NON-COMMUTATIVE QUADRICS.

MICHEL VAN DEN BERGH

Abstract. In this paper we describe non-commutative versions of $\mathbb{P}^1 \times \mathbb{P}^1$. These contain the class of non-commutative deformations of $\mathbb{P}^1 \times \mathbb{P}^1$.

CONTENTS

1. Introduction 1
2. Acknowledgment 4
3. Reminder on $I$-algebras 4
4. Artin-Schelter regular $\mathbb{Z}$-algebras 7
5. Non-commutative quadrics 10
6. Non-commutative quadrics as hypersurfaces. 21
7. The translation principle 23
8. Non-commutative quadrics as deformations of commutative quadrics 25
References 29

1. INTRODUCTION

Throughout $k$ is an algebraically closed field of characteristic zero. In this paper we perform the $\mathbb{Z}$-algebra version of the classification of 3-dimensional cubic regular algebras in [2, 3, 4] (see below for unexplained terminology). In doing so we were inspired by [9] which essentially treats the quadratic case.

Our main motivation for doing this classification is to describe the non-commutative deformations of $\mathbb{P}^1 \times \mathbb{P}^1$ (in the sense of [15, 16]).

Recall [2] that an AS-regular algebra is a graded $k$-algebra $A = k + A_1 + A_2 + \cdots$ satisfying the following conditions

1. $\dim A_i$ is bounded by a polynomial.
2. The projective dimension of $k$ is finite.
3. There is exactly one $i$ for which $\operatorname{Ext}_A^i(k, A)$ is non-vanishing and for this $i$ we have $\dim \operatorname{Ext}_A^i(k, A) = 1$.

Three dimensional regular algebras generated in degree one were classified in [2, 3, 4] and in general in [24, 25]. It was discovered that they are intimately connected to plane elliptic curves.

There are two possibilities for a three dimensional regular algebra $A$ generated in degree one.

1991 Mathematics Subject Classification. Primary 19S99; Secondary 16S32.

Key words and phrases. non-commutative geometry, non-commutative quadrics, deformations.

The author is a senior researcher at the FWO.
(1) \( A \) is defined by three generators satisfying three quadratic relations (the "quadratic case").

(2) \( A \) is defined by two generators satisfying two cubic relations ("the cubic case").

In [2] it is shown that all 3-dimensional regular algebras are obtained by specialization from a number "generic" regular algebras. These generic regular algebras depend on at most two parameters.

If \( A \) is a non-commutative graded algebra then one defines \( \text{QGr}(A) \) as the category of graded right \( A \)-modules modulo finite dimensional ones [1]. One should think of \( \text{QGr}(A) \) as (the category of quasi-coherent sheaves over) the non-commutative Proj of \( A \).

If \( A \) is a quadratic 3-dimensional regular algebra then it has the Hilbert series of a polynomial ring in three variables. Therefore it is reasonable to define a non-commutative \( \mathbb{P}^2 \) as \( \text{QGr}(A) \) for such an \( A \). There are good reasons why this is the correct definition. See e.g. [9] and also the discussion below.

For a cubic 3-dimensional regular algebra \( A \) it is convenient to look at its 2-Veronese \( A^{(2)} = \bigoplus_{n} A_{2n} \). One has \( \text{QGr}(A) \cong \text{QGr}(A^{(2)}) \) (see e.g. Lemma 3.5 for a more general version). Thus \( A \) and \( A^{(2)} \) describe the same non-commutative space.

The 2-Veronese of \( A \) has Hilbert series \( (1 - t^4)/(1 - t)^4 \) which is the same as the Hilbert series of the homogeneous coordinate ring of \( \mathbb{P}^1 \times \mathbb{P}^1 \) for the Plucker embedding \( \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^4 \). Thus one may wonder if it is correct to define a non-commutative \( \mathbb{P}^1 \times \mathbb{P}^1 \) (or quadric) as \( \text{QGr}(A) \) for a cubic 3-dimensional regular algebra. We claim this is not so and we will now give some motivation for this.

If \( X \) is quasi-compact quasi-separated scheme then according to [15, 16] the obstruction theory for the deformation theory of \( \text{Qch}(X) \) is given by the Hochschild cohomology groups \( \mathbb{H}^2(X), \mathbb{H}^3(X) \) of \( X \) where \( \mathbb{H}^i(X) = \text{Ext}^i_{\mathcal{O}_{X \times X}}(\mathcal{O}_X, \mathcal{O}_X) \). If \( X \) is a smooth \( k \)-variety then the Hochschild cohomology of \( X \) can be computed using the HKR decomposition: \( \mathbb{H}^n(X) = \bigoplus_{i+j=n} H^i(X, \wedge^j T_X) \). A trite computation shows

\[
\dim \mathbb{H}^2(\mathbb{P}^2) = 10, \quad \dim \mathbb{H}^3(\mathbb{P}^2) = 0
\]

and

\[
\dim \mathbb{H}^2(\mathbb{P}^1 \times \mathbb{P}^1) = 9, \quad \dim \mathbb{H}^3(\mathbb{P}^1 \times \mathbb{P}^1) = 0
\]

Thus in both cases the deformation theory is unobstructed. To estimate the actual numbers of parameters we have to subtract the dimensions of automorphism groups of \( \mathbb{P}^2 \) and \( \mathbb{P}^1 \times \mathbb{P}^1 \) which are respectively 8 and 6. So the expected number of parameters for a non-commutative \( \mathbb{P}^2 \) is \( 10 - 8 = 2 \) and for a non-commutative \( \mathbb{P}^1 \times \mathbb{P}^1 \) it is \( 9 - 6 = 3 \). Hence whereas in the case of \( \mathbb{P}^2 \), 3-dimensional regular algebras have the required amount of freedom this is not the case for \( \mathbb{P}^1 \times \mathbb{P}^1 \).

The solution to this problem is presented in this paper. The idea (taken from [9]) is that instead of graded algebras we should look at "\( \mathbb{Z} \)-algebras". I.e. algebras \( A = \bigoplus_{ij \in \mathbb{Z}} A_{ij} \) satisfying \( A_{ij} A_{jk} \subset A_{ik} \) and possessing local units in \( A_{ij} \) (see §3 below). If \( B \) is a \( \mathbb{Z} \)-graded algebra then we may define a corresponding \( \mathbb{Z} \)-algebra \( \hat{B} \) via \( \hat{B}_{ij} = B_{j-i} \). Conversely for a \( \mathbb{Z} \)-algebra to be obtained from a graded algebra it is necessary and sufficient for it to be "1-periodic". I.e. there should be an identification \( A_{ij} \cong A_{i+1,j+1} \) compatible with the multiplication.

The basic definitions and results from the theory of graded rings extend readily to \( \mathbb{Z} \)-algebras and in particular we may define \( \mathbb{Z} \)-algebra analogues of 3-dimensional
are parametrized by the $j$-invariant. For a non-commutative $P$-algebra both in the quadratic and cubic case we obtain the expected number of parameters for a three-dimensional Artin-Schelter regular algebra $D$. There exists a $D$-regular algebra. Proposition 1.3. (Proposition 5.1.2 in the text) The three-dimensional cubic regular algebras are classified in terms of quadruples $(C, L_0, L_1)$ where either:

1. $(C, L_0, L_1) \cong (\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}(1))$, (the “linear” case); or else

2. $C$ is a curve which is embedded as a divisor of degree 3 in $\mathbb{P}^2$ by the global sections of $L_0$ and $L_1$.
   - (a) $\deg(L_0 | E) = \deg(L_1 | E)$ for every irreducible $E$ component of $C$.
   - (c) $L_0 \not\cong L_1$ (the “elliptic” case).

Proposition 1.2. (Proposition 5.1.2 in the text) The three-dimensional cubic regular $Z$-algebras are classified in terms of quadruples $(C, L_0, L_1, L_2)$ where either:

1. $(C, L_0, L_1, L_2) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,0))$ (the “linear” case); or else

2. $C$ is a curve which is embedded as a divisor of degree $(2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ by the global sections of both $(L_0, L_1)$ and $(L_1, L_2)$.
   - (b) $\deg(L_0 | E) = \deg(L_2 | E)$ for every irreducible component $E$ of $C$ (the “elliptic case”).
   - (c) $L_0 \not\cong L_2$.

Assume that $C$ is a smooth elliptic curve. Isomorphism classes of elliptic curves are parametrized by the $j$-invariant. Furthermore $\dim \text{Aut}(C) = 1$, $\dim \text{Pic}(C) = 1$. Hence the number of parameters in the quadratic case is $1[j(C)] + 1[\text{Pic}(C)] + 1[\text{Pic}(C)] - 1[\text{Aut}(C)] = 2$. In the cubic case we find that the number is 3. Thus both in the quadratic and cubic case we obtain the expected number of parameters for a non-commutative $\mathbb{P}^2$, resp. $\mathbb{P}^1 \times \mathbb{P}^1$.

If $k$ is algebraically closed of characteristic different from three then it is shown in [9] that a quadratic $Z$-algebra is $1$-periodic (see Theorem 4.2.2 below) and hence is obtained from a graded algebra. Hence there is no added generality in working with $Z$-algebras. However there is no analogous result for cubic algebras (see §5.6). So in this case we really need $Z$-algebras.

Nonetheless we have the following result.

Proposition 1.3. (see Proposition 6.6 below) Assume that $k$ is an algebraically closed field of characteristic different from two. Let $A$ be a cubic 3-dimensional regular $Z$-algebra. Let $A'$ be the 2-Veronese of $A$ defined by $A'_{ij} = A_{2i,2j}$. Then there exists a $Z$-graded algebra $B$ such that $B \cong A'$. Furthermore there exists a 4-dimensional Artin-Schelter regular algebra $D$ with Hilbert series $1/(1-t)^4$ together with a regular normal element $C \in D_2$ such that $B \cong D/(C)$.

If we think of $\text{QGr}(D)$ as a non-commutative $\mathbb{P}^3$ then we have embedded our non-commutative $\mathbb{P}^1 \times \mathbb{P}^1$ (represented by $A$) as a divisor in a non-commutative $\mathbb{P}^3$ (represented by $D$).

We will also discuss a translation principle for non-commutative $\mathbb{P}^1 \times \mathbb{P}^1$'s. If $A$ is a cubic 3-dimensional regular algebra with associated quadruple $(C, L_0, L_1, L_2)$ (see Proposition 1.2) then following a similar definition in [9] we define the elliptic helix
associated to \((C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)\) as a sequence of line-bundles \((\mathcal{L}_i)_{i \in \mathbb{Z}}\) on \(C\) satisfying the relation
\[
\mathcal{L}_i \otimes \mathcal{L}_{i+1}^{-1} \otimes \mathcal{L}_{i+2}^{-1} \otimes \mathcal{L}_{i+3} = \mathcal{O}_C
\]
To simplify the exposition let us assume that \(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2\) are sufficiently generic such that there are no equalities of the form \(\mathcal{L}_2i \sim \mathcal{L}_2j+1\). One may then check that all quadruples of the form \((C, \mathcal{L}_0, \mathcal{L}_{2n+1}, \mathcal{L}_2)\) satisfy the hypotheses of Proposition 1.2 and hence they define a cubic 3-dimensional regular \(\mathbb{Z}\)-algebra which we denote by \(T^n\). Furthermore the elliptic helix associated to \((C, \mathcal{L}_0, \mathcal{L}_{2n+1}, \mathcal{L}_2)\) is \((C, \ldots, \mathcal{L}_{2n-1}, \mathcal{L}_0, \mathcal{L}_{2n+1}, \mathcal{L}_2, \ldots)\) with \(\mathcal{L}_0\) occurring in position zero. In other words we have shifted the odd part of the original elliptic helix \(2n\) places to the left.

**Theorem 1.4.** *(a simplified version of Theorem 7.1 below)* Let \(A\) be a cubic 3-dimensional regular \(\mathbb{Z}\)-algebra. Then \(\text{QGr}(T^nA) \cong \text{QGr}(A)\).

This paper fits in an ongoing program to understand non-commutative Del Pezzo surfaces. In the commutative case \(\mathbb{P}^1 \times \mathbb{P}^1\) is special as it is not obtained by blowing up \(\mathbb{P}^2\). For some approaches to non-commutative Del Pezzo surfaces see [6, 18, 28].

In the final section of this paper we make rigid the heuristic deformation theoretic arguments presented above by proving the following result

**Theorem 1.5.** *(a compressed version of Theorems 8.1.1 and 8.1.2 below).* Let \((R, m)\) be a complete commutative noetherian local ring with \(k = R/m\) if \(C = \text{coh}(\mathbb{P}^2_k)\) and \(D\) is an \(R\)-deformation of \(C\) then \(D = \text{qgr}(A)\) where \(A\) is an \(R\)-family of three dimensional quadratic regular algebras. Similarly if \(C = \text{coh}(\mathbb{P}^1_k \times \mathbb{P}^1_k)\) and \(D\) is an \(R\)-deformation of \(C\) then \(D = \text{qgr}(A)\) where \(A\) is an \(R\)-family of three dimensional cubic regular algebras.

The undefined notions in the statement of this result will of course be introduced below. In particular we will have to make sense of the notion of a deformation of an abelian category. Infinitesimal deformations of abelian categories where defined in [16] and from this one may define non-infinitesimal deformations by a suitable limiting procedure. The theoretical foundation of this is the exposé by Jouanolou [14]. The details are provided in [26].

### 2. Acknowledgment

This is a slightly updated version of a paper written in 2001 which was circulated privately. Some of the results were announced in [22].

The author wishes to thank Michael Artin, Alexey Bondal, Paul Smith and Toby Stafford for useful discussions.

### 3. Reminder on \(I\)-algebras

This section has some duplication with similar basic material in the recent papers [11, 12] and [20].

Let \(I\) be a set. Abstractly an \(I\)-algebra [9] \(A\) is a pre-additive category whose objects are indexed by \(I\).

It will be convenient for us to spell this definition out concretely. We view an \(I\)-algebra as a ring \(A\) (without unit) together with a decomposition \(A = \bigoplus_{i,j \in I} A_{ij}\) (here \(A_{ij} = \text{Hom}_A(j, i)\)) such that the multiplication has the property \(A_{ij}A_{jk} \subset A_{ik}\).
and \( A_{ij}A_{kl} = 0 \) if \( j \neq k \). The identity morphisms \( i \to i \) yield local units, denote by \( e_i \), such that if \( a \in A_i \) then \( e_i a = a = a e_j \). In the sequel we denote the category of \( I \)-algebras by \( \text{Alg}(I) \).

In the same vein a right \( A \)-module will be an ordinary right \( A \)-module \( M \) together with a decomposition \( M = \oplus_i M_i \) such that \( M_i A_{ij} \subset M_j \), \( e_i \) act as unit on \( M_i \) and \( M_i A_{jk} = 0 \) if \( i \neq j \). We denote the category of right \( A \) modules by \( \text{Gr}(A) \). It is easy to see that \( \text{Gr}(A) \) is a Grothendieck category [23]. We will write \( \text{Hom}_A(\cdot,\cdot) \) for \( \text{Hom}_{\text{Gr}(A)}(\cdot,\cdot) \).

The obvious definitions of related concepts such as left modules, bimodules, ideals, etc... which we will use below are left to the reader.

Let \( A \in \text{Alg}(I) \). Then for \( i \in I \) we put \( P_{i,A} = e_i A \in \text{Gr}(A) \). If \( A \) is clear from the context then we write \( P_i \) for \( P_{i,A} \). Obviously \( \text{Hom}_A(P_i,M) = M_i \) and \( \text{Hom}_A(P_i,P_j) = A_{ij} \). In particular \( P_i \) is projective. It is easy to see that \( (P_i)_{i \in I} \) is a set of projective generators for \( \text{Gr}(A) \).

Let \( J \subset I \) be an inclusion of sets. The \( J \)-Veronese of \( A \) is defined as \( B = \oplus_{i,j \in J} A_{ij} \). The restriction functor \( \text{Res} : \text{Gr}(A) \to \text{Gr}(B) \) is defined by \( \oplus_{i \in I} M_i \to \oplus_{i \in J} M_i \). Its left adjoint is denoted by \( - \otimes_B A \).

It is easy to see that the composition \( \text{Res}( - \otimes_B A ) \) is the identity. Applying this to \( K \)-projective complexes [21] we find that \( \text{Res}( - \otimes_B A ) \) is the identity on \( D(\text{Gr}(A)) \).

**Definition 3.1.** \( A \) is noetherian if \( \text{Gr}(A) \) is a locally noetherian Grothendieck category or, equivalently, if all \( P_i \) are noetherian objects in \( \text{Gr}(A) \).

**Convention 3.2.** In this paper we will use the convention that if \( Xyz(\cdots) \) is an abelian category then \( xyz(\cdots) \) denotes the full subcategory of \( Xyz(\cdots) \) whose objects are given by the noetherian objects.

Following this convention we let \( \text{gr}(A) \) stand for the category of noetherian \( A \)-modules for a noetherian \( I \)-algebra \( A \).

If \( I = G \) is a group then we denote by \( \text{GrAlg}(G) \) the category of \( G \)-graded algebras. There is an obvious functor \( (\cdot) : \text{GrAlg}(G) \to \text{Alg}(G) \) which sends a \( G \)-graded algebra \( A = \oplus_{g \in G} A_g \) to the \( G \)-algebra \( \hat{A} \) with \( \hat{A}_{gh} = A_{g^{-1}h} \). In this case we call \( A \) a realization of \( B \). Note that trivially \( \text{Gr}(\hat{A}) = \text{Gr}(A) \).

It follows that \( I \)-algebras are generalizations of \( I \)-graded rings when \( I \) is a group. In fact most general result for graded rings generalize directly to \( I \)-algebras. We use such results without further comment.

It will often happen that \( \hat{A} \cong \hat{B} \) as \( G \)-algebras whereas \( A \cong B \) as graded rings. This is closely related to the notion of a Zhang twist [29]. Recall that if \( A \) is a \( G \)-graded algebra then a Zhang-system is a set of graded isomorphisms \( (\tau_g)_{g \in G} : A \to A \) of abelian groups satisfying \( \tau_g(a \tau_h(b)) = \tau_g(a) \tau_{gh}(b) \) for homogeneous elements \( a, b \in A \) with \( a \in A_h \). A Zhang twist allows one to define a new multiplication on \( A \) by \( a \cdot b = a \tau_b(b) \) for \( a \in A_g, b \in A \). One denotes the resulting graded ring by \( A_\tau \) and calls it the Zhang-twist of \( A \) with respect to \( \tau \).

**Proposition 3.3.** Assume that \( A, B \) are two \( G \)-graded rings. Then \( \hat{A} \cong \hat{B} \) if and only if \( B \) is isomorphic to a Zhang twist of \( A \).

**Proof.** See [20]. □
If $A$ is an $I$-algebra and $G$ is a group which acts on $I$ then for $g \in G$ we define $A(g)$ by $A(g)_{ij} = A_{g(i),g(j)}$. $A(g)$ is again an $I$-algebra. Similarly if $M \in \text{Gr}(A)$ then we define $M(g) = M_{g(i)}$ and with this definition $M(g) \in \text{Mod}(A(g))$. If $A$, $B$ are $I$-algebras then a morphism $A \to B$ of degree $g$ is a morphism of $I$-algebras $A \to B(g)$. $A$ is said to be $g$-periodic if it possesses an automorphism of degree $g$.

Now we will assume that $I = \mathbb{Z}$ and we let $\mathbb{Z}$ acts on itself by translation. If $B$ is a $\mathbb{Z}$-graded ring then clearly $\hat{B}$ is 1-periodic. The following lemma shows that the converse is true.

**Lemma 3.4.** Let $A$ be a 1-periodic $\mathbb{Z}$-algebra. Then $A$ is of the form $\hat{B}$ for a $\mathbb{Z}$-graded ring $B$.

**Proof.** Let $\phi : A \to A(1)$ be an isomorphism. We view $\phi$ as a map $A_{ij} \to A_{i+1,j+1}$ for $i,j \in \mathbb{Z}$. Hence $\phi^\circ$ becomes a map $A_{ij} \to A_{i+n,j+n}$.

We define $B_i = A_0,i$ and $B = \oplus_i B_i$: We make $B$ into a graded ring by defining the multiplication $b_i b_j = b_i \phi^i(b_j)$ for $b_i \in B_i$, $b_j \in B_j$.

Now we claim $\hat{B} \cong A$. One has $B_{ij} = B_{j-i} = A_{0,j-i}$. We define $\psi : \hat{B} \to A$ as $\phi^i$ on $\hat{B}_{ij}$. It is easy to check that $\psi$ is an isomorphism. \qed

Let $A \in \text{Alg}(\mathbb{Z})$ and assume that $A$ is noetherian. We borrow a number of definitions from [1]. Let $M \in \text{Gr}(A)$. We say that $M$ is left, resp. right bounded if $M_i = 0$ for $i \ll 0$ resp. $i \gg 0$. We say that $M$ is bounded if $M$ is both left and right bounded. We say $M$ is torsion if it is a direct limit of right bounded objects. We denote the corresponding category by $\text{Tors}(A)$. Following [1] we also put $\text{QGr}(A) = \text{Gr}(A) / \text{Tors}(A)$. If $A$ is noetherian then following Convention 3.2 we introduce $\text{qgr}(A)$ and $\text{tors}(A)$. Note that if $M \in \text{tors}(A)$ then $M$ is right bounded, just as in the ordinary graded case. It is also easy to see that $\text{qgr}(A)$ is equal to $\text{gr}(A) / \text{tors}(A)$.

We put $A_{\geq 0} = \bigoplus_{j \geq 0} A_{ij}$ and similarly $A_{\leq 0} = \bigoplus_{j \leq 0} A_{ij}$. These are both $\mathbb{Z}$-subalgebras of $A$. We say that $A$ is positively graded if $A = A_{\geq 0}$. In the sequel we will only be concerned with positively graded $\mathbb{Z}$-algebras.

If $A$ is $k$-linear then we say that $A$ connected if it is positively graded, each $A_{ij}$ if finite dimensional and $A_{ii} = k$ for all $i$. In that case we let $S_i = S_i,A$ be the unique simple quotients of $P_i$ (we write $S_i = S_i,A$ if $A$ is not in doubt). Note that $S_i$ is naturally an $A$-bimodule. We say that $A$ is generated in degree one if it is positively graded and generated as $\mathbb{Z}$-algebra by $(A_{i,i+1})_i$.

We will use the following result.

**Lemma 3.5.** Let $A \in \text{Alg}(\mathbb{Z})$ be noetherian and generated in degree one and let $J$ be an infinite subset of $I = \mathbb{Z}$ which is not bounded above. Let $B$ be the $J$-Veronese of $A$. Then $B$ is also noetherian and furthermore the functors $\text{Res} : \text{Gr}(A) \to \text{Gr}(B)$ and $- \otimes_B A$ defines inverse equivalences between $\text{QGr}(A)$ and $\text{QGr}(B)$ (for the notation $\text{QGr}(B)$ to make sense we identify $J$ with $\mathbb{Z}$ or $\mathbb{N}$ as an ordered set).

**Proof.** We already know that $\text{Res}( - \otimes_B A)$ is the identity. From this we easily deduce that $B$ is noetherian and that $\text{Res}$ preserves noetherian objects.

To prove that $- \otimes_B A$ gives a well defined functor $\text{QGr}(B) \to \text{QGr}(A)$ we need to prove that $\text{Tor}_i^B(-,A)$ preserves torsion objects for $i = 0,1$. Looking at projective resolutions we see $\text{Res}(\text{Tor}_i^B(-,A)) = 0$ for $i > 0$ and $\text{Res}( - \otimes_B A) = \text{id}_{\text{Gr}(B)}$. Hence is sufficient to prove the following sublemma.
**Sublemma.** Assume that $M \in \text{Gr}(A)$ is such that $\text{Res}(M)$ is torsion. Then $M$ itself is torsion.

**Proof.** Since everything is compatible with filtered colimits, it suffices to consider the case that $M$ is noetherian. Then $\text{Res}(M)$ is also noetherian and hence right bounded.

Let $m \in M_k$. We need to show that $mA_{kl} = 0$ for $l \gg 0$. Since $\text{Res}(M)$ is right bounded there exists $j \in J$, $j \geq k$ such that $mA_{kj} = 0$. Thus we have $mA_{kj}A_{jl} = 0$ for all $l$. If $l \geq j$ then since $A$ is generated in degree one we have $A_{kj}A_{jl} = A_{kl}$ which proves what we want. □

To prove the asserted equivalence of categories we must show that for any $M \in \text{Gr}(A)$ the canonical map $\text{Res}(M) \otimes_B A \to M$ has torsion kernel and cokernel. Since both the kernel and cokernel have zero restriction this follows from the sublemma. □

### 4. Artin-Schelter regular $\mathbb{Z}$-algebras

#### 4.1. Definition and motivation

Let $k$ be a field and let $A$ be a $\mathbb{Z}$-algebra defined over $k$. We make the following tentative definition. We say that $A$ is (AS-)regular if the following conditions are satisfied.

1. $A$ is connected.
2. $\dim A_{ij}$ is bounded by a polynomial in $j - i$.
3. The projective dimension of $S_i,A$ is finite and bounded by a number independent of $i$.
4. For every $i$, $\sum_{l,k} \dim \text{Ext}^j_{\text{Gr}(A)}(S_k,A, P_i,A) = 1$.

It is clear that this definition generalizes the notion of AS-regularity for ordinary graded algebras [2] in the sense that if $B$ is a graded algebra then it is AS-regular if and only if $\hat{B}$ is AS-regular in the above sense. Below we write $S_i = S_i,A, P_i = P_i,A$.

**Definition 4.1.1.** Let $A$ be a regular algebra. Inspired by [9] we say that $A$ is a **three dimensional quadratic regular algebra** if the minimal resolution of $S_i$ has the form

$$0 \to P_{i+3} \to P_{i+2}^3 \to P_{i+1}^3 \to P_i \to S_i \to 0.$$  

We say that $A$ is a **three dimensional cubic regular algebra** if the minimal resolution of $S_i$ has the form

$$0 \to P_{i+4} \to P_{i+3}^2 \to P_{i+1}^2 \to P_i \to S_i \to 0.$$  

**Definition 4.1.2.** A non-commutative $\mathbb{P}^2$ is a category of the form $Q\text{Gr}(A)$ with $A$ a three dimensional quadratic regular $\mathbb{Z}$-algebra. A non-commutative $\mathbb{P}^1 \times \mathbb{P}^1$ (or quadric) is a category of the form $Q\text{Gr}(A)$ with $A$ a three dimensional cubic regular $\mathbb{Z}$-algebra.

For the motivation of this definition we refer to the introduction.

#### 4.2. The work of Bondal and Polishchuk

As an introduction to non-commutative quadrics we first discuss (and slightly generalize) some results from [9]. Some of the arguments will only be sketched since we will repeat them in greater detail for quadrics.

From (4.1) one easily obtains the following property
(P1) \[ \dim A_{ij} = \begin{cases} \frac{(j-i+1)(j-i+2)}{2} & \text{if } j \geq i \\ 0 & \text{otherwise} \end{cases} \]

In addition closer inspection of (4.1) also yields:

(P2) Define \( V_i = A_{i,i+1} \). Then \( A \) is generated by the \( (V_i)_i \).

(P3) Put \( R_i = \ker(V_i \otimes V_{i+1} \rightarrow A_{i,i+2}) \). Then the relations between the \( V_i \) in \( A \) are generated by the \( R_i \).

(P4) Dimension counting reveals that \( \dim R_i = 3 \). Put \( W_i = R_i \otimes V_{i+2} \cap V_i \otimes R_{i+1} \).

Using dimension counting once again we find that \( \dim W_i = 1 \). Then \( W_i \) is a non-degenerate tensor, both as element of \( R_i \otimes V_{i+2} \) and as element of \( V_i \otimes R_{i+1} \).

Proposition 4.2.1. (see [9]) The three dimensional regular quadratic \( \mathbb{Z} \)-algebras are classified in terms of triples \((C, L_0, L_1)\) where either

1. \((C, L_0, L_1) \cong (\mathbb{P}^2, \mathcal{O}(1), \mathcal{O}(1)), \) (the “linear” case); or else
2. (a) \( C \) is embedded as a divisor of degree 3 in \( \mathbb{P}^2 \) by the global sections of \( L_0 \) and \( L_1 \).
   (b) \( \deg(L_0 | E) = \deg(L_1 | E) \) for every irreducible \( E \) component of \( C \).
   (c) \( L_0 \not\cong L_1 \) (the “elliptic” case).

Let us recall how \( A \) is constructed from a triple \((C, L_0, L_1)\). First we construct line bundles \( (L_i)_{i \in \mathbb{Z}} \) on \( C \) via the relation

(4.3) \[ L_i \otimes L_{i+1} \otimes L_{i+2} \cong \mathcal{O}_C \]

The sequence of line bundles \( (L_i)_i \) is the so-called “elliptic helix” associated to \((C, L_0, L_1)\). Note that (4.3) is equivalent to

(4.4) \[ L_n = L_0 \otimes (L_1 \otimes L_0^{-1})^\oplus n \]

In other words the \( (L_i)_i \) form an arithmetic progression.

We put \( V_i = H^0(C, L_i) \) and

(4.5) \[ R_i = \ker(H^0(C, L_0) \otimes H^0(C, L_1) \rightarrow H^0(C, L_0 \otimes L_1)) \]

Then we let \( A \) be the \( \mathbb{Z} \)-algebra generated by the \( V_i(= A_{i,i+1}) \) subject to the relations \( R_i \subset V_i \otimes V_{i+1} \).

Conversely if we have a three-dimensional quadratic regular \( \mathbb{Z} \)-algebra then the triple \((C, L_0, L_1)\) is constructed using a similar procedure as in [3]. To be more precise let \( (V_i)_i, (R_i)_i \) be as above. Then \( R_0 \) defines a closed subscheme \( C \) of \( \mathbb{P}(V_0^*) \times \mathbb{P}(V_1^*) \cong \mathbb{P}^2 \times \mathbb{P}^2 \). We let \( L_0, L_1 \) be the inverse images \( \mathcal{O}(1) \) under the two projections \( C \rightarrow \mathbb{P}^1 \).

Theorem 4.2.2. Assume that the ground field \( k \) is algebraically closed of characteristic different from three and that \( B \) is a three dimensional quadratic regular \( \mathbb{Z} \)-algebra. Then \( B \cong A \) for a quadratic three-dimensional regular algebras \( A \).

This result is proved in [9] in characteristic zero using some case by case analysis. We give a streamlined proof based upon the following result from [4].

Theorem 4.2.3. [4, Cor. 5.7, lemma 5.10] Let \( C \) be a cubic divisor in \( \mathbb{P}^3 \) or a divisor of bidegree \((2, 2)\) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). Let \( \text{Pic}^0(C) \) be the connected component of the identity in \( \text{Pic}(C) \), i.e. those line bundles which have degree zero on every
component of $C$. Then there is a morphism of algebraic groups $\eta : \text{Pic}^0(C) \to \text{Aut}(C)$ with the following property: for $A \in \text{Pic}^0(C)$ and $B \in \text{Pic}(C)$.

\begin{equation}
\eta(A)^*(B) = B \otimes A^{-b}
\end{equation}

where $b$ is the total degree on $B$.

**Proof of Theorem 4.2.2.** Let $B$ be as in the statement of the theorem. According to lemma 3.4 we must check that $B$ is isomorphic to $B(1)$. In order to be able to do this we must know how to recognize when two $\mathbb{Z}$-algebras $A, B$ are isomorphic, given their associated triples. The answer to this question is given by the following easily proved lemma.

**Sublemma.** Assume that $A, B$ are quadratic regular $\mathbb{Z}$-algebras. Let $(C, \mathcal{L}_0, \mathcal{L}_1)$ and $(F, \mathcal{M}_0, \mathcal{M}_1)$ be their associated triples. Then $A \cong B$ if and only if there exists an isomorphism $\sigma : C \to F$ such that $\sigma^* \mathcal{M}_i = \mathcal{L}_i$.

Hence to prove that $B$ and $B(1)$ are isomorphic we must know the triple associated to $B(1)$. By construction (see 4.5) if $(C, (\mathcal{L}_i)_i)$ is the elliptic helix associated to $B$ then $(C, (\mathcal{L}_{i+1})_i)$ is the elliptic helix associated to $B(1)$. Hence it follows from lemma [9] and the construction of elliptic helices that $B \cong B(1)$ if and only if there exist $\sigma \in \text{Aut}(C)$ such that

\begin{equation}
\mathcal{L}_{i+1} \cong \sigma^* \mathcal{L}_i
\end{equation}

for all $i$. It is easy to verify that this is equivalent to the following conditions for the triple $(C, \mathcal{L}_0, \mathcal{L}_1)$ associated to $B$.

\begin{align}
\mathcal{L}_1 &= \sigma^* (\mathcal{L}_0) \\
\sigma^* \mathcal{L}_0 \otimes (\sigma^* \mathcal{L}_0)^{-2} \otimes \mathcal{L}_0 &= \mathcal{O}_C
\end{align}

Hence one has to prove that there is an automorphism $\sigma$ of $C$ satisfying the conditions (4.8-4.9) above. If $C = \mathbb{P}^2$ then we may take $\sigma = \text{id}_C$. If $C$ is a curve we will take $\sigma$ to be of the form $\eta(A)$ (with notations from Theorem 4.2.3) for suitable $A \in \text{Pic}^0(C)$. Then according to (4.6) the first condition (4.8) translates into

\begin{equation}
\mathcal{L}_1 = \mathcal{L}_0 \otimes A^{-3}
\end{equation}

and if this holds then the second condition (4.9) is satisfied automatically.

By Proposition 4.2.1.2(c) $\mathcal{L}_1$ and $\mathcal{L}_0$ have the same degree on every component of $C$. Hence $\mathcal{L}_1 \otimes \mathcal{L}_0^{-1} \in \text{Pic}^0(C)$. Thus we must be able to divide by three in $\text{Pic}^0(C)$. The only possible problem arises when $\text{Pic}^0(C) = k^+$ and $\text{char} k = 3$ but this cases is excluded by the hypotheses.

Hence an $A$ as in (4.10) exists. This finishes the proof. \hfill \Box

**Remark 4.2.4.** The condition that the ground field is algebraically closed is necessary for Theorem 4.2.2 to be true as the following example shows.

Assume that $k$ is not algebraically closed and let $C$ be a smooth elliptic curve over $k$ which has no complex multiplication over $k$. Assume that $C$ has a rational point $o$. We may use $o$ to identify $C$ with $\text{Pic}^0(C)$ via the map $p \mapsto \mathcal{O}(o - (p))$. In this way $C$ becomes an algebraic group. Then $\text{Pic}(C)$ may be identified with $\text{Pic}(C)_{\text{Gal}(\overline{k}/k)}$. The automorphisms of $C$ are of the form $\sigma_t : p \mapsto t + p$ and $\tau_t : p \mapsto t - p$ for $t \in C$. If $d$ is a divisor on $C$ then we denote by $|d|$ the sum of $d$ as an element of $C$. 

If \( \mathcal{L}_0 = \mathcal{O}_C(d_0) \), \( \mathcal{L}_1 = \mathcal{O}_C(d_1) \) with \( \deg d_i = 3 \) then \( \sigma_1^* \mathcal{L}_0 = \mathcal{L}_1 \) if and only if \( |d_0| - 3t = |d_1| \). Similarly we will have \( \tau_t^* \mathcal{L}_0 = \mathcal{L}_1 \) if and only if \( 3t - |d_0| = |d_1| \).

If we now choose \( d_0 \) and \( d_1 \) in such a way that neither \( |d_0| - |d_1| \) nor \( |d_0| + |d_1| \) is divisible by 3 in \( C \) then a suitable \( t \) cannot exist and hence \( \mathcal{L}_0 \) and \( \mathcal{L}_1 \) are not in the same \( \text{Aut}(C) \) orbit. A concrete example can be made by assuming that \( k \) is a number field. Then \( C \) is a finitely generated abelian group and hence there exists \( q \in C \) which is not divisible by 3. Now take \( d_0 = 3(o) \) and \( d_1 = 2(o) + (q) \).

Remark 4.2.5. The hypothesis on the characteristic of \( k \) is also necessary. Assume that \( k \) is algebraically closed of characteristic 3 and let \( C \) be a cuspidal elliptic curve. One may show that \( \text{Aut}(C) \) is isomorphic to the ring of matrices \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \) where \( a \in k^*, \ b \in k^* \) and \( \text{Pic}(C) \) is isomorphic to the abelian group of column vectors \( \left( \begin{smallmatrix} \mu \\ n \end{smallmatrix} \right) \) with \( \mu \in k^* \) and \( n \in \mathbb{Z} \). The integer \( n \) is the degree of the corresponding line bundle. The action \( \text{Aut}(C) \) on \( \text{Pic}(C) \) is given by matrix multiplication. From this it is clear that the action of \( \text{Aut}(C) \) on \( \text{Pic}^3(C) \) is not transitive. An exhaustive enumeration of all possibilities shows that this is in fact the only counter example (in the algebraically closed case). So we could replace the hypotheses of Theorem 4.2.2 by \( \text{char } k \neq 3 \) or the triple corresponding to \( A \) is not cuspidal.

4.3. Some more properties. We state without proof some additional properties of 3-dimensional quadratic AS-regular \( \mathbb{Z} \)-algebras. These results can be deduced easily from the methods in \([3, 4, 5, 9]\) and will be discussed in more detail in the context of quadrics.

Assume that \( (C, (\mathcal{L}_i)_i) \) is the elliptic helix associated to a three dimensional quadratic regular \( \mathbb{Z} \)-algebra \( A \). Define

\[
B_{ij} = \Gamma(C, \mathcal{L}_i \otimes \cdots \otimes \mathcal{L}_{j-1})
\]

Then we have an obvious multiplication map \( B_{ij} \times B_{jk} \to B_{ik} \) and in this way \( B = \oplus_{i,j} B_{ij} \) becomes a \( \mathbb{Z} \)-algebra. Furthermore the construction of \( A \) from its elliptic helix (see (4.5)) yields a canonical map \( A \to B \) with kernel \( K \).

Proposition 4.3.1. (1) The canonical map \( A \to B \) is surjective.

(2) \( K_{i,i+3} \) is one dimensional. Choose non-zero elements \( g_i \in K_{i,i+3} \) elements. Then \( K \) is generated by the \( (g_i)_i \) both as left and as right ideal.

(3) The \( g_i \) are normalizing elements in \( A \) in the sense that there is an isomorphism \( \phi : A \to A(3) \) such that for \( a \in A_{i,j} \) we have \( ag_j = g_i\phi(a) \).

(4) \( A \) and \( B \) are noetherian.

(5) \( \text{qgr}(B) \cong \text{coh}(E) \).

(6) \( \text{qgr}(A) \) is Ext-finite.

5. Non-commutative quadrics

5.1. Generalities. To classify cubic 3-dimensional AS-regular algebras we follow the program already outlined the quadratic case.

Let \( A \) be a three-dimensional cubic regular \( \mathbb{Z} \)-algebra. From (4.2) one easily obtains the following properties

(Q1) The following holds for the dimensions of \( A_{ij} \)

\[
\dim A_{i,i+n} = \begin{cases} 
0 & \text{if } n < 0 \\
(k + 1)^2 & \text{if } n = 2k \text{ and } n \geq 0 \\
(k + 1)(k + 2) & \text{if } n = 2k + 1 \text{ and } n \geq 0 
\end{cases}
\]
We start by defining a prequadric as a pair $(V_i)\) (5.2).

Going from quadrics to quintuples.

A cubic regular algebra. We will refer to the algebra $A$ in terms of quadruples $(V_i)$. Other three dimensional cubic regular algebras will be called $Qch(V)$. The motivation for this terminology will become clear below.

The three-dimensional cubic regular algebra $A$ is the $A$ such that the relations in $(5.1)$ are as follows:

\[
A = \bigoplus_{i,j} \text{Hom}_X(O_X(-j), O_X(-i)).
\]

According to [19] (see also [1]) we have $\text{Qch}(X) \cong \text{QGr}(A)$. Furthermore it is easy to check that $A$ is a three-dimensional cubic regular algebra. We will refer to the algebra $A$ as the linear quadric. The other three dimensional cubic regular algebras will be called elliptic quadrics. The motivation for this terminology will become clear below.

**Example 5.1.1.** Let $A$ be a linear quadric. Then we may choose bases $x_i, y_i$ for $V_i$ such that the relations in $A$ are given by

\[
x_i x_{i+1} y_{i+2} - y_i x_{i+1} x_{i+2} = 0
\]

\[
x_i y_{i+1} y_{i+2} - y_i y_{i+1} x_{i+2} = 0
\]

The $w_i$ corresponding to these relations is given by

\[
x_i x_{i+1} y_{i+2} y_{i+3} - y_i x_{i+1} x_{i+2} y_{i+3} - x_i y_{i+1} y_{i+2} x_{i+3} + y_i y_{i+1} x_{i+2} x_{i+3}
\]

Our aim this section is to prove an analogue for Proposition 4.2.1.

**Proposition 5.1.2.** The three-dimensional cubic regular $\mathbb{Z}$-algebras are classified in terms of quadruples $(C, L_0, L_1, L_2)$ where either:

1. $(C, L_0, L_1, L_2) \cong (\mathbb{P}^1 \times \mathbb{P}^1, O(0, 1), O(0, 1), O(1, 0))$ (the “linear” case); or

2. $(C, L_0, L_1, L_2)$ is a curve which is embedded as a divisor of degree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ by the global sections of both $(L_0, L_1)$ and $(L_1, L_2)$.

(b) $\deg(L_0 \mid E) = \deg(L_2 \mid E)$ for every irreducible component $E$ of $C$ (the “elliptic case”).

(c) $L_0 \not\cong L_2$.

This result follows from Theorem 5.5.10 below.

5.2. Going from quadrics to quintuples. From now on we follow closely [9].

We start by defining a prequadric as a pair $(A, (W_i))$ with the following properties.

1. $A$ is a connected $\mathbb{Z}$-$k$-algebra, generated in degree one. Put $V_i = A_{i,i+1}$ and $R_i = \ker(V_i \otimes V_{i+1} \otimes V_{i+2} \to A_{i,i+2})$. We require that $\dim V_i = \dim R_i = 2$ and furthermore that the $R_i$ generate the relations of $A$. 

(2) For all $i \in \mathbb{Z}$, $W_i = kw_i$ is a one-dimensional subspace of $V_i \otimes R_{i+1} \cap R_i \otimes V_{i+3}$ such that $w_i$ is non-degenerate, both as a tensor in $V_i \otimes R_i$ and as a tensor in $R_i \otimes V_{i+3}$.

It is clear that if $A$ is a quadric then it is also a prequadric in a unique way. Fix a prequadric $(A, (W_i)_i)$ together with non-zero elements $w_i$ in the one-dimensional spaces $W_i$. By hypotheses $w_i$ can be written as $\sum_j r_{ij} \otimes v_{i+3,j}$ and as $\sum v_{ij} \otimes r_{i+1,j}$ where $(r_{ij})_i$, $(v_{i+3,j})_i$, $(v_{ij})_i$, $(r_{i+1,j})_i$ are bases of $R_i$, $V_{i+3}$, $V_i$, $R_{i+1}$ respectively. There are unique invertible linear maps $\theta_i : V_i \rightarrow V_{i+4}$ and $\theta_i : R_i \rightarrow R_{i+4}$ with the following properties.

$$w_{i+3} = \sum_j v_{i+3,j} \otimes \theta_i(r_{ij})$$

$$w_{i+1} = \sum_j r_{i+1,j} \otimes \theta_i(v_{ij})$$

We may use $\theta_i$ to identify $V_{i+4}$ with $V_i$ and $R_{i+4}$ with $R_i$. In this way $\theta_i$ becomes the identity.

For $v_i \in V_i$ define $R(v_i \otimes v_{i+1} \otimes v_{i+2} \otimes v_{i+3}) = v_{i+1} \otimes v_{i+2} \otimes v_{i+3} \otimes v_i$ and extend this to a linear map $V_i \otimes V_{i+1} \otimes V_{i+2} \otimes V_{i+3} \rightarrow V_{i+1} \otimes V_{i+2} \otimes V_{i+3} \otimes V_i$.

Then for $n \in \mathbb{Z}$ we clearly have

$$w_{i+n} = R^n w_i$$

Since the $w_i$ determine the relations it follows (with the current identifications) that $A = A(4)$. Thus in particular $\theta = (\theta_i)_i$ defines an automorphism of degree 4 of $A$.

Let us define a “quintuple” as a quintuple of vector space $Q = (V_0, V_1, V_2, V_3, W)$ where the $V_i$’s are two dimensional vector spaces and $W$ is a one-dimensional subspace of $V_0 \otimes \cdots \otimes V_3$. In the sequel we will sometimes identify a quintuple by a non-zero element $w$ of $W$.

We say that an element $w$ of $V_0 \otimes \cdots \otimes V_3$ is strongly non-degenerate if $w$ is a non-degenerate tensor when considered as an element of $V_j \otimes (V_0 \otimes \cdots \otimes V_j \otimes \cdots \otimes V_3)$ for $j = 0, \ldots, 3$. We say that $Q$ is non-degenerate if $W = kw$ with $w$ a strongly non-degenerate tensor. Note the following trivial lemma.

**Lemma 5.2.1.** Let $Q = (V_0, V_1, V_2, V_3, W)$ be a non-degenerate quintuple. Write $W = kw$, $w = r_1 \otimes v_1 + r_2 \otimes v_2$ where $v_1, v_2$ is a basis for $V_3$. Then $R = kr_1 + kr_2$ is a two-dimensional subspace of $V_0 \otimes V_1 \otimes V_2$, independent of the choice of $w, v_1, v_2$ and $Q$ is up to isomorphism determined by $R \subset V_0 \otimes V_1 \otimes V_2$.

**Proof.** That $R$ is independent of the choice of $w, v_1, v_2$ is clear. Furthermore it is also clear that $Q$ is isomorphic to $(V_0, V_1, V_2, R^*, r_1 \otimes r_1^* + r_2 \otimes r_2^*)$. \hfill $\Box$

**Theorem 5.2.2.** Let $F$ be the functor which associates to a prequadric $(A, (W_i)_i)$ the quintuple $(V_0, V_1, V_2, V_3, W_0)$. Then $F$ defines an equivalence of categories between prequadrics and non-degenerate quintuples (both equipped with isomorphisms as maps).

**Proof.** Let $(A, (W_i)_i)$ be a prequadric. We have $W_i = kw_i$ and $w_i = R^i w_0$ (after choosing suitable bases for the $V_i$). The non-degeneracy of $w_i$ implies the strong non-degeneracy of $w_0$.

Conversely assume that we are given a non-degenerate quintuple $Q = (V_0, V_1, V_2, V_3, kw)$. We look for a prequadric $(A, (W_i)_i)$ such that $F(A) = Q$. If $i \in \mathbb{Z}$ then let us denote
by \(i\) the unique element of \(\{0, \ldots, 3\}\) congruent to \(i\) modulo 4. We put \(V_i = V_1,\)

\[ w_i = R^i w \text{ and } W_i = kw_i. \]

Then the strong non-degeneracy of \(w\) implies the non-degeneracy of \(w_i\). It is clear from the above discussion that any other prequadratic yielding \(Q\) will be isomorphic to the prequadratic we have constructed. \(\square\)

**Corollary 5.2.3.** A quadric \(A\) is determined up to isomorphism by its “truncation” \(A' = \oplus_{i,j=0,\ldots,3} A_{ij}\).

**Proof.** By Theorem 5.2.2 we already know that \(A\) is uniquely determined by a generator \(w\) of \(W_0\). Now according to lemma 5.2.1 \(w\) (or rather its associated quintuple) is up to isomorphism determined by \(R_0\). Clearly \(R_0\) is determined by \(A'\).

If \(A\) is a quadric then in the sequel we will write \(F(A)\) for \(F(A_1 (W_1)_{ij})\) for \((A_1, (W_1)_{ij})\) the unique prequadric associated to \(A\). We will say that a quintuple \(Q\) is linear if it is of the form \(F(A)\) where \(A\) is a linear quadric.

From Example 5.1.1 we obtain.

**Lemma 5.2.4.** \(Q\) is linear if and only if we may choose bases \(x_i, y_i\) for \(V_i\) such that \(w\) is given by the tensor

\[ w = x_0 x_1 y_2 y_3 - y_0 x_1 x_2 y_3 - x_0 y_1 y_2 x_3 + y_0 y_1 x_2 x_3. \]

### 5.3. Geometric quintuples

Our next aim is to recognize among the non-degenerate quintuples those that correspond to quadrics, and not only to prequadrics. Following [9] we introduce the following definition.

**Definition 5.3.1.** Let \(V_i, i = 0, \ldots, 3\) be two-dimensional vectorspaces and let \(w\) be a non-zero element of \(V_0 \otimes \cdots \otimes V_3\). Then we say that \(w\) is geometric if for all \(j \in \{0, \ldots, 3\}\) and for all non-zero \(\phi_j \in V_j^*\), \(\phi_j+1 \in V_{j+1}^*\) the tensor \(\langle \phi_j \otimes \phi_{j+1}, w \rangle\) is non-zero. Indices are taken modulo 4 here. A quintuple \((V_0, V_1, V_2, V_3, kw)\) is geometric if \(w\) is geometric.

It is easy to see that a linear quintuple is geometric. A geometric quintuple that it not linear will be called elliptic.

Note the following:

**Lemma 5.3.2.** If \(w \in V_0 \otimes \cdots \otimes V_3\) is geometric then it is strongly non-degenerate.

**Proof.** Assume that \(w\) is not strongly non-degenerate. By rotating \(w\) we may assume that \(w = u \otimes v\) where \(u \in V_0\) and \(v \in V_1 \otimes V_2 \otimes V_3\). Choose \(\phi_0 \in V_0^*\) such that \(\phi_0(u) = 0\) and \(\phi_1 \in V_1^*\) arbitrary. Then clearly \(\langle \phi_0 \otimes \phi_1, w \rangle = 0\). Hence \(w\) is not geometric. \(\square\)

We will eventually show (see Theorem 5.5.10) that quadrics are classified by geometric quintuples. In this section we start by proving one direction.

**Lemma 5.3.3.** Assume that \(A\) is a quadric. Then \(F(A)\) is geometric.

**Proof.** We use the notations \((V_i)_{ij}\), \((R_i)_{ij}\), etc... with their usual interpretations. Assume \(w_0\) is not geometric. Replacing \(A\) by \(A(n)\) for some \(n \in \mathbb{Z}\) and using (5.2) we may assume that there exist non-zero \(\phi_0 \in V_0^*\), \(\phi_1 \in V_1^*\) such that \(\langle \phi_0 \otimes \phi_1, w \rangle = 0\).

We choose bases for \(x_i, y_i\) for \(V_i\). Then \(w_0 = fx_3 + gy_3\). Changing coordinates we may assume that \(\phi_0\) and \(\phi_1\) represent the points \((1, 0)\), \((0, 1)\). Thus \(f((1, 0), (0, 1), -) = 0\) and \(g((1, 0), (0, 1), -) = 0\).
Writing
\[ f = ax_0 x_1 x_2 + bx_0 x_1 y_2 + cx_0 y_1 x_2 + dx_0 y_1 y_2 + ey_0 x_1 x_2 + fy_0 x_1 y_2 + gy_0 y_1 x_2 + hy_0 y_1 y_2 \]
\[ g = a' x_0 x_1 x_2 + b' x_0 x_1 y_2 + c' x_0 y_1 x_2 + d' x_0 y_1 y_2 + e' y_0 x_1 x_2 + f' y_0 x_1 y_2 + g' y_0 y_1 x_2 + h' y_0 y_1 y_2 \]
this yields \( c = d = c' = d' = 0 \). Let \( k = \mathbb{Q}(a, b, e, f, g, h, a', b', e', f', g', h') \). A Groebner basis computation yields that in this case \( \dim A_{1,8} = 22 \) whereas the correct value for a quadric is 20. Using semi-continuity we deduce that \( \dim A_{1,8} \geq 22 \) over any base field. This yields a contradiction. \( \square \)

5.4. Going from quintuples to quadruples. Below a quadruple \( U = (C, L_0, L_1, L_2) \) will be a quadruple \( U = (C, L_0, L_1, L_2) \) where \( C \) is a \( k \)-scheme and \( L_i \) are line bundles on \( C \). An isomorphism of quadruples \( U = (C, L_0, L_1, L_2) \to U' = (C', L'_0, L'_1, L'_2) \) will a quadruple \( (\psi, t_0, t_1, t_2) \) where \( \psi : C \to C' \) is an isomorphism and the \( t_i \) are isomorphisms \( L_i \to \psi^*(L'_i) \).

We will say that \( U \) is linear if it is isomorphic to \( (\mathbb{P}_1 \times \mathbb{P}_1, O(0, 0), O(0, 1), O(1, 0)) \). We will say that \( U \) is elliptic if \( C \) is a curve of arithmetic genus one, the \( (L_i) \), are line bundles whose global sections define morphisms \( p_i : C \to \mathbb{P}(V_i^*) \) of degree two such that the pairs \( (p_i, p_{i+1}) \) for \( i = 0, 1 \) define closed embeddings of \( C \) in \( \mathbb{P}_1 \times \mathbb{P}_1 \).

Deriving the properties of an elliptic quadruple depends on the Riemann-Roch theorem. However, as is pointed out in [3], if \( C \) is not irreducible, then there will often be non-trivial line bundles \( L \) on \( C \) such that both \( H^0(L) \) and \( H^1(L) \) are non-zero. This complicates the application of the Riemann-Roch theorem. In order to circumvent this difficulty one introduces as in [3] the notion of a tame sheaf.

**Definition 5.4.1.** Assume that \( C \) is a divisor of bidegree \((2, 2)\) in \( \mathbb{P}_1 \times \mathbb{P}_1 \). Then a line bundle \( \mathcal{M} \) on \( C \) is tame if either \( H^0(\mathcal{M}) = 0 \), \( H^1(\mathcal{M}) = 0 \) or \( \mathcal{M} \cong O_C \).

The usefulness of this definition stems from the fact that various criteria can be obtained for showing that a line bundle is tame. See [3, Proposition 7.12]. The following lemma is an extract of that proposition.

**Proposition 5.4.2.** [3] Let \( C \) be as in Definition 5.4.1 and let \( \mathcal{M} \) be a line bundle on \( C \). If \( \mathcal{M} \) has non-negative degree on every component (for example if \( \mathcal{M} \) is generated by global sections) then \( \mathcal{M} \) is tame.

One easily deduces (see [3]).

**Lemma 5.4.3.** Let \( C \) be as in Definition 5.4.1. If \( \mathcal{L}, \mathcal{M} \) are line bundles on \( C \) such that \( \mathcal{L} \) is tame of non-negative total degree and \( \mathcal{M} \) is generated by global sections then \( \mathcal{L} \otimes \mathcal{M} \) is tame of non-negative total degree.

**Lemma 5.4.4.** Let \( C \) be as in Definition 5.4.1. Then \( O_C(1, -1) \) is tame. More generally if \( \mathcal{L}, \mathcal{M} \) are line bundles such that the degrees of \( \mathcal{L}, \mathcal{M} \), when restricted to the irreducible components of \( C \) are the same as the degrees of \( O_C(1, 0) \) and \( O_C(0, 1) \) then \( \mathcal{L} \otimes \mathcal{M}^{-1} \) is also tame.

**Proof.** The proof of the lemma in the case of \( O_C(1, -1) \) is contained in the proof of [3, Lemma 7.18]. If we look at that proof then we see that it is purely numerical. Hence it is also valid for \( \mathcal{L} \otimes \mathcal{M}^{-1} \).

Tameness of \( \mathcal{L} \otimes \mathcal{M}^{-1} \) is important as can be seen from the following lemma.
Lemma 5.4.5. Let $C$ be as in Definition 5.4.1 and let $\mathcal{L}$, $\mathcal{M}$ be line bundles of degree two on $C$, the second one generated by global sections. Assume in addition that $\mathcal{L} \otimes \mathcal{M}^{-1}$ is tame. Then the natural map

$$H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L} \otimes \mathcal{M})$$

is an isomorphism if $\mathcal{L} \not\cong \mathcal{M}$ and otherwise it has one dimensional kernel.

Proof. Since $\mathcal{M}$ is generated by global sections it is tame by Proposition 5.4.2 and hence by Riemann-Roch $\dim H^0(\mathcal{M}) = 2$. We have a surjective map $H^0(\mathcal{M}) \otimes_k \mathcal{O}_C \to \mathcal{M}$. Looking at exterior powers we find that its kernel is given by $\mathcal{M}^{-1}$. Tensoring with $\mathcal{L}$ yields an exact sequence

$$0 \to \mathcal{L} \otimes \mathcal{M}^{-1} \to \mathcal{L} \otimes_k H^0(\mathcal{M}) \to \mathcal{L} \otimes \mathcal{M} \to 0$$

Now applying the long exact sequence for $H^*$ and using the tameness of $\mathcal{L} \otimes \mathcal{M}^{-1}$ yields what we want. $\square$

The following result is a partial converse to lemma 5.4.4.

Lemma 5.4.6. Let $C$ be as in Definition 5.4.1. Let $\mathcal{L}$, $\mathcal{M}$ be distinct line bundles on $C$ of degree two which are generated by global sections and which have in addition the following properties.

1. $\mathcal{L} \otimes \mathcal{M}^{-1}$ is tame.
2. $\mathcal{L} \otimes \mathcal{M}$ is ample.

Then $(\mathcal{L}, \mathcal{M})$ defines an embedding of $C$ in $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. By Proposition 5.4.2 $\mathcal{L}$, $\mathcal{M}$ are tame, whence by Riemann-Roch $\dim H^0(\mathcal{L}) = H^0(\mathcal{M}) = 2$. So $\mathcal{L}$, $\mathcal{M}$ define maps $p, q : C \to \mathbb{P}^1$. We have to show that $(p, q)$ defines a closed embedding $C \to \mathbb{P}^1 \times \mathbb{P}^1$. Note that $\mathbb{P}^1 \times \mathbb{P}^1$ is itself embedded in $\mathbb{P}^3$ by the global sections of $\mathcal{O}(1, 1)$. The composed morphism $C \to \mathbb{P}^3$ is given by the global sections of $H^0(\mathcal{L} \otimes \mathcal{M})$ which are in the image $H^0(\mathcal{L}) \otimes H^0(\mathcal{M})$. Now it follows from lemma 5.4.5 that actually $H^0(\mathcal{L} \otimes \mathcal{M}) = H^0(\mathcal{L}) \otimes H^0(\mathcal{M})$.

Thus it suffices to show that $H^0(C, L \otimes M)$ generates $\oplus_n H^0(C, L^\otimes n \otimes M^\otimes n)$ as ring. A variant of the proof of Lemma 5.4.5 shows that in fact the map $H^0(C, L^\otimes n \otimes M^\otimes n) \to H^0(C, L^\otimes n \otimes M^\otimes n)$ is surjective. This finishes the proof. $\square$

The following result is another useful addition to our toolkit.

Proposition 5.4.7. [3, Proposition 7.13] Let $C$ be as in Definition 5.4.1 and let $\mathcal{M}$ be a line bundle on $C$. If $\deg \mathcal{M} \geq 2$ and furthermore $\mathcal{M}$ has positive degree on every component of $C$ then $\mathcal{M}$ is generated by global sections.

Lemma 5.4.8. If $U$ is an elliptic quadruple then $\mathcal{L}_0 \not\cong \mathcal{L}_1 \not\cong \mathcal{L}_2$. In addition there are isomorphisms

\begin{align*}
(5.4) & \quad \Gamma(\mathcal{L}_0) \otimes \Gamma(\mathcal{L}_1) \to \Gamma(\mathcal{L}_0 \otimes \mathcal{L}_1) \\
(5.5) & \quad \Gamma(\mathcal{L}_1) \otimes \Gamma(\mathcal{L}_2) \to \Gamma(\mathcal{L}_1 \otimes \mathcal{L}_2)
\end{align*}

and furthermore there is a surjection:

\begin{align*}
(5.6) & \quad \Gamma(\mathcal{L}_0) \otimes \Gamma(\mathcal{L}_1) \otimes \Gamma(\mathcal{L}_2) \to \Gamma(\mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \mathcal{L}_2)
\end{align*}

Proof. If $\mathcal{L}_0 \cong \mathcal{L}_1$ then $(\mathcal{L}_0, \mathcal{L}_1)$ does not define an inclusion $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. The same is true if $\mathcal{L}_1 \cong \mathcal{L}_2$.

(5.4) and (5.5) follow directly from lemma 5.4.5 and (5.6) is proved in a similar (but easier) way. $\square$
If $U$ is elliptic then we will say that $U$ is \textit{prelinear} if $\mathcal{L}_0 \cong \mathcal{L}_2$. We will say that $U$ is \textit{regular} if for all components $C_1$ of $C$ we have $\deg(\mathcal{L}_0 \mid C_1) = \deg(\mathcal{L}_2 \mid C_1)$.

We will say that $U$ is admissible if it is elliptic, regular and not prelinear.

**Lemma 5.4.9.** Assume that $U = (C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ is an elliptic quadruple. Then the following are equivalent.

1. $U$ is not regular.
2. (a) There are non-zero $u \in \Gamma(\mathcal{L}_0)$, $v \in \Gamma(\mathcal{L}_1) \otimes \Gamma(\mathcal{L}_2)$ with $uv = 0$ in $\Gamma(\mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \mathcal{L}_2)$ or (b) there are non-zero $s \in \Gamma(\mathcal{L}_0) \otimes \Gamma(\mathcal{L}_1)$, $t \in \Gamma(\mathcal{L}_2)$ with $st = 0$ in $\Gamma(\mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \mathcal{L}_2)$.

**Proof.** We start by observing that $U$ is not regular if and only there exists a component $C_1$ such that either $\deg(\mathcal{L}_0 \mid C_1) = 0$ or $\deg(\mathcal{L}_2 \mid C_1) = 0$ but not both. One direction is trivial. For the other direction assume that there is a component $C' \subset C$ such that for example $\deg(\mathcal{L}_0 \mid C') = 1$ and $\deg(\mathcal{L}_2 \mid C') = 2$. Then $C'$ must have another component $C''$ with $\deg(\mathcal{L}_0 \mid C'') = 1$ and now $\deg(\mathcal{L}_2 \mid C'') = 0$. So we take $C_1 = C''$.

(1) $\Rightarrow$ (2) Assume that $C_1 \subset C$ is such that $\deg(\mathcal{L}_0 \mid C_1) = 0$ and $\deg(\mathcal{L}_2 \mid C_1) \geq 1$. Then $C_1$ is in a fiber of $p_0$. Since $(p_0, p_1)$ defines a closed embedding of $C$ in $\mathbb{P}^1 \times \mathbb{P}^1$ it follows that $C_1$ is isomorphic to $\mathbb{P}^1$ and has bidegree $(0, 1)$. In other words $\deg(\mathcal{L}_1 \mid C_1) = 1$.

Pulling back a non-zero section of $\mathcal{O}_{\mathbb{P}^1}(1)$ vanishing on $p_0(C_1)$ yields a non-zero $u \in \Gamma(\mathcal{L}_0)$ such that $u \mid C_1 = 0$. Let $C_2 = C - C_1$ (as divisors). Then $C_1 \cdot C_2 = 2$ and $\deg(\mathcal{L}_1 \otimes \mathcal{L}_2 \mid C_1) \geq 2$. Hence $\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{O}_C(-C_2)$ is supported on $C_1$ and has non-negative degree. Thus $\Gamma(\mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{O}_C(-C_2)) \neq 0$ which yields a non-zero section $v$ of $\mathcal{L}_1 \otimes \mathcal{L}_2$, vanishing on $C_2$. Combining this with lemma 5.4.8 yields that (2a) holds.

(2) $\Rightarrow$ (1) Assume that there exist $u, v$ such as in (2a). Put $C_1 = V(u)$, $C_2 = V(v)$. Again $C_1$ has bidegree $(0, 1)$ if we consider $C$ as being embedded in $\mathbb{P}^1 \times \mathbb{P}^1$ by $(p_0, p_1)$. Assume now that $C_1$ is not a double component. Since $v$ is non-vanishing on $C$ it follows $v \mid C_1$ is a non-zero section of $\mathcal{L}_1 \otimes \mathcal{L}_2 \mid C_1$. Hence $\deg(\mathcal{L}_1 \otimes \mathcal{L}_2 \mid C_1) \geq C_1 \cdot C_2 = 2$. Since $\deg(\mathcal{L}_1 \mid C_1) = 1$ it follows that $\deg(\mathcal{L}_2 \mid C_1) \geq 1$. Since $\deg(\mathcal{L}_0 \mid C_1) = 0$ this case is done.

If $C_1$ is a double component but $v \mid C_1 \neq 0$ then we may use the same argument. If on the other hand $v \mid C_1 = 0$ then $v = v'u$ where $v'$ is now a section of $\mathcal{L}_1 \otimes \mathcal{L}_2$ which is non-vanishing on $C_1$. We now use the same argument with $v'$ replacing $v$. \qed

Let $Q = (V_0, V_1, V_2, V_3, kw)$ be a geometric quintuple. Choose bases $x_i, y_i$ for $V_i$. Then $w = fx_3 + gy_3$. To $Q$ we associate the variety $\Gamma_{012} \subset \mathbb{P}(V_0^*) \times \mathbb{P}(V_1^*) \times \mathbb{P}(V_2^*)$ defined by the equations \{f, g\}. Let $p_i : \Gamma_{012} \rightarrow \mathbb{P}(V_i^*)$ be the projections. We write $L_i = p_i^{*}(\mathcal{O}(1))$ and $\Gamma_{i, i+1} = (p_i, p_{i+1})(\Gamma_{012})$ for $i = 0, 1$. Note that the geometricity of $Q$ implies that $(p_0, p_1)$ and $(p_1, p_2)$ are closed embeddings.

We write $E$ for the functor which associates to $Q$ the quadruple $U = (\Gamma_{012}, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$. It is clear that there are two cases. In the first case we have $\Gamma_{012} \cong \Gamma_{01} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and with this identification $\mathcal{L}_0 \cong \mathcal{O}(1,0)$ and $\mathcal{L}_1 \cong \mathcal{O}(0,1)$. Furthermore $(\mathcal{L}_1, \mathcal{L}_2)$ defines an isomorphism $\Gamma_{012} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Inspecting the Picard group of $\Gamma_{012}$ this will only be true if $\mathcal{L}_2 = \mathcal{O}(1,0)$. Whence $U$ is linear.
In the second case \( \Gamma_{01} \) is defined by the vanishing of a \( 2 \times 2 \) determinant with bilinear entries. So it is a curve of bidegree \((2,2)\) in \( \mathbb{P}^1 \times \mathbb{P}^1 \). It follows from the adjunction formula that \( \Gamma_{012} \cong \Gamma_{01} \) has arithmetic genus one. Furthermore by Proposition 5.4.2 together with Riemann-Roch \( \dim \Gamma(L) = 2 \) and hence the projection maps \( \Gamma_{012} \to \mathbb{P}^1 \) are given by the global sections of \( L \). In addition the projections define inclusions \( \Gamma_{012} \cong \Gamma_{1,i+1} \to \mathbb{P}^1 \times \mathbb{P}^1 \), \( i = 0,1 \). Hence it follows that \( U \) is an elliptic quadruple.

**Lemma 5.4.10.** \( Q \) is linear if and only \( E(Q) \) is linear.

**Proof.** It is easy to check that the \( w \) defined by (5.3) yields a linear quadruple. Conversely assume that \( E(Q) \cong (\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,0)) \). Under this isomorphism we have \( \Gamma_{01} = \Gamma_{12} = \mathbb{P}^1 \times \mathbb{P}^1 \) and

\[
\Gamma_{012} = \{(p,q,p) \mid p,q \in \mathbb{P}^1\}
\]

We know that \( w \) is of the form \( fx_3 + gy_3 \) where \( f,g \in \Gamma(\mathcal{O}(1,1)) \) vanish on \( \Gamma_{012} \). All such functions are multiples of \( x_0y_2 - y_0x_2 \). From this we deduce what we want. \( \square \)

**Lemma 5.4.11.** Assume that \( Q \) is a geometric quintuple. Then \( Q \) is determined up to isomorphism by \( E(Q) \).

**Proof.** Put \( U = E(Q) = (C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) \). \( U \) is linear then this lemma follows from lemma 5.4.10. So we assume that \( U \) is elliptic. By lemma 5.4.8 \( \mathcal{L}_0 \not\cong \mathcal{L}_1 \) and \( \mathcal{L}_1 \not\cong \mathcal{L}_2 \).

Let \( Q = (V_0, V_1, V_2, V_3, W) \). We already know by lemma 5.2.1 that \( Q \) is determined up to isomorphism by the corresponding two dimensional subspace \( R \subset V_0 \otimes V_1 \otimes V_2 \).

Now from the construction of \( U \) it follows that there is a complex

\[(5.7) \quad 0 \to R \to \Gamma(\mathcal{L}_0) \otimes \Gamma(\mathcal{L}_1) \otimes \Gamma(\mathcal{L}_2) \to \Gamma(\mathcal{L}_0 \otimes \mathcal{L}_1 \otimes \mathcal{L}_2) \]

and by dimension counting as well as surjectivity of the right most map (lemma 5.4.8) we obtain that this complex is actually an exact sequence. Hence \( R \) is uniquely determined by \( U \). This finishes the proof. \( \square \)

**Lemma 5.4.12.** Assume that \( Q \) is an elliptic geometric quintuple. Then \( E(Q) \) is an admissible quadruple.

**Proof.** As in the proof of lemma 5.4.11 we have \( U = E(Q) = (C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) \). Again \( \mathcal{L}_0 \not\cong \mathcal{L}_1 \) and \( \mathcal{L}_1 \not\cong \mathcal{L}_2 \).

We use again the exact sequence (5.7). If \( \mathcal{L}_0 \cong \mathcal{L}_2 \) and if \( x, y \) is a basis for \( \Gamma(\mathcal{L}_0) \cong \Gamma(\mathcal{L}_2) \) then \( R \) contains \( xx_1y - yx_1x \) and \( xy_1y - yy_1x \). From this one deduces that \( Q \) is linear, contradicting the hypotheses. So \( U \) is not prelinear.

Assume now that \( U \) is not regular. Then according lemma 5.4.9, \( R \) contains a relation of the form \( uw = 0 \) with \( \deg u = 1 \) or \( \deg v = 1 \). The two cases being similar, we assume that we are in the first case. The defining tensor \( w \) for \( Q \) is now of the form \( uwx_3 + hy_3 \) (after choosing suitable bases). We now choose \( \phi_0, \phi_3 \) in such a way that \( \phi_0(u) = 0, \phi_3(y_3) = 0 \). Then we have \( \langle \phi_0 \otimes \phi_3, w \rangle = 0 \), contradicting the fact that \( U \) is geometric. \( \square \)
5.5. Twisted homogeneous coordinate algebras. Let us introduce the following adhoc terminology inspired [9]. If \( C \) is a (proper) curve of arithmetic genus one then a **cubic elliptic helix** on \( C \) is a sequence of line-bundles \( (\mathcal{L}_i)_i \) of degree two on \( C \) satisfying the relation
\[
\mathcal{L}_i \otimes \mathcal{L}_{i+1}^{-1} \otimes \mathcal{L}_{i+2}^{-1} \otimes \mathcal{L}_{i+3} = \mathcal{O}_E
\]
Below we will always use cubic elliptic helices so we will drop the adjective “cubic”.

The quadruple associated to an elliptic helix will be \((C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)\). Note that an elliptic helix is determined up to isomorphism by its associated quadruple.

If we start with the linear quadruple \((\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,0))\) then (5.8) defines a sequence
\[
\begin{cases}
\mathcal{L}_i = \mathcal{O}(1,0) & \text{if } i \text{ is even} \\
\mathcal{L}_i = \mathcal{O}(0,1) & \text{if } i \text{ is odd}
\end{cases}
\]
For uniformity we refer to this sequence as the elliptic helix on \( \mathbb{P}^1 \times \mathbb{P}^1 \).

If \( U \) is a linear or elliptic quadruple and \((C, (\mathcal{L}_i)_i)\) its elliptic helix we define \( B = B(U) \) as the \( \mathbb{Z} \)-algebra given by \( B_{ij} = \Gamma(C, \mathcal{L}_i \otimes \cdots \otimes \mathcal{L}_{j-1}) \). If \( U \) is linear then a direct computation shows that \( B \) is a linear quadric.

We denote by \( R^iU \) the quadruple \((C, \mathcal{L}_i, \mathcal{L}_{i+1}, \mathcal{L}_{i+2})\). First note the following.

**Lemma 5.5.1.**

1. If \( U \) is elliptic regular then so is \( R^iU \) for all \( i \).
2. If \( U \) is admissible then so is \( R^iU \) for all \( i \).

**Proof.** (2) follows from (1) so we concentrate on (1). By induction it suffices to do the cases \( i = 1 \) and \( i = -1 \). Both these cases are similar so we do \( i = 1 \). We have \( \mathcal{L}_3 = \mathcal{L}_2 \otimes \mathcal{L}_1 \otimes \mathcal{L}_0^{-1} \). Since \( U \) is regular we have that \( \deg(\mathcal{L}_3 | D) = \deg(\mathcal{L}_1 | D) \geq 0 \) for every component \( D \) of \( C \). So provided \( R^1U \) is elliptic, it is clearly regular.

Since \( \deg \mathcal{L}_3 = 2 \) it follows from Proposition 5.4.7 that \( \mathcal{L}_3 \) is generated by global sections. So by Proposition 5.4.2 \( \mathcal{L}_3 \) is tame, whence by Riemann-Roch \( \dim H^0(\mathcal{L}_3) = 2 \). It follows that \( \mathcal{L}_3 \) defines a map \( p_3 : C \to \mathbb{P}^1 \). We are left with showing that \((p_2, p_3)\) defines a closed embedding \( C \to \mathbb{P}^1 \times \mathbb{P}^1 \). Now \( \mathcal{L}_2 \otimes \mathcal{L}_3^{-1} = \mathcal{L}_0^{-1} \otimes \mathcal{L}_1 \) is clearly tame by lemma 5.4.4. Furthermore by the discussion in the first paragraph of this proof, the degrees of the restrictions of \( \mathcal{L}_2 \otimes \mathcal{L}_3 \) are the same as those of \( \mathcal{L}_0 \otimes \mathcal{L}_1 \), whence they are strictly positive. We can now invoke lemma 5.4.6.

Assume now that \( U \) is an admissible quadruple and let \( B = B(U) \) Put \( V_i = B_{i,i+1} \) and let \( T \) be the “tensor algebra” of the \( V_i \). That is
\[
T_{ij} = \begin{cases}
V_i \otimes \cdots \otimes V_j & \text{if } j > i \\
k & \text{if } i = j \\
0 & \text{otherwise}
\end{cases}
\]
There is a canonical map \( T \to B \). Let \( J \) be the kernel of this map. From lemma 5.4.8 we obtain

**Lemma 5.5.2.** \( J_{ij} = 0 \) for \( j \leq i + 2 \) and \( \dim J_{i,i+3} = 2 \).

The following is an easy application of Riemann-Roch together with Proposition 5.4.2.

**Lemma 5.5.3.** One has \( \dim B_{i,i+n} = 2n \) (for \( n > 0 \)).

An easy modification of the proof of lemma 5.4.8 yields
Lemma 5.5.4. $B$ is generated in degree one.

The following theorem encodes more subtle properties of $B$. It is proved in the same way as [3, Theorem 6.6] except that one must replace $\mathcal{L}^\sigma$ by $\mathcal{L}_i$.

**Theorem 5.5.5.** Let $U$ be an admissible quadruple and put $B = B(U)$.

1. If $b \in B_{ij}$ is such that $V_{i-1}b = 0$ or $bV_j = 0$ then $b = 0$.
2. For $j-i \geq 5$ one has $J_{i,j} = T_{i,i+1}J_{i+1,j} + J_{i,i+3}T_{i+3,j} = T_{i,j-3}J_{j-3,j} + J_{i,j-1}T_{j-1,j}$.
3. Let $U_i = J_{i,i+4}/(T_{i,i+1}J_{i+1,i+4} + J_{i,i+3}T_{i+3,i+4})$ and $W_i = T_{i,i+1}J_{i+1,i+4} \cap J_{i,i+3}T_{i+3,i+4}$. Then $\dim U_i = \dim W_i = 1$.
4. $W_i$ is a non-degenerate subspace of both $T_{i,i+1}J_{i+1,i+4}$ and of $J_{i,i+3}T_{i+3,i+4}$.

If $U$ is admissible then we define $A(U)$ as the quotient of $T$ by the ideal generated by $(J_i)_{i \in \mathbb{Z}}$ where $R_i = J_{i,i+3}$. If $U$ is linear then we put $A(U) = B(U)$. By construction $A(U)$ is connected.

Following [3, Theorem 6.8] one can now prove the following results.

**Theorem 5.5.6.** Let $U = (C, L_0, L_1, L_2)$ be an admissible quadruple. Put $A = A(U)$, $B = B(U)$.

1. The canonical map $A \to B$ is surjective. Let $K$ be the kernel of this map. Then $K_{i,i+3}$ is one dimensional. Furthermore if $g_i$ are non-zero elements of $K_{i,i+4}$ then $K$ is generated by these $g_i$’s as left and as right ideal.
2. $A$ is an elliptic quadric.
3. The elements $(g_i)_i$ defined in (1) are non-zero divisors in $A$ in the sense that left multiplication by $g_i$ defines injective maps $A_{i+4} \to A_{i,j}$ and right multiplication by $g_j$ defines injective maps $A_{i,j} \to A_{i,j+4}$.

In [3] Theorem 5.5.6 is proved jointly with the following one.

**Theorem 5.5.7.** Let $U$ be either an admissible quadruple, or a linear quadruple. Put $A = A(U)$. Let $V_i, R_i, W_i$ be as above and let $P_i = P_i(A)$. $S_i = S_i(A)$ have their usual meaning.

1. The complexes of right modules with obvious maps

$$0 \to W_i \otimes P_{i+4} \to R_i \otimes P_{i+3} \to V_i \otimes P_{i+1} \to P_i \to S_i \to 0$$

are exact.

2. One has

$$\text{Ext}^n(S_i, P_j) = \begin{cases} k & \text{if } n = 3 \text{ and } j = i + 4 \\ 0 & \text{otherwise} \end{cases}$$

In particular one deduces:

**Corollary 5.5.8.** Let $U$ be as in the previous theorem. Then $A(U)$ is a 3-dimensional cubic AS-regular regular $\mathbb{Z}$-algebra.

**Proof.** This follows easily from the previous theorem. \qed

**Corollary 5.5.9.** Let $U$ be either an admissible quadruple, or a linear quadruple. Then

1. $A(U)$ and $B(U)$ are noetherian;
2. $\text{aggr}(B(U)) \cong \text{coh}(C)$;
(3) \( \text{qgr}(A(U)) \) is Ext-finite.

Proof. Let \((C, (L_i)_i)\) be the elliptic helix associated to \(U\). That \(B(U)\) is noetherian and \(\text{qgr}(B(U)) \cong \text{coh}(C)\) can e.g. be proved like in [5], replacing \(L^a\) by \(L_i\).

Alternatively we can invoke a \(Z\)-algebra version of the Artin-Zhang theorem [1]. Since \(B(U) = A(U)/(g_i)\) the fact that \(B(U)\) is noetherian easily implies that \(A(U)\) is noetherian.

That \(\text{qgr}(A(U))\) is Ext-finite follows from the AS-regular property in the same way as in the graded case [1]. □

We now finish the classification of quadrics.

**Theorem 5.5.10.** All functors in the following diagram are equivalences

\[
\begin{array}{ccc}
\{\text{quadrics}\} & \xrightarrow{F} & \{\text{admissible quadruples and linear quadruples}\} \\
A & \xrightarrow{E} & \{\text{geometric quintuples}\}
\end{array}
\]

In this diagram we have only indicated the objects of the categories in question. It is understood that the only homomorphisms we admit are isomorphisms.

Proof. By Theorem 5.2.2 and lemma 5.4.11 we already know that \(E\) and \(F\) are fully faithful. So it suffices to show that \(EFA\) is naturally equivalent to the identity functor. This is trivial in the linear case so let \(Q = (C, L_0, L_1, L_2)\) be an admissible quadruple. Then \(Q' = EFA(Q) = (C', L'_0, L'_1, L'_2)\) with \(C' \hookrightarrow \mathbb{P}(V_0^{*}) \times \mathbb{P}(V_1^{*}) \times \mathbb{P}(V_2^{*})\) where \(V_i = H^0(C, L_i)\) is defined by \(R = \ker(H^0(C, L_0) \otimes H^0(C, L_1) \otimes H^0(C, L_2) \to H^0(C, L_0 \otimes L_1 \otimes L_2))\) and \(L'_i\) is the inverse image of the projections \(C' \to \mathbb{P}(V_i^{*})\). Thus obviously \(C \subset C'\) and \(L_i = (L'_i)_{C'}\). Hence we need to show that \(C = C'\). The only possible problem is that perhaps \(Q'\) is linear. But if \(EFA(Q)\) is linear then so is \(Q\), contradicting the hypotheses. □

**Corollary 5.5.11.** There is a commutative diagram

\[
\begin{array}{ccc}
\{\text{elliptic quadrics}\} & \xrightarrow{F} & \{\text{non-linear geometric quintuples}\} \\
A & \xrightarrow{E} & \{\text{admissible quadruples}\}
\end{array}
\]

in which all functors are equivalences.

Proof. This follows from Theorem 5.5.10 if we take out the linear quadrics. □

### 5.6. Comparison with the graded case.

We now ask ourselves if we could have defined non-commutative quadrics using only graded algebras. After all this is what happened for non-commutative projective \(\mathbb{P}^2\)’s (see Theorem 4.2.2).

So the question is when a quadric \(B\) is of the form \(\hat{A}\) for a graded algebra \(A\). Following the same strategy as in the proof of Theorem 4.2.2 we see that this will be the case if and only if \(B \cong B(1)\) if and only if \((C, L_0, L_1, L_2) \cong (C, L_1, L_2, L_3)\) where \((C, (L_i)_i)\) is the elliptic helix associated to \(B\). This is then equivalent to the
following condition on the quadruple associated to $B$: there exists $\sigma \in \text{Aut}(E)$ such that

$$L_1 = \sigma^*(L_0)$$
(5.10)

$$L_2 = \sigma'^2(L_0)$$

$$\sigma^* L_0 \otimes (\sigma'^2 L_0)^{-1} \otimes (\sigma^* L_0)^{-1} \otimes L_0 = \mathcal{O}_E$$

It is now easy to see that if $C$ is a smooth elliptic curve and $L_0, L_1, L_2$ are generic line-bundles then there will be no $\sigma$ satisfying (5.10). Thus there is no analogue for Theorem 4.2.2 and hence our quadrics are genuinely more general that cubic three-dimensional regular algebras.

On the other hand the following is true.

**Proposition 5.6.1.** Assume that $B$ is a quadric and $k$ is algebraically closed of characteristic different from two. Then $B \cong B(2)$.

**Proof.** We assume that $B$ is elliptic since otherwise the claim is trivial. Let $(C, (\mathcal{L}_i)_i)$ be the elliptic helix associated to $C$. Now it should be true that $(C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2) \cong (C, \mathcal{L}_2, \mathcal{L}_3, \mathcal{L}_4)$. We have $\mathcal{L}_3 = \mathcal{L}_1 \otimes \mathcal{L}_2 \otimes \mathcal{L}_0^{-1}$ and $\mathcal{L}_4 = \mathcal{L}_2^2 \otimes \mathcal{L}_0^{-1}$. Since in addition we have $\mathcal{L}_2 = \mathcal{L}_0 \otimes \mathcal{L}_2 \otimes \mathcal{L}_0^{-1}$ it follows that it is sufficient to find $\sigma \in \text{Aut}(E)$ such that $\sigma^*(B) = B \otimes \mathcal{L}_2 \otimes \mathcal{L}_0^{-1}$ for all $B \in \text{Pic}^3(C)$.

Since $(C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ is a quadruple associated to a helix we have that $\mathcal{L}_2 \otimes \mathcal{L}_0^{-1} \in \text{Pic}^3(C)$. Hence if we take $A$ in $\text{Pic}^3(E)$ such that $A^{\otimes -2} = \mathcal{L}_2 \otimes \mathcal{L}_0^{-1}$ (this is possible since the characteristic is not two) then $\eta(A)^*(B) = B \otimes A^{\otimes -2} = B \otimes \mathcal{L}_2 \otimes \mathcal{L}_0^{-1}$ (see Theorem 4.2.3). This finishes the proof.

We will denote the Veronese of $A$ associated to the subset $2\mathbb{Z} \subset \mathbb{Z}$ by $A^e$ ("$e$" = even). We now have the following result.

**Corollary 5.6.2.** Assume that $k$ is algebraically closed of characteristic different from two and let $A$ be a quadric. Then there exists a $\mathbb{Z}$-graded algebra $B$ such that $B \cong A^e$. In particular $\text{QGr}(A) \cong \text{QGr}(B)$.

**Proof.** Since $A \cong A(2)$, we also have $A^e \cong A^e(1)$. Hence the first claim follows from Proposition 5.6.1. The second claim follows from lemmas 3.4 and 3.5. □

6. Non-commutative quadrics as hypersurfaces.

In this section we show that a non-commutative quadric can be obtained as a hypersurface in a non-commutative $\mathbb{P}^4$.

We will say that a AS-regular $\mathbb{Z}$-algebra is standard of dimension $n$ if the minimal resolution of the simples has the form

$$0 \leftarrow S_i \leftarrow P_t \leftarrow P_{n+t-1} \leftarrow P_{t+1}^{(n)} \leftarrow \cdots \leftarrow P_{t+n-1}^{(n)} \leftarrow P_{t+n} \leftarrow 0$$

We employ the same terminology for graded algebras. As in Definition 4.1.2 it makes sense to think of $\text{QGr}(A)$, with $A$ a standard noetherian AS-regular $\mathbb{Z}$-algebra of dimension $n$, as a non-commutative $\mathbb{P}^{n-1}$.

Let $\alpha$ be an automorphism of degree $-n$ of a $\mathbb{Z}$-algebra $A$. A sequence of normalizing elements inducing $\alpha$ is a sequence of regular elements $C_i \in A_{i+n}$ such that we have $C_i x = \alpha(x) C_j$ for all $i, j$ and for every $x \in A_{i+j}$. We say that $(C_i)_i$ is regular if left and right multiplication by $C_i$ define injections $A_{i+n+j} \rightarrow A_{ij}$ and $A_{ij} \rightarrow A_{i+j+n}$.
Assume now that $A$ is a $k$-$\mathbb{Z}$-algebra. If $\lambda_i \in k$ are arbitrary non-zero scalars then sending $a \in A_{ij}$ to $\lambda_i \lambda_j^{-1} a$ defines an automorphism of $A$. We call such automorphisms scalar. Two automorphisms of degree $n$ are said to be equivalent if they differ by a scalar automorphism.

Now let $A$ be quadric. We define a hull of $A'$ as a surjective homomorphism of $2\mathbb{Z}$-algebras $\phi : D \to A'$ where $D$ is a four-dimensional standard AS-regular $2\mathbb{Z}$-algebra and the kernel of $\phi$ is generated by sequence of regular normalizing elements $(C_{2i})_i \in D_{2i, 2i+2}$.

If $\phi : D \to A'$ is a hull then the $C_{2i}$ induce an automorphism $\alpha$ of degree $-2$ of $D$. Since the $C_{2i}$ are only determined up to a scalar, $\alpha$ is only determined up to equivalence. We will write $a(\phi)$ for the equivalence class of $\alpha$.

An interesting problem is to classify the hulls of $A'$. This problem was partially solved in [8] in the graded case and it turns out that the approach in loc. cit. generalizes in a straightforward way to $\mathbb{Z}$-algebras. On obtains the following

**Theorem 6.1.** (See [8, Prop. 3.2, Rem. 3.3]) Assume that $\alpha$ is an automorphism of degree $-4$ of $A$ and denote its restriction to $A'$ by the same letter. Then up to isomorphism there is at most one hull $\phi$ of $A'$ such that $a(\phi) \sim \alpha$. If we take $\alpha = \theta^{-1}$ (where $\theta$ is as defined in §5.4) then an associated hull exists.

**Remark 6.2.** Note that since $A$ is noetherian (see Corollary 5.5.9) so is $A'$ and from this we deduce that any hull is noetherian as well.

From Theorem 6.1 we easily obtain the following result.

**Corollary 6.3.** Let $X = \text{QGr}(A)$ be a non-commutative quadric. Then $X$ can be embedded as a divisor [13] in a non-commutative $\mathbb{P}^3$.

Note following result.

**Lemma 6.4.** Let $\alpha, \beta$ be an automorphisms of $A$ of degrees $-4, n$ respectively, which commute up to equivalence. Let $\phi : D \to A'$ be a hull associated to $\alpha$. Then there is an automorphism $\beta' : D \to D(n)$ and a commutative diagram

\[
\begin{array}{ccc}
D & \xrightarrow{\phi} & A' \\
\beta' \downarrow & & \downarrow \beta \\
D(n) & \xrightarrow{\phi(n)} & A'(n)
\end{array}
\]

**Proof.** This follows from the uniqueness of $\phi$, up to isomorphism.

We combine this with the following lemma.

**Lemma 6.5.** Let $\gamma$ be an arbitrary automorphism of degree $n$ of $A$. Then $\gamma$ commutes with $\theta$, up to equivalence.

**Proof.** This is of course because $\theta$ is canonical, up to equivalence. The actual proof is an easy verification.

Combining everything we obtain the following result

**Proposition 6.6.** Assume that $k$ is an algebraically closed field of characteristic different from two. Let $A$ be a cubic 3-dimensional regular $\mathbb{Z}$-algebra. Then there exists a $\mathbb{Z}$-graded algebra $B$ such that $B \cong A'$. Furthermore there exists a 4-dimensional Artin-Schelter regular algebra $D$ with Hilbert series $1/(1-t)^4$ together with a regular normal element $C \in D_2$ such that $B \cong D/(C)$. 
Proof. By Corollary 5.6.2 $A^e = \hat{B}$. Let $E$ be the hull of $A^e$ given by $\alpha = \theta^{-1}$ (see Theorem 6.1). By Lemma 6.4 $E$ is 1-periodic, i.e. it is of the form $\hat{D}$ by Lemma 3.4. One easily verifies that the surjective map $E \to A^e$ yields a surjective map of $\mathbb{Z}$-graded algebra $D \to B$ and that its kernel is given by a normalizing element $C$. \hfill $\square$

Corollary 6.7. Assume that $k$ is an algebraically closed field of characteristic different from two and let $X = \text{QGr}(A)$ be a quadric. Then there exists a four-dimensional standard noetherian Artin-Schelter graded algebra $D$ together with a regular normalizing element $C \in D_2$ such that $X = \text{QGr}(D/(C))$.

Proof. With notations as in the previous proposition we have (using Lemma 3.5) $\text{QGr}(A) = \text{QGr}(A^e) = \text{QGr}(B) = \text{QGr}(D/(C))$. \hfill $\square$

7. THE TRANSLATION PRINCIPLE

In this section we prove the following theorem.

Theorem 7.1. Let $A$ be a quadric with quadruple $U = (C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$. Fix $n \in \mathbb{Z}$ and assume that $\mathcal{L}_0 \not\sim \mathcal{L}_{2n+1} \not\sim \mathcal{L}_2$. Then $(C, \mathcal{L}_0, \mathcal{L}_{2n+1}, \mathcal{L}_2)$ is admissible or linear. Denote the associated quadric by $T^n A$.

Assume that for all odd $m$ between 1 and $2n+1$ (inclusive) we have $\mathcal{L}_0 \not\sim \mathcal{L}_m \not\sim \mathcal{L}_2$. Then $\text{QGr}(T^n A) \cong \text{QGr}(A)$.

If $(C, (\mathcal{L}_i)_i)$ is the elliptic helix associated to $A$ then the elliptic helix associated to $T^n A$ is given by $(C, \ldots, \mathcal{L}_{2n-1}, \mathcal{L}_0, \mathcal{L}_{2n+1}, \mathcal{L}_2, \ldots)$ with $\mathcal{L}_0$ occurring in position zero. In other words we have shifted the odd part of $A$’s elliptic helix $2n$ places to the left. This is the translation principle alluded to in the title of this section.

Assume that the conditions of the theorem are satisfied and assume that $k$ is algebraically closed of characteristic different from 2. By Corollary 5.6.2 we have $A^e = \hat{B}$ for a $\mathbb{Z}$-graded algebra $B$. Similarly we have $(T^n A)^e = \hat{C}$ for a $\mathbb{Z}$-graded algebra $C$. It follows from Lemma 3.5 that $\text{QGr}(B) = \text{QGr}(C)$. However one may show that in general $B \not\cong C$. This is similar to the situation [27]. The exact relation between the translation principle in [27] and the current one will be discussed elsewhere.

Theorem 7.1 is trivial to prove in the linear case so we assume first that $A$ is elliptic.

Lemma 7.2. Let $U = (C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)$ be an admissible quadruple and let $V = (C, \mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_2)$ be a quadruple such that

1. $\mathcal{M}_0 \not\sim \mathcal{M}_1 \not\sim \mathcal{M}_2 \not\sim \mathcal{M}_0$
2. $\deg(\mathcal{L}_i | E) = \deg(\mathcal{M}_i | E)$ for $i = 0, 1, 2$ and for every irreducible component $E$ of $C$.

Then $V$ is admissible.

Proof. That $V$ is regular and not prelinear is clear from the hypotheses. So we only need to show that $V$ is elliptic, i.e. $(\mathcal{M}_0, \mathcal{M}_1)$ and $(\mathcal{M}_1, \mathcal{M}_2)$ define embeddings $C \hookrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. We already know by Proposition 5.4.7 that all $\mathcal{M}_i$ are generated by global sections. It now suffices to verify the hypothesis for Lemma 5.4.6 for $(\mathcal{M}_0, \mathcal{M}_1)$ and $(\mathcal{M}_1, \mathcal{M}_2)$. Both cases are similar so we only look at the first one.
We already have $M_0 \not\cong M_1$ by hypotheses. Next we need that $M_0 \otimes M_2^{-1}$ is tame. This follows from lemma 5.4.4. Finally we need that $M_0 \otimes M_2$ is ample but this is clear since the condition for ampleness on a curve is purely numerical.

Now starting from the quadric $A$ we will construct another quadric $A^\omega$ with the property $A^\omega \cong A^{\omega,\epsilon}$. Thus in particular $\text{QGr}(A) \cong \text{QGr}(A^\omega)$. This will be the first step in the proof of Theorem 7.1.

Let us assume that the elliptic helix of $A$ satisfies

(7.1) \[ \mathcal{L}_1 \not\cong \mathcal{L}_2 \]

Then by lemma 7.2 combined with lemma 5.5.1 we easily see that the quadruple $U^\omega \overset{\text{def}}{=} (C, \mathcal{L}_1, \mathcal{L}_2, \mathcal{L}_1)$ is admissible. We define $A^\omega$ as the quadric associated to $U^\omega$. A direct verification using (5.8) shows that the elliptic helix associated to $A^\omega$ is of the form

(7.2) \[ (C, \ldots, \mathcal{L}_-2, \mathcal{L}_-3, \mathcal{L}_0, \mathcal{L}_{-1}, \mathcal{L}_2, \mathcal{L}_1, \ldots) \]

with $\mathcal{L}_-$ occurring in position zero. I.e the odd part of the elliptic helix of $A$ is shifted one place to the right and the even part is shifted one place to the left.

**Lemma 7.3.** $A^\epsilon \cong A^{\omega,\epsilon}$.

**Proof.** To prove this we let the notations $V_i, R_i$ have their usual meaning and we use the corresponding notations $V_i^\omega, R_i^\omega$ for $A^\omega$. Note that $A^\epsilon$ is generated by $(A_{2i, 2i+2})_i$, with relations $R_{2i} \otimes V_{2i+3} + V_{2i} \otimes R_{2i+1}$. A similar statement holds for $A^{\omega,\epsilon}$.

We start by defining maps $\phi_{2i} : A_{2i} \rightarrow A_{2i}^\omega$ as the composition

\[
A_{2i, 2i+2} \cong H^0(C, \mathcal{L}_0) \otimes H^0(C, \mathcal{L}_{2i+1}) \\
\cong H^0(C, \mathcal{L}_{2i} \otimes \mathcal{L}_{2i+1}) \\
\cong H^0(C, \mathcal{L}_{2i} \otimes \mathcal{L}_{2i+1}) \\
\cong (1) A_{2i} \cong A_{2i}^\omega
\]

The canonical isomorphisms marked (1),(3) are obtained from Lemma 5.4.8. The isomorphism marked (2) is obtained from (5.8). To make it canonical we assume that we have fixed explicit isomorphisms in (5.8).

To prove that $(\phi_{2i})_i$ defines an isomorphism between $A$ and $A^\omega$ we have to show

\[(\phi_{2i} \otimes \phi_{2i+2})(R_{2i} \otimes V_{2i+3} + V_{2i} \otimes R_{2i+1}) = (R_{2i} \otimes V_{2i+3}^\omega + V_{2i}^\omega \otimes R_{2i+1}^\omega)\]

To this end it is sufficient to show

\[(\phi_{2i} \otimes 1)(R_{2i} \otimes V_{2i+3}) = V_{2i}^\omega \otimes R_{2i+1}^\omega\]

and

\[(\phi_{2i} \otimes 1)(V_{2i} \otimes R_{2i+1}) = V_{2i}^\omega \otimes R_{2i+1}^\omega\]

Both equalities are similar, so we only look at the first one. This equality is equivalent to

\[(\phi_{2i} \otimes 1)(R_{2i}) \otimes V_{2i+3} = V_{2i}^\omega \otimes (1 \otimes \phi_{2i+2})^(-1)(R_{2i+1})\]
Now note that \( \phi_{2i} \otimes 1 \) defines an isomorphism

\[
H^0(C, \mathcal{L}_{2i}) \otimes H^0(C, \mathcal{L}_{2i+1}) \otimes H^0(C, \mathcal{L}_{2i+1}) \cong H^0(C, \mathcal{L}_{2i-1}) \otimes H^0(C, \mathcal{L}_{2i+2}) \otimes H^0(C, \mathcal{L}_{2i+3})
\]

We claim that the image of \( R_{2i} \) under this isomorphism is given by \( V_{2i-1} \otimes \Lambda^2 V_{2i+2} \).

Admitting this claim we find that

\[
(\phi_{2i} \otimes 1)(R_{2i}) \otimes V_{2i+3} = V_{2i-1} \otimes \Lambda^2 V_{2i+2} \otimes V_{2i+3}
\]

and a dual argument shows that we get the same result for \((1 \otimes \phi^{-1}_{2i+2})(R_{2i+1})\).

To prove our claim we consider the following commutative diagram.

\[
\begin{array}{cccccc}
0 & \longrightarrow & R_{2i} & \longrightarrow & H^0(C, \mathcal{L}_{2i}) \otimes H^0(C, \mathcal{L}_{2i+1}) \otimes H^0(C, \mathcal{L}_{2i+2}) & \longrightarrow & 0 \\
\downarrow \phi_{2i} \otimes 1 \mid \cong & & & \cong & & \downarrow \\
0 & \longrightarrow & V_{2i-1} \otimes \Lambda^2 V_{2i+2} & \longrightarrow & H^0(C, \mathcal{L}_{2i-1}) \otimes H^0(C, \mathcal{L}_{2i+2}) \otimes H^0(C, \mathcal{L}_{2i+3}) & \longrightarrow & 0
\end{array}
\]

The rows in this diagram is are exact. Hence the left most arrow is an isomorphism.

This proves our claim.

\[\Box\]

**Corollary 7.4.** Assume that \((C, \mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2)\) is admissible. Then \(\text{QGr}(TA) \cong \text{QGr}(A)\).

**Proof.** This is clear by lemma 7.3 since \(TA = A(1)^\omega\) and hence \((TA)^e \cong A^e\) \((^\omega \text{odd})\). It suffices to invoke Lemma 3.5. \(\Box\)

**Proof of Theorem 7.1.** The linear case is clear. The elliptic case follows from repeated application of Corollary 7.4. \(\Box\)

8. Non-commutative quadrics as deformations of commutative quadrics

8.1. Introduction. We start by giving a convenient definition of certain “families”.

**Definition 8.1.** Let \(R\) be a commutative noetherian ring. An \(R\)-family of three dimensional quadratic regular algebras is an \(R\)-flat noetherian \(\mathbb{Z}\)-algebra \(A\) such that for any map to a field \(R \rightarrow K\) we have that \(A_K\) is a three dimensional quadratic regular algebra in the sense of Definition 4.1.1. An \(R\)-family of three dimensional cubic regular algebras is defined similarly.

Let \((R, m)\) be a commutative noetherian complete local ring with \(k = R/m\). Our aim in this section is to prove the following results.

**Theorem 8.1.1.** Put \(C = \text{coh}(\mathbb{P}^2_k)\) and let \(D\) be an \(R\)-deformation of \(C\). Then \(D = \text{qgr}(A)\) where \(A\) is an \(R\)-family of three dimensional quadratic regular algebras.

**Theorem 8.1.2.** Put \(C = \text{coh}(\mathbb{P}^1_k \times \mathbb{P}^1_k)\) and let \(D\) be an \(R\)-deformation of \(C\). Then \(D = \text{qgr}(A)\) where \(A\) is an \(R\)-family of three dimensional cubic regular algebras.

To make sense of these statements we must understand what we mean by a “deformation” of \(C\). Infinitesimal deformations of abelian categories where defined in [16]. From this one may define non-infinitesimal deformations by a suitable limiting procedure. The theoretical foundation of this is Jouanolou’s exposé in SGA 5 [14]. The details are provided in [26]. In the next section we state the results we need. If the reader is willing to accept that deformations of abelian categories behave in the “expected way” he/she should go directly to §8.3.

The proofs of Theorems 8.1.1 and 8.1.2 are based on the fact that \(\mathbb{P}^2\) and \(\mathbb{P}^1 \times \mathbb{P}^1\) have ample sequences consisting of exceptional objects. Such sequences can be lifted
to any deformation (see §8.2). This idea is basically due to Bondal and Polishchuk and is described explicitly in [22, §11.2]. See also the recent paper [10].

8.2. Deformations of abelian categories. For the convenience of the reader we will repeat the main statements from [26]. We first recall briefly some notions from [16]. Throughout $R$ will be a commutative noetherian ring and $\text{mod}(R)$ is its category of finitely generated modules.

Let $C$ be an $R$-linear abelian category. Then we have bifunctors $- \otimes_R - : C \times \text{mod}(R) \to C$, $\text{Hom}_R(-, -) : \text{mod}(R) \times C \to C$ defined in the usual way. These functors may be derived in their $\text{mod}(R)$-argument to yield bi-delta-functors $\text{Tor}_i^R(-, -)$, $\text{Ext}_i^R(-, -)$. An object $M \in C$ is $R$-flat if $M \otimes_R -$ is an exact functor, or equivalently if $\text{Tor}_i^R(M, -) = 0$ for $i > 0$.

By definition (see [16, §3]) $C$ is $R$-flat if $\text{Tor}_i^R$ or equivalently $\text{Ext}_i^R$ is effaceble in its $C$-argument for $i > 0$. This implies that $\text{Tor}_i^R$ and $\text{Ext}_i^R$ are universal $\partial$-functors in both arguments.

If $f : R \to S$ is a morphism of commutative noetherian rings such that $S/R$ is finitely generated and $C$ is an $R$-linear abelian category then $C_S$ denotes the (abelian) category of objects in $C$ equipped with an $S$-action. If $f$ is surjective then $C_S$ identifies with the full subcategory of $C$ given by the objects annihilated by $\ker f$. The inclusion functor $C_S \to C$ has right and left adjoints given respectively by $\text{Hom}_R(S, -)$ and $- \otimes_R S$.

Now assume that $J$ is an ideal in $R$ and let $\hat{R}$ be the $J$-adic completion of $R$. Recall that an abelian category $D$ is said to be noetherian if it is essentially small and all objects are noetherian. Let $D$ be an $R$-linear noetherian category and let $\text{Pro}(D)$ be its category of pro-objects. We define $\hat{D}$ as the full subcategory of $\text{Pro}(D)$ consisting of objects $M$ such that $M/MJ^n \in D$ for all $n$ and such that in addition the canonical map $M \to \text{proj lim}_n M/MJ^n$ is an isomorphism. The category $\hat{D}$ is $\hat{R}$-linear. The following is basically a reformulation of Jouanolou’s results [14].

**Proposition 8.2.1.** (see [26, Prop. 2.2.5]) $\hat{D}$ is a noetherian abelian subcategory of $\text{Pro}(D)$.

There is an exact functor

\[
\Phi : D \to \hat{D} : M \mapsto \text{proj lim}_n M/MJ^n
\]

and we say that $\hat{D}$ is complete if $\Phi$ is an equivalence of categories. In addition we say that $D$ is formally flat if $D_{R/J^n}$ is $R/J^n$-flat for all $n$.

**Definition 8.2.2.** Assume that $C$ is an $R/J$-linear noetherian flat abelian category. Then an $R$-deformation of $C$ is a formally flat complete $R$-linear abelian category $D$ together with an equivalence $D_{R/J} \cong C$.

In general, to simplify the notations, we will pretend that the equivalence $D_{R/J} \cong C$ is just the identity.

Thus below we consider the case that $D$ is complete and formally flat and $C = D_{R/J}$. The following definition turns out to be natural.

**Definition 8.2.3.** (see [26, (1.1)]) Assume that $E$ is a formally noetherian $R$-linear abelian category. Let $E_{\ell}$ be the full subcategory of $E$ consisting of objects
annihilated by a power of $J$. Let $M, N \in \mathcal{E}$. Then the completed Ext-groups between $M, N$ are defined as

\[ \hat{\text{Ext}}^i_\mathcal{E}(M, N) = \text{Ext}^i_{\text{proj}(\mathcal{E})}(M, N) \]

An $R$-linear category $\mathcal{E}$ is said to be Ext-finite if $\hat{\text{Ext}}^i_\mathcal{E}(M, N)$ is a finitely generated $R$-module for all $i$ and all objects $M, N \in \mathcal{E}$. Assuming Ext-finiteness the completed Ext-groups become computable.

**Proposition 8.2.4.** [26, Prop. 2.5.3] Assume that $\mathcal{E}$ is a formally flat noetherian $R$-linear abelian category and that $\mathcal{E}_{R/J}$ is Ext-finite. Then $\hat{\text{Ext}}^i_\mathcal{E}(M, N) \in \text{mod}(R)$ for $M, N \in \hat{\mathcal{E}}$ and furthermore

\[ \hat{\text{Ext}}^i_\mathcal{E}(M, N) = \text{proj lim}_{k} \text{lim}_{i} \text{Ext}^i_{\mathcal{E}_{R/J}}(M/MJ^k, N/NJ^k) \]

If $M$ is in addition $R$-flat then

\[ \hat{\text{Ext}}^i_\mathcal{E}(M, N) = \text{proj lim}_{k} \text{Ext}^i_{\mathcal{E}_{R/J}}(M/MJ^k, N/NJ^k) \]

The results below allow one to lift properties from $C$ to $D$. They follow easily from the corresponding infinitesimal results ([16, Prop. 6.13], [17, Theorem A], [26]).

**Proposition 8.2.5.** Let $M \in C$ be a flat object such that $\text{Ext}^i_C(M, M \otimes R/J J^n/J^{n+1}) = 0$ for $i = 1, 2$ and $n \geq 1$. Then there exists a unique $R$-flat object (up to non-unique isomorphism) $\overline{M} \in D$ such that $\overline{M}/MJ \cong M$.

**Proposition 8.2.6.** Let $\overline{M}, \overline{N} \in D$ be flat objects and put $\overline{M}/MJ = M, \overline{N}/NJ = N$. Assume that for all $X$ in $\text{mod}(R/J)$ we have $\text{Ext}^i_C(M, N \otimes R/J X) = 0$ for a certain $i > 0$. Then we have $\hat{\text{Ext}}^i_D(\overline{M}, \overline{N} \otimes_R X) = 0$ for all $X \in \text{mod}(R)$.

**Proposition 8.2.7.** Let $\overline{M}, \overline{N} \in D$ be flat objects and put $\overline{M}/MJ = M, \overline{N}/NJ = N$. Assume that for all $X$ in $\text{mod}(R/J)$ we have $\text{Ext}^i_C(M, N \otimes R/J X) = 0$. Then $\text{Hom}_D(\overline{M}, \overline{N})$ is $R$-flat and furthermore for all $X$ in mod($R$) we have $\text{Hom}_D(\overline{M}, \overline{N} \otimes_R X) = \text{Hom}_D(\overline{M}, \overline{N}) \otimes_R X$.

Let us also mention Nakayama’s lemma [26].

**Lemma 8.2.8.** Let $M \in D$ be such that $MJ = 0$. Then $M = 0$.

It is convenient to strengthen the standard notion of (categorical) ampleness. Let $\mathcal{E}$ a noetherian abelian category.

**Definition 8.2.9.** A $(\mathcal{O}(n))_{n \in \mathbb{Z}}$ of objects in a noetherian category $\mathcal{E}$ is strongly ample if the following conditions hold

- (A1) For all $M \in \mathcal{E}$ and for all $n$ there is an epimorphism $\oplus_{i=1}^{n} \mathcal{O}(-n_i) \to M$ with $n_i \geq n$.
- (A2) For all $M \in \mathcal{E}$ and for all $i > 0$ one has $\text{Ext}^i_D(\mathcal{O}(-n), M) = 0$ for $n \gg 0$.

A strongly ample sequence $(\mathcal{O}(n))_{n \in \mathbb{Z}}$ in $\mathcal{E}$ is ample in the sense of [19]. Hence using the methods of [1] or [19] one obtains $\mathcal{E} \cong \text{qgr}(A)$ if $\mathcal{E}$ is Hom-finite, where $A = \bigoplus_j \text{Hom}_E(\mathcal{O}(-j), \mathcal{O}(-i))$. The functor realizing the equivalence is $M \mapsto \pi \left( \bigoplus \text{Hom}_E(\mathcal{O}(-i), M) \right)$ where $\pi : \text{gr}(A) \to \text{qgr}(A)$ is the quotient functor.

The following result is a version of “Grothendieck’s existence theorem.”
Proposition 8.2. (see [26, Prop. 4.1]). Assume that $R$ is complete and let $\mathcal{E}$ be an Ext-finite $R$-linear noetherian category with a strongly ample sequence $(O(n))_n$. Then $\mathcal{E}$ is complete and furthermore and furthermore is $\mathcal{E}$ is flat then we have for $M, N \in \mathcal{E}$:

$$\text{Ext}^i_R(M, N) = \text{Ext}^i_R(M, N)$$

(8.2)

The following results shows that the property of being strongly ample lifts well.

Theorem 8.2.10. (see [26, Thm. 4.2]) Assume that $R$ is complete and that $C$ is Ext-finite and let $O(n)_n$ be a sequence of $R$-flat objects in $\mathcal{D}$ such that $(O(n)/O(n).J)_n$ is strongly ample. Then

1. $O(n)_n$ is strongly ample in $\mathcal{D}$;
2. $\mathcal{D}$ is flat (instead of just formally flat);
3. $\mathcal{D}$ is Ext-finite as $R$-linear category.

8.3. Proofs of Theorem 8.1.1 and 8.1.2. We consider first Theorem 8.1.1. We will show that $\mathcal{D} = \text{qgr}(A)$ where $A = \bigoplus_{ij} A_{ij}$ is a noetherian Z-algebra such that all $A_{ij}$ are projective of finite rank over $R$; $A_{ii} = R$, $V_i = A_{ii+1}$ is a free $R$-module of rank 3; $R_i = \ker(V_i \otimes R V_{i+1} \rightarrow A_{ii+2})$ is also a free $R$-module of rank 3, $W_i = R_i \otimes V_{i+2} \cap V_i \otimes R_{i+1}$ is free of rank one and finally the canonical complex

$$0 \rightarrow W_i \otimes R P_{i+2} \rightarrow R_i \otimes R P_{i+2} \rightarrow V_i \otimes R P_{i+1} \rightarrow P_i \rightarrow S_i \rightarrow 0$$

(8.3)

where $P_i = e_i A$ and $S_i = A_{ii}$ is exact. Since all $A$-modules appearing in this exact sequence are $R$-projective it is split as $R$-modules and hence remains exact after tensoring with an arbitrary $R$-module. From this we conclude that $A$ is an $R$-family of three dimensional quadratic regular algebras

Step 1 Put $L_i = O_{z^2}(i)$. Then $L_i$ is a strongly ample sequence on $C$. Let $A = \bigoplus_{ij} A_{ij}$ with $A_{ij} = \text{Hom}_C(L_{i,j}, L_{-i})$ be the associated Z-algebra. Since we have $\text{Ext}^n_R(L_i, L_j) = 0$ for $n > 0$ and $j \geq i$ we have by Propositions 8.2.5, 8.2.6 and 8.2.7 that $(L_i)_i$ can be lifted to $R$-flat objects $(\mathcal{L}_i)_i$ in $\mathcal{D}$ such that $A_{ij} = \text{Hom}_D(L_{i,j}, L_{-i})$ is a finitely generated projective $R$-module such that $A_{ij} \otimes_R R/m = A_{ij}$. Furthermore by Theorem 8.2.10, $(\mathcal{L}_i)_i$ is a strongly ample sequence in $\mathcal{D}$. Thus by [7] we have $\mathcal{D} = \text{qgr}(A)$.

Step 2 Now we prove the remaining assertions about $A$. That $V_i = A_{ii+1}$ is projective of rank 3 is clear. We also have that $\mu : V_i \otimes_R V_{i+1} \rightarrow A_{ii+1}$ is an epimorphism after tensoring with $R/m$. By Nakayama’s lemma (see Lemma 8.2.8) it follows that it was already an epimorphism originally. Since $A_{ii+1}$ is projective it follows that $\mu$ is split. Since $V_i \otimes_R V_{i+1}$ is finitely generated projective we obtain that $R_i$ is finitely generated projective of rank 3 as well. Furthermore the formation of $R_i$ is compatible with base change.

The following complex of right $A$-modules, which consists of projective $R$-modules, is exact after tensoring with $R/m$.

$$R_i \otimes_R P_{i+2} \rightarrow V_i \otimes_R P_{i+2} \rightarrow P_i \rightarrow S_i \rightarrow 0$$

(8.4)

It is then again an easy consequence of Nakayama’s lemma (see Lemma 8.2.8) that it is exact. We deduce from this that the following “slice” of (8.4) is exact.

$$0 \rightarrow W_i \rightarrow R_i \otimes_R V_{i+2} \rightarrow V_i \otimes A_{i+1,i+2} \rightarrow A_{i,i+2} \rightarrow 0$$
(the kernel of the middle map is \( W_i \) by our definition of \( W_i \) above). It follows that \( W_i \) is finitely generated projective and one computes that it has rank 1. If follows that \( W_i \) is compatible with base change as well.

Finally we note that (8.4) is exact after tensoring with \( R/m \). Hence it was exact originally as well.

The proof of Theorem 8.1.2 is completely similar but now we start with

\[
L_i = \begin{cases} 
\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, k) & \text{if } i = 2k \\
\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(k, k+1) & \text{if } i = 2k + 1 
\end{cases}
\]

References

[1] M. Artin and J. Zhang, Noncommutative projective schemes, Adv. in Math. 109 (1994), no. 2, 228–287.
[2] M. Artin and W. Schelter, Graded algebras of global dimension 3, Adv. in Math. 66 (1987), 171–216.
[3] M. Artin, J. Tate, and M. Van den Bergh, Some algebras associated to automorphisms of elliptic curves, The Grothendieck Festschrift, vol. 1, Birkhäuser, 1990, pp. 33–85.
[4] M. Artin, J. Tate, and M. Van den Bergh, Modules over regular algebras of dimension 3, Invent. Math. 106 (1991), 335–388.
[5] M. Artin and M. Van den Bergh, Twisted homogeneous coordinate rings, J. Algebra 188 (1990), 249–271.
[6] D. Auroux, L. Katzarkov, and D. Orlov, Mirror symmetry for del Pezzo surfaces: vanishing cycles and coherent sheaves, Invent. Math. 166 (2006), no. 3, 537–582.
[7] M. Auslander and I. Reiten, Cohen-Macaulay modules for graded Cohen-Macaulay rings and their completions, Commutative algebra (Berkeley, CA, 1987) (New York), Math. Sci. Res. Inst. Publ., vol. 15, Springer, New York, 1989, pp. 21–31.
[8] K. Bauwens and M. Van den Bergh, Normalizing extensions of the two-Veronese of a three-dimensional Artin-Schelter regular algebra on two generators, J. Algebra 205 (1998), no. 2, 368–390.
[9] A. Bondal and A. Polishchuk, Homological properties of associative algebras: the method of helices, Russian Acad. Sci. Izv. Math 42 (1994), 219–260.
[10] O. De Decker and W. Lowen, Abelian and derived deformations in the presence of Z-generating geometric helices, arXiv:1001.4265.
[11] I. Gordon and J. T. Stafford, Rational Cherednik algebras and Hilbert schemes, Adv. Math. 198 (2005), no. 1, 222–274.
[12] I. Gordon and J. T. Stafford, Rational Cherednik algebras and Hilbert schemes. II. Representations and sheaves, Duke Math. J. 132 (2006), no. 1, 73–135.
[13] P. Jørgensen, Intersection theory on non-commutative surfaces, Trans. Amer. Math. Soc. 352 (2000), no. 12, 5817–5854.
[14] J. P. Jouanolou, Systèmes projectifs J-adiques, Cohomologie l-adique et fonctions L, SGA5 (Berlin), Lecture notes in mathematics, vol. 589, Springer Verlag, Berlin, 1977.
[15] W. Lowen and M. Van den Bergh, Hochschild cohomology of abelian categories and ringed spaces, Adv. Math. 198 (2005), no. 1, 172–221.
[16] W. Lowen, Deformation theory of abelian categories, Trans. Amer. Math. Soc. 358 (2006), no. 12, 5441–5483.
[17] W. Lowen, Obstruction theory for objects in abelian and derived categories, Comm. Algebra 33 (2005), no. 9, 3195–3223.
[18] E. P. and G. V., Noncommutative del pezzo surfaces and calabi-yau algebras, arXiv:0709.3593.
[19] A. Polishchuk, Noncommutative proj and coherent algebras, Math. Res. Lett. 12 (2005), no. 1, 63–74.
[20] S. Sierra, G-algebras, twistings, and equivalences of graded categories, arXiv:math/0608791.
[21] N. Spaltenstein, Resolutions of unbounded complexes, Compositio Math. 65 (1988), no. 2, 121–154.
[22] J. T. Stafford and M. Van den Bergh, Noncommutative curves and noncommutative surfaces, Bull. Amer. Math. Soc. (N.S.) 38 (2001), no. 2, 171–216.
[23] B. Stenström, Rings of quotients, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, vol. 217, Springer Verlag, Berlin, 1975.
[24] D. R. Stephenson, Artin–Schelter regular algebras of global dimension three, J. Algebra 183 (1996), 55–73.
[25] ———, Algebras associated to elliptic curves, Trans. Amer. Math. Soc. 349 (1997), 2317–2340.
[26] M. Van den Bergh, Notes on formal deformations of abelian categories, arXiv:1002.0259.
[27] ———, A translation principle for Sklyanin algebras, J. Algebra 184 (1996), 435–490.
[28] ———, Blowing up of non-commutative smooth surfaces, Mem. Amer. Math. Soc. 154 (2001), no. 734, x+140.
[29] J. J. Zhang, Twisted graded algebras and equivalences of graded categories, Proc. London Math. Soc. (3) 72 (1996), no. 2, 281–311.

Universiteit Hasselt, Universitaire Campus, 3590 Diepenbeek
E-mail address: michel.vandenbergh@uhasselt.be