Decoupling Gating from Linearity

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Abstract

ReLU neural-networks have been in the focus of many recent theoretical works, trying to explain their empirical success. Nonetheless, there is still a gap between current theoretical results and empirical observations, even in the case of shallow (one hidden-layer) networks. For example, in the task of memorizing a random sample of size $m$ and dimension $d$, the best theoretical result requires the size of the network to be $\tilde{\Omega}(\frac{m^2 d}{T})$ \cite{18}, while empirically a network of size slightly larger than $\frac{m d}{T}$ is sufficient. To bridge this gap, we turn to study a simplified model for ReLU networks. We observe that a ReLU neuron is a product of a linear function with a gate (the latter determines whether the neuron is active or not), where both share a jointly trained weight vector. In this spirit, we introduce the Gated Linear Unit (GaLU), which simply decouples the linearity from the gating by assigning different vectors for each role. We show that GaLU networks allow us to get optimization and generalization results that are much stronger than those available for ReLU networks. Specifically, we show a memorization result for networks of size $\tilde{\Omega}(\frac{m^d}{T})$, and improved generalization bounds. Finally, we show that in some scenarios, GaLU networks behave similarly to ReLU networks, hence proving to be a good choice of a simplified model.

1 Introduction

ReLU neural-networks attracted vast interest in recent years due to their empirical success. This interest has sparked many theoretical works aiming to explain the behavior of learning ReLU networks with gradient-based algorithms. While the theoretical research greatly advanced in the last few years, there are still many open questions and gaps between our theoretical understanding and empirical observations. Even in the case of shallow (one hidden-layer) ReLU networks, current theoretical results do not seem to apply in practice. Take for example the simple task of memorizing a random sample of $m$ examples sampled from a $d$-dimensional Gaussian distribution. As far as we know, the best result in the literature shows that a ReLU neural-network can memorize such sample when the number of neurons is $\tilde{\Omega}(\frac{m^2}{T})$ \cite{18}. Other results assume far worse dependence on the number of examples, requiring the number of neurons to be polynomial in $m$ (refer to Table 1 for a comparison of the results). In practice, on the other hand, a neural network needs only slightly more than $\frac{m d}{T}$ neurons to memorize a sample of size $m$ (observe the experiments in \cite{18}).

To understand why there is such a significant gap between theoretical and empirical results, we briefly review the main theoretical works on ReLU networks. Most theoretical results in this context rely on the concept of Random Features. Random feature schemes are in fact two-layer neural-networks, where the first layer is fixed (after random initialization), and the second is trained. These “networks” have been shown to approximate various kernels, proving to be more efficient than kernel methods \cite{19}. While the original works on random features did not consider ReLU activations, it has been shown that similar results can be given for many network architecture and activation functions.

\footnote{We use $\tilde{\Omega}$ to hide constant and logarithmic factors.}

Preprint. Under review.
The work of [8] shows that when only the last layer of a neural-network is trained, it can approximate functions from the kernel space induced by the activation function and architecture. However, when assuming that only the last layer is trained, the parameter utilization is by definition very low. Indeed, observe that a one hidden-layer network with $k$ hidden neurons and output in $\mathbb{R}$, has $dk$ parameter in the first layer but only $k$ parameters in the second. Hence, training only the last layer is sub-optimal (in terms of parameter utilization) by at least a factor of $d$.

In practice, however, all layers of the neural-network are trained. To this end, there are many recent works analyzing this typical setting, where gradient-descent updates all layers of the network [27, 7, 9, 17, 1, 2, 5, 15, 15, 1, 2, 5, 18, 15, 12]. While the details of each work vary, the key idea in all of these works is the following: when the network is large enough, the weights of the network change very little during the training process. Hence, training a neural-network is “almost” a random features scheme, as the activation are governed completely by their value upon initialization. Since in order to apply such argument the neural-network is required to be rather large, the results obtained in this fashion are also very far from being tight.

One approach for closing the gap between theory and practice is to try harder: apply more complex theoretical tools, perform tedious analysis and hope to get improved results for ReLU networks. Another approach is to study simplified models, that are different than those used in practice, but can nonetheless provide significant insights on ReLU networks. A primary example for such simplified model is linear networks - neural-networks with the linear activation function. Indeed, there is a growing body of work providing various results on optimization of linear networks, showing different convergence properties [20, 11, 14, 3, 4]. While these are very far from neural-networks used in practice, and in fact do not offer any improvement over simple linear classifiers, they exhibit some phenomena that are also observed in ReLU networks. Another example of a simplified model is networks with quadratic activation function ($\sigma(x) = x^2$) or polynomial activation. Although such networks are not used in practice, they are studied in theoretical works [13, 16, 24].

Simplified models are attractive from a theoretical perspective, as they are obviously simpler to analyze. However, it is often not clear whether the results obtained for simple models are relevant for the cases that are of real interest. Linear networks, for example, implement only linear functions and therefore cannot account for learnability of complex non-linear functions learned by ReLU networks. Networks with polynomial activations can implement only low-degree polynomials, and therefore are very different from ReLU network, even from an expressivity point-of-view.

In this work, we introduce a new simplified model that enjoys the best of both worlds: it is simple to analyze, and yet maintains great similarity to ReLU networks. This simple model arises from the observation that the output of a ReLU neuron is a product of a linear function with a gating mechanism. That is, we can write $[x \cdot u]_+ = (1_{x \geq 0}) \cdot (x \cdot u)$. Notice that both the gate and the linear function share the same parameter $u$. Our simplified model is in fact a generalization of the ReLU neuron, in which the gating and the linear function are determined by two different parameters. This gives rise to a neural-network composed of Gated Linear Units (GaLU network), where each unit is a function $f_{u,w}(x) = (1_{x \geq 0}) \cdot (x \cdot w)$. Note that the gradient of this function with respect to the gate $u$ is always zero, so we cannot use gradient-descent to learn the gates. Instead, these gates are randomly initialized, and stay constant throughout the training process.

Since a GaLU network is a generalization of a ReLU network, its expressive power is at least as good as that of a ReLU network. As noted, other simple models are essentially weaker than ReLU networks in terms of expressivity. On the other hand, GaLU networks are indeed simpler to analyze than ReLU networks, since their gates remain fixed throughout the training process. Using this fact allows us to give optimization and generalization results for GaLU networks, that are much stronger than those available for ReLU networks. Specifically, we show that for the memorization task mentioned above, a GaLU network needs only $\Omega(d^2)$ neurons, which is essentially the minimal possible number of neurons needed for this task. Furthermore, we prove generalization results for GaLU network that improve on the equivalent results for ReLU networks. Finally, we show that in some scenarios, GaLU networks exhibit great similarity to ReLU networks. All these results indicate that GaLU networks are a good simplified model for ReLU networks, and we believe they can be used to provide further results that will contribute to our understanding of ReLU networks.

As a final remark, it should be emphasized that we do not claim that ReLU and GaLU networks are equivalent from the optimization point of view. Indeed, in some problems, the fact that in ReLU networks the weight vectors of the gate and linear part are shared steers the optimization problem to
a better direction. What we claim is that GaLU networks are a simpler model, that often performs similarly to ReLU networks and hence can shed light on the performance of ReLU networks as well.

2 GaLU Networks

Consider a neuron with ReLU activation. It is a function \( f_w(x) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) such that:

\[
f_w(x) = \max \{ x^\top w, 0 \} = (1_{x \cdot w \geq 0}) \cdot (x^\top w).
\]

The latter formulation demonstrates that the parameter vector \( w \) plays two roles in determining the value of the neuron. It decides whether the output is 0 or not: it acts as a filter for some gating mechanism. It also determines the value of the neuron, assuming that the neuron is active. In this role the parameter \( w \) acts as the linear weights of the neuron. It is not immediately clear why it makes sense for the two roles to be filled by a single parameter. There are some intuitive explanations, and it is partially motivated by neuroscience, but essentially the justification for using ReLU neurons comes from the practical success of ReLU networks.

This work starts from the assumption that the connection between those two roles doesn’t have a strong theoretical justification. We propose, at least tentatively, to consider a generalization of the ReLU neurons, that we call GaLU neurons (GaLU for “Gated Linear Unit”). A GaLU neuron is a function \( g_{w,u}(x) : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) such that:

\[
g_{w,u}(x) = (1_{x \cdot u \geq 0}) \cdot (x^\top w).
\]

GaLU networks are networks built from GaLU neurons. Note that GaLU is not, strictly speaking, an activation function: activation functions are generally \( \mathbb{R} \to \mathbb{R} \) functions that are composed with a linear function to create a neuron. In this sense, GaLU breaks the common paradigm, but that shouldn’t be taken too seriously: gated units appeared in the deep learning literature before.

GaLU neurons, and therefore GaLU networks, are at least as expressive as their ReLU counterparts, since \( f_w = g_{w,u} \). So every expressivity result on ReLU networks is immediately also an expressivity result on GaLU networks. The expressive power is potentially much greater.

However, this shouldn’t convince anyone that the research of GaLU networks is of any relevance. To anyone who is familiar with deep learning practices, GaLU networks should seem highly suspicious. The parameters \( u \) of the networks cannot be trained using gradient based optimization. As \( \nabla_u g_{w,u}(x) = 0 \) at every point, attempting to use gradient based algorithm would simply leave them intact. As gradient based algorithm are the common optimization tool in deep learning, finding the optimal solution seems to be completely hopeless.

In the following section we show that randomly initializing the gates and fixing them throughout the optimization process is enough. In other words, the gradient based optimization is only important for learning the linear weights, while the random initialization gives the model enough expressive power. In fact, for such training scheme we get optimization and generalization results that are essentially stronger than current results that appear in the literature of ReLU networks.

3 Theoretical Results for GaLU Networks

Consider a GaLU network with a single hidden layer of \( k \) neurons: \( \mathcal{N}(x) = \sum_{j=1}^{k} \alpha_j g_{w_j,u_j}(x) \).

A convenient property of a GaLU neuron is that it is linear in the weights \( w_j \), hence, \( \alpha_j g_{w_j,u_j}(x) = g_{\alpha_j w_j,u_j}(x) \). It means that the network can be rewritten as:

\[
\mathcal{N}(x) = \sum_{j=1}^{k} \alpha_j g_{w_j,u_j}(x) = \sum_{j=1}^{k} g_{\alpha_j w_j,u_j}(x) = \sum_{j=1}^{k} g_{\tilde{w}_j,u_j}(x)
\]

with \( \tilde{w}_j = \alpha_j w_j \). Because we want to optimize over the weights \( w_1, \ldots, w_k, \alpha_1, \ldots, \alpha_k \), we might as well optimize over the reparameterization \( \tilde{w}_1, \ldots, \tilde{w}_k \) without losing expressive power. It means that in a GaLU network of this form, it is sufficient to train the first layer of the network, as the readout layer adds nothing to the expressiveness of the network (as long all the weights are non-zero).
We can then write the optimization problem as:

$$\arg\min_{U, w} L_S(\mathcal{N}) = \arg\min_{U, w} \sum_{i=1}^{m} (\mathcal{N}(x_i) - y_i)^2 = \arg\min_{U, w} \sum_{i=1}^{m} (\Phi_U(x_i)^\top w - y_i)^2$$

As noted, this is a convex optimization problem. We can rewrite this optimization problem as follows: let $X \in \mathbb{R}^{m \times d}$ be the examples matrix (each example is a row in $X$). Denote:

$$\tilde{X}^{(i)} = \begin{bmatrix} (1_{u_1 \geq 0} \cdot x_1) \\ \vdots \\ (1_{u_m \geq 0} \cdot x_m) \end{bmatrix} \in \mathbb{R}^{m \times d}, \quad \tilde{X} = \begin{bmatrix} \tilde{X}^{(1)} \\ \vdots \\ \tilde{X}^{(k)} \end{bmatrix} \in \mathbb{R}^{m \times dk}$$

We can then write the optimization problem as: $\arg\min_{U, w} ||\tilde{X}w - y||^2$.

Now, from standard results for linear regression, we know that if $\text{rank}(\tilde{X}) = m$ (or alternatively, if the minimal singular value of $\tilde{X}$ satisfies $\sigma_{\min}(\tilde{X}) > 0$), then the solution $w^* = (\tilde{X}^\top \tilde{X})^{-1} y$ achieves zero loss. Since for this convex problem, gradient-descent converges to the optimal solution, it is enough to show that $\sigma_{\min}(\tilde{X}) > 0$ to guarantee the convergence to zero loss solution.

Note that the matrix $\tilde{X}$ depends on the examples $X$ and on the randomly initialized gates $U$. In general, we cannot guarantee that it will have full row rank. If there are two identical examples in the sample, then $\tilde{X}$ will have two identical rows, and thus will not be full rank. Similarly, if many of the gates in $U$ are similar, then we may have dependence between columns in the matrix, which will also limit the rank.

To overcome this problem, we assume that the data is “nice” enough, i.e. that it does not contain examples that are very similar. Then, by initializing the gates from a normal distribution, we can confirm that the matrix $X$ will have full rank with high probability. So throughout the paper we will...
While the above analysis applies for cases where the number of parameters is larger than the number of examples, it is also interesting to observe situations where this is not the case. In these cases, we cannot guarantee convergence to zero loss without further assumptions on the labels. On the other hand, we can still give an estimation of the loss, using the results we have shown so far. The assumption \( u_j \sim N(0, I_d) \). To formalize our assumption on the data, we denote:

\[
\lambda(X) = \lambda_{\min}\left(\frac{1}{k}E_{u_1, \ldots, u_k \sim N(0, I_d)}[\bar{X}^\top \bar{X}]\right)
\]

In our theoretical analysis, we assume that \( \lambda(X) > 0 \). Note that this value depends only on the data, and not on the choice of gates. We use the same notation as in [18] (which gives an equivalent definition of \( \lambda(X) \)), and note that many other results for ReLU networks make the same assumption (\([9, 5]\)). In the work of [27], the behavior of \( \lambda(X) \) is studied, and it is shown that typically, it is indeed strictly positive. Given this assumption, we get that for a large enough GaLU network, the matrix \( \bar{X} \) is full rank with high probability:

**Lemma 1** Assume \( \lambda(X) > 0 \) and fix \( \delta > 0 \). If \( k \geq \frac{8\|X\|^2_2}{\lambda(X)} \log(\frac{m}{\delta}) \) then with probability at least \( 1 - \delta \) we have: \( \sigma_{\min}(X)^2 \geq \frac{1}{2} \lambda(X) \).

To apply this lemma, the number of neurons \( k \) needs to be on the order of \( \frac{\|X\|^2_2}{\lambda(X)} \) (up to logarithmic factors). In [18] is shown that when the data is Gaussian, we get that w.h.p. \( \|X\| = O(\sqrt{\frac{d}{m}}) \) and that \( \lambda(X) \) behaves like a constant. Therefore, the number of neurons in this case is \( \Omega(m^2) \):

**Lemma 2** Assume \( x_i \overset{i.i.d.}{\sim} \text{Uni}(S^{d-1}) \) and assume \( u_i \sim N(0, 1) \). Then there exist \( \gamma_1, \gamma_2, c_1, c_2 > 0 \) such that for \( d \leq m \leq c_2 d^2 \), if \( k \geq \frac{64 \pi d}{\gamma_1 \sqrt{c_1}} \log(\frac{m}{\delta}) \), then we have \( \sigma_{\min}(\bar{X}) \geq \frac{\gamma_2}{\sqrt{\gamma_1 \pi}} > 0 \) with probability of at least \( 1 - me^{-\gamma_1 \sqrt{m} - \frac{1}{m} - (2m + 1)e^{-\gamma_2 d} - \delta} \).

Notice that the number of trainable parameters in a GaLU network is \( kd \). Therefore, the number parameters required for our result to hold scales linearly (up to logarithmic factors) with the number of examples. Generally speaking, to fit an arbitrary sample we need the number of parameters to be at least the number of samples, so in this sense our result is almost optimal. To the best of our knowledge, this is the first result that shows convergence to zero loss, when the number of parameters scales only linearly with the number of examples. For comparison, the best result for a ReLU network requires that the number of parameters scales with \( m^2 \). Table 1 shows a comparison between our result and previous optimization results that are directly comparable.

To finish the optimization analysis, we turn to analyzing the behavior of gradient-descent when optimizing a GaLU network. We showed that a very mild over-parametrization is sufficient for \( X \) to be of rank \( m \). Now, in this case, from standard results from convex optimization we get that gradient descent converges linearly to \( w^* \):

**Theorem 1** Assume \( \lambda(X) > 0 \) and fix \( \delta > 0, \epsilon > 0 \). Let \( k \geq \frac{8\|X\|^2_2}{\lambda(X)} \log(\frac{m}{\delta}) \), and assume we initialize a GaLU network with \( k \) neurons. Fix \( \eta = \frac{m}{\|X\|} \log(\frac{1}{\epsilon}) \). Then with probability at least \( 1 - \delta \) on the initialization of the gates, after \( t \geq \frac{2\|X\|^2_2}{\lambda(X)} \log(\frac{\|X\|^2_2\|w_0 - w^*\|^2_2}{m\epsilon}) \) iterations of gradient-descent with step size \( \eta \), the value of the loss function is bounded by \( \epsilon \).

While the above analysis applies for cases where the number of parameters is larger than the number of examples, it is also interesting to observe situations where this is not the case. In these cases, we cannot guarantee convergence to zero loss without further assumptions on the labels. On the other hand, we can still give an estimation of the loss, using the results we have shown so far. The
following theorem estimates the loss achieved by a GaLU network, when the number of parameters is not necessarily large enough to guarantee zero loss:

**Theorem 2** Assume that \( y_1, \ldots, y_n \sim N(0, 1) \). Define the expected squared loss on the training set, for weights \( w \), as \( L_S(w) \). Then we have: 
\[
\mathbb{E}[\min_w L_S(w)] = 1 - \frac{\text{rank}(X)}{m}.
\]

Now, when there are not enough parameters, we get that \( \text{rank}(X) \approx kd \), so the loss behaves like 
\[
1 - \frac{dk}{m}.
\]
Therefore, we get a characterization of the loss which holds in the under-parametrized case. This is shown formally in the following Corollary:

**Corollary 1** There exist some absolute constants \( \gamma_1, \gamma_2, c_1, c_2 > 0 \) such that the following holds: Fix \( \delta > 0 \) and \( k > 0 \), denote \( m' = \lceil kd \frac{c_1^2}{\delta^2} \log^{-1} \left( \frac{c_2^2}{\delta^2} \right) \rceil \), and assume \( d \leq m' \leq c_2d^2 \). Assume \( \mathbf{x}_i \overset{i.i.d.}{\sim} \text{Uni}(S^{d-1}), y_i \sim N(0, 1) \) and assume \( \mathbf{u}_i \sim N(0, 1) \). Then with probability of at least 
\[
1 - m'e^{-\gamma_1\sqrt{m'}} - \frac{1}{m'} - (2m' + 1)e^{-\gamma_2d} - \delta
\]
we have: 
\[
\mathbb{E}[\min_w L_S(w)] \leq 1 - \frac{m'}{m}.
\]

In this section we considered a pure memorization task, where the labels may be independent of the input examples. While this is an interesting task from a theoretical point of view, it is not immediately clear why this result is relevant in practice. However, we note that in many cases memorization is an important tool in solving various complex problems. For example, when the data is highly clustered around a few cluster centers, memorizing the labels of the cluster centers is a simple technique that is often used in practice. We show that our results can also be applied for highly clustered data. In this case we require that the number of neurons scales with the number of cluster centers, and does not depend on the number of examples. For lack of space, we leave this analysis to appendix B.

### 3.2 Generalization in the Over-Parametrized Case

In this section, we give a generalization bound for learning GaLU networks. Before we do so, let us review the main approach used for analyzing ReLU networks:

1. Define a kernel associated with the ReLU network, and observe functions with large-margin in the induced Hilbert space. These functions are learnable via standard kernel learning.
2. Show that the defined kernel can be approximated using a random-features scheme. Hence, large-margin functions can be learned using random features.
3. Show that when training a large enough ReLU network, the weights stay close to the initialization point. Since the weights are randomly initialized, this shows that a ReLU network essentially implements a random-features scheme.

We take a similar approach when analyzing the generalization of GaLU networks. We study the kernel associated with the GaLU network, and show that a GaLU network can learn functions from the Hilbert space induced by this kernel. In fact, we observe that the kernel of the GaLU network is the same kernel used for the analysis of ReLU network. That said, notice that there is a crucial difference between the analysis of GaLU networks and that of ReLU networks. While for the analysis of ReLU networks it is essential to show that the network’s weights stay close to their initial value, this property is not required for GaLU networks. Since the gates of GaLU stay fixed through the entire training process, the non-linear part of the network is defined upon initialization, and does not change. Therefore, step 3 in the scheme above becomes trivial for GaLU networks.

We begin with a few definitions. To simplify the analysis, we consider the normalized GaLU network:

\[
\mathcal{N}(x) = \frac{1}{\sqrt{k}} \sum_{j=1}^{k} y_{w,j} \Phi_{L,U}(x) = \frac{1}{\sqrt{k}} \Phi_{L,U}(x)^\top w
\]

We define the following kernel:

\[
\kappa(x, y) = \mathbb{E}_{u \sim N(0, I_d)} \left[ (1_{u^\top x \geq 0} \cdot (1_{u^\top y \geq 0})(x, y) \right] = \left( \frac{1}{2} - \frac{\arccos(\langle x, y \rangle)}{2\pi} \right) \langle x, y \rangle
\]

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We can define similarly the equivalent ReLU network:

These results depend on some convenient properties of GaLU neurons, that make their analysis much simpler than their ReLU counterparts. However, since ReLU networks are extremely popular, we show two results in this context. First, we observe that failure cases of GaLU, i.e. cases where the optimization of a GaLU network fails upon initialization, immediately imply that a ReLU network will fail on the same data, and vice versa. Second, we show that in some cases, the best GaLU network with fixed random gates is competitive with the best ReLU network.

So we can think of a GaLU network as a random-features scheme approximating the kernel $\kappa$. Let $H_\kappa$ be the RKHS induced by this kernel, and denote $\| \cdot \|_{\kappa}$ the norm of $H_\kappa$. We denote $B_\kappa(M) = \{ f \in H_\kappa : \| f \|_{\kappa} \leq M \}$, the set of function in $H_\kappa$ with norm bounded by $M$. Let $D$ be a distribution over $\mathcal{X} \times [-1, 1]$ that is separable by $B_\kappa(M)$, i.e., there is $f^* \in B_\kappa(M)$, such that if $(x, y) \sim D$ then $y = f^*(x)$ with probability 1. Then we have the following generalization bound:

**Theorem 3** Assume $\lambda(X) > 0$, and fix $\delta > 0$. Let $k \geq \left( \frac{m}{M^2 \lambda(X)} + 1 \right)^2 \log(m/\delta)$, with probability at least $1 - 2\delta$, the generalization error of the GaLU network is bounded by

$$C \left( \frac{2M^2 \log^3 m + (2M^2 + \sqrt{2}M) \log(1/\delta)}{m} \right).$$

Compare this result to the generalization bound presented in the recent work by [5]. In this result, generalization bound is obtained when the network size grows with $m^7$, while our bound requires a more modest (yet admittedly large) dependence on the number of examples. Furthermore, our generalization bound decays with $\frac{1}{m}$, where the bound shown in [5] decays with $\frac{1}{\sqrt{m}}$.

## 4 Relation to ReLU

So far, we showed various results analyzing optimization and generalization of GaLU networks. These results depend on some convenient properties of GaLU neurons, that make their analysis much simpler than their ReLU counterparts. However, since ReLU networks are extremely popular, and achieve remarkable performance empirically, it would be beneficial to account for the relation between GaLU and ReLU networks. In this section, we aim to understand to what extent results shown for GaLU networks can be applied for ReLU, and vice-versa. As in any algorithmic research field, there are two types of results on ReLU networks: positive results, that show cases where ReLU networks succeed in a given task, and negative results, that present interesting failure cases. To this end, we wish to show that for both positive and negative results, GaLU networks are a good proxy for ReLU networks. We show two results in this context. First, we observe that failure cases of GaLU, i.e. cases where the optimization of a GaLU network fails upon initialization, immediately imply that a ReLU network will fail on the same data, and vice-versa. Second, we show that in some cases, the best GaLU network with fixed random gates is competitive with the best ReLU network.

We begin by reviewing some notations that will allow us to compare GaLU networks to ReLU networks. Given a set of weights $W = \{w_1, \ldots, w_k\}$, a set of gates $U = \{u_1, \ldots, u_k\}$ and a set of scalars $\alpha = \{\alpha_1, \ldots, \alpha_k\}$, a normalized GaLU network is defined as:

$$N^{G}_{W, U, \alpha}(x) = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \alpha_i g_{w_i, u_i}(x)$$

We can define similarly the equivalent ReLU network:

$$N^{R}_{U, \alpha}(x) = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \alpha_i f_{u_i}(x) = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \alpha_i g_{u_i, u_i}(x)$$

### 4.1 Failure of GaLU vs. Failure of ReLU

In this part we use the hinge loss $\ell(y, \hat{y}) = \max\{1 - y\hat{y}, 0\}$, instead of the square loss, to simplify the analysis. Notice that the optimization results in section 3 depend on the data being “nice” enough (which is captured by the assumption that $\lambda(X) > 0$). However, we might encounter extreme cases where the data doesn’t behave “nicely”. These cases can cause the optimization to fail, and achieve large train loss. In fact, in some extreme cases the failure may happen upon initialization. In these cases, the gradient will be very small with high probability upon the initialization. Refer to [23][22].
for examples of such cases. We show that in these failure cases, the behavior of GaLU and ReLU are similar: GaLU fails if and only if ReLU fails.

**Theorem 4** Let $N^R_{W, \alpha}$ be a ReLU network, and let $N^G_{W, U, \alpha}$ be a GaLU network, both initialized such that $N^R_{W, \alpha}(B_1), N^G_{W, U, \alpha}(B_1) \subseteq [-1, 1]$. Then $||\frac{\partial}{\partial W} L_S(N^G_{W, U, \alpha})|| \leq \epsilon$ with probability $1 - \delta$ upon initialization if and only if $||\frac{\partial}{\partial U} L_S(N^G_{U, \alpha})|| \leq \epsilon$ with probability $1 - \delta$ upon initialization.

4.2 GaLU Networks are Competitive with Large ReLU Networks

As mentioned, various previous results show that when training a large ReLU network, gradient-descent reaches a stationary point with zero loss with high probability.\cite{22, 27, 9, 12, 1, 5, 18, 15, 12}

All of these results rely on the key observation that when the network is large enough, the weights of the network barely change from their initial value. In this part we show that if this is the case, i.e. if the value of the weights of the ReLU network changes very little, then the best GaLU network (with randomly initialized gates) achieves loss that is competitive with the best ReLU network.

To formalize this, let $D$ be a distribution over $\mathcal{X} \times \mathcal{Y}$ and assume we initialize $u_1, \ldots, u_k \sim N(0, I_d)$. Fix some $L$-Lipschitz loss $\ell : \mathbb{R} \times \mathcal{Y} \to \mathbb{R}$, and observe the loss on the distribution $L_D(f) = \mathbb{E}_{(x, y) \sim D} [\ell(f(x), y)]$. Let $N^{G}_{W^*, U, \alpha^{**}}$ be the optimal GaLU network with respect to $L_D$ (with gates $u_1, \ldots, u_k$ fixed), so $W^*, \alpha^{**} = \arg\min_{W, \alpha} L_D(N^{G}_{W, U, \alpha})$. Let $N^{R}_{U^*, \alpha^{*}}$ be the optimal ReLU network with respect to $L_D$ satisfying that $\|u^*_i - u_i\| \leq \epsilon$ for all $i \in [k]$ (small distance from initialization), so $U^*, \alpha^{*} = \arg\min_{V, \alpha} \|V - u_i\| \leq \epsilon L_D(N^{R}_{V, \alpha})$. Then we get:

**Theorem 5** Fix $\delta > 0$, let $k \geq \frac{3 \sqrt{3 \log(\frac{1}{\delta})}}{\sqrt{2\pi}}$, and assume we assume $d > \log(2k/\delta)$. Then with probability at least $1 - \delta$, we have:

$$L_D(N^{G}_{W^*, U, \alpha^{**}}) \leq L_D(N^{R}_{U^*, \alpha^{*}})+ L \max_{i} \|u^*_i\|$$

This result means that GaLU networks with randomly initialized gates are competitive with ReLU networks with small distance from initialization. Therefore, GaLU networks are indeed a good simplified model for ReLU networks, when the distance from initialization is small.

5 Discussion

In this paper we introduced a new neural-network model - the GaLU network. Since optimization of a GaLU network is a convex problem, these networks allow us to get strong theoretical results with much simpler tools. Indeed, we showed theoretical results for GaLU networks that are significantly better than equivalent results in the literature of ReLU networks. Furthermore, since current analysis of ReLU networks assumes that the weights of the network stay close to their initial value, we note that in some sense current ReLU analysis is implicitly an analysis of GaLU networks.

However, we do not claim that GaLU networks fully capture the behavior of ReLU networks, nor do we claim that they are a preferable model to use in practice. Indeed, we perform various experiments, covering cases where the behavior of GaLU and ReLU networks is similar, but also cases where they differ. Due to the lack of space, these experiments are detailed in appendix D. What we do claim is that a GaLU network is a better simplified model, compared to other simplified models that appear in the literature, such as linear networks or networks with polynomial activation. These simplified models allow theoretical research to gain insights on various aspects of neural-networks, and we believe that GaLU networks would prove to be another useful tool in the theoretician’s toolbox.

Finally, we note that the scope of this work is limited only to the analysis of one-hidden layer networks with output in $\mathbb{R}$. While this is a rich research area, there is still much more to say about neural-networks in general. Specifically, the analysis of shallow networks with vector-valued output, as well as the research of deep networks and convolutional networks, is not covered in this paper. We leave these promising research directions to future work.

**Acknowledgements:** This research is supported by the European Research Council (TheoryDL project).
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A Proofs of section 3.1

Proof of Lemma 1 We use the following notations:

\[ H(i) = \bar{X}(i)(\bar{X}(i))^T; \quad H = \bar{X}\bar{X}^T = \sum_{i=1}^{k} H(i) \]

Notice that \( \lambda(X) = \lambda_{\text{min}}(\mathbb{E}[H(i)]) \). Denote \( R := \|X\|^2 \), and observe that we have: \( \lambda_{\text{max}}(H(i)) \leq \|X\|^2 = R \), so \( H(i) \) are i.i.d. random positive semi-definite self-adjoint matrices with bounded norm. Notice that \( \mu_{\text{min}} := \lambda_{\text{min}}(\mathbb{E}[H(i)]) = k\lambda(X) \). Now, we can use matrix Chernoff bound (26) and get that:

\[
\mathbb{P}\left[ \lambda_{\text{min}}(\sum_{i=1}^{k} H(i)) \leq (1 - \epsilon)\mu_{\text{min}} \right] \leq m \cdot \left( \frac{e^{-\epsilon}}{(1 - \epsilon)^{1-\epsilon}} \right)^{\mu_{\text{min}}/R} = m \cdot \left( \frac{e^{-\epsilon}}{(1 - \epsilon)^{1-\epsilon}} \right)^{k\lambda(X)/R}
\]

Now, if we take \( \epsilon = \frac{1}{2} \), we get:

\[
\mathbb{P}\left[ \lambda_{\text{min}}(H) \leq \frac{k}{2} \lambda(X) \right] = \mathbb{P}\left[ \lambda_{\text{min}}(\sum_{i=1}^{k} H(i)) \leq \frac{k}{2} \lambda(X) \right] \leq m \cdot \left( \frac{e^{-\epsilon}}{(1 - \epsilon)^{1-\epsilon}} \right)^{k\lambda(X)/R} \leq \delta
\]

Proof of Lemma 2 A recent work gives the following bound on \( \lambda(X) \) (Lemma 6.4 in [18]):

\[
\lambda(X) \geq \frac{1}{2\pi} \sigma_{\text{min}}^2 (X*X)
\]

Where \( X*X \) is the Khatri-Rao product. Following a similar proof to Corollary 2.2 in [18], we have:

\[
\|X\| \leq 2\sqrt{\frac{m}{d}}
\]

with probability of at least \( 1 - e^{-\gamma_2d} \). We also have:

\[
\sigma_{\text{min}}(X*X) \geq c_1
\]

with probability of at least \( 1 - ne^{-\gamma_1\sqrt{m}} - \frac{1}{m} - 2me^{-\gamma_2d} \). Assuming that both of these hold, we get from what we have shown:

\[
\mathbb{P}\left[ \lambda_{\text{min}}(H) \leq \frac{k\sigma_{\text{min}}^2}{4\pi} \right] \leq m \cdot \left( \frac{e^{-\epsilon}}{(1 - \epsilon)^{1-\epsilon}} \right)^{k\lambda(X)/2m} \leq m \cdot \left( \frac{e^{-\epsilon}}{(1 - \epsilon)^{1-\epsilon}} \right)^{k\lambda(X)/2m} \leq \delta
\]

Observing that \( \sigma_{\text{min}}(\bar{X}) = \sqrt{\lambda_{\text{min}}(H)} \) and using union bound completes the proof.

Proof of Theorem 1 Denote \( H = \bar{X}\bar{X}^T \) and \( H(i) = \bar{X}(i)(\bar{X}(i))^T \). Now, assuming rank \( (\bar{X}) = m \), observe the objective of the optimization of the GaLU network. From what we developed previously, this objective is given by:

\[
F(w) = \frac{1}{2m} \|\bar{X}w - y\|^2 = \frac{1}{2m} (w^T H w - 2y^T \bar{X}w + ||y||^2)
\]
Since $\bar{X}$ is full-rank, we can define the optimum of $F$ by $w^* = \bar{X}^\top H^{-1} y$, and we get: $F(w^*) = 0$.

Notice that $\lambda_{\max}(H) = \lambda_{\max}(\sum_{i=1}^k H^{(i)}) \leq k \|X\|^2$. From [1] with probability at least $1 - \delta$ we have: $\lambda_{\min}(H) \geq \frac{\delta}{2} \lambda(X)$. Therefore, applying Theorem 6.3 in [10] gives:

$$||w_t - w^*||^2 \leq \exp\left(-\frac{t\lambda(X)}{2\|X\|^2}\right) ||w_0 - w^*||^2$$

Now, we have $\nabla^2 F(w) = \frac{1}{m} H$, so:

$$||\nabla F(w_t)|| = ||\nabla F(w_t) - \nabla F(w^*)||$$

$$\leq ||\nabla^2 F(w)|| ||w_t - w^*||$$

$$= \frac{H}{m} ||w_t - w^*||$$

$$\leq \frac{k ||X||^2}{m} ||w_t - w^*||$$

Using the fact that $F$ is convex, we get:

$$F(w_t) = |F(w_t) - F(w^*)| \leq \|\nabla F(w_t)|| ||w_t - w^*|| \leq \frac{k ||X||^2}{m} ||w_t - w^*||^2$$

Using what we previously showed, we get that w.p at least $1 - \delta$ we have:

$$F(w_t) \leq \exp\left(-\frac{t\lambda(X)}{2\|X\|^2}\right) \frac{k ||X||^2}{m} ||w_0 - w^*||^2 \leq \epsilon$$

**Proof** of Theorem 2. Every vector $y = (y_1, \ldots, y_m) \in \mathbb{R}^m$ can be decomposed to a sum $y = a + b$ where $a$ is in the span of the columns of $X$ and $b$ is in the null space of $X$. It follows that $\min_w L_S(w) = \|b\|^2/m$. The claim follows because if $y \sim N(0, I_m)$ then the expected value of $\|b\|^2$ is $m - \text{rank}(X)$.

**Proof** of Corollary 1. Observe the sub-sample $S' \subseteq S$, which is simply the first $m'$ examples from $S$. Denote $X' \in \mathbb{R}^{m' \times d}$ the corresponding sub-matrix of $X$, and $\bar{X}' \in \mathbb{R}^{m' \times dk}$ the corresponding sub-matrix of $\bar{X}$. Then, from Lemma [2] with probability at least $1 - m'e^{-\frac{1}{m'} - \frac{1}{m'} - (2m' + 1)e^{-\gamma d} - \delta}$, the matrix $\bar{X}'$ has maximal rank, so $\text{rank} \bar{X}' = m'$. Therefore, it must hold that $\text{rank} \bar{X} \geq m'$, so the result follows from Theorem 2.

### B Highly Clustered Piecewise Linear Data

In the optimization analysis presented in section 3.1, we saw that GaLU networks can achieve zero training loss when the number of parameters grows with the number of examples. However, in practice neural-networks can achieve low train error with relatively small amount of parameters. To account for this gap, observe that in our optimization analysis we did not depend on the value of the labels. That is, the same analysis can be applied for random labels and for labels that depend on the input examples. Naturally, we would like to show that when the labels depend on the inputs, we can get better guarantees from an optimization point of view. In this section, we analyze a model where the data is sampled from a distribution over $n$ clusters, such that on each cluster the label is generated by a distinct linear function. In such case, we show that to reach zero loss, the number of neurons in the network depends only on the number of clusters, with no dependency on the number of examples. This can potentially give much better bounds on the required network width under this model.

We start by formalizing our model. We are going to consider a distribution that is very clustered around $n$ cluster centers, and that within each cluster, the label $y$ is a linear function of the input $x$. 
Fix \( n \in \mathbb{N} \) to be the number of clusters, \( r \in \mathbb{R} \) the radius of each cluster, and \( n \) linear transformations \( \ell_1, \ldots, \ell_n \in \mathbb{R}^d \). Let \( v_1, \ldots, v_n \in \mathbb{S}^{d-1} \) be \( n \) cluster centers. Let \( H \in \mathbb{R}^{n \times n} \) be such that \( H_{ij} = \frac{1}{2} - \frac{\arccos(v_i \cdot v_j)}{2\pi} \), and denote \( \mu = \lambda_{\min}(H) \). We shall assume \( \mu > 0 \) (and we will soon justify this assumption). Pick \( \delta > 0 \) and \( k \geq \frac{8n}{\mu} \log \left( \frac{\mu}{\delta} \right) \). Denote \( r = \frac{\delta}{nk\sqrt{d}} \).

Define the distribution \( D \) over \( \mathbb{S}^{d-1} \times \mathbb{R} \) by the following random process. First, pick \( q \sim Q \) where \( Q \) is some distribution over \([n]\). Then, pick \( x \sim D_q \), where \( D_q \) is a distribution over \( \mathbb{S}^{d-1} \) such that \( \Pr(\|x - v_q\|_2 > r) = 0 \). Finally, return \((x, x^\top \ell_q)\).

For this model, we get much better results than in the general case. Specifically, we show that when the number of neurons grows with the number of clusters, a GaLU network achieves zero loss. Notice that in the previous results, we required that the number of parameters grows with the number of examples, which typically can be much larger than the number of cluster centers. This is captured in the following theorem:

**Theorem 6** Pick \( \epsilon > 0 \), and set \( m = \frac{e^{nd\log(2/d) + \log(2/\delta) \epsilon}}{\epsilon} \) (\( \epsilon \) is a global constant). Let \( W^* \) be the result of training a GaLU network with \( k \) neurons on an i.i.d. sample from \( D \). Then, with probability \( \geq 1 - 3\delta \), the training loss of the network on the sample is 0, and the test loss is \( \leq \epsilon \).

Note that from the previous lemma, we get that the value of \( k \) is governed by \( \frac{\pi}{\mu} \). The value of \( \mu \) depends only on the choice of the cluster centers \( v_1, \ldots, v_n \), and we would like to show that it is typically not too small. In fact, we will show that when the dimension is large enough, namely \( d = \Omega(n^2) \), and when \( v_i \)-s are chosen randomly, then \( \mu \) is a constant.

**Lemma 3** Fix \( \delta > 0 \). Assume \( d \geq \frac{2\pi^2}{\epsilon} \log \left( \frac{2\pi^2}{\epsilon} \right) \), and assume we choose \( v_i \sim \text{Uni}(\{\pm \frac{1}{\sqrt{d}}\}^d) \). Then with probability at least \( 1 - \delta \) we have that \( \mu \geq \frac{1}{8} \).

### B.1 Proof of Theorem 6

The theorem follows from the following deterministic claim. Let \( m_1, \ldots, m_n \in \mathbb{N} \) be \( n \) cluster sizes, and for every \( i \in [n] \) let \( S_i = \{(x_{ip}, y_{ip})\}_{p=1}^{m_i} \) be such that for every \( p \in [m_i] \), \( \|x_{ip} - v_i\| < r \) and \( y_{ip} = x_{ip}^\top \ell_i \). Define \( S = \bigcup_{i=1}^n S_i \). In addition, pick \( q \in [n] \) and \( \tilde{x}, \tilde{y} \) such that:

1. \( \tilde{x} \in \text{span}\{x_{ip}\}_{p=1}^{m_q} \).
2. \( \|\tilde{x} - v_q\| < r \).
3. \( \tilde{y} = \tilde{x}^\top \ell_q \).

**Theorem 7** W.p. \( \geq 1 - 2\delta \) over the choice of gates, there is an exact solution when training a GaLU network with \( k \) neurons on \( S \). Moreover, any such solution would correctly predict the example \((\tilde{x}, \tilde{y})\).

Let \( u_1, \ldots, u_k \overset{i.i.d.}{\sim} \text{Uni}(\mathbb{S}^{d-1}) \) be the gates of the network. Let \( A = [a_{ij}] \in \mathbb{R}^{n \times k} \) be such that 

\[
a_{ij} = \mathbf{1}_{u_j^\top v_i \geq 0}.
\]

For every \( i \in [n] \), Let \( X_i = \begin{bmatrix}
x_{i1}^\top \\
x_{i2}^\top \\
\vdots \\
x_{im_i}^\top
\end{bmatrix} \).

**Lemma 4** With probability of at least \( 1 - \delta \), \( \text{rank}(A) = n \).

**Proof** We shall show the stronger claim \( \sigma_{\min}(A)^2 > \frac{\mu}{2} \). Denote \( B_i = A_i^\top A_i \in \{0, 1\}^{n \times n} \), and notice that \( B_i \) are i.i.d. random self-adjoint positive semi-definite matrices. Note that \( \|B_i\| \leq n \), and that \( \lambda_{\min}(\sum_{i=1}^k E[B_i]) = k\mu \). Therefore, by using matrix Chernoff bound, we get that:

\[
P\left[ \lambda_{\min}(\sum_{i=1}^k B_i) \leq (1 - \epsilon)k\mu \right] \leq n \cdot \left[ \frac{e^{-\epsilon}}{(1 - \epsilon)^{(1-\epsilon)}} \right]^{\frac{\mu}{2}}
\]
Taking $\epsilon = \frac{1}{2}$ we get that:
\[
P \left[ \lambda_{\min} \left( \sum_{i=1}^{k} B_i \right) \leq \frac{k}{2} \mu \right] \leq n \cdot \left( \frac{\epsilon}{2} - \frac{\mu}{2} \right) \leq \delta
\]

Since we have $A^\top A = \sum_{i=1}^{k} B_i$ and $\sigma_{\min} (A)^2 = \lambda_{\min} (A^\top A)$, this completes the proof.

The next two lemmas show that for this model, none of the $n$ clusters are split by any of the $k$ filters, with probability $> 1 - \delta$.

**Lemma 5** Let $u \sim \text{Uni}(S^{d-1})$, $x \in \mathbb{S}^{d-1}$ and $r > 0$. Define $z = \langle u, x \rangle$. Then $\Pr(-r \leq z \leq r) \leq r \sqrt{d}$.

**Proof** Let $t = \frac{z+1}{2}$. It is well known that $t \sim \text{Beta} \left( \frac{d-1}{2}, \frac{d-1}{2} \right)$. We shall start by bounding the Beta function at $B \left( \frac{d-1}{2}, \frac{d-1}{2} \right)$ with the following version of Stirling’s approximation:
\[
B \left( \frac{d-1}{2}, \frac{d-1}{2} \right) = \frac{\Gamma \left( \frac{d-1}{2} \right)^2}{\Gamma (d-1)} \geq \frac{\left( \sqrt{\frac{4\pi}{d-1}} \left( \frac{d-1}{2e} \right)^{\frac{d-1}{2}} \right)^2}{\sqrt{\frac{2\pi}{d-1}} \left( \frac{d-1}{2e} \right)^{d-1} e^{-\frac{1}{2}}} = 2 \sqrt{\frac{2\pi}{d-1}} \left( \frac{1}{2} \right)^{d-2} e^{-\frac{1}{12(d-1)}}
\]

And so,
\[
\Pr (-r \leq z \leq r) = \Pr \left( \frac{1-r}{2} \leq t \leq \frac{1+r}{2} \right) \leq t \left( \frac{1}{2} \right)^{d-3} \left( \frac{1}{2} \right)^{d-3} \leq \frac{t}{B \left( \frac{d-1}{2}, \frac{d-1}{2} \right)} \leq \frac{r}{\sqrt{\frac{2\pi}{d-1}} \left( \frac{1}{2} \right)^{d-2} e^{-\frac{1}{12(d-1)}}} = r \sqrt{d} \frac{2}{\sqrt{2\pi} e^{-\frac{1}{12(d-1)}}} \leq r \sqrt{d}
\]

Where the last inequality is easily verified numerically.

**Lemma 6** Fix $i \in [n]$, and let $u \sim \text{Uni}(S^{d-1})$. Then, with probability of at least $1 - \frac{1}{k^2} \delta$, $\forall (x, y) \in S_i$, $\text{sign} (u^\top x) = \text{sign} (u^\top y)$.
Lemma 8 There is at least one solution to the above equation set. Because \( \operatorname{rank}(A) = n \), there is a matrix \( B = [b_{ij}] \in \mathbb{R}^{n \times k} \) such that \( AB^\top = I_n \). Equivalently, for every \( i, i' \in [n], \sum_{j=1}^k b_{ij} a_{ij} = 1_{i=i'} \). For every \( j \in [k] \), let \( w_j = \sum_{i'=1}^n b_{ij} \ell_{i'} \). Now, for every \( i \in [n] \),

\[
\sum_{j=1}^k a_{ij} X_i w_j = X_i \left( \sum_{j=1}^k a_{ij} w_j \right) = X_i \left( \sum_{j=1}^k a_{ij} \left( \sum_{i'=1}^n b_{ij} \ell_{i'} \right) \right) = X_i \sum_{i'=1}^n \left( \sum_{j=1}^k a_{ij} b_{ij} \right) \ell_{i'} = X_i \sum_{i'=1}^n 1_{i=i'} \ell_{i'} = X_i \ell_i
\]

Lemma 9 Every exact solution \( w_1, \ldots, w_k \) gives the correct prediction for \( \hat{x} \).
\textbf{Proof} Because $w_1, \ldots, w_k$ is an exact solution,
\[\sum_{j=1}^k a_{qj} X_q w_j = X_q \left( \sum_{j=1}^k a_{qj} w_j \right) = X_q \ell_q\]
Because $\tilde{x}^\top \in \text{rowspan}(X_q)$,
\[\sum_{j=1}^k a_{qj} \tilde{x}^\top w_j = \tilde{x}^\top \left( \sum_{j=1}^k a_{qj} w_j \right) = \tilde{x}^\top \ell_q = \tilde{y}\]
As required.

\section*{B.2 Proof of Lemma \[3\]}
\textbf{Proof} Denote $\sigma(x) := 1_{x \geq 0}$. Let $\epsilon = \frac{1}{2n}$. Fix some $i \neq j$. Notice that using Hoeffding’s inequality, we get that:
\[\Pr \left[ |\langle v_i, v_j \rangle| \geq \epsilon \right] \leq 2 \exp\left(-8d \epsilon^2 \right) \leq \frac{\delta}{n^2}\]
Using the union bound we get that with probability at least $1 - \delta$, for all $i \neq j$, we have $|\langle v_i, v_j \rangle| \leq \epsilon$.
We assume that this property holds.
Now, we have:
\[H_{i,j} = E_{u \sim N(0, I_d)} [\sigma(u^\top v_i)\sigma(u^\top v_j)] = \frac{1}{2} - \frac{\arccos(v_i, v_j)}{2\pi}\]
Therefore, $H_{i,i} = \frac{1}{2}$, and also:
\[|H_{i,j} - \frac{1}{4}| = \left| \frac{1}{4} - \frac{\arccos(v_i, v_j)}{2\pi} \right| = \frac{1}{2\pi} \left| \frac{\pi}{2} - \arccos(v_i, v_j) \right| \leq \frac{1}{2\pi} \frac{\pi}{2} |\langle v_i, v_j \rangle| \leq \frac{1}{4} \epsilon\]
Where we use $|\arccos(x) - \frac{x}{2}| \leq \frac{\epsilon}{2}$. Denote $T = \frac{1}{4} I + \frac{1}{2} 11^\top$, and we therefore have:
\[\|H - T\| \leq \|H - T\| \leq \frac{n \epsilon}{4} \leq \frac{1}{8}\]
Notice that $T$ is invertible, and $T^{-1} = 4I - \frac{1}{4d+1} 11^\top$ (this is easy to check). By simple calculation we get that $\|T^{-1}\| = 4$ (see below). Therefore, we get that $\|H - T\| \leq \|T^{-1}\|^{-1}$, so $H$ is invertible, and we have:
\[\left\| H^{-1} \right\| = \left\| \sum_{j=0}^\infty (T^{-1}(T - H)) \right\| \leq \|T^{-1}\| \sum_{j=0}^\infty (\|T^{-1}\| |T - H|)^j \leq \|T^{-1}\| \sum_{j=0}^\infty (\frac{1}{2})^j \leq 8\]
Therefore $\mu_{\min} = \lambda_{\min}(H) \geq \frac{1}{8}$.

\section*{Lemma \[10\]}
\[\|I - \frac{1}{d+1} 11^\top\| = 4.\]

\textbf{Proof} Let $x$ be a unit vector. Denote $x = x_{11^\top} + x'$ such that $x_{11^\top} \in \text{span}\{1\} \text{ and } \langle x', 1 \rangle = 0$.
Now,
\[\left\| (I - \frac{1}{d+1} 11^\top) x \right\| = \left\| (I - \frac{1}{d+1} 11^\top)x_{11^\top} + x' \right\| = \left\| x' - \frac{d}{d+1} x_{11^\top} \right\| \leq \left\| x' \right\| + \frac{d}{d+1} \left\| x_{11^\top} \right\| \leq \left\| x' \right\| + \left\| x_{11^\top} \right\| = \left\| x \right\| = 1\]
With equality iff $\left\| x_{11^\top} \right\| = 0$. 

C Proof of Theorem [3]

Proof of Theorem [3] Let $S = \{(x_1, y_1), \ldots, (x_m, y_m)\}$ where $S \sim \mathcal{D}^m$. We denote $H^\infty$ the Gram matrix such that $H_{i,j}^\infty = \kappa(x_i, x_j)$. Observe that $\lambda_{\min}(H^\infty) = \lambda(X) > 0$, so $H^\infty$ is full-rank. Define $\varphi: \mathcal{X} \rightarrow \mathcal{H}_\infty$ such that $\varphi(x) = \kappa(\cdot, x)$. Observe the minimization problem:

$$
\hat{f} = \arg \min \sum_{i=1}^m \frac{1}{2}(f(x_i) - y_i)^2
$$

s.t. $f(x) = \sum_{j=1}^m \hat{w}_j \varphi(x_i)$

The solution to this minimization problem is given by:

$$
\hat{w} = (H^\infty)^{-1} y
$$

Now, calculating the norm of $\hat{f}$ we get:

$$
\|\hat{f}\|_\kappa^2 = \left( \sum_{i=1}^m \hat{w}_i \varphi(x_i), \sum_{j=1}^m \hat{w}_j \varphi(x_j) \right)
$$

$$
= \sum_{i,j=1}^m \hat{w}_i \hat{w}_j \langle \varphi(x_i), \varphi(x_j) \rangle
$$

$$
= (\hat{w})^T H^\infty \hat{w} = y^T (H^\infty)^{-1} y
$$

Observe that $\hat{f}$ is the projection of $f^*$ onto the space spanned by $\{\varphi(x_1), \ldots, \varphi(x_m)\}$ (since the loss of this projection on the space must be zero, and the only choice for such function is $\hat{f}$). Therefore:

$$
\sqrt{y^T (H^\infty)^{-1} y} = \|\hat{f}\|_\kappa \leq \|f^*\|_\kappa \leq M
$$

Now, observe the GaLU optimization problem (where $\hat{X}, \hat{H}$ are as defined previously):

$$
\hat{w}^* = \arg \min \sum_{i=1}^m \frac{1}{2}(\hat{X}_i w - y_i)^2
$$

The solution is given by:

$$
\hat{w}^* = \hat{X}(\hat{X} \hat{X}^T)^{-1} y
$$

So we have:

$$
\|\hat{w}^*\|^2 = y^T (\hat{X} \hat{X}^T)^{-1} y = y^T \hat{H}^{-1} y
$$

To finish the argument, we need to relate $H^{-1}$ to $(H^\infty)^{-1}$. To do this, we start by bounding $\|H - H^\infty\|$. Recall that we define $H = \frac{1}{k} \sum_{i=1}^k H^{(i)}$, and that $H^\infty = E[H] = E[H^{(i)}]$. We also have $\|H^{(i)}\| \leq \|X\|^2 := R$ and therefore $\|H^\infty\| \leq \|H^{(i)}\| \leq R$. Now, denote $Y^{(i)} = \frac{1}{R} H^{(i)} - \frac{1}{R} H^\infty$ so we have $\|Y^{(i)}\| \leq \frac{2}{R} R$. Also, we have $E[Y^{(i)}] = 0$, and $Y^{(i)}$ are i.i.d random self-adjoint matrices, so we can use Matrix Hoeffding inequality and get for every $r > 1$:

$$
P\left[\|H - H^\infty\| \geq \frac{1}{r} \lambda(X)\right] = P\left[\left\|\sum_{i=1}^k Y^{(i)}\right\| \geq \frac{1}{r} \lambda(X)\right]
$$

$$
\leq m \cdot \exp\left(-\frac{k \lambda(X)^2}{32r^2 R^2}\right)
$$

Therefore, if we take $k \geq \frac{32r^2 \|X\|^4}{\lambda^2(X)} \log(m/\delta)$ we get that the above happens w.p at most $1 - \delta$. So from now we assume that $\|H - H^\infty\| \leq \frac{1}{r} \lambda(X)$. 

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Now, recall the following property: for two square matrices $A, B$ such that $A$ is invertible, if $\|B - A\| \leq \|A^{-1}\|^{-1}$ then $B$ is invertible and $B^{-1} = A^{-1} \sum_{n=0}^{\infty} (B - A) A^{-1}$. In our case, we know (assume) that $H^\infty$ is invertible, and we showed that w.h.p:

$$\|H - H^\infty\| \leq \frac{1}{r} \lambda(X) = \frac{1}{r} \lambda_{\min}(H^\infty) = \frac{1}{r} \|H^\infty\|^{-1}$$

therefore we get:

$$\|H^{-1} - (H^\infty)^{-1}\| = \| (H^\infty)^{-1} \left( \sum_{n=0}^{\infty} (H - H^\infty) (H^\infty)^{-1} \right) \|$$

$$\leq \| (H^\infty)^{-1} \| \sum_{n=1}^{\infty} (\|H - H^\infty\| \|H^\infty\|^{-1})^n$$

$$\leq \| (H^\infty)^{-1} \| \sum_{n=1}^{\infty} r^{-n}$$

$$= \frac{1}{r^{-1}} \| (H^\infty)^{-1} \| = \frac{1}{(r-1) \lambda(X)}$$

Combining this with what we have shown previously we get:

$$\|w^*\|^2 = y^\top H^{-1} y$$

$$\leq y^\top (H^\infty)^{-1} y + \|y\| \|H^\infty - H^{-1}\| \|y\|$$

$$\leq M^2 + \frac{m}{(r-1) \lambda(X)}$$

Now if we choose $r \geq \frac{m}{M^2 \lambda(X)} + 1$ we get that $\|w^*\|^2 \leq 2M^2$. Denote:

$$H_{\sqrt{m}} = \{ \frac{1}{\sqrt{k}} \sum_{j=1}^{k} g w_j, u_j | \|w\| = \sqrt{\sum_{j=1}^{k} \|w_j\|^2} \leq \sqrt{2M} \}$$

This is the hypothesis class of (normalized) GaLU networks with norm bounded by $\sqrt{2M}$. The Rademacher complexity of $H_{\sqrt{m}}$ is given by:

$$\mathcal{R}_m(H_{\sqrt{m}}) = \sup_{x_1, \ldots, x_m \in \mathcal{X}} \mathbb{E}_{\sigma \sim U(\{\pm 1\}^m)} \left[ \sup_{h \in H_{\sqrt{m}}} \frac{1}{m} \sum_{i=1}^{m} h(x_i) \sigma_i \right]$$

$$= \sup_{x_1, \ldots, x_m \in \mathcal{X}} \mathbb{E}_{\sigma \sim U(\{\pm 1\}^m)} \left[ \frac{1}{m} \sum_{i=1}^{m} \frac{1}{\sqrt{k}} \sum_{j=1}^{k} \Phi_{u_j}(x_i)^\top w \right]$$

Notice that we have $\left\| \frac{1}{\sqrt{k}} \Phi_{w_j}(x_i) \right\|^2 = \frac{1}{k} \sum_{j=1}^{k} \|1_{u_j, x_i \geq 0\|}^2 \leq \|x_i\| \leq 1$. Therefore, from standard Rademacher analysis for linear functions with bounded norm (for example in [21]), we get that $\mathcal{R}_m(H_{\sqrt{m}}) \leq \frac{\sqrt{m}}{\sqrt{2m}}$. Notice that the square loss function $\ell(y, \hat{y}) = \frac{1}{2} (y - \hat{y})^2$ is 1-smooth. Since for every $h \in H_{\sqrt{m}}$ and $x \in \mathcal{X}$ we have $|h(x)| \leq \sqrt{2M}$, and we assume that $Y \subseteq [-1, 1]$, we can assume that the loss function $\ell$ is defined over $[-\sqrt{2M}, \sqrt{2M}] \times [-1, 1]$. Then for $y, \hat{y} \in [-\sqrt{2M}, \sqrt{2M}], y \in [-1, 1]$ we have:

$$|\ell(y, \hat{y}) - \ell(\hat{y}, y)| = \frac{1}{2} |\hat{y} - y - \hat{y}^2 - y\hat{y}| \leq 2M^2 + \sqrt{2M}$$

Now, for $\hat{h} \in H_{\sqrt{m}}$, the GaLU network with weights $w^*$, using Theorem 1 in [25], we get with probability at least $1 - \delta$ a generalization bound of:

$$L_D(\hat{h}) \leq C \left( \frac{2M^2 \log^3 m + (2M^2 + \sqrt{2M}) \log(1/\delta)}{m} \right)$$

For some constant $C$.  

\[\blacksquare\]
D Proofs of Section 4

D.1 Proof of Theorem 4

Proof Denote \( \sigma(x) = 1_{x \geq 0} \) the gate of the Galu network and \( \phi(x) = |x|_+ = 1_{x \geq 0} \cdot x \) the ReLU activation. By our assumption, the output of the network is bounded in \([-1, 1]\) upon initialization, so:

\[
L_S(N_{W,U,\alpha}^G) = -\frac{1}{m} \sum_{i=1}^{m} y_i N_{U,\alpha}^R(x_i)
\]

Therefore we get for every \( j \):

\[
\frac{\partial}{\partial w_j} L_S(N_{W,U,\alpha}^G) = -\frac{1}{m\sqrt{k}} \sum_{i=1}^{m} y_i \alpha_j \sigma(x_i^T u_j) x_i
\]

\[
= -\frac{1}{m\sqrt{k}} \sum_{i=1}^{m} y_i \alpha_j \phi'(x_i^T u_j) x_i
\]

\[
= \frac{\partial}{\partial u_j} L_S(N_{U,\alpha}^R)
\]

And the result immediately follows. \(\blacksquare\)

D.2 Proof of Theorem 5

Recall that for some vectors \( u_1, \ldots, u_k \in \mathbb{R}^d \), we denote \( \Phi : \mathcal{X} \rightarrow \mathbb{R}^{dk} \) where \( \Phi_U(x) = \sqrt{k}[1_{u_1 \geq 0} x, \ldots, 1_{u_k \geq 0} x] \). For some \( W = [w_1, \ldots, w_k] \) where \( w_i \in \mathbb{R}^d \), and \( \alpha = [\alpha_1, \ldots, \alpha_k] \) where \( \alpha_i \in \mathbb{R} \), we define a vector \( v(W, \alpha) = [\alpha_1 w_1 \ldots \alpha_k w_k] \in \mathbb{R}^{dk} \). Now, we can write:

\[
N_{W,U,\alpha}^G(x) = \frac{1}{\sqrt{k}} \Phi_U(x)^T v(W, \alpha), \quad N_{U,\alpha}^R(x) = \frac{1}{\sqrt{k}} \Phi_U(x)^T v(U, \alpha)
\]

We start with the following lemma:

Lemma 11 Fix \( \delta > 0 \), let \( k \geq \frac{\delta^2}{2d \log(d)} \), and we assume \( d > \log(2k/\delta) \). Assume we draw \( u_1, \ldots, u_k \sim N(0, I_d) \). Let \( w_1, \ldots, w_k \) be some vectors such that for all \( j \in [k] \) we have \( \|u_j - w_j\| \leq \epsilon \), for some \( \epsilon > 0 \). Then with probability at least \( 1 - \delta \), we have \( \|\Phi_U - \Phi_W\|_\infty \leq \sqrt{\frac{2d\epsilon^2}{\sqrt{2\pi}}}. \)

Proof Fix \( B = \sqrt{6d} \), and from Lemma B.12 in [21], we have that:

\[
P \left[ \|u_j\|^2 \geq \sqrt{6d} \right] \leq e^{-d} \leq \frac{\delta}{2k}
\]

Using the union bound, we have with probability at least \( 1 - \frac{\delta}{2k} \), for all \( j \in [k] \) we have \( \|u_j\| \leq B \), so we assume this holds. Let \( \delta' = \frac{1}{2}(3\epsilon)^{-d}\delta \). Fix some \( x \in \mathcal{X} = S^{d-1} \), and fix some \( j \in [k] \).

Notice that \( u_j^T x \sim N(0, 1) \), and therefore:

\[
P_{u_j \sim N} \left[ |u_j^T x| \leq 2B\epsilon \right] \leq \frac{4B\epsilon}{\sqrt{2\pi}}
\]

Denote \( S_x = \frac{1}{r} \sum_{j=1}^{r} 1_{|u_j^T x| \leq 2B\epsilon} \), so \( E[S_x] \leq \frac{4B\epsilon}{\sqrt{2\pi}} \), and from Hoeffding’s inequality we have:

\[
P \left[ S_x \geq \frac{5B\epsilon}{\sqrt{2\pi}} \right] = P \left[ S_x \leq E[S_x] + \frac{B\epsilon}{\sqrt{2\pi}} \right] \leq \exp \left( -\frac{2k^2\epsilon^2}{2\pi} \right) \leq \delta'
\]

For every \( x' \in \mathcal{X} \) with \( \|x - x'\| \leq \epsilon \), if \( |u_j^T x| > 2B\epsilon \) then we have:

\[
|u_j^T x'| \geq |u_j^T x| - \|x - x'\| \|u_j\| \geq |u_j^T x| - B\epsilon > B\epsilon \geq \epsilon
\]
For such \( x' \) we have \( |u_j^\top x' - w_j^\top x'| \leq \|u_j - w_j\| \|x'\| \leq \epsilon \), so \( \text{sign}(u_j^\top x') \neq \text{sign}(w_j^\top x') \) only if \( |u_j^\top x'| \leq \epsilon \).

Therefore, w.p at least 1\( - \delta' \) we have for every \( x' \in X \) with \( \|x - x'\| \leq \epsilon \):

\[
\|\Phi_U(x') - \Phi_W(x')\|^2 = \frac{1}{k} \sum_{j=1}^{k} 1_{\text{sign}(u_j^\top x') \neq \text{sign}(w_j^\top x')} \|x\|^2 \\
\leq \frac{1}{k} \sum_{j=1}^{k} 1_{|u_j^\top x'| \leq \epsilon} \\
\leq \frac{1}{k} \sum_{j=1}^{k} 1_{|u_j^\top x| \leq 2 \epsilon} \\
= S_x \leq \frac{5B\epsilon}{\sqrt{2\pi}}
\]

Now, there is an \( \epsilon \)-net of \( X \) of size at most \( (3/\epsilon)^d \), and we denote this net by \( N \subseteq X \). From the union bound we get that with probability at least 1\( - (3/\epsilon)^d \delta' = 1 - \delta/2 \) we have for all \( x \in N \), and for every \( x' \in X \) with \( \|x - x'\| \leq \epsilon \), that:

\[
\|\Phi_U(x') - \Phi_W(x')\| \leq \sqrt{\frac{5B\epsilon}{2\pi}}
\]

In this case, the above inequality holds for every \( x' \in X \), and we get the required.

The above shows that small perturbation in \( u_j \)'s implies small perturbation of the map \( \Phi_U \). Now, fix some \( L \)-Lipschitz loss \( \ell : \mathbb{R} \times \mathbb{Y} \rightarrow \mathbb{R} \), and denote \( L_D(f) = \mathbb{E}_{(x,y) \sim D} [\ell(f(x), y)] \). Fix some \( v \in \mathbb{R}^{dk} \) and observe the two functions \( h_{u,v}(x) = \Phi_u(x)^\top v \) and \( h_{w,v}(x) = \Phi_w(x)^\top v \). Then we have:

\[
L_D(h_{u,v}) - L_D(h_{w,v}) = |\mathbb{E}_{(x,y) \sim D} [\ell(h_{u,v}(x), y)] - \mathbb{E}_{(x,y) \sim D} [\ell(h_{w,v}(x), y)]| \\
\leq |\mathbb{E}_{(x,y) \sim D} [\ell(h_{u,v}(x), y)] - \mathbb{E}_{(x,y) \sim D} [\ell(h_{w,v}(x), y)]| \\
\leq |\mathbb{E}_{(x,y) \sim D} [\ell(h_{u,v}(x), y) - \ell(h_{w,v}(x), y)]| \\
\leq \mathbb{E}_{(x,y) \sim D} [L|h_{u,v}(x) - h_{w,v}(x)|] \\
= \mathbb{E}_{(x,y) \sim D} [L|\Phi_u(x)^\top v - \Phi_w(x)^\top v|] \\
\leq L \|v\| \|\Phi_u - \Phi_w\|_\infty
\]

And this gives the following:

\[
L_D(N_{W^*,U^*}^G) \leq L_D(N_{U^*,U^*}^G) \\
\leq L_D(N_{U^*,U^*}^G) + L \left\| \frac{1}{\sqrt{k}} v (U^*, \alpha^{**}) \right\| \|\Phi_{U^*} - \Phi_U\|_\infty \\
= L_D(N_{U^*,U^*}^G) + L \left\| \frac{1}{\sqrt{k}} \sum_{i=1}^{k} ||\alpha_i^{**} u_i^* ||^2 \right\| \|\Phi_{U^*} - \Phi_U\|_\infty \\
\leq L_D(N_{U^*,U^*}^G) + L \max_i ||\alpha_i^{**} u_i^* || \|\Phi_{U^*} - \Phi_U\|_\infty
\]

Now, observing that \( g_{U^*,U^*} = f_{U^*} \), and using Lemma 11 completes the proof.
E Experiments

We showed theoretical results that establish the relation between ReLU and GaLU. To complete the picture, we now turn to evaluate this relation empirically. We start with a memorization experiment, where the task at hand is to memorize a randomly generated sample (as described in §3.1). In this experiment, we draw $m$ examples in dimension $d$, where both the input and the label are sampled from a Gaussian distribution. Recall that for this case, we theoretically showed that a GaLU network needs $\tilde{\Omega}(\frac{m}{d})$ neurons to reach zero loss. We train both GaLU and ReLU network on this task, with Adam optimizer, batch size 128 and learning rate of 0.001 for 100$k$ iterations. Using binary search, we find the minimal $k$ to reach MSE loss < 0.01. Each experiment is repeated 5 times. We see that for both the ReLU and GaLU networks we get $k \approx \frac{m}{d}$, for different sample sizes. The results of this experiments are shown in figure 1.

Next, we turn to observing a memorization task in the under-parametrized case. In this experiment we observe the loss of the network different choices of $k$ where $k \leq \frac{m}{d}$. In this case, the loss of the GaLU network behaves like $1 - \frac{kd}{m}$, as predicted by our theoretical analysis. A ReLU network, on the other hand, achieves slightly better performance than the GaLU network in this regime, but its loss is lower bounded by $1 - \frac{2kd}{m}$. In other words, a GaLU network with $2k$ neurons achieves the same performance as a ReLU network with $k$ neurons, so a ReLU network gives only a constant gain in parameter utilization. The results of this experiments are shown in figure 2.

Going beyond a pure memorization task, we observe the behavior of ReLU and GaLU on linearly separable data. It has been shown [6] that linearly separable data is learnable by neural-networks, with sample complexity similar to a linear classifier. Therefore, this task is an interesting benchmark to compare the performance of ReLU and GaLU networks. In this experiment we draw examples from a Gaussian distributions in $\mathbb{R}^{100}$ and uniformly choose a vector $w$ on the sphere, to be the linear separator. We use 50$k$ examples for train and 10$k$ examples for test, filtering only examples with margin $\geq 0.01$. Here we train both GaLU and ReLU networks with the Adam optimizer, using learning rate 0.001, for 100$k$ iterations and batch size 128, comparing different network widths. Each experiment is repeated 3 times, and the results are averaged over the exper-
Next, we observe the performance of GaLU and ReLU on MNIST and Fashion-MNIST datasets. Training is performed as described previously. A comparison of the performance of various network widths on the test data is shown in Figure 3. Again, we observe similar behavior, with ReLU networks performing slightly better than GaLU.

Finally, we move to observing a failure case. We test GaLU and ReLU networks on the parity task, which is known to be a hard task for neural-networks in general [22]. In this task, we draw uniformly examples s.t. $x \sim Uni(\{-1,1\})^{100}$, and setting the labels to be $y = \prod_{i=1}^{100} x_i$. So the label of the example $x$ is 1 if the number of $-1$-s in the example is even. Using again $50k$ examples for a training set and $10k$ examples as a test set, with a training scheme similar to before, we observe that both GaLU and ReLU networks completely fail in this task, achieving only chance-level performance. This is shown in Figure 4.