Superconducting Order Parameter for the Even-denominator Fractional Quantum Hall Effect

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One of the most intriguing phenomena in nature is the fractional quantum Hall effect (FQHE) observed in the half-filled second Landau level [1] which, arising in even-denominator filling factors, \( \nu = 5/2 \) and \( 7/2 \), is completely different from other FQHEs in its origin, all of which, except for those two filling factors, occur in odd-denominator fractions. Usually formulated in terms of a trial wave function called the Moore-Read Pfaffian wave function [2], current leading theories [3–7] attribute the origin of the 5/2 FQHE to the formation of Cooper pairs, not of electron, but of the true quasi-particle of the system known as composite fermion [8]. The nature of superconductivity resulting from such Cooper pairing is particularly puzzling in the sense that it apparently coexists with strong magnetic fields, which poses an interesting dilemma since the Meissner effect is the most important defining property of superconductivity. This apparent dilemma is resolved by the fact that composite fermions do not respond to external magnetic field at even-denominator filling factors. To provide direct evidence that it is composite fermions that actually form the superconducting condensate, here, we develop a numerically exact method of creating a Cooper pair of composite fermions and explicitly compute the superconducting order parameter as a function of real space coordinates. As results, in addition to direct evidence for superconductivity, we obtain quantitative predictions for superconducting coherence length. Obtaining such theoretical predictions can serve as an important step toward fault-tolerant topological quantum computation [9, 10].

We begin by emphasizing that no assumption is made throughout this work regarding the nature of the 5/2 ground state. Our point of view is that, if it exists, the superconducting order parameter should arise naturally when computed via exact diagonalization. Computing the superconducting order parameter, however, explicitly depends on our choice of the quasi-particle in the ground state. In the usual Fermi liquid, the quasi-particle is identified with something very close to the free electron so that the superconducting order parameter is computed in terms of the “free” electron operator. The situation is less clear for strongly correlated electron systems. It is, thus, useful to provide independent, unbiased evidence for superconductivity that is not susceptible to our knowledge of the quasi-particle nature. Such evidence is obtained in the ground state energy itself. Since a prerequisite for superconductivity is the pairing between quasi-particles, the ground state energy must oscillate depending on whether the particle number is even or odd.

In order to obtain the FQHE ground state energy, it is convenient to use a compact geometry with periodic boundary condition. In this study, we use the so-called spherical geometry where electrons are confined on the surface of a sphere which has a magnetic monopole at its center [11, 12]. The total magnetic flux piercing through the spherical surface is given by \( 2Q \phi_0 \) with \( \phi_0 \), called the flux quantum, being equal to \( 2\pi \hbar c/e \) and with \( Q \), called the monopole strength, being equal to either an integer or a half integer. Angular momentum eigenstates in this geometry are known as the monopole harmonics which form a basis set to span the single-particle Hilbert space of each Landau level. Many-particle FQHE ground states are obtained by numerically diagonalizing the inter-particle interaction matrix within specific Landau levels; for example, the diagonalization is done within the half-filled second Landau level for the 5/2 FQHE. It is important to note that, due to the nature of the monopole harmonics, the ratio between the number of particles, \( N \), and that of flux quanta, \( 2Q \), is not exactly equal to the thermodynamic filling factor, \( \nu \), in finite-size systems; \( N/2Q \) approaches \( \nu \) as \( N \rightarrow \infty \). It is known from various studies that the 5/2 FQHE state occurs at \( 2Q = 2N - 3 \) with an apparent “flux shift” of \( -3 \) [3, 6].

Figure 1 shows the results of the ground state energy per particle for the Coulomb interaction, \( H_C \), of the half-filled second-Landau-level (SLL) in comparison with that of the half-filled lowest-Landau-level (LLL) as well as that of a model two-body interaction, \( H_2 \), for which the Moore-Read Pfaffian wave function is known to be very close to the exact ground state [14]. As one can see, even-odd effects are clearly visible for both SLL \( H_C \) and \( H_2 \) (top panels of Fig. 1), while the same is not true for...
FIG. 1: Even-odd effect The ground state energies per particle are computed for the Pfaffian two-body interaction, $H_2$, [top left panel], the Coulomb interaction, $H_C$, projected in the second-Landau-level (SLL) [top right panel] and the lowest-Landau-level (LLL) [bottom two panels]. For the Coulomb interaction, the energy is given in units of $e^2/l E_\infty$ (where $l_{\text{c}} = l_0 \sqrt{2eQ/N}$ with $l_0 = \sqrt{\hbar eB}$ and the usual charge background correction is taken into account. In the case of the LLL $H_C$, two different sectors of flux quantum number, $2Q$, are studied for a given particle number, $N$: the Pfaffian flux of $2Q = 2N - 3$ and the composite-fermion-sea flux of $2Q = 2N - 2$.

LLL $H_C$ (bottom panels). In fact, the behavior of the LLL $H_C$ ground state energy is quite different from the SLL counterpart, showing a peculiar pattern of repeating local minima at $N = 4, 9,$ and $16$ (bottom right panel of Fig. 1). The energy minima would repeatedly appear whenever $N$ becomes $n^2$ with $n$ being a positive integer if numerical diagonalization is possible for bigger systems.

This behavior can be understood as follows. It is well established by now that the ground state of the half-filled lowest Landau level is the composite fermion (CF) sea which, while being a compressible state in the thermodynamic limit, has a shell structure in terms of the CF Landau levels in the finite-size spherical geometry. The ground state energy exhibits local minima whenever energy shells below the chemical potential become completely filled, which occurs at $N = n^2$ with $n$ being a positive integer $13, 16$. Note that the correct flux sector for the CF sea is obtained at $2Q = 2N - 2$ for reasons provided by the CF theory. One may wonder what happens if the same flux sector studied for SLL $H_C$, i.e., $2Q = 2N - 3$, is investigated in the LLL. The LLL results are plotted in the bottom left panel of Fig. 1 where it is shown that energy minimum positions are a little bit different from those of the CF sea flux sector, while still not exhibiting the even-odd effect. Here, energy minima occur for a combination of two reasons. Now that the flux is shifted from that of the CF sea, composite fermions are subject to small, residual field, resulting in two ways of minimizing the total energy. The first is to minimize the kinetic energy of composite fermions by filling the similar shell structure as before, but with a different particle number sequence of $N = n(n + 1)$ with $n$ being a positive integer. The second is to reduce any residual interaction energy by forming Wigner crystals of composite fermions, which occur at $N = 4, 6, 8, 12,$ and $20$ with $N$ corresponding to the number of vertices in regular polyhedrons of Euclid. The energy reduction due to the second effect is seen via a weak local minimum at $N = 8$ and a kink at $N = 4$. The conclusion is that the SLL $H_C$ definitely exhibits the even-odd effect, which strongly supports the existence of superconductivity in a complete analogy with what happens in small superconducting grains. On the other hand, there is no such evidence for the lowest Landau level in any flux sectors. The clear signature of the even-odd effect in the half-filled SLL, but not in the LLL, is one of the most important results in this work. It is worthwhile to mention that the difference between the second and lowest Landau-level physics stems from very subtle quantitative changes in the Coulomb matrix elements parameterized by the Haldane pseudopotential.

Emboldened by strong support from the even-odd effect, we now set out to address the main issue in this paper: what are the quasi-particles that are being paired and what would be the appropriate order parameter for them? At this stage, it is important to distinguish between two types of the quasi-particles in the system. The first is the aforementioned composite fermions that, as shown in the following, form Cooper pairs in a similar fashion as electrons in the BCS theory with exactly the same statistics. The second is vortices that, in a direct analogy to their counterpart in the $p + ip$ superconductivity, may possess non-Abelian statistics. In this study, we are interested in the superconducting nature of composite fermions.

The superconducting order parameter is conventionally defined in the BCS theory as the expectation value of two fermion creation operators, say, $c^\dagger_{\mathbf{k}}$ and $c^\dagger_{\mathbf{k}'}$, in the fixed-phase coherent ground state. For fixed-number systems, an alternative but physically identical formulation of the BCS superconducting order parameter is given in terms of the matrix element between the ground states in the $N$ and $N + 2$ systems. In this work, we use such formulation and compute the superconducting order parameter in a numerically exact manner without any (BCS or otherwise) approximation. Instead of the usual momentum space representation, it is much more convenient here to compute the superconducting order parameter in real space: $F(r) = (N + 2)c^\dagger(r)c^\dagger(0)|N\rangle$ where $|N + 2\rangle$ and $|N\rangle$ are the ground states in the $N + 2$ and $N$ systems, respectively. The preceding definition, however, is not yet appropriate for the $5/2$ FQHE since the true elementary quasi-particle is not an electron, but a composite fermion. A necessary condition for creating a composite fermion is that the magnetic flux should be increased by $2\phi_0$, i.e., $Q \rightarrow Q + 1$, at the same time as
an electron is created. For this reason, a correct definition for the 5/2 FQHE superconducting order parameter may be given as follows:

$$F_c(r) = \langle N + 2, Q + 2 | c_\uparrow(r)c_\downarrow(0) | N, Q \rangle,$$

(1)

where $|N + 2, Q + 2\rangle$ and $|N, Q\rangle$ are the ground states for the particle-flux sector of $(N + 2, Q + 2)$ and $(N, Q)$, respectively. In the above, $c_\uparrow$ indicates the composite fermion creation operator accompanied with an appropriate increase in flux. We emphasize that Eq. (1) is considered an appropriate order parameter since it is proportional to the superconducting gap within the BCS theory [20]. It vanishes in the normal state not by symmetry reasons, but due to gap vanishing. Finally, it is interesting to note that the composite fermion pair operator used in Eq. (1) is conceptually similar to the composite boson operator considered by Read [22] and by Rezayi and Hal-dane [23], which exhibits Bose-Einstein condensation at odd-denominator fractions.

The composite fermion creation, however, requires more than just an increase in total flux: two flux quanta should be locally bound to an electron. In fact, the reason for the apparent absence of the Meissner effect is that composite fermions do not respond to external magnetic fields at half filling since all magnetic fields are captured away in the very process of forming composite fermions. Recalling that a flux quantum is actually equivalent to a correlation hole with $2\pi$ phase twist, what one has to do is to make sure that appropriate correlation holes are bound to each added electron. Unfortunately, however, it is close to impossible to derive the CF creation operator as a function of that of electrons in a closed analytic form since a composite fermion is an extended object containing full many-body correlations. In this study, we take a different approach. Since we do not know a priori how correlation holes are attached to the constituent electron inside a composite fermion, we let the system evolve to reach the lowest energy configuration. In other words, we perform exact diagonalization in the increased particle-flux sector of $(N + 2, Q + 2)$ under the constraint that an electron is pinned at the origin and the other at the position of $r$. Let us call the lowest energy state in this situation $|\psi_{N+2,Q+2}^\uparrow(r,0)\rangle$. Relegating the discussion for details to the following paragraph, we show the density profile of $|\psi_{N+2,Q+2}^\uparrow(r,0)\rangle$ in Fig. 2.

Let us now discuss how to obtain $|\psi_{N+2,Q+2}^\uparrow(r,0)\rangle$. To this end, it is necessary to know the LLL-projected operator for creating an electron at the position of $r$, which is given by:

$$c_{\text{proj}}^\dagger(r) = \sqrt{\frac{4\pi}{2Q+1}} \sum_m Y_{QQm}(\theta, \phi) c_m^\dagger,$$

(2)

where $r = R(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta - 1)$ with the radius $R = l_B \sqrt{Q}$ and the magnetic length $l_B = \sqrt{\hbar c/eB}$. $Y_{Qlm}$ is a complex conjugate of the monopole harmonics with the monopole strength, $Q$, the total angular momentum, $l$, and the $z$-component angular momentum, $m$. $c_m^\dagger$ is the usual operator for creating an electron in the $m$-eigenstate. The LLL-projected density operator is then defined by $\hat{\rho}_{\text{proj}}(r) = c_{\text{proj}}^\dagger(r)c_{\text{proj}}(r)$ with normalization that the maximum value of density is unity. Equipped with the LLL-projected density operator, one can obtain $|\psi_{N+2,Q+2}^\uparrow(r,0)\rangle$ by using the Lagrange multiplier method. That is, the constraint of pinning electrons at the origin and the $r$ position are imposed by introducing the Lagrange multiplier term in the Hamiltonian:

$$H = H_{\text{int}} - \lambda_1 [\hat{\rho}_{\text{proj}}(r) - 1] - \lambda_2 [\hat{\rho}_{\text{proj}}(0) - 1],$$

(3)

where $H_{\text{int}}$ is the inter-particle interaction Hamiltonian. The Lagrange multipliers, $\lambda_1$ and $\lambda_2$, are determined via imposing the saddle-point conditions for the ground state energy: $\partial e_{\text{gr}}/\partial \lambda_1 = \partial e_{\text{gr}}/\partial \lambda_2 = 0$. In the end,
$|\psi_r^{N+2,Q+2}(r,0)\rangle$ is given as the ground state satisfying the above saddle-point conditions. It is worthwhile to mention that, when two electrons are separated across the north and south poles, the constraint can be imposed precisely without recourse to the Lagrange multiplier method. We have verified that the results from the Lagrange multiplier method are completely consistent with that of the precise method.

The final ingredient for computing the superconducting order parameter of the 5/2 FQHE is the realization that $|\psi_r^{N+2,Q+2}(r,0)\rangle$ obtained from exact diagonalization is actually a normalized wave function:

$$|\psi_r^{N+2,Q+2}(r,0)\rangle = \frac{c_r^\dagger(r)c_r^\dagger(0)|N,Q\rangle}{\sqrt{\langle N,Q|c_r(0)c_r(r)c_r^\dagger(r)c_r^\dagger(0)|N,Q\rangle}} \quad (4)$$

where normalization is insured by the denominator which is nothing but the square root of the pair distribution function for composite fermion. Arguing that the CF pair distribution function should be similar to the electronic counterpart, $g(r)$, we arrive at the following conclusion: $c_r^\dagger(r)c_r^\dagger(0)|N,Q\rangle = \sqrt{g(r)}|\psi_r^{N+2,Q+2}(r,0)\rangle$. The CF superconducting order parameter is finally written as follows:

$$F_s(r) = \sqrt{g(r)}|N+2, Q+2|\psi_r^{N+2,Q+2}(r,0)\rangle. \quad (5)$$

Figure 3 shows $F_s(r)$ for both $H_2$ and the SLL $H_C$ obtained from exact diagonalization of $N = 8-18$ systems, which are compared with that of the LL $H_C$. As one can see, for $H_2$, the superconducting order parameter exhibits a rather smooth curve, being robust across various finite-size systems while, for the SLL $H_C$, there are somewhat more fluctuations even though the overall behavior is similar. The superconducting order parameter is essentially negligible for the LL $H_C$, which, along with the absence of the even-odd effect, is consistent with the recent conclusion of Storni, Morf, and Das Sarma [24]. It is emphasized that all our numerical results taken together convincingly establish that the real 5/2 FQHE state has underlying superconductivity. In addition to providing concrete theoretical evidence for superconductivity, an important lesson from $F_s(r)$ is that the superconducting order parameter is rather long-ranged, being sizable at least up to ten magnetic lengths. This is important since the superconducting coherence length sets a natural length scale for performing coherent quantum operations.

Now, let us make a connection between our findings and previous results from field theories. The composite fermion (CF) creation operator defined in our paper creates a bound state between an electron and a correlation hole under the condition that the magnetic field is increased by two flux quanta whenever an additional electron is added. In the lowest Landau level, a correlation hole is always concomitant with the formation of a vortex. Since the magnetic field is increased by two flux quanta, the created bound state is actually equivalent to that between an electron and a double vortex. Meanwhile, in the composite fermion Chern-Simons (CS) gauge field theory, it is generally accepted that binding a double vortex is adiabatically connected to the attachment of an infinitesimally thin tube of two flux quanta, i.e., Chern-Simons flux, into an electron. In this sense, our CF creation operator essentially plays the same role as creating CS-flux attached electrons. According to the theory of Halperin, Lee, and Read [23], and also Kalmeyer and Zhang [20], the ground state at $\nu = 1/2$ is the Fermi sea state of the CS-flux attached electrons and therefore, in this state, the superconducting order parameter must vanish when measured in terms of the CS-flux attached electrons. On the other hand, the Moore-Read Pfaffian state, which is the exact ground state of $H_3$, by construction contains the superconducting order in terms of the CS-flux attached electrons. It is for the first time in this work that these expectations are explicitly shown.
to be true in realistic Hamiltonians.

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