Bayesian Wavelet Shrinkage with Beta Priors

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Abstract

We present a Bayesian approach for wavelet shrinkage in the context of non-parametric curve estimation with the use of the beta distribution with symmetric support around zero as the prior distribution for the location parameter in the wavelet domain in models with additive Gaussian errors. Explicit formulas of shrinkage rules for particular cases are obtained, statistical properties such as bias, classical and Bayesian risk of the rules are analyzed and performance of the proposed rules is assessed in simulations studies involving standard test functions. Application to Spike Sorting real data set is provided.

1 Introduction

Wavelet-based methods are widely applied in a range of fields, such as mathematics, signal and image processing, geophysics, and many others. In statistics, applications of wavelets arise mainly in the areas of non-parametric regression, density estimation, functional data analysis and stochastic processes. These methods basically utilize the possibility of representing functions that belong to certain functional spaces as expansions in wavelet bases, similar to others expansions such as splines or Fourier (among others) for example. However, wavelet expansions have characteristics that make them quite useful from the point of view of
function representation: they are localized in both time and scale in an adaptive way. Their
coefficients are typically sparse, they can be obtained by fast computational algorithms, and
the magnitudes of coefficients are linked with the smoothness properties of the functions
they represent. These properties enable time/frequency data analysis, bring computational
advantages, and allow for statistical data modeling at different resolution scales.

Grossmann and Morlet (1984), Daubechies (1988, 1992) and Mallat (1998) are standard
early references about wavelets in general and Ogden (1997), Vidakovic (1999), Percival and
Walden (2000), Nason (2008), Antoniadis (1997), Morettin (1999) and Morettin et al (2017)
are well-cited references about wavelet techniques in statistics.

Wavelet shrinkage methods are used to estimate the coefficients associated with the rep-
resentation of the function in the wavelet domain by reducing the magnitude of the observed
(empirical) coefficients that are obtained by the wavelet transform of the the original data.
There are in fact several shrinkage techniques available in the literature. The main works
in this area are of Donoho and Johnstone (1994, 1995, 1998), but also Donoho et al. (1995,
1996a, 1996b), Johnstone e Silverman (1997), Vidakovic (1998, 1999) and Antoniadis et al.
(2002) can be cited. For more details of shrinkage methods, see Vidakovic (1999) and Jansen
(2012).

It is well known that optimal rules in Bayesian models are shrinkage rules, except for
pathological cases. Bayesian approach offers theoretical paradigm for statistical formaliza-
tion of wavelet shrinkage. In addition to their controllable shrinkage properties, Bayesian
methods have also been extensively studied for the possibility of incorporating, by means of
a prior probabilistic distributions, prior information about the regression smoothness,
neighboring coefficients and other parameters to be estimated. Bayesian models in the wavelet
domain have showed to be capable of incorporating prior information about the unknown re-
gression function such as smoothness, periodicity, self-similarity, degree of “noisiness” (SNR),
and, for some particular bases (e.g., Haar), monotonicity.

In this sense, the choice of the prior distribution in the statistical model describing
wavelet coefficients is extremely important to achieve meaningful results.
Several priors models in the wavelet domain were proposed since 1990s, see Chipman et all (1997), Abramovich, Sapatinas and Silverman (1998), Abramovich and Sapatinas (1999), Vidakovic (1998), Vidakovic and Ruggeri (2001), Angelini and Vidakovic (2004), Johnstone and Silverman (2004, 2005), among others.

The standard statistical models in which shrinkage techniques are applied assume Gaussian additive errors. These models are important not only from their applicability to a range of different problems, but also from the mathematical point of view since the Gaussian additive errors remain both Gaussian and additive after the wavelet transformation.

In this paper we propose and explore a beta distribution symmetric around zero as a prior distribution for the location parameter in a Gaussian model on wavelet coefficients. As traditionally done in this kind of analysis the prior is in fact a distribution contaminated at 0. This added point mass at zero to the spread part of the prior facilitates thresholding. The flexibility of the beta distribution, as a spread part of the prior, is readily controlled by convenient choice of its parameters. Moreover, we show that there is an interesting relationship between the prior (hyper) parameters and the degree of wavelet shrinkage, which is useful in practical applications.

In this paper we aim to incorporate prior belief on the boundedness of the energy of the signal (the $L^2$-norm of the regression function). The prior information on the energy bound often exists in real life problems and it can be modeled by the assumption that the location parameter space is bounded, which in our context relates to estimating a bounded normal mean. Estimation of a bounded normal mean has been considered in Miyasawa (1953), Bickel (1981), Casella and Strawderman (1981), and Vidakovic and DasGupta (1996). In the context of Bayesian modeling, if the structure of the prior can be supported by the analysis of the empirical distribution of the wavelet coefficients, the precise elicitation of the prior hyperparameters cannot be done without some kind of guidance, especially in the Empirical Bayes spirit. Of course, when prior knowledge on the signal-to-noise ratio ($SNR$) is available, then any 0-contaminated symmetric and unimodal distribution supported on the bounded set, say $[-m, m]$, could be a possible candidate for the prior. If the problem is
rescaled so that the size of the noise (its variance) is 1, then $m$ can be taken as $SNR$.

This paper is organized as follows: Section 2 defines the considered model in time and wavelet domain and the proposed prior distribution, Section 3 presents the shrinkage rule, shows statistical properties such as variance, bias and risks and explicit formulas of two particular cases of the shrinkage rule under beta prior and of an extension the shrinkage rule under triangular prior. Section 4 is dedicated to prior elicitation. To verify the strength of the proposed approach simulation studies are performed in Section 5, and the shrinkage rule is applied in a Spike Sorting real data set in Section 6. Section 7 provides conclusions.

2 The Model

2.1 The Symmetric Around Zero Beta Distribution

In Statistics, the beta distribution is extensively used to model variables in the $[0,1]$ domain. This distribution is extremely flexible in shape controlled by a convenient choices of its parameters. In our framework, it is convenient to use its version shifted and rescaled to the interval $[-m,m]$ as well as choosing its parameters to keep it symmetric about 0. Therefore, we propose the use of beta distribution with symmetric support around zero as a prior distribution for the wavelet coefficients. Its density function is

$$g(x; a, m) = \frac{(m^2 - x^2)^{(a-1)}}{(2m)^{(2a-1)}B(a,a)} I_{[-m,m]}(x),$$

(2.1)

where $B(\cdot, \cdot)$ is the standard beta function, $a > 0$ and $m > 0$ are the parameters of the distribution, and $I_{[-m,m]}(\cdot)$ is an indicator function equal to 1 for its argument in the interval $[-m, m]$ and 0 else.

For $a > 1$, the density function (2.1) is unimodal around zero and as $a$ increases, the density becomes more concentrated around zero. This is an important feature for wavelet shrinkage methods, since high values of $a$ imply higher levels of shrinkage of the empirical coefficients, which results in sparse estimated coefficients. Figure 2.1 shows the beta density function (2.1) for some selected values of $a$ and $m = 3$. 

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2.2 Beta Distribution as Prior for the Wavelet Coefficients

We start with the nonparametric regression problem of the form

\[ y_i = f(x_i) + e_i, \quad i = 1, \ldots, n = 2^J, J \in \mathbb{N}, \]  

(2.2)

where \( f \in L^2(\mathbb{R}) = \{ f : \int f^2 < \infty \} \) and \( e_i, i = 1, \ldots, n, \) are zero mean independent normal random variables with unknown variance \( \sigma^2. \) In vector notation, we have

\[ y = f + e, \]  

(2.3)

where \( y = (y_1, \ldots, y_n)', f = (f(x_1), \ldots, f(x_n))' \) and \( e = (e_1, \ldots, e_n)' \). The goal is to estimate the unknown function \( f. \) After applying a discrete wavelet transform (DWT) on (2.3), given by the orthogonal matrix \( W, \) we obtain the following model, in the wavelet domain,

\[ d = \theta + \epsilon, \]  

(2.4)

where \( d = Wy, \theta = Wf \) and \( \epsilon = We. \)

Due to the independence of the random errors and the orthogonality of the \( W \) transform, the model in the wavelet domain remains additive and the errors are i.i.d. normal.
Because the strong decorrelating property of wavelets we can study one coefficient at a time. Of course, wavelet coefficients are “almost decorrelated,” but never fully decorrelated, except for special cases, and strategies for involvement of neighboring coefficients in the shrinkage policy are common in the literature. However, these strategies increase the complexity of the shrinkage for very modest improvements, and still favored approach, especially among the practitioners, is to consider shrinkage coefficient-by-coefficient.

For the $i$th component of the vector $d$, we have a simple model

$$d_i = \theta_i + \epsilon_i,$$

(2.5)

where $d_i$ is the empirical wavelet coefficient, $\theta_i \in [-m, m]$ is the coefficient to be estimated and $\epsilon_i \sim N(0, \sigma^2)$ is the normal random error with unknown variance $\sigma^2$. For simplicity of notation, we suppress the subindices of $d$, $\theta$ and $\epsilon$. Note that, according to the model (2.5), $d|\theta \sim N(\theta, \sigma^2)$ and then, the problem of estimating a function $f$ becomes a normal mean estimation problem in the wavelet domain for each coefficient.

To complete the Bayesian model, we propose the following prior distribution for $\theta$,

$$\pi(\theta; \alpha, a, m) = \alpha \delta_0(\theta) + (1 - \alpha)g(\theta; a, m),$$

(2.6)

where $\alpha \in (0, 1)$, $\delta_0(\theta)$ is the point mass function at zero and $g(\theta; a, m)$ is the beta distribution (2.1) in $[-m, m]$. The proposed prior distribution has $\alpha \in (0, 1)$, $a > 0$ and $m > 0$ as hyperparameters and their choices are directly related to the degree of shrinkage of the empirical coefficients. It will be shown that as $a$ or $\alpha$ (or both of them) increase, the degree of shrinkage increases as well.

3 The Shrinkage Rule

The shrinkage rule $\delta(\cdot)$ for Bayesian estimation of the wavelet coefficient $\theta$ of model (2.5) depends on the choice of location of the posterior (mean, mode, or median) and the loss function. Under square error loss function $L(\delta, \theta) = (\delta - \theta)^2$, it is well known that the Bayes rule is the posterior expected value of $\theta$, i.e., $\delta(d) = E_{\pi}(\theta \mid d)$ minimized Bayes risk. The
Proposition 3.1 gives an expression of the shrinkage rule under a mixture prior consisting of a point mass at zero and a density function with support in \([-m, m]\).

**Proposition 3.1.** If the prior distribution of \(\theta\) is of the form 
\[
\pi(\theta; \alpha, m) = \alpha \delta_0(\theta) + (1 - \alpha) g(\theta),
\]
where \(g\) is a density function with support in \([-m, m]\), then the shrinkage rule under the quadratic loss function is given by
\[
\delta(d) = \frac{(1 - \alpha) \int \frac{m - d}{m - u} (\sigma u + d) g(\sigma u + d) \phi(u) du}{\alpha \frac{1}{\sigma} \phi\left(\frac{d}{\sigma}\right) + (1 - \alpha) \int \frac{m - d}{m - u} g(\sigma u + d) \phi(u) du}
\]
where \(\phi(\cdot)\) is the standard normal density function.

**Proof.** If \(L(\cdot | \theta)\) is the likelihood function, we have that
\[
\delta(d) = E_\pi(\theta | d) = \frac{\int_\Theta \theta [\alpha \delta_0(\theta) + (1 - \alpha) g(\theta)] L(d | \theta) d\theta}{\int_\Theta [\alpha \delta_0(\theta) + (1 - \alpha) g(\theta)] L(d | \theta) d\theta}
\]
\[
= \frac{(1 - \alpha) \int_{-m}^m \sigma g(\theta) \frac{1}{\sqrt{2\pi} \sigma} \exp\left\{-\frac{1}{2} \left(\frac{d - \theta}{\sigma}\right)^2\right\} d\theta}{\alpha \frac{1}{\sigma} \phi\left(\frac{d}{\sigma}\right) + (1 - \alpha) \int_{-m}^m \sigma g(\theta) \frac{1}{\sqrt{2\pi} \sigma} \exp\left\{-\frac{1}{2} \left(\frac{d - \theta}{\sigma}\right)^2\right\} d\theta}
\]
\[
= \frac{(1 - \alpha) \int \frac{m - d}{m - u} (\sigma u + d) g(\sigma u + d) \phi(u) du}{\alpha \frac{1}{\sigma} \phi\left(\frac{d}{\sigma}\right) + (1 - \alpha) \int \frac{m - d}{m - u} g(\sigma u + d) \phi(u) du}.
\]
to be estimated belong to the range $[-m, m]$, so that empirical coefficients outside this range occur due to the presence of noise.

Figure 3.1: Shrinkage rules and their variances under beta prior distribution with hyperparameters $m = 3$ and $\alpha = 0.9$.

Figure 3.2 shows the bias and classical risks respectively for the shrinkage rules under beta prior distribution with hyperparameters $m = 3$ and $\alpha = 0.9$ showed in Figure 2.1. Observe that, as expected, the rules have smaller variances and biases for values of $\theta$ near zero, reaching minimum values in both graphs when the wavelet coefficient is zero. It is also noted that as hyperparameter $a$ increases, the bias of the estimator increases and the variance decreases. The classical risk decreases as $\theta$ tends to zero and that for high values of $\theta$, the risk is larger for rules with large values of $a$. These features are justified by the fact that the degree of shrinkage increases as the hyperparameter $a$ increases, so if the value of the wavelet coefficient is far from zero, such rules with larger values of $a$ tend to underestimate $\theta$ than rules with small values of $a$. 
Figure 3.2: Bias and classical risks of the shrinkage rules under beta prior distribution with hyperparameters \( m = 3 \) and \( \alpha = 0.9 \).

Table 3.1 shows Bayes risks in terms of the hyperparameter \( a \) of the shrinkage rules considered. As expected, Bayes risk decreases as \( a \) increases.

| \( a \) | 0.5 | 1   | 2   | 5   |
|-------|-----|-----|-----|-----|
| \( r \) | 0.226 | 0.188 | 0.137 | 0.008 |

Table 3.1: Bayes risks of the shrinkage rules under beta prior distribution with hyperparameters \( m = 3 \) and \( \alpha = 0.9 \).

### 3.1 Particular Cases

In this section we show two particular cases of the shrinkage rule under beta prior and an extension of the beta prior, the triangular prior. Angelini and Vidakovic (2004) studied in details the shrinkage rule under uniform prior, an interesting particular case of beta distribution with the hyperparameter \( a = 1 \). The proofs of Propositions 3.2, 3.3 and 3.4 use the Lemma 3.1 whose proofs are in the Appendix.
Lemma 3.1. Let \( \phi(\cdot) \) and \( \Phi(\cdot) \) be the standard normal density and cumulative distribution functions respectively and \( a, b \in \mathbb{R} \). Then,

1. \( \int_a^b x \phi(x) dx = \phi(a) - \phi(b) \),
2. \( \int_a^b x^2 \phi(x) dx = [a \phi(a) - b \phi(b)] + [\Phi(b) - \Phi(a)] \),
3. \( \int_a^b x^3 \phi(x) dx = (a^2 + 2a) \phi(a) - (b^2 + 2b) \phi(b) \).

3.1.1 Shrinkage Rule for \( a = 2 \)

Proposition 3.2. The shrinkage rule under prior distribution of the form \( \pi(\theta; \alpha, a, m) = a \delta_0(\theta) + (1 - a) g(\theta; a, m) \), where \( g(\cdot; a, m) \) is the beta distribution in \([-m, m]\) with the hyperparameter \( a = 2 \) is

\[
\delta(d) = \frac{(1 - \alpha) \{N_1(d)[\Phi(q_2) - \Phi(q_1)] + N_2(d) \phi(q_1) + N_3(d) \phi(q_2)\}}{\frac{4m^3a}{3\sigma} \phi\left(\frac{d}{\sigma}\right) + (1 - \alpha) \{D_1(d)[\Phi(q_2) - \Phi(q_1)] + D_2(d) \phi(q_1) + D_3(d) \phi(q_2)\}},
\]

where \( q_1 = -\frac{m - d}{\sigma}, q_2 = \frac{m - d}{\sigma}, N_1(d) = -\frac{3}{\sigma^2}d + dm^2 - d^3, N_2(d) = -\frac{\sigma^3}{2}(q_1^2 + 2) - 3\sigma^2dq_1 + \sigma m^2 - 3\sigma d^2, N_3(d) = \frac{\sigma}{2}(q_1^2 + 2) + 3\sigma^2dq_2 + 3\sigma d^2 - \sigma m^2, D_1(d) = m^2 - d^2 - \sigma^2, D_2(d) = -2\sigma d - \sigma^2 q_1 \) and \( D_3(d) = 2\sigma d + \sigma^2 q_2 \).

Proof. In the Proposition 3.1, we do \( g(\theta) \) as the beta distribution \[(2.1) \] with \( a = 2 \). Then,

\[
\delta(d) = \frac{(1 - \alpha) \int_{-\frac{m-d}{\sigma}}^{\frac{m-d}{\sigma}} (\sigma u + d) \frac{[m^2 - (\sigma u + d)^2]^{2-1}}{(2m)^2 B(2,2)} \phi(u) du}{\frac{4m^3a}{3\sigma} \phi\left(\frac{d}{\sigma}\right) + (1 - \alpha) \int_{-\frac{m-d}{\sigma}}^{\frac{m-d}{\sigma}} \frac{[m^2 - (\sigma u + d)^2]^{2-1}}{(2m)^2 B(2,2)} \phi(u) du}
\]

\[
= \frac{(1 - \alpha) I_1}{C \phi\left(\frac{d}{\sigma}\right) + (1 - \alpha) I_2},
\]

where \( C = (2m^3) B(2,2) \alpha \frac{1}{\sigma} = 4m^3 \alpha / 3 \sigma \), \( q_1 = (-m - d) / \sigma \), \( q_2 = (m - d) / \sigma \), \( I_1 \) and \( I_2 \) are the integrals of the numerator and denominator, respectively.

\[
I_1 = \int_{q_1}^{q_2} (\sigma u + d)[m^2 - (\sigma u + d)^2] \phi(u) du = N_1(d) [\Phi(q_2) - \Phi(q_1)] + N_2(d) \phi(q_1) + N_3(d) \phi(q_2),
\]

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where \( N_1(d) = -3\sigma^2 + dm^2 - d^3 \), \( N_2(d) = -\sigma^3(q_1^2 + 2) - 3\sigma^2 dq_1 + \sigma m^2 - 3\sigma d^2 \), \( N_3(d) = \sigma^3(q_2^2 + 2) + 3\sigma^2 dq_2 + 3\sigma d^2 - \sigma m^2 \). Similarly,

\[
I_2 = \int_{q_1}^{q_2} [m^2 - (\sigma u + d)^2] \phi(u) du \\
= D_1(d)[\Phi(q_2) - \Phi(q_1)] + D_2(d)\phi(q_1) + D_3(d)\phi(q_2),
\]

where \( D_1(d) = (m^2 - d^2 - \sigma^2) \), \( D_2(d) = -(2\sigma d + \sigma^2 q_1) \) e \( D_3(d) = (2\sigma d + \sigma^2 q_2) \).

Figure 3.3 shows the shrinkage rules (3.2), bias, variance and classical risk for the hyperparameters \( m = 3 \) and for \( \alpha = 0.8, 0.9, 0.99 \).

Figure 3.3: Shrinkage rules, bias, variances and classical risks of these rules under beta prior for \( a = 2 \) and \( m = 3 \). Green color for \( \alpha = 0.99 \), red color for \( \alpha = 0.9 \) and black color for \( \alpha = 0.8 \).
As expected, the rule presents a higher degree of shrinkage as the hyperparameter $\alpha$ increases. Similarly to the previous section results, the bias increases and the classical risk increases and the variance decreases as $\alpha$ increases. Observe that both bias, variance and classical risk increase as $\theta$ moves away from zero and approaches $m = 3$.

Table 3.2 presents the Bayes risks of the rules (3.2) for the hyperparameters $m = 3$ and for $\alpha = 0.8, 0.9, 0.99$.

| $\alpha$ | 0.8   | 0.9   | 0.99  |
|----------|-------|-------|-------|
| $r$      | 0.241 | 0.137 | 0.016 |

Table 3.2: Bayes risks of the shrinkage rules under beta prior with the hyperparameters $a = 2$ and $m = 3$

The Bayes risk decreases as $\alpha$ increases, since increasing $\alpha$ implies a greater degree of shrinkage of empirical coefficients.

### 3.1.2 Shrinkage Rule for $a = \frac{1}{2}$

**Proposition 3.3.** The shrinkage rule under prior distribution of the form $\pi(\theta; \alpha, a, m) = \alpha \delta_0(\theta) + (1 - \alpha) g(\theta; a, m)$, where $g(\cdot; a, m)$ is the beta distribution on $[-m, m]$ with the hyperparameter $a = \frac{1}{2}$ can be approximated by

$$\delta(d) \approx \frac{(1 - \alpha)[P_1(d)\phi\left(\frac{m-1-d}{\sigma}\right) - P_2(d)\phi\left(\frac{1-m-d}{\sigma}\right)]}{C\phi\left(\frac{d}{\sigma}\right) + (1 - \alpha)[P_3(d)\phi\left(\frac{m-1-d}{\sigma}\right) - P_4(d)\phi\left(\frac{1-m-d}{\sigma}\right)]},$$

(3.3)

where

$$C = 2\pi\alpha\sigma^2\sqrt{2m - 1},$$

$$P_1(d) = 2\sigma^2(m - 1) + (m - 1)d + \sigma^2 - (m - 1)^2 + \sigma^2(m - 1)^2(2m - 1)^{-1},$$

$$P_2(d) = 2\sigma^2(m - 1) - (m - 1)d + \sigma^2 - (m - 1)^2 + \sigma^2(m - 1)^2(2m - 1)^{-1},$$

$$P_3(d) = 2\sigma^2 - (m - 1) + d + \sigma^2(m - 1)(2m - 1)^{-1},$$

and

$$P_4(d) = -2\sigma^2 + (m - 1) + d - \sigma^2(m - 1)(2m - 1)^{-1}.$$
Proof. In Proposition 3.1 we take \( g(\theta) \) as the beta distribution (2.1) with \( a = 0.5 \). Then,

\[
\delta(d) = \frac{(1 - \alpha) \int_{-m}^{m} \theta (2\sigma^2)^{-\frac{1}{2}} \frac{1}{\sigma \sqrt{2\pi}} \exp\left\{-\frac{(\theta - d)^2}{2\sigma^2}\right\} d\theta}{\alpha \frac{1}{\sigma} \phi\left(\frac{d}{\sigma}\right) + (1 - \alpha) \int_{-m}^{m} \theta (2\sigma^2)^{-\frac{1}{2}} \frac{1}{\sigma \sqrt{2\pi}} \exp\left\{-\frac{(\theta - d)^2}{2\sigma^2}\right\} d\theta}
\]

where \( C = \alpha \pi \sqrt{2\pi} \), \( I_1 \) e \( I_2 \) are the integrals of the numerator and denominator respectively.

In fact, we approximate these integrals using second order Taylor series.

The following approximation of the integral in the numerator can be obtained around a value \( c \):

\[
T_1(c, \theta) = \frac{\exp\left\{-\frac{(c-d)^2}{2\sigma^2}\right\}}{2(m^2 - c^2)^{-\frac{1}{2}} \sigma^2} \{2\sigma^2 (\theta - c) + [cd + \sigma^2 - c^2 + \sigma^2 (m^2 - c^2)^{-1}](\theta - c)^2\}.
\]

Therefore,

\[
I_1 \approx T_1(m - 1, m) - T_1(-m + 1, -m) = \frac{\exp\left\{-\frac{(1-m-d)^2}{2\sigma^2}\right\} P_1(d) - \exp\left\{-\frac{(1-m-d)^2}{2\sigma^2}\right\} P_2(d)}{2\sigma^2(2m - 1)^{-\frac{1}{2}}}.\]

In the same way, the approximation of the integral in the denominator around \( c \) is

\[
T_2(c, \theta) = \frac{\exp\left\{-\frac{(c-d)^2}{2\sigma^2}\right\}}{2(m^2 - c^2)^{-\frac{1}{2}} \sigma^2} \{2\sigma^2 (\theta - c) - [c + d + \sigma^2 (m^2 - c^2)^{-1}](\theta - c)^2\}.
\]

Then,

\[
I_2 \approx T_2(m - 1, m) - T_2(-m + 1, -m) = \frac{\exp\left\{-\frac{(1-m-d)^2}{2\sigma^2}\right\} P_3(d) - \exp\left\{-\frac{(1-m-d)^2}{2\sigma^2}\right\} P_4(d)}{2\sigma^2(2m - 1)^{-\frac{1}{2}}}.\]

Therefore,

\[
\delta(d) \approx \frac{(1 - \alpha) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1-m-d)^2}{2\sigma^2}\right\} P_1(d) - \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1-m-d)^2}{2\sigma^2}\right\} P_2(d)}{2\alpha \pi \sigma^2 (2m - 1)^{-\frac{1}{2}} \phi\left(\frac{d}{\sigma}\right) + (1 - \alpha) \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1-m-d)^2}{2\sigma^2}\right\} P_3(d) - \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1-m-d)^2}{2\sigma^2}\right\} P_4(d)}.
\]

Figure 3.4 shows the Bayes rule and the approximation given in Proposition 3.3 for the case \( m = 3 \) and \( \alpha = 0.9 \).
3.1.3 Shrinkage Rule for Triangular Prior

We present the triangular prior distribution for the wavelet coefficients as an extension of the beta distribution. In fact, the triangular distribution in $[-m, m]$ is the convolution of two uniform distributions in $[-m/2, m/2]$.

The triangular distribution in the interval $[-m, m]$ symmetric around zero has density function given by

$$g_T(x; m) = \begin{cases} 
\frac{x+m}{m^2}, & \text{se } -m < x < 0, \\
\frac{m-x}{m^2}, & \text{se } 0 \leq x < m \\
0, & \text{se } c.c.
\end{cases} \quad (3.4)$$

The following proposition provides an explicit formula for the shrinkage rule under triangular prior.

**Proposition 3.4.** The shrinkage rule under prior distribution of the form $\pi(\theta; \alpha, a, m) = \alpha \delta_0(\theta) + (1 - \alpha)g_T(\theta; m)$, where $g_T(\cdot; m)$ is the triangular distribution over $[-m, m]$, is
\[
\delta_T(d) = \frac{(1 - \alpha) S_1(d)}{\alpha \frac{m^2}{\sigma} \Phi \left( \frac{d}{\sigma} \right) + (1 - \alpha) S_2(d)},
\]  
(3.5)

where
\[
S_1(d) = d \sigma [\phi \left( \frac{m + d}{\sigma} \right) + \phi \left( \frac{m - d}{\sigma} \right) - 2 \phi \left( \frac{d}{\sigma} \right)] + (d^2 + \sigma^2 + dm) \Phi \left( \frac{m + d}{\sigma} \right) + (d^2 + \sigma^2 - dm) \Phi \left( \frac{m - d}{\sigma} \right) - 2(d^2 + \sigma^2) \Phi \left( \frac{d}{\sigma} \right)
\]
and
\[
S_2(d) = \sigma [\phi \left( \frac{m + d}{\sigma} \right) + \phi \left( \frac{m - d}{\sigma} \right) - 2 \phi \left( \frac{d}{\sigma} \right)] + (d + m) \Phi \left( \frac{m + d}{\sigma} \right) + (d - m) \Phi \left( \frac{m - d}{\sigma} \right) - 2d \Phi \left( \frac{d}{\sigma} \right).
\]

**Proof.** Applying Proposition 3.1, we get
\[
\delta_T(d) = \frac{(1 - \alpha) \int_{-\sigma}^{\frac{m - d}{\sigma}} (\sigma u + d) g_T(\sigma u + d) \phi(u) du}{\alpha \frac{1}{\sigma^2} \phi \left( \frac{d}{\sigma} \right) + (1 - \alpha) \int_{-\frac{m}{\sigma}}^{\frac{m - d}{\sigma}} g_T(\sigma u + d) \phi(u) du}
\]
\[
= \frac{(1 - \alpha) I_2}{\alpha \frac{1}{\sigma^2} \phi \left( \frac{d}{\sigma} \right) + (1 - \alpha) I_1}.
\]

For \( I_1 \), the integral in the denominator, we have that
\[
I_1 = \int_{-\sigma}^{\frac{m - d}{\sigma}} g_T(\sigma u + d) \phi(u) du
\]
\[
= \int_{-\sigma}^{\frac{m - d}{\sigma}} \left[ (\sigma u + d) + m \right] \phi(u) du + \int_{-\frac{m}{\sigma}}^{\frac{m - d}{\sigma}} \frac{|m - (\sigma u + d)|}{m^2} \phi(u) du
\]
\[
= I_1^* + I_1^{**}.
\]

We calculate \( I_1^* \) e \( I_1^{**} \) separately, i.e,
\[
I_1^* = \int_{-\sigma}^{\frac{m - d}{\sigma}} \left[ (\sigma u + d) + m \right] \frac{m^2}{\sigma^2} \phi(u) du
\]
\[
= \frac{1}{m^2} \left\{ \sigma \left[ \phi \left( \frac{m + d}{\sigma} \right) - \phi \left( \frac{d}{\sigma} \right) \right] + (d + m) \left[ \Phi \left( \frac{-d}{\sigma} \right) - \Phi \left( \frac{-m - d}{\sigma} \right) \right] \right\}.
\]
\[
I_1^{**} = \int_{-\sigma}^{\frac{m - d}{\sigma}} \frac{|m - (\sigma u + d)|}{m^2} \phi(u) du
\]
\[
= \frac{1}{m^2} \left\{ (m - d) \left[ \Phi \left( \frac{m - d}{\sigma} \right) - \Phi \left( \frac{-d}{\sigma} \right) \right] - \sigma \left[ \phi \left( \frac{d}{\sigma} \right) - \phi \left( \frac{m - d}{\sigma} \right) \right] \right\}.
\]

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For $I_2$, the integral in the numerator,

$$I_2 = \sigma \int_{-\sigma}^{\sigma} u g_T(\sigma u + d) \phi(u) du + d \int_{-\sigma}^{\sigma} g_T(\sigma u + d) \phi(u) du = \sigma I_2^* + d I_1.$$ 

$$I_2^* = \int_{-\sigma}^{\sigma} u \left[ \frac{(\sigma u + d) + m}{m^2} \right] \phi(u) du + \int_{-\sigma}^{\sigma} u \left[ \frac{m - (\sigma u + d)}{m^2} \right] \phi(u) du$$

$$= \frac{\sigma}{m^2} \left[ 2 \Phi \left( \frac{-d}{\sigma} \right) - \Phi \left( \frac{-m - d}{\sigma} \right) - \Phi \left( \frac{m - d}{\sigma} \right) \right].$$

In this way, we obtain $I_2$ as

$$I_2 = \frac{1}{m^2} \left\{ d \sigma \left[ \phi \left( \frac{m + d}{\sigma} \right) + \phi \left( \frac{md}{\sigma} \right) - 2 \phi \left( \frac{d}{\sigma} \right) \right] + (\sigma^2 + dm + d^2) \Phi \left( \frac{m + d}{\sigma} \right) + (\sigma^2 - dm + d^2) \Phi \left( \frac{d - m}{\sigma} \right) - 2(\sigma^2 + d^2) \Phi \left( \frac{d}{\sigma} \right) \right\}. $$

Finally, substituting the integrals $I_1$ and $I_2$ into (???), we have

$$\delta_T(d) = \frac{(1 - \alpha) S_1(d)}{\frac{am^2}{\sigma^3} \phi \left( \frac{d}{\sigma} \right) + (1 - \alpha) S_2(d)}.$$

4 Default Prior Hyperparameters

Methods and criteria for determination of the involved parameters and hyperparameters to estimate the coefficients are important in Bayesian procedures. In the framework of Bayesian shrinkage with beta prior, the choices of the $\sigma$ parameter of the random error distribution and the hyperparameters $\alpha$, $m$ and $a$ of the beta prior distribution of the wavelet coefficient are required. We present the methods and criteria already available in the literature for such choices and used in simulation and application studies.

Based on the fact that much of the noise information present in the data can be obtained on the finest resolution scale, for the robust $\sigma$ estimation, Donoho and Johnstone (1994) suggested robust estimator

$$\delta_T(d) = \frac{(1 - \alpha) S_1(d)}{\frac{am^2}{\sigma^3} \phi \left( \frac{d}{\sigma} \right) + (1 - \alpha) S_2(d)}.$$

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\[ \hat{\sigma} = \frac{\text{median}\{|d_{j-1,k}| : k = 0, \ldots, 2^{J-1}\}}{0.6745}. \] (4.1)

Angelini et al. (2004) suggested the hyperparameters \( \alpha \) and \( m \) be dependent on the level of resolution \( j \) according to the expressions

\[ \alpha = \alpha(j) = 1 - \frac{1}{(j - J_0 + 1)^\gamma} \] (4.2)

and

\[ m = m(j) = \max_k \{|d_{jk}|\}, \] (4.3)

where \( J_0 \leq j \leq J - 1 \), \( J_0 \) is the primary resolution level and \( \gamma > 0 \). They also suggested that in the absence of additional information, \( \gamma = 2 \) can be adopted.

Chaloner et al. (1983) proposed a method for choosing the hyperparameters of the beta distribution as a priori distribution for the probability of success in each Bernoulli assay of the binomial distribution. Duran and Booker (1988) propose the percentile method. For \( k \in [-m, m] \) and \( p \in (0, 1) \) fixed, \( a \) is chosen so that

\[ P(\theta \leq k) = p \] (4.4)

i.e.

\[ \int_{-m}^{k} \frac{(m^2 - \theta^2)^{(a-1)}}{(2m)^{(2a-1)}B(a,a)} d\theta = p. \] (4.5)

Thus, the choice of \( a \) is made by determining the probability of occurrence of a particular event \( \{\theta \leq k\} \). This procedure is interesting because of the greater facility of subjective determination of a \( p \) probability, that is, it is simpler to cognitively assign a probability to a certain event than to directly assign a value to the parameter of a probability distribution.

5 Simulation Studies

Simulation studies were done to evaluate the performance of the shrinkage rules under beta prior distribution for the particular cases in which the hyperparameter \( a \) assumes the fixed values \( a = 0.5, 1, 2, 5 \) and triangular distribution and to compare them with the performances
of some available shrinkage methods in the literature, namely, Universal Soft Thresholding (UNIV), False Discovery Rate (FDR), Cross Validation (CV), and Bayesian shrinkage rule under normal prior (BNormal). We also considered the shrinkage rule under Bickel prior, suggested by Angelini and Vidakovic (2004). They proved that the shrinkage rule under this prior is approximately $\Gamma$-minimax for the class of all symmetric unimodal priors bounded on $[-m, m]$, $\Gamma_{SU[-m, m]}$. When $m$ increases, the weak limit of the least Bickel (1981) proved that the least favorable prior in $\Gamma_{SU[-m, m]}$ is approximately (in sense of weak distributional limits) $g_m(\theta) = \frac{1}{m} \cos^2 \left( \frac{\pi \theta}{2m} \right) I_{[-m, m]}(\theta)$. Applying this result in our context, we have that the Bickel shrinkage is induced by prior

$$
\pi(\theta) = \alpha \delta_0 + (1 - \alpha) \frac{1}{m} \cos^2 \left( \frac{\pi \theta}{2m} \right) I_{[-m, m]}(\theta).
$$

The corresponding Bayes rule does not have a simple analytical form and needs to be numerically computed. The hyperparameters $m$ and $\alpha$ were selected according to the proposals described in Section 4.

To perform the simulation, the rules were applied in the Donoho-Johnstone (DJ) test functions. These functions, shown in Figure 5.1, are widely used in the literature for comparison of wavelet-based methods. These are four functions, called Bumps, Blocks, Doppler and Heavisine, which represent some characteristics of curves in real problems.
For each test function, three sample sizes were selected, \( n = 512, 1024 \) and 2048, and for each point, a normal error with zero mean and variance \( \sigma^2 \) was added, where \( \sigma^2 \) was selected according to three signal to noise ratio (SNR), 3, 5 and 7.

We used the mean squared error (MSE), \( MSE = \frac{1}{n} \sum_{i=1}^{n} [\hat{f}(x_i) - f(x_i)]^2 \) as performance measure of the shrinkage rules. For each function, the process was repeated \( M = 500 \) times and a comparison measure, the average of the obtained MSEs, \( AMSE = \frac{1}{M} \sum_{j=1}^{M} MSE_j \), was calculated for each rule as shown in Tables 5.1 and 5.2. Boxplots of the MSEs of the shrinkage rules for \( n = 1024 \) and SNR=5 are shown in Figure 5.2.

Figure 5.1: Donoho-Johnstone (DJ) test functions.
| Signal | n  | Method | SNR=3 | SNR=5 | SNR=7 | Signal | n  | Method | SNR=3 | SNR=5 | SNR=7 |
|--------|----|--------|-------|-------|-------|--------|----|--------|-------|-------|-------|
| Bumps  | 512| UNIV   | 11.073| 5.168 | 3.035 | Blocks | 512| UNIV   | 6.898 | 3.667 | 2.258 |
|        |    | CV     | 11.482| 9.441 | 6.329 |        |    | CV     | 2.546 | 1.248 | 0.840 |
|        |    | FDR    | 9.300 | 4.369 | 2.642 |        |    | FDR    | 5.855 | 2.913 | 1.742 |
|        |    | Normal | 7.723 | 5.169 | 4.429 |        |    | Normal | 2.178 | 1.040 | 0.662 |
|        |    | a = 0 | 3.165 | 1.307 | 0.730 |        |    | a = 0 | 3.230 | 1.327 | 0.769 |
|        |    | a = 1 | 3.005 | 1.236 | 0.699 |        |    | a = 1 | 1.794 | 0.750 | 0.483 |
|        |    | a = 2 | 2.887 | 1.189 | 0.670 |        |    | a = 2 | 2.865 | 1.306 | 0.870 |
|        |    | a = 3 | 2.827 | 1.158 | 0.657 |        |    | a = 3 | 2.713 | 1.187 | 0.669 |
|        |    | Triang | 2.840 | 1.160 | 0.658 |        |    | Triang | 2.763 | 1.273 | 0.815 |
|        |    | Bickel | 2.827 | 1.157 | 0.656 |        |    | Bickel | 2.765 | 1.222 | 1.656 |
|        |    | 1024   | 7.571 | 3.570 | 2.133 |        |    | 1024   | 4.851 | 2.484 | 1.548 |
|        |    | CV     | 2.934 | 1.933 | 1.727 |        |    | CV     | 1.789 | 0.840 | 0.536 |
|        |    | FDR    | 5.593 | 2.536 | 1.471 |        |    | FDR    | 3.899 | 1.882 | 1.129 |
|        |    | Normal | 4.514 | 1.172 | 0.684 |        |    | Normal | 1.616 | 0.679 | 0.400 |
|        |    | a = 0 | 1.990 | 0.794 | 0.423 |        |    | a = 0 | 1.794 | 0.750 | 0.483 |
|        |    | a = 1 | 1.900 | 0.755 | 0.404 |        |    | a = 1 | 1.687 | 0.711 | 0.437 |
|        |    | a = 2 | 1.834 | 0.726 | 0.387 |        |    | a = 2 | 1.623 | 0.830 | 0.403 |
|        |    | a = 5 | 1.799 | 0.724 | 0.448 |        |    | a = 5 | 1.571 | 0.649 | 0.379 |
|        |    | Triang | 1.798 | 0.711 | 0.384 |        |    | Triang | 1.568 | 0.832 | 0.404 |
|        |    | Bickel | 1.800 | 0.712 | 0.380 |        |    | Bickel | 1.571 | 1.641 | 0.383 |
|        |    | 2048   | 5.056 | 2.349 | 1.394 |        |    | 2048   | 3.412 | 1.768 | 1.104 |
|        |    | CV     | 1.609 | 0.738 | 0.462 |        |    | CV     | 1.298 | 0.588 | 0.354 |
|        |    | FDR    | 5.567 | 1.587 | 0.918 |        |    | FDR    | 2.682 | 1.286 | 0.767 |
|        |    | Normal | 1.285 | 0.489 | 0.287 |        |    | Normal | 1.175 | 0.443 | 0.242 |
|        |    | a = 0 | 1.333 | 0.579 | 0.356 |        |    | a = 0 | 1.331 | 0.755 | 2.257 |
|        |    | a = 1 | 1.276 | 0.557 | 0.325 |        |    | a = 1 | 1.283 | 0.652 | 1.916 |
|        |    | a = 2 | 1.233 | 0.539 | 0.353 |        |    | a = 2 | 1.445 | 0.617 | 1.656 |
|        |    | a = 5 | 1.220 | 0.755 | 1.101 |        |    | a = 5 | 1.182 | 0.589 | 2.463 |
|        |    | Triang | 1.215 | 0.523 | 0.324 |        |    | Triang | 1.424 | 0.626 | 1.579 |
|        |    | Bickel | 1.213 | 0.520 | 0.311 |        |    | Bickel | 1.560 | 0.632 | 1.825 |

Table 5.1: AMSE of the shrinkage/thresholding rules in the simulation study for Bumps and Blocks DJ test functions.
| Signal  | n  | Method  | SNR=3 | SNR=5 | SNR=7 | Signal  | n  | Method  | SNR=3 | SNR=5 | SNR=7 |
|--------|----|---------|-------|-------|-------|--------|----|---------|-------|-------|-------|
| Doppler| 512| UNIV    | 2.663 | 1.406 | 0.885 | Heavisine| 512| UNIV    | 0.569 | 0.403 | 0.302 |
|        |    | CV      | 1.285 | 0.642 | 0.446 |        |    | CV      | 0.508 | 0.278 | 0.175 |
|        |    | FDR     | 2.543 | 1.255 | 0.759 |        |    | FDR     | 0.597 | 0.435 | 0.309 |
|        |    | BNormal | 1.170 | 0.528 | 0.316 |        |    | BNormal | 0.475 | 0.244 | 0.165 |
| a = 0.5|    |         | 1.161 | 0.575 | 0.319 | a = 0.5|    |         | 0.734 | 0.542 | 0.458 |
| a = 1  |    |         | 1.124 | 0.558 | 0.311 | a = 1  |    |         | 0.786 | 0.543 | 0.445 |
| a = 2  |    |         | 1.103 | 0.530 | 0.301 | a = 2  |    |         | 0.831 | 0.547 | 0.442 |
| a = 5  |    |         | 1.108 | 0.515 | 0.289 | a = 5  |    |         | 0.971 | 0.670 | 0.726 |
| Triang |    |         | 1.089 | 0.523 | 0.296 | Triang |    |         | 0.839 | 0.566 | 0.439 |
|        |    | Bickel  | 1.094 | 0.518 | 0.287 |        |    | Bickel  | 0.892 | 0.583 | 0.439 |
| 1024   |    | UNIV    | 1.606 | 0.842 | 0.533 |        |    | UNIV    | 0.462 | 0.314 | 0.230 |
|        |    | CV      | 0.797 | 0.366 | 0.218 |        |    | CV      | 0.373 | 0.200 | 0.127 |
|        |    | FDR     | 1.502 | 0.744 | 0.453 |        |    | FDR     | 0.506 | 0.324 | 0.225 |
|        |    | BNormal | 0.825 | 0.283 | 0.150 |        |    | BNormal | 0.292 | 0.142 | 0.099 |
| a = 0.5|    |         | 1.274 | 0.486 | 0.325 | a = 0.5|    |         | 1.016 | 0.832 | 0.799 |
| a = 1  |    |         | 1.156 | 0.464 | 0.307 | a = 1  |    |         | 1.020 | 0.832 | 0.783 |
| a = 2  |    |         | 0.994 | 0.478 | 0.405 | a = 2  |    |         | 1.035 | 0.842 | 0.813 |
| a = 5  |    |         | 0.969 | 0.436 | 0.305 | a = 5  |    |         | 1.142 | 1.062 | 1.474 |
| Triang |    |         | 0.968 | 0.474 | 0.390 | Triang |    |         | 1.042 | 0.829 | 0.817 |
|        |    | Bickel  | 0.986 | 0.447 | 0.840 |        |    | Bickel  | 1.067 | 0.848 | 0.794 |
| 2048   |    | UNIV    | 1.163 | 0.583 | 0.368 |        |    | UNIV    | 0.359 | 0.231 | 0.165 |
|        |    | CV      | 0.560 | 0.255 | 0.148 |        |    | CV      | 0.263 | 0.141 | 0.088 |
|        |    | FDR     | 1.043 | 0.491 | 0.297 |        |    | FDR     | 0.389 | 0.228 | 0.154 |
|        |    | BNormal | 0.402 | 0.153 | 0.084 |        |    | BNormal | 0.185 | 0.099 | 0.064 |
| a = 0.5|    |         | 0.437 | 0.248 | 0.202 | a = 0.5|    |         | 0.475 | 0.366 | 0.315 |
| a = 1  |    |         | 0.439 | 0.240 | 0.182 | a = 1  |    |         | 0.479 | 0.351 | 0.319 |
| a = 2  |    |         | 0.425 | 0.294 | 0.181 | a = 2  |    |         | 0.481 | 0.423 | 0.303 |
| a = 5  |    |         | 0.441 | 0.216 | 0.150 | a = 5  |    |         | 0.626 | 0.441 | 0.314 |
| Triang |    |         | 0.419 | 0.263 | 0.165 | Triang |    |         | 0.486 | 0.400 | 0.316 |
|        |    | Bickel  | 0.432 | 0.791 | 0.149 |        |    | Bickel  | 0.515 | 0.562 | 1.255 |

Table 5.2: AMSE of the shrinkage/thresholding rules in the simulation study for Doppler and Heavisine DJ test functions.
In general, when comparing the beta, triangular, and Bickel prior shrinkage rules with the methods present in the literature, for the three SNR scenarios and the three sample sizes considered, it performed better for the Bumps function, somehow better for Blocks and Doppler functions and worse for the Heavisine function. Specifically, in the simulations involving the Blocks and Doppler functions, the beta rules and their extensions were better in relation to the Universal Thresholding and FDR rules and had performances close to CV and BNormal rules.

Regarding the hyperparameter $a$, it can be observed that the performance of the rule under beta prior improved as $a$ increased, that is, the AMSE decreased as $a$ increased. This
is related to the degree of rule shrinkage. High values of $a$ imply a greater degree of shrinkage of the applied rule, reducing the effect of random noise on the simulated data.

6 Application in Spike Sorting Data Set

Spike sorting is a classification procedure of action potentials (spikes) emitted by neurons according to their different forms and amplitudes. Typically, action potentials data for sorting by spike sorting are collected extracellularly by means of electrodes connected at certain locations in the head of animals. It is a method of extreme relevance in Neuroscience due to the possibility of studies on which neurons are present in certain regions of the brain and how they interact.

Once the raw data of action potentials is collected, the first step of the spike sorting procedure is to filter through noise reduction data to facilitate visualization of spikes and misclassification of noise as spike. Among several methods used for spike sorting, the method of reducing noise by welds is one of the most used. For more details on spike sorting and statistical methods involved in the analysis of characteristic data, one has Pouzat et al. (2002), Lewicki (1994), Shoham et al. (2003), Pettersen et al. (2012), Kadir et al. (2013), Einevoll et al. (2012) among others. Applications of wavelets in spike sorting occur in the works of Quiroga et al. (2004), Letelier and Webber (2000) and Shalchyan et al. (2012).

The purpose here is to apply DWT to the data and use the beta and triangular shrink rule for noise reduction.

The original data set, presented in Figure 6.1, has 20000 neuronal action potentials (tit spikes) observed over time. For the application of DWT, it was considered $n = 2^{14} = 16384$. The data set is a courtesy of Kenneth Harris, of Institute of Neurology, Faculty of Brain Sciences, University College London and it is available at https://ifcs.boku.ac.at/repository/data/spike-sorting/index.html.
To the empirical coefficients obtained, the shrinkage rule was applied under beta and triangular prior. The hyperparameters chosen for $\sigma$, $m$ e $\alpha$ were given according to Section 4, with $\hat{\sigma} = 19913$ e $\hat{a} = 2$ for the beta distribution. Figure 6.2 presents the estimated functions and Figure 6.3 shows the empirical wavelet coefficients after the application of DWT (Daubechies base $N = 10$) and estimated by the rule under beta prior.
(a) Estimated action potentials - beta prior shrinkage rule with $a = 2$.

(b) Estimated action potentials - triangular prior shrinkage rule.

Figure 6.2: Estimated action potentials - beta prior shrinkage rule with $a = 2$ (a) and triangular prior shrinkage rule (b).

(a) Empirical coefficients.

(b) Estimated coefficients - beta prior shrinkage rule with $a = 2$.

Figure 6.3: Empirical coefficients (a) and estimated coefficients - beta prior shrinkage rule with $a = 2$ (b) of the Spike Sorting data set.
7 Conclusion

The paper proposes the use of beta distribution, as well as the triangular distribution, as a prior distribution for the wavelet coefficients and, in fact, the results indicate that the shrinkage rule associated with such distribution perform better than the shrinkage techniques already used in the practice for most of the cases and test functions. In addition, the explicit formulas for shrinkage rules in particular cases where \( a = 2 \), \( a = 0.5 \) and for the rule associated with the triangular distribution can facilitate the computational implementation of such rules and allow mathematical analysis.

Further extensions, generalizations, and new results are planned. The performance of the shrinkage rules on statistical models with other distributions with symmetric support for the random error or even generalizations for asymmetric (but 0-mean) distributions could be considered. The impact of using different wavelet bases in such rules may also be of great interest and were not considered here. As improvement and consolidation of the proposed technique, the use of other performance measures in simulation studies ns comparisons against the state of art techniques, especially for a low SNR will be of interest.

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Appendix

Lemma 7.1. Let $\phi(\cdot)$ and $\Phi(\cdot)$ be the standard normal density and cumulative distribution functions respectively and $a, b \in \mathbb{R}$. Then,

1. $\int_a^b x\phi(x)dx = \phi(a) - \phi(b)$,
2. $\int_a^b x^2\phi(x)dx = [a\phi(a) - b\phi(b)] + [\Phi(b) - \Phi(a)]$,
3. $\int_a^b x^3\phi(x)dx = (a^2 + 2)\phi(a) - (b^2 + 2)\phi(b)$.

Proof. In fact, for the first integral, we have

$$\int x\phi(x)dx = \int x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = -\frac{1}{\sqrt{2\pi}} \int e^{ux} du = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = -\phi(x) + C.$$ The substitution was done using $u = -\frac{x^2}{2}$. Then,

$$\int_a^b x\phi(x)dx = -\phi(b) - (-\phi(a)) = \phi(a) - \phi(b).$$

For the second one, we have

$$\int_a^b x^2\phi(x)dx = \int_a^b x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_a^b x x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$
Let $u = x$ and $dv = x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$, using integration by parts,

$$
\int_a^b x^2 \phi(x) dx = \left[ x^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \right]_a^b + \int_a^b \phi(x) dx = [a \phi(a) - b \phi(b)] + [\Phi(b) - \Phi(a)].
$$

And for the third integral,

$$
\int_a^b x^3 \phi(x) dx = \int_a^b x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int_0^\infty x^3 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \int 2u \frac{1}{\sqrt{2\pi}} e^u du = \frac{2}{\sqrt{2\pi}} e^u (u - 1) = \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} (-\frac{x^2}{2} - 1) = -(x^2 + 2) \phi(x) + C.
$$

In the substitution, we took $u = -\frac{x^2}{2}$. Then,

$$
\int_a^b x^3 \phi(x) dx = -(b^2 + 2) \phi(b) - (-a^2 + 2) \phi(a)) = (a^2 + 2) \phi(a) - (b^2 + 2) \phi(b).
$$