NON-RECURSIVE MULTIPLICITY FORMULAS FOR $A_N$ LIE ALGEBRAS

H. R. Karadayi
Dept. Physics, Fac. Science, Tech.Univ.Istanbul
80626, Maslak, Istanbul, Turkey
Internet: karadayi@sariyer.cc.itu.edu.tr

Abstract

It is shown that there are infinitely many formulas to calculate multiplicities of weights participating in irreducible representations of $A_N$ Lie algebras. On contrary to the recursive character of Kostant and Freudenthal multiplicity formulas, they provide us systems of linear algebraic equations with N-dependent polynomial coefficients. These polynomial coefficients are in fact related with polynomials which represent eigenvalues of Casimir operators.
I. INTRODUCTION

In a previous paper [1], we establish a general method to calculate eigenvalues of Casimir operators of any order. Although it is worked for \( A_N \) Lie algebras in ref.(1), the method works well also for other Classical and Exceptional Lie algebras because any one of them has a subgroup of type \( A_N \). Let \( I(s) \) be a Casimir operator of degree \( s \) and

\[
s_1 + s_2 + \ldots + s_k \equiv s \quad (I.1)
\]

be the partitions of \( s \) into positive integers on condition that

\[
s_1 \geq s_2 \geq \ldots \geq s_k \quad (I.2)
\]

The number of all these partitions of \( s \) is given by the partition function \( p(s) \) which is known to be defined by

\[
\prod_{n=1}^{\infty} \frac{1}{(1 - x^n)} \equiv \sum_{s=0}^{\infty} p(s) x^s.
\]

We also define \( \kappa(s) \) to be the number of partitions of \( s \) into positive integers except 1. We know then that we have at least a \( \kappa(s) \) number of ways to represent the eigenvalues of \( I(s) \) in the form of polynomials

\[
P_{s_1,s_2,\ldots s_k}(\Lambda^+, N) \quad (I.3)
\]

where \( \Lambda^+ \) is the dominant weight which specifies the representation \( R(\Lambda^+) \) for which we calculate eigenvalues. In ref.(1), we give, for orders \( s=4,5,6,7 \) all the polynomials (I.3) explicitly and show that they are in coincidence with the ones calculated directly as traces of the most general Casimir operators [2]. This, on the other hand, brings out another problem of computing multiplicities of all other weights participating within the representation \( R(\Lambda^+) \). This problem has been solved some forty years ago by Kostant [3] and also Freudenthal [4]. In spite of the fact that they are quite explicit, the multiplicity formulas of Kostant and Freudenthal expose serious difficulties in practical calculations due especially to their recursive characters. It is therefore worthwhile to try for obtaining some other non-recursive formulas to calculate weight multiplicities. Let us recall here that, since the last twenty years [5], there are efforts which give us weight multiplicities in the form of tables, one by one.

We will give in this work general formulas with the property that they depend only on multiplicities and rank \( N \). This will give us the possibility to obtain, by induction on values \( N=1,2,\ldots \), as many equations as we need to solve all unknown parameters, i.e. multiplicities.

II. WEYL ORBITS AND IRREDUCIBLE REPRESENTATIONS

We refer the book of Humphreys [6] for Lie algebraic technology which we need in the sequel. The indices like \( i_1, i_2, \ldots \) take values from the set \( I_0 \equiv 1, 2, \ldots N \) while \( I_1, I_2, \ldots \) from \( S_0 \equiv 1, 2, \ldots N, N+1 \). The essential figures are simple roots \( \alpha_i \), fundamental dominant weights \( \lambda_i \) and the fundamental weights defined by

\[
\begin{align*}
\mu_1 &\equiv \lambda_1 \\
\mu_I &\equiv \mu_{I-1} - \alpha_{I-1}, \quad I = 2, 3, \ldots N + 1 \quad (II.1)
\end{align*}
\]

Any dominant weight then has the decompositions

\[
\Lambda^+ = \sum_{i=1}^{N} r_i \lambda_i \quad , \quad r_i \in Z^+ \quad (II.2)
\]

where \( Z^+ \) is the set of positive integers including zero. (II.2) can be expressed equivalently in the form

\[
\Lambda^+ = \sum_{i=1}^{\sigma} q_i \mu_i \quad , \quad \sigma = 1, 2, \ldots N \quad (II.3)
\]
together with the conditions
\[ q_1 \geq q_2 \geq \ldots \geq q_s > 0 \quad \text{(II.4)} \]

The weight space decomposition of an irreducible representation \( R(\Lambda^+) \) has now the form
\[ R(\Lambda^+) = \Pi(\Lambda^+) + \sum_{\lambda^+ \in \text{Sub}(\Lambda^+)} m(\lambda^+ < \Lambda^+) \Pi(\lambda^+) \quad \text{(II.5)} \]

where \( m(\lambda^+ < \Lambda^+) \) is the multiplicity of weight \( \lambda^+ \) within the representation \( R(\Lambda^+) \) and \( \Pi(\lambda^+) \) represents its corresponding Weyl orbit. \( \text{Sub}(\Lambda^+) \) here is the set of all sub-dominant weights of \( \Lambda^+ \). Although the concept of Casimir eigenvalue is known to be defined for representations of Lie algebras, an extension for Weyl orbits has been made, in ref. (1), with the aid of definition
\[ ch_s(\Pi) \equiv \sum_{\mu \in \Pi} (\mu)^s \quad \text{(II.6)} \]

This allows us to make the decompositions
\[ ch_s(\Pi) = \sum_{s_1, s_2, \ldots, s_k} \mu(s_1) \mu(s_2) \ldots \mu(s_k) \cofs_{s_1, s_2, \ldots, s_k} \quad \text{(II.7)} \]

in terms of generators \( \mu(s) \) defined by
\[ \mu(s) \equiv \sum_{i=1}^{N+1} (\mu_i)^s \quad \text{(II.8)} \]

We consider here the partitions (I.1) for any order \( s \). It is then known that the coefficients in (II.7) are expressed in the form
\[ \cofs_{s_1, s_2, \ldots, s_k} \equiv \cofs_{s_1, s_2, \ldots, s_k} (\lambda^+, N) \]

i.e. as \( N \)-dependent polynomials for a Weyl orbit \( \Pi(\lambda^+) \) of \( A_N \) Lie algebras. One further step is to propose the existence of some polynomials \( P_{s_1, s_2, \ldots, s_k} (\lambda^+, N) \) satisfying following equations:
\[ P_{s_1, s_2, \ldots, s_k} (\lambda^+, N) \equiv \frac{\cofs_{s_1, s_2, \ldots, s_k} (\lambda^+, N)}{\cofs_{s_1, s_2, \ldots, s_k} (\lambda_k, N)} \frac{\dim R(\lambda^+, N)}{\dim R(\lambda_k, N)} P_{s_1, s_2, \ldots, s_k} (\lambda_k, N) \quad \text{(II.9)} \]

Note here that
\[ \cofs_{s_1, s_2, \ldots, s_k} (\lambda_i, N) \equiv 0 \quad , \quad i < k \quad \text{(II.10)} \]

and also
\[ \dim R(\lambda_k, N) = \frac{(N + 1)!}{k! (N + 1 - k)!} \quad , \quad k = 1, 2, \ldots N. \quad \text{(II.11)} \]

where \( \dim R(\Lambda^+) \) is the number of weights within the representation \( R(\Lambda^+) \), i.e. its dimension. By examining (II.9) for a few simple representations, a \( \kappa(s) \) number of polynomials can be obtained for each particular value of order \( s \). The ones for \( s = 4, 5, 6, 7 \) are given in ref. (1) explicitly.

### III. THE MULTIPlicity FORMULAs

Our strategy here is to use the equations (II.9) in the form of
\[ \Phi^{s_1, s_2, \ldots, s_k} (\lambda^+, N) \equiv P_{s_1, s_2, \ldots, s_k} (\lambda^+, N) \dim R(\lambda^+, N) \cofs_{s_1, s_2, \ldots, s_k} (\lambda_k, N) - P_{s_1, s_2, \ldots, s_k} (\lambda_k, N) \dim R(\lambda_k, N) \cofs_{s_1, s_2, \ldots, s_k} (\lambda^+, N) \quad \text{(III.1)} \]

with which we obtain, for each particular partition (I.1) of degree \( s \), a multiplicity formula
\[ \Phi^{s_1, s_2, \ldots, s_k} (\lambda^+, N) \equiv 0 \quad \text{(III.2)} \]
for weight multiplicities within the representation $R(\lambda^+)$ and for any values of rank $N > \sigma$. We think $\lambda^+$ here as in the form of (II.3). In addition to the ones given in (II.10), the following expressions can be borrowed from ref.(1):

$$
\begin{align*}
\text{cof}_s(\lambda_1, n) &= 1, \\
\text{cof}_{s_1, s_2}(\lambda_2, n) &= \frac{(s_1 + s_2)!}{s_1! s_2!}, \quad s_1 > s_2, \\
\text{cof}_{s_1, s_2}(\lambda_2, n) &= \frac{(s_1 + s_2)!}{2! s_1! s_2!}, \\
\text{cof}_{s_1, s_2, s_3}(\lambda_3, n) &= \frac{(s_1 + s_2 + s_3)!}{s_1! s_2! s_3!}, \quad s_1 > s_2 > s_3, \\
\text{cof}_{s_1, s_2, s_2}(\lambda_3, n) &= \frac{(s_1 + s_2 + s_2)!}{2! s_1! s_2! s_2!}, \quad s_1 > s_2, \\
\text{cof}_{s_1, s_2, s_2}(\lambda_3, n) &= \frac{(s_1 + s_2 + s_2)!}{3! s_1! s_1! s_1!}.
\end{align*}

(III.3)

These are sufficient to give

$$
\sum_{s=4}^{7} \kappa(s) = 12
$$

different multiplicity formulas originating from the same form (III.2). To proceed further let us take (II.5) in the form

$$
R(\Lambda^+) \equiv \sum_{\alpha=1}^{p(h_{\Lambda^+})} m(\alpha) \Pi(\rho_\alpha), \quad \rho_\alpha \in \text{Sub}(\Lambda^+)
$$

(III.4)

where we usually define the height

$$
h_{\Lambda^+} \equiv \sum_{i=1}^{\sigma} q_i
$$

(III.5)

for (II.3) and $p(h_{\Lambda^+})$ is just the partition function defined above. A further focus here is to make a gradation for elements of the set $\text{Sub}(s_1 \lambda_1)$ by assigning, for each one of them, a grade

$$
\gamma(s_1, s_2, ..., s_k) = 1, 2, ..., p(s)
$$

(III.6)

as being in line with the conditions (I.2). Then, it is clear in (III.4) that

$$
m(\alpha) \equiv 0, \quad \gamma(\rho_\alpha) > \gamma(\Lambda^+)
$$

(III.7)

Note also that all dominant weights within a $\text{Sub}(\Lambda^+)$ must have the same height.

In view of (III.4), one knows both $\text{cof}_{s_1, s_2, ..., s_k}(\lambda^+, N)$ and also $\dim R(\lambda^+, N)$ as linear superpositions of multiplicities $m(\alpha)$ with N-dependent polynomial coefficients. It is noteworthy here that dimensions of Weyl orbits are already known due to a permutational lemma given in ref.(1). We, hence, give in the following our results for 12 multiplicity formulas extracted from (III.2) for $s=4,5,6,7$:

$$
\Phi^7(\lambda^+, N) = \text{cof}_7(\lambda^+, N) g(N) + \\
\dim R(\lambda^+, N) \left( - 720 f^7_7(N) \Theta(7, \lambda^+, N) + 5040 f^7_5(N) \Theta(5, \lambda^+, N) \Theta(2, \lambda^+, N) + 5040 f^7_4(N) \Theta(4, \lambda^+, N) \Theta(3, \lambda^+, N) - 10080 f^7_3(N) \Theta(3, \lambda^+, N) \Theta(2, \lambda^+, N)^2 \right)
$$

(III.8)
\[
\Phi^{52}(\lambda^+, N) = \text{cof}_{52}(\lambda^+, N) \ N \ (N + 2) \ g(N) + \\
\dim R(\lambda^+, N) \ ( \ 5040 \ f_7^{52}(N) \ \Theta(7, \lambda^+, N) \\
- 504 \ f_5^{52}(N) \ \Theta(5, \lambda^+, N) \ \Theta(2, \lambda^+, N) \\
- 5040 \ f_{43}^{52}(N) \ \Theta(4, \lambda^+, N) \ \Theta(3, \lambda^+, N) \\
+ 2520 \ f_{322}^{52}(N) \ \Theta(3, \lambda^+, N) \ \Theta(2, \lambda^+, N)^2 \\
+ 42 \ f_5^{52}(N) \ \Theta(5, \lambda^+, N) \\
- 210 \ f_{32}^{52}(N) \ \Theta(3, \lambda^+, N) \ \Theta(2, \lambda^+, N) \ )
\]

\[
\Phi^{43}(\lambda^+, N) = \text{cof}_{43}(\lambda^+, N) \ 12 \ N \ (N + 2) \ g(N) + \\
\dim R(\lambda^+, N) \ ( \ 60480 \ f_7^{43}(N) \ \Theta(7, \lambda^+, N) \\
- 60480 \ f_5^{43}(N) \ \Theta(5, \lambda^+, N) \ \Theta(2, \lambda^+, N) \\
- 5040 \ f_{43}^{43}(N) \ \Theta(4, \lambda^+, N) \ \Theta(3, \lambda^+, N) \\
+ 5040 \ f_{322}^{43}(N) \ \Theta(3, \lambda^+, N) \ \Theta(2, \lambda^+, N)^2 \\
- 7 \ f_5^{43}(N) \ \Theta(3, \lambda^+, N) \ )
\]

\[
\Phi^{322}(\lambda^+, N) = \text{cof}_{322}(\lambda^+, N) \ 24 \ N \ (N + 2) \ g(N) + \\
\dim R(\lambda^+, N) \ ( \ -241920 \ f_7^{322}(N) \ \Theta(7, \lambda^+, N) \\
+ 60480 \ f_5^{322}(N) \ \Theta(5, \lambda^+, N) \ \Theta(2, \lambda^+, N) \\
+ 10080 \ f_{43}^{322}(N) \ \Theta(4, \lambda^+, N) \ \Theta(3, \lambda^+, N) \\
- 5040 \ f_{322}^{322}(N) \ \Theta(3, \lambda^+, N) \ \Theta(2, \lambda^+, N)^2 \\
- 5040 \ f_5^{322}(N) \ \Theta(5, \lambda^+, N) \\
- 840 \ f_{32}^{322}(N) \ \Theta(3, \lambda^+, N) \ \Theta(2, \lambda^+, N) \\
- 7 \ f_5^{322}(N) \ \Theta(3, \lambda^+, N) \ )
\]

\[
\Phi^6(\lambda^+, N) = \text{cof}_6(\lambda^+, N) \ 252 \ g_6(N) + \dim R(\lambda^+, N) \ (N + 1) \ g(N) + \\
\dim R(\lambda^+, N) \ ( \ -30240 \ f_7^6(N) \ \Theta(6, \lambda^+, N) \\
+ 181440 \ f_5^6(N) \ \Theta(4, \lambda^+, N) \ \Theta(2, \lambda^+, N) \ ) \ (III.12) \\
+ 30240 \ f_{43}^6(N) \ \Theta(3, \lambda^+, N)^2 \\
- 211680 \ f_{222}^6(N) \ \Theta(2, \lambda^+, N)^3 \ )
\]

\[
\Phi^{42}(\lambda^+, N) = \text{cof}_{42}(\lambda^+, N) \ 672 \ N \ (N + 1) \ (N + 2) \ g_6(N) + \\
\dim R(\lambda^+, N) \ N \ (N + 1) \ (N + 2) \ (7 \ N^2 + 14 \ N + 47) \ g_6(N) + \\
\dim R(\lambda^+, N) \ ( \ 483840 \ f_7^{42}(N) \ \Theta(6, \lambda^+, N) \\
- 60480 \ f_{43}^{42}(N) \ \Theta(4, \lambda^+, N) \ \Theta(2, \lambda^+, N) \\
- 1209600 \ f_{322}^{42}(N) \ \Theta(3, \lambda^+, N)^2 \\
+ 60480 \ f_{222}^{42}(N) \ \Theta(2, \lambda^+, N)^3 \\
+ 5040 \ f_5^{42}(N) \ \Theta(4, \lambda^+, N) \\
- 5040 \ f_3^{42}(N) \ \Theta(2, \lambda^+, N)^2 \\
- 84 \ f_3^{42}(N) \ \Theta(2, \lambda^+, N) \ )
\]
\[ \Phi^{33}(\lambda^+, N) = \text{cof}_{33}(\lambda^+, N) 
126 \ N \ (N + 1) \ (N + 2) \ g_6(N) - 
\text{dim} R(\lambda^+, N) \ 5 \ N \ (N + 1) \ (N + 2) \ g_6(N) + 
\text{dim} R(\lambda^+, N) \ (15120 \ f_{6}^{33}(N) \ \Theta(6, \lambda^+, N) 
- 226800 \ f_{42}^{33}(N) \ \Theta(4, \lambda^+, N) \ \Theta(2, \lambda^+, N) 
- 5040 \ f_{42}^{33}(N) \ \Theta(3, \lambda^+, N)^2 
+ 60480 \ f_{222}^{33}(N) \ \Theta(2, \lambda^+, N)^3 ) \] (III.14)

\[ \Phi^{22}(\lambda^+, N) = \text{cof}_{22}(\lambda^+, N) 
576 \ N \ (N + 1) \ (N + 2) \ g_6(N) + 
\text{dim} R(\lambda^+, N) \ N \ (N + 1)^2 \ (N + 2) \ (5 \ N^2 + 10 \ N + 23) \ g_6(N) + 
\text{dim} R(\lambda^+, N) \ (483840 \ f_{6}^{22}(N) \ \Theta(6, \lambda^+, N) 
- 51840 \ f_{42}^{22}(N) \ \Theta(4, \lambda^+, N) \ \Theta(2, \lambda^+, N) 
- 276480 \ f_{42}^{22}(N) \ \Theta(3, \lambda^+, N)^2 
+ 8640 \ f_{42}^{22}(N) \ \Theta(2, \lambda^+, N)^3 
+ 4320 \ f_{42}^{22}(N) \ \Theta(4, \lambda^+, N) 
- 2160 \ f_{42}^{22}(N) \ \Theta(2, \lambda^+, N)^2 
+ 36 \ f_{222}^{22}(N) \ \Theta(2, \lambda^+, N) ) \] (III.15)

\[ \Phi^5(\lambda^+, n) = \text{cof}_{5}(\lambda^+, n) \ g_5(n) 
- \text{dim} R(\lambda^+, n) \ (24 \ f^5_5(n) \ \Theta(5, \lambda^+, n) 
- 120 \ f^5_{42}(n) \ \Theta(3, \lambda^+, n) \ \Theta(2, \lambda^+, n) ) \] (III.16)

\[ \Phi^{32}(\lambda^+, n) = \text{cof}_{32}(\lambda^+, n) \ 3 \ g_5(n) + 
\text{dim} R(\lambda^+, n) \ (360 \ f_{32}^{32}(n) \ \Theta(5, \lambda^+, n) 
- 60 \ f_{42}^{32}(n) \ \Theta(3, \lambda^+, n) \ \Theta(2, \lambda^+, n) 
+ 5 \ f_{32}^{32}(n) \ \Theta(3, \lambda^+, n) ) \] (III.17)

\[ \Phi^4(\lambda^+, N) = -\text{cof}_{4}(\lambda^+, N) \ 120 \ g_4(N) + \text{dim} R(\lambda^+, N) \ (N + 1) \ g_4(N) + 
\text{dim} R(\lambda^+, N) \ (720 \ f^4_4(N) \ \Theta(4, \lambda^+, N) 
- 720 \ f_{42}^4(N) \ \Theta(2, \lambda^+, N)^2 ) \] (III.18)

\[ \Phi^{22}(\lambda^+, N) = \text{cof}_{22}(\lambda^+, N) \ 240 \ (N + 1) \ g_4(N) - (N + 1) \ g_{22}(N) \text{dim} R(\lambda^+, N) + 
\text{dim} R(\lambda^+, N) \ (1440 f_{224}(N) \Theta(4, \lambda^+, N) 
- 720 f_{222}(N) \Theta(2, \lambda^+, N)^2 
+ 120 f_{222}(N) \Theta(2, \lambda^+, N) ) \] (III.19)

where the quantities defined by

\[ \Theta(s, \lambda^+, N) = \sum_{i=1}^{N+1} (\theta_i)^s \]

can be calculated explicitly via re-definitions

\[ 1 + r_i \equiv \theta_i - \theta_{i+1} , \ i \in I_o \]

of the parameters \( r_i \) in (II.2). Note here that \( \Theta(1, \lambda^+, N) \equiv 0 \). All coefficient polynomials are given in appendix.
IV. AN EXAMPLE : $R(\lambda_1 + \lambda_2 + \lambda_6)$

Now it will be instructive to demonstrate the idea in an explicit example, chosen, say, from the set $\text{Sub}(9, \lambda_1)$ with the gradation (III.6) from 1 to $p(9) = 30$ for its 30 elements all having the same height ($= 9$). In the notation of parameters $q_i$ defined in (II.3), (III.4) turns out to be

$$R(\lambda_1 + \lambda_2 + \lambda_6) = m(0) \Pi(3, 2, 1, 1, 1, 1) + m(1) \Pi(2, 2, 1, 1, 1) + m(2) \Pi(3, 1, 1, 1, 1, 1) + m(3) \Pi(2, 2, 1, 1, 1, 1) + m(4) \Pi(2, 1, 1, 1, 1, 1) + m(5) \Pi(1, 1, 1, 1, 1, 1, 1)$$

with

$$\dim R(\lambda_1 + \lambda_2 + \lambda_6) = \frac{m(0)}{24} (N - 4) (N - 3) (N - 2) (N - 1) N (N + 1) + \frac{m(1)}{36} (N - 4) (N - 3) (N - 2) (N - 1) N (N + 1) + \frac{m(2)}{720} (N - 5) (N - 4) (N - 3) (N - 2) (N - 1) N (N + 1) + \frac{m(3)}{240} (N - 5) (N - 4) (N - 3) (N - 2) (N - 1) N (N + 1) + \frac{m(4)}{5040} (N - 6) (N - 5) (N - 4) (N - 3) (N - 2) (N - 1) N (N + 1) + \frac{m(5)}{362880} (N - 7) (N - 6) (N - 5) (N - 4) (N - 3) (N - 2) (N - 1) N (N + 1).$$

and with a straightforward computation

$$\Theta(2, \lambda_1 + \lambda_2 + \lambda_6, N) = \frac{3}{g(2, N)} (-1152 - 70N + 113 N^2 + 4 N^3 + N^4)$$

$$\Theta(4, \lambda_1 + \lambda_2 + \lambda_6, N) = \frac{1}{g(4, N)} (-7925760 - 3447368 N - 69144 N^2 + 191516 N^3 + 11947 N^4 - 2052 N^5 + 1154 N^6 + 24 N^7 + 3 N^8)$$

$$\Theta(6, \lambda_1 + \lambda_2 + \lambda_6, N) = \frac{1}{g(6, N)} (-9704669184 - 9453386848 N - 3436715360 N^2 + 155802792 N^3 + 289888824 N^4 + 644322 N^5 - 10826973 N^6 + 375224 N^7 + 259141 N^8 - 7110 N^9 + 4445 N^{10} + 36 N^{11} + 3 N^{12})$$

$$\Theta(3, \lambda_1 + \lambda_2 + \lambda_6, N) = \frac{432}{g(3, N)} (448 + 98 N - 31 N^2 - 4 N^3 + N^4)$$

$$\Theta(5, \lambda_1 + \lambda_2 + \lambda_6, N) = \frac{432}{g(5, N)} (663040 + 458420 N + 96638 N^2 - 22556 N^3 - 8441 N^4 + 608 N^5 + 144 N^6 - 16 N^7 + 3 N^8)$$

$$\Theta(7, \lambda_1 + \lambda_2 + \lambda_6, N) = \frac{288}{g(7, N)} (1099055104 + 1399367648 N + 708562192 N^2 + 61441320 N^3 - 51346940 N^4 - 8957530 N^5 + 2041581 N^6 + 268592 N^7 - 85565 N^8 - 1730 N^9 + 1875 N^{10} - 60 N^{11} + 9 N^{12})$$

where

$$g(s, N) = 3 2^s (s + 1) (N + 1)^{s-1}.$$
In this example, we have suppressed explicit $N$-dependences though we recall that all expressions are valid for $N > 6$. It is thus seen that all the formulas given from (III.8) to (III.19) above give rise to the same result

\[
\begin{align*}
    m(1) &= 2 m(0) \\
    m(2) &= 5 m(0) \\
    m(3) &= 10 m(0) \\
    m(4) &= 35 m(0) \\
    m(5) &= 105 m(0)
\end{align*}
\]

for which one always knows that $m(0) = 1$. As can be easily investigated by Weyl dimension formula, this also leads us to the result

\[
\dim R(\lambda_1 + \lambda_2 + \lambda_6) = \frac{1}{3456} (N - 4) (N - 3) (N - 2) (N - 1) N (N + 1)^2 (N + 2) (N + 3)
\]

V. CONCLUSIONS

We obtained here 12 multiplicity formulas for a weight within an irreducible representation of $A_N$ Lie algebras. The method is based essentially on some polynomials representing eigenvalues of Casimir operators. These polynomials has been given for degrees $s=4,5,6,7$ in a previous work. If one further considers the Casimir operators of higher degrees, it is clear that one can obtain, in essence, an infinite number of multiplicity formulas.

On the other hand, Casimir eigenvalues of any other Classical or Exceptional Lie algebra having a subalgebra of type $A_N$ can be obtained by the aid of these polynomials. It could therefore be expected that the multiplicity formulas given above are to be generalized further to these Lie algebras. Another point which seems to be worthwhile to study is the hope that such a framework will prove useful also for Lie algebras beyond the finite ones. To this end, the crucial point will be to fix a convenient subalgebra which underlies the infinite dimensional one. For instance, one can think that an $E_8$ multiplicity formula could be reformulated in terms of its subalgebra $A_7$ or more suitably $A_8$. But what is more intriguing is to consider the same possibility, say, for hyperbolic Lie algebra $E_{10}$. It is clear that to investigate these possibilities shed some light on the multiplicity problems of infinite Lie algebras for which quite little is known about their multiplicity formulas in general.

REFERENCES

[1] Karadayi H.R and Gungormez M: Explicit Construction of Casimir Operators and Eigenvalues:II , physics/mathematical methods in physics/9611002, submitted to Jour.Math.Phys.
[2] Karadayi H.R and Gungormez M: Explicit Construction of Casimir Operators and Eigenvalues:I , hep-th/9609060, submitted to Jour.Math.Phys.
[3] Kostant B. ; Trans.Am.Math.Soc. 93 (1959) 53-73
[4] Freudenthal H. ; Indag.Math. 16 (1954) 369-376 and 487-491
Freudenthal H. ; Indag.Math. 18 (1956) 511-514
[5] Patra J. and Sankoff D. : Tables of Branching Rules for Representations of Simple Lie Algebras, L’Universite de Montreal, 1973
McKay W. and Patra J. : Tables of Dimensions, Indices and Branching Rules for Representations of Simple Algebras, Dekker, NY 1981
Slansky R : Group Theory for Unified Model Building, Physics Reports
[6] Humphreys J.E: Introduction to Lie Algebras and Representation Theory, Springer-Verlag (1972) N.Y.
APPENDIX

We give here N-dependent coefficient polynomials encountered in the multiplicity formulas given in section (III).

\[ f_2^7(N) = N^6 + 6 N^5 + 50 N^4 + 160 N^3 + 309 N^2 + 314 N + 120 \]
\[ f_{52}^2(N) = N^5 + 5 N^4 + 21 N^3 + 43 N^2 - 70 N - 96 \]
\[ f_{52}^3(N) = N^5 + 5 N^4 + 9 N^3 + 7 N^2 + 62 N + 60 \]
\[ f_{322}^3(N) = 2 N^4 + 8 N^3 - 5 N^2 - 26 N - 15 \]
\[ f_{32}^2(N) = N^7 + 7 N^6 + 31 N^5 + 85 N^4 + 16 N^3 - 236 N^2 - 192 N \]
\[ f_{52}^2(N) = N^8 + 8 N^7 + 32 N^6 + 80 N^5 + 515 N^4 + 1676 N^3 + 1648 N^2 + 72 N - 10080 \]
\[ f_{13}^2(N) = 6 N^6 + 36 N^5 - N^2 + 13 N^4 - 188 N^3 + 470 N + 840 \]
\[ f_{32}^3(N) = N^7 + 7 N^6 - 70 N^4 + 217 N^3 + 987 N^2 - 134 N - 840 \]
\[ f_{13}^2(N) = 2 N^7 + 14 N^6 + 133 N^5 + 525 N^4 - 553 N^3 - 3647 N^2 + 1510 N + 4200 \]
\[ f_{3}^{33}(N) = (N - 5) (N - 4) (N - 3) (N - 2) N (N + 1)^3 (N + 2) (N + 4) (N + 5) (N + 6) (N + 7) \]
\[ f_{22}^3(N) = 2 N^6 + 12 N^5 + 11 N^4 - 36 N^3 - 67 N^2 - 30 N \]
\[ f_{52}^2(N) = N^7 + 7 N^6 - 70 N^4 + 217 N^3 + 987 N^2 - 134 N - 840 \]
\[ f_{13}^2(N) = 2 N^7 + 14 N^6 + 133 N^5 + 525 N^4 - 553 N^3 - 3647 N^2 + 1510 N + 4200 \]
\[ f_{32}^3(N) = N^8 + 8 N^7 - 3 N^6 - 130 N^5 + 109 N^4 + 1452 N^3 + 5113 N^2 + 6890 N - 4200 \]
\[ f_{5}^{32}(N) = (N - 5) (N - 4) N (N + 1)^2 (N + 2) (N + 6) (N + 7) (N^2 + 2 N - 1) \]
\[ f_{3}^{32}(N) = -(N - 4) (N - 5) N (N + 1) (N + 2) (N + 6) (N + 7) (N^4 + 4 N^3 + 6 N^2 + 4 N + 25) \]
\[ f_{3}^{22}(N) = (N - 5) (N - 4) N (N + 1) (N + 2) (N + 6) (N + 7) (5 N^7 + 35 N^6 - 14 N^5 - 420 N^4 - 445 N^3 + 625 N^2 + 2014 N + 1320) \]
\[ f_{0}^{6}(N) = N^5 + 5 N^4 + 25 N^3 + 55 N^2 + 58 N + 24 \]
\[ f_{52}^6(N) = N^4 + 4 N^3 + 7 N^2 + 6 N - 18 \]
\[ f_{32}^5(N) = 3 N^4 + 12 N^3 + 7 N^2 - 10 N + 72 \]
\[ f_{22}^6(N) = N^3 + 3 N^2 - 4 N - 6 \]
\[ f_{12}^6(N) = N^7 + 7 N^6 + 21 N^5 + 35 N^4 + 14 N^3 - 42 N^2 - 36 N \]
\[ f_{12}^7(N) = N^8 + 8 N^7 + 28 N^6 + 56 N^5 + 169 N^4 + 452 N^3 + 762 N^2 + 684 N - 2160 \]
\[ f_{12}^{12}(N) = N^6 + 6 N^5 + 5 N^4 - 20 N^3 - 20 N^2 + 16 N + 96 \]
\[ f_{12}^{22}(N) = 2 N^7 + 14 N^6 - 3 N^5 - 155 N^4 + 163 N^3 + 1221 N^2 - 162 N - 1080 \]
\[ f_{12}^4(N) = N^{11} + 11 N^{10} + 14 N^9 - 204 N^8 - 747 N^7 - 189 N^6 + 3716 N^5 + 9334 N^4 + 10696 N^3 + 6168 N^2 + 1440 N \]
\[ f_{12}^{22}(N) = 2 N^{10} + 20 N^9 + 3 N^8 - 456 N^7 - 1008 N^6 + 1680 N^5 + 7327 N^4 + 7036 N^3 + 1236 N^2 - 720 N \]
\[ f_2^3 = (N + 1)^2 g_0(N) \]
\[ f_{6}^{33}(N) = 3 N^7 + 21 N^6 + 49 N^5 + 35 N^4 + 56 N^3 + 196 N^2 + 144 N \]
\[ f_{6}^{44}(N) = N^6 + 6 N^5 + 5 N^4 - 20 N^3 - 20 N^2 + 16 N + 96 \]
\[ f_{6}^{33}(N) = N^8 + 8 N^7 - 112 N^5 + 127 N^4 + 1404 N^3 + 580 N^2 - 2032 N - 3840 \]
\[ f_{6}^{33}(N) = 4 N^5 + 20 N^4 - 19 N^3 - 137 N^2 + 78 N + 180 \]
\[ f_{6}^{22}(N) = N^6 + 6 N^5 + 7 N^4 - 12 N^3 - 26 N^2 - 12 N \]
\[ f_{6}^{44}(N) = 2 N^7 + 14 N^6 - 3 N^5 - 155 N^4 + 163 N^3 + 1221 N^2 - 162 N - 1080 \]
\[ f_{6}^{44}(N) = 4 N^5 + 20 N^4 - 19 N^3 - 137 N^2 + 78 N + 180 \]
\[ f_{6}^{22}(N) = N^8 + 8 N^7 - 7 N^6 - 154 N^5 - 79 N^4 + 860 N^3 + 1777 N^2 + 1338 N - 3240 \]
\[ f_{6}^{44}(N) = 2 N^{10} + 20 N^9 + 3 N^8 - 456 N^7 - 1008 N^6 + 1680 N^5 + 7327 N^4 + 7036 N^3 + 1236 N^2 - 720 N \]
\[ f_{6}^{22}(N) = N^{11} + 11 N^{10} + 7 N^9 - 267 N^8 - 687 N^7 + 1407 N^6 + 5543 N^5 + 157 N^4 - 6664 N^3 + 6252 N^2 + 9360 N \]
\[ f_{6}^{22}(N) = 5 N^{14} + 70 N^{13} + 186 N^{12} - 240 N^{11} - 7964 N^{10} - 1320 N^9 + 65098 N^8 + 121616 N^7 - 67617 N^6 - 437030 N^5 - 422284 N^4 + 127992 N^3 + 432576 N^2 + 190080 N \]
\[ f_{6}^5(N) = N^4 + 4 N^3 + 11 N^2 + 14 N + 6 \]
\[ f_{6}^7(N) = N^3 + 3 N^2 + N - 1 \]
\[ f_{6}^{32}(N) = N^3 + 3 N^2 + N - 1 \]
\[ f_{6}^{22}(N) = N^4 + 4 N^3 + 6 N^2 + 4 N + 25 \]
\[ f_{6}^{44}(N) = (N - 3) (N - 2) (N + 1)^3 (N + 4) (N + 5) \]
\[ f_{6}^{44}(N) = N^3 + 3 N^2 + 4 N + 2 \]
\[ f_{6}^{22}(N) = 2 N^2 + 4 N - 1 \]
\[ f_{6}^{22}(N) = 2 N^3 + 6 N^2 + 3 N - 1 \]
\[ f_{6}^{22}(N) = N^4 + 4 N^3 - 8 N + 13 \]
\[ f_{6}^{22}(N) = N^7 + 7 N^6 + 8 N^5 - 30 N^4 - 59 N^3 - N^2 + 50 N + 24 \]