Analytical study of heat and mass transfer in MHD flow of chemically reactive and thermally radiative Casson nanofluid over an inclined stretching cylinder

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Abstract

The main purpose of this study is to give a mathematical analysis of heat and mass transfer in a boundary layer flow of Casson fluid over an inclined stretching cylinder in the presence of magnetic nanoparticles. The effects of Casson parameter, curvature of the cylinder, angle of inclination, Buoyancy force, external magnetic field, thermal radiation, Joule heating, viscous dissipation, heat source and chemical reaction are taken into account. Appropriate transformations are incorporated to convert the governing partial differential equations and the boundary conditions suitable for computation. The elegant optimal homotopy analysis method is used to obtain analytic approximations for the resulting system of nonlinear differential equations. The features of flow characteristics such as velocity, temperature and concentration profiles in response to the variations of the emerging parameters are simulated and examined in detail. Extensive analysis is also made to explore the influences of relevant constraints on the rates of momentum, heat and mass transfer near the surface of the cylinder. Among the many outputs of the study, it is found that increasing the non-Newtonian Casson parameter can slowdown the flow velocity and enhance the temperature and concentration profiles. It is also revealed that significant enhancement of wall friction and mass transfer rate can be achieved by increasing the curvature of the cylinder. Further, the analytic approximations obtained by implementing the optimal homotopy analysis method to the present model are in close agreements with previous studies under common assumptions.

Nomenclature

| Symbol | Description |
|--------|-------------|
| Bi     | Biot Number |
| B_o    | Magnetic field strength (NmA$^{-1}$) |
| C      | Concentration in boundary layer |
| C_f    | Skin friction coefficient |
| c_p   | Heat capacity (Jkg$^{-1}$K$^{-1}$) |
| C_w    | Wall nanoparticle concentration |
| C_∞    | Concentration in ambient flow |
| D_B    | Brownian diffusion coefficient (m$^2$s$^{-1}$) |
| D_T    | Thermophoretic diffusion coefficient |
| E_c    | Eckert number |
| e_i    | Deformation rate |
| f      | Dimensionless stream function |
Gravitational acceleration $g$

Mass Grashof number $Gm$

Thermal Grashof number $Gr$

Hartman number $Ha$

Heat transfer coefficient $h_f$

Curvature of cylinder $K$

Rate of chemical reaction $K_r$

Thermal absorption coefficient $K^*$

Brownian motion parameter $Nb$

Thermophoresis parameter $Nt$

Nusselt number $Nu_z$

Prandtl number $Pr$

Heat generation/absorption parameter $Q$

Coefficient of heat source $Q_0$

Mass flux of the nanofluid (Kgm$^2$s) $q_m$

Surface heat flux (Wm$^{-2}$) $q_w$

Radius of cylinder $R$

Radiation parameter $Rd$

Local Reynolds number $Re_z$

Sherwood number $Sh_z$

Schmidt number $Sc$

Temperature (K) $T$

Temperature of heated fluid (K) $T_f$

Ambient fluid temperature (K) $T_\infty$

Reference length $L$

Constant velocity (ms$^{-1}$) $U_0$

Velocity of cylindrical surface (ms$^{-1}$) $U_w$

Velocity components (ms$^{-1}$) $(u, w)$

Angle of inclination from horizontal $\alpha$

Thermal diffusivity of the nanofluid $\alpha_f$

Casson parameter $\beta$

Concentration expansion coefficient $\beta_c$

Thermal expansion coefficient $\beta_T$

Chemical reaction parameter $\gamma$

Residual error $\varepsilon$

Similarity variable $\eta$

Dimensionless temperature $\theta$

Thermal conductivity (Wm$^{-1}$K$^{-1}$) $\kappa$

Plastic dynamic viscosity $\mu_B$

Critical value of deformation rate $\pi_c$

Density of nanofluid (Kgm$^{-3}$) $\rho$

Heat capacity of the fluid $(\rho c)_f$

Heat capacity of the nanoparticles $(\rho c)_p$

Electric conductivity $\sigma$

Stefan-Boltzmann constant $\sigma^*$

Ratio of heat capacities $\tau$
1. Introduction

Heat and mass transfer in boundary layer flow of fluids are theoretically and practically useful physical phenomena that occur in nature and industry. In manufacturing industries, the rate of heat and mass transfer has significant effects on cost of production and quality of products. In particular, effective heating and/or cooling are some of the top technical challenges facing many high-tech industries and technological devices. In 1993, Masuda et al. [1] notified that saturation of ultra-fine particles in conventional fluids remarkably increase the thermal conductivity of the fluid. Then this idea was further extended in 1995 by Choi [2] in which he emphasized that the thermal capabilities of fluids can be enhanced by dispersion of small amount of nanometer-sized particles (1–100 nm), called nanoparticles in a large amount of hosting fluid. The resulting fluid with uniform and stable suspension of nanoparticles is called a nanofluid. According to Eastman et al. [3], a fluid with nanoparticle mixture can have a thermal conductivity of about 40% greater than the corresponding fluid without nanoparticles. In general, nanofluids possess outstanding thermal, mechanical, electrical, optical and magnetic properties that have fascinated a number of researchers. In 2006, a comprehensive study for the reason behind the extraordinary thermal conductivity of nanofluids was reported by Buongiorno [4]. He concluded that Brownian diffusion and thermophoresis are important factors contributing for the enhancement of thermal conductivity in nanofluids. Further, he proposed the Buongiorno’s model for the conservation of momentum, heat and concentration of nanoparticles in nanofluid flows. Since then, the nanofluids become the working fluids in a number of industrial and biomedical applications such as microelectronics, fuel cells, spray coating, pharmaceutical procedures and many other advanced cooling and thermal management systems. Some of the recent studies on nanofluid flow characteristics are presented in [5–8].

In fluid dynamics, the study of flows over stretching surfaces play decisive roles in manufacturing industries, power generation, metallurgy and transportation. In 1961, Sakiadis [9] initiated the study of boundary layer flow over a stretched surface and he formulated the boundary layer equations for two dimensional and axisymmetric flows. Later, Crane [10] presented an exact solution of a two-dimensional Navier–Stokes equations for a steady boundary layer flow due to stretching surface. Wang [11] extended the two-dimensional stretching sheet problem to a three-dimensional setting and he obtained exact similarity solutions of the Navier–Stokes equations. The boundary layer flow of nanofluid over stretched surface was investigated by Khan and Pop [12]. Following these pioneer works, a number of researchers have reported their studies on boundary layer flow whose results agree well with the experimental observations [13]. Recently, Kumar et al. [14] studied melting heat transfer of a nanofluid over a stretching sheet embedded in a porous medium while Misra and Kamatam [15] focused on nanofluid flow over a nonlinear stretching sheet. Alizadeh et al. [16] and Giri et al. [17] reported their studies on boundary layer flows over cylindrical surfaces.

On the other hand, most of the fluids processed in manufacturing industries are non-Newtonian; that is, the relation between the shear stress and the rate of deformation is not linear. The highly diversified nature of such fluids provide interesting challenges to scientists and engineers. Consequently, several constitutive equations have been proposed to specify the rheological properties of such fluids. Review of non-Newtonian fluids and their constitutive models are briefly presented in [18, 19]. Recently, the boundary layer flow of Maxwell nanofluid over a stretching cylinder is studied by Islam et al. [20]. They analyzed the thermal effect for a mixed convection flow of Maxwell nanofluid spinning motion produced by rotating and bidirectional stretching cylinder. The effect of viscous dissipation on Williamson nanofluid over stretching/shrinking wedge is examined by Ibrahim and Negera [21]. In 1959, Casson [22] introduced another non-Newtonian fluid model to simulate the flow behavior of pigment-oil in the formation of silicon suspensions and printing inks. He also examined the validity of the model in characterizing blood flow properties subjected to low shear rates. Nowadays, the Casson fluid model becomes one of the most useful models to give more precise characterization and accurate predictions for the flow of human blood in narrow blood vessels under low shear rate. It also helps to analyze the flow of many other biological and industrial fluids including chocolate melts, concentrated fruit

\[ \tau_w \quad \text{Wall shear stress (Pa)} \]
\[ \tau_0 \quad \text{Yield stress} \]
\[ \tau_w \quad \text{Wall shear stress} \]
\[ \nu \quad \text{Kinematic viscosity} \]
\[ \varphi \quad \text{Dimensionless concentration function} \]
\[ \phi \quad \text{Homotopy approximation} \]
\[ \psi \quad \text{Stokes stream function} \]
\[ h \quad \text{Convergence-control parameter} \]
juices, liquid detergents, lubricants, paper pulp, paints, gypsum pastes, cosmetics and many other shear thinning liquids and viscoelastic materials. The Casson fluid model is also utilized by petroleum engineers to examine the characteristics of cement slurry and to forecast high shear-rate viscosities. The boundary layer flow of Casson nanofluid over a vertical stretching cylinder was analyzed by Malik et al [23]. The study reported that increasing the value of the Casson parameter leads to a decreasing velocity profile. The convective flow of Casson fluid over a horizontal cylinder was reported by Tamoor et al [24]. They revealed that velocity and thermal fields have reverse behavior for larger magnetic field parameter. Recently, Murthy et al [25] considered boundary layer flow of Casson fluid over a horizontal cylinder. The study revealed that the velocity profile and the corresponding boundary-layer thickness decay for increasing values of the Casson parameter. The study also emphasized that the coefficient of skin friction increases with stronger magnetic field effects and the local Nusselt number is enhanced for increasing values of the curvature parameter. Further studies on Casson nanofluid flow phenomena are reported by Shah et al [26] and Khan et al [27].

The above survey of related literature reflects that even though plenty of investigations have been reported and considerable progresses have been achieved in the area under consideration, still the field remains an active area of research with a number of study gaps. To the best of the authors’ knowledge, no analytical study of magnetohydrodynamic flow of chemically reactive and thermally radiative Casson nanofluid due to an inclined stretching cylinder is reported in open literature. Thus, the main goal of the present study is to give a more comprehensive mathematical model describing the boundary layer flow phenomena of Casson nanofluid generated by an inclined stretching cylinder. On the other hand, a relatively recent and a more reliable method, namely the optimal homotopy analysis method, is applied to obtain analytic approximations for the solution of the nonlinear differential equations. The computational tasks due to this method are well accomplished and highly facilitated by using the BVPh2.0 package implemented in Mathematica environment. Verification on the convergence of the method is indicated by plotting both the h-curves and the graph of average squared residual error. Further validation of the results is made by comparative analysis with previous studies in the absence of the additional assumptions. Detailed investigation is also carried out to explore the variations of the boundary layer flow profiles in response to different thermo-physical effects. The influences of the governing parameters on the quantities of engineering interest are also studied in terms of the local wall friction coefficient, Nusselt number and Sherwood number respectively. The results have been illustrated in graphical or tabular forms followed by brief discussion and physical interpretations.

The findings of this study are believed to contribute in the efforts made by the scientific community to give more realistic descriptions and more accurate predictions to such flow phenomena. In particular, the results of this study are expected to have potential use in fiber technology, wire drawing, material insulation, crude oil refinement, metallic coating, advanced cooling, polymer processing, crude oil extraction, food processing, waste disposal or in many other industrial and engineering activities.
2. Model assumptions and mathematical formulations

Assume an inclined circular cylinder is placed in an incompressible viscous fluid in which both the cylinder and the fluid are initially at rest with the same temperature, say \( T_\infty \). Then the cylinder stretches promptly with a velocity \( U_0(z) = \frac{U_0}{z} \), where \( U_0 \) and \( L \) denote the constant reference values of velocity and length of the cylinder respectively.

At the instant the cylinder starts moving, the surface of the cylinder is also heated by a hot fluid which suddenly rises the temperature of the nano fluid and the unknown value of the nano fluid temperature in the thermal boundary layer, \( k_f \) and \( h_f \) are the coefficients of thermal conductivity and heat convection respectively. The cylinder is assumed to be sufficiently long and its end effects are neglected in the flow system. It is also anticipated that the fluid is electrically conducting, consisting of magnetic nanoparticles and the flow is considered to be steady, laminar and axisymmetric in the vicinity of the stretching cylinder. The boundary layer flow is exposed to an external magnetic field \( B = B_0 \) which is applied along the radial direction \( (r - \text{axis}) \) as indicated in figure 1.

The rheological equation for an incompressible Casson fluid as defined by Casson [22] is given by:

\[
\tau_{ij} = \begin{cases} 
2 \left( \mu_B + \frac{\rho}{2\gamma_f} \right) \sigma_{ij}, & \pi > \pi_c \\
2 \left( \mu_B + \frac{\rho}{2\gamma_f} \right) \varepsilon_{ij}, & \pi < \pi_c 
\end{cases}
\]

where \( \tau_{ij} \) is the shear stress, \( \pi_c \) is a finite value of shear stress (called the Casson yield stress), \( \mu_B \) is called the coefficient of plastic dynamic viscosity and \( \gamma = \varepsilon_{ij} \sigma_{ij} \) is the product of deformation rate \( \varepsilon_{ij} \) with itself, \( \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \) is the rate of strain tensor with \( u_i \) denoting the components of fluid velocity and \( \pi_c \) is the critical value of \( \pi \). Further, the kinematic viscosity \( v \) of the Casson fluid is given by

\[
v = \frac{\mu_B}{\rho} \left( 1 + \frac{1}{\beta} \right),
\]

where \( \rho \) is the fluid density and \( \beta = \frac{\mu_B}{\rho} \frac{1}{2\gamma_f} \) is called the Casson parameter, which is the apparent viscosity coefficient for the non-Newtonian Casson fluid. Also, for \( \pi > \pi_c \), the shear stress can be expressed as

\[
\tau_{ij} = \mu_B \left( 1 + \frac{1}{\beta} \right) (2\varepsilon_{ij}) = \mu_B \left( 1 + \frac{1}{\beta} \right) \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

Here, it is important to note that as the value of the Casson parameter is very large (\( \beta \to \infty \)), the yield stress becomes very small and the characteristic of the fluid can be approximated by the nature of a Newtonian fluid.

Now, with all the above assumptions and taking the impacts of Casson fluid parameter, curvature of the cylinder, angle of inclination, buoyancy force, external magnetic field, thermal radiation, viscous dissipation, Joule heating, heat source and chemical reaction into account, the governing equations for the conservation of mass, linear momentum, thermal energy and nanoparticle concentration can be extended from Mahdy [28] and Murthy et al [25].
represents the angle of inclination of the cylinder measured from the horizontal; $\beta_c$ and $\beta_T$ are coefficients of volumetric concentration and thermal expansion respectively. Further, the non-Newtonian Casson fluid parameter $(\beta)$; thermal diffusivity ($\alpha_f$), the ratio ($\tau$) of heat capacities of nanoparticle and the base fluid are defined respectively as $\alpha_f = \frac{k_f}{\rho_f c_p}$, and $\tau = \frac{(\rho_p c_p)}{(\rho_f c_p)}$ where $c_p$ is the heat capacity at constant pressure.

The following relevant boundary conditions at the surface of the cylinder and in the ambient region are considered:

\[ A t\quad r = R, \quad u = \frac{U_0 z}{L}, \quad w = 0, \quad -k_f \frac{\partial T}{\partial r} = \frac{h_f(T_f - T)}{r} \quad \text{and} \quad C = C_w \quad (8) \]
\[ As\quad r \to \infty, \quad u \to 0, \quad T \to T_\infty \quad \text{and} \quad C \to C_\infty \quad (9) \]

where $R$ is the radius of the cylinder, $C_w$ is the constant value of nanoparticle concentration at the surface of the cylinder.

In order to further simplify the flow model in equations (4)–(9), the following dimensionless radial variable is introduced:

\[ \eta = \sqrt{\frac{L}{U_0}} \left( \frac{r^2 - R^2}{2R} \right) \quad (10) \]

Let the Stokes stream function $\psi$ be defined by

\[ \psi(r, z) = \sqrt{\frac{U_0}{L}} z R f(\eta). \quad (11) \]

Then using the Cauchy-Riemann equations, $\psi$ is related to the velocity components as:

\[ u = \frac{1}{r} \frac{\partial \psi}{\partial r} \quad \text{and} \quad w = -\frac{1}{r} \frac{\partial \psi}{\partial z}. \quad (12) \]

This gives

\[ u = \frac{U_0 z}{L} f'(\eta) \quad (13) \]

and

\[ w = -\frac{R}{r} \sqrt{\frac{U_0}{L}} f(\eta). \quad (14) \]

where the function $f$ and the derivative $f'$ represent the dimensionless stream function and velocity profile respectively.

We also need

\[ \theta(\eta) = \frac{T - T_\infty}{T_f - T_\infty} \quad (15) \]

and

\[ \varphi(\eta) = \frac{C - C_\infty}{C_w - C_\infty}. \quad (16) \]

Computing the required partial derivatives of the dimensionless functions and upon substitution, it is straightforward to verify that the continuity equation in equation (4) is automatically satisfied. Similarly, the momentum conservation in equation (5) takes the form

\[ \left( 1 + \frac{1}{\beta} \right) [(1 + 2K\eta)f'' + 2Kf''] + f' - f'^2 - Ha^2 f' + [G, \theta + G_m \varphi] \cos(\alpha) = 0, \quad (17) \]

where the prime $'$ indicates differentiation with respect to $\eta$, $K = \sqrt{\frac{L}{U_0}}$ is the parameter for curvature of the cylinder, $Ha = \sqrt{\frac{g \rho_c L}{\eta U_0}}$ is the magnetic field parameter, $G_r = \frac{g^2(T_f - T_\infty) L}{\rho c_p}$ and $G_m = \frac{g^2(C_w - C_\infty) L^2}{\rho c_p}$ are respectively the thermal and mass Grashof numbers for buoyancy effect.

The energy equation in equation (6) is also re-written as

\[ \frac{1}{Pr} \left[ 1 + \frac{4}{3} \frac{Rd}{Pr} \right] [(1 + 2K\eta) \theta'' + 2K \theta'] + \theta' + (1 + 2K\eta)(N_0 \theta' \varphi' + N_i \theta'^2) \]
\[ + \left( 1 + \frac{1}{\beta} \right) (1 + 2K\eta) Ec \varphi'' - Ha^2 Ec \varphi'^2 + Q \theta = 0, \quad (18) \]
where the prime indicates differentiation with respect to \( \eta \); \( R_p = \frac{\nu}{\alpha} \) is the Prandtl number; \( Rd = \frac{4 \nu^2 T_{\infty}}{3 k^2} \) is the thermal radiation parameter; \( N_b = \frac{2 \nu_0 (C_w - C_{\infty})}{c} \) and \( N_t = \frac{2 R_t (T_1 - T_{\infty})}{\nu^2 T_{\infty}} \) correspond to the Brownian motion and the thermophoresis parameters respectively; \( Ec = \frac{U_o}{(\nu_0 k (T_1 - T_{\infty}))} \) is the Eckert number representing dissipation effects; \( Q = \frac{2 \nu_0}{(\nu_0 k (T_1 - T_{\infty}))} \) is the heat generation \((Q > 0)\) or absorption \((Q < 0)\) parameter.

The nanoparticle volume fraction given in equation (7) can be written as

\[
(1 + 2K\eta)\varphi'' + 2K\varphi' + \frac{N_t}{N_b}[1 + 2K\eta]\theta'' + 2K\theta' + Sc(f \varphi' - \gamma \varphi) = 0, \tag{19}
\]

where \( Sc = \frac{\nu}{\nu_0} \) is the Schmid number and \( \gamma = \frac{\nu \nu_0}{\nu_0 k} \) is the chemical reaction parameter with \( \gamma > 0 \) and \( \gamma < 0 \) representing the destructive and generative chemical reaction rates respectively.

The boundary conditions can also be reduced as follows:

\[
f(\eta) = 0, \quad f'(\eta) = 1, \quad \theta'(\eta) = -Bi[1 - \theta(\eta)], \quad \varphi(\eta) = 1 \quad \text{at} \quad \eta = 0 \tag{20}
\]

\[
f'(\eta) = 0, \quad \theta(\eta) = 0, \quad \varphi(\eta) = 0 \quad \text{as} \quad \eta \to \infty, \tag{21}
\]

where \( Bi = \frac{\nu_0}{k U_0} \) denotes the Biot number for convective heat transfer.

Also, in many practical applications, it is essential to get insight on the feature of the three most important physical quantities of interest, namely the rates of momentum, heat and mass transfer. This can be done by analyzing the skin friction coefficient \( C_f \), local Nusselt number \( Nu_t \), and Sherwood number \( Sh_z \) defined as follows:

\[
C_f = \tau_w \rho_f U_w^2, \quad Nu_t = \frac{2q_w}{\kappa (T_f - T_{\infty})} \quad \text{and} \quad Sh_z = \frac{2q_m}{D_B (C_w - C_{\infty})} \tag{22}
\]

where

\[
\tau_w = \left( \frac{\nu_0}{\sqrt{2\pi}} \right) \frac{\partial q_w}{\partial R}_{R=R} \quad \text{and} \quad q_w = -\left( \kappa + \frac{16\nu^2 T_{\infty}}{3(\rho C_p) k^4} \right) \frac{\partial T}{\partial R}_{R=R} \quad \text{and} \quad q_m = -D_B \left( \frac{\partial C}{\partial R} \right)_{R=R} \tag{23}
\]

are respectively the shear stress, heat flux and mass flux quantities at the surface of the stretching cylinder. By substituting the expressions of equation (23) into equation (22) and simplifying gives the following expressions:

\[
C_f = \left( 1 + \frac{1}{\beta} \right) Re_{\infty}^{-1/2} f''(0), \quad Nu_t = -Re_{\infty}^{1/2} \left( 1 + \frac{4}{3} Re \right) \theta'(0), \quad \text{and} \quad Sh_z = -Re_{\infty}^{1/2} \varphi'(0) \tag{24}
\]

where \( Re_{\infty} = \frac{U_0 U_0}{\nu} \) is the local Reynolds number, \( f''(0), \theta'(0) \) and \( \varphi'(0) \) are the boundary derivatives for the gradients of the dimensionless velocity, temperature and concentration respectively.

### 3. Method of solution

The strongly nonlinear and coupled model equations given in equations (17)–(19) along with the boundary conditions (20)–(21) are solved by the robust optimal homotopy analysis method. The homotopy analysis method (HAM) was first introduced in 1992 by Shijun Liao. Since HAM is independent of small/large parameters, it is more general method valid for both weakly and strongly nonlinear problems. Also, this method avoids discretization and other restrictive assumptions so that the method is easy for computation, free from rounding off errors and valid in the whole domain of the problem. Using the concept of homotopy in topology, HAM provides us extremely large freedom and flexibility to choose a better solution from the family of solution expressions. Further, by using the so-called convergence control parameter, we can adjust and control the region and rate of convergence of the series solution. In general, the homotopy analysis method combines the advantages of high accuracy of analytical methods and flexibility of numerical methods. Details of the method are available in [29]. Due to its efficiency, a number of researchers have implemented the method to solve nonlinear equations in science, engineering and finance. For instance, the recent studies of Khan et al [8], Tesfaye et al [30] and Gupta et al [31] revealed the successful application of the method in their flow analysis of certain non-Newtonian fluid models. Since hand calculation is not efficient and effective way to compute all the required solutions, we prefer to employ the HAM-based Mathematica package, BVPh2.0 contributed by Zhao and Liao [32].

In order to implement the homotopy analysis method in the present study, we first choose a set of basis functions in the form

\[
(1 + 2K\eta)\varphi'' + 2K\varphi' + \frac{N_t}{N_b}[1 + 2K\eta]\theta'' + 2K\theta' + Sc(f \varphi' - \gamma \varphi) = 0, \tag{19}
\]
where \( C_m, n \) are constant coefficients to be determined. Then the auxiliary linear operators, denoted by \( L_f, L_\theta \) and \( L_\varphi \), are also selected in such a way that the solutions of the homogeneous equations

\[
L_f[f(\eta)] = 0, \quad L_\theta[\vartheta(\eta)] = 0, \quad L_\varphi[\varphi(\eta)] = 0
\]

can be expressed as a linear combination of the base functions. So, we define the following auxiliary linear operators:

\[
L_f(f) = \frac{d^2f}{d\eta^2} + \frac{d^2f}{d\eta^2}, \quad L_\theta(\theta) = \frac{d^2\theta}{d\eta^2} + \frac{d\theta}{d\eta}, \quad L_\varphi(\varphi) = \frac{d^2\varphi}{d\eta^2} + \frac{d\varphi}{d\eta}
\]

satisfying the conditions \( L_f[C_1 + C_2\eta + C_3e^{-\eta}] = 0, \ L_\theta[C_4 + C_5e^{-\eta}] = 0, \) and \( L_\varphi[C_6 + C_7e^{-\eta}] = 0, \) where \( C_i (i = 1 - 7) \) are the subjective constants to be determined from the boundary conditions. The corresponding initial approximations \( f_0(\eta), \ \theta_0(\eta) \) and \( \varphi_0(\eta) \) are determined as follows:

\[
f_0(\eta) = 1 - e^{-\eta}, \quad \theta_0(\eta) = \frac{Bi}{1 + Bi}e^{-\eta}, \quad \varphi_0(\eta) = e^{-\eta}.
\]

The auxiliary functions can also be selected as

\[
H_f(\eta) = H_\theta(\eta) = H_\varphi(\eta) = e^{-\eta}.
\]

According to Liao [29], the convergence of the HAM solutions will also be ensured by proper selection of the so-called convergence-control parameters. Using the built in functions of the package along with the default values \( \beta = 5, \ \alpha = \frac{2}{7}, \ K = N_i = Rd = Q = 0.1, \ Pr = 2, \ Sc = 2.2, \)
The relevant squared residual errors, \( N_{b} \), some of these possible values of convergent-control parameters, the best values of the parameters can be any value from these intervals can be taken to get convergent solutions and of course by comparing the results at convergence control parameters are obtained:

\[ \text{Order of approx: } -f'(0), -\theta'(0), -\varphi'(0) \]

Table 1. Convergence of some HAM solutions over the order of approximations.

| Order of approx | \(-f'(0)\) | \(-\theta'(0)\) | \(-\varphi'(0)\) | \(\varepsilon_f\) | \(\varepsilon_{\theta}\) | \(\varepsilon_{\varphi}\) |
|-----------------|-------------|-----------------|-----------------|----------------|----------------|----------------|
| 2               | 1.2374      | 0.2659          | 1.1238          | 4.34 \times 10^{-6} | 7.83 \times 10^{-7} | 9.66 \times 10^{-7} |
| 6               | 1.2358      | 0.2497          | 1.1501          | 4.23 \times 10^{-7} | 1.84 \times 10^{-7} | 2.12 \times 10^{-7} |
| 10              | 1.2357      | 0.2474          | 1.1518          | 2.64 \times 10^{-7} | 8.66 \times 10^{-8} | 1.34 \times 10^{-7} |
| 14              | 1.2356      | 0.2470          | 1.1520          | 2.08 \times 10^{-7} | 5.53 \times 10^{-8} | 9.78 \times 10^{-8} |
| 18              | 1.2354      | 0.2467          | 1.1520          | 1.68 \times 10^{-7} | 4.27 \times 10^{-8} | 7.69 \times 10^{-8} |
| 22              | 1.2352      | 0.2464          | 1.1520          | 1.38 \times 10^{-7} | 3.55 \times 10^{-8} | 6.16 \times 10^{-8} |
| 26              | 1.2351      | 0.2462          | 1.1520          | 1.14 \times 10^{-7} | 3.04 \times 10^{-8} | 5.08 \times 10^{-8} |
| 30              | 1.2350      | 0.2461          | 1.1520          | 9.70 \times 10^{-8} | 2.63 \times 10^{-8} | 4.34 \times 10^{-7} |

Table 2. Comparisons showing computational values of \(-f'(0)\)
when \( Pr = 10, Sc = 11, N_{b} = 0.2, N_{t} = 0.1, B_{i} = 0.5, K = G_{c} = G_{r} = \alpha = B_{d} = E_{c} = Q = \gamma = 0 \), and \( \beta \to \infty \) against some values of \( H_{u} \).

| \( H_{u} \) | Fang et al [33] | Murthy et al [25] | Present Study |
|-------------|----------------|----------------|--------------|
| 0.0          | —              | 1.0000         | 1.0000       |
| 0.2          | —              | 1.0198         | 1.0197       |
| 0.5          | 1.1180         | 1.1180         | 1.1180       |
| 0.8          | —              | 1.2606         | 1.2605       |
| 1.0          | —              | 1.4142         | 1.4142       |

\( Nb = G_{r} = \gamma = 0.2, H_{u} = 1, G_{m} = 0.3, E_{c} = 0.01 \) and \( B_{i} = 0.5 \), the admissible values of the convergence-control parameters can be determined from the so-called \( h \)-curves. Thus, it can be seen from figure 2 that the \( h \)-curves are found to be nearly horizontal in the ranges

\[-1.4 < h_{f} < -0.1, \quad -1.8 < h_{\theta} < -0.2 \quad \text{and} \quad -1.5 < h_{\varphi} < -0.2.\]

That is, these intervals are the valid regions for which the HAM solutions converge to the exact solutions. So, any value from these intervals can be taken to get convergent solutions and of course by comparing the results at some of these possible values of convergent-control parameters, the best values of the parameters can be determined. Still, this approach is not technically effective. Thus, a more systematic way of finding the optimal values of the parameters involves minimizing the average squared residual errors, given by

\[ \varepsilon_{f}(h_{i}) \approx \frac{1}{N + 1} \sum_{j=0}^{N} k_{i} \left( \sum_{n=0}^{k} \phi_{n}^{(i)}(\eta_{n;j};\eta_{j};\eta_{j}) \right)^{2} \] (29)

The relevant squared residual errors, \( \varepsilon_{\theta}, \varepsilon_{\varphi} \) are shown in table 1. Also, the following optimal values for the convergence control parameters are obtained:

\[ h_{f} \approx -0.81, \quad h_{\theta} \approx -1.79 \quad \text{and} \quad h_{\varphi} \approx -1.15. \]

In table 1, evaluation of the boundary derivatives, \(-f'(0), -\theta'(0)\) and \(-\varphi'(0)\) and the corresponding squared residual errors are presented. It is shown that the values of \(-f'(0), -\theta'(0)\) and \(-\varphi'(0)\) converge at the 10th, 18th and 14th-order approximations respectively with error tolerance less than 0.0001. So it is reasonable to take the 18th order of HAM approximations for computing the remaining results of the study. Besides, it is clearly seen from the table that the individual squared residual errors are decreasing.

It is revealed in table 2 that the results of the present study are in a close agreement. It is clearly indicated in figure 3 that the total error is decaying very fast for increasing orders of the first few iterations. This guarantees convergence of the method for the flow problem under investigation.

The accuracy of the method is further ensured by comparing the results of the present study with respect to some previous reports under common considerations.

It is revealed that the results of the present study are in a close agreement with the findings of Fang et al [33] and Murthy et al [25]. The verification on convergence of the method and agreement of its results with some previous works give confidence and pave the way for computing and analyzing the remaining major tasks of the study.
Figure 4. Effects of $\beta$ on Velocity Profile.

Figure 5. Effects of $\beta$ on Temperature Distribution.

Figure 6. Effects of $\beta$ on Nanoparticle Volume Fraction.
4. Results and discussion

In this section, a mathematical analysis of heat and mass transfer in a boundary layer flow of an electrically conducting Casson nanofluid over an inclined cylinder is presented. An in depth investigation is made on the
behavior of the flow profiles as well as the physical quantities of practical interest against the variations of the emerging parameters. Analytic approximations for the solutions of the governing system of nonlinear differential equations are obtained in the frame of optimal homotopy analysis method. The BVPh 2.0 Mathematica package is used to generate the desired graphical outputs and tabular values. Efforts are also made to give brief discussion and interpretations for the corresponding outputs of the study.

Figures 4–6 are plotted to outline the influences of the Casson fluid parameter $\beta$ on dimensionless velocity $f'(\eta)$, temperature $\theta(\eta)$ and nanoparticle concentration $\varphi(\eta)$ profiles. It is observed in figures 4–6 that with the increment in Casson fluid parameter ($\beta$), the velocity $f'(\eta)$ declines while temperature $\theta(\eta)$ and concentration $\varphi(\eta)$ grow in the boundary layer region. This is due to the direct relationship between the Casson parameter and the plastic dynamic viscosity of the nano fluid. Physically speaking, the increasing trends of the Casson fluid parameter has a tendency to rise the viscosity and hence internal resistance of the fluid to motion. This in turn cause a retarded fluid velocity and enhanced temperature and concentration profiles.

The study on the effects of curvature parameter ($K$) on boundary layer flow profiles for both Newtonian ($\beta \to \infty$) and non-Newtonian ($\beta = 0.5$) Casson nanofluids are plotted in figures 7–9. Here, it is interesting to note from the relation, $K = \frac{\alpha}{\sqrt{\beta}}$ that the curvature parameter $K$ is inversely proportional to the radius $R$ of the cylinder. That is, if the radius of the cylinder is getting very big ($R \to \infty$), the curvature parameter becomes very small ($K \to 0$) which turns the shape of cylinder to the shape of a flat sheet. Conversely, larger values of $K$ corresponds to narrower cylindrical surfaces having very small diameter.

It is observed in figures 7 and 8 that the velocity ($f'(\eta)$) and concentration ($\varphi(\eta)$) profiles are increasing functions of the curvature parameter ($K$) in which the variation of ($\varphi(\eta)$) is more prominent in the middle of the
boundary layer region. This behavior will also happen physically since larger value of the curvature parameter corresponds to smaller surface area of the cylinder which reduces the resistance of the solid surface to the motion of fluid. It is also evident from figure 9 that for increasing values of $K$, the temperature ($\theta(\eta)$) distribution is increasing near the boundary surface and decreasing away from the boundary surface. This is because a cylinder with higher curvature has a smaller contact surface area which allows the transfer of less amount of heat from the solid surface to the nearby fluid. The results in figures 7–9 also revealed that though the variations of the respective profiles are almost similar for both the Newtonian and non-Newtonian cases, it is noticed that the non-Newtonian Casson nanofluid has larger boundary layer thicknesses than the corresponding Newtonian case.

The influences of angle of inclination ($\alpha$) on boundary layer flow profiles for both Newtonian ($\beta \to \infty$) and non-Newtonian ($\beta = 0.5$) Casson nanofluids are exhibited in figures 10–12. It is observed in figures 10–12 that the flow velocity is slowed down while temperature and concentration profiles are enhanced as the cylinder is moved from horizontal ($\alpha = 0$) to vertical ($\alpha = \frac{\pi}{2}$) orientation. Actually, this is physically meaningful as increasing angle of inclination is the cause for increasing the effect of gravity in the boundary layer flow system. Unlike the momentum boundary layer thickness, the thermal and concentration boundary layer thicknesses of non-Newtonian fluid are more prominent than the Newtonian cases.

Since the flow is subjected to an external magnetic field, it is appropriate to predict its impact on the boundary layer flow phenomena. Figures 13–15 are sketched to outline the influences of external magnetic field upon the boundary layer flow profiles for both Newtonian and non-Newtonian Casson nanofluids. The Hartman number $Ha$ is a quantity that is used to estimate the intensity of magnetic field strength.
Figure 14. Effects of $Ha$ on Temperature Profile.

Figure 15. Effects of $Ha$ on Concentration Profile.

Figure 16. Effects of $Rd$ on Velocity, Temperature and Concentration Profiles.
It is revealed in figures 13–15 that as the Hartman number increases, the velocity profile falls while temperature and concentration profiles grow. This is due to the fact that increasing the magnetic field strength causes the development of the so-called Lorentz force which acts against the direction of the flow field. Also, the additional work required to drag the nano fluid against the action of the magnetic field leads to the dissipation of kinetic energy as thermal energy and serves to agitate species diffusion. The variations of the flow profiles for Newtonian and non-Newtonian cases is almost similar except that the boundary layer thicknesses of the Newtonian fluid is relatively greater than that of the non-Newtonian case.

For flows that occur at a very high temperature, thermal radiation ($Rd$) has its own impact on boundary layer flow profiles. The impacts of $Rd$ in the boundary layer flow phenomena of the present study is analyzed and depicted in figure 16.

It is demonstrated in figure 16 that a considerable influence of $Rd$ is noticed for temperature profile. It is found that temperature profile is uplifted with increasing values of thermal radiation. The reason behind this observation is that the presence of thermal radiation is associated to enhanced heat supply to the nano fluid flow system. On the other hand, the effect of $Rd$ is relatively insignificant for velocity and concentration fields.

The work done against a viscous fluid is usually measured in terms of the dissipation parameter, called the Eckert number ($Ec$). The present analysis on the behavior of flow field profiles in response to the dissipation parameter $Ec$ is exhibited in figure 17.

It is noted that for larger values of the Eckert number, significant enhancement of temperature profile and the associated thermal boundary layer thickness are observed in the vicinity of the cylinder. This holds mainly...
because increasing the values of $Ec$ has a tendency to augment the friction between fluid elements and the collision between fluid particles which helps to store thermal energy and contributes for the rise in temperature distribution.

The impacts of heat source ($Q$) on the flow field profiles has also been investigated for the present flow problem as shown in figures 18 and 19.

It is shown in figure 18 that the temperature of the nanofluid is increasing drastically for increased values of the heat generation effect. This impact of the parameter is very small for velocity and concentration profiles. In the case of heat sink ($Q < 0$), a significant reduction of temperature profile is noticed whereas the effect of heat sink on velocity and concentration profiles is insignificant as depicted in figure 19. This observation is in agreement with the fact that heat generation and heat sink play the roles of adding and removing heat in the flow system respectively.

The convective boundary condition for heat transfer can be estimated by using a dimensionless quantity, called the Biot number $\left( Bi = \frac{h_s}{\kappa U_f} \right)$, which is the ratio of heat convection to the heat conduction between the the solid boundary and the surrounding fluid. Here, $h_s$ and $\kappa$ stand for the coefficient of heat transfer and thermal conductivity of the nanofluid respectively. The variations of the flow profiles in response to the Biot number are depicted in figure 20.

It is shown that a remarkable enhancement of temperature profile is observed for increasing values of $Bi$ whereas the concentration and velocity profiles remain the same with the changes of $Bi$. This holds from the relation that $Bi$ is directly proportional to the intensity of convective heating. That is, the increase in $Bi$ causes the
transfer of more heat from the heated cylinder to the boundary layer fluid and thereby enhancing the temperature distribution.

The variations of flow profiles in response to both destructive ($\gamma > 0$) and constructive ($\gamma < 0$) chemical reaction effects are analyzed and presented in figures 21 and 22. The impacts of chemical reaction is found more prominent in changing the concentration profile than the velocity and temperature profiles.

It is indicated in figure 21 that the nanoparticle concentration reduces for larger values of the destructive chemical reaction parameter ($\gamma > 0$). On the other hand, figure 22 verify that the concentration profile is enhanced for increasing values of the constructive chemical reaction parameter ($\gamma < 0$). This happens because in destructive and constructive chemical reaction processes, chemically reactive species are consumed whereas in constructive chemical reactions, chemically reactive species are created.

The differences in temperature and concentration between the solid surface and the ambient region flow result in the buoyancy forces for temperature and concentration. To predict the impacts of thermal and mass buoyancy forces in the boundary layer flow configuration, the quantity $N_r = \frac{\dot{m}_T (\bar{T}_w - \bar{T}_0)}{\dot{m} \gamma}$ which is the ratio of the buoyancy forces is used. The influence of this ratio on velocity, temperature and concentration profiles is illustrated in figures 23.

It is shown in figure 23 that the increase in the buoyancy ratio parameter causes a gradual enhancement of velocity and reduction of concentration profiles.
Finally, we present the impacts of some pertinent parameters on the rates of momentum, heat and mass transfer processes. This can be estimated in terms of the boundary derivatives, \(-f''(0), -\theta'(0)\) and \(-\phi'(0)\) of velocity, temperature and concentration at the surface of the stretching cylinder. In this study, the features of the aforementioned boundary derivatives against some ranges of Casson parameter \((\beta)\), curvature \((K)\), magnetic field parameter \((Ha)\), angle of inclination \((\alpha)\), heat source parameter \((Q)\) and chemical reaction parameter \((\gamma)\) are presented in table 3.

Table 3. Variation of boundary derivatives against some distinct values of pertinent parameters.

| \(\beta\) | \(K\) | \(Ha\) | \(\alpha\) | \(Bi\) | \(Gr\) | \(Gm\) | \(-f'(0)\) | \(-\theta'(0)\) | \(-\phi'(0)\) |
|---|---|---|---|---|---|---|---|---|---|
| 2.0 | 1.164734 | 0.081717 | 1.183866 |
| 3.0 | 1.309287 | 0.081503 | 1.174472 |
| 4.0 | 1.268841 | 0.081370 | 1.169250 |
| 0.2 | 1.312524 | 0.082234 | 1.207580 |
| 0.3 | 1.268841 | 0.083067 | 1.246265 |
| 2.0 | 1.164734 | 0.077441 | 1.172379 |
| 3.0 | 2.971165 | 0.070688 | 1.122435 |
| \(\pi/6\) | 2.971165 | 0.070688 | 1.122435 |
| \(\pi/4\) | 2.977161 | 0.070621 | 1.121960 |
| \(\pi/3\) | 2.984991 | 0.070528 | 1.121323 |
| 0.3 | 2.979135 | 0.152707 | 1.122098 |
| 0.5 | 2.975101 | 0.177742 | 1.083476 |
| 0.3 | 2.966676 | 0.178800 | 1.093330 |
| 0.4 | 2.959612 | 0.199424 | 1.124243 |
| 0.2 | 2.948378 | 0.200172 | 1.124747 |
| 0.3 | 2.937555 | 0.205220 | 1.125367 |

The results in table 3 depicts that the skin friction increases by increasing the values of \(\beta, K, Ha, \alpha\) and \(Bi\). Also, the rate of heat and mass transfer near the stretching surface can be facilitated by increasing the values of \(K, Gr\) and \(Gm\).

5. Conclusions

In this study, a steady laminar flow of an incompressible, viscous, electrically conducting, thermally radiative and chemically reactive Casson nanofluid over an inclined stretching cylinder has been investigated. Efforts have been made to examine the characteristics of flow field profiles in response to the effects of various thermophysical parameters by employing the homotopy analysis method.

- The flow velocity can be maximized by increasing the curvature of the cylinder. This profile can also be maximized by decreasing the values of the Casson parameter, angle of inclination or the effect of magnetic field strength.

Figure 23. Effects of Buoyancy ratio \((Nr)\) on Velocity, Temperature and Concentration Profiles.
• The temperature distribution in the boundary layer region can be enhanced by increasing the Casson parameter, angle of inclination, magnetic field, thermal radiation, Eckert number, heat generation or the Biot number. This profile can also be enhanced by decreasing the curvature of the cylinder or heat absorption.

• The volume fraction of the nanoparticles can be enriched in the boundary layer region by increasing the Casson parameter, angle of inclination, magnetic field, curvature of the cylinder or the constructive chemical reaction effect. Clearly, the destructive chemical reaction has the reverse effect on the concentration profile.

• The skin friction coefficient grows for greater values of the Casson parameter, curvature of the cylinder, Hartman number or angle of inclination. This quantity can also be enhanced by reducing the effects of heat source, thermal radiation, Eckert number, buoyancy parameters or Biot number.

• The Nusselt number can be maximized by increasing the radiation, thermal buoyancy parameter or Biot number. This quantity can also be maximized by reducing the effects of heat source or Eckert number.

• The Sherwood number can be improved by increasing the curvature of the cylinder, chemical reaction parameter, the heat source parameter, thermal radiation, Eckert number, buoyancy parameters or Biot number.

• The validity of the homotopy analysis method has been ensured by verifying the convergence of the solutions and accuracy of the findings as compared to previously published study results.

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