ON CERTAIN GENERALIZED ISOMETRIES OF THE SPECIAL ORTHOGONAL GROUP

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ABSTRACT. In this paper we explore the structure of certain generalized isometries of the special orthogonal group $SO(n)$ which are transformations that leave any member of a large class of generalized distance measures invariant. This gives us a far-reaching generalization of the former structural result of T. Abe, S. Akiyama and O. Hatori concerning $c$-spectral isometries of the special orthogonal group.

1. Introduction

Let $SO(n)$ and $\mathbb{K}_n(\mathbb{R})$ denote the special orthogonal group and the associated Lie algebra consisting of the set of all skew-symmetric real matrices, respectively. The symbols $\|\cdot\|$ and $\|\cdot\|_F$ stand for the operator norm and the Frobenius norm, respectively. For any $X \in \mathbb{K}_4(\mathbb{R})$ we denote by $\tilde{X}$ the matrix which is obtained from $X$ by interchanging its $(1,4)$ and $(2,3)$ entries, and interchanging the $(4,1)$ and $(3,2)$ entries, respectively. If $M$ is a metric space and $d : M \times M \to [0, +\infty[$ is a function satisfying $d(x, y) = 0$ if and only if $x = y$, then $d$ is termed to be a generalized distance measure. Clearly, a generalized distance measure is not a metric in the usual sense because we require neither the symmetry nor the triangle inequality.

In [1] the structure of isometries of $SO(n)$ with respect to the metric induced by any $c$-spectral norm were determined. At the end of that paper the authors proposed an open problem: how one can describe the form of isometries with respect to any unitary invariant norm? The main purpose of this paper is to make steps towards in this direction. More precisely, we consider those kind of generalized distance measures on the manifold $SO(n)$ which are given of the form

$$d_{N,f}(A, B) = N(f(A^{-1}B))$$

where $N$ is any unitary invariant norm and $f : \mathbb{T} \to \mathbb{C}$ is a bounded function satisfying

1. $f(z) = 0$ if and only if $z = 1$;
2. $f$ is conformal in a neighbourhood of 1;
3. $f$ is continuous on $\mathbb{T} \setminus \{-1\}$;
4. $f(-1) = \lim_{t \uparrow \pi} f(e^{it})$;

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and determine the structure of those maps (called generalized isometries) which preserve the above type generalized distance measures between the elements of $SO(n)$. This problem is slightly more general than our original one.

Unfortunately, if $n > 3$ we are not able to give the complete description of the above type generalized isometries with respect to any unitary invariant norm, but we are with respect to any unitary invariant norm which is not a scalar multiple of the Frobenius norm. Our results give us a far reaching generalization of the beautiful result [1, Theorem 1] and also include the characterization of geodesic distance preserving maps on $SO(3)$. As recent literature on investigations the structures of isometries and certain generalized isometries on matrix classes the reader is referred to [2], [3], [8], [10], [11].

We remark that the conditions (f2)-(f4) concerning the numerical function $f$ came from the requirement that we want to cover the functions $z \mapsto z - 1$ and $z \mapsto \log z$ which correspond to the cases of the norm distance and the geodesic distance, respectively (see the examples below).

**Example 1.** If $f(z) = z - 1$, then we have
\[ d_{N,f}(A, B) = N(A - B) \]
which is just the usual norm distance.

**Example 2.** If $f(z) = \log z$ and $N(.) = \| . \|_F$, then we have
\[ d_{N,f}(A, B) = \| \log (A^{-1}B) \|_F \]
which is the geodesic distance (we apply the convention $\log(-1) = i\pi$).

These generalized distance measures are commonly used in robotics, computer vision, computer graphics and in the medical sciences, the reader can consult e.g. [4], [7] for further details.

2. Results

Our first result concerning in the previously section discussed type generalized distance measures reads as follows.

**Theorem 1.** Let $N(.)$ be a unitary invariant norm which is not a constant multiple of the Frobenius norm and assume that $f: \mathbb{T} \to \mathbb{C}$ is a bounded function satisfying (f1)-(f4). The map $\phi: SO(n) \to SO(n)$ is a generalized isometry with respect to the generalized distance measure $d_{N,f}(., .)$, i.e.,
\[ d_{N,f}(\phi(A), \phi(B)) = d_{N,f}(A, B), \quad A, B \in SO(n) \]
if and only if there exists an orthogonal matrix $Q \in O(n)$ such that $\phi$ is of one of the following forms:

(a) $\phi(A) = \phi(I)QAQ^{-1}$ for all $A \in SO(n)$;
(b) $\phi(A) = \phi(I)QA^{-1}Q^{-1}$ for all $A \in SO(n)$;
(c) \( n = 4 \) and \( \phi(A) = \phi(I)Q\exp(\tilde{X})Q^{-1} \) for all \( A \in SO(4) \), where \( X \in \mathbb{K}_4(\mathbb{R}) \) such that \( \exp(X) = A \);

(d) \( n = 4 \) and \( \phi(A) = \phi(I)Q\exp(-\tilde{X})Q^{-1} \) for all \( A \in SO(4) \), where \( X \in \mathbb{K}_4(\mathbb{R}) \) such that \( \exp(X) = A \).

Our second theorem reads as follows.

\textbf{Theorem 2.} Let \( N(\cdot) \) be a constant multiple of the Frobenius norm and assume that \( f : \mathbb{T} \to \mathbb{C} \) is a bounded function satisfying (f1)-(f4). The map \( \phi : SO(3) \to SO(3) \) is a generalized isometry with respect to the generalized distance measure \( d_{N,f}(\cdot,\cdot) \) if and only if there exists an orthogonal matrix \( Q \in O(3) \) such that \( \phi \) is of either of the form (a) or (b) in Theorem 1.

For the proof we need some more preliminaries. We begin with the following observations: If \( \phi : SO(n) \to SO(n) \) is a generalized isometry with respect to the generalized distance measure \( d_{N,f}(\cdot,\cdot) \), then \( \phi(I)^{-1}\phi(\cdot) \) is a map which holds the same preserver property as \( \phi(\cdot) \) and sends the unit to the unit. It means that in the sequel without loss of generality we may and do assume that \( \phi(I) = I \). Furthermore, it is not difficult to verify that every unital Jordan triple map is compatible with the inverse and the power operations, i.e., \( \phi(A^k) = \phi(A)^k \) \((k \in \mathbb{N})\) and \( \phi(A^{-1}) = \phi(A)^{-1} \) holds for every \( A \in SO(n) \).

Since \( SO(n) \) is a compact connected Lie group the exponential map defined by

\[
\exp : \mathbb{K}_n(\mathbb{R}) \to SO(n), \quad X \mapsto \sum_{k=0}^{\infty} \frac{X^k}{k!}
\]

is surjective. Moreover, due to the Lie group-Lie algebra correspondence we have that if \( \gamma(t) \) is a one-parameter subgroup, i.e.,

\[
\gamma(t+s) = \gamma(t) \cdot \gamma(s), \quad t, s \in \mathbb{R},
\]

then there exists an \( X \in \mathbb{K}_n(\mathbb{R}) \), the generator of \( \gamma(t) \), for which \( \gamma(t) = \exp(tX) \).

It is apparent that if \( X, Y \in \mathbb{K}_n(\mathbb{R}) \), then we have a special orthogonal matrix \( BCH(X,Y) \) such that \( \exp(X)\exp(Y) = \exp(BCH(X,Y)) \). According to the famous Baker-Campbell-Hausdorff formula we have

\[
BCH(X,Y) = X + Y + \frac{1}{2} (XY - YX) + \\
\frac{1}{12} \left( X^2Y + XY^2 - 2XYX + Y^2X + YX^2 - 2YXY \right) + ...
\]

for the first three terms of the series expansion of \( BCH(X,Y) \). The following lemma appeared in [1].

\textbf{Lemma 3.} [1, Lemma 8.] For any \( X, Y \in \mathbb{K}_4(\mathbb{R}) \) the eigenvalues of \( BCH(X,Y) \) and \( BCH(\tilde{X},\tilde{Y}) \) coincide with each other.
Next we recall the Youla-decomposition of skew-symmetric matrices. Clearly, any skew-symmetric matrix is normal and thus unitary diagonalizable. Since all the nonzero eigenvalues of a skew-symmetric matrix are located on the imaginary axis it cannot be diagonalized by a real orthogonal matrix. Nevertheless, if the eigenvalues of the matrix $X \in \mathbb{K}_n(\mathbb{R})$ are $\{\pm i\lambda_1, \pm i\lambda_2, \ldots, \pm i\lambda_r, 0, 0, \ldots, 0\}$, then there is a real orthogonal matrix such that $X$ can be decomposed as $X = Q\Sigma Q^{-1}$ where $$\Sigma = \left( \begin{array}{cccccccc} 0 & \lambda_1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ -\lambda_1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 0 & -\lambda_2 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & \lambda_r & 0 \\ 0 & 0 & 0 & 0 & \ldots & -\lambda_r & 0 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \end{array} \right)$$ The above decomposition is called the Youla-decomposition of $X$.

3. PROOF

The proof broadly follows the common general approach of [1] and [3], however, the details are slightly different at some points. The proof based on the so-called commutative diagram argument (CDA) which works in general as follows. The method is to reduce the problem of finding the (generalized) isometries of a Lie group to the characterization of certain linear maps on the corresponding Lie algebra. More precisely, let $G$ be a Lie group and $\mathfrak{g}$ be the associated Lie algebra. If $\phi$ is an isometry on $G$, then by applying the one-parameter subgroup technique developed by Sakai (see [12]) we obtain that there exists a linear map $h$ on the corresponding Lie algebra for which the following diagram commutes:

$$\begin{array}{ccc}
G & \xrightarrow{\phi} & G \\
\exp & \downarrow & \exp \\
\mathfrak{g} & \xrightarrow{h} & \mathfrak{g}
\end{array}$$

If the form of $h$ is known, then it can be applied to derive the complete description of $\phi$.

Furthermore, we rely heavily on the following general Mazur-Ulam type result, as well, which appeared in [8].

**Proposition 4.** [8, Proposition 20] Assume that $G$ and $H$ are groups equipped with generalized distance measures $d$ and $\rho$, respectively. Select
ON CERTAIN GENERALIZED ISOMETRIES OF THE SPECIAL ORTHOGONAL GROUP

Let $a, b \in G$ and set

$$L_{a,b} := \{ x \in G : d(a, x) = d(x, ba^{-1}b) = d(a, b) \},$$

and assume the following:

1. $d(bx^{-1}b, bx'^{-1}b) = d(x', x)$ holds for every $x, x' \in G$;
2. $\sup\{d(x, b) : x \in L_{a,b} \} < \infty$;
3. there is a constant $K > 1$ such that $d(x, bx^{-1}b) \geq K d(x, b)$, $x \in L_{a,b}$;
4. $\rho(\phi^{-1}c', \phi^{-1}c') = \rho(y', y)$ holds for every $y, y', c, c' \in H$.

Then for any surjective map $\phi : G \rightarrow H$ satisfying

$$\rho(\phi(x), \phi(x')) = d(x, x') \quad (x, x' \in G)$$

we necessarily have

$$\phi(ba^{-1}b) = \phi(b)\phi(a)^{-1}\phi(b).$$

We are now in a position to prove the following auxiliary lemma.

**Lemma 5.** If $\phi : SO(n) \rightarrow SO(n)$ is a generalized isometry with respect to the generalized distance measure $d_{N,f}(\cdot,\cdot)$, then $\phi$ is a continuous Jordan-triple endomorphism, i.e., a continuous map with the property that

$$\phi(ABA) = \phi(A)\phi(B)\phi(A), \quad A, B \in SO(n).$$

**Proof.** We first show that $\phi$ is continuous with respect to the operator norm topology. Consider a fixed $A \in SO(n)$ and a sequence $(A_i)_{i \in \mathbb{N}} \in SO(n)$ such that $A_i \rightarrow A$ in the operator norm. Clearly, this implies that $A_i^{-1}A \rightarrow I$. By the continuity of $f$ and the property (f1) we infer that $f(A_i^{-1}A) \rightarrow 0$ in the operator norm. Since on a finite dimensional normed space every norm generate the same topology we have

$$N(f(A_i^{-1}A)) = d_{N,f}(A_i, A) \rightarrow 0.$$

Since $\phi$ preserves the generalized distance measure $d_{N,f}$ we also have

$$N( f(\phi(A_i)^{-1}\phi(A)) ) = d_{N,f}(\phi(A_i), \phi(A)) \rightarrow 0.$$

It follows that $f(\phi(A_i)^{-1}\phi(A)) \rightarrow 0$ in the operator norm. By the continuity of $f$ and the property (f1) we necessarily have $\phi(A_i)^{-1}\phi(A) \rightarrow I$, and thus $\phi(A_i) \rightarrow \phi(A)$ in the operator norm.

Our aim is now to show that $\phi$ is surjective. Assuming on the contrary implies that there exists an $A \in SO(n)$ which is not contained in the image of $\phi$. This yields

$$\inf\{\|\phi(B) - A\| : B \in SO(n)\} > 0$$

and, similar to the above discussed argument relating to the continuity, we infer that

$$c := \inf\{d_{N,f}(\phi(B), A) : B \in SO(n)\} > 0.$$
Set $A_0 := A$ and define the sequence $(A_i)_{i \in \mathbb{N}} \in SO(n)$ by the recursion $A_i := \phi(A_{i-1})$. Since $\phi$ is a generalized isometry we obtain that

$$d_{N,f}(A_i, A_j) = d_{N,f}(\phi^i(A_0), \phi^j(A_0)) = d_{N,f}(\phi^{i-j}(A), A) \geq c > 0$$

holds for every $i, j \in \mathbb{N}$ with $i > j$. This implies that $\|A_i - A_j\|$ is bounded away from zero. We notice that $\phi(SO(n))$, as an image of the compact set $SO(n)$ under the continuous map $\phi$, is compact. Hence one can choose a convergent subsequence of $(A_i)_{i \in \mathbb{N}}$, and thus the quantity $\|A_i - A_j\|$ cannot be bounded away from zero. Therefore, $\phi$ is a surjective generalized isometry.

We now intend to prove that $\phi$ preserves the inverted Jordan triple product locally. In order to do so, it is sufficient to show that the conditions (c1)-(c4) appearing in Proposition 4 are satisfied in the following setting: $G = H = SO(n)$ and $d = \rho = d_{N,f}$. As for the condition (c1), we calculate

$$d_{N,f}(BX^{-1}B, BX'^{-1}B) = N\left(f(B^{-1}XB^{-1}BX'^{-1}B)\right) = N\left(f(XX'^{-1})\right) = N\left(f(X'^{-1}X)\right) = d_{N,f}(X', X)$$

where the third equality follows from Jacobson’s lemma which claims that the spectrum of $XX'^{-1}$ and $X'^{-1}X$ coincide with each other.

The fact that (c2) is also valid is an immediate consequence of the boundedness of $f$ and the equivalence of the norm $N(.)$ to the operator norm.

We next show that (c3) is satisfied when $A$ and $B$ are close enough to each other in the operator norm. By the definition of conformal maps we have that the limit

$$\lim_{z \to 1} \frac{f(z) - f(1)}{z - 1}$$

exists and is different from zero. It is written on p.2 in [3] that this property guarantees that the condition $|f(z^2)| \geq K |f(z)|$ is satisfied automatically with some positive constant $K > 1$ for every $z \in \mathbb{T}$ from a neighbourhood of 1. Clearly, this implies that

$$N(f(C^2)) \geq K \cdot N(C)$$

holds whenever $C \in SO(n)$ is close enough to $I$ in the operator norm. Select $A, B \in SO(n)$, which are close enough to each other in the operator norm, and pick an arbitrary $X \in L_{A,B}$. Since the norm $N(.)$ is equivalent with the operator norm, by the definition of $L_{A,B}$ and the property (f1) of $f$ we obtain that the quantity

$$N\left(f(X^{-1}B)\right) = d_{N,f}(X, B) = d_{N,f}(A, B) = N\left(f(A^{-1}B)\right)$$

is small. Since $SO(n)$ is compact this gives us that $X^{-1}B$ is close enough to the identity. According to (2) we have

$$d_{N,f}(X, BX^{-1}B) = N\left(f\left((X^{-1}(BX^{-1}B))\right)\right) = N\left(f\left((X^{-1}B)^2\right)\right) \geq K \cdot N\left(f\left(X^{-1}B\right)\right) = K \cdot d_{N,f}(X, B).$$

This verifies the property (c3) for close enough $A, B \in SO(n)$. 

Relating to the condition (c4), similarly to the argument we have presented concerning the condition (c1), we compute

\[
d_{N,f}(CY^{-1}C', CY'-1C') = N \left( f(C'^{-1}YC^{-1}CY'-1C') \right) = N \left( f(YY'-1) \right) = N \left( f(Y'-1Y) \right) = d_{N,f}(Y', Y).
\]

Taking all the information what we have into account we conclude that \( \phi \) preserves the inverted Jordan triple product for close enough elements. In order to complete the proof, [2, Lemma 7] can be applied for obtaining the global inverted Jordan triple product preserver property if it holds for close enough elements. To check that the assumptions of [2, Lemma 7] are satisfied one can argue by literally following the second part of the proof of [1, Lemma 3]. Since \( \phi \) is unital, and thus compatible with the inverse operation, we conclude that \( \phi \) preserves (globally) the Jordan triple product, as well. The reader should consult with [1] and [2] for a more detailed argument. \( \square \)

The following technical lemma concerning continuous Jordan triple endomorphisms we shall also need.

**Lemma 6.** If \( \phi \) is a continuous unital Jordan triple endomorphism on \( SO(n) \), then \( \phi \) is necessarily Lipschitz-continuous.

**Proof.** The argument used in the proof is an adaption of that in [9, Lemma 6]. Nevertheless, for the sake of understandability, completeness and the reader’s convenience we present all the necessary details.

Select arbitrary \( A \in SO(n) \). Since the exponential map is surjective there exist an \( X \in \mathbb{K}_n(\mathbb{R}) \) such that \( A = \exp(X) \). By a particular choice of \( X \) we can ensure that \( \|X\| \leq \pi \). Let us denote the operator norm \( \|X\| \) of \( X \) by \( \Theta(A) \). It is apparent that the inequalities

\[
\|A - I\| \leq \Theta(A) \leq 2\|A - I\|
\]

hold. Furthermore, if \( k \) is a positive integer for which \( k \cdot \Theta(A) < \pi \) we have \( \Theta(A^k) \leq k \cdot \Theta(A) \).

In order to prove the statement of the lemma it is sufficient to show that there exists a positive real number \( M \) such that

\[
\|\phi(A) - I\| \leq M\|A - I\|, \quad A \in SO(n).
\]

Indeed, consider any \( A, B \in SO(n) \) and select a \( T \in SO(n) \) for which \( T^2 = B \). The existence of such \( T \) follows from the fact that every special orthogonal matrix can be written of the form as the exponential of a skew-symmetric real matrix. Then we can find an \( S \in SO(n) \) such that \( TST = A \). Hence we deduce

\[
\|\phi(A) - \phi(B)\| = \|\phi(TST) - \phi(T^2)\| = \|\phi(T)\phi(S)\phi(T) - \phi(T)\phi(T)\| = \|\phi(S) - I\| \leq M\|S - I\| = M\|TST - T^2\| = M\|A - B\|
\]
implying that \( \phi \) is Lipschitz-continuous. Therefore, in the remaining part of the proof we intend to show the existence of a positive real number \( M \) which satisfies (3).

To this end, assuming on the contrary implies that we have a sequence \((A_i)_{i \in \mathbb{N}} \in SO(n)\) and a sequence of positive integers \( k_i \) such that \( \lim_{i \to \infty} k_i = \infty \) and

\[
\| \phi(A_i) - I \| > k_i \| A_i - I \|, \quad i \in \mathbb{N}.
\]

Since \( SO(n) \) is compact the sequence \((A_i)_{i \in \mathbb{N}}\) has a convergent subsequence. Without loss of generality the original sequence \((A_i)_{i \in \mathbb{N}}\) may and do considered to be convergent. According to (4) if its limit were different from \( I \), then we would have \( \lim_{i \to \infty} \| \phi(A_i) - I \| = \infty \) which contradicts the fact that \( \| \phi(A_i) - I \| < 2 \). Therefore, \( \lim_{i \to \infty} A_i = I \) and thus by the continuity of \( \phi \) we also have \( \lim_{i \to \infty} \phi(A_i) = I \).

Let \( d_i := \| \phi(A_i) - I \| \). It is apparent that for a sufficient large \( i \) the inequality \( d_i < 1/2 \) holds. Then choose positive integers \( l_i \) for which

\[
\frac{1}{l_i + 1} \leq d_i < \frac{1}{l_i}.
\]

Plainly, \( l_i \geq 2 \) and due to the fact that \( d_i > k_i \| A_i - I \| \) we infer \( \| A_i - I \| < d_i/k_i \) implying that

\[
\Theta(A_i) \leq 2 \| A_i - I \| < \frac{2d_i}{k_i}.
\]

On the other hand, we conclude

\[
\frac{2ld_i}{k_i} < \frac{2}{k_i} < \pi
\]

and thus \( l_i \Theta(A_i) < 2/k_i \). This yields

\[
\| A_i^{l_i} - I \| \leq l_i \cdot \Theta(A_i) < \frac{2}{k_i}
\]

and hence \( A_i^{l_i} \to I \). As \( A_i \to I \) we obtain that \( A_i^{l_i+1} \to I \) as \( i \) tends to infinity.

Since \( d_i < 1/l_i \) and \( l_i \geq 2 \) we have

\[
2d_i(l_i + 1) < 2\frac{l_i + 1}{l_i} \leq \frac{3}{2} < \pi.
\]

Combining this with the inequalities

\[
\| \phi(A_i) - I \| \leq \Theta(\phi(A_i)) \leq 2\| \phi(A_i) - I \| = 2d_i
\]

and (5) implies that

\[
1 = \frac{1}{d_i} \cdot \| \phi(A_i) - I \| \leq (l_i + 1) \cdot \| \phi(A_i) - I \| \leq (l_i + 1) \cdot \Theta(\phi(A_i)) \leq 2\| \phi(A_i^{l_i+1}) - I \|.
\]

It means that the sequence \( \phi(A_i^{l_i+1}) \) do not converge to the identity matrix and this is a contradiction. The proof is complete. \( \square \)
ON CERTAIN GENERALIZED ISOMETRIES OF THE SPECIAL ORTHOGONAL GROUP

Now we are in a position to prove our first result.

Proof of Theorem 1. Let us begin with the sufficiency part. It is not difficult to verify that if $\phi$ is of the form (a) or (b), then it preserves the quantity $d_{N,f}(. , .)$ between the elements of $SO(n)$. So, we are concerned about the case when $\phi$ is of the form (c) or (d). Let us assume that $\phi$ is of the form (c); the case when $\phi$ is of the form (d) can be handled similarly. By Lemma 3 and the spectral mapping theorem we conclude that the eigenvalues of $f \left( \exp(BCH(-X, Y)) \right)$ and $f \left( \exp(BCH(\bar{X}, \bar{Y})) \right)$ coincide with each other, and thus their Youla decompositions are of the form $R_1 \Sigma R_1^{-1}$ and $R_2 \Sigma R_2^{-1}$, respectively. Since $N$ is unitary invariant we have

$$d_{N,f} (\phi(\exp(X)), \phi(\exp(Y))) = N \left( f \left( \exp(BCH(-X, Y)) \right) \right) = N \left( \Sigma \right) = N \left( f \left( \exp(BCH(-X, Y)) \right) \right) = d_{N,f} (\exp(X), \exp(Y))$$

It is now apparent that any map of the form (c) and (d) is a generalized isometry.

As for the necessity, assume that $\phi$ is a generalized isometry with respect to the generalized distance measure $d_{N,f}(. , .)$. We have learnt in Lemma 5 that $\phi$ is a Jordan triple endomorphism, as well. We intend to show that $\phi$ maps a one-parameter subgroup to another one. In order to do so, for a fixed $X \in \mathbb{K}_n(\mathbb{R})$ consider the one-parameter subgroup generated by $X$, that is, $e^{tX}$ where $t \in \mathbb{R}$. We claim $\gamma(t) := \phi(e^{tX})$ is a one-parameter subgroup of $SO(n)$, as well. To verify this property it is sufficient to show that (1) holds in the case where $t, s \in \mathbb{Q}$. Indeed, then (1) follows from the facts that the rationals are dense in $\mathbb{R}$ and $\phi$ is continuous. So, consider the numbers $t = p/q$ and $s = r/m$ where $p, q, r$ and $m$ are integers. Then we compute

$$\gamma(t + s) = \phi \left( e^{t+s}X \right) = \phi \left( e^{\left( \frac{pm+rq}{qm} \right)X} \right) =$$

$$\phi \left( e^{\frac{pm}{qm}} \right)^{pm+rq} \cdot \phi \left( e^{\frac{rq}{qm}} \right) =$$

$$\phi \left( e^{\frac{pm+rq}{qm}} \right) = \gamma(t) \cdot \gamma(s)$$

and this verifies our claim.

Since the exponential map is surjective on the Lie-algebra $\mathbb{K}_n(\mathbb{R})$ we obtain that there exists an $Y \in \mathbb{K}_n(\mathbb{R})$, the generator of $\gamma(t)$, such that $\gamma(t) = e^{tY}$. We constitute a map $h: \mathbb{K}_n(\mathbb{R}) \to \mathbb{K}_n(\mathbb{R}), X \mapsto Y$ such that $\phi(e^{tX}) = e^{th(X)}$ holds for every $t \in \mathbb{R}$. In what follows, just as in [10, Lemma 6] we show that $h$ is a linear transformation. Select arbitrary $X,Y,Z \in \mathbb{K}_n(\mathbb{R})$ and calculate

$$e^{\frac{t}{2}X} e^{tY} e^{\frac{t}{2}X} - e^{tZ} =$$

$$\left( e^{\frac{t}{2}X} - I \right) e^{tY} e^{\frac{t}{2}X} + (e^{tY} - I) e^{\frac{t}{2}X} + \left( e^{\frac{t}{2}X} - I \right) - (e^{tZ} - I) .$$
Therefore, we obtain
\[
\lim_{t \to 0} \frac{e^{\frac{t}{2}X}e^{tY}e^{\frac{t}{2}X} - e^{tZ}}{t} = 0 \iff \frac{X}{2} + Y + \frac{X}{2} - Z = X + Y - Z = 0.
\]
As a consequence of the Lipschitz property of \(\phi\) (see the previous Lemma) we have
\[
\lim_{t \to 0} \frac{\phi \left( e^{\frac{t}{2}X}e^{tY}e^{\frac{t}{2}X} \right) - \phi \left( e^{tZ} \right)}{t} = 0
\]
whenever \(Z = X + Y\) holds. Since \(\phi\) is a Jordan triple automorphism in this case we infer
\[
\phi \left( e^{\frac{t}{2}X}e^{tY}e^{\frac{t}{2}X} \right) - \phi \left( e^{tZ} \right) = \phi \left( e^{\frac{t}{2}X} \right) \phi \left( e^{tY} \right) \phi \left( e^{\frac{t}{2}X} \right) - \phi \left( e^{tZ} \right) = e^{\frac{t}{2}h(X)}e^{th(Y)}e^{\frac{t}{2}h(X)} - e^{th(Z)}.
\]
Therefore, on the one hand we have
\[
\lim_{t \to 0} \frac{e^{\frac{t}{2}h(X)}e^{th(Y)}e^{\frac{t}{2}h(X)} - e^{th(Z)}}{t} = 0,
\]
but on the other hand, similar to the above discussed argument, we deduce
\[
\lim_{t \to 0} \frac{e^{\frac{t}{2}h(X)}e^{th(Y)}e^{\frac{t}{2}h(X)} - e^{th(Z)}}{t} = 0 \iff h(Z) = h(X) + h(Y)
\]
implying that \(\phi\) is additive. To obtain the homogeneity we compute
\[
e^{\lambda h(X)} = \phi \left( e^{\lambda X} \right) = e^{th(\lambda X)} \quad (\lambda, t \in \mathbb{R}).
\]
This yields \(\lambda h(X) = h(\lambda X)\) holds for every \(\lambda \in \mathbb{R}\). Consequently, \(h\) is a linear transformation on the Lie algebra \(\mathbb{K}_n(\mathbb{R})\).

Next we prove that \(h : \mathbb{K}_n(\mathbb{R}) \to \mathbb{K}_n(\mathbb{R})\) is an isometry with respect to the unitary invariant norm \(N(.)\). Since \(f\) is conformal in a neighbourhood of \(1\) with the property that \(f(1) = 0\) we obtain that \(f\) has locally a power series expansion of the form
\[
f(z) = \sum_{k=1}^{\infty} a_k(z - 1)^k
\]
with \(a_1 \neq 0\). Let us consider any skew-symmetric matrices \(X, Y \in \mathbb{K}_n(\mathbb{R})\). It is apparent that
\[
e^{-tX}e^{tY} = I + (Y - X)t + \mathcal{O}(t^2)
\]
and thus
\[
d_{N,f} \left( e^{tX}, e^{tY} \right) = N \left( a_1(Y - X)t + \mathcal{O}(t^2) \right).
\]
Similarly, we have
\[
d_{N,f} \left( \phi \left( e^{tX} \right), \phi \left( e^{tY} \right) \right) = d_{N,f} \left( e^{th(X)}, e^{th(Y)} \right) = N \left( a_1 h(Y - X)t + \mathcal{O}(t^2) \right).
\]
Since \(\phi\) is a generalized isometry on \(SO(n)\) we conclude that
\[
(6) \quad N \left( Y - X + \mathcal{O}(t) \right) = N \left( h(Y - X) + \mathcal{O}(t) \right).
\]
Taking the limit $t \downarrow 0$ in (6) yields that $h : \mathbb{K}_n(\mathbb{R}) \to \mathbb{K}_n(\mathbb{R})$ is a linear isometry with respect to the metric induced by the unitary invariant norm $N(\cdot)$. The general form of linear isometries on $K_n(\mathbb{R})$ is well-known (see for example [5],[6]) in the case where $N$ is any unitary similarity invariant norm which is not a constant multiple of the Frobenius norm. We have that there exist a real number $\eta \in \{-1, 1\}$ and an orthogonal matrix $Q \in O(n)$ such that

(aa) $h(X) = \eta QXQ^{-1}$ for every $X \in \mathbb{K}_n(\mathbb{R})$;
(bb) $n = 4$ and $h(X) = \eta Q\tilde{X}Q^{-1}$ for every $X \in \mathbb{K}_n(\mathbb{R})$.

From this we infer that

$$\phi(\exp X) = \exp h(X) = \exp (\eta QXQ^{-1}) = Q(\exp \eta X)Q^{-1}$$

and this results (a) and (b) when $h$ is of the form (aa) and $A = \exp X$. The case (bb) can be handled by a similar way. The proof is complete. 

We continue with the proof of our second theorem.

**Proof of Theorem 2.** Parallel with the proof of Theorem 1, we obtain that

$$\phi(A) = \exp(h(X)), \quad X \in \mathbb{K}_3(\mathbb{R})$$

where $A = \exp X$ and $h : \mathbb{K}_3(\mathbb{R}) \to \mathbb{K}_3(\mathbb{R})$ is a linear isometry with respect to the Frobenius norm. The Frobenius norm is a $c$-spectral norm on $\mathbb{K}_3(\mathbb{R})$ with $c = (\sqrt{2},0,0)$. This gives us that there exist a real number $\eta \in \{-1, 1\}$ and an orthogonal matrix $Q \in O(3)$ such that $h$ is of the form (aa) (see [6, Theorem 4.2]). Now, just as at the end of the proof of Theorem 1, we conclude that $\phi$ is of the desired form. 

4. **Concluding remarks**

We close this paper with some concluding remarks. A natural question arises what happens in the "missing case" when $N(\cdot)$ is a multiple of the Frobenius norm and $n > 3$. By following our argument one can deduce the following necessary condition for being a discussed type generalized isometry on $SO(n)$: $\phi(\exp X) = \phi(I)\exp(O(X))$ with some orthogonal transformation $O$ on $\mathbb{K}_n(\mathbb{R})$. Clearly, there are abundance of such transformation and we do not know anything about their structure. It means that determining the structure of generalized isometries with respect to the generalized distance measure $d_{N,f}(\cdot, \cdot)$ when $N(\cdot)$ is a constant multiple of the Frobenius norm require another approach. For example, if the structure of continuous Jordan triple endomorphisms on $SO(n)$ were known, we would be able to answer the question. We propose this an open problem for further research but in this paper we finish here.
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