Turing patterns resulting from a Sturm-Liouville problem

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Pattern formation in reaction-diffusion systems where the diffusion terms correspond to a Sturm-Liouville problem are studied. These correspond to a problem where the diffusion coefficient depends on the spatial variable: \( \nabla \cdot (D(x)\nabla u) \). We found that the conditions for Turing instability are the same as in the case of homogeneous diffusion but the nonlinear analysis must be generalized to consider general orthogonal eigenfunctions instead of the standard Fourier approach. The particular case \( D(x) = 1 - x^2 \), where solutions are linear combinations of Legendre polynomials, is studied in detail. From the developed general nonlinear analysis, conditions for producing stripes and spots are obtained, which are numerically verified using the Schaneknberg system. Unlike to the case with homogeneous diffusion, and due to the properties of the Legendre polynomials, stripped and spotted patterns with variable wavelength are produced, and a change from stripes to spots is predicted when the wavelength increases. The patterns obtained can model biological systems where stripes or spots accumulate close to the boundaries and the theory developed here can be applied to study Turing patterns associated to other eigenfunctions related with Sturm-Liouville problems.

Keywords: Turing patterns | Sturm-Liouville problem | Inhomogeneous diffusion

I. INTRODUCTION

In a ground-breaking work from 1952 Alan Turing laid the foundations of chemical morphogenesis by proposing a reaction-diffusion theory for pattern formation [1]. Turing showed that two or more chemical substances, called morphogens, that react and diffuse in a medium such as a tissue, can produce stable periodic patterns through a linear instability of a spatially uniform state. The general form of a two-chemicals reaction diffusion system is:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= D_u \nabla^2 u + f(u, v), \\
\frac{\partial v}{\partial t} &= D_v \nabla^2 v + g(u, v),
\end{align*}
\]

where \( u \) and \( v \) represent the concentration of the two morphogens, \( f \) and \( g \) the reaction kinetics and \( D_u \) and \( D_v \) the diffusion coefficients of \( u \) and \( v \), respectively. Neutral (Neumann) boundary conditions are assumed. Turing’s important discovery is that, under certain conditions, a stable spatially uniform state, in the absence of diffusion, can become unstable under nonuniform perturbations (introduced as random initial conditions) due to diffusion. This is the so-called Turing instability [2].

Conditions for Turing, or diffusion driven, instability are obtained using linear analysis [2]. If \( (u_0, v_0) \) is a steady state, then stability in the absence of diffusion is obtained if

\[
\begin{align*}
f_u + g_v &< 0, \\
f_u g_v - f_v g_u &> 0,
\end{align*}
\]

and the instability due to diffusion arises if

\[
\begin{align*}
D_v f_u + D_u g_v &> 0, \\
(D_v f_u + D_u g_v)^2 &> 4 D_u D_v (f_u g_v - f_v g_u).
\end{align*}
\]

In all the cases, the partial derivatives of \( f \) and \( g \) are evaluated at \( (u_0, v_0) \). A key step to deduce [2] is to notice that the eigenfunctions of the Laplace operator \( \nabla^2 \) are \( W_k = e^{i k \cdot r} \), with eigenvalues (wavenumbers) \( k = ||k||^2 \), thus
solutions are look in the form

$$\sum_k c_ke^{\lambda t}e^{ikr}$$

(3)

where the constants $c_k$ are determined by Fourier expanding the initial conditions in terms of $W_k$ [2]. By doing this, the dispersion relation $\lambda(k^2)$ is obtained from where, looking for the instability condition $\text{Re}\left(\lambda(k^2)\right) > 0$, Equations [2] are deduced.

A step forward is to predict the type of pattern obtained and its stability. Standard non-linear analysis has been used to show that the selection of stripes or spots depends on the non-linear terms and can not be inferred from the linear analysis [3]. An important conclusion of this non-linear approach is that stripes and spots can not coexist.

A step further is to generalize the theory for pattern formation introduced by Turing to consider more realistic situations. For instance, in all the above, it has been assumed that $D_u$ and $D_v$ are constant diffusion coefficients. Experimental evidence that in some biological systems spatial inhomogeneities are important to regulate patterns lead to several generalizations of the reaction diffusion problem: when one of the diffusion coefficients either depends on the spatial variables [4–5] or is discontinuous [4–6]; time-dependent [7] or concentration-dependent diffusion coefficients [8–10]; spatially varying parameters [11, 12]; reaction diffusion system in a channel with the projected Fick-Jacobs-Zwanzig operator (with a diffusion coefficient that depends on the longitudinal coordinate) [13], as well as pattern formation with superdiffusion [14] or anomalous diffusion [15].

In this work, we also consider the problem of the reaction diffusion equation when the diffusion coefficient depends explicitly on the space variables but following a different approach that consist into state the problem in such a way that the diffusion operator corresponds to a Sturm-Liouville eigenvalue problem with eigenfunctions $e^{ikr}$. We then recast the linear and non-linear analysis of the reaction-diffusion system using orthogonal functions $W_n(x)$ instead of the eigenfunctions of the Laplace operator $e^{ikr}$. In particular the operator $\partial_x((1-x^2)\partial_x)$ is worked out. The eigenfunctions of this operator are the Legendre polynomials $P_n(x)$, with eigenvalues $n(n+1)$. From the developed general nonlinear analysis, parameter regions for producing stripes or spots can be identified. The Schnakenberg system is used to verify the predictions of the non-linear analysis.

From this study we report new kind of patterns where the wavelength is not constant but related with the positions of the zeros of the Legendre polynomials. Also, a transition from stripes to spots is predicted when a parameter of the system varies. In the particular case of the Schnakenberg system this transition is observed when the domain size grows, which can be of interest in some biological systems.

The generalization of the standard non-linear analysis using orthogonal functions $W_n(x) = P_n(x)$ developed here can be also of interest in the field of pattern formation.

II. THE PROBLEM

Consider the following dimensionless two species Turing model in one space dimension:

$$\frac{\partial u}{\partial t} = d \frac{\partial}{\partial x} \left( D(x) \frac{\partial u}{\partial x} \right) + \eta f(u,v)$$

(4a)

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x} \left( D(x) \frac{\partial v}{\partial x} \right) + \eta g(u,v)$$

(4b)

$$D(x) \frac{\partial u}{\partial x} = 0, \quad D(x) \frac{\partial v}{\partial x} = 0 \quad \text{on the boundaries,}$$

(4c)

where $D(x)$ is a function defined below, $d$ is a constant and $\eta$ is a scale factor.

By following the standard procedure [2], the linear system associated to this problem is

$$\dot{w} = \eta A w + \mathbb{D} \frac{\partial}{\partial x} \left( D(x) \frac{\partial w}{\partial x} \right),$$

(5)

where $w = (u - u_0, v - v_0)$, $\mathbb{D} = \text{Diag}[d, 1]$ and

$$A = \begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix}_{(u_0, v_0)}$$

is the Jacobian matrix evaluated at the equilibrium point $u_0 = (u_0, v_0)$. 
Now, we now look for solutions $w(x,t)$ of (5) in the form
\[ w(x,t) = \sum_k c_k e^{\lambda t} W_k(x), \] (6)
where $c_k$ are determined by expanding the initial conditions in terms of the eigenfunctions $W_k(x)$. These eigenfunctions $W_k(x)$ are the solutions of the following problem:
\[
\frac{\partial}{\partial x} \left( D(x) \frac{\partial W_k}{\partial x} \right) + l_k W_k = 0, \\
D(x) \frac{\partial W_k}{\partial x} = 0,
\] (7a)
that we identify as a Sturm-Liouville eigenvalue problem \[16\] and where in general $l_k = l_k(x)$.

Notably, the linear analysis follows closely the results of the homogeneous diffusion case. By substituting (6) in (5), using (7) and cancelling $e^{\lambda t}$, we get, that for each $k$,
\[
\lambda W_k = \eta A W_k + D \frac{\partial}{\partial x} \left( D(x) \frac{\partial W_k}{\partial x} \right)
\]
(8)
\[
= \eta A W_k - D l_k W_k
\]
(9)

Nontrivial solutions for $W_k$ are obtained provided that
\[
|\lambda - \eta A + l_k| = 0.
\] (10)

From here, we get the eigenvalues $\lambda(k)$ depending on the wavenumber $k$ as the roots of the dispersion relation $\lambda^2 + \alpha(l_k)\lambda + \beta(l_k) = 0$, where
\[
\alpha(l_k) = (l_k(1 + d) - \eta (f_u + g_v)), \\
\beta(l_k) = d l_k^2 - \eta (f_u + d g_v)l_k + \eta^2|A|.
\] (11)
(12)

The critical value of $d$ and the critical wavenumber $l_c$ are obtained from the conditions $\beta(l_c) = 0$ and $d\beta(l_c)/dl_c = 0$, respectively, yielding
\[
d_c = \frac{(f_u g_v - 2 f_v g_u) - \sqrt{-f_v g_u |A|}}{g_v^2},
\] (13)
and
\[
l_c = \eta \frac{f_u + d_c g_v}{2d_c} = \eta \left( \frac{|A|}{d_c} \right)^{1/2}.
\] (14)

The conditions for a Turing instability turn out to be the same as in the case of homogeneous diffusion, that is (1), and (2), where $D_u = d$ and $D_v = 1$.

**The Legendre equation**

Consider $D(x) = (1 - x^2)$. In this case, the Sturm-Liouville eigenvalue problem \[7\] is
\[
\frac{\partial}{\partial x} \left( (1 - x^2) \frac{\partial P_k(x)}{\partial x} \right) + k(k + 1) P_k(x) = 0, \\
(1 - x^2) \frac{\partial P_k(x)}{\partial x} = 0,
\] (15a)
(15b)
where the eigenfunctions $P_k(x)$ are Legendre polynomial of degree $k \in \mathbb{N}$. The domain is the interval $(-1,1)$ and since $D(x) = (1 - x^2) = 0$ at the boundaries $x = \pm 1$, the problem (15) is singular \[17\].
Since $l_k = k(k + 1)$, from [14] we have $k(k + 1) = \eta(|\lambda|/d_c)^{1/2}$, thus the critical wave number is

$$k_c = \left\{ \sqrt{1 + 4\eta(|\lambda|/d_c)^{1/2}} - 1 \right\}/2. \quad (16)$$

For two dimensional problems, we define $D(x, y) = \begin{pmatrix} (1 - x^2) & 0 \\ 0 & (1 - y^2) \end{pmatrix}$, on the square $(x, y) \in [-1, 1] \times [-1, 1]$. In this case, the eigenfunctions are

$$W_k = W_{ij}(x, y) = P_i(x)P_j(y), \quad (18)$$

with eigenvalues

$$l = l(i, j) = i(i + 1) + j(j + 1). \quad (19)$$

We notice that the main difference in the analysis as compared with the case of homogeneous diffusion is the value of the critical wave number (16). This quantity will be useful to predict the wavelength and amplitudes, which in this case are both variable, as can be inferred if one considers the zeros of the Legendre polynomials (see for example [16]); the zeros accumulate at the ends of the domain as the order of the polynomial increases. It implies that the wavelengths of the resulting Turing patterns will be smaller close to the boundaries.

Another important difference is that the transition from stripes to spots is controlled by the scale factor $\eta$, as will be shown in the nonlinear analysis section.

In the system (15) the diffusion term is nonhomogeneous. According to the function $D(x) = (1 - x^2)$, the chemical concentrations move faster when they are close to the origin than close to the boundaries $-1$ and $+1$.

III. NONLINEAR ANALYSIS

Here, the methodology developed in References [3, 18, 19] is generalized for the orthogonal functions $P_n$ and conditions for the generation of stripes and spots are deduced. The general guidelines of the nonlinear analysis is presented in this Section and a more detailed analysis is presented in the Appendix.

The general form of the reaction diffusion equation can be written in vector form as

$$\frac{\partial u}{\partial t} = D \nabla \cdot (D(x)\nabla u) + F(u; p), \quad (20)$$

where $F(u; p) = \eta(f(u; p), g(u; p))^T$ and $p$ is a parameter of the system. Now $\lambda$ is perturbed around its bifurcation value $\lambda_c$ as $\lambda = \lambda_c + \epsilon^2$. Close to the bifurcation point we assume that

$$u = u_0 + \hat{u} = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \ldots \quad (21)$$

The purpose is to determine the non-linear effects near the branch $\lambda_c$ obtained from the roots of $|\lambda - (\eta A + l_c D)| = 0$, where $l_c = (i_c, j_c)$ satisfies the diffusion-driven instability conditions. The case $0 < \epsilon << 1$ is considered and a slow time scale is introduced:

$$T = \tilde{w} t. \quad (22)$$

where

$$\tilde{w} = \epsilon w_1 + \epsilon^2 w_2 + \cdots \quad (23)$$

By expanding $F$ in (20) in a Taylor series around the stable equilibrium point $u_0$ and substituting (21), yields

$$F(u) = \eta A \hat{u} + Q(\hat{u}, \hat{u}) + C(\hat{u}, \hat{u}, \hat{u}) + \cdots, \quad (24)$$

where $A$, $Q$ and $C$ are, respectively, the lineal quadratic and cubic terms.

Then substituting (21) and (22) into (20), we get
\[
\dot{\mathbf{u}} = \mathcal{D}(\mathbf{x}) \nabla \mathbf{u} + \eta \ddot{\mathbf{u}} + \mathbf{Q}(\hat{\mathbf{u}}) + \mathbf{C}(\tilde{\mathbf{u}}, \hat{\mathbf{u}}, \ddot{\mathbf{u}}) + \cdots,
\]

If the parameter \( p \) of the model is perturbed about its value in a critical set:
\[
p = p_c + (\epsilon p_1 + \epsilon^2 p_2 + \cdots),
\]
by expanding \( \mathbf{u} \) in a Taylor series around \( p_c \), yields
\[
\dot{\mathbf{u}} = (\mathcal{D}(\mathbf{x}) \nabla)^c \mathbf{u} + \eta \ddot{\mathbf{u}} + \mathbf{Q}(\hat{\mathbf{u}}) + \mathbf{C}(\tilde{\mathbf{u}}, \hat{\mathbf{u}}, \ddot{\mathbf{u}}) + \cdots + \hat{\rho}((\mathcal{D}(\mathbf{x}) \nabla)^p \mathbf{u} + \eta \ddot{\mathbf{u}} + \mathbf{Q}_p(\hat{\mathbf{u}}) + \cdots) \tag{26}
\]

If a solution of the form \( \mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 \) is proposed, we obtain the following hierarchy of linear differential equations:
\[
\begin{align*}
0(\epsilon) : & \quad \mathbf{L} \mathbf{u}_1 = 0, \\
0(\epsilon^2) : & \quad \mathbf{L} \mathbf{u}_2 = \mathbf{Q} + p_1 \eta \mathbf{u}_1 - w_1 \frac{\partial \mathbf{u}_1}{\partial T}, \tag{27} \\
0(\epsilon^3) : & \quad \mathbf{L} \mathbf{u}_3 = \mathbf{Q} + \mathbf{C} + p_1 \eta \mathbf{u}_2 + p_1 \mathbf{Q}_p \mathbf{u}_1 + p_1 \mathbf{Q}_p \mathbf{u}_2 + \cdots - w_1 \frac{\partial \mathbf{u}_2}{\partial T} - w_2 \frac{\partial \mathbf{u}_1}{\partial T},
\end{align*}
\]
where \( \mathbf{L} = (-\eta \mathbf{u}^c - \mathbf{D}(\mathbf{x}) \nabla) \).

For \( 0(\epsilon) \), we look at the special case where the solution is the sum of two spatial modes
\[
\mathbf{u}_1 = V^{(1)} \alpha(T) P_{l_c}(x) + V^{(1)} \bar{\alpha}(T) P_{j_c}(y),
\]
where \( l_c = l(\epsilon, 0) = l(0, j_c) \) satisfies the diffusion-driven instability conditions. This type of solution is proposed in the Turing pattern theory \( \text{[2][13][20]} \) to study under which conditions on the parameters, lines parallel to the y-axis \((\alpha(T) = 0)\) or spots \((\alpha(T) \text{ and } \bar{\alpha}(T) \text{ different from zero})\) are produced. The expressions for \( V^{(1)} \) and \( \bar{V}^{(1)} \) are given in the Appendix.

To solve the second order approximation \( 0(\epsilon^2) \) the Fredholm Alternative is applied. In the Appendix it is shown that if \( i_c \) and \( j_c \) are odd, the alternative of Fredholm is fulfilled when the value of the parameters \( p_1 \) and \( w_1 \) are zero. A key step here is to expand the products of the Legendre polynomials as linear functions of Legendre polynomials \( \text{[21]}, \) that is
\[
(P_l)^2 = \sum_{n=0}^{M} \rho_n P_n \quad \text{and} \quad P_l P_s = \sum_{n=0}^{M} \rho_n^{(q)} P_n, \tag{29}
\]
where the subindex \( q \) in the coefficients stands for the quadratic terms. The main difference with respect to the case of homogeneous diffusion where the Fourier basis is used is that in the expansions of the products of functions \( e^{i\mathbf{k} \cdot \mathbf{r}} \) is that all the coefficients of the series have the same fixed values. Here, on the contrary, the appearance of coefficients with different values contribute importantly to the amplitude equations and introduce asymmetries when varying its critical wavenumber, a situation that does not occur with homogeneous diffusion.

Once the Fredholm Alternative is applied, for the solution of the second order approximation the following form is assumed:
\[
\mathbf{u}_2 = \left( \sum_{x=0}^{M} V_s^{(2)}(x) P_s(x) + V_s^{(2)}(\bar{x})^2 P_s(y) \right) + V_{ij} \alpha(T) \bar{\alpha}(T) P_{i_c}(x) P_{j_c}(y), \tag{30}
\]
where \( V_{ij} = V_{ij}(x, y) \), and \( V_s^{(2)}, V_s^{(2)} \) and \( V_{ij} \) depends on \( V^{(1)} \) and \( \bar{V}^{(1)} \). The values of the coefficients \( V_s^{(2)}, V_s^{(2)} \) and \( V_{ij} \) are listed in the Appendix.

Once that \( \mathbf{u}_2 \) is known, the term \( 0(\epsilon^3) \) in \( \text{[27]} \) is worked using the Fredholm alternative, leading finally to the Stuart-Landau equations:
\[
\begin{align*}
\frac{d|\alpha(T)|^2}{dT} &= \alpha |\alpha(T)|^4 + \beta |\alpha(T)|^2 |\bar{\alpha}(T)|^2 + \theta |\alpha(T)|^2, \tag{31a} \\
\frac{d|\bar{\alpha}(T)|^2}{dT} &= \alpha |\alpha(T)|^4 + \beta |\alpha(T)|^2 |\bar{\alpha}(T)|^2 + \theta |\bar{\alpha}(T)|^2. \tag{31b}
\end{align*}
\]
where

\[ E = \frac{1}{2} \langle V^* | V^{(1)} \rangle \]  

(32a)

\[ \alpha = \frac{1}{E} \left\{ V^* \left| \sum_{s=0}^{M} \left( Q(V^{(1)}, V_s^{(2)}) \rho_s^{(g)} \right) + \left( C(V^{(1)}, V^{(1)}) \rho_s^{(c)} \right) \right. \right\}, \]  

(32b)

\[ \beta = \frac{1}{E} \left\{ V^* \left| \left( Q(V^{(1)}, V_{ij}) \rho_0 \right) + Q(V^{(1)}, \bar{V}_0^{(2)}) + 3 \left( C(V^{(1)}, V^{(1)}) \rho_0 \right) \right. \right\}, \]  

(32c)

\[ \theta = \frac{1}{E} \left\{ V^* \left| \nu BV^{(1)} \right. \right\}. \]  

(32d)

A phase plane analysis is applied to the system in (31), with which the conditions given in Table I for the linear stability are obtained.

| Steady state | Conditions for linear stability | Spatial pattern |
|--------------|-------------------------------|----------------|
| $|a|^2 = |\bar{a}|^2 = 0$ | $\theta < 0$ | None |
| $|a|^2 = 0, |\bar{a}|^2 = \frac{\theta}{\alpha}$ | $\theta > 0$ and $\frac{\alpha}{\beta} > 1$ | Stripes |
| $|a|^2 = \frac{\theta}{\alpha}, |\bar{a}|^2 = 0$ | $\theta > 0$ and $\frac{\alpha}{\beta} > 1$ | Stripes |
| $|a|^2 = |\bar{a}|^2 = \frac{\theta}{\alpha+\beta}$ | $\theta > 0$ and $\alpha < -|\beta| < 0$ | Spots |

TABLE I: Steady states and conditions for linear stabilities of the amplitude functions.

A. Example

As a particular example, consider the Schnakenberg reaction diffusion system, which has been frequently used due to its simple structure. In dimensionless form, we consider the following system:

\[
\frac{\partial u}{\partial t} = d \nabla \cdot (D(x) \nabla u) + \eta \left( a - u + u^2 v \right) \]  

(33a)

\[
\frac{\partial v}{\partial t} = \nabla \cdot (D(x) \nabla u) + \eta \left( b - u^2 v \right) \]  

(33b)

\[D(x) \nabla u \cdot n = 0, \quad D(x) \nabla v \cdot n = 0 \text{ on the boundaries}, \]  

(33c)

where $a$ and $b$ are positive constants. The system has a single equilibrium state at $(u_0, v_0) = (a + b, b/(a + b)^2)$ and the conditions for a Turing instability (1) and (2) become

\[ \frac{b - a}{a + b} - (a + b)^2 < 0, \]  

(34a)

\[ (a + b)^2 > 0, \]  

(34b)

\[ \frac{b - a}{a + b} - d(a + b)^2 > 0, \]  

(34c)

\[ \left( \frac{b - a}{a + b} - d(a + b)^2 \right) - 4d(a + b)^2 > 0. \]  

(34d)

The critical values (13) and (14) become

\[ d_c = \frac{(a + b)(a + 3b) - 2 \sqrt{2b(a + b)^3}}{(a + b)^4}, \]  

(35)

and

\[ l_c = \frac{\eta(a + b) \left( (a + b)^2 + \sqrt{2b(a + b)^3} \right)}{b - a}. \]  

(36)

The results of the non linear analysis, summarized in Table I, can be used to divide the parameter space $(a, b)$ into domains corresponding to different spatial patterns. Some results are shown in Fig. I where the regions where
FIG. 1: Parameter space of the Schnakenberg model with homogeneous diffusion (A) and with Legendre diffusion (B) with spatial mode $i_c = 3$ in both cases. Yellow color represents stripes, orange color represents spots, red color represents patterns that cannot be predicted by the non linear analysis, purple and black colors represent the region where the stability and instability Turing conditions are not fulfilled, respectively.

spatial patterns are stripes, spots, or when the non linear analysis cannot predict a particular pattern are indicated with white, yellow and purple, respectively. When either the stability or instability Turing conditions are not fulfilled, is indicated with red and black colors, respectively. Fig. 1A corresponds to the case of homogeneous diffusion, where a spatial mode $k_c = 4\pi$ was chosen (see below). Fig. 1B corresponds to the case of Legendre diffusion with spatial modes $i_c = 3$. The procedure to generate the plots is as follows. A spatial mode $i_c (k_c)$ in which we are interested is set. With this value of $i_c (k_c)$, for each mapped point $(a, b)$, a suitable value of $\eta$ is calculated using (36) and $\alpha$ is calculated using (35). With these values, the parameters $\alpha, \beta$ and $\theta$ are evaluated and the conditions listed in the second column of Table I are used to determinate the type of pattern that is generated.

It is observed that the region of patterns that cannot be predicted by the linear analysis is reduced in the case of Legendre diffusion as compared with homogeneous diffusion. On the contrary, the region where the Turing stability condition is not fulfilled is enlarged.

IV. NUMERICAL SIMULATIONS

Henceforth the Schnakenberg system will be solved numerically with the general purposes finite element COMSOL Multiphysics software. Zero flux boundary conditions and random initial conditions around the equilibrium point are considered. Except otherwise specified, the range of the random perturbation (related with the perturbation $\epsilon$ of the non-linear analysis) will be 0.1.

A. 1D problem

Numerical simulations of the one-dimensional Schnakenberg system are performed with parameter values $a = 0.289$, $b = 1.49$, which gives from (35) $d_c = 0.02735$.

In Fig. 2 patterns obtained at $T = 1000$ seconds with Legendre diffusion ($D(x) = 1 - x^2$) are compared with those obtained with homogeneous diffusion ($D(x) = 1$). Three values of $k_c$ were considered, 3, 11 and 19, which from (36) produces $\eta = 1.11571$, $\eta = 12.2729$ and $\eta = 35.3310$, respectively.

Notice that for $\eta = 1.11571$ and $\eta = 12.2729$ no pattern with homogeneous diffusion is formed, but for Legendre diffusion patterns with the expected spatial modes are produced. For $\eta = 35.3310$ both patterns are formed with the expected spatial mode and the changes in wavelength in the case of Legendre diffusion is noticeable.
FIG. 2: Patterns obtained at $T = 1000$ seconds with Legendre diffusion (continuous black line) are compared with those obtained with homogeneous diffusion (green dots). Also, the second order approximation of the nonlinear analysis, $u_2$, is shown with green dots, for the values $\eta = 1.11571$ (A), $\eta = 12.2729$ (B), and $\eta = 35.3310$ (C).
The two-dimensional Schnakenberg system is solved to verify numerically the predictions in Fig. 1 for different values of the parameters $a$ and $b$. Since the nonlinear analysis is performed at critical values, once a pair $(a, b)$ is picked, $d$ is chosen to satisfy \( (35) \) and $\eta$ is obtained from \( (36) \) to satisfy a given $i_c$. Since the critical value of $d$ is used, for generating a initial growth of spatial pattern parameters $a$ and $b$ are perturbed by $\Delta a = 0.001$ and $\Delta b = 0.01$.

Three sets of parameters $(a, b)$ were chosen: $(0.1, 1.0)$ in the region of stripes, $(0.16, 1.0)$ in the region were a transition to stripes to spots occurs, and $(0.3, 1.0)$ in the region of spots. The corresponding values of $d_c$ are $0.10032$, $0.07284$ and $0.03418$. In all the cases, $\eta$ was fixed to ensure $i_c = 14$, so that the reference plot is Fig. 1(B), and from \( (36) \), we obtain in each case $\eta = 60.466$, $48.858$ and $29.866$. In Fig. 3 the resulting patterns are shown. We observe that the linear and nonlinear predictions are numerically validated in the three cases. The effects of the boundaries $(+1$ and $-1)$, where the zeros of the Legendre polynomials are closer is clearly appreciated.

1. Pattern transitions

The main difference with the case of homogeneous diffusion is the appearance of the Fourier coefficients $\rho_n^{(s)}$, $\rho_n^{(q)}$ and $\rho_n^{(c)}$, which results when products of eigenfunctons $P_n$ are represented as expansions of themselves (see Eq. 29). This produces important contributions to the amplitude equations and asymmetries by varying its critical wavenumber. This situation does not occur with homogeneous diffusion, where the Fourier coefficients $\rho_s$, $\rho_h^{(q)}$ and $\rho_h^{(c)}$ remain constant with the critical wave number. In figure 4, we show the value of $\beta/\alpha$ when the wave number $(i_c)$ in the Schnakenberg model increases. According to Table I, spots are produced if $0 < \beta/\alpha < 1$ and stripes if $\beta/\alpha > 1$, therefore a transition from stripes to spots is observed when the wave number increases.

V. DISCUSSION

Pattern formation with reaction diffusion equations based on a Sturm-Liouville problem are discussed. This problem corresponds to diffusion coefficients with explicit dependence on the spatial variables. In particular the case of the operator $\partial_x ((1 - x^2) \partial_x)$, where eigenfunctons are the Legendre polynomials $P_n(x)$, with eigenvalues $n(n + 1)$, was worked out. This case required to recast the standard linear and non-linear analysis using the orthogonal functions $P_n(x)$, instead of the eigenfunctons of the Laplace operator, namely, $e^{ik \cdot r}$.

The Schnakenberg reaction diffusion system was used as an example and from the nonlinear analysis, conditions for the formation of stripes or spots were deduced and compared with those of homogeneous (constant) diffusion. These prediction were validated by solving numerically the Schnakenberg system. The main difference with the case of homogeneous diffusion, is that due to the properties of the Legendre polynomials, stripped and spotted patterns with variable wavelength are produced and that a change from stripes to spots is predicted when the wavelength increases.

The results presented here enriches the field of pattern formation. A new kind of patterns where the wavelength
FIG. 4: $\beta/\alpha$ as function of the spatial mode $i_c$. A transition from stripes to spots occurs at $i_c = 13$.

is not constant but related with the positions of the zeros of the Legendre polynomials are produced. Also, the possibility of a transition from stripes to spots is predicted when a parameter of the system (the wavelength in the case of the Schnakenberg system) changes. Finally, the generalization of the standard non-linear analysis using orthogonal functions $W_n(x)$ developed here can be also of interest.

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**Appendix: Nonlinear analysis**

**Preeliminaries**

**Product of vector fields and transpose**

The product of two vector fields is defined as follows

$$\langle F|G\rangle = \int_{\Omega} (f_1g_1 + f_2g_2)dx.$$  \hfill (A.1)
The transpose of an operator $L$ is defined as the operator $L^T$ such that

$$ (F|LG) = \langle L^T F|G \rangle. $$

**Cuadratic and cubic terms of the Taylor expansion**

The Taylor series in several variables around $u^*$, for the field $F = \eta(f,g)^T$, is given by:

$$ F(u) = F^* + \eta \Lambda u + Q(u,u) + C(u,u) + \cdots. $$  \hfill (A.2)

$F^* = 0$ and the linear, quadratic and cubic terms are:

$$ \eta \Lambda = \eta \begin{pmatrix} f_u^* & f_v^* \\ g_u^* & g_v^* \end{pmatrix}, $$

$$ Q(X,Y) = \frac{\eta}{2} \begin{pmatrix} f_{uu}^* x_1 y_1 + f_{uv}^* x_1 y_2 + f_{vu}^* x_2 y_1 + f_{vv}^* x_2 y_2 \\ g_{uu}^* x_1 y_1 + g_{uv}^* x_1 y_2 + g_{vu}^* x_2 y_1 + g_{vv}^* x_2 y_2 \end{pmatrix}. $$  \hfill (A.3)

$$ C(X,Y) = \frac{\eta}{6} \begin{pmatrix} (x_1^2 y_1) f_{u3}^* + (2 x_1 y_1 x_2 + x_2^2 y_2) f_{u2}^* + (2 x_2 y_2 x_1 + x_1^2 y_1) f_{u1}^* + (x_2^2 y_2) f_{v1}^* \\ (x_1^2 y_1) g_{u3}^* + (2 x_1 y_1 x_2 + x_2^2 y_2) g_{u2}^* + (2 x_2 y_2 x_1 + x_1^2 y_1) g_{u1}^* + (x_2^2 y_2) g_{v1}^* \end{pmatrix}. $$  \hfill (A.4)

**Multiscale method**

In what follows we generalize the standard linear analysis using the orthogonal functions $P_n(x)$, instead of the eigenfunctions of the Laplace operator, $e^{i k \cdot r}$.

Consider the reaction diffusion equation in vector form:

$$ \frac{\partial u}{\partial t} = \nabla \cdot \left( D(x) \nabla u \right) + F(u;p), $$  \hfill (A.5)

where $F(u;p) = \eta(f(u;p),g(u;p))^T$, $p$ is a parameter of the system, and $\eta$ is a scale factor.

The wavelength $\lambda$ is perturbed around its bifurcation value $\lambda_c$ as $\lambda = \lambda_c + \epsilon \lambda^2$, and close the bifurcation point it is assumed that the solution of the equation is

$$ u = c u_1 + \epsilon^2 u_2 + \epsilon^3 u_3 + \cdots. $$  \hfill (A.6)

The case $0 < \epsilon < 1$ is considered and a slow time scale is introduced:

$$ T = \tilde{w} t. $$  \hfill (A.7)

where

$$ \tilde{w} = \epsilon w_1 + \epsilon^2 w_2 + \cdots. $$  \hfill (A.8)

A Taylor series expansion of $F$ in (A.5), around the stable equilibrium point $u_0$, produces

$$ F(u) = \eta \Lambda \hat{u} + Q(\hat{u},\hat{u}) + C(\hat{u},\hat{u},\hat{u}) + \cdots, $$  \hfill (A.9)

where $\Lambda$, $Q$ and $C$ are, respectively, the lineal quadratic and cubic terms.

By substituting (A.6) and (A.7) into (A.5), one obtains

$$ \tilde{w} \frac{\partial \hat{u}}{\partial T} = \nabla \cdot D(x) \nabla \hat{u} + \eta \Lambda \hat{u} + Q(\hat{u},\hat{u}) + C(\hat{u},\hat{u},\hat{u}) + \cdots, $$  \hfill (A.10)

If additionally the parameter $p$ of the model is perturbed about its value in a critical set:

$$ p = p_c + (\epsilon p_1 + \epsilon^2 p_2 + \cdots), $$
then the expansion of $[A.10]$ in a Taylor series around $p_c$, produces

$$
\hat{w} \frac{\partial \hat{u}}{\partial T} = (\nabla \cdot D(x) \nabla)^c \hat{u} + \eta \hat{A}^c \hat{u} + Q^c(\hat{u}, \hat{u}) + \cdots + \hat{p}(\nabla \cdot D(x) \nabla)^c \hat{u} + \eta \hat{A}^c \hat{u} + Q^c_p(\hat{u}, \hat{u}) + \cdots)
$$

(A.11)

By proposing a solution of the form $[A.6]$, this perturbation method produces a hierarchy of linear differential equations:

\begin{align}
0(\epsilon) : & \quad (-\eta \hat{A} - \nabla \cdot D(x) \nabla)u_1 = 0, \\
0(\epsilon^2) : & \quad (-\eta \hat{A} - \nabla \cdot D(x) \nabla)u_2 = Q(u_1, u_1) + p_1 \eta \hat{A}^c_p u_1 - w_1 \frac{\partial u_1}{\partial T}, \\
0(\epsilon^3) : & \quad (-\eta \hat{A} - \nabla \cdot D(x) \nabla)u_3 = Q(u_1, u_2) + C(u_1, u_1, u_1) + p_2 \eta \hat{A}^c_p u_1 + p_1 \eta \hat{A}^c_p u_2 + p_1 Q^c(u_1, u_1) \\
& \quad - w_1 \frac{\partial u_2}{\partial T} - w_2 \frac{\partial u_1}{\partial T},
\end{align}

(A.12) - (A.14)

which will be solved in what follows.

\textbf{$O(\epsilon)$ equations}

In the presence of a Turing instability $l_c = l(i_c, 0) = l(0, j_c)$, the solution of the linear problem can be written as the sum of $N$ spatial modes. Here we consider the special case when two spatial modes are present ($N = 2$):

$$
u_1 = V^{(1)}a(T)P_{i_c}(x) + \bar{V}^{(1)}a(T)P_{j_c}(y),
$$

(A.15)

where $P_k$ is the Legendre polynomial of degree $k$ and $l_c = (i_c, j_c)$ satisfies the diffusion-driven instability conditions. Then, we have

$$
(-\eta \hat{A} + \nabla l_c)\bar{V}^{(1)} = 0,
$$

(A.16)

and the same for $\bar{V}^{(1)}$. The solution of the above equation can be obtained and chosen to be a unitary vector this is:

$$\bar{V}^{(1)} = \frac{1}{\sqrt{(-\eta g_v + dl_c)^2 + \eta^2 g_a^2}} \left( -\eta g_v + dl_c \right)
$$

(A.17)

since $i_c = j_c$, then $\bar{V}^{(1)} = \bar{V}^{(1)}$.

In what follows, to simplify notation, we assume that $P_{i_c} = P_{x_c}(x)$, $P_{j_c} = P_{j_c}(y)$, $a = a(T)$ and $\bar{a} = \bar{a}(T)$.

\textbf{$O(\epsilon^2)$ equations}

To solve the second equation, we begin by applying the Fredholm Alternative.

$$
\left\langle u^* | Q(u_1, u_1) + p_1 \eta \hat{A}^c_p u_1 - w_1 \frac{\partial u_1}{\partial T} \right\rangle = 0.
$$

(A.18)

Where $u^*$ is in the nullspace of the adjoint. Since the solution of the adjoint problem has the same form as the solution of the linear case, we propose

$$u^* = V^* aP_{i_c}, \quad \text{or} \quad u^* = \bar{V}^* \bar{a} \bar{P}_{j_c}.
$$

(A.19)

The explicit form of $Q(u_1, u_1)$ can be obtained by if $X = u_1$ and $Y = u_1$ in $[A.3]$. By doing this, we get:

$$Q(u_1, u_1) = Q(V^{(1)}, V^{(1)})a^2(P_{i_c})^2 + 2Q(V^{(1)}, \bar{V}^{(1)})\bar{a}P_{i_c} \bar{P}_{j_c} + Q(\bar{V}^{(1)}, \bar{V}^{(1)})\bar{a}^2(\bar{P}_{j_c})^2.
$$

(A.20)

A key step here is to expand the powers and products of $P_{i_c}$ in the same basis functions, that is

$$
(P_{i_c})^2 = \sum_{n=0}^{M} \rho_n P_n
$$

(A.21)
By substituting (A.21), (A.20) in (A.18) and applying the orthogonality properties of the eigenfunctions of the Sturm-Liouville operators, we obtain

\[
\rho_s (V^* | Q(V^{(1)}, V^{(1)})) a^3 + \rho_1 \eta (V^* | \Delta_p V_1) a^2 - w_1 (V^* | V_1) a \frac{\partial a}{\partial T} = 0
\]  
(A.22)

In the case of odd Legendre polynomials it turns out that \( \rho_{ci} = 0 \) is zero, hence it is enough to guarantee that the perturbation parameter and the time scale are canceled to ensure stable non-null patterns, that is, \( p_1 = 0 \) and \( w_1 = 0 \).

Once the Fredholm-Alternative is fulfilled, for the solution the second order system the following form is assumed:

\[
u_2 = \left( \sum_{s=0}^{M} V_s^{(2)} a^2 P_s + V_s^{(2)} a^2 \bar{P}_s \right) + V_{ij} a \bar{a} P_i \bar{P}_j.
\]  
(A.23)

where \( V_{ij} = V_{ij}(x, y) \).

Substituting (A.23) and (A.15) into (A.13), and collecting terms, we obtain the coefficient vectors:

\[
V_s^{(2)} = (-1)^s (-\eta + l_{\alpha} \mathbb{D})^{-1} \rho_s Q(V^{(1)}, V^{(1)})
\]  
(A.24)

\[
\bar{V}_s^{(2)} = (-1)^s (-\eta + l_{\alpha} \mathbb{D})^{-1} \rho_s Q(V^{(1)}, \bar{V}^{(1)})
\]  
(A.25)

\[
V_{ij} = 2(-1)^s (-\eta + l_{ij} \mathbb{D})^{-1} Q(V^{(1)}, V^{(1)})
\]  
(A.26)

\( O(\varepsilon^3) \) equations

We start again applying the Fredholm Alternative:

\[
\left\langle u^s | Q(u_1, u_2) + C(u_1, u_1, u_1) + p_2 \eta \Delta_p \bar{a}_n u_1 - w_2 \frac{\partial u_1}{\partial T} \right\rangle = 0.
\]  
(A.27)

If \( X = u_1 \) and \( Y = u_2 \) in (A.3) and \( X = u_1 \) and \( Y = u_1 \) in (A.4), the quadratic and cubic terms of (A.27) can be obtained:

\[
Q(u_1, u_2) = \sum_{s=0}^{M} Q(V^{(1)}, V_s^{(2)}) a^3 P_s + Q(V^{(1)}, \bar{V}_s^{(2)}) a^2 \bar{P}_s + Q(V^{(1)}, V_s^{(2)}) a \bar{a} \bar{P}_s + Q(V^{(1)}, \bar{V}_s^{(2)}) a \bar{a} \bar{P}_s + Q(V^{(1)}, \bar{V}_s^{(2)}) a^2 P_s
\]  
(A.28)

and

\[
C(u_1, u_1, u_1) = C(V^{(1)}, V^{(1)}) a^3 P_i^3 + 3C(V^{(1)}, V^{(1)}) a^2 \bar{a} P_i^2 \bar{P}_j + 3C(V^{(1)}, V^{(1)}) a \bar{a} \bar{a} \bar{P}_j
\]  
(A.29)

once again the product of eigenfunctions can be represented as an expansion of the same eigenfunctions:

\[
P_s P_s = \sum_{n} \rho_n^{(s)} P_n
\]  
(A.30)

\[
P_i^3 = \sum_{n} \rho_i^{(s)} P_n
\]  
(A.31)

Thus, substituting (A.30) and (A.31) in (A.28) and (A.29), we obtain

\[
Q(u_1, u_2) = \sum_{s=0}^{M} \sum_{s=0}^{M} (Q(V^{(1)}, V_s^{(2)}) \rho_n^{(s)} a^3 P_n + (Q(V^{(1)}, \bar{V}_s^{(2)}) \rho_{\eta}^{(s)} a^3 \bar{P}_n + (Q(V^{(1)}, V_{ij}) \rho_n a^2 \bar{a} P_n \bar{P}_j + (Q(V^{(1)}, V_{ij}) \rho_n a \bar{a} \bar{a} \bar{P}_j P_s + Q(V^{(1)}, V_s^{(2)}) a^2 \bar{a} \bar{P}_j P_s,
\]  
(A.32)
where

\[ C(u) = \sum_{n=0}^{M} (C(V^{(1)}, V^{(1)}) \rho_n^a) u^n + 3(C(V^{(1)}, V^{(1)}) \rho_0) a^2 \bar{P}_n + 3(C(V^{(1)}, V^{(1)}) \rho_n) a^3 \bar{P}_n + (C(V^{(1)}, V^{(1)}) \rho_n^c) a^3 \bar{P}_n. \]  

(A.33)

Now, (A.32), (A.33) and \( u^* = V^* aW_i \) are replaced into (A.27) to get

\[
\begin{align*}
&\langle V^*| \sum_{s=0}^{M} (Q(V^{(1)}, V^{(2)}_s) \rho_i^*) \rangle a^4 \delta_{i0} + \langle V^*| (Q(V^{(1)}, V_{ij}) \rho_0) \rangle a^2 \bar{a}^2 \delta_{i0} + \langle V^*| (Q(V^{(1)}, V^{(2)}_0) \rho_i^*) \rangle a^2 \bar{a}^2 \delta_{i0} + \langle V^*| 3(C(V^{(1)}, V^{(1)}) \rho_0) \rangle a^2 \delta_{i0} - \\
&\langle V^*| V^{(1)} \rangle a \frac{da}{dT} \delta_{i0} = 0, \quad \text{(A.34)}
\end{align*}
\]

where \( \delta_{i0} = \int_0^\infty (P_i \bar{P}_i) dx dy \).

Equation (A.34) can be recast as

\[
\begin{align*}
\frac{1}{2} \langle V^*| V^{(1)} \rangle \frac{d|a|^2}{dT} &= \left( \langle V^*| \sum_{s=0}^{M} (Q(V^{(1)}, V^{(2)}_s) \rho_i^*) + (C(V^{(1)}, V^{(1)}) \rho_i^*) \rangle a^4 + \\
&\langle V^*| (Q(V^{(1)}, V_{ij}) \rho_0) + Q(V^{(1)}, V^{(2)}_0) \rangle a^2 \bar{a}^2 + \langle V^*| \nu B V^{(1)} \rangle a^2 \right) \quad \text{(A.35)}
\end{align*}
\]

By repeating the same procedure but for for the null vector \( u^* = V^* \bar{a} \bar{W}_j \), the Stuart-Landau amplitude equations are obtained:

\[
\begin{align*}
\frac{d|a|^2}{dT} &= \alpha |a|^4 + \beta |a(T)|^2 |\bar{a}|^2 + \theta |a|^2, \\
\frac{d|\bar{a}|^2}{dT} &= \alpha |\bar{a}|^4 + \beta |a|^2 |\bar{a}|^2 + \theta |\bar{a}|^2, \quad \text{(A.36)}
\end{align*}
\]

where

\[
\begin{align*}
E &= \frac{1}{2} \langle V^*| V^{(1)} \rangle \\
\alpha &= \frac{1}{E} \left( \langle V^*| \sum_{s=0}^{M} (Q(V^{(1)}, V^{(2)}_s) \rho_i^*) + (C(V^{(1)}, V^{(1)}) \rho_i^*) \rangle \right) \quad \text{(A.37)} \\
\beta &= \frac{1}{E} \langle V^*| (Q(V^{(1)}, V_{ij}) \rho_0) + Q(V^{(1)}, V^{(2)}_0) \rangle + 3(C(V^{(1)}, V^{(1)}) \rho_0) \rangle \quad \text{(A.38)} \\
\theta &= \frac{1}{E} \langle V^*| \nu B V^{(1)} \rangle \quad \text{(A.39)}
\end{align*}
\]

**Stability of the Stuart-Landau equations**

The system (A.36) has 4 equilibrium points:

\[
\begin{align*}
1) \quad |a|^2 = 0, & \quad |\bar{a}|^2 = 0 \quad \text{(A.40)} \\
2) \quad |a|^2 = 0, & \quad |\bar{a}|^2 = -\frac{\theta}{\alpha} \quad \text{(A.41)} \\
3) \quad |a|^2 = -\frac{\theta}{\alpha}, & \quad |\bar{a}|^2 = 0 \quad \text{(A.42)} \\
4) \quad |a|^2 = -\frac{\theta}{\alpha + \beta}, & \quad |\bar{a}|^2 = -\frac{\theta}{\alpha + \beta} \quad \text{(A.43)}
\end{align*}
\]
We are interested in the parameter regions where the equilibrium points are stable so that the Turing patterns are maintained for long times.

By means of the corresponding Jacobian matrices, conditions for stability of each equilibrium point can be established to obtain the results summarized in Table I.

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