Question Number Two: Is the function

\[ f(x) = \begin{cases} 
0, & \text{if } x = 0 \\
1, & \text{if } x \geq 1 
\end{cases} \]  

continuous at 0? If we open any elementary calculus textbook in use, the answer will almost invariably be “no” to both questions. Moreover with some textbooks, these are the only “discontinuities” various functions seem to have. The daily calculus teaching practice follows the same party line. We will give correct answers to these questions in the subsequent pages.

First let us briefly look into the development of the ideas and the concepts.

What is a continuous function?

It is worth noting that the concept of a function as we know it today was arrived at only through numerous historical meanderings. Historically, this concept has been closely intertwined with that of continuity. [Ferraro, 2000], for instance, gives a detailed account of the development of this concept with Euler. A function for Euler was by definition continuous and differentiable and expandable into a Taylor series. The functions were intrinsically continuous, for “the variables varied in a continuous way”. In the eighteenth century a function was restricted to a quantity definable by a single formula, and Euler followed the same practice. The following was not considered to be a function:

\[ f(x) = \begin{cases} 
 x^2, & \text{if } x \leq 0, \\
 5x, & \text{if } x > 0 
\end{cases} \]  

By this definition, \(1/x\) is a function, and a continuous one since it is defined by a unique formula; the fact that its graph consisted of two pieces was of no consequence to Euler. By the same token the integer part function \([\ ]\) would likely be considered to be a continuous function by Euler. [Cauchy, 1844] quickly points out some difficulties with Euler’s view by noting that the function \(\sqrt{x^2}\) would be considered a continuous function by Euler, but
the same function written in two pieces: \( = x \), if \( x \geq 0 \) and \( = -x \), if \( x \leq 0 \) would not be considered to be a function. Only in 1837 did Dirichlet get rid of this notion of function interpreted as a “unique” formula.

The notion of function has nowadays crystallized into a well-founded concept. A function is no longer only a formula (or a compact formula at that), but is a triple \( f : A \rightarrow B \) where \( A \) is the domain for its variables, \( f \) is a relation with unique second component value, for every independent variable, and \( B \) is the codomain (containing the range); see here [Bourbaki, 1939], for example. Thus, the functions should be taught properly: domain, codomain and the assignment rule(s) with certain properties make up a function. Precisely speaking, a function \( f \) from \( A \) to \( B \) is the ordered triple \( (f, A, B) \), (also denoted by \( f : A \rightarrow B \), where \( f \subseteq A \times B \) is a relation (i.e. a subset), with the property that, for every \( a \in A \) there is a unique \( b \in B \) such that \( (a, b) \in f \). Thus, two functions are equal, if all three of their corresponding parts (the formula, domain and codomain) are equal. One might say that using relations to define functions may be a bit overreaching in a basic calculus course and perhaps it is; on the other hand, I see textbooks on “precalculus” discuss the subject of relations. I encounter students’ puzzlement, or even confusion when they hear the “new word” ‘codomain’ (some also use the word ‘target’ for the same thing). They’ve heard of the range, but what is this? Codomain is as important as the domain and is perfectly dual to it (and that is a very important point). Given a formula, we at first simply do not know what the range will be (in any but trivial cases), but can often give a “rough” target.

Consider the following example: For a square black-and-white photograph, for every point on it (and one tends to modernize terminology, not quite accurately and say “to every pixel”), assign a value to it in the interval \([0, 1]\) that denotes the amount of black at that point; thus 1 would be assigned to a perfectly black point and 0 to a perfectly white. The interval \([0, 1]\) is a natural codomain, although we could go up to \( \mathbb{R} \) if we are only talking real functions; anything in between can also be taken to be a codomain; as we change the codomains we get different function each time, although the assignment rule (the formula) would stay the same. On the other hand \([-1, 0)\) would not be a valid codomain. We don’t know the range and it would be rather difficult to find – very dark photos would have the range in the upper half of the interval \([0, 1]\), the light ones in the bottom half. We know however that the range is the smallest of all the codomains. Dually, given a formula, we speak of the domain to fit the formula, whereas we really mean the largest domain for which the formula works. Any subset of the maximal domain can also be a domain. As we change these (without changing the formula) we do get different functions. Thus, for the same formula \( f(x) = x^2 \), we have various possibilities

a) \( f : \mathbb{R} \rightarrow \mathbb{R} \), b) \( f : \mathbb{R}^+ \rightarrow \mathbb{R} \), c) \( f : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), d) \( f : \mathbb{R} \rightarrow \mathbb{R}^+ \) (3)
which give us different functions with different properties: the first is neither one-to-one nor onto (i.e. range is not equal to its codomain), the second one is one-to-one but not onto while the function in c) has an inverse (it is one-to-one and onto), unlike the rest of them; the function in d) is onto but not one-to-one. In this way, not only is \(1/x\) not a function at 0, but we can point out to our students some crucial facts, such as that the property of being one-to-one depends on the domain (such as in functions (3) above). This then helps better understand the notion of inverse function, the relationship between the range and the codomain, etc.\(^1\)

Now a definition of continuity: Given a function \(f : A \to \mathbb{R}\) on a real domain \(A \subseteq \mathbb{R}\), it is continuous at \(a \in A\), if\(^2\)

\[
\lim_{x \to a} f(x) = f(\lim_{x \to a} x) = f(a). \tag{4}
\]

Implicitly, this definition is in the form of an implication: If \(a \in A\), then the function \(f\) and the limit as \(x \to a\) can be commuted. This is fairly close to how Bolzano defines continuity in his privately published manuscript on what we today call the intermediate value theorem \(\text{[Bolzano, 1817]}\): “A function \(f(x)\) varies according to the law of continuity for all values of \(x\) which lie inside or outside certain limits, is nothing other than this: If \(x\) is any such value, the difference \(f(x+\omega) - f(x)\) can be made smaller than any given quantity, if one makes \(\omega\) as small as one wishes.” For \([\text{Euler, 1748}]\), this is a consequence of what he considers to be a function.

Years later, \([\text{Weierstrass, 1874}]\) gave a similar, somewhat more formal definition: “Here we call a quantity \(y\) a continuous function of \(x\), if upon taking a quantity \(\epsilon\), the existence of \(\delta\) can be proved, such that for any value between \(x_0 - \delta...x_0 + \delta\), the corresponding value of \(y\) lies between \(y_0 - \epsilon...y_0 + \epsilon\).” (The difference \(f(x) - f(x_0)\) can be made arbitrarily small, if the difference \(x - x_0\) is made sufficiently small.)

In our modern quantifier notation, \(f\) is continuous at \(x_0 \in \text{Dom}f\), if

\[
\forall \epsilon > 0 \exists \delta > 0 \forall x \in A(|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon). \tag{5}
\]

It is not clear whether Bolzano’s paper had been well known in his own time and in particular whether Cauchy was familiar with the content of this paper; see \([\text{Grattan-Guinness, 1969}]\) on this. Whatever the answer, \([\text{Cauchy, 1821, p.43}]\) gives his own definition of continuity:

“... \(f(x)\) will be called a continuous function, if...the numerical values of the difference \(f(x + \alpha) - f(x)\) decrease indefinitely with those of \(\alpha\)....”

\(^1\)To be fair, better, or more advanced books on analysis often have a correct treatment of the notion of function, and in particular that of an “onto” function and a codomain; see here for instance \([\text{Rudin, 1976}]\) or \([\text{Mardedˇsić, 1974}]\).

\(^2\)Here we do not require that the limit is taken over a non-isolated point – \(x\)’s may equal \(a\),
(infinitesimally small changes in $x$ should lead to infinitesimally small changes in $f$).

On a slightly more fancy level, if $f : A \rightarrow B$ is a function between two metric spaces, then it is continuous at $a \in A$, if, for every (open) ball $V$ in $B$ centered at $f(a)$ there is an (open) ball centered at $a$ completely mapped into $V$.

Thus, the mid-eighteenth century notion of “continuity” referred to uniformity (wholeness) of the formula used to define the function; piecewise defined functions, such as (2), were not deemed to be functions (or to be “continuous”) under this notion. D’Alembert was one of the proponents of defining (continuous) functions in this restricted sense. This was challenged by Daniel Bernoulli, after d’Alembert’s work in 1747 on the motion of a vibrating string (d’Alembert, 1747; see here for instance [Struik, 1969, pp.351–368] and Truesdell’s introduction to [Euler, 1960] titled: “The rational mechanics of flexible or elastic bodies, 1638–1788”). D’Alembert’s solution to the partial differential equation describing motion of an elastic vibrating string was of the form: $z(x,t) = f(t+x) + F(t-x)$, which was unusual in that the solution was a combination of two arbitrary functions, thus could not be considered to be a (continuous) function. [Euler, 1765] resolved this problem by change of terminology and interpretations of the constants that come out in integrating the given pde’s [Feraro, 2000].

The alternative notion of “continuity” that was coming into greater prominence referred to functions that “can be produced by a free motion of the hand.” This notion of being able to draw the graph of a function without lifting a pencil is that of contiguity as formulated in 1791 by Louis Arbogast: “The law of continuity is again broken when the different parts of a curve do not join to one another... We will call curves of this kind discontiguous curves, because all their parts are not contiguous, and similarly for discontiguous functions” (see [Jourdain, 1913], pp.675–676). The interest in (dis)continuities was further heightened by Fourier in his celebrated work on heat [Fourier, 1822]. A definite disturbance regarding the concept of continuity and a need to fix the concept was induced by Peter Lejeune-Dirichlet who gave, in 1829, his example of the function equal to a constant over the set of rational numbers and equal to another constant on the set of the irrationals; this function was not continuous even at a single point.3 Bolzano added more excitement in 1834 by giving the first example of a nowhere differentiable continuous function. As much as works on continuity by Bolzano and Cauchy were ignored, the world seem to have started paying attention to the definition of continuity given in [Darboux, 1875], where its local nature is finally underlined.

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3Another nice example was given by Riemann: $f : [0, 1] \rightarrow [0, 1]$ is defined to be zero on irrational numbers and $= 1/q$, for rational numbers in the reduced form $p/q$; this function is discontinuous at every rational point and continuous at every irrational number.
Answers.
Let’s look closer into the questions we posed at the beginning.

A Socratic dialog.
“Do you love your cat?,” Arthuro asked.
“Oh, yes, definitely... and then, no, not at all, most definitely,” Gwendolin replied.
“How can that be, aren’t you contradicting yourself?”
“Oh no, no, I simply do not have a cat!” exclaimed Gwendolin, and added (philosophically and pensively): “I cannot decide on whether it may be easier not to love a cat you do not have than to love a cat you do not have...”
“Hmmm,” Arthuro sighed unhappily, for the lack of a definite answer.
“But I will give you a cat, just please give me definitely an answer whether you love your cat or not.”
“O.K. Arthuro, this would then be a different situation, from the previous one... Also, do not forget that my having a cat may not guarantee that I will be able to decide whether I love it or not...”

Arthuro is puzzled, but something seems to be happening, for his eyes are wide open and he seems to be thinking with intensity...

When I ask my students whether they love their cats, those who do not have a cat would never answer the question with a “yes” or “no,” but would rather say that they do not have a cat. This phenomenon of a question with a false presumption seems to be well recognized in the folk wisdom; the question of the same kind is a question of the type “Why do you beat your wife?” (while both the beating and the wife may not exist).

Recently, I heard a folk child riddle: A rooster is standing on top of a roof and it lays an egg. Which way will the egg go? [The child is supposed to recognize that the answer is “not applicable,” since roosters do not lay eggs.]

The question of continuity of \(1/x\) at \(a = 0\) is the same as that of continuity of this function at \(a = \text{red tomato}\); or for that matter the question of Gwendolin loving “her” cat that she does not have. The premise part of the definition of continuity is not satisfied, since it is vacuous – there is no function at 0. For this reason, saying that \(1/x\) is continuous at 0 is equally (in)valid as saying that it is continuous at 0. The answer to the title question is thus “N/A” (namely the question of “continuity at 0” is not applicable for the function \(1/x\)). In the same manner, we define various kinds of discontinuities of a function (within its domain; such as done for instance in [Rudin, 1976, p.94]).

\(^4\)See also [Mardesić, 1974, p.194]. A more general result holds: Let \(X\) be a Banach algebra with the unity \(e\) and let \(I\) denote the set of all invertible (regular) elements in \(X\). Then \(I\) is an open set and the function \(f: I \rightarrow X, f(x) = x^{-1}\) is continuous.
As for question number two, we need to find \( \lim_{x \to 0} f(x) \). But what are the \( x \) with \( x \to 0 \)? In fact the only such \( x \)'s are \( x = 0 \), thus we indeed have (4) and (5) satisfied (sufficiently small balls centered at 0 coincide with its center). It will always be like this if \( a \) is not a limit point, if, for instance, it is an isolated point as in our case. The function (1) is continuous (at all points of its domain) and so is \( 1/x \). One is tempted to dismiss the “pathological cases” of isolated points or non-compact domains in teaching non-mathematicians. (Un)fortunately discussing continuity only on compact domains would be tantamount to simply replacing continuity by a stronger notion of uniform continuity. The field of applied mathematics (aka “the sciences”) however largely consists of singularities, isolated points, non-compact domains and limits that do not exist or are infinity – all the staple brushed away in usual calculus courses.

Why does this myth of \( 1/x \) not being continuous at zero linger on so persistently with teachers of calculus and authors of calculus textbooks? If we take aside a small population of people who maintain that incorrect mathematics is O.K. (for whatever “higher pedagogical goals”), the constraints to open thinking about continuity have historical as well as “objective” roots.

The question of contiguity clearly depends on the space into which the domain is embedded. For contiguity purposes, we look at the domain as a part of \( \mathbb{R} \). If we somehow imagine the domain \((-\infty, 0) \cup (0, \infty) \) of \( 1/x \) not to be embedded in \( \mathbb{R} \), but standing on its own, we can perhaps imagine that the graph is contiguous – that we can draw it in one go, without lifting up the pencil (after fusing the parts together?)... The same thinking applies to the domain \( \{0\} \cup [1, \infty) \) of the function (1) in question number two. An interesting point of view is brought about in [Burgess, 1990], where it is argued that continuity and contiguity are synonymous in some cases. Thus, if \( f : A \to \mathbb{R} \) is a function such that its domain \( A = \mathbb{R} \), a closed interval, or is a closed ray in \( \mathbb{R} \), and such that its graph is closed in \( \mathbb{R}^2 \), then \( f \) is continuous iff the graph of \( f \) is connected in \( \mathbb{R}^2 \).

Another “objective” reason not seeing continuity for what it is, is the desire (and sometimes necessity) to extend functions (their domains) so that the new functions become continuous extensions of the starting functions; this is the moment when Arthuro wants to give Gwendolin a cat. We cannot extend \( 1/x \) so as to make it continuous at 0, but we can extend \( x \sin(1/x) \), by defining its value at 0 to be the limit there, namely 0. Notice that if \( f : [-1, 0) \cup (0, 1] \to \mathbb{R} \), \( f(x) = 1/x \), even those who say that this \( f \) is discontinuous at 0, would not say that it is discontinuous at 5, for instance.

I do however want to pause for a moment from being a devil’s advocate: Many discussions on the nature of the notions of function and continuity did spill into the early 20th century and beyond, for instance in the papers

\[ \text{See also [Rudin, 1976, p.86]} \]
of Borel, Lebesgue, Brower, Baer, etc.; some of these discussions had added dimension of set-theoretical considerations that led to deep discoveries in mathematical logic and developments of “new kinds” of mathematics, such as intuitionism, etc. It must be said also that there is often a considerable delay in adoption and application of fundamental notions in mathematics.

**Conclusions.**

The notion of continuity is subtle, but it had undergone its evolution through the historical birthing process. The subtleties eluded Cauchy into making a nice error when he “proved” that a multi-variable function is continuous if it is continuous in each of its variables. The notion of continuity has been demystified however, and there is no much reason for dragging the old fog surrounding it into modern day textbooks and classrooms.

It is crucial to see functions as ordered triples $f : A \rightarrow B$, for not only the formula, but both domain and codomain determine what exact properties the function has. This also helps the students understand that extending a function continuously refers to extension of its domain, etc. One aesthetic consequence is the quotient rule: if $f$ and $g$ are continuous then $f/g$ is continuous (the quotient is not a function when the denominator is zero, thus we do not need to discuss continuity at those points).

While contiguity has its own merits because it is directly related to path connectedness, it simply is not identical to the notion of continuity. As a stronger requirement, it implies continuity. We cannot however use the two synonymously – there are, for instance, certain assumptions about a function that would imply continuity of the function (but certainly not contiguity, etc.). Once we let go of interpretation of continuity as contiguity we can concentrate on what continuity actually is: the commutation of the function with limits as in (4). Why can we plug numbers into expression whose limits we are finding? Students would invariably plug in $x = 1$ when finding the limit $\lim_{x \to 1} \sqrt{x^2 + 1}$, and an opportunity is missed to point out to them that this natural move is possible because of continuity of the function. In fact plugging in numbers when finding limits is possible because most of the functions given by formulas in elementary calculus (e.g. all elementary functions such as algebraic or trigonometric functions) are continuous. Interestingly intuitionists, like Brower, have it that every function $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous (by their own definitions of functions and continuity).

Additional discussion on historical development of the concepts of function and continuity can be found in [Monna, 1972], [Edwards, 1979] and [Hairer and Wanner, 1995].

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References

Jean-B. le Rond d’Alembert: Recherches sur la courbe que forme corde tendue mise en vibration, Mémoires de l’Académie des Sciences de Berlin, 3(1747), 214–219.

B. Bolzano: Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwei Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege. (Translation: Purely analytical proof of the theorem, that between each two values with the opposing signs, at least one real root of the equation lies.) Prag 1817; Ostwald’s Klassiker #153, 1905 [See also O. Stolz: B. Bolzano’s Bedeutung in der Geschichte der Infinitesimalrechnung. Math. Ann. 18(1879), 255-279].

N. Bourbaki: Les structures fondamentales de l’analyse, Théorie des ensembles (fascicule de résultats). Paris, 1939.

A.-L. Cauchy: Cours d’analyse algébrique. Œuvres série 2, vol. III., 1821 (Alternatively: Cours D’analyse de l’Ecole Royale Polytechnique, Oeuvres, Ser.2, Vol. 3, Paris, Gauthier-Villars, 1897).

A.-L. Cauchy: Mémoire sur les fonctions continues, C. R. Acad. Sci. Paris, 18(1844), 116; see Œuvres Complètes, Paris, 1882.

G. Darboux: Mémoire sur les fonctions discontinues, Ann. Ecole Norm. Sup., 2e Série IV (1875), 57–112.

G.L. Dirichlet: Über die Darstellung ganz willkürlicher Functionen durch Sinus- and Cosinusreihen, Repertorium der Physik, Bd. I(1837), 152–174. In Gesammelte Werke, Berlin (1889–1897), I, 133–160. Also in Ostwald’s Klassiker, No. 116, H. Liebmann, Ed., 1900, 3–34.

C.H. Edwards, Jr.: The Historical Development of the Calculus, Springer-Verlag, New York, 1979.

L. Euler: Opera Omnia, Ser. 2, Vol. 11, Part 2, Series II, 1960.

L. Euler: Introductio in analysin infinitiorum, Lausannae: M.M. Bousquet et Soc., 1748 (or Opera omnia (1), 8–9).

L. Euler: De usu functionum discontinuarum in Analysi, Novi Commentarii academicae scientiarum Petropolitanae, 11(1765), 3-27.

Ferraro, Giovanni: Functions, functional relations, and the laws of continuity in Euler, Historia Mathematica, 27(2000), 107–132.

J. Fourier: La théorie analytique de la chaleur, Paris, 1822 (The Analytical Theory of Heat (translated by A. Freeman). New York, Dover (reprint), 1955).

I. Grattan-Guinness: Bolzano, Cauchy and the ‘New analysis’ of the early nineteenth century. Arch. Hist. Exact Sci. 6(1969/70), 372–400.

E. Hairer, G. Wanner: Analysis by Its History, Springer-Verlag, New York, 1995.

P.E.B. Jourdain: The origin of Cauchy’s conceptions of a definite integral and of the continuity of a function. Isis 1(1913), 661–703.
S. Mardešić: Matematička Analiza. U $n$-dimenzionalnom realnom prostoru. Školska knjiga, Zagreb, 1974.

A.F. Monna: The concepts of function in the 19th and 20th centuries, in particular with regard to the discussions between Baire, Borel and Lebesgue, *Arch. Hist. Exact Sci.*, 9(1972), No.1, 57–84.

W. Rudin: *Principles of Mathematical Analysis*, Third ed., McGraw-Hill, Inc., New York, 1976.

D.J. Struik: *A Source Book in Mathematics 1200–1800*. Harvard University Press, Cambridge, MA, 1969.

K. Weierstrass: Theorie der analytischen Funktionen, Vorlesung an der Univ. Berlin 1874, manuscript (ausgearbeitet von G. Valentin), Math. Bibl. Humboldt Universität Berlin, 1874.