BRST Inner Product Spaces and the Gribov Obstruction

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Abstract
A global extension of the Batalin–Marnelius proposal for a BRST inner product to gauge theories with topologically nontrivial gauge orbits is discussed. It is shown that their (appropriately adapted) method is applicable to a large class of mechanical models with a semisimple gauge group in the adjoint and fundamental representation. This includes cases where the Faddeev–Popov method fails. Simple models are found also, however, which do not allow for a well-defined global extension of the Batalin–Marnelius inner product due to a Gribov obstruction. Reasons for the partial success and failure are worked out and possible ways to circumvent the problem are briefly discussed.

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1 Introduction and Overview

Canonical quantization of gauge theories leads, in general, to an ill-defined scalar product for physical states. In the Dirac approach physical states are selected by the quantum constraints. Assuming that the theory under consideration is of the Yang–Mills type with its gauge group acting on some configuration space, the total Hilbert space may be realized by square integrable functions on the configuration space and the quantum constraints imply gauge invariance of these wave functions. Thus the physical wave functions must be constant along the orbits traversed by gauge transformations in the configuration space. Consequently, the norm of the physical states is infinite, if the gauge orbits are noncompact or if the number of nonphysical degrees of freedom is infinite as in gauge field theories.

The problem is similar to that which occurs also in the path integral quantization of gauge theories where the integral over gauge field configurations diverges because of the gauge invariance of the action. In their seminal work Faddeev and Popov proposed a solution based on the idea that in order for the path integral measure to be finite, only one representative of each gauge orbit should be taken into account. They provided a systematic way of implementing gauge conditions in the integral so that, by inclusion of an appropriate additional contribution to the measure, namely the Faddeev–Popov (FP) determinant, the resulting integral becomes independent of the choice of gauge and effectively ranges over the physical degrees of freedom only. However, if the gauge orbits possess a nontrivial topology, as often happens in physically interesting theories, a good choice of gauge turns out to be impossible. This deficiency is known as the Gribov obstruction. It can be illustrated with simple mechanical models that ignoring global deficiencies of a particular gauge can result in explicitly wrong predictions of the corresponding path integral quantization.
The FP path integral measure specifies uniquely the measure in the scalar product. In fact, the norm of the Dirac states can be made finite by reducing the initial measure to the gauge fixing surface and inserting the corresponding FP determinant to maintain the gauge invariance of the physical amplitudes. The Gribov obstruction will again be present, if the gauge orbits have a non-trivial topology. So, the FP approach would, in general, lead to an ill-defined scalar product. In many of these cases, one may, however, further modify the resulting FP inner product so as to obtain a finally reasonable norm for physical states. In the simplest case this is effected, e.g., by restricting the domain of integration along the gauge fixing surface to its modular domain (i.e. to a region which contains no points that are still gauge equivalent to others on that surface, their so-called Gribov copies).

More recent suggestions to handle the norm regularization problem within the Dirac approach include a redefinition of the scalar product along the lines suggested in [7, 8] or the transition to the coherent state representation for constrained systems as performed in [9, 10].

In gauge field theories and especially in their path integral formulation, an explicit Lorentz invariance of the quantum theory is desired, which is not available in the Dirac Hamiltonian approach. The Lorentz invariance can, however, be achieved within the (nonminimal) BRST quantization program (see, e.g., [11]).

BRST quantization is based on the extension of the original phase space by Lagrange multiplier and ghost sectors. For a set of first-class constraints $G_a$ one introduces the Lagrange multipliers $y^a$ and their canonical momenta $p_{y^a}$ and adds canonical pairs $(C^a, P_a)$ and $(\bar{C}^a, \bar{P}_a)$ of fermionic (Grassmann) ghost and antighost variables, respectively. The extended phase space has a natural grading with respect to the ghost number operator $N$: $[N, C^a] = C^a$, $[N, \bar{C}^a] = -\bar{C}^a$, etc. Finally, the BRST charge $Q$ is constructed. It is a
hermitian nilpotent operator \((Q^\dagger = Q, Q^2 = 0)\) such that, at least generically, the Dirac physical subspace formed by functions on the orbit space is isomorphic to the subspace composed of elements of BRST–cohomology classes (usually at ghost number zero). That is, one looks for wave functions that are annihilated by \(Q\), called BRST–closed, and identifies those differing by elements of the image of this operator (identification by BRST–exact states). Formally different representatives chosen from an equivalence class yield the same physical answers since

\[
\langle s_1 | (|s_2\rangle + Q|p\rangle) = \langle s_1 | s_2\rangle ,
\]

(1)

where we made use of \(Q|s_1\rangle = 0\) and the hermiticity of \(Q\). However, it turns out that in practice the physical states (among others) often do not have a well–defined norm in the original (indefinite) Hilbert space. Typically, the norm is proportional to the meaningless factor \(\infty \cdot 0\). The infinity comes from the integration over the Lagrange multipliers, while zero results from the Berezin integral over the ghosts and antighosts (cf., e.g., [11] or Sec. 2 below, providing a simple illustration). So, such as in Dirac quantization, also in canonical BRST quantization there is a problem with the inner product [12], which, moreover, has not been resolved in generality up to present day.

In this work we shall discuss an approach due to Batalin and Marnelius (B & M) to this problem [13]. Their main idea is to not alter the original inner product, but to single out specifically chosen representatives in the BRST cohomology classes which then have a well–defined inner product among each other. They provide a scheme to construct a (hermitian) gauge fixing fermion \(\Psi\) and a space of so–called auxiliary states \(|s\rangle_0\), which, as we will see, resemble the physical (gauge invariant) states obtained in the ghost–free Dirac approach. The representatives of the cohomology which have a
well–defined inner product are then provided by the BRST singlets

$$|s\rangle := e^{[Q, \Psi]+} |s\rangle_0 .$$  \hspace{1cm} (2)

More precisely, Batalin and Marnelius define the inner product of physical states, which are in a one–to–one correspondence with the states $|s\rangle_0$, by means of

$$\langle s|s'\rangle = \langle s|\exp(2[Q, \Psi]+) |s'\rangle_0 ,$$  \hspace{1cm} (3)

which follows formally from the above representation of $|s\rangle$ and the analogous one for $|s'\rangle$ when using (naive) hermiticity of $Q$ and $\Psi$. Note that the auxiliary states $|s\rangle_0$ have a specific dependence on the ghost and nonphysical variables. So they are also called the ghost– and gauge–fixed state $s$. The BRST transformed states (2) with a generic $\Psi$ yield, at least up to global issues, the whole cohomology class represented by $|s\rangle_0$. The conventional BRST inner product between quantum states is not always well–defined. The goal of Batalin and Marnelius was to develop a formalism for selecting a set of representatives $|s\rangle_0$ together with an adapted $\Psi$ such that the resulting representatives $|s\rangle$ have a well–defined inner product with one another. The reason for introducing the states $|s\rangle_0$ on an intermediate level is that generically they are much simpler than the states $|s\rangle$, containing, e.g., no ghosts in an appropriate polarization and sometimes even coinciding literally with the states found in a Dirac quantization (cf. also the examples below in this paper).

Arriving at formula (3) in this way, one implicitly has defined an inner product between the BRST cohomology classes (represented by $|s\rangle_0$ or $|s\rangle$). The B&M solution (2) of the BRST inner product cohomologies is local [13]. So the question of a global extendibility of their formalism arises.

As follows from (3), the choice of gauge conditions — or, equivalently, the choice of the gauge fixing fermion $\Psi$ — is an essential ingredient of the
B&M approach, such as it is in the FP procedure for defining an inner product between Dirac quantum states. In gauge theories with a nontrivial topology of the gauge orbits in the configuration space, there is, in general, a Gribov obstruction to a (globally well-defined) choice of a gauge condition, which can cause serious deficiencies of the FP inner product. Therefore one might expect an analogous problem within the B&M procedure.

The aim of the paper is to investigate possible global obstructions to the B&M construction that might occur through a non-Euclidean gauge orbit space geometry [14], which has not yet been addressed. In the B&M formalism, one of the conditions placed on $\Psi$ and $|s\rangle_0$ is that the FP determinant of the gauge underlying the choice of $\Psi$ is nonvanishing everywhere. In the presence of the Gribov obstruction this condition cannot be met anymore. We will study possible consequences of fulfilling this condition only almost everywhere in the configuration space, i.e. the associated FP determinant is nonzero everywhere in configuration space except for some region of lower dimension. It then will turn out that in some cases the B&M method still provides a useful recipe for constructing an inner product between BRST cohomologies, while in others it will not. Among these cases we will find examples where the FP method fails, while the B&M procedure works.

To single out the crucial difference between gauge systems with and without the Gribov problem, we shall apply the B&M procedure to simple mechanical gauge models in which the Gribov obstruction is evident and compare them with similar models where the latter is absent. The models chosen are simple enough to be analyzed by less sophisticated methods to full accuracy, thus allowing for a first test of the B&M version of the BRST approach. In particular, the configuration space of the models will be finite dimensional always in this paper (which is in contrast to field theories) and, moreover, in most (but not all) of the models, the gauge group is compact and of fi-
nite volume. Correspondingly, in all those miniature models with a compact
gauge group, irrespective of the presence of a Gribov problem, the physical
wave functions in the Dirac quantization will have a well-defined, nonsin-
gular, and physically reasonable inner product already with respect to the
measure defined in the original Hilbert space. So, in these cases the FP
method (or similarly the B&M method) for constructing an inner product
is superfluous. However, we are still free to apply these methods also to
such simple models and then compare the result to the one obtained by the
Dirac procedure, which we then may use as the touchstone for a correct inner
product, at least up to unitary equivalence.

In the following section (Sec. 2), we will briefly recapitulate the idea of
the B&M construction of the inner product at the example of the simplest
possible "gauge theory" without Gribov obstruction, the gauge orbits being
straight lines in a two-dimensional Euclidean configuration space. In Sec. 3
we apply the B&M recipe to a model with gauge group $SO(2)$, the gauge
orbits of the previous model having been compactified to circles. The seem-
ingly small deficiency of a vanishing FP determinant at a set of zero mea-
Sure in the configuration space, which is a consequence of the Gribov pro-
blem, turns out to be a decisive obstruction to the B&M method in this case
(for a specific choice of the gauge fixing fermion the inner product vanishes
identically). Studying, on the other hand, the analogous model with gauge
group $SO(3)$ in Sec. 4, the gauge orbits being spheres in an IR$^3$ now, the
B&M procedure is found to provide a well-defined inner product, equivalent
to the covariant result of Dirac quantization. To be precise, in order for the
latter statement to hold, some additional new condition in the construction
of the BRST operator $Q$ has to be met, which is not present in the work of
Batalin and Marnelius. Yet, also the gauge fixing fermion is to be restricted
in a certain way, discussed in more detail further below.
In Sec. 5 we then see that the successful application of the B&M approach to the $SO(3)$–model can be extended to models with an arbitrary semisimple Lie group acting in its adjoint representation. The condition on the operator $Q$ will be specified and further clarified in this context.

Much of the remainder of the paper is then devoted to the question, why the proposal of Batalin and Marenlius, refined by the aforementioned condition on the form of $Q$, works for the models discussed in Sec. 4 and 5, while it fails for the simple $SO(2)$–model.

The first and most near–at–hand ansatz to answering this question is the following observation: For the models studied up to that point, the FP method works and fails in precisely the same cases as the B&M method does. This is not the full answer, however. As we will show in Sec. 6.1, there are models for which the FP method fails, while the B&M approach still works! These models are obtained from another generalization of the $SO(3)$–model: Interpreting the action of the $SO(3)$ group on the configuration space $\mathbb{R}^3$ not as the adjoint action in the Lie algebra as in the generalization of Sec. 5 (in which case the configuration space variables are somewhat similar to gauge fields in realistic Yang–Mills theories) but as the fundamental action, it is most straightforward to generalize the $SO(2)$– and $SO(3)$–model simultaneously to obtain a model with gauge group $SO(N)$ acting in its fundamental representation on $\mathbb{R}^N$. This is interesting because it may be seen that the FP method works for odd $N$ while it fails for even $N$, producing a gauge dependent norm in the latter instance which, in the worst case, may even vanish. All the more it comes somewhat as a surprise that the (appropriately refined) B&M method yields a good inner product (equivalent to “the correct” one in the original $N$–dimensional configuration space) provided only

\[ \text{For reducible theories (discussed further below) the FP determinant is defined with respect to a subset of locally independent constraints.} \]
that $N \geq 3$.

Another obvious difference between $SO(2)$ on the one hand and $SO(3)$ with all its successful generalizations on the other hand is that the gauge group of the former model is abelian, while the gauge groups of all the other models are semisimple, which, from a group theoretical point of view (cf., e.g., [15]), is something like the extreme opposite of abelian. The model studied in Sec. 6.2 provides an example to this guess demonstrating the opposite: Considering more than just one particle in a three-dimensional configuration space with the rotational group $SO(3)$ acting on all of them simultaneously, the B&M procedure is found to fail again.

From all of these studies it appears to us that it could be the reducibility of the constraints that allows for a successful application of the B&M construct, while theories with irreducible constraints generically will lead to unacceptable results in the presence of a Gribov obstruction. Here reducibility of (first-class) constraints $G_a \approx 0$ means that they are not independent from one another, i.e. there exists at least one relation $Z^a G_a \equiv 0$ for some functions $Z^a$ on the phase space of the theory. Clearly any theory formulated in an irreducible manner can be reformulated by means of a reducible set of constraints. So, in the above, “reducibility” should be specified to what one might call “essential reducibility”, by which we mean a constrained Hamiltonian system with reducible constraints which cannot be replaced globally by a set of irreducible first-class constraints. The prototype of such a theory is the initially mentioned $SO(3)$–model, the constraints being the three components of the angular momentum in the phase space $T^* (\mathbb{R}^3)$. The reducibility of the $SO(3)$–model is lost when the number of particles is increased above one, as is done in Sec. 6.2.

We remark at this point that the condition mentioned above to refine the B&M version of BRST quantization is one placed on the functions $Z^a$.
expressing the mutual dependence of the constraints. Classically there is a large ambiguity or freedom in choosing such functions. Only an appropriately chosen subset of those functions will lead to a good inner product of the quantum theory, while others turn out to be unacceptable in the end.

For the models of Sec. 4 and 5, with the (semisimple and compact) gauge group acting in the adjoint representation, we observe that the (refined) B&M construction yields a BRST inner product that reproduces the one found in the Dirac quantization (which, as remarked already above, is also well-defined in these mechanical toy models). This result holds at least for all choices of the BRST gauge fixing fermion Ψ which correspond to a gauge that is linear in the configuration space variables. As the Dirac inner product certainly is independent of any possible choice of gauge, we may conclude also gauge independence of the B&M procedure, at least within the above class of gauge fixing fermions. However, this apparent gauge independence is by no means complete. In Sec. 7 we will see at the example of the helix model [9, 16, 17, 18] that the B&M procedure fails for gauge conditions with a nonconstant number of Gribov copies. Moreover, it is found that if the number of Gribov copies diverges — in Sec. 8 we will argue by means of an example why this is of relevance in physical theories —, then also the B&M inner product diverges! This effect was not observed in Secs. 4 and 5 since there the number \( N_W \) of copies was finite; the resulting inner product comes out proportional to \( N_W \), but any finite proportionality constant drops out from an inner product by normalization.

Sec. 8 contains our conclusions and a discussion of possible ways to circumvent the global topological obstructions for constructing the proper BRST inner product in physical theories.
2 The Batalin–Marnelius procedure: a simple example

In this section we illustrate the B&M procedure \[13\] by means of a simple example, namely a particle on \( \mathbb{R}^2 \) parameterized by coordinates \( x \) and \( y \) with the translational gauge symmetry along the \( y \)-direction. The gauge orbits are straight lines parallel to the \( y \)-axis. So there is no Gribov problem, although the gauge orbits have infinite volume, thus indicating already the divergence of a naive BRST scalar product. The model will allow us to recall the main idea and ingredients of the B&M construction. Denoting the momenta by \( p_x \) and \( p_y \), the constraint is simply \( p_y \approx 0 \), while the Hamiltonian \( H \) has to be independent of \( y \) due to the translational symmetry. The nonminimal BRST charge is \( Q = p_y \mathcal{C} + \mathcal{P} \pi \), where \((\mathcal{C}, \mathcal{P})\) and \((\mathcal{C}, \mathcal{P})\) are canonical pairs of fermionic ghost and antighost variables, respectively, and \( \pi \) is the momentum conjugate to the Lagrange multiplier \( \lambda \), which enforces the constraint within the Hamiltonian action. The nonminimality of the BRST scheme means that the canonical pair \((\lambda, \pi)\), supplemented by the additional first–class constraint \( \pi \approx 0 \), is added to the phase space. The reason for doing this is the analogue with gauge field theories where the addition of the Lagrange multipliers to the BRST multiplet allows for explicit Lorentz covariance (in contrast to the minimal Hamiltonian approach where the Lagrange multipliers are excluded before quantization).

The BRST invariant quantum states, \( Q|\psi\rangle = 0 \), modulo shifts on \( Q \)-exact states, \(|\psi\rangle \rightarrow |\psi\rangle + Q|\phi\rangle\), form a space that is isomorphic to the Dirac physical subspace determined by the gauge invariance condition \( p_y|\psi\rangle = 0 \). Since zero eigenvalue of the operator \( p_y \) lies in the continuum spectrum, the \( L^2 \)-norm of the physical states is infinite. In the coordinate representation, a function \( \psi_0(x) \) is annihilated by the constraint operator \( p_y = -i\partial_y \), but
it clearly does not have finite norm in the original Hilbert space with inner
product given by $\int_{\mathbb{R}^2} dx dy$. Similarly, in the $p_y$-polarization the physical wave
functions $\psi_0(x) \delta(p_y)$ lead to the ill–defined square of a delta function. It is
obvious also from the form of the BRST charge $Q$ that, in an appropriate
polarization of the wave functions, $\psi_0(x)$ is also BRST–closed. The norm
then contains the infinity obtained in the Dirac approach, multiplied here by
a zero from the (Berezin) integration of the Grassmann variables, and thus
is ill–defined as well, as already mentioned in the Introduction.

Let us now apply the B&M procedure to the model. We first have to
pick a gauge condition. This is trivial in the present case, let us choose
$y = 0$ as the simplest possibility. Similarly, in the nonminimal sector we
choose $\lambda = 0$. Following the recipe of B&M, one then has to construct two
hermitian operator sets, subject to some consistency conditions (cf., e.g.,
[13]). In the present case a possible choice of these two sets is:

$$D_{(1)} := \{(y, C), (i\bar{C}, \pi)\}, \quad D_{(2)} := \{(\lambda, \bar{P}), (i\mathcal{P}, p_y)\}.$$  \hspace{1cm} (4)

Each set of operators consists of so–called BRST–doublets, which means that
the second operator in each round bracket is — up to a possible prefactor of
$i$ ensuring hermiticity — the BRST–transformed (graded commutator $[Q, \cdot]$)
of the first operator in the respective round bracket. So each of the two sets
$D_{(i)}$ consists of four operators in the present example, which in turn may be
grouped into two doublets. We remark here that the two sets $D_{(i)}$, $i = 1, 2$,
are not independent from one another. It is a generic feature of the two sets
that one of them contains the gauge condition of the minimal sector and the
constraint of the nonminimal sector and vice versa for the other set; together
with the doublet structure this is the basic principle behind the consistency
conditions required for the sets.

Next, one has to decide for one of the two sets, say $D_{(2)}$, and to deter-
mine its kernel, i.e. the simultaneous kernel of all four operators of this set.
Choosing a polarization such that all momenta are represented by derivative operators except for $\pi$, which we take as multiplication operator, this kernel is spanned by the BRST–closed functions $\psi_0(x)$. These are the so–called auxiliary states denoted by $|s\rangle_0$ in the Introduction. In the present polarization the coincide with the physical wave functions of a Dirac quantization.

To obtain inner product states $|s\rangle$, the following (B & M ) procedure is applied: Multiply the respectively first entry of each of the two doublets of the other operator set, i.e., of $D_{(1)}$ in our case, to obtain a gauge fixing fermion $\Psi \equiv iy\bar{C}$ and define $|s\rangle := \exp\{Q,\Psi\}|s\rangle_0$. Since $Q$ and $\Psi$ are hermitian, one finds

$$\langle s|s\rangle = _0\langle s|\exp\{2i[Q,\Psi]\}|s\rangle_0.$$  

In explicit terms the above formula reads

$$\langle s|s\rangle \propto \int dx\,dy\,d\tilde{\pi} \,d\bar{C} \,d\bar{C} \,\psi^*_0(x) \left[\exp\left(-2i\tilde{\pi}y\right) \left(1 + 2\bar{C}\bar{C}\right)\right] \psi_0(x)$$

$$\propto 2\int dx\,dy \,\psi^*_0(x) \delta(2y) \psi_0(x) \equiv \int dx|\psi_0(x)|^2. \quad (6)$$

Here $\tilde{\pi} \equiv i\pi \in \mathbb{R}$ as $\pi$ has to be quantized indefinitely [20, 21]. This latter fact is also the reason for the above phase in front of $|s\rangle_0$ to add up in the inner product rather than to drop out from it: The original inner product for the indefinitely quantized variable $\pi$ is of the form $\langle f|g\rangle = \int d\tilde{\pi} \, f^*(\pi^*)g(\pi)$, where the wave functions are understood as functions of the spectrum of the hermitian operator $\pi$, which is purely imaginary in this case. In fact, the ghost degree of freedom $\bar{C}$ has also to be quantized indefinitely; however, for a Grassmann variable this makes no difference in the end.

So, in comparison to the Dirac procedure the upshot of the B & M inner product is to get rid of the gauge group volume $\int dy = \infty$ by effectively introducing an appropriate delta function for this integral (cf. second line

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6 But cf. also the last paragraph in this section and the comments to Eq. (3) in the Introduction.
in Eq. (6), leaving the ordinary Lebesgue measure for the single physical variable $x$.

It is worthwhile to have a look at the same analysis in the momentum representation of the constraint $p_y$. After switching to the momentum representation (only for this variable for simplicity), the kernel of the operator set $D^{(2)}$ again coincides with the Dirac physical wave functions of the respective polarization (no additional ghost terms occur, as is the case for other polarizations of the wave functions): $\psi(x) \delta(p_y)$. By means of Eq. (4) one finds now

$$\langle s | s \rangle \propto \int dx \, dp_y \, d\tilde{\pi} \, (\psi_0^*(x) \delta(p_y)) \exp(-2\tilde{\pi} \frac{d}{dp_y}) \psi_0(x) \delta(p_y)$$

$$= \int dx \, dp_y \, d\tilde{\pi} \, \psi_0^*(x) \delta(p_y) \psi_0(x) \delta(p_y - 2\tilde{\pi}) \propto \int dx \, |\psi_0(x)|^2 . \quad (7)$$

Thus, the meaningless square of the delta–function in the original inner product for the Dirac states becomes “regularized” through the B&M procedure by the point splitting in the product of the $\delta$–functions (cf. second line in Eq. (7)), again leading to a well–defined $L^2$ norm over the physical variable.

Summarizing, we see that the B&M scheme indeed resolves the aforementioned problem of the inner product for physical states. An essential ingredient of their procedure is the choice of an appropriate gauge condition, which underlies the construction of the operator sets (4) and eventually specifies the norm (5). In realistic gauge theories, one often is plagued by the Gribov problem, excluding the existence of global gauge conditions. Strictly speaking, this excludes also the existence of two operator sets $D_{(i)}$ fulfilling all the requirements of [13]. In the sequel we study, with examples of simple gauge theories with a Gribov problem, whether the local B&M procedure, which ignores subtleties of the gauge fixing, can be extended to a global level.

Finally we remark that in the argumentation within this section we in part remained quite formal, e.g., when speaking of hermiticity or cohomol-
ogy classes of various operators — we never specified domains of definition of them. In the following, however, we intend to take equation (3) as a definition, analyzing its consequences with care and accuracy.

3 An irreducible abelian model with Gribov obstruction: The $SO(2)$–model

In this section we consider a two–dimensional model with gauge group $SO(2)$. The gauge orbits are concentric circles (one–spheres), generated by the constraint $G \equiv l = x_1 p_2 - x_2 p_1 \approx 0$ \cite{22, 23}. Obviously, this is the angular momentum in two dimensions. The reduced phase space is the half–plane $\mathbb{R}_+ \times \mathbb{R}$ with the identification $(0, p) \sim (0, -p)$, which is homeomorphic to a cone \cite{23, 6}. We draw attention to the nontrivial topology of the gauge orbits as in contrast to the model of the previous section where the gauge orbits are just parallel straight lines. This fact gives rise to the non–Euclidean geometry of the reduced phase space and will play a crucial role in the subsequent analysis.

The classical Hamiltonian of the model is simply

$$H = \frac{1}{2} p^2 + V(x^2) + \lambda l,$$

where $V(x^2)$ is some gauge invariant potential and $\lambda$ a Lagrange multiplier enforcing the constraint $l = 0$. Here we shall again adopt the nonminimal BRST scheme and treat $\lambda$ as a dynamical variable with conjugate momentum $\pi$. The extended model has the further constraint $\pi \approx 0$, generating orbits isomorphic to $\mathbb{R}$. Then the nilpotent BRST charge becomes $Q = Cl + \pi \mathcal{P}$.

The Gribov obstruction arises from the nontrivial topology of the gauge orbits and is already obvious at this stage: There exists no single–valued, globally regular function $\chi(x_1, x_2)$ such that the gauge fixing curve $\chi = 0$
intersects each gauge orbit precisely once. In the B&M procedure, one has to specify two operator sets of the BRST doublets satisfying some consistency conditions. Among these is a condition that essentially states that the Faddeev–Popov determinant of the underlying gauge conditions has to be nonzero. This requirement cannot be met. However, the deficiency may be localized to a single point on the gauge fixing line in the configuration space, spanned by \((x_1, x_2)\), namely to the origin \(x_1 = x_2 = 0\). Let us ignore, for a moment, this seemingly small deficiency and proceed with the B&M construction of a scalar product. The two sets of operator doublets are chosen to be

\[
D^{(1)} = \{ (x_2, x_1 \mathcal{C}), (i \bar{\mathcal{C}}, \pi) \},
\]

\[
D^{(2)} = \{ (i \mathcal{P}, l), (\lambda, \bar{\mathcal{P}}) \}.
\]

Now we determine the kernel of the set \(D^{(2)}\). In a convenient polarization it reads \(\langle x, \pi, \mathcal{C}, \bar{\mathcal{C}} | s \rangle_0 = \psi_0(x^2)\). The hermitian gauge fixing fermion is \(\Psi = ix_2 \mathcal{C}\) in this case. With these ingredients we may now apply Eq. (5) to obtain the following inner product between two physical states:

\[
\langle s | s' \rangle \propto \int_{\mathbb{R}} dx_1 x_1 \bar{\psi}_0(x_1^2) \psi_0'(x_1^2) \equiv 0.
\]

Thus, the B&M procedure does not lead to a well-defined physical scalar product here. This can also be verified for other polarizations of the wave function.

In the particular polarization chosen here it is possible to obtain a scalar product by some simple additional manipulation, e.g. by restricting the range of integration to the positive axis or by replacing the integration measure \(x_1\) by \(|x_1|\). However, this would not be in the spirit of the B&M procedure: As outlined in Secs. 1 and 2, the idea was to keep the original inner product and
to just select appropriate BRST-representatives in order to yield well-defined amplitudes. The original measure was not to be altered.

One could try to think of another gauge fixing fermion $\Psi$ in (3) that would lead to a more appropriate measure like $|x_1|$ or $x_1\theta(x_1)$ with $\theta(x_1)$ being the characteristic function of the Gribov domain $x_1 > 0$. However, it is not hard to convince oneself that such a gauge fixing fermion cannot be of the conventional form $\Psi = i\chi(x_1, x_2)C$ for any smooth single-valued function $\chi$. So, the vanishing of the FP determinant even at a single point appears to be quite an obstacle to a naive global extension of the B&M procedure. We shall see, however, that in the reducible case the situation turns out to be better.

4 A reducible nonabelian model with Gribov obstruction: The $SO(3)$-model

In this section we apply the B&M procedure to a mechanical model with gauge group $SO(3)$ [24, 23]. The new feature of this model, besides being nonabelian, is the reducibility of the constraints generating the gauge orbits. The constraints $G_a$ are given by the three components of the angular momentum $G_a \equiv l_a = \varepsilon_{abc}x^b p_c$. It is easy to see that the $G_a$ can be combined nontrivially to zero: $x^a G_a \equiv 0$. The reducibility arises from the fact that the gauge orbits, which are two-spheres, are not parallelizable. They do not admit one globally nonvanishing vector field. In general, irreducible theories can be turned easily into reducible ones by adding constraints that are not independent of the original ones, but, as demonstrated already by the above example, not necessarily vice versa. This is what we called “essential reducibility” in the Introduction.

In the context of BRST-quantization, the reducibility of the constraints
is taken into account by the introduction of additional ghost–of–ghost variables. The total number of variables in a BRST–quantization blows up considerably with increasing rank of the Lie algebra of the considered model, especially when one deals with nonminimally extended models. The dependences of the constraints can be given by $Z^a_A G_a = 0$ with phase space functions $Z^a_A(x, p)$, $a = 1, \ldots, d$; $A = 1, \ldots, r$. In what follows the functions $Z^a_A$, which exhaust all possible reducibilities, turn out to be independent. So there will be no need for ghost–of–ghosts of higher rank, unless explicitly stated otherwise as in the models of Sec. 6.1 below.

The $r$ sets of functions (vectors) $Z^a_A$ may always be multiplied by some nonvanishing functions $f_A(x)$ without spoiling their characteristic feature of specifying all independent reducibilities of the constraints $G_a$. We will see that, upon following the B & M procedure, this arbitrariness in the choice of the functions $f_A$ will yield different measures in the physical scalar products. Moreover, only for a subset of such functions $f_A$ or, equivalently, for a subset of vectors $Z^a_A(x, p)$, the resulting measure will make sense (i.e. will be physically acceptable). Still, also in these cases the measure will depend explicitly on the specification of those functions. However, the algebra of observables will then turn out to be modified accordingly so that the representations with different choices of $f_A$ (from the allowed subset) turn out to be equivalent (and in particular also unitarily equivalent to the covariant Dirac result).

In the present model we found one relation ($r = 1$) between the constraints $G_a$ with $Z^a_1 \equiv x^a$ as a possible choice. In this case the “allowed subset” of nonvanishing functions $f_1(x^a)$ will turn out to comprise the (nonvanishing) gauge invariant functions of $x^a$, i.e. $f_1 = f(x^2)$.[7] In the following we will first restrict ourselves to this “gauge invariant” parameterization of

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[7] In fact the characteristic feature of the allowed subset is merely to have $|f_1|$ invariant under $x_1 \rightarrow -x_1$. We will come back to this issue.
Discussing changes that occur when allowing for more general functions \( f_1 \) only at the end.

In the \( SO(3) \)-model the BRST-operator \( Q \) is given by

\[
Q = C_0^a G_a + i f(x^2) C_1^a x^a P_{0a} - \frac{1}{2} \varepsilon_{ab}^c C_0^a C_0^b P_{0c} + \pi_{0a} \tilde{P}_{0}^a + \pi_{11} \tilde{P}_{1}^{11} + \pi_{11} \tilde{P}_{1}^{11} , \tag{11}
\]

where indices \( a, b, c \) run from 1 to 3, \( \varepsilon_{ab}^c \) are the structure constants of the Lie algebra \( so(3) \) of the group \( SO(3) \), and \( f(x^2) \) is some nonvanishing function of \( x^2 \). The first term in the BRST-charge contains the constraints \( G_a \), the second term reflects their mutual dependence with \( f(x^2) \) parameterizing the arbitrariness mentioned above, the third term, cubic in the ghosts, is standard for nonabelian groups \[11\], and the last three terms are a result of nonminimal extension \( \text{including the ghost-of-ghost sector} \ [25] \). The full extended phase space of the model consists of nine bosonic and eight fermionic pairs of canonically conjugate variables. The bosonic pairs are \((x^a, p_a), (\lambda^a, \pi_a), (C_1^a, P_{11}), (\bar{C}_{11}, \bar{P}_{11})\) and \((\lambda_{11}^a, \pi_{11}^a)\), while the fermionic pairs are \((C_0^a, P_{0a}), (\bar{C}_{0a}, \bar{P}_{0a}), (\lambda_{1}^a, \pi_{1}^a)\) and \((\bar{C}_{11}^{1}, \bar{P}_{11}^{1})\). The number of bosonic pairs exceeds the number of fermionic ones by one. So the model possesses one bosonic physical degree of freedom. The unphysical sector has eight fermionic and eight bosonic degrees of freedom. In quantum theory both the bosonic and the fermionic degrees of freedom of the unphysical sector have to be quantized with half positive and half indefinite metric states.

Following the B&M procedure we construct two sets of hermitian BRST doublets

\[
D_{(1)} = \{(0, 0), (x^2, C_1^a x^a - C_3^0 x^1), (x^3, C_0^2 x^1 - C_1^0 x^2), (\lambda_{11}^1, \bar{P}_{11}^{11})\}, \tag{12}
\]

\[
D_{(2)} = \{(0, 0), (iP_{02}, G_2 - \varepsilon_{2b}^{c} C_0^b P_{0c}), (iP_{03}, G_3 - \varepsilon_{3b}^{c} C_0^b P_{0c}), \}
\]

\[
(P_{11}, i f(x^2) x^a P_{0a}), (i \lambda_{1}^1, \bar{P}_{11}^{1}), (\lambda_{0}^a, \bar{P}_{00}), (i \bar{C}_{11}^{1}, \bar{P}_{11}^{1})\} .
\]

\[8\]Here the notation conforms to the one used in \[25\].
Here a short comment about the admittedly somewhat strange doublet \((0, 0)\) is in place: The first three doublets of the set \(D^{(1)}\) comprise essentially three gauge fixing conditions \(\chi^i\) together with their BRST transforms \([Q, \chi^i] = iM^i_jC^j\), \(i, j = 1, \ldots, 3\). Because of the reducibility of the model the FP matrix \(M^i_j\) is at most of rank two. So it is possible and convenient to make the trivial choice \(\chi^1 \equiv 0\) for the first of the gauge fixing functions. (In other words, locally, i.e. up to some regions in phase space of a lower dimension, only two of the three constraints are essential and thus only two gauge conditions, \(\chi^2 = 0\) and \(\chi^3 = 0\), are necessary). An analogous reasoning applies to the trivial doublets \((0, 0)\) in the set \(D^{(2)}\) above as well as in the operator sets of the models to be discussed below.

The following steps are like those in Sec. 2. First, we evaluate the kernel of the operator set \(D^{(2)}\). As before it is given by gauge invariant functions \(\psi_0(x^2)\) as the physical wave functions in the Dirac quantization. They are the auxiliary states \(|s\rangle_0\). The hermitian gauge fixing fermion \(\Psi\) is bilinear in those operators of the set \(D^{(1)}\) which are not BRST invariant (first entries of doublets). A possible choice is

\[
\Psi = i\bar{\xi}_{02}x^2 + i\bar{\xi}_{03}x^3 + \bar{\xi}_{11}C^1_0 + i\bar{\xi}_{01}\lambda_{11}^{11}.
\]

Now we can compute the norm of a physical state by means of Eq. (5). Again one has to keep in mind that half of the unphysical variables must be quantized with indefinite metric. After some calculation one finds

\[
\langle s|s\rangle \propto \int dx_1 \frac{|x_1|}{|f(x^2_1)|} \psi_0^*(x^2_1)\psi_0(x^2_1).
\]

In an intermediate step in deriving (14), we made use of the formula

\[
\int_{-\infty}^{\infty} dx dy \delta(xy)x^2\varphi(x) = \int_{-\infty}^{\infty} dx |x|\varphi(x),
\]

\(\text{(15)}\)
where $\varphi$ is some smooth (test) function. This relation may be obtained, e.g., by interpreting $\delta(z)$ as the limit of an arbitrary delta sequence $\delta_n(z)$. The above integral equality is then understood as a limit of the sequence of the integrals in the l.h.s. with $\delta(xy) \to \delta_n(xy)$. The limit does not depend on a particular choice of the delta sequence. This may be proven by going over to new integration variables $\tilde{x} = xy, \tilde{y} = x/y$ (after splitting the integral into four integrals over regions with a definite sign of $x$ and $y$ to make the coordinate transformation well-defined) and by making use of the characteristic properties of the delta sequence $\delta_n(\tilde{x})$. Alternatively, the relation (15) may also be obtained by the means of [26] (cf. chapter III.4.5).

In a final step we may now rewrite the r.h.s. of Eq. (14) identically by switching to a new integration variable $C := (x^1)^2$ as follows:

$$\int_{\mathbb{R}^+} \frac{dC}{|f(C)|} |\psi_0(C)|^2 . \quad (16)$$

Note that $C$ may be interpreted also as the gauge invariant Casimir polynomial $C \equiv x^2$, expressed in the gauge $x^2 = x^3 = 0$ (we therefore use the same symbol). The result (16) for the norm of a physical state $|s\rangle$ represented by $\psi_0(C)$ is now a well-defined ($f$ was required to be nonvanishing), physically sensible (states are represented by functions of the only independent gauge invariant quantity $C$; the ambiguity in $f$ will be discussed shortly) positive definite inner product.

It is now in place to discuss changes that are induced by a more general choice for the function $f_1$ in $Z_1^a$. Had we chosen $f_1$ as an arbitrary nonvanishing function of $x^a$, $f_1 = f_1(x^a)$, all the steps leading to Eq. (14) may be repeated, we only need to replace $f((x^1)^2)$ by $f_1(x^1, 0, 0)$ in this formula. Now we arrive at the curious conclusion that only if $|f_1|$ is invariant under a change of sign of $x_1$, we obtain a physically acceptable inner product.\footnote{This is a necessary and sufficient condition in the present ansatz for quantizing the}
otherwise, with a noninvariant function $f_1$, the *gauge equivalent* positive and negative half–axis of $x_1$ are weighted differently, which is incompatible with the principle of gauge invariance.

Restricting ourselves to the allowed subset of invariant functions $f_1$ or $f$ as discussed above, there is still the apparent ambiguity of the result (16) in choosing $f$, which may seem puzzling at first sight. To shed some light on this issue, we briefly illustrate the situation with the example of the model discussed in Sec. 2, turning it (artificially) into a reducible theory by counting the constraint $p_y = 0$ twice: $G_1 = G_2 = p_y$. This reducibility may be built in by means of the relation $f(x) (G_1 - G_2) = 0$ with some arbitrary nonvanishing function $f$. Adapting the steps of Sec. 2 to the reducible case, one finds that Eq. (16) becomes replaced by $\langle s|s \rangle \propto \int dx |\psi_0(x)|^2/|f(x)|$ — in complete analogy with (14).

The presence of the function $f$ in the measure poses no problem per se; after all, we may absorb it by redefining the wave functions $\psi_0(x): \psi_0 \rightarrow \psi := \psi_0/\sqrt{|f|}$. Such an ambiguity of the measure goes hand in hand with changes in the representation of the momentum operator: If, as usual, $p\psi = -i (d/dx) \psi$ for the standard measure $dx$, then it has to take the form $-i |f|^{1/2} (d/dx) |f|^{-1/2} = -i d/dx + i (\ln |f|)'(x)/2$ when applied to $\psi_0$. Moreover, it is only this latter expression that allows for hermiticity of $p$ in the space of square integrable functions with the measure $dx/|f(x)|$. With this definition of $p$ both representations of the quantum theory become unitarily equivalent and thus physical amplitudes are unaffected by the choice of $f$.

Now we return to the inner product defined by (16). We will show that also here the physical amplitudes are independent of the choice of $f$. The restriction to gauge invariant prefactors $f$ was just made for convenience, which would, in particular, simplify a similar discussion, when starting with the operator set $D_{(2)}$ instead of $D_{(1)}$ chosen for the present treatment.
and coincide with those in the gauge invariant approach of Dirac. As the \( SO(3) \)-model has one physical degree of freedom, we have to find one further independent gauge invariant observable beside the Casimir \( C = x^2 \).

Restricting ourselves to an observable that is at most linear in derivatives in the \( x \)-representation, the simplest hermitian choice is

\[
\mathcal{O} := \frac{1}{2} (xp + px) .
\]

(17)

Its commutation relation with \( C \) is \([C, \mathcal{O}] = 2iC\), forming an affine algebra. Note that \( \mathcal{O} \) is an algebraically well-defined object, while a gauge invariant canonical conjugate to \( C \) does not exist globally; such a conjugate operator would be \( \mathcal{O}/2C \), but \( C \) may have zeros or, on the operator level, it is not invertible.

To find the action of \( \mathcal{O} \) on the auxiliary state \( \psi_0(x) = \langle x|s \rangle_0 \), we have to apply this operator to \( \langle x|s \rangle \), on which the operator \( \mathcal{O} \) acts according to the definition (17) where \( p_j \) is represented by \(-i\partial/\partial x^j\), and pull it through the operator \( \exp (\mathcal{Q}, \Psi) \), cf. Eq. (5). Integrating out all variables except for \( x_1 \) or \( C \), respectively, a straightforward calculation yields

\[
\langle s|\mathcal{O}|s \rangle \propto \int_{\mathbb{R}^+} \frac{dC}{|f(C)|} \psi_0^*(C)(C P_C + P_C C) \psi_0(C)
\]

(18)

with \( P_C \equiv -i|f(C)|^{1/2} \frac{d}{dC}|f(C)|^{-1/2} \).

Thus, we observe that, first, the action of \( \mathcal{O} \) on \( \psi_0 \) depends on \( f \). Second, by construction \( \mathcal{O} \) still has the correct commutation relations with \( C \). Moreover, upon an appropriate choice of boundary conditions for the physical wave functions \( \psi_0(C) \), it is hermitian with respect to the effective inner product of Eq. (16). In complete analogy with the reducible version of the

\[\text{Recall that in this simple model the gauge group is compact (and of finite volume 4}\pi\text{) and thus the inner product of the original Dirac quantum space remains well-defined also for physical states.}\]
model with a translational gauge symmetry, one then easily establishes that all physical results are independent of the choice of $f$. The nontrivial representation of $O$ for a given function $f$ is essential in this context, however. Note also that all gauge invariant observables of the original theory may be expressed in terms of $C$ and $O$ and, up to usual operator ordering ambiguities, can be represented as operators of the theory defined on the positive real axis $C \in \mathbb{R}_+$. We finally want to specialize the result (16) to two particularly nice choices for $f$. For $f := 1$ the measure in the Casimir variable $C$ becomes trivial and the operator $P_C$ reduces simply to $-id/dC$. The choice $f(C) := 1/\sqrt{C}$ with the simultaneous change of variables to the “radial” coordinate $r := \sqrt{C}$ leads to the measure $\int_{\mathbb{R}_+} r^2 dr$, which one might favour as stemming from a spherical reduction of $\int d^3x$, while the operator $O$ may be shown to turn into $O = (rp_r + p_r r)/2$ where $p_r = -i\frac{d}{dr}r$ is the radial momentum operator, $p_r = (x \cdot p + p \cdot x)/2r$. Up to an irrelevant factor of $4\pi$, the latter measure is the one found in the Dirac quantization, thus proving unitary equivalence of the B&M result (16) and the covariant Dirac result as promised. Despite the similarity of the Gribov problem in both the models studied in this and the previous section, the topological obstruction to the global extension of the B&M procedure appears to be not that fatal in the present $SO(3)$–model as it was in the $SO(2)$–model. As already remarked in the Introduction, much of the motivation for studying further models in this paper is to find possible reasons for why the B&M procedure fails for the $SO(2)$–model while it works for the $SO(3)$–model. Moreover, in the latter case it worked only when some restrictions were placed on the functions characterizing the dependences of the constraints; we also want to find the analogous restrictions in more general models where the B&M procedure works despite the Gribov obstruction.
5 Mechanical models with a semisimple gauge group in the adjoint representation

In this section we want to extend the considerations of the previous section to arbitrary semisimple gauge groups. For this purpose we interpret the action of $SO(3)$ on the three–dimensional configuration space $\mathbb{R}^3$ as the adjoint action of $SO(3)$ on its Lie algebra. To define the model we then only have to replace $SO(3)$ by a general compact semisimple Lie group $G$. In a way these models are $(0+1)$–dimensional nonabelian Yang–Mills theories, cf. [6, 27, 28] for a definition of these models on the Lagrangian level.

For pedagogical reasons the analysis is carried through in the detail for the $SU(3)$–model first (Sec. 5.1). The generalization to arbitrary $G$ is then straightforward and contained in Sec. 5.2.

5.1 The $SU(3)$–model

Let the $3 \times 3$ matrices $\tau_a$, $a = 1, \ldots, 8$, be the generators of the $su(3)$ Lie algebra satisfying $[\tau_a, \tau_b] = i f_{abc} \tau^c$. The generators $\tau_1$ and $\tau_2$ are chosen to be diagonal. They generate the Cartan subalgebra of $su(3)$. Within our conventions the Cartan–Killing metric is trivial, $g_{ab} = \text{tr}(\tau_a \tau_b) = \delta_{ab}$, and the totally symmetric (ad–)invariant tensor $d_{abc}$ is defined by

$$\{\tau_a, \tau_b\} = \frac{2}{3} \delta_{ab} \mathbf{I}_3 + d_{abc} \tau^c. \quad (19)$$

The configuration space coincides with the Lie algebra itself, and the physical motion is subject to eight first–class constraints: $G_a \equiv f_{abc} x^b p_c \approx 0$.

The constraints are not independent from one another and satisfy the relations $Z_A^a G_a = 0$, $A = 1, 2$, making the model reducible. The functions $Z_A^a$ are chosen to be

$$Z_1^a = f_1(C_1, C_2) x^a, \quad Z_2^a = f_2(C_1, C_2) d_{bc} x^b x^c, \quad (20)$$
where \( f_1 \) and \( f_2 \) are arbitrary nonvanishing functions of the two independent invariant (Casimir) polynomials
\[
C_1 = \delta_{ab} x^a x^b \quad \text{and} \quad C_2 = d_{abc} x^a x^b x^c.
\] (21)
Again at this point we could allow for arbitrary nonvanishing functions \( f_A \) on the configuration space. However, only a subset of these, containing the gauge invariant functions chosen above, will provide a reasonable inner product in the end. We will discuss this issue further below.

It is readily seen that \( Z_1^a \) provides a dependence among the \( G_a \). For \( Z_2^a \) this follows from the relation \( f_{(a|b}^c d_{c|de)} = 0 \), where \( (a| \cdots |de) \) means symmetrization with respect to the indices \( a, d, e \). Alternatively, the relations \( Z_A^a G_a = 0 \) may be inferred from the \( \text{ad} \)-invariance of the Casimir polynomials \( C_A \).

The existence of two relations amongst the eight gauge constraints implies six dimensional gauge orbits, leaving two physical degrees of freedom. The BRST charge \( Q \) has the same structure as for the \( SO(3) \)-model:
\[
Q = C_0^a \pi_a + \frac{1}{2} f_{ab}^c C_0^a C_0^b \mathcal{P}_{0c} + i C_1^A Z^a_A \mathcal{P}_{0a} + \pi_{0a} \mathcal{P}_0^a + \pi_{1A} \mathcal{P}_1^A + \pi_{11}^1 \mathcal{P}_{11}^1.
\] (22)
The extended phase space of this model is similar to the one in the preceding model, just with more variables. There are 42 pairs of canonically conjugate variables now, only two of which represent physical degrees of freedom. The rest of the phase space comprises both 20 bosonic and fermionic unphysical degrees of freedom.

Performing the B&M procedure along the lines already explained above, one obtains for the scalar product of two physical states after some tedious computation
\[
\langle s|s' \rangle \propto \int d x_1 d x_2 \frac{|x_1|}{|f_1(u,v)|} \frac{|3 x_2^2 - x_1^2|}{|f_2(u,v)|} \psi_0^*(u,v) \psi_0'(u,v).
\] (23)
Here we used the abbreviations \( u \equiv x_1^2 + x_2^2 \) and \( v \equiv x_2 (3x_1^2 - x_2^2) \), which respectively equal the Casimirs \( C_1 \) and \( C_2 \) in the gauge \( x_i = 0, i = 3, \ldots, 8 \) chosen to construct the gauge fixing fermion.

Let us remark first of all that the appearance of absolute value signs around \( x_1 \) and \( 3x_2^2 - x_1^2 \) seems quite noteworthy to us. Irrespective of the fact that anyway a scalar product necessarily has to be positive, without these absolute value signs the inner product would vanish identically! The reason is that the Casimir functions \( u \) and \( v \) exhibit some residual gauge invariance, known as the Weyl group \( W \) (more on this below). E.g. they are obviously invariant under \( x_1 \to -x_1 \).

A similar situation was encountered in the previous section: In contrast to the \( so(2) \)-model, in the \( so(3) \) case the \( x_1 \) in the measure appeared as absolute value, ensuring nonvanishing of the inner product. The symmetry \( x_1 \to -x_1 \) related precisely those points on the gauge fixing surface \( x_2 = 0 = x_3 \) which were still gauge equivalent.

We are thus led to study the residual gauge invariance in the \((x_1, x_2)\)-plane in the gauge \( x_i = 0, i = 3, \ldots, 8 \). As illustrated in the left hand side of Fig. 1 a generic point has five gauge equivalent “Gribov copies”. The six gauge equivalent points may be related to one another by (multiple) reflections with respect to the lines \( x_1 = 0 \) and \( x_2 = x_1 / \sqrt{3} \). This \( \mathbb{Z}_6 \) is known as the Weyl group \( W \) of \( su(3) \). Due to the absolute value signs the measure is invariant under the full group \( W \) and the inner product does not vanish due to this symmetry.

The analogy with \( so(3) \) goes even further: Similarly to the transition from Eq. (14) to Eq. (16), also the right hand side of Eq. (23) can be expressed as an integral over the Casimirs only, namely as

\[
\int_{\text{Im}K^+} \frac{dC_1}{|f_1(C_1, C_2)|} \frac{dC_2}{|f_2(C_1, C_2)|} \psi^*(C_1, C_2) \psi_0'(C_1, C_2) . \tag{24}
\]
Here the Casimir polynomials $C_1$ and $C_2$ first arise in the change of variables from $(x_1, x_2)$ to $(C_1 := u(x_1, x_2), C_2 := v(x_1, x_2))$, but may also be identified with the Casimirs (21) of the original gauge invariant formulation of the theory. Due to the Weyl invariance of the functions $u$ and $v$, the map to the new coordinates is not bijective: Each of the six modular domains, one representative of which we denote by $K^+$, is mapped to one and the same region in the $(u, v)$–plane, Im $K^+$ (cf. hatched regions in Fig. 1).

![Fig.1: The left hand side of the picture shows the residual gauge freedom in the case of su(3): The six dots are gauge equivalent. Upon the transition to Casimir coordinates they are all mapped to the one point on the right hand side of the picture.](image)

We remark that the functional determinant of the map from $(x_1, x_2)$ to $(u, v)$, or $(C_1, C_2)$, includes a factor of $(1/6)$, which cancels precisely against the degree of the map (the number of Gribov copies). As we will clarify in subsequent sections (cf. in particular Sec. 7), this feature is rather accidental and not characteristic for the B&N procedure. By a different normalization of the Casimirs there is no factor of $(1/6)$ anymore and only a multiplicative
factor equal to the number of Gribov copies remains. If this factor diverges, the B&M inner product diverges as well, as will become most transparent in studying the helix model in Sec. 7 below.

As for the $SO(3)$–model, one can supplement the operators $C_{1,2}$ by two additional observables $O_1 = p_a x_a$ and $O_2 = d_{abc} p_a x_b x_c$ (hermitized appropriately) and prove that the physical amplitudes do not depend on the choice of the functions $f_1$ and $f_2$.

Also now we are in the position to analyze modifications that occur when replacing the functions $f_1, f_2$ of the Casimir coordinates in Eq. (20) by arbitrary nonvanishing functions of the Lie algebra coordinates $x^a$. As before, the transition from Eq. (23), where now the arguments of $f_1$ and $f_2$ are replaced by $(x^1, x^2, 0, 0, 0, 0, 0, 0)$, to Eq. (24) will no more be possible in general. A necessary condition for this transition is that $f_1$ and $f_2$ are invariant with respect to the residual gauge freedom $W$ left by our choice $x^3 = \ldots = x^8 = 0$. Otherwise gauge equivalent sectors in the $(x^1, x^2)$–plane would receive different weights, yielding an unacceptable inner product.

Still the above condition in the freedom of choosing $Z^a_A$ depends on the gauge. The necessary and sufficient gauge independent condition on the functions $f_1$ and $f_2$ is that they are (nonvanishing) functions of the Casimir polynomials $C_1, C_2$ only, as in our original ansatz in Eq. (20). Reformulating this condition directly for the vectors $Z^a_A$ (instead of just for the functions $f_1, f_2$ defined through Eq. (20)), one obtains that the most general form of these vectors that produces a well–defined and acceptable inner product within the B&M version presented here is:

$$Z^a_A = f_A(C_B) \frac{\partial F_A(C_B)}{\partial x^a}, \quad (25)$$

where the $f_A$ are nonvanishing functions of the Casimir polynomials in Eq. (21) and $\det(\partial F_A(C_B)/\partial x^a) \neq 0$. Actually, this parameterization of the $Z^a_A$
results from our previous ansatz (20) by a change of coordinates in the space of Casimirs from $C_A$ to $F_A(C_B)$.

5.2 Generalization to arbitrary semisimple groups

The $SU(3)$–model studied in detail in the previous subsection can be generalized to models of point particles transforming in the adjoint representation of arbitrary semisimple Lie algebras $g$. We will see that increasing the number of physical degrees of freedom does not affect the conclusion of the previous section: the (appropriately refined) B&M inner product is well–defined for the reducible case despite the presence of a Gribov obstruction.

Now the variable $x = x^a \tau_a$, $a = 1, \ldots, d = \dim g$, takes values in a semisimple Lie algebra $g$. Here $\tau_a$ denotes a basis in $g$, where we choose the convention that the first $r = \text{rank } g$ generators span a Cartan subalgebra $H$ of $g$: $\tau_a = (\tau_{\mu}, \tau_i)$, $\mu = 1, \ldots, r$; $i = r + 1, \ldots, d$. The variable $x$ transforms according to the adjoint action of the respective Lie group $G$. This action is generated by the $d$ first–class constraints $G_a = f_{abc} x^b p_c$ with $f_{abc}$ being the structure constants of $g$ and $p_a$ the momenta canonically conjugate to $x^a$.

The constraints fulfill $r = \dim H$ independent relations $Z^a_A G_a = 0$, $A = 1, \ldots, r$. The functions $Z^a_A$ read

$$Z^a_A = f_A C_{A, a_{b_2 \ldots b_{d(A)}}}^a x^{b_2} \ldots x^{b_{d(A)}}. \quad (26)$$

Here $C_{A, a_1 \ldots a_{d(A)}}$ denote $r$ ad–invariant, symmetric, irreducible tensors of rank $d(A)$ on the Lie algebra and the $f_A$ are $r$ arbitrary nonvanishing functions of the Casimir polynomials

$$C_A = C_{A, a_1 \ldots a_{d(A)}} x^{a_1} \ldots x^{a_{d(A)}}. \quad (27)$$

For every semisimple group there is a polynomial of second order, $d(1) = 2$. For groups of rank 2, the degree of the second invariant polynomial is $d(2) = \ldots$
3, 4 and 6 for $SU(3)$, $SO(4)$ and $G_2$, respectively. The degrees of the Casimir polynomials for groups of higher ranks can be found, e.g., in [29]. We remark also that there is no sum over the index $A$ in the right hand side of Eq. (26).

As in the previous subsection the relations $Z_A^a G_a = 0$ follow directly from the ad–invariance of the symmetric tensors $C_A; a_1 \ldots a_d(A)$. Our choice (26) of the $Z_A^a$ is obtained from the general ansatz of Eq. (25) by setting $F_A := C_A$ and the restriction of the $f_A$ to depend only on the $C_A$ is justified by the same reasoning as in the $SO(3)$– and the $SU(3)$–model.

The BRST–charge $Q$ is given by Eq. (22), with the indices running over the appropriate ranges now. Performing the B&M procedure in the gauge $x^i = 0$, $i = r + 1, \ldots, d$, one obtains for the inner product of two states

$$
\langle s \mid s' \rangle \propto \int_{H \times \mathbb{R}^r} dx^\mu d\tilde{C}_{1\mu} \psi^*(C_1(x), \ldots, C_r(x)) \left( \prod_{\alpha > 0} \alpha \cdot x \right)^2 \times \prod_{A=1}^r \delta(\tilde{C}_{1\mu} f_A C'_A; \nu_{2 \ldots d(A)} x^{\nu_{2 \ldots d(A)}}) \psi'(C_1(x), \ldots, C_r(x)).
$$

Here $\alpha > 0$ are positive roots of $g$. They entered the calculation through the structure constants present in the BRST charge, which have the form $f_{\mu \alpha} = \delta^\alpha_\alpha \mu$, $\mu = 1, \ldots, r$, if we assume $\tau^a = (\tau^\mu, \tau^\alpha)$ to be the Cartan–Weyl basis [29, 30]. However, the result does not depend on the choice of the basis. The quantity $C_{A; \mu_1 \ldots \mu_d(A)}$ is the pullback of the respective Casimir tensor $C_{A; a_1 \ldots a_d(A)}$ under the embedding of the chosen Cartan subalgebra $H$ (with coordinates $x^\mu$) into the Lie algebra $g$. Note that as the Casimirs are ad–invariant, the tensors on $H$ are independent of the embedding of $H$ into $g$, since any two Cartan subalgebras within $g$ are related to one other by an adjoint transformation. Also Eq. (28) is independent of the specific choice of the ad–invariant tensors on $g$: A redefinition $C_{A; a_1 \ldots a_d(A)}$ by $C_{A; a_1 \ldots a_d(A)} + C_{B; (b_1 \ldots b_d(B)) C_{C; c_1 \ldots c_d(C)}}$, where $d(A) = d(B) + d(C)$ and the bracket indicates symmetrization over the smaller case indices, is easily seen.
to have no effect.

With an appropriate normalization of the structure constants and irreducible invariant symmetrical tensors, the following relation holds

$$\prod_{\alpha > 0} (\alpha \cdot x) = \text{det} \left( C_{\alpha;\mu\mu_2...\mu_d(\alpha)} x^{\mu_2} ... x^{\mu_d(\alpha)} \right),$$  \hspace{1cm} (29)

where the determinant is taken with respect to the two free indices $A$ and $\mu$, both of which range from one to $r$. We now substitute (29) into (28), make use of the multidimensional generalization of Eq. (15),

$$\int dx dy \prod_{m} \delta(a_{mn}(x) y_n) (\text{det} a_{mn}(x))^2 \varphi(x) = \int dx |\text{det} a_{mn}(x)| \varphi(x),$$

and then change the integration variables from $x^\mu$ to $C_A$. This yields the generalization

$$\int_{\text{Im} K^+} \prod_{A=1}^r \frac{dC_A}{f_A(C_1, \ldots, C_r)} \psi^*(C_1, \ldots, C_r) \psi'(C_1, \ldots, C_r)$$ \hspace{1cm} (31)

of formula (24) for the right hand side of Eq. (28).

Like in the $SO(3)-$ and $SU(3)-$case, imposing $x^{r+1} = \ldots = x^d = 0$ does not fix the gauge completely, but leaves some discrete residual gauge freedom. In the context of Lie algebras the above “gauge fixing” corresponds to a projection of the Lie algebra to some representative of the respective Cartan subalgebra $H$ while the residual gauge freedom is identified with the Weyl group $W$ \cite{27, 6}. The Weyl group consists of elements that are obtained by all inequivalent compositions of reflections in the hyperplanes orthogonal to simple roots of the Lie algebra. A modular domain of $W$ on $H$ is called Weyl chamber $K^+ = H/W$. A possible representative of $K^+$ is $K^+ = \{x \in H | (\alpha \cdot x) > 0 \ \forall \alpha > 0\}$. For the special case of $SU(3)$ $W$ is generated by \( \hat{s}_1 : (x_1, x_2) \to (-x_1, x_2) \) and \( \hat{s}_2 : (x_1, x_2) \to \left( \frac{1}{2} \sqrt{3} x_2 - x_1, \frac{1}{2} [-x_2 - \sqrt{3} x_1] \right) \)
and \( K_+ \) may be identified with a sector of angle \( \pi/6 \) in the two–plane, cf. Fig. 1 of the previous subsection.

The reduction of the integration domain from the Cartan subalgebra \( H \) to the Weyl chamber \( K^+ \), performed implicitly as one of the steps in bringing (29) into the form (31), is possible\(^{11}\) since the number \( N_W \) of modular domains or of Gribov copies is finite. In fact, as Fig. 1 illustrates for the case of \( SU(3) \), in this reduction the number \( N_W \), which equals six in the particular case of Fig. 1 but in general may be identified with \( d - r \), \( d \equiv \dim G \), appears as a multiplicative factor to the inner product. Given our normalization of the latter polynomials in Eq. (27), this number drops out from the final result (31) due to an exact cancelation\(^{12}\) with the Jacobian of the map \( x^\mu \rightarrow C_A(x^\mu) \), performed in a subsequent step. However, if the number of Gribov copies were infinite, such a subsequent step would be impossible. Indeed, in Sec. 7 we will verify explicitly by means of an example with an infinite number of Gribov copies that in such a case the B & M inner product becomes divergent.

Concluding, we observe that, up to this stage, all the results obtained from the B&M method coincide with those obtained from the Faddeev–Popov (FP) method (using the respectively same gauge conditions and the FP determinant being defined as mentioned in the first footnote in the Introduction). Indeed, for the \( SO(2) \)–model \( \det \{ x_2, l \} \equiv x^1 \) in coincidence with the measure found in formula (10). For the subsequent models, on the other hand, it is not difficult to convince oneself that the FP determinant is \((\prod_{\alpha > 0} \alpha \cdot x)^2\), evaluated in the gauge chosen. This is nonnegative (cf. also [27, 28] for details) and yields an inner product that coincides with the one

\(^{11}\)Besides the fact that the dependences were chosen in accordance with Eq. (25) cer-

\(^{12}\)Use \( N_W = \prod_{A=1}^r d(A) \).
found in Eq. (31) upon an appropriate choice of the functions $f_A$.

Thus the question arises, whether the B&M procedure works only in those cases where the FP method does (despite a Gribov obstruction). The models considered in the following will show that this is not the case. There are also theories in which the B&M method works despite the failure of the FP method.

6 Models in the fundamental representation

For the reducible models we have studied so far, the B&M procedure has provided us with a well-defined inner product for physical states in the BRST formalism, even in the presence of a Gribov obstruction and for any finite number of physical degrees of freedom. As just mentioned, these models exhibited the specific feature that the FP determinant is nonnegative in the gauge used to construct the inner product measure. Now we are going to demonstrate in Sec. 6.1 that this latter feature is not crucial for the existence of the global extension of the B&M inner product for reducible gauge models. However, when the reducibility is removed, as will be done in Sec. 6.2 by adding more degrees of freedom while keeping the gauge group fixed, the B&M inner product becomes ill-defined due the Gribov topological obstruction.

6.1 $SO(N)$–model in the fundamental representation

Here we study a point particle model with gauge group $SO(4)$ in the fundamental representation and its generalization to $SO(N)$. We have in mind to get a better understanding of the fact that the B&M procedure yields an ill-defined inner product in the case $N = 2$ and a well-defined one for $N = 3$. From simple spherical reduction, performed in the Dirac quantization
after restriction to (rotationally invariant) physical states, one would expect a measure \( r^{N-1}, \quad r^2 := x^2 \). Unfortunately, this measure cannot be obtained by naive application of the FP method. In the gauge \( x^2 = \ldots = x^N = 0 \) the FP determinant is \( (x^1)^{N-1} \). For even \( N \) this is not positive definite on \( \mathbb{R} \) and leads to a vanishing inner product for gauge invariant wave functions.

We begin with the discussion of the \( SO(4) \)–model which contains all essential features of the general one with gauge group \( SO(N) \).

Let the motion of a point particle in the configuration space \( \mathbb{R}^4 \) be subject to the constraints \( G_a = O_a^{ij} p_i x^j = 0, \quad i, j = 1, \ldots, 4; \quad a = 1, \ldots, 6 =: \Gamma_0 \) which are the angular momentum components in the eight–dimensional phase space.\(^{13} \) \( O_a^{ij} \) form a basis of real antisymmetric \( 4 \times 4 \)-matrices. The constraints are not independent, but fulfill four relations \( Z_a^A G_a = 0, \quad A = 1, \ldots, 4 =: \Gamma_1 \). The four 6-vectors \( Z_a^A \) are chosen to be linear in the configuration space variables \( x^i \). It is not hard to see that such a choice is always possible. Certainly, again it would be possible to multiply the vectors \( Z_a^A \) by nonvanishing gauge invariant functions \( f_A(r) \); for simplicity they are set to one in the following. The linearity of the functions \( Z_a^A \) ensures that they are defined on the whole configuration space, but, on the other hand, has the consequence, that the \( Z_a^A \) are not independent: they combine to zero via a relation \( Z_1^A Z_a^A = 0 \), where also \( Z_1^A \) may be chosen linear in the \( x^i \). It is easy to see that we have locally \( 6 - 4 + 1 = 3 =: \gamma_0 \) independent constraints, \( 4 - 1 = 3 =: \gamma_1 \) independent relations between the constraints, and \( 1 =: \Gamma_2 = \gamma_2 \) independent relation of second stage. Thus, in contrast to the other reducible gauge models discussed so far, the ghost–of–ghosts of higher rank must be introduced in order to describe the dependence of the functions \( Z_a^A \), called also null–eigenvectors of the constraints \(^{25} \).

\(^{13} \) \( \Gamma_0 \) as well as the subsequent \( \Gamma \)s and \( \gamma \)s are introduced for later convenience when we generalize to \( SO(N) \)
Following the general procedure proposed in [25], we obtain the nonminimal BRST–charge

\[ Q = C_0^a G_a - \frac{1}{2} f_{ab} c^a_c c^b_c P_{0c} + i c_1^A Z_1^a P_{0a} + C_2^1 Z_1^A P_{1A} + \]
\[ + \pi_{0a} \bar{P}_0^a + \pi_{1A} \bar{P}_1^A + \pi_{21} \bar{P}_2^1 + \pi_{14} \bar{P}_1^{1A} + \pi_{21} \bar{P}_2^{11} + \pi_{21}^{21} \bar{P}_2^{21}. \]

The first line of (33) is the standard expression for a reducible gauge model of rank two (i.e., with two stages of reducibility) in the minimal BRST approach. The second line contains the nonminimal sector of \( Q \). According to [25] we have several canonical pairs of unphysical Lagrange multiplier, antighost, and extraghost variables: The fermionic pairs \((C_{0a}, \bar{P}_0^a), (\lambda_1^A, \pi_{1A}), (\bar{C}_1^A, \bar{P}_1^A), (\lambda_2^{11}, \pi_{21}), (\bar{C}_2^{21}, \bar{P}_2^{21})\), together with the bosonic pairs \((\lambda_0^a, \pi_{0a}), (\bar{C}_1^A, \bar{P}_1^A), (\lambda_2^{11}, \pi_{21}), (\lambda_1^A, \pi_{1A}), (\bar{C}_2^{21}, \bar{P}_2^{21})\).

After choosing two consistent and convenient sets of hermitian operator doublets, the B&M procedure is straightforward but tedious. The auxiliary states come out to be gauge invariant states independent of the ghost degrees of freedom, i.e., \( \psi_0 = \psi_0(x^2) \). For the gauge \( x^2 = x^3 = x^4 = 0 \), one obtains the inner product as an integral over the real \( x \)-axis of two wave functions which depend on one variable \( x^2 = (x^1)^2 \). The resulting measure may be constructed along the following lines: for every locally independent constraint take a factor of \( x \), for every locally independent relation between the constraints a factor of \( |x|^{-1} \) (this stems from a \( \delta \)-function mechanism similar to the one in the previous sections), while one independent relation between the \( Z_{a0} \) gives rise to another factor of \( x \). So the measure is \( x^3 \cdot |x|^{-3} \cdot x = |x| \) which is positive definite on the entire real axis!

Let us now see, how this procedure can be generalized to the group \( SO(N) \). Here we have \( \Gamma_0 = \binom{N}{2} \) constraints \( G_{a0} = \mathcal{O}_{a0}^i p_i x^j = 0; \ i, j = 1, \ldots, N \). These fulfill \( \Gamma_1 = \binom{N}{3} \) relations \( Z_{a1}^{a2} G_{a0} = 0 \). The null–eigenvectors \( Z_{a1}^{a2} \) of the constraints are not independent and possess null–eigenvectors of
their own, which may also be linearly dependent, etc. The procedure goes on up to the \((N - 2)\)-nd stage where there are no more linearly dependent null–eigenvectors \([25]\). On the \(i\)th stage of reducibility we have \(\Gamma_i = \binom{N}{i+2}\) relations \(Z_{a_{i-1}}^a Z_{a_{i-2}}^a = 0\). All the \(Z\)s may be chosen to be linear in the configuration space variables \(x^i\). \(\Gamma_i\) is the smallest possible number for which the \(Z_{a_{i-1}}^a\) are well–defined on the whole configuration space. Out of the \(\Gamma_i\) relations \(\gamma_i = \sum_{k=i+2}^{N} (-1)^{i+k} \binom{N}{k} = \binom{N-1}{i+1}\) are locally independent. Especially, \(\gamma_0 = \binom{N-1}{1} = N - 1\) constraints are locally independent and so we have \(N - (N - 1) = 1\) physical degree of freedom, which may be identified with the radial coordinate.

Now a generalization of the B &M inner product measure of the \(SO(4)\)-model is straightforward. For every locally independent relation of even stage (these include the constraints \(G_a\)) the measure is provided with the factor \(x\) and for every locally independent relation of odd stage we have a factor of \(|x|^{-1}\). So one obtains for the measure

\[
\frac{x^{(N-1)_1 + (N-1)_3 + \ldots}}{|x|^{(N-1)_2 + (N-1)_4 + \ldots}} = \frac{x^{2N-2}}{|x|^{2N-2-1}} = |x|, \quad \forall N > 2.
\] (33)

More generally, multiplying the linear sets \(Z_{a_{i-1}}^a\) by nonvanishing gauge invariant (i.e. rotationally invariant) functions, the measure always takes the form \(|f(x^2)x|\) (provided \(N > 2\)). Any (nonvanishing) function \(f\) may be obtained in this way leading to unitarily equivalent quantum theories (cf. foregoing sections).

We see that the B &M procedure leads for all \(N > 2\) to a well–defined inner product. It does so not only for odd \(N\) like the FP method but also for even \(N\) where the latter failed. Moreover, the Dirac result \(x^{N-1}\) for the measure is reproduced upon the choice \(f(x^2) = |x^{N-2}|\).

Concerning our question why B &M works in the \(SO(3)\)-model but not in the \(SO(2)\)-model we are led to the following conclusion: The failure of the
B&M approach in the $SO(2)$–model was not due to the inherent failure of the FP method for all $SO(N)$–models with even $N$ in the fundamental representation. The reason for the failure is assumed to be the combination of the Gribov topological obstruction and the irreducibility of the constraint(s). We will study this question in the next subsection where the essential reducibility of the constraints in the $SO(3)$–model is lost by increasing the number of physical degrees of freedom.

6.2 Yang–Mills quantum mechanics

In [13] Batalin and Marnelius have shown that their approach is equivalent to the Faddeev–Popov procedure for models with irreducible constraint algebras. As an example we have considered a particle on the plane with the translational gauge symmetry and the $SO(2)$–model. In the presence of the Gribov obstruction, the Faddeev–Popov procedure suffers from nonpositivity of the FP determinant and the subsequent vanishing of the inner product for the conventional gauge fixing fermion. In models with a reducible constraint algebra the B&M procedure provided us with a mechanism to obtain a positive measure in the inner product, which enabled us, under the assumption of a finite number of Gribov copies, to construct a well–defined scalar product. The transition from a reducible model to an irreducible one can be made by adding more degrees of freedom subject to simultaneous gauge transformations, while keeping the gauge group fixed. In doing so, we observe that the positivity of the measure is lost, thus leading to an ill–defined inner product when the Gribov topological obstruction is present.

We illustrate the statement by means of the example of Yang–Mills mechanics. The model is obtained from the four–dimensional Yang–Mills field
theory by setting all the gauge potentials to be homogeneous in space. So, the configuration space consists of three copies of a Lie algebra. The gauge group acts in each copy of the Lie algebra simultaneously in the adjoint representation. We take the SO(3)–model discussed in Sec. 4 and add two more particles \( x_2 \) and \( x_3 \) to the first one \( x_1 \) (\( \equiv x \) from Sec. 4). Simultaneous rotations of the position vectors are generated by the constraint being the sum of all three angular momenta \( l = l_1 + l_2 + l_3 \approx 0 \). These are three irreducible constraints. The B&M treatment of this model is equivalent to the Faddeev–Popov approach. Indeed, in the gauge \( y_1 = z_1 = z_2 = 0 \) one obtains the inner product as the integral over the remaining six variables with the measure \( x_1^2 y_2 \), which is obviously not positive definite. The physical wave functions depend on the six Casimirs \( x_i \cdot x_j, \ 1 \leq i, j \leq 3 \) of the model in the gauge chosen. The model suffers from the Gribov obstruction because the gauge cannot be fixed completely \[31\]: Here we have four copies obtained by applying the discrete gauge transformations \((x_1, x_2) \rightarrow (-x_1, -x_2)\) and \( y_2 \rightarrow -y_2 \). The physical amplitudes vanish.

So, we conclude that in the presence of a Gribov obstruction (and in cases where the FP method fails) the reducibility of the constraints is crucial for the existence of the global extension of the B&M procedure. The point which is left and yet to be discussed is the effect of an infinite number of Gribov copies in the B&M inner product. We now turn to this issue.

7 The helix model

In this section we study a model in which the gauge orbits are (noncompact) helices \[16, 17, 9, 18\]. The configuration space of the model is a three–dimensional Euclidean space in which gauge transformations are generated by the constraint \( G = p_3 + x^1 p_2 - x^2 p_1 = 0 \). So they are simultaneous translations
along the third axis and SO(2)–rotations in the plane spanned by $x^1, x^2$. The purpose of studying this model is to see what happens to the B&M inner product if the number of Gribov copies becomes infinite or may even depend on the position on the gauge fixing surface.

Before we proceed, let us make a remark concerning the Gribov problem in the model. The topology of the gauge orbits in the model is that of the real line and thus trivial. There is no topological obstruction to find a unique single–valued gauge fixing condition. In fact, e.g. the plane $x^3 = 0$ intersects each gauge orbit precisely once. No Gribov ambiguity occurs in contrast to the models with topologically nontrivial gauge orbits studied above. So the Gribov problem here can be artificially created by a bad choice of the gauge. For example, with the choice $x^2 = 0$ we have infinitely many Gribov copies. Indeed, the plane $x^2 = 0$ intersects each helix winding around the third axis at the points related to one another by transformations $x^1 \rightarrow (-1)^n x^1$, $x^3 \rightarrow x^3 + \pi n$ with $n$ being any integer. The modular domain on the gauge fixing surface in configuration space is therefore a half–strip $x^1 \geq 0$, $x^3 \in [-\pi, \pi)$. Note also that the absence of a global topological obstruction allows one to construct a well–defined BRST scalar product in the helix model via the Fock space representation [18].

The physical amplitudes should not depend on the choice of the gauge. On the other hand, the B&M inner product explicitly depends on the BRST gauge fixing fermion. Thus, the independence of physical amplitudes from the gauge fixing fermion may turn out to be nontrivial to prove in the presence of a Gribov problem. We will see that the B&M inner product does not provide an interpolation between the two choices of the gauge with no and an infinite number of Gribov copies, respectively, thus leading to a general

\footnote{Such gauges are easy to find even in electrodynamics. — “Artificial reducibility” is to be contrasted with what we called essential reducibility in the Introduction.}
gauge dependence of the physical amplitudes.

The BRST–charge of the model is

\[ Q = GC + \pi \bar{P} \]  \hspace{1cm} (34)

The model is irreducible and so it is clear that we have two physical degrees of freedom. We first take the good gauge \( x^3 = 0 \). The corresponding sets of BRST–doublets read

\[ D_{(1)} = \{(x^3, C), (\bar{C}, \pi)\}, \quad D_{(2)} = \{(P, G), (\lambda, \bar{P})\} . \]  \hspace{1cm} (35)

From this the hermitian gauge fixing fermion is obtained

\[ \Psi = x^3 \bar{C}, \quad [Q, \Psi] = C \bar{C} + \pi x^3 . \]  \hspace{1cm} (36)

The auxiliary states are functions of the two Casimirs

\[ C_1 = x^1 \cos x^3 + x^2 \sin x^3 , \]  \hspace{1cm} (37)
\[ C_2 = x^1 \sin x^3 - x^2 \cos x^3 . \]  \hspace{1cm} (38)

The B&M scalar product is now easy to derive

\[ \langle s|s' \rangle \propto \int dx^1 dx^2 \psi^*(x^1, x^2) \psi'(x^1, x^2) . \]  \hspace{1cm} (39)

Here the arguments of the wave functions are the Casimirs in the gauge chosen. The scalar product is well–defined as has been expected since the model does not exhibit any topological obstruction and with \( x^3 = 0 \) a good choice of gauge was used.

Let us now calculate the inner product with the bad choice of gauge \( x^2 = 0 \). The sets of BRST–doublets are

\[ D_{(1)} = \{(x^2, x^1 C), (\bar{C}, \pi)\}, \quad D_{(2)} = \{(P, G), (\lambda, \bar{P})\} , \]  \hspace{1cm} (40)
which lead to the gauge fixing fermion
\[ \Psi = x^2 \bar{\mathcal{C}} , \quad [Q, \Psi] = x^1 \mathcal{C} \bar{\mathcal{C}} + \pi x^2 . \] (41)

The auxiliary states are given by gauge invariant functions as above. After simple algebraic computations we find the B&M inner product
\[ \langle s | s' \rangle \propto \int dx^1 dx^3 x^1 \psi^* (x^1 \cos x^3, x^1 \sin x^3) \psi'(x^1 \cos x^3, x^1 \sin x^3) . \] (42)

It is readily seen that due to the periodicity in \( x^3 \) of the integrand, the integral diverges. The periodicity is nothing but the residual gauge symmetry in the gauge chosen and the infinite factor occurring in physical amplitudes is simply related to the infinite number of Gribov copies.

One can take an interpolating gauge \( \xi x^2 + x^3 = 0 \). When the parameter \( \xi \) vanishes we have a good gauge condition without the Gribov problem and the B&M inner product is well–defined. The bad gauge is attained when \( \xi \) approaches infinity. Let \( \omega \) be a parameter of the gauge transformations. To obtain the residual gauge transformations in the gauge chosen, one has to find all nontrivial values of the parameter \( \omega \) for which the gauge transformed configurations belong to the gauge fixing surface. So we have to solve the system
\[ \xi x^2 + x^3 = 0 , \]
\[ \xi [x^2 \cos \omega - x^1 \sin \omega] + x^3 - \omega = 0 . \] (43)

For \( \xi \neq \infty \), the number of solutions of this system is finite and so is the number of Gribov copies. The sets of BRST–doublets in this gauge read
\[ D_{(1)} = \{ (\xi x^2 + x^3), (\xi x^1 + 1)\mathcal{C}), (\bar{\mathcal{C}}, \pi) \} , \quad D_{(2)} = \{ (\mathcal{P}, \mathcal{G}), (\lambda, \bar{\mathcal{P}}) \} . \] (44)

For the gauge fixing fermion we find
\[ \Psi = (\xi x^2 + x^3)\bar{\mathcal{C}} , \quad [Q, \Psi] = (\xi x^1 + 1)\mathcal{C} \bar{\mathcal{C}} + \pi(ax^2 + x^3) . \] (45)
Then the B&M scalar product becomes

$$\langle s|s' \rangle \propto \int dx^1 dx^2 (\xi x^1 + 1) \psi^* \psi' , \quad (46)$$

where $\psi, \psi'$ are functions of the Casimirs $C_1, C_2$ in the gauge $x^3 = -\xi x^2$:

$$C_1 = x^1 \cos(\xi x^2) + x^2 \sin(\xi x^2) , \quad C_2 = x^2 \cos(\xi x^2) - x^1 \sin(\xi x^2) . \quad (47)$$

The integration over the entire plane in (46) poses a problem. The physical states are labeled by the values of the Casimir functions. Since the range of values for the variables $x^1, x^2$ is the whole plane, there is no one-to-one correspondence between $(x^1, x^2)$ and $(C_1, C_2)$, as one may see from (47). For each pair $(C_1, C_2)$, one can find several pairs $(x^1, x^2)$ that satisfy (47). All these points on the plane are Gribov copies of one another. While the wave functions are invariant under the Gribov residual gauge transformations, the measure in (46) is not. Moreover the number of copies appears to be a function of the point on the gauge fixing surface. Therefore for all $\xi \neq 0$, we cannot simply factor out this number as in the reducible case, thus shrinking the integration domain in (46) to the modular domain (i.e., to the set of $x^1, x^2$ for which (47) is a one-to-one map).

So we conclude that an infinite number of Gribov copies leads, in general, to a divergence of the B&M inner product. In the irreducible case, moreover, the physical amplitude may be gauge dependent, if the Gribov problem exists in the gauge chosen to construct the measure of the inner product.

### 8 Conclusion and Outlook

We have seen that in the case of irreducible constraints that generate topologically nontrivial gauge orbits a naive global extension of the B&M inner product encounters substantial problems and, hence, requires a modification.
In the gauge models with reducible constraints the B&M construction may apply globally, provided the dependences are parameterized appropriately within the BRST operator (cf. the discussion in Secs. 4 and 5). It is expected that the positivity of the measure will not be sufficient for the global extension in the case of gauge field theories. The reason is that the number of copies is typically infinite in the physically interesting gauges. This infinite number appears as a factor in the B&M inner product. Though we have made this conclusion from the study of the helix model where, in fact, no topological obstruction to the unique gauge fixing exists, we expect it to be valid for the models where such an obstruction does exist.

An example for an infinite number of Gribov copies is provided already by a Yang–Mills theory on a two–dimensional cylindrical spacetime (space is compactified into a circle of length \( L \)) \[32, 33\]. The model has a finite number of physical degrees of freedom which equals the rank of the gauge group. They can be described by constant connections taking their values in the Cartan subalgebra. The residual gauge transformations that specify the gauge equivalent configurations in the Cartan subalgebra (Gribov copies) form the affine Weyl group \[33\]. The latter is a semi-direct product of the Weyl group \( W \) encountered already in Sec. 5 and the group of translations in the group unit lattice which consists of Cartan subalgebra elements whose exponential is the group unit. This additional gauge translational symmetry makes the modular domain compact. The modular domain lies in the Weyl chamber and is called the Weyl cell. For \( SU(3) \), the Weyl cell is an equilateral triangle. As the simplest example we consider \( SU(2) \). The affine Weyl group consists of reflections on the real line, \( x \rightarrow -x \), and translations, \( x \rightarrow x + 2nL \), where \( n \) is an integer. So the only independent Casimir function is \( C = \cos(\pi x/L) \). It is the character of a group element defined by the Polyakov loop in the fundamental representation in the gauge chosen. The number of
Gribov copies is infinite. We have seen that the B&M procedure does not provide us with a mechanism for the reduction of the integration domain to the modular domain, only the positivity of the measure can be expected. So the infinite number of copies should appear as a divergent factor in the physical amplitudes.

In the case of irreducible constraints there seem to be essentially two obstructions to the global extension of the B&M inner product. The first one is again a possible infinite number of copies and the second one is the noninvariance of the measure under residual (Gribov) transformations. The latter problem may lead to a gauge dependence of the physical amplitudes or even to their identical vanishing.

Though these conclusions may sound discouraging because in realistic models none of these obstructions seem easy to circumvent, we would like to stress that the possibility of unconventional gauge fixing fermions has not been explored in our work. In this respect we would like to mention recent works [34] where a modified BRST path integral for continuous and lattice gauge theories has been proposed to resolve the Gribov problem. It might be possible to make a similar modification of the B&M procedure to achieve its global extension.

The problem of the infinite number of copies in Yang–Mills theory can, in principle, be circumvented by imposing a gauge condition in the momentum space [35]. Since the momenta transform in the adjoint representation, we can use the gauge fixing procedure from Sec. 5 so that the number of copies would be finite. A rigorous study of this approach would require a lattice regularization of the theory in order to give a meaning to spatially local Weyl transformations, and this goes beyond the scope of this paper.

Finally, it is worth mentioning that in our paper we have addressed only kinematical aspects of the global extension of the B&M inner product.
Global obstructions in the BRST formalism occur not only at the kinematical level (constructing physical states and the proper inner product), but also on the dynamical level \[^3\]. This in turn may lead to additional restrictions (or conditions) on the existence of a global extension of the B&M inner product formalism.

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