Topological characteristics of lattice Dirac operators

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Abstract

We show that even if a lattice Dirac operator satisfies the conditions consisting of locality, free of species doublings, correct continuum behavior, $\gamma_5$-hermiticity and the Ginsparg-Wilson relation, it does not necessarily have exact zero modes in nontrivial gauge backgrounds. This implies that each lattice Dirac operator has its own topological characteristics which cannot be fixed by these conditions. The role of topological characteristics in the axial anomaly is derived explicitly.

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1 Introduction

It seems to have been commonly regarded that if a lattice Dirac operator $D$ satisfies the conditions: (i) locality, (ii) free of species doublings, (iii) having correct continuum behavior, (iv) $\gamma_5$-hermiticity and (v) the Ginsparg-Wilson relation, then $D$ would possess exact zero modes with definite chirality, obeying the Atiyah-Singer index theorem \[1\] for smooth gauge backgrounds with non-zero integer topological charge. Explicitly, these conditions are:

(i) $D$ is local.
   \[
   (|D^{ab}_{\alpha\beta}(x,y)| \sim \exp(-|x-y|/l) \text{ with } l \sim a \text{ and } l \ll L \text{ where } L = Na \text{ is the size of the lattice; or } D(x,y) = 0 \text{ for } |x-y| > z \text{ with } z \ll L. )
   \]

(ii) In the free fermion limit, $D$ is free of species doublings.
   \[\text{ (The free fermion propagator } D^{-1}(p) \text{ has only one simple pole at the origin } p = 0 \text{ in the Brillouin zone. )}\]

(iii) In the free fermion limit, $D$ has the correct continuum behavior.
   \[\text{ (In the limit } a \to 0, \text{ } D(p) \sim i\gamma_\mu p_\mu \text{ around } p = 0. )\]

(iv) $D$ is $\gamma_5$-hermitian. \[D^\dagger = \gamma_5 D \gamma_5.\]

(v) $D$ breaks the chiral symmetry according to the Ginsparg-Wilson relation \[2\]. \[D\gamma_5 + \gamma_5 D = 2raD\gamma_5 D, \text{ where } r \text{ is a positive real number and } a \text{ is the lattice spacing. }\]

However, in this paper, we show that this folklore may not be justified. We demonstrate that even if a lattice Dirac operator satisfies all above five conditions (i)-(v), it does not necessarily produce any exact zero modes in nontrivial gauge backgrounds. Only for topologically trivial gauge backgrounds with zero topological charge, these five conditions (i)-(v) can guarantee that the continuum axial anomaly can be recovered on the lattice.

The basic motivation of constructing a lattice Dirac operator satisfying these five conditions (i)-(v) is to formulate a nonperturbatively regularized theory for massless Dirac fermion interacting with a background gauge field, as a first step towards the full theory including gauge dynamics. These five conditions together provide a bypass to the Nielson-Ninomiya no-go theorem \[3\], and constitute the necessary conditions for $D$ to reproduce the correct physics in the free fermion limit as well as in the weak coupling perturbation regime. However, they are not sufficient to guarantee that any $D$ satisfying these five conditions must have the correct index in topologically nontrivial gauge fields. So far, among all $D$ satisfying these five conditions (i)-(v), the Neuberger-Dirac operator \[4\] is the only one which reproduces the correct index (as well as other important quantities such as the fermion determinant ratio).

In the next paragraph, we show how this could be possible in principle, and
then present two explicit counterexamples in Section 3. Some basic properties of the GW Dirac operator pertinent to our discussions are collected in the Appendix A.

It has been proved in ref. [5] that for any lattice Dirac operator \( D \) satisfying the \( \gamma_5 \)-hermiticity (iv) and the GW relation (v), the necessary condition for it to have nonzero index in the topologically nontrivial gauge background is

\[
\text{det}(I - arD) = 0.
\] (1)

In other words, if \( D \) does not satisfy (1), then the index of \( D \) must be zero. This condition (1) is equivalent to that \( D_c = D(I - raD)^{-1} \) (which is chirally invariant and \( \gamma_5 \)-hermitian) has singularities. Thus, if \( D_c \) is well-defined and does not have any singularities, then the condition (1) is violated, and the GW Dirac operator \( D = D_c(I + arD_c)^{-1} \) must have zero index, i.e., topologically trivial.

If one can construct a well-defined \( D_c \) satisfying the four conditions (ii)-(iv) (note that any \( D_c \) satisfying conditions (ii) and (iii) must be nonlocal, as a consequence of Nielson-Ninomiya theorem [3]), then the GW Dirac operator \( D = D_c(I + arD_c)^{-1} \) is topologically trivial and satisfies the four conditions (ii)-(v). Now if there exists a range of \( r \) such that \( D \) is local, then \( D \) satisfies all five conditions (i)-(v) but has zero index, hence, a counterexample to the folklore. We have at least two explicit counterexamples outlined in Section 3. In general, we can argue that there exists a range of \( r \) such that \( D \) is local. Our argument is as follows. First, we note that in the free fermion limit \( D_c \) has a zero mode at \( p = 0 \) in the Brillouin zone, the condition (ii). However, unlike any genuine zero modes (in nontrivial gauge fields) which are stable against local fluctuations of the gauge fields, this zero mode can be elevated by turning on a trivial gauge background with \( A_\mu(x) \) equal to a tiny constant. Henceforth we shall assume that this trivial zero mode of \( D_c \) has been elevated in this manner. Since both \( D \) and \( D_c \) are topologically trivial, \( D_c \) does not have any zero modes in nontrivial gauge fields. Thus \( D_c \) does not have any (non-)trivial zero modes. Then, for fixed lattice spacing \( a \), in the limit \( ra \gg 1 \), \( D \approx (ra)^{-1}I + \) higher order corrections, a highly local operator. On the other hand, in the limit \( ra \to 0 \), \( D \to D_c \), which is nonlocal. Thus, between these two extreme cases, there must exist a range of \( r \) such that \( D \) is local.

Further, since the index of \( D = D_c(I + raD_c)^{-1} \) is invariant for any \( r \) [Eq. (12)], the locality condition (i) should not be relevant to the question whether \( D \) has the correct index or not. Only in the case \( D \) has the correct index (i.e., the sum of the axial anomaly over all sites is correct), then the locality of \( D \) comes into play to ensure that the axial anomaly is also correct at each site.

So far, we do not know of any additional constraints which can guarantee that \( D \) has the correct index satisfying the Atiyah-Singer index theorem. However, it is obvious that if such a criterion exists, it cannot be imposed only
in the perturbation regime or in the free fermion limit like conditions (ii) and (iii), since it must be pertinent to the topologically nontrivial sectors.

Nevertheless, for any $D$ satisfying the GW relation (v), its zero modes have definite chirality. So we can classify any GW Dirac operator $D$ according to the ratio of its index ($n_- - n_+$) to the nonzero integer topological charge $Q$ of the gauge background, where $n_+$ ($n_-$) denotes the number of zero modes of positive (negative) chirality. The virtue of such a classification depends on whether this ratio is robust with respect to the topological charge $Q$, the size of the lattice ($L = Na$) and the lattice spacing ($a$). It turns out to be the case provided that the number of sites $N$ in each direction is sufficiently large (equivalently, the lattice spacing $a = L/N$ is sufficiently small) such that the absolute value of the topological charge inside any unit cell of volume $a^4$ is less than a small number $\epsilon_1$,

$$a^4|\tilde{\rho}(x)| < \epsilon_1 \quad \forall x,$$

where

$$\tilde{\rho}(x) = \frac{1}{a^4} \prod_{\eta=1}^{4} \int_{x-\eta/a/2}^{x+\eta/a/2} dy_{\eta} \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \text{tr}(F_{\mu\nu}(y) F_{\lambda\sigma}(y)),$$

and $F$ is the field tensor defined in (7). If one uses $c[D]$ to denote this ratio, then

$$\text{index}(D) = n_- - n_+ = c[D] Q,$$

where $c[D]$ is an integer constant provided that (2) is satisfied. Evidently the value of $\epsilon_1$ in (2) is dependent on $D$.

In topologically trivial gauge backgrounds ($Q = 0$), $D$ does not have any exact zero modes. Thus $c[D]$ cannot be defined by Eq. (4). Nevertheless, we can define $c[D] = 1$ for the trivial sector. Then the formal definition of $c[D]$ is

$$c[D] = \begin{cases} (n_- - n_+)/Q, & Q \neq 0 \\ 1, & Q = 0 \end{cases}$$

(5)

With gauge backgrounds satisfying the topological bound (2), we can classify any GW Dirac operator according to its topological characteristics $c[D]$ in nontrivial backgrounds as follows. If $c[D] = 1$, then $D$ is called topologically proper, else if $c[D] = 0$, then $D$ is called topologically trivial, otherwise $D$ is called topologically improper. Evidently, we are only interested in topologically proper $D$, since this is a prerequisite for $D$ to reproduce the continuum axial anomaly on the lattice. There are many examples of topologically trivial $D$, however, so far, there is only one topologically proper $D$ which can be written down explicitly, namely, the Neuberger-Dirac operator [4]

$$D = m_0(1 + V), \quad V = D_w(D_w^\dagger D_w)^{-1/2}, \quad 0 < m_0 < 2a^{-1},$$

$$D_w = -m_0 + D_W, \quad D_W : \text{massless Wilson-Dirac operator}$$

(6)
and its GW generalization

\[ D = 2m_0(\mathbb{I} + V)[(\mathbb{I} - V + 2m_0ra(\mathbb{I} + V))]^{-1}. \]

Note that we do not have any genuine examples of topologically improper \( D \) except by setting \( m_0 > 2a^{-1} \) in (1) (3).

The outline of this paper is as follows. In Section 2, we argue that if a lattice Dirac operator does not have the correct index on any finite lattices, then it cannot have the correct index in the continuum limit. Then we demonstrate that if \( D \) has exact zero modes in a nontrivial gauge background, then its topological characteristics can be revealed on a finite lattice, sometimes even on the smallest lattice of size \( 3^4 \). We also estimate the topological bound \( \epsilon_1 \) for the Neuberger-Dirac operator. In Section 3, we present two examples of lattice Dirac operators which satisfy the five conditions (i)-(v), but do not have any exact zero modes in any nontrivial gauge backgrounds. In Section 4, we derive a general expression for the axial anomaly of lattice Dirac operator on a finite lattice, in which the role of its topological characteristics \( c[D] \) is displayed explicitly. In Section 5, we conclude and discuss. In Appendix A, we collect some basic properties of the GW Dirac operator pertinent to our discussions. In Appendix B, we show that nonlocal Dirac operators can have well-defined indices.

## 2 Robustness of the exact zero modes

It seems to have been commonly believed that a lattice Dirac operator may possess exact zero modes in the classical continuum limit even if it does not have any zero modes in nontrivial gauge backgrounds on any finite lattices. However, our view is that if a lattice Dirac operator (satisfying (i)-(v)) does not have the correct index on any finite lattices, then it will never have the correct index in the continuum limit.

First, we set up the notations for our discussions.

Consider a smooth gauge configuration \( \{ A_\mu(x) \} \) with nonzero integer topological charge \( Q \) in the 4-dimensional Euclidean spacetime which is temporarily truncated to a hypercube of size \( L^4 \) with periodic boundary conditions, i.e., a 4-d flat torus, \( T^4 \). (The infinite volume limit \( L \to \infty \) is presumably to be taken at the end of all calculations.)

The covariant differential operator \( \mathcal{D}_\mu \) acting on the matter fields is defined as

\[ \mathcal{D}_\mu = \partial_\mu + iA_\mu. \]

The field tensor \( F_{\mu\nu} \) is defined by the commutator,

\[ [\mathcal{D}_\mu, \mathcal{D}_\nu] = iF_{\mu\nu}. \quad (7) \]
The topological charge density of the gauge background is
\[ \rho(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \text{tr}(F_{\mu\nu}(x)F_{\lambda\sigma}(x)) \] (8)
where \( \text{tr} \) denotes the trace over the gauge group space. Then the topological charge
\[ Q = \int_{T^4} d^4x \rho(x) = \int_{T^4} d^4x \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \text{tr}(F_{\mu\nu}(x)F_{\lambda\sigma}(x)) \] (9)
is an integer.

The massless Dirac operator in continuum is
\[ \mathcal{D} = \gamma_\mu D_\mu = \gamma_\mu (\partial_\mu + iA_\mu) , \]
which is chirally invariant (\( \mathcal{D}\gamma_5 + \gamma_5 \mathcal{D} = 0 \)) and antihermitian (\( \mathcal{D}^\dagger = -\mathcal{D} \)). The Dirac matrices are defined by
\[ \gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu^\dagger & 0 \end{pmatrix} \] (10)
where
\[ \sigma_\mu\sigma_\nu^\dagger + \sigma_\nu\sigma_\mu^\dagger = 2\delta_{\mu\nu} \] (11)
Explicitly, we choose \( \sigma_1, \sigma_2, \sigma_3 \) to be the Pauli matrices, and \( \sigma_4 = i\mathbb{1} \) where \( \mathbb{1} \) is the \( 2 \times 2 \) unit matrix. Then the chirality operator is
\[ \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} . \] (12)

In a smooth background gauge field with nonzero integer topological charge \( Q \), \( \mathcal{D} \) has zero eigenvalues and the corresponding eigenfunctions are chiral. The Atiyah-Singer index theorem \( \mathbb{I} \) asserts that the difference of the number of left-handed and right-handed zero modes is equal to the topological charge of the gauge configuration,
\[ n_- - n_+ = Q . \] (13)

The lattice regularization of the gauge background amounts to discretize the 4-d torus into a hypercubical lattice with \( N \) sites in each direction (i.e., \( L = Na \)), and then transcribe the gauge fields \( \{A_\mu(x)\} \) to the link variables \( \{U_\mu(x)\} \), where \( U_\mu(x) \) is the link variable defined on the link \( (x, x + a\hat{\mu}) \). We choose the lattice spacing \( a \) sufficiently small (i.e., \( N \) is sufficiently large for fixed \( L \)) such that the variations of \( \{A_\mu(x)\} \) can be tracked by the link variables \( \{U_\mu(x)\} \). For any lattice Dirac operator \( D \) (satisfying the five conditions (i)-(v)), our primary concern is whether \( D \) can produce exact zero modes satisfying the Atiyah-Singer index theorem \( \mathbb{I} \). If the index of \( D \) always agrees
with the topological charge \( Q \) for any gauge background satisfying the bound (2), then \( D \) is called topologically proper. It is evident that the value of \( \epsilon_1 \) depends on \( D \). A rough estimate of \( \epsilon_1 \) can be obtained by the following simple prescription.

Consider the following gauge configuration with constant field tensor and topological charge \( Q = 1 \),

\[
\begin{align*}
A_1^c(x) &= -\frac{2\pi x_2}{L^2} t^\alpha, \\
A_2^c(x) &= 0, \\
A_3^c(x) &= -\frac{2\pi x_4}{L^2} t^\alpha, \\
A_4^c(x) &= 0
\end{align*}
\]

where \( t^\alpha \) is any one of the generators of the gauge group with normalization \( \text{tr}(t^\alpha t^\beta) = \delta^{\alpha\beta} \). The nonzero components of the field tensor are

\[
F_{12}(x) = F_{34}(x) = \frac{2\pi}{L^2} t^\alpha \equiv F. \tag{14}
\]

Then the corresponding link variables on the lattice ( \( x_\mu = 0, \cdots, (N-1)a \) ) are:

\[
\begin{align*}
U_1(x) &= \exp \left[ -i\frac{2\pi a x_2}{L^2} t^\alpha \right] \tag{15} \\
U_2(x) &= \exp \left[ i\delta_{x_2,(N-1)a} \frac{2\pi x_1}{L} t^\alpha \right] \tag{16} \\
U_3(x) &= \exp \left[ -i\frac{2\pi a x_4}{L^2} t^\alpha \right] \tag{17} \\
U_4(x) &= \exp \left[ i\delta_{x_3,(N-1)a} \frac{2\pi x_3}{L} t^\alpha \right]. \tag{18}
\end{align*}
\]

Using the notation

\[
U_{\mu\nu}(x) = U_\mu(x)U_\nu(x + a\hat{\mu})U_\mu^\dagger(x + a\hat{\nu})U_\nu^\dagger(x)
\]

(19)

to stand for the path-ordered product of link variables around the plaquette \( p = (x, \hat{\mu}, \hat{\nu}) \), we obtain

\[
U_{12}(x) = U_{34}(x) = \exp \left[ i\frac{2\pi a^2}{L^2} t^\alpha \right] = \exp \left[ ia^2 F \right] \tag{20}
\]

and \( U_{13}(x) = U_{14}(x) = U_{23}(x) = U_{24}(x) = 1 \).

Then we test whether \( D \) has exact zero modes on finite lattices by successively increasing the number of sites \( N \) in the sequence \( N = 3, 4, 5, \cdots \).

If \( D \) is topologically proper, then the exact zero mode would emerge after \( N > N_1 \) for a certain finite integer \( N_1 \), ( i.e., for the strength of the nonzero
field tensor inside each plaquette, $a^2|F_{12}| = a^2|F_{34}| = 2\pi/N^2$, is less than $2\pi/N^2$). This yields a rough estimate of $\epsilon_1 \approx 1/(N_1 + 1)^4$ for the topological bound (2) such that $c[D]$ is an integer constant. A more optimal estimate of $\epsilon_1$ can be obtained by introducing local fluctuations which change the topological charge density locally but maintain the total topological charge fixed.

On the other hand, if $D$ is topologically trivial, then $D$ does not have any exact zero modes for any finite lattice spacing $a = L/N$ (as $N$ is increased successively). Suppose that in the limit $a = 0$ ($N = \infty$), the exact zero mode emerges, however, in our view, $D$ still does not have the correct index in the continuum limit. Our argument is as follows.

[Proof]:

First, let the numbers of exact zero modes of $D$ be $k_+$ and $k_-$ in a nontrivial gauge background with topological charge $Q$ on a finite lattice, and they do not yield the correct index. (i.e., $\text{index}(D) = k_- - k_+ \neq Q$.) Suppose that the index of $D$ remains the same for any finite lattice spacing $a = L/N$ as $N$ is increased successively, except at $a = 0$ where the correct numbers of exact zero modes ($n_{\pm}$) emerge and the index becomes $n_- - n_+ = Q$. This implies that the index of $D$ must undergo a discontinuous transition from the integer $k_- - k_+$ to another integer $n_- - n_+$ at $a = 0$, and correspondingly, $D$ becomes $D_1$. Thus the index($D_1$) = $n_- - n_+$ does not constitute a limiting point of the index($D$) = $k_- - k_+$, because a discontinuity has occurred. Since the index of $D_1$ is not equal to the index of $D$, they are in two different topological phases, or in other words, not in the same universality class. It follows that $D_1$ cannot be the continuum limit of $D$. This completes the proof.

In short, if $D$ does not have the correct index at finite lattice spacing, then $D$ cannot have the correct index in the continuum limit.

Therefore we conclude that the index of a lattice Dirac operator (satisfying the conditions (i)-(v)) is invariant for any nontrivial gauge backgrounds with the same topological charge provided that the topological bound (2) is satisfied.

Next we demonstrate that if a lattice Dirac operator is topologically proper, then it can produce the correct index on a finite lattice, sometimes even on a very small lattice. For example, on the $3^4$ lattice, the Neuberger-Dirac operator (6) with $m_0 = 1$ has one exact zero mode in a gauge background with $Q = 1$. We also estimate the value of its $\epsilon_1$ in the topological bound (2).

For simplicity, we consider the following $U(1)$ background gauge fields on the 4-d flat torus ($x_\mu \in [0, L], \mu = 1, \cdots, 4$):

\begin{align*}
aA_1(x) &= -\frac{2\pi q_1 a x_2}{L^2} + A_1^{(0)} \sin \left( \frac{2\pi n_2}{L} x_2 \right) \tag{21} \\
aA_2(x) &= A_2^{(0)} \sin \left( \frac{2\pi n_1}{L} x_1 \right) \tag{22} \\
aA_3(x) &= -\frac{2\pi q_2 a x_4}{L^2} + A_3^{(0)} \sin \left( \frac{2\pi n_4}{L} x_4 \right) \tag{23}
\end{align*}
\[
aA_4(x) = A_4^{(0)} \sin \left( \frac{2\pi n_3}{L} x_3 \right)
\]

(24)

where \( q_1 \) and \( q_2 \) are integers. The global part of the gauge background is characterized by the topological charge

\[
Q = \frac{1}{32\pi^2} \int d^4x \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x) = q_1 q_2.
\]

The local parts are chosen to be sinusoidal fluctuations with amplitudes \( A_1^{(0)}, A_2^{(0)}, A_3^{(0)} \) and \( A_4^{(0)} \), and frequencies \( \frac{2\pi n_1}{L}, \frac{2\pi n_2}{L}, \frac{2\pi n_4}{L} \) and \( \frac{2\pi n_3}{L} \) where \( n_1, n_2, n_3 \) and \( n_4 \) are integers. The discontinuity of \( A_1(x) \) (\( A_3(x) \)) at \( x_2 = L \) (\( x_4 = L \)) due to the global part only amounts to a gauge transformation. The components of the field tensor are continuous on the flat torus. To transcribe the background gauge field to link variables on the lattice, we take the lattice sites at \( x_\mu = 0, a, \ldots, (N-1)a \), where \( a \) is the lattice spacing and \( L = Na \) is the lattice size. Then the link variables are

\[
U_1(x) = \exp \left[ ia A_1(x) \right]
\]

(25)

\[
U_2(x) = \exp \left[ ia A_2(x) + i\delta_{x_2,(N-1)a} \frac{2\pi q_1 x_1}{L} \right]
\]

(26)

\[
U_3(x) = \exp \left[ ia A_3(x) \right]
\]

(27)

\[
U_4(x) = \exp \left[ ia A_4(x) + i\delta_{x_4,(N-1)a} \frac{2\pi q_2 x_3}{L} \right]
\]

(28)

First, we consider the simplest nontrivial gauge configuration with constant field tensor at each plaquette, i.e.,

\[
a^2 F_{\mu\nu}(x) = -a^2 F_{\nu\mu}(x) = \begin{cases} 
2\pi q_1/N^2, & \{\mu, \nu\} = \{1, 2\}; \\
2\pi q_2/N^2, & \{\mu, \nu\} = \{3, 4\}; \\
0, & \text{otherwise}. 
\end{cases}
\]

(29)

This amounts to setting the local sinusoidal fluctuations to zero (\( A_i^{(0)} = 0, i = 1, \ldots, 4 \)) in Eqs. (21)-(24).

In Table 1, we list the number of exact zero modes of each chirality versus the topological charge of the simplest nontrivial gauge background, for the Neuberger-Dirac operator on the \( 3^4 \) lattice. The Atiyah-Singer index theorem \((13)\) is satisfied exactly for \( Q = 1 \), but not for \( Q = 2 \). This yields a rough estimate of \( \epsilon_1 \) in \((3)\): \( 1/3^4 < \epsilon_1 < 2/3^4 \). It is obvious that the Neuberger-Dirac operator is nonlocal on such a small lattice. This demonstrates that even a nonlocal Dirac operator can have exact zero modes satisfying the Atiyah-Singer index theorem. However, the axial anomaly of a nonlocal lattice Dirac operator must disagree with the topological charge density of the gauge background.

Besides the zero modes, the +2 eigenmodes of the Neuberger-Dirac operator are also chiral \([7]\), and the numbers of both chiralities are listed in the last
The Atiyah-Singer index theorem $Q = n_- - n_+$ is satisfied only for $|Q| = 1$. The chirality sum rule (30) is satisfied in all cases. The last column is the topological characteristics, $c[D] = (n_- - n_+)/Q$. The fractional value of $c[D]$ indicates that topological bound is violated.

Table 1: The exact zero modes versus the topological charge, for the Neuberger-Dirac operator (3) with $m_0 = 1$, on the $3^4$ lattice. 

| $q_1$ | $q_2$ | $Q = q_1q_2$ | $n_+$ | $n_-$ | $n_2$ | $c[D]$ |
|-----|-----|-----|-----|-----|-----|-----|
| -1  | -1  | 1   | 0   | 1   | 1   | 0   |
| -1  | 1   | -1  | 1   | 0   | 0   | 1   |
| 1   | 1   | 1   | 0   | 1   | 0   | 1   |
| -2  | 1   | -2  | 1   | 0   | 0   | 1   |
| 2   | -1  | -2  | 1   | 0   | 0   | 1   |
| -2  | -1  | -2  | 0   | 1   | 1   | 0   |
| 2   | 1   | 2   | 0   | 1   | 1   | 0   |

Note that the exact zero modes (as well as the +2 chiral modes) are very stable under local fluctuations of the gauge background provided that the topological bound (2) is satisfied. (Otherwise, they are not regarded as genuine zero modes of a lattice Dirac operator in a nontrivial gauge background.) To demonstrate the robustness of the zero modes, as well as to obtain a better estimate of $\epsilon_1$, we turn on the local sinusoidal fluctuations in (21).

With $Q = 1$ on the $3^4$ lattice, we increase the amplitude $A_1^{(0)}$ gradually until the exact zero mode disappears. At this point, the average of $a^4|\bar{\rho}(x)|$ (3) over all sites gives a conservative estimate of $\epsilon_1$. In Table 2, the amplitude $A_1^{(0)}$ is listed in the first column. The frequency of the sinusoidal fluctuation is $2\pi/L$. The maximum and the average of $a^4|\bar{\rho}(x)|$ are listed in the second and third columns respectively. We note that for $0 \leq A_1^{(0)} < 1.022$, the index is $n_- - n_+ = 1$, equal to the topological charge $Q = 1$. However, as soon as $A_1^{(0)}$ reaches 1.022, the exact zero mode disappears and the index becomes zero. As we further increase $A_1^{(0)}$ up to 1.090, $\bar{D}$ enters into another topological phase with index equal to −1. Thus, at $A_1^{(0)} = 1.022$, the average of $a^4|\bar{\rho}(x)|$ over all sites yields a conservative estimate of $\epsilon_1 < 0.025$. To be secure, we take $\epsilon_1 = 0.02$. For various gauge configurations on larger lattices, we have checked
Table 2: The exact zero modes versus the amplitude of the local sinusoidal fluctuations, \( A_{1}(0) \) in (21), for the Neuberger-Dirac operator on the 3\( ^4 \) lattice. The topological charge is \( Q = 1 \). The Atiyah-Singer index theorem \( Q = n_{-} - n_{+} \) is satisfied for \( A_{1}(0) < 1.022 \), or equivalently, \( a^{4} \lvert \bar{\rho}(x) \rvert < 0.028 \). The last column is the topological characteristics, \( c[D] = (n_{-} - n_{+})/Q \).

| \( A_{1}(0) \) | \( \max(a^{4} \lvert \bar{\rho}(x) \rvert) \) | \( < a^{4} \bar{\rho}(x) > \) | \( n_{+} \) | \( n_{-} \) | \( n_{+}^{2} \) | \( n_{-}^{2} \) | \( c[D] \) |
|---|---|---|---|---|---|---|---|
| 0.000 | 0.0123 | 0.0123 | 0 | 1 | 1 | 0 | 1 |
| 0.400 | 0.0185 | 0.0123 | 0 | 1 | 1 | 0 | 1 |
| 0.600 | 0.0215 | 0.0164 | 0 | 1 | 1 | 0 | 1 |
| 0.800 | 0.0246 | 0.0205 | 0 | 1 | 1 | 0 | 1 |
| 1.000 | 0.0277 | 0.0245 | 0 | 1 | 1 | 0 | 1 |
| 1.020 | 0.0279 | 0.0249 | 0 | 1 | 1 | 0 | 1 |
| 1.022 | 0.0280 | 0.0250 | 0 | 0 | 0 | 0 | 0 |
| 1.080 | 0.0289 | 0.0262 | 0 | 0 | 0 | 0 | 0 |
| 1.085 | 0.0289 | 0.0263 | 0 | 0 | 0 | 0 | 0 |
| 1.090 | 0.0290 | 0.0264 | 1 | 0 | 0 | 1 | -1 |
| 1.100 | 0.0292 | 0.0266 | 1 | 0 | 0 | 1 | -1 |

that the index of the Neuberger-Dirac operator with \( m_{0} = 1 \) is always equal to the topological charge of the background provided that the topological bound

\[
a^{4} \lvert \bar{\rho}(x) \rvert < 0.02 \quad \forall \; x
\]

is satisfied.

We note that the topological bound (2) can be satisfied if the link configuration \( \{U_{\mu}(x)\} \) fulfils

\[
\lvert \text{Re tr}(\mathbb{1} - U_{\mu\nu}(x)) \rvert < \frac{2\pi^{2} \epsilon_{1}}{3} \quad \text{for all plaquettes},
\]

where \( U_{\mu\nu}(x) \) is the path-ordered product of link variables around the plaquette \( p = (x, \hat{\mu}, \hat{\nu}) \), as defined in (19). However, the relevant parameter for the topological phase transition of \( D \) is the topological charge density \( a^{4} \lvert \bar{\rho}(x) \rvert \) rather than the plaquette action \( \lvert \text{Re tr}(\mathbb{1} - U_{\mu\nu}(x)) \rvert \), since a topologically proper \( D \) always has \( c[D] = 1 \) for any gauge configuration satisfying (2) even if it violates (32) for some plaquettes.

Finally, we note that the topological bound (2) or (32) does not guarantee that \( D \) is local, but only that \( c[D] \) is an integer constant. In general, if a lattice Dirac operator is local in the free fermion limit, then it would maintain its localness in a gauge background with sufficiently small field strength at each plaquette, i.e.,

\[
\lvert \lvert \mathbb{1} - U_{\mu\nu}(x) \rvert \lvert < \epsilon \quad \text{for all plaquettes},
\]

\[10\]
where the matrix norm $||M||$ can be defined to be the square root of the maximum eigenvalue of $M^\dagger M$, i.e., $||M|| = \sqrt{\lambda_{\text{max}}(M^\dagger M)}$. Evidently the value of $\epsilon$ is dependent on $D$.

The locality bound (33) implies that the topological charge inside any unit cell satisfies

$$a^4|\bar{\rho}(x)| < \epsilon_2 \quad \forall \ x,$$

$$\epsilon_2 = \frac{3}{4\pi^2} \text{tr}(\mathbb{I}) \epsilon^2$$

where $\text{tr}(\mathbb{I}) = N_c$ for link variables in the fundamental representation of the gauge group $SU(N_c)$. For the Neuberger-Dirac operator with $m_0 = 1$, $\epsilon = 1/6(2 + \sqrt{2}) \approx 0.0488$ [3], which gives $\epsilon_2 \approx 5.43 \times 10^{-4}$ for $N_c = 3$. This shows that $\epsilon_2$ is much less than $\epsilon_1 \approx 0.02$ in the topological bound. Evidently, for any topologically proper $D$, the inequality $\epsilon_2 < \epsilon_1$ must hold, since the topological bound only requires that the index (i.e., the sum of the axial anomaly over all sites) is equal to the topological charge, while the locality bound further guarantees that the axial anomaly $A_L(x)$ at each site is in good agreement with the topological charge density $\rho(x)$ of the gauge background.

We summarize the main theme of this section as follows. Consider a fixed gauge background with nonzero integer topological charge $Q$ on the 4-torus of size $L^4$. Then we impose a lattice on the 4-torus with lattice spacing $a$, and the number of sites in each direction, $N$ (i.e., $L = Na$). Given any lattice Dirac operator $D$, we can test whether the index of $D$ agrees with $Q$ by successively increasing $N$ (equivalently, decreasing lattice spacing $a = L/N$) in the sequence $N = 3, 4, \ldots$. If $D$ is topologically proper, then there exists a finite integer $N_1$ such that for $N > N_1$ ($a < L/N_1$), the index of $D$ is equal to $Q$. However, when $N$ is only slightly larger than $N_1$ (equivalently, $a$ is slightly less than $L/N_1$), $D$ may not be local yet. In general, if $D$ is local in the free fermion limit, then there exists another integer $N_2 > N_1$ such that for $N > N_2$ ($a < L/N_2$), $D$ is local. However, if $D$ is topologically trivial, then $D$ does not have any zero modes for any finite $N$, even though it has become local for $N > N_2$. Whether $D$ has any zero modes at $N = \infty$ ($a = 0$) really does not matter at all, since in case it has, then it cannot be the continuum limit of $D$.

So far, among the lattice Dirac operators which satisfy the five conditions (i)-(v), the Neuberger-Dirac operator is the only one which is topologically proper for gauge configurations satisfying the topological bound: $a^4|\bar{\rho}(x)| < 0.02$. In the next section, we present two explicit examples which satisfy these five conditions (i)-(v), but do not have any exact zero modes in nontrivial gauge backgrounds.
3 Topologically trivial lattice Dirac operators

It has been proved in ref. [5] that for any chirally symmetric and $\gamma_5$-hermitian Dirac operator $D_c$, if it is well-defined (without any singularities), then the GW Dirac operator $D = D_c(\mathbb{1} + raD_c)^{-1}$ has zero index in any nontrivial gauge background.

Therefore, if one can construct a well-defined $D_c$ satisfying the three conditions (ii)-(iv) (note that any $D_c$ satisfying (ii)-(iii) must be nonlocal, as a consequence of no-go theorem), then the GW Dirac operator $D = D_c(\mathbb{1} + raD_c)^{-1}$ is topologically trivial and satisfies all five conditions (i)-(v). The locality of $D$ in the free fermion is ensured by choosing $r$ in the proper range. Then it would be local in a gauge background which satisfies the locality bound (33). In this section, we present two examples from recent studies [10, 11].

In ref. [10], a well-defined $D_c$ satisfying the three conditions (ii)-(iv) has been constructed,

$$D_c \equiv \begin{bmatrix} 0 & -D_L^\dagger \\ D_L & 0 \end{bmatrix},$$

(36)

where

$$D_L = (\sigma \cdot t)^{-1} \left[ w - \sqrt{w^2 \sqrt{1 + w^{-1} t^2}} \right],$$

(37)

$$\sigma \cdot t = \sum_\mu \sigma_\mu t_\mu,$$

(38)

$$t_\mu(x, y) = \frac{1}{2} \left[ U_\mu(x)\delta_{x+\bar{\mu}, y} - U_\mu^\dagger(y)\delta_{x-\bar{\mu}, y} \right],$$

(39)

$$t^2 = -\sigma \cdot t(\sigma^\dagger \cdot t),$$

(40)

$$\sigma_\mu \sigma^\dagger_\nu + \sigma_\nu \sigma^\dagger_\mu = 2\delta_{\mu\nu},$$

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma^\dagger_\mu & 0 \end{pmatrix},$$

$$\gamma_5 = \begin{pmatrix} 0 & \sigma_5 \\ \sigma^\dagger_5 & 0 \end{pmatrix},$$

and

$$w(x, y) = \delta_{x,y} - \frac{1}{2} \sum_\mu \left[ 2\delta_{x,y} - U_\mu(x)\delta_{x+\bar{\mu}, y} - U_\mu^\dagger(y)\delta_{x-\bar{\mu}, y} \right].$$

(41)

It is evident that $D_c$ (36) is $\gamma_5$-hermitian since it is chirally symmetric and antihermitian. It has been shown [10] that in the free fermion limit, $D_c$ (36) is free of species doublings at finite lattice spacing and agrees with $\gamma_\mu \partial_\mu$ in the classical continuum limit. So, $D_c$ (36) satisfies the three conditions (ii)-(iv).

Now we examine whether $D_c$ (36) is well-defined. Since the naive lattice Dirac fermion operator $\gamma_\mu t_\mu$ does not have any exact zero modes in nontrivial
gauge backgrounds, its left-handed fermion propagator \((\sigma \cdot t)^{-1}\) is well-defined. Therefore, the first factor of \(D_L\) \((37)\) is well defined. Next we examine the second factor of \(D_L\) \((37)\). We see that the second factor is well-defined except for the gauge configurations which give \(\det(w) = 0\). However, these exceptional configurations have zero measure, thus they would never be encountered in any practical calculations. Therefore we have confirmed that \(D_c\) \((36)\) is well defined and satisfies the three conditions \((ii)-(iv)\). Then the GW Dirac operator \(D = D_c(1 + aD_c)^{-1}\) satisfies the five conditions \((i)-(v)\), where the locality of \(D\) is ensured by choosing \(r\) in the proper range. Since \(D_c\) is well defined, \(D = D_c(1 + aD_c)^{-1}\) is topologically trivial, according to the theorem proved in ref. \([5]\). Explicitly,

\[
D = \begin{bmatrix} r(bb^\dagger + r^2)^{-1} & -b(b^\dagger b + r^2)^{-1} \\ b^\dagger(bb^\dagger + r^2)^{-1} & r(b^\dagger b + r^2)^{-1} \end{bmatrix}
\]

(42)

where

\[
b = \frac{1}{2} \left( w - \sqrt{w^2 + w^{-1} t^2} \right)^{-1} (\sigma \cdot t)
\]

(43)

Another example of well-defined \(D_c\) which satisfies the three conditions \((ii)-(iv)\) is constructed in ref. \([11]\),

\[
D_c = \gamma_\mu t_\mu - W (\gamma_\mu t_\mu)^{-1} W.
\]

(44)

where \(t_\mu\) is defined in \((39)\) and \(W\) is the Wilson term \([21]\]

\[
W(x, y) = \frac{1}{2} \sum_\mu \left[ 2\delta_{x,y} - U_\mu(x)\delta_{x+\hat{\mu},y} - U_\mu^\dagger(y)\delta_{x-\hat{\mu},y} \right].
\]

(45)

The \(D_c\) in \((44)\) is \(\gamma_5\)-hermitian, since it is chirally symmetric and antihermitian. It is nonlocal due to the factor \((\gamma_\mu t_\mu)^{-1}\) in the second term. But it is well-defined since the naive fermion operator \(\gamma_\mu t_\mu\) does not have any exact zero modes in nontrivial gauge backgrounds. The free fermion propagator of \(D_c\) \((44)\) in momentum space is

\[
D_c^{-1}(p) = (\gamma_\mu t_\mu)^{-1} \frac{t^2}{w^2 + t^2}
\]

(46)

where \(t_\mu = ia^{-1}\sin(p_\mu a), t^2 = a^{-2}\sum_\mu \sin^2(p_\mu a)\) and \(w = 2a^{-1}\sum_\mu \sin^2(p_\mu a/2)\). The doubled modes are decoupled completely due to the vanishing of the factor \(t^2/(w^2 + t^2)\) at the \(2^d - 1\) corners of the Brillouin zone. So the fermion propagator \(D_c^{-1}(p)\) is free of species doublings. In the limit \(a \to 0\), Eq. \((46)\) gives \(D_c(p) = i\gamma_\mu p_\mu\) around \(p = 0\). Now we have confirmed that \(D_c\) \((44)\) is well-defined, and it satisfies the conditions \((ii)-(iv)\).
Then we substitute (44) into the formula 
\[ D = D_c(1 + arD_c)^{-1} \]
to obtain a GW Dirac operator
\[ D = \begin{bmatrix} rC\dagger C(1 + r^2CC\dagger)^{-1} & -C\dagger(1 + r^2CC\dagger)^{-1} \\ C(1 + r^2CC\dagger)^{-1} & r CC\dagger(1 + r^2CC\dagger)^{-1} \end{bmatrix} \] (47)
where
\[ C = (\sigma^\dagger t) - W(\sigma t)^{-1}W. \] (48)

Then \[ D \] (47) satisfies all five conditions (i)-(v), where the locality of \[ D \] is ensured by choosing \( r \) in the proper range. Since \( D_c \) is antihermitian and well-defined, the index of \( D = D_c(1 + rD_c) \) must be zero [5].

We have explicitly checked that the GW Dirac operators in (42) and (47) do not have any exact zero modes in a constant nontrivial gauge background with \( Q = 1 \), [ Eqs. (15)-(18) ], on different lattices with sizes ranging from \( N = 3 \) to \( N = 6 \).

We note that even though these two GW Dirac operators (42) and (47) do not have any exact zero modes in nontrivial gauge backgrounds, the five conditions (i)-(v) do guarantee that they can reproduce the correct axial anomaly in trivial gauge backgrounds. This will be demonstrated explicitly in the next section.

4 The axial anomaly and topological characteristics

In this section, we derive a general expression for the axial anomaly of lattice Dirac operator \( D \) on a finite lattice, in which the role of the topological characteristics \( c[D] \) is displayed explicitly.

4.1 In continuum

First, we discuss the axial anomaly of the massless Dirac operator in a gauge background in the continuum, i.e., the 4-d flat torus ( \( T^4 \) ) with size \( L^4 \). The anomaly equation can be written as
\[ \langle \partial_\mu j_\mu^5(x) \rangle = 2A(x) + 2 \sum_{s=1}^{n_+} [\phi^s_+(x)]^\dagger \phi^s_+(x) - 2 \sum_{t=1}^{n_-} [\phi^t_-(x)]^\dagger \phi^t_-(x) \] (49)
where \( j_\mu^5(x) = \bar{\psi}(x)\gamma_\mu\gamma_5\psi(x) \) is the axial vector current, \( \langle \partial_\mu j_\mu^5(x) \rangle \) is the fermionic average of the divergence of axial vector current, \( \phi^s_+ \) and \( \phi^t_- \) are normalized eigenfunctions of the zero modes with chirality +1 and −1 respectively, and \( A(x) \) is the axial anomaly. Integrating the anomaly equation (49)
over the 4-torus, the l.h.s. vanishes automatically, and the r.h.s. gives

\[ \int_{T^4} d^4 x \, A(x) = n_- - n_+ \]. \hspace{1cm} (50)

Using the Atiyah-Singer index theorem \[13\], Eq. (50) becomes

\[ \int_{T^4} d^4 x \, A(x) = Q = \int d^4 x \, \rho(x) \], \hspace{1cm} (51)

where \( \rho(x) \) is the topological charge density defined in \[8\]. Then (51) yields

\[ \int_{T^4} d^4 x \, (A(x) - \rho(x)) = 0 \]. \hspace{1cm} (52)

According to the divergence theorem, (52) implies that

\[ A(x) = \rho(x) + \partial_\mu g_\mu(x) \], \hspace{1cm} (53)

where the current \( g_\mu(x) \) is gauge invariant and continuous on the 4-torus.

For weak gauge backgrounds ( \( |A_\mu(x)| \ll 1/L \) ) with zero topological charge, \( A(x) \) can be evaluated using weak-coupling perturbation theory, and the result is equal to the topological charge density of the gauge background,

\[ A(x) = \rho(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \text{tr}(F^\mu_\nu(x)F^\lambda_\sigma(x)) \]. \hspace{1cm} (54)

Therefore \( \partial_\mu g_\mu(x) = 0 \) in this case.

Next we consider a nontrivial gauge background with constant field tensor \( F^\mu_\nu(x) = F^0_\mu\nu \). Since field tensor is constant, \( A(x) \) must be constant. Thus \( A(x) \) can be solved directly from Eq. (51),

\[ A(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \text{tr}(F^0_\mu F^0_\nu \lambda \sigma) = \rho_0 \]. \hspace{1cm} (55)

Again, \( \partial_\mu g_\mu(x) = 0 \) in this case.

It can be shown that \( \partial_\mu g_\mu(x) = 0 \) for any smooth gauge backgrounds.

### 4.2 On a finite lattice

Next we regularize the theory by discretizing the 4-torus into a hypercubical lattice with \( N \) sites in each direction (i.e., \( L = Na \)). The gauge fields \( \{A_\mu(x)\} \) are transcribed to link variables \( \{U_\mu(x)\} \) on the lattice. For any lattice Dirac operator \( D \), the anomaly equation on the finite lattice is \[12\],

\[ a^4 \langle \partial_\mu j_\mu^5(x) \rangle = 2a^4 A_L(x) + 2 \sum_{s=1}^{n_+} [\phi_+^s(x)]^\dagger \phi_+^s(x) - 2 \sum_{t=1}^{n_-} [\phi_-^t(x)]^\dagger \phi_-^t(x) \] \hspace{1cm} (56)
where the axial anomaly is

\[ a^4 A_L(x) = \lim_{m \to 0} \frac{1}{4} \text{tr} \left[ (B \hat{D}^{-1})(x,x) + (\hat{D}^{-1}B)(x,x) \right]. \]  

(57)

Here \( \hat{D} = D + m \), and \( B \) is an irrelevant operator denoting the chirality breaking of \( D \),

\[ D\gamma_5 + \gamma_5 D = B. \]  

(58)

In particular, for GW Dirac operators, \( B = 2raD\gamma_5D \), then (57) reduces to

\[ a^4 A_L(x) = r a \text{ tr} \left[ \gamma_5 D(x,x) \right]. \]  

(59)

Summing the anomaly equation (56) over all sites of the finite lattice, then the l.h.s. vanishes and the r.h.s. gives

\[ n_- - n_+ = \sum_x a^4 A_L(x). \]  

(60)

From (61) and (62), we obtain

\[ \sum_x a^4 A_L(x) = c[D] Q. \]  

(61)

Now we can write

\[ Q = \sum_x a^4 \rho_L(x), \]  

(62)

where \( \rho_L(x) \) is the topological charge density of the gauge background on the finite lattice. Note that we do not need an explicit expression of \( \rho_L(x) \), which is supposed to be expressed in terms of link variables.

Then from (61) and (62), we obtain

\[ \sum_x (A_L(x) - c[D] \rho_L(x)) = 0. \]  

(63)

The general solution of (63) is

\[ A_L(x) = c[D] \rho_L(x) + g(x), \]  

(64)

where \( g(x) \) satisfies \( \sum_x g(x) = 0 \). We can rewrite \( g(x) \) in a form analogous to \( \partial_\mu g_\mu(x) \) in (53),

\[ g(x) = \sum_\mu [g_\mu(x) - g_\mu(x - a\hat{\mu})] \equiv \partial_\mu g_\mu(x), \]  

(65)

where \( g_\mu(x) \) satisfies the periodic boundary conditions: \( g_\mu(x + L\hat{\nu}) = g_\mu(x) \) for \( \mu, \nu = 1, \cdots, 4 \).
Now we write

\[ \rho_L(x) = \bar{\rho}(x) + f(x), \tag{66} \]

where \( \bar{\rho}(x) \) is the topological charge density inside the unit cell of volume \( a^4 \) centered at the site \( x \), defined in Eq. (3), and \( f(x) \) denotes the difference between \( \rho_L(x) \) and \( \bar{\rho}(x) \). Summing Eq. (66) over all sites and using the relation \( \sum_x a^4 \rho_L(x) = \sum_x a^4 \bar{\rho}(x) = Q \), we obtain

\[ \sum_x f(x) = 0. \tag{67} \]

Therefore, we can rewrite \( f(x) \) in terms of a total divergence term

\[ f(x) = \sum_{\mu} \left[ f_\mu(x) - f_\mu(x - a\hat{\mu}) \right] \equiv \partial_\mu f_\mu(x), \tag{68} \]

where \( f_\mu(x) \) satisfies the periodic boundary conditions: \( f_\mu(x + L\hat{\nu}) = f_\mu(x) \) for \( \mu, \nu = 1, \cdots, 4 \). Substituting (66) into (64), we obtain a general expression for \( A_L(x) \),

\[ A_L(x) = c[D]\bar{\rho}(x) + \partial_\mu G_\mu(x), \tag{69} \]

where

\[ \partial_\mu G_\mu(x) = \sum_{\mu} \left[ G_\mu(x) - G_\mu(x - a\hat{\mu}) \right] = c[D]f(x) + g(x). \]

If the variation of \( \rho(x) \) is very small across the unit cell \( a^4 \) centered at \( x \) (which must be the case in the limit \( N \to \infty \) ( \( a \to 0 \) ) for a smooth background), then \( \bar{\rho}(x) \simeq \rho(x) \), and \( A_L(x) \) becomes

\[ A_L(x) \simeq c[D] - \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \text{tr}(F_{\mu\nu}(x)F_{\lambda\sigma}(x)) + \partial_\mu G_\mu(x). \tag{70} \]

where \( \text{tr} \) denotes the trace over the gauge group space.

Equation (69) is a general expression for the axial anomaly of any lattice Dirac operator on a finite lattice. Note that we have not imposed any conditions on \( D \) except assuming that the exact zero modes (if any) of \( D \) have definite chirality (this is the case for GW Dirac operators) which has been used in deriving the anomaly equation (56).

From (70), we immediately see that \( A_L(x) \) can recover the continuum axial anomaly if and only if \( c[D] = 1 \) and \( \partial_\mu G_\mu(x) = 0 \) for all \( x \). However, satisfying the five conditions (i)-(v) does not guarantee that \( c[D] = 1 \) and \( \partial_\mu G_\mu(x) = 0 \) in nontrivial gauge backgrounds.
Next we try to understand the total divergence term $\partial_\mu G_\mu(x)$. Let us consider a nontrivial gauge configuration with constant field tensor. Then the topological charge density is constant and the topological charge is,

$$Q = \sum_x a^4 \rho_L(x) = N^4 a^4 \rho_0 ,$$

$$\rho_0 = \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \text{tr}(F^0_{\mu\nu} F^0_{\lambda\sigma}) . \quad (71)$$

An example is given in Eqs. (15)-(18).

Since $\rho_L(x)$ is constant, one may expect that $A_L(x)$ is constant too. However, this is true only if $D$ is local. In general, if $D$ is local in the free fermion limit, then it would be local in a gauge background which satisfies the locality bound (33). If we assume that is the case, then $A_L(x)$ is constant and it can be solved directly from Eq. (61),

$$A_L(x) = c[D] \rho_L(x) = c[D] \rho_0 . \quad (72)$$

Comparing (72) to (70), we obtain $\partial_\mu G_\mu(x) = 0$ provided that $c[D] \neq 0$. If $c[D] = 0$, then Eq. (61) becomes

$$\sum_x A_L(x) = 0 , \quad (73)$$

which implies that $A_L(x) = 0$ if $A_L(x)$ is constant. However, in a gauge background with nonzero constant field tensor, $A_L(x)$ cannot be zero at all sites, since $D$ must at least have some responses to the gauge background except for $D \equiv 0$. Therefore, in the case $c[D] = 0$, $A_L(x)$ cannot be identically zero. Thus (73) implies that $A_L(x)$ can be written as a total divergence,

$$A_L(x) = \partial_\mu g_\mu(x) . \quad (74)$$

Therefore, in a nontrivial gauge background with constant field tensor, the axial anomaly on a finite lattice is

$$A_L(x) = \begin{cases} c[D] \rho_0 , & c[D] \neq 0 \\ \partial_\mu g_\mu(x) , & c[D] = 0 \end{cases} \quad (75)$$

Thus a topologically trivial $D$ cannot have the correct axial anomaly in nontrivial gauge sectors.

Now suppose $D$ is topologically proper. If we introduce very small local fluctuations of $\rho(x)$ on top of the constant $\rho_0$ such that $\sum_x \delta \rho(x) = 0$, then evidently the axial anomaly of $D$ becomes

$$A_L(x) = \bar{\rho}(x) . \quad (76)$$

Further, one can deduce that Eq. (76) holds for any gauge configurations satisfying the locality bound (33).
In the trivial sector ($Q = 0$), for weak gauge backgrounds ($|a A_\mu(x)| \ll 1/N$) at finite lattice spacing or smooth backgrounds in the classical continuum limit, the axial anomaly $A_L(x)$ can be evaluated using perturbation theory (For examples, see ref. [15]-[20]). If $D$ satisfies the five conditions (i)-(v), then $A_L(x)$ is equal to the topological charge density of the gauge background,

$$A_L(x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} \text{tr}(F_{\mu\nu}(x) F_{\lambda\sigma}(x)).$$

(77)

Thus even if $D$ is topologically trivial in nontrivial gauge backgrounds, it can have correct axial anomaly in the trivial gauge sector.

4.3 Examples

In the following, we illustrate the results of (75) and (77) by explicit examples.

First, for the purpose of comparison, we also consider the Wilson-Dirac lattice fermion operator [21] which does not satisfy the Ginsparg-Wilson relation (v) but the other four conditions (i)-(iv). Explicitly, the massless Wilson-Dirac operator can be written as

$$D_W = \gamma_\mu t_\mu + W,$$

(78)

where $t_\mu$ and $W$ are defined in Eqs. (39) and (45) respectively. According to formula (57), the axial anomaly of $D_W$ can be written

$$A_W(x) = \frac{1}{4} \sum_\mu \left( 4 \text{tr}[D_w^{-1}(x, x) \gamma_5] - \text{tr}[D_w^{-1}(x, x + a\hat{\mu}) \gamma_5 U_\mu(x)] - \text{tr}[D_w^{-1}(x + a\hat{\mu}, x) \gamma_5 U_\mu(x) - a\hat{\mu})] - \text{tr}[D_w^{-1}(x, x - a\hat{\mu}) \gamma_5 U_\mu(x) - a\hat{\mu})] \right).$$

(79)

where tr denotes the traces over the Dirac space and the gauge group space. It is well known that $D_W$ does not possess any exact zero modes in topologically nontrivial gauge fields, thus $c[D_W] = 0$. Therefore its axial anomaly must disagree with the topological charge density in nontrivial gauge backgrounds, according to (70). However, one may wonder whether $D_W$ can produce the correct axial anomaly in trivial gauge backgrounds satisfying the locality bound. For this purpose, it suffices to consider the trivial $U(1)$ gauge backgrounds with local sinusoidal fluctuations on a 2-dimensional flat torus of size $L \times L$:

$$a A_1(x) = \frac{2\pi h_1 a}{L} + A_1^{(0)} \sin \left( \frac{2\pi n_2}{L} x_2 \right)$$

(80)

$$a A_2(x) = \frac{2\pi h_2 a}{L} + A_2^{(0)} \sin \left( \frac{2\pi n_1}{L} x_1 \right)$$

(81)
The corresponding link variables on the 2d lattice \((x_{\mu} = 0, a, ..., (N-1)a)\) are
\[
U_1(x) = \exp[iaA_1(x)], \quad U_2(x) = \exp[iaA_2(x)].
\]
The topological charge density on the 2d torus is
\[
\rho(x) = \frac{1}{2\pi} F_{12}(x) = A_2^{(0)} \frac{n_1}{aL} \cos \left( \frac{2\pi n_1}{L} x_1 \right) - A_1^{(0)} \frac{n_2}{aL} \cos \left( \frac{2\pi n_2}{L} x_2 \right)
\]
(82)
The topological charge density inside the square \(a^2\) centered at \(x\) is
\[
\bar{\rho}(x) = \frac{1}{a^2} \int_{x_1-a/2}^{x_1+a/2} dx_1 \int_{x_2-a/2}^{x_2+a/2} dx_2 \rho(x)
\]
(83)
\[
= \frac{1}{a^2 \pi} \left[ A_2^{(0)} \sin \left( \frac{\pi n_1}{L} x_1 \right) \cos \left( \frac{2\pi n_1}{L} x_1 \right) - A_1^{(0)} \sin \left( \frac{\pi n_2}{L} x_2 \right) \cos \left( \frac{2\pi n_2}{L} x_2 \right) \right]
\]
(84)
In Fig. 1 we plot \(A_W(x)\) for each site on a \(12 \times 12\) lattice with lattice spacing \(a = 1\), comparing with the topological charge density \(\bar{\rho}(x)\), in a trivial gauge background [Eqs. (80) and (81)] with parameters \(A_1^{(0)} = 0.3, A_2^{(0)} = 0.4, n_1 = n_2 = 1, h_1 = 0.1\) and \(h_2 = 0.2\). Note that in this case, the difference between \(\rho(x)\) and \(\bar{\rho}(x)\) is very small, always less than one percent. The position of a site with coordinates \((x_1, x_2)\) is represented by an integer \(x = 12(x_2 - 1) + x_1\), as the x-coordinate in Fig. 1. The axial anomaly \(A_W(x)\) is denoted by squares. The topological charge density \(\bar{\rho}(x)\) of the gauge background is denoted by circles. The line segments between circles are inserted only for the visual purpose. Evidently \(A_W(x)\) disagrees with the topological charge density \(\bar{\rho}(x)\). The discrepancies are due to the presence of the fermion doublets which decouple completely only in the limit \(a \to 0\).

The deviation of the axial anomaly of a lattice Dirac operator in a gauge background can be measured in terms of
\[
\delta = \frac{1}{N_s} \sum_x \frac{|A_L(x) - \bar{\rho}(x)|}{|\bar{\rho}(x)|}
\]
(85)
where \(N_s\) is the total number of sites of the lattice, and \(\bar{\rho}(x)\) is the topological charge density inside the unit cell of volume \(a^d\) centered at \(x\).

For the axial anomaly of the Wilson-Dirac operator as shown in Fig. 1, the deviation is \(\delta_W = 0.45\). One expects that the deviation \(\delta_W\) goes to zero only in the limit \(N \to \infty\) (or \(a \to 0\)).

Next we consider the lattice Dirac operators \([12]\) and \([17]\) in Section 3, which satisfy all five conditions (i)-(v), but both are topologically trivial. According to \([57]\), the axial anomaly of \([12]\) is
\[
A_i(x) = r^2 \text{ tr} \left[ (bb^t + r^2)^{-1} - (b^*b + r^2)^{-1} \right] (x, x),
\]
(86)
and that of (47) is

\[ \mathcal{A}_{wc}(x) = r^2 \text{tr} \left[ C^\dagger C (\mathbb{1} + r^2 C^\dagger C)^{-1} - CC^\dagger (\mathbb{1} + r^2 CC^\dagger)^{-1} \right] (x, x), \]  

(87)

where tr denotes the traces over the Dirac space and the gauge group space. The axial anomalies \( \mathcal{A}_t(x) \) and \( \mathcal{A}_{wc}(x) \) are plotted in Fig. 2 and Fig. 3 respectively, for the same gauge background as in Fig. 1. The deviation for the axial anomaly \( \mathcal{A}_t(x) \) shown in Fig. 2 is \( \delta = 0.121 \), and for \( \mathcal{A}_{wc}(x) \) shown in Fig. 3 is \( \delta = 0.068 \). (For comparison, the deviation for the axial anomaly of the Neuberger-Dirac operator with \( m_0 = 1 \) is \( \delta = 0.049 \) for the same gauge background in Fig. 1). Evidently both \( \mathcal{A}_t(x) \) and \( \mathcal{A}_{wc}(x) \) agree with \( \bar{\rho}(x) \) for the trivial gauge background. They provide explicit demonstrations of (77) which holds for any \( D \) satisfying the five conditions (i)-(v), and for any trivial gauge backgrounds satisfying the locality bound. It is instructive to compare the axial anomalies of the GW Dirac operators in Fig. 2 and Fig. 3 to that of the Wilson-Dirac operator in Fig. 1. The former ones can reproduce the continuum axial anomaly even on a finite lattice while the latter cannot.

In Fig. 4, we plot the axial anomaly in a nontrivial gauge background (\( Q = 2 \)) with constant field strength on a 12 × 12 lattice, for the topologically proper Neuberger-Dirac operator, as well as the topologically improper GW Dirac operators (42) and (47), respectively. It is clear that the axial anomaly of the Neuberger-Dirac operator agrees with the constant topological charge density \( \rho_0 = 1/72 \) at each site, while those of the GW Dirac operators (42) and (47) are in complete disagreement with \( \rho_0 \). This provides an explicit demonstration of Eq. (75) for two different cases \( c[D] = 1 \) and \( c[D] = 0 \). For the case \( c[D] = 0 \), it shows explicitly that the axial anomaly is in the form \( \partial_\mu g_\mu(x) \), which oscillates around zero with its sum over all sites equal to zero.

In passing, we note that the topological bound of the Neuberger-Dirac operator with \( m_0 = 1 \) in two dimensions is

\[ a^2|\bar{\rho}(x)| < \epsilon_1 \simeq 0.28 \quad \forall \ x, \]

while the locality bound [8] is

\[ || \mathbb{1} - U_{\mu\nu}(x) || < \epsilon = \left. \frac{1}{2 + \sqrt{2}} \right\} \simeq 0.29 \quad \text{for all plaquettes} , \]

which is transcribed into

\[ a^2|\bar{\rho}(x)| < \epsilon_2 \simeq 0.046 \quad \forall \ x. \]

Again, \( \epsilon_2 \) is much less than \( \epsilon_1 \).
5 Summary and Discussions

Given any lattice Dirac operator $D_0$ satisfying the four conditions (i)-(iv), one can construct a chirally symmetric lattice Dirac operator $D_c$ satisfying the three conditions (ii)-(iv), i.e.,

$$D_c = 2\gamma_5 D_0 (\gamma_5 D_0 - D_0 \gamma_5)^{-1} D_0,$$

which must be nonlocal as a consequence of the Nielson-Ninomiya theorem [3]. An example of such $D_c$ is given in Eq. (44). Then the GW Dirac operator $D = D_c (1 + ra D_c)^{-1}$ satisfies all five conditions (i)-(v), where the locality of $D$ is ensured by choosing $r$ in the proper range. It is obvious that one can replace (88) by any $D_c$ which satisfies the three conditions (ii)-(iv). An example is given in Eqs. (36)-(37).

If $D_c$ is well-defined (without any singularities) in nontrivial gauge backgrounds, then the index of $D = D_c (1 + ra D_c)^{-1}$ is zero [4]. In other words, even if $D$ satisfies all five conditions (i)-(v), $D$ does not necessarily have exact zero modes in nontrivial gauge fields. Two explicit examples have been presented in Section 3. At present, we do not know how to formulate additional constraints (in useful forms) such that $D$ can produce exact zero modes satisfying the Atiyah-Singer index theorem. Even though such conditions will be found in the future, it is still convenient for us to introduce the topological characteristics $c[D]$ associated with each lattice Dirac operator, as defined in (5). For a given $D_c$, $c[D]$ is an integer constant provided that the topological charge inside any unit cell is small enough, i.e., $a^4 |\bar{\rho}(x)| < \epsilon_1$. (For Neuberger-Dirac operator with $m_0 = 1$, $\epsilon_1 \simeq 0.02$). However, in general, one does not need to go to the limit $a = 0$ (or $N = \infty$) in order to reveal the topological characteristics of any $D$. Our view is that if a lattice Dirac operator (satisfying (i)-(v)) does not have the correct index on any finite lattices, then it will never have the correct index in the continuum limit. An argument has been presented in Section 2.

Since the index of $D = D_c (1 + ra D_c)^{-1}$ is invariant for any $r$, the topological characteristics $c[D]$ is also invariant from $r = 0$, $D = D_c$ (a nonlocal operator) up to $ra \gg 1$, $D \simeq (ra)^{-1} 1 + \text{higher order corrections}$ (a highly local operator). Thus the locality condition (i) does not form a constraint to $c[D]$, even though it is crucial to recover the continuum axial anomaly on the lattice when $c[D] = 1$. Further, one can always transform any lattice Dirac operator into a GW Dirac operator with the same index [4], so the GW relation (v) also does not form a constraint to $c[D]$. Therefore, we conclude that among these five conditions (i)-(v), only three of them [i.e., (ii), (iii) and (iv)] provide constraints to $c[D]$. However, satisfying all of them does not guarantee that $c[D] = 1$. Additional constraints are needed. They should be formulated explicitly with exact solutions rather than a set of equations with only implicit or approximate solutions. From this viewpoint, the Neuberger-Dirac operator
is *much more* than just a solution of the GW relation, since the former has the proper topological characteristics, while the latter does not necessarily have. This seems to indicate that the Overlap formalism \[23, 24\] (and the Domain-Wall fermion \[25\]) indeed plays the fundamental role in solving the problem of chiral fermions on the lattice.

It is instructive to compare Lüscher’s definition of topological charge \[22\] to that defined by the index of the Neuberger-Dirac operator. The former definition relies on the fact that if the gauge field satisfies the bound

\[
|\text{tr}(\mathbb{1} - U_{\mu \nu}(x))| < \epsilon \quad \text{for all plaquettes}, \tag{89}
\]

then any link configuration (with nonzero integer topological charge) cannot deform into the trivial configuration (with all link variables equal to the identity), and vice versa. Thus the gauge link configurations can be decomposed into disconnected topological sectors as in the continuum, and the topological charge is a well-defined integer even on a lattice. For gauge fields satisfying the bound (89), Lüscher derived an expression for the topological charge density on the lattice $\rho_L(x)$ (i.e., $a^{-4}q(n)$ of Eq. (32) in ref. \[22\]), which satisfies $Q = \sum_x a^4\rho_L(x)$, and it tends to $\rho(x)$ smoothly in the continuum limit. That is, even on a finite lattice, $\rho_L(x)$ is already in good agreement with $\rho(x)$. On the other hand, the axial anomaly of a topologically proper $D$ (assuming it is local in the free fermion limit) is $\mathcal{A}_L(x) = \tilde{\rho}(x) \simeq \rho(x)$ provided that $D$ is local, i.e., the gauge background satisfies the locality bound (33). Since $\mathcal{A}_L(x) \simeq \rho_L(x)$ for a topologically proper $D$ [see Eq. (64) with $c[D] = 1$, and $g(x) \simeq 0$], it follows that the locality bound (33) of the Neuberger-Dirac operator is compatible with Lüscher’s bound (89). However, if one is only interested in the topological charge, then the index of the Neuberger-Dirac operator is already equal to the topological charge for gauge configurations satisfying the topological bound (2) or (32) which is less restrictive than the locality bound (33). This seems to suggest that if one uses the index of the Neuberger-Dirac operator \[4\] to define the topological charge of a gauge background, then the lattice size $N$ can be smaller (or the lattice spacing $a$ can be larger) than that using Lüscher’s definition \[22\].

A general expression (69) for the axial anomaly of any lattice Dirac operator has been derived, in which the role of topological characteristics is displayed explicitly. For a topologically proper $D$, its axial anomaly is equal to the topological charge density $\rho(x)$ for smooth gauge background satisfying the locality bound. Thus, lattice QCD with the Neuberger-Dirac quarks has the correct axial anomaly associated with the global chiral symmetry. For chiral gauge theories such as the standard model, the gauge anomaly must exactly cancel at finite lattice spacing before one is sure the existence of a nonperturbative regularization of the theory. For an anomaly-free fermion multiplet, it has been argued \[23\] that the nonabelian gauge anomaly cancellation (in the trivial sector) holds to all orders of an expansion in powers of the lattice spacing.
while the abelian gauge anomaly cancellation [27, 28] is exact at finite lattice spacing. In continuum, the non-abelian gauge anomaly in 4-dimensional space may be obtained from the axial anomaly in 6-dimensional space [29, 30, 31]. Thus its form and normalization follow from the Atiyah-Singer index theorem for a certain 6-dimensional Dirac operator. Presumably, a similar analysis can be performed on a finite lattice with the axial anomaly in the form of (76) for a topologically proper $D$ in a gauge background satisfying the locality bound (33). It seems that the gauge anomaly can exactly cancel at finite lattice spacing. We intend to return to this question in a later publication.

To conclude, we emphasize that $c[D] = 1$ is a stringent requirement for any construction of lattice Dirac operator $D$ in $2n$-dimensional space. It seems that this requirement could not be satisfied by any approximate solutions to the GW relation plus other physical constraints in the $2n$-dimensional space. Unless such $2n$-dimensional $D$ can possess exact zero modes and the correct index in nontrivial gauge backgrounds on finite lattices, otherwise its axial anomaly will never recover the correct result in the continuum limit.

A Some basic properties of GW Dirac operators

In this appendix, we review some basic properties of the GW Dirac operator, which are pertinent to our discussions in this paper. Most of these properties can be found in refs. [7, 5]. Here we assume that $D$ satisfies the five conditions (i)-(v) listed in Section 1.

The general solution to the GW relation (v) can be written as [5]

$$D = D_c(\mathbb{I} + arD_c)^{-1} = (\mathbb{I} + arD_c)^{-1}D_c$$

(90)

where $D_c$ is any chirally symmetric ($D_c \gamma_5 + \gamma_5 D_c = 0$) Dirac operator which must violate at least one of the three conditions (i)-(iii) above, according to the Nielson-Ninomiya no-go theorem. Now we require $D_c$ to satisfy (ii) and (iii), but violate (i) (i.e., $D_c$ is nonlocal), since (90) can transform the nonlocal $D_c$ into a local $D$ on a finite lattice for $r$ in the proper range, while the properties (ii)-(iv) are preserved. Then $D$ satisfies all five conditions (i)-(v).

Moreover, the zero modes and the index of $D_c$ are invariant under the transformation (90) [5]. That is, a zero mode of $D_c$ is also a zero mode of $D$ and vice versa, i.e.,

$$D\phi_\pm = 0 \Leftrightarrow D_c\phi_\pm = 0,$$

where $\gamma_5 \phi_\pm = \pm \phi_\pm$. Then the number of zero modes for each chirality must be the same for $D$ and $D_c$, thus the index of the local $D$ is equal to the index
of the nonlocal $D_c$,

$$n_+(D_c) = n_+(D), \quad n_-(D_c) = n_-(D), \quad (91)$$

$$\text{index}(D_c) = n_-(D_c) - n_+(D_c) = n_-(D) - n_+(D) = \text{index}(D). \quad (92)$$

Therefore, a nonlocal Dirac operator can have well-defined index, at least for those obtained by the topologically invariant transformation

$$D_c = D(\mathbb{1} - arD)^{-1} = (\mathbb{1} - arD)^{-1}D, \quad (93)$$

which is the inverse transform of (90). From the definition of the topological characteristics $c[D]$, (4), we have

$$c[D] = c[D_c].$$

The $\gamma_5$-hermiticity of $D$ (iv) is equivalent to the $\gamma_5$-hermiticity of $D_c$,

$$D_c^\dagger = \gamma_5 D_c \gamma_5. \quad (94)$$

Then the chiral symmetry of $D_c$ together with its $\gamma_5$-hermiticity implies that $D_c$ is antihermitian ($D_c^\dagger = -D_c$). This last property is in agreement with the massless Dirac fermion operator in continuum. Then there exists one to one correspondence between $D_c$ and a unitary operator $V$ such that

$$D_c = M(\mathbb{1} + V)(\mathbb{1} - V)^{-1}, \quad V = (D_c - M)(D_c + M)^{-1}. \quad (95)$$

where $M$ is a mass scale and $V$ also satisfies the $\gamma_5$-hermiticity $V^\dagger = \gamma_5 V \gamma_5$. Then the general solution (90) can be written as

$$D = M(\mathbb{1} + V)[(\mathbb{1} - V) + r Ma(\mathbb{1} + V)]^{-1} \quad (96)$$

Due to the $\gamma_5$-hermiticity, the eigenvalues of $D$ are either real or come in complex conjugate pairs. Furthermore, from (90), since $r$ is a positive real number, the lower bound of real eigenvalues of $D$ is zero, thus det($D$) is real and nonnegative, and is amenable to Hybrid Monte Carlo simulation with any number of flavors of dynamical fermions. For the GW relation (v) with any $r$ (not restricted to 1/2), the analysis in ref. Eqs. (24)-(41) goes through with trivial modification. The main results are:

(a) The eigenvalues of $D$ fall on a circle with center at $1/2r$, and radius $1/2r$, and have the reflection symmetry with respect to the real axis.

(b) The real eigenmodes (if any) at 0 and $1/r$ have definite chirality +1 or −1.

(c) The chirality of any complex eigenmodes is zero.
Total chirality of all eigenmodes must vanish.
\[ \text{Tr}(\gamma_5) = \sum_s \phi_s^\dagger \gamma_5 \phi_s = n_+ - n_- + N_+ - N_- = 0 \]
where \( n_+(n_-) \) denotes the number of zero modes of positive (negative) chirality, and \( N_+(N_-) \) the number of 1/r modes of positive (negative) chirality. From (\( \delta \)), we immediately see that any zero mode must be accompanied by a real 1/r mode with opposite chirality, and the index of \( D \) is
\[ \text{index}(D) \equiv n_- - n_+ = -(N_- - N_+) \quad (97) \]
Now the central problem is to construct the chirally symmetric\( D_c \) which is nonlocal, and satisfies (iii), (iv), and (94). Furthermore we also require that\( D_c \) is topologically proper (i.e., satisfying the Atiyah-Singer index theorem) for any smooth gauge background satisfying the topological bound (2). These constitute the necessary requirements [16] for \( D_c \) to enter (90) such that \( D \) could provide a nonperturbative regularization for a massless Dirac fermion interacting with a background gauge field. Explicitly, these necessary requirements are:

(a) \( D_c \) is antihermitian (hence \( \gamma_5 \)-hermitian) and it agrees with \( \gamma_\mu (\partial_\mu + igA_\mu) \) in the classical continuum limit.

(b) \( D_c \) is free of species doubling.

(c) \( D_c \) is nonlocal.

(d) \( D_c \) is well defined in topologically trivial background gauge field.

(e) \( D_c \) has zero modes as well as simple poles in topologically non-trivial background gauge fields (each zero mode of \( D_c \) must be accompanied by a simple pole of \( D_c \)). Furthermore, the zero modes of \( D_c \) satisfy the Atiyah-Singer index theorem for any smooth gauge background satisfying the bound (2).

The general solution of \( D_c \) satisfying these requirements had been investigated in ref. [10]. However, in general, given any lattice Dirac operator \( D \), there exists a transformation \( \mathcal{T}(R_c) \) for \( D \) such that the transformed Dirac operator \( D_c = \mathcal{T}(R_c)[D] \) is chirally symmetric [12, 11].

### B Indices of nonlocal Dirac operators

In this appendix, we note that the locality condition (i) may not be relevant to the exact zero modes of a Dirac operator. Consider a massless fermion in a background gauge field, the Dirac operator in continuum is
\[ D = \gamma_\mu (\partial_\mu + iA_\mu) \quad (98) \]
which is local, chirally invariant \((D \gamma_5 + \gamma_5 D = 0)\) and antihermitian \((D^\dagger = -D)\). Now we can define a new Dirac operator

\[
D' = D(1 - rbD)^{-1} = (1 - rbD)^{-1}D,
\]

(99)

where \(r\) is a positive real number and \(b\) denotes a regulator with mass dimension \(M^{-1}\). One can always choose the value of \(r\) such that \(D'\) is nonlocal. Now it is evident that any zero mode of \(D\) must be a zero mode of \(D'\), and vice versa, i.e.,

\[
D \phi_\pm = 0 \iff D' \phi_\pm = 0,
\]

where \(\gamma_5 \phi_\pm = \pm \phi_\pm\). Then the number of zero modes for each chirality must be the same for \(D\) and \(D'\), thus the index of the local Dirac operator \(D\) is equal to the index of the nonlocal Dirac operator \(D'\),

\[
\begin{align*}
n_+ (D') &= n_+ (D), \\
n_- (D') &= n_- (D)
\end{align*}
\]

\[
\text{index}(D') = n_- (D') - n_+ (D') = n_- (D) - n_+ (D) = \text{index}(D).
\]

(100)

Therefore, the nonlocal Dirac operator \(D'\) has well-defined index, and it also satisfies the Ginsparg-Wilson relation

\[
D' \gamma_5 + \gamma_5 D' = -2rbD' \gamma_5 D'.
\]

(101)

It is interesting to note that the chiral limit (i.e., \(r = 0\)) of the nonlocal \(D'\) is the local operator \(D\) in (98), while on the lattice, the chiral limit of a local GW Dirac operator \(D\) satisfying (v) is

\[
D_c = D(1 - arD)^{-1} = (1 - arD)^{-1}D,
\]

(102)

which must be nonlocal if \(D_c\) is free of species doubling and has the correct continuum behavior, according to the Nielson-Ninomiya no-go theorem. Again, the index of \(D_c\) is equal to the index of \(D\). Therefore, no matter in continuum or on a lattice, a nonlocal Dirac operator can have well-defined index, at least for those obtained by the topologically invariant transformations, (99) and (102). In other words, if a nonlocal lattice Dirac operator does not have exact zero modes in topologically nontrivial gauge backgrounds, then the cause may not be due to its nonlocalness, but its topological characteristics.

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Figure 1: The axial anomaly $A_W(x)$ [Eq. (79)] of the massless Wilson-Dirac operator $D_w$ [Eq. (78)] in a trivial gauge background on a $12 \times 12$ lattice. The background $U(1)$ gauge field [Eqs. (80)-(81)] is specified by the parameters $h_1 = 0.1, h_2 = 0.2, A_1^{(0)} = 0.3, A_2^{(0)} = 0.4$ and $n_1 = n_2 = 1$. The axial anomaly of the Wilson-Dirac operator, $A_W(x)$, is denoted by squares. The topological charge density $\bar{\rho}(x)$ [Eq. (83)] of the gauge background is denoted by circles. The line segments between circles are inserted only for the visual purpose. Evidently $A_W(x)$ disagrees with $\bar{\rho}(x)$.
Figure 2: The axial anomaly $A_t(x)$ [Eq. (86)] of the massless GW Dirac operator $D$ [Eq. (42)] in a trivial gauge background on a $12 \times 12$ lattice. The value of $r$ in $D$ has been set to 0.4, and there is no significant changes to $A_t(x)$ for any $r$ in the range $\sim 0.2$ to $\sim 0.8$. The background $U(1)$ gauge field is the same as that in Fig. 1. The axial anomaly $A_t(x)$ is denoted by triangles. The topological charge density $\bar{\rho}(x)$ [Eq. (83)] of the gauge background is denoted by circles. The line segments between circles are inserted only for the visual purpose.
Figure 3: The axial anomaly $A_{wc}(x)$ [ Eq. (87) ] of the massless GW Dirac operator $D$ [ Eq. (47) ] in a trivial gauge background on a $12 \times 12$ lattice. The value of $r$ in $D$ has been set to 0.5, and there is no significant changes to $A_{wc}(x)$ for any $r$ in the range $\sim 0.2$ to $\sim 0.8$. The background $U(1)$ gauge field is the same as that in Fig. 1. The axial anomaly $A_{wc}(x)$ is denoted by triangles. The topological charge density $\bar{\rho}(x)$ [ Eq. (83) ] of the gauge background is denoted by circles. The line segments between circles are inserted only for the visual purpose.
Figure 4: The axial anomalies of GW Dirac operators in a nontrivial gauge background with constant field tensor $F_{12} = \pi/36$ (topological charge $Q = 2$) on a $12 \times 12$ lattice. The topological charge density ($\rho(x) = 1/72$) of the gauge background is denoted by the horizontal line. The axial anomaly $A_N(x)$ of the Neuberger-Dirac operator agrees with $1/72$ excellently, lying on the horizontal line. The axial anomaly $A_{wc}(x)$ of (47) is denoted by diamonds, while $A_t(x)$ of (42) is denoted by triangles. Evidently $A_{wc}(x)$ and $A_t(x)$ both disagree with the topological charge density $\rho(x) = 1/72$. 

\[ 12 (x_2 - 1) + x_1 \]