Feynman diagrams as a weight system: four-loop test of a four-term relation

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Abstract

At four loops there first occurs a test of the four-term relation derived by the second author in the course of investigating whether counterterms from subdivergence-free diagrams form a weight system. This test relates counterterms in a four-dimensional field theory with Yukawa and $\phi^4$ interactions, where no such relation was previously suspected. Using integration by parts, we reduce each counterterm to massless two-loop two-point integrals. The four-term relation is verified, with $\langle G_1 - G_2 + G_3 - G_4 \rangle = 0 - 3\zeta_3 + 6\zeta_3 - 3\zeta_3 = 0$, demonstrating non-trivial cancellation of the trefoil knot and thus supporting the emerging connection between knots and counterterms, via transcendental numbers assigned by four-dimensional field theories to chord diagrams. Restrictions to scalar couplings and renormalizable interactions are found to be necessary for the existence of a pure four-term relation. Strong indications of richer structure are given at five loops.

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1. Introduction

In [1] one of us (DK) formulated an argument leading to the conclusion that a four-term relation is obeyed by a class of subdivergence-free counterterms obtainable by conventional perturbative expansions of \textit{bona fide} field theories, thus extending consideration of four-term relations from the rarefied realm of topological [2] field theory to the concrete workbench of calculational [3] techniques, of practical value in four-dimensional spacetime.

There are two avenues opened up by the argument of [1]. The first concerns the mapping [4, 5] from knots to numbers, realized [6, 7, 8, 9] by counterterms. We remark that the discovery of a four-term relation offers a prospect of deriving a knot-to-number connection from the abstract properties of the resulting weight system. It may thus provide \textit{post hoc} clarification of the field-theoretic successes [6, 7, 8, 9, 10, 11, 12, 13, 14, 15] of the ideas in [4, 5]. The second, here addressed, concerns tests of the four-term relation and investigation of whether it fails when the stipulations in [1] are not met.

In Section 2 we prosecute a successful test in a combined Yukawa and $\phi^4$ theory, at four loops. Sections 3 and 4 confirm the expectations [1] that a pure four-term relation is vitiated by vector couplings, and by non-renormalizable interactions. Section 4 also considers a specific three-term relation, derived in [1]. Section 5 offers conclusions.

2. Four terms, four loops, and four dimensions

Fig. 1 shows the four subgraphs that generate every four-term relation. In each of the four cases, three arcs of a circle are indicated, with a chord connecting the upper pair. These arcs form part of a hamiltonian circuit that passes through every vertex of each diagram. The connections of vertices on other parts of the hamiltonian circuit need not yet concern us. From the bottom arc, connections are made, in turn, to the four parts of the hamiltonian circuit that are adjacent to the chord. We assume that the four terms:

(i) are free of subdivergences;
(ii) differ only by the subgraphs of Fig. 1;
(iii) have trivial vertices, involving no vectorial (or higher tensorial) structure;
(iv) involve no propagator with spin $s > \frac{1}{2}$;
(v) modify one of the dimensionless couplings of a renormalizable theory.

\textbf{Fig. 1} Every four-term relation contains these subgraphs.

\begin{figure}[h]
\begin{center}
\includegraphics[width=\textwidth]{fig1.pdf}
\end{center}
\end{figure}
The necessity of this set of provisos is not established. In [1] it is, however, claimed to be sufficient to derive the four-term relation

\[ \langle G_1 - G_2 + G_3 - G_4 \rangle = 0 \]  

(1)

where \( \langle G_k \rangle \) is the corresponding counterterm, i.e. the coefficient of overall logarithmic divergence of the \( k \)-th of the four diagrams, numbered in cyclic order, as in Fig. 1. These counterterms may be calculated by nullifying external momenta and internal masses, and cutting the diagram wheresoever one pleases, since infrared problems are excluded by the provisos. Thus if we can find a non-trivial case, with less than five loops, in four dimensions, the machinery of [3] suffices to test the prediction (1), without any subtleties of infrared rearrangement.

The four-term relation of Fig. 1 necessarily operates on counterterms with at least three loops, since it entails a hamiltonian circuit, a chord, and a connection from an origin to one of the four parts of the hamiltonian circuit that are adjacent to that chord. To prevent subdivergences in four dimensions, there must be at least one further loop. Indeed we have found only one four-loop four-dimensional case in which the above conditions are satisfied. There are several five-loop cases, but their computation lies beyond what is systematically achievable by the algorithms of [3].

**Fig. 2** To generate four terms, at four loops, connect O to each blob, in turn.

To generate the four-loop test, consider Fig. 2, whose four blobs indicate the connections that will be made to the origin O. The horizontal double line represents the propagation of a Dirac fermion field, \( \psi \), with a Yukawa coupling, \( \overline{\psi} \phi \psi \), to a scalar boson field, \( \phi \). At X there is a Yukawa coupling to an external boson, which prevents subdivergences. The asymmetry which it introduces also guarantees non-triviality of the four-term relation. Now we connect the origin O to each of the four blobs, in turn, so that O becomes a \( \phi^4 \) vertex. Masses are then set to zero, and the external momenta at A, B and X are nullified, to give the four terms of Fig. 3. Each has a (possible) overall logarithmic divergence, since it is a contribution to the renormalization of the Yukawa coupling, which, like the \( \phi^4 \) coupling, is dimensionless. After nullification, we cut the four diagrams at convenient places, marked by | in Fig. 3. The value of each counterterm is thus given by a finite three-loop massless two-point function. Moreover, the counterterms \( \langle G_1 \rangle \) and \( \langle G_4 \rangle \) factorize into products of one-loop and two-loop functions. Hence we obtain \( \langle G_{1,4} \rangle \) as two-loop integrals and \( \langle G_{2,3} \rangle \) as three-loop integrals. The latter may be reduced to two-loop integrals, using integration by parts [3].
Fig. 3 The four terms, after nullification, with cuts at convenient places.

Explicit expressions for the four counterterms may be compactly written using

\[ d\mu_n = \frac{(p_0^2)^{1+n\varepsilon}}{(p_n - p_0)^2} \prod_{k=1}^{n} \frac{d^D p_k}{\pi^{D/2}} \frac{G(1 + \varepsilon)}{[G(1)]^2} \frac{1}{p_k^2} \frac{1}{(p_{k-1} - p_k)^2} \]  

as a $n$-loop integration measure in $D \equiv 4 - 2\varepsilon$ euclidean dimensions, with $p_0$ as the cut momentum, and $G(\alpha) \equiv \Gamma(D/2 - \alpha)/\Gamma(\alpha)$. The four terms of Fig. 3 are given by

\[ \langle G_1 \rangle = \frac{1}{4} \lim_{\varepsilon \to 0} \text{Tr} \int d\mu_2 \frac{1}{p_1^2} \frac{1}{p_{02}} \]  
\[ \langle G_2 \rangle = \frac{1}{4} \lim_{\varepsilon \to 0} \text{Tr} \int d\mu_3 \frac{1}{p_{10}^2} \frac{1}{p_{12}} \frac{1}{p_{30}} \]  
\[ \langle G_3 \rangle = \frac{1}{4} \lim_{\varepsilon \to 0} \text{Tr} \int d\mu_3 \frac{1}{p_{10}^2} \frac{1}{p_{12}} \frac{1}{p_{30}} \]  
\[ \langle G_4 \rangle = \frac{1}{4} \lim_{\varepsilon \to 0} \text{Tr} \int d\mu_2 \frac{1}{p_1^2} \frac{1}{p_{12}} \]

with $p_{ij} \equiv p_i - p_j$.

To proceed, we use the following properties of the measure (2):

\[ \int d\mu_1 = -\frac{\varepsilon}{\varepsilon} \]  
\[ \lim_{\varepsilon \to 0} \int d\mu_n = \left( \frac{2n}{n} \right) \zeta_{2n-1} \]  
\[ \int d\mu_2 \frac{p_0 \cdot p_1}{p_1^2} = \frac{1 + 2\varepsilon}{2} \int d\mu_2 \]  
\[ \int d\mu_2 \frac{p_1 \cdot p_2}{p_1^2} = \frac{1 + \varepsilon}{2} \int d\mu_2 \]

with (7) resulting from the choice of normalization in (2), and (8,10) from integration by parts. The $n$-loop result (8), with $n > 1$, was proved in [16], by analysis of the wheel diagram with $n + 1$ spokes in $D$ dimensions. It generates the transcendentals associated [4].
with the \((2n - 1, 2)\) torus knots [17], via crossed ladder diagrams [3, 18] that are obtained from wheel diagrams by conformal transformation [19]. A purely four-dimensional derivation of \([8]\) was given in [19, 20], using Chebyshev-polynomial techniques [21].

These results lead to immediate evaluation of \([3]\), for which \([8, 10]\) give \(\langle G_1 \rangle = 0\) and \(\langle G_4 \rangle = 3\zeta_3\). The four-term relation \([1]\) thus requires \(\langle G_3 - G_2 \rangle\) to evaluate to \(3\zeta_3\), which is a strong prediction for the three-loop two-point functions of \([1, 3]\), unexpected prior to \([1]\). We shall show that each term evaluates to a multiple of the trefoil-knot transcendental, \(\zeta_3 = \sum_{n>0} 1/n^3\), and that the four-term relation is indeed satisfied.

To complete the experimentum crucis, we use integration by parts [3] on the central triangles of \(\langle G_2 \rangle\) and \(\langle G_3 \rangle\) in Fig. 3. Each term so generated lacks a fermion propagator. Subintegration then reduces the integrals to combinations of terms of two-loop form, each with a propagator raised to a non-integer power. This method is intrinsically \(D\)-dimensional; at \(\varepsilon = 0\) separate contributions diverge. Performing the subintegrations and relabelling momenta, we obtain

\[
\int d\mu_3 \frac{1}{p_{10}^1 p_k^1 p_{30}^1} = \int d\mu_2 \frac{1}{p_{10}^1} H_k \frac{1}{p_{20}^1}
\]

for \(k = 2, 3\), with

\[
(D - 3)H_2^i = \frac{\phi_1 (\phi_1 + \phi_2) (E_{10} - E_{12})}{\varepsilon} + (\phi_0 \phi_1 + \phi_1 \phi_2) E_{10}
\]

\[
(D - 4)H_3^i = \frac{2\phi_1 \phi_2 (E_{10} - E_{12})}{\varepsilon} + 2\phi_0 \phi_2 E_{10}
\]

and \(E_{ij} \equiv (p_i^2 / p_j^2)^{\varepsilon}\). Evaluation of \(\langle G_3 \rangle\), from \(H_3\), thus requires one to expand two-loop integrals to \(O(\varepsilon^2)\). However, we found that this did not generate \(\zeta_5\), whose absence is required by the four-term relation, since no other term may generate it. Knot theory [4] alone is insufficient to show the absence of \(\zeta_5\), since the momentum flow in \(\langle G_{2,3} \rangle\) is identical to that in the four-loop zig-zag diagram [3] for renormalization of \(\phi^4\) theory, which yields \([3]\) \(20\zeta_5\), corresponding [4] to the \((5,2)\) torus knot [17].

It remains to perform the two-loop integrals \([11]\) and take the limit \(\varepsilon \to 0\) in \([3, 8]\). This is easily accomplished with the aid of \([8]\), where we showed how to reduce two-loop two-point integrals to Saalschützian \(_3F_2\) series, when there are two adjacent propagators with integer exponents. Such series were also encountered in [22, 23] and are exploited in [3, 24, 25, 26]. Their \(\varepsilon\)-expansions are easily developed [3], using identities systematized in [24]. Only at the level relevant to six-loop renormalization [3] does one first [27, 28] encounter a transcendental that is an irreducible multiple zeta value [29], of the type studied by [1, 3, 8, 10, 24], in the context of field theory and knot theory, and by [30, 31, 32, 33, 34, 35] in the context of number theory. Up to five loops, all counterterms are believed to be reducible to \(\{\zeta_n \mid 3 \leq n \leq 7\}\), though an algorithm for achieving this reduction is established only up to four loops [3].

Having already exploited integration by parts, in \([11]\), before taking traces, we found it more economical to use the hypergeometric recurrence relations of [8], instead of the full machinery of [3, 36]. To check our results, we used the Reduce [37] program Slicer [33], which implements [3] by slicing four-loop bubble diagrams, and hence avoids the proliferation of three-loop two-point cases that are handled in separate programs by Mincer [34]. Each method yields \(\langle G_2 \rangle = 3\zeta_3\) and \(\langle G_3 \rangle = 6\zeta_3\).
We hence verify the four-term relation of [1], in its sole non-trivial appearance below five loops, where the four-loop diagrams of Fig. 3 give

\[ \langle G_1 - G_2 + G_3 - G_4 \rangle = 0 - 3\zeta_3 + 6\zeta_3 - 3\zeta_3 = 0 \]  

(14)
demonstrating cancellation of the trefoil knot in a manner that could scarcely have been anticipated before the analysis of [1].

3. Vector couplings and vector propagators

In [1] the derivation of (1) made an explicit stipulation that propagators adjacent to the chord have no tensor structure. To investigate whether this restriction is indeed necessary, we consider the case that \( \times \) in Fig. 3 represents a \( \gamma_\mu \) coupling to an external vector boson. This modifies the four terms as follows

\[ \langle \tilde{G}_1 \rangle = \frac{1}{16} \lim_{\epsilon \to 0} \text{Tr} \int d\mu \gamma_\mu \gamma_\mu \psi_{12} \psi_{12} \psi_{02} = \frac{3\zeta_3 - 1}{2} \]  

(15)

\[ \langle \tilde{G}_2 \rangle = \frac{1}{16} \lim_{\epsilon \to 0} \text{Tr} \int d\mu \gamma_\mu \gamma_\mu \psi_{10} \psi_{10} \psi_{20} = \frac{3\zeta_3 + 1}{2} \]  

(16)

\[ \langle \tilde{G}_3 \rangle = \frac{1}{16} \lim_{\epsilon \to 0} \text{Tr} \int d\mu \gamma_\mu \gamma_\mu \psi_{10} \psi_{10} \psi_{20} = -3\zeta_3 \]  

(17)

\[ \langle \tilde{G}_4 \rangle = \frac{1}{16} \lim_{\epsilon \to 0} \text{Tr} \int d\mu \gamma_\mu \gamma_\mu \psi_{12} \psi_{12} \psi_{02} = -\frac{3}{2} \zeta_3 \]  

(18)

which, as allowed by the provisos, fail to satisfy a four-term relation.

Similarly, we find that there is no four-term relation when the chord is a vector boson, with any rational choice of the gauge parameter \( a \) in its propagator \( g_{\mu\nu}/k^2 + (a-1)k_\mu k_\nu/k^4 \).

4. Indications of richer structure at five loops

There is one class of five-loop subdivergence-free counterterms that may be obtained [16] from integration by parts: those whose momentum flow is that of the wheel with five spokes. Consider a putative four-term relation, generated by Fig. 4.

Fig. 4 To generate four terms, at five loops, connect O to each blob, in turn.

Each term is a radiative correction to a \( \overline{\psi} \phi^2 \psi \) coupling, induced by Yukawa couplings and a non-renormalizable \( \phi^5 \) interaction, thereby violating proviso (v) of Section 2. After
systematic implementation of integration by parts for five-spoke wheels, via recurrence relations on 15 exponents of Lorentz scalars, we found that the counterterms

\[
\langle G_1 \rangle = \frac{1}{4} \lim_{\varepsilon \to 0} \operatorname{Tr} \int d\mu_3 \frac{1}{p_{30}} = -2\zeta_3
\]

(19)

\[
\langle G_2 \rangle = \frac{1}{4} \lim_{\varepsilon \to 0} \operatorname{Tr} \int d\mu_4 \frac{1}{p_{10}} \frac{1}{p_{10}} \frac{1}{p_{40}} = 4\zeta_3
\]

(20)

\[
\langle G_3 \rangle = \frac{1}{4} \lim_{\varepsilon \to 0} \operatorname{Tr} \int d\mu_4 \frac{1}{p_{10}} \frac{1}{p_{10}} \frac{1}{p_{40}} = 20\zeta_5
\]

(21)

\[
\langle G_4 \rangle = \frac{1}{4} \lim_{\varepsilon \to 0} \operatorname{Tr} \int d\mu_3 \frac{1}{p_{10}} \frac{1}{p_{10}} = 10\zeta_5
\]

(22) fail to satisfy a four-term relation. This failure (discovered by DJB) was the origin of proviso (v) in [1] and indicates how closely the pure four-term relation is associated with renormalizable field theory.

Remarkably, a four-term relation is obtained, if one moves the external vertex Y, on the $p_4$ line of $\langle G_2 \rangle$, to the $p_3$ line where X resides, giving

\[
\langle G^*_2 \rangle = \frac{1}{4} \lim_{\varepsilon \to 0} \operatorname{Tr} \int d\mu_4 \frac{1}{p_{10}} \frac{1}{p_{10}} \frac{1}{p_{40}} \frac{1}{p_{40}} = 10\zeta_5 - 2\zeta_3
\]

(23) and hence non-trivial five-loop cancellation

\[
\langle G_1 - G^*_2 + G_3 - G_4 \rangle = -2\zeta_3 - (10\zeta_5 - 2\zeta_3) + 20\zeta_5 - 10\zeta_5 = 0
\]

(24) of both the (3, 2) and (5, 2) torus knots. Efforts are in hand to derive the modified four-term relation (24) from mixing of $\bar{\psi}\phi^2\psi$ radiative corrections with $(\bar{\psi}\psi)^2$ corrections that are indistinguishable from the former, after nullification. For the present, we adduce it as an indication of richer structure that may be deducible from the extension of [1] to cases in which the provisos are relaxed.

Finally, we remark on a specific three-term relation, derived in [1]. It is possible that such relations, called STU relations in the theory of chord diagrams [2], impose even stronger constraints upon the structure of field-theory counterterms. Here we give a single intriguing example. It involves the five-loop counterterms $\langle G_{3,4} \rangle$, above, which are related, via $\langle G_3 - G_4 \rangle = \langle I_{\text{sub}} \rangle$, to a four-loop counterterm, $\langle I_{\text{sub}} \rangle$, that occurs because of a subdivergence in the contour integrals that were devised in [1] to derive the four-term relation [1]. Note that the counterterm $\langle G_{4} \rangle = 10\zeta_5$ is also reducible to a four-loop diagram, since it contains a trivial convergent one-loop subintegration. Moreover, the identity $\langle G_{4} \rangle \equiv \langle I_{\text{sub}} \rangle$ is obtainable purely at the diagrammatic level, without need of four-loop integration. Hence the three-term relation of [1] tells us that

\[
\langle G_3 \rangle = \langle G_4 \rangle + \langle G_4 \rangle
\]

(25) which is indeed confirmed by the highly non-trivial calculations (24,22). We are still recovering from our surprise at this successful prediction of the five-loop counterterm [21]. Before the advent of [4, 5], it might have been expected to involve $\zeta_3$, $\zeta_5$ and $\zeta_7$, in any rational combination.

We expect that the source of the findings above as well as the restrictions summarized in the provisos can be ultimately explained by the presence of a modified STU relation. We expect this relation to connect the difference between two three-point couplings to a four-point coupling. We will report progress along these lines elsewhere [18].
5. Conclusions

We have used the methods of \[3, 8\] to verify the sole predicted \[1\] non-trivial four-term relation between subdivergence-free four-dimensional counterterms with less than five loops. Analytical tools \[39\] do not yet exist \[40\] to investigate pure four-term relations in four-dimensional renormalizable theories at five loops and beyond, where trivalent couplings frustrate the progress that we achieved to seven loops \[9\] in pure $\phi^4$ theory. Nor can the all-order methods of \[8\] be turned to account, at present, since these derive from large-$N$ methods, where subdivergences are of the essence. The situation is somewhat similar to that in quenched QED, where the cancellation of transcendentals that is predicted to all orders by knot theory \[10\] has been confirmed at four loops \[41\], with little immediate prospect of progressing to five loops.

In a forthcoming book \[42\], the relation to three-dimensional Chern-Simons theory will be discussed. From the calculational point of view, this topological theory (with apparently no possibility of observable particles) appears to have little to offer in terms of multi-loop perturbative results, which is ironical in view of the fact that much discussion of four-term relations has so far been grounded in its chord diagrams \[2, 43\]. We note that the rational three-dimensional two-loop beta-function in \[44\] is in full accord with the expectations of \[4, 5\]. A promising avenue of multi-loop inquiry concerns the cosmological constant generated by Yukawa and $\phi^4$ couplings, whose four-loop analysis proved tractable in three dimensions \[45\], where it is a logarithmically divergent quantity, because of the super-renormalizability of the theory.

If one is prepared to progress to five loops, with purely trivalent couplings, then renormalization of $\phi^3$ theory in six dimensions \[15\] is the cleanest case to study, being free of any tensorial complication, and extremely benign in the infrared. Here, one suspects that the first tests of four-term relations will be made by approximation methods, rather than analytically.

More generally, we have an intuition that worldline techniques \[46\] may illuminate cancellation of knot-transcendentals between counterterms, in both the quenched-QED analysis of \[14\] and also the four-term analysis of \[11\], since each is concerned with subdivergence-free combinations of chord diagrams. The results in \[17\] indicate that the incorporation of diagrams with subdivergences is not out of reach.

In conclusion, it is gratifying that the prediction of \[1\] is borne out in the only non-trivial test that we have been able to devise, without exorbitant labour. It is, however, still frustrating to lack further definitive case law, beyond \(14\) and the expected failures of Sections 3 and 4. The modified four-term relation \[24\] and the remarkable STU-type relation \(23\) indicate that an even richer structure of counterterms awaits discovery. We hope that colleagues will exercise ingenuity to progress the issue, either by developing \[16\] calculational techniques that may prove more powerful than the four-loop methods of \[3\], and less restricted than the $n$-loop methods of \[16\], or by analyzing how the four-term relation is modified by subdivergences, by vector couplings, by vector propagators, and by dimensionful coupling constants. Some progress along these lines will be reported elsewhere \[18\].

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