On Finsler spacetimes with a timelike Killing vector field

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Abstract
We study Finsler spacetimes and Killing vector fields taking care of the fact that the generalised metric tensor associated to the Lorentz–Finsler function $L$ is in general well defined only on a subset of the slit tangent bundle. We then introduce a new class of Finsler spacetimes endowed with a timelike Killing vector field that we call stationary splitting Finsler spacetimes. We characterize when a Finsler spacetime with a timelike Killing vector field is locally a stationary splitting. Finally, we show that the causal structure of a stationary splitting is the same of one of two Finslerian static spacetimes naturally associated to the stationary splitting.

Keywords: spacetime, Finsler, Killing, stationary, causality

1. Introduction

The main feature of Finsler geometry is that the associated generalised metric tensor $\tilde{g}$ (also called fundamental tensor) has a dependence on the directions. More precisely, at any point $p$ of a spacetime $\tilde{M}$ there are infinitely many scalar products $g_v$, one for each direction $v$ where $\tilde{g}$ is defined. In this respect, a gravitation theory based on Finsler geometry is a metric one and it can include, as its isotropic case, general relativity. There have indeed been several works in relativistic physics where Finsler geometry is used$^4$. We recall here the pioneering work of Randers [58] about asymmetry of time intervals, where it is introduced a class of Finsler metrics, nowadays called Randers metrics, together with its connection with 5D Kaluza–Klein theory; the work of Bogoslovsky on Lorentz symmetry violation (see e.g. [9–11]) and the finding by Gibbons et al [30] that general very special relativity is the group of transformations

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$^{4}$ For applications of Finsler geometry to non-relativistic physics and biology we recommend [5].
that leave invariant a Finsler metric introduced by Bogoslovsky; the work of Brandt about maximal proper acceleration (see e.g. [12]) where generalised metric on the tangent bundle of the spacetime are used; the extension of the Fermat’s principle to Finsler spacetime by Perlick [56], motivated by optics in anisotropic non-dispersive media; the applications to quantum gravity by Girelli et al [32]. More recently, mathematical models where Finsler geometry replaces Lorentzian one have been considered in gravitation, see e.g. [1, 7, 26, 42, 45, 50, 57], cosmology [34, 54, 62, 64] and in the so-called standard model extension (see e.g. [23, 40, 41, 59, 61]).

In this work we study Finsler spacetimes endowed with a timelike Killing vector field and we introduce a new class of Finsler spacetimes that can be viewed as a Finslerian extension of the class of the standard stationary Lorentzian manifolds (see, e.g. [24, 38]).

A Finsler spacetime is here defined as a smooth, connected, paracompact manifold \( \tilde{M} \) of dimension \( n + 1, n \geq 1 \), endowed with a generalised metric tensor \( \tilde{g} \), defined on an open subset \( A \subset T\tilde{M} \), having index 1 for each \( v \in A \) and which is the Hessian w.r.t. to the velocities of a Lorentz–Finsler function \( L \) (see definition 1.1 below). This is a function \( L : T\tilde{M} \to \mathbb{R} \) which is positively homogeneous of degree 2 in the velocities, i.e. \( L(p, \lambda v) = \lambda^2 L(p, v) \), \( \forall \lambda > 0 \).

The domain \( A \) where \( \tilde{g} \) is well defined and has index 1 is in general, a smooth, cone subset of \( T\tilde{M} \setminus 0 \), where 0 denotes the zero section in \( T\tilde{M} \). By ‘smooth cone subset’ we mean that \( \tilde{\pi}(A) = \tilde{M} \), where \( \tilde{\pi} : T\tilde{M} \to \tilde{M} \) is the natural projection from the tangent bundle \( T\tilde{M} \) to \( \tilde{M} \), and, for every \( p \in M, A_p := A \cap T_p\tilde{M} \) is an open linear cone (without the vertex \( \{0\} \)) of the tangent space \( T_p\tilde{M} \), i.e. if \( v \in A_p \) then \( \lambda v \in A_p \) for each \( \lambda > 0 \). Moreover \( A_p \) varies smoothly with \( p \in M \) in the sense that \( A_p \) is defined by the union of the solutions of a finite number of systems of inequalities

\[
\begin{align*}
E_{1,k}(p, v) &> 0 \\
\vdots \\
E_{m,k}(p, v) &> 0
\end{align*}
\]

where, for each \( k \in \{1, \ldots, l\} \), \( E_{1,k}, \ldots, E_{m,k} : \tilde{T}\tilde{M} \to \mathbb{R} \) are \( m_k \) smooth functions on \( \tilde{T}\tilde{M} \), positively homogeneous of degree 1 in \( v \).

In our paper, below section 2, \( A_p \) will be equal to \( T^+_p\tilde{M} \setminus T_p \) or, in some cases, to \( T^+_p\tilde{M} \setminus \tilde{T}_p \), where \( T^+_p\tilde{M} \) is a half-space in \( T^+_p\tilde{M} \) whose boundary is an hyperplane passing through \( \{0\} \) and \( T_p \) is a 1D subspace intersecting \( T^+_p\tilde{M} \). We will denote the set \( A = \bigcup_{p \in \tilde{M}} A_p \) by \( T^+\tilde{M} \setminus \tilde{T} \) in the former case and with \( T\tilde{M} \setminus \tilde{T} \) in the latter. Indeed, the cone subsets \( T^+\tilde{M} \setminus \tilde{T} \) and \( T\tilde{M} \setminus \tilde{T} \) are the natural candidate domains for a generalised metric tensor \( \tilde{g} \) if one asks for a Lorentz–Finsler function \( L \) on a product manifold \( \tilde{M} = \mathbb{R} \times M \) such that, on \( T\tilde{M} \), \( L \) reduces to the square of a classical Finsler metric. In fact, \( L \) cannot be smoothly extended to vectors which project on 0 in \( T\tilde{M} \), because the square of a classical Finslerian metric is not twice differentiable on zero vectors.

More generally, as suggested in [44], \( L \) could be smooth only on \( T\tilde{M} \setminus \tilde{Z} \) where \( \tilde{Z} \) is a zero measure subset in \( T\tilde{M} \). It is worth to recall that it would be possible to define \( L \) on a cone subset \( A \) where \( L \) is negative and, at each point \( p \in M, A_p \) is a convex salient cone and \( L \) is extendible and smooth on a cone subset around the set of lightlike vectors \( \{v \in T\tilde{M} \setminus 0 : L(v) = 0\} \) which defines the boundary of \( A \) in \( T\tilde{M} \setminus 0, \{1\} \) (\( A \) is then the cone subset of the Finslerian future pointing timelike vectors).

Let us now give some further details about the generalised metric tensor \( \tilde{g} \). Let \( A \subset T\tilde{M} \) be a cone subset as above and let \( \pi : A \to \tilde{M} \) be the restriction of the canonical projection, \( \pi : T\tilde{M} \to \tilde{M} \), to \( A \). Moreover, let \( \pi^*(T^*\tilde{M}) \) the pull-back cotangent bundle
over $A$. We consider the tensor product bundle $\pi^*(T^*\tilde{M}) \otimes \pi^*(T^*\tilde{M})$ over $A$ and a section $\tilde{g}: v \in A \mapsto \tilde{g}_v \in T_{\pi(v)}^*\tilde{M} \otimes T_{\pi(v)}^*\tilde{M}$. We say that $\tilde{g}$ is symmetric if $\tilde{g}_v$ is symmetric for all $v \in A$. Analogously, $\tilde{g}$ is said non-degenerate if $\tilde{g}_v$ is non-degenerate for each $v \in A$ and its index will be the common index of the symmetric bilinear forms $\tilde{g}_v$; moreover, $\tilde{g}$ will be said homogeneous if, for all $\lambda > 0$ and $v \in A$, $\tilde{g}_{v\lambda} = \tilde{g}_v \cdot \lambda$. A smooth, symmetric, homogeneous, non-degenerate section $\tilde{g}$ of the tensor bundle $\pi^*(T^*\tilde{M}) \otimes \pi^*(T^*\tilde{M})$ over $A$ will be said a generalised metric tensor.

**Definition 1.1.** A Finsler spacetime is a smooth $(n + 1)$-dimensional manifold $\tilde{M}$, $n \geq 1$, endowed with a generalised metric tensor $\tilde{g}$, defined on a (maximal) cone subset $A \subset T\tilde{M} \setminus 0$, such that $\tilde{g}_{(p,v)}$ has index 1, for each $(p,v) \in A$, and it is the fiberwise Hessian of a Lorentz–Finsler function $L$:

(i) $L : T\tilde{M} \to \mathbb{R}$, $L \in C^0(T\tilde{M}) \cap C^1(A)$,

(ii) $L(p, \lambda v) = \lambda L(p, v)$, for all $(p, v) \in T\tilde{M}$,

(iii) $\tilde{g}_{(p,v)}(u_1, u_2) := \frac{1}{2} \frac{\partial^2 L}{\partial s_1 \partial s_2}((p,v + s_1 u_1 + s_2 u_2)|_{(s_1, s_2) = (0, 0)})$, \hspace{1cm} (1)

for all $(p,v) \in A$. Moreover, there exists a smooth vector field $Y$ such that $Y_p \in \mathcal{A}_p$ and $L(p, Y_p) < 0$, for all $p \in \tilde{M}$, where $\mathcal{A}_p$ is the closure of $\mathcal{A}_p$ in $T_p\tilde{M} \setminus \{0\}$.

We denote a Finsler spacetime by $(\tilde{M}, L)$; in some circumstances, to emphasize that $\tilde{g}$ is defined and has index 1 only on $A \subset T\tilde{M} \setminus 0$, we denote it by $(\tilde{M}, L, A)$.

**Remark 1.2.** Observe that, whenever $\tilde{g}$ is defined on $T\tilde{M} \setminus 0$, definition 1.1 coincides with that in [49, definition 3]. We emphasize that $A$ has to be intended as the maximal open domain in $T\tilde{M} \setminus 0$ where $\tilde{g}$ is well defined and has index 1. We do not assume a priori that the connected component of $\mathcal{A}_p$ that contains $Y_p$ is convex and that all the lightlike vectors $(v \in T\tilde{M} \setminus 0)$, such that $L(v) = 0$ in such a component belong also to $\mathcal{A}_p$; anyway both these properties should hold for obtaining reasonable local and global causality properties (see [1, 49, 50]) and indeed they are satisfied by the class of stationary Finsler spacetimes that we introduce below. On the other hand, some Finslerian models do not satisfy the above second requirement: for example, in (deformed) very special relativity minus the square of the line element (see [26, 30]) is given by

$L(v) := -(-g(v,v))^{1-b}(-\omega(v))^{2b}$,

where $g$ is a Lorentzian metric on $\tilde{M}$ admitting a global smooth timelike vector field $Y$, which gives to $(\tilde{M}, g)$ a time orientation, and $\omega$ is a one-form on $\tilde{M}$ which is equivalent, w.r.t the metric $g$, to a future-pointing lightlike vector field. According to the value of the parameter $b \neq 1$, the fundamental tensor $\tilde{g}$ of $L$ is not defined or vanishing at $u$, which is also lightlike for $L$, while for all the timelike future-pointing vectors $v$ of $g$, we have that $L(v) < 0$.

**Remark 1.3.** We could allow more generality by not prescribing the existence of a Lorentz–Finsler function. This is a quite popular generalisation of classical Finsler geometry, see, e.g. [5, section 3.4.2] or [46, 48], and the references therein, where such structures are indeed called generalised metrics. Anyway as observed above and in [22, remark 2.11] the existence of a good (in the sense explained in remark 1.2) Lorentz–Finsler function avoids the occurrence of some causality issues.
Henceforth we will omit the dependence from the point on the manifold \( \tilde{M} \), writing simply \( L(v), \tilde{g}_v, \) etc, unless to reintroduce it in necessary cases (as in the statement of theorem 4.8 where we use the notation \( L(z, \cdot) \) to denote the map from \( T_z \tilde{M} \to \mathbb{R} \) obtained from \( L \) by fixing \( z \in \tilde{M} \)).

2. Killing vector fields

In this section we extend the notion of Killing vector field to Finsler spacetimes following the approach of [46], with the difference that the base space in our setting is the open subset \( A \subset TM \), while in [46] it is the standard one for Finsler geometry, i.e. the slit tangent bundle. We will only consider Killing vector fields that are vector fields on \( \tilde{M} \) by passing to their complete lifts on \( TM \) and then restricting them to the open base space \( A \). Clearly a more general approach is possible by considering generalised vector field, i.e. sections of \( \pi^*(TM) \), as in [53]. Another interesting approach is given in [37, section 2.9].

Let us give some preliminary notions. Let \( f: \tilde{M} \to \mathbb{R} \) be a smooth function. The complete lift of \( f \) on \( TM \) is the function \( f^c \) defined as \( f^c(v) := v(f) \) for any \( v \in TM \). Let now \( X \) be a vector field on \( \tilde{M} \) and set \( X^c \) the complete lift of \( X \) to \( TM \), defined by

\[
X^c(f \circ \pi) := X(f), \quad \text{for all smooth functions } f \text{ on } \tilde{M},
\]

\[
X^c(f^c) := (X(f))^c.
\]

Observe that if \((x^0, \ldots, x^n)\) are local coordinates on \( \tilde{M} \) and \((x^0, \ldots, y^0, \ldots, y^n)\) are the induced ones on \( TM \) (by an abuse of notation we denote the induced coordinate \( x^i = \tilde{x}^i \) again by \( x^i \)), then \((x^i)^c = y^i\), for all \( i = 0, \ldots, n \); so it is easy to check that in local coordinates \((X^c)^i_{(x,y)}\) is given by

\[
X^c_i(x) = \frac{\partial X^h(x)}{\partial x^i} + \frac{\partial X^h(x)}{\partial y^i} \frac{\partial}{\partial y^h},
\]

where we have used the Einstein summation convention; here \((x,y) \in TM\) has coordinates \((x^0, \ldots, x^n, y^0, \ldots, y^n)\), and \(X^h(x), h = 0, \ldots, n\), are the components of \(X\) w.r.t. \( \left( \frac{\partial}{\partial y^h} \right)_{h \in \{0, \ldots, n\}} \).

**Remark 2.1.** It is worth to observe that complete lifts on \( A \) are well defined by restricting functions and fields to the open subset \( A \) and, in the following, we will consider such restrictions and we will denote them, with an abuse of notation, always by \( f^c \) and \( X^c \).

The canonical vertical bundle map between \( \tilde{x}^*(TM) \) and \( T(TM) \) induces an injective bundle map \( i: \pi^*(TM) \to T(A) \); in local coordinate \((x^i, y^i)\) of \( TM \), if \( z_j = z^j(x,y) \frac{\partial}{\partial y^j} |_{(x,y)} \), \((x,y) \in A\), it holds

\[
i(z_j) = z^i(x,y) \frac{\partial}{\partial y^i} |_{(x,y)}.
\]

Observe that the map \( i \) induces also a injective homomorphism between \( x(\pi) \) and \( x(A) \), denoted always by \( i \), where \( x(\pi) \) and \( x(A) \) are the sets of smooth sections of \( \pi^*(TM) \) (over \( A \)) and, respectively, of \( T(A) \). In analogous way, a map \( j: T(A) \to \pi^*(TM) \) can be defined as
\[ j(w) := d\pi_i(w), \text{ for every } w \in T_A. \text{ Observe that } i(\pi^*(\overline{T^*M})) = \ker j \text{ and we have the following exact sequence} \]

\[ 0 \rightarrow \pi^*(\overline{T^*M}) \xrightarrow{j} T(A) \xrightarrow{i} \pi^*(\overline{T^*M}) \rightarrow 0. \]

Thus another homomorphism between \( \mathcal{X}(A) \) and \( \mathcal{X}(\pi) \), denoted always by \( j \), is defined and it holds the following exact sequence

\[ 0 \rightarrow \mathcal{X}(\pi) \xrightarrow{j} \mathcal{X}(A) \xrightarrow{i} \mathcal{X}(\pi) \rightarrow 0. \]

The vertical vectors fields are the elements of \( i(\mathcal{X}(\pi)) \). We define the Lie derivative relative to any smooth vector field \( X \) on \( \overline{M} \) on the tensor product bundles \( \pi^*(\overline{T^*M}) \) and \( \pi^*(\overline{T^*\overline{M}}) \) over \( A \) such that:

\[ \mathcal{L}_X : = X^f, \quad \mathcal{L}_X Y := i^{-1}([X^f, i(Y)]), \quad (3) \]

for any smooth function \( f \) on \( A \) and any \( Y \in \mathcal{X}(\pi) \), where \([\cdot, \cdot]\) is the Lie bracket on \( A \) (recall remark 2.1). Then \( \mathcal{L}_X \) is extended to any section of the tensor product bundles of the pull-back bundles \( \pi^*(\overline{T^*M}) \) and \( \pi^*(\overline{T^*\overline{M}}) \) over \( A \) by the generalised Willmore’s theorem for tensor derivations (see, e.g. [63, section 1.32]). Observe that the second equation in (3) is well posed, namely \( [X^f, i(Y)] \) is vertical. In fact, it is almost immediate to see that the Lie bracket of any vector field \( X^f \) and any vertical vector field is vertical; in local coordinates \((x', y')\) of \( \overline{T^*M} \), if \( Y = Y^k(x, y) \frac{\partial}{\partial x^k} \) and \( X = X^h(x) \frac{\partial}{\partial x^h} \), we have indeed

\[ [X^f, i(Y)] = \left( X^h \frac{\partial Y^k}{\partial x^h} + \frac{\partial X^h}{\partial x^k} Y^k - Y^h \frac{\partial X^k}{\partial x^h} \right) \frac{\partial}{\partial x^k}. \quad (4) \]

The Lie derivative \( \mathcal{L}_X \) on \( \pi^*(\overline{T^*\overline{M}}) \otimes \pi^*(\overline{T^*M}) \) is then,

\[ \mathcal{L}_X \tilde{g}(Y, Z) := X^f(\tilde{g}(Y, Z)) - \tilde{g}(\mathcal{L}_X Y, Z) - \tilde{g}(Y, \mathcal{L}_X Z), \quad (5) \]

for any \( \tilde{g} \in \pi^*(\overline{T^*\overline{M}}) \otimes \pi^*(\overline{T^*M}) \) and for every \( Y, Z \in \mathcal{X}(\pi) \). Observe that in a local base \( \left( \frac{\partial}{\partial x^0}, \ldots, \frac{\partial}{\partial x^n} \right) \) of \( \mathcal{X}(\pi) \), \( \frac{\partial}{\partial x^i} := \frac{\partial}{\partial x^i} \circ \pi \), for each \( i \in \{0, \ldots, n\} \), we have:

\[ \mathcal{L}_X \tilde{g} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = X^f(\tilde{g}_{ij}) - \tilde{g} \left( \mathcal{L}_X \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) - \tilde{g} \left( \frac{\partial}{\partial x^i}, \mathcal{L}_X \frac{\partial}{\partial x^j} \right) \]

\[ = X^f(\tilde{g}_{ij}) + \tilde{g} \left( \frac{\partial X^h}{\partial x^i} \frac{\partial}{\partial x^h}, \frac{\partial}{\partial x^j} \right) + \tilde{g} \left( \frac{\partial}{\partial x^i}, \frac{\partial X^h}{\partial x^j} \frac{\partial}{\partial x^h} \right) \]

\[ = X^f(\tilde{g}_{ij}) + \frac{\partial X^h}{\partial x^i} \tilde{g}_{j0} + \frac{\partial X^h}{\partial x^j} \tilde{g}_{ih}. \quad (6) \]

where \( \tilde{g}_{ij} := \tilde{g} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \), for all \( i, j \in \{0, \ldots, n\} \); here, in the second equality, we have used (4) and the fact that \( i(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} \).

**Definition 2.2.** Let \((\overline{M}, L, A)\) be a Finsler spacetime, \( K \) be a smooth vector field on \( \overline{M} \) and \( \psi \) its flow. We say that \( K \) is a **Killing vector field of** \((\overline{M}, L, A)\) if \( \mathcal{L}_K \tilde{g} = 0 \).

The following characterization of Killing vector fields holds:
Proposition 2.3. Let \((\bar{M}, L, A)\) be a Finsler spacetime (hence \(L \in C^0(\bar{T}\bar{M}) \cap C^1(A)\), according to definition 1.1), then \(K\) is a Killing vector field, if and only if \(K^c(L)|_A = 0\).

Proof. Observe that from (2) we have

\[
K^c(L)(x, y) = K^h(x) \frac{\partial L}{\partial x^h}(x, y) + \frac{\partial K^h}{\partial y^i}(x) \frac{\partial L}{\partial y^i}(x, y) \\
= K^h(x) \frac{\partial}{\partial x^h}(\tilde{g}_{ij}(x, y)y^i y^j) + \frac{\partial K^h}{\partial y^i}(x) y^i \frac{\partial \tilde{g}_{ij}(x, y)}{\partial y^i} y^j \\
= \left( K^c(\tilde{g}_{ij})(x, y) + \frac{\partial K^h}{\partial y^i}(x) \tilde{g}_{ij}(x, y) + \frac{\partial K^h}{\partial y^i}(x) \tilde{g}_{lh}(x, y) \right) y^i y^j
\]

for every \((x, y) \in A\). Thus, if \(K\) is Killing, by (6), \(K^c(L)(x, y) = \left( \mathcal{L}_{K\tilde{g}} \right)_{(x,y)}(y, y) = 0\), for every \((x, y) \in A\).

Let us now assume that \(K^c(L)|_A = 0\). Observe that if there exists an open subset \(U\) of \(\bar{M}\) where \(K\) vanishes then also \(K^c|^U = 0\) and, from (6), \(\left( \mathcal{L}_{K\tilde{g}} \right)_{(x,y)} = 0\) for all \((x, y) \in TU \cap A\). Let, then, \(p \in M\) such that \(K_p \neq 0\) and let us consider, in a neighborhood \(V \subset \bar{M}\) of \(p\), a coordinate system \(\left(x^1, \ldots, x^n\right)\) such that \(\frac{\partial}{\partial x^0} = K\). In the induced coordinates on \(\bar{T}\bar{M}\) we have that \(K^c = \frac{\partial}{\partial x^0} \circ \tilde{\pi}\) (recall (2)) and, so, \(K^c(L)(x, y) = \frac{\partial L}{\partial x^0}(x, y) = 0\) for all \((x, y) \in TV \cap A\).

Hence,

\[
\frac{\partial^3 L}{\partial y^i \partial y^j \partial x^0}(x, y) = \frac{\partial^3 L}{\partial y^i \partial y^j \partial x^0}(x, y) = 2 \frac{\partial \tilde{g}_{ij}}{\partial x^0}(x, y) = 0,
\]

for all \((x, y) \in TV \cap A\) and for all \(i, j \in \{0, \ldots, n\}\); from (6), \(\left( \mathcal{L}_{K\tilde{g}} \right)_{(x,y)} = 0\) for each \((x, y) \in TV \cap A\). Thus, \(\left( \mathcal{L}_{K\tilde{g}} \right)_{(p,y)} = 0\) for any \((p, y) \in A\) such that \(K_p \neq 0\). By continuity, \(\left( \mathcal{L}_{K\tilde{g}} \right)_{(q,y)} = 0\) for any \((q, y) \in A\) such that \(q\) belongs to the closure of \(\{ p \in \bar{M} : K_p \neq 0 \}\) and then \(\mathcal{L}_{K\tilde{g}} = 0\) everywhere in \(A\). \(\square\)

Let us now see how the flow of \(K^c\) behaves w.r.t. \(A\).

Lemma 2.4. Let \(\bar{M}\) be a manifold and \(A\) a cone subset of \(\bar{T}\bar{M}\) (according to the definition in the Introduction). Let \(X\) be a smooth vector field on \(\bar{M}\). Then for each \(\bar{p} \in \bar{M}\) there exists an interval \(I_p, 0 \in I_p\), and a neighborhood \(U\) of \(\bar{p}\) in \(\bar{M}\) such that the flow \(\tilde{\psi}\) of \(X\) is well defined on \(I_p \times TU\) and \(\tilde{\psi}(I_p \times (TU \cap A)) \subset A\).

Proof. Let us denote by \(\tilde{\psi}\) the flow of \(X\). It is well known that for any \(v \in \bar{T}\bar{M}\) there exists a neighborhood \(V \subset \mathbb{R} \times \bar{T}\bar{M}\) of \((0, v)\) such that

\[
\tilde{\psi} : V \rightarrow \bar{T}\bar{M}, \quad \tilde{\psi}(t, v) = (\psi(t, \bar{p}, d\psi_t(v)),
\]

\(p = \tilde{\pi}(v)\), is a local flow of \(X\). In fact,

\[
\frac{\partial \tilde{\psi}}{\partial t}(t, v) = \left( \frac{\partial \psi}{\partial t}(t, \bar{p}), \frac{\partial}{\partial t} (\partial_t \psi(t, \bar{p})(v)) \right),
\]

where \(\partial_t \psi(t, \bar{p})(v)\) denotes the partial differential of \(\psi\) w.r.t. the second variable at \((t, \bar{p})\), evaluated in \(v\) (hence \(\partial_t \psi(t, \bar{p})(v) = d\psi_t(v)\)). Thus, in local coordinates on \(\bar{T}\bar{M}\), we have
\[
\begin{align*}
\frac{\partial \tilde{\psi}}{\partial t}(t,v) &= X^h(\psi(t,p)) \frac{\partial}{\partial x^b} + \frac{\partial}{\partial t} \left( \frac{\partial \psi^h}{\partial x^j}(t,p) \right) v^j \\
&= X^h(\psi(t,p)) \frac{\partial}{\partial x^b} + \frac{\partial^2 \psi^h}{\partial t \partial x^j}(t,p) v^j \\
&= X^h(\psi(t,p)) \frac{\partial}{\partial x^b} + \frac{\partial^2 \psi^h}{\partial t \partial x^j}(t,p) v^j \\
&= X^h(\psi(t,p)) \frac{\partial}{\partial x^b} \left( X^h(\psi(t,p)) \right) v^j \\
&= X^c(\psi(t,p)).
\end{align*}
\]

As \( \frac{\partial^d}{\partial t^d}(0,p) = \delta^d_l \) for all \( p \in M \), where \( \delta^d_l \) are the Kronecker symbols, and \( \psi \) is smooth, we have that for any \( \tilde{p} \in \hat{M} \) any \( \epsilon > 0 \) there exists an interval \( I_{\tilde{p}} \) centered at 0, and a neighborhood \( U \) of \( \tilde{p} \) in \( \hat{M} \) such that \( \psi \) is well defined in \( I_{\tilde{p}} \times U \) and

\[
\left| \frac{\partial \psi^d}{\partial x^l}(t,p) - \delta^d_l \right| < \epsilon,
\]

for all \( (t,p) \in I_{\tilde{p}} \times U \) and each \( l, j \in \{0, \ldots, n\} \). Hence, for any \( u = (u^0, \ldots, u^n) \in \mathbb{R}^n \) such that \( |u| = 1 \) we have

\[
\left| \frac{\partial \psi^d}{\partial x^l}(t,p)u^l - \delta^d_l |u| \right| < (n + 1)\epsilon,
\]

for each \( l \in \{0, \ldots, n\} \). Being \( A \) open and \( A_p \) a cone for all \( p \in M \), we then conclude that the vector \( \left( \frac{\partial^d}{\partial t^d}(t,p)v^j, \ldots, \frac{\partial^d}{\partial t^d}(t,p)v^n \right) \in A_{\psi(t,p)} \) for all \( (t,p) \in I_{\tilde{p}} \times U \), provided that \( (v^0, \ldots, v^n) \in A_p \). Thus the flow of \( X^c \) is well defined on \( I_{\tilde{p}} \times TU \) and \( \hat{\psi}(I_{\tilde{p}} \times (TU \cap A)) \subset A \).

From propositions 2.3 and 2.4 it follows that \( L \) is invariant under the flow of \( K^c \). In fact, \( 0 = (K^c(L))(\psi_s(v)) = \frac{d}{ds}L(\psi_s(v)) \big|_{s=0} \), for all \( v \in A \) and \( s \in I_{\pi(v)} \), hence \( s \in I_{\pi(v)} \mapsto L(\tilde{\psi}_s(v)) \) is constant. From this observation we get that Killing vector fields are also the infinitesimal generators of local \( \hat{g} \)-isometries:

**Proposition 2.5.** Let \( (\hat{M}, L, A) \) be a Finsler spacetime, \( K \) be a smooth vector field on \( \hat{M} \) and let us denote by \( \psi \) the flow of \( K \). Then \( K \) is a Killing vector field if and only if for each \( v \in A \) and for all \( v_1, v_2 \in T_{\psi(v)}\hat{M} \), we have

\[
\tilde{g}_{\psi(t,v)}(dv_1(\psi(v)), dv_2(\psi(v))) = \tilde{g}_v(v_1, v_2),
\]

for all \( t \in I_{\tilde{p}} \), where \( I_{\tilde{p}} \subset \mathbb{R} \) is an interval containing \( \theta \) such that the stages \( \psi_t \) are well defined in a neighbourhood \( U \subset M \) of \( p = \pi(v) \) and \( dv_1(\psi(v)) \in A \), for each \( t \in I_{\tilde{p}} \).

**Proof.** Let \( v \in A \) and \( p = \pi(v) \). From lemma 2.4, the flow \( \tilde{\psi} \) of \( K^c \) is well defined in \( I_{\tilde{p}} \times TU \), for an interval \( I_{\tilde{p}} \) containing \( 0, \tilde{\psi}(0,v) = v \), and a neighborhood \( U \) of \( p \) in \( M \) and, moreover, \( \tilde{\psi}(I_{\tilde{p}} \times (TU \cap A)) \subset A \). If (7) holds, then in particular \( L(dv_1(\tilde{\psi}(v))) = g_{\psi(t,v)}(dv_1(\psi(v)), dv_1(\psi(v))) = g_v(v, v) = L(v) \), for all \( t \in I_{\tilde{p}} \). Hence \( 0 = \frac{d}{dt}L(dv_1(\psi(v))) \big|_{s=0} = (K^c(L))(v) \) and then we conclude using proposition 2.3. The converse
follows observing that, being \( L \) invariant under the flow of \( K^s \),
\[
\begin{align*}
g_{(\psi^s)(v)}(d\psi^s(v_1), d\psi^s(v_2)) &= \frac{1}{2} \frac{\partial^2 L}{\partial s_1 \partial s_2} \left( d\psi^s(v) + s_1 d\psi^s(v_1) + s_2 d\psi^s(v_2) \right) \big|_{(s_1, s_2) = (0, 0)} \\
&= \frac{1}{2} \frac{\partial^2 L}{\partial s_1 \partial s_2} (d\psi^s(v + s_1 v_1 + s_2 v_2)) \big|_{(s_1, s_2) = (0, 0)} \\
&= \frac{1}{2} \frac{\partial^2 L}{\partial s_1 \partial s_2} (v + s_1 v_1 + s_2 v_2) \big|_{(s_1, s_2) = (0, 0)} = g^s(v_1, v_2).
\end{align*}
\]

\[\square\]

3. Stationary splitting Finsler spacetimes

As in the Lorentzian setting, we say that a Finsler spacetime \( (\tilde{M}, L, A) \) is stationary if it admits a timelike Killing vector field. Here timelike means that \( L(K_p) < 0 \) for all \( p \in \tilde{M} \).

A particular type of stationary Lorentzian manifolds (called standard stationary) can be obtained starting from a product manifold \( M = \mathbb{R} \times M \), a Riemannian metric \( g \), a one-form \( \omega \) and a positive function \( \Lambda \) on \( M \), by considering the Lorentzian metric:
\[
\tilde{g} = -\Lambda dt^2 + \omega \otimes dt + dt \otimes \omega + g.
\]

It is well known (see e.g. [29, appendix C]) that any stationary Lorentzian spacetime is locally isometric to a standard one.

Looking at the quadratic form associated to the Lorentzian metric (8) with the aim of introducing a Finslerian analogue, we are led to the Lagrangian \( L : TM \to \mathbb{R} \),
\[
L(\tau, v) = -\Lambda \tau^2 + 2B(v)\tau + F^2(v),
\]
where \( \Lambda \) is a positive function on \( M \), \( B : TM \to \mathbb{R} \) is a fiberwise positively homogeneous of degree 1 Lagrangian which is at least \( C^3 \) on \( TM \setminus 0 \) and \( F : TM \to [0, +\infty) \), \( F \in C^0(TM) \cap C^1(TM \setminus 0) \) is a classical Finsler metric on \( M \), i.e. it is fiberwise positively homogeneous of degree 1 and
\[
g_0(u, u) := \frac{1}{2} \frac{\partial^2 F^2}{\partial s_1 \partial s_2} (v + s_1 u + s_2 u) \big|_{(s_1, s_2) = (0, 0)} > 0
\]
for all \( v \in TM \setminus 0 \) and all \( u \in T_{\pi_M(v)}M \), where \( \pi_M \) is the canonical projection \( \pi_M : TM \to M \).

Let us now introduce some notation. We will denote coordinates \((t, x^1, \ldots, x^n)\), in \( \tilde{M} = \mathbb{R} \times M \) by \( z \), i.e. \( z = (t, x^1, \ldots, x^n) \). Natural coordinates in \( TM \), will be then denoted by \((z, \dot{z})\), that is \((z, \dot{z}) = (t, x^1, \ldots, x^n, \tau, y^1, \ldots, y^n)\). For a Lagrangian \( A : TM \to \mathbb{R} \) and \( v \in TM \setminus 0 \), let us denote by \((\partial A)_v\), and \((\partial^2 A)_v\), respectively, the fiberwise differential and Hessian of \( A \) at \( v \), i.e. for all \( u, u_1, u_2 \in TM \)
\[
(\partial A)_v(u) := \frac{d}{ds} (A(v + su)) \big|_{s=0},
\]
\[
(\partial^2 A)_v(u_1, u_2) := \frac{\partial^2}{\partial s_1 \partial s_2} (A(v + s_1 u_1 + s_2 u_2)) \big|_{(s_1, s_2) = (0, 0)}.
\]

\[5\] Analogously a vector \( v \in TM \) is said lightlike (resp. spacelike; causal) if \( L(v) = 0 \) (resp. either \( L(v) > 0 \) or \( v = 0 \); \( L(v) \leq 0 \)); anyway observe that, being \( g \) defined only on \( A \), \( L(v) = \tilde{g}(v, v) \) only for vectors \( v \in A \), so whenever \( v \notin A \) this causal character is purely formal, and it is no way related to the generalised metric \( \tilde{g} \).
These are respectively sections of the pull-back bundles $\pi^*_M(T^*M)$ and $\pi^*_M(T^*M) \otimes \pi^*_M(T^*M)$ over $TM \setminus 0$.

The analogous fiberwise derivatives, for a Lagrangian $L : TM \to \mathbb{R}$ on $\tilde{M}$, are denoted by $(\partial_t L)_w$ and $(\partial_{t \tau} L)_w$, $w \in TM$, and when $L$ is a Lorentz–Finsler function on $\tilde{M}$ then $\frac{1}{2}(\partial_{t \tau} L)_w$ is, then, the generalised metric tensor $\tilde{g}_w$ already introduced in (1).

Let us denote by $\tilde{T}$ the trivial line subbundle of $TM$ defined by the vector field $\partial_t$. Let $z = (t, x) \in \tilde{M}$ and let us denote by $T^+_p \tilde{M}$ and $T^-_p \tilde{M}$ respectively the open half-spaces of $T_p \tilde{M}$ given by $T^+_p \tilde{M} := \{(t, v) \in T_p \tilde{M} : \tau > 0\}$ and $T^-_p \tilde{M} := \{(t, v) \in T_p \tilde{M} : \tau < 0\}$; moreover let $T^+_p \tilde{M}$ and $T^-_p \tilde{M}$ be their closures in $T_p \tilde{M}$. Let us then denote by $T^+_\tilde{M} \setminus \tilde{T}$ (resp. $T^-_\tilde{M} \setminus \tilde{T}$) the open cone subset of $TM$ given by $T^+_\tilde{M} \setminus \tilde{T} := \cup_{p \in \tilde{M}} T^+_p \tilde{M} \setminus T_p$ (resp. $T^-_\tilde{M} \setminus \tilde{T}$) (resp. $T^-_\tilde{M} \setminus \tilde{T}$) the open cone subset defined by $T^+_\tilde{M} \setminus \tilde{T} := \cup_{p \in \tilde{M}} T^+_p \tilde{M} \setminus T_p$ (resp. $T^-_\tilde{M} \setminus \tilde{T} := \cup_{p \in \tilde{M}} T^-_p \tilde{M} \setminus T_p$). Finally, let us denote by $\tilde{TM} \setminus \tilde{T}$ the open cone subset of $\tilde{TM}$ defined as $\tilde{TM} \setminus \tilde{T} := \cup_{p \in \tilde{M}} T^+_p \tilde{M} \setminus T_p$.

**Remark 3.1.** Notice that $L$ is continuous on $\tilde{TM}$ and at least $C^3$ on $\tilde{TM} \setminus \tilde{T}$. Since, in general, $B$ is not differentiable at the zero section of $TM$, $L$ is not differentiable at vectors $y \in \tilde{T}$. An exception is when $B$ reduces to a one-form on $M$ (being, so, differentiable at 0 too) so that $L$ is $C^3$ on $\tilde{TM} \setminus 0$.

**Proposition 3.2.** Let $L : TM \to \mathbb{R}$ defined as in (9) and $w \in T_{(t,x)}M$ such that $L(w) = \frac{1}{2} \Lambda(x)(\tau + sp)^2 + 2B_s(sv)(\tau + sp) + F^2(sv)$. Hence the right and left derivative at $s = 0$ of $L(w + su)$ do exist and are respectively equal to $-2\Lambda(x)\tau + 2B_s(sv)\tau$ and $-2\Lambda(x)\tau + 2B_s(-sv)\tau$; thus they are equal if and only if $B_s(v) = -B_s(-v)$. In this case, being $B_s$ positively homogeneous of degree 1, $(\partial_{t \tau} L)_w(u) = -2\Lambda(x)\tau + 2B_s(sv)\tau$ is linear in $u \in \mathbb{R}$ and only if $B_s$ is linear.

We characterize now when, for $L$ in (9), $(\partial_{t \tau} L)_w$ has index 1 for all $w \in T^+_\tilde{M} \setminus \tilde{T}$, and for all $w \in TM \setminus \tilde{T}$.

**Proposition 3.3.** Let $L : TM \to \mathbb{R}$ defined as in (9), then $(\partial_{t \tau} L)_w$ has index 1 at $w \in T^+_\tilde{M} \setminus \tilde{T}$ (resp. $w \in T^-_\tilde{M} \setminus \tilde{T}$) if, and only if, $(\partial_{v \tau} B)_v$ is positive semi-definite (resp. $(\partial_{v \tau} B)_v$ is negative semi-definite). Conversely, if there exists $\tilde{\tau} > 0$ (resp. $\tilde{\tau} < 0$) such that $(\partial_{v \tau} L)_{(t, v)}$ has index 1 for $v \in TM \setminus 0$ and for all $\tau > \tilde{\tau}$ (resp. $\tau < \tilde{\tau}$) then $(\partial_{v \tau} B)_v$ is positive semi-definite (resp. $(\partial_{v \tau} B)_v$ is negative semi-definite).

**Proof.** Let us prove the sufficient condition in the first equivalence. For $(\tau, v) \in T^+_\tilde{M} \setminus \tilde{T}$, the fiberwise Hessian of $L$ is given by

$$
(\partial_{t \tau}^2 L)_{(\tau, v)} = -\Delta v^2 + (\partial_{t \tau} B)_v \otimes dv + dv \otimes (\partial_{t \tau} B)_v + \tau(\partial_{v \tau}^2 B)_v + g_v.
$$

First notice that, if $(\partial_{t \tau} B)_v = 0$ then, being $\Lambda$ positive and $\tau$ non-negative, we immediately get that $(\partial_{t \tau}^2 L)_{(\tau, v)}$ has index 1. Assume now that $(\partial_{t \tau} B)_v \neq 0$ and consider the vector $v_1 \in T_{v \tau}(v)M$
representing \((\partial B)\), w.r.t. the scalar product \(\langle \cdot, \cdot \rangle_{\nu, \tau}\) defined by \(g_{\nu} + \tau(\partial^2 B)_{\nu}\). Now take a \(((\cdot, \cdot)_{\nu, \tau})\)-orthonormal basis \(v_1, v_2, \ldots, v_0\) of \(\text{Ker}(\partial B)\). The matrix of \((\partial^2 L)_{(\tau, \nu)}\) relative to the basis \(e, v_1, v_2, \ldots, v_0\) of \(T_{\pi((\tau, \nu))}M\), where \(e = \frac{\partial}{\partial \nu}\), is given by

\[
\begin{pmatrix}
-\Lambda & \langle \partial B \rangle_{\nu}(v_1) & 0 & \cdots & 0 \\
\langle \partial B \rangle_{\nu}(v_1) & \langle v_1, v_1 \rangle_{\nu, \tau} & 0 & \cdots & 0 \\
0 & 0 & & & \\
\vdots & \vdots & & & I \\
0 & 0 & & & 
\end{pmatrix}
\]

Since \(-\Lambda(v_1, v_1)_{\nu, \tau} - \langle \partial B \rangle_{\nu}(v_1) \rangle < 0\), we conclude that \((\partial^2 L)_{(\tau, \nu)}\) has index 1. Conversely, let us assume that \((\partial^2 L)_{(\tau, \nu)}\) has index 1, for any \((\tau, \nu) \in T^+ M \setminus T\) with \(\tau > \tilde{\tau}\). For a fixed \((\tau, \nu) \in T^+ M \setminus T\), with \(\tau > \tilde{\tau}\), consider the Lorentzian metric \(\tilde{g}_{(\tau, \nu)} := (\partial^2 L)_{(\tau, \nu)}\) and the \(g_{(\tau, \nu)}\)-orthogonal complement, \(D_{(\tau, \nu)}\), in \(T_{\pi((\tau, \nu))}M\) of the 1D subspace generated by the vector \((1, 0)\). This is given by vectors \((\tau_1, v_1)\) with \(\tau_1 = \frac{\langle \partial B \rangle_{\nu}(v_1)}{\Lambda}\), for each \(v_1 \in TM\). Thus

\[
\tilde{g}_{(\tau, \nu)} \left( \left( \frac{\langle \partial B \rangle_{\nu}(v_1)}{\Lambda}, v_1 \right), \left( \frac{\langle \partial B \rangle_{\nu}(v_1)}{\Lambda}, v_1 \right) \right) = -\left( \frac{\langle \partial B \rangle_{\nu}(v_1)}{\Lambda} \right)^2 + \tau \left( \frac{\partial^2 B}{\partial y^2} \right)_{\nu}(v_1, v_1) + g_{\nu}(v_1, v_1) > 0
\]

for all \(v_1 \in TM \setminus 0\). If there exists \(v_1 \in TM \setminus 0\) such that \((\partial^2 B)_{\nu}(v_1, v_1) < 0\), then, being \(D_{(\tau, \nu)}\) independent of \(\tau\), for any fixed \(v \in TM\), we can consider the limit as \(\tau \to +\infty\) in the left-hand side of (11), obtaining a negative quantity for \(\tau\) big enough, in contradiction with (11).

It is easy to check that the analogous statement involving \((\partial^2 B)\), negative semi-definite, holds as the previous one with obvious modifications.

\begin{corollary}
Let \(L: T\hat{M} \to \mathbb{R}\) defined as in (9) and \(x \in M\). Then \((\partial^2 L)_{(\tau, \nu)}\) has index 1 for all \(\nu \in T_x M \setminus \{0\}\) and all \(\tau \in \mathbb{R}\) if and only if \(B(x, \cdot)\) is linear form on \(T_x M\).
\end{corollary}

\begin{proof}
It is trivial to check that if \(B(x, \cdot)\) is a linear form on \(T_x M\) then \((\partial^2 L)_{(\tau, \nu)}\) has index 1 for all \(\tau \in \mathbb{R}\) and all \(v \in T_x M \setminus \{0\}\). Conversely, from proposition 3.3 \((\partial^2 B)\), must vanish on \(T_x M \setminus 0\) and, being \(B\) fiberwise positively homogeneous of degree 1, it must be linear on \(T_x M\).
\end{proof}

As the fiberwise Hessian of a classical Finsler metric is positive semi-definite, from proposition 3.3 we immediately get:

\begin{corollary}
Let \(L: T\hat{M} \to \mathbb{R}\) defined as in (9) with \(B = \omega F_1\) (resp. \(B = \omega - F_1\)), where \(\omega\) and \(F_1\) are, respectively, a one-form and a Finsler metric on \(M\). Then \((\partial^2 L)_{\omega}\) has index 1 for all \(w \in T^+ M \setminus T\) (resp. \(w \in T^+ M \setminus T\)).
\end{corollary}

\begin{remark}
We have found that if the conditions of proposition 3.3 and corollary 3.4 hold on the whole \(T^+ M \setminus T\) (resp. \(T^- M \setminus T\); \(TM \setminus T\)) then \((\mathbb{R} \times M, L, T^+ M \setminus T\) (resp. \((\mathbb{R} \times M, L, T^- M \setminus T\); \((\mathbb{R} \times M, L, TM \setminus T)\)) is a Finsler spacetime. Notice, indeed, that the
role of the vector field $Y$, such that $Y_p \in A_p$ for all $p \in \mathbb{R} \times M$, can be taken by $\partial_i$ in the first and in the last case and by $-\partial_i$ in the second one.

**Remark 3.7.** We could consider the case when the assumptions on $B$ hold pointwise for all $x \in M$ being all the three cases possible. Anyway, take into account that we would not get a Finsler spacetime due to the impossibility of fulfilling the assumption about the existence of a smooth vector field $Y$ such that $Y_p \in A_p$ and $L(p, Y_p) < 0$, for all $p \in M$. On the other hand, the case when $(\partial_i, B)$ is either positive or negative semi-definite for all $v \in TM$ include also the possibility that $B(x, \cdot)$ is a linear form on $T_x M$ for some $x \in M$.

Henceforth, we will denote by $(\tilde{M}, L)$, $\tilde{M} = \mathbb{R} \times M$, each of the Finsler spacetimes $(\tilde{M}, L, T^+\tilde{M} \setminus T)$, $(\tilde{M}, L, T^-\tilde{M} \setminus T)$, $(\tilde{M}, L, T\tilde{M} \setminus T)$, associated to $L$ given in (9), implicitly assuming that if $\partial^2_{ij} B$ is positive semi-definite (resp. $\partial^2_{ij} B$ is negative semi-definite; $B$ is a one-form on $M$) then $\tilde{g}$ is defined and has index 1 on the cone subset $A$ given by $T^+\tilde{M} \setminus T$ (resp. $T^-\tilde{M} \setminus T$; $T\tilde{M} \setminus T$).

**Remark 3.8.** In analogy with the Lorentzian case, we say that a causal vector $w^\dot{i}$, with $w \in A \setminus 0$ is future-pointing (resp. past-pointing) if $g_{w,w}(w, Y) < 0$ (resp. $g_{w,w}(w, Y) > 0$), whenever $w \in A$, or $w$ is causal and belongs to the closure of the set of future-pointing vectors in $A$. In the case of a stationary splitting Finsler spacetime, taking into account that when $A = T^-\tilde{M} \setminus T$ we pick $-\partial_i$ as the vector field $Y$, we have that a causal vector $(\tau, v)$ of $(\tilde{M}, L, T^+\tilde{M} \setminus T)$ (resp. of $(\tilde{M}, L, T^-\tilde{M} \setminus T)$; $(\tilde{M}, L, T\tilde{M} \setminus T)$) with $(\tau, v) \in T^+\tilde{M} \setminus T$ (resp. $(\tau, v) \in T^-\tilde{M} \setminus T$; $(\tau, v) \in T\tilde{M} \setminus T$) is future-pointing if $-\Lambda \tau + (\partial_j B)(v) > 0$ (resp. $-\Lambda \tau + (\partial_j B)(v) < 0$). By homogeneity, the first (resp. second) inequality becomes $-\Lambda \tau + B(v) < 0$ (resp. $-\Lambda \tau + B(v) > 0$). Being $B(0) = 0$, the vectors of the type $(\tau, 0)$, $\tau > 0$, (resp. $\tau < 0$) are then also timelike and future-pointing (resp. past-pointing). We will see in remark 6.2 that the causal future-pointing causal vectors of $(\tilde{M}, L, T^-\tilde{M} \setminus T)$ (resp. $(\tilde{M}, L, T^-\tilde{M} \setminus T)$) at $p \in \tilde{M}$ are all and only the causal vectors in $T^+\tilde{M} := \bigcup_{p \in \tilde{M}} T^+_p \tilde{M}$ (resp. $T^-\tilde{M} := \bigcup_{p \in \tilde{M}} T^-_p \tilde{M}$).

**Proposition 3.9.** Assume that $(\tilde{M}, L)$, $\tilde{M} = \mathbb{R} \times M$ and $L$ as in (9), is a Finsler spacetime, then $\partial_i$ is timelike and Killing.

**Proof.** We consider the fundamental tensor $\tilde{g}$ of $L$ in (10). Let us prove that $L_{\partial_i \tilde{g}}(\partial_j, \partial_k) = 0$ for all $i, j \in \{0, \ldots , n\}$, where $z^0 = t$ and $z^i = x^i$ for all $i \in \{1, \ldots , n\}$. From (6), it is enough to prove that $(\partial_i)^2(\tilde{g}_{ij}) = 0$, for all $i, j \in \{0, \ldots , n\}$. From (2), we have $-\partial_i^2(\tilde{g}_{ij}) = \partial_i \tilde{g}_{ij} = 0$. \(\square\)

Proposition 3.3, corollary 3.4, remark 3.6 and proposition 3.9 justify the following definition:

**Definition 3.10.** Let $\tilde{M} = \mathbb{R} \times M$, $L: T\tilde{M} \to \mathbb{R}$ defined as in (9), such that $\partial^2_{ij} B$ is positive semi-definite on $TM \setminus 0$ (resp. $\partial^2_{ij} B$ is negative semi-definite on $TM \setminus 0$) then we call $(\tilde{M}, L)$ a stationary splitting Finsler spacetime.

---

*See footnote 5.*
4. On the local structure of stationary Finsler spacetimes

An important role in several geometric properties of stationary Lorentzian manifolds is played by the distribution $D$ orthogonal to the Killing vector field $K$. For example, since $K$ is time-like, $D$ is spacelike and then, in a standard stationary Lorentzian manifold $(\mathbb{R} \times M, \tilde{g})$, $\tilde{g}$ given in (9), it is the horizontal distribution of the semi-Riemannian submersion $\pi: \mathbb{R} \times M \to M$, where the Riemannian metric on $M$ is equal to $g + \frac{\ Lambda}{\pi} \omega$ (see, e.g. [21]). Moreover, if $D$ is integrable then a stationary Lorentzian manifold $(\tilde{M}, \tilde{g})$ is said static (see [52, definition 12.35]) and is locally isometric to a warped product $(a, b) \times S$, with the metric $-\Lambda d\tau^2 + g$, where $S$ is an integral manifold of the distribution, $(a, b) \subset \mathbb{R}$, $\phi_\tau(\partial_\tau) = K$, being $\phi: (a, b) \times S \to M$ a local isometry, $(g(\phi^*u, \phi^*v) = g(u, v)$ for all $u, v \in D$ (see [52, proposition 12.38]).

A natural generalisation of the orthogonal distribution to $K$ in the Finsler setting is the distribution $\ker(\partial_\lambda L(K))$ where $\partial_\lambda L(K)$ denotes the one-form on $M$ given by $\frac{\partial g}{\partial \lambda}(K)du^i$.

**Remark 4.1.** In order to get a well defined distribution $\ker(\partial_\lambda L(K))$, we need that $L$ is differentiable at $K$, for all $z \in M$. Thus, we assume that $L$ is $C^1$ on $TM$ whenever we need to consider such a distribution, as in theorem 4.8. Recall that for a stationary splitting Finsler spacetime $(\mathbb{R} \times M, L)$, this assumption implies that $B$ reduces to a one-form on $M$ (proposition 3.2) that we will denote, in this section, by $\omega$. Recall that from corollary 3.4, $\tilde{g}$ is then defined on $TM \setminus T$.

Following [22], we introduce the next two definitions:

**Definition 4.2.** We say that a Finsler spacetime $(\tilde{M}, L, A)$ is static if there exists a timelike Killing vector field $K$ such that the distribution of hyperplanes $\ker(\partial_\lambda L(K))$ is integrable.

**Definition 4.3.** We say that a Finsler spacetime $(\tilde{M}, L, TM \setminus T)$, where $T$ is a line sub-bundle of $TM$, is standard static if there exist a smooth non vanishing global section $K$ of $T$, a Finsler manifold $(M, F)$, a positive function $\Lambda$ on $M$ and a smooth diffeomorphism $f: \mathbb{R} \times M \to \tilde{M}$, $f = f(t, x)$, such that $\partial_t = f^*(K)$ and $L(f_\tau(t, x)) = -\Lambda \tau^2 + F^2(\psi)$, for all $(\tau, \psi) \in T(\mathbb{R} \times M)$.

In relation to the local structure of a stationary Finsler spacetime, we introduce also the following definition:

**Definition 4.4.** A stationary Finsler spacetime $(\tilde{M}, L, TM \setminus T)$, where $T$ is a line subbundle of $\tilde{M}$, with timelike Killing field $K$, $K_z \in T_z$, for all $z \in \tilde{M}$, is locally a standard stationary splitting if for any point $z \in \tilde{M}$ there exists a neighborhood $U_z \subset \tilde{M}$ of $z$ and a diffeomorphism $\phi: I_z \times S_z \to U_z$, where $I_z = (-\varepsilon_z, \varepsilon_z)$ is an interval in $\mathbb{R}$ and $S_z$ a manifold, such that, named $t$ the natural coordinate of $I_z$, $\phi_\tau(\partial_\tau) = K|_{U_z}$, and for all $(\tau, \psi) \in T(I_z \times S_z)$, $L \circ \phi_\tau(\tau, \psi) = -\Lambda \tau^2 + \omega(\tau) \psi + F^2(\psi)$, where $\Lambda$, $\omega$ and $F$ are respectively a positive function, a one-form and a Finsler metric on $S_z$. Moreover, we say that $(\tilde{M}, L)$ is locally standard static if for any $z \in \tilde{M}$ there exists a map $\phi$ as above such that $\omega = 0$.

**Remark 4.5.** We observe that, although $L$ might be not twice differentiable along vectors $w \in T$, its fiberwise second derivative at $w \in T \setminus 0$, evaluated at any couple of vectors $u_1, u_2 \in T_{\pi(w)}$, $\frac{\partial^2 L}{\partial u_1 \partial u_2}(u_1, u_2) := \frac{\partial}{\partial s_1} \frac{\partial}{\partial s_2} L(w + s_1 u_1 + s_2 u_2)|_{(s_1, s_2) = (0, 0)}$, does exist. Indeed, let $\lambda_1, \lambda_2 \in \mathbb{R}$ such that $u_1 = \lambda_1 w$, $u_2 = \lambda_2 w$, then by homogeneity, we have
\[ L(w + s_1 u_1 + s_2 u_2) = L((1 + s_1 \lambda_1 + s_2 \lambda_2)w) = (1 + s_1 \lambda_1 + s_2 \lambda_2)^2 L(y), \]

for small \( s_1, s_2 \in \mathbb{R} \). Thus \((\partial^2_{z^2} L)_{w_0}(u_1, u_2) = 2\lambda_1 \lambda_2 L(w)\). This fact will be used in the following propositions, where we aim to characterize stationary and static Finsler spacetimes which are locally standard.

Recalling remark 4.1, we assume that \( L \in C^1(T\tilde{M}) \) and we denote by \( D \) the distribution of hyperplanes in \( T\tilde{M} \) given by \( \ker(\partial L(K)) \).

**Remark 4.6.** If \( L \) is differentiable on \( T\tilde{M} \) and it is fiberwise positively homogeneous of degree 2, we have that \((\partial L)_K(K) = 2L(K) < 0\) hence \((\partial L)_K \neq 0\), \( K \notin D_z \) and \( T_z\tilde{M} = D_z \oplus [K_z] \), for all \( z \in \tilde{M} \).

Let us define the map

\[ \bar{B}(w) := w \in T\tilde{M} \mapsto \frac{1}{2}(\partial L)_w(K_{\pi(w)}). \]

**Lemma 4.7.** Let \((\tilde{M}, L, A)\) be a stationary Finsler spacetime with timelike Killing vector field \( K \). Assume that \( L \in C^1(T\tilde{M}) \), \( L(K) = L(-K) \) and

\[ L(w \pm K_{\pi(w)}) = L(w) + L(K_{\pi(w)}), \]

for all \( w \in D \). Then

\[ \bar{B}(w) = \frac{1}{2}(L(w + K_{\pi(w)}) - L(w) - L(K_{\pi(w)})), \]

for all \( w \in T\tilde{M} \). Moreover, \( \bar{B} \) is fiberwise linear.

**Proof.** For each \( w \in D \), let \( w_D \in D \) and \( \lambda_w \in \mathbb{R} \) be such that \( w = w_D + \lambda_w K_{\pi(w)} \) (recall remark 4.6). Moreover, let \( \epsilon(x) = \text{sign}(x) \), if \( x \in \mathbb{R} \setminus \{0\} \), and \( \epsilon(0) = 0 \). By definition and our assumptions, we obtain

\[ \bar{B}(w) = \frac{1}{2} \frac{d}{ds}L(w + sK_{\pi(w)}) \big|_{s=0} = \frac{1}{2} \frac{d}{ds}L(w_D + \lambda_w K_{\pi(w)} + sK_{\pi(w)}) \big|_{s=0} \]

\[ = \frac{1}{2} \frac{d}{ds} \left( L(w_D) + (\lambda_w + s)^2 L(\epsilon(\lambda_w + s)K_{\pi(w)}) \right) \big|_{s=0} \]

\[ = \frac{1}{2} \frac{d}{ds} \left( L(w_D) + (\lambda_w + s)^2 L(K_{\pi(w)}) \right) \big|_{s=0} \]

\[ = \lambda_w L(K_{\pi(w)}). \]  

(12)

On the other hand,

\[ \frac{1}{2} \left( L(w + K_{\pi(w)}) - L(w) - L(K_{\pi(w)}) \right) \]

\[ = \frac{1}{2} \left( L(w_D + \lambda_w K_{\pi(w)} + K_{\pi(w)}) - L(w_D + \lambda_w K_{\pi(w)}) - L(K_{\pi(w)}) \right) \]

\[ = \frac{1}{2} \left( L(w_D) + (\lambda_w + 1)^2 L(K_{\pi(w)}) - L(w_D) - (\lambda_w)^2 L(K_{\pi(w)}) - L(K_{\pi(w)}) \right) \]

\[ = \lambda_w L(K_{\pi(w)}). \]  

(13)
This proves the first part of the lemma. As \( \hat{B}(w) = \lambda_w L(K) \) we immediately get that \( \hat{B} \) is linear in \( w \).

**Theorem 4.8.** Let \( \mathcal{T} \) be a line subbundle of \( T\bar{M} \) and \((\bar{M}, L, T\bar{M} \setminus \mathcal{T})\) be a stationary Finsler spacetime with timelike Killing vector field \( K \). Then \((\bar{M}, L, T\bar{M} \setminus \mathcal{T})\) is locally a standard stationary splitting if and only if the following conditions are satisfied:

(a) \( L \in C^1(T\bar{M}) \cap C^2(T\bar{M} \setminus \mathcal{T}) \) and \( K_z \in \mathcal{T}_z \) at every point \( z \in \bar{M} \) where \( L(z, \cdot) \) is not twice differentiable on \( T_z \bar{M} \) (so, at these points \( z \), \( L(z, \cdot) \) is not the the quadratic form defined by a Lorentzian metric on \( T_z \bar{M} \)).

(b) \( L(K) = L(-K) \);

(c) \( L(w \pm K_{\pi(w)}) = L(w) + L(K_{\pi(w)}) \), for all \( w \in D \).

Furthermore, it is locally standard static if and only if (a), (b) and (c) hold and \( D \) is integrable.

**Proof.** \((\Rightarrow)\) Let \( z \in \bar{M} \), \( U_z \subset \bar{M} \) be a neighborhood of \( z \), and \( \phi : I_z \times S_0 \rightarrow \bar{U}_z \) be a diffeomorphism such that \( \phi_\tau(\tilde{x}) = K_{\tilde{x}} \) and \( L \circ \phi_\tau(\tau, v) = -\Lambda \tau^2 + \omega(v) \tau + F^2(v) \) (recall definition 4.4). If \( \tilde{x} \in S_0 \) is such that \( F^2(\tilde{x}, \cdot) \) is not the square of the norm defined by a Riemannian metric on \( T_{\tilde{x}}S_0 \), \( L \circ \phi_\tau((t, \tilde{x}), (\cdot, \cdot)) \), \( t \in I_z \), is not twice differentiable at any vector \( (\tau, 0) \in \mathbb{R} \times T_{\tilde{x}}S_0 \). As \( \phi_{(1,\cdot)}(\tilde{t}) = \phi_{(1,\cdot)}(1, 0) = K_{\phi_{(1,\cdot)}} \) we deduce that \( K_{\phi_{(1,\cdot)}} \) must belong to \( \mathcal{T}_{\phi_{(1,\cdot)}} \) for all \( t \in I_z \) and this proves (a). For (b), let \( z \in \bar{M} \) and take a map \( \phi \) as above (with \( z = \phi(0, x) \)); then

\[
L(K_z) = L(d\phi_{(0, x)}(1, 0)) = -\Lambda(x) = L(d\phi_{(0, x)}(-1, 0)) = L(-K_z).
\]

In order to prove (c), let \( y \in D \), and \( (\tau, v) \in \mathbb{R} \times T_\bar{M} \) be such that \( y = d\phi_{(0, x)}(\tau, v) \). Hence

\[
0 = (\partial_1 L)_{K_z}(y) = (\partial_1 L)_{d\phi_{(0, x)}(1, 0)}(d\phi_{(0, x)}(\tau, v))
\]

\[
= \frac{d}{ds} L(d\phi_{(0, x)}(1, 0) + s d\phi_{(0, x)}(\tau, v))|_{s=0}
\]

\[
= \frac{d}{ds} L(d\phi_{(0, x)}(1 + s\tau, sv))|_{s=0}
\]

\[
= \frac{d}{ds} \left( -\Lambda(x)(1 + s\tau)^2 + 2s\omega(v)(1 + s\tau) + F^2(sv) \right)|_{s=0}
\]

\[
= 2\left(-\tau \Lambda(x) + w(v)\right);
\]

thus \( \tau = \omega(v)/\Lambda(x) \) and

\[
L(y \pm K_z) = L\left(d\phi_{(0, x)}(\frac{\omega(v)}{\Lambda(x)} \pm 1, v)\right) = \frac{\omega^2(v)}{\Lambda(x)} + F^2(v) - \Lambda(x)
\]

\[
L\left(d\phi_{(0, x)}\left(\frac{\omega(v)}{\Lambda(x)} v\right)\right) + L(d\phi_{(0, x)}(1, 0)) = L(y) + L(K_z).
\]

\((\Leftarrow)\) Let \( \tilde{z} \in \bar{M} \) and \( S_\tilde{z} \) be a small smooth hypersurface in \( \bar{M} \) such that \( \tilde{z} \in S_\tilde{z} \) and \( T_{\tilde{z}}S_\tilde{z} = \mathcal{D}_\tilde{z} \). Recalling remark 4.6, we can assume that \( K_z \) is transversal to \( S_\tilde{z} \), i.e. \( T_{\tilde{z}}\bar{M} = T_{\tilde{z}}S_\tilde{z} \oplus [K_z] \), for all \( x \in S_\tilde{z} \). From (b) and (c) we get, for any \( y, u \in T_{\tilde{z}}S_\tilde{z} \),
\[ \tilde{g}_y(K_\lambda, K_\lambda) = \frac{1}{2} \left( \partial_2 L \right)_y(K_\lambda, K_\lambda) = \frac{1}{2} \frac{\partial^2}{\partial s_1 \partial s_2} L(y + (s_1 + s_2) K_\lambda) \big|_{(s_1, s_2) = (0, 0)} = \frac{1}{2} \frac{\partial^2}{\partial s_1 \partial s_2} \left( L(y) + (s_1 + s_2)^2 L(K_\lambda) \right) \big|_{(s_1, s_2) = (0, 0)} = L(K_\lambda) < 0, \tag{14} \]

and
\[ \tilde{g}_y(u, K_\lambda) = \frac{1}{2} \left( \partial_2 L \right)_y(u, K_\lambda) = \frac{1}{2} \frac{\partial^2}{\partial s_1 \partial s_2} L(y + s_1 u + s_2 K_\lambda) \big|_{(s_1, s_2) = (0, 0)} = \frac{\partial^2}{\partial s_1 \partial s_2} \left( L(y + s_1 u) + s_2^2 L(K_\lambda) \right) \big|_{(s_1, s_2) = (0, 0)} = 0, \tag{15} \]

that is \( K_\lambda \) is timelike w.r.t. the Lorentzian scalar product \( \tilde{g}_y \) on \( T_y \tilde{M} \) and \( T_y S_\lambda \) is a spacelike hyperplane, for all \( y \in T_y \tilde{S}_\lambda \). Let \( UT \tilde{S}_\lambda \) be the unit tangent bundle of \( \tilde{S}_\lambda \) (with respect to any auxiliary Riemannian metric on \( \tilde{M} \)). As \( \tilde{g}_y \) is positively homogeneous of degree 0 in \( y \), by continuity of the map \( y \mapsto \tilde{g}_y \), we get that (up to consider a smaller hypersurface \( S \)) \( K_\lambda \) is timelike and \( T_y \tilde{S}_\lambda \) is spacelike w.r.t. \( \tilde{g}_y \), for each \( y \in T_y \tilde{S}_\lambda \) and for any \( x \in S \).

Let now \( w \in T_x \tilde{M} \setminus T_x, x \in S \), and \( w_\lambda \in T_x S_\lambda \), \( \tau_\lambda \in \mathbb{R} \) such that \( w = w_\lambda + \tau_\lambda K_\lambda \). Let us evaluate \( \tilde{g}_w(u, u) \), for any \( u \in T_x \tilde{S}_\lambda \). From lemma \( 4.7 \) we have
\[ \tilde{g}_w(u, u) = \frac{1}{2} \frac{\partial^2}{\partial s_1 \partial s_2} L(w + (s_1 + s_2) u) \big|_{(s_1, s_2) = (0, 0)} = \frac{1}{2} \frac{\partial^2}{\partial s_1 \partial s_2} \left( L(w_\lambda + \tau_\lambda K_\lambda + (s_1 + s_2) u) \right) \big|_{(s_1, s_2) = (0, 0)} = \frac{1}{2} \frac{\partial^2}{\partial s_1 \partial s_2} \left( L(w_\lambda + (s_1 + s_2) u) \right. \\
+ \left. \tau_\lambda^2 L(K_\lambda) + \tilde{B}(w_\lambda + (s_1 + s_2) u) \right) \big|_{(s_1, s_2) = (0, 0)}. \]

Since \( \tilde{B} \) is linear, we have that \( \frac{\partial^2}{\partial s_1 \partial s_2} \tilde{B}(w_\lambda + (s_1 + s_2) u) \big|_{(s_1, s_2) = (0, 0)} = 0 \) and then
\[ \tilde{g}_w(u, u) = \tilde{g}_w(u, u) > 0. \tag{16} \]

By using \( w = w_\lambda + \lambda_\lambda K_\lambda \) and \( w_\lambda = (w_\lambda)_\lambda + \lambda_\lambda K_\lambda \), we also get, as in (13),
\[ \tilde{g}_w(K_\lambda, K_\lambda) = L(K_\lambda) = \tilde{g}_w(K_\lambda, K_\lambda). \]

Moreover, recalling (12), we obtain
\[ \tilde{g}_w(u, K_\lambda) = \frac{1}{2} \frac{\partial^2}{\partial s_1 \partial s_2} L(w + s_1 u + s_2 K_\lambda) \big|_{(s_1, s_2) = (0, 0)} = \frac{1}{2} \frac{\partial^2}{\partial s_1 \partial s_2} \left( L(w_\lambda + \lambda_\lambda K_\lambda + s_1 u_\lambda + (s_1 + s_2) K_\lambda) \right) \big|_{(s_1, s_2) = (0, 0)} = \frac{1}{2} \frac{\partial^2}{\partial s_1 \partial s_2} \left( L(w_\lambda + s_1 u_\lambda + \lambda_\lambda K_\lambda + (s_1 + s_2)^2 L(K_\lambda)) \right) \big|_{(s_1, s_2) = (0, 0)} = \lambda_\lambda L(K_\lambda) = \tilde{B}(u). \tag{17} \]
Let $I_z = (-\varepsilon_z, \varepsilon_z)$ be an interval such that the map $\phi : I_z \times S_z \to \tilde{M}, \phi(t,x) = \psi_t(x)$, where $\psi$ is the flow of $K$, is a diffeomorphism onto a neighborhood $U_z$ of $z$ in $M$. Consider a non vanishing smooth section $W : S_z \to TM$, such that $W_x \notin T_x$, for all $x \in S_z$. Set $Y_z = (d\psi)_x(W_x)$, with $z = \phi(t,x)$. So $Y$ is a non vanishing smooth vector field in $U_z$. The evaluation $\tilde{g}_y$ of the fundamental tensor of $L$ in $Y$ becomes, then, a Lorentzian metric on $U_z$ (and, by definition of $Y$, $K$ is a Killing vector field for $\tilde{g}_y$). In particular, $\tilde{g}_y(w_1, w_2) = \tilde{g}_w(v_1, v_2)$, for all $z \in U_z$, $z = \phi(t,x)$ and $w_i = (d\phi)(t,x)(v_i), i = 1, 2$. Thus, $\phi^* \tilde{g}_y$ in $I_z \times S_z$ is given by

$$
\phi^* \tilde{g}_y((\tau, v), (\tau, v)) = -\tilde{g}_w(K_x, K_x)\tau^2 + 2\tilde{g}_w(v, K_x)\tau + \tilde{g}_w(v, v)
$$

for all $(t,x) \in I_z \times U_z$ and $(\tau, v) \in \mathbb{R} \times T_xS_z$. From (15)–(17) we then obtain:

$$
\phi^* \tilde{g}_y((\tau, v), (\tau, v)) = -\tilde{g}(w_x)(K_x, K_x)\tau^2 + 2\tilde{g}(w_x)(v, K_x)\tau + \tilde{g}(w_x)(v, v)
$$

$$
-\Lambda(x)\tau^2 + \omega(v)\tau + F^2(v),
$$

where $-\Lambda(x) = g(w_x)(K_x, K_x) = L(K_x)$ and $\omega$ is the one-form on $S_z$ defined by $\omega = \tilde{B}|_{TS_z}$. Thus, for all $(\tau, v) \in \mathbb{R} \times T_xS_z \setminus 0$,

$$
L \circ \phi_x(\tau, v) = \tilde{g}(\phi_x(\tau, v), \phi_x(\tau, v)) = \Lambda\tau^2 + \omega(v)\tau + F^2(v),
$$

where $F$ is the Finsler metric on $S_z$ defined by $F(v) = \sqrt{\tilde{g}(v, v)} = \sqrt{L(v)}$ and

$$
L \circ \phi_x(\tau, 0) = \tau^2L(K_x) = -\Lambda(x)\tau^2.
$$

Hence, for all $(\tau, v) \in \mathbb{R} \times T_xS_z$,

$$
L \circ \phi_x(\tau, v) = \Lambda\tau^2 + \omega(v)\tau + F^2(v).
$$

This concludes the proof of the implication to the left. For the last part of the theorem, it is enough to observe that in a standard static splitting $(0, v) \in D$, for all $v \in TS_z$ since $\omega = 0$. On the other hand, if $D$ is integrable then we can take an integral manifold $S_z$ and then as in (14) we get $\tilde{g}_w(u, K_x) = 0$ for all $u \in T_xS_z$ and all $x \in S_z$.

5. The optical metrics of a stationary splitting Finsler spacetime

Generally speaking, by optical metric is meant a metric tensor that comes into play in the description of the optical geometry of a curved spacetime (see, e.g. [33]). For static and stationary Lorentzian spacetimes, it usually denotes a Riemannian metric which is conformal to the one induced by the spacetime metric on the space of orbits of the Killing field [3]. For a standard stationary Lorentzian manifold $(\mathbb{R} \times M, \tilde{g}), \tilde{g}$ given in (8), it becomes the Riemannian metric on $M$ given by $\omega/\Lambda \otimes \omega/\Lambda + g/\Lambda$. The role attributed to this metric seems to come from the static case ($\omega = 0$) where the metric $g/\Lambda$ does fully describe its optical geometry, in the sense that light rays in $(\mathbb{R} \times M, \tilde{g})$ project geodesics of $(M, g/\Lambda)$. The same is not true in the more general stationary case where the equation satisfied by the projected curves is the one of a unit positive or negative charged test particle moving on the Riemannian manifold $(M, \omega/\Lambda \otimes \omega/\Lambda + g/\Lambda)$ under the action of the magnetic field $B = d(\omega/\Lambda)$ (the positive charge correspond to future-pointing lightlike geodesics, the negative one to past-oriented). This equation (actually, these two equations) can effectively be interpreted as the equation of geodesics, parametrized with constant velocity w.r.t. $\omega/\Lambda \otimes \omega/\Lambda + g/\Lambda$, of a Finsler metric of Randers type on $M$ and of its reverse metric.
Several results about lightlike and timelike geodesics in the standard stationary spacetime can then be deduced by studying geodesics of such Finsler metrics [8, 13, 15–18]. The properties of these Randers metrics encode also the causal structure [15, 19] and the topological lensing [14, 65] in a standard stationary Lorentzian spacetime, moreover they give information about its c-boundary [25] and its curvature [31].

Such correspondence between spacetime geometry and Finsler geometry has also been extended to more general Lorentzian spacetimes introducing some generalised Finsler-type structures [20]. Our aim in this section is to prove that the correspondence still holds for a wide class of stationary splitting Finsler spacetimes \((\mathbb{R} \times M, L)\).

Observe that if \(\gamma = (\theta, \sigma)\) is a lightlike curve on \((\mathbb{R} \times M, L)\), then, by definition, it satisfies the equation

\[
0 = L(\dot{\gamma}) = -\Lambda(\sigma) \dot{\theta}^2 + 2B(\dot{\sigma}) \dot{\theta} + F^2(\dot{\sigma}),
\]

so

\[
\dot{\theta} = \frac{B(\dot{\sigma})}{\Lambda(\sigma)} \pm \sqrt{\frac{B(\dot{\sigma})^2}{\Lambda(\sigma)^2} + \frac{F^2(\dot{\sigma})}{\Lambda(\sigma)}}.
\]

Solving the above equation w.r.t. to \(\dot{\theta}\), we get the following non-negative and fiberwise positively homogeneous of degree 1 Lagrangians on \(TM\) associated to the stationary splitting Finsler spacetime \((\mathbb{R} \times M, L)\):

\[
F_B = \frac{B}{\Lambda} + \sqrt{\frac{B^2}{\Lambda^2} + \frac{F^2}{\Lambda}},
\]

\[
F_{-B} = -\frac{B}{\Lambda} + \sqrt{\frac{B^2}{\Lambda^2} + \frac{F^2}{\Lambda}}.
\]

The same assumptions ensuring that \(L\) is a Lorentz–Finsler function on \(\tilde{M} = \mathbb{R} \times M\), plus a definite sign of \(B\) when it does not reduce to a one-form on \(M\), give that \(F_B\) and \(F_{-B}\) are Finsler metrics on \(M\).

**Theorem 5.1.** Let \(L: T\tilde{M} \to \mathbb{R}\) defined as in (9). Assume that

1. \((\partial^2_{\partial\sigma} B)_v\) is positive semi-definite (resp. \((\partial^2_{\partial\sigma} B)_v\) is negative semi-definite), for all \(v \in TM \setminus 0\); 
2. for each \(x \in M\) either \(B(x, v) \geq 0\) (resp. \(B(x, v) \leq 0\)) for all \(v \in T_xM\) or \(B(x, \cdot)\) is linear on \(T_xM\);

then \(F_B\) (resp. \(F_{-B}\)) is a Finsler metric on \(M\).

**Proof.** Let us prove the statement for \(F_B\), i.e. in the case when \((\partial^2_{\partial\sigma} B)_v\) is positive semi-definite and either \(B(x, \cdot)\) is non-negative or it is linear. The only non trivial part of the proof is to show that the fiberwise Hessian of the square of \(F_B\) is positive definite. Let us define

\[
G = \sqrt{B^2 + \Lambda F^2},
\]

so that \(F_B = \frac{1}{2}(B + G)\). Let us equivalently compute the fiberwise Hessian of \(\frac{1}{2}(\Lambda F_B)^2\) at \(v \in TM \setminus 0\):

\[
\frac{\Lambda^2}{2} \left(\partial^2_{\partial\sigma} F_B\right)_v = \left((\partial_i B)_v + (\partial_i G)_v\right) \otimes \left((\partial_j B)_v + (\partial_j G)_v\right) + (B(v) + G(v)) \left((\partial^2_{\partial\sigma^2} B)_v + (\partial^2_{\partial\sigma^2} G)_v\right).
\]
Let us now show that \( G \) is a Finsler metric. Clearly, \( G \) is non-negative and vanishes only at zero vectors, it is continuous in \( TM \) and smooth outside the zero section, moreover it is fiberwise positively homogeneous of degree 1. It remains to prove that the fiberwise Hessian of \( \frac{1}{2} G^2 \) is positive definite on \( TM \). Let us evaluate,

\[
\frac{1}{2} (\partial^2 G^2)_v = (\partial_v B)_v \otimes (\partial_v B)_v + B(v)(\partial^2 B)_v + \frac{1}{2} (\partial^2 v, B^2)_v.
\]

As \((\partial_v B)_v \otimes (\partial_v B)_v + \frac{1}{2} (\partial^2 v, B^2)_v\) is positive definite, we see that \( \frac{1}{2} (\partial^2 G^2)_v \) is positive definite provided that \( B \) is linear on \( T_{\pi u}(v) M \) (thus, \( (\partial^2 B)_v = 0 \) for all \( v \in T_{\pi u}(v) M \setminus 0 \) or \( B(v) \geq 0 \)) and \((\partial^2 v, B^2)_v\) is positive semi-definite.

Since \( G \) is a Finsler metric, we know that \((\partial^2 v, G)_v\) is positive semi-definite and \((\partial^2 v, G)_v(u, u) = 0\) if and only if \( u = v \). Thus, as \( B(v) + G(v) \leq 0 \) and it vanishes only if \( v = 0 \), from (20), we see \( \frac{1}{2} (\partial^2 v, F^2)_v\) is positive semi-definite. Let us then assume, by contradiction that there exist \( u \in T_{\pi u}(v) M, u \neq 0\), such that \( \frac{1}{2} (\partial^2 v, F^2)_v(u, u) = 0\). This implies that \((\partial^2 v, G)_v(u, u) = 0\) and then \( u = v \). Hence, by homogeneity, \((\partial^2 v, B)_v(v, v) = 0\) and

\[
0 = (\partial_v B)_v \otimes (\partial_v B)_v + (\partial_v G)_v(v, v) = (B(v) + G(v))^2,
\]

which implies that \( v = 0 \), a contradiction.

If \((\partial^2 v, B)_v\) is negative semi-definite and \( B(x, \cdot) \) is non-positive or linear, we get that \( F_B^2 \) is a Finsler metric on \( M \) as above taking into account that \( F_B^2 = \frac{1}{2}(G - B) \).

**Remark 5.2.** A part from the case when \( B \) is a one-form on \( M \), a significant class of maps satisfying the assumptions of theorem 5.1 is given (up to the sign) by Randers variations of Finsler metrics: \( B = \pm(\omega + F_1) \), where \( F_1 \) is a Finsler metric on \( M \) and \( \omega \) is a one-form such that \(|\omega(v/F_1(v))| < 1\) for all \( v \in TM \setminus 0 \).

### 6. Trivial isocausal static Finsler spacetimes

The optical metrics have been already introduced in [22] for a standard static Finsler spacetime. Since in this case \( B = 0 \), both \( F_B \) and \( F_B^2 \) reduce to the Finsler metric \( F / \Lambda \) on \( M \). The causal properties of a standard static Finsler spacetime can then be described in terms of metric properties of \( F / \Lambda \). Thanks to the metrics \( F_B \) and \( F_B^2 \), we easily see that the causal properties of a stationary splitting Finsler spacetime coincides with the ones of a couple of standard static Finsler spacetimes and therefore they can be described using \( F_B \) and \( F_B^2 \). Indeed, under the assumptions of theorem 5.1, we can consider the Lorentz–Finsler functions on \( \tilde{M} \) given by

\[
L_B(\tau, v) := -\tau^2 + F_B^2(v), \quad L_B^-(\tau, v) := -\tau^2 + (F_B^2)^2(v),
\]

and the Finsler spacetimes \(( \mathbb{R} \times M, L_B, T^+ \tilde{M} \setminus \mathcal{T} ), ( \mathbb{R} \times M, L_B^-, T^- \tilde{M} \setminus \mathcal{T} )\) that we call **trivial isocausal static Finsler spacetimes** associated to \( (\tilde{M}, L) \). Isocausality is a relation between Lorentzian spacetimes introduced in [28]. If \( V \) and \( W \) are spacetimes they are said **isocausal** if there exists two diffeomorphisms \( \varphi : V \rightarrow W \) and \( \psi : W \rightarrow V \) such that \( \varphi \) and \( \psi \) map future-pointing causal vectors into future-pointing causal vectors. Clearly this notion make sense also for Finsler spacetimes. However, notice that in the Finsler setting future and past-pointing causal vectors are in general not related by the symmetry \( v \mapsto -v \) and so one should consider
separately causal relations involving future and past, provided that both the future and past causal cones are defined. In our case, we have that \((\bar{M}, L, T^+ \bar{M} \setminus T)\) and \((\bar{M}, L, T^- \bar{M} \setminus T)\) are both trivially related (i.e. \(\varphi = \psi = \bar{r}_{\mathbb{R} \times \bar{M}}\)) respectively to \((\mathbb{R} \times M, L_B, T^+ M \setminus T)\) and \((\mathbb{R} \times M, L_B, T^- M \setminus T)\) as the following proposition shows.

**Proposition 6.1.** Let \((\bar{M}, L)\) be a stationary splitting Finsler spacetime satisfying the assumptions of theorem 5.1. Then \((\tau, v) \in T\bar{M}, \) with \(\tau > 0\) (resp. \(\tau < 0\)) is a causal vector of \((\bar{M}, L)\) if and only if it is a causal, future-pointing (resp. past-pointing) vector of \((\bar{M}, L_B)\) (resp. \((\bar{M}, L_B^-)\)).

**Proof.** By definition, a non-zero vector \((\tau, v) \in T\bar{M}\) is causal for \((\bar{M}, L)\) if and only if \(L(\tau, v) \leq 0\) i.e. \(-\Lambda \tau^2 + 2B(v) \tau + F^2(v) \leq 0\) which is equivalent to \(\tau \geq F_B(v)\) or \(\tau \leq -F_B^-(v)\). Then the equivalence follows recalling that (see remark 3.8), a causal, future-pointing (resp. past-pointing) vector in an ultra-static standard Finsler spacetime \((\bar{M}, L_1)\), \(L_1(\tau, v) = -\tau^2 + F_1^2(v),\) such that \(Y = \partial_\tau,\) is a non-zero vector \((\tau, v)\) such that \(\tau \geq F_1(v)\) (resp. \(\tau \leq -F_1(v)\)).

**Remark 6.2.** In particular, the statement of proposition 6.1 holds with the words ‘time-like’ or ‘lightlike’ replacing ‘causal’. Notice also that the causal future-pointing vectors of \((\bar{M}, L, T^+ \bar{M} \setminus T)\) (resp. of \((\bar{M}, L, T^- \bar{M} \setminus T)\)) are all and only the causal vectors in \(\bar{T}^+ \bar{M} := \cup_{p \in \bar{M}} \bar{T}_p^+ \bar{M}\) (resp. \(\bar{T}^- \bar{M} := \cup_{p \in \bar{M}} \bar{T}_p^- \bar{M}\)). In fact, \((\tau, v) \in T^+ \bar{M}\) (resp. \((\tau, v) \in T^- \bar{M}\)) is causal for \((\bar{M}, L, T^+ \bar{M} \setminus T)\) (resp. \((\bar{M}, L, T^- \bar{M} \setminus T)\)) if and only \(\tau \geq F_B(v)\) (resp. \(\tau \leq -F_B^-(v)\)). As \(F_B(v) = \frac{1}{2}B(v) + G(v) > \frac{B(v)}{\Lambda}\) (resp. \(-F_B^-(v) = \frac{1}{2}B(v) - G(v) < \frac{B(v)}{\Lambda}\), provided that \(v \neq 0\), we conclude by recalling remark 3.8. When \(B\) is a one-form on \(\bar{M}\) both the sets of future and past-pointing vectors of \((\bar{M}, L, T \bar{M} \setminus T)\) are non empty and, being in this case \(Y = \partial_\tau,\) they coincide with with the causal vectors belonging respectively in \(T^+ \bar{M}\) and \(T^- \bar{M}\).

**Remark 6.3.** As in lemma 2.21 in [22], the epigraph (resp. hypograph) of the function \(\tau = F_B(v)\) (resp. \(\tau = -F_B^-(v)\)) in \(T_p \bar{M}\) is connected and convex, for all \(p \in \bar{M},\) i.e. the set of the future-pointing causal vectors of \((\bar{M}, L, T^+ \bar{M} \setminus T)\) (resp. \((\bar{M}, L, T^- \bar{M} \setminus T)\)) at \(p \in \bar{M}\) is connected and convex. Moreover, for each \(c > 0\), the set of the future-pointing timelike vectors \((\tau, v)\) of \((\bar{M}, L, T^+ \bar{M} \setminus T)\) (resp. \((\bar{M}, L, T^- \bar{M} \setminus T)\)) in \(T_p \bar{M}\) such that \(L(\tau, v) \leq -c\) is also connected and strictly convex.

Let \(p_0 \in \bar{M}\) and let us denote by \(I^+(p_0)\) (resp. \(I^-(p_0)\)) the subset of \(\bar{M}\) given by all the points \(p \in \bar{M}\) such that there exists a timelike future-pointing curve of \((\bar{M}, L)\) connecting \(p_0\) to \(p\) (resp. \(p\) to \(p_0\)). From proposition 6.1 and remark 6.2, it follows that these sets coincide with those of the corresponding trivial isocausal static Finsler spacetime. Thus from [22, proposition 3.2] we obtain:

**Proposition 6.4.** Let \((\bar{M}, L, T^+ \bar{M} \setminus T)\) be a stationary splitting Finsler spacetime (thus \((\partial^2_\tau B)\) is positive semi-definite, for all \(v \in T \bar{M} \setminus 0\)) such that, for each \(x \in \bar{M},\) either \(B(x, v) \geq 0,\) for all \(v \in T_x \bar{M},\) or \(B(x, \cdot)\) is linear on \(T_x \bar{M}\). Then, for all \(p_0 = (t_0, x_0) \in \bar{M}\) we have:

\[
I^+(p_0) = \bigcup_{r > 0} \left\{ (t_0 + r) \times B^+(x_0, r) \right\}, \quad I^-(p_0) = \bigcup_{r > 0} \left\{ (t_0 - r) \times B^-(x_0, r) \right\},
\]
where $B^+(x_0, r)$ and $B^-(x_0, r)$ denote, respectively, the forward and the backward open ball of centre $x_0$ and radius $r$ (see, e.g. [6] for the definitions of these balls) of the Finsler metric $F_B$. Moreover, $I^\pm(p_0)$ are open subsets of $\bar{M}$.

**Remark 6.5.** Taking into account that, in $(\bar{M}, L, T^-\bar{M} \setminus T)$, under the assumptions $(\partial^2_v B)_v$ is negative semi-definite, for all $v \in TM \setminus 0$ and, for each $x \in \bar{M}$, either $B(x, v) \leq 0$, for all $v \in T_xM$, or $B(x, \cdot)$ is linear on $T_xM$, a timelike future-pointing vector is past-pointing for the standard static Finsler spacetime $(\bar{M}, L_{B^-})$, an analogous proposition holds for a stationary splitting spacetime of the type $(\bar{M}, L, T^-\bar{M} \setminus T)$ by considering the forward and backward ball of the metric $F_{\bar{B}}$ and replacing $t_0 + r$ with $t_0 - r$ in the first equality and $t_0 - r$ with $t_0 + r$ in the second one.

Analogously, using also the Fermat’s principle (see appendix B), from the results in section 3 of [22], we get the following proposition which extends to stationary splitting Finsler spacetimes some results obtained in [19] for standard stationary Lorentzian spacetimes (we refer to [22] and to [51] for the definitions of the causality properties involved in its statement, while for the notions of forward and backward completeness of a Finsler metric we refer to [6]).

**Proposition 6.6.** Under the assumptions of theorem 5.1, let $(\bar{M}, L, T^+\bar{M} \setminus T)$ (resp. $(\bar{M}, L, T^-\bar{M} \setminus T)$) be a stationary splitting Finsler spacetime. Then the following propositions hold true:

1. $(\bar{M}, L, T^+\bar{M} \setminus T)$ (resp. $(\bar{M}, L, T^-\bar{M} \setminus T)$) is causally simple if and only if for any $x, y \in \bar{M}$ there exists a geodesic of $F_{\bar{B}}$ (resp. $F_{\bar{B}}^{-}$) joining $x$ to $y$, with length equal to the distance associated to $F_{\bar{B}}$ (resp. $F_{\bar{B}}^{-}$);
2. a slice (and then any slice) $S_t = \{t\} \times \bar{M}$ is a Cauchy hypersurface if and only the Finsler manifold $(\bar{M}, F_{\bar{B}})$ (resp. $(\bar{M}, F_{\bar{B}}^{-})$) is forward and backward complete;
3. $(\bar{M}, L, T^+\bar{M} \setminus T)$ (resp. $(\bar{M}, L, T^-\bar{M} \setminus T)$) is globally hyperbolic if and only if $B^+(x, r) \cap \bar{B}^-(y, s)$ is compact, for every $x, y \in \bar{M}$ and $r, s > 0$, where $B^+(x_0, r_0)$ are the closure of the forward and backward balls on $\bar{M}$ associated to metric $F_{\bar{B}}$ (resp. $F_{\bar{B}}^{-}$).

7. Conclusions

In this work, we have introduced a Lorentz–Finsler function, equation (9), on $\mathbb{R} \times \bar{M}$ which admits a timelike Killing vector field and can be considered as a natural generalisation of the quadratic form of a standard stationary Lorentzian metric. We have characterized when a Finsler spacetime with a timelike Killing vector field is locally of the type introduced here. Moreover, we have seen that the optical geometry of these Finsler spacetimes can be described by (at least) one of two classical Finsler metrics on $M$, equation (18), leading to causal relations between this class of stationary Finsler spacetimes and $\mathbb{R} \times \bar{M}$ endowed with two possible static Lorentz–Finsler functions, equation (21), corresponding respectively to some sign assumptions on $B$ and its fiberwise Hessian. These relations hold also for a standard stationary spacetime, equation (8), as its optical metrics are Finslerian too: $F_B$ is a Randers metric and $F_{B}^{-}$ is its reverse metric (see [15]), showing that isocausality can hold between Lorentzian and Finsler spacetimes as well.

The isocausality between stationary splitting Finsler spacetimes and static Finsler spacetimes allowed us to deduce some results about the former class from already known ones valid for the latter [22], as shown in propositions 6.4 and 6.6. In particular these results hold whenever the map $B$ reduces to a one form on $M$. Thus, imitating the modification of the
Schwarzschild metric in [44], a Finslerian perturbation of the Kerr metric given (in geometric units) as
\[
L(\tau, \dot{r}, \dot{\theta}, \dot{\varphi}) := -\left(1 - \frac{2Mr}{\rho^2} + \psi_0(r)\right)\tau^2 - \frac{2Mra\sin^2 \theta}{\rho^2}\tau \dot{\varphi} + F^2(\dot{r}, \dot{\theta}, \dot{\varphi}),
\]
where
\[
F(\dot{r}, \dot{\theta}, \dot{\varphi}) := \left((\frac{\rho^4}{\Delta} + \psi_1(r)\right)\dot{r}^4 + (\rho^4 + \psi_2(r))\dot{\theta}^4
+ (r^2 + a^2 + \frac{2Ma^2\sin^2 \theta}{\rho^2})^2 \sin^4 \theta + \psi_3(r)\right)^{1/4},
\]
\[
\rho^2 := r^2 + a^2 \cos^2 \theta, \quad \Delta := r^2 - 2Mr + a^2, \quad (r, \theta, \varphi) \text{ are spherical coordinates on } \mathbb{R}^3 \text{ and }
(\tau, \dot{r}, \dot{\theta}, \dot{\varphi}) \text{ the induced ones on } T_p\mathbb{R}^4, \text{ for each } p = (t, r, \theta, \varphi) \in \mathbb{R}^4, M > 0 \text{ and } a \neq 0, \text{ belongs (on the region where } 1 - \frac{2M}{\rho^2} + \psi_0(r) > 0 \text{ and for small enough functions } \psi_i(r) \text{ to the class of Lorentz–Finsler functions for which our results hold. Nevertheless, more general Lorentz–Finsler functions } L \text{ can be considered by changing the Finsler metric } F \text{ on } \mathbb{R}^3. \text{ In particular, as } F \text{ can be taken non-reversible } (F(\dot{r}, \dot{\theta}, \dot{\varphi}) \neq F(-\dot{r}, -\dot{\theta}, -\dot{\varphi}), \text{ the frame dragging effect, which emphasizes the bending angle of light rays that propagate in the direction of rotation of the Kerr black hole [2, 35, 36], might be fine-tuned by a suitable choice of a Finsler metric } F \text{ depending in a non symmetric way from } \dot{\varphi}.

An example of a stationary splitting Finsler spacetime appeared as a solution of the Finslerian gravitational field equations proposed by Rutz in [60] as a generalisation of the Einstein field equations in vacuum. It is described by the spherical symmetric Lorentz–Finsler function (compare also with [7, equation (37)])
\[
L(\tau, \dot{r}, \dot{\theta}, \dot{\varphi}) := -\left(1 - \frac{2M}{r}\right)\tau^2 + \left(1 - \frac{2M}{r}\right)^{-1} \dot{r}^2 + r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2)
+ \epsilon \left(1 - \frac{2M}{r}\right)\tau B(\dot{r}, \dot{\theta}, \dot{\varphi}),
\]
where \(B(\dot{r}, \dot{\theta}, \dot{\varphi}) := (\dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2)^{1/2}\). Observe that when the parameter \(\epsilon\) is equal to 0, \(L\) reduces to the quadratic form associated to the Schwarzschild metric. The map \(B\) satisfies the assumption of theorem 5.1 only on the open region defined by \(0 < \theta < \pi \) and, for each \((r, \theta, \varphi) \in (0, +\infty) \setminus \{2M\} \times (0, \pi) \times [0, 2\pi], \text{ on the cone } \dot{\theta}^2 + \sin^2 \theta \dot{\varphi}^2 > 0 \text{ in } \mathbb{R}^3, \text{ where } B \text{ admits fiberwise Hessian. This example suggests that it would be interesting to weaken our setting allowing that } B \text{ is twice differentiable only on a cone subset } A_M \setminus TM \setminus 0, \text{ giving rise to two conic (in the sense of [39]) optical metrics } F_B \text{ and } F_{\bar{B}}, \text{ smooth only on } A_M. \text{ This can be further generalized starting from a conic Finsler metric } F \text{ or to a Killing vector field } \theta, \text{ which is not timelike everywhere [20]. However, the study of causality, geodesics and Fermat’s principle in the corresponding Lorentz–Finsler spacetime would be much more delicate than the case where } F \text{ is a standard Finsler metric on } M.

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Appendix A. Geodesics

The fact that the Lorentz–Finsler function of a Finsler spacetime is regular (in the sense of the calculus of variations) only on the cone subset $A \subset T\tilde{M}$ poses some problems for the definition of geodesics. A shortcut leading to the geodesics equation as the Euler–Lagrange equation of the energy functional $E = \gamma \mapsto \frac{1}{2} \int_a^b L(\dot{\gamma})ds$ consists in considering only smooth curves $\gamma : [a, b] \to \tilde{M}$, with velocity vector field $\dot{\gamma}(s) \in A$ for all $s \in [a, b]$ and smooth variations (recall that $A$ is open, thus for small values of the parameter of the variation the ‘longitudinal’ curves have velocity vectors in $A$). Then we can give the following:

**Definition A.1.** Let $(\tilde{M}, L, A)$ be a Finsler spacetime. A smooth curve $\gamma : [a, b] \to \tilde{M}$ such $\dot{\gamma}(s) \in A$ for all $s \in [a, b]$ is a (affinely parametrized) geodesic if in local natural coordinates $(x^0, \ldots, x^i, y^0, \ldots, y^n)$ of $TM$ it satisfies the equations

$$\frac{\partial L}{\partial x^i}(\gamma(s), \dot{\gamma}(s)) = \frac{d}{ds} \left( \frac{\partial L}{\partial y^i}(\gamma(s), \dot{\gamma}(s)) \right), \quad i = 0, \ldots, n. \tag{A.1}$$

Since $\frac{\partial^2 L}{\partial y^i \partial y^j}(x, y)$ is non-degenerate for all $x \in \tilde{M}$ and $y \in A$, (A.1) can be put in normal form (as a second order ODE) and then we have also that for each initial condition $(x_0, y_0) \in A$ there exists one and only geodesic defined in a neighborhood of $0 \in \mathbb{R}$ and such that $\gamma(0) = x_0, \dot{\gamma}(0) = y_0$.

Moreover, from $s$-independence of the Lagrangian $L$ and the fact that it is positively homogeneous of degree 2, we know that there exists a constant $C_\gamma \in \mathbb{R}$ such that $L(\dot{\gamma}(s)) = (\partial_i L)(\gamma(s)) \dot{\gamma}(s) - L(\gamma(s)) = C_\gamma$. Hence geodesics in Finsler spacetime have a well defined causal character: they are timelike, lightlike or spacelike according to $C_\gamma < 0$, $C_\gamma = 0$, $C_\gamma > 0$.

However, in many geometrical and analytical problems, considering only smooth curves is not optimal being $H^1$ or ‘piecewise smooth’ curves more convenient. In general, a key ingredient in order to prove that a $H^1$ or a piecewise smooth critical point of the energy is smooth is the injectivity of the Legendre map $v \in TM \mapsto (\partial_i L)_v \in T^*M$. Injectivity could be proved for timelike or lightlike vectors ($L(v) \leq 0, v \in A$) adapting [49, theorem 5], being then enough to prove regularity of lightlike of timelike geodesics but not for spacelike ones ($L(v) > 0, v \in A$). We observe that in a stationary splitting Finsler spacetime, if $\tilde{B}$ is not a one-form, the Legendre map is not defined on $T$ and a proof of its global injectivity following [49, theorem 6] does not seem to extend immediately to our setting due both to the possible degeneration of $\tilde{g}$ outside $A$ and the lack of compactness of $S^1 := (T_x\tilde{M} \setminus T_x) \cap \mathbb{S}^n$, where $\mathbb{S}^n$ is the sphere in $T_x\tilde{M}, x \in \tilde{M}$. Nevertheless, we would like to write down the geodesics equations taking into account the splitting $\tilde{M} = \mathbb{R} \times M$. Let us then introduce the following setting.

Let $\tilde{M} = \mathbb{R} \times M$ and $(\tilde{M}, L, 2)$ be a stationary splitting spacetime. Let $x_0, x_1 \in M$ and $\Omega'_{[a,b]}(M)$ be the set of the continuous, piecewise smooth, regular curves $\sigma$ on $M$, parametrized on a given interval $[a, b] \subset \mathbb{R}$ and connecting $x_0$ to $x_1$ (i.e. $\sigma(a) = x_0, \sigma(b) = x_1$). By regular here we mean that the left and right derivatives $\sigma^+(s)$ are different from 0 for all $s \in [a, b]$. Let $t_0, t_1 \in \mathbb{R}$ and let $\Omega_{[a,b]}(\mathbb{R})$ be the set of the continuous, piecewise smooth functions $t$ defined on $[a, b]$ such that $t(a) = t_0$ and $t(b) = t_1$. For $p_0, p_1 \in \tilde{M}, p_0 = (x_0, t_0), p_1 = (x_1, t_1)$, let us set
\[ \Omega'_{\text{rep}}(\tilde{M}) := \Omega_{\text{init}}(\mathbb{R}) \times \Omega_{\text{quat}}(M). \]

If \( \gamma \in \Omega'_{\text{rep}}(\tilde{M}) \), we call a (proper) variation of \( \gamma \) a continuous two-parameter map \( \psi: (-\varepsilon, \varepsilon) \times [a, b] \to \tilde{M} \) such that \( \psi(0, s) = \gamma(s) \), for all \( s \in [a, b] \), \( \psi(r, \cdot) \) is a continuous curve between \( p_0 \) and \( p_1 \), for all \( r \in (-\varepsilon, \varepsilon) \), and there exists a subdivision \( a = s_0 < s_1 < \ldots < s_k = b \) of the interval \([a, b] \) for which \( \psi|_{(-\varepsilon, \varepsilon) \times [s_{j-1}, s_j]} \) is smooth for all \( j \in \{1, \ldots, k\} \). Clearly, we can define classes of proper variations of \( \gamma \) as those sharing the same variational vector field \( Z \). This is, by definition, a continuous piecewise smooth vector field along \( \gamma \) such that \( Z(a) = 0 = Z(b) \) and \( Z(s) = \frac{\partial \psi}{\partial s}(0, s) \). By considering any auxiliary Riemannian metric \( h \) on \( \tilde{M} \), we see that each variational vector field \( Z \) along \( \gamma \) individuates a variation (and then also a class of them) by setting \( \psi(w, s) := \exp_{\gamma(s)}(w Z(s)) \), for \( |w| < \varepsilon \) small enough. Moreover, as \( \sigma \) is regular and \( \psi|_{(-\varepsilon, \varepsilon) \times [s_{j-1}, s_j]} \) is smooth, we can assume, up to consider a smaller \( \varepsilon \), that all the curves \( \psi(r, \cdot) \) have component on \( M \) which is a regular curve. Of course, if \( \gamma \) is a geodesic of the stationary splitting spacetime \((\tilde{M}, L)\) then it is a critical point of \( E \) in \( \Omega'_{\text{rep}}(\tilde{M}) \), i.e. \( \frac{d}{d \varepsilon}(E(\psi(r, \cdot)))|_{\varepsilon=0} = 0 \), for all proper variations \( \psi \) of \( \gamma \). Let us study the converse implication.

**Proposition A.2.** Let \((\tilde{M}, L)\) be a stationary splitting Finsler spacetime and \( \gamma \in \Omega'_{\text{rep}}(\tilde{M}) \), \( \gamma(s) = (\theta(s), \sigma(s)) \), be a critical point of \( E \) in \( \Omega'_{\text{rep}}(\tilde{M}) \) then:

1. there exists a constant \( c_\gamma \) such that
   \[ -\Lambda(\sigma)\dot{\theta} + B(\dot{\sigma}) = c_\gamma; \tag{A.2} \]
2. assume that \( n = \dim(M) \geq 3 \), \( B \geq 0 \), \( (\partial^2_B) \) is positive semi-definite for all \( v \in TM \setminus 0 \) and \( c_\gamma \leq 0 \) (resp. \( B \leq 0 \), \( (\partial^2_B) \) is negative semi-definite for all \( v \in TM \setminus 0 \) and \( c_\gamma \geq 0 \)) then \( \sigma \) is smooth (then from (A.2) also \( \theta \) is smooth) and, in local natural coordinates \((x^1, \ldots, x_n, y^1, \ldots, y^n)\) on \( TM \), the following equations are satisfied:
   \[ -\frac{1}{2} \left( \frac{c_\gamma}{\Lambda(\sigma)} \right)^\frac{2}{\gamma} \frac{\partial \Lambda}{\partial x^\gamma}(\sigma) + \frac{\partial H_\gamma}{\partial y^\gamma}(\sigma) - \frac{d}{ds} \left( \frac{\partial H_\gamma}{\partial y^\gamma}(\sigma) \right) = 0, \quad i = 1, \ldots, n \tag{A.3} \]

where \( H_\gamma = -c_\gamma \frac{\ddot{\theta}}{\dot{\theta}} + \frac{1}{2} \left( \frac{\ddot{\theta}^2}{\dot{\theta}} + F^2 \right) \).

**Remark A.3.** In a stationary splitting Finsler spacetime equations (A.2) and (A.3) are equivalent to (A.1) if we consider smooth curves \( \gamma = (\theta, \sigma) \), with regular components \( \sigma \), and smooth variation. The assumptions \( B \geq 0 \), \( (\partial^2_B) \), positive semi-definite (resp. \( B \leq 0 \), \( (\partial^2_B) \), negative semi-definite) for all \( v \in TM \setminus 0 \) are compatible with the case where \( B \) is not a one-form and \( A = T^* \tilde{M} \setminus T \) (resp. \( A = T^* \tilde{M} \setminus T \))—recall proposition 3.3. The additional assumption \( c_\gamma \leq 0 \) (resp. \( c_\gamma \geq 0 \)) ensures that \( \dot{\theta} \geq 0 \) (resp. \( \dot{\theta} \leq 0 \)) and therefore, apart from the case when \( c_\gamma = 0 \) (where \( \dot{\theta}(s) \) could vanish at some instants) \( \gamma \) in proposition A.2 is a geodesic of \((\tilde{M}, L, A)\). Recall that from remark 6.2, any causal curve \( \gamma = (\theta, \sigma) \) of \((\tilde{M}, L, T^* \tilde{M} \setminus T)\) \((\tilde{M}, L, T^* \tilde{M} \setminus T)\)) with \( \dot{\gamma}(s) \in T^* \tilde{M} \setminus T \) (resp. \( \dot{\gamma}(s) \in T^* \tilde{M} \setminus T \)) satisfies \(-\Lambda(\sigma)\dot{\theta} + B(\dot{\sigma}) < 0 \) (resp. \(-\Lambda(\sigma)\dot{\theta} + B(\dot{\sigma}) > 0 \)) and therefore we have that any causal curve \( \gamma \in \Omega'_{\text{rep}}(\tilde{M}) \) which is a critical point of \( E \), is a geodesic.

Before proving proposition A.2 we need the following lemma:
Lemma A.4. Let \( \alpha \in \mathbb{R} \) and \( H_\alpha = -\alpha \frac{\partial}{\partial x} + \frac{1}{2} \left( \frac{y^2}{x} + F^2 \right) \). If \( B \geq 0, \) \( (\partial_{\alpha}^2 B)_{\nu} \) is positive semi-definite, for all \( \nu \in TM \setminus 0, \) and \( \alpha \leq 0 \) (resp. \( B \leq 0, \) \( (\partial_{\alpha}^2 B)_{\nu} \) is negative semi-definite for all \( \nu \in TM \setminus 0 \) and \( \alpha \geq 0, \) then \( (\partial_{\alpha}^2 H)_{\nu} \) is positive definite for all \( \nu \in TM \setminus 0. \) Moreover, if \( n = \dim(M) \geq 3, \) the map
\[
\mathcal{L} = \nu \in TM \setminus 0 \mapsto (\partial_{\alpha} H_\alpha)_{\nu} \in T^* M \setminus 0
\]
is a diffeomorphism.

Proof. Let us prove the statement under the assumptions \( B \geq 0, \) \( (\partial_{\alpha}^2 B)_{\nu} \) positive semi-definite, \( \alpha \leq 0, \) being the other case analogous. Observe that if there exists \( \nu \in TM \setminus 0 \) such that \( (\partial_{\alpha} H_\alpha)_{\nu} = 0 \) then by homogeneity \( 0 = (\partial_{\alpha} H_\alpha)_{\nu} (\nu) = -\frac{\alpha}{x} B(\nu) + \frac{1}{2} B^2(\nu) + F^2(\nu) \) hence, being \( -\frac{\alpha}{x} B(\nu) \geq 0, \) it must be \( F(\nu) = 0 \) and then \( \nu = 0. \) Thus \( \mathcal{L} \) is a continuous map from \( TM \setminus 0 \) into \( T^* M \setminus 0. \) Let us observe that \( B^2/\alpha + F^2 = G^2/\alpha \) where \( G \) is the map in (19). As \( (\partial_{\alpha}^2 H)_{\nu} = -\frac{\alpha}{x} (\partial_{\alpha}^2 B)_{\nu} + \frac{1}{2} (\partial_{\alpha}^2 G^2)_{\nu}, \) and \( G \) is a Finsler metric (recall the proof of theorem 5.1) we get that \( \mathcal{L} \) is a local diffeomorphism.

Moreover, since for any \( x \in M \) the map \( \nu \in T_x M \setminus \{0\} \mapsto (\partial_{\alpha} B)_{\nu} \in T^*_x M \) is positively homogeneous of degree 0 and \( \nu \in T_x M \setminus \{0\} \mapsto (\partial_{\alpha} G^2)_{\nu} \in T^*_x M \) is positively homogeneous of degree 1, we get that \( \mathcal{L}(x, \cdot) \) is a bijection and therefore \( \mathcal{L} \) is a diffeomorphism from \( TM \setminus 0 \) onto \( T^* M \setminus 0. \)

Proof of theorem A.2. We observe first that
\[
L(\tau, \nu) = -\Lambda \left( \tau - \frac{B(\nu)}{\Lambda} \right)^2 + \frac{B^2(\nu)}{\Lambda} + F^2(\nu).
\]
(1) By considering variational vector fields \( Z \) which are of the type \( (Y, 0) \) we deduce, by a standard argument, that (A.2) is satisfied on \([a, b]\) for some constant \( c_r.\)

(2) Let \( I \) be an interval where \( \sigma \) is smooth and \( Z \) a variational vector field along \( \gamma \) of the type \((0, W), \) with \( W \) having compact support in \( I. \) Then, in local natural coordinates \((x^1, \ldots, x^n, y^1, \ldots, y^n)\) of \( TM, \sigma \) satisfies, in such interval, (A.3). At the instants \( s_j, j \in \{0, \ldots, k\}, \) where \( \sigma \) has a break, by taking any vector \( w_j \in T_{\sigma(s_j)}M, j \in \{1, \ldots, k-1\}, \) and a variational vector field \((0, W_j), \) such that \( W_j(s_j) = w_j \) and \( W_j \equiv 0 \) outside a small neighbourhood of \( s_j, \) we get, using that (A.3) is satisfied both in \([s_{j-1}, s_j]\) and \([s_j, s_{j+1}],\)
\[
\frac{\partial H}{\partial y^j}(\sigma^-(s_j))w_j = \frac{\partial H}{\partial y^j}(\sigma^+(s_j))w_j,
\]
hence \( (\partial_{\alpha} H)_{\sigma^-(s_j)} = (\partial_{\alpha} H)_{\sigma^+(s_j)}. \) From lemma A.4, \( \dot{\sigma}^-(s_j) = \dot{\sigma}^+(s_j). \) Thus, \( \sigma \) is a \( C^1 \) curve on \([a, b]\) and, from (A.2), also \( \theta \) is a \( C^1 \) function. By putting (A.3) in normal form on each interval \( I \) where \( \sigma \) is smooth (recall again lemma A.4) we deduce that also the second derivatives of \( \sigma \) must agree at the instants \( s_j \) and then both \( \sigma \) and \( \theta \) are smooth curves.

If \( B \) reduces to a one-form on \( M \) then we can avoid to consider only regular curves on \( M \) and we can drop the assumption about the dimension of \( M. \) Indeed, let us now define, for \( x_0, x_1 \in M, \) \( \Omega_{x_0, x_1}(M) \) as the set of the continuous, piecewise smooth, curves \( \sigma \) on \( M, \) parametrized on a given interval \([a, b] \subset \mathbb{R} \) and connecting \( x_0 \) to \( x_1. \) The analogous space of paths between two points \( p_0 = (t_0, x_0), p_1 = (t_1, x_1) \) in \( M \) is given by
\[ \Omega^\text{ppp}(\tilde{M}) = \Omega^\text{ini}(\mathbb{R}) \times \Omega^\text{ext}(M). \]

Taking into account that in this case \( B \) is smooth as a map defined on \( TM \) and \((\partial_t B)\), is equal to \( B \) and then it is independent from \( v \), we obtain, arguing as in the proof of [22, theorem 2.13] and using that the Legendre map associated to \( F^2 \), \( v \in TM \mapsto (\partial_t F^2)_v \in T^*M \), is a homeomorphism diffeomorphism (whatever the dimension of \( M \) is) we get the following:

**Proposition A.5.** Let \((\tilde{M}, L)\) be a stationary splitting Finsler spacetime where \( B \) is a one form on \( M \). A curve \( \gamma \in \Omega_{\text{ppp}}(\tilde{M}) \) is a critical point of \( E \) in \( \Omega_{\text{ppp}}(\tilde{M}) \) if and only if equations (A.2) and (A.3) are satisfied. Any critical point \( \gamma = (\theta, \sigma) \) of \( E \) in \( \Omega_{\text{ppp}}(\tilde{M}) \) is then at least \( C^1 \) on \([a, b]\) and there exists a constant \( C \in \mathbb{R} \) such that \( L(\gamma(s)) = C \), for all \( s \in [a, b] \).

Moreover, if \( \gamma \) is non-constant then: (a) if it is spacelike or lightlike (i.e. \( C > 0 \)) then \( \sigma \) never vanishes and \( \gamma \) is smooth; (b) if \( \sigma \) is constant equal to \( x_0 \in M \) on the whole interval \([a, b]\) then \( C < 0, d\Lambda(x_0) = 0 \) and \( \theta \) is constant too; vice versa, if \( d\Lambda(x_0) = 0 \) then, for each \( \theta_0 \in \mathbb{R} \) and \( m \neq 0 \), the curve \( s \in [a, b] \mapsto (\theta_0 + m(s-a), x_0) \in M \) is a timelike geodesic.

**Remark A.6.** As in the previous case about \( B \), any geodesic of \((\tilde{M}, L)\) connecting \( p_0 \) to \( p_1 \), according to definition A.1, is a critical point of \( E \) on \( \Omega^\text{ppp}(\tilde{M}) \). On the other hand, proposition A.5 allows us to define geodesics also when \( \gamma \not\in A \); for example, we can say that constant curves are geodesics as they satisfy equations (A.2) and (A.3); analogously, we can say when a flow line of \( \partial_t \) is a geodesic and when a curve whose velocity vector is collinear to \( \partial_t \) at some instants is a geodesic. Technically, these cases are not covered by definition A.1 or by introducing geodesics as auto parallel curves with respect to a Finslerian connection because, whatever this connection is, it cannot be computed at vector \( v \in TM \) where the fiberwise Hessian of \( L \) is not defined.

**Appendix B. Fermat’s principle**

Fermat’s principle characterizes light rays as the critical points of the travel time. Thanks to an ingenious adaptation of the notion of arrival time, it remains valid in general relativity [43, 55]. Its extension to Finsler spacetimes has been proved in [56] (see also [27] for timelike geodesics). In this appendix we prove that, under the assumption of theorem 5.1 lightlike geodesics of a stationary splitting Finsler spacetime \((\mathbb{R} \times M, L)\) project on \( M \) as pregeodesics of the corresponding Fermat metric. By pregeodesic, it is meant a curve that is an arbitrary reparametrization of a geodesic. We point out that this result could be also somehow deduced from the Fermat’s principle for Finsler spacetime in [56] since the arrival time coincides, in our setting, with the value of the coordinate \( t \) on \( \mathbb{R} \times M \) at the final point of a lightlike curve and therefore, up to a constant, with the length of its component w.r.t. one of the Fermat metrics. Anyway, we give here a simple proof which is tailored on the splitting structure of \( \tilde{M} \) and gives also a precise information about the parametrization of the projection of the lightlike geodesic.

**Proposition B.1.** Under the assumptions of theorem 5.1, let \((\tilde{M}, M, T^+M \setminus T)\) (resp. \((\tilde{M}, M, T^\times M \setminus T)\)) be a stationary splitting Finsler spacetime. A curve \( \gamma : [a, b] \to \tilde{M}, \gamma = (\theta, \sigma) \), is a lightlike geodesic if and only if \( \sigma \) is a (non-constant) pregeodesic of the Fermat metric \( F_B \) (resp. \( F_B^{-} \)) parametrized with \( G(\sigma) = -c_\gamma \) (resp. \( G(\sigma) = c_\gamma \)), where \( G \) is defined in (19), and \( \theta(s) = \theta(a) + \int_a^s F_B(\sigma)d\tau \) (resp. \( \theta(s) = \theta(a) - \int_a^s F_B^{-}(\sigma)d\tau \)).
Proof. From (A.4), we get that for a lightlike curve \( \dot{\theta} - B(\dot{\theta}) \) equals \( G^2(\dot{\theta}) \). Moreover, as \( \gamma \) is lightlike, \( \dot{\gamma}(s) \neq 0 \) for all \( s \in [a,b] \). Therefore, from remark A.3 and observing that any curve \( (\theta, \sigma) \) such that \( \theta(s) = \theta(a) + \int_a^s F_B(\dot{\sigma}) d\tau \) (resp. \( \theta(s) = \theta(a) - \int_a^s F_B(\dot{\sigma}) d\tau \)) is lightlike (recall proposition 6.1) we have that the statement is equivalent to prove that (A.3) is the equation of geodesics of \( (M, F_B) \). From (A.4), we get that for a lightlike curve \( (\theta, \sigma) \) such that \( \theta(s) = \theta(a) + \int_a^s F_B(\dot{\sigma}) d\tau \), \( \dot{\gamma}(s) = c \) is lightlike, \( \dot{\sigma}(s) \neq 0 \) for all \( s \in [a,b] \). Therefore, from remark A.3 and observing that (A.3) is the equation of geodesics of \( (M, F_B) \) parametrized with \( G(\dot{\sigma}) = \text{const.} \) (such a constant is equal to \( -c_\gamma \) for \( (\dot{M}, L, T^+ M \setminus T) \) and to \( c_\gamma \) for \( (\dot{M}, L, T^- M \setminus T) \), \( c_\gamma \) being negative in the first case and positive in the second one). Notice that (A.3) is equivalent to

\[
- \frac{c_\gamma^2}{2 \Lambda^2(\sigma)} \frac{\partial \Lambda}{\partial \sigma}(\sigma) - c_\gamma \frac{\partial}{\partial \sigma} \left( \frac{B}{\Lambda} \right)(\sigma) + c_\gamma \frac{d}{ds} \left( \frac{\partial}{\partial y^i} \left( \frac{B}{\Lambda} \right)(\sigma) \right) + \frac{1}{2} \frac{\partial G^2}{\partial \sigma}(\sigma) - \frac{1}{2} \left( \frac{\partial}{\partial y^i} \left( \frac{B}{\Lambda} \right)(\sigma) \right) = 0, \quad i = 1, \ldots, n
\]

where \( \tilde{G}^2 = G^2/\Lambda \). Hence, if \( G(\dot{\sigma}) = -c_\gamma \) (the other case \( G(\dot{\sigma}) = c_\gamma \) is analogous) we get equivalently

\[
- c_\gamma \left( \tilde{G}(\dot{\sigma}) \frac{\partial}{\partial \sigma} \left( \frac{1}{\sqrt{\Lambda}} \right)(\sigma) + \frac{1}{\sqrt{\Lambda(\sigma)}} \frac{\partial \tilde{G}}{\partial \sigma}(\sigma) - \frac{d}{ds} \left( \frac{1}{\sqrt{\Lambda(\sigma)}} \frac{\partial \tilde{G}}{\partial y^i}(\sigma) \right) \right) + \frac{\partial}{\partial \sigma} \left( \frac{B}{\Lambda} \right)(\sigma) - \frac{d}{ds} \left( \frac{\partial}{\partial y^i} \left( \frac{B}{\Lambda} \right)(\sigma) \right) = 0, \quad i = 1, \ldots, n
\]

which is, up to the constant factor \( c_\gamma \), the equation of pregeodesics of \( F_B \), in local natural coordinates on \( TM \).

Remark B.2. In particular when \( B \) is a one-form on \( M \), lightlike future-pointing geodesics project to pregeodesics of \( F_B \) and past-pointing ones to pregeodesics of \( F_B^- \).

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References

[1] Aazami A B and Javaloyes M A 2016 Penrose’s singularity theorem in a Finsler spacetime Class. Quantum Grav. 33 025003
[2] Aazami A B, Keeton C R and Petters A O 2011 Lensing by Kerr black holes. II: analytical study of quasi-equatorial lensing observables J. Math. Phys. 52 102501
[3] Abramowicz M, Carter B and Lasota J 1988 Optical reference geometry for stationary and static dynamics Gen. Relativ. Gravit. 20 1173–83
[4] Ambrosetti A and Prodi G 1993 A Primer of Nonlinear Analysis (Cambridge Studies in Advanced Mathematics) (Cambridge: Cambridge University Press)
[5] Antonelli P L, Ingar den R S and Matsumoto M 1993 The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology (Fundamental Theories of Physics) (Dordrecht: Kluwer)
[6] Bao D, Chern S-S and Shen Z 2000 An Introduction to Riemann–Finsler Geometry (Graduate Text in Mathematics vol 200) (New York: Springer)
[7] Barletta E and Dragomir S 2012 Gravity as a Finslerian metric phenomenon *Found. Phys.* **42** 436–53
[8] Biliotti L and Javaloyes M A 2011 t-periodic light rays in conformally stationary spacetimes via Finsler geometry *Houst. J. Math.* **37** 127–46
[9] Bogoslovsky G Y 1977 A special-relativistic theory of the locally anisotropic space-time. I: the metric and group of motions of the anisotropic space of events *IL Nuovo Cimento B* **40** 99–115
[10] Bogoslovsky G Y 1977 A special-relativistic theory of the locally anisotropic space-time. II: mechanics and electrodynamics in the anisotropic space *IL Nuovo Cimento B* **40** 116–34
[11] Bogoslovsky G Y 1994 A viable model of locally anisotropic space-time and the Finslerian generalization of the relativity theory *Fortschr. Phys.* **42** 143–93
[12] Brandt H E 1999 Finslerian fields in the spacetime tangent bundle *Chaos Solitons Fractals* **10** 267–82
[13] Caponio E 2010 The index of a geodesic in a Randers space and some remarks about the lack of regularity of the energy functional of a Finsler metric *Acta Math. Acad. Paedagogicai Nyireghahaziensis* **26** 265–74
[14] Caponio E, Germinario A and Sánchez M 2016 Convex regions of stationary spacetimes and Randers spaces. Applications to lensing and asymptotic flatness *J. Geom. Anal.* **26** 791–836
[15] Caponio E, Javaloyes M A and Masiello A 2011 On the energy functional on Finsler manifolds and applications to stationary spacetimes *Math. Ann.* **351** 365–92
[16] Caponio E, Javaloyes M A and Masiello A 2010 Finsler geodesics in the presence of a convex function and their applications *J. Phys. A: Math. Theor.* **43** 135207
[17] Caponio E, Javaloyes M A and Masiello A 2010 Morse theory of causal geodesics in a stationary spacetime via Morse theory of geodesics of a Finsler metric *Ann. Inst. Henri Poincare* **27** 857–76
[18] Caponio E, Javaloyes M A and Masiello A 2013 Morse theory of causal geodesics in a stationary spacetime via Morse theory of geodesics of a Finsler metric (2010 *Ann. Inst. Henri Poincare* **27** 857–76) *Ann. Acad. Inst. Henri Poincare* **30** 961–8 (addendum to)
[19] Caponio E, Javaloyes M A and Sánchez M 2011 On the interplay between Lorentzian causality and Finsler metrics of Randers type *Rev. Mat. Iberoamericana* **27** 919–52
[20] Caponio E, Javaloyes M A and Sánchez M 2015 Wind Finslerian structures: from Zermelo’s navigation to the causality of spacetimes (arXiv:1407.5494 [math.DG])
[21] Caponio E, Javaloyes M A and Piccione P 2010 Maslov index in semi-Riemannian submersions *Ann. Global Anal. Geom.* **38** 57–75
[22] Caponio E and Stancarone G 2016 Standard static Finsler spacetimes *Int. J. Geom. Methods Mod. Phys.* **13** 1650040
[23] Colladay D and McDonald P 2012 Classical Lagrangians for momentum dependent Lorentz violation *Phys. Rev.* **D** **85** 044042
[24] Fortunato D, Giannoni F and Masiello A 1996 A Fermat principle for stationary space-times and applications to light rays *J. Geom. Phys.* **15** 159–88
[25] Flores J L, Herrera J and Sánchez M 2013 Gromov, Cauchy and causal boundaries for Riemannian, Finslerian and Lorentzian manifolds *Mem. Am. Math. Soc.* **1064** vi–4–76
[26] Fuster A and Pabst C U 2016 Finsler pp-waves *Phys. Rev.* **D** **94** 104072
[27] Gallego Torromé R, Piccione P and Vitório H 2012 On Fermat’s principle for causal curves in time oriented Finsler spacetimes *J. Math. Phys.* **53** 123511
[28] García-Parrado A and Senovilla J M M 2003 Causal relationship: a new tool for the causal characterization of Lorentzian manifolds *Class. Quantum Grav.* **20** 625–64
[29] Giannoni F and Piccione P 1999 An intrinsic approach to the geodesical connectedness of stationary Lorentzian manifolds *Commun. Anal. Geom.* **7** 157–97
[30] Gibbons G W, Gosm J and Pope C N 2007 General very special relativity is Finsler geometry *Phys. Rev.* **D** **76** 081701
[31] Gibbons G W, Herdeiro C A R, Warnick C M and Werner M C 2009 Stationary metrics and optical Zermelo–Randers–Finsler geometry *Phys. Rev.* **D** **79** 044022
[32] Girelli F, Liberati S and Sindoni L 2007 Planck-scale modified dispersion relations and Finsler geometry *Phys. Rev.* **D** **75** 064015
[33] Hehl F W and Obukhov Y N 2003 *Foundations of Classical Electrodynamics. Charge, Flux, and Metric* (Boston, MA: Birkhäuser)
[34] Hohmann M and Pfeifer C 2017 Geodesics and the magnitude-redshift relation on cosmologically symmetric Finsler spacetimes *Phys. Rev.* **D** **95** 104021
[35] Iyer S V and Hansen E C 2009 Light’s bending angle in the equatorial plane of a Kerr black hole Phys. Rev. D 80 124023
[36] Iyer S V and Hansen E C 2009 Strong and weak deflection of light in the equatorial plane of a Kerr black hole (arXiv:0908.0085 [gr-qc])
[37] Javaloyes M A 2016 Anisotropic tensor calculus (arXiv:1602.05492v1 [math.DG])
[38] Javaloyes M A and Sánchez M 2008 A note on the existence of standard splittings for conformally stationary spacetimes Class. Quantum Grav. 25 168001
[39] Javaloyes M A and Sánchez M 2014 On the definition and examples of Finsler metrics Ann. Scuola Norm. Super. Pisa Cl. Sci. XIII 813–58
[40] Kostelecký V A 2011 Riemann–Finsler geometry and Lorentz-violating kinematics Phys. Lett. B 701 137
[41] Kostelecký V A, Russell N and Tso R 2012 Bipartite Riemann–Finsler geometry and Lorentz violation Phys. Lett. B 716 470
[42] Kouretsis A P, Stathakopoulos M and Stavrinos P C 2012 Covariant kinematics and gravitational bounce in Finsler space-times Phys. Rev. D 86 124025
[43] Kovner I 1990 Fermat principles for arbitrary space-times Astrophys. J. 351 114–20
[44] Lämmerzahl C, Perlick V and Hasse W 2012 Observable effects in a class of spherically symmetric static Finsler spacetimes Phys. Rev. D 86 104042
[45] Li X and Chang Z 2014 Exact solution of vacuum field equation in Finsler spacetime Phys. Rev. D 90 064049
[46] Lovas R L 2004 On the Killing vector fields of generalized metrics SUT J. Math. 40 133–56
[47] Masiello A 1994 Variational Methods in Lorentzian Geometry (Pitman Research Notes in Mathematics Series vol 309) (London: Longman)
[48] Mestdag T, Szilasi J and Tóth V 2003 On the geometry of generalized metrics Publ. Math. Debrecen 62 511–45
[49] Minguzzi E 2015 Light cones in Finsler spacetime Commun. Math. Phys. 334 1529–51
[50] Minguzzi E 2015 Raychaudhuri equation and singularity theorems in Finsler spacetimes Class. Quantum Grav. 32 185008
[51] Minguzzi E and Sánchez M 2008 The causal hierarchy of spacetimes Recent Developments in Pseudo-Riemannian Geometry (Zürich: European Mathematical Society) pp 299–358
[52] O’Neill B 1983 Semi-Riemannian Geometry (Pure and Applied Mathematics vol 103) (New York: Academic)
[53] Ootsuka T, Yáhagi R and Ishida M 2017 Killing symmetry on the Finsler manifold Class. Quantum Grav. 34 095002
[54] Papagiannopoulos G, Basilakos S, Paliafitou S, Savvidou S and Stavrinos P 2017 Finsler–Randers cosmology: dynamical analysis and growth of matter perturbations Class. Quantum Grav. 34 225008
[55] Perlick V 1990 On Fermat’s principle in general relativity. I. The general case Class. Quantum Grav. 7 1319–31
[56] Perlick V 2006 Fermat principle in Finsler spacetimes Gen. Relativ. Gravit. 38 365–80
[57] Pfeifer C and Wohlfarth M N R 2011 Causal structure and electrodynamics on Finsler spacetimes Phys. Rev. D 84 044039
[58] Randers G 1941 On an asymmetrical metric in the fourspace of general relativity Phys. Rev. 59 195–9
[59] Russell N 2015 Finsler-like structures from Lorentz-breaking classical particles Phys. Rev. D 91 045008
[60] Rutz S F 1993 A Finsler generalisation of Einstein’s vacuum field equations Gen. Relativ. Gravit. 25 1139–58
[61] Schreck M 2016 Classical Lagrangians and Finsler structures for the nonminimal fermion sector of the standard model extension Phys. Rev. D 93 105017
[62] Stavrinos P and Vacaru S 2013 Cyclic and ekpyrotic universes in modified Finsler osculating geometry on tangent Lorentz bundles Class. Quantum Grav. 30 055012
[63] Szilasi J 2003 A setting for Spray and Finsler geometry Handbook of Finsler Geometry (Dordrecht: Kluwer)
[64] Vacaru S 2012 Principles of Einstein–Finsler gravity and perspectives in modern cosmology Int. J. Mod. Phys. D 21 1250072
[65] Werner M C 2012 Gravitational lensing in the Kerr–Randers optical geometry Gen. Relativ. Gravit. 44 3047–57

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