Local colourings and monochromatic partitions in complete bipartite graphs

Richard Lang and Maya Stein

Universidad de Chile
Santiago, Chile
rlang@dim.uchile.cl, mstein@dim.uchile.cl

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Abstract

We show that for any 2-local colouring of the edges of a complete bipartite graph, its vertices can be covered with at most 3 disjoint monochromatic paths. And, we can cover almost all vertices of any complete or complete bipartite r-locally coloured graph with $O(r^2)$ disjoint monochromatic cycles.

MSC: 05C38, 05C55.

1 Introduction

The problem of partitioning a graph into few monochromatic paths or cycles, first formulated explicitly in the beginning of the 80’s [10], has lately received a fair amount of attention. Its origin lies in Ramsey theory and its subject are complete graphs (later substituted with other types of graphs), whose edges are coloured with $r$ colours. Call such a colouring an $r$-colouring; note that this need not be a proper edge-colouring. The challenge is now to find a small number of disjoint monochromatic paths, which together cover the vertex set of the underlying graph. Or, instead of disjoint monochromatic paths, we might ask for disjoint monochromatic cycles. Here, single vertices and edges count as cycles. Such a cover is called a monochromatic path partition, or a monochromatic cycle partition, respectively. It is not difficult to construct $r$-colourings that do not allow for partitions into less than $r$ paths, or cycles.

At first, the problem has been studied mostly for $r = 2$, and the complete graph $K_n$ as the host graph. In this situation, a partition into two disjoint paths always exists [9], regardless of the size of $n$. Moreover, these paths have different colours. An extension of this fact, namely that every 2-colouring of $K_n$ has a partition into two monochromatic cycles of different colours was conjectured by

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1For instance, take vertex sets $V_1, \ldots, V_r$ with $|V_i| = 2^i$, and for $i \leq j$ give all $V_i-V_j$ edges colour $i$. 
Lehel, and verified by Bessy and Thomassé [3], after preliminary work for large 
$n$ [1 20].

A generalisation of these two results for other values of $r$, i.e. that any 
$r$-coloured $K_n$ can be partitioned into $r$ monochromatic paths, or into $r$ mono-
chromatic cycles, was conjectured by Gyárfás [11] and by Erdős, Gyárfás and 
Pyber [7], respectively. The conjecture for cycles was recently disproved by 
Pokrovskiy [22]. He gave counterexamples for all $r \geq 3$, but he also showed 
that the conjecture for paths is true for $r = 3$. Gyárfás, Ruszinkó, Sárközy and 
Szemerédi [14] showed that any $r$-coloured $K_n$ can be partitioned into $O(r \log r)$ 
monochromatic cycles, improving an earlier bound from [7].

Monochromatic path/cycle partitions have also been studied for bipartite 
graphs, mainly for $r = 2$. A 2-colouring of $K_{n,n}$ is called a split colouring if 
there is a colour-preserving homomorphism from the edge-coloured $K_{n,n}$ to a 
properly edge-coloured $K_{2,2}$. Note that any split colouring allows for a partition 
into three paths, but not always into two. However, split colourings are the only 
‘problematic’ colourings, as the following result shows.

**Theorem 1.1.** [22] Let the edges of $K_{n,n}$ be coloured with 2 colours; then $K_{n,n}$ 
can be partitioned into two paths of distinct colours or the colouring is split.

Split colourings can be generalised to more colours [22]. This gives a lower 
bound of $2r - 1$ on the path/cycle partition number for $K_{n,n}$. For $r = 3$, 
this bound is asymptotically correct [18]. For an upper bound, Peng, Rödl 
and Ruciński [21] showed that any $r$-coloured $K_{n,n}$ can be partitioned into 
$O(r^2 \log r)$ monochromatic cycles, improving a result of Haxell [17].

The problem has also been studied for multipartite graphs [27], and for 
arbitrary graphs [2, 26]. There are also variants replacing paths or cycles with 
other graphs [8, 24, 25].

We will give new bounds for the size of cycle partitions with respect to local 
colourings. Local colourings are a natural way to generalise $r$-colourings. A 
colouring is $r$-local if no vertex is adjacent to more than $r$ edges of distinct 
colours. Many results of Ramsey theory have been reproved in terms of local 
colourings [4, 5, 12, 13, 23, 28, 29, 30]. With respect to monochromatic path or cycle partitions, Conlon and Stein [6] recently generalised some of the 
above mentioned results to $r$-local colourings. They show that for any $r$-local 
colouring of $K_n$, there is a partition into $O(r^2 \log r)$ monochromatic cycles, and, 
if $r = 2$, then two cycles (of different colours) suffice.

In this paper we improve their general bound for complete graphs. We also 
give a bound for monochromatic cycle partitions in bipartite graphs. In both 
cases, $O(r^2)$ cycles suffice.

**Theorem 1.2.** For every $r \geq 1$ there is an $n_0$ such that for $n \geq n_0$ the following 
holds.

(a) If $K_n$ is $r$-locally coloured, then all its vertices can be covered with at most 
$2r^2$ disjoint monochromatic cycles.

(b) If $K_{n,n}$ is $r$-locally coloured, then all its vertices can be covered with at most 
$4r^2$ disjoint monochromatic cycles.

Our second result is a generalisation of Theorem 1.1 to local colourings:
Theorem 1.3. Let the edges of $K_{n,n}$ be coloured 2-locally. Then $K_{n,n}$ can be partitioned into 3 or less monochromatic paths.

We prove Theorem 1.2 in Section 2 and Theorem 1.3 in Section 3. These proofs are totally independent of each other.

In view of the results from [6] and our Theorems 1.2 and 1.3, it might seem that in terms of path- or cycle-partitions, $r$-local colourings are not very different from $r$-colourings. Let us give an example where they do behave differently, even for $r = 2$.

It is shown in [27] that any 2-coloured complete tripartite graph can be partitioned into at most 2 monochromatic paths, provided that no part of the tripartition contains more than half of the vertices. This is not true for 2-local colourings: Let $G$ be a complete tripartite graph with triparts $U$, $V$ and $W$ such that $|U| = 2|V| = 2|W| \geq 6$. Pick vertices $u \in U$, $v \in V$ and $w \in W$ and write $U' = U \setminus \{u\}$, $V' = V \setminus \{v\}$ and $W' = W \setminus \{w\}$. Now colour the edges of $[W' \cup \{v\}, U']$ red, $[V' \cup \{w\}, U']$ green and the remaining edges blue. This is a 2-local colouring. However, since no monochromatic path can cover all vertices of $U'$, we need at least 3 monochromatic paths to cover all of $G$.

Note that in our example, the graph $G$ contains a 2-locally coloured balanced complete bipartite graph. This shows that in the situation of Theorem 1.3, we might need 3 paths even if the 2-local colouring is not a split colouring (and thus a 2-colouring). Blowing this example up, and adding some smaller sets of vertices seeing new colours, one obtains examples of $r$-local colourings of balanced complete bipartite graphs requiring $2r - 1$ monochromatic paths.

2 Proof of Theorem 1.3

In this section we will prove our bounds for monochromatic cycle partitions, given by Theorem 1.2. The heart of this section is Lemma 2.1. This lemma enables us to use induction on $r$, in order to prove new bounds for the number of monochromatic matchings needed to cover an $r$-locally coloured graph. In particular, we find these bounds for the complete and the complete bipartite graph. All of this is the topic of Subsection 2.1.

To get from monochromatic cycles to the promised cycle cover, we use a nowadays standard approach, which was first introduced in [19]. We find a large robust hamiltonian graph, regularise the rest, find monochromatic matchings covering almost all, blow them up to cycles, and the absorb the remainder with the robust hamiltonian graph. The interested reader may find a sketch of this well-known method in Subsection 2.2.

2.1 Monochromatic matchings

Given a graph $G$ with an edge colouring, a monochromatic connected matching is a matching in a connected component of the subgraph that is induced by the edges of a single colour.

Lemma 2.1. For $k \geq 2$, let the edges of a graph $G$ be coloured $k$-locally. Suppose there are $m$ monochromatic components that together cover $V(G)$, of colours $c_1, \ldots, c_m$. Then there are $m$ vertex-disjoint monochromatic connected matchings $M_1, \ldots,$
$\text{M}_n$, of colours $c_1, \ldots, c_m$, such that the inherited colouring of $G - V(\bigcup_{i=1}^m M_i)$ is a $(k-1)$-local colouring.

**Proof.** Let $G$ be covered by $m$ monochromatic components $C_1, \ldots, C_m$ of colours $c_1, \ldots, c_m$. Let $M_1 \subset C_1$ be a maximum matching in colour $c_1$. For $2 \leq i \leq m$ we iteratively pick maximum matchings $M_i \subset C_i - V(\bigcup_{j<i} M_j)$ in colour $c_i$. Set $M := \bigcup_{j=m} M_j$.

Now let $v$ be any vertex in $H := G - V(M)$. Say $v \in V(C_i - V(M))$. In particular, vertex $v$ sees colour $c_i$ in $G$. However, by maximality of $M_i$, vertex $v$ does not see the colour $c_i$ in $H$. Thus in $H$, vertex $v$ sees at most $k-1$ colours. Hence, the inherited colouring of $H$ is a $(k-1)$-local colouring, which is as desired.

**Corollary 2.2.** If $K_n$ is $r$-locally edge coloured, and $H$ is obtained from $K_n$ by deleting $o(n^2)$ edges, then

(a) $V(K_n)$ can be covered with at most $r(r+1)/2$ monochromatic connected matchings, and

(b) all but $o(n)$ vertices of $H$ can be covered with at most $r(r+1)/2$ monochromatic connected matchings.

Note that the matchings from (b) are connected in $H$.

**Proof.** The proof is based on the following easy observation. In any colouring of $K_n$, the closed monochromatic neighbourhoods of any vertex $v$ together cover $K_n$. Since the colouring is a $k$-local colouring, we can cover all of $V(K_n)$ with $k$ components. Now apply Lemma 2.1 successively to obtain the bound from (a).

For (b), it suffices to observe that we can choose at each step a vertex $v$ that has $o(n)$ non-neighbours in the current graph. For, if at some step, there is no such vertex, then a simple calculation shows we have already covered all but $o(n)$ of $V(K_n)$, and can hence abort the procedure.

**Corollary 2.3.** If $K_{n,n}$ is $r$-locally edge coloured, and $H$ is obtained from $K_{n,n}$ by deleting $o(n^2)$ edges, then

(a) $V(K_{n,n})$ can be covered with at most $(2r-1)r$ monochromatic connected matchings, and

(b) all but $o(n)$ vertices of $H$ can be covered with at most $(2r-1)r$ monochromatic connected matchings.

Note that the matchings from (b) are connected in $H$.

**Proof.** The proof very similar to the proof Corollary 2.2. We only note that in any colouring of $K_{n,n}$ the two closed monochromatic neighbourhoods of any edge form a vertex cover of size at most $2r - 1$.

### 2.2 From matchings to cycles

#### 2.2.1 Regularity

Regularity is the key for expanding our partition of an $r$-locally coloured $K_n$ or $K_{n,n}$ into monochromatic connected matchings to a partition of almost all
vertices into monochromatic cycles. We follow an approach introduced by Luczak [19], which has become a standard method for cycle embeddings in large graphs. We will focus on the parts where our proof differs from other applications of this method (see [14, 16, 18]).

The main result of this section is:

**Lemma 2.4.** If $K_n$ and $K_{n,n}$ are $r$-locally edge coloured, then

(a) all but $o(n)$ vertices of $K_n$ can be covered with at most $r(r + 1)/2$ monochromatic cycles.

(b) all but $o(n)$ vertices of $K_{n,n}$ can be covered with at most $(2r - 1)r$ monochromatic cycles.

Before we start, we need a couple of regularity preliminaries. For a graph $G$ and disjoint subsets of vertices $A, B \subset V(G)$ we denote by $[A, B]$ the bipartite subgraph with biparts $A$ and $B$ and edge set $\{ab \in E(G) : a \in A, b \in B\}$. We write $\deg_G(A, B)$ for the number of edges in $[A, B]$. If $A = \{a\}$ we write shorthand $\deg_G(a, B)$.

The subgraph $[A, B]$ is $(\epsilon, G)$-regular if $|\deg_G(X, Y) - \deg_G(A, B)| < \epsilon$ for all $X \subset A, Y \subset B$ with $|X| > \epsilon|A|$, $|Y| > \epsilon|B|$. Moreover, $[A, B]$ is $(\epsilon, \delta, G)$-super-regular if it is $(\epsilon, G)$-regular and $\deg_G(a, B) > \delta|B|$ for each $a \in A$ and $\deg_G(b, A) > \delta|A|$ for each $b \in B$.

A vertex-partition $\{V_0, V_1, \ldots, V_l\}$ of the vertex set of a graph $G$ into $l + 1$ clusters is called $(\epsilon, G)$-regular, if

(i) $|V_1| = |V_2| = \ldots = |V_l|;

(ii) $|V_0| < \epsilon n$;

(iii) apart from at most $\epsilon\binom{l}{2}$ exceptional pairs, the graphs $[V_i, V_j]$ are $(\epsilon, G)$-regular.

The following version of Szemerédi’s regularity lemma is well-known. The given prepartition will only be used for the bipartition of the graph $K_{n,n}$ in Lemma 2.4 (b). The colours on the edges are represented by the graphs $G_i$.

**Lemma 2.5** (Regularity lemma with prepartition and colours). For every $\epsilon > 0$ and $m, l \in \mathbb{N}$ there are $M, n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ the following holds.

For all graphs $G_0, G_1, G_2, \ldots, G_t$ with $V(G_0) = V(G_1) = \ldots = V(G_t) = V$ and a partition $A_1 \cup \ldots \cup A_s = V$, where $r \geq 2$ and $|V| = n$, there is a partition $V_0 \cup V_1 \cup \ldots \cup V_l$ of $V$ into $l + 1$ clusters such that

(a) $m \leq l \leq M$;

(b) for each $1 \leq i \leq l$ there is a $1 \leq j \leq s$ such that $V_i \subset A_j$;

(c) $V_0 \cup V_1 \cup \ldots \cup V_l$ is $(\epsilon, G_i)$-regular for each $0 \leq i \leq t$. 

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Observe that the regularity lemma provides regularity only for a number of colours bounded by the input parameter $t$. However, the total number of colours of an $r$-local colouring is not bounded by any function of $r$ (for an example, see Section 3.1). Luckily, it turns out that it suffices to focus on the $t$ colours of largest density, where $t$ depends only on $r$ and $\varepsilon$. This is guaranteed by the following result from [13].

**Lemma 2.6.** Let a graph $G$ with average degree $a$ be $r$-locally coloured. Then one colour has at least $\frac{a^2}{2r^2}$ edges.

**Corollary 2.7.** For all $\varepsilon > 0$ and $r \in \mathbb{N}$ there is a $t = t(\varepsilon, r)$ such that for any $r$-local colouring of $K_n$ or $K_{n,n}$, there are $t$ colours such that all but at most $\varepsilon n^2$ edges use these colours.

**Proof.** We only prove the corollary for $K_{n,n}$, as the proof for $K_n$ is very similar.

Let $t := \lceil -\frac{2r^2}{\varepsilon} \log \varepsilon \rceil$. We iteratively take out the edges of the colours with the largest number of edges. We stop either after $t$ steps, or before, if the remaining graph has density less than $\varepsilon$. At each step Lemma 2.6 ensures that at least a fraction of $\varepsilon^2 r^2$ of the remaining edges has the same colour. Hence we can bound the number of edges of the remaining graph by

$$\left(1 - \frac{\varepsilon}{2r^2}\right)n^2 \leq e^{-\varepsilon^2 r^2} n^2 \leq \varepsilon n^2.$$  

\[\Box\]

### 2.2.2 Proof of Lemma 2.4

We only prove part (b) of Lemma 2.4 since the proof of part (a) is very similar and actually simpler. For the sake of readability, we assume that $n_0 \gg 0$ is sufficiently large and $0 < \varepsilon \ll 1$ is sufficiently small without calculating exact values.

Let the edges of $K_{n,n}$ with biparts $A_1$ and $A_2$ be coloured $r$-locally and encode the colouring by edge-disjoint graphs $G_1, \ldots, G_s$ on the vertex set of $K_{n,n}$. By Corollary 2.7, there is a $t = t(\varepsilon, r)$ such that the union of $G_1, \ldots, G_t$ covers all but at most $\varepsilon n^2/8r^2$ edges of $K_{n,n}$. We merge the remaining edges into $G_0 := \bigcup_{i=t+1}^s G_i$. Note that the colouring remains $r$-local and by the choice of $t$, we have

$$|E(G_0)| \leq \varepsilon n^2/8r^2. \quad (1)$$

For $\varepsilon$, $t$ and $m := 1/\varepsilon$, the regularity lemma (Lemma 2.5) provides $n_0$ and $M$ such for all $n \geq n_0$ there is a vertex-partition $V_0, V_1, \ldots, V_l$ of $K_{n,n}$ satisfying Lemma 2.5(a)–(c) for $G_0, G_1, \ldots, G_t$.

As usual, we define the reduced graph $R$ which has a vertex $v_i$ for each cluster $V_i$ for $1 \leq i \leq l$. We place an edge between vertices $v_i$ and $v_j$ if the subgraph $[V_i, V_j]$ of the respective clusters is non-empty and forms an $(\varepsilon, G_q)$-regular subgraph for all $0 \leq q \leq t$. Thus, $R$ is a balanced bipartite graph with at least $(1-\varepsilon)\binom{l}{2}$ edges.

Finally, the colouring of the edges of $K_{n,n}$, induces a majority colouring of the edges of $R$. More precisely, we colour each edge $v_iv_j$ of $R$ with the colour

\[2\] Here we use that in a balanced bipartite graph $H$ with $2n$ vertices, $m$ edges, average degree $a$ and density $d$ we have $a^2 = \frac{4m^2}{4n^2} = dm$. 


from \( \{0, 1, \ldots, t\} \) that appears most on the edges of the subgraph \( [V_i, V_j] \subset G \) (in case of a tie, pick any of the densest colours). Note that if \( v_i, v_j \) is coloured \( i \) then by Lemma 2.6

\[
[V_i, V_j] \text{ has at least } \frac{1}{2r^2} \left( \frac{2}{3t} \right)^2 \text{ edges of colour } i. \tag{2}
\]

Our next step is to verify that the majority colouring is an \( r \)-local colouring of \( R \). To this end we need the following easy and well-known fact about regular graphs.

**Fact 2.8.** Let \( [A, B] \) be an \( \varepsilon \)-regular graph of density \( d > \varepsilon \). Then at most \( \varepsilon |A| \) vertices from \( A \) have no neighbours in \( B \).

**Claim 2.9.** The colouring of the reduced graph \( R \) is \( r \)-local.

**Proof.** Assume otherwise. Then there is a vertex \( v_i \in V(R) \) that sees \( r + 1 \) different colours in \( R \). By Fact 2.8 all but at most \( (r + 1)\varepsilon |V_i| < |V_i| \) of the vertices in \( V_i \) see \( r + 1 \) different colours in \( K_n,n \), contradicting the \( r \)-locality of our colouring.

By (1), and by (2), we know that \( R \) has at most \( |E(G_0)| \frac{4t^2 \cdot 2r^2}{n^2} \leq \varepsilon |E| \) edges of colour 0. Delete these edges and use Corollary 2.3 to cover all but \( o(l) \) vertices of \( R \) with \( (2r - 1)r \) vertex-disjoint monochromatic matchings \( M_1, \ldots, M_{(2r - 1)r} \) of spectrum \( 1, \ldots, t \).

We finish by applying Luczak’s technique for blowing up matching to cycles [19]. This is done by using the following (by now well-known) lemma.

**Lemma 2.10.** Let \( t \geq 1 \) and \( \gamma > 0 \) be fixed. Suppose \( R \) is the edge-coloured reduced graph of an edge-coloured graph \( H \), for some \( \gamma \)-regular partition, such that each edge \( uv \) of \( R \) corresponds to a \( \gamma \)-regular pair of density at least \( \sqrt{\gamma} \) in the colour of \( uv \). If all but at most \( \gamma |V(R)| \) vertices of \( R \) can be covered with \( t \) disjoint connected monochromatic matchings, then there is a set of at most \( t \) monochromatic disjoint cycles in \( H \), which together cover all but at most \( 10\sqrt{\gamma} |V(H)| \) vertices of \( H \).

For completeness, let us outline a proof of Lemma 2.10.

**Sketch of a proof of Lemma 2.10.** We first connect in \( H \) the pairs corresponding to matching edges with monochromatic paths, following their connections in \( R \). We do this simultaneously for all matchings. Note that, as \( t \) is fixed, in total, these paths consume only a constant number of vertices of \( H \). Then we connect the monochromatic paths using the matching edges, blowing up the edges to long paths, where regularity and density ensure that we can fill up all but a small fraction of the corresponding pairs. This gives the desired cycles.

A more detailed explanation of this argument can be found in the proof of the main result of [15].

### 2.3 The absorbing method

In this subsection we prove Theorem 1.2. We apply a well known absorbing argument introduced in [7]. To this end we need a few tools.
Call a balanced bipartite subgraph $H$ of a $2n$-vertex graph $\varepsilon$-hamiltonian, if any balanced bipartite subgraph of $H$ with at least $2(1 - \varepsilon)n$ vertices is hamiltonian. The next lemma is a combination of results from [17] [21] and can be found in [18] in the following explicit form.

**Lemma 2.11.** For any $1 > \gamma > 0$, there is an $n_0 \in \mathbb{N}$ such that any balanced bipartite graph on $2n \geq 2n_0$ vertices and of edge density at least $\gamma$ has a $\gamma/4$-hamiltonian subgraph of size at least $\gamma^{3024/\gamma}n/3$.

The following lemma is taken from [6].

**Lemma 2.12.** Suppose that $A$ and $B$ are vertex sets with $|B| \leq |A|/r^{r+3}$ and the edges of the complete bipartite graph between $A$ and $B$ are $r$-locally coloured. Then all vertices of $B$ can be covered with at most $r^2$ disjoint monochromatic cycles.

**Sketch of a proof of Theorem 1.2** Here we only prove part (b) of Theorem 1.2 since the proof of (a) is almost identical. The differences are discussed at the end of the section.

Let $A$ and $B$ be the two partition classes of the $r$-locally edge coloured $K_{n,n}$. We assume that $n \geq n_0$, where we specify $n_0$ later. Pick subsets $A_1 \subseteq A$ and $B_1 \subseteq B$ of size $|n/2|$ each. Say red is the majority colour of $[A_1, B_1]$. Then by Lemma 2.12 there are at least $n^2/8r^2$ red edges in $[A_1, B_1]$.

Lemma 2.11 applied with $\gamma = 1/10r^2$ yields a red $\gamma/4$-hamiltonian subgraph $[A_2, B_2]$ of $[A_1, B_1]$ with

$$|A_2| = |B_2| \geq \gamma^{3024/\gamma}|A_1|/3 \geq \gamma^{3024/\gamma}n/7.$$

Set $H := G - (A_2 \cup B_2)$, and note that each bipart of $H$ has order at least $|n/2|$. Let $\delta := \gamma^{3000/\gamma}$. Assuming $n_0$ is large enough, Lemma 2.12(b) provides $(2r-1)r$ monochromatic vertex-disjoint cycles covering all but at most $2\delta n$ vertices of $H$. Let $X_A \subseteq A$ (resp. $X_B \subseteq B$) be the set of uncovered vertices in $A$ (resp. $B$). Since we may assume none of the monochromatic cycles is an isolated vertex, we have $|X_A| = |X_B| \leq \delta n$.

By the choice of $\delta$, and since we assume $n_0$ to be sufficiently large, we can apply Lemma 2.12 to the bipartite graphs $[A_2, X_B]$ and $[B_2, X_A]$. This gives $2r^2$ vertex-disjoint monochromatic cycles that together cover $X_A \cup X_B$. Again, we assume none of these cycles is trivial. As $|X_A| = |X_B| \leq \delta n$, we know that the remainder of $[A_2, B_2]$ contains a red Hamilton cycle. Thus, in total, we found a cover of $G$ with at most $(2r-1)r + 2r^2 + 1 \leq 4r^2$ vertex-disjoint monochromatic cycles.

As claimed above, the proof of Theorem 1.2(a) is very similar. The main difference is that instead of an $\varepsilon$-hamiltonian subgraph we use a large red triangle cycle. A triangle cycle $T_k$ consists of a cycle on $k$ vertices $\{v_1, \ldots, v_k\}$ and $k$ additional vertices $A = \{a_1, \ldots, a_k\}$, where $a_i$ is joined to $v_i$ and $v_{i+1}$ (modulo $k$). Note that $T_k$ remains hamiltonian after the deletion of any subset of vertices of $A$.

We use some basic Ramsey theory to find a large monochromatic triangle cycle $T_k$ in an $r$-locally coloured $K_n$, as shown in [6]. Next, Lemma 2.12(a) guarantees we can cover most vertices of $K_n - T_k$ with $r(r+1)/2$ monochromatic cycles. We finish by absorbing the remaining vertices $B$ into $A$ with only one application of Lemma 2.12 thus producing $r^2$ additional cycles. As noted above, the remaining part of $T_k$ is hamiltonian and so we have partitioned $K_n$ into $r(r+1)/2 + r^2 + 1 \leq 2r^2$ monochromatic cycles. \hfill $\Box$
3 Proof of Theorem 1.3

This section is dedicated to the proof of Theorem 1.3. We split our proof into two parts. After some preliminaries in Subsection 3.1, we specify the structure of the colouring in Subsection 3.2 and then, we find the monochromatic path partition in Subsection 3.3.

3.1 Simple colourings

Let \( G \) be any graph, and let the edges of \( G \) be coloured arbitrarily. We say that a vertex \( c \) sees a colour \( i \), if \( x \) is adjacent to an edge of colour \( i \). A subgraph \( X \) sees a colour \( i \), if it has a vertex that sees \( i \).

Consider a recolouring of a monochromatic component \( X \) of colour \( i \) with colour \( j \) by changing the colour of all the edges of \( H \) from \( i \) into \( j \). Such a recolouring is permissible, if in the original colouring, \( X \) does not see \( j \). An edge colouring can be simplified, if the number of total colours can be reduced by a sequence of permissible recolourings. An edge colouring is simple, if it can not be further simplified. Observe that in a simple colouring each colour has a non-trivial component that that sees all other colours.

We will see in the next subsection that the total number of colours in simple 2-local colourings of \( K_{n,n} \) is bounded by 4. However, in general colourings can have an arbitrary large total number of colours: Take a \( t \times t \) grid \( G \) and colour the edges of the column \( i \) and row \( i \) with colour \( i \) for \( 1 \leq i \leq t \). Then add edges of a new colour \( t+1 \) until \( G \) is complete (or complete bipartite) and observe that \( G \) is 3-locally edge coloured, but the total number of used colours is \( t \). Note that this colouring can not be simplified (see Section 3 for simple colourings).

3.2 The structure of the colouring

In this subsection we will show that the colouring takes one of three specific forms. Let the edges of \( K_{n,n} \) be coloured 2-locally. As we are not interested in the actual colours of the monochromatic paths, we may assume that the colouring is simple. Moreover, we can assume the total number of colours is at least 3, since otherwise Theorem 1.1 applies.

Claim 3.1. The colouring has one of the following forms (modulo colour swapping and modulo swapping \( K_{n,n} \) with \( K_{n,n} \)):

(a) There are red, green and blue components \( R \), \( G \) and \( B \) such that
\[
K_{n,n} = B \cap R \cup G \cap R \cup B \cap G \quad \text{and} \quad K_{n,n} = B \cap G \cup R,
\]

(b) there are a red, a green and two blue components \( R \), \( G \), \( B_1 \) and \( B_2 \) such that
\[
K_{n,n} = B_1 \cap R \cup G \cap R \cup B_2 \quad \text{and} \quad K_{n,n} = B_2 \cap R \cup G \cap R \cup B_1 \cap G
\]
or
(c) there are red, green, blue and yellow components \( R \), \( G \), \( B \) and \( Y \) such that
\[
K_{n,n} = G \cap R \cup B \cap R \cup G \cap Y \cup B \cap Y \quad \text{and} \quad K_{n,n} = R \cap Y \cup B \cap G.
\]
Some of these components might be empty.

Before we can prove Claim 3.1 we need an auxiliary result.

Claim 3.2. If there is a colour component that spans one of the partition classes of $K_{n,n}$, then the colouring is of type (a).

Proof. Say the red component $R$ contains $K_{n,n}$. (All other cases are analogous.) If $R$ also contains $K_{n,n}$, then all other colour components are pairwise disjoint. As the colouring is simple, there are thus only two colours, a contradiction to our assumptions. So assume there is a vertex $v \in K_{n,n} \setminus R$.

If $v$ only sees one colour, say green, then all of $K_{n,n}$ sees green and red. Hence the colouring has only two colours, again contradicting our assumptions. On the other hand, if $v$ belongs to two non-trivial colour components, say to $G$ and $B$, then $K_{n,n}$ is the disjoint union of the non-empty sets $R \cap B$ and $R \cap G$. Thus, $K_{n,n} = B \cap G \cup R$, and the colouring is as desired for (a).

Claim 3.3. If for each colour component $C$ it holds that $\overline{C}$ and $C$ each see only one colour apart from the colour of $C$, then the colouring is of type (a).

Proof. Since the colouring is simple, there is a red component $R$ that sees all other colours (and there are at least two other colours). By our assumption, $\overline{R}$ and $R$ each see only one colour apart from red, suppose $\overline{R}$ sees green, and $R$ sees blue. Setting $X := K_{n,n} - R$, we may assume that $X \neq \emptyset \neq \overline{X}$ by Claim 3.2. Then $R \cup X$ is contained in a green component $G$, and $X \cup \overline{R}$ is contained in a blue component $B$. If $X \setminus B$ and $\overline{X} \setminus G$ are both empty, then our colouring is as desired for (a). So assume otherwise, say $X \setminus B \neq \emptyset$. 

Figure 1: The edge colourings of Claim 3.1.
By Claim 3.2, each vertex sees two colours, and thus, by our assumption, there is a non-green colour $c$ such that each vertex in $X$ sees $c$ and green. Since $X \setminus B$ does not see blue, colour $c$ is not blue. Hence $X \cap B$ is empty.

Moreover, $X$ sees $c$, and so, all vertices in $X$ see blue and $c$. Therefore, the components $B$ and $G$ are disjoint. Furthermore, $B$ and $G$ are the only non-trivial blue/green components, a contradiction to our colouring being simple.

We are now ready to prove Claim 3.1.

Proof of Claim 3.1. Since the colouring is simple, there is a non-trivial red component $R$ seeing all other colours. Set $X := K_{n,n} - R$. By Claim 3.2, we may assume that $X \neq \emptyset = X$. Also by Claim 3.2, we can assume that each vertex sees exactly two colours.

By Claim 3.3, we may assume that there are a non-trivial blue component $B$ and a non-trivial green component $G$ meeting the same partition class of $R$, say both meet $R$. Then $X \subset G \cap B$. Moreover, as we are working with a 2-local colouring, any vertex in $X$ sees only blue and green, and thus $K_{n,n} \subset G \cup B$.

If $X \setminus G$ and $X \setminus B$ are both empty, we are in situation (a). So assume $X \setminus G \neq \emptyset$. Since each vertex sees exactly 2 colours, there is a colour $c$ such that all edges between $X \setminus G$ and the non-empty set $R \setminus B$ have colour $c$, and all edges between $X \setminus G$ and the (possibly empty) set $R \cap B$ are blue. Actually, all edges between $X$ and $R \setminus B$ have colour $c$, and all edges between $X$ and $R \cap B$ are blue.

Note that $c$ is neither red nor blue. Further note that if $c$ is green, then all of $X \cap G$ is empty, and after swapping green for blue, the colouring is as in (b). So $c$ is a fourth colour, yellow say. Then $X \cap B \cap G$ is empty. Also, since the colouring is simple, the set $X \setminus B$ is not empty, and is contained in the same yellow component as $R \setminus B$. But then $R \cap B$ is empty, and we have arrived at situation (c).}

3.3 Finding the paths

We shall now find the desired partition into monochromatic paths. To continue, we distinguish between the three colourings (a), (b), and (c) and in the first case, we also distinguish whether $R \setminus (B \cup G)$ is empty or not. For convenience let us divide type (a) colourings into the following subtypes:

- type (a) (I) are the type (a) colourings with $R \subset B \cup G$, and
- type (a) (II) the type (a) colourings with $R \not\subset B \cup G$.

Colourings of type (a) (II) are very similar to colourings of type (b); the only difference is the number of blue components. We shall see both can be treated much in the same way. We start with two general observations.

Claim 3.4. If there is a balanced monochromatic path $P$ covering all of $K_{n,n}$ but

- one of the sets $R \cap B$, $R \cap G$, $B \cap G$ for type (a),
- one of the sets $B_2 \cap R$, $R \cap G$, $B_1 \cap G$ for type (b), or
- $B \cap G$ for type (c),
then there is a partition into three monochromatic paths.

Proof. Let $P$ be as in the assumptions and in addition of maximum length. Then $K_{n,n} - P$ is 2-coloured. By Theorem 1.1 we are done unless the colouring of $G - P$ is split. This rules out type (c) colourings, and it also rules out type (b) colourings where $R \cap G$ is covered by $P$. So we can assume that the colouring is of type (a) or (b), and $P$ covers all but $R \cap G \neq \emptyset$.

Clearly, if $p \notin R \cap G$, then $p$ either sends only red or only green edges to $R \cap G \setminus V(P)$. On the other hand, if $p \in R \cap G$, then $P$ is either green or red. So, by maximality of $P$, again $p$ either sends only red or only green edges to $R \cap G \setminus V(P)$. By symmetry, this allows us to assume that the edges between $p$ and $R \cap G \setminus V(P)$ are red.

Since we assume $G - P$ to have a split colouring, we can find a path partition $P_1, P_2, P_3$ of $K_{n,n} - P$ and such that $P_1, P_2$ are red and balanced. Let $p_1$ and $p_2$ be the endpoints of $P_1$ and $P_2$ that belong to $K_{n,n}$. As the edges $p_1p$ and $p_2p$ are red, we can use $p$ to connect $P_1$ and $P_2$. This new path together with $P - p$ and $P_3$ gives a path partition as desired.

Now the case of type (c) is easily solved; by symmetry we can assume that $|Y| \leq |\overline{Y}|$, there exists a balanced yellow path $P$ covering $Y$ and hence Claim 3.4 immediately yields:

**Claim 3.5.** If the colouring is of type (c), then there is a partition into three monochromatic paths.

**Claim 3.6.** Suppose $K_{n,n}$ has a colouring of type (a) or (b). If there is a balanced monochromatic path $P$, and a colour $c$, such that the graph $K_{n,n} - P$ is connected in colour $c$, then there is a partition into three monochromatic paths.

Proof. Say colour $c$ is blue. Note that then $P$ covers all of $R \cap G$. Simplify the 3-colouring of $K_{n,n} \setminus V(P)$ to a 2-colouring by merging red and green. The new colouring is not a split colouring since blue has only one component. Hence Theorem 1.1 applies, and we are done.

We are now ready to deal with colourings of type (a) (I).

**Claim 3.7.** If our colouring is of type (a) (I), then there is a partition into three monochromatic paths of distinct colours.

Proof. Observe that since the colouring is 2-local, $G \cap B$ is empty. If $|G \cap \overline{B}| > |R \cap B|$, we can choose a balanced blue path that contains all vertices of $R \cap B$ and apply Claim 3.4. Otherwise, let $P$ be a balanced blue path between $G \cap B$ and $|R \cap B|$ that covers all vertices of $G \cap B$. Since $K_{n,n} - P$ is connected in red, Claim 3.4 applies and we are done.

We can now deal with colourings of type (a) (II). First, let us see how Claim 3.4 implies that we easily find the three paths if one of the components is complete.

**Claim 3.8.** If our colouring is of type (a) (II), and one of the components $R$, $B$, $G$ is complete bipartite, then there is a partition into three monochromatic paths.
Proof. Say $R$ is complete bipartite. Take out a longest balanced red path in $R$. This leaves us either with only $B \cap G$ in the bottom partition class, or with only $\overline{B} \cap G$ in the top partition class. We may thus apply Claim 3.4, after possibly switching top and bottom parts. This gives three paths as desired. \qed

**Claim 3.9.** If our colouring is of type \( \mathbb{II} \), (II), then there is a partition into three monochromatic paths.

**Proof.** Note that if $X \cap Y = \emptyset = X \cap Z$ for any permutation $(X, Y, Z)$ of $(R, B, G)$, then it is easy to find one monochromatic path covering all but one of the sets $R \cap B$, $R \cap G$, $B \cap G$, and we can apply Claim 3.4. So assume in none of these pairs, both sides are empty.

Consider the four sets $R \cap B$, $R \cap G$, $B \cap G$, and $R \cap G$. Observe that after possibly swapping colours green and blue, and/or switching partition classes of $K_{n,n}$, we have one of the following situations:

1. $|R \cap B| \leq |R \cap G|$ and $|R \cap B| \leq |R \cap G|$, or
2. $|R \cap B| \leq |R \cap G|$ and $|R \cap G| \leq |R \cap B|$.

In situation (i) we proceed as follows. Since $R$ is a connected component, and since we assume $R \cap Y = \emptyset = R \cap Z$ does not hold for either permutation $(Y, Z)$ of $(B, G)$, there is a red edge $e_R$ in the graph that is induced by the vertices of either $R \cap B$ or $R \cap G$. Similarly, there is a green edge $e_G$ in the graph that is induced by the vertices $B \cap G$ or $R \cap G$. By Claim 3.8 we may assume that $B$ is not complete bipartite, and thus it is possible to choose $e_R$ and $e_G$ independent (i.e. non-incident).

Extend $e_R$ to a red path $P$ covering all of $R \cap B$, using (apart from $e_R$) only edges from $[R \cap B, R \cap G]$ and from $[R \cap G, R \cap B]$, while avoiding the endvertices of $e_G$, if possible. We can assume that $P$ is balanced, since otherwise, $P$ covers either all of $K_{n,n} \setminus B \cap G$ or all of $K_{n,n} \setminus B \cap G$, and Claim 3.4 applies.

Similarly, if we had to use one of the endvertices of $e_G$, then $P$ either covers all of $R \cap G$ or all of $R \cap G$. In either case we may apply Claim 3.4 and are done. On the other hand, if we could avoid both endvertices of $e_G$ for $P$, then Claim 3.6 applies, and we are done as before.

So let us now assume we have situation (ii). Since at least one of $R \cap B$, $R \cap G$ has a red edge, we can find a balanced red path covering all of $R \cap B \cup R \cap G$. Now Claim 3.4 applies, and we are done. \qed

As noted above, colourings of type \( \mathbb{I} \) are very similar to those of type \( \mathbb{II} \). The subtle difference is that in colourings of type \( \mathbb{II} \), colours are less exchangeable.

**Claim 3.10.** If our colouring is of type \( \mathbb{I} \), then there is a partition into three monochromatic paths.

**Proof.** By symmetry we can assume that $|R \cap B_2| \leq |R \cap B_1|$. Further, we have $|R \cap B_2| < |R \cap B_1| + |R \cap G|$, since otherwise we could find a balanced red path that covers all of $R \cap B_1 \cup R \cap G$, and use Claim 3.4. So we can choose a red balanced red path $P$, alternating between $R \cap B_2$ and $R \cap B_1$. Thus $K_{n,n} - P$ is connected in green and Claim 3.6 applies. \qed

Claims 3.1, 3.5, 3.7, 3.9 and 3.10 together prove Theorem 1.3.
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