Lie superbialgebra structures on the Lie superalgebra \((C^3 + A)\) and deformation of related integrable Hamiltonian systems

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Abstract

Admissible structure constants related to the dual Lie superalgebras of particular Lie superalgebra \((C^3 + A)\) are found by straightforward calculations from the matrix form of super Jacobi and mixed super Jacobi identities which are obtained from adjoint representation. Then, by making use of the automorphism supergroup of the Lie superalgebra \((C^3 + A)\), the Lie superbialgebra structures on the Lie superalgebra \((C^3 + A)\) are obtained and classified into inequivalent 31 families. We also determine all corresponding coboundary and bi-r-matrix Lie superbialgebras. The quantum deformations associated with some Lie superbialgebras \((C^3 + A)\) are obtained, together with the corresponding deformed Casimir elements. As an application of these quantum deformations, we construct a deformed integrable Hamiltonian system from the representation of the Hopf superalgebra \(U^\epsilon_\lambda(C^2_{p=1} \oplus A_{1,1})((C^3 + A))\).

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1 Introduction

From the mathematical standpoint, the study of Lie bialgebra structures provides a primary classification of possible quantum deformations of a given Lie algebra. Many interesting examples of Lie bialgebras based on complex semisimple Lie algebras have been given by Drinfeld [1]. In the case of simple Lie algebras, all Lie bialgebras of the coboundary type have been classified [2] (see, also, [3]) and by using this classification, all constant solutions of the classical Yang-Baxter equation have been obtained. In the case of non-semisimple, only some of low-dimensional examples have been studied [4–6]. A complete classification of Lie bialgebras with reduction was given in [7]. However, a classification of Lie bialgebras is out of reach, with similar reasons as for the classification of Lie algebras. On the other hand, from the physical standpoint, the theory of classical integrable systems naturally relates to the geometry and representation theory of Poisson-Lie groups and the corresponding Lie bialgebras and their classical r-matrices (see, e.g., [8]). In the same way, Lie superbialgebras [9], as the underlying symmetry algebras, play an important role in the integrable structure of AdS/CFT correspondence [10]. Considering that there is a universal quantization for Lie superbialgebras [11], one can assign an important role to the classification of Lie superbialgebras (especially low-dimensional Lie superbialgebras) from both physical and mathematical point of view. Until now there were distinguished and non-systematic ways for obtaining low-dimensional Lie superbialgebras (see, for instance, [12, 13]). We have recently presented a systematic way for obtaining and classifying low-dimensional Lie superbialgebras by using the adjoint representation of Lie superalgebras [14] and then have applied this method to classify the Lie superbialgebras $gl(1|1)$ [15]. In the present paper, we will try to perform the classification of the Lie superbialgebras $(C^3 + A)$. The reason of focusing on this classification is that we have already begun a study on super Poisson-Lie symmetry in WZW model based on the Lie supergroup $(C^3 + A)$ [16]. In this way, we have shown that the dual model to the $(C^3 + A)$ WZW model is itself a WZW model on a Lie supergroup whose Lie superalgebra is isomorphic to $(C^3 + A)$. However, to make that more completed, that is, to obtain a hierarchy of $(C^3 + A)$ WZW models related to the super Poisson-Lie T-duality, first we must obtain and classify all Lie superbialgebras $(C^3 + A)$. Moreover, as explained in [13], it has been noticed that the Lie superalgebras $gl(1|1)$ and $(C^3 + A)$ are each the two-dimensional Lie superbialgebras, i.e., they are isomorphic to four-dimensional Drinfeld superdoubles of the type $(2|2)$ [17].

On the other hand, it is well known that quantum groups as a new kind of symmetry related to the integrability of some quantum models appeared in the context of quantum inverse scattering methods [18]. A direct and systematic method to construct $N$-particle completely integrable Hamiltonian systems from representations of coalgebras with Casimir element has been presented in Ref. [17]. Their construction shows that quantum deformations can be interpreted as generating structures for deformations of integrable Hamiltonian systems with coalgebra symmetry. In this paper, we construct a deformed integrable Hamiltonian system from a convenient representation
of the quantum Lie superalgebra \((C^3 + A)\) with the corresponding Casimir element. This system is constructed on a supersymplectic flat supermanifold of the superdimension-(4|4) as the phase superspace.

In section 2, we recall some properties of \(Z_2\)-graded vector spaces and basic definitions concerning Lie superbialgebras. Section 3 is initiated by a representation of the indecomposable and decomposable Lie superalgebras of the type \((2|2)\) whose representatives are given by Tables 1 and 2, respectively. Then, in order to obtain the Lie superbialgebra structures on \((C^3 + A)\), the automorphism supergroup of \((C^3 + A)\) is derived at the end of this section. In section 4, we obtain the solutions of the super Jacobi and mixed super Jacobi identities by making use of the adjoint representations of the Lie superalgebras \(g\) and \(\tilde{g}\), then we find 31 families of inequivalent Lie superbialgebra structures on \((C^3 + A)\) whose representatives are classified in Table 3. In section 5, we list coboundary and bi-r-matrix Lie superbialgebras of \((C^3 + A)\) (triangular or quasi-triangular) with their corresponding classical r-matrices in Table 4. Making use of the Lyakhovsky and Mudrov formalism \[20\], the Hopf superalgebras related to some Lie superbialgebras \((C^3 + A)\) of Table 3 are obtained as three Propositions in section 6. As an application, we get at the end of this section a family of quantum integrable Hamiltonian systems that can be constructed from a convenient representation of the Lie supercoalgebra \((C^3 + A)\) with the corresponding Casimir element. The paper is closed by a final section that includes some remarks.

2 Basic definitions and notation

In order to make the paper somewhat self-contained, let us first recall some properties of \(Z_2\)-graded vector spaces and Lie superbialgebras on an appropriate field.

If \(V\) is a \(Z_2\)-graded vector space, then \(V = V_B \oplus V_F\), and we refer to \(V_B\) and \(V_F\) as the even and odd subspaces of \(V\), respectively. We define the gradation index \(|\cdot| : V \to \{0, 1\}\) for the homogenous elements of \(V\) by \[21\]

\[
|x| := \begin{cases} 0 & x \in V_B \\ 1 & x \in V_F . \end{cases} \tag{2.1}
\]

The dual graded vector space \(V^* = V^*_B \oplus V^*_F\) of \(V\) inherits a natural \(Z_2\)-gradation.

A Lie superalgebra \(g\) is a \(Z_2\)-graded vector space, thus admitting the decomposition \(g = g_B \oplus g_F\), equipped with a bilinear superbracket structure \(\{.,.\} : g \otimes g \to g\) satisfying the requirements of (graded) antisymmetry and super Jacobi identity. In order to express them, it is useful to introduce a basis in \(g\), \(\{X_i\} \subset g_B \cup g_F\), and structure constants

\[
[X_i, X_j] = f_{ij}^k X_k. \tag{2.2}
\]
Structure constants have to satisfy the following super Jacobi identity:

\[
(-1)^{(j+k)} f^m_{ji} f^l_{kj} + f^m_{il} f^l_{kj} + (-1)^{(i+j)} f^m_{kl} f^l_{ij} = 0,
\]

(2.3)

where

\[
f^k_{ij} = (-1)^{ij} f^k_{ji},
\]

(2.4)

and \( f^k_{ij} = 0 \) whenever \( \text{grade}(i) + \text{grade}(j) \neq \text{grade}(k) \).

We see that the superdimension of Lie superalgebra \( g \) is \( (m|n) \) if \( \dim g_B = m \) and \( \dim g_F = n \). It is also useful to define the supersymmetry bilinear form. A bilinear form \( < . , . > \) on \( g \) is called supersymmetry if only if for any \( x, y \in g \)

\[
< x, y > = (-1)^{xy} < y, x >,
\]

(2.5)

and it is called super ad-invariant if and only if

\[
< [x, y], z > = < x, [y, z] >,
\]

(2.6)

for all \( x, y, z \in g \).

Let \( g = g_B \oplus g_F \) be a finite dimensional Lie superalgebra, and consider its dual \( g^* = g^*_B \oplus g^*_F \).

By definition, an element \( x^* \in g^* \) is a linear functional on \( g \), i.e., \( x^*(y) = < x^*, y > \) for all \( y \in g \). It is obvious that we can extend \( < . , . > \) to \( (g \otimes g)^* \otimes (g \otimes g) \) (in the present case, \( (g \otimes g)^* = g^* \otimes g^* \)) by setting

\[
< x^* \otimes y^*, x \otimes y > = (-1)^{y^* x} < x^*, x > < y^*, y >,
\]

(2.7)

for elements \( x^*, y^* \in g^* \) and \( x, y \in g \).

A \textit{Lie superbialgebra} structure on a Lie superalgebra \( g \) is a linear map \( \delta : g \rightarrow g \otimes g \), called the super cocommutator, such that

1. \( \delta \) is a super one-cocycle on \( g \) with values in \( g \otimes g \), i.e., for \( x, y \in g \),

\[
\delta([x, y]) = [x \otimes 1 + 1 \otimes x , \delta(y)] - (-1)^{xy}[y \otimes 1 + 1 \otimes y , \delta(x)].
\]

(2.8)

2. The dual map \( [, , ]_* : g^* \otimes g^* \rightarrow g^* \) defines a Lie superbracket on \( g^* \), i.e., a super skew-symmetric bilinear map on \( g^* \) satisfying the super Jacobi identity. By definition, let us set

\[
< [x^*, y^*]_*, x > = < \delta(x) , x^* \otimes y^* >,
\]

(2.9)

for \( x^*, y^* \in g^* \) and \( x \in g \). The Lie superbialgebra defined in this way will be denoted by \( (g, g^*) \) or \( (g, \delta) \) [9,14].
Let \( r \) be an element of \( \mathfrak{g} \otimes \mathfrak{g} \). The super commutator given by
\[
\delta(x) = [x \otimes 1 + 1 \otimes x, \ r], \quad x \in \mathfrak{g},
\]
defines a coboundary Lie superbialgebra if only if \( r \) fulfills the modified graded classical Yang-Baxter equation (GCYBE)
\[
[x \otimes 1 + 1 \otimes x \otimes 1 + 1 \otimes 1 \otimes x, \ [r, r]] = 0, \quad x \in \mathfrak{g},
\]
where the graded Schouten bracket is defined by
\[
[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}],
\]
and, if \( r = r^{ij}X_i \otimes X_j \), we have denoted \( r_{12} = r^{ij}X_i \otimes X_j \otimes 1 \), \( r_{13} = r^{ij}X_i \otimes 1 \otimes X_j \) and \( r_{23} = r^{ij}1 \otimes X_i \otimes X_j \). A solution of the GCYBE is often called a classical r-matrix (in the following we call it an r-matrix). Using the fact that \( r \) has even Grassmann parity and Grassmann parity of \( r^{ij} \) comes from indices, one can show that
\[
[r_{12}, r_{13}] = (-1)^{(k+l)+jl} r^{ij} r^{kl} [X_i, X_k] \otimes X_j \otimes X_l,
\]
\[
[r_{12}, r_{23}] = (-1)^{(i+j)(k+l)} r^{ij} r^{kl} X_i \otimes [X_j, X_k] \otimes X_l,
\]
\[
[r_{13}, r_{23}] = (-1)^{(k+l)+jl} r^{ij} r^{kl} X_i \otimes X_k \otimes [X_j, X_l].
\]

Coboundary Lie superbialgebras can be of two different types: when the r-matrix is a super skew-symmetric solution of the (GCYBE), so that \( [[r, r]] = 0 \), we shall say the coboundary Lie superbialgebra is a triangular one. In contrast, a super skew-symmetric solution \( r \) of equation (2.11) with non-vanishing graded Schouten bracket
\[
[[r, r]] = \omega, \quad \omega \in \wedge^3 \mathfrak{g},
\]
will give rise to a so-called quasi-triangular Lie superbialgebra \([9]\). We note that if \( \mathfrak{g} \) is a Lie superbialgebra then \( \mathfrak{g}^* \) is also a Lie superbialgebra , but this is not always true for the coboundary property.

Suppose that \( \mathfrak{g} \) is a coboundary Lie superbialgebra with one-coboundary (2.10) and \( \mathfrak{g}^* \) be its dual coboundary Lie superbialgebra with the one-coboundary
\[
\delta^*(x^*) = [x^* \otimes 1 + 1 \otimes x^*, \ r^*], \quad x^* \in \mathfrak{g}^*, \quad r^* \in \mathfrak{g}^* \otimes \mathfrak{g}^*,
\]
where \( \delta^*: \mathfrak{g}^* \to \mathfrak{g}^* \otimes \mathfrak{g}^* \) is a super one-cocycle on \( \mathfrak{g}^* \) defined by \( r^* \). Then, we will call the pair
(g, g*) a bi-r-matrix superbialgebra [5] if the graded Lie brackets [., .]' on g defined by \( \delta^* \)

\[
< \delta^*(x^*), x \otimes y > = < x^*, [x, y]' >, \quad x, y \in g, \quad x^* \in g^*, \quad (2.15)
\]

are equivalent to the original ones

\[
[x, y]' = A^{-1}[Ax, Ay], \quad x, y \in g, \quad A \in Aut(g), \quad (2.16)
\]

where \( Aut(g) \) stands for the automorphism supergroup of the Lie superalgebra \( g \).

A Manin supertriple is a triple of Lie superalgebras \((D, g, \tilde{g})\) together with a nondegenerate ad-invariant supersymmetric bilinear form (natural scalar product) \(< . , . >\) on \( D \), such that

1. \( g \) and \( \tilde{g} \) are Lie subsuperalgebras of \( D \),
2. \( D = g \oplus \tilde{g} \) as a supervector space,
3. \( g \) and \( \tilde{g} \) are isotropic with respect to the scalar product \(< . , . >\). By definition, an isotropic subspace is that the scalar product vanishes on it, i.e., for basis \( \{X_i\} \in g \) and \( \{\tilde{X}^i\} \in \tilde{g} \),

\[
< X_i, X_j > = < \tilde{X}^i, \tilde{X}^j > = 0, \quad \delta_{ij} = < X_i, \tilde{X}^j > = (-1)^{ij} < \tilde{X}^j, X_i > = (-1)^{ij} \delta_{ij}. \quad (2.17)
\]

The Lie superbracket on \( \tilde{g} \) defines a Lie superbracket on \( g^* \). Also, to see it defines a Lie superbialgebra structure on \( g \), we use the super Jacobi identity in \( D \) and the invariance of the scalar product. Thus, there is a one-to-one correspondence between Lie superbialgebra \((g, g^*)\) and Manin supertriple \((D, g, \tilde{g})\) with \( \tilde{g} \cong g^* \) [8]. Consider the structure constants of Lie superalgebras \( g \) and \( \tilde{g} \) as

\[
[X_i, X_j] = f^k_{ij} X_k, \quad [\tilde{X}^i, \tilde{X}^j] = \tilde{f}^{ij}_k \tilde{X}^k, \quad (2.18)
\]

then, super ad-invariance of the bilinear form \(< . , . >\) on \( D = g \oplus \tilde{g} \) implies [14]

\[
[X_i, \tilde{X}^j] = (-1)^i j \tilde{f}^{jk}_i X_k + (-1)^i j f^j_{ki} \tilde{X}^k. \quad (2.19)
\]

The Lie superbrackets (2.18) and (2.19) define a Lie superalgebra structure on the vector space \( D \). In this case, we say that the Lie superalgebra \( D \) is the Drinfeld superdouble of \( g \) (or, equivalently, of \( \tilde{g} \)). In order to get an applicable result we use equations (2.7), (2.9), (2.17) together with equation (2.18) to obtain

\[
\delta(X_i) = (-1)^{jk} \tilde{f}^{jk}_i X_j \otimes X_k. \quad (2.20)
\]

Utilizing this relation in the super one-cocycle condition (2.8), one can, respectively, obtain the super Jacobi identities for the dual Lie superalgebra and the mixed super Jacobi identities as follows:

\[
(-1)^{i(j+k)} \tilde{f}^{ji}_m \tilde{f}^{ki}_j + \tilde{f}^{il}_m \tilde{f}^{jk}_l + (-1)^{k(i+j)} \tilde{f}^{kl}_m \tilde{f}^{ij}_l = 0, \quad (2.21)
\]

5
\[ f^m_{jk} f^i_{il} = f^i_{mk} f^m_{jl} + f^l_{jm} f^i_{im} + (-1)^j f^i_{jm} f^m_{l} + (-1)^{j} f^l_{mk} f^m_{i} \quad (2.22) \]

3 Lie superalgebras of the type \( (2|2) \)

To present the notation and for the self-consistency of the paper, we use the list of four-dimensional Lie superalgebras of the type \( (2|2) \) of Ref. [23]. In that classification, Lie superalgebras are divided into two types: trivial and nontrivial Lie superalgebras for which the fermion-fermion commutations are, respectively, zero or non-zero. The results are presented in Table 1. Because we use the DeWitt notation and standard basis here, the structure constants \( f^B_{EF} \) must be purely imaginary [22]. Note that in [23], only the indecomposable Lie superalgebras have classified. The decomposable Lie superalgebras of the type \( (2|2) \) have been recently obtained in [15], and here are presented in Table 2. As can be seen from the Tables 1 and 2, the Lie superalgebras have two bosonic generators \( \{X_1, X_2\} \) and two fermionic ones \( \{X_3, X_4\} \). In labeling the trivial Lie superalgebras, the letters \( A, B, C \) and \( D \) denote the equivalence classes of Lie superalgebras of dimension \( d \), where \( d = 1, 2, 3 \) and 4, respectively, for \( A, B, C \) and \( D \). The superscript \( i \) and real subscripts \( p \) denote the respective number of non-isomorphic Lie superalgebras and the Lie superalgebra parameter. For the nontrivial Lie superalgebras, we add an integer superscript and a real subscript to parentheses around the symbol of the corresponding trivial Lie superalgebra, where necessary.

Recently, we have classified all four-dimensional Drinfeld superdoubles of the type \( (2|2) \) as a theorem in [17]. We have shown that there are just two classes of non-isomorphic Drinfeld superdoubles of the type \( (2|2) \) so that they are isomorphic to the Lie superalgebras \( gl(1|1) \cong (C_2 + A) \) and \( (C_3 + A) \) of Table 1. So, four-dimensional Drinfeld superdoubles of the type \( (2|2) \) have no new results for Tables 1 and 2. However, there are just 36 families of non-isomorphic four-dimensional Lie superalgebras of the type \( (2|2) \) which have presented in Tables 1 and 2. We also note that some of the Lie superalgebras in the list contained at Table 1, such as \((2A_{1,1} + 2A)^2\), \((2A_{1,1} + 2A)^3\), and \((2A_{1,1} + 2A)^4\), can be considered to be relevant for the \( AdS_2/CFT_1 \) correspondence. Because they, as the sub-superalgebras of the centrally-extended \( psu(1|1) \) Lie superalgebra, correspond to the algebra controlling the exact S-matrix theory of magnons transforming in a centrally-extended \( psu(1|1) \) superalgebra [24] (see, also, [25]).

3.1 The Lie superalgebra \((C^3 + A)\) and its automorphism supergroup

The Lie superalgebra \((C^3 + A)\) is spanned by the set of generators \( \{X_1, X_2, X_3, X_4\} \) with grading \( grade(X_1) = grade(X_2) = 0 \) and \( grade(X_3) = grade(X_4) = 1 \), which in the standard basis [22] fulfill the following (anti)commutation relations [23]

\[ [X_1, X_4] = X_3, \quad \{X_4, X_4\} = iX_2, \quad [X_2, \ ] = 0, \quad [X_3, \ ] = 0. \quad (3.1) \]
In section 4, we will obtain all the dual Lie superalgebras (the super cocommutators) related to the Lie superalgebra \((\mathcal{C}^3 + \mathcal{A})\). In this respect, we consider two Lie supercoalgebra structures \(\delta\) and \(\delta'\) equivalent if one can be obtained from the other by means of a change of basis which is an automorphism \(A\) of the Lie superalgebra preserving the parity of the generators and the structure constants \(f^i_{jk} (A : \mathfrak{g} \rightarrow \mathfrak{g})\). Therefore it is crucial for our further considerations to obtain the automorphism supergroup of the particular Lie superalgebra \((\mathcal{C}^3 + \mathcal{A})\).

| \(\mathcal{D}^5\) | \([X_1, X_3] = X_3, \ [X_1, X_4] = X_4, \ [X_2, X_4] = X_3\) |
| \(\mathcal{D}^6\) | \([X_1, X_3] = X_3, \ [X_1, X_4] = X_4, \ [X_2, X_4] = -X_4, \ [X_2, X_3] = X_3\) |
| \(\mathcal{D}^1_{pq}\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = pX_3, \ [X_1, X_4] = qX_4\) \(pq \neq 0, \ p \geq q\) |
| \(\mathcal{D}^8_p\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = pX_3, \ [X_1, X_4] = X_3 + pX_4\) \(p \neq 0\) |
| \(\mathcal{D}^9_{pq}\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = pX_3 - qX_4, \ [X_1, X_4] = qX_3 + pX_4\) \(q > 0\) |
| \(\mathcal{D}^{10}_p\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = (p + 1)X_3, \ [X_1, X_4] = pX_4, \ [X_2, X_4] = X_3\) |
| \(\mathcal{D}^{7/2}_1\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = \frac{1}{2}X_3, \ [X_1, X_4] = \frac{1}{2}X_4, \ \{X_3, X_4\} = iX_2, \ {X_4, X_4} = iX_2\) |
| \(\mathcal{D}^{7/2}_2\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = \frac{1}{2}X_3, \ [X_1, X_4] = \frac{1}{2}X_4, \ \{X_3, X_4\} = iX_2, \ {X_4, X_4} = -iX_2\) |
| \(\mathcal{D}^{7/2}_3\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = \frac{1}{2}X_3, \ [X_1, X_4] = \frac{1}{2}X_4, \ \{X_3, X_4\} = iX_2, \ {X_4, X_4} = iX_2\) |
| \(\mathcal{D}^{7/2}_1-p\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = pX_3, \ [X_1, X_4] = (1 - p)X_4, \ \{X_3, X_4\} = iX_2, \ {X_4, X_4} = iX_2\) \(p \leq \frac{1}{2}\) |
| \(\mathcal{D}^{7/2}_p\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = \frac{1}{2}X_3 - pX_4, \ [X_1, X_4] = pX_3 + \frac{1}{2}X_4, \ \{X_3, X_4\} = p > 0\) |
| \(\mathcal{D}^{(10)}_0\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = X_3, \ [X_2, X_4] = X_3, \ \{X_3, X_4\} = iX_1, \ {X_3, X_3} = -iX_2\) |
| \(\mathcal{D}^{(10)}_2\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = 3, \ [X_2, X_4] = X_3, \ \{X_4, X_4\} = -iX_1, \ {X_4, X_4} = iX_2\) |
| \((2A_{1,1} + 2A)^2\) | \([X_3, X_3] = iX_1, \ \{X_4, X_4\} = iX_2, \ \{X_3, X_4\} = iX_1\) \text{Nilpotent} |
| \((2A_{1,1} + 2A)^3_p\) | \([X_3, X_3] = iX_1, \ \{X_4, X_4\} = iX_2, \ \{X_3, X_4\} = ip(X_1 + X_2)\) \(p > 0\) \text{Nilpotent} |
| \((2A_{1,1} + 2A)^4_p\) | \([X_3, X_3] = iX_1, \ \{X_4, X_4\} = iX_2, \ \{X_3, X_4\} = ip(X_1 - X_2)\) \(p > 0\) \text{Nilpotent} |
| \((C^1 + A)\) | \([X_1, X_2] = X_2, \ [X_1, X_3] = X_3, \ \{X_3, X_4\} = iX_2\) |
| \((C^2 + A)\) | \([X_1, X_3] = X_3, \ [X_1, X_4] = -X_4, \ \{X_3, X_4\} = iX_2\) \text{Jordan-Wigner quantization} |
| \((C^3 + A)\) | \([X_1, X_4] = X_3, \ \{X_4, X_4\} = iX_2\) \text{Nilpotent} |

Since the Lie superalgebra is generated by the bosonic and fermionic generators, every automorphism is generated by a linear transformation within the bosonic and fermionic sectors \(\mathfrak{g}_B\) and
\( g \), respectively. On the other hand, because of the preserving of grading the automorphisms of Lie superalgebra cannot mix fermionic with bosonic. Thus, the action of the automorphism \( A \) on \( g = g_0 \oplus g_F \) is given by the block diagonal matrix \( A = \text{diag}(A_B, A_F) \), that is, in the standard basis \( \mathcal{B} \) \( X_i' = \begin{pmatrix} X_i' \mid g_0 \end{pmatrix} \) and \( X_i = \begin{pmatrix} X_i \mid g_F \end{pmatrix} \), we have the following transformation

\[
X_i' = A(X_i) = (-1)^j A_i^j X_j,
\]

(3.2)

Table 2. Decomposable Lie superalgebras of the type \((2, 2)\).

| \( \mathfrak{g} \) | Non-zero commutation relations | Comments |
|-----------------|-------------------------------|---------|
| \( \mathcal{B} \) | All of the (anti)commutation relations are zero. |         |
| \( B \oplus B \) [22] | \([X_1, X_3] = X_3, \ [X_2, X_4] = X_4\) |         |
| \( C_{i} \oplus A \) [22] | \([X_1, X_2] = X_2, \ [X_1, X_3] = pX_3\) | \( p \neq 0\) |
| \( C_{p=0} \oplus A \) [22] | \([X_1, X_2] = X_2\) |         |
| \( B \oplus A \oplus A_{1,1} \) | \([X_1, X_3] = X_3\) | \( \equiv C_{p=0} \oplus A_{1,1} \) |
| \( C^{3} \oplus A_{1,1} \) [22] | \([X_1, X_4] = X_3\) | Nilpotent |
| \( C^{4} \oplus A_{1,1} \) [22] | \([X_1, X_4] = X_3 + X_4\) |         |
| \( (2A_{1,1} + 2A)^{0} \) | \(\{X_3, X_3\} = iX_1\) | \( \equiv (A_{1,1} + A) \oplus A \oplus A_{1,1}, \) Nilpotent |
| \( (2A_{1,1} + 2A)^{1} \) | \(\{X_3, X_3\} = iX_1, \ \{X_4, X_4\} = iX_2\) | \( \equiv (A_{1,1} + A) \oplus (A_{1,1} + A), \) Nilpotent |
| \( (A \oplus (A_{1,1} + A)) \) | \([X_1, X_3] = X_3, \ \{X_4, X_4\} = iX_2\) |         |
| \( ((A_{1,1} + 2A)^{0} \oplus A_{1,1}) \) | \(\{X_3, X_3\} = iX_1, \ \{X_4, X_4\} = iX_1\) | Nilpotent |
| \( ((A_{1,1} + 2A)^{2} \oplus A_{1,1}) \) | \(\{X_3, X_3\} = iX_1, \ \{X_4, X_4\} = -iX_1\) | Nilpotent |

We recall that the Lie superalgebra \( A \) is one-dimensional Abelian Lie superalgebra with one fermionic generator while Lie superalgebra \( A_{1,1} \) is its bosonization.

where \( X_i' \) are the changed basis by the automorphism \( A \). As mentioned above, the automorphism preserves the structure constants, so the basis \( X_i' \) must obey the same (anti)commutation relations as \( X_i \), i.e.,

\[
[X_i', X_j'] = f^{k}_{ij} X_k' .
\]

(3.3)

Inserting the transformation (3.2) into (3.3), we obtain the following matrix equation for the elements of automorphism supergroup [14]:

\[
(-1)^{ij + mk} A^{jk} A^{il} = Y^{j} A_{i}^{k},
\]

(3.4)
where $m$ denotes the column of matrix $A^st$ in the left hand side, and the indices $i$ and $j$ correspond to the row and column of matrix $Y^k$, respectively. Here, $(Y^i)_jk = -f^i_{jk}$ are the adjoint representations of the Lie superalgebra $g$. For $(C^3 + A)$ one can use (3.1) to get

$$
\begin{align*}
Y^1 &= 0, \\
Y^2 &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -i & 0
\end{pmatrix}, \\
Y^3 &= \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}, \\
Y^4 &= 0,
\end{align*}
$$

(3.5)

then, using (3.4) the automorphism $A$ can be easily obtained. The result is given by the following statement.

**Proposition 1:** The automorphism supergroup of the Lie superalgebra $(C^3 + A)$ is expressed as matrices in basis $\{X_1, X_2; X_3, X_4\}$ as

$$
\text{Aut}
((C^3 + A)) = \left\{ A^j_i = \begin{pmatrix} a & c & 0 & 0 \\ 0 & b^2 & 0 & 0 \\ 0 & 0 & ab & 0 \\ 0 & 0 & d & b \end{pmatrix} ; \ a, b \neq 0 \right\}.
$$

(3.6)

4 The Lie superbialgebra structures on $(C^3 + A)$

As mentioned in [14], because of tensorial form of super Jacobi and mixed super Jacobi identities (2.21) and (2.22), working with them is not so easy and we suggest writing these equations as matrix forms using the following adjoint representations for Lie superalgebras $g$ and $\tilde{g}$

$$
(\tilde{X}^i)_{jk} = -\tilde{f}^i_{jk}, \quad (Y^i)_{jk} = -f^i_{jk}.
$$

(4.1)

Then the matrix forms of identities (2.21) and (2.22) become, respectively, as follows:

$$
\begin{align*}
(\tilde{X}^i)_l \tilde{X}^l - \tilde{X}^j \tilde{X}^j + (-1)^{ij} \tilde{X}^i \tilde{X}^j &= 0, \\
(\tilde{X}^i)_l Y^l - (-1)^k (\tilde{X}^{st})^i Y^j \tilde{X}^i - (-1)^{ij} Y^i \tilde{X}^j + (-1)^{k+ij} (\tilde{X}^{st})^i Y^j &= 0,
\end{align*}
$$

(4.2) (4.3)

where index $k$ in the right hand side of equation (4.3) represents the column of matrix $\tilde{X}^{st}$ and superscript $st$ stands for supertranspose.

To obtain the Lie superbialgebra structures on the Lie superalgebra $(C^3 + A)$, we must first solve equations (4.2) and (4.3) in structure constants. The initial steps of the analysis are made using a computer. We recall that $Y^i$ in equation (4.3) are the adjoint representations of the Lie superalgebra $(C^3 + A)$, which have presented in (3.5). Thus by solving equations (4.2) and (4.3), we obtain the general form of the structure constants related to the dual Lie superalgebras to $(C^3 + A)$.

The possibilities found by means of the computer are given by the following four cases:
**Theorem 1:** The solutions of super Jacobi and mixed super Jacobi identities \([2,21]\) and \([2,22]\) for the Lie superalgebra \((C^3 + A)\) are given by the following four cases:

- **Case (1):** \( \tilde{f}_1^{12} = \tilde{f}_1^{12}, \quad \tilde{f}_2^{23} = \tilde{f}_2^{23}, \quad \tilde{f}_2^{24} = \tilde{f}_2^{24}, \quad \tilde{f}_4^{24} = \tilde{f}_1^{12} + \tilde{f}_3^{23}. \)

- **Case (2):** \( \tilde{f}_2^{23} = \tilde{f}_2^{23}, \quad \tilde{f}_4^{24} = -\tilde{f}_2^{23}, \quad \tilde{f}_1^{12} = -2\tilde{f}_2^{23}, \quad \tilde{f}_2^{24} = \tilde{f}_2^{24}, \quad \tilde{f}_3^{31} = \tilde{f}_3^{31}. \)

- **Case (3):** \( \tilde{f}_3^{33} = \tilde{f}_3^{33}, \quad \tilde{f}_3^{34} = \tilde{f}_3^{34}, \quad \tilde{f}_2^{23} = \tilde{f}_2^{23}, \quad \tilde{f}_1^{12} = -\tilde{f}_2^{23} + \frac{i}{2}\tilde{f}_3^{34}, \quad \tilde{f}_2^{24} = -\frac{i}{2}\tilde{f}_3^{34}. \)

- **Case (4):** \( \tilde{f}_3^{33} = \tilde{f}_3^{33}, \quad \tilde{f}_3^{34} = \tilde{f}_3^{34}, \quad \tilde{f}_3^{32} = \tilde{f}_3^{32}, \quad \tilde{f}_1^{12} = \frac{i}{2}\tilde{f}_3^{34}, \quad \tilde{f}_3^{34} = \frac{i}{4}\tilde{f}_3^{32}, \quad \tilde{f}_2^{24} = -\frac{i}{4}\tilde{f}_3^{32}, \quad \tilde{f}_2^{24} = -\frac{i}{2}\tilde{f}_3^{34}. \)

In the next step, we must find out that the above solutions are isomorphic with which one of the Lie superalgebras listed in Tables 1 and 2 (the detail of method has been explained in Ref. \[14\]). We have found that the solution of case (1) is isomorphic to the Lie superalgebras \( C_{p=\pm 1}^1 \oplus A, C_{p=1}^2 \oplus A_{1,1}, C^3 \oplus A_{1,1}, \) and \( C^4 \oplus A_{1,1} \) of Table 2 and \( D_{p,q=p-1}^1 \) of Table 1. The solution of case (2) is isomorphic to the Lie superalgebras \( D_{p=\pm 1}^1 \) and \( (C^3 + A) \) of Table 1 and \( (2, A_{1,1} + 2A)^0 \) of Table 2; moreover, this case is also isomorphic to \( C^3 \oplus A_{1,1} \) with the same conditions of case (1). For case (3), we see that this solution is isomorphic to the Lie superalgebras \( (D_{p=\pm 1}^1)^2, (D_{p=\pm p}^1), (C^1 + A) \) and \( (C^2_{-1} + A) \) of Table 1. Finally, we find that the solution of case (4) is isomorphic to the Lie superalgebras \( (D_{1-p}^1)^2, (D_0^{10})^1, (D_{10}^1)^2 \) and \( (C^3 + A) \) of Table 1. Also, case (4) is isomorphic to the Lie superalgebra \( (C^1_{-1} + A) \) with the same conditions of case (3). Hitherto, we have found the general forms of the Lie superbialgebra structures on \( (C^3 + A) \). But now, one question raises: which of these Lie superbialgebra structures are equivalent? In order to answer this question, one has to use the automorphism supergroup of \( (C^3 + A) \) \([3,6]\) and the following proposition.

**Proposition 2:** If there exists an automorphism \( A \) of \( g \) such that

\[
\delta' = (A \otimes A) \circ \delta \circ A^{-1},
\]

then the super one-cocycles \( \delta \) and \( \delta' \) of the Lie superalgebra \( g \) are equivalent. In this case, the two Lie superbialgebras \( (g, \delta) \) and \( (g, \delta') \) are equivalent \([26]\).

From the action of both sides of relation \([4.4]\) on basis \( \{X_i\} \) of \( g \) and then using relations \([3.2]\) and \([2.20]\) we arrive at

\[
(-1)^{jk+im} A^{-st} \tilde{Y}_i A^{-1} = (A^{-1})_i^n \tilde{Y}_n,
\]

where \( m \) denotes the row of matrix \( A^{-st} \) in the left hand side, and the indices \( j \) and \( k \) correspond to the row and column of matrix \( \tilde{Y}_i \), respectively. Note that the bilinear form of the graded brackets
[2.17] is invariant with respect to the transformations

\[ X'_i = (-1)^j A_i^j X_i, \quad \tilde{X}'_{ij} = (A^{-st})^j_m \tilde{X}^m. \]

Inserting the second transformation into

\[ [\tilde{X}^n, \tilde{X}^m] = f^{nij}_{\ k} \tilde{X}^k, \]

one can recover relation \[1150\]. Finally we obtain 31 families of inequivalent Lie superalgebra structures on \((C^3 + \mathcal{A})\) whose representatives have classified in Table 3.

### Table 3. Dual Lie superalgebras to the Lie superalgebra \((C^3 + \mathcal{A})\), \(\epsilon = \pm 1\).

| \(\mathfrak{g}\) | Non-zero (anti)commutation relations | Comments |
|------------------|----------------------------------|----------|
| \(\mathcal{I}_{(2,2)}\) | All of the (anti)commutation relations are zero. | |
| \(C^{1,e}_{p=1} \oplus \mathcal{A}\) | \([\tilde{X}^1, \tilde{X}^2] = \epsilon \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = -\epsilon \tilde{X}^3\) | |
| \(C^{1,e}_{p=-1} \oplus \mathcal{A}\) | \([\tilde{X}^1, \tilde{X}^2] = \epsilon \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^4] = \epsilon \tilde{X}^4\) | |
| \(C^{2,e}_{p=1} \oplus \mathcal{A}_{1,1}\) | \([\tilde{X}^2, \tilde{X}^3] = \epsilon \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \epsilon \tilde{X}^4\) | |
| \(C^{3} \oplus \mathcal{A}_{1,1}\) | \([\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4\) | |
| \(C^{4,e} \oplus \mathcal{A}_{1,3}\) | \([\tilde{X}^2, \tilde{X}^3] = \tilde{X}^4, \quad [\tilde{X}^2, \tilde{X}^4] = \epsilon \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \epsilon \tilde{X}^4\) | |
| \(D^{1,e}_{p=1} \oplus \mathcal{A}\) | \([\tilde{X}^1, \tilde{X}^2] = -2 \epsilon \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = \epsilon \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \epsilon (p - 1) \tilde{X}^4\) | \(p \neq 0, 1\) |
| \((D^{1,e}_{p=1})^2\) | \([\tilde{X}^1, \tilde{X}^2] = -2 \epsilon \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = \epsilon \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \epsilon \tilde{X}^4, \quad [\tilde{X}^3, \tilde{X}^4] = 2 \epsilon \tilde{X}^1\) | |
| \((D^{1,e}_{p=1})^4\) | \([\tilde{X}^1, \tilde{X}^2] = - \epsilon \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = \epsilon \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = - \epsilon (p - 1) \tilde{X}^4\) | \(p < \frac{1}{2}, p \neq 0\) |
| \((D^{1,e}_{p=1})^2\) | \([\tilde{X}^1, \tilde{X}^2] = - \epsilon \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = - \epsilon (p - 1) \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \epsilon \tilde{X}^4\) | \(p < \frac{1}{2}, p \neq 0\) |
| \((D^{1,e}_{p=1})^4\) | \([\tilde{X}^1, \tilde{X}^2] = - \epsilon \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = - \epsilon \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \epsilon \tilde{X}^4\) | \(p < \frac{1}{2}, p \neq 0\) |
| \((D^{1,e}_{p=1})^2\) | \([\tilde{X}^1, \tilde{X}^2] = - \epsilon \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = \epsilon \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = \epsilon \tilde{X}^4\) | \(p < \frac{1}{2}, p \neq 0\) |
| \((D^{1,e}_{p=1})^4\) | \([\tilde{X}^1, \tilde{X}^2] = - \epsilon \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^3] = \epsilon \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = - \epsilon \tilde{X}^2\) | \(p < \frac{1}{2}, p \neq 0\) |
| \((2A_{1,1} + 2A)\) | \([\tilde{X}^3, \tilde{X}^3] = i \tilde{X}^1\) | |
| \((C^{1} + \mathcal{A})\) | \([\tilde{X}^1, \tilde{X}^2] = - \frac{1}{2} \tilde{X}^1, \quad [\tilde{X}^2, \tilde{X}^4] = \frac{1}{2} \tilde{X}^4\) | \(p < \frac{1}{2}, p \neq 0\) |
| \((C^{2} + \mathcal{A})\) | \([\tilde{X}^1, \tilde{X}^2] = \epsilon \tilde{X}^3, \quad [\tilde{X}^2, \tilde{X}^4] = - \epsilon \tilde{X}^4\) | \(p < \frac{1}{2}, p \neq 0\) |
| \((C^{3} + \mathcal{A})\) | \([\tilde{X}^1, \tilde{X}^2] = - \frac{1}{2} \tilde{X}^4\) | \(p < \frac{1}{2}, p \neq 0\) |
| \((C^{3} + \mathcal{A})\) | \([\tilde{X}^2, \tilde{X}^3] = i \tilde{X}^4\) | \(k > 0\) |
5 The coboundary Lie superbialgebras \((C^3 + A)\)

The aim of this section is to find coboundary Lie superbialgebras of the Lie superalgebra \((C^3 + A)\) with coboundary duals. As mentioned in section 2, since such structures can be specified (up to automorphism) by pairs of r-matrices, so they are called bi-r-matrix Lie superbialgebras. For determining the coboundary Lie superbialgebras of the Lie superalgebra \((C^3 + A)\) we must find \(r = r^{ij} X_i \otimes X_j \in (C^3 + A) \otimes (C^3 + A)\). To this end, one can use relations (2.2) and (2.20) to rewrite relation (2.10) as

\[
\hat{\delta}_i = X_i^{st} r + (-1)^i r X_i, \quad (5.1)
\]

where the superscript \(l\) corresponds to the row of the matrix \(X_i\). In this manner, we determine which of the Lie superbialgebras \((C^3 + A)\) are coboundary and obtain the corresponding r-matrices. On the other hand, by using the super one-cocycle (2.14) for some r-matrix \(\tilde{r} \in \hat{g} \otimes \hat{g}\) and by considering \(\tilde{\delta}(\hat{X}^i) = (-1)^{ik} f_{jk} \hat{X}^j \otimes \hat{X}^k\) we get

\[
\hat{\gamma}^i = \hat{X}^{i\text{st}} \tilde{r} + (-1)^i \tilde{r} \hat{X}^i. \quad (5.2)
\]

Note that the super skew-symmetric part of \(r\) defined by \(\hat{r}^{ij} = \frac{1}{2} (r^{ij} - (-1)^{ij} r^{ji})\) yields the same \(\hat{f}^{ij}_k\), so we will assume that \(r \in \hat{g} \wedge \hat{g}\) with \(r^{ij} = (-1)^{ij} r^{ji}\). In the same way, the mix part of \(r\) as an element of \(\hat{g}_B \wedge \hat{g}_F\) subspace of \(\hat{g} \wedge \hat{g}\) cannot influence \(\hat{f}^{ij}_k\) without violating the condition \(\hat{f}^{ij}_k = 0\), if \(i + j + k \neq 0\). Therefore, we get even r-matrix as \(r \in \hat{g}_B \wedge \hat{g}_B \oplus \hat{g}_F \wedge \hat{g}_F\), i.e., \(r^{ij} = 0\) if \(i \neq j\). To solve equations (5.1) and (5.2), we first obtain all adjoint representations of the Lie superbialgebras \((C^3 + A)\) (listed in Table 3). Then we find that the Lie superbialgebras \(((C^3 + A), I_{(2,2)})\), \(((C^3 + A), C^3 \oplus A_{1,1,1}, i)\), \(((C^3 + A), (2A_{1,1} \oplus 2A^0, i))\) and \(((C^3 + A), (C^3 + A)_k)\) are coboundary. Among these, only the Lie superbialgebras \(((C^3 + A), C^3 \oplus A_{1,1,1}, i)\) and \(((C^3 + A), (C^3 + A)_k)\) have coboundary duals. In the following, we give the solution of equations (5.1) and (5.2) for the coboundary Lie superbialgebras.

• For the Lie superbialgebra \(((C^3 + A), I_{(2,2)})\), equation (5.1) can be solved and the general solution has the following form

\[
r = -a_1 (X_1 \otimes X_2 + X_2 \otimes X_1 - X_3 \otimes X_4 + X_4 \otimes X_3) + a_2 X_2 \otimes X_2 + \frac{a_3}{2} X_3 \wedge X_3, \quad (5.3)
\]

where \(a_i\) \((i = 1, 2, 3)\) are some the constant values. Note that, here and the following, the constants \(a_i, b_i, \ldots\) are c-numbers [22]. For the above solution, the graded Schouten bracket is then read

\[
[[r, r]] = -\frac{a_1^2}{2} X_2 \wedge X_3 \wedge X_3. \quad (5.4)
\]

Triangular solutions are obtained when \(a_1\) and \(a_2\) are zero. As mentioned above, in this case, the dual Lie superalgebra is not coboundary.
The corresponding r-matrix to the coboundary Lie superbialgebra \(((C^3 + A), C^3 \oplus A_{1,1})\) from equation (5.1) is obtained to be of the form

\[
r = X_1 \wedge X_2 - b_1 (X_1 \otimes X_2 + X_2 \otimes X_1 - X_3 \otimes X_4 + X_4 \otimes X_3) + b_2 X_2 \otimes X_2 + \frac{b_3}{2} X_3 \wedge X_3, \tag{5.5}
\]

for which the graded Schouten bracket is

\[
[r, r] = -\frac{b_1^2}{2} X_2 \wedge X_3 \wedge X_3. \tag{5.6}
\]

Then the condition \(b_1 = b_2 = 0\) gives rise to triangular solutions. In this case we have a bi-r-matrix Lie superbialgebra, i.e., by solving equation (5.2) we obtain an r-matrix \(\tilde{r} \in \tilde{g} \otimes \tilde{g}\) (\(\tilde{g}\) refers to \(C^3 \oplus A_{1,1}\)) as

\[
\tilde{r} = -\tilde{X}^1 \wedge \tilde{X}^2 + c_1 \tilde{X}^1 \otimes \tilde{X}^1 + c_2 \tilde{X}^3 \otimes \tilde{X}^1 - (1 + c_2) \tilde{X}^4 \otimes \tilde{X}^3 + \frac{c_3}{2} \tilde{X}^4 \wedge \tilde{X}^4, \tag{5.7}
\]

which induces on \(g = (C^3 + A)\) the original superbrackets (3.1). For this solution, one can obtain

\[
[[\tilde{r}, \tilde{r}]] = \frac{1}{2} \tilde{X}^1 \wedge \tilde{X}^4 \wedge \tilde{X}^4, \tag{5.8}
\]

such that quasi-triangular solutions are obtained when \(c_1 = 0\) and \(c_2 = -\frac{1}{2}\).

- The general solution of equation (5.1) for the Lie superbialgebra \(((C^3 + A), (2A_{1,1} \oplus 2A^0, i))\) is given by

\[
r = -d_1 (X_1 \otimes X_2 - X_3 \otimes X_4) - (1 + d_1) (X_2 \otimes X_1 + X_4 \otimes X_3) + d_2 X_2 \otimes X_2 + \frac{d_3}{2} X_3 \wedge X_3, \tag{5.9}
\]

for which, we obtain

\[
[r, r] = -\frac{d_1 (1 + d_1)}{2} X_2 \wedge X_3 \wedge X_3. \tag{5.10}
\]

Therefore if \(d_1 = -\frac{1}{2}\) and \(d_2 = 0\), we have quasi-triangular solutions.

- For the Lie superbialgebra \(((C^3 + A), (C^3 + A)^c)\), equations (5.1) and (5.2) can be also solved and the general solutions are, respectively, read

\[
r = (\epsilon - e_1) X_1 \otimes X_2 - (\epsilon + e_1 + k) X_2 \otimes X_1 + e_2 X_2 \otimes X_2 + \frac{e_3}{2} X_3 \wedge X_3 \nonumber \\
+ e_1 X_3 \otimes X_4 - (k + e_1) X_4 \otimes X_3, \tag{5.11}
\]

\[
\tilde{r} = f_1 \tilde{X}^1 \otimes \tilde{X}^1 + \epsilon (-1 + f_2 k) \tilde{X}^1 \otimes \tilde{X}^2 + (k + \frac{1}{\epsilon} + \frac{f_2 k}{\epsilon}) \tilde{X}^2 \otimes \tilde{X}^1 + \frac{f_3}{2} \tilde{X}^4 \wedge \tilde{X}^4 \nonumber \\
+ f_2 \tilde{X}^3 \otimes \tilde{X}^4 - (\frac{1}{\epsilon} + f_2) \tilde{X}^4 \otimes \tilde{X}^3. \tag{5.12}
\]
The corresponding graded Schouten brackets are then obtained to be

\[ [[r, r]] = -\frac{1}{2}(e_1^2 + e_1 k - e k)X_2 \wedge X_3 \wedge X_3 \]  

\begin{equation}
(5.13)
\end{equation}

and

\[ [[\tilde{r}, \tilde{r}]] = \frac{1}{2} \left( \frac{1}{\epsilon} - k f_2^2 - e k f_2 \right) \tilde{X}_1 \wedge \tilde{X}_4 \wedge \tilde{X}_4. \]

\begin{equation}
(5.14)
\end{equation}

The super skew-symmetric solutions of (5.11) are given by \( e_1 = -\frac{k}{2} \) and \( e_2 = 0 \). Thus, since \( k \) is positive for \( \epsilon = -1 \) and \( k = 4 \), we have triangular solutions, otherwise we are considering quasi-triangular ones. Similarly, by putting \( f_1 = 0 \) and \( f_2 = -\frac{1}{2k} \) into (5.12), we obtain the super skew-symmetric solutions. In the same way, triangular solutions are obtained only when \( \epsilon = -1 \) and \( k = 4 \). We give the super skew-symmetry solutions of equations (5.11) and (5.12) along with the corresponding graded Schouten brackets in Table 4.

| \((\mathfrak{g}, \tilde{\mathfrak{g}})\) | \(r, \tilde{r}\) | [[r, r]], [[\tilde{r}, \tilde{r}]] |
|---|---|---|
| \((C^3 + \mathcal{A}), T_{(2, 3)}\) | \(r = \frac{a_2}{2} X_3 \wedge X_3\) | [[r, r]] = 0 |
| \((C^3 + \mathcal{A}), C^3 \oplus A_{11, 1}\) | \(r = X_1 \wedge X_2 + \frac{a_3}{2} X_3 \wedge X_3\) | [[r, r]] = 0 |
| \((C^3 + \mathcal{A}), (2A_{11, 1} \oplus A^0_{11, 1})\) | \(r = \frac{1}{2} (X_1 \wedge X_2 - X_3 \wedge X_4 + a_3 X_3 \wedge X_3)\) | [[r, r]] = \frac{1}{2} X_2 \wedge X_3 \wedge X_3 |
| \((C^3 + \mathcal{A}), (C^3 + \mathcal{A})^k\) | \(r = (\epsilon + \frac{k}{4}) X_1 \wedge X_2 + \frac{a_3}{2} X_3 \wedge X_3 - \frac{1}{2} X_3 \wedge X_4\) | [[r, r]] = k(\epsilon + \frac{k}{4}) X_2 \wedge X_3 \wedge X_3 |
| \(\tilde{r} = - (\epsilon + \frac{k}{4}) \tilde{X}_1 \wedge \tilde{X}_2 - \frac{1}{2} \tilde{X}_3 \wedge \tilde{X}_4 + \frac{a_3}{2} \tilde{X}_4 \wedge \tilde{X}_4\) | [[\tilde{r}, \tilde{r}]] = \frac{1}{2} (\epsilon + \frac{k}{4}) \tilde{X}_1 \wedge \tilde{X}_4 \wedge \tilde{X}_4 |

6 The quantum deformation of \((C^3 + \mathcal{A})\) and deformation of related integrable Hamiltonian systems

In this section, we quantize the Lie superalgebra \((C^3 + \mathcal{A})\) by making use of the Lyakhovsky and Mudrov formalism [20]. For this purpose and self-containing of the paper, we first review the main result of this formalism as a following proposition. Then, we use this method in order to build up the Hopf superalgebras related to some Lie superbialgebras \((C^3 + \mathcal{A})\) of Table 3. Nevertheless, as an application, we get at the end of this section a family of quantum integrable Hamiltonian systems that can be constructed from a convenient representation of the quantum Lie superalgebra \((C^3 + \mathcal{A})\) with the corresponding Casimir element.
6.1 The Hopf superalgebras corresponding to some Lie superbialgebras \((C^3 + A)\)

**Proposition 3:** \([20]\) Let \(1 I, H_i (i = 1, \cdots, n)\) and \(X_m (l = 1, \cdots, m)\) be a basis of an associative unital algebra \(A\) over the field \(C\), and \(\mu_i, \nu_j (i, j = 1, \cdots, n)\) be a set of \(m \times m\) complex matrices such that they are commute with together. In addition, let \(\vec{X}\) be a column\((row)\) vector with component \(X_l (l = 1, \cdots, m)\). The coproduct

\[
\Delta(\vec{X}) = \exp(\sum_{i=1}^{n} \mu_i H_i) \otimes \vec{X} + \sigma(\exp(\sum_{i=1}^{n} \nu_i H_i) \otimes \vec{X})
\]

\[
\Delta(H_i) = 1 I \otimes H_i + H_i \otimes 1 I, \\
\Delta(1 I) = 1 I \otimes 1 I,
\]

endow \((A, \Delta, \epsilon, \gamma)\) with a Hopf algebra structure if the generators \(H_i\) commute with together.

Let \(P\) be an \(m \times m\) matrix with entries \(p_{kl} \in A\), then the \(k\)th component of \(P \otimes \vec{X}\) is defined as \((P \otimes \vec{X})_k = \sum_{l=1}^{m} p_{kl} \otimes X_l\). Also, notice that here \(\sigma\) is the flip operator \(\sigma(X_l \otimes X_m) = X_m \otimes X_l\).

The deformation parameters in the resulting coalgebra are the entries of the matrices \(\mu_i\) and \(\nu_j\). If we can find a compatible multiplication with the coproduct \((6.1)\) we will have finally obtained a quantum algebra. In fact, the role of the matrices \(\mu_i\) and \(\nu_j\) is to reflect the Lie bialgebra underlying a given quantum deformation. This can be clearly appreciated by taking the first order of \((6.1)\)

\[
\Delta_{(1)}(\vec{X}) = (\sum_{i=1}^{n} \mu_i H_i) \otimes \vec{X} + \sigma(\sum_{i=1}^{n} \nu_i H_i \otimes \vec{X}).
\]

On the other hand, since the cocommutator \(\delta\) corresponds to the co-antisymmetric part of \((6.4)\), it can be written as

\[
\delta(\vec{X}) = \Delta_{(1)}(\vec{X}) - \sigma \circ \Delta_{(1)}(\vec{X}).
\]

In this formalism, elements \(H_i\) are called primitive generators. These elements must be chosen such that \(\delta(\vec{X}_i)\) does not contain terms of the form \(H_i \wedge H_j\). We note that the same cocommutator \((6.5)\) can be obtained from different choices of the matrices \(\mu_i\) and \(\nu_j\), i.e., the different sets of matrices lead to right quantization of \(U(\mathfrak{g})\). When the algebra \(A\) is a Lie algebra \(\mathfrak{g}\), then \(H_i\) generate an
Abelian Lie subalgebra, and with this condition the deformed commutation relations in $U_\lambda(g)$ with $\lambda$ being the deformation parameter are given by \[20\]

$$[X_l, X_p] = [X_l, X_p]_0 + \phi_{lp}(\mu_i, \nu_j, H_k),$$

(6.6)

where $[X_l, X_p]_0$ is the classical commutation relation and the deforming functions $\phi_{lp}$ are the power series of $H_k$'s. Note that after determining of $\phi_{lk}$, the Jacobi identity for (6.6) must be checked.

The above formalism was presented for the Lie algebras and one can use this formalism for Lie superalgebras by keeping that the graded tensor product law must be taken into account \[27\]

$$(F \otimes G)_{ijkl} = (-1)^{ij} F_{ik} G_{jl}.$$  

(6.7)

If $U(g)$ be a quantum superalgebra, the extension of the coproduct $\Delta : U(g) \rightarrow U(g) \otimes U(g)$ to products of generators should be substituted by $(a \otimes b)(c \otimes d) = (-1)^{bc} ac \otimes bd$ for all $a, ..., d$ in $U(g)$; moreover, the flip operator $\sigma$ should be written as $\sigma(a \otimes b) = (-1)^{ab} b \otimes a$.

This quantization procedure can be applied to the Lie superbialgebras of Table 3 to quantize the Lie superalgebra $(C^3 + A)$. We shall write the supercoproduct and the deformed (anti)commutation rule, as the supercounit is always trivial and the superantipode can be easily deduced by means of the Hopf superalgebra axioms. Deformed Casimir operator, which is essential for the construction of integrable systems, is also explicitly given.

Let us first denote the Lie superalgebra $(C^3 + A)$ with the generators $Z, H, Q_+$ and $Q_-$ instead of $X_1, X_2, X_3$ and $X_4$, respectively. Then, in the non-standard basis, the (anti)commutation relations (3.1) are written as

$$[Z, Q_-] = Q_+, \quad \{Q_-, Q_-\} = H, \quad [H, .] = 0, \quad [Q_+, .] = 0.$$  

(6.8)

The relevant invariant supersymmetric bilinear form for $(C^3 + A)$ is

$$< Z, H > = < H, Z > = < Q_-, Q_+ > = - < Q_+, Q_+ > = 1.$$  

(6.9)

For the above choice of metric, the quadratic Casimir is found to be of the form

$$C^{(2)} = 2(ZH - Q_+ Q_-).$$  

(6.10)

Now we proceed to obtain the Hopf superalgebra corresponding to the Lie superbialgebra $((C^3 + A), C_{p=1}^{2, \epsilon} \oplus A_{1, 1})$. Commutation relations of the dual Lie superalgebra $C_{p=1}^{2, \epsilon} \oplus A_{1, 1}$ have displayed in Table 3. Using relation \[2.20\], one can write the super cocommutators as

$$\delta(Z) = 0, \quad \delta(H) = 0,$$

$$\delta(Q_+) = \epsilon H \wedge Q_+, \quad \delta(Q_-) = \epsilon H \wedge Q_-.$$  

(6.11)
We see that there does not exist the term of the type $Z \wedge H$ within the super cocommutators and $[Z, H] = 0$. So, we denote the primitive generators $H_i$ by $H_1 = Z$ and $H_2 = H$. Hence, by considering $\epsilon = \lambda$ the super cocommutators for the non-primitive generators $Q_+$ and $Q_-$ can be written as

$$
\delta \left( \begin{array}{c} Q_+ \\ Q_- \end{array} \right) = \left( \begin{array}{cc} \lambda H & 0 \\ 0 & \lambda H \end{array} \right) \lambda \left( \begin{array}{c} Q_+ \\ Q_- \end{array} \right). \tag{6.12}
$$

In view of this expression, the matrices $\mu_i$ and $\nu_j$ can be chosen as

$$
\mu_1 = 0, \quad \mu_2 = \left( \begin{array}{cc} \frac{\lambda}{2} & 0 \\ 0 & \frac{\lambda}{2} \end{array} \right), \quad \nu_1 = 0, \quad \nu_2 = \left( -\frac{\lambda}{2} & 0 \\ 0 & -\frac{\lambda}{2} \right). \tag{6.13}
$$

Clearly, this choice satisfies relation (6.5). Now we can get the supercoproducts

$$
\Delta \left( \begin{array}{c} Q_+ \\ Q_- \end{array} \right) = \exp \left\{ \left( \begin{array}{cc} \frac{\lambda}{2} H & 0 \\ 0 & \frac{\lambda}{2} H \end{array} \right) \right\} \otimes \left( \begin{array}{c} Q_+ \\ Q_- \end{array} \right) + \sigma \left( \exp \left\{ \left( \begin{array}{cc} -\frac{\lambda}{2} H & 0 \\ 0 & -\frac{\lambda}{2} H \end{array} \right) \right\} \otimes \left( \begin{array}{c} Q_+ \\ Q_- \end{array} \right) \right). \tag{6.14}
$$

Suppose that in the classical (anti)commutation relations (6.8), only the composition $\{ Q_-, Q_- \}$ is deformed

$$
\{ Q_-, Q_- \} = \phi(\lambda, H).
$$

Imposing the conditions $\Delta(\{ Q_-, Q_- \}) = \{ \Delta Q_-, \Delta Q_- \}$ and $\Delta\phi(\lambda, H) = \phi(\lambda, 1 \otimes H + H \otimes 1)$, one easily obtains the relation

$$
e^{\lambda H} \otimes \phi(\lambda, H) + \phi(\lambda, H) \otimes e^{-\lambda H} = \phi(\lambda, 1 \otimes H + H \otimes 1).
$$

The solution is

$$
\phi(\lambda, H) = \frac{\sinh(\lambda H)}{\lambda}.
$$

Finally, the results of quantum deformation for the Lie superbialgebra $((C^3 + A), C_{p=1}^{2,\epsilon} \oplus A_{1,1})$ are given by the following statement.

**Proposition 4:** The quantum superalgebra which quantizes the Lie superbialgebra $((C^3 + A), C_{p=1}^{2,\epsilon} \oplus A_{1,1})$ has Hopf structure denoted by $U_\lambda^{C_{p=1}^{2,\epsilon} \oplus A_{1,1}}((C^3 + A))$ and is, respectively, characterized by
the following supercoproduct, supercounit and superantipode \cite{28}
\[
\Delta(\hat{Z}) = 1 \otimes \hat{Z} + \hat{Z} \otimes 1, \quad \Delta(\hat{H}) = 1 \otimes \hat{H} + \hat{H} \otimes 1, \\
\Delta(\hat{Q}_+) = e^{\frac{\hat{H}}{2}} \otimes \hat{Q}_+ + \hat{Q}_+ \otimes e^{-\frac{\hat{H}}{2}}, \quad \Delta(\hat{Q}_-) = e^{\frac{\hat{H}}{2}} \otimes \hat{Q}_- + \hat{Q}_- \otimes e^{\frac{\hat{H}}{2}},
\]
(6.15)
\[
\epsilon(1) = 1, \quad \epsilon(\hat{X}) = 0, \quad \hat{X} \in \{\hat{Z}, \hat{H}, \hat{Q}_+, \hat{Q}_-\},
\]
(6.16)
\[
\gamma(\hat{Z}) = -\hat{Z}, \quad \gamma(\hat{H}) = -\hat{H}, \\
\gamma(\hat{Q}_+) = -\hat{Q}_+, \quad \gamma(\hat{Q}_-) = -\hat{Q}_-,
\]
(6.17)
together with the (anti)commutation relations
\[
[\hat{Z}, \hat{Q}_-] = \hat{Q}_+, \quad \{\hat{Q}_-, \hat{Q}_-\} = \frac{\sinh(\lambda \hat{H})}{\lambda}, \quad [\hat{H}, .] = 0, \quad [\hat{Q}_+, .] = 0.
\]
(6.18)

The quantum Casimir belonging to the centre of $U_\lambda^{(C^3 + A)}((C^3 + A))$ (whose classical limit is \cite{6.10}) is generated by
\[
\hat{C}_\lambda^{(2)} = 2(Z \frac{\sinh(\lambda \hat{H})}{\lambda}) - \hat{Q}_+ \hat{Q}_-).
\]
(6.19)

This quantization procedure can be applied to the remaining types of Lie superbialgebras in the same way. We also quantize the Lie superbialgebras $((C^3 + A), C^{4,e} \oplus A_{1,1})$ and $((C^3 + A), C^{1,e} \oplus A)$. The results are given by the following statements.

**Proposition 5:** The supercoproduct $\Delta$, supercounit $\epsilon$, superantipode $\gamma$

\[
\Delta(\hat{Z}) = 1 \otimes \hat{Z} + \hat{Z} \otimes 1, \quad \Delta(\hat{Q}_+) = 1 \otimes \hat{Q}_+ + \hat{Q}_+ \otimes e^{-\lambda \hat{H}}, \\
\Delta(\hat{H}) = 1 \otimes \hat{H} + \hat{H} \otimes 1, \quad \Delta(\hat{Q}_-) = 1 \otimes \hat{Q}_- + \hat{Q}_- \otimes e^{-\lambda \hat{H}} - \hat{Q}_+ \otimes \hat{H} e^{-\lambda \hat{H}},
\]
(6.20)
\[
\epsilon(1) = 1, \quad \epsilon(\hat{X}) = 0, \quad \hat{X} \in \{\hat{Z}, \hat{H}, \hat{Q}_+, \hat{Q}_-\},
\]
(6.21)
\[
\gamma(\hat{Z}) = -\hat{Z}, \quad \gamma(\hat{H}) = -\hat{H}, \\
\gamma(\hat{Q}_+) = -\hat{Q}_+ e^{\lambda \hat{H}} - \hat{H} \hat{Q}_- e^{\lambda \hat{H}}, \quad \gamma(\hat{Q}_-) = -\hat{Q}_- e^{\lambda \hat{H}},
\]
(6.22)

and the (anti)commutation relations
\[
[\hat{Z}, \hat{Q}_-] = \hat{Q}_+, \quad \{\hat{Q}_-, \hat{Q}_-\} = \frac{1 - e^{-2\lambda \hat{H}}}{2\lambda}, \quad [\hat{H}, .] = 0, \quad [\hat{Q}_+, .] = 0.
\]
(6.23)
determine a Hopf superalgebra denoted by $U_\lambda^{(C^3 + A)}((C^3 + A))$ which quantizes the Lie superbialgebra $((C^3 + A), C^{4,e} \oplus A_{1,1})$.

The deformed Casimir that commutes with all the generators of the quantum superalgebra , in this case, reads
\[
\hat{C}_\lambda^{(2)} = \frac{1}{\lambda} \hat{Z}(1 - e^{-2\lambda \hat{H}}) - 2\hat{Q}_+ \hat{Q}_-.
\]
(6.24)
Proposition 6: The Hopf superalgebra denoted by $U^{(C_1, \{p = -1 \oplus A\})}_\lambda ((C^3 + A))$ which quantizes the Lie superbialgebra $((C^3 + A), C_1, \epsilon_{p = -1} \oplus A)$ has, respectively, the supercoproduct, the supercounit and the superantipode

$$\Delta(\hat{Z}) = 1 \otimes \hat{Z} + \hat{Z} \otimes e^{\lambda \hat{H}},$$
$$\Delta(\hat{Q}_+) = 1 \otimes \hat{Q}_+ + \hat{Q}_+ \otimes 1,$$
$$\Delta(\hat{H}) = 1 \otimes \hat{H} + \hat{H} \otimes 1,$$
$$\Delta(\hat{Q}_-) = 1 \otimes \hat{Q}_- + \hat{Q}_- \otimes e^{-\lambda \hat{H}},$$

(6.25)

$$\epsilon(1) = 1,$$
$$\epsilon(\hat{X}) = 0,$$
$$\hat{X} \in \{\hat{Z}, \hat{H}, \hat{Q}_+, \hat{Q}_-\},$$

(6.26)

$$\gamma(\hat{Z}) = -\hat{Z}e^{-\lambda \hat{H}},$$
$$\gamma(\hat{H}) = -\hat{H},$$
$$\gamma(\hat{Q}_+) = -\hat{Q}_+,$$
$$\gamma(\hat{Q}_-) = -\hat{Q}_-e^{\lambda \hat{H}},$$

(6.27)

with the same (anti)commutation rules and the Casimir element derived in (6.23) and (6.24), respectively.

6.2 Integrable deformation of Hamiltonian systems

Let $\mathcal{A}$ be a the Hopf superalgebra. Then the pair of $(\mathcal{A}, \Delta)$ is called a supercoalgebra. Given any supercoalgebra $(\mathcal{A}, \Delta)$ with the corresponding Casimir element, each of its representations gives rise to a family of completely integrable Hamiltonians [19]. In this subsection, we construct a deformed quantum integrable Hamiltonian system from the representation of the Lie supercoalgebra $(C^3 + A)$ given by Proposition 4, along with the corresponding deformed Casimir element. This system is constructed on a supersymplectic flat supermanifold of the superdimension-(4|4) as the phase superspace.

Consider $\mathcal{M}$ be a supermanifold with a non-degenerate supersymplectic form $\omega = \frac{(-1)^{\lambda \Gamma}}{2} \omega_{\gamma \Lambda} d\Phi^\gamma \\
\wedge d\Phi^\Lambda$ where $\Phi^\gamma$ are the local coordinates of $\mathcal{M}$. The coordinates $\Phi^\gamma$ include the bosonic and the fermionic coordinates, and the label $\gamma$ runs over $\mu = 0, \cdots, d_B - 1$ and $\alpha = 1, \cdots, d_F$ where $d_B$ and $d_F$ indicate the dimension of the bosonic coordinates and the fermionic coordinates, respectively.

We define the graded Poisson bracket structure on supermanifold $\mathcal{M}$ for two arbitrary functions $F, G \in C^\infty(\Phi^\gamma)$ as [29]

$$\{F, G\}_{P.B.} = \frac{F_{\gamma} \overleftarrow{\partial}}{\partial \Phi^\gamma} \omega^{\gamma \Lambda} \frac{G_{\Lambda}}{\partial \Phi^\Lambda} = (-1)^{\lambda(\gamma + [F])} \omega^{\gamma \Lambda} \overleftarrow{\partial} F \overleftarrow{\partial} G,$$

(6.28)

where $\omega^{\gamma \Lambda}$ is the superinverse of $\omega_{\gamma \Lambda}$. Here we assume that $\mathcal{M}$ is a flat supermanifold $R^4_c \times R^4_a$ of the superdimension-(4|4) with the local coordinates $(q^\mu, p_\mu; \xi^\alpha, \pi_\alpha)$, $(\mu, \alpha = 1, 2)$ and the supersymplectic structure

$$\omega = dq^1 \wedge dp_1 + dq^2 \wedge dp_2 - d\xi^1 \wedge d\pi_1 - d\xi^2 \wedge d\pi_2.$$

(6.29)
Hence, the canonical graded Poisson brackets are given by

\[
\{q^\mu, p_\nu\}_{P.B.} = -\{p_\nu, q^\mu\}_{P.B.} = \delta^\mu_\nu,
\{\xi^\alpha, \pi_\beta\}_{P.B.} = \{\pi_\beta, \xi^\alpha\}_{P.B.} = \delta^\alpha_\beta,
\{q^\mu, \xi^\alpha\}_{P.B.} = \{p_\mu, \pi_\alpha\}_{P.B.} = \{\xi^\alpha, p_\mu\}_{P.B.} =\{q^\mu, \pi_\alpha\}_{P.B.} = 0, \tag{6.30}
\]

where \(p_\mu = -\frac{\partial}{\partial q^\mu}\) and \(\pi_\alpha = \frac{\partial}{\partial \xi^\alpha}\) are the conjugate to \(q^\mu\) and \(\xi^\alpha\) momentums, respectively.

We now consider that the Lie superalgebra \(g\) is realized by means of smooth functions on the phase superspace \(R^4_c \times R^4_a\) with the local coordinates \((q^\mu, p_\mu; \xi^\alpha, \pi_\alpha)\)

\[
S(X_i) = X_i(q^\mu, p_\mu; \xi^\alpha, \pi_\alpha). \tag{6.31}
\]

This means that under the canonical graded Poisson bracket

\[
\{F, G\}_{P.B.} = \sum_{\mu=1}^{2} \left( \frac{\partial F}{\partial q^\mu} \frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu} \frac{\partial G}{\partial q^\mu} \right) - (-1)^{|F|} \sum_{\alpha=1}^{2} \left( \frac{\partial F}{\partial \xi^\alpha} \frac{\partial G}{\partial \pi_\alpha} - \frac{\partial F}{\partial \pi_\alpha} \frac{\partial G}{\partial \xi^\alpha} \right), \tag{6.32}
\]

the dynamical variables \(S(X_i)\) close the initial Lie superalgebra

\[
\{S(X_i), S(X_j)\}_{P.B.} = f^k_{ij} S(X_k). \tag{6.33}
\]

Clearly, using relation (6.32), (6.33) is reduced to the following system of partial differential equations (PDEs)

\[
\sum_{\mu=1}^{2} \left( \frac{\partial S(X_i)}{\partial q^\mu} \frac{\partial S(X_j)}{\partial p_\mu} - \frac{\partial S(X_i)}{\partial p_\mu} \frac{\partial S(X_j)}{\partial q^\mu} \right) - (-1)^i \sum_{\alpha=1}^{2} \left( \frac{\partial S(X_i)}{\partial \xi^\alpha} \frac{\partial S(X_j)}{\partial \pi_\alpha} + \frac{\partial S(X_i)}{\partial \pi_\alpha} \frac{\partial S(X_j)}{\partial \xi^\alpha} \right) - f^k_{ij} S(X_k) = 0. \tag{6.34}
\]

Here the generators \(X_i\) are \(Z, H, Q_+\) and \(Q_-\), and \(f^k_{ij}\) stand for the structure constants of \((C^3 + A)\). Note that two different realizations (6.31) (two different solutions of the PDEs (6.34)) will be equivalent if there exists a canonical transformation that maps one into the other. A convenient realization linked to \((C^3 + A)\) (a solution for the above system of PDEs) given by

\[
S(Z) = -q^1 p_2 - \xi^1 \pi_2, \quad S(H) = -q^1 p_2 + \xi^1 \pi_2,
S(Q_+) = q^1 \pi_2, \quad S(Q_-) = -\xi^1 p_2 + \frac{1}{2}(q^1 \pi_1 + q^2 \pi_2). \tag{6.35}
\]
This realization can be easily deformed:

\[ S(\hat{Z}) = -q^1 p_2 - \xi^1 \pi_2, \quad S(\hat{H}) = -\frac{1}{\lambda} \sinh(\lambda q^1 p_2) + \xi^1 \pi_2 \cosh(\lambda q^1 p_2), \]
\[ S(\hat{Q}_+) = q^1 \pi_2, \quad S(\hat{Q}_-) = -\frac{\xi^1}{\lambda q^1} \sinh(\lambda q^1 p_2) + \frac{1}{2}(q^1 \pi_1 + q^2 \pi_2). \] (6.36)

These phase superspace functions close a quantum superalgebra \((C^3 + A)\) (6.18) under the canonical graded Poisson bracket (6.32). Under the realization (6.35), the undeformed Casimir element of \((C^3 + A)\) (given by relation (6.10)) is represented by

\[ S(C^{(2)}) = 2(q^1)^2(p_2)^2 - 2q^1 p_2 \xi^1 \pi_2 + (q^1)^2 \pi_1 \pi_2. \] (6.37)

One can easily show that \(S(C^{(2)})\) will always be in involution with the functions \(S(X_i)\) (6.35) in such a way that it can be considered as a common constant of motion. Therefore, a particular subset of undeformed integrable Hamiltonian can be found by setting

\[ \mathcal{H} = S(C^{(2)}) + \mathcal{F}(S(H)) \]
\[ = 2(q^1)^2(p_2)^2 - 2q^1 p_2 \xi^1 \pi_2 + (q^1)^2 \pi_1 \pi_2 + \mathcal{F}(\frac{q^1 \pi_1}{2} + (q^1 \xi^1 \pi_2). \] (6.38)

where \(\mathcal{F}(S(H))\) is any smooth function of the dynamical variable \(S(H)\).

We recall that the deformed Casimir element for the Hopf superalgebra \(U^{(c_2 \oplus A_1)}(C^3 + A)\) has given by relation (6.19). In terms of the deformed realization (6.36), it reads

\[ S(C^{(2)}_\lambda) = 2\left\{ S(\hat{Z}) \frac{\sinh(\lambda S(\hat{H}))}{\lambda} - S(\hat{Q}_+) S(\hat{Q}_-) \right\}. \] (6.39)

If we consider the deformed (anti)commutation rules (6.18) as graded Poisson brackets, then we get that (6.39) Poisson-commutes with any function of (6.36). An example of the deformed Hamiltonian is provided by (6.38) where the phase superspace functions are now replaced by their deformed counterparts

\[ \hat{\mathcal{H}}_\lambda = S(C^{(2)}_\lambda) + \mathcal{F}(S(\hat{H})), \] (6.40)

by construction, \(\hat{\mathcal{H}}_\lambda\) is in involution with the deformed functions of (6.36). Of course, in order to obtain the Casimir as a Hamiltonian, we should consider \(\hat{\mathcal{H}}_\lambda \equiv S(C^{(2)}_\lambda)\); then, \(\mathcal{F}(S(\hat{H}))\) can be taken as the remaining integral of the motion in involution.
7 Conclusion

In summary, as mentioned in the Introduction, the importance of the classification of Lie superbialgebras \((C^3 + A)\) lies in the fact that it helps us to obtain a hierarchy of \((C^3 + A)\) WZW models related to the super Poisson-Lie T-duality \([16]\). We have performed a complete classification of Lie superbialgebras \((C^3 + A)\) and obtained 31 families of inequivalent Lie superbialgebra structures on \((C^3 + A)\) in such a way that all results are new and applicable. We have also classified all corresponding coboundary Lie superbialgebras (triangular or quasi-triangular) with coboundary duals and their corresponding classical \(r\)-matrices. The generalization of the Lyakhovsky and Mudrov formalism \([20]\) to Lie superalgebras was firstly done in Ref. \([15]\). Making use of this formalism, we have obtained new quantum deformations of \((C^3 + A)\). In this way, we have gotten the Hopf superalgebras which quantize the Lie superbialgebras \(\left((C^3 + A), C^2_{p=1} \oplus A_{1,1}\right)\), \(\left((C^3 + A), C^{4,\ell} \oplus A_{1,1}\right)\) and \(\left((C^3 + A), C^{1,\ell}_{p=1} \oplus A\right)\) of Table 3. Finally, as an application of these quantum deformations, a deformed integrable Hamiltonian system has been constructed by the representation of the Hopf superalgebra \(U^\Lambda_{(C^2_{p=1} \oplus A_{1,1})\left((C^3 + A)\right)}\)

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We call \((-1)^{|x|}\) the parity of \(x\). Here and in the following we identify grading of indices by the same indices in the power of \((-1)^x\), i.e., we put \((-1)^x\) instead of \((-1)^{|x|}\); this notation has used by Dewitt in [22].

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[29] *Notation*: If \(f\) be a differentiable function on \(\mathbb{R}_c^m \times \mathbb{R}_a^n\) (\(\mathbb{R}_c^m\) are subset of all real numbers with dimension \(m\) while \(\mathbb{R}_a^n\) are subset of all odd Grassmann variables with dimension \(n\)), then, the relation between the left partial differentiation and right one is given by \(\Upsilon f := \frac{\partial f}{\partial \Phi} = (-1)^{|f|+1} f \frac{\partial}{\partial \Phi} \), where \(|f|\) indicates the grading of \(f\) [22].