On the Definitions of Fractional Sum and Difference on Non-uniform Lattices

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Abstract
As is well known, the idea of a fractional sum and difference on uniform lattice is more current, and gets a lot of development in this field. But the definitions of fractional sum and fractional difference of \( f(z) \) on non-uniform lattices \( x(z) = c_1z^2 + c_2z + c_3 \) or \( x(z) = c_1q^z + c_2q^{-z} + c_3 \) seem much more difficult and complicated. In this article, for the first time we propose the definitions of the fractional sum and fractional difference on non-uniform lattices by two different ways. The analogue of Euler’s Beta formula, Cauchy’ Beta formula on on non-uniform lattices are established, and some fundamental theorems of fractional calculus, the solution of the generalized Abel equation and fractional central difference equations on non-uniform lattices are obtained etc.

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1 Introduction

The definition of non-uniform lattices date back to the approximation of the following differential equation of hypergeometric type:

\[
\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0, \tag{1}
\]

where \( \sigma(z) \) and \( \tau(z) \) are polynomials of degrees at most two and one, respectively, and \( \lambda \) is a constant. Its solutions are some types of special functions of mathematical physics, such as the classical orthogonal polynomials, the hypergeometric and cylindrical functions, see G. E. Andrews, R. Askey, R. Roy \[6\] and Z. X. Wang \[12\], A. F. Nikiforov, V. B. Uvarov and S. K. Suslov \[35, 36\].
generalized Eq. (1) to a difference equation of hypergeometric type case and studied the Nikiforov-Uvarov-Suslov difference equation on a lattice \( x(s) \) with variable step size \( \Delta x(s) = x(s+1) - x(s) \), \( \nabla x(s) = x(s) - x(s-1) \) as

\[
\tilde{\sigma}[x(s)] \frac{\Delta}{\Delta x(s-1/2)} \left( \nabla y(s) \frac{\Delta}{\nabla x(s)} \right) + \frac{1}{2} \tilde{\tau}[x(s)] \left( \frac{\Delta y(s)}{\Delta x(s)} + \frac{\nabla y(s)}{\nabla x(s)} \right) + \lambda y(s) = 0, \tag{2}
\]

where \( \tilde{\sigma}(x) \) and \( \tilde{\tau}(x) \) are polynomials of degrees at most two and one in \( x(s) \), respectively, \( \lambda \) is a constant, \( \Delta y(s) = y(s+1) - y(s) \), \( \nabla y(s) = y(s) - y(s-1) \), and \( x(s) \) is a lattice function that satisfies

\[
\frac{x(s+1) + x(s)}{2} = ax(s) + \frac{1}{2} + \beta, \quad \alpha, \beta \text{ are constants}, \tag{3}
\]

\( x^2(s+1) + x^2(s) \) is a polynomial of degree at most two w.r.t. \( x(s+\frac{1}{2}) \). \( \tag{4} \)

It should be pointed out that the difference equation (2) obtained as a result of approximating the differential equation (1) on a non-uniform lattice is of independent importance and arises in a number of other questions. Its solutions essentially generalized the solutions of the original differential equation and are of interest in their own right \[7, 8, 9\].

**Definition 1** \([35, 36]\) Two kinds of lattice functions \( x(s) \) are called non-uniform lattices which satisfy the conditions in Eqs. (3) and (4) are

\[
x(s) = \tilde{c}_1 s^2 + \tilde{c}_2 s + \tilde{c}_3, \tag{5}
\]

\[
x(s) = c_1 q^s + c_2 q^{-s} + c_3, \tag{6}
\]

where \( c_i, \tilde{c}_i \) are arbitrary constants and \( c_1 c_2 \neq 0, \tilde{c}_1 \tilde{c}_2 \neq 0 \). When \( c_1 = 1, c_2 = c_3 = 0 \), or \( \tilde{c}_1 = 1, \tilde{c}_1 = \tilde{c}_3 = 0 \), these two kinds of lattice functions \( x(s) \)

\[
x(s) = s \tag{7}
\]

\[
x(s) = q^s \tag{8}
\]

are called uniform lattices.

Let \( x(s) \) be a non-uniform lattice, where \( s \in \mathbb{C} \). For any real \( \gamma \), \( x_\gamma(s) = x(s + \frac{\gamma}{2}) \) is also a non-uniform lattice. Given a function \( F(s) \), define the difference operator with respect to \( x_\gamma(s) \) as

\[
\nabla_\gamma F(s) = \frac{\nabla F(s)}{\nabla x_\gamma(s)},
\]

and

\[
\nabla_\gamma^k F(z) = \frac{\nabla}{\nabla x_\gamma(z)} \left( \frac{\nabla}{\nabla x_{\gamma+1}(z)} \left( \ldots \frac{\nabla F(z)}{\nabla x_{\gamma+k-1}(z)} \right) \right), \quad (k = 1, 2, \ldots)
\]

The following equalities can be verified straightforwardly.
Proposition 2  Given two functions $f(s), g(s)$ with complex variable $s$, the following difference equalities hold

\[
\Delta_{\nu}(f(s)g(s)) = f(s + 1)\Delta_{\nu}g(s) + g(s)\Delta_{\nu}f(s) \\
= g(s + 1)\Delta_{\nu}f(s) + f(s)\Delta_{\nu}g(s),
\]

\[
\Delta_{\nu}\left(\frac{f(s)}{g(s)}\right) = \frac{g(s + 1)\Delta_{\nu}f(s) - f(s + 1)\Delta_{\nu}g(s)}{g(s)g(s + 1)} \\
= \frac{g(s)\Delta_{\nu}f(s) - f(s)\Delta_{\nu}g(s)}{g(s)g(s + 1)},
\]

\[
\nabla_{\nu}(f(s)g(s)) = f(s - 1)\nabla_{\nu}g(s) + g(s)\nabla_{\nu}f(s) \\
= g(s - 1)\nabla_{\nu}f(s) + f(s)\nabla_{\nu}g(s),
\]

\[
\nabla_{\nu}\left(\frac{f(s)}{g(s)}\right) = \frac{g(s - 1)\nabla_{\nu}f(s) - f(s - 1)\nabla_{\nu}g(s)}{g(s)g(s - 1)} \\
= \frac{g(s)\nabla_{\nu}f(s) - f(s)\nabla_{\nu}g(s)}{g(s)g(s - 1)}.
\]

The notions of fractional calculus date back to Euler, and in the last decades the fractional calculus had a remarkable development as shown by many mathematical volumes dedicated to it [37, 34, 39, 33, 38, 21, 32], but the idea of a fractional difference on uniform lattice (7) and (8) is more current.

Some of the more extensive papers on the fractional difference on uniform lattice (7), Diaz and Osler [20], Granger and Joyeux [27], Hosking [29] have employed the definition of the $\alpha$-th order fractional difference by

\[
\nabla^{\alpha}f(x) = \sum_{k=0}^{\infty}(-1)^k \binom{\alpha}{k} f(x - k),
\]

(10)

where $\alpha$ is any real number and the notation $\nabla^{\alpha}$ is used since this definition is natural extension of the backward difference operator.

H. H. Gray and N. F. Zhang [28] gave the following new definition of the fractional sum and difference:

**Definition 3**  ([28]) For $\alpha$ any complex number, and $f$ defined over the integer set $\{a, a + 1, ..., x\}$, the $\alpha$-th order sum over $\{a, a + 1, ..., x\}$ is defined by

\[
S_{a}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \sum_{k=a}^{x}(x - k + 1)_{\alpha-1}f(k).
\]

(11)
For any complex number \( \alpha \) and \( \beta \) let \((\alpha)\beta\) be defined as follows:

\[
(\alpha)\beta = \begin{cases} 
\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)}, & \text{when } \alpha \text{ and } \beta \text{ are neither zero nor negative integers,} \\
1, & \text{when } \alpha = \beta = 0, \\
0, & \text{when } \alpha = 0, \beta \text{ is not zero or a negative integer} \\
\text{undefined otherwise}
\end{cases}
\]

and when \( n \in \mathbb{N} \), \((\alpha)n\) denotes the Pochhammer symbol.

**Definition 4** \([28]\) For \( \alpha \) any complex number, the \( \alpha \)-th order difference over \( \{a, a+1, \ldots, x\} \) is defined by

\[
\nabla_{a}^{\alpha}f(x) = S_{a}^{-\alpha}f(x).
\]  \hspace{1cm} (12)

J. F. Cheng \([16]\) independently gave the following definitions of the fractional sum and difference, which are consistent with Definition 3 and Definition 4, and are well defined for any real or complex number \( \alpha \).

**Definition 5** \([16]\) For \( \alpha \) complex number, \( \text{Re}\alpha > 0 \) and \( f \) defined over the integer set \( \{a, a+1, \ldots, x\} \), the \( \alpha \)-th order sum over \( \{a, a+1, \ldots, x\} \) is defined by

\[
\nabla_{a}^{-\alpha}f(x) = \sum_{k=a}^{x} \left[ \binom{\alpha}{x-k} f(k) \right],
\]  \hspace{1cm} (13)

where \( \binom{\alpha}{n} = \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!} \).

**Definition 6** \([16]\) For \( \alpha \) any complex number, \( n-1 \leq \text{Re}\alpha < n \), the Riemann-Liouville type \( \alpha \)-th order difference over \( \{a, a+1, \ldots, x\} \) is defined by

\[
\nabla_{a}^{\alpha}f(x) = \nabla^{n}\nabla_{a}^{\alpha-n}f(x),
\]  \hspace{1cm} (14)

and Caputo type \( \alpha \)-th order difference over \( \{a, a+1, \ldots, x\} \) is defined by

\[
\nabla_{a}^{\alpha}f(x) = \nabla_{a}^{\alpha-n}\nabla^{n}f(x).
\]  \hspace{1cm} (15)

In the case of uniform lattice uniform \([8]\), a \( q \)-analogue of the Riemann-Liouville fractional sum operator is introduced in \([2]\) by Al-Salam through

\[
I_{q}^{\alpha}f(x) = x^{\alpha-1} \frac{d}{d_{q}} \frac{\Gamma(\alpha)}{\Gamma(q(\alpha))} \int_{0}^{x} (qt/x; q)_{\alpha-1} f(t) d_{q}(t). \]  \hspace{1cm} (16)

The \( q \)-analogue of the Riemann-Liouville fractional difference operator is also given independently by Agarwal \([1]\), who defined the \( q \)-fractional difference to be

\[
D_{q}^{\alpha}f(x) = I_{q}^{-\alpha}f(x) = x^{-\alpha-1} \frac{d}{d_{q}} \frac{\Gamma(-\alpha)}{\Gamma(-q(\alpha))} \int_{0}^{x} (qt/x; q)_{-\alpha-1} f(t) d_{q}(t).
\]  \hspace{1cm} (17)
Althought the discrete fractional calculus on uniform lattice (7) and (8) are more current, but great development has been made in this field. In the recent monographs, J. F. Cheng [16], C. Goodrich and A. Peterson [25] provided the comprehensive treatment of the discrete fractional calculus with up-to-date references, and the developments in the theory of fractional q-calculus had been well reported by M. H. Annaby and Z. S. Mansour [3].

But we should mention that, in the case of nonuniform lattices (7) or (8), even when \( n \in \mathbb{N} \), the formula of \( n \)-order difference on non-uniform lattices is a remarkable job, since it is very complicated and difficult to be obtained. In fact, in [35, 36], A. Nikiforov, V. Uvarov, S. Suslov obtained the formula of \( n \)-th difference \( \nabla_1^{(n)}[f(s)] \) as follows:

**Definition 7** ([35, 36]) Let \( n \in \mathbb{N}^+ \), for nonuniform lattices (5) or (6), then

\[
\nabla_1^{(n)}[f(s)] = \sum_{k=0}^{n} \frac{(-1)^{n-k} \Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \sum_{l=0}^{n} \nabla x[s + k - (n - 1)/2] f(s - n + k)
\]

\[
= \sum_{k=0}^{n} \frac{(-1)^{n-k} \Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} \sum_{l=0}^{n} \nabla x_n + 1(s - k) f(s - k),
\]

(18)

where \( \Gamma(s)_q \) is modified q-gamma function which is defined as

\[
\Gamma(s)_q = q^{-(s-1)(s-2)/4} \Gamma_q(s),
\]

and function \( \Gamma_q(s) \) is called the q-gamma function; it is a generalization of Euler’s gamma function \( \Gamma(s) \). It is defined by

\[
\Gamma_q(s) = \begin{cases} 
\frac{\prod_{k=0}^{\infty} (1-q^k s^{(1-q^k)})^{-1}}{\prod_{k=0}^{\infty} (1-q^k s^{(1-q^k)}):} & \text{when } |q| < 1; \\
q^{-(s-1)(s-2)/2} \Gamma_1/s^{1/q}(s) & \text{when } |q| < 1.
\end{cases}
\]

(19)

After further transformations, A. Nikiforov, V. Uvarov, S. Suslov in [35] rewrittred the formula of \( n \)-th difference \( \nabla_1^{(n)}[f(s)] \) as follows:

**Definition 8** ([35]) Let \( n \in \mathbb{N}^+ \), for nonuniform lattices (5) or (6), then

\[
\nabla_1^{(n)}[f(s)] = \sum_{k=0}^{n} \frac{([-n]_q)k}{[k]_q!} \frac{l^{(2s-k+c)}_q}{l^{(2s-k+n+1+c)}_q} f(s-k)\nabla x_{n+1}(s-k),
\]

where

\[
\mu_q = \begin{cases} 
\frac{q^2 - q^{s+c}}{q^2 - q^s} & \text{if } x(s) = c_1 q^s + c_2 q^{-s} + c_3; \\
\mu, & \text{if } x(s) = c_1 s^2 + c_2 s + c_3,
\end{cases}
\]

(20)
and
\[ c = \begin{cases} 
\frac{\log \bar{c}_2}{\log q}, & \text{when } x(s) = c_1 q^s + c_2 q^{-s} + c_3, \\
\bar{c}_2, & \text{when } x(s) = \bar{c}_1 s^2 + \bar{c}_2 s + \bar{c}_3.
\end{cases} \]

Now there exist two important and challenging problems that need to be further discussed:

(1) Assume that \( g(s) \) be a known function, \( f(s) \) be an unknown function, which satisfies the following generalized difference equation on non-uniform lattices

\[ \nabla^{[\alpha]}_s [f(s)] = g(s), \tag{21} \]

how to solve generalized difference equation (21)?

(2) However, the related definitions of \( \alpha \)-order fractional sum and \( \alpha \)-order fractional difference on non-uniform lattices are very difficult and interesting problems, they have not been appeared since the monographs [35, 36] were published. Can we give reasonable definitions of fractional sum and difference on non-uniform Lattices?

We think that as the most general discrete fractional calculus on non-uniform Lattices, they will have an independent meaning and lead to many interesting new theories about them. They are the important extension and development of Definition 7, 8 and the discrete fractional calculus.

It is the purpose of this paper to inquire into the feasibility of establishing a fractional calculus of finite difference on nonuniform lattices. In this article, for the first time we propose the definitions of the fractional sum and fractional difference on non-uniform lattices. Then give some fundamental theorems of fractional calculus, such as the analogue of Euler’s Beta formula, Cauchy’ Beta formula, Taylor’s formula on non-uniform lattices are established, and the solution of the generalized Abel equation and fractional central difference equations on non-uniform lattices are obtained etc. The results we obtain are essentially new and appeared in the literature for the first time only recently.

2 Integer sum and Fractional Sum on Non-uniform Lattices

Let \( x(s) \) be a non-uniform lattice, where \( s \in \mathbb{C} \). let \( \nabla x_F(s) = f(s) \). Then

\[ F(s) - F(s - 1) = f(s) [x_s(s) - x_s(s - 1)] \]

Choose \( z, a \in \mathbb{C} \), and \( z - a \in \mathbb{N} \). Summing from \( s = a + 1 \) to \( z \), we have

\[ F(z) - F(a) = \sum_{s=a+1}^{z} f(s) \nabla x_s(s). \]
Thus, we define
\[
\int_{a+1}^{z} f(s)dv x_\gamma(s) = \sum_{s=a+1}^{z} f(s)\nabla x_\gamma(s).
\]

It is easy to verify that

**Proposition 9** Given two function \(F(z), f(z)\) with complex variable \(z, a \in C\), and \(z - a \in N\), we have

(1) \(\nabla_\gamma \left[ \int_{a+1}^{z} f(s)dv x_\gamma(s) \right] = f(z)\),

(2) \(\int_{a+1}^{z} \nabla_\gamma F(s)dv x_\gamma(s) = F(z) - F(a)\).

A generalized power \([x(s) - x(z)]^{(n)}\) on nonuniform lattice is given by

\[ [x(s) - x(z)]^{(n)} = \prod_{k=0}^{n-1} [x(s) - x(z - k)], (n \in N^+) \]

and a more formal definition and further properties of the generalized powers \([x_\nu(s) - x_\nu(z)]^{(\alpha)}\) on nonuniform lattice are very important, which are defined as follows:

**Definition 10** (See [11, 12, 40]) Let \(\alpha \in C\), the generalized powers \([x_\nu(s) - x_\nu(z)]^{(\alpha)}\) are defined by

\[
[x_\nu(s) - x_\nu(z)]^{(\alpha)} = \begin{cases}
\Gamma(s+\alpha) \Gamma(s+z) / \Gamma(s+\alpha+z+1), & \text{if } x(s) = s, \\
\Gamma(s-z+\alpha) \Gamma(s+z+\nu+1) / \Gamma(s-z) \Gamma(s+z+\alpha+1), & \text{if } x(s) = s^2, \\
(q-1)^{\alpha} q^{\alpha(\nu-\alpha+1)/2} \Gamma(s+\alpha) / \Gamma(q(s-z)), & \text{if } x(s) = q^s, \\
\Gamma(\alpha) \prod_{n=1}^{\infty} \left(1 - \frac{s}{n}\right) \Gamma(n+s+\nu+1) / \Gamma(n+s+\alpha+1), & \text{if } x(s) = q^{s+\nu}.
\end{cases}
\]

For the quadratic lattices of the form \((2, 2)\), with \(c = \frac{c^2}{c^2}\),

\[
[x_\nu(s) - x_\nu(z)]^{(\alpha)} = c \alpha \Gamma(s-z+\alpha) \Gamma(s+z+\nu+1) / \Gamma(s-z) \Gamma(s+z+\nu-\alpha+1+1); \tag{22}
\]

For the \(q\)-quadratic lattices of the form \((2, 2)\), with \(c = \frac{\log q}{\log q}\),

\[
[x_\nu(s) - x_\nu(z)]^{(\alpha)} = c_1 (1-q)^{\alpha(\nu+1)/2} \Gamma(q(s-z+\alpha)) / \Gamma(q(s-z)) \Gamma(q(s+z+\nu-\alpha+1+1)); \tag{23}
\]

where \(\Gamma(s)\) is Euler Gamma function and \(\Gamma_q(s)\) is Euler \(q\)-Gamma function which is defined as \([13]\).
Proposition 11 \[11, 12, 40\]. For \( x(s) = c_1q^s + c_2q^{-s} + c_3 \) or \( x(s) = \tilde{c}_1s^2 + \tilde{c}_2s + \tilde{c}_3 \), the generalized power \( [x_\nu(s) - x_\nu(z)]^{(\alpha)} \) satisfy the following properties:

\[
[x_\nu(s) - x_\nu(z)][x_\nu(s) - x_\nu(z - 1)]^{(\mu)} = [x_\nu(s) - x_\nu(z)]^{(\mu)}[x_\nu(s) - x_\nu(z - \mu)]
\]

\[= [x_\nu(s) - x_\nu(z)]^{(\mu + 1)}; \quad (25)\]

\[
[x_\nu-1(s + 1) - x_\nu-1(z)]^{(\mu)}[x_\nu-\mu(s) - x_\nu-\mu(z)]
\]

\[= [x_\nu-\mu(s + \mu) - x_\nu-\mu(z)][x_\nu-1(s) - x_\nu-1(z)]^{(\mu)} = [x_\nu(s) - x_\nu(z)]^{(\mu + 1)}; \quad (27)\]

\[
\frac{\Delta_z}{\Delta x_{\nu-\mu+1}(z)}[x_\nu(s) - x_\nu(z)]^{(\mu)} = -\frac{\nabla_z}{\nabla x_{\nu+1}(s)}[x_\nu+1(s) - x_\nu+1(z)]^{(\mu)}
\]

\[= -[\mu]_q[x_\nu(s) - x_\nu(z)]^{(\mu - 1)}; \quad (28)\]

\[
\frac{\nabla_z}{\nabla x_{\nu-\mu+1}(z)}\left\{ \frac{1}{[x_\nu(s) - x_\nu(z)]^{(\mu)}} \right\} = -\frac{\Delta_z}{\Delta x_{\nu-1}(s)}\left\{ \frac{1}{[x_\nu-1(s) - x_\nu-1(z)]^{(\mu)}} \right\}
\]

\[= \frac{[\mu]_q}{[x_\nu(s) - x_\nu(z)]^{(\mu + 1)}}, \quad (30)\]

where \([\mu]_q\) is defined as \([20]\).

Now let us first define the integer sum on non-uniform lattices \( x_\gamma(s) \) in detail, which is very helpful for us to define fractional sum on non-uniform lattices \( x_\gamma(s) \).

For \( \gamma \in R \), the 1-th order sum of \( f(z) \) over \( \{a + 1, a + 2, ..., z\} \) on non-uniform lattices \( x_\gamma(s) \) is defined by

\[
y_1(z) = \nabla^{-1}_\gamma f(z) = \int_{a+1}^{z} f(s) d\nabla x_\gamma(s), \quad (32)\]

then by Proposition \[9\] we have

\[
\nabla^1_\gamma \nabla^{-1}_\gamma f(z) = \frac{\nabla y_1(z)}{\nabla x_\gamma(z)} = f(z), \quad (33)\]

and 2-th order sum of \( f(z) \) over \( \{a + 1, a + 2, ..., z\} \) on non-uniform lattices \( x_\gamma(s) \) is defined by
Meanwhile, we have

\[ y_2(z) = \nabla_{\gamma}^{-2} f(z) = \nabla_{\gamma+1}^{-1} [\nabla_{\gamma}^{-1} f(z)] = \int_{a+1}^{z} y_1(s) \, d\varphi x_{\gamma+1}(s) \]
\[ = \int_{a+1}^{z} d\varphi x_{\gamma+1}(s) \int_{a+1}^{z} f(t) \, d\varphi x_{\gamma}(t) \]
\[ = \int_{a+1}^{z} f(t) \, d\varphi x_{\gamma}(t) \int_{t}^{z} d\varphi x_{\gamma+1}(s) \]
\[ = \int_{a+1}^{z} [x_{\gamma+1}(z) - x_{\gamma+1}(t-1)] \, f(s) \, d\varphi x_{\gamma}(s). \]  \quad (34)

and 3-th order sum of \( f(z) \) over \( \{a+1, a+2, \ldots, z\} \) on non-uniform lattices \( x_{\gamma}(s) \) is defined by

\[ y_3(z) = \nabla_{\gamma}^{-3} f(z) = \nabla_{\gamma+2}^{-1} [\nabla_{\gamma}^{-2} f(z)] = \int_{a+1}^{z} y_2(s) \, d\varphi x_{\gamma+2}(s) \]
\[ = \int_{a+1}^{z} d\varphi x_{\gamma+2}(s) \int_{a+1}^{z} [x_{\gamma+1}(s) - x_{\gamma+1}(t-1)] \, f(t) \, d\varphi x_{\gamma}(t) \]
\[ = \int_{a+1}^{z} f(t) \, d\varphi x_{\gamma}(t) \int_{t}^{z} [x_{\gamma+1}(s) - x_{\gamma+1}(t-1)] \, d\varphi x_{\gamma+2}(s). \]

In View of Proposition \([1]\) one has

\[ \nabla_{\gamma+2}(z) [x_{\gamma+2}(s) - x_{\gamma+2}(t-1)](2) = [2]_{q} [x_{\gamma+1}(s) - x_{\gamma+1}(t-1)], \]  \quad (36)

then by the use of Proposition \([9]\) we have

\[ \frac{[x_{\gamma+2}(z) - x_{\gamma+2}(t-1)](2)}{[2]_{q}} = \int_{t}^{z} [x_{\gamma+1}(s) - x_{\gamma+1}(t-1)] \, d\varphi x_{\gamma+2}(s). \]  \quad (37)

Therefore, we obtain that 3-th order sum of \( f(z) \) over \( \{a+1, a+2, \ldots, z\} \) on non-uniform lattices \( x_{\gamma}(s) \) is

\[ y_3(z) = \nabla_{\gamma}^{-3} f(z) = \nabla_{\gamma+2}^{-1} [\nabla_{\gamma}^{-2} f(z)] \]
\[ = \frac{1}{[1(3)]_{q}} \int_{a+1}^{z} [x_{\gamma+2}(z) - x_{\gamma+2}(t-1)](2) \, f(s) \, d\varphi x_{\gamma}(s). \]  \quad (38)
Mean while, it is easy to know that
\[
\nabla_\gamma^3 \nabla_\gamma^{-3} f(z) = \frac{\nabla}{\nabla x_\gamma(z)} \left( \frac{\nabla}{\nabla x_{\gamma+1}(z)} \left( \frac{\nabla y_3(z)}{\nabla x_{\gamma+2}(z)} \right) \right) = f(z).
\]

(39)

More generally, by the induction, we can define the \( k \)-th order sum of \( f(z) \) over \( \{a + 1, a + 2, \ldots, z\} \) on non-uniform lattices \( x_\gamma(s) \) as
\[
y_k(z) = \nabla_\gamma^{-k} f(z) = \nabla_\gamma^{-1}(\nabla_\gamma^{-(k-1)} f(z)) = \int_{a+1}^{z} y_{k-1}(s) d\nabla x_{\gamma+k-1}(s)
\]
\[
= \frac{1}{[\Gamma(k)]_q} \int_{a+1}^{z} [x_{\gamma+k-1}(z) - x_{\gamma+k-1}(t-1)]^{(k-1)} f(t) d\nabla x_\gamma(t), (k = 1, 2, \ldots)
\]
(40)

where
\[
[\Gamma(k)]_q = \left\{ \begin{array}{ll}
q^{-(k-1)(k-2)} \Gamma_q(k), & \text{if } x(s) = c_1 q^s + c_2 q^{-s} + c_3; \\
\Gamma(\alpha), & \text{if } x(s) = \tilde{c}_1 s^2 + \tilde{c}_2 s + \tilde{c}_3,
\end{array} \right.
\]
which satisfies the following
\[
[\Gamma(k+1)]_q = [k]_q [\Gamma(k)]_q, \quad [\Gamma(2)]_q = [1]_q [\Gamma(1)]_q = 1.
\]

And then we have
\[
\nabla_\gamma^k \nabla_\gamma^{-k} f(z) = \frac{\nabla}{\nabla x_\gamma(z)} \left( \frac{\nabla}{\nabla x_{\gamma+1}(z)} \left( \frac{\nabla y_k(z)}{\nabla x_{\gamma+k-1}(z)} \right) \right) = f(z), (k = 1, 2, \ldots)
\]
(41)

It is noted that the right hand side of (40) is still meaningful when \( k \in C \), so we can give the definition of fractional sum of \( f(z) \) on non-uniform lattices \( x_\gamma(s) \) as follows

**Definition 12 (Fractional sum on non-uniform lattices)** For any \( \Re \alpha \in \mathbb{R}^+ \), the \( \alpha \)-th order sum of \( f(z) \) over \( \{a + 1, a + 2, \ldots, z\} \) on non-uniform lattices \( x_\gamma(s) \) and (3) is defined by
\[
\nabla_\gamma^\alpha f(z) = \frac{1}{[\Gamma(\alpha)]_q} \int_{a+1}^{z} [x_{\gamma+\alpha-1}(z) - x_{\gamma+\alpha-1}(t-1)]^{(\alpha-1)} f(s) d\nabla x_\gamma(s),
\]
(42)

where
\[
[\Gamma(\alpha)]_q = \left\{ \begin{array}{ll}
q^{-(s-1)(s-2)} \Gamma_q(\alpha), & \text{if } x(s) = c_1 q^s + c_2 q^{-s} + c_3; \\
\Gamma(\alpha), & \text{if } x(s) = \tilde{c}_1 s^2 + \tilde{c}_2 s + \tilde{c}_3,
\end{array} \right.
\]
which satisfies the following
\[
[\Gamma(\alpha+1)]_q = [\alpha]_q [\Gamma(\alpha)]_q.
\]
3 The Analogue of Euler Beta Formula on Non-uniform Lattices

Euler Beta formula is well known as

\[
\int_0^1 (1-t)^{\alpha-1} t^{\beta-1} dt = B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}, \quad (\text{Re} \alpha > 0, \text{Re} \beta > 0)
\]

or

\[
\int_a^z (z-t)^{\alpha-1} (t-a)^{\beta-1} \frac{dt}{\Gamma(\alpha)} = \frac{(z-a)^{\alpha+\beta-1}}{\Gamma(\alpha + \beta)} \quad (\text{Re} \alpha > 0, \text{Re} \beta > 0)
\]

In this section, we obtain the analogue Euler Beta formula on non-uniform lattices. It is very crucial for us to propose several definitions in this manuscript. And it is also of independent importance.

**Theorem 13** (Euler Beta formula on non-uniform lattices) For any \(\alpha, \beta \in \mathbb{C}\), then for non-uniform lattices \(x(s)\), we have

\[
\int_a^z \frac{[x_{\beta}(z) - x_{\beta}(t-1)]^{(\beta-1)} [x(t) - x(a)]^{(\alpha)}}{[\Gamma(\beta)]_q [\Gamma(\alpha + 1)]_q} d\gamma x_1(t) = \frac{[x_{\beta}(z) - x_{\beta}(a)]^{(\alpha+\beta)}}{[\Gamma(\alpha + \beta + 1)]_q}.
\]

The proof of **Theorem 13** should use some lemmas.

**Lemma 14** For any \(\alpha, \beta\), we have

\[
[\alpha + \beta]_q x(t) - [\alpha]_q x_{\beta}(t) - [\beta]_q x_{\alpha}(t) = \text{const.}
\]

**Proof.** If we set \(x(t) = \tilde{c}_1 t^2 + \tilde{c}_2 t + \tilde{c}_3\), then left hand side of Eq.(44) is

\[
\text{LHS} = \frac{\tilde{c}_1 [(\alpha + \beta) t^2 - \alpha (t - \beta)^2 - \beta (t + \frac{\alpha}{2})^2]}{\tilde{c}_2 [(\alpha + \beta) t - \alpha (t - \beta) - \beta (t + \frac{\alpha}{2})]} = \frac{-\alpha \beta}{4} (\alpha + \beta) \tilde{c}_1 = \text{const.}
\]

If we set \(x(t) = c_1 q^t + c_2 q^{-t} + c_3\), then left hand side of Eq.(44) is

\[
\text{LHS} = c_1 \left[ \frac{q^{\frac{\alpha+\beta}{2}} - q^{-\frac{\alpha+\beta}{2}}}{q^t - q^{-\frac{1}{2}}} q^t - \frac{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}}{q^t - q^{-\frac{1}{2}}} q^{t - \frac{t}{2}} - \frac{q^{\frac{\beta}{2}} - q^{-\frac{\beta}{2}}}{q^t - q^{-\frac{1}{2}}} q^{t + \frac{t}{2}} \right] + c_2 \left[ \frac{q^{\frac{\alpha+\beta}{2}} - q^{-\frac{\alpha+\beta}{2}}}{q^t - q^{-\frac{1}{2}}} q^{-t} - \frac{q^{\frac{\alpha}{2}} - q^{-\frac{\alpha}{2}}}{q^t - q^{-\frac{1}{2}}} q^{-t - \frac{t}{2}} - \frac{q^{\frac{\beta}{2}} - q^{-\frac{\beta}{2}}}{q^t - q^{-\frac{1}{2}}} q^{-t + \frac{t}{2}} \right] = 0.
\]
Lemma 15. For any $\alpha, \beta$, we have

$$[\alpha + 1]q[x_\beta(z) - x_\beta(t - \beta)] - [\beta]q[x_{1-\alpha}(t + \alpha) - x_{1-\alpha}(a)]$$

$$= [\alpha + 1]q[x_\beta(z) - x_\beta(a - \alpha - \beta)]$$

$$- [\alpha + \beta + 1]q[x(t) - x(a - \alpha)].$$

(49)

Proof. (49) is equivalent to

$$[\alpha + \beta + 1]q[x(t) - [\alpha + 1]q x_\beta(t - \beta) - [\beta]q x_{1-\alpha}(t + \alpha)$$

$$= [\alpha + \beta + 1]q x(a - \alpha) - [\alpha + 1]q x_\beta(a - \alpha - \beta) - [\beta]q x_{1-\alpha}(a).$$

(50)

Set $\alpha + 1 = \bar{\alpha}$, we only to prove that

$$[\bar{\alpha} + \beta]q x(t) - [\bar{\alpha}]q x_\beta(t - \beta) - [\beta]q x_{2-\bar{\alpha}}(t + \bar{\alpha} - 1)$$

$$= [\bar{\alpha} + \beta]q x(a - \bar{\alpha} + 1) - [\bar{\alpha}]q x_\beta(a - \bar{\alpha} + 1 - \beta) - [\beta]q x_{2-\bar{\alpha}}(a).$$

(51)

That is

$$[\bar{\alpha} + \beta]q x(t) - [\bar{\alpha}]q x_{-\beta}(t) - [\beta]q x_{-\bar{\alpha}}(t)$$

$$= [\bar{\alpha} + \beta]q x(a - \bar{\alpha} + 1) - [\bar{\alpha}]q x_{-\beta}(a - \bar{\alpha} + 1) - [\beta]q x_{-\bar{\alpha}}(a).$$

(52)

By Lemma 14, Eq. (52) holds, and then Eq. (49) holds. □

Using Proposition 11 and Lemma 15, now it is time for us to prove Theorem 13.

Proof of Theorem 13. Set

$$\rho(t) = [x(t) - x(a)]^{(\alpha)}[x_\beta(z) - x_\beta(t - 1)]^{(\beta-1)},$$

(53)

and

$$\sigma(t) = [x_{1-\alpha}(t + \alpha) - x_{1-\alpha}(a)] [x_\beta(z) - x_\beta(t)].$$

(54)

By Proposition 11 since

$$[x_{1-\alpha}(t + \alpha) - x_{1-\alpha}(a)] [x(t) - x(a)]^{(\alpha)} = [x_{1}(t) - x_{1}(a)]^{(\alpha+1)}$$

(55)

and

$$[x_\beta(z) - x_\beta(t)] [x_\beta(z) - x_\beta(t - 1)]^{(\beta-1)} = [x_\beta(z) - x_\beta(t)]^{(\beta)},$$

(56)

so that we obtain

$$\sigma(t) \rho(t) = [x_{1}(t) - x_{1}(a)]^{(\alpha+1)} [x_\beta(z) - x_\beta(t)]^{(\beta)}.$$

(57)

Making use of

$$\nabla_t [f(t)g(t)] = g(t - 1) \Delta_t [f(t)] + f(t) \nabla_t [g(t)],$$

where

$$f(t) = [x_{1}(t) - x_{1}(a)]^{(\alpha+1)}, g(t) = [x_\beta(z) - x_\beta(t)]^{(\beta)},$$

(58)
let’s calculate the $\frac{\nabla_{x_1(t)} \sigma(t)}{\nabla x_1(t)}$.

From Proposition [11] we have

$$\frac{\nabla_t}{\nabla x_1(t)} [x_1(t) - x_1(a)]^{(\alpha + 1)} = [\alpha + 1]_q [x(t) - x(a)]^{(\alpha)},$$

and

$$\frac{\nabla_t}{\nabla x_1(t)} [x_\beta(z) - x_\beta(t)]^{(\beta)}$$

$$= \frac{\Delta_t}{\Delta x_1(t - 1)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta)}$$

$$= -[\beta]_q [x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)}.$$

These yield

$$\frac{\nabla_t}{\nabla x_1(t)} [x_1(t) - x_1(a)]^{(\alpha + 1)} [x_\beta(z) - x_\beta(t)]^{(\beta)}$$

$$= [\alpha + 1]_q [x(t) - x(a)]^{(\alpha)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta)}$$

$$- [\beta]_q [x_1(t) - x_1(a)]^{(\alpha + 1)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)}$$

$$= ([\alpha + 1]_q [x_\beta(z) - x_\beta(t - \beta)] - [\beta]_q [x_1 - \alpha(x + \alpha) - x_1 - \alpha(a)]) \rho(t)$$

$$\equiv \tau(t) \rho(t),$$

(58)

where

$$\tau(t) = [\alpha + 1]_q [x_\beta(z) - x_\beta(t - \beta)] - [\beta]_q [x_1 - \alpha(x + \alpha) - x_1 - \alpha(a)].$$

(59)

This is due to

$$[x_\beta(z) - x_\beta(t - 1)]^{(\beta)} = [x_\beta(z) - x_\beta(t - \beta)][x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)}.$$

Then from Lemma [13] it yields

$$\tau(t) = [\alpha + 1]_q [x_\beta(z) - x_\beta(a - \alpha - \beta)] - [\alpha + \beta + 1]_q [x(t) - x(a - \alpha)].$$

(60)

So that one gets

$$\frac{\nabla_t}{\nabla x_1(t)} [x_1(t) - x_1(a)]^{(\alpha + 1)} [x_\beta(z) - x_\beta(t)]^{(\beta)}$$

$$= ([\alpha + 1]_q [x_\beta(z) - x_\beta(a - \alpha - \beta)]$$

$$- [\alpha + \beta + 1]_q [x(t) - x(a - \alpha)]) \rho(t).$$

Or

$$\nabla_t [x_1(t) - x_1(a)]^{(\alpha + 1)} [x_\beta(z) - x_\beta(t)]^{(\beta)}$$

$$= ([\alpha + 1]_q [x_\beta(z) - x_\beta(a - \alpha - \beta)]$$

$$- [\alpha + \beta + 1]_q [x(t) - x(a - \alpha)])$$

$$\cdot [x(t) - x(a)]^{(\alpha)} [y_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)} \nabla x_1(t).$$

(61)
Summing from $a + 1$ to $z$, we have

$$
\sum_{t=a+1}^{z} \nabla_t \left\{ [x_1(t) - x_1(a)]^{(\alpha+1)} [x_\beta(z) - x_\beta(t)]^{(\beta)} \right\}
$$

$$
= \int_{a+1}^{z} \left\{ (\alpha + 1)_q [x_\beta(z) - x_\beta(a - \alpha - \beta)] 
- [\alpha + \beta + 1]_q [x(t) - x(a - \alpha)] \right\} 
\cdot [x(t) - x(a)]^{(\alpha)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta-1)} d_\varphi x_1(t).
$$

(62)

Set

$$
I(\alpha) = \int_{a+1}^{z} [x_\beta(z) - x_\beta(t - 1)]^{(\beta-1)} [x(t) - x(a)]^{(\alpha)} d_\varphi x_1(t),
$$

and

$$
I(\alpha + 1) = \int_{a+1}^{z} [x_\beta(z) - x_\beta(t - 1)]^{(\beta-1)} [x(t) - x(a)]^{(\alpha+1)} d_\varphi x_1(t).
$$

(63)

(64)

Then from (62) and by the use of Proposition III, one has

$$
\sum_{t=a+1}^{z} \nabla_t \left\{ [x_1(t) - x_1(a)]^{(\alpha+1)} [x_\beta(z) - x_\beta(t)]^{(\beta)} \right\}
$$

$$
= [\alpha + 1]_q [x_\beta(z) - x_\beta(a - \alpha - \beta)] \int_{a+1}^{z} [x(t) - x(a)]^{(\alpha)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta-1)} d_\varphi x_1(t)
- [\alpha + \beta + 1]_q \int_{a+1}^{z} [x(t) - x(a - \alpha)] [x(t) - x(a)]^{(\alpha)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta-1)} d_\varphi x_1(t)
$$

$$
= [\alpha + 1]_q [x_\beta(z) - x_\beta(a - \alpha - \beta)] \int_{a+1}^{z} [x(t) - x(a)]^{(\alpha)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta-1)} d_\varphi x_1(t)
- [\alpha + \beta + 1]_q \int_{a+1}^{z} [x(t) - x(a)]^{(\alpha+1)} [x_\beta(z) - x_\beta(t - 1)]^{(\beta-1)} d_\varphi x_1(t)
$$

$$
= [\alpha + 1]_q [x_\beta(z) - x_\beta(a - \alpha - \beta)] I(\alpha) - [\alpha + \beta + 1]_q I(\alpha + 1).
$$

Since

$$
\sum_{t=a+1}^{z} \nabla_t \left\{ [x_1(t) - x_1(a)]^{(\alpha+1)} [x_\beta(z) - x_\beta(t)]^{(\beta)} \right\} = 0,
$$

(65)

therefore, we have prove that

$$
\frac{I(\alpha + 1)}{I(\alpha)} = \frac{[\alpha + 1]_q [x_\beta(z) - x_\beta(a - \alpha - \beta)]}{[\alpha + \beta + 1]_q}.
$$

(66)

From (66), one has

$$
\frac{I(\alpha + 1)}{I(\alpha)} = \frac{\frac{\Gamma(\alpha+2)}{\Gamma(\alpha+\beta+2)}_q [x_\beta(z) - x_\beta(a)]^{(\alpha+\beta+1)}}{\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)}_q [x_\beta(z) - x_\beta(a)]^{(\alpha+\beta)}}.
$$
So that we can set
\[
I(\alpha) = k \frac{[\Gamma(\alpha + 1)]_q}{[\Gamma(\alpha + \beta + 1)]_q} [x_\beta(z) - x_\beta(a)]^{(\alpha + \beta)},
\]
(67)
where \( k \) is undetermined.

Set \( \alpha = 0 \), then
\[
I(0) = k \frac{1}{[\Gamma(\beta + 1)]_q} [x_\beta(z) - x_\beta(a)]^{(\beta)},
\]
(68)
From (63), one has
\[
I(0) = \int_{a+1}^{z} [x_\beta(z) - x_\beta(t - 1)]^{(\beta - 1)} d_{\nabla} x_1(t)
\]
\[
= \frac{1}{[\beta]_q} [x_\beta(z) - x_\beta(a)]^{(\beta)},
\]
(69)
From (68) and (69), one gets
\[
k = \frac{[\Gamma(\beta + 1)]_q}{[\beta]_q} = [\Gamma(\beta)]_q.
\]
Hence, we obtain that
\[
I(\alpha) = \frac{[\Gamma(\beta)]_q [\Gamma(\alpha + 1)]_q [x_\beta(z) - x_\beta(a)]^{(\alpha + \beta)}},
\]
(70)
and the proof of Theorem 13 is completed.

4 Generalized Abel Equation and Fractional Difference on Non-uniform Lattices

The definition of fractional difference of \( f(z) \) on non-uniform lattices \( x_\gamma(s) \) seems more difficult and complicated. Our idea is to start by solving the generalized Abel equation on non-uniform lattices. In detail, an important question is: Let \( m - 1 < \text{Re}\alpha \leq m \), \( f(z) \) over \( \{a + 1, a + 2, ..., z\} \) be a known function, \( g(z) \) over \( \{a + 1, a + 2, ..., z\} \) be an unknown function, which satisfies the following generalized Abel equation
\[
\nabla^{-\alpha} g(z) = \int_{a+1}^{z} \frac{[x_{\gamma+a-1}(z) - x_{\gamma+a-1}(t - 1)]^{(\alpha - 1)} \Gamma(t)}{[\Gamma(\alpha)]_q} g(t) d_{\nabla} x_\gamma(t) = f(t),
\]
(71)
how to solve generalized Abel equation (71)?

In order to solve equation (71), we should use the fundamental analogue of Euler Beta Theorem 13 on non-uniform lattices.
Theorem 16 (Solution 1 for Abel equation) Set functions $f(z)$ and $g(z)$ over \( \{a + 1, a + 2, \ldots, z\} \) satisfy

\[
\nabla_{\gamma}^{-\alpha} g(z) = f(z), 0 < m - 1 < \text{Re } \alpha \leq m,
\]

then

\[
g(z) = \nabla_{\gamma}^m \nabla_{\gamma + \alpha}^{-m+\alpha} f(z) \tag{72}
\]

holds.

**Proof.** We only need to prove that

\[
\nabla_{\gamma}^{-m} g(z) = \nabla_{\gamma + \alpha}^{-m-\alpha} f(z).
\]

That is

\[
\nabla_{\gamma + \alpha}^{-[m-\alpha]} f(z) = \nabla_{\gamma + \alpha}^{-[m-\alpha]} \nabla_{\gamma}^{-\alpha} g(z) = \nabla_{\gamma}^{-m} g(z).
\]

In fact, by **Definition 12** we have

\[
\nabla_{\gamma + \alpha}^{-[m-\alpha]} f(z) = \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(t-1)]^{[m-\alpha-1]}}{[\Gamma(m-\alpha)]_q} f(t) d_{\gamma} x_{\gamma+\alpha}(t)
\]

\[
= \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(t-1)]^{[m-\alpha-1]}}{[\Gamma(m-\alpha)]_q} d_{\gamma} x_{\gamma+\alpha}(t)
\]

\[
\cdot \int_{a+1}^{z} \frac{[x_{\gamma+\alpha-1}(t) - x_{\gamma+\alpha-1}(s-1)]^{[\alpha-1]}}{[\Gamma(\alpha)]_q} g(s) d_{\gamma} x_{\gamma}(s)
\]

\[
= \int_{a+1}^{z} g(s) \nabla_{\gamma} x_{\gamma}(s) \int_{s}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(t-1)]^{[m-\alpha-1]}}{[\Gamma(m-\alpha)]_q} d_{\gamma} x_{\gamma+\alpha}(t)
\]

\[
\cdot \frac{[x_{\gamma+\alpha-1}(t) - x_{\gamma+\alpha-1}(s-1)]^{[\alpha-1]}}{[\Gamma(\alpha)]_q} d_{\gamma} x_{\gamma+\alpha}(t).
\]

In **Theorem 13**, replacing $a+1$ with $s$; $\alpha$ with $\alpha - 1$; $\beta$ with $m-\alpha$, and replacing $x(t)$ with $x_{\nu+a-1}(t)$, then $x_{\beta}(t)$ with $x_{\nu+m-1}(t)$, we can obtain the following equality

\[
\int_{s}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(t-1)]^{[m-\alpha-1]}}{[\Gamma(m-\alpha)]_q} \frac{[x_{\gamma+\alpha-1}(t) - x_{\gamma+\alpha-1}(s-1)]^{[\alpha-1]}}{[\Gamma(\alpha)]_q} d_{\gamma} x_{\gamma+\alpha}(t)
\]

\[
= \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{[-m-1]}}{[\Gamma(m)]_q},
\]

therefore, we have

\[
\nabla_{\gamma + \alpha}^{-[m-\alpha]} f(z) = \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{[-m-1]}}{[\Gamma(m)]_q} g(s) d_{\gamma} x_{\gamma}(s) = \nabla_{\gamma}^{-m} g(z),
\]

which yields

\[
\nabla_{\gamma}^m \nabla_{\gamma + \alpha}^{-[m-\alpha]} f(z) = \nabla_{\gamma}^m \nabla_{\gamma}^{-m} g(z) = g(z).
\]

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Inspired by Theorem 16, this is natural that we give the $\alpha$-th order ($0 < m - 1 < \text{Re}\alpha \leq m$) Riemann-Liouville difference of $f(z)$ as follows:

**Definition 17** (Riemann-Liouville fractional difference 1) Let $m$ be the smallest integer exceeding $\text{Re}\alpha$, $\alpha$-th order Riemann-Liouville difference of $f(z)$ over \{a + 1, a + 2, ..., z\} on non-uniform lattices is defined by

$$\nabla^\alpha f(z) = \nabla^m(\nabla^\alpha f(z)). \tag{73}$$

Formally, in Definition 12 if $\alpha$ is replaced by $-\alpha$, then the RHS of (42) become

$$\int^z_{a + 1} \frac{[x_{\gamma - \alpha - 1}(z) - x_{\gamma - \alpha - 1}(t - 1)]^{(-\alpha - 1)}}{\Gamma(-\alpha)_q} f(t) d\nabla x_\gamma(t) \tag{74}$$

$$= \nabla^\alpha \nabla^{\alpha + n - 1} f(z) = \nabla^\alpha f(z). \tag{75}$$

From (74), we can also obtain $\alpha$-th order difference of $f(z)$ as follows

**Definition 18** (Riemann-Liouville fractional difference 2) Let $\text{Re}\alpha > 0$, $\alpha$-th order Riemann-Liouville difference of $f(z)$ over \{a + 1, a + 2, ..., z\} on non-uniform lattices can be defined by

$$\nabla^\alpha_{\gamma - \alpha} f(z) = \int^z_{a + 1} \frac{[x_{\gamma - \alpha - 1}(z) - x_{\gamma - \alpha - 1}(t - 1)]^{(-\alpha - 1)}}{\Gamma(-\alpha)_q} f(t) d\nabla x_\gamma(t). \tag{76}$$

Replacing $x_{\gamma - \alpha}(t)$ with $x_\gamma(t)$. Then

$$\nabla^\alpha f(z) = \int^z_{a + 1} \frac{[x_{\gamma - 1}(z) - x_{\gamma - 1}(t - 1)]^{(-\alpha - 1)}}{\Gamma(-\alpha)_q} f(t) d\nabla x_{\gamma + \alpha}(t), \tag{77}$$

where $\alpha \not\in \mathbb{N}$.

## 5 Caputo fractional Difference on Non-uniform Lattices

In this section, we give suitable definition of Caputo fractional difference on non-uniform lattices.
Theorem 19 (Sum by parts formula) Given two functions \( f(s), g(s) \) with complex variable \( s \), then
\[
\int_{a+1}^{z} g(s)\nabla_{\gamma} f(s) d\varphi x_{\gamma}(s) = f(z)g(z) - f(a)g(a) - \int_{a+1}^{z} f(s-1)\nabla_{\gamma} g(s) d\varphi x_{\gamma}(s),
\]
where \( z, a \in C \), and \( z - a \in N \).

**Proof.** Make use of Proposition\(^2\) one has
\[
g(s)\nabla_{\gamma} f(s) = \nabla_{\gamma} [f(z)g(z)] - f(s-1)\nabla_{\gamma} g(s),
\]
it yields
\[
g(s)\nabla_{\gamma} f(s)\nabla x_{\gamma}(s) = \nabla_{\gamma} [f(z)g(z)]\nabla x_{\gamma}(s) - f(s-1)\nabla_{\gamma} g(s)\nabla x_{\gamma}(s).
\]
Summing from \( a+1 \) to \( z \) with variable \( s \), then we get
\[
\int_{a+1}^{z} g(s)\nabla_{\gamma} f(s) d\varphi x_{\gamma}(s) = \int_{a+1}^{z} \nabla_{\gamma} [f(z)g(z)]\nabla x_{\gamma}(s) - \int_{a+1}^{z} f(s-1)\nabla_{\gamma} g(s) d\varphi x_{\gamma}(s)
\]
\[
= f(z)g(z) - f(a)g(a) - \int_{a+1}^{z} f(s-1)\nabla_{\gamma} g(s) d\varphi x_{\gamma}(s).
\]

The idea of the definition of Caputo fractional difference on non-uniform lattices is also inspired by the the solution of generalized Abel equation \([71]\). In section 4, we have obtained that the solution of the generalized Abel equation
\[
\nabla_{\gamma}^{-\alpha} g(z) = f(z), 0 < m - 1 < \alpha \leq m,
\]
is
\[
g(z) = \nabla_{\gamma}^{\alpha} f(z) = \nabla_{\gamma}^{m} \nabla_{\gamma}^{-m+\alpha} f(z).
\]
Now we will give a new expression of \((78)\) by parts formula. In fact, we have
\[
\nabla_{\gamma}^{\alpha} f(z) = \nabla_{\gamma}^{m} \nabla_{\gamma}^{-m+\alpha} f(z) = \nabla_{\gamma}^{m} \nabla_{\gamma}^{m-\alpha} f(z)
\]
\[
= \nabla_{\gamma}^{m} \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{m-\alpha}}{[\Gamma(m-\alpha)]_{q}} f(s) d\varphi x_{\gamma+m-1}(s).
\]

In view of the identity
\[
\nabla_{(s)}[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha)} \nabla x_{\gamma+m-1}(s) = \Delta_{(s)}[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha)} \Delta x_{\gamma+m-1}(s-1)
\]
\[
= -[m-\alpha]_{q}[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha-1)},
\]
then the expression
\[
\int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(m-\alpha-1)}}{[\Gamma(m-\alpha)]_{q}} f(s) d\varphi x_{\gamma+m-1}(s)
\]
can be written as

\[
\int_{a+1}^{\gamma} f(s) \nabla_{(s)} \left\{ \frac{-[x_{\gamma} m - 1(z) - x_{\gamma} m - 1(s)](m-\alpha)}{[\Gamma(m - \alpha + 1)]_q} \right\} d\varphi x_{\gamma} m - 1(s).
\]

Further, consider

\[
\int_{a+1}^{\gamma} f(s) \nabla_{\gamma m - 1} \left\{ \frac{-[x_{\gamma} m - 1(z) - x_{\gamma} m - 1(s)](m-\alpha)}{[\Gamma(m - \alpha + 1)]_q} \right\} d\varphi x_{\gamma} m - 1(s).
\]

Summing by parts formula, we get

\[
\int_{a+1}^{\gamma} f(s) \nabla_{\gamma m - 1} \left\{ \frac{-[x_{\gamma} m - 1(z) - x_{\gamma} m - 1(s)](m-\alpha)}{[\Gamma(m - \alpha + 1)]_q} \right\} d\varphi x_{\gamma} m - 1(s)
= f(a) \frac{[x_{\gamma} m - 1(z) - x_{\gamma} m - 1(a)](m-\alpha)}{[\Gamma(m - \alpha + 1)]_q}
+ \int_{a+1}^{\gamma} \left\{ \frac{[x_{\gamma} m - 1(z) - x_{\gamma} m - 1(s - 1)](m-\alpha)}{[\Gamma(m - \alpha + 1)]_q} \right\} \nabla_{\gamma m - 1}[f(s)] d\varphi x_{\gamma} m - 1(s).
\]

Therefore, it reduce to

\[
\int_{a+1}^{\gamma} \left\{ \frac{[x_{\gamma} m - 1(z) - x_{\gamma} m - 1(s - 1)](m-\alpha)}{[\Gamma(m - \alpha + 1)]_q} \right\} \nabla_{\gamma m - 1}[f(s)] d\varphi x_{\gamma} m - 1(s).
\] (80)

Further, consider

\[
\int_{a+1}^{\gamma} \left\{ \frac{[x_{\gamma} m - 1(z) - x_{\gamma} m - 1(s - 1)](m-\alpha)}{[\Gamma(m - \alpha + 1)]_q} \right\} \nabla_{\gamma m - 1}[f(s)] d\varphi x_{\gamma} m - 1(s).
\] (81)

By the use of the identity

\[
\nabla_{(s)}[x_{\gamma} m - 1(z) - x_{\gamma} m - 1(s)](m-\alpha + 1) = \frac{\Delta_{(s)}[x_{\gamma} m - 1(z) - x_{\gamma} m - 1(s - 1)](m-\alpha + 1)}{\Delta x_{\gamma} m - 1(s - 1)} = -[m - \alpha + 1]_q [x_{\gamma} m - 1(z) - x_{\gamma} m - 1(s - 1)](m-\alpha),
\]

the expression (81) can be written as

\[
\int_{a+1}^{\gamma} \nabla_{\gamma m - 1}[f(s)] \nabla_{(s)} \left\{ \frac{[x_{\gamma} m - 1(z) - x_{\gamma} m - 1(s - 1)](m-\alpha + 1)}{[\Gamma(m - \alpha + 2)]_q} \right\} d\varphi s
= \int_{a+1}^{\gamma} \nabla_{\gamma m - 1}[f(s)] \nabla_{\gamma m - 2} \left\{ \frac{[x_{\gamma} m - 1(z) - x_{\gamma} m - 1(s - 1)](m-\alpha + 1)}{[\Gamma(m - \alpha + 2)]_q} \right\} d\varphi x_{\gamma} m - 2(s).
\]
Summing by parts formula, we have

\[
\int_{a+1}^{z} \nabla_{\gamma+\alpha-1}[f(s)]\nabla_{\gamma+\alpha-2}\left(\frac{x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)}{\Gamma(m - \alpha + 2)}\right) d\nabla_{\gamma+\alpha-2}(s) \\
= \nabla_{\gamma+\alpha-1}f(a)\frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)](m-\alpha+1)}{\Gamma(m - \alpha + 2)} + \\
+ \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)](m-\alpha+1)}{\Gamma(m - \alpha + 2)} \nabla_{\gamma+\alpha-1}f(s)d\nabla_{\gamma+\alpha-1}(s) \\
= \nabla_{\gamma+\alpha-1}f(a)\frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)](m-\alpha+1)}{\Gamma(m - \alpha + 2)} + \\
+ \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)](m-\alpha+1)}{\Gamma(m - \alpha + 2)} \nabla_{\gamma+\alpha-2}^2f(s)d\nabla_{\gamma+\alpha-2}(s) \\
\quad \quad (82)
\]

Therefore, we conclude that

\[
\int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)](m-\alpha)}{\Gamma(m - \alpha + 1)} \nabla_{\gamma+\alpha-1}f(s)d\nabla_{\gamma+\alpha-1}(s) \\
= \nabla_{\gamma+\alpha-1}f(a)\frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)](m-\alpha+1)}{\Gamma(m - \alpha + 2)} + \\
+ \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)](m-\alpha+1)}{\Gamma(m - \alpha + 2)} \nabla_{\gamma+\alpha-2}^2f(s)d\nabla_{\gamma+\alpha-2}(s) \\
\quad \quad (82)
\]

In the same way, by mathematical induction we can obtain

\[
\int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)](m-\alpha+k-1)}{\Gamma(m - \alpha + k)} \nabla_{\gamma+\alpha-k}f(s)d\nabla_{\gamma+\alpha-k}(s) \\
= \nabla_{\gamma+\alpha-k}f(a)\frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)](m-\alpha+k)}{\Gamma(m - \alpha + k + 1)} + \\
+ \int_{a+1}^{z} \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)](m-\alpha+k)}{\Gamma(m - \alpha + k + 1)} \nabla_{\gamma+\alpha-(k+1)}^2f(s)d\nabla_{\gamma+\alpha-(k+1)}(s). \\
\quad \quad (83)
\]

\[(k = 0, 1, \ldots, m - 1)\]
Substituting (80), (82) and (83) into (79), we get

\[
\begin{align*}
\nabla_\gamma^\alpha f(z) &= \nabla_\gamma^m \left\{ f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(m-\alpha)}}{\Gamma(m - \alpha + 1)} \right\} + \\
&+ \nabla_\gamma^{\alpha - 1} f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(m-\alpha+1)}}{\Gamma(m - \alpha + 2)} + \\
&+ \nabla_\gamma^{\alpha - k} f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(m-\alpha+k)}}{\Gamma(m - \alpha + k + 1)} + \\
&+ \ldots + \nabla_\gamma^{m-1} f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(2m-\alpha-1)}}{\Gamma(2m - \alpha)} + \\
&+ \int_a^z \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(s-1)]^{(2m-\alpha-1)}}{\Gamma(2m - \alpha)} \nabla_\gamma^{m} f(s) d\nabla x_{\gamma+m}(s) \\
&= \nabla_\gamma^m \left\{ \sum_{k=0}^{m-1} \nabla_\gamma^{\alpha - k} f(a) \frac{[x_{\gamma+m-1}(z) - x_{\gamma+m-1}(a)]^{(m-\alpha+k)}}{\Gamma(m - \alpha + k + 1)} + \\
&+ \nabla_\gamma^{\alpha - 2m} \nabla_\gamma^{m} f(z) \right\} \\
&= \sum_{k=0}^{m-1} \nabla_\gamma^{\alpha - k} f(a) \frac{[x_{\gamma-1}(z) - x_{\gamma-1}(a)]^{(-\alpha+k)}}{\Gamma(-\alpha + k + 1)} + \nabla_\gamma^{\alpha - m} \nabla_\gamma^{m} f(z).
\end{align*}
\]

As a result, we have the following

**Theorem 20 (Solution for Abel equation)** Set functions \( f(z) \) and \( g(z) \) over \( \{a + 1, a + 2, \ldots, z\} \) satisfy

\[
\nabla_\gamma^{-\alpha} g(z) = f(z), \quad 0 < m - 1 < \text{Re} \alpha \leq m,
\]

then

\[
g(z) = \sum_{k=0}^{m-1} \nabla_\gamma^{\alpha - k} f(a) \frac{[x_{\gamma-1}(z) - x_{\gamma-1}(a)]^{(-\alpha+k)}}{\Gamma(-\alpha + k + 1)} + \nabla_\gamma^{\alpha - m} \nabla_\gamma^{m} f(z)
\]

(84)

holds.

Inspired by **Theorem 20** This is also natural that we give the \( \alpha \)-th order Caputo fractional difference of \( f(z) \) as follows:

**Definition 21 (Caputo fractional difference)** Let \( m \) be the smallest integer exceeding \( \text{Re} \alpha \), \( \alpha \)-th order Caputo fractional difference of \( f(z) \) over \( \{a + 1, a + 2, \ldots, z\} \) on non-uniform lattices is defined by

\[
C^\alpha \nabla_\gamma^m f(z) = \nabla_\gamma^{\alpha - m} \nabla_\gamma^{m} f(z).
\]

(85)
6 Some Propositions and Theorems

Some fundamental Propositions, Theorems and Taylor formula on non-uniform lattices are very important. We will take effort to establish it in this section. First, it is easy to prove that

**Lemma 22** Let \( \alpha > 0 \), then

\[
\nabla^{-\alpha_1} = \frac{[x_{\gamma+\alpha-1}(z) - x_{\gamma+\alpha-1}(a)]^{(\alpha)}}{[\Gamma(\alpha + 1)]_q}.
\]

**Proof.** By the use of Proposition 11, one has

\[
\nabla_{-t} \left[ x_{\gamma+\alpha-1}(z) - x_{\gamma+\alpha-1}(t) \right]^{(\alpha)} \nabla_{x_{\gamma}} f(t) = -[\alpha]_q [x_{\gamma+\alpha-1}(z) - x_{\gamma+\alpha-1}(t-1)]^{(\alpha-1)}. \quad (86)
\]

It is easy to know that

\[
\nabla^{-\alpha_1} = \sum_{t=a+1}^{z} \frac{[x_{\gamma+\alpha-1}(z) - x_{\gamma+\alpha-1}(t-1)]^{(\alpha-1)}}{[\Gamma(\alpha)]_q} \nabla_{x_{\gamma}} f(t)
\]

\[
= - \sum_{t=a+1}^{z} \nabla_{t} [x_{\gamma+\alpha-1}(z) - x_{\gamma+\alpha-1}(t)]^{(\alpha)}
\]

\[
= \frac{[x_{\gamma+\alpha-1}(z) - x_{\gamma+\alpha-1}(a)]^{(\alpha)}}{[\Gamma(\alpha + 1)]_q}. \quad (87)
\]

**Theorem 23** (Taylor Theorem) Let \( k \in \mathbb{N} \), then

\[
\nabla^{-k} \nabla_{x_{\gamma}}^k f(z) = f(z) - f(a) - \nabla_{\gamma}^1 f(a) [x_{\gamma+1}(z) - x_{\gamma+1}(a)]
\]

\[
- \frac{1}{[2]_q!} \nabla_{\gamma+2}^k f(a) [x_{\gamma+2}(z) - x_{\gamma+2}(a)]^{(2)}
\]

\[
- \cdots - \frac{1}{[k-1]_q!} \nabla_{\gamma+k-1}^1 f(a) [x_{\gamma+k-1}(z) - x_{\gamma+k-1}(a)]^{(k-1)}
\]

\[
= f(z) - \sum_{j=0}^{k-1} \frac{1}{[j]_q!} \nabla_{\gamma+k-j}^j f(a) [x_{\gamma+k-j}(z) - x_{\gamma+k-j}(a)]^{(j)}. \quad (88)
\]

**Proof.** When \( k = 1 \), we should prove that

\[
\nabla_{\gamma}^{-1} \nabla_{x_{\gamma}}^1 f(z) = f(z) - f(a). \quad (89)
\]

In fact, one has

\[
LHS = \sum_{s=a+1}^{z} \nabla_{\gamma}^1 f(s) \nabla_{x_{\gamma}} f(s) = \sum_{s=a+1}^{z} \nabla f(s) = f(z) - f(a).
\]
When $k = 2$, we should prove that
\[
\nabla^2_\gamma \nabla^2_\eta f(z) = f(z) - f(a) - \nabla^1_\eta f(a)[x_{\gamma+1}(z) - x_{\gamma+1}(a)].
\]
(90)

Actually, we have
\[
\nabla^2_\gamma \nabla^2_\eta f(z) = \nabla^1_\gamma \nabla^1_\eta \nabla^1_\gamma \nabla^1_\eta f(z) = \nabla^1_\gamma [\nabla^1_\gamma \nabla^1_\eta] \nabla^1_\gamma \nabla^1_\eta f(z),
\]
by the use of (89) and Lemma 22 we have
\[
\nabla^{-1}_{\gamma+1}[\nabla^{-1}_{\gamma} \nabla^1_{\gamma+1}] \nabla^1_{\gamma+1} f(z) = \nabla^{-1}_{\gamma+1}[\nabla^1_{\gamma+1} f(z) - \nabla^1_{\gamma+1} f(a)]
= f(z) - f(a) - \nabla^{-1}_{\gamma+1}[\nabla^1_{\gamma+1} f(a)]
= f(z) - f(a) - \nabla^1_{\gamma+1} f(a)[x_{\gamma+1}(z) - x_{\gamma+1}(a)].
\]
(91)

Assume that when $n = k$, (88) holds, then for $n = k + 1$, we should prove that
\[
\nabla^{-(k+1)} \nabla^1_{\gamma} f(z) = f(z) - \sum_{j=0}^{k} \frac{1}{[j]!q^j} \nabla^j_{\gamma+k-j} f(a)[x_{\gamma+k}(z) - x_{\gamma+k}(a)]^{[j]}.
\]
(92)

In fact, we have
\[
\nabla^{-(k+1)} \nabla^1_{\gamma} f(z) = \nabla^{-1}_{\gamma+k} \nabla^1_{\gamma} \nabla^1_{\gamma+k} \nabla^1_{\gamma+k} f(z) = \nabla^{-1}_{\gamma+k} [\nabla^{-1}_{\gamma} \nabla^1_{\gamma+k}] \nabla^1_{\gamma+k} f(z)
= \nabla^{-1}_{\gamma+k} [\nabla^1_{\gamma+k} f(z) - \sum_{j=0}^{k-1} \frac{1}{[j]!q^j} \nabla^j_{\gamma+k-j} \nabla^1_{\gamma+k} f(a)[x_{\gamma+k-1}(z) - x_{\gamma+k-1}(a)]^{[j]}]
= f(z) - f(a) - \sum_{j=0}^{k-1} \frac{1}{[j+1]!q^j} \nabla^j_{\gamma+k-j} \nabla^1_{\gamma+k} f(a)[x_{\gamma+k-1}(z) - x_{\gamma+k-1}(a)]^{[j+1]}
\]
(93)
\[
= f(z) - \sum_{j=0}^{k} \frac{1}{[j]!q^j} \nabla^j_{\gamma+k-j} f(a)[x_{\gamma+k}(z) - x_{\gamma+k}(a)]^{[j]},
\]
(94)
the last equation holds is due to
\[
\frac{\nabla}{\nabla x_{\gamma+k}(z)} \nabla^{-1}_{\gamma+k} [x_{\gamma+k-1}(z) - x_{\gamma+k-1}(a)]^{[j]} = [x_{\gamma+k-1}(z) - x_{\gamma+k-1}(a)]^{[j]}
\]
\[
= \frac{1}{[j+1]q} \frac{\nabla}{\nabla x_{\gamma+k}(z)} [x_{\gamma+k}(z) - x_{\gamma+k}(a)]^{[j+1]},
\]
(95)
hence it holds that
\[
\nabla^{-1}_{\gamma+k} [x_{\gamma+k-1}(z) - x_{\gamma+k-1}(a)]^{[j]} = \frac{1}{[j+1]q} [x_{\gamma+k}(z) - x_{\gamma+k}(a)]^{[j+1]}.
\]
(96)
**Proposition 24** For any $\Re \alpha, \Re \beta > 0$, we have

$$\nabla_{\gamma + \alpha}^{-\beta} \nabla_{\gamma}^{-\alpha} f(z) = \nabla_{\gamma + \alpha}^{-\beta} f(z) = \nabla_{\gamma}^{-(\alpha + \beta)} f(z). \quad (97)$$

**Proof.** By Definition [12] we have

$$\nabla_{\gamma + \alpha}^{-\beta} \nabla_{\gamma}^{-\alpha} f(z) = \sum_{t=\alpha+1}^{z} \frac{[x_{\gamma + \alpha + \beta - 1}(z) - x_{\gamma + \alpha + \beta - 1}(t - 1)]^{(\beta - 1)}}{[\Gamma(\beta)]_q} \nabla_{\gamma}^{-\alpha} f(t) \nabla_{\gamma + \alpha}(t)$$

$$= \sum_{t=\alpha+1}^{z} \frac{[x_{\gamma + \alpha + \beta - 1}(z) - x_{\gamma + \alpha + \beta - 1}(t - 1)]^{(\beta - 1)}}{[\Gamma(\beta)]_q} \nabla_{\gamma + \alpha}(t)$$

$$+ \sum_{s=\alpha+1}^{t} \frac{[x_{\gamma + \alpha - 1}(t) - x_{\gamma + \alpha - 1}(s - 1)]^{(\alpha - 1)}}{[\Gamma(\alpha)]_q} f(s) \nabla_{\gamma}(s)$$

$$= \sum_{s=\alpha+1}^{z} \frac{[x_{\gamma + \alpha - 1}(t) - x_{\gamma + \alpha - 1}(s - 1)]^{(\alpha - 1)}}{[\Gamma(\alpha)]_q} f(s) \nabla_{\gamma + \alpha}(t).$$

In Theorem [13], replacing $a + 1$ with $s; \alpha$ with $\alpha - 1$; and replacing $x(t)$ with $x_{\nu + \alpha - 1}(t)$, then $x_{\beta}(t)$ with $x_{\nu + \alpha + \beta - 1}(t)$, we get that

$$\sum_{s=\alpha+1}^{z} \frac{[x_{\gamma + \alpha + \beta - 1}(z) - x_{\gamma + \alpha + \beta - 1}(t - 1)]^{(\beta - 1)}}{[\Gamma(\beta)]_q} \nabla_{\gamma + \alpha}(t)$$

$$\cdot \frac{[x_{\gamma + \alpha - 1}(t) - x_{\gamma + \alpha - 1}(s - 1)]^{(\alpha - 1)}}{[\Gamma(\alpha)]_q} \nabla_{\gamma + \alpha}(t)$$

$$= \frac{[x_{\gamma + \alpha + \beta - 1}(z) - x_{\gamma + \alpha + \beta - 1}(s - 1)]^{(\alpha + \beta - 1)}}{[\Gamma(\alpha + \beta)]_q}$$

it yields

$$\nabla_{\gamma + \alpha}^{-\beta} \nabla_{\gamma}^{-\alpha} f(z) = \sum_{s=\alpha+1}^{z} \frac{[x_{\gamma + \alpha + \beta - 1}(z) - x_{\gamma + \alpha + \beta - 1}(s - 1)]^{(\alpha + \beta - 1)}}{[\Gamma(\alpha + \beta)]_q} f(s) \nabla_{\gamma}(s)$$

$$= \nabla_{\gamma}^{-(\alpha + \beta)} f(z).$$

**Proposition 25** For any $\Re \alpha > 0$, we have

$$\nabla_{\gamma}^{\alpha} \nabla_{\gamma}^{-\alpha} f(z) = f(z). \quad (98)$$

**Proof.** By Definition [12] we have

$$\nabla_{\gamma}^{\alpha} \nabla_{\gamma}^{-\alpha} f(z) = \nabla_{\gamma}^{m(\nabla_{\gamma}^{-m})} \nabla_{\gamma}^{-\alpha} f(z). \quad (99)$$
In view of Proposition 24, one gets
\[ \nabla_{\gamma}^{m-m} \nabla_{\gamma}^{-\alpha} f(z) = \nabla_{\gamma}^{-m} f(z). \]
Therefore, we obtain
\[ \nabla_{\gamma}^{\alpha} \nabla_{\gamma}^{-\alpha} f(z) = \nabla_{\gamma}^{m} \nabla_{\gamma}^{-m} f(z) = f(z). \]

\[ \textbf{Proposition 26} \]
Let \( m \in N^+ \), \( \alpha > 0 \), then
\[ \nabla_{\gamma}^{m} \nabla_{\gamma+m}^{-\alpha} f(z) = \begin{cases} \nabla_{\gamma}^{m-\alpha} f(z), & \text{when } m - \alpha < 0 \\ \nabla_{\gamma}^{-\alpha} f(z), & \text{when } m - \alpha > 0 \end{cases} \] (100)

\[ \textbf{Proof.} \] If \( 0 \leq \alpha < 1 \), set \( \beta = m - \alpha \), then \( 0 \leq m - 1 < \beta \leq m \). By Definition 17, one has
\[ \nabla_{\gamma}^{\beta} f(z) = \nabla_{\gamma}^{m} \nabla_{\gamma+\beta}^{-\alpha} f(z), \]
that is
\[ \nabla_{\gamma}^{m} \nabla_{\gamma+m-\alpha}^{-\alpha} f(z) = \nabla_{\gamma}^{-\alpha} f(z). \] (101)
If \( k \leq \alpha < k + 1 \), \( k \in N^+ \), set \( \tilde{\alpha} = \alpha - k \), then \( 0 \leq \tilde{\alpha} < 1 \), one has
\[ \nabla_{\gamma}^{m} \nabla_{\gamma+m-\alpha}^{-\alpha} f(z) = \nabla_{\gamma}^{m} \nabla_{\gamma+m-k-\tilde{\alpha}}^{-\tilde{\alpha}} f(z) = \nabla_{\gamma}^{m} \nabla_{\gamma+m-k}^{-k} \nabla_{\gamma+m-k-\tilde{\alpha}}^{-\tilde{\alpha}} f(z) \]
When \( m - k > 0 \), we have
\[ \nabla_{\gamma}^{m} \nabla_{\gamma+m-k}^{-k} \nabla_{\gamma+m-k-\tilde{\alpha}}^{-\tilde{\alpha}} f(z) = \nabla_{\gamma}^{m-k} \nabla_{\gamma+m-k-\tilde{\alpha}}^{-\tilde{\alpha}} f(z), \]
since \( m - k - \tilde{\alpha} = m - \alpha > 0 \), From (101), one has
\[ \nabla_{\gamma}^{m-k} \nabla_{\gamma+m-k-\tilde{\alpha}}^{-\tilde{\alpha}} f(z) = \nabla_{\gamma}^{-\alpha} f(z) = \nabla_{\gamma}^{m-\alpha} f(z). \]
When \( m - k < 0 \), we have
\[ \nabla_{\gamma}^{m-k} \nabla_{\gamma+m-k}^{-k} \nabla_{\gamma+m-k-\tilde{\alpha}}^{-\tilde{\alpha}} f(z) = \nabla_{\gamma}^{m-k} \nabla_{\gamma+m-k-\tilde{\alpha}}^{-\tilde{\alpha}} f(z), \]
since \( m - k - \tilde{\alpha} = m - \alpha < 0 \), From (101), we obtain
\[ \nabla_{\gamma}^{m-k} \nabla_{\gamma+m-k-\tilde{\alpha}}^{-\tilde{\alpha}} f(z) = \nabla_{\gamma}^{m-\alpha} f(z) = \nabla_{\gamma}^{m-\alpha} f(z). \]
Obviously, \( m - k > 0 \) or \( m - k > 0 \) is equivalent to \( m - \alpha > 0 \) or \( m - \alpha > 0 \), hence it yields
\[ \nabla_{\gamma}^{m} \nabla_{\gamma+m-\alpha}^{-\alpha} f(z) = \begin{cases} \nabla_{\gamma}^{m-\alpha} f(z), & \text{when } m - \alpha < 0 \\ \nabla_{\gamma}^{-\alpha} f(z), & \text{when } m - \alpha > 0 \end{cases} \]
Proposition 27 Let $\alpha > 0, \beta > 0$. We have

$$\nabla_\gamma^\beta \nabla_\gamma^{\alpha} f(z) = \begin{cases} \nabla_\gamma^{\beta-\alpha} f(z), & (\text{when } \beta - \alpha < 0) \\ \nabla_\gamma^{-\alpha} f(z), & (\text{when } \beta - \alpha > 0) \end{cases}$$

Proof. Let $m$ be the smallest integer exceeding $\beta$, then by Definition 17 we have

$$\nabla_\gamma^m \nabla_\gamma^{\alpha-m} f(z) = \begin{cases} \nabla_\gamma^{\beta-\alpha} f(z), & (\text{when } \beta - \alpha < 0) \\ \nabla_\gamma^{-\alpha} f(z), & (\text{when } \beta - \alpha > 0) \end{cases}$$

Proposition 28 (Fractional Taylor formula) Let $\alpha > 0$, $k$ be the smallest integer exceeding $\alpha$, then

$$\nabla_\gamma^{-\alpha} \nabla_\gamma^{\alpha} f(z) = f(z) - \sum_{j=0}^{k-1} \nabla_\gamma^{j-k+\alpha} f(a) \frac{[x_{\gamma+a-1}(z) - x_{\gamma+a-1}(a)]^{(a-k+j)}}{[\Gamma(\alpha - k + j + 1)]_q}.$$  

Proof. By Definition 24 Definition 17 and Proposition 23 we obtain

$$\nabla_\gamma^{-\alpha} \nabla_\gamma^{\alpha} f(z) = \nabla_\gamma^{-\alpha+k} \nabla_\gamma^{-k} \nabla_\gamma^{\alpha-k} f(z)$$

$$= \nabla_\gamma^{-\alpha+k} \{ \nabla_\gamma^{\alpha-k} f(z) - \frac{1}{[j]_q} \sum_{j=0}^{k-1} \nabla_\gamma^{j} \nabla_\gamma^{k-j+1} \nabla_\gamma^{\alpha-k} f(a) [x_{\gamma+k-1}(z) - x_{\gamma+k-1}(a)]^{(j)} \},$$

From Lemma 26 the use of

$$\nabla_\gamma^{j} \nabla_\gamma^{k-j+1} \nabla_\gamma^{\alpha-k} f(a) = \begin{cases} \nabla_\gamma^{\alpha-k} f(a), & (\text{when } j = 0) \\ \nabla_\gamma^{j-k+\alpha} f(a). & (\text{when } j > 0) \end{cases}$$

and

$$\nabla_\gamma^{-\alpha+k} \frac{[x_{\gamma+k-1}(z) - x_{\gamma+k-1}(a)]^{(j)}}{[\Gamma(j+1)]_q} = \nabla_\gamma^{-\alpha+k} \nabla_\gamma^{-j} f(1) = \nabla_\gamma^{-\alpha+k-j} f(1)$$

$$= \frac{[x_{\gamma+k-1}(z) - x_{\gamma+k-1}(a)]^{(a-k+j)}}{[\Gamma(\alpha - k + j + 1)]_q},$$

reduce to

$$\nabla_\gamma^{-\alpha} \nabla_\gamma^{\alpha} f(z) = f(z) - \sum_{j=0}^{k-1} \nabla_\gamma^{j-k+\alpha} f(a) \frac{[x_{\gamma+a-1}(z) - x_{\gamma+a-1}(a)]^{(a-k+j)}}{[\Gamma(\alpha - k + j + 1)]_q}.$$  

(102)

26
Theorem 29 (Caputo type Fractional Taylor formula) Let $0 < k - 1 < \alpha \leq k$, then
\[
\nabla_{\gamma}^{-\alpha} [C \nabla_{\gamma}^\alpha] f(z) = f(t) - \sum_{j=0}^{k-1} (a \nabla_{\gamma+a-j}^k) f(a) \frac{[x_{\gamma+a-(j+1)}(z) - x_{\gamma+a-(j+1)}(a)]^{(j)}}{\Gamma(j+1)}.
\]  
(103)

Proof. By Definition 21 and Proposition 23 we have
\[
\nabla_{\gamma}^{-\alpha} [C \nabla_{\gamma}^\alpha] f(z) = \nabla_{\gamma}^{-\alpha} \nabla_{\gamma+\alpha-k}^{\alpha-k} \nabla_{\gamma+\alpha-k}^k f(z)
\]
\[
= \nabla_{\gamma+\alpha-k}^k f(z)
\]
\[
= f(t) - \sum_{j=0}^{k-1} (a \nabla_{\gamma+a-j}^j) f(a) \frac{[x_{\gamma+a-(j+1)}(z) - x_{\gamma+a-(j+1)}(a)]^{(j)}}{\Gamma(j+1)}.
\]  
(104)

The relationship between Riemann-Liouville fractional difference and Caputo fractional difference is

Proposition 30 Let $m$ be the smallest integer exceeding $\alpha$, we have
\[
C a \nabla_{\gamma}^\alpha f(z) = [a \nabla_{\gamma}^\alpha] \{f(t) - \sum_{k=0}^{m-1} (a \nabla_{\gamma+\alpha-k}^k) f(a) \frac{[x_{\gamma+a-(k+1)}(z) - x_{\gamma+a-(k+1)}(a)]^{(k)}}{\Gamma(k+1)}\}.
\]  

Proof. We have
\[
C a \nabla_{\gamma}^\alpha f(z) = [a \nabla_{\gamma+\alpha-m}^{\alpha-m} (a \nabla_{\gamma+\alpha-m}^{\alpha-m})] f(z) = [(a \nabla_{\gamma}^\alpha) (a \nabla_{\gamma+\alpha-m}^{\alpha-m})] f(z)
\]
\[
= [a \nabla_{\gamma}^\alpha] \{f(t) - \sum_{k=0}^{m-1} (a \nabla_{\gamma+\alpha-k}^k) f(a) \frac{[x_{\gamma+a-(k+1)}(z) - x_{\gamma+a-(k+1)}(a)]^{(k)}}{\Gamma(k+1)}\}.
\]  
(105)

Proposition 31 Let $\alpha > 0$, we have
\[
(a \nabla_{\gamma}^\alpha) (a \nabla_{\gamma}^{-\alpha}) f(z) = f(z).
\]  
(106)

Proof. Set
\[
g(z) = (a \nabla_{\gamma}^{-\alpha}) f(z) = \int_{a+1}^{z} \frac{[x_{\gamma+a-1}(z) - x_{\gamma+a-1}(t-1)]^{(\alpha-1)}}{\Gamma(\alpha)} f(t) d\varphi x_{\gamma}(t),
\]
then we have $g(a) = 0$. 
And
\[
(a \nabla_{\gamma+a-1}) g(z) = \int_{a+1}^{z} \frac{[x_{\gamma+a-2}(z) - x_{\gamma+a-2}(t-1)]^{(\alpha-2)}}{\Gamma(\alpha-1)} f(t) d\varphi x_{\gamma}(t),
\]
then we have \((a\nabla_{\gamma+\alpha-1})g(a) = 0\).
In the same way, we have
\[(a\nabla_{\gamma+\alpha-k}^k)g(a) = 0, k = 0, 1, \ldots, m - 1.\]
Therefore, by Proposition 30 we obtain \((a\nabla_{\gamma}^\alpha)g(z) = (a\nabla_{\gamma}^\alpha)g(z) = f(z).\)

7 Complex Variable Approach for Riemann-Liouville Fractional Difference On Non-uniform Lattices

In this section, we represent \(k \in N^+\) order difference and \(\alpha \in C\) order fractional difference on non-uniform lattices in terms of complex integration.

**Theorem 32** Let \(n \in N, \Gamma\) be a simple closed positively oriented contour. If \(f(s)\) is analytic in simple connected domain \(D\) bounded by \(\Gamma\) and \(z\) is any nonzero point lies inside \(D\), then
\[
\nabla^n_{\gamma-n+1}f(z) = \frac{[n]!}{2\pi i} \log q \int_\Gamma \frac{f(s)\nabla x_{\gamma+1}(s)ds}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \int_\Gamma [x_{\gamma}(s) - x_{\gamma}(z)]^{(n+1)},
\]
where \(\Gamma\) enclosed the simple poles \(s = z, z - 1, \ldots, z - n\) in the complex plane.

**Proof.** Since the set of points \(\{z - i, i = 0, 1, \ldots, n\}\) lie inside \(D\). Hence, from the genaralized Cauchy’s integral formula, we obtain
\[
f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(s)x'_{\gamma}(s)ds}{[x_{\gamma}(s) - x_{\gamma}(z)]},
\]
and it yields
\[
f(z - 1) = \frac{1}{2\pi i} \int_\Gamma \frac{f(s)x'_{\gamma}(s)ds}{[x_{\gamma}(s) - x_{\gamma}(z - 1)]}. \quad (109)
\]
Substituting with the value of \(f(z)\) and \(f(z - 1)\) into \(\nabla f(z) = \nabla x_{\gamma}(z)\), then we have
\[
\nabla f(z) \nabla x_{\gamma}(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(s)x'_{\gamma}(s)ds}{[x_{\gamma}(s) - x_{\gamma}(z)][x_{\gamma}(s) - x_{\gamma}(z - 1)]}
\]
\[
= \frac{1}{2\pi i} \int_\Gamma \frac{f(s)x'_{\gamma}(s)ds}{[x_{\gamma}(s) - x_{\gamma}(z)]^{(2)}}.
\]
Substituting with the value of \(\nabla f(z)\) and \(\nabla f(z - 1)\) into \(\nabla x_{\gamma}(z) - \nabla x_{\gamma}(z - 1)\), then we have
\[
\frac{\nabla f(z)}{x_{\gamma}(z) - x_{\gamma}(z - 2)} = \frac{1}{2\pi i} \int_\Gamma \frac{f(s)x'_{\gamma}(s)ds}{[x_{\gamma}(s) - x_{\gamma}(z)]^{(3)}}.
\]
In view of

\[ x_\gamma(z) - x_\gamma(z - 2) = [2]_q \nabla x_{\gamma - 1}(z), \]

we obtain

\[ \nabla \left( \frac{\nabla f(z)}{\nabla x_{\gamma - 1}(z)} \right) = \frac{[2]_q}{2\pi i} \oint_{|x_\gamma(s) - x_\gamma(z)|^3} f(s) x_\gamma'(s) ds. \]

More generally, by the induction, we can obtain

\[ \nabla \left( \frac{\nabla \nabla \nabla x_{\gamma - n + 1}(z)}{\nabla x_{\gamma - n + 2}(z)} \right) = \frac{[n]_q!}{2\pi i} \oint_{|x_\gamma(s) - x_\gamma(z)|^{n+1}} f(s) x_\gamma'(s) ds. \]

where

\[ [x_\gamma(s) - x_\gamma(z)]^{(n+1)} = \prod_{i=0}^{n} [x_\gamma(s) - x_\gamma(z - i)]. \]

And last, by the use of identity

\[ x_\gamma'(s) = \frac{\log q}{q^\frac{1}{2} - q^{-\frac{1}{2}}} \nabla x_{\gamma + 1}(s), \]

we have

\[ \nabla^{n}_{\gamma - n + 1} f(z) = \frac{[n]_q!}{2\pi i} \frac{\log q}{q^\frac{1}{2} - q^{-\frac{1}{2}}} \oint_{|x_\gamma(s) - x_\gamma(z)|^{(n+1)}} f(s) \nabla x_{\gamma + 1}(s) ds. \]  \hspace{1cm} (110)

Inspired by formula (110), so we can give the definition of fractional difference of \( f(z) \) over \( \{a + 1, a + 2, ..., z\} \) on non-uniform lattices as follows

**Definition 33 (Complex fractional difference on non-uniform lattices)** Let \( \Gamma \) be a simple closed positively oriented contour. If \( f(s) \) is analytic in simple connected domain \( D \) bounded by \( \Gamma \), assume that \( z \) is any nonzero point inside \( D \), \( a + 1 \) is a point inside \( D \), and \( z - a \in N \), then for any \( \alpha \in R^+ \), the \( \alpha \)-th order fractional difference of \( f(z) \) over \( \{a + 1, a + 2, ..., z\} \) on non-uniform lattices is defined by

\[ \nabla^{\alpha}_{\gamma - a + 1} f(z) = \frac{[\Gamma(\alpha + 1)]_q}{2\pi i} \frac{\log q}{q^\frac{1}{2} - q^{-\frac{1}{2}}} \oint_{|x_\gamma(s) - x_\gamma(z)|^{(\alpha+1)}} f(s) \nabla x_{\gamma + 1}(s) ds. \]  \hspace{1cm} (111)

where \( \Gamma \) enclosed the simple poles \( s = z, z - 1, ..., a + 1 \) in the complex plane.

We can calculate the integral (111) by Cauchy’s residue theorem. In detail, we have

**Theorem 34 (Fractional difference on non-uniform lattices)** Assume \( z, a \in C, z - a \in N, \alpha \in R^+ \).
(1) Let \( x(s) \) be quadratic lattices \([3]\), then the \( \alpha \)-th order fractional difference of \( f(z) \) over \( \{a + 1, a + 2, \ldots, z\} \) on non-uniform lattices can be rewritten by

\[
\nabla_{\gamma+1-\alpha}^\alpha [f(z)] = \sum_{k=0}^{z-(a+1)} f(z-k) \frac{\Gamma(2z-k+\gamma+1-\alpha) \nabla x_{\gamma+1}(z-k)}{\Gamma(2z+\gamma+1-k)} k! ; \quad (112)
\]

(2) Let \( x(s) \) be quadratic lattices \([4]\), then the \( \alpha \)-th order fractional difference of \( f(z) \) over \( \{a + 1, a + 2, \ldots, z\} \) on non-uniform lattices can be rewritten by

\[
\nabla_{\gamma+1-\alpha}^\alpha [f(z)] = \sum_{k=0}^{z-(a+1)} f(z-k) \frac{\Gamma(2z-k+\gamma+1-\alpha) \nabla x_{\gamma+1}(z-k)}{\Gamma(2z+\gamma+1-k)} \left(\frac{1}{k}\right)_q. \quad (113)
\]

**Proof.** From \([11]\), in quadratic lattices \([5]\), one has

\[
\nabla_{\gamma+1-\alpha}^\alpha [f(z)] = \frac{\Gamma(a+1)}{2\pi i} \int_{\Gamma} \frac{f(s) \nabla x_{\gamma+1}(s)}{\Gamma(s-z+\alpha+1)\Gamma(s-z+\gamma+1)} ds
\]

According to the assumption of **Definition 33**, \( \Gamma(s-z) \) has simple poles at \( s = z-k, k = 0, 1, 2, \ldots, z-(a+1) \). The residue of \( \Gamma(s-z) \) at the point \( s = z-k \) is

\[
\lim_{s \to z-k} (s-z-k)\Gamma(s-z) = \lim_{s \to z-k} \frac{(s-z)(s-z+1)\ldots(s-z+k-1)(s-z-k)\Gamma(s-z)}{(s-z)(s-z+1)\ldots(s-z+k-1)\Gamma(s-z+k+1)} = \frac{1}{(-k)(-k+1)\ldots(-1)} = \frac{(-1)^k}{k!}.
\]

Then by the use of Cauchy’s residue theorem, we have

\[
\nabla_{\gamma+1-\alpha}^\alpha [f(z)] = \Gamma(a+1) \sum_{k=0}^{z-(a+1)} f(z-k) \frac{\Gamma(2z-k+\gamma+1-\alpha) \nabla x_{\gamma+1}(z-k)}{\Gamma(a+1-k)\Gamma(2z+\gamma+1-k)} (-1)^k k!.
\]

Since

\[
\frac{\Gamma(a+1)}{\Gamma(a+1-k)} = \alpha(\alpha-1)\ldots(\alpha-k+1),
\]

and

\[
\alpha(\alpha-1)\ldots(\alpha-k+1)(-1)^k = (-\alpha)_k,
\]

therefore, we get

\[
\nabla_{\gamma+1-\alpha}^\alpha [f(z)] = \sum_{k=0}^{z-(a+1)} f(z-k) \frac{\Gamma(2z-k+\gamma+1-\alpha) \nabla x_{\gamma+1}(z-k)}{\Gamma(2z+\gamma+1-k)} (-\alpha)_k k!.
\]

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From (111), in quadratic lattices (6), we have
\[
\nabla_{\gamma - \alpha + 1} f(z) = \frac{[\Gamma(\alpha + 1)]_q}{2\pi i} \log q \frac{\log q}{q^z - q^{-\frac{1}{2}}} \oint f(s) \nabla x_{\gamma + 1}(s) ds
\]

Therefore, we obtain that
\[
\nabla_{\gamma - \alpha + 1} f(z) = \frac{[\Gamma(\alpha + 1)]_q}{2\pi i} \log q \frac{\log q}{q^z - q^{-\frac{1}{2}}} \oint f(s) \nabla x_{\gamma + 1}(s) [\Gamma(s - \alpha + 1)]_q [\Gamma(s + \gamma - \alpha + 1)]_q ds
\]

From the assumption of Definition 33, \([\Gamma(s - \alpha + 1)]_q\) has simple poles at \(s = z - k, k = 0, 1, 2, ..., z - (\alpha + 1)\). The residue of \([\Gamma(s - \alpha + 1)]_q\) at the point \(s - \alpha = -k\) is
\[
\lim_{s \to z - k} (s - z + k)[\Gamma(s - \alpha + 1)]_q
\]
\[
= \lim_{s \to z - k} \frac{s - z + k}{[s - z + k]_q}[s - z + k][\Gamma(s - \alpha + 1)]_q
\]
\[
= \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{\log q} \lim_{s \to z - k} \frac{[s - z + k][s - z + 1]_q...[s - z + k - 1]_q[s - z + k - 1]_q[\Gamma(s - \alpha + 1)]_q}{(s - z)(s - z + 1)...(s - z + k)}
\]
\[
= \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{\log q} \lim_{s \to z - k} \frac{[\Gamma(s - z + k + 1)]_q}{[s - z + 1]_q...[s - z + k - 1]_q} = \frac{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}{\log q} [k + 1]_q^k.
\]

Then by the use of Cauchy’s residue theorem, we have
\[
\nabla_{\gamma + 1 - \alpha} f(z) = \frac{[\Gamma(\alpha + 1)]_q}{[\Gamma(\alpha + 1 - k)]_q} \sum_{k=0}^{z-(\alpha+1)} f(z-k)[\Gamma(2z - k + \gamma - \alpha)]_q \nabla x_{\gamma + 1}(z - k) (-1)^k
\]

Since
\[
\frac{[\Gamma(\alpha + 1)]_q}{[\Gamma(\alpha + 1 - k)]_q} = [\alpha]_q[\alpha - 1]_q...[\alpha - k + 1]_q,
\]
and
\[
[\alpha]_q[\alpha - 1]_q...[\alpha - k + 1](-1)^k = ([-\alpha])_k,
\]
therefore, we obtain that
\[
\nabla_{\gamma + 1 - \alpha} f(z) = \sum_{k=0}^{z-(\alpha+1)} f(z-k) \frac{[\Gamma(2z - k + \gamma - \alpha)]_q \nabla x_{\gamma + 1}(z - k) \frac{([\alpha]_k)_k}{k!}}{[\Gamma(2z + \gamma + 1 - k)]_q}.
\]
So far, with respect to the definition of the R-L fractional difference on non-uniform lattices, we have given two kinds of definitions, such as Definition 17 or Definition 18 in section 4 and Definition 33 or Definition 34 in section 7 through two different ideas and methods. Now let’s compare Definition 18 in section 4 and Definition 34 in section 7.

Here follows a theorem connecting the R-L fractional difference (77) and the complex generalization of fractional difference (111):

**Theorem 35** For any \( \alpha \in \mathbb{R}^+ \), let \( \Gamma \) be a simple closed positively oriented contour. If \( f(s) \) is analytic in simple connected domain \( D \) bounded by \( \Gamma \), assume that \( z \) is any nonzero point inside \( D \), \( a + 1 \) is a point inside \( D \), such that \( z - a \in \mathbb{N} \), then the complex generalization fractional integral (111) equals the R-L fractional difference (77) or (78):

\[
\nabla_{\gamma+1-a}^\alpha [f(z)] = \sum_{k=a+1}^{z} \frac{[x_{\gamma-a}(z) - x_{\gamma-a}(k-1)]^{(a-1)}}{[\Gamma(-\alpha)]_q} f(k) \nabla_{\gamma+1}(k).
\]

**Proof.** By Theorem 34 we have

\[
\nabla_{\gamma+1-a}^\alpha [f(z)] = \sum_{k=0}^{z-(a+1)} \frac{(-\alpha)_k q [\Gamma(2z - k + \gamma - \alpha)]_q}{[\Gamma(2z - k + \gamma + 1)]_q} f(z - k) \nabla_{\gamma+1}(z - k).
\]

\[
\begin{align*}
&= \sum_{k=0}^{z-(a+1)} \frac{[\Gamma(k - \alpha)]_q}{[\Gamma(-\alpha)]_q} \frac{[\Gamma(2z - k + \gamma - \alpha)]_q}{[\Gamma(2z - k + \gamma + 1)]_q} f(z - k) \nabla_{\gamma+1}(z - k) \\
&= \sum_{k=0}^{z-(a+1)} \frac{[x_{\gamma-a}(z) - x_{\gamma-a}(z - k - 1)]^{(a-1)}}{[\Gamma(-\alpha)]_q} f(z - k) \nabla_{\gamma+1}(z - k) \\
&= \sum_{k=a+1}^{z} \frac{[x_{\gamma-a}(z) - x_{\gamma-a}(k-1)]^{(a-1)}}{[\Gamma(-\alpha)]_q} f(k) \nabla_{\gamma+1}(k).
\end{align*}
\]

So that the two Definition 18 and Definition 34 are consistent. \( \square \)

Set \( \alpha = \gamma \) in Theorem 34 we obtain

**Corollary 36** Assume that conditions of Definition 33 hold, then

\[
\nabla_{\gamma}^1 [f(z)] = \frac{[\Gamma(\gamma + 1)]_q \log q}{2\pi i} \int_{C} f(s) \nabla_{\gamma+1}(s) ds
\]

\[
= \sum_{k=0}^{z-(\alpha+1)} f(z - k) \frac{[\Gamma(2z + \mu - k)]_q \nabla_{\gamma+1}(z - k) ([-\alpha]_q)_k}{[\Gamma(2z + \gamma + \mu + 1 - k)]_q}.
\]

where \( \Gamma \) enclosed the simple poles \( s = z, z - 1, ..., a + 1 \) in the complex plane.
Remark 37  When $\gamma = n \in N^+$, we have

$$\nabla_n^n[f(z)] = \frac{\Gamma(n+1)q}{2\pi i} \log q \frac{\Gamma(\beta + 1) - \Gamma(\alpha + 1)}{z - \beta} f(s)\nabla x_{\gamma+1}(s) ds$$

$$= \sum_{k=0}^{n} f(z - k) \frac{\Gamma(2z + \mu - k)q\nabla x_{n+1}(z - k) \left(\frac{-\mu - qk}{k!}\right)}{\Gamma(2z + n + \mu + 1 - k)q}, \quad (115)$$

where $\Gamma$ enclosed the simple poles $s = z, z - 1, ..., z - n$ in the complex plane.

This is consistent with Definition 39 proposed by Nikiforov. A, Uvarov. V, Suslov. S in [35].

For complex integral of Riemann-Liouville fractional difference on non-uniform lattices, we can establish an important Cauchy Beta formula as follows:

Theorem 38 (Cauchy Beta formula) Let $\alpha, \beta \in C$, and assume that

$$\int_{\Gamma} \Delta \{\frac{1}{[x_\beta(z) - x_\beta(t)]^{(\beta)}}, \frac{1}{[x_{\alpha}(t) - x_{\alpha}(a)]^{(\alpha)}}\} dt = 0,$$

then

$$\frac{1}{2\pi i q^{\beta} - q^{\beta}} \int_{\Gamma} \frac{[\Gamma(\beta + 1)]_q}{[x_\beta(z) - x_\beta(t)]^{(\beta)}} \frac{[\Gamma(\alpha)]_q}{[x_{\alpha}(t) - x_{\alpha}(a)]^{(\alpha)}} = \frac{[\Gamma(\alpha + \beta)]_q}{[x_\beta(z) - x_\beta(a)]^{(\alpha + \beta)}},$$

where $\Gamma$ be a simple closed positively oriented contour, a lies inside $C$.

In order to prove Theorem 38, we first give a lemma.

Lemma 39 For any $\alpha, \beta$, then we have

$$[1 - \alpha]_q[x_\beta(z) - x_\beta(t - \beta)] + [\beta]_q[x_{\alpha}(t + \alpha - 1) - x_{\alpha}(a)]$$

$$= [1 - \alpha]_q[x_\beta(z) - x_\beta(a + 1 - \alpha - \beta)] + [\alpha + \beta - 1]_q[x(t) - x(a + 1 - \alpha)]. \quad (116)$$

Proof. (116) is equivalent to

$$[\alpha + \beta - 1]_q[x(t) + [1 - \alpha]_q x_\beta(t - \beta) - [\beta]_q x_{\alpha}(t + \alpha - 1)$$

$$= [\alpha + \beta - 1]_q x(a + 1 - \alpha) + [1 - \alpha]_q x_\beta(a + 1 - \alpha - \beta) - [\beta]_q x_{\alpha}(a). \quad (117)$$

Set $\alpha - 1 = \bar{\alpha}$, then (117) can be written as

$$[\bar{\alpha} + \beta]_q x(t) - [\bar{\alpha}]_q x_{-\beta}(t) - [\beta]_q x_{\bar{\alpha}}(t)$$

$$= [\bar{\alpha} + \beta]_q x(a - \bar{\alpha}) - [\bar{\alpha}]_q x_{-\beta}(a - \bar{\alpha}) - [\beta]_q x_{\bar{\alpha}}(a - \bar{\alpha}). \quad (118)$$

By the use of Lemma 15, then Eq. (118) holds, and then Eq. (116) holds.  

Proof of Theorem 38 Set

$$\rho(t) = \frac{1}{[x_\beta(z) - x_\beta(t)]^{(\beta+1)}} \frac{1}{[x(t) - x(a)]^{(\alpha)}},$$

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\[
\sigma(t) = [x_{\alpha-1}(t + \alpha - 1) - x_{\alpha-1}(a)][x_{\beta}(z) - x_{\beta}(t)],
\]

Since
\[
[x_{\beta}(z) - x_{\beta}(t)]^{(\beta + 1)} = [x_{\beta}(z) - x_{\beta}(t - 1)]^{(\beta)}[x_{\beta}(z) - x_{\beta}(t)],
\]

and
\[
[x(t) - x(a)]^{(\alpha)} = [x_{-1}(t) - x_{-1}(a)]^{(\alpha-1)}[x_{\alpha-1}(t + \alpha - 1) - x_{\alpha-1}(a)],
\]

these reduce to
\[
\sigma(t)\rho(t) = \frac{1}{[x_{\beta}(z) - x_{\beta}(t)]^{(\beta)}} \frac{1}{[x_{-1}(t) - x_{-1}(a)]^{(\alpha-1)}}.
\]

Making use of
\[
\Delta_t[f(t)g(t)] = g(t + 1)\Delta_t[f(t)] + f(t)\Delta_t[g(t)],
\]

where
\[
f(t) = \frac{1}{[x_{-1}(t) - x_{-1}(a)]^{(\alpha-1)}}, g(t) = \frac{1}{[x_{\beta}(z) - x_{\beta}(t - 1)]^{(\beta)}},
\]

and
\[
\frac{\Delta_t}{\Delta x_{-1}(t)} \left\{ \frac{1}{[x_{-1}(t) - x_{-1}(a)]^{(\alpha-1)}} \right\} = \frac{[1 - \alpha]_q}{[x(t) - x(a)]^{(\alpha)}},
\]

\[
\frac{\Delta_t}{\Delta x_{-1}(t)} \left\{ \frac{1}{[x_{\beta}(z) - x_{\beta}(t - 1)]^{(\beta)}} \right\}
\]

\[
= \nabla_t \left\{ \frac{1}{[x_{\beta}(z) - x_{\beta}(t)]^{(\beta)}} \right\}
\]

\[
= \nabla x_{1}(t) \left\{ \frac{1}{[x_{\beta}(z) - x_{\beta}(t)]^{(\beta)}} \right\}
\]

\[
= \frac{\Delta_t}{\Delta x_{-1}(t)} \left\{ [\beta]_q \right\}
\]

then, we have
\[
\frac{\Delta_t}{\Delta x_{-1}(t)} \left\{ \sigma(t)\rho(t) \right\}
\]

\[
= \frac{1}{[x_{\beta}(z) - x_{\beta}(t)]^{(\beta)}} \frac{[1 - \alpha]_q}{[x(t) - x(a)]^{(\alpha)}},
\]

\[
+ \frac{1}{[x_{-1}(t) - x_{-1}(a)]^{(\alpha-1)}[x_{\beta}(z) - x_{\beta}(t)]^{(\beta+1)}}
\]

\[
= \{[1 - \alpha]_q[x_{\beta}(z) - x_{\beta}(t - \beta)] + [\beta]_q[x_{1-\alpha}(t + \alpha - 1) - x_{1-\alpha}(a)]\}
\]

\[
\times \frac{1}{[x(t) - x(a)]^{(\alpha)}} \frac{1}{[x_{\beta}(z) - x_{\beta}(t)]^{(\beta+1)}}
\]

\[
= \tau(t)\rho(t),
\]

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where

\[ \tau(t) = [1 - \alpha]q[x_\beta(z) - x_\beta(t - \beta)] + [\beta]q[x_1-\alpha(t + \alpha - 1) - x_1-\alpha(a)], \]

this is due to

\[ [x_\beta(z) - x_\beta(t)]^{(\beta+1)} = [x_\beta(z) - x_\beta(t)]^{(\beta)}[x_\beta(z) - x_\beta(t - \beta)]. \]

From Proposition 119 one has

\[
\frac{\Delta_t}{\Delta x_{-1}(t)} \{ \sigma(t)\rho(t) \} \\
= \{ [1 - \alpha]q[x_\beta(z) - x_\beta(a + 1 - \alpha - \beta)] + [\alpha + \beta - 1]q[x(t) - x(a + 1 - \alpha)] \} \\
\cdot \frac{1}{[x_\beta(z) - x_\beta(t)]^{(\beta+1)}} \frac{1}{[x(t) - x(a)]^{(\alpha)\cdot}}.
\]

Or

\[
\Delta_t \{ \sigma(t)\rho(t) \} \\
= \{ [1 - \alpha]q[x_\beta(z) - x_\beta(a + 1 - \alpha - \beta)] + [\alpha + \beta - 1]q[x(t) - x(a + 1 - \alpha)] \} \\
\cdot \frac{1}{[x_\beta(z) - x_\beta(t)]^{(\beta+1)}} \frac{1}{[x(t) - x(a)]^{(\alpha)}} \Delta x_{-1}(t).
\] (119)

Set

\[
I(\alpha) = \frac{1}{2\pi i} \frac{\log q}{q^\frac{1}{2} - q^{-\frac{1}{2}}} \oint_D \frac{1}{[x_\beta(z) - x_\beta(t)]^{(\beta+1)}} \frac{1}{[x(t) - x(a)]^{(\alpha)\cdot}} \nabla y_1(t) \, dt,
\] (120)

and

\[
I(\alpha - 1) = \frac{1}{2\pi i} \frac{\log q}{q^\frac{1}{2} - q^{-\frac{1}{2}}} \oint_D \frac{1}{[x_\beta(z) - x_\beta(t)]^{(\beta+1)}} \frac{1}{[x(t) - x(a)]^{(\alpha - 1)\cdot}} \nabla y_1(t) \, dt.
\]

Since

\[ [x(t) - x(a)]^{(\alpha - 1)[x(t) - x(a + 1 - \alpha)]} = [x(t) - x(a)]^{(\alpha)}, \]

then

\[
I(\alpha - 1) = \frac{1}{2\pi i} \frac{\log q}{q^\frac{1}{2} - q^{-\frac{1}{2}}} \oint_D \frac{1}{[x_\beta(z) - x_\beta(t)]^{(\beta+1)}} \frac{[x(t) - x(a + 1 - \alpha)]\nabla y_1(t) \, dt}{[x(t) - x(a)]^{(\alpha)}}.
\]

Integrating both sides of equation (119), then we have

\[
\oint_D \Delta_t \{ \sigma(t)\rho(t) \} \, dt = [1 - \alpha]q[x_\beta(z) - x_\beta(a + 1 - \alpha - \beta)]I(\alpha) \\
- [\alpha + \beta - 1]qI(\alpha - 1).
\]

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If
\[ \oint_{\Gamma} \Delta_t \{ \sigma(t) \rho(t) \} dt = 0, \]
then we obtain that
\[ \frac{I(\alpha - 1)}{I(\alpha)} = \frac{[\alpha - 1]_q}{[\alpha + \beta - 1]_q} [y_\beta(z) - y_\beta(a + 1 - \alpha - \beta)]. \]

That is
\[ I(\alpha - 1) = \frac{[\Gamma(\alpha + \beta - 1)]_q}{[\Gamma(\alpha - 1)]_q} \frac{1}{x_\beta(z) - x_\beta(a)}^{(\alpha + \beta - 1)} \left( I(\alpha) \right). \tag{121} \]

From (121), we set
\[ I(\alpha) = k \frac{[\Gamma(\alpha + \beta)]_q}{[\Gamma(\alpha)]_q} \frac{1}{x_\beta(z) - x_\beta(a)}^{(\alpha + \beta)}, \tag{122} \]
where \( k \) is undetermined.

Set \( \alpha = 1 \), one has
\[ I(1) = k \frac{[\Gamma(1 + \beta)]_q}{[\Gamma(1)]_q} \frac{1}{x_\beta(z) - x_\beta(a)}^{(1 + \beta)}, \tag{123} \]
and from (120) and generalized Cauchy residue theorem, one has
\[
I(1) = \frac{1}{2\pi i} \frac{\log q}{q^\frac{1}{\beta} - q^{-\frac{1}{\beta}}} \oint_{\Gamma} \frac{1}{x_\beta(z) - x_\beta(t)}^{(\beta + 1)} \frac{\nabla x_1(t)}{x'(t)} dt
= \frac{1}{2\pi i} \oint_{\Gamma} \frac{1}{x_\beta(z) - x_\beta(t)}^{(\beta + 1)} \frac{1}{x(t) - x(a)} \frac{\nabla x_1(t) dt}{x'(t)}
= \frac{1}{[x_\beta(z) - x_\beta(a)]^{(\beta + 1)}}, \tag{124}
\]

From (123) and (124), we get
\[ k = \frac{1}{[\Gamma(1 + \beta)]_q}. \]

Therefore, we obtain that
\[ I(\alpha) = \frac{[\Gamma(\alpha + \beta)]_q}{[\Gamma(\beta + 1)]_q [\Gamma(\alpha)]_q} \frac{1}{x_\beta(z) - x_\beta(a)}^{(\alpha + \beta)}, \]
and Theorem 38 is completed.
8 Fractional Central Sum and Difference on Non-uniform Lattices

Next we will give the definition of fractional central sum and fractional central difference on Non-uniform Lattices. Let us first give the integral central sum of $f(z)$ on non-uniform lattices $x(s)$.

$1 - th$ central sum of $f(z)$ on non-uniform lattices $x(s)$ is defined by

$$
\delta_{-1}^{0} f(z) = y_1(z) = \sum_{s=a+\frac{1}{2}}^{z-\frac{1}{2}} f(s) \delta x(s) = \int_{a+\frac{1}{2}}^{z-\frac{1}{2}} f(s) d_\delta x(s),
$$

where $f(s)$ is defined in $\{a + \frac{1}{2}, \text{mod}(1)\}$ and $y_1(z)$ is defined in $\{a + 1, \text{mod}(1)\}$.

Then we have $2 - th$ central sum of $f(z)$ on non-uniform lattices $x(s)$ as follows

$$
\delta_{-2}^{0} f(z) = y_2(z) = \int_{a+\frac{1}{2}}^{z-\frac{1}{2}} \delta_{-1}^{0} f(s) = \int_{a+\frac{1}{2}}^{z-\frac{1}{2}} \int_{a+\frac{1}{2}}^{s-\frac{1}{2}} f(t) d_\delta x(t)
$$

$$
= \int_{a+\frac{1}{2}}^{z-\frac{1}{2}} \int_{a+\frac{1}{2}}^{\frac{s-1}{2}} f(t) d_\delta x(t) d_\delta x(s)
$$

$$
= \int_{a+\frac{1}{2}}^{z-1} \int_{a+\frac{1}{2}}^{\frac{s-1}{2}} \int_{a+\frac{1}{2}}^{x(s) - x(t)} f(t) d_\delta x(t) d_\delta x(s)
$$

$$
= \int_{a+\frac{1}{2}}^{z-2} \int_{a+\frac{1}{2}}^{\frac{s-1}{2}} \int_{a+\frac{1}{2}}^{\frac{(s-1)-1}{2}} \int_{a+\frac{1}{2}}^{x(s) - x(t)} f(t) d_\delta x(t) d_\delta x(s)
$$

where $y_1(z)$ is defined in $\{a + 1, \text{mod}(1)\}$ and $y_2(z)$ is defined in $\{a + \frac{3}{2}, \text{mod}(1)\}$.

and $3 - th$ central sum of $f(z)$ on non-uniform lattices $x(s)$ is

$$
\delta_{-3}^{0} f(z) = y_3(z) = \int_{a+\frac{3}{2}}^{z-\frac{1}{2}} \delta_{-2}^{0} f(s) = \int_{a+\frac{3}{2}}^{z-\frac{1}{2}} \int_{a+\frac{3}{2}}^{z-\frac{1}{2}} \int_{a+\frac{1}{2}}^{x(s) - x(t)} f(t) d_\delta x(t) d_\delta x(s)
$$

$$
= \int_{a+\frac{3}{2}}^{z-2} \int_{a+\frac{3}{2}}^{z-2} \int_{a+\frac{1}{2}}^{x(s) - x(t)} f(t) d_\delta x(t) d_\delta x(s)
$$

$$
= \int_{a+\frac{3}{2}}^{z-3} \int_{a+\frac{3}{2}}^{z-3} \int_{a+\frac{1}{2}}^{x(s) - x(t)} f(t) d_\delta x(t) d_\delta x(s)
$$

where $y_2(z)$ is defined in $\{a + \frac{3}{2}, \text{mod}(1)\}$ and $y_3(z)$ is defined in $\{a + 2, \text{mod}(1)\}$.

More generalaly, we have $k - th$ central sum of $f(z)$ on non-uniform lattices $x(s)$ as follows

$$
\delta_{-k}^{0} f(z) = y_k(z) = \int_{a+\frac{k}{2}}^{z-\frac{1}{2}} \delta_{-k+1}^{0} f(s) = \int_{a+\frac{k}{2}}^{z-\frac{1}{2}} \int_{a+\frac{k}{2}}^{z-\frac{1}{2}} \int_{a+\frac{k-1}{2}}^{x(s) - x(t)} f(t) d_\delta x(t) d_\delta x(s)
$$

$$
= \int_{a+\frac{k}{2}}^{z-k} \int_{a+\frac{k}{2}}^{z-k} \int_{a+\frac{k-1}{2}}^{x(s) - x(t)} f(t) d_\delta x(t) d_\delta x(s)
$$

$$
= \int_{a+\frac{k}{2}}^{z-k} \int_{a+\frac{k}{2}}^{z-k} \int_{a+\frac{k-1}{2}}^{x(s) - x(t)} f(t) d_\delta x(t) d_\delta x(s)
$$

where $y_k(z)$ is defined in $\{a + \frac{k}{2}, \text{mod}(1)\}$ and $y_3(z)$ is defined in $\{a + k, \text{mod}(1)\}$. 


\[ \delta^{-k}f(z) = y_k(z) = \int_{a + \frac{k}{2}}^{z - \frac{1}{2}} y_{k-1}(s) d\delta x(s) \]
\[ = \int_{a + \frac{k}{2}}^{z - \frac{1}{2}} \frac{(x(z) - x_{k-2}(t))^{k-1}}{[\Gamma(k)]_q} f(t) d\delta x(t). \]  
(125)

where \( y_{k-1}(s) \) is defined in \( \{ a + \frac{k}{2}, \text{mod}(1) \} \), and \( y_k(z) \) is defined in \( \{ a + \frac{k+1}{2}, \text{mod}(1) \} \).

**Definition 40** For any \( \text{Re} \alpha \in \mathbb{R}^+ \), the \( \alpha \)-th fractional central sum of \( f(z) \) on non-uniform lattices \( x(s) \) is defined by

\[ \delta^{-\alpha} f(z) = \int_{a + \frac{1}{2}}^{z - \frac{1}{2}} \frac{(x(z) - x_{\alpha-2}(t))^{(\alpha-1)}}{[\Gamma(\alpha)]_q} f(t) d\delta x(t). \]
(126)

where \( \delta^{-\alpha} f(z) \) is defined in \( \{ a + \frac{\alpha+1}{2}, \text{mod}(1) \} \), and \( f(t) \) is defined in \( \{ a + \frac{1}{2}, \text{mod}(1) \} \).

**Definition 41** Let \( \delta f(z) = f(z + \frac{1}{2}) - f(z - \frac{1}{2}) \) and \( \delta x(z) = x(z + \frac{1}{2}) - x(z - \frac{1}{2}) \), the central difference of \( f(z) \) on \( x(z) \) is defined by

\[ \delta_0 f(z) = \frac{\delta f(z)}{\delta x(z)} = \frac{f(z + \frac{1}{2}) - f(z - \frac{1}{2})}{x(z + \frac{1}{2}) - x(z - \frac{1}{2})}. \]
(127)

and

\[ \delta_0^m f(z) = \delta_0[\delta_0^{m-1} f(z)], m = 1, 2, ... \]
(128)

**Definition 42** Let \( m \) be the smallest integer exceeding \( \alpha \), the Riemann-Liouville central fractional difference is defined by

\[ \delta_0^\alpha f(z) = \delta_0^m(\delta_0^{\alpha-m} f(z)). \]
(129)

where \( f(z) \) is defined in \( \{ a + \frac{1}{2}, \text{mod}(1) \} \), \( (\delta_0^{\alpha-m} f(z)) \) is defined in \( \{ a + \frac{m-\alpha+1}{2}, \text{mod}(1) \} \), and \( \delta_0^\alpha f(z) \) is defined in \( \{ a + \frac{\alpha+1}{2}, \text{mod}(1) \} \).

Let us calculate the right hand of Eq. (129). First, by **Definition 40** and **Definition 41**, one has
Then, one gets further

\[
\delta_0(\delta_0^{\alpha-m} f(z)) = \frac{\delta}{\delta x(z)} \int_{a+\frac{1}{2}}^{z-m} \frac{[x(z) - x_{m-\alpha-2}(t)](m-\alpha-1)}{[\Gamma(m-\alpha)]} f(t) d_\delta x(t)
\]

\[
= \frac{1}{\delta x(z)} \left\{ \int_{a+\frac{1}{2}}^{z+\frac{1}{2}} \frac{[x(z+\frac{1}{2}) - x_{m-\alpha-2}(t)](m-\alpha-1)}{[\Gamma(m-\alpha)]} f(t) d_\delta x(t) \right\}
\]

\[
- \int_{a+\frac{1}{2}}^{z+\frac{1}{2}} \frac{[x(z - \frac{1}{2}) - x_{m-\alpha-2}(t)](m-\alpha-1)}{[\Gamma(m-\alpha)]} f(t) d_\delta x(t) \}
\]

\[
\delta_0^2(\delta_0^{\alpha-m} f(z)) = \delta_0[\delta_0(\delta_0^{\alpha-m} f(z))]
\]

\[
= \frac{\delta}{\delta x(z)} \int_{a+\frac{1}{2}}^{z-m-(\alpha+1)} \frac{[x(z) - x_{m-(\alpha+1)-2}(t)](m-(\alpha+1)-1)}{[\Gamma(m-(\alpha+1))]} f(t) d_\delta x(t),
\]

In the same way, we obtain

\[
\delta_0(\delta_0^{\alpha-m} f(z)) = \frac{\delta}{\delta x(z)} \int_{a+\frac{1}{2}}^{z-m-(\alpha+2)} \frac{[x(z) - x_{m-(\alpha+2)-2}(t)](m-(\alpha+2)-1)}{[\Gamma(m-(\alpha+2))]} f(t) d_\delta x(t).
\]

That is

\[
\delta_0(\delta_0^{\alpha-m} f(z)) = \int_{a+\frac{1}{2}}^{z-m-(\alpha+2)} \frac{[x(z) - x_{m-(\alpha+2)-2}(t)](m-(\alpha+2)-1)}{[\Gamma(m-(\alpha+2))]} f(t) d_\delta x(t).
\]

And, by induction, we conclude that

\[
\delta_0(\delta_0^{\alpha-m} f(z)) = \int_{a+\frac{1}{2}}^{z-m} \frac{[x(z) - x_{m-\alpha-2}(t)](m-\alpha-1)}{[\Gamma(m-\alpha)]} f(t) d_\delta x(t).
\]
\[
\delta^m_0 (\delta^{a-m}_0 f(z)) = \int_{a+\frac{1}{2}}^{z-\frac{m+1}{2}} \frac{[x(z) - x_{m-(\alpha+2)}(t)]^{(m-(\alpha+m)-1)}}{[\Gamma[m-(\alpha+m)]]q} f(t) d_3 x(t) \\
= \int_{a+\frac{1}{2}}^{z+\frac{1}{2}} \frac{[x(z) - x_{-\alpha-2}(t)]^{(-\alpha-1)}}{[\Gamma(-\alpha)]q} f(t) d_3 x(t). \tag{130}
\]

Therefore from Eq. (130), we can give the following equivalent definition of the \(\alpha\)-th Riemann-Liouville central fractional difference:

**Definition 43** Assume that \(\alpha \notin N\), let \(m\) be the smallest integer exceeding \(\alpha > 0\), the \(\alpha\)-th Riemann-Liouville central fractional difference can be defined by

\[
\delta^a_0 f(z) = \int_{a+\frac{1}{2}}^{z+\frac{1}{2}} \frac{[x(z) - x_{-\alpha-2}(t)]^{(-\alpha-1)}}{[\Gamma(-\alpha)]q} f(t) d_3 x(t), \tag{131}
\]

where \(f(z)\) is defined in \(\{a+\frac{1}{2}\mod(1)\}\), \(\delta^a_0 f(z)\) is defined in \(\{a+\frac{-\alpha+1}{2}\mod(1)\}\).

We can also give the definition of the Caputo central fractional difference as follows:

**Definition 44** Let \(m\) be the smallest integer exceeding \(\alpha\), the \(\alpha\)-th Caputo central fractional difference is defined by

\[
C \delta^a_0 f(z) = \delta^{a-m}_0 \delta^m_0 f(z). \tag{132}
\]

We should mention that it is also important to establish the analogue Euler Beta formula on non-uniform lattices with respect to the central fractional sum.

**Theorem 45** For any \(\alpha, \beta\), we have

\[
\int_{a+\frac{1}{2}+\frac{\alpha}{2}}^{z-\frac{\beta}{2}} \frac{[x(z) - x_{\beta-2}(t)]^{(\beta-1)}}{[\Gamma(\beta)]q} \cdot \frac{[x(t) - x_{\alpha-1}(a)]^{(\alpha)}}{[\Gamma(\alpha+1)]q} d_3 x(t) = \frac{[x(z) - x_{\alpha+\beta-1}(a)]^{(\alpha+\beta)}}{[\Gamma(\alpha+\beta+1)]q}. \tag{133}
\]

**Proof.** Since

\[
a + \frac{1}{2} + \frac{\alpha}{2} \leq t \leq \frac{\beta}{2},
\]

we have

\[
a + 1 \leq t + \frac{1}{2} - \frac{\alpha}{2} \leq z + \frac{1}{2} - \frac{\alpha+\beta}{2}.
\]

Set

\[
\begin{cases}
t + \frac{1}{2} - \frac{\alpha}{2} = \overline{7} \\
z + \frac{1}{2} - \frac{\alpha+\beta}{2} = \overline{7}
\end{cases}
\]

then

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The LHS of Eq. (133) is equivalent to

\[
\int_{a+1}^{\infty} \frac{[x(z + \frac{\alpha+\beta+1}{2}) - x_{\alpha+1}(a)](\alpha+\beta)}{[\Gamma(\alpha+\beta+1)]q} d\delta x(t + \frac{1}{2}),
\]

and the RHS is equivalent to

\[
\frac{[x(z + \frac{\alpha+\beta+1}{2}) - x_{\alpha+1}(a)](\alpha+\beta)}{[\Gamma(\alpha+\beta+1)]q} = \frac{[x_{\alpha+\beta-1}(z) - x_{\alpha+\beta-1}(a)](\alpha+\beta)}{[\Gamma(\alpha+\beta+1)]q}.
\]

By the use of Euler Beta Theorem [13] on non-uniform lattices, Theorem 45 is completed. ■

Proposition 46 For any Re $\alpha, \text{Re} \beta > 0$, we have

\[
\delta_0^{-\beta} \delta_0^{-\alpha} f(z) = \delta_0^{-(\alpha+\beta)} f(z).
\]

where $f(z)$ is defined in $\{a+\frac{1}{2}, \text{mod}(1)\}$, $\delta^{-\alpha} f(z)$ is defined in $\{a+\frac{\alpha+1}{2}, \text{mod}(1)\}$, and $(\delta^{\alpha} \delta^{-\beta}) f(z)$ is defined in $\{a + \frac{\alpha+\beta+1}{2}, \text{mod}(1)\}$.

Proof. By Definition [10] we have

\[
\int_{a+\frac{1}{2}}^{z} \frac{[x(z) - x_{\alpha-2}(t)](\beta-1)}{[\Gamma(\beta)]q} \nabla_{\gamma}^{-\alpha} f(t) d\delta(t) = \int_{a+\frac{1}{2}}^{z} \frac{[x(z) - x_{\alpha-1}(t)](\beta-1)}{[\Gamma(\beta)]q} f(t) d\delta(t) = \int_{a+\frac{1}{2}}^{t} \frac{[x(t) - x_{\alpha-2}(s)](\alpha-1)}{[\Gamma(\alpha)]q} f(s) d\delta(s) = \int_{a+\frac{1}{2}}^{z} \frac{f(s) d\delta(s)}{[\Gamma(\alpha)]q} \int_{s+\frac{1}{2}}^{z} \frac{[x(z) - x_{\alpha-2}(t)](\beta-1)}{[\Gamma(\beta)]q} d\delta(t).
\]
In viewer of Theorem 45, one has
\[
\int_{z+\frac{\beta}{2}}^{z-\frac{\beta}{2}} \frac{[x(z) - x_{\beta - 2}(t)]^{(\beta - 1)}}{[\Gamma(\beta)]_q} \frac{[x(t) - x_{\alpha - 2}(s)]^{(\alpha - 1)}}{[\Gamma(\alpha)]_q} d_\beta(t)
= \frac{[x(z) - x_{\alpha - \beta - 2}(s)]^{(\alpha + \beta - 1)}}{[\Gamma(\alpha + \beta)]_q},
\]
it yields
\[
\delta_0^\beta \delta_0^{-\alpha} f(z) = \int_{a+\frac{1}{2}}^{z+\frac{1}{2}} \frac{[x(z) - x_{\alpha + \beta - 2}(s)]^{(\alpha + \beta - 1)}}{[\Gamma(\alpha + \beta)]_q} f(s) \nabla x_\gamma(s)
= \delta_0^{-(\alpha + \beta)} f(z).
\]

Proposition 47 For any \( \Re \alpha > 0 \), we have
\[
\delta_0^\alpha \delta_0^{-\alpha} f(z) = f(z).
\]

Proof. By Definition 42, we have
\[
\delta_0^\alpha \delta_0^{-\alpha} f(z) = \delta_0^m (\delta_0^{\alpha - m}) \delta_0^{-\alpha} f(z).
\]
In view of Theorem 46, one has
\[
\delta_0^{\alpha - m} \delta_0^{-\alpha} f(z) = \delta_0^{-m} f(z),
\]
it yields that
\[
\delta_0^\alpha \delta_0^{-\alpha} f(z) = \delta_0^m \delta_0^{-m} f(z) = f(z).
\]

Proposition 48 Let \( k \in N \), then
\[
\delta_0^{-k} \delta_0^k f(z) = f(z) - \sum_{j=0}^{k-1} \frac{\delta_0^j f(a)}{[j]_q} [x(z) - x_{j-1}(a)]^{(j)}.
\] (135)

Proof. When \( k = 1 \), we have
\[
\delta_0^{-1} \delta_0^1 f(z) = \sum_{a+\frac{1}{2}}^{z+\frac{1}{2}} \delta_0^1 f(s) \delta x(s)
= \sum_{a+\frac{1}{2}}^{z+\frac{1}{2}} \delta^1 f(s) = f(z) - f(a).
\]
Assume that when \( n = k \), (135) holds, then for \( n = k + 1 \), we conclude that
\[
\delta_0^{-(k+1)} \delta_0^{k+1} f(z) = \delta_0^{-1} [\delta_0^{-k} \delta_0^{k+1} f(z)]
= \delta_0^{-1} \{ \delta_0^j f(a) - \sum_{j=0}^{k-1} \frac{\delta_0^j [\delta_0 f(a)]}{[j]_q!} [x(z) - x_{j-1}(a)]^{(j)} \}
= f(z) - f(a) - \sum_{j=0}^{k-1} \frac{\delta_0^j f(a)}{[j]_q!} [x(z) - x_{j-1}(a)]^{(j+1)}
= f(z) - \sum_{j=0}^{k} \frac{\delta_0^j f(a)}{[j]_q!} [x(z) - x_{j-1}(a)]^{(j)}.
\]

Therefore, by the induction, the proof of (135) is completed. \( \blacksquare \)

**Proposition 50** Let \( 0 < k - 1 < \alpha \leq k \), then
\[
\delta_0^{-\alpha} \delta_0^\alpha f(z) = f(z) - \sum_{j=0}^{k-1} \delta_0^{j-k+\alpha} f(a) \frac{[x(z) - x_{\alpha+j-k-1}(a)]^{(j+\alpha-k)}}{[\Gamma(j + \alpha - k + 1)]_q}.
\] (136)

**Proof.** Since
\[
\delta_0^{-\alpha} \delta_0^\alpha f(z) = \delta_0^{-\alpha+k} \delta_0^{-k} \delta_0^{k+\alpha} f(z),
\]
then by the use of (135), one has
\[
\delta_0^{-\alpha} \delta_0^\alpha f(z) = \delta_0^{-\alpha+k} \{ \delta_0^{j-k+\alpha} f(z) - \sum_{j=0}^{k-1} \frac{\delta_0^j \delta_0^{j-k+\alpha} f(a)}{[j]_q!} [x(z) - x_{j-1}(a)]^{(j)} \}
= f(z) - \sum_{j=0}^{k-1} \delta_0^{j-k+\alpha} f(a) \frac{[x(z) - x_{\alpha+j-k-1}(a)]^{(j+\alpha-k)}}{[\Gamma(j + \alpha - k + 1)]_q}.
\]
\( \blacksquare \)

**Proposition 50** Let \( 0 < k - 1 < q \leq k \), then
\[
\delta_0^{-p} \delta_0^q f(z) = \delta_0^{-p} f(z) - \sum_{j=1}^{k} \delta_0^{j} \frac{[x(z) - x_{p-j-1}(a)]^{(p-j)}}{[\Gamma(p-j+1)]_q}.
\] (137)

**Proof.** Since
\[
\delta_0^{-p} \delta_0^q f(z) = \delta_0^{-p+q} \delta_0^{-q} \delta_0^q f(z),
\]
then by the use of (135), one has
\[
\delta_0^{-p+q} \delta_0^{-q} \delta_0^q f(z) = \delta_0^{-p+q} \{ f(z) - \sum_{j=1}^{k} \delta_0^{j-p+q} \frac{[x(z) - x_{p-j-1}(a)]^{(q-j)}}{[\Gamma(q-j+1)]_q} \}
= \delta_0^{-p+q} f(z) - \sum_{j=1}^{k} \delta_0^{j-p+q} \frac{[x(z) - x_{p-j-1}(a)]^{(p-j)}}{[\Gamma(p-j+1)]_q},
\]
the equality (137) is completed. \( \blacksquare \)
Proposition 51 Let \( 0 < k - 1 < q \leq k, \ p > 0 \), then

\[
\delta_0^k \delta_0^q f(z) = \delta_0^{k+q} f(z) - \sum_{j=1}^{k} \delta_0^{q-j} f(a) \frac{[x(z) - x_{p-j-1}(a)]^{(p-j)}}{[\Gamma(-p-j+1)]_q}.
\]  

(138)

Proof. Let \( m - 1 < p \leq m \), in view of

\[
\delta_0^m \delta_0^q f(z) = \delta_0^{m-q} \delta_0^q f(z),
\]

then by the use of (137), we have

\[
\delta_0^m \delta_0^q f(z) = \delta_0^{m-q} \delta_0^q f(z) - \sum_{j=1}^{k} \delta_0^{q-j} f(a) \frac{[x(z) - x_{m-p-j-1}(a)]^{(m-p-j)}}{[\Gamma(-p-j+1)]_q},
\]

the equality (138) is completed. ■

The relationship between Riemann-Liouville fractional difference and Caputo fractional difference is

Proposition 52 We have

\[
C \delta_0^\alpha f(z) = \delta_0^\alpha \left\{ f(z) - \sum_{j=1}^{m-1} \frac{\delta_0^j f(a)}{[j]_q!} [x(z) - x_{j-1}(a)]^{(j)} \right\}.
\]

Proof. According to Definition 44 and Proposition 48, we have

\[
C \delta_0^\alpha f(z) = \delta_0^{\alpha-m} \delta_0^m f(z) = \delta_0^\alpha \delta_0^{-m} \delta_0^m f(z)
\]

\[
= \delta_0^\alpha \left\{ f(z) - \sum_{j=1}^{m-1} \frac{\delta_0^j f(a)}{[j]_q!} [x(z) - x_{j-1}(a)]^{(j)} \right\}.
\]

■

Proposition 53 Let \( 0 < m - 1 < \alpha \leq m \), then

\[
C \delta_0^\alpha \delta_0^{-\alpha} f(z) = f(z).
\]

(139)

Proof. Set

\[
g(z) = \delta_0^{-\alpha} f(z),
\]

then we know that

\[
g(a) = \delta_0 g(a) = ... = \delta_0^{m-1} g(a) = 0,
\]

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Assume that when central difference on non-uniform, which is defined as

\[ \text{Definition 55} \]

Proposition 54 Let \( 0 < m - 1 < \alpha \leq m \), then

\[ \delta_0^{-\alpha} f(z) = f(z) - \sum_{j=1}^{m-1} \frac{\delta^j f(a)}{j!} (x(z) - x_{j-1}(a))^{(j)}. \]  

Pro. By Definition 54, one has

\[ \delta_0^{-\alpha} [C \delta_0^\alpha] f(z) = \delta_0^{-\alpha} \delta_0^{-(m-\alpha)} \delta_0^{m} f(z) = \delta_0^{-m} \delta_0^m f(z) \]  

Therefore, one has

\[ \delta_0^{-\alpha} \delta_0^{-\alpha} f(z) = \delta_0^{-\alpha} f(z) = f(z). \]  

\[ \text{Proposition 54} \]

For sequential fractional central difference on non-uniform, we can obtain Taylor formula.

\[ S \delta_0^{k\alpha} f(z) = \delta_0^{\alpha} \delta_0^{\alpha} \ldots \delta_0^{\alpha} f(z). (k - \text{multiple}) \]  

Theorem 56 Let \( 0 < \alpha \leq 1, k \in N \), then

\[ \delta_0^{-\alpha} [S \delta_0^{k\alpha}] f(z) = f(z) - \sum_{j=0}^{k-1} \frac{S \delta_0^{j\alpha} f(a)}{[\Gamma(j + 1)]_q} (x(z) - x_{ja-1}(a))^{ja}. \]  

Proof. When \( k = 1 \), from Proposition 54 we have

\[ \delta_0^{-\alpha} \delta_0^\alpha f(z) = f(z) - f(a). \]

Assume that when \( n = k \), \( 145 \) holds, then for \( n = k+1 \), we conclude that

\[ r_{k+1}(z) = \delta_0^{-(k+1)\alpha} [S \delta_0^{(k+1)\alpha}] f(z) = \delta_0^{-\alpha} \delta_0^{-k\alpha} [S \delta_0^{k\alpha}] \delta_0^\alpha f(z) \]

\[ = \delta_0^{-\alpha} \delta_0^\alpha f(a) - \sum_{j=0}^{k-1} \frac{S \delta_0^{j\alpha} \delta_0^{j\alpha} f(a)}{[\Gamma(j + 1)]_q} (x(z) - x_{ja-1}(a))^{ja} \]

\[ = f(z) - f(a) - \sum_{j=0}^{k-1} \frac{S \delta_0^{j+1\alpha} f(a)}{[\Gamma(j + 1)\alpha + 1)]_q} (x(z) - x_j(a))^{(j+1)\alpha} \]

Therefore, by the induction, the proof of \( 145 \) is completed. ■
Theorem 57  The following Taylor series:

\[ f(z) = \sum_{k=0}^{\infty} \left[ S^\frac{k\alpha}{\delta_0} f(a) \right] \frac{(x(z) - x_{ka-1}(a))^{(k\alpha)}}{\Gamma(k\alpha + 1)} \]

holds if and only if

\[ \lim_{k \to \infty} r_k(z) = \lim_{k \to \infty} \delta_0^{k\alpha} \left[ S^\frac{k\alpha}{\delta_0} f(z) \right] = 0. \]

Proof. This is a direct consequence of Theorem 56. 

9 Applications: Series Solution of Fractional Difference Equations

Next we will give the solution of the fractional central difference equation on nonuniform lattices as follows:

\[ C^\delta_0^\alpha f(z) = \lambda f(z), (0 < \alpha \leq 1) \]

(146)

Theorem 58  The solution of Eq. (146) is

\[ f(z) = \sum_{k=0}^{\infty} \lambda^k \frac{(x(z) - x_{(k-1)\alpha}(a))^{(k\alpha)}}{\Gamma(k\alpha + 1)} \]

(147)

Proof. Using the generalized sequence Taylor’s series, assuming that the solution \( f(z) \) can be written as

\[ f(z) = \sum_{k=0}^{\infty} c_k \frac{(x(z) - x_{(k-1)\alpha}(a))^{(k\alpha)}}{\Gamma(k\alpha + 1)} \]

(148)

From the equality

\[ C^\delta_0^\alpha \frac{(x(z) - x_{(k-1)\alpha}(a))^{(k\alpha)}}{\Gamma(k\alpha + 1)} = [C^\delta_0^\alpha] \delta_{0}^{\alpha} \frac{(1)}{\delta_0^{k\alpha}} = \delta^{-1}_{0} \delta_{0}^{k\alpha} (1) \]

\[ = \delta_{0}^{k\alpha} (1) \frac{(1)}{\delta_0^{(k-1)\alpha}} = \delta_{0}^{(k-1)\alpha} (1) \]

\[ = \frac{(x(z) - x_{(k-1)\alpha}(a))^{(k\alpha)}}{\Gamma((k-1)\alpha + 1)} \]

we obtain

\[ C^\delta_0^\alpha f(z) = \sum_{k=1}^{\infty} c_k \frac{(x(z) - x_{(k-1)\alpha}(a))^{((k-1)\alpha)}}{\Gamma((k-1)\alpha + 1)} \]

(149)
Substituting (148) and (149) into (146) yields
\[
\sum_{k=1}^{\infty} c_{k+1} \frac{[x(z) - x_{k\alpha-1}(a)]^{(k\alpha)}}{\Gamma(k\alpha + 1)}_q - \lambda \sum_{k=0}^{\infty} c_k \frac{[x(z) - x_{k\alpha-1}(a)]^{(k\alpha)}}{\Gamma(k\alpha + 1)}_q = 0. \tag{150}
\]
Equating the coefficient of 
\([x(z) - x_{k\alpha-1}(a)]^{(k\alpha)}\) to zero in (150), we get
\[
c_{k+1} = \lambda c_k, \tag{151}
\]
that is
\[
c_k = \lambda^k c_0.
\]
Therefore, we obtain the solution of (146) is
\[
f(z) = c_0 \sum_{k=0}^{\infty} \lambda^k \frac{[x(z) - x_{k\alpha-1}(a)]^{(k\alpha)}}{\Gamma(k\alpha + 1)}_q.
\]

\begin{definition}
The basic \(\alpha\)-order fractional exponential function is defined by
\[
e(\alpha, z) = \sum_{k=0}^{\infty} \frac{[x(z) - x_{k\alpha-1}(a)]^{(k\alpha)}}{\Gamma(k\alpha + 1)}_q, \tag{152}
\]
and
\[
e(\alpha, \lambda, z) = \sum_{k=0}^{\infty} \lambda^k \frac{[x(z) - x_{k\alpha-1}(a)]^{(k\alpha)}}{\Gamma(k\alpha + 1)}_q. \tag{153}
\end{definition}

\begin{remark}
When \(\alpha = 1\) in (153), the basic 1-order fractional exponential function on a \(q\)-quadric lattices was originally introduced by Ismail, Zhang [31], and Suslov [41] with different notation and normalization, which was very important for Basic Fourier analytic. Definition 59 is an natural extension of it.
\end{remark}

\begin{example}
Let us consider a general \(n\alpha\)-order sequence fractional difference equation with coefficients on nonuniform lattices of the form:
\[
[a_n(S^{n\alpha}) + a_{n-1}(S^{(n-1)\alpha}) + \ldots + a_1(S^{\alpha}) + a_0(S^{0})]f(z) = 0 \tag{154}
\]
\end{example}

\begin{proof}
As in the classical case, substituting
\[
f(z) = e(\alpha, \lambda, z),
\]
into Eq. (154), one can obtain
\[
a_n \lambda^n + a_{n-1} \lambda^{n-1} + \ldots + a_1 \lambda + a_0 = 0. \tag{155}
\]
\]
Assume Eq. (155) have different roots \( \lambda_i, i = 1, 2, ..., n \), then one can get \( n \) linearly independent solutions

\[
f_i(z) = e(\alpha, \lambda_i, z), \quad i = 1, 2, ..., n
\]


**Example 62** Let \( \omega > 0 \), consider \( 2\alpha \)-order sequence fractional difference equation for harmonic motion of the form:

\[
S_0^\alpha S_0^\alpha f(z) + \omega^2 f(z) = 0, \quad (0 < \alpha \leq 1)
\]

and its solutions are related to the generalized basic trigonometric functions.

**Proof.** Set

\[
f(z) = e(\alpha, \lambda, z),
\]

substituting (157) into Eq. (156), then we have

\[
\lambda^2 + \omega^2 = 0,
\]

which has two solutions

\[
\lambda_1 = i\omega, \lambda_2 = -i\omega.
\]

So the solutions of Eq. (154) are

\[
f_1(z) = e(\alpha, i\omega, z) = \sum_{k=0}^{\infty} (i\omega)^k \frac{[x(z) - x_{k\alpha-1}(a)]^{k\alpha}}{[\Gamma(k\alpha + 1)]_q}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \omega^{2n} \frac{[x(z) - x_{2n\alpha-1}(a)]^{(2n\alpha)}}{[\Gamma(2n\alpha + 1)]_q} + \sum_{n=0}^{\infty} (-1)^n \omega^{2n+1} \frac{[x(z) - x_{2n\alpha+1\alpha-1}(a)]^{(2n+1)\alpha}}{[\Gamma((2n+1)\alpha + 1)]_q},
\]

and

\[
f_2(z) = e(\alpha, -i\omega, z) = \sum_{k=0}^{\infty} (-i\omega)^k \frac{[x(z) - x_{k\alpha-1}(a)]^{(k\alpha)}}{[\Gamma(k\alpha + 1)]_q}
\]

\[
= \sum_{n=0}^{\infty} (-1)^n \omega^{2n} \frac{[x(z) - x_{2n\alpha-1}(a)]^{(2n\alpha)}}{[\Gamma(2n\alpha + 1)]_q} - \sum_{n=0}^{\infty} (-1)^n \omega^{2n+1} \frac{[x(z) - x_{2n\alpha+1\alpha-1}(a)]^{(2n+1)\alpha}}{[\Gamma((2n+1)\alpha + 1)]_q},
\]

Using the notation of Euler, we denote
\[
\cos(\alpha, \omega, z) = \sum_{n=0}^{\infty} \frac{(-1)^n \omega^{2n} [x(z) - x_{2n\alpha-1}(a)]^{(2n\alpha)}}{\Gamma(2n\alpha + 1)}
\]

and
\[
\sin(\alpha, \omega, z) = \sum_{n=0}^{\infty} (-1)^n \omega^{2n+1} \frac{[x(z) - x_{(2n+1)\alpha-1}(a)]^{((2n+1)\alpha)}}{\Gamma((2n+1)\alpha + 1)}
\]

Then it holds that
\[
\cos(\alpha, \omega, z) = \frac{e(\alpha, i\omega, z) + e(\alpha, -i\omega, z)}{2},
\]
\[
\sin(\alpha, \omega, z) = \frac{e(\alpha, i\omega, z) - e(\alpha, -i\omega, z)}{2i},
\]

and
\[
\cos^2(\alpha, \omega, z) + \sin^2(\alpha, \omega, z) = e(\alpha, i\omega, z)e(\alpha, -i\omega, z).
\]

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