\textbf{Abstract}

In this note, we consider $M$-curves of odd degree with real scheme of the form $\langle J \amalg \alpha \amalg 1 \amalg \beta \rangle$. With help of complex orientations, we prove that for $m = 9$, $\alpha \geq 2$, and for $m = 11$, $\alpha \geq 3$.

\section{Introduction}

Let $A$ be a real algebraic non-singular plane curve of degree $m$, its complex part $\mathbb{C}A \subset \mathbb{C}P^2$ is a Riemannian surface with genus $g = (m-1)(m-2)/2$ and its real part $\mathbb{R}A$ is a set of $L \leq g + 1$ circles embedded in $\mathbb{R}P^2$. A circle embedded in $\mathbb{R}P^2$ is called oval or pseudo-line depending on whether it realizes the class 0 or 1 of $H_1(\mathbb{R}P^2)$. If $m$ is even, all of the circles are ovals, if $m$ is odd, the curve has one unique pseudo-line, denoted by $J$. An oval separates $\mathbb{R}P^2$ into a Möbius band and a disc. The latter is called the interior of the oval. An oval of $\mathbb{R}A$ is empty if its interior contains no other oval. One calls exterior oval an oval that is surrounded by no other oval. We say that $A$ is an $M$-curve if $L = g + 1$. A curve is dividing if its real part disconnects its complex part: $\mathbb{C}A \setminus \mathbb{R}A$ has then two homeomorphic halves that are exchanged by the complex conjugation. One can endow $\mathbb{R}A$ with a complex orientation induced by the orientation of one of these halves, note that the complex orientation is defined only up to complete reversion. The $M$-curves are dividing. Two ovals form an injective pair if one of them lies in the interior of the other one. One can provide all the injective pairs of $\mathbb{R}A$ with a sign as follows: such a pair is positive if and only if the orientations of its two ovals induce an orientation of the annulus that they bound in $\mathbb{R}P^2$. Let $\Pi_+$ and $\Pi_-$ be the numbers of positive and negative injective pairs of $A$. If $A$ has odd degree, each oval of $\mathbb{R}A$ can be endowed with a sign: given an oval $O$ of $\mathbb{R}A$, consider the Möbius band $M$ obtained by cutting away the interior of $O$ from $\mathbb{R}P^2$. The classes $[O]$ and $[2J]$ of $H_1(M)$ either coincide or are opposite. In the first case, we say that $O$ is negative; otherwise $O$ is positive. Let $\Lambda_+$ and $\Lambda_-$ be respectively the numbers of positive and negative ovals of $\mathbb{R}A$. Let us call the isotopy type of $\mathbb{R}A \subset \mathbb{R}P^2$ the real scheme of $A$. We consider here curves of odd degree with one unique non-empty oval, their real schemes are...
denoted by \( \langle J \Pi \alpha \Pi 1(\beta) \rangle \). If \( A \) is dividing, its complex scheme is obtained by enriching the real scheme with the complex orientation. The complex schemes of our curves will be encoded by \( \langle J \Pi \alpha_+ \Pi \alpha_- \Pi 1(\beta_+ \Pi \beta_-) \rangle \) where \( \epsilon \in \{+, -\} \) is the sign of the non-empty oval, \( \alpha_+, \alpha_- \) are the numbers of positive and negative ovals among the \( \alpha \) exterior empty ovals, \( \beta_+, \beta_- \) are the numbers of positive and negative ovals among the \( \beta \) interior ovals. Let \( A \) be a dividing curve of degree \( m = 2k + 1 \), endowed with a complex orientation.

**Rokhlin-Mishachev formula:** If \( m = 2k + 1 \), then
\[
2(\Pi_+ - \Pi_-) + (\Lambda_+ - \Lambda_-) = L - 1 - k(k + 1) = k^2 - 2k
\]

**Fiedler theorem:** Let \( L_t = \{ L_t, t \in [0, 1] \} \) be a pencil of real lines based at a point \( P \) of \( \mathbb{R}P^2 \). Consider two lines \( L_{t_1} \) and \( L_{t_2} \) of \( L_t \), which are tangent to \( RA \) at two points \( P_1 \) and \( P_2 \), such that \( P_1 \) and \( P_2 \) are related by a pair of conjugated imaginary arcs in \( CA \cap (\bigcup L_t) \).

Orient \( L_{t_1} \) coherently to \( RA \) in \( P_1 \), and transport this orientation through \( L_t \) to \( L_{t_2} \). Then this orientation of \( L_{t_2} \) is compatible to that of \( RA \) in \( P_2 \).

A pencil turning in some direction, gives rise to two types of tangency points, depending on whether the number of real intersection points decreases or increases by two. A sequence of ovals that are connected one to the next by pairs of conjugated imaginary arcs will be called a Fiedler chain, see Figure 1. To form a chain, one allows actually two consecutive ovals to be connected indirectly via a fold due to some component of the curve as shown in Figure 2. A fold is a pair of tangency points of each type on the component. To apply the theorem as a restriction tool, we try to distribute the ovals of the curve under consideration in chains, in which the ovals have alternating orientations with respect to some pencil of lines. We can safely ignore the folds.

Let \( C_m \) be an \( M \)-curve of degree \( m \). Given an empty oval \( X \) of \( C_m \), we often have to consider one point chosen in the interior of \( X \). For simplicity, we call this point also \( X \). Let \( m \) be odd, for any two empty ovals \( X, Y \), we shall denote by \( [XY] \) and \( [XY]' \) the segments of the line \( XY \) that cut the pseudo-line \( J \) an even and an odd number of times respectively. We say that \( [XY] \) is the principal segment determined by \( X, Y \). Let \( X, Y \) and \( Z \) be three interior ovals of \( C_m \). The corresponding three points give rise to four triangles in \( \mathbb{R}P^2 \). We denote by \( XYZ \) the triangle bounded by \( [XY], [XZ], [YZ] \) and call it the principal triangle. The complete pencil of lines based at \( X \) is divided in two portions by the lines \( XY \) and \( XZ \). We say that the portion sweeping out \( [XY]' \) has a \( J \)-jump. For example, assume that the folds in Figure 2 are on \( J \). The pencil of lines between the two ovals has a \( J \)-jump in the left picture, and none in the right picture.

## 2 Results

Let \( C_m \) be a curve of odd degree \( m \), with one unique non-empty oval, consider the union of all principal triangles whose vertices are interior ovals. If this union
is a disc bounded by a polygon, we say that it is the convex hull of the interior ovals, and call it $\Delta$. It was first observed by S. Orevkov that for $m \geq 13$, the interior ovals may have no convex hull.

**Lemma 1** If $m \leq 11$, then the interior ovals have a convex hull $\Delta$.

*Proof:* Assume there exist four interior ovals $A, B, C, D$, such that $D$ lies in the principal triangle $ABC$. One has either $ABC = ABD \cup BCD \cup ACD$, or $J$ cuts an odd number of times each of the three segments joining $D$ to $A, B$ and $C$ in $ABC$. Assume the latter case occurs. Then a piece of $J$ and a piece of the non-empty oval $O$ in $ABC$ realize one of the six positions displayed in Figure 3, each piece cuts $[AB]$ twice. Note that as $A, B$ are interior ovals, the segment $[AB]$ has two supplementary intersection points with $O$. The segment $[AB]'$ cuts $J$ at least once, and $O$ at least twice. The line $AB$ cuts $C_{m}$ at 13 points or more, so $m \geq 13$. For $m \leq 11$, one constructs $\Delta$ inductively, adding one oval after the other. Start with the principal triangle $\Delta_3$ determined by three ovals. Assume one has obtained with $n$ ovals a disc $\Delta_n$ bounded by a polygon. Add a new oval $X$, if $X$ is in $\Delta_n$, then $\Delta_{n+1} = \Delta_n$. Otherwise, consider the pencil of lines $F_X$ sweeping out $\Delta_n$, the extreme ovals $Y, Z$ met by the pencil are vertices of $\Delta_n$. One has $\Delta_{n+1} = \Delta_n \cup XYZ$. $\Box$

**Lemma 2** Let $C_m$ be a dividing curve of odd degree $m = 2k + 1$ with one unique non-empty oval. If the interior ovals have a convex hull, then:

$$1 - k - \alpha \leq \Pi_+ - \Pi_- \leq k - 1 + \alpha - \epsilon$$
where \( \epsilon = 0 \) if \( \alpha = 0 \), \( \epsilon = 1 \) otherwise.

**Lemma 3** Let \( C_m \) be an M-curve of odd degree \( m \geq 7 \) with one unique non-empty oval and at least one exterior empty oval.

1. If \( O \) is negative, then: 
   \[ \alpha_+ - \alpha_- \equiv (k-1)^2 \pmod{3} \]

2. Assume that the interior ovals have a convex hull. If \( O \) is positive, then:
   \[ k^2 - 3k + 1 \leq 2\alpha_+ \leq 2\alpha \]
   If \( O \) is negative, then:
   \[ k^2 - 5k + 7 \leq 2\alpha_+ + 2\alpha \leq 4\alpha \]

In both cases, one has:
   \[ \alpha \geq (k^2 - 5k + 7)/4 \]

**Remark:** In degree 7, there exist M-curves realizing 14 different real schemes: \( \langle J \Pi \alpha \Pi 1(\beta) \rangle \) (1 \( \leq \alpha \leq 13 \)) and \( \langle J \Pi 15 \rangle \). The classification of the complex schemes is presented in [2], it was established using the restrictions and constructions from [1], [2], [5], [7] and [9]. One has actually for the M-curves with a non-empty oval: \( \Pi_+ - \Pi_- \in \{0, +1, -1, 2\} \).

**Proof of Lemmas 2 and 3:** Consider the convex hull \( \Delta \) of the interior ovals. Let \( B \) be an oval placed at a vertex. The pencil \( F_B \) sweeping out \( \Delta \) sweeps out all of the interior ovals and possibly some of the \( \alpha \) exterior ovals, giving rise to \( k-2 \) chains. We consider here not the complete chains, but the longest subchains whose extremities are interior ovals. There are no \( J \)-jumps between the interior ovals, two consecutive interior ovals in a chain have thus alternating orientations with respect to \( J \). A chain has \( j \) jumps if it is formed of \( 2j + 1 \) subchains with alternatively interior and exterior ovals. A chain with \( j \) jumps brings a contribution to \( \Pi_+ - \Pi_- \) whose absolute value is less or equal to \( j + 1 \). The total number of jumps is at most \( \alpha \). Taking into account the contribution of the last oval \( B \) one gets the inequality: \( |\Pi_+ - \Pi_-| \leq k - 1 + \alpha \). Assume there are some exterior ovals, let \( A \) be any one of them. Consider the pencil of lines \( F_A \) sweeping out \( O \). The interior ovals are distributed in \( k-2 \) chains and the total number of jumps is at most \( \alpha - 1 \). If this pencil has no \( J \)-jumps between the interior ovals, then \( |\Pi_+ - \Pi_-| \leq k + \alpha - 3 \). Assume now that one has the equality \( |\Pi_+ - \Pi_-| = k - 1 + \alpha \). For any choice of \( A \), the pencil \( F_A \)
has \(J\)-jumps between some interior ovals. The exterior ovals lie all inside of \(\Delta\).
(Note that the ovals vertices of \(\Delta\) have all the same orientation.) Some edge of \(\Delta\) must intersect the non-empty oval \(O\). Let \(B\) be a vertex of \(\Delta\), extremity of such an edge. The pencil of lines based at \(B\) sweeping out \(\Delta\) meets all of the other empty ovals, these ovals are distributed in \(k - 2\) chains whose extremal ovals are all interior ovals having the same orientation as \(B\). The last oval of one chain must be connected with a point on \(O\), this oval forms therefore a negative pair with \(O\), see Figure 4. Thus, \(\Pi_+ - \Pi_- = 1 - k - \alpha\). This finishes the proof of Lemma 2.

The Rokhlin-Mishachev formula may be rewritten for the \(M\)-curves with one unique non-empty oval:

- **\(O\) positive:** \(\Pi_+ - \Pi_- + 1 + \alpha_+ - \alpha_- = k^2 - 2k\)
- **\(O\) negative:** \(3(\Pi_+ - \Pi_-) - 1 + \alpha_+ - \alpha_- = k^2 - 2k\)

Lemma 3 follows immediately from these formulas, combined with the inequality \(\Pi_+ - \Pi_- \leq k - 2 + \alpha\). □

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**Figure 4:** The last oval of one chain is connected with \(O\)

### Proposition 1
There doesn’t exist \(M\)-curves of degree \(m = 2k + 1\) \((k = 3, 4\) or \(5\)), having one unique non-empty oval that contains all others in its interior. Otherwise stated, the real schemes: \(\langle J \sqcup 1 \langle 14 \rangle \rangle \) \((k = 3)\), \(\langle J \sqcup 1 \langle 27 \rangle \rangle \) \((k = 4)\), \(\langle J \sqcup 1 \langle 44 \rangle \rangle \) \((k = 5)\) are not realizable.

**Proof:** Assume there exists some curve \(C_m\) contradicting the proposition, let \(O\) be its non-empty oval. Denote by \(\beta_\pm\) the numbers of positive and negative interior ovals, and let \(\lambda = \beta_+ - \beta_-\). For \(O\) positive, \(\lambda = 1 - k^2 + 2k\); for \(O\) negative \(\lambda = (k - 1)^2/3\). By Lemma 2, \(|\lambda| = |\Pi_+ - \Pi_-| \leq k - 1\). By Lemma 3, the non-empty oval can be negative only if \(k \equiv 1 \pmod{3}\), so \(k = 4\) The only possibility is a curve of degree 9 with complex scheme \(\langle J \sqcup 1 \langle 15_+ 12_- \rangle \rangle\). If \(O\) is positive and \(k > 3\), then \(\lambda = 1 - k^2 + 2k < 1 - k < 0\), contradiction. The only possibility is a curve of degree 7 with complex scheme \(\langle J \sqcup 1 \langle 6_+ 8_- \rangle \rangle\). The real scheme \(\langle J \sqcup 1 \langle 14 \rangle \rangle\) is not realizable [9]. [1] and the complex scheme
⟨\mathcal{J} II 1\langle15, II 12\rangle⟩ has been excluded more recently in [3]. We present here an alternative method to exclude both complex schemes. Let \(C_m, m = 7\) or 9 realize one of these complex schemes. Assume there exists some bitangent to the non-empty oval \(O\) and consider a point \(P\) lying in the zone bounded by the principal segment joining the tangency points, and an arc of \(O\), and outside of \(\triangle\). The pencil of lines based at \(P\) gives rise to \(k-1\) chains of interior ovals, each chain brings a contribution +1 to \(\Pi^+ - \Pi^-\), see Figure 5. The last oval of one chain should be connected with a point on \(O\), contradiction. Therefore, the non-empty oval \(O\) must be convex. Let \(P, Q\) be two points on \(O\). We will consider the pencils of lines based at a mobile point \(P\) percursoring the segment \([PQ]'\), exterior to \(O\). In each pencil, there are two lines tangent to \(O\). For \(P'\) close to \(P\), the two points of tangency are connected by a small circle in \(CA\), with center \(P\). Moving away from \(P\), one gets a family of growing concentric circles on \(CA\). At the other extremity \(Q\), we have similarly a family of concentric circles, centered this time at \(Q\). At some moment, there must have been some bifurcation in the set of conjugated imaginary arcs, after which the two tangency points with \(O\) are no longer connected one to the other (see also [1] where a similar argument is used in a different proof). Each tangency point is then connected to an interior oval whose orientation (with respect to \(\mathcal{J}\)) coincides with that of \(O\), this oval is the starting oval of one of the \(k-1\) interior chains determined by the pencil. But for any choice of base point outside of \(O\), the interior ovals are distributed in \(k-1\) chains whose ovals have alternating orientations with respect to \(\mathcal{J}\), each chain bringing a contribution +1 to \(\Pi^+ - \Pi^-\). The bifurcation is impossible, see Figure 6. \(\square\)

![Figure 5: M-curves ⟨\mathcal{J} II 1\langle15, II 12\rangle⟩ and ⟨\mathcal{J} II 1\langle15, II 12\rangle⟩, contradiction if \(O\) non-convex](image)  

**Proposition 2** There doesn’t exist M-curves of degree \(2k+1\) with the following real schemes: \(⟨\mathcal{J} II 1\mathfrak{H} \langle26\rangle⟩\) (\(k = 4\)), \(⟨\mathcal{J} II 1 \mathfrak{H} \langle43\rangle⟩\) (\(k = 5\)), \(⟨\mathcal{J} II 2 \mathfrak{H} \langle42\rangle⟩\) (\(k = 5\)).

**Proof:** For \(k = 5\), one has by Lemma 3: \(\alpha \geq 2\) if \(O\) is negative, and \(\alpha \geq 6\) if \(O\) is positive. Assume \(O\) is negative and \(\alpha = 2\). As \(\alpha_+ - \alpha_- \equiv 1 \pmod{3}\), one must have \(\alpha_+ = 0, \alpha_- = -2\). Hence, \(2\alpha_+ + 2\alpha = 4 < k^2 - 5k + 7\), contradiction. For \(k = 4\), one has by Lemma 2: \(\alpha_+ \geq 3\) or \(\alpha_+ - \alpha_- \equiv 0 \pmod{3}\). Neither condition is realized if \(\alpha = 1\). \(\square\)
Note that for $m = 7$, the real scheme $\langle \mathcal{J} II 1 \ 1 \ (13) \rangle$ is realizable, with the two admissible complex schemes: $\langle \mathcal{J} II 1_+ \ II \ 1_+ (6_+ \ II \ 7_-) \rangle$ and $\langle \mathcal{J} II 1_+ \ II \ 1_- (7_+ \ II \ 6_-) \rangle$, see [10], [5], [2]. For $m = 9$, the non-realizability of the schemes $\langle \mathcal{J} II 1 \ (27) \rangle$, $\langle \mathcal{J} II \ II \ (26) \rangle$ proved above had previously been announced in [1], but unfortunately the proofs were never published and went lost. Ninth degree $M$-curves $\langle \mathcal{J} II \ (\alpha \ II \ (\beta)) \rangle$ with $1 \leq \beta \leq 19$, $\beta = 22, 23$ have been constructed by A. Korchagin [6]. By Lemma 3, the only admissible schemes for $\alpha = 2$ and $3$ are: $\langle \mathcal{J} III_+ III_+ (14_+ III_1) \rangle$, $\langle \mathcal{J} III_+ III_+ (10_+ III_4) \rangle$, $\langle \mathcal{J} III_4 III_1 (13_+ III_1) \rangle$, $\langle \mathcal{J} II 3_- II 1_- (14_+ II 10_-) \rangle$. Finally, note that the proofs and results presented here are valuable also for pseudo-holomorphic curves.

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