Types, Codes and TFTs

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Abstract

We establish a relation between fully extended 2-dimensional TQFTs and recognisable weighted formal languages, rational biprefix codes and lattice TFTs. We show the equivalence of 2D closed TFTs and rational exchangeable series and we discuss the important special case of finite groups. Finally, we outline a reformulation in terms of a restricted version of second order monadic logic.

Contents

1 Preliminaries 1
   1.1 Rational series .......................... 2
   1.2 Codes .................................. 3
   1.3 Frobenius algebras ....................... 4
   1.4 Bicategories ................................ 4
   1.5 Automaticity ................................ 5

2 Fully extended two-dimensional TFTs 6
   2.1 Group algebras ............................ 8

3 Two-cobordism and second order monadic logic 8

1 Preliminaries

Let $k$ be a field of characteristic zero. We consider the following categories: $\text{Alg}_k$, of associative and unital $k$-algebras, $\text{cAlg}_k$, associative, commutative and unital $k$-algebras and $\text{fGrp}$, of finite groups. We assume all algebras to be unital and, if necessary, $k$ also to be algebraically closed, which we shall indicate.

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1.1 Rational series

We recall a few facts about formal series and rational languages from [3, 4, 7, 19]. Let \( X \) be a finite set, called alphabet, and \( X^\ast \) the free monoid generated by \( X \), i.e. the set of all possible words \( w \) over \( X \), including the empty word \( 1 \). A map \( S : X^\ast \to k, w \mapsto S(w) =: S_w \) is called a formal series, and the \( k \)-algebra of all formal series is \( k \langle \langle X \rangle \rangle \). The subalgebra of polynomials is \( k \langle X \rangle \).

Equivalently, \( k \langle X \rangle \) is the free associative algebra generated by \( X \) and \( k \langle \langle X \rangle \rangle \) its linear dual, i.e. every formal series \( S \) defines a linear functional \( S : k \langle X \rangle \to k \) by \( k \)-linear extension of the map \( w \mapsto S_w \).

The free commutative monoid \( X^+ \) is defined as \( X^\ast / \sim \) where the congruence \( \sim \) is generated by the relation \( xy \sim yx \) for \( x, y \in X, x \neq y \). Let \( \alpha : X^\ast \to X^+ \) denote the canonical epimorphism from the free non-commutative monoid \( X^\ast \) onto the free commutative monoid \( X^+ \), where \( \alpha(w) \) denotes the commutative image of a word \( w \in X^\ast \). Then, by abuse of notation, \( \alpha : k \langle \langle X \rangle \rangle \to k [[X]] \) is the induced extension.

The notion of an exchangeable series was introduced by M. Fliess [7] and it is closely related to the notion of exchangeable stochastic processes and Markov chains.

**Definition 1.1.** Let \( S \in k \langle \langle X \rangle \rangle \). \( S \) is called exchangeable if for all \( v, w \in X^\ast \), with \( \alpha(v) = \alpha(w) \), \( S_v = S_w \) holds.

**Definition 1.2.** For \( A \in \text{Alg}_k \), \( X \) a finite set and \( S \in k \langle \langle X \rangle \rangle \),

- the syntactic ideal of \( S \), denoted \( I_S \), is defined as
  \[
  I_S := \sup_{J \subset \ker(S)} \{ J \mid \text{two-sided ideal} \},
  \]
  i.e. it is the maximal two-sided ideal contained in \( \ker(S) \subset k \langle X \rangle \).

- The algebra
  \[
  \mathfrak{A}_S := k \langle X \rangle / I_S,
  \]
  is called the syntactic algebra of \( S \), with \( \pi : k \langle X \rangle \to \mathfrak{A}_S \) the canonical algebra epimorphism.

- If \( A \cong \mathfrak{A}_S \), for some \( S \) then \( A \) is called syntactic.

- If \( A \cong \mathfrak{A}_S \), for some \( S \) and \( \dim_k(\mathfrak{A}_S) < \infty \), then we call \( A \) rational syntactic.

A codimension one linear subspace \( H \) of a \( k \)-vector space \( V \) is called a hyperplane, i.e. \( \dim_k(V/H) = 1 \). In particular \( H \) contains the origin, i.e. \( 0 \in H \). For the classical relation between linear functionals, their nullspaces and hyperplanes, cf. e.g. the monograph [2].

The following fundamental statements have originally been established by Ch. Reutenauer and correspond to [[19] Proposition I.2.4 and Théorème II.1.2.]

**Theorem 1.3** (Reutenauer [19]). Let \( X \) be a finite alphabet and \( A \in \text{Alg}_k \), a finitely generated algebra, \( A \neq 0 \). Then the following statements hold:

1. \( A \) is syntactic iff there exists a hyperplane \( H \subset A \), which contains no two-sided ideal other than \( \{0\} \).
2. A formal series $S \in k\langle X \rangle^*$ is rational iff $\dim_k(\mathfrak{A}_S) < \infty$, i.e. the syntactic algebra $\mathfrak{A}_S$ of $S$ is a finite-dimensional $k$-vector space.

In the proof of the first statement above the following auxiliary facts are used which we shall spell out explicitly.

**Lemma 1.4.** If $A, B \in \text{Alg}_k$, $J \subset A$ a two-sided ideal and $\varphi : A \to B$ a surjective algebra morphism then $\varphi(J) \subset B$ is a two-sided ideal.

**Lemma 1.5.** For $A \in \text{Alg}_k$, $A \neq 0$, let $A$ be syntactic and finitely generated, i.e. $A = k[a_1, \ldots, a_n]$, for some $a_i \in A$, $i = 1, \ldots, n$. Then there exists a formal series $S \in k\langle x_1, \ldots, x_n \rangle^*$ such that $A \cong k\langle x_1, \ldots, x_n \rangle / I_S$, i.e. the representing alphabet $X$ can be assumed in this case to be finite.

**Proof.** As $A$ is syntactic, by Theorem 1.3 1., there exists a hyperplane $H \subset A$ and a linear functional $\lambda_H : A \to k$, induced by $H$, such that $\ker(\lambda_H) = H$. Let $j_a : k\langle x_1, \ldots, x_n \rangle \to A$ be the algebra morphism onto $A$, defined by $j_a(x_i) := a_i$, $i = 1, \ldots, n$. Define the linear functional $S : k\langle x_1, \ldots, x_n \rangle \to k$ by $S := \lambda_H \circ j_a$ with corresponding syntactic ideal $I_S$, as shown in the diagram below:

\[
\begin{align*}
&\xymatrix{
k\langle x_1, \ldots, x_n \rangle \ar[r]^-{j_a} & A = k[a_1, \ldots, a_n] \ar[r]^-{\lambda_H} & k \\
& k\langle x_1, \ldots, x_n \rangle / I_{I_S} \ar[u]_{\pi} & k \ar[u] \\
S & A = k[a_1, \ldots, a_n] & k 
}\end{align*}
\]

Now, $\ker(j_a)$ is a two-sided ideal, $\pi$ the canonical algebra epimorphism and by the “First Isomorphism Theorem for Rings”, it follows that $k\langle x_1, \ldots, x_n \rangle / \ker(j_a) \cong A$. Let us show that

\[
\ker(j_a) = I_S,
\]

from which

\[
\mathfrak{A}_S = \frac{k\langle x_1, \ldots, x_n \rangle}{I_S} = \frac{k\langle x_1, \ldots, x_n \rangle}{\ker(j_a)} \cong A,
\]

and hence, the claim follows. Let us show the equality (1).

We have $\ker(S) = j_a^{-1}(\lambda_H^{-1}(0)) = j_a^{-1}(H)$ and hence $j_a^{-1}(0) = \ker(j_a) \subset I_S \subset \ker(S)$.

By Lemma 1.4, $j_a(I_S) \subset H$ is a two-sided ideal and by assumption $H$ does not contain a non-trivial two-sided ideal and hence $j_a(I_S) = 0$. Therefore $I_S \subset \ker(j_a)$ from which (1) follows. \qed

### 1.2 Codes

We recall the following facts from [3, 4, 20]. Every language $L \subset X^*$ over $X$ induces a linear functional $\chi_L$, the characteristic series of $L$, defined as $\chi_L : k\langle X \rangle \to k$, by $k$-linear extension of the map $w \mapsto 1$ if $w \in L$ and 0 otherwise.

Let $X$ be a finite alphabet and $C$ a language, i.e. $C \subset X^*$. Then $C$ is a code if the submonoid $C^*$ generated by $C$ is free, with base $C$. The syntactic algebra $\mathfrak{A}_{C^*}$ of a code $C$ is the syntactic
algebra of $C^*$. A code $C$ is called rational if its syntactic algebra satisfies $\dim_k(\mathfrak{A}_{C^*}) < \infty$, i.e. it is a finite-dimensional $k$-vector space.

A language $L \subset X^*$ is called prefix if for all $u, v \in X^*$ with $u, uv \in L$ it follows that $v = 1$, and similarly, $L$ is called suffix, if $v, uv \in L$ implies $u = 1$. The language $L$ is called biprefix if it is both prefix and suffix.

**Theorem 1.6** ([20] Theorem 1). If a code $C$ is rational and biprefix then its syntactic algebra $\mathfrak{A}_{C^*}$, is a finite-dimensional semi-simple $k$-algebra.

1.3 Frobenius algebras

Let us recall the following classes of algebras, cf. e.g. [14] and also [4, 15].

**Definition 1.7.** Let $A \in \text{Alg}_k$ and $\dim_k(A) < \infty$. Then $A$ is

1. simple if $\{0\}$ and $A$ are the only two-sided ideals in $A$,

2. semi-simple if $A \cong M_{n_1}(k) \times \cdots \times M_{n_r}(k)$, $r, n_i \in \mathbb{N}^*$, $i \in \{1, \ldots, r\}$, or alternatively, if $I \leq A$ is a nilpotent left ideal, then $I = 0$.

3. Frobenius if it satisfies one of the following equivalent characterisations:
   - There exists a bilinear form $B : A \times A \to k$ which is non-degenerate and associative, i.e. which satisfies $B(ab, c) = B(a, bc)$ $\forall a, b, c \in A$.
   - There exists a linear functional $\lambda : A \to k$ whose kernel $\ker(\lambda)$ (nullspace) contains no left or right ideal other than zero.
   - There exists a hyperplane $H \subset A$, $(0 \in H)$, which contains no nonzero right ideal (left ideal).

4. symmetric Frobenius if $\lambda(ab) = \lambda(ba)$, for all $a, b \in A$, i.e. $\lambda$ is a trace.

**Remark 1.** If $A$ is Frobenius then $\ker(\lambda)$ does not contain a non-trivial two-sided ideal as every two-sided ideal is both a left and right ideal.

**Proposition 1.8.** If $A \in \text{Alg}_k$, $\dim_k(A) < \infty$ and $A$ is semi-simple then it can be endowed with the structure of a symmetric Frobenius algebra.

**Proof.** The Wedderburn-Artin Theorem is an essential part of the proof; for the remaining details cf. [[14] Exercise 12. p. 114] or [[13], Exercise 9. p. 106].

1.4 Bicategories

For this subsection we use as references [11, 14, 17, 21].

**Definition 1.9.** The symmetric monoidal bicategory $\text{Alg}^2_k$ over $k$ is given by the following data:

**Objects:** $\text{Alg}_k$, i.e. associative and unital $k$-algebras,
1-morphisms: \((A,B)\)-bimodules, \(A,B \in \text{Alg}_k\), and composition is given by the tensor product of bimodules,

2-morphisms: bimodule morphisms.

In [[17] 2.3.] J. Lurie discusses the notion of fully dualisable objects in a symmetric monoidal \((\infty,n)\)-category. For \(A \in \text{Alg}_k^2\), \(A\) is called fully dualisable if it is separable.

**Proposition 1.10** ([17, 21]). Let \(k\) be algebraically closed. Then the fully dualisable objects \(\text{fdAlg}_k^2\) in \(\text{Alg}_k^2\) correspond to the finite-dimensional semi-simple \(k\)-algebras.

The notion of Morita contexts is discussed in [[14] §18C], [[21] 3.8.4] or [[11] Section 2.].

**Definition 1.11.** The bicategory \(\text{sFrob}_k\) of semi-simple symmetric Frobenius algebras is given by the following data:

- **Objects:** semi-simple, symmetric Frobenius algebras,
- **1-morphisms:** compatible Morita contexts,
- **2-morphisms:** isomorphisms of Morita contexts.

**1.5 Automaticity**

Here we show that the algebras we considered arise as syntactic algebras of recognisable power series, i.e. there exists a weighted finite-state automaton which recognises any such formal series.

**Proposition 1.12.** Let \(A \in \text{Alg}_k\) and \(\dim_k(A) =: N < \infty\). Then, if \(A\) is

1. simple, or
2. semi-simple, or
3. Frobenius

then \(A\) is rational syntactic, i.e. there exists a recognisable series \(S \in k\langle x_1, \ldots, x_N \rangle^*,\) such that \(A \cong k\langle x_1, \ldots, x_N \rangle/I_S.\)

**Remark 2.** In [[19] Examples 1., p. 452] Ch. Reutenauer lists simple and (symmetric) Frobenius algebras as examples for syntactic algebras. However, the stronger statements in Proposition 1.12 seem to be absent from the literature.

**Proof.** Let us first show 3. By assumption \(A = ka_1 \oplus \cdots \oplus ka_N,\) and hence \(\{a_1, \ldots, a_N\}\) is a finite generating set for \(A\). Therefore, the claim follows from Lemma 1.5. In more detail: define the surjective algebra morphism \(j_a : k\langle x_1, \ldots, x_N \rangle \to A\) by \(x_i \mapsto a_i,\) for \(i = 1, \ldots, N\) and the linear
functional \( S := \lambda \circ j_a \), where \( \lambda : A \to k \) is the Frobenius form as in Definition 1.7 3., cf. the diagram below:

Then as in the proof of Lemma 1.5, we have \( \ker(j_a) = I_S \) and hence \( \mathfrak{A}_S \cong A \) which shows finite dimensionality. Therefore, by Theorem 1.3, the formal series \( S \) is rational.

In order to show 1. and 2. we remark that simple implies semi-simple. Then Proposition 1.8 shows that any such algebra can be endowed with the structure of a symmetric Frobenius algebra from which the claim follows from 3.

The following is an opposite statement for commutative syntactic algebras.

**Proposition 1.13.** Let \( X \) be a finite set and \( S \in k\langle\langle X\rangle\rangle \). If \( \dim_k(\mathfrak{A}_S) < \infty \) and \( \mathfrak{A}_S \in \mathsf{cAlg}_k \) then \( \mathfrak{A}_S \in \mathsf{cFrob}_k \), i.e. it is a commutative Frobenius algebra.

**Proof.** By assumption, \( \mathfrak{A}_S \) is rational-syntactic and by Proposition 1.3, there exists a hyperplane \( H \subset \mathfrak{A}_S \) which contains no nontrivial two-sided ideal. As \( \mathfrak{A}_S \) is commutative every ideal, left or right, is two-sided. Therefore by Definition 1.7, 3., the claim follows. \( \square \)

Types of languages form pseudo-varieties, and by extension of the Eilenberg Variety Theorem, they correspond to pseudo-varieties of finite algebras \([19] \) Théorème III.1.1.\)

It is shown in \([19] \) 2. Exemples de variétés c. p. 472 that finite-dimensional commutative syntactic algebras correspond to exchangeable rational series, cf. Definition 1.1. Therefore we have

**Corollary 1.14.** There is a (bijective) correspondence between \( \mathsf{cFrob}_k \) and the \( s \)-variety of rational exchangeable series.

**Remark 3.** The above statement implies that the tangent space to a Frobenius manifold can be considered as having a weighted finite-state automaton located at every point of the manifold whose associated syntactic algebra corresponds to the Frobenius algebra at that point.

2 Fully extended two-dimensional TFTs

Previously with T. Kato \([8] \) we outlined how several (classes) of phenomena related to enumerative problems in geometry or integrable systems are principally governed by cellular automata. Here we relate weighted finite-state automata to open-closed string theories.

Throughout this section we assume \( k \) to be algebraically closed and of characteristic zero.

(Oriented) open-closed cobordisms, have been studied e.g. by A. Lauda and H. Pfeiffer \([15] \) with the help of \textsl{knowledgedale Frobenius algebra} which they introduced, and J. Morton \([18] \) considered weak 2-functors from \( \mathsf{nCob}_2 \) to \( \mathsf{Vect}_2 \). J. Lurie’s fundamental work \([17] \) aims at classifying all
TFTs which he achieves by developing new mathematical tools and the corresponding language in order to reformulate the **cobordism hypothesis** within this framework and to outline its proof. The main theorem for the **oriented two-dimensional cobordism hypothesis** was given by Ch. Schommer-Pries [21], which we recall in a combined form with the succinct formulation [[11] Theorem 2.10].

**Theorem 2.1** ([21] Theorem 3.52). Let \( k \) be an algebraically closed field with \( \text{char}(k) = 0 \). There exists a weak 2-functor:

\[
\text{Fun}_\otimes(\text{Cob}^{or}_{2,1,0}, \text{Alg}_k^2) \to \text{sFrob}_k
\]

\[
Z \mapsto Z(*_+),
\]

i.e. there exists an equivalence of the bicategory of two-dimensional oriented fully extended TFTs with values in \( \text{Alg}_k^2 \) and the bicategory of semi-simple Frobenius algebras \( \text{sFrob}_k \).

In String Theory, independently M. Fukuno, S. Hosono and H. Kawai [10] and C. Bachas and M. Petropolous [1] investigated **lattice TFTs** (LTFT) and constructed **state sums** based on triangulations of ordinary two-dimensional cobordisms. Their results can be rephrased as follows.

**Theorem 2.2** ([1, 10]). Let \( k \) be an algebraically closed field with \( \text{char}(k) = 0 \). The class of LTFTs is equivalent to \( \text{fdAlg}_k^2 \), i.e. the fully dualisable objects in \( \text{Alg}_k^2 \). For \( A \in \text{fdAlg}_k^2 \), the centre \( \mathcal{Z}(A) \), corresponds to the closed string states and \( \text{dim}_k(\mathcal{Z}(A)) \) corresponds to the number of independent physical operators.

Before we proceed further, let us summarise the chain of equivalences, which results from [11, 17, 21]:

\[
\text{fdAlg}_k^2 \leftrightarrow \text{sFrob}_k \leftrightarrow \text{Calabi-Yau category}
\]

The following relation holds between rational languages and closed string theory.

**Theorem 2.3.** The category of oriented 2-dimensional TQFTs is equivalent to the \( s \)-variety of exchangeable rational power series.

**Proof.** The classic equivalence between 2D TQFTs and \( \text{cFrob}_k \), cf. e.g. [13], combined with Corollary 1.14, yields the statement. \( \square \)

The second relation between rational series and open-closed string theory is given next.

**Theorem 2.4.** Let \( X \) be a finite alphabet and \( C \) a regular biprefix code. Then the following correspondences hold:

\[
\begin{array}{ccc}
\{\text{C code: regular, biprefix}\} & \longrightarrow & \text{fdAlg}_k^2 \\
\downarrow & \downarrow & \downarrow \\
\{\text{s-variety: rational exchangeable series}\} & \longrightarrow & \text{cFrob}_k \\
\end{array}
\]

where \( \mathcal{Z} \) is the **centraliser**, and \( \text{cFrob}_k \) is given by the centres \( \mathcal{Z}(A) \) of the finite-dimensional semi-simple algebras \( A \).
2.1 Group algebras

Group algebras of finite groups have been of particular interest in string theory, cf. e.g. [1, 10, 12, 15, 18].

The content of [[20] Remarks: 1. p.455] can be stated as follows.

For \( G \in \text{fGr} \), a finite group, let \( X := |G| \) be the underlying set (alphabet) with elements (letters) \( |g| \). Let \( j_G : k\langle |G| \rangle \to G \) be the canonical monoid epimorphism given by \( |g| \mapsto g \), and define

\[
C^*_G := j_G^{-1}(e),
\]

where \( e \in G \) is the neutral element.

**Proposition 2.5 ([20]).** Let \( G \in \text{fGrp} \) and \( C^*_G \) as in (2). Then \( C^*_G \) is generated by a rational biprefix code \( C_G \) with the syntactic algebra \( \mathfrak{A}_{C^*_G} \) being isomorphic to \( k[G] \), where \( k[G] \) is the group algebra of \( G \), i.e. we have the commutative diagram

\[
\begin{array}{ccc}
G & \longrightarrow & C_G \\
\downarrow & & \downarrow \\
k[G] & \longrightarrow & \mathfrak{A}_{C^*_G} : \text{semisimple, finite-dimensional}
\end{array}
\]

We have the following statement.

**Proposition 2.6.** Let \( k \) be an algebraically closed field of characteristic zero. The \( k \)-functor \( F : \text{fGrp} \to \text{LTFT}_k \) factorises through the category of rational biprefix codes, i.e. the diagram

\[
\begin{array}{ccc}
\text{fGrp} & \longrightarrow & \text{LTFT}_k \\
\downarrow & \swarrow & \\
\{\text{code: rational, biprefix}\}
\end{array}
\]

commutes.

**Remark 4.** Let \( A \in \text{Frob} \) such that \( \lambda(1_A) = 1_k \). Then \( (A, \lambda) \) defines a non-commutative probability space. In particular, \( (k[G], \phi) \) is a non-commutative probability space with a faithful trace \( \phi \). Further, if we restrict to permutation groups then we obtain a natural relation with free probability theory, cf. [9].

3 Two-cobordism and second order monadic logic

Here we show that the rational series describing the algebras which are equivalent to the topological field theories are describable by a restricted version of monadic second order logic (MSO). This is possible by an extension of the Büchi-Elgot Theorem given by M. Droste and P. Gastin [6]. For the necessary facts cf. [5, 6, 16] and in particular for the definition of restricted weighted second order monadic logic (rwMSO) [6] or the lecture notes [5].

**Theorem 3.1 ([6]).** Let \( X \) be a finite alphabet and \( \text{char}(k) = 0 \). Then for \( S \in k\langle\langle X \rangle\rangle \) the following equality holds: \( S \) is rwMSO\((k,X)\)-definable iff \( S \) is rational (recognisable).
Corollary 3.2. Let $X$ be a finite alphabet and $S \in k\langle X \rangle^*$ a formal power series. Then the following equivalences hold:

$$\dim_k(A_S) < \infty \iff S \text{ is rational} \iff S \text{ is rwMSO}(k, X) \text{ definable}.$$ 

The relevance with respect to two-dimensional lattice topological field theories is given by:

**Proposition 3.3.** Every LTFT is rwMSO($k$)-definable, i.e. let $A \in \text{fdAlg}_k^2$ be the finite-dimensional semi-simple algebra corresponding to an LTFT and $(X, S)$ a formal power series with syntactic algebra $A_S = A$. Then $S$ is rwMSO($k$)-definable.

Let us conclude with following observations and remarks.

The relations we have established in a first step between weighted finite-state automata, second order monadic logic and fully extended two-dimensional topological quantum field theories are at an algebraic level. However, the automata theoretic but also model theoretic part can be described in more intrinsic, i.e. (higher) categorical, terms which is necessary in order to extend and generalise the present results.

Further, it appears that the relation between logic and 2D TFTs should generalise to higher dimensions. Namely, the order of the logic / type theory should parallel the dimension of the cobordisms involved, i.e. we have:

$$n\text{-cobordism} \leftrightarrow n\text{-order logic/type theory}.$$ 

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