EXOTIC BAILEY-SLATER SPT-FUNCTIONS I: GROUP A

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Abstract. We introduce several spt-type functions that arise from Bailey pairs. We prove simple Ramanujan type congruences for these functions which can be explained by a spt-crank-type function. The spt-crank-type functions are constructed by adding an extra variable $z$ into the generating functions. We find dissections when $z$ is a certain root of unity, as has been done for many rank and crank difference formulas of various partition type objects. Our formulas require an identity of Chan [8] on generalized Lambert series.

1. Introduction and statement of Results

We recall a partition of a positive integer $n$ is a non-increasing sequence of positive integers that sum to $n$, we let $p(n)$ denote the number of partitions of $n$. For example $p(3) = 3$ since the partitions of 3 are just 3, 2 + 1, and 1 + 1 + 1. In [1] Andrews introduced a weighted count on the partitions of $n$ given by counting each partition of $n$ by the number of times the smallest part occurs. We call this weighted count $spt(n)$ and note $spt(3) = 5$.

In [3] Andrews, Garvan, and Liang gave combinatorial refinements of congruences for $spt(n)$ by considering $S(z, q)$, a two variable generalization of the generating function of $spt(n)$. They then applied Bailey’s Lemma to recognize $S(z, q)$ as the difference between the generating functions for the rank and crank of ordinary partitions. Based on information on the rank and crank of a partition, they were able to deduce results for $spt(n)$.

This method was used again by Garvan and the author in [11] to give combinatorial refinements and prove new congruences for the number of smallest parts in the overpartitions of $n$, the number of smallest parts in the overpartitions of $n$ with smallest part even, the number of smallest parts in the overpartitions of $n$ with smallest part odd, and the number of smallest parts in the partitions of $n$ with smallest part even and distinct odd parts. The process has two key steps. The first is to use a Bailey pair and Bailey’s lemma to see the difference of a rank and crank function, the second is to use dissection formulas for the rank and crank function to deduce congruences. In the cases of [11], the rank functions had been previously considered by Lovejoy and Osburn in [15] and [16].

Here we first look to Bailey pairs to get the two variable generating function and from that deduce what are the partition functions, rather than starting with a partition function and trying to find a related Bailey pair. We apply Bailey’s Lemma to the two variable generalizations of each partition function to get the difference of a series we can dissect and the generating function for the crank of ordinary partitions. We then dissect the series at roots of unity using methods similar to those used by Atkin and Swinnerton-Dyer in [4] for the rank of partitions. This method has also been used by Ekin in studying the crank of partitions [9], Lewis and Santa-Gadea in studying the rank and crank of partitions [14], and by Lovejoy and Osburn in studying the rank of overpartitions [15], the $M_2$-rank of overpartitions [17], and the $M_2$-rank of partitions without repeated odd parts [16].

We use the product notation

$$(z; q)_n = \prod_{j=0}^{n-1} (1 - zq^j), \quad (z; q)_\infty = \prod_{j=0}^{\infty} (1 - zq^j), \quad [z; q]_\infty = (z, q/z; q)_\infty,$$

$$(z_1, \ldots, z_k; q)_n = (z_1; q)_n \cdots (z_k; q)_n, \quad (z_1, \ldots, z_k; q)_\infty = (z_1; q)_\infty \cdots (z_k; q)_\infty,$$

$$_[z_1, \ldots, z_k; q]_\infty = [z_1; q]_\infty \cdots [z_k; q]_\infty.$$

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We recall a pair of sequences \((\alpha, \beta)\) is a Bailey pair relative to \((a, q)\) if

\[
\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q; q)_{n-k} (aq; q)_{n+k}}.
\]

A limiting case of Bailey’s Lemma gives that if \((\alpha, \beta)\) is a Bailey pair relative to \((a, q)\) then

\[
\sum_{n=0}^{\infty} (\rho_1, \rho_2; q)_n \left( \frac{aq/\rho_1, \rho_2}{(aq, \rho_1, \rho_2; q)_{\infty}} \right)^n \beta_n = \sum_{n=0}^{\infty} (\rho_1, \rho_2; q)_n \left( \frac{aq/\rho_1, \rho_2}{(aq, \rho_1, \rho_2; q)_{\infty}} \right)^n \alpha_n.
\]

We start with four Bailey pairs. Each \((\beta^i, \alpha^i)\) is a Bailey pairs relative to \((1, q)\), these are out of group \(A\) of [21]. For each \(i\), \(\beta_0^i = 1\) and for \(n \geq 1\) the \(\alpha_n^i\) and \(\beta_n^i\) are defined by

\[
\begin{align*}
\beta_1^n &= \frac{1}{(q)_n}, & \alpha_1^n &= q^{6n^2-n} + q^{6n^2+n}, & \alpha_{3n \pm 1}^1 &= -q^{6n^2 \pm 5n + 1}, \\
\beta_2^n &= \frac{n}{(q)_n}, & \alpha_2^n &= q^{6n^2-2n} + q^{6n^2+2n}, & \alpha_{3n \pm 1}^2 &= -q^{6n^2 \pm 2n}, \\
\beta_3^n &= \frac{n^2}{(q)_n}, & \alpha_3^n &= q^{3n^2-n} + q^{3n^2+n}, & \alpha_{3n \pm 1}^3 &= -q^{3n^2 \pm n}, \\
\beta_4^n &= \frac{n^2}{(q)_n}, & \alpha_4^n &= q^{3n^2-2n} + q^{3n^2+2n}, & \alpha_{3n \pm 1}^4 &= -q^{3n^2 \pm 4n + 1}.
\end{align*}
\]

For each \(\beta^i\) we define a corresponding series,

\[
\begin{align*}
PP_1(z, q) &= \frac{(q; q)_{\infty}}{(z, z^{-1}; q)_{\infty}} \sum_{n=0}^{\infty} (z, z^{-1}; q)_n \beta_1^n q^n - \frac{(q; q)_{\infty}}{(z, z^{-1}; q)_{\infty}} \sum_{n=1}^{\infty} q^n (q^{2n+1}; q)_{\infty} \\
PP_2(z, q) &= \frac{(q; q)_{\infty}}{(z, z^{-1}; q)_{\infty}} \sum_{n=0}^{\infty} (z, z^{-1}; q)_n \beta_2^n q^n - \frac{(q; q)_{\infty}}{(z, z^{-1}; q)_{\infty}} \sum_{n=1}^{\infty} q^{2n} (q^{2n+1}; q)_{\infty} \\
PP_3(z, q) &= \frac{(q; q)_{\infty}}{(z, z^{-1}; q)_{\infty}} \sum_{n=0}^{\infty} (z, z^{-1}; q)_n \beta_3^n q^n - \frac{(q; q)_{\infty}}{(z, z^{-1}; q)_{\infty}} \sum_{n=1}^{\infty} q^{2n+1} (q^{2n+1}; q)_{\infty} \\
PP_4(z, q) &= \frac{(q; q)_{\infty}}{(z, z^{-1}; q)_{\infty}} \sum_{n=0}^{\infty} (z, z^{-1}; q)_n \beta_4^n q^n - \frac{(q; q)_{\infty}}{(z, z^{-1}; q)_{\infty}} \sum_{n=1}^{\infty} q^{2n+2} (q^{2n+1}; q)_{\infty}.
\end{align*}
\]

We then set \(z = 1\) and simplify to get the series

\[
\begin{align*}
PP_1(q) &= \sum_{n=1}^{\infty} pp_1(n) q^n = \frac{q^n}{(q^n; q)^{\infty}} \frac{q^n}{(q^n; q)^{\infty}} \\
PP_2(q) &= \sum_{n=1}^{\infty} pp_2(n) q^n = \frac{q^{2n}}{(q^n; q)^{\infty}} \frac{q^{2n}}{(q^n; q)^{\infty}} \\
PP_3(q) &= \sum_{n=1}^{\infty} pp_3(n) q^n = \frac{q^{n^2+n}}{(q^n; q)^{\infty}} \frac{q^{n^2+n}}{(q^n; q)^{\infty}} \\
PP_4(q) &= \sum_{n=1}^{\infty} pp_4(n) q^n = \frac{q^{n^2}}{(q^n; q)^{\infty}} \frac{q^{n^2}}{(q^n; q)^{\infty}}.
\end{align*}
\]

We now interpret the \(pp_i(n)\) in terms of the smallest parts of certain partition pairs and as partition pairs. For a partition \(\pi\) we let \(\ell(\pi)\) denote the largest part, \(s(\pi)\) the smallest part, and \(|\pi|\) the sum of the parts. We say a pair of partitions \((\pi_1, \pi_2)\) is a partition pair of \(n\) if \(|\pi_1| + |\pi_2| = n\).

We note that

\[
\frac{q^n}{(1 - q^n)^2} = \sum_{m=1}^{\infty} mq^{nm}.
\]
Thus \(1 - q^{11} \sum_{i=0}^{\infty} q^{11i} \) is the generating function for the number of occurrences of the smallest part in partitions with smallest part \( n \).

We see \( pp_1(n) \) is the number of partition pairs \((\pi_1, \pi_2)\) of \( n \), counted by the number of times \( s(\pi_1) \) occurs, where either \( \pi_2 \) is empty or \( s(\pi_1) < s(\pi_2) \) and \( \ell(\pi_2) \leq 2s(\pi_1) \). Alternatively, we can interpret \( pp_1(n) \) as the number of partition pairs \((\pi_1, \pi_2)\) of \( n \), with \( \pi_2 \) allowed to be empty but if it is not empty then \( s(\pi_1) \leq s(\pi_2) \) and \( \ell(\pi_2) \leq 2s(\pi_1) \).

Similarly we see \( pp_2(n) \) is the number of partition pairs \((\pi_1, \pi_2)\) of \( n \) where the smallest part of \( \pi_1 \) occurs at least twice, counted by the number of times \( s(\pi_1) \) occurs past the first, where either \( \pi_2 \) is empty or \( s(\pi_1) < s(\pi_2) \) and \( \ell(\pi_2) \leq 2s(\pi_1) \). Alternatively, we can interpret \( pp_2(n) \) as the number of partition pairs \((\pi_1, \pi_2)\) where the smallest part of \( \pi_1 \) occurs at least twice, with \( \pi_2 \) allowed to be empty but if it is not empty then \( s(\pi_1) \leq s(\pi_2) \) and \( \ell(\pi_2) \leq 2s(\pi_1) \).

We see \( pp_3(n) \) is the number of partition pairs \((\pi_1, \pi_2)\) of \( n \) where the smallest part of \( \pi_1 \) occurs more than enough times to form a square in the Ferrers diagram, counted by the number of times \( s(\pi_1) \) occurs past the first \( s(\pi_1) \) times, where either \( \pi_2 \) is empty or \( s(\pi_1) < s(\pi_2) \) and \( \ell(\pi_2) \leq 2s(\pi_1) \). Alternatively, \( pp_3(n) \) is the number of partition pairs \((\pi_1, \pi_2)\) where the smallest part of \( \pi_1 \) occurs more than enough times to form a square in the Ferrers diagram, with \( \pi_2 \) allowed to be empty but if it is not empty then \( s(\pi_1) \leq s(\pi_2) \) and \( \ell(\pi_2) \leq 2s(\pi_1) \).

We see \( pp_4(n) \) is the number of partition pairs \((\pi_1, \pi_2)\) of \( n \) where the smallest part of \( \pi_1 \) occurs enough times to at least form a square in the Ferrers diagram, counted by the number of times \( s(\pi_1) \) occurs past the first \( s(\pi_1) - 1 \) times, where either \( \pi_2 \) is empty or \( s(\pi_1) < s(\pi_2) \) and \( \ell(\pi_2) \leq 2s(\pi_1) \). Alternatively, \( pp_4(n) \) is the number of partition pairs \((\pi_1, \pi_2)\) where the smallest part of \( \pi_1 \) occurs enough times to at least form a square in the Ferrers diagram, with \( \pi_2 \) allowed to be empty but if it is not empty then \( s(\pi_1) \leq s(\pi_2) \) and \( \ell(\pi_2) \leq 2s(\pi_1) \).

There are of course other ways to interpret the series \( PP_i(q) \). It is the partition pair interpretations that allow us to easily define cranks in the same fashion of the crank defined in Section 3 of [11].

We will prove the following congruences.

**Theorem 1.1.** For \( n \geq 0 \),

\[
pp_1(3n) \equiv 0 \pmod{3},
\]
\[
pp_2(3n + 1) \equiv 0 \pmod{3},
\]
\[
pp_2(5n + 1) \equiv 0 \pmod{5},
\]
\[
pp_3(5n + 4) \equiv 0 \pmod{5},
\]
\[
pp_3(7n + 1) \equiv 0 \pmod{7},
\]
\[
pp_4(5n + 4) \equiv 0 \pmod{5}.
\]

We use \( PP_i(z, q) \) to prove these congruences as follows. We write

\[
PP_i(z, q) = \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} M_i(m, n)z^m q^n,
\]

and define for any positive integer \( t \)

\[
M_i(k, t, n) = \sum_{m \equiv k \pmod{t}} M_i(m, n).
\]

Thus for any \( t \) we have

\[
pp_i(n) = \sum_{m=-\infty}^{\infty} M_i(m, n) = \sum_{k=0}^{t-1} M_i(k, t, n)
\]

and for \( \zeta \) a \( t \)th root of unity we have

\[
PP_i(\zeta, q) = \sum_{n=1}^{\infty} q^n \sum_{k=0}^{t-1} \zeta^k M_i(k, t, n).
\]
If \( \ell \) is prime and \( \zeta_\ell \) is a primitive \( \ell^{th} \) root of unity, then the minimal polynomial for \( \zeta_\ell \) is \( \sum_{k=1}^{\ell-1} x^k \). If the coefficient of \( q^n \) in \( PP_i(\zeta_\ell, q) \) is zero we would then have \( M_i(1, \ell, N) = M_i(2, \ell, N) = \cdots = M_i(\ell-1, \ell, N) \). That is, if the coefficient of \( q^n \) in \( PP_i(\zeta_\ell, q) \) is zero then \( pp_i(N) = \ell \cdot M_1(1, \ell, N) \) and \( pp_i(N) \) is clearly divisible by \( \ell \).

To prove the congruences of Theorem 1 we then need to prove that the coefficients of \( q^{3n}, q^{3n+1}, q^{5n+1}, q^{5n+4}, q^{7n+1}, \) and \( q^{5n+4} \) are zero in \( PP_1(\zeta_3, q), PP_2(\zeta_3, q), PP_2(\zeta_5, q), PP_3(\zeta_7, q), PP_3(\zeta_7, q), \) and \( PP_4(\zeta_5, q) \) respectively. That these coefficients are zero is not immediately obvious.

We are actually proving something stronger than just the congruences because we are saying how to split up the numbers \( pp_i(n) \). While we can immediately read off a combinatorial interpretation of each \( M_i(m, n) \) in terms of partition triples, this is not particularly satisfying. We conclude this paper by defining a crank on \( \ell \) in \( \mathbb{Z} \) and form a Bailey

We begin by applying Bailey’s Lemma to each \( PP_i(z, q) \). We note in general that if \( \alpha \) and \( \beta \) form a Bailey pair with \( \alpha_0 = \beta_0 = 1 \) then

\[
\frac{(q; q)_\infty}{(z; z-1; q)_\infty} \sum_{n=0}^{\infty} (z, z^{-1}; q)^n q^n \alpha_n - \frac{(q; q)_\infty}{(z; z^{-1}; q)_\infty} = \frac{1}{(1-z)(1-z^{-1})(q; q)_\infty} \left( 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^n \alpha_n}{(1-qz^n)(1-z^{-1}q^n)} \right) - \frac{(q; q)_\infty}{(z, z^{-1}; q)_\infty}.
\]

We note

\[
\frac{1}{(1-zq^{-3n-1})(1-z^{-1}q^{-3n-1})} = \frac{q^{6n+2}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})},
\]

\[
\frac{1}{(1-zq^{-3n})(1-z^{-1}q^{-3n})} = \frac{q^{6n}}{(1-zq^{3n})(1-z^{-1}q^{3n})}.
\]

Thus

\[
\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})q^n \alpha_n}{(1-zq^n)(1-z^{-1}q^n)} = \frac{1}{(1-zq^{3n})(1-z^{-1}q^{3n})} - \frac{1}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} - \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{6n^2+2n}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} - \sum_{n=0}^{\infty} \frac{(1-z)(1-z^{-1})q^{6n^2+8n+2}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})}
\]

so that

\[
PP_1(z, q) = \frac{1}{(q; q)_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{6n^2+2n}}{(1-zq^{3n})(1-z^{-1}q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{6n^2+8n+2}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} \right) - \frac{(q; q)_\infty}{(z, z^{-1}; q)_\infty}.
\]

In the same fashion, we find that

\[
PP_2(z, q) = \frac{1}{(q; q)_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{6n^2+n}}{(1-zq^{3n})(1-z^{-1}q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{6n^2+5n+1}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} \right) - \frac{(q; q)_\infty}{(z, z^{-1}; q)_\infty},
\]

\[
PP_3(z, q) = \frac{1}{(q; q)_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+2n}}{(1-zq^{3n})(1-z^{-1}q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+4n+1}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} \right) - \frac{(q; q)_\infty}{(z, z^{-1}; q)_\infty},
\]

\[
PP_4(z, q) = \frac{1}{(q; q)_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+n}}{(1-zq^{3n})(1-z^{-1}q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+7n+2}}{(1-zq^{3n+1})(1-z^{-1}q^{3n+1})} \right) - \frac{(q; q)_\infty}{(z, z^{-1}; q)_\infty}.
\]

We determine dissection formulas for the series without \( \frac{(q; q)_\infty}{(z, z^{-1}; q)_\infty} \) at roots of unity. Then we use known formulas for \( \frac{(q; q)_\infty}{(z, z^{-1}; q)_\infty} \) at roots of unity to give formulas for the \( PP_i(z, q) \). Although it would be possible
to use the methods of this paper to determine all of the dissections of \( PP_1(z, q) \), \( PP_2(z, q) \), \( PP_3(z, q) \), and \( PP_4(z, q) \) at \( z = \zeta_3, \zeta_5, \) and \( \zeta_7 \), we only do so for the cases that lead to a congruence.

We define

\[
S(z, w, q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n(n+1)}w^n}{1 - q^{2n}}, \quad S^*(w, q) = \sum_{n\neq 0} \frac{q^{2n(n+1)}w^n}{1 - q^n}.
\]

Although they do not appear in the statement of our theorems, we will also need the following series,

\[
T(z, w, q) = \sum_{n=-\infty}^{\infty} \frac{q^{n(n+1)}w^n}{1 - q^{n}}, \quad T^*(w, q) = \sum_{n\neq 0} \frac{q^{n(n+1)}w^n}{1 - q^n}.
\]

For integers \( a, b, c \) we use the abuse of notation \( S(a, b, c) = S(q^a, q^b, q^c) \), \( S^*(b, c) = S^*(q^b, q^c) \), \( T(a, b, c) = T(q^a, q^b, q^c) \), and \( T^*(b, c) = T^*(q^b, q^c) \).

**Theorem 1.2.**

\[
\frac{1}{(q; q)_\infty} \left( \sum_{n=-\infty}^{\infty} (1 - \zeta_3 q^{3n})(1 - \zeta_3^{-1} q^{3n}) \right) - \sum_{n=-\infty}^{\infty} (1 - \zeta_3 q^{3n})(1 - \zeta_3^{-1} q^{3n+1}) = A_{130}(q^3) + q A_{131}(q^3) + q^2 A_{132}(q^3),
\]

where

\[
A_{130}(q) = \frac{1}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^4; q^9 \right]_{\infty} \right) + \frac{2}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^2; q^9 \right]_{\infty} \right),
\]

\[
A_{131}(q) = \frac{1}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^4; q^9 \right]_{\infty} \right) - \frac{1}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^2; q^9 \right]_{\infty} \right),
\]

\[
A_{132}(q) = -\frac{q^2}{(q^3; q^9)_{(q; q)_\infty}} \left( S(3, -2, 9) - q^2 S(4, 2, 9) \right) + \frac{1}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^4; q^9 \right]_{\infty} \right) - \frac{2}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^2; q^9 \right]_{\infty} \right).
\]

**Theorem 1.3.**

\[
\frac{1}{(q; q)_\infty} \left( \sum_{n=-\infty}^{\infty} (1 - \zeta_3 q^{3n})(1 - \zeta_3^{-1} q^{3n}) \right) - \sum_{n=-\infty}^{\infty} (1 - \zeta_3 q^{3n+1})(1 - \zeta_3^{-1} q^{3n+1}) = A_{230}(q^3) + q A_{231}(q^3) + q^2 A_{232}(q^3),
\]

where

\[
A_{230}(q) = \frac{1}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^2; q^9 \right]_{\infty} \right) \left( S(3, -2, 9) - q^2 S(4, 2, 9) \right) - \frac{2}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^4; q^9 \right]_{\infty} \right) - \frac{1}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^2; q^9 \right]_{\infty} \right),
\]

\[
A_{231}(q) = -\frac{2}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^4; q^9 \right]_{\infty} \right) + \frac{1}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^2; q^9 \right]_{\infty} \right),
\]

\[
A_{232}(q) = \frac{q^2}{(q^3; q^9)_{(q; q)_\infty}} \left( S(3, -2, 9) - q^2 S(4, 2, 9) \right) + \frac{1}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^4; q^9 \right]_{\infty} \right) + \frac{1}{3} \left( \frac{q^3; q^3}_{(q; q)_\infty} \left[ q^2; q^9 \right]_{\infty} \right).
\]
Theorem 1.5.

\[
\frac{1}{(q; q)_\infty} \left( \sum_{n=\infty}^{\infty} \frac{q^{6n^2+n}}{(1 - \zeta_5 q^{3n})(1 - \zeta^{-1}_5 q^{3n})} - \sum_{n=\infty}^{\infty} \frac{q^{6n^2+5n+1}}{(1 - \zeta_5 q^{3n+1})(1 - \zeta^{-1}_5 q^{3n+1})} \right) = A_{250}(q^5) + qA_{251}(q^5) + q^2A_{252}(q^5) + q^3A_{253}(q^5) + q^4A_{254}(q^5),
\]

where

\[
A_{250}(q) = -\frac{q^{-3}}{(q^{15}; q^{15})_\infty [q; q^{15}]_\infty} (S(-3, -28, 15) - q^{28} S(11, 28, 15))
+ (\zeta_5 + \zeta^{-1}_5) \frac{q^2}{(q^{15}; q^{15})_\infty [q^2; q^{15}]_\infty} (S(3, -14, 15) - q^{14} S(10, 14, 15))
+ \frac{3 + \zeta_5 + \zeta^{-1}_5}{5} \left( \frac{q^4}{[q^4; q^5]^2_\infty} - \frac{q^{15}}{[q^2; q^3][q^5; q^5]^2_\infty} \right) - (\zeta_5 + \zeta^{-1}_5) q^2 \left( \frac{q^{15}; q^{15}}{[q^4; q^4; q^5; q^5]^2_\infty} \right),
\]

\[
A_{251}(q) = \frac{-2 + \zeta_5 + \zeta^{-1}_5}{5} \left( \frac{q^5}{[q^5; q^5]^2_\infty} \right),
\]

\[
A_{252}(q) = \frac{1 - 3(\zeta_5 + \zeta^{-1}_5)}{5} \left( \frac{q^6}{[q^6; q^6]^2_\infty} \right),
\]

\[
A_{253}(q) = \frac{q^{-2}}{(q^{15}; q^{15})_\infty [q^2; q^3]^2_\infty} (S(-3, -34, 15) - q^{34} S(14, 34, 15))
+ \frac{-1 + 3(\zeta_5 + \zeta^{-1}_5)}{5} \left( \frac{q^3}{[q^3; q^4]^2_\infty} \right) - (\zeta_5 + \zeta^{-1}_5) q \left( \frac{q^{15}; q^{15}}{[q^4; q^4; q^5; q^5]^2_\infty} \right),
\]

\[
A_{254}(q) = \frac{q^2}{(q^{15}; q^{15})_\infty [q^4; q^5]^2_\infty} (S(3, -8, 15) - q^8 S(7, 8, 15)) + \left( \frac{q^{15}; q^{15}}{[q^4; q^4; q^5; q^5]^2_\infty} \right).
\]

Theorem 1.5.

\[
\frac{1}{(q; q)_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+2n}}{(1 - \zeta_5 q^{3n})(1 - \zeta^{-1}_5 q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+4n+1}}{(1 - \zeta_5 q^{3n+1})(1 - \zeta^{-1}_5 q^{3n+1})} \right) = (q^{25}; q^{25})_\infty (A_{350}(q^5) + qA_{351}(q^5) + q^2A_{352}(q^5) + q^3A_{353}(q^5)) ,
\]

where

\[
A_{350}(q) = \frac{3 + \zeta_5 + \zeta^{-1}_5}{5} \left[ \frac{q^2}{[q^2; q^3]^2_\infty} \right] - q \left[ \frac{q^{15}}{[q^4; q^4; q^5; q^5]^2_\infty} \right],
\]

\[
A_{351}(q) = \frac{-2 + \zeta_5 + \zeta^{-1}_5}{5} \left[ \frac{1}{[q^3; q^3]^2_\infty} \right] - (\zeta_5 + \zeta^{-1}_5) q \left[ \frac{1}{[q^4; q^4; q^5; q^5]^2_\infty} \right],
\]

\[
A_{352}(q) = \frac{1 + 2(\zeta_5 + \zeta^{-1}_5)}{5} \left[ \frac{1}{[q^2; q^3]^2_\infty} \right] - (\zeta_5 + \zeta^{-1}_5) q \left[ \frac{q^{15}}{[q^4; q^4; q^5; q^5]^2_\infty} \right],
\]

\[
A_{353}(q) = \frac{-1 + 3(\zeta_5 + \zeta^{-1}_5)}{5} \left[ \frac{q^4}{[q^2; q^3]^2_\infty} \right] + (-1 + \zeta_5 + \zeta^{-1}_5) q \left[ \frac{q^{15}}{[q^4; q^4; q^5; q^5]^2_\infty} \right].
\]

Theorem 1.6.

\[
\frac{1}{(q; q)_\infty} \left( \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+n}}{(1 - \zeta_5 q^{3n})(1 - \zeta^{-1}_5 q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+7n+2}}{(1 - \zeta_5 q^{3n+1})(1 - \zeta^{-1}_5 q^{3n+1})} \right) = (q^{25}; q^{25})_\infty (A_{450}(q^5) + qA_{451}(q^5) + q^2A_{452}(q^5) + q^3A_{453}(q^5)) ,
\]

where

\[
A_{450}(q) = \frac{3 + \zeta_5 + \zeta^{-1}_5}{5} \left[ \frac{q^2}{[q^5]^2_\infty} \right] - q(1 + \zeta_5 + \zeta^{-1}_5) \left[ \frac{q^{15}}{[q^4; q^4; q^5; q^5]^2_\infty} \right].
\]
Theorem 1.7.

\[
\frac{1}{(q; q)_\infty}\left(\sum_{n=-\infty}^{\infty} \frac{q^{3n^2+2n}}{(1-\zeta_7q^{3n})(1-\zeta_7^{-1}q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+4n+1}}{(1-\zeta_7q^{3n+1})(1-\zeta_7^{-1}q^{3n+1})}\right) = \frac{(q^{49}; q^{49})}{(q; q)_\infty}\left( A_{370}(q^7) + qA_{371}(q^7) + q^2A_{372}(q^7) + q^3A_{373}(q^7) + q^4A_{374}(q^7) + q^5A_{375}(q^7) + q^6A_{376}(q^6) \right),
\]

where

\[
A_{370}(q) = \frac{6 + 3(\zeta_7 + \zeta_7^6) + (\zeta_7^2 + \zeta_7^5)}{7} \frac{[q^6; q^7]_\infty}{[q^6, q^7, q^8; q^{21}]_\infty} - q^2(1 + \zeta^6) \frac{[q^6, q^7, q^{10}; q^{21}]_\infty}{[q^6, q^7, q^8; q^{21}]_\infty},
\]

\[
A_{371}(q) = \frac{1 - 3(\zeta_7 + \zeta_7^6) + (\zeta_7^2 + \zeta_7^5)}{7} \frac{1}{[q; q^7]_\infty},
\]

\[
A_{372}(q) = \frac{5 - (\zeta_7 + \zeta_7^6) + 2(\zeta_7^2 + \zeta_7^5)}{7} \frac{[q^2; q^7]_\infty}{[q^2, q^4; q^{10}]_\infty} - q(1 + \zeta^2 + \zeta^5) \frac{1}{[q^4, q^7, q^{10}; q^{21}]_\infty},
\]

\[
A_{373}(q) = \frac{-3 + 2(\zeta_7 + \zeta_7^6) - 4(\zeta_7^2 + \zeta_7^5)}{7} \frac{1}{[q^4, q^7]_\infty},
\]

\[
A_{374}(q) = \frac{2 + (\zeta_7 + \zeta_7^6) + 5(\zeta_7^2 + \zeta_7^5)}{7} \frac{1}{[q^4; q^7]_\infty},
\]

\[
A_{375}(q) = \frac{(\zeta + \zeta^6)}{7} \frac{\left[q^5, q^{21}\right]_\infty}{\left[q^5, q^7, q^8, q^{10}, q^{21}\right]_\infty} + q^2(\zeta + \zeta^6) \frac{\left[q^2, q^{21}\right]_\infty}{\left[q^2, q^5, q^7, q^{10}, q^{21}\right]_\infty},
\]

\[
A_{376}(q) = \frac{-4 - 2(\zeta_7 + \zeta_7^6) - 3(\zeta_7^2 + \zeta_7^5)}{7} \frac{1}{\left[q^2, q^5; q^7\right]_\infty} + \frac{1 + (1 + \zeta_7^2 + \zeta_7^5)}{\left[q^4, q^7, q^8; q^{21}\right]_\infty}.
\]

We recognize \(\frac{(q;q)_\infty}{(\zeta_3 q; \zeta_3^{-1} q)_\infty}\) as the generating function for the crank of partitions. The dissections for the crank at roots of unity are well known by the work of Garvan [12]. We start with \(\zeta_3\). Since

\[
\frac{1}{(\zeta_3 q; \zeta_3^{-1} q)_\infty} = \frac{(q;q)_\infty}{(q^3; q^7)_\infty},
\]

and by Euler's Pentagonal Numbers Theorem along with the Jacobi Triple Product Identity

\[
(q; q)_\infty = (q^{27}; q^{27})_\infty \left( [q^{12}; q^{27}]_\infty - q [q^6; q^{27}]_\infty - q^2 [q^3; q^{27}]_\infty \right),
\]

we have

\[
\frac{(q; q)_\infty}{(\zeta_3 q; \zeta_3^{-1} q; q)_\infty} = \frac{(q^{27}; q^{27})^2}{(q^3; q^7)^2} \left( [q^{12}; q^{27}]^2_\infty + 2q^3 [q^3, q^6, q^{27}]_\infty - 2q [q^6, q^{12}, q^{27}]_\infty + q^4 [q^3, q^{27}]^2_\infty - 2q^2 [q^3, q^{12}, q^{27}]_\infty + q^2 [q^6, q^{27}]^2_\infty \right).
\]

As in (3.8) and Theorem 5.1 of [12] we have

\[
\frac{(q; q)_\infty}{(\zeta_5 q; \zeta_5^{-1} q; q)_\infty} = (q^{25}; q^{25}) \left( \frac{[q^{10}; q^{25}]_\infty}{[q^5; q^{25}])^2_\infty} + (\zeta_5 + \zeta_5^4 - 1)q \frac{1}{[q^5; q^{25}]_\infty} - (\zeta_5 + \zeta_5^4 + 1)q^2 \frac{1}{[q^{10}; q^{25}]_\infty} \
- (\zeta_5 + \zeta_5^4)q^3 \frac{[q^5; q^{25}]_\infty}{[q^{10}; q^{25}]_\infty},
\]

\[
\frac{(q; q)_\infty}{(\zeta_7 q; \zeta_7^{-1} q; q)_\infty} = (q^{49}; q^{49}) \left( \frac{[q^{21}; q^{49}]_\infty}{[q^7, q^{14}, q^{49}]_\infty} + (\zeta_7 + \zeta_7^6 - 1)q \frac{1}{[q^7; q^{49}]_\infty} + (\zeta_7 + \zeta_7^6)q^2 \frac{[q^{14}; q^{49}]_\infty}{[q^7, q^{21}; q^{49}]_\infty}
\right).
Theorem 1.12.

Remark 1.9. The congruence \( \text{pp}_1(3n) \equiv 0 \pmod{3} \) then follows as \( \text{PP}_1(\zeta_3, q) \) has no \( q^{3n} \) terms.

Theorem 1.10.

Remark 1.11. The congruence \( \text{pp}_2(3n + 1) \equiv 0 \pmod{3} \) then follows as \( \text{PP}_2(\zeta_3, q) \) has no \( q^{3n+1} \) terms.

Theorem 1.12.

Remark 1.13. The congruence \( \text{pp}_2(5n + 1) \equiv 0 \pmod{5} \) then follows as \( \text{PP}_2(\zeta_5, q) \) has no \( q^{5n+1} \) terms.

Theorem 1.14.

\[(\zeta_3^2 + \zeta_3^4 + 1)q^3 \left[ \frac{1}{q^{14}; q^{48}} \right] - (\zeta_7 + \zeta_7^6)q^4 \left[ \frac{1}{q^{21}; q^{49}} \right] - (\zeta_5^2 + \zeta_5^4 + 1)q^6 \left[ \frac{q^7; q^{49}}{q^{14}; q^{21}; q^{49}} \right].
\]

Dividing (1.11) by \((1 - \zeta_3)(1 - \zeta_3^{-1})\), (1.12) by \((1 - \zeta_5)(1 - \zeta_5^{-1})\), and (1.13) by \((1 - \zeta_7)(1 - \zeta_7^{-1})\) along with Theorems 1.2 through 1.7 gives the following formulas for \( \text{PP}_1(z, q) \), \( \text{PP}_2(z, q) \), \( \text{PP}_3(z, q) \), and \( \text{PP}_4(z, q) \).
Remark 1.15. The congruence \( pp_3(5n + 4) \equiv 0 \pmod{5} \) then follows as \( PP_3(\zeta_5, q) \) has no \( q^{5n+4} \) terms.

Theorem 1.16.

\[
PP_4(\zeta_5, q) = (q^{25}; q^{25})_\infty \left( -(1 + \zeta_5 + \zeta_5^4)q^5 \frac{\left[ q^{10}, q^{75} \right]_\infty}{\left[ q^{5}; q^{25}, q^{30}; q^{75} \right]_\infty} + \frac{q}{3 \left[ q^{10}, q^{15}, q^{25}; q^{75} \right]_\infty} \right.
+ (1 + \zeta_5 + \zeta_5^4) \frac{q^2}{\left[ q^{10}; q^{25} \right]_\infty} - \frac{q^2}{\left[ q^5, q^{25}, q^{30}; q^{75} \right]_\infty} - (\zeta_5 + \zeta_5^4)q^8 \frac{\left[ q^{5}; q^{75} \right]_\infty}{\left[ q^{10}, q^{15}, q^{25}, q^{35}; q^{75} \right]_\infty}.
\]

Remark 1.17. The congruence \( pp_4(5n + 4) \equiv 0 \pmod{5} \) then follows as \( PP_4(\zeta_5, q) \) has no \( q^{5n+4} \) terms.

Theorem 1.18.

\[
PP_3(\zeta_7, q) = (q^{49}; q^{49})_\infty \left( -(1 + \zeta_7 + \zeta_7^6)q^9 \frac{q^{14}}{\left[ q^{42}, q^{49}, q^{56}, q^{147} \right]_\infty} + q^2 \frac{\left[ q^{14}; q^{49} \right]_\infty}{\left[ q^7, q^{21}; q^{49} \right]_\infty} \right.
+ (1 + \zeta_7 + \zeta_7^2 + \zeta_7^5 + \zeta_7^6) \frac{q^4}{\left[ q^{15}; q^{49}, q^{49}; q^{147} \right]_\infty} \right.
+ (\zeta_7 + \zeta_7^3)q^5 \frac{q^3}{\left[ q^{14}; q^{147} \right]_\infty} + (1 + \zeta_7 + \zeta_7^2 + \zeta_7^5 + \zeta_7^6) \frac{q^4}{\left[ q^{21}; q^{49} \right]_\infty} \right.
+ (2 + \zeta_7^2 + \zeta_7^6) \frac{q^6}{\left[ q^{14}, q^{49}, q^{53}; q^{47} \right]_\infty}.
\]

Remark 1.19. The congruence \( pp_3(7n + 1) \equiv 0 \pmod{7} \) then follows as \( PP_3(\zeta_7, q) \) has no \( q^{7n+1} \) terms.

The rest of the paper is organized as follows. In section 2 we develop the necessary identities to eventually express the series in Theorems 1.12 through 1.17 in terms of products. In the sections thereafter we use these formulas to reduce the proofs to verifying an identity between products, which follows by checking that the equality holds in the first few terms of the \( q \)-expansions. In Section 9 we interpret the coefficients \( M_i(m, n) \) in terms of cranks defined on the partition pairs counted by \( pp_i \). We then end with a few remarks and questions.

2. Preliminary Identities

Although we are only concerned with the cases \( \ell = 3, 5 \) and 7, all of the formulas we state and prove in this section are valid for all odd \( \ell > 1 \). To begin we define

\[
U_\ell^n(a, b) = \sum_{n=-\infty}^{\infty} \frac{q^{an^2 + bn}}{\left( q^\ell \right)^{3n+1}},
\]

\[
V_\ell^n(a, b) = \sum_{n \neq 0} \frac{q^{an^2 + bn}}{\left( q^\ell \right)^n}.
\]

We express our series in terms of \( U_\ell^n(a, b) \) and \( V_\ell^n(a, b) \) by using the fact that

\[
1 - x^\ell = \prod_{j=0}^{\ell-1} (1 - \zeta_\ell^j x).
\]

With \( \zeta_3 \) a primitive third root of unity we have

\[
\sum_{n=-\infty}^{\infty} \frac{q^{6n^2 + 2n}}{(1 - \zeta_3 q^{3n})(1 - \zeta_3^{-1} q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{6n^2 + 8n + 2}}{(1 - \zeta_3 q^{3n+1})(1 - \zeta_3^{-1} q^{3n+1})} = \frac{1}{3} + V_3^6(2) - V_3^6(5) - q^2 U_3^6(8) + q^3 U_3^6(11),
\]

(2.1)
the $1/3$ is from the $n = 0$ term. We also have

$$
\sum_{n=-\infty}^{\infty} \frac{q^{6n^2+n}}{(1 - \zeta_5 q^{3n})(1 - \zeta_5^{-1} q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{6n^2+5n+1}}{(1 - \zeta_5 q^{3n+1})(1 - \zeta_5^{-1} q^{3n+1})}
= \frac{1}{3} + V_5^q(1) - V_5^q(4) - qU_5^q(5) + q^2T_5^q(8).
$$

(2.2)

With $\zeta_5$ a primitive fifth root of unity we find

$$
\sum_{n=-\infty}^{\infty} \frac{q^{6n^2+n}}{(1 - \zeta_5 q^{3n})(1 - \zeta_5^{-1} q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{6n^2+5n+1}}{(1 - \zeta_5 q^{3n+1})(1 - \zeta_5^{-1} q^{3n+1})}
= \frac{1}{(1 - \zeta_5)(1 - \zeta_5^4)} + V_5^q(1) + (\zeta_5 + \zeta_5^4)V_5^q(4) - (\zeta_5 + \zeta_5^4)V_5^q(7) - V_5^q(10) - qU_5^q(5) - (\zeta_5 + \zeta_5^4)q^2U_5^q(8)
$$

$$
+ (\zeta_5 + \zeta_5^4)q^3U_5^q(11) + q^4U_5^q(14),
$$

(2.3)

$$
\sum_{n=-\infty}^{\infty} \frac{q^{3n^2+2n}}{(1 - \zeta_5 q^{3n})(1 - \zeta_5^{-1} q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+4n+1}}{(1 - \zeta_5 q^{3n+1})(1 - \zeta_5^{-1} q^{3n+1})}
= \frac{1}{(1 - \zeta_5)(1 - \zeta_5^4)} + V_5^q(2) + (\zeta_5 + \zeta_5^4)V_5^q(5) - (\zeta_5 + \zeta_5^4)V_5^q(8) - V_5^q(11) - qU_5^q(4) - (\zeta_5 + \zeta_5^4)q^2U_5^q(7)
$$

$$
+ (\zeta_5 + \zeta_5^4)q^3U_5^q(10) + q^4U_5^q(13),
$$

(2.4)

$$
\sum_{n=-\infty}^{\infty} \frac{q^{3n^2+n}}{(1 - \zeta_5 q^{3n})(1 - \zeta_5^{-1} q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+7n+2}}{(1 - \zeta_5 q^{3n+1})(1 - \zeta_5^{-1} q^{3n+1})}
= \frac{1}{(1 - \zeta_5)(1 - \zeta_5^4)} + V_5^q(1) + (\zeta_5 + \zeta_5^4)V_5^q(4) - (\zeta_5 + \zeta_5^4)V_5^q(7) - V_5^q(10) - q^2U_5^q(7) - (\zeta_5 + \zeta_5^4)q^3U_5^q(10)
$$

$$
+ (\zeta_5 + \zeta_5^4)q^4U_5^q(13) + q^5U_5^q(16).
$$

(2.5)

Lastly with $\zeta_7$ a primitive seventh root of unity, we find

$$
\sum_{n=-\infty}^{\infty} \frac{q^{3n^2+2n}}{(1 - \zeta_7 q^{3n})(1 - \zeta_7^{-1} q^{3n})} - \sum_{n=-\infty}^{\infty} \frac{q^{3n^2+4n+1}}{(1 - \zeta_7 q^{3n+1})(1 - \zeta_7^{-1} q^{3n+1})}
= \frac{1}{(1 - \zeta_7)(1 - \zeta_7^6)} + V_7^q(2) + (\zeta_7 + \zeta_7^6)V_7^q(5) + (1 + \zeta_7^2 + \zeta_7^5)V_7^q(8) - (1 + \zeta_7^2 + \zeta_7^5)V_7^q(11)
$$

$$
- (\zeta_7 + \zeta_7^6)V_7^q(14) - V_7^q(17) - qU_7^q(4) - (\zeta_7 + \zeta_7^6)q^2U_7^q(7) - (1 + \zeta_7^2 + \zeta_7^5)q^3U_7^q(10)
$$

$$
+ (1 + \zeta_7^2 + \zeta_7^5)q^4U_7^q(13) + (\zeta_7 + \zeta_7^6)q^5U_7^q(16) + q^6U_7^q(19).
$$

(2.6)

Rearranging the $S$ and $T$ series with $n \mapsto -n$ and $n \mapsto n+1$ we have

$$
S(z, w, q) = -z^{-1}S(1, z^{-1}w^{-1}q^{-3}, q),
$$

(2.7)

$$
S(z, w, q) = wz^3S(zq, wz^4 q),
$$

(2.8)

$$
S(z, w, q) = -z^{-1}w^{-1}qS(z^{-1}q, w^{-1}q, q),
$$

(2.9)

$$
S^+(w, q) = -S^+(w^{-1}q^{-3}, q),
$$

(2.10)

$$
T(z, w, q) = -z^{-1}T(z^{-1}, w^{-1}q^{-1}, q),
$$

(2.11)

$$
T(z, w, q) = wz^2T(zq, wz^2 q),
$$

(2.12)

$$
T(z, w, q) = -z^{-1}w^{-1}qT(z^{-1}q, w^{-1}q, q),
$$

(2.13)

$$
T^+(w, q) = -T^+(w^{-1}q^{-1}, q).
$$

(2.14)

We will also often rearrange infinite products without mention by

$$
[z: q]_\infty = -z [zq: q]_\infty = -z [z^{-1}: q]_\infty.
$$

To prove the dissection formulas, we first find general formulas that express certain differences of $V^q_5(b)$ and $U^q_5(b)$ primarily in terms of products. For this we express $V^q_5(b)$ and $U^q_5(b)$ in terms of $S(z, w, q)$ and $V^q_5(b)$
Lemma 2.3. For any integer $b$, as each $V_g$ and similarly do so for certain combinations of $T(z, w, q)$. The main tools for this are two specializations of Theorem 2.1 of [3].

For $PP_1$ and $PP_2$ we use $r = 0$ and $s = 4$ to get the following identity.

Lemma 2.1.

\[
\frac{(q; q)_\infty^2}{[b_1, b_2, b_3, b_4; q]_\infty} = \frac{1}{[b_2/b_1, b_3/b_1, b_4/b_1; q]_\infty} S \left( b_1, \frac{b_3^3}{b_2 b_3 b_4}, q \right) + \frac{1}{[b_1/b_2, b_3/b_2, b_4/b_2; q]_\infty} S \left( b_2, \frac{b_3^3}{b_1 b_3 b_4}, q \right) + \frac{1}{[b_1/b_2, b_3/b_2, b_4/b_3; q]_\infty} S \left( b_3, \frac{b_3^3}{b_1 b_2 b_4}, q \right) + \frac{1}{[b_1/b_2, b_3/b_2, b_4/b_3; q]_\infty} S \left( b_4, \frac{b_3^3}{b_1 b_2 b_3}, q \right).
\]

Next we define

\[
g(z, q) = -z^2 S \left( \frac{z^2}{q^3}, q \right) + \frac{[z; q]_\infty [z^6; q]_\infty}{[z^3; q]_\infty} S \left( \frac{z^2}{q^3}, q \right) + \frac{1}{[w^2 z^{-1}, w^2 z^{-1}; q]_\infty} S \left( \frac{z^2}{q^3}, q \right) + \sum_{n \neq 0} \frac{q^{2n(n+1)} z^{-2n}}{1 - q^n}.
\]

The definition of $g(z, q)$ is motivated as follows. We would like to set one of the $b_i$ in Lemma 2.1 equal to 1, as each $V_i^q(b)$ will contribute a $S^*(w, q)$. For this we let $b_1 = w$, $b_2 = zw$, $b_3 = z/w$, and $b_4 = 1/w$, multiply both sides by the product $[w^2 z^{-1}, w^2 z^{-1}; q]_\infty$, subtract the $n = 0$ term from $S(z w^{-1}, z^2 w^4, q)$, and let $w \to z$. In particular, we also have

\[
g(z, q) = \lim_{w \to z} \left( \frac{w^2 z^{-1}, w^2 z^{-1}; q}{w^2 w^{-1}, w^2 w^{-1}; q} - \frac{1}{1 - z/w} \right).
\]

For $PP_3$ and $PP_4$ we use $r = 0$ and $s = 2$ and simplify to get the following identity.

Lemma 2.2.

\[
\frac{(q; q)_\infty^2}{[b_1, b_2; q]_\infty} = T(b_1, b_1 b_2^{-1}, q) - b_2 b_1^{-1} T(b_2, b_2 b_1^{-1}, q).
\]

We define

\[
h(z, q) = T^*(z^{-1}, q) - zT(z, q).
\]

We note that this function arises from the the right hand side of the equation in Lemma 2.2 by removing the $n = 0$ term of $T(b_1, b_1 b_2^{-1}, q)$, setting $b_2 = z$, and letting $b_1 \to 1$.

Lemma 2.3. For any integer $b$,

\[
V_\ell^q(3b + 2) - q^{b+1} U_\ell^q(3b + 4)
\]

\[
= h(\ell(3\ell - 3b - 2), 3\ell^2) + \sum_{k=1}^{\ell-1} q^{3k^2 + 2k + 3bk} \left( \frac{q^{3\ell^2}; q^{3\ell^2}}{q^{3\ell k}, q^{(3\ell - 3k - 2)}} \right)_\infty - \sum_{n \neq 0} q^{3n^2 + (3b + 4)n}.
\]

Proof. We have

\[
V_\ell^q(3b + 2) - q^{b+1} U_\ell^q(3b + 4)
\]

\[
= \sum_{n \neq 0} \frac{q^{3n^2 + (3b + 2)n}}{1 - q^{3n^2}} - q^{b+1} \sum_{n \neq 0} \frac{q^{3n^2 + (3b + 4)n}}{1 - q^{3n^2 + 3k}}
\]

\[
= \sum_{k=0}^{\ell-1} q^{3k^2 + 2k + 3bk} \sum_{n \neq 0} \frac{q^{3n^2 + (n+1)^2 - 3\ell^2 + 3bn + 6\ell kn + 2\ell n}}{1 - q^{3n^2 + 3\ell k}}
\]

\[
- \sum_{k=0}^{\ell-1} q^{3k^2 + 2k + 3bk - 6\ell k - 3\ell b - 2\ell + 3\ell^2} \sum_{n \neq 0} \frac{q^{3n^2 + 6\ell^2 n - 3\ell bn - 6\ell kn - 2\ell n}}{1 - q^{3n^2 - 3\ell b + 3\ell^2 - 3\ell - 3\ell k}}.
\]
\[= \sum_{k=0}^{\ell-1} q^{3k^2+2k+3bk} T(3\ell k, \ell(3b + 6k + 2 - 3\ell), 3\ell^2) \]
\[= h(\ell(3\ell - 3b - 2), 3\ell^2) + \sum_{k=1}^{\ell-1} q^{3k^2+2k+3bk} \left( \frac{q^{3\ell^2}; q^{3\ell^2}}{q^{3\ell k}, q^{(3\ell - 3b - 2) k}; q^{3\ell^2}} \right)_\infty \]
\[= h(\ell(3\ell - 3b - 2), 3\ell^2) + \sum_{k=1}^{\ell-1} q^{3k^2+2k+3bk} \left( \frac{q^{3\ell^2}; q^{3\ell^2}}{q^{3\ell k}, q^{(3\ell - 3b - 2) k}; q^{3\ell^2}} \right)_\infty . \]

In \( V^3_r(3b + 2) \) we have replaced \( n \) by \( \ell n + k \) in \( U^3_r(3b + 4) \) we have replaced \( n \) by \( \ell n - b - k - \ell - 1 \). The last equality follows by Lemma 2.2.

**Lemma 2.4.** For any integer \( b \),
\[ V^6_r(6b + 1) - q^{2b+1} U^6_r(6b + 5) \]
\[= -g(q^{3\ell^2}; q^{3\ell^2}) - \sum_{k=2}^{\ell-1} q^{6bk+6k^2+k-3kl} \left( \frac{q^{3\ell^2}; q^{3\ell^2}}{q^{3\ell k}, q^{3\ell^2} k q^{3\ell^2}; q^{3\ell^2}} \right)_\infty \]
\[+ q^{3\ell^2+k+3bk} \left( \frac{q^{3\ell^2}; q^{3\ell^2}}{q^{3\ell^2} k q^{3\ell^2}; q^{3\ell^2}} \right)_\infty \]
\[+ (-1)^{b+1} q^{3\ell^2+k-3\ell} \left( \frac{q^{3\ell^2}; q^{3\ell^2}}{q^{3\ell^2} k q^{3\ell^2}; q^{3\ell^2}} \right)_\infty \]
\[\times \left( S(-3\ell,-3\ell^2-6\ell b-13\ell,3\ell^2) - q^{3\ell^2+6b+13\ell} S \left( \frac{3\ell^2 + 7\ell}{2} + 3b\ell, 3\ell^2 + 6\ell b + 13\ell, 3\ell^2 \right) \right) . \]

*Proof.* The proof is similar to Lemma 2.3, but a little more involved. In \( V^6_r(6b + 1) \) we use \( n \to \ell n + k \) and rearrange the terms by \( 2.27 \) and \( 2.10 \); in \( U^6_r(6b + 5) \) we use \( n \to \ell n - b - k + \frac{\ell-1}{3} \) and rearrange the terms by \( 2.9 \). We isolate the \( k = 0 \) and \( k = 1 \) summands and apply Lemma 2.1 so that we only have products and a term involving the \( k = 1 \) summand.

\[ V^6_r(6b + 1) - q^{2b+1} U^6_r(6b + 5) \]
\[= \sum_{n \geq 0} q^{6bn^2+(6b+1)n} - q^{2b+1} \sum_{n=-\infty}^{\infty} q^{6bn^2+(6b+5)n} \]
\[= \sum_{k=0}^{\ell-1} q^{6bk+6k^2+k} S(3\ell k,-6\ell^2+6\ell b+12k\ell+\ell,3\ell^2) \]
\[\quad - \sum_{k=0}^{\ell-1} q^{3\ell^2-\ell+k+6bk+6k^2-6k\ell-3\ell^2} S(-3k\ell-3b\ell+\frac{3\ell^2-\ell}{2},-6\ell b-12k\ell-\ell,3\ell^2) \]
\[= - \sum_{k=0}^{\ell-1} q^{6bk+6k^2+k-3k\ell} S(-3k\ell,-3\ell^2-6\ell b-12k\ell-\ell,3\ell^2) \]
\[\quad + \sum_{k=0}^{\ell-1} q^{3\ell^2+6b\ell+9k\ell+6bk+6k^2+k} S \left( \frac{3\ell^2 + \ell}{2} + 3b\ell, 3\ell^2 + 6\ell b + 12k\ell + \ell, 3\ell^2 + 6b\ell + 13\ell, 3\ell^2 \right) \]
\[= -q^{6b+7-3\ell} S(-3\ell^2-6\ell b-\ell,3\ell^2) + q^{3\ell^2+6b\ell+\ell} S \left( \frac{3\ell^2 + \ell}{2} + 3b\ell, 3\ell^2 + 6b\ell + 13\ell, 3\ell^2+6b\ell+13\ell, 3\ell^2 \right) \]
Applying Lemma 2.1 with \( q \mapsto q^{3\ell^2} \), \( b_1 = q^{-3\ell^2} \), \( b_2 = \frac{q^{3\ell^2+6\ell+3b\ell}}{q^{3\ell^2+6\ell+3b\ell}} \), \( b_3 = q^{-3k\ell^2} \), \( b_4 = \frac{q^{3\ell^2+6k\ell+3b\ell}}{q^{3\ell^2+6k\ell+3b\ell}} \) and simplifying gives

\[
-S \left( -3\ell^2 - 6\ell b - 12k\ell - \ell, 3\ell^2 \right) + q^{3\ell^2+6\ell+12k\ell+\ell} S \left( \frac{3\ell^2 + \ell}{2} + 3k\ell + 3b\ell, 3\ell^2 + 6\ell b + 12k\ell + \ell, 3\ell^2 \right)
\]

We would like to add in the terms to see \( g(\frac{3\ell^2+6\ell+3b\ell}{q^{3\ell^2}}, q^{3\ell^2}) \) and apply Lemma 2.1 again, however these terms are not well defined when \( 6b \equiv -1 \pmod{\ell} \). Instead we first apply Lemma 2.1 with \( q \mapsto q^{3\ell^2} \), \( b_2 = -q^{-3\ell^2-3\ell} \), \( b_3 = zq^{3\ell^2} \), \( b_4 = z^{-2}q^{3\ell^2} \) and simplify to obtain

\[
-S \left( -3\ell^2 - 6\ell b - 13\ell^2, 3\ell^2 \right) + q^{3k+3\ell^2+6b\ell+10} S \left( \frac{3\ell^2 + 7\ell}{2} + 3b\ell, 3\ell^2 + 6b\ell + 13\ell, 3\ell^2 \right)
\]

Thus by the definition of \( g(z, q^{3\ell^2}) \) we have

\[
-S^* \left( -2, q^{3\ell^2} \right) + z^2 S \left( z, z^2, q^{3\ell^2} \right)
\]

and here it is now valid to set \( z = \frac{q^{3\ell^2+6\ell+3b\ell}}{q^{3\ell^2}} \). Thus

\[
V_\ell^b(6b+1) - q^{2b+1} V_\ell^b(6b+5)
\]

\[
= -g \left( \frac{q^{3\ell^2+6\ell+3b\ell}}{z}, q^{3\ell^2} \right) - \sum_{k=2}^{\ell-1} q^{0b+k+6k^2+k-3k\ell} \left[ q^{3k\ell-3\ell^2} \frac{q^{3\ell^2+6k\ell+3b\ell}}{q^{3\ell^2+6k\ell+3b\ell}} \right] \left[ z, q^{3\ell^2} \right] \left[ q^{3\ell^2+6k\ell+3b\ell} \right]
\]
\[ + \frac{q^{2\ell+\ell} + 3b\ell}{q^{3\ell^2} ; q^{3\ell^2}} \cdot \left( q^{-6b\ell-4\ell} \right)^{\infty} \left[ q^{-6b\ell-\ell} ; q^{3\ell^2} \right] \cdot \boxed{q^{3\ell^2} ; q^{3\ell^2} ; q^{3\ell^2} ; q^{3\ell^2} ; q^{3\ell^2}} \]

+ \left( S(-3\ell, -3\ell^2 - 6b\ell - 13\ell, 3\ell^2) - q^{3\ell^2 + 6b\ell + 13\ell} S \left( \frac{3\ell^2 + 7\ell}{2} + 3b\ell, 3\ell^2 + 6b\ell + 13\ell, 3\ell^2 \right) \right)

where

\[ X = -q^{6b + 7 - 3\ell} - q^{-3\ell} \cdot \left( q^{3\ell^2 + 3b\ell} ; q^{3\ell^2} \right)_{\infty} - \sum_{k=2}^{\ell-1} q^{6b + 6k^2 + k - 3\ell} \cdot \left( q^{\frac{3\ell^2 + 3b\ell}{2} + 3b\ell} ; q^{3\ell^2} \right)_{\infty}. \]

We note in fact

\[ X = -\sum_{k=0}^{\ell-1} q^{6b + 6k^2 + k - 3\ell} \cdot \left( q^{\frac{3\ell^2 + 3b\ell}{2} + 3b\ell} ; q^{3\ell^2} \right)_{\infty}. \]

But by Euler’s Pentagonal Numbers Theorem and the Jacobi Triple Product identity we have

\[ (q; q)_{\infty} = \sum_{n=-\infty}^{\infty} (-1)^n \frac{(n+1)}{n} \]

\[ = \frac{\ell - 1}{\ell} \cdot \sum_{n=-\infty}^{\infty} (-1)^{\ell n + 2k + b} \cdot \frac{\ell n + 2k + b}{2} \cdot \frac{\ell n + 2k + b + 1}{2} \]

\[ = (-1)^b \cdot \sum_{k=0}^{\ell-1} q^{\frac{3k^2 + 1}{2}} \cdot \sum_{n=-\infty}^{\infty} (-1)^n q^{3b n + 6k + 1} \cdot \sum_{n=-\infty}^{\infty} \left( q^{3\ell^2} ; q^{3\ell^2} \right)_{\infty} = (-1)^b \cdot \sum_{n=-\infty}^{\infty} q^{3b n + 6k + 1} \cdot \sum_{n=-\infty}^{\infty} \left( q^{3\ell^2} ; q^{3\ell^2} \right)_{\infty}.

Thus

\[ X = (-1)^{b+1} \cdot q^{-\frac{3k^2 + 1}{2} - 3\ell} \cdot \left( q^{3\ell^2} ; q^{3\ell^2} \right)_{\infty} \cdot \left( q^{3\ell^2} ; q^{3\ell^2} \right)_{\infty} \cdot \boxed{q^{3\ell^2} ; q^{3\ell^2} ; q^{3\ell^2} ; q^{3\ell^2}}.

\]

Lemma 2.5. For any integer \( b \),

\[ V_{\ell}^{b}(6b + 4) - q^{2b+1} V_{\ell}^{b}(6b + 8) \]

\[ = \frac{g(3\ell^2 - 3b\ell - 2\ell, q^{3\ell^2}) + \sum_{k=2}^{\ell} q^{6k^2 + 4k + 6k} \cdot \left( q^{3\ell^2} ; q^{3\ell^2} \right)_{\infty}^{2} \cdot \left[ q^{-3k\ell + 3\ell} ; q^{3\ell^2 - 3b\ell - 3b\ell - 5\ell} ; q^{3\ell^2 - 3b\ell - 6k\ell - 2\ell} ; q^{3\ell^2} \right]_{\infty}}{\left[ q^{3\ell^2} ; q^{3\ell^2 - 3b\ell - 5\ell} ; q^{3b\ell} ; q^{3\ell^2 - 3b\ell - 3b\ell - 2\ell} ; q^{3\ell^2} \right]_{\infty}} \]

\[ - \frac{\left( q^{3\ell^2} ; q^{3\ell^2} \right)_{\infty}^{2} \cdot \left[ q^{3\ell^2 - 3b\ell - 5\ell} ; q^{3\ell^2} ; q^{3\ell^2 - 6b\ell - 7\ell} ; q^{3\ell^2} \right]_{\infty}}{\left[ q^{3\ell^2} ; q^{3\ell^2 - 3b\ell - 5\ell} ; q^{3\ell^2} ; q^{3\ell^2 - 6b\ell - 4\ell} ; q^{3\ell^2} \right]_{\infty}} \]

\[ + (-1)^{\ell+1} \cdot q^{4b^2 + 36b^2 - 3b - 3b^2 + 5} \cdot \left( q^{3\ell^2} ; q^{3\ell^2} \right)_{\infty}^{2} \cdot \left[ q^{3b\ell + 8\ell} ; q^{3\ell^2} \right]_{\infty} \cdot \boxed{q^{3b\ell + 8\ell} ; q^{3\ell^2} ; q^{3\ell^2} ; q^{3\ell^2}} \]

\[ \times \left( S(3\ell, 6b + 16\ell - 6b^2, 3\ell^2) - q^{6\ell^2 - 6b\ell - 16\ell} \cdot S(3\ell^2 - 3b\ell - 5\ell, 6\ell^2 - 6b\ell - 16\ell, 3\ell^2) \right). \]

Proof. For \( V_{\ell}^{b}(6b + 4) \) we use \( n \mapsto \ell n + k \) and in \( U_{\ell}^{b}(6b + 8) \) we use \( n \mapsto \ell n - k - \ell - b - 1. \)

\[ V_{\ell}^{b}(6b + 4) - q^{2b+1} U_{\ell}^{b}(6b + 8) \]
\[
= \sum_{n \neq 0} q^{6n^2 + (6b+4)n} \frac{1}{1 - q^{3n\ell}} - q^{2k+2} \sum_{n = -\infty}^{\ell-1} q^{6n^2 + (6b+8)n} \frac{1}{1 - q^{3n\ell + \ell}} \\
= \sum_{k=0}^{\ell-1} q^{6k^2 + 4k + 6bk} S (3k\ell, 6b\ell + 12k\ell + 4\ell - 6\ell^2, 3\ell^2) \\
- \sum_{k=0}^{\ell-1} q^{6\ell^2 - 6b\ell - 12k\ell + 6bk^2 - 4\ell + 4k} S (-3k\ell + 3\ell^2 - 3b\ell - 2\ell, 6\ell^2 - 6\ell - 12k\ell - 4\ell, 3\ell^2) \\
= S^* (6b\ell + 4\ell - 6\ell^2, 3\ell^2) - q^{6\ell^2 - 6b\ell - 4\ell} S (3\ell^2 - 3b\ell - 2\ell, 6\ell^2 - 6\ell - 4\ell, 3\ell^2) \\
+ q^{10 + 6b} S (3\ell, 6b\ell + 16\ell - 6\ell^2, 3\ell^2) - q^{10 + 6b + 6\ell^2 - 6b\ell - 16\ell} S (3\ell^2 - 3b\ell - 5\ell, 6\ell^2 - 6\ell - 16\ell, 3\ell^2) \\
+ \sum_{k=2}^{\ell-1} q^{6k^2 + 4k + 6bk} S (3k\ell, 6b\ell + 12k\ell + 4\ell - 6\ell^2, 3\ell^2) \\
- \sum_{k=2}^{\ell-1} q^{6\ell^2 - 6b\ell - 12k\ell + 6bk^2 - 4\ell + 4k} S (-3k\ell + 3\ell^2 - 3b\ell - 2\ell, 6\ell^2 - 6\ell - 12k\ell - 4\ell, 3\ell^2) .
\]

We apply Lemma 2.2 with \( q \mapsto q^{3\ell^2}, b_1 = q^{3\ell^2}, b_2 = q^{3\ell^2 - 3b \ell - 5\ell}, b_3 = q^{3\ell^2}, b_4 = q^{3\ell^2 - 3k \ell - 3b \ell - 2\ell} \) and simplify to find that
\[
S (3k\ell, 6b\ell + 12k\ell + 4\ell - 6\ell^2, 3\ell^2) - q^{6\ell^2 - 6b\ell - 12k\ell - 4\ell} S (3\ell^2 - 3k\ell - 3b\ell - 2\ell, 6\ell^2 - 6\ell - 12k\ell - 4\ell, 3\ell^2) \\
= \left( \frac{q^{3\ell^2}; q^{3\ell^2}}{q^{3\ell^2}; q^{3\ell^2}} \right)^2 \left[ q^{-3k\ell + 3\ell}, q^{3\ell^2 - 3b\ell - 3k\ell - 5\ell}, q^{3\ell^2 - 3b\ell - 6k\ell - 2\ell}; q^{3\ell^2} \right]_{\infty} \\
+ q^{3\ell - 3k\ell} \left[ q^{3\ell^2 - 3b\ell - 5\ell}; q^{3\ell^2} \right]_{\infty} S (3\ell, 6b\ell + 16\ell - 6\ell^2, 3\ell^2) \\
- q^{6\ell^2 - 6b\ell - 3k\ell - 13\ell} \left[ q^{3\ell^2 - 3b\ell - 6k\ell - 2\ell}; q^{3\ell^2} \right]_{\infty} S (3\ell^2 - 3b\ell - 5\ell, 6\ell^2 - 6\ell - 16\ell, 3\ell^2) .
\]

This time there is an issue with \( g(q^{3\ell^2 - 3b\ell - 2\ell}, q^{3\ell^2}) \) when when \( 3b \equiv -2 \pmod{\ell} \). To avoid this we first apply Lemma 2.1 with \( q \mapsto q^{3\ell^2}, b_1 = q^{3\ell^2}, b_2 = zq^{-3\ell}, b_3 = z^{-1}, b_4 = z^2 \) and find that
\[
\left[ z; q^{3\ell^2} \right]_{\infty} S (z, z^6 q^{-9\ell}, q^{3\ell^2}) + z \left[ z; q^{3\ell^2} \right]_{\infty} S (z^2, z^6, q^{3\ell^2}) \\
= \left( \frac{q^{3\ell^2}; q^{3\ell^2}}{q^{3\ell^2}; q^{3\ell^2}} \right)^2 \left[ z^2 q^{-3\ell}, z^6 q^{3\ell^2}; q^{3\ell^2} \right]_{\infty} \\
- q^{3\ell} \left[ z; q^{3\ell^2} \right]_{\infty} S (z, z^2 q^{12\ell}, q^{3\ell^2}) + z^2 q^{-9\ell} \left[ z; q^{3\ell^2} \right]_{\infty} S (zq^{-3\ell}, z^2 q^{12\ell}, q^{3\ell^2}) .
\]

We then have
\[
S^* (z^{-2}, q^{3\ell^2}) - z^2 S (z, z^2, q^{3\ell^2}) \\
= g(z, q^{3\ell^2}) - \left( \frac{q^{3\ell^2}; q^{3\ell^2}}{q^{3\ell^2}; q^{3\ell^2}} \right)^2 \left[ z^2 q^{-3\ell}, z^6 q^{3\ell^2}; q^{3\ell^2} \right]_{\infty} \\
+ q^{3\ell} \left[ z; q^{3\ell^2} \right]_{\infty} S (z^2, z^2 q^{-12\ell}, q^{3\ell^2}) - z^2 q^{-12\ell} S (zq^{-3\ell}, z^{2} q^{-12\ell}, q^{3\ell^2})
\]
and here we can set \( z = q^{3\ell^2 - 3b\ell - 2\ell} \).

Thus

\[
V(6b + 4) - q^{2b+2}U(6b + 8)
= g(3\ell^2 - 3b\ell - 2\ell, q^{3\ell^2}) + \sum_{k=2}^{\ell-1} q^{6k^2+4k+6bk} \frac{\left(q^{3\ell^2}; q^{3\ell^2}\right)_\infty^2 \left[q^{-3k\ell+3\ell}, q^{3\ell^2-3b\ell-3k\ell-5\ell}; q^{3\ell^2-3b\ell-2\ell}; q^{3\ell^2}\right]_\infty}{\left[q^{3\ell^2}, q^{3\ell^2-3b\ell-5\ell}; q^{3\ell^2-3b\ell-2\ell}; q^{3\ell^2}\right]_\infty}
- \left(q^{3\ell^2}; q^{3\ell^2}\right)_\infty^2 \left[q^{3\ell^2-3b\ell+\ell}, q^{6\ell^2-6b\ell-7\ell}; q^{3\ell^2}\right]_\infty
+ Y \cdot \left(S(3\ell, 6b\ell + 16\ell - 6\ell^2, 3\ell^2) - q^{6\ell^2-6b\ell-16\ell} S(3\ell^2 - 3b\ell - 5\ell, 6\ell^2 - 6b\ell - 16\ell, 3\ell^2)\right)
\]

where

\[
Y = q^{10+6b} + q^{3\ell^2} \frac{q^{3\ell^2-3b\ell-2\ell}; q^{3\ell^2}}{q^{3\ell^2-3b\ell-8\ell}; q^{3\ell^2}} \frac{q^{3\ell^2-3b\ell+3\ell}; q^{3\ell^2}}{q^{3\ell^2+8\ell}; q^{3\ell^2}} + q^{10+6b} + q^{3\ell^2} \frac{q^{3\ell^2-3b\ell-2\ell}; q^{3\ell^2}}{q^{3\ell^2-3b\ell-8\ell}; q^{3\ell^2}} \frac{q^{3\ell^2-3b\ell+3\ell}; q^{3\ell^2}}{q^{3\ell^2+8\ell}; q^{3\ell^2}}.
\]

We note

\[
Y = \sum_{k=0}^{\ell-1} q^{6k^2+4k+6bk+3\ell} \frac{q^{3b\ell+6k+2\ell}; q^{3\ell^2}}{q^{3\ell^2+8\ell}; q^{3\ell^2}}.
\]

In Euler’s Pentagonal Numbers Theorem we replace \( n \) by \( \ell n - \frac{\ell+1}{2} + 2k + b \) to obtain

\[
(q; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)}
= (-1)^{\frac{\ell+1}{2} + b} q^{\ell^2 - 3\ell - 3b\ell + 2b + \frac{3\ell^2 + 5}{2} - 2b - \frac{3\ell^2 + 5}{2}} \frac{(q; q)_\infty}{(q^{3\ell^2}; q^{3\ell^2})_\infty} \frac{(q; q)_\infty}{(q^{3\ell^2}; q^{3\ell^2})_\infty}
\]

so that

\[
Y = (-1)^{\frac{\ell+1}{2} + b} q^{\ell^2 - 3\ell - 3b\ell + 2b + \frac{3\ell^2 + 5}{2} - 2b - \frac{3\ell^2 + 5}{2}} \frac{(q; q)_\infty}{(q^{3\ell^2}; q^{3\ell^2})_\infty} \frac{(q; q)_\infty}{(q^{3\ell^2}; q^{3\ell^2})_\infty}.
\]

Next we need identities for \( g(z, q) \) and \( h(z, q) \). From the limit definitions, we find that \( g(z, q) \) and \( h(z, q) \) are basically logarithmic derivatives of theta functions and are surprisingly related.

**Lemma 2.6.**

\[
g(z, q) = 1 - \sum_{n=0}^{\infty} \frac{z^2 q^n}{1 - z^2 q^n} + \sum_{n=1}^{\infty} \frac{z^{-2} q^n}{1 - z^{-2} q^n},
\]

\[
h(z, q) = - \sum_{n=0}^{\infty} \frac{z q^n}{1 - z q^n} + \sum_{n=1}^{\infty} \frac{z^{-1} q^n}{1 - z^{-1} q^n}.
\]

**Proof.** We have

\[
g(z, q) = \lim_{w \to z} \left( \left[\frac{w^2 z^{-1}, z^{-1}, w^2; q}{w, w^{-1}, z w, z w^{-1}; q}\right]_\infty (q; q)^2_\infty (q; q)^2_\infty \frac{1}{1 - z/w} - 1 \right)
= \lim_{w \to z} \left( \frac{\frac{w^2 z^{-1}, z^{-1}, w^2; q}{w, w^{-1}, z w; q}_\infty (z w^{-1} q, z^{-1} w q; q)_\infty}{1 - z/w} \right)
= \lim_{w \to z} \frac{\partial}{\partial w} \left[\frac{w^2 z^{-1}, z^{-1}, w^2; q}{w, w^{-1}, z w; q}_\infty (z w^{-1} q, z^{-1} w q; q)_\infty \right].
\]
We handle the partial derivative with a logarithmic derivative. We have

\[
\frac{[w, w^{-1}, zw; q]_\infty (zw^{-1}q, z^{-1}wq; q)_\infty}{[w^2 z^{-1}, z^{-1}, w^2; q]_\infty (q; q)_\infty^2} \frac{\partial}{\partial w} \frac{[w^2 z^{-1}, z^{-1}, w^2; q]_\infty (z^{-1}w^{-1}, q; q)_\infty^2}{[w, w^{-1}, zw; q]_\infty (zw^{-1}q, z^{-1}wq; q)_\infty}
= -2 \sum_{n=0}^\infty \frac{wz^{-1}q^n}{1-w^{-2}z^{-1}q^n} + 2 \sum_{n=0}^\infty \frac{w^{-3}zq^n}{1-w^{-2}zq^n} - 2 \sum_{n=0}^\infty \frac{wq^n}{1-w^{-2}zq^n} + 2 \sum_{n=0}^\infty \frac{w^{-3}zq^n}{1-w^{-1}z^{-1}q^n} + \sum_{n=0}^\infty \frac{q^n}{1-wq^n} - \sum_{n=1}^\infty \frac{w^{-2}z^{-1}q^n}{1-w^{-1}z^{-1}q^n} - \sum_{n=1}^\infty \frac{w^{-2}zq^n}{1-w^{-1}zq^n} + \sum_{n=1}^\infty \frac{z^{-1}q^n}{1-wzq^n}.
\]

Multiplying by \(z\), letting \(w \to z\), and simplifying then gives the result. The proof for \(h(z, q)\) is similar. \(\square\)

Here we see taking \(b_4 = 1/b_1\) gives nice cancellations. Since \(h(z, q) = g(z^{1/2}, q) - 1\), we see we need only prove identities for one of the functions. Using Lemma 2.6, we can quickly deduce the following identities.

**Lemma 2.7.**

\[
g(z, q) - g(zq, q) = 2,
g(z, q) + g(q/z, q) = 1,
h(z, q) - h(zq, q) = 1,
h(z, q) + h(z/q, q) = 0.
\]

We will also need formulas that turn \(g(z, q)\) and \(h(z, q)\) into products. For the 3-dissections, we can actually just use a product identity for a similar function from [4]. We use this formula only for \(g\) and not \(h\).

**Lemma 2.8.**

\[
3 - 2g(z, q) - g(z^2, q) + g(z^4, q) = \frac{[z^6; q]_\infty^2 (q; q)_\infty^2}{[z^2; q]_\infty^2 [z^4; q]_\infty}.
\]

**Proof.** We let

\[
k(z, q) = \lim_{w \to z} \frac{1}{1-z/w - [z, zw, zw^{-1}; q]_\infty}\]

With Lemma 7 of [4] (which is also followed by a specialization of Theorem 2.1 [5]), we find that

\[
k(z, q) = 2 - g(z^{1/2}, q) - g(z, q).
\]

Replacing \(z\) by \(z^2\) in equation (5.6) of [4] we have

\[
2k(z^2, q) - k(z^4, q) + 1 = \frac{[z^6; q]_\infty^2 (q; q)_\infty^2}{[z^2; q]_\infty^2 [z^8; q]_\infty}.
\]

But also \(2k(z^2, q) - k(z^4, q) + 1 = 3 - 2g(z, q) - g(z^2, q) + g(z^4, q)\) and so we are done. \(\square\)

For the 5 and 7-dissections, we need the following product formulas.
Lemma 2.9.

\[ 4g(z, q) - 2g(z^2, q) = 3 - \frac{z^2 (q; q)^2}{[z; q]_\infty} \left[ z^2, z^8, q \right]_\infty - \frac{(q; q)^2}{[z; q]_\infty} \left[ z^4; q \right]_\infty^3, \]

\[ 4h(z, q) - 2h(z^2, q) = 1 - \frac{z (q; q)^2}{[z; q]_\infty^2} \left[ z, z^4, q \right]_\infty - \frac{(q; q)^2}{[z; q]_\infty} \left[ z^2; q \right]_\infty^3. \]

Proof. We need only prove the formula for \( h(z, q) \), the formula for \( g(z, q) \) then follows.

By Lemma 2.7 we find that \( 4h(z, q) - 2h(z^2, q) \) is invariant under \( z \mapsto zq \), as are the products \( \frac{z(q; q)^2}{[z; q]_\infty} \left[ z^2; q \right]_\infty^3 \) and \( \frac{(q; q)^2}{[z; q]_\infty^2} \left[ z^4; q \right]_\infty^3 \). We define

\[ F(z) := F(z, q) = 4h(z, q) - 2h(z^2, q) + \frac{z (q; q)^2}{[z; q]_\infty^2} \left[ z, z^4, q \right]_\infty + \frac{(q; q)^2}{[z; q]_\infty} \left[ z^2; q \right]_\infty^3 - 1. \]

By Lemma 2 of [4], if \( F(z) \) is not identically zero, then \( F \) has exactly as many poles and zeros in the region \( |q| < |z| \leq 1 \). Our proof is then to show all the poles cancel out and find a single zero of \( F(z) \).

First we show \( q^{1/3} \) is a zero of \( F \), to make the calculations cleaner we use \( q \mapsto q^3 \) and \( z = q \). By Lemma 2.6 we have

\[ 4h(q^3, q^3) - 2h(q^2, q^2) = -6 \sum_{n=0}^\infty \frac{q^{3n+1}}{1 - q^{3n+1}} + 6 \sum_{n=0}^\infty \frac{q^{3n+2}}{1 - q^{3n+2}}. \]

For the products we have

\[ \lim_{z \to q} \frac{z (q; q)^2}{[z; q]\_\infty^2} \left[ z, z^4, q \right]_\infty + \frac{(q; q)^2}{[z; q]_\infty} \left[ z^2; q \right]_\infty^3 = \frac{1}{1 - q^3/3^3}. \]

We note that

\[ \lim_{z \to q} \frac{z (q; q)^2}{[z; q]\_\infty^2} \left[ z, z^4, q \right]_\infty = -1, \quad \lim_{z \to q} \frac{(q; q)^2}{[z; q]_\infty} \left[ z^2; q \right]_\infty^3 = 1, \]

so logarithmic differentiation yields

\[ \frac{q}{z} \lim_{z \to q} \frac{\partial}{\partial z} \frac{z (q; q)^2}{[z; q]\_\infty^2} \left[ z, z^4, q \right]_\infty = -1 + \sum_{n=1}^\infty \frac{q^{3n+1}}{1 - q^{3n+1}} - \sum_{n=1}^\infty \frac{q^{3n-1}}{1 - q^{3n-1}} + 4 \sum_{n=1}^\infty \frac{q^{3n+4}}{1 - q^{3n+4}} - 4 \sum_{n=1}^\infty \frac{q^{3n-4}}{1 - q^{3n-4}} \]

\[ = -4 \sum_{n=0}^\infty \frac{q^{3n+2}}{1 - q^{3n+2}} + \sum_{n=1}^\infty \frac{q^{3n-2}}{1 - q^{3n-2}} - 3 \sum_{n=0}^\infty \frac{q^{3n+3}}{1 - q^{3n+3}} + 3 \sum_{n=2}^\infty \frac{q^{3n-3}}{1 - q^{3n-3}} \]

\[ = 3 + 6 \sum_{n=0}^\infty \frac{q^{3n+1}}{1 - q^{3n+1}} - 6 \sum_{n=0}^\infty \frac{q^{3n+2}}{1 - q^{3n+2}}, \]

and similarly

\[ \frac{q}{z} \lim_{z \to q} \frac{\partial}{\partial z} \frac{(q; q)^2}{[z; q]_\infty^3} \left[ z, z^4, q \right]_\infty^3 = 9 \sum_{n=0}^\infty \frac{q^{3n+1}}{1 - q^{3n+1}} - 9 \sum_{n=0}^\infty \frac{q^{3n+2}}{1 - q^{3n+2}}. \]

We then have

\[ \lim_{z \to q} \frac{z (q; q)^2}{[z; q]^2} \left[ z^2, z^3, q \right]_\infty^3 + (q; q)^2 \left[ z^2; q \right]_\infty^3 = 1 + 6 \sum_{n=0}^\infty \frac{q^{3n+1}}{1 - q^{3n+1}} - 6 \sum_{n=0}^\infty \frac{q^{3n+2}}{1 - q^{3n+2}}, \]
and so $q^{1/3}$ is a zero of $F(z)$.

We see $F(z)$ has at worst simple poles when $z$, $z^2$, or $z^3$ is an integral power of $q$. Elementary manipulations show that the poles between the two products cancel out for $z^3 = q$ and $q^2$ and for $z$ a primitive third root of unity. Thus we need only compute residues for $z = 1, -1, q^{1/2}$, and $-q^{1/2}$. For $h(z, q)$ these residues are 1, 0, 0, and 0; for $h(z^2, q)$ these residues are $\frac{1}{2}, -\frac{1}{2}, \frac{1}{2^{1/2}},$ and $\frac{1}{2^{1/2}}$; for $z((q,q)\frac{z}{z^{3}})$ these residues are $\frac{1}{3}$, $-1, q^{1/2}$ and $-q^{1/2}$; and for $\frac{(q,q)\frac{z}{z^{3}}}{z(q,q)\frac{z}{z^{3}}}$ these residues are $\frac{1}{3}, 0, 0$ and 0. Thus at all four points the residues cancel out in $F(z)$ so that $F(z)$ has no poles in $|q| < |z| \leq 1$. Thus $F$ is identically zero and the theorem holds. 

We can use Lemma 2.9 to expand $g(q^a, q^{3\ell^2})$ into products as follows. Suppose $n$ is a positive integer with $2^n \equiv 1 \pmod{3\ell}$ and $2^n - 1 = 3\ell d$. Then by applying Lemma 2.7 $ba$ times, with $q \mapsto q^{3\ell^2}$ and $z = q^a$, we have

$$g(q^a, q^{3\ell^2}) = g(q^{a + 3\ell^2 ba}; q^{3\ell^2}) + 2ba,$$

and so

$$(2^{n+1} - 2)g(q^a, q^{3\ell^2}) = 2^{n+1}g(q^a, q^{3\ell^2}) - 2g(q^{a + 3\ell^2 ba}; q^{3\ell^2}) - 4ba$$

$$= 2^{n+1}g(q^a, q^{3\ell^2}) - 2g(q^{2a}; q^{3\ell^2}) - 4ba$$

$$= -4ba + \sum_{k=0}^{n-1} 2^{n-k-1} \left( 4g(q^{2ak}; q^{3\ell^2}) - 2g(q^{2ak+1}; q^{3\ell^2}) \right)$$

$$= -4ba + \sum_{k=0}^{n-1} 2^{n-k-1} \left( 3 - \frac{q^{2ak+1}(q^{3\ell^2}; q^{3\ell^2})^2}{[q^{ak+1}; q^{3\ell^2}]^3} \right)$$

$$- \frac{(q^{3\ell^2}; q^{3\ell^2})^2}{[q^{ak+1}; q^{3\ell^2}]^3}.$$
where \( q = e^{2\pi i \tau} \) and \( P(t) = \{t\}^2 - \{t\} + \frac{1}{q} \). So \( \eta_{\delta, \theta}(\tau) = \eta(\delta \tau)^2 \) and \( \eta_{\delta, \theta}(\tau) = q^{P(g/\delta)/2} \left[ q^\theta; q^\delta \right]_\infty \) for \( 0 < g < \delta \). We use Theorem 3 of [20] to determine when a quotient of \( \eta_{\delta, \theta}(\tau) \) is a modular function with respect to a congruence subgroup \( \Gamma_1(N) \) and use Theorem 4 of [20] to determine the order at the cusps.

We recall some facts about modular functions as in [19] and use the notation in [3]. Suppose \( f \) is a modular function with respect to the congruence subgroup \( \Gamma \) of \( \Gamma_0(1) \). For \( A \in \Gamma_0(1) \) we have a cusp given by \( \zeta = A^{-1} \infty \). The width of the cusp \( N := N(\Gamma, \zeta) \) is given by

\[
N(\Gamma, \zeta) = \min\{k > 0 : \pm A^{-1}T^k A \in \Gamma\},
\]

where \( T \) is the translation matrix

\[
T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

If

\[
f(A^{-1} \tau) = \sum_{m=m_0}^{\infty} b_m q^{m/N}
\]

and \( b_{m_0} \neq 0 \), then we say \( m_0 \) is the order of \( f \) at \( \zeta \) with respect to \( \Gamma \) and we denote this value by \( \text{Ord}_f(\Gamma ; \zeta) \).

By \( \text{ord}(f; \zeta) \) we mean the invariant order of \( f \) at \( \zeta \) given by

\[
\text{ord}(f; \zeta) = \frac{\text{Ord}_f(f; \zeta)}{N}.
\]

For \( z \) in the upper half plane \( \mathcal{H} \), we write \( \text{ord}(f; z) \) for the order of \( f \) at \( z \) as an analytic function in \( z \).

We define the order of \( f \) at \( z \) with respect to \( \Gamma \) by

\[
\text{Ord}_f(f; z) = \frac{\text{ord}(f; z)}{m},
\]

where \( m \) is the order of \( z \) as a fixed point of \( \Gamma \).

The valence formula for modular functions is as follows. Suppose a subset \( \mathcal{F} \) of \( \mathcal{H} \cup \{\infty\} \cup \mathbb{Q} \) is a fundamental region for the action of \( \Gamma \) along with a complete set of inequivalent cusps, if \( f \) is not the zero function then

\[
\sum_{z \in \mathcal{F}} \text{Ord}_f(f; z) = 0.
\]

(2.15)

We can verify an identity between sums of generalized eta quotients as follows. Suppose we are to show

\[
a_1 f_1 + a_2 f_2 + \cdots + a_k f_k = a_{k+1} f_{k+1} + a_{k+2} f_{k+2} + \cdots + a_{k+m} f_{k+m},
\]

where each \( a_i \in \mathbb{C} \) and each \( f_i \) is of the form

\[
f_i = \prod_{j=1}^{m} \eta_{\delta_j, \theta_j}(\tau)^{r_j}.
\]

We verify each \( f_i \) is a modular function with respect to a common \( \Gamma_1(N) \), so that \( f = a_1 f_1 + \cdots + a_k f_k - a_{k+1} f_{k+1} - \cdots - a_{k+m} f_{k+m} \) is a modular function with respect to \( \Gamma_1(N) \). Although \( f \) may have zeros at points other than the cusps, the poles must occur only at the cusps. At each cusp \( \zeta \), not equivalent to \( \infty \), we compute a lower bound for \( \text{Ord}_f(f; \zeta) \) by taking the minimum of the \( \text{Ord}_f(f; \zeta) \), we call this lower bound \( B_\zeta \). We then use the \( q \)-expansion of \( f \) to find \( \text{Ord}_f(f; \infty) \) is larger than \( -\sum_{\zeta \in C} B_\zeta \), where \( C \) is a set of cusps with a representative of each cusp not equivalent to \( \infty \). By the valence formula we have \( f \equiv 0 \) since \( \sum_{z \in \mathcal{F}} \text{Ord}_f(f; z) > 0 \).

For reference we list the product formulas for the various \( g(z, q) \) and \( h(z, q) \) we use in our calculations,

\[
h(q^5, q^{75}) = g(q^{10}, q^{75}) = \frac{13}{30} - \frac{4}{15} q^{5} \left( q^{75}; q^{75} \right)^2_\infty \left[ q^5; q^{20}; q^{75} \right]_\infty - \frac{4}{15} \left( q^{75}; q^{75} \right)^2_\infty \left[ q^{10}; q^{75} \right]_\infty.
\]

\[
- \frac{2}{15} q^{10} \left( q^{75}; q^{75} \right)^2_\infty \left[ q^{10}; q^{35}; q^{75} \right]_\infty - \frac{2}{15} \left( q^{75}; q^{75} \right)^3_\infty \left[ q^{20}; q^{75} \right]_\infty.
\]
\[ h(q^{28}, q^{147}) = \frac{13}{42} - \frac{16}{63} q^{28} - \frac{1}{63} q^{28} + \frac{8}{63} q^{21} + \frac{1}{126} q^{21} \]

\[ h(q^{112}, q^{147}) = \frac{11}{42} + \frac{16}{63} q^{28} + \frac{8}{63} q^{21} + \frac{1}{126} q^{21} \]

\[ h(q^{91}, q^{147}) = \frac{5}{42} + \frac{16}{63} q^{28} + \frac{8}{63} q^{21} + \frac{1}{126} q^{21} \]
We let $\chi_0$ By [13] this is a weight 1 modular form with respect to $\Gamma_0(3)$ and character $\chi$. We note we also have $V_{\chi,1}(\tau) = -g(q^2, q^3) + \frac{1}{6} = h(q^2, q^3) + \frac{1}{6}$.

3. PROOF OF THEOREM 1.2

We note $V_3^6(2) = -V_3^6(7)$ and $V_3^6(5) = -V_3^6(4)$ and so we set

$$A = \frac{1}{3} + V_3^6(4) - q^2 U_3^6(8) - V_3^6(7) + q^3 U_3^6(11).$$

By [2.1] the left hand side of Theorem 1.2 is $\frac{4}{q; q}_\infty$, so we must show

$$\frac{A}{(q; q)_\infty} = \left(\frac{q^{27}; q^{27}}{q^{3}; q^{3}}\right)_\infty \left(\frac{q^{12}; q^{12}}{q^{3}; q^{3}}\right)_\infty \left(\frac{q^{3}; q^{3}}{q^{3}; q^{3}}\right)_\infty \left(\frac{q^{12}; q^{12}}{q^{3}; q^{3}}\right)_\infty \left(\frac{q^{15}; q^{15}}{q^{3}; q^{3}}\right)_\infty \left(\frac{q^{15}; q^{15}}{q^{3}; q^{3}}\right)_\infty$$

$$+ \frac{q^8}{(q^{27}; q^{27})_\infty (q^{3}; q^{3})_\infty} (S(9, -6, 27) - q^6 S(12, 6, 27))$$

By [2.1] the left hand side of Theorem 1.2 is $\frac{4}{q; q}_\infty$, so we must show

$$A = \frac{1}{3} + g(q^{21}, q^{27}) + g(q^{24}, q^{27}) - q^2 \left(\frac{q^{27}; q^{27}}{q^{3}; q^{3}}\right)_\infty \left(\frac{q^6; q^{27}}{q^{3}; q^{3}}\right)_\infty + q^6 \left(\frac{q^{27}; q^{27}}{q^{3}; q^{3}}\right)_\infty \left(\frac{q^6; q^{27}}{q^{3}; q^{3}}\right)_\infty$$

But by Lemmas [2.4] and [2.5]

$$A = \frac{1}{3} + g(q^{21}, q^{27}) + g(q^{24}, q^{27}) - q^2 \left(\frac{q^{27}; q^{27}}{q^{3}; q^{3}}\right)_\infty \left(\frac{q^6; q^{27}}{q^{3}; q^{3}}\right)_\infty + q^6 \left(\frac{q^{27}; q^{27}}{q^{3}; q^{3}}\right)_\infty \left(\frac{q^6; q^{27}}{q^{3}; q^{3}}\right)_\infty$$
By (2.2) the left hand side of Theorem 1.3 is 
\[ \frac{A}{(q; q)_{\infty}} (q; q)_{\infty} \]

Each term is a modular function with respect to $\Gamma(q)$ valence formula we need only verify the identity in the

\[ 1 + 3g(q^{24}, q^{27}) + 3g(q^{21}, q^{27}) = 6 - 2g(q^{3}, q^{27}) - 3g(q^{6}, q^{27}) + g(q^{24}, q^{27}) \]
\[ = 3 - 2g(q^{3}, q^{27}) - g(q^{6}, q^{27}) + g(q^{12}, q^{27}) \]
\[ + 2g(q^{6}, q^{27}) - g(q^{12}, q^{27}) + g(q^{24}, q^{27}) \]
\[ = \frac{(q^{27}; q^{27})_{\infty} [q^{9; q^{27}}]_{\infty}}{[q^{9; q^{27}2}]_{\infty} [q^{9; q^{27}}]_{\infty}} - q^{3} \frac{(q^{27}; q^{27})_{\infty} [q^{9; q^{27}}]_{\infty}}{[q^{9; q^{27}2}]_{\infty} [q^{9; q^{27}}]_{\infty}}. \]

Combining the products we find it only remains to show that
\[ -q^{2} \frac{(q^{27}; q^{27})_{\infty} [q^{9; q^{27}}]_{\infty}}{[q^{9; q^{27}2}]_{\infty} [q^{9; q^{27}}]_{\infty}} + q^{6} \frac{(q^{27}; q^{27})_{\infty} [q^{9; q^{27}}]_{\infty}}{[q^{9; q^{27}2}]_{\infty} [q^{9; q^{27}}]_{\infty}} - q^{5} \frac{(q^{27}; q^{27})_{\infty} [q^{9; q^{27}}]_{\infty}}{[q^{9; q^{27}2}]_{\infty} [q^{9; q^{27}}]_{\infty}} \]
\[ + q^{4} \frac{(q^{27}; q^{27})_{\infty} [q^{9; q^{27}}]_{\infty}}{[q^{9; q^{27}2}]_{\infty} [q^{9; q^{27}}]_{\infty}} + \frac{1}{3} \frac{(q^{27}; q^{27})_{\infty} [q^{9; q^{27}}]_{\infty}}{[q^{9; q^{27}2}]_{\infty} [q^{9; q^{27}}]_{\infty}} - \frac{1}{3} q^{3} \frac{(q^{27}; q^{27})_{\infty} [q^{9; q^{27}}]_{\infty}}{[q^{9; q^{27}2}]_{\infty} [q^{9; q^{27}}]_{\infty}} \]
\[ = \frac{(q; q)_{\infty} (q^{27}; q^{27})_{\infty}^{2}}{(q^{3}; q^{27})_{\infty}^{2}} \left( \frac{1}{3} [q^{12}; q^{27}]_{\infty}^{2} + \frac{2}{3} q^{3} [q^{6}; q^{27}]_{\infty}^{2} + \frac{1}{3} q^{3} [q^{6}; q^{12}; q^{27}]_{\infty}^{2} + \frac{1}{3} q^{4} [q^{3}; q^{27}]_{\infty}^{2} \right) \]
\[ + \frac{1}{3} q^{2} [q^{3}; q^{12}; q^{27}]_{\infty}^{2} - \frac{2}{3} q^{2} [q^{6}; q^{27}]_{\infty}^{2} \right) \].

(3.1)

If we multiply both sides of (3.1) by $q^{-2} \frac{(q^{3}; q^{27})_{\infty}}{(q^{27}; q^{27})_{\infty}}$ we find it is equivalent to
\[ -1 + \frac{1}{3} \frac{\eta_{27, 3}(\tau) \eta_{27, 12}(\tau)^{2}}{\eta_{27, 6}(\tau) \eta_{27, 12}(\tau)} + \frac{1}{3} q^{3} \frac{\eta_{27, 3}(\tau) \eta_{27, 12}(\tau)^{2}}{\eta_{27, 6}(\tau) \eta_{27, 12}(\tau)} + \frac{1}{3} q^{3} \frac{\eta_{27, 3}(\tau) \eta_{27, 12}(\tau)^{2}}{\eta_{27, 6}(\tau) \eta_{27, 12}(\tau)} - \frac{1}{3} q^{3} \frac{\eta_{27, 3}(\tau) \eta_{27, 12}(\tau)^{2}}{\eta_{27, 6}(\tau) \eta_{27, 12}(\tau)} \]
\[ = \frac{1}{3} \frac{\eta_{27, 3}(\tau) \eta_{27, 12}(\tau)^{2}}{\eta_{27, 6}(\tau)} + \frac{1}{3} \frac{q^{3} \eta_{27, 3}(\tau) \eta_{27, 12}(\tau)^{2}}{\eta_{27, 6}(\tau)} + \frac{1}{3} \frac{q^{3} \eta_{27, 3}(\tau) \eta_{27, 12}(\tau)^{2}}{\eta_{27, 6}(\tau)} - \frac{1}{3} \frac{q^{3} \eta_{27, 3}(\tau) \eta_{27, 12}(\tau)^{2}}{\eta_{27, 6}(\tau)} \]

Each term is a modular function with respect to $\Gamma(27)$ and by the reasoning explained in Section 2 by valence formula we need only verify the identity in the $q$-expansion past $q^{25}$. Thus Theorem 1.2 holds.

4. PROOF OF THEOREM 1.3

We set
\[ A = \frac{1}{3} + V_{3}^{6}(1) - q V_{3}^{6}(5) - V_{3}^{6}(4) + q^{2} U_{3}^{6}(8). \]

By (2.2) the left hand side of Theorem 1.3 is $A = \frac{4}{(q; q)_{\infty}}$, so we must show
\[ \frac{A}{(q; q)_{\infty}} = \frac{(q^{27}; q^{27})_{\infty}^{2}}{(q^{3}; q^{27})_{\infty}^{2}} \left( \frac{2}{3} [q^{12}; q^{27}]_{\infty}^{2} - \frac{1}{3} q^{3} [q^{3}; q^{27}]_{\infty}^{2} - \frac{2}{3} q^{3} [q^{6}; q^{27}]_{\infty}^{2} + \frac{1}{3} q^{4} [q^{3}; q^{27}]_{\infty}^{2} + \frac{1}{3} q^{2} [q^{3}; q^{12}; q^{27}]_{\infty}^{2} + \frac{1}{3} q^{2} [q^{6}; q^{27}]_{\infty}^{2} \right) \]

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Again each term is a modular function with respect to \( \Gamma \). By Lemmas 2.4 and 2.5,

\[
A = \frac{1}{3} - \eta_{q^{27}}(\tau) + \frac{\eta_{q^{27}}(\tau)}{\eta_{q^{27}}(\tau)} - \eta_{q^{27}}(\tau) \eta_{q^{27}}(\tau) + \frac{\eta_{q^{27}}(\tau) \eta_{q^{27}}(\tau)}{\eta_{q^{27}}(\tau)} + \frac{1}{3} \eta_{q^{27}}(\tau) \eta_{q^{27}}(\tau) + \frac{1}{3} \eta_{q^{27}}(\tau) \eta_{q^{27}}(\tau) - \frac{2}{3} \eta_{q^{27}}(\tau) \eta_{q^{27}}(\tau) - \frac{1}{3} \eta_{q^{27}}(\tau) \eta_{q^{27}}(\tau) + \frac{1}{3} \eta_{q^{27}}(\tau) \eta_{q^{27}}(\tau) + \frac{1}{3} \eta_{q^{27}}(\tau) \eta_{q^{27}}(\tau).
\]

We now have only an identity to verify between products. Multiplying (4.1) by \( q^{-2}(q; q)_{\infty} q^{q^{27} + q^{27}} \) we find we must show

\[
1 - 3g(q^{27}) + 3g(q^{27}) - 3g(q^{27}) + 3g(q^{27}) - 3g(q^{27}) = -3 + 2g(q^{27}) + g(q^{27}) + g(q^{27}) - g(q^{27}) - g(q^{27}) = q^{27} \frac{(q^{27}; q^{27})_{\infty}^2 [q^{27}; q^{27}]_{\infty}}{[q^{27}; q^{27}]_{\infty}^2 [q^{27}; q^{27}]_{\infty}} + \frac{(q^{27}; q^{27})_{\infty}^2 [q^{27}; q^{27}]_{\infty}}{[q^{27}; q^{27}]_{\infty}^2 [q^{27}; q^{27}]_{\infty}} + \frac{(q^{27}; q^{27})_{\infty}^2 [q^{27}; q^{27}]_{\infty}}{[q^{27}; q^{27}]_{\infty}^2 [q^{27}; q^{27}]_{\infty}}.
\]

5. PROOF OF THEOREM 1.4

First we set

\[
A = \frac{3}{5} + V_{5}^{6}(1) - qU_{5}^{6}(5) - V_{5}^{6}(10) - q^{4}U_{5}^{6}(14),
\]

\[
B = \frac{1}{5} + V_{5}^{6}(4) - q^{2}U_{5}^{6}(8) - V_{5}^{6}(7) + q^{4}U_{5}^{2}(11).
\]

By 2.8 the left hand side of Theorem 1.4 is \( \frac{A + (q + q^{2})B}{(q; q)_{\infty}} \), so we must show

\[
\frac{A}{(q; q)_{\infty}} = \left( q^{27}; q^{27} \right)_{\infty} \left( \frac{3}{5} \left[ q^{10}; q^{25} \right]_{\infty}^{2} - \frac{1}{5} \left[q^{10}; q^{25}\right]_{\infty}^{2} \right) - \frac{1}{5} \left[q^{5}; q^{25}\right]_{\infty}^{2} + q^{4} \left[q^{5}; q^{25}\right]_{\infty}^{2} - \frac{1}{5} \left[q^{75}; q^{27}\right]_{\infty}^{2} \left(S(-15, -140, 75) - q^{140}S(55, 140, 75) \right)
\]
\[ B_{(q; q)_{\infty}} = \frac{q^{14}}{(q^{75}; q^{75})_{\infty} (q^{20}; q^{75})_{\infty}} \left( S(15, -40, 75) - q^{40} S(35, 40, 75) \right), \quad (5.1) \]

\[ \frac{q^{10}}{(q^{75}; q^{75})_{\infty}} \left( S(15, -70, 75) - q^{70} S(50, 70, 75) \right) \]

By Lemmas 2.4 and 2.5

\[ A = \frac{3}{5} - g(q^{40}, q^{75}) - g(q^{50}, q^{75}) - q^4 \frac{(q^{75}; q^{75})^2}{[q^5, q^{20}, q^{30}, q^{75}]_{\infty}} + q^2 \frac{(q^{75}; q^{75})^2}{[q^5, q^{10}, q^{75}]_{\infty}} - \frac{q^8}{[q^{15}, q^{15}, q^{20}, q^{25}, q^{75}]_{\infty}} - \frac{q^7}{[q^{75}, q^{75}]_{\infty}} \]

Multiplying by \( q^{40} \frac{(q; q)_{\infty}}{(q^{75}; q^{75})_{\infty} [q^{20}, q^{30}, q^{75}]_{\infty}} \), we find (1.1) to be equivalent to

\[ -1 + \eta_{75,5} (\tau) \eta_{75,20} (\tau) \eta_{75,30} (\tau) \eta_{75,10} (\tau) \eta_{75,15} (\tau) \eta_{75,25} (\tau) \eta_{75,35} (\tau) - \eta_{75,20} (\tau) \eta_{75,30} (\tau) - \eta_{75,10} (\tau) \eta_{75,15} (\tau) \eta_{75,25} (\tau) \eta_{75,35} (\tau) \]

\[ = \frac{1}{15} \eta_{75,10} (\tau) \eta_{75,15} (\tau) \eta_{75,25} (\tau) \eta_{75,35} (\tau) - \frac{1}{15} \eta_{75,10} (\tau) \eta_{75,15} (\tau) \eta_{75,25} (\tau) \eta_{75,35} (\tau) \]

\[ = \eta_{75,5} (\tau) \eta_{75,20} (\tau) \eta_{75,25} (\tau) \eta_{75,20} (\tau) \eta_{75,25} (\tau) \eta_{75,20} (\tau) \eta_{75,25} (\tau) \]

\[ = \eta_{75,5} (\tau) \eta_{75,20} (\tau) \eta_{75,25} (\tau) \eta_{75,20} (\tau) \eta_{75,25} (\tau) V_{X,1}(25\tau) \]

\[ = \frac{3}{5} \eta_{75,5} (\tau) \eta_{75,20} (\tau) \eta_{75,25} (\tau) \eta_{75,30} (\tau) \eta_{75,10} (\tau) \]

\[ = \frac{1}{5} \eta_{75,0} (\tau) \eta_{75,0} (\tau) \eta_{75,10} (\tau) \eta_{75,25} (\tau) \eta_{75,30} (\tau) \]

\[ = \frac{1}{5} \eta_{75,0} (\tau) \eta_{75,0} (\tau) \eta_{75,10} (\tau) \eta_{75,25} (\tau) \eta_{75,30} (\tau) \]

\[ = \frac{1}{5} \eta_{75,0} (\tau) \eta_{75,0} (\tau) \eta_{75,10} (\tau) \eta_{75,25} (\tau) \eta_{75,30} (\tau) \]

\[ = \frac{1}{5} \eta_{75,0} (\tau) \eta_{75,0} (\tau) \eta_{75,10} (\tau) \eta_{75,25} (\tau) \eta_{75,30} (\tau) \]

\[ = \frac{1}{5} \eta_{75,0} (\tau) \eta_{75,0} (\tau) \eta_{75,10} (\tau) \eta_{75,25} (\tau) \eta_{75,30} (\tau) \]

\[ = \frac{1}{5} \eta_{75,0} (\tau) \eta_{75,0} (\tau) \eta_{75,10} (\tau) \eta_{75,25} (\tau) \eta_{75,30} (\tau) \]

\[ = \frac{1}{5} \eta_{75,0} (\tau) \eta_{75,0} (\tau) \eta_{75,10} (\tau) \eta_{75,25} (\tau) \eta_{75,30} (\tau) \]
However each term is a modular function with respect to $\Gamma(75)$ and by the valence formula it is sufficient to verify the identity in the $q$-expansion past $q^{199}$. For the term with $V_{\chi,1}(25\tau)$ we use that

$$\frac{\eta_{75,5}(\tau) \eta_{75,20}(\tau) \eta_{75,30}(\tau)}{\eta_{75,10}(\tau) \eta_{75,25}(\tau) \eta_{75,0}(\tau)} V_{\chi,1}(25\tau)$$

$$= \frac{\eta_{75,5}(\tau) \eta_{75,20}(\tau) \eta_{75,30}(\tau) \eta_{1,0}(\tau)^{1/2} \eta_{3,0}(\tau)^{1/2}}{\eta_{75,10}(\tau) \eta_{75,25}(\tau) \eta_{75,0}(\tau)} \frac{\eta(5\tau)^2}{\eta(\tau)\eta(3\tau)\eta(75\tau)^2} V_{\chi,1}(25\tau),$$

to see a product of a modular function, a meromorphic modular form of weight $-1$, and a holomorphic modular form of weight $1$ all with respect to $\Gamma(75)$. We can compute the order of the eta quotient at the cusps of $\Gamma(75)$ based on the formulas for the order at the cusps of $\Gamma_0(75)$ (Theorem 1.65 of [18]) and using

$$\text{Ord}_{\Gamma_0(75)}(f; \zeta) = \text{ord}(f; \zeta) N(\Gamma_1(75), \zeta) = \frac{\text{Ord}_{\Gamma_0(75)}(f; \zeta') N(\Gamma_1(75), \zeta)}{N(\Gamma_0(75), \zeta')},$$

where $\zeta$ is a cusp of $\Gamma_1(75)$ and $\zeta'$ is a cusp of $\Gamma_0(75)$ that is $\Gamma_0(75)$-equivalent to $\zeta$. We can ignore the contribution of $V_{\chi,1}(25\tau)$ since it can only possibly increase the order at a cusp.

By Lemmas 2.4 and 2.5

$$B = \frac{1}{5} + g(q^{55}, q^{75}) + g(q^{55}, q^{75}) - q^{17}(q^{75}; q^{75})_\infty^2 [q^5, q^{30}; q^{75}]_\infty + q^{11}(q^{75}; q^{75})_\infty^2 [q^5, q^{75}]_\infty$$

$$- q^2(q^{75}; q^{75})_\infty^2 [q^{10}, q^{20}, q^{30}; q^{75}]_\infty + q^{0}(q^{75}; q^{75})_\infty^2 [q^5, q^{35}, q^{75}]_\infty$$

$$- q^5(q^{75}; q^{75})_\infty^2 [q^{35}, q^{75}]_\infty + q^{1}(q^{75}; q^{75})_\infty^2 [q^{20}, q^{30}, q^{75}]_\infty$$

$$+ q^{10}(q^{75}; q^{75})_\infty^2 [q^{25}; q^{75}]_\infty (S(15, -70, 75) - q^{70}S(50, 70, 75))$$

Multiplying by $q^{-17}(q^{99}; q^{75})_\infty^2 [q^{25}, q^{30}, q^{35}, q^{75}]_\infty$ we find (5.2) to be equivalent to

$$- 1 + \frac{\eta_{75,25}(\tau) \eta_{75,30}(\tau) \eta_{75,35}(\tau)}{\eta_{75,15}(\tau) \eta_{75,20}(\tau)^2} - \frac{\eta_{75,10}(\tau) \eta_{75,30}(\tau)^2 \eta_{75,35}(\tau)}{\eta_{75,5}(\tau)^2 \eta_{75,15}(\tau)^2} + \frac{\eta_{75,30}(\tau) \eta_{75,35}(\tau)^2}{\eta_{75,15}(\tau) \eta_{75,20}(\tau)^2}$$

$$- \frac{\eta_{75,25}(\tau)^2 \eta_{75,35}(\tau)^2}{\eta_{75,5}(\tau)^2 \eta_{75,10}(\tau) \eta_{75,20}(\tau)} - \frac{\eta_{75,30}(\tau) \eta_{75,35}(\tau)^2}{\eta_{75,5}(\tau)^2 \eta_{75,15}(\tau) \eta_{75,20}(\tau)} + \frac{\eta_{75,25}(\tau)^2 \eta_{75,30}(\tau)^2}{\eta_{75,5}(\tau)^2 \eta_{75,15}(\tau)^2}$$

$$+ \frac{\eta_{75,25}(\tau)^2 \eta_{75,30}(\tau) \eta_{75,35}(\tau)}{\eta_{75,5}(\tau)^2 \eta_{75,15}(\tau) \eta_{75,20}(\tau)} - \frac{3 \eta_{75,25}(\tau) \eta_{75,30}(\tau)^2}{10 \eta_{75,15}(\tau) \eta_{75,35}(\tau)^2} + \frac{2 \eta_{75,10}(\tau) \eta_{75,25}(\tau) \eta_{75,35}(\tau)^2}{5 \eta_{75,25}(\tau) \eta_{75,25}(\tau) \eta_{75,35}(\tau)^2}$$

$$+ \frac{1 \eta_{75,25}(\tau) \eta_{75,30}(\tau) \eta_{75,35}(\tau)}{5 \eta_{75,10}(\tau)^2 \eta_{75,15}(\tau)} + \frac{1 \eta_{75,10}(\tau) \eta_{75,25}(\tau) \eta_{75,35}(\tau)^2}{10 \eta_{75,15}(\tau)^2 \eta_{75,20}(\tau)^2} + \frac{3 \eta_{75,25}(\tau) \eta_{75,30}(\tau) \eta_{75,35}(\tau) \theta}{10 \eta_{75,5}(\tau) \eta_{75,15}(\tau) \eta_{75,20}(\tau)}$$

$$= \frac{\eta_{1,0}(\tau)^{1/2} \eta_{75,25}(\tau)^2 \eta_{75,30}(\tau) \eta_{75,35}(\tau) \eta_{25,10}(\tau)}{\eta_{75,5}(\tau)^{1/2} \eta_{75,15}(\tau) \eta_{75,20}(\tau)^2} - \frac{\eta_{1,0}(\tau)^{1/2} \eta_{75,30}(\tau) \eta_{75,35}(\tau)}{\eta_{75,5}(\tau)^{1/2} \eta_{75,15}(\tau) \eta_{75,20}(\tau)^2}$$

$$+ \frac{1 \eta_{1,0}(\tau)^{1/2} \eta_{75,25}(\tau)^2 \eta_{75,30}(\tau) \eta_{75,35}(\tau) \eta_{25,10}(\tau)}{5 \eta_{75,0}(\tau)^{1/2} \eta_{75,5}(\tau) \eta_{75,20}(\tau) \eta_{25,5}(\tau)} + \frac{3 \eta_{1,0}(\tau)^{1/2} \eta_{75,25}(\tau)^2 \eta_{75,30}(\tau) \eta_{75,35}(\tau) \eta_{25,5}(\tau)}{5 \eta_{75,0}(\tau)^{1/2} \eta_{75,5}(\tau) \eta_{75,20}(\tau) \eta_{25,5}(\tau)}$$

$$+ \frac{3 \eta_{1,0}(\tau)^{1/2} \eta_{75,25}(\tau)^2 \eta_{75,30}(\tau) \eta_{75,35}(\tau) \eta_{25,5}(\tau)}{\eta_{75,0}(\tau)^{1/2} \eta_{75,5}(\tau) \eta_{75,20}(\tau) \eta_{25,10}(\tau)^2} - \frac{\eta_{1,0}(\tau)^{1/2} \eta_{75,25}(\tau) \eta_{75,30}(\tau) \eta_{75,35}(\tau) \eta_{25,5}(\tau)}{\eta_{75,0}(\tau)^{1/2} \eta_{75,5}(\tau)^2 \eta_{75,15}(\tau)^2}.$$
However each term is a modular function with respect to $\Gamma_1(75)$ and by the valence formula we find it is sufficient to verify the identity in the $q$-expansion past $q^{201}$. Thus Theorem 1.4 holds.

6. Proof of Theorem 1.5

First we set

$$A = \frac{3}{5} + V(2) - V(11) - qU(4) + q^4U(13),$$

$$B = \frac{1}{5} + V(5) - V(8) - q^2U(7) + q^3U(10).$$

By \textbf{2.24} the left hand side of Theorem 1.5 is $\frac{4+(\zeta_5+3)B}{(q;\tau)\infty}$, so we must show

$$A = (q^{25};q^{25})_{\infty}\left(\frac{3\left[q^{10};q^{25}\right]_{\infty}}{5\left[q^3;q^{25}\right]_{\infty}^2} - q^3\left[q^{10};q^{75}\right]_{\infty} - \frac{2}{5}q^2 + \frac{1}{5}q^{10}\right)_{\infty}$$

$$- \frac{1}{5}q^3\left[q^5;q^{25}\right]_{\infty} - \frac{q^3}{\frac{5q^{10};q^{25}}{q^{15},q^{25},q^{30},q^{75};q^{25}}}_{\infty}, \quad (6.1)$$

$$B = (q^{25};q^{25})_{\infty}\left(\frac{1\left[q^{10};q^{25}\right]_{\infty}}{5\left[q^3;q^{25}\right]_{\infty}^2} + \frac{1}{5}\left[q^{15};q^{25}\right]_{\infty} - q^6\right)_{\infty} + \frac{2}{5}q^2$$

$$- q^2\left[q^4,q^{25},q^{30},q^{75};q^{25}\right]_{\infty} + \frac{3}{5}q^3\left[q^{10},q^{25}\right]_{\infty} + q^8\left[q^{10},q^{15},q^{25},q^{35},q^{75};q^{25}\right]_{\infty}. \quad (6.2)$$

By Lemma \textbf{2.3}

$$A = \frac{3}{5} + h(q^{65},q^{75}) - h(q^{20},q^{75}) + q^2\left(q^{75};q^{75}\right)_{\infty}^2\left[q^{35};q^{75}\right]_{\infty} + q^{16}\left[q^{75};q^{75}\right]_{\infty}^2\left[q^{5};q^{75}\right]_{\infty}$$

$$- q^8\left[q^{75};q^{75}\right]_{\infty}^2\left[q^{25};q^{75}\right]_{\infty} - q^4\left[q^{75};q^{75}\right]_{\infty}^2\left[q^{15};q^{75}\right]_{\infty}$$

$$- q^{4}\left[q^{75};q^{75}\right]_{\infty}^2\left[q^{35};q^{75}\right]_{\infty} - q^{15}\left[q^{75};q^{75}\right]_{\infty}^2\left[q^{5};q^{75}\right]_{\infty} + q^{2}\left[q^{75};q^{75}\right]_{\infty}^2\left[q^{25};q^{75}\right]_{\infty}.$$

Multiplying (6.1) by $q^{-1}\left(q^{45};q^{45}\right)_{\infty}\left[q^{10};q^{25}\right]_{\infty}$, we are to show

$$- 1 + \frac{\eta_{75,5}(\tau)\eta_{75,35}(\tau)}{\eta_{75,20}(\tau)\eta_{75,25}(\tau)} - \frac{\eta_{75,5}(\tau)\eta_{75,15}(\tau)\eta_{75,20}(\tau)\eta_{75,25}(\tau)}{\eta_{75,20}(\tau)^4\eta_{75,30}(\tau)} + \frac{\eta_{75,5}(\tau)^2\eta_{75,15}(\tau)}{\eta_{75,20}(\tau)\eta_{75,30}(\tau)\eta_{75,35}(\tau)}$$

$$+ \frac{\eta_{75,10}(\tau)}{\eta_{75,20}(\tau)} - \frac{\eta_{75,5}(\tau)\eta_{75,15}(\tau)\eta_{75,35}(\tau)}{\eta_{75,10}(\tau)\eta_{75,20}(\tau)\eta_{75,30}(\tau)} + \frac{\eta_{75,5}(\tau)\eta_{75,20}(\tau)\eta_{75,25}(\tau)}{\eta_{75,20}(\tau)\eta_{75,35}(\tau)} - \frac{\eta_{75,5}(\tau)^2\eta_{75,15}(\tau)}{\eta_{75,20}(\tau)\eta_{75,25}(\tau)\eta_{75,30}(\tau)}$$

$$+ \frac{3}{10}\eta_{75,5}(\tau)\eta_{75,10}(\tau)\eta_{75,15}(\tau)\eta_{75,35}(\tau) + \frac{3}{10}\eta_{75,5}(\tau)\eta_{75,20}(\tau)\eta_{75,15}(\tau)\eta_{75,35}(\tau) - \frac{2}{5}\eta_{75,5}(\tau)^2 \eta_{75,15}(\tau)\eta_{75,30}(\tau)$$

$$+ \frac{2}{5}\eta_{75,5}(\tau)^3 \eta_{75,30}(\tau) - \frac{1}{5}\eta_{75,5}(\tau)^3 \eta_{75,15}(\tau)\eta_{75,30}(\tau) + \frac{1}{5}\eta_{75,5}(\tau)^4 \eta_{75,15}(\tau)\eta_{75,30}(\tau)$$

$$+ \frac{1}{10}\eta_{75,10}(\tau)^3 + \frac{1}{10}\eta_{75,5}(\tau)^2 \eta_{75,20}(\tau)$$

$$= \frac{3}{5}\eta_{1,0}(\tau)^{1/2} \eta_{75,5}(\tau)\eta_{75,15}(\tau)\eta_{75,25}(\tau)\eta_{25,10}(\tau) - \frac{\eta_{1,0}(\tau)^{1/2}}{\eta_{75,0}(\tau)^{1/2} \eta_{75,20}(\tau)\eta_{25,5}(\tau)^2} \eta_{75,5}(\tau)^2 \eta_{75,30}(\tau)$$

$$- \frac{2}{5}\eta_{1,0}(\tau)^{1/2} \eta_{75,5}(\tau)\eta_{75,15}(\tau)\eta_{75,25}(\tau) + \frac{1}{5}\eta_{1,0}(\tau)^{1/2} \eta_{75,5}(\tau)\eta_{75,15}(\tau)\eta_{75,25}(\tau)$$
Each term is a modular function with respect to $\Gamma_1(75)$ and by the valence formula the identity holds if we verify the $q$-expansion past $q^{199}$.

By Lemma 2.3

$$B = \frac{1}{5} + h(q^{50}, q^{75}) - h(q^{35}, q^{75}) + q^8 \left( q^{75}; q^{75} \right)_\infty^2 \left( q^{20}; q^{75} \right)_\infty - q^{12} \left( q^{75}; q^{75} \right)_\infty^2 \left[ q^{10}; q^{75} \right]_\infty^2$$

$$- q^{2} \left( q^{75}; q^{75} \right)_\infty^2 \left[ q^{35}; q^{75} \right]_\infty^2 + q^8 \left( q^{75}; q^{75} \right)_\infty^2 \left[ q^{25}; q^{75} \right]_\infty^2 - q^{11} \left( q^{75}; q^{75} \right)_\infty^2 \left[ q^{15}; q^{75} \right]_\infty^2$$

$$+ q^4 \left( q^{75}; q^{75} \right)_\infty^2 \left[ q^{25}; q^{75} \right]_\infty^2 - q^6 \left( q^{75}; q^{75} \right)_\infty^2 \left[ q^{20}; q^{75} \right]_\infty^2 + q^{10} \left( q^{75}; q^{75} \right)_\infty^2 \left[ q^{10}; q^{75} \right]_\infty^2.$$

Multiplying (3.2) by $q^{-2} \left( q^{75}; q^{75} \right)_\infty^2 \left[ q^{35}; q^{75} \right]_\infty^2$ we are to show

$$- \frac{1}{5} \eta_{75,25}(\tau) \eta_{75,30}(\tau) + \frac{\eta_{75,25}(\tau) \eta_{75,20}(\tau) \eta_{75,30}(\tau)}{\eta_{75,35}(\tau)} - \frac{\eta_{75,5}(\tau \eta_{75,15}(\tau) \eta_{75,20}(\tau) \eta_{75,30}(\tau)}{\eta_{75,35}(\tau)}$$

$$- \frac{\eta_{75,5}(\tau) \eta_{75,10}(\tau) \eta_{75,20}(\tau) \eta_{75,30}(\tau)}{\eta_{75,35}(\tau) \eta_{75,15}(\tau) \eta_{75,25}(\tau) \eta_{75,35}(\tau)} + \frac{\eta_{75,5}(\tau) \eta_{75,10}(\tau) \eta_{75,30}(\tau)}{\eta_{75,35}(\tau) \eta_{75,15}(\tau) \eta_{75,25}(\tau) \eta_{75,35}(\tau)}$$

$$+ \frac{\eta_{75,5}(\tau) \eta_{75,30}(\tau)}{\eta_{75,15}(\tau) \eta_{75,35}(\tau)} + \frac{4 \eta_{75,10}(\tau)}{15} - \frac{4 \eta_{75,5}(\tau)}{15} \eta_{75,30}(\tau) + \frac{2 \eta_{75,5}(\tau) \eta_{75,20}(\tau)}{15} \eta_{75,30}(\tau)$$

$$- \frac{2 \eta_{75,5}(\tau) \eta_{75,30}(\tau)}{15} \eta_{75,35}(\tau) + \frac{1 \eta_{75,5}(\tau) \eta_{75,10}(\tau)}{15} \eta_{75,30}(\tau)$$

$$+ \frac{1 \eta_{75,5}(\tau) \eta_{75,20}(\tau)}{30} \eta_{75,30}(\tau) + \frac{1 \eta_{75,5}(\tau) \eta_{75,30}(\tau)}{30} \eta_{75,15}(\tau) \eta_{75,35}(\tau)$$

$$= \frac{1}{5} \eta_{1,0}(\tau \eta_{1,0}(\tau \eta_{25,10}(\tau) + \frac{1}{5} \eta_{1,0}(\tau) \eta_{1,0}(\tau) \eta_{75,25}(\tau) \eta_{75,30}(\tau)$$

$$- \frac{1}{5} \eta_{1,0}(\tau) \eta_{75,0}(\tau) \eta_{75,25}(\tau) \eta_{75,30}(\tau)$$

$$+ \frac{2 \eta_{1,0}(\tau) \eta_{75,0}(\tau) \eta_{75,25}(\tau) \eta_{75,30}(\tau)}{5} \eta_{75,0}(\tau) \eta_{75,10}(\tau)$$

$$+ \frac{2 \eta_{1,0}(\tau) \eta_{75,0}(\tau) \eta_{75,30}(\tau)}{5} \eta_{75,0}(\tau) \eta_{75,10}(\tau) \eta_{75,35}(\tau)$$

Each term is a modular functions with respect to $\Gamma_1(75)$ and by the valence formula we need only verify the equality holds in the $q$-expansion past $q^{198}$. Here we handle the term involving $V_{X,1}(25\tau)$ as in the proof of Theorem 1.4. Thus Theorem 1.5 holds.

7. PROOF OF THEOREM 1.6.

We note that $V_3^5(4) = -V_3^5(11)$, $V_3^5(7) = -V_3^5(8)$, and $V_3^5(10) = -V_3^5(5)$ so we set

$$A = \frac{3}{5} + V_3^5(1) + V_3^5(5) - q^2U_3^5(7) + q^7U_3^5(16),$$

$$B = \frac{1}{5} - V_3^5(11) + V_3^5(8) - q^3U_3^5(10) + q^4U_3^5(13).$$

By 2.5 the left hand side of Theorem 1.6 is $\frac{A + (5\zeta_5 + 2\zeta_2)B}{(q;q)_\infty}$, so we must show

$$\frac{A}{(q;q)_\infty} = \left( q^{25}; q^{25} \right)_\infty \left( 3 \left[ \left[ q^{10}; q^{25} \right]_\infty - \frac{1}{5} \left[ q^{10}; q^{75} \right]_\infty - \left[ q^{15}; q^{25}; q^{25}, q^{75} \right]_\infty \right) - \frac{2}{5} \left[ q^{5}; q^{25} \right]_\infty + q^{10} \left[ q^{10}; q^{15}; q^{25}; q^{75} \right]_\infty.$$
\[ B(q) = \sum \frac{q^n}{(q^5)^n} \]

By Lemma 2.3, we have

\[ A = \frac{3}{5} + h(q^{30}; q^{75}) - h(q^5, q^{75}) + q^8 \left( \frac{q^{75}; q^{75}}{q^{35}; q^{75}} \right)^2 \left( \frac{q^{20}; q^{75}}{q^{30}; q^{75}} \right) - q^{12} \left( \frac{q^{75}; q^{75}}{q^{30}; q^{75}} \right)^2 \left( \frac{q^{10}; q^{75}}{q^{30}; q^{75}} \right) - q^2 \left( \frac{q^{75}; q^{75}}{q^{30}; q^{75}} \right)^2 \left( \frac{q^{25}; q^{75}}{q^{30}; q^{75}} \right) - q^4 \left( \frac{q^{75}; q^{75}}{q^{30}; q^{75}} \right)^2 \left( \frac{q^{35}; q^{75}}{q^{30}; q^{75}} \right) \]

and with this we find (1.1) is equivalent to

\[
1 - \frac{\eta_{75,10} (\tau) \eta_{75,15} (\tau) \eta_{75,35} (\tau)}{\eta_{75,20} (\tau)^2 \eta_{75,30} (\tau)} - \frac{\eta_{75,15} (\tau) \eta_{75,35} (\tau) \eta_{75,30} (\tau)}{\eta_{75,20} (\tau) \eta_{75,30} (\tau)} + \frac{\eta_{75,5} (\tau) \eta_{75,35} (\tau) \eta_{75,30} (\tau)}{\eta_{75,15} (\tau) \eta_{75,30} (\tau)} \\
- \frac{\eta_{75,35} (\tau) \eta_{75,30} (\tau)}{\eta_{75,20} (\tau)^2} - \frac{4}{15} \eta_{75,20} (\tau) + \frac{4}{15} \eta_{75,30} (\tau) - \frac{1}{15} \eta_{75,5} (\tau) - \frac{1}{15} \eta_{75,35} (\tau) \\
+ \frac{1}{15} \eta_{75,35} (\tau) \eta_{75,30} (\tau) - \frac{1}{30} \eta_{75,35} (\tau) \eta_{75,30} (\tau) + \frac{1}{30} \eta_{75,20} (\tau) \eta_{75,30} (\tau) \eta_{75,35} (\tau) + \frac{q^{22}}{30} \eta_{75,5} (\tau) \eta_{75,30} (\tau) \eta_{75,35} (\tau) \\
+ \frac{\eta_{75,15} (\tau) \eta_{75,35} (\tau) \eta_{75,30} (\tau) \eta_{75,35} (\tau)}{\eta_{75,30} (\tau) \eta_{75,35} (\tau) \eta_{75,30} (\tau) \eta_{75,35} (\tau)} \]

However each term is a modular function with respect to \( \Gamma_0(75) \), where we have treated the term with \( V_{\lambda,1}(25\tau) \) as before. This time by the valence formula it suffices to verify the identity in the \( q \)-expansion past \( q^{192} \).

Next we have by Lemma 2.3

\[ B = \frac{1}{5} + h(q^{35}; q^{75}) - h(q^{20}; q^{75}) + q^{11} \left( \frac{q^{75}; q^{75}}{q^{15}; q^{75}} \right)^2 \left( \frac{q^{20}; q^{75}}{q^{15}; q^{75}} \right)^2 - q^3 \left( \frac{q^{75}; q^{75}}{q^{30}; q^{75}} \right)^2 \left( \frac{q^{25}; q^{75}}{q^{30}; q^{75}} \right) \]
and with this we find (7.2) to be equivalent to

\[ 1 - \frac{2 \eta_{75,10}(\tau) \eta_{75,20}(\tau) \eta_{75,25}(\tau)}{\eta_{75,5}(\tau) \eta_{75,10}(\tau) \eta_{75,20}(\tau)} - \frac{2 \eta_{75,10}(\tau) \eta_{75,20}(\tau) \eta_{75,25}(\tau)}{\eta_{75,5}(\tau) \eta_{75,15}(\tau) \eta_{75,25}(\tau)} = 3 \frac{\eta_{75,5}(\tau)^3}{\eta_{75,5}(\tau) \eta_{75,15}(\tau) \eta_{75,20}(\tau)}, \]

Again each term is a modular function with respect to \( \Gamma_1(75) \) and we need only verify the \( q \)-expansion past \( q^{189} \). Thus Theorem 1.6 holds.

8. PROOF OF THEOREM 1.7

First we set

\[ A = \frac{3}{7} + V_7^3(2) - V_7^3(17) - q U_7^3(4) + q^6 U_7^3(19), \]
\[ B = \frac{3}{7} + V_7^3(5) - V_7^3(14) - q^2 U_7^3(7) + q^6 U_7^3(16), \]
\[ C = \frac{1}{7} + V_7^3(8) - V_7^3(11) - q^3 U_7^3(10) + q^6 U_7^3(13). \]

By (8.1) the left hand side of Theorem 1.7 is \( A + (5 \tau + c_1^2) B + (1 + c_2^2 + c_1^2) C \), so we must show

\[
\frac{A}{(q;q)_\infty} = (q^{49}, q^{49})_\infty \left( \frac{5}{7} [q^{21}, q^{49}]_\infty - q^{14} \frac{1}{[q^{42}, q^{49}, q^{56}; q^{147}]_\infty} - \frac{1}{7} q^3 \frac{1}{[q^{47}; q^{49}]_\infty} - \frac{3}{7} q^2 \frac{[q^{14}; q^{49}]_\infty}{[q^{7}, q^{21}; q^{49}]_\infty} \right),
\]

\[
\frac{B}{(q;q)_\infty} = (q^{49}, q^{49})_\infty \left( \frac{3}{7} [q^{21}, q^{49}]_\infty - q^{14} \frac{1}{[q^{42}, q^{49}, q^{56}; q^{147}]_\infty} + \frac{1}{7} q^3 \frac{1}{[q^{47}; q^{49}]_\infty} - \frac{1}{7} q^2 \frac{[q^{14}; q^{49}]_\infty}{[q^{7}, q^{21}; q^{49}]_\infty} \right),
\]

\[
\frac{C}{(q;q)_\infty} = (q^{49}, q^{49})_\infty \left( \frac{2}{7} [q^{21}, q^{49}]_\infty + q^{14} \frac{1}{[q^{42}, q^{49}, q^{56}; q^{147}]_\infty} + \frac{3}{7} q^3 \frac{1}{[q^{47}; q^{49}]_\infty} + \frac{1}{7} q^2 \frac{[q^{14}; q^{49}]_\infty}{[q^{7}, q^{21}; q^{49}]_\infty} \right),
\]
\[-2 \frac{q^6}{\eta} \binom{q^7; q^{49}}{q^{14}, q^{21}; q^{49}} \bigg|_{\infty}, \quad (8.2)\]

\[C = \frac{q^{49}; q^{49}}{\eta} \bigg( \frac{1}{7} \frac{q^{21}; q^{49}}{\eta} \bigg|_{\infty} + \frac{1}{7} \frac{q^{49}}{\eta} \bigg|_{\infty} + \frac{2}{7} \frac{q^{49}}{\eta} \bigg|_{\infty} - q^5 \frac{q^{49}}{\eta} \bigg|_{\infty} \bigg) + \frac{4}{7} q^3 - \frac{3}{7} q^6 \binom{q^7; q^{49}}{q^{14}, q^{21}; q^{49}} \bigg|_{\infty} + q^6 \frac{q^{49}}{\eta} \bigg|_{\infty} \bigg). \quad (8.3)\]

We have

\[A = \frac{5}{7} + h(q^{13}, q^{147}) - h(q^{28}, q^{147}) + q^5 \binom{q^{147}; q^{147}}{q^{21}, q^{51}; q^{47}} \bigg|_{\infty} + q^6 \binom{q^{147}; q^{147}}{q^{19}, q^{63}; q^{147}} \bigg|_{\infty} + q^8 \binom{q^{147}; q^{147}}{q^{28}, q^{42}; q^{147}} \bigg|_{\infty} + q^9 \binom{q^{147}; q^{147}}{q^{49}, q^{28}; q^{147}} \bigg|_{\infty} + q^{10} \binom{q^{147}; q^{147}}{q^{35}; q^{49}} \bigg|_{\infty} \bigg), \]

\[B = \frac{3}{7} + h(q^{11}, q^{147}) - h(q^{28}, q^{147}) + q^6 \binom{q^{147}; q^{147}}{q^{21}, q^{56}; q^{147}} \bigg|_{\infty} + q^8 \binom{q^{147}; q^{147}}{q^{28}, q^{42}; q^{147}} \bigg|_{\infty} + q^9 \binom{q^{147}; q^{147}}{q^{49}, q^{28}; q^{147}} \bigg|_{\infty} + q^{10} \binom{q^{147}; q^{147}}{q^{35}; q^{49}} \bigg|_{\infty} \bigg), \]

\[C = \frac{1}{7} + h(q^{91}, q^{147}) - h(q^{147}, q^{147}) + q^9 \binom{q^{147}; q^{147}}{q^{21}, q^{56}; q^{147}} \bigg|_{\infty} + q^{10} \binom{q^{147}; q^{147}}{q^{28}, q^{42}; q^{147}} \bigg|_{\infty} + q^{11} \binom{q^{147}; q^{147}}{q^{49}, q^{28}; q^{147}} \bigg|_{\infty} + q^{12} \binom{q^{147}; q^{147}}{q^{35}; q^{49}} \bigg|_{\infty} \bigg). \]
Multiplying both sides of (8.1) by $q^{-5}[q, q^{35}; q^{147}]_{\infty}$, we find it is equivalent to proving

$$
1 + \frac{\eta_{47.21}(\tau)\eta_{47.35}(\tau)\eta_{47.49}(\tau)}{\eta_{47.42}(\tau)\eta_{47.56}(\tau)} + \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)\eta_{47.70}(\tau)} - \frac{\eta_{47.21}(\tau)\eta_{47.35}(\tau)\eta_{47.70}(\tau)}{\eta_{47.42}(\tau)\eta_{47.56}(\tau)\eta_{47.63}(\tau)} + \frac{\eta_{47.14}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} \\
+ \frac{\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.42}(\tau)\eta_{47.56}(\tau)\eta_{47.70}(\tau)} + \frac{\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.42}(\tau)\eta_{47.56}(\tau)\eta_{47.70}(\tau)} + \frac{\eta_{47.7}(\tau)\eta_{47.35}(\tau)}{\eta_{47.42}(\tau)\eta_{47.56}(\tau)\eta_{47.70}(\tau)} + \frac{\eta_{47.14}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} \\
+ \frac{\eta_{47.14}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} + \frac{\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.42}(\tau)\eta_{47.56}(\tau)\eta_{47.70}(\tau)} + \frac{\eta_{47.14}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} \\
- \frac{\eta_{47.21}(\tau)\eta_{47.35}(\tau)\eta_{47.70}(\tau)}{\eta_{47.42}(\tau)\eta_{47.56}(\tau)\eta_{47.63}(\tau)} + \frac{\eta_{47.14}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} \\
+ \frac{\eta_{47.21}(\tau)\eta_{47.35}(\tau)\eta_{47.70}(\tau)}{\eta_{47.42}(\tau)\eta_{47.56}(\tau)\eta_{47.63}(\tau)} + \frac{\eta_{47.14}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} \\
- \frac{\eta_{47.21}(\tau)\eta_{47.35}(\tau)\eta_{47.70}(\tau)}{\eta_{47.42}(\tau)\eta_{47.56}(\tau)\eta_{47.63}(\tau)} + \frac{\eta_{47.14}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} \\
+ \frac{\eta_{47.21}(\tau)\eta_{47.35}(\tau)\eta_{47.70}(\tau)}{\eta_{47.42}(\tau)\eta_{47.56}(\tau)\eta_{47.63}(\tau)} + \frac{\eta_{47.14}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} \\
+ \frac{\eta_{47.21}(\tau)\eta_{47.35}(\tau)\eta_{47.70}(\tau)}{\eta_{47.42}(\tau)\eta_{47.56}(\tau)\eta_{47.63}(\tau)} + \frac{\eta_{47.14}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)} - \frac{\eta_{47.7}(\tau)\eta_{47.21}(\tau)\eta_{47.35}(\tau)}{\eta_{47.56}(\tau)\eta_{47.63}(\tau)}$$. 

Each term is a modular function with respect to $\Gamma_1(147)$ and by the valence formula it suffices to verify the identity in the $q$-expansion past $q^{779}$. 

Multiplying both sides by $q^{-8}[q^{21}, q^{56}, q^{147}]_{\infty}$, we find (8.2) is equivalent to
immediately apparent as in the 5-dissections. However, we can write working with this larger congruence subgroup, we must verify the id entity in the

\[ - \frac{2}{63} \eta_{147,21}(\tau) \eta_{147,42}(\tau) \eta_{147,70}(\tau) \eta_{147,56}(\tau)^2 + \frac{16}{63} \eta_{147,21}(\tau) \eta_{147,56}(\tau) \eta_{147,70}(\tau) \eta_{147,70}(\tau) - \frac{2}{63} \eta_{147,21}(\tau) \eta_{147,42}(\tau)^3 \eta_{147,70}(\tau) - \frac{1}{63} \eta_{147,21}(\tau) \eta_{147,56}(\tau) \eta_{147,70}(\tau) + \eta_{147,70}(\tau) \eta_{147,56}(\tau) \eta_{V_{r,1}(49\tau)} \]

\[ + \frac{3}{7} \eta_{1,0}(\tau)^{1/2} \eta_{147,21}(\tau) \eta_{147,56}(\tau) \eta_{147,49}(\tau) \eta_{49,21}(\tau) - \frac{3}{7} \eta_{1,0}(\tau)^{1/2} \eta_{147,56}(\tau) \eta_{147,49}(\tau) \eta_{49,21}(\tau) \]

\[ + \frac{1}{7} \eta_{1,0}(\tau)^{1/2} \eta_{147,21}(\tau) \eta_{147,56}(\tau) \eta_{147,49}(\tau) \eta_{49,21}(\tau) + \eta_{1,0}(\tau)^{1/2} \eta_{147,21}(\tau) \eta_{147,56}(\tau) \eta_{147,49}(\tau) \eta_{147,49}(\tau) \]

While it is true that the term involving \( V_{r,1}(49\tau) \) is a modular function with respect \( \Gamma_1(147) \), it is not immediately apparent as in the 5-dissections. However, we can write

\[ \eta_{147,21}(\tau) \eta_{147,56}(\tau) \eta_{147,70}(\tau) \eta_{147,49}(\tau) \eta_{49,21}(\tau) = \frac{\eta_{147,21}(\tau) \eta_{147,56}(\tau) \eta_{1,0}(\tau)^{1/2}}{\eta_{147,70}(\tau) \eta_{147,21}(\tau) \eta_{49,14}(\tau) \eta_{49,21}(\tau)} \eta(\tau) \eta(147\tau)^2 \]

to see a modular function and meromorphic modular form of weight \(-1\) with respect to \( \Gamma_1(447) \). Instead working with this larger congruence subgroup, we must verify the identity in the \( q \)-expansion past \( q^{7804} \).

Multiplying by \( \frac{q^{-1}}{\eta(q)^3 \eta(q^7)^{147}} \) We then find \((8,3)\) is equivalent to

\[ 1 + \eta_{147,7}(\tau) \eta_{147,21}(\tau) \eta_{147,42}(\tau) \eta_{147,49}(\tau)^2 - \frac{\eta_{147,21}(\tau) \eta_{147,35}(\tau) \eta_{147,70}(\tau) \eta_{147,70}(\tau)}{\eta_{147,28}(\tau) \eta_{147,49}(\tau) \eta_{147,63}(\tau) \eta_{147,63}(\tau)} - \frac{\eta_{147,21}(\tau) \eta_{147,70}(\tau)^2}{\eta_{147,7}(\tau) \eta_{147,49}(\tau) \eta_{147,49}(\tau)^2} \]

\[ + \frac{5}{42} \eta_{147,42}(\tau) \eta_{147,49}(\tau) \eta_{147,70}(\tau) \eta_{147,35}(\tau) - \frac{2}{21} \eta_{147,42}(\tau) \eta_{147,70}(\tau) \eta_{147,70}(\tau) \eta_{147,35}(\tau)^2 + \frac{1}{21} \eta_{147,21}(\tau) \eta_{147,28}(\tau) \eta_{147,56}(\tau) \eta_{147,70}(\tau) \eta_{147,70}(\tau) \eta_{147,70}(\tau) - \frac{1}{21} \eta_{147,42}(\tau)^2 \eta_{147,49}(\tau) \eta_{147,49}(\tau) \]

\[ - \frac{42}{42} \eta_{147,49}(\tau) \eta_{147,56}(\tau) \eta_{147,35}(\tau)^2 \eta_{147,70}(\tau) - \frac{5}{42} \eta_{147,49}(\tau) \eta_{147,42}(\tau) \eta_{147,49}(\tau) \eta_{147,35}(\tau)^2 - \frac{5}{42} \eta_{147,49}(\tau) \eta_{147,42}(\tau) \eta_{147,49}(\tau) \eta_{147,35}(\tau)^2 - \frac{2}{21} \eta_{147,42}(\tau) \eta_{147,70}(\tau) \eta_{147,70}(\tau) \eta_{147,70}(\tau) \eta_{147,70}(\tau) - \frac{2}{21} \eta_{147,7}(\tau) \eta_{147,70}(\tau) \eta_{147,70}(\tau) \]

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However each term is a modular function with respect to $\Gamma_1(147)$ and by the valence formula it is sufficient to verify the identity in the $q$-expansion past $q^{773}$. Thus Theorem 17 holds.

9. Partition Pair Cranks

We use $\ell(\pi)$ for the largest part of a partition, $s(\pi)$ for the smallest part of a partition, $\#(\pi)$ for the number of parts of a partition, and $spt(\pi)$ for the number of occurrences of the smallest part of a partition. For a partition pair $(\pi_1, \pi_2)$ we let $k(\pi_1, \pi_2)$ denote the number of parts of $\pi_1$ that are larger than $s(\pi_1) + \#(\pi_2)$. We note that $\pi_2$ is empty that $k(\pi_1, \pi_2)$ is the number of parts of $\pi_1$ larger than the smallest part, so that

$$spt(\pi_1) + k(\pi_1, \pi_2) = \#(\pi_1).$$

We define the following cranks on partition pairs,

$$\text{paircrank}_1((\pi_1, \pi_2)) = spt(\pi_1) - 1 + k(\pi_1, \pi_2) - \#(\pi_2),$$
$$\text{paircrank}_2((\pi_1, \pi_2)) = spt(\pi_1) - 2 + k(\pi_1, \pi_2) - \#(\pi_2),$$
$$\text{paircrank}_3((\pi_1, \pi_2)) = spt(\pi_1) - s(\pi_1) - 1 + k(\pi_1, \pi_2) - \#(\pi_2),$$
$$\text{paircrank}_4((\pi_1, \pi_2)) = spt(\pi_1) - s(\pi_1) + k(\pi_1, \pi_2) - \#(\pi_2).$$

We let $PP_i$ denote the set of partition pairs counted by $pp_i$. That is, we first let $PP_1$ denote the set of partition pairs $(\pi_1, \pi_2)$ where $\pi_1$ is non-empty and if $\pi_2$ is non-empty then $s(\pi_1) \leq s(\pi_2)$ and $\ell(\pi_2) \leq 2s(\pi_1)$. Next $PP_2$ is the subset of $PP_1$ where $spt(\pi_1) \geq 2$, $PP_3$ is the subset of $PP_1$ where $spt(\pi_1) \geq s(\pi_1) + 1$, and $PP_4$ is the subset of $PP_1$ where $spt(\pi_1) \geq s(\pi_1)$. We claim $M_i(m, n)$ is the number of partition pairs of $n$ from $PP_i$ with paircrank equal to $m$. We prove this for $i = 1$ and see the other three cases follow similarly.

The only rearrangement of $q$-series we need is the $q$-binomial theorem. This is in a similar fashion to the rearrangements for the original crank as in [2] and for $\text{crank}$ from Section 3.2 of [11].

We have

$$PP_1(z, q) = \sum_{n=1}^{\infty} \frac{q^n (q^{2n+1}; q)_\infty}{(zq^n; q)_\infty} \sum_{k=0}^{\infty} \frac{(zq^{n+1}; q)_k z^{-k}q^{nk}}{(q; q)_k},$$

$$= \sum_{n=1}^{\infty} \frac{q^n (zq^n; q)_\infty}{(q; q)_\infty} \sum_{k=1}^{\infty} \frac{(1 - zq^n) (zq^{n+k+1}; q)_\infty (q; q)_k}{(q^n; q)_\infty (q; q)_k} \frac{q^n}{(q; q)_k (q; q)_n}.$$
\( \pi_1 \) that are either the smallest part (past the first occurrence of the smallest part) or are at least \( n+k+1 \) in size. Since \( \frac{(q;q)_n}{(q^2;q^2)_n} \) is well known to be the generating function for partitions into at most \( k \) parts with largest part at most \( n \), we have \( z^{-k}q^{kn}(q\pi)_n^{a+1} \) is the generating function for partitions \( \pi_2 \) into exactly \( k \) parts with smallest part at least \( n \) and largest part no more than \( 2n \) with the power of \( q \) counting the number being partitioned by \( \pi_2 \) and the power of \( z \) counting the negative of the number of parts of \( \pi_2 \). Thus the second series corresponds to paircrank\(_1(\pi_1, \pi_2)\) when \( \pi_2 \) is non-empty.

10. Remarks

We see this method is rather powerful, as from a single Bailey pair we get a partition type function, congruences for that function, and a combinatorial refinement of those congruences. There is still the question of how these functions behave from a modular perspective. It is not clear if they also naturally arise from considering certain weak harmonic mass forms as has been seen for the spt functions as in \([6]\) and \([7]\). In another direction there is also the question of whether or not the series we have dissected also arise as ranks for some types of partitions.

In a coming paper we find that these functions \( PP_i(z, q) \) have representations as Hecke-Rogers type double series, as was done in \([10]\) by Garvan for other spt cranks. These double series can also be used to prove most of the congruences of this paper. Also in that paper we prove the corresponding results for the Bailey pairs in groups C and E of Slater \([21]\). In papers after that, we will handle groups B, F, and G.

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