Temperature dependence of the vortices number for rotating superfluid Bose-Einstein condensates in a two-dimensional lattice

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We developed an approach to investigate the temperature dependence of the vortices number for the rotating superfluid Bose-Einstein condensates in a two-dimensional lattice. Our approach enabled us to parametrize the thermal average of the quantum mechanics expectation value and the vortices number in terms of the thermodynamic potential \( q(T) \) for the system under consideration. The effects of the interatomic interaction, the finite number of atoms and optical potential depth on the vortices number are investigated. The calculated results show that the vortices number as a function of the rotation rate depends on the normalized temperature, interatomic interaction, optical potential depth and the number of atoms in the \( \ell \) size. These results provide solid theoretical foundations for the measured experimental data for rotating condensate boson in optical lattice such as Williams et. al experiment [R. A. Williams, S. Al-Assam, and C. J. Foot, Phys. Rev. Lett. 104 (2010) 050404].

I. INTRODUCTION

Rotating superfluid Bose-Einstein-condensation (BEC) in optical lattice is probably the simplest system that have been developed to simulate the properties of the physics of condensed matter systems such as the periodic superfluids, type-II superconductors and quantum Hall effect. For such systems, special attention have been devoted to calculate the critical rotation rate needed to reach the vortices configuration as well as the vortices number created in this configuration. These parameters are of crucial importance in their own right.

Great theoretical effort using quantum mechanics arguments have been performed using the zero-temperature Bogoliubov approximation, which neglects the effects of the thermal atoms. In spite of the Popov approximation of the Hartree-Fock-Bogoliubov theory taking into account the thermal atoms component in self-consistent manner, a consistent mean-field theories turn out to be difficult to formulate. However, the challenge to develop a computationally feasible theory still remains. So, it is important to calculate these parameters at finite temperature under a real experimental condition, in order to be able to illustrate the experimental data for such parameters from the theoretical viewpoint.

Indeed, we have shown previously the effects of thermal atoms on the critical rotation rate for nucleation of vortices. This study was realized in a quantum statistical approach based on the semiclassical approximation. Moreover, this approach enabled us to study the effects of finite size of atoms, the impact of the periodic optical potential, as well as the effect of repulsive interaction provided by the mean-field theory approach. In this work, we extend our approach to investigate the effect of the thermal average on the number of vortices at finite temperature. The latter is parametrized in terms of the thermodynamic potential \( q(T, \alpha) \), with \( \alpha \) being the rotation rate, of the system under consideration.

The calculated results show that, the vortices number decreases with increasing the interatomic interaction and increases with increasing the atoms number. Moreover the vortices number depends crucially on the condensate temperature at temperature \( \tau < 0.3 \). The optical potential depth increases the vortices number.

This paper is formatted as follows. In the first section, we describe the physics of the single-boson atom model. In section two, the thermal average of the quantum expectation value is given. Number of vortices is calculated in section three. Results and discussion are given in section four. Conclusion is summerized in the last section.
II. PHYSICAL MODEL

We consider a system of $N$ bosons trapped in a combined potential $V_{\text{com}}$,

$$V_{\text{com}} = V(r, z) + V_{\text{opt}}(x, y)$$  \hfill (1)

where

$$V(r, z) = \frac{1}{2}M(\omega_{x}^2 r^2 + \omega_{z}^2 z^2)$$  \hfill (2)

is a cylindrically symmetric harmonic potential, and $V_{\text{opt}}(x, y)$ is a 2D deep optical potential,

$$V_{\text{opt}}(x, y) = V_{0}\left[\sin^2\left(\frac{\pi x}{d}\right) + \frac{1}{2}M[\omega_{latt}^2(x^2 + y^2)] = \frac{1}{2}M[4\omega_{latt}^2(x^2 + y^2)]\right]$$  \hfill (3)

where $\{\omega_{x}, \omega_{z}\}$ are the radial and axial harmonic frequencies, $r^2 = x^2 + y^2$, $M$ is the mass of the boson atom, $d$ is the lattice spacing and $V_{0}$ is the optical potential depth. In Eq. (3) the harmonic frequency approximation for the optical potential,

$$\omega_{latt}^2 = \frac{2\pi^2}{M d^2} V_{0}$$  \hfill (4)

In a rotating frame at an angular velocity $\Omega$ around the $z-$ axis, the Hamiltonian (time independent) of the system has the form$^{36,37}$,

$$\mathcal{H} = \frac{\mathbf{p}^2}{2M} + \frac{1}{2}M[(1 + \gamma^2)\omega_{r}^2 r^2 + \omega_{z}^2 z^2] - \Omega L_z$$

where $\gamma = \frac{\omega_{latt}}{\omega_{z}}$ gives the ratio between the lattice frequency and the harmonic frequency. Completing the square we obtain an equivalent form for $\mathcal{H}$,

$$\mathcal{H} = \frac{(\mathbf{p} - M \Omega \times \mathbf{r})^2}{2M} + \frac{p_z^2}{2M} + \frac{1}{2}M[(1 + \gamma^2 - \alpha^2)\omega_{r}^2 r^2 + \omega_{z}^2 z^2]$$  \hfill (6)

where $p^2 = p_x^2 + p_y^2$, $L_z = xp_y - yp_x$ is the angular momentum of the $z$-components, and $\alpha = \frac{\Omega}{\omega_{r}}$ is the rotation rate. In a rotating frame at an angular velocity $\Omega$ around the $z-$ axis, the Hamiltonian (time independent) of the system has the form$^{36,37}$,

$$\mathcal{H} = \frac{(p - M \Omega \times r)^2}{2M} + \frac{p_z^2}{2M} + \frac{1}{2}M[(1 + \gamma^2 - \alpha^2)\omega_{r}^2 r^2 + \omega_{z}^2 z^2]$$  \hfill (6)

where $p^2 = p_x^2 + p_y^2$, $L_z = xp_y - yp_x$ is the angular momentum of the $z$-components, and $\alpha = \frac{\Omega}{\omega_{r}}$ is the rotation rate.

Introducing the well-known dimensionless creation, annihilation and number operators which obey the usual commutation relation for this system$^{36,37}$,

$$a_{x,y} = \frac{1}{\sqrt{2}}\left(\frac{x y}{d_{\perp}} + \frac{i p_{x,y} d_{\perp}}{\hbar}\right), a_{x,y}^\dagger = \frac{1}{\sqrt{2}}\left(\frac{x y}{d_{\perp}} - \frac{i p_{x,y} d_{\perp}}{\hbar}\right)$$

$$a_z = \frac{1}{\sqrt{2}}\left(\frac{z}{d_{z}} + \frac{i p_z d_{z}}{\hbar}\right), \quad a_z^\dagger = \frac{1}{\sqrt{2}}\left(\frac{z}{d_{z}} - \frac{i p_z d_{z}}{\hbar}\right)$$

$$a_{\pm} = \frac{1}{\sqrt{2}}(a_{x,y} \mp ia_{y,x}) \quad a_{\pm} = \frac{1}{\sqrt{2}}(a_{x} \pm ia_{y})$$  \hfill (7)

where $d_{\perp} = \sqrt{\hbar/M \omega_{\perp} \sqrt{(1 + \gamma^2)}}$ and $d_{z} = \sqrt{\hbar/M \omega_{z}}$ is the radial and axial oscillator length, respectively. In terms of the number operators, the Hamiltonian in Eq. (6) becomes,

$$\mathcal{H} = \hbar \omega_{\perp}(a^\dagger_{+} a_{+} + a^\dagger_{-} a_{-} + 1) + \hbar \omega_{z}(a^\dagger_{z} a_{z} + \frac{1}{2}) - \hbar \Omega(a^\dagger_{+} a_{+} - a^\dagger_{-} a_{-})$$  \hfill (8)

The single particle energy eigenvalues for the Hamiltonian (8) is given by

$$E_{n_{+},n_{-},n_{z}} = n_{+}\hbar \omega_{\perp} + n_{-}\hbar \omega_{\perp} + n_{z}\hbar \omega_{z} + (\hbar \omega_{\perp} + \frac{1}{2}\hbar \omega_{z})$$  \hfill (9)
where $\omega_{\pm} = \omega_\perp \sqrt{1 + \gamma^2} \mp \Omega$, $n_+, n_-$ are integers number. In the limit of rapid rotation $\Omega \to \omega_\perp$ the energy eigenvalue given in Eq.(9) is essentially independent on $n_+$, in this case $n_-$ becomes the Landau level index and $n_+$ gives the vibrational quantum number along $z$. For rotation frequency $\Omega < \omega_\perp$ it is convenient to introduce a different set of quantum numbers $n = n_+ + n_-$ for energy, and $m = n_+ - n_-$ for the angular momentum. Using the new set in Eq.(9) the energy eigenvalues become

$$E_{n,m,n_\pm} = n\hbar \omega_\perp \sqrt{1 + \gamma^2} - m\hbar \Omega + n_\pm \hbar \omega_z + \left(\hbar \omega_\perp + \frac{1}{2}\hbar \omega_z\right)$$

with $n, n_\pm = 0, 1, 2, 3, \ldots$ and $m = 0, \pm 1, \pm 2, \pm 3, \ldots$.

## III. THERMAL AVERAGE OF THE ANGULAR MOMENTUM

Let us suppose that the Hamiltonian $\mathcal{H}$ depends on a set of parameters $\Omega, \omega_\perp, \omega_\perp \ldots$ etc such as that:

$$\mathcal{H}|n, m, n_z\rangle = E_{n,m,n_z}|n, m, n_z\rangle$$

where $E_{n,m,n_z}$ and $|n, m, n_z\rangle$ are the energy eigenvalues and eigenstate of $\mathcal{H}$ respectively. Now, since $\langle n, m, n_z|n, m, n_z\rangle = 1$, we have

$$\frac{\partial}{\partial \lambda}\langle n, m, n_z|\mathcal{H}|n, m, n_z\rangle = \langle n, m, n_z|\frac{\partial E_{n,m,n_z}}{\partial \lambda}|n, m, n_z\rangle$$

$$\langle n, m, n_z|\frac{\partial \mathcal{H}}{\partial \lambda}|n, m, n_z\rangle = \frac{\partial E_{n,m,n_z}}{\partial \lambda}$$

Formula in Eq.(12), which is known as Hellmann-Feynman theorem, is the exact result for the Hamiltonian $\mathcal{H}$. This theorem analyzes the variation of bound state energy with respect to the dynamic parameter involved in the Hamiltonian.

The thermal average of the quantum mechanical expectation values can be calculated for the basis in which the Hamiltonian $\mathcal{H}(\lambda)$ is diagonal. When the system has $N$ particles and in a state $\{n, m, n_z\}$, the thermal average of $\langle L_z\rangle_{th}$, $(r^2)_{th}$ or $(\bar{z}^2)_{th}$ at finite non zero temperature can be extracted from the relation,

$$\langle \frac{\partial \mathcal{H}(\lambda)}{\partial \lambda}\rangle_{th} = \frac{1}{Z}\text{Tr}[e^{-\beta \mathcal{H}}(\frac{\partial \mathcal{H}(\lambda)}{\partial \lambda})]$$

$$= \frac{1}{Z} \sum_{n_\pm=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{N} \langle n, m, n_z|Z^N e^{-\beta N E_{n,m,n_z}} \frac{\partial \mathcal{H}(\lambda)}{\partial \lambda}|n, m, n_z\rangle$$

$$= \frac{1}{Z} \sum_{n_\pm=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{N} Z^N e^{-\beta N E_{n,m,n_z}} \frac{\partial E_{n,m,n_z}}{\partial \lambda}$$

$$= -\frac{1}{N\beta} \sum_{n_\pm=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{N} \frac{1}{Z} \left(\frac{\partial}{\partial \lambda}\ln Z\right) Z^N e^{-\beta N E_{n,m,n_z}}$$

$$= -\frac{1}{N\beta} \frac{\partial}{\partial \lambda} \sum_{n_\pm=0}^{\infty} \sum_{m=-\infty}^{\infty} \sum_{N} \ln Z$$

(13)

where $Z$ is the grand canonical partition function for $N$ particles, and $z$ is the effective fugacity, $z = e^{\beta(\mu(s, \alpha) - (\hbar \omega_z + \frac{1}{2}\hbar \omega_z))}$.

$$Z = \sum_{N} (Ze^{-\beta E_{n,m,n_z}})^N = \frac{1}{(1 - Ze^{-\beta E_{n,m,n_z}})}$$

where $\beta = \frac{1}{k_B T}$, and $(\mu(s, \alpha)$ is the chemical potential,

$$\mu(s, \alpha) = \mu_0 \left(1 - \alpha^2\right)^{2/5} \left(\frac{\pi}{2}\right)^{2/5} s^{1/5}$$

(14)
with \( s = 0.1 \frac{v_c}{\hbar \omega} \), \( \mu_0 = \frac{\hbar^2}{2a} \left( \frac{15N_a}{\alpha_{ah}} \right) \) is the chemical potential calculated at \( T = 0 \) in Thoms-Fermi approximation, \( a \) is the s-wave scattering length, \( \alpha_{ah} = \sqrt{\frac{\hbar}{M\omega}} \) is the harmonic oscillator length.

Result in Eq.(13) provides an important relation for calculating the thermal average of the quantum mechanics expectation value in terms of the thermodynamic potential \( q(s, \alpha) \) of the system,

\[
\langle \frac{\partial H(\lambda)}{\partial \lambda} \rangle_{th} = \frac{1}{N\beta} \frac{\partial q(T, \alpha)}{\partial \lambda}
\]

where

\[
q(T, \alpha) = - \sum_{n,n_z=0}^{\infty} \sum_{m=-\infty}^{\infty} \ln Z = \ln(1-z) + \sum_{n,n_z=1}^{\infty} \sum_{m=-\infty}^{\infty} \ln(1-z e^{-\beta E_{n,m,n_z}}) = \ln(1-z) + q_{th}(T, \alpha)
\]

for later convenient the ground state contribution, which provides the thermodynamic potential of the ground state, is separated out. The second term in Eq.(16) represents the grand potential for the thermal atoms in the excited states.

Now it is straightforward to calculate the thermal average for both the angular momentum and the in situ size. Substituting by \( \lambda = \Omega \) into Eq.(15), the thermal average of the angular momentum\(^{46}\),

\[
\langle L_z(T) \rangle = - \frac{1}{N\beta} \frac{\partial q(T, \alpha)}{\partial \alpha} = - \frac{1}{N\beta} \frac{\partial q_{th}(T, \alpha)}{\partial \Omega} = - \frac{1}{N\beta} \frac{\partial q_{th}(T, \alpha)}{\partial \Omega}
\]

where Eq.5 for \( \mathcal{H} \) is used here and \( \frac{\partial \Omega}{\partial T} = 0 \) is considered. Eq.(17) can be written in the form,

\[
\langle L_z(T) \rangle = - \frac{1}{N\beta} \frac{\partial q_{th}(T, \alpha)}{\partial \alpha}
\]

Using the same procedures the effective average value of \( \langle r^2 \rangle \equiv \langle r^2 \rangle \) for the condensate, (in situ size), can be calculated using \( \lambda \equiv \omega_\perp \) in Eq.(15),

\[
\langle r^2 \rangle = \frac{1}{M\omega_\perp (1 + \gamma^2)} \frac{1}{N\beta} \frac{\partial q(T, \alpha)}{\partial \omega_\perp} = \frac{1}{M\omega_\perp (1 + \gamma^2)} \frac{1}{N\beta} \left[ \frac{\partial}{\partial \omega_\perp} \ln(1-z) - \frac{\alpha}{\omega_\perp} \frac{\partial q_{th}(T, \alpha)}{\partial \omega_\perp} \right] = - \frac{\hbar}{M\omega_\perp (1 + \gamma^2)} \frac{1}{N} \left[ N_0 + \alpha \frac{k_B T}{\hbar} \frac{\partial q_{th}(T, \alpha)}{\partial \alpha} \right]
\]

where Eq.6 for \( \mathcal{H} \) is used here and \( N_0 = \frac{\pi^2}{2} \) is the number of the condensate atoms.

At zero temperature both of \( \langle L_z(T) \rangle \) and \( \langle r^2 \rangle \) provide the quantum mechanics expectation values of the observable. Generally in the limit \( T \to 0 (\beta \to \infty) \) all the \( e^{-\beta E_{n,m,n_z}} \) reach zero fast, the slowest one to reach zero is the lowest \( E_{n,m,n_z} \), i.e. the ground state energy. However, the quantum mechanics expectation value of the angular momentum at zero temperature is given by\(^{47}\),

\[
\langle L_z(T) \rangle_{T \to 0} = \lim_{T \to 0} \left[ \frac{1}{N\beta} \frac{\partial q(T, \alpha)}{\partial \Omega} \right] = \lim_{T \to 0} \frac{1}{N\beta} \frac{\partial}{\partial \Omega} \sum_{n,m,n_z} \ln \left( 1 - ze^{-\beta E_{n,m,n_z}} \right) = - \lim_{T \to 0} \left[ \frac{1}{N\beta} \sum_{n,m,n_z} \sum_{j=1}^{\infty} z^j j e^{-\beta E_{n,m,n_z}} \right]
\]
where \( \ln(1 - y) = -\sum_{j=1}^{\infty} \frac{y^j}{j} \) is used here. Result in Eq. (20) is agreement with the result calculated by cooper\(^{36}\).

IV. NUMBER OF VORTICES

In order to consider the temperature dependence of the number of vortices, we calculate the total number of vortices within the average value of \( \langle \rho^2 \rangle \) at finite temperature. Since the average vortex density is\(^{4,15,22}\)

\[
\begin{align*}
N_v &= \frac{2M}{\hbar} \frac{F_v - F_0}{\langle L_z(T) \rangle_v - \langle L_z(T) \rangle_0} \\
&= \frac{2M}{\hbar} \frac{F_v - F_0}{\langle L_z(T) \rangle_v - \langle L_z(T) \rangle_0} \langle \rho^2 \rangle
\end{align*}
\]

(22)

where \( F_v, F_0 \) and \( \langle L_z(T) \rangle_v, \langle L_z(T) \rangle_0 \) are the free energy and the thermal average of the angular momentum of the configurations with and without vortex. where \( \langle L_z(T) \rangle \) is the thermal average of the angular momentum. In Eq. (22) the average free energy per atoms is given by,

\[
F_v = \frac{1}{N} [E - TS]
\]

\[
= \frac{1}{N} \left[ q(T, \alpha) + N \frac{\mu(s, \alpha)}{k_B T} \right]
\]

(23)

where \( S = k_B \left[ -q(T, \alpha) + \frac{E}{k_B T} - N \frac{\mu(s, \alpha)}{k_B T} \right] \) is the entropy of the system, \( E = -k_B T^2 \left( \frac{\partial \ln \bar{Z}}{\partial \beta} \right)_T \). The last term in Eq. (23) for the chemical potential is given by\(^{28,46}\),

\[
\frac{\mu(s, \alpha)}{k_B T} = \frac{\eta}{k_B T^2} \left[ \frac{1 - \alpha^2}{1 - \mathcal{T}^3} \right]^{2/5} \left[ \frac{T^2 s}{4} \right]^{1/5} \mathcal{T}^{-1}
\]

(24)

and \( \mathcal{T} = \frac{T}{T_0} \) is the reduced temperature, with

\[
T_0 = \frac{(\hbar \omega_g)}{k_B} \left( \frac{N}{\zeta(3)} \right)^{1/3}
\]

(25)

is the BEC transition temperature predicted by the non-interacting model with \( \omega_g = \left( \omega_{\perp} \omega_z \right)^{1/3} \) in the absence of rotation. The parameter \( \eta \) is given by the ratio between the chemical potential \( \mu \) at temperature \( T = 0 \) calculated in Thomas-Fermi approximation and the transition temperature for the non-interacting particles in the same trap, i.e. \( \eta = \frac{\mu(N,T=0)}{k_B T_0} \).

Substitution from Eqs. (18), (19), (23) into (22), we have the number of vortices in terms of \( q(T, \alpha) \),

\[
N_v = \frac{1}{N} \left[ q_{th}(T, \alpha) - q_{th}(T,0) \right] + \frac{N}{\hbar \omega_{\perp}} \left( \frac{\partial \mu(s, \alpha)}{\partial \alpha} - \frac{\partial q_{th}(T,0)}{\partial \alpha} \right) \left[ N_0 + \alpha \frac{k_B T \partial q_{th}(T, \alpha)}{\hbar \omega_{\perp} \partial \alpha} \right]
\]

(26)
In this section we will consider these two issues separately. Hartree-Fock approximation providing a correction term which accounts for the interatomic interaction effect term which accounting for the finite size effect 29,51,52 between the atoms is very weak.

V. RESULTS AND DISCUSSION

Once \( q(T, \alpha) \) has been determined, the number of vortices can be calculated. A possible way to calculate \( q(T, \alpha) \), is to converted the sum over \( n, m, n_z \) in Eq. (10) into an integral over the phase space by replacing the discrete \( E_{n,m,n_z} \) with a continuous variable, \( c(\mathbf{r}, \mathbf{p}) \), corresponds to the classical energy associated with the single-particle Hamiltonian for the system given in Eq. (10) 32,39,46. As well as for a finite number of atoms the sum over \( n, m, n_z \) can be approximated into an integral weighted by a density of states (DOS), \( \rho(E) \) 29,35,51. Both of these approximations provide the same thermodynamic potential for the ideal system. However, using an appropriate approximation providing a correction term which accounting for the finite size effect 29,31,52. While performing the integral over the phase space using the Hartree-Fock approximation providing a correction term which accounts for the interatomic interaction effect 29,39,53,54. In this section we will consider these two issues separately.

A. Vortices number for interacting system

Generally for interacting system 32,39,46, the thermodynamic potential in Eq. (10) is given by

\[
q^{\text{int}}(T, \alpha) = \ln(1 - z) + q_{\text{th}}(T, \alpha)
\]

\[
= \ln(1 - z) + A \left( \frac{k_0 T}{\hbar \omega_k} \right)^3 \left[ g_4(z) + \frac{\mu(s, \alpha)}{k_0 T} g_3(z) \right]
\]

where

\[
A = \frac{1}{(1 + \gamma^2 - \alpha^2)}
\]

thus when \( \Omega = \omega \) the confinement potential is not compensated by the centrifugal force due to the stiffer potential in the radial direction.

Substituting from Eq. (27) into Eq. (26) we have,

\[
N_v = - \left[ (A - A_0) \frac{\zeta(4)}{\zeta(3)} r^3 + \eta (AB - A_0 B_0) r^2 + \eta (B - B_0)/r \right] \left[ \frac{N_0}{N} + \frac{2 \alpha AC}{(1 + \gamma^2 - \alpha^2)} \frac{\zeta(4)}{\zeta(3)} r^4 + \frac{3}{5} \eta B r^3 \right]
\]

\[
A_0 = \frac{1}{(1 + \gamma^2)}, \quad B = \left[ (1 - \alpha^2)(1 - r^3) \right]^{2/5} \left[ \frac{\pi^2 s}{4} \right]^{1/5}, \quad B_0 = \left[ (1 - r^3) \right]^{2/5} \left[ \frac{\pi^2 s}{4} \right]^{1/5}
\]

and

\[
C = \left( \frac{\omega_2}{\omega} \right)^{1/3} \frac{N}{\zeta(3)}
\]

The condensate fraction is given by 33-35,

\[
\frac{N_0}{N} = 1 - T^3 - \eta B \frac{\zeta(2)}{\zeta(3)} T^2
\]

for \( T < 1 \). At \( T \sim 1 \) the last term in Eq. (29) can be omitted due to the dilution of the boson gas, where the interaction between the atoms is very weak.

Results calculated from Eq. (28) is represented graphically for various vortices configuration. In Fig. (1) we show the vortex number \( N_v \) within the radius determined by the expectation value \( \langle r^2 \rangle \) (in situ size) in the equilibrium state as a function of the rotation rate \( \alpha \) for several interatomic interaction parameter \( \eta \). In order to reproduce the experimental setup of Williams et al. 12 we use the following parameters: the trapping frequencies \( (\omega_\perp, \omega_z) = 2\pi (20.1, 53.0) \) Hz, \( d_\perp = 2.4 \) \( \mu \)m, and the lattice spacing \( d = 2 \) \( \mu \)m. For the above mentioned parameters \( \gamma \sim 0.3 \) and \( s = 0.1 \frac{\hbar}{\omega_\perp} \) ranging from 10 to 60. The vortex number \( N_v \) increases monotonically with increasing \( \alpha \). In contrast to the experiment of Williams et al. 22, the vortex number \( N_v \) grows faster than a linear function, until it reaches a maximum value and then decreases with the increase of the rotation rate. The number of vortices decreases with increasing the interatomic interaction parameter. Fig. (2) is devoted to illustrate the optical potential depth on the vortex number. This figure shows that the increase of the optical potential depth leads to the increase of the normalized temperature. Effect of the normalized temperature on the vortex number is considered in Fig. (3). This figure shows that the increase of the normalized temperature toward the BEC transition temperature leads to the decrease of the vortex number. However, the decrease of the vortex number is due to the decreasing of the condensate atoms number.
FIG. 1: The vortex number as a function of the rotation rate for various values of interatomic interaction: $\eta = 0.1$ for black line, 0.2 for red line and 0.3 for blue line. The number of particles $N = 1.0 \times 10^3$, $s = 20.16$ ($s = 0.1 \frac{v_0}{\hbar \omega}$) and the normalized temperature is $\tau = 0.2$.

FIG. 2: The vortex number as a function of the rotation rate for various values of the optical potential depth: $s = 20$ for black line, 60 for red line and 90 for blue line. The trap parameters of figure 1 is used.

FIG. 3: The vortex number as a function of the rotation rate for various values of the normalized temperature: $\tau = 0.15$ for black line, 0.17 for red line and 0.2 for blue line. The trap parameters of figure 1 is used.
FIG. 4: The vortex number as a function of the rotation rate for various values of number of particles : \( N = 1.0 \times 10^3 \) for black line, \( 1.0 \times 10^4 \) for red line and \( 1.0 \times 10^5 \) for blue line. The number of particles \( \eta = 1.0, s = 30.0 (s = 0.1 \frac{m}{\hbar \omega}) \) and the normalized temperature is \( \tau = 0.2 \).

**B. Vortices number for ideal system with finite size of atoms**

For system with finite number of atoms, the thermodynamic potential is given by\(^\text{46}\)
\[
g^f(z)(T, \alpha) = \ln(1 - z) + A \left\{ \left( \frac{k_B T}{\hbar \omega_g} \right)^3 g_3(z) + \frac{3 \bar{\omega}}{2 \omega_g \hbar \omega_g} \left( \frac{k_B T}{\hbar \omega_g} \right)^2 g_3(z) \right\}
\]

Substituting from Eq.(27) into Eq.(26) we have,
\[
N_v = - \frac{[A - A_0] \zeta(4) \tau^3 + \eta(AB - A_0B_0)\tau^2 + \eta(B - B_0)/\tau]}{2\pi A \zeta(4) \zeta(3) \tau^3} \left[ \frac{N_0}{N} + \frac{2\alpha AC \zeta(4)}{(1 - \alpha^2) \zeta(3) \tau^4} \right]
\]

with condensate fraction given by,
\[
\frac{N_0}{N} = 1 - \frac{3}{2} \left( \frac{\zeta(2)}{\zeta(3)} \right)^{1/3} \tau^2
\]

in the thermodynamic limit \( N \to \infty \), the last term in Eq.\(^\text{32}\) goes to zero.

Results calculated from Eq.\(^\text{31}\) for the vortices number as a function of the rotation rate is represented graphically in figures \( 4 \) for various values of \( N \), \( 5 \) for various number of the optical potential depth \( s \) and \( 6 \) at different values for the normalized temperature \( \tau \). These figure show that the vortices number increases with increasing the number of particles the optical potential depth and deceases with increasing the normalized temperature.

**VI. CONCLUSION**

In this paper, the temperature dependence of the vortices number for rotating condensate boson in optical lattice is investigated. The vortices number is parametrized in terms of the thermodynamic potential. Within the semiclassical approximation, calculation of the thermodynamic potential enabled the prediction of the interatomic interaction and the finite size effect. In conclusion, the number of vortices as a function of stirring frequency shows a peak. Moreover, the obtained results confirmed that the vortices number for this systems depends significantly on the interatomic interaction parameter, the number of particles and the optical potential depth. Interatomic interaction deceases the number of vortices while increasing the number of particles increases the vortices number. Increasing the optical potential depth increases the number of vortices. The vortices number depends strong on the condensate temperature at \( \tau < 0.3 \).
FIG. 5: The vortex number as a function of the rotation rate for various at different values of the optical potential depth $s = 10.0$ for black line, 30.0 for red line and 50 for blue line. The number of particles $\eta = 1.0$, $s = 30.0$ and the normalized temperature is $\tau = 0.2$.

FIG. 6: The vortex number as a function of the rotation rate for various values of the normalized temperature :$\tau = 0.15$ for black line, 0.17 for red line and 0.2 for blue line. The trap parameters of figure (4) is used.

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