Three-magnon bound states in exactly rung-dimerized spin ladders

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Abstract

Three magnon bound state problem is studied within Bethe Ansatz for an exactly rung-dimerized spin ladder. It is shown that contrary to the general three-magnon problem the solvability is less sensitive to non integrability. All obtained wave functions are presented in explicit forms.

1 Introduction

During the last century a wide number of methods were developed for analysis of quantum integrable systems \[1, 2, 3\]. However almost all physically interesting models are non integrable. It is always assumed that in this case all the analytical methods (which may be successfully applied for integrable models) should fail. In the present paper we show that the situation is not so extremely hopeless. Treating the general model of exactly rung-dimerized spin ladder \[4, 5\] we obtain general conditions for solvability of three-magnon bound state problem within Bethe Ansatz. Surprisingly it turned out that these conditions are essentially weaker than the corresponding integrability ones.

Bethe Ansatz approach to three-magnon states in an exactly rung-dimerized spin ladder \[4\] was previously studied by the author \[5\]. It was shown that the general problem
is completely solvable only in five special integrable cases. Solvability of a more special bound state problem at the first glance also implies integrability. Indeed a bound state may be obtained \[1, 2, 3\] from scattering ones by analytic continuation of the corresponding wave numbers. However one may choose another way suggesting Bethe Ansatz for bound states as a separate problem when only bounded exponents should be leaved. Surprisingly this approach is less sensitive to non integrability.

## 2 Spin ladder Hamiltonian

Spin ladder Hamiltonian acts on an infinite tensor product of spaces related to the ladder rungs (numered here by an index \(n\))

\[
\mathcal{H} = \prod_n \otimes h_n. \tag{1}
\]

For each \(n\) one has \(h_n = \mathbb{C}^4 = \mathbb{C}^2 \otimes \mathbb{C}^2\) where the two \(\mathbb{C}^2\)-factors are representation spaces for two triples of \(S = 1/2\) spin operators \(S_{j,n} (j = 1, 2)\) associated with the \(n\)-th rung

\[
S^a_{j,n} S^b_{j,n} = \frac{i}{2} \varepsilon_{abc} S^c_{j,n}, \quad [S^a_{1,n}, S^b_{2,n}] = 0, \quad a, b, c = 1, 2, 3. \tag{2}
\]

With respect to action of the total rung-spin operator

\[
S_n = S_{1,n} + S_{2,n}. \tag{3}
\]

each space \(h_n\) may be decomposed into three-dimensional rung-triplet and one-dimensional rung-singlet subspaces

\[
h_n = h^t_n \oplus h^s_n. \tag{4}
\]

In the present paper we use the following representation for Hamiltonian of an \textit{exactly} rung-dimerized spin ladder \[5\]

\[
\hat{H} = \sum_n J_1 Q_n + J_2 (\Psi_n \cdot \Psi_{n+1} + \bar{\Psi}_n \cdot \Psi_{n+1}) + J_3 Q_n Q_{n+1} + J_4 S_n \cdot S_{n+1} + J_5 (S_n \cdot S_{n+1})^2. \tag{5}
\]

Here

\[
Q_n = \frac{1}{2} S_n^2, \tag{6}
\]

is rung-triplet projector

\[
Q_n |h^s_n = 0, \quad Q_n |h^t_n = 1. \tag{7}
\]

\[2\]
The two Hermitian conjugated su(2)-covariant operator triples \( [5] \)

\[
\Psi_n = \frac{1}{2}(S_{1,n} - S_{2,n}) - i[S_{1,n} \times S_{2,n}],
\]

\[
\bar{\Psi}_n = \frac{1}{2}(S_{1,n} - S_{2,n}) + i[S_{1,n} \times S_{2,n}],
\]

(8)

additionally to

\[
[S^a_n, \Psi^b_n] = i\varepsilon_{abc} \Psi^c_n, \quad [S^a_n, \bar{\Psi}^b_n] = i\varepsilon_{abc} \bar{\Psi}^c_n, \quad a, b, c = 1, 2, 3,
\]

(9)

satisfy relations

\[
S^a_n \Psi^b_n = 0, \quad \bar{\Psi}^a_n S^b_n = 0.
\]

(10)

Both (9) and (10) may be proved from (2). According to (6), (9) and (10)

\[
[Q_n, \bar{\Psi}_n] = \bar{\Psi}_n, \quad [Q_n, \Psi_n] = -\Psi_n.
\]

(11)

So \( \bar{\Psi}_n \) and \( \Psi_n \) may be treated as (neither Bose no Fermi) creation-annihilation operators for rung-triplets. The subspace \( h^*_n \) is generated by a single vector \( |0\rangle_n \) for which

\[
\Psi_n |0\rangle_n = 0, \quad S_n |0\rangle_n = 0, \quad Q_n |0\rangle_n = 0.
\]

(12)

We shall utilize the following two basises of the space \( h^*_n \). The covariant basis

\[
|1\rangle^a_n = \bar{\Psi}^a_n |0\rangle_n, \quad a = 1, 2, 3,
\]

(13)

for which according to (9)

\[
S^a_n |1\rangle^b_n = i\varepsilon_{abc} |1\rangle^c_n,
\]

(14)

will be used for representation of three-magnon states with total spin 0 and 1. The standard \( S^3_n \)-eigenbasis

\[
S^\pm |\bar{1}\rangle^\pm_n = 0, \quad S^\pm |\bar{1}\rangle^\mp_n = \sqrt{2} |\bar{1}\rangle^0_n, \quad S^\pm |\bar{1}\rangle^0_n = \sqrt{2} |\bar{1}\rangle^\mp_n,
\]

\[
S^3_n |\bar{1}\rangle^\pm_n = \pm |\bar{1}\rangle^\pm_n, \quad S^3_n |\bar{1}\rangle^0_n = 0,
\]

(15)

will be used for representation of three-magnon states with total spin 2 and 3. According to (9) one can suggest the following representation

\[
|\bar{1}\rangle^\pm_n = \pm \bar{\Psi}^\pm_n |0\rangle_n, \quad |\bar{1}\rangle^0_n = |1\rangle^3_n.
\]

(16)

where

\[
\Psi^\pm_n \equiv \frac{1}{\sqrt{2}}(\Psi^1_n \mp i\Psi^2_n), \quad \bar{\Psi}^\pm_n \equiv \frac{1}{\sqrt{2}}(\Psi^1_n \pm i\Psi^2_n).
\]

(17)
The formulas
\[ \Psi_n^a |1\rangle_n^b = \delta_{ab} |0\rangle_n, \quad a, b = 1, 2, 3, \] (18)
may be obtained from (13) by Hermitian conjugation. Analogously for the operators \( \Psi_n^\pm \) one have
\[ \Psi_n^\pm |\bar{1}\rangle_n^+ = \pm |0\rangle_n, \quad \Psi_n^\pm |\bar{1}\rangle_n^- = \Psi_n^\pm |\bar{1}\rangle_n^0 = 0. \] (19)

Eqs. (7) and (11) result in the commutation relation
\[ [\hat{H}, \hat{Q}] = 0, \] (20)
where
\[ \hat{Q} = \sum_n Q_n, \] (21)
is (according to Eq. (7)) the number operator for rung-triplets. As it follows from (20) an infinite tensor product of rung-singlets
\[ |0\rangle_{r-d} = \prod_n \otimes |0\rangle_n, \] (22)
is an eigenvector for \( \hat{H} \). It was already mentioned \[4, 5\] that for rather big \( J_1 > 0 \) it is the ground state of the system. In this (exactly rung-dimerized) case the physical Hilbert space is an infinite direct sum of multi-magnon sectors
\[ \mathcal{H}^{\text{phys}} = \sum_{m=0}^{\infty} \mathcal{H}^m, \quad \hat{Q}|\mathcal{H}^m = m. \] (23)

Interpretation of the Hamiltonian (5) in this case is clear. The first term describes the chemical potential of an excited rung-triplet. The second one corresponds to rung-triplets kinetic energy. We shall imply that \( J_2 \neq 0 \). The last three terms describe a spin-dependent interaction between two neighboring rung-triplets.

3 General properties of three-magnon wave functions

We shall use the notation \( |S, k\rangle \) for a three-magnon state with total spin \( S \). At \( S > 0 \) it will be supplied by an upper index. For representation of a \( S = 1 \) state \( |1, k\rangle^a \) will be used the basis (13) (so in this case \( a = 1, 2, 3 \)). At \( S = 2, 3 \) using the basis (16) we shall
represent only the vectors $|S, k\rangle^S$ with $S^z = S$. Namely we shall treat the following states

\[ |0, k\rangle = \epsilon_{abc} \sum_{m<n<p} e^{ik(m+n+p)/3} b_0(k, n-m, p-n) \ldots |1\rangle_m^a \ldots |1\rangle_n^b \ldots |1\rangle_p^c \ldots, \]

\[ |1, k\rangle^a = \sum_{m<n<p} e^{ik(m+n+p)/3} \left[ b_1^{(1)}(k, n-m, p-n) \ldots |1\rangle_m^a \ldots |1\rangle_n^b \ldots |1\rangle_p^c \ldots \right] + b_1^{(2)}(k, n-m, p-n) \ldots |1\rangle_m^a \ldots |1\rangle_n^b \ldots |1\rangle_p^c \ldots \]

\[ |2, k\rangle^2 = \sum_{m<n<p} e^{ik(m+n+p)/3} b_2^{(1)}(k, n-m, p-n) \ldots |\tilde{1}\rangle_m^+ \ldots (|\tilde{1}\rangle_n^0 - |\tilde{1}\rangle_n^0 \ldots |\tilde{1}\rangle_p^0) \ldots + b_2^{(2)}(k, n-m, p-n) \ldots (|\tilde{1}\rangle_m^+ \ldots |\tilde{1}\rangle_n^0 - |\tilde{1}\rangle_n^0 \ldots |\tilde{1}\rangle_p^0) \ldots\]

\[ |3, k\rangle^3 = \sum_{m<n<p} e^{ik(m+n+p)/3} b_3(k, n-m, p-n) \ldots |\tilde{1}\rangle_m^+ \ldots |\tilde{1}\rangle_n^+ \ldots |\tilde{1}\rangle_p^+ \ldots (24) \]

All the (reduced to center mass) wave functions $b_j(k, m, n)$ ($j = 0, 1, 2, 3$) have a physical sense only at $m, n > 0$. However within the Bethe Ansatz approach [3] they should be continued into the two unphysical boundary regions ($m = 0, n > 0$) and ($m > 0, n = 0$). At $m, n > 1$ the corresponding Schrödinger equations have an identical form [5]

\[ 6J_1 b_j(k, m, n) + J_2 [e^{-ik/3} b_j(k, m+1, n) + e^{ik/3} b_j(k, m-1, n) + e^{-ik/3} b_j(k, m-1, n+1) + e^{ik/3} b_j(k, m+1, n-1) + e^{-ik/3} b_j(k, m-1, n+1) + e^{ik/3} b_j(k, m, n+1)] = E b_j(k, m, n) \]

for all $j = 0, 1, 2, 3$. Requiring correctness of (25) in the boundary regions ($m = 1, n > 0$) and ($m > 0, n = 1$) one immediately obtains the following systems of Bethe conditions [5]

\[ e^{ik/3} b_0(k, 0, n) + e^{-ik/3} b_0(k, 0, n+1) = 2\Delta b_0(k, 1, n), \]
\[ e^{-ik/3} b_0(k, m, 0) + e^{ik/3} b_0(k, m+1, 0) = 2\Delta b_0(k, m, 1), \]
\[ e^{ik/3} b_1^{(1)}(k, 0, n) + e^{-ik/3} b_1^{(1)}(k, 0, n+1) = (\Delta_2 + \Delta_1) b_1^{(1)}(k, 1, n) + (\Delta_2 - \Delta_1) b_1^{(2)}(k, 1, n), \]
\[ e^{ik/3} b_1^{(2)}(k, 0, n) + e^{-ik/3} b_1^{(2)}(k, 0, n+1) = (\Delta_2 + \Delta_1) b_1^{(2)}(k, 1, n) + (\Delta_2 - \Delta_1) b_1^{(1)}(k, 1, n), \]
\[ e^{-ik/3} b_1^{(2)}(k, m, 0) + e^{ik/3} b_1^{(2)}(k, m+1, 0) = (\Delta_2 + \Delta_1) b_1^{(2)}(k, m, 1) + (\Delta_2 - \Delta_1) b_1^{(3)}(k, m, 1), \]
\[ e^{-ik/3} b_1^{(3)}(k, m, 0) + e^{ik/3} b_1^{(3)}(k, m+1, 0) = (\Delta_2 + \Delta_1) b_1^{(3)}(k, m, 1) + (\Delta_2 - \Delta_1) b_1^{(2)}(k, m, 1), \]
\[ e^{ik/3} b_1^{(3)}(k, 0, n) + e^{-ik/3} b_1^{(3)}(k, 0, n+1) = \frac{2}{3} (\Delta_0 - \Delta_2) [b_1^{(1)}(k, 1, n) + b_1^{(2)}(k, 1, n)] + 2\Delta b_1^{(3)}(k, 1, n), \]

5
\[
e^{-i k/3}b_1^{(1)}(k, m, 0) + e^{i k/3}b_1^{(1)}(k, m + 1, 0) = \frac{2}{3} (\Delta_0 - \Delta_2)[b_1^{(2)}(k, m, 1) + b_1^{(3)}(k, m, 1)] + 2\Delta_0 b_1^{(1)}(k, m, 1),
\]

\[
e^{i k/3}b_2^{(2)}(k, 0, n) + e^{-i k/3}b_2^{(2)}(k, 0, n + 1) = 2\Delta_1 b_2^{(2)}(k, 1, n) + (\Delta_2 - \Delta_1)b_2^{(1)}(k, 1, n),
\]

\[
e^{i k/3}b_2^{(1)}(k, 0, n) + e^{-i k/3}b_2^{(1)}(k, 0, n + 1) = 2\Delta_2 b_2^{(1)}(k, 1, n),
\]

\[
e^{-i k/3}b_2^{(1)}(k, m, 0) + e^{i k/3}b_2^{(1)}(k, m + 1, 0) = 2\Delta_1 b_2^{(1)}(k, m, 1) + (\Delta_2 - \Delta_1)b_2^{(2)}(k, m, 1),
\]

\[
e^{-i k/3}b_2^{(2)}(k, m, 0) + e^{i k/3}b_2^{(2)}(k, m + 1, 0) = 2\Delta_2 b_2^{(2)}(k, m, 1),
\]

where

\[
\Delta_0 = \frac{J_3 - 2J_4 + 4J_5}{2J_2}, \quad \Delta_1 = \frac{J_3 - J_4 + J_5}{2J_2}, \quad \Delta_2 = \frac{J_3 + J_4 + J_5}{2J_2}.
\]

The system related to \(b_3(k, m, n)\) has the form identical identical to Eq. (26) however with \(\Delta_1\) replaced on \(\Delta_2\).

The systems (26)-(28) are invariant under the set of corresponding duality transformations

\[
D(b_{0,3}(k, m, n)) = \bar{b}_{0,3}(k, n, m), \quad D(b_1^{(j)}(k, m, n)) = \bar{b}_1^{(4-j)}(k, n, m),
\]

\[
D(b_2^{(j)}(k, m, n)) = \bar{b}_2^{(3-j)}(k, n, m).
\]

Since in the present paper we are interesting only in bound states we imply a normalization condition

\[
\sum_{m, n>0} |b_j(k, m, n)|^2 < \infty, \quad j = 0, 1, 2, 3.
\]

One may ask a question in what interval lies \(k\). On the one hand there should be

\[
0 \leq k < 2\pi,
\]

on the other a substitution \(k + 2\pi\) evidently changes the exponential factors in (24). In order to clarify the situation we notice that this substitution together with simultaneous multiplication of the wave function on the factor \(e^{2\pi i(m-n)/3}\) retains invariant all the states in (24).

## 4 Bound states at \(S = 0\) (\(S = 3\))

The following substitution

\[
b_0(k, m, n) = z_+^m z_-^n,
\]
solves Eq. (25) resulting in a dispersion

\[ E(k) = 6J_1 + J_2 \left[ e^{ik/3} \left( \frac{z_+}{z_-} + \frac{1}{z_+} + z_- \right) + e^{-ik/3} \left( \frac{z_-}{z_+} + \frac{1}{z_-} + z_+ \right) \right]. \]  

A substitution of (33) into Eq. (26) gives

\[ 2\Delta_1 z_\pm = e^{\pm ik/3} + e^{\mp ik/3} z_\mp. \]  

From Eq. (35) readily follows

\[ 2\Delta_1 (z_- - \bar{z}_+) = e^{ik/3} (z_+ - \bar{z}_-). \]  

In the general case

\[ 2|\Delta_1| \neq 1, \]  

Eq. (36) results in an autoduality (with respect to (30)) condition

\[ \bar{z}_- = z_+ \equiv z, \]  

and the system (35) reduces to an equation

\[ G_1 = 0, \]  

where

\[ G_j \equiv 2\Delta_j z - e^{ik/3} - e^{-ik/3} \bar{z}. \]  

Eqs. (39), (40) result in

\[ z = \frac{2\Delta_1 e^{ik/3} + e^{-2ik/3}}{2\Delta_1^2 - 1}. \]  

A substitution of (38) and (41) into (34) gives

\[ E(k) = 6J_1 + \frac{2J_2}{4\Delta_1^2 - 1} \left( 8\Delta_1^3 + \cos k \right). \]  

Condition (31) is satisfied only if \(|z| < 1\) or according to (41)

\[ \Delta_1 \cos k < (4\Delta_1^2 - 3)\Delta_1^2. \]  

The non-autodual states at \(2|\Delta_1| = 1\) will be studied in a separate paper. Here we shall treat only autodual wave functions.

As it was mentioned above all calculations for \(S = 3\) may be performed in the similar manner after the replacement \(\Delta_1\) on \(\Delta_2\).
5 Bound states at $S = 1$

Treating an autodual substitution

\[ b_1^{(j)}(k, m, n) = B^{(j)} z^m \bar{z}^n, \quad B^{(3)} = \bar{B}^{(1)}, \quad (44) \]

one gets from Eq. (27)

\[
\begin{pmatrix}
G_1 + G_2 & G_2 - G_1 & 0 \\
G_2 - G_1 & G_1 + G_2 & 0 \\
G_0 - G_2 & G_0 - G_2 & 3G_0
\end{pmatrix}
\begin{pmatrix}
B^{(1)} \\
B^{(2)} \\
B^{(3)}
\end{pmatrix} = 0. \quad (46)
\]

Since determinant of the matrix in Eq. (46) is proportional to $G_0 G_1 G_2$ the system (46) is solvable in three cases

\[ G_j = 0, \quad j = 0, 1, 2. \quad (47) \]

The corresponding solutions in general do not satisfy Eq. (45). However this problem may be solved if the system (46) has two different solutions. Taking into account that the latter is possible only if $\Delta_j = \Delta_l$ for $j \neq l$ we conclude that a three-magnon bound state related to the wave function exists at three cases

\[ \begin{align*}
\Delta_0 &= \Delta_1, \quad B = [1, -1, 1], \quad (48) \\
\Delta_0 &= \Delta_2, \quad B = [1, 1, 1], \quad (49) \\
\Delta_1 &= \Delta_2, \quad B = [1, -4, 1], \quad (50)
\end{align*} \]

where each $B \equiv [B^{(1)}, B^{(2)}, B^{(3)}]$ represents the corresponding autodual solution of Eq. (46).

6 Bound states at $S = 2$

An autodual substitution

\[ b_2^{(j)}(k, m, n) = C^{(j)} z^m \bar{z}^n, \quad C^{(2)} = \bar{C}^{(1)}, \quad (51) \]

8
results in
\[
\begin{pmatrix}
  G_2 & 0 \\
  G_2 - G_1 & 2G_1
\end{pmatrix}
\begin{pmatrix}
  C^{(1)} \\
  C^{(2)}
\end{pmatrix} = 0.
\] (53)

The system (53) is solvable under one of the conditions (47) taken for \( j = 1, 2 \). However autodual solutions \([1, 1]\) and \([i, -i]\) exist only at \( \Delta_2 = \Delta_1 \).

7 Summary and discussion

As it was previously shown in Ref. 5 the system related to Hamiltonian (5) is integrable only in five special cases

\[
\begin{align*}
\Delta_0 &= \Delta_1 = \Delta_2, \quad (54) \\
\Delta_0 &= \Delta_2 = \pm 1, \quad \Delta_1 = 0, \quad (55) \\
\Delta_0 &= \Delta_2 = 0, \quad \Delta_1 = \pm 1, \quad (56) \\
\Delta_1 &= \Delta_2 = \pm \frac{3}{2}, \quad \Delta_0 = 0, \quad (57) \\
\Delta_1 &= \Delta_2 = 0, \quad \Delta_0 = \pm \frac{3}{2}. \quad (58)
\end{align*}
\]

We see that restrictions (48)-(50) are essentially weaker than (54)-(58). So the bound states of Bethe form exist even for a rather big number of non integrable models.

In order to clarify a relevance of conditions (48)-(50) to physical applications we use an equivalent representation of the Hamiltonian (5) \[5\]

\[
H_{n,n+1} = J_r H^r_{n,n+1} + J_l H^l_{n,n+1} + J_d H^d_{n,n+1} + J_{rr} H_{n,n+1}^{rr} + J_{ll} H_{n,n+1}^{ll} + J_{dd} H_{n,n+1}^{dd}, \quad (59)
\]

where

\[
\begin{align*}
H^r_{n,n+1} &= \frac{1}{2}(S_{1,n} \cdot S_{2,n} + S_{1,n+1} \cdot S_{2,n+1}), \quad H^l_{n,n+1} = S_{1,n} \cdot S_{1,n+1} + S_{2,n} \cdot S_{2,n+1}, \\
H^d_{n,n+1} &= S_{1,n} \cdot S_{2,n+1} + S_{2,n} \cdot S_{1,n+1}, \quad H_{n,n+1}^{rr} = (S_{1,n} \cdot S_{2,n})(S_{1,n+1} \cdot S_{2,n+1}), \\
H_{n,n+1}^{ll} &= (S_{1,n} \cdot S_{1,n+1})(S_{2,n} \cdot S_{2,n+1}), \quad H_{n,n+1}^{dd} = (S_{1,n} \cdot S_{2,n+1})(S_{2,n} \cdot S_{1,n+1}). \quad (60)
\end{align*}
\]

A correspondence between the coupling constants of (5) and (59) is given by Eq. (20) of Ref. 5. We imply also the condition \[4, 5\]

\[
J_{ll} - J_{dd} = 4(J_l - J_d), \quad (61)
\]
which guarantees the commutation relation (20). According to Eq. (32) of Ref. 5

\[ \Delta_0 = \Delta_1 \iff 12J_l - 8J_d - 5J_d = 0, \]  
\[ \Delta_0 = \Delta_2 \iff 4J_l - J_d = 0, \]  
\[ \Delta_1 = \Delta_2 \iff 4J_d + J_d = 0. \]  

(62)  
(63)  
(64)

It is assumed [6] that for physically relevant models there should be

\[ |J_{rr}|, |J_{ll}|, |J_{dd}| < |J_l|, \]  

(65)

so Eq. (61) may be satisfied only for highly frustrated ladders with

\[ J_d \approx J_l. \]  

(66)

In this context the case (62) seems to be physically more reliable.

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