Higher Dimensional Charged Rotating Solutions in (A)dS Space-times

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Abstract

We present a class of solutions to the Einstein-Maxwell equations in \( d \)-dimensions, all of which are asymptotically (anti)–de Sitter space-times. They describe electrically charged rotating solutions, which are generalizations of those found by Lemos (gr-qc/9404041). These solutions have toroidal, planar or cylindrical horizons and can be interpreted as black holes, or black strings/branes. We calculate the inverse temperature and entropy, and then we use the Brown–York stress–tensor to calculate mass and angular momenta of these solutions.

1 Introduction

The AdS/CFT correspondence [1]–[5], has created lots of interest in asymptotically anti–de–Sitter(AAdS) space-times and their thermodynamics. Several gauge field theory phenomena such as confinement, confinement/deconfinement phase transitions, and conformal anomalies, have been shown to be encoded in the semi–classical physics of AAdS black holes [2]–[6]. Following these works many AAdS black configurations have been studied, in the context of the AdS/CFT correspondence to test this very interesting duality as well as to understand, more, the physics of the strongly coupled gauge theory on the boundary (see for example [5]). Beside the essential role played by AAdS in the above duality, AAdS enjoys several attractive features. For example, having a negative cosmological constant \( \Lambda < 0 \) allows for various types of topologies for black holes horizons in contrast to the asymptotically flat case with \( \Lambda = 0 \). These horizons can be spherical, hyperbolic or planar, which by global identifications can lead to tori, or cylinders (for the planar case) and Riemann surfaces with genus \( g > 1 \) (for the hyperbolic case), in four dimensions (see for example [23]). Another important property for AAdS is that they are locally thermodynamically stable, for certain ranges of their parameters. Large Schwarzschild-AdS black holes— with horizons \( r_+ > |\Lambda|^{-1/2} \) — have positive specific heat because the AdS effectively acts as a box with reflecting walls provided by its boundaries [16]. Another example is the classical stability of Kerr–AdS black holes, which is not shared by asymptotically flat Kerr black holes. The reason for their stability is different from the Schwarzschild case. Kerr–AdS black holes have rotating Killing vectors which are time–like everywhere outside the horizon. Following the orbits of these rotating Killing vectors, thermal radiation can rotate around the black hole with the same angular velocity of the hole, implying that super-radiance is not allowed, and the black hole will be in
thermal equilibrium with the thermal gas around it \cite{21,22} (see also \cite{18} for other interesting phenomena in Kerr–AdS space-times).

Since AAdS space–times play such an important role in the above duality, it is interesting to find new AAdS configurations and study their semi–classical physics and, if possible, relate it to the physics of the boundary field theory. In this paper we present a class of solutions in asymptotically (anti)–de–Sitter space-times which are electrically charged and rotating. These solutions are generalizations of the solution found by Lemos \cite{9,10} to higher dimensions. One can interpret these solutions, depending on their global identifications, either as black holes with planar horizons in \( d \)-dimensions, or as black strings/branes. The paper is organized as follows; first we present the solutions after writing down the general Einstein–Maxwell action. Next we discuss the physical properties of these solutions such as, types of horizons and the extremality condition. We then calculate the inverse temperature and entropy of the solutions. Finally we use Brown–York divergence free stress–tensor to calculate the total mass and angular momenta of the solutions, after which we give some final remarks.

2 Rotating Electrically charged \( d \)–dimension Black Holes

The action for Einstein-Maxwell theory in \( d \)–dimensions for asymptotically (anti)–de–Sitter space-times is

\[
I_d = -\frac{1}{16\pi G_d} \int_M d^d x \sqrt{-g} \left( R + \frac{(d-1)(d-2)}{l^2} - F_{\mu\nu} F^{\mu\nu} \right),
\]

(2.1)

where \( \Lambda = \frac{(d-1)(d-2)}{2l^2} \) is the cosmological constant, \( G_d \) is Newton’s constant in \( d \)–dimensions, and \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \), where \( A_{\mu} \) is the vector potential. Varying the action with respect to the metric and the vector field \( A_{\mu} \) we get the following field equations

\[
G_{\mu\nu} - \Lambda g_{\mu\nu} = 2 T_{\mu\nu},
\]

\[
\partial_{\mu} (\sqrt{-g} F^{\mu\nu}) = 0,
\]

(2.2)

where

\[
T_{\mu\nu} = F_{\mu\rho} F^{\rho}_{\nu} - \frac{1}{4} g_{\mu\nu} F^{\rho\sigma} F_{\rho\sigma}. \]

(2.3)

Lemos found a four dimensional rotating charged black hole solution with a flat horizon \cite{9,10}, which can be written in the following form

\[
ds^2 = -f(r) (\Xi dt - a d\phi)^2 + \frac{dr^2}{f(r)} + \frac{r^2}{l^2} (a dt - \Xi l^2 d\phi)^2 + r^2 dz^2
\]

(2.4)

where \( f(r) \) is given by

\[
f(r) = \frac{r^2}{l^2} - \frac{m}{r} + \frac{q^2}{r^2}.
\]

(2.5)

The coordinates \( \phi \) and \( z \) assume the values, \( 0 \leq \phi < 2\pi \) and \( -\infty < z < \infty \). The gauge potential has the form

\[
A = -\frac{q \Xi}{r} (dt - \frac{a}{\Xi} d\phi).
\]

(2.6)

Here \( a \) is the rotation parameter, \( m \) is the mass parameter, \( q \) is the electric charge and \( \Xi = \sqrt{1 + a^2/l^2} \). This solution has been studied in \cite{9,10,11,12,13} and it is known to preserve some supersymmetry.
One can observe that there is a coordinate transformation which relates the rotating solution to the non–rotating solution, namely,

\[ t' = \Xi t - a \phi \quad \quad \phi' = \frac{a}{l^2} t - \Xi \phi \quad (2.7) \]

This coordinate transformation is allowed locally but not globally \[9\] since it mixes the compactified \( \phi \) coordinate with the time coordinate. As Stachel \[15\] (see also Lemos \[9\]) has shown, if the first Betti number of the manifold is non-vanishing, there are no global diffeomorphisms that can map the two metrics \[9, 15\] and the new manifold will be parameterized globally by \( a \). It is easy to show that the first Betti number is one for the above solution when we have cylindrical or toroidal horizons. In higher dimensions we can have non–vanishing first Betti number by compactifying certain numbers of coordinates in a \((d - 2)\)-dimensional submanifold of the solution. These are angular coordinates, which we will call \( \phi_i \), describe the rotation of the solutions. The higher dimensional non–rotating charged solutions, \( i.e. \ d > 4 \), have been discussed together with their thermodynamics in \[17\]. In fact, some of the above solutions can be constructed as near horizon limit to spinning D3 or M2 branes \[19\].

The rotation group in \( d \)-dimensions is \( SO(d-1) \) and the number of independent rotation parameters for a localized object is equal to the number of Casimir operators, which is \( [(d-1)/2] \), where \( [x] \) is the integer part of \( x \). We now present our solution which is the generalization of the above solution found by Lemos \[9, 10\] to arbitrary dimensions with all rotation parameters. The metric for this solution is given by

\[ ds^2 = -f(r) \left( \Xi dt - \sum_{i=1}^{n} a_i d\phi_i \right)^2 + \frac{r^2}{l^4} \sum_{i=1}^{n} (a_i dt - \Xi l^2 d\phi_i)^2 + \frac{dr^2}{f(r)} \]

\[ -\frac{r^2}{l^2} \sum_{i<j} \left( a_i d\phi_j - a_j d\phi_i \right)^2 + r^2 d\Omega^2 \quad (2.8) \]

where \( n = [(d-1)/2] \) is the number of rotation parameters \( a_i \), \( \Xi = \sqrt{1 + \sum_i^n a_i^2/l^2} \), \( d\Omega^2 = dy^k dy^k \) is the Euclidean metric on \((d-2-n)\)-dimensions submanifold with volume \( \omega_{d-2} \) and \( k = 1, ..., d-2-n \). Here \( f(r) \) is

\[ f(r) = \frac{r^2}{l^2} - \frac{m}{r^{d-3}} + \frac{q^2}{r^{2d-6}}, \quad (2.9) \]

and the gauge potential is given by

\[ A = -\frac{q \Xi}{c_d r^{d-3}} \left( dt - \frac{1}{\Xi} \sum_{i}^{n} a_i d\phi_i \right). \quad (2.10) \]

The coordinates \( \phi_i \) assume the values \( 0 \leq \phi_i < 2\pi \). In the above expression, \( c_d \) is a constant given by

\[ c_d = \sqrt{\frac{2d-6}{d-2}}. \quad (2.11) \]

The asymptotically de–Sitter version of these solutions can be obtained by simply taking \( l \rightarrow il \) which will be left for future work.
3 Physical Properties

3.1 Horizons and singularities

Similar to higher dimensional Kerr solutions in asymptotically flat space-times the above metric has two types of hypersurfaces or horizons [27]: a stationary limit surface and an event horizon. The stationary limit surface is the boundary of the region in which an observer travelling a long time-like curve can follow the orbits of the asymptotic time translation Killing vector $\partial/\partial t$ and so remain stationary with respect to infinity. Beyond this surface any observer following the orbits of $\partial/\partial t$ must go faster than the speed of light, i.e., he cannot remain stationary with respect to an observer at infinity. On this surface the Killing vector $\partial/\partial t$ is null. They are surfaces of infinite redshift, and are the real solutions of

$$g_{tt} = -\Xi^2 f(r) + \frac{r^2}{l^2} \sum_{i=1}^{n} \frac{a_i^2}{l^2} = 0. \quad (3.12)$$

On the other hand, surfaces at which the radial coordinate $r$ flips its signature is an event horizon, and defined as

$$g^{rr} = f(r) = 0. \quad (3.13)$$

If we call the largest solution of (3.12) $r_{s+}$, and the outermost event horizon $r_+$. One can see that for the physically interesting case with $q < m$, there is a region between $r_{s+}$ and $r_+$ in which any observer following a time-like curve cannot remain stationary. To see this for this solution, let us first simplify the discussion by keeping only one rotation parameter. Parameterizing the time-like curve using a proper time $\tau$, we calculate the norm-square of the tangent $u^a = dx^a/d\tau$ to be negative, i.e.

$$g_{tt}(dt/d\tau)^2 + 2g_{t\phi}(dt/d\tau)(d\phi/d\tau) + g_{\phi\phi}(d\phi/d\tau)^2 < 0. \quad (3.14)$$

Notice here that in the region between $r_{s+}$ and $r_+$ all terms in this expression are positive but the second term, this can be seen from the fact that $r^2/l^2 > f(r)$ in this region. This leads, also, to $g_{t\phi} < 0$ and consequently to $d\phi/d\tau > 0$, i.e., any observer is forced to rotate in the direction rotation of the black hole. These two types of horizons coincide for static limit of these solutions, i.e., as $a \to 0$, similar to the Kerr solution in asymptotically flat space-time.

The singularities appeared in the metric by setting $g_{tt} = 0$, and $g^{rr} = 0$ are coordinate singularities which can be removed using more appropriate set of coordinates. One way to check the existence of physical singularity is to calculate curvature scalars, such as the square of the Riemann tensor. Calculating this, we find,

$$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} \sim F(r)/r^{(4d-8)}, \quad (3.15)$$

where $F(r)$ is some polynomial in $r$. Therefore the conclusion is that a physical singularity occurs at $r = 0$.

There is a critical value of the black hole mass parameter,

$$m_{ext} = \left(\frac{2d - 4}{d - 3}\right) l^{\frac{d-1}{d-2}} \left(\frac{d - 1}{d - 3}\right)^{\frac{d-1}{2d-4}} q^{\frac{d-1}{2d-2}} \quad (3.16)$$

at which the horizon is degenerate and the black hole is extremal. We notice here that the extremal solutions are not the supersymmetric ones. For the solution to be supersymmetric (in $a = 0$ case) we must set $m = 0 [17]$. 


3.2 Physical Quantities; Temperature Mass and Angular Momenta

By taking $t \rightarrow i\tau$, we define the Euclidean section of the metric (2.8) which requires that the period of Euclidean time, i.e. the inverse temperature, have the form

$$\beta = \frac{4\pi \Xi l^2 r_+^{2d-5}}{[(d-1)r_+^{2d-4} - (d-3)q^2 l^2]},$$

(3.17)

where $r_+$ is the largest root of $f(r)$. Upon Euclidean continuation, we must take $a_j \rightarrow i a_j$. The identification $\tau \sim \tau + \beta$ leads to the identification $\phi_j \sim \phi_j + i\beta \Omega_j \ [21]$ as well, where

$$\Omega_j = \frac{a_j}{2l^2}$$

(3.18)

are the angular velocities at the horizon. The entropy for this metric is

$$S = \frac{A_H}{4G_d} = \frac{\omega_{d-2} r_+^{d-2} \Xi}{4G_d}.$$  

(3.19)

To calculate various thermodynamical quantities we first add the Gibbons–Hawking boundary term to action. We get Einstein equations upon varying the bulk metric with fixed boundary metric; the boundary term is

$$I_b = -\frac{1}{8\pi G} \int_{\partial M} d^{d-1}x \sqrt{-h} K.$$  

(3.20)

Here $h_{ab}$ is the boundary metric and $K$ is the trace of the extrinsic curvature $K^{ab}$ of the boundary. We use the counterterm subtraction method [6, 7] to render the action finite, and then use it to calculate different physical quantities. In this method we add a finite number of local surface integrals to leave the action finite. For example, the following counterterm can define a finite action up to seven dimensions;

$$I_{ct} = \frac{1}{8\pi G} \int_{\partial M} d^{d-1}x \sqrt{-h} \left[ \frac{(d-2)}{l} - \frac{lR}{2(d-3)} \right] + \ldots$$

(3.21)

Here $R$ and $R_{ab}$ are the Ricci scalar and tensor for the boundary metric $h$. Having a finite action we can define a divergence free stress–tensor which has been introduced by Brown and York [8]

$$T^{ab} = \frac{1}{8\pi G_d} \left[ K^{ab} - h^{ab} K - \frac{(d-2)}{l} h^{ab} + \frac{lG^{ab}}{(d-3)} + \ldots \right].$$

(3.22)

The above stress–tensor is divergence free for dimensions less than six, but we can always add more counterterms to have a finite actions in higher dimensions (see for example [20]). In the case of AAdS solutions with flat horizons, the only non–vanishing counterterm is the first term in (3.19). Using the above stress–tensor we can define a conserved charges for any given AAdS space–time. The conserved charges associated to a Killing vector $\xi^a$ as follows:

$$Q_\xi = \int_{\Sigma} d^{d-1}x \sqrt{\sigma} u^a T_{ab} \xi^b,$$  

(3.23)

where $T_{ab}$ is Brown–York stress–tensor, $u_a = -N t_a$, while $N$ and $\sigma$ are the lapse function and the spacelike metric which appear in the ADM–like decomposition of the boundary metric

$$ds^2 = -N^2 dt^2 + \sigma_{ab}(dx^a + N^a dt)(dx^b + N^b dt).$$

(3.24)
The convention we use for the Killing vectors is the following; $\partial_t$ is the Killing vector conjugate to $t$ and $\partial_\phi$ is the Killing vector conjugate to $\phi$.

Using the above definition for conserved charges, we find the total energy of the solution to be given by

$$M = \frac{\omega d - 2}{16\pi G_d} [(d - 1) \Xi^2 - 1] m,$$

while the angular momenta are given by

$$J_i = \frac{(d - 1) \omega d - 2}{16\pi G_d} \Xi a_i m. \quad (3.26)$$

We notice here that the total mass reduces to the mass of charged $d$–dimensions solutions \cite{17} upon taking the limit $a \to 0$, and for $d = 4$ it reduces to the mass and angular momenta calculated in \cite{10, 11}.

4 Final Remarks

We have presented a class of charged rotating solutions in $d$–dimensions which are generalizations to the solutions found by Lemos. They have toroidal, planar, or cylindrical horizons. They can also be interpreted as black strings/branes. We have used the definition of conserved charge provided by the quasilocal stress–tensor introduced by Brown and York to calculate the total energy and angular momenta of these solutions. We have also calculated the inverse temperature and the entropy of these space–times. It would be interesting to study the thermodynamics of such solutions and its relevance in the AdS/CFT correspondence.

Another important aspect of these string/branes solutions which deserves investigation is their classical stability to linearized perturbations. It known that some extended black configurations are not classically stable against linearized perturbations \cite{24}, but there are also stable extended configurations (see for example \cite{26}). A first step toward a better understanding of this issue has been provided by Gubser and Mitra \cite{25}; they conjectured that a black string/brane with a noncompact translational symmetry is classically stable if and only if it is locally thermodynamically stable. In fact the four dimensional case given by the metric (2.4) is known to be locally thermodynamically stable as has been shown in \cite{13, 11}, which leads according to Gubser and Mitra to the classical stability of the solution. Thus the solutions presented here can also be used to test the Gubser–Mitra conjecture, as we hope to discuss in a future work.
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