Pairs of Lie-type and large orbits of group actions on filtered modules

(A characteristic-free approach to finite determinacy)

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Abstract
Finite determinacy for mappings has been classically thoroughly studied in numerous scenarios in the real- and complex-analytic category and in the differentiable case. It means that the map-germ is determined, up to a given equivalence relation, by a finite part of its Taylor expansion. The equivalence relation is usually given by a group action and the first step is always to reduce the determinacy question to an “infinitesimal determinacy”, i.e., to the tangent spaces at the orbits of the group action. In this work we formulate a universal, characteristic-free approach to finite determinacy, not necessarily over a field, and for a large class of group actions. We do not restrict to pro-algebraic or Lie groups, rather we introduce the notion of “pairs of (weak) Lie type”, which are groups together with a substitute for the tangent space to the orbit such that the orbit is locally approximated by its tangent space, in a precise sense. This construction may be considered as a kind of replacement of the exponential resp. logarithmic maps. It is of independent interest as it provides a general method to pass from the tangent space to the orbit of a group action in any characteristic. In this generality we establish the “determinacy versus infinitesimal determinacy” criteria, a far reaching generalization of numerous classical and recent results, together with some new applications.

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Contents

1 Introduction .................................................................................. 2417
  1.1 Prologue .................................................................................. 2417
  1.2 Goals and methods .................................................................. 2417
  1.3 Main construction .................................................................. 2418
  1.4 Main results ........................................................................... 2419
  1.5 Remarks ................................................................................ 2420
2 Preparations ................................................................................. 2422
  2.1 Notations and assumptions ..................................................... 2422
     2.1.1 Typical rings .................................................................... 2423
     2.1.2 Implicit function theorem with unit linear part, IFT ........................ 2423
  2.2 Filtered modules and group actions ........................................... 2424
     2.2.1 Induced filtrations on End_k(M) and GL_k(M) ....................... 2424
     2.2.2 The structure of GL_k(1)(M) and GL_k(1)(M) ....................... 2424
     2.2.3 Actions involving ring automorphisms ............................... 2425
  2.3 The condition z + w ∈ Gz as implicit function equation .......... 2427
  2.4 The relevant approximation theorems ..................................... 2428
     2.4.1 The passage from an order-by-order solution to a formal solution 2429
     2.4.2 The passage from a formal solution to an ordinary solution for polynomial or analytic equations ........................................... 2429
     2.4.3 The C∞-case ..................................................................... 2430
     2.4.4 Transition from order-by-order determinacy to G-determinacy 2430
3 Pairs of Lie type .......................................................................... 2432
  3.1 Definitions and basic examples ............................................... 2432
     3.2 The pair (Der_k(1)(R), Aut_k(1)(R)) ....................................... 2435
        3.2.1 The case k ⊇ Q .............................................................. 2435
        3.2.2 The pair (Der_k(1)(R), Aut_k(1)(R)) for the case of arbitrary k 2439
        3.2.3 The subgroup Aut_k,a(R) ⊂ Aut_k(R) .............................. 2441
  3.3 Constructing new Lie pairs from old ones .................................. 2442
     3.3.1 Diagonal action ................................................................ 2442
     3.3.2 A group generated by two groups ..................................... 2443
     3.3.3 The pair (T_{G(1)}, M_1) + T_{H(1)}, M_1, G(1), H(1)) in the case k ⊇ Q 2444
     3.3.4 The case of direct product, G × H ..................................... 2446
4 The general criteria of determinacy ..................................... 2447
  4.1 \overset{\circ}{\varphi}(G(1), M_1) vs \overset{\circ}{G}(1)z for pairs of pointwise Lie type 2447
  4.2 \overset{\circ}{\varphi}(G(1), M_1) vs \overset{\circ}{G}(1)z for pairs of pointwise weak Lie type 2447
  4.3 Sharpness of results ............................................................... 2449
  4.4 Finite determinacy in terms of infinitesimal stability ............... 2449
  4.5 The passage from Gz to Gz in the C∞-case .................................... 2450
5 Applications and examples ....................................................... 2452
  5.1 Right determinacy of germs of functions .................................. 2453
  5.2 Right (in)determinacy of germs of maps .................................. 2455
  5.3 Contact determinacy of germs of maps .................................... 2456
  5.4 Determinacy for maps relative to a germ ................................... 2458
  5.5 Relative determinacy for non-isolated singularities of function germs 2458
  5.6 Relative algebraization ............................................................ 2459
  5.7 Finite determinacy of matrices ................................................ 2460
  5.8 Determinacy of families .......................................................... 2460
References ................................................................................... 2461
1 Introduction

1.1 Prologue

Let $f$ be the germ at the origin of a real- or complex-analytic function or a $C^\infty$-function of several variables $x = (x_1, \ldots, x_p)$. Famous results of [43–48, 62] and many others on finite determinacy of function-germs bound the order of determinacy in terms of the jacobian ideal of $f$:

- if $m^2 \cdot Jac(f) \supseteq m^{N+1}$ then $f$ is $N$-right-determined,
- if $m^2 \cdot Jac(f) + m(f) \supseteq m^{N+1}$ then $f$ is $N$-contact-determined.

Here $m = \langle x_1, \ldots, x_p \rangle$ is the maximal ideal and $Jac(f)$ is the ideal generated by the partials of $f$. A function $f$ is $N$-right (resp. $N$-contact) determined if every $g$ whose Taylor expansion up to order $N$ coincides with that of $f$ lies in the same $R$- (resp. $K$-) orbit as $f$. Here $R$ (resp. $K$) is the right group (resp. contact group) acting on the ring of germs by analytic or $C^\infty$-coordinate change (resp. additionally by multiplication with a unit).

These results have been generalized to numerous group actions and rings in the following way. Let $M$ be a space of maps, usually a filtered module over a ring, together with a fixed action $G \curvearrowright M$ of a nice subgroup $G$ of the contact group $K$. The classical statements compare the tangent space $T_{(Gf, f)}$ to the group orbit $Gf$ at $f$ with the filtration of $M$ given by the subspaces $m^i \cdot M$. Let $f \in M$ be some element and consider the statement:

(i) Suppose the tangent space to the orbit of $f$ satisfies: $m \cdot T_{(Gf, f)} \supseteq m^{N+1} \cdot M$.

(ii) Then the orbit of $f$ is large in the sense: $Gf \supseteq \{f\} + m^{d_N+1} \cdot M$,

where $d_N$ is some integer depending on $N$. Whenever the statement (1)ii. holds one says that $f$ is $d_N$-determined with respect to the $G$-action.

A statement like (1) can be rephrased in saying “a large tangent space implies a large orbit”. To prove such a statement basically two different methods have been used. Primarily the integration of vector fields and the use of the exponential map in characteristic zero with the space of maps $M$ involving formal or analytic power series or germs of $C^\infty$-maps and with $G$ an algebraic group or a Lie group (after reduction to a finite dimensional parameter space). Secondly, power series methods with $M$ involving formal power series over a field of positive characteristic. However, in different scenarios for different kinds of $M$ and groups, these methods had always to be adapted and modified. One of the aims of our paper is to give a unified approach.

1.2 Goals and methods

The goals of our current work are three-fold:

(A) To extend the whole theory to arbitrary filtered modules $M$ over a base ring $k$ of any characteristic, not necessarily a field.

(B) To broaden the class of admissible group actions in a characteristic-free way, by combining the characteristic zero approach and the use of the exponential map with the power series approach in positive characteristic.

(C) To show how the general theory may be applied, not only to recover most of the previously known results, but also to obtain some new ones.

To deduce the inclusion as in (1)(ii) from the assumption (1)(i) we study the orbit $Gz$ of an element $z \in M$. For a given “higher order” element $w \in M$ we want to prove $z + w \in Gz$,
i.e., to solve the equation $z + w = gz$, for the unknown $g \in G$. This is done in two steps, as follows.

**Step 1.** First one establishes an “order-by-order” solution, i.e., a sequence $\{g_n\}$ of elements of $G$ satisfying $g_n z \rightarrow z + w$. The convergence is taken in the filtration topology with the limit being an element in the closure of the orbit, $z + w \in \overline{Gz}$. This step involves the main new construction.

**Step 2.** To pass from an order-by-order solution $\{g_n\}$ to an ordinary solution $g \in G$, we use various approximation results. For example, if all the equations involve power series we invoke first the Theorem of Popescu to ensure a formal solution over the completion (see Theorem 2.4) and then we use Artin-type approximations to ensure an ordinary solution. For $C^\infty$-equations we use an approximation Theorem of Tougeron-type.

### 1.3 Main construction

Fix a base ring $k$ and a filtered $k$-module $M = M_0 \supset M_1 \supset M_2 \cdots$ (usually not finitely generated over $k$) with the filtration topology, i.e., $\{M_i\}$ is a fundamental system of neighbourhoods of $0 \in M$ (the $M_i$ are both open and closed). Denote the group of $k$-linear automorphisms of $M$ by $GL_k(M)$. The filtration of $M$ induces a natural descending filtration of $GL_k(M)$ by the normal subgroups $GL_k^{(i)}(M)$, consisting of elements $g$ that preserve the filtration and such that $g$ and $g^{-1}$ are of the form $I + \phi$ with $\phi$ an element of

$$
\text{End}_k^{(i)}(M) := \{ \phi \in \text{End}_k(M) \mid \phi(M_j) \subseteq M_{j+i}, \forall j \geq 0 \},
$$

with $\text{End}_k^{(0)}(M)$ the endomorphisms of $M$ that respect the filtration.

With the filtration topology we get as topological closure

- for a submodule $\Lambda \subseteq M$, $\overline{\Lambda} = \bigcap_{i \geq 1} (\Lambda + M_i)$;
- for a subgroup $G \subseteq GL_k(M)$, $\overline{G} = \bigcap_{i \geq 1} \left(G \cdot GL_k^{(i)}(M)\right)$;
- for a subgroup $G \subseteq GL_k(M)$ and an element $z \in M$, $\overline{Gz} = \bigcap_{i \geq 1} (Gz + M_i)$;
- for a submodule $T \subseteq \text{End}_k(M)$ and $z \in M$, $\overline{T(z)} = \bigcap_{i \geq 1} (T(z) + M_i)$.

Since the filtration topology is first-countable, the closure $\overline{X}$ of a subset $X \subseteq M$ consist of the points $x \in M$ for which there exists a sequence $x_n \in X$ converging to $x$. The same holds for subsets of $GL_k(M)$. If $k$ is Noetherian and $M$ finitely generated (over $k$) then any submodule of $M$ is already closed.

Any subgroup $G$ of $GL_k(M)$ carries the induced filtration $G^{(i)} := G \cap GL_k^{(i)}(M)$. A special role here plays the following subgroup of $GL_k^{(1)}(M)$:

$$
G^{(1)} := G \cap GL_k^{(1)}(M).
$$

We call $G^{(1)}$ the (topologically) unipotent part of $G$. Note that if $\bigcap M_j = 0$ (which will be the case in most of our applications) then $g \in G^{(1)}$ satisfies $\lim_{i \to \infty} (g - I)^i = 0$, but we use the term “topologically unipotent” also if $\bigcap M_j \neq 0$ (e.g. in the $C^\infty$-case $\bigcap m^i$ contains the flat functions).

Since any element of $G^{(1)}$ is of the form $I + \phi$ with $\phi \in \text{End}_k^{(1)}(M)$, $G^{(1)}$ induces the identity on the “linear” part $M_1/M_2$. 

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The key notion, which is introduced in this paper, is that of a “pair of (weak) Lie-type” for a subgroup \( G \subset GL_k(M) \). It is a pair
\[
(T_{(G^{(1)}, M)}, G^{(1)}),
\]
where \( T_{(G^{(1)}, M)} \) is a certain (not unique) submodule of \( \text{End}_k(M) \) that will be a substitute of the tangent space to the orbit of the action of \( G^{(1)} \) on \( M \). The pair \((T_{(G^{(1)}, M)}, G^{(1)})\) is called of (weak) Lie type if there exists a (weak) substitution of the classical exponential and logarithmic maps \( T_{(G^{(1)}, M)} \xrightarrow{\sim} G^{(1)} \). The choice of such substitutions is part of the data. For a precise definition of (weak) Lie type and the even more general notions of pointwise (weak) Lie type we refer to Sect. 3.

If \( k \) contains the subring \( \mathbb{Q} \) (e.g., if \( k \) is a field of characteristic zero) then many groups admit the standard exponential and logarithmic maps and they are trivially of Lie-type. In positive characteristic however, the standard exponential map cannot be defined, but nevertheless many groups are of weak Lie type. For example, for \( R = k[[x]] \), the ring of power series over an arbitrary field \( k \) in finitely many variables \( x \), the group of \( k \)-algebra automorphisms \( \text{Aut}_k(R) \subset GL_k(M) \), acting on \( M = R^n \) component-wise by coordinate change, gives a pair of weak Lie-type (cf. Example 3.21). We mention that the group of \( R \)-module automorphisms, \( GL_R(M) \subset GL_k(M) \), gives a pair of Lie type for any ring \( k \), in any characteristic (see Example 3.4).

### 1.4 Main results

Let \( k \) be a ring, \( M \) a filtered \( k \)-module and \( G \subset GL_k(M) \) a subgroup with induced filtration. We set
\[
T_{(G^{(i)}, M)} := T_{(G^{(1)}, M)} \cap \text{End}_k^{(i)}(M),
\]
and prove the following general criterion for finite determinacy (under the slightly weaker assumptions of “pointwise Lie type”, see Theorems 4.1 and 4.3). Fix some \( z \in M \) and \( i, N \geq 0 \).

(i) If \((T_{(G^{(1)}, M)}, G^{(1)})\) is a pair of Lie type and \( M_{N+1} \subseteq T_{(G^{(i+1)}, M)}(z) \) then \( \{z\} + M_{N+1} \subseteq G^{(i+1)}z \).

(ii) If \((T_{(G^{(1)}, M)}, G^{(1)})\) is a pair of weak Lie type and \( M_{N+k} \subseteq T_{(G^{(i+k)}, M)}(z) \) for any \( k > 0 \), then \( \{z\} + M_{N+k} \subseteq G^{(i+k)}z \) for any \( k > \max(0, N - 2i - \text{ord}(z)) \).

Here \( \text{ord}(z) := \sup\{j \mid z \in M_j\} \) is the order of \( z \). This statement linearizes the determinacy question and reduces it to the level of the tangent space. Part i. implies that \( z \) is \( N \)-determined and generalizes the classical results in characteristic zero as in (1) (if we take \( i = 0 \)). Part ii. implies that \( z \) is \((2N - \text{ord}(z))\)-determined and generalizes the known results in positive characteristic as in (6) below.

The assumption in (4.ii.) is however a condition for all \( k > 0 \), which is stronger than some known cases, but the conclusion is also stronger since it implies determinacy of \( z + (\text{terms of order } N + k) \) already by the smaller group \( G^{(k)} \subset G^{(1)} \) (for \( i = 0 \)). In many cases the assumption in (4.ii.) for all \( k > 0 \) is already implied by the assumption for \( k = 1 \) (cf. Corollary 4.4 and Example 4.5).

The upper bound of 4.ii on the order of determinacy is \((2N - 2i - \text{ord}(z))\), which is weaker than \( N \) of 4.i. This \((2N - 2i - \text{ord}(z))\) cannot be significantly improved, see Sect. 4.3.
Statement (4) may be rephrased as “large tangent space implies determinacy”. We also prove the converse statement, “determinacy implies large tangent space” (under slightly weaker assumptions, cf. Theorems 4.1 and 4.3), which reads:

(i) Let \((T_{(G^{(1)}, M)}, G^{(1)})\) be a pair of Lie type and suppose \(z \in M\) satisfies \(\{z\} + M_{N+1} \subseteq G^{(1)}z\). Then \(M_{N+1} \subseteq T_{(G^{(1)}, M)}(z)\).

(ii) Let \((T_{(G^{(1)}, M)}, G^{(1)})\) be a pair of weak Lie type and suppose \(z \in M\) satisfies \(\{z\} + M_{N+k} \subseteq G^{(k)}z\) for any \(k \geq 1\). Then \(M_{N+k} \subseteq T_{(G^{(k)}, M)}(z)\) holds for any \(k > N - \text{ord}(z)\).

In Sect. 5 we couple these statements with the approximation theorems of Popescu, Artin and Tougeron to get from an order-by-order solution an algebraic, resp. analytic, resp. \(C^\infty\) solution. The proposed generality allows to recover finite determinacy statements for many particular scenarios, e.g., for germs of functions, of maps on smooth and non-smooth spaces and of matrices. In particular,

(i) when \(k\) is a field of characteristic zero this recovers numerous classical results e.g., by Mather, Gaffney, Bruce–Du Plessis–Wall, Damon, and many others;

(ii) when \(k\) is a field of positive characteristic this gives other known results, e.g., those of Boubakri–Greuel–Markwig and Greuel–Pham;

(iii) the notion of weak Lie-type might be potentially useful not only in prime characteristic, but also in mixed characteristic, as we do not impose any kind of restriction on the base \(k\);

(iv) beyond this we get new results, e.g on relative determinacy results for non-isolated singularities, Sects. 5.4 and 5.5.

1.5 Remarks

In order to put our results into perspective, we finish this introduction by giving references to previous results (by far not complete).

1.5.1. Investigations on determinacy had been classically restricted to the real resp. complex case, with \(M\) being formal or analytic power series, or to germs of \(C^\infty\)-maps. In order to apply methods and results from Lie groups or algebraic groups, the setting was immediately reduced to a finite dimensional parameter space (either a finite jet space or the parameter space of a semi-universal deformation) by assuming some kind of “isolated singularity”. The proofs used essentially complex or real analysis, integration of vector fields, and topology. See [22, 65] for a short introduction and [8, section 2.7.2] for some more recent history. It was observed in [17] that in fact the essential ingredient for a statement like (1) is the unipotency of the group action. In [8] this idea was used to extend (1) to Henselian rings over a field of characteristic zero and to filtered groups possessing a (formal) exponential and logarithmic map, or at least an “order-by-order” version of these maps. These exponential and logarithmic maps were the basis of the construction, thus the method seemed to be inapplicable to the case when the base ring does not contain the rational numbers. Furthermore, to define the tangent space one had to restrict to some particular class of group actions, though broad enough to include most of the known scenarios.

1.5.2. Another direction of generalization was to positive characteristic. This study was initiated (to the best of our knowledge) in [29] and then continued in [14, 16, 30, 56]. In [31] the authors considered the case of matrices over the ring \(R = k[[x]]\), \(k\) an arbitrary field of any characteristic, with \(M = Mat_m \times R\), and the group \(G = GL(m, R) \times GL(n, R) \times \ldots \)
Pairs of Lie-type and large orbits of group...

$Aut_k(R)$. The proved result was ($m = \langle x_1, \ldots, x_p \rangle$) the maximal ideal, $A \in Mat_{m \times n}(R)$:

If $m \cdot \tilde{T}_G(A) \supseteq m^{N+1} \cdot Mat_{m \times n}(R)$ then $GA \supseteq \{A\} + m^{2N+1 - \text{ord}(A)} \cdot Mat_{m \times n}(R)$. In particular, $A$ is then $(2N - \text{ord}(A))$ determined. (6)

Here $\tilde{T}_G(A)$ is the tangent image, i.e., the image of the tangent map of the orbit map $G \to Mat_{m \times n}(R)$, $g \mapsto gA$.

In characteristic zero $\tilde{T}_G(A)$ coincides with the tangent space $T_G(A)$ to the orbit $GA$ at $A$, but in positive characteristic $\tilde{T}_G(A)$ differs in general from $T_G(A)$. In our general framework the module $T((G^{(1)}, M))(z)$ of a pair of weak Lie-type is a generalization of the tangent image. For $m = 1 = n$ we get the contact determinacy of function germs, recovering [16, Theorem 3].

Note that these bounds coincide with those of Eq. (4).

1.5.3. The approximation step (Step 2 in Sect. 1.2) is more or less standard, we repeat it briefly in Sect. 2.4. The case $C^\infty$ is more involved, we treat it in 4.5.

In some cases no approximation theorems are possible. For example, analytic questions of dynamical systems or differential equations are notoriously difficult when compared to the formal ones. Even in such cases, establishing that “two objects are order-by-order equivalent” is a significant result.

1.5.4. In this paper we address only the topologically unipotent part $G^{(1)}$ of the group $G$. However, this has no significant impact on the finite determinacy, as for most “reasonable” groups over a local ring $R$ we have $m \cdot T_G \subseteq T(G^{(1)}, M) \subseteq T_G$ for the maximal ideal $m \subset R$. Accordingly, the orders of determinacy under $G$ and $G^{(1)}$ differ at most by one in the case of a Lie type pair, and by two in the case of a weak Lie type pair.

1.5.5. We do not consider the group of left-right or $A$-equivalences. This group is not a subgroup of $GL_k(R^n) \rtimes Aut_k(R)$, as the $A$-action is not $k$-linear. Thus our method does not apply directly. Certain modifications are needed to establish a (weak) Lie-pair structure for this group and to obtain the “determinacy vs. infinitesimal determinacy” statements. We hope to report on this soon.

1.5.6. In many cases a result of type (1) is not yet a complete solution. The tangent module can be rather complicated, and to check the condition $m \cdot T_{(Gf, f)} \supseteq m^{N+1} \cdot M$ in particular cases can be a difficult task (although, when $R$ is the ring of power series, standard basis methods provide effective algorithms, cf. [33] or [2]). For example, for matrices over local rings and various groups acting on them, one gets non-trivial questions on the annihilators of quotient modules, see [7, 10, 39].

1.6 Contents of the paper

• Section 2 is preparatory, we review the relevant facts about filtered rings, the Implicit Function Theorem (with the “unit main part”) filtered modules, and the associated filtration on $GL_k(M)$ and its subgroups. In Sect. 2.4 we recall some relevant facts on Artin approximation and their extensions to non-polynomial equations.

• Sections 3 and 4 form the core of the paper. In Sect. 3 we introduce the pairs of (point-wise/weak) Lie type. As was briefly mentioned in Sect. 1.3, these are groups together with a substitute for the tangent space at the orbit of the action. The orbit is locally approximated by its tangent space and there are some substitutions for the classical exponential/logarithmic maps.
We show that the class of such pairs is rich enough. It contains the main interesting subgroups of $GL_k(M)$, in particular the groups $Aut_k(R)$ (under certain assumptions on $R$), $GL_R(M)$, the groups of contact equivalences for matrices, and the (semi-)direct products of these groups (cf. Theorem 3.6).

- Section 4 contains the main results of this paper (as indicated in Sect. 1.4), the Finite Determinacy Theorems 4.1 and 4.3. These results establish the “determinacy versus infinitesimal determinacy” criteria in a characteristic free way. The determinacy bounds for pairs of weak Lie type are weaker than those for pairs of Lie type. We show in Sect. 4.3 that these weaker bounds are often sharp.

In Sect. 4.4 we translate the finite determinacy into the infinitesimal stability in the traditional way: an element $z \in M$ is finitely determined iff its fibres are infinitesimally stable on the punctured neighborhood $Spec(R)^\times$.

The ring $C^\infty(\mathbb{R}^p, 0)$ does not have the Artin approximation property, thus we cannot pass from $Gz$ to $Gz$ by using the general/standard theory of Sect. 2.4. However, we prove the relevant approximation statement by using the Lie type notions, Sect. 4.5.

- Finally, in Sect. 5 we couple the Finite Determinacy Theorems 4.1 and 4.3 with various approximation results and apply these to several particular scenarios. We recover and extend numerous classical results and obtain some new results (in arbitrary characteristic), e.g. the right determinacy of germs of functions, the right indeterminacy of germs of maps, the contact determinacy of germs of maps, determinacy of maps relative to a space-germ, relative determinacy of non-isolated singularities, relative algebraization of power series, determinacy of matrices.

2 Preparations

2.1 Notations and assumptions

In this paper all rings are assumed associative, commutative and unital, with subrings having the same identity element as the ambient ring. We fix a base ring $\mathbb{k}$, not necessarily a field, of any characteristic. Consider a $\mathbb{k}$-module filtered by submodules, $M = M_0 \supset M_1 \supset M_2 \cdots$, usually not finitely generated over $\mathbb{k}$.

We consider also a (commutative, associative) $\mathbb{k}$-algebra $R$, filtered by a chain of ideals, $R = I_0 \supset I_1 \supset \cdots$ with $I_j \cdot I_k \subseteq I_{j+k}$, e.g., $I_j = I^j$ for some ideal $I$.

If $R$ is local (i.e., a ring with the unique maximal ideal $m$) then the classically considered filtration is by the powers of $m$.

If $M$ is also an $R$-module, not just a $\mathbb{k}$-module, then the filtrations of $R$, $M$ are supposed to be compatible, i.e., $I_j M_i \subseteq M_{i+j}$ for any $i$, $j$.

When a group $G$ acts on $M$ one fixes some action $G \circ R$, possibly trivial, and then the action $G \circ M$ is assumed to be $R$-multiplicative, i.e., $g(f \cdot z) = g(f) \cdot g(z)$ for any $g \in G$, $f \in R$, $z \in M$.

On any filtered object we use the filtration topology. Thus $\{M_i\}$ is a fundamental system of neighbourhoods of $0 \in M$, the $M_i$ are both open and closed. (See e.g. [13, Chapter III, section 2, no.1, Corollary to Proposition 4].) The completion of $M$ with respect to this topology is $\hat{M} := \lim_{\leftarrow} M/M_i$. The orbit closure of an element $z$ is $\overline{Gz} = \cap_i (Gz + M_i)$. 

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2.1.1 Typical rings

Let \( R \) be a \( k \)-algebra of any characteristic (e.g., \( k \) a field, the classically considered case). We use the multi-variable notation (always finitely many variables), \( x = (x_1, \ldots, x_p) \). Typical examples for \( R \) are:

- the formal power series, \( k[[x]] \);
- the algebraic power series, \( k(x) \);
- the convergent power series, \( k[x] \), when \( k \) is a complete normed ring;
- the germs of \( C^\infty \)-functions, \( C^\infty(\mathbb{R}^p, 0) \);
- the quotients of these rings by some ideal, \( k[[x]]/J, k[x]/J, k(x)/J, C^\infty(\mathbb{R}^p, 0)/J \).

More generally, we often assume the condition: \( R \subseteq k[[x]]/J \) and \( R \) contains the images of \( \{x_i\} \subseteq k[[x]] \). In this case the completion map \( R \hookrightarrow \hat{R} \) is injective and \( \hat{R} \) is of Cohen-type. This is satisfied by many rings, not necessarily over a field, see e.g., page 214 in [42].

2.1.2 Implicit function theorem with unit linear part, IFT

We often need to solve an equation \( z + w = g z \) for the unknown \( g \in G \). In many cases this can be stated as a system of implicit function equations. For example, let \( R \subseteq k[[x]], M = R, G = \text{Aut}_k(R) \), then given \( f \in R \) and its perturbation \( h \in R \) we want to resolve \( f(y) = f(x) + h(x), y = gx \) for the unknown \( g \in G \).

Moreover (and this is one of the main results of Sect. 3), in certain cases this implicit function equation can be transformed to a simpler form (with \( x \in \mathbb{R}^n \), and \( y = gx \) an \( n \)-tuple of unknowns):

\[
y + h(y, x) = x, \quad h(y, x) \in \langle y^2, x \cdot y \rangle R^n := \langle \{ y_i \cdot y_j, x_i \cdot y_j \} \rangle R^n \subset R[[y]]^n. \quad (7)
\]

Here the second condition means that \( h \) is “of higher order”. Usually implicit function equations are studied with the assumption that \( R \) contains a field. This assumption is not needed for equation (7).

**Definition 2.1** (cf. [9, section 3.4]) We say that \( IFT_1 \) holds over \( R \) if the equation \( y + h(y, x) = x \) has a solution \( y(x) \in R^n \) for any \( h(y, x) \) satisfying \( h(y, x) \in \langle y^2, x \cdot y \rangle R^n \), and moreover \( y(x) - x \in \langle \{ x_i \cdot x_j \} \rangle R^n \subset R^n \).

**Example 2.2** Examples of rings for which \( IFT_1 \) holds include:

(i) \( k[[x]]/J, k(x)/J, \) and \( k[x]/J \) for \( k \) a normed field, complete w.r.t. its norm.

(ii) More generally, let \( k \) be either a field or a discrete valuation ring, and \( \{ W_n \}_{n \geq 1} \) be a Weierstrass system of rings over \( k \) [23, page 2]. Then, for any \( n \geq 1 \), the ring \( W_n/J \) satisfies \( IFT_1 \) [23, page 4].

(iii) \( C^\infty(\mathbb{R}^p, 0)/J \). For \( J = (0) \) see [58, page 79]. Otherwise one can assume \( J \subseteq (x)^2 \).

Then one lifts the equations to \( C^\infty(\mathbb{R}^p, 0), \) resolves there and sends the solutions back to \( C^\infty(\mathbb{R}^p, 0)/J \).
2.2 Filtered modules and group actions

2.2.1 Induced filtrations on $\text{End}_k(M)$ and $\text{GL}_k(M)$

Let $k$ be a commutative ring.

Fix a filtered $k$-module, $M = M_0 \supset M_1 \supset \cdots$ and consider the set of all $k$-linear endomorphisms, $\text{End}_k(M)$. The filtration of $M$ induces a filtration of $\text{End}_k(M)$ by $k$-submodules

$$\text{End}_k(M) \supset \text{End}_k^{(0)}(M) \supset \text{End}_k^{(1)}(M) \supset \cdots,$$

where

$$\text{End}_k^{(i)}(M) := \{\phi \in \text{End}_k(M) | \phi(M_j) \subseteq M_{j+i}, \forall j \geq 0\}. \tag{8}$$

Here $\text{End}_k^{(0)}(M)$ is the module of endomorphisms that preserve the filtration, while we call $\text{End}_k^{(1)}(M)$ the module of topologically nilpotent endomorphisms (see the remark after equation (3)).

The order of an element $z \in M$ is defined as $\text{ord}(z) := \sup\{j | z \in M_j\}$. In particular, $\text{ord}(z) = \infty$ if $z \in \cap_{j=1}^{\infty} M_j$. Similarly, for an endomorphism $\phi \in \text{End}_k(M)$ the order is $\text{ord}(\phi) := \sup\{j | \phi \in \text{End}_k^{(j)}(M)\}$.

Denote the group of all $k$-linear automorphisms of $M$ by $\text{GL}_k(M)$. Define the subgroup of automorphisms that preserve the filtration,

$$\text{GL}_k^{(0)}(M) := \{g | g, g^{-1} \circ M_i, \forall i \subseteq \text{GL}_k(M)\}. \tag{9}$$

It is filtered by its subgroups,

$$\text{GL}_k^{(i)}(M) := \text{GL}_k^{(0)}(M) \cap \left(\mathbb{I} + \text{End}_k^{(i)}(M)\right), \quad i \geq 1, \tag{10}$$

i.e., the subgroups of $\text{GL}_k^{(0)}(M)$, such that the elements and their inverses are of the form $\mathbb{I} + \phi$ with $\phi \in \text{End}_k^{(i)}(M)$. $\text{GL}_k^{(1)}(M)$ is the subgroup of topologically unipotent automorphisms.

One readily checks that $\text{GL}_k^{(i)}(M)$ is indeed a subgroup of $\text{GL}_k^{(0)}(M)$. For example, if $g \in \text{GL}_k^{(i)}(M)$ then $g^{-1} \in \text{GL}_k^{(i)}(M)$: since $g|M_{j/M_{i+j}} = \mathbb{I}|M_{j/M_{i+j}}$ for all $j$, $g^{-1}|M_{j/M_{i+j}} = \mathbb{I}|M_{j/M_{i+j}}$ and thus $g^{-1} - \mathbb{I} \in \text{End}_k^{(i)}(M)$.

As the simplest case take $M = \mathcal{O}_n = \mathbb{C}(x), M_i = \langle x \rangle^i$ and $G = \mathcal{R}$, the right group acting by $\phi \cdot f = f \circ \phi^{-1}$ for $\phi \in \mathbb{C}(\mathbb{C}^n, 0)$. Then $\text{GL}_k^{(i)}(M)$ is the subgroup $\mathcal{R}^{(i)}$ leaving the $i + 1$-jet of every $f \in \mathcal{O}_n$ unchanged.

The group $\text{GL}_k^{(0)}(M)$ will be always taken as the ambient group. For any subgroup $G \subseteq \text{GL}_k^{(0)}(M)$ we get the induced filtration,

$$G^{(i)} := G \cap \text{GL}_k^{(i)}(M), \quad i = 0, 1, \ldots. \tag{11}$$

These are always normal subgroups, $G^{(0)} \triangleright G^{(1)} \triangleright \cdots$, as their action on the quotient $M/M_{i+1}$ is trivial.

2.2.2 The structure of $\text{GL}_k^{(i)}(M)$ and $\text{GL}_k^{(i)}(M)$

In (10) we define $\text{GL}_k^{(i)}(M)$ as a subset of $\mathbb{I} + \text{End}_k^{(i)}(M)$, where $k$ can be any commutative ring. We give now examples of group actions where both sets coincide.
Pairs of Lie-type and large orbits of group . . .

(i) Suppose $M$ is complete with respect to the filtration $\{M_i\}$. Then we have for all $i \geq 1$:

$$GL_k^{(i)}(M) = \{1\} + End_k^{(i)}(M) = \{1 + \phi, \ \phi \in End_k^{(i)}(M)\}. \quad (12)$$

The inclusion $GL_k^{(i)}(M) \subseteq \{1\} + End_k^{(i)}(M)$ follows just from the definition, equation (10). To show $GL_k^{(i)}(M) \supseteq \cdots$ it is enough to check that any endomorphism of the form $1 + \phi$, for $\phi \in End_k^{(1)}(M)$, is invertible. Indeed, its inverse, $\sum_{j=0}^{\infty}(-\phi)^j$, is a well defined $k$-linear operator on $M$ (by completeness of $M$ and the topological nilpotence of $\phi$).

If $M$ is not complete then statement (12) does not necessarily hold, regardless of how nice is $k$. For example, suppose $k$ is a field and $M = k[x]$. Consider the operator $(1 + x) \in End_k(M)$, acting by $p(x) \mapsto (1 + x)p(x)$. Though it is topologically unipotent, it is not invertible.

(ii) Now consider a ring extension $k \subseteq R$ such that $M$ is also a filtered module over the larger ring $R$. Suppose that $GL_k^{(1)}(M) = \{1\} + End_k^{(1)}(M)$ holds for the $k$-module $M$, then we have also $GL_R^{(1)}(M) = \{1\} + End_R^{(1)}(M)$.

(iii) Take a filtered ring $R$ and let $M$ be a finitely generated $R$-module with a filtration $M = M_0 \supset M_1 \supset \cdots$. Let $J(R)$ be the Jacobson radical of $R$. (If $R$ is local then $J(R)$ is the maximal ideal.) Assume $M_j \subseteq J(R) \cdot M$, for $j \gg 1$. Then the group of $R$-linear automorphisms, $GL_R(M)$, is related to the module of $R$-linear endomorphisms $End_R(M)$:

$$\forall \ i \geq 1: \ GL_R^{(i)}(M) = \{1\} + End_R^{(i)}(M). \quad (13)$$

As before, the inclusion $\subseteq$ follows from the definition. For $\supseteq$ it is enough to check that $1 + \phi$ is invertible for $\phi \in End_R^{(1)}(M)$. This can be proved as follows. By our assumption there is an integer $d$ with $\phi^d(M) \subseteq M_d \subseteq J(R) \cdot M$. Now notice:

$$\left(1 - \phi\right)\left(1 + \phi + \cdots + \phi^{d-1}\right) = 1 - \phi^d \equiv 1 \mod J(R) \cdot End_R(M). \quad (14)$$

Setting $\psi := 1 - \phi^d$ we get $M \subseteq \psi(M) + J(R) \cdot M$. By Nakayama’s Lemma $M = \psi(M)$, i.e. $\psi$ is surjective and hence bijective ([49, Theorem 2.4]). Therefore $1 - \phi$ is invertible, with $(1 - \phi)^{-1} = \left(1 + \phi + \cdots + \phi^{d-1}\right)(1 - \phi^d)^{-1}$.

### 2.2.3 Actions involving ring automorphisms

Let $k$ be a ring and $R$ be a (not necessarily Noetherian) filtered $k$-algebra, e.g., one of the rings in Sect. 2.1.1.

(i) Considering $R$ as a $k$-module we get the group $GL_k(R)$ and its filtration. Consider the subgroup $Aut_k(R) \subseteq GL_k(R)$ of $k$-linear automorphisms of this ring. This group is naturally filtered,

$$Aut_k^{(i)}(R) := Aut_k(R) \cap GL_k^{(i)}(R). \quad (15)$$

Every $g \in Aut_k^{(i)}(R), \ i \geq 1$, is of the form $1 + \phi$ for some $\phi \in End_k^{(i)}(R)$, by equation (10). The multiplicativity property of $g \in Aut_k(R)$ imposes the “almost Leibniz rule” for $\phi$,

$$\phi(f_1 f_2) = \phi(f_1) f_2 + f_1 \phi(f_2) + \phi(f_1) \phi(f_2). \quad (16)$$
We claim: if $IFT_1$ holds over $R$ (see §2.1.2) then (16) is the only condition on $\phi$. Namely,

$$
\text{Aut}^{(i)}_R(R) = \left\{ \mathbb{I} + \phi \mid \phi \in \text{End}^{(i)}_R(R), \ \phi(f_1 f_2) = \phi(f_1) f_2 + f_1 \phi(f_2) + \phi(f_1) \phi(f_2), \ \forall \ f_1, f_2 \in R \right\}.
$$

(17)

Indeed, $\mathbb{I} + \phi$ is $k$-linear, while the almost Leibniz rule ensures the multiplicativity. And any topologically unipotent endomorphism is invertible, as the equation $x = y + \phi(y)$ is solvable over $R$, by $IFT_1$.

(ii) Let $R$ be a $k$-subalgebra of $k[[x]]/J$ and assume that $R$ contains the images of all $x_i \in k[[x]]$. Consider elements of $R$ as power series, $f(x) \in R$. Then any topologically unipotent automorphism $\phi \in \text{Aut}_k(R)$ acts as a coordinate change. Namely, fix its action on the generators $x_i \in k[[x]]$,

$$
x_i \mapsto \phi(x_i) = x_i + h_i(x), \quad h_i(x) \in m^2 \subset k[[x]],
$$

and consider $\{x_i\}$ as coordinates on $\text{Spec}(R)$. Then $\phi(f(x)) = f(\phi(x))$ and $\phi(J) = J$. (See e.g., [1, 2, 11], [34, Lemma I.1.23] or [8, Lemma 3.1]. In that lemma $k$ was assumed to be a field of zero characteristic, but the proof is characteristic-free.)

Because of this, the group $\text{Aut}_k(R)$ (resp. $\text{Aut}_k^{(1)}(R)$) is also called the right group (resp. unipotent right group) or the group of right equivalences of function germs.

For more general rings there exist “non-geometric” automorphisms, not arriving from coordinate changes as above. Yet, if the subring $R \subseteq C^\infty(\mathbb{R}^n, 0)/J$ admits the Taylor expansion up to any order, then the coordinate changes are dense inside all the automorphisms. Therefore, for $R \subseteq C^\infty(\mathbb{R}^n, 0)/J$, (following [62]) we denote by $\text{Aut}_k(R)$ the subgroup of all automorphisms $\phi$ of $R$ arriving from coordinate changes, i.e., $\phi(f(x)) = f(\phi(x))$, with $\phi = (\phi_1, \ldots, \phi_n)$ a $C^\infty$-coordinate change of $(\mathbb{R}^n, 0)$ satisfying $\phi(J) = J$. As before, $\text{Aut}_k^{(1)}(R)$ denotes the subgroup of coordinate changes satisfying $\phi(x) = x + h(x)$ with $h(x) \in m^2$.

(iii) Let $M$ be a (not necessarily free) $R$-module and $G$ a subgroup of $GL_R(M) \rtimes \text{Aut}_k(R)$.

We assume some prescribed $R$-multiplicative action $G \circ M$. Namely, for $g = (W, \phi) \in G \subseteq GL_R(M) \rtimes \text{Aut}_k(R)$ one has

$$
g(a \cdot z) = \phi(a) \cdot g(z), \quad a \in R, \ z \in M,
$$

(19)

where $\phi(a)$ is the standard action $\text{Aut}_k(R) \circ R$.

For example, for a free module we fix a basis, i.e., identify $M = R^n$. Then we have the action $GL_R(R^n) \rtimes \text{Aut}_k(R) \circ R^n$, where $\text{Aut}_k(R)$ acts component-wise. This recovers the classical right and contact equivalence of maps.

If $M$ is not a free module then it is a quotient of a free module and the action must of course respect the relations. In particular, the possible actions $\text{Aut}_k(R) \ni \phi \circ M$ are restricted. For example, the support and the annihilator of $M$ must be preserved by $\phi$.

Indeed, if $f \cdot M = 0$ then $\phi(f) \cdot M = \phi(f) \cdot \phi(M) = \phi(f \cdot M) = 0$.

We consider the following group actions, with $M = \text{Mat}_{m \times n}(R)$ the free $R$-module of rectangular matrices $A = [a_{ij}]$:

- $GL(m, R) \circ \text{Mat}_{m \times n}(R)$, by $(U, A) \rightarrow UA$;
- $GL(n, R) \circ \text{Mat}_{m \times n}(R)$, by $(V, A) \rightarrow AV^{-1}$;
- $\text{Aut}_k(R) \circ \text{Mat}_{m \times n}(R)$ by $(\phi, A) \rightarrow \phi(A) = \{\phi(a_{ij})\}$;
- products of these groups, e.g. $GL(m, R) \times GL(n, R) \rtimes \text{Aut}_k(R)$.
• (for $m = n$) the conjugation $A \rightarrow UA U^{-1}$, the congruence $A \rightarrow UA U^t$, etc.

The filtration $R = I_0 \supset I_1 \supset \cdots$ induces the filtration $\{Mat_{m \times n}(I_j)\}$. Accordingly one gets the subgroups $G^{(j)} \leq G$.

We remark: if the action $G \circ Mat_{m \times n}(R)$ is $R$-linear and preserves the subset of degenerate matrices then $G$ is contained in the group of left-right multiplications, $G_{fr} := GL(m, R) \times GL(n, R)$. (See [10, §3.6] for the precise statement.)

2.3 The condition $z + w \in Gz$ as implicit function equation

For the initial determinacy question (whether $Gz \supseteq \{z\} + M_{N+1}$) we should resolve the equation (for the unknown $g \in G$):

$$z + w = gz,$$

for any given $w \in M_{N+1}$. (20)

Let us explain in detail how these conditions are presentable as a finite system of implicit function equations, $z + w - gz = F(g) = 0$, for some power series equations $F(y) \in R[[y]]^p$, if $M$ is finitely presented as $R$-module.

Fix any finite presentation $R^q \xrightarrow{A} R^p \xrightarrow{\alpha} M \xrightarrow{0}$ and consider the $R$-multiplicative action of some group element $g$ on $M$,

$$GL_R(M) \times Aut_k(R) \ni g = (W, \phi) \circ M.$$

The easiest way to see that the action of $g = (W, \phi)$ lifts to the presentation (and is determined by it) is by describing it by an $R$-linear map as follows. For $\phi \in Aut_k(R)$ denote by $M_\phi$ the $R$-module, which is equal to $M$ as $k$-module but with the new scalar multiplication

$$a \ast z \equiv a \ast_\phi z := \phi(a) \cdot z, \; a \in R, \; z \in M,$$

also called an $R$-linear homomorphism over $\phi$. Then the map

$$W_\phi : M \rightarrow M_\phi \text{ with } W_\phi(z) \equiv (W, \phi) \cdot z$$

is $R$-linear since it is $k$-linear and satisfies $W_\phi(a z) = a \ast W_\phi(z)$. The filtration $R = I_0 \supset I_1 \supset \cdots$ induces the filtrations $I_j M$ and $I_j \ast M_\phi$ and $W_\phi$ is a map of filtered $R$-modules, assuming that $\phi$ preserves $\{I_j\}$.

If $A = [a_{i,j}]$, then $R^q \xrightarrow{A_\phi} R^p \xrightarrow{\alpha_\phi} M_\phi \xrightarrow{0}$ is a presentation of $M_\phi$, with $A_\phi := \phi^{-1}(A) = [\phi^{-1}(a_{i,j})]$ and $\alpha_\phi(\sum_j a_j e_j) = \sum_j a_j \ast_\phi(e_j), \{e_j\}$ the canonical basis of $R^p$. As is well known, the linear map $W_\phi : M \rightarrow M_\phi$ lifts to a morphism of the presentations of $M$ and $M_\phi$ (unique up to homotopy of complexes) and we get a commutative diagram

$$R^q \xrightarrow{A} R^p \xrightarrow{\alpha} M \xrightarrow{\phi} 0$$

of filtered modules ($R'$ being filtered by $I_i R'$). Since $W_\phi$ is an isomorphism, it follows that the linear maps $U = [u_{i,j}]$ and $V = [v_{i,j}]$ (depending on the action of $g = (W, \phi)$) are isomorphisms. To see this we may assume that $R$ is local (just localize diagram (22) w.r.t. the maximal ideals of $R$). For local rings the result follows from (the proof of) [27, Theorem 20.2]), since in our situation the free modules of the presentation have the same rank. The
action on $M \cong R^p/im(A)$ is hence induced by an action on $R^p$ preserving $im(A)$. Conversely, for a given $\phi$ the left-hand square of (22) determines a commutative diagram as above with an isomorphism $W_\phi : M := \text{coker}(A) \rightarrow \text{coker}(A_\phi) =: M_\phi$ and hence a $k$-linear action of $(W, \phi)$ on $M$ (note that $M$ and $M_\phi$ coincide as $k$-modules).

We will mainly work with the action of $(U, V, \phi) \in GL_R(R^p) \times GL_R(R^q) \times \text{Aut}_k(R)$ on a presentation of $M$. If $z = \sum z_i \alpha(e_i)$ and $w = \sum w_i \alpha(e_i)$ are elements in $M$ set $\tilde{z} := \sum z_i e_i$ and $\tilde{w} := \sum w_i e_i$ for the $\alpha$-preimages of $z$ and $w$. Then $\phi^{-1}(\tilde{z}) = \sum \phi^{-1}(z_i) e_i$ and $\phi^{-1}(\tilde{w}) = \sum \phi^{-1}(w_i) e_i$ are $\alpha_\phi$-preimages in $R^p$ of $z$ and $w$. By diagram (22) the condition $z + w = g z$ reads then: $\phi^{-1}(\tilde{z}) + \phi^{-1}(\tilde{w}) = U \tilde{z} + \phi^{-1}(A) \tilde{v}$ for some $v \in R^q$ and $UA = \phi^{-1}(A)V$. After applying $\phi$ and renaming we get the system of equations

$$\tilde{z} + \tilde{w} = U \cdot \phi(\tilde{z}) + A \cdot v, \quad U \cdot \phi(A) = A \cdot V,$$

with $\tilde{z}, \tilde{w}, A$ given and (the entries of) $U, V, \phi, v$ as unknowns.

To this explicit form (23) we add the condition “$g(J) = J$” (if $R$ is a quotient ring mod $J$) and the condition “$g \in G$”, if $G$ is a subgroup of $GL_R(R^p) \times GL_R(R^q) \times \text{Aut}_k(R)$, as follows.

(i) Assume $R \subseteq S/J$, where $S = k[[x]]$ or $C^\infty(\mathbb{R}^p, 0)$, thus $\phi \in \text{Aut}_k(R)$ is induced by a coordinate change $\phi \in \text{Aut}_k(S)$ that satisfies $\phi(J) = J$. Fix some generators $\{f_j\}$ of $J$, then the condition $\phi(J) = J$ becomes the system of (formal) power series (or $C^\infty$-germs) equations $f_j(\phi(x)) = \sum_i t_{j,i} f_i$, in the unknowns $\phi(x) = (\phi_1, \ldots, \phi_n)$ and $\{f_{j,i}\}$.

(ii) Assume that $G \subseteq GL_R(M) \times \text{Aut}_k(R)$ is defined by (formal) power series. That is, for a presentation of $M$ (as above) and the lift of $G$ to a subgroup of $GL_R(R^p) \times GL_R(R^q) \times \text{Aut}_k(R)$, the conditions on $(U, V, \phi) \in GL_R(R^p) \times GL_R(R^q) \times \text{Aut}_k(R)$ can be written as some (formal) power series equations in the unknowns $U, V, \phi$.

Altogether, the equations i. for $J$, ii. for $G$ and the equations (23) give a finite system of implicit equations

$$F(y) = 0, \quad F \in R[[y]]^s,$$

in the unknowns $\{u_{i,j}\}$, $\{w_{i,j}\}$, $\{\phi_i\}$, $\{v_i\}$ (the entries of $U, V, \phi, v$) and $\{t_{i,j}\}$ denoted all together by $y = (\gamma_1, \ldots, \gamma_q)$. (In the $C^\infty$-case the equations are of $C^\infty$-type.)

**Remark 2.3** (i) Suppose the ideal $J \subset k[[x]]$ admits polynomial generators, the group $G \subseteq GL_R(M) \times \text{Aut}_k(R)$ is defined by polynomial equations, and $\{z_i\}$, $\{w_i\}$ in the expansion $z = \sum z_i \cdot \alpha(e_i)$, $w = \sum w_i \cdot \alpha(e_i)$ as well as the entries $\{a_{i,j}\}$ of the presentation matrix $A$ are polynomials. Then all the equations in (24) are polynomial, i.e., $F(y) \in R[y]^s$.

(ii) Similarly, if the data of $A, J, G, z, w$ are algebraic resp. analytic power series, then $F(y) \in R(y)^s$ resp. $F(y) \in R[y]^s$.

### 2.4 The relevant approximation theorems

The determinacy criteria of Sects. 4.1, 4.2 provide only an order-by-order solution to the condition $gz = z + w$, $g \in G$. However, in many cases these are implicit function equations, see Sect. 2.3. Then we can use fundamental approximation theorems, to achieve ordinary solutions. We explain this in detail.
2.4.1 The passage from an order-by-order solution to a formal solution

When the coefficient ring is complete we use the following important result (due to Pfister and Popescu).

**Theorem 2.4** Let $(R, m)$ be a complete Noetherian local ring (of arbitrary characteristic, not necessarily over a field). Fix any $F(y) \in R[[y]]^p$, $y = (y_1, \ldots, y_q)$, and suppose that the system of equations $F(y) = 0$ has an order-by-order solution. (That is, there exists a sequence $\{y_n \in m : R^q\}_{n \geq 1}$ such that $F(y_n) \equiv 0 \pmod{m^n}$.) Then there exists an ordinary solution, i.e., $y \in R^q$ such that $F(y) = 0$.

A stronger version of this theorem first appeared as [55, Theorem 2.5] for the particular case $R = \mathbb{k}[[x]]$, where $\mathbb{k}$ is either a field or a complete DVR, of zero characteristic. For the case of arbitrary characteristic see [23, Theorem 7.1]. The generalization to the case where $(R, m)$ is a complete Noetherian local ring is done as follows (we thank D. Popescu for this explanation):

(i) First one observes that if this property holds for $R$ then it holds for any finite algebra over $R$ [41, page 82, Satz 1.2].

(ii) Then one uses the Cohen Structure Theorem for complete Noetherian local rings, [21, Theorem 12].

2.4.2 The passage from a formal solution to an ordinary solution for polynomial or analytic equations

If the equations $F(y) = 0$ are polynomial, i.e., $F(y) \in R[y]^p$, then Artin’s approximation theorem [5] can be used whenever $R$ has the Artin approximation property.

**Definition 2.5** The Artin approximation property, AP, holds for a ring $R$ with filtration $\{I_j\}$ if for every finite system of polynomial equations over $R$, a solution in the completion $\hat{R}^{(I_j)}$ implies a solution in $R$, which moreover can be chosen arbitrarily close to the formal solution in the filtration topology.

The famous characterization of such rings reads:

**Theorem 2.6** [57, Remark 2.15] A commutative local Noetherian filtered ring with the filtration $\{m^l\}$ has the Artin approximation property if and only if it is excellent and Henselian.

This result implies AP for rather general filtrations due to the following observation:

**Lemma 2.7** [12] Suppose a commutative, unital ring $R$ (not necessarily local or Noetherian) has AP for a filtration $\{I_j\}$. Then $R$ has AP for any filtration by finitely generated ideals, $\{a_j\}$, such that $a_j \subseteq I_{d_j}$, for some sequence satisfying $\lim_{j \to \infty} d_j = \infty$.

**Example 2.8** The most important rings that have AP (assuming $\{I_j\}$ satisfy $I_j \subseteq m^{d_j}$, with $\lim_{j \to \infty} d_j = \infty$):

- $\mathbb{k}[[x]]/J$ (trivially);
- $\mathbb{k}(x)/J$, with $\mathbb{k}$ a field or a discrete valuation ring or a multi-variable formal power series ring;
- $\mathbb{k}(x)/J$, with $\mathbb{k}$ a complete normed field.
Remark 2.9 Sometimes the equations are not polynomial, e.g. for \( k(x)/J, k[x]/J \) one has algebraic/\( k \)-analytic equations. In this case the approximation property of Definition 2.5 still holds:

- If \( k \) is a valued field then the analytic Artin Approximation theorem holds over \( k[x]/J \), if and only if the completion of \( k \) with respect to the absolute value is separable over \( k \), see [60]. This condition is satisfied if either \( k \) is a field of characteristic zero, or a perfect field of prime characteristic.
- For \( k \)-algebraic approximation (and more generally, for W-systems), we refer to [23, Theorem 1.1]. The proofs in these papers are given for the filtration \( \{m^j\} \), but the generalization to \( \{I_j\} \) is done as in lemma 2.7.

2.4.3 The \( C^\infty \)-case

The ring \( C^\infty(\mathbb{R}^p, 0) \) is not Noetherian and has no Artin approximation property, due to the ideal \( m^\infty \) of flat functions, whose Taylor series are identically zero. For example, consider the equation \( x_1 \cdot y = \tau(x) \), here \( \tau(x) \in m^\infty \), but \( \tau(x) \not\in (x_1) \). The completion of this equation has the solution, \( y = 0 \), but the equation has no continuous solutions.

Yet, for \( \mathbb{R} \)-analytic equations there exists a stronger approximation property by \( C^\infty \)-functions. (The Taylor series of the smooth solution does not only approximate the formal solution but coincides with it):

Theorem 2.10 [63, Theorem 1.2] Let \( F(x, y) \in \mathbb{R}\{x, y\}^{\infty} \) and suppose the equation \( F(x, y) = 0 \) has a formal solution, \( \hat{y}(x) \in \mathbb{R}\{[x]\}^{\infty} \). Then there exists a smooth solution \( y(x) \in (C^\infty(\mathbb{R}^p, 0))^{\infty} \), whose Taylor series is equal to \( \hat{y} \).

For \( C^\infty \)-equations there is another approximation result:

Theorem 2.11 [9, Theorem 5.3] Let \( F(x, y) \in C^\infty(\mathbb{R}^p \times \mathbb{R}^n, 0) \) and suppose the m-completion \( \hat{F}(x, y) = 0 \) has a formal solution \( \hat{y}_0(x) \). Take a germ \( y_0 \in (C^\infty(\mathbb{R}^p, 0))^{\infty} \), whose Taylor series is \( \hat{y}_0(x) \). Suppose \( \det \left[ F'(x, y_0)(F'(x, y_0))^t \right] \cdot m^\infty = m^\infty \). Then there exists a \( C^\infty \)-solution, \( y(x) \in C^\infty(\mathbb{R}^p, 0)^{\infty} \), \( F(x, y(x)) \equiv 0 \), whose Taylor series at the origin is precisely \( \hat{y}_0(x) \).

For the case of a general filtration, \( \{I_j\} \), and for more statements of Artin-Tougeron type, see [B.B.K.19a].

2.4.4 Transition from order-by-order determinacy to G-determinacy

We want to pass from the stage “\( \overline{Gz} \supseteq \{z\} + M_{N+1} \)” to the stage “\( Gz \supseteq \{z\} + M_{N+1} \)”.

Equations (24) are often non-polynomial, thus one cannot use Artin approximation directly. However, with some natural weak assumptions one can ensure ordinary solutions.

Below \( k \) is a ring, and \( R \subseteq k[[x]]/J \) is a \( k \)-subalgebra that contains the images of \( \{x_j\} \). Let \( M \) be a finitely presented \( R \)-module, with a filtration \( \{M_j\} \) satisfying: \( M_{N_j} \subseteq (x)^j \cdot M \) for any \( j \) and some \( N_j \not< \infty \). Then a subgroup \( G \subseteq GL_R(M) \times Aut_k(R) \) acts on \( M \).

Proposition 2.12 Suppose \( G \subseteq GL_R(M) \times Aut_k(R) \) is defined by power series, in the sense of condition ii. of the explicit form (23) at the end of Sect. 2.3. Suppose \( \overline{Gz} \supseteq \{z\} + M_{N+1} \), i.e., \( z \in M \) is order-by-order \( N \)-determined.

1. The \( m \)-adic completions satisfy: \( \widehat{Gz} \supseteq \{z\} + \widehat{M}_{N+1} \), i.e., \( z \) is formally \( N \)-determined.
2. Suppose $R$ has the Artin approximation property, $J \subset \mathbb{k}[[x]]$ has polynomial generators, and $G$ is defined by polynomial equations. Suppose also that the $G$-action preserves $M_{N+1}$ and $z \in M$ is polynomially presented modulo $M_{N+1}$ (i.e., there exists a presentation $\mathbb{R}^q \overset{A}{\to} \mathbb{R}^p \overset{\alpha}{\to} M \to 0$ with $A \in \text{Mat}_{p \times q}(\mathbb{k}[x]/J)$, and an approximation $\tilde{z}_{pol} \in (\mathbb{k}[x]/J)^p$ such that $\alpha(\tilde{z}_{pol}) - z \in M_{N+1}$). Then $Gz \supseteq \{z\} + M_{N+1}$, i.e., $z$ is $N$-determined.

3. Let $R = \mathbb{k}\{x\}/J$, with $\mathbb{k}$ a complete normed ring. Suppose $G \subseteq GL_R(M) \rtimes Aut_k(R)$ is defined by analytic equations. Then $Gz \supseteq \{z\} + M_{N+1}$, i.e., $z$ is analytically $N$-determined.

4. Let $R = \mathbb{k}\{x\}/J$, with $\mathbb{k}$ a field or a DVR. Suppose $G \subseteq GL_R(M) \rtimes Aut_k(R)$ is defined by algebraic power series equations. Then $Gz \supseteq \{z\} + M_{N+1}$, i.e., $z$ is algebraically $N$-determined.

**Proof** We use notations from Sect. 2.3 and use the equations in (24) to present the condition $z + w = g z$, $w \in M_{N+1}$ as a system of implicit function equations, $F(y) = 0$, for $F(y) \in R[[y]]^s$. Let $z = \alpha(\tilde{z})$ and $w = \alpha(\tilde{w})$.

1. As the equations $F(y) = 0$ are (formal) power series equations, the existence of an order-by-order solution implies a formal solution $y \in m \cdot \hat{R}^q$ by Popescu’s Theorem 2.4.

2. Let us assume first that $z \in M$ admits a polynomial presentation with $A$ and $\tilde{z}$ having polynomials components. Then $F(y)$ is polynomial in $y$ and we can use the Artin Approximation (for the filtration $\{m^l\}$) to get an ordinary solution $y \in m \cdot R^q$. This implies $Gz \supseteq \{z\} + M_{N+1}$.

   The general case, when the components of $\tilde{z}$ are not polynomials, is reduced to the polynomial case as follows. By the assumption $z$ is polynomial modulo $M_{N+1}$, so we fix some $\tilde{z}_{pol} \in (\mathbb{k}[x]/J)^p$ satisfying $\alpha(\tilde{z}_{pol}) - z \in M_{N+1}$. The conditions $z = g \cdot \alpha(\tilde{z}_{pol})$, $g \in G$, are polynomial equations in the entries of $g$. By part (1) these equations are formally solvable, so there exists $\hat{g} \in \hat{G}$ such that $z = \hat{g} \cdot \alpha(\tilde{z}_{pol})$. Thus, by Artin approximation, there exists an ordinary solution, $z = g_0 \cdot \alpha(\tilde{z}_{pol})$, $g_0 \in G$, i.e., $z \in G\alpha(\tilde{z}_{pol})$. Now the $G$-orbits of $z$ and $\alpha(\tilde{z}_{pol})$ coincide and we can continue with $\alpha(\tilde{z}_{pol})$ (using that $G$ preserves $M_{N+1}$).

3. In this case the Eq. (24) are analytic, $F(y) \in R[y]^s$. (Note that the entries of $A$ and of $\tilde{z}$ are in $\mathbb{k}\{x\}/J$.) Thus we apply the analytic approximation (Remark 2.9) to get a solution $y \in R^q$ analytic in $x$.

4. In this case the equations (24) are algebraic, $F(y) \in R[y]^s$. Apply the approximation of remark 2.9.

**Remark 2.13** (i) The assumptions of part 2 are rather weak:

- If $G$ is one of the groups $GL_R(M)$, $Aut_k(R)$, $GL_R(M) \rtimes Aut_k(R)$, or of the groups of Sect. 2.2.3 iv., then its defining equations are polynomial.
- Suppose the submodule $M_{N+1} \subset M$ satisfies $M_{N+1} \supseteq m^n \cdot M$, for $n \gg 1$. Then any element $z \in M$ is polynomial mod $M_{N+1}$, for any presentation. More generally, if $z \in I \cdot M$ then a sufficient condition (for polynomiality mod $M_{N+1}$) is: $M_{N+1} \supseteq m^n \cdot I \cdot M$, for $n \gg 1$.

(ii) The ring $C^\infty(\mathbb{R}^p, 0)$ has no Artin approximation property, see Sect. 2.4.3. Yet, we obtain the relevant approximation result in Sect. 4.5, using the structure of Lie pairs.

(iii) In Sect. 5.4, we study determinacy relative to a germ. Besides equations (24) this involves the condition $\phi(I) = I$, for an automorphism $\phi \in Aut_k(R)$ and some ideal $I \subset R$. This
As in Sect. 2.2.1, the first intersection means that assign to each derivation \(\xi \in \text{Der}_{\mathbb{k}}(R)\) a local coordinate change, i.e. an automorphism \(\phi_{\xi} \in \text{Aut}_{\mathbb{k}}(R)\). However each term \(\exp\) is well defined, even if each \(\frac{\xi^j}{j!}\) is not well defined, unless \(\mathbb{k} \supseteq \mathbb{Q}\). Even if each \(\frac{\xi^j}{j!}\) is well defined, there is no convergence notions for the infinite sum, unless \(R\) is complete. Therefore we are looking for weak replacements of the exponential \(\exp\) and the logarithmic map \(\ln\), preserving the passage \(\text{Der}_{\mathbb{k}}(R) \rightleftharpoons \text{Aut}_{\mathbb{k}}(R)\). The would-be-exponential map should begin as \(\xi \to \mathbb{1} + \xi + \ldots\), and then one should specify the condition on the higher order terms “\(\ldots\)”. We give four versions of this condition, they all mimic the vector field integration, and each is important in particular examples.

Given a pair \((T_{(G^{(i)},M)}, G^{(i)})\) we define the pairs \(\{(T_{(G^{(i)},M)}, G^{(i)})\}_i\):

\[
G^{(i)} := G^{(1)} \cap \text{GL}_{\mathbb{k}}^{(i)}(M) \quad \text{and} \quad T_{(G^{(i)},M)} := T_{(G^{(i)},M)} \cap \text{End}_{\mathbb{k}}^{(i)}(M), \quad i \geq 2.
\]

As in Sect. 2.2.1, the first intersection means that \(g\) and \(g^{-1}\) are of the form \(\mathbb{1} + \phi\) with \(\phi \in \text{End}_{\mathbb{k}}^{(i)}(M)\).

**Definition 3.1** 1. The pair \((T_{(G^{(i)},M)}, G^{(i)})\) is called of Lie type if the following holds:

(i) For any \(\xi \in T_{(G^{(1)},M)}\) there exists \(g \in G^{(1)}\), such that for any \(z \in M\),

\[
\text{if } \text{ord}(\xi \cdot z) < \infty \text{ then } \text{ord}((g - \mathbb{1} - \xi) \cdot z) > \text{ord}(\xi \cdot z).
\]

(ii) For any \(g \in G^{(1)}\) there exists \(\xi \in T_{(G^{(1)},M)}\), such that for any \(z \in M\),

\[
\text{if } \text{ord}((g - \mathbb{1})z) < \infty \text{ then } \text{ord}((g - \mathbb{1} - \xi) \cdot z) > \text{ord}((g - \mathbb{1})z).
\]

2. The pair \((T_{(G^{(1)},M)}, G^{(1)})\) is called of weak Lie type if the following holds:

(i) For any \(\xi \in T_{(G^{(1)},M)}\) there exists \(g \in G^{(1)}\), such that \(\text{ord}((g - \mathbb{1} - \xi) \geq 2\text{ord}(\xi)\).
(ii) For any \( g \in G^{(1)} \) there exists \( \xi \in T_{(G^{(1)}, M)} \), such that \( ord((g - \mathbb{I} - \xi)) \geq 2ord(g - \mathbb{I}) \).

3. The pair \((T_{(G^{(1)}, M)}, G^{(1)})\) is called **pointwise of Lie type** if the following holds:

(i) For any \( \xi \in T_{(G^{(1)}, M)} \) and \( z \in M \) there exists \( g \in G^{(1)} \) satisfying:

\[
if \ ord(\xi \cdot z) < \infty \ then \ ord((g - \mathbb{I} - \xi)z) > \ ord(\xi \cdot z).
\]

(ii) For any \( g \in G^{(1)} \) and \( z \in M \) there exists \( \xi \in T_{(G^{(1)}, M)} \) satisfying:

\[
if \ ord(g - \mathbb{I})z < \infty \ then \ ord((g - \mathbb{I} - \xi)z) > \ ord(g - \mathbb{I})z).
\]

4. The pair \((T_{(G^{(1)}, M)}, G^{(1)})\) is called **pointwise of weak Lie type** if the following holds for any \( i \geq 1 \):

(i) For any \( \xi \in T_{(G^{(i)}, M)} \) and \( z \in M \) there exists \( g \in G^{(i)} \) satisfying:

\[
if \ ord(\xi \cdot z) < \infty \ then \ ord((g - \mathbb{I} - \xi)z) \geq 2ord(\xi) + ord(z).
\]

(ii) For any \( g \in G^{(i)} \) and \( z \in M \) there exists \( \xi \in T_{(G^{(i)}, M)} \) satisfying:

\[
if \ ord(g - \mathbb{I})z < \infty \ then \ ord((g - \mathbb{I} - \xi)z) \geq 2ord(g - \mathbb{I}) + ord(z).
\]

**Remark 3.2**

(i) Being of Lie type implies pointwise Lie type. In most cases “Lie type” implies also “weak Lie type”.

Note that “\( ord(\xi \cdot z) \geq ord(\xi \cdot z) \) for any \( z \in M \)” implies “\( ord(\xi) \geq ord(\xi) \)” The converse does not hold, as for some \( z \) there may be cancelation among the lowest terms in \( \xi \cdot z \) causing \( ord(\xi \cdot z) > ord(\xi) + ord(z) \). Thus in many cases “weak Lie type” is weaker than “Lie type”.

(ii) In the definition of pointwise weak Lie type we ask: “for any \( \xi \in T_{(G^{(i)}, M)} \) exists \( g \in G^{(i)} \)” rather than “for any \( \xi \in T_{(G^{(i)}, M)} \) exists \( g \in G^{(1)} \)” and similarly from \( g \) to \( \xi \). This condition for arbitrary \( i \) is needed for the determinacy theorems in Sect. 4.2.

(iii) Suppose \((T_{(G^{(i)}, M)}, G^{(1)})\) is a pair of pointwise weak Lie type and the following two conditions hold:

- for any \( \xi \in T_{(G^{(i)}, M)}, z \in M \), with \( ord(\xi \cdot z) < \infty \), exists \( \tilde{\xi} \in T_{(G^{(i)}, M)} \) satisfying:

\[
ord(\tilde{\xi} \cdot z - \xi \cdot z) > ord(\xi \cdot z) \quad \text{and} \quad ord(\tilde{\xi} \cdot z) = ord(\tilde{\xi}) + ord(z);
\]

- for any \( g \in G^{(i)}, z \in M \), with \( ord((g - \mathbb{I})z) < \infty \), exists \( \tilde{g} \in G^{(i)} \) satisfying:

\[
ord((\tilde{g} - g) \cdot z) > ord((g - \mathbb{I}) \cdot z)
\]

and

\[
ord((\tilde{g} - g) \cdot z) = ord(\tilde{g} - g) + ord(z).
\]

Then (by direct check), \((T_{(G^{(i)}, M)}, G^{(1)})\) is a pair of pointwise Lie type.

(vi) The classical Lie groups (over \( \mathbb{R}, \mathbb{C} \)) admit the exponential and logarithmic maps

\[
exp, \ln : T_{(G^{(i)}, M)} \cong G^{(1)}, \quad exp(\xi) = \sum_{j=0}^{\infty} \frac{\xi^j}{j!}, \quad ln(g) = \sum_{j=0}^{\infty} (-1)^j \frac{(g - \mathbb{I})^j}{j}.
\]

In our general context these maps do not exist, both because of non-convergence and because of the denominators, e.g. when \( char(k) > 0 \). The natural weaker versions could be of the form

\[
\psi^{(exp)}, \psi^{(ln)} : T_{(G^{(i)}, M)} \cong G^{(1)}, \quad \psi^{(exp)}(\xi) = \mathbb{I} + \xi + F^{ex}(\xi), \quad \psi^{(ln)}(\xi) = (g - \mathbb{I}) + F^{ln}(g - \mathbb{I}).
\]
Here $F^{(exp)}(\xi)$, $F^{(ln)}(g-\mathbb{I})$ should represent the higher order terms in the following sense: for any $\xi \in T_{(G^{(1)},M)}$, $g \in G^{(1)}$ and $z \in M$ holds

$$ord(F^{(exp)}(\xi) \cdot z) \geq 2ord(\xi \cdot z), \quad ord(F^{(ln)}(g-\mathbb{I}) \cdot z) \geq 2ord((g-\mathbb{I})z). \quad (28)$$

Even this condition is too restrictive and does not hold e.g. for the group $Aut_k(R)$. The conditions of Definition 3.1 are further weakening of (27) that yet ensure the needed tight relation $T_{(G^{(1)},M)} \hookrightarrow G^{(1)}$. See [11] for some rings/derivations that admit the full exponential.

Recall that in differential geometry (over $\mathbb{R}$, $\mathbb{C}$) the map $\Psi^{(exp)}$ integrates a vector field to a flow, and is transcendental. Therefore, even if one assumes weaker conditions on $\Psi^{(exp)}$, in most cases one cannot expect $\Psi^{(exp)}$ to be algebraic.

**Example 3.3** (The classical characteristic zero case). Suppose $k \supseteq \mathbb{Q}$ and assume that $G^{(1)} \subseteq GL_k^{(1)}(M)$ and let $T_{(G^{(1)},M)} \subseteq End_k^{(1)}(M)$ be the tangent space. They often admit “order-by-order” exponential and logarithmic maps, i.e., for any $\xi \in T_{(G^{(1)},M)}$ and $g \in G^{(1)}$ we have for $n \gg 1$:

$$\Psi_n^{(exp)} : \xi \rightarrow \mathbb{I} + \xi + \frac{\xi^2}{2!} + \cdots + \frac{\xi^n}{n!} \in G^{(1)} \cdot GL_k^{(n+1)}(M),$$

$$\Psi_n^{(ln)} : g \rightarrow (g - \mathbb{I}) - \frac{(g-\mathbb{I})^2}{2} + \cdots + (-1)^n \frac{(g-\mathbb{I})^n}{n} \in T_{(G^{(1)},M)} + End_k^{(n+1)}(M). \quad (29)$$

Then $(T_{(G^{(1)},M)}, G^{(1)})$ is a pair of Lie type. (The verification of condition (I) in Definition 3.1 is immediate.) The simplest case when this happens is when $G^{(1)}$, $T_{(G^{(1)},M)}$ are complete with respect to their filtrations. Then, instead of the order-by-order maps, we take just the standard exponential and logarithmic maps, $exp, ln : T_{(G^{(1)},M)} \rightarrow G^{(1)}$.

**Example 3.4** (i) Suppose $GL_k^{(1)}(M) = \{\mathbb{I}\} + End_k^{(1)}(M)$, see Sect. 2.2.2. Here one can take just the simplest maps: $\Psi^{(exp)}(\xi) = \mathbb{I} + \xi$ and $\Psi^{(ln)}(g) = g - \mathbb{I}$. The pair $(End_k^{(1)}(M), GL_k^{(1)}(M))$ is then of Lie type.

(ii) If $M$ happens to be a filtered module over a larger filtered ring $R$, and one has $GL_R^{(1)}(M) = \{\mathbb{I}\} + End_R^{(1)}(M)$, then the pair $(End_R^{(1)}(M), GL_R^{(1)}(M))$ is of Lie type with maps $\Psi^{(exp)}$ and $\Psi^{(ln)}$ the same as in (i).

(iii) More generally, suppose $G^{(1)} \subseteq GL_k^{(1)}(M)$ is of the form $\{\mathbb{I}\} + \Lambda$, for some $k$-submodule $\Lambda \subseteq End_k^{(1)}(M)$. Then the same maps, $\xi \mapsto \mathbb{I} + \xi$, $g \mapsto g - \mathbb{I}$, provide the pair of Lie type $(\Lambda, G^{(1)})$.

For this example the maps $\Psi^{(exp)}$ and $\Psi^{(ln)}$ are non-unique, e.g., for $k \supseteq \mathbb{Q}$ we could take the exponential and the logarithmic maps or some of their approximations.

**Example 3.5** Suppose a pair $(T_{(G^{(1)},M)}, G^{(1)})$ is of (pointwise) (weak) Lie type. It follows directly from the conditions of Definition 3.1 that the pairs $(T_{(G^{(1)},M)}, G^{(1)})$ are of (pointwise) (weak) Lie type as well, i.e. the properties in Definition 3.1 hold also for $G^{(1)}$ instead of $G^{(1)}$.

If moreover the pair is given with some maps, $\{\Psi_n^{(exp)}, \Psi_n^{(ln)}\}$, satisfying the relevant conditions of Definition 3.1, then the maps restrict to the filtration,

$$T_{(G^{(1)},M)} \xrightarrow{\Psi_n^{(exp)}} G^{(1)} \cdot GL_k^{(n+1)}(M), \quad G^{(1)} \xrightarrow{\Psi_n^{(ln)}} T_{(G^{(1)},M)} + End_k^{(n+1)}(M).$$
3.2 The pair \((\text{Der}_k^{(1)}(R), \text{Aut}_k^{(1)}(R))\)

We continue part i. and ii. of Sect. 2.2.3. Let \(k\) be a ring, \(R\) be a \(k\)-subalgebra of \(k[[x]]/J\) or of \(C^\infty(\mathbb{R}^p, 0)/J, x = (x_1, \ldots, x_p)\), that contains the images of \([x_i]\). Accordingly we consider the elements of \(R\) as power series or function-germs and denote them \(f(x)\). Take the maximal ideal \(m = \langle x \rangle\). Fix a filtration \(\{I_j\}\) satisfying \(I_j \cdot I_k \subseteq I_{j+k}, I_1 \subseteq m\). This induces the filtration on the module of \(k\)-linear derivations,

\[
\text{Der}_k^{(j)}(R) := \text{Der}_k(R) \cap \text{End}_k^{(j)}(R) = \{\xi \in \text{End}_k(R) \mid \xi(ab) = \xi(a)b + a\xi(b), \forall a : \xi(I_j) \subseteq I_{j+i}\}.
\]

We take the module \(\text{Der}_k^{(1)}(R)\) as the tangent space for the (topologically unipotent) group \(\text{Aut}_k^{(1)}(R)\).

3.2.1 The case \(k \supseteq \mathbb{Q}\)

**Theorem 3.6** Let \(k\) be a commutative ring, \(k \supseteq \mathbb{Q}\).

1. If \(R\) is complete with respect to filtration \(\{I_j\}\) then the pair \((\text{Der}_k^{(1)}(R), \text{Aut}_k^{(1)}(R))\) is of Lie type.
2. Suppose \(R\) is one of \(k(x)/J, k[x]/J, \) or \(C^\infty(\mathbb{R}^p, 0)/J\). Suppose the filtration \(\{I_j\}\) satisfies: \(\text{Der}_k^{(1)}(R)(x) \subseteq \langle x \rangle^2\) and \(I_{N+1} \cdot \text{Der}_k(R) \subseteq \text{Der}_k^{(1)}(R)\) for \(N \gg 1\). For \(R = C^\infty(\mathbb{R}^p, 0)\) we assume also:
   - \(J\) is analytically generated;
   - \(\cap_{j=1}^\infty I_j \supseteq m^\infty\) and \(\{I_j\}\) are analytically generated mod \(m^\infty\).

Then the pair \((\text{Der}_k^{(1)}(R), \text{Aut}_k^{(1)}(R))\) is pointwise of Lie type.

**Proof** 1. As \(R\) is complete with respect to \(\{I_j\}\), and \(k \supseteq \mathbb{Q}\), we can use the classical exponential and logarithmic maps,

\[
\xi \rightarrow \exp(\xi) := \sum_{j=0}^\infty \frac{\xi^j}{j!}, \quad g \rightarrow \ln(g) := \sum_{j=0}^\infty \frac{(-1)^{j+1}(g - \mathbb{I})^j}{j}.
\] (30)

Note that \(\exp(\xi), \ln(g)\) are well defined self-maps of \(R\), because \(R\) is complete and \(\xi, (g - \mathbb{I})\) are topologically nilpotent.

The map \(\exp(\xi) \circ R\) is \(k\)-linear and topologically unipotent. Moreover, it is multiplicative (by repeatedly applying the Leibniz rule for \(\xi\)). Finally, this map is invertible, as \(\exp(-\xi) \cdot \exp(\xi) = 1d = \exp(\xi) \cdot \exp(-\xi)\). Hence \(\exp(\xi) \in \text{Aut}_k^{(1)}(R)\).

Similarly, \(\ln(g) \circ R\) is topologically nilpotent, \(k\)-linear. Moreover, it satisfies the Leibniz rule (by repeatedly applying the multiplicativity of \(g\)). Therefore \(\ln(g) \in \text{Der}_k^{(1)}(R)\).

Finally we check the conditions of Definition 3.1. We get immediately:

(i) If \(\text{ord}(\xi \cdot f) < \infty\) then \(\text{ord}(\xi \cdot f) < \text{ord}(\exp(\xi) - \mathbb{I} - \xi f)\);
(ii) If \(\text{ord}((g - \mathbb{I}) \cdot f) < \infty\) then \(\text{ord}((g - \mathbb{I} - \ln(g)) \cdot f) < \text{ord}((g - \mathbb{I} - \ln(g)) \cdot f)\).

Therefore the maps of Eq. (30) equip the pair \((\text{Der}_k^{(1)}(R), \text{Aut}_k^{(1)}(R))\) with the Lie-type structure.
2. First we record the useful form of the maps \( \exp(\xi) \), \( \ln(g) \) of Eq. (30) in the particular case, \( R = \mathbb{k}[[x]]/J \). For any power series \( f(x) = \sum a_m x^m \) and any \( \xi \in \text{Der}^{(1)}_k(R) \), \( g \in \text{Aut}^{(1)}_k(R) \), we have (by part 1):

\[
\exp(\xi)(f(x)) = \sum \exp(\xi)(a_m x^m) = \sum a_m \left( \exp(\xi)(x) \right)^m = f(\exp(\xi)(x)),
\]

\[
\ln(g)(f(x)) = \sum_i \ln(g)(x_i) h_i f(x).
\]

(i) We check condition 3.i. of Definition 3.1. Let \( f \in R \) and \( \xi \in \text{Der}^{(1)}_k(R) \) such that \( \text{ord}(\xi \cdot f) < \infty \). Fix \( N \geq 1 \) satisfying:

\[
I_{N+1} \cdot \text{Der}_k(R) \subseteq \text{Der}^{(1)}_k(R), \quad N > \text{ord}(\xi \cdot f),
\]

\[
I_{N+1} \cdot \text{Der}_k(R)(f) \subseteq I_{\text{ord}(\xi \cdot f)+1}.
\]

- First consider the case \( J = 0 \). We define \( g \in \text{Aut}^{(1)}_k(R) \) by \( q(x) \to q\left( \sum_{j=0}^{N} \frac{\xi j(x)}{j!} \right) \in R \).

In more detail, write \( h(x) = \sum_{j=1}^{N} \frac{\xi j(x)}{j!} \in R^P \). The map \( q(x) \to q(x + h(x)) \) is \( \mathbb{k} \)-linear and multiplicative. It is invertible, as the equation \( y + h(y) = x \) is resolvable over \( R \). (Note that \( h(y) \in (y)^2 \) and apply IFT.) Therefore \( g \in \text{Aut}_k(R) \).

We claim: \( g \in \text{Aut}^{(1)}_k(R) \). For any \( q(x) \in I_1 \) we should verify: \( q(\sum_{j=0}^{N} \frac{\xi j(x)}{j!}) - q(x) \in I_{1+i} \), i.e. this difference goes to 0 \( \in R/I_{1+i} \). For \( R \) one of \( \mathbb{k}(x) \), \( \mathbb{k}\{x\} \), or \( C^\infty(\mathbb{R}^P, 0) \) this can be checked at the level of Taylor power series of \( q(x) \), i.e. we can take the completion and consider \( q(x) \in \mathbb{k}[[x]] \). (For the case \( R = C^\infty(\mathbb{R}^P, 0) \) we use: \( \cap I_j \supseteq m^\infty \cdot \).

Now, for \( q(x) \in \mathbb{k}[[x]] \) we can use the formal exponent:

\[
q\left( \sum_{j=0}^{N} \frac{\xi j(x)}{j!} \right) = q\left( \exp(\xi)(x) - \sum_{j=N+1}^{\infty} \frac{\xi j(x)}{j!} \right) \in \sum_{j=0}^{N} \frac{\xi j(q)}{j!} + I_{N+1} \text{Der}_k(R)(q).
\]

By the assumption, the right hand side here is contained in \( I_{J \cdot N+1} \cdot \text{Der}^{(i)}_k(R)(q) \). Therefore the map \( q(x) \to q(x + h(x)) \) defines an element \( g \in \text{Aut}^{(1)}_k(R) \).

Finally we compare the orders, as in the condition 3.i. of Definition 3.1. We note, by (31):

\[
(g - I - \xi)(f) = f\left( \sum_{j=0}^{N} \frac{\xi j(x)}{j!} \right) - f(x) - \xi(f) \in \left\{ \sum_{j=2}^{N} \frac{\xi j}{j!}(f) \right\} + I_{N+1} \cdot \text{Der}_k(R)(q).
\]

This follows by checking the Taylor series, as in the Eq. (33). Therefore, for the chosen \( \xi \), \( f \) and the constructed \( g \), holds: \( \text{ord}((g - I - \xi)(f)) > \text{ord}(\xi \cdot f) \). Note that \( g \) depends on \( \xi \) and \( f \), via \( N \).

- For the general case, \( J \neq 0 \), the map \( q(x) \to q\left( \sum_{j=0}^{N} \frac{\xi j(x)}{j!} \right) \) is not well defined, as it does not necessarily preserve \( J \subseteq R \), see Example 3.9. We adjust this map by higher order terms as follows.

Let \( S \) be one of \( \mathbb{k}(x) \), \( \mathbb{k}\{x\} \), \( C^\infty(\mathbb{R}^P, 0) \), and take the full preimages of \( \{I_j\} \) to
get the filtration \( \{ \tilde{I}_j \} \) of \( S \). For \( C^\infty(\mathbb{R}^p, 0) \) one has \( \tilde{I}_j \supseteq J + m^\infty \), for any \( j \), and \( \tilde{I}_j \) are analytically generated \textit{mod}(m^\infty).

For \( \xi \in \text{Der}^{(i)}_{k}(R) \) take its representative \( \tilde{\xi} \in \text{Der}^{(i)}_{k}(S) \) then \( \tilde{\xi}(J) \subseteq J \).

In the \( C^\infty(\mathbb{R}^p, 0) \) case the coefficients of \( \tilde{\xi} = \sum \tilde{\xi}_i \hat{\alpha}_i \) are smooth functions. But there exists an analytic derivation, \( \tilde{\xi} \in \text{Der}^{(i)}_{\mathbb{R}}(\mathbb{R}[x]) \), satisfying: \( \tilde{\xi}(J) \subseteq J \) and \( \tilde{\xi} \in \text{Der}^{(2N)}_{\mathbb{R}}(C^\infty(\mathbb{R}^p, 0)) \). Indeed, fix some analytic generators, \( \{ q_{\alpha}^{(i)} \} \) of \( J \subset S \), and expand \( \tilde{\xi} = \sum \tilde{\xi}_i \hat{\alpha}_i \). Then the condition \( \tilde{\xi}(J) \subseteq J \) amounts to the \( \mathbb{R}[x] \)-linear equations \( \sum \tilde{\xi}_i \hat{\alpha}_i q_{\alpha}^{(i)} = \sum t_{\beta} q_{\beta}^{(i)} \). (Here the unknowns are \( \{ \tilde{\xi}_i \} \), \( \{ t_{\beta} \} \).) These equations posses the formal solution, the image of \( \tilde{\xi} \) under the completion. Therefore, by Artin approximation, there exists an analytic solution, \( \tilde{\xi} \tilde{\xi} \), that approximates \( \tilde{\xi} \), see Remark 2.9. We replace \( \tilde{\xi} \) by this analytic derivation \( \tilde{\xi} \).

Now we construct the map \( \tilde{\phi} \in \text{Aut}^{(i)}_{k}(S) \) by \( q(x) \rightarrow q \left( \sum_{j=0}^{N} \frac{\xi_j(x)}{j!} + \tilde{h}(x) \right) \), where \( \tilde{h}(x) \in \tilde{I}_{N+1} \cdot S^p \) and \( \tilde{\phi}(J) = J \). Fix some (finite) sets of analytic generators, \( \{ q_{\alpha}^{(i)} \} \) of \( J \), and \( \{ q_{\beta}^{(i)} \} \) of \( I_{N+1} \). Expand \( \tilde{h}(x) \) as \( \sum t_{\beta} q_{\beta}^{(i)} \), for the new variables \( \{ t_{\beta} \} \). Then the condition \( \tilde{\phi}(J) = J \) amounts to:

\[
q_{\alpha}^{(J)} \left( \sum_{j=0}^{N} \frac{\xi_j(x)}{j!} + \sum_{\beta} t_{\beta} q_{\beta}^{(i)} \right) = \sum_{\beta} w_{\alpha\beta} q_{\beta}^{(J)}, \quad \forall \alpha. \tag{35}
\]

This is a finite system of implicit function equations on the variables \( \{ w_{\alpha\beta} \} \), \( \{ t_{\beta} \} \). The equations are algebraic/analytic power series. (Also in the \( C^\infty \) case!)

Thus, by the relevant approximation property, it suffices to demonstrate a formal solution. Note that \( \sum_{j=N+1}^{\infty} \frac{\xi_j(x)}{j!} \in I_{N+1} \), thus we can take \( \{ t_{\beta} \} \) satisfying \( \sum_{j=N+1}^{\infty} \frac{\xi_j(x)}{j!} = 0 \). This gives the formal solution,

\[
q_{\alpha}^{(J)} \left( \sum_{j=0}^{N} \frac{\xi_j(x)}{j!} + \sum_{\beta} t_{\beta} q_{\beta}^{(i)} \right) = q_{\alpha}^{(J)}(\exp(\tilde{\xi})(x)) = \exp(\tilde{\xi})(q_{\alpha}^{(i)}(x)) = \sum_{\beta} w_{\alpha\beta} q_{\beta}^{(J)} \in J \cdot \tilde{\xi}, \quad \forall \alpha. \tag{36}
\]

Now, by approximation, we get the needed analytic/algebraic solution.

Altogether, we have constructed a coordinate change \( \phi_{\xi} : q(x) \rightarrow q \left( \sum_{j=0}^{N} \frac{\xi_j(x)}{j!} + \tilde{h}(x) \right) \) that preserves \( J \subset S \). It descends to the element \( \phi_{\tilde{\xi}} \in \text{Aut}_{k}(R) \). As in the case \( J = 0 \), one checks that \( \phi_{\tilde{\xi}} \) is topologically unipotent and \( \text{ord}((\phi_{\tilde{\xi}} - I - \tilde{\xi})f) > \text{ord}(\xi \cdot f) \).

ii. (We check condition 3.ii. of Definition 3.1.) Let \( f \in R \) and \( g \in \text{Aut}^{(i)}_{k}(R) \) such that \( \text{ord}((g - \tilde{I})f) < \infty \). Fix \( N \gg 1 \) satisfying:

\[
I_{N+1} \cdot \text{Der}_{k}(R) \subseteq \text{Der}^{(i)}_{k}(R), \quad N > \text{ord}((g - \tilde{I})f),
\]

\[
I_{N+1} \cdot \text{Der}_{k}(R)(f) \subseteq I_{\text{ord}((g - \tilde{I})f)+1}. \tag{37}
\]
First assume $J = 0$. We define $\xi \in Der^{(i)}_k(R)$ by

$$\xi(q(x)) = \sum_{j=1}^{N} \frac{(-1)^{j+1}(g - 1)^j(x)}{j} \cdot q'(x)$$

$$= \sum_{j=1}^{N} \sum_{i} \frac{(-1)^{j+1}(g - 1)^j(x_i)}{j} \cdot \frac{\partial q(x)}{\partial x_i}. \tag{38}$$

This is a derivation by construction. (Note that $k(x)$ is closed under differentiation.

To check the topological nilpotence we should verify: if $q(x) \in I_l$ then $\xi(q(x)) \in I_{l+i}$, i.e. $\xi(q(x))$ goes to $0 \in R/I_{l+i}$. In the same way as in the verification of topological unipotence, we can pass to completion and use the formal relation $ln(g)q(x) = ln(g)(x) \cdot q'(x)$, of Eq. (31). (For $C^\infty$ case we use $\cap I_x \supseteq m^\infty$, as before.) Therefore

$$\sum_{j=1}^{N} \frac{(-1)^{j+1}(g - 1)^j(x)}{j} \cdot q'(x) \in End^{(i)}_k(R)(q) + I_{N+1} \cdot Der_k(R)(q). \tag{39}$$

Finally, for the condition 3.ii. of Definition 3.1 we note, using Eq. (30):

$$(g - \mathbb{I} - \xi)f = \sum_{j=2}^{N} \frac{(-1)^{j+1}(g - 1)^j(x)}{j} f'(x)$$

$$= \ln(g) f' - (g - \mathbb{I}) f - \sum_{j=N+1}^{\infty} \frac{(-1)^{j+1}(g - 1)^j(f)}{j} \tag{40}.$$

And the later belongs to $\left\{ln(g) f' - (g - \mathbb{I}) f \right\} + I_{N+1}$ i.e. to $\left\{\sum_{j=2}^{N} \frac{(-1)^{j+1}(g - 1)^j}{j} (f) \right\} + I_{N+1}$.

Thus we have constructed $\xi \in Der^{(i)}_k(R)$ satisfying $ord((g - \mathbb{I} - \xi)f) > ord((g - \mathbb{I}) f)$. Note that $\xi$ depends on $g$, $f$ (via $N$).

For the general case, $J \neq \{0\}$, we cannot use $\left( \sum_{j=1}^{N} \frac{(-1)^{j+1}(g - 1)^j(x)}{j} \right) q'(x)$ as this does not necessarily preserve $J$. Thus we adjust this derivation,

$$\xi(q(x)) := \left( \sum_{j=1}^{N} \frac{(-1)^{j+1}(g - 1)^j(x)}{j} + h(x) \right) q'(x). \tag{41}$$

Here $h \in I_{N+1}$ is chosen to ensure $\xi(J) \subseteq J$.

First we provide a formal solution for this condition, $\hat{h}(x) = \sum_{j=N+1}^{\infty} \frac{(-1)^{j+1}(g - 1)^j(x)}{j}$. Now as in the step 2.i., for the case $R = k(x)/J$, $k[x]/J$ we use the relevant approximation directly. For $C^\infty$-case we first modify $\xi$ by high enough terms to ensure that it is an analytic/polynomial derivation.

Thus the needed $h(x) \in R$ exists in all the cases and we get $\xi \in Der^{(i)}_k(R)$. The condition $ord\left((g - \mathbb{I} - \xi)f\right) > ord((g - \mathbb{I}) f)$ is checked as before.
Example 3.7 As a special case of Theorem 3.6 we get: Let $\mathbb{k} \supseteq \mathbb{Q}$ (e.g. $\mathbb{k}$ a field of characteristic 0) and $R$ one of $\mathbb{k}[x]/J$, $\mathbb{k}[x]/J$, $\mathbb{k}[x]/J$ and $C^\infty(\mathbb{R}^p, 0)/J$ ($J$ analytically generated), with filtration \{m/\}j. Then the pair $(\text{Der}_{\mathbb{k}}^{(1)}(R), \text{Aut}_{\mathbb{k}}^{(1)}(R))$ is of pointwise Lie type.

Remark 3.8 The constructions in part 2 of this proof are somewhat involved, for various reasons.

(i) We could not use the full power series $\exp(\xi)$, $\ln(g)$, as they do not always act on rings like $\mathbb{k}[x]$ and $C^\infty(\mathbb{R}^p, 0)$. For example, for any derivation $D$ there exists a smooth function $f$ such that $(\sum \frac{D^j}{j!}) f$ diverges at each point off the origin, [11]. Thus we cannot define the action $\exp(D) \cap C^\infty(\mathbb{R}^p, 0)$ via $\exp(D) = \sum \frac{D^j}{j!}$.

(ii) Let $\xi \in \mathbb{k}$ and $\mathbb{k}$ analytically generated), such that $\ln(\xi)$ is not necessarily multiplicative, and for small $n$ not necessarily invertible. For example, let $R = \mathbb{C}[x]$ and $\xi = x^2 \partial_x$. Then the solution to $(1 + \xi)y = x$ is $y = \sum n!(1 - 1)^n x^{n+1}$, which is diverging off the origin. Therefore instead of this map, we had to use the map $f \mapsto f(\Psi_n^{(exp)}(\xi)(x))$, for $\Psi_n^{(exp)}$ as above, and then to adjust it in the case of $R = S/J$.

Remark 3.9 Another problem is due to non-regular rings. Let $\mathbb{k} \supseteq \mathbb{Q}$ and $R = \mathbb{k}[[x, y]]/(x^p - y^q)$, $p < q$, filtered by \{m/\}.

(i) Let $\xi = x(q \cdot x \partial_x + p \cdot y \partial_y)$, so that $\xi \cdot (x^p - y^q) \in m \cdot (x^p - y^q)$. Thus $\xi \in \text{Der}_{\mathbb{k}}^{(1)}(R)$.

However, for $f(x, y) = x^p - y^q$,

$$f((x, y) + \xi(x, y)) = f(x + qx^2, y + px y) = x^p (1 + qx)^p - y^q (1 + px)^q \notin (x^p - y^q).$$

Thus $f(x, y) \mapsto f((x, y) + \xi(x, y))$ is not an automorphism of $R$.

(ii) Let $g : (x, y) \mapsto (x + y^q, uy)$, where $u \in \mathbb{k}[[x, y]]$ is defined by the condition $u^q = 1 + \frac{(x + y^q)^p - x^p}{y^q}$, $u(0, 0) = 1$. Then $g(f) \in (f)$ thus $g \in \text{Aut}_{\mathbb{k}}^{(1)}(R)$. But $\xi := \sum (g - \mathbb{I})(x) \partial_i$ is not a derivation of $R$, as $\xi(f) = y^q(-px^{p-1} + (u - 1)q) \notin (f) \in \mathbb{k}[[x, y]]$.

Remark 3.10 As one sees from the proof, the statement of part 2 of Theorem 3.6 holds for more general rings. We only use in the proof the following: $R$ is a ring over a field of zero characteristic, $R$ is closed under differentiation, and the approximation property in $R$ holds for the particular class of equations over $R$ that we need to resolve. For instance, our proof works when $R = W_n/J$, where $\{W_n\}_{n \geq 1}$ is either a Weierstrass system over $\mathbb{k}$, see [23, Definition of page 2], or a convergent Weierstrass system over $\mathbb{R}$, [64, page 798]. (The filtration is \{m/\}.)

3.2.2 The pair $(\text{Der}_{\mathbb{k}}^{(1)}(R), \text{Aut}_{\mathbb{k}}^{(1)}(R))$ for the case of arbitrary $\mathbb{k}$

In the previous Sect. 3.2.1 we considered the case $\mathbb{k} \supseteq \mathbb{Q}$ and proved conditions for pointwise Lie type in Theorem 3.6. Now we do not assume that the ring $\mathbb{k}$ contains $\mathbb{Q}$ and establish conditions to guarantee weak Lie type under certain Assumptions 3.11.
Let $R$ be a filtered subring of $\mathbb{k}[[x]]/J$, $x = (x_1, \ldots, x_p)$. We assume that the filtration satisfies $I_1 \subseteq m = (x)$ and $\text{Der}_{\mathbb{k}}^{(1)}(R)(x) \subseteq (x)^2$. Moreover, in this section we make the following assumptions about $R$ and the filtration:

**Assumptions 3.11**

(i) Elements of $\text{Der}_{\mathbb{k}}^{(1)}(R)$ induce coordinate changes: for any $\xi \in \text{Der}_{\mathbb{k}}^{(1)}(R)$ there exists $h^{(\xi)}(x) = (h_1(x), \ldots, h_p(x))$, with $\text{ord}(h_i(\xi)) > 2\text{ord}(\xi(x_i)))$, such that $f(x) \to f(x + \xi(x) + h(x))$ is a well defined self-map of $R$, i.e. maps $J$ to $J$.

(ii) Elements of $\text{Aut}_{\mathbb{k}}^{(1)}(R)$ induce derivations: for any $g \in \text{Aut}_{\mathbb{k}}^{(1)}(R)$ there exists $h^{(g)}(x) = (h_1(x), \ldots, h_p(x))$, with $\text{ord}(h_i^{(g)}(x)) > 2\text{ord}((g - \mathbb{I})(x_i))$, such that $\sum_i ((g - \mathbb{I})x_i + \xi)^{h_i^{(g)}} \partial_i \in \text{Der}^{(1)}_{\mathbb{k}}(R)$.

(iii) $R$ admits Taylor expansion up to second order for coordinate changes, i.e., for any $\xi \in \text{Der}_{\mathbb{k}}^{(1)}(R)$ and a corresponding $h^{(\xi)}(x)$ and any $f(x) \in R$ holds $f(x + \xi(x) + h^{(\xi)}(x)) = f(x) + \xi \cdot f(x) + F(\xi, f, x)$, where $F$ satisfies $F(\text{Der}_{\mathbb{k}}^{(1)}(R), I_i, x) \subseteq I_{2j+i}$ for any $i \geq 1, j \geq 1$.

(iv) $IFT_1$ holds over $R$, i.e., any equation $y + F(y) = x$, with $F(y) \in (y)^2$ is solvable for $y$ over $R$.

**Example 3.12** These assumptions hold for the rings of Theorem 3.6. (Assumption i. is verified after Eq. (37) and assumption ii. is verified after Eq. (41), and $IFT_1$ holds for those rings.)

We verify the assumptions also for $R$ one of $\mathbb{k}[[x]]/J$, $\mathbb{k}(x)/J$, $\mathbb{k}(x)/J$, where $J \subseteq m^2$ and $\mathbb{k}$ is a (complete normed) ring.

(i) Condition i. holds trivially when $J = 0$ (i.e. regular rings), with any filtration. One just takes $h^{(\xi)} = 0$. If $J \neq 0$ then condition i. is non-trivial and the non-zero correction $h^{(\xi)}$ is needed, see Remark 3.9.

(ii) Condition ii. holds trivially when $J = 0$ (one takes $h^{(g)} = 0$) for the filtrations $\{m^j\}$ or $I_j = a^j \cdot I_1$, $I_1 \subseteq a^2 \subseteq m^2$. More generally, a sufficient condition on the filtration is: $(I_1 \cap (x)^2) \partial I_j \subseteq I_{j+1}$. If $J \neq 0$ then condition ii. is non-trivial and the non-zero correction $h^{(g)}$ is needed, see Remark 3.9.

(iii) Condition iii. holds for our rings $\mathbb{k}[[x]]/J$, $\mathbb{k}(x)/J$, $\mathbb{k}(x)/J$, see Example 2.2.

(iv) Condition iv. holds for filtrations $\{I_j = m^j\}$ and $\{I_{j+1} = a^j \cdot I_1\}$, with $I_1 \subseteq a^2 \subseteq m$. More generally, it holds when the filtration $\{I_j\}$ satisfies $I_j \cdot I_{j+1} \subseteq \text{Der}_{\mathbb{k}}(R)(\text{Der}_{\mathbb{k}}(R)(I_j)) \subseteq I_{2j+i}$. In particular, Assumptions 3.11 hold for $\mathbb{k}[[x]]$, $\mathbb{k}(x)$, $\mathbb{k}(x)$ and for the filtrations $\{m^j\}$ or $I_{j+1} = a^j \cdot I_1$, $I_1 \subseteq a^2 \subseteq m^2$.

**Example 3.13** Here are more examples of rings satisfying the assumptions, for the filtration $\{m^j\}$. Let $\mathbb{k}$ be either a field or a discrete valuation ring, and $\{W_n\}_{n \geq 1}$ a Weierstrass system over $\mathbb{k}$. Then, for any integer $n \geq 1$, the ring $W_n$ satisfies 3.11.

Indeed, $R$ is closed under compositions, see [23, page 3], and admits Taylor expansions up to second order, see [23, page 4]. The $IFT_1$ holds over $R$, see Example 2.2. Finally, $R$ is a $\mathbb{k}$–subalgebra of $\mathbb{k}[[x]]$, therefore the topologically unipotent $\mathbb{k}$–automorphisms of $R$ come as coordinate changes. Thus condition iv. holds for $R$. 

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Proposition 3.14 Suppose $R$ satisfies Assumptions 3.11. Then the pair $(\text{Der}_k^{(1)}(R),\text{Aut}_k^{(1)}(R))$ is of weak Lie type.

Proof (i) (Verifying condition 2.i. of Definition 3.1)
For any $\xi \in \text{Der}_k^{(1)}(R)$ we take the corresponding coordinate change, $f(x) \rightarrow f(x + \xi(x) + h^{(\xi)}(x))$. This is a well defined (set-theoretic) self-map of $R$, by assumption i. It is $k$-linear and multiplicative by construction. It is invertible by $IFT_1$. Thus we get an element $g_\xi \in \text{Aut}_k(R)$.

By Taylor-expansion (Assumption 3.11 iii.), we have: $f(x + \xi(x)) = f(x) + \xi \cdot f(x) + F(\xi, f, x)$. Here, for $f \in I_i, \xi \in \text{Der}_k^{(j)}(R)$ we have: $F(\xi, f, x) \subseteq I_{i+2j}$, thus $g - I - \xi \in \text{End}_k^{(2j)}(R)$. In particular, $g_\xi$ is topologically unipotent and $\text{ord}(g - I - \xi) \geq 2\text{ord}(\xi)$.

(ii) (Verifying condition 2.ii. of definition 3.1) To any $g \in \text{Aut}_k^{(1)}(R)$ we associate $\xi_g \in \text{Der}_k^{(1)}(R)$ by
\[
\xi_g \cdot f(x) := \sum ((g - I)(x_i) + h_i^{(g)}(x_i)) \partial_i f(x).
\]
(43)

This is a derivation, $\xi \in \text{Der}_k^{(1)}(R)$, by Assumption 3.11 ii.

Now we check:
\[
(g - I)f(x) - \xi_g \cdot f(x) = f(x + (g - I)(x)) - f(x) - \xi_g \cdot f(x)
= F((g - I)(x), f, x) - \sum h_i^{(g)}(x) \partial_i f.
\]
(44)

Thus, by Assumption 3.11 ii., $\text{ord}((g - I - \xi_g)f) \geq 2\text{ord}((g - I)f)$. As this holds for any $f$ we get: $\text{ord}(g - I - \xi_g) \geq 2\text{ord}(g - I)$.

\[\square\]

Example 3.15 For $k$ a field, the rings $k[[x]], k\{x\}, k(x)$ as well as Weierstrass systems and functions of the Denjoy Carleman class, with filtration $\{m^j\}$ or $\{a^j\}$, $I_1 \subseteq a^2 \subseteq m^2$, satisfy the Assumptions 3.11. Therefore for these rings the pair $(\text{Der}_k^{(1)}(R), \text{Aut}_k^{(1)}(R))$ is of weak Lie type.

3.2.3 The subgroup $\text{Aut}_{k,a}(R) \subset \text{Aut}_k(R)$

Sometimes one considers only those automorphisms of the ring that preserve a given ideal $a$,
\[
\text{Aut}_{k,a}(R) := \{\phi \in \text{Aut}_k(R) | \phi(a) = a\}.
\]
(45)

Accordingly, one considers the module of $a$-logarithmic derivations (also known as the module of $a$-preserving derivations [19, Definition 2.1.1]), $\text{Der}_{k,a}(R) := \{\xi \in \text{Der}_k(R) | \xi(a) \subseteq a\}$. This is an $R$-submodule of $\text{Der}_k(R)$.

As before, we get the induced filtrations of the group $\{\text{Aut}_{k,a}^{(1)}(R)\}$ and of the module $\{\text{Der}_{k,a}^{(1)}(R)\}$.

Proposition 3.16 Suppose $k \supseteq \mathbb{Q}$.

1. If $R = k[[x]]/J$ then $(\text{Der}_{k,a}^{(1)}(R), \text{Aut}_{k,a}^{(1)}(R))$ is a pair of Lie type for any filtration.
2. Suppose $R$ is one of $k[[x]], k[x]/J$, or $C\infty(\mathbb{R}^p, 0)/J$. Suppose the filtration $\{I_j\}$ satisfies: $\text{Der}_{k,a}^{(1)}(R)(x) \subseteq (x)^2$ and $I_{N+1} \cdot \text{Der}_k(R) \subseteq \text{Der}_{k}^{(1)}(R)$ for $N \gg 1$. For $R = C\infty(\mathbb{R}^p, 0)$ we assume also:
• J, α are analytically generated;
• ⋂Ij ⊇ m∞ and {Ij} are analytically generated mod m∞.

Then the pair \((\text{Der}^{(1)}_{k,α}(R), \text{Aut}^{(1)}_{k,α}(R))\) is pointwise of Lie type.

The proof is the same as for Theorem 3.6. The correspondence \(\text{Der}^{(1)}_{k,α}(R) \ni ξ \mapsto g ∈ \text{Aut}^{(1)}_{k,α}(R)\) is defined in the same way, one just imposes the condition that the images preserve α. (This results in algebraic, resp. analytic equations).

Remark 3.17 (i) The natural homomorphism of groups \(\text{Aut}_{k,α}(R) \rightarrow \text{Aut}_k(R/α)\) is not necessarily injective or surjective. Neither is the natural map \(\text{Der}_{k,α}(R) \rightarrow \text{Der}_k(R/α)\), though it is known to be surjective in some particular cases, e.g., for \(R\) one of \(\Bbbk[x], \Bbbk[[x]]\), [19, Lemma 2.1.2], [51, p. 2717] or when \(R\) is a regular local ring of zero characteristic with some technical conditions (see [49, Theorems 30.6 and 30.8] or [38, Theorem 2.1]). Therefore the determinacy problem for the pair \((\text{Der}^{(1)}_{k,α}(R), \text{Aut}^{(1)}_{k,α}(R))\) is in general not reduced to the pair \((\text{Der}^{(1)}_k(R/α), \text{Aut}^{(1)}_k(R/α))\). Thus we need this special proposition.

(ii) The analogue of Proposition 3.16 for weak Lie type is more technical. We need to assume the conditions 3.11 for \(R\) and for \(R/α\), and moreover some lifting properties. Therefore we omit it.

3.3 Constructing new Lie pairs from old ones

3.3.1 Diagonal action

Suppose a pair of Lie type (resp. pointwise and/or weak Lie type) acts on several modules. Namely, we have a collection of (not necessarily injective) homomorphisms

\[
\left\{\left(T_{(G^{(1)},M)}, G^{(1)}\right) \stackrel{ψ_α}{\rightarrow} (\text{End}^{(1)}_{k}(M_α), GL^{(1)}_{k}(M_α))\right\}_α,
\]

such that the conditions of Definition 3.1 are satisfied for the pairs \((ψ_α(T_{(G^{(1)},M)}), ψ_α(G^{(1)}))\)α.

Take the diagonal action, \((T_{(G^{(1)},M)}, G^{(1)}) \circ \oplus_α M_α\), by \((ξ, g) \cdot \{z_α\} := (ψ_α(ξ)z_α),\ (ψ_α(g)z_α))\). This corresponds to the diagonal homomorphism, \((T_{(G^{(1)},M)}, G^{(1)}) \xrightarrow{Δ} \oplus_α (T^{(1)}_{(G^{(1)},M)}, G^{(1)})\).

Lemma 3.18 Suppose each pair \((ψ_α(T_{(G^{(1)},M)}), ψ_α(G^{(1)}))\) is of Lie type (resp. pointwise and/or weak Lie type). Then the same holds for the pair \((Δ(T_{(G^{(1)},M)}), Δ(G^{(1)}))\).

Proof By the assumption to every \(ψ_α(ξ)\) is assigned \(ψ_α(g)\) and vice versa. To verify the (relevant) conditions of Definition 3.1 we should compare the orders of \(ψ_α(g - I - ξ)\) to the orders of \(ψ_α(g - I), ψ_α(ξ)\). (For the pointwise case we should compare the order \(ψ_α(g - I - ξ)(z)\) to the orders of \(ψ_α(g - I)(z), ψ_α(ξ)(z)\).

For each case it is enough to observe: \(ord(ψ_α) = min\{ord(w_α)\}\).

Example 3.19 Suppose the pair \((\text{Der}^{(1)}_k(R), \text{Aut}^{(1)}_k(R))\) is of Lie type (resp. pointwise and/or weak Lie type) for the action on \(R\). Then the same holds for the diagonal actions \(\text{Aut}^{(1)}_k(R) \circ R^{∞n}, \text{Aut}^{(1)}_k(R) \circ \text{Mat}_{m×n}(R)\).
3.3.2 A group generated by two groups

Fix some subgroups \( G^{(1)}, H^{(1)} \subseteq GL_{k}^{(1)}(M) \) and denote the subgroup they generate by \( \langle G^{(1)}, H^{(1)} \rangle \). The Lie-type structures of \( G^{(1)}, H^{(1)} \) often induce the one on \( \langle G^{(1)}, H^{(1)} \rangle \) as we show now.

In this subsection we assume: \( \langle G^{(1)}, H^{(1)} \rangle = G^{(1)} \cdot H^{(1)} \), i.e., any element of \( \langle G^{(1)}, H^{(1)} \rangle \) is presentable in the form \( g \cdot h \), for some \( g \in G^{(1)}, h \in H^{(1)} \). (This holds e.g., for semi-direct products.)

**Lemma 3.20** Suppose \( \langle G^{(1)}, H^{(1)} \rangle = G^{(1)} \cdot H^{(1)} \). Suppose moreover that for any \( i \geq 1 \) the following holds:

\[
\begin{align*}
(T_{G^{(i)}, M} + T_{H^{(i)}, M}) \cap \End_{k}^{(i)}(M) &= T_{G^{(i)}, M} + T_{H^{(i)}, M}, \quad (G^{(1)} \cdot H^{(1)}) \cap GL_{k}^{(i)}(M) = G^{(i)} \cdot H^{(i)}. \\
\end{align*}
\]

1. If the pairs \( (T_{G^{(i)}, M}, G^{(1)}) \) and \( (T_{H^{(i)}, M}, H^{(1)}) \) are of (pointwise) weak Lie type, then \( (T_{G^{(i)}, M} + T_{H^{(i)}, M}, G^{(1)}, H^{(1)}) \) is a pair of (pointwise) weak Lie type.

2. Let \( (T_{G^{(i)}, M}, G^{(1)}) \) and \( (T_{H^{(i)}, M}, H^{(1)}) \) be pairs of pointwise Lie type. Suppose that the following holds: for any \( z \in M \) and any \( u \in T_{G^{(i)}, M} + T_{H^{(i)}, M} \) there exist \( \xi \in T_{G^{(i)}, M} \), \( \eta \in T_{H^{(i)}, M} \), and \( g \in G^{(i)}, h \in H^{(i)} \), such that \( u = \xi + \eta \) and

\[
\ord(u \cdot z) < \min \left\{ \ord((g - I - \xi)z), \ord((h - I - \eta)z), \ord((g - I - \xi)\eta \cdot z), \ord(\xi \cdot \eta \cdot z) \right\}.
\]

Then \( (T_{G^{(i)}, M} + T_{H^{(i)}, M}, G^{(1)} \cdot H^{(1)}) \) is a pair of pointwise Lie type.

3. Let \( (T_{G^{(i)}, M}, G^{(1)}) \) and \( (T_{H^{(i)}, M}, H^{(1)}) \) be pairs of Lie type. Suppose the following holds: for any \( u \in T_{G^{(i)}, M} + T_{H^{(i)}, M} \) there exist \( \xi \in T_{G^{(i)}, M} \), \( \eta \in T_{H^{(i)}, M} \), and \( g \in G^{(i)} \), \( h \in H^{(i)} \), such that \( u = \xi + \eta \) and for any \( z \in M \) holds:

\[
\ord(u \cdot z) < \min \left\{ \ord((g - I - \xi)z), \ord((h - I - \eta)z), \ord((g - I - \xi)\eta \cdot z), \ord(\xi \cdot \eta \cdot z) \right\}.
\]

Then \( (T_{G^{(i)}, M} + T_{H^{(i)}, M}, G^{(1)} \cdot H^{(1)}) \) is a pair of Lie type.

Note that the long condition in part 2. is a weakening of the conditions

\[
\begin{align*}
(T_{G^{(i)}, M}z + T_{H^{(i)}, M}z) \cap M_j &= \left( T_{G^{(i)}, M}z \cap M_j \right) + \left( T_{H^{(i)}, M}z \cap M_j \right), \\
(G^{(1)} \cdot H^{(1)}z - \{z\}) \cap M_j &= \left( (G^{(1)}z - \{z\}) \cap M_j \right) + \left( (H^{(1)}z - \{z\}) \cap M_j \right). \quad (47)
\end{align*}
\]

**Proof** 1. For the pairs of weak Lie type, take \( u \in T_{G^{(i)}, M} + T_{H^{(i)}, M} \) and choose (any) presentation \( u = \xi + \eta \) such that \( \ord(u) \leq \ord(\xi), \ord(\eta) \).

By the assumption of weak Lie pairs we fix \( g \in G^{(1)}, h \in H^{(1)} \) satisfying: \( \ord(g - I - \xi) \geq 2\xi, \ord(h - I - \eta) \geq 2\eta \). Finally we present

\[
gh - I - \xi - \eta = (g - I - \xi)(h - I - \eta) + (g - I - \xi) + (h - I - \eta) + \xi \cdot \eta.
\]

\[
\begin{align*}
&= (g - I - \xi)\eta + \xi(h - I - \eta) + \xi \cdot \eta.
\end{align*}
\]
Thus we get \( ord(gh - I - \xi - \eta) \geq 2ord(u) \), i.e., the bound of part 3.i. of Definition 3.1.

For the bound of part ii. we take any element \( a \in G^{(1)} \cdot H^{(1)} \) and present it as \( a = gh \), with \( ord(g - I) \geq ord(a - I) \leq ord(h - I) \). By the assumption of weak Lie pairs we fix \( \xi, \eta \) satisfying: \( ord(g - I - \xi) \geq 2(g - I), ord(h - I - \eta) \geq 2(h - I) \). Now, the bound \( ord(gh - I - \xi - \eta) \geq 2ord(\tilde{g} - I) \) follows from Eq. (48). This verifies condition 3.ii. of definition 3.1.

For the pairs of pointwise weak Lie type the proof is the same, just apply all the formulas to some \( z \in M \), and note: if \( \xi, \eta \in T(G^{(i)}, M), \eta \in T(G^{(i)}, M) \) then \( g \cdot h \in G^{(i)} \cdot H^{(i)} \) (and vice versa).

2. Take \( z \in M \) and \( u \in T(G^{(1)}, M) + T(H^{(1)}, M) \) and choose any presentation \( u = \xi + \eta \) satisfying the assumptions. Then Eq. (48) ensures \( ord((gh - I - \xi - \eta)z) \geq ord(uz) \).

Vice versa, take \( \tilde{g} \in G^{(1)} \cdot H^{(1)} \) and choose a presentation \( \tilde{g} = gh \) satisfying the assumptions. Then Eq. (48) ensures \( ord((gh - I - \xi - \eta)z) > ord((\tilde{g} - I)z) \).

3. The proof for pairs of Lie type is similar. \(\square\)

**Example 3.21** (Continuing part iii. of Sect. 2.2.3) Let \( R \) be a filtered ring over a ring \( k \), and \( M \) a filtered \( R \)-module. Fix a system of generators \( \{e_j\} \) of \( M \), and some \( R \)-multiplicative action \( \text{Aut}_{k}^{(1)}(R) \circ M \), by \( \phi(\sum a_j e_j) = \sum \phi(a_j) e_j \). Thus \( \text{Aut}_{k}^{(1)}(R) \) preserves \( \{e_j\} \) and we put \( \text{Der}_{k}^{(1)}(R)(e_j) = 0 \). Take some subgroups \( G \subseteq GL_R(M), H \subseteq Aut_{k}^{(1)}(R) \) such that the pairs \( (T(G^{(1)}, M), G^{(1)}), (T(H^{(1)}, M), H^{(1)}) \) are of weak Lie type. We claim: the pair \( (T(G^{(1)}, M) + T(H^{(1)}, M), G^{(1)} \cdot H^{(1)}) \) is of weak Lie type.

**Proof** The assumptions of part 1 of the last lemma are satisfied. Indeed, suppose \( (\xi + \eta) \in (T(G^{(1)}, M) + T(H^{(1)}, M)) \cap End_{k}^{(1)}(M) \). Apply this to the generators \( \{e_j\} \) to get: \( \xi(e_j) \in M_i \).

Thus \( \xi \in T(G^{(1)}, M) \) and hence \( \eta \in T(H^{(1)}, M) \). Similarly, suppose \( g \cdot h \in (G \cdot H) \cap GL_{k}^{(1)}(M) \) as \( h(e_j) = e_j \), we get: \( g(e_j) - e_j \in M_i \), hence \( g \in G^{(i)} \). But then \( h \in H^{(i)} \). \(\square\)

As a particular case take the action of the group \( G = GL(m, R) \times GL(n, R) \times Aut_{k}(R) \) on matrices \( M = Mat_{m \times n}(R) \). We get: the pair

\[
\left(\text{End}_{R}^{(1)}(m) \oplus \text{End}_{R}^{(1)}(n) \oplus \text{Der}_{k}^{(1)}(R), GL^{(1)}(m, R) \times GL^{(1)}(n, R) \times Aut_{k}^{(1)}(R)\right)
\]  

is of weak Lie type.

### 3.3.3 The pair \( (T(G^{(1)}, M) + T(H^{(1)}, M), G^{(1)} \cdot H^{(1)}) \) in the case \( k \supseteq \mathbb{Q} \)

While Lemma 3.20 is rather general, the assumptions of its parts 2 and 3 are often difficult to check. However, in the case \( k \supseteq \mathbb{Q} \) we have a much simpler statement (see Lemma 3.22).

Suppose \( (T(G^{(1)}, M), G^{(1)}), (T(H^{(1)}, M), H^{(1)}) \) are pairs of pointwise Lie type. Suppose for any \( z \in M \) the correspondences of Definition 3.1, \( T(G^{(1)}, M) \supseteq G^{(1)}, T(H^{(1)}, M) \supseteq H^{(1)} \) are realized by maps

\[
\Psi^{(exp)}(\xi, z) = \sum_{j=0}^{N} \frac{\xi e_j}{j!} + \phi_{N,z}^{(exp)}, \quad \Psi^{(ln)}(g, z) = \sum_{j=1}^{N} \frac{(-1)^{j+1}(g - I)^j}{j} + \phi_{N,z}^{(ln)}.
\]  

Here \( \phi_{N,z}^{(exp)}, \phi_{N,z}^{(ln)} \in End_{k}^{(N+1)}(M) \) are the higher order terms, with \( N \geq \max(\text{ord}(\xi z), \text{ord}(g - I)z) \), when this \( \max \) is finite. (These \( \phi_{N,z}^{(exp)}, \phi_{N,z}^{(ln)} \) are not necessarily the same for \( G, H \)).

\(\square\) Springer
For example, this holds for $GL_R(M)$ and in many cases for $Aut_k(R)$, see the proof of theorem 3.6.

**Lemma 3.22** Assume $k \supseteq \mathbb{Q}$ and suppose $\langle G^{(1)}, H^{(1)} \rangle = G^{(1)} \cdot H^{(1)}$. Moreover, assume for the commutator $[T_{G^{(1)}(M)}, T_{H^{(1)}(M)}] \subseteq T_{H^{(1)}(M)} \subseteq \text{End}_k^{(1)}(M)$. Then the pair $(T_{G^{(1)}(M)} + T_{H^{(1)}(M)}), G^{(1)} \cdot H^{(1)})$ is of pointwise Lie type.

**Proof** (i) (Verifying condition 3.i. of Definition 3.1) Take an element $\xi + \eta \in T_{G^{(1)}(M)} + T_{H^{(1)}(M)}$. Assume $N := \text{ord}(\langle \xi + \eta \rangle z) < \infty$. To define the corresponding element in $G \cdot H$ we recall the Baker–Campbell–Hausdorff formula,

$$
exp(\xi)exp(\eta) = \exp\left(\xi + \eta + \frac{[\xi, \eta]}{2} + \frac{[\xi, [\xi, \eta]] + [\eta, [\xi, \eta]]}{12} + \cdots\right). \quad (51)
$$

As we do not assume completeness of $M$, we truncate this formula at order $N$ to get:

$$
\Psi^{(\exp)}(\xi, z) \cdot \Psi^{(\exp)}(\eta, z) \in \Psi^{(\exp)}(\xi + \eta + \cdots, z) \cdot GL_k^{(N+1)}(M). \quad (52)
$$

Fix $\tilde{\eta} \in T_{H^{(1)}(M)}$ by the condition $\tilde{\eta} = \xi + \eta + \frac{[\xi, \eta]}{2} + \frac{[\xi, [\xi, \eta]] + [\eta, [\xi, \eta]]}{12} + \cdots \in \{\eta\} + \text{End}_k^{(N+1)}(M)$. Such $\tilde{\eta}$ is obtained iteratively, one puts $\tilde{\eta} = \eta - \frac{[\xi, \eta]}{2} - \cdots$, then adjusts the higher order terms, and so on. Each summand here belongs to $T_{H^{(1)}(M)}$, because $[T_{G^{(1)}(M)}, T_{H^{(1)}(M)}] \subseteq T_{H^{(1)}(M)}$. And we stop at order $N$.

Finally, associate to $\xi + \eta \in T_{G^{(1)}(M)} + T_{H^{(1)}(M)}$ the element $g \cdot h := \Psi^{(\exp)}(\xi, z) \cdot \Psi^{(\exp)}(\tilde{\eta}, z) \in G^{(1)} \cdot H^{(1)}$. By construction we have:

$$
g \cdot h = \sum_{j=2}^{N} \frac{\langle \xi + \eta \rangle^j}{j!} + \text{End}_k^{(N+1)}(M) z. \quad (53)
$$

Therefore $\text{ord}(g \cdot h - \mathbb{I} - (\xi + \eta) z) > \text{ord}(\langle \xi + \eta \rangle z)$.

(ii) (Verifying condition 3.ii. of Definition 3.1) For any element of $G^{(1)} \cdot H^{(1)}$ choose a presentation $g \cdot h$. Assume $N := \text{ord}(\langle g \cdot h - \mathbb{I} \rangle z) < \infty$. Let $\xi := \Psi^{(\text{ln})}(g, z)$ and $\eta := \Psi^{(\text{ln})}(h, z)$. Define $\tilde{\eta} = \eta + \frac{[\xi, \eta]}{2} + \frac{[\xi, [\xi, \eta]] + [\eta, [\xi, \eta]]}{12} + \cdots \in T_{H^{(1)}(M)}$, the summation up to order $N$. Then $g \cdot h \in \Psi^{(\text{exp})}(\xi + \tilde{\eta}, z) \cdot GL_k^{(N+1)}(M)$. Associate to $g \cdot h$ the element $\xi + \tilde{\eta} \in T_{G^{(1)}(M)} + T_{H^{(1)}(M)}$. By construction holds: $\text{ord}(g \cdot h - \mathbb{I} - \xi - \tilde{\eta} z) > \text{ord}(\langle g \cdot h - \mathbb{I} \rangle z)$.

□

**Example 3.23** (i) Let $k \supseteq \mathbb{Q}$ and $R$ be one of $k[[x]]/J, k(x)/J, k[x]/J$, or $C^\infty(\mathbb{R}^p, 0)/J$. Suppose the filtration satisfies the assumptions of theorem 3.6. Take $H = GL_R(M)$ and $G = Aut_k(R)$ and suppose $GL_R(M) \rtimes Aut_k(R)$ acts on $M$, see Sect 2.2.3.ii. We claim: the pair $(\text{End}_R^{(1)}(M) + \text{Der}_k^{(1)}(R), GL_R^{(1)}(M) \rtimes Aut_k^{(1)}(R))$ is of pointwise Lie type.

**Proof** The pairs $(\text{End}_R^{(1)}(M), GL_R^{(1)}(M))$ and $(\text{Der}_k^{(1)}(R), Aut_k^{(1)}(R))$ are of pointwise Lie type, with the maps $\Psi^{(\text{exp})}(\xi, z), \Psi^{(\text{ln})}(\xi, z)$ as in Eq. (50), see the proof of theorem 3.6. In addition there holds $[\text{Der}_k^{(1)}(R), \text{End}_R^{(1)}(M)] \subseteq \text{End}_R^{(1)}(M)$. Indeed, for any $\xi \in \text{Der}_k^{(1)}(R), \eta \in \text{End}_R^{(1)}(M)$ and some generators $e_i$ of $M$ one has: $[\xi, \eta]\sum_i c_i e_i = \sum_{ij} c_i \eta_{ij} e_j$ where $\eta_{ij}$ is the presentation matrix of $\eta$. Now apply the last lemma.

□

(ii) In particular, let $M = R^n$, then the group of contact equivalences, $\mathcal{K} := GL(n, R) \rtimes Aut_k(R)$, gives a pair of pointwise Lie type.
(iii) In the same way one gets, for $M = \text{Mat}_{m \times n}(R)$, the pair of pointwise Lie type:

$$(\text{End}_R^{(1)}(R^m) + \text{End}_R^{(1)}(R^n) + \text{Det}_R^{(1)}(R), \ GL^{(1)}(m, R) \times GL^{(1)}(n, R) \rtimes \text{Aut}_k^{(1)}(R))$$

(54)

For square matrices, $m = n$, we get pointwise Lie pairs for the groups $G_{\text{congr}} \rtimes \text{Aut}_k(R)$, $G_{\text{con}} \rtimes \text{Aut}_k(R)$, etc., see §2.2.3.iv. (The verification of $[T(G^{(1)}, M), T(H^{(1)}, M)] \subseteq T(H^{(1)}, M)$ is immediate.)

**Example 3.24** In some cases one can take the full Taylor expansions in Eq. (50). This holds, e.g. when $M$ is complete for its filtration. (For example, $R$ is complete wrt. $\{I_j\}$ and $M_j = I_j \cdot M$.) Then the maps $\Psi^{(\text{exp})}(\xi, z), \Psi^{(\text{ln})}(\xi, z)$ do not depend on $z$, and one gets: the pair $(T(G^{(1)}, M), T(H^{(1)}, M), G^{(1)} \cdot H^{(1)})$ is of Lie type. (In the proof of lemma 3.22 one puts $N = \infty$, and all the formulas become exact, not just up to $GL^{(N+1)}_k (M), \text{End}_k^{(N+1)}(M)$.)

### 3.3.4 The case of direct product, $G \times H$

We treat this special case separately since it requires different assumptions.

**Lemma 3.25** Fix two pairs of Lie type, $T(G^{(1)}, M), T(H^{(1)}, M)$, $\Psi^{(\text{exp})}_G, \Psi^{(\text{exp})}_H$ are power series, with the same coefficients, i.e.

$$\Psi^{(\text{exp})}_G(\xi) = \mathbb{I} + \xi + \sum_{j \geq 2} a_j \xi^j, \quad \Psi^{(\text{exp})}_H(\eta) = \mathbb{I} + \eta + \sum_{j \geq 2} a_j \eta^j.$$

Suppose that the same holds also for the maps $\Psi^{(\text{ln})}_G, \Psi^{(\text{ln})}_H$. Then the pair $(T(G^{(1)}, M) \oplus T(H^{(1)}, M), G^{(1)} \times H^{(1)})$ is of Lie type for the maps

$$\Psi^{(\text{exp})}_{G \times H}(\xi, \eta) = \Psi^{(\text{exp})}_G(\xi) \cdot \Psi^{(\text{exp})}_H(\eta)^{-1}, \quad \Psi^{(\text{ln})}_{G \times H}(g, h) = \Psi^{(\text{ln})}_G(g) - \Psi^{(\text{ln})}_H(h^{-1}).$$

(Here $\Psi^{(\text{exp})}_H(\eta)^{-1}$ is the inverse of $\Psi^{(\text{exp})}_H(\eta)$ inside $H$.)

**Proof** The map $\Psi^{(\text{exp})}_{G \times H}$ is well defined and one has: $\Psi^{(\text{exp})}_{G \times H}(\xi, \eta) = \mathbb{I} + \xi + \eta + F(\xi, \eta)$, with $F(\xi, \eta) \in (\xi^2, \eta^2, \xi \cdot \eta)$.

Moreover, $\Psi^{(\text{ln})}_{G \times H}(\xi, -\xi) = \mathbb{I}$, therefore $F(\xi, \eta) = \tilde{F}(\xi, \eta) \cdot (\xi + \eta)$, where $\tilde{F}(\xi, \eta) \in (\xi, \eta)$. Therefore, for any $z \in M$ holds: $\text{ord} (F(\xi, \eta) \cdot (\xi + \eta)) > \text{ord}(\xi \cdot z + \eta \cdot z)$.

Similarly, for the logarithmic map we have:

$$\Psi^{(\text{ln})}_{G \times H}(g, h) = (gh - \mathbb{I}) + F_G(g - \mathbb{I}) - F_H(h^{-1} - \mathbb{I}) - (g - \mathbb{I})(h - \mathbb{I}) - h^{-1}(h - \mathbb{I})^2,$$

$$\Psi^{(\text{ln})}_{G \times H}(g, h^{-1}) = \mathbb{O}.$$  

(55)

Thus the total higher order expression, $F_G(g - \mathbb{I}) - F_H(h^{-1} - \mathbb{I}) - (g - \mathbb{I})(h - \mathbb{I}) - h^{-1}(h - \mathbb{I})^2$, is presentable in the form $\tilde{F}(g, h) \cdot (gh - \mathbb{I})$, where $\tilde{F}(g, h) \in (g - \mathbb{I}) + (h - \mathbb{I})$. In this way we get the needed bounds of Definition 3.1.

**Example 3.26** We continue Example 3.4. Let $(R, m)$ be a local ring and $GL(n, R) \cup R^n$.

The pair $(\text{Mat}_{n \times n}(m), GL^{(1)}(n, R))$ is of Lie type for the map $A \rightarrow A + I$. Therefore the pair

$$\text{Mat}_{m \times m}(m) \oplus \text{Mat}_{n \times n}(m) \rightarrow GL^{(1)}(m, R) \times GL^{(1)}(n, R)$$
is of Lie type for $\Psi^{(\exp)}(A, B) = (I + A)(I - B)^{-1}$. (And not just of weak Lie type, as was proved in Example 3.21).

4 The general criteria of determinacy

Fix a filtered action $G \bowtie M$ and a pair of pointwise (weak) Lie type, $(T_{G^{(1)}}, G^{(1)})$. By definition, $T_{G^{(1)}}, M) \subseteq \text{End}^1(M)$, this defines the action $T_{G^{(1)}}, M \bowtie M$. For any $z \in M$ we take the corresponding orbits $T_{G^{(1)}}, M}(z)$ resp. $G^{(1)}z \subseteq M$ and their closures $\overline{T_{G^{(1)}}, M}(z)$ resp. $\overline{G^{(1)}z}$, in the filtration topology. We prove the statements of the type $\overline{T_{G^{(1)}}, M}(z)$ is large iff $\overline{G^{(1)}z}$ is large”.

Recall that $z \in M$ is order-by-order $N$-determined iff $\{z\} + M_{N+1} \subseteq \overline{G^{(1)}z}$. Thus the statements read roughly “finite determinacy is equivalent to large tangent space”.

4.1 $\overline{T_{G^{(1)}}, M}(z)$ vs $\overline{G^{(1)}z}$ for pairs of pointwise Lie type

**Theorem 4.1** Suppose $(T_{G^{(1)}}, G^{(i)})$ is pointwise of Lie type, for some $i \geq 1$ (i.e., the properties in Definition 3.1 hold for $G^{(i)}$ instead of $G^{(1)}$). Then for any $z \in M$ holds:

$$\overline{T_{G^{(1)}}, M}(z) \supseteq M_{N+1} \text{ iff } \overline{G^{(i)}z} \supseteq \{z\} + M_{N+1}.$$ 

Thus, $z$ is order-by-order $N$-determined iff it is “infinitesimally” order-by-order $N$-determined.

**Proof** “$\Rightarrow$”: Take any $w_{N+1} \in M_{N+1}$. We construct a sequence $\{g_n \in G^{(i)}\}$ satisfying:

$$g_n^{-1} \cdots g_1^{-1}(z + w_{N+1}) \in \{z\} + M_{N+1+n}.$$ 

By the assumption $w_{N+1} \in \xi \cdot z + M_{N+2}$ for some $\xi \in T_{G^{(1)}}, M)$. Since $(T_{G^{(1)}}, G^{(i)})$ is pointwise of Lie type there is $g_1 \in G^{(i)}$, satisfying $\text{ord}((g_1 - I - \xi)z) > \text{ord}(\xi \cdot z)$. Then $g_1z - z - w_{N+1} \in M_{N+2}$, i.e., $g_1^{-1}(z + w_{N+1}) = z + w_{N+2}$, for some $w_{N+2} \in M_{N+2} \subseteq M_{N+1}$. By assumption $w_{N+2} \in \xi \cdot z + M_{N+3}$ for some $\xi \in T_{G^{(1)}}, M)$ and there is a $g_2 \in G^{(i)}$, satisfying $g_2z - z - w_{N+2} \in M_{N+3}$, i.e., $g_2^{-1}(z + w_{N+2}) = z + w_{N+3}$, $w_{N+3} \in M_{N+3}$. Proceed inductively.

“$\Leftarrow$”: Let $w_{N+1} \in M_{N+1}$, we construct a sequence $\{\xi_n \in T_{G^{(1)}}, M)\}$ satisfying: $w_{N+1} - (\sum_{i=1}^n \xi_i)z \in M_{N+1+n}$. By the assumption, $g_1z - z - w_{N+1} \in M_{N+2}$ for some $g \in G^{(i)}$. Again, since $(T_{G^{(1)}}, G^{(i)})$ is pointwise of Lie type there is $\xi_1 \in T_{G^{(1)}}, M)$, satisfying: $\text{ord}((g - I - \xi_1)z) > \text{ord}((g - I)z)$. Thus $w_{N+1} \in \xi_1 \cdot z + M_{N+2}$, i.e. $w_{N+1} - \xi_1 \cdot z = w_{N+2} \in M_{N+2}$ and we can proceed inductively.

**Remark 4.2** One is naturally tempted to a stronger statement, “Suppose $(T_{G^{(1)}}, G^{(1)})$ is of Lie type, then $\{z\} + T_{G^{(1)}}, M) \subseteq \overline{G^{(k)}z}$ for some $k \geq 1$.” This does not hold, see [8, Remark 2.3].

4.2 $\overline{T_{G^{(1)}}, M}(z)$ vs $\overline{G^{(1)}z}$ for pairs of pointwise weak Lie type

**Theorem 4.3** Suppose $(T_{G^{(1)}}, G^{(1)})$ is a pair of pointwise weak Lie type. Then for any $z \in M$ holds:
1. If $\overline{T_{(G^{(k)},M)}}z \supseteq M_{N+k+\text{ord}(z)}$ for any $k \geq 1$ then $G^{(k)}z \supseteq \{z\} + M_{N+k+\text{ord}(z)}$ for any $k > N$.

2. If $G^{(k)}z \supseteq \{z\} + M_{N+k+\text{ord}(z)}$ for any $k \geq 1$ then $\overline{T_{(G^{(k)},M)}}z \supseteq M_{N+k+\text{ord}(z)}$ for any $k > N$.

In the first part we get: $z$ is order-by-order $(2N + \text{ord}(z))$-determined.

**Proof.** 1. For any $w \in M_{N+k+\text{ord}(z)}$, with $k > N$, we construct a sequence $\{g_n \in G^{(k)}\}$ satisfying: $g_n^{-1} \cdots g_1^{-1} (z + w) \in \{z\} + M_{N+k+n+\text{ord}(z)}$.

By assumption we have $w \in \xi \cdot z + M_{N+k+1+\text{ord}(z)}$, for some $\xi \in T_{(G^{(k)},M)}$. By the pointwise weak Lie type assumption there exists $g_1 \in G^{(k)}$ satisfying: $(g_1 - \Pi - \xi)z \in M_{2\text{ord}(\xi)+\text{ord}(z)}$, with $z \geq k \geq N + 1$.

Thus $g_1z - z - w \in M_{N+k+1+\text{ord}(z)}$. Hence $g_1^{-1}(z + w) = z + w_1$ with $w_1 \in M_{N+k+1+\text{ord}(z)}$. We continue now with $w_1$ and with $k$ replaced by $k + 1$. Then we have $w' \in \xi \cdot z + M_{N+k+2+\text{ord}(z)}$, for some $\xi \in T_{(G^{(k+1)},M)}$, i.e. $\text{ord}(\xi) \geq k + 1$.

The pointwise weak Lie type assumption implies that there exists $g_2 \in G^{(k+1)} \subseteq G^{(k)}$ satisfying: $(g_2 - \Pi - \xi)z \in M_{2(k+1)+\text{ord}(z)}$. Thus $g_2z - z - w' \in M_{N+k+2+\text{ord}(z)}$ and $g_2^{-1}(z + w') = z + w_2$ with $w_2 \in M_{N+k+2+\text{ord}(z)}$. Proceed inductively.

2. For any $w \in M_{N+k+\text{ord}(z)}$, with $k > N$, we construct a sequence $\{\xi_n \in T_{(G^{(k)},M)}\}$ satisfying: $\sum_1^n \xi_n(z) - w \in M_{N+k+n+\text{ord}(z)}$.

By assumption, $g_1z - z - w \in M_{N+k+1+\text{ord}(z)}$ for some $g \in G^{(k)}$. Take the corresponding $\xi \in T_{(G^{(k)},M)}$, satisfying: $\text{ord}((g - \Pi - \xi)z) \geq 2k + \text{ord}(z)$. Thus $w - \xi \cdot z \in M_{N+k+1+\text{ord}(z)}$. We proceed inductively as in part 1.

\[ \square \]

**Corollary 4.4** Suppose $(T_{(G^{(1)},M)}, G^{(1)})$ is a pair of pointwise weak Lie type.

1. Assume that $M_{N+1+\text{ord}(z)} \subseteq \overline{T_{(G^{(1)},M)}}(z)$ implies $M_{N+k+\text{ord}(z)} \subseteq \overline{T_{(G^{(k)},M)}}(z)$ for all $k \geq 1$. If $M_{N+1} \subseteq T_{(G^{(1)},M)}(z)$ then $z + M_{2N+1+\text{ord}(z)} \subseteq \overline{G^{(N+1+\text{ord}(z))}}(z)$.

2. Assume that $\{z\} + M_{N+1+\text{ord}(z)} \subseteq \overline{G^{(1)}}(z)$ implies $\{z\} + M_{N+k+\text{ord}(z)} \subseteq \overline{G^{(k)}}(z)$ for all $k \geq 1$. If $\{z\} + M_{N+k} \subseteq G^{(k)}z$ then $\overline{T_{(G^{(N+1)},M)}}(z) \supseteq M_{2N+\text{ord}(z)}+1$.

The assumption of part 1. holds for many pairs of pointwise weak Lie type, see the following Example 4.5 i. The assumption of part 2. is more subtle.

**Example 4.5** (i) Suppose a ring $R$ is filtered by ideals $\{I_j\}$, satisfying the condition $(I_{i+j} : I_j) \cdot I_j \supseteq I_{i+j}$ for any $i, j, k \geq 1$. (The filtrations $\{I_j\}$, $[a^j : I_1]$ satisfy this condition.) Let $M$ be $R$-module with the induced filtration $M_j = I_j \cdot M$. Suppose $(T_{(G^{(1)},M)}, G^{(1)})$ is a pointwise weak Lie pair and $M_{N+1+\text{ord}(z)} \subseteq \overline{T_{(G^{(i+1)},M)}}(z)$, for some $i \geq 0$. Then $M_{N+k+\text{ord}(z)} \subseteq \overline{T_{(G^{(i+k)},M)}}(z)$ for any $k \geq 1$. Indeed, we observe:

\[ T_{(G^{(i+k)},M)} = T_{(G^{(i+1)},M)} \cap \text{End}_k^{(i+k)}(M) \supseteq (I_{i+k} : I_{i+1}) \cdot T_{(G^{(i+1)},M)}. \]

Here the embedding holds because $(I_{i+k} : I_{i+1}) \cdot \text{End}_k^{(i+1)}(M) \subseteq \text{End}_k^{(i+k)}(M)$. Thus

\[ \overline{T_{(G^{(i+k)},M)}}(z) \supseteq (I_{i+k} : I_{i+1}) \cdot \overline{T_{(G^{(i+1)},M)}}(z) \supseteq (I_{i+k} : I_{i+1}) \cdot M \supseteq I_{N+k+\text{ord}(z)} \cdot M. \]

\[ \square \]
(ii) If the filtration of $M$ does not satisfy some condition similar to $(I_{i+k} : I_j) \supseteq I_{i+j}$, as above, then the assumptions of the last corollary do not hold. For example, let $M = R = \mathbb{k}[[x]]$ filtered by

$$I_1 = m^n \supset I_2 = m^{2n} \supset I_3 = m^{2n+1} \supset I_4 = m^{2n+2} \supset \cdots .$$

Let $f(x) = x^{n+1}$ and assume $\text{char}(\mathbb{k}) \neq (n + 1)$. Then $T_{(\text{Aut}_k^1(R), M)}(f) = (x)^{2n} = I_1$, but we have $T_{(\text{Aut}_k^2(R), M)}(f) = (x)^{3n} = I_{n+2}$. Similarly $\text{Aut}_k^1(R)(f) \supseteq \{ f \} + (x)^{2n}$ but $\text{Aut}_k^2(R)(f) \not\supseteq \{ f \} + (x)^{2n+1}$.

### 4.3 Sharpness of results

In Theorems 4.1, 4.3 we see essential difference between the Lie type case (typically when $\mathbb{k}$ is a field of characteristic zero) and the weak Lie type case (e.g., when $\mathbb{k}$ is of positive characteristic). The natural question is whether the bounds in the weak Lie case can be improved, brought closer to the bounds for the Lie-type case. The following example shows that the bound of Theorem 4.3 in the weak Lie-type case is close to being sharp. Let $\mathbb{k}$ a field of characteristic $p > 0$. Take $R = \mathbb{k}[[x]]$, filtered by $\{ (x)^i \}$, and $G = \text{Aut}_k^1(R)$. Then $T_{(G^{(1)}, M)} = \text{Der}_k^1(R) = (x^2 \partial_x)$. Let $f = x^p + x^{p+N}$, with $N > 2p, \gcd(p, N) = 1$. Then $T_{(G^{(1)}, M)}(f) = (x^{p+N+1})$, so one would naively expect the order of determinacy to be close to $(p + N)$, which is however not the case.

In fact, for any univariate $f \in \mathbb{k}[[x]]$ the order of determinacy is computed in [52, Proposition 2.8], assuming $\mathbb{k} = \bar{\mathbb{k}}$, and for this $f$ the exact order of determinacy is $p + N + \lceil \frac{N}{p-1} \rceil - 1$. In particular, for $p = 2$ we get: the order of determinacy equals $p + 2N = 2(p + N) - \text{ord}(f)$, which has the same order as the bound of part 1 of Theorem 4.3.

For a multivariate example (and contact equivalence) see Remark 2.3 (b) of [16]. The following examples illustrate the non-triviality of the prime characteristic case.

**Example 4.6**

(i) Let $\mathbb{k}$ be an algebraically closed field, $\text{char}(\mathbb{k}) = p$. Let $R = \mathbb{k}[[x]]/(x^{pN+1})$, with $N > 1$. Then $f(x) = x^p \in R$ is clearly $(pN - 1)$-determined.

But $T_{(\text{Aut}_k^1(R), M)}(f) = \{0\}$.

(ii) Over a field of zero characteristic we often have: if $\{z\} \cup (M_N \setminus M_{N+1}) \subseteq G^{(1)}z$ then $\{ z \} + M_N \subseteq G^{(1)}z$. This does not hold in prime characteristic. For example, let $\mathbb{k}$ be algebraically closed, $\text{char}(\mathbb{k}) = p$, $R = \mathbb{k}[[x]]$ and $f(x) = x^p \in R$. Then $\{ f \} + (x^{pn}) \setminus (x^{pn+1}) \subseteq \text{Aut}_k^1(R) \cdot f$. But $(f) + (x^{pn+1}) \not\subseteq \text{Aut}_k^1(R) \cdot f$.

### 4.4 Finite determinacy in terms of infinitesimal stability

In this subsection we assume $R$ to be a local Noetherian ring, filtered by ideals $\{ I_j \}$, and $M$ a finitely generated $R$-module, filtered by $\{ I_j \cdot M \}$. Theorem 4.1 and Nakayama lemma give the following corollary.

**Corollary 4.7** Let $(T_{(G^{(1)}, M)}, G^{(1)})$ be a pair of pointwise Lie type. Then $z \in M$ is $G^{(1)}$-finitely order-by-order determined if and only if the quotient module $M/(T_{(G^{(1)}, M)}(z))$ is annihilated by some $I_n$, for $n \gg 1$.

Geometrically this means: the module $M/(T_{(G^{(1)}, M)}(z))$ is not supported outside of $V(I_n) \subseteq \text{Spec}(R)$.
In many cases the elements of the module $M$ may be considered as functions on the scheme $\text{Spec}(R)$, e.g., this happens for $M = \text{Maps}((\mathbb{k}^n, 0), (\mathbb{k}^m, 0))$ or $M = \text{Mat}_{m\times n}(R)$. Then we can evaluate $T_{(G^{(1)}, M)}(z)$ at points of $\text{Spec}(R)$ and compare it with the ambient module. More precisely, for any prime ideal $p \in \text{Spec}(R)$ we take the generic point of the corresponding (irreducible) locus, $V(p) \subset \text{Spec}(R)$, i.e., pass to the field of fractions, $\text{Frac}(R/p)$. Accordingly, we pass from modules over $R$ to vector spaces over $\text{Frac}(R/p)$,

$$T_{(G^{(1)}, M)}(z) \otimes_R \text{Frac}(R/p) \subseteq M \otimes_R \text{Frac}(R/p).$$

Then the condition “the module $M/(T_{(G^{(1)}, M)}(z))$ is not supported outside of $V(I_n)$” can be stated as:

if $p \not\supset I_n$ then $T_{(G^{(1)}, M)}(z) \otimes \text{Frac}(R/p) = M \otimes \text{Frac}(R/p).$

Geometrically this means: the condition $T_{(G^{(1)}, M)}(z)|_{pt} = M|_{pt}$ holds for any point $pt \in \text{Spec}(R) \setminus V(I_n)$. In the classical terminology this equality of vector spaces is called “infinitesimal stability at a given point”. Therefore in the classical language we get:

**Corollary 4.8** With the assumptions as before, $z \in M$ is $G^{(1)}$-finitely order-by-order determined if and only if $z$ is infinitesimally stable at any closed point of $\text{Spec}(R) \setminus V(I_n)$.

For the rings like $\mathbb{C}\{x\}$ or $\mathbb{R}\{x\}$, and the classical groups like right or contact equivalence, this recovers the classically known criteria, see e.g., [65, Theorem 2.1].

### 4.5 The passage from $\overline{Gz}$ to $Gz$ in the $C^\infty$-case

The results of Sects. 4.1–4.4 are of order-by-order type, $\overline{G^{(1)}z} \supseteq \{z\} + M_{N+1}$, i.e., they address the orbit closures. To get the ordinary determinacy criteria, i.e., of type $G^{(1)}z \supseteq \{z\} + M_{N+1}$, we use the approximation results of Sect. 4.4.4.

In Proposition 2.12 we did not consider $C^\infty$-case, as this ring has no Artin approximation. Now, with the notion of Lie type pairs, we can extend the approximation results to the $C^\infty$-case, and we obtain the usual (not just order-by-order) determinacy statements.

Let $R = C^\infty(\mathbb{R}^p, 0)/J$ with the maximal ideal $m = \langle x_1, \ldots, x_n \rangle$. Let $M$ be a finitely presented filtered $R$-module.

**Proposition 4.9** Suppose the ideal $J \subset C^\infty(\mathbb{R}^p, 0)$ is generated by analytic power series. Suppose the module $M$ admits a presentation matrix with analytic entries and satisfies $M_{N+1} \supseteq m^{\tilde{N}} \cdot M$, for some $\tilde{N} \gg N$. Suppose the group $G \subseteq \text{GL}_R(M) \times \text{Aut}_R(R)$ is defined by analytic equations and $(T_{(G^{(1)}, M)}, G^{(1)})$ is a pair of pointwise Lie type. If $G^{(1)}z \supseteq \{z\} + M_{N+1}$ or $\overline{T_{(G^{(1)}, M)}z} \supseteq \overline{M_{N+1}}$ then $G^{(1)}z \supseteq \{z\} + M_{N+1}$.

**Proof** (The proof below is based on the initial proof of G. Belitskii [6].)

Fix some $w \in M_{N+1}$. By theorem 4.1 $\overline{T_{(G^{(1)}, M)}z} \supseteq M_{N+1}$ implies $G^{(1)}z \supseteq \{z\} + M_{N+1}$, meaning that the condition $gz = z + w$, $g \in G$ is resolvable order-by-order. We prove that an order-by-order solution implies an ordinary solution. First we show a formal solution of the completion, $\hat{T} \hat{z} = \hat{z} + \hat{w}$. Then we check the surjectivity of completion map, $G \to \hat{G}$. Finally, we establish a solution $g \in G$.

We remark that $\overline{T_{(G^{(1)}, M)}z} \supseteq M_{N+1}$ implies $T_{(G^{(1)}, M)}z \supseteq M_{N+1}$. (Apply Nakayama and use $M_{N+1} \supseteq m^{\tilde{N}} \cdot M$.)
Step 1. Take the m-adic completion \( R \to \hat{R} = \mathbb{R}[[x]]/J \). In this case \( R \) is not Noetherian and \( R \to \hat{R} \) is surjective, by Borel’s lemma. The kernel of \( R \to \hat{R} \) is \( \mathfrak{m}^\infty \), the ideal of flat functions. Accordingly we have the maps

\[
R^n \to \hat{R}^n, \quad M \to \hat{M} \cong M \otimes_R \hat{R}, \quad G \to \hat{G} \subseteq GL(\hat{M}) \times Aut_\hat{R}(\hat{R}). \tag{61}
\]

As in the proof of proposition 2.12 we lift the condition \( g_z = z + w, g \in G \) to the presentation of \( M \), to get implicit function equations, (24). By our assumption they admit order-by-order solution, thus their completion admits a formal solution. Therefore we have: \( \hat{G} \hat{z} \supseteq \{ \hat{z} \} + \hat{M}_{N+1} \). (Here \( \hat{M}_{N+1} \) is the image of \( M_{N+1} \) in \( \hat{M} \).

Step 2. We verify the surjectivity of the completion map \( G \to \hat{G} \). Using the diagram (22) any \( \hat{g} \in \hat{G} \) is presentable as \((\hat{U}, \hat{V}, \hat{\phi}) \in GL_R(\hat{R}^p) \times GL_R(\hat{R}^q) \times Aut_\hat{R}(\hat{R})\). To find a corresponding preimage \( g \in G \) we should find \((U, V, \phi) \in GL_R(R^p) \times GL_R(R^q) \times Aut_R(R)\) that satisfy \( UA = \phi(A) V, \phi(A) = [a_{ij}((j))] \) (and further equations of \( G \)) and whose Taylor expansions are \((\hat{U}, \hat{V}, \hat{\phi})\). As before, we get the system of implicit function equations, (24). By the assumption \( J \) is generated by analytic series (hence finitely generated), the entries of \( A \) are analytic and the equations of \( G \subseteq GL_R(M) \times Aut_R(R) \) are analytic. And we are given a formal solution \((\hat{U}, \hat{V}, \hat{\phi})\). Therefore, Tougeron approximation (Theorem 2.10) implies the existence of \( C^\infty \)-solution, \((U, V, \phi)\), whose Taylor expansion is precisely \((\hat{U}, \hat{V}, \hat{\phi})\). This is the needed preimage \( g \in G \).

By restricting to the unipotent part we get the surjection \( G^{(1)} \to \hat{G}^{(1)} = \hat{G}(1) \).

Step 3. For \( z \in M \) and any \( w \in M_{N+1} \) we want to resolve the condition \( g_z = z + w, g \in G \).

By Step 1 this has a formal solution, \( \hat{g}_z = \hat{z} + \hat{w} \), with \( \hat{g} \in \hat{G} \). By Step 2, there is a preimage \( g \in G \) of \( \hat{g} \), and it satisfies: \( g_z = z + w \in \cap \alpha(m^j \cdot R^p) = \alpha(\mathfrak{m}^\infty \cdot R^p) = \mathfrak{m}^\infty \cdot M \), for \( \alpha \) as in §2.3. Therefore it is enough to prove: \( \hat{G} \hat{z} \supseteq \{ z \} + \mathfrak{m}^\infty \cdot M \).

Fix \( z \) and vary \( g, w \) then we get the (lifted) orbit \( \{(g, w) \mid g_z = z + w \} \subset G \times M \). We recall its defining equations, as in 2.3. Fix the canonical basis \( \{ e_i \} \) of \( R^p \), thus \( z = \sum z_i \alpha(e_i) \), \( w = \sum w_i \alpha(e_i) \), where \( z_i, w_i \in R \). A group element is \( g = (U, V, \phi) \in G \) and the explicit definition of this orbit is:

\[
\{(U, V, \phi, \{w_i\}, \{t_j\}) \mid \sum_j U_{ij}z_j(\phi(x)) = z_i + w_i + \sum_j V_{ij}t_j, \forall i\}. \tag{62}
\]

These are \( C^\infty \)-implicit function equations, \( F(y, x, w) = 0 \). (Here \( y \) is the tuple of the unknowns \( U, V, \phi, \{t_j\} \).) In our case \( w_i \in \mathfrak{m}^\infty \) thus the completion of these equations has the trivial solution, \( y_0 = (Id, Id, x, 0) \). To ensure an ordinary \( C^\infty \) solution we use Theorem 2.11.

In our case \( \partial_y F \mid_{y_0} \) is exactly the tangent space map, \( T_{(G^{(1)}, M)} \to T_{(G^{(1)}, M)} z \subseteq M, \xi \to \xi(z) \). By the assumption, the image of this map contains \( \mathfrak{m}^N \cdot M \). Thus the ideal of maximal minors of \( \partial_y F \mid_{y_0} \) contains \( \mathfrak{m}^N \) for some \( N < N < \infty \).

Finally we estimate \( \det \left( F'_y(x, y_0)(F'_y(x, y_0))^t \right) \).

(i) Recall the Cauchy-Binet formula: if \( L \in Mat_{m \times n}(R), m \leq n \), then \( \det(LL^t) = \sum det(L_{ij})^2 \), where \( L_{ij} \) is a maximal \((m \times m)\) minor and the sum goes over all the maximal minors.

(ii) Recall the statement: if the ideal generated by \( \{f_i\} \) contains \( \mathfrak{m}^N \subseteq C^\infty(R^p, 0) \), for some \( N < \infty \), then \( (\sum f_i^2) \cdot \mathfrak{m}^\infty = \mathfrak{m}^\infty \). Indeed, for any \( g \in \mathfrak{m}^\infty \) the vanishing order of \( \frac{g}{\sum f_i} \) at \( 0 \in R^p \) is \( \infty \). Therefore this fraction extends to a \( C^\infty \) function on \( (R^p, 0) \).
Altogether we get det \(F'_y(x, y_0)(F'_y(x, y_0))^t\) \(m^\infty = m^\infty\), thus Theorem 2.11 ensures the \(C^\infty\) solution. This resolves the condition \(gz = z + w_\infty, g \in G\).

**Remark 4.10** Proposition 4.9 does not hold if the ideal \(J \subset C^\infty(\mathbb{R}^p, 0)\) is not generated by analytic functions. For example, let \(R = C^\infty(\mathbb{R}^2, 0)/(x_2)^\infty\), then \(\hat{R} = \mathbb{R}[[x_1, x_2]]\) and the projection \(\text{Aut}(R) \rightarrow \text{Aut}(\hat{R})\) is non surjective. The order of \(\text{Aut}_R(\hat{R})\)-determinacy of the element \(\hat{x}_2 \in \mathbb{R}[[x_1, x_2]]\) is 1, but the element \(x_2 \in C^\infty(\mathbb{R}^2, 0)/(x_2)^\infty\) is not finitely \(\text{Aut}_R(\hat{R})\)-determined. For example, \(x_2 \sim x_2 + x_1^N\) for any \(N\).

In addition, if \(J\) is not generated by analytic functions then it might be not finitely generated, as \(C^\infty(\mathbb{R}^p, 0)\) is not Noetherian. Then condition \(\phi(J) = J\) brings infinitely many equations.

Finally we state the criterion for the orbits, not just their closures. Suppose the group \(G \subseteq GL_R(M) \times \text{Aut}_k(R)\) acts on \(M\). We assume that the filtration satisfies \(\text{Der}_{\mathbb{k}}(1)(R)(x) \subseteq (x)^2\) and \(I_N \cdot \text{Der}_k(R) \subseteq \text{Der}_{\mathbb{k}}(1)(R)\) for \(N \gg 1\).

As a consequence of the previous results we state the following finite determinacy result which covers positive characteristic, characteristic 0 and the \(C^\infty\)-case (see Remark 2.9 and Sect. 2.4.3).

**Corollary 4.11**

1. Let \(\mathbb{k}\) be any field, and let \(R\) be one of \(\mathbb{k}[[x]]/J, \mathbb{k}(x)/J, \) or \(\mathbb{k}(x)/J\) (for the latter case assume that \(\mathbb{k}\) is a valued field such that its completion with respect to the absolute value is separable over \(\mathbb{k}\)). Suppose the conditions 3.11 are satisfied. Suppose \(G \subseteq GL_R(M) \times \text{Aut}_k(R)\) is defined by formal/algebraic/analytic power series. If \(T_{(G^{(1)}, M)}z \supseteq MN_1 + J\) then \(G^{(k)}z \supseteq \{z\} + MN_1 + k\), for any \(k > \max(0, N - \text{ord}(z))\).
2. Suppose \(\mathbb{k} \supseteq \mathbb{Q}\) and \(R\) is one of \(\mathbb{k}[[x]]/J, \mathbb{k}(x)/J, \mathbb{k}(x)/J\). Suppose \(G \subseteq GL_R(M) \times \text{Aut}_k(R)\) is defined by formal/algebraic/analytic equations. Then \(T_{(G^{(1)}, M)}z \supseteq MN_1 + J\) if \(G^{(1)}z \supseteq \{z\} + MN_1 + J\).
3. Suppose \(R = C^\infty(\mathbb{R}^p, 0)/J\) with \(J\) analytically generated, \(M\) admits a presentation matrix with analytic entries, and \(G \subseteq GL_R(M) \times \text{Aut}_k(R)\) is defined by analytic equations. Suppose \(MN_1 + J \supseteq m^\infty M\) for some \(\tilde{N} \gg N\). Then \(T_{(G^{(1)}, M)}z \supseteq MN_1 + J\) if \(G^{(1)}z \supseteq \{z\} + MN_1 + J\).

**Proof** By Theorem 3.6 and Proposition 3.14 we have the (pointwise) (weak) Lie type pairs.

For part 1 use theorem 4.3 to get \(\overline{G^{(k)}z} \supseteq \{z\} + MN_1 + J\). For parts 2,3 use theorem 4.1, to get:

\[
\overline{T_{(G^{(1)}, M)}z} \supseteq MN_1 + J
\]

if and only if \(G^{(1)}z \supseteq \{z\} + MN_1 + J\) \(\forall J\). Then use Propositions 2.12 and 4.9 to remove the closures.

**5 Applications and examples**

In this section we derive several explicit determinacy statements. We work under the following assumptions:

**Assumptions 5.1**

(i) Let \(\mathbb{k}\) be any field and, when \(R\) is \(\mathbb{k}(x)/J\), we assume that \(\mathbb{k}\) is a complete valued field (of any characteristic).

(ii) \(R \subseteq \mathbb{k}[[x]]/J\) or \(R = C^\infty(\mathbb{R}^p, 0)/J\), with filtration by some ideals \(\{I_j\}\), satisfying:

\[
\text{Der}_{\mathbb{k}}(1)(R)(x) \subseteq (x)^2, \text{ and } I_{N_1} \cdot \text{Der}_k(R) \subseteq \text{Der}_{\mathbb{k}}(1)(R) \text{ for } N \gg 1.
\]
(iii) • For $R = C^\infty(\mathbb{R}^p, 0)/J$ we assume $J$ is analytically generated, $\cap I_j \supseteq m^\infty$, and $\{I_j\}$ are analytically generated mod $m^\infty$.
• If $R$ is not one of the rings $\mathbb{k}[x]/J$, $\mathbb{k}(x)/J$, $C^\infty(\mathbb{R}^p, 0)/J$, then we also assume that $R$ has the Artin approximation property; $\{I_j\}$, $J$ are polynomially generated and the objects for which we derive the determinacy (e.g. $f \in R$ or $f \in R^n$ or $f \in Mat_{m \times n}(R)$) is polynomial mod $I_N$, for some (specified) $N$.
• If $\mathbb{k} \not\supseteq \mathbb{Q}$ then in addition we assume conditions 3.11.

These conditions are needed to get the (pointwise resp. weak) Lie type pair and to apply the approximation results.

Recall that the filtrations $\{m^j\}$, $\{I_{j+1} = I_j^i \cdot \alpha\}$, for some ideals $I_1 \subseteq \alpha^2$ satisfy these assumptions.

Recall that $ord(f)$ is defined in Sect. 2.2.1.

For a group action $G \circlearrowleft M$ and a filtration $\{M_j\}$ of $M$ we say that $z \in M$ is infinitesimally $d$-determined if $T_{(G^{(i)},M)}z \supseteq M_{d+1}$.

5.1 Right determinacy of germs of functions

We formulate our results for the classical case of function germs and show how they specialize to classically known results on finite right-determinacy (our results are more general with respect to the allowed rings and filtrations).

**Corollary 5.2** Suppose $R$ with filtration $\{I_j\}$ satisfy the Assumptions 5.1. For $f \in R$ the following holds:

1. (i) If $\text{Der}^{(1)}_{\mathbb{k}}(R)f \supseteq I_{N+1}$ then $\{f\} + I_{2N+1-ord(f)} \subseteq \text{Aut}_{\mathbb{k}}^{(N+1-ord(f))}(R) \cdot f$.
   (Thus $f$ is $(2N - ord(f))$-determined.)

(ii) If $\{f\} + I_{N+k} \subseteq \text{Aut}_{\mathbb{k}}^{(k)}(R) \cdot f$ for any $k \geq 1$, then $I_{2N+1-ord(f)} \subseteq \text{Der}^{(1)}_{\mathbb{k}}(R)(f)$.
   (Thus $f$ is infinitesimally $(2N - ord(f))$-determined.)

2. Suppose $\mathbb{k} \supseteq \mathbb{Q}$ and $R$ is one of $\mathbb{k}[[x]]/J$, $\mathbb{k}(x)/J$, $\mathbb{k}(x)/J$, $C^\infty(\mathbb{R}^p, 0)/J$. Then $I_{N+1} \subseteq \text{Der}^{(1)}_{\mathbb{k}}(R)(f)$ if $\{f\} + I_{N+1} \subseteq \text{Aut}_{\mathbb{k}}^{(1)}(R) \cdot f$.
   Thus $f$ is right-N-determined if it is infinitesimally right-N-determined.

**Proof** 1. By Proposition 3.14 the pair $(\text{Der}^{(1)}_{\mathbb{k}}(R), \text{Aut}^{(1)}_{\mathbb{k}}(R))$ is of weak Lie type. For the chosen filtration the condition $I_{N+1} \subseteq \text{Der}^{(1)}_{\mathbb{k}}(R)(f)$ implies $I_{N+k} \subseteq \text{Der}^{(k)}_{\mathbb{k}}(R)(f)$ for any $k \geq 1$. Then part 1. of Theorem 4.3 gives:

$$\{f\} + I_{2N+1-ord(f)} \subseteq \text{Aut}_{\mathbb{k}}^{(N+1-ord(f))}(R) \cdot f.$$  \hfill (64)

Now apply Proposition 2.12 to remove the closure.
Part ii. is proved similarly, using part 2. of Theorem 4.3.

2. By Theorem 3.6 the pair $(\text{Der}^{(1)}_{\mathbb{k}}(R), \text{Aut}^{(1)}_{\mathbb{k}}(R))$ is of pointwise Lie type. Then, by Theorem 4.1 we have: $I_{N+1} \subseteq \text{Der}^{(1)}_{\mathbb{k}}(R)(f)$ if $\{f\} + I_{N+1} \subseteq \text{Aut}^{(1)}_{\mathbb{k}}(R) \cdot f$. Finally, apply Propositions 2.12, 4.9 to get the result.

**Example 5.3** Suppose $\mathbb{k}$ is a field and $R = \mathbb{k}[[x]]$, or $\mathbb{k}(x)$ (\mathbb{k} perfect), or $\mathbb{k}(x)$, or $C^\infty(\mathbb{R}^p, 0)$, filtered by $\{m^j = (x)^j\}$. Then $\text{Der}^{(1)}_{\mathbb{k}}(R)(f) = m^2 \cdot (\partial_1 f, \ldots, \partial_p f)$, and Corollary 5.2 gives the classical criteria (for right equivalence).
(i) For \( k \in \mathbb{C}, \mathbb{R} \) and \( R = \mathbb{R}[x] \) or \( \mathbb{k}(x) \) see either [34, Theorem 2.23] or [65, Theorem 1.2]. If \( k \) is a field of characteristic zero this is [14, Theorem 3.1.13].

(ii) For \( R = \mathbb{k}[x] \) with \( \mathbb{k} \) algebraically closed of arbitrary characteristic, our Corollary 5.2 i. is [16, Theorem 3, part 1].

**Example 5.4** (i) More generally, suppose \( k \) is a field and \( R \) is one of \( \mathbb{k}[[x]]/J, \mathbb{k}(x)/J \) (\( k \) perfect), \( \mathbb{k}(x)/J, \mathbb{C}^\infty(\mathbb{R}^p, 0)/J \), filtered by \( \{m^j\} \). Then the assumptions of 3.11 are satisfied and we obtain the determinacy criteria for functions on singular germs. For \( R = \mathbb{C}(x)/J \) this was obtained in [22], see also [18, Theorem 2.2. i.] and [24, Proposition 1.4].

(ii) If the ring is non-regular then (even for \( k \supseteq \mathbb{Q} \)) finite determinacy does not imply that the singularity is isolated. For example, let \( f(x, y, z, w) = w \in \mathbb{k}[[x, y, z, w]]/(xy) \). Then \( f \) is obviously 1-determined (for \( w \mapsto w + h \), but the hypersurface \( f^{-1}(0) \) has a non-isolated singularity.

(iii) We remark that the group \( Aut_k^{(1)}(R) \) can be rather small when the ideal \( J \) is complicated, and similarly for the module \( Der_k^{(1)}(R) \). In such cases there are no finitely right determined functions at all.

**Example 5.5** (Newton filtrations, \( k \supseteq \mathbb{Q} \)) Let \( R \) be one of \( \mathbb{k}[[x]], \mathbb{k}(x), \mathbb{k}(x) \). For any Newton diagram \( \Gamma \subseteq \mathbb{R}^p_{\geq 0} \) (of some element \( f \in R \) take the corresponding ideal \( I(\Gamma) \), generated by monomials lying on or above the diagram. Take a sequence of decreasing Newton diagrams, i.e. \( \{ \Gamma_i \} \supseteq \{ \Gamma_{i+1} \} \supseteq \{ \Gamma_{i+2} \} \supseteq \{ \Gamma_{i+3} \} \). Then the ideals \( \{ I(\Gamma_j) \} \) form a decreasing filtration. Suppose this filtration satisfies Assumptions 5.1, e.g.

- To ensure \( Der_k^{(1)}(R)(x) \subseteq (x)^2 \) it is enough to assume that no point of \( \Gamma_1 \) lies under the hyperplane \( \{ \sum x_i = 2 \} \subseteq \mathbb{R}^p \).
- To ensure \( I_N \cdot Der_k^{(1)}(R) \subseteq Der_k^{(1)}(R) \), for \( N \gg 1 \), it is enough to assume that the distances between the successive diagrams are bounded from above.

Then we get the determinacy criterion for Newton filtration (see also [4, 15, 66] for a detailed treatment of piecewise filtrations and a refined determinacy criterion in [15, Corollary 4.8]).

**Example 5.6** Let \( R \) be one of \( \mathbb{k}[[x]], \mathbb{k}(x), \mathbb{k}(x), \mathbb{C}^\infty(\mathbb{R}^p, 0), \) with filtration \( \{m^j\} \). Another way to control the finite determinacy is to bound it by the Milnor number, \( \mu(f) := dim_k R/Der_k(R)(f) \). Using the obvious bound \( m^\mu(f) \subseteq Der_k(R)(f) \) we get

\[
m^\mu(f)^{+2} \subseteq m^2 \cdot Der_k(R)(f) \subseteq Der_k^{(1)}(R)(f).
\]

Therefore we get from Corollary 5.2:

(i) If \( \mu(f) < \infty \) then \( f \) is \((2\mu(f) - ord(f) + 2)\)-right-determined. This is [16, Corollary 1, part 1].

(ii) If \( k \supseteq \mathbb{Q}, \) (e.g., \( k \) is a field of characteristic zero) then \( f \) is \((\mu(f) + 1)\)-right-determined. For \( k = \mathbb{C} \) this is [34, Corollary 2.24 (1)]. If \( f \) is \( N\)-right determined and \( k \supseteq \mathbb{Q} \) then \( \mu(f) \leq \binom{N+n}{n} \).

(iii) For positive characteristic we get: if \( \{ f \} + m^{N+k} \subseteq Aut_k^{(1)}(R) \cdot f \) for any \( k \geq 1 \), then \( \mu(f) \leq \binom{2N-ord(f)+n}{n} \). In particular, \( f \) has at most an isolated singularity.

Note that in [16, Theorem 4.1] (with a correction in [32, Theorem 4.13]) the conclusion was proved with the weaker assumption: \( \{ f \} + m^{N+1} \subseteq Aut_k^{(1)}(R) \cdot f \), and hence, finite determinacy is equivalent to isolated singularity. The proof in [32] required a special construction in positive characteristic, and it is not clear if such a construction is available in our general setting.
5.2 Right (in)determinacy of germs of maps

We consider the free \( R \)-module \( R^n \), which can be identified with the space of maps \( \text{Spec}(R) \to (k^n, 0) \). The group \( G = \text{Aut}_k(R) \) acts componentwise on \( R^n \). In this section we assume \( k \supseteq \mathbb{Q} \). For simplicity assume that the filtration of \( R \) is by powers of the maximal ideal.

**Proposition 5.7** Let \((R, m)\) be a local \( k \)-algebra, \( k \supseteq \mathbb{Q} \), filtered by \( m^j \). Suppose that the \( m \)-adic completion \( \hat{R} \) of \( R \) is a complete Noetherian ring of positive Krull dimension with \( \hat{R}/m \hat{R} = k \). If \( n > 1 \) and a tuple \( f = (f_1, \ldots, f_n) \in m \cdot R^n \) is \( \text{Aut}_k^{(1)}(R) \)-finitely determined then the following holds.

1. For any associated prime \( m \neq p \in \text{Ass}(R) \) holds: \( \dim(R/p) \geq n \) and the image of \( f \) in \((R/p)^n\) is 1-determined.
2. The images of \( \{f_i\} \) in the vector space \( m/m^2 \) are linearly independent.

In particular we get: if \( f_i \in m^2 \) for some \( i \), then \( f \) is not \( \text{Aut}_k^{(1)}(R) \)-finitely determined. (See also Proposition 4.5 and Corollary 4.6 in [31] for the case \( R = k[[x]] \)).

**Proof Step 1.** We reduce the proof to the particular case: \( R \) is a complete local Noetherian domain (over a field of zero characteristic).

(a) Take the \( m \)-adic completion, \( R \to \hat{R} \). If \( f \in R^n \) is \( N \)-determined for \( \text{Aut}_k^{(1)}(R) \), then \( \hat{f} \in \hat{R}^n \) is \( N \)-determined for \( \text{Aut}_k^{(1)}(\hat{R}) \). Indeed, fix any \( \hat{h} \in m^{N+1} \cdot \hat{R}^n \) and take a representing sequence \( \{h_j \in m^{N+1} \cdot \hat{R}^n\} \). It satisfies: \( h_{i+j} - h_j \in m^j \cdot R^n \) and the image of this sequence in \( \hat{R}^n \) converges to \( \hat{h} \). Then \( \hat{f} + \hat{h} \in \text{Aut}_k^{(1)}(\hat{R})(\hat{f}) \), thus \( \hat{f} + \hat{h} \in \text{Aut}_k^{(1)}(\hat{R})(\hat{f}) \), by completeness. Thus it is enough to prove the statement for \( R \)-complete local Noetherian.

(b) Let \( \{p_i\} \) be the associated primes of \( R \). Any automorphism \( \phi \in \text{Aut}_k^{(1)}(R) \) acts on the set \( \{p_i\} \). Moreover, by the unipotence, \( \phi \) preserves each \( p_i \). Thus, for any such \( p \), we have the natural homomorphism of groups, \( \text{Aut}_k^{(1)}(R) \to \text{Aut}_k^{(1)}(R/p) \). Therefore, if \( f \in R^n \) is \( N \)-determined for \( \text{Aut}_k^{(1)}(R) \), then its image in \((R/p)^n\) is \( N \)-determined for \( \text{Aut}_k^{(1)}(R/p) \). As \( p \) is a prime ideal, \( R/p \) is a domain. It is of positive dimension iff \( p \neq m \). Thus it is enough to prove the statement for \( R \) a local complete Noetherian domain.

**Step 2.**

(a) Let \( R \) be a local complete Noetherian domain, containing \( \mathbb{Q} \). The pair \((\text{Der}_k^{(1)}(R), \text{Aut}_k^{(1)}(R))\) is of Lie type, by Theorem 3.6. Thus (Theorem 4.1) the finite determinacy of \( f \in R^n \) implies: \( \text{Der}_k^{(1)}(R)(f) \supseteq m^{N+1} \cdot R^n \), for some \( N \). In particular, \( \text{Der}_k(R)(f) \supseteq m^{N+1} \cdot R^n \), i.e. \( m \) is the support of the module \( R^n / \text{Der}_k(R)(f) \) and is the only associated prime of \( \text{Ann}(R^n / \text{Der}_k(R)(f)) \). Hence \( \text{Ann}(R^n / \text{Der}_k(R)(f)) \) is \( m \)-primary and its height must be \( \dim(R) \).

Fix some generators \( \{D_{a}\} \) of \( \text{Der}_k(R) \) and consider the matrix \( \{D_{a}(f_i)\}_{i,\alpha} \), which is a presentation matrix of \( R^n / \text{Der}_k(R)(f) \). By [27, Proposition 20.7] \( \text{Ann}(R^n / \text{Der}_k(R)(f)) \) and the \( n \)th determinantal ideal \( I_n[D_{a}(f_i)] \) have the same radical. We want to check whether \( I_n[D_{a}(f_i)] \supseteq m^{N} \) for some \( N \).

(b) Recall that for a local complete Noetherian domain the rank of the module of derivations equals \( \dim(R) \). Therefore for \( D_1 \ldots D_{\dim(R)+1} \in \text{Der}_k(R) \) holds: \( \sum a_i D_i = 0 \) for
some \( a_i \in R \), where not all \( a_i \)’s are zero. Therefore any \( \dim(R) + 1 \) columns of \([D_a(f_i)]\) are \( R \)-linearly dependent. Thus it suffices to consider only some block of \( \dim(R) \) columns. So, we assume the matrix \([D_a(f_i)]\) is of size \( n \times \dim(R) \).

If \( \dim(R) < n \) then \( I_n[D_a(f_i)] = 0 \) and no power of \( m \) is 0 since \( \dim(R) > 0 \). Thus we assume \( \dim(R) \geq n \). Evaluate this matrix at the origin and check its rank, over the field \( R/m \). Suppose \( r := \text{rank}[D_a(f_i)]_0 < n \), then the matrix is left-right equivalent to \( \mathbb{I}_{r \times r} \oplus B \), where \( B \in \text{Mat}_{(n-r) \times (\dim(R)-r)}(m) \). Thus \( I_n[D_a(f_i)] = I_{n-r}(B) \). But the later ideal has height at most \( (\dim(R) - r) - (n - r) + 1 < \dim(R) \), see [20, (2.1)]. Thus \( I_{n-r}(B) \) cannot contain any power of \( m \).

Therefore the finite determinacy of \( f \) implies: \( \text{rank}[D_a(f_i)]_0 = n \). Then \([D_a(f_i)]\) is left-right equivalent to \( [\mathbb{I}_{n \times n}] \oplus [\text{dim}(\dim(R) - n)] \), the unit and the zero matrices. But then \( \text{Der}_k(R)(f) = R^n \) and hence

\[
\text{Der}^{(1)}_k(R)(f) \supseteq m \cdot \text{Der}_k(R)(f) \supseteq m \cdot R^n. \tag{65}
\]

Thus, by theorem 4.1, \( f \in R^n \) is 1-determined.

(c) Finally, if over the initial ring the images of \( \{f_i\} \) in \( m/m^2 \) are not \( \mathbb{k} \)-linearly independent, then they are not linearly independent also after the completion and the projection to \( R/p \), for an associated prime \( p \). But then \( I_n[D_a(f_i)] \) cannot contain any power of \( m \).

(The row operations correspond to \( \mathbb{k} \)-linear operations on \( \{f_i\} \), thus we can assume \( f_1 \in m^2 \) and get: \( \text{rank}[D_a(f_i)]_0 < n \).)

\( \square \)

**Example 5.8** (i) Suppose \((R, m)\) is regular Noetherian and a tuple \((f_1 \ldots f_n)\) is finitely \( \text{Aut}_k(R) \) determined, \( n > 1 \). Then the \( f_i \) are linearly independent mod \( m^2 \) and hence can be extended to local coordinates on \( \text{Spec}(R) \), i.e. a regular system of parameters of \( R \) (by Nakayama). For \( \mathbb{C}[x] \) this was proved by Mather, see proposition 2.3 of [65].

(ii) For non-regular rings the tuple \( \{f_i\} \) can be finitely determined without even being a regular sequence. For example, let \( R = k[[x, y]]/(x \cdot (x, y)) \). Here \( x, y \) are multi-variables. Then the sequence \( \{y_i\} \) is linearly independent mod \( m^2 \), hence finitely determined (in fact 1-determined). But \( y_i \) is a zero-divisor in \( R \) and \( \{y_i\} \) is not a regular sequence.

**Remark 5.9** The statement of Proposition 5.7 is restricted to the case \( \text{char}(k) = 0 \), because only in this case we have established the implication (Theorem 4.1)

\[
\overline{G^{(1)z}} \supseteq \{z\} + M_{N+1} \Rightarrow \overline{T_{(G^{(1)}, \mathcal{M})} z} \supseteq M_{N+k}, \text{ for some } k < \infty. \tag{66}
\]

### 5.3 Contact determinacy of germs of maps

In this subsection we consider the determinacy under the action of the contact group, \( \mathcal{X} := GL(n, R) \rtimes \text{Aut}_k(R) \ltimes R^n \). In this case the tangent space \( T_{(\mathcal{X}^{(1)}, \mathcal{M})} \) is generated by \( \text{Der}^{(1)}_k(R) \) and \( \text{End}^{(1)}_R(R^n) = \text{Mat}_{n \times n}(I_1) \), where \( \{I_j\} \) is the filtration of \( R \). Denote the ideal generated by a tuple \( f = (f_1, \ldots, f_n) \in R^n \) by \( (f) \). Denote by \( \text{Der}^{(1)}_k(R)(f) \subseteq R^n \) the submodule obtained by applying the derivations to the tuple \( f \). The pair \((T_{(\mathcal{X}^{(1)}, \mathcal{M})}, \mathcal{X}^{(1)})\) is of pointwise (weak) Lie type, see Examples 3.21, 3.23. As in the case of right equivalence we have:

**Corollary 5.10** Suppose \( R \) with filtration \( \{I_j\} \) satisfies the assumptions 5.1.
1. (i) If $I_{N+1} \cdot R^n \subseteq Der^{(1)}_{k}(R)(f) + (f) \cdot I_1 \cdot R^n$ then $(f) \cdot R^n + I_{2N+1 - ord(f)} \cdot R^n \subseteq \mathcal{X}(N+1-ord(f)) \cdot f$.

   (Thus $f$ is $(2N - ord(f))$-contact-determined.)

   (ii) Suppose $(f) \cdot R^n + I_{N+k} \cdot R^n \subseteq \mathcal{X}(k)$ for any $k \geq 1$, then $I_{2N+1 - ord(f)} \cdot R^n \subseteq Der^{(1)}_{k}(R)(f) + (f) \cdot I_1 \cdot R^n$.

   (Thus $f$ is infinitesimally $(2N - ord(f))$-contact-determined.)

2. Let $k \supseteq \mathbb{Q}$. Then $I_{N+1} \cdot R^n \subseteq Der^{(1)}_{k}(R)(f) + (f) \cdot I_1 \cdot R^n$ iff $(f) \cdot R^n + I_{N+1} \cdot R^n \subseteq \mathcal{X}(1) \cdot f$.

   ($(N+1)$-contact-determinacy of $f$ vs infinitesimal $(2N+1)$-contact-determinacy of $f$)

The proof is the same as for corollary 5.2.

Example 5.11 (For function germs, $n = 1$) For $R$ one of $k[[x]], k{\{x\}}, k{\langle x \rangle}, C^\infty(\mathbb{R}^p, 0)$, with filtration $\{m^j\}$, we get the classical statements, with $Jac(f)$ the Jacobian ideal of $f$.

   (i) $(k \supseteq \mathbb{Q}) m^{N+1} \subseteq m^2 \cdot Jac(f) + m \cdot (f) + m^{N+1} \subseteq \mathcal{X}(1)$. For $k = \mathbb{C}$ see [65, Theorem 1.2], [34, Theorem 2.23]. For $k$ a field of characteristic zero this is [14, Theorem 3.1.13] and [16, Theorem 4.2].

   (ii) In positive characteristic, for $R = k[[x]]$, part i.ii is [16, Theorem 3, part 1], see also [29, Lemma 2.6]. Part i.ii. was claimed in [16], but the proof was incomplete. The full (quite involved) proof was given in Theorem 4.6 and 4.8 in [32], for the case that $k$ is infinite, but assuming only: $(f) + I_{N+1} \subseteq \mathcal{X}(1) \cdot f$.

   For maps, $n \geq 1$, cf. Lemma 6.22 and Theorem 6.27 in [26].

Example 5.12 (For maps, $n \geq 1$) Let $R$ be one of $k[[x]]/J, k{\{x\}}/J, k{\langle x \rangle}/J, C^\infty(\mathbb{R}^p, 0)/J$, with filtration $\{m^j\}$. We get the determinacy criteria of maps from $Spec(R)$ to $(k^p, 0)$.

   (i) For $k \supseteq \mathbb{Q}$ and $J = 0$ (regular rings) the condition for finite $\mathcal{X}$-determinacy is in general not implied by the condition that the variety $V(f)$ has an isolated singularity, except if $(f)$ is a minimal generating set of a complete intersection ideal (in which case both conditions are equivalent, cf. [32, Theorem 4.6] and Example 5.14). For example, take the map $f = (xy, xz, yz)$, then $V(f)$ has an isolated singularity but the map $f$ is not finitely determined by Corollary 5.10.

   (ii) As we have observed in Example 5.4, if $Spec(R)$ has a “complicated” singularity, then the group $Aut_k(R)$ can be very small, and there might be no finitely determined germs at all.

Example 5.13 Let $R$ be one of $k[[x]], k{\{x\}}, k{\langle x \rangle}, C^\infty(\mathbb{R}^p, 0)$, filtered by $\{m^j\}$. As before, we can control the finite determinacy of function germs $f \in R$ in terms of the Tjurina number $\tau(f) := dim_k(R/Der_k(R)(f) + (f))$. Use the obvious bound $m^{\tau(f)} \subseteq Der_k(R)(f) + (f)$ to get:

$$m^{\tau(f)+1} \subseteq m \cdot Der_k(R)(f) + m \cdot (f) \subseteq Der^{(1)}_k(R)(f) + m \cdot (f).$$

Therefore we get:

   (i) If $\tau(f) < \infty$ then $f$ is $(2\tau(f) - ord(f) + 2)$-contact-determined. This is [16, Corollary 1, part 2].

   (ii) If $k \supseteq \mathbb{Q}$, (e.g., $k$ is a field of characteristic zero) then $f$ is $(\tau(f) + 1)$-contact-determined. For $k = \mathbb{C}$ this is [34, Corollary 2.24], for $k$ a field of zero characteristic this is [14, Theorem 3.1.13].
(iii) If \( k \supseteq \mathbb{Q} \) and \( f \in R \) is \( N \)-contact determined then \( \tau(f) \leq \binom{N+n}{n} \).

**Example 5.14** (Finite determinacy of complete intersection ideals) Let \( R \) be one of \( \mathbb{k}[[x]]/J \), \( \mathbb{k}[x]/J \), \( \mathbb{k}(x)/J \), \( C^\infty(\mathbb{R}^p, 0)/J \), filtered by \( \{m^j\} \). A complete intersection ideal \( J \subset R \) is called \( N \)-determined if for any minimal generating sequence, \( f = (f_1 \ldots f_n) \), and any sequence \( (g_1 \ldots g_n) \in m^{N+1} \) holds: \( (f_1 + g_1, \ldots, f_n + g_n) \). An ideal is called infinitesimally \( N \)-determined if \( m^{N+1} R^n \subseteq \text{Der}^{(1)}_{\mathbb{k}}(f) + m \cdot I \cdot R^n \). (This condition does not depend on the choice of \( f \).) For \( \mathbb{k} \supseteq \mathbb{Q} \) Corollary 5.10 can be stated: \( I \) is infinitesimally \( N \)-determined if it is \( N \)-determined. See also [32, Theorem 4.6 (2)], where this is proved for arbitrary infinite fields. For the case \( R = \mathbb{C}(x)/J \), with \( J \) a complete intersection, and \( I \subset R \) a principal ideal, we get proposition 1.3 of [25].

### 5.4 Determinacy for maps relative to a germ

Another typical scenario is when the ambient space contains a special subscheme, \( V(\mathfrak{a}) \subset \text{Spec}(R) \). Then one studies the maps of \( \text{Spec}(R) \) up to right or contact transformations that preserve \( V(\mathfrak{a}) \). Thus we use the group of relative right transformations, \( \mathcal{R}_\mathfrak{a} := \text{Aut}_{\mathbb{k},\mathfrak{a}}(R) \), and the group of relative contact transformations, \( \mathcal{K}_\mathfrak{a} := GL(n, R) \rtimes \text{Aut}_{\mathbb{k},\mathfrak{a}}(R) \), see Sect. 3.2.3.

As was explained in §3.2.3, this case cannot be reduced to the case \( \text{Aut}_{\mathbb{k},\mathfrak{a}}(R/a) \).) The pair \( (\text{Der}^{(1)}_{\mathbb{k},\mathfrak{a}}(R), \text{Aut}^{(1)}_{\mathbb{k},\mathfrak{a}}(R)) \) is of (pointwise) Lie type, and similarly for the pair \( (\text{Der}^{(1)}_{\mathbb{k},\mathfrak{a}}(R) \oplus m \cdot R^n, \mathcal{K}_\mathfrak{a}^{(1)}) \). Theorems 4.1, 4.3 give the immediate corollaries:

**Corollary 5.15** Suppose \( \mathbb{k} \supseteq \mathbb{Q} \) and \( R \) with filtration \( \{I_j\} \) satisfies the assumptions of 5.1. For \( R = C^\infty/J \) we assume that \( \mathfrak{a} \) is analytically generated.

1. \( n = 1 \), right determinacy \( I_{N+1} \subseteq \text{Der}^{(1)}_{\mathbb{k},\mathfrak{a}}(R)(f) + I_{N+1} \subset \mathcal{K}_\mathfrak{a}^{(1)}(f) \).

2. \( n \geq 1 \), contact determinacy \( I_{N+1} R^n \subseteq \text{Der}^{(1)}_{\mathbb{k},\mathfrak{a}}(R)(f) + I_{N+1} R^n \subset \mathcal{K}_\mathfrak{a}^{(1)}(f) \).

**Example 5.16** The classical studied cases are: \( R = O(\mathbb{k}^n, 0) \), for \( \mathbb{k} \) a field of zero characteristic, and \( I \subset R \) - a radical ideal. See e.g., [37, Theorem 2.2], [24, Proposition 1.4], [28, Theorem 3.5, Corollaries 3.6 and 3.7], [22], [36, Propositions 2.3, 2.4 and 2.5]. Suppose \( \mathbb{k} \) is one of \( \mathbb{R}, \mathbb{C} \) and \( R \) is one of \( \mathbb{k}[[x]], \mathbb{k}[x], \mathbb{k}(x), C^\infty(\mathbb{R}^p, 0) \). Then we recover, e.g., [53, Theorem 3.6] (for \( \mathcal{R}_I \)) and [53, Lemma 3.11] (for \( \mathcal{K}_I \)). The converse statement (finite determinacy implies large tangent space) is e.g., [53, Theorem 3.5].

### 5.5 Relative determinacy for non-isolated singularities of function germs

Suppose an element \( f \in R \) defines a non-isolated singularity, i.e., one of the ideals \( \text{Der}_{\mathbb{k}}(R)(f) \subseteq R, (f) + \text{Der}_{\mathbb{k}}(R)(f) \subseteq R \) has infinite colength. Then no finite determinacy is possible for the filtration \( \{m^j\} \). In such cases one restricts the possible deformations, taking only those that preserve the singular locus of \( f \) (with its multiplicity). This corresponds to filtration \( \{m^j : \mathfrak{a}\} \). Here the ideal \( \mathfrak{a} \) is usually non-radical, it defines the relevant singularity scheme. Accordingly, instead of the groups \( \text{Aut}_{\mathbb{k}}(R), GL_{\mathbb{k}}(n) \rtimes \text{Aut}_{\mathbb{k}}(R) \), one considers the subgroups \( \mathcal{R}_\mathfrak{a} = \text{Aut}_{\mathbb{k},\mathfrak{a}}(R), \mathcal{K}_\mathfrak{a} = GL_{\mathbb{k}}(n) \rtimes \text{Aut}_{\mathbb{k},\mathfrak{a}}(R) \), see §3.2.3. As before, one has the \( \text{rel}(\mathfrak{a}) \) notions of determinacy. For simplicity we restrict to the case \( \mathbb{k} \supseteq \mathbb{Q} \).

**Corollary 5.17** Suppose \( \mathbb{k} \supseteq \mathbb{Q} \) and \( R, \{I_{N+1} = m^N : \mathfrak{a}\} \) satisfy the condition 5.1.
1. \((n = 1) \ I_{N+1} \subseteq \text{Der}^{(1)}_{K,a}(R)(f) \iff f + I_{N+1} \subseteq \mathcal{R}^{(1)}_a(f)\).
2. \((n \geq 1) \ I_{N+1} \cdot R^n \subseteq \text{Der}^{(1)}_{K,a}(R)(f) + a \cdot (f) \cdot R^n \iff f + I_{N+1} \cdot R^n \subseteq \mathcal{X}^{(1)}_a(f)\).

The module of \(a\)-logarithmic derivations, \(\text{Der}^{(1)}_{K,a}(R)\), is in general complicated, but it often contains a simpler module,

\[
\text{Der}^{(1)}_{K,a}(R) \supseteq \sqrt{a} \cdot \text{Der}(R) + \text{Ann}_{\text{Der}}(R)(a).
\]

(The latter summand here denotes all the derivations that annihilate \(a\).) This leads to a weaker statement, but with the condition easier to check.

**Example 5.18** Suppose \(R = \mathbb{k}[[x_1, \ldots, x_n]]\) and take the ideal \(a = (x_1, \ldots, x_l)^g\). Then

\[
\text{Der}^{(1)}_{K,a}(R) = \langle \partial_l+1, \ldots, \partial_n \rangle + (x_1, \ldots, x_l) \cdot \langle \partial_1, \ldots, \partial_l \rangle.
\]

Then for \(f \in (x_1, \ldots, x_l)^g \setminus (x_1, \ldots, x_l)^{g+1}\) we get:

(i) Suppose \(\mathbb{k} \supseteq \mathbb{Q}\). If \(m^2 (\partial_1+1, \ldots, \partial_n)(f) + m \cdot \sqrt{a} \cdot (\partial_1, \ldots, \partial_n)(f) \supseteq I \cdot m^{N+1}\) then \(f\) is \(N\)-right\(e_{rel}(a)\)-determined. (And similarly for the contact determinacy.) For the case \(a = (x_1, \ldots, x_l)^2\) this goes in the style of results of [54, Theorem 6.5 and Corollary 6.6], see also [61, Proposition 1.5 and Corollary 1.6] and [28, Theorem 3.5 and Corollary 3.9).

(ii) For an arbitrary \(\mathbb{k}\) we have: if \(m^2 (\partial_1+1, \ldots, \partial_n)(f) + m \cdot \sqrt{a} \cdot (\partial_1, \ldots, \partial_n)(f) \supseteq a \cdot m^{N+1}\) then \(f\) is \((2N - \text{ord}(f))\)-right\(e_{rel}(a)\)-determined. (And similarly for the contact determinacy.) This is [35, Theorem 3.2].

### 5.6 Relative algebraization

If \(f \in \mathbb{k}[[x]]\) is finitely (\(\mathcal{R}\) resp. \(\mathcal{X}\)) determined then, in particular, it is (\(\mathcal{R}\) resp. \(\mathcal{X}\)) equivalent to a polynomial. Such an algebraization does not hold for non-isolated critical points resp. singularities. For example, \(f(x, y, z) = xy(x+y)(x-zy)(x-e^z y) \in \mathbb{C}[x, y, z]\) is not \(\mathcal{X}\)-equivalent to a polynomial, see example 14.1 of [67]. (More references are in [40].) Certain non-isolated singularities can still be converted to a polynomial, [50].

We prove now that in the non-isolated case \(f\) can be converted to a polynomial “in the direction transversal to the critical resp. singular locus”.

**Proposition 5.19** Let \(\mathbb{k}\) be a field and \(f \in (x)^2 \subset \mathbb{k}[[x]]\). Suppose the ideal \(I := \sqrt{\text{Jac}(f)}\) (resp. \(I := \sqrt{f} + \text{Jac}(f)\) is of height \(c\). Suppose the projection \(V(I) \rightarrow V(x_1, \ldots, x_c) \subset \text{Spec} \mathbb{k}[[x]]\) by \((x_1, \ldots, x_p) \rightarrow (x_{c+1}, \ldots, x_p)\) is a finite morphism. Then \(f\) is \(\mathcal{R}\) (resp. \(\mathcal{X}\))-equivalent to an element of \(\mathbb{k}[[x_{c+1}, \ldots, x_p]][x_1, \ldots, x_c]\).

This result extends (in the irreducible case) Theorem 1.1 of [40] to an arbitrary field.

**Proof** Denote \(I := \sqrt{\text{Jac}(f)}\), resp. \(I := \sqrt{f} + \text{Jac}(f)\). As \(I\) is finitely-generated we have: \(I^{N+1} \subseteq I \cdot \text{Jac}(f)\), resp. \(I^{N+1} \subseteq I \cdot (f) + \text{Jac}(f)\), for \(N \gg 1\).

Take the filtration of \(\mathbb{k}[[x]]\) by \(\{I^n\}\). Note that \(\{I^n\}\) satisfies Assumptions 5.1, and (in the case \(\mathbb{k} \supseteq \mathbb{Q}\)) Assumptions 3.11. Thus \(I^{N+1} + \{f\} \subset \mathcal{R}^{(1)} f\), resp. \(I^{N+1} + \{f\} \subset \mathcal{X}^{(1)} f\).

As the scheme \(V(I)\) of co-dimension \(c\) and the projection \(V(I) \rightarrow V(x_1, \ldots, x_c)\) is finite, the intersection \(V(I) \cap V(x_{c+1}, \ldots, x_p)\) is a zero-dimensional scheme. Therefore \(I + (x_{c+1}, \ldots, x_p) \supseteq (x_1, \ldots, x_p)^N\), for some \(N \gg 1\). But then we have also \(I^{N} + (x_{c+1}, \ldots, x_p) \supseteq (x_1, \ldots, x_p)^{N-N'}\). And this implies \(I^{N} + \mathbb{k}[[x_{c+1}, \ldots, x_p]][x_1, \ldots, x_c] = \mathbb{k}[[x_1, \ldots, x_p]]\), for any \(N\).

Combining the two steps we have: \(\mathcal{R}^{(1)} f \cap \mathbb{k}[[x_{c+1}, \ldots, x_p]][x_1, \ldots, x_c] \neq \emptyset\), or respectively \(\mathcal{X}^{(1)} f \cap \mathbb{k}[[x_{c+1}, \ldots, x_p]][x_1, \ldots, x_c] \neq \emptyset\).\(\square\)
Example 5.20 (Generalizing Corollary 1.3 of [40] to an arbitrary field) Any element \( f \in \mathbb{k}[\{x\}] \) is \( \mathcal{X} \)-equivalent to an element of \( \mathbb{k}[x_1, \ldots, x_p] \). Indeed, take the irreducible decomposition, \( f = \prod f_i^{n_i} \). The scheme \( \text{Sing}(\prod f_i) \) is of codimension at least 2. Now apply the proposition.

In particular, any element \( f \in \mathbb{k}[x_1, x_2] \) is \( \mathcal{X} \)-equivalent to a polynomial. If \( \text{char}(\mathbb{k}) = 0 \) then the critical and singular loci coincide, i.e., \( \sqrt{\text{Jac}(f)} = \sqrt{(f)} + \text{Jac}(f) \), and thus \( f \in \mathbb{k}[x_1, x_2] \) is \( \mathcal{X} \)-equivalent to a polynomial.

5.7 Finite determinacy of matrices

Take as \( M \) the \( R \)-module of matrices, \( \text{Mat}_{m \times n}(R) \), with the filtration \( \{ \text{Mat}_{m \times n}(I_j) \} \). In this section the group \( G \) will be one of \( GL(m, R) \), \( GL(n, R) \), \( \text{Aut}_\mathbb{k}(R) \), or their (semi-)direct products. They are of (pointwise) (weak) Lie type, their tangent spaces are written down in Examples 3.21 and 3.23. Thus Theorems 4.1, 4.3 imply:

Corollary 5.21 Suppose \( R, \{ I_j \} \) satisfy the Assumption 5.1.

1. (i) If \( \text{Mat}_{m \times n}(I_{N+1}) \subseteq T_{G^{(1)}, M}(A) \) then \( A + \text{Mat}_{m \times n}(I_2N+1-\text{ord}(A)) \subseteq G^{(N+1-\text{ord}(A))}(A) \) \hfill ((2N - \text{ord}(A))-contact-determinacy)

(ii) Suppose \( A + \text{Mat}_{m \times n}(I_N + k) \not\subseteq T_{G^{(1)}, M}(A) \),

for any \( k \geq 1 \), then \( \text{Mat}_{m \times n}(I_2N+1-\text{ord}(A)) \not\subseteq T_{G^{(1)}, M}(A) \).

(infinitesimal \( (2N - \text{ord}(A))-contact-determinacy) \)

2. For \( \mathbb{k} \subseteq \mathbb{Q} \): \( \text{Mat}_{m \times n}(I_{N+1}) \subseteq T_{G^{(1)}, M}(A) \) iff \( A + \text{Mat}_{m \times n}(I_{N+1}) \subseteq G^{(1)}(A) \).

\( (N + 1)-contact-determinacy \ vs \ infinitesimal \ N + 1-contact-determinacy) \)

For \( \mathbb{k} \) a field of characteristic zero this statement was proved in [8, Corollary 2.9]. For \( \mathbb{k} \) a field of positive characteristic and \( R = \mathbb{k}[\{x\}] \), part (1.i) of the statement gives [31, Theorem 3.2].

For \( R = \mathbb{k}[\{x\}] \), \( \mathbb{k} \) an arbitrary field, a more general statement for matrices was proved in [31, Proposition 4.2].

Remark 5.22 In the case of matrices the submodule \( T_{G^{(1)}, M}(A) \subset \text{Mat}_{m \times n}(R) \) can be rather complicated. And a bound like \( \text{Mat}_{m \times n}(I_{N+1}) \subseteq T_{G^{(1)}, M}(A) \) can be difficult to verify. Here one faces a purely commutative algebra question, to compute or bound the support of the quotients module, \( \text{Mat}_{m \times n}(R) / T_{G^{(1)}, M}(A) \), i.e., its annihilator ideal. An algorithm to compute \( T_{G^{(1)}, M}(A) \) is described in [33] for the filtration \( \{m^j\} \). The module \( T_{G^{(1)}, M} \) is usually close to \( T_{G, M} \), with the simple bound \( \mathfrak{m} \cdot T_{G, M} \subseteq T_{G^{(1)}, M} \subseteq T_{G, M} \). And the quotient

\[
T_{\text{Mat}_{m \times n}(R), G, A}^1 := \text{Mat}_{m \times n}(R) / T_{G, M}(A)
\]

(68)

is usually better behaved, thus one first studies this quotient. In [7, 10, 39] this quotient was extensively studied for several group actions with useful bounds for the annihilator of the module \( T_{\text{Mat}_{m \times n}(R), G, A}^1 \). These led to the simple bounds on the order of determinacy and to full control of finite determinacy.

5.8 Determinacy of families

Suppose \( R \) is an \( S \) algebra, \( S \) an algebra over a field \( \mathbb{k} \), e.g., \( S = \mathbb{k}[\{I\}], \mathbb{k}[t], \mathbb{k}[t] \). We consider the elements of \( R, R^n, \text{Mat}_{m \times n}(R) \) as families of some objects over the base space.
Spec(S). Fixing a section $\mathrm{Spec}(S) \to \mathrm{Spec}(R)$ we have the (local) families of elements in the families of modules, $\{z_t \in M_t\}$. The groups $\mathrm{Aut}_k(R)$, $GL(n, R) \rtimes \mathrm{Aut}_k(R)$, $GL(m, R) \rtimes GL(n, R) \rtimes \mathrm{Aut}_k(R)$, etc., induce equivalence of such families. (The equivalence acts as identity on the base $\mathrm{Spec}(S)$ and maps fibers to fibers.) Then one speaks about the order of determinacy of “families of elements inside families of modules, under the action of some group families”.

A family $\{z_t \in M_t\}$ is $N$-determined under the action of the family $\{G_t\}$ if $\{\{z_t\} + M_{N+1,t} \subseteq G_t z_t\}$, for some $N$. \hfill (69)

Then, as in all the examples of this section, we get criteria for finite determinacy and for bounds of the order of determinacy for right/contact/etc. equivalence of families. Note that the determinacy is in general not semicontinuous in a flat family (cf. Example 1.2.2.4 of [34]), but it can usually be bounded by a semicontinuous invariant. This follows from the semicontinuity theorem [32, Proposition 3.4] and its application in [32, Theorem 4.6].

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