ON RATIONALLY PARAMETRIZED MODULAR EQUATIONS

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Abstract. Many rationally parametrized elliptic modular equations are derived. Each comes from a family of elliptic curves attached to a genus-zero congruence subgroup \( \Gamma_0(N) \), as an algebraic transformation of elliptic curve periods, parametrized by a Hauptmodul (function field generator). The periods satisfy a Picard–Fuchs equation, of hypergeometric, Heun, or more general type; so the new modular equations are algebraic transformations of special functions. When \( N = 4, 3, 2 \), they are modular transformations of Ramanujan’s elliptic integrals of signatures 2, 3, 4. This gives a modern interpretation to his theories of integrals to alternative bases: they are attached to certain families of elliptic curves. His anomalous theory of signature 6 turns out to fit into a general Gauss–Manin rather than a Picard–Fuchs framework.

1. Introduction

1.1. Context and overview. The theory of elliptic modular equations is classical, and predates Gauss’s introduction of the homogeneous modular group \( PSL(2, \mathbb{Z}) \). It can be traced to Landen’s transformation of the first complete elliptic integral

\[
K(\alpha) = \frac{\sin(\pi/\alpha)}{\pi} \int_0^1 x^{-1/\alpha}(1-x)^{-1+1/\alpha}(1-\alpha x)^{-1/\alpha} \, dx,
\]

which is defined on \( 0 < \alpha < 1 \), and extends to a single-valued analytic function on the Riemann sphere \( \mathbb{P}^1(\mathbb{C})\), slit between \( \alpha = 1 \) and \( \alpha = \infty \). (Without the slit, it would be multivalued.) Landen’s transformation is

\[
K(\alpha) = \frac{2}{\alpha}(1 - \sqrt{1 - \alpha}) K(\beta),
\]

where \( \alpha, \beta \) are constrained by the modular relation

\[
\alpha^2 (1 - \beta)^2 - 16(1 - \alpha)\beta = 0.
\]

A uniformized version of (1.2) is

\[
K\left( \frac{t(t+8)}{(t+4)^2} \right) = 2 \left[ \frac{t+4}{t+8} \right] K\left( \frac{t^2}{(t+8)^2} \right),
\]

where \( t \) is an auxiliary parameter. Landen’s transformation is a special function identity; in particular, a quadratic hypergeometric transformation, as \( K_r(\cdot) \) equals 2/\( \pi \) times the Gauss hypergeometric function \( _2F_1\left( \frac{1}{2}, 1 - \frac{1}{r}; 1; \cdot \right) \).

The transformation theory of \( K := K_2 \) was of intense interest to nineteenth-century mathematicians, and led to the modern theory of elliptic curves. Ramanujan’s alternative integrals \( K_r \) (for \( r = 3, 4, 6 \), in addition to \( r = 2 \)), which he

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introduced while deriving rapidly convergent series for \( \pi \), have until recently been much less well understood. For most of the twentieth century, elliptic integrals were best known to applied mathematicians.

In recent years, interest in elliptic integrals and modular equations among pure mathematicians has revived [7] [8] [33]. In part, it has been stimulated by a desire to understand Ramanujan’s modular equations: to derive them algorithmically [5, 6], and also place them in a modern conceptual framework. The proper setting of identities such as (1.2) or (1.3) is now felt to be a general one, a Gauss–Manin connection on a two-dimensional period bundle over the base curve \( E \) and (ii) elliptic modular families, parametrized by the base curve, which may not be of genus zero.

This setting subsumes the classical (Picard–Fuchs) situation where \( \mathfrak{E} \to X \) is an elliptic modular surface, i.e., (i) \( X = \Gamma \setminus \mathcal{H}^* \), the quotient of the compactified upper half plane \( \mathcal{H}^* \ni \tau = \tau_1/\tau_2 \) by a subgroup \( \Gamma \subset \text{PSL}(2,\mathbb{Z}) \) of finite index, and (ii) \( \mathfrak{E} = \mathfrak{E}_\tau \), the family of elliptic curves attached to \( \Gamma \). The Picard–Fuchs equation for \( \mathfrak{E}_\tau \to \Gamma \setminus \mathcal{H}^* \), which has elliptic curve periods as its solutions, defines a Gauss–Manin connection on a two-dimensional period bundle over \( X \).

The classical Picard–Fuchs theory is the natural setting of Landen’s transformation in the form (1.2). The parameter \( \alpha \) can be viewed as a Hauptmodul (rational parameter) for the genus-zero modular curve \( X_0(4) \cong \mathbb{P}^1(\mathbb{C}) \), the quotient of \( \mathcal{H}^* \) by the Hecke subgroup \( \Gamma_0(4) \). Likewise, \( K(\alpha) \) is a weight-1 modular form (with character) for \( \Gamma_0(4) \). The second-order differential equation satisfied by \( K \) as a function of \( \alpha \), i.e., the Gauss hypergeometric equation, is the Picard–Fuchs equation attached to \( \Gamma_0(4) \). It defines a flat connection on a 2-plane period bundle over \( X_0(4) \). Landen’s transformation is a relation on \( \mathfrak{E}_{\Gamma_0(4)} \xrightarrow{\pi} X_0(4) \cong \mathbb{P}^1(\mathbb{C}) \), which ties together fibres (elliptic curves) over distinct points \( \alpha, \beta \in X_0(4) \) if and only if they are related by a 2-isogeny. The ‘multiplier’ \( K(\alpha)/K(\beta) \) is a quotient of period ratios. It is automorphic of weight 0 and must be rational in \( \alpha \); or more accurately (taking characters into account) finite-valued, i.e., algebraic in \( \alpha \), as one sees from (1.2a).

The computational theory of modular equations for Gauss–Manin connections is still in an incomplete state, even for elliptic-modular families \( \mathfrak{E}_\tau \to \Gamma \setminus \mathcal{H}^* \); and even in the genus-zero case, when the differential equation defining the flat connection on the 2-plane bundle over the base is, in effect, a Fuchsian equation on \( \mathbb{P}^1(\mathbb{C}) \). We recently began the systematic derivation of such modular equations, viewed as special function identities, i.e., algebraic transformations of \( 2F_1 \) [40]. We began with certain genus-0 elliptic families, attached to subgroups \( \Gamma < \text{PSL}(2,\mathbb{Z}) \); and more generally to their extensions by Atkin–Lehner involutions, which are subgroups not of \( \text{PSL}(2,\mathbb{Z}) \) but of \( \text{PSL}(2,\mathbb{R}) \). It soon became clear that the family of transformations of \( 2F_1 \) of modular origin is larger than previous treatments had revealed. Rational transformation of \( 2F_1 \) were investigated in the nineteenth century, most systematically by Goursat; though Vidiūnas [62] has recently shown that Goursat’s classification was incomplete. Algebraic transformations of \( 2F_1 \) are much more numerous, and our treatment in Ref. [40] only scratched the surface.

Our ultimate goal is determining which transformations of \( 2F_1 \) come ‘from geometry’. This includes known \( 2F_1 \) and special function identities, such as Ramanujan’s. As a first step, in this article we go beyond Landen’s transformation by systematically working out all rationally parametrized modular equations associated with the
14 modular curves $X_0(M)$ that are of genus zero. The parameter in the degree-$N$
modular equation is a parameter for $X_0(NM)$; so such an equation exists if and only
if $\Gamma_0(NM)$ like $\Gamma_0(M)$ is of genus zero. Our equations include (i) correspondences
between elliptic curves related by $N$-isogenies, i.e., between points on the base of
an elliptic family $\mathcal{E}_M \overset{\pi_M}{\to} X_0(M)$, which are analogous to the $\alpha-\beta$ relation (1.2a);
and (ii) full modular equations, analogous to (1.2a). We express these in terms of
a canonical Hauptmodul $t_M = t_M(\tau)$ and a weight-1 modular form $h_M = h_M(\tau)$, treated initially as a (multivalued) function of $t_M$, i.e., $h_M = h_M(t_M)$. Our key
Theorem 8.1 says essentially the following. If $\nabla_M, \nabla_N$ are the Gauss–Manin
connections coming from the Picard–Fuchs equations for $\mathcal{E}_M, \mathcal{E}_N$, then pulling back $\nabla_M$ along the maps $t_{NM}(\tau) \mapsto t_M(\tau)$ and $t_{NM}(\tau) \mapsto t_M(N\tau)$ yields the same
connection; namely, $\nabla_{NM}$. The equality between the two yields a rationally para-
meterized degree-$N$ modular equation for the multivalued function $h_M$, analogous to
(1.3).

To facilitate the interpretation of our modular equations as special function iden-
tities, we first express each weight-1 modular form $h_M$ (regarded as a function $h_M$
of the corresponding Hauptmodul $t_M$) in terms of $2F_1$, and give the Picard–Fuchs
equation that $h_M$ satisfies. To place the identities in context, we also express each
Hauptmodul $t_M$ and modular form $h_M$ in terms of the Dedekind eta function, and
give explicit $q$-expansions where appropriate. These $q$-expansions yield combinatorial
identities resembling those of Fine [22]. The $q$-expansions of the modular forms
tend to be simple, but those of the Hauptmoduln are complicated, and we mostly omit them. In fact, in deriving Picard–Fuchs and modular equations, we do not rely on $q$-series at all. This contrasts with recent work of Lian and Wiczer [37],
who derived Picard–Fuchs equations for 175 genus-zero subgroups of $PSL(2, \mathbb{R})$ by
$q$-series manipulations.

The culmination of this article is §9–10, where we succeed in placing Ramanu-
jan’s theories of elliptic integrals to alternative bases, which have been developed
by Berndt, Bhargava, and Garvan [5] among others, in a modern setting. We show
that Ramanujan’s modular equations for his elliptic integral $K_r$, where the
signature $r$ equals 2, 3, 4, come from elliptic families parametrized by $X_0(4), X_0(3),
X_0(2)$, respectively. In fact, the (multivalued) functions $K_2, K_3, K_4$ can be viewed
as defining weight-1 modular forms $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ for $\Gamma_0(4), \Gamma_0(3), \Gamma_0(2)$, which are
modified versions of $h_4, h_3, h_2$.

Modular interpretations of Ramanujan’s theories were pioneered by the Bor-
weins [7, 8], but this new interpretation is very fruitful. It leads to new param-
eterized modular equations for $K_3, K_4$. (See our Table 13 which is likely to be of
broad interest.) We also find that his somewhat mysterious theory of signature 6
is associated to a nonclassical elliptic family $\mathcal{E} \overset{\pi}{\to} X$ that is not an elliptic modular
family: the base curve $X$ is not of the form $\Gamma \setminus \mathcal{H}$. Hence, his theory of signature 6
fits into a general Gauss–Manin rather than a Picard–Fuchs framework.

1.2. Detailed overview. Any elliptic curve over $\mathbb{C}$ is necessarily of the form
$\mathbb{C}/(\mathbb{Z}\tau_1 \oplus \mathbb{Z}\tau_2) \cong \mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z})$, where the period ratio $\tau = \tau_1/\tau_2$ lies in the upper
half plane $\mathcal{H}$. Any $(a \ b \ c \ d)^t \in \Gamma(1) := SL(2, \mathbb{Z})$, the inhomogeneous modular
group, acts projectively by $\tau \mapsto \frac{a\tau + b}{c\tau + d}$, giving an action of $\Gamma(1) := PSL(2, \mathbb{Z})$ on $\mathcal{H}$. The
space of isomorphism classes of elliptic curves over $\mathbb{C}$ is the quotient $\Gamma(1) \setminus \mathcal{H}$. Its
one-point compactification is the modular curve $X(1) := \Gamma(1) \setminus \mathcal{H}^*$, where
\( \mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}) = \mathcal{H} \cup \mathbb{Q} \cup \{i\infty\} \) includes cusps. The function field of \( X(1) \) is generated by the \( j \)-invariant, which as traditionally defined has \( q \)-expansion 
\[ q^{-1} + 744 + 196884q + O(q^2) \] about the infinite cusp \( \tau = i\infty \). Here \( q := e^{2\pi i \tau} \).

For any integer \( N > 1 \), the algebraic relation \( \Phi_N(j, j') = 0 \), with \( j'(\tau) := j(N\tau) \), is called the \textit{classical modular equation of degree} \( N \). The polynomial \( \Phi_N \) is symmetric and in \( \mathbb{Z}[j, j'] \). Its degree is
\[ \psi(N) := N \prod_{\substack{p \mid N \ \text{prime}}} \left( 1 + \frac{1}{p} \right), \quad (1.4) \]
a multiplicative function of \( N \). If \( N \) is prime, the \( N + 1 \) roots \( j' \in \mathbb{C} \) of the modular equation correspond to \( \tau' = N\tau \), and to \( \tau/N, (\tau+1)/N, \ldots, (\tau+N-1)/N \).

In general the \( \psi(N) \) roots correspond to values \( \tau' = \frac{a\tau+b}{c\tau+d} \), where the matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z}) \) are reduced level-\( N \) modular correspondences, i.e., they satisfy \( ad = N, \ c = 0, \ 0 \leq b < d, \ (a, b, d) = 1 \). They are bijective with the \( \psi(N) \) (isomorphism classes of) unramified \( N \)-sheeted coverings of a general elliptic curve \( E = \mathbb{C}/(\mathbb{Z}\tau \oplus \mathbb{Z}) \) by an elliptic curve \( \mathbb{C}/(\mathbb{Z}\tau' \oplus \mathbb{Z}) \); or equivalently, with the order-\( N \) cyclic subgroups of its group of \( N \)-division points \( E_N \cong C_N \times C_N \).

The coefficients of \( \Phi_N \) are large, even for small \( N \). E.g., \( \Phi_2(j, j') = 0 \) is
\[ (j^3 + j^3) - j^2 j^2 + 2^4 \cdot 3 \cdot 31(j^2 j' + j j'^2) - 2^{12} \cdot 3^5 5^3 (j^2 + j'^2) + 3^4 5^3 4027 j j' + 2^8 3^7 5^6 (j + j') - 2^{12} 3^9 5^9 = 0. \quad (1.5) \]

Due to the difficulty of computing \( \Phi_N \), modular equations of alternative forms have long been of interest. One approach begins by viewing \( \Phi_N(j, j') = 0 \) as a singular plane model of an algebraic curve over \( \mathbb{C} \), the function field of which is \( \mathbb{C}(j, j') \). The curve is \( X_0(N) \), the quotient of \( \mathcal{H}^* \) by the level-\( N \) Hecke subgroup \( \Gamma_0(N) \) of \( \Gamma(1) \), comprising all \( \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \) with \( c \equiv 0 \pmod{N} \). This is because the reduced level-\( N \) modular correspondences are bijective with the \( \psi(N) \) cosets of \( \Gamma_0(N) \). If \( X_0(N) \) is of genus zero, like \( X(1) \), then its function field too will be singly generated, i.e., will be generated by some univalent function (‘Hauptmodul’) \( t_N \in \mathbb{C}(j, j') \); and \( j, j' \) will be rational functions of \( t_N \). An example is the case \( N = 2 \). Expressions for \( j, j' \) in terms of an appropriate \( t_2 \) turn out to be
\[ j = \frac{(t_2 + 16)^3}{t_2}, \quad j' = \frac{(t_2 + 256)^3}{t_2^2}. \quad (1.6) \]

These constitute a degree-2 \textit{rationally parametrized} modular equation, more pleasant and understandable than \( (1.5) \).

In §\[4\] and §\[5\] we begin by tabulating parametrized modular equations for \( j \) that are of degrees \( N = 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25 \). (See Tables \[4\] and \[5\]) These \( N \) are the ones for which \( X_0(N) \) is of genus zero, and the parameters are Hauptmoduli \( t_N \), normalized in accordance with a convention that we introduce and follow consistently. The rational formulas \( j = j(t_N) \), or certain more general relations \( \Psi_N(t_N, j) = 0 \) that can be derived when \( X_0(N) \) is of positive genus, have been called ‘canonical modular equations’ \[44\], since the formulas \( j' = j'(t_N) \) can be deduced from them. The cases \( N = 2, 3, 5, 7, 13 \), where \( N \) is a prime (and \( N - 1 \mid 12 \)) were treated by Klein \[32\], and his formulas \( j = j(t_N) \) are frequently reproduced. (See, e.g., \[19\] §4.) Composite values of \( N \) were treated by Gierster \[27\], and magisterially, by Fricke \[25\] II. Abschnitt, 4. Kap., but their findings are not
reproduced in recent references. Our Tables 4 and 5 are based on their results, but are more consistently presented.

Better known than modular equations for \( j \) are modular equations for the \( \lambda \)-invariant, or equivalently for the \( \alpha \)-invariant, which were extensively investigated in the nineteenth century. Any elliptic curve \( E/\mathbb{C} \) has a Legendre model \( y^2 = x(x - 1)(x - \lambda) \), the parameter \( \lambda \in \mathbb{C} \setminus \{0, 1\} \) of which is the \( \lambda \)-invariant. Like the \( j \)-invariant, it can be chosen to be a single-valued function of \( \tau \in \mathcal{H} \). Its \( q \)-expansion about the infinite cusp is \( 2^4 \cdot [q^2 - 8q^2 + 44q^3 - 192q^4 + \cdots] \), where the bracketed series has integer coefficients, and \( q_2 := \sqrt{q} = e^{\pi i \tau} \). The \( j \)-invariant can be expressed in terms of \( \lambda \) by

\[
j(\tau) = 2^8 \left( \frac{\lambda^2 - \lambda + 1}{\lambda(\lambda - 1)} \right)^3 (\tau) = 2^4 \left( \frac{\lambda^2 + 14\lambda + 1}{\lambda(\lambda - 1)} \right)^3 (2\tau).
\]

Any elliptic curve \( E/\mathbb{C} \) also has a quartic Jacobi model \( y^2 = (1 - x^2)(1 - \alpha x^2) \), with parameter \( \alpha \in \mathbb{C} \setminus \{0, 1\} \). From its birational equivalence to the Legendre model one can deduce that \( \lambda = \frac{1 - \alpha^4}{(1 + \alpha^4)} \), where \( \alpha = k^2 \). In fact, one can choose \( \alpha(\tau) = \lambda(2\tau) \); so the \( \alpha \)-invariant equals \( 2^4 \cdot [q^2 - 8q^2 + 44q^3 - 192q^4 + \cdots] \).

Many large-\( N \) modular equations for \( j \) assume a simple form when written as an (unparametrized) algebraic relation between \( \lambda := \lambda(\tau) \) and \( \mu := \lambda(N\tau) \), or equivalently \( \alpha := \alpha(\tau) \) and \( \beta := \alpha(N\tau) \); or alternatively, between the square roots \( k := k(\tau) \) and \( l := k(N\tau) \). Classical work on modular equations focused on deriving modular equations of the \( k-l \) type, or the related \( u-v \) type [28].

In [4] we follow a different path: we compute rationally parametrized modular equations for the Hauptmoduln \( t_M \). (See Table 6 and for related ‘factored’ modular equations for \( j \), see Table 7) A rational parametrization of the equation of degree \( N \) at level \( M \) is possible iff \( NM \), as well as \( M \), is one of the numbers 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25. The connection between parametrized modular equations of this type, and the \( \alpha-\beta \) or \( k-l \) type, is not distant. The invariant \( \lambda \) is a Hauptmodul of the modular curve \( X(2) \), the quotient of \( \mathcal{H}^* \) by the level-2 principal congruence subgroup \( \Gamma(2) \). The subgroup relation \( \Gamma(2) < \Gamma(1) \) induces an injection of the function field of \( X(1) \) into that of \( X(2) \), so \( j \) must be a rational function of \( \lambda \); which is where (1.7) comes from. But the groups \( \Gamma(2), \Gamma_0(4) < \Gamma(1) \) are conjugated to each other in \( \mathrm{PSL}(2, \mathbb{R}) \) by a 2-isogeny, so the quotients \( X(2), X_0(4) \) are isomorphic. Their canonical Hauptmoduln \( \lambda, t_4 \), the latter to be defined below, are closely related. In fact, \( \lambda(\tau) = [t_4/(t_4 + 16)](\tau/2) \); so \( \alpha = t_4/(t_4 + 16) \), revealing that the Jacobi model is more closely associated with \( \Gamma_0(4) \) than with \( \Gamma(2) \). In consequence, any degree-\( N \) modular equation of the \( \alpha-\beta \) type, etc., is equivalent to an algebraic relation between \( t_4(\tau) \) and \( t_4(N\tau) \).

The connection between the traditional approach and ours is illustrated by Landen’s transformation, which is based on a \( \alpha-\beta \) modular equation of degree 2, namely Eq. (1.2b), with \( \alpha := \alpha(\tau), \quad \beta := \alpha(2\tau) \). Its more familiar \( k-l \) counterpart is

\[
l = (1 - k')/(1 + k'), \quad k' := \sqrt{1 - k^2}.
\]

If Eq. (1.2b) is converted to an \( t_4(\tau)-t_4(2\tau) \) relation, it becomes the degree-2 modular equation at level 4 that will be derived in [4] with rational parameter \( t_8 \); namely,

\[
t_4 = t_8(t_8 + 8), \quad t'_4 = \frac{t_8^2}{t_8 + 4}.
\]
where \( t_4', t'_8 \) signify \( t_4(\tau), t_4(2\tau) \). This is the level-4 counterpart of (1.6). The parameter \( t_8 \) will be interpreted as the canonical Hauptmodul of \( X_0(8) \).

In §3.4, 4 and 5 we go beyond Hauptmodul relations, and derive modular equations for families of elliptic curves. Along with each degree-\( N \) modular equation for a Hauptmodul of \( \Gamma < \Gamma(1) \), e.g., (1.6) and (1.9), there is a degree-\( N \) modular equation for the elliptic family \( \mathcal{E}_F \rightarrow \Gamma \backslash \mathcal{H}^* \). It is a functional equation satisfied by a certain canonical weight-1 modular form for \( \Gamma \), viewed as a function of the Hauptmodul. It connects the periods \( \tau_1, \tau_2 \) of elliptic curves, i.e., fibres, over related points on the base, and is really a modular equation for a Gauss–Manin connection.

To see all this, consider the case \( \Gamma = \Gamma(1) \), where the Hauptmodul is \( j \) and the family is the universal family of elliptic curves. Since \( j \) is a Fuchsian function of the first kind on \( \mathcal{H} \ni \tau \), it follows from a theorem on conformal mapping [24] that any branch of the multivalued function \( \tau \) on the curve \( X(1) \) can be written locally as the ratio of two solutions of a second-order differential equation, with independent variable \( j \). This equation, used for constructing the unitizing variable \( \tau \), is the classical Picard–Fuchs equation. As Stiller [59] notes, it is best to use \( J := 1/J = 12^3/j \) as the independent variable. The Picard–Fuchs equation will then be hypergeometric, with solution space \( h_1(J)(\mathbb{C}_\tau + \mathbb{C}) \), where \( h_1(J) \) is the Gauss hypergeometric function \( {}_2F_1(\frac{1}{12}, \frac{5}{12}; 1; J) \). The associated hypergeometric differential operator will be a Gauss–Manin connection for the universal family. Near the cusp \( \tau = \infty \), where \( J = 0 \), \( h_1(J(\tau)) \) is the fourth root of the Eisenstein sum \( E_4(\tau) \), a weight-4 modular form for \( \Gamma(1) \). So \( h_1(\tau) := h_1(J(\tau)) \) is formally a weight-1 modular form for \( \Gamma(1) \) (formally only, because globally on \( \mathcal{H} \), it is not single-valued).

By pulling back (i.e., lifting) the Picard–Fuchs equation and its solutions from \( X(1) \) to each genus-zero curve \( X_0(N) \equiv \mathbb{P}^1(\mathbb{C})_{t_N} \), we derive a rationally parametrized degree-\( N \) modular equation for \( h_1 \). For instance, when \( N = 2 \) this equation is

\[
\hat{h}_1(12^3t_2/(t_2 + 16)^3) = 2[(t_2 + 256)/(t_2 + 16)]^{-1/4} \hat{h}_1(12^3t_2^2/(t_2 + 256)^3),
\]

where the arguments of \( \hat{h}_1 \) on the two sides are \( \hat{J} = 12^3/j \) and \( \hat{J}' = 12^3/j' \), written in terms of the Hauptmodul \( t_2 \) of \( X_0(2) \), as in (1.6). This functional equation is the degree-2 modular equation for the universal family of elliptic curves parametrized by \( j \) (or \( J \)). It relates the periods of two elliptic curves with \( j \)-invariants related by (1.5), i.e., by (1.6). Because \( \hat{h}_1(\cdot) = 2F_1(\frac{1}{12}, \frac{5}{12}; 1; \cdot) \), each such modular equation is an algebraic hypergeometric transformation: it relates a \( 2F_1 \) to another \( 2F_1 \) with an algebraically transformed argument.

At each level \( M > 1 \) with a genus-zero \( X_0(M) \), there is a rational elliptic family \( \mathcal{E}_M \rightarrow X_0(M) \equiv \mathbb{P}^1(\mathbb{C})_{t_M} \). By pulling back along \( X_0(M) \rightarrow X(1) \), one can derive a Gauss–Manin connection for it. We make this concrete by pulling back the function \( h_1 = \hat{h}_1(\hat{J}) \) to a function \( h_M = h_M(t_M) \), and working out eta product and \( q \)-series representations for the weight-1 modular form \( h_M(\tau) := h_M(t_M(\tau)) \) on \( \mathcal{H} \). (See Tables 8 and 10). The Picard–Fuchs equation for the level-\( M \) family has solution space \( h_M(t_M)(\mathbb{C}_\tau + \mathbb{C}) \). Our key theorem on further pullings back of the modular form \( h_M(t_M(\tau)) \), Theorem S.1 efficiently produces a degree-\( N \) modular equation for \( h_M \) if \( NM \), as well as \( M \), is one of 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25. These equations are listed in Table 17. The most important single table of this article (though its corollary Table 18 is more immediately understandable).
They may be viewed as (i) algebraic transformations of Picard–Fuchs equations, or (ii) transformations of the special functions that satisfy them (i.e., the functions $h_M = h_M(t_M)$), or (iii) transformations of certain integer sequences defined by recurrences (i.e., the coefficient sequences of the functions $h_M$, expanded in $t_M$).

The cases $M = 2, 3, 4$ are especially interesting, partly because in these cases the multiplier system of the modular form $h_M(\tau) = h_M(t_M(\tau))$ is nontrivial: only if $M \geq 5$ is it given by a Dirichlet character mod $M$. When $M = 2, 3, 4$, the Picard–Fuchs equation for $\Gamma_0(M)$ is of hypergeometric type, with three regular singular points on $X_0(M) \cong \mathbb{P}^1(\mathbb{C})_{t_M}$; namely, the fixed points of $\Gamma_0(M)$. For instance, the level-4 function $h_4 = h_4(t_4)$, pulled back from $\hat{h}_1 = \hat{h}_1(\hat{J})$, is $\text{2F1}(\frac{1}{2}, \frac{3}{2}; 1; -t_4/16)$. The degree-2 modular equation for the elliptic family parametrized by $t_4$ is

$$h_4(t_8(t_8 + 8)) = 2(t_8 + 4)^{-1/2}h_4(t_8^2/(t_8 + 4)),$$

(1.10)

where the arguments of $h_4$ on the two sides are $t_4$ and $t_4'$ written in terms of a Hauptmodul $t_8$ for $\Gamma_0(8)$, as in (1.9). This is another algebraic hypergeometric transformation, as are all our modular equations at levels $M = 2, 3, 4$.

If $M = 5, 6, 7, 8, 9$, then the Picard–Fuchs equation is of Heun type, with four singular points on $X_0(M) \cong \mathbb{P}^1(\mathbb{C})_{t_M}$; as before, the fixed points of $\Gamma_0(M)$. Hence, its solutions can be expressed in terms of the canonical ‘local Heun’ function $Hl$ (51). (For basic facts on equations of hypergeometric and Heun type, see the Appendix.) Our elliptic-family modular equations on levels 5, 6, 7, 8, 9 are thus algebraic Heun transformations: functional equations satisfied by $Hl$. By expanding $Hl$ in its argument and focusing on the resulting coefficient sequence, one can view them as identities satisfied by integer sequences defined by three-term recurrences. The functional equations of Proposition 8.4, which are satisfied by the generating function for the Franel numbers $\sum_{\ell=0}^{n} \binom{n}{\ell}^3$, $n \geq 0$, are an example. The theory of such Heun transformations, whether viewed combinatorially or not, is entirely undeveloped.

Our modular equations at levels 4, 3, 2 fit into Ramanujan’s theories of elliptic integrals in signatures $r = 2, 3, 4$, respectively. In special function terms, this is because his complete elliptic integral in signature $r$, denoted $K_r(\alpha_r)$ here, is proportional to $\text{2F1}(\frac{1}{4}, 1 - \frac{1}{4}; 1; \alpha_r)$. By employing Pfaff’s transformation of $\text{2F1}$, given in the Appendix as (A.3), one can convert identities involving $h_4, h_3, h_2$ to ones involving $K_2, K_3, K_4$. For instance, the modular equation (1.10) becomes

$$K_2(4p/(p + 1)^2) = (1 + p) K_2(p^2).$$

(1.11)

This is a parametric form of Landen’s transformation, since $r = 2$ is the classical base and $K_2$ equals $K$, the traditional complete elliptic integral. It is equivalent to (A.3). Like $h_4$, $K_2$ can be viewed as defining a weight-1 modular form for $\Gamma_0(4)$, and the modular equation (1.11) expresses $K_2$ at a general point $\tau \in \mathcal{H}$ in terms of its value at $\tau' = 2\tau$.

In §5 by systematically converting our modular equations for the elliptic families parametrized by $t_4, t_3, t_2$, we compute parametrized modular equations of degrees $N = 2, 3, 4$ in Ramanujan’s theory of signature 2; of degrees $N = 3, 4, 5, 6, 8, 9$ in that of signature 3; and of degrees $N = 2, 3, 4, 5, 6, 8, 9$ in that of signature 4. (See Table 18; the underlined ones are new to the literature.) We also derive equations relating $K_2, K_3, K_4$, arising from commensurability of the subgroups $\Gamma_0(4), \Gamma_0(3)$,
The modular curve $X_0(N) = \Gamma_0(N) \backslash \mathcal{H}$ classifies each cyclic $N$-isogeny $\phi : E \to E/C$ up to isomorphism, where an isomorphism $(E_1, C_1) \cong (E_2, C_2)$ is an isomorphism of elliptic curves $E_1, E_2$ that takes $C_1$ to $C_2$. Since $X_0(N)$ is defined over $\mathbb{Q}$, it classifies up to isomorphism over $\mathbb{Q}$. As noted, its function field is $\mathbb{C}(j, j')$ where $j' = j(N\tau)$. The Fricke involution $W_N : (E, C) \to (E/C, E_N/C)$, which on $\mathcal{H}$ is the map $\tau \mapsto -1/N\tau$, interchanges $j, j'$.

The cusps of $X_0(N)$ are as follows. The set of cusps of $\mathcal{H} \ni \tau$, i.e., $\mathbb{P}^1(\mathbb{Q})$, is partitioned into equivalence classes under $\Gamma_0(N)$. A system of representatives, i.e., a choice of one from each, may be taken to comprise certain fractions $\tau = \frac{a}{d}$ for each $d \mid N$, with $1 \leq a < d$ and $(a, d) = 1$. Here $a$ is reduced modulo $(d, N/d)$ while remaining coprime to $d$, so there are $\varphi((d, N/d))$ possible values of $a$, and hence $\varphi((d, N/d))$ inequivalent cusps in $\mathbb{P}^1(\mathbb{Q})$ associated to $d$. (In this statement $\langle \cdot, \cdot \rangle$ is the greatest common divisor function, and $\varphi(\cdot)$ is the Euler totient function.) So in all, $X_0(N)$ has

$$
\sigma_\infty(N) := \sum_{d \mid N} \varphi((d, N/d))
$$

cusps, which are involuted by $W_N$. The rational cusps $\frac{a}{d}$ are those for which $\varphi((d, N/d)) = 1$, i.e., the ones with $(d, N/d) = 1$ or 2.

The covering $j : X_0(N) \to \mathbb{P}^1(\mathbb{C}) \cong X(1)$ is $\psi(N)$-sheeted, since the index $[\Gamma(1) : \Gamma_0(N)]$ equals $\psi(N)$. It is ramified only above the cusp $j = \infty$ and the elliptic fixed points $j = 0, 12^3$, corresponding to equianharmonic and lemniscatic elliptic curves; i.e., only above the three vertices $\tau = i\infty, \zeta_3 := e^{2\pi i/3}, i$ of the fundamental half-domain of $\Gamma(1)$. The fibre above $j = \infty$ includes (the equivalence class of) each cusp $\tau = a/d$ with multiplicity equal to its width $\epsilon_{d,N} := N/d(N, d/N)$. To indicate that a cusp is an equivalence class, the notation $[\frac{a}{d}]$ or $[\frac{a}{d}]_N$ will be used as appropriate.

The fibre above $j = 0$ (resp. $12^3$) includes $\epsilon_2(N)$ cubic elliptic points (resp. $\epsilon_2(N)$ quadratic ones), each with unit multiplicity; other points on the fibre appear with cubic (resp. quadratic) multiplicity $\frac{1}{3}$. Here

$$
\epsilon_2(N) := \begin{cases} 
\prod_{p \mid N, \ p \text{ prime}} \left(1 + \left(\frac{-1}{p}\right)\right), & 4 \nmid N, \\
0, & 4 \mid N,
\end{cases}
$$

$$
\epsilon_3(N) := \begin{cases} 
\prod_{p \mid N, \ p \text{ prime}} \left(1 + \left(\frac{-3}{p}\right)\right), & 9 \nmid N, \\
0, & 9 \mid N,
\end{cases}
$$

with $\left(\cdot\right)$ the Legendre symbol. Any elliptic curve $\mathbb{C}/(\mathbb{Z} \tau_1 \oplus \mathbb{Z} \tau_2)$ that is lemniscatic has $\epsilon_2(N)$ non-isomorphic self-$N$-isogenies, performed by complex multiplication of the period lattice by a Gaussian integer; so $\epsilon_2(N)$ simply counts the ways of representing $N$ as the sum of two squares. $\epsilon_3(N)$ has a similar interpretation, in terms of Eisenstein integers. It follows from the Hurwitz formula that $X_0(N)$ has genus

$$
g = 1 + \frac{\psi(N)}{12} - \frac{\sigma_\infty(N)}{2} - \frac{\epsilon_2(N)}{4} - \frac{\epsilon_3(N)}{3}.
$$
Table 1. Basic data on each genus-0 group $\Gamma_0(N) < \Gamma(1)$.

| $N$ | $\psi(N)$ | $\sigma_\infty(N)$ | Cusps (τ values) | Cusp widths | $\epsilon_2(N)$ | $\epsilon_3(N)$ |
|-----|-----------|--------------------|------------------|-------------|----------------|----------------|
| 2   | 3         | 2                  | $\frac{1}{1}$   | 2;1         | 1              | 0             |
| 3   | 4         | 2                  | $\frac{1}{1}$   | 3;1         | 0              | 1             |
| 4   | 6         | 3                  | $\frac{1}{1}$   | 4;1;1      | 0              | 0             |
| 5   | 6         | 2                  | $\frac{1}{1}$   | 5;1         | 2              | 0             |
| 6   | 12        | 4                  | $\frac{1}{1}$   | 6;3;2;1    | 0              | 0             |
| 7   | 8         | 2                  |                 | 7;1         | 0              | 2             |
| 8   | 12        | 4                  | $\frac{1}{1}$   | 8;2;1;1    | 0              | 0             |
| 9   | 12        | 4                  | $\frac{1}{1}$   | 9;1;1;1    | 0              | 0             |
| 10  | 18        | 4                  | $\frac{1}{1}$   | 10;5;2;1   | 2              | 0             |
| 12  | 24        | 6                  | $\frac{1}{1}$   | 12;4;3;3;1;1 | 0           | 0             |
| 13  | 14        | 2                  |                 | 13;1        | 2              | 2             |
| 16  | 24        | 6                  | $\frac{1}{1}$   | 16;4;1;1;1;1 | 0           | 0             |
| 18  | 36        | 8                  | $\frac{1}{1}$   | 18;9;2;2;2; | 1,1;1         | 0             |
| 25  | 30        | 6                  | $\frac{1}{1}$   | 25;1,1,1,1;1 | 2             | 0             |

In Table 1 the basic data (number of fixed points, cusp locations, etc.) are listed for each of the 14 genus-zero curves $X_0(N)$.

3. Hauptmoduln and Parametrized Modular Equations

To make the covering $j : X_0(N) \to \mathbb{P}^1(\mathbb{C}) \cong X(1)$ more concrete, one needs (i) an explicit formula for a Hauptmodul $t_N$ for $\Gamma_0(N)$, so that $X_0(N)$ like $X(1)$ can be identified with $\mathbb{P}^1(\mathbb{C})$, and (ii) an expression for the covering map, as a degree-$\psi(N)$ rational function of $t_N$. For most of the above 14 values of $N$, items (i) and (ii) were worked out by Klein, Gierster and Fricke. But because the classical derivation was somewhat unsystematic, ours is de novo.

3.1. Canonical Hauptmoduln. For each $N$, we can specify the Hauptmodul $t_N$ uniquely by requiring that (I) $t_N$ have a simple zero at the cusp $\frac{1}{N}$ and a simple pole at the cusp $\frac{1}{1}$, and (II) the function $(t_N|W_N)(\tau) := t_N(-1/N\tau)$ have a Fourier expansion on $H^*$ that begins $1 \cdot q^{-1} + O(q^2)$, where $q := e^{2\pi i \tau}$. Since the Fricke involution $W_N : \tau \mapsto -1/N\tau$ interchanges $\frac{1}{N}$ and $\frac{1}{1}$, the product $t_N(\tau)t_N(-1/N\tau)$ necessarily equals some constant function of $\tau$, to be denoted $\kappa_N$, and condition (II) fixes this constant. Imposing this normalization condition will simplify the modular equations to be deduced, e.g., by forcing many polynomial factors to be monic.

A $q$-product, such as the one used to define the Dedekind eta function, is the natural way of defining each $t_N$. Though Hauptmoduln of general genus-zero congruence subgroups cannot be expressed in terms of the eta function, it turns out to be possible for each genus-zero $\Gamma_0(N)$, as we explain. Recall that $\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n)$ satisfies $\eta(\tau + 1) = \zeta_{24} \eta(\tau)$ and $\eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau)$. One first defines an unnormalized Hauptmodul $\hat{t}_N$ by

$$\hat{t}_N = \prod_{\delta | N} \eta(\delta \tau)^{\epsilon_3}, \quad (3.1)$$
where \( \delta \mapsto r_3 \in \mathbb{Z} \) is an appropriate ‘generalized permutation’. An obvious requirement is that the Hauptmodul have the correct order of vanishing at each cusp \([33]\).

Since \( \eta(\delta \tau) \) has order \( \frac{1}{24} (\delta, d)^2 / \delta \) at any cusp \( \tau = \frac{a}{d} \) on \( \mathcal{H}^* \), this function \( \hat{t}_N(\tau) \) has order \( \frac{1}{24} \sum_{\delta | N} r_3(\delta, d)^2 / \delta \) there. To compute its order at \( \frac{N}{d} \in X_0(N) \), one must multiply by the cusp width \( e_{d,N} = N/d(d, N/d) \). The resulting order of vanishing should be +1 if \( \left[ \frac{a}{d} \right] = \frac{1}{\delta}, \) -1 if \( \left[ \frac{a}{d} \right] = \frac{1}{\delta}, \) and 0 at each of the other \( \sigma_\infty(N) - 2 \) cusps.

The generalized permutation \( \delta \mapsto r_3 \) must also be such that \( \hat{t}_N \) is a single-valued function on \( X_0(N) \). A sufficient condition for this was given by Newman \([45]\). If it is the case that \( \sum_{\delta | N} r_3 \equiv 0 \) (mod 24), \( \sum_{\delta | N} \delta r_3 \equiv 0 \) (mod 24), and \( \sum_{\delta | N} (N/\delta) r_3 \equiv 0 \) (mod 24), and also \( \prod_{\delta | N} \delta r_3 = \kappa_N^2 \) for some natural number \( \kappa_N \), then \( \hat{t}_N \) will be single-valued on \( X_0(N) \). Moreover, \( \hat{t}_N |_{W_N} \) will equal \( \kappa_N^{-1} \prod_{\delta | N} \eta(\delta \tau)^{r_3 / \delta} \).

It follows that to satisfy condition (II), one should let \( t_N := \kappa_N \cdot \hat{t}_N \). If that choice is made, \( t_N |_{W_N} = \prod_{\delta | N} \eta(\delta \tau)^{r_3 / \delta} \) will have a Fourier expansion beginning \( 1 \cdot q^{-1} + O(q^0) \).

For each of the 14 values of \( N \), a unique map \( \delta \mapsto r_3 \) satisfying the preceding conditions can be found by inspection. The resulting normalised eta-product expressions, \( t_N := \kappa_N \cdot \prod_{\delta | N} \eta(\delta \tau)^{r_3 / \delta} \), are listed in the second column of Table 2.

Fine’s compact notation \([\delta] \) for the function \( \eta(\delta \tau) \) on \( \mathcal{H}^* \equiv \tau \) is used. These eta products were worked out by Fricke, with the exception of those for \( N = 4, 16, 18 \), and also with the exception of the one for \( N = 2 \). (In an unfortunate confusion that can be traced to Klein \([32\), p. 143\], Fricke’s analogue of \( t_2(\tau) \) is proportional to \( t_2(2\tau) \).) It is well known that \([2\), Ch. 4\] \begin{equation}
\tag{3.2}
t_N = N^{12/(N-1)} \cdot [N]^{24/(N-1)} / [1]^{24/(N-1)}, \quad \text{if } N - 1 \mid 24.
\end{equation}

But in the six cases when \( N - 1 \nmid 24 \), the prefactors \( \kappa_N \) seem not to have been tabulated before. Several of those given by Klein–Giester–Fricke differ from ours, since they did not consistently impose condition (II). Shih \([55]\) reproduces and uses their \( \kappa_N \)’s, each of which is less than ours by a factor equal to a rational square. Fortunately, for Shih’s application inconsistently scaled \( \kappa_N \)’s are adequate.

For each \( N, \) if \( \zeta_\infty \) is a distinct rational cusp for the genus-zero subgroup \( \Gamma_0(N) \), there is a Hauptmodul \( \hat{t}_N \) on \( X_0(N) \) with a simple zero at \( \zeta_0 \) and a simple pole at \( \zeta_\infty \), which is a rational (over \( \mathbb{Q} \)) degree-1 function of the canonical Hauptmodul \( t_N \). In Table 3 the alternative Hauptmodul \( \hat{t}_N \) for which \( \zeta_0 = \left[ \frac{N}{2} \right] \equiv 1 \infty \) are listed. They were extracted from a long list of Hauptmoduln of genus-zero groups obtained by Ford et al. \([23]\) by manipulating \( q \)-series, but can also be verified individually by examining behavior at each cusp. The eta-product representation \( 2^{4} \cdot [1]^8 [4]^{16} / [2]^{24} \) of the alternative Hauptmodul \( \alpha = t_4/(t_4 + 16) \) for \( X_0(4) \) should be noted. Combined with the known expression \( 2^{4} \cdot [1]^8 [2]^{16} / [1]^{24} \) for the \( \lambda \)-invariant, it confirms that \( \lambda(\tau) = [t_4/(t_4 + 16)](\tau/2) = \alpha(\tau/2) \), as mentioned in \([12]\).

One can find \( q \)-series for most of the Hauptmoduln in Tables 2 and 3 in Sloane’s Encyclopedia \([57]\). In a neighborhood of the infinite cusp, i.e., of the point \( q = 0 \), each can be written as prefactor \( [g + \sum_{n=2}^{\infty} a_n q^n] \), where the \( a_n \) are integers. Thus \( t_4 = 2^{8} \cdot [g + 8q^2 + 44q^3 + 192q^4 + \cdots] \), which is consistent with the \( q \)-expansion of \( \alpha = t_4/(t_4 + 16) \) given in \([12]\). It is worth remarking that with the exception of prefactors, the \( q \)-series for \( t_4 \) and \( \alpha = t_4/(t_4 + 16) \) are related by \( q \mapsto -q \), i.e., \( \tau \mapsto \tau + \frac{1}{2} \), as are those for \( t_8 \) and \( t_8/(t_8 + 4) \), and those for \( t_{16} \) and \( t_{16}/(t_{16} + 2) \).
Table 2. The canonical Hauptmodul $t_N = \kappa_N \cdot \hat{t}_N$ for $\Gamma_0(N)$, and the fixed points of $\Gamma_0(N)$ on $X_0(N)$. (The polynomial $p_{25}(t)$ equals $t^4 + 5t^3 + 15t^2 + 25t + 25$.)

| $N$ | $t_N(\tau) = \kappa_N \cdot \hat{t}_N(\tau)$ | Cusps ($t_N$ values) | Elliptic points ($t_N$ values) |
|-----|---------------------------------|------------------|-------------------------------|
| 2   | $2^{12} \cdot [2]^{24}/[1]^{24}$ | $\infty; 0$ | $-64$ [quadratic] |
| 3   | $3^{8} \cdot [3]^{12}/[1]^{12}$  | $\infty; 0$ | $-27$ [cubic] |
| 4   | $2^{8} \cdot [4]^{8}/[1]^{8}$   | $\infty; -16, 0$ | $-11 \pm 2\sqrt{3}$ [quadratic] |
| 5   | $5^{4} \cdot [5]^{6}/[1]^{6}$   | $\infty; 0$ | $-13\pm 3\sqrt{5}$ [cubic] |
| 6   | $2^{3} \cdot [2]^{6}/[1]^{6}[3]$ | $\infty; -8; -9, 0$ | $-14 \pm 2\sqrt{3}$ [quadratic] |
| 7   | $7^{2} \cdot [7]^{4}/[1]^{4}$   | $\infty; 0$ | $-13\pm 3\sqrt{5}$ [cubic] |
| 8   | $2^{5} \cdot [2]^{8}/[1]^{4}[4]^2$ | $\infty; -4; -8, 0$ | $-14 \pm 2\sqrt{3}$ [quadratic] |
| 9   | $3^{3} \cdot [9]^{3}/[1]^{3}$   | $\infty; -2\pm 3\sqrt{2}, 0$ | $-3 \pm 2\sqrt{3}$ [quadratic] |
| 10  | $2^{5} \cdot [2]^{10}/[1]^{5}[5]$ | $\infty; -4; -5, 0$ | $-14 \pm 2\sqrt{3}$ [quadratic] |
| 12  | $2^{3} \cdot [2]^{12}/[1]^{5}[3]^2/[1]^{4}[4]^6[6]^2$ | $\infty; -3; -2, -4; -6, 0$ | $-14 \pm 2\sqrt{3}$ [quadratic] |
| 13  | $13 \cdot [13]^{4}/[1]^{4}$    | $\infty; 0$ | $-14 \pm 2\sqrt{3}$ [quadratic], $-5\pm 3\sqrt{5}$ [cubic] |
| 16  | $2^{3} \cdot [2]^{16}/[1]^{2}[8]$ | $\infty; -2; -2 \pm 2\sqrt{3}, -4, 0$ | $-14 \pm 2\sqrt{3}$ [quadratic] |
| 18  | $3 \cdot [2]^{3}[18]^{2}/[1]^{2}[6]^{9}$ | $\infty; -2; -3 \pm 2\sqrt{3}, -3, -3 \pm 3\sqrt{3}, 0$ | $-14 \pm 2\sqrt{3}$ [quadratic] |
| 25  | $5 \cdot [25]^{1}/[1]$       | $\infty; 0$ | $-1 \pm 2\sqrt{3}$ [quadratic] |

Table 3. Alternative Hauptmoduln $\hat{t}_N$ for $\Gamma_0(N)$.

| $N$ | $\hat{t}_N$ | pole ($\tau$ class) | eta product representation |
|-----|----------------|------------------|----------------------------|
| 4   | $t_4/(t_4 + 16)$ | $\frac{1}{2}$ | $2^{4} \cdot [1]^{8}[4]^{16} / [2]^{24}$ |
| 6   | $t_6/(t_6 + 8)$  | $\frac{1}{2}$ | $3^{2} \cdot [1]^{4}[6]^{8}/[2]^{2}[3]^{4}$ |
| 6   | $t_6/(t_6 + 9)$  | $\frac{1}{2}$ | $3^{2} \cdot [1]^{4}[6]^{9}/[2]^{3}[3]^{9}$ |
| 8   | $t_8/(t_8 + 4)$  | $\frac{1}{2}$ | $2^{3} \cdot [1]^{4}[4]^{8}/[2]^{10}$ |
| 8   | $t_8/(t_8 + 8)$  | $\frac{1}{2}$ | $2^{2} \cdot [2]^{1}[8]^{8}/[4]^{12}$ |
| 10  | $t_{10}/(t_{10} + 4)$ | $\frac{1}{2}$ | $5 \cdot [1]^{10}[10]^{8}/[2]^{3}[5]^{2}$ |
| 10  | $t_{10}/(t_{10} + 5)$ | $\frac{1}{2}$ | $2^{2} \cdot [1]^{10}[5]^{5}/[2]^{3}[5]^{5}$ |
| 12  | $t_{12}/(t_{12} + 2)$ | $\frac{1}{2}$ | $2 \cdot [1]^{1}[4]^{8}[6]^{12}[2]^{7}[3]$ |
| 12  | $t_{12}/(t_{12} + 3)$ | $\frac{1}{2}$ | $2^{2} \cdot [1]^{2}[12]^{6}/[3]^{3}[4]$ |
| 12  | $t_{12}/(t_{12} + 4)$ | $\frac{1}{2}$ | $3 \cdot [2]^{7}[2]^{1}[4]^{6}[6]^{2}$ |
| 12  | $t_{12}/(t_{12} + 6)$ | $\frac{1}{2}$ | $2 \cdot [2]^{3}[3]^{5}[12]^{10}/[1]^{3}[4]^{3}[6]^{9}$ |
| 16  | $t_{16}/(t_{16} + 2)$ | $\frac{1}{2}$ | $2^{2} \cdot [1]^{6}[4]^{2}[16]^{2}/[2]^{9}[8]$ |
| 16  | $t_{16}/(t_{16} + 4)$ | $\frac{1}{2}$ | $2 \cdot [4]^{7}[16]^{4}/[8]^{16}$ |
| 18  | $t_{18}/(t_{18} + 2)$ | $\frac{1}{2}$ | $3 \cdot [1]^{18} + [2]^{2}[9]$ |
| 18  | $t_{18}/(t_{18} + 3)$ | $\frac{1}{2}$ | $2 \cdot [3]^{1}[18]^{2}[6][9][2]$ |

The q-series coefficients $a_n$ for the Hauptmoduln of Table 2 are always non-negative.

Remark. Many of these Hauptmoduln, or their q-series, have cropped up in the literature. For instance, the alternative Hauptmoduln $3[1]^{18} / [6][9][3]$ for $\Gamma_0(18)$, i.e., $t_{18}/2(t_{18} + 3)$, was expanded in an interesting continued fraction by both Ramanujan and Selberg. (Duke [17] (9.13]) gives this continued fraction, with $q \mapsto q^{1/3}$.)
Table 4. The degree-\(\psi(N)\) covering map from 
\(X_0(N) \cong \mathbb{P}^1(\mathbb{C})_{t_N}\) to \(X(1) \cong \mathbb{P}^1(\mathbb{C})_j\).

\[\begin{array}{cc}
N & j(\tau) \text{ as a function of } t_N(\tau) \\
\hline
2 & \frac{(1+16)^3}{t} = 12^3 + \frac{(t+64)(t-8)^2}{t} \\
3 & \frac{(1+27)(t+3)^3}{t} \\
4 & \frac{(t^2+18t-27)^2}{t} \\
5 & \frac{(t^2+16t+16)^3}{t(t+16)} \\
6 & \frac{(t^2+10t+5)^3}{t} \\
7 & \frac{(t^2+22t+12t^2+4t+24)^3}{t(t+8)^3(t+9)^2} \\
8 & \frac{(t^2+24t+24)^2(t^4+243^2+192t^2+504t-72)^2}{t(t+8)^3(t+9)^4} \\
9 & \frac{(t^2+13t+49)(t^2+5t+1)^3}{t(t+8)^3(t+9)^4} \\
10 & \frac{(t^2+14t^3)^2+63t^2+70t-7)^2}{t(t+8)^4(t+9)^4} \\
11 & \frac{(t^2+16t^3+80t^2+128t+16)^3}{t(t+8)^4(t+9)^4} \\
12 & \frac{(t^2+8t+8)^2(t^4+16t^3+80t^2+128t-8)^2}{t(t+8)(t+9)^4} \\
13 & \frac{(t^2+9t+27)^2}{t(t+8)^4(t+9)^5} \\
14 & \frac{(t^2+18t^4)^2+504t^3+891(t^4+486t^3-27)^2}{t(t+8)^4(t+9)^5} \\
15 & \frac{(t^2+20t^5+160t^4+60t^3+1209t^2+1040t+60)^3}{t(t+8)^4(t+9)^5} \\
16 & \frac{(t^2+6t+6)^3(t^6+18t^5+12t^4+432t^3+732t^2+504t+24)^3}{t(t+8)^4(t+9)^5} \\
17 & \frac{(t^2+12t^3+484t^2+722+2415^3(t^8+24t^7+2410^8+12968^5+4080^4+7488^3+7416^2+30241^1-72)^3}{t(t+8)^4(t+9)^5} \\
18 & \frac{(t^2+5t+13)(t^6+7t^5+20t^4+19t+1)^3}{t(t+8)^4(t+9)^5} \\
19 & \frac{(t^2+6t+13)(t^6+10t^5+46t^4+108t^3+122t^2+38t-1)^2}{t(t+8)^4(t+9)^5} \\
20 & \frac{(t^2+16t^7+112t^6+448t^5+1104t^4+1664t^3+1408t^2+512t+16)^3}{t(t+8)^4(t+9)^5} \\
21 & \frac{(t^2+8t+3)^4(t^2+4t+8)^3}{t(t+8)^4(t+9)^5} \\
22 & \frac{(t^2+12t^3+484t^2+722+2415^3(t^8+24t^7+2410^8+12968^5+4080^4+7488^3+7416^2+30241^1-72)^3}{t(t+8)^4(t+9)^5} \\
23 & \frac{(t^2+12t^3+484t^2+722+2415^3(t^8+24t^7+2410^8+12968^5+4080^4+7488^3+7416^2+30241^1-72)^3}{t(t+8)^4(t+9)^5} \\
24 & \frac{(t^2+5t+13)(t^6+7t^5+20t^4+19t+1)^3}{t(t+8)^4(t+9)^5} \\
25 & \frac{(t^2+16t^7+112t^6+448t^5+1104t^4+1664t^3+1408t^2+512t+16)^3}{t(t+8)^4(t+9)^5} \\
\hline
\end{array}\]

Also, the alternative Hauptmodul \([4]^2[16]^4/[8]^6\) for \(\Gamma_0(16)\), i.e., \(t_{16}/2(t_{16} + 4)\), is the so-called \(\varepsilon\)-invariant, in which Weierstrass developed the nome \(q\) as a power series. That is,

\[q = q(\varepsilon) = \varepsilon + \sum_{k=1}^{\infty} \delta_k \varepsilon^{4k+1}. \quad (3.3)\]

The problem of efficiently calculating the sequence of positive integers \(\{\delta_k\}_{k=1}^{\infty}\) by reverting the \(q\)-series for \(\varepsilon\) has generated a significant literature \(21\).

3.2. Coverings and parametrized modular equations. An explicit formula for each cover \(X_0(N)/X(1)\), i.e., for the covering map \(t_N \mapsto j\) of degree \(\psi(N)\),
on Rationally Parametrized Modular Equations

Table 5. The degree-ψ(N) map \( t_N \mapsto j' \).

| \( N \) | \( j'(\tau) := j(N\tau) \) as a function of \( t_N(\tau) \) |
|---|---|
| 2 | \( \frac{(t+256)^3}{t^2} \) |
| 3 | \( \frac{12^3 + \left(\frac{t+64}{t-64}\right)(t-512)^2}{t^2} \) |
| 4 | \( \frac{12^3 + \left(\frac{t^2-486t-19683}{t^2+16}\right)^2}{t^3} \) |
| 5 | \( \frac{12^3 + \left(\frac{t^2+128}{t^2+16}\right)^2}{t^4(t+16)} \) |
| 6 | \( \frac{12^3 + \left(\frac{t^2+22t+125}{t^2-500t-15625}\right)^2}{t^6(t+16)} \) |
| 7 | \( \frac{12^3 + \left(\frac{t^2+22t+125}{t^2-500t-15625}\right)^2}{t^7(t+16)} \) |
| 8 | \( \frac{12^3 + \left(\frac{t^2+22t+125}{t^2-500t-15625}\right)^2}{t^8(t+16)} \) |
| 9 | \( \frac{12^3 + \left(\frac{t^2+22t+125}{t^2-500t-15625}\right)^2}{t^9(t+16)} \) |
| 10 | \( \frac{12^3 + \left(\frac{t^2+22t+125}{t^2-500t-15625}\right)^2}{t^{10}(t+16)} \) |
| 11 | \( \frac{12^3 + \left(\frac{t^2+22t+125}{t^2-500t-15625}\right)^2}{t^{11}(t+16)} \) |
| 12 | \( \frac{12^3 + \left(\frac{t^2+22t+125}{t^2-500t-15625}\right)^2}{t^{12}(t+16)} \) |

is given in Table 3. The rational expressions originated with Klein and Gierster, but have been modified to agree with our Hauptmodul normalization convention, which ensures monicity of each polynomial factor. The factored expressions add much detail to the ramification data of Table II For any \( N \), the roots of the denominator are the \( t_N \)-values of cusps; and rational and irrational cusps are easy to distinguish. The multiplicity of each root is the width of the corresponding cusp. Each numerator contains one or more cubic factors, and possibly a polynomial that is not cubed; if the latter is present, its roots are the cubic fixed points of \( \Gamma_0(N) \) on \( X_0(N) \). Similarly, an unsquared factor (if any) of the numerator of \( j - 12^3 \) has quadratic fixed points as its roots. The cusp and elliptic point locations read off from Table IV are listed in the final two columns of Table II.
The covering maps of Table 4 can be trusted, since they agree with the known ramification data for \( t_N \mapsto j \). For each \( N \), the ramification data constitute a schema, such as the \( N = 5 \) schema, which is

\[
\psi(5) = 6 = 5 + 1 = 3 + 3 = 2 + 2 + 1 + 1 \tag{3.4}
\]

(in an obvious notation). The final three members specify multiplicities on the fibres over \( j = \infty, 0, 12^4 \), respectively. Ordinarily a covering of \( \mathbb{P}^1(\mathbb{C}) \) by \( \mathbb{P}^1(\mathbb{C}) \), unramified over the complement of three points, is not uniquely specified by its ramification schema. Rather, by the Grothendieck correspondence it is specified by its associated dessin d'enfants [53]. But it is the case that for each prime \( N \) in the table, there is only a single dessin compatible with the schema. This is a combinatorial statement, in fact a graph-theoretic one, which can be verified for each \( N \) individually. There is also a deeper, Galois-theoretic reason why it is true. Within the symmetric group \( \mathfrak{S}_{\psi(N)} \) of permutations of the sheets of the covering, the ramification schema specifies a disjoint cycle decomposition of the monodromy generators \( g_{\infty}, g_0, g_{12} \) associated to loops around \( j = \infty, 0, 12^4 \); that is, it specifies each of them up to conjugacy. For all \( N \), it can be shown that this triple of conjugacy classes is rigid: there is essentially only one way of embedding it in \( \mathfrak{S}_{\psi(N)} \) that is compatible with the constraint \( g_{\infty}g_0g_{12} = 1 \). This is the same as saying that only one dessin is possible. For a discussion, see Elkies [19, p. 49] and Malle and Matzat [41, Chap. I, §7.4].

Each ‘canonical modular equation’ \( j = j(t_N) \) of Table 4 immediately yields a parametrization of the corresponding classical modular equation, i.e., of the equation \( \Phi_N(j, j') = 0 \), where \( j'(\tau) = j(N\tau) \). Since \( j|_{W_N} = j' \) and \( t_N|_{W_N} = \kappa_N/t_N \), applying \( W_N \) to the formula \( j = j(t_N) \) yields the other formula \( j' = j'(t_N) \). The rational expressions for \( j' \) as a function of \( t_N \), and also for \( j' - 12^3 \), are given in fully factored form in Table 5. A few expressions for \( j' - 12^3 \), as for \( j - 12^3 \), are omitted on account of length.

It should be noted that combined with the formula for the \( \lambda \)-invariant, namely \( \lambda(\tau) = \{t_4/(t_4 + 16)\}^{(\tau/2)} \), the formulas for \( j, j' \) in terms of \( t_4 \) yield Eq. (1.7) of [12]. It should also be noted that an uncubed factor appears in the numerator of any \( j(t_N) \) if and only if the same factor appears in that of \( j'(t_N) \); and that similarly, an unsquared factor appears in the numerator of \( j(t_N) - 12^3 \) if and only if it appears in that of \( j'(t_N) - 12^3 \). Such factors (i.e., their roots) correspond to non-isomorphic self-\( N \)-isogenies of equianharmonic and lemniscatic elliptic curves, respectively. For instance, the roots of \( t_5^2 + 22t_5 + 125 \), the common unsquared factor of the numerators of \( j(t_5) - 12^3 \), \( j'(t_5) - 12^3 \), are bijective with the representations of \( N = 5 \) as a sum of two squares, i.e., \( 5 = 1^2 + 2^2 \) and \( 5 = 2^2 + 1^2 \).

Taken together, Tables 4 and 5 comprise the 14 rationally parametrized modular equations of level 1. When \( N \) is composite, functional decompositions of the coverings \( j = j(t_N), j' = j'(t_N) \) are valuable; these will be given in the next section. But for prime \( N \), the tables are useful as they stand. In the spirit of Mathews [42] and Fricke, one can use them to compute singular moduli and class invariants. The case \( N = 5 \) is typical. Factoring \( j(t_5) - j'(t_5) \) yields some factors that one can show are extraneous, together with the polynomial \( t_5^2 - 125 \), which has roots \( t_5 = \pm 5\sqrt{5} \). There are two because the discriminant \( D = -5 \) has class number 2. The associated values of \( j \), computed from the formula \( j = j(t_5) \), have minimal polynomial over \( \mathbb{Q} \) equal to \( j^2 - 2^25^579j - 880^4 \), which is the class polynomial of \( \mathbb{Q}(\sqrt{-5}) \).
4. Parametrized Modular Equations at Higher Levels

To generate rationally parametrized modular equations at higher levels, one reasons as follows. The invariants \( j, j' \) of the last section, where \( j' = j(N\tau) \), are Hauptmoduln for \( \Gamma(1) \) and the conjugated subgroup \( \Gamma(1)' = w_N^2 \Gamma w_N \) of \( PSL(2,\mathbb{R}) \), where \( w_N = \left( \begin{smallmatrix} N & 0 \\ 0 & 1 \end{smallmatrix} \right) \) is the Fricke involution. Since \( \Gamma(1) \cap \Gamma(1)' = \Gamma_0(N) \), if \( \Gamma_0(N) \) is of genus zero with Hauptmodul \( t_N \) then one must have \( j, j' \in \mathbb{C}(t_N) \).

In fact, quotienting \( \mathcal{H}^* \) yields a pair of covers \( X_0(N)/X(1), X_0(N)/X(1)' \), the projections of which were given in Tables 4 and 5. Similarly, for any \( 1 < d | N \) one has \( \Gamma_0(N) < \Gamma_0(d) \), and the pair of groups \( \Gamma_0(d), \Gamma_0(d)' := w_N^{-1} \Gamma_0(d) w_N \), with Hauptmoduln \( t_d, t'_d \) where \( t'_d(\tau) = t_d((N/d)\tau) \), have intersection \( \Gamma_0(N) \). One must have \( t_d, t'_d \in \mathbb{C}(t_N) \), and a pair of covers \( X_0(N)/X_0(d), X_0(N)/X_0(d)' \).

For the nine composite \( N \) for which \( \Gamma_0(N) \) is of genus zero, the corresponding rational maps \( t_d = t_d(t_N) \) were derived by Gierster [27]; not all are in Fricke. They are listed in the third column of Table 6, modified to agree with our normalization convention. Like the covers \( X_0(N)/X(1), X_0(N)/X(1)' \) for prime \( N \), one can show that the covers \( X_0(N)/X_0(d) \) are uniquely determined by ramification data. This uniqueness enabled Gierster and Fricke to work them out from ‘pictorial’ ramification data, i.e., from figures showing how a fundamental region of \( \Gamma_0(N) \) comprises \( \psi(N)/\psi(d) \) ones of \( \Gamma_0(d) \). Knopp [33, §7.6] gives a linear-algebraic, nonpictorial derivation of the projection map \( t_5 = t_5(t_{25}) \) of the cover \( X_0(25)/X_0(5) \), but most of the others are not well known.

**Proposition 4.1.** For all \( 1 < d | N \) listed in Table 6, the Hauptmodul \( t_d \) is a polynomial (rather than merely rational) function of the Hauptmodul \( t_N \) iff all primes that divide \( N \) also divide \( d \). This occurs if \( \Gamma_0(N) \) is a normal subgroup of \( \Gamma_0(d) \), though the converse does not hold.

**Proof.** For any \( 1 < d | N \), the cusps \( \left[ \frac{1}{1} \right]_d, \left[ \frac{1}{1} \right]_N \) have widths \( d, N \), so if \( \Gamma_0(d), \Gamma_0(N) \) are of genus zero with Hauptmoduln \( t_d, t_N \) having poles at \( \tau = 0 \), the rational function \( t_d = t_d(t_N) \) must take \( t_N = \infty \) to \( t_d = \infty \) with multiplicity \( N/d \). This function has degree \( \psi(N)/\psi(d) \), which equals \( N/d \) iff all primes that divide \( N/d \) also divide \( d \). The final sentence follows from a result of Cummins [15, Prop. 7.1]: \( \Gamma_0(N) \) is normal in \( \Gamma_0(d) \) iff \( (N/d) \not| (d, 24) \). The cases \( d = 2, N = 8, 16 \) show that the converse does not hold.

We computed each of the coverings \( X_0(N)/X_0(d)' \) shown in Table 6 i.e., each formula \( t'_d = t'_d(t_N) \) in the fourth column, by applying \( W_N \) to the corresponding formula \( t_d = t_d(t_N) \), noting that just as \( t_N|_{W_N} = \kappa_N/t_N \), so \( t_d|_{W_N} = \kappa_d/t_d((N/d)\tau) \). The several functional decompositions appearing in columns 3 and 4 have an intuitive explanation. They come from composite covers: e.g., the formula \( (t(t + 16) \circ t(t + 8) \circ t(t + 4))(t_{12}) \) for \( t_{12} \) is associated with the composite cover \( X_0(16)/X_0(8)/X_0(4)/X_0(2) \). It is the composition \( t_{10} \rightarrow t_8 \rightarrow t_4 \rightarrow t_2 \). Distinct paths from \( \Gamma_0(d) \) to \( \Gamma_0(N) \) in the subgroup lattice yield distinct composite covers. For instance, the two distinct representations given for the rational function \( t_2 = t_2(t_{12}) \) come from \( X_0(12)/X_0(6)/X_0(2) \) and \( X_0(12)/X_0(4)/X_0(2) \), respectively. In the same way, the two representations given for \( t_2(6\tau) \), the Hauptmodul for \( \Gamma_0(2)' = w_{12}^{-1} \Gamma_0(2) w_{12} \), in terms of \( t_{12} \), the Hauptmodul for \( \Gamma_0(12) \), come from \( t_{12}(\tau) \rightarrow t_4(3\tau) \rightarrow t_2(6\tau) \) and \( t_{12}(\tau) \rightarrow t_4(2\tau) \rightarrow t_2(6\tau) \).

Taken together, columns 3, 4 of Table 6 list all rationally parametrized modular equations of level greater than unity (the level being \( d \), and the degree \( N/d \). As
The parameter \( \alpha \) of \( \alpha - \beta \) type mentioned in \([1,2]\) Using the formula \( \alpha = t_4/(t_4 + 16) \), and defining \( \alpha := \alpha(\tau), \beta := \alpha((N/d)\tau) \), converts the three equations with \( d = 4 \) (and degrees \( N/d = 2, 3, 4 \)) respectively to

\[
\alpha = \frac{t + 8}{(t + 4)^2}, \quad \beta = \frac{t^2}{(t + 8)^2}; \quad (4.1a)
\]

\[
\alpha = \frac{t + 4}{(t + 2)^3}, \quad \beta = \frac{t^3(t + 4)}{(t + 2)(t + 6)^3}; \quad (4.1b)
\]

\[
\alpha = \frac{t + 8}{(t + 4)^2} \circ t(t + 4), \quad \beta = \frac{t^2}{(t + 8)^2} \circ \frac{t^2}{t + 2}. \quad (4.1c)
\]

The parameter \( t \) signifies \( t_8, t_{12}, t_{16} \), respectively. Equation \( (4.1a) \), of degree 2, is a parametrization of the \( \alpha - \beta \) relation \([1,2]\), and hence of Landen's transformation;
Equation (4.1), of degree 3, is also classical; it was discovered by Legendre and rediscovered by Jacobi. The derivation of Cayley [10, §265] is perhaps the most accessible (His uniformizing parameter is not our $t_{12}$, but rather the alternative Hauptmodul $t_{12}/(t_{12} + 6)$ for $\Gamma_0(12)$; cf. Table 3) Equation (4.1c), of degree 4, is classical too, though it may not have appeared in this form before; it is the basis of the little-known quartic (i.e., biquadratic) arithmetic–geometric mean iteration [7, p. 17]. Cayley’s method of deriving a modular equation of prime degree $p$ for the $\alpha$-invariant is difficult to apply when $p > 3$, and one now sees why: if $p > 3$ then $\Gamma_0(4p)$ is of positive genus, and no rational parametrization exists.

The parametrizations of the classical modular equations $\Phi_N(j, j') = 0$ given in Tables 4 and 5 were quite complicated, and by exploiting Table 6 one can write them in more understandable form. For each $d \mid N$, $j$ is rationally expressible in terms of $t_d$, which in turn is rationally expressible in terms of $t_N$. This expresses $j = j(t_N)$ as a composition, and $j'$ (i.e., $j(N\tau)$) can be similarly expressed. Taking into account the many pairs $d, N$ of Table 6 one obtains Table 7 an improved version of Tables 4 and 5 that displays each functional composition. For example, the three representations given for the projection $j = j(t_{12})$ of $X_0(12)/X(1)$ come from the composite covers $X_0(12)/X_0(6)/X_0(2)/X(1)$, $X_0(12)/X_0(6)/X_0(3)/X(1)$, and $X_0(12)/X_0(4)/X_0(2)/X(1)$, respectively.

No table resembling Table 7 has appeared in print before, and it may prove useful, e.g., in the numerical computation of transformed invariants $j'$. In many cases it clarifies the relation between $j, j'$. It is clear from a glance that the modular equation $\Phi_N(j, j') = 0$ has solvable Galois group for many of the listed values of $N$, but that the group is not solvable if $N = 5, 10, 25$.

The table lists only $j(\tau)$ and $j(N\tau)$ as a function of $t_N(\tau)$, but in fact $j(d\tau)$ is rational in $t_N(\tau)$ for all $d \mid N$, the rational map being the composition $t_N(\tau) \mapsto t_d(\tau) \mapsto j(d\tau)$. The formulas for the cases $1 < d < N$ are easily worked out, and include two that throw light on Weber’s functions. Recall that his functions

$$\gamma_2(\tau) := j^{1/3}(\tau), \quad \gamma_3(\tau) := (j - 12^3)^{1/2}(\tau) \tag{4.2}$$

are useful in the computation of singular moduli. By group theory one can prove that $\gamma_2(3\tau), \gamma_3(2\tau)$ are automorphic under $\Gamma_0(9), \Gamma_0(4)$, respectively [14, §12A]. But by direct computation one obtains from Tables 6 and 7

$$\gamma_2(3\tau) = \frac{(t_2 + 3)(t_2 + 9)(t_2 + 27)}{t_3(t_2^2 + 9t_2 + 27)}, \quad \gamma_3(2\tau) = \frac{(t_4 - 16)(t_4 + 8)(t_4 + 32)}{t_4(t_4 + 16)} \tag{4.3}$$

These are more precise statements about modularity, since they indicate the locations of zeroes and poles.

The first equation in (1.3) also clarifies the modular setting of the Hesse–Dixon family of elliptic curves. Any elliptic curve $E/\mathbb{C}$ has a cubic Hesse model, the function field of which was studied by Dixon [16]; namely,

$$x^3 + y^3 + 1 - (\gamma + 3)xy = 0. \tag{4.4}$$

Here $\gamma \in \mathbb{C} \setminus \{0, 3(\zeta_3 - 1), 3(\zeta_3^2 - 1)\} = \mathbb{C} \setminus \{0, -\frac{9+3\sqrt{3}}{2}\}$ is the Hesse–Dixon parameter. This $\gamma$-invariant may be chosen to be a single-valued function of $\tau \in \mathcal{H}$.

---

1Cayley’s equation relates the value of his invariant $k^2$ at any point $\tau \in \mathcal{H}$ to its value at $3\tau \in \mathcal{H}$. The formula he gives for the Klein invariant $J = j/12^3$ in terms of $k^2$ (see his §300) makes clear that $k^2$ is to be interpreted as our $\alpha$-invariant, i.e., $\alpha(\tau) = \lambda(2\tau)$. Cf. Eq. (1.3).
Table 7. Rationally parametrized modular equations for the $j$-invariant, of all degrees $N$ for which the curve $X_0(N)$ is of genus zero. Here $j$, $j'$ and $t$ signify $j(\tau)$, $j(N\tau)$ and $t_N(\tau)$.

| $N$ | $j$ | $j'$ |
|-----|-----|-----|
| 2   | $(t+16)^3$ | $(t+256)^3$ |
| 3   | $(t+27)(t+9)^3$ | $(t+27)(t+243)^3$ |
| 4   | $(t+16)^3 \circ t(t+16)$ | $(t+256)^3 \circ t^2$ |
| 5   | $(t^2+10t+5)^3$ | $(t^2+250t+3125)^3$ |
| 6   | $\frac{(t+16)^3}{t} \circ \frac{(t+8)^3}{t+9} \circ \frac{(t+9)^2}{t+8}$ | $\frac{(t+27)(t+243)^3}{t^2} \circ \frac{(t^2+250t+3125)^3}{t^2}$ |
| 7   | $\frac{(t^2+13t+49)(t^2+5t+1)^3}{t^2} \circ \frac{(t+8)^3}{t+9} \circ t(t+8)$ | $(t^2+13t+49)(t^2+245t+2401)^3 \circ t^2$ |
| 8   | $\frac{(t+27)(t+9)^3}{t} \circ t(t+27)$ | $(t^2+250t+3125)^3 \circ \frac{t^2}{t+16}$ |
| 9   | $\frac{(t+16)^3}{t} \circ \frac{(t+4)^3}{t+5} \circ t(t+8)$ | $\frac{(t+256)^3}{t^2} \circ \frac{t^2}{t+16} \circ \frac{t^2}{t+4}$ |
| 10  | $\frac{(t+16)^3}{t} \circ \frac{(t+8)^3}{t+9} \circ \frac{(t+9)^2}{t+8}$ | $\frac{(t+27)(t+243)^3}{t^2} \circ \frac{t^2}{t+16} \circ \frac{t^2}{t+4} \circ \frac{t^2}{t+2}$ |
| 11  | $\frac{(t+16)^3}{t} \circ \frac{(t+13)(t^2+20t+19t+11)^3}{t+3} \circ \frac{(t+4)^3}{t+3}$ | $\frac{(t+256)^3}{t^2} \circ \frac{t^2}{t+16} \circ \frac{t^2}{t+4} \circ \frac{t^2}{t+2}$ |
| 12  | $\frac{(t+16)^3}{t} \circ \frac{(t+8)^3}{t+9} \circ \frac{(t+9)^2}{t+8}$ | $\frac{(t+27)(t+243)^3}{t^2} \circ \frac{t^2}{t+16} \circ \frac{t^2}{t+4} \circ \frac{t^2}{t+2}$ |
| 13  | $\frac{(t+16)^3}{t} \circ \frac{(t+13)(t+20t+19t+11)^3}{t+3} \circ \frac{(t+4)^3}{t+3}$ | $\frac{(t+256)^3}{t^2} \circ \frac{t^2}{t+16} \circ \frac{t^2}{t+4} \circ \frac{t^2}{t+2}$ |
| 14  | $\frac{(t+16)^3}{t} \circ \frac{(t+8)^3}{t+9} \circ \frac{(t+9)^2}{t+8}$ | $\frac{(t+27)(t+243)^3}{t^2} \circ \frac{t^2}{t+16} \circ \frac{t^2}{t+4} \circ \frac{t^2}{t+2}$ |
| 15  | $\frac{(t+16)^3}{t} \circ \frac{(t+13)(t+20t+19t+11)^3}{t+3} \circ \frac{(t+4)^3}{t+3}$ | $\frac{(t+256)^3}{t^2} \circ \frac{t^2}{t+16} \circ \frac{t^2}{t+4} \circ \frac{t^2}{t+2}$ |

and the $j$-invariant expressed in terms of it by

$$j(\tau) = \frac{(\gamma + 3)^3(\gamma + 9)^3(\gamma^2 + 27)^3}{\gamma^3(\gamma^2 + 9\gamma + 27)^3}(\tau) = \frac{(\gamma + 3)^3(\gamma^3 + 9\gamma^2 + 27\gamma + 3)^3}{\gamma(\gamma^2 + 9\gamma + 27)}(3\tau). \quad (4.5)$$

In fact, one may take $\gamma(\tau) = t_9(\tau/3)$. (To see all this, compute the $j$-invariant of the model $[4.3]$, obtaining the first equality in (4.3), and notice the equivalence to (4.3).) Since $t_9(\tau/3)$ is a Hauptmodul for a subgroup conjugated to $\Gamma_0(9)$ in $\text{PSL}(2, \mathbb{R})$ by a 3-isogeny, namely $\Gamma(3)$, the Hesse–Dixon model is associated to $\Gamma(3)$, just as the Legendre and Jacobi models are to $\Gamma(2)$ and $\Gamma_0(4)$. This was shown by Beauville [3], but the present derivation is more concrete than his. Since
In terms of radicals, for the case $\tau$ relation cannot be rationally parametrized, since $P_2$ ties satisfied by Gauss–Manin connections, or in classical language, transformation including ‘multipliers’) for the corresponding elliptic families. These are really identi-
are Hauptmodul relations. We now begin the derivation of modular eq uations (in-
t point certain weight-1 modular forms $h$.

cial function identities, we shall develop them in a very concrete way , by defining

genus zero.

\section{The Canonical Weight-1 Modular Form $h_N(\tau)$ for $\Gamma_0(N)$}

The preceding sections focused on the canonical Hauptmoduln $t_N$ for the genus-
zero congruence subgroups $\Gamma_0(N)$ of $\Gamma(1)$, and on $t_N-t'_N$ modular equations, which are Hauptmodul relations. We now begin the derivation of modular equations (in-
cluding ‘multipliers’) for the corresponding elliptic families. These are really identi-
ties satisfied by Gauss–Manin connections, or in classical language, transformation laws for Picard–Fuchs equations. But because of our interest in $2F_1$ and other special function identities, we shall develop them in a very concrete way, by defining certain weight-1 modular forms $h_N(\tau)$ that are (multivalued) functions of the corre-
spending Hauptmoduln, according to $h_N(\tau) = h_N(t_N(\tau))$. Each $h_N$ is a solution of a normal-form Picard–Fuchs equation. Each modular form $h_N$, or equivalently the multivalued function $h_N$, is defined by a pullback along $X_0(N) \to X(1)$.

\begin{definition}
If $\Gamma < \Gamma(1)$ is of genus zero with Hauptmodul $t$, and $j$ equals $P(t)/Q(t)$ with $P, Q \in \mathbb{C}[t]$ having no factor of positive degree in common, and $t$ equals zero at the cusp $\tau = i\infty$ (so that $Q(0) = 0$), then in a neighborhood of the point $t = 0$ on the quotient $\Gamma \setminus \mathcal{H}^*$, the holomorphic function $h_{\Gamma,t}$ is defined by

$$h_{\Gamma,t}(t) = [P(t)/P(0)]^{-1/12} 2F_1 \left( \frac{1}{12}, \frac{5}{12}; 1; 12^3Q(t)/P(t) \right).$$

This definition of $h_{\Gamma,t}$ is unaffected by the Hauptmodul $t$ being replaced by any nonzero scalar multiple, i.e., $h_{\Gamma,t} = h_{\Gamma,\alpha t}$ for any nonzero $\alpha$.

For useful facts on the Gauss function $2F_1$, see the appendix. Here it suffices to know that $2F_1(\frac{1}{12}, \frac{5}{12}; 1; z)$ equals unity at $z = 0$ and is holomorphic on the unit disk. It has a branch point of square-root type at $z = 1$, but can be holomorphically extended to $\mathbb{P}^1(\mathbb{C})_z$, slit along the positive real axis from $z = 1$ to $z = \infty$.

\begin{definition}
For notational simplicity, let $\hat{h}_1 := h_{\Gamma(1),J}$ where $J := 1/J = 12^3/j$ is the abovementioned alternative Hauptmodul for $\Gamma(1)$, which equals zero at the cusp $\tau = i\infty$. Similarly, let $h_N := h_{\Gamma_0(N),t_N}$. So if $j = P_N(t_N)/Q_N(t_N)$, then

$$\hat{h}_1(J(\tau)) = 2F_1 \left( \frac{1}{12}, \frac{5}{12}; 1; J(\tau) \right),$$

$$h_N(t_N(\tau)) = [P_N(t_N(\tau))/P_N(0)]^{-1/12} \hat{h}_1(J(\tau)),$$

on a neighborhood of the point $\tau = i\infty$ in $\mathcal{H}^*$.
By examining Table 4 one sees that for each \( N \), the function \( j = j(t_N) \) maps the positive real axis \( t_N > 0 \) into the interval \( 12^3 \leq j < \infty \). That is, if \( 0 < t_N < \infty \) then \( \hat{J} = 12^3/j \) satisfies \( 0 < \hat{J} \leq 1 \). Moreover, at each point \( t_N = t'_N > 0 \) at which \( j = 12^3 \), i.e., \( \hat{J} = 1 \), the behavior of \( j - 12^3 \) is quadratic. (The point \( t_2 = 8 \) on \( X_0(2) \) is an example.) It follows that \( h_N = h_N(t_N) \), which as defined above is holomorphic at \( t_N = 0 \) and equal to unity there, has a real holomorphic continuation along the positive real axis \( t_N > 0 \).

A connection with differential equations is made by Theorems 5.1 and 5.2 below, which are standard [24, 59]. First, recall some facts on Fuchsian differential operators and equations. A second-order Fuchsian differential operator of \( \hat{L}u = \alpha_m \) differences may be reduced to one in normal form. Any substitution \( u = f^{-\alpha} \hat{u} \), where \( f \in \mathbb{C}(t) \) and \( \alpha \in \mathbb{C} \), will transform the equation \( \hat{L}u = 0 \) to \( \hat{L} \hat{u} = 0 \), where \( \hat{L} \) has transformed coefficients \( \hat{A}, \hat{B} \in \mathbb{C}(t) \). Any such 'index transformation' leaves exponent \emph{differences} invariant: the exponents of \( \hat{L} \) at any finite point \( t = t_0 \) at which \( f \) has order of vanishing \( m \) will be those of \( \hat{L} \), shifted down by \( \alpha m \). By index transformations, any Fuchsian operator on \( \mathbb{C}(t) \) may be reduced to one in normal form.

**Theorem 5.1.** Let \( \Gamma \) be a Fuchsian subgroup \emph{(of the first kind)} of the automorphism group \( \text{PSL}(2, \mathbb{R}) \) of \( \mathcal{H} \supset \tau \) for which the quotient curve \( X := \Gamma \backslash \mathcal{H}^* \) is of genus zero, and let \( t \) denote a Hauptmodul. Then in a neighborhood of any point on \( X \), any branch of \( \tau \), which can be viewed as a multivalued function on \( X \), will equal the ratio of two independent solutions of some second-order Fuchsian differential equation \( \hat{L}u := (D_t^2 + \hat{A} \cdot D_t + \hat{B})u = 0 \), where \( \hat{A}, \hat{B} \in \mathbb{C}(t) \). The space of
local solutions of this Picard–Fuchs equation will be \( [C \tau(\cdot) + C] H(\cdot) \), where \( H \) is some particular local solution. One can choose \( L \) so that its singular points are the fixed points of \( \Gamma \) on \( X \), with the difference of characteristic exponents equaling \( 1/k \) at each fixed point of order \( k \), and zero at each parabolic fixed point (i.e., cusp).

This is a special (e.g., genus zero) case of a classical theorem dealing with Fuchsian automorphic functions of the first kind [24, §44, Thm. 15]. It does not require that \( \Gamma \) be a subgroup of \( \Gamma(1) = \text{PSL}(2, \mathbb{Z}) \). The following theorem is also classical.

**Theorem 5.2.** For any first-kind Fuchsian subgroup \( \Gamma \) and Hauptmodul \( t \), if two Picard–Fuchs equations of the form \((D_t^2 + A \cdot D_t + B)u = 0\) have the same characteristic exponents (not merely exponent differences) at each singular point, then they must be equal.

**Corollary 5.2.1.** Requiring the Picard–Fuchs equation mentioned in the last sentence of Theorem 5.1 to be in normal form determines it uniquely.

**Proof.** If any finite point \( t \in \Gamma \setminus \mathbb{H}^* \cong \mathbb{P}^1(\mathbb{C}) \) is a cusp, its exponents will be \( 0, 0 \); if it is a quadratic (resp. cubic) elliptic fixed point, they will be \( 0, 1/2 \) (resp. \( 0, 1/3 \)). The exponents at \( t = \infty \) are uniquely determined by Fuchs’s relation: the sum of all \( 2k \) characteristic exponents of any second-order Fuchsian differential equation with \( k \) singular points on \( \mathbb{P}^1(\mathbb{C}) \) must equal \( k - 2 \).

**Theorem 5.3.** If \( \Gamma = \Gamma(1) \), so that \( X = X(1) \), and the Hauptmodul \( t \) equals \( \hat{J} \), then in a neighborhood of the cusp \( \hat{J} = 0 \) (i.e., \( \tau = i\infty \)), the unique normal-form Picard–Fuchs equation is the Gauss hypergeometric equation with parameters \( a = \frac{1}{12}, b = \frac{1}{12}, c = 1 \) and independent variable \( \hat{J} \), i.e.,

\[
\hat{L}_1 u := \hat{L}_{\hat{J}, \frac{1}{12}, \frac{1}{12}, 1} u = \left\{ D_{\hat{J}}^2 + \left[ \frac{1}{2} + \frac{1}{2(\hat{J}-1)} \right] D_{\hat{J}} + \frac{5/144}{\hat{J}(\hat{J}-1)} \right\} u = 0.
\]

The fundamental local solution \( H = H(\hat{J}) \) can be taken to be \( \hat{h}_1(\hat{J}) \) as defined above, i.e., \( 2F_1\left(\frac{1}{12}, \frac{1}{12}, 1; \hat{J}\right) \), the unique local solution of \( \hat{L}_1 u = 0 \) which is holomorphic at \( \hat{J} = 0 \), up to normalization. Also, the following connection to the theory of modular forms exists: in a neighborhood of the cusp \( \tau = i\infty \),

\[
E_4(\tau) = \hat{h}_4(\hat{J}(\tau)),
\]

\[
E_6(\tau) = [1 - \hat{J}(\tau)]^{1/2} \hat{h}_6(\hat{J}(\tau)),
\]

\[
\Delta(\tau) = (2\pi)^{12}12^{-3}(E_4^3 - E_6^2)(\tau) = (2\pi)^{12}12^{-3}\hat{J}(\tau) \hat{h}_4^3(\hat{J}(\tau)),
\]

where \( E_4, E_6, \Delta \) are the classical Eisenstein sums and modular discriminant.

**Proof.** The Fuchsian equation \( \hat{L}_1 u = 0 \) is in normal form, with exponents \( 0, 0 \) at \( \hat{J} = 0 \), \( 0, \frac{1}{2} \) at \( \hat{J} = 1 \), and \( \frac{1}{12}, \frac{5}{12} \) at \( \hat{J} = \infty \); so its exponent differences are \( 0 \) at the cusp \( \tau = i\infty \), \( \frac{1}{2} \) at the quadratic elliptic point \( \tau = i \), and \( \frac{1}{12} \) at the cubic elliptic point \( \tau = \zeta_3 \), in agreement with Theorem 5.1. Up to shifts of exponents, any hypergeometric equation is uniquely determined by its exponent differences, since it contains no accessory parameters; hence the first sentence of the theorem follows. The remaining two are due to Stiller [59] (another derivation of the expression for \( \Delta \) in terms of \( \hat{J} \) and \( \hat{h}_1 \) will be mentioned below).

**Theorem 5.4.** If \( \Gamma = \Gamma_0(N) \), so that \( X = X_0(N) \), and the Hauptmodul \( t \) equals \( t_N \), then in a neighborhood of the cusp \( t_N = 0 \) (i.e., \( \tau = i\infty \)), the unique normal-form Picard–Fuchs equation \( \mathcal{L}_N u = 0 \) has
(1) one singular point with characteristic exponents $\frac{1}{12}\psi(N), \frac{1}{12}\psi(N)$, viz., the

cusp $t_N = \infty$ (i.e., $\tau = 0$);

(2) $\sigma_\infty(N) - 1$ singular points with exponents 0, 0, viz., the remaining cusps,

including $t_N = 0$ (i.e., $\tau = i\infty$);

(3) $\epsilon_2(N)$ singular points with exponents $0, \frac{1}{2}$, viz., the order-2 elliptic fixed

points;

(4) $\epsilon_3(N)$ singular points with exponents $0, \frac{1}{3}$, viz., the order-3 elliptic fixed

points;

and is the equation on $X_0(N)$ obtained by (i) pulling back $\hat{L}_1 u = 0$ to $X_0(N)$ along

the covering map $j = P_N(t_N)/Q_N(t_N)$ of $X_0(N)/X(1)$, and (ii) performing the

substitution $\hat{u} = P_N(t_N)^{-1/12} u$. The local solution $H = H(t_N)$ can be taken to be

$h_N = h_N(t_N)$ as defined above. It is the unique local solution of $\mathcal{L}_N u = 0$ that is

holomorphic at the cusp $t_N = 0$ and equals unity there.

Proof. Pulling back from $X(1)$ to $X_0(N)$ and performing the indicated substitution

will not remove the property that the Picard–Fuchs equation should have; namely,

that any branch of $\tau$ should equal the ratio of two of its solutions. So, all that needs

to be proved are the statements about the exponents of the resulting operator $\mathcal{L}_N$;

and the final two sentences of the theorem.

At any point $t_N \in X_0(N)$ at which $J = J(t_N) = 12^3 Q_N(t_N)/P_N(t_N)$, the

covering map, has ramification index $k$, the pullback $(\hat{L}_1)^* \cdot \hat{L}_1$ has exponents

equal to $k$ times those of $\hat{L}_1$ at $J(t_N)$. So at each cusp of $X_0(N)$, i.e., at each point

on the fibre above $J = 0$, the pulled-back operator will have exponents 0, 0.

The points on $X_0(N)$ above $J = 1$ are partitioned into the $\epsilon_2(N)$ order-2 elliptic

fixed points of $\Gamma_0(N)$ (at which $k = 1$), and non-fixed points at which $k = 2$ [54].

The corresponding exponents will be $0, \frac{1}{2}$ and 0, 1, so the latter will be ordinary

(non-singular) points of $(\hat{L}_1)^*$. Similarly, the points on $X_0(N)$ above $J = 0$ are

partitioned into the $\epsilon_3(N)$ order-3 elliptic fixed points of $\Gamma_0(N)$ (at which $k = 1$)

and non-fixed points at which $k = 3$. The corresponding exponents will be $\frac{1}{12}, \frac{5}{12}

and \frac{1}{2}, \frac{2}{3}$. Since the polynomial $P_N$ has a simple root at each of the former and

a triple root at each of the latter, performing the substitution $\hat{u} = P_N(t_N)^{-1/12} \hat{u}$

will shift these exponents to $0, \frac{1}{2}$ and 0, 1 respectively; so the non-fixed points will

become ordinary. It will also, since $\deg P_N = \psi(N)$, shift the exponents at $t_N = \infty$

from 0, 0 to $\frac{1}{12} \psi(N), \frac{1}{12} \psi(N)$.

If the operator $\mathcal{L}_N$ resulting from the substitution is taken to be monic, then

$\mathcal{L}_N = P_N(\cdot)^{-1/12} (\hat{L}_1)^* P_N(\cdot)^{1/12}$. But by the definition of a pullback,

$$(\hat{L}_1)^* \left[ 2F_1(\frac{1}{12}, \frac{5}{12}; 1; J(\cdot)) \right] = 0.$$  \hspace{1cm} (5.2)

So

$$\mathcal{L}_N[h_N] = \mathcal{L}_N \left[ P_N^{-1/12}(\cdot) \left[ 2F_1(\frac{1}{12}, \frac{5}{12}; 1; J(\cdot)) \right] \right] = 0.$$  \hspace{1cm} (5.3)

The final sentence of the theorem now follows by the general theory of Fuchsian differen-
tial equations: since $\mathcal{L}_N$ has exponents 0, 0 at $t_N = 0$, its space of holomorphic

solutions there is one-dimensional, and must accordingly be $\mathbb{C} h_N$. The penultimate

sentence of the theorem follows by examination. \qed

Theorems 5.3 reveals why the prefactor $[P(t)/P(0)]^{-1/12}$ was included in Definition 5.1.

If it were absent, then the Picard–Fuchs equation satisfied by $h_N$ would not be in normal form.
In the theory of conformal mapping, there is another type of normal-form Picard–Fuchs equation in widespread use \([24,29]\). If \(t\) is a Hauptmodul for a first-kind Fuchsian subgroup of \(PSL(2,\mathbb{R})\) as in Theorem 5.1, any branch of \(\tau\) will equal the ratio of two independent solutions of the second-order Fuchsian differential equation 
\[D^2_t + Q(t)\] 
v = 0, where \(Q \in C(t)\) is defined by \(Q := \frac{1}{2}\{\tau, t\}\), with \(\{\cdot, \cdot\}\) the Schwarzian derivative. This is a self-adjoint Picard–Fuchs equation, and its space of local solutions is \([C\tau(-) + C](dt/dr)^{1/2}\). The reader may wonder why we introduced a non-self-adjoint normal form, instead. It is because a Picard–Fuchs equation in our normal form is more convenient for deriving special function identities. Though asymmetric, it does permit an elegant modular interpretation of its fundamental solution \(h_N\), as the following theorem and corollaries indicate.

**Theorem 5.5.** For each \(N\) with \(\Gamma_0(N)\) of genus zero, if \(h_N = h_N(t_N)\) is the holomorphic function defined above on a neighborhood of the point \(t_N = 0\), and \(h_N = h_N(\tau)\) is defined on a neighborhood of the infinite cusp \(\tau = i\infty\) by \(h_N(\tau) := h_N(t_N(\tau))\), then \(h_N\) extends to \(H^* \ni \tau\) by continuation, yielding a weight-1 modular form for \(\Gamma_0(N)\), with some multiplier system. In particular,

\[
h_N(\tau) := h_N(t_N(\tau)) = P_N(0)^{1/12}Q_N(t_N(\tau))^{-1/12}\eta^2(\tau). \tag{5.4}
\]

This modular form is regular and non-vanishing at each cusp in \(P^1(\mathbb{Q})\) other than those in the class \([\frac{1}{1}]_N \ni 0\), at each of which its order of vanishing is \(\psi(N)/12N\).

**Proof.** In a neighborhood of the infinite cusp, it is a striking fact that the holomorphic function \(h_1 = \frac{2}{3}F_1(\frac{1}{2}, \frac{1}{2}; 1; J)\) of \(J\) can be expressed in terms of its \(J\)-invariant and the eta function, as \(12^{1/4}J^{-1/12}\eta^2\). This representation was known to Dedekind \([11, \text{p. 137}]\), and was rediscovered by Stiller \([59]\). Formula (5.4) follows from it, since \(J(\tau) = 12^4Q_N(t_N(\tau))/P_N(t_N(\tau))\). (Incidentally, the representation for the discriminant \(\Delta = \Delta(\tau)\) in terms of \(J\) and \(h_1\) given in Theorem 5.3 also follows from it, since \(\Delta = (2\pi)^{12}\eta^{24}\).) Since the factors \(Q_N(t_N(\tau))^{-1/12}\) and \(\eta^2(\tau)\) are holomorphic on \(H^* \ni \tau\), \(h_N(t_N(\tau))\) can be continued from the neighborhood of \(\tau = i\infty\) on which it was originally defined, to all of \(H\). As a modular form it has weight 1, since \(h\) has weight 1/2.

The roots of the polynomial \(Q_N\) correspond to the cusps of \(\Gamma_0(N)\) on \(X_0(N)\) other than \(t_N = \infty\), i.e., to the cusp classes \([\frac{0}{1}]_N \subset P^1(\mathbb{Q})\) other than \([\frac{1}{1}]_N\). Any root \(t_N = t^*\) corresponding to \([\frac{0}{1}]_N\) appears in \(Q_N\) with multiplicity equal to the cusp width \(c_{d,N} = N/d(N, N/d)\). But this is also the multiplicity with which \(t_N = t^*\) is mapped to \(X(1) \cong P^1(\mathbb{C})\) by the covering map \(j = P_N(t_N)/Q_N(t_N)\). So the order of vanishing of \(Q_N(t_N(\tau))\) at any cusp not in \([\frac{1}{1}]_N\) is unity, and (as the order of \(\eta(\tau)\) at any cusp \(\tau \in P^1(\mathbb{Q})\) is \(\frac{1}{12}\)), the order of \(Q_N(t_N(\tau))^{-1/12}\eta^2(\tau)\) will be \(\frac{1}{12} + 2 \cdot \frac{1}{12} = 0\). Definition 5.2 implies that at any cusp in \([\frac{1}{1}]_N\), the order of \(h_N(t_N(\tau))\) will be \(\frac{1}{12}(\deg P)/e_{1,N} = \psi(N)/12N\).

**Corollary 5.5.** \(t_N, h_N\) have divisors \([\frac{1}{1}]_N \rightarrow \frac{1}{12}([\frac{1}{1}]_N), \frac{\psi(N)}{12}([\frac{1}{1}]_N), \frac{\psi(\tau)}{12}([\frac{1}{1}]_N), \) if viewed as functions on \(H^* \ni \tau\). If viewed respectively as a univalent and a multivalued function on \(X_0(N)\), they have divisors \((t_N = 0) \rightarrow (t_N = \infty)\) and \(\frac{\psi(N)}{12}, (t_N = \infty)\).

**Remark.** The term ‘divisor’ is used in a generalized sense here, referring to an element of a free \(\mathbb{Q}\)-module, rather than a free \(\mathbb{Z}\)-module. The coefficients of a generalized divisor are orders of vanishing, which may not be integers.
Proof. That the divisor of \( h_\tau(t) = h_N(t_N(\tau)) \) is \( \frac{6(N)}{2\pi i} \cdot (\mathbf{1}_N) \) follows from the last sentence of the theorem, since \( h_N \) has no zeroes or poles on \( \mathcal{H} \). That \( t_\tau \), as a function on \( X_0(N) \), has the stated divisor is trivial. The remaining statements follow by taking the cusps widths \( e_{1,N} = N \) and \( e_{N,N} = 1 \) into account.

Corollary 5.5.2. For each \( N \) with \( \Gamma_0(N) \) of genus zero, the weight-1 modular form \( h_N(\tau) = h_N(t_N(\tau)) \) for \( \Gamma_0(N) \) has the alternative representation

\[
\prod_{i=1}^{\epsilon_2(N)} \left[ 1 - \frac{t_N(\tau)}{t_{N,i}^{(2)}} \right]^{-1/4} \prod_{i=1}^{\epsilon_3(N)} \left[ 1 - \frac{t_N(\tau)}{t_{N,i}^{(3)}} \right]^{-1/3} \prod_{i=1}^{\sigma_\infty(N)-2} \left[ 1 - \frac{t_N(\tau)}{t_{N,i}^{(\infty)}} \right]^{-1/2} \times t_N^{-1/2(\tau)} \left[ \frac{1}{2\pi i} \frac{dt_N(\tau)}{d\tau} \right]^{1/2},
\]

in which the three products run over (i) the quadratic fixed points \( t_N = t_{N,i}^{(2)} \), \( i = 1, \ldots, \epsilon_2(N) \), (ii) the cubic fixed points \( t_N = t_{N,i}^{(3)} \), \( i = 1, \ldots, \epsilon_3(N) \), and (iii) the cusps \( t_N = t_{N,i}^{(\infty)} \), \( i = 1, \ldots, \sigma_\infty(N)-2 \), other than the two distinguished cusps \( t_N = 0 \) (i.e., \( \tau = \infty \)) and \( t_N = \infty \) (i.e., \( \tau = 0 \)). These fixed points on \( X_0(N) \) are given in Table 2.

Remark. This result clarifies the connection between two Picard–Fuchs equations for \( \Gamma_0(N) \): the self-adjoint one noted above, of the form \( [D_N^2 + \mathcal{Q}_N(t_N)]v = 0 \), and the normal-form one \( \mathcal{L}_Nu = 0 \). They are related by a similarity transformation, and in fact are projectively equivalent, since their respective solution spaces are \([\mathbb{C}\tau(\cdot) + \mathbb{C}][dt_N/d\tau]^{1/2}(\cdot)\) and \([\mathbb{C}\tau(\cdot) + \mathbb{C}]h_N(\cdot)\). The given representation for \( h_N(t_N(\tau)) \) yields a formula for \( u/v \), the quotient of their solutions.

Proof. Despite the disconcerting presence of fractional powers, the given expression extends from a neighborhood of \( \tau = \infty \) to a single-valued function of \( \tau \) in \( \mathcal{H} \), without zeroes or poles. To see this, begin by examining \( \tau = \zeta_3 \), at which \( t_N = t_{N,i}^{(3)} \), some cubic fixed point on \( X_0(N) \). To leading order one has \( j \sim C(\tau - \zeta_3)^3 \) near \( \tau = \zeta_3 \), and also \( j \sim C'(t_N - t_{N,i}^{(3)}) \). So \( [dt_N/d\tau]^{1/2} \sim C''(\tau - \zeta_3) \) and \([1 - t_N/t_{N,i}^{(3)}]^{-1/3} \sim C'''(\tau - \zeta_3)^{-1} \), implying finiteness and single-valuedness of their product near \( \tau = \zeta_3 \). This extends to points \( \tau \in \mathcal{H} \) congruent to \( \zeta_3 \). Quadratic fixed points can be handled similarly.

The expression also has a nonzero, non-infinite limit as any cusp \( \tau \in \mathbb{P}^1(\mathbb{Q}) \) not congruent to \( \tau = 0 \) under \( \Gamma_0(N) \) is approached. The limit as \( \tau \to \infty \) is easily seen to be unity, since \( t_N \sim C^{(\infty)}q \) to leading order. Cusps \( \tau \in \mathbb{P}^1(\mathbb{Q}) \) ‘above’ the \( \sigma_\infty(N) - 2 \) parabolic fixed points \( t_{N,i}^{(\infty)} \), i.e., cusps not congruent to \( \tau = \infty \) or \( \tau = 0 \), are almost as easy to handle: one expands to leading order, as in the last paragraph, and in each case finds a nonzero, non-infinite limit.

When \( \tau \to 0, t_N \to \infty \), yielding leading-order behavior \( \sim C^{(\nu)}t_N^{-a} \), where the exponent \( a \) equals \( \frac{1}{4}\epsilon_2(N) + \frac{1}{3}\epsilon_3(N) + \frac{1}{3}\sigma_\infty(N) - 1 \). From the genus formula \( \langle 2,3 \rangle \), one deduces \( a = \frac{1}{12}\psi(N) \). By the preceding corollary, the quotient of the given expression and \( h_N(t_N(\tau)) \) must tend to a nonzero, non-infinite limit as any cusp congruent to \( \tau = 0 \) is approached; and by the last paragraph, the same is true of any other cusp. The quotient must therefore be a nonzero constant. By considering the limit \( \tau \to \infty \), one sees that it equals unity.

□
It is natural to wonder whether the function $h_N = h_N(t_N)$ alone, rather than the two-dimensional space of solutions $[C\tau(\cdot) + C]h_N(\cdot)$ of the Picard–Fuchs equation, supplies a means of computing the multivalued function $\tau$ on $X_0(N)$. It is easiest to consider the half-line $t_N > 0$, since $h_N$ extends holomorphically to it. For each of the 14 values of $N$, $t_N(\tau) \sim \kappa_N \cdot q$ to leading order as $\tau \to \infty$, and accordingly $\tau \sim \frac{1}{N} \log(1/t_N)$ as $t_N \to 0^+$. By choosing the principal branch of the logarithm one obtains a unique function $\tau = \tau(t_N)$, $t_N > 0$. It is evident that $\tau/i$ is positive and monotone decreasing on $t_N > 0$. In fact it decreases to zero as $t_N \to \infty$, since $\tau$ must satisfy $\tau(\kappa_N/t_N) = -1/[N\tau(t_N)]$, which follows from the identity $t_N(\tau)t_N(-1/N\tau) = \kappa_N$ introduced in \cite{3}

Theorem 5.7 below reveals how the function $\tau = \tau(t_N)$, $t_N > 0$, can be computed from the corresponding function $h_N = h_N(t_N)$ on the half-line.

Lemma 5.6. For each $N$, the Picard–Fuchs equation $\mathcal{L}_Nu = 0$ is invariant under the substitution $\tilde{u}(t_N) = t_N^{-\psi(N)/12}u_N(\kappa_N/t_N)$. That is, $\mathcal{L}_N\tilde{u} = 0$.

Proof. Suppose that applying the substitution to $\mathcal{L}_Nu = 0$ yields a transformed equation $\tilde{\mathcal{L}}\tilde{u} = 0$. Since $t_N(\tau)t_N(-1/N\tau) = \kappa_N$, the transformed equation has the defining property of a Picard–Fuchs equation; namely, that any branch of the multivalued function $\tau$ is locally a quotient of two of its solutions. But the map $t_N \mapsto \kappa_N/t_N$ (i.e., $\tau \mapsto -1/N\tau$) is the Fricke involution, which separately involves cusps, quadratic elliptic points, and cubic elliptic points. So the transformed operator $\tilde{\mathcal{L}}_N = D_N^2 + \tilde{A}(t_N)D_{t_N} + \tilde{B}(t_N)$ must have, like $\mathcal{L}_N$, exponents 0, 0 at each cusp other than $t_N = \infty$, exponents $0, \frac{1}{2}$ at each quadratic point, and $0, \frac{1}{2}$ at each cubic point. (The factor $t_N^{-\psi(N)/12}$ adjusts the exponents at $t_N = 0$ to be 0, 0, and those at $t_N = \infty$ to be $\frac{1}{12}\psi(N), \frac{1}{12}\psi(N)$.) Since the exponents of $\tilde{\mathcal{L}}_N$ and $\mathcal{L}_N$ agree, $\tilde{\mathcal{L}}_N = \mathcal{L}$ by Theorem 5.2.

Theorem 5.7. For each $N$ with $\Gamma_0(N)$ of genus zero, on $t_N > 0$ one has that

$$\tau(t_N)/i = A_N t_N^{-\psi(N)/12} \frac{h_N(\kappa_N/t_N)}{h_N(t_N)},$$

for some $A_N > 0$. Here $h_N, \tau/i$ are the single-valued holomorphic continuations introduced above, which are real and positive on the half-line $t_N > 0$.

Proof. Let $\tau_1 := \tau h_N$ and $\tau_2 := h_N$. These are solutions of the Picard–Fuchs equation $\mathcal{L}_Nu = 0$, with $\tau = \tau_1/\tau_2$. Since $\mathcal{L}_N$ has exponents 0, 0 at $t_N = 0$, solutions of $\mathcal{L}_Nu = 0$ must be asymptotic to const \times $\log(1/t_N)$ or to const, as $t_N \to 0^+$. The solutions $\tau_1, \tau_2$ are of these two types, respectively.

Similarly, since $\mathcal{L}_N$ has exponents $\frac{1}{12}\psi(N), \frac{1}{12}\psi(N)$ at $t_N = \infty$, solutions must be asymptotic to $\text{const} \times t_N^{-\psi(N)/12} \log t_N$ or to $\text{const} \times t_N^{-\psi(N)/12}$ as $t_N \to \infty$. Consider in particular $\tilde{\tau}_i(t_N) := t_N^{-\psi(N)/12} \tau_i(\kappa_N/t_N)$, for $i = 1, 2$, which by Lemma 5.6 are solutions. Their ratio $\tilde{\tau}_1/\tilde{\tau}_2(t_N) = \tau(\kappa_N/t_N) = -1/[N\tau(t_N)]$ converges to zero as $t_N \to 0^+$, so it must be the case that $\tilde{\tau}_1(t_N)$ \sim const and $\tilde{\tau}_2(t_N)$ \sim const \times $\log(1/t_N)$, as $t_N \to 0^+$. Hence $\tau_1(t_N) \sim \text{const} \times t_N^{-\psi(N)/12}$ and $\tau_2(t_N) \sim \text{const} \times t_N^{-\psi(N)/12} \log t_N$ as $t_N \to \infty$. 

\[ \text{(Proof completes)} \]
last, we obtain interesting combinatorial identities that extend those of Fine [22].

In fact specified by a Dirichlet character \( \chi \) to the modulus \( N \). In this section, we make \( \eta_N(\tau) \) much more concrete by (i) expressing it as an eta product, (ii) determining (with much labor!) its multiplier system, and (iii) working out its \( q \)-expansion. As a byproduct of the last, we obtain interesting combinatorial identities that extend those of Fine [22].

For \( \tau(\kappa_N/t_N) = -1/|N \tau(t_N)| \) to be true, it must in fact be the case that

\[
\tau_1(t_N) = i A_N \tilde{\tau}_2(t_N) = i A_N t_N^{-\psi(t_N)/2} \tau_2(\kappa_N/t_N),
\]

\[
\tau_2(t_N) = B_N \tilde{\tau}_1(t_N) = B_N t_N^{-\psi(t_N)/2} \tau_1(\kappa_N/t_N),
\]

with \( A_N = B_N/N \), where by examination \( A_N, B_N \) are real and positive. Since \( \tau = \tau_1/\tau_2 \) and \( \tau_2 = h_N \), the claimed identity follows.

6. The Modular Form \( \eta_N(\tau) \): Explicit Computations (I)

In [5] the canonical weight-1 modular form \( \eta_N(\tau) = h_N(t_N(\tau)) \) was defined for each genus-zero group \( \Gamma_0(N) \). In this section, we make \( \eta_N(\tau) \) much more concrete by (i) expressing it as an eta product, (ii) determining (with much labor!) its multiplier system, and (iii) working out its \( q \)-expansion. As a byproduct of the last, we obtain interesting combinatorial identities that extend those of Fine [22].

For each \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \Gamma_0(N) \), the inhomogeneous version of \( \Gamma_0(N) \), one must have \( (h_N \circ t_N) \left( \frac{a \tau + b}{c \tau + d} \right) = \tilde{\chi}_N(a, b, c, d) \cdot (\tau + d) (h_N \circ t_N)(\tau) \), where the function \( \tilde{\chi}_N : \tilde{\Gamma}_0(N) \to U(1) \) is the multiplier system. The simplest case is when \( \tilde{\chi}_N \) depends only on \( d \), i.e., \( \tilde{\chi}_N(a, b, c, d) = \chi_N(d) \), and \( \chi_N \) has period \( N \). In this case \( \chi_N \) is a homomorphism of \( (\mathbb{Z}/N) \) to \( U(1) \), i.e., is a Dirichlet character to the modulus \( N \). It must be an odd function, since the Möbius transformation \( \tau \mapsto \frac{a \tau + b}{c \tau + d} \) is unaltered by negating the matrix \( \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \). So, e.g., \( \chi_N(-1) \) must equal \(-1 \).

What remains to be determined is whether the multiplier system of each \( \eta_N \) is in fact specified by a Dirichlet character \( \chi_N \); which may be imprimitive, with its fundamental period, or conductor, equal to a proper divisor of \( N \).
Theorem 6.1. For each $N$ with $\Gamma_0(N)$ of genus zero, the eta product representations of the canonical Hauptmodul $t_N$ and weight-1 modular form $h_N$ are as shown in Table 3. When $N = 6, 8, 9, 12, 16, 18$, the multiplier system of $h_N$ is given by the unique odd, $\pm 1$-valued Dirichlet character with the stated conductor. When $N$ is one of the other eight values, it is not of Dirichlet type.

Remark. If $N = 6, 9, 12, 18$, then $\chi_N(d) = \left(\frac{-1}{d}\right) = \left(\frac{d}{N}\right)$; and if $N = 8, 16$, then $\chi_N(d) = \left(\frac{-3}{d}\right) = \left(\frac{1}{d}\right)$. The non-Dirichlet cases are $N = 2, 3, 4$; and $N = 5, 7, 10, 13, 25$, when the eta product for $h_N$ contains fractional powers.

Proof. The eta products for the $t_N$ are taken from Table 2 but those for the $h_N$ are new. They come from Theorem 5.3. The polynomial $Q_N(t_N)$ in that theorem is simply the denominator of the rational expression for $j$ in terms of $t_N$, given in Table 4. If $\Gamma_0(N)$ has no irrational cusps then $Q_N$ factors over $\mathbb{Q}$ into linear factors, eta products for which follow from Tables 2 and 3. For example, $Q_4(t_4) = t_4(t_4 + 16)$, and by Tables 2 and 3 on $H^+$ the functions $t_4, t_4 + 16$ have respective eta-representations $2^8 \cdot [4]^6/[1]^{10}, 2^4 \cdot [2]^{24}/[1]^{16}[4]^8$, yielding $h_4(t_4(t_4)) = [1]^{10}/[2]^2$ as claimed.

When $\Gamma_0(N)$ has irrational cusps, i.e., when $N = 9, 16, 18, 25$, one must work harder, since $Q_N(t_N)$ contains at least one higher-degree factor. For all but $N = 25$ these are bivalent functions on $X_0(N)$ with zeroes at a conjugate pair of irrational cusps. Fortunately, an eta product for each factor follows from Table 6. For example, the factor $t_9^3 + 9t_9 + 27$ of $Q_9(t_9)$ equals $t_9^3/t_9$, i.e., $3^6 \cdot [3]^{12}/[1]^{10}[9]^3$. In all, the multivalent functions that appear are:

\[
\begin{align*}
& t_9^2 + 9t_9 + 27 = 3^3 \cdot [3]^{12}/[1]^{10}[9]^3, \\
& t_{16}^2 + 4t_{16} + 8 = 2^3 \cdot [4]^6/[1]^{12}[2]^2[8]^4, \\
& t_{18}^2 + 6t_{18} + 12 = 2^2 \cdot [6]^6/[1]^{18}[3]^2[18]^2, \\
& t_{23}^2 + 3t_{23} + 3 = 3 \cdot [2]^{24}/[1]^{25}[3]^8, \\
& t_{25}^2 + 5t_{25} + 15 = 5^2 \cdot [5]^6/[1]^{25}[25].
\end{align*}
\]

To continue with the $N = 9$ example, $Q_9(t_9) = t_9(t_9^2 + 9t_9 + 27)$, which equals $3^6 \cdot [3]^{12}/[1]^{12}$, yielding $h_9(t_9(t_9)) = [1]^{10}/[3]$ as claimed. The remaining three ‘irrational’ values of $N$ are handled similarly.

To determine if the multiplier system of $h_N$ is given by an (odd) Dirichlet character mod $N$, reason as follows. Let $g_1, \ldots, g_l \in \Gamma_0(N)$ be a generating set for $\Gamma_0(N)$, and let $a(g_i), b(g_i), c(g_i), d(g_i)$ denote the matrix elements of $g_i$. For each $1 \leq i \leq l$, the character $\chi_N(a(g_i), b(g_i), c(g_i), d(g_i))$ can be computed from the eta product for $h_N$ given in Table 3 using the transformation formula for $\eta$, which is 50:

\[
\eta^{\left(\frac{a}{c}\right)_{\mathfrak{c} + d}} = \left\{ \begin{array}{ll}
\left(\frac{d}{c}\right) \zeta^{3(1-c)+bd(1-c^2)+c(a+d)}[-i(c\tau + d)]^{1/2} \eta(\tau), & c \text{ odd,} \\
\left(\frac{d}{c}\right) \zeta^{3d+ac(1-a^2)+d(b-c)}[-i(c\tau + d)]^{1/2} \eta(\tau), & d \text{ odd,}
\end{array} \right.
\]

if $c > 0$, with $\left(\frac{c}{d}\right) := \left(\frac{d}{c}\right)$ by convention. If there is a unique Dirichlet character mod $N$, say $\chi$, with $\chi(d(g_i)) = \chi_N(a(g_i), b(g_i), c(g_i), d(g_i))$ for $1 \leq i \leq l$, then one must have $\chi_N = \chi$. If on the other hand no such Dirichlet character exists, then the multiplier system of $h_N$ cannot be so encoded.

This procedure requires, for each genus-zero $\Gamma_0(N)$, an explicit generating set. The cardinality of any minimal generating set is $(\sigma_\infty + \epsilon_2 + \epsilon_3)(N) - 1$, the number of fixed points minus 1, but the generating set need not be minimal. For prime $N,$
a standard result is available. Rademacher [49] showed that the set of matrices comprising $\pm \left( \frac{k'}{k} \right)$ and $\pm \left( \frac{1}{-kk'+1-k} \right)$, $k = 1, \ldots, N-1$, where $k' = k'(k)$ is determined by the condition $kk' \equiv -1 \pmod{N}$, generates $\Gamma_0(N)$; and also worked out a minimal generating set for each prime $N$ up to 31 (reproduced by Apostol [2]). For general $N$, one can use the algorithm of Kulkarni [34], which is provided by the Magma system, or that of Lascurain Orive [36].

Actually, we shall need generating sets for $\Gamma_0(N)$ only in the cases $N = 3$ and $N = 4, 6, 8, 9$. For the latter small composite values, we shall use the (minimal) sets given by Harnad and McKay [29], who obtained them by ad hoc methods.

For the 14 values of $N$, whether or not the multiplier system of $\mathcal{h}_N$ is of Dirichlet type is determined as follows.

- $N = 2$. There is no odd Dirichlet character mod 2.
- $N = 3$. A generating set for $\Gamma_0(3)$, from [49], is $\left( \frac{1}{0}, \frac{1}{1} \right)$, $\left( \frac{-1}{-3}, \frac{1}{-2} \right)$, up to sign. These have $d \equiv 1, 1 \pmod{3}$ respectively. The power of $\zeta_{24}$ appearing in the transformation formula for $\mathcal{h}_3 = [1]^3/[3]$ is computed to be 0, 20 respectively. Since $0 \neq 20$, the multiplier system is non-Dirichlet.
- $N = 4$. A generating set for $\Gamma_0(4)$, from [29], is $\left( \frac{1}{0}, \frac{1}{1} \right)$, $\left( \frac{1}{4}, \frac{-1}{3} \right)$, up to sign. These have $d \equiv 1, 1 \pmod{4}$ respectively. The power of $\zeta_{24}$ appearing in the transformation formula for $\mathcal{h}_4 = \mathcal{h}_2 = [1]^4/[2]^2$ is computed to be 0, 12 respectively. Since $0 \neq 12$, the multiplier system is non-Dirichlet.
- $N = 6, 12, 18$. A generating set for $\Gamma_0(6)$, from [29], is $\left( \frac{0}{1}, \frac{1}{0} \right)$, $\left( \frac{5}{12}, \frac{-3}{7} \right)$, $\left( \frac{5}{18}, \frac{-2}{7} \right)$, up to sign. These have $d \equiv 1, 5, 5 \pmod{6}$. The power of $\zeta_{24}$ appearing in the transformation formula for $\mathcal{h}_6 = \mathcal{h}_2 = [1]^6/[2]^3[3]^2$ is computed to be 0, 12, 12 respectively. Hence $\chi_6 : (\mathbb{Z}/6\mathbb{Z})^* \to U(1)$ is the odd Dirichlet character $d \mapsto \left( \frac{-1}{d} \right)$, which takes 1, 5 to 1, −1. Since $\Gamma_0(12), \Gamma_0(18)$ are subgroups of $\Gamma_0(6)$, and $\mathcal{h}_{12}, \mathcal{h}_{18}$ equal $\mathcal{h}_6$, the multiplier systems when $N = 12, 18$ must also be of Dirichlet type, with the same character.
- $N = 8, 16$. A generating set for $\Gamma_0(8)$, from [29], is $\left( \frac{0}{1}, \frac{1}{0} \right)$, $\left( \frac{3}{8}, \frac{-5}{2} \right)$, $\left( \frac{3}{16}, \frac{-1}{5} \right)$, up to sign. These have $d \equiv 1, 3, 3 \pmod{9}$. The power of $\zeta_{24}$ appearing in the transformation formula for $\mathcal{h}_8 = \mathcal{h}_2 = [1]^4/[2]^2$ is computed to be 0, 12, 12 respectively. Hence $\chi_8 : (\mathbb{Z}/8\mathbb{Z})^* \to U(1)$ is the odd Dirichlet character $d \mapsto \left( \frac{-1}{d} \right)$, which takes 1, 3, 5, 7 to 1, −1, 1, −1. Since $\Gamma_0(16)$ is a subgroup of $\Gamma_0(8)$, and $\mathcal{h}_{16}$ equals $\mathcal{h}_8$, the multiplier system when $N = 16$ must also be of Dirichlet type, with the same character.
- $N = 9$. A generating set for $\Gamma_0(9)$, from [29], is $\left( \frac{0}{1}, \frac{1}{0} \right)$, $\left( \frac{5}{9}, \frac{-4}{3} \right)$, $\left( \frac{5}{9}, \frac{-1}{3} \right)$, up to sign. These have $d \equiv 1, 2, 5 \pmod{9}$. The power of $\zeta_{24}$ appearing in the transformation formula for $\mathcal{h}_9 = \mathcal{h}_3 = [1]^3/[3]$ is computed to be 0, 12, 12 respectively. Hence $\chi_9 : (\mathbb{Z}/9\mathbb{Z})^* \to U(1)$ is the odd Dirichlet character $d \mapsto \left( \frac{-1}{d} \right)$, which takes 1, 2, 4, 5, 7, 8 to 1, −1, −1, −1, −1, −1.
- $N = 5, 7, 10, 13, 25$. Careful numerical computation of $\mathcal{h}_N$, using the infinite product representation for the eta function, reveals that in these cases, the character $\hat{\chi}_N(N-1, 1, -N, -1)$ is respectively $\zeta_{24}, \zeta_{24}, \zeta_{24}, \zeta_{24}, \zeta_{24}, \zeta_{24}$. In each case this is inconsistent with $\chi_N(-1) = \zeta_{24}^2 = -1$, so in none is the multiplier system of $\mathcal{h}_N$ given by a Dirichlet character mod $N$. \qed
Table 9. Eta product representations for $\hat{\ell}_N$ and $(qd/dq)\hat{\ell}_N$, in the cases when $\Gamma_0(N)$ has no elliptic fixed points.

| $N$ | $\hat{\ell}_N$ | $(qd/dq)\hat{\ell}_N$ |
|-----|----------------|-------------------|
| 4   | $[4]^6/[1]^8$ | $2^2/[1]^4$       |
| 6   | $[2][6]^3/[1]^5[3]$ | $[2]^4/[1]^6$ |
| 8   | $[2]^2[8]^4/[1]^4[4]^2$ | $[2][4]^4/[1]^8$ |
| 9   | $[3]^3/[1]^3$ | $[3]^8/[1]^9$     |
| 12  | $[2]^2[3][12]^3/[1]^3[4][6]^2$ | $[2][4]^3/[1]^5$ |
| 16  | $[2][16]^2/[1]^2[8]$ | $[2][4]^4/[1]^4$ |
| 18  | $[2][3][18]^2/[1]^2[6][9]$ | $[2][3]^3/[1]^4$ |

Corollary 6.1.1. For each $N$ for which $\Gamma_0(N)$ is of genus zero and has no elliptic fixed points, there is an eta product representation not only for the canonical Hauptmodul $t_N = \kappa_N \cdot \hat{\ell}_N$, but also one for the weight-2 modular form $|(2\pi i)^{-1} d/d\tau|t_N$, i.e., for $(qd/dq)t_N$, as given in Table 9.

Remark. An eta product for the derivative $(qd/dq)\hat{\ell}_N$, when $\hat{\ell}_N$ is similarly expressed, constitutes a combinatorial identity. Several of the identities in Table 9 are new, though Fine [22, §33] gives the eta products for $(qd/dq)\hat{\ell}_4$, $(qd/dq)\hat{\ell}_6$ (in effect), and $(qd/dq)\hat{\ell}_8$, and Cooper [12] gives the one for $(qd/dq)\hat{\ell}_9$. By using the chain rule, one can also work out an eta product for the derivative of each non-canonical Hauptmodul in Table 3. For instance, consider the invariant $\alpha = t_4/(t_4 + 16)$, which equals $2^4 \cdot [1]^8[4][16]/[2]^{24}$. One easily deduces that $(qd/dq)\alpha = 2^4 \cdot [1][16][4][16]/[2]^{28}$.

Proof. If $\Gamma_0(N)$ has no elliptic fixed points, the alternative representation for the modular form $h_N(\tau) = h_N(t_N(\tau))$ given in Corollary 5.5.2 implies

$$\frac{1}{2\pi i} \frac{dt_N}{d\tau}(\tau) = t_N(\tau) h_N^2(t_N(\tau)) \prod_{i=1}^{\sigma_\infty(N)-2} \left[ 1 - \frac{t_N(\tau)}{t_{N,i}^{(\infty)}} \right]. \quad (6.6)$$

The formulas in Table 9 follow from this, with the aid of the eta product representations given in Tables 3 and 8. When $N = 9, 16, or 18$, not all the $\sigma_\infty(N) - 2$ cusps $t_N = t_{N,i}^{(\infty)}$ are rational, and for the right-hand side one needs also special eta product formulas: \(\{6,1\}, \{6,2\}, or \{6,3\} \cdot \{6,4\}\).

One can derive alternative (non-canonical) modular forms $\tilde{h}_N$ from $h_N$, just as one derives the alternative Hauptmodul $\tilde{\ell}_N$ of Table 3 from the canonical Hauptmodul $t_N$. For each $N$, if $c_0 \neq i\infty$ is a cusp of $\Gamma_0(N)$, there is a Hauptmodul $\tilde{\ell}_N := t_N/(t_N - t^*)$, where $t^*$ is the value of $t_N$ at $c_0$, as listed in Table 2. It has divisor $(t_N = 0) - (t_N = t^*)$ on $X_0(N) \cong \mathbb{P}^1(\mathbb{C})_{t_N}$. If the cusp is rational, then $\tilde{\ell}_N$ is a rational (over $\mathbb{Q}$) function of $t_N$. The associated alternative $\tilde{h}_N$ to $h_N$ is naturally taken to be $[(t_N - t^*)/t^*]^{\psi(N)/12} \tilde{h}_N$. By Corollary 5.5.1 this $\tilde{h}_N$, regarded as a (multivalued) function on $X_0(N)$, will have divisor $\psi(N)/12 \cdot (t_N = t^*)$.

One example will suffice. The $\alpha$-invariant, i.e., the alternative Hauptmodul $\tilde{\ell}_4 = t_4/(t_4 + 16)$ for $\Gamma_0(4)$, equals $2^4 \cdot [1]^8[4][16]/[2]^{24}$ and corresponds via the 2-isogeny $\tau \mapsto \tau/2$ to the $\lambda$-invariant $2^4 \cdot [1][4][16]/[2]^{24}$, a Hauptmodul for $\Gamma(2)$. The associated weight-1 modular form $\tilde{h}_4 = [(t_4 + 16)/16]^{1/2} \tilde{h}_4$ equals, by direct
TABLE 10. Weight-1 modular forms $\tilde{h}_N(\tau) = \tilde{h}_N(t_N(\tau))$ on $\Gamma_0(N)$ with integer-coefficient $q$-series, including the canonical one, $h_N(\tau) = h_N(t_N(\tau))$. Here, $\sum$ signifies $\sum_{n=1}^{\infty}$.

| $N$ | $h_N = \tilde{h}_N(t_N)$ | $\eta$-prod. for $\tilde{h}_N(t_N(\tau))$ | $q$-series for $\tilde{h}_N(t_N(\tau))$ |
|-----|-------------------------|------------------------------------------|------------------------------------------|
| 2   | $h_2$                   | $[1]^1 / [2]^2$                         | $1 + 4 \sum E_1(n;4)(-q)^n$             |
| 3   | $h_3$                   | $[1]^3 / [3]$                           | $1 - 3 \sum [E_1(n;3) - 3E_1(n;3;3)] q^n$ |
| 4   | $(t_4 + 16)^{1/2} h_4$  | $[2]^{10} / [1]^4[4]^4$                 | $1 + 4 \sum E_1(n;4)q^n$                |
| 6   | $h_6$                   | $[1]^6[6] / [2]^3[3]^2$                 | $1 - 6 \sum [E_1(n;3) - 2E_1(n;2;3)] q^n$ |
| 8   | $(t_8 + 4)/4 h_8$       | $[2]^{10} / [1]^4[4]^4$                 | $1 + 4 \sum E_1(n;4)q^n$                |
| 9   | $h_9$                   | $[4]^{10} / [2]^4[8]^4$                 | $1 + 4 \sum E_1(n;4)q^{2n}$             |
| 12  | $h_{12}$                | $[2]^{10} [12]^2 / [1]^6[4]^6[6]^5$     | $1 - 6 \sum [E_1(n;3) - 2E_1(n;2;3)](-q)^n$ |
| 16  | $h_{16}$                | $[2]^{10} / [1]^4[4]^4$                 | $1 + 4 \sum E_1(n;4)q^n$                |
| 18  | $h_{18}$                | $[2]^{10} / [4]^4[16]^4$                | $1 + 4 \sum E_1(n;4)q^{4n}$             |

computation, the eta product $[2]^{10} / [1]^4[4]^4$. Under the 2-isogeny this corresponds to $[1]^{10}/[2]^4[2]^4$, a weight-1 modular form for $\Gamma(2)$.

If the eta product for $h_N(\tau)$ contains fractional powers of $\eta$, i.e., if $N = 5, 7, 10, 13, 25$, its $q$-expansion about the infinite cusp $q = 0$, say $\sum_{n=0}^{\infty} a_n(N) q^n$, turns out to contain coefficients $a_n(N)$ that are not integers (but of course $a_n(N) \in \mathbb{Q}$ for all $n$). The $q$-series is nonetheless ‘almost integral’: one can show that the associated scaled sequence $\alpha_n a_n(N)$, $n \geq 0$, where

$$\alpha_N := 2^2, 3^2, 2, 2^23^2, 2^2,$$  \quad \text{for } N = 5, 7, 10, 13, 25, \quad (6.7)

is an integer sequence. The number-theoretic interpretation of the integers $\alpha_n a_n(N)$ is unclear. The integral $q$-series for the nine remaining $h_N$, and for the alternatives $h_N$ derived from them, can more easily be expressed in closed form.

**Theorem 6.2.** The canonical and alternative weight-1 modular forms $h_N, \tilde{h}_N$ for $\Gamma_0(N)$, when $N = 2, 3, 4, 6, 8, 9, 12, 16, 18$, have the eta product and $q$-series representations given in Table 10. In the $q$-series, $E_{r,s,...}(n;k)$ denotes the excess
of the number of divisors of \( n \) congruent to \( r, s, \ldots \) (mod \( k \)) over the number congruent to \(-r, -s, \ldots \) (mod \( k \)); or zero, if \( n \) is not an integer.

**Proof.** The eta products for the \( \tilde{h}_N \) follow from those for \( t_N, h_N \) given in Table 8 Several of these eta products were expanded in multiplicative \( q \)-series by Fine [22 §32], and \( q \)-expansions of the remainder follow by applying such transformations as \( q \mapsto q^2 \) and \( q \mapsto -q \). Under \( q \mapsto -q \), i.e., \( \tau \mapsto \tau + \frac{1}{2} \), the function \([m] \) on \( \mathcal{H} \ni \tau \) is taken to itself if the integer \( m \) is even, and to \([2m]^2/|m||4m|\) if it is odd.

One can also derive from each \( h_N \) a weight-1 modular form \( \tilde{h}_N \) for \( \Gamma_0(N) \) that has a zero at the infinite cusp, rather than at one of the finite ones. This can be accomplished by applying the Fricke involution \( W_N \), which on the half plane \( \mathcal{H} \) is the map \( \tau \mapsto -1/N\tau \). Equivalently (up to a constant factor), one can let \( \tilde{h}_N := (t_N/\kappa_N)^{\psi(N)/12} h_N \). Defined thus, \( \tilde{h}_N \) will have divisor \( \psi(N)/12 \cdot (t_N = 0) \) on \( X_0(N) \), and hence will equal zero at \( \tau = i\infty \). The nine modular forms \( \tilde{h}_N(\tau) \) that have integral \( q \)-series are listed in Table 11 There are only three fundamental ones, the others being obtained by modular substitutions \( \tau \mapsto \ell \tau \), i.e., \( q \mapsto q^\ell \). The \( q \)-expansion of \( h_6(\tau) \) is due to M. Somos (unpublished); the others, to Fine.

The reader will recall that each of the canonical modular forms \( h_N(\tau) = h_N(t_N(\tau)) \) was originally pulled back from \( \tilde{h}_1(J(\tau)) = E_4(\tau)^{1/4}, \) the fourth root of a weight-4 modular form for \( \Gamma(1) \). Due to multivaluedness this is not a true modular form, but it does have an integral \( q \)-expansion about \( \tau = i\infty \). By a useful result of Heninger et al. [30], a \( q \)-series \( f \) in \( 1 + q\mathbb{Z}[[q]] \) has the property that it equals \( g^k \), for some \( g \) in \( 1 + q\mathbb{Z}[[q]] \), iff the reduction of \( f \) mod \( \mu_k \) has the same property, where \( \mu_k := k \prod_{p|k} p \). Since the Eisenstein sum \( E_4 \) equals \( 1 - 240 \sum \sigma_3(n)q^n \), it follows that its fourth and eighth roots must be in \( 1 + q\mathbb{Z}[[q]] \). Hence, not only does \( \tilde{h}_1(J(\tau)) \) have an integral \( q \)-expansion, but so does its square root. The \( q \)-expansion of \( \tilde{h}_1(J(\tau)) \) is \( 1 - 60[q + 99q^2 + 14248q^3 + \cdots] \).

7. The Modular Form \( h_N(\tau) \): Explicit Computations (II)

We now return to treating each canonical weight-1 form \( h_N(\tau) \) as a function of the corresponding Hauptmodul \( t_N \) for \( \Gamma_0(N) \), i.e., as a (multivalued) function \( h_N \)

| \( N \) | \( h_N = \tilde{h}_N(t_N) \) | \( \eta \)-prod. for \( \tilde{h}_N(t_N(\tau)) \) | \( q \)-series for \( \tilde{h}_N(t_N(\tau)) \) |
|---|---|---|---|
| 2 | \((t_2/2)^{1/4} h_2 \) | \(2^{4}/[1]^2 \) | \(q^{1/4}[1 + \sum E_1(4n + 1; 4)q^n] \) |
| 3 | \((t_3/3)^{1/3} h_3 \) | \(3^{3}/[1] \) | \(q^{1/3}[1 + \sum E_1(3n + 1; 3)q^n] \) |
| 4 | \((t_4/2)^{1/2} h_4 \) | \(= \tilde{h}_2(t_2(2\tau)) \) | \(q\{1 + \sum [E_1(n; 6) - 2E_1(n/2; 3)]q^n \} \) |
| 6 | \((t_6/2^3)^{2} h_6 \) | \(1[6]^6/[2^2][3]^3 \) | \(q\{1 + \sum [E_1(3n; 6) - 2E_1(n/2; 3)]q^n \} \) |
Table 12. The (multivalued) functions $h_N = h_N(t_N)$ that define the canonical modular forms $h_N(\tau) = h_N(t_N(\tau))$, expressed in terms of the special function $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \cdot \right)$. Here, $t := t_N$.

| $N$ | $h_N(t_N)$ |
|-----|-------------|
| 2   | $[16^{-1}(t + 16)]^{-1/4} \text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12^4 t}{(t + 16)^3} \right)$ |
| 3   | $[3^{-6}(t + 27)(t + 3)]^{-1/12} \text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12^3 t}{(t + 27)(t + 3)^3} \right)$ |
| 4   | $[16^{-1}(t + 16) \circ t(t + 16)]^{-1/4} \text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12^3 t}{(t + 16)^3} \circ t(t + 16) \right)$ |
| 5   | $[5^{-1}(t^2 + 10t + 5)]^{-1/4} \text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12^2 t}{(t + 10t + 5)^3} \right)$ |
| 6   | $[144^{-1}(t + 6)(t^3 + 18t^2 + 84t + 24)]^{-1/4}$ \times $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12^2 t}{(t + 16)^3} \circ t(t + 9)^2 \right)$ |
| 7   | $[49^{-1}(t^2 + 13t + 49)(t^2 + 5t + 1)^3]^{-1/12}$ \times $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 27)(t + 3)^3} \right)$ |
| 8   | $[16^{-1}(t + 16) \circ t(t + 8)]^{-1/4}$ \times $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 16)^3} \circ t(t + 8) \right)$ |
| 9   | $[3^{-3}(t + 27)(t + 3)^6 \circ t(t^2 + 9t + 27)^{-1/12}$ \times $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 27)^2(t + 3)^3} \right)$ |
| 10  | $[80^{-1}(t^6 + 20t^5 + 160t^4 + 640t^3 + 1280t^2 + 1040t + 80)]^{-1/4}$ \times $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 16)^3} \circ t(t^2 + 9t + 27) \right)$ |
| 12  | $[144^{-1}(t + 6)(t^3 + 18t^2 + 84t + 24) \circ t(t + 6)]^{-1/4}$ times any of: $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 16)^3} \circ t(t + 6) \right)$, $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 27)^2(t + 3)^3} \circ t(t + 9)^2 \right)$, $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 16)^3} \circ t(t + 16) \right)$ |
| 13  | $[13^{-1}(t^2 + 5t + 13)(t^3 + 7t^3 + 20t^2 + 19t + 1)^3]^{-1/12}$ \times $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 27)^2(t + 3)^3} \right)$ |
| 16  | $[16^{-1}(t + 16) \circ t(t + 8)]^{-1/4}$ \times $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 16)^3} \circ t(t + 8) \right)$ |
| 18  | $[144^{-1}(t + 6)(t^3 + 18t^2 + 84t + 24) \circ t(t^2 + 6t + 12)]^{-1/4}$ times any of: $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 16)^3} \circ t(t^2 + 6t + 12) \right)$, $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 27)(t + 3)^3} \circ t(t^2 + 9t + 27) \right)$ |
| 25  | $[5^{-1}(t^2 + 10t + 5) \circ t(t^4 + 5t^3 + 15t^2 + 25t + 25)]^{-1/4}$ \times $\text{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; \frac{12 t}{(t + 16)^3} \circ t(t^4 + 5t^3 + 15t^2 + 25t + 25) \right)$ |

on the genus-0 curve $X_0(N) = \Gamma_0(N) \backslash H^* \cong \mathbb{P}^1(\mathbb{C})_{t_N}$. The Picard–Fuchs equations that the functions $h_N$ satisfy, and their $t_N$-expansions, will be computed. These two pieces of information will place the $h_N$ firmly in the world of special functions. We pay particular attention to closed-form expressions and recurrences for the coefficients of their $t_N$-expansions, since a functional equation for $h_N$ can be viewed as a series identity. It turns out that many of the coefficient sequences have cropped up in other contexts, and are listed in Sloane’s Encyclopedia.
Table 13. Differential operators \( \mathcal{L}_N \) in the normal-form Picard–Fuchs equations \( \mathcal{L}_N u = 0 \) satisfied by the modular forms \( h_N \) for \( \Gamma_0(N) \), when viewed as functions of \( t := t_N \). In the four sections of the table, \( \mathcal{L}_N \) has respectively 3, 4, 6, 8 singular points on the curve \( X_0(N) \cong \mathbb{P}^1(\mathbb{C})_{t_N} \).

| \( N \) | Differential operator \( \mathcal{L}_N \) |
|-------|---------------------------------|
| 2     | \( D_2^2 + \frac{1}{t} + \frac{1}{2(t+6)} \) \( D_t + \frac{1}{16(t+64)} \) |
| 3     | \( D_3^2 + \frac{1}{t} + \frac{2}{(t+2)^2} \) \( D_t + \frac{1}{9(t+4)} \) |
| 4     | \( D_4^2 + \frac{1}{t} + \frac{1}{t+16} \) \( D_t + \frac{1}{4t(t+16)} \) |
| 5     | \( D_5^2 + \frac{1}{t} + \frac{t+1}{t+16} \) \( D_t + \frac{1}{4t(t+16)} \) |
| 6     | \( D_6^2 + \frac{1}{t} + \frac{1}{t+8} + \frac{1}{t+9} \) \( D_t + \frac{1}{t(t+8)(t+9)} \) |
| 7     | \( D_7^2 + \frac{1}{t} + \frac{t+2}{t+24} \) \( D_t + \frac{1}{4t(t+24)} \) |
| 8     | \( D_8^2 + \frac{1}{t} + \frac{1}{t+8} + \frac{1}{t+9} \) \( D_t + \frac{1}{t(t+8)(t+9)} \) |
| 9     | \( D_9^2 + \frac{1}{t} + \frac{2t+9}{t^2+9t+27} \) \( D_t + \frac{1}{t^2+9t+27} \) |
| 10    | \( D_{10}^2 + \frac{1}{t} + \frac{t+2}{t+5} + \frac{1}{t+3} + \frac{1}{t+4} + \frac{1}{t+6} + \frac{1}{t+9} + \frac{1}{t+16} + \frac{1}{t+24} \) \( D_t + \frac{4t^3+9t^2+8t+20}{4t^3(t+4)(t+5)(t+6)(t+9)(t+16)(t+24)} \) |
| 12    | \( D_{12}^2 + \frac{1}{t} + \frac{t+2}{t+7} + \frac{1}{t+3} + \frac{1}{t+4} + \frac{1}{t+6} + \frac{1}{t+9} + \frac{1}{t+16} + \frac{1}{t+24} \) \( D_t + \frac{4t^3+9t^2+8t+20}{4t^3(t+4)(t+5)(t+6)(t+9)(t+16)(t+24)} \) |
| 16    | \( D_{16}^2 + \frac{1}{t} + \frac{t+2}{t+7} + \frac{1}{t+3} + \frac{1}{t+4} + \frac{1}{t+6} + \frac{1}{t+9} + \frac{1}{t+16} + \frac{1}{t+24} \) \( D_t + \frac{4t^3+9t^2+8t+20}{4t^3(t+4)(t+5)(t+6)(t+9)(t+16)(t+24)} \) |
| 18    | \( D_{18}^2 + \frac{1}{t} + \frac{t+2}{t+7} + \frac{1}{t+3} + \frac{1}{t+4} + \frac{1}{t+6} + \frac{1}{t+9} + \frac{1}{t+16} + \frac{1}{t+24} \) \( D_t + \frac{4t^3+9t^2+8t+20}{4t^3(t+4)(t+5)(t+6)(t+9)(t+16)(t+24)} \) |
| 25    | \( D_{25}^2 + \frac{1}{t} + \frac{4t^3+15t^2+20t+25}{t^3+6t^2+15t+20} + \frac{1+t+1}{t^3+6t^2+15t+20} \) \( D_t + \frac{25t^3+5t^2+15t^2+25t+10}{4t^3(t+2)(t+5)(t+6)(t+9)(t+16)(t+24)} \) |

Theorem 7.1. In a neighborhood of the point \( t_N = 0 \) (i.e., the infinite cusp \( \tau = i\infty \) ), each function \( h_N = h_N(t_N) \) can be expressed in terms of the Gauss hypergeometric function \( _2F_1(\frac{1}{12}, \frac{5}{12}; 1; \cdot) \), as given in Table 12.

Proof. This follows from Definition 5.2, the many composite rational expressions \( F_N(t_N)/Q_N(t_N) \) for the \( j \)-invariant being taken from Table 7.

Remark. The hypergeometric recurrence (A.3) yields a \( \hat{J} \)-expansion of the underlying function \( \hat{h}_1 = \hat{h}_1(\hat{J}) = 2F_1(\frac{1}{12}, \frac{5}{12}; 1; \hat{J}) \). If \( \hat{h}_1(\hat{J}) = \sum_{n=0}^{\infty} c_n(1) j^n \) then the sequence \( c_n(1) = 123nC_n(1) \), \( n \geq 0 \), is integral, the first few terms being 1, 60, 39780, 3845400, 43751038500. This is Sloane’s sequence A092870.

For most purposes the explicit formulas of Table 12 are far less useful than the Fuchsian differential equations that the functions \( h_N \) satisfy, and representations that can be deduced from them. Recall that by Theorem 5.3, the function \( h_N \) is the unique holomorphic solution (up to normalization) of a normal-form Picard–Fuchs equation \( \mathcal{L}_N u = 0 \), in a neighborhood of the point \( t_N = 0 \). The full space of local solutions is \( h_N(\cdot)[\mathcal{C}\tau(\cdot) + \mathbb{C}] \), where \( \tau = \tau(t_N) \) means any branch of what is, in reality, an infinite-valued function on \( X_0(N) \cong \mathbb{P}^1(\mathbb{C})_{t_N} \).
Theorem 7.2. For each $N$, the normal-form Fuchsian differential operator $L_N = \mathcal{D}_N^2 + A_N(t_N) \mathcal{D}_N + B(t_N)$ in the Picard–Fuchs equation $L_N u = 0$ of Theorem 5.3, which has holomorphic local solution $u = h_N$, is as given in Table 13.

Proof. Each operator $L_N$ is computed as in Theorem 5.4. This is the procedure: (i) Pull back the Gauss hypergeometric operator $L_1 = L_{12} + \frac{1}{t_1}$ along $X_0(N) \to X(1)$, or equivalently, along $\mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, using the formula for the degree-$\psi(N)$ covering map $j = P_N(t_N)/Q_N(t_N)$ given in Table 4 and (ii) Make the substitution $\tilde{u} = P_N(t_N)^{-1/12} u$, to reduce the resulting operator to normal form. The computations are lengthy but rewarding.

Remark. By comparing each $L_N$ of Table 13 with the normal form (5.1), one sees that its singular points and exponents are in agreement with Theorem 5.3. In each $L_N$ the coefficient $A_N$ of $\mathcal{D}_N$ has only simple poles, and they are located at the $t_N$-values of the fixed points of $\Gamma_0(N)$ on $X_0(N)$ other than the cusp $t_N = \infty$ (i.e., $\tau = 0$). The locations agree with those given in Table 2. For example, the three (of a total of four) singular points on $X_0(5)$ evident in the rational coefficient $\frac{t_5 + 10}{t_5 + t_5^2 + 10}$ of $D_{\tau}$ in $L_5$ are located at (i) the infinite cusp $t_5 = 0$, and (ii) the pair of quadratic elliptic points $t_5 = -11 \pm 2i$. (As was noted in (7.2), the latter are naturally bijective with the representations $1^2 + 2^2, 2^2 + 2^2$ of $N = 5$ as a sum of two squares.) For each $N$, the coefficient $A_N$ of $\mathcal{D}_N$ has residue at each cusp equal to 1, at each quadratic elliptic point to 1/2, and at each cubic one to 2/3.

Remark. The equation $L_N u = 0$ is equivalent to the Picard–Fuchs equation derived (in matrix form) by Dwork [13, §8] as the equation satisfied by the periods of the Hesse–Dixon elliptic family (4.3), parametrized by $\gamma = \gamma(\tau) = t_0(\tau/3)$.

Remark. For each of the 14 values of $N$, a Picard–Fuchs equation in the self-adjoint form $[D_N^2 + Q_N(t_N)] v = 0$, projectively equivalent to the normal-form equation $L_N u = 0$ of Table 13, has been derived by Lian and Wiczer [37], as part of a larger computational project on the 175 genus-zero Conway–Norton subgroups of $\text{PSL}(2, \mathbb{R})$. They derive each $Q_N \in \mathbb{Q}(t_N)$ in a heuristic way from the formula $Q_N := \frac{1}{2} \{\tau, \tau_N\}$, starting with a $q$-expansion for each $t_N$. (For the first two dozen terms in each expansion, see [23, Table 3].) Each of their Hauptmoduln is of the McKay–Thompson type, with $q$-expansion $q^{-1} + 0 \cdot q^1 + O(q^1)$; so it is not identical to our Hauptmodul $h_N$, but rather related to it by a linear fractional transformation over $\mathbb{Q}$. Their Picard–Fuchs equations are indexed by Conway–Norton classes; for the correspondence to $N$, see, e.g., [23, Table 4].

Remark. From the Picard–Fuchs equations $L_N h_N = 0$, one can deduce that

$$h_2(t_2) = 1/\text{AG}_2(1, \sqrt{1 + \sqrt{1 + t_2/64}}) \tag{7.1a}$$
$$h_4(t_4) = 1/\text{AG}_2(1, \sqrt{1 + t_4/16}) \tag{7.1b}$$
$$h_8(t_8) = 1/\text{AG}_2(1, 1 + t_8/4), \tag{7.1c}$$
$$h_{16}(t_{16}) = 1/\text{AG}_2(1, (1 + t_{16}/2)^2), \tag{7.1d}$$
Table 14. Recurrences satisfied by the coefficients \( \{ c_n(N) \} \) of the series expansion \( h_N(t_N) = \sum_{n=0}^{\infty} c_n(N) t_N^n \), in the cases when \( \Gamma_0(N) \) has 3, 4, 6, 8 fixed points on \( \mathcal{X}_0(N) \).

| \( N \) | recurrence |
|---|---|
| 2 | \((4n - 3)^2 c_{n-1} + 1024n^2 c_n = 0\) |
| 3 | \((3n - 2)^2 c_{n-1} + 243n^2 c_n = 0\) |
| 4 | \((2n - 1)^2 c_{n-1} + 64n^2 c_n = 0\) |
| 5 | \((2n - 1)^2 c_{n-1} + 2(44n^2 + 22n + 5) c_n + 500(n + 1)^2 c_{n+1} = 0\) |
| 6 | \(n^2 c_{n-1} + (17n^2 + 17n + 6) c_n + 72(n + 1)^2 c_{n+1} = 0\) |
| 7 | \((3n - 1)^2 c_{n-1} + 3(39n^2 + 26n + 7) c_n + 441(n + 1)^2 c_{n+1} = 0\) |
| 8 | \(n^2 c_{n-1} + 4(3n^2 + 3n + 1) c_n + 32(n + 1)^2 c_{n+1} = 0\) |
| 9 | \(n^2 c_{n-1} + 3(3n^2 + 3n + 1) c_n + 27(n + 1)^2 c_{n+1} = 0\) |
| 10 | \((2n + 1)^2 c_{n-1} + (68n^2 + 152n + 95) c_n + 4(112n^2 + 392n + 365) c_{n+1} + 80(17n^2 + 81n + 99) c_{n+2} + 1600(n + 3)^2 c_{n+3} = 0\) |
| 12 | \((n + 1)^2 c_{n-1} + 3(5n^2 + 15n + 12) c_n + 16(5n^2 + 20n + 21) c_{n+1} + 36(5n^2 + 25n + 32) c_{n+2} + 144(n + 3)^2 c_{n+3} = 0\) |
| 13 | \((6n + 1)^2 c_{n-1} + 3(132n^2 + 232n + 117) c_n + 2(2016n^2 + 6384n + 5395) c_{n+1} + 78(66n^2 + 302n + 353) c_{n+2} + 6084(n + 3)^2 c_{n+3} = 0\) |
| 16 | \((n + 1)^2 c_{n-1} + 2(5n^2 + 15n + 12) c_n + 8(5n^2 + 20n + 21) c_{n+1} + 16(5n^2 + 25n + 32) c_{n+2} + 64(n + 3)^2 c_{n+3} = 0\) |
| 18 | \((n + 2)^2 c_{n-1} + 2(7n^2 + 35n + 45) c_n + 12(7n^2 + 42n + 65) c_{n+1} + 39(7n^2 + 49n + 88) c_{n+2} + 72(7n^2 + 56n + 114) c_{n+3} + 72(7n^2 + 63n + 143) c_{n+4} + 216(n + 5)^2 c_{n+5} = 0\) |
| 25 | \((2n + 3)^2 c_{n-1} + (28n^2 + 116n + 125) c_n + 5(24n^2 + 128n + 179) c_{n+1} + 5(64n^2 + 416n + 701) c_{n+2} + 25(24n^2 + 184n + 361) c_{n+3} + 50(14n^2 + 124n + 277) c_{n+4} + 500(n + 5)^2 c_{n+5} = 0\) |

where \( \text{AG}_3(\cdot, \cdot) \) is the quadratic arithmetic–geometric mean (AGM) function of Gauss [7], and that

\[
\begin{align*}
\text{h}_3(t_3) &= 1/ \text{AG}_3(1, \sqrt{1 + t_3/27}), \\
\text{h}_9(t_9) &= 1/ \text{AG}_3(1, 1 + t_9/3),
\end{align*}
\]

(7.2a)

(7.2b)

where \( \text{AG}_3(\cdot, \cdot) \) is the cubic AGM function of the Borweins [8].

**Theorem 7.3.** For each \( N \), the sequence \( \{ c_n^{(N)} \}_{n=0}^{\infty} \) of coefficients of the \( t_N \)-expansion of the modular form \( h_N \) about the infinite cusp \( t_N = 0 \) satisfies the recurrence in Table 14, initialized by \( c_0^{(N)} = 1 \) and \( c_1^{(N)} = 0, n < 0 \). The number of terms is the number of fixed points of \( \Gamma_0(N) \), minus 1. The scaled sequence \( d_n^{(N)} := (\alpha_N \kappa_N)^n c_n^{(N)} \), \( n \geq 0 \), is an integral sequence. Here \( \kappa_N \) is as in Table 8, and \( \alpha_N \) equals \( 2^2, 3^2, 2, 2^2 3^2 \) if \( N = 5, 7, 10, 13, \) and 1 otherwise.

**Remark.** That a factor \( \alpha_N \kappa_N^2 \) is needed for integrality is due to the definition of each Hauptmodul \( l_N \) as \( \kappa_N : l_N \), where \( l_N \) is an eta product. The factor \( \alpha_N^2 \kappa_N^2 \) appeared in [3], as the superimposed geometric growth that forces the \( q \)-series of \( h_N(\tau) \) to become integral. (See Eq. [67].)

**Proof.** Substitute \( u = h_N = \sum_{n=0}^{\infty} c_n^{(N)} t_N^n \) into the equation \( \mathcal{L}_N u = 0 \), and extract and analyse the recurrence equation satisfied by the coefficients. □
Table 15. Alternative expressions for the modular forms $h_N$ as functions of $t := t_N$, in terms of $2F_1$ (when $\Gamma_0(N)$ has three fixed points) or the special function $H_I$ (when it has four).

| $N$ | $h_N(t)$ |
|-----|----------|
| 2   | $2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; -t/64\right)$ |
| 3   | $2F_1\left(\frac{1}{3}, \frac{1}{3}; 1; -t/27\right)$ |
| 4   | $2F_1\left(\frac{1}{4}, \frac{1}{4}; 1; -t/16\right)$ |
| 5   | $H_I\left(\frac{-11+2\sqrt{11}}{11+2\sqrt{11}}, \frac{1}{3}(-11 \mp 2i); \frac{1}{2}, \frac{1}{2}, \frac{1}{2}; t/[-11 \mp 2i]\right)$ |
| 6   | $H_I\left(\frac{1}{3}, \frac{1}{3}; 1, 1, 1, 1; -t/8\right) = H_I\left(\frac{1}{2}, \frac{1}{2}; 1, 1, 1, 1; -t/9\right)$ |
| 7   | $H_I\left(\frac{-11+2\sqrt{11}}{11+2\sqrt{11}}, \frac{1}{3}(-11 \mp 2i); \frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \frac{1}{3}; t/\left[-11 \mp 2i\right]\right)$ |
| 8   | $H_I\left(2, 1; 1, 1, 1, 1; -t/4\right) = H_I\left(\frac{1}{3}, \frac{1}{3}; 1, 1, 1, 1; -t/8\right)$ |
| 9   | $H_I\left(\frac{-9+2\sqrt{11}}{9+2\sqrt{11}}, \frac{1}{3}(-9 \mp 2i); 1, 1, 1, 1; t/\left[-9 \mp 2i\right]\right)$ |

The integral sequences $d_n^{(N)}$, $n \geq 0$, for $N = 2, 3$, do not currently appear in Sloane’s Encyclopedia [57], perhaps because they are not expressible in binomial form. Their first few terms are $1, -4, 100, -3600, 152100, -7033104, 344622096; \text{and}$ $1, -3, 36, -588, 11025, -223587, 4769856$, respectively. The integral sequences $d_n^{(N)}$, $n \geq 0$, for $N = 4, 6, 8, 9$, appear in the Encyclopedia as A002894, A093388, A081085, A006077. One easily sees that $d_n^{(4)} = 2^{n^2}c_n^{(4)} = (-)^n \left(\binom{2n}{n}\right)^2$. For $N = 6, 8, 9$, the recurrence satisfied by $d_n^{(N)}$ was derived by Coster [13] in an investigation of modular curves $\Gamma \setminus \mathcal{H}^*$, where $\Gamma < PSL(2, \mathbb{R})$ has four cusps, no elliptic points, and is of genus 0. (The possible $\Gamma$ include $\Gamma_0(6)$, $\Gamma_0(8)$, $\Gamma_0(9)$, and three others [3].) It was later shown that

\begin{align*}
d_n^{(6)} &= 72^n c_n^{(6)} = \sum_{k=0}^{n} \binom{n}{k} (-8)^{n-k} \sum_{j=0}^{n-k} \binom{n-k}{j}, \\
d_n^{(8)} &= 2^{5n} c_n^{(8)} = (-)^n \sum_{k=0}^{n} \binom{n}{k} \left(\frac{2n-2k}{2k}\right) \left(\frac{2k}{k}\right), \\
d_n^{(9)} &= 3^{3n} c_n^{(9)} = \sum_{k=0}^{\lfloor n/3 \rfloor} (-)^k 3^{n-3k} \binom{n}{3k} \binom{3k}{k} \binom{2k}{k},
\end{align*}

respectively by Verrill [60]; by Larcombe and French [35] Thm. 3] and V. Jovović; and by Zagier and Verrill.

**Theorem 7.4.** In the cases when $\Gamma_0(N)$ has exactly 3 fixed points on $X_0(N)$ (i.e., $N = 2, 3, 4$), and 4 fixed points (i.e., $N = 6, 7, 8, 9$), the modular form $h_N$ as a function of the Hauptmodul $t_N$ can be expressed in terms of the special functions $2F_1$ and $H_I$, as shown in Table 15.

**Remark.** When $N = 2, 3, 4$, one can write, compactly and elegantly,

$$h_N(t_N) = 2F_1\left(\frac{1}{12} \psi(N), \frac{1}{12} \psi(N); 1; -t_N/\kappa_N^{1/2}\right),$$

since $\kappa_N = 2^{12}, 3^{6}, 2^{8}$, respectively.
Remark. Using the equivalence of the expressions of Table 15 to those of Table 12 in the cases $N = 2, 3, 4$ one obtains transformation formulas for $2F_1$. They are respectively cubic, quartic, and sextic, and are special cases of Goursat’s transformations. In the cases $N = 5, 6, 7, 8, 9$ one obtains reductions of $Hl$ to $2F_1$, which are new (such reductions have never been classified).

Proof. Compare the recurrences of Table 14 with those for $2F_1, Hl$ given in the Appendix; or more directly, compare the differential operators of Table 13 with the differential equations satisfied by $2F_1, Hl$, also given in the Appendix. The awkwardness of the expressions for $h_5, h_7, h_9$ (which are not over $\mathbb{Q}$) comes from the three singular points of each of $L_N$, $N = 5, 7, 9$, other than the cusp $t_N = \infty$, not being collinear on $\mathbb{C} \ni t_N$. (Cf. Table 2)

Corollary 7.4.1. When $N = 2, 3, 4$, on the half-line $t_N > 0$ one has the identity

$$
\tau(t_N)/i = A_N t_N^{-\psi(N)/12} \frac{h_N(k_N/t_N)}{h_N(t_N)},
$$

with $A_N = 2, \sqrt{3}, 2$, respectively. Here $h_N, \tau$ are the single-valued holomorphic continuations to $t_N > 0$ discussed in [5] which are real and positive.

Proof. This continues the proof of Theorem 5.7. It is known [20, §2.10] that

$$
2F_1(a, a; 1; z) \sim \frac{\sin(\pi a)}{\pi} (-z)^{-1} \left[ \log(-z) + h_0(a) \right]
$$

(7.7)

to leading order as $z \to -\infty$, with $h_0(a) := 2\Psi(1) - \Psi(a) - \Psi(1 - a)$, where $\Psi$ is the Euler digamma function, i.e., $\Psi(a) = d(\log \Gamma(a))/da$. Substituting the values of $\Psi(a), \Psi(1 - a)$ for $a = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, which are known [18, vol. 2, app. II.3], reveals that the expressions in Table 15 for $h_N(t_N), N = 2, 3, 4$, have leading-order behavior

$$
h_N(t_N) \sim B_N t_N^{-\psi(N)/12} \log t_N
$$

(7.8)
as $t_N \to \infty$, where $B_N = 4, 3\sqrt{3}, 8$ respectively. One then uses $A_N = B_N/N$, shown in the proof of Theorem 5.7.

The Picard–Fuchs equations $L_N u = 0$ of Table 13 and the recurrences of Table 11, though ‘canonical’, are not unique. As was explained in [6] to each alternative Hauptmodul $\hat{t}_N$ of Table 3 there is associated an alternative weight-1 modular form $\hat{h}_N \circ t_N$. For convenience, write $t_N = \hat{\phi}(\hat{t}_N)$, where $\hat{\phi}$ is the appropriate Möbius transformation. The alternative modular form will then be $(\hat{h}_N(\hat{\phi}))(\hat{t}_N(\tau))$. The function $\hat{h}_N \circ \hat{\phi}$ will satisfy its own normal-form Picard–Fuchs equation, with independent variable $\hat{t}_N$. When $N \leq 9$, the unique holomorphic solution equal to unity at $\hat{t}_N = 0$ will be expressible in terms of $2F_1$ or $Hl$, providing an easy route to the recurrence satisfied by the coefficients of its $\hat{t}_N$-expansion. But unlike the recurrences of Table 14 this recurrence will contain at least one minus sign.

The following pair of examples shows how in the just-described way, one may expand $h_N(t_N(\tau))$ as a function of $\hat{t}_N(\tau)$, sometimes obtaining a combinatorially interesting series. In the case $N = 6$, consider the Hauptmoduln $\hat{t}_6, \hat{\tilde{t}}_6$ defined by

$$
\hat{t}_6 = t_6/(t_6 + 8), \quad t_6 = \hat{\phi}(t_6) := 8 \hat{t}_6/(1 - \hat{t}_6) \quad (7.9a)
$$

$$
\hat{\tilde{t}}_6 = t_6/(t_6 + 9), \quad t_6 = \hat{\phi}(t_6) := 9 \hat{\tilde{t}}_6/(1 - \hat{\tilde{t}}_6) \quad (7.9b)
$$
employing (A.7), the generalized Pfaff transformation of $H_l$

The scaled coefficients $\tilde{c}_n$ show that $\tilde{c}_3 = 0$.

Expressing $h_6$ in terms of $Hl$ is as in Table 15 and employing (A.7), the generalized Pfaff transformation of $Hl$, yields

$$\tilde{h}_6 = \frac{Hl(t_6)}{(t_6 + 8)/8},$$
$$h_6 = \frac{Hl(t_6)}{(t_6 + 9)/9}.$$  \hfill (7.10a, 7.10b)

with a zero at $[\frac{1}{3}]_6$, resp. $[\frac{1}{4}]_6$.

If $\tilde{h}_6$ and $\tilde{h}_6$ are expanded about the infinite cusp, as \[\sum_{n=0}^{\infty} \tilde{c}_n t^n\] and \[\sum_{n=0}^{\infty} \tilde{c}_n t^n\], the coefficients will satisfy the three-term recurrences

$$n^2 \tilde{c}_{n-1} = (10n^2 + 10a + 3) \tilde{c}_n + 9(n+1)^2 \tilde{c}_{n+1} = 0,$$
$$n^2 \tilde{c}_n = (7n^2 + 7n + 2) \tilde{c}_n - 8(n+1)^2 \tilde{c}_{n+1} = 0.$$  \hfill (7.12a, 7.12b)

The scaled coefficients $\tilde{d}_n := 9^n \tilde{c}_n$, $\tilde{d}_n := 8^n \tilde{c}_n$ will be integral, and one can show that $\tilde{d}_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{k}{k}$, and that the integers $\tilde{d}_n$, $n \geq 0$, are the so-called Franel numbers, i.e., $\tilde{a}_n = \sum_{k=0}^{n} \binom{n}{k}^3$. These recurrences and closed-form solutions were first derived by Stienstra and Beukers [58], in a somewhat different framework (see also Verrill [60]). In Sloane’s Encyclopedia the sequence $\tilde{a}_n$, $n \geq 0$, is listed as A002893, and $\tilde{d}_n$, $n \geq 0$, as A000172.

8. Modular Equations for Elliptic Families

Now that the weight-1 modular forms $h_M = h_M \circ t_M : \mathcal{H}^* \to \mathbb{P}^1(\mathbb{C})$ for the genus-zero modular subgroups $\Gamma_0(M) < \Gamma(1)$ have been thoroughly examined, we proceed to derive rationally parametrized modular equations for them. When $\Gamma_0(NM)$ is of genus zero, there is such a modular equation of degree $N$, which links $h_M(t_M(\tau))$, $h_M(t_M(N\tau))$. That is, it links $h_M(t_M), h_M(t_M')$, where the relation between $t_M, t_M' \in X_0(M) \cong \mathbb{P}^1(\mathbb{C})$ is uniformized by the Hauptmodul $t_{NM} \in X_0(NM)$, according to rational maps $t_M = t_M(t_{NM}), t'_M = t'_M(t_{NM})$ that can be found in Table 6. The modular equation is of the form $h_M(t_M) = \mathcal{M} \cdot h_M(t_M')$, where the multiplier $\mathcal{M}$ is (a root of) a function in $\mathbb{Q}(t_{NM})$.

Such modular equations can be derived mechanically, with the aid of q-series representations for $h_M(t_M(\tau))$ and $t_{NM}(\tau)$. That is not our approach, since it ignores the algebraic-geometric context. Our key Theorem 8.8 which efficiently produces such modular equations (as transformations of special functions), is really a theorem on pullbacks of Gauss–Manin connections along maps between rational elliptic modular surfaces. The following remarks place it in context.

An elliptic surface is a flat morphism $\mathcal{E} \to \mathbb{P}^1(\mathbb{C})$, the generic fibre of which is an elliptic curve $E/\mathbb{C}$. An example is the universal family of elliptic curves over $\mathbb{C}$, denoted $\mathfrak{E}_1 \cong X(1)$ here, with base equal to the j-line. It has one singular fibre, over $j = \infty$. For any genus-zero $\Gamma < \Gamma(1)$, a rational elliptic surface $\mathcal{E}_\Gamma \to \Gamma \setminus \mathcal{H}^*$, by definition a rational elliptic modular surface, can be constructed from the quotiented half-plane $\Gamma \setminus \mathcal{H}$ and elliptic fibres over it, by resolving singularities [58]. If $\Gamma$ has
Hauptmodul $t$, with

$$j = \frac{P^3(t)S(t)}{R(t)} = 12^3 + \frac{Q^2(t)T(t)}{R(t)}$$  \tag{8.1}

(in lowest terms, as in Table 4, then this surface, regarded as an elliptic family parametrized by $t$, will have Weierstrass presentation

$$y^2 = 4x^3 - 3P(t)S(t)T(t)x - Q(t)S(t)T^2(t).$$  \tag{8.2}

There will be a singular fibre over every fixed point of $\Gamma$ on $\Gamma \setminus H^*$. The surfaces $E_{NM} \xrightarrow{\pi_M} X_0(NM)$ attached to genus-zero $\Gamma_0(NM)$ are of primary interest. For instance, $E_5 \xrightarrow{\pi} X_0(5)$, with parameter $t = t_5$ on the base, has presentation

$$y^2 = 4x^3 - 3(t^2 + 10t + 5)(t^2 + 22t + 125)x - (t^2 + 4t - 1)(t^2 + 22t + 125)^2.$$  

This rational elliptic family, with four singular fibres, can be found in the classification of Herfurtner [31]. (The singular ones are of Kodaira types $I_1, I_5, III, III$, located above the two cusps $t = 0, \infty$ and the two quadratic elliptic fixed points $t = -11 \pm 2i$; cf. Table 2.) The Hesse–Dixon family (4.4) yields an elliptic surface with base parameter $\gamma = \gamma(\tau) = t_0(\tau/3)$, and four singular fibres. It is attached to $\Gamma(3)$, which is conjugated to $\Gamma_0(9)$ by a 3-isogeny. Verrill [61] gives additional examples.

For any $M, N$ for which $\Gamma_0(M)$ and $\Gamma_0(NM)$ are of genus zero, there is a commutative diagram of rational elliptic surfaces,

$$
\begin{array}{cccc}
E_{NM} & \longrightarrow & E_M & \longrightarrow & \hat{E}_1 \\
\pi_{NM} & & \pi_M & & \pi_1 \\
X_0(NM) & \longrightarrow & X_0(M) & \longrightarrow & X(1)
\end{array}
$$  \tag{8.3}

where the maps $X_0(NM) \rightarrow X_0(M)$ and $X_0(M) \rightarrow X(1)$ are given by the rational functions $t_{NM} \mapsto t_M$ and $t_M \mapsto j$, listed in Tables 4 and 5 respectively, and the fibre maps are induced by them. (If $t_{NM} \mapsto t_M$ is replaced by $t_{NM} \mapsto t_M'$, i.e., $t_{NM}(\tau) \mapsto t_M(N\tau)$, or $t_M \mapsto j$ by $t_M \mapsto j'$, i.e., $t_M(\tau) \mapsto j(M\tau)$, then the resulting diagram will still commute.)

Now consider the Picard–Fuchs equations attached to the genus-zero subgroups $\Gamma_0(M)$. At any generic point $t_M$ on the base of $E_M \xrightarrow{\pi_M} X_0(M)$, the fibre (an elliptic curve) will have period module $\mathbb{Z}_{\tau_1} \oplus \mathbb{Z}_{\tau_2}$, where $\tau_1/\tau_2$ and $\tau_2$ are what we have been calling $\tau$ and $h_M(t_M(\tau))$. The space of local solutions of the Picard–Fuchs equation $\mathcal{L}_M u = 0$ on $X_0(M) \cong \mathbb{F}^1(C)_{t_M}$ is $h_M(\cdot) [C\tau(\cdot) + \mathbb{C}]$, i.e., $C\tau_1(\cdot) + C\tau_2(\cdot)$.

Any element $\omega \in H^1(\pi^{-1}_M(t_M))$ of the de Rham cohomology group of a fibre of $E_M$ above a generic point $t_M \in X_0(M)$ is a period, i.e., a point in $C\tau_1 + C\tau_2$. So the second-order differential equation $\mathcal{L}_M u = 0$ defines a flat connection $\nabla_M$ on a 2-dimensional period bundle, each fibre being a de Rham cohomology group; viz., a Gauss–Manin connection.

In [47] each Picard–Fuchs operator $\mathcal{L}_M$ was constructed explicitly, by pulling back along $X_0(M) \rightarrow X(1)$ the Gauss hypergeometric operator $\mathcal{L}_1$ that defines a Gauss–Manin connection for the universal family $E_1 \xrightarrow{\mathcal{L}_1} X(1)$. (Much of the work went into keeping $\mathcal{L}_M$ in normal form, to permit the derivation of $2F_1$ and other special function identities.) So by construction, the connection $\nabla_1$ on $E_1$ pulls back to $\nabla_M$ on $E_M$, which in turn pulls back to $\nabla_{NM}$ on $E_{NM}$.
Theorem 8.1.  

(1) If \( \Gamma_0(M) \) is of genus zero, and if one has (in lowest terms) that \( j = P(t_M)/Q(t_M), \ j' = P'(t_M)/Q'(t_M) \), then the identity
\[
h_M(t_M) = [P(t_M)/P(0)]^{-1/12} \hat{h}_1(P(t_M)/Q(t_M))
\]
holds near the point \( t_M = 0 \) on \( X_0(M) \), i.e., near the infinite cusp.

(2) If \( \Gamma_0(M), \Gamma_0(NM) \) are of genus zero, and if (in lowest terms) \( t_M = P(t_{NM})/Q(t_{NM}), \ t'_M = P'(t_{NM})/Q'(t_{NM}) \), then the identity
\[
h_{NM}(t_{NM}) = [Q(t_{NM})/Q(0)]^{-\psi(M)/12} h_M(P(t_{NM})/Q(t_{NM}))
\]
holds in a neighborhood of the point \( t_{NM} = 0 \) on \( X_0(NM) \), i.e., of the infinite cusp.

Proof. First, recall that by Theorems 5.3 and 5.4, \( \hat{h}_1 = \hat{h}_1(J) \) and \( h_M = h_M(t_M) \) are the unique local solutions of the canonical Picard–Fuchs equations \( \hat{L}_1u = 0 \) for \( X(1) \) and \( \mathcal{L}_M u = 0 \) for \( X_0(M) \) that are holomorphic at the infinite cusp (i.e., at \( J = 0 \), resp. \( t_M = 0 \)), and are normalized to equal unity there.

The first equality in part 1 is true by definition (see Definition 5.2). To prove the first in part 2, pull back \( \mathcal{L}_M \) along \( X_0(NM) \rightarrow X_0(M) \), i.e., along \( t_M = t_M(t_{NM}) \), which takes 0 to 0. The pulled-back operator \( (\mathcal{L}_M)^* \) will be a Picard–Fuchs operator for \( \Gamma_0(NM) \), but it may not be the unique normal-form one \( \mathcal{L}_{NM} \). By Theorem 5.4 \( \mathcal{L}_M \) has exponents \( \frac{1}{12} \psi(M), \frac{1}{12} \psi(M) \) at the cusp \( t_M = \infty \) (i.e., \( \tau = 0 \)) and 0, 0 at other cusps. If the function \( t_M = t_M(t_{NM}) \) is not a polynomial, there will be a cusp other than \( t_{NM} = \infty \) in the inverse image of \( t_M = \infty \), and there will be a disparity in exponents: at any such cusp, the operator \( \mathcal{L}_{NM} \) will have exponents 0, 0, but \( (\mathcal{L}_M)^* \) will have exponents \( \frac{k}{12} \psi(M), \frac{k}{12} \psi(M) \), where \( k \) is the multiplicity with which the cusp appears in the fibre. Applying a similarity transformation to \( (\mathcal{L}_M)^* \), by performing the change of variable (substitution) \( \hat{u} = [Q(t_{NM})/Q(0)]^{-\psi(M)/12} u \), will remove this disparity, for all such cusps. By Theorem 5.2, the resulting transformed operator must equal \( \mathcal{L}_{NM} \), which has unique normalized holomorphic solution \( h_{NM} \) in a neighborhood of \( t_{NM} = 0 \): so the left and right sides are equal as claimed.

The second equality in each of parts 1, 2 is proved in a related way. Consider \( \mathcal{L}'_1 u = 0 \) and \( \mathcal{L}'_{M} u = 0 \), the Fuchsian differential equations on \( X(1)' \) and \( X_0(M) \) obtained by formally substituting \( J' \) (or \( j' \)) and \( t'_M \) for \( J \) (or \( j \)) and \( t_M \), in \( \hat{L}_1u = 0 \) and \( \mathcal{L}_M u = 0 \) respectively. Any ratio of independent solutions of \( \hat{L}_1u = 0 \) or \( \mathcal{L}_M u = 0 \) is of the form \( (a \tau + b)/(c \tau + d) \), where \( a, b, c, d \in \mathbb{C} \) with \( ad \neq bc \); so the same is true of \( \hat{L}_1 u = 0 \) and \( \mathcal{L}_M u = 0 \). (In effect, adding primes multiplies the coefficients \( a, c \) by \( N \).) The same is necessarily true of their respective pullbacks to \( X_0(M) \) and \( X_0(NM) \), by the definition of a pullback. But the pullbacks may
Corollary 8.1.1. If all primes that divide $N$ also divide $M$, then the modular forms $h_M = h_M \circ t_M$ and $h_{NM} = h_{NM} \circ t_{NM}$ are equal.

Proof. Under the divisibility assumption, $t_M = P(t_{NM})/Q(t_{NM})$ reduces to a polynomial map $t_M = P(t_{NM})$ by Proposition 8.1 so by part 2 of the theorem, $h_{NM}(t_{NM}(\tau)) = h_N(P(t_{NM}(\tau))) = h_M(t_M(\tau))$. □

Remark. This corollary reveals why there are duplications in column 3 of Table 16 which lists the eta product representations of the fourteen modular forms $h_M = h_M \circ t_M$. The seven duplicates are due to the seven identities

\begin{align*}
h_4(t) &= h_2(t(t + 16)), \quad (8.4a) \\
h_8(t) &= h_2(t(t + 16) \circ t(t + 8)), \quad (8.4b) \\
h_{16}(t) &= h_2(t(t + 16) \circ t(t + 8) \circ t(t + 4)), \quad (8.4c) \\
h_9(t) &= h_3(t(t^2 + 9t + 27)), \quad (8.5a) \\
h_{12}(t) &= h_6(t(t + 6)), \quad (8.6a) \\
h_{18}(t) &= h_6(t(t^2 + 6t + 12)), \quad (8.6b) \\
h_{25}(t) &= h_5(t(t^4 + 5t^3 + 15t^2 + 25t + 25)). \quad (8.7a)
\end{align*}

The identities \((8.4a)\)–\((8.4c)\) and \((8.5a)\) clarify the relations among the quadratic AGM representations, Eqs. \((7.1a)\)–\((7.1d)\), and the cubic ones, Eqs. \((7.2a)\)–\((7.2d)\), respectively.

Corollary 8.1.2. For each $N$ with $\Gamma_0(N)$ of genus zero, the modular discriminant $\Delta$ satisfies the modular equations given in Table 16 which express $\Delta(N\tau)/\Delta(\tau)$ in terms of the Hauptmodul $t_N$ for $\Gamma_0(N)$.
Proof. For each \(N\), the modular equation for \(E_4 = (\hat{h}_1 \circ \hat{J})^4\) follows from part 1 of Theorem 8.1 if one uses the formulas for \(j = j(t_N), j' = j'(t_N)\) given in Tables 4 and 5. Also, by Theorem 5.3 \(E_6 = (1 - \hat{J})^{1/2} \hat{h}_1 \circ \hat{J}^6\) and \(\Delta = (2\pi)^{12-3} j \hat{h}_1 \circ \hat{J}^{12}\), so the modular equations for \(E_6, \Delta\) follow from some elementary further manipulations, using the fact that \(\hat{J} = 12^3/j\). The modular equations for \(\Delta\) are given in the table; those for \(E_4, E_6\) are omitted. \(\square\)

Remark. The expressions for \(\Delta(N\tau)/\Delta(\tau)\) in Table 16 seem not to have been tabulated before. It is well known that if \(N - 1 \mid 24\), then \(N^{12} \Delta(N\tau)/\Delta(\tau)\) equals our canonical Hauptmodul \(t_N\). (See Apostol [2 Ch. 4].) But the table reveals what happens in the six cases when \(N - 1 \nmid 24\).

Each expression for \(\Delta(N\tau)/\Delta(\tau)\) in the table can alternatively be verified by rewriting it as an eta product, since \(\Delta(t_N)/\Delta(\tau) = [N]^{24}/[1]^{24}\). To do this, one would exploit the eta product representations given in Tables 2 and 3. To handle the case \(N = 18\), one would also need the eta product representations of \(\Delta\) given in Table 8, and those for Hauptmoduln and alternative Hauptmoduln given in Tables 2 and 3. To handle the cases \(M = 9, 16, 18, 25\), one also needs the special eta product formulas for certain non-univalent functions of \(t_M\) given in (6.1)–(6.5). The computations are straightforward and are left to

The following theorem is a slight extension of Theorem 8.1. It is proved in the same way.

Theorem 8.1’.

(1) If \(\Gamma_0(N)\) is of genus zero, and if for some \(d \mid N\), the function \(j(d\tau)\) can be expressed rationally in terms of \(t_N(\tau)\) (in lowest terms) by the formula

\[
\hat{h}_N(t_N) = \frac{[Q(t_N)/Q(0)]^{-1/2} \hat{h}_1(P(t_N)/Q(t_N))}{h_N(t_N) = [Q(t_N)/Q(0)]^{-1/2} \hat{h}_1(P(t_N)/Q(t_N))}
\]

holds in a neighborhood of the point \(t_N = 0\).

(2) If \(\Gamma_0(M), \Gamma_0(NM)\) are of genus zero, and if for some \(d \mid N\), the function \(t_M(d\tau)\) can be expressed rationally in terms of \(t_{NM}(\tau)\) by the formula

\[
t_M(t_M) = \frac{[Q(t_M)/Q(0)]^{-\psi(M)/2} h_N(P(t_M)/Q(t_M))}{h_{NM}(t_{NM}) = [Q(t_{NM})/Q(0)]^{-\psi(M)/2} h_N(P(t_{NM})/Q(t_{NM}))}
\]

holds in a neighborhood of the point \(t_{NM} = 0\).

Theorem 8.2. For each \(M\) with \(\Gamma_0(M)\) of genus zero, the function \(h_M\) defined in a neighborhood of \(t_M = 0\) on \(X_0(M)\) (i.e., of the infinite cusp) satisfies the rationally parametrized degree-\(N\) modular equations given in Table 17 of the form

\[
h_M(t_M(t_{NM})) = M_{M,N}(t_{NM}) \cdot h_M(t_{NM}).
\]

Proof. For each \(M, N\), the claimed modular equation follows from Theorem 8.1 (part 2), if one takes into account the formulas \(t_M = t_M(t_{NM}), t'_M = t'_M(t_{NM})\) given in Table 6. \(\square\)

Alternative Proof. For each \(M, N\), evaluate both sides as eta products, and verify that they are the same. One needs the eta product representations for the modular forms \(h_M\) given in Table 8 and those for Hauptmoduln and alternative Hauptmoduln given in Tables 2 and 3. To handle the cases \(M = 9, 16, 18, 25\), one also needs the special eta product formulas for certain non-univalent functions of \(t_M\) given in (6.1)–(6.5). The computations are straightforward and are left to
Table 17. Rationally parametrized modular equations for the weight-1 modular forms $h_M(\tau)$ for $\Gamma_0(M)$, viewed as functions $h_M = h_M(t_M)$ of the Hauptmoduln $t_M$. In each, $t := t_{NM}$.

| $M$ | $N$ | Modular equation for $h_M$, of degree $N$ |
|-----|-----|------------------------------------------|
| 2   | 2   | $h_2(t(t+16)) = 2(t+16)^{1/4} h_2 \left( \frac{t^2}{t^2+16} \right)$ |
| 2   | 3   | $h_2 \left( \frac{(t+8)^2}{t+9} \right) = 3(t+9)^{-1/2} h_2 \left( \frac{t^2}{t+9} \right)$ |
| 2   | 4   | $h_2(t(t+16) \circ t(t+8)) = 4 \left( [t+4](t+8) \right)^{-1/4} h_2 \left( \frac{t^2}{t+4} \circ \frac{t^2}{t+8} \right)$ |
| 2   | 5   | $h_2 \left( \frac{(t+4)^3}{t+2} \right) = 5(t+5)^{-1} h_2 \left( \frac{t^3}{t+5} \right)$ |
| 2   | 6   | $h_2 \left( \frac{(t+8)^3}{t+9} \circ t(t+6) \circ t(t+4) \right) = 6 \left( [t+2](t+3)(t+6)^3 \right)^{-1/4} h_2 \left( \frac{t^3}{t+2} \circ \frac{t^2}{t+3} \circ \frac{t^2}{t+6} \right)$ |
| 2   | 8   | $h_2(t(t+16) \circ t(t+8) \circ t(t+4)) = 8 \left( [t+2](t+4)^4(t^2+4t+8) \right)^{-1/4} h_2 \left( \frac{t^2}{t+2} \circ \frac{t^2}{t+4} \circ \frac{t^2}{t+8} \right)$ |
| 2   | 9   | $h_2 \left( \frac{(t+8)^3}{t+9} \circ t(t^2+6t+12) \right) = 9(t+3)^{-2} h_2 \left( \frac{t^3}{t+3} \circ \frac{t^2}{t+2} \circ \frac{t^2}{t+6} \right)$ |
| 3   | 2   | $h_3 \left( \frac{(t+9)^2}{t+8} \right) = 2(t+8)^{-1/3} h_3 \left( \frac{t^2}{t+8} \right)$ |
| 3   | 3   | $h_3 \left( \frac{(t+9+4t^2+27)}{t+8} \right) = 3(t^2+9t+27)^{-1/3} h_3 \left( \frac{t^3}{t+2} \circ \frac{t^2}{t+8} \right)$ |
| 4   | 4   | $h_4 \left( t(t+8) \circ t(t+6) \circ t(t+4) \right) = 4 \left( [t+2](t^2+4t+8) \right)^{-1/2} h_4 \left( \frac{t^2}{t+2} \circ \frac{t^2}{t+4} \right)$ |
| 5   | 2   | $h_5 \left( \frac{(t+5)^2}{t+4} \right) = 2(t+4)^{-1/2} h_5 \left( \frac{t^2}{t+4} \right)$ |
| 5   | 5   | $h_5 \left( t(t^4+5t^3+15t^2+25t+25) \right) = 5 \left( [t^2+5t^3+15t^2+25t+25] \right)^{-1/2} h_5 \left( \frac{t^5}{t^2+5t^3+15t^2+25t+25} \right)$ |
| 6   | 2   | $h_6(t(t+6)) = 2(t+2)^{-1} h_6 \left( \frac{t^2}{t+2} \right)$ |
| 6   | 3   | $h_6(t(t^2+6t+12)) = 3(t^2+3t+3)^{-1} h_6 \left( \frac{t^3}{t^2+3t+3} \right)$ |

Remark. At a general point $\tau \in \mathcal{H}$, each equation in Table 17 evaluates to

$$(h_M \circ t_M)(\tau) = M_{M,N}(t_{NM}(\tau)) \cdot (h_M \circ t_M)(N\tau),$$

and can be called an $N : 1$ modular equation. The degree-$N$ multiplier $M_{M,N} = M_{M,N}(t_{NM})$ specifies the relationship between the period modules of the fibres.
(elliptic curves) over $t_M, t'_M \in X_0(M)$, for the elliptic surface $E_M \rightarrow X_0(M)$. It is algebraic over each of the related points $t_M, t'_M$ on the base.

**Remark.** Each equation in Table 14 holds for all $t > 0$, if one interprets $h_M = h_M(t_M)$ as the unique holomorphic continuation along the positive $t_M$-axis. The two argument functions in each equation, i.e., $t = t_N M \rightarrow t_M$ and $t = t_N M \rightarrow t'_M$, by examination take $(0, \infty)$ bijectively to $(0, \infty)$.

There are two additional modular equations that one can derive, using not Theorem 8.1 but its enhancement Theorem 8.1'.

**Theorem 8.3.** The functions $h_2 = h_2(t_2), h_3 = h_3(t_3)$, in neighborhoods of the points $t_2 = 0, t_3 = 0$, satisfy respective functional equations

\[
h_2 \left( \frac{t^2}{t^2 + 1} \right) \circ \left( \frac{t(t+4)^3}{t^3 + 1} \right) = \left( \frac{t(t+8)^3}{t^3 + 1} \right) \circ \left( \frac{t^2}{t^2 + 1} \right) = \left( \frac{t+3}{t^2 + 1} \right) \circ \left( \frac{t(t+4)}{t^2 + 1} \right) \equiv \left( \frac{t(t+16)}{t^2 + 1} \right) \circ \left( \frac{t^3(t+8)}{t^2 + 1} \right),
\]

\[
h_3 \left( \frac{t(t^2 + 9t + 27)}{t^2 + 1} \right) \circ \left( \frac{t(t+9)^3}{t^3 + 1} \right) = \left( \frac{t(t^2 + 9t + 27)}{t^2 + 1} \right) \circ \left( \frac{t(t^2 + 6t + 12)}{t^2 + 1} \right) = \left( \frac{t(t+3)^3}{t^2 + 1} \right) \circ \left( \frac{t^2}{t^2 + 1} \right) \equiv \left( \frac{t(t+9)^3}{t^2 + 1} \right) \circ \left( \frac{t(t^2 + 9t + 27)}{t^2 + 1} \right),
\]

which can be called modular equations of degree $3:2$, since they relate $(h_2 \circ t_2)(2\tau)$ to $(h_2 \circ t_2)(3\tau)$, and $(h_3 \circ t_3)(2\tau)$ to $(h_3 \circ t_3)(3\tau)$.

**Proof.** In these two equations, the parameter $t$ signifies respectively the Hauptmoduln $t_12$ and $t_18$. The first equation follows from the $M = 2, N = 6$ case of Theorem 8.1 (part 2), by equating the expressions for $h_{12} = h_{12}(t_{12})$ obtained from the cases $d = 2, 3$. The second follows similarly from the $M = 3, N = 6$ case, by equating expressions for $h_{18} = h_{18}(t_{18})$. \qed

Each modular equation for $h_M$ in Table 17 (and Theorem 8.3) with $M = 2, 3, 4$ can be written as an algebraic transformation of $2F_1$, with the aid of the expressions for $h_M$ given in Table 15. These transformations will be discussed in §3. The modular equations with $M = 5, 6, 7, 8, 9$ can similarly be written as algebraic transformations of the local Heun function $H_l$. As special function identities, they are quite novel, since there is not even a rudimentary theory of Heun transformations (unlike the classical theory of transformations of $2F_1$). It is possible to restate them as functional equations for combinatorial generating functions, without explicit reference to $H_l$. The following striking proposition illustrates this.

**Proposition 8.4.** Let $F = F(x) = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the Fresnel numbers $a_n = \sum_{k=0}^{n} \binom{n}{k}^3$, $n \geq 0$. Then $F$, which is defined on the disk $|x| < 1/8$, satisfies the quadratic and cubic functional equations

\[
F \left( \frac{t(t+6)}{8(t+3)} \right) = 2 \left[ \frac{t+1}{t+6} \right] F \left( \frac{t^2}{8(t+3)(t+6)} \right),
\]

\[
F \left( \frac{t(t^2 + 9t + 27)}{8(t+3)(t+9)} \right) = 3 \left[ \frac{t^2 + 3(t+3)}{t^2 + 9(t+3)} \right] F \left( \frac{t^3}{8(t+3)^3} \right),
\]

for $|t|$ sufficiently small, and also for all $t > 0$.\]
Proposition 8.5. Let parameters we shall give a direct modular interpretation of Ramanujan’s theories defined by modular correspondence that doubles the period ratio \( \tau \) or base \( t \). treatment in this section is on the level of special function manipulations. In we derive the full set of such modular equations. Most are shown in Table 18. Our the ones proved by Berndt, Bhargava, and Garvan [5], and others. In this section stories of elliptic integrals to alternative bases. These include but are not limited to \( M \) parametrized by \( \gamma \). The Franel numbers \( \tau \) that (changing the dummy variable \( t \) to \( t_0 \)) \[ F(t_0) = (\bar{h}_6 \circ \bar{\phi})(8t_0) = (1 - 8t_0)^{-1}h_6\left(\frac{\tau_{2t_0}}{1 - 8t_0}\right) \] \[ = H(\frac{1}{8}, \frac{1}{8}, 1, 1, 1, 1; -t_0). \] (8.9)

The functional equations accordingly follow from the modular equations for \( h_6 \) of degrees \( N = 2, 3 \) in Table 17. The parameter \( t \) is respectively \( t_{12}, t_{18} \).

The lone modular equation for \( h_9 \) in Table 17 of degree \( N = 2 \), implies the following proposition on the behavior of the periods of the Hesse–Dixon family \[ x^3 + y^3 + 1 - (\gamma + 3)xy = 0, \] (8.10) parametrized by \( \gamma \in \mathbb{C} \setminus \{0, 3(\zeta_3 - 1), 3(\zeta_3^2 - 1)\} \), under the action of the level-2 modular correspondence that doubles the period ratio \( \tau = \tau_1/\tau_2 \).

**Proposition 8.5.** Let \( E \) and \( E' \) be Hesse–Dixon elliptic curves with respective parameters \( \gamma \) and \( \gamma' \), which are related parametrically by \[ \gamma = \frac{t(t+3)^2}{t+2}, \quad \gamma' = \frac{t^2(t+3)}{(t+2)^2}. \]

Then their respective fundamental periods \( \tau_1, \tau_2 \) and \( \tau'_1, \tau'_2 \) will be related by \( \tau'_1 = (t+2)\tau_1, \quad \tau'_2 = [(t+2)/2]\tau_2. \)

9. Ramanujan’s Modular Equations in Signatures 2, 3, 4

The modular equations derived in [13] for the elliptic families \( E_M \sim \mathbb{Z} \) \( X_0(M) \), \( M = 2, 3, 4 \), yield rationally parametrized modular equations in Ramanujan’s theories of elliptic integrals to alternative bases. These include but are not limited to the ones proved by Berndt, Bhargava, and Garvan [7], and others. In this section we derive the full set of such modular equations. Most are shown in Table 18. Our treatment in this section is on the level of special function manipulations. In \([14]\) we shall give a direct modular interpretation of Ramanujan’s theories.

His complete elliptic integral (of the first kind) in the theory of signature \( r \) (or base \( r \)), where \( r > 1 \), is \( K_r : (0, 1) \rightarrow \mathbb{R}^+, \) a monotone increasing function defined by

\[ K_r(\alpha_r) = \frac{\pi}{2} 2F_1\left(\frac{1}{2}, 1 - \frac{1}{r}; 1; \alpha_r\right) \] (9.1a)

\[ = \frac{\sin(\pi/r)}{2} \int_0^1 x^{-1/r}(1-x)^{-1+1/r}(1-\alpha_rx)^{-1/r} dx. \] (9.1b)

Its limits as \( \alpha_r \to 0^+, 1^- \) are \( \pi/2, \infty \), and in the latter limit it diverges logarithmically. The representation \( (9.1b) \) comes from Euler’s integral formula for \( 2F_1 \) [20 §2.1.3]. By the Schwartz–Christoffel theory, \( K_r(\alpha_r) \) can be interpreted [4] as the side length of a certain parallelogram with angles \( \frac{\pi}{r}, \pi(1 - \frac{1}{r}) \), its aspect ratio being such that it can be mapped conformally onto the upper half plane, with its vertices taken to \( 0, 1, \alpha_r^{-1}, \infty \).

In Ramanujan’s theory of signature \( r \), an \( \alpha_r-\beta_r \) modular equation of degree \( N \) is an explicit relation between \( \alpha_r, \beta_r \in (0, 1) \) induced by

\[ \frac{K'_r(\beta_r)}{K_r(\beta_r)} = N \frac{K'_r(\alpha_r)}{K_r(\alpha_r)}, \] (9.2)
where $K'_r(\alpha_r) := K_r(1 - \alpha_r)$ is the so-called complementary complete elliptic integral, and $N \in \mathbb{Q}^+$ is the degree. A degree-$N$ modular equation in signature $r$, interpreted more strongly, should also include an expression for the ‘multiplier’ $K_r(\alpha_r)/K_r(\beta_r)$ from which $\frac{2}{\psi}K_r(\alpha_r)$ may be recovered. If $N = N_1/N_2$ in lowest terms, a degree-$N$ modular equation will optionally be referred to here as a modular equation of degree $N_1 : N_2$. In all closed-form signature-$r$ modular equations found to date, $r = 2, 3, 4$, or $6$, and the $\alpha_r - \beta_r$ relation is algebraic.

To see that modular equations for the families of elliptic curves attached to $\Gamma_0(M)$, $M = 2, 3, 4$, can be converted to modular equations in the theories of signature $r = 4, 3, 2$, where $r = 12/\psi(M)$, recall from (9.3) that when $M = 2, 3, 4$,

$$h_M(t_M) = 2F_1\left(\frac{12}{12}\psi(M), \frac{12}{12}\psi(M); 1; -t_M/\kappa_M^{1/2}\right),$$

(9.3)

with $\kappa_M = 2^{12}, 3^{6}, 2^{8}$, respectively. The function $h_M$ has a holomorphic continuation from a neighborhood of $t_M = 0$ to the half-line $t_M > 0$. Let a bijection between $(0, \infty) \ni t_M$ and $(0, 1) \ni \alpha_r$, where $r = 12/\psi(M)$, be given by

$$\alpha_r = \alpha_r(t_M) = t_M/(t_M + \kappa_M^{1/2}),$$

(9.4a)

$$t_M = t_M(\alpha_r) = \kappa_M^{1/2} \alpha_r/(1 - \alpha_r).$$

(9.4b)

Then one can write

$$\frac{2}{\psi}K_r(\alpha_r) = (1 - \alpha_r)^{-\psi(M)/12} h_M(t_M(\alpha_r)),$$

(9.5a)

$$h_M(t_M) = (1 + t_M/\kappa_M^{1/2})^{-\psi(M)/12} \frac{2}{\psi}K_r(\alpha_r(t_M)).$$

(9.5b)

These follow immediately from (A.5), Pfaff’s transformation of $2F_1$.

**Theorem 9.1.** If $M = 2, 3, 4$, on $t_M > 0$ the ratio $iK'_r(\alpha_r(t_M))/K_r(\alpha_r(t_M))$ equals $C_M \tau(t_M)$, where $\tau(\cdot)$ is the $\mathbb{R}$-valued branch of the period ratio $\tau$ that was introduced in (8.8). In all three cases, the prefactor $C_M$ equals $M^{1/2}$.

**Proof.** Rewrite the representation for $\tau(t_M)$ given in Corollary 7.4.1 in terms of $K_r(\alpha_r(t_M))$, with the aid of (9.5b). \(\square\)

**Remark.** By reviewing the proof of Corollary 7.4.1 one discovers that the simple formula $C_M = M^{1/2}$ for the prefactor comes ultimately from a subtle result (Eq. (7.7)) on the asymptotic behavior of the Gauss hypergeometric function, together with evaluations of the Euler digamma function $\Psi(a)$ at the rational points $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $a = 1 - \frac{1}{2}, 1 - \frac{1}{3}, 1 - \frac{1}{4}$. This is a remarkably complicated proof of a seemingly simple result.

**Corollary 9.1.1.** In the cases $M = 2, 3, 4$, any degree $N_1 : N_2$ modular equation for the function $h_M = h_M(t_M)$ associated to the group $\Gamma_0(N)$ may be converted to a degree $N_1 : N_2$ modular equation for Ramanujan’s complete elliptic integral of the first kind, $K_r = K_r(\alpha_r)$, with $r = 12/\psi(M)$.

**Theorem 9.2.** In the theories of signature $r = 2, 3, 4$, one has the $N : 1$ modular equations of Table 18. Each holds for all $t > 0$ and incorporates a rationally parametrized $\alpha_r - \beta_r$ modular equation of degree $N$, the pair $\alpha_r, \beta_r$ being the arguments of $K_r$ on the left and right sides. In addition, $K_3$ and $K_4$ satisfy
Table 18. Rationally parametrized modular equations in the theories of signature \( r = 2, 3, 4 \). The parameter \( t \) signifies respectively \( t_{4N}, t_{3N}, t_{2N} \), i.e., the canonical Hauptmodul for \( \Gamma_0(4N), \Gamma_0(3N), \Gamma_0(2N) \).

| \( r \) | \( N \) | Modular equation for \( K_r \), of degree \( N \) |
|-----|-----|--------------------------------------------------|
| 2   | 2   | \( K_2 \left( \frac{t(t+4)}{(t+8)^2} \right) = 2 \left( \frac{t^2}{t+8} \right) K_2 \left( \frac{t^2}{t+8} \right) \) |
| 3   | 2   | \( K_2 \left( \frac{t(t+4)}{(t+8)^2} \right) = 3 \left( \frac{t^2}{t+8} \right) K_2 \left( \frac{t^2}{t+8} \right) \) |
| 4   | 2   | \( K_2 \left( \frac{t(t+4)}{(t+8)^2} \right) = 4 \left( \frac{t^2}{t+8} \right)^2 K_2 \left( \frac{t^2}{t+8} \right)^2 \) |
| 3   | 3   | \( K_3 \left( \frac{t(t+9)^2}{(t+6)^2} \right) = 3 \left( \frac{t^3}{t+9} \right) K_3 \left( \frac{t^3}{t+9} \right) \) |
| 4   | 3   | \( K_3 \left( \frac{t(t+9)^2}{(t+6)^2} \right) = 4 \left( \frac{t^3}{t+9} \right)^3 K_3 \left( \frac{t^3}{t+9} \right)^3 \) |
| 6   | 3   | \( K_3 \left( \frac{t(t+9)^2}{(t+6)^2} \right) \circ t(t+6) = 6 \left( \frac{t^3}{t+9} \right)^3 \circ t(t+6) \) |
| 4   | 4   | \( K_4 \left( \frac{t(t+16)}{(t+8)^2} \right)^2 = 4 \left( \frac{t^4}{t+8} \right)^2 K_4 \left( \frac{t^4}{t+8} \right)^2 \) |
| 5   | 4   | \( K_4 \left( \frac{t(t+16)}{(t+8)^2} \right)^3 = 5 \left( \frac{t^5}{t+8} \right)^3 K_4 \left( \frac{t^5}{t+8} \right)^3 \) |
| 6   | 4   | \( K_4 \left( \frac{t(t+16)}{(t+8)^2} \right) \circ t(t+6) = 6 \left( \frac{t^6}{t+8} \right)^3 \circ t(t+6) \) |
| 8   | 4   | \( K_4 \left( \frac{t(t+16)}{(t+8)^2} \right) \circ t(t+4) = 8 \left( \frac{t^8}{t+8} \right)^3 \circ t(t+4) \) |
| 9   | 4   | \( K_4 \left( \frac{t(t+16)}{(t+8)^2} \right) \circ t(t^2+6t+12) = 9 \left( \frac{t^9}{t+8} \right)^3 \circ t(t^2+6t+12) \) |

\[
K_3 \left( \frac{t^2}{(t+9)^2} \right) \circ t(t+9) = \frac{t^2}{(t+9)^2} \circ t(t^2+6t+12)
\]

\[
\left( \frac{2}{2} \right) \left( \frac{t^2}{(t+9)^2} \right) \circ t(t+9) \left( \frac{t^3}{(t+9)^2} \right) \circ t(t+9) = \frac{t^3}{(t+9)^2} \circ t(t^3+3t^2+3t+1)
\]

\[
K_4 \left( \frac{t^2}{(t+3)^2} \right) \circ t(t+3) = \frac{t^2}{(t+3)^2} \circ t(t+3)
\]

\[
\left( \frac{2}{2} \right) \left( \frac{t^2}{(t+3)^2} \right) \circ t(t+3) \left( \frac{t^3}{(t+3)^2} \right) \circ t(t+3) = \frac{t^3}{(t+3)^2} \circ t(t^3+3t^2+3t+1)
\]

\[
\times K_4 \left( \frac{t(t+16)}{(t+8)^2} \right) \circ t(t+6) = \frac{t^3}{(t+8)^2} \circ t(t+6)
\]

which are modular equations of degree 3 : 2, i.e., of degree \( \frac{3}{2} \).
Proof. Rewrite the modular equations given in Theorems 8.2 and 8.3 for $h_M$, $M = 2, 3, 4$, in terms of $K_r$, $r = 4, 3, 2$, with the aid of \(9.5^{11}\).

One can also derive modular equations of a certain mixed type from the elliptic-family modular equations of \(8\). These are $\alpha_r$-$\beta_s$ relations, induced by

$$\frac{K_r'(\beta_r)}{K_r(\beta_r)} = N \frac{K_r'(\alpha_r)}{K_r(\alpha_r)},$$

(9.6)

where $r \neq s$. If the degree $N \in \mathbb{Q}^+$ equals $N_1/N_2$ in lowest terms, such an algebraic relation may be called an $\alpha_r$-$\beta_s$ modular equation of degree $N_1 : N_2$; or if accompanied by an explicit multiplier, a modular transformation of degree $N_1 : N_2$ from $K_r$ to $K_s$. The source of such combined modular equations, when $r, s \in \{2, 3, 4\}$, is commensurability: the fact that the intersection of any two of the subgroups $\Gamma_0(4), \Gamma_0(3), \Gamma_0(2)$ has finite index in both.

**Theorem 9.3.** The full collection of rationally parametrized $\alpha_r$-$\beta_s$ modular equations, for $r \neq s$ with $r, s \in \{2, 3, 4\}$, includes

- $\alpha_2$-$\beta_3$ modular equations of degrees $4, 2, \frac{4}{3}, 1, \frac{2}{3}, \frac{1}{3}$;
- $\alpha_2$-$\beta_2$ modular equations of degrees $8, 6, 4, 3, 2, 1, \frac{2}{3}, \frac{1}{3}, \frac{1}{4}$;
- $\alpha_3$-$\beta_3$ modular equations of degrees $9, \frac{9}{2}, 6, 3, \frac{3}{2}, 2, 1, \frac{1}{2}, \frac{1}{3}$, $\frac{1}{4}$.

Each such $\alpha_r$-$\beta_s$ relation is the basis of a rationally parametrized modular transformation from $K_r$ to $K_s$.

**Proof.** For any $\mathcal{N}$ with $\Gamma(\mathcal{N})$ of genus zero, $t_M(d\tau)$ may be expressed rationally in terms of $t_{\mathcal{N}}(\tau)$ if $M \mid \mathcal{N}$ and $d \mid (\mathcal{N}/M)$. So if $M_i \mid \mathcal{N}$ and $d_i \mid (\mathcal{N}/M_i)$, $i = 1, 2$, there is an $t_{M_i}(d_i \tau) - t_{M_2}(d_2 \tau)$ modular equation, i.e., an $t_{M_i} - t_{M_2}$ modular equation of degree $d_2/d_1$. By Theorem \(8.1^{11}\) (part 2), it yields a modular transformation from $h_{M_1}$ to $h_{M_2}$. But the Hauptmodul $t_{M_i}$ and function $h_{M_i} = h_{M_i}(t_{M_i})$, when $M_i = 2, 3, 4$, correspond to the invariant $\alpha_r$ and elliptic integral $K_r = K_r(\alpha_r)$, where $r = 4, 3, 2$. The given lists are computed by considering all relevant $\mathcal{N}$ (i.e., $\mathcal{N} = 4, 6, 8, 12, 16, 18$), and enumerating divisors $M_i \in \{2, 3, 4\}$, and $d_i$ such that $d_i \mid (\mathcal{N}/M_i)$. \(\Box\)

**Remark.** The collection mentioned in the theorem consists of 28 modular equations, but at most 14 are independent. As Berndt et al. have observed, for any $r \neq s$ there is an involution of the set of $\alpha_r$-$\beta_s$ modular equations, called ‘reciprocation.’ It consists in applying the simultaneous substitutions $\alpha_r \mapsto 1 - \alpha_r$, $\beta_s \mapsto 1 - \beta_s$. If $r = 12/\psi(M_1)$ and $s = 12/\psi(M_2)$ with $M_1, M_2 \in \{2, 3, 4\}$, it is not difficult to see that the procedure transforms an $\alpha_r$-$\beta_s$ modular equation of degree $N$ to one of degree $(M_1/M_2)/N$. For the three types of modular equation mentioned in the theorem, the degree involutions are $N \mapsto \frac{1}{2}/N$, $N \mapsto 2/N$, and $N \mapsto 2^2/N$.

The extent to which the modular equations derived in this section are new can now be discussed. To begin with, the modular equations for $K_2$ in Table \(18\) of degrees $N = 2, 3, 4$, are classical. This is because $r = 2$ is the classical base, and $K_2 = K_2(\alpha_2)$ is identical to $K = K(\alpha)$, the classical complete elliptic integral, which is a single-valued function of the $\alpha$-invariant in a neighborhood of the infinite cusp, at which $\alpha = 0$. The argument $\alpha_2$ of $K_2$ is now seen to have a modular interpretation: being identical to $\alpha$, it is a Hauptmodul for $\Gamma_0(4)$. The three underlying $\alpha_2$-$\beta_2$ modular equations, i.e., $\alpha$-$\beta$ modular equations, were discussed
The $N = 2$ equation is Landen’s transformation, in the form already given in Eq. (1.3).

Of the remaining rationally parametrized equations in the table, Ramanujan found the degree-2, 3, 4 modular equations for $K_3$ and $K_4$. That is, he found the underlying $\alpha_3-\beta_3$ and $\alpha_4-\beta_4$ modular equations, and expressions for the multipliers. He also found the degree-$\frac{2}{3}$ modular equation for $K_3$, given in Theorem 9.2, and several of the transformations mentioned in Theorem 9.3, those from $K_2$ to $K_3$ of degrees $1, \frac{2}{3}$, those from $K_2$ to $K_4$ of degrees $1, 2$, and that from $K_3$ to $K_4$ of degree $1$. Presumably he was aware of the reciprocation principle.

Proofs of his results have been constructed by the Borweins, Berndt, Garvan, and others. The Borweins derived the transformations of degree 1 among $K_3, K_3, K_4$ [7 §5.5], and the degree-3 modular equation in the theory of signature 3 [8]. Berndt, Bhargava and Garvan [5] (see also Berndt [4 Chap. 33]) systematically derived Ramanujan’s remaining modular equations and transformations in the theories of signature 3 and 4. Garvan [26, Eq. (2.34)] additionally derived the transformation $\kappa$ in Eq. (1.3).

Recently, Berndt, Chan and Liaw [6] have conducted further investigations into the theory of signature 4. However, the signature-4 modular equations in Table 18 of degrees greater than 4 are new, as is the signature-4 degree-$\frac{4}{3}$ equation; and also the signature-3 degree-6 equation. Of the $14 = 28/2$ independent rationally parametrized transformations of $K_r$ to $K_s$ identified in Theorem 9.3 all of which can readily be worked out explicitly if needed, 10 are new.

10. A Modular Approach to Signatures 2, 3, 4 (and 6)

The derivation in [3] of modular equations in Ramanujan’s theories of signature $r = 2, 3, 4$ relied on Pfaff’s transformation of $2F_1$, and took place on the level of special functions. In this section we give a direct modular interpretation of the complete elliptic integral $K_r$ and its argument $\alpha_r$ (the ‘$\alpha_r$-invariant’). Modular interpretations were pioneered by the Borweins [7] [8], but our new interpretation is very concise, and lends itself to extension. Like the classical complete elliptic integral, each $K_r(\alpha_r)$ is simply a weight-1 modular form, expressed as a function of a Hauptmodul. We close by discussing the underdeveloped, and very interesting, theory of signature 6. It does not fit into the Picard–Fuchs framework of this article, but we indicate a slightly more general Gauss–Manin framework into which it fits.

When $M = 2, 3, 4$ and correspondingly $r = 12/\psi(M) = 4, 3, 2$, consider the subgroup $\Gamma_0(M) < \Gamma(1)$. The function field of $X_0(M) = \Gamma_0(M) \setminus \mathcal{H}^*$ is generated by the canonical Hauptmodul

$$t_M = \kappa_M \cdot [M]^{24/(M-1)}/[1]^{24/(M-1)},$$

(10.1a)

$$\kappa_M := M^{12/(M-1)},$$

(10.1b)

and there are exactly three fixed points on $X_0(M)$. (See Table 2) These are the cusp $t_M = 0$ (i.e., $\tau \in \left[\frac{1}{M}\right]$, including the infinite cusp $\tau = i\infty$), the cusp $t_M = \infty$ (i.e., $\tau \in \left[\frac{1}{M}\right]$, including $\tau = 0$), and the third fixed point $t_M = -\kappa_M^{1/2}$. When $M = 2, 3, 4$, the third point is respectively a quadratic elliptic point, a cubic one, and a cusp (namely, $\tau \in \left[\frac{1}{M}\right]$). To focus on this third fixed point, which is stabilized by the Fricke involution $t_M \mapsto \kappa_M/t_M$, one defines an alternative Hauptmodul

$$\alpha_r := t_M/(t_M + \kappa_M^{1/2})$$
that is zero at the infinite cusp, like the canonical Hauptmodul \( t_M \), but has its pole at the third point rather than at the cusp \( \tau = 0 \). At the latter cusp, \( \alpha_r = 1 \).

In terms of \( \alpha_r \), the Fricke involution is the map \( \alpha_r \mapsto 1 - \alpha_r \).

For each signature \( r = 2, 3, 4 \), associated to \( M = 4, 3, 2 \) by \( r = 12/\psi(M) \), define also a triple of weight-1 modular forms \( A_r, B_r, C_r \) for \( \Gamma_0(M) \) thus:

\[
A_r := (1 - \alpha_r)^{-1/r} B_r, \quad B_r := h_M, \quad C_r := \left[ \frac{1}{\alpha_r} \right]^{-1/r} B_r,
\]

where \( h_M = h_M \circ t_M \) is the canonical weight-1 modular form. By Corollary 5.3.1, \( B_r \) has divisor \( \psi(M)/12 (t_M = \infty) = \frac{1}{r} (\alpha_r = 1) \) on \( X_0(M) \). As a consequence of their definition, \( C_r \) has divisor \( \psi(M)/12 (t_M = 0) = \frac{1}{r} (\alpha_r = 0) \) and \( A_r \) has divisor \( \psi(M)/12 (t_M = -\kappa_M^{1/2}) = \frac{1}{r} (\alpha_r = \infty) \). Also, the triple satisfies (by definition)

\[
A_r^r = B_r^r + C_r^r,
\]

which in each case is an equality between two weight-\( r \) modular forms.

**Theorem 10.1.** For \( r = 2, 3, 4 \), corresponding to \( M = 4, 3, 2 \) by \( r = 12/\psi(M) \), the Hauptmodul \( \alpha_r \) and triple of weight-1 modular forms \( A_r, B_r, C_r \) for \( \Gamma_0(M) \) have the eta product representations and \( q \)-expansions shown in Table \( \text{Table 10} \).

**Proof.** The formulas for \( t_M \) and \( B_r := h_M = h_M \circ t_M \) are taken from Table \( \text{Table 8} \) and those for \( C_r \) from Table \( \text{Table 11} \) since \( C_r = M^{\psi(M)/2(M-1)} h_M \circ t_M \), in that table’s notation. Since \( A_2 \) equals \( [(t_4 + 16)/16]^{1/2} (h_4 \circ t_4) \), it is an alternative weight-1 modular form of the sort considered in \( \text{[6]} \) with its (single) zero located at a finite cusp other than \( \tau = 0 \); so its eta product and \( q \)-series can be found in Table \( \text{Table 10} \).

The \( q \)-series representations for \( A_3 \) and \( A_4^2 \) were discovered by Ramanujan. \( \square \)

**Remark.** As is clear from its definition as \( t_4/(t_4 + 16) \) and its eta product representation, the \( \alpha_3 \)-invariant is the \( \alpha \)-invariant, an alternative Hauptmodul for \( \Gamma_0(4) \).

Ramanujan’s theory of signature 2 is Jacobi’s classical theory of elliptic integrals. The \( r = 2 \) identity \( A_2^2 = B_2^2 + C_2^2 \) is, in fact, the Jacobi theta identity \( \vartheta_3(0)^4 = \vartheta_4(0)^4 + \vartheta_2(0)^4 \), as is evident from the \( q \)-series for \( A_2, B_2, C_2 \).

**Remark.** It also follows by examining the \( q \)-series for \( A_r, B_r, C_r \), that the identities \( A_r^r = B_r^r + C_r^r \), \( r = 3, 4 \), are explicitly modular restatements of \( q \)-series identities previously obtained by the Borweins \( \text{[8]} \).

**Remark.** The modular form \( A_4 \) for \( \Gamma_0(2) \) is anomalous: no number-theoretic interpretation of the coefficients of its \( q \)-series \( 1 + 12q - 5q^2 + 64q^3 - 917q^4 + \cdots \) is known, unlike that of \( A_4^2 \), which is the theta series of the \( D_4 \) lattice packing. Its coefficient sequence \( 1, 12, -60, 768, -11004, \ldots \) is Sloane’s \( \text{A108096} \). For some apocryphal remarks on how roots of \( q \)-series with integer coefficients may unexpectedly turn out to have the same property, see Heninger et al. \( \text{[30]} \).

By applying Pfaff’s transformation of \( 2F1 \), one can immediately transform the known fact (see \( \text{[17]} \) that for \( M = 4, 3, 2 \),

\[
h_M(t_M) = 2F1 \left( \frac{1}{12} \psi(M), \frac{1}{12} \psi(M); 1; -t_M/\kappa_M^{1/2} \right),
\]

into the statement that \( A_r = \check{K}_r \circ \alpha_r \) when \( r = 12/\psi(M) = 2, 3, 4 \), with

\[
\check{K}_r(\alpha_r) := \frac{2}{\alpha_r} K_r(\alpha_r) = 2F1 \left( \frac{1}{r}, 1 - \frac{1}{r}; 1; \alpha_r \right).
\]
Table 19. For $M = 2, 3, 4$, the Hauptmodul $t_M$ and weight-1 modular forms $h_M$, for $\Gamma_0(M)$; and in the corresponding theory of signature $r = 12/\psi(M) = 4, 3, 2$, the Hauptmodul $\alpha_r$ and modular forms $A_r, B_r, C_r$, which satisfy $A_r^r = B_r^r + C_r^r$. Here $K_r$ signifies $(2/\pi)K_r$, and $\sum \text{ means } \sum_{n=1}^{\infty}.$

| $M$ | $r$ | Hauptmodulin and modular forms |
|-----|-----|--------------------------------|
| 2   | 4   | $t_2 = [2]^{12}/[1]^{24}$    |
|     |     | $h_2 = h_2 \circ t_2 = [1]^{1}/[2]^{2} = 1 + 4 \sum E_1(n; 4) (-q)^n$ |
|     |     | $\alpha_2 = 2^6 [2]^{24}/(2^6 [2]^{24} + [1]^{24})$ |
|     |     | $1 - \alpha_2 = [1]^{12}/(2^6 [2]^{24} + [1]^{24})$ |
|     |     | $A_2 = K_2 \circ \alpha_2 = (2^6 [2]^{24} + [1]^{24})^{1/2}/[1]^{2}[2]^{2}$ |
|     |     | $= \sqrt{1 + 24 \sum \left(\sum_{d|n, \text{ odd}} d\right) q^n}$ |
|     |     | $B_2 = [1]^{1}/[2]^{2} = 1 + 4 \sum E_1(n; 4) (-q)^n$ |
|     |     | $C_2 = 2^{3/2} \cdot [2]^{4}/[1]^{2} = 2^{3/2} q^{1/4} [1 + \sum E_1(4n + 1; 4) q^n]$ |
| 3   | 3   | $t_3 = [3]^{[3]^{12}/[1]^{12}}$ |
|     |     | $h_3 = h_3 \circ t_3 = [1]^{3}/[3] = 1 - 3 \sum [E_1(n; 3) - 3E_1(n/3; 3)] q^n$ |
|     |     | $\alpha_3 = 3 [3]^{[3]^{12}/[1]^{12}}$ |
|     |     | $1 - \alpha_3 = [1]^{12}/(3^3 [3]^{12} + [1]^{12})$ |
|     |     | $A_3 = K_3 \circ \alpha_3 = (3^3 [3]^{12} + [1]^{12})^{1/3}/[1]^{3}$ |
|     |     | $= 1 + 6 \sum E_1(n; 3) q^n$ |
|     |     | $B_3 = [1]^{3}/[3] = 1 - 3 \sum [E_1(n; 3) - 3E_1(n/3; 3)] q^n$ |
|     |     | $C_3 = 3 \cdot [3]^{3}/[1] = 3 q^{1/3} [1 + \sum E_1(3n + 1; 3) q^n]$ |
| 4   | 2   | $t_4 = [4]^{[4]^{8}/[1]^{8}}$ |
|     |     | $h_4 = h_4 \circ t_4 = [1]^{4}/[2]^{2} = 1 + 4 \sum E_1(n; 4) (-q)^n$ |
|     |     | $\alpha_2 = 2^4 [4]^{8}/(2^4 [4]^{8} + [1]^{8}) = 2^4 \cdot [1]^{8}[4]^{16}/[2]^{24}$ |
|     |     | $1 - \alpha_2 = [1]^{8}/(2^4 [4]^{8} + [1]^{8}) = [1]^{16}[4]^{8}/[2]^{24}$ |
|     |     | $A_2 = K_2 \circ \alpha_2 = (2^4 [4]^{8} + [1]^{8})^{1/2}/[2]^{2} = [2]^{10}/[1]^{4}[4]^{4}$ |
|     |     | $= 1 + 4 \sum E_1(n; 4) q^n$ |
|     |     | $B_2 = [1]^{1}/[2]^{2} = 1 + 4 \sum E_1(n; 4) (-q)^n$ |
|     |     | $C_2 = 2^2 \cdot [4]^{4}/[2]^{2} = 4 q^{1/2} [1 + \sum E_1(2n + 1; 4) q^n]$ |

That is, Ramanujan’s complete elliptic integral $K_r(\alpha_r), r = 2, 3, 4$, arises naturally in a modular context, when the noncanonical weight-1 modular form $A_r$ is expressed as a function of the alternative Hauptmodul $\alpha_r$. This has not previously been realized. Ramanujan may, in fact, have made a good choice in focusing on $A_r$, rather than on its canonical counterpart $B_r$, since there are ways in which $A_r$ is ‘nicer’ than $B_r$. It can be shown that the multiplier systems of $A_2, A_3$ are given by Dirichlet characters (of conductors 4 and 3 respectively), unlike those of $B_2, B_3$; cf. Table 8. Also, it can be shown that the squares $A_3^2, A_4^2$ are the unique (normalized) weight-2 modular forms with trivial character for $\Gamma_0(3), \Gamma_0(2)$, respectively.

The preceding interpretation of Ramanujan’s theories to alternative bases does not cover his theory of signature 6. But on a purely formal level, one can develop that theory in parallel with those of signatures 2, 3, 4. His complete elliptic integral
$K_6 = K_6(\alpha_6)$ is defined by (9.13), i.e., by
\[
K_6(\alpha_6) = \frac{2}{\pi} \, _2F_1 \left( \frac{1}{6}, \frac{5}{6}; 1; \alpha_6 \right).
\] (10.5)
To recover it, one introduces a ‘Hauptmodul’ $y_6$, defined implicitly by
\[
j = \frac{(y_6 + 432)^2}{y_6} = 12^3 + \frac{(y_6 - 432)^2}{y_6}.
\] (10.6)
By mechanically following the prescription of Definition 5.1, one defines an associated function $H_6 = H_6(y_6)$ by
\[
H_6(y_6) = \left[ 432^{-1}(y_6 + 432) \right]^{-1/6} \, _2F_1 \left( \frac{1}{12}, \frac{5}{12}; 1; \frac{12^3 y_6}{(y_6 + 432)^2} \right).
\] (10.7)
One can show that $H_6$ satisfies a normal-form ‘Picard–Fuchs equation’ of hypergeometric type, with three singular points ($y_6 = 0$ and $\infty$, which by (10.6) are formally over the cusp $j = \infty$ on $X(1)$; and $y_6 = -432$, over the cubic elliptic point $j = 0$). This leads to the alternative hypergeometric representation
\[
H_6(y_6) = 2\, _2F_1 \left( \frac{1}{6}, \frac{1}{6}; 1; -y_6/432 \right),
\] (10.8)
which closely resembles the formulas of Table 16. As in $\overline{9}$ one introduces a ‘non-canonical Hauptmodul’ $\alpha_6$, related to $y_6$ by
\[
\alpha_6 = \alpha_6(y_6) = y_6/(y_6 + 432),
\] (10.9a)
\[
y_6 = y_6(\alpha_6) = 432 \, \alpha_6/(1 - \alpha_6).
\] (10.9b)
Then one readily deduces
\[
\frac{2}{\pi} K_6(\alpha_6) = (1 - \alpha_6)^{-1/6} \, H_6(y_6(\alpha_6)),
\] (10.10a)
\[
H_6(y_6) = (1 + y_6/432)^{-1/6} \frac{2}{\pi} K_6(\alpha_6(y_6)),
\] (10.10b)
from Pfaff’s transformation of $\_2F_1$. In this way the function $K_6$ is recovered.

One can go further, and work out modular transformations of $H_6$ and $K_6$. Starting with (14.5), the degree-2 classical modular equation $\Phi_2(j, j') = 0$, by a lengthy process of polynomial elimination one first derives the $y_6$-$y_6'$ modular equation of degree 2. This can be rationally parametrized with the aid of the MAPLE $\text{algcurves}$ package, or other software. A further analysis yields the degree-2 modular transformation of $H_6$ that incorporates the parametrized $y_6$-$y_6'$ modular equation, namely
\[
H_6 \left( \frac{(s+60)^2(s+72)^2(s+96)}{(s+48)^2(s+80)^2(s+120)^2} \right)
= 2 \left[ \frac{(s+60)(s+80)(s+96)^2}{(s+48)(s+120)^2} \right]^{-1/6} \, H_6 \left( \frac{s^2(s+48)^2(s+72)^2(s+120)}{(s+60)^2(s+80)^2(s+96)} \right).
\] (10.11)
Expressing $H_6$ in terms of $K_6$, and replacing $s$ by $12s$, then yields
\[
K_6 \left( \frac{(s+5)^2(s+6)^2(s+8)}{(s^2+10s+20)^4} \right) = 2 \left[ \frac{s^2+10s+20}{s^2+20s+80} \right]^{1/2} K_6 \left( \frac{s^2(s+4)^2(s+6)(s+10)}{(s^2+20s+80)^4} \right),
\] (10.12)
which incorporates a rational parametrization of the $\alpha_6$-$\beta_6$ modular equation of degree 2. Here $\alpha_6, \beta_6$ are of course the arguments of the left and right $K_6$’s. This uniformization of the $\alpha_6$-$\beta_6$ relation agrees with the explicit expression for the algebraic function $\beta_6 = \beta_6(\alpha_6)$ obtained by Borwein, Borwein, and Garvan $\overline{9}$, using radicals. The intriguing identity (10.12) is a parametrized signature-6 counterpart of Landen’s transformation, Eq. (12a). Ramanujan did not discover it, but he would surely have appreciated it.
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The just-sketched manipulations are however quite formal, valid only near the
infinite cusp $\tau = \infty$, where $y_6 = 0$ and $\alpha_6 = 0$. The ‘Hauptmodul’ $y_6$, algebraic
over $j$, does not extend to a single-valued function on the upper half $\tau$-plane, as is
clear from its definition \[(10.10)\]. Also, the Gauss hypergeometric equation satisfied
by $H_6 = H_6(y_6)$ cannot be the normal-form Picard–Fuchs equation attached to any
first-kind Fuchsian subgroup $\Gamma < PSL(2, \mathbb{Z})$. By \[(10.8)\], the exponent differences at
its singular points $y_6 = 0, \infty, -432$ are 0, 0, $\frac{2}{3}$ respectively; and by Theorem 6.1 a
difference of $\hat{\tau}$ should not be present.

A recasting of the theory of signature 6 in algebraic–geometric terms should be
based not on the elliptic modular surface $\mathcal{E}_\Gamma \to \Gamma \setminus \mathcal{H}^*$ attached to some $\Gamma$, but
rather on an elliptic non-modular surface $\mathcal{E} \xrightarrow{\pi} \mathbb{P}^1(\mathbb{C})_{y_6}$, the base curve of which
is parametrized by the pseudo-Hauptmodul $y_6$. Its functional invariant $j = j(y_6)$,
in Kodaira’s terminology, is given by \[(10.6)\] Each of its fibres is an elliptic curve
over $\mathbb{C}$, except for those above the singular points $y_6 = 0, \infty, -432$. This elliptic
surface is one of the rational ones, with nonconstant $j$-invariant, three singular
fibres, and a section, that were classified by Schmickler-Hirzebruch \[52\]. It is her
‘Fall 8’, and up to a broad notion of equivalence, it is the only such surface that
does not come from $\mathcal{H}^*$ and some $\Gamma$ in the classical way. Its singular fibres are of
Kodaira types $I_1, I_1^*, IV$. The hypergeometric equation satisfied by $H_6 = H_6(y_6)$ is
its normal-form ‘Picard–Fuchs equation’, though the term is not really appropriate,
suggesting as it does close ties to the classical theory of automorphic functions.

The parameter $s$ in the degree-2 modular equation for $H_6$, Eq. (10.11), is best
viewed as the coordinate on the base of a second rational elliptic surface, $\hat{\mathcal{E}} \xrightarrow{\pi} \mathbb{P}^1(\mathbb{C})_s$, which parametrizes the modular equation. There is a commutative diagram

$$
\begin{array}{ccc}
\hat{\mathcal{E}} & \xrightarrow{s} & \mathcal{E} & \xrightarrow{j} & \hat{\mathcal{E}}_1 \\
\pi \downarrow & & \pi \downarrow & & \pi_1 \downarrow \\
\mathbb{P}^1(\mathbb{C})_s & \xrightarrow{\pi} & \mathbb{P}^1(\mathbb{C})_{y_6} & \xrightarrow{\pi_1} & X(1)
\end{array}
$$

\[(10.13)\]
in which the lower map $s \mapsto y_6$ is the degree-6 rational function of $s$ appearing
on the left-hand side of \[(10.11)\]. By pulling back the ‘Picard–Fuchs equation’ for
$\mathcal{E} \xrightarrow{\pi} \mathbb{P}^1(\mathbb{C})_{y_6}$ along $\mathbb{P}^1(\mathbb{C})_s \to \mathbb{P}^1(\mathbb{C})_{y_6}$, one readily computes a (formal!) Picard–
Fuchs equation for $\mathcal{E}$. It turns out to have 10 singular points, two of which are only
apparent, i.e., have trivial monodromy. So $\hat{\mathcal{E}} \xrightarrow{s} \mathbb{P}^1(\mathbb{C})_s$ has 8 singular fibres. They
are located above

$$s = 0, \infty, -48, -60, -72, -80, -96, -120,$$

as one would expect from \[(10.11)\]. Having so many singular fibres, this elliptic
surface is more complicated than those classified by Schmickler-Hirzebruch or by
Herfurtner \[31\].

A theory of modular equations for general rational elliptic surfaces must pro-
duce algebraic transformation laws, of arbitrarily high degree, for Gauss–Manin
connections pulled back from the universal elliptic family. It is clear from the
above analysis that such modular equations should be parametrized by elliptic sur-
faces, which may sometimes themselves be rational. In the absence of a general
theory, whether there are additional rationally parametrized modular equations in
Ramanujan’s theory of signature 6 remains unclear.
Appendix. Hypergeometric and Heun Equations

Any second-order Fuchsian differential equation on $\mathbb{P}^1(\mathbb{C})$ can be reduced to the normal form

$$D^2 u + \left[ \frac{c}{t} + \frac{d}{t-1} \right] D_t u + \left[ \frac{ab}{t(t-1)} \right] u = 0, \quad (A.1)$$

if it has exactly three singular points, and to the normal form

$$D^2 u + \left[ \frac{\gamma}{t} + \frac{\delta}{t-1} + \frac{\epsilon}{t-a} \right] D_t u + \left[ \frac{\alpha\beta t - q}{t(t-1)(t-a)} \right] u = 0, \quad (A.2)$$

if it has exactly four. The reduction employs a Möbius transformation to reposition (three of) the singular points to $t = 0, 1, \infty$. (In the latter case the fourth singular point moves to some location $a \in \mathbb{C} \setminus \{0, 1\}$.) It also employs index transformations, to shift one characteristic exponent at each of the finite singular points to zero. The exponents of (A.1) and (A.2) are respectively

$$0, 1 - c; 0, 1 - d; a, b \quad \text{at} \quad t = 0, 1, \infty$$
$$0, 1 - \gamma; 0, 1 - \delta; 0, 1 - c; \alpha, \beta \quad \text{at} \quad t = 0, 1, a, \infty.$$

Fuchs’s relation (the sum of the $2k$ exponents of a Fuchsian differential equation with $k$ singular points on $\mathbb{P}^1(\mathbb{C})$ equating $k - 2$) implies that $d = a + b - c + 1$, resp. $\epsilon = \alpha + \beta - \gamma - \delta + 1$. The quantity $q \in \mathbb{C}$ in (A.2) is an accessory parameter, which does not affect the local monodromy about each singular point.

Each of (A.1), (A.2) has a unique local solution holomorphic at $t = 0$ and equal to unity there. For (A.1) this solution is the Gauss hypergeometric function $\, _2F_1(a, b; c; \cdot)$, and for (A.2) it is the local Heun function [51], widely denoted $\, Hl(a, q; \alpha, \beta, \gamma, \epsilon; \cdot)$. These have $t$-expansions $\sum_{n=0}^{\infty} c_n t^n$, which converge on $|t| < 1$, resp. $|t| < \min(1, |a|)$. The coefficients satisfy respective recurrences

$$(n + a - 1)(n + b - 1) c_{n-1} - n(n + c - 1) c_n = 0 \quad (A.3)$$

and

$$(n + \alpha - 1)(n + \beta - 1) c_{n-1}$$
$$- \left\{ n \left[ (n + \gamma + \delta - 1)a + (n + \gamma + \epsilon - 1) \right] + q \right\} c_n$$
$$+ (n + 1)(n + \gamma)a c_{n+1} = 0, \quad (A.4)$$

initialized by $c_0 = 1$ (and $c_{-1} = 0$ in the latter case).

In the spirit of Kummer, one can derive functional equations satisfied by the parametrized functions $\, _2F_1$ and $\, Hl$ by applying Möbius and index transformations to their defining differential equations. The idea is that one should transform the equation to itself, with altered parameters and argument [39]. If there are $k$ singular points ($k = 3, 4$ here), there will be a subgroup of the Möbius group, isomorphic to $\mathfrak{S}_k$, that permutes them. Also, there will be a subgroup of the group of index transformations, isomorphic to $(C_2)^{k-1}$, that negates the nonzero exponents of the $k - 1$ finite singular points. The first group normalizes the latter, so the group of composite transformations, i.e., the transformation group that they generate, is a semidirect product $C_2^{k-1} \rtimes \mathfrak{S}_k$. By examination, this semidirect product is isomorphic to the group of even-signed permutations of $k$ objects, which is an index-2 subgroup of the wreath product $C_2 \wr \mathfrak{S}_k$, the group of signed permutations of $k$ objects.
The action of the order-24 group \([C_2 \wr S_3]_{\text{even}}\) on \((A.1)\) yields 24 local solutions, each expressed in terms of \(_2F_1\). These are the famous 24 solutions of Kummer. The action of the order-192 group \([C_2 \wr S_4]_{\text{even}}\) on \((A.2)\) yields 192 local solutions, each expressed in terms of \(Hl\), which are less well known than Kummer’s (but see Refs. [39, 61]). If the transformation in \([C_2 \wr S_k]_{\text{even}}\) applied to \((A.1)\), resp. \((A.2)\), is to yield a function equal to \(_2F_1\), resp. \(Hl\), then it should stabilize the singular point \(t = 0\), and perform no index transformation there. Hence, the subgroup of allowed transformations is isomorphic to \([C_2 \wr S_{k-1}]_{\text{even}}\).

So, there is a group of transformations of \(_2F_1\) isomorphic to \([C_2 \wr S_2]_{\text{even}}\), i.e., to the four-group \(C_2 \times C_2\). It is generated by classical transformations named after Euler and Pfaff. Pfaff’s is used in this article. It is

\[
_2F_1(a, b; c; t) = (1 - t)^{-a} _2F_1(a, c - b; c; \frac{t}{t - 1}), \quad (A.5)
\]

and arises from the Möbius transformation that interchanges \(t = 1, \infty\).

The group of transformations of \(Hl\) is isomorphic to \([C_2 \wr S_3]_{\text{even}}\), which by examination is isomorphic to the octahedral group \(S_4\). Two of the transformations in this group are used in this article. They are the relatively trivial one

\[
Hl(a, q; \alpha, \beta, \gamma, \delta; t) = Hl(\frac{\alpha}{a}, \frac{q}{a}; \alpha, \beta, \gamma, \alpha + \beta - \gamma - \delta + 1; \frac{t}{t - 1}), \quad (A.6)
\]

and the generalized Pfaff transformation

\[
Hl(a, q; \alpha, \beta, \gamma, \delta; t) = (1 - t)^{-a} Hl(\frac{a}{a - 1}, \frac{-q + \gamma a}{a - 1}; \alpha, \alpha - \delta + 1, \gamma, \alpha - \beta + 1; \frac{t}{t - 1}). \quad (A.7)
\]

They arise respectively from the Möbius transformation that in effect interchanges \(t = 1, a\) (i.e., takes \(a\) to \(1\) and \(1\) to \(a^\prime := 1/a\)), and the one that interchanges \(t = 1, \infty\) (and less importantly, takes \(t = a\) to \(t = a^\prime := a/(a - 1)\)).

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