Unsteady MHD free convection flow of rotating Jeffrey fluid embedded in a porous medium with ramped wall temperature

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Abstract. The effect of radiative heat transfer on unsteady magnetohydrodynamic (MHD) free convection flow of rotating Jeffrey fluid past an infinite vertical plate saturated in a porous medium with ramped wall temperature is investigated. The incompressible fluid is taken electrically conducting under influence of transverse magnetic field which perpendicular to the flow. An appropriate dimensionless variables are employed to the governing equations and solved analytically by Laplace transform technique. The results of several controlling parameters for both ramped wall temperature and an isothermal plate are presented graphically with comprehensive discussions. It has been observed that, an increase in rotation parameter, reduced the primary velocity, but an opposite behaviour is noticed for the secondary velocity. Moreover, large values of Hartmann number tends to retard the fluid flow due to the Lorentz force.

1. Introduction
Investigation of rotating flows which directly governed by the action of Cariolis force has drawn considerable attentions of large number of researchers owing to its overwhelming and important applications in various areas of science and engineering such as meteorology, geophysics, astrophysics, aeronautics, oceanography and etc. Keeping in view the importance of this fact, Ahmed and Sinha [1] discussed analytically the mass transfer effect on unsteady MHD free convection flow of a viscous fluid past an accelerated vertical plate under influence of rotation and thermal radiation using Laplace transform technique. Govindarajan et al. [2] used perturbation technique to present the problem of unsteady MHD free convection flow of a viscous incompressible fluid past an infinite vertical porous medium in a rotating system with chemical reaction. Also, a detailed discussions on rotating viscous fluid for various reactions can be found in articles [3-5].

Instead of viscous fluid or Newtonian fluid, there are other researchers who studied the rotation effect in the non-Newtonian fluid including Olajuwon and Oahimire [6] who examined the influences of thermo diffusion on unsteady MHD free convection flow of non-newtonian micropolar fluid past a vertical porous plate in a rotating frame of reference with thermal radiation and suction. The problem are solved analytically by perturbation technique. Comprehensive review of radiation absorption and Hall current effects on unsteady free convection flow of micropolar fluid past a semi infinite vertical porous plate in a rotating system with uniform transverse magnetic field and chemical reaction are
well presented by Satya Narayana et al. [7] using regular perturbation method. On the other hand, Ismail et al. [8] proposed the analytic solutions for the unsteady MHD free convection heat and mass transfer flow of rotating second grade fluid in a porous medium past an infinite inclined plate in the presence of heat absorption and ramped wall temperature using Laplace transform method. Veerakrishna and Swarnalathamma [9] applied the perturbation technique to analyze the interaction of hall current, heat source/sink and chemical reaction effects on unsteady MHD free convection flow of second grade fluid through a porous medium over an infinite vertical plate fluctuating in a rotating system.

Mathematical model for unsteady MHD free convection flow of non-Newtonian casson fluid past a semi-infinite moving vertical plate in a rotating system with convective boundary conditions has been constructed by Sugunamma et al. [10]. This problem is generated by the effects of hall current and constant heat source, and hence solved by using perturbation technique. Using the same methodology as in [10], Pushpalatha et al. [11] obtained an analytical solutions for radiative heat transfer in unsteady MHD free convection flow of rotating casson fluid with convective boundary conditions and chemical reaction. Some important studies related to rotation effect on unsteady free convection flow of non-Newtonian fluids are cited in references [12-16]. In view of the above analysis and up to the best author’s knowledge, no attempt has been made to investigate the influence of rotation on unsteady MHD free convection flow of non-Newtonian Jeffrey fluid through a porous medium past a vertical plate with ramped wall temperature. The effects of magnetic field and thermal radiation are also considered in the present problem. Dimensionless variables are employed to the governing equations and solved analytically by Laplace transform technique. The expression of velocity profile are presented graphically and discussed in details for several pertinent parameters.

2. Mathematical formulation

Consider the unsteady MHD free convection flow of rotating Jeffrey fluid past a vertical plate situated in the \((x,y)\) plane of Cartesian coordinate system \(x, y\) and \(z\). A uniform transverse magnetic field of strength \(B_0\) are applied parallel to the axis of rotation \((0,0,B_0)\), where the induced magnetic and external electric fields are neglected under the small magnetic Reynolds number assumption. Both fluid and plate are in a state solid body rotation with a constant angular velocity, \(\Omega\) about the \(z-\)axis.

Initially, at time \(t \leq 0\) both plate and fluid are at stationary condition with constant temperature, \(T_\infty\). Subsequently, at time \(t > 0\), the temperature of the plate is raised or lowered to \(T_\infty + (T_u - T_\infty)\frac{t}{t_0}\) when \(t \leq t_0\). Thereafter, at \(t > t_0\) it is maintained at uniform temperature, \(T_u\).

Under the foregoing assumptions and usual Boussinesq approximation, the momentum and energy equations for unsteady MHD free convection flow of Jeffrey fluid through a porous medium in a rotating system are governed by

\[
\frac{\partial F}{\partial t} + 2\Omega F = \frac{\nu}{1 + \lambda_2} \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) \frac{\partial^2 F}{\partial z^2} - \frac{\sigma B_0^2 F}{\rho} - \frac{\nu \Phi}{\kappa (1 + \lambda_1)} \left(1 + \lambda_2 \frac{\partial}{\partial t}\right) F + g \beta_e (T - T_\infty),
\]

\[
\rho c_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial z^2} - \frac{\partial q_r}{\partial z},
\]

with initial and boundary conditions

\[
F(z,0) = 0, \quad T(z,0) = T_\infty; \quad z \geq 0,
\]

\[
F(0,t) = 0; \quad T(0,t) = T_\infty + (T_u - T_\infty)\frac{t}{t_0}; \quad 0 < t < t_0, \quad T(0,t) = T_u; \quad t \geq t_0,
\]

\[
F(\infty,t) = 0, \quad T(\infty,0) = T_\infty; \quad t > 0,
\]
in which \( F = u + iv \) are complex velocity where \( u \) and \( v \) are real and imaginary part respectively, \( \nu \) is the kinematic viscosity, \( \lambda_{1,2} \) are the material parameter of Jeffrey fluid, where \( \lambda_1 \) is the ratio of relaxation to retardation times and \( \lambda_2 \) is the retardation time, \( \sigma \) is the electric conductivity, \( \rho \) is the constant density of the fluid, \( \phi \) is the porosity, \( \kappa \) is the permeability of the porous medium, \( g \) is the acceleration due to gravity, \( \beta_r \) is the volumetric coefficient of heat transfer, \( T \) is the fluid temperature, \( c_p \) is the specific heat capacity, \( k \) is the thermal conductivity and \( q_r \) is the radiation heat flux, respectively. Using Rosseland approximation, the radiation heat flux is simplified as

\[
q_r = -\frac{4\sigma^* \partial T^4}{3k_1},
\]

where \( \sigma^* \) is the Stefan-Boltzman constant and \( k_1 \) is the absorption coefficient. Assuming that, the temperature differences within the flow are sufficiently small, such that \( T^4 \) is linearized by expanding in Taylor series about \( T_w \). Hence, neglecting the higher order terms gives

\[
T^4 \approx 4T_w^2 T - 3T_w^2.
\]

Making use of equations (4) and (5), then equation (2) reduces to

\[
\rho c_p \frac{\partial T}{\partial t} = k \left( 1 + \frac{16\sigma^* T_w^2}{3k_1} \right) \frac{\partial^2 T}{\partial z^2}.
\]

The following dimensionless variables are introduced

\[
F^* = \frac{F}{U_0}, \quad z^* = \frac{zU_0}{\nu}, \quad t^* = \frac{tU_0^2}{\nu}, \quad \theta = \frac{T - T_w}{T_w - T_\infty}.
\]

The system of equations (1)-(3) are reduced to (* notations are dropped for simplicity)

\[
\frac{\partial F}{\partial t} + 2irF = \frac{1}{1 + \lambda_1} \left( 1 + \lambda_2 \frac{\partial}{\partial t} \right) \frac{\partial^2 F}{\partial z^2} - HaF - \frac{1}{K(1 + \lambda_1)} \left( 1 + \lambda_2 \frac{\partial}{\partial t} \right) F + Gr\theta,
\]

\[
Pr \frac{\partial \theta}{\partial t} = (1 + Rd) \frac{\partial^2 \theta}{\partial z^2},
\]

\[
F(z,0) = 0, \quad \theta(z,0) = 0; \quad z \geq 0,
\]

\[
F(0,t) = 0, \quad \theta(0,t) = t; \quad 0 < t \leq 1, \quad \theta(0,t) = 1; \quad t > 1,
\]

\[
F(\infty,0) = 0, \quad \theta(\infty,t) = 0; \quad t > 0,
\]

Here \( r \) is the rotation parameter, \( \lambda \) is the Jeffrey fluid parameter, \( Ha \) is the Hartmann number, \( K \) is the permeability parameter, \( Gr \) is the Grashof number, \( Pr \) is the Prandtl number and \( Rd \) is the radiation parameter which are defined as follows
3. Solution of the problem

In order to solve the above system of partial differential equations (8)-(9), the Laplace transform technique is used. Thus, by applying the Laplace transform into equations (8)-(10), and using the initial conditions, yield the following transformed solutions in \((z,q)\)-plane

\[
\mathcal{F}(z, q) = \frac{d_4}{(q + d_5)^2 - d_6^2} \left(1 - e^{-q}\right) \left( e^{-\sqrt{d_5^2 + q^2}} - e^{-\sqrt{d_6^2 + q^2}} \right),
\]

\[
\mathcal{\bar{\theta}}(z, q) = \left(1 - e^{-q}\right) \frac{1}{q} e^{-\sqrt{d_6^2}},
\]

where

\[
d_4 = \frac{Gr}{\gamma_1}, \quad d_5 = \frac{a_1 \gamma_2 - \gamma_1}{2 a_4 \gamma_3}, \quad d_6 = \sqrt{\frac{(a_1 \gamma_2 - \gamma_1)^2 + 4 a_4 \gamma_3}{2 a_4 \gamma_3}}, \quad \gamma_1 = 1 + \frac{\lambda}{K (1 + \lambda)};
\]

\[
\gamma_2 = \frac{1}{1 + \lambda_1}, \quad \gamma_3 = \frac{\lambda}{1 + \lambda_1}, \quad \gamma_4 = Ha + \frac{1}{K (1 + \lambda_1)} + 2 ir, \quad a_i = \frac{Pr}{1 + Rd}.
\]

By taking the inverse Laplace transform of equations (12) and (13), the expressions of velocity and temperature distributions can be written as

\[
F(z,t) = F_R(z,t) - F_R(z,t-1)H(t-1),
\]

\[
\theta(z,t) = \theta_R(z,t) - \theta_R(z,t-1)H(t-1),
\]

where

\[
F_R(z,t) = \frac{d_4}{d_6^2} \int_0^t e^{d_6^2(t-\ell)} \sinh\left[d_6(t-\ell)\right] d\ell
\]

\[
+ \frac{d_4}{2d_6} \sqrt{\frac{2}{\pi}} \int_0^{\frac{d_6^2}{4}} \int_0^{d_6^2(t-s)} e^{-\frac{d_6^2}{4}(t-s)} \sinh\left[d_6(t-\ell)\right] I_1(2\sqrt{d_6^2 \mu s}) du ds d\ell
\]

\[
- \frac{d_4}{d_6} \int_0^t \left( \frac{a_i z^2}{2} + \ell \right) \text{erfc}\left(\frac{z}{2\sqrt{\ell}}\right) e^{d_6^2(t-\ell)} \sinh\left[d_6(t-\ell)\right] d\ell
\]

\[
+ \frac{d_4}{d_6} \sqrt{\frac{2}{\pi}} \int_0^{d_6^2(t-z^2)} e^{d_6^2(t-\ell)} \sinh\left[d_6(t-\ell)\right] d\ell,
\]
\[ \theta_s(z,t) = \left( \frac{a_z z^2}{2} + t \right) \text{erfc} \left( \frac{z}{2 \sqrt{\frac{a_z}{t}}} \right) - z \sqrt{\frac{a_z}{\pi t}} e^{-\frac{z^2}{4t}}. \]  

(18)

\[ \xi_4 = \frac{\gamma_2}{\gamma_3}, \quad \xi_5 = \frac{\gamma_4}{\gamma_3}, \quad \xi_6 = \frac{\gamma_4^2 - \gamma_4}{\gamma_3}. \]  

(19)

4. Solution for an isothermal plate

In order to highlight the effect of ramped wall temperature on the fluid flow, it is worth to compare the obtained results of equations (15) and (16) with an isothermal plate case or uniform temperature. Thus, the solution of velocity and temperature profiles for an isothermal plate are given by

\[ F(z,t) = \frac{d_4}{d_6} e^{d_4(t-\ell)} \int e^{d_6(t-\ell)} \sinh \left[ d_6(t-\ell) \right] d\ell \]

\[ + \frac{d_4}{2d_6} \frac{z}{\pi} \int e^{d_4(t-\ell)} \int e^{d_6(t-\ell)} \frac{z}{4\ell} \sinh \left[ d_6(t-\ell) \right] I_1 \left( 2\sqrt{\xi_3 \xi_5} \right) dudsd\ell \]

\[ - \frac{d_4}{d_6} e^{d_4(t-\ell)} \text{erfc} \left( \frac{z}{2 \sqrt{\frac{a_z}{t}}} \right) d\ell, \]

\[ \theta(z,t) = \text{erfc} \left( \frac{z}{2 \sqrt{\frac{a_z}{t}}} \right). \]  

(20)

5. Results and discussions

This section is prepared to study the influence of several pertinent parameters on velocity field components which displayed graphically in figures 1-4. The panels (a) and (b) in each plot are indicates the variations of real and imaginary part of velocity respectively. Figures 1a and 1b portrayed the effect of \( \lambda \) on velocity profiles. It is viewed that, an increase of \( \lambda \) tends to decelerate the fluid flow for both velocities components. Physically, this is because of \( \lambda \) is a viscoelastic parameter where viscoelasticity is a property of materials combination of both elastic and viscous behaviour when undergoing deformation. Hence, the increases of viscosity and elasticity will always resist the fluid velocity.

The influence of \( r \) on velocity components is elucidated from figures 2a and 2b. Clearly, the rotation retard the fluid flow in primary flow direction in figure 2a and the opposite behaviour is noted in the secondary flow direction shown in figure 2b. The velocity profile in secondary flow direction accelerates as \( r \) increases in the region near the plate, and later reduces in the region away from the plate due to the fact that Coriolis force is dominant in the region near to the axis of rotation. Also, it is worth to mention that, the Coriolis force acts as a constraints in the fluid flow in the primary flow direction which induces the secondary fluid velocity in the flow field.

Finally, the graphical result for \( Ha \) are presented in figures 3a and 3b respectively. These figures indicates that an increase in the values of \( Ha \) tends to reduce the velocity components monotonically. This situation is due to the Lorentz force which similar to the drag force acts as an agent to retard the fluid flow. Therefore, when \( Ha \) increases, then the Lorentz force is technically increased and more resistance is given to the motion of the fluid which lead to decelerate the primary and secondary velocities.
Figure 1a. Primary velocity profile for different values of $\lambda$ when $r = 0.5$, $K = 1$, $\lambda_0 = 2$, $Ha = 1$, $Gr = 1$, $Pr = 0.71$, $Rd = 1$ and $t = 0.5$.

Figure 1b. Secondary velocity profile for different values of $\lambda$ when $r = 0.5$, $K = 1$, $\lambda_0 = 2$, $Ha = 1$, $Gr = 1$, $Pr = 0.71$, $Rd = 1$ and $t = 0.5$.

Figure 2a. Primary velocity profile for different values of $r$ when $K = 1$, $\lambda_0 = \lambda = 1$, $Ha = 1$, $Gr = 1$, $Pr = 0.71$, $Rd = 1$ and $t = 0.5$.

Figure 2b. Secondary velocity profile for different values of $r$ when $K = 1$, $\lambda_0 = \lambda = 1$, $Ha = 1$, $Gr = 1$, $Pr = 0.71$, $Rd = 1$ and $t = 0.5$.

Figure 3a. Primary velocity profile for different values of $Ha$ when $r = 0.5$, $K = 1$, $\lambda_0 = \lambda = 1$, $Gr = 1$, $Pr = 0.71$, $Rd = 1$ and $t = 0.5$.

Figure 3b. Secondary velocity profile for different values of when $r = 0.5$, $K = 1$, $\lambda_0 = \lambda = 1$, $Gr = 1$, $Pr = 0.71$, $Rd = 1$ and $t = 0.5$. 
6. Conclusion
The unsteady MHD free convection flow of rotating Jeffrey fluid embedded in a porous medium with ramped wall temperature are studied analytically by Laplace transform technique. The expression of velocity profile are plotted graphically and discussed for several embedded parameters. It is observed that, the velocity profile decreases with an increase in $\lambda$ for both components. Meanwhile, increasing $r$ reduced the primary velocity, but enhanced the fluid motion for secondary velocity. As expected, a rise in $Ha$ tends to decelerate the fluid flow for both components due to the Lorentz force effect. Lastly, it is noticed that, the boundary layer thickness for ramped wall temperature is always less than isothermal plate.

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