THE DIFFERENTIAL-ALGEBRAIC AND BI-HAMILTONIAN INTEGRABILITY ANALYSIS OF THE RIEMANN TYPE HIERARCHY REVISITED

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Abstract. A differential-algebraic approach to studying the Lax type integrability of the generalized Riemann type hydrodynamic hierarchy is revisited, its new Lax type representation is constructed in exact form. The related bi-Hamiltonian integrability and compatible Poissonian structures of the generalized Riemann type hierarchy are also discussed by means of the gradient-holonomic and geometric methods.

1. Introduction

Recently new mathematical approaches, based on differential-algebraic and differential geometric methods and techniques, were applied in works \cite{1,15,17} for studying the Lax type integrability of nonlinear differential equations of Korteweg-de Vries and Riemann type. In particular, a great deal of analytical studies \cite{2,4,5,7,1,8} were devoted to finding the corresponding Lax-type representations of the infinite Riemann type hydrodynamical hierarchy

\begin{equation}
D_t^N u = 0, \quad D_t := \partial/\partial t + u_1 \partial/\partial x,
\end{equation}

where $N \in \mathbb{Z}_+$, $(x,t)^T \in \mathbb{R}^2$ and $u \in C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$. It was found that the related dynamical system

\begin{equation}
D_t u_1 = u_2, \ldots, D_t u_j = u_{j+1}, \ldots, D_t u_N = 0,
\end{equation}

defined on a $2\pi$-periodic infinite-dimensional smooth functional manifold $M_N \subset C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^N)$, possesses \cite{3,17} for an arbitrary integer $N \in \mathbb{Z}_+$ a suitable Lax type representation

\begin{equation}
D_x f = l_N[u; \lambda] f, \quad D_t f = q_N(\lambda) f
\end{equation}

with $\lambda \in \mathbb{C}$ being a complex spectral parameter and $f \in L^\infty(\mathbb{R}; \mathbb{C}^N)$ and matrices $l_N[u; \lambda], q_N(\lambda) \in \text{End}\mathbb{C}^2$. Here, by definition, $u_1 := u \in C^\infty(\mathbb{R}^2;\mathbb{R})$ and the differentiations

\begin{equation}
D_t := \partial/\partial t + u_1 \partial/\partial x, \quad D_x := \partial/\partial x
\end{equation}

satisfy on the manifold $M_N$ the following commutation relationship:

\begin{equation}[[D_x, D_t]] = (D_x u_1) D_x.
\end{equation}
In particular, for the cases \( N = 2, 3 \) and \( N = 4 \) the following exact matrix expressions

\[
I_2[u; \lambda] = \begin{pmatrix} \lambda u_{1,x} & u_{2,x} \\ -2\lambda^2 & -\lambda u_{1,x} \end{pmatrix}, \quad q_2(\lambda) = \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix},
\]

(1.6)

\[
I_3[u; \lambda] = \begin{pmatrix} \lambda^2 u_{1,x} & -\lambda u_{2,x} & u_{3,x} \\ 3\lambda^3 & -2\lambda^2 u_{1,x} & \lambda u_{3,x} \\ 6\lambda^4 \tau_3^{(1)}[u] & -3\lambda^3 & \lambda^2 u_{1,x} \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 \\ \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{pmatrix},
\]

\[
I_4[u; \lambda] = \begin{pmatrix} -\lambda^2 u_{1,x} & \lambda^2 u_{2,x} & -\lambda u_{4,x} & u_{5,x} \\ -4\lambda^3 & 3\lambda^3 u_{1,x} & -2\lambda^2 u_{2,x} & \lambda u_{5,x} \\ -10\lambda^5 \tau_4^{(1)}[u] & 6\lambda^4 & -3\lambda^3 u_{1,x} & \lambda^2 u_{5,x} \\ -20\lambda^6 \tau_4^{(2)}[u] & 10\lambda^5 \tau_4^{(1)}[u] & -4\lambda^4 & \lambda^3 u_{1,x} \end{pmatrix}, \quad q(\lambda) := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \end{pmatrix},
\]

(1.7)

These results are mainly based on a new differential-algebraic approach, devised recently in work [17] and contain complicated enough [3] functional expressions \( r_N^{(j)}, j = 1, N - 2 \), which satisfy the recurrent differential-functional equations

\[
\begin{align*}
D_1 r_N^{(1)} + r_N^{(1)} D_x u^{(1)} &= 1, \\
D_1 r_N^{(2)} + r_N^{(2)} D_x u^{(1)} &= r_N^{(1)}, \\
D_1 r_N^{(j+1)} + r_N^{(j+1)} D_x u^{(1)} &= r_N^{(j)}, & j &= 1, N - 3, \\
D_1 r_N^{(N-2)} + r_N^{(N-2)} D_x u^{(1)} &= r_N^{(N-3)}.
\end{align*}
\]

on the functional manifold \( M_N \). Recently Popowicz [4] has found a new Lax representation for the generalized Riemann equation \( \text{(1.2)} \) at arbitrary \( N \in \mathbb{Z}_+ \). In the present article, being strongly based on the methods devised in works [17, 3], we will revisit the differential-algebraic approach to the Riemann type hydrodynamical hierarchy \( \text{(1.1)} \) and construct its new Lax type representation

\[
I_N[u; \lambda] = \begin{pmatrix} \lambda u_{N-1,x} & u_{N,x} & 0 & \ldots & 0 \\ 0 & \lambda u_{N-1,x} & 2u_{N,x} & \ddots & \ldots \\ \ldots & \ldots & \ddots & \ddots & \ldots \\ 0 & \ldots & 0 & \lambda u_{N-1,x} & (N - 1)u_{N,x} \\ -N\lambda^N & -\lambda^{N-1}u_{1,x} & \ldots & -\lambda^2 Nu_{N-2,x} & \lambda(1 - N)u_{1,x} \end{pmatrix},
\]

(1.8)

\[
q_N(\lambda) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 0 \end{pmatrix},
\]

in a very simple and useful for applications form for any \( N \in \mathbb{Z}_+ \), which is scale equivalent to that found by Popowicz [4]. Moreover, we prove that this Riemann type hydrodynamical hierarchy generates the bi-Hamiltonian flows on the manifold \( M_N \) and, finally, we analyze for the cases \( N = 2, 3 \) and \( N = 4 \) the corresponding compatible [23, 25, 27] Poissonian structures, following within the gradient-holonomic scheme from these new Lax type representations. A mathematical nature of the Lax type representations \( \text{(1.6), (1.7)} \) presents from the differential-algebraic point of view a very interesting question, an answer to which may be useful for the integrability theory, remains open and needs additional investigations.
It is also worth to mention that the methods devised in this work can be successfully applied to other interesting for applications \[16\,20\,21\,19\,22\] nonlinear dynamical systems, such as Burgers, Korteweg-de Vries and Ostrovsky-Vakhnenko equations

\[
D_tu = D_x^2u, \quad D_tu = D_x^3u, \quad D_xD_tu = -u,
\]
on the 2π-periodic functional manifold \(M_1 \subset C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R})\), new infinite hierarchies \([6, 9]\) of Riemann type hydrodynamic systems

\[
D_t^{N-1}u = D_x^{4-\bar{z}}, \quad D_t\bar{z} = 0,
\]
and

\[
D_t^{N-1}u = \bar{z}^2, \quad D_t\bar{z} = 0
\]
on a smooth 2π-periodic functional manifold \(M_N \subset C^{(\infty)}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^N)\) for \(N \in \mathbb{N}\) and many others.

2. Differential-algebraic approach revisiting

We will consider the ring \(K := \mathbb{R}\{\{x, t\}\}, (x, t)^t \in \mathbb{R}^2\), of the convergent germs of real-valued smooth functions from \(C^{(\infty)}(\mathbb{R}^2; \mathbb{R})\) and construct \([10\,11\,12\,13\,14]\) the associated differential polynomial ring \(K\{u\} := K[\Theta u]\) with respect to a functional variable \(u_1 := u \in \mathcal{K}\), where \(\Theta\) denotes the standard monoid of the all commuting differentiations \(\partial/\partial x\) and \(\partial/\partial t\). The ideal \(I\{u\} \subset K\{u\}\) is called differential if the condition \(I\{u\} = \Theta I\{u\}\) holds.

In the differential ring \(K\{u\}\), interpreted as an invariant differential ideal in \(\mathcal{K}\), there are naturally defined two differentiations

\[
D_t, \ D_x : \mathcal{K}\{u\} \to \mathcal{K}\{u\},
\]
satisfying the Lie-commutator relationship \([15]\). For a general function \(u \in \mathcal{K}\) there exists the only representation of \([1.5]\) in the ideal \(\mathcal{K}\{u\}\) of form \([1.4]\). Nonetheless, if the additional constraint \([1.1]\) holds, its linear finite dimensional matrix representation in the corresponding functional vector space \(K\{u\}^N\) for \(N \in \mathbb{Z}_+\), related with some finitely generated invariant differential ideal \(\mathcal{I}\{u\} \subset \mathcal{K}\{u\}\), may exists being thereby equivalent to the corresponding Lax type representation for the Riemann type dynamical system \([1.2]\). To make this scheme analytically feasible, we consider in detail the cases \(N = 1, 4\) and construct the corresponding linear finite dimensional matrix representations of the Lie-commutator relationship \([1.5]\), polynomially depending on an arbitrary spectral parameter \(\lambda \in \mathbb{C}\).

2.1. The case \(N = 1\). Aiming to find the corresponding representation vector space for the Lie-algebraic relationship \([1.5]\) at \(N = 1\), we need firstly to construct \([17]\) a so called invariant generating Riemann differential ideal \(R\{u\} \subset \mathcal{K}\{u\}\) as

\[
R\{u\} := \{ \sum_{j \in \mathbb{Z}_+} \sum_{n \in \mathbb{Z}_+} \lambda^{-(n+j)}f_n^{(j+1)}D_t^jD_x^n u \in R\{u\} : f_n^{(j+1)} \in \mathcal{K}, \ j, n \in \mathbb{Z}_+ \},
\]
where \(\lambda \in \mathbb{R}\setminus\{0\}\) is an arbitrary parameter. The differential ideal \([2.2]\) is, evidently, invariant and characterized by the following lemma.

**Lemma 2.1.** The kernel \(\text{Ker}D_t \subset R\{u\}\) of the differentiation \(D_t : \mathcal{K}\{u\} \to \mathcal{K}\{u\}\) is generated by elements \(f^{(j)} \in \mathbb{R}\{\{x, t\}\}, j \in \mathbb{Z}_+\), satisfying the linear differential-functional relationships

\[
D_tf^{(j+1)} = -\lambda f^{(j)},
\]
where, by definition,

\[
f^{(j+1)} := f^{(j+1)}(\lambda) = \sum_{n \in \mathbb{Z}_+} f_n^{(j+1)}\lambda^{-n}
\]
for \(\lambda \in \mathbb{R}\setminus\{0\}\) and \(j \in \mathbb{Z}_+\).
Taking now into account the invariant reduction of the differential ideal \( \{2.2\} \) subject to the condition \( D_t u = 0 \) to the ideal
\[
\mathcal{R}^{(1)} \{ u \} := \left\{ \sum_{n \in \mathbb{Z}_+} \lambda^{-n} f^{(1)}_n D_x^n u \in \mathcal{R} \{ u \} : D_t u = 0, \ f^{(1)}_n \in \mathcal{K}, \ n \in \mathbb{Z}_+ \right\}
\]
one can simultaneously reduce the hierarchy of relationships \( \{2.3\} \) to the simple expression
\[
\{2.6\} \quad D_t f^{(1)}(\lambda) = 0,
\]
where \( f^{(1)} \in \mathcal{K}_1 \{ u \} := \mathcal{K} \{ u \} |_{D_t u = 0} \). Now, to formulate the Lax type integrability condition for the case \( N = 1 \), it is necessary to construct the corresponding \( D_t \)-invariant Lax differential ideal
\[
\{2.7\} \quad \mathcal{L}^{(1)} \{ u \} := \left\{ g_1 f^{(1)}(\lambda) \in \mathcal{K}_1 \{ u \} : D_t f^{(1)}(\lambda) = 0, \ g_1 \in \mathcal{K} \right\}
\]
and to check its invariance subject to the differentiation \( D_x : \mathcal{K}_1 \{ u \} \to \mathcal{K}_1 \{ u \} \).

Lemma 2.2. The Lax differential ideal \( \{2.7\} \) is \( D_x \)-invariant, if the linear equality
\[
\{2.8\} \quad D_x f^{(1)}(\lambda) = (\lambda u_x + \mu u_x^{-2} u_{xx}) f^{(1)}(\lambda)
\]
holds for arbitrary parameters \( \lambda, \mu \in \mathbb{R} \), where the subscript "\( x \)" means the usual \( D_x \)-differentiation.

Proof. The condition \( \{2.8\} \) makes the Lax ideal \( \{2.7\} \) to be also \( D_x \)-invariant if the following condition on the linear representation of the \( D_x \)-differentiation
\[
\{2.9\} \quad D_x f^{(1)}(\lambda) = l_1[u; \lambda] f^{(1)}(\lambda)
\]
holds:
\[
\{2.10\} \quad D_t l_1[u; \lambda] + l_1[u; \lambda] D_x u_1 = 0
\]
for any parameter \( \lambda \in \mathbb{R} \). Differential-functional equation \( \{2.10\} \) upon substitution
\[
\{2.11\} \quad l_1[u; \lambda] := D_x a_1[u; \lambda]
\]
reduces easily to the equation
\[
\{2.12\} \quad D_t a_1[u; \lambda] = \lambda
\]
for any parameter \( \lambda \in \mathbb{R} \). Taking into account the condition \( D_t u_1 = 0 \) one can easily find that
\[
\{2.13\} \quad a_1[u; \lambda] = \lambda x + \mu / D_x u_1,
\]
where \( \mu \in \mathbb{R} \) is arbitrary. Having now substituted expression \( \{2.13\} \) into \( \{2.11\} \) and taking into account that \( D_t x = u_1 \), the result \( \{2.8\} \) follows. \( \Box \)

Thereby, having naturally extended the differential ring \( \mathcal{K} \) to the ring \( \mathbb{C} \{ \{ x, t \} \} \), one can formulate the following proposition.

Proposition 2.3. The Lax type representation for the Riemann hydrodynamical equation
\[
\{2.14\} \quad D_t u = 0
\]
is given by a set of the linear compatible equations
\[
\{2.15\} \quad D_t f^{(1)}(\lambda) = 0, \ D_x f^{(1)}(\lambda) = (\lambda u_x + \mu u_x^{-2} u_{xx}) f^{(1)}(\lambda)
\]
for \( f^{(1)}(\lambda) \in \mathcal{L}_\infty(\mathbb{R}^2; \mathbb{C}) \) and for any \( \lambda, \mu \in \mathbb{C} \).
2.2. The case $N = 2$. For the case $N = 2$ the respectively reduced invariant Riemann differential ideal (2.2) is given by the set

$$R^{(2)}\{u\} := \{ \sum_{j=0}^{n} \sum f_{n}^{(j+1)} f_{n}^{(j+2)} + D_{t}^{2} u : D_{t}^{2} u = 0, \}$$

(2.16)

and characterized by the kernel $\text{Ker} D_{t} \subset R^{(2)}\{u\}$ of the $D_{t}$-differentiation, which is generated by the following linear differential relationships:

$$D_{t} f^{(1)}(\lambda) = 0, \quad D_{t} f^{(2)}(\lambda) = -\lambda f^{(1)}(\lambda),$$

(2.17)

where $f^{(j)}(\lambda) \in K_{2}\{u\} := K\{u\}|_{D_{t}^{2} u = 0}, j = \overline{0,1}$, and $\lambda \in \mathbb{R}$ is an arbitrary parameter.

The condition (2.17) can be rewritten in a compact vector form as

$$D_{t} f(\lambda) = q_{2}(\lambda) f(\lambda), \quad q_{2}(\lambda) := \begin{pmatrix} 0 & 0 \\ \lambda & 0 \end{pmatrix},$$

(2.18)

where $f(\lambda) := (f^{(1)}(\lambda), f^{(2)}(\lambda))^T \in K_{2}\{u\}^{2}$.

Now we can construct the invariant Lax differential ideal

$$L^{(2)}\{u\} := \{ \sum_{j=1}^{2} g_{j} f^{(j)}(\lambda) \in K_{2}\{u\} : D_{t} f^{(1)}(\lambda) = 0, \}$$

(2.19)

where $f^{(j)}(\lambda) \in K_{2}\{u\} := K\{u\}|_{D_{t}^{2} u = 0}, j = \overline{1,2}$,

and to check its invariance subject to a linear representation of the $D_{x}$-differentiation in the form:

$$D_{x} f(\lambda) = l_{2}[u; \lambda] f(\lambda)$$

(2.20)

for some matrix $l_{2}[u; \lambda] \in \text{End} \mathbb{R}^{2}$.

The following lemma holds.

**Lemma 2.4.** The Lax differential ideal (2.19) is $D_{x}$-invariant if the matrix

$$l_{2}[u; \lambda] = \begin{pmatrix} \lambda u_{1,x} & u_{2,x} \\ -2\lambda^{2} & -\lambda u_{1,x} \end{pmatrix}.$$ 

(2.21)

**Proof.** The $D_{x}$-invariant condition for the Lax ideal (2.19) forces the matrix $l_{2}[u; \lambda] \in \text{End} \mathbb{R}^{2}$ to satisfy the differential-functional relationship

$$D_{t} l_{2} + l_{2} D_{x} u_{1} = [q_{2}, l_{2}],$$

(2.22)

where we have put by definition $l_{2} := l_{2}[u; \lambda], q_{2} := q_{2}(\lambda) \in \text{End} \mathbb{R}^{2}$. Making the substitution

$$l_{2} = D_{x} a_{2}$$

(2.23)

for some matrix $a_{2} := a_{2}[u; \lambda] \in \text{End} \mathbb{R}^{2}$, we can easily reduce equation (2.22) to the equivalent one in the form

$$D_{t} a_{2} = [q_{2}, a_{2}],$$

(2.24)

To solve equation (2.24) it is useful to take into account that

$$D_{t}^{2} a_{2} = k_{2} q_{2}, \quad D_{t}^{3} a_{2} = 0$$

(2.25)

for some constant $k_{2} \in \mathbb{R}$. Taking into account the relationship (2.24) one easily obtains the exact matrix representation

$$a_{2} = \begin{pmatrix} \lambda u_{1} & u_{2} \\ -2\lambda^{2} & -\lambda u_{1} \end{pmatrix},$$

(2.26)

entailing the result (2.21) $\square$.

Thus, having as above extended the differential ring $K$ to the ring $\mathbb{C}\{x,t\}$, one can formulate the next proposition.
Proposition 2.5. The Lax representation for the Riemann type hydrodynamical system

\( D_t u_1 = u_2, \quad D_t u_2 = 0 \)

is given by a set of the linear compatible equations

\[
D_t f(\lambda) = q_2(\lambda) := \begin{pmatrix} 0 & 0 \\ -\lambda & 0 \end{pmatrix} f(\lambda), \quad D_x f(\lambda) = \begin{pmatrix} \lambda u_{1,x} & u_{2,x} \\ -2\lambda^2 & -\lambda u_{1,x} \end{pmatrix} f(\lambda)
\]

for \( f(\lambda) \in L_\infty(\mathbb{R}^2; \mathbb{C}^2) \) and arbitrary complex parameter \( \lambda \in \mathbb{C} \).

2.3. The case \( N = 3 \). Similarly to the above, for the case \( N = 3 \) the respectively reduced invariant Riemann differential ideal \( \mathcal{R}_2 \) is given by the set

\[
R^{(3)} \{ u \} := \left\{ \sum_{n=0}^{2} \sum_{j=0,2} \lambda^{-(j+n)} f_n^{(j+1)} D_t^j D_x^3 D^u : D_t^3 u = 0, f_n^{(j+1)} \in \mathcal{K}, n \in \mathbb{Z}_+, j = 0,2 \right\}
\]

and characterized by the kernel \( \text{Ker} D_t \subset R^{(3)} \{ u \} \), generated by the following differential relationships:

\[
D_t f^{(1)}(\lambda) = 0, \quad D_t f^{(2)}(\lambda) = -\lambda f^{(1)}(\lambda), \quad D_t f^{(3)}(\lambda) = -\lambda f^{(2)}(\lambda),
\]

where \( f^{(j)}(\lambda) \in \mathcal{K}_3 \{ u \} := \mathcal{K} \{ u \} |_{D_t^3 u = 0}, j = 0,2 \), and \( \lambda \in \mathbb{R} \) is an arbitrary parameter. The system \( \mathcal{R}_2 \) can be equivalently rewritten in the matrix form

\[
D_t f(\lambda) = q_3(\lambda) f(\lambda), \quad q_3(\lambda) = \begin{pmatrix} 0 & 0 & 0 \\ -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \end{pmatrix},
\]

where \( f(\lambda) := (f^{(1)}, f^{(2)}, f^{(3)}(\lambda)) \in \mathcal{K}_3 \{ u \} \).

Now we can construct the corresponding invariant Lax differential ideal

\[
L_3 \{ u \} := \left\{ \sum_{j=1}^{3} g_j f^{(j)}(\lambda) \in \mathcal{K}_3 \{ u \} : D_t f^{(1)}(\lambda) = 0, \quad D_t f^{(2)}(\lambda) = -\lambda f^{(1)}(\lambda), \quad D_t f^{(3)}(\lambda) = -\lambda f^{(2)}(\lambda), g_j \in \mathcal{K}, j = 1,3 \right\},
\]

whose \( D_x \)-invariance is looked for in the linear matrix form

\[
D_x f(\lambda) = l_3[u; \lambda] f(\lambda)
\]

for some matrix \( l_3[u; \lambda] \in \text{End} \mathbb{R}^3 \). The latter satisfies the differential-functional equation

\[
D_t l_3 + l_3 D_x u_1 = [q_3, l_3],
\]

where we put, by definition, \( l_3 := l_3[u; \lambda], q_3 := q_3(\lambda) \in \text{End} \mathbb{R}^3 \). Making use of the derivative representation

\[
l_3 = D_x a_3,
\]

where \( a_3 := a_3[u; \lambda] \in \text{End} \mathbb{R}^2 \), equation \( \mathcal{R}_2 \) reduces to an equivalent one in the form

\[
D_t a_3 = [q_3, a_3].
\]

To solve matrix equation \( \mathcal{R}_2 \), we take into account that

\[
D_t^3 a_3 = k_3 q_3, \quad D_t^3 a_3 = 0
\]

for some constant \( k_3 \in \mathbb{R} \). Based both on \( \mathcal{R}_2 \) and on the component wise form of \( \mathcal{R}_3 \) one easily finds that

\[
a_3[u; \lambda] = \begin{pmatrix} \lambda u_2 & u_3 & 0 \\ 0 & \lambda u_2 & 2u_3 \\ -3\lambda^3 x & -3\lambda^2 u_1 & -2\lambda u_2 \end{pmatrix},
\]
entailing the matrix

\[
(2.39) \quad l_3[u; \lambda] = \begin{pmatrix}
\lambda u_{2,x} & u_{3,x} & 0 \\
0 & \lambda u_{2,x} & 2u_{3,x} \\
-3\lambda^3 & -3\lambda^2 u_{1,x} & -2\lambda u_{2,x}
\end{pmatrix}.
\]

Thereby, having naturally extended the differential ring \( K \) to the ring \( \mathbb{C} \{x, t\} \), one can formulate the next proposition.

**Proposition 2.6.** The Lax representation for the Riemann type hydrodynamical system

\[
(2.40) \quad D_1u_1 = u_2, \quad D_1u_2 = u_3, \quad D_1u_3 = 0
\]

is given by a set of linear compatible equations

\[
(2.41) \quad D_1f(\lambda) = \begin{pmatrix}
0 & 0 & 0 \\
-\lambda & 0 & 0 \\
0 & -\lambda & 0
\end{pmatrix} f(\lambda), \quad D_xf(\lambda) = \begin{pmatrix}
\lambda u_{2,x} & u_{3,x} & 0 \\
0 & \lambda u_{2,x} & 2u_{3,x} \\
-3\lambda^3 & -3\lambda^2 u_{1,x} & -2\lambda u_{2,x}
\end{pmatrix} f(\lambda),
\]

where \( f(\lambda) \in L_\infty(\mathbb{R}^2; \mathbb{C}^3) \) and \( \lambda \in \mathbb{C} \) is an arbitrary complex parameter.

As one can easily observe, the scheme of finding the Lax representation for the Riemann type hydrodynamical equations at \( N = \frac{1}{1}, 3 \) can be easily generalized for arbitrary \( N \in \mathbb{Z}_+ \), that is a topic of the next section.

### 2.4. The general case \( N \in \mathbb{Z}_+ \).

Consider at arbitrary \( N \in \mathbb{Z}_+ \) the constraint \( D_1^N u = 0 \) and the respectively reduced invariant Riemann differential ideal \( (2.2) \), which is given by the set

\[
(2.42) \quad R^{(N)}\{u\} := \{ \sum_{j=0}^{N-1} \sum_{n \in \mathbb{Z}_+} \lambda^{-(j+n)} f_{n}^{(j+1)} D_t^j D_x^n u \in K\{u\} : D_t^N u = 0 \}.
\]

The related kernel \( \text{Ker}D_t \subset R^{(N)}\{u\} \) of the \( D_t \)-differentiation is, owing to \( (2.3) \) and \( (2.42) \), generated by the relationships

\[
(2.43) \quad D_1 f^{(1)}(\lambda) = 0, \quad D_1 f^{(2)}(\lambda) = -\lambda f^{(1)}(\lambda), \quad \ldots, \quad D_1 f^{(N)}(\lambda) = -\lambda f^{(N-1)}(\lambda),
\]

which can be rewritten in a matrix form as

\[
(2.44) \quad D_t f(\lambda) = q_N(\lambda) f(\lambda), \quad q_N(\lambda) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
-\lambda & 0 & 0 & 0 \\
0 & -\lambda & \ddots & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & -\lambda
\end{pmatrix},
\]

where, by definition, a vector \( f(\lambda) := (f^{(1)}, f^{(2)}, \ldots, f^{(N)})^T \in K_N\{u\}^N := K\{u\}^N |_{D_t^N u = 0} \). The latter generates the invariant Lax differential ideal

\[
(2.45) \quad L^{(N)}\{u\} := \{ \sum_{j=1}^{N} g_j f^{(j)}(\lambda) \in K_N\{u\} : g_j \in K, \quad j = 1, N \},
\]

whose \( D_x \)-invariance holds, if its linear matrix representation in the space \( K^{(N)}\{u\}^N \)

\[
(2.46) \quad D_x f(\lambda) = l_N[u; \lambda] f(\lambda)
\]

is compatible with \( (2.43) \) for some matrix \( l_N := l_N[u; \lambda] \in \text{End} \mathbb{R}^N \) and arbitrary \( \lambda \in \mathbb{R} \). As a result we obtain the compatibility condition

\[
(2.47) \quad D_t l_N + l_N D_x u = [q_N, l_N].
\]

Having represented the matrix \( l_N \in \text{End} \mathbb{R}^N \) in the derivative form

\[
(2.48) \quad l_N := D_x a_N,
\]
we can reduce the functional-differential equation \(2.47\) to
\[
D_t a_N = [q_N, a_N].
\]

Now, taking into account that
\[
D_t^N a_N = k_N q_N, \quad D_t^{N+1} a_N = 0
\]
for some constant \(k_N \in \mathbb{R}\), one can easily obtain by means of simple calculations the following solution to equation \(2.49\):
\[
a_N[u; \lambda] = \begin{pmatrix}
\lambda u_{N-1} & u_N & 0 & \cdots & 0 \\
0 & \lambda u_{N-1} & 2u_N & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \lambda u_{N-1} & (N-1)u_{N-1} \\
-N\lambda^N & -\lambda^{N-1}Nu_1 & \cdots & -\lambda^2Nu_{N-2} & \lambda(1-N)u_N
\end{pmatrix},
\]

entailing the exact matrix expression
\[
l_N[u; \lambda] = \begin{pmatrix}
\lambda u_{N-1,x} & u_{N,x} & 0 & \cdots & 0 \\
0 & \lambda u_{N-1,x} & 2u_{N,x} & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \lambda u_{N-1,x} & (N-1)u_{N,x} \\
-N\lambda^N & -\lambda^{N-1}Nu_{1,x} & \cdots & -\lambda^2Nu_{N-2,x} & \lambda(1-N)u_{N-1,x}
\end{pmatrix}.
\]

Thereby, having naturally extended the differential ring \(\mathcal{K}\) to the ring \(\mathbb{C}\{\{x, t\}\}\), one can formulate the following general proposition.

**Proposition 2.7.** The Lax representation for the generalized Riemann type hydrodynamical system
\[
D_t u_1 = u_2, \quad D_t u_2 = u_3, \ldots, \quad D_t u_{N-1} = u_N, \quad D_t u_N = 0
\]
is given for any arbitrary \(N \in \mathbb{Z}_+\) by a set of linear compatible equations
\[
D_t f(\lambda) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & -\lambda & 0 & 0 & 0 \\
0 & 0 & \ddots & 0 & 0 \\
0 & 0 & 0 & -\lambda & 0 \\
0 & 0 & 0 & 0 & -\lambda
\end{pmatrix} f(\lambda),
\]
\[
D_x f(\lambda) = \begin{pmatrix}
\lambda u_{N-1,x} & u_{N,x} & 0 & \cdots & 0 \\
0 & \lambda u_{N-1,x} & 2u_{N,x} & \cdots & \cdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 0 & \lambda u_{N-1,x} & (N-1)u_{N,x} \\
-N\lambda^N & -\lambda^{N-1}Nu_{1,x} & \cdots & -\lambda^2Nu_{N-2,x} & \lambda(1-N)u_{N-1,x}
\end{pmatrix} f(\lambda),
\]

where \(f(\lambda) \in L_\infty(\mathbb{R}^2; \mathbb{C}^N)\) and \(\lambda \in \mathbb{C}\) is an arbitrary complex parameter. Moreover, the relationships \(2.54\) realize a linear matrix representation of the commutator condition \(1.3\) in the vector space \(\mathcal{K}^{(N)}\{u\}_N\).

The general Lax type representation \(2.54\), obtained above for arbitrary \(N \in \mathbb{Z}_+\), looks essentially simpler than those obtained before for \(N = 1, 4\) in [3, 17, 8] and given by differential-matrix expressions \(1.6\), depending at \(N \geq 3\) on the solutions to the set of differential-functional equations \(1.7\). As it was already mentioned above, it was first found in a similar form up to a scaling parameter \(\lambda \in \mathbb{C}\) in [4] by means of another approach. Taking this simplicity into account, one can apply to the Lax type representation \(2.54\) the gradient-holonomic approach and find the corresponding bi-Hamiltonian
structures, responsible for the Lax type integrability of the Riemann type hierarchy of hydrodynamical systems \((2.46)\).

3. **The bi-Hamiltonian structures for cases \(N = \frac{7}{3}, 3\) and \(N = 4\)**

To proceed with proving the related bi-Hamiltonian integrability of the generalized Riemann type dynamical system \((2.54)\) at arbitrary \(N \in \mathbb{Z}_+\), and with finding the naturally associated with the Lax representations \((2.41)\) Poissonian structures, we need preliminarily to study with respect to the gradient-holonomic approach \([2, 25, 28]\) the corresponding spectral properties of the linear differential system \((2.53)\). Below will analyze the next cases: \(N = \frac{7}{3}\) and \(N = 4\).

3.1. **The case \(N = 2\).** Making use of the exact matrix expression \((2.21)\), one can define for the second linear equation of \((2.28)\) the corresponding monodromy matrix \(S(x; \lambda) \in \text{End} \, \mathbb{C}^4, x \in \mathbb{R}\), which satisfies \([24, 26, 25, 28, 29]\) the Novikov-Marchenko commutator equation

\[
D_x S(x; \lambda) = [l_2[u; \lambda], S(x; \lambda)],
\]

Since the trace-functional \(\gamma(\lambda) := \text{tr}S(x; \lambda)\) is, owing to the Lax representation \((2.41)\), invariant with respect to the \(\mathbb{R} \ni x, t\)-evolutions on the manifold \(M_2\), its gradient \(\varphi(x; \lambda) := \text{grad} \gamma(\lambda) \in T^*(M_2), x \in \mathbb{R}\), depending parametrically on the spectral parameter \(\lambda \in \mathbb{C}\), equals

\[
\varphi(x; \lambda) = \text{tr}(l_2^* S(x; \lambda))
\]

and satisfies the well known gradient relationship

\[
(3.2) \quad \varphi(x; \lambda) = \text{grad} \gamma(\lambda) := \text{tr}(l_2^* S(x; \lambda))
\]

for some meromorphic function \(z : \mathbb{C} \to \mathbb{C}\) and \(x \in \mathbb{R}\), where \(\vartheta, \eta : T^*(M_2) \to T(M_2)\) are compatible \([23, 25, 27]\) Poissonian structures on the manifold \(M_2\), the dash sign \(\text{tr}^*\) means the usual Frechet derivative and \(\text{tr}^*\) means the corresponding conjugation with respect to the standard bilinear form on \(T^*(M_2) \times T(M_2)\). The latter naturally follows from the Lie-algebraic theory \([27, 23, 25]\) of Lax type integrable dynamical systems, treating them as the corresponding Lie-Poisson flows on the adjoint space \(\mathcal{G}^*(\lambda)\) to a suitably constructed centrally extended affine Lie algebra \(\mathcal{G}(\lambda), \lambda \in \mathbb{C}\). By simple enough calculations one find that

\[
(3.4) \quad \varphi(x; \lambda) := (\varphi_1, \varphi_2) = (\lambda(S^{(11)}_x - S^{(22)}_x), S^{(21)}_x)^\top,
\]

where we put, by definition,

\[
(3.5) \quad S(x; \lambda) := \begin{pmatrix} S^{(11)}_x & S^{(12)}_x \\ S^{(21)}_x & S^{(22)}_x \end{pmatrix},
\]

Taking into account that equation \((3.1)\) can be rewritten as the system

\[
(3.6) \quad S^{(11)}_x = -u_{2,x}S^{(22)} + 2\lambda^2 S^{(12)}_x,
\]

\[
S^{(12)}_x = -u_{2,x}S^{(11)} - S^{(22)} + 2\lambda u_{1,x}S^{(12)}_x,
\]

\[
S^{(21)}_x = -2\lambda^2 u_{1,x}(S^{(11)}_x - S^{(22)}_x) - 2\lambda u_{1,x}S^{(21)}_x,
\]

\[
S^{(22)}_x = -2\lambda^2 S^{(12)}_x + u_{2,x}S^{(21)}_x,
\]

by means of simple enough calculations, consisting in substituting \((3.4)\) into \((3.6)\), one easily finds the gradient relationship \((3.7)\) in the exact differential-matrix form:

\[
(3.7) \quad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = 2\lambda \begin{pmatrix} \partial^{-1} \partial^{-1}u_{1,x} & u_{1,x}\partial^{-1} \\ -u_{2,x}\partial^{-1} + \partial^{-1}u_{2,x} \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix},
\]

where, by definition, \(z(\lambda) := 2\lambda, \lambda \in \mathbb{C}\), and

\[
(3.8) \quad \vartheta := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \eta := \begin{pmatrix} \partial^{-1} \partial^{-1}u_{1,x} & u_{1,x}\partial^{-1} \\ -u_{2,x}\partial^{-1} + \partial^{-1}u_{2,x} \end{pmatrix}
\]

constitute the corresponding compatible Poissonian structures on the functional manifold \(M_2\). Thus, one can formulate the following proposition, before stated in \([7, 18, 5, 3]\) using different approaches.
Proposition 3.1. The Riemann type hydrodynamical system (1.2) at \( N = 2 \) is a Lax integrable bi-Hamiltonian flow with respect to the compatible on the functional manifold \( M_2 \) Poissonian pair (3.8).

As a consequence of Proposition (3.1) easily one states that there exists [23, 27, 25] an associated infinite hierarchy of commuting to each other integrable bi-Hamiltonian flows on \( M_2 \)

\[
\frac{d(u_1, u_2)^T}{dt_n} := -(\eta \vartheta^{-1})^n (u_{1,x}, u_{2,x})^T,
\]

where \( t_n \in \mathbb{R}, n \in \mathbb{Z}_+ \), are the corresponding evolution parameters.

3.2. The cases \( N = 3 \) and 4. For the case \( N = 3 \) there was proved in [1, 2, 5] that the corresponding Riemann type hydrodynamical system (2.20) is a Hamiltonian dynamical system on the 2\( \pi \)-periodic functional manifold \( M_3 := \{(u_1, u_2, u_3)^T \in C^{(\infty)}(\mathbb{R}/(2\pi Z); \mathbb{R}^3)\} \), whose exact Poissonian structure is given by the differential-matrix expression

\[
\eta = \left( \begin{array}{ccc}
\vartheta^{-1} & u_{1,x} \vartheta^{-1} & 0 \\
\vartheta^{-1} u_{1,x} & u_{2,x} \vartheta^{-1} + \vartheta^{-1} u_{2,x} & \vartheta^{-1} u_{3,x} \\
0 & u_{3,x} \vartheta^{-1} & 0
\end{array} \right),
\]

acting as a linear mapping \( \eta : T^*(M_3) \to T(M_3) \) from the cotangent space \( T^*(M_3) \) to the tangent space \( T(M_3) \). Making use of the exact matrix expression (2.39), one can define for the linear equation (2.33) the corresponding monodromy matrix \( S(x; \lambda) \in \text{End} \mathbb{C}^3; x \in \mathbb{R} \), which satisfies the Novikov-Marchenko commutator equation

\[
dS(x; \lambda)/dx = [l_3[u; \lambda], S(x; \lambda)].
\]

Since the trace-functional \( \gamma(\lambda) := \text{tr} S(x; \lambda) \) is invariant with respect to the \( \mathbb{R} \supset x \), \( t \)-evolutions on the manifold \( M_3 \), its gradient \( \varphi(x; \lambda) := \text{grad} \gamma(\lambda) \in T^*(M_3), x \in \mathbb{R} \), depending parametrically on the spectral parameter \( \lambda \in \mathbb{C} \), equals

\[
\varphi(x; \lambda) = \text{tr}(l_3^* S(x; \lambda)) = (-3\lambda^2 S_x^{(23)}, \lambda(S_x^{(11)} + S_x^{(22)}) - 2S_x^{(23)}, S_x^{(21)} + 2S_x^{(32)})^T
\]

and satisfies the gradient relationship

\[
\vartheta \varphi(x; \lambda) = z(\lambda) \eta \varphi(x; \lambda)
\]

for some meromorphic function \( z : \mathbb{C} \to \mathbb{C} \) and all \( x \in \mathbb{R} \), where \( \vartheta : T^*(M_3) \to T(M_3) \) is a second compatible with (3.10) Poisson structure on the manifold \( M_3 \). In this case the standard calculations, consisting in substituting (3.3) into (3.2) and reducing the result to the form (3.3), give rise by means of slightly cumbersome calculations to the equivalent to (3.13) relationship

\[
\varphi(x; \lambda) = \lambda^2 \vartheta^{-1} \eta \varphi(x; \lambda),
\]

where

\[
\vartheta^{-1} =
\begin{pmatrix}
\vartheta^{-1} u_{3,x} & u_3 \partial & -u_{2,x} - u_2 \partial \\
u_2 \partial & \vartheta^{-1} u_{1,x} & -2u_1 \partial + 2\partial \frac{u_{3,x}}{u_{3,x}} \\
-2u_1 + 2\frac{u_{3,x}}{u_{3,x}} \partial & 2 - 2\frac{u_{3,x}}{u_{3,x}} \partial & -\partial \left( \frac{u_{3,x}}{u_{3,x}} + 2u_{3,x} \right)
\end{pmatrix}
\]

is the second compatible with (3.10) co-Poissonian structure on the functional manifold \( M_3 \).

A completely similar results hold in the case \( N = 4 \). Namely, the corresponding Novikov-Marchenko equation

\[
dS(x; \lambda)/dx = [l_4[u; \lambda], S(x; \lambda)].
\]

jointly with the expression

\[
\varphi(x; \lambda) = \text{tr}(l_4^* S(x; \lambda)) = (-4\lambda^3 S_x^{(24)}, -4\lambda^2 S_x^{(34)}, \lambda(S_x^{(11)} + S_x^{(22)}) - 3S_x^{(33)} - 3S_x^{(44)}, S_x^{(21)} + 2S_x^{(32)} + 3S_x^{(43)})^T,
\]

is reduced by means of simple but cumbersome calculations to the gradient relationship

\[
\varphi(x; \lambda) = \lambda^3 \vartheta^{-1} \eta \varphi(x; \lambda),
\]
where the Poissonian structures \( \eta \) and \( \vartheta : T^*(M_4) \to T(M_4) \) are, respectively, inverse to the co-

Poissonian operators

\[
\eta^{-1} = \begin{pmatrix}
0 & 0 & -\partial & \frac{\partial u_2}{u_1} \\
0 & \partial & 0 & -\frac{\partial u_2}{u_3} \\
-w_3 \partial & -\frac{u_2}{u_4} \partial & w_4 \partial & \frac{1}{2}(u_4^2 - 2u_1u_3)\partial + \\
-w_3 \partial & -\frac{u_2}{u_4} \partial & w_4 \partial & +\partial(u_2^2 - 2u_1u_3)u_4^{-2}
\end{pmatrix}
\]

and

\[
\vartheta^{-1} = \begin{pmatrix}
0 & -3u_4 \partial & u_3 \partial & -\vartheta u_2 - u_2 \vartheta - \\
3u_4 \partial & 0 & u_2 \partial & \vartheta u_1 + u_1 \vartheta + \\
\vartheta u_3 & \vartheta u_2 & -u_1 \vartheta - u_1 \vartheta & -5 + \vartheta u_4 \vartheta \\
-u_1 \vartheta u_4 \partial + u_2 \vartheta u_4 \partial & +u_2 \vartheta u_4 \partial + u_3 \partial & u_4 \vartheta - u_4 \vartheta + u_3 \vartheta - u_3 \vartheta + & -\vartheta u_4 \vartheta
\end{pmatrix}
\]

on the functional manifold \( M_4 \). Below we will reanalyze the bi-Hamiltonian structures of the generalized

Riemann type hierarchy \( \text{(1.1)} \) from the geometric point of view, described in detail in \( \text{(25, 28, 3)} \).

3.3. Poissonian structures: the geometric approach. To construct the Hamiltonian structures,
related with the general dynamical system \( \text{(1.2)} \), by means of the geometric approach \( \text{it is first necessary} \)

to present it in the following equivalent form:

\[
\begin{align*}
du_1/\!\!d t &= u_2 - u_1 u_{1,x} \\
du_2/\!\!d t &= u_3 - u_1 u_{2,x} \\
&\ldots \\
du_j/\!\!d t &= u_{j+1} - u_1 u_{j,x} \\
&\ldots \\
du_N/\!\!d t &= -u_1 u_{N,x}
\end{align*}
\]

where \( K : M_N \to T(M_N) \) is the corresponding vector field on the functional manifold \( M_N \) for a fixed

\( N \in \mathbb{Z}_+ \). Then the following proposition \( \text{(25, 28, 3)} \) holds.

Proposition 3.2. Let \( \psi \in T^*(M_N) \) be, in general, a quasi-local functional-analytic vector, which satisfies
the condition \( \psi' \neq \psi'^+ \) and solves the Lie-Lax equation

\[
L_K \psi := \psi_t + K'^{\ast} \psi = \text{grad} \, \mathcal{L},
\]

for some smooth functional \( \mathcal{L} \in D(M_N) \). Then the dynamical system \( \text{(3.21)} \) allows the Hamiltonian
representation

\[
K[u] = -\eta \text{ grad } H_{\psi(\eta)}[u], \\
H_{\eta} = (\psi(\eta), K) - \mathcal{L}(\psi(\eta), \eta^{-1} = \psi(\eta) - \psi(\eta)).
\]
where $\eta : T^*(M_N) \to T(M_N)$ is the corresponding Poissonian structure, invertible on $M_N$. Otherwise, if the operator $\eta^{-1} : T(M_N) \to T^*(M_N)$ is not invertible, the following relationship

\begin{equation}
\text{grad } H(\eta)[u] = -\eta^{-1}K[u]
\end{equation}

holds on the manifold $M_N$.

**Remark 3.3.** Concerning a general solution to the Lie-Lax equation (3.22) it is easy to observe that it possesses the following representation:

\begin{equation}
\psi = \bar{\psi} + \text{grad } \bar{\mathcal{L}},
\end{equation}

where

\begin{equation}
L_K \bar{\psi} := \bar{\psi}_t + K_t^* \bar{\psi} = 0, \quad L_K \bar{\mathcal{L}} = \mathcal{L}.
\end{equation}

The related co-Poissonian operator

\begin{equation}
\eta^{-1} := \psi' - \psi'^* = \bar{\psi}' - \bar{\psi}'^*,
\end{equation}

owing to the Volterra symmetry condition $(\text{grad } \bar{\mathcal{L}})' = (\text{grad } \mathcal{L})^t^*$, persists to be unchangeable.

Based on Proposition 3.2, it is enough to search for non-symmetric functional-analytic solutions to (3.22) making use for this, for instance, of special computer-algebraic algorithms.

**3.3.1. The case $N = 3$.** The corresponding dynamical system

\begin{equation}
\begin{align*}
\frac{du_1}{dt} &= u_2 - u_1 u_{1,x}, \\
\frac{du_2}{dt} &= u_3 - u_1 u_{2,x}, \\
\frac{du_3}{dt} &= -u_1 u_{3,x},
\end{align*}
\end{equation}

is defined on the functional manifold $M_3$. A vector $\psi := (\psi_1, \psi_2, \psi_3)^T \in T^*(M_3)$ satisfies owing to (3.22) a set of functional-differential equations

\begin{equation}
\begin{align*}
\frac{d\psi_1}{dt} + u_1 \psi_{1,x} - u_{2,x} \psi_2 - u_{3,x} \psi_3 &= \delta \mathcal{L}/\delta u_1, \\
\frac{d\psi_2}{dt} + \psi_1 - (u_1 \psi_{2,x}) &= \delta \mathcal{L}/\delta u_2, \\
\frac{d\psi_3}{dt} + \psi_2 + (u_1 \psi_{3,x}) &= \delta \mathcal{L}/\delta u_3
\end{align*}
\end{equation}

for some smooth functional $\mathcal{L} \in D(M_3)$. System (3.28) can be slightly simplified, if to use the differentiation $D_t : M_3 \to T(M_3)$:

\begin{equation}
\begin{align*}
D_t \psi_1 - u_{2,x} \psi_2 - u_{3,x} \psi_3 &= \delta \mathcal{L}/\delta u_1, \\
D_t \psi_2 + \psi_1 + u_{1,x} \psi_2 &= \delta \mathcal{L}/\delta u_2, \\
D_t \psi_3 + \psi_2 + u_{1,x} \psi_3 &= \delta \mathcal{L}/\delta u_3,
\end{align*}
\end{equation}

which is equivalent to the scalar functional-analytic expression

\begin{equation}
D_t^2(\alpha \psi_1) = D_t(\alpha \delta \mathcal{L}/\delta u_1) + (D_t \alpha) \delta \mathcal{L}/\delta u_1 + (D_t^2 \alpha) \delta \mathcal{L}/\delta u_2 + \delta \mathcal{L}/\delta u_3
\end{equation}

on the alone function $\psi_1 \in K\{u\}$, where we have denoted the function $\alpha := u_{3,x}^{-1}$; satisfying the useful differential relationships

\begin{equation}
D_t \alpha = u_{1,x} \alpha, \quad D_t^2 \alpha = u_{2,x} \alpha, \quad D_t^3 \alpha = 1, \quad D_t^4 \alpha = 0.
\end{equation}

The corresponding functions $\psi_2, \psi_3 \in K\{u\}$ are given by the functional-operator expressions

\begin{equation}
\begin{align*}
\psi_2 &= \alpha^{-1}D_t^{-1}(-\alpha \psi_1 + \alpha \delta \mathcal{L}/\delta u_2), \\
\psi_3 &= \alpha^{-1}D_t^{-1}(-\alpha \psi_2 + \alpha \delta \mathcal{L}/\delta u_3),
\end{align*}
\end{equation}

easily following from (3.30). In the special case $\mathcal{L} = 0$ equation (3.31) reduces to

\begin{equation}
D_t^2(\alpha \psi_1) = 0,
\end{equation}

whose functional-analytic solutions can be found analytically by means of both differential-algebraic tools, devised in [17], and modern computer-algebraic algorithms. In particular, making use of the
differential-algebraic approach of the work [17], one can easily enough to find, amongst many others, the following solutions to (3.31):

\begin{equation}
\psi_1 = \frac{u_{1,x}}{2}, \quad \mathcal{L}_{(\eta)} = \frac{1}{2} \int_0^{2\pi} (2u_3 + u_2u_{1,x}) dx;
\end{equation}

\begin{equation}
\psi_1 = u_{1,x}u_3 - u_{1,x}u_3, \quad \mathcal{L}_{(\eta)} = 0;
\end{equation}

and

\begin{equation}
\psi_1 = -\frac{u_{3,x}}{2}, \quad \mathcal{L}_{(\theta_0)} = 0;
\end{equation}

\begin{equation}
\psi_1 = u_2u_3, \quad \mathcal{L}_{(\theta_1)} = 0;
\end{equation}

\begin{equation}
\psi_1 = u_{1,x}u_2 - 2u_3 - \left(\frac{u_1u_2}{u_3} - \frac{u_3^3}{3u_3^2}\right)u_3, \quad \mathcal{L}_{(\theta_2)} = 0;
\end{equation}

\begin{equation}
\psi_1 = u_{1,x} - \frac{u_2^2 u_3}{2u_3}, \quad \mathcal{L}_{(\theta_3)} = 0;
\end{equation}

\begin{equation}
\psi_1 = u_{1,x} - \frac{u_1 u_3 u_3}{u_3}, \quad \mathcal{L}_{(\theta_4)} = 0;
\end{equation}

giving rise to the generating vectors

\begin{equation}
\psi_{(\eta)} = (\frac{u_{1,x}}{2}, 0, -u_{3,x}^{-1} u_{1,x}^2/2 + u_{3,x}^{-1} u_{2,x})^\top, \quad \mathcal{L}_{(\eta)} = \frac{1}{2} \int_0^{2\pi} (2u_3 + u_2u_{1,x}) dx;
\end{equation}

\begin{equation}
\psi_{(\theta)} = (u_{1,x}u_3 - u_{1,x}u_3, \ u_1 u_2 - u_{1,x}u_2, \ 2u_2 - u_{1,x}u_3 + \frac{u_2 u_3^2}{2u_3})^\top, \quad \mathcal{L}_{(\theta)} = 0;
\end{equation}

\begin{equation}
\psi_{(\theta_0)} = (-u_{3,x}/2, \ u_{2,x}/2, \ u_{1,x}/2 - u_{3,x}^{-1} u_{2,x}/2)^\top, \quad \mathcal{L}_{(\theta_0)} = 0;
\end{equation}

\begin{equation}
\psi_{(\theta_1)} = (u_2 u_3, \ u_1 u_2 u_3 u_3^1 - u_2^2 u_3^2 / 3)^\top, \quad \mathcal{L}_{(\theta_1)} = 0;
\end{equation}

\begin{equation}
\psi_{(\theta_2)} = (u_{1,x}u_2 - 2u_3 - (u_1u_2 - \frac{u_3^3}{3u_3^2})u_3, \ u_3, \ u_2 - \frac{u_3^2 u_2}{2u_3} - \frac{u_3 u_2^2 + u_3 u_2}{3u_3^2} + \frac{u_3 u_3^2}{2u_3})^\top;
\end{equation}

\begin{equation}
\psi_{(\theta_3)} = (u_{1,x} + \frac{u_3 u_3^2}{2u_3}, \ u_3^2 u_3^2 - \frac{u_3 u_3^3 - 3u_3^3}{6u_3^2} + \frac{u_3 u_3^3 - 6u_3^3}{12u_3^2} u_3^2 + \frac{u_3 u_3^3 - 6u_3^3}{3u_3^2})^\top, \quad \mathcal{L}_{(\theta_3)} = 0;
\end{equation}

\begin{equation}
\psi_{(\theta_4)} = (u_{1,x} - \frac{u_3 u_3^2}{2u_3}, \ u_3^2 u_3^2 - \frac{u_3 u_3^3 - 3u_3^3}{6u_3^2} + \frac{u_3 u_3^3 - 6u_3^3}{12u_3^2} u_3^2 + \frac{u_3 u_3^3 - 6u_3^3}{3u_3^2})^\top, \quad \mathcal{L}_{(\theta_4)} = 0;\end{equation}
As a result of (3.39) and expressions (3.37) one easily obtains the corresponding co-Poissonian operators:

$\eta^{-1} := \psi'_{(\eta)} - \psi'^*_{(\eta)} = \left( \begin{array}{ccc} \partial & 0 & -\partial u_{1,x}u_{3,x}^{-1} \\ 0 & 0 & -u_{1,x}u_{3,x}^{-1}\partial \\ -u_{1,x}u_{3,x}^{-1}\partial & u_{3,x}^{-1}\partial & 1/2(u_{1,x}^2u_{3,x}^{-2}\partial + \partial u_{1,x}^2u_{3,x}^{-2}) - \\ & & -(u_{2,x}u_{3,x}^{-2}\partial + \partial u_{2,x}u_{3,x}^{-2}) \end{array} \right)$, 

$\theta^{-1} := \psi'_{(\theta)} - \psi'^*_{(\theta)} = \left( \begin{array}{ccc} u_{3}\partial + \partial u_{3} & -u_{2,x} - \partial u_{2} & -2u_{1}\partial + 2\partial u_{3}u_{3,x}^{-1} \\ -u_{2,x} - \partial u_{2} & u_{3}\partial + \partial u_{3} & -2 - 2\partial u_{3}u_{3,x}^{-1} \\ -2\partial u_{1} + 2u_{3}\partial u_{3,x}^{-1}u_{3,x} & 2 - 2\partial u_{3}u_{3,x}^{-1} & -\partial\frac{u_{3}\partial u_{3}^2 + 2u_{3}\partial u_{3,x}u_{3}u_{3,x}^{-1}}{u_{3,x}^2} \end{array} \right)$, 

and

$\theta^{-1}_0 := \psi'_{(\theta_0)} - \psi'^*_{(\theta_0)} = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & -2u_{3,x} \\ -2u_{3,x} & 0 & u_{2}\partial - u_{2}u_{3}u_{3,x}^{-1}/u_{3} \\ & & -u_{1}\partial - u_{1}u_{3}u_{3,x}^{-1}/u_{3} + u_{3}\partial u_{3}^{-1}/u_{3}^{-1} \end{array} \right)$, 

and so on. The second expression of (3.39) coincides exactly with the co-Poissonian operator (3.15), satisfying the gradient relationship (3.14), which proves the bi-Hamiltonicity of the Riemann type equation (3.28). Concerning the next two expressions (3.39) and (3.40) it is easy to observe that they are not strictly invertible, since the translation vector field $d/dx : M_3 \rightarrow T(M_3)$ with components $(u_{1,x}, u_{2,x}, u_{3,x})^T \in T(M_3)$ belongs to their kernels, that is $\eta^{-1}(u_{1,x}, u_{2,x}, u_{3,x})^T = 0 = \theta^{-1}_0(u_{1,x}, u_{2,x}, u_{3,x})^T$. Nonetheless, upon formal inverting the operator expression (3.39), we obtain by means of simple enough, but slightly cumbersome, direct calculations, that the Poissonian $\eta$-operator looks as

$\eta = \left( \begin{array}{ccc} \partial^{-1} & u_{1,x}\partial^{-1} & 0 \\ u_{2,x}\partial^{-1} + \partial^{-1}u_{2,x} & \partial^{-1}u_{3,x} & 0 \\ 0 & u_{3,x}\partial^{-1} & 0 \end{array} \right)$, 

coinciding with expression (3.10), and the corresponding Hamiltonian function equals

$H(\eta) := (\psi(\eta), K) - L(\eta) = \int_0^{2\pi} dx(u_{1,x}u_{2} - u_{3})$.

Concerning the operator expression (3.40) the corresponding Hamiltonian function

$H(\theta_0) := (\psi(\theta_0), K) = \int_0^{2\pi} dx(u_{3}u_{2,x})$.

It is evident that these Hamiltonian functions (3.42) and (3.43) are conservation laws for the dynamical system (3.28), which were also before found in [8].

Remark 3.4. It is worth to mention here that as the sum of vectors $\sum_{j=0}^{4} s_j \psi(\theta_j) \in T^*(M_3)$ with arbitrary $s_j \in \mathbb{R}$, $j = 0, 4$, solves the determining equation (3.22), the corresponding operator

$\theta^{-1} := \sum_{j=0}^{4} s_j \theta_j^{-1}$

will also be a co-Poissonian operator for the dynamical system (3.28)
3.3.2. The case $N = 4$. The Riemann type hydrodynamical equation at $N = 4$ is equivalent to the nonlinear dynamical system

$$
\begin{align*}
\frac{du_1}{dt} &= u_2 - u_1 u_{1,x}, \\
\frac{du_2}{dt} &= u_3 - u_1 u_{2,x}, \\
\frac{du_3}{dt} &= u_4 - u_1 u_{3,x}, \\
\frac{du_4}{dt} &= -u_1 u_{4,x},
\end{align*}
$$

where $K : M_4 \rightarrow T(M_4)$ is a suitable vector field on the smooth $2\pi$-periodic functional manifold $M_4 := C^\infty(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^4)$. To study its possible Hamiltonian structures, we need to find functional solutions to the determining Lie-Lax equation (3.22):

$$
\psi_t + K^{\ast} \psi = \text{grad} \mathcal{L}
$$

for $\psi \in T^*(M_4)$ and some smooth functional $\mathcal{L} \in D(M_4)$, where

$$
K = \begin{pmatrix}
-\partial u_1 & 1 & 0 & 0 \\
-u_2 & -u_1 \partial & 1 & 0 \\
-u_3 & 0 & -u_1 \partial & 1 \\
-u_4 & 0 & 0 & -u_1 \partial
\end{pmatrix}, \quad K^{\ast} = \begin{pmatrix}
u_1 \partial & -u_2 & -u_3 & -u_4 \\
1 & \partial u_1 & 0 & 0 \\
0 & 1 & \partial u_1 & 0 \\
0 & 0 & 1 & \partial u_1
\end{pmatrix}
$$

are, respectively, the Frechet derivative of the mapping $K : M_4 \rightarrow T(M_4)$ and its conjugate. The small parameter method [25, 44], applied to equation (3.46), gives rise to the following its exact solution:

$$
\psi_{(\eta)} = \left(-u_3, u_2/2, 0, -\frac{u_2^2}{2u_4} + \frac{u_1 u_3}{u_4}\right)^T, \quad \mathcal{L}_{(\eta)} = \int_0^{2\pi} (u_4 u_{1,x} - u_2 u_{3,x}/2) dx.
$$

As a result, we obtain right away from (3.23) that dynamical system (3.45) is a Hamiltonian system on the functional manifold $M_4$, that is

$$
K = -\partial \text{grad} H_{(\eta)},
$$

where the Hamiltonian functional equals

$$
H_{(\eta)} := (\psi_{(\eta)}, K) - \mathcal{L}_{(\eta)} = \int_0^{2\pi} (u_1 u_{4,x} - u_2 u_{3,x}) dx
$$

and the co-implicte operator equals

$$
\eta^{-1} := \psi_{(\eta)} - \psi_{(\eta)}^{\ast} = \begin{pmatrix}
0 & 0 & -\partial & \frac{\partial u_3}{u_4} \\
0 & \partial & 0 & -\frac{\partial u_2}{u_4} \\
-\partial & 0 & 0 & \frac{\partial u_1}{u_4} \\
\frac{u_3}{u_4} \partial & -\frac{u_2}{u_4} \partial & \frac{u_1}{u_4} \partial & \frac{1}{2}(u_{4,x}^2 - u_{2,x}^2 - 2u_{1,x} u_{3,x}) \partial + \frac{1}{2}(u_{2,x}^2 - 2u_{1,x} u_{3,x})^2 + \partial (u_{3,x}^2 - 2u_{1,x} u_{3,x} u_{4,x})^2
\end{pmatrix}.
$$

The latter is degenerate: the relationship $\eta^{-1}(u_1, u_2, u_3, u_4)^T = 0$ exactly on the whole manifold $M_4$, but the inverse to (3.51) exists and can be calculated analytically.

Proceed now to constructing other solutions to the generating equation (3.46) at the reduced case $\mathcal{L} = 0$, having rewritten it in the following form:

$$
\begin{align*}
D_t \psi_1 - u_2 x \psi_2 - u_3 x \psi_3 - u_4 x \psi_4 &= 0, \\
D_t \psi_2 + \psi_1 + u_{1,x} \psi_2 &= 0, \\
D_t \psi_3 + \psi_2 + u_{1,x} \psi_3 &= 0, \\
D_t \psi_4 + \psi_3 + u_{1,x} \psi_4 &= 0,
\end{align*}
$$

where $\psi := (\psi_1, \psi_2, \psi_3, \psi_4)^T \in T^*(M_4)$. The latter is equivalent to the system of differential-functional relationships

$$
\begin{align*}
D_t (\alpha \psi_1) - \psi_1 D_t \alpha - \psi_2 D_t^2 \alpha - \psi_3 D_t^3 \alpha - \psi_4 D_t^4 \alpha &= 0, \\
D_t (\alpha \psi_2) + \alpha \psi_1 &= 0, D_t (\alpha \psi_3) + \alpha \psi_2 &= 0, D_t (\alpha \psi_4) + \alpha \psi_3 &= 0,
\end{align*}
$$

where $\alpha := \alpha_1, \alpha_2, \alpha_3, \alpha_4$.
where we have denoted the function $\alpha := u_{4,x}^{-1}$. From (3.53) one ensues, by means of simple calculations, the following generating differential-functional equation

\[(3.54) \quad D_t^4(\alpha \psi_1) = 0\]

on the manifold $M_4$. As above in the case $N = 3$, the obtained equation (3.54) can be easily enough solved by means of the differential-algebraic approach, devised before in [3], which based on the following observation: owing to the equality $D_t^4 = 0$ the set

\[(3.55) \quad A(u) := \{f_0\alpha + f_1 D_\alpha + f_2 D_t^2 \alpha + f_3 D_t^4 \alpha + f_4 D_t^4 \alpha \in K_4\{u\} : f_j \in K_4\{u\}, j = 0, 4\}\]

generates a finite dimensional invariant differential ideal of the differential ring $K_4\{u\} := K\{u\}_{D_t^4}$. As a result of constructing solutions to the functional-differential equation (3.54), as non-symmetric elements of the ideal (3.54), one finds the following expression:

\[(3.56) \quad \psi_\vartheta = (u_{4,x}, u_4 - u_{2,x}, u_{4,x}, u_4 - u_1, 2u_4 - (u_1u_3 - u_2^2/2), u_1 - u_2, u_1, u_{4,x} - \frac{u_1u_3 - u_2^2}{2}x, 7u_3 - u_2u_1, x + u_{4,x} + \frac{u_1u_3 - u_2^2}{2}x), \quad \mathcal{L}_\vartheta = 0.\]

Now taking into account the expression (3.23) one easily obtains the second co-Poissonian operator $\vartheta^{-1} = \psi_\vartheta - \psi_\vartheta^*$ in the following matrix form:

\[(3.57) \vartheta^{-1} = \begin{pmatrix}
0 & -3u_{4,x} & u_3 \partial u_3 \\
3u_{4,x} & 0 & u_2 \partial u_2 \\
\partial u_3 & \partial u_2 & -u_1 \partial - u_1 \partial u_1 \\
-\partial u_2 - u_2 \partial u_2 & \partial u_1 + u_1 \partial u_1 & 5 + (\frac{u_1u_3 - u_2^2}{u_4,x}) \partial - \frac{u_1u_3}{u_4,x} \partial u_1 + \frac{u_1u_3}{u_4,x} \partial u_1 \\
-\frac{u_1u_3}{u_4,x} \partial u_3 & +\frac{u_1u_3}{u_4,x} \partial u_2 & -\frac{u_3}{u_4,x} \partial u_1 \\
-\frac{u_2}{u_4,x} \partial u_3 & +\frac{u_2}{u_4,x} \partial u_2 & -\frac{u_3}{u_4,x} \partial u_1 \\
-\frac{u_3}{u_4,x} \partial u_3 & -\frac{u_3}{u_4,x} \partial u_1 & -\frac{u_3}{u_4,x} \partial u_1
\end{pmatrix},\]

satisfying jointly with (3.51) the gradient relationship (3.18).

4. Conclusion

A differential-algebraic approach, elaborated in this article for revisiting the integrability analysis of generalized Riemann type hydrodynamical equation (1.1), made it possible to construct new and more simpler Lax type representation for the general case $N \in \mathbb{Z}_+$, contray to those constructed before in [17, 3]. This representation, obtained by means of the suggested differential-algebraic approach, proves also to coincide up to the scaling parameter $\lambda \in \mathbb{C}$ with that first obtained by Z. Popowicz in [4]. It is worth to mention that the corresponding Lax type representations for the generalized Riemann type hierarchy are well fitting for constructing the related compatible Poissonian structures and proving their corresponding bi-Hamiltonian integrability. The corresponding calculations are fulfilled in details for case $N = 1, 2$ and preliminary results are obtained for cases $N = 3, 4$ by means of the geometric method, devised in [25, 28, 3]. It was also demonstrated that the corresponding differential-functional relationships give rise to the suitable compatible Poissonian structures and present a very interesting mathematical problem from the differential-algebraic point of view, which is planned to be studied in other work.
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