Topological interface states due to spontaneous symmetry breaking in a chain of anharmonic oscillators

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Abstract. In this work, we propose a one-dimensional system of coupled anharmonic oscillators with nonlinear coupling and study a spontaneous formation of interface excitations in the linearized spectrum of such periodic array. We show that during its evolution such system can undergo a topological transition from the disordered and topologically trivial phase into the nontrivial one. The topological transition is accompanied by the formation of the interface state in the spectrum of linearized excitations of the equilibrium state. Employing the technique based on the calculation of mean chiral displacement we demonstrate that the in-gap interface state exhibits a topological nature and is analogous to that in the well-known Su-Schrieffer-Heeger model. Our results demonstrate the formation of topological interface state induced by spontaneous symmetry breaking mechanism and open novel possibilities for designing various nonlinear electronic, photonic and polaritonic systems with disorder-robust control of information flows.

Topological edge or interface states of electrons \cite{1}, light \cite{2, 3, 4} and sound \cite{5} have recently received much attention due to their prospects for realization of disorder-robust one-way transport of information. Presently, the interest is shifting towards topological states in nonlinear and interacting systems promising higher tunability and richer fundamental physics \cite{6, 7, 8, 9, 10, 11, 12, 13, 14}. However, there is still no clear recipe to realize a nonlinear system with edge or interface states between topologically distinct domains appearing due to spontaneous symmetry breaking. This work addresses an interesting question raised in this regard: is it possible to realize a topological phase transition from a trivial to a gapped nontrivial phase accompanied by appearance of the edge or interface states between topologically distinct domains?

To this end, we consider a system depicted in Fig. 1(a), which is based on the array of identical anharmonic oscillators with double-well on-site potential and nonlinear coupling between the nearest neighbors. The entire array is described by potential function

\begin{equation}
U = \sum_{n=1}^{N} (a_2 y_n^2 + a_4 y_n^4) + \sum_{n=1}^{N-1} \left[ b_2 (y_n - y_{n+1})^2 + b_4 (y_n - y_{n+1})^4 \right],
\end{equation}

where $y_n$ is the displacement of the $n$th oscillator, $a_2$, $a_4$ and $b_2$, $b_4$ are on-site and inter-site force constants, respectively, and $N$ is the number of oscillators. The evolution of bulk oscillators are...
then described with the following equation of motion:

\[ \ddot{y}_n + 2\gamma \dot{y}_n = -2a_2 y_n - 4a_4 y_n^3 - 2b_2 (y_n - y_{n-1}) - 2b_2 (y_n - y_{n+1}) - 4b_4 [(y_n - y_{n-1})^3 + (y_n - y_{n+1})^3], \]  

(2)

where \( \gamma \) term describes the friction. Equations of motion for the first and \( N \)th oscillators can be obtained from (2) by omitting the brackets with \( y_n \) and \( y_{n+1} \), respectively. The equilibrium states of the system \( y_n(0) \) can be found from the system of equations (2) with zero left side. Our analysis reveals that one of the stable static states of such system is the tetramer stationary state in the form \( y_{4n}(0) = y_{4n+1} = -y_{4n+2} = -y_{4n+3} = v_T \). The equilibrium state of this system has a gap in linear dispersion.

The interest in the tetramer state stems from the behaviour of small oscillations \( z_n(t) \) near this equilibrium state. The system of linear equations for amplitudes \( z_n \) can be written down in the frequency domain, after substituting \( y_n(t) = y_n(0) + z_n \exp(-i\omega t) \) into the equation (2) and linearizing it. A linear dispersion of an infinite chain then reads:

\[
\begin{cases}
(\omega^2 - \omega_R^2)A + (J_1 + J_2 e^{-i k})B = 0, \\
(J_1 + J_2 e^{i k})A + (\omega^2 - \omega_R^2)B = 0,
\end{cases}
\]

(3)

where \( \omega_R^2 = \omega_R^2 + J_1 + J_2, \omega_R^2 = 2(a_2 + 6a_4 v_T^2) \), \( J_1 = 2b_2, J_2 = 2b_2 + 48b_4 v_T^2 \), \( A, B \) are the amplitudes of the two oscillators in the unit cell [see Fig. 1(a)], and \( k \) is the normalized Bloch wavenumber. The interaction constants for the tetramer state differ by \( |J_{n1} - J_{n2}| = 48b_4 v_T^2 \), which makes this model similar to the Su-Shrieffer-Heeger one. Consequently, the solution of this system has a gap in linear dispersion.

In general, the edge oscillators have different static displacements than the bulk ones. However, it can be shown that the tetramer state still persists at the edges, if the following condition is fulfilled: \( b_2/a_2 = 4b_4/a_4 \). This condition along with the constraints imposed by the requirement of existence and stability of the tetramer solution defines the range of possible values of the force constants. In our calculations we have chosen \( a_2 = -20, b_2 = -2, a_4 = 1 \) and \( b_4 = 1/40 \).

In contrast to the linear Su-Shrieffer-Heeger model, such nonlinear system does not possess edge states, which is due to the nonlinear detuning of the resonance frequencies of oscillators. Therefore, we focus on the interface states which can exist even in the nonlinear system at the boundary between two different tetramer phases [16] [see Fig. 1(b)]. In order to probe the emergence of the expected topological interface state, we analyze the dynamics of \( N = 80 \) oscillators by directly solving full dynamic equations with small friction term \( \gamma = 0.03 \) included for convergence. Initial displacements of the oscillators were deviated from the static tetramer state with the boundary at the 40th oscillator [the scheme is shown in Fig. 1(b)] by a random value in the range \((-\delta_y, \delta_y)\), with maximum deviation \( \delta_y = 0.3|v_0| \) (the static displacement was

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**Figure 1.** (a) Scheme of an infinite chain in the tetramer state. (b) Scheme of the possible interface configurations between two semi-infinite chains in different tetramer phases. The brackets indicate the unit cell choice in left and right domains.
|ν₀| ≈ 3.16). Similarly, initial velocities were defined by random values in the range (− δy, δy), with δy = 1.5. At each moment of time t the calculated yₙ(t) were considered as stationary displacements, and the spectrum of small oscillations was evaluated. In this way we recovered the evolution of the spectrum presented in Fig. 2(a). It is seen that during the evolution the spectrum of the system becomes gapped, and two edge states appear, the in-gap state and the state above the allowed bands. We are interested in the former state, corresponding to the topological zero-energy state in the SSH model [15]. The calculated displacement distribution depicted in Fig. 2(b) confirms that this state is indeed localized at the interface.

Since in the considered system the elements of the interaction matrix that describes the dispersion of the linearized system are random in the beginning of system evolution (due to random initial conditions) and converge to stationary values only in the limit of t → ∞, at each moment of time the system is not periodic, and consequently the topological properties of the system cannot be derived from direct calculation of the Zak phase. Thus, we employ the technique based on the calculation of mean chiral displacement [17], which allows for determining of the topological phase of the not necessarily periodic systems. For a given system one can calculate the MCD of a freely evolved state Ψ as a function of time delay τ as follows:

$$\text{MCD}(t, \tau) = \sum (\mathbf{\Gamma} \mathbf{n}) \Psi(\tau + t),$$

where Γ is the matrix of the chiral operator, n is the matrix of the position operator, so that $\mathbf{\Gamma} \mathbf{n} = \text{diag}(..., -1, 1, 0, 0, 1, -1, 2, -2, ...)$; and the vector of displacements $\Psi = [..., y_1, y_2, ...]^T$ is localized in an arbitrary unit cell for τ = 0. For the SSH model the value of MCD at large times τ converges to either 0 or 0.5 depending on the choice of the unit cell, which corresponds to the values of the Zak phase $\gamma = 0$ or $\gamma = \pi$, respectively [17]. However, for non-periodic systems time dependence of the MCD might not converge to any certain value. Such behaviour can be observed in Fig. 3 where we plotted the limit values of MCD at $\tau = \infty$ as a function of time t. At small times all oscillators possess relatively large random displacements, and consequently the calculated values of the MCD are also random, i.e. the system does not exhibit nontrivial topological properties. However, at large times $t \gtrsim 150$ the values of MCD($\tau \rightarrow \infty, t$) converge to 0 and 0.5 for the left and the right sides of the chain, respectively, indicating the formation of topologically different phases and consequently the topological origin of the interface state.

To summarize, our findings prove that nonlinear systems can dynamically switch from the disordered regime to the regime with non-zero-frequency topological edge states due to the spontaneous symmetry breaking mechanism. We believe that the fundamental link between
Figure 3. Mean chiral displacement calculated for the left (red dots) and right (blue dots) tetramer domains, see Fig. [1]b).

spontaneous symmetry breaking and dynamical topological states demonstrated here on a simple example of mechanical system is much more general being applicable to a wide variety of nonlinear electronic, photonic and polaritonic systems.

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