Six-dimensional regularization of chiral gauge theories on a lattice I

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1. Introduction
History of lattice chiral symmetry

1981 Nielsen-Ninomiya’s no go theorem
1992 Domain-wall fermion (Kaplan)
1997 Fixed point action (Hasenfratz)
1998 Overlap fermion (Neuberger)

1999-2001 Luescher’s proof for existence of $U(1)$ chiral gauge invariant regularization
Kikukawa-Nakayama: $SU(2)\times U(1)$ also O.K.

2015 Grabowska & Kaplan: manifestly gauge invariant construction of chiral gauge theory
What’s new in Grabowska-Kaplan, PRL 116 (2016) no.21 211602?

[Grabowska Theory Wed, Kaplan, plenary Sat]

**Before GK** (Neuberger, Luescher, Kikukawa, Suzuki…):

1. Break gauge sym. explicitly,
2. Find a **counter-term if anomaly free**, 
3. (Mainly) studied w/ 4D overlap fermions.

**Grabowska & Kaplan** PRL 116 (2016) no.21 211602:

1. Keep gauge sym. explicitly,
2. If **not anomaly free**, no 4D local action, 
3. 5D construction is essential.
The key is gradient flow (again).

They put

\[ U_\mu(x, t) = \begin{cases} 
\text{flow time t configuration} & (\mu = 1, 2, 3, 4) \\
1 & (\mu = 5)
\end{cases} \]

\[ S_{DW} = \int d^4 x dt \overline{\Psi} (D_{\text{Wilson}}^{5D} - \Lambda \epsilon(t)) \Psi \]

\[ \epsilon(t) = \begin{cases} 
+1 & (t \geq 0) \\
-1 & (t < 0)
\end{cases} \]

Gauge d.o.f. do not flow!

Links at different t have the same 4D gauge invariance.

Figure by Kaplan
They successfully reproduced a picture: gauge anomaly = gauge current missing in extra dim, [Callan & Harvey 1985] keeping total gauge invariance in 5D. Absorbed by 5D Chern-Simons term.
Global anomaly [Witten 1982]:
Gauge anomaly $\text{SU}(2) = \text{mod } 2$ index of 5D Dirac operator w/ 5-th direction

$A_\mu(x_\mu, x_5) = (1 - x_5)A_\mu(x) + x_5 A_\mu^g(x_\mu)$

$\times 5$

- Mod 2 instanton flips the sign of partition function.

How about global anomaly?
Extra-dim. is essential for global anomaly, too.

Witten’s claim at Strings 2015:
We need “extension” of global anomaly.

Not only for “mapping torus”:
\[ A_\mu(x_\mu, x_5) = (1 - x_5) A_\mu(x) + x_5 A_\mu^g(x_\mu) \]

but also for ANY D+1 manifold with D-dim. boundary Weyl fermions, if the determinant has a phase \[ \exp(i\pi\eta) \neq 1 \],
then the theory has a global anomaly.

Anomaly cannot be understood within 4-dim !
We need extra dimension(s)!

We want combine them.

1. Grabowska-Kaplan’s 5D
   - 4D boundary
   - Gauge inv. gradient flow $\rightarrow$ cannot detect global anomaly.

2. Witten’s 5D
   - 4D boundary
   - Gauge non-invariant flow $\rightarrow$ cannot keep gauge symmetry.

But how?
Our proposal = 6D with 2 different domain-walls

W domain-wall
Linear flow

GK domain-wall
YM gradient flow

We (4dim) are here.
GK domain-wall contains Stora-Zumino anomaly descent equations

$\delta_g(\text{phase}) = \int_{x_5 < 0} d^5x \delta_g(CS)$

$\int d^5x (CS) = \int d^6x F \wedge F \wedge F$

[Stora 1983, Zumino 1983, Alvarez-Gaume & Ginsparg 1984, Sumitani 1984]
We (4dim) are here.

SU(2) example

global anomaly

$\pi_4(SU(2)) = \mathbb{Z}_2$

$\pi_5(SU(2)) = \mathbb{Z}_2$
Our 6-dim formulation has

1. Stora-Zumino anomaly ladder:
   6D $U(1)_A$ index $\rightarrow$ gauge anomaly.

2. Global anomaly ladder (new finding):
   6D *exotic* index $\rightarrow$ global anomaly.

3. Anomaly free condition = sign-problem free condition in 6D:
   $\rightarrow$ If anomaly free, 6D determinant is real positive. $\rightarrow$ Monte Carlo is O.K.!
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My talk

Next talk by Ryo Yamamura
2. “Parity” and axial U(1) anomalies in 6D
Two anomalous symmetries

1. Axial $U(1)$ symmetry
\[ \psi \rightarrow e^{i\alpha\gamma_7}\psi, \quad \bar{\psi} \rightarrow \bar{\psi}e^{i\alpha\gamma_7} \]

2. Parity’ symmetry
   (reflection in 5th direction)
\[ P'\psi(x_1,\cdots,4,5,6) = i\gamma_5 R_5\psi(x_1,\cdots,4,5,6) \]
\[ = i\gamma_5\psi(x_1,\cdots,4,-5,6) \]
\[ P'^2 = -1 \]

Mass-term is not invariant
Parity anomaly

Cf. Usual parity (only in even-dim.):

\[ P \psi(x_1, x_2, \ldots, 6) = \gamma_1 \psi(x_1, -x_2, \ldots, 6) \]

mass is allowed since \( P^2 = 1 \)

\( P' \) (in any dim.) has anomaly:

massless fermion action is invariant, but (zero-mode part of) fermion measure is NOT:

\[ D \bar{\psi}_0 P' D P' \psi_0 = -D \bar{\psi}_0 D \psi_0. \]
Two mass terms

\[ M \bar{\psi} \psi \quad : \text{U}(1)_A \text{ and } P' \text{ asymmetric.} \]

\[ \mu \bar{\psi} (i \gamma_6 \gamma_7 R_5 R_6) \psi \quad : \text{odd in } P' \text{ but } \text{U}(1)_A \text{ invariant.} \]

\[ \Rightarrow \text{Dirac fermion w/ periodic boundary} \]

\[ \det \left( \frac{D^{6D} - M - i \mu \gamma_6 \gamma_7 R_5 R_6}{D^{6D} + M + i \mu \gamma_6 \gamma_7 R_5 R_6} \right) = (-1)^{P+I} \]

\[ P : \text{U}(1)_A \text{ index } (\rightarrow \text{perturbative anomaly}) \]

\[ I : \text{exotic index } (\rightarrow \text{global anomaly}) \]
3. Two domain-walls
Two domain-walls

Let’s consider a 6D Dirac fermion

\[ \epsilon(x) = \frac{x}{|x|} \]

\[
\det \left( \frac{D^{6D} + M \epsilon(x_6) + i\mu \epsilon(x_5)\gamma_6\gamma_7 R_5 R_6}{D^{6D} + M + i\mu \gamma_6 \gamma_7 R_5 R_6} \right)
\]

where we assume \( M > 0, \mu > 0 \)

\[ A_5 = A_6 = 0, \]

\[ A_{\mu=1,\ldots,4}(x) \text{ is symmetric under } x_5 \rightarrow -x_5, x_6 \rightarrow -x_6 \]

(* later, gauge field is given by gradient & linear flows)
Fermion determinant is still real!

\[
\det \left( \frac{D^{6D} + M \epsilon(x_6) + i \mu \epsilon(x_5) \gamma_6 \gamma_7 R_5 R_6}{D^{6D} + M + i \mu \gamma_6 \gamma_7 R_5 R_6} \right) \propto (-1)^{\mathcal{P} + \mathcal{I}}
\]

Determine has \(\gamma_5 R_5\) Hermiticity. Indices become non-trivial

\[\mathcal{P} : \text{APS index through GK domain-wall} \quad \rightarrow \quad \text{Perturbative anomaly in 4D}\]

\[\mathcal{I} : \text{APS index through W domain-wall} \quad \rightarrow \quad \text{global anomaly in 4D}\]

[Atiyah-Patodi-Singer 1975]
Massless Weyl fermion appears!

Dirac equation

\[(D^{6D} + M \epsilon(x_6) + i \mu \epsilon(x_5) \gamma_6 \gamma_7 R_5 R_6) \psi(x) = 0\]

has a localized solution at \( x_5 = x_6 = 0 \) as

\[\psi(x) = e^{-M|x_6|} e^{-\mu|x_5|} \phi(\bar{x}),\]

\[D^{4D} \phi(\bar{x}) = 0, \quad \bar{x} = (x_1, x_2, x_3, x_4)\]

\[\gamma_6 \phi(\bar{x}) = \phi(\bar{x}), \quad \begin{pmatrix} \bar{\gamma}_5 & 0 \\ 0 & 0 \end{pmatrix} \phi(\bar{x}) = +\phi(\bar{x})\]

* Opposite chiral mode appears if \( M < 0, \mu < 0 \)
4. Anomaly ladder through GK domain-wall
Bulk/edge decomposition

Simple example without W domain-wall

\[
\det \left( \frac{D^{6D} + M \epsilon(x_6)}{D^{6D} + M} \right) \left[ \propto (-1)^3 \right] \\
= \det \left( \frac{D^{6D} + M \epsilon(x_6) + iM_2 \gamma_6 \gamma_7 R_6}{D^{6D} + M} \right) \left[ \propto \exp(i\phi_{6D}) \right] \\
\times \det \left( \frac{D^{6D} + M \epsilon(x_6) + iM_2 \gamma_6 \gamma_7 R_6}{D^{6D} + M \epsilon(x_6) + iM_2 \gamma_6 \gamma_7 R_6} \right) \left[ \propto \exp(i\phi_{5D}) \right]
\]

where we assume \( M \gg M_2 \gg 0 \)

Imaginary part \( \rightarrow \) \[\pi \tilde{\mathcal{J}} = \phi_{6D} + \phi_{5D}\]
Atiyah-Patodi-Singer index

6D bulk $\rightarrow$ Axial U(1) anomaly

$$\phi_{6D} = \pi \int d^6 x \frac{1 - \epsilon(x_6)}{2} \frac{1}{6(4\pi)^3} \epsilon^{\mu_1 \cdots \mu_6} \text{tr}[F_{\mu_1 \mu_2} F_{\mu_3 \mu_4} F_{\mu_5 \mu_6}]$$

Fujikawa’s method $\uparrow$

$$= \pi \mathcal{P}^{6D}_{x_6 < 0} + \pi CS$$

$$CS \equiv - \int_{x_6=0} d^5 x \frac{2}{3(4\pi)^3} \epsilon^{\mu_1 \cdots \mu_5} \text{tr} \left[ \frac{1}{2} A_{\mu_1} F_{\mu_2 \mu_3} F_{\mu_4 \mu_5} - \frac{i}{2} A_{\mu_1} A_{\mu_2} A_{\mu_3} F_{\mu_4 \mu_5} - \frac{1}{5} A_{\mu_1} A_{\mu_2} A_{\mu_3} A_{\mu_4} A_{\mu_5} \right]$$

2nd determinant $\rightarrow$ 5D Dirac fermion

$$\lim_{M \to \infty} \det \left( \frac{D^{6D} + M \epsilon(x_6)}{D^{6D} + M \epsilon(x_6) + i M_2 \gamma_6 \gamma_7 R_6} \right) = \det \left( \frac{\bar{D}^{5D}}{\bar{D}^{5D} + M_2} \right) = \det \left( \frac{\bar{D}^{5D}}{\bar{D}^{5D} + M_2} \right) e^{-i\pi \eta/2}$$

$$\mathcal{J} = \mathcal{P}^{6D}_{x_6 < 0} + CS - \frac{\eta_{5D}}{2}$$

Integer $=$ non-integer + non-integer

[Atiyah-Patodi-Singer 1975]
With $W$ domain-wall and $M \gg \mu \gg 0$

$$\det \left( \frac{D^{6D} + M \epsilon(x_6) + i \mu \epsilon(x_5) \gamma_6 \gamma_7 R_5 R_6}{D^{6D} + M + i \mu \gamma_6 \gamma_7 R_5 R_6} \right) \left[ \alpha (-1)^J \right]$$

No change in 6D bulk. (U(1)A cannot feel W-DW.)

$$\propto \exp \left( i \pi \left( \mathcal{P} x_6 < 0 + CS \right) \right)$$

Weyl fermion!

* 5D/4D decomposition is (almost) the same as Grabowska & Kaplan.
Stora-Zumino anomaly ladder

To summarize what we have computed,

\[ \mathcal{I} = \mathcal{P} + \mathcal{I} = \mathcal{P}_{x_6<0}^D + CS - \frac{\eta_{5D}}{2} \]

(integer)

6D U(1) anomaly → 5D parity anomaly

\[ \frac{1}{2} \eta_{5D} = CS^{(x_5<0)} - \frac{\phi_{\text{anom}}}{\pi} + \text{gauge invariant phase} \]

→ 4D gauge anomaly

\[ \mathcal{I} \text{ is hidden. (Next talk)} \]

[Stora 1983, Zumino 1983, Alvarez-Gaume & Ginsparg 1984, Sumitani 1984]
4D perturbative anomaly

6D Axial U(1) anomaly

5D Chern-Simons 1

5D Chern-Simons 2

GK domain-wall (YM gradient flow)

\[ \delta_g(\text{phase}) = \int_{x_5 < 0} d^5 x \delta_g(CS) \]

\[ \int d^5 x (CS) = \int d^6 x F \wedge F \wedge F \]

[Stora 1983, Zumino 1983, Alvarez-Gaume & Ginsparg 1984, Sumitani 1984]
Summary of part I

Our 6D determinant w/ 2-different DWs

\[
\begin{align*}
\det \left( \frac{D^{6D} + M \epsilon(x_6) + i \mu \epsilon(x_5) \gamma_6 \gamma_7 R_5 R_6}{D^{6D} + M + i \mu \gamma_6 \gamma_7 R_5 R_6} \right) [\propto (-1)^J] \\
1. \text{is real,} \quad \epsilon(x) = x/|x| \\
2. \text{has a Weyl fermion at 4D junction,} \\
3. \text{gauge anomaly originates from 6D U(1)\text{\textsubscript{A}} index} [\text{Stora-Zumino anomaly ladder}].
\end{align*}
\]

\[
\pi \mathcal{I} = \phi^{6D} + \phi^{5D} + \phi^{4D}
\]

6D U(1)\text{\textsubscript{A}} anomaly → 5D parity anomaly → 4D gauge anomaly
Next talk

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My talk

Next talk by
Ryo
Yamamura
Claiming anomaly free $\Leftrightarrow$ sign problem free.

Although I do not claim a complete proof, I believe that there is a general answer for when a theory with fermions is completely consistent and anomaly-free, meaning that the path integral on a general manifold can be defined in a way that is anomaly-free and consistent with all principles of unitarity, locality and cutting and pasting. The condition is just that

$$e^{i\pi \eta} = 1$$

for all $D + 1$-manifolds $Y$, not just for mapping tori. Anomaly cancellation gives the same condition just for mapping tori.
Back-up slides
Possible applications

• 4D→2D: Doubly gapped (M and $\mu$) topological insulator can exist?
• Higgs: $\rightarrow$ definition of standard model?
• Higher dim theory: Our world is really 6D?
What are really essential?

- **6D**: Yes. Stora-Zumino’s solution for consistent anomaly is *unique*.
- **Two domain-walls**: Yes. At least, need to distinguish U(1)$_A$ and $P'$
- **Gradient flow**: we don’t know. no imaginary part even without it.
- **Non-locality $(R_5,R_6)$**: probably no. but analysis is easier with them.
Phase of 5D determinant

\[
\det \left( \frac{\bar{D}^{5D} + \mu \epsilon(x_5)}{\bar{D}^{5D} + \mu} \right) \propto \exp(-i\pi \eta^{5D})
\]

\[\pi \eta^{5D} = \pi CS + \phi \text{gauge non invariant} + \phi \text{gauge invariant}\]

Perturbative anomaly

global anomaly (old definition)

global anomaly new def. by Witten 2015: no local 4D action to express the phase.

\[\text{Anomaly-free} \rightarrow \eta^{5D} \text{ must be zero!}\]
Why 6D?

$\eta^{5D}$ can be determined only relatively (direct computation is ill-defined due to UV div.).

$\eta^{5D} = \int_0^1 du \frac{d\eta^{5D}(u)}{du}$

[Alvarez-Gaume et al. 1986]

$u$ is our 6th coordinate! $\rightarrow$ We need 5th direction to separate L/R chiral modes, 6th direction to determine $\eta^{5D}$
CP restoration

Complex phase of 5D determinant = CP violating lattice artifact (w/o CKM)

Our 6D construction may be automatically giving a counter-term to keep the CP symmetry at finite lattice spacing.

[Fujikawa-Ishibashi-Suzuki 2002, Hasenfratz 2005]
Global anomaly classification

- SU(2) global anomaly: O.K.
- Other groups on 4-dim torus: Maybe. Index $\mathbb{Z}, \mathbb{Z}_2$ can be detected by $P'$. But higher dim: we don’t know. For example, $\pi_6(SU(2)) = \mathbb{Z}_{12}$ may require quite non-trivial treatment.
Parity anomaly on a lattice

• $P'$ has an anomaly.
• On the lattice, we may need Ginsparg-Wilson-type relation for $P'$ symmetry.
• The $U(1)_A$ invariant mass term in the kernel of overlap Dirac operator?
Anomaly free condition

1. Axial U(1) cancelation in 6D:

$$\sum_{L} \text{tr} T_{L}^{a} \{ T_{L}^{b}, T_{L}^{c} \} - \sum_{R} \text{tr} T_{R}^{a} \{ T_{R}^{b}, T_{R}^{c} \} = 0$$

cancels perturbative anomaly.

2. “Parity” anomaly cancelation :

# fundamental rep. = even

cancels global anomaly.

⇒ Our determinant is real positive !
Together with anti-domain-wall, it becomes

\[
\det \left( \frac{\bar{D}^{5D} + \mu \epsilon(x_5) \epsilon(x_5 - L_5) R_5}{\bar{D}^{5D} + \mu} \right) = \text{Det} \left( \frac{(x-x') (\bar{D}^{5D} + \mu \epsilon(x_5) \epsilon(x_5 - L_5) R) + x_5^5}{\delta(x-x')(\bar{D}^{5D} + \mu)} \right) \times \text{Det} \left( \frac{(x-x') (\bar{D}^{5D} + \mu \epsilon(x_5) \epsilon(x_5 - L_5) R_5)}{\delta(x-x')(\bar{D}^{5D} + \mu \epsilon(x_5) \epsilon(x_5 - L_5) R_5) + \mu} \right)
\]

Another CS on 5D

\[
-\pi \int_{x_6=0} d^5x \frac{4}{3(4\pi)^3} \frac{1 - \epsilon(x_5) \epsilon(x_5 - L_5)}{2} \epsilon^{\mu_1 \ldots \mu_5} \text{tr} \left[ \frac{1}{2} A_{\mu_1} F_{\mu_2 \mu_3} F_{\mu_4 \mu_5} - \frac{i}{2} A_{\mu_1} A_{\mu_2} A_{\mu_3} A_{\mu_4} A_{\mu_5} \right]
\]

\[
\frac{1}{2} \eta^{5D} = CS(x_5 < 0) + \frac{1}{2} \eta^{4D} - \frac{\phi_{\text{anom}}}{\pi} - \frac{\phi'}{\pi},
\]

\[
\det \frac{D}{D + \mu_2} \times \text{det} (R_5^{\text{bulk}})
\]

\[
D = P^5 \bar{D}^{4D} P^5 + P^5 \bar{\delta}^{4D} P^5
\]
Massless Weyl fermion appears!

Dirac equation

\[ (D^{6D} + M \epsilon(x_6) + i \mu \epsilon(x_5) \gamma_6 \gamma_7 R_5 R_6) \psi(x) = 0 \]

has a localized solution at \( x_5 = x_6 = 0 \) as

\[ \psi(x) = e^{-M |x_6|} e^{-\mu |x_5|} \phi(\bar{x}), \]

\[ D^{4D} \phi(\bar{x}) = 0, \quad \bar{x} = (x_1, x_2, x_3, x_4) \]

\[ \gamma_6 \phi(\bar{x}) = \phi(\bar{x}), \]

\[ i \gamma_5 \gamma_6 \gamma_7 R_5 R_6 \phi(\bar{x}) = \phi(\bar{x}) \]

\[ \begin{pmatrix} \bar{\gamma}_5 & 0 \\ 0 & 0 \end{pmatrix} \phi(\bar{x}) = +\phi(\bar{x}) \]

* Opposite chiral mode appears if \( M < 0, \mu < 0 \)
Summary of part 1 and 2: Our 6D formulation has

1. Stora-Zumino anomaly ladder:
   6D $U(1)_A$ index $\rightarrow$ gauge anomaly

2. Global anomaly ladder:
   6D *exotic* index $\rightarrow$ global anomaly.

3. Gradient flow in $x_5$ + linear interpolation in $x_6$ $\rightarrow$ mirror fermions are decoupled.

4. Anomaly free condition = sign-problem free condition in 6D:
   Monte Carlo is O.K.!