On Inhibition of Rayleigh–Taylor Instability by a Horizontal Magnetic Field in Non-resistive MHD Fluids: the Viscous Case

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Abstract

It is still open whether the phenomenon of inhibition of Rayleigh–Taylor (RT) instability by a horizontal magnetic field can be mathematically verified for a non-resistive viscous magnetohydrodynamic (MHD) fluid in a two-dimensional (2D) horizontal slab domain, since it was roughly proved in the linearized case by Wang in [41]. In this paper, we prove such inhibition phenomenon by the (nonlinear) inhomogeneous, incompressible, viscous case with Navier (slip) boundary condition. More precisely, we show that there is a critical number of field strength $m_C$, such that if the strength $|m|$ of a horizontal magnetic field is bigger than $m_C$, then the small perturbation solution around the magnetic RT equilibrium state is algebraically stable in time. In addition, we also provide a nonlinear instability result for the case $|m| \in [0, m_C)$. The instability result presents that a horizontal magnetic field can not inhibit the RT instability, if it’s strength is too small.

Keywords: Non-resistive viscous MHD fluids; Rayleigh–Taylor instability; algebraic decay-in-time; stability.

1. Introduction

The equilibrium of a heavier fluid on top of a lighter one, subject to gravity, is unstable. In fact, small disturbances acting on the equilibrium will grow and lead to the release of potential energy, as the heavier fluid moves down under the gravity, and the lighter one is displaced upwards. This phenomenon was first studied by Rayleigh [36] and then Taylor [39], is called therefore the Rayleigh–Taylor (RT) instability. In the last decades, RT instability had been extensively investigated from mathematical, physical and numerical aspects, see [3, 4, 6, 40] for examples. It has been also widely analyzed how physical factors, such as elasticity [29, 32], rotation [2, 3], internal surface tension [14, 18, 43], magnetic field [21, 27, 28] and so on, influence the dynamics of the RT instability.

In this paper we are interested in the phenomenon of inhibition of RT instability by magnetic fields. This topic goes back to the theoretical work of Kruskal and Schwarzchild [33]. They analyzed the effect of the (impressed) horizontal magnetic field upon the growth of the RT instability in a horizontally periodic motion of a completely ionized plasma with zero resistance in three dimensions in 1954, and pointed out that the curvature of the magnetic lines can influence the

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development of instability, but can not inhibit the growth of the RT instability. The inhibition of RT instability by the vertical magnetic field was first verified for the inhomogeneous, incompressible, non-resistive magnetohydrodynamic (MHD) fluids in three dimensions by Hide [3, 17].

In 2012, Wang also noticed that the horizontal magnetic field can inhibit the RT instability in a non-resistive MHD fluid in two dimensions [41]. Later, Jiang–Jiang further found that impressed magnetic fields always inhibit the RT instability, if a non-slip velocity boundary-value condition is imposed in the direction of magnetic fields [20]. Such boundary condition is called the “fixed condition” for the sake of simplicity.

All results mentioned above are based on the linearized non-resistive MHD equations. Thanks to the multi-layers method developed in the well-posedness theory of surface wave problems [15], recently the phenomenon of inhibition of RT instability by magnetic fields has been rigorously proved based on the (nonlinear) non-resistive viscous MHD equations under the fixed condition, for example, Wang verified the inhibition phenomenon by the non-horizontal magnetic field in a stratified incompressible viscous MHD fluid in a 2D/3D slab domain [42]. Moreover, he also proved that the horizontal magnetic field can not inhibit the RT instability for the horizontally periodic motion for the 3D case [42], but can inhibit the RT instability based on a 2D linearized motion equations in [41]. Similar results can be also found in other magnetic inhibition phenomena, see [22] for the Parker instability and [25] for the thermal instability. The previous nonlinear stability/instability results in the non-resistive viscous magnetic RT problem can be summarized as in the following table.

Can an impressed horizontal/vertical magnetic field inhibit the RT instability in a non-resistive viscous MHD fluid in a slab domain?

|         | horizontal | vertical |
|---------|------------|----------|
| 2D      | No clear   | Yes      |
| 3D      | No         | Yes      |

In [24] Jiang–Jiang further established a so-called magnetic inhibition theory in viscous non-resistive MHD fluids, which reveals the physical effect of the fixed condition in magnetic inhibition phenomena. Roughly speaking, let us consider an element line along an impressed magnetic field in the rest state of a non-resistive MHD fluid, then the element line of fluids can be regarded as an elastic string. The two endpoints of the element line are fixed due to the fixed condition. Thus, the bent element line will restore to its initial location under the magnetic tension as well as viscosity.

By the magnetic inhibition mechanism in non-resistive MHD fluids, the positive assertions in the table above seem to be obvious; moreover, we easily predict the phenomenon of inhibition of RT instability by a horizontal magnetic field in a non-resistive viscous MHD fluid in a 2D slab domain, see [41] for the linear case. However, it is still an open problem to rigorously prove this prediction based on the nonlinear motion equations. Recently the authors noted that this prediction can be mathematically verified by the inviscid case with velocity damping, i.e. the viscous term is replaced by the velocity damping term. Under such case, some difficulties arising from the viscous term can be avoided, when we exploit our estimates. In this paper, we further find that such inhibition phenomenon can be also proved in the inhomogeneous, incompressible, viscous and non-resistive MHD fluids with Navier (slip) boundary condition in two-dimensions, and thus move a first step to this open problem with the viscous case. More precisely, there exists a critical number $m_C$, such that if the strength $|m|$ of a horizontal magnetic field is bigger than $m_C$, then the small perturbation solution around the magnetic RT equilibrium state is
algebraically stable in time, i.e. the RT instability can be inhibited by a horizontal magnetic field in a 2D slab domain. Finally we further mention our stability result.

(1) Ren–Xiang–Zhang proved the existence of the global(-in-time) small perturbation solutions of a non-resistive viscous MHD fluid with a horizontal magnetic field and with a Navier boundary condition in a 2D slab domain \[37\]. However, they cannot obtained the decay-in-time for the global solutions. As a by-product of our stability result, we further provide the algebraical decay-in-time behavior for the solutions.

(2) Wang mathematically verified the the inhibition phenomenon by the non-horizontal magnetic field in a stratified incompressible viscous MHD fluid in a 2D/3D slab domain \[42\] and also obtained unique global solutions with decay-in-time. It should be noted the decay-in-time plays an important role to derive the existence of global solution in Wang’s result. However, our proof for the existence of global-in-time solutions is independent of the decay-in-time. We provide the additional derivation of decay-in-time in our result, since it may be useful in the further investigation of the case of the non-slip boundary condition in future.

(3) Our stability result can be viewed as a continuation of the previous work of the inviscid case with velocity damping in \[31\]. However, we develop a new idea to capture the high-order normal (spacial) derivatives of the deviation function of fluid particles from the viscosity term under the Navier boundary condition, and the details will be further discussed after introducing our stability result in Theorem \[1.1\].

(4) By the magnetic inhibition mechanism in non-resistive MHD fluids, the horizontal magnetic field plays a role of tension in the horizontal direction, and thus can inhibit the RT instability as well as the surface tension \[18, 43\]. Our result mathematically verifies such physical phenomenon. It seems that our proof idea can be extended to verify that the horizontal magnetic field can also inhibit other flow instabilities, such as thermal instability in \[25\]. In addition, we will further use the basic idea in this paper to prove that the RT instability can be also inhibited by other stabilizing forces in the horizontal direction, such as capillary action on RT instability in capillary fluids in a forthcoming paper.

1.1. Mathematical formulation for the magnetic RT problem

Before stating our results in details, we shall mathematically formulate the physical problem of inhibition of RT instability by a horizontal magnetic field. The governing equations of an inhomogeneous, incompressible, viscous, non-resistive MHD fluid in the presence of a uniform gravitational field in a 2D slab domain \(\Omega\) read as follows.

\[
\begin{align*}
\rho_t + v \cdot \nabla \rho &= 0, \\
\rho v_t + \rho v \cdot \nabla v + \nabla (P + \lambda |M|^2/2) - \mu \Delta v &= \lambda M \cdot \nabla M - \rho g e_2, \\
M_t + v \cdot \nabla M &= M \cdot \nabla v, \\
\text{div} v &= \text{div} M = 0.
\end{align*}
\tag{1.1}
\]

Below, we explain the mathematical notations in the system \[(1.1)\].

The unknowns \(\rho := \rho(x, t), v := v(x, t), M := M(x, t)\) and \(P := P(x, t)\) denote the density, velocity, magnetic field and kinetic pressure of a MHD fluid, resp.. \(x \in \Omega \subset \mathbb{R}^2\) and \(t > 0\) are the spatial and temporal variables resp.. The constants \(\lambda, g > 0\) and \(\mu > 0\) stand for the permeability of vacuum, the gravitational constant and the viscosity coefficient, resp.. \(e_2 = (0, 1)^T\) represents the normal (or vertical) unit vector, and \(-\rho g e_2\) the gravity, where the superscript \(T\) denotes the transposition.
Since we consider horizontally periodic motion solutions of (1.1), we define a horizontally periodic domain:

$$\Omega := 2\pi LT \times (0, h),$$

where \(T = \mathbb{R}/\mathbb{Z}\) and \(L > 0\). For the horizontally periodic domain \(\Omega\), the 1D periodic domain \(2\pi LT \times \{0, h\}\), denoted by \(\partial \Omega\), which customarily is regarded as a boundary of \(\Omega\). For the well-posedness of the system (1.1), we impose the following initial and boundary conditions:

$$\begin{align*}
\left( \rho, v, M \right)|_{t=0} &= \left( \rho^0, v^0, M^0 \right), \\
v|_{\partial \Omega} \cdot \vec{n} &= 0, \quad 2(\nabla v|_{\partial \Omega})\vec{n})_{\kappa} = 0,
\end{align*}$$

where \(\vec{n} = (\vec{n}_1, \vec{n}_2)^T\) denotes the outward normal unit vector on \(\partial \Omega\), \(\nabla\) the strain tensor, and the subscript “\(\kappa\)” the tangential component of a vector (for example \(v_{\kappa} = v - (v \cdot \vec{n}) \vec{n}\) \cite{7, 8, 34, 38}). Here and in what follows, we always use the superscript 0 to emphasize the initial data.

We call the boundary conditions in (1.4) the Navier (slip) boundary condition. Since \(\Omega\) is a slab domain, the Navier boundary condition is equivalent to

$$(v_2, \partial_2 v_1)|_{\partial \Omega} = 0. \tag{1.5}$$

Now, we choose a RT density profile \(\bar{\rho} := \bar{\rho}(x_2)\), which is independent of \(x_1\) and satisfies

$$\bar{\rho} \in C^2(\Omega), \quad \inf_{x \in \Omega} \bar{\rho} > 0, \quad \bar{\rho}'|_{x_2=y_2} > 0 \quad \text{for some } y_2 \in \{x_2 \mid (x_1, x_2)^T \in \Omega\},$$

where \(\bar{\rho}' := d\bar{\rho}/dx_2\) and \(\Omega := \mathbb{R} \times [0, h]\). We remark that the second condition in (1.6) prevents us from treating vacuum, while the condition in (1.7) is called the RT condition, which assures that there is at least a region in which the density is larger with increasing height \(x_2\), thus leading to the classical RT instability, see \cite{19}, Theorem 1.2.

With the RT density profile in hand, we further define a magnetic RT equilibria \(r_M := (\bar{\rho}, 0, \bar{M})\), where \(\bar{M} = (m, 0)^T\) with \(m\) being a constant. Usually, \(\bar{M}\) is called an impressed horizontal magnetic field (or horizontal magnetic field for the sake of simplicity). The pressure profile \(\bar{P}\) under the equilibrium state is determined by the relation

$$\nabla \bar{P} = -\bar{\rho}g e_2 \text{ in } \Omega. \tag{1.8}$$

Denoting the perturbation around the magnetic RT equilibrium by

$$\varrho = \rho - \bar{\rho}, \quad v = v - 0, \quad N = M - \bar{M}$$

and then using the relation (1.8), we obtain the system of perturbation equations from (1.1):

$$\begin{cases}
\varrho_t + v \cdot \nabla (\varrho + \bar{\rho}) = 0, \\
(\varrho + \bar{\rho})v_t + (\varrho + \bar{\rho})v \cdot \nabla v + \nabla \beta - \mu \Delta v = \lambda (N + \bar{M}) \cdot \nabla N - g \varrho e_2, \\
N_t + v \cdot \nabla N = (N + \bar{M}) \cdot \nabla v, \\
div v = div N = 0,
\end{cases} \tag{1.9}$$
where $\beta := P - \bar{P} + \lambda |M|^2 - |\bar{M}|^2)/2$ is called the total perturbation pressure. The corresponding initial and boundary conditions read as follows.

\[
(\varrho, v, N)|_{t=0} = (\varrho^0, v^0, N^0), \quad (1.10)
\]

\[
(v_2, \partial_2 v_1)|_{\partial \Omega} = 0 \text{ on } \partial \Omega. \quad (1.11)
\]

We call the initial-boundary value problem \((1.9)-(1.11)\) the magnetic RT problem for the sake of simplicity. Obviously, to mathematically prove the inhibition of RT instability by a horizontal magnetic field in a 2D slab domain, it suffices to verify the stability in time for the solutions of the magnetic RT problem with some non-trivial initial data.

1.2. Reformulation in Lagrangian coordinates

To proceed, as in \([22, 25, 42]\), we shall first reformulate the magnetic RT problem in Lagrangian coordinates. Let the flow map $\zeta$ be the solution to the initial value problem:

\[
\begin{aligned}
\partial_t \zeta(y,t) &= v(\zeta(y,t), t), \\
\zeta(y,0) &= \zeta_0(y),
\end{aligned} \quad (1.12)
\]

where the invertible mapping $\zeta_0 := \zeta_0(y)$ maps $\Omega$ to $\Omega$, and satisfies

\[
\begin{aligned}
J^0 := \det \nabla \zeta_0 &= 1, \\
\zeta_0^2 &= y_2 \text{ on } \partial \Omega.
\end{aligned} \quad (1.13)
\]

Here and in what follows, “det” denotes the determinant of a matrix. In our results, we will see that the flow map $\zeta$ satisfies, for each fixed $t > 0$,

\[
\begin{aligned}
\zeta|_{y_2=r} : \mathbb{R} \to \mathbb{R} \text{ is a } C^1(\mathbb{R})\text{-diffeomorphism mapping for } r = 0, h, \\
\zeta : \overline{\Omega} \to \overline{\Omega} \text{ is a } C^1(\overline{\Omega})\text{-diffeomorphism mapping.}
\end{aligned} \quad (1.15)
\]

Since $v$ satisfies the divergence-free condition and non-slip boundary condition $v_2|_{\partial \Omega} = 0$, we can deduce from \((1.12)-(1.14)\) that

\[
\begin{aligned}
J := \det \nabla \zeta &= 1, \\
\zeta_2 &= y_2 \text{ on } \partial \Omega.
\end{aligned} \quad (1.14)
\]

We define the matrix $A := (A_{ij})_{2\times2}$ via

\[
A^T = (\nabla \zeta)^{-1} = (\partial_j \zeta_i)^{-1}_{2\times2}.
\]

Then we further define the differential operators $\nabla_A$, $\text{div}_A$ and $\text{curl}_A$ as follows: for a scalar function $f$ and a vector function $X := (X_1, X_2)^T$,

\[
\nabla_A f := (A_{1k}\partial_k f, A_{2k}\partial_k f)^T, \quad \text{div}_A(X_1, X_2)^T := A_{lk}\partial_k X_l
\]

and

\[
\text{curl}_A X := A_{1k}\partial_k X_2 - A_{2k}\partial_k X_1,
\]

where we have used the Einstein convention of summation over repeated indices, and $\partial_k := \partial_{y_k}$. In particular, $\text{curl} X := \text{curl}_I X$, where $I$ represents an identity matrix.
Defining the Lagrangian unknowns:

\[(\vartheta, u, Q, B)(y, t) = (\rho, v, P + \lambda|\mathbf{M}|^2/2, M)(\zeta(y, t), t)\] for \((y, t) \in \Omega \times \mathbb{R}_0^+\),

then in Lagrangian coordinates, the initial-boundary value problem (1.1), (1.3) and (1.5) can be rewritten as follows:

\[
\begin{aligned}
\zeta_t &= u, \quad \vartheta_t = 0, \quad \text{div}_A u = 0, \\
\vartheta u_t + \nabla_A Q - \mu \Delta_A u &= \lambda B \cdot \nabla_A B - \vartheta g e_2, \\
B_t &= B \cdot \nabla_A u, \quad \text{div}_A B = 0, \\
(\zeta, \vartheta, u, B)|_{t=0} &= (\zeta^0, \vartheta^0, u^0, B^0), \\
(\zeta_2 - y_2, u_2, A_2 \partial_1 u_1)|_{\partial \Omega} &= 0,
\end{aligned}
\]  

(1.17)

where \((\vartheta^0, u^0, B^0) := (\rho^0(\zeta^0), v^0(\zeta^0), M^0(\zeta^0))\). In addition, the relation (1.8) in Lagrangian coordinates reads as follows.

\[
\nabla_A \bar{P}(\zeta_2) = -\bar{\rho}(\zeta_2) g e_2. \tag{1.18}
\]

Let \(\eta = \zeta - y, \eta^0 = \zeta^0 - y, q = Q - \bar{P}(\zeta_2) - \lambda|\mathbf{M}|^2/2, \mathbf{A} = (I + \nabla \eta)^{-T}\) and 
\[
G_\eta := g(\bar{\rho}(\eta_2(y, t) + y_2) - \bar{\rho}(y_2)).
\]

In particular, we can calculate that

\[
\mathbf{A} = \begin{pmatrix}
1 + \partial_2 \eta_2 & -\partial_1 \eta_2 \\
-\partial_2 \eta_1 & 1 + \partial_1 \eta_1
\end{pmatrix}.
\]

If \(\eta^0, \vartheta^0\) and \(B^0\) additionally satisfy

\[
\vartheta^0 = \bar{\rho}(y_2) \quad \text{and} \quad B^0 = m \partial_1 (\eta^0 + y),
\]

then the initial-boundary value problem (1.17), together with the relation (1.18), implies that

\[
\begin{aligned}
\eta_t &= u, \\
\bar{\rho} u_t + \nabla_A q - \mu \Delta_A u &= \lambda m^2 \partial_1^2 \eta + G_\eta e_2, \\
\text{div}_A u &= 0, \\
(\eta, u)|_{t=0} &= (\eta^0, u^0), \\
(\eta_2, u_2)|_{\partial \Omega} &= 0, \\
((1 + \partial_1 \eta_1) \partial_2 u_1 - \partial_2 \eta_1 \partial_1 u_1)|_{\partial \Omega} &= 0
\end{aligned}
\]  

(1.19)

and

\[
\partial = \bar{\rho}(y_2), \quad B = m \partial_1 (y + \eta),
\]

(1.22)

please refer to [21] for the derivation. We mention that the term \(\lambda m^2 \partial_1^2 \eta\) physically represents the magnetic tension, which can inhibit flow instabilities [24]. It should be remarked that (1.19)–(1.22) also implies (1.17), and that \(q\), still called the perturbation pressure for simplicity, is in fact the sum of the perturbation pressure and perturbation magnetic pressure in Lagrangian coordinates.
Unfortunately, it seems to be difficult to capture the estimates of high-order normal derivatives of \( \eta \) due to the absence the boundary condition of \( \eta_1 \). Thus we shall pose an additional boundary condition
\[
\partial_2 \eta_1 |_{\partial \Omega} = 0, \tag{1.23}
\]
which, together with (1.19), formally yields
\[
\partial_2 u_1 |_{\partial \Omega} = 0.
\]
It is easy to see that (1.21) automatically holds under the above two boundary conditions. Hence we use (1.23) to replace (1.21), and thus pose the new boundary condition
\[
(\eta_2, \partial_2 \eta_1, u_2, \partial_2 u_1)|_{\partial \Omega} = 0 \tag{1.24}
\]
From now on, we call the initial-boundary value problem (1.19) and (1.24) \textit{the transformed MRT problem}. The stability problem of the magnetic RT problem reduces to investigating the stability of the transformed MRT problem.

We mention that the boundary condition
\[
(\eta_2, \partial_2 \eta_1)|_{\partial \Omega} = 0 \tag{1.25}
\]
is called the characteristic boundary condition. Indeed, if the initial data \( \eta^0 \) satisfies \( (\eta^0, \partial_2 \eta^0_1)|_{\partial \Omega} = 0 \), then \( \eta \) automatically satisfies (1.25) due to the facts (1.19) and the boundary condition
\[
(u_2, \partial_2 u_1)|_{\partial \Omega} = 0. \tag{1.26}
\]
It should be noted that the boundary condition (1.25) automatically implies
\[
\text{cur}_A \partial^i \eta |_{\partial \Omega} = 0 \text{ for } i = 0, 1, \tag{1.27}
\]
which will plays an important to capture the high-order normal estimates for \( \eta \), see Lemma 2.6. This is also a key idea in the mathematical proof for the magnetic inhibition phenomenon under the horizontal field in our paper.

1.3. Notations

Before stating our main results on the transformed MRT problem, we shall introduce simplified notations throughout this paper.

(1) Simplified basic notations: \( e_1 := (1,0)^T, I_a := (0,a) \) denotes a time interval, in particular, \( I_\infty = \mathbb{R}^+ \). \( \overline{S} \) denotes the closure of a set \( S \subset \mathbb{R}^n \) with \( n \geq 1 \), in particular, \( \overline{I_T} = [0,T] \) and \( \overline{I_\infty} = \mathbb{R}^+_0 \). \( \Omega_t := \Omega \times I_t, \int := \int_{(0,2\pi L) \times (0,h)}, (u)_{\Omega} \) denotes the mean value of \( u \) in a periodic cell \( (0,2\pi L) \times (0,h) \). \( a \lesssim b \) means that \( a \leq cb \) for some constant \( c > 0 \). If not stated explicitly, the positive constant \( c \) may depend on \( \mu, g, \lambda, m, \bar{\rho} \) and \( \Omega \) in the transformed MRT problem, and may vary from one place to other place. Sometimes, we use \( c_i \) to replace \( c \) in order to emphasize that \( c_i \) is a fixed value for \( 1 \leq i \leq 3 \). The letter \( \alpha \) always denotes the multi-index with respect to the variable \( y, |\alpha| = \alpha_1 + \alpha_2 \) is called the order of multi-index, \( \partial^\alpha := \partial_1^{\alpha_1} \partial_2^{\alpha_2} \) and \( [\partial^\alpha, \phi] \varphi := \partial^\alpha (\phi \varphi) - \phi \partial^\alpha \varphi \).
(2) Simplified Banach spaces, norms and semi-norms:

\[ L^p := L^p(\Omega) = W^{0,p}(\Omega), \quad H^2 := W^{1,2}(\Omega), \quad H^2_\gamma := \{ w \in H^2 | \omega_{\partial \Omega} = 0 \} \]

\[ H^1_\gamma := \{ w \in H^1_\gamma | \operatorname{div} w = 0 \}, \quad H^3_\gamma := \{ w \in H^3_\gamma | \| \nabla w \|_2 \leq \gamma \}, \]

\[ \mathcal{H}_s^k := \{ w \in H^k_\gamma | \partial_w 1 |_{\partial \Omega} = 0 \}, \quad H^1_\gamma := \{ w \in H^1_\gamma | \det(\nabla w + I) = 1 \}, \]

\[ 0X := \{ w \in X | (\tilde{\omega} w)_{\partial \Omega} = 0 \}, \quad w_1 \text{ is the first compont of } w, \]

\[ X := \{ w \in X | (w)_{\Omega} = 0 \}, \quad \mathcal{H}_s^k := \mathcal{H}_s^k \cap H^1_\gamma, \quad \mathcal{H}_s^{3,8} := H^3_\gamma \cap H^3_\gamma, \]

\[ \mathcal{H}_s^\infty := \bigcap_{n=2}^{\infty} \mathcal{H}_s^\infty, \quad \| \cdot \|_i := \| \cdot \|_{H^i}, \quad \| \cdot \|_{l,i} := \| \partial_{l,i} \cdot \|_i, \quad \| \cdot \|_{L,i} := \sqrt{\sum_0 \| \cdot \|_{n,i}}; \]

where \( 1 \leq p \leq \infty, \ i, \ l \geq 0, \ j \geq 1, \ k \geq 2, \ X \) denotes a general Banach space and \( \gamma \in (0, 1) \) is the constant in Lemma A.6. It should be noted that if \( w \in H^3_\gamma \), then \( \psi := w + y \) (after possibly being redefined on a set of measure zero with respect to variable \( y \)) satisfies the same diffeomorphism properties as \( \zeta \) in (1.15) and (1.16) by Lemma A.6. In addition, for simplicity, we denote \( \sqrt{\sum_{i=0}^{n \leq j} \| f^k \|^2_X} \) by \( \| (f^1, \ldots, f^n) \|_X \), where \( \| \cdot \|_X \) represents a norm or a semi-norm, and \( f^k \) may be a scalar function, a vector or a matrix for \( 1 \leq n \leq j \).

(3) Simplified spaces of functions with values in a Banach space:

\[ L^p_\gamma X := L^p(I_T, X), \quad \mathcal{U}_T = \{ u \in C^0(\mathcal{I}_T, H^2_\gamma) \cap L^2_\gamma H^3 | u_t \in C^0(\mathcal{I}_T, L^2) \cap L^2_\gamma H^1_\gamma \}, \]

\[ \tilde{\mathcal{S}}^3_\gamma := \{ \eta \in C^0(\mathcal{I}_T, H^3_\gamma) | \eta(t) \in H^3_\gamma \text{ for each } t \in \mathcal{I}_T \}. \]

It should be noted that \( L^2_\gamma L^2 = L^2(\Omega_T) \).

(4) A functional of potential energy: for any given \( w \in H^1 \),

\[ E(w) := g \int \rho w^2 \, dy - \lambda \| \omega_1 w \|_0^2. \]

(5) Energy and dissipation functionals (generalized):

\[ \mathcal{E} := \| \eta \|_2^2 + \| u \|_2^2 + \| u_t \|_2^2 + \| q \|_2^2, \]

\[ \mathcal{D} := \| \partial_t \eta \|_2^2 + \| \eta_t \|_2^2 + \| u \|_2^2 + \| u_t \|_1^2 + \| q \|_2^2. \]

We call \( \mathcal{E} \), resp. \( \mathcal{D} \) the total energy, resp. dissipation functionals.

(6) Other notations for decay-in-times:

\[ \mathcal{E} := (t) \sum (\| \partial^2_t \eta_t \|_2^2 + \| \partial^2_t \eta_t \|_2^2) + (t)^2 (\| \partial^2_t \eta_t \|_2^2 + \| \partial_t \eta_t \|_2^2) \]

\[ + (t)^3 (\| \partial^2 \eta_t \|_2^2 + \| \partial_t \eta_t \|_2^2), \]

\[ \mathcal{D} := (t) \sum (\| \partial^2 \eta_t \|_2^2 + \| u \|_2^2) + (t)^2 (\| \partial_t \eta_t \|_2^2 + \| \partial_t \eta_t \|_2^2) \]

\[ + (t)^3 (\| \partial_t u \|_2^2 + \| u_t \|_2^2). \]

1.4. Main results

Now, we introduce the stability result for the transformed MRT problem.

**Theorem 1.1** (Stability). Let \( \tilde{\rho} \) satisfy (1.6), (1.7) and

\[ |m| > m_c := \sqrt{\frac{\sup_{w \in H^2_\gamma} g \int \rho w^2 \, dy}{\lambda \| \omega_1 w \|_0^2}}. \]
Further assume \((\eta^0, u^0) \in {\mathcal{H}}^{3} \times {\mathcal{H}}^{2} \) and \(\text{div}{\mathcal{A}}u^0 = 0\), where \(\mathcal{A} := (\nabla \eta^0 + I)^{-T}\). Then there is a sufficiently small constant \(\delta > 0\), such that for any \((\eta^0, u^0)\) satisfying
\[
\| (\nabla \eta^0, u^0) \|_2 \leq \delta,
\]
the transformed MRT problem \((1.19)\) and \((1.24)\) admits a unique global strong solution \((\eta, u, q)\) in the function class \(\tilde{S}_{1,\infty}^{1,3} \times \mathcal{U}_{\infty} \times \left( {\mathcal{C}}_{0}^{0}(\mathbb{R}^{+}, {\mathcal{H}}^{1}) \cap L_{\infty}^{2} {\mathcal{H}}^{2}\right)\). Moreover, the solution enjoys the following properties:

1. Stability estimate of total energy: for a.e. \(t > 0\),
\[
\mathcal{E}(t) + \int_0^t \mathcal{D}(\tau) d\tau \lesssim \| (\nabla \eta^0, u^0) \|_2^2,
\]

2. Algebraic decay-in-time: for a.e. \(t > 0\),
\[
\mathcal{E}(t) + c \int_0^t \mathcal{D}(\tau) d\tau \lesssim \| (\nabla \eta^0, u^0) \|_2^2,
\]
\[
\| \eta_1(t) - \eta_1^{\infty} \|_2^2 \lesssim (\| \nabla \eta_0 \|_2^2 + \| u_0 \|_2^2)(t),
\]
where \(\eta_1^{\infty} \in {\mathcal{H}}^{2}\) only depends on \(y_2\).

Remark 1.1. By the assumptions of \(\bar{\rho}\), we easily find that
\[
0 < m_C \leq \frac{h}{\pi} \sqrt{g \| \bar{\rho}' \|_{L_{\infty}^{\infty}} / \lambda} < (1.34)
\]
please refer to (4.25) and Lemma 4.6 in [24]. Thus, in view of Theorem 1.1 we see that the horizontal field can inhibit the RT instability, if the field strength is properly large. Since (1.34) also holds for the domain \(\Omega = \mathbb{R} \times (0, h)\), we naturally believe the conclusion that the properly large horizontal field can also inhibit the RT instability in the domain \(\Omega = \mathbb{R} \times (0, h)\). Such conclusion will be further investigated in an independent paper in future.

Remark 1.2. It is easy to see from the proof of Theorem 1.1 that

1. if the assumptions (1.7) and (1.30) are replaced by
\[
\bar{\rho}' \leq 0 \text{ in } \Omega \text{ and } |m| > 0,
\]
then the conclusions in Theorem 1.1 still hold.

2. Theorem 1.1 is also valid for the case \(g = 0\).

Remark 1.3. For each fixed \(t \in \mathbb{R}_0^{+}\), the solution \(\eta(y, t)\) in Theorem 1.1 belongs to \(H_\gamma^3\). Let \(\zeta = \eta + y\), then \(\zeta\) satisfies (1.15) and (1.16) for each \(t \in \mathbb{R}_0^{+}\) by Lemma A.6. We denote the inverse transformation of \(\zeta\) by \(\zeta^{-1}\), and then define that
\[
(p, v, N, \beta)(x, t) := (\bar{\rho}(y_2) - \bar{\rho}(\zeta_2), u(y, t), m \partial_1 \eta(y, t), q(y, t))|_{y = \zeta^{-1}(x, t)}.
\]
Consequently, \(p, v, N, \beta\) is a strong solution of the magnetic RT problem (1.9)–(1.11) and enjoys stability estimates, which are similar to (1.31)–(1.32) for sufficiently small \(\delta\).
Remark 1.4. In Theorem 1.1 we have assumed

$$(\bar{\rho} \eta_1^0)_{\Omega} = (\bar{\rho} u_1^0)_{\Omega} = 0.$$  

If $(\bar{\rho} \eta_1^0)_{\Omega}$, $(\bar{\rho} u_1^0)_{\Omega} \neq 0$, we define $\bar{\eta}_1^0 := \eta_1^0 - (\bar{\rho} \eta_1^0)_{\Omega} (\bar{\rho})_{\Omega}^{-1}$, $\bar{u}_1^0 := u_1^0 - (\bar{\rho} u_1^0)_{\Omega} (\bar{\rho})_{\Omega}^{-1}$ and $(\bar{\eta}_2^0, \bar{u}_2^0) := (\eta_2^0, u_2^0)$. Then, by virtue of Theorem 1.1, there exists a unique global strong solution $(\bar{\eta}, \bar{u}, \bar{q})$ to the transformed MRT problem with initial data $(\bar{\eta}^0, \bar{u}^0)$. It is easy to verify that $(\eta_1, \eta_2, u_1, u_2, q) := (\bar{\eta}_1 + t(\bar{\rho} u_1^0)_{\Omega} (\bar{\rho})_{\Omega}^{-1} + (\bar{\rho} \eta_1^0)_{\Omega} (\bar{\rho})_{\Omega}^{-1}, \bar{\eta}_2, \bar{u}_1 + (\bar{\rho} u_1^0)_{\Omega} (\bar{\rho})_{\Omega}^{-1}, \bar{u}_2, q)$ is just the unique strong solution of the transformed MRT problem with initial data $(\eta^0, u^0)$.

Remark 1.5. If additionally, the initial data $(\eta^0, u^0)$ in Theorem 1.1 satisfies the odevity conditions:

$$(\eta_1^0, u_1^0)(y_1, y_2) = -(\eta_1^0, u_1^0)(-y_1, y_2),$$

$$(\eta_2^0, u_2^0)(y_1, y_2) = (\eta_2^0, u_2^0)(-y_1, y_2).$$

then the solution $(\eta, u, q)$ established in Theorem 1.1 also satisfies the odevity conditions:

$$(\eta_1, u_1)(y_1, y_2, t) = -(\eta_1, u_1)(-y_1, y_2, t),$$

$$(\eta_2, u_2)(y_1, y_2, t) = (\eta_2, u_2)(-y_1, y_2, t), q(y_1, y_2, t) = q(-y_1, y_2, t).$$

Hence we have $\|\eta_1\|_0 \lesssim \|\eta_1\|_{1,0}$, which, together with (1.32), yields $\eta_1^\infty = 0$ in (1.33). This presents that all particles of the fluid restore to their initial locations, and thus the odevity conditions also strengthen the stabilizing effect of horizontal magnetic field as well as the fixed condition in [42].

Now we roughly sketch the proof of Theorem 1.1 and the details will be presented in Section 2. The key step in the existence proof for global small solutions is to derive an a priori energy inequality (1.31). To this purpose, let $(\eta, u)$ be a solution to the transformed MRT problem, satisfying that, for some $T > 0$,

$$(\bar{\rho} \eta_1)_{\Omega} \equiv 0 \text{ for any } t \in T_T, \quad (1.36)$$

$$\det(I + \nabla \eta) = 1 \text{ in } \Omega \times T_T, \quad (1.37)$$

$$\|\nabla \eta(t)\|_2 \leq C(t \in T_T). \quad (1.38)$$

For sufficiently small $\delta$, similarly to [42] where Wang verified that the vertical magnetic field can inhibit the RT instability in a stratified incompressible viscous MHD fluid in a 3D slab domain, the first step in our proof is also to derive the tangential energy inequality (i.e. (2.73) including the estimates of the both horizontal derivatives and temporal derivative). The next step is to capture the estimates of high-order normal derivatives of $\eta$. For the vertical magnetic field considered by Wang in [42], the magnetic tension in 3D case is given by $\lambda m^2 \partial_3^2 \eta$, which can be rewritten as follows

$$\lambda m^2 \Delta \eta - \lambda m^2 (\partial_1^2 + \partial_2^2) \eta.$$  

Thus the normal estimates of $\nabla \eta$ (not only includes the horizontal derivatives $\partial_1 \eta$ and $\partial_2 \eta$, but also the normal derivative $\partial_3 \eta$) can be converted into the tangential estimates by exploiting the regularity theory of Stokes equations. Obviously, this key idea fails to the horizontal magnetic field, and thus we shall seek a new idea.
In view of the first two equations in (1.19), we easily consider other two roads to capture the high-order normal estimates for $\eta$: one is to use the transport equation (1.19)$_1$, and the other one is to exploit the viscosity term $\Delta_A u$ in the momentum equation (1.19)$_2$. Since the first road seems to be more difficult, we naturally turn to the second one. By careful analysis of the structure of (1.19)$_2$, we find that the energy estimates of $\nabla_A \partial_1 \mathbf{curl}_A \eta$ and $\nabla_2 \mathbf{curl}_A \eta$ (associated with the dissipation estimates of $\partial_1 \mathbf{curl}_A \partial_1 \eta$ and $\partial_2 \mathbf{curl}_A \partial_1 \eta$, resp.) can be established under the Navier boundary condition, see Lemma 2.6. Thus we further derive the normal estimates of $\eta$ by using the curl estimates of $\eta$, the nonlinear estimates of $\text{div} \eta$ and Hodge-type elliptic estimate.

Summing up the tangential energy inequality and the curl-estimates of $\eta$, we can arrive at the total energy inequality

$$\frac{d}{dt} \tilde{\mathcal{E}} + \mathcal{D} \lesssim \sqrt{\mathcal{E} \mathcal{D}}$$

for some energy functional $\tilde{\mathcal{E}}$, which is equivalent to $\mathcal{E}$ under the stability condition (1.30). In particular, (1.39) further implies

$$2 \frac{d}{dt} \tilde{\mathcal{E}} + \mathcal{D} \leq 0,$$

which yields the priori stability estimate (1.31). Thanks to the priori estimate (1.31) and the unique local (in-time) solvability of the transformed MRT problem in Proposition 2.2, we immediately get the unique global solvability for the transformed MRT problem. We mention that the derivation for the a priori stability estimate strongly depends on the 2D structures of $\text{div} \eta$ and $\text{div} u$.

The decay-in-time estimate (1.32) can be easily observed from linear analysis. However the rigorous derivation is very complicated due to the nonlinear terms. In [26], Jiang–Jiang investigated the decay-in-time of solutions to the incompressible non-resistive viscous MHD equations in two-dimensional periodic domains, and used a bootstrap method in decay-in-time to obtain the higher rate of decay-in-time of solutions, similarly to the method to improve the regularity of solutions of elliptic equations. However such method is too complicated to be applied our problem. To simplify the proof, we fore an additional a priori assumption

$$\langle t \rangle^2 (\|\eta\|^2_{L^2} + \|u\|^2_{L^2}) \leq \delta.$$  (1.41)

Then we can also follow the idea in [26] with simplified derivation to quickly establish (1.32). It should be noted the derivation for the decay-in-time of $\|u(t)\|_2$ in (1.32) is different to the one in [26]. In fact, Jiang–Jiang obtained the rate of decay-in-time $\langle t \rangle^{-1}$ for $\|u(t)\|_2$ by directly using the momentum equation (1.19)$_2$ [26]. However, we further get the better rate of decay-in-time $\langle t \rangle^{-3/2}$ for $\|u(t)\|_2$ by using the estimate of temporal derivative of $u$ and the Stoke estimates, see (2.88). Finally, we eaily further get (1.33) from (1.32) by an asymptotic analysis method.

We can not expect the stability result for the transformed MRT problem under the condition $|m| \in [0, m_C)$. In fact, this condition results in the RT instability.

**Theorem 1.2 (Instability).** Let $\bar{\rho}$ satisfy (1.6) and (1.7). If $|m| \in [0, m_C)$, then the equilibria $(\bar{\rho}, 0, \bar{M})$ is unstable in the Hadamard sense, that is, there are positive constants $\varpi$, $\epsilon$, $\delta_0$, and $(\bar{\eta}^0, \eta^r, \bar{u}^0, u^r) \in H^3$, such that for any $\delta \in (0, \delta_0]$ and the initial data

$$(\eta^0, u^0) := \delta (\bar{\eta}^0, \bar{u}^0) + \delta^2 (\eta^r, u^r),$$
there exists a unique strong solution \((\eta, u, q)\) to the transformed MRT problem \((1.19)\) and \((1.24)\), where \((\eta, u, q) \in \tilde{S}^{1,3}_{\gamma,T} \times \mathcal{U}_r \times (C^0(\tilde{T}_r, H^1) \cap L^2 H^2)\) for any \(\tau \in I_{T_{\max}}\) and \(T_{\max}\) denotes the maximal time of existence of the solution. However, the solution satisfies for some escape time \(T^\delta := \Lambda^{-1} \ln(2\epsilon/\omega\delta) \in I_T\), where \(i = 1, 2\) and \(\chi\) means \(\eta\) or \(u\).

**Remark 1.6.** Following the arguments of Theorem 1.2 and [30, Corollary 2.2], it is easy to check that the corresponding 3D transformed MRT problem is always unstable for any \(|m| \geq 0\).

**Remark 1.7.** By the inverse transformation of Lagrangian coordinates in \((1.35)\) in Remark 1.3 and the instability relation in \((1.42)\), we easily obtain the instability expressions in Eulerian coordinates: for \(i = 1, 2\),

\[
\|\rho(T^\delta)\|_{L^1}, \|v_i(T^\delta)\|_{L^1}, \|\partial_1 v_i(T^\delta)\|_{L^1}, \|\partial_2 v_i(T^\delta)\|_{L^1} \geq \epsilon
\]

and

\[
\|N_i(T^\delta)\|_{L^1} \geq m \epsilon.
\]

The proof of Theorem 1.2 is based on the so-called bootstrap instability method. The bootstrap instability method has its origin in [12, 13], and adapted and generalized by many authors to investigate other flow instabilities, see [10, 11, 30] for examples. In particular, recently Jiang–Jiang–Zhan proved the existence of the RT instability solution under \(L^1\)-norm for the stratified viscous, non-resistive MHD fluids [30]. In this paper, we will adapt the version of the bootstrap instability method in [30] to prove Theorem 1.2. For the completeness, we will present the detailed proof in Section 3.

The rest of this paper is organized as follows. In Sections 2–3, we provide the proofs for Theorems 1.1 and 1.2 in sequence. Finally, in Appendix A, we list some mathematical results, which will be used in Sections 2–3.

### 2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. The key step in the proof is to a priori derive the total energy estimate \((1.31)\) and the decay-in-time \((1.32)\) for the transformed MRT problem \((1.19)\) and \((1.24)\). To this end, let \((\eta, u, q)\) be a solution to the transformed MRT problem, and satisfy \((1.36)\)–\((1.38)\), where \(\delta\) is sufficiently small, and the smallness of \(\delta\) depends on \(\mu, g, \lambda, m, \bar{\rho}\) and \(\Omega\). It should be noted that \(m\) and \(\bar{\rho}\) satisfy the assumptions in Theorem 1.1. Next, we start with a priori estimates.

#### 2.1. Preliminary estimates

First, we shall establish some preliminary estimates involving \((\eta, u)\).

**Lemma 2.1** (Nonlinear estimates). For any given \(t \in \tilde{T}_T\), we have

\[ (1) \] the estimates of \(\text{div} \eta\):

\[
\|\text{div} \eta\|_i \lesssim \|\nabla \eta\|_{1,i} \text{ for } 0 \leq i \leq 2,
\]

\[
\|\text{div} \eta\|_{i,0} \lesssim \begin{cases} \|\nabla \eta\|_2 \|\eta\|_{1+i,0} & \text{for } i = 0, 1; \\ \|\eta\|_{3,0} \|\nabla \eta\|_2 + \|\eta\|_{2,1}^2 & \text{for } i = 2. \end{cases}
\]
Moreover, for any given \( \delta \),
\[
\|G\|_{1,0} \lesssim \|\eta_2\|_0 \|\eta_2\|_{2,1}
\]  
(2.3)
where \( G := \eta - g\bar{\rho}\eta_2 \).

(3) the estimates involving \( \text{curl} \):
\[
\|\text{curl}_{\partial_t} \eta\|_2 \lesssim \|\eta\|_{1,2} \|\nabla \eta\|_2,
\]
(2.4)
\[
\|\text{curl}_{\partial_t} \eta\|_0 \lesssim \|\eta\|_{2,1}^2.
\]
(2.5)

Remark 2.1. Here and in what follows, we define \( \tilde{A} := A - I \). Then
\[
\tilde{A} = \begin{pmatrix}
\partial_2 \eta_2 & -\partial_1 \eta_2 \\
-\partial_2 \eta_1 & \partial_1 \eta_1
\end{pmatrix}.
\]

Proof. (1) Recalling (1.37), we can calculate that
\[
\text{div} \eta = \partial_1 \eta_2 \partial_2 \eta_1 - \partial_1 \eta_1 \partial_2 \eta_2
\]
and
\[
\partial_1 \text{div} \eta = \partial_1 \eta_2 \partial_2 \eta_1 + \partial_2^2 \eta_2 \partial_2 \eta_1 - \partial_1 \eta_1 \partial_2 \partial_2 \eta_2 - \partial_1^2 \eta_1 \partial_2 \eta_2.
\]
(2.6)
Exploiting the product estimates (A.3), (A.4) and Poincaré’s inequality (A.8), it is easy to see from the above two relations that (2.1) and (2.2) hold for \( i = 0, 1 \). If we further apply \( \partial_1 \) to the identity (2.0), we also check that \( \|\text{div} \eta\|_{2,0} \) satisfies (2.2) with \( i = 2 \).

(2) By virtue of (1.38) and Lemma A.6, \( \zeta := \eta + y \) satisfies the diffeomorphism properties (1.15) and (1.16) for sufficiently small \( \delta \). Thus, \( \bar{\rho}^{(j)}(y_2 + \eta_2) \) for any \( y \in \Omega \) makes sense, and
\[
\bar{\rho}^{(j)}(y_2 + \eta_2) - \bar{\rho}^{(j)}(y_2) = \int_0^{\eta_2} \bar{\rho}^{(j+1)}(y_2 + z) \mathrm{d} z \text{ for } j = 0, 1.
\]
(2.7)
Moreover, for any given \( t \in \Omega_T \),
\[
\sup_{y \in \Omega} \sup_{z \in \Psi} |\bar{\rho}^{(j+1)}(y_2 + z)| \lesssim 1,
\]
(2.8)
where \( \Psi := \{ \tau \mid 0 \leq \tau \leq \eta_2 \} \) for \( \eta_2 \geq 0 \) and \( := (\eta_2, 0) \) for \( \eta_2 < 0 \).

Making use of (2.7), (2.8), (A.4) and the relation
\[
\bar{\rho}(y_2 + \eta_2) - \bar{\rho}(y_2) = \bar{\rho}'(y_2) \eta_2 + \int_0^{\eta_2} (\eta_2(y, t) - z) \bar{\rho}''(y_2 + z) \mathrm{d} z,
\]
it is easy to estimate that
\[
\|G\|_0 = g \left\| \int_0^{\eta_2} (\eta_2(y, t) - z) \bar{\rho}''(y_2 + z) \mathrm{d} z \right\|_0 \lesssim \|\eta_2\|_0 \|\eta_2\|_{2,1}
\]
and
\[
\|G\|_{1,0} = g \left\| (\bar{\rho}'(y_2 + \eta_2) - \bar{\rho}') \partial_1 \eta_2 \right\|_0 \lesssim \|\eta_2 \partial_1 \eta_2\|_0 \lesssim \|\eta_2\|_0 \|\partial_1 \eta_2\|_{1,1}.
\]
(2.9)
Thanks to the above two estimates, we immediately obtain (2.3).

(3) Noting that, for \( 0 \leq k + l \leq 2 \),
\[
\text{curl}_{\partial_t} \eta = (\partial_2 \partial_l^k \eta_1 \partial_1 - \partial_1 \partial_l^k \eta_1 \partial_2) \partial_l \eta_1 + (\partial_2 \partial_l^k \eta_2 \partial_1 - \partial_1 \partial_l^k \eta_2 \partial_2) \partial_l \eta_2,
\]
thus it is easy to check that (2.4) and (2.5) hold by following the arguments of (2.1) and (2.2). \( \square \)
Lemma 2.2. We have

1. the estimate of $\eta_2$:

$$\|\eta_2\|_i \lesssim \begin{cases} \|\eta\|_{1,0} & \text{for } i = 0, 1; \\ \|\eta\|_{1,i-1} & \text{for } i = 2, 3. \end{cases}$$

(2.10)

2. Poincaré inequality for $\eta$, $u$ and $u_t$: for $j = 1$ and 2,

$$\|\eta_j\|_1 \lesssim \|\nabla \eta_j\|_0,$$

(2.11)

$$\|u_j\|_1 \lesssim \|\nabla u_j\|_0,$$

(2.12)

$$\|\partial_t u_j\|_1 \lesssim \|\nabla \partial_t u_j\|_0.$$

(2.13)

3. the estimate involving the gravity term: for sufficiently small $\delta$,

$$\|G\|_1 \lesssim \|\eta_2\|_0,$$

(2.14)

4. curl estimates: for sufficiently small $\delta$,

$$\|\eta\|_{k,3-k} \lesssim \|\text{curl} \eta\|_{k,2-k} \text{ where } 0 \leq k \leq 2.$$

(2.15)

Proof. (1) Noting that $\eta_2|_{\partial \Omega} = 0$ and $\partial_2 \eta_2 = \text{div} \eta - \partial_1 \eta_1,$

we use (1.38), (2.1), (2.16) and (A.6) to get

$$\|\eta_2\|_0 \lesssim \|(\partial_1 \eta_1, \text{div} \eta)\|_0 \lesssim \|\eta\|_{1,0},$$

$$\|\eta_2\|_1 \lesssim \|\eta\|_0 + \|\nabla \eta_2\|_0 \lesssim \|(\partial_1 \eta, \text{div} \eta)\|_0 \lesssim \|\eta\|_{1,0},$$

$$\|\eta_2\|_i \lesssim \|\eta\|_0 + \|\eta_2\|_{1,i-1} + \|\partial_2 \eta_2\|_{i-1} \lesssim \|\eta\|_{1,i-1} \text{ for } i = 2, 3.$$ (2.17)

Thus, we immediately get (2.10) from the above four estimates.

(2) By (1.19) and (1.36), it is easy to see that

$$\bar{\rho} u_1|_{\partial \Omega} = 0$$

(2.18)

Thus the estimates (2.11) and (2.12) obviously hold due to (1.36), (2.18), (A.6), Lemma A.9 and the boundary condition $\eta_2, u_2)|_{\partial \Omega} = 0$.

Multiplying (1.19) by $e_1$ in $L^2$, and then using the integral by parts and the relation

$$\partial_j (\partial^k_i A_{ij} f) = \partial^k_i A_{ij} \partial_j f \text{ for } k = 0, 1,$$

we get

$$\int \bar{\rho} \partial_t u_1 \, dy + \int_{\partial \Omega} \bar{n}_2 (A_{12} q - \mu A_{12} A_{ij} \partial_j u_1) \, dy = 0,$$

which, together with the boundary condition (1.24), yields

$$\bar{\rho} \partial_t u_1|_{\partial \Omega} = 0.$$

(2.20)

Thanks to (2.20) and Lemma A.9, thus we have (2.13) for $j = 1$. Noting that $\partial_t u_2|_{\partial \Omega} = 0$, thus, by (A.6), we also have (2.13) for $j = 2$. Hence (2.13) holds.
Moreover, by the boundary condition (1.24), we have
\[ \| \partial_2 G \|_0 = g \| (\rho'(y_2 + \eta_2) - \rho') (1 + \partial_2 \eta_2) - \rho'' \eta_2 \|_0 \]
\[ \lesssim \| (\eta_2, \eta_2 \partial_2 \eta_2) \|_0 \lesssim \| \eta_2 \|_0, \]
which, together with (2.3), yields (2.14).

(4) Making use of (2.1), (2.2), (2.11), (A.8) and (A.9), we have
\[ \| \eta \|_{k,3-k} \lesssim \| \nabla \eta \|_{k,2-k} \lesssim \| (\text{curl} \eta, \text{div} \eta) \|_{k,2-k} \lesssim \| \text{curl} \eta \|_2 \| \eta \|_{k,3-k}, \]
which yields (2.15) for sufficiently small \( \delta \).

\[ \square \]

2.2. Tangential estimates

This section is devoted to establishing the tangential estimates by the following three lemmas, which include the estimates of horizontal derivatives of \( (\eta, u) \) and temporal derivative of \( u \).

**Lemma 2.3.** For sufficiently small \( \delta \), it holds that, for \( 0 \leq i \leq 2 \),
\[
\frac{d}{dt} \left( \int \rho \partial_i^1 \eta \cdot \partial_i^1 u \, dy + \frac{\mu}{2} \| \nabla \partial_i^1 \eta \|_0^2 \right) - E(\partial_i^1 \eta) \\
\leq \| \sqrt{\rho} \partial_i^1 u \|_0^2 + \| \nabla \eta \|_2 \| \eta \|_3,0(\| \eta_2 \|_0 + \| u \|_1^2 + \| q \|_{-1}) \\
+ \| \eta \|_2 \| q \|_1 + \left\{ \begin{array}{ll}
\| \eta \|_{1,1} \| \nabla \eta \|_1 \| u \|_1 & \text{for } i = 0; \\
0 & \text{for } i = 1, 2.
\end{array} \right.
\]

**Proof.** We apply \( \partial_i^1 \) to (1.19) and (1.24), and then use the relation (2.19) to derive that
\[
\begin{cases}
\partial_i^1 \eta_t = \partial_i^1 u, \\
\partial_i^1 (\rho u_t - \mu \Delta u) = \partial_i^1 (\lambda m^2 \partial_i^2 \eta + g \rho' \eta_2 e_2 + G e_2 + N^\mu - \nabla_A q), \\
[\partial_i^1, A_{kl}] \partial_i u_k + A_{kl} \partial_i^1 \partial_i^1 u_k = 0, \\
\partial_i^1 (\eta_2, u_2, \partial_2(\eta_1, u_1)) \mid_{\partial \Omega} = 0,
\end{cases}
\]
where
\[
N^\mu = \partial_i (N^\mu_{1,i}, N^\mu_{2,i}), \\
N^\mu_{1,i} = \mu (A_{kl} \tilde{A}_{km} + \tilde{A}_{m1}) \partial_m u_j \\
= \mu ((2 \partial_2 \eta_2 + (\partial_2 \eta_1)^2 + (\partial_2 \eta_2)^2) \partial_1 u_j - \Theta \partial_2 u_j), \\
N^\mu_{2,i} = \mu (A_{kl} \tilde{A}_{km} + \tilde{A}_{m2}) \partial_m u_j \\
= \mu ((2 \partial_1 \eta_1 + (\partial_1 \eta_1)^2 + (\partial_1 \eta_2)^2) \partial_2 u_j - \Theta \partial_1 u_j)
\]
and
\[
\Theta := \partial_1 \eta_2 + \partial_2 \eta_1 + \partial_1 \eta_2 \partial_2 \eta_2 + \partial_1 \eta_1 \partial_2 \eta_1.
\]
Moreover, by the boundary condition (1.24), we have
\[
\partial_i^1 N^\mu_{1,2} \mid_{\partial \Omega} = 0.
\]
Let $0 \leq i \leq 2$. Multiplying (2.22) by $\partial_i^4 \eta$, then we use the integral by parts, (2.22), and (2.22) to obtain

$$
\frac{d}{dt} \left( \int \tilde{\rho} \partial_i^4 \eta \cdot \partial_i^4 u \, dy + \frac{\mu}{2} \| \nabla \partial_i^4 \eta \|_0^2 \right) - E(\partial_i^4 \eta) = \| \sqrt{\tilde{\rho}} \partial_i^4 u \|_0^2 + \sum_{j=1}^4 I_{j,i},
$$

(2.24)

where we have defined that

$$I_{1,i} := \int \partial_i \mathcal{G} \partial_i^2 \eta \, dy, \quad I_{2,i} := - \int \partial_i^2 \mathcal{N}_j \partial_i^{j+1} \eta \, dy$$

$$I_{3,i} := \int \partial_i \mathcal{N}_j \partial_i^{j+1} \eta \, dy \quad \text{and} \quad I_{4,i} := - \int \partial_i \nabla q \cdot \partial_i^4 \eta \, dy.$$

Next we estimate for the above four integrals $I_{1,i} - I_{4,i}$ in sequence.

1. By the integral by parts, Hölder’s inequality and (2.3), we infer that

$$I_{1,i} \leq \begin{cases} \| \eta_2 \|_0 \| \mathcal{G} \|_0 \lesssim \| \eta_2 \|_0^2 \| \eta_2 \|_{L^2} & \text{for } i = 0; \\ \| \eta_2 \|_{L^{i+1,0}} \| \mathcal{G} \|_{L^{i-1,0}} \lesssim \| \eta_2 \|_0 \| \eta_2 \|_{L^2} \| \eta_2 \|_{L^{i+1,0}} & \text{for } i = 1, 2. \end{cases}
$$

(2.25)

2. Exploiting Hölder’s inequality, (1.38), (A.3), and (A.4), we can see that

$$I_{2,i} := \mu \int \partial_i^{j+1} \eta \cdot \partial_i (\Theta \partial_2 u - (2 \partial_2 \eta_2 + (\partial_2 \eta_1^2 + (\partial_2 \eta_2)^2) \partial_1 u) \, dy$$

$$\lesssim \| \nabla \eta \|_{L^2} \| \eta \|_{L^{i+1,0}} \| u \|_{L^2}.
$$

(2.26)

3. Using the integral by parts, (2.22), (2.23) and the product estimate (A.3), we can estimate

$$I_{3,0} := \mu \int \partial_2 \eta \cdot (\Theta \partial_1 u - (2 \partial_1 \eta_1 + (\partial_1 \eta_1)^2 + (\partial_1 \eta_2)^2) \partial_2 u) \, dy$$

$$= \mu \int (\partial_2 \eta \cdot (2 \partial_1 \eta_1 + \partial_1 \eta_1 \partial_2 \eta_2 + \partial_1 \eta_1 \partial_2 \eta_1) \partial_1 u - (2 \partial_1 \eta_1 + (\partial_1 \eta_1)^2 + (\partial_1 \eta_2)^2) \partial_2 u)$$

$$+ \partial_1 \eta_1 \partial_2 \eta_1 \partial_1 \eta \cdot u - \partial_2 \eta_1 \partial_1 \eta_1 \partial_1 \eta \cdot u \, dy$$

$$\lesssim \| \eta \|_{L^1} \| \nabla \eta \|_{L^1} \| u \|_{L^1}.
$$

(2.27)

Similarly to (2.26), we also have

$$I_{3,i} := \mu \int \partial_i^{j+1} \eta \cdot \partial_2 \partial_i^{j-1} (\Theta \partial_1 u - (2 \partial_1 \eta_1 + (\partial_1 \eta_1)^2 + (\partial_1 \eta_2)^2) \partial_2 u) \, dy$$

$$\lesssim \| \nabla \eta \|_{L^2} \| \eta \|_{L^{i+1,0}} \| u \|_{L^2} \quad \text{for } i = 1, 2.
$$

(2.28)

4. Finally we bound the last integral $I_{4,i}$. Noting

$$\text{div} \tilde{\mathcal{A}} \eta = 2(\partial_1 \eta_1 \partial_2 \eta_2 - \partial_1 \eta_2 \partial_2 \eta_1),$$

making use of the above identity, the integral by parts, (2.2), (2.19), (2.22), (A.4), and (A.8), we can estimate that

$$I_{4,0} = \int \text{div} \tilde{\mathcal{A}} \eta \eta \, dy + \int \text{div} \eta q \, dy \lesssim \| \eta \|_{L^1} \| \nabla \eta \|_{L^2} \| q \|_0
$$

(2.29)
and, for $i = 1, 2$,

$$I_{4,i} = \int \partial_{1}^{i+1} \eta \cdot \partial_{1}^{-1} \nabla q dy + \int \text{div} \partial_{1} \eta \partial_{1} q dy$$

$$\lesssim \| \eta \|_{i+1,0} \| \nabla \partial_{1} q \|_{i-1,0} + \| \text{div} \eta \|_{i,0} \| q \|_{i,0}$$

$$\lesssim (\| \eta \|_{3,0} \| \nabla \eta \|_{2,1} + \| \eta \|_{2,1}^{2}) \| q \|_{L,1}.$$  \hspace{1cm} (2.30)

Consequently, putting $(2.25)$ - $(2.30)$ into $(2.24)$, and then using $(2.10)$ and $(A.8)$, we arrive at $(2.21)$. This completes the proof. \hspace{1cm} \qed

**Lemma 2.4.** For sufficiently small $\delta$, it holds that, for $0 \leq i \leq 2$,

$$\frac{d}{dt} \left( \| \sqrt{\rho u} \|_{i,0}^{2} - E(\partial_{1}^{i} \eta) \right) + c \| u \|_{i,0}^{2}$$

$$\lesssim \| \eta \|_{2,1} (\| \eta \|_{2,0} + \| u \|_{1,2}) \| u \|_{i,1} + (\| \eta \|_{3,0} \| u \|_{1,2} + \| \eta \|_{2,1} \| u \|_{2,1}) \| q \|_{L,1}.$$  \hspace{1cm} (2.31)

**Proof.** Multiplying $(2.22)_2$ by $\partial_{1}^{i} u$ in $L^{2}$, and then using the integrate by parts, $(2.22)_1$ and the boundary conditions $(2.22)_4$ and $(2.23)$, we have

$$\frac{1}{2} \frac{d}{dt} \left( \| \sqrt{\rho u} \|_{i,0}^{2} - E(\partial_{1}^{i} \eta) \right) + \mu \| \nabla u \|_{i,0}^{2}$$

$$= \int \partial_{1} \mathcal{G} \partial_{1} u_{2} d y - \int \partial_{1} \mathcal{N}_{j} \partial_{1} u_{j} d y - \int \partial_{1} \nabla \cdot \partial_{1} q u d y =: \sum_{j=5}^{7} I_{j,i}. \hspace{1cm} (2.32)$$

Similarly to $(2.25)$, we have

$$I_{5,i} \lesssim \begin{cases} 
\| \eta \|_{0} \| \eta \|_{2,1} \| u \|_{0} & \text{for } i = 0; \\
\| \eta \|_{0} \| \eta \|_{2,1} \| u \|_{1+i,0} & \text{for } i = 1, 2.
\end{cases} \hspace{1cm} (2.33)$$

By $(A.3)$ and $(A.4)$, we can estimate that

$$I_{6,i} = \mu \int (\partial_{1}^{i} ((2 \partial_{2} \eta_{2} + (\partial_{2} \eta)_{0})^{2} + (\partial_{2} \eta)_{2}^{2}) \partial_{1} u - \Theta \partial_{2} u) \cdot \partial_{1}^{i+1} u$$

$$\quad + \partial_{1}^{i} ((2 \partial_{1} \eta_{1} + (\partial_{1} \eta)_{0})^{2} + (\partial_{1} \eta)_{2}^{2}) \partial_{2} u - \Theta \partial_{2} u) \cdot \partial_{2} \partial_{1} u) dy$$

$$\lesssim \| \partial_{1} \eta \|_{L,1} \| u \|_{L,2} \| u \|_{i,1} + \| \nabla \eta \|_{2} \| u \|_{i,1}^{2}.$$  \hspace{1cm} (2.34)

Similarly, for $i=1, 2$,

$$I_{7,i,1} := - \int [\partial_{1}, \mathcal{A}_{kl}] \partial_{1} u \partial_{1} q u_{k} dy \lesssim \| \partial_{1} \eta \|_{L,1} \| u \|_{i,1} \| q \|_{L,1}$$

and

$$I_{7,i,2} = - \int [\partial_{1}, \mathcal{A}_{kl}] \partial_{1} u_{k} \partial_{1} q dy$$

$$= \int ([\partial_{1}, \partial_{1} \eta_{2}] \partial_{2} u_{1} - [\partial_{1}, \partial_{2} \eta_{2}] \partial_{1} u_{1} + [\partial_{1}, \partial_{2} \eta_{1}] \partial_{1} u_{2} - [\partial_{1}, \partial_{1} \eta_{1}] \partial_{2} u_{2}) \partial_{1} q dy$$

$$\lesssim (\| \partial_{1} \eta \|_{2,0} \| u \|_{L,2} + \| \partial_{1} \eta \|_{L,1} \| \partial_{1} u \|_{L,1}) \| q \|_{L,0}.$$
Making use of the integral by parts, (2.19), (2.22), (A.8) and the above two estimates, we have

\[ I_{7,i} = \begin{cases} 
0 & \text{for } i = 0; \\
I_{7,i,1} + I_{7,i,2} & \text{for } i = 1, 2 
\end{cases} \]

\[ \lesssim (\|\partial_1 \eta\|_{2,0} \|u\|_{1,2} + \|\partial_1 \eta\|_{1,1} \|\partial_1 u\|_{1,1}) \|q\|_{1,1}. \]

(2.35)

Consequently, putting (2.33–2.35) into (2.32), and then using (1.38), (2.10), (2.12) and (A.8), we arrive at (2.31) for sufficiently small \( \delta \). This completes the proof. □

**Lemma 2.5.** For sufficiently small \( \delta \), we have

\[ \frac{d}{dt} \|\nabla_A u\|_{0}^2 + c \|u_t\|_{0}^2 \lesssim \|\eta\|_{2,0}^2 + \|u\|_{2}^3 + \|u\|_{2}^2 \|q\|_{1} \]

(2.36)

and

\[ \frac{d}{dt} (\|\sqrt{\bar{\rho}} \psi\|_{0}^2 - E(u)) + c \|u_t\|_{1}^2 \lesssim (\|\eta\|_{2,0} + \|u\|_{2}) \|u\|_{2}^3, \]

(2.37)

where \( \psi := u_t - u \cdot \nabla_A u \).

**Proof.** (1) By (1.19)3, we see that

\[ \text{div}_A u_t = -\text{div}_A u. \]

Multiplying (1.19)2 by \( u_t \) in \( L^2 \), and then using the integral by parts, (1.24), (2.19) and the above relation, we obtain

\[ \frac{\mu}{2} \frac{d}{dt} \|\nabla_A u\|_{0}^3 + \|\sqrt{\bar{\rho}} u_t\|_{0}^3 = I_8, \]

(2.38)

where we have defined that

\[ I_8 := \int (\lambda m^2 \partial_1^2 \eta + (g \bar{\rho}' \eta_2 + \mathcal{G})e_2) \cdot u_t dy + \int (\mu \nabla_A u : \nabla_A u + \nabla q \cdot (A^T u)) dy. \]

Exploiting (2.3) and (A.3), we get

\[ I_8 \lesssim \|(\partial_1^2 \eta, \eta_2)\|_0 \|u_t\|_0 + \|u\|_3^3 + \|u\|_2^2 \|q\|_1. \]

Putting the above estimate into (2.38), and then using (2.10) and (A.8), we get (2.36).

(2) Let

\[ I_9 := \int ((\mu A_{il} \partial_t (A_{ik} \partial_k u) + \lambda m^2 \partial_1^2 \eta + (\mathcal{G} + g \bar{\rho}' \eta_2)) e_2 \\
- \bar{\rho} u \cdot \nabla_A u - \bar{\rho} \psi \cdot (u \cdot \nabla_A \psi) - \partial_t (\bar{\rho} u \cdot \nabla_A u) \cdot \psi) dy, \]

\[ I_{10} := \int ((\lambda m^2 \partial_1^2 u + g \bar{\rho}' (y_2 + \eta_2) u_2 e_2) \cdot \psi - \mu \partial_t (A_{il} A_{ik} \partial_k u) \cdot \partial_t \psi) dy. \]

Recalling the derivation of (4.7) in Section 4 and the relation

\[ \frac{1}{2} \int |\psi|^2 u \cdot \nabla_A \bar{\rho} dy = - \int \bar{\rho} u \cdot \nabla_A \psi \cdot \psi dy, \]

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we can get the following identity (i.e. taking $J = 1$, $w = u$ and $f = \lambda m^2 \partial_t^2 \eta + (G + g \rho' \eta_2)e_2$ in (4.7)) from (1.19):

$$
\frac{1}{2} \frac{d}{dt} \| \sqrt{\rho} \psi \|_0^2 = I_9 + I_{10}.
$$

(2.39)

The integral term $I_{10}$ can be further rewritten as follows:

$$
I_{10} = \frac{1}{2} \frac{d}{dt} E(u) - \mu \| \nabla u_t \|_0^2 + \bar{I}_{10},
$$

where

$$
\bar{I}_{10} := \int (g(\rho'(y_2 + \eta_2) - \rho'(y_2))u_2 \partial_t u_2 - (\lambda m^2 \partial_t^2 u + g \rho'(y_2 + \eta_2)u_2 e_2) \cdot (u \cdot \nabla_A u) + \mu(\partial_t(A_{ik}A_{ik}) \cdot \partial_t(u \cdot \nabla_A u) - \partial_t(A_{it}A_{it} + \tilde{A}_{it} \partial_t u) \cdot \partial_t \partial_t u) dy.
$$

Making use of (1.38), (1.19), (2.7), (2.14) and (A.8), the above estimate and Young's inequality, we immediately get (2.37) for sufficiently small $\delta$. This completes the proof.

2.3. Curl estimates for $\eta$

This section is devoted to establishing the curl estimates of $\eta$ for the normal estimates of $\eta$.

Lemma 2.6. Let the multiindex $\alpha$ satisfy $|\alpha| \leq 1$. We have

$$
\frac{d}{dt} \left( \frac{\mu}{2} \| \nabla_A \partial^\alpha \text{curl}_A \eta \|_0^2 + \int \partial^\alpha \text{curl}_A \eta \partial^\alpha \text{curl}_A (\bar{\rho} u) \ dy \right) + c \| \partial^\alpha \text{curl}_A \partial_1 \eta \|_0^2
\lesssim \| \eta \|_{1,0} \| \eta \|_{1+|\alpha|,1} + \| u \|_1 \| u \|_3 + \| \nabla \eta \|_2 (\| \eta \|_2^2 + |\alpha|,1
+ \| \eta \|_{1,2} \| u \|_{1,2} + \| u \|_2^2) \text{ for } \alpha_2 \neq 1
$$

(2.41)

and

$$
\frac{d}{dt} \left( \frac{\mu}{2} \| \nabla_2 \text{curl}_A \eta \|_0^2 + I_2 \right) + c \| \nabla_2 \text{curl}_A \partial_1 \eta \|_0^2
\lesssim \| \eta \|_{1,0} \| \eta \|_{1,2} + \| u \|_1 \| u \|_3 + \sqrt{ED},
$$

(2.42)

where

$$
I_2 := \int (\mu \partial_2^2 (\partial_1 \eta \partial_2 \eta) \partial_2 \text{curl}_A \eta - \partial_2^2 \text{curl}_A \eta \partial_2 \text{curl}_A (\bar{\rho} u))dy - I_1,
$$

$$
I_1 := \int \partial_2 \text{curl}_A \eta (\partial_2 (\partial_2 \text{curl}_A \partial_1 \eta) + \partial_2 \eta (\partial_2 \text{curl}_A \partial_1 \eta - \partial_2 \eta \partial_2 \text{curl}_A \partial_1 \eta))dy.
$$

Proof. In view of (1.18), we have

$$
\text{curl}_A (gG_\eta e_2) = \text{curl}_A (-g \bar{\rho} e_2) = -gA_{1j} \partial_j \bar{\rho} = g \rho' \partial_1 \eta_2.
$$
Therefore, applying \( \text{curl}_A \) to (1.19) and then using the fact
\[
\text{curl}_A \Delta_A u = \Delta_A \text{curl}_A u,
\]
we get
\[
\partial_t \text{curl}_A(\hat{\rho} u) - \mu \Delta_A \text{curl}_A u - \lambda m^2 \partial_t \text{curl}_A \partial_1 \eta = g \rho' \partial_t \eta - \lambda m^2 \text{curl}_A \partial_1 \eta + \text{curl}_A(\hat{\rho} u).
\] (2.43)

Let \( \alpha \) be a multi-index. Applying \( \partial^\alpha \) to (2.43) yields
\[
\partial_t \partial^\alpha \text{curl}_A(\hat{\rho} u) - \mu \partial^\alpha \Delta_A \text{curl}_A u - \lambda m^2 \partial_t \partial^\alpha \text{curl}_A \partial_1 \eta = g \rho' \partial^\alpha \partial_1 \eta - \lambda m^2 \partial^\alpha \text{curl}_A \partial_1 \eta + \partial^\alpha \text{curl}_A(\hat{\rho} u).
\] (2.44)

(1) We take \( \alpha = (i, 0) \) and \( 0 \leq i \leq 1 \). Multiply (2.44) by \( \partial_t \text{curl}_A \eta \) in \( L^2 \) and then using the integral by parts, we have
\[
\frac{d}{dt} \int \partial_t^i \text{curl}_A(\hat{\rho} u) \partial_t^i \text{curl}_A \eta dy + \lambda m^2 \| \partial_t^i \text{curl}_A \partial_1 \eta \|_0^2 = \sum_{j=1}^4 J_{j,i},
\] (2.45)

where
\[
J_{1,i} := \int (\partial_t^i \text{curl}_A(\hat{\rho} u) \partial_t^i \text{curl}_A u - g \rho' \partial_t \eta_2 \partial_t^1 \text{curl}_A \eta) dy,
J_{2,i} := \int (\partial_t^i \text{curl}_A(\hat{\rho} u) \partial_t^i \text{curl}_A \partial_1 \eta + \partial_t^i \text{curl}_A \partial_1 \eta \partial_t^i \text{curl}_A \eta) dy,
J_{3,i} := -\lambda m^2 \int (\partial_t \text{curl}_A \partial_1 \eta \partial_t^i \text{curl}_A u + \partial_t \text{curl}_A \partial_1 \eta \partial_t^i \text{curl}_A \eta) dy,
J_{4,i} := \mu \int \partial_t^i \Delta_A \text{curl}_A u \partial_t^i \text{curl}_A \eta dy.
\]

Making use of (1.38), (2.3), (A.3) and the integral by parts, we have
\[
J_{1,i} \lesssim \| \partial_t^i \text{curl}_A(\hat{\rho} u) \|_0 \| \partial_t^i \text{curl}_A u \|_0 + \| \partial_t \eta_2 \|_0 \| \partial_t^1 \text{curl}_A \eta \|_0
\lesssim \| u \|_2 + \| \eta_2 \|_{i,0} \| \eta \|_{1+i,1},
\] (2.46)
\[
J_{2,i} \lesssim \| \nabla \eta \|_2 \| u \|_2.
\] (2.47)

and
\[
J_{3,i} \leq c \| \nabla \eta \|_2 \| \eta \|_{1+i,1}^2 + \begin{cases} 0 & \text{for } i = 0; \\ \lambda m^2 \int \text{curl}_A \partial_1 \eta \partial_t^2 \text{curl}_A \eta dy & \text{for } i = 1 \\ 0 & \text{for } i = 0; \\ \lambda m^2 \int \text{curl}_A \partial_1 \eta \partial_t^2 \text{curl}_A \eta dy & \text{for } i = 1 \\ \leq c \| \nabla \eta \|_2 \| \eta \|_{1+i,1}^2. \end{cases}
\] (2.48)

Next we turn to the estimate of \( J_{4,i} \).

Using the integral by parts, the boundary condition (1.27) and (2.19), we have
\[
J_{4,i} = \frac{1}{2} \sum_{j=1}^3 J_{4,i,j} - \mu \frac{d}{dt} \| \nabla_A \partial_t^i \text{curl}_A \eta \|_0^2,
\] (2.49)
where

\[ J_{4,i,1} := \mu \int \nabla_A \partial_1^i \text{curl}_A \eta \cdot \nabla_A \partial_1^i \text{curl}_A \eta \, dy, \]

\[ J_{4,i,2} := \mu \int \nabla_A \partial_1^i \text{curl}_A \eta \cdot \nabla_A \partial_1^i \text{curl}_A \eta \, dy, \]

\[ J_{4,i,3} := \begin{cases} 0 & \text{for } i = 0; \\ -\mu \int \partial_1 (A_{kl} A_{kn}) \partial_n \text{curl}_A u \cdot \partial_i \partial_1 \text{curl}_A \eta \, dy & \text{for } i = 1. \end{cases} \]

Exploiting (A.3) and (A.4), we easily get

\[ \sum_{j=1}^{3} J_{4,1,j} \lesssim \| \eta \|_{1,2} \| \nabla \eta \|_{2} \| u \|_{1,2}. \]

In addition,

\[ J_{4,0,1} = \mu \int \nabla_A \text{curl}_A \eta \cdot \nabla_A (\partial_1 \eta \partial_2 u_1 + \partial_1 \eta_2 \partial_2 u_2 - \partial_2 \eta_1 \partial_1 u_1 - \partial_2 \eta_2 \partial_1 u_2) \, dy \]

\[ \leq c \| \nabla \eta \|_1 \| (\partial_1 \eta, \partial_2 \eta_2) \|_2 \| u \|_1 + \mu \int (\nabla_A \text{curl}_A \eta \cdot (\nabla_A (\partial_1 \partial_2 \eta_1 u_1) + \nabla_A \text{curl}_A \eta \cdot \nabla_A (\partial_2 \eta_1 u_1)) \, dy \]

\[ \lesssim \| \nabla \eta \|_1 \| (\partial_1 \eta, \partial_2 \eta_2) \|_2 \| u \|_2. \]

and

\[ J_{4,0,2} = \mu \int (A_{1i} \partial_2 \text{curl}_A \eta (\partial_1 \text{curl}_A \eta \partial_2 u_2 - \partial_2 \text{curl}_A \eta \partial_1 u_2) \]

\[ + A_{2i} \partial_1 \text{curl}_A \eta (\partial_2 \text{curl}_A \eta \partial_1 u_1 - \partial_1 \text{curl}_A \eta \partial_2 u_1)) \, dy \]

\[ \lesssim \| \nabla \eta \|_1 \| \eta \|_{1,2} \| u \|_2. \]

Thanks to the above three estimates and (2.10), we can infer from (2.49) that

\[ J_{4,i} \leq c \| \eta \|_{1,2} \| \nabla \eta \|_{2} \| u \|_{1,2} - \frac{\mu}{2} \frac{d}{dt} \| \nabla_A \partial_1^i \text{curl}_A \eta \|_2^2. \quad (2.50) \]

Inserting the estimates (2.46)–(2.50) into (2.45), and then using the interpolation inequality (A.2), we immediately obtain (2.41).

(2) Now we turn to the derivation of (2.42). Multiplying (2.43) by \(-\partial_2^2 \text{curl}_A \eta\), and then using the integral by parts, and the boundary conditions (1.25), (1.27) and \(\partial_1 \text{curl}_A u|_{\partial \Omega} = 0\), we have

\[ \frac{d}{dt} \left( \frac{\mu}{2} \| \nabla \partial_2 \text{curl}_A \eta \|_2^2 - \int \partial_2^2 \text{curl}_A \eta \text{curl}_A (\bar{\rho} u) \, dy \right) + \lambda m^2 \| \partial_2 \text{curl}_A \partial_1 \eta \|_2^2 = \sum_{j=5}^{9} J_j, \quad (2.51) \]

where

\[ J_5 := - \int (\bar{\rho} \partial_2 \eta_1 \partial_1 \text{curl}_A \eta + \text{curl}_A (\bar{\rho} u) \partial_2^2 \text{curl}_A \eta) \, dy, \]

\[ J_6 := - \int (\text{curl}_A (\bar{\rho} u) \partial_2^2 \text{curl}_A \eta + \text{curl}_A (\bar{\rho} u) \partial_2^2 \text{curl}_A \eta) \, dy, \]

\[ J_7 := \lambda m^2 \int (\text{curl}_A \partial_1 \eta \partial_2^2 \text{curl}_A \eta - \partial_2 \text{curl}_A \partial_1 \eta \partial_2 \text{curl}_A \eta) \, dy, \]

\[ J_8 := \mu \int \nabla \partial_2 \text{curl}_A \eta \cdot \nabla \partial_2 \text{curl}_A \eta \, dy \]

and

\[ J_9 := \mu \int (\Delta - \Delta_A) \text{curl}_A u \partial_2^2 \text{curl}_A \eta \, dy. \]
It is easy to see that

\[
\begin{aligned}
J_5 & \lesssim \|\partial_2 \eta_2\|_0 \|\eta\|_{1,2} + \|u\|_1 \|\nabla u\|_2, \\
J_6 & \lesssim \|\nabla \eta\|_2 \|\nabla u\|_2 \|u\|_1 \lesssim \sqrt{\mathcal{E}D}, \\
J_7 & \lesssim \|\nabla \eta\|_2 \|\eta\|_{1,2}^2 \lesssim \sqrt{\mathcal{E}D}.
\end{aligned}
\]
(2.52)

Obviously

\[
J_8 = -\mu \frac{d}{dt} \int \partial_2^2 (\partial_1 \eta_1 \partial_2 \eta_1) \partial_2^2 \text{curl} \eta dy + \tilde{J}_8,
\]
where we have defined that

\[
\tilde{J}_8 := \mu \int (\partial_2 \partial_1 \text{curl}_A \eta \partial_2 \partial_1 \text{curl}_A \eta + \partial_2^2 \text{curl}_A \eta \partial_2^2 \text{curl}_A \eta) dy \\
+ \mu \int \partial_2^2 \text{curl} \eta \partial_2^2 (\partial_1 \eta_1 \partial_2 \eta_1 u_1 + \partial_1 \eta_2 \partial_2 \eta_2 u_2 - \partial_2 \eta_2 \partial_1 \eta_1 u_2) dy \\
+ \mu \int \partial_2^2 \text{curl} \eta \partial_2^2 (\partial_1 \eta_1 \partial_2 \eta_1 u_1) dy + \mu \int \partial_2^2 (\partial_1 \eta_1 \partial_2 \eta_1) \partial_2^2 \text{curl} \eta dy.
\]

Exploiting (2.4) and (A.3), we have

\[
\tilde{J}_8 \lesssim (\|\eta\|_{1,2} + \|\partial_2 \eta_2\|_2) \|\nabla \eta\|_2 \|\nabla u\|_2 \lesssim \sqrt{\mathcal{E}D}.
\]
(2.54)

The integral \(J_9\) can be rewritten as follows:

\[
J_9 := \mu \int \partial_2^2 \text{curl}_A \eta ((1 + \partial_2 \eta_2) \partial_1 ((\partial_1 \eta_2 \partial_2 - \partial_2 \eta_2 \partial_1) \text{curl}_A u) \\
+ (1 + \partial_1 \eta_1) \partial_2 ((\partial_2 \eta_1 \partial_1 - \partial_1 \eta_1 \partial_2) \text{curl}_A u) + \partial_1 \eta_2 \partial_2 ((1 + \partial_2 \eta_2) \partial_1 \\
- \partial_1 \eta_2 \partial_2 \text{curl}_A u) + \partial_2 \eta_1 \partial_1 ((1 + \partial_1 \eta_1) \partial_2 \\
- \partial_2 \eta_1 \partial_1 \text{curl}_A u)) - (\partial_1 \eta_1 \partial_2^2 + \partial_2 \eta_2 \partial_1^2) \text{curl}_A u) dy = J_{9,1} + J_{9,2},
\]

where

\[
J_{9,1} := \mu \int \partial_2^2 \text{curl}_A \eta ((1 + \partial_2 \eta_2) \partial_1 ((\partial_1 \eta_2 \partial_2 - \partial_2 \eta_2 \partial_1) \text{curl}_A u) - \partial_2 \eta_2 ((1 + \partial_2 \eta_2) \partial_1 \text{curl}_A u) \\
- \partial_2 \eta_2 (1 + \partial_2 \eta_2) \partial_1^2 \text{curl}_A u + \partial_2 (\partial_2 \eta_1 \text{curl}_\partial_1 \text{curl}_A u) \\
+ \partial_1 \eta_1 \partial_2 (\partial_2 \eta_1 \partial_1 \text{curl}_A u) - (1 + \partial_1 \eta_1) \partial_2 (\partial_1 \eta_1 \partial_2 \text{curl}_A u) \\
+ \partial_1 \eta_2 \partial_2 ((1 + \partial_2 \eta_2) \partial_1 - \partial_1 \eta_2 \partial_2) \text{curl}_A u) + \partial_2 \eta_1 \partial_2 \partial_1 \text{curl}_A u \\
+ \partial_1 \eta_1 \partial_2 \eta_1 \partial_1 \text{curl}_A u + \partial_2 \eta_1 \partial_2 \text{curl}_\partial_1 \text{curl}_A u - \partial_2 \eta_1 \partial_2 \text{curl}_A \eta_1 \partial_1 \text{curl}_A u \\
- (\partial_2 \eta_1)^2 (\text{curl}_\partial_1^2 \text{curl}_A u + \text{curl}_\partial_1 \text{curl}_A \partial_1 u) - (\partial_1 \eta_1 \partial_2^2 + \partial_2 \eta_2 \partial_1^2) \text{curl}_A u) dy,
\]
\[
J_{9,2} := \mu \int \partial_2^2 \text{curl}_A \eta (\partial_2 (\partial_2 \eta_1 \text{curl}_A \partial_1 u) - \partial_2 \eta_2 (1 + \partial_2 \eta_2) \text{curl}_A \partial_1^2 u \\
+ \partial_2 \eta_1 (\partial_2 \text{curl}_A \partial_1 u - \partial_2 \eta_1 \text{curl}_A \partial_1^2 u)) dy.
\]

We further rewrite the integral \(J_{9,2}\) as follows:

\[
J_{9,2} := \tilde{J}_{9,2} - \frac{d}{dt} T_1,
\]
(2.56)
where
\[ J_{0.2} := \mu \int (\partial_2 \eta_2 (1 + \partial_2 \eta_2) \text{curl}_A \partial_1^2 \eta - \partial_2 (\partial_2 \eta_1 \text{curl}_A \partial_1 \eta) - \partial_2 \eta_1 (\partial_2 \text{curl}_A \partial_1 \eta) \\
- \partial_2 \eta_1 (\text{curl}_A \partial_1^2 \eta)) \partial_t \partial_2^2 \text{curl}_A \eta - \partial_2 \eta_1 (\text{curl}_A \partial_1 \eta) \eta_2 u_1 \\
+ \partial_2 \eta_1 (\text{curl}_A \partial_1 \eta) + (\partial_2 \text{curl}_A \partial_1 \eta - \partial_2 \eta_1 \text{curl}_A \partial_1^2 \eta) \partial_2 u_1 \\
+ \partial_2 \eta_1 (\partial_2 \text{curl}_A \partial_1 \eta - \partial_2 u_1 \text{curl}_A \partial_1^2 \eta - \partial_2 \eta_1 \text{curl}_A \partial_1^2 \eta)) dy. \]

It is easy to estimate that
\[ J_{0.1} + J_{0.2} \lesssim (\| \eta \|_{1,2} + \| \partial_2 \eta_2 \|_2) \| \nabla \eta \|_2 \| \nabla u \|_2. \]

Inserting (2.56) into (2.55) and then using the above estimate, we get
\[ J_{0} \leq c \sqrt{\mathcal{E} \mathcal{D}} - \frac{d}{dt} I_1. \quad (2.57) \]

Finally, putting (2.53) and (2.57) into (2.51), and then using the estimates (2.10), (2.52) and (2.54), we arrive at (2.42). This completes the proof. \( \square \)

2.4. Stabilizing estimates

Now we further establish the stabilizing estimates for \( E(\partial_1^i \eta) \) and \( E(u) \) appearing in Lemmas 2.3, 2.5.

Lemma 2.7. We have
\[
\begin{align*}
\| \eta \|_{1,0}^2 &\lesssim -E(\partial_1^i \eta) + \begin{cases} 0 & \text{for } i = 0, 1; \\ \| \eta \|_{2,1}^2 & \text{for } i = 2, \end{cases} \\
\| u \|_{1,0}^2 &\lesssim -E(u) + \| \nabla \eta \|_2 \| u \|_{1,0}^2.
\end{align*}
\]

Proof. By virtue of the definition of \( m_C \), it is easy to see that
\[ -g \int \rho^2 u_2^2 dy \geq -\lambda m_C \| \partial_1 w \|_0^2 \]
for any \( w \in H_0^1 \), which, together with the stability condition \( |m| > m_C \), implies
\[ \| w \|_{1,0}^2 \leq \lambda (m^2 - m_C^2) \| \partial_1 w \|_0^2 \lesssim -E(w) \quad \text{for any } w \in H_0^1. \] (2.60)

For applying the above estimate to \( \eta \), we shall modify \( \eta \). Let \( 0 \leq i \leq 2 \) be given. By Lemma A.5, there exists a Bogovskii’s operator \( \mathcal{B} : \partial_1^i \text{div} \eta \in L^2 \to H_0^1 \) such that
\[ \text{div} \mathcal{B}(\partial_1^i \text{div} \eta) = \partial_1^i \text{div} \eta \quad \text{and} \quad \| \mathcal{B}(\partial_1^i \text{div} \eta) \|_1 \lesssim \| \partial_1^i \text{div} \eta \|_0. \] (2.61)

Now we use \( \partial_1^i \eta - \mathcal{B}(\partial_1^i \text{div} \eta) \) to rewrite \( E(\partial_1^i \eta) \) as follows.
\[ E(\partial_1^i \eta) = E(\partial_1^i \eta - \mathcal{B}(\partial_1^i \text{div} \eta)) + E(\mathcal{B}(\partial_1^i \text{div} \eta)) - K_{1+i}, \]
where
\[ K_{1+i} := 2 \lambda m^2 \int \partial_1^{1+i} \eta \cdot \partial_1 \mathcal{B}(\partial_1^i \text{div} \eta) dy - 2g \int \rho^2 \partial_1^i \eta_2 \mathcal{B}(\partial_1^i \text{div} \eta) \cdot e_2 dy. \]
Recalling $\partial_1^i \eta - \mathcal{B}(\partial_1^i \text{div} \eta) \in H^1_0$, we use \eqref{2.63} to get
\[
\|\partial_1^i \eta - \mathcal{B}(\partial_1^i \text{div} \eta)\|_{1,0}^2 \lesssim -E(\partial_1^i \eta - \mathcal{B}(\partial_1^i \text{div} \eta)),
\]
which, together with \eqref{2.62} and Young’s inequality, gives
\[
\|\eta\|_{i+1,0}^2 \lesssim E(\mathcal{B}(\partial_1^i \text{div} \eta)) - E(\partial_1^i \eta) - K_{1+i} + \|\mathcal{B}(\partial_1^i \text{div} \eta)\|_{1,0}^2. \tag{2.63}
\]

Exploiting \eqref{2.61}, we find that
\[
E(\mathcal{B}(\partial_1^i \text{div} \eta)) - K_{1+i} + \|\mathcal{B}(\partial_1^i \text{div} \eta)\|_{1,0}^2
\lesssim \|\eta\|_{i+1,0}\|\mathcal{B}(\partial_1^i \text{div} \eta)\|_{1,0} + \|\eta\|_{i,0}\|\mathcal{B}(\partial_1^i \text{div} \eta)\|_{0} + \|\mathcal{B}(\partial_1^i \text{div} \eta)\|_{i,0}^2
\lesssim (\|\eta\|_{i+1,0} + \|\eta\|_{i,0})\|\text{div} \eta\|_{i,0} + \|\text{div} \eta\|_{i,0}^2.
\]
Finally, putting the above estimate into \eqref{2.63}, and then using \eqref{2.2}, \eqref{2.10}, \eqref{A.8} and Young’s inequality, we get \eqref{2.58}.

Similarly we can easily follow the argument of \eqref{2.58} with $i = 0$ to establish \eqref{2.59} by further using the relation
\[
\text{div} u = -\text{div} \tilde{A} u \tag{2.64}
\]
and the estimate
\[
\|\text{div} \tilde{A} u\|_0 \lesssim \|\nabla \eta\|_2 \|u\|_1.
\]
This completes the proof. \hfill \Box

### 2.5. Stokes estimates

In this section, we use the regularity theory of the Stokes problem (with Navier boundary condition) to derive the estimates of normal estimates of $u$ and the equivalence estimates for $\mathcal{E}$.

**Lemma 2.8.** For sufficiently small $\delta$, we have
\[
\|u\|_{2,2+j} + \|q\|_{2,1+j} \lesssim \|\partial_1^i \eta, u_i\|_{2,j} \text{ for } 0 \leq i + j \leq 1, \tag{2.65}
\]
\[
\mathcal{E} \text{ and } \|\nabla \eta, u\|_2^2 \text{ are equivalent to each other}, \tag{2.66}
\]
where the equivalent coefficients in \eqref{2.66} are independent of $\delta$.

**Proof.** To begin with, we derive \eqref{2.65}. By \eqref{2.19}, \eqref{2.22}, and \eqref{2.64}, we have the following Stokes problem
\[
\begin{cases}
\partial_1^i (\nabla q - \mu \Delta u) = \partial_1^i (\lambda m^2 \partial_1^2 \eta + g \tilde{\rho} \eta_2 e_2 - \tilde{\rho} u_i + \tilde{G} e_2 + \tilde{N}^\mu - \nabla \tilde{A} q), \\
\partial_1^i \text{div} u = -\partial_1^i \text{div} (\tilde{A}^T u), \\
\partial_1^i (u_2, \partial_2 u_1)|_{\partial \Omega} = 0,
\end{cases} \tag{2.67}
\]
where $i = 0, 1$. By Remark \ref{A.5}, we can apply the regularity estimate \eqref{A.34} to above Stokes problem to get
\[
\|u\|_{i,2+j} + \|q\|_{i,1+j} \lesssim \|\partial_1^i \eta, \eta_2, u_i\|_{i,j} + K_4, \tag{2.68}
\]
where $0 \leq i + j \leq 1$ and
\[
K_4 := \|\tilde{G}, \tilde{N}^\mu, \nabla \tilde{A} q, \tilde{A}^T u\|_{i,j} + \|\text{div} (\tilde{A}^T u)\|_{i,1+j}.
\]
It is easy to estimate that
\[
\|G\|_{i,j} \lesssim \|G\|_1 \lesssim \eta_2 \|1 \lesssim \eta_{2,0},
\]
\[
\|\nabla Aq\|_{i,j} \lesssim \|\nabla \eta\|_2 \|\nabla q\|_{i,j},
\]
\[
\|\hat{A}^T u\|_{i,j} \lesssim \|\nabla \eta\|_2 \|u\|_{i,j},
\]
\[
\|\mathcal{N}^\mu\|_{i,j} + \|\text{div}(\hat{A}^T u)\|_{i,1+j} = \|\mathcal{N}^\mu\|_{i,j} + \|\text{div}\hat{A} u\|_{i,1+j} \lesssim \|\nabla \eta\|_2 \|u\|_{i,1+j},
\]
where we shall use (2.10) and (2.14) in (2.69). Putting the above estimates into (2.68) yields the desired estimate (2.65).

Next we turn to the derivation of (2.66). By (2.11), the Stokes estimate (2.65) with \(i = j = 0\) and the definition of \(E\), we easily see that
\[
\|((\nabla \eta, u))\|_2^2 \lesssim E \lesssim \|((\nabla \eta, u))\|_2^2 + \|u_t\|_0^2.
\]

Obviously, to complete the proof, it suffices to prove that
\[
\|u_t\|_0 \lesssim \|\eta\|_{2,0} + \|u\|_2.
\]

To this end, we multiply (1.19) by \(u_t\) in \(L^2\) to obtain
\[
\|\sqrt{\rho} u_t\|_0^2 = \int ((\lambda m^2 \partial^2 \eta + (G + g\bar{\rho}\eta_2)e_2 + \mu \Delta_A u) \cdot u_t + \nabla q \cdot (\hat{A}^T u)) dy =: K_5.
\]

Making use of (2.3), (2.10), (2.65) and (A.8), we obtain
\[
K_5 \lesssim (\|\eta_2\|_0 + \|\eta\|_{2,0} + \|u\|_2) \|u_t\|_0 + \|\nabla u\|_0 \|u\|_2 \|\nabla q\|_0 \\
\lesssim (\|\eta\|_{2,0} + \|u\|_2 + \|\nabla u\|_0 \|u\|_2) \|u_t\|_0 + \|\eta\|_{2,0} \|u\|_2^2.
\]

Putting the above estimate into (2.72), and then using (1.38) and Young’s inequality, we arrive at (2.71). This completes the proof.

2.6. A priori stability estimates

Now we are in a position to establish the a priori stability estimate (1.31) under the assumptions (1.36)–(1.38).

We can derive from Lemmas 2.3, 2.4 and the estimate (2.37) that there exist two different constants \(c\), such that for any sufficiently large \(\chi \geq 1\) (depending on \(\mu\) and \(\hat{\rho}\)), and for any sufficiently small \(\delta \in (0, 1]\) (independent of \(\chi\)), the following tangential energy inequality holds:
\[
\frac{d}{dt} E_{\text{tan}} + c D_{\text{tan}} \lesssim \chi \sqrt{E D},
\]
where
\[
E_{\text{tan}} := \sum_{0 \leq i \leq 2} \left( \int \bar{\rho} \partial^i \eta \cdot \partial^i u dy + \frac{\mu}{2} \|\nabla \partial^i \eta\|_0^2 \right) \\
+ \chi \left( \|\sqrt{\rho} u\|_{2,0}^2 - \sum_{0 \leq i \leq 2} E(\partial^i \eta) \right) + \|\sqrt{\rho} \psi\|_0^2 - E(u),
\]
\[
D_{\text{tan}} := \chi \|u\|_{2,1}^2 - \sum_{0 \leq i \leq 2} E(\partial^i \eta) + \|u_t\|_1^2.
\]

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Making use of Lemma 2.7, (2.11), (2.65) and Young’s inequality, we get, for any sufficiently large \( \chi \) and for any sufficiently small \( \delta \in (0, \chi^{-1}) \),

\[
||\eta||_{L^2}^2 + ||u||_{L^2}^2 + ||u_t||_{L^2}^2 + ||q||_{L^2}^2 + \chi ||(\partial_t \eta, u)||_{L^2}^2 \lesssim \mathcal{E}_{\tan} \lesssim \chi \mathcal{E}.
\]

and

\[
||\partial_1 \eta||_{L^2}^2 + \chi ||u||_{L^2}^2 + ||u_t||_{L^2}^2 \lesssim \mathcal{D}_{\tan} + ||\eta||_{L^2}^2.
\]

Obviously,

\[
||\text{curl}\eta||_{L^2}^2 \lesssim ||\text{curl}_A \partial_1 \eta||_{L^2}^2 + ||\eta||_{L^2}^2 ||\nabla\eta||_{L^2}^2,
\]

\[
||\text{curl}\eta||_{L^2}^2 \lesssim ||\partial_1 \text{curl}_A \partial_1 \eta||_{L^2}^2 + ||\eta||_{L^2}^2 ||\nabla\eta||_{L^2}^2,
\]

\[
||\partial_2 \text{curl}\eta||_{L^2}^2 \lesssim ||\partial_2 \text{curl}_A \partial_1 \eta||_{L^2}^2 + ||\eta||_{L^2}^2 ||\nabla\eta||_{L^2}^2.
\]

Thanks to the above three estimates, we can further derive the total energy inequality from Lemma 2.6 and (2.73):

\[
\frac{d}{dt} \tilde{\mathcal{E}} + c \mathcal{D} \lesssim ||\eta||_{L^2} ||\eta||_{L^2} + ||u||_{L^2} ||u||_{L^2} + \chi^2 \sqrt{\mathcal{E}} \mathcal{D},
\]

where

\[
\tilde{\mathcal{E}} := \mathcal{I}_3 + \frac{\mu}{2} ||(\nabla_A \text{curl}_A \eta, \nabla_A \partial_1 \text{curl}_A \eta, \nabla_2 \text{curl}_A \eta)||_{L^2}^2 + \chi \mathcal{E}_{\tan},
\]

\[
\mathcal{D} := ||\text{curl}\eta||_{L^2}^2 + \chi \mathcal{D}_{\tan},
\]

\[
\mathcal{I}_3 := \mathcal{I}_2 + \int (\text{curl}_A \eta \text{curl}_A (\rho u) - \partial_1 \text{curl}_A \eta \text{curl}_A (\rho u)) dy.
\]

Moreover, by (2.10), (2.15) and (2.75), it holds that, for any sufficiently small \( \delta \in (0, \chi^{-1}) \),

\[
\mathcal{D} \lesssim ||\eta||_{L^2}^2 + ||\eta||_{L^2}^2 + ||u||_{L^2}^2 + ||q||_{L^2}^2 + \chi ||(\partial_1 \eta)||_{L^2}^2
\]

\[
\lesssim \mathcal{D}.
\]

Noting that

\[
||\text{curl}\eta||_{L^2}^2 \lesssim ||\nabla\eta||_{L^2}^2 + ||\nabla \text{curl}\eta||_{L^2}^2
\]

\[
\lesssim ||(\nabla \eta, \nabla_A \text{curl}_A \eta, \nabla_A \partial_1 \text{curl}_A \eta, \nabla_2 \text{curl}_A \eta, \nabla_A \text{curl}_A \eta, \nabla A \partial_1 \text{curl}_A \eta, \nabla A \partial_2 \text{curl}_A \eta)||_{L^2}^2
\]

\[
\lesssim ||(\nabla \eta, \nabla_A \text{curl}_A \eta, \nabla_A \partial_1 \text{curl}_A \eta, \nabla_2 \text{curl}_A \eta)||_{L^2}^2 + ||\nabla\eta||_{L^2}^2,
\]

thus, we get from (2.15) and the above estimate that

\[
||\eta||_{L^2}^2 \lesssim ||(\nabla \eta, \nabla_A \text{curl}_A \eta, \nabla_A \partial_1 \text{curl}_A \eta, \nabla_2 \text{curl}_A \eta)||_{L^2}^2.
\]

In addition, it is easy to check that

\[
||\mathcal{I}_3|| \lesssim ||\nabla\eta||_{L^2} ||u||_{L^2} + ||\nabla\eta||_{L^2}^2.
\]

Exploiting (2.68), (2.74) and the above two estimates, we obtain that, for any sufficiently large \( \chi \) and for any sufficiently small \( \delta \),

\[
\mathcal{E} \lesssim ||\eta||_{L^2}^2 + \chi (||u||_{L^2}^2 + ||u||_{L^2}^2 + ||q||_{L^2}^2) \lesssim \tilde{\mathcal{E}} \lesssim \chi^2 \mathcal{E} \lesssim \chi^2 ||(\nabla \eta, u)||_{L^2}^2.
\]
Using Young's inequality, (2.78) and the last inequality (2.80), we further derive from (2.77) with some sufficiently large $\chi$ that
\[
\frac{d}{dt}\tilde{E} + cD \leq 0 \text{ for any sufficiently small } \delta,
\] (2.81)
where $\tilde{E}$ satisfies (2.80). Integrating the above inequality over $(0, t)$, and then using (2.80), we arrive at, for some $c_1 \geq 1$,
\[
E(t) + \int_0^t D(\tau)d\tau \leq c_1\|\nabla \eta^0, u^0\|^2.
\] (2.82)

2.7. Decay-in-time estimates

This section is devoted to the derivation of decay-in-time estimate (1.32) under the additional a priori assumption (1.41).

Exploiting the integral by parts, we can derive from Lemmas 2.3 and 2.4 that, for $i = 1, 2$,
\[
\frac{d}{dt}\left(\langle t \rangle \left(\int \rho \partial^i_1 \eta \cdot \partial_1^i u dy + \frac{\mu}{2} \|\nabla \partial^i_1 \eta\|_0^2 + \gamma \left(\|\sqrt{\rho} u\|_{1,0}^2 - E(\partial_1^i \eta)\right)\right) + c\langle t \rangle \gamma \|u\|^2_{1,1} - E(\partial_1^i \eta)\right) < \langle t \rangle^{-1} \left(\|\nabla \partial^i_1 \eta, \partial_1^{-1} u\|_0^2 + \gamma \|\eta\|^2_{1,0} - E(\partial_1^i \eta)\right) + K_6,
\] (2.83)
where $\gamma \geq 1$ is a sufficiently large constant (may depending on $\mu$, $g$, $\lambda$, $m$, $\hat{\rho}$ and $\Omega$) and
\[
K_6 := \langle t \rangle^2 \left(\|\nabla \eta\|_2 \|\eta\|_{3,0} \|\eta\|_0 + \|u\|_{1,2} + \|q\|_{4,1} + \|\eta\|_{2,1} \|\eta\|_{1,1}\right) + \gamma \langle t \rangle^3 \left(\|\nabla \partial^i_1 \eta, \partial_1^{-1} u\|_0^2 + \gamma \|\eta\|^2_{1,0} + \|\eta\|^3_{2,0} + \|\eta\|_{1,0} \|\eta\|_{2,1} + \|u\|_1 \|u\|_3\right) + \langle t \rangle^2 \left(\|u\|^2_{2,0} + \|\eta\|^2_{3,0}\right) + K_7,
\] (2.84)
where
\[
\mathcal{E}_D := \gamma \sum_{i=1}^2 \left(\langle t \rangle \left(\int \rho \partial^i_1 \eta \cdot \partial_1^i u dy + \frac{\mu}{2} \|\nabla \partial^i_1 \eta\|_0^2 + \gamma \left(\|\sqrt{\rho} u\|_{1,0}^2 - E(\partial_1^i \eta)\right)\right) + \langle t \rangle^2 \|\sqrt{\rho} u\|_{2,0} - E(\partial_1^i \eta)\right) + \gamma \langle t \rangle \mu \|\nabla A \partial_1 \text{curl}_A \eta\|_{0,2}^2/2,
\]
\[
\mathcal{D}_D := \gamma \sum_{i=1}^2 \langle t \rangle^3 \left(\|u\|^2_{1,1} + \|\eta\|^2_{2,1,0}\right) + \langle t \rangle^3 \|u\|^2_{2,1} + \gamma \langle t \rangle \|\text{curl} \eta\|_{2,0}^2,
\]
\[
K_7 := (1 + \gamma)K_6 + \gamma \langle t \rangle \|\nabla \eta\|_2 \left(\|\eta\|^2_{2,1} + \|\eta\|_{1,2} \|u\|_{1,2} + \|u\|^2_{2,1}\right) + \gamma \langle t \rangle^2 \|\eta\|^2_{1,1}.
\]
Following the argument of (2.79), we have
\[
\|\text{curl} \eta\|^2_{1,1} \lesssim \|\nabla \eta\|^2_{1,0} + \|\text{curl} \eta\|^2_{1,0}
\]
\[
\lesssim \|\nabla \partial_1 \eta, \nabla A \partial_1 \text{curl}_A \eta, \nabla \partial_1 \text{curl}_A \eta, \nabla \partial_1 \text{curl}_A \eta\|_0^2
\]
\[
\lesssim \|\nabla \partial_1 \eta, \nabla A \partial_1 \text{curl}_A \eta\|_0^2 + \|\eta\|^2_{1,2} \|\nabla \eta\|^2_{2,1}.
\]
which, together with (2.15), implies that

\[ \| \eta \|_{\mathcal{L}^2}^2 \lesssim \| (\nabla \partial_1 \eta, \nabla \partial_1 \text{curl}_A \eta) \|_{2}^2. \]

Thanks to (2.15), (2.58), (A.8) and the above estimate, we easily further see that, for any given sufficiently large \( \gamma \),

\[
\begin{aligned}
&\left\{ \langle t \rangle \| \partial_2^2 \eta \|_{1,0}^2 + \langle t \rangle^2 \| \partial_2 \partial_1 \eta \|_{1,0}^2 + \langle t \rangle^3 \| \partial_1 \eta \|_{2,0}^2 \\
&- c(1 + \gamma^2) \langle t \rangle^3 \| \eta \|_{2,1}^4 \lesssim \mathcal{D}_D \lesssim \gamma^2 \langle t \rangle^3 \| (\nabla \eta, u) \|_{2}^2, \\
&\gamma^2 \langle t \rangle \| \partial_1 \eta \|_{2,1}^2 + \gamma \langle t \rangle^2 \| \partial_1 \eta \|_{2,0}^2 + (\gamma^2 \langle t \rangle^2 + \langle t \rangle^3) \| \partial_1 u \|_{2,1}^2 \lesssim \mathcal{D}_D. \end{aligned}
\]

(2.85)

In addition,

\[
\left| \int \partial_1 \text{curl}_A \eta \partial_1 \text{curl}_A (\tilde{\rho} u) \, d\tau \right| \lesssim \| \eta \|_{1,1} \| \nabla u \|_1.
\]

Thus integrating (2.84) with some sufficiently large \( \gamma \) over \((0, t)\) and then using (2.82), (2.85), the above estimate and Young’s inequality, we arrive at, for any sufficiently small \( \delta \),

\[
\begin{aligned}
\langle t \rangle \| \partial_2^2 \eta \|_{1,0}^2 &+ \langle t \rangle^2 \| \partial_2 \partial_1 \eta \|_{1,0}^2 + \langle t \rangle^3 \| \partial_1 \eta \|_{2,0}^2 \\
&+ \int_0^t \left( \langle \tau \rangle \| \partial_2 \partial_1 \eta \|_{1,0}^2 + \langle \tau \rangle^2 \| \partial_1 \eta \|_{2,0}^2 + \langle \tau \rangle^3 \| \partial_1 u \|_{2,1}^2 \right) d\tau \\
&\lesssim \| (\nabla \eta^0, u^0) \|_{2}^2 + \langle t \rangle^3 \| \eta \|_{2,1}^4 + \int_0^t \langle \tau \rangle \| u \|_1 \| u \|_3 + K \tau \rangle \, d\tau. 
\end{aligned}
\]

(2.86)

It is easy see from (2.36) and (2.37) that, for any sufficiently large \( \alpha \) (may depending on \( \mu, g, \lambda, m, \tilde{\rho} \) and \( \Omega \)),

\[
\frac{d}{dt} \left[ \langle \alpha \langle t \rangle^2 \| \nabla A u \|_{0}^2 + \langle t \rangle^3 \| (\psi \|_{0}^2 - E(u)) \rangle + c \langle \alpha \langle t \rangle^2 \| u \|_{0}^2 + \langle t \rangle^3 \| u \|_{1}^2 \rangle \right] \\
\lesssim \langle \alpha \langle t \rangle^2 \| \nabla A u \|_{0}^2 + \langle t \rangle^2 (\alpha \| \eta \|_{2,0}^4 + \| u \|_{2}^4 + \| u \|_{2}^4 \| q \|_{1}) + E(u) + \| u \cdot \nabla A u \|_{0}^2 \rangle + \langle t \rangle^3 (\| \eta \|_{2,0}^4 + \| u \|_{2}^4 \| u \|_{2}^4) \rangle \right) d\tau.
\]

Integrating the resulting inequality over \((0, t)\) yields

\[
\begin{aligned}
&\langle t \rangle^2 \| \nabla A u \|_{0}^2 + \langle t \rangle^3 \| \eta \|_{0}^2 - E(u) \rangle + c \int_0^t \langle \alpha \langle \tau \rangle^2 \| u \|_{0}^2 + \langle \tau \rangle^3 \| u \|_{2}^4 \rangle \rangle \, d\tau \\
&\lesssim \alpha \| \nabla A u \|_{0}^2 \rangle + \| u \|_{0}^2 + \langle t \rangle^3 \| u \cdot \nabla A u \|_{2}^2 + \int_0^t \langle \alpha \langle \tau \rangle^2 \| \nabla A u \|_{0}^2 \rangle + \langle \tau \rangle^2 (\alpha \| \eta \|_{2,0}^4 + \| u \|_{2}^4 \| q \|_{1}) + E(u) + \| u \cdot \nabla A u \|_{0}^2 \rangle + \langle \tau \rangle^3 (\| \eta \|_{2,0}^4 + \| u \|_{2}^4 \| u \|_{2}^4) \rangle \rangle \, d\tau, \\
&\lesssim \alpha \| (\nabla \eta^0, u^0) \|_{2}^2 + \langle t \rangle^{3} \| \eta \|_{2,0}^4 + \int_0^t \langle \tau \rangle \| u \|_{1}^2 + \langle \tau \rangle^2 (\alpha \| \eta \|_{2,0}^4 + \| u \|_{2}^4 \| q \|_{1}) + \| u \|_{2}^2 + \langle \tau \rangle^3 \| \eta \|_{2,0}^4 + \| u \|_{2}^4 \| u \|_{2}^4 \rangle \rangle \, d\tau. 
\end{aligned}
\]

(2.88)

where we have used (1.38), (2.66) and (A.3) in the last inequality above. Exploiting (1.38),

(2.39), (2.65) and Young’s inequality, we further derive from (2.88) with some sufficiently large \( \alpha \) that

\[
\begin{aligned}
&\langle t \rangle^3 (\| u \|_{2}^2 + \| q \|_{2}^2 + \| u \|_{3}^2) + c \int_0^t \langle \tau \rangle \| u \|_{3}^2 + \langle \tau \rangle^2 (\| u \|_{2,1}^2 + \| q \|_{2,1}^2) \rangle + \langle \tau \rangle^3 \| u \|_{1}^2 \rangle \rangle \, d\tau \\
&\lesssim \| (\nabla \eta^0, u^0) \|_{2}^2 + \langle t \rangle^{3} \| \eta \|_{2,0}^4 + \int_0^t \langle \tau \rangle \| u \|_{2}^2 + \langle \tau \rangle^2 (\| u \|_{2,1}^2 + \| q \|_{2,1}^2) \rangle + \langle \tau \rangle^3 \| u \|_{1}^2 \rangle \rangle \, d\tau.
\end{aligned}
\]
Finally, using (2.10) and Young’s inequality, we further derive from (2.82), (2.86) and the above inequality that
\[
\mathcal{E}(t) + c \int_0^t \mathcal{D}(\tau) d\tau \lesssim \|\nabla \eta^0, u^0\|_2^2 + \langle t \rangle^3 \|\eta\|_{2,1}^4 + \int_0^t (K_7 + \langle \tau \rangle^3 (\|\eta\|_{2,0}^2 + \|u\|_2 \|u\|_2^2)) d\tau,
\]
which together with (1.38), (1.41) and Young’s inequality, yields
\[
\mathcal{E}(t) + \int_0^t \mathcal{D}(\tau) d\tau \leq c_2 \|\nabla \eta^0, u^0\|_2^2, \tag{2.89}
\]
see (1.28) and (1.29) for the definitions of \(\mathcal{E}\) and \(\mathcal{D}\).

Now we sum up the a priori estimates (2.82) and (2.89) as follows.

**Proposition 2.2.** Let \((\eta, u, q)\) be a solution to the transformed MRT problem (1.19) and (1.24), and satisfy (1.36)–(1.38). If \(m\) and \(\bar{\rho}\) satisfy the assumptions in Theorem 1.1, there exists a constant \(\delta_1\), depending on \(\mu, \lambda, m, \bar{\rho} \) and \(\Omega\), such that the solution \((\eta, u, q)\) enjoys the a priori estimates (2.82) and (2.89) for any \(\delta \leq \delta_1\).

### 2.8. Proof of Theorem 1.1

Now we state the local well-posedness result for the transformed MRT problem.

**Proposition 2.2.** Let \(b > 0\) be a constant and \(\gamma > 0\) the same constant in Lemma A.6. Assume that \(\bar{\rho}\) satisfies (1.6), \((\eta^0, u^0) \in \mathcal{H}_s^3 \times \mathcal{H}_s^2, \|\nabla \eta^0, u^0\|_2 \leq b\) and \(\text{div}_A u^0 = 0\), where \(A^0 := (\nabla \zeta_0)^{-T}\) and \(\zeta_0 = \eta^0 + y\). Then, there is a sufficiently small constant \(\delta_2 \leq \gamma/2\), such that if \(\eta^0\) satisfies
\[
\|\nabla \eta^0\|_2 \leq \delta_2,
\]
the transformed MRT problem (1.19) and (1.24) admits a unique local-in-time classical solution \((\eta, u, q) \in C^0(I_T, H^3) \times U_T \times (C^0(I_T, H^1) \cap L^2_I H^2)\) for some \(T > 0\). Moreover, \((\eta, u)\) satisfies
\[
\sup_{t \in I_T} \|\nabla \eta\|_2 \leq 2\delta_2.
\]

Here \(\delta_2\) and \(T\) may depend on \(\mu, g, \lambda, m, \bar{\rho} \) and \(\Omega\), while \(T\) further depends on \(b\).

**Proof.** The proof of Proposition 2.2 will be provided in Section 4. \(\square\)

**Remark 2.2.** If the initial data \((\eta^0, u^0)\) in Proposition 2.2 additionally satisfies \(\text{det}(\nabla \eta^0 + I) = 1\) and \((\rho u^0)_{\Omega} = (\rho u^1)_{\Omega} = 0\), then \((\eta, u) \in \tilde{S}^{1,3}_{\gamma, T} \times U_T\).

Thanks to the priori estimate (2.82) and Proposition 2.2, we can easily establish the global solvability in Theorem 1.1. Next, we briefly describe the proof.

Let \((\eta^0, u^0)\) satisfy the assumptions in Theorem 1.1
\[
\|\nabla \eta^0, u^0\|_2 \leq \delta/\sqrt{2}, \quad \delta = \min \{\delta_1, \delta_2\}/\sqrt{\gamma} \text{ and } c_3 = c_1 + c_2 \geq 1,
\]
where \(c_1\) and \(c_2\) are the same constants in (2.82) and (2.89), resp.. By virtue of Proposition 2.2 and Remark 2.2, there exists a unique local solution \((\eta, u, q)\) to the transformed MRT problem (1.19) and (1.24) with a maximal existence time \(T^{\text{max}}\), which satisfies

---

1 Here the uniqueness means that if there is another solution \((\tilde{u}, \tilde{\eta}, \tilde{q}) \in C^0(I_T, H^3) \times U_T \times (C^0(I_T, H^1) \cap L^2_I H^2)\) satisfying \(0 < \inf_{(y, t) \in \Omega \times T} \text{det}(\nabla \eta + I)\), then \((\tilde{\eta}, \tilde{u}, \tilde{q}) = (\eta, u, q)\) by virtue of the smallness condition “\(\sup_{t \in I_T} \|\nabla \eta\|_2 \leq 2\delta_2\)”. In addition, we have, by the fact “\(\sup_{t \in I_T} \|\nabla \eta\|_2 \leq \gamma\)” and Lemma A.6
\[
\inf_{(y, t) \in \Omega \times T} \text{det}(\nabla \eta + I) \geq 1/4.
\]
• for any $a \in I_{T_{\max}}$, the solution $(\eta, u, q)$ belongs to $\tilde{D}^{1,3}_{\gamma,a} \times 0U_a \times (C^0(\overline{T_a}, H^1) \cap L^2_a H^2)$, $\sup_{t \in I_a} \|\nabla \eta\|_2 \leq 2\delta_2$;

• $\limsup_{t \to T_{\max}} \|\nabla \eta(t)\|_2 > \delta_2$ or $\limsup_{t \to T_{\max}} \|\nabla \eta(t)\|_2 = \infty$, if $T_{\max} < \infty$.

Let

$$T^* := \sup\{\tau \in I_{T_{\max}} \mid \|\nabla \eta(t)\|_2 + \|u(t)\|_2 \leq c_3 \delta^2 \text{ for any } t \leq \tau\}.$$  

It is easy to see that the definition of $T^*$ makes sense. Thus, to show the existence of a global solution, it suffices to verify $T^* = \infty$. We shall prove this fact by contradiction below.

Assume $T^* < \infty$, then, by Proposition 2.2,

$$T^* \in (0, T_{\max})$$  \hspace{1cm} (2.90)

and

$$(\|\nabla \eta(t)\|_2^2 + \|u(t)\|_2^2) = c_3 \delta^2.$$  

Noting that

$$\sup_{t \in T_{\max}}(\|\nabla \eta(t)\|_2^2 + \|u(t)\|_2^2) = c_3 \delta^2 \leq \delta_1,$$  \hspace{1cm} (2.91)

thus, using (2.91) and a standard regularization method, we can follow the same arguments as in the derivation of (2.82) and (2.89) to verify that

$$E(t) + E(0) + \int_0^{T^*} (D(\tau) + D(\tau)) d\tau \leq c_3 \|\nabla \eta(0, u(0))\|_2 \leq c_3 \delta^2 / 2.$$  

In particular,

$$\sup_{t \in T_{\max}}(\|\nabla \eta(t)\|_2^2 + \|u(t)\|_2^2) = c_3 \delta^2 / 2.$$  \hspace{1cm} (2.92)

By (2.90), (2.92) and the strong continuity $(\nabla \eta, u) \in C^0([0, T_{\max}], H^2)$, there exists $\tilde{T} \in (T^*, T_{\max})$ such that

$$\sup_{t \in \tilde{T}}(\|\nabla \eta(t)\|_2^2 + \|u(t)\|_2^2) = c_3 \delta^2,$$

which contradicts with the definition of $T^*$. Hence, $T^* = \infty$ and thus $T_{\max} = \infty$. This completes the uniqueness result of the local solutions in Proposition 2.2 and the fact $\sup_{t \geq 0} \|\nabla \eta\|_2 \leq 2\delta_2$.

To complete the proof of Theorem 1.1, we have to show that the solution $(\eta, u, q)$ satisfies the properties (1.31)–(1.33). Recalling the derivation of (2.82) and (2.89), we easily verify that the global solution $(\eta, u)$ enjoys (1.31) and (1.32) by a standard regularization method. Hence next it suffices to show (1.33).

From (1.32) we get

$$\partial_t \eta(t) \to 0 \text{ in } H^2 \text{ as } t \to \infty$$  \hspace{1cm} (2.93)

and

$$\left\| \int_0^t u d\tau \right\|_2 \lesssim \int_0^t \|u\|_2 d\tau \lesssim \sqrt{\|\nabla \eta(0)\|_2^2 + \|u(0)\|_2^2}$$  \hspace{1cm} (2.94)
for any $t > 0$. Due to (2.94), there are a subsequence $\{t_n\}_{n=1}^\infty$ and some function $\eta_1^\infty \in H^2$, such that
\[
\int_0^{t_n} u_1 \, d\tau \to \eta_1^\infty - \eta_1^0 \text{ weakly in } H^2.
\]
Utilizing (1.19), (1.32), and the weakly lower semi-continuity, we conclude
\[
\|\eta(t) - \eta_1^\infty\|_2 \leq \liminf_{t_n \to \infty} \left\| \int_t^{t_n} u_1 \, d\tau \right\|_2
\leq \sqrt{\|\eta^0\|_{2,1}^2 + \|u^0\|_2^2} \liminf_{t_n \to \infty} \int_t^{t_n} \langle \tau \rangle^{-3/2} \, d\tau
\leq \sqrt{\|\nabla \eta^0\|_2^2 + \|u^0\|_2^2(t)^{-1/2}},
\]
which, combined with (2.93), yields (1.33) holds and that $\eta_1^\infty$ depends on $y_2$ only. This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

The existence of RT instability solutions had been widely investigated, see [19, 30] for examples. In particular, Jiang et.al. proved the existence of RT instability solutions for the stratified incompressible viscous resistive MHD fluids with Dirichlet boundary conditions on the both of upper and lower boundaries of a slab domain [30]. We can easily establish the instability result of the transformed MRT problem in Theorem 1.2 by following the proof frame in [30]. Next we briefly sketch the proof for the completeness. In what follows, the fixed positive constant $c_i$ for $i \geq 1$ may depend on $\mu, g, \lambda, m, \bar{\rho}$ and the domain $\Omega$.

To begin with, we introduce the instability result for the linearized MRT problem under the instability condition $|m| \in [0, m_C]$.

**Proposition 3.1.** Let $\mu > 0$ and $\bar{\rho}$ satisfy (1.6) and (1.7). If $|m| \in [0, m_C)$, then the zero solution is unstable to the linearized MRT problem
\[
\begin{align*}
\eta_t &= u, \\
\bar{\rho}u_t + \nabla q - \mu \Delta u &= \lambda m^2 \partial_1^2 \eta + g \bar{\rho}' \eta_2 e_2, \\
\text{div} u &= 0, \\
(\eta_2, \partial_2 \eta_1, u_2, \partial_2 u_1)|_{\partial \Omega} &= 0.
\end{align*}
\]
That is, there is an unstable solution $(\eta, u, q) := e^{\Lambda t}(w/\Lambda, w, \beta)$ to the above problem (3.1), where $(w, \beta) \in 0 \mathcal{H}_s^3 \times H^2$ solves the boundary-value problem
\[
\begin{align*}
\Lambda^2 \bar{\rho} w + \Lambda \nabla \beta - \Lambda \mu \Delta w &= m^2 \partial_1^2 w + g \bar{\rho}' w_2 e_2, \\
\text{div} w &= 0, \\
(w_2, \partial_2 w_1)|_{\partial \Omega} &= 0.
\end{align*}
\]
with some growth rate $\Lambda > 0$, where $\Lambda$ satisfies
\[
E(v) \leq \Lambda^2 \|\sqrt{\bar{\rho}} v\|_0^2 + \Lambda \mu \|\nabla v\|_0^2 \text{ for any } v \in H^1_\sigma.
\]

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In addition,
\[ \int \tilde{\rho}'|w_2|^2g\|w_i\|_0\|\partial_1 w_i\|_0\|\partial_2 w_i\|_0 \neq 0 \text{ for } i = 1, 2. \] (3.3)

**Proof.** We can use the modified variational method as in [30, Proposition 3.1] and Lemma A.11 to easily verify Proposition 3.1 and thus omit the trivial proof. \(\square\)

Then we follow the derivation of a priori stability estimate (2.82) in Section 2.6 with some slight modifications to establish the following Gronwall-type energy inequality for the solutions of the transformed MRT problem.

**Proposition 3.2.** Let \(\Lambda > 0\) be the same as in Proposition 3.1 and \((\eta, u, q)\) be the local solution constructed by Proposition 2.2 with initial condition \((\eta^0, u^0) \in H^{3,\gamma}_1 \times H^2_0\). There are two constants \(\delta_1^0, c_1^0 > 0\), such that if \(\|\nabla (\eta, u)\|_2 \leq \delta_1^0\) in some time interval \(I_\tilde{T} \subset I_T\) where \(I_T\) is the existence time interval of \((\eta, u, q)\), then \((\eta, u, q)\) satisfies the Gronwall-type energy inequality: for any \(t \in I_\tilde{T}\),

\[ \mathcal{E}(t) + c \int_0^t D(\tau) d\tau \leq c_1^0 \left( \|\nabla \eta^0, u^0\|_2^2 + \int_0^t \|\eta_2, u_2(\tau)\|_0^2 d\tau \right), \] (3.4)

where the constants \(\delta_1^0\) may depend on \(\mu, g, \lambda, m, \tilde{\rho}\) and \(\Omega\).

**Proof.** Let \((\eta, u, q)\) be the local solution constructed by Proposition 2.2. Then \((\eta, u) \in H^{1,3}_\gamma \times H^2_0\). We further assume

\[ \|\nabla (\eta, u)\|_2 \leq \delta \in (0, 1] \text{ for any } t \in I_\tilde{T} \subset I_T. \]

Recalling the derivation of (2.81) and using the regularity of \((\eta, u, q)\), we easily verify that, for sufficiently small \(\delta\),

\[ \frac{d}{dt} \Xi + cD \lesssim \|\eta_2, u_2\|^2_0, \] (3.5)

where \(\Xi\) has been defined by \(\tilde{\mathcal{E}}\) with \(g = 0\) and satisfies

\[ \Xi, \mathcal{E} \text{ and } \|\nabla (\eta, u)\|^2_2 \text{ are equivalent to each other.} \] (3.6)

It should be noted that the equivalent coefficients in (3.6) are independent of \(\delta\).

By the interpolation inequality (A.2), we have, for any \(\varepsilon \in (0, 1]\),

\[ \|\chi_2\|_{k,0} \lesssim \begin{cases} \varepsilon^{-1} \|\chi_2\|_0 + \varepsilon \|\chi_2\|_2 & \text{for } k = 1; \\ \varepsilon^{-1} \|\chi_2\|_{1,0} + \varepsilon \|\chi_2\|_{1,2} & \text{for } k = 2, \end{cases} \] (3.7)

where \(\chi = \eta\) or \(u\). Therefore, with the help of (3.6)–(3.7), we easily deduce (3.4) from (3.5) for sufficiently small \(\delta\). The proof is complete. \(\square\)

For any given \(\delta > 0\), let

\[ (\eta^a, u^a, q^a) = \delta e^M(\tilde{\eta}^0, \tilde{u}^0, \tilde{q}^0), \] (3.8)
where \((\tilde{\eta}^0, \tilde{u}^0, \tilde{q}^0) := (w/\Lambda, w, \beta)\), and \((w, \beta, \Lambda)\) is given by Proposition \ref{prop:linearization}. Then \((\eta^a, u^a, q^a)\) is also a solution to the linearized MRT problem \eqref{eq:linearized_MRT}, and enjoys the estimate: for \(j \geq 0\),

\[
\|\partial_t^j (\eta^a, u^a)\|_3 + \|\partial_t^j q^a\|_2 = \Lambda^j \delta e^{\Lambda t} (\| (\tilde{\eta}^0, \tilde{u}^0)\|_3 + \| \tilde{q}^0\|_2) \lesssim \Lambda^j \delta e^{\Lambda t}. \tag{3.9}
\]

In addition, we have by \eqref{eq:rho_chi} that

\[
\|\bar{\rho}' \chi_2\|_0 \|\chi_i\|_0 \|\partial_1 \chi_i\|_0 > 0, \tag{3.10}
\]

where \(\chi = \tilde{\eta}^0\) or \(\tilde{u}^0\), and \(i = 1, 2\).

Since the initial data of the solution \((\eta^a, u^a, q^a)\) to the linearized MRT problem may not satisfy the necessary compatibility conditions required by initial data of the corresponding nonlinear transformed MRT problem. So, we shall modify the initial data of the linearized problem as in \cite[Proposition 5.1]{30}, such that the modified initial data approximates the original initial data of the linearized problem, and satisfies the compatibility conditions for the corresponding nonlinear problem.

**Proposition 3.3.** Let \((\tilde{\eta}^0, \tilde{u}^0) := (w/\Lambda, w)\) be the same as in \eqref{eq:initial_data_linearized}. There is a constant \(\delta_1^2 \in (0, 1]\), such that for any \(\delta \in (0, \delta_1^2]\), there exists \((\eta^r, u^r) \in \mathcal{H}_s^3\) enjoying the following properties:

1. The modified initial data

\[(\eta_0^\delta, u_0^\delta) := \delta (\tilde{\eta}^0, \tilde{u}^0) + \delta^2 (\eta^r, u^r)
\]

belongs to \(\mathcal{H}^{3,s}_1 \times \mathcal{H}^3_s\) and satisfies the compatibility condition

\[
\text{div}_{A_0^\delta} u_0^\delta = 0 \text{ in } \Omega,
\]

where \(A_0^\delta\) is defined as \(A\) with \(\eta_0^\delta\) in place of \(\eta\).

2. Uniform estimate:

\[
\|(\eta^r, u^r)\|_3 \leq c_2^r, \tag{3.11}
\]

where the positive constant \(c_2^r\) is independent of \(\delta\).

**Proof.** Thanks to Lemma \ref{lem:compatibility} and Remark \ref{rem:compatibility}, we can easily establish Proposition \ref{prop:modified_initial_data} by following the argument of \cite[Proposition 5.1]{30}, and thus omit the trivial proof. \(\square\)

Let \((\eta_0^\delta, u_0^\delta) \in \mathcal{H}^{3,s}_1 \times \mathcal{H}^3_s\) be constructed by Proposition \ref{prop:modified_initial_data}

\[
c_3^l = \|(\tilde{\eta}^0, \tilde{u}^0)\|_3 + c_2^l > 0 \tag{3.12}
\]

and

\[
\delta_0 = \frac{1}{2c_3^l} \min \{ \gamma, \delta_1^l, \delta_2, 2c_3^l \delta_2 \} \leq 1. \tag{3.13}
\]

From now on, we assume that \(\delta \leq \delta_0\). Since \(\delta \leq \delta_1^l\), we can use Proposition \ref{prop:modified_initial_data} to construct \((\eta_0^\delta, u_0^\delta)\) that satisfies

\[
\|(\eta_0^\delta, u_0^\delta)\|_3 \leq c_3^l \delta \leq \delta_2.
\]
By Proposition 2.2, there exists a unique solution \((\eta, u, q)\) of the transformed MRT problem (1.19) and (1.24) with initial value \((\eta^0, u^0)\) in place of \((\eta^0, u^0)\), where \((\eta, u, q) \in \tilde{S}_{\gamma, \tau}^{1.3} \times \mathcal{U}_\tau \times (C^0(T, H^1) \cap L^2 T H^2)\) for any \(\tau \in I_{T_{\text{max}}}\) and \(T_{\text{max}}\) denotes the maximal time of existence.

Let \(\epsilon_0 \in (0, 1]\) be a constant, which will be given in \((3.42)\). We define

\[
T^\delta := \Lambda^{-1}\ln(\epsilon_0/\delta) > 0, \text{ i.e. } \delta e^{\Lambda T^\delta} = \epsilon_0, \tag{3.14}
\]

\[
T^* := \sup \left\{ t \in I_{T_{\text{max}}} \left| \sup_{\tau \in [0, t]} \sqrt{\|\eta(\tau)\|_3^2 + \|u(\tau)\|_2^2} \leq 2c^j_3 \delta_0 \right. \right\}, \tag{3.15}
\]

\[
T^{**} := \sup \left\{ t \in I_{T_{\text{max}}} \left| \sup_{\tau \in [0, t]} \|\eta(\tau)\|_0 \leq 2c^j_3 \delta e^{\Lambda t} \right. \right\}. \tag{3.16}
\]

Since

\[
\sqrt{\|\eta(t)\|_3^2 + \|u(t)\|_2^2} \bigg|_{t=0} = \sqrt{\|\eta_0\|_3^2 + \|u_0\|_2^2} \leq c^j_3 \delta < 2c^j_3 \delta, \tag{3.17}
\]

we have \(T^* > 0, T^{**} > 0\). Obviously,

\[
T^* = T_{\text{max}} = \infty \text{ or } T^* < T_{\text{max}}, \tag{3.18}
\]

\[
\|\eta(T^{**})\|_0 = 2c^j_3 \delta e^{\Lambda T^{**}}, \text{ if } T^{**} < T_{\text{max}}, \tag{3.19}
\]

\[
\sqrt{\|\eta(T^*)\|_3^2 + \|u(T^*)\|_2^2} = 2c^j_3 \delta_0, \text{ if } T^* < T_{\text{max}}. \tag{3.20}
\]

From now on, we define

\[
T_{\text{min}} := \min\{T^\delta, T^*, T^{**}\}. \tag{3.21}
\]

Noting that \(\sup_{t \in [0, T_{\text{min}}]} \sqrt{\|\eta(t)\|_3^2 + \|u(t)\|_2^2} \leq \delta_t\), thus, by Proposition 3.2 \((\eta, u, q)\) enjoys the Gronwall-type energy inequality (3.4) for any \(t \in I_{T_{\text{min}}}\). Making use of this fact, (3.15)–(3.17), Lemma A.6 and the condition \(\|\eta\|_3 < \gamma\), we see that, for any \(t \in [0, T_{\text{min}}]\),

\[
\mathcal{E}(t) + c \int_0^t D(\tau) d\tau \leq c^j_4 \delta^2 e^{2\Lambda t} \tag{3.21}
\]

and

\[
\left\| \int_0^{\tau} (\eta_2 - z) \hat{\rho}''(y_2 + z) d\tau \right\|_{L^1} \leq \left( c^j_4 \delta e^{\Lambda t} \right)^2. \tag{3.22}
\]

Next we further establish the estimates for the errors between \((\eta, u)\) and \((\eta^a, u^a)\) as in \([30, \text{ Proposition 6.1}].

**Proposition 3.4.** Let \((\eta^d, u^d, q^d) = (\eta, u, q) - (\eta^a, u^a, q^a)\), then there is a constant \(c^j_5\), such that for any \(\delta \in (0, \delta_0]\) and for any \(t \in I_{T_{\text{min}}}\),

\[
\|\rho^d \partial^i \chi^d\|_x + \|u^d_t\|_0 \leq c^j_5 \sqrt{\delta^3 e^{3\Lambda t}}, \tag{3.23}
\]

\[
\|(A_{ik} \partial_k \chi - \partial_i \chi^a_j)(t)\|_{L^1} \leq c^j_5 \sqrt{\delta^3 e^{3\Lambda t}}, \tag{3.24}
\]

where \(i, j = 1, 2, \chi = \eta \text{ or } u, x = W^{1,1} \text{ or } H^1\), and \(c^j_5\) is independent of \(T_{\text{min}}\).
Proof. Let \( K^1 = K^1_L + K^1_N, \)
\[
K^1_L = \lambda m^2 \partial_1^2 \eta^a + g \rho' \eta^2 e_2 + \mu \Delta u^a
\]
and
\[
K^1_N = \nabla \eta (\lambda m^2 \partial_1^2 \eta^a + g \rho' \eta^2 e_2) + \rho \nabla uu^a + \mu \nabla \eta \Delta u^a + \nabla_A q^a - \nabla \eta \nabla q^a.
\]
It is easy to see that \((\eta^a, u^a, q^a)\) satisfies that
\[
\begin{aligned}
\eta^a_t &= u^a, \\
\bar{\rho}(\nabla \zeta u^a)_t + \nabla_A q^a &= K^1, \\
\text{div}_A(\nabla \zeta u^a) &= \text{div} u^a = 0, \\
(X_2, \partial_2 X_1)|_{\partial \Omega} &= 0,
\end{aligned}
\]
where \( \zeta := \eta + y \), and \( X = \eta^a, u^a \) or \( \nabla \zeta u^a \). Subtracting the both of the transformed MRT problem and the above problem, we get
\[
\begin{aligned}
\eta^d_t &= u^d, \\
\bar{\rho} \bar{u}^d_t + \nabla_A q^d - \mu \Delta_A u &=: K^2, \\
\text{div}_A \bar{u}^d &= \text{div}_A u = 0, \\
(Y_2, \partial_2 Y_1)|_{\partial \Omega} &= 0, \\
(n^d, u^d)|_{t=0} &= \delta^2(y, u^t),
\end{aligned}
\]
where \( \bar{u}^d := u - \nabla \zeta u^a, \) \( K^2 = \lambda m^2 \partial_1^2 \eta + G_\eta e_2 - K^1 \), and \( Y = \eta^d, u^d \) or \( \bar{u}^d \).

Let \( \Phi = \bar{u}^d_t - u \cdot \nabla_A \bar{u}^d, \)
\[
I_1' = \int \left( (\mu A_{ij} \partial_i (A_{ik} \partial_k u) + \bar{\rho} u \cdot \nabla_A \bar{u}^d + K^2 + \bar{\rho} \Phi) \cdot (u \cdot \nabla_A \Phi) - \partial_t (\bar{\rho} \cdot \nabla_A \bar{u}^d) \cdot \Phi \right) dy
\]
and
\[
I_2' = \int (K^2 \cdot \Phi - \mu \partial_t (A_{ij} A_{ik} \partial_k u) \cdot \partial_i (\Phi)) dy.
\]
Following the argument of (4.7) in Section 4 we can deduce from (3.26)\(_2\)–(3.26)\(_4\) with \( X = \bar{u}^d \) and the boundary conditions of \((\eta, u)\) that
\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\bar{\rho}} \Phi\|^2_0 = I_1' + I_2'.
\]
Obviously the integral \( I_2' \) can be rewritten as follows:
\[
I_2' := \frac{1}{2} \frac{d}{dt} E(u^d) - \mu \|\nabla u^d\|^2_0 + \bar{I}_2',
\]
where
\[
\bar{I}_2' = \int \left( \mu \partial_t (A_{ij} A_{ik} \partial_k u) \cdot \partial_t (u \cdot \nabla_A \bar{u}^d + \partial_t (\nabla \eta u^a)) \\
- (\lambda m^2 \partial_1^2 u + g \rho' (\eta_2 + y_2) u_2 e_2 - K^1) \cdot (u \cdot \nabla_A \bar{u}^d \\
+ \partial_t (\nabla \eta u^a)) - \mu \partial_t (\bar{\eta} A_{ij} \partial_k u + \bar{A}_{ik} \partial_k u) \cdot \partial_t u^d_t \\
+ (g (\rho' (\eta_2 + y_2) - \rho' (y_2)) u_2 e_2 - \partial_t K^1) \cdot u^d_t \right) dy,
\]

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Putting (3.28) into (3.27) yields
\[ \frac{1}{2} \frac{d}{dt} \left( \| \sqrt{\bar{\rho}} \Phi \|^2 \right) - \mu \int_0^t \| \nabla u^d \|^2 d\tau = I_1^t + I_2^t. \] (3.29)

Integrating the above identity in time from 0 to \( t \), we get
\[ \| \sqrt{\bar{\rho}} \Phi \|^2 - E(u^d) + 2\mu \int_0^t \| \nabla \mathcal{A} u^d \|^2 d\tau = \| \sqrt{\bar{\rho}} \Phi \|^2 \big|_{\tau=0} - E(u^d)_{\tau=0} + 2 \int_0^t (I_1^\tau + I_2^\tau) (\tau) d\tau. \] (3.30)

Making use of (3.9), (3.21), the fact \( \delta e^{\lambda t} \leq 1 \) and the initial condition \( u^d(0) = \delta^2 u^r \) in (3.26)\(_5\), we easily estimate that
\[ \| \sqrt{\bar{\rho}} u^d \|^2 \big|_{\tau=0} - \| \sqrt{\bar{\rho}}(u \cdot \nabla \mathcal{A} u^d + \partial_t(\nabla \eta u^a)) \|^2 \big|_{\tau=0} + 2 \int_0^t \rho(u \cdot \nabla \mathcal{A} u^d + \partial_t(\nabla \eta u^a)) \cdot u^d dy \leq \| \sqrt{\bar{\rho}} \Phi \|^2 + \delta^3 e^{3\lambda t}, \] (3.31)
\[ E(u^d)_{\tau=0} \leq \delta^4 \| u^d \|^2 \leq \delta^3 e^{3\lambda t}, \] (3.32)
\[ \int_0^t (I_1^\tau + I_2^\tau) (\tau) d\tau \leq \delta^3 e^{3\lambda t}. \] (3.33)

Next we shall estimate for \( \| u^d \|^2 \big|_{\tau=0} \). Noting that
\[ \text{div} u^d = -\text{div} \partial_t(\mathcal{A}^T u) \]
and
\[ \bar{\rho} u^d = \mu \Delta u^d + \lambda m^2 \partial_1^2 \eta^d + g \bar{\rho} \eta^d e_2 + \mathcal{N}^\mu + \mathcal{G} e_2 - \nabla q - \nabla \mathcal{A} q \]
(see (2.22) for the definition of \( \mathcal{N}^\mu \)), thus, multiplying the above identity by \( u^d \) in \( L^2 \), and then using the integration by parts, we have
\[ \int \bar{\rho} |u^d|^2 dy = \int \left( \mu \Delta u^d + \lambda m^2 \partial_1^2 \eta^d + g \bar{\rho} \eta^d e_2 + \mathcal{N}^\mu + \mathcal{G} e_2 - \nabla q - \nabla \mathcal{A} q \right) \cdot u^d dy + \int \nabla q \partial_t(\mathcal{A}^T u) dy. \]

Thus we immediately derive from the above estimate that
\[ \| u^d \|^2 \big|_{\tau=0} \leq \| (\eta^d, u^d) \|^2 + \delta^3 e^{3\lambda t}, \]
which, together with the initial data (3.26)\(_5\), implies that
\[ \| u^d \|^2 \big|_{\tau=0} \leq \delta^3 e^{3\lambda t}. \] (3.35)

Consequently, putting (3.31)–(3.35) into (3.30) yields
\[ \| \sqrt{\bar{\rho}} u^d \|^2 + 2\mu \int_0^t \| \nabla \mathcal{A} u^d \|^2 d\tau \leq E(u^d) + c\delta^3 e^{3\lambda t}. \] (3.36)
Since \( u^d \) does not satisfy the divergence-free condition (i.e., \( \text{div} u^d = 0 \)), thus we can not use (3.2) to deal with \( E(u^d) \). To overcome this trouble, we consider the following Stokes problem

\[
\begin{aligned}
\left\{ \begin{array}{ll}
\nabla \psi - \Delta \tilde{u} = 0, & \text{div} \tilde{u} = \text{div} u = -\text{div} \tilde{A} u, \\
(\tilde{u}_2, \partial_2 \tilde{u}_1)|_{\partial \Omega} = 0.
\end{array} \right.
\end{aligned}
\]

Then there exists a solution \((\tilde{u}, \psi) \in H^2_s \times H^1\) to the above Stokes problem for given \((\eta, u)\) by Lemma A.11. Moreover,

\[
||\tilde{u}||_2 \lesssim ||\text{div} \tilde{A} u||_1 \lesssim \delta^2 e^{2\Lambda t}.
\]  

(3.37)

It is easy to check that \( v^d := u^d - \tilde{u} \in H^2_\sigma \).

Now, we can apply (3.2) to \( E(v^d) \), and get

\[
E(v^d) \leq \Lambda^2 ||v^d||_0^2 + \Lambda \mu ||\nabla v^d||_0^2,
\]

which, together with (3.37), immediately implies

\[
E(u^d) \leq \Lambda^2 ||\sqrt{\rho} u^d||_0^2 + \Lambda \mu ||\nabla u^d||_0^2 + c\delta^3 e^{3\Lambda t}.
\]

Inserting it into (3.36), we arrive at

\[
\int_0^t ||\sqrt{\rho} u^d||_0^2 d\tau + 2\mu \int_0^t ||\nabla u^d||_0^2 d\tau \leq \Lambda^2 ||\sqrt{\rho} u^d||_0^2 + \Lambda \mu ||\nabla u^d||_0^2 + c\delta^3 e^{3\Lambda t}.
\]

(3.38)

In addition,

\[
\int_0^t ||\nabla u^d||_0^2 d\tau \leq \int_0^t ||\nabla \tilde{A} u^d||_0^2 d\tau + ||A||_2 ||u^d||_0^2 + c\delta^3 e^{3\Lambda t}.
\]

Putting the above estimate into (3.38) yields that

\[
\int_0^t ||\sqrt{\rho} u^d||_0^2 d\tau + 2\mu \int_0^t ||\nabla u^d||_0^2 d\tau \leq \Lambda^2 ||\sqrt{\rho} u^d||_0^2 + \Lambda \mu ||\nabla u^d||_0^2 + c\delta^3 e^{3\Lambda t}.
\]

(3.39)

Then we can further deduce from the above estimate that

\[
||u^d||_1^2 + ||u^d||_0^2 \lesssim \delta^3 e^{3\Lambda t}.
\]

(3.40)

We turn to derive the error estimate for \( \eta^d \). It follows from (3.26) \( 1 \) that

\[
\frac{d}{dt} ||\eta^d||_1^2 \lesssim ||u^d||_1 ||\eta^d||_1.
\]

Therefore, using (3.40) and the initial condition \( ||\eta^d||_{t=0} = \delta^2 \eta^\circ \) in (3.26) \( 5 \), it follows that

\[
||\eta^d||_1 \lesssim \int_0^t ||u^d||_1 d\tau + \delta^2 ||\eta^\circ||_1 \lesssim \sqrt{\delta^3 e^{3\Lambda t}}.
\]

(3.41)

Noting that \( H^1 \rightarrow W^{1,1} \), then we can derive (3.23) from (3.40) and (3.41). Finally, it is easy to see that

\[
||(A_{ik} \partial_k x_j - \partial x_j^k)(t)||_{L^1} \lesssim ||A_{ik} \partial_k x_j||_{L^1} + ||\partial x_j^d(t)||_{L^1} \lesssim \sqrt{\delta^3 e^{3\Lambda t}},
\]

which yields (3.24). This completes the proof of Proposition 3.4. \( \square \)
Let
\[
\varpi := \min_{\chi = 0, a_0} \{ \| \chi_1 \|_{L^1}, \| \partial_1 \chi_1 \|_{L^1}, \| \partial_2 \chi_1 \|_{L^1}, \| \chi_2 \|_{L^1}, \| \partial_1 \chi_2 \|_{L^1}, \| \partial_2 \chi_2 \|_{L^1} \}.
\]
Then \( \varpi > 0 \) by (3.10).
Now defining
\[
\epsilon_0 = \min \left\{ \left( \frac{c'_3}{2c'_5} \right)^2, \frac{c'_4 \delta_0}{c'_4}, \frac{\varpi^2}{4(c'_5 + |c'_4|^2)^2}, 1 \right\} > 0,
\]
we claim that
\[
T^\delta = T^{\min} = \min \{ T^\delta, T^*, T^{**} \} \neq T^* \text{ or } T^{**},
\]
which can be shown by contradiction as follows.

(1) If \( T^{\min} = T^* \), then \( T^* < T^{\max} \) by (3.18). Noting that \( \sqrt{\epsilon_0} \leq c'_3/2c'_5 \), we see that by (3.8), (3.12), (3.14) and (3.23),
\[
\| \eta(T^*) \|_0 \leq \| \eta^a(T^*) \|_0 + \| \eta^d(T^*) \|_0 \leq \delta e^{AT^{**}} (c'_3 + c'_5 \sqrt{\delta e^{AT^{**}}}) \leq \delta e^{AT^{**}} (c'_3 + c'_5 \sqrt{\epsilon_0}) \leq 3c'_3 \delta e^{AT^{**}}/2 < 2c'_3 \delta e^{AT^{**}},
\]
which contradicts to (3.19). Hence \( T^{\min} \neq T^{**} \).

(2) If \( T^{\min} = T^* \), then \( T^* < T^{**} \). Recalling \( \epsilon_0 \leq c'_4 \delta_0/c'_4 \), we deduce from (3.24) that for any \( t \in I_{T^{\min}} \),
\[
\sqrt{\| \eta(t) \|_3^2 + \| u(t) \|_2^2} \leq c'_4 \delta e^{AT^\delta} \leq c'_3 \delta_0 < 2c'_3 \delta_0,
\]
which contradicts (3.20). Hence \( T^{\min} \neq T^* \).

Since \( T^\delta \) satisfies (3.43), the inequalities (3.23)–(3.24) hold to \( t = T^\delta \). Using this fact, (3.8), (3.14), (3.22) and the condition \( \epsilon_0 \leq \varpi^2/4(c'_3 + |c'_4|^2)^2 \), we find the following instability relations: for \( i, j = 1, 2 \),
\[
\| A_{ik} \partial_k \chi_j(T^\delta) \|_{L^1} \geq \| \partial_1 \chi_j(T^\delta) \|_{L^1} - \| A_{ik} \partial_k \chi_j(T^\delta) - \partial_1 \chi_j(T^\delta) \|_{L^1} \geq \delta e^{AT^\delta} (\| \partial_1 \chi_j(0) \|_{L^1} - c'_3 \sqrt{\delta e^{AT^\delta}}) \geq \epsilon := \varpi \epsilon_0/2
\]
and
\[
\| \tilde{\rho} - \tilde{\rho}(\chi_2(y, T^\delta) + y_2) \|_{L^1} \geq \| \tilde{\rho}' \chi_2(T^\delta) \|_{L^1} - \| \tilde{\rho}' \chi_2(T^\delta) \|_{L^1} - \| \int_0^\chi_2(y, T^\delta) (\chi_2(y, T^\delta) - z) \tilde{\rho}''(y_2 + z)dz \|_{L^1} \geq \delta e^{AT^\delta} (\| \tilde{\rho}' \chi_2(0) \|_{L^1} - c'_5 \sqrt{\delta e^{AT^\delta}}) \geq \varpi \epsilon_0/2,
\]
where \( \chi = \eta \) or \( u \). Similarly, we can also verify that \( (\eta, u) \) satisfies the rest instability relations in (1.42) by using (3.23). This completes the proof of Theorem 1.2.
4. Local well-posedness

This section is devoted to the proof of the local well-posedness results in Propositions 2.2. To begin with we shall establish the existence of strong solutions to the following linear initial value problem:

\[
\begin{aligned}
&\bar{p}u_t + \nabla_A q - \mu \Delta_A u = f, \\
&\text{div}_A u = 0, \\
&t=0, u_0 = u^0, \\
&\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = 0,
\end{aligned}
\]  
(4.1)

where \(\mu > 0, (\eta^0, u^0, w)\) are given,

\[A = (\nabla \zeta)^{-T}\text{ and } \zeta = y + \eta^0 + \int_0^t w dy,\]  
(4.2)

In what follows, we shall use the following notations.

\*X = the dual space of X, \(\cdot, \cdot\) denotes the dual product,

\[G_T := \{ f \in C^0(\mathcal{T}_T, L^2) \mid (f, f_l) \in L^2_T H^1 \times L^2_T H^1 \},\]

\[\|v\|_{u_T} := \sqrt{\|v\|_{C^0(\mathcal{T}_T, H^2)}^2 + \|v_l\|_{C^0(\mathcal{T}_T, L^2)}^2 + \sum_{0 \leq j \leq 1} \|\partial^j v\|_{L^2_T H^2(1-j+1)}^2},\]

\[a \lesssim_0 b \text{ means } a \leq c_0 b, \quad A \lesssim_L B \text{ means } A \leq c^L B,\]

where \(X\) denotes a Banach space, \(c_0\) a generic positive constant at most depending on the domain \(\Omega\), and \(c^L\) a generic positive constant depending on \(\mu, \lambda, m, \bar{p}\) and \(\Omega\), and may vary from line to line (if not stated explicitly).

**Proposition 4.1.** Let \(B > 0, \delta > 0, (\eta^0, u^0) \in \mathcal{H}_s^3 \times \mathcal{H}_s^2, \mathcal{A}^0 = (I + \nabla \eta^0)^{-T}, w \in \mathcal{U}_T, f \in \mathcal{G}_T, \mathcal{A} \text{ and } \zeta \text{ be defined by } (4.2), \eta = \zeta - y\) and

\[T := \min\{1, (\delta/B)^4\}.\]  
(4.3)

Assume that

\[\|\nabla \eta^0\|_2 \leq \delta, \quad \text{div}_A u^0 = 0, \quad w|_{t=0} = u^0, \]

\[\sqrt{\|\nabla w\|_{C^0(\mathcal{T}_T, H^1)}^2 + \|\nabla w\|_{L^2_T H^2}^2 + \|\nabla w_l\|_{L^2(\Omega_T)}^2} \leq B,\]  
(4.4)

then there is a sufficiently small constant \(\delta^L \in (0, 1]\) independent of \(\mu\), such that for any \(\delta \leq \delta^L\), there exists a unique local strong solution \((u, q) \in \mathcal{U}_T \times (C^0(\mathcal{T}_T, H^1) \cap L^2_T H^2)\) to the initial boundary value problem (4.1). Moreover, the solution enjoys the following estimates:

\[1 \leq 2 \det \zeta \leq 3, \quad \|\nabla \eta\|_2 \leq 2\delta,\]  
(4.5)

\[\|u\|_{u_T} + \|q\|_{C^0(\mathcal{T}_T, H^1)} + \|q\|_{L^2_T H^2} \lesssim_L \sqrt{\mathcal{B}(u^0, f)},\]  
(4.6)

where

\[\mathcal{B}(u^0, f) := \|u^0\|_2^2 + \|u^0\|_4^4 + \|f\|_{C^0(\mathcal{T}_T, L^2)}^2 + \|f\|_{L^2_T H^1}^2 + \|f_l\|_{L^2_T H^1}^2 + \|\nabla w\|_{C^0(\mathcal{T}_T, H^1)} (1 + \|u^0\|_1) (\|u^0\|_2^2 + \|f\|_{L^2(\Omega_T)}^2),\]
Moreover, for a.e. \( t \in I_T \),
\[
\frac{1}{2} \frac{d}{dt} \| \sqrt{\rho} J\Psi \|_0^2 = \int ((J(\mu A_{ik}\partial_i(A_{ik}\partial_k u) - \rho w \cdot \nabla A u + f) \cdot w \cdot \nabla A \Psi + |\Psi|^2 w \cdot \nabla A \tilde{\rho}/2)
\]
\[- \mu \partial_t(J A_{ik} A_{ik} \partial_k u) \cdot \partial_t \Psi - \partial_t(J \rho w \cdot \nabla A u) \cdot \Psi + f \Psi J) dy + < f_t, \Psi J >_{H^1, H^1}, \tag{4.7}
\]
where \( \Psi := u_t - w \cdot \nabla A u \).

**Proof.** We shall break up the proof into three steps.

1. **Existence of local strong solutions**
   
   Recalling that \( \| \nabla \eta \|_2 \leq \delta \), the definition (4.3) and the relation
   \[
   \eta = \eta^0 + \int_0^t w d\tau,
   \tag{4.8}
   \]
   we make use of the regularity of \( w \), (4.3) and (4.4) to find that \( \eta \in C^0(T, H^3_\sigma) \), \( \eta_t = w \) and
   \[
   \| \nabla \eta(t) \|_2 \leq \delta + \sqrt{t} \| \nabla w \|_{L^3_2 H^2} \leq 2\delta \text{ for all } t \in \overline{T}.
   \]
   Obviously
   \[
   \| \nabla \eta(t) \|_{L^\infty} \lesssim_0 \| \nabla \eta(t) \|_2 \lesssim_0 \delta \quad \text{for any } t \in \overline{T}.
   \tag{4.9}
   \]
   Thanks to the estimates (4.9) and (A.1), we have for sufficiently small \( \delta \) that \( 1 \leq 2J \leq 3 \), where and in what follows, \( J := \det \nabla \zeta \). Therefore, \( A \) makes sense and is given by the following formula:
   \[
   A = J^{-1} \begin{pmatrix} 1 + \partial_2 \eta_2 & -\partial_3 \eta_2 \\ -\partial_2 \eta_1 & 1 + \partial_3 \eta_1 \end{pmatrix}.
   \]
   We remark that the smallness of \( \delta \) (independent of \( \mu \)) will be often used in the derivation of some estimates and conclusions later, and we shall omit to mention it for the sake of simplicity.

   Inspired by the proof in [16, Theorem 4.3], we next solve the linear problem (4.1) by applying the Galerkin method. Let \( \{ \varphi_i \}_{i=1}^\infty \subset H^1_\sigma \) be a countable orthogonal basis in \( H^1_\sigma \) by Lemma A.12. For each \( i \geq 1 \) we define \( \psi^i = \psi^i(t) := \nabla \zeta \varphi^i \). Let \( \mathcal{F}(t) = \{ v \in H^2_s \mid \text{div} A v = 0 \} \) and \( \mathcal{M}(t) = \{ v \in H^1_s \mid \text{div} A v = 0 \} \). Then \( \psi^i(t) \in \mathcal{F}(t) \) and \( \{ \psi^i(t) \}_{i=1}^\infty \) is a basis of \( \mathcal{M}(t) \) for each \( t \in \overline{T} \). Moreover,
   \[
   \psi^i_t = R \psi^i,
   \tag{4.10}
   \]
   where \( R := \nabla w A^T \in C^0(\overline{T}, H^1) \cap L^2 H^2 \). Obviously, by (A.14), \( R_t = \nabla w A^T + \nabla w A T \in L^2 L^2 \).

   For any integer \( n \geq 1 \), we define the finite-dimensional space \( \mathcal{F}^n(t) := \text{span} \{ \psi^1, \ldots, \psi^n \} \subset \mathcal{F}(t) \), and write \( \mathcal{P}^n(t) : \mathcal{F}(t) \rightarrow \mathcal{F}^n(t) \) for the \( \mathcal{F} \) orthogonal projection onto \( \mathcal{F}^n(t) \). Clearly, for each \( v \in \mathcal{F}(t) \), \( \mathcal{P}^n(t) v \rightarrow v \) in \( H^1_\sigma \) as \( n \rightarrow \infty \) and \( \| \mathcal{P}^n(t) v \|_2 \leq \| v \|_2 \).

   Now, we define an approximate solution
   \[
   u^n(t) = a^n_j(t) \psi^j \text{ with } a^n_j : \overline{T} \rightarrow \mathbb{R} \text{ for } j = 1, \ldots, n,
   \]
   where \( n \geq 1 \) is given. We want to choose the coefficients \( a^n_j \), so that for any \( 1 \leq i \leq n \),
   \[
   \int \rho w a^n_i \cdot \psi^i dy + \mu \int \nabla A u^n : \nabla A \psi^i dy = \int f \cdot \psi^i dy
   \tag{4.11}
   \]
with initial data \( u^n(0) = P^n u_0 \in \mathcal{H}^n \).

Let
\[
X = (a^n_i)_{n \times 1}, \quad X = \left( \int f \cdot \psi^i dy \right)_{n \times 1}, \quad \mathcal{C}^1 = \left( \int \bar{\rho} \psi^i \cdot \psi^j dy \right)_{n \times n},
\]
\[
\mathcal{C}^2 = \left( \int R \psi^i \cdot \psi^j dy + \mu \int \nabla \psi^i \cdot \nabla \psi^j dy \right)_{n \times n}.
\]

Recalling the regularity of \((\eta, \psi^i, R, f)\), Lemma A.7 and (A.13), we easily verify that
\[
\mathcal{C}^1 \in C^{1,1/2}(\mathcal{I}_T), \quad \mathcal{C}^2 \in C^{0,1/2}(\mathcal{I}_T), \quad \mathcal{H} \in C^{0,1/2}(\mathcal{I}_T) \text{ and } (\mathcal{C}^2, \mathcal{H}) \in L^2(\mathcal{I}_T).
\]  

(4.12)

Noting that \(\mathcal{C}^1\) is invertible, we can rewrite (4.11) as follows.
\[
X_t + (\mathcal{C}^1)^{-1}(\mathcal{C}^2 X - \mathcal{H}) = 0
\]

(4.13)

with initial data
\[
X|_{t=0} = \left( \int P^n u^0 \cdot \psi^i dy \right)_{n \times 1},
\]

where \((\mathcal{C}^1)^{-1}\) denotes the inverse matrix of \(\mathcal{C}^1\). By virtue of the well-posedness theory of ODEs (see [44, Section 6 in Chapter II]), the equation (4.13) has exactly one solution \(X \in C^1(\mathcal{I}_T)\). Moreover, by (4.12) and (4.13), we have
\[
\hat{a}^n_j(t) \in L^2(\mathcal{I}_T).
\]

(4.14)

Thus, we have established the existence of the approximate solution \(u^n(t) = a^n_j(t) \psi^j \in C^0(\mathcal{I}_T, \mathcal{H}_s^3)\).

Next, we derive uniform-in-\(n\) estimates for \(u^n\).

Due to (4.9), we easily get from (4.11) with \(u^n\) in place of \(\psi\) that for sufficiently small \(\delta\),
\[
\frac{d}{dt} \| \sqrt{\rho} u^n \|^2_0 + c^5 \| u^n \|^2_1 \lesssim_L \| f \|^2_0.
\]

(4.15)

By (4.10),
\[
u^n_t - Ru^n = \hat{a}^n_j \psi^j \in C^0(\mathcal{I}_T, \mathcal{H}_s^1) \cap L^2_H H^2.
\]

(4.16)

Obviously, we can replace \(\psi\) by \(\hat{a}^n_j \psi^j\) in (4.11) and use (4.16) to deduce that
\[
\| \sqrt{\rho} u^n \|^2_0 + \mu \int \nabla \psi^n \cdot \nabla \psi^n dy
= \int \bar{\rho} u^n \cdot (Ru^n) dy + \mu \int \nabla \psi^n \cdot \nabla \psi^n dy + \int f \cdot (u^n_t - Ru^n) dy.
\]

(4.17)

Thanks to (4.9), (A.4), (A.13) and (A.14), we can further obtain from (4.17) that
\[
\mu \frac{d}{dt} \| \nabla \psi^n \|^2_0 + \| \sqrt{\rho} u^n \|^2_0 \lesssim_L \| \nabla u^n \|_0 \| R \|_{L^\infty} \| u^n \|_0 + \| A_t \|_{L^\infty} \| \nabla u^n \|_0 + \int |\nabla u^n| \nabla |R| |u^n| dy + ||f, Ru^n||^2_0 \lesssim_L \| \nabla w \|_2 \| u^n \|_0^2 + \| \nabla \psi^n \|^2_0 + \| f \|^2_0
\]

(4.18)

where \(\| \nabla \psi^n \|_0^2 \in AC(\mathcal{I}_T)\).
With the help of Gronwall’s lemma, \((4.3)\) and \((4.9)\), we infer from \((4.15)\) and \((4.18)\) that for any \(t \in I_T\),

\[
\|u^n\|_1^2 + \int_0^t \|u_\tau\|_0^2 d\tau \lesssim L \left( \left\| P^n u^0 \right\|^2_1 + \int_0^t \|f\|^2_0 d\tau \right) e^{\int_0^t c_\tau (\|\nabla w\|_2 + \|\nabla u\|_1 \|\nabla w\|_2) d\tau} \\
\lesssim L \|u^0\|_2^2 + \|f\|^2_{L^2(\Omega)}.
\]

(4.19)

By the regularity of \((f, \psi^i)\) and \((A.13)\), we have

\[
\frac{d}{dt} \int f(t) \cdot \psi^i(t) dy = \langle f_\tau, \psi^i \rangle_{H^1, H^1} + \int f \cdot \psi^i dy \quad \text{for a.e. } t \in I_T.
\]

(4.20)

Recalling \((4.14)\) and \((4.16)\), we see that \(u^n_{tt} \in L^2(\Omega_T)\) makes sense. So, with the help of \((4.10)\) and \((4.20)\), we get from \((4.11)\) that

\[
\int \tilde{\rho} u^n_{tt} \cdot \psi^i dy + \mu \int \nabla_A u^n_t : \nabla_A \psi^i dy \\
= \langle f_\tau, \psi^i \rangle_{H^1, H^1} + \int (f - \tilde{\rho} u^n_t) \cdot (R \psi^i) dy \\
- \mu \int (\nabla_A u^n : \nabla_A \psi^i + \nabla_A (R \psi^i)) dy \quad \text{a.e. in } I_T.
\]

(4.21)

Noting that, by \((A.10)\),

\[
\frac{1}{2} \| \sqrt{\tilde{\rho}} u^n_t \|_0^2 - \int \tilde{\rho} u^n_t \cdot (R u^n) dy - \left( \int \tilde{\rho} u^n_t \cdot (R u^n) dy \right) \big|_{t=0} \\
= \int_0^t \left( \int \tilde{\rho} u^n_t \cdot (u^n_t - R u^n) dy - \int \tilde{\rho} u^n_t \cdot (R u^n) dy \right) d\tau
\]

and

\[
\int f(\tau) \cdot (R u^n)(\tau) dy \bigg|_{\tau=0}^{\tau=t} = \int_0^t \left( \langle f_\tau, R u^n \rangle_{H^1, H^1} + \int f \cdot (R u^n) dy \right) d\tau,
\]

we utilize \((4.16)\) and the above two identities to infer from \((4.21)\) with \(\psi^i\) replaced by \((u^n_t - R u^n)\) that

\[
\frac{1}{2} \| \sqrt{\tilde{\rho}} u^n_t \|_0^2 - \int \tilde{\rho} u^n_t \cdot (R u^n) dy + \int f \cdot (R u^n) dy + \mu \int_0^t \| \nabla_A u^n_t \|_0^2 d\tau \\
= \left( \frac{1}{2} \| \sqrt{\tilde{\rho}} u^n_t \|_0^2 - \int \tilde{\rho} u^n_t \cdot (R u^n) dy + \int f \cdot (R u^n) dy \right) \bigg|_{t=0}^{t} + I_L,
\]

(4.22)

where

\[
I_L := \int_0^t \left( \langle f_\tau, u^n_t \rangle_{H^1, H^1} + \int \left( f \cdot (2 R u^n + R u^n - R^2 u^n) - \tilde{\rho} u^n_t \cdot (R u^n_t - R u^n) \right) dy - \int \tilde{\rho} u^n_t \cdot (R u^n) dy - \mu \int (\nabla_A u^n : (\nabla_A (u^n_t - R u^n)) + \nabla_A u^n : \nabla_A (u^n_t - R u^n) - \nabla_A u^n_t : \nabla_A (R u^n)) dy \right) d\tau.
\]
Keeping in mind that $H^1 \hookrightarrow L^6$ and
\[
\|\nabla w\|_0 \leq \int_0^t \|\nabla \tau\|_0 d\tau + \|\nabla u^0\|_0,
\]
we get from (4.22) that
\[
\|\sqrt{\rho} u^n_t\|_0^2 + \int_0^t \mu \|\nabla u^n_t\|_0^2 d\tau \\
\leq L \|\nabla w\|_0 \|\nabla w\|_1 \|u^n_t\|_2^2 + \|u^0\|_2^4 + \|f\|_{C^0(T_T, L^2)}^2 + \|u^n_t\|_{t=0}^2 + I^L \\
\leq L \|\nabla w\|_1 (1 + \|u^0\|_1) \left(\|u^0\|_2^2 + \|f\|_{L^2(\Omega)}^2\right) \\
+ \|u^0\|_2^4 + \|f\|_{C^0(T_T, L^2)}^2 + \|u^n_t\|_{t=0}^2 + I^L,
\]
where we have used (4.3), (4.19) and (4.23) in the second inequality. Below, we shall bound the last two terms in (4.24).

Replacing $\psi^i$ by $(u^n_t - Ru^n)$ in (4.11), we see that
\[
\|\sqrt{\rho} u^n_t\|_0^2 = \int f \cdot (u^n_t - Ru^n) dy + \mu \int \Delta_A u^n \cdot (u^n_t - Ru^n) dy + \int \bar{\rho} u^n \cdot (Ru^n) dy,
\]
which implies
\[
\|\sqrt{\rho} u^n_t\|_0^2 \leq L \|f\|_0^2 + \|u^n\|_2^4 + \|\nabla w\|_1^2 \|u^n\|_2^2.
\]
In particular,
\[
\|\sqrt{\rho} u^n_t\|_{t=0}^2 \leq L \|u^0\|_2^2 + \|u^0\|_2^4 + \|f^0\|_0^2.
\]
In addition, the last term on the right-hand side of (4.24) can be estimated as follows:
\[
I^L \leq L \int_0^t \left(\|f\|_{L^1} \|u^n\|_1 + \|u^n\|_1 \left(\|f\|_0 \|\nabla w\|_1 \sqrt{\|\nabla w\|_1} \|\nabla w\|_2 + \sqrt{\|f\|_0 \|f\|_1 \|\nabla w\|_0} \right) \\
+ \|f\|_1 \|\nabla w\|_1 \|u^n\|_0 + \|u^n\|_2 \|\nabla w\|_1 \|\nabla w\|_2 \\
+ \|u^n\|_0 (\|\nabla w\|_2 \|u^n\|_0 + \|\nabla w\|_2 \|u^n\|_1) \\
+ \|\nabla w\|_1 \|u^n\|_1 \sqrt{\|u^n\|_0 \|u^n\|_1 \|\nabla w\|_1 \|\nabla w\|_2} d\tau \right) \\
\leq L \|u^0\|_2^2 + \|f\|_{L^\infty}^2 + \|f\|_{L^2 H^1}^2 + \int_0^t \|f\|_{L^2 H^1} \|u^n\|_1 d\tau \\
+ (\|u^0\|_2^2 + \|f\|_{L^2 H^1}^2) \left(\|u^n\|_{L^\infty L^2}^2 + \|\nabla u^n\|_{L^2(\Omega)}^2 + \delta \|u^n\|_{L^\infty L^2}^2 \right),
\]
where we have used (A.3) and the embedding $H^1 \hookrightarrow L^6$ in the first inequality, and (4.3) in the second inequality.

Substituting (4.26) and (4.27) into (4.24), and applying Young’s inequality, we arrive at
\[
\|u^n\|_{L^\infty L^2}^2 + \|u^n\|_{L^2 H^1}^2 \lesssim L \mathcal{B}(u^0, f).
\]
Summing up (4.19) and (4.28), we conclude
\[
\|u^n\|_{L^\infty H^1}^2 + \|u^n\|_{L^\infty L^2}^2 + \|u^n\|_{L^2 H^1}^2 \lesssim L \mathcal{B}(u^0, f).
\]
In view of (1.29), the Banach–Alaoglu and Arzelà–Ascoli theorems, up to the extraction of a subsequence (still labelled by $u^n$), we have, as $n \to \infty$, that

\[
(u^n, u^n_t) \rightharpoonup (u, u_t) \quad \text{weakly-* in } L_T^\infty H^1_\delta \times L_T^\infty L^2,
\]

$u^n_t \rightharpoonup u_t$ weakly in $L^2_T H^1_\delta$,

$u^n \rightharpoonup u$ strongly in $C^0(\overline{T_T}, L^2)$,

\[
\text{div}_A u = 0 \text{ a.e. in } \Omega_T \text{ and } u(0) = u^0,
\]

where $u$ and $u_t$ are measurable functions defined on $\Omega_T$. Moreover,

\[
\|u\|_{L_T^\infty H^1} + \|u_t\|_{L_T^\infty L^2} + \|u_t\|_{L^2_T H^1} \lesssim L \sqrt{\mathcal{B}(u^0, f)}.
\]

Therefore, we can take to the limit in (4.11) as $n \to \infty$, and obtain that there exists a zero-measurable set $\mathcal{B}$ such that, for any $t \in I_T \setminus \mathcal{B}$,

\[
\int \bar{\rho} u_t \cdot \zeta \, dy + \mu \int \nabla_A u : \nabla_A \zeta \, dy = \int f \cdot \zeta \, dy, \quad \forall \zeta \in \mathcal{B}(t).
\]

Now, we begin to show spatial regularity of $u$. Let us further assume that $\delta \in (0, \gamma)$ is so small that $\eta$ satisfies (1.15) and (1.16) by virtue of Lemma A.6 Denoting $F := f - \bar{\rho} u_t$, $\tilde{F} := F(\zeta^{-1}, t)$ and $\tilde{J} := J(\zeta^{-1}, t)$, we see that $\tilde{F}$ has the same regularity as that of $F$ by (A.18), i.e.,

\[
\|\tilde{F}\|_{L_T^\infty L^2} + \|\tilde{F}\|_{L^2_T H^1} < \infty.
\]

Moreover,

\[
\int F(y, t) \, dy = \int \tilde{F} \tilde{J}^{-1} \, dx.
\]

Applying the regularity theory of the Stokes problem with Navier boundary condition, we see that there is a unique strong solution $\alpha \in L_T^\infty \mathcal{H}^2_\sigma \cap L_T^2 H^3$ with a unique associated function $P \in L_T^\infty H^1_\delta \cap L_T^2 H^2$, such that

\[
\nabla P - \mu \Delta \alpha = \tilde{F} \quad \text{and} \quad (\alpha_1)_{\Omega} = 0.
\]

Let $\varpi = \alpha(\zeta, t)$ and $q = P(\zeta, t) - (P(\zeta, t))_\Omega$, then, by (A.17) and the regularity $\eta \in C^0(\overline{T_T}, \mathcal{H}^3_\delta)$, $(\varpi, q) \in (L_T^\infty H^2_\delta \cap L_T^2 H^3) \times (L_T^\infty H^1_\delta \cap L_T^2 H^2)$ satisfies the following boundary value problem: for a.e. $t \in I_T$,

\[
\begin{cases}
\nabla_A q - \mu \Delta_A \varpi = F,
\div_A \varpi = 0, \\
(\varpi_2, \partial_2 \varpi_1) = 0.
\end{cases}
\]

Recalling that $\{\psi^i(t)\}_{i=1}^\infty \subset \mathcal{B}(t)$ is a basis of $\mathcal{M}(t)$, thus the identity (4.31) also holds for any $\zeta \in \mathcal{M}(t)$. This fact, together with (4.34), implies

\[
\nabla u = \nabla \varpi.
\]

Exploiting Lemma A.11, we easily derive from (4.34) that, for sufficiently small $\delta$,

\[
\|\varpi\|_{2+i} + \|q\|_{1+i} \lesssim \|(u_t, f)\|_i \quad \text{for } i = 0, 1.
\]
So, it follows from \((4.30), (4.35)\) and \((4.36)\) that
\[
\| (u, u_t, q) \|_{L^\infty(I_T; H^2 \times L^2 \times H^1)} + \| (u, u_t, q) \|_{L^2(I_T; H^3 \times H^2 \times H^2)} \lesssim L \sqrt{\mathcal{B}(u_0, f)}.
\]
(4.37)

This completes the existence of local strong solutions. Moreover, a strong solution, which enjoys the regularity of \((\eta, u)\) constructed above, is obviously unique.

2) Strong continuity of \((u, u_t, q)\).

Since \(u \in L^2 T^3\) and \(u_t \in L^2 T^1\), \(u \in C^0(\overline{T_T}, H^2)\). By Lemma \(\text{A.10}\) for each \(t \in \overline{T_T}\), there exists a unique weak solution \(\bar{q} \in H^1\) such that
\[
\int \bar{\rho}^{-1} \nabla A \bar{q} \cdot \nabla A \vartheta \, dy = \int \bar{\rho}^{-1} (f + \nu \Delta \bar{u}) \cdot \nabla A \vartheta \, dy + \int A_T^T u \cdot \nabla \vartheta \, dy
\]
for any \(\vartheta \in H^1\), and
\[
\sup_{t \in T_T} \| \bar{q} \|_1 \lesssim \| (\bar{\rho}^{-1} A^T (f + \nu \Delta \bar{u}) + A_T^T u) \|_{C^0(\overline{T_T}; L^2)} < \infty.
\]
(4.39)

Moreover, it is easy to check that \(q \in C^0(\overline{T_T}, H^1)\) by \((4.38)\) and \((4.39)\).

Multiplying \((4.34)\) by \(\bar{\rho}^{-1} \nabla A \vartheta\) in \(L^2\) and using the integral by parts and \((4.35)\), we get, for a.e. \(t \in I_T\),
\[
\int \bar{\rho}^{-1} \nabla A q \cdot \nabla A \vartheta \, dy = \int \bar{\rho}^{-1} (f + \mu \Delta \bar{u}) \cdot \nabla A \vartheta \, dy + \int A_T^T u \cdot \nabla \vartheta \, dy.
\]
(4.40)

We immediately see \(q = \bar{q} \in C^0(\overline{T_T}, H^1)\) from \((4.38)\) and the above identity for sufficiently small \(\delta\).

Finally, we can further derive \(u_t \in C^0(\overline{T_T}, L^2)\) from \((4.34)\). Hence, \(u \in U_T\). Thanks to the strong continuity of \((u, u_t, q)\), we immediately get \((1.6)\) from \((4.37)\).

3) Verification of the identity \((4.7)\).

For any given \(\varphi \in H^1\), let \(\psi = \varphi(\zeta(y, t))\). Noting \(J_t = J_{\text{div} A w}, \partial_j (J A_{ij}) = 0\) and \(\nabla \varphi|_{x=\zeta(y,t)} = \nabla A \psi\), we deduce from \((4.34)\) and \((4.35)\) that for any \(\phi \in C^0_0(I_T)\),
\[
- \int_0^t \phi_t \int \bar{\rho} u \cdot \psi J \, dy \, d\tau = \int_0^t \phi \int (f + \mu \Delta A u - \nabla A q - \bar{\rho} w \cdot \nabla A u - w \cdot \nabla A \bar{\rho} u) \cdot \psi J \, dy \, d\tau.
\]
(4.41)

Let \(\rho = \bar{\rho}(\zeta^{-1}(x, t))\) and \(v = u(\zeta^{-1}(x, t), t)\). Using Lemma \(\text{A.8}\) we can check that
\[
\rho \in C^0(\overline{T_T}, H^2), \quad v \in C^0(\overline{T_T}, H^2) \cap L^2 H^3,
\]
(4.42)
\[
\rho_t = -(w \cdot \nabla A \bar{\rho})|_{y=\zeta^{-1}(x, t)} \in L^\infty_T H^1,
\]
(4.43)
\[
v_t = (u_t - w \cdot \nabla A u)|_{y=\zeta^{-1}(x, t)} \in L^\infty_T L^2 \cap L^2 H^1.
\]
(4.44)

Thanks to the regularity \((4.42)\), we derive from \((4.41)\) that
\[
- \int_0^t \phi_t \int \rho v \cdot \varphi \, dx \, d\tau = \int_0^t \phi \int (f + \mu \Delta A u - \nabla A q - \bar{\rho} w \cdot \nabla A u - w \cdot \nabla A \bar{\rho} u) \cdot \psi J \, dy \, d\tau.
\]
(4.45)
In addition, by (4.43) and (4.44), we have

\[- \int_0^t \phi \int \rho v \cdot \varphi dy d\tau = \int_0^t \phi \int (\rho t v_\eta + \rho v_t) \cdot \varphi dy d\tau\]

\[= \int_0^t \phi \int \rho v_t \cdot \varphi dy d\tau - \int_0^t \phi \int w \cdot \nabla_A \rho u \cdot \psi J dy d\tau.\]

Inserting the above identity into (4.45) yields

\[\int_0^t \phi \int \rho v_t \cdot \varphi dx d\tau\]

\[= \int_0^t \phi \int (f + \mu \Delta_A u - \nabla_A q - \rho w \cdot \nabla_A u) \cdot \psi J dy d\tau.\]

Now let us further assume \(\varphi \in H^1_\sigma\), then \(\text{div}_A \psi = 0\) and \(\psi_t|_{x=\zeta} = w \cdot \nabla_A \psi\). The above identity further implies

\[\frac{d}{dt} \int \rho v_t \cdot \varphi dy = \int (J((\mu A_{ik}\partial_i(A_{ik}\partial_k u) - \rho w \cdot \nabla_A u + f)w \cdot \nabla_A \psi)
\]

\[- \mu \partial_t(JA_{ik}A_{ik}\partial_k u) \cdot \partial_t \psi - \partial_t(J \rho w \cdot \nabla_A u) \cdot \psi
\]

\[+ f \psi J_t) dy + < f_t, \psi J >_{H^1_\sigma, H^1} =: < \chi, \varphi >_{H^1_\sigma, H^1} \cdot (4.46)\]

Recalling the definition of \(< \chi, \varphi >_{H^1_\sigma, H^1}\), \(\|\psi\|_1 \lesssim_0 \|\varphi\|_1 \lesssim_0 \|\psi\|_1\) for any \(t \in I_T\) and \(H^1_\sigma\) is a reflexive Banach space, we immediately see that (referring to \([35, \text{Lemma 1.66}]\))

\[(\rho v_t)_t = \chi \in L^2_T H^1_\sigma. \quad (4.47)\]

Exploiting the regularity \(\rho, u_t\) and (4.47), by means of a classical regularization method, we have

\[\frac{d}{dt} \int \rho |v_t|^2 dx = 2 < (\rho v_t)_t, v_t >_{H^1_\sigma, H^1} - \int \rho |v_t|^2 dx \text{ for a.e. in } I_T.\]

Consequently, making use of (4.43), (4.44) and the second identity in (4.46), we get (4.7) from the above identity. This completes the proof of Proposition 4.1. \(\square\)

Now we are in a position to show Proposition 2.2. To start with, let \((\eta^0, u^0)\) satisfy all the assumptions in Proposition 2.2 and \(\|\nabla \eta^0\|_2^2 \leq \delta \leq \delta^1\). We should remark here that the smallness of \(\delta\) (independent of \(\lambda\) and \(m\)) will be frequently used in the calculations that follow.

Let \(f = \partial_t^2 \eta + G_\eta e_2\) with \(\eta_t = w\) in Proposition 4.1. Then, by (4.3)–(4.5), we have

\[\|f\|_{C^0(T, L^2)} + \|f\|_{L^2_T H^1} + \|f_t\|_{L^2_T H^1} + \|f\|_{L^2(\Omega_T)} \lesssim 1,\]

which implies that

\[\sqrt{\text{B}(u^0, \partial_t^2 \eta + G_\eta e_2) \lesssim 1 + \|u^0\|_2^2 + \sqrt{\|\nabla w\|_{C^0(T, H^1)}(1 + \|u^0\|_2^3/2))}. \quad (4.48)\]

Thus, from (4.6) and (4.48) we get

\[\|u\|_{L^\infty_T} + \|q\|_{C^0(T, H^1)} + \|q\|_{L^2_T H^2} \leq c^1(1 + \|u^0\|_2^3) + \|\nabla w\|_{L^\infty H^1}/2. \quad (4.49)\]

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Denote
\[ B := 2c^L(1 + \|u_0^0\|_2^3), \]
where the constant \(c^L\) is the same as in (4.49). By (4.49) and Proposition 4.1 with \(B\) defined by (4.50), we can construct a function sequence \(\{u^k, q^k\}_{k=1}^\infty\) defined on \(\Omega_T\) with \(T\) satisfying (4.3). Moreover,

- for \(k \geq 1\), \((u^{k+1}, q^{k+1}) \in \mathcal{U}_T \times (C^0(\overline{T}, H^1) \cap L^2_T H^2)\), \((\bar{\rho}u_1^{k+1})_\Omega = (\bar{\rho}u_1^{k+1})_{\|t=0\text{ and}}
\begin{equation}
\begin{cases}
\eta^k = \int_0^t u^k d\tau + \eta^0, \\
\bar{\rho}u_t^{k+1} + \nabla A_k q^{k+1} - \mu \Delta A_k u^{k+1} = \lambda m^2 \partial_t^2 \eta^k + g G_{\eta^k} e_2, \\
\text{div}_A u^{k+1} = 0, \\
u^{k+1}|_{|t=0} = u^0, \\
(\eta^k_2, \partial_2 \eta^k_1, u^{k+1}_2, \partial_2 u^{k+1}_1)|_{\partial \Omega} = 0 
\end{cases}
\end{equation}
with initial condition \(u^{k+1}|_{|t=0} = u^0\), where \(A_k := (\nabla \eta^k + I)^{-T}\);

- \((u^1, q^1)\) is constructed by Proposition 4.4 with \(w = 0\) and with \((\partial_1^2 \eta^0 + g G_{\eta^0} e_2)\) in place of \(f\);

- the solution sequence \(\{u^k, q^k\}_{k=1}^\infty\) satisfies the following uniform estimates: for all \(k \geq 1\),
\begin{equation}
1 \leq 2 \det(\nabla \eta^k + I) \leq 3, \quad \|\nabla \eta^k\|_2 \leq 2\delta \quad \text{for all } t \in \overline{T},
\end{equation}
\begin{equation}
\|u^k\|_{\mathcal{U}_T} + \|q^k\|_{C^0(\overline{T}, H^1)} + \|q^k\|_{L^2_T H^2} \leq B.
\end{equation}

In order to take the limit in (4.51) as \(k \to \infty\), we have to show that \(\{u^k, q^k\}_{k=1}^\infty\) is a Cauchy sequence. To this end, we define for \(k \geq 2\),
\[(\eta^k, \bar{u}^{k+1}, \bar{A}^k, \bar{q}^{k+1}) := (\eta^k - \eta^{k-1}, u^{k+1} - u^{k}, \bar{A}^k - \bar{A}^{k-1}, q^{k+1} - q^k),\]
which satisfies \((\bar{\rho} \eta^k_2)_{\|t=0\text{ and}} = 0\) and
\begin{equation}
\begin{cases}
\bar{\eta}^k = \int_0^t \bar{u}^k d\tau, \\
\bar{\rho}u_t^{k+1} + \nabla \bar{q}^{k+1} - \mu \Delta \bar{u}^{k+1} = \lambda m^2 \partial_t^2 \eta^k = \mathcal{N}^k, \\
\text{div} \bar{u}^{k+1} = -(\text{div}_A u^{k+1} + \text{div}_A k^{k+1}), \\
\bar{u}^{k+1}|_{|t=0} = 0, \\
(\bar{\eta}^k_2, \partial_2 \bar{\eta}^k_1, \bar{u}^{k+1}_2, \partial_2 \bar{u}^{k+1}_1)|_{\partial \Omega} = 0 
\end{cases}
\end{equation}
Here and in what follows, \(\bar{A}^k = \bar{A}^k - I\) and
\[ \mathcal{N}^k := \mu (\text{div}_A \nabla A_k u^{k+1} + \text{div}_A \bar{A}_k u^{k+1} + \text{div}_A \bar{A}_{k-1} \bar{u}^{k+1} + \text{div}_A \bar{A}_{k-1} \bar{u}^{k+1}) - \mu \Delta \bar{u}^{k+1} - \nabla \bar{A}_k q^{k+1} - \nabla \bar{A}_{k-1} \bar{q}^{k+1} + g (\bar{\rho} \eta^k_2(y, t) + y_2)\bar{\rho} \eta^k_2(y, t) + y_2) e_2.\]
Thanks to (4.3), (4.52) and (4.53), it is easy to check that
\begin{equation}
\|A^{k-1}\|_2 \leq 0, \quad \|\bar{A}^{k-1}\|_2 \leq 0, \quad \|\bar{A}^{k-1}\|_1 \leq 0, \quad \|\nabla u^{k-1}\|_1, \\
\|\bar{A}^k\|_1 \leq 0, \quad \|\nabla \eta^k\|_1 \leq 0, \quad T^{1/2} \|\nabla \bar{u}^k\|_{L^2_T H^1}, \quad \|\bar{A}^k\|_0 \leq 0, \quad \|\nabla \bar{u}^k\|_0 + B \|\nabla \eta^k\|_1, \\
\|\mathcal{N}^k\|_{L^2(\Omega_T)} \leq c T^{1/4} \|\nabla \bar{u}^k\|_{L^2_T H^1} + c_0 \delta (\mu \|\nabla u^{k+1}\|_{L^2_T H^1} + \|q^{k+1}\|_{L^2_T H^1}).
\end{equation}
By Lemmas A.9 and A.11, we can derive from (4.54)₂, (4.54)₃ and (4.54)₅ that
\[
\mu \|\bar{u}^{k+1}\|_2 + \|q^{k+1}\|_1 \lesssim_0 \|(\bar{p} u_t^{k+1}, \lambda \mu (\nabla \bar{\eta}, \nabla \bar{q}^{k+1}, \div \bar{A} u^{k+1}, \div \bar{A}_{\bar{A}} = \bar{u}^{k+1})\|_0.
\]  
(4.57)

Multiplying (4.54)₂ by \(\bar{u}^{k+1}\), resp. \(\bar{u}_t^{k+1}\) in \(L^2\), we get
\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} \bar{u}^{k+1}\|_0^2 + \mu \|\nabla \bar{u}^{k+1}\|_0^2 = \int (\lambda \mu \partial_t \bar{\eta} + \nabla \bar{q}^{k+1}) \bar{u}^{k+1} \, dy
\]
and
\[
\frac{1}{2} \frac{d}{dt} \|\nabla \bar{u}^{k+1}\|_0^2 + \|\sqrt{\rho} \bar{u}_t^{k+1}\|_0^2 = \int (\lambda \mu \partial_t \bar{\eta} + \nabla \bar{q}^{k+1}) \bar{u}_t^{k+1} \, dy.
\]  
(4.58)

(4.59)

Noting that
\[
\int \nabla \bar{q}^{k+1} \cdot \bar{u}^{k+1} \, dy = \int (\div \bar{A} u^{k+1} + \div \bar{A}_{\bar{A}} \bar{u}^{k+1}) \bar{q}^{k+1} \, dy \lesssim (B \|\bar{A}^k\|_1 + \delta \|\bar{u}^{k+1}\|_2) \|\bar{q}^{k+1}\|_1
\]
and
\[
\int \nabla \bar{q}^{k+1} \cdot \bar{u}_t^{k+1} \, dy = \int \partial_t (\div \bar{A} u^{k+1} + \div \bar{A}_{\bar{A}} \bar{u}^{k+1}) \bar{q}^{k+1} \, dy
\]
\[
= - \int \partial_t ((\bar{A}^k)^T u^{k+1} + (\bar{A}_{\bar{A}}^k)^T \bar{u}^{k+1}) \cdot \nabla \bar{q}^{k+1} \, dy
\]
\[
\lesssim (\|\bar{A}^k\|_1 \|u_t^{k+1}\|_1 + B(\|\bar{A}^k\|_0 + \|u^{k+1}\|_1) + \delta \|u_t^{k+1}\|_0) \|\bar{q}^{k+1}\|_1,
\]
thus putting (4.58) and (4.59) together, and then using (4.57), the above two estimates and Young’s inequality, we get, for sufficiently small \(\delta\),
\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} \bar{u}^{k+1}\|_0^2 + \|\nabla \bar{u}^{k+1}\|_0^2 + c(\|\bar{u}^{k+1}\|_2^2 + \|\bar{u}_t^{k+1}\|_2^2 + \|\bar{q}^{k+1}\|_2^2)
\]
\[
\lesssim_1 (\|\partial_t \bar{\eta}\|_{H^1} + \|\bar{A}^k\|_1 \|u^{k+1}\|_{L^2} + \|\bar{A}_{\bar{A}}\|_1 \|\bar{u}^{k+1}\|_1) + T \|\nabla \bar{u}^{k+1}\|_{L_t^2 H^1}.
\]  
(4.60)

Integrating the above inequality over \((0, t)\), and then using (4.3), (4.54)₁, (4.55) and (4.56), we get
\[
\|\bar{\eta}^k\|_{L_t^\infty H^2} + \|\bar{u}^{k+1}\|_{L_t^\infty H^1} + \|\bar{u}^{k+1}\|_{L_t^2 H^2} + \|\bar{u}_t^{k+1}\|_{L_t^2 H^1} + \|\bar{q}^{k+1}\|_{L_t^2 H^1} \lesssim_L T^{1/4} (\|\bar{u}^{k+1}\|_{L_t^\infty H^1} + \|\nabla \bar{u}^{k+1}\|_{L_t^\infty L^2} + \|\nabla \bar{u}^{k+1}\|_{L_t^2 H^1}).
\]

Hence, for sufficiently small \(T\) (depending possibly on \(B, \mu, \lambda, m, \rho\), and \(\Omega\),
\[
\{\eta^k, u^k, \bar{u}^k, q^k\}_{k=1}^\infty \text{ is a Cauchy sequence in}
\]
\[
L_T^\infty H^2 \times (L_T^\infty H^1 \cap L_T^2 H^2) \times L_T^2 H^1 \times L_T^2 H^1.
\]
(4.61)

By (4.53), up to the extraction of a subsequence, we have, as \(k \to \infty\), that
\[
(n^{n_k}, u^{n_k}, q^{n_k}) \rightharpoonup (n, u_t, q) \text{ weakly-* in } L_T^\infty H^3_s \times L_T^\infty L^2 \times L_T^\infty H^1_0,
\]
(4.62)
\[
(u^{n_k}, u_t^{n_k}, q^{n_k}) \rightharpoonup (u, u_t, q) \text{ weakly in } L_T^2 H^3_s \times L_T^2 H_1^s \times L_T^2 H^2,
\]
(4.63)
\[
u^{n_k} \to u \text{ strongly in } C^0(T_T, H^2_s) \text{ with } u|_{t=0} = u_0,
\]
(4.64)
where

\[ \eta := \eta^0 + \int_0^t u \mathrm{d}\tau. \quad (4.65) \]

In addition, by (4.52) and (4.61), we further obtain

\[ (\eta^k, 1/J^k, u^k, q^k) \rightarrow (\eta, u, J^{-1}, u_t, q) \text{ strongly in} \]

\[ L_T^\infty H^2 \times L_T^\infty H^1 \times (L_T^\infty H^1 \cap L_T^2 H^2) \times L_T^2 H^1 \times L_T^2 H^1, \quad (4.66) \]

where \( J = \det(\nabla \eta + I) \).

Remembering that (4.65) implies \( \eta_t = u \), we infer from (4.51) and (4.62)–(4.66) that the limit \((\eta, u, q)\) is a solution to the initial value problem (1.19) and (1.24); moreover, the solution \((\eta, u, q)\) belongs to \( C^0(T, H^3_s) \times U_T \times (C^0(T, H^1) \cap L^2 T H^2) \) by following the argument of the regularity of \((u, q)\) in the proof of Proposition 4.1. The uniqueness of solutions to (1.19) and (1.24) is easily verified by using Gronwall’s lemma and a similar energy method to derive (4.60), and its proof will be omitted here. We complete the proof of Proposition 2.2.

Appendix A. Analysis tools

This appendix is devoted to providing some mathematical results, which have been used in previous sections. We should point out that \( \Omega \) resp. the simplified notations appearing in what follows are defined by (1.2) resp. the same as in Section 1.3. In addition \( H^1_0 := \{ \upsilon \in H^1 | \upsilon|_{\partial \Omega} = 0 \} \) and \( a \lesssim b \) still denotes \( a \leq cb \) where the positive constant \( c \) depends on the parameters and the domain in lemmas in which \( c \) appears.

**Lemma A.1.** (1) **Embedding inequality (see [1, 4.12 Theorem]):** Let \( D \subset \mathbb{R}^2 \) be a domain satisfying the cone condition, then

\[ \| f \|_{C^0(D)} = \| f \|_{L^\infty(D)} \lesssim \| f \|_{H^2(D)}. \quad (A.1) \]

(2) **Interpolation inequality in \( H^j \) (see [1, 5.2 Theorem]):** Let \( D \) be a domain in \( \mathbb{R}^2 \) satisfying the cone condition, then for any given \( 0 \leq j < i \),

\[ \| f \|_{H^j(D)} \lesssim \| f \|^{(i-j)/i}_{L^2(D)} \| f \|^{j/i}_{H^i(D)} \lesssim \varepsilon^{-j/(i-j)} \| f \|_{L^2(D)} + \varepsilon \| f \|_{H^i(D)}, \quad \forall \varepsilon > 0, \quad (A.2) \]

where the two estimate constants in (A.2) are independent of \( \varepsilon \).

(3) **Product estimates (see Section 4.1 in [23]):** Let \( D \in \mathbb{R}^2 \) be a domain satisfying the cone condition, and the functions \( \varphi, \psi \) defined in \( D \). Then

\[ \| \varphi \psi \|_{H^i(D)} \lesssim \begin{cases} \| \varphi \|_{H^1(D)} \| \psi \|_{H^i(D)} & \text{for } i = 0; \\ \| \varphi \|_{H^i(D)} \| \psi \|_{H^i(D)} & \text{for } 0 \leq i \leq 2. \end{cases} \quad (A.3) \]

(4) **Anisotropic product estimates (please refer to [26, Lemma 3.1]):** Let the functions \( \varphi \) and \( \psi \) be defined in \( \Omega \). Then

\[ \| \varphi \psi \|_0 \lesssim \begin{cases} \sqrt{\| \varphi \|_0} \| \psi \|_1, \quad & \| \varphi \|_0 \sqrt{\| \varphi \|_0} \| \psi \|_1. \end{cases} \quad (A.4) \]
Lemma A.2. Friedrich’s inequality (see [33, Lemma 1.42]): Let $1 \leq p < \infty$, $n \geq 2$ and $D \subset \mathbb{R}^n$ be a bounded Lipschitz domain. Let $f$ be a measurable function with respect to the $(n-1)$-dimensional measure $\mu := \text{meas}_{n-1}$ defined on $\partial D$ and let $\text{meas}_{n-1}(\Gamma) > 0$. Then
\[
\|w\|_{W^{1,p}(D)} \lesssim \|\nabla w\|_{L^p(D)}
\] (A.5)
for any $w \in W^{1,p}(D)$ with $u|_{\Gamma} = 0$ in the sense of trace.

Remark A.1. By Lemma A.2, we easily see that
\[
\|w\|_{W^{1,p}(D)} \lesssim \|w'\|_{L^p(D)}
\]
for any $w \in W^{1,p}(D)$ with $w(0) = 0$ or $w(T) = 0$, where $D := (0, T)$. Hence, we further obtain
\[
\|\varpi\|_0 \lesssim \|\partial_2 \varpi\|_0 \text{ for any } \varpi \in H^1_0.
\] (A.6)

Lemma A.3. Poincaré’s inequality (see [33, Lemma 1.43]): Let $1 \leq p < \infty$, and $D$ be a bounded Lipschitz domain in $\mathbb{R}^n$ for $n \geq 2$ or a finite interval in $\mathbb{R}$. Then for any $w \in W^{1,p}(D)$,
\[
\|w\|_{L^p(D)} \lesssim \|\nabla w\|_{L^p(D)} + \left| \int_D w \, dy \right|^p.
\] (A.7)

Remark A.2. By Poincaré’s inequality, we have that for any given $i \geq 0$,
\[
\|w\|_{1,i} \lesssim \|w\|_{2,i} \text{ for any } w \text{ satisfying } \partial_1 w, \partial_2^2 w \in H^2.
\] (A.8)

Lemma A.4. Hodge-type elliptic estimates (see [33, Lemma A.4]): If $w \in H^1$ with $i \geq 1$, then
\[
\|\nabla w\|_{i-1} \lesssim \|(\text{curl} w, \text{div} w)\|_{i-1}.
\] (A.9)

Lemma A.5. Bogovskii’s operator in the standing-wave form (see (2.52) in [21]): There exists Bogovskii’s operator $B : f \in L^2 \to H^1_0$. Moreover, $B(f)$ satisfies
\[
\text{div} B(f) = f,
\]
\[
\|B(f)\|_1 \lesssim \|f\|_0 \text{ and } (B(f))(2\pi nL) = 0
\]
for any integer $n$.

Lemma A.6. Diffeomorphism mapping theorem (see [33, Lemma A.8]): There exists a sufficiently small constant $\gamma \in (0, 1)$, depending on $\Omega$, such that for any $\zeta \in H^3$ satisfying $\|\nabla \zeta\|_2 \leq \gamma$, $\psi := \zeta + y$ (after possibly being redefined on a set of measure zero with respect to variable $y$) satisfies the same diffeomorphism properties as $\zeta$ in (1.15) and (1.16), and $\inf_{y \in \Omega} \text{det}(\nabla \zeta + I) \geq 1/4$.

Lemma A.7. Integration by parts for the functions with values in Banach spaces (see [33, Theorem 1.67]): Let $H$ be a Hilbert space and $V \hookrightarrow H$ be dense in $H$. If $u, v \in L^p_T V$ with $T \in \mathbb{R}^+$, $1 < p < \infty$ and $u_t, v_t \in L^p_T V$, $p^{-1} + q^{-1} = 1$, then $u, v \in C(I_T, H)$ and
\[
(u(t), v(t)) - (u(s), v(s)) = \int_s^t \langle u_t(\tau), v(\tau) \rangle + \langle v_t(\tau), u(\tau) \rangle \, d\tau,
\] (A.10)
where $s, t \in I_T$ and $\langle \cdot, \cdot \rangle$ is the duality between $V$ and $V^*$.  

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Remark A.3. The integral at the right hand of the identity (A.10) presents that
\[ < u_t(\tau), v(\tau) > + < v_t(\tau), u(\tau) \in L^1(I_T). \] (A.11)

Thus, by the property of absolutely continuous function [35, Lemma 1.7] and (A.10), we easily see that
\[ \frac{d}{dt}(u(t), v(t)) = < u_t, v > + < v_t, u > \text{ a.e. in } I_T. \] (A.12)

In particular, we easily derive from (A.12) that
- for \( H = L^2 \),
  \[ \frac{d}{dt} \int u(t)v(t)dy = < u_t, v > + < v_t, u > \text{ a.e. in } I_T. \] (A.13)
- for \( H = V = L^2 \),
  \[ \partial_t(u(t)v(t)) = u_t v + v_t u. \] (A.14)

Lemma A.8. Properties of composite functions with values in Banach spaces: Let \( T > 0 \), integers \( i \geq 0 \) be given and \( 1 \leq p \leq \infty \). Let \( \varphi = \xi(y, t) + y \) and \( \xi \in \{ \psi \in C^0(I_T, H^3) \mid \psi \in H^3, \inf_{(y, t) \in \Omega_T} \det(\nabla \psi + I) \geq 1/4 \} \).

(1) If \( f \in C^0(I_T, H^i) \) or \( f \in L^p_T H^i \) with \( 0 \leq i \leq 3 \), then
\[ F := f(\varphi, t) \in C^0(I_T, H^i) \text{ or } L^p_T H^i \] (A.15)

and
\[ \mathcal{F} := f(\varphi^{-1}, t) \in C^0(I_T, H^i) \text{ or } L^p_T H^i. \] (A.16)

Moreover,
\[ \| F \|_{L^p_T H^i} \lesssim P(\| \xi \|_{L^\infty_T H^3}) \| f \|_{L^p_T H^i}, \] (A.17)
\[ \| \mathcal{F} \|_{L^p_T H^i} \lesssim P(\| \xi \|_{L^\infty_T H^3}) \| f \|_{L^p_T H^i}. \] (A.18)

(2) If \( \xi \) additionally satisfies \( \xi_t \in L^\infty_T H^2 \), then for any \( f \in L^p_T H^i \) satisfying \( f_t \in L^p_T H^{i-1} \) with \( 1 \leq i \leq 3 \),
\[ F_t = (f_t(x, t) + \xi_t \cdot \nabla f(y, t))|_{x=\varphi} \in L^p_T H^{i-1} \] (A.19)

and
\[ \mathcal{F}_t = (f_t(y, t) - (\nabla \varphi)^{-1}\xi_t \cdot \nabla f(y, t))|_{y=\varphi^{-1}} \in L^p_T H^{i-1}. \] (A.20)

Proof. Please refer to [31, Lemma A.10].
Lemma A.9. Generalized Korn–Poincaré inequality: Let $D \subset \mathbb{R}^2$ be a bounded domain satisfying the cone condition. Assume that $p > 1$, $u \in H^1(D)$,

$$\chi \geq 0, \ 0 < a \leq \|\chi\|_{L^1(D)}, \ \|\chi\|_{L^p(D)} \leq b. \quad (A.21)$$

Then

$$\|u\|_{L^2(D)} \lesssim \|\nabla u\|_{L^2(D)} + \left|\int_D \chi u dy\right|. \quad (A.21)$$

Proof. We prove the lemma by contradiction. Suppose that the conclusion of Lemma A.9 fails, then there would be a sequence $\{\chi_n\}_{n=1}^\infty$ of non-negative functions satisfying (A.21) with $\chi_n$ in place of $\chi$ for any $n \geq 1$ and a sequence $\{u_n\}_{n=1}^\infty \subset H^1(D)$, such that

$$\|u_n\|_{L^2(D)} \geq a_n \left(\|\nabla u_n\|_{L^2(D)} + \left|\int_D \chi_n u_n dy\right|\right) \text{ and } a_n \to +\infty. \quad (A.22)$$

Setting $w_n = u_n\|u_n\|_{L^2(D)}^{-1}$, making use of the compactness embedding $H^1(D) \hookrightarrow \hookrightarrow L^p(D)$ and (A.22), we find that

$$w_n \to w = |D|^{-1/2} \text{ in } L^q(D), \quad (A.23)$$

where $q = p/(p-1)$.

In addition, there exists a function $\tilde{\chi} \geq 0$ satisfying

$$0 < a \leq \|\tilde{\chi}\|_{L^1(\Omega)}, \ \|\tilde{\chi}\|_{L^p(D)} \leq b \quad (A.24)$$

and

$$\int_D \chi_n \varphi dy \to \int_D \tilde{\chi} \varphi dy \text{ for any } \varphi \in L^q(D). \quad (A.25)$$

Thus, by virtue of (A.23) and (A.24), we have

$$\lim_{n \to \infty} \int_D (\chi_n w_n - \tilde{\chi} w) dy = \lim_{n \to \infty} \int_D \chi_n (w_n - w) dy + \lim_{n \to \infty} \int_D (\chi_n - \tilde{\chi}) w dy = 0. \quad (A.26)$$

The identity (A.26), together with (A.23) and (A.24), yields

$$\lim_{n \to \infty} \int_D \chi_n w_n dy = \int \tilde{\chi} w dy > 0. \quad (A.27)$$

Finally, (A.22) implies

$$\lim_{n \to \infty} \int_D \chi_n w_n dy = 0, \quad (A.28)$$

which contradicts with (A.27). Therefore, the conclusion of Lemma A.9 remains true.

Lemma A.10. Elliptic estimates: Let $a > 0$, $\delta \in (0, 1]$, $\chi \in L^\infty$, $\chi \geq a$,

$$\|A - I\|_2 \lesssim \delta. \quad (A.29)$$

If $f \in L^2$, for sufficiently small $\delta$, there exists a unique weak solution $p \in H^1$ such that

$$\int \chi \nabla Aq : \nabla \psi dy = \int f \cdot \nabla \psi dy \text{ for any } \psi \in H^1. \quad (A.30)$$

Moreover, $q$ satisfies

$$\|q\|_1 \lesssim \|f\|_0. \quad (A.31)$$
Proof. We define an inner-product of $H^1$ by

$$(\varphi, \phi)_{H^1} := \int \chi \nabla A \varphi \cdot \nabla A \phi \, dy$$

for $\varphi, \phi \in H^1$. and the corresponding norm by $\|\varphi\|_x := \sqrt{(\varphi, \varphi)_{H^1}}$. Obviously, by (A.1), the Poincaré inequality (A.7) and the smallness condition (A.29), we obtain

$$\|\varphi\|_1 \lesssim \|\varphi\|_x \lesssim \|\varphi\|_1.$$ 

Defining the functional

$$F(\varphi) := \int f \cdot \nabla \varphi \, dy$$

for $\varphi \in H^1$, we easily see that $F$ is a bounded linear functional on $H^1$. By virtue of the Riesz representation theorem, there is a unique $q \in H^1$ such that

$$(q, \varphi)_{H^1} = F(\varphi)$$

for any $\varphi \in H^1$; (A.32)

$$\|q\|_1 \lesssim \|q\|_x \lesssim \|f\|_0.$$ 

For $\psi \in H^1$, we denote $\varphi = \psi - (\psi)_{\Omega}$. Then, $\varphi \in H^1$. Putting this $\varphi$ in (A.32), we get (A.30). This completes the proof. □

Lemma A.11. Stokes estimates: Let $i \geq 0$, $(f, \varpi) \in H^i \times H^{2+i}$ and $\varpi_2|_{\partial \Omega} = 0$, the Stokes problem with Navier boundary condition

$$\begin{cases}
\nabla P - \Delta v = f & \text{in } \Omega, \\
\text{div } v = \text{div } \varpi & \text{in } \Omega, \\
(v_2, \partial_2 v_1) = 0 & \text{on } \partial \Omega
\end{cases}$$

(A.33)

admits a unique solution $(v_1, P) \in H^{2+i}_2 \times H^{1+i}$, and the solution satisfies

$$\|v\|_{2+i} + \|P\|_{1+i} \lesssim \|(f, \varpi)\|_i + \|\text{div } \varpi\|_{1+i}.$$ 

(A.34)

Remark A.4. Obviously, the above lemma with $^0H^{2+i}_2$ in place of $H^{2+i}_2$ also holds.

Proof. (1) We first consider the case $i = 0$. Noting that $\varpi_2|_{\partial \Omega} = 0$, the boundary value problem

$$\begin{cases}
\Delta \theta = -\text{div } \varpi, \\
\partial_2 \theta|_{\partial \Omega} = 0
\end{cases}$$

(A.35)

admits a unique solution $\theta \in H^3$, which satisfies

$$\|\theta\|_3 \lesssim \|\varpi\|_0 + \|\text{div } \varpi\|_1,$$ 

(A.36)

please refer to [35, Lemma 4.27] for the proof.

Let $\psi = f + \nabla \text{div } \varpi \in L^2$, then the Stokes problem with Navier boundary condition

$$\begin{cases}
\nabla P - \Delta w = \psi, \\
\text{div } w = 0, \\
(w_2, \partial_2 w_1)|_{\partial \Omega} = 0
\end{cases}$$

(A.37)
also admits a unique solution \((w, P) \in \mathcal{H}_2^2 \times \mathcal{H}_1^1\), which satisfies
\[
\|w\|_2 + \|P\|_1 \lesssim \|\psi\|_0. \tag{A.38}
\]
please refer to [38, Theorem 5.10] for the proof. Thus let \(v = w - \nabla \theta\), we easily see that \(v\) satisfies (A.33) and (A.34) from (A.35)–(A.38).

(2) Now we turn to the proof of the case \(i > 1\) by induction. We assume that the problem (A.33) admits a unique solution \((v_1, P) \in \mathcal{H}_{2+j}^2 \times \mathcal{H}_{1+j}^1\), and the solution satisfies
\[
\|v\|_{2+j} + \|P\|_{1+j} \lesssim \|(f, \varpi)\|_j + \|\text{div} \varpi\|_{1+j}, \tag{A.39}
\]
where \(0 \leq j < i\). Obviously, to get the desired conclusion, next it suffices to prove that
\[
\|v\|_{3+j} + \|P\|_{2+j} \lesssim \|(f, \varpi)\|_{1+j} + \|\text{div} \varpi\|_{2+j}. \tag{A.40}
\]
By the standard method of difference quotient in [31, Lemma A.7], it is easy to see that
\[
\|v\|_{1,2+j} + \|P\|_{1,1+j} \lesssim \|(f, \varpi)\|_{1,j} + \|\text{div} \varpi\|_{1,1+j}. \tag{A.41}
\]
We can rewrite (A.33)\(_1\) as the following boundary value problem:
\[
\begin{cases}
\Delta v_1 = \partial_1 P - f, \\
v_1 = v_1|_{\partial \Omega}.
\end{cases}
\]
Applying the classical regularity of elliptic equation to the above boundary value problem, and then using the trace theorem, we further get
\[
\|v_1\|_{3+j} \lesssim \|f\|_{1+j} + \|P\|_{1,1+j} + \|v_1\|_{H^{5/2+j}} \lesssim \|f\|_{1+j} + \|P\|_{1,1+j} + \|v_1\|_{2,1+j}. \tag{A.42}
\]
In addition, thanks to (A.33)\(_2\), we further have
\[
\|\partial_2 v_2\|_{2+j} \lesssim \|(\partial_1 v_1, \text{div} \varpi)\|_{2+j}. \tag{A.43}
\]
Putting (A.39) and (A.41)–(A.43) together yields (A.40). This completes the proof. \(\Box\)

**Lemma A.12.** Existence of orthogonal basis in \(\mathcal{H}_1^1\): There exists a countable orthogonal basis \(\{\varphi^i\}_{i=1}^\infty \subset \mathcal{H}_1^\infty\) to \(\mathcal{H}_1^1\). Moreover \(\{\varphi^i\}_{i=1}^\infty\) is an orthonormal basis to \(L_2^2 := \{w \in L^2 \mid \text{div}w = 0\}\).

**Proof.** Please refer to [4, Lemma 2.2], [34, Lemma 3.2] or [9, Theorem 1 in Section 6.5]. \(\Box\)

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