Double Integral of the Product of the Exponential of an Exponential Function and a Polynomial Expressed in Terms of the Lerch Function

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Abstract: In this work, the authors use their contour integral method to derive an application of the Fourier integral theorem given by \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{m(x-y)} - e^{x+y} (\log(a) + x - y) dx dy \) in terms of the Lerch function. This integral formula is then used to derive closed solutions in terms of fundamental constants and special functions. Almost all Lerch functions have an asymmetrical zero distribution. There are some useful results relating double integrals of certain kinds of functions to ordinary integrals for which we know no general reference. Thus, a table of integral pairs is given for interested readers. All of the results in this work are new.

Keywords: Fourier integral theorem; double integral; Lerch function; contour integral; exponential function

1. Statement of Significance

In 1906, Niels Nielsen [1] produced his famous book on the Gamma function. In this work, the authors use their contour integral method and apply it to an interesting integral in the book of Nielsen [1] to yield a double integral and express its closed form in terms of the Lerch function. This derived integral formula is then used to provide formal derivations and new formulæ in the form of a summary table of integrals, Table 1. The Lerch function being a special function has the fundamental property of analytic continuation, which enables the expansion of the range of evaluation for the parameters involved in the definite integral.

Double integrals over a real line are used in very interesting areas in mathematics. Some areas of high interest are namely in the use of the Fourier integral theorem in Electromagnetic Theory of Propagation, Interference, and Diffraction of Light [2], evaluation of two-dimensional Gaussian integrals in the constructions of representation theory and related topics of differential geometry and analysis [3], and the implementation of Cahn’s scheme for simulating the morphology of isotropic spinodal decomposition [4].

2. Introduction

In 1882, Joseph Fourier (1768–1830) discovered a double integral representation [5] of a non-periodic function \( f(x) \) for all real \( x \), which is universally known as the Fourier Integral Theorem in the form

\[
\begin{align*}
  f(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \left( \int_{-\infty}^{\infty} f(a) e^{-ika} da \right) dk
\end{align*}
\]

(1)

Throughout the nineteenth and twentieth centuries, mathematicians and mathematical physicists recognized the significance of this theorem. It is regarded as one of the most fundamental representation theorems of modern mathematical analysis according to Lord Kelvin (1824–1907) and Peter Guthrie Tait (1831–1901).
In this work, we will derive the Fourier integral theorem applied to a function involving the product of the exponential of an exponential function and a polynomial and express this integral in terms of the Lerch function. The application of the Fourier integral theorem is in the form of a double integral over a real line. The definite integral derived in this manuscript is given by

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{mx-my-e^{\theta}+y} (\log(a) + x - y)^k dx dy$$

(2)

where the parameters $k$ and $a$ are general complex numbers and $0 < Re(m) < 1$. In the book of Titchmarsh [6], examples on the Fourier integral theorem are applied to a vast number of functions and real-world applications are showcased. This work is important because the authors were unable to find similar derivations in current literature. The number of functions and real-world applications are showcased. This work is important.

3. Definite Integral of the Contour Integral

We use the method in [7]. The variable of integration in the contour integral is $t = m + w$. The cut and contour are in the second quadrant of the complex $t$-plane. The cut approaches the origin with zero radius and is on opposite sides of the cut. Using Equation (3), we replace $y$ by $x - y + \log(a)$ and then multiply by $e^{mx-my-e^{\theta}+y}$. Next, we take the definite integral over $x \in (-\infty, \infty)$ and $y \in (-\infty, \infty)$ to obtain

$$\frac{y^k}{\Gamma(k+1)} = \frac{1}{2\pi i} \int_C \frac{e^{wv}}{w^{k+1}} dw.$$  

(3)

where $C$ is, in general, an open contour in the complex plane, where the bilinear concomitant has the same value at the end points of the contour. This method involves using a form of Equation (3), then multiplying both sides by a function, and then taking a definite integral of both sides. This yields a definite integral in terms of a contour integral. A second contour integral is derived by multiplying Equation (3) by a function and performing some substitutions so that the contour integrals are the same.

4. The Lerch Function

We use Equation (1.11.3) in [9], where $\Phi(z, s, v)$ is the Lerch function, which is a generalization of the Hurwitz zeta $\zeta(s, v)$ and Polylogarithm functions $Li_n(z)$. The Lerch function has a series representation given by

$$\Phi(z, s, v) = \sum_{n=0}^{\infty} (v+n)^{-s}z^n$$

(5)

where $|z| < 1, v \neq 0, -1, -2, -3, ..$, and is continued analytically by its integral representation given by
\[\Phi(z, s, v) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-vt}}{1 - ze^{-t}} dt = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}e^{-(v-1)t}}{e^t - z} dt \quad (6)\]

where \(Re(v) > 0\) and either \(|z| \leq 1, z \neq 1, Re(s) > 0\), or \(z = 1, Re(s) > 1\).

5. Infinite Sum of the Contour Integral

In this section, we again use Cauchy’s integral formula from Equation (3) and take the infinite sum to derive equivalent sum representations for the contour integrals. We proceed by using Equation (3), replacing \(y\) by \(\log(a) + i\pi(2y + 1)\), multiplying both sides by \(-2i\pi e^{i\pi m(2y+1)}\), and simplifying to obtain

\[\frac{-2i\pi e^{\frac{1}{2}i\pi(k+4my+2m)}(-i\log(a) + 2\pi y + \pi)^k}{\Gamma(k+1)} = -\frac{1}{2\pi i} \int_C 2i\pi a^w w^{-k-1}e^{i\pi(2y+1)(m+w)} dw \quad (7)\]

Next, we take the infinite over \(y \in [0, \infty)\) and simplify it using the Lerch function to obtain

\[-\frac{(2i\pi)^{k+1} e^{i\pi m} \Phi\left(e^{2i\pi m}, -k, \frac{\pi - i\log(a)}{2\pi}\right)}{\Gamma(k+1)} = -\frac{1}{2\pi i} \sum_{y=0}^\infty \int_C 2i\pi a^w w^{-k-1}e^{i\pi(2y+1)(m+w)} dw \]

\[= -\frac{1}{2\pi i} \int_C \sum_{y=0}^\infty 2i\pi a^w w^{-k-1}e^{i\pi(2y+1)(m+w)} dw \]

\[= \frac{1}{2\pi i} \int_C \pi a^w w^{-k-1} \csc(\pi(m + w)) dw \quad (8)\]

from (1.232.3) in [8] and \(Im(m + w) > 0\) for convergence of the sum.

6. Definite Integral in Terms of the Lerch Function

**Theorem 1.** For all \(k, a \in \mathbb{C}, 0 < Re(m) < 1,\)

\[\int_{-\infty}^\infty \int_{-\infty}^\infty e^{\mu x - my - e^x + y(\log(a) + x - y)^k} dxdy = (2i\pi)^{k+1} \left(-e^{i\pi m}\right) \Phi\left(e^{i\pi m}, -k, \frac{\pi - i\log(a)}{2\pi}\right) \quad (9)\]

**Proof.** Since the right-hand sides of Equations (4) and (8) are equal, we can equate the left-hand sides and simplify the factorial to achieve the stated result. \[\square\]

**Main Results and Table of Integrals**

In this section, we evaluate Equation (9) for various values of the parameters in terms of special functions and fundamental constants and create a table of integrals. Some of the fundamental constants evaluated are Aprey’s constant \(\xi(3)\) from Section 1.6 in [10]; Catalan’s constant \(C\), Equation (9.73) in [8]; Euler’s constant \(\gamma\), Equation (9.73) in [8]; and the Glaisher–Kinkelin constant \(A\), Section 2.15 in [10]. Some special functions used are the polylogarithm function \(Li_v(z)\) from Equation (64:12:2) in [11] and the hypergeometric function \(_2F_1(a, b; c; z)\) from Equation (9.559) in [8].
7. Derivation of the Degenerate Case

7.1. Derivation of Entry (1)

**Lemma 1.** For $0 < \text{Re}(m) < 1$,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{mx-ny-e^{\varphi}+y} \, dx \, dy = \pi \csc(\pi m)
\] (10)

**Proof.** Use Equation (9), set $k = 0$, and simplify using entry (2) from the Table below (64:12:7) in [11]. □

7.2. Derivation of Entry (2)

**Lemma 2.**
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{1/2\left(-2(e^{x}+e^{y})+y\right)} \frac{x-y}{(x-y)^2 + \pi^2} \, dx \, dy = \log(2) / \pi
\] (11)
and
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{1/2\left(-2(e^{x}+e^{y})+y\right)} \frac{(x-y)(x-y)\log(a) + x - y}{(x-y)^2 + \pi^2} \, dx \, dy = 0
\] (12)

**Proof.** Use Equation (9), set $k = -1$, $a = -1$, $m = 1/2$, rationalize the denominator, compare the real and imaginary parts, and simplify using entry (3) from the table below (64:12:7) in [11]. □

7.3. Derivation of Entry (3)

**Theorem 2.** For all $k, a \in \mathbb{C}, 0 < \text{Re}(m) < 1, 0 < \text{Re}(n) < 1$,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{y\left(-\left(m+n-1\right) - e^{x} - e^{y}\left(e^{my+nx} - e^{mx+ny}\right)\log(a) + x - y\right)} \, dx \, dy
\]
\[= (2i\pi)^{k+1}\left(e^{in\pi}, -k, \frac{\pi - i \log(a)}{2\pi}\right) - e^{in\pi}\Phi\left(e^{2in\pi}, -k, \frac{\pi - i \log(a)}{2\pi}\right)
\] (13)

**Proof.** Use Equation (9) to form a second equation by replacing $m$ by $n$, and take their difference. □

7.4. Derivation of Entry (4)

**Lemma 3.** For all $0 < \text{Re}(m) < 1, 0 < \text{Re}(n) < 1$,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{y\left(-\left(m+n-1\right) - e^{x} - e^{y}\left(e^{my+nx} - e^{mx+ny}\right)\log(a) + x - y\right)} \frac{x-y}{(x-y)^2 + \pi^2} \, dx \, dy
\]
\[= \log\left(\cot\left(\frac{\pi n}{2}\right) \tan\left(\frac{\pi n}{2}\right)\right)
\] (14)

**Proof.** Use Equation (13), set $k = -1, a = 1$, and simplify using entry (5) from the table below (64:12:7). □

7.5. Derivation of Entry (5)

**Theorem 3.** For $k \in \mathbb{C}, 0 < \text{Re}(m) < 1$,
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - y + i\pi)^{k} e^{mx-ny-e^{\varphi}+y} \, dx \, dy
\]
\[= (2i\pi)^{k+1}\left(-e^{-i\pi m}\right) Li_{-k}\left(e^{2i\pi}\right)
\] (15)

**Proof.** Use Equation (9), set $a = -1$, and simplify using Equation (64:12:2) in [11]. □
7.6. Derivation of Entry (6)

**Lemma 4.** For \( a \in \mathbb{C} \),
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(-2(e^x + e^y) + x + y)} \log(ia + x - y)\,dx\,dy = \pi \log \left( \frac{4i/\pi \Gamma \left( \frac{3}{4} + \frac{i}{8\lambda} \right)^2}{\Gamma \left( \frac{1}{4} + \frac{i}{8\lambda} \right)^2} \right) \tag{16}
\]

**Proof.** Use Equation (9), and set \( m = 1/2 \). Next, take the first partial derivative with respect to \( k \), set \( k = 0 \), and simplify using entry (4) from the table below (64:12:7) and from Equation (64:10:2) in [11]. \( \Box \)

7.7. Derivation of Entry (7)

**Lemma 5.**
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(-2(e^x + e^y) + x + y)} \log \left( x - y - \frac{1}{2} \right)\,dx\,dy = \pi \log \left( \frac{4i/\pi \Gamma \left( \frac{3}{4} + \frac{i}{8\lambda} \right)^2}{\Gamma \left( \frac{1}{4} + \frac{i}{8\lambda} \right)^2} \right) \tag{17}
\]

**Proof.** Use Equation (16), set \( a = i/2 \), and simplify. \( \Box \)

7.8. Derivation of Entry (8)

**Lemma 6.**
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(-6(e^x + e^y) + 2x + 3y)} \left( e^{y/6} - e^{x/6} \right) \log(x - y)\,dx\,dy
\]
\[
= 2(-1)^{5/6} \pi \Phi \left( \frac{1}{2}, \frac{1}{3} \right) - \frac{i\pi^2}{2} + \frac{i\pi^2}{\sqrt{3}} + \frac{\pi \log(4)}{\sqrt{3}}
\]
\[
- \pi \log(\pi) + \frac{2\pi \log(\pi)}{\sqrt{3}} + \pi \log \left( \frac{4\Gamma \left( \frac{1}{4} \right)^2}{\Gamma \left( \frac{3}{4} \right)^2} \right) \tag{18}
\]

**Proof.** Use Equation (13), set \( m = 1/2, n = 1/2, a = 1 \), take the first partial derivative with respect to \( k \), set \( k = 0 \), and simplify using entry (4) from the table below (64:12:7) in [11]. \( \Box \)

7.9. Derivation of Entry (9)

**Lemma 7.**
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(-6(e^x + e^y) + 2x + 3y)} \left( e^{y/6} - e^{x/6} \right) \log(x - y)\,dx\,dy = -\frac{\log(3)}{2} \tag{19}
\]

**Proof.** Use Equation (13), set \( k = -1, a = 1, m = 1/2, n = 1/3 \), and simplify using entry (1) from the table below (64:12:7) in [11]. \( \Box \)

7.10. Derivation of Entry (10)

**Lemma 8.**
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(-6(e^x + e^y) + 2x + 3y)} \left( e^{y/6} - e^{x/6} \right) (x - y)\,dx\,dy
\]
\[
= \frac{1}{3} \pi \left( (-1)^{5/6} \pi \left( 12i\Phi \left( \frac{1}{2}, \frac{1}{3} \right) \right) + \left( 2\pi \right) + 12iC - i\pi^2 \right) \tag{20}
\]

**Proof.** Use Equation (13), set \( a = 1, m = 1/2, n = 1/3 \), and take the first partial derivative with respect to \( k \), set \( k = 1 \), and simplify using Equation (21) in [12]. \( \Box \)
7.11. Derivation of Entry (11)

Lemma 9.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{6}(x^6 + e^y + 2x + 3y)} \left( e^{x/6} - e^{y/6} \right) (\pi - i(x - y))^2 \log(x - y + i\pi) dxdy = 8\pi^3 \Phi' \left( \frac{(-1)^{2/3} - 2, 1}{\sqrt{3}} \right)
- \frac{8(-1)^{2/3} \pi^3 \Phi' \left( \frac{(-1)^{2/3} - 2, 1}{\sqrt{3}} \right)}{\sqrt{3}}
- 14\pi \zeta(3) + \frac{8\pi^3 \log(2i\pi)}{3\sqrt{3}} - \frac{8\sqrt{-1}\pi^3 \log(2i\pi)}{3\sqrt{3}}
\]

(21)

Proof. Use Equation (13), set \(a = -1, m = 1/2, n = 1/3\), and take the first partial derivative with respect to \(k\), then set \(k = 2\), and simplify using Equation (19) in [12].

7.12. Derivation of Entry (12)

Lemma 10.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{8}(x^8 + e^y + 2x + 3y)} \left( e^{y/8} - e^{x/8} \right) (x - y)^2 \log(x - y + i\pi) dxdy = 4(-1)^{2/3} \pi^2 L_{-1} \left( \frac{(-1)^{2/3}}{2} \right) + 12i\pi^2 \log(A)
- i\pi^2 - \frac{1}{3}i\pi^2 \log(16) - \left( \frac{2}{3} + i \right) \pi^2 \log(2i\pi) + \frac{2i\pi^2 \log(2i\pi)}{\sqrt{3}}
\]

(22)

Proof. Use Equation (13), set \(a = -1, m = 1/2, n = 1/3\), take the first partial derivative with respect to \(k\), then set \(k = 1\), and simplify using Equation (18) in [12].

7.13. Derivation of Entry (13)

Lemma 11.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{8}(x^8 + e^y + 2x + 3y)} (x - y) \sinh \left( \frac{x - y}{8} \right) dxdy = \frac{\pi^2}{\sqrt{2}}
\]

(23)

Proof. Use Equation (13), set \(k = 1, a = 1, m = 1/4, n = 1/2\), and simplify using entry (1) from the table below (64:12:7) in [11].

7.14. Derivation of Entry (14)

Lemma 12.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{8}(x^8 + e^y + 2x + 3y)} \sinh \left( \frac{x - y}{8} \right) dxdy = -\frac{1}{2} \left( \sqrt{2} - 1 \right) \pi
\]

(24)

Proof. Use Equation (13), set \(k = 0, a = 1, m = 1/4, n = 1/2\), and simplify using entry (2) from the table below (64:12:7) in [11].
7.15. Derivation of Entry (15)

Lemma 13.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{\frac{x}{6}(x^2+e^y)+2x+3y}(e^x/6-e^{y/6})\log(x-y)}{x-y} \, dx \, dy
\]

\[
= -\sqrt{3} \Phi'\left(\frac{1}{3}, 1, 1\right) + i\gamma \pi \frac{1}{2} + \frac{1}{4} \log(9) \log(2\pi) + \frac{1}{4} i\pi \log \left(\frac{192 \pi^6}{\Gamma \left(\frac{1}{8}\right)}\right)
\]

Proof. Use Equation (13), set \(a = 1, n = 1/2, m = 1/3\), take the first partial derivative with respect to \(k\), then set \(k = -1\), and simplify using entry (1) from the table below (64:12:7) in [11]. □

7.16. Derivation of Entry (16)

Theorem 4. For \(k \in \mathbb{C}\),

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{x}{6}(x^2+e^y)+2x+3y} (x - y + i\pi)k \, dx \, dy = -i^k \left(2^{k+1} - 1\right)(2\pi)^{k+1} \xi(-k)
\]

Proof. Use Equation (9), set \(a = -1, m = 1/2\), and simplify using entry (4) from the table below (64:12:7) and entry (2) from the table below (64:7) in [11]. □

7.17. Derivation of Entry (17)

Lemma 14.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{\frac{x}{6}(x^2+e^y)+2x+3y}(e^x/6-e^{y/6})}{x-y} \, dx \, dy = \frac{\log(3)}{2}
\]

Proof. Use Equation (13), set \(k = -1, a = 1, n = 1/2, m = 1/3\), and simplify using entry (1) from the table below (64:12:7) in [11]. □

7.18. Derivation of Entry (18)

Theorem 5. For \(0 < \text{Re}(m) < 1, 0 < \text{Re}(n) < 1\),

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{\left(-m(m+n-1)\right)-e^{-e^y}(e^{my+nx} - e^{nx+my})}}{2x - 2y + i\pi} \, dx \, dy
\]

\[
= \frac{2}{5} \left(\text{\,}_2F_1\left(\frac{3}{4}, 1; \frac{7}{4}; e^{2in\pi}\right) - \text{\,}_2F_1\left(\frac{3}{4}, 1; \frac{7}{4}; e^{2in\pi}\right)\right)
\]

Proof. Use Equation (13), set \(k = -1, a = i\), and simplify using Equation (9.559) in [8]. □

7.19. Derivation of Entry (19)

Theorem 6. For \(0 < \text{Re}(m) < 1, 0 < \text{Re}(n) < 1\),

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{\left(-m(m+n-1)\right)-e^{-e^y}(e^{my+nx} - e^{nx+my})}}{x-y+i\pi} \, dx \, dy
\]

\[
= e^{-imn} \log\left(1 - e^{2in\pi}\right) - e^{-imn} \log\left(1 - e^{2in\pi}\right)
\]

Proof. Use Equation (13), set \(k = -1, a = -1\), and simplify using Equation (9.559) in [8]. □
7.20. Derivation of Entry (20)

Lemma 15.
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(2x^2+y^2)} (x-y) \log(x-y) \, dx \, dy = -4i\pi C \]  
(30)

Proof. Use Equation (9), set \( m = 1/2 \), and simplify using entry (4) below Table (64:12:7) in [11]. Then, take the first partial derivative with respect to \( k \), set \( k = 1, a = 1 \), and simplify using Equation (9.73) in [8].

7.21. Derivation of Entry (21)

Lemma 16.
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(2x^2+y^2)} (x-y + i\pi) \log(x-y + i\pi) \, dx \, dy = \frac{1}{6} \pi^2 \left( -3\pi + 2i \log \left( \frac{128\sqrt{3} \pi^{3/2}}{A^{36}} \right) \right) \]  
(31)

Proof. Use Equation (9), set \( m = 1/2 \), and simplify using entry (4) below Table (64:12:7) in [11]. Then, take the first partial derivative with respect to \( k \), set \( k = 1, a = -1 \), and simplify using 2.15 in [10].

7.22. Derivation of Entry (22)

Lemma 17.
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(2x^2+y^2)} (x-y) \log(x-y) \, dx \, dy = \pi \log(4) + i\pi \frac{3}{2} \zeta \left( -\frac{1}{2} \right) \]  
(33)

Proof. Use Equation (9) set \( m = 1/2 \) and simplify using entry (4) below Table (64:12:7) in [11]. Then, take the first partial derivative with respect to \( k \); set \( a = -1 \); apply L'Hopital’s rule to the right-hand side as \( k \to -1 \); and simplify using Equations (1.7.7), (44.7.7), and (64.4.1) in [11].

7.23. Derivation of Entry (23)

Lemma 18.
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(2x^2+y^2)} (x-y + i\pi) \log(x-y + i\pi) \, dx \, dy = \pi \log(4) + i\pi \frac{\pi^2}{2} \]  
(33)

Proof. Use Equation (9) set \( m = 1/2 \) and simplify using entry (4) below Table (64:12:7) in [11]. Then, take the first partial derivative with respect to \( k \), set \( k = 0, a = -1 \), and simplify.

7.24. Derivation of Entry (24)

Lemma 19.
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{x^2-x^2-y^2}{2}} \sqrt{x-y+i\pi} \, dx \, dy = \left( \sqrt{2} - 1 \right) e^{\frac{3\pi}{2}} \pi \zeta \left( \frac{1}{2} \right) \]  
(34)

Proof. Use Equation (9), set \( k = -1/2, m = 1/2, a = -1 \), and simplify using Equation (64:12:1) and entry (2) from the table below (64:7) in [11] and 1.5 in [13].

7.25. Derivation of Entry (25)

Lemma 20.
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{x^2-x^2-y^2}{2}} \sqrt{x-y+i\pi} \, dx \, dy = 2\sqrt{2} \left( 1 - 2\sqrt{2} \right) e^{\frac{3\pi}{4}} \pi \zeta \left( \frac{1}{2} \right) \]  
(35)
Proof. Use Equation (9), set \( k = 1/2, m = 1/2, a = -1 \), and simplify using Equation (64:12:1) and entry (2) from the table below (64:7) in [11].

7.26. Derivation of Entry (26)

Lemma 21.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(-2(e^{x}+e^{y})+x+y)} \frac{((x-y)^2 - \pi^2)}{((x-y)^2 + \pi^2)^2} \, dx \, dy = -\frac{\pi}{24}
\]  

(36)

and

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\frac{1}{2}(-2(e^{x}+e^{y})+x+y)} \frac{(x-y)}{((x-y)^2 + \pi^2)^2} \, dx \, dy = 0
\]  

(37)

Proof. Use Equation (9), set \( k = -1 \), then take the first partial derivative with respect to \( a \), set \( a = -1, m = 1/2 \), rationalize the denominator, compare the real and imaginary parts, and simplify using entry (3) from the table below (64:12:7) in [11].

7.27. Derivation of Entry (27)

Lemma 22.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{\frac{1}{2}(-2(e^{x}+e^{y})+x+y)}((x-y+2)(x-y)^3 + 6\pi^2(y-x) - \pi^4)}{((x-y)^2 + \pi^2)^3} \, dx \, dy = -\frac{\pi}{24}
\]  

(38)

and

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\pi e^{\frac{1}{2}(-2(e^{x}+e^{y})+x+y)}((x-y+3)(x-y)^2 + \pi^2(x-y-1))}{((x-y)^2 + \pi^2)^3} \, dx \, dy = -\frac{3\zeta(3)}{16\pi^2}
\]  

(39)

Proof. Use Equation (9), set \( k = -1 \), then take the second partial derivative with respect to \( a \), set \( a = -1, m = 1/2 \), rationalize the denominator, compare the real and imaginary parts, and simplify using entries (1) and (3) from the table below (64:12:7) and entry (2) from the table below (64:7) in [11].

7.28. Derivation of Entry (28)

Lemma 23.

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{\frac{1}{2}(-4(e^{x}+e^{y})+x+3y)}(x-y)}{((x-y)^2 + \pi^2)^2} \, dx \, dy = \frac{\pi^2 - 48C}{192\sqrt{2}\pi^2}
\]  

(40)

and

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{\frac{1}{2}(-4(e^{x}+e^{y})+x+3y)}((x-y)^2 - \pi^2)}{((x-y)^2 + \pi^2)^2} \, dx \, dy = -\frac{48C + \pi^2}{96\sqrt{2}\pi}
\]  

(41)

Proof. Use Equation (9), set \( k = -1 \), then take the first partial derivative with respect to \( a \), set \( a = -1, m = 1/4 \), rationalize the denominator, compare the real and imaginary parts, and simplify using entry (3) from the table below (64:12:7) in [11].

8. Summary Table of Results

In this section we will summarize the evaluation of Equation (9) from the previous section.
Table 1. Summary Table of Results.

| $f(x, y)$ | $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy$ |
|-----------|--------------------------------------------------|
| $e^{mx-ny-e^{x}+e^{y}}$ | $\pi \arccsch(\pi m)$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $\log(\cot(\frac{\pi m}{2}))$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $(2i\pi)^{k+1} (-e^{-i\pi m}) \text{Li}_{-k}(e^{2i\pi})$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $\pi \log \left( \frac{4i\pi \Gamma \left( \frac{\pi}{\pi m} \right)^{2}}{(\frac{\pi}{\pi m} + i\pi)^{2}} \right)$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $-\log(3)$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $\frac{\pi^2}{2}$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $\frac{1}{2} (\sqrt{2} - 1) \pi$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $-i k (2k+1-1) (2\pi)^{k+1} \zeta(-k)$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $\log(3)$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $\frac{3}{2} (e^{i\pi n} \text{$_2F_1$}(\frac{3}{4}, 1; \frac{1}{4}; e^{2i\pi}) - e^{i\pi n} \text{$_2F_1$}(\frac{3}{4}, 1; \frac{1}{4}; e^{2i\pi}))$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $e^{-i\pi m} \log(1 - e^{2i\pi}) - e^{-i\pi m} \log(1 - e^{2i\pi})$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $-4i\pi C$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $\frac{1}{2} \log(2)(2i\gamma + \pi + i\log(8\pi^2))$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $\pi \log(4) + \frac{i\pi}{2}$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $\frac{1}{2} (\sqrt{2} - 1) e^{i\pi} \sqrt{2\pi} \zeta\left(\frac{1}{2}\right)$ |
| $\frac{1}{2} \left( -2(e^{x}+e^{y}) + x + y \right)$ | $2\sqrt{2} (1 - 2\sqrt{2}) e^{i\pi} \pi^{3/2} \zeta\left(-\frac{1}{2}\right)$ |

9. Discussion

In this work, the authors derived a double integral formula in terms of the Lerch function. This integral formula was then used to derive special cases in terms of fundamental constants and special functions. A table of integrals featuring some of the integral results was presented for the benefit of interested readers. We used Wolfram Mathematica to numerically verify the formulas for various ranges of the parameters for real and imaginary values. We will use our contour integral method to derive other double integrals and produce more tables of integrals in future work.

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