Extendibility limits the performance of quantum processors

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Resource theories in quantum information science are helpful for the study and quantification of the performance of information-processing tasks that involve quantum systems. These resource theories also find applications in other areas of study; e.g., the resource theories of entanglement and coherence have found use and implications in the study of quantum thermodynamics and memory effects in quantum dynamics. In this paper, we introduce the resource theory of unextendibility, which is associated to the inability of extending quantum entanglement in a given quantum state to multiple parties. The free states in this resource theory are the k-extendible states, and the free channels are k-extendible channels, which preserve the class of k-extendible states. We make use of this resource theory to derive non-asymptotic, upper bounds on the rate at which quantum communication or entanglement preservation is possible by utilizing an arbitrary quantum channel a finite number of times, along with the assistance of k-extendible channels at no cost. We then show that the bounds we obtain are significantly tighter than previously known bounds for both the depolarizing and erasure channels.

Introduction—Recent years have seen progress in the development of programmable quantum computers and information processing devices; several groups are actively developing superconducting quantum processors [1] and satellite-to-ground quantum key distribution [2]. It is thus pertinent to establish benchmarks on the information-processing capabilities of quantum devices that are able to process a finite number of qubits reliably. Experimentalists can then employ these benchmarks to evaluate how far they are from achieving the fundamental limitations on performance.

In this paper, we first develop a resource theory of unextendibility and then apply it to bound the performance of quantum processors. In particular, the resource theory of unextendibility leads to non-asymptotic upper bounds on the rate at which entanglement can be preserved when using a given quantum channel a finite number of times. We then apply this general bound to the case of depolarizing and erasure channels, which are common models of noise in quantum processors. For these channels, we find that our bounds are significantly tighter than previously known non-asymptotic bounds from [3, 4].

The resource theory of unextendibility can be understood as a relaxation of the well known resource theory of entanglement [5, 6], and it is a relaxation alternative to the resource theory of negative partial transpose states from [7, 8], in which the free states are the positive partial transpose (PPT) states and the free channels are the PPT-preserving channels. In the resource theory of entanglement, the free states are the separable states, those not having any entanglement at all and denoted by SEP(A:B). Any separable state σ_{AB} can be written as σ_{AB} = \sum_x p(x) \tau_A^x \otimes \omega_B^x, where p(x) is a probability distribution and \{\tau_A^x\}_x and \{\omega_B^x\}_x are sets of states; the free channels are those that can be performed by local operations and classical communication (LOCC) [5, 9]. An LOCC channel \mathcal{L}_{AB\rightarrow A'B'} is a separable superoperator (although the converse is not true), and can hence be written as \mathcal{L}_{AB\rightarrow A'B'} = \sum_y \mathcal{E}_{A\rightarrow A'}^y \otimes \mathcal{F}_{B\rightarrow B'}^y, where \{\mathcal{E}_{A\rightarrow A'}^y\}_y and \{\mathcal{F}_{B\rightarrow B'}^y\}_y are sets of completely positive (CP) maps such that \mathcal{L}_{A\rightarrow A'}^y is trace preserving. A special kind of LOCC channel is a one-way (1W-) LOCC channel from A to B, in which Alice performs a quantum instrument, sends the classical outcome received from Bob, who then performs a quantum channel conditioned on the classical outcome received from Alice. As such, any 1W-LOCC channel takes the form stated above, except that \{\mathcal{E}_{A\rightarrow A'}^y\}_y is a set of CP maps such that the sum map \sum_y \mathcal{E}_{A\rightarrow A'}^y is trace preserving, while \{\mathcal{F}_{B\rightarrow B'}^y\}_y is a set of quantum channels.

The set of free states in the resource theory of unextendibility is larger than the set of free states in the resource theory of entanglement. By relaxing the resource theory of entanglement in this way, we obtain tighter, non-asymptotic bounds on the entanglement transmission rates of a quantum channel.

Before we begin with our development, we note here that detailed proofs of all statements that follow are given in the supplementary material.

Resource theory of unextendibility—In the resource theory of unextendibility, there is implicitly a positive integer k ≥ 2, with respect to which the theory is defined. The free states in this resource theory are the k-extendible states [10–12], a prominent notion in quantum information and entanglement theory that we recall now. For a positive integer k ≥ 2, a bipartite state \rho_{AB} is k-extendible with respect to system B if

1. (State Extension) There exists a state \omega_{AB_1...B_k} that extends \rho_{AB}, so that Tr_{B_2...B_k} \{\omega_{AB_1...B_k}\} = \rho_{AB},
with systems $B_1$ through $B_k$ each isomorphic to system $B$ of $\rho_{AB}$.

2. (Permutation Invariance) The extension state $\omega_{AB_1\cdots B_k}$ is invariant with respect to permutations of the $B$ systems, in the sense that $\omega_{AB_1\cdots B_k} = W^\pi_{B_1\cdots B_k} \omega_{AB_1\cdots B_k} W^\pi_{B_1\cdots B_k}$, where $W^\pi_{B_1\cdots B_k}$ is a unitary representation of the permutation $\pi \in S_k$, with $S_k$ denoting the symmetric group.

Given the above definition of $k$-extendible states and the fact that they are the free states, it is then clear that postulates I–V from [13] apply to the resource theory of unextendibility.

To give some physical context to the definition of a $k$-extendible state, suppose that Alice and Bob share a bipartite state and that Bob subsequently mixes his system and the vacuum state at a 50:50 beamsplitter. Then the resulting state of Alice’s system and one of the outputs of the beamsplitter is a two-extendible state by construction.

As a generalization of this, suppose that Bob sends his system through the $N$-splitter of [14, Eq. (10)], with the other input ports set to the vacuum state. Then the state of Alice’s system and one of the outputs of the $N$-splitter is $N$-extendible by construction. One could also physically realize $k$-extendible states in a similar way by means of quantum cloning machines [15].

It is worthwhile to mention that there are free states in the resource theory of unextendibility that are not free in the resource theory of entanglement. For example, if we send one share of the maximally entangled state $\Phi_{AB}$ through a 50% erasure channel [16], then the resulting state $\frac{1}{2}(\Phi_{AB} + I_A/2 \otimes |\epsilon\rangle\langle\epsilon|_B)$ is a two-extendible state, and is thus free in the resource theory of unextendibility for $k = 2$. However, this state has distillable entanglement via LOCC [17], and so it is not free in the resource theory of entanglement.

Let $\text{EXT}_k(A; B)$ denote the set of $k$-extendible states, where with this notation and as above, we take it as implicit that the system $B$ is being extended. The $k$-extendible states are a relaxation of the set of separable (unentangled) states, in the sense that a separable state is $k$-extendible for any positive integer $k \geq 2$. Furthermore, if a state $\rho_{AB}$ is entangled, then there exists some $k$ for which $\rho_{AB}$ is not $k$-extendible, and $\rho_{AB}$ is not $\ell$-extendible for all $\ell \geq k$ [11, 12].

We define the free channels in the resource theory of unextendibility to satisfy two constraints that generalize those given above for the free states. A bipartite channel $\mathcal{N}_{AB\rightarrow\!A'B'}$ is $k$-extendible if

1. (Channel Extension) There exists a quantum channel $\mathcal{M}_{AB_1\cdots B_k \rightarrow\!A'B'_1\cdots B'_k}$ that extends $\mathcal{N}_{AB\rightarrow\!A'B'}$, in the sense that the following equality holds for all quantum states $\theta_{AB_1\cdots B_k}$:

$$\text{Tr}_{B'_1\cdots B'_k} \left\{ \mathcal{M}_{AB_1\cdots B_k \rightarrow\!A'B'_1\cdots B'_k} (\theta_{AB_1\cdots B_k}) \right\} = \mathcal{N}_{AB\rightarrow\!A'B'}(\theta_{AB}),$$

with $B_1 \cdots B_k$ each isomorphic to $B$, and $B'_1 \cdots B'_k$ each isomorphic to $B'$ [18].

2. (Permutation Covariance) The extension channel $\mathcal{M}_{AB_1\cdots B_k \rightarrow\!A'B'_1\cdots B'_k}$ is covariant with respect to permutations of the input $B$ and output $B'$ systems, in the sense that the following equality holds for all quantum states $\theta_{AB_1\cdots B_k}$:

$$\mathcal{M}_{AB_1\cdots B_k \rightarrow\!A'B'_1\cdots B'_k} (W^\pi_{B_1\cdots B_k} \theta_{AB_1\cdots B_k} W^\pi_{B_1\cdots B_k}) = W^\pi_{B'_1\cdots B'_{k'}} \mathcal{M}_{AB_1\cdots B_k \rightarrow\!A'B'_1\cdots B'_k} (\theta_{AB_1\cdots B_k}) W^\pi_{B'_1\cdots B'_{k'}}$$

where $W^\pi_{B_1\cdots B_k}$ and $W^\pi_{B'_1\cdots B'_{k'}}$ are unitary representations of the permutation $\pi \in S_k$.

The first condition above can be alternatively understood as a no-signaling condition. That is, it implies that it is impossible for the parties controlling the $B_2\cdots B_k$ systems to communicate to the parties holding systems $A'B'_1$.

We advocate that our definition above is a natural channel generalization of state extendibility, since the reduced channel $\mathcal{N}_{AB\rightarrow\!A'B'}$ of the channel extension $\mathcal{M}_{AB_1\cdots B_k \rightarrow\!A'B'_1\cdots B'_k}$ is defined in an unambiguous way only when we impose a no-signaling constraint (cf. [19]). Furthermore, the above definition is quite natural in the resource theory of unextendibility developed here, as evidenced by the following theorem:

**Theorem 1** Let $\rho_{AB} \in \text{EXT}_k(A; B)$, and let $\mathcal{N}_{AB\rightarrow\!A'B'}$ be a $k$-extendible channel. Then the output state $\mathcal{N}_{AB\rightarrow\!A'B'}(\rho_{AB})$ is $k$-extendible.

The above theorem is a fundamental statement for the resource theory of unextendibility, indicating that the $k$-extendible channels are free, as they preserve the free states.

There are several interesting classes of $k$-extendible channels that we can consider. Even if it might seem trivial, we should mention that a particular kind of $k$-extendible channel is in fact a $k$-extendible state, in which the input systems $A$ and $B$ are trivial. Thus, $k$-extendible channels can generate $k$-extendible states.

Any 1W-LOCC channel is $k$-extendible for all $k \geq 2$, similar to the way in which any separable state is $k$-extendible for all $k \geq 2$. Thus, a 1W-LOCC channel is free in the resource theory of unextendibility. The fact that a 1W-LOCC channel takes a $k$-extendible input state to a $k$-extendible output state had already been observed for the special case $k = 2$ in [20]. See [21] for a discussion of other $k$-extendible channels.

Quantifying unextendibility—In any resource theory, it is pertinent to quantify the resourcefulness of the resource states and channels. It is desirable for any quantifier to be non-negative, attain its minimum for the free states and channels, and be monotone under the action of a free channel [13]. With this in mind, we define the $k$-unextendible generalized divergence of an arbitrary density operator $\rho_{AB}$ as follows:

$$E_k(A; B)_{\rho} = \inf_{\sigma_{AB} \in \text{EXT}_k(A; B)} \mathbf{D}(\rho_{AB}||\sigma_{AB}),$$

where $\mathbf{D}(\rho||\sigma)$ denotes a generalized divergence [22, 23], which is any quantifier of the distinguishability of states.
\( \rho \) and \( \sigma \) that is monotone under the action of a quantum channel. Special cases of the quantifier in (1) were previously defined in [20, 24] (relative entropy to two-extendible states and to \( k \)-extendible states, respectively), [25] (best two-extendible approximation, related to max-relative entropy of unextendibility defined here), and [26] (maximum \( k \)-extendible fidelity).

Particular examples of generalized divergences between states \( \rho \) and \( \sigma \) are the \( \varepsilon \)-hypothesis-testing divergence \( D_{k}^{\varepsilon}(\rho||\sigma) [27, 28] \), and the max-relative entropy \( D_{\text{max}}(\rho||\sigma) [29, 30] \), where for \( \varepsilon \in (0, 1] \),

\[
D_{k}^{\varepsilon}(\rho||\sigma) := -\log_{2} \inf_{\Lambda \in [0,1]} \{ \text{Tr} \{ \Lambda \sigma \} : \text{Tr} \{ \Lambda \rho \} \geq 1 - \varepsilon \},
\]

and \( D_{\text{max}}(\rho||\sigma) := \inf \{ \lambda : \rho \leq 2^{\lambda} \sigma \} \) in the case that \( \text{supp}(\rho) \subseteq \text{supp}(\sigma) \), and otherwise \( D_{\text{max}}(\rho||\sigma) = +\infty \).

Information-processing tasks—Now that we have established the free states and channels in the resource theory of unextendibility, we are ready to discuss tasks that can be performed in it. We consider two main tasks here: entanglement distillation and quantum communication with the assistance of \( k \)-extendible states.

The goal of these protocols is to use many copies of a bipartite state or many invocations of a quantum channel, along with the free assistance of \( k \)-extendible channels, in order to generate a high-fidelity maximally entangled state with as much entanglement as possible. This kind of task was defined and developed in [31], albeit with the assistance of a particular kind of \( k \)-extendible channel and only the case \( k = 2 \) was considered there, generalizing the usual notion of entanglement distillation and quantum communication protocols from [5, 32–38].

Let \( n, M \in \mathbb{Z}^{+} \) and \( \varepsilon \in [0, 1] \). Let \( \rho_{AB} \) be a bipartite state. An \( (n, M, \varepsilon) \) entanglement distillation protocol assisted by a \( k \)-extendible channel begins with Alice and Bob sharing \( n \) copies of \( \rho_{AB} \), to which they apply a \( k \)-extendible channel \( \mathcal{K}_{A^{n}B^{n} \rightarrow M_{A}M_{B}} \). The resulting state satisfies the following performance condition:

\[
F(\mathcal{K}_{A^{n}B^{n} \rightarrow M_{A}M_{B}}(\rho_{AB} \otimes \rho_{AB} \otimes \cdots \otimes \rho_{AB})), \Phi_{M_{A}M_{B}}) \geq 1 - \varepsilon,
\]

where \( \Phi_{M_{A}M_{B}} := \frac{1}{M} \sum_{m,m'} |m\rangle \langle m'|_{M_{A}} \otimes |m\rangle \langle m'|_{M_{B}} \) is a maximally entangled state of Schmidt rank \( M \) and \( F(\omega, \tau) := \| \sqrt{\omega} \tau \|^2 \) is the quantum fidelity [39]. Let \( D^{(k)}(\rho_{AB}, n, \varepsilon) \) denote the non-asymptotic distillable entanglement with the assistance of \( k \)-extendible channels; i.e., \( D^{(k)}(\rho_{AB}, n, \varepsilon) \) is equal to the maximum value of \( \frac{1}{n} \log_{2} M \) such that there exists an \( (n, M, \varepsilon) \) protocol for \( \rho_{AB} \) as described above.

We define two different variations of quantum communication, with one simpler and one more involved. Let \( \mathcal{N}_{A \rightarrow B} \) denote a quantum channel. In the simpler version, an \( (n, M, \varepsilon) \) entanglement transmission protocol assisted by a \( k \)-extendible post-processing begins with Alice preparing a maximally entangled state \( \Phi_{RA'} \) of Schmidt rank \( M \). She applies an encoding channel \( \mathcal{E}_{A' \rightarrow A_{n}} \), which leads to a state \( \rho_{RA_{n}} := \mathcal{E}_{A' \rightarrow A_{n}}(\Phi_{RA'}) \). She transmits the systems \( A_{n} := A_{1} \cdots A_{n} \) using the channel \( \mathcal{N}_{A_{n} \rightarrow B} \). Alice and Bob then perform a \( k \)-extendible channel \( \mathcal{K}_{RB^{n} \rightarrow M_{A}M_{B}} \), such that

\[
F(\mathcal{K}_{RB^{n} \rightarrow M_{A}M_{B}}(\rho_{RA_{n}})), \Phi_{M_{A}M_{B}}) \geq 1 - \varepsilon.
\]

Let \( Q^{(k)}_{AB}(n, M, \varepsilon) \) denote the non-asymptotic quantum capacity assisted by a \( k \)-extendible post-processing; i.e., \( Q^{(k)}_{AB}(n, M, \varepsilon) \) is the maximum value of \( \frac{1}{n} \log_{2} M \) such that there exists an \( (n, M, \varepsilon) \) protocol for \( \mathcal{N}_{A \rightarrow B} \).

For the cases of entanglement distillation and the simpler version of entanglement transmission, note that an \( (n, M, \varepsilon) \) entanglement distillation protocol for the state \( \rho_{AB} \) is a \((1, M, \varepsilon)\) protocol for the state \( \rho_{AB} \) and vice versa. Similarly, an \((n, M, \varepsilon)\) entanglement transmission protocol for the channel \( \mathcal{N}_{A \rightarrow B} \) is a \((1, M, \varepsilon)\) protocol for the channel \( \mathcal{N}_{A \rightarrow B} \) and vice versa.

In the more involved version of entanglement transmission, every channel use is interleaved with a \( k \)-extendible channel, similar to the protocols considered in [40–42]. Specifically, the protocol is a special case of one discussed in [42] for general resource theories [43]. We do not discuss these protocols in detail here, but we simply note that, for an \((n, M, \varepsilon)\) quantum communication protocol assisted by \( k \)-extendible channels, the performance criterion is that the final state of the protocol should have fidelity \( \geq 1 - \varepsilon \) to a maximally entangled state \( \Phi_{M_{A}M_{B}} \) of Schmidt rank \( M \). Let \( Q^{(k)}_{AB}(N_{A \rightarrow B}, n, \varepsilon) \) denote the non-asymptotic quantum capacity assisted by \( k \)-extendible channels; i.e., \( Q^{(k)}_{AB}(N_{A \rightarrow B}, n, \varepsilon) \) is the maximum value of \( \frac{1}{n} \log_{2} M \) such that there exists an \((n, M, \varepsilon)\) protocol for \( \mathcal{N}_{A \rightarrow B} \) as described for the more involved case above.

**Theorem 2** The following bound holds for all \( k \geq 2 \) and for any \((1, M, \varepsilon)\) entanglement transmission protocol that uses a channel \( \mathcal{N} \) assisted by a \( k \)-extendible post-processing:

\[
-\log_{2} \left[ \frac{1}{M} + \frac{1}{k} - \frac{1}{Mk} \right] \leq \sup_{\psi_{RA}} E^{(k)}_{\mathcal{N}}(R; B),
\]

where \( E^{(k)}_{\mathcal{N}}(R; B) := \inf_{\sigma_{RB} \in \text{EXT}_{n}(R; B)} D^{(k)}(\rho_{RB}||\sigma_{RB}), \quad \tau_{RB} := \mathcal{N}_{A \rightarrow B}(\psi_{RA}), \quad \text{and the optimization is with respect to pure states } \psi_{RA} \text{ such that } R \simeq A. \) The following bound holds for all \( k \geq 2 \) and for any \((1, M, \varepsilon)\) entanglement distillation protocol that uses a quantum state \( \rho_{AB} \) assisted by a \( k \)-extendible post-processing:

\[
-\log_{2} \left[ \frac{1}{M} + \frac{1}{k} - \frac{1}{Mk} \right] \leq E^{(k)}_{\mathcal{N}}(A; B)_{\rho}.
\]

The proof of the above theorem follows by employing the fact that \( E^{(k)}_{\mathcal{N}} \) does not increase under the action of a \( k \)-extendible channel, because the extendibility of a \( k \)-extendible state does not change under the action of \( U \) for a unitary \( U \), and by employing [44, Theorem III.8].

**Theorem 3** The following bound holds for all \( k \geq 2 \) and for any \((n, M, \varepsilon)\) quantum communication protocol employing \( n \) uses of a channel \( \mathcal{N} \) interleaved by \( k \)-extendible
channels:

\[-\log_2 \left[ \frac{1}{M} + \frac{1}{k} - \frac{1}{Mk} \right] \leq nE^\max_k(N) + \log_2 \left( \frac{1}{1 - \varepsilon} \right),\]

where

\[E^\max_k(N) := \sup_{\psi_{RA}} \inf_{\sigma_{RB} \in \text{EXT}_{A} (R; B)} D_{\max}(\tau_{RB} \| \sigma_{RB}),\]

\[\tau_{RB} := \mathcal{N}_{A \rightarrow B}(\psi_{RA}),\]

and the optimization is with respect to pure states \(\psi_{RA}\) with \(|R| = |A|\).

We note here that special cases of the entanglement distillation and quantum communication protocols described above occur when the \(k\)-extendible assisting channels are taken to be 1W-LOCC channels. As such, \(D^{(k)}(\rho_{AB}, n, \varepsilon), Q_{I}^{(k)}(\mathcal{N}_{A \rightarrow B}, n, \varepsilon)\), and \(Q_{II}^{(k)}(\mathcal{N}_{A \rightarrow B}, n, \varepsilon)\) are upper bounds on the non-asymptotic distillable entanglement and capacities when 1W-LOCC channels are available for assistance.

Pretty strong converse for antidegradable channels—As a direct application of Theorem 3, we revisit the “pretty strong converse” of [45] for antidegradable channels. Recall that a channel \(\mathcal{N}_{A \rightarrow B}\) is antidegradable [46, 47] if the output state \(\mathcal{N}_{A \rightarrow B}(\rho_{RA})\) is two-extendible for any input state \(\rho_{RA}\). Due to this property, antidegradable channels have zero asymptotic quantum capacity [17, 48]. Theorem 3 implies the following bound for the non-asymptotic case:

**Corollary 1** Fix \(\varepsilon \in (0, 1/2)\). The following bound holds for any \((n, M, \varepsilon)\) quantum communication protocol employing \(n\) uses of an antidegradable channel \(\mathcal{N}\) interleaved by two-extendible channels:

\[
\frac{1}{n} \log_2 M \leq \frac{1}{n} \log_2 \left(1 - \frac{1}{2 \varepsilon} \right). \tag{6}
\]

We conclude from (6) that, for an antidegradable channel, there is a strong limitation on its ability to generate entanglement whenever the error parameter \(\varepsilon < \frac{1}{2}\), as is usually the case in applications for quantum computation. We also remark that the bound in (6) is tighter than related bounds given in [45], and furthermore, the bound applies to quantum communication protocols assisted by interleaved two-extendible channels, which were not considered in [45].

Limitations on quantum devices—In practice, the evolution affected by quantum processors is never a perfect unitary process. There is always some undesirable interaction with the environment, the latter of which is inaccessible to the processor. Furthermore, there are practical limitations on the ability to construct perfect quantum gates [49]. The depolarizing and erasure channels are two classes of noisy models for qubit quantum processors that are widely considered (see [50–52]).

Both families of channels mentioned above are covariant channels [53]: i.e., these channels are covariant with respect to a group \(G\) with representations given by a unitary one-design. Thus, these channels can be simulated using 1W-LOCC with the Choi states as the resource states [54, Section VII]. Using this symmetry and the monotonicity of the unextendible generalized divergence under 1W-LOCC, we conclude that the optimal input state to a covariant channel \(\mathcal{N}\), with respect to the upper bound in Theorem 2, is a maximally entangled state \(\Phi_{RA}\). Also, for any \((n, M, \varepsilon)\) quantum communication protocol conducted over a covariant channel and assisted by any \(k\)-extendible channel, the optimal input state is \(\Phi_{RA}^{\otimes n}\) and \(Q_{II}^{(k)}(\mathcal{N}_{A \rightarrow B}, n, \varepsilon) = Q_{II}^{(k)}(\mathcal{N}_{A \rightarrow B}, n, \varepsilon); i.e., an upper bound on non-asymptotic quantum capacity \(Q_{II}^{(k)}\) is given by Theorem 2.

A qubit depolarizing channel acts on an input state \(\rho\) as \(D^{\rho}_{A \rightarrow B}(\rho) = (1 - p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z)\), where \(p \in [0, 1]\) is the depolarizing parameter, and \(X, Y, Z\) are Paulis. The best known upper bound on the asymptotic quantum capacity of this channel for values of \(p \in [0, \frac{1}{3}]\) was recently derived in [55, 56], and this channel has zero asymptotic quantum capacity for \(p \in [\frac{1}{3}, 1]\) [57, 58].

With the goal of bounding the non-asymptotic quantum capacity of \(D_{p}\), we make a particular choice of the \(k\)-extendible state for \(E_{k}^{\max}\) (which need not be optimal) to be a tensor power of the isotropic states \(\sigma_{AB}^{(t, 2)}\), which is similar to what was done in [3]. The inequality in Theorem 2 then reduces to

\[
\frac{1}{n} \log_2 M \leq \frac{1}{n} \log_2 \left(1 - \frac{1}{k} \right) - \frac{1}{n} \log_2 \left(f(\varepsilon, p, t) - \frac{1}{k} \right), \tag{7}
\]

where \(f(\varepsilon, p, t) = 2^{-D^{\rho}_{p}\left(\left(0 \rightarrow p\right)^{\otimes n}\left|\{1 \rightarrow t\}^{\otimes n}\right)\right)}\) and \(\{1 - p, p\}\) denotes a Bernoulli distribution. The optimal measurement (Neyman-Pearson test) for the resulting hypothesis testing relative entropy between Bernoulli distributions is then well known [59] (see also [60]), giving an explicit upper bound on the rate \(\frac{1}{n} \log_2 M\). Figure 1 compares various upper bounds on the number of qubits that can be reliably transmitted over a depolarizing channel with \(p = 0.15\), and \(\varepsilon = 0.05\). The red dashed line is the bound from Theorem 2. The green dash-dotted and blue dotted lines are upper bounds from [3] and [4], respectively.
FIG. 2. Upper bounds on the number of qubits that can be reliably transmitted over an erasure channel with \( p = 0.35 \), and \( \varepsilon = 0.05 \). The red dashed line is the bound from Theorem 2. The green dash-dotted line is an upper bound from [3].

A qubit erasure channel acts on an input state \( \rho \) as \( E^p_{A \rightarrow B}(\rho_A) = (1-p)\rho_B + p|e\rangle\langle e|_B \) [16], where \( p \in [0,1] \) is the erasure probability, and the erasure state \( |e\rangle\langle e| \) is orthogonal to the input Hilbert space. We employ the symmetries of the erasure channel to make a particular choice of the \( k \)-extendible state for \( E^p_k \). Theorem 2 gives upper bounds on the number of qubits that can be reliably transmitted over \( n \) uses of the erasure channel. The bounds that we obtain are not necessarily optimal, but they still are significantly tighter than those from [3]. See Figure 2.

Discussion—In this paper, we developed the resource theory of unextendibility and discussed limits that it places on the performance of finite-sized quantum processors. The free states in this resource theory are \( k \)-extendible states, and the free channels are the \( k \)-extendible channels. We determined non-asymptotic upper bounds on the rate at which qubits can be transmitted over a finite number of uses of a given quantum channel. The bounds coming from the resource theory of unextendibility are significantly tighter than those in [3, 4] for depolarizing and erasure channels.

It would be interesting to explore the resource theory of unextendibility further. One plausible direction would be to use this resource theory to obtain non-asymptotic converse bounds on the entanglement distillation rate of bipartite quantum interactions and compare with the bounds obtained in [62]. Another direction is to analyze the bounds in Theorem 2 for other noise models that are practically relevant. Finally, it remains open to link the bounds developed here with the open problem of finding a strong converse for the quantum capacity of degradable channels [45]. To solve that problem, recall that one contribution of [45] was to reduce the question of the strong converse of degradable channels to that of establishing the strong converse for symmetric channels.

Note—We noticed the related work “Optimising practical entanglement distillation” by Rozpedek et al. recently posted as arXiv:1803.10111, which like us uses extendibility to address entanglement distillation, and which presents results that are complementary to ours.

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**Supplementary material for “Extendibility limits the performance of quantum processors”**

by Eneet Kaur, Siddhartha Das, Mark M. Wilde, and Andreas Winter

In this supplementary material, we provide detailed mathematical proofs for all claims made in the main text. In Appendix A, we review preliminary notions that are relevant for the other appendices. This includes basic notions of quantum information theory, $k$-extendibility, entropies and information measures, generalized divergences, entanglement measures, and channels with symmetry. In Appendix B, we provide more details of the resource theory of unextendibility, including details of $k$-extendible channels and measures of $k$-unextendibility. We also calculate several of these measures for isotropic and Werner states, and we prove several properties of the relative entropy of unextendibility, including uniform continuity, faithfulness, subadditivity, non-extensivity, and convexity. We finally prove in Appendix B that amortization does not enhance the max-$k$-unextendibility of a quantum channel, analogous to the finding from [65], and we show how to use symmetries to figure out the form of optimal input states for the generalized unextendibility of a quantum channel, along the lines of [66, Proposition 2]. Appendix C provides detailed proofs for the last two theorems in the main text, regarding upper bounds on entanglement distillation and quantum communication protocols that make use of $k$-extendible channels for free. Appendices D and E provide detailed proofs for the upper bounds on the non-asymptotic quantum capacity of the depolarizing and erasure channels, respectively.

**Appendix A: Preliminaries**

We begin here by establishing some notation and reviewing some definitions needed in the rest of the supplementary material.
1. States, channels, isometries, and k-extendibility

Let $\mathcal{B}(\mathcal{H})$ denote the algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$. For the majority of our developments, we restrict to finite-dimensional Hilbert spaces. However, some of the claims apply to separable, infinite-dimensional Hilbert spaces, and in what follows, we clarify which ones do. The subset of $\mathcal{B}(\mathcal{H})$ containing all positive semi-definite operators is denoted by $\mathcal{B}_+(\mathcal{H})$. We denote the identity operator as $I$ and the identity superoperator as id. The Hilbert space of a quantum system $A$ is denoted by $\mathcal{H}_A$. The state of a quantum system $A$ is represented by a density operator $\rho_A$, which is a positive semi-definite operator with unit trace. Let $\mathcal{D}(\mathcal{H})$ denote the set of density operators, i.e., all elements $\rho_A \in \mathcal{B}_+(\mathcal{H}_A)$ such that $\text{Tr}\{\rho_A\} = 1$. The Hilbert space for a composite system $RA$ is denoted as $\mathcal{H}_{RA}$ where $\mathcal{H}_{RA} = \mathcal{H}_R \otimes \mathcal{H}_A$. The density operator of a composite system $RA$ is defined as $\rho_{RA} \in \mathcal{D}(\mathcal{H}_{RA})$, and the partial trace over $A$ gives the reduced density operator for system $R$, i.e., $\text{Tr}_A(\rho_{RA}) = \rho_R$ such that $\rho_R \in \mathcal{D}(\mathcal{H}_R)$. The notation $A^n := A_1A_2\cdots A_n$ indicates a composite system consisting of $n$ subsystems, each of which is isomorphic to Hilbert space $\mathcal{H}_A$. A pure state $\psi_A$ of a system $A$ is a rank-one density operator, and we write it as $\psi_A = |\psi\rangle\langle\psi|_A$ for $|\psi\rangle_A$ a unit vector in $\mathcal{H}_A$. A purification of a density operator $\rho_A$ is a pure state $\psi_{EA}$ such that $\text{Tr}_E(\psi_{EA}) = \rho_A$, where $E$ is known as a purifying system. $\pi_A := I_A/\dim(\mathcal{H}_A) \in \mathcal{D}(\mathcal{H}_A)$ denotes the maximally mixed state. The fidelity of $\tau, \sigma \in \mathcal{B}_+(\mathcal{H})$ is defined as $F(\tau, \sigma) = \|\sqrt{\tau}\sqrt{\sigma}\|_1^2$ [39], where $\|\|_1$ denotes the trace norm.

The adjoint $M^\dagger : \mathcal{B}(\mathcal{H}_B) \rightarrow \mathcal{B}(\mathcal{H}_A)$ of a linear map $M : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B)$ is the unique linear map that satisfies

$$\langle Y_B, M(X_A) \rangle = \langle M^\dagger(Y_B), X_A \rangle, \quad \forall X_A \in \mathcal{B}(\mathcal{H}_A), Y_B \in \mathcal{B}(\mathcal{H}_B)$$

(A1)

where $\langle C, D \rangle = \text{Tr}(C^\dagger D)$ is the Hilbert-Schmidt inner product. An isometry $U : \mathcal{H} \rightarrow \mathcal{H}'$ is a linear map such that $U^\dagger U = I_{\mathcal{H}}$.

The evolution of a quantum state is described by a quantum channel. A quantum channel $\mathcal{M}_{A \rightarrow B}$ is a completely positive, trace-preserving (CPTP) map $\mathcal{M} : \mathcal{B}_+(\mathcal{H}_A) \rightarrow \mathcal{B}_+(\mathcal{H}_B)$. Let $U^M_{A \rightarrow BE}$ denote an isometric extension of a quantum channel $\mathcal{M}_{A \rightarrow B}$, which by definition means that

$$\text{Tr}_E \left\{ U^M_{A \rightarrow BE} \rho_A \left( U^M_{A \rightarrow BE} \right)^\dagger \right\} = \mathcal{M}_{A \rightarrow B}(\rho_A), \forall \rho_A \in \mathcal{D}(\mathcal{H}_A),$$

(A2)

along with the following conditions for $U^M$ to be an isometry:

$$(U^M)^\dagger U^M = I_A, \quad \text{and} \quad U^M(U^M)^\dagger = \Pi_{BE},$$

(A3)

where $\Pi_{BE}$ is a projection onto a subspace of the Hilbert space $\mathcal{H}_{BE}$.

The Choi isomorphism represents a well known duality between channels and states. Let $\mathcal{M}_{A \rightarrow B}$ be a quantum channel, and let $|\Upsilon\rangle_{RA}$ denote the following maximally entangled vector:

$$|\Upsilon\rangle_{RA} := \sum_i |i\rangle_R |i\rangle_A,$$

(A4)

where $\dim(\mathcal{H}_R) = \dim(\mathcal{H}_A)$, and $\{|i\rangle_R\}_i$ and $\{|i\rangle_A\}_i$ are fixed orthonormal bases. We extend this notation to multiple parties with a given bipartite cut as

$$|\Upsilon\rangle_{RA:RB:AB} := |\Upsilon\rangle_{RA:A} \otimes |\Upsilon\rangle_{RB:B}.\quad \quad (A5)$$

The maximally entangled state $\Phi_{RA}$ is denoted as

$$\Phi_{RA} = \frac{1}{|A|} |\Upsilon\rangle \langle \Upsilon|_{RA},$$

(A6)

where $|A| = \dim(\mathcal{H}_A)$. The Choi operator for a channel $\mathcal{M}_{A \rightarrow B}$ is defined as

$$J^M_{RA} = (\text{id}_R \otimes \mathcal{M}_{A \rightarrow B}) (|\Upsilon\rangle \langle \Upsilon|_{RA}),$$

(A7)

where $\text{id}_R$ denotes the identity map on $R$. For $A' \simeq A$, the following identity holds

$$\langle \Upsilon|_{A':R} \rho_{SA'} \otimes J^M_{RB} |\Upsilon\rangle_{A':R} = \mathcal{M}_{A \rightarrow B}(\rho_{SA}),$$

(A8)

where $A' \simeq A$. The above identity can be understood in terms of a post-selected variant [67] of the quantum teleportation protocol [68]. Another identity that holds is

$$\langle \Upsilon|_{R:A} |Q_{SR} \otimes I_A\rangle |\Upsilon\rangle_{R:A} = \text{Tr}_R \{Q_{SR}\},$$

(A9)
for an operator $Q_{SR} \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_R)$.

Let $\text{SEP}(A : B)$ denote the set of all separable states $\sigma_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B)$, which are states that can be written as

$$
\sigma_{AB} = \sum_x p(x) \omega^x_A \otimes \tau^x_B,
$$

(A10)

where $p(x)$ is a probability distribution, $\omega^x_A \in \mathcal{D}(\mathcal{H}_A)$, and $\tau^x_B \in \mathcal{D}(\mathcal{H}_B)$ for all $x$. This set is closed under the action of the partial transpose maps $T_A$ or $T_B$ [69, 70]. Generalizing the set of separable states, we can define the set $\text{PPT}(A : B)$ of all bipartite states $\rho_{AB}$ that remain positive after the action of the partial transpose $T_B$. A state $\rho_{AB} \in \text{PPT}(A : B)$ is also called a PPT (positive under partial transpose) state. We then have the containment $\text{SEP} \subset \text{PPT}$. A special kind of LOCC channel is a one-way (1W-) LOCC channel from $A$ to $B$, in which Alice performs a quantum instrument, sends the classical outcome to Bob, who then performs a quantum channel conditioned on the classical outcome received from Alice. As such, any 1W-LOCC channel takes the form in (A11), except that $\{E^y_{A \to A'}\}_y$ is a set of CP maps such that the sum map $\sum_y E^y_{A \to A'}$ is trace preserving, while $\{F^y_{Y \to Y'}\}_y$ is a set of quantum channels.

$$
\mathcal{L}_{AB \to A'B'} = \sum_y E^y_{A \to A'} \otimes F^y_{B \to B'},
$$

(A11)

where $\{E^y_{A \to A'}\}_y$ and $\{F^y_{B \to B'}\}_y$ are sets of completely positive (CP) maps such that $\mathcal{L}_{AB \to A'B'}$ is trace preserving.

A local operations and classical communication (LOCC) channel $\mathcal{L}_{AB \to A'B'}$ can be written as

$$
\mathcal{L}_{AB \to A'B'} = \sum_y E^y_{A \to A'} \otimes F^y_{B \to B'},
$$

(A11)

2. Entropies and information

The quantum entropy of a density operator $\rho_A$ is defined as [71]

$$
S(A)_\rho := S(\rho_A) = - \text{Tr}[\rho_A \log_2 \rho_A].
$$

(A12)

The quantum relative entropy of two quantum states is a measure of their distinguishability. For $\rho \in \mathcal{D}(\mathcal{H})$ and $\sigma \in \mathcal{B}_+(\mathcal{H})$, it is defined as [72]

$$
D(\rho \| \sigma) := \begin{cases} 
\text{Tr}\{\rho \log_2 \rho - \log_2 \sigma\}, & \text{supp}(\rho) \subseteq \text{supp}(\sigma) \\
+\infty, & \text{otherwise}.
\end{cases}
$$

(A13)

The quantum relative entropy is non-increasing under the action of positive trace-preserving maps [73], which is the statement that $D(\rho \| \sigma) \geq D(M(\rho) \| M(\sigma))$ for any two density operators $\rho$ and $\sigma$ and a positive trace-preserving map $M$ (this inequality applies to quantum channels as well [74], since every completely positive map is also a positive map by definition).

3. Generalized divergence and relative entropies

Let $D$ be a function from $\mathcal{D}(\mathcal{H}) \times \mathcal{L}_+(\mathcal{H})$ to $\mathbb{R}$. Then $D$ is a generalized divergence [22, 23] if it satisfies the following monotonicity (data-processing) inequality for all density operators $\rho$ and $\sigma$, and quantum channels $\mathcal{N}$:

$$
D(\rho \| \sigma) \geq D(\mathcal{N}(\rho) \| \mathcal{N}(\sigma)).
$$

(A14)

It is a measure of distinguishability of the states $\rho$ and $\sigma$. As a direct consequence of the above inequality, any generalized divergence satisfies the following two properties for an isometry $U$ and a state $\tau$ [75]:

$$
\begin{align*}
D(\rho \| \sigma) &= D(U\rho U^\dagger \| U\sigma U^\dagger), \quad (A15) \\
D(\rho \| \sigma) &= D(\rho \otimes \tau \| \sigma \otimes \tau). \quad (A16)
\end{align*}
$$

The sandwiched Rényi relative entropy [75, 76] is denoted as $\tilde{D}_\alpha(\rho \| \sigma)$ and defined for $\rho \in \mathcal{D}(\mathcal{H})$, $\sigma \in \mathcal{B}_+(\mathcal{H})$, and $\forall \alpha \in (0, 1) \cup (1, \infty)$ as

$$
\tilde{D}_\alpha(\rho \| \sigma) := \frac{1}{\alpha - 1} \log_2 \text{Tr}\left\{\left(\sigma^{\frac{1-\alpha}{\alpha}} \rho \sigma^{\frac{1-\alpha}{\alpha}}\right)^\alpha\right\},
$$

(A17)
but it is set to $+\infty$ for $\alpha \in (1, \infty)$ if $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$. The sandwiched Rényi relative entropy obeys the following “monotonicity in $\alpha$” inequality \cite{76}:

\[
\tilde{D}_\alpha(\rho\|\sigma) \leq \tilde{D}_\beta(\rho\|\sigma) \text{ if } \alpha \leq \beta, \text{ for } \alpha, \beta \in (0, 1) \cup (1, \infty).
\]

The following lemma states that the sandwiched Rényi relative entropy $\tilde{D}_\alpha(\rho\|\sigma)$ is a particular generalized divergence for certain values of $\alpha$.

**Lemma 1** (\cite{77, 78}) Let $\mathcal{N} : \mathcal{B}_+(\mathcal{H}_A) \to \mathcal{B}_+(\mathcal{H}_B)$ be a quantum channel and let $\rho_A \in \mathcal{D}(\mathcal{H}_A)$ and $\sigma_A \in \mathcal{B}_+(\mathcal{H}_A)$. Then,

\[
\tilde{D}_\alpha(\rho\|\sigma) \geq \tilde{D}_\alpha(\mathcal{N}(\rho)\|\mathcal{N}(\sigma)), \quad \forall \alpha \in [1/2, 1) \cup (1, \infty).
\]

In the limit $\alpha \to 1$, the sandwiched Rényi relative entropy $\tilde{D}_\alpha(\rho\|\sigma)$ converges to the quantum relative entropy \cite{75, 76}:

\[
D(\rho\|\sigma) = D_1(\rho\|\sigma) := \lim_{\alpha \to 1} \tilde{D}_\alpha(\rho\|\sigma).
\]

In the limit $\alpha \to \infty$, the sandwiched Rényi relative entropy $\tilde{D}_\alpha(\rho\|\sigma)$ converges to the max-relative entropy, which is defined as \cite{29, 30}

\[
D_{\max}(\rho\|\sigma) = \inf\{\lambda : \rho \leq 2^\lambda \sigma\},
\]

and if $\text{supp}(\rho) \not\subseteq \text{supp}(\sigma)$ then $D_{\max}(\rho\|\sigma) = \infty$.

Another generalized divergence is the $\varepsilon$-hypothesis-testing divergence \cite{27, 28}, defined as

\[
D^\varepsilon(\rho\|\sigma) := -\log_2 \inf_{\Lambda} \left\{ \text{Tr}\{\Lambda\sigma\} : 0 \leq \Lambda \leq I \land \text{Tr}\{\Lambda\rho\} \geq 1 - \varepsilon \right\},
\]

for $\varepsilon \in (0, 1)$, $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, and $\sigma \in \mathcal{B}_+(\mathcal{H})$.

The following inequality relates $D^\varepsilon(\rho\|\sigma)$ and $\tilde{D}_\alpha(\rho\|\sigma)$, where $\rho, \sigma \in \mathcal{D}(\mathcal{H})$, for $\alpha \in (1, \infty)$ and $\varepsilon \in (0, 1)$ \cite{79, Lemma 5} (see also \cite{80–82}):

\[
D^\varepsilon(\rho\|\sigma) \leq \tilde{D}_\alpha(\rho\|\sigma) + \frac{\alpha}{\alpha - 1} \log_2 \left( \frac{1}{1 - \varepsilon} \right).
\]

### 4. Entanglement measures

Let $\text{Ent}(A; B)_\rho$ denote an entanglement measure \cite{6} that is evaluated for a bipartite state $\rho_{AB}$. The basic property of an entanglement measure is that it should be an LOCC monotone \cite{6}, i.e., non-increasing under the action of an LOCC channel. Given such an entanglement measure, one can define the entanglement $\text{Ent}(\mathcal{N})$ of a channel $\mathcal{N}_{A\to B}$ in terms of it by optimizing over all pure, bipartite states that can be input to the channel:

\[
\text{Ent}(\mathcal{N}) = \sup_{\psi_{RA}} \text{Ent}(R; B)_\omega,
\]

where $\omega_{RB} = \mathcal{N}_{A\to B}(\psi_{RA})$. Due to the properties of an entanglement measure and the well known Schmidt decomposition theorem, it suffices to optimize over pure states $\psi_{RA}$ such that $R \simeq A$ (i.e., one does not achieve a higher value of $\text{Ent}(\mathcal{N})$ by optimizing over mixed states with an unbounded reference system $R$). In an information-theoretic setting, the entanglement $\text{Ent}(\mathcal{N})$ of a channel $\mathcal{N}$ characterizes the amount of entanglement that a sender $A$ and a receiver $B$ can generate by using the channel if they do not share entanglement prior to its use.

Alternatively, one can consider the amortized entanglement $\text{Ent}_A(\mathcal{N})$ of a channel $\mathcal{N}_{A\to B}$ as the following optimization \cite{42} (see also \cite{83–87}):

\[
\text{Ent}_A(\mathcal{N}) := \sup_{\rho_{RA}AR_B} \left[ \text{Ent}(R_A; BR_B)_\tau - \text{Ent}(R_A; A; R_B)_\rho \right],
\]

where $\tau_{RA}RR_B = \mathcal{N}_{A\to B}(\rho_{RA}AR_B)$ for a state $\rho_{RA}AR_B$, with $R_A$ and $R_B$ reference systems. The supremum is with respect to all input states $\rho_{RA}AR_B$, and the systems $R_A, R_B$ are finite-dimensional but could be arbitrarily large. Thus, in general, $\text{Ent}_A(\mathcal{N})$ need not be computable. The amortized entanglement quantifies the net amount of entanglement that can be generated by using the channel $\mathcal{N}_{A\to B}$, if the sender and the receiver are allowed to begin with some initial entanglement in the form of the state $\rho_{RA}AR_B$. That is, $\text{Ent}(R_A; R_B)_\rho$ quantifies the entanglement of the initial state $\rho_{RA}AR_B$, and $\text{Ent}(R_A; BR_B)_\tau$ quantifies the entanglement of the final state produced after the action of the channel.
5. Channels with symmetry

Consider a finite group $G$. For every $g \in G$, let $g \rightarrow U_A(g)$ and $g \rightarrow V_B(g)$ be projective unitary representations of $g$ acting on the input space $\mathcal{H}_A$ and the output space $\mathcal{H}_B$ of a quantum channel $\mathcal{N}_{A\rightarrow B}$, respectively. A quantum channel $\mathcal{N}_{A\rightarrow B}$ is covariant with respect to these representations if the following relation is satisfied [88–90]:

$$\mathcal{N}_{A\rightarrow B}\left(U_A(g)\rho_A U_A^\dagger(g)\right) = V_B(g)\mathcal{N}_{A\rightarrow B}\left(\rho_A\right)V_B^\dagger(g), \quad \forall \rho_A \in \mathcal{D}(\mathcal{H}_A) \text{ and } \forall g \in G. \tag{A26}$$

In our paper, we define covariant channels in the following way:

**Definition 1 (Covariant channel)** A quantum channel is covariant if it is covariant with respect to a group $G$ for which each $g \in G$ has a unitary representation $U(g)$ acting on $\mathcal{H}_A$, such that $\{U(g)\}_{g \in G}$ is a unitary one-design; i.e., the map $\frac{1}{|G|} \sum_{g \in G} U(g)(\cdot)U(g)^\dagger$ always outputs the maximally mixed state for all input states.

The notion of teleportation simulation of a quantum channel first appeared in [5], and it was subsequently generalized in [91, Eq. (11)] to include general LOCC channels in the simulation. It was developed in more detail in [92] and used in the context of private communication in [93] and [61, 94].

**Definition 2 (Teleportation-simulable channel [5, 91])** A channel $\mathcal{N}_{A\rightarrow B}$ is teleportation-simulable if there exists a resource state $\omega_{RB} \in \mathcal{D}(\mathcal{H}_{RB})$ such that for all $\rho_A \in \mathcal{D}(\mathcal{H}_A)$

$$\mathcal{N}_{A\rightarrow B}(\rho_A) = \mathcal{L}_{RAB\rightarrow B}(\rho_A \otimes \omega_{RB}), \tag{A27}$$

where $\mathcal{L}_{RAB\rightarrow B}$ is an LOCC channel (a particular example of an LOCC channel could be a generalized teleportation protocol [95]).

**Lemma 2 ([54])** All covariant channels (Definition 1) are teleportation-simulable with respect to the resource state $\mathcal{N}_{A\rightarrow B}(\Phi_{RA})$.

Appendix B: Framework for the resource theory of $k$-unextendibility

Any quantum resource theory is framed around two ingredients [13]: the free states and the restricted set of free channels. The resource states by definition are those that are not free. The resource states or channels are useful and needed to carry out a given task. Resource states cannot be obtained by the action of the free channels on the free states. Free channels are incapable of increasing the amount of resourcefulness of a given state, whereas free states can be generated for free (without any resource cost).

1. $k$-extendible states

To develop a framework for the quantum resource theory of $k$-unextendibility, specified with respect to a fixed subsystem ($B$) of a bipartite system ($AB$), let us first recall the definition of a $k$-extendible state [10–12]:

**Definition 3 ($k$-extendible state)** For integer $k \geq 2$, a state $\rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB})$ is $k$-extendible if there exists a state $\omega_{AB^k} := \omega_{AB_1B_2\ldots B_k} \in \mathcal{D}(\mathcal{H}_{AB_1B_2\ldots B_k})$ that satisfies the following two criteria:

1. The state $\omega_{AB_1B_2\ldots B_k}$ is permutation invariant with respect to the $B$ systems, in the sense that $\forall \pi \in S_k$

$$\omega_{AB_1B_2\ldots B_k} = W^\pi_{B_1\ldots B_k} \omega_{AB_1B_2\ldots B_k}, \tag{B1}$$

where $W^\pi$ is the unitary permutation channel associated with $\pi$.

2. The state $\rho_{AB}$ is the marginal of $\omega_{AB_1\ldots B_k}$, i.e.,

$$\rho_{AB} = \text{Tr}_{B_2\ldots B_k} \{\omega_{AB_1\ldots B_k}\}. \tag{B2}$$

Note that, due to the permutation symmetry, the second condition above is equivalent to

$$\forall i \in [k] : \rho_{AB} = \omega_{AB_i}, \tag{B3}$$

where $[k] := \{1, \ldots, k\}$, and for all $i \in [k]$, $\mathcal{H}_{B_i} \simeq \mathcal{H}_B$ and $\omega_{AB_i} = \text{Tr}_{B^i\setminus B_i} \{\omega_{AB_1\ldots B_k}\}$. 
Definition 4 (Unextendible state) A state that is not $k$-extendible by Definition 3 is called $k$-unextendible.

For simplicity and throughout this work, if we mention “extendibility,” “extendible,” “unextendibility,” or “extendible,” then these terms should be understood as $k$-extendibility, $k$-extendible, $k$-unextendibility, or $k$-unextendible, respectively, with an implicit dependence on $k$.

Let $\text{EXT}_k(A : B)$ denote the set of all states $\sigma_{AB} \in D(\mathcal{H}_{AB})$ that are $k$-extendible with respect to system $B$. A $k$-extendible state is also $\ell$-extendible, where $\ell \leq k$. This follows trivially from the definition.

2. $k$-extendible channels

In order to define $k$-extendible channels, we need to generalize the notions of permutation invariance and marginals of quantum states to quantum channels. First, permutation invariance of a state gets generalized to permutation covariance of a channel. Next, the marginal of a state gets generalized to the marginal of a channel, which includes a no-signaling constraint, in the following sense:

Definition 5 ($k$-extendible channel) A bipartite channel $\mathcal{N}_{AB \to A'B'}$ is $k$-extendible if there exists a quantum channel $\mathcal{M}_{AB_1 \cdots B_k \to A'B'_1 \cdots B'_k}$ that satisfies the following two criteria:

1. The channel $\mathcal{M}_{AB_1 \cdots B_k \to A'B'_1 \cdots B'_k}$ is permutation covariant with respect to the $B$ systems. That is, $\forall \pi \in S_k$ and for all states $\theta_{AB_1 \cdots B_k}$, the following equality holds
   \[ \mathcal{M}_{AB_1 \cdots B_k \to A'B'_1 \cdots B'_k}(\mathcal{W}_{B_1 \cdots B_k}^\pi(\theta_{AB_1 \cdots B_k})) = \mathcal{W}_{B'_1 \cdots B'_k}^\pi(\mathcal{M}_{AB_1 \cdots B_k \to A'B'_1 \cdots B'_k}(\theta_{AB_1 \cdots B_k})), \]
   where $\mathcal{W}_{B}^\pi$ is the unitary permutation channel associated with $\pi$.

2. The channel $\mathcal{N}_{AB \to A'B'}$ is the marginal of $\mathcal{M}_{AB_1 \cdots B_k \to A'B'_1 \cdots B'_k}$ in the following sense:
   \[ \forall \theta_{AB_1 \cdots B_k} : \mathcal{N}_{AB \to A'B'}(\theta_{AB_1 \cdots B_k}) = \text{Tr}_{B'_1 \cdots B'_k}(\mathcal{M}_{AB_1 \cdots B_k \to A'B'_1 \cdots B'_k}(\theta_{AB_1 \cdots B_k})). \]

A channel $\mathcal{M}_{AB_1 \cdots B_k \to A'B'_1 \cdots B'_k}$ satisfying the above conditions is called a $k$-extension of $\mathcal{N}_{AB \to A'B'}$.

Equivalently, the condition in (B5) can be formulated as
\[ \text{Tr}_{B'_1 \cdots B'_k}(\mathcal{M}_{AB_1 \cdots B_k \to A'B'_1 \cdots B'_k}(X_{AB_1} \otimes Y_{B_2 \cdots B_k})) = 0 \]
for all $X_{AB_1}, Y_{B_2 \cdots B_k}$ such that $\text{Tr}(Y_{B_2 \cdots B_k}) = 0$ [96]. The condition in (B5) corresponds to a one-way no-signaling (semi-casual) constraint on the extended $(k - 1)$ subsystems $B^{k-1} := B^k \setminus B_i$ for all $i \in [k]$ (cf., [96, Proposition 7]).

Classical $k$-extendible channels were defined in a somewhat similar way in [63], and so our definition above represents a quantum generalization of the classical notion. We also note here that $k$-extendible channels were defined in a slightly different way in [64], but our definitions reduce to the same class of channels in the case that the input systems $B_1$ through $B_k$ and the output systems $A'$ are trivial.

We can reformulate the constraints on the $k$-extendible channels in terms of the Choi state $\Gamma^N_{A'AB'B'B^k}$ of the extension channel of $\mathcal{N}_{AB \to A'B'}$ as follows:

\[ \text{Tr}_{A'B'B^k}(\Gamma^N_{A'AB'B'B^k}) = 0, \quad \text{(completely positive)} \]
\[ \text{Tr}_{A'B'B^k}(\Gamma^N_{A'AB'B'B^k}) = I_{AB^k}, \quad \text{(trace-preserving)} \]
\[ \left[ W_{B_1 \cdots B_k}^\pi \otimes W_{B'_1 \cdots B'_k}^\pi, \Gamma^N_{A'AB'B'B^k} \right] = 0, \quad \forall \pi \in S_k, \quad \text{(covariance)} \]
\[ \text{Tr}_{B^k \setminus B^i}(\Gamma^N_{A'AB'B'B^k} Y_{B^k \setminus B^i}^T) = 0, \quad \forall \text{Tr}(Y) = 0, \quad \forall i \in [k], \quad \text{($A'B'_i \leftrightarrow B^{k-1}$)} \]

where $Y$ is an arbitrary Hermitian operator and the last constraint need only be verified on a Hermitian matrix basis of $B^{k-1}$. The key to deriving these constraints is the following well known "transpose trick":
\[ (M_R \otimes I_A) |\Psi\rangle_{RA} = (I_A \otimes M_A^T) |\Psi\rangle_{RA}, \quad \text{(B11)} \]

where $M^T$ is the transpose of $M$ with respect to the basis in (A4).

The following theorem is the key statement that makes the resource theory of unextendibility, as presented above, a consistent resource theory:
Theorem 4  For a bipartite $k$-extendible channel $\mathcal{N}_{AB\to A'B'}$ and a $k$-extendible state $\sigma_{AB}$, the output state $\mathcal{N}_{AB\to A'B'}(\sigma_{AB})$ is $k$-extendible.

Proof. Let $\sigma_{A_{1}\ldots A_{k}}$ be a $k$-extension of $\sigma_{AB}$. Let $\mathcal{M}_{A_{1}\ldots A_{k}\to A'B_{1}'\ldots B_{k}'}$ be a channel that extends $\mathcal{N}_{AB\to A'B'}$. Then the following state is a $k$-extension of $\mathcal{N}_{AB\to A'B'}(\sigma_{AB})$:

$$\mathcal{M}_{A_{1}\ldots A_{k}\to A'B_{1}'\ldots B_{k}'}(\sigma_{A_{1}\ldots A_{k}}).$$

To verify this statement, consider that $\forall \pi \in S_{k}$, the following holds by applying (B4) and the fact that $\sigma_{A_{1}\ldots A_{k}}$ is a $k$-extension of $\rho_{AB}$:

$$W_{B_{1}'\ldots B_{k}'}(A_{1}\ldots A_{k}\to A'B_{1}'\ldots B_{k}')(\sigma_{A_{1}\ldots A_{k}}) = \mathcal{M}_{A_{1}\ldots A_{k}\to A'B_{1}'\ldots B_{k}'}(W_{B_{1}'\ldots B_{k}'}(\sigma_{AB})) = \mathcal{M}_{A_{1}\ldots A_{k}\to A'B_{1}'\ldots B_{k}'}(\sigma_{A_{1}\ldots A_{k}}).$$

Due to (B5), it follows that $\mathcal{N}_{AB\to A'B'}(\sigma_{AB})$ is a marginal of $\mathcal{M}_{A_{1}\ldots A_{k}\to A'B_{1}'\ldots B_{k}'}(\sigma_{A_{1}\ldots A_{k}})$. ■

With the above framework in place, we note here that postulates I–V of [13] apply to the resource theory of unextendibility. The $k$-extendible channels are the free channels, and the $k$-extendible states are the free states.

An important and practically relevant class of $k$-extendible channels are 1W-LOCC channels:

Example 1 (1W-LOCC) An example of a $k$-extendible channel is a one-way local operations and classical communication (1W-LOCC) channel. Consider that a 1W-LOCC channel $\mathcal{N}_{AB\to A'B'}$ can be written as

$$\mathcal{N}_{AB\to A'B'} = \sum_{x} E_{A\to A'}^{x} \otimes F_{B\to B'}^{x},$$

where $\{E_{A\to A'}^{x}\}_{x}$ is a collection of completely positive maps such that $\sum_{x} E_{A\to A'}^{x}$ is a quantum channel and $\{F_{B\to B'}^{x}\}_{x}$ is a collection of quantum channels. A $k$-extension $\mathcal{M}_{A_{1}\ldots A_{k}\to A'B_{1}'\ldots B_{k}'}$ of the channel $\mathcal{N}_{AB\to A'B'}$ can be taken as follows:

$$\mathcal{M}_{A_{1}\ldots A_{k}\to A'B_{1}'\ldots B_{k}'} = \sum_{x} E_{A\to A'}^{x} \otimes F_{B_{1}\to B_{1}'}^{x} \otimes F_{B_{2}\to B_{2}'}^{x} \otimes \cdots \otimes F_{B_{k}\to B_{k}'}^{x}.$$  

It is then clear that the condition in (B4) holds for $\mathcal{M}_{A_{1}\ldots A_{k}\to A'B_{1}'\ldots B_{k}'}$ as chosen above. Furthermore, the condition in (B5) holds because each $F_{B_{i}\to B_{i}'}$ is a channel for $i \in \{1, \ldots, k\}$.

A 1W-LOCC channel can also be represented as

$$\mathcal{D}_{C'A'B''B''} \circ \mathcal{P}_{C\to C'} \circ \mathcal{M}_{C\to C} \circ \mathcal{E}_{A\to A'C},$$

where $\mathcal{E}_{A\to A'C}$ is an arbitrary channel, $\mathcal{M}_{C\to C}$ is a measurement channel, $\mathcal{P}_{C\to C'}$ is a preparation channel, such that $C$ is a classical system, and $\mathcal{D}_{C'A'B''B''}$ is an arbitrary channel. A measurement channel followed by a preparation channel realizes an entanglement breaking (EB) channel [97].

a. Subclass of extendible channels

We now define a subclass of $k$-extendible channels. These channels are inspired by 1W-LOCC channels and are realized as follows: Alice performs a quantum channel $\mathcal{E}_{A\to A'C}$ on her system $A$ and obtains systems $A'C$. Then, Alice sends $C$ to Bob over a $k$-extendible channel $A_{k}^{C\to C'}$. This channel is a special case of the bipartite $k$-extendible channel $\mathcal{N}_{AB\to A'B'}$ considered in Definition 5, in which we identify the input $C$ with $A$ of $\mathcal{N}_{AB\to A'B'}$, the output $C'$ with $B'$ of $\mathcal{N}_{AB\to A'B'}$ and the systems $B$ and $A'$ are trivial. Finally, Bob applies the channel $\mathcal{D}_{C'A'B''B''}$ on system $C'$ and his local system $B$ to get $B'$. Denoting the overall channel by $K_{AB\to A'B'}^{k}$, it is realized as follows:

$$K_{AB\to A'B'}^{k}(\cdot) := (\mathcal{D}_{C'A'B''B''} \circ \mathcal{A}_{C\to C'}^{k} \circ \mathcal{E}_{A\to A'C})(\cdot).$$

Due to their structure, we can place an upper bound on the distinguishability of a channel in the subclass described above and the set of 1W-LOCC channels, as quantified by the diamond norm [98]. This upper bound allows us to conclude that the subclass of channels discussed above converges to the set of 1W-LOCC channels in the limit $k \to \infty$. Before stating it, recall that the diamond norm of the difference of two channels $\mathcal{N}$ and $\mathcal{M}$ is given by

$$||\mathcal{N} - \mathcal{M}||_{\diamond} := \max_{\psi_{RA}} ||\text{id}_{R} \otimes (\mathcal{N} - \mathcal{M})(\psi_{RA})||_{1},$$

where the optimization is with respect to pure-state inputs $\psi_{RA}$, with $R$ a reference system isomorphic to the channel input system $A$. 
Proposition 1 The diamond distance between the channel $K_{AB\rightarrow A'B'}^k$ in (B18) and a 1W-LOCC channel is bounded from above as

$$
\min_{L_{AB\rightarrow A'B'}\in 1W-LOCC} \| K_{AB\rightarrow A'B'}^k - L_{AB\rightarrow A'B'} \|_\diamond \leq |C| \frac{2|C|^2}{|C|^2 + k}, \tag{B20}
$$

where 1W-LOCC denotes the set of all 1W-LOCC channels acting on input systems $A'B'$. \hfill $\blacksquare$

**Proof.** Letting $S_{C\rightarrow C'_1\cdots C'_k}$ denote an extension channel for $A^k_{C\rightarrow C'}$, observe that

$$
\min_{L_{AB\rightarrow A'B'}\in 1W-LOCC} \| K_{AB\rightarrow A'B'}^k - L_{AB\rightarrow A'B'} \|_\diamond
\leq \min_{P_{\Omega}, M} \| \text{Tr}_{C^{k-1}} \circ S_{C\rightarrow C'_1\cdots C'_k} \circ E_{A\rightarrow A'C} - \mathcal{P}_{C\rightarrow C'} \circ M_{C\rightarrow \bar{C}} \circ E_{A\rightarrow A'C} \|_\diamond \tag{B21}
$$

$$
= \min_{P_{\Omega}, \psi_{RA}} \| \text{Tr}_{C^{k-1}} \circ S_{C\rightarrow C'_1\cdots C'_k} \circ E_{A\rightarrow A'C}(\psi_{RA}) - \mathcal{P}_{C\rightarrow C'} \circ M_{C\rightarrow \bar{C}} \circ E_{A\rightarrow A'C}(\psi_{RA}) \|_1 \tag{B22}
$$

$$
\leq \min_{P_{\Omega}, M} \| \text{Tr}_{C^{k-1}} \circ S_{C\rightarrow C'_1\cdots C'_k} \circ E_{A\rightarrow A'C}(\psi_{RA}) - \mathcal{P}_{C\rightarrow C'} \circ M_{C\rightarrow \bar{C}} \|_\diamond. \tag{B23}
$$

The first inequality follows from (B18), by choosing a particular 1W-LOCC and from the monotonicity of trace norm with respect to quantum channels. The first equality follows from the definition of diamond distance. The second inequality follows from the definition of diamond distance, which has an implicit maximization over all the input states. We now observe that

$$
\min_{P_{\Omega}, M} \| \text{Tr}_{C^{k-1}} \circ S_{C\rightarrow C'_1\cdots C'_k} \circ E_{A\rightarrow A'C}(\psi_{RA}) - \mathcal{P}_{C\rightarrow C'} \circ M_{C\rightarrow \bar{C}} \|_\diamond
\leq |C| \min_{J_{RC'}^S, J_{RC'}^E} \| J_{RC'}^S/|C| - J_{RC'}^E/|C| \|_1 \tag{B24}
$$

$$
\leq |C| \frac{2|C|^2}{|C|^2 + k}, \tag{B25}
$$

where

$$
J_{RC'}^S/|C| = \text{Tr}_{C^{k-1}} \circ S_{C\rightarrow C'_1\cdots C'_k} \circ E_{A\rightarrow A'C}(\Phi_{RC}) \in \text{EXT}_k(R:C'), \tag{B26}
$$

$$
J_{RC'}^E/|C| = \mathcal{P}_{C\rightarrow C'} \circ M_{C\rightarrow \bar{C}}(\Phi_{RC}) \in \text{SEP}(R:C'). \tag{B27}
$$

The first inequality follows from bounding the diamond distance between the two channels by the trace norm between the corresponding Choi states (see, e.g., [99, Lemma 7]). The last inequality follows from [100, Eq. (11)], which in turn built on the developments in [101]. \hfill $\blacksquare$

3. Quantifying $k$-unextendibility

In any resource theory, it is pertinent to quantify the resourcefulness of the resource states and the resourceful channels. Based on the resource theory of unextendibility, any measure of the $k$-unextendibility of a state should possess the following two desirable properties:

1. monotonicity: non-increasing under the action of $k$-extendible channels,
2. attains minimum value if the state is $k$-extendible.

Here we present a rather general measure of unextendibility, based on the notion of generalized divergence recalled in Section A3, and which satisfies both criteria discussed above:

Definition 6 (Unextendible generalized divergence) The $k$-unextendible generalized divergence of a bipartite state $\rho_{AB}$ is defined as

$$
E_k(A;B)_\rho = \inf_{\sigma_{AB} \in \text{EXT}_k(A:B)} D(\rho_{AB}\|\sigma_{AB}), \tag{B28}
$$

where $D(\rho\|\sigma)$ denotes the generalized divergence from Section A3.
We can extend the definition above to define the unextendible generalized divergence of a quantum channel, in order to quantify how well a quantum channel can preserve unextendibility.

**Definition 7 (Unextendible generalized channel divergence)** The $k$-unextendible generalized divergence of a quantum channel $\mathcal{N}_{A \rightarrow B}$ is defined as

$$E_k(\mathcal{N}) = \sup_{\psi_{RA} \in \mathcal{D}(H_{RA})} \inf_{\sigma_{RB} \in \text{EXT}_k(A:B)} D(\mathcal{N}_{A \rightarrow B}(\psi_{RA})\|\sigma_{RB}),$$

where $D(\cdot\|\cdot)$ is a generalized divergence and the optimization is over all pure states $\psi_{RA} \in \mathcal{D}(H_{RA})$ with $R \simeq A$.

In the above definition, we could have taken an optimization over all mixed-state inputs with the reference system $R$ arbitrarily large. However, due to purification, data processing, and the Schmidt decomposition theorem, doing so does not result in a larger value of the quantity, so that it suffices to restrict the optimization as we have done above.

In Definitions 6 and 7, we can take the generalized divergence to be the quantum relative entropy $D$, $\varepsilon$-hypothesis-testing divergence $D_\varepsilon$, $\alpha$-sandwiched-Rényi divergence $\tilde{D}_\alpha$, the traditional Rényi divergence, the trace distance, etc., in order to have various $k$-unextendible measures of states and channels.

**a. $k$-unextendible divergences for isotropic and Werner states**

In this section, we evaluate some unextendible divergences for two specific classes of states: isotropic and Werner states.

**Definition 8 (Isotropic state [102])** An isotropic state $\rho_{AB}^{(t,d)}$ is $U \otimes U^*$-invariant for an arbitrary unitary $U$, where $\dim(H_A) = \dim(H_B) = d$. Such a state can be written in the following form for $t \in [0,1]$:

$$\rho_{AB}^{(t,d)} = t \Phi^d_{AB} + (1 - t) I^d_{AB} - \Phi^d_{AB} (d^2 - 1),$$

where $\Phi^d_{AB}$ denotes a maximally entangled state of Schmidt rank $d$.

**Lemma 3 ([44])** An isotropic state $\rho_{AB}^{(t,d)}$ written as in (B30) is $k$-extendible if and only if $t \leq 1 + (d - 1) / k$.  

**Proof.** This is a direct application of [44, Theorem III.8], and we provide details for completeness. Isotropic states are parametrized in [44] for $y \in [0,d]$ as

$$\frac{d}{d^2 - 1} \left[(d - y) \frac{I_{AB}}{d^2} + \left(y - \frac{1}{d}\right) \Phi^d_{AB}\right].$$

There, as shown in [44, Theorem III.8], an isotropic state is $k$-extendible if and only if

$$y \leq 1 + (d - 1) / k.$$  

(B32)

Translating this to the parametrization in (B30), we find that

$$\frac{d}{d^2 - 1} \left[(d - y) \frac{I_{AB}}{d^2} + \left(y - \frac{1}{d}\right) \Phi^d_{AB}\right] = \frac{d}{d^2 - 1} \left[\frac{d - y}{d^2} (I_{AB} - \Phi^d_{AB}) + \left(d - y \frac{I_{AB}}{d^2} + y - \frac{1}{d}\right) \Phi^d_{AB}\right] = \frac{d - y}{d} \frac{I_{AB} - \Phi^d_{AB}}{d^2 - 1} + \frac{y}{d} \Phi^d_{AB}. $$

(B33)

Using the fact that $t = y/d$ to translate between the two different parametrizations of isotropic states, the condition in (B32) translates to

$$t \leq \frac{1}{d} \left(\frac{d - 1}{k} + 1\right).$$

(B35)

This concludes the proof.
**Definition 9 (Werner state [103])** Let $A$ and $B$ be quantum systems, each of dimension $d$. A Werner state is defined for $p \in [0, 1]$ as

$$W_{AB}^{(p,d)} := (1 - p) \frac{2}{d(d+1)} \Pi_{AB}^+ + p \frac{2}{d(d-1)} \Pi_{AB}^-,$$

(B36)

where $\Pi_{AB}^+ := (I_{AB} + F_{AB})/2$ are the projections onto the symmetric and antisymmetric subspaces of $A$ and $B$, with $F_{AB}$ denoting the swap operator.

**Lemma 4 ([44])** A Werner state $W_{AB}^{(p,d)}$ is $k$-extendible if and only if $p \in [0, \frac{1}{2} \left( \frac{d-1}{k} + 1 \right)]$.

**Proof.** This is a direct application of [44, Theorem III.7], and we provide details for completeness. Werner states are parametrized in [44] for $q \in [-1, 1]$ as

$$\frac{d}{d^2 - 1} \left[ (d - q) \frac{I_{AB}}{d^2} + \left( q - \frac{1}{d} \right) \frac{F_{AB}}{d} \right].$$

(B37)

There, as shown in [44, Theorem III.7], a Werner state is $k$-extendible if and only if

$$q \geq -(d - 1)/k.$$  

(B38)

Translating this to the parametrization in (B36), and using that

$$I_{AB} = \Pi_{AB}^+ + \Pi_{AB}^-,$$

$$F_{AB} = \Pi_{AB}^+ - \Pi_{AB}^-,$$

we find that

$$\frac{d}{d^2 - 1} \left[ (d - q) \frac{I_{AB}}{d^2} + \left( q - \frac{1}{d} \right) \frac{F_{AB}}{d} \right] = \frac{d}{d^2 - 1} \left[ \frac{d - q}{d^2} \left( \Pi_{AB}^+ + \Pi_{AB}^- \right) + \left( q - \frac{1}{d} \right) \left( \Pi_{AB}^+ - \Pi_{AB}^- \right) \right] = \frac{1 + q}{2} \frac{2}{d(d + 1)} \Pi_{AB}^+ + \frac{1 - q}{2} \frac{2}{d(d + 1)} \Pi_{AB}^-.$$  

(B41)

(B42)

(B43)

Using the fact that $p = (1 - q)/2$ to translate between the two different parametrizations of Werner states, the condition in (B38) translates to

$$p \leq \frac{1}{2} \left( \frac{d - 1}{k} + 1 \right).$$

(B44)

This concludes the proof. 

For $p, q \in [0, 1]$ and for any generalized divergence $D$, we make the following abbreviation:

$$D(p\|q) := D(\kappa(p)\|\kappa(q)),$$

(B45)

where

$$\kappa(x) = x|0\rangle\langle 0| + (1 - x)|1\rangle\langle 1|.$$  

(B46)

We then have the following:

**Proposition 2** The $k$-unextendible generalized divergence of a Werner state $W_{AB}^{(p,d)}$ and an isotropic state $\rho_{AB}^{(1,d)}$ are respectively equal to

$$E_k(A; B)_{W_{AB}^{(p,d)}} = \inf_{q \in [0, \frac{1}{2} \left( \frac{d - 1}{k} + 1 \right)]} D(p\|q),$$

(B47)

$$E_k(A; B)_{\rho_{AB}^{(1,d)}} = \inf_{q \in [0, \frac{1}{2} \left( \frac{d - 1}{k} + 1 \right)]} D(t\|q).$$

(B48)
Proof. By definition, $E_k(A; B)_{W^p}$ involves an infimum with respect to all possible $k$-extendible states. It is monotone with respect to all $1W$-LOCC channels, and one such choice is the full bilateral twirl:

$$\omega_{AB} \rightarrow T_{AB}^W(\omega_{AB}) := \int d\mu(U)[U_A \otimes U_B] \omega_{AB} [U_A \otimes U_B]^\dagger. \quad (B49)$$

Note that this can be implemented by a unitary two-design [104]. The Werner state is invariant with respect to this channel, whereas any other $k$-extendible state $\sigma_{AB}$ becomes a Werner state under this channel. Let $\sigma_{AB}$ denote an arbitrary $k$-extendible state. We thus have

$$D(W_{AB}^{(p,d)}\|\sigma_{AB}) \geq D(T_{AB}^W(W_{AB}^{(p,d)})) \| T_{AB}^W(\sigma_{AB})) \quad (B50)$$

$$= D(W_{AB}^{(p,d)}\|\sigma_{AB})) \quad (B51)$$

$$= D(W_{AB}^{(r,d)}\|W_{AB}^{(r,d)}), \quad (B52)$$

where in the last line, we have noted that $T_{AB}^W(\sigma_{AB})$ is a Werner state and can thus be written as $W_{AB}^{(r,d)}$ for some $r \in [0, 1]$. Furthermore, by Theorem 4, $W_{AB}^{(r,d)}$ is a $k$-extendible state since $\sigma_{AB}$ is by assumption. Thus, it suffices to consider only $k$-extendible Werner states in the optimization of $E_k(A; B)_{W(p,d)}$. Next, the following equality holds

$$D(W_{AB}^{(p,d)}\|W_{AB}^{(r,d)}) = D(p\|r), \quad (B53)$$

because the quantum-to-classical channel

$$\omega_{AB} \rightarrow \text{Tr}\{\Pi_{AB}^+\omega_{AB}\} |0\rangle \langle 0| + \text{Tr}\{\Pi_{AB}^-\omega_{AB}\} |1\rangle \langle 1| \quad (B54)$$

takes a Werner state $W_{AB}^{(p,d)}$ to $(1 - p) |0\rangle \langle 0| + p |1\rangle \langle 1|$ and the classical-to-quantum channel

$$\tau \rightarrow (0|\tau|0) + 2d \Pi_{AB}^+ + (1|\tau|1) - 2 \Pi_{AB}^- \quad (B55)$$

takes $(1 - p) |0\rangle \langle 0| + p |1\rangle \langle 1|$ back to $W_{AB}^{(p,d)}$. Finally, we can conclude the first equality in the statement of the theorem.

The reasoning for the second equality is exactly the same, but we instead employ the bilateral twirl

$$T_{AB}^I(\omega_{AB}) := \int d\mu(U)[U_A \otimes U_B^*] \omega_{AB} [U_A \otimes U_B^*]^\dagger. \quad (B56)$$

This is a $k$-extendible channel, the isotropic states are invariant under this twirl, and all other states are projected to isotropic states under this twirl. Also, the channel

$$\omega_{AB} \rightarrow \text{Tr}\{\Phi_{AB}\omega_{AB}\} |0\rangle \langle 0| + \text{Tr}\{(I_{AB} - \Phi_{AB})\omega_{AB}\} |1\rangle \langle 1| \quad (B57)$$

takes an isotropic state $\rho_{AB}^{(t,d)}$ to $t|0\rangle \langle 0| + (1 - t) |1\rangle \langle 1|$ and the classical-to-quantum channel

$$\tau \rightarrow (0|\tau|0) \Phi_{AB} + (1|\tau|1) I_{AB} - \Phi_{AB} \quad \frac{d}{d^2 - 1} \quad (B58)$$

allows for going back. These statements allow us to conclude the second inequality. ■

The following two lemmas are helpful in establishing an explicit formula for the $k$-unextendible relative entropy and Rényi relative entropy.

Lemma 5 Let $1 > p > q > 0$. Then the relative entropy $D(p\|q)$ is a monotone decreasing function of $q$ for $p > q > 0$. That is, for $1 > p > q > r > 0$, the following inequality holds

$$D(p\|r) > D(p\|q). \quad (B59)$$

Proof. To prove the statement, we show that the derivative of $D(p\|q)$ with respect to $q$ is negative. The derivative of $D(p\|q)$ with respect to $q$ is equal to

$$\frac{d}{dq} D(p\|q) = \frac{1 - p}{1 - q} - \frac{p}{q}. \quad (B60)$$
The condition that \( \frac{d}{dq} D(p||q) < 0 \) is thus equivalent to the condition
\[
\frac{q}{1-q} < \frac{p}{1-p}.
\]
(B61)

This latter condition holds because the function \( x/(1-x) \) is a monotone increasing function on the interval \( x \in (0, 1) \). That this latter claim is true follows because the derivative of \( x/(1-x) \) with respect to \( x \) is given by
\[
\frac{d}{dx} \left( \frac{x}{1-x} \right) = \frac{1}{1-x} + \frac{x}{(1-x)^2},
\]
(B62)

which is positive for \( x \in (0, 1) \). ■

**Lemma 6** Let \( 1 > p > q > 0 \) and let \( \alpha \in (0, 1) \cup (1, \infty) \). Then the Rényi relative entropy \( D_\alpha(p||q) \) is a monotone decreasing function of \( q \) for \( p > q > 0 \). That is, for \( 1 > p > q > r > 0 \), the following inequality holds
\[
D_\alpha(p||r) > D_\alpha(p||q).
\]
(B63)

**Proof.** To prove the statement, we show that the derivative of \( D_\alpha(p||q) \) with respect to \( q \) is negative. The derivative of \( D_\alpha(p||q) \) with respect to \( q \) is equal to
\[
\frac{d}{dq} D_\alpha(p||q) = \left[ 1 - q + \frac{1}{\left( \frac{q}{1-q} \right) \left( \frac{p}{1-p} \right)^\alpha - 1} \right]^{-1}.
\]
(B64)

\[
= \frac{\left( \frac{q}{1-q} \right) \left( \frac{p}{1-p} \right)^\alpha - 1}{\left( \frac{q}{1-q} \right) \left( \frac{p}{1-p} \right)^\alpha - 1} [1-q] + 1.
\]
(B65)

Since \( \frac{p}{1-p} > \frac{q}{1-q} \) for \( 1 > p > q > 0 \) (as shown in the previous proof), it follows that
\[
\left( \frac{q}{1-q} \right) \left( \frac{p}{1-p} \right)^\alpha - 1 < 0
\]
for all \( \alpha \in (0, 1) \cup (1, \infty) \). We would then like to prove that
\[
\left[ \left( \frac{q}{1-q} \right) \left( \frac{p}{1-p} \right)^\alpha - 1 \right] [1-q] + 1 > 0.
\]
(B67)

Note that this is equivalent to
\[
\left[ 1 - \left( \frac{q}{1-q} \right) \left( \frac{p}{1-p} \right)^\alpha \right] [1-q] < 1,
\]
(B68)

which follows because
\[
1 - \left( \frac{q}{1-q} \right) \left( \frac{p}{1-p} \right)^\alpha \in (0, 1)
\]
and \( 1-q \in (0, 1) \). Thus, we can conclude that \( \frac{d}{dq} D_\alpha(p||q) < 0 \) for \( 1 > p > q > 0 \), and the statement of the lemma follows. ■

With all of the above, we conclude the following:

**Proposition 3** The \( k \)-unextendible relative entropy of a Werner state \( W_{AB}^{(p,d)} \) and an isotropic state \( \rho_{AB}^{(t,d)} \) are respectively equal to
\[
E_k(A;B)_{W^{(p,d)}} = \left\{ \begin{array}{ll}
D(p\|\frac{1}{2} (\frac{d-1}{k} + 1)) & \text{if } p \in \left[ 0, \frac{1}{2} \right] (\frac{d-1}{k} + 1] \\
\text{else} & \end{array} \right.,
\]
(B70)

\[
E_k(A;B)_{\rho^{(t,d)}} = \left\{ \begin{array}{ll}
D(t\|\frac{1}{2} (\frac{d-1}{k} + 1)) & \text{if } p \in \left[ 0, \frac{1}{2} \right] (\frac{d-1}{k} + 1] \\
\text{else} & \end{array} \right.,
\]
(B71)

Similarly, the \( k \)-unextendible Rényi divergences are given for \( \alpha \in (0, 1) \cup (1, \infty) \) by
\[
E_k^\alpha(A;B)_{W^{(p,d)}} = \left\{ \begin{array}{ll}
D_\alpha(p\|\frac{1}{2} (\frac{d-1}{k} + 1)) & \text{if } p \in \left[ 0, \frac{1}{2} \right] (\frac{d-1}{k} + 1] \\
\text{else} & \end{array} \right.,
\]
(B72)

\[
E_k^\alpha(A;B)_{\rho^{(t,d)}} = \left\{ \begin{array}{ll}
D_\alpha(t\|\frac{1}{2} (\frac{d-1}{k} + 1)) & \text{if } p \in \left[ 0, \frac{1}{2} \right] (\frac{d-1}{k} + 1] \\
\text{else} & \end{array} \right.,
\]
(B73)
b. Properties of $k$-unextendible divergences of a bipartite state

In this section, we discuss some of the properties of an unextendible generalized divergence, focusing first on the quantity derived from quantum relative entropy. The $k$-unextendible relative entropy of a state $\rho_{AB}$ is given by Definition 6, by replacing $D$ with the quantum relative entropy $D$.

We begin by proving the uniform continuity of unextendible relative entropy. In order to do so, we use the following result [105] concerning the relative entropy distance with respect to any closed, convex set $C$ of states, or more generally positive semi-definite operators $B_+ (H_A)$:

$$D_C(\rho) = \min_{\gamma \in C} D(\rho \parallel \gamma). \quad \text{(B74)}$$

**Lemma 7 ([105])** For a closed, convex, and bounded set $C$ of positive semi-definite operators, containing at least one of full rank, let

$$\kappa := \sup_{\tau, \tau'} [D_C(\tau) - D_C(\tau')]$$

be the largest variation of $D_C$. Then, for any two states $\rho$ and $\sigma$ for which $\frac{1}{2} \| \rho - \sigma \|_1 \leq \varepsilon$, with $\varepsilon \in [0, 1]$, we have that

$$|D_C(\rho) - D_C(\sigma)| \leq \varepsilon \kappa + g(\varepsilon), \quad \text{(B76)}$$

where $g(\varepsilon) := (\varepsilon + 1) \log_2 (\varepsilon + 1) - \varepsilon \log_2 \varepsilon$.

**Lemma 8 (Uniform continuity of unextendible relative entropy)** For any two bipartite states $\rho_{AB}$ and $\sigma_{AB}$ acting on the composite Hilbert space $H_A \otimes H_B$, with $d = \min \{|A|, |B|\}$, and $\frac{1}{2} \| \rho_{AB} - \sigma_{AB} \| \leq \varepsilon \in [0, 1]$, we have that

$$|E_k (A; B)_\rho - E_k (A; B)_\sigma| \leq \varepsilon \log_2 \min \{d, k\} + g(\varepsilon). \quad \text{(B77)}$$

**Proof.** This follows directly from Lemma 7. To see this, observe that we have the following inequalities holding for any states $\tau_{AB}$ and $\tau'_{AB}$:

$$E_k (A; B)_{\tau'} \geq 0, \quad \text{(B78)}$$

$$E_k (A; B)_{\tau} \leq E_R (A; B)_{\tau} \leq \min \{S(A)_{\tau}, S(B)_{\tau} \} \leq \log d, \quad \text{(B79)}$$

where $E_R (A; B)_{\tau}$ denotes the relative entropy of entanglement [6, 106].

Finally, we obtain the log$_2 k$ upper bound on $E_k (A; B)_{\tau}$ by picking the $k$-extendible state for $E_k (A; B)_{\tau} = \inf_{\sigma_{AB} \in \text{EXT}_k (A; B)} D(\tau_{AB} \parallel \sigma_{AB})$ as

$$\sigma_{AB} = \frac{1}{k} \tau_{AB} + \left(1 - \frac{1}{k} \right) \tau_A \otimes \tau_B. \quad \text{(B80)}$$

Such a state is $k$-extendible with a $k$-extension given by

$$\sigma_{AB_1 \cdots B_k} = \frac{1}{k} \sum_{i=1}^{k} \tau_{B_1} \otimes \cdots \otimes \tau_{B_{i-1}} \otimes \tau_{AB_i} \otimes \tau_{B_{i+1}} \otimes \cdots \otimes \tau_{B_k}. \quad \text{(B81)}$$

Then by using the facts that $D(\rho \parallel \sigma) \geq D(\rho \parallel \sigma')$ for $0 \leq \sigma \leq \sigma'$ and $D(\rho \parallel c\sigma) = D(\rho \parallel \sigma) - \log_2 c$ for $c > 0$, we find that

$$E_k (A; B)_{\tau} \leq \inf_{\sigma_{AB} \in \text{EXT}_k (A; B)} D(\tau_{AB} \parallel \sigma_{AB}) \quad \text{(B82)}$$

$$\leq D \left( \tau_{AB} \left| \left| \frac{1}{k} \tau_{AB} + \left(1 - \frac{1}{k} \right) \tau_A \otimes \tau_B \right| \right. \right) \quad \text{(B83)}$$

$$\leq D(\tau_{AB} \parallel \tau_{AB}) - \log_2 (1/k) = \log_2 k. \quad \text{(B84)}$$

This concludes the proof. 

**Lemma 9 (Faithfulness)** Fix $\varepsilon \in [0, 1]$. The $k$-unextendible relative entropy $E_k (A; B)_\rho$ of an arbitrary state $\rho_{AB}$ is a faithful measure, in the sense that

$$E_k (A; B)_\rho \leq \varepsilon \implies \min \limits_{\sigma_{AB} \in \text{EXT}_k (A; B)} \| \rho_{AB} - \sigma_{AB} \|_1 \leq \sqrt{\varepsilon \cdot 2 \ln 2}, \quad \text{(B85)}$$

$$\min \limits_{\sigma_{AB} \in \text{EXT}_k (A; B)} \frac{1}{2} \| \rho_{AB} - \sigma_{AB} \|_1 \leq \varepsilon \implies E_k (A; B)_\rho \leq \varepsilon \log_2 \min \{d, k\} + g(\varepsilon), \quad \text{(B86)}$$

with $d = \min \{|A|, |B|\}$.
**Proof.** The proof of the first statement follows directly from the quantum Pinsker inequality [107, Theorem 1.15]. The second statement follows directly from Proposition 8. ■

The following lemma provides a strong limitation on the $k$-unextendible relative entropy of any multipartite product state, and a related observation was made in [24, Section 4.4].

**Lemma 10 (Subadditivity and non-extensivity)** For a state $\rho_{A_1B_1A_2B_2\cdots A_nB_n} := \omega_{A_1B_1}^{(1)} \otimes \omega_{A_2B_2}^{(2)} \otimes \cdots \otimes \omega_{A_nB_n}^{(n)}$, the $k$-unextendible relative entropy is sub-additive and non-extensive, in the sense that

$$E_k(A_1A_2 \cdots A_n; B_1B_2 \cdots B_n) \leq \min \left\{ \log_2 k, \sum_{i=1}^n E_k(A_i; B_i)_{\omega(i)} \right\}. \quad \text{(B87)}$$

In fact, the non-extensivity bound $E_k(A_1A_2 \cdots A_n; B_1B_2 \cdots B_n) \leq \log_2 k$ applies to an arbitrary state $\rho_{A_1B_1A_2B_2\cdots A_nB_n}$.

**Proof.** The subadditivity proof is straightforward. We show it for a tensor product of two states and note that the general statement follows from induction:

$$E_k(A_1A_2; B_1B_2) = \min_{\sigma_{A_1A_2B_1B_2} \in \text{EXT}_{k}(A_1A_2B_1B_2)} D(\omega_{A_1B_1} \otimes \tau_{A_2B_2} |\sigma_{A_1A_2B_1B_2}) \quad \text{(B88)}$$

$$\leq \min_{\sigma_{A_1B_1} \otimes \sigma_{A_2B_2} \in \text{EXT}_{k}(A_1A_2B_1B_2)} D(\omega_{A_1B_1} |\sigma_{A_1B_1}) + \min_{\sigma_{A_2B_2} \in \text{EXT}_{k}(A_1A_2B_1B_2)} D(\tau_{A_2B_2} |\sigma_{A_2B_2}) \quad \text{(B89)}$$

$$= E_k(A_1; B_1) + E_k(A_2; B_2). \quad \text{(B90)}$$

The first equality follows from the definition. The first inequality follows from a particular choice of $\sigma_{A_1A_2B_1B_2}$. The second inequality follows from additivity of relative entropy with respect to tensor-product states.

The proof of the non-extensivity upper bound of $\log_2 k$ follows from the same reasoning as in (B82)–(B84). ■

**Lemma 11 (Convexity)** Let $\rho_{AB} = \sum_{x \in X} p_X(x) \rho_{AB}^x$ be a bipartite state, where $p_X(x)$ is a probability distribution and $\{\rho_{AB}^x\}_x$ is a set of quantum states. Then the $k$-unextendible relative entropy is convex, in the sense that

$$E_k(A; B) = \sum_{x \in X} p_X(x) E_k(A; B)_{\rho^x}. \quad \text{(B92)}$$

**Proof.** Let $\sigma_{AB}^x$ be the $k$-extendible state that achieves the minimum for $\rho_{AB}^x$ in $E_k(A; B)_{\rho^x}$. Then,

$$E_k(A; B) = \min_{\sigma_{AB} \in \text{EXT}_{k}(A; B)} D(\rho_{AB} |\sigma_{AB}) \quad \text{(B93)}$$

$$\leq D \left( \sum_x p_X(x) \rho_{AB}^x |\sum_x p_X(x) \sigma_{AB}^x \right) \quad \text{(B94)}$$

$$\leq \sum_x p_X(x) D(\rho_{AB}^x |\sigma_{AB}^x) \quad \text{(B95)}$$

$$= \sum_x p_X(x) E_k(A; B)_{\rho^x}. \quad \text{(B96)}$$

The second inequality follows from the joint convexity of quantum relative entropy. ■

The following lemmas have straightforward proofs, following from reasoning above, properties of sandwiched Rényi relative entropy, making use of the additivity of sandwiched Rényi relative entropy with respect to tensor-product states, as well as its joint quasi-convexity:

**Lemma 12 (Subadditivity and non-extensivity)** For a state $\rho_{A_1B_1A_2B_2\cdots A_nB_n} := \omega_{A_1B_1}^{(1)} \otimes \omega_{A_2B_2}^{(2)} \otimes \cdots \otimes \omega_{A_nB_n}^{(n)}$ and $\alpha \in (0,1) \cup (1,\infty)$, the $k$-unextendible $\alpha$-sandwiched-Rényi divergence is sub-additive and non-extensive, in the sense that

$$E_k^\alpha(A_1A_2 \cdots A_n; B_1B_2 \cdots B_n) \leq \min \left\{ \log_2 k, \sum_{i=1}^n E_k^\alpha(A_i; B_i)_{\omega(i)} \right\}. \quad \text{(B97)}$$
In fact, the non-extensivity bound \( \tilde{E}_k^\alpha(A_1 A_2 \cdots A_n; B_1 B_2 \cdots B_n)_\rho \leq \log_2 k \) applies to an arbitrary state \( \rho_{A_1 B_1 A_2 B_2 \cdots A_n B_n} \).

**Lemma 13** The \( k \)-unextendible \( \alpha \)-sandwiched-Rényi divergence is quasi-convex: i.e., if \( \rho_{AB} \in \mathcal{D}(\mathcal{H}_{AB}) \) decomposes as \( \rho_{AB} = \sum_{x \in X} p_X(x) \rho_{AB}^x \), where \( \{p_X(x)\}_x \) is a probability distribution and each \( \rho_{AB}^x \in \mathcal{D}(\mathcal{H}_{AB}) \), then

\[
\tilde{E}_k^\alpha(A; B)_\rho \leq \sup_x \tilde{E}_k^\alpha(A; B)_{\rho^x}.
\]

(89)

**4. Amortization does not enhance the max-\( k \)-unextendibility of a channel**

The purpose of this section is to prove that the unextendible max-relative entropy of a quantum channel does not increase under amortization. Similar results are known for the squashed entanglement of a channel [40], a channel’s max-relative entropy of entanglement [85], and the max-Rains information of a quantum channel [65]. Our proof of this result is strongly based on the approach given in [65], which in turn made use of some of the developments in [4].

We begin by establishing equivalent forms for the unextendible max-relative entropy of a state and a channel. Let \( \text{EXT}_k(A; B) \) denote the cone of all \( k \)-extendible operators. This set is defined in the same way as the set of \( k \)-extendible states, but there is no requirement for a \( k \)-extendible operator to have trace equal to one. Then we have the following alternative expression for the max-relative entropy of unextendibility:

**Lemma 14** Let \( \rho_{AB} \in \mathcal{D}(\mathcal{H}_A \otimes \mathcal{H}_B) \). Then

\[
E_k^{\max}(A; B)_\rho = \log_2 W_k(A; B)_\rho,
\]

where

\[
W_k(A; B)_\rho := \min_{X_{AB} \in \text{EXT}_k(A; B)} \{\text{Tr}\{X_{AB}\} : \rho_{AB} \leq X_{AB}\}.
\]

(B100)

**Proof.** Employing the definition of \( k \)-unextendible max-relative entropy, consider that

\[
E_k^{\max}(A; B)_\rho = \min_{\sigma_{AB} \in \text{EXT}_k(A; B)} D_{\max}(\rho_{AB} \| \sigma_{AB})
\]

(B101)

\[
= \log_2 \min_{\mu, \sigma_{AB}} \{\mu : \rho_{AB} \leq \mu \sigma_{AB}, \sigma_{AB} \in \text{EXT}_k(A; B)\}
\]

(B102)

\[
= \log_2 \min_{X_{AB}} \{\text{Tr}\{X_{AB}\} : \rho_{AB} \leq X_{AB}, X_{AB} \in \text{EXT}_k(A; B)\}.
\]

(B103)

This concludes the proof. \( \blacksquare \)

Let \( E_k^{\max}(\mathcal{N}) \) denote the unextendible max-relative entropy of a channel \( \mathcal{N} \), as given in Definition 7, but with the generalized divergence \( D \) replaced by the max-relative entropy \( D_{\max} \). We can write \( E_k^{\max}(\mathcal{N}) \) in an alternate way, by employing similar reasoning as given in the proof of [79, Lemma 6]:

\[
E_k^{\max}(\mathcal{N}) = \max_{\rho_{SB} \in \mathcal{D}(\mathcal{H}_S)} \min_{\sigma_{SB} \in \text{EXT}_k(S; B)} D_{\max}(\rho_{SB}^{1/2} J_{SB}^{1/2} \rho_{SB}^{1/2} \| \sigma_{SB}),
\]

(B104)

where \( J_{SB}^{1/2} \) is the Choi operator for the channel \( \mathcal{N} \).

An alternative expression for the unextendible max-relative entropy \( E_k^{\max}(\mathcal{N}) \) of the channel \( \mathcal{N} \) is given by the following lemma:

**Lemma 15** For any quantum channel \( \mathcal{N}_{A \rightarrow B} \),

\[
E_k^{\max}(\mathcal{N}) = \log_2 \Sigma_k(\mathcal{N}),
\]

(B105)

where

\[
\Sigma_k(\mathcal{N}) := \min_{Y_{SB} \in \text{EXT}_k(S; B)} \{\|\text{Tr}_B\{Y_{SB}\}\|_\infty : J_{SB}^{1/2} \leq Y_{SB}\},
\]

(B106)

and \( J_{SB}^{1/2} \) is the Choi operator for the channel \( \mathcal{N}_{A \rightarrow B} \).
Theorem 5 (Amortization inequality) Let \( \rho_{RA} \) be a state, and let \( \mathcal{N}_{A \to B} \) be an arbitrary quantum channel. Then the following inequality holds for the \( k \)-unextendible max-relative-entropy of a channel \( \mathcal{N} \):

\[
E_k^{\max}(RA; BR_B) \omega \leq E_k^{\max}(RA; R_B) + E_k^{\max}(N),
\]

where \( \omega_{RA BR_B} := \mathcal{N}_{A \to B}(\rho_{RA}) \).

Proof. We adapt the proof steps of [65, Proposition 8] to show that amortization does not enhance the unextendible max-relative entropy of an arbitrary channel.

By removing logarithms and applying Lemmas 14 and 15, the desired inequality is equivalent to the following one:

\[
W_k(R_A; BR_B) \omega \leq W_k(R_A A; R_B) \rho \cdot \Sigma_k(N),
\]

and so we aim to prove this one. Exploiting the identity in Lemma 14, we find that

\[
W_k(R_A A; R_B) \rho = \min \text{Tr}\{C_{RA BR_B}\},
\]

subject to the constraints

\[
C_{RA BR_B} \in \text{EXT}_k(R_A A; B),
\]

\[
C_{RA BR_B} \geq \rho_{RA AB},
\]

while the identity in Lemma 15 gives that

\[
\Sigma_k(N) = \min \|\text{Tr}_B\{Y_{SB}\}\|_\infty,
\]

subject to the constraints

\[
Y_{SB} \in \text{EXT}_k(S; B),
\]

\[
Y_{SB} \geq J_{SB}^N.
\]

The identity in Lemma 14 implies that the left-hand side of (B108) is equal to

\[
W_k(R_A; BR_B) \omega = \min \text{Tr}\{E_{RA BR_B}\},
\]

subject to the constraints

\[
E_{RA BR_B} \in \text{EXT}_k(R_A; BR_B),
\]

\[
E_{RA BR_B} \geq \mathcal{N}_{A \to B}(\rho_{RA AB}).
\]

Once we have these optimizations, we can now show that the inequality in (B108) holds by making an appropriate choice for \( E_{RA BR_B} \). Let \( C_{RA AB} \) be optimal for \( W_k(R_A A; R_B) \rho \), and let \( Y_{RA BR_B} \) be optimal for \( \Sigma(N) \). Let \( \{\Upsilon\}_{SA} \) be the maximally entangled vector. Choose

\[
E_{RA BR_B} = \langle \Upsilon |_{SA} C_{RA AB} \otimes Y_{SB} | \Upsilon \rangle_{SA}.
\]

We need to prove that \( E_{RA BR_B} \) is feasible for \( W_k(R_A A; BR_B) \omega \). To this end, we have

\[
\langle \Upsilon |_{SA} C_{RA AB} \otimes Y_{SB} | \Upsilon \rangle_{SA} \geq \langle \Upsilon |_{SA} \rho_{RA AB} \otimes J_{SB}^N | \Upsilon \rangle_{SA} = \mathcal{N}_{A \to B}(\rho_{RA AB}).
\]

Now, since \( C_{RA AB} \in \text{EXT}_k(R_A A; R_B) \) and \( Y_{SB} \in \text{EXT}_k(S; B) \), it immediately follows that \( \langle \Upsilon |_{SA} C_{RA AB} \otimes Y_{SB} | \Upsilon \rangle_{SA} \in \text{EXT}_k(R_A A; R_B B) \).

Consider that

\[
\text{Tr}\{E_{RA BR_B}\} = \text{Tr}\{\langle \Upsilon |_{SA} (C_{RA AB} \otimes Y_{SB}) | \Upsilon \rangle_{SA}\}
\]

\[
= \text{Tr}\{C_{RA AB} T_A(Y_{AB})\}
\]

\[
= \text{Tr}\{C_{RA AB} T_A(Y_{AB})\}
\]

\[
\leq \text{Tr}\{C_{RA AB} \|T_A(Y_{AB})\|_\infty\}
\]

\[
= \text{Tr}\{C_{RA AB} \|T_B(Y_{AB})\|_\infty\}
\]

\[
= W_k(R_A A; R_B) \rho \cdot \Sigma(N).
\]
The inequality is a consequence of Hölder’s inequality [108]. The final equality follows because the spectrum of a positive semi-definite operator is invariant under the action of a full transpose (note, in this case, \(T_A\) is the full transpose as it acts on reduced positive semi-definite operator \(Y_A\)).

Therefore, we can infer that our choice of \(E_{RA, BR_B}\) is feasible for \(W_k(R_A; BR_B)\). Since \(W_k(R_A; BR_B)\) involves a minimization over all \(E_{RA, BR_B}\) satisfying (B116) and (B117), this concludes our proof of (B108).

**Remark 1** We briefly remark here that if a channel \(\mathcal{N}_{A \to B}\) can be simulated by the action of a k-extendible channel \(\mathcal{K}_{ARB' \to B}\) on the channel input \(\rho_A\) as well as a resource state \(\omega_{RB'}\) (i.e., \(\mathcal{N}_{A \to B}(\rho_A) = \mathcal{K}_{ARB' \to B}(\rho_A \otimes \omega_{RB'})\)), then the k-unextendible divergence of that channel does not increase under amortization, for divergences that are subadditive with respect to tensor-product states. This is a special case of the more general observation put forward in [42, Section 7] for general resource theories.

5. Exploiting symmetries

The following lemma is helpful in determining the form of a state that optimizes the unextendible generalized channel divergence of a quantum channel that has some symmetry. Its proof is identical to that given for [66, Proposition 2], but we give it here for completeness.

**Lemma 16** Let \(\mathcal{N}_{A \to B}\) be a covariant channel with respect to a group \(G\), as in Definition 1. Let \(\rho_A \in \mathcal{D}(\mathcal{H}_A)\), and let \(\psi_{RA}^\rho\) be a purification for it. Define \(\rho_{RB} := \mathcal{N}_{A \to B}(\psi_{RA}^\rho)\) and \(\bar{\rho}_A := \frac{1}{|G|} \sum_{g \in G} U_A(g) \rho_A U_A^\dagger(g)\). Let \(\phi_{RA}^\rho\) be a purification of \(\bar{\rho}_A\) and \(\bar{\rho}_{RB} := \mathcal{N}_{A \to B}(\phi_{RA}^\rho)\). Then

\[
E_{k}(R; B)_{\bar{\rho}} \geq E_{k}(R; B)_{\rho},
\]

**Proof.** Define

\[
|\phi\rangle_{PRA} := \frac{1}{\sqrt{|G|}} \sum_{g \in G} |g\rangle_{P} [I_R \otimes U_A(g)] |\psi\rangle_{RA},
\]

so that \(\phi_{PRA}\) is a purification of \(\bar{\rho}_A\). Let \(\tau_{PRB} \in \text{EXT}_k(PR; B)\), and, given that a local channel is a k-extendible channel, observe that

\[
\sum_{g \in G} |g\rangle \langle g|_P \tau_{PRB} |g\rangle \langle g|_P = \sum_{g \in G} p(g) |g\rangle \langle g|_P \otimes \tau_{RB}^g \in \text{EXT}_k(PR; B),
\]

where \(\tau_{RB}^g = \frac{1}{p(g)} \langle g| \tau_{PRB} |g\rangle_\rho\) and \(p(g) = \text{Tr} \{ |g\rangle \text{\tau}_{PRB} |g\rangle_\rho\}. Then

\[
\mathcal{D}(\mathcal{N}_{A \to B}(\phi_{PRA}) \| \tau_{PRB})
\]

\[
\geq D \left( \sum_{g, g' \in G} \frac{1}{|G|} |g\rangle \langle g'|_P \otimes [I_R \otimes U_A(g)] \phi_{RA}^\rho [I_R \otimes U_A^\dagger(g')] \right) \| \tau_{PRB}^g
\]

\[
\geq \bar{D} \left( \sum_{g \in G} \frac{1}{|G|} |g\rangle \langle g|_P \otimes \mathcal{N}_{A \to B}(\psi_{RA}^\rho) \otimes \bar{\rho}_{RB} \right) \| \sum_{g \in G} p(g) |g\rangle \langle g|_P \otimes \tau_{RB}^g
\]

\[
= \bar{D} \left( \sum_{g \in G} \frac{1}{|G|} |g\rangle \langle g|_P \otimes \mathcal{N}_{A \to B}(\psi_{RA}^\rho) \otimes \bar{\rho}_{RB} \right) \| \sum_{g \in G} p(g) |g\rangle \langle g|_P \otimes \tau_{RB}^g
\]

\[
\geq \bar{D} \left( \mathcal{N}_{A \to B}(\psi_{RA}^\rho) \right) \| \sum_{g \in G} p(g) |g\rangle \langle g|_P \otimes \tau_{RB}^g
\]

\[
\geq \inf_{\tau_{RB} \in \text{EXT}_k(R; B)} \mathcal{D}(\mathcal{N}_{A \to B}(\psi_{RA}) \| \tau_{RB})
\]

\[
= E_{k}(R; B)_{\rho}.
\]
The first inequality follows because any general divergence is monotonically non-increasing under the action of a quantum channel, which in this case is the completely dephasing channel \( \cd_{N} \). The second equality follows because the channel \( N \) is covariant according to Definition 1. To arrive at the third equality, we use the fact that any generalized divergence is invariant under the action of isometries. To get the second inequality, we apply the partial trace over the classical register \( P \), which is a quantum channel. The last inequality follows because the state \( \sum_{g \in G} p(g) V_B^\dagger(g) \cd_{PR}^n V_B(g) \) is \( k \)-extendible, given that it arises from the action of a 1W-LOCC channel on the \( k \)-extendible state \( \cd_{PR} \). Noticing that the chain of inequalities holds for arbitrary \( \cd_{PR} \in \text{EXT}_k(P; B) \), we can then take an infimum over all possible \( \cd_{PR} \in \text{EXT}_k(P; B) \), and we arrive at the following inequality:

\[
E_k(\rho; P; B)_{N(\emptyset)} \geq E_k(\rho; R; B)_{\rho} \tag{B136}
\]

The desired inequality in the statement of the lemma then follows because all purifications of a given state are related by an isometry acting on the purifying system, and the unextendible generalized divergence is invariant under the action of a local isometry. □

Appendix C: Unextendibility, non-asymptotic one-way distillable entanglement, and non-asymptotic quantum capacity

In this section, we use the resource theory of unextendibility to derive non-asymptotic converse bounds on the rate at which entanglement can be transmitted over a finite number of uses of a quantum channel. We do the same for the non-asymptotic, one-way distillable entanglement of a bipartite state.

1. Entanglement transmission codes and one-way entanglement distillation protocols

An \((n, M, \varepsilon)\) entanglement transmission protocol accomplishes the task of entanglement transmission over \( n \) independent uses of a quantum channel \( N_{A \rightarrow B} \). The case \( n = 1 \) is known as “one-shot entanglement transmission,” given that we are considering just a single use of a channel in this case. However, note that a given \((n, M, \varepsilon)\) entanglement transmission protocol for the channel \( N_{A \rightarrow B} \) can be considered as \((1, M, \varepsilon)\) entanglement transmission protocol for the channel \( N_{A^{\otimes n} \rightarrow B^{\otimes n}} \).

An entanglement transmission code for \( N \), is specified by a triplet \( \{M, E, D\} \), where \( M = \dim(\mathcal{H}_R) \) is the Schmidt rank of a maximally entangled state \( \Phi_{RA'}^{\otimes n} \), one share of which is to be transmitted over \( N \). The quantum channels \( E_{A' \rightarrow A^n} \) and \( D_{B^n \rightarrow A} \) are encoding and decoding channels, respectively. An \((n, M, \varepsilon)\) code is such that

\[
F(\Phi_{RA}^{\otimes n}, \omega_{RA}) \geq 1 - \varepsilon,
\]

where

\[
\omega_{RA} = (D_{B^n \rightarrow A} \circ N_{A^{\otimes n} \rightarrow B} \circ E_{A' \rightarrow A^n}) (\Phi_{RA'}^{\otimes n}).
\]

We note that the criterion \( F(\Phi_{RA}^{\otimes n}, \omega_{RA}) \geq 1 - \varepsilon \) is equivalent to

\[
\text{Tr}(\Phi_{RA}^{\otimes n} \omega_{RA}^{\otimes n}) \geq 1 - \varepsilon.
\]

The non-asymptotic quantum capacity \( Q(N_{A \rightarrow B}, n, \varepsilon) \) of a quantum channel \( N_{A \rightarrow B} \) is equal to the largest value of \( \log_2 M \) for which there exists an \((n, M, \varepsilon)\) protocol as described above [27].

We can also consider a modification of the above protocol in which the final decoding is a \( k \)-extendible channel \( D_{R^n B^n \rightarrow RA} \), acting on the input systems \( R \): \( B^n \) and outputting the systems \( R \): \( A \). See Figure 3 for a depiction of such a modified protocol. We call such a protocol entanglement transmission assisted by a \( k \)-extendible post-processing, and the resulting non-asymptotic quantum capacity is denoted by \( Q_k(N_{A \rightarrow B}, n, \varepsilon) \).

Another kind of protocol to consider is a one-way entanglement distillation protocol. An \((n, M, \varepsilon)\) one-way entanglement distillation protocol begins with Alice and Bob sharing \( n \) copies of a bipartite state \( \rho_{AB} \). They then act with a 1W-LOCC channel \( L_{A^n B^n \rightarrow MA MB} \) on \( \rho_{AB}^{\otimes n} \), and the resulting state satisfies

\[
F(L_{A^n B^n \rightarrow MA MB}(\rho_{AB}^{\otimes n}), \Phi_{MA MB}) \geq 1 - \varepsilon,
\]

where \( \Phi_{MA MB} \) is a maximally entangled state of Schmidt rank \( M \). We can also modify this protocol to allow for a \( k \)-extendible channel instead of a 1W-LOCC channel, and the resulting protocol is a \((n, M, \varepsilon)\) entanglement distillation protocol assisted by a \( k \)-extendible channel. Let \( D(k)(\rho_{AB}, n, \varepsilon) \) denote the non-asymptotic distillable entanglement with the assistance of \( k \)-extendible channels; i.e., \( D(k)(\rho_{AB}, n, \varepsilon) \) is equal to the maximum value of \( \frac{1}{k} \log_2 M \) such that there exists an \((n, M, \varepsilon)\) protocol for \( \rho_{AB} \) as described above.
2. Bounds on non-asymptotic quantum capacity and one-way distillable entanglement in terms of $k$-extendible divergence

We now provide a proof for the second theorem claimed in the main text, regarding a bound on non-asymptotic quantum capacity in terms of the unextendible hypothesis testing divergence.

**Theorem 6** The following bound holds $\forall k \in \mathbb{N}$ and for any $(1, M, \varepsilon)$ entanglement transmission protocol conducted over a quantum channel $\mathcal{N}$ and assisted by a $k$-extendible post-processing:

$$-\log_2 \left[ \frac{1}{M} + \frac{1}{k} - \frac{1}{Mk} \right] \leq \sup_{\psi_{RA}} E^\varepsilon_k(R; B)_\tau,$$

where

$$E^\varepsilon_k(R; B)_\tau := \inf_{\sigma_{RB} \in \text{EXT}_k(R; B)} D^\varepsilon_k(\tau_{RB} \parallel \sigma_{RB})$$

is the $k$-unextendible $\varepsilon$-hypothesis-testing divergence, $\tau_{RB} := \mathcal{N}_{A \rightarrow B}(\psi_{RA})$, and the optimization in (C5) is with respect to pure states $\psi_{RA}$ such that $R \simeq A$. Similarly, the following bound holds for any $(1, M, \varepsilon)$ entanglement distillation protocol for a state $\rho_{AB}$, which is assisted by a $k$-extendible post-processing:

$$-\log_2 \left[ \frac{1}{M} + \frac{1}{k} - \frac{1}{Mk} \right] \leq E^\varepsilon_k(A; B)_\rho.$$

**Proof.** Suppose that there exists a $(1, M, \varepsilon)$ entanglement transmission protocol, assisted by a $k$-extendible post-processing, that satisfies the condition given in (C1). Let $\sigma_{R\hat{A}} \in \text{EXT}_k(R; \hat{A})$, and let $\Phi_{R\hat{A}}$ denote a maximally entangled state. Then the following chain of inequalities holds

$$D^\varepsilon_k(\omega_{R\hat{A}} \parallel \sigma_{R\hat{A}}) \geq -\log_2 \text{Tr}\{\Phi_{R\hat{A}} \sigma_{R\hat{A}}\}$$

$$\quad = -\log_2 \text{Tr}\left\{ \int d\mu(U) \left(U_R \otimes U^*_\hat{A}\right) \Phi_{R\hat{A}} \left(U_R \otimes U^*_\hat{A}\right)^\dagger \sigma_{R\hat{A}} \right\},$$

$$\quad = -\log_2 \text{Tr}\left\{ \Phi_{R\hat{A}} \int d\mu(U) \left(U_R \otimes U^*_\hat{A}\right)^\dagger \sigma_{R\hat{A}} \left(U_R \otimes U^*_\hat{A}\right) \right\}.$$  \hspace{1cm} \text{(C8)}

The first inequality follows because the condition in (C3) implies that we can relax the measurement operator $\Lambda$ in (A22) to be equal to $\Phi_{R\hat{A}}$. The first equality is due to the “transpose trick” property of the maximally entangled state, which leads to its $U \otimes U^*$ invariance. For the last equality, we use the cyclic property of the trace.

Let $\sigma_{R\hat{A}} := \int d\mu(U)\left(U_R \otimes U^*_\hat{A}\right)^\dagger \sigma_{R\hat{A}}\left(U_R \otimes U^*_\hat{A}\right)$. The state $\sigma_{R\hat{A}}$ is $k$-extendible because $\sigma_{R\hat{A}}$ is and because the unitary twirl can be realized as a 1W-LOCC channel. The symmetrized state $\overline{\sigma}_{R\hat{A}}$ is furthermore isotropic because it is invariant under the action of a unitary of the form $U \otimes U^*$. From Lemma 3, we find that

$$\overline{\sigma}_{R\hat{A}} = t\Phi_{R\hat{A}} + (1-t)\frac{I_{R\hat{A}} - \Phi_{R\hat{A}}}{M^2 - 1},$$  \hspace{1cm} \text{(C11)}

FIG. 3. Depiction of an entanglement transmission protocol assisted by a $k$-extendible post-processing channel. The quantum channel $\mathcal{N}$ is used $n$ times, in conjunction with an encoding channel $\mathcal{E}_{A' \rightarrow A^n}$ and a $k$-extendible post-processing decoding channel $\mathcal{K}_{RB^n \rightarrow R\hat{A}}$, in order to establish entanglement shared between Alice and Bob.
for some \( t \in \left[ 0, \frac{1}{M} + \frac{1}{k} - \frac{1}{Mk} \right] \). Combining (C11) with (C10) leads to

\[
D_h^\varepsilon(\omega_{RA} || \sigma_{RA}) \geq -\log_2 t \geq -\log_2 \left[ \frac{1}{M} + \frac{1}{k} - \frac{1}{Mk} \right].
\] (C12)

Since the above bound holds for an arbitrary state \( \sigma_{RA} \in \text{EXT}_k(R; \hat{A}) \), we conclude that

\[
E^\varepsilon_k(R; \hat{A})_\omega = \inf_{\sigma_{RA} \in \text{EXT}_k(R; \hat{A})} D_h^\varepsilon(\omega_{RA} || \sigma_{RA}) \geq -\log_2 \left[ \frac{1}{M} + \frac{1}{k} - \frac{1}{Mk} \right].
\] (C13)

Let \( \rho_{RB} := N_{A \rightarrow B}(\rho_{RA}) \), where \( \rho_{RA} := E_{A' \rightarrow A}(\Phi_{RA'}) \), and let \( \sigma_{RB} \in \text{EXT}_k(R; B) \). Then for a \( k \)-extendible post-processing channel \( D_{RB \rightarrow RA} \), we have that

\[
D_h^\varepsilon(\rho_{RB} || \sigma_{RB}) \geq D_h^\varepsilon(D_{RB \rightarrow RA}(\rho_{RB}) || D_{RB \rightarrow RA}(\sigma_{RB}))
\]

\[
= D_h^\varepsilon(\omega_{RA} || \sigma_{RA})
\]

\[
\geq E^\varepsilon_k(R; \hat{A})_\omega.
\] (C16)

The first inequality follows from the data processing inequality for the hypothesis testing relative entropy. The channel \( D_{RB \rightarrow RA} \) is a \( k \)-extendible channel, and given that \( \sigma_{RB} \in \text{EXT}_k(R; B) \), Theorem 4 implies that \( \sigma_{RA} \in \text{EXT}_k(R; \hat{A}) \). The last inequality follows from the definition in (C6). Since this inequality holds for all \( \sigma_{RB} \in \text{EXT}_k(R; B) \), we conclude that

\[
E^\varepsilon_k(R; B)_\rho \geq E^\varepsilon_k(R; \hat{A})_\omega.
\] (C17)

We now optimize \( E^\varepsilon_k \) with respect to all inputs \( \rho_{RA} \) to the channel \( N_{A \rightarrow B} \):

\[
\sup_{\rho_{RA}} E^\varepsilon_k(R; B)_{N(\rho)} \geq E^\varepsilon_k(R; B)_{N(\rho)}.
\] (C18)

Using purification, the Schmidt decomposition theorem, and the data processing inequality of \( E^\varepsilon_k(R; B)_\rho \), we find that

\[
\sup_{\rho_{RA}} E^\varepsilon_k(R; B)_{N(\rho)} = \sup_{\psi_{RA}} E^\varepsilon_k(R; B)_{N(\psi)}.
\] (C19)

for a pure state \( \psi_{RA} \) with \( |R| = |A| \). Combining (C13), (C17), and (C19), we conclude the bound in (C5).

By employing similar reasoning as above, we arrive at the bound in (C7).

**Remark 2** Note that Theorem 6 applies in the case that the channel \( N \) is an infinite-dimensional channel, taking input density operators acting on a separable Hilbert space to output density operators acting on a separable Hilbert space. In claiming this statement, we are supposing that an entanglement transmission protocol begins with a finite-dimensional space, the encoding then maps to the infinite-dimensional space, the channel \( N \) acts, and then finally the decoding channel maps back to a finite-dimensional space. Furthermore, an entanglement distillation protocol acts on infinite-dimensional states and distills finite-dimensional maximally entangled states from them. We arrive at this conclusion because the \( \varepsilon \)-hypothesis testing relative entropy is well defined for infinite-dimensional states.

**Remark 3** Due to the facts that \( D_h^\varepsilon(\rho || \sigma) \geq D_h^\varepsilon(\rho || \sigma') \) for \( 0 \leq \sigma \leq \sigma' \), \( D_h^\varepsilon(\rho || c\sigma) = D_h^\varepsilon(\rho || \sigma) - \log_2 c \) for \( c > 0 \) \cite{109}, Lemma 7], \( D_h^\varepsilon(\rho || \rho) = \log_2 \left( \frac{1}{1-\varepsilon} \right) \), and by applying the same reasoning as in (B82)–(B84), we conclude that

\[
\sup_{\psi_{RA}} E^\varepsilon_k(R; B)_\tau \leq \log_2 \left( \frac{1}{1-\varepsilon} \right) + \log_2 k,
\] (C20)

which provides a limitation on the \((\varepsilon, k)\)-unextendibility of any quantum channel.

By turning around the bound in (C5), we find the following alternative way of expressing it:

**Remark 4** The number of ebits \( \log_2 M \) transmitted by a \((1, M, \varepsilon)\) entanglement transmission protocol conducted over a quantum channel \( N \) and assisted by a \( k \)-extendible post processing is bounded from above as

\[
\log_2 M \leq \log_2 \left( \frac{k-1}{k} \right) - \log_2 \left( 2^{-\sup_{\psi_{RA}} E^\varepsilon_k(R; B)_\tau} - \frac{1}{k} \right).
\] (C21)

where \( E^\varepsilon_k(R; B)_\tau \) is defined in (C6).
a. On the size of the extendibility parameter $k$ versus the error $\varepsilon$

By observing the form of the bound in Remark 4, we see that it is critical for the inequality

$$2^{-\sup_{\psi_{RA}} E_k^\varepsilon(R;B)} - \frac{1}{k} > 0$$

(C22)

to hold in order for the bound to be non-trivial. Related, we see that this inequality always holds in the limit $k \to \infty$, and in this limit, we recover the $\varepsilon$-relative entropy of entanglement bound from [3, 61]. Here, we address the question of how large $k$ should be in order to ensure that the inequality in (C22) holds.

**Proposition 4** For a fixed $\varepsilon \in (0, 1)$, the following inequality holds

$$2^{-E_k^\varepsilon(N)} - \frac{1}{k} > 0,$$

or equivalently, that

$$E_k^\varepsilon(N) < \log_2 k,$$

as long as

$$k > 2^{I_k^\varepsilon(N)} \varepsilon + 1,$$

where

$$I_k^\varepsilon(N) \equiv \sup_{\psi_{RA}} D_k^\varepsilon(N_{A \to B}(\psi_{RA}) \| \psi_{RA} \otimes N_{A \to B}(\psi_{A})).$$

(C26)

is the channel’s $\varepsilon$-mutual information.

**Proof.** This follows because the condition in (C24) is equivalent to

$$E_k^\varepsilon(N) = \sup_{\psi_{RA}} \inf_{\sigma_{RB} \in \text{EXT}_k(R;B)} D_k^\varepsilon(N_{A \to B}(\psi_{RA}) \| \sigma_{RB}) < \log_2 k.$$  

(C27)

We can pick the $k$-extendible state $\sigma^\psi_{RB}$, for a fixed $\psi_{RA}$, as follows:

$$\sigma^\psi_{RB} = \frac{1}{k} N_{A \to B}(\psi_{RA}) + \left(1 - \frac{1}{k}\right) \psi_{R} \otimes N_{A \to B}(\psi_{A}),$$

(C28)

implying that

$$E_k^\varepsilon(N) \leq \sup_{\psi_{RA}} D_k^\varepsilon(N_{A \to B}(\psi_{RA}) \| \sigma^\psi_{RB}).$$

(C29)

The choice $\sigma^\psi_{RB}$ is $k$-extendible because the following state constitutes its $k$-extension:

$$\sigma^\psi_{RB_1 \cdots B_k} = \frac{1}{k} \sum_{i=1}^{k} N_{A \to B_1}(\psi_{A}) \otimes \cdots \otimes N_{A \to B_{i-1}}(\psi_{A}) \otimes N_{A \to B_i}(\psi_{RA}) \otimes N_{A \to B_{i+1}}(\psi_{A}) \otimes \cdots \otimes N_{A \to B_k}(\psi_{A}).$$

(C30)

The optimal measurement operator $\Lambda^\star$ for $D_k^\varepsilon(N_{A \to B}(\psi_{RA}) \| \sigma^\psi_{RB})$ satisfies

$$\text{Tr}\{\Lambda^\star N_{A \to B}(\psi_{RA})\} \geq 1 - \varepsilon,$$

(C31)

which means that

$$\text{Tr}\{\Lambda^\star \sigma^\psi_{RB}\} \geq \frac{1}{k} \text{Tr}\{\Lambda^\star N_{A \to B}(\psi_{RA})\} + \left(1 - \frac{1}{k}\right) \text{Tr}\{\Lambda^\star (\psi_{R} \otimes N_{A \to B}(\psi_{A}))\}$$

$$\geq \frac{1}{k} [1 - \varepsilon] + \left(1 - \frac{1}{k}\right) 2^{-I_k^\varepsilon(N)},$$

(C32)

(C33)
and in turn that

$$D_h^k(N_{\mathcal{A}\rightarrow\mathcal{B}}(\psi_{\mathcal{RA}})\|\rho^\psi_{\mathcal{RB}}) \leq -\log_2 \left( \frac{1}{k} [1 - \varepsilon] + \left( 1 - \frac{1}{k} \right) 2^{-I_k^k(\mathcal{N})} \right).$$

(C34)

The goal is to have the right-hand side above less than $\log_2 k$ for all $\psi_{\mathcal{RA}}$, and this condition is equivalent to

$$-\log_2 \left( \frac{1}{k} [1 - \varepsilon] + \left( 1 - \frac{1}{k} \right) 2^{-I_k^k(\mathcal{N})} \right) < \log_2 k.$$

(C35)

Rewriting this, it is the same as

$$\frac{1}{k} [1 - \varepsilon] + \left( 1 - \frac{1}{k} \right) 2^{-I_k^k(\mathcal{N})} > \frac{1}{k},$$

which is in turn the same as

$$-\frac{\varepsilon}{k} + \left( 1 - \frac{1}{k} \right) 2^{-I_k^k(\mathcal{N})} > 0$$

(C37)

$$\Leftrightarrow (k - 1) 2^{-I_k^k(\mathcal{N})} > \varepsilon$$

(C38)

$$\Leftrightarrow k > 2^{I_k^k(\mathcal{N})} \varepsilon + 1.$$  

(C39)

This concludes the proof. ■

**Remark 5** We note that the lower bound on $k$ from Proposition 4 is not necessarily optimal and certainly could be improved. For example, when $\varepsilon < 1/2$ and the channel $\mathcal{N}$ is a two-extendible channel, $k = 2$ suffices in order for the bound from Theorem 6 to apply, and thus the bound in Proposition 4 can be very loose. The value of Proposition 4 is simply in knowing that a finite lower bound on $k$ exists for every channel, such that one can always find a finite $k$ for and beyond which our bound on entanglement transmission rates applies.

### 3. Non-asymptotic quantum capacity assisted by $k$-extendible channels

In this subsection, we define another kind of non-asymptotic quantum capacity, in which a quantum channel is used $n$ times, and between every channel use, a $k$-extendible channel is employed for free to assist in the goal of entanglement transmission. Such a protocol is similar to those that have been discussed in the literature previously [5, 40, 42, 65, 92], but we review the details here for completeness.

In such a protocol, a sender Alice and a receiver Bob are spatially separated and connected by a quantum channel $\mathcal{N}_{\mathcal{A}\rightarrow\mathcal{B}}$. They begin by performing a $k$-extendible channel $\mathcal{K}^{(1)}_{A_1' \rightarrow A_1 A_2'}$, which leads to a $k$-extendible state $\rho^{(1)}_{A_1' A_2' B_1}$, where $A_1'$ and $B_1$ are systems that are finite-dimensional but arbitrarily large. The system $A_1$ is such that it can be fed into the first channel use. Alice sends system $A_1$ through the first channel use, leading to a state $\sigma_{A_1' B_1}^{(1)} := \mathcal{N}_{A_1 \rightarrow B_1}(\rho^{(1)}_{A_1' A_1 B_1})$. Alice and Bob then perform the $k$-extendible channel $\mathcal{K}^{(2)}_{A_2' B_1' \rightarrow A_2 A_3 B_2'}$, which leads to the state

$$\rho^{(2)}_{A_2' A_2 B_2'} := \mathcal{K}^{(2)}_{A_2' A_2 B_2'} \sigma^{(1)}_{A_1' B_1}.$$  

(C40)

Alice sends system $A_2$ through the second channel use $\mathcal{N}_{A_2 \rightarrow B_2}$, leading to the state $\rho^{(2)}_{A_2' B_2 B_2'} := \mathcal{N}_{A_2 \rightarrow B_2}(\rho^{(1)}_{A_2' A_2 B_2'})$. This process iterates: the protocol uses the channel $n$ times. In general, we have the following states for all $i \in \{2, \ldots, n\}$:

$$\rho^{(i)}_{A_i' A_i B_i'} := \mathcal{K}^{(i)}_{A_i' A_i B_i' \rightarrow A_i A_i B_i'} \sigma^{(i-1)}_{A_i' A_i B_i' \rightarrow A_i A_i B_i'},$$

(C41)

$$\sigma^{(i)}_{A_i' A_i B_i'} := \mathcal{N}_{A_i \rightarrow B_i}(\rho^{(i)}_{A_i' A_i B_i'}).$$

(C42)

where $\mathcal{K}^{(i)}_{A_i' A_i B_i', i+1, B_i i+1, B_i i+1, B_i'}$ is a $k$-extendible channel. The final step of the protocol consists of a $k$-extendible channel $\mathcal{K}^{(n+1)}_{A_n' B_n' \rightarrow M_A M_B}$, which generates the systems $M_A$ and $M_B$ for Alice and Bob, respectively. The protocol's final state is as follows:

$$\omega_{MA MB} := \mathcal{K}^{(n+1)}_{A_n' B_n' \rightarrow M_A M_B} \sigma^{(n)}_{A_n' B_n' B_n'},$$

(C43)
The goal of the protocol is that the final state $\omega_{M_AM_B}$ is close to a maximally entangled state. Fix $n, M \in \mathbb{N}$ and $\varepsilon \in [0, 1]$. The original protocol is an $(n, M, \varepsilon)$ protocol if the channel is used $n$ times as discussed above, $|M_A| = |M_B| = M$, and if

$$F(\omega_{M_AM_B}, \Phi_{M_AM_B}) = \langle \Phi|_{M_AM_B} \omega_{M_AM_B} \Phi \rangle_{M_AM_B}$$

$$\geq 1 - \varepsilon.$$  \hfill (C44)

Figure 4 depicts such a protocol.

Let $Q_{1/k}(\mathcal{N}_{A\rightarrow B}, n, \varepsilon)$ denote the non-asymptotic quantum capacity assisted by $k$-extendible channels; i.e., $Q_{1/k}(\mathcal{N}_{A\rightarrow B}, n, \varepsilon)$ is the maximum value of $\frac{1}{n} \log_2 M$ such that there exists an $(n, M, \varepsilon)$ protocol for $\mathcal{N}_{A\rightarrow B}$ as described above.

A rate $R$ is achievable for $k$-extendible-assisted quantum communication if for all $\varepsilon \in (0, 1]$, $\delta > 0$, and sufficiently large $n$, there exists an $(n, 2^n(R - \delta), \varepsilon)$ protocol. The $k$-extendible-assisted quantum capacity of a channel $\mathcal{N}$, denoted as $Q_{1/k}(\mathcal{N})$, is equal to the supremum of all achievable rates.

**Proposition 5** The following converse bound holds for all integer $k \geq 2$ and for any $(n, M, \varepsilon)$ $k$-extendible assisted quantum communication protocol over $n$ uses of a quantum channel $\mathcal{N}$:

$$- \frac{1}{n} \log_2 \left[ \frac{1}{M} + \frac{1}{k} - \frac{1}{Mk} \right] \leq E_{1/k}^{\max}(\mathcal{N}) + \frac{1}{n} \log_2 \left( \frac{1}{1 - \varepsilon} \right),$$

where $E_{1/k}^{\max}(\mathcal{N})$ is the $k$-unextendible max-relative entropy of the channel $\mathcal{N}$, as defined in (B104).

**Proof.** The above bound can be derived by invoking Theorem 5 and following arguments similar to those given in the proof of [65, Theorem 3]. \hfill \blacksquare

Similar to the observation in Remark 4, by turning around the bound in (C46), we find the following alternative way of expressing it:

**Remark 6** The number of ebits ($\log_2 M$) transmitted by a $(1, M, \varepsilon)$ entanglement transmission protocol conducted over a quantum channel $\mathcal{N}$ and assisted by a $k$-extendible post processing is bounded from above as

$$\log_2 M \leq \log_2 \left( \frac{k - 1}{k} \right) - \log_2 \left( 2^{-nE_{1/k}^{\max}(\mathcal{N})} [1 - \varepsilon] - \frac{1}{k} \right),$$

where $E_{1/k}^{\max}(\mathcal{N})$ is the $k$-unextendible max-relative entropy of the channel $\mathcal{N}$, as defined in (B104).

Related to the discussion in Section C2a, it is necessary for the inequality $2^{-nE_{1/k}^{\max}(\mathcal{N})} [1 - \varepsilon] - \frac{1}{k} > 0$ to hold in order for the bound in (C47) to be non-trivial. The following proposition gives a sufficient condition on the size of $k$ in order for the inequality in (C47) to hold. This condition can be checked numerically.

**Proposition 6** Fix $\varepsilon \in (0, 1)$, a channel $\mathcal{N}$, and $n \geq 1$. The following inequality holds

$$2^{-nE_{1/k}^{\max}(\mathcal{N})} [1 - \varepsilon] - \frac{1}{k} > 0,$$  \hfill (C48)
or equivalently,

\[ nE_k^{\max}(\mathcal{N}) + \log_2 \left( \frac{1}{1 - \varepsilon} \right) < \log_2 k, \]  

(C49)

as long as

\[ k > 2^{I_{\max}(\mathcal{N})} \left[ \frac{1}{1 - \varepsilon} \right]^{1/n} \left( 1 - 2^{-I_{\max}(\mathcal{N})} \right) \]  

(C50)

where

\[ I_{\max}(\mathcal{N}) \equiv \sup_{\psi_{RA}} D_{\max}(\mathcal{N}_{A\rightarrow B}(\psi_{RA}) \| \psi_{R} \otimes \mathcal{N}_{A\rightarrow B}(\psi_{A})) \]  

(C51)

is the channel’s max-mutual information.

**Proof.** The condition in (C49) is equivalent to

\[ E_k^{\max}(\mathcal{N}) = \sup_{\psi_{RA}} \inf_{\sigma_{RB} \in \text{EXT}_{A \rightarrow B}(R:B)} D_{\max}(\mathcal{N}_{A\rightarrow B}(\psi_{RA}) \| \sigma_{RB}) < \log_2 k. \]  

(C52)

We can pick the $k$-extendible state $\sigma_{RB}^{\psi}$, for a fixed $\psi_{RA}$, as follows:

\[ \sigma_{RB}^{\psi} = \frac{1}{k} \mathcal{N}_{A\rightarrow B}(\psi_{RA}) + \left( 1 - \frac{1}{k} \right) \psi_{R} \otimes \mathcal{N}_{A\rightarrow B}(\psi_{A}), \]  

(C53)

implying that

\[ E_k^{\max}(\mathcal{N}) \leq \sup_{\psi_{RA}} D_{\max}(\mathcal{N}_{A\rightarrow B}(\psi_{RA}) \| \sigma_{RB}^{\psi}). \]  

(C54)

Now defining, for a fixed $\psi_{RA}$,

\[ \lambda(\psi) \equiv I_{\max}(R:B)_{\psi} \equiv D_{\max}(\mathcal{N}_{A\rightarrow B}(\psi_{RA}) \| \psi_{R} \otimes \mathcal{N}_{A\rightarrow B}(\psi_{A})), \]  

(C55)

we find that

\[ \sigma_{RB}^{\psi} = \frac{1}{k} \mathcal{N}_{A\rightarrow B}(\psi_{RA}) + \left( 1 - \frac{1}{k} \right) \psi_{R} \otimes \mathcal{N}_{A\rightarrow B}(\psi_{A}) \]  

(C56)

\[ \geq \frac{1}{k} \mathcal{N}_{A\rightarrow B}(\psi_{RA}) + \left( 1 - \frac{1}{k} \right) 2^{-\lambda(\psi)} \mathcal{N}_{A\rightarrow B}(\psi_{RA}) \]  

(C57)

\[ = \left[ \frac{1}{k} + \left( 1 - \frac{1}{k} \right) 2^{-\lambda(\psi)} \right] \mathcal{N}_{A\rightarrow B}(\psi_{RA}). \]  

(C58)

Now exploiting the fact that $D_{\max}(\rho \| \sigma) \leq D_{\max}(\rho \| \sigma')$ for $\sigma \geq \sigma' \geq 0$, as well as $D_{\max}(\rho \| \sigma) = D_{\max}(\rho \| \sigma) - \log_2 c$ for $c > 0$, we find that

\[ \sup_{\psi_{RA}} D_{\max}(\mathcal{N}_{A\rightarrow B}(\psi_{RA}) \| \sigma_{RB}^{\psi}) \]  

\[ \leq \sup_{\psi_{RA}} \left[ D_{\max}(\mathcal{N}_{A\rightarrow B}(\psi_{RA}) \| \mathcal{N}_{A\rightarrow B}(\psi_{RA})) - \log_2 \left( \frac{1}{k} + \left( 1 - \frac{1}{k} \right) 2^{-\lambda(\psi)} \right) \right] \]  

(C59)

\[ = \sup_{\psi_{RA}} \left[ - \log_2 \left( \frac{1}{k} + \left( 1 - \frac{1}{k} \right) 2^{-\lambda(\psi)} \right) \right] \]  

(C60)

\[ = - \log_2 \left( \frac{1}{k} + \left( 1 - \frac{1}{k} \right) 2^{-I_{\max}(\mathcal{N})} \right) \]  

(C61)

\[ = - \log_2 \left( 2^{-I_{\max}(\mathcal{N})} + \frac{1}{k} \left( 1 - 2^{-I_{\max}(\mathcal{N})} \right) \right). \]  

(C62)
The goal is to have the inequality in (C49) holding, and, by the above analysis, this results if the following inequality holds
\[- n \log_2 \left( \left[ 2^{2^{-H_{\text{max}}(N)}} + \frac{1}{k} \left( 1 - 2^{2^{-H_{\text{max}}(N)}} \right) \right] \right) + \log_2 \left( \frac{1}{1 - \varepsilon} \right) < \log_2 k. \] (C63)

Rewriting this, it is the same as
\[\left[ 2^{2^{-H_{\text{max}}(N)}} + \frac{1}{k} \left( 1 - 2^{2^{-H_{\text{max}}(N)}} \right) \right]^n \left[ 1 - \varepsilon \right] > \frac{1}{k} \] (C64)

\[\Leftrightarrow \left[ 2^{2^{-H_{\text{max}}(N)}} + \frac{1}{k} \left( 1 - 2^{2^{-H_{\text{max}}(N)}} \right) \right] [1 - \varepsilon]^{1/n} > \frac{1}{k^{1/n}} \] (C65)

\[\Leftrightarrow [k2^{-H_{\text{max}}(N)} + \left( 1 - 2^{-H_{\text{max}}(N)} \right)] [1 - \varepsilon]^{1/n} > k^{1-1/n} \] (C66)

\[\Leftrightarrow k^{1-1/n} \left[ \frac{1}{1 - \varepsilon}^{1/n} \right] - \left( 1 - 2^{-H_{\text{max}}(N)} \right) \] (C67)

\[\Leftrightarrow k > 2^{H_{\text{max}}(N)} \] (C68)

This concludes the proof. ■

A similar comment as in Remark 5 applies to Proposition 6.

**Definition 10 (k-simulable channels)** A channel \( \mathcal{N}_{A \rightarrow B} \) is \( k \)-simulable, if there exists a resource state \( \omega_{RB} \in \mathcal{D}(\mathcal{H}_{RB}) \), such that for all \( \rho \in \mathcal{D}(\mathcal{H}_A) \)

\[\mathcal{N}_{A \rightarrow B}(\rho_A) = \mathcal{K}_{RAB \rightarrow B}(\rho_A \otimes \omega_{RB}), \] (C69)

where \( \mathcal{K}_{RAB \rightarrow B} \) is a \( k \)-extendible channel.

Note that a teleportation-simulable channel, as given in Definition 2, is a particular example of a \( k \)-simulable channel, whenever the LOCC channel in (A27) is a 1W-LOCC channel.

For a \( k \)-simulable channel, an \((n, M, \varepsilon)\) quantum communication protocol assisted by \( k \)-extendible channels simplifies in such a way that it is equivalent to an \((n, M, \varepsilon)\) entanglement distillation protocol starting from the resource state \( \omega_{RB}^{\otimes n} \) and assisted by a \( k \)-extendible post-processing channel. This observation was made in [5, 92] and extended to any resource theory in [42]. See Figure 5 of [42] for a summary of the reduction that applies to our case of interest here. We then have the following:

**Corollary 2** Let \( \mathcal{N} \) be a \( k \)-simulable channel as in Definition 10. The following bound holds \( \forall k \in \mathbb{N} \) and for any \((n, M, \varepsilon)\) quantum communication protocol conducted over the quantum channel \( \mathcal{N} \) and assisted by \( k \)-extendible channels:

\[- \log_2 \left[ \frac{1}{M} + \frac{1}{k} \right] - \log_2 \left( \frac{1}{1 - \varepsilon} M \right) \leq E_k^n(\mathcal{R}^n; \hat{B}^n)_{\omega_{RB}^{\otimes n}},\] (C70)

where \( \omega_{RB} \) is the resource state in Definition 10.

### 4. Pretty strong converse for antidegradable channels

In this subsection, we provide a proof for the pretty strong converse bound for the non-asymptotic quantum capacity of antidegradable channels, when they are assisted by two-extendible channels. We also examine a generalization of this result to channels that output only \( k \)-extendible states.

We recall the first statement here:

**Corollary 3** Fix \( \varepsilon \in [0, 1/2) \). The following bound holds for any \((n, M, \varepsilon)\) quantum communication protocol employing \( n \) uses of an antidegradable channel \( \mathcal{N} \) interleaved by two-extendible channels:

\[\frac{1}{n} \log_2 M \leq \frac{1}{n} \log_2 \left( \frac{1}{1 - 2\varepsilon} \right),\] (C71)
Proof. Let $\mathcal{N}_{A\rightarrow B}$ be an antidegradable channel, and suppose that $\rho_{RA}$ is a state input to the channel. Then the output state $\mathcal{N}_{A\rightarrow B}(\rho_{RA})$ is always a two-extendible state (due to anti-degradability) [47]. As a direct consequence of the third theorem in the main text, the following bound applies to any $(n, M, \varepsilon)$ quantum communication protocol employing $n$ uses of an antidegradable channel $\mathcal{N}$ interleaved by two-extendible channels:

$$-\frac{1}{n} \log_2 \left[ \frac{1}{M} + \frac{1}{2} - \frac{1}{2M} \right] \leq \frac{1}{n} \log_2 \left( \frac{1}{1 - \varepsilon} \right).$$  \hspace{1cm} (C72)

This follows by setting $k = 2$ and noticing that $\sup_{\psi_{RA}} E^\max_{(R;B)\tau} = 0$, where $\tau_{RB} := \mathcal{N}_{A\rightarrow B}(\psi_{RA})$, for such antidegradable channels. After some basic algebraic steps, for $\varepsilon < \frac{1}{2}$, we can rewrite this bound as

$$-\frac{1}{n} \log_2 M \leq -\frac{1}{n} \log_2 \left[ \frac{1}{2(1 - \varepsilon) - 1} \right].$$  \hspace{1cm} (C73)

These steps are as follows:

$$-\frac{1}{n} \log_2 \left[ \frac{1}{M} + \frac{1}{2} - \frac{1}{2M} \right] \leq -\frac{1}{n} \log_2 \left( \frac{1}{1 - \varepsilon} \right)$$

$\Leftrightarrow \log_2 \left[ \frac{2M}{M + 1} \right] \leq \log_2 \left( \frac{1}{1 - \varepsilon} \right)$  \hspace{1cm} (C74)

$\Leftrightarrow \frac{2}{1 + 1/M} \leq \frac{1}{1 - \varepsilon}$  \hspace{1cm} (C75)

$\Leftrightarrow 2 (1 - \varepsilon) - 1 \leq 1/M$  \hspace{1cm} (C76)

$\Leftrightarrow 1 - 2\varepsilon \leq 1/M$.  \hspace{1cm} (C77)

This concludes the proof. $\blacksquare$

Thus, for a fixed $\varepsilon \in [0, 1/2)$, we conclude that the rate of quantum communication for an antidegradable channel decays to zero as $n \to \infty$. Related, if the communication rate for a sequence of codes used over such a channel is strictly greater than zero, then it must be the case that the error in communication is greater than or equal to $1/2$. As a consequence, we have established a tighter bound for the pretty strong converse of antidegradable channels, when compared to that given in [45].

More generally, if the output of the channel is always a $k$-extendible state, then we have the following bound:

**Corollary 4** Fix $\varepsilon \in [0, 1 - 1/k)$. Let $\mathcal{N}_{A\rightarrow B}$ be a $k$-extendible channel, in the sense that $\mathcal{N}_{A\rightarrow B}(\rho_{RA})$ is $k$-extendible for any input state $\rho_{RA}$. Then the following bound holds for any $(n, M, \varepsilon)$ quantum communication protocol employing $n$ uses of the channel $\mathcal{N}$ interleaved by $k$-extendible channels:

$$-\frac{1}{n} \log_2 \left[ \frac{1}{M} + \frac{1}{k} - \frac{1}{Mk} \right] \leq -\frac{1}{n} \log_2 \left( \frac{1}{1 - \varepsilon} \right).$$  \hspace{1cm} (C79)

**Proof.** This follows by the same reasoning as in the previous proof. If the output of the channel is $k$-extendible, then employing the third theorem in the main text gives that

$$-\frac{1}{n} \log_2 \left[ \frac{1}{M} + \frac{1}{k} - \frac{1}{Mk} \right] \leq -\frac{1}{n} \log_2 \left( \frac{1}{1 - \varepsilon} \right).$$  \hspace{1cm} (C80)
We then employ the following algebraic steps:

\[ -\frac{1}{n} \log_2 \left( \frac{1}{M} + \frac{1 - \frac{1}{k}}{Mk} \right) \leq -\frac{1}{n} \log_2 \left( \frac{1}{1 - \varepsilon} \right) \]

\[ \frac{kM}{k - 1 + M} \leq 1 - \frac{1}{1 - \varepsilon} \]

\[ \frac{(k - 1)/M + 1}{k} \leq 1 - \varepsilon \]

\[ \frac{k(1 - \varepsilon) - 1}{k - 1} \leq 1/M \]

\[ 1 - \frac{k}{k - 1} \varepsilon \leq 1/M. \]

We then get that

\[ -\frac{1}{n} \log_2 M \leq -\frac{1}{n} \log_2 \left( \frac{1}{1 - \frac{k}{k - 1} \varepsilon} \right). \]

This concludes the proof. \[\square\]

Thus, for a fixed \( \varepsilon \in [0, 1 - 1/k] \), we conclude that the rate of quantum communication for a single-sender single-receiver \( k \)-extendible channel decays to zero as \( n \to \infty \). Related, if the communication rate for a sequence of codes used over such a channel is strictly greater than zero, then it must be the case that the error in communication is greater than or equal to \( 1 - 1/k \), which is a higher jump than discussed in the previous case. An example of a channel for which this effect occurs is a quantum erasure channel with erasure probability \( 1 - \varepsilon \). Thus, for a fixed \( \varepsilon \), we conclude that the rate of quantum communication for a single-sender single-receiver \( k \)-extendible channel decays to zero as \( n \to \infty \). Related, if the communication rate for a sequence of codes used over such a channel is strictly greater than zero, then it must be the case that the error in communication is greater than or equal to \( 1 - 1/k \), which is a higher jump than discussed in the previous case. An example of a channel for which this effect occurs is a quantum erasure channel with erasure probability \( 1 - \varepsilon \).

Another example of a channel for which the bound in Corollary 4 holds is the universal cloning machine channel (a \( 1 \to k \) universal quantum cloner followed by a partial trace over \( k - 1 \) of the clones) [15]. When the dimension of the channel input is \( M \), the bound in Corollary 4 is in fact saturated, as observed in the proof of [44, Theorem III.8].

Appendix D: Depolarizing Channel

In this section, we establish an upper bound on the non-asymptotic quantum capacity of a depolarizing channel assisted by \( k \)-extendible channels.

The action of a qubit depolarizing channel \( \mathcal{D}^p_{A \to B} \) on an input state \( \rho \) is as follows:

\[ \mathcal{D}^p_{A \to B}(\rho) = (1 - p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z), \]

where \( p \in [0, 1] \) is the depolarizing parameter and \( X, Y, \) and \( Z \) are the Pauli operators. A depolarizing channel is a covariant channel for all \( p \in [0, 1] \), which is a fact that is easy to see after expressing its action as \( \mathcal{D}^p_{A \to B}(\rho) = (1 - q)\rho + qI/2 \), for \( q = 4p/3 \). This property is crucial to obtain an upper bound on the unextendible \( \varepsilon \)-hypothesis-testing divergence of the depolarizing channel.

To this end, we first argue that the optimal input state for \( n \) independent uses of the depolarizing channel is an \( n \)-fold tensor product of the maximally entangled state \( \Phi_{RA} = \frac{1}{2} \sum_{i,j \in \{0,1\}} |i\rangle_R \otimes |j\rangle_A \). For tensor-product channels, we can restrict the input state to be invariant under permutations of the input systems, due to Lemma 16. Also, for covariant channels, the input states which optimize the \( k \)-extendible relative entropy are of the form given in Lemma 16. Therefore, it suffices to restrict the input state to be a tensor-power maximally entangled state; i.e., we conclude that

\[ E_k^\varepsilon([\mathcal{D}^p]^\otimes n) = \min_{\sigma_{Ra^nB^n} \in \text{EXT}_k(R^{n\cdot B^n})} D_k^\varepsilon([\mathcal{D}^p_{A \to B}(\Phi_{RA})]^\otimes n \parallel \sigma_{Ra^nB^n}). \]

We make a particular choice of the \( k \)-extendible state \( \sigma_{Ra^nB^n} \) above (which is not necessarily optimal) to be a tensor product of the isotropic states \( \sigma_{AB}^{(t, 2)} \), as defined in (B30). Note that the action of \( \mathcal{D}^p \) on a maximally entangled state
FIG. 5. Upper bounds on the number of qubits that can be reliably transmitted over a depolarizing channel with \( p = 0.24 \), and \( \varepsilon = 0.05 \). The red dashed line is the bound from Theorem 2. The green dash-dotted and blue dotted lines are upper bounds from [3] and [4], respectively.

results in an isotropic state \( \sigma^{(p,2)}_{AB} \) parametrized by \( p \). Since the states \( \left( \sigma^{(p,2)}_{AB} \right)^{\otimes n} \) and \( \left( \sigma^{(t,2)}_{AB} \right)^{\otimes n} \) are diagonal in the same basis, the \( \varepsilon \)-hypothesis testing relative entropy between the two states is equal to the \( \varepsilon \)-hypothesis testing relative entropy between the product Bernoulli probability distributions \( \{1 - p, p\}^{\otimes n} \) and \( \{t, 1 - t\}^{\otimes n} \). We therefore obtain the following bound on the number of ebits transmitted by \( n \) channel uses of the depolarizing channel:

\[
\frac{1}{n} \log_2 M \leq \frac{1}{n} \log_2 \left( \frac{k - 1}{k} \right) - \frac{1}{n} \log_2 \left( 2^{D_{\varepsilon}^{\otimes n}(\{1 - p, p\}^{\otimes n} \| \{t, 1 - t\}^{\otimes n})} - \frac{1}{k} \right). \tag{D3}
\]

Due to this channel’s covariance, the upper bound holds for both \( Q_1^{(k)}(N_{A \rightarrow B}, n, \varepsilon) \) and \( Q_1^{(k)}(N_{A \rightarrow B}, n, \varepsilon) \). The resulting classical hypothesis testing relative entropy between the product Bernoulli distributions can be distinguished exactly by the optimal Neyman-Pearson test [59].

Note that (D3) converges to the upper bound given in [3] in the limit as \( k \rightarrow \infty \). Please refer to Figure 1 for a comparison of various upper bounds on the non-asymptotic quantum capacity of the depolarizing channel. For tensor products of the isotropic states \( \sigma^{(t,2)}_{AB} \), the numerics suggest that the minimizing state is either \( k = 2 \) extendible or a separable state. If the minimizing state is a separable state, then the bound in (D3) is equal to the TBR bound from [3].

Appendix E: Erasure channel

In this section, we establish an upper bound on the non-asymptotic quantum capacity of an erasure channel assisted by \( k \)-extendible channels.

The action of a qubit erasure channel [16] on an input density operator \( \rho \) is as follows:

\[
\mathcal{E}_{A \rightarrow B}^p(\rho_A) = (1 - p)\rho_B + p|e\rangle\langle e|_B , \tag{E1}
\]
where $p \in [0, 1]$ is the erasure parameter and $|e\rangle\langle e|$ is a pure state, orthogonal to any input state. The optimal input state for $n$ uses of erasure channel, when considering its unextendible generalized divergence, is the $n$-fold tensor product the maximally entangled state $\Phi_{A^n}$. This follows also from the covariance of the erasure channel and Lemma 16. The upper bounds we establish here thus hold for both $Q(k)_{N_{A\rightarrow B}}(n, \varepsilon)$ and $Q_{\mathbb{E}}(k)_{N_{A\rightarrow B}}(n, \varepsilon)$.

Our goal is to obtain upper bounds on the entanglement transmission rate when using the erasure channel $n$ times. Consider sending $n$ maximally entangled states $\Phi_{A^n}$ over $n$ uses of the erasure channel $\mathcal{E}_{A^n \rightarrow B^n}^{(k)}$. The output state $\rho_{A^n_1 A^n_2 B^n_1 \cdots A^n_n B^n_n}$ has the form

$$\rho_{A^n_1 A^n_2 B^n_1 \cdots A^n_n B^n_n} = \sum_{x^n \in \{0, 1\}^n} p(x^n) \left( \bigotimes_{j=1}^n \tau_j^{x_j} \right),$$

(E2)

where for all $j \in [n]$, $\tau_j^{x_j} \in \{ \Phi_{A_j B_j}, \pi_{A_j} \otimes |e\rangle\langle e|_{B_j} \}$, and for all $x^n \in \{0, 1\}^n$, $p(x^n) \in [0, 1]$ is a product distribution such that $\sum_{x^n \in \{0, 1\}^n} p(x^n) = 1$. Due to an i.i.d. application of the channels, we find that the probabilities $p(x^n)$ corresponding to a state $\tau_j^{x_j}$ with the same number of erasure symbols should be equal. The total probability for having $\ell$ erasure symbols in the state $\rho_{A^n_1 A^n_2 B^n_1 \cdots A^n_n B^n_n}$ is equal to $\binom{n}{\ell}(1-p)^{n-\ell}p^\ell$, where $\ell \in [0, n]$.

Without loss of generality, the block-diagonal form of the output state of $n$ uses of an erasure channel, when inputting a tensor-power maximally entangled state, allows us to restrict the class of $k$-extendible states $\sigma \in \text{EXT}_k(A^n; B^n)$, over which we optimize the unextendible $\varepsilon$-hypothesis testing relative entropy, to be of the form in (E2), except with $p(x^n)$ a probability distribution that is not necessarily product and chosen such that the state is $k$-extendible. This follows because the state $\rho_{A^n_1 A^n_2 B^n_1 \cdots A^n_n B^n_n}$ is invariant under $n$ independent bilateral twirls, along with $n$ independent and incomplete measurements of the form $\{(0\rangle\langle 0), |1\rangle\langle 1|, |e\rangle\langle e|\}$ by Bob, while such a 1W-LOCC channel symmetrizes the $k$-extendible state to have the aforementioned form. We let $\sigma_{A^n_1 A^n_2 B^n_1 \cdots A^n_n B^n_n}$ be of the form in (E2) with coefficients (probabilities) set to $q(x^n)$. Furthermore, we note that $\rho_{A^n_1 A^n_2 B^n_1 \cdots A^n_n B^n_n}$ is permutation invariant under Alice and Bob perform a coordinated random permutation channel on their composite systems locally. This allows us to restrict the form of $\sigma_{A^n_1 A^n_2 B^n_1 \cdots A^n_n B^n_n}$ to be permutation invariant under such a symmetrizing permutation channel because it is a $k$-extendible channel.

From the above argument, we find that the minimizing state has the block structure given in (E2), and the coefficients for states in the sum with the same number of erasure symbols are equal. We now want to obtain conditions on the probabilities $q(x^n)$, where $x^n \in \{0, 1\}^n$ from the $k$-extendibility of the state $\sigma_{A^n_1 A^n_2 B^n_1 \cdots A^n_n B^n_n}$. The constraints that we impose on $q(x^n)$ are not unique. That is, there could exist other constraints such that the state $\sigma_{A^n_1 A^n_2 B^n_1 \cdots A^n_n B^n_n}$ is still $k$-extendible.

Let us first consider $n = 2$ channel uses. By what we discussed above, the minimizing $k$-extendible state $\sigma_{A^n_1 A^n_2 B^n_2}$ then has the following form

$$\sigma_{A^n_1 A^n_2 B^n_2} := c_0 \Phi_{A^n_1} \otimes \Phi_{A^n_2 B^n_2} + c_1 \left( \Phi_{A^n_1} \otimes \pi_{A^n_2} \otimes |e\rangle\langle e|_{B^n_2} + \Phi_{A^n_2 B^n_2} \otimes \pi_{A^n_1} \otimes |e\rangle\langle e|_{B^n_1} \right) + c_2 \pi_{A^n_1} \otimes |e\rangle\langle e|_{B^n_1} \otimes \pi_{A^n_2} \otimes |e\rangle\langle e|_{B^n_2},$$

(E3)

where $\{c_i\}$, for $i \in \{0, 1, 2\}$ is a probability distribution such that $c_0 + 2c_1 + c_2 = 1$. Focusing on the special case $k = 2$, we now want to obtain constraints on each $c_i$ such that $\sigma_{A^n_1 A^n_2 B^n_2}$ is a two-extendible state. To this end, we replace all the terms $\Phi_{A^n_j}$ in the above state with the two-extendible state $\frac{1}{2} \Phi_{A^n_j} + (1 - \frac{1}{2}) \pi_{A^n_j} \otimes |e\rangle\langle e|_{B^n_j}$. We obtain the following state, which is guaranteed to be two-extendible by construction:

$$\frac{c_0}{4} \Phi_{A^n_1} \otimes \Phi_{A^n_2} + \left( \frac{c_0}{4} + \frac{c_1}{2} \right) \left( \Phi_{A^n_1} \otimes \pi_{A^n_2} \otimes |e\rangle\langle e|_{B^n_2} + \pi_{A^n_1} \otimes |e\rangle\langle e|_{B^n_1} \otimes \Phi_{A^n_2} \right) + \left( \frac{c_0}{4} + c_1 + c_2 \right) \left( \pi_{A^n_1} \otimes |e\rangle\langle e|_{B^n_1} \otimes \pi_{A^n_2} \otimes |e\rangle\langle e|_{B^n_2} \right).$$

(E4)

Abbreviating the new coefficients as $b_0$, $b_1$, and $b_2$, the above approach leads to the following constraint on them such that the state $\sigma_{A^n_1 A^n_2 B^n_2}$ is two-extendible:

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 2 & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}. $$

(E5)

We now generalize the above procedure of obtaining two-extendible states for two uses of channel.
extendible states for \( n \) channel uses. We obtain the following condition on the coefficients \( b_i \):

\[
\begin{bmatrix}
  b_0 \\
  b_1 \\
  b_2 \\
  \vdots \\
  b_n
\end{bmatrix} = \mathbf{M}
\begin{bmatrix}
  \binom{n}{0}c_0 \\
  \binom{n}{1}c_1 \\
  \binom{n}{2}c_2 \\
  \vdots \\
  \binom{n}{n}c_n
\end{bmatrix},
\]

(E6)

where the general form of the matrix \( \mathbf{M}_{(n+1)\times(n+1)} = [m_{u,v}] \) is given as

\[
m_{u,v} = \begin{cases} 
(n - v)(1 - \frac{1}{k})^{u - v} & \text{if } u \geq v \\
0 & \text{otherwise}
\end{cases},
\]

(E7)

if \( u \geq v \) and otherwise \( m_{u,v} = 0 \), where \( n \) is the number of channel uses and \( u, v \in [0, n] \). The coefficients are such that \( c_0, c_1, \ldots, c_n \in [0, 1] \) and \( \sum_{j=0}^{n} \binom{n}{j} c_j = 1 \). We then have that

\[
\min_{\sigma'_{A_1B_1\ldots A_nB_n} \in \text{EXT}_k} D_{h}^k (\rho_{A_1B_2\ldots A_nB_n} \| \sigma'_{A_1B_1\ldots A_nB_n}) \leq \min_{b_0, b_1, \ldots, b_n} D_{h}^k (\{a_0, a_1, \ldots, a_n\} \| \{b_0, b_1, \ldots, b_n\}),
\]

(E8)

where the distribution \( \{a_0, a_1, \ldots, a_n\} \) is induced by measuring the number of erasures in \( \rho_{A_1B_2\ldots A_nB_n} \) and the coefficients \( \{b_0, b_1, \ldots, b_n\} \) are chosen as discussed above. The inequality follows from restricting the form of the minimizing state. By exploiting the dual formulation of the hypothesis testing relative entropy [110], we can now write the expression in (E8) as the following linear program:

\[
\min_{c_0, c_1, \ldots, c_n} D_{h}^k (\{a_0, a_1, \ldots, a_n\} \| \{b_0, b_1, \ldots, b_n\}) = -\log_2 \left( \max_{\{c_0, c_1, \ldots, c_n\}, \{\alpha_i\}, \rho(1 - \varepsilon) - \sum_{i=0}^{n} \alpha_i} \right),
\]

(E9)

such that

\[
\forall i \in [0, n], \quad \alpha_i - y a_i + b_i \geq 0,
\]

(E10)

\[
b_i = \sum_{j=0}^{n} m_{i,j} c_j,
\]

(E11)

\[
0 \leq c_i \leq 1,
\]

(E12)

\[
y \geq 0, \quad \alpha_i \geq 0,
\]

(E13)

\[
\sum_{j=0}^{n} \binom{n}{j} c_j = 1.
\]

(E14)

For the plot in Figure 6, we have taken \( \sigma_{A_1B_1A_2B_2\ldots A_nB_n} \) to be in a particular set of extendible states as defined above. Within this set, we have optimized over at most \( k = 10 \) extendible states.
FIG. 6. Upper bounds on the number of qubits that can be reliably transmitted over an erasure channel with $p = 0.49$, and $\varepsilon = 0.05$. The red dashed line is the bound from Theorem 2. The green dash-dotted line is an upper bound from [3].