On the asymptotics of visible elements and homogeneous equations in surface groups

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Abstract

Let \( F \) be a group whose abelianization is \( \mathbb{Z}^k \), \( k \geq 2 \). An element of \( F \) is called visible if its image in the abelianization is visible, that is, the greatest common divisor of its coordinates is 1.

In this paper we compute three types of densities, annular, even and odd spherical, of visible elements in surface groups. We then use our results to show that the probability of a homogeneous equation in a surface group to have solutions is neither 0 nor 1, as the lengths of the right- and left-hand side of the equation go to infinity.

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1 Introduction

Let \( F \) be a group whose abelianization is \( \mathbb{Z}^k \), with \( k \geq 2 \). An element of \( F \) is called visible with respect to a basis of \( \mathbb{Z}^k \) if its image in the abelianization is visible, that is, the greatest common divisor of its coordinates is 1. Being visible is, in fact, independent of the basis of \( \mathbb{Z}^k \) (see Definition 2.3), and we therefore omit the references to the basis henceforth.

Let \( \Sigma \) be a compact connected orientable surface of genus \( r \), \( r \geq 2 \). If \( \Sigma \) has no boundary, then a presentation for the fundamental group of \( \Sigma \), which we call the surface group of genus \( r \), is \( \langle a_1, b_1, \ldots, a_r, b_r \mid [a_1, b_1] \cdots [a_r, b_r] \rangle \). If \( \Sigma \) has boundary, then the fundamental group of \( \Sigma \) is simply a free group of finite rank. For a group \( G \), a positive integer \( n \), and a fixed generating set \( A \), one defines the sphere of radius \( n \) to be the set of elements of length \( n \), with respect to \( A \), in \( G \). Then the spherical density of a set \( S \) of elements in \( G \) measures the proportion of elements of length \( n \) in \( S \) in the sphere of radius \( n \), as \( n \) goes to infinity (see Section 2). The annular density of a set \( S \) records the proportions of \( S \) in two successive spheres.

While the spherical density of visible elements does not exist for the groups we consider, one can instead look at the ‘odd spherical density’ and ‘even spherical density’ of visible elements of odd and even length, respectively. In this paper we compute the annular, odd and even spherical densities of visible elements in a class of groups containing the surface groups of compact connected orientable surfaces, with or without boundary. In [4] the annular density of visible elements was computed for all free groups of finite rank ([4], Theorem A), and odd and even spherical density values were also given for the free group of rank two ([4], Theorem 3.7).

Since the limits we obtain are different from 0 and from 1, this shows that visible elements form a set of intermediate density in the groups we study. Intermediate density of sets in groups...
has been displayed for the first time in [4], and this tends to be a relatively rare behaviour for many combinatorial and algebraic properties encountered in group theory. Most of the properties studied in the literature (see for example [5]) turned out to be negligible or generic, that is, with density equal to 0 or 1, respectively.

We would also like to mention the results of [6], where densities of sets of conjugacy classes in free and surface groups are investigated. More precisely, the density considered in [6] is the asymptotic density of sets of root-free conjugacy classes of hyperbolic elements in surface groups, and for free groups, the density is similar to the annular density, but records the proportion in two successive balls instead of two successive spheres.

A consequence of our results is the fact that the solvability of homogeneous equations in the class of groups we study is a non-negligible and non-generic property. Let $G$ be a finitely generated group, $A$ a fixed generating set, and $X = \{X_1, \ldots, X_n\}$, $n \geq 1$, a set of variables. An equation in variables $X_1, \ldots, X_n$ with coefficients $g_1, \ldots, g_{m+1}$ in $G$ is a formal expression given by

$$g_1X_{i_1}^{\varepsilon_1}g_2X_{i_2}^{\varepsilon_2} \ldots X_{i_m}^{\varepsilon_m}g_{m+1} = 1,$$

where $m \geq 1$, $\varepsilon_j \in \{1, -1\}$ for all $1 \leq j \leq m$, and $i_j \in \{1, \ldots, n\}$. An equation is homogeneous if the variables are on the left-hand side of the equation and the constants are on the right-hand side of the equation:

$$X_{i_1}^{\varepsilon_1}X_{i_2}^{\varepsilon_2} \ldots X_{i_m}^{\varepsilon_m} = w,$$

where $w \in G$. We say that the equation (1.1) is a homogeneous equation of type $(m, |w|_A)$ or an $(m, |w|_A)$-homogeneous equation, where $|w|_A$ denotes the length of $w$ with respect to $A$.

We will be interested in the asymptotic behavior of $(m, |w|_A)$-homogeneous equations when $G$ is a surface or a free group, and $m$ and $|w|_A$ go to infinity. Our study of the asymptotics of homogeneous equations was motivated by two related questions: firstly, how often does a homogeneous equation in a free or surface group have solutions, and secondly, how likely is it, for two random words $u$ and $v$ in the group to have that $v$ is an endomorphic image of $u$? The second question was partly inspired by the work of Kapovich, Schupp and Shpilrain ([5]). They show that the probability of two elements $u$ and $v$ in $F_k$ to be in the same automorphic orbit is 0 as the lengths of $u$ and $v$ go to infinity. The following paragraph clarifies the relation between the two questions.

Suppose that $z(X_1, \ldots, X_n)$ is the word in $X_1, \ldots, X_n$ representing the left-hand side of (1.1), i.e. $z(X_1, \ldots, X_n) = X_{i_1}^{\varepsilon_1}X_{i_2}^{\varepsilon_2} \ldots X_{i_m}^{\varepsilon_m}$. Let $F_n$ be the free group of rank $n$ on generators $x_1, \ldots, x_n$. Notice that the equation (1.1) has solutions if and only if there exists an endomorphic homomorphism $\phi: F_n \rightarrow G$ such that $\phi(z(x_1, \ldots, x_n)) = w$, where $z$ is written in the generators $x_1, \ldots, x_n$. The following ratios quantify the pairs of elements of the form $(z, w)$.

**Definitions 1.1.** Let $F$, $G$ be countable groups and $l_F: F \rightarrow \mathbb{N}$ and $l_G: G \rightarrow \mathbb{N}$ be length functions, as defined in Definition 2.1.

1. The $(s, t)$-mapping ratio $e_{\rho}(F, G, s, t)$ is the ratio of the pairs of elements $(f, g) \in F \times G$ such that $l_F(f) \leq s$, $l_G(g) \leq t$ and with the property that $g$ is a homomorphic image of $f$, among all pairs $(f, g) \in F \times G$ with $l_F(f) \leq s$, $l_G(g) \leq t$, that is,
1. Let $S \subseteq F$ and $n \geq 0$. Then
\[ \rho_{l_F}(n, S) = \sharp \{ x \in S : l_F(x) \leq n \}, \]
and
\[ \gamma_{l_F}(n, S) = \sharp \{ x \in S : l_F(x) = n \}. \]

2. The spherical $(s,t)$-mapping ratio $e_{\gamma}(F, G, s, t)$ is the ratio of the pairs of elements $(f,g) \in F \times G$ such that $l_F(f) = s$, $l_G(g) = t$ and with the property that $g$ is a homomorphic image of $f$ among all pairs $(f,g) \in F \times G$ with $l_F(f) = s$, $l_G(g) = t$, that is,
\[
e_{\gamma}(F, G, s, t) = \frac{\sharp \{(f,g) \in F \times G : l_F(f) = s, l_G(g) = t, \phi(f) = g \text{ for some } \phi \in \text{Hom}(F,G)\}}{\sharp \{(f,g) \in F \times G : l_F(f) \leq s, l_G(g) \leq t\}}
\]
denote the cardinality of the intersection of $S$ with the ball and sphere of radius $n$ in $F$, respectively.

2. Let $S \subseteq F$. The \textit{asymptotic density} of $S$ in $F$ is

$$\bar{\rho}_F(S) = \limsup_{n \to \infty} \frac{\rho_{l_F}(n, S)}{\rho_{l_F}(n, F)}.$$ 

If the limit exists, then we denote it by $\rho_F(S)$ and we call it the \textit{strict asymptotic density}.

3. Let $S \subseteq F$. The \textit{spherical density} of $S$ in $F$ is

$$\bar{\gamma}_F(S) = \limsup_{n \to \infty} \frac{\gamma_{l_F}(n, S)}{\gamma_{l_F}(n, F)}.$$ 

If the limit exists, then we denote it by $\gamma_F(S)$ and we call it the \textit{strict spherical density}.

4. Let $S \subseteq F$. The \textit{annular density} of $S$ in $F$ is

$$\bar{\sigma}_F(S) = \limsup_{n \to \infty} \frac{1}{2} \left( \frac{\sharp \{ x \in S : l_F(x) = n - 1 \}}{\sharp \{ x \in F : l_F(x) = n - 1 \}} + \frac{\sharp \{ x \in S : l_F(x) = n \}}{\sharp \{ x \in F : l_F(x) = n \}} \right).$$ 

If the limit exists, then we denote it by $\sigma_F(S)$ and we call it the \textit{strict annular density}.

When $F$ is a group, finitely generated by $A$, and $l_F = | \cdot |_A$, the word length, we will just write $\rho_A, \gamma_A$ and $\sigma_A$. Similarly if $F = \mathbb{Z}^r$ and $l_F = l_p$, the restriction of the $p$-norm, we will just write $\rho_p, \gamma_p$ and $\sigma_p$.

\textbf{Definitions 2.3.} For a nonzero element $z \in \mathbb{Z}^r$ we denote by gcd($z$) the greatest common divisor of its coordinates. If $z = (0, \ldots, 0) \in \mathbb{Z}^r$ we set gcd($z$) = $\infty$. Note that gcd is invariant under the action of Aut($\mathbb{Z}^r$) = SL$(r, \mathbb{Z})$. Hence, for all $z \in \mathbb{Z}^r$, gcd($z$) does not depend on the basis of $\mathbb{Z}^r$.

An element of $z \in \mathbb{Z}^r$ is called \textit{visible} if gcd($z$) = 1. If gcd($z$) = $t$, then we call the element \textit{t-visible}.

We denote by $F_{ab}$ the abelianization of the group $F$, that is, $F_{ab} = F/[F,F]$. Suppose that $F_{ab}$ is a free-abelian group of finite rank and let ab : $F \to F_{ab}$ be the abelianization map. We say that an element $f \in F$ is \textit{visible} (resp. \textit{t-visible}) if ab($f$) is visible (resp. \textit{t-visible}) in $F_{ab}$.

\section{Densities of visible elements in $\mathbb{Z}^r$}

Let $r \geq 2$ be an integer and let $U_t$ denote the set of all \textit{t-visible} elements in $\mathbb{Z}^r$. For a complex number $k$, recall that the Riemann zeta function is given by

$$\zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k}, \quad \Re(k) > 1.$$ 

A classical result in number theory provides the value for the strict asymptotic density of \textit{t-visible} elements in $\mathbb{Z}^r$. 

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Proposition 3.1 ([2]). For any integer $t \geq 1$

$$\rho_\infty(U_t) = \frac{1}{\nu \zeta(r)}.$$  

\[ \square \]

By [4, Theorem A (1)] or Remark 4.7, one can substitute $\rho_\infty$ by $\rho_p$ for the sets $U_t$:

Proposition 3.2. [4, Theorem A (1)] For any integer $t \geq 1$ and any $p$, $1 \leq p \leq \infty$, $\rho_p(U_t) = \rho_\infty(U_t)$.  

The following lemma shows that homomorphisms between groups with free-abelian abelianization (of finite rank) send $t$-visible elements to $tm$-visible elements, where $t, m$ are positive integers. The second part of the lemma shows that a visible element in a group can be mapped to any element in the image via a homomorphism.

Lemma 3.3. Let $F, G$ be groups whose abelianization is free-abelian of finite rank. Let $f \in F$.

(i). Let $g = \phi(f)$. Then $(ab(g))_j = \sum_{i=1}^{n} (ab(f))_i (\phi(e_i))_j$.

Thus each $(ab(g))_j$ is a multiple of $gcd(ab(f))$, since each $(ab(f))_i$ is a multiple of $gcd(ab(f))$.

(ii). Since $gcd(ab(f)) = 1$, then $gcd((ab(f))_1, \ldots, (ab(f))_n) = 1$ and therefore there exist integers $p_1, \ldots, p_n$ such that $\sum_{i=1}^{n} (ab(f))_i p_i = 1$. Consider the homomorphism $\psi_1: F_{\text{ab}} \to \langle x \mid \rangle$ which sends $e_i$ to $x^{p_i}$ for all $1 \leq i \leq n$. It follows that $\psi_1(ab(f)) = x$.

Let $\psi_2: \langle x \mid \rangle \to G$ be any homomorphism sending $x$ to $g$. This shows that the composition of $ab$, $\psi_1$ and $\psi_2$ produces a homomorphism $\phi: F \to G$ such that $\phi(f) = g$.

\[ \square \]

Corollary 3.4. Let $\mathbb{Z}^n$ and $\mathbb{Z}^k$ be the free-abelian groups of ranks $n$ and $k$, respectively. Then the following inequalities hold with respect to $l_p$ for $1 \leq p \leq \infty$:

$$\frac{1}{\zeta(n)} \leq \liminf_{s \to \infty, t \to \infty} e_p(\mathbb{Z}^n, \mathbb{Z}^k, s, t),$$

$$\limsup_{s \to \infty, t \to \infty} e_p(\mathbb{Z}^n, \mathbb{Z}^k, s, t) \leq 1 - \frac{1}{\zeta(k)} \left( 1 - \frac{1}{\zeta(n)} \right).$$
Proof. We fix some $p$, $1 \leq p \leq k$. Let $e_{ab}(s, t) := e_p(Z^n, Z^k, s, t)$ with respect the length $l_p$
and let $|u| = l_p(u)$.

By Lemma 3.3(ii)

$$e_{ab}(s, t) \geq \frac{(u, v) \in Z^n \times Z^k : |u| \leq s, |v| \leq t, \gcd(u) = 1}{\rho_p(s, Z^n)\rho_p(t, Z^k)} = \frac{\{u \in Z^n : |u| \leq s, \gcd(u) = 1\}}{\rho_p(s, Z^n)}.$$ 

Taking limits, we obtain (3.2) by Propositions 3.1 and 3.2.

By Lemma 3.3(i)

$$e_{ab}(s, t) \leq 1 - \frac{(u, v) \in Z^n \times Z^k : |u| \leq s, |v| \leq t, \gcd(u) \neq 1, \gcd(v) = 1}{\rho_p(s, Z^n)\rho_p(t, Z^k)} = 1 - \left(1 - \frac{\{u \in Z^n : |u| \leq s, \gcd(u) = 1\}}{\rho_p(s, Z^n)}\right) \frac{\{v \in Z^k : |v| \leq t, \gcd(v) = 1\}}{\rho_p(t, Z^k)}.$$ 

Taking limits, we obtain (3.2) by Propositions 3.1 and 3.2. \hfill \square

One of the key ingredients needed to extend the previous result to the analogue for surface groups is determining the asymptotic density of elements of even length in $Z^k$. This was done in [4, Proposition 3.6] for $k = 2$, and we now compute the value for a general $k$.

**Proposition 3.5.** Let $k \geq 2$, and let $U_{1}^{ev} = \{ z \in U_1 : l_1(z) \text{ is even} \}$ denote the set of visible elements of even length in $Z^k$. Then

$$\rho_\infty(U_{1}^{ev}) = \frac{2^{k-1} - 1}{2^k - 1} \rho_\infty(U_1) = \frac{2^{k-1} - 1}{(2^k - 1)(k)}.$$ 

Proof. Let $n$ be a positive integer and let $[0, n] = \{0, 1, \ldots, n\}$. For $X_1, \ldots, X_k \in \{A, O, E\}$ we denote by $X_1X_2 \ldots X_k(n)$ the number of all $z = (z_1, \ldots, z_k) \in U_1$ such that $z_i \in [0, n]$ and the parity of $z_i$ is $X_i$. Here $A$ stands for “any”, $E$ stands for “even” and $O$ stands for “odd”.

We will use the convention $X \underbrace{X \ldots X}_k = X^k$, for any $X \in \{A, O, E\}$ and $k \geq 1$.

Note that $X_1X_2 \ldots X_k(n) = X_{s(1)}X_{s(2)} \ldots X_{s(k)}(n)$, for any permutation $s$ of $\{1, \ldots, k\}$, and that $E^k(n) = 0$ for any $k, n \geq 1$.

The total number of elements in $U_1$ in $[0, n]^k$ is

$$A^k(n) = \sum_{i=1}^k \binom{k}{i} E^{k-i}O^i(n). \quad (3.3)$$

Let $U_{1}^{ev}(n)$ be the set $U_{1}^{ev} \cap [0, n]^k$. Then

$$|U_{1}^{ev}(n)| = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2i} E^{k-2i}O^{2i}(n). \quad (3.4)$$

We claim that:
\[ \mathcal{E}^{k-i} \mathcal{O}^i(n) = \mathcal{O}^k(n) + o(n^k) \text{ for all } 1 \leq i \leq k. \quad (3.5) \]

Assume first that (3.5) holds. From (3.4) and (3.5) we get

\[ |U_{1e}^e (n)| = \sum_{i=1}^{\left[ \frac{k}{2} \right]} \binom{k}{2i} \mathcal{O}^k(n) + o(n^k), \]

and since \( \sum_{i=1}^{\left[ \frac{k}{2} \right]} \binom{k}{2i} = 2^{k-1} - 1 \), we get that

\[ |U_{1e}^e (n)| = (2^{k-1} - 1) \mathcal{O}^k(n) + o(n^k). \]

Since \( \sum_{i=1}^{k} \binom{k}{i} = 2^k - 1 \), from (3.3) and (3.5) we get

\[ \mathcal{O}^k(n)(2^k - 1) = A_k(n) + o(n^k), \]

and hence

\[ |U_{1e}^e (n)| = \frac{2^{k-1} - 1}{2^k - 1} A_k(n) + o(n^k). \]

Since \( \rho_\infty(U_1) = \lim_{n \to \infty} \frac{A_k(n)}{n^k} = \frac{1}{\zeta(k)} \), we get that

\[ \rho_\infty(U_{1e}^e) = \limsup_{n \to \infty} \frac{|U_{1e}^e (n)|}{n^k} \]

\[ = \lim_{n \to \infty} \frac{\frac{2^{k-1} - 1}{2^k - 1} A_k(n) + o(n^k)}{n^k} \]

\[ = \frac{2^{k-1} - 1}{(2^k - 1) \zeta(k)} \rho_\infty(U_1) \]

This completes the proof of the proposition. We now show (3.5). Notice first that

\[ \mathcal{O}^i \mathcal{E}^{k-i-1} A(n) = \mathcal{O}^i \mathcal{E}^{k-i}(n) + \mathcal{O}^{i+1} \mathcal{E}^{k-i-1}(n). \]

Hence it is enough to show

\[ \mathcal{O}^i \mathcal{E}^{k-i-1} A(n) = 2 \mathcal{O}^i \mathcal{E}^{k-i}(n) + o(n^k) \text{ for all } 1 \leq i \leq k. \quad (3.6) \]

Let \( \mu : \mathbb{N} \to \{-1, 0, 1\} \) denote the Möbius function and recall that \( \sum_{d|n} \mu(d) \) is equal to 1, if \( n = 1 \) and 0 otherwise. Hence

\[ \mathcal{O}^i \mathcal{E}^{k-i-1} A(n) = \sum_{0 \leq x_1 \leq n, 2|x_j} \sum_{0 \leq x_j \leq n, 2|x_j} \sum_{0 \leq x_k \leq n} \sum_{d|\gcd(x_1, \ldots, x_k)} \mu(d) \]

and
\[ O^iE^{k-i}(n) = \sum_{0 \leq y_j \leq n/d, 0 \leq x_j \leq n, 2 \mid y_j \mid x_j \leq n, 2 \nmid x_j} \sum_{d \mid \gcd(x_1, \ldots, x_k)} \mu(d). \]

Now we switch the order in the summation. We rearrange the terms depending on \( d \mid \gcd(x_1, \ldots, x_k) \), writing \( x_i = y_i d \). Since there is an odd coordinate, \( 2 \nmid d \). We obtain that

\[ O^iE^{k-i-1}A(n) = \sum_{2 \mid d} \mu(d) \sum_{0 \leq y_j \leq n/d} \sum_{j=1, \ldots, i} \sum_{j=i+1, \ldots, k-1} 1 \]

and

\[ O^iE^{k-i}(n) = \sum_{2 \mid d} \mu(d) \sum_{0 \leq y_j \leq n/d, 2 \nmid y_j \mid x_j \leq n/d} \sum_{j=1, \ldots, i} \sum_{j=i+1, \ldots, k} 1. \]

Hence \( O^iE^{k-i-1}A(n) - 2O^iE^{k-i}(n) \) is equal to

\[ \sum_{2 \mid d} \mu(d) \sum_{0 \leq y_j \leq n/d, 2 \nmid y_j \mid x_j \leq n/d} \sum_{j=1, \ldots, i} \sum_{j=i+1, \ldots, k-1} \left( \left\lceil \frac{n}{d} \right\rceil - 2 \left\lfloor \frac{n}{2d} \right\rfloor \right). \quad (3.7) \]

The term in parenthesis is either 0 or 1, and it is always 0 for \( d > n \). Thus the asymptotic behavior of (3.7) is of type

\[ O(\sum_{d \leq n} \sum_{0 \leq y_j \leq n/d} 1) \leq O(\sum_{d=1}^{n} (n/d)^{k-1}) \]

\[ = O(n^{k-1}) \left( \frac{1}{k-2} - \frac{1}{(k-2)n^{k-2}} \right) \]

\[ = O(n^{k-1}) \subset o(n^k) \]

4 Densities of visible elements in surface groups

The main result of this section is an extension of [4, Theorem A] that allows us to compute densities of visible elements in free and surface groups. We need to fix some notation.

Notation 4.1. For \( k \geq 2 \), we denote by \( F_k \) the free group of rank \( k \) and by \( S_k \) the surface group of genus \( k \).

We will work with the standard presentation for \( F_k \),

\[ \langle a_1, \ldots, a_k \mid \rangle, \]

and let \( A = \{ a_1, \ldots, a_k \}^{\pm 1} \).
A presentation for $S_k$ has the form

$$\langle a_1, b_1, \ldots, a_k, b_k \mid [a_1, b_1] \cdots [a_k, b_k] \rangle.$$  

In this case we let $A = \{a_1, b_1, \ldots, a_k, b_k\}^{\pm 1}$.

Let $r$ denote the rank of the abelianization, that is $r = k$ for $F_k$, and $r = 2k$ for $S_k$.

Our main result is based on the following local limit theorem of Sharp in [8].

**Theorem 4.2.** (see Theorems 1, 3, 4 in [8]) Let $F$ be $F_k$ or $S_k$, and $A$ and $r$ be the corresponding generating set and rank of the abelianization of $F$, as in notation 4.1.

Let $ab : F \to \mathbb{Z}^r$ be the abelianization map. Then there exists a symmetric positive definite real matrix $D$ such that

$$\lim_{n \to \infty} \left| (\det D)^{1/2} n^{r/2} \left( \frac{\gamma_A(n, ab^{-1}(\alpha))}{\gamma_A(n, F)} + \frac{\gamma_A(n + 1, ab^{-1}(\alpha))}{\gamma_A(n + 1, F)} \right) - \frac{2}{(2\pi)^{r/2}} e^{-\langle \alpha, D^{-1}\alpha \rangle / 2n} \right| = 0,$$

uniformly in $\alpha \in \mathbb{Z}^r$.

*Proof.* For $F = S_k$ this is exactly [8, Theorem 4] with $g = r/2$. For $F = F_k$ and $D$ the diagonal matrix with all entries equal to $\sigma^2$, one obtains exactly [8, Theorem 1].

Since the proof of the main theorem of this section does not use the fact that $F$ is a free or surface group, but only the conclusions of Theorem 4.2, we will fix the following Hypothesis.

**Hypothesis 4.3.** Let $F$ be a group generated by a finite set $A$ such that $F_{ab} \cong \mathbb{Z}^r$ and $D$ be a symmetric positive definite real matrix such that the limit (4.1) goes to zero uniformly in $\alpha \in \mathbb{Z}^r$.

By Theorem 4.2, the free group $F_k$ and the surface group $S_k$ of Notation 4.1 satisfy the Hypothesis 4.3.

**Definition 4.4.** Let $G_r$ be the set of all $M \in \text{SL}(r, \mathbb{Z})$ such that $M = I_r$ in $\text{SL}(r, \mathbb{Z}/2\mathbb{Z})$. Then $G_r$ is a finite-index subgroup of $\text{SL}(r, \mathbb{Z})$.

**Definition 4.5.** We say that a bounded open subset of $\mathbb{R}^r$ is *nice* if its boundary is piecewise smooth.

**Proposition 4.6.** [4, Proposition 3.3.] Let $S \subseteq \mathbb{Z}^r$ be a $G_r$-invariant subset such that $\delta = \rho_\infty(S)$ exists. Let $\Omega \subseteq \mathbb{R}^r$ be a nice bounded open set and for $t \in \mathbb{R}$, $t > 0$, let

$$\mu_{t, S}(\Omega) := \frac{\#(S \cap t\Omega)}{t^r}.$$  

Then we have

$$\lim_{t \to \infty} \mu_{t, S}(\Omega) = \delta \lambda(\Omega),$$

where $\lambda$ is the Lebesgue measure.
Although [4] indicates that the proof is similar to that of [4, Proposition 2.3], we include here a proof for Proposition 4.6 for the sake of completeness.

**Proof.** Each $\mu_{t,S}$ can be regarded as a measure on $\mathbb{R}^r$. We prove the result by showing that $\mu_{t,S}$ weakly converge to $\delta \lambda$ as $t \to \infty$.

By Helly’s theorem (see, for instance, [1, Thm 25.9]), there exists a sequence $\{t_i\}$ with $\lim_{i \to \infty} t_i = \infty$ such that the sequence $\mu_{t_1,S}$, $\mu_{t_2,S}$, ... is weakly convergent to some limiting measure. We now identify this measure by showing that for every convergent subsequence of $\mu_{t,S}$ the limiting measure is equal to $\delta \lambda$.

Indeed, we assume that $\eta = \{t_i\}$ is a sequence with $\lim_{i \to \infty} t_i = \infty$ such that the sequence $\mu_{t_i,S}$ converges to the limiting measure $\mu_\eta = \lim_{i \to \infty} \mu_{t_i,S}$. Every $\mu_{t,S}$ is invariant with respect to the $G_r$-action on $\mathbb{R}^r$. Therefore the limiting measure $\mu_\eta$ is also $G_r$-invariant. Moreover, the measures $\mu_{t,S}$ are dominated by the measures $\lambda_t$ defined as $\lambda_t(\Omega) = \frac{\sharp(\mathbb{Z}^r \cap \Omega)}{t^r}$.

It is well known that if $\Omega \subseteq \mathbb{R}^r$ is a nice bounded open set, then the measures $\lambda_t$ converge to the Lebesgue measure $\lambda$. It follows that $\mu_\eta$ is absolutely continuous with respect to $\lambda$. It is also known that the natural action of $G_r$ on $\mathbb{R}^r$ is ergodic with respect to $\lambda$ (see [9] for the proof of ergodicity). Therefore $\mu_\eta$ is a constant multiple $c\lambda$ of $\lambda$. The constant $c$ can be computed for a set such as the open unit ball $B$ in the $|| \cdot ||_\infty$ norm on $\mathbb{R}^r$ defining the length function $l_\infty$ on $\mathbb{Z}^r$. By assumption we know that

$$\rho_\infty(S) = \lim_{t \to \infty} \frac{\sharp\{z \in \mathbb{Z}^r : z \in S \cap tB\}}{t^r} = \delta.$$ 

We also have

$$\lim_{t \to \infty} \frac{\sharp\{z \in \mathbb{Z}^r : z \in tB\}}{t^r} = \lambda(B)$$

and hence

$$\lim_{t \to \infty} \frac{\sharp\{z \in \mathbb{Z}^r : z \in tB\}}{t^r} = \frac{\sharp\{z \in \mathbb{Z}^r : z \in S \cap tB\}}{t^r} = \frac{\sharp\{z \in \mathbb{Z}^r : z \in S \cap tB\}}{t^r} = \delta \lambda(B).$$

Therefore $c = \delta$ and $\mu_\eta = \delta \lambda$. The above argument shows in fact that every convergent subsequence of $\mu_{t,S}$ converges to $\delta \lambda$ and $\lim_{t \to \infty} \mu_{t,S} = \delta \lambda$. \qed

**Remark 4.7.** (see [4, Theorem A]) Let $1 \leq p \leq \infty$. The sets $U_q$ of $q$-visible elements in $\mathbb{Z}^r$ are $G_r$-invariant and

$$\rho_p(U_q) = \rho_\infty(U_q).$$

**Proof.** Let $\Omega$ be an $l_p$ ball of radius 1. It is well known that

$$\lambda(\Omega) = \lim_{t \to \infty} \frac{\sharp(\mathbb{Z}^r \cap t\Omega)}{t^r}.$$
Then
\[
\rho_p(U_q) = \lim_{n \to \infty} \frac{\# \{ x \in U_q : l_p(x) \leq n \}}{\# \{ x \in \mathbb{Z}^r : l_p(x) \leq n \}}
\]
\[
= \lim_{n \to \infty} \frac{\# \{ x \in U_q : l_p(x) \leq n \}}{\# \{ x \in \mathbb{Z}^r : l_p(x) \leq n \}} \cdot \lim_{t \to \infty} \frac{\# (Z_r \cap t \Omega)}{\lambda(\Omega) t^r}
\]
\[
= \lim_{t \to \infty} \frac{\# (U_q \cap t \Omega)}{\lambda(\Omega) t^r}
\]
\[
= \frac{\delta \lambda(\Omega)}{\lambda(\Omega) t^r}
\]
\[
= \rho_\infty(U_q).
\]

\(\square\)

**Definition 4.8.** Let \(F\) be a group generated by the finite set \(A\) such that \(F_{ab} \cong \mathbb{Z}^r\). 

For an integer \(n \geq 1\) and a point \(x \in \mathbb{R}^r\), let \(p_n\) be given by
\[
p_n(x) = \frac{1}{2} \left( \frac{\gamma_A(n-1, \{ g \in F : ab(g) = x \sqrt{n} \})}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, \{ g \in F : ab(g) = x \sqrt{n} \})}{\gamma_A(n, F)} \right). \tag{4.3}
\]

This is a distribution supported on finitely many points of \(\frac{1}{\sqrt{n}} \mathbb{Z}^r\).

We need the following results from \([7, 8]\) about the sequence of distributions \(p_n\).

In our context, we need to restate our Hypothesis 4.3

**Proposition 4.9.** ([7, 8, 4]) Let \(F, A, r\) satisfy Hypothesis 4.3. Then there exists a normal distribution \(N\) with density \(n\) such that:

(a) The sequence of distributions \(p_n\) converges weakly to \(n\) and we have
\[
\sup_{x \in \mathbb{R}^r} |n^{r/2} p_n(x) - n(x)| \to 0, \text{ as } n \to \infty. \tag{4.4}
\]

(b) For \(c > 0\), let \(\Omega_c := \{ x \in \mathbb{R}^r : \|x\| \geq c \}\). Then
\[
\lim_{c \to \infty} \lim_{n \to \infty} \sum_{x \in \Omega_c} p_n(x) = 0. \tag{4.5}
\]

**Proof.** Let \(D\) be the matrix of Hypothesis 4.3, and let \(n(x) = e^{-(x, D^{-1} x)/2} / (2\pi)^{r/2} |\det D|^{1/2}\), the density of a normal distribution \(N\). Firstly, we prove the limit in (4.4).
After performing some easy computations,
\[
\lim_{n \to \infty} \frac{1}{2(\det D)^{1/2}} \left| (\det D)^{1/2} n^{r/2} \right|
\]
\[
\times \left( \frac{\gamma_A(n-1, ab^{-1}(x \sqrt{n}))}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, ab^{-1}(x \sqrt{n}))}{\gamma_A(n, F)} \right) - \frac{2}{(2\pi)^r} e^{-\left(\frac{\alpha \cdot \rho^{-1} \alpha}{\gamma_d} \right)^2/2}
\]
\[
= \frac{1}{2(\det D)^{1/2}} \left| (\det D)^{1/2} n^{r/2} \right|
\]
\[
\times \left( \frac{\gamma_A(n-1, ab^{-1}(\alpha))}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, ab^{-1}(\alpha))}{\gamma_A(n, F)} \right) - \frac{2}{(2\pi)^r} e^{-\left(\alpha, D^{-1} \alpha \right)/2}
\]
\[
= \frac{1}{2(\det D)^{1/2}} \left| (\det D)^{1/2} n^{r/2} \right|
\]
\[
\times \left( \frac{\gamma_A(n-1, ab^{-1}(\alpha))}{\gamma_A(n-1, F)} + \frac{\gamma_A(n, ab^{-1}(\alpha))}{\gamma_A(n, F)} \right) - \frac{2}{(2\pi)^r} e^{-\left(\alpha, D^{-1} \alpha \right)/2},
\]
using the limit (4.1) of the Hypothesis 4.3, and the fact that this limit is uniform in \( \alpha = x \sqrt{n} \), we obtain the desired result.

In order to show that the sequence of probability distributions \( \{p_n\} \) converges weakly to \( n \), we use [1, Thm 25.8], that is, it is necessary and sufficient that for every bounded continuous function \( f(x) \) on \( \mathbb{R}^r \)
\[
\lim_{n \to \infty} \int_{\mathbb{R}^r} f(x)p_n(x)d\lambda(x) = \int_{\mathbb{R}^r} f(x)n(x)d\lambda(x).
\] (4.6)

We can write:
\[
\left| \int_{\mathbb{R}^r} f(x)p_n(x)d\lambda(x) - \int_{\mathbb{R}^r} f(x)n(x)d\lambda(x) \right| = \left| \int_{\mathbb{R}^r} f(x)(p_n(x) - n(x))d\lambda(x) \right|
\]
\[
\leq \int_{\mathbb{R}^r} |f(x)||n^{r/2}p_n(x) - n(x)|d\lambda(x).
\]

Given that \( f \) is a bounded continuous function and by the limit (4.4) proved above, the hypothesis of the Dominated Convergence Theorem is satisfied. Applying this last result, we obtain that
\[
\left| \int_{\mathbb{R}^r} f(x)p_n(x)d\lambda(x) - \int_{\mathbb{R}^r} f(x)n(x)d\lambda(x) \right| \xrightarrow{n \to \infty} 0,
\]
and the weak convergence of the sequence \( \{p_n\} \) is proved.

We now prove (b). For \( c > 0 \), let \( \Omega_c = \{ x \in \mathbb{R}^r : \|x\| < c \} \), and denote by \( \Omega_c \) the complement of \( \Omega_c \). Then, by the weak convergence of the \( p_n \) to \( n \), we have that
\[
\lim_{c \to \infty} \left( \lim_{n \to \infty} \sum_{x \in \Omega_c} p_n(x) \right) = \lim_{c \to \infty} \left( 1 - \lim_{n \to \infty} \sum_{x \in \Omega_c} p_n(x) \right)
\]
\[
= 1 - \lim_{c \to \infty} \int_{x \in \Omega_c} n(x)d\lambda(x) = 0.
\]
\[\square\]
Theorem 4.10. Let $\Omega \subseteq \mathbb{R}^r$ be a nice bounded open set. Let $S \subseteq \mathbb{Z}^r$ be a $G_r$-invariant subset such that $\delta = \rho_\infty(S)$ exists. Then there exists a normal distribution $\mathfrak{n}$ such that

$$
\lim_{n \to \infty} \sum_{x \in S \cap \sqrt{n}\Omega} p_n(x/\sqrt{n}) = \delta \mathfrak{n}(\Omega).
$$

Proof. Note that the proof is the same as that of Theorem 3.4 in \cite{4}. The only difference lies in the use of Proposition 4.6.

There exists a normal $\mathfrak{n}$ distribution with density $\nu$ satisfying the conclusions of Proposition 4.9.

We have

$$
\sum_{x \in S \cap \sqrt{n}\Omega} p_n(x/\sqrt{n}) = \sum_{y \in \frac{1}{\sqrt{n}}S \cap \Omega} p_n(y) = n^{-r/2} \sum_{y \in \frac{1}{\sqrt{n}}S \cap \Omega} n(y) + n^{-r/2} \sum_{y \in \frac{1}{\sqrt{n}}S \cap \Omega} (n^{r/2}p_n(y) - n(y)).
$$

The local limit theorem of Proposition 4.9(a) tells us that, as $n \to \infty$, each summand $n^{-r/2}p_n(y) - n(y)$ of the sum in the last line above converges to zero, and hence so does their Cesaro mean.

Using the following convergence of the measures defined in Proposition 4.6,

$$
\lim_{n \to \infty} \mu_{\sqrt{n},S}(\Omega) = \delta \lambda(\Omega),
$$

(recall that $\mu_{\sqrt{n},S}(\Omega) := \frac{\sharp(S \cap \sqrt{n}\Omega)}{\sqrt{n^r}}$), we have that

$$
\lim_{n \to \infty} \sum_{x \in S \cap \sqrt{n}\Omega} \frac{1}{(\sqrt{n})^r}n(x/\sqrt{n}) = \int_{\Omega} n(y)\delta d\lambda(y) = \delta \mathfrak{n}(\Omega).
$$

We obtain the main result of this section by basically following \cite[Theorem A]{4}. Our theorem provides the formula for the ‘spherical densities’ of visible elements in groups that satisfy Hypothesis 4.3, which include free groups of all finite ranks and surface groups.

Theorem 4.11. (see also \cite[Theorem A]{4})

Let $F, A, r$ satisfy Hypothesis 4.3, $S \subseteq \mathbb{Z}^r$ be a $G_r$-invariant subset and $\tilde{S} = ab^{-1}(S)$.

(i). The strict annular density $\sigma_A(\tilde{S})$ exists and, moreover, $\sigma_A(\tilde{S}) = \rho_\infty(S)$. 

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(ii). Let $U_1$ denote the set of visible elements in $\mathbb{Z}^r$ and $V_1 = ab^{-1}(U_1)$ denote the visible elements in $F$. Let $U_1^{ev} = \{ z \in U_1 : l_1(z) \text{ is even} \}$ denote the visible elements of even length. If $ab^{-1}(U_1^{ev}) = \{ v \in V_1 : |v|_A \text{ is even} \}$, then

$$
\lim_{m \to \infty} \gamma_A(2m, V_1) = 2\rho_\infty(U_1^{ev}) = \frac{2^r - 2}{(2^r - 1)\zeta(r)},
\lim_{m \to \infty} \gamma_A(2m - 1, V_1) = 2\rho_\infty(U_1) - 2\rho_\infty(U_1^{ev}) = \frac{(2^r - 1)\zeta(r)}{2^r - 1}.
$$

Proof. For $c > 0$ let $\Omega_c := \{ x \in \mathbb{R}^r : \|x\| < c \}$ and let $\overline{\Omega_c}$ be the complement of $\Omega_c$. Then

$$(4.7) \quad \lim_{c \to \infty} \mathfrak{M}(\Omega_c) = 1$$

Let $\epsilon > 0$ be arbitrary. By (4.7) and Proposition 4.9 (b) we can choose $c > 0$ such that

$$\mathfrak{M}(\Omega_c) - 1 \leq \epsilon/3$$

and

$$\lim_{n \to \infty} \sum_{x \in \Omega_c} p_n(x) \leq \epsilon/6.$$

Let $S$ be a $G_r$-invariant subset of $\mathbb{Z}^r$. By Theorem 4.10 and the above formula there is some $n_0 \geq 1$ such that for all $n \geq n_0$ we have

$$\left| \sum_{x \in S \cap \sqrt{n}\overline{\Omega_c}} p_n(x/\sqrt{n}) - \rho_\infty(S)\mathfrak{M}(\Omega_c) \right| \leq \epsilon/3,$$  

and

$$\sum_{x \in \Omega_c} p_n(x) \leq \epsilon/3.$$  

Let

$$Q(n) := \frac{\gamma_A(n - 1, ab^{-1}(S))}{2\gamma_A(n - 1, F)} + \frac{\gamma_A(n, ab^{-1}(S))}{2\gamma_A(n, F)}.$$

For $n \geq n_0$ we let

$$Q(n) = \left( \frac{\#\{g \in F : ab(g) \in S, |g|_A = n - 1 \text{ and } \|ab(g)\| < c\sqrt{n}\}}{2\gamma_A(n - 1, F)} + \frac{\#\{g \in F : ab(g) \in S, |g|_A = n \text{ and } \|ab(g)\| < c\sqrt{n}\}}{2\gamma_A(n - 1, F)} + \frac{\#\{g \in F : ab(g) \in S, |g|_A = n - 1 \text{ and } \|ab(g)\| \geq c\sqrt{n}\}}{2\gamma_A(n - 1, F)} + \frac{\#\{g \in F : ab(g) \in S, |g|_A = n \text{ and } \|ab(g)\| \geq c\sqrt{n}\}}{2\gamma_A(n - 1, F)} \right) \leq \sum_{x \in S \cap \sqrt{n}\overline{\Omega_c}} p_n(x/\sqrt{n}) + \sum_{x \in S \cap (\mathbb{R}^r \setminus \sqrt{n}\overline{\Omega_c})} p_n(x/\sqrt{n}).$$
In the last line of the above equation, by (4.8), the first sum differs from $\rho_\infty(S)\Omega(\Delta_c)$ by at most $\epsilon/3$ since $n \geq n_0$, and by (4.9), the second sum is $\leq \epsilon/3$ given the choice of $c$ and $n_0$.

Therefore, again by the choice of $c$, we have $|Q(n) - \rho_\infty(S)| \leq \epsilon$. Since $\epsilon$ is arbitrary, this proves (i).

We now prove (ii). First notice that since $U_1$ is $SL(r,\mathbb{Z})$-invariant, it is also $G_r$-invariant. We check that $U_1^{ev}$ is $G_r$-invariant as well. Let $u \in \mathbb{Z}$. Then $u \in U_1^{ev}$ if and only if $\sum_{1 \leq i \leq r} (u)_i \mod 2 = 0$ and $\gcd(u) = 1$. Let $M \in G_r$. As $M \in SL(r,\mathbb{Z})$, $\gcd(Mu) = \gcd(u) = 1$. Also, as $M = I_r$ in $SL(r,\mathbb{Z}/2\mathbb{Z})$,

$$\sum_{1 \leq i \leq r} (Mu)_i \mod 2 = \sum_{1 \leq i \leq r} (u)_i \mod 2 = 0.$$ 

Hence, $U_1^{ev}$ is $G_r$-invariant.

We now take $S = U_1^{ev}$, for $n \geq 2$ even. Then

$$Q(n) = \frac{\gamma_A(n-1,ab^{-1}(U_1^{ev}))}{2\gamma_A(n-1,F)} + \frac{\gamma_A(n,ab^{-1}(U_1^{ev}))}{2\gamma_A(n,F)} = \frac{\gamma_A(n,ab^{-1}(U_1^{ev}))}{2\gamma_A(n,F)}.$$ 

The latter equality follows from the fact that $ab^{-1}(U_1^{ev}) = \{v \in V_1 : |v|_A \text{ is even}\}$.

By (i),

$$\lim_{m \to \infty} \frac{\gamma_A(2m,V_1)}{\gamma_A(2m,F)} = 2 \lim_{m \to \infty} Q(2m) = 2\rho_\infty(U_1^{ev}).$$ 

Thus $\lim_{m \to \infty} \frac{\gamma_A(2m-1,V_1)}{\gamma_A(2m-1,F)} = 2\rho_\infty(U_1) - 2\rho_\infty(U_1^{ev})$. By Proposition 3.1 and Proposition 3.5, we obtain the desired results.

We now focus on surface and free groups.

**Corollary 4.12.** Let $k \geq 2$ and let $F$ be a free group of rank $k$ or a surface group of genus $k$. Let $A$ and $r$ be as in Notation 4.1. Then

(i). $\lim_{m \to \infty} \frac{\gamma_A(2m,V_1)}{\gamma_A(2m,F)} = \frac{2^r - 2}{(2^r - 1)\zeta(r)}.$

(ii). $\lim_{m \to \infty} \frac{\gamma_A(2m-1,V_1)}{\gamma_A(2m-1,F)} = \frac{2^r}{(2^r - 1)\zeta(r)}.$

**Proof.** By Theorem 4.3, $F, A$ and $r$ satisfy the Hypothesis of Theorem 4.11. It only remains to show that $ab^{-1}(U_1^{ev}) = \{v \in V_1 : |v|_A \text{ is even}\}$. Let $f$ be an element of $F$ such that $ab(f) = 0 \in \mathbb{Z}^r$. Then any word representing $w$ has the same number of $a$ and $a^{-1}$ and thus it has even length.

Since $ab$ maps elements of $A$ to unit vectors, for $u \in U_1^{ev}$ there exists $v \in ab^{-1}(U_1^{ev})$ of even length. If $ab(v) = \bar{ab}(v')$, then $ab(v'v^{-1}) = 0$. Hence $v'v^{-1}$ has even length, and so does $v'$. Thus Theorem 4.11(ii) applies. 

$\square$
5 Asymptotic behavior of homogeneous equations in surface groups

We now study the asymptotic behavior of \( e_\epsilon(G_n, G_k, s, t) \) when \( G_n \) and \( G_k \) are surface or free groups, or more generally, satisfy the hypothesis of Theorem 4.11 (ii).

**Theorem 5.1.** Let \( G_k \) and \( G_n \) be free or surface groups and let \( A, B \) be their respective generating sets, as in Notation 4.1. Let \( r(k) \) and \( r(n) \) denote the ranks of the abelianization of \( G_k \) and \( G_n \), respectively. Let \( \epsilon, \delta \in \{0, 1\} \). Then the following inequalities hold:

\[
\frac{2^{r(n)} - 2(1 - \epsilon)}{(2^{r(n)} - 1)\zeta(r(n))} \leq \liminf_{s \to \infty, t \to \infty} e_\epsilon(G_n, G_k, 2s + \epsilon, 2t + \delta),
\]

\[
\limsup_{s \to \infty, t \to \infty} e_\epsilon(G_n, G_k, 2s + \epsilon, 2t + \delta) \leq 1 - \frac{2^{r(k)} - 2(1 - \delta)}{(2^{r(k)} - 1)\zeta(r(k))} \left( 1 - \frac{2^{r(n)} - 2(1 - \epsilon)}{(2^{r(n)} - 1)\zeta(r(n))} \right).
\]

**Proof.** Let \( V_t \) and \( W_t \) denote the sets of \( t \)-visible elements in \( G_n \) and \( G_k \), respectively. Let

\[ E(s, t) = \{(u, v) \in G_n \times G_k : |u|_A = s, |v|_B = t, \phi(u) = v \text{ for some } \phi \in \text{Hom}(G_n, G_k)\}. \]

Then \( e_\epsilon(G_n, G_k, s, t) = \frac{|E(s, t)|}{\gamma_B(s, G_n)\gamma_A(t, G_k)} \).

By Lemma 3.3 we have the following inequalities:

\[
\gamma_B(s, W_1)\gamma_A(t, G_k) \leq |E(s, t)| \leq \gamma_B(s, G_n)\gamma_A(t, G_k) - \sum_{r \neq 1} \gamma_B(s, W_r)\gamma_A(t, V_1).
\]

The left inequality holds because every element \( v \) in \( G_k \) is the homomorphic image of a visible element in \( G_n \). The right inequality holds because no visible element in \( G_k \) is the homomorphic image of an \( r \)-visible element in \( G_n \), if \( r \neq 1 \).

By dividing both sides by \( \gamma_B(s, G_n)\gamma_A(t, G_k) \), we get

\[
\frac{\gamma_B(s, W_1)}{\gamma_B(s, G_n)} \leq e_\epsilon(G_n, G_k, s, t) \leq 1 - \frac{\sum_{r \neq 1} \gamma_B(s, W_r)\gamma_A(t, V_1)}{\gamma_B(s, G_n)\gamma_A(t, G_k)} = f(s, t),
\]

where

\[
f(s, t) = 1 - \frac{\gamma_A(t, V_1)\gamma_B(s, G_n) - \gamma_B(s, W_1)}{\gamma_A(t, G_k)\gamma_B(s, G_n)}.
\]

Let us use \( \beta_{m,k} \) to denote the limits, which depend on the parity of \( m \) and the rank of the abelianization of \( G_n \) and \( G_k \), found in Corollary 4.12. That is, \( \beta_{m,k} = \frac{2^{r(k)} - 2}{(2^{r(k)} - 1)\zeta(r(k))} \) if \( m \) is even, and \( \beta_{m,k} = \frac{2^{r(k)}}{(2^{r(k)} - 1)\zeta(r(k))} \) if \( m \) is odd. In order to simplify the exposition we will abuse the fact that \( \beta_{m,k} \) depends on the parity of \( m \) and for the next paragraph ignore the parities of \( s \) and \( t \).

Then

\[
\lim_{s \to \infty, t \to \infty} f(s, t) = 1 - \beta_{t,k}(1 - \beta_{s,n}),
\]

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and we get the following inequalities

\[ \beta_{s,n} \leq \liminf_{s \to \infty, t \to \infty} e_{\gamma}(G_n, G_k, s, t) \leq \limsup_{s \to \infty, t \to \infty} e_{\gamma}(G_n, G_k, s, t) \leq 1 - \beta_{t,k}(1 - \beta_{s,n}). \tag{5.1} \]

Now taking into account the parities of \( s \) and \( t \) we get the inequalities in the statement of the theorem.

Thus the probability of an \((s, t)\)-homogeneous equation to be solvable is neither 0 nor 1 as \( s, t \) go to infinity. One sees this by choosing \( G_n \) to be the free group on \( n \) generators and \( G_k \) a surface group of genus \( g \geq 2 \) or a free group of rank \( \geq 2 \) in Theorem 5.1.

**Corollary 5.2.** Let \( G \) be a surface group of genus \( g \geq 2 \) or a free group of rank \( \geq 2 \). Let

\[ A(s, t) = \frac{\sharp \{ \text{solvable } (s, t)\text{-homogeneous equations in } G \text{ in } n \text{ variables} \}}{\sharp \{ (s, t)\text{-homogeneous equations in } G \text{ in } n \text{ variables} \}}. \]

Then

\[ 0 < \liminf_{s \to \infty, t \to \infty} A(s, t) \leq \limsup_{s \to \infty, t \to \infty} A(s, t) < 1. \]

Similarly, by choosing both \( G_n \) and \( G_k \) in Theorem 5.1 to be surface groups one obtains the following.

**Corollary 5.3.** Let \( \Sigma \) be an orientable closed surface of genus \( k \geq 2 \). We fix a presentation for \( \pi_1(\Sigma) \), \( \langle a_1, b_1, \ldots, a_k, b_k \mid [a_1, b_1] \cdots [a_k, b_k] \rangle \). For a closed curve \( \gamma \) in \( \Sigma \) we denote by \( [\gamma] \) the image of \( \gamma \) in \( \pi_1(S) \) and by \( ||\gamma|| \) the length of \( [\gamma] \) with respect to \( \{a_1, b_1, \ldots, a_k, b_k\} \).

We say that \( \gamma_2 \) is the image of \( \gamma_1 \), if it is the image of \( \gamma_1 \) under a continuous map \( S \to S \). Let

\[ B(s, t) = \frac{\sharp \{ ([\gamma_1], [\gamma_2]) \in \pi_1(S)^2, \ (||[\gamma_1]|, ||[\gamma_2]|) = (s, t) \text{ with } \gamma_2 \text{ the image of } \gamma_1 \}}{\sharp \{ ([\gamma_1], [\gamma_2]) \in \pi_1(S)^2, \ (||[\gamma_1]|, ||[\gamma_2]|) = (s, t) \}}. \]

Then

\[ 0 < \liminf_{s \to \infty, t \to \infty} B(s, t) \leq \limsup_{s \to \infty, t \to \infty} B(s, t) < 1. \]

Thus for a fixed orientable surface \( \Sigma \), the probability of a closed curve in \( \Sigma \) to be the image of another closed curve in \( \Sigma \) by a continuous map is neither 0 nor 1, as the curves get more and more “complicated.”

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