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Original articles

Quantifying chaos in stock markets before and during COVID-19 pandemic from the phase space reconstruction

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Abstract

From a methodology in the reconstruction scheme, applicable to chaotic time series of economic indices, this paper presents an analysis of the underlying dynamics of stock markets of North America, Europe and Asia. The same global fit model and reconstruction parameters—employed to study the time evolution of S&P 500, NASDAQ Composite, IBEX 35, EURONEXT 100, Nikkei 225 and SSE Composite Index—led a convenient simplification in the analysis. The tools chosen to analyse the time dependence of the level of chaos concerning weeks of economic activity were scatter plots, histograms and sample Spearman correlation coefficients. The results permit to evaluate the impact of the pandemic in the underlying dynamics of different stock markets and to compare them to one another.

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1. Introduction

At the beginning of this century, the impact of the coronavirus pandemic for all countries has been demanding intense activity of scientists to find solutions for this unprecedented crisis worldwide. The improvement of the health care system and the trial of vaccines are actions of direct managerial implication. Epidemiological studies, social and economic consequences of the present scenario in the world public health also represent areas of knowledge of great relevance for researchers.

To look at the evolution of economic indices from stock markets of different continents is worth in this context. Fortunately, the actual stage of the knowledge opens up the possibility to face the complexity of the real-world from cross-disciplinary methodologies. An emblematic structure in the description of a physical system that attends since the classical two-dimensional harmonic oscillator to the formulation of Quantum Statistical Mechanics is the Phase Space. It can still be a powerful tool in new implementations nowadays, e.g., development of the optimal control theory [15]. In a study on the time series from the stock market indices, an assumption is that the underlying dynamics is unknown. On the other hand, Takens’ Theorems give mathematical support for the phase space reconstruction and constitute the theoretical basis for the analyses presented in this paper [22,27,35].

Nonlinear approaches and Time Series Analysis have been enlightening in the research on economic activity successfully. Relevant contributions look at implied volatility indexes, price transmission, international reserves,
stock market volatility, managerial confidence deeply [1,28,29,36,38]. This work points in this direction following the line of chaos approach and complexity in Econophysics [19,21,25]. Here, the idea is to quantify the level of the chaos of different stock markets, before and during the pandemic. It permits an evaluation of the effect of the crisis on the dynamic characteristic of the indices. To contribute to the literature, the proposal of the present study is to determine changes in the level of chaos in financial time series due to pandemic.

A recent complex quantifier of chaos able to attend the demand above admits a practical computational implementation from a stable algorithm. It takes part in the new method to identify the dynamic characteristic when a time series is available [9]. Dynamic analysis is a stage of research to study time series from implementations in a computer algebra system (CAS) [14,18]. The others are forecasting [5–8,13] and applications in Econophysics [3, 4,10]. All implementations are based on the global approach to map a reconstructed state vector into the expected value for an observable. This paper aims to evaluate the impact of the COVID-19 pandemic in the underlying dynamics of different stock markets and compare them to one another.

This work is structured as follows. Section 2 describes the methodology. Section 3 presents the stock markets analysed and the corresponding results. Discussions are the content of Section 4. Finally, Section 5 is devoted to managerial implications and comparison of the results with other works.

2. Methodology

2.1. Scheme of reconstruction and global fitting

In the theory of dynamical systems, we can consider the time evolution of a \( k \)-dimensional state of a system, defined by the variables \((X[1], X[2], \ldots, X[k])\) as governed by a from a set of differential equations, like

\[
\begin{align*}
\frac{dX[1]}{dt} &= f_1 (X[1], X[2], \ldots, X[k]) \\
\frac{dX[2]}{dt} &= f_2 (X[1], X[2], \ldots, X[k]) \\
&\vdots \\
\frac{dX[k]}{dt} &= f_k (X[1], X[2], \ldots, X[k])
\end{align*}
\]

(1)

and the initial conditions [27]. On the other hand, there are dynamical systems where the researcher does not know the differential equations. Fortunately, even if only a time series of observable is available, it is possible to recover the time evolution without considering the underlying dynamics. The Takens’ Theorem proves that a \( m \)-dimensional manifold is an embedding in a \( d_E \)-dimensional Euclidean space if \( d_E \geq 2m + 1 \) [22,35]. It is the solid mathematical basis of the methodology presented in this section.

To reconstruct the \( k \)th state vector represented below,

\[
|\theta_k \rangle = \begin{bmatrix}
X (\Delta \phi (k + d_E T)) \\
X (\Delta \phi (k + 2T)) \\
\vdots \\
X (\Delta \phi (k + T))
\end{bmatrix}
\]

(2)

in a \( d_E \)-dimensional Euclidean space, it suffices to pick, as its components, elements of the time series \( S \) with \( \eta \) observables. The time interval of data acquisition and the delay time are \( \Delta \phi \) and \( T \), respectively [16,32].

\[
S = \{X(0), X(\Delta \phi), \ldots, X((\eta - 1) \Delta \phi)\}
\]

(3)

In this scheme, the parameters of reconstruction \( \{d_E, T\} \), the cardinality of the time series \( \eta \) and the reconstructed vectors’ order \( k \) preserve the inequality \( k \leq \eta - (1 + d_E T) \). Then, the size of the reconstruction’s set, denoted by \( \mathcal{N} \), also obeys it, i.e., \( \mathcal{N} \leq \eta - (1 + d_E T) \).

In the next step of the global approach, the goal is to map a reconstructed state vector \( |\theta_k \rangle \), from the predictor \( \mathcal{P}_\tau \), as it is shown as follows,

\[
X_{k+t} \simeq \mathcal{P}_\tau |\theta_k \rangle
\]

(4)
into the expected value for the observable corresponding to the order \(k + \tau\) — i.e., \(X_{k+\tau} \approx X(\Delta \varphi (k + \tau + d \xi T))\), where the parameter \(\tau\) is an integer. This method admits only linear global fit models — denoted below by \(f_i(\{\theta_k\})\) — in the adjustment parameters, given by \(\{\beta_0, \beta_1, \ldots, \beta_{n-1}\}\). It is important to note that the method admits nonlinear terms in the variables. An example, in a three-dimensional Euclidean space, is \(\ln(1 + \frac{\cos(X_1X_2X_3)}{10})\). On the contrary, the method does not admit terms like \(X^{\beta_1}_1\). Then the predictor assumes the functional form

\[
P_\tau |\theta_k\rangle = \beta_0 + \sum_{i=1}^{n-1} \beta_i f_i(\{\theta_k\}) .
\]  

Beyond determining \(\{\beta_0, \beta_1, \ldots, \beta_{n-1}\}\), the routine also returns the sample standard deviation in the global fit, denoted by \(\sigma_t\).

\[
\sigma_t = \left\{ \frac{\sum_{k=1}^{N+\tau-1} \left[ X_{1k} - \left( \beta_0 + \sum_{i=1}^{n-1} \beta_i f_i(\{\theta_k\}) \right) \right]^2}{N-1} \right\}^{\frac{1}{2}}
\]

Above, \(X_{1k}\) denotes the first component of the \(k\)th reconstructed vector, i.e., \(X_{1k} \equiv X(\Delta \varphi (k + d \xi T))\). This deviation assumes a fundamental role in the statistical quantities of interest. The next section will show how it takes part in the construction of the quantifiers of chaos.

### 2.2. Analysis of the underlying dynamics and quantifiers of chaos

Recent works demonstrate satisfactory performance of the new quantifier of chaos in the analysis of time series of economic indices, as a feasible alternative to Lyapunov Exponents [3,4]. This point deserves consideration. A positive Lyapunov Exponent indicates the presence of chaos in dynamical systems [27]. However, accurate and valid estimations impose subtle requirements on the algorithms. Another trouble is the identifying of spurious exponents in the spectrum of Lyapunov in the reconstructed phase space. This sort of problem has stimulated research in a branch of nonlinear science to diversify methodologies [30].

The deviation \(\sigma_t\), calculated for different values of integer parameter \(\tau\), allows the identification of the dynamic characteristic. Chaotic time series require that deviation increase with the prediction time, i.e., \(\sigma_1 < \sigma_2 < \cdots < \sigma_N\), and so on. Non-chaotic systems do not obey this inequality. For the dozen sorted in a lottery, for instance, the relationships between the large deviations are close to \(\sigma_1 \approx \sigma_2 \approx \cdots \approx \sigma_N\). The magnitudes for a non-chaotic periodic signal preserve a relationship similar to the random numbers, but \(\sigma_\tau\) is ever small. These are the main ideas of this new methodology to characterise the dynamics and quantify chaos detailed discussed in the Refs. [9,10]. Here, the following paragraphs revisit this complex quantity, denoted by \(Z_{\text{dyn}}\). It is defined from the statistical magnitudes \(A_\tau\) and \(D_\tau\) as follows.

\[
A_\tau = \frac{\sum_{k=1}^{N} |X_{1k}|}{3N\sigma_t}
\]

\[
D_\tau = \frac{\sigma_t}{\sigma_1},
\]

\[
Z_{\text{dyn}} = A_1 + i \lambda_{\text{dyn}}
\]

Above, \(i = \sqrt{-1}\) is the imaginary unit. Denoting the maximum value for the parameter \(\tau\) in the analysis by \(M\), e.g., \(M = 12\), the average

\[
\lambda_{\text{dyn}} = \frac{\sum_{\tau=1}^{M-1} A_\tau - A_{\tau+1}}{M-1}
\]

---

1. The method employs a set of reconstructed vectors of size \(N\) to determine the coefficients of the predictor. Each parameter \(\tau\) corresponds to a set of reconstructed vectors with \(N\) vectors, i.e., the routine employs different subsets of the reconstruction’s set in this method. The collection of vectors used for the fit is a subset of the set of reconstruction. It is the motivation for the change in the notation \(k \to \hat{k}\). To obtain a predictor, the selection of parameters must attend the inequality \(N \geq k > \tau\). Typically, in satisfactory fits, the smallest integer assumed by \(k\) need to be, at least, a half hundred of times larger than \(\tau\).
determines the imaginary part. To determine \( A_1 \), it suffices to substitute \( \tau = 1 \) into Eq. (7).

In addition to the inequality \( \lambda_{\text{dyn}} \geq 0 \), a graphical representation, called Diagrams Accuracy-Deviation, marks the presence of chaos from a time series [9]. There are two vertical axes in the diagram, \( A \) (Accuracy) on the left and \( D \) (Deviation), on the right, against the parameter \( \tau \), at the horizontal axis. The coordinates of the Accuracy (symbol \( \bullet \)) and Deviation (symbol \( \circ \)) curves are \( (\tau, \ln(A_i)) \) and \( (\tau, \ln(D_i)) \), respectively.

The decreasing of Accuracy and increasing of Deviation is the signature of chaos in these diagrams. Fig. 1 shows a time series for the variable \( X \) (see Fig. 1(a)) and a typical chaotic diagram for the Lorenz system\(^2\), abbreviated as LS (see Fig. 1(b)).

Substituting the time series of interest for another of identical size, generated by numerical integration from the Lorenz system\(^3\), and repeating all steps and parameters, the routine determines the quantity \( Z_{LS} \). It is necessary to construct the statistic called Chaoticity in the Lorenz’s sense, defined by the rate

\[
\tilde{Z} = \frac{|Z_{LS}|}{Z_{\text{dyn}}},
\]

which permits a direct comparison – for the level of chaos – between the time series and a benchmark model [3,4].

In this context, a usual formula

\[
(\delta \tilde{Z})^2 \approx (\delta A_1)^2 \left( \frac{\partial \tilde{Z}}{\partial A_1} \right)^2 + (\delta \lambda)^2 \left( \frac{\partial \tilde{Z}}{\partial \lambda} \right)^2 + (\delta A_{1LS})^2 \left( \frac{\partial \tilde{Z}}{\partial A_{1LS}} \right)^2 + (\delta \lambda_{LS})^2 \left( \frac{\partial \tilde{Z}}{\partial \lambda_{LS}} \right)^2
\]

\[\text{Eq. (13)}\]

\(^2\) This emblematic dynamical system is the set of differential equations below [27].

\[
\begin{align*}
\frac{dX[1]}{dt} &= \sigma (X[2] - X[1]) \\
\frac{dX[2]}{dt} &= \rho X[1] - X[2] - X[1]X[3] \\
\frac{dX[3]}{dt} &= -bX[3] + X[1]X[2]
\end{align*}
\]

\[\text{Eq. (11)}\]

Above, \( \{X[1], X[2], X[3]\} \) are variables and \( \{\sigma, \rho, b\} \) are the physical parameters.

\(^3\) The use of mathematical software, like TISEAN package [20], is a convenient alternative to generate time series of reference. Strictly speaking, the routine permits the selection of parameters and returns the time series. An example of the output is an ordered set for the dynamical variable \( X[1] \) from Eqs. (11).
Fig. 2. Chaotic voltage and its dynamic characteristic. (a) The electric signal, measured from a capacitor in an electronic circuit, is ordered in time [12,26]. (b) Both parameters of reconstruction and functional form of the predictor follow the previous example for the Lorenz System (see Fig. 1). The quantifiers of chaos, in this application, have magnitudes $Z_{\text{dyn}} = 4.34 + 0.39i$ and $\bar{Z} = 0.988$.

in Experimental Physics is proper [11]. It evaluates the error propagation of uncertainties of $Z_{\text{dyn}}$, i.e., $\{\delta A_1, \delta \lambda\}$ and $\{Z_{\text{LS}}, \delta A_{1\text{LS}}, \delta \lambda_{\text{LS}}\}$ in the determination of the expected error for $\bar{Z}$, given by $\delta \bar{Z}$.

This paragraph presents an instructive interpretation of the statistic $\bar{Z}$. If the level of chaos grows, one expects that the statistical quantity $A_\tau(7)$ decreases because $\sigma_\tau(6)$ becomes larger. The mean $\lambda_{\text{dyn}}(10)$ accompanies this behaviour due to differences $A_\tau - A_{\tau+1}$ also decays. For a high degree of chaos, the complex quantifier of chaos approaches of the asymptotic behaviour typical of the random time series, i.e., $Z_{\text{dyn}} \to 0 + 0i$ [9]. Inserting it in the chaoticity $\bar{Z}$, we have the asymptotic behaviour $\bar{Z} \to +\infty$ when chaos becomes too large [3,4]. Then, one may interpret, for instance, that a time series associated with a statistic $\bar{Z} = 3.0$ has three times the chaoticity of the Lorenz system. If $\bar{Z} = 0.5$, then the series has half the chaoticity of LS, and so on.

Strictly speaking, we are at a point to establish a connection between the Lyapunov exponents and the chaos quantifiers. The reasoning is simple. If the quantifiers do not manifest the asymptotic behaviours, then it is possible to associate the Largest Lyapunov Exponent (LLE) of the dynamical system as positive. For instance, the Lyapunov exponents of the time series generated for the Lorenz System employed in the graphics of Fig. 1, from the output of the TISEAN package [20], are $\{0.9659, -0.0467, -13.9133\}$ in 1/unit time. So the outputted values $\{Z_{\text{dyn}} = 4.27 + 0.39i, \bar{Z} = 1.00\}$ suggest the existence of a positive LLE, calculated as 0.9659 in this case. This association between these distinct indicators of chaos also may be found for other benchmark models of chaotic systems, e.g., the Rössler and Chua attractors [2].

To illustrate the chaos quantifiers from an application physically intuitive, Fig. 2 presents the chaotic signal measured from a capacitor in an electronic circuit associated with an attractor similar to the so-called Chua circuit [12,26]. The chaotic voltage follows chaotic time evolution (see Fig. 2(a)). Its Diagram Accuracy Deviation is very similar to the Lorenz System (see Figs. 1(b) and 2(b)). It is worth noting that a physical magnitude, e.g., voltage, receives the same treatment as variables determined from numerical integration, economic indices, and so on. In other words, there is no distinction between the nature of the time series in this methodology.

2.3. Computational assistance

Many fits are necessary to get results of the next section. In this study – as an illustrative example – each determination of the chaos quantifier requires twelve global mappings for time series containing approximately
Table 1
Quantifier of chaos and correlations for the stock markets studied before (BP) and during COVID-19 pandemic (DP). From many runnings of the routine for different sizes of the series studied, the upper estimating for the error bars in the quantifier $Z_{\text{dyn}}$ are $[\delta a = \pm 0.08, \delta A_1 = \pm 0.01i]$. To evaluate their impact in the quantifier $Z$, the formula for errors propagation (13) gives the uncertainties $\delta Z$. Beyond the $p$-value for the sample Spearman correlation coefficient $r_s$ between the chaos quantifier $Z$ and the week analysed of order $\eta$, this table also presents the corresponding critical values $r_C$, for accurate coefficients from the Edgeworth approximation [31], at a level of significance $\alpha$. The research hypothesis is the existence of an association between $Z$ and time. So the statistical evidence in favour of it must permit the rejection of the null hypothesis of absence of correlation from p-values $\leq \alpha$. Below, the level of significance for the hypothesis test is $\alpha = 0.05$.

| Stock market | Index | Level of chaos | Spearman correlation $(\bar{Z}\eta)$ |
|--------------|-------|----------------|-------------------------------------|
|              |       | $Z_{\text{dyn}}$ | $\bar{Z}$ | $r_s$ | $p$-value | $r_C$ | $\alpha$ |
| North America |        |                |            |       |         |       |     |
| GSPC         | BP    | $3.77 + 0.11i$ | $1.15 \pm 0.02$ | 0.077 | 0.64 | – | – |
|              | DP    | $3.18 + 0.09i$ | $1.35 \pm 0.03$ | 0.635 | $< 0.01$ | 0.519 | 0.001 |
| IXIC         | BP    | $3.56 + 0.11i$ | $1.20 \pm 0.03$ | 0.079 | 0.63 | – | – |
|              | DP    | $3.03 + 0.09i$ | $1.41 \pm 0.04$ | 0.645 | $< 0.01$ | 0.519 | 0.001 |
| Europe       |        |                |            |       |         |       |     |
| IBEX         | BP    | $3.47 + 0.11i$ | $1.24 \pm 0.03$ | $-0.393$ | 0.01 | 0.373 | 0.02 |
|              | DP    | $3.51 + 0.13i$ | $1.22 \pm 0.03$ | 0.292 | 0.08 | – | – |
| N100         | BP    | $3.60 + 0.11i$ | $1.19 \pm 0.03$ | $-0.682$ | $< 0.01$ | 0.512 | 0.001 |
|              | DP    | $3.64 + 0.13i$ | $1.19 \pm 0.03$ | 0.593 | $< 0.01$ | 0.519 | 0.001 |
| Asia         |        |                |            |       |         |       |     |
| 000001.SS    | BP    | $3.30 + 0.11i$ | $1.30 \pm 0.03$ | $-0.292$ | 0.07 | – | – |
|              | DP    | $3.38 + 0.11i$ | $1.27 \pm 0.03$ | $-0.714$ | $< 0.01$ | 0.519 | 0.001 |
| N225         | BP    | $3.41 + 0.12i$ | $1.26 \pm 0.03$ | $-0.142$ | 0.39 | – | – |
|              | DP    | $3.41 + 0.12i$ | $1.25 \pm 0.03$ | 0.127 | 0.45 | – | – |

1300 data. The analysis for a stock market repeats this procedure for 77 runnings. Then, taking into account six case studies, the computer runs the algorithm 5544 times.

The programming logistics applies the least-squares method to minimise the cost functions. The fundamental condition for this efficiency is to choose linear global fit models in the adjustment parameters as the restrict form for the predictor, states in (5), establishes. This restriction leads to a significant reduction in the running time [5,6,8].

Previous works in this line of research employed a set of procedures to qualify the predictors. These applications had been demonstrating that the algorithm ensures robust results in economic time series [3,4,10]. So the routine incorporated them to characterise the underlying dynamics properly.

3. Case studies and results

Here, the proposal is to analyse the time evolution of economic indexes in North America, Europe and Asia extracted from the Yahoo Finance website [17]. They are listed below.

- S&P 500 (‘GSPC), United States, currency in USD.
- NASDAQ Composite (‘IXIC), United States, currency in USD.
- IBEX 35 (‘IBEX), Spain, currency in EUR.
- EURONEXT 100 (‘N100) France, currency in EUR.
- SSE Composite Index (000001.SS), China, currency in CNY.
- Nikkei 225 (‘N225), Japan, currency in JPY.

The period of economic activity in analysis starts on 3 January 2020 and ends on 25 September 2020. It covers the possible effects of COVID-19 pandemic on the stock markets studied until the current date. In this regard, there is a point in the methodology employed that deserves consideration. It is necessary to include many data before pandemic to perform global fittings. The time series may have a minimum of days in economic activity close to 1000. So the chaos quantifier connects information on the working of the stock market before and during the pandemic, i.e., the time series analysis for the first week of January 2020 requires stock market indices since April 2015.
Fig. 3. Global fittings for North American stock markets (parameter $\tau = 1$). The overlapping of the predicted values with real observables illustrates the satisfactory quality of the results from the least-squares method in the reconstruction scheme.

(a) Index "GSPC before COVID-19 pandemic. (b) Index "GSPC during COVID-19 pandemic.

(c) Index "IXIC before COVID-19 pandemic. (d) Index "IXIC during COVID-19 pandemic.

After preliminary global fittings, the fit model proper for the dynamics’ analysis is the same for all stock markets studied and the parameters $\tau$, i.e., $P_1 = P_2 = \ldots = P_M$. The parameters of reconstruction are $\{d_E = 3, T = 3\}$ and the functional form is

$$P_t |_{\theta_k} = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3.$$  \hspace{1cm} (14)

Besides the level of chaos, it also instructive to associate changes in it with the time evolution. The Spearman Correlation – a rank statistic that measures the degree of association between two variables [34] – is an adequate statistic to evaluate this interrelation. For a general analysis, Table 1 presents both the quantifier of chaos before during COVID-19 pandemic and the Spearman correlation between the magnitudes $\bar{Z}$ and time (in weeks).

Graphical representations show the performance of the method to recover the underlying dynamics of stock markets in North America, Europe and Asia, before and during the pandemic, in Figs. 3–5. The corresponding Diagrams Accuracy-Deviation are presented in Figs. 6–8. Finishing the presentation of results, Figs. 9, 10, 11, 12,
Fig. 4. Global fittings for European stock markets (parameter $\tau = 1$).

13 and 14 are devoted to Graphical analysis for the impact of the crisis in the quantifiers of chaos, one for each case study.

4. Discussions

Fortunately, it was possible to analyse all the time series from the same fit model and the parameters of reconstruction. It simplifies the analysis because the statistical fluctuations due to differences in functional forms, in association with distinct choices for $\{d_E, T\}$, do not need to be considered. Besides the possibility of a direct comparison between results, this favourable mathematical peculiarity was able to perform global mapping of high quality (see Figs. 3(a), 3(b), 3(c), 3(d), 4(a), 4(b), 4(c), 4(d), 5(a), 5(b), 5(c) and 5(d)). The chaos’ detection also was well-succeed, for all stock markets, by the increasing deviation curves and decreasing accuracy in the Diagrams Accuracy Deviation (see Figs. 6(a), 6(b), 6(c), 6(d), 7(a), 7(b), 7(c), 7(d), 8(a), 8(b), 8(c) and 8(d)).

The quantities $Z_{\text{dyn}}$ and $\bar{Z}$, in Table 1, agree with diagrams on the presence of chaos. NASDAQ Composite (‘IXIC) index presented the superior level of chaos, given by $Z_{\text{dyn}} = (3.03 \pm 0.08) + (0.09 \pm 0.01)i, \bar{Z} =$
An interesting finding is that this same stock market presented the smaller magnitude of $\bar{Z}$ before the crisis. The change of the chaoticity of the NASDAQ Composite (\textsuperscript{\textcopyright}IXIC) index was $+0.21 \pm 0.04$ whereas the growing for S&P 500 was $0.20 \pm 0.03$. European and Asian indices did not experiment growing with this intensity. Then, looking only for the results in the borderlines stages, i.e., before and during COVID-19 pandemic, already it is possible to detect significant differences in the underlying dynamics between the North American stock markets and the other continents.

Moving forward with the analysis of the North American indices, the stage on the left of the 0th week presents a clear distinction both in scatter plots and histograms. It is possible to visualise the preferential occupation of the upper region in graphics (see Figs. 9(a) and 10(a)) and the displacement of the bars of frequency to the right (see Figs. 9(b), 9(c), 10(b) and 10(c)). After the beginning of the pandemic, the chaos quantifier presents statistical evidence in favour of its association with time. In this issue, the results of the hypothesis tests, for the Spearman correlation coefficients, are significant at $\alpha \leq 0.001$ (see Table 1).

Indices of the European stock markets studied were quite similar concerning their underlying dynamics in the year 2019. However, from January 2020 to September 2020, there are differences in the time evolution of \textsuperscript{\textcopyright}IBEX.
Fig. 6. Diagrams Accuracy-Deviation for North American stock markets.

(a) Index ‘GSPC before COVID-19 pandemic
\(Z_{\text{dyn}} = 3.17 + 0.11i, \bar{Z} = 1.15\).

(b) Index ‘GSPC during COVID-19 pandemic
\(Z_{\text{dyn}} = 3.18 + 0.09i, \bar{Z} = 1.35\).

(c) Index ‘GSPC before COVID-19 pandemic
\(Z_{\text{dyn}} = 3.56 + 0.11i, \bar{Z} = 1.20\).

(d) Index ‘GSPC during COVID-19 pandemic
\(Z_{\text{dyn}} = 3.03 + 0.09i, \bar{Z} = 1.41\).

Even though the chaoticity of the Asian stock markets is close before and during the pandemic, the same behaviour did not appear in the time evolution of the magnitude \(\bar{Z}\). The scattering on the right and left sides of the plot in Fig. 14(a) are quite similar for the \(^{\prime}\text{N225}\) index. However, there is a decay for the 000001.SS index with the time, in both intervals \([-38, 0]\) and \([0, 38]\), as Fig. 13(a) shows. The displacement of the bar of frequency agree with scatter plots, i.e., it moves to the left for the 000001.SS index and practically maintains the same position in and \(^{\prime}\text{N100}\) index. For the first case, the scattering of points of the graphic for the chaos quantifier has practically the same distribution on both sides (see Fig. 11(a)). The bar of frequency also does not change significantly (see Figs. 11(b) and 11(c)). However, the graphical analysis for the EURONEXT 100 stock market shows increasing in the filling of the upper region of the scatter plot beyond a displacement of the frequency’s bar to the right in the histogram (see Figs. 12(a)–12(c)). These two study cases diverge if one looks at the association between the level of chaos and time. For the first index, the hypothesis test’s result is not significant at \(\alpha \leq 0.05\) whereas it is significant at \(\alpha \leq 0.001\) in the other case (see Table 1).
the other case (see Figs. 14(b), 14(c), 13(b) and 13(c)). On the other hand, the correlation analysis presents p-values $> 0.05$ for \{N225 BP, N225 DP, 000001.SS BP\} and the negative coefficient, significant at $\alpha \leq 0.001$, for the SSE Composite Index during the pandemic (see Table 1). It is interesting to note that the decaying of chaoticity of the 000001.SS index during the crisis differ from all stock markets studied in the same period.

5. Managerial implications and final considerations

The results suggest that the riskiest investment are S&P 500, NASDAQ Composite and EURONEXT 100 because their level of chaos increased due to pandemic. Contrariwise, the SSE Composite Index presented a reduction of its magnitude of chaos. So this stock market is the safest option for investment. The IBEX 35 and Nikkei 225 stock markets do not require guidance because their underlying dynamics remains essentially the same before and during the crisis.
Besides the direct implications for investments, a potential application of the present study deserves consideration. The methodology developed and its computational implementation may take part in an unsupervised learning routine to monitor economic indices. At this point, the idea is to offer magnitudes of chaos in real-time, i.e., a new tool to aid investors and traders in their decision-making process.

At this stage, it is instructive to look at results from recent papers on the impact of the COVID-19 pandemic in the financial markets and to draw a parallel between them and the present study. The continuous wavelet transforms applied to the US stock market revealed unprecedented sensitivity during the pandemic [33]. It is compatible with the increasing level of chaos detected for the two indices of North America presented above. Both country-specific risks and systemic risks in the global financial markets for the crisis were subject of analysis in a perspective of the potential consequence of policy interventions [37]. Fortunately, the magnitudes of the quantifiers of chaos, found from the present methodology, were able to detect similarity and divergences between markets under different regional politics. It is a point in favour of the compatibility of this paper with other approaches and methodologies.

Fig. 8. Diagrams Accuracy-Deviation for Asian stock markets.
Fig. 9. Time evolution of the level of chaos for \( \hat{G}_{SPC} \) index. The histograms (b) and (c) work as a visual aid to analyse the influence of COVID-19 pandemic on the underlying dynamics of the historical prices.

To compare the recommendations to investors from different estimators for the complexity, one can consider the S&P 500 stock market as risky markets both by the Renyi entropy [24] and the quantifier of chaos showed in Table 1 and Fig. 9. Moving forward in this direction, the Largest Lyapunov Exponents (LLE) admit an instructive comparison with the Chaoticity in the Lorenz’s sense. To do it, it suffices to take into account the results of a study on the impact of the pandemic on financial markets and Table 1. From sixteen equity markets worldwide,
the worthy paper [23] did not detect statistical alterations of the average level of the LLE during the pandemic. Following the same level of significance employed for the Lyapunov Exponents, equal to 5%, the Two-tailed T-test for the statistical quantities from Table 1 requires p-value < 0.05 to reject the Null-Hypothesis. Considering that the mean of $\bar{Z}$ is equal before ($\langle \bar{Z} \rangle_{BP}$) and during the pandemic ($\langle \bar{Z} \rangle_{DP}$) as the null hypothesis, the t-statistic and the p-value are $-1.44313$ and $0.17957$, respectively, for $\langle \bar{Z} \rangle_{BP} = 1.22 \pm 0.01$ and $\langle \bar{Z} \rangle_{DP} = 1.28 \pm 0.03$. So the

Fig. 10. Time evolution of the level of chaos for 'IXIC index.
Fig. 11. Time evolution of the level of chaos for IBEX index.

(a) Quantifier of chaos $Z(Y) \times$ the order of week studied $\eta(X)$.

(b) Frequency before the pandemic (blue).

(c) Frequency during the pandemic (red).
Fig. 12. Time evolution of the level of chaos for $\hat{N}_{100}$ index.

(a) Quantifier of chaos $\hat{Z}(Y) \times$ the order of week studied $\eta(X)$.

(b) Frequency before the pandemic (blue).

(c) Frequency during the pandemic (red).
Fig. 13. Time evolution of the level of chaos for 000001.SS index.

(a) Quantifier of chaos $Z(Y)$ × the order of week studied $η(X)$.

(b) Frequency before the pandemic (blue).

(c) Frequency during the pandemic (red).
Fig. 14. Time evolution of the level of chaos for N225 index.

(a) Quantifier of chaos $Z(Y) \times$ the order of week studied $\eta(X)$.

(b) Frequency before the pandemic (blue).

(c) Frequency during the pandemic (red).
statistical evidence is against the rejection of the null hypothesis. Again, distinct indicators of chaos, applied to different samples, gave compatible results.

6. Conclusion

The new quantifiers of chaos and Diagrams Accuracy-Deviation, in the reconstruction scheme, have been presenting conclusive results on chaotic time series from complex systems of the real-world. In the analysis of the underlying dynamics of stock market indices before and during COVID-19 pandemic, it was possible to employ the same global fit model and reconstruction parameters to study the time evolution of S&P 500, NASDAQ Composite, IBEX 35, EURONEXT 100, Nikkei 225 and SSE Composite Index. This simplification led to a direct comparison between stock markets without the influence of the statistical fluctuations due to functional forms, the dimension of reconstruction and delay time. The complex chaos quantifiers and the Diagrams Accuracy-Deviation permitted the identification of the dynamics of the time series for 77 weeks, from the last 38 weeks of the year 2019 to September 2020.

Scatter plots, histograms and sample Spearman correlation coefficients were the tools chosen to analyse the time dependence of the level of chaos concerning weeks of economic activity. North American stock markets presented a growing in the level of chaos during the pandemic from the analysis of the indices \( \hat{G}_{SPC} \) and \( \hat{G}_{IXIC} \). It was greater than that of other indices. The filling of the upper region of the scatter plots increases considerably after the first week of January 2020. The corresponding histograms suffered displacement of the bars of frequency to the right. Statistical evidence in favour of the association between chaoticity and time is significant at the level of 0.001.

The European index \( \hat{G}_{IBEX} \) and the Asian index \( \hat{G}_{N225} \) did not reveal changes in their chaoticity after the first week of January 2020 from scatter plots and histograms. It is according to the graphical representations for these stock markets. On the other hand, the Spearman correlation is significant at the level of 0.001 for the EURONEXT 100 stock market. The scatter plot and histogram also suggest that the chaoticity increases during COVID-19 pandemic. This kind of agreement deserves brief consideration. In this work, Spearman correlation, scatter plots, histograms and chaos quantifiers point to the same conclusion on the time evolution of the level of chaos. It is a strong point in favour of the robustness of the methodology employed.

The degree of chaos for the Asian index 000001.SS decays in the scatter plot. In the histogram, the displacement of the bar of frequency to the left suggests the same conclusion. Another statistical evidence in this direction is the negative Spearman correlation between the quantifier of chaos and time, significant at the level 0.001. There are sufficient reasons to conclude that this index had the behaviour opposite to the others.

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