Ground-state properties of artificial bosonic atoms, Bose interaction blockade and the single-atom pipette

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We analyze the ground-state properties of an artificial atom made out of repulsive bosons attracted to a center for the case that all the interactions are short-ranged. Such bosonic atoms could be created by optically trapping ultracold particles of alkali vapors; we present the theory describing how their properties depend on experimentally adjustable strength of “nuclear” attraction and interparticle repulsion. The binding ability of the short-range potential increases with space dimensionality - only a limited number of particles can be bound in one dimension, while in two and three dimensions the number of bound bosons can be chosen at will. Particularly in three dimensions we find an unusual effect of enhanced resonant binding: for not very strong interparticle repulsion the equilibrium number of bosons bound to a nuclear potential having a sufficiently shallow single-particle state increases without bound as the nuclear potential becomes less attractive. As a consequence of the competing nuclear attraction enhanced by the Bose statistics and interparticle repulsions, the dependence of the ground-state energy of the atom on the number of particles has a minimum whose position is experimentally tunable. This implies a staircase dependence of the equilibrium number of bound bosons on external parameters which may be used to create a single-atom pipette - an arrangement which allows the transport of atoms into and out of a reservoir, one at a time.

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I. INTRODUCTION

The experimental observation of Bose-Einstein condensation (BEC) of trapped alkali vapors [1] has triggered a tremendous volume of experimental and theoretical work in this field [2]. One of the attractive features of BECs which makes them an active area of research is that they represent well-controlled ensembles of atoms that can be used to study novel aspects of many-body physics.

In this work we will describe an effect and its device implementation which will farther improve control over the number of particles in a trap: we explain how to build a quantum pipette enabling single-atom manipulation. The operation of the pipette is based on the discrete nature of the particles combined with their mutual repulsion. The pipette is a bosonic cousin of the single-electron box [3], however the differing statistics (Bose versus Fermi) and the nature of the interactions (short-range repulsion among bosons versus long-range Coulomb repulsion between the electrons) require a separate analysis.

A way to create a quantum tweezer for atoms was put forward recently [4]. Even though the underlying physics is related to what follows, the proposed extraction scheme cannot work. It was claimed that by moving an attractive potential out of the BEC at certain speeds, a chosen number of atoms could be extracted by means of Landau-Zener tunneling [5]. It was assumed that the superfluidity of the BEC helps prevent excitations which otherwise would be induced by the motion; this restricts the speeds to be smaller than the sound velocity. However the local sound velocity vanishes at the edge of the condensate [6]. Thus it is impossible to exit the BEC at finite velocity without incurring nonsuperfluid effects. Moreover Ref. [4] relies on a model which, as will be explained below, cannot be the starting point of a realistic computation.

We call our arrangement a pipette because it allows both uploading and downloading at the single-particle level. As in Ref. [4] the role of the pipette is played by an external short-range attractive potential which can be created at the focus of a red-detuned laser beam [7]. In order to understand the mechanism by which single-particle manipulation becomes possible, we need to know the ground-state energy $E_n$ of $n$ repulsive bosons bound by the potential.

To set the conventions, select the zero of the external potential at infinity and assume that it only has one single-particle state of energy $E_1 < 0$. For noninteracting bosons, all $n$ particles will populate this state thus leading to a ground-state energy that decreases linearly with the number of particles: $E_n = -|E_1| n$. The effect of interparticle repulsion may be accounted for heuristically by noticing that there are $n(n-1)/2$ pair interactions among $n$ particles. Then the ground-state energy can be estimated as $E_n = -|E_1| n + \nu n(n-1)/2$, where $\nu > 0$ is proportional to the amplitude of the interparticle repulsion.

Imagine that the pipette is placed in the vicinity of a BEC reservoir formed by a potential which does not vary significantly on the spatial scale of the pipette, as shown in Fig.1. Then the overlap of the reservoir and pipette potentials shifts the single-particle level of the pipette to $E_1 + U_{\text{res}}(r_p)$ where the reservoir potential $U_{\text{res}}(r)$ is
evaluated at the location of the pipette. The ground-state energy calculated relative to the chemical potential \( \mu \) of the reservoir, \( E'_n = E_n + \left[ U_{res}(r_p) - \mu \right] n \), is given by [4]

\[
E'_n = -|E_1| + U_{res}(r_p) - \mu |n + \nu(n-1)/2 \quad (1)
\]

Eq. (1) overlooks some important aspects of the interacting problem to be discussed below, but it correctly captures the basic physics of the effect, and will be used to motivate our study. Since the pipette is outside the BEC \( U_{res}(r_p) > \mu \), they are separated by a tunneling barrier whose strength can be tuned by varying the distance between the pipette and the reservoir. The particles of the reservoir will still be able to tunnel into the pipette, and there can be particles in the pipette whenever the chemical potential of the BEC \( \mu \) is higher than the single-particle level \( |E_1 + U_{res}(r_p)| \) shown in Fig. 1. So for the particle extraction to occur in the presence of the tunneling barrier, the dimensionless parameter \( \gamma = |E_1| + \mu - U_{res}(r_p) |/|E_1| \) should lie between zero and unity.

![Potential Energy Landscape](potential_energy.png)

**FIG. 1.** Potential energy landscape where the smooth external potential confining a BEC, \( U_{res}(r) \), overlaps with that of a sharp short-range well (the pipette) as a function of position. The pipette is placed some distance away from the BEC edge, \( U_{res}(r) = \mu \), so that a tunneling barrier with the condensate exists. The particles can be transported in and out of the pipette one at a time by tuning both the depth of the short-range potential and the strength of interparticle repulsion. The short horizontal segment inside the pipette indicates its single-particle state.

When \( \gamma \) is close to unity, the pipette is nearly inside the condensate. The coupling to the reservoir is strong, so that the particles can easily tunnel in and out of the pipette. Under these conditions the pipette will not be in a state of definite particle number: it will be in a quantum superposition state, with amplitudes for different integer pipette populations. This means that the particle discreteness is irrelevant, and as far as expectation values are concerned, \( n \) in (1) can be treated as a continuous variable. The average number of bound bosons is then given by the minimum of (1), which occurs for \( n_x = 1/2 + |E_1|/\nu \). We note that the parameter \( n_x \) can be controlled experimentally: the single-particle energy \( E_1 \) can be varied by changing the depth of the pipette potential which in turn can be accomplished by adjusting the power of the focused laser beam. For fixed pipette position this also affects the tunneling barrier. The strength of the interparticle repulsion \( \nu \) can also be tuned by applying an external magnetic field which alters the proximity of the Feschbach resonance [7]. We will use \( n_x \) as the independent variable in what follows to represent these ways of controlling the system.

The opposite limit of weak coupling to the reservoir will be our main interest, because then single-atom manipulation is possible. Although the external parameter \( n_x = 1/2 + |E_1|/\nu \) is generally not an integer, the pipette population will nearly always take on fixed integer values [8]. Half-integer values of \( n_x \) play a special role because now two states of the pipette with \( n = n_x \pm 1/2 \) bosons have the same energy. Tunnelling mixes these two states of the system so that as \( n_x \) is slowly varied, the \( n \) particle state adiabatically evolves into an \( n \pm 1 \) particle state. When the tunnelling is small, the increase or decrease in particle number occurs over a small range of \( n_x \), so that the change seems almost discontinuous. In what follows we will refer to these changes as population transitions keeping in mind that the notion of the transition is strictly valid only in the limit of zero tunneling and correspondingly infinitesimally slow variation of \( n_x \).

![n_x Dependence](nx_dependence.png)

**FIG. 2.** The \( n_x \) dependence of the ground-state energy \( \Delta E'_n \) for a series of integer \( n \) for the model (1). The lower bold curve corresponds to the ground-state. Gray scale is used for metastable states.

To help visualize the population transitions it is useful to look at the ground-state energy computed relative to its minimum, \( \Delta E'_n = E'_n - E'_0 = \nu(n - n_x)^2 / 2 \). The loci of the population transitions are then given by \( \Delta E'_n = \)
shown in Fig. 2 where we plot the $n_x$ dependence of $\Delta E'_n$ for a series of integer $n$. This would give us a series of intersecting parabolas, each corresponding to a fixed number of bound bosons given by its minimum. Similar to the electronic case [3], mixing of the states of different particle number replaces the intersections of the energy curves at half-integer values of $n_x$ by small energy gaps as shown in Fig. 2. As the external parameter $n_x$ changes adiabatically, the system follows the lowest energy curve shown in bold. In the limit of zero tunneling the switches between different parabolic segments can be regarded as first-order population transitions because the slopes of the intersecting energy curves do not coincide.

As the barrier between the pipette and reservoir changes from barely penetrable to almost transparent, the equilibrium staircase $\langle n \rangle = [n_x]$ evolves into a linear dependence $\langle n \rangle = n_x$ as shown in Fig. 3. We note that a finite BEC temperature also contributes into smearing of the dependence of $\langle n \rangle$ on $n_x$.

![Fig. 3. The $n_x$ dependence of the equilibrium population of the pipette $\langle n \rangle$ for different levels of transparency of the barrier with the BEC: barely penetrable (bold solid staircase), almost transparent (gray $\langle n \rangle = n_x$ line), and in between (thin solid curve).](image)

The role of a finite tunneling rate is twofold: First it leads to the rounding of staircase dependence shown in Fig. 3: the discontinuities at degeneracy points are replaced by rapid variations near half-integer $n_x$. Second, it constrains the rate of change of the external parameter $n_x$ to be slower than the tunneling rate to allow for the single-particle extraction to take place. Otherwise the system will not follow the lowest energy path shown in Fig. 2: the dependence of $\langle n \rangle$ on $n_x$ will exhibit metastability and hysteresis.

Here is an example of the pipette operation. Imagine turning on the laser beam adiabatically, focussed at a fixed place outside the BEC, but not very far from its edge, as depicted in Fig. 1. As the laser power increases, so does the amplitude of the pipette potential, and when it reaches a critical depth, a single-particle bound state appears [9]. Further increase of the power moves this (yet unpopulated) state down in energy until it reaches the BEC chemical potential $\mu$ (see Fig. 1). At that point the states with zero and one particles in the pipette become degenerate, $E'_0 = E'_1$, and upon infinitesimal increase of the power a particle tunnels into the pipette. As the laser power increases further, the pipette remains singly occupied until the energy gain of having two extracted particles overwhelms the penalty paid in increase of inter-particle repulsion: the second particle enters the pipette at $E'_1 = E'_2$. A further increase of the pipette population in unit increments happens at special values of the laser power as it goes up: single-particle tunneling out of the BEC reservoir occurs whenever the degeneracy condition $E'_0 = E'_n$ is met. The blocking of single-particle tunneling by repulsive interactions among bosons – the Bose interaction blockade – parallels the phenomenon of the Coulomb blockade in electronic systems [3].

After a desired number of particles has been extracted, the focus of the laser can be translated nonadiabatically to somewhere else [10], and then the pipette can be emptied, releasing the stored particles one by one by adiabatically decreasing the beam power. To verify that the correct number of atoms has been extracted, the atoms could be released into a magneto-optical trap (MOT). Atoms in a MOT scatter light at a well-defined rate, so by measuring the scattered power small numbers of atoms can be accurately counted [11,12].

Let us now critically assess the physical content of the model. The functional form (1) tells us that $n$ bosons are condensed into the ground state of the pipette potential and that $n((n-1)/2)$ pair interactions are distributed over a spatial scale which purports to be independent of the number of particles $n$. This scale can only be the localization length of the single-particle ground state. Therefore the size of the many-body state corresponding to (1) is the same as that of the single-particle state: the model (1) only attempts to account for the increase of energy due to inter-particle repulsion but overlooks the simultaneous “swelling” of the ground-state. As the number of bound particles increases, so does the localization length of the ground state. This in turn decreases the zero-point energy per particle thus effectively introducing an $n$ dependence into $E_1$. At the same time, the inter-particle repulsions are weakened by being distributed over a length scale which grows with $n$. We conclude that the model of interacting particles populating a single-particle state [as described by Eq. (1)] is valid only for extremely weak inter-particle repulsion $\nu(n-1)/|E_1| \ll 1$, when the swelling of the ground state can be neglected. With this in mind, the conclusion that $E'_n$ has a minimum is actually beyond the range of applicability of (1). Below we will exhibit a series of counterexamples to the functional form (1) including the practically important case
of a three-dimensional system.

Our goal now is to compute $E_n$ in a controlled fashion and convince the reader that parts of the physics based on Eq. (1) are in fact correct. We start from the more fundamental problem of calculating the many-body ground-state energy $E_n$ in the presence of a confining potential that vanishes at infinity. In view of the recent achievement of an all-optical BEC [13] where the trapping potential is short-ranged, solving this problem has an importance of its own. The ground-state energy $E_n$ calculated relative to the BEC chemical potential and accounting for the overlap with the reservoir potential then simply follows as $E_n' = E_n + [U_{res}(r_p) - \mu]n = E_n + (1 - \gamma)|E_1|n$.

The rest of the paper is organized as follows. In Section II we introduce the self-consistent (Hartree-Fock) approximation which is our main calculational tool and pose the problem using a language resembling the Coulomb terminology of atomic physics. In Section III a one-dimensional problem is studied where we start by discussing an exact minimizer of the Hartree-Fock theory (Section IIIA). In Section IIIB we consider a model of repulsive bosons on a half-line attracted to the origin. Though this problem represents a considerable deviation from that treated in Section IIIA, it has a benefit of exact solvability thus allowing us to clarify certain aspects of the self-consistent treatment. Higher-dimensional cases are investigated in Sections IV (between one and two dimensions) and V (two dimensions) where variational solutions to the Hartree-Fock theory are given. A variational solution for a three-dimensional problem is described in Section VI. Not only is this case the practically most relevant, but it is also somewhat unusual in terms of physics: we find that there is a range of parameters when the number of bound particles can increase as the trapping potential gets less attractive. In Section VII we present a Landau-Zener [5] type theory which explains tunneling smearing of the boson staircase sketched in Fig.3. In Section VIII we discuss feasibility of experimental implementation of the single-particle manipulation. Since the single-particle effects are enhanced in traps of small range, first (Section VIIIIA) we describe a method of creating an optical trap which is considerably tighter than those in current use. We conclude (Sections VIIIIB and VIIIIC) by discussing experimental limitations set by nonzero temperature of the condensate and finite time of tunneling.

II. FORMULATION OF THE PROBLEM

A controlled analytical calculation of the ground-state properties of a system of $n$ repulsive bosons attracted to a center $U(r)$ largely avoiding model assumptions is possible in the limit that the interaction range of the trapping potential $a$ is significantly smaller than the localization length $R$ of the ground state. Assume in addition that the $d$-dimensional trapping potential is radically symmetric, $U(r) = U(|r|)$, and decays sufficiently rapidly so that $S_dV = -\int U(r)d^dr = -\int_0^\infty (U(r)S_d)^{d-1}dr > 0$ is finite.

Then the condition $R \gg a$ implies that $n$ bosons are effectively bound to an attractive delta-function center of strength $S_dV$ where $S_d$ is the area of the $d$-dimensional unit sphere.

We call this system an artificial bosonic “atom” (with the pipette potential playing a role of the nucleus), since the inequality $R \gg a$ parallels the situation in the physics of real atoms, where the size of the nucleus is significantly smaller than the atomic size. In contrast to the atom of fermions wherein the Pauli principle dictates most of the properties and interactions play secondary role, the physics of the bosonic atom is dominated by interparticle repulsion [14], and even a nucleus with only one shallow single-particle state may bind more than one particle.

For general $n$ and $d$ the many-body properties cannot be computed exactly in a system with broken translational symmetry, and we have to resort to approximate methods. Then the permutation symmetry of the problem implies that the ground-state wave function can be approximated by the Hartree-Fock product ansatz: $\Psi(r_1, ..., r_n) = \prod_{i=1}^n (\psi(r_i)/\sqrt{n})$ where the “single-particle” wave function $\psi(r)$ minimizes the energy functional:

$$E = \int d^dx \left[ \frac{\hbar^2}{2m} (\nabla \psi)^2 + \frac{g(n-1)}{2n} \psi^4 \right] - S_dV\psi^2(0) \quad (2)$$

Here $m$ is the particle mass, $g$ [proportional to $\nu$ in (1)] accounts for the short-range repulsion among the bosons, and the wave function is normalized so that $\psi$ is the particle density, i.e. $n = \int d^dx \psi^2(r)$. The interaction term of (2) tells us that each particle moves in a self-consistent potential provided by the remaining $n-1$ particles, and the simplified form of the last term describing interaction with the nucleus is due to the condition $R \gg a$.

For $n = 1$ minimization of Eq.(2) is equivalent to the variational principle of quantum mechanics while for $n \gg 1$ the $n$-dependence drops out of (2) and we arrive at the Gross-Pitaevskii theory [6,15]. When substituted back in the Hartree-Fock-Gross-Pitaevskii (HFGP) energy functional (2), the radially-symmetric wave function $\psi(r) = \psi(r)$ minimizing Eq.(2) provides us with the ground-state energy $E_n$.

In what follows we will often use dimensionless parameters that contain information about various aspects of the problem:

$$\xi = \frac{mVa^{2-d}}{\hbar^2}, \quad \mu = \frac{mU_0a^2}{\hbar^2}, \quad Z = \frac{S_dV}{g}, \quad \nu = \frac{\xi}{Z}, \quad \lambda = \frac{n-1}{Z} \quad (3)$$

The strength of the nuclear attraction is parametrized by $\xi$, which is the typical depth of the nuclear well $U_0$.
measured in units of the zero-point energy of a particle localized within the spatial scale \( a \). The relative strength of the nuclear attraction and interparticle repulsion is characterized by the reduced nuclear “charge” \( Z \) which in a real atom is the number of protons in the nucleus. The dimensionless ratio \( \epsilon \) parametrizes interparticle repulsions alone. The parameter \( \lambda \) characterizes the degree of “ionicity”; in a standard atom (ion) it tells us about the relative number of protons and electrons. Given that all the particles are neutral, the Coulomb language is used figuratively.

Since the binding properties of a short-range potential depend on the spatial dimensionality, we will consider the \( d = 1, 2 \) and 3 cases separately.

III. ONE-DIMENSIONAL CASE

In one dimension at distances satisfying \( |x| \gg a \) the single-particle ground-state wave function decays as [16]

\[
\psi(x) \sim e^{-|x|/R_1} \quad \text{where} \quad R_1 = \hbar^2/2mV
\]

plays the role of an “atomic” unit of length. This behavior can be continued to the origin without introducing a divergence which implies that we can set \( a = 0 \) from the beginning. The consistency condition \( R_1 \gg a \) reduces to the requirement of a shallow well, \( \xi \ll 1 \).

A. Self-consistent treatment

For \( a = 0 \) the minimization of (2) is mathematically identical to a problem solved by Lieb and collaborators [17] in the context of the ground-state properties of an ordinary atom placed in an extremely strong magnetic field. For general \( n \) the normalized wave function minimizing (2) is given by [17]

\[
\psi(x) = \frac{\sqrt{n}(2 - \lambda)}{2\sqrt{R_1}\lambda \sinh[\frac{2 - \lambda}{2R_1} (|x| + C)]}
\]

Here tanh \( C = (2 - \lambda)/\lambda \), and the minimum of (2) exists only for \( \lambda \leq 2 \). For \( \lambda = 2 \) Eq. (4) reduces to \( \psi(x) = \sqrt{(Z + \frac{1}{2})R_1/(|x| + R_1)} \). The localization length of the many-body ground state follows from (4) as \( R_n = R_1/(1 - \frac{1}{2}) \). In “atomic” units this becomes

\[
\rho_n = R_n/R_1 = \left( 1 - \frac{n - 1}{2Z} \right)^{-1}
\]

This result implies that a given particle of the atom effectively sees a nuclear charge \( Z \) screened by an amount proportional to the number of remaining \( n - 1 \) bosons.

Eqs. (4) and (5) show that the number of bound particles increases, so does the size of the atom, eventually diverging at \( n = n_{\text{max}} = 2Z + 1 \), which is the maximum number of bosons that can be bound. The latter can be viewed as a critical phenomenon: for \( n = n_{\text{max}} \) the wave function is no longer well-localized but still normalizable.

The reduced ground-state energy \( \epsilon_n = E_n/E_1 \), measured in “atomic” units of the single-particle energy \( |E_1| = 2mV^2/\hbar^2 \) [16], is found to be [17]

\[
\epsilon_n = -n \left[ 1 - \frac{n - 1}{2Z} + \frac{(n - 1)^2}{12Z^2} \right]
\]

We note that even at \( n = n_{\text{max}} = 2Z + 1 \) the ground-state energy remains negative.

![FIG. 4. The dependence of the ground-state energy of a one-dimensional Bose atom on the number of bound particles \( n \) for \( Z = 4.8 \). The minimum of the curve indicated by the cross is an average population of the atom strongly coupled to a zero chemical potential bosonic reservoir. In the opposite limit of a well-isolated atom the allowed integer occupation states are shown by the circles - the ground state (solid circle) and metastable states (gray and open circles).](image-url)

A self-consistent treatment based on (2) is valid in one dimension only when the system is sufficiently dense [18], so that direct interactions are more important than zero-point motion. For a fixed number of bound bosons \( n \), however, the size of the atom (5) increases as \( Z \) approaches the critical condition \( n = n_{\text{max}} = 2Z + 1 \): the particle density everywhere decreases. Here the relevant quantity to look at is the reduced density \( \hbar^2\psi^2(x)/mg = Z(Z + \frac{1}{2})/(|x| + 1)^2 \) which must be significantly larger than unity [18]. Applying this condition first to the center of the density distribution, we find that \( \hbar^2\psi^2(0)/mg \gg 1 \) provided \( Z \gg 1 \). As one moves away from the center, the density decreases and at \( |x| \approx ZR_1 \gg R_1 \) the parameter \( \hbar^2\psi^2(x)/mg \) becomes of order unity. It is then straightforward to verify that in the tails of the distribution \( |x| \approx ZR_1 \) there is roughly one particle out of \( 2Z + 1 \gg 1 \) total. Thus for all practical purposes the tail of the distribution is irrelevant, the particles are not farther away from the center than \( ZR_1 \), and the self-consistent treatment is justified. The latter conclusion also holds away from the critical condition \( n = n_{\text{max}} \). Rigorous analysis [17] concludes that the
self-consistent treatment based on (2) produces asymptotically exact results, Eqs.(4)-(6), for $n, Z \gg 1$ with the ratio $n/Z$ fixed.

Beyond the range of their asymptotic accuracy Eqs.(4)-(6) can be used as interpolation formulas: for $n = 1$ they automatically reproduce the exact results; the second term of (6) can be also recognized as a result of the first-order perturbation theory in $(n-1)/Z \ll 1$.

The population dependence of the ground-state energy (6) typical to the case of weakly-repulsive particles (not very small nuclear charge $Z$) is sketched in Fig.4.

For $n \leq n_{\text{max}} = 2Z + 1$ the ground-state energy has a minimum at $n = n_x = 2Z + \frac{2}{3} - \sqrt{(6Z + 1)/9}$. The position of this minimum can be externally controlled, and it determines the average population of the atom connected through an easily penetrated barrier to a zero chemical potential bosonic reservoir (see Fig.4). We note that the minimum becomes more shallow as $Z$ increases.

In the opposite limit of weak coupling to the reservoir, the particle discreteness is important and the atomic population is quantized. The equilibrium number of bound bosons is determined by an integer $n$ such that the energy $\epsilon_n$ is closest to $\epsilon_{n_x}$. In Fig.4 this is indicated by the bold solid circle. All other integer-valued population states are metastable and shown by gray and open circles. We make a distinction between metastable atoms whose population is smaller (gray circles) or larger (open circles) than the optimal because the latter can lower their energy by ejecting a particle while the former cannot. As a result, well-isolated underpopulated atoms are more stable than their overpopulated counterparts. In equilibrium particle exchange takes place when the degeneracy between the states with $n$ and $n-1$ particles occurs, $\epsilon_n = \epsilon_{n-1}$, i. e. at $n = n_1 = 2Z + \frac{2}{3} - \sqrt{(24Z + 1)/36}$ which is a first-order transition. We note that the inequality $n_x < n_1 < n_{\text{max}}$ always holds.

We also note that in the limit $Z \to \infty$ the difference $n_1 - n_x$ approaches 1/2. This has a simple interpretation: if one is only interested in the population transitions, then for $Z \gg 1$ the energy dependence (6) can be approximated by a parabola centered at $n = n_x$. As a result we arrive at a model similar to (1) with half-integer $n_x$ playing a special role.

In order to understand the implications of these results to the operation of the pipette, we need to look at the ground-state energy calculated relative to the BEC chemical potential and accounting for the overlap with the reservoir potential $E'_n = E_n + [U_{\text{res}}(r_\rho) - \mu]n = E_n + (1 - \gamma)|E_1|n$. Its dimensionless counterpart $\epsilon'_n = E'_n/|E_1| = \epsilon_n + (1 - \gamma)n$ follows from (6)

$$\epsilon'_n = -n\left[\gamma - \frac{n - 1}{2Z} + \frac{(n - 1)^2}{12Z^2}\right]$$  \hspace{1cm} (7)

The differences of the binding properties of the pipette from those of the atom are due to the mismatch of the chemical potential $\mu$ and the trapping potential of the BEC at the pipette location $U_{\text{res}}(r_\rho)$. In the regime of interest when the particle extraction takes place in the presence of the tunneling barrier, $0 < \gamma < 1$, the energy gain of condensation is diminished. The analysis of (7) is similar to that of (6), and leads to similar results.

B. Exactly solvable example

The self-consistent treatment of the previous Section predicts that population transitions are discontinuous. However for the exactly solvable case of two point repulsive bosons in the presence of a delta-function potential well [19] a transition from two to one bound particles is known to be continuous. Below we obtain an exact solution of a one-dimensional many-body problem where all population transitions are continuous. In addition to clarifying the nature of the population transitions this solution provides us with an alternative derivation of the $n \gg 1$ limit of Eqs.(5) and (6). Solving the many-body problem exactly becomes possible due to a choice of the pipette potential $U(x)$ which allows application of the Bethe ansatz methods [20].

The many-body Hamiltonian has the form

$$\hat{H} = \sum_{i=1}^{n} \left[ -\frac{\hbar^2}{2m} \frac{d^2}{dx_i^2} + U(x_i) \right] + g \sum_{i<j} \delta(x_i - x_j)$$  \hspace{1cm} (8)

We select the trapping potential $U(x)$ to have an attractive part and an impenetrable core causing the wave function of a particle immediately next to it to fall off as $\exp(-x/R_1)$ - now the particles are only allowed on the half-line $x > 0$. The potential can be heuristically pictured as a delta-function well located right next to an impenetrable wall. We select the decay length $R_1$ as in Section IIIA: $R_1 = \hbar^2/2mV$, i. e. as if it were due to a delta-function well of strength $2V$. Shortly it will become clear why this choice is reasonable.

An attractive, $g < 0$, version of the many-body problem (8) has been solved by Kardar [21] in the classical context of the line depinning transition in two dimensions in the presence of quenched impurities. In pure system the transfer-matrix technique reduces this problem to the quantum mechanics of a particle in a trapping potential. With disorder present, the replicated problem corresponds to $n$ mutually attracting bosons in the same trapping potential. The interesting physics pertinent to the random problem is revealed in the limit of zero number of replicas, $n \to 0$ [21].

Although Kardar focused on attractive interactions [21], his arguments do not depend to the sign of the interparticle interaction. Some of the results given below can be deduced from those of Ref. [21] by simply reversing the sign of interparticle interaction.

6
For each permutation P of the particles with 0 < x_{p1} < x_{p2} < ... < x_{pn}, the bosonic wave function is written as a product of exponentials:

\[ \Psi \sim \exp \left( -\sum_{i=1}^{n} \kappa_i x_{p_i} \right) \]  
(9)

Selecting the “momenta” \( \kappa_i \) to satisfy

\[ \kappa_i = \frac{1}{R_1} - \frac{mg(i-1)}{\hbar^2} = \frac{1}{R_1} \left( 1 - \frac{i-1}{Z} \right) \]  
(10)

will guarantee the correct derivative discontinuities upon two-particle exchange. Also for the particle closest to the attractive center the wave function has the required \( \exp(-x_{p1}/R_1) \) fall-off. Since the wave function (9) is an eigenstate of the Hamiltonian (8) with no nodes, it describes the ground state of the problem.

Eqs.(9) and (10) imply that the reduced size of the ground state is given by

\[ \rho_n = R_n/R_1 = \left( 1 - \frac{n-1}{Z} \right)^{-1} \]  
(11)

This result shows that as the number of bound particles increases, so does the size of the ground state, eventually diverging at \( n = n_{\text{max}} = Z + 1 \) which is the maximal number of bosons that can be bound. Similar to Eq.(5) screening of the nuclear attraction is the mechanism responsible for the divergence of the localization length (11).

The ground-state energy can be computed as \( E_n = -\sum_{i=1}^{n} \hbar^2 \kappa_i^2/2m \). For the reduced ground-state energy \( \epsilon_n = E_n/|E_1| \) measured in units of the single-particle energy \( |E_1| = \hbar^2/(2mR_1^2) \) we find

\[ \epsilon_n = -n \left( 1 - \frac{n-1}{Z} + \frac{(n-1)(2n-1)}{6Z^2} \right) \]  
(12)

We note that for \( n = n_{\text{max}} = Z + 1 \) the ground-state energy remains negative.

The problem posed by the Hamiltonian (8) can be viewed as a simplified version of that of a one-dimensional Bose atom treated in Section IIIA. Indeed consider two sets of \( n/2 \) point repulsive bosons each constrained to occupy positive and negative half-lines and attracted to a delta-function center of strength \( 2V \) at the origin. Since the sets are noninteracting, this is merely two systems described by the Hamiltonian (8) with \( n \) replaced by \( n/2 \) put “back to back” to each other. It is then natural for the wave function for \( x < 0 \) to be a mirror of the wave function for \( x > 0 \). This explains why it was reasonable to require that the wave function of the particle next to the origin decays as \( \exp(-x/R_1) \) with \( R_1 = \hbar^2/(2mV) \) because now we built a discontinuity in the derivative of the wave function at \( x = 0 \) which is appropriate for a delta-function of strength \( 2V \). The reduced size of the Bose-atom can be deduced from (11) as \( \rho_n = [1 - (n-2)/(2Z)]^{-1} \) where we simply replaced \( n \) by \( n/2 \). This result is very similar to Eq.(5): they have common \( n \gg 1 \) limit and for \( Z \gg 1 \) predict the same maximal number \( 2Z \) of bound bosons. Similarly, the ground-state energy of the one-dimensional Bose-atom can be deduced from (12) as \( \epsilon_n = -n[1 - (n-2)/(2Z) + (n-1)(n-2)/(12Z^2)] \) where we replaced \( n \) by \( n/2 \) and doubled the outcome since there are two sets each containing \( n/2 \) particles. This result is very similar to Eq.(6) and they have common \( n \gg 1 \) limit. It is unclear whether these \( \rho_n \) and \( \epsilon_n \) improve on Eqs.(5) and (6) as the exact solution to the one-dimensional Bose-atom problem should account for arbitrary partitions of \( n \) particles.

![FIG. 5. Dependence of the ground-state energy of a one-dimensional model, Eq.(8), on the number of bound particles \( n \), Eq.(12), for \( Z = 9.6 \). The styling is the same as in Fig.4 and the inset shows the behavior of the energy curve in the vicinity of its minimum.](attachment:fig5.png)

Typical dependence of the ground-state energy (12) on the number of bound bosons \( n \) is shown in Fig.5. In producing Fig.5 we intentionally chose \( Z = 9.6 \) which is twice the value used to draw Fig.4 because according to the argument of the previous paragraph this allows us to make a comparison with the self-consistent treatment. The difference between Figs.4 and 5 is hardly perceivable to the eye and manifests itself in the vicinity of the minimum of the energy function shown in the inset of Fig.5.

For \( n \leq n_{\text{max}} = Z + 1 \) the ground-state energy (12) has a minimum at \( n = n_x = Z + \frac{1}{2} - \sqrt{\frac{3}{2}} \) and a maximum at \( n = Z + \frac{1}{2} + \sqrt{\frac{3}{2}} \) with fixed \( Z \)-independent distance of \( \sqrt{3}/3 < 1 \) between them. The position of the minimum, indicated in Fig.5 by the cross, determines the average number of bound bosons provided the trap is connected through easily penetrable barrier to a zero chemical potential bosonic reservoir. Similar to the case
of the one-dimensional Bose-atom, Section IIIA, as interparticle interactions weaken, i.e. \( Z \) increases, the minimum becomes increasingly more shallow.

In the opposite limit of weak coupling to the reservoir the particle discreteness is important and the number of bound bosons is determined by an integer \( n \) for which the energy \( \epsilon_n \) is closest to \( \epsilon_{n-1} \). The particle exchange takes place when the degeneracy between the states with \( n \) and \( n-1 \) particles occurs, i.e. \( \epsilon_n = \epsilon_{n-1} \). It is straightforward to verify that this condition coincides with the condition of divergence of the localization length (11): \( n = n_1 = n_{\text{max}} = Z + 1 \). Moreover at the transition point the derivative of the ground-state energy with respect to \( Z \) is continuous, \( \partial \epsilon_{n=Z+1}/\partial Z = \partial \epsilon_{n=Z}/\partial Z = -(Z^2 - 1)/(3Z^2) \). These facts imply that the population transitions are continuous.

\[ Z^2 \Delta \epsilon_n \]

![Graph](image_url)

**FIG. 6.** Dependences of \( Z^2 \Delta \epsilon_n \) of the model (8) on the interaction parameter \( Z \) for a series of integer \( n \leq Z + 1 \). The styling is similar to that in Fig.2. Gray lines are continuations of \( Z^2 \Delta \epsilon_n \) on \( Z > n \), while \( Z^2 \Delta \epsilon_n \) does not exist for \( Z < n - 1 \).

The continuity of the population transitions is easier to visualize by looking at the ground-state energy computed relative to its minimum, \( \Delta \epsilon_n = \epsilon_n - \epsilon_{n-1} \). The transitions take place at \( \Delta \epsilon_n = \Delta \epsilon_{n-1} \), and Fig.6 in a fashion similar to Fig.2 shows the \( Z \) dependences of \( Z^2 \Delta \epsilon_n \) for a series of integer \( n \leq Z + 1 \). As the interaction parameter \( Z \) changes adiabatically, the system follows the lowest energy path shown in bold. Although Fig. 6 resembles Fig. 2, there is some difference in their origin: in the model (8) the states of differing \( n \) are not coupled by tunneling. Nonetheless, the switches between neighboring energy branches are smooth, and the lowest energy state (represented this way) is a strictly periodic function with unit period.

In order to understand why for the model (8) the population transitions are continuous, we note that the Hartree-Fock wave function assumes that the ground state is Bose-condensed - every particle is assigned the same function. On the other hand, in the exact wavefunction (9) each particle is described by a different function. As a result the ground-state has a shell-like structure - there are \( n \) localization lengths in the problem. The particles that are farther out act like they are bound by a weaker potential screened by all the particles of inner shells. This implies that the nuclear potential is screened more effectively than self-consistent treatment would predict. The way the system evolves from an \( n \) particle wavefunction to an \( n-1 \) particle wavefunction is by making the wave function of the outermost particle extremely extended in space - only the radius of the largest shell diverges while all other remain finite. Therefore at the point of the population transition the ground-state energy \( E_n \) merges without discontinuity with that of the \( n-1 \) particle wavefunction.

Since the exactly solvable model (8) has basic ingredients of any model of repulsive bosons in the presence of a center, we argue that (8) defines a universality class: the shell structure of the ground-state is generic and screening is the mechanism leading to continuous population transitions.

In trying to apply this reasoning to higher-dimensional cases we note that only in one dimension arranging the repulsive particles in certain order with respect to the attractive center necessarily lowers the energy relative to that of the Hartree-Fock state. In higher dimensions this does not have to be the case as there is more “room” around the nucleus and thus the screening is less effective. As a result the ground state is most likely to be Bose-condensed, and the Hartree-Fock single-shell wave function adequately represents physics.

We note that in the vicinity of the particle reservoir the conditions of degeneracy are modified due to the mismatch of the BEC chemical potential and the trapping potential evaluated at the pipette location. As a result the transitions when the pipette population changes are discontinuous regardless of the nature of the population transitions of Bose-atoms.

**IV. VARIATIONAL TREATMENT: 1 \leq D < 2**

For general dimension \( d > 1 \) we could not find a closed form minimizer of the HFGP energy functional (2), and instead used a variational method. The idea behind selecting a good trial wave function is to have it resemble as much as possible the single-particle wave function.

To prepare the reader for the practically important two- and three-dimensional cases, and to assess the accuracy of the variational treatment in general, first we approximate the trial wave function by a simple exponential, \( \psi(r) \sim e^{-r/R} \), with \( R \) being the variational size of the ground state. Normalizing it and substituting in (2) we arrive at the variational energy
\[ E(R) = n \left[ \frac{\hbar^2}{2mR^2} - \frac{V(2^{d+1} - \lambda)}{2\Gamma(d)R^d} \right] \tag{13} \]

The first term of (13) represents the zero-point energy cost of having \( n \) bosons localized within the range \( R \), while the second term is the potential energy consisting of energy gain of attraction to the nucleus and the energy cost of interparticle repulsion. Since \( R \) is the only scale in the problem, all the \( R \) dependences in (13) can be recovered heuristically from dimensionality arguments: apart from numerical coefficients any function with localization scale \( R \) will lead to (13).

The \( E(R) \) dependence (13) has a minimum provided \( 0 < d < 2 \) and \( \lambda < 2^{d+1} \), i.e., for \( n < n_{\text{max}} = 1 + 2^{d+1}Z \). For \( n = 1 \) minimizing \( E(R) \) reproduces the correct answers for the localization length \( R_L = [\hbar^2\Gamma(d)/2^{d}mdV]^{1/(2-d)} \) and the ground-state energy \( E_1 = E(R_L) = [-\hbar^2(2-d)/(2mdV)]^{2/(2-d)} \) [22]; the consistency condition \( R_L \gg a \) becomes \( \xi \ll 1 \). For general \( n \) minimizing (13) we find the reduced size of the ground-state

\[ \rho_n = R_n/R_1 = \left(1 - \frac{n - 1}{2^{d+1}Z} \right)^{1/(2-d)} \tag{14} \]

Substituting this back into (13) provides us with dimensionless ground-state energy \( \epsilon_n = E_n/|E_1| \)

\[ \epsilon_n = -n \left(1 - \frac{n - 1}{2^{d+1}Z} \right)^{2/(2-d)} \tag{15} \]

In one dimension Eqs. (14) and (15) reduce to the results of Kadomtsev and Kudryavtsev [23] obtained in the context of the ground-state properties of ordinary atoms placed in a supermagnetic field. They are numerically just slightly less accurate than Eqs. (5) and (6) and predict very similar physics. Indeed, for \( d = 1 \) the ground-state energy (15), \( \epsilon_n = -n[1 - (n-1)/(2Z) + (n-1)^2/(16Z^2)] \), is only different from (6) in the numerical coefficient of the last term (1/16 versus 1/12). Moreover apart from the fact that the variational treatment predicts \( n_{\text{max}} = 4Z + 1 \) (instead of \( n_{\text{max}} = 2Z + 1 \)), the divergences of the localization length, Eqs. (5) and (14), near \( n_{\text{max}} \) are identical.

For general \( d \) the result (14) can also be interpreted using the language of a screening whose effect diminishes as the space dimensionality grows. Equivalently we can say that the higher the dimensionality of space, the more particles can be bound. This is physically plausible as higher dimensionality implies more “room” around the nucleus. The numerical accuracy of the variational predictions degrades as the space dimensionality increases; only for \( d = 1 \) does the exponential dependence correctly solve the single-particle problem.

For the marginal two-dimensional case when the potential energy part of (13) has the same \( R^{-2} \) dependence as the kinetic energy, variational predictions are not even qualitatively correct: for \( n = 1 \) (\( \lambda = 0 \)) Eq. (13) predicts a threshold for the appearance of the bound state. This contradicts the known fact [24] that there always is a bound state in a two-dimensional radially symmetric potential well.

For \( d > 2 \) Eq. (13) predicts a minimum as \( R \to 0 \) which cannot be made consistent with the condition \( R \gg a \).

The reason why well-localized single-scale wave functions fail to describe the Bose-atom in \( d \geq 2 \) lies in our assumption that we can take \( a = 0 \). Indeed, for \( d \geq 2 \) the true large-distance behavior of the bound-state single-particle wave function cannot be continued to the origin without introducing divergences [22]. Therefore in order to recover the correct physics we need both to keep the range of the nuclear potential \( a \) finite and to improve the variational function.

V. TWO-DIMENSIONAL CASE

In two dimensions the form of the trial wave function is suggested by the solution of the single-particle problem [24]: we choose \( \psi(r) = A \ln(R/a) \) for \( 0 \leq r < a \), \( \psi(r) = A\ln(R/r) \) for \( a \leq r < R \), and \( \psi(r) = 0 \) for \( r \geq R \) where \( A \) is normalization constant. With this choice the variational energy becomes

\[ E(R) = \frac{2\hbar^2 n}{mR^2} \left[ \ln(R/a) - 2\xi \ln^2(R/a) + 3\xi\lambda \right] \tag{16} \]

Apart from logarithmic factors (which are expected in marginal dimensionality), Eq. (16) has the same structure as Eq. (13) for \( d = 2 \).

For \( n = 1 \) (\( \lambda = 0 \)) we find that the minimum of (16) occurs for \( R_1/a = \exp([1 + 2\xi + \sqrt{1 + 4\xi^2})/4\xi] \). The consistency condition \( R_1 \gg a \) becomes equivalent to the requirement of shallow nuclear well, \( \xi \ll 1 \). In this limit we find \( R_1 \simeq a e^{1/2\xi} \) thus reproducing the known result [24]. The expression for the ground-state energy \( E_1 = -(\hbar^2/mR_1^2)\left(2\xi + \sqrt{1 + 4\xi^2} \right) \) also reduces to the correct answer [24] \( E_1 \simeq -(\hbar^2/ma^2)e^{-1/\xi} \) in the \( \xi \ll 1 \) limit.

As the trapping potential gets deeper, i.e., \( \xi \) increases, the size of the single-particle bound state \( R_1 \) monotonically decreases. In the limit of very deep well we find \( R_1/a = e[1 + 1/(4\xi)] > 1 \) which meets expectation as for \( \xi \gg 1 \) the size of the ground state must be of the order of the well size. In a very deep well the ground-state energy is expected to be greater than the negative of the well depth (the classical answer) by the zero-point energy of a particle localized inside the well, of order \( \hbar^2/ma^2 \). Apart from a numerical factor of order unity, the expression for the single-particle energy \( E_1 = -2U_0/e^2 + \hbar^2/ma^2 \) (specified to the case of a rectangular well of depth \( U_0 \)) conforms with the physics. Since this is a variational result, the exact ground-state energy is lower which is reflected in the inequality \( 2/e^2 < 1 \). We note that the
localization length $R_1$ is at least $e$ times larger than $a$, and binding more repelling particles can only increase the size of the ground-state. We conclude that the trial wave function which led to (16) correctly captures the physics of the two-dimensional Bose-atom for any value of $\xi$. Below we restrict ourselves to the case of shallow well, $\xi \ll 1$, when it supports only one single-particle state.

For general $n$ minimization of (16) provides us with the reduced size of the ground-state

$$
\rho_n = R_n/R_1 = \exp \left[ \frac{\sqrt{1 + 24\zeta^2 \lambda} - 1}{4\xi} \right],
$$

where the $n$-dependence enters through $\lambda = (n - 1)/Z$, Eq.(3). In contrast to the lower-dimensional cases, $1 \leq d < 2$, for finite values of $n$, $Z$, and $\xi \ll 1$, the two-dimensional localization length (17) never diverges. This implies that in two dimensions even a shallow well, $\xi \ll 1$, can bind an arbitrary number of bosons. We note that the size of the ground-state is very sensitive to the number of bound particles. For example, for $24\zeta^2(n - 1)/Z \ll 1$, Eq.(17) simplifies to $\rho_n \simeq \exp[3\xi(n - 1)/Z]$. In this limit the dependence on the nuclear strength effectively drops out as $v = \xi/Z$ parameterizes the interparticle repulsion alone [see Eq.(3)]. In the opposite limit $24\zeta^2(n - 1)/Z \gg 1$, Eq.(17) reduces to $\rho_n \simeq \exp[3(n - 1)/(2Z)]$, and the dependence on the nuclear strength $\xi$ drops out.

The corresponding reduced ground-state energy, $\epsilon_n = E_n/E_1$, is given by

$$
\epsilon_n = -n\sqrt{1 + 24\zeta^2 \lambda} \exp \left[ \frac{1 - \sqrt{1 + 24\zeta^2 \lambda}}{2\xi} \right]
$$

For $n \geq 1$ and not very strong interparticle repulsion, the function $\epsilon_n$ first decreases, while for large $n$ it asymptotically approaches zero, thus implying that it has a minimum. The minimum, indicated in Fig.7 by the cross, provides us with the equilibrium population of an atom strongly coupled to the zero chemical potential bosonic reservoir. The position of this minimum can be found in closed form in the limit $24\zeta^2(n - 1)/Z \ll 1$. Then Eq.(18) simplifies to $\epsilon_n \simeq -n \exp[-6\xi(n - 1)/Z]$ which has a minimum at $n = n_x = Z/6\zeta$. Even though for fixed $Z$ and $\xi \ll 1$ we find $n_x \gg 1$, it is straightforward to verify that the condition $24\zeta^2(n - 1)/Z \ll 1$ still holds. For the argument to work, $n_x$ must exceed unity thus implying $Z > 6\zeta$; this tells us how weak the inter-particle interaction should be for the minimum to occur at $n > 1$.

When the coupling to the reservoir is weak, the population of the atom is quantized. In Fig.7 the corresponding ground state is shown by a solid circle. The population transitions take place whenever the degeneracy condition is met, i. e. $\epsilon_n = \epsilon_{n-1}$: in the $24\zeta^2(n - 1)/Z \ll 1$ limit this is solved by $n_1 = (1 - (e^{-6\xi/Z})^{-1}) n_x$. Similar to what we found in the one-dimensional case, in the $n_x \gg 1$ limit the difference $n_1 - n_x$ approaches 1/2. This implies that in order to understand the population transitions we can approximate the true $\epsilon_n$ dependence by a parabola centered at $n = n_x$ thus arriving at a model similar to (1).

For fixed $\xi \ll 1$ as the nuclear charge $Z$ gets smaller, there will be a value for which $n_1 = 2$, i. e. the transition from the doubly to singly occupied atom happens. This takes place at $6\xi/Z = \ln 2$; for $Z < 6\xi/\ln 2$ the interparticle repulsions are too strong, and only single-bosonic atom is stable.

If a two-dimensional shallow well is used as a pipette to extract particles out of the BEC, then the relevant quantity to look at is the reduced ground-state energy, $\epsilon_n = \epsilon_n + (1 - \gamma)n$. For not very strong interparticle repulsions this function has a minimum whose position can be experimentally tuned to extract particles out of the BEC.

VI. THREE-DIMENSIONAL CASE

In three dimensions the form of the trial wave function is also suggested by the solution of the single-particle problem [9]: we select $\psi(r) = (A/a)e^{-a/r}$ for $0 \leq r < a$ and $\psi(r) = (A/r)e^{-r/R}$ for $r \geq a$ where $A$ is normalization constant. Let us first look at the single-particle case $n = 1$. Substituting this function into the HFGP functional (2) we arrive at

$$
E(R) = \frac{\hbar^2}{ma^2} \left[ \frac{8\xi - 1}{6(R/a)^2} - \frac{2\xi - 1}{(R/a)^2} \right]
$$

For $n \geq 1$ and not very strong interparticle repulsion, the function $\epsilon_n$ first decreases, while for large $n$ it asymptotically approaches zero, thus implying that it has a minimum. The minimum, indicated in Fig.7 by the cross, provides us with the equilibrium population of an atom strongly coupled to the zero chemical potential bosonic reservoir. The position of this minimum can be found in closed form in the limit $24\zeta^2(n - 1)/Z \ll 1$. Then Eq.(18) simplifies to $\epsilon_n \simeq -n \exp[-6\xi(n - 1)/Z]$ which has a minimum at $n = n_x = Z/6\zeta$. Even though for fixed $Z$ and $\xi \ll 1$ we find $n_x \gg 1$, it is straightforward to verify that the condition $24\zeta^2(n - 1)/Z \ll 1$ still holds. For the argument to work, $n_x$ must exceed unity thus implying $Z > 6\zeta$; this tells us how weak the inter-particle interaction should be for the minimum to occur at $n > 1$.

When the coupling to the reservoir is weak, the population of the atom is quantized. In Fig.7 the corresponding ground state is shown by a solid circle. The population transitions take place whenever the degeneracy condition is met, i. e. $\epsilon_n = \epsilon_{n-1}$: in the $24\zeta^2(n - 1)/Z \ll 1$ limit this is solved by $n_1 = (1 - (e^{-6\xi/Z})^{-1}) n_x$. Similar to what we found in the one-dimensional case, in the $n_x \gg 1$ limit the difference $n_1 - n_x$ approaches 1/2. This implies that in order to understand the population transitions we can approximate the true $\epsilon_n$ dependence by a parabola centered at $n = n_x$ thus arriving at a model similar to (1).

For fixed $\xi \ll 1$ as the nuclear charge $Z$ gets smaller, there will be a value for which $n_1 = 2$, i. e. the transition from the doubly to singly occupied atom happens. This takes place at $6\xi/Z = \ln 2$; for $Z < 6\xi/\ln 2$ the interparticle repulsions are too strong, and only single-bosonic atom is stable.

If a two-dimensional shallow well is used as a pipette to extract particles out of the BEC, then the relevant quantity to look at is the reduced ground-state energy, $\epsilon_n = \epsilon_n + (1 - \gamma)n$. For not very strong interparticle repulsions this function has a minimum whose position can be experimentally tuned to extract particles out of the BEC.
This function has a minimum provided \( \xi > \xi_c = 1/2 \) which corresponds to the well-known fact that in a three-dimensional well the first bound state appears only for a sufficiently deep well [9]. As an assessment of the accuracy of the trial function, we note that for the exactly solvable case of rectangular well of radius \( a \) and depth \( U_0 \) [9] the corresponding \( \xi_c = \pi^2/24 \approx 0.4112 \) is fairly close to the variational threshold of 1/2.

Minimizing Eq.(19) for \( \xi > \xi_c = 1/2 \) we arrive at the expressions for the localization length and the ground-state energy, respectively

\[
\frac{R_1}{a} = \frac{8 - \xi^{-1}}{3(2 - \xi^{-1})}, \quad E_1 = -\frac{3\hbar^2\xi (2 - \xi^{-1})^2}{2ma^2 - 8 - \xi^{-1}} \tag{20}
\]

which define a system of units used below.

Requiring \( R_1 \gg a \) puts us in the vicinity of the critical value \( \xi = \xi_c \). Introducing the reduced distance from the threshold, \( \Delta = 1 - 1/(2\xi) \), and specifying to the \( \Delta \ll 1 \) limit we find \( R_1 = a/\Delta \) and \( E_1 = -\hbar^2 \Delta^2/2ma^2 \) both correctly reproducing the critical behavior near the threshold [9].

For arbitrary \( n \) the variational energy is given by

\[
E(R) = \frac{\hbar^2 n^2\xi}{ma^2} \left[ \frac{8 - \xi^{-1} + 16\lambda}{6(R/a)^2} \right. \\
- \frac{2 - \xi^{-1}}{(R/a)^2} + \frac{8\lambda \ln(R/a)}{(R/a)^2} \left. \right] \tag{21}
\]

where some of the \( n \)-dependence comes through \( \lambda = (n - 1)/Z \), Eq.(3). Even though the last term of (21) is of higher order in \( R/a \gg 1 \), keeping it helps clarify the physics of weak interparticle repulsions, as will become clear shortly.

The variational energy has a minimum whose existence is not affected by the last term of (21). Let us temporarily drop it and look at the outcome. Minimizing (21) and using \( R_1 \) from (20) as a unit of length we find

\[
\rho_n = R_n/R_1 = 1 + Z_c(n - 1)/Z \tag{22}
\]

where we introduced a new parameter

\[
Z_c = \frac{16\xi}{8\xi - 1} \tag{23}
\]

whose significance will become clear shortly.

Similar to the two-dimensional case, for finite values of the parameters the size of the ground-state never diverges, thus implying that an arbitrary number of bosons can be bound. The localization length (22) does not grow with \( n \) as fast as its two-dimensional counterpart (17) indicating a stronger ability to bind particles than in two dimensions. In units of \( E_1 \) [Eq.(20)], the reduced ground-state energy \( \epsilon_n = E_n/|E_1| \) is given by

\[
\epsilon_n = -\frac{n}{1 + Z_c(n - 1)/Z} \tag{24}
\]

The interesting feature of this result is that the ground-state energy has a finite negative limit of \(-Z/Z_c\) as \( n \to \infty \). For \( n \geq 1 \) and \( Z < Z_c \) the ground-state energy is a monotonically increasing function of \( n \) approaching the asymptotic limit of \(-Z/Z_c\) from below. This implies that only the single-bosonic atom is stable when the nuclear well is weakly coupled to the zero chemical potential bosonic reservoir: the interparticle repulsions are too strong and all the states with \( n > 1 \) are metastable. As the reduced nuclear strength \( \xi \) increases from its threshold value of 1/2 to infinity, the critical parameter \( Z_c \) varies insignificantly: it monotonically decreases from 8/3 to 2.

On the other hand, Eq.(24) predicts that the ground-state energy is a monotonically decreasing function of \( n \) for \( Z > Z_c \), which approaches its asymptotic limit of \(-Z/Z_c\) from above. This would imply that the lowest energy state would have an infinite number of bound bosons. We arrived at this conclusion however neglecting the last negative term of (21): now we demonstrate that it actually introduces a minimum in \( \epsilon_n \) for \( Z > Z_c \).

Indeed substituting (22) in (21) we find a more accurate expression for the reduced ground-state energy

\[
\epsilon_n = -\frac{n}{\rho_n} - \frac{9Z_c^2\Delta(n - 1) \ln(8\rho_n/3Z_c\Delta)}{8Z\rho_n^2} \tag{25}
\]

where \( \Delta = 1 - 1/(2\xi) \) is the dimensionless distance from the threshold. The second term of (25) does not affect our earlier conclusion that for \( Z < Z_c \) the ground-state energy is a monotonically increasing function of \( n \) approaching \(-Z/Z_c\) from below.

For \( Z > Z_c \), Eq.(25) has a minimum whose position in the \( n \gg 1 \) limit is given by

\[
n = n_x \simeq v^{-1}(2\xi - 1) \exp \left( \frac{8\xi - 116\nu}{18\xi - 9} \right) \tag{26}
\]

where \( v = \xi/Z \) parametrizes interparticle repulsions, and we used \( \Delta = 1 - 1/(2\xi) \) and the definition (23). The ground-state energy evaluated at \( n_x \) is lower than \(-Z/Z_c\) and for \( n > n_x \) it monotonically increases approaching \(-Z/Z_c\) from below. The minimum \( n = n_x \) provides us with the average number of bound bosons provided the nuclear well is strongly coupled to the zero chemical potential bosonic reservoir.

In Fig.8 we sketch the dependence of the ground-state energy on the number of bound bosons for the cases of strong \((Z < Z_c, \text{Fig.}8a)\) and weak \((Z > Z_c, \text{Fig.}8b)\) interparticle interaction.

If the strength of the nuclear attraction \( \xi > \xi_c = 1/2 \) is kept constant and the interparticle repulsion goes to zero \((v \to 0)\), or if the interparticle repulsion \( v \) is fixed while the depth of the trapping potential goes to infinity \((\xi \to \infty)\), the number of bound bosons (26) increases to infinity. These conclusions are in accordance with physical expectations.
The effect of enhanced binding due to resonant interaction with an attractive center bears qualitative similarity with the Efimov effect occurring in a system of three bosons [25] where resonant two-body forces trigger a formation of an arbitrarily large number of loosely bound levels in a three-particle system. Specifically, the Efimov states occur due to an effective long-range interparticle attraction induced by the proximity of the two-body resonance. This interaction, decaying as an inverse square of distance, is universal in a sense that it does not depend on the shape of the two-body potential; both at small and large distances it is cut off by the range of two-body forces and the size of the two-particle bound state, respectively.

Formally our effect occurs due to the last negative term of Eq. (21), and the fact that it is proportional to the number of pair interactions \( n(n - 1)/2 \) tempts us to interpret it as originating from an effective long-range attraction of the \( \hbar^2 \alpha \ln(r/a)/mr^3 \) form where we expressed \( g \) in terms of the low-energy \( s \)-wave scattering length \( a_s \) [6,15]: \( g = 4\pi\hbar^2a_s/m \). This interpretation is not entirely implausible as the particle attraction to the resonant nuclear well, occuring at distances significantly exceeding the scattering length \( a_s \), also pulls the bosons towards each other thus inducing an effective long-range attraction. Similar to the Efimov effect, the \( \hbar^2a_s \ln(r/a)/mr^3 \) interaction is universal: it has its origin in the large-distance behavior of the wave function and it is characterized by a very weak (logarithmic) dependence on the size of the nuclear well \( a \). At large distances the induced interaction is cut off by the size of the many-body state while at small distances it vanishes at \( r \approx a \).

We note that our effective interaction is different from the Efimov \( 1/r^2 \) form. The difference will not appear that surprising if we look at two identical repulsive bosons attracted to a much heavier particle mimicking the nucleous of our problem. It turns out that if the mass of the heavy particle approaches infinity, the effective \( 1/r^2 \) attraction and thus the Efimov effect disappear [25]. This makes us think that our effect of enhanced binding to the resonant nucleous is not a manifestation of the original Efimov effect in a many-body system. It is curious that for the three-boson problem the correction to the \( 1/r^2 \) Efimov attraction is also universal and has the \( \hbar^2a_s/mr^3 \) form [26]. Apart from logarithmic factor this is what we find.

The arguments of the last two paragraphs are no more than suggestive - the translationally invariant three-body problem is somewhat different from that of the \( n \)-boson atom where the translational symmetry is broken, and complementary work is necessary to better understand the physics of our effect.
This effectively cuts off the resonant binding singularity as long as $\xi > 0$; $v = 0.00275$ was used to construct the graph.

In Fig. 9 we sketch the dependence of the number of bound bosons on the dimensionless nuclear strength $\xi$, given by Eq. (26), for the case of sufficiently weak ($Z > Z_c$) interparticle repulsion. The enhanced resonant binding corresponds to unlimited growth of $n_x$ as $\xi$ approaches $\xi_c = 1/2$ from above.

In the limit of weak coupling to the zero chemical potential bosonic reservoir, the population of the atom is quantized and first-order population transitions take place whenever the condition of degeneracy holds ($\epsilon_n = \epsilon_{n-1}$). In the limit $n_x \gg 1$ when the $\epsilon_n$ dependence can be approximated by a parabola centered at $n = n_x$, the transitions happen at half-integer $n_x$.

When a three-dimensional trapping potential is used as a pipette to extract particles out of the BEC, the single-particle ground state remains unoccupied as long as it is higher in energy than the BEC chemical potential. This effectively cuts off the resonant binding singularity at $\xi = \xi_c$. As a result the $\xi$ dependence of the equilibrium pipette population may or may not have a segment where the particle number decreases as the pipette potential gets deeper. Below we ignore this possibility because the resonant region is rather narrow, due to the exponential dependence in (26). The reduced ground-state energy, $\epsilon'_n = \epsilon_n + (1 - \gamma)n$, is given by:

$$\epsilon'_n = \frac{n}{1 + Z_c(n-1)/Z} + (1 - \gamma)n$$

(27)

Since adding the $(1 - \gamma)n$ term to (24) inevitably introduces a minimum in $\epsilon'_n$, we can ignore the last term of (25). It amounts to the assumption that the minimum of (25) lies at a substantially larger value of $n$ than that of (27) which can be always accomplished.

The ground-state energy (27) has a minimum at

$$n = n_x = 1 + (Z/Z_c) \left[ \sqrt{1 - Z_c/Z} / \left(1 - \gamma\right) - 1 \right]$$

(28)

whose position can be experimentally tuned to extract particles out of the BEC. In the vicinity of $n_x$ the energy (27) can be represented as $\epsilon'_n = \epsilon'_n + Z_c(1 - \gamma)^{3/2}(n - n_x)^2 / Z \sqrt{1 - Z_c/Z}$. In the original units this introduces an energy scale

$$E_c = |E_1| \frac{Z_c(1 - \gamma)^{3/2}}{Z \sqrt{1 - Z_c/Z}}$$

(29)

Since the divergence of $E_c$ at $Z = Z_c$ corresponds to $n_x = 0$, it does not reflect any physical singularity. The energy scale (29) parallels the charging energy of electronic systems [3] and should exceed the temperature $T$ for the single-particle effects to be observed. In the limit of weak interactions ($Z \gg Z_c$) when $E_c \sim Z_c/Z$ and $n_x$ the condition $E_c > T$ inevitably breaks down - finite condensate temperature will make it impossible to control precisely a very large number of particles. On the other hand as $Z$ approaches $Z = Z_c$ from above, the inequality $E_c > T$ can be satisfied, implying that a precise few-particle manipulation is realizable in practice despite the finite condensate temperature.

VII. Rounding of the Boson Staircase by Tuning

In the discussion above we ignored particle discreteness in the regime of strong tunneling, while for weak tunneling the quantization of the atom (pipette) population played an important role. Here we develop a quantitative theory of weak finite tunneling which provides a link between the two limiting cases.

In the preceding discussion we assumed that the total number of particles in the system and the number in the pipette are both good quantum numbers. The energy spectrum for the system was derived from the spectra for the condensate reservoir and the pipette potential, treated as separate problems. Tunneling couples states of the system having a fixed total number of particles and differing numbers in the pipette. This is a small effect that can be treated perturbatively when the states of different $n$ have significantly different energies, but becomes important in case of degeneracy. The discussion of the separated systems shows that we need consider only the case of a simple degeneracy of states with $n - 1$ and $n$ particles ($E'_n = E'_n$), and then all other population states can be ignored. Let these two states be coupled by tunneling with a matrix element $M$. The Hamiltonian of the system reduces to a $2 \times 2$ matrix $H_{ik}$ where the diagonal entries $H_{11} = E'_n - H_{22} = E'_n$ are the previously computed ground-state energies and the off-diagonal element $H_{12} = H_{21} = M$ is estimated below. The matrix
elements all depend on the external parameter $n_x$ and as it changes slowly from the region where $E'_{n-1} < E'_n$, to $E'_{n-1} > E'_n$, the $n-1$ particle state adiabatically evolves into an $n$ particle state.

The Hamiltonian $H_{ik}$ has eigenvalues
\[ E' = (E'_{n-1} + E'_n)/2 \pm \sqrt{(E'_{n-1} - E'_n)^2/4 + M^2} \] (30)
which show that finite tunneling replaces the crossing of the energy levels by the avoided crossing. At the former crossing point, $E'_{n-1} = E'_n$, there is an energy gap of magnitude $2M$ between the ground state (the lower root) and the first excited state (the upper root). Fig. 2 of the Introduction is the illustration of Eq.(30) where we chose $E_n \sim (n-n_x)^2$ and then periodically continued the graph.

The average population of the pipette can be found as $< n > = (n-1)a_1^2 + na_2^2$ where $a_1$ and $a_2$ are the probability amplitudes to be in the $n-1$ and $n$ particle states, respectively. The amplitudes $a_{1,2}$ are given by the components of the normalized eigenvector corresponding to the ground-state eigenvalue in (30). The result of this calculation is
\[ < n > = \frac{n-1 + n(\delta + \sqrt{\delta^2 + 1})^2}{2(\delta^2 + \delta \sqrt{\delta^2 + 1} + 1)} \] (31)
where $\delta = (E'_{n-1} - E'_n)/2M$ measures distance between unperturbed energy curves in units of the energy gap $2M$. The $\delta$-dependence of the equilibrium pipette population $< n >$ (31) has a form of a rounded step which interpolates between $n-1$ and $n$ as $\delta \to \infty$, respectively, and has inversion symmetry about the step center $(0; n-1/2)$. This implies that for small $M$ except near the crossing $E'_{n-1} = E'_n$, the average population of the pipette is either nearly $n-1$ ($E'_{n-1} < E'_n$) or nearly $n$ ($E'_{n-1} > E'_n$), i.e. there is very little mixing of the two states. At the crossing $E'_{n-1} = E'_n$, the average population is always half-integer $n-1/2$ as expected. As the tunnelling matrix element $M$ gets larger, the mixing between the two states affects a larger parameter region around the crossing $E'_{n-1} = E'_n$ which results in progressively rounded staircases.

Using the two-state approximation to trace a range of $n_x$ (as is done in Fig. 2) will lead to small anomalies (quadratic in $M$) at the points where we change point of view on which pair of states are being mixed; this minor defect could be removed by enlarging the matrix $H_{ik}$ to include more values of $n$. However, the two-state approximation is entirely adequate in describing the physically interesting region of the vicinity of the level crossing $E'_{n-1} = E'_n$ where we find $< n > = -(n-1/2) = (E'_{n-1} - E'_n)/4M$. In previous Sections it was demonstrated that if one is only concerned with the population transitions, then for not very strong interparticle repulsion the energy curves can be approximated by $E'_n = const + E_c(n-n_x)^2/2$. With this choice we find $< n > = -(n-1/2) = (E_c/4M)[n_x - (n-1/2)]$, i.e. the slope $d < n > /dn_x$ of the $< n >$ on $n_x$ dependence at the degeneracy point $n_x = n-1/2$ is $E_c/4M$.

The system will remain in the lowest energy state if the parameter $n_x$ is varied sufficiently slowly. In an experimental realization, the pipette parameters will be changed on a finite time scale $\tau$. According to the Landau-Zener theory [5], the adiabatic behavior will be observed provided that the energy gap $E_g$ satisfies the condition $E_g \gg \hbar/\tau$. Thus it becomes important to have an estimate of the size of the energy gap. According to Eq.(30) this is determined by the the matrix element of the trap potential that mixes the pipette state with the condensate wavefunction
\[ M = \int \psi_{pipette}(\mathbf{r}) |U(\mathbf{r})| \psi_{BEC}(\mathbf{r}) dV \approx U_0a^3 \psi_{pipette}(\mathbf{r}_p) \psi_{BEC}(\mathbf{r}_p) \] (32)
where the wave functions are normalized at unity. Since the pipette is located in the classically forbidden region, the magnitude of the energy gap will depend strongly on the distance between the pipette and the edge of the condensate droplet. The discussion of Section VI implies that the pipette wave function evaluated at the pipette location can be estimated as $\psi_{pipette}(\mathbf{r}_p) \simeq 1/(a\sqrt{R_n})$ where $R_n$ is the localization length of the pipette ground state.

VIII. EXPERIMENTAL FEASIBILITY

Below we address some issues of practical matter, namely, we explain how to build a pipette suitable for single-particle manipulation, and discuss experimental limitations set by nonzero temperature of the condensate and finite time of tunneling. We restrict ourselves to exploring the feasibility of observing single-particle effects in three-dimensional systems.

A. Trap implementation

Very small atom traps can be created using optical dipole forces [27], in which atoms are attracted to the intense region of a focused laser beam. Tight three-dimensional confinement has been demonstrated at the intersection of two beams, each with a diameter of about $15 \mu m$ [28–30]. However, for a trap of this size, the zero-point energy $\hbar^2/ma^2$ is below $1 nK$ which is lower than achievable BEC temperatures. This implies that the finite temperature of the condensate is sufficient for excitation away from the single-particle ground state. Tighter confinement could be achieved with stronger focusing of the beams. In principle, beam sizes comparable to
the laser wavelength can be achieved, with typical wavelengths around 1 μm. At this value, the zero-point energy is about 20 nK which is within the range of observed condensate temperatures [31,32]. Unfortunately, achieving such tight focussing is very difficult. Alternative approaches can be considered, including holographic techniques [33,34], optical superlattices [35,36], and near-field techniques [37].

We also describe a novel approach which may be particularly well suited for the atom pipette [38]. Rather than using a laser, an optical trap can be created at an interference fringe produced by a broadband lamp which has been optically filtered to remove wavelengths blue of the principal atomic transition. If light from the lamp is split into two beams and the beams directed to be counterpropagating, then each frequency component creates an independent optical standing wave. Ordinarily the effect of the different standing waves averages out, but at precisely the point where both beams have travelled the same distance from the splitter, the standing waves have a uniform phase and an observable interference pattern is produced. For a source with frequency bandwidth ∆ν the interference will extend for a distance c/∆ν, so by using a broad source, a narrow fringe can be obtained. We estimate that trap sizes below 1 μm can be achieved using light from a low-power Hg arc lamp, with trap depths of up to several μK.

We assume, then, that a method such as these is used to generate a trap with a ≈ 1 μm, which is adjacent to a BEC of 23Na atoms. Using the definition of the pseudopotential $g = 4\pi\hbar^2a_s/m[6,15]$ and Eq.(3) the reduced interaction $v = \xi/Z$ parametrizing the interparticle repulsion turns into $v = a_s/a$. For 23Na particles we have $a_s = 2.75nm$ [6] which leads to $v = 0.00275$; this is the value used to produce Fig. 9. In principle the strong interaction regime $v ≃ 1$ can be achieved by applying a magnetic field, which increases the scattering length via bringing the system in the vicinity of the Feschbach resonance [39].

### B. Finiteness of the BEC temperature

One of the main practical obstacles to the realization of the effects we have predicted is the finite BEC temperature which, to be definite, will be set at 4nK. The zero-point energy $\hbar^2/ma^2$ of a localized $^{23}$Na atom is of order $20nK$ which is sufficient to neglect its thermal agitation. The maximal depth of our trap will be assumed to be $1 \mu K$, thus implying $\xi = 50$. The minimal depth is determined from the condition that the magnitude of the single-particle energy $E_1$, Eq.(20) exceeds the BEC temperature; this leads to $\xi ≃ 1$. Therefore by changing the laser power so that the reduced nuclear strength $\xi$ varies between 1 and 50, the finiteness of the BEC temperature can be made negligible as far as single-particle excitations are concerned. On the other hand, by looking at Fig. 9 we conclude that finite BEC temperature prevents us from seeing the enhanced resonant binding which is limited to the vicinity of the single-particle threshold.

If a trap is used as a pipette, then to observe single-particle effects we also need to make sure that the change of the system energy upon adding or removal of one particle is larger than the temperature. This is equivalent to requiring that the “charging” energy (29) exceeds the BEC temperature. For $v = 0.00275$ and $\xi$ varying between 1 and 50, we find that as the laser power goes up, the reduced nuclear charge $Z = \xi/v$ increases from about 350 to 18000. Substituting these values in Eq. (29) and using the expression for the single-particle energy from (20) we find that for $\gamma ≃ 0$ the charging energy is an order of magnitude smaller than the temperature. However the inequality $E_c > T$ can be achieved by magnetically increasing the scattering length.

### C. Adiabaticity

Another aspect that needs to be considered is adiabaticity. Tunnelling of particles from the condensate mixes states with different numbers of particles in the pipette; instead of there being a degeneracy for certain values of $n_x$ there is an energy gap $E_g$.

In order to estimate the condensate wave function at the pipette location we assume that the condensate is confined by a harmonic potential of frequency $\omega$. Outside the BEC the nonlinearity of the corresponding Gross-Pitaevskii equation [6,15] is negligible so that in the classically forbidden region the wave function will have the form of a harmonic oscillator function. The function we should choose corresponds to the $l$th excited state where $l$ is determined by the BEC chemical potential, $\mu = \hbar\omega(l + 1/2)$. In other words it is a product of the Gaussian, $\exp[-r^2/(2a_{ho}^2)]$, and $l$th Hermite function where $a_{ho} = (\hbar/\sqrt{m\omega})^{1/2}$ is the harmonic oscillator length [6]. This behavior is only valid outside the condensate droplet - as we move in, instead of all of the oscillations of the $l$th Hermite polynomial we will get just the slow variation indicated by the Thomas-Fermi result [6]. Outside the condensate, the $l$th Hermite function is dominated by its highest term $r^l$, so that we find $\psi_{BEC} \propto r^l \exp[-r^2/(2a_{ho}^2)]$ where the exponent $l$ depends on the number of the condensate particles $N$. Since this is a solution to the linear differential equation, the overall prefactor is not determined by it, and this requires matching the behavior inside the BEC to that outside. At the classical edge of the BEC droplet the wave function should satisfy $\psi_{BEC}(R_{TF}) \approx 1/R_{TF}^{3/2}$ where $R_{TF} = a_{ho}(15Na_s/a_{ho})^{1/5}$ is the Thomas-Fermi
radius of the condensate [6]. The harmonic-oscillator-type function that agrees with this has the form
\[ \psi_{\text{BEC}}(r) \simeq R_{TF}^{-3/2} \exp[-(r^2 - R_{TF}^2)/(2a_{ho}^2)] \]
(33)
Evaluation of this function at the pipette location \( r_p = R_{TF} + D \) leads to \( \psi_{\text{BEC}}(r_p) \simeq R_{TF}^{-3/2} \exp(-D R_{TF}/a_{ho}^2) \) where we used the fact that the distance between the BEC edge and the pipette \( D \) is significantly smaller than the Thomas-Fermi radius of the condensate, \( R_{TF} \). This result together with \( \psi_{\text{pipette}}(r_p) \simeq 1/(a/R_n) \) provides us with the estimate for the energy gap \( E_g = 2M \), Eq. (32):
\[ E_g \simeq U_0(a/R_{TF})^{3/2}(a/R_n)^{1/2} \exp(-D/L) \]
(34)
where the length scale is \( L = a_{ho}^2/R_{TF} \). It is desirable to have \( L \) fairly short, so that the pipette will rapidly decouple from the condensate droplet as it is moved away; however, it should also be large enough that there is a significant tunnelling rate when the pipette is at closest approach. For \( a_{ho} = 10^{-5}m \) and \( R_{TF} = 10^{-4}m \) we find \( L = 10^{-6}m \). With the choices \( U_0 = 1\mu K \) and \( a = 10^{-6}m \), this requires only that the time scale over which the pipette parameters are changed satisfies \( \tau > 10^{-2}(R_p/a)^{1/2} \exp(D/L) \) seconds. This is not very restrictive unless the number of particles in the pipette is too large or \( D \gg L \).

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