On Geometry of Flat Complete Strictly Causal Lorentzian Manifolds

V.M. Gichev, E.A. Meshcheryakov

Abstract

A flat complete causal Lorentzian manifold is called strictly causal if the past and the future of each of its point are closed near this point. We consider strictly causal manifolds with unipotent holonomy groups and assign to a manifold of this type four nonnegative integers (a signature) and a parabola in the cone of positive definite matrices. Two manifolds are equivalent if and only if their signatures coincide and the corresponding parabolas are equal (up to a suitable automorphism of the cone and an affine change of variable). Also, we give necessary and sufficient conditions, which distinguish parabolas of this type among all parabolas in the cone.

1 Introduction

A flat complete Lorentzian manifold can be realized as a quotient space \( M = M_n/\Gamma \), where \( \Gamma \cong \pi_1(M) \) is a discrete subgroup of Poincare group \( P_n \) of affine automorphisms of Minkowski spacetime \( M_n \) which acts on \( M_n \) freely and properly. There are several equivalent definitions: \( M \) is a geodesically complete Lorentzian manifold with vanishing torsion and curvature; \( M \) admits an atlas of coordinate charts in \( M_n \) with coordinate transformations in \( P_n \) such that any affine mapping of a segment in \( \mathbb{R} \) to \( M \) extends to an affine mapping of \( \mathbb{R} \) (a complete affine manifold with a compatible Lorentzian metric). The Lorentzian metric defines a pair of closed convex round cones in each tangent space \( T_p M \). Choosing one of them, we get locally a cone field. It can be extended to a global cone field on \( M \) or on a two-sheet covering space of \( M \); we assume that the field is defined on \( M \). This is equivalent to the assumption that linear parts of transformations in \( \Gamma \) do not transpose the cones of the past and the future in \( M_n \). If \( M \) admits no closed timelike curves, then it is called causal. For \( \Gamma \), this means that the orbit of any point \( v \) does not meet the cone at \( v \). Fixing origin \( o \in M_n \), we may identify \( M_n \) with a real vector space \( V \) endowed with a Lorentzian metric \( \ell \) of the signature \((+,-\ldots,-)\); the causal structure is determined by a cone \( C \) (one of the two cones defined by the inequality \( \ell(v, v) \geq 0 \)). We say that \( M \) is strictly causal if the past and the future of any point \( p \in M \) are closed near \( p \) (they are not closed globally in general). In paper [7], these manifolds were found up to finite coverings. Here is a brief description: \( M \) is the total
space of a topologically trivial vector bundle with a bounded linear holonomy group, whose base is $M_k/\Gamma$, $k \leq n$, where the action of $\Gamma$ on $M_k$ is affine and unipotent. This reduces the problem to the case of unipotent $\Gamma$. There are two types of them. The first (elliptic) is trivial: $\Gamma$ is the group of translations by vectors in a uniform lattice in a linear subspace $T \subset V$ such that $T \cap C = 0$. We consider the second (parabolic) type. The least dimension of such a manifold is 4; moreover, all these manifolds are mutually homothetic in dimension 4. In this paper, we assign to a manifold of this type (of any dimension) a parabola in the cone $P_T$ of positive definite quadratic forms on a Euclidean space $T$. It is parametrized by a quadratic polynomial with matrix coefficients. The manifold $M$ is completely determined by this parabola (considered up to automorphisms of the cone) and a lattice $\Gamma \subset T$. The parabolas corresponding to the manifolds do not exhaust all parabolas in the cone; we characterize them and describe some their invariants.

The object of this paper lies in the intersection of the two well-explored fields: flat complete affine manifolds (see recent surveys [1] and [4]) and the causality in the Lorentzian manifolds; the studying of the latter was mainly stimulated by general relativity (see [4, 5]). In their common part, we know only some articles of A. D. Alexandrov’s chronogeometric school (see [8] for references) and papers [7], [6], [2]. The paper [6] contains a characterization of two-ended causal 4-manifolds which can be realized as $H/\Gamma$, where $H$ is a subgroup of the Poincare group whose action on Minkowski spacetime is simply transitive, $\Gamma$ is its discrete subgroup. Most of them are not strictly causal while the nontrivial strictly causal manifolds are never globally hyperbolic ([2],[7]); the latter class of flat Lorentzian (not necessarily complete) manifolds was considered in [2]. This paper continues article [7], where some parametric description was found for the flat complete strictly causal Lorentzian manifolds (we give it below).

2 Preliminaries and statement of results

Fixing the origin $o \in M_n$, we realize $M_n$ as a real vector space $V$ equipped with a Lorentzian form $\ell$ of the signature $(+, -, \ldots, -)$. The set $\ell(v, v) \geq 0$ is the union of two closed convex round cones in $V$. Let $C$ be one of them. The group $\Gamma$ acts freely and properly in $V$ by affine transformations whose linear parts preserve $\ell$ and $C$ invariant. We denote by $\kappa$ the quotient mapping $\kappa : M_n \rightarrow M = V/\Gamma$ and define the past $P_p$ and the future $F_p$ of $p \in M$ by

$$P_p = \kappa(v - C), \quad F_p = \kappa(v + C), \quad v \in \kappa^{-1}(p).$$

Clearly, $P_p$ and $F_p$ do not depend on the choice of $v$. Hence, the projection of the cone field to $M$ is well-defined:

$$C_p = d_v \kappa(v + C) \subset T_p M, \quad v \in \kappa^{-1}(p).$$

The manifold $M$ is said to be causal if $M$ admits no closed piecewise smooth timelike paths. A smooth path $\eta$ is called timelike if $\eta'(t) \in C_{\eta(t)}$ for all $t$; for
lightlike paths, $\eta'(t) \in \partial C_{\eta(t)}$ (note that lightlike paths are timelike). Clearly, each timelike curve in $M$ can be lifted to a timelike curve in $V$ and the projection into $M$ of a timelike curve in $V$ is timelike. We say that an isometry of Lorentzian manifolds is causal if it preserved the orientation of timelike curves. Then the manifolds are said to be causally isometric. An affine manifold is a manifold with affine coordinate transformations. A complete affine manifold can be realized as $V/\Gamma$ where $\Gamma$ is a free and proper group of affine transformations of $V$; in our setting, they belong to $P_n$. Given $\gamma \in \Gamma$, put

$$
\gamma(v) = \lambda(\gamma)v + \tau(\gamma), \quad \text{where} \quad \lambda(\gamma) \in O(\ell), \quad \tau(\gamma) \in V;
$$

$$
G = \lambda(\Gamma),
$$

where $O(\ell)$ denotes the group of all linear transformations of $V$ preserving $\ell$. Clearly, the mapping $\lambda: \Gamma \to O(\ell)$ is a homomorphism and for all $\gamma, \nu \in \Gamma$

$$
\tau(\gamma \circ \nu) = \lambda(\gamma)\tau(\nu) + \tau(\gamma).
$$

According to [7, Theorem 1], each strictly causal flat complete Lorentzian manifold is finitely covered by the total space of a vector bundle with (arbitrary) bounded holonomy group and unipotent base (the latter means that $\Gamma$ consists of affine transformations with unipotent linear parts). Thus we consider only the unipotent case. By [7, Theorem 2], a unipotent manifold of this type can be finitely covered by a manifold described below.

**The Main Construction.** Let $v_0, v_1 \in \partial C$ satisfy $\ell(v_0, v_1) = 1$ and put

$$
L = \mathbb{R}v_0, \quad W = L^\perp, \quad N = W \cap v_1^\perp;
$$

$$
l_0(v) = \ell(v_0, v).
$$

The hyperplane $W = N \oplus L$ is tangent to $\partial C$ at $v_0$, while the set $W \cap C$ is a ray. The form $\ell$ is nonpositive and degenerate in $W$ and nondegenerate and negative in $N$. Let $T \subseteq N$ be a linear subspace and put $\bar{T} = T + L$. We will often identify $T$ with $T/L$. Let $\Gamma$ be a lattice (cocompact discrete subgroup of the additive group) in $T$ and $a$ be an $\ell$-symmetric linear mapping:

$$
a: T \to N, \quad \ell(ax, y) = \ell(x, ay), \quad x, y \in T.
$$

The affine action $x \to \gamma_x$ of $T$ in $V$ is defined by formulas

$$
\lambda(x)v = v + l_0(v)ax - \left(\ell(ax, v) + \frac{1}{2} l_0(v)\ell(ax, ax)\right)v_0, \quad (4)
$$

$$
\tau(x) = x - \frac{1}{2} \ell(ax, x)v_0; \quad (5)
$$

$$
\gamma_x(v) = \lambda(x)v + \tau(x).
$$

The following condition is necessary and sufficient for the action of $T$ to be free and for action of $\Gamma$ to be free and proper [7, Lemma 19]:

$$
\ker(1 + sa) = 0 \quad \text{for all} \quad s \in \mathbb{R}, \quad (6)
$$
where 1 is the identical mapping.

The quotient mappings \( V \rightarrow V/L \) and \( V \rightarrow V/\Gamma \) are denoted by \( \phi \) and \( \kappa \), respectively. Some simple properties of the above action are stated in next lemma.

**Lemma 1.** If (6) is true then the following hold for the action (2)–(5).

1. If \( x \in T \) and \( ax \neq 0 \), then the line \( L \) is precisely the set of all fixed points of \( \lambda(x) \) in \( C \cup (-C) \); translations by vectors in \( L \) commute with \( \gamma_x \) for all \( x \in T \).

2. The action of \( \Gamma \) in the quotient space \( V/L \) is free and proper. Every hyperplane \( W_s = \{ v \in V : l_0(v) = s \} \subset V \) is \( \Gamma \)-invariant, and \( \Gamma \) acts by pure translations in \( W_s/L \subset V/L \).

3. The mapping \( \phi \) is one-to-one on every \( T \)-orbit in \( V \).

The set of common fixed points of \( G = \lambda(\Gamma) \) in \( V \) may be greater than \( L \). If \( a = 0 \) then \( \Gamma \) and \( T \) act by translations: \( \gamma_x(v) = v + x \). In [7] this case was called **elliptic** and considered separately. In this paper, we combine elliptic and **parabolic** (\( a \neq 0 \)) cases.

The affine structure makes it locally possible to decide whether two vectors are parallel or not. Hence, the parallel transport of vectors along curves is well-defined. Applying this to loops at \( p \in M \), we obtain the holonomy representation \( \lambda_p : \pi_1(M, p) \cong \Gamma \rightarrow GL(T_pM) \). We have the natural identification \( \lambda_p(\pi_1(M, p)) = \lambda(\Gamma) = G \). The following theorem is an observation that refines [7, Theorem 2], where an analogous assertion was proved up to finite coverings and without mentioning of holonomy. A linear group is said to be **unipotent** if, in some linear base, it can be realized by triangular matrices whose diagonal entries are equal to 1. We say that \( \Gamma \subset Aff(V) \) is unipotent if \( G = \lambda(\Gamma) \) has this property (it can be considered as a unipotent linear group in the space \( V \oplus \mathbb{R} \)).

**Theorem 1.** A flat complete strictly causal Lorentzian manifold admits a realization above if and only if its holonomy group is unipotent.

We say that manifolds of Theorem 1 are **unipotent**.

**Corollary 1.** The fundamental group of a unipotent manifold is isomorphic to \( \mathbb{Z}^m \).

Let \( \hat{\Gamma} \) denote the algebraic (i.e. in the Zariski topology) closure of \( \Gamma \) in the group \( Aff(V) \) of all affine transformations of \( V \) (clearly, \( \hat{\Gamma} \subset P(V) \)).

**Proposition 1.** The algebraic closure \( \hat{\Gamma} \) of \( \Gamma \) coincides with the image of \( T \) under the embedding to \( P(V) \) defined by (4) and (5); in particular, \( \hat{\Gamma} \) is isomorphic to the vector group \( T \) and \( \hat{\Gamma}/\Gamma \) is a torus. Moreover, all orbits of \( \hat{\Gamma} \) in \( V \) are Zariski closed.
We assume that $\hat{\Gamma}$ is a subgroup of $\mathcal{P}_n$ and $T$ is a linear subspace of $V$ in the sequel. The orbits of $\hat{\Gamma}$ in $M$ are tori $\hat{\Gamma}/\Gamma$. If $a = 0$ then they are affine submanifolds of $M$; otherwise, there is the compatible affine structure on line bundles over these tori, with lines parallel to $L$ (see Lemma 1, (2)). The torus $\hat{\Gamma}/\Gamma$ acts freely on $M$ in both cases, $M/\hat{\Gamma}$ is homeomorphic to a vector space and $M$ is homeomorphic to the product $\hat{\Gamma}/\Gamma \times M/\hat{\Gamma}$ ([7, Theorem 2]).

The manifold $M$ is completely determined by parameters $v_0, v_1, T, a, \Gamma$. They are not independent (for example, $T$ is the linear span of $\Gamma$). The vectors $v_0$ and $v_1$ define $W, N, L$ by (2). If $v_0$ is fixed then every choice of a complementary to $L$ subspace $N$ in $W$ determines $v_1$ uniquely: the space $N^\perp$ is two dimensional and intersects $\partial(C \cup (-C))$ by two lines, $L$ and $\mathbb{R}v_1$, and the position of $v_1$ in the second line is defined by $\ell(v_0, v_1) = 1$. We write

$$M = M(v_0, v_1, T, a, \Gamma) = M(v_0, N, T, a, \Gamma)$$

omitting parameters sometimes. Most of them have a natural geometrical meaning; the following proposition clarifies it. Let $M$ be as in Theorem 1, $p \in M$, $\Gamma = \pi_1(M, p)$. We define some subspaces of $T_p M$ in the notation which agrees with the main construction:

- $T$: the tangent space to the orbit $\hat{\Gamma}p$;
- $H$: the linear span of vectors $(\lambda_p(x) - 1)v$, where $x \in \Gamma$, $v \in T_p M$;
- $U = T + H$, $L = H \cap H^\perp$, $W = L^\perp$.

If $M$ is elliptic then $H = L = 0$, $U = T$, $W = V$. Since the action of $\hat{\Gamma}/\Gamma$ in $V$ is free, $T$ may be identified with the Lie algebra of the torus $\hat{\Gamma}/\Gamma$. This defines the exponential mapping $\exp : T \to \hat{\Gamma}p$. For a flat complete affine manifold $M$ and each $p \in M$, another exponential mapping $\exp_p : T_p M \to M$ is uniquely defined by following conditions:

1. $\frac{d}{dt}\big|_{t=0} \exp_p(tu) = u$ for each $u \in T_p M$,
2. the mapping $t \to \exp_p(tu)$ is affine.

The two exponential mappings are not equal but their $\phi$-projections coincide on $T$ by Lemma 1, (2). Let $\pi_X$ denote the $\ell$-orthogonal projection to a subspace $X$ (the definition is sound if $\ell$ is nondegenerate in $X$).

**Proposition 2.** Let $M$ be a unipotent nonelliptic manifold, $p \in M$, and $T, H, U, L, W$ be as above. Then $\dim L = 1$ and $L$ consists of fixed points of the holonomy representation.

1. Every choice of a generating vector $v_0 \in \partial C_p$ for $L$, an isotropic vector $v_1 \perp U$ such that $\ell(v_0, v_1) = 1$, defines the action (2)–(5) of $T$ on the setting $N = v_1^\perp \cap W$ by

$$ax = \pi_X \lambda(\exp(x))v_1, \quad x \in T,$$

with $T_p M$ as $V$. 

5
(2) The mapping $\exp_p$ satisfies the condition $\exp_p(\gamma(v)) = \exp_p(v)$ for all $v \in T_pM$ and identifies $M$ with $T_pM/\Gamma$.

(3) Put $E = U^\perp \cap N$. Then $M = E \times M'$, where $M' = \exp_p(E^\perp)$ is a nonelliptic unipotent submanifold of $M$.

Each $\ell$-symmetric mappings has an evident structure: if $a$ satisfies (3) then it admits the unique decomposition

$$a = a' + a'', \quad (7)$$

where $a': T \to T$ is the self-adjoint transformation of $T$ corresponding to the symmetric bilinear form $\ell(ax, y)$:

$$\ell(ax, y) = \ell(a'x, y) = \ell(x, a'y), \quad x, y \in T,$$

and $a''$ is an arbitrary linear mapping

$$a'' : T \to T^\perp \cap N.$$  

Put

$$R = a''T.$$  

The condition (6) can be rewritten as follows:

$$t \in \mathbb{R}, \quad a'x = tx \neq 0 \implies a''x \neq 0. \quad (8)$$

In other words, $a''$ is nondegenerate in all eigenspaces of $a'$ but $\ker a'$ (note that $a'$ has only real eigenvalues and is semisimple since $a'$ is self-adjoint). The space

$$\ker a = \ker a' \cap \ker a'' \subseteq T$$

acts by pure translations. It is a trivial summand for the action of $T$ but this fails in general for $\Gamma$.

There are three natural steps in the construction of $M$:

(A) fix $v_0, v_1$, define $L, W, N$ by (2) and choose $T \subseteq N$;

(B) pick an $\ell$-symmetric linear operator $a' : T \to T$, and, for each its eigenspace $\Lambda_j$, a linear operator $a''_j : \Lambda_j \to T^\perp \cap N$ (which must be nondegenerate if $\Lambda_j \neq \ker a'$); and set $a'' = \sum_j a''_j$, $a = a' + a''$;

(C) choose a linear basis for $T$ and define $\Gamma$ as the subgroup of the vector group $T$ generated by it.

The first step provides a frame for the second and the third which are independent of one another. For example, if $T = N$, then $a = 0$ and we get an elliptic manifold; if $\dim R = 1$, then $a''$ has rank 1 and can be nondegenerate in all eigenspaces of $a'$ only if they are one-dimensional. We say that $a'$ and $M(a)$ have the simple spectrum if each eigenvalue of $a'$ has multiplicity $\leq 1$. 

6
Remark 1. It is not difficult to construct a unipotent manifold with the simple spectrum and prescribed eigenvalues and eigenvectors of $a'$ (any orthonormal linear base in $T$). A manifold with the simple spectrum is determined up to an isometry by $m$ real numbers (the spectrum of $a'$) and the Gram matrix of $m$ vectors ($a''$-images of eigenvectors) of rank $r \geq 1$ which must have nonzero diagonal elements (this is not a classification since these parameters do not distinguish some isometric manifolds).

The causally isometric manifolds of this type admit realizations (2)–(5) with identical parameters. We say that manifolds are almost causally isometric if they admit realizations that differ only on the step (C) of the construction above.

To all $p \in M$ and $x \in \pi_1(M, p)$, there is a realization of the loop $x$ as a straight line segment. Precisely, this is the projection into $M$ of the segment with endpoints $\gamma_x(v)$ and $v$, where $\kappa(v) = p$ and $\gamma_x \in \Gamma$ is the affine transformation corresponding to $x$. Put

$$q_v(x) = -\ell(\gamma_x(v) - v, \gamma_x(v) - v). \quad (9)$$

This is a function on the group $\pi_1(M, p) = \Gamma$ (the squared $\ell$-length of the segment mentioned above). If $a = 0$ then $q_v(x) = -\ell(x, x)$ does not depend on $v$ since $\Gamma$ acts by pure translations. In general, $\Gamma$ acts by translations in each hyperplane $W/L$ (Lemma 1, (2)). Since $W \perp L$, this means that $q_v$ depends only on $s = l_0(v)$. By a straightforward calculation with (4), (5) we obtain

$$q_v(x) = q_s(x) = -\ell((1 + sa)x, (1 + sa)x), \quad \text{where } s = l_0(v). \quad (10)$$

Hence $\{q_v : v \in V\}$ is one-parameter family of quadratic forms on $T$. Since $\ell$ is negative definite on $T$, it follows from (6) that all forms $q_s$ are positive definite on $\tilde{\Gamma}$. Thus, we arrive at a curve in the cone of positive definite quadratic forms on $\tilde{\Gamma}$ which is a parametrized by $s = l_0(v)$. By (9) and (10), the change of origin is equivalent to the shift of the parameter:

$$s \rightarrow s - s_0, \quad s_0 = l_0(\tilde{\delta}), \quad (11)$$

where $\tilde{\delta}$ is new origin. For all $t > 0$, replacing $v_0, v_1$ by $tv_0, v_1/t$, respectively, we come to the same formulas with $tl_0, a/t$ instead of $l_0, a$. This corresponds to the change of variable

$$s \rightarrow ts. \quad (12)$$

If $t < 0$ then the time reverses. Thus, we will consider the curve $s \rightarrow q_s$ up to orientation-preserving affine changes of the variable $s$.

We will identify $\Gamma$ with $\mathbb{Z}^m$ and $\tilde{\Gamma}$ with $\mathbb{R}^m$. More precisely, let $\iota : \mathbb{R}^m \rightarrow T$ be a linear isomorphism such that $\iota \mathbb{Z}^m = \Gamma$ and let $\langle \ , \ \rangle$ be the standard inner product in $\mathbb{R}^m$. By (10), $q_\sigma$ is quadratic in $s$. Hence, there exist symmetric $m$-matrices $A, B, C$ such that

$$q_s(x) = \langle (A + 2sB + s^2C)z, z \rangle \quad \text{for all } z \in \mathbb{R}^m, \quad \text{where } x = \iota z \in T. \quad (13)$$
The inequality $S > 0$ ($S \geq 0$), where $S$ is a matrix, means that $S$ is symmetric positive definite (respectively, nonnegative). We denote the set of positive matrices by $P_m$; it is a homogeneous space of the group $\text{GL}(m, \mathbb{R})$ acting by

$$S \rightarrow X^\top SX,$$

where $^\top$ stands for the transposition. Moreover, the involution $S \rightarrow S^{-1}$ defines on $P_m$ the structure of a symmetric space. The condition

$$Q(s) = A + 2sB + s^2C > 0 \quad \text{for all } s \in \mathbb{R}$$

is necessary (but not sufficient) for the matrix valued quadratic polynomial $Q$ to satisfy (13). It implies $A > 0$, $C \geq 0$ (note that $B = C = 0$ if $a = 0$).

**Remark 2.** The affine span of a generic curve of this type is at most two-dimensional: the linear subspace parallel to it is spanned by matrices $B$ and $C$. Thus, it is a parabola, which may degenerate into a ray if $B = 0$ and into a point if $B = C = 0$ (the case $C = 0$, $B \neq 0$ cannot occur due to (14)). Up to an affine change of the variable $s$, there is only one quadratic parametrization of a parabola in a plane. Thus, we come to a geometrical object, a characteristic curve of $M$.

A linear change of the variable $z \in \mathbb{R}^m$ defined by a real $m$-matrix $X \in \text{GL}(m, \mathbb{R})$, induces come translation of this curve in the symmetric space $P_m$:

$$Q(s) \rightarrow X^\top Q(s)X.$$  

We say that $Q(s)$ is a characteristic polynomial of $M$ and denote it by $Q_M$, considering $Q_M$ up to affine changes of the variable $s$. Put

$$n = \dim M, \quad m = \dim T, \quad r = \dim R; \quad k = \dim \ker a;$$

the 4-tuple $(n, m, r, k)$ will be called a signature of $M$. It follows from Proposition 2 that the signature does not depend on the realization of $M$ in the form (2)–(5). Clearly, these numbers satisfy inequalities

$$m + r + 2 \leq n, \quad r + k \leq m.$$

**Theorem 2.** Let manifolds $M$ and $\tilde{M}$ be as in Theorem 1. They are causally isometric if and only if their signatures coincide and

$$Q_M(s) = X^\top Q_{\tilde{M}}(as + \beta)X$$

for some $X \in \text{GL}(m, \mathbb{Z})$, $\alpha > 0$, $\beta \in \mathbb{R}$.

In other words, the manifolds with equal signatures are isometric if and only if projections of their characteristic curves to $P_m/\text{GL}(m, \mathbb{Z})$ coincide. Note that $P_m/\text{GL}(m, \mathbb{Z})$ is the modulus space for Euclidean structures in $m$-tori. It appears naturally since Euclidean structures induced by $-\ell$ in the orbits of the torus $\hat{\Gamma}/\Gamma$ in $M/L$ depends only on $s = l_0(v)$ (the group $L \cong \mathbb{R}$ acts on $M = V/\Gamma$ since translations by vectors in $L$ commute with $\Gamma$).
Remark 3. Replacing the containment \( X \in \text{GL}(m, \mathbb{Z}) \) by \( X \in \text{GL}(m, \mathbb{R}) \), we come to a criterion for \( M \) and \( \tilde{M} \) to be almost casually isometric. Indeed, the manifolds are almost isometric if and only if they admit realizations with equal parameters, except for \( \Gamma \), but all lattices in \( \mathbb{R}^m \) are linearly equivalent. For \( m = 1 \) and \( a \neq 0 \), we have an ordinary quadratic polynomial, which is equivalent to \( A + s^2, A > 0 \). If \( k = 0 \), then \( \dim M = 4 \), and the main theorem implies that nonelliptic manifolds in this dimension form a one-parameter family. There is a more precise version of this result [7, Theorem 1]: all flat complete strictly casual nonelliptic Lorentzian 4-manifolds are nomothetic to the manifold of the following example.

Example 1. Let \( \Gamma = \mathbb{Z} \) be the cyclic infinite group generated by the affine transformation \( v \to \lambda v + \tau \) which is defined in the basis \( e_0, \ldots, e_3 \) by relations

\[
\begin{align*}
\lambda e_0 &= e_0, \quad \lambda e_1 = e_1, \quad \lambda e_2 = e_2 + e_0, \quad \lambda e_3 = e_3 + e_2 + \frac{1}{2} e_0, \quad \tau = e_1; \\
\ell(v, v) &= 2v_0v_3 - v_1^2 - v_2^2.
\end{align*}
\]

The manifold \( M = V/\Gamma \) is strictly casual. Let \( u = (u_0, \ldots, u_3) \in V, u_3 > 0 \). A straightforward calculation (see [7]) shows that the past of \( u \) contains the open halfplane \( x_3 < -\frac{1}{u_3} \), meets the hyperplane \( x_3 = -\frac{1}{u_3} \), but does not include it. In particular, this implies that the past of \( u \) contains a straight line and is not closed.

Our aim is now to describe the quadratic polynomials \( Q(s) \) in (14) such that \( Q = Q_M \) for some \( M \). We will achieve in the two steps: in the first, we reduce the problem to the case of nondegenerate matrix \( C \); in the second, we describe the polynomials with \( C > 0 \).

Proposition 3. Let \((n, m, r, k)\) be the signature of a manifold \( M \) and let \( k > 0 \). Then there exists \( X \in \text{GL}(m, \mathbb{R}) \) such that the matrix \( X^\top Q_M(s)X \) admits block-diagonal realization with blocks of size \( k \times k \) and \((m-k) \times (m-k)\), where \( k \)-block does not depend on \( s \) and \((m-k)\)-block is a characteristic polynomial for a manifold \( \tilde{M} \) of the signature \((n-k, m-k, r, 0)\); \( M \) is almost casually isometric to the product of \( \tilde{M} \) and a flat \( k \)-torus. Moreover,

\[
m - k = \text{rank } C. \tag{18}
\]

Theorem 3. The polynomial \( Q(s) \) in (14), where \( A, B, C \) are \( m \)-matrices, defines a characteristic curve of manifold \( M \) with signature \((n, m, r, 0)\) if and only if (14) holds, \( m + r + 2 \leq n \), and

\[
C - BA^{-1}B \geq 0, \tag{19}
\]

\[
r = \text{rank} (C - BA^{-1}B). \tag{20}
\]

If \( m = r \) then (14) may be replaced by a weaker condition \( A > 0 \).

It follows from (14), (19), and (20) that \( r > 0 \) (see remark at the end of the paper). If (19) is true and \( A > 0 \), then (14) can be formulated in term of the eigenvectors of some matrices as in (8).
Remark 4. The conditions (19) is not a consequence of (14) as well as (14) does not follow from (19) even under the additional assumption $A > 0$. For example, let $m = 2$, $A = B = 1$, and

$$C = \begin{pmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{pmatrix}, \quad 0 < \varepsilon < 1.$$  

Then (14) is true but (19) is false. If $m = 2$, $A = C = 1$, $B$ is diagonal with entries 1 and 0 in the diagonal then $Q(-1)$ is degenerate; hence (14) is false but (19) is true in this case.

The condition (19) means that $Q(s)$ is, roughly speaking, a sum of squares (see the proof of the theorem). If $m = 1$, then $BA^{-1}B - C$ is the discriminant (divided by $A$) of the quadratic polynomial $Q(s)$.

The characteristic curve implicitly contains some invariants of $M$. An essential instance is given in the following proposition. It does not determine $M$ completely, in particular, it does not distinguish almost isometric and homothetic manifolds. The spectrum of a matrix $X$ is denoted by $\text{sp}(X)$.

**Proposition 4.** Let polynomials $Q_M(s) = A + 2sB + s^2C$ and $\tilde{Q}_M(s) = \tilde{A} + 2s\tilde{B} + s^2\tilde{C}$ be characteristic for almost causally isometric manifolds $M$, $\tilde{M}$, and let $C > 0$, $\tilde{C} > 0$. Then eigenvalues of $BC^{-1}$, $\tilde{B}\tilde{C}^{-1}$ are real and

$$\text{sp}(BC^{-1}) = \alpha(\text{sp}(\tilde{B}\tilde{C}^{-1}))$$  

for some $\alpha \in \text{Aff}(\mathbb{R})$.

**Remark 5.** Using results above, it is not difficult to describe the flat complete strictly causal manifolds in small dimensions. For $n = 4$ there is exactly one, up to a homothety, nonelliptic manifold (see (see Example 1). Let $n = 5$, $e_1, \ldots, e_5$ be the standard basis for $V$,

$$\ell(u, u) = 2u_4u_5 - u_1^2 - u_2^2 - u_3^2.$$  

If $m + r + 2 = 5$ and $k = 0$, then $m = 2$ and $r = 1$. By (8), $a'$ must have a simple spectrum. Putting $ae_1 = te_3$, $ae_2 = e_2 + re_3$, where $t, r \neq 0$, $v_0 = e_5$, $v_1 = e_4$, and choosing $\Gamma$ in the linear span of $e_1, e_2$, we get all manifolds of the signature $(5, 2, 1, 0)$. Other signatures can be reduced to less dimensions. The number of variants grows rather fast with $n$.

### 3 Proof of results

**Proof of Lemma 1.** Since $ax \notin L$, $\lambda(x)v = v$ and (4) imply $l_0(v) = 0$. Hence, $v \in W$. Further, we have $W \cap (C \cup (-C)) = L$ since $W$ is tangent to $\partial C$ at $v_0$. The same relations imply the second assertion of (1) in the lemma. Since $v_0, x, ax \in W$, all hyperplanes $W_x$ are $\Gamma$-invariant by (4) and (5). Putting $v_0 = 0$ and $l_0(v) = s$ in (4) and (5), we obtain the formula for the action of $\Gamma$ in $W_x/L$: $\gamma_x(v) = v + sax + x( \text{ mod } L)$. Thus, $T$ acts by pure translations in $W_x/L$, and the action is free and proper if (6) is true. Then it is free and proper in $W_x$. This proves (3) and (2) of the lemma. \qed
Proof of Proposition 1. Let \( p \) be a polynomial on \( V \). Suppose that \( p(\gamma x(v)) \) is independent of \( x \in \Gamma \) for some \( v \in V \). It follows from (4) and (5) that \( p(\gamma x(v)) \) is a polynomial on \( x \). Hence, \( p(\gamma x(v)) \) is constant on \( T \). Therefore, the Zariski closure of \( \Gamma \) includes \( Tv \). To prove the reverse inclusion, note that the projection of a \( T \)-orbit to \( V/L \) is the affine subspace \( \phi((1 + sa)T + v) \), where \( s = l_0(v) \). Hence, \( T \)-orbit has codimension 1 in the affine subspace \( X = L + (1 + sa)T + v \). By Lemma 1, (3), it has the form \( \{v + x + sax + f(x)v_0 : x \in T\} \) for some function \( f \) on \( T \). One can find \( f \) by simple straightforward calculation with (4) and (5), which shows also that the orbit is distinguished by an algebraic equation in \( X \). Thus, every \( T \)-orbit is Zariski-closed. Clearly, the image of \( T \) under the embedding defined by (4) and (5) is also closed. Since the action of \( T \) is free by (6), the same is true for \( \tilde{\Gamma} \).

Proof of Theorem 1. It follows from (4) that

\[
(\lambda(x) - 1)V \subset W, \quad (\lambda(x) - 1)W \subset L, \quad (\lambda(x) - 1)L = 0.
\]

Hence, every \( M(a, \Gamma) \) has unipotent holonomy group. To prove the converse, note that \( M \) can be finitely covered by \( M(a, \Gamma) \) for some \( a, \Gamma \) by [7, Theorem 2]. Then \( M = V/\Gamma \), where \( \Gamma \subset P(V) \) is a unipotent group that contains \( \Gamma \) as a subgroup of finite index and acts in \( V \) freely and properly. Then \( \Gamma \) has finite index in the algebraic closure \( \hat{\Gamma} \) of \( \Gamma \). Each unipotent matrix \( U \) lies in a one-parameter group \( \exp(tX) \), where \( X \) is nilpotent and is a polynomial of \( U - 1 \); moreover, \( \exp(tX) \) is polynomial in \( t \). Hence, the Zariski closure of the cyclic group generated by \( U \) includes \( \exp(\mathbb{R}X) \). Therefore, every Zariski-closed unipotent linear group is connected. This implies that \( \hat{\Gamma} = \tilde{\Gamma} \) and \( \Gamma \subset \hat{\Gamma} \). By Proposition 1, \( \hat{\Gamma} \) is a discrete subgroup of the vector group \( \hat{\Gamma} \) including the uniform lattice \( \Gamma \). Hence, \( \hat{\Gamma} \) is a uniform lattice itself; thus, \( M = M(a, \Gamma) \). □

Proof of Proposition 2. By Theorem 1, we may assume \( M = M(v_0, v_1, T, a, \Gamma) \). Suppose first that \( p = \kappa(o) \). Then we may identify \( V \) and \( T \), \( a, \Gamma \). It follows from (5) that \( d_0(p) = x \). Hence, the tangent space at \( p \) to the orbit of \( T \) coincides with \( T \) under this identification. By Proposition 1, the two definitions of \( T \) agree. Since \( M \) is nonelliptic, \( ax \neq 0 \) for some \( x \in T \). Furthermore, \( \ell(ax, ax) \neq 0 \) because \( \ell \) is negative definite on \( T \). By (4), for all sufficiently large \( t > 0 \) and all \( v \in V \) such that \( l_0(v) \neq 0 \), we have \( \ell(txv - v, -txv - v) = rv_0 \), where \( r \neq 0 \). Hence, \( v_0 \in H \) and we see that \( H = aT + \mathbb{R}v_0 \) as an immediate consequence of (4). Since \( v_0 \) is isotropic and \( \ell \) is nondegenerate on \( aT \), \( L = H \cap H^\perp = \mathbb{R}v_0 \) and \( \dim L = 1 \). Let \( \tilde{v}_0 = \frac{1}{r}v_0 \) for some \( r > 0 \), \( \tilde{v}_1 = rv_1 + w \), where \( w \in T^\perp \cap W \) is such that \( \tilde{v}_1 \) is isotropic. Then, according to (4),

\[
\lambda(x)\tilde{v}_1 = \lambda(x)(rv_1 + w) = rax + \xi(x, w, r)v_0
\]

for some function \( \xi \). Put \( a = ra \). Then \( \ell(ax, x)v_0 = \ell(\tilde{a}x, x)\tilde{v}_0 \). Since the kernel of the orthogonal projection in \( V \) to every subspace complementary to \( L \) in \( W \) always intersects \( W \) by \( L \) and \( ax \in U \), we come to the formulas (2)–(5) for the
action of $T$, with $v_0, v_1$, and $a$ replaced by $\tilde{v}_0, \tilde{v}_1$, and $\tilde{a}$. The embedding of $\Gamma$ to $T$ satisfies the equality

$$\Gamma = \pi_T(\exp_p^{-1}(p))$$

and is completely determined by the latter. Evidently, $\exp_p(\gamma(v)) = \exp_p(v)$ for all $v \in V$ and $\gamma \in \Gamma$. Further, the inclusion $E \subseteq N$ implies that $\ell$ is nondegenerate in $E$. Hence, $V = E \oplus E^\perp$. The decomposition is $\Gamma$-invariant since $E^\perp \supseteq T + aT + L$. Therefore, $E$ is a direct factor in $M = V/\Gamma$. Since the action of $\Gamma$ in $E^\perp$ is subject to the same formulas as in $V$, it follows that $M'$ satisfies the proposition.

To prove the assertion for any $p \in M$, it is sufficient to remove origin to an arbitrary point $\tilde{o} \in V$ and find parameters that realize the action in the form (2)–(5). A pure translation does not change the linear parts of affine transformations. Therefore, $H$ and $L$ are the same as above. By the first part of the proof, we may preserve $v_1$. Then $N$ does not change. Consequently,

$$\lambda(x) = \lambda(\tilde{x}),$$
$$\tilde{a}\tilde{x} = \pi_N\lambda(\tilde{x})v_1 = \pi_N\lambda(x)v_1 = ax$$

(the tilde distinguishes new parameters), where $\tilde{x}$ is the point in the tangent space $T$ to the orbit of $\tilde{o}$ at $\tilde{o}$ that satisfies $\phi(\tilde{x}) = \phi(\tilde{x})$, where

$$\tilde{\tau}(\tilde{x}) = \gamma_x(\tilde{o}) - \tilde{o} = \tau(x) + (\lambda(x) - 1)\tilde{o}, \quad x \in T. \quad (22)$$

Differentiating by $x$ at $x = 0$ we find

$$\tilde{T} = \{x + sax - \ell(ax, \tilde{o})v_0 : x \in T\}, \quad s = l_0(\tilde{o});$$
$$\tilde{x} = x + sax - \ell(ax, \tilde{o})v_0.$$

The latter is true since $\phi(\tilde{x}) = \phi(x + sax)$ by (22) and (4), (5). If $s = 0$ (equivalently, if $\tilde{o} \in W$), then $\phi(\tilde{T}) = \phi(T)$; $U = T + H$ does not change since $L \subset H$. Inserting this into (4), (5), we obtain the same formulas with new parameters. Let $\tilde{o} = sv_1$, where $s \in \mathbb{R} \setminus \{0\}$. Then $l_0(\tilde{o}) = s, \ell(ax, \tilde{o}) = 0$, and we find

$$\tilde{\tau}(\tilde{x}) = \tau(x) + (\lambda(x) - 1)\tilde{o} = x - \frac{1}{2}\ell(ax, x)v_0 + sax - \frac{1}{2}\ell(ax, ax)v_0 =$$
$$\tilde{x} - \frac{1}{2}\ell(\tilde{a}\tilde{x}, \tilde{x})v_0.$$

Since each translation in $V$ is a composition of a translation along $W$ and $\mathbb{R}v_1$, the action can be realized in the form (2)–(5) for any choice of origin.

Proof of Theorem 2. The assertion on signatures is clear. The left-hand side of (9) is the squared length of the unique segment in $V$ that represents $x \in \pi_1(M, p)$. Hence, an isometry identify $q_v, \tilde{q}_v$ as functions on $\pi_1(M, p)$. Identifying $\Gamma$ with $\mathbb{Z}^n$, we arrive at quadratic polynomials with matrix coefficients by (10) and (13). Clearly, the transformations of the coefficients which are
induced by a change of parameters are subject to (17). If images of polynomials \( Q_M \) and \( Q_{\delta \tilde{\gamma}} \) (which can be a parabola, a ray, or a single point) coincide, then there exists an increasing affine change of the variable \( s \) that identifies them. Since any change of this type can be realized by shifting the origin along \( \mathbb{R} v_1 \) and scaling the vectors \( v_0, v_1 \) (see the proof of Proposition 2 above and (11), (12)), we arrive at equal curves in \( P_m/\text{GL}(m, \mathbb{Z}) \) if the manifolds are causally isometric. Hence, (17) is true.

Conversely, let (17) hold. It is sufficient to prove the existence of a transformation in \( P_m \) equivariant with respect to the action of the fundamental groups of \( M \) and \( \tilde{M} \) in \( V \) assuming that they are subject to (2)–(5). Applying a transformation in \( \mathcal{P}(V) \), we may assume \( v_0 = \tilde{v}_0, v_1 = \tilde{v}_1 \) (hence, \( N = \tilde{N} \) and \( W = \tilde{W} \)); and also that \( T = \tilde{T} \) and \( R = \tilde{R} \) (since the signatures coincide). Thus, it is sufficient to prove that \( a \) and \( \tilde{a} \) are conjugated by an \( \ell \)-orthogonal linear transformation of \( N \) preserving \( T \) and \( R \). Using (10) and the decomposition (7), by (9) and (13) we find

\[
\begin{align*}
\langle Az, z \rangle &= \ell(x, x), \\
\langle Bz, z \rangle &= \ell(\alpha' x, x), \\
\langle Cz, z \rangle &= \ell(\alpha'' x, \alpha'' x),
\end{align*}
\]

where \( x = \iota z \in T, z \in \mathbb{R}^m \); similar equalities hold for \( Q_{\delta \tilde{\gamma}} \). Thus, for instance, (23) means that \( \ell(x, x) = \ell(\tilde{x}, \tilde{x}) \), where \( \tilde{x} = \iota \tilde{z} \). It follows from (23) that \( \iota = \xi i \) for some \( \xi \in O(\ell|T) \). Then \( \alpha' = \xi^{-1} \alpha' \xi \) by (24) (note that the quadratic form on the left-hand side uniquely determines \( \alpha' \) and \( \tilde{\alpha}' \) since they are \( \ell \)-symmetric). This implies \( \ell(\alpha' x, \alpha' x) = \ell(\tilde{\alpha}' \tilde{x}, \tilde{\alpha}' \tilde{x}) \) for all \( x \in T \). Then \( \ell(\alpha'' x, \alpha'' x) = \ell(\tilde{\alpha}'' \tilde{x}, \tilde{\alpha}'' \tilde{x}) \) by (25). Therefore, \( \tilde{\alpha}'' = \xi^{-1} \alpha'' \xi \) for some \( \xi \in O(\ell|R) \). Thus, the transformation which is equal to \( \xi \) on \( T \), equal to \( \xi \) on \( R \), and is identical on \( U^\perp \), identifies parameters for the two actions (including the embedding of \( \Gamma \)).

\[ \square \]

Proof of Proposition 3. Since \( a \) is \( \ell \)-symmetric, \( \ker a \perp aT \). The form \( \ell \) is negative definite on \( N \); hence \( \ker a \cap aT = 0 \). There exists \( X \in \text{GL}(m, \mathbb{R}) \) such that \( \iota X \iota^{-1} \) identifies the decomposition \( T = \ker a \oplus \tilde{T} \), where \( \tilde{T} = T \cap (\ker a)^\perp \), with the natural decomposition: \( \mathbb{R}^m = \mathbb{R}^k \oplus \mathbb{R}^{m-k} \). It follows from (10) that \( g_s(x) \) is independent of \( s \) if \( x \in \ker a \). Put \( \tilde{V} = (\ker a)^\perp \) and \( \tilde{\Gamma} = \mathbb{Z}^{m-k} \). Since \( \tilde{V} \) contains \( \tilde{T} \), \( aT = a\tilde{T} \), and \( L \); it is \( \tilde{\Gamma} \)-invariant by (2)–(5), and the action is subject to the same formulas. Put \( \tilde{M} = \tilde{V}/\tilde{\Gamma} \). Clearly, \( M \) is almost causally isometric to the product of \( \tilde{M} \) and the torus \( \mathbb{R}^k/\mathbb{Z}^k \) (note that \( \gamma_s(v) = v + x \) if \( x \in \ker a \)). Comparing the coefficients of \( s^2 \) in (10) and (13), we find \( \text{rank} C = \text{rank} a \). This proves (18). Remaining assertions are obvious.

\[ \square \]

In what follows, the fractional powers of nonnegative matrices are supposed nonnegative.

**Lemma 2.** Let \( Q \) be as in (14), \( C > 0 \). Any transformation (15) with \( X \in \text{GL}(m, \mathbb{R}) \) does not change \( \text{sp} \left( C^{-\frac{k}{2}} B C^{-\frac{k}{2}} \right) \) and \( \text{sp} \left( A^{-\frac{k}{2}} B A^{-\frac{k}{2}} \right) \).
Proof. If two quadratic polynomials of the type \( A + 2sB + s^21 \) are conjugated by a transformation \( X \) as in (15), then \( X \) is orthogonal. Therefore, the spectrum of the coefficient of \( s \) does not depend on the choice of the polynomial with coefficient \( 1 \) at \( s^2 \) in the \( GL(m, \mathbb{R}) \)-orbit under the transformations (15) of the polynomial \( A + 2sB + s^2C \). Analogous arguments show that spectra of the coefficients of \( s \) of all polynomials of the type \( 1 + 2sB + s^2C \) in this orbit are equal. Thus, the equalities

\[
C^{-\frac{1}{2}}(A + 2sB + s^2C)C^{-\frac{1}{2}} = C^{-\frac{1}{2}}AC^{-\frac{1}{2}} + 2sC^{-\frac{1}{2}}BC^{-\frac{1}{2}} + s^21, \\
A^{-\frac{1}{2}}(A + 2sB + s^2C)A^{-\frac{1}{2}} = 1 + 2sA^{-\frac{1}{2}}BA^{-\frac{1}{2}} + s^2A^{-\frac{1}{2}}CA^{-\frac{1}{2}}
\]

prove the lemma.

Proof of Proposition 4. Spectra of \( C^{-\frac{1}{2}}BC^{-\frac{1}{2}} \) and \( BC^{-1} \) are identical since these matrices are conjugated; they are real because the first matrix is symmetric. Replacing \( s \) by \( t(s - s_0) \) in \( A + 2sB + s^2C \), we find coefficients \( t^2C \) and \( 2(tB - t^2s_0C) \) at \( s^2 \) and \( s \), respectively. So, \( BC^{-1} \) corresponds to \( \frac{1}{2}BC^{-1} - s_01 \). It remains to apply Theorem 2 and Lemma 2.

Proof of Theorem 3. Let \( Q(s) = Q_M(s) \) for some \( M \). Put \( \tilde{B} = A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \), \( \tilde{C} = A^{-\frac{1}{2}}CA^{-\frac{1}{2}} \). Then

\[
Q(s) = A^\frac{1}{2}(1 + 2s\tilde{B} + s^2\tilde{C})A^\frac{1}{2} = A^\frac{1}{2}((1 + s\tilde{B})^2 + s^2(\tilde{C} - \tilde{B}^2))A^\frac{1}{2}. 
\]

On the other hand, \( q_\epsilon(x) = \ell((1 + sa')x, (1 + sa')x) + s^2\ell(a''x, a''x) \). Comparing this with (27), we find \(-\ell(a''x, a''x) = \langle (\tilde{C} - \tilde{B}^2)z, z \rangle \), where \( x = \epsilon A^\frac{1}{2}z, z \in \mathbb{R}^m \). Therefore,

\[
C - BA^{-1}B = A^\frac{1}{2}(\tilde{C} - \tilde{B}^2)A^\frac{1}{2} \geq 0, \\
r = \text{rank}(A^\frac{1}{2}) = \text{rank}(C - BA^{-1}B).
\]

The condition (6) holds if and only if \( \text{rank}(1 + sa) = m \) for all \( s \in \mathbb{R} \); in other words, it is true if and only if the form on the left-hand side of (10) is positive definite, which is equivalent to (14).

Conversely, let \( Q(s) \) satisfy (14) and (19). Then the manifold \( M \) may be constructed following (not word for word) the procedure (A)–(C) Proposition 2.

(A) Put \( V = \mathbb{R}^n \), where \( n \geq m + r + 2 \) and \( r \) is defined by (20). Let \( e_1, \ldots, e_n \) be the standard basis for \( \mathbb{R}^n \),

\[
\ell(z, z) = 2z_nz_{n-1} - z_1^2 - \cdots - z_{n-2}^2,
\]

\( v_0 = e_{n-1}, v_1 = e_n \), and let \( L, W, N \) be as in (2). Define special subspaces that were introduced above by the following decomposition:

\[
\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^r \oplus \mathbb{R}^{n-m-r-2} \oplus \mathbb{R} \oplus \mathbb{R} = T \oplus R \oplus E \oplus \mathbb{R}v_1 \oplus L.
\]
(B) Put \( a' = \tilde{B} \) and \( a'' = J(\tilde{C} - \tilde{B}^2)^{\frac{1}{2}} \), where \( J \) is a linear isometry of the range of \( \tilde{C} - \tilde{B}^2 \) onto \( R \).

(C) Define \( \iota : \mathbb{R}^m \to T \) as \( \iota = A^{-\frac{1}{2}} \).

The condition \( m = r \) is equivalent to \( C - BA^{-1}B > 0 \); if \( A > 0 \) then \( \tilde{B} \) and \( \tilde{C} \) are well defined. Furthermore, (28) taken together with (27), implies (14).

It follows from (27) that the equality \( C - BA^{-1}B \) hold if and only if \( Q(s) \) admits representation in the form \( X(1 + sY)^2X \), where \( X \) and \( Y \) are symmetric matrices. If \( Y \neq 0 \) then \( Q(s) \) is degenerate for some \( s \in \mathbb{R} \). Thus, (14), (19), and (20) imply that \( r > 0 \).

References

[1] Abels, H.: Properly Discontinuous Groups of Affine Transformations: A Survey, Geometriae Dedicata 87 (2001), 309–333.

[2] Barbot, T.: Globally hyperbolic flat space–times, Journal of Geometry and Physics 53 (2005), 123—165.

[3] Beem, J.K., Ehrlich, P.E.: Global Lorentzian geometry, Monographs and Textbooks in Pure and Applied Mathematics, 2nd ed., vol. 202, Marcel Dekker, New York, 1996.

[4] Charette, V., Drumm, T., Goldman, W. and Morill M.: Complete Flat Affine and Lorentzian Manifolds, Geometriae Dedicata 97 (2003), 187–198.

[5] Ellis, G., Hawking, S.: The large scale structure of space–time, Cambridge Monographs on Mathematical Physics, No. 1, Cambridge University Press, London, New York, 1973.

[6] Fried, D.: Flat Spacetimes, J. Diff. Geom. 26 (1987), 385–396.

[7] Gichev V.M., Morozov O.S., On Flat Complete Causal Lorentzian Manifolds, Geometriae Dedicata 116 (2005), 37–59.

[8] Guts A.K, Semigroups in foundations of geometry and axiomatic theory of space-time, in: Semigroups in Algebra, Geometry and Analysis, (editors K.-H. Hofmann, J.D. Lawson, E.B. Vinberg), Walter de Gruyter, Berlin, New York, 1995, p. 56–76.