Circuit Lower Bounds for the p-Spin Optimization Problem

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Abstract

We consider the problem of finding a near ground state of a p-spin model with Rademacher couplings by means of a low-depth circuit. As a direct extension of the authors’ recent work [GJW20], we establish that any poly-size n-output circuit that produces a spin assignment with objective value within a certain constant factor of optimality, must have depth at least log n/(2 log log n) as n grows. This is stronger than the known state of the art bounds of the form Ω(log n/(k(n) log log n)) for similar combinatorial optimization problems, where k(n) depends on the optimality value. For example, for the largest clique problem k(n) corresponds to the square of the size of the clique [Ros10]. At the same time our results are not quite comparable since in our case the circuits are required to produce a solution itself rather than solving the associated decision problem. As in our earlier work [GJW20], the approach is based on the overlap gap property (OGP) exhibited by random p-spin models, but the derivation of the circuit lower bound relies further on standard facts from Fourier analysis on the Boolean cube, in particular the Linial-Mansour-Nisan Theorem.

To the best of our knowledge, this is the first instance when methods from spin glass theory have ramifications for circuit complexity.

1 Introduction

Boolean circuits constitute one of the standard models for understanding algorithmic tractability and hardness in combinatorial optimization problems [AB09, Sip97, AS04]. One potential route to proving the widely-believed conjecture $P \neq NP$ would be to show non-existence of polynomial-size Boolean circuits solving problems in $NP$. A large body of literature in circuit complexity is devoted to establishing limits on the power of circuits with bounds on its depth, specifically the power of constant depth circuits known as $AC^0$ circuits and its immediate extension – circuits with nearly logarithmic (in problem size) depth. Many hardness results have been obtained by working on models with random inputs. A classical result is one by Håstad [Has86] who established a $\log n/(\log \log n + O(1))$ lower bound on the depth of any poly-size circuit that

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computes the $n$-parity function. A lot of focus is on circuit complexity for computing various natural combinatorial optimization problems. For example Rossman [Ros10] has shown that poly-size circuits detecting the presence of a $k(n)$-size clique in a graph must have depth at least $\Omega(\log n / (k^2(n) \log \log n))$, where $n$ is the number of graph nodes and $k(n)$ is any function growing in $n$. This was obtained by working with the sparse random Erdős-Rényi graph model with the edge probability tuned so that the largest “naturally occurring” clique is of size smaller than $k(n)$, with high probability as $n$ increases. In fact, for combinatorial optimization problems this represents the state of the art lower bound on circuit depth for polynomial-size circuits, though many extensions exist for subgraph homomorphism existence problems [Ros18, LRR17].

In this paper we establish a lower bound of $\log n / (2 \log \log n)$ on the depth of any polynomial-size Boolean circuit that solves another natural combinatorial optimization problem. It is a core computational problem in statistical physics of finding a near ground state of a p-spin model with i.i.d. Rademacher couplings. Specifically, we show that any polynomial-size $n$-output Boolean circuit that produces a $\pm 1$ spin assignment with objective value within a certain constant factor of optimality, has depth at least $\log n / (2 \log \log n)$. In particular, unlike the aforementioned result for cliques, our lower bound does not depend on the objective value. The result though, strictly speaking, does not improve upon [Ros10], since our circuits are supposed to produce the entire solution, as opposed to just solving the associated YES/NO decision problem. Nevertheless, to the best of our knowledge, our result presents the strongest known circuit depth lower bounds for combinatorial optimization problems.

Our result is obtained as a fairly direct extension of our recent work [GJW20], where limits are obtained for algorithms based on low-degree polynomials for the same problem (though with Gaussian as opposed to Rademacher couplings). The associated optimization problem admits a polynomial-time approximation scheme for the case $p = 2$ [Mon21] and also for some so-called mixed spin models [AMS20], as well as for some so-called spherical spin glass models [Sub21] associated with optimizing over the sphere as opposed to the binary cube. But as shown in [GJW20], algorithms based on low-degree polynomials fail in general for the problem of finding near ground states in $p$-spin models.

As in our earlier paper we employ the overlap gap property (OGP) exhibited by this and many other combinatorial optimization and constraint satisfaction problems with random inputs. In the present context, the OGP says that every two spin assignments that achieve value within a certain multiplicative factor away from optimality have to either agree on many coordinates (high overlap) or disagree on many coordinates (low overlap). More specifically, as in the aforementioned reference, we employ the so-called ensemble overlap gap property (e-OGP), which we formally introduce in the next section. At the same time, employing standard circuit insensitivity results, specifically the Linial-Mansour-Nisan Theorem [LMN93], we show that vectors produced by the outputs of $n$ low-depth circuits can produce two spin assignments with overlap value ruled out by the e-OGP, thus obtaining a contradiction. The Linial-Mansour-Nisan Theorem states that low-depth polynomial-size circuits are well approximated by the low-degree terms in their Fourier expansion. The version we use in this paper is found on page 93 in [O'D14]. The tightest known bound of this kind was obtained by Tal in [Tal17].

We believe that the approach used in this paper can be extended to many other models exhibiting OGP and its variants (see [GJW20] for a more thorough treatment of the subjects). However, additional analysis seems necessary for validating such extensions, since many such models are defined on sparse random graphs, and the variant of the Linial-Mansour-Nisan Theorem that is valid in this setting is not as tight.
orem for sparse graph models (see for example Lemma 9 in [FJS91]) appears unfortunately too weak to rule out depths of order \( \log n / \log \log n \). Similarly, it is an open question whether OGP-based methods can be used to rule out decision-type (single output) low-depth circuits for these problems.

2 Models, Assumptions and Results

2.1 \( p \)-spin model and its ground state

We begin by describing the optimization problem of finding the ground state value of a \( p \)-spin model. Let \( J \in \{\pm 1\}^{n^p} \) denote a \( p \)-tensor on \( \mathbb{R}^n \) with \( \{\pm 1\} \)-valued entries. That is, \( J = (J_{i_1,\ldots,i_p}, 1 \leq i_1, i_2, \ldots, i_p \leq n) \) and \( J_{i_1,\ldots,i_p} = \pm 1 \) for each \( p \)-tuple \( i_1, \ldots, i_p \). For every \( \sigma \in \{\pm 1\}^n \), we let

\[
\langle J, \sigma \rangle \triangleq \sum_{1 \leq i_1, i_2, \ldots, i_p \leq n} J_{i_1,i_2,\ldots,i_p} \sigma_{i_1} \sigma_{i_2} \cdots \sigma_{i_p}.
\]

The optimal value of the optimization problem

\[
n^{-\frac{p+1}{2}} \max_{\sigma \in \{\pm 1\}^n} \langle J, \sigma \rangle
\]

is called the ground state energy of the \( p \)-spin model with coupling \( J \). Any optimizer \( \sigma \) achieving the optimal value is called the ground state. We will assume that the entries of \( J \) are i.i.d. equiprobable \( \pm 1 \) (that is, Rademacher) entries. In this setting, we denote the optimal value of (1) by \( \eta^*_n,p \), which is a random variable. The normalization by \( n^{\frac{p+1}{2}} \) in the objective function is chosen so that the ground state energy is typically order 1. In particular, we recall here the following well-known concentration result.

**Theorem 2.1.** For every \( p \), there exists \( \eta^*_p > 0 \) such that for every \( \epsilon > 0 \)

\[
\mathbb{P}(|\eta^*_n,p - \eta^*_p| \geq \epsilon) \leq \exp(-\gamma n),
\]

for some \( \gamma > 0 \) and all large enough \( n \).

The existence of the limit above follows from a rather simple interpolation argument based essentially on Gaussian comparison inequality techniques. This was shown first by Guerra and Toninelli [GT02]. The value of \( \eta^*_p \) can be computed to any desired precision using the powerful Parisi variational representation as was done first by Parisi [Par80], and rigorously verified later by Talagrand [Tal06]. See Panchenko [Pan13] for a book treatment of the subject.

We now state the ensemble overlap gap property (e-OGP) exhibited by the \( p \)-spin model above. For this purpose we need to consider an ensemble of random tensors constructed as follows. Let \( J \) and \( \bar{J} \) be two independent copies of the random \( p \)-spin model above with Rademacher entries constructed as follows. Let \( \leq_R \) denote the (random) total order on \([n^p]\), given by drawing points from that set uniformly at random without replacement. We call this the uniform at random order. Let \( I_t \in [n^p] \) be the point that was drawn at time \( t \). Starting with \( J_0 = J \), let \( J_t \) denote the random tensor obtained from \( J_{t-1} \), by resampling the entry indexed by \( I_t \) according to the Rademacher distribution, independently. Define \( \bar{J} = J_{n^p} \). Note that for each fixed \( t \), the values \( J_t, J \) and \( \bar{J} \) are identically distributed (but not independent). We now formally state the e-OGP.
Theorem 2.2. For every even \( p \geq 4 \), there exists \( 0 < \eta_{\text{OGP},p} < \eta_p^* \) and \( 0 < \nu_{1,p} < \nu_{2,p} < 1 \), and \( \gamma > 0 \) such that for all sufficiently large \( n \), with probability at least \( 1 - \exp(-\gamma n) \), the following holds:

(a) For every \( 0 \leq t_1 \leq t_2 \leq np \) and every pair \( \sigma_1, \sigma_2 \in \{\pm 1\}^n \) satisfying \( \langle J_{t_j}, \sigma_j \rangle \geq \eta_{\text{OGP},p} n^{\frac{p+1}{2}} \) for \( j = 1, 2 \), it is the case

\[
\frac{|\langle \sigma_1, \sigma_2 \rangle|}{n} \notin (\nu_{1,p}, \nu_{2,p}),
\]

(b) In the special case \( t_1 = 0, t_2 = np \), in fact

\[
\frac{|\langle \sigma_1, \sigma_2 \rangle|}{n} \leq \nu_{1,p}.
\]

The proof is given in Section 3.2. Informally, e-OGP states that for every pair of instances in the interpolated family, including the case of identical instances (that is \( t_1 = t_2 \)), every two solutions achieving multiplicative factor \( \eta_{\text{OGP},p}/\eta^*_p \) to optimality in those two instances are either close or far from each other. Furthermore, for the case of independent instances (that is \( t_1 = 0, t_2 = np \)), only the latter is possible. Theorem 2.2 was proved in [GJ21] for the case of Gaussian as opposed to Rademacher distribution, but its adaptation to the latter case follows from standard universality-type arguments based on Lindeberg’s approach, which can be found in the original paper [GT02] which established the existence of the limit \( \eta^*_p \). For a recent work that covers many examples of such arguments see [Sen18] and for a textbook presentation of this and related spin glass results, see [Pan13].

The core of the proof in [GJ21] is the case of a single-instance (non-ensemble) OGP for the Gaussian case which was done in [CGPR19]. Its adaptation to the ensemble case follows in a rather straightforward way using the chaos property exhibited by the \( p \)-spin models. Adaptation to non-Gaussian distribution, as we said, uses standard universality arguments.

2.2 Boolean circuits

We now turn to the discussion of Boolean circuits. Formal definitions of those can be found in most standard textbooks on algorithms and computation [AB09, Sip97, AS04]. Informally, these are functions from \( \{0,1\}^M \rightarrow \{0,1\}^n \) obtained by considering a directed graph with \( M \) input nodes which have in-degree zero, and \( n \) output nodes which have out-degree zero. Each intermediate node corresponds to one of three standard Boolean operation \( \lor, \land \) or \( \neg \). The size of the circuit is the number of nodes in the associated graph and its depth is the length of its longest directed path. Equivalently, one may consider instead functions from \( \{\pm 1\}^M \rightarrow \{\pm 1\}^n \) by adopting the appropriate logical functions in gates. We also adopt this assumption for convenience. In particular, when \( M = np \), such a Boolean function maps any \( p \)-tensor with \( \pm 1 \) entries into an \( n \)-vector with \( \pm 1 \) entries. Thus given an \( n \)-output, \( M = np \)-input Boolean circuit \( C \) and \( p \)-tensor \( J \) with \( \pm 1 \) entries, we denote by \( C(J) \) the binary vector in \( \{\pm 1\}^n \) produced when \( C \) takes input \( J \). We denote by \( C_j \) the single-output Boolean circuit associated with the \( j \)-th component of \( C \) so that \( C(J) = (C_j(J), 1 \leq j \leq n) \).

As stated in the introduction, we are interested in Boolean circuits with bounds on their size and depth. Fix any two \( n \)-dependent positive integer-valued sequences \( s(n), d(n) \) and a constant
0 < \rho < 1. We denote by \( C_p(n, s(n), d(n), \rho) \) the family of all \( n^p \)-input, \( n \)-output Boolean circuits \( C \) which satisfy the following properties:

(a) The size (the number of gates) of \( C \) is at most \( s(n) \) and its depth is at most \( d(n) \).

(b) For every tensor \( J \in \{ \pm 1 \}^{n^p} \) such that the value in (1) is non-negative, the output \( C(G) \) satisfies

\[ \langle J, C(J) \rangle \geq \rho \max_{\sigma \in \{ \pm 1 \}^n} \langle J, \sigma \rangle. \]

If the value in (1) is negative, the output \( C(J) \) can be arbitrary.

In other words, this is a family of circuits which is required to produce a solution with value which is at least a multiplicative constant \( \rho \) away from optimality when the value of the optimization problem is non-negative. Clearly a family of such circuits can be empty if either \( s(n) \) or \( d(n) \) is too restrictive.

We now state our main result.

**Theorem 2.3.** For every even \( p \geq 4, \alpha > 0 \) and \( \rho > \eta_{\text{OGP},p}/\eta_p^* \), for all sufficiently large \( n \), either the associated family of circuits \( C_p(n, s(n), d(n), \rho) \) is empty, or \( s(n) \geq n^\alpha \), or \( d(n) \geq \log n/(2 \log \log n) \).

The result above essentially rules out poly-size circuits with the stated bound on depth that obtain a solution with value at least \( \rho \) times the optimum value, where \( \rho \) is any constant larger than \( \eta_{\text{OGP},p}/\eta_p^* \). The proof is given in Section 3.1. The constant 2 in the depth lower bound can be slightly improved, although it appears that it has to be larger than 1 for our analysis to go through. We note that \( \alpha \) does not appear in the claimed lower bound on the circuit depth because the bound holds even for mildly super-polynomially sized circuits. We will not attempt to find the optimal growth rate.

### 3 Proofs

#### 3.1 Proof of Theorem 2.3

We fix any \( \alpha > 0 \) and any circuit \( C \in C_p(n, n^\alpha, d(n), \rho) \). Recall that the output of the Boolean circuit \( C \) acting on any input \( J \in \{ \pm 1 \}^{n^p} \) is a vector in \( \{ \pm 1 \}^n \) and thus has \( \| \cdot \|_2 \) norm equal \( n \). Fix a constant \( \kappa > 0 \). Consider any two tensors \( J_1, J_2 \in \{ \pm 1 \}^{n^p} \) which differ in at most one entry. Namely, there exists \( 1 \leq i_1, \ldots, i_p \leq n \) such that \( J_1 \) and \( J_2 \) are identical at all multi-indices \( (i_1', \ldots, i_p') \) that are distinct from the \( p \)-tuple \( (i_1, \ldots, i_p) \). We say that the pair \( (J_1, J_2) \) is \( \kappa \)-bad if

\[ \| C(J_1) - C(J_2) \|_2^2 \geq \kappa n. \]  

(2)

Fix two independent copies \( J \) and \( \tilde{J} \) of a random \( p \)-tensor with i.i.d. \( \pm 1 \) entries and consider the discrete interpolation \( J = J_0, J_1, J_2, \ldots, J_{n^p} = \tilde{J} \) associated with the statement of Theorem 2.2. The following “stability” result is a direct analogue of Theorem 4.2 in [GJW20] (but for circuits instead of low-degree polynomials).
Theorem 3.1. The probability that no pair \((J_t, J_{t+1}), 0 \leq t \leq n^p - 1\) is \(\kappa\)-bad is at least
\[
\exp\left(-(\log n)^{1.2d(n)}\right),
\]
for all large enough \(n\).

The constants \(\kappa, \alpha, \eta_{\text{OGP,p}}\) will be subsumed by \(\log n\) as we will see. Note that this probability converges to zero if \(d(n)\) is bounded away from zero. For our purposes, we will need the rate of convergence in this bound to be no faster than exponential. (In particular, this requirement will control our depth bound.)

The proof of Theorem 3.1 (which we include for completeness) is similar to Theorem 4.2 of [GJW20] but with two minor changes: first we observe that [GJW20] applies not just to functions of bounded degree but more generally to functions of bounded total influence; we then use the Linial-Mansour-Nisan Theorem to bound the total influence of any low-depth circuit.

Let’s first see how Theorem 3.1 implies the result.

Proof of Theorem 2.3. We fix \(\epsilon > 0\) with value specified later. By Theorem 2.1, with probability at least \(1 - (n^p + 1)\exp(-\gamma n)\), the objective values associated with instances \(J_t\) are all at least \((1 - \epsilon)\eta^*_p n^{\frac{2\kappa+1}{2}}\). By Theorem 2.2 with probability at least \(1 - \exp(-\gamma n)\) the e-OGP holds with parameters \(\eta_{\text{OGP,p}}, \nu_{1,p} < \nu_{2,p}\). Call the intersection of these two events \(\mathcal{A}\) and assume the same constant \(\gamma\) for convenience, so that \(\mathbb{P}(\mathcal{A}) \geq 1 - \exp(-\gamma n)\) and the multiplier \(n^p\) in the union bound is absorbed by making \(\gamma\) smaller. Fix any \(\kappa \leq (\nu_{2,p} - \nu_{1,p})^2\). Denote the event described in Theorem 3.1—namely the event that no pair \((J_t, J_{t+1}), 0 \leq t \leq n^p - 1\) is \(\kappa\)-bad—by \(\mathcal{B}\). We claim that \(\mathcal{B} \subset \mathcal{A}^c\). Assuming this claim, we obtain
\[
\exp\left(-(\log n)^{1.2d(n)}\right) \leq \mathbb{P}(\mathcal{B}) \leq \mathbb{P}(\mathcal{A}^c) \leq \exp(-\gamma n),
\]
and the desired lower bound on depth \(d(n)\) is obtained by changing the constant from 1.2 to 2 in order to subsume the term depending on \(\gamma\).

It remains to prove the claim \(\mathcal{B} \subset \mathcal{A}^c\). By way of contradiction, suppose that \(\mathcal{B} \cap \mathcal{A}\) is non-empty. (Note that the event \(\mathcal{A}\) implies in particular that the value of the optimization problem [11] is non-negative for every \(J_t\).) By assumption, the circuit \(C\) must produce solutions \(C(J_t), 1 \leq t \leq n^p\) with value at least a factor of \(\rho\) from optimality. On the other hand, on the event \(\mathcal{A}\), we have that the optimal value is at least \((1 - \epsilon)\eta^*_p n^{\frac{2\kappa+1}{2}}\). So if we choose \(\epsilon\) small enough that \(\rho(1 - \epsilon)\eta^*_p > \eta_{\text{OGP,p}}\), we have that the solutions \(C(J_t)\) yield an objective value above the onset of OGP, that is, \(\langle J_t, C(J_t) \rangle \geq \eta_{\text{OGP,p}} n^{\frac{2\kappa+1}{2}}\).

On \(\mathcal{A}\) we have by part (b) of the e-OGP that \(\langle C(J_0), C(J_{n^p}) \rangle \leq \nu_{1,p} n\). Let \(1 \leq t_0 \leq n^p\) be the smallest index for which \(\langle C(J_0), C(J_{t_0}) \rangle \leq \nu_{1,p} n\), and thus \(\langle C(J_0), C(J_{t_0-1}) \rangle > \nu_{1,p} n\). (Note that this includes the possibility \(t_0 = 1\).) On \(\mathcal{A}\), part (a) of the e-OGP applied in the case \(t_1 = 0, t_2 = t_0 - 1\) implies that in fact \(\langle C(J_0), C(J_{t_0-1}) \rangle \geq \nu_{2,p} n\). By Cauchy-Schwarz,
\[
\|C(J_0)\|_2 \cdot \|C(J_{t_0-1}) - C(J_{t_0})\|_2 \geq |\langle C(J_0), C(J_{t_0-1}) - C(J_{t_0}) \rangle| \\
= |\langle C(J_0), C(J_{t_0-1}) \rangle - \langle C(J_0), C(J_{t_0}) \rangle| \\
\geq \nu_{2,p} n - \nu_{1,p} n
\]
and so, since \( \|C(J_0)\|_2 = \sqrt{n} \),

\[
\|C(J_{t_0-1}) - C(J_{t_0})\|_2^2 \geq (\nu_{2,p} - \nu_{1,p})^2 n \geq \kappa n,
\]

and thus \((J_{t_0-1}, J_{t_0})\) is a \(\kappa\)-bad pair, contradicting the definition of the event \(\mathcal{B}\). \(\Box\)

Before turning to the proof of Theorem 3.1, let us briefly recall the following notions from Boolean analysis/Fourier analysis on \(\{\pm 1\}^m\); see e.g. [O’D14] for a reference. (In our setting, we will always work with the specific case \(m = n^p\).) Consider the standard Fourier expansion of functions on \(\{\pm 1\}^m\) associated with the uniform measure on \(\{\pm 1\}^m\). The basis of this expansion are monomials of the form \(x_S \triangleq \prod_{i \in S} x_i, S \subset [m]\). For every function \(g : \{\pm 1\}^m \to \mathbb{R}\), the associated Fourier coefficients are

\[
\hat{g}_S = \mathbb{E}[g(x)x_S], \quad S \subset [m],
\]

where the expectation is with respect to the uniform measure on \(x = (x_1, \ldots, x_m) \in \{\pm 1\}^m\). Then the Fourier expansion of \(g\) is \(g = \sum_{S \subset [m]} \hat{g}_S x_S\), and the Parseval (or Walsh) identity states that \(\sum_S \hat{g}_S^2 = \mathbb{E}[g(x)^2]\). For \(i \in [m]\), let \(L_i\) denote the Laplacian operator:

\[
L_i g(x) = \sum_{S \ni i} \hat{g}_S x_S,
\]

and let \(I(g)\) be the total influence

\[
I(g) = \sum_{i \in [m]} \mathbb{E}[L_i g(x)^2] = \sum_{S \subset [m]} |S| \hat{g}_S^2. \quad (4)
\]

Also note that

\[
\mathbb{E}[L_i g(x)^2] = \frac{1}{2} (\mathbb{E}[L_i g(x_{-i, -1})^2] + \mathbb{E}[L_i g(x_{-i, 1})^2]) \quad (5)
\]

where for \(\ell = \pm 1\), \(x_{-i, \ell} \in \{\pm 1\}^m\) is obtained from \(x\) by fixing the \(i\)-th coordinate of \(x\) to \(\ell\).

**Proof of Theorem 3.1.** Fix \(\delta_n = n^{-\beta}\) with some constant \(\beta > p\). Let \(\hat{C}_{S,j}, S \subset [n^p]\) be the Fourier coefficients of the function \(C_j\), which we recall is the \(j\)-component of \(C\). Fix a constant \(c > 0\) and let

\[
D_n = \log(1/\delta_n) \left( c \log \frac{s(n)}{\delta_n} \right)^{d(n) - 1} = \beta \log n \left(c(\alpha + \beta) \log n \right)^{d(n) - 1} \leq (\log n)^{1.1d(n)} \quad (6)
\]

for all sufficiently large \(n\). We use the Linial-Mansour-Nisan Theorem [LMN93] (the version we take here is the version from page 93 in [O’D14]) which states for some universal constant \(c\) the spectrum of a depth-\(d(n)\) circuit with size \(s(n)\) is \(\delta_n\)-concentrated on degree \(\leq D_n\). More precisely, in our context, it states that for each \(j = 1, 2, \ldots, n\),

\[
\sum_{S \subset [n^p] : |S| > D_n} \left| \hat{C}_{S,j} \right|^2 \leq \delta_n.
\]

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Thus for the influence function we have for each $j$

$$I(C_j) = \sum_{S \subseteq [np]} |S| \left( \hat{C}_{S,j} \right)^2 \leq \sum_{S : |S| \leq D_n} |S| \left( \hat{C}_{S,j} \right)^2 + n^p \delta_n \leq D_n \sum_{S : |S| \leq D_n} \left( \hat{C}_{S,j} \right)^2 + n^p \delta_n \leq D_n \sum_S \left( \hat{C}_{S,j} \right)^2 + n^p \delta_n = D_n + n^p \delta_n,$$

(7)

where in the last equation we use the fact $C_j = \pm 1$ and thus $1 = \mathbb{E}[C_j^2] = \sum_S \hat{C}_{S,j}^2$.

Let $\mathcal{E}_t$ be event that the pair $(J_t, J_{t+1})$ is $\kappa$-bad (as defined in (2)). The following is similar to Lemma 4.3 in [GJW20].

**Lemma 3.2.** The following holds for any fixed order $\leq_R$ on the $n^p$ entries of the tensor:

$$\frac{\kappa}{4} \sum_{0 \leq t \leq n^p-1} \mathbb{P}(\mathcal{E}_t) \leq D_n + n^p \delta_n. \quad (8)$$

As $\beta > p$ we have $n^p \delta_n = o(1)$ and we obtain that for all large enough $n$,

$$\frac{\kappa}{4} \sum_{0 \leq t \leq n^p-1} \mathbb{P}(\mathcal{E}_t) \leq 2D_n. \quad (9)$$

For any fixed $x \in \{\pm 1\}^{np}$ let $q(x)$ be the probability of the event $\cap_t \mathcal{E}_t^c$ when we condition on $J_0 = x$. Namely $q(x)$ is the probability of there being no $\kappa$-bad pairs when the initial tensor of the interpolation sequence is $x$. The following is a special case of Lemma 4.4 in [GJW20]; we will include the proof for completeness.

**Lemma 3.3.** The following bound holds for any fixed order $\leq_R$ on the $n^p$ entries of the tensor:

$$-\mathbb{E}[\log q(J)] \leq 2 \log 2 \sum_{0 \leq t \leq n^p-1} \mathbb{P}(\mathcal{E}_t),$$

where $J \in \{\pm 1\}^{np}$ is i.i.d.

We now combine these two results to complete the proof of Theorem 3.1. Note that the probability of no $\kappa$-bad pairs is

$$\mathbb{P}(\cap_t \mathcal{E}_t^c) = \mathbb{E}[q(J)],$$

where the expectation is with respect to the starting sample $J = J_0$ and $\leq_R$. By Lemma 3.3 we have

$$-\log \mathbb{E}[q(J)] \leq -\mathbb{E}[\log q(J)] \leq 2 \log 2 \sum_{0 \leq t \leq n^p-1} \mathbb{P}(\mathcal{E}_t)$$
Applying this is at most $16(\log 2) D_n/\kappa$. Exponentiating, we obtain 
$$\mathbb{P}(\cap_t \mathcal{E}_t) \geq \exp\left(-16(\log 2) D_n/\kappa\right).$$

Recalling bound (6) and since $\kappa$ is a constant, we obtain the claim by increasing the exponent $1.1d(n)$ in (6) to $1.2d(n)$.

**Proof of Lemma 3.2.** Combining (7), (4) and (5) we obtain for each $j$,
$$D_n + n^p \delta_n \geq \sum_{i \in [n^p]} \frac{1}{2} \left( \mathbb{E}[L_i C_j(x_{i-1})^2 + L_i C_j(x_{i+1})^2] \right).$$

Using the inequality $a^2 + b^2 \geq \frac{1}{2} (a - b)^2$, the right-hand side is at least
$$\frac{1}{4} \sum_{i \in [n^p]} \mathbb{E}[(L_i C_j(x_{i-1}) - L_i C_j(x_{i+1}))^2].$$

Note that for any $g$,
$$L_i g(x_{i-1}) - L_i g(x_{i+1}) = g(x_{i-1}) - g(x_{i+1}).$$

Summing over $j$ we obtain
$$(D_n + n^p \delta_n)n \geq \frac{1}{4} \sum_{i \in [n^p]} \sum_{1 \leq j \leq n} \mathbb{E}\left[ (C_j(x_{i-1}) - C_j(x_{i+1}))^2 \right]$$

$$= \frac{1}{4} \sum_{i \in [n^p]} \mathbb{E}\left[ \|C(x_{i-1}) - C(x_{i+1})\|_2^2 \right]$$

$$\geq \frac{1}{4} \sum_{t \in [n^p]} \kappa n \mathbb{P}(\mathcal{E}_t),$$

using (2).

**Proof of Lemma 3.3.** Fix any order $\leq_R$ with respect to which the sequence $J_0, J_1, \ldots, J_{n^p}$ is generated. We refer to coordinates $1, 2, \ldots, n^p$ as coordinates associated with this order. In particular, coordinate 1 is resampled when moving from $J_0$ to $J_1$. Coordinate 2 is resampled when moving from $J_1$ to $J_2$, etc.

For every $0 \leq m \leq n^p$ and tensor $x \in \{\pm\}^{n^p}$ let $q_m(x)$ be the probability that the truncated interpolation path $J_0, J_1, \ldots, J_m$ does not contain any $\kappa$-bad pairs $(J_t, J_{t+1}), t \leq m - 1$, when conditioned on $J_0 = x$. We claim the following stronger bound, from which the required claim follows as a special case $m = n^p$: for every $m = 0, 1, \ldots, n^p$,
$$-\mathbb{E}[\log q_m(J_0)] \leq 2 \log 2 \sum_{0 \leq t \leq m-1} \mathbb{P}(\mathcal{E}_t).$$

The proof is by induction on $m$. In the base case $m = 0$, the inequality above is in fact an equality with both sides equal to 0.
Assume the assertion holds for \( m' \leq m - 1 \). We now establish it for \( m \). For any tensor \( x \in \{-1, 1\}^n \) let \( q_{1,m}(x) \) be the probability of the event that the interpolation path \( J_1, J_2, \ldots, J_m \) does not contain any \( \kappa \)-bad pairs \( (J_t, J_{t+1}) \), \( 1 \leq t \leq m - 1 \), when conditioned on \( J_1 = x \). Observe that by our inductive assumption we have

\[
-\mathbb{E}[\log q_{1,m}(J_1)] \leq 2 \log 2 \sum_{1 \leq t \leq m-1} \mathbb{P}(\mathcal{E}_t). \tag{10}
\]

For any tensor \( x \in \{-1, 1\}^n \) let \( x_\pm \) be the tensor obtained from \( x \) by forcing coordinate 1 to be \( \pm \).

We have

\[
-\mathbb{E}[\log q_{m}(J_0)] = -\mathbb{E}[(1/2) \log q_{m}(J_{0,+}) + (1/2) \log q_{m}(J_{0,-})]. \tag{11}
\]

Let \( \mathcal{F} \) be the event that changing the value of the coordinate 1 is \( \kappa \)-bad. Note that this event is measurable (determined) by the realization of \( J_0 \), and furthermore \( \mathcal{E}_0 = \mathcal{F} \) on the event that coordinate 1 of \( J_1 \) (and therefore \( J_2, \ldots, J_m \)) is different from one of \( J_0 \), and \( \mathcal{E}_0 = \emptyset \) otherwise. In particular, \( \mathbb{P}(\mathcal{E}_0) = (1/2)\mathbb{P}(\mathcal{F}) \).

We claim

\[
q_{m}(J_{0,+}) = (1/2)q_{1,m}(J_{1,+}) + (1/2)q_{1,m}(J_{1,-})1(\mathcal{F}^c). \tag{12}
\]

We justify this identity as follows. With probability 1/2 the coordinate 1 of \( J_1 \) is +1, namely \( J_1 = J_{1,\pm} \). Conditioned on this event we have \( q_{m}(J_{0,+}) = q_{1,m}(J_{1,+}) \). On the other hand, with probability 1/2 coordinate 1 of \( J_1 \) is \( -1 \). In this case no bad pair occurs if \( \mathcal{F}^c \) takes place and furthermore no bad pairs occur during the remaining \( m - 1 \) resamplings, the probability of which is \( q_{1,m}(J_{1,-}) \).

Similarly,

\[
q_{m}(J_{0,-}) = (1/2)q_{1,m}(J_{1,-}) + (1/2)q_{1,m}(J_{1,+})1(\mathcal{F}^c). \tag{13}
\]

If the event \( \mathcal{F}^c \) occurs, the right-hand sides of the expressions (12) and (13) are identical, so on this event we obtain by the concavity of \( \log \)

\[
(1/2) \log q_{m}(J_{0,+}) + (1/2) \log q_{m}(J_{0,-}) = \log ( (1/2)q_{1,m}(J_{1,+}) + (1/2)q_{1,m}(J_{1,-})) \geq (1/2) \log (q_{1,m}(J_{1,+})) + (1/2) \log (q_{1,m}(J_{1,-})).
\]

On the other hand, if the event \( \mathcal{F} \) occurs then

\[
\log q_{m}(J_{0,+}) = \log ( (1/2)q_{1,m}(J_{1,+})),
\]

and

\[
\log q_{m}(J_{0,-}) = \log ( (1/2)q_{1,m}(J_{0,-})).
\]

In this case

\[
(1/2) \log q_{m}(J_{0,+}) + (1/2) \log q_{m}(J_{0,-}) = (1/2) \log ( (1/2)q_{1,m}(J_{1,+}) + (1/2)q_{1,m}(J_{1,-})) = -\log 2 + (1/2) \log (q_{1,m}(J_{1,+})) + (1/2) \log (q_{1,m}(J_{1,-})).
\]
Recalling \( P \) for some \( K, C, c > 0 \), we obtain

\[
(1/2) \log q_m(J_{0,+}) + (1/2) \log q_m(J_{0,-}) \\ \geq -\log 2 \, 1(F) + (1/2) \log (q_{1,m}(J_{1,+})) + (1/2) \log (q_{1,m}(J_{1,-})) .
\]

Applying (11) we obtain

\[
-\mathbb{E}[\log q_m(J_0)] \leq (\log 2) \mathbb{P}(F) - (1/2)\mathbb{E}[\log q_{1,m}(J_{1,+})] - (1/2)\mathbb{E}[\log q_{1,m}(J_{1,-})] \\ = (\log 2) \mathbb{P}(F) - \mathbb{E}[\log q_{1,m}(J_1)].
\]

Recalling \( \mathbb{P}(F) = 2\mathbb{P}(\mathcal{E}_0) \) and applying (10) we obtain the claim. \( \square \)

### 3.2 Proof of Theorem 2.2

The proof of Theorem 2.2 is similar to that in the Gaussian case after a coupling argument where we couple the path \( J_i \) above to a discretized \( \hat{J}_i \) induced by correlated Rademacher random vectors. We explain here the key steps that are different. In the following for two mean zero random vectors we say that \( X \sim \rho Y \), if \( (X_i, Y_i) \) and \( (X_j, Y_j) \) are independent for \( i \neq j \) and \( \mathbb{E}[X_iY_i] = \rho \) for each \( i \).

Let us begin with the following construction of \( \leq_R \) which corresponds to said coupling. Fix \( \delta > 0 \) so that \( 1/\delta \) is an integer, and let \( 0 = \rho_0 < \rho_1 < \ldots < \rho_{1/\delta} = 1 \) be \( \rho_k = k\delta \). We now construct a sequence of correlated instances of Rademacher random vectors as follows. Let \( \hat{J}_0 \) be an i.i.d. Ber(1/2) random vector. Now for each entry of \( \hat{J}_0 \), we resample it independently with probability \( \rho_1 \) and leave it as it was with probability \( 1 - \rho_1 \), to obtain \( \hat{J}_1 \). Notice that \( \hat{J}_1 \sim_{\rho_1} \hat{J}_0 \). Now for those entries of \( \hat{J}_1 \) that have not been resampled, resample them with probability \( (\rho_2 - \rho_1)/(1 - \rho_1) \). Repeat this process \( 1/\delta \) times to obtain a sequence \( \hat{J}_k \) of vectors each of which has i.i.d. Ber(1/2) entries and are such that if \( \rho_k > \rho_{k-1} \), then \( \hat{J}_{k} \sim_{\rho_k - \rho_{k-1}} \hat{J}_{k-1} \). Furthermore, notice that as \( \rho_{1/\delta} = 1 \), in the last step of this sequence, all of the remaining entries are resampled (independently) with probability \( 1 \), so that \( \hat{J}_{1/\delta} \) is an independent copy of \( \hat{J}_0 \). Now for each \( k = 1, \ldots, 1/\delta \), let \( \mathcal{I}_k \) be indices that have been resampled when going from \( \hat{J}_{k-1} \) to \( \hat{J}_k \). We now define \( \leq_R \) in the obvious way: if \( I \in \mathcal{I}_k \), \( I' \in \mathcal{I}_\ell \), then \( I \leq_R I' \) if \( k \leq \ell \). Within \( \mathcal{I}_k \) we let \( \leq_R \) be the uniform at random order for \( \mathcal{I}_k \). Note that \( \{\mathcal{I}_k\} \) is a partition of \([n^p]\). Evidently the law of \( \leq_R \) is that of the uniform at random order. Notice that if we construct \((J_k)\) in the corresponding way and let \( \tau_k \) be such that \( \tau_0 = 0 \) and \( \tau_k - \tau_{k-1} = |\mathcal{I}_k| \), then \( J_{\tau_k} = \hat{J}_k \).

Now let us note the following continuity argument that allows us to prove the OGP statement for \((J_{\tau_k})\). To this end, let us note the following lemma whose proof we defer momentarily:

**Lemma 3.4.** We have that

\[
H_{t,s}(\sigma) = \frac{1}{n^p} \langle \langle J_t, \sigma \rangle - \langle J_s, \sigma \rangle \rangle
\]

has

\[
P(\max_{\sigma} |H_{t,s}(\sigma)| \geq K \cdot \sqrt{t - sn}) \leq C e^{-cn},
\]

for some \( K, C, c > 0 \) depending only on \( p \).
With this in hand, note that there are at most $O(n^{2p})$ pairs of such $t, s$, so that by a union bound and this lemma, we have that

$$\max_{k \leq \frac{1}{C}} \max_{\tau_{k-1} \leq t \leq s \leq \tau_k} |H_{t,s}(\sigma)| \leq K\sqrt{\delta n}.$$ 

From here, the result follows from the following theorem which is essentially Theorem 2.2 but for the $\rho$-correlated instances. The proof of this theorem is found in [GJ21] for the case of Gaussian distribution of the entries of $J$. Its extension for the case of Rademacher distribution is obtained by standard universality type arguments and is omitted. See for example [CH06, AC16, Sen18, CGPR19] for similar arguments.

**Theorem 3.5.** Let $(X_t)$ be i.i.d. bounded random variables with

$$\mathbb{E}[X_t] = 0 \quad \mathbb{E}[X_t^2] = 1 \quad \|X_t\|_{\infty} < M$$

for some $M > 0$. Let $(X_t^\rho)$ be such that $X^\rho \sim \rho X$. Let $H(\sigma)$ denote the Hamiltonian corresponding to $X$ and $H_\rho$ denote that corresponding to $X^\rho$. Then for every $\epsilon > 0$ and $\rho \in (0, 1)$, there exists $C, \tilde{\mu} > 0$ such that with probability $1 - \exp(-Cn)/C$, for every $\sigma_1, \sigma_2 \in \{\pm 1\}^n$ satisfying $H(\sigma_1)/\rho \geq \mathbb{E}[\eta_n] - \tilde{\mu}, H^\rho(\sigma_2)/\rho \geq \mathbb{E}[\eta_n] - \tilde{\mu}$ we have that $|\langle \sigma_1, \sigma_2 \rangle| \leq n\epsilon$. If $\rho = 1$, the same result holds except we have that there are $0 \leq a < b \leq 1$ such that $|\langle \sigma_1, \sigma_2 \rangle| \leq 0, a \cup [b, 1]$.

**Proof of Lemma 3.4.** First, by symmetrization, it suffices to prove the same bound for $H_{t,s}(\sigma)$. In this case, note that conditionally on $\leq_R$, the map

$$\tilde{J}_t - \tilde{J}_s \mapsto \frac{1}{n} \max_{\sigma} H_{t,s}(\sigma)$$

is uniformly $\sqrt{2p|\tau_k - \tau_s|/n^{p+1}}$-Lipschitz. On the other hand, conditionally on $\leq_R$, we have that $Y = J_t - J_s$ is a Rademacher vector of length at most $|I_{t,s}|$ where $I_{t,s}$ are those indices resample between time $t$ and $s$. Note that $|I_{t,s}| = |\tau_k - \tau_s|$. Thus applying McDairmids inequality, we have that, conditionally on $\leq_R$, this maximum,

$$M = \frac{1}{n} \max_{\sigma} H_{t,s}(\sigma)$$

has

$$P(|M - \mathbb{E}[M]| \geq \sqrt{|\tau_k - \tau_s|\eta}) \leq \exp(-2\eta^2|\tau_k - \tau_s|).$$

Now, since $H_{t,s}$ is conditionally sub-gaussian, we may apply Talagrand’s comparison inequality [Ver18, Cor 8.6.2], to obtain

$$\mathbb{E}[M] \leq C\mathbb{E}[\tilde{M}]$$

for some universal constant where $\tilde{M}$ is obtained from $M$ by replacing $(X_t, X_t^\rho)$ with gaussian random vectors correlated in the same fashion. By a standard application of Slepian’s inequality, we obtain,

$$\mathbb{E}[\tilde{M}] \leq C(p)\sqrt{|\tau_k - \tau_s|},$$

where we have used here that the variance of $H_{t,s}$ is at most $2(t - s)$. Finally note that with probability $1 - \exp(-cn^p)$, we have $|\tau_k - \tau_s| \leq 2(t - s)$. Combining these bounds yields the desired result. 

\[\square\]
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