consider the subspace $\{Hx_n\} \subseteq H\setminus X_n$. By Theorem 1.1 and Proposition 2.9 we have

$$\text{asdim } \{X_n\}_{n \in \mathbb{N}} = \text{asdim } S(\{X_n\}, \{f(n)\}) = \text{asdim } H\setminus S(\{X_n\}, \{f(n)\}) = \text{asdim } S(\{H\setminus X_n\}, \{f(n)\}) = \text{asdim } \{H\setminus X_n\}_{n \in \mathbb{N}}.$$  

□

**Corollary 1.4** directly follows from Theorem 1.1 and Lemma 2.10.

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