OSCILLATORY SOLITONS OF U(1)-IN Variant MKDV EQUATIONS II:
ASYMPTOTIC BEHAVIOR AND CONSTANTS OF MOTION

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Abstract. The Hirota equation and the Sasa-Satsuma equation are $U(1)$-invariant integrable generalizations of the modified Korteweg-de Vries equation. These two generalizations admit oscillatory solitons, which describe harmonically modulated complex solitary waves parameterized by their speed, modulation frequency, and phase. Depending on the modulation frequency, the speeds of oscillatory waves (1-solitons) can be positive, negative, or zero, in contrast to the strictly positive speed of ordinary solitons. When the speed is zero, an oscillatory wave is a time-periodic standing wave. Oscillatory 2-solitons with non-zero wave speeds are shown to describe overtake collisions of a fast wave and a slow wave moving in the same direction, or head-on collisions of two waves moving in opposite directions. When one wave speed is zero, oscillatory 2-solitons are shown to describe collisions in which a moving wave overtakes a standing wave. An asymptotic analysis using moving coordinates is carried out to show that, in all collisions, the speeds and modulation frequencies of the individual waves are preserved, while the phases and positions undergo a shift such that the center of momentum of the two waves moves at a constant speed. The primary constants of motion as well as some other features of the nonlinear interaction of the colliding waves are discussed.

1. Introduction

Complex $U(1)$-invariant modified Korteweg-de Vries (mKdV) equations

\[ u_t + (\alpha u \bar{u}_x + \beta u_x \bar{u})u + \gamma u_{xxx} = 0 \]  
(1.1)

(where $\alpha, \beta, \gamma$ are real constants) arise in many physical applications, such as short wave pulses in optical fibers [1, 2] and deep water waves [3, 4]. Of particular mathematical and physical interest are two integrable equations in this class, given by the Hirota equation [5]

\[ u_t + \beta |u|^2 u_x + \gamma u_{xxx} = 0, \]  
(1.2)

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and the Sasa-Satsuma equation [6]
\[ u_t + \alpha(u\bar{u}_x + 3u_x\bar{u})u + \gamma u_{xxx} = 0. \]
(1.3)
The integrability properties of these two equations consist of multi-soliton solutions, a Lax pair, a bilinear formulation, a bi-Hamiltonian structure, and an infinite hierarchy of symmetries and conservation laws.

Both equations possess ordinary soliton solutions of the form
\[ u(t,x) = \exp(i\phi)f(x-ct) \]
(1.4)
which are solitary waves with speed \( c > 0 \) and phase angle \( -\pi \leq \phi \leq \pi \). Collisions of two or more solitary waves are described by multi-soliton solutions. In all collisions, the net effect on the solitary waves is to shift in their asymptotic positions, while their asymptotic phase angles stay unchanged in the case of the Hirota equation (2.1) but become shifted in the case of the Sasa-Satsuma equation (2.2). The actual nonlinear interaction of these solitary waves during a collision exhibits interesting features which depend on the speed ratios and relative phase angles of the waves, as studied in previous work [7]. (See the animations at http://lie.math.brocku.ca/~sanco/solitons/mkdv_solitons.php)

In a recent paper [8], we began a comprehensive study of a more general type of soliton solution with the form
\[ u(t,x) = \exp(i\phi)\exp(i\nu t)\tilde{f}(x-ct) \]
(1.5)
which is a harmonically modulated solitary wave, called an oscillatory soliton, where the temporal modulation frequency \( \nu \neq 0 \) and the speed \( c \) obey the kinematic condition
\[ (c/3)^3 + (\nu/2)^2 > 0. \]
(1.6)
In contrast to an ordinary soliton, the speed \( c \) of an oscillatory soliton can be positive, negative, or zero. Consequently, these solitons have three different types of collisions: (1) right-overtake — where a faster right-moving soliton overtakes a slower right-moving soliton or a stationary soliton; (2) left-overtake — where a faster left-moving soliton overtakes a slower left-moving soliton or a stationary soliton; (3) head-on — where a right-moving soliton collides with a left-moving soliton. All of these collisions are described by oscillatory 2-soliton solutions
\[ u(t,x) = \exp(i\phi_1)\exp(i\nu_1 t)\tilde{f}_1(x-c_1 t, x-c_2 t) \]
\[ + \exp(i\phi_2)\exp(i\nu_2 t)\tilde{f}_2(x-c_1 t, x-c_2 t), \quad c_1 \neq c_2 \]
(1.7)
whose temporal frequencies \( \nu_1, \nu_2 \) and speeds \( c_1, c_2 \) satisfy the kinematic conditions
\[ (c_1/3)^3 + (\nu_1/2)^2 > 0, \quad (c_2/3)^3 + (\nu_2/2)^2 > 0. \]
(1.8)
In the present paper we will study the asymptotic features of colliding oscillatory solitons (1.7) for both the Hirota equation (2.1) and the Sasa-Satsuma equation (2.2). These collisions can be expected to exhibit highly interesting new features compared to collisions of ordinary solitons.

In section 2, we recall some details of the oscillatory 1-soliton and 2-soliton solutions derived in Ref. [8] for the Hirota equation and the Sasa-Satsuma equation.

In section 3, we carry out an asymptotic analysis using moving coordinates to show that the 2-soliton solutions (1.7) for the Hirota equation and the Sasa-Satsuma equation reduce to a superposition of two 1-soliton solutions (1.5) with speeds \( c_1, c_2 \) and temporal frequencies
$\nu_1$, $\nu_2$ in the asymptotic past and future. This analysis rigorously establishes that these 2-soliton solutions describe collisions between two oscillatory waves when both $c_1$ and $c_2$ are non-zero, or collisions of an oscillatory wave with a standing wave, when one of the speeds $c_1$ or $c_2$ is zero.

In section 4, for the 1-soliton and 2-soliton solutions, we discuss the primary constants of motion arising from the conservation laws for momentum, energy, and Galilean energy admitted by [9] the Hirota equation and the Sasa-Satsuma equation. In particular, from conservation of Galilean energy, we show that the center of momentum for the 2-soliton solutions moves at a constant speed throughout a collision.

Our main results are obtained in section 5. For overtake and head-on collisions described by the 2-soliton solutions, we first show that the net effect of a collision is to shift the asymptotic positions and phases of the two waves while the speed and the temporal frequency of each wave remains unchanged. Explicit formulas for these asymptotic shifts are presented in terms of the speeds and temporal frequencies of the two waves in the collision. Next, from these formulas, we find that for overtake collisions the faster wave gets shifted forward relative to its direction of motion while the slower or stationary wave gets shifted in the backward direction. In contrast, for head-on collisions, we find that both waves get shifted forward relative to their directions of motion. Finally, for all collisions, we show that the position shifts of the two oscillatory waves are related by the property that the center of momentum of the waves is preserved in the collision.

In section 6, we discuss a few interesting features of the nonlinear interactions that occur for oscillatory waves and standing waves during collisions, and we make some concluding remarks.

Previous work on soliton solutions of the Hirota equation and Sasa-Satsuma equation appears in Ref. [10, 11, 12, 13]. This work amounts to deriving the 1-soliton and 2-soliton formulas in a mathematically equivalent but less physically useful envelope form, without any analysis of the asymptotic behaviour and the constants of motion for these solutions.

All computations in the present paper have been carried out by use of Maple. Hereafter, by scaling variables $t, x, u$, we will put

$$\alpha = 6, \quad \beta = 24, \quad \gamma = 1$$

for convenience.

### 2. Oscillatory soliton solutions

For the Hirota equation

$$u_t + 24|u|^2u_x + u_{xxx} = 0$$

and the Sasa-Satsuma equation

$$u_t + 6(u\bar{u}_x + 3u_x\bar{u})u + \gamma u_{xxx} = 0$$

we will first summarize the expressions for the respective travelling-wave functions $\tilde{f}(x - ct)$ in the oscillatory 1-soliton solutions (1.5).
Let
\[
 k = \frac{\sqrt{3}}{2} \left( \sqrt[3]{\sqrt{(c/3)^3 + (\nu/2)^2} - \nu/2} + \sqrt[3]{\sqrt{(c/3)^3 + (\nu/2)^2} + \nu/2} \right) \\
 \kappa = \frac{1}{2} \left( \sqrt[3]{\sqrt{(c/3)^3 + (\nu/2)^2} - \nu/2} - \sqrt[3]{\sqrt{(c/3)^3 + (\nu/2)^2} + \nu/2} \right)
\]  
(2.3)
(2.4)
where \(c\) and \(\nu\) \(\neq 0\) obey the relation (1.6), which implies the properties \(k > 0\) and \(\kappa \neq 0\).

**Proposition 1.** The Hirota and Sasa-Satsuma oscillatory 1-soliton solutions
\[
 u(t, x) = \exp(i\phi) \exp(i\nu t) \tilde{f}(\xi), \quad \xi = x - ct
\]  
expressed using a travelling wave coordinate are given by
\[
 \tilde{f}(\xi) = \exp(i\kappa \xi) U(\xi)
\]  
in terms of the respective envelope functions
\[
 U_H(\xi) = \frac{k}{2 \cosh(k \xi)},
\]  
\[
 U_{SS}(\xi) = \frac{k(2|\kappa|)^{1/2}(k^2 + \kappa^2)^{1/4} \cosh(k \xi + i\lambda/2)}{|\kappa| \cosh(2k \xi) + (k^2 + \kappa^2)^{1/2}}, \quad \lambda = \arg(k(\kappa + ik)).
\]  
(2.7)
(2.8)
When \(c = 0\), these 1-soliton solutions are harmonically modulated standing waves.

We next summarize the expressions for the travelling-wave functions \(\tilde{f}_1(x - c_1 t, x - c_2 t)\) and \(\tilde{f}_2(x - c_1 t, x - c_2 t)\) in oscillatory 2-soliton solutions (1.7) for the Hirota equation (2.1) and the Sasa-Satsuma equation (2.2).

Let
\[
 k_1 = \sqrt{3}(\beta_{1-} + \beta_{1+})/2, \quad \kappa_1 = (\beta_{1-} - \beta_{1+})/2,
\]  
\[
 k_2 = \sqrt{3}(\beta_{2-} + \beta_{2+})/2, \quad \kappa_2 = (\beta_{2-} - \beta_{2+})/2,
\]  
(2.9)
(2.10)
with
\[
 \beta_{1\pm} = \sqrt[3]{\sqrt{(c_1/3)^3 + (\nu_1/2)^2} \pm \nu_1/2}, \quad \beta_{2\pm} = \sqrt[3]{\sqrt{(c_2/3)^3 + (\nu_2/2)^2} \pm \nu_2/2},
\]  
(2.11)
where \(c_1, c_2, \nu_1 \neq 0, \nu_2 \neq 0\) obey the relations (1.8). We then have the properties
\[
 k_1 > 0, \quad k_2 > 0,
\]  
(2.12)
\[
 \kappa_1 \neq 0, \quad \kappa_2 \neq 0.
\]  
(2.13)

**Proposition 2.** As expressed using travelling wave coordinates \(\xi_1 = x - c_1 t\) and \(\xi_2 = x - c_2 t\) when \(c_1 \neq c_2\), the Hirota and Sasa-Satsuma oscillatory 2-soliton solutions
\[
 u(t, x) = \exp(i\phi_1) \exp(i\nu_1 t) \tilde{f}_1(\xi_1, \xi_2) + \exp(i\phi_2) \exp(i\nu_2 t) \tilde{f}_2(\xi_1, \xi_2)
\]  
are given by
\[
 \tilde{f}_1(\xi_1, \xi_2) = \exp(i\kappa_1 \xi_1)V_1(\xi_1, \xi_2)/W(\xi_1, \xi_2), \quad \tilde{f}_2(\xi_1, \xi_2) = \exp(i\kappa_2 \xi_2)V_2(\xi_1, \xi_2)/W(\xi_1, \xi_2)
\]  
(2.14)
(2.15)
in terms of the respective envelope functions

\[
V_{1H}(\xi_1, \xi_2) = k_1 \cosh(k_2\xi_2 + i\gamma_2),
\]

\[
V_{2H}(\xi_1, \xi_2) = k_2 \cosh(k_1\xi_1 + i\gamma_1),
\]

\[
W_{1H}(\xi_1, \xi_2) = \sqrt{\Gamma} \cosh(k_1\xi_1 + k_2\xi_2) + \frac{1}{\sqrt{\Gamma}} \cosh(k_1\xi_1 - k_2\xi_2)
\]

\[
- \frac{4k_1k_2}{\sqrt{\Gamma}} \cos(\kappa_1\xi_1 - \kappa_2\xi_2 + \mu(\xi_2 - \xi_1) + \phi_1 - \phi_2),
\]

in the Hirota case, and in the Sasa-Satsuma case

\[
V_{1SS}(\xi_1, \xi_2) = k_1(k_1^2 + k_2^2)^{1/4}|2\kappa_1|^{1/2}\left(|\kappa_2|\left(\sqrt{\Delta T} \cosh(k_1\xi_1 + 2k_2\xi_2 + i(\alpha_2 + \gamma_2))
\right.ight.
\]

\[
+ \left.\frac{1}{\sqrt{\Delta T}} \cosh(k_1\xi_1 - 2k_2\xi_2 + i(\nu_2 - \gamma_2))\right)
\]

\[
+ \left(\kappa_2^2 - \kappa_1^2\right)^{1/2} \left( -8k_2^2\kappa_2s \frac{1}{\sqrt{\Omega T}} \cosh(k_1\xi_1 + i(\omega_1 + \gamma_1))
\right.
\]

\[
+ \left.\sqrt{\frac{\Gamma}{\Delta}} \cosh(k_1\xi_1 + i(\alpha_2 - \gamma_2)) + \sqrt{\frac{\Delta}{\Gamma}} \cosh(k_1\xi_1 + i(\nu_2 + \gamma_2))\right)
\]

\[
+ k_1k_2s\sqrt{\frac{\Omega}{T}} \left( k_1(k_1^2 + k_2^2)^{1/4}|8\kappa_2|^{1/2}s_2 \cosh(k_1\xi_1 + i(\omega_1 - \gamma_1))
\]

\[
- k_1(k_2^2 + k_2^2)^{1/4}|32\kappa_1|^{1/2}s_1 \text{Re}\left( \cosh(k_2\xi_2 + i(\gamma_2 - \omega_2)) \times \exp(i(\kappa_1\xi_1 - \kappa_2\xi_2 + \mu(\xi_2 - \xi_1) + \phi_1 - \phi_2))\right)\right),
\]

\[
V_{2SS}(\xi_1, \xi_2) = k_2(k_1^2 + k_2^2)^{1/4}|2\kappa_2|^{1/2}\left(|\kappa_2|\left(\sqrt{\Delta T} \cosh(k_2\xi_2 + 2k_1\xi_1 + i(\alpha_1 + \gamma_1))
\right.ight.
\]

\[
+ \left.\frac{1}{\sqrt{\Delta T}} \cosh(k_2\xi_2 - 2k_1\xi_1 + i(\nu_1 - \gamma_1))\right)
\]

\[
+ \left(\kappa_1^2 + \kappa_2^2\right)^{1/2} \left( -8k_1^2\kappa_1s \frac{1}{\sqrt{\Omega T}} \cosh(k_2\xi_2 + i(\omega_2 + \gamma_2))
\right.
\]

\[
+ \left.\sqrt{\frac{\Gamma}{\Delta}} \cosh(k_2\xi_2 + i(\alpha_1 - \gamma_1)) + \sqrt{\frac{\Delta}{\Gamma}} \cosh(k_2\xi_2 + i(\nu_1 + \gamma_1))\right)
\]

\[
+ k_1k_2s\sqrt{\frac{\Omega}{T}} \left( k_1(k_1^2 + k_2^2)^{1/4}|8\kappa_1|^{1/2}s_1 \cosh(k_2\xi_2 + i(\omega_2 - \gamma_2))
\]

\[
- k_2(k_1^2 + k_1^2)^{1/4}|32\kappa_2|^{1/2}s_2 \text{Re}\left( \cosh(k_1\xi_1 + i(\gamma_1 - \omega_1)) \times \exp(i(\kappa_2\xi_2 - \kappa_1\xi_1 + \mu(\xi_1 - \xi_2) + \phi_2 - \phi_1))\right)\right),
\]
\[
W_{SS}(\xi_1, \xi_2) = |\kappa_1 \kappa_2| \left( \Delta \Gamma \cosh(2(k_1 \xi_1 + k_2 \xi_2)) + \frac{1}{\Delta \Gamma} \cosh(2(k_1 \xi_1 - k_2 \xi_2)) \right)
+ 2(k_1^2 + \kappa_1^2)^{1/2} |\kappa_2| \cosh(2k_2 \xi_2) + 2(k_2^2 + \kappa_2^2)^{1/2} |\kappa_1| \cosh(2k_1 \xi_1)
+ 4k_1^2 k_2^2 s_1 s_2 \frac{\Omega}{\Upsilon} \cos(2(\kappa_1 \xi_1 - \kappa_2 \xi_2 + \mu(\xi_2 - \xi_1)) + 2(\phi_1 - \phi_2))
+ (k_1^2 + \kappa_1^2)^{1/2}(k_2^2 + \kappa_2^2)^{1/2} \left( \frac{\Gamma}{\Delta} + \frac{\Delta}{\Gamma} + 64k_1^2 k_2^2 \kappa_1 \kappa_2 \frac{1}{\Omega \Upsilon} \right)
- 16k_1 k_2 |\kappa_1 \kappa_2|^{1/2} \text{Re} \left( \exp(i(\kappa_1 \xi_1 - \kappa_2 \xi_2 + \mu(\xi_2 - \xi_1) + \phi_1 - \phi_2)) \times \right.
\left( \sqrt{\frac{\Gamma}{\Upsilon}} \cosh(k_1 \xi_1 + k_2 \xi_2 + i(\varpi_1 - \varpi_2))
+ \frac{1}{\sqrt{\Gamma \Upsilon}} \cosh(k_1 \xi_1 - k_2 \xi_2 + i(\varpi_1 + \varpi_2)) \right) \right)
\]

(2.21)

where, in both cases,

\[\mu = (\nu_1 - \nu_2)/(c_1 - c_2),\]

(2.22)

\[\Omega = \sqrt{((k_1 + k_2)^2 + (\kappa_1 + \kappa_2)^2)((k_1 - k_2)^2 + (\kappa_1 + \kappa_2)^2)},\]

(2.23)

\[\Upsilon = ((k_1 - k_2)^2 + (\kappa_1 - \kappa_2)^2)((k_1 + k_2)^2 + (\kappa_1 - \kappa_2)^2),\]

(2.24)

\[\Delta = \sqrt{(k_1 - k_2)^2 + (\kappa_1 + \kappa_2)^2},\]

(2.25)

\[\Gamma = \frac{(k_1 - k_2)^2 + (\kappa_1 - \kappa_2)^2}{(k_1 + k_2)^2 + (\kappa_1 + \kappa_2)^2},\]

(2.26)

\[\alpha_1 = (\lambda_2 + \delta_1)/2, \quad \alpha_2 = (\lambda_1 + \delta_2)/2,\]

(2.27)

\[\nu_1 = (\lambda_2 - \delta_1)/2, \quad \nu_2 = (\lambda_1 - \delta_2)/2,\]

(2.28)

\[\varpi_1 = (\lambda_1 - \delta_1)/2, \quad \varpi_2 = (\lambda_2 - \delta_2)/2,\]

(2.29)

\[\gamma_1 = \text{arg}(k_1^2 - k_2^2 - (\kappa_1 - \kappa_2)^2 + i2k_1(\kappa_1 - \kappa_2)),\]

(2.30)

\[\gamma_2 = \text{arg}(k_2^2 - k_1^2 - (\kappa_1 - \kappa_2)^2 - i2k_2(\kappa_1 - \kappa_2)),\]

(2.31)

\[\delta_1 = \text{arg}(k_2^2 - k_1^2 + (\kappa_1 + \kappa_2)^2 + i2k_1(\kappa_1 + \kappa_2)),\]

\[\delta_2 = \text{arg}(k_1^2 - k_2^2 + (\kappa_1 + \kappa_2)^2 + i2k_2(\kappa_1 + \kappa_2)),\]

\[\lambda_1 = \text{arg}(\kappa_1(\kappa_1 + ik_1)), \quad \lambda_2 = \text{arg}(\kappa_2(\kappa_2 + ik_2)).\]

(2.32)

\[s_1 = \text{sgn}(\kappa_1), \quad s_2 = \text{sgn}(\kappa_2),\]

(2.33)

\[s = \begin{cases} 1, & |k_1| \neq |k_2| \\ \text{sgn}(\kappa_1 + \kappa_2), & |k_1| = |k_2| \end{cases}.\]

(2.34)
For convenience, we also write out the half-angle expressions needed in equations (2.27)–(2.29):

$$\lambda_1/2 = \arg \left( \sqrt{1 + k_1^2/k_2^2} + 1 + is_1 \right),$$
$$\lambda_2/2 = \arg \left( \sqrt{1 + k_2^2/k_2^2} + 1 + is_2 \right),$$

and

$$\delta_1/2 = \arg \left( (e^2\epsilon_+ + (1 - e^2\epsilon_-)\text{sgn}(\kappa_1 + \kappa_2)) \sqrt{k_2^2 - k_1^2 + (\kappa_1 + \kappa_2)^2 + \Omega} 
+ is_1(1 + e^2\epsilon_-(\text{sgn}(\kappa_1 + \kappa_2) - 1)) \sqrt{k_2^2 - k_1^2 + (\kappa_1 + \kappa_2)^2 - \Omega} \right),$$

$$\delta_2/2 = \arg \left( (e^2\epsilon_+ + (1 - e^2\epsilon_-)\text{sgn}(\kappa_1 + \kappa_2)) \sqrt{k_1^2 - k_2^2 + (\kappa_1 + \kappa_2)^2 + \Omega} 
+ is_2(1 + e^2\epsilon_+(\text{sgn}(\kappa_1 + \kappa_2) - 1)) \sqrt{k_1^2 - k_2^2 + (\kappa_1 + \kappa_2)^2 - \Omega} \right),$$

where

$$\epsilon_\pm = (1 + \epsilon)/2, \quad \epsilon = \text{sgn}(|k_1| - |k_2|) = \begin{cases} 1, & |k_1| > |k_2| \\ -1, & |k_1| < |k_2| \\ 0, & |k_1| = |k_2| \end{cases}.$$

3. Asymptotic analysis

An oscillatory wave (1.5) having phase angle $\phi$, temporal frequency $\nu$, and speed $c$ reduces to an ordinary travelling wave (1.4) when (and only when) $\nu = 0$. In this case the kinematic condition (1.6) implies $c > 0$, showing that all ordinary travelling wave solutions of the Hirota equation (2.1) and the Sasa-Satsuma equation (2.2) are right-moving.

In contrast, when $\nu \neq 0$, the condition (1.6) is equivalent to the kinematic relation

$$c > -(3/\sqrt{4})(\sqrt{\nu})^2 \neq 0$$

which allows $c < 0$ and $c = 0$, in addition to allowing $c > 0$. Correspondingly, oscillatory 1-soliton solutions from Proposition 1 for the Hirota equation (2.1) and the Sasa-Satsuma equation (2.2) consist of right-moving oscillatory waves when $c > 0$, left-moving oscillatory waves when $c < 0$, and standing waves when $c = 0$.

We will now show that the oscillatory 2-soliton solutions with $c_1 \neq c_2$ from Proposition 2 for the Hirota equation (2.1) and the Sasa-Satsuma equation (2.2) reduce in both the asymptotic past ($t \to -\infty$) and future ($t \to +\infty$) to a linear superposition of oscillatory 1-soliton solutions whose speeds are precisely $c_1$ and $c_2$. Since the solutions are symmetric under simultaneously interchanging $c_1 \leftrightarrow c_2$, $\nu_1 \leftrightarrow \nu_2$, $\phi_1 \leftrightarrow \phi_2$, we will assume

$$c_1 > c_2$$

hereafter without loss of generality.

To proceed, we first note

$$\xi_1 = x - c_1 t, \quad \xi_2 = x - c_2 t$$

are moving coordinates centered at positions $x = c_1 t$ and $x = c_2 t$, respectively. Consider

$$\varepsilon = \xi_2 - \xi_1 = (c_1 - c_2)t$$

(3.4)
with \( c_1 - c_2 > 0 \) whereby \( \xi_1 \) is the rightmost coordinate and \( \xi_2 \) is the leftmost coordinate.

Asymptotic expansions for \( t \to \pm \infty \) then correspond to asymptotic expansions given by \( \varepsilon \to \pm \infty \). In each expansion, we separately hold fixed the coordinates \( \xi_1 \) and \( \xi_2 \).

### 3.1. Moving-coordinate expansion of the Hirota oscillatory 2-soliton

We begin by holding the rightmost coordinate fixed, and expressing the leftmost coordinate in terms of the expansion parameter \( \varepsilon \) from equation (3.4), so thus

\[
\xi_1 = \text{const.}, \quad \xi_2 = \xi_1 + \varepsilon \to \pm \infty \quad (3.5)
\]

as \( \varepsilon \to \pm \infty \). In the Hirota oscillatory 2-soliton solution from Proposition 2, applying the expansion (3.5) to the functions (2.16), (2.17), (2.18) and neglecting subdominant terms, we find

\[
e^{i\kappa_1 \xi_1} V_{1H}(\xi_1, \xi_1 + \varepsilon) \sim e^{\frac{\varepsilon^2}{2k_1}} \left( \frac{1}{2} k_1 e^{\pm i\gamma_2 \varepsilon} e^{i\kappa_1 \xi_1} e^{\pm k_2 \xi_1 i} \right) + O(1) \quad (3.6)
\]

\[
e^{i\kappa_2(\xi_1 + \varepsilon)} V_{2H}(\xi_1, \xi_1 + \varepsilon) \sim O(1) \quad (3.7)
\]

\[
W_H(\xi_1, \xi_1 + \varepsilon) \sim e^{k_2|\varepsilon|} \left( \frac{1}{2} k_1 e^{\pm k_2 \xi_1} \left( \sqrt{1} e^{\mp k_1 \xi_1} + \frac{1}{\sqrt{1}} e^{\mp k_1 \xi_1} \right) \right) + O(1) \quad (3.8)
\]

(with \( k_2 > 0 \) due to equation (2.12)). Up to the exponential factor \( e^{\frac{\varepsilon^2}{2k_1}} e^{\pm k_2 \xi_1} \), the asymptotic functions (3.6) and (3.8) resemble the form of the numerator and denominator in the oscillatory 1-soliton solution (2.6) and (2.7) for the Hirota equation. Specifically, the expansion (3.8) can be expressed as

\[
e^{\pm k_2 \xi_1} W_H(\xi_1, \xi_1 + \varepsilon) \sim \cosh(k_1 \xi_1^\pm), \quad \varepsilon \to \pm \infty \quad (3.9)
\]

where

\[
\xi_1^\pm = \xi_1 \mp x_1, \quad x_1 = -\frac{\ln \Gamma}{2k_1} > 0 \quad (3.10)
\]

is a shifted moving coordinate. Then the expansion (3.6) can be written in terms of this coordinate, which gives

\[
e^{\pm k_2 \xi_1} e^{i\kappa_1 \xi_1} V_{1H}(\xi_1, \xi_1 + \varepsilon) \sim \frac{1}{2} k_1 e^{i(\pm \gamma_2 \pm \kappa_1 x_1)} e^{i\kappa_1 \xi_1^\pm}, \quad \varepsilon \to \pm \infty \quad (3.11)
\]

Hence we have

\[
e^{i\kappa_1 \xi_1} V_{1H}(\xi_1, \xi_1 + \varepsilon)/W_H(\xi_1, \xi_1 + \varepsilon) = \tilde{f}_{1H}(\xi_1, \xi_1 + \varepsilon) \sim e^{i(\pm \gamma_2 \pm \kappa_1 x_1)} e^{i\kappa_1 \xi_1^\pm} U_H(\xi_1^\pm), \quad \varepsilon \to \pm \infty \quad (3.12)
\]

where \( \tilde{f}_{1H} \) is the function (2.6) and \( U_H \) is the function (2.7). Finally, the phase factor \( e^{i(\pm \gamma_2 \pm \kappa_1 x_1)} \) can be combined with \( e^{i\phi_1} \) to get a shifted phase angle

\[
\phi_1^\pm = \phi_1 \pm \eta_1, \quad \eta_1 = \gamma_2 + \kappa_1 x_1. \quad (3.13)
\]

In a similar way, the expansion (3.7) gives

\[
e^{\pm k_2 \xi_1} e^{i\kappa_2(\xi_1 + \varepsilon)} V_{2H}(\xi_1, \xi_1 + \varepsilon) \sim 0, \quad \varepsilon \to \pm \infty \quad (3.14)
\]

whence

\[
e^{i\kappa_2(\xi_1 + \varepsilon)} V_{2H}(\xi_1, \xi_1 + \varepsilon)/W_H(\xi_1, \xi_1 + \varepsilon) = \tilde{f}_{2H}(\xi_1, \xi_1 + \varepsilon) \sim 0, \quad \varepsilon \to \pm \infty. \quad (3.15)
\]
Combining equations (3.12) and (3.15) with equation (2.15), we see that the asymptotic expansion of the Hirota oscillatory 2-soliton solution with respect to its leftmost coordinate is given by

\[ u(t, x) \sim e^{i\phi_1^+} e^{i\nu t} e^{i\kappa_1^+} U_H(\xi_1^+) = u_1^+(t, x), \quad \xi_1 = \text{const.,} \quad \varepsilon = \xi_2 - \xi_1 \to \pm \infty. \quad (3.16) \]

Next, we hold the leftmost coordinate fixed, and express the rightmost coordinate in terms of the expansion parameter \( \varepsilon \) from equation (3.4), so now

\[ \xi_2 = \text{const.}, \quad \xi_1 = \xi_2 - \varepsilon \to \mp \infty \quad (3.17) \]
as \( \varepsilon \to \pm \infty \). Applying this expansion (3.17) to the functions (2.16), (2.17), (2.18), we obtain

\[ e^{i\kappa_1(\xi_2 - \varepsilon)} V_{1H}(\xi_2 - \varepsilon, \xi_2) \sim O(1) \quad (3.18) \]
\[ e^{i\kappa_2\xi_2} V_{2H}(\xi_2 - \varepsilon, \xi_2) \sim e^{k_1|\varepsilon|} \left( \frac{1}{2} k_2 e^{i\eta_1} e^{i\kappa_2\xi_2} e^{i\kappa_1\xi_2} \right) + O(1) \quad (3.19) \]
\[ W_H(\xi_2 - \varepsilon, \xi_2) \sim e^{k_1|\varepsilon|} \left( \frac{1}{2} k_2 e^{i\eta_1} \left( \sqrt{1} e^{i\kappa_2\xi_2} + \frac{1}{\sqrt{1}} e^{i\kappa_2\xi_2} \right) \right) + O(1) \quad (3.20) \]

(with \( k_1 > 0 \) due to equation (2.12)). The expansions (3.20) and (3.19) can be expressed as

\[ e^{-k_1|\varepsilon|} e^{\pm k_1\xi_2} W_{1H}(\xi_2 - \varepsilon, \xi_2) \sim \cosh(k_2\xi_2^\pm), \quad \varepsilon \to \pm \infty \quad (3.21) \]

and

\[ e^{-k_1|\varepsilon|} e^{\pm k_1\xi_2} e^{i\kappa_2\xi_2} V_{2H}(\xi_2 - \varepsilon, \xi_2) \sim \frac{1}{2} k_2 e^{i(\mp \gamma_1 \pm k_2\xi_2)} e^{i\kappa_2\xi_2^\pm}, \quad \varepsilon \to \pm \infty \quad (3.22) \]

where

\[ \xi_2^\pm = \xi_2 \mp x_2, \quad x_2 = \frac{\ln \Gamma}{2k_2} < 0 \quad (3.23) \]
is a shifted moving coordinate. Hence we have

\[ e^{i\kappa_2\xi_2} V_{2H}(\xi_2 - \varepsilon, \xi_2)/W_{1H}(\xi_2 - \varepsilon, \xi_2) = \tilde{f}_{1H}(\xi_2 - \varepsilon, \xi_2) \sim e^{i(\mp \gamma_1 \pm k_2\xi_2)} e^{i\kappa_2\xi_2^\pm} U_H(\xi_2^\pm), \quad \varepsilon \to \pm \infty. \quad (3.24) \]

The phase factor \( e^{i(\mp \gamma_1 \pm k_2\xi_2)} \) can be combined with \( e^{i\phi_2} \) to get a shifted phase angle

\[ \phi_2^\pm = \phi_2 \pm \eta_2, \quad \eta_2 = -\gamma_1 + k_2 x_2. \quad (3.25) \]

Similarly, the expansion (3.18) gives

\[ e^{-k_1|\varepsilon|} e^{\pm k_1\xi_2} e^{i\kappa_1(\xi_2 - \varepsilon)} V_{1H}(\xi_2 - \varepsilon, \xi_2) \sim 0, \quad \varepsilon \to \pm \infty \quad (3.26) \]

whence

\[ e^{i\kappa_1(\xi_2 - \varepsilon)} V_{1H}(\xi_2 - \varepsilon, \xi_2)/W_H(\xi_2 - \varepsilon, \xi_2) = \tilde{f}_{1H}(\xi_2 - \varepsilon, \xi_2) \sim 0, \quad \varepsilon \to \pm \infty. \quad (3.27) \]

Combining equations (3.27) and (3.24) with equation (2.15), we see that the asymptotic expansion of the Hirota oscillatory 2-soliton solution with respect to its rightmost coordinate is given by

\[ u(t, x) \sim e^{i\phi_2^+} e^{i\nu_2 t} e^{i\kappa_2\xi_2^+} U_H(\xi_2^+) = u_2^+(t, x), \quad \xi_2 = \text{const.}, \quad \varepsilon = \xi_2 - \xi_1 \to \pm \infty. \quad (3.28) \]
3.2. Moving-coordinate expansion of the Sasa-Satsuma oscillatory 2-soliton. For
the Sasa-Satsuma oscillatory 2-soliton solution from Proposition 2, we apply the asymptotic
expansions (3.5) and (3.17) to the functions (2.19), (2.20), (2.21).

First using the expansion (3.5) and neglecting subdominant terms for \( \varepsilon \to \pm \infty \), we find

\[
\begin{align*}
    e^{ik_1 \xi} V_{\text{SS}}(\xi_1, \xi_1 + \varepsilon) &\sim e^{2k_2 |\varepsilon|} \left( k_1(k_1^2 + \kappa_1^2)^{1/4} |\kappa_1/2|^{1/2} |\kappa_2| e^{ik_1 \xi} e^{2k_2 \xi} \times 
    \left( \sqrt{\Delta \Gamma} e^{i(\alpha_2 + \gamma_2) \varepsilon} e^{k_1 \xi_1} + \frac{1}{\sqrt{\Delta \Gamma}} e^{-k_1 \xi_1} e^{i(\gamma_2 - \nu_2) \varepsilon} \right) \right) + O(e^{k_2 |\varepsilon|}) \\
    e^{-ik_2 (\xi_1 + \varepsilon)} V_{2\text{SS}}(\xi_1, \xi_1 + \varepsilon) &\sim O(e^{k_2 |\varepsilon|}) \\
    W_{\text{SS}}(\xi_1, \xi_1 + \varepsilon) &\sim e^{2k_2 |\varepsilon|} \left( |\kappa_2| e^{\pm 2k_2 \xi} \left( \frac{1}{2} |\kappa_1| \right) \left( \Delta \Gamma e^{\pm 2k_1 \xi} + \frac{1}{\Delta \Gamma} e^{\mp 2k_1 \xi} \right) 
    + (k_1^2 + \kappa_1^2)^{1/2} \right) + O(e^{k_2 |\varepsilon|}) 
\end{align*}
\]

(3.29) (3.30) (3.31)

(with \( k_2 > 0 \) due to equation (2.12)). The expansion (3.31) can be expressed as

\[
    e^{-2k_2 |\varepsilon|} e^{\mp 2k_2 \xi} W_{\text{SS}}(\xi_1, \xi_1 + \varepsilon) \sim |\kappa_2| \left( \cosh(2k_1 \xi^\pm) + (k_1^2 + \kappa_1^2)^{1/2} \right), \quad \varepsilon \to \pm \infty \quad (3.32)
\]

where

\[
    \xi^\pm = \xi_1 \mp x_1, \quad x_1 = -\frac{\ln(\Delta \Gamma)}{2k_1} > 0 \quad (3.33)
\]

is a shifted moving coordinate. Then writing the expansion (3.29) in terms of this coordinate,
and using the relations (2.27)–(2.28), we get

\[
\begin{align*}
    e^{-2k_2 |\varepsilon|} e^{\mp 2k_2 \xi_1} e^{ik_1 \xi} V_{\text{SS}}(\xi_1, \xi_1 + \varepsilon) \\
    \sim k_1(k_1^2 + \kappa_1^2)^{1/4} |\kappa_1/2|^{1/2} |\kappa_2| e^{\pm i(\gamma_2 + \frac{1}{4} \delta_2 + \kappa_1 x_1)} e^{ik_1 \xi^\pm} \cosh(k_1 \xi^\pm + \frac{1}{2} i \lambda_1), \quad \varepsilon \to \pm \infty. 
\end{align*}
\]

(3.34)

These functions (3.32) and (3.34) resemble the denominator and numerator in the oscillatory
1-soliton solution (2.6) and (2.8) for the Sasa-Satsuma equation. Specifically, we have

\[
    e^{ik_1 \xi} V_{\text{SS}}(\xi_1, \xi_1 + \varepsilon) / W_{\text{SS}}(\xi_1, \xi_1 + \varepsilon) = \tilde{f}_{\text{SS}}(\xi_1, \xi_1 + \varepsilon) \sim e^{\pm i(\gamma_2 + \frac{1}{4} \delta_2 + \kappa_1 x_1)} e^{ik_1 \xi^\pm} U_{\text{SS}}(\xi^\pm), \quad \varepsilon \to \pm \infty \quad (3.35)
\]

where \( \tilde{f}_{\text{SS}} \) is the function (2.6) and \( U_{\text{SS}} \) is the function (2.8). Finally, the phase factor
\( e^{\pm i(\gamma_2 + \frac{1}{4} \delta_2 + \kappa_1 x_1)} \) can be combined with \( e^{i\phi_1} \) to get a shifted phase angle

\[
    \phi_1^\pm = \phi_1 \pm \eta_1, \quad \eta_1 = \gamma_2 + \frac{1}{2} \delta_2 + \kappa_1 x_1. \quad (3.36)
\]

In a similar way, the expansion (3.30) gives

\[
    e^{-2k_2 |\varepsilon|} e^{\mp 2k_2 \xi_1} e^{ik_2 (\xi_1 + \varepsilon)} V_{2\text{SS}}(\xi_1, \xi_1 + \varepsilon) \sim 0, \quad \varepsilon \to \pm \infty \quad (3.37)
\]

whence

\[
    e^{ik_2 (\xi_1 + \varepsilon)} V_{2\text{SS}}(\xi_1, \xi_1 + \varepsilon) / W_{\text{SS}}(\xi_1, \xi_1 + \varepsilon) = \tilde{f}_{2\text{SS}}(\xi_1, \xi_1 + \varepsilon) \sim 0, \quad \varepsilon \to \pm \infty. \quad (3.38)
\]
Hence we have
\[ u(t, x) \sim e^{i\phi_2} e^{i\nu_1 t} e^{i\kappa_1 t} \mathcal{U}_{SS}(\xi_1^\pm) = u_1^\pm(t, x), \quad \xi_1 = \text{const.}, \quad \varepsilon = \xi_2 - \xi_1 \to \pm \infty. \] (3.39)

Next using the expansion (3.17) and neglecting subdominant terms for \( \varepsilon \to \pm \infty \), we find
\[ e^{i\nu_2(\xi_2 - \varepsilon)} V_{SS}(\xi_2 - \varepsilon, \xi_2) \sim O(\varepsilon^{k_1 |\varepsilon|}) \] (3.40)

\[ e^{i\kappa_2 \xi_2} V_{SS}(\xi_2 - \varepsilon, \xi_2) \sim e^{2k_1 |\varepsilon|} \left( k_2 (k_2^2 + \kappa_2^2)^{1/4} |\kappa_2/2|^{1/2} |\kappa_1| e^{i\kappa_2 \xi_2} e^{i\kappa_2 \xi_2} \times \left( \sqrt{\Delta \Gamma} e^{-i(\alpha_1 + \gamma_1)} e^{k_2 \xi_2} + \frac{1}{\sqrt{\Delta \Gamma}} e^{-k_2 \xi_2} e^{i(\nu_1 - \gamma_1)} \right) \right) + O(\varepsilon^{k_1 |\varepsilon|}) \] (3.41)

\[ W_{SS}(\xi_2 - \varepsilon, \xi_2) \sim e^{2k_1 |\varepsilon|} \left( |\kappa_1| e^{i\kappa_2 \xi_2} \left( \frac{1}{2} |\kappa_2| \left( \Delta \Gamma e^{i\kappa_2 \xi_2} + \frac{1}{\Delta \Gamma} e^{-i\kappa_2 \xi_2} \right) + (k_2^2 + \kappa_2^2)^{1/2} \right) \right) + O(\varepsilon^{k_1 |\varepsilon|}) \] (3.42)

(with \( k_1 > 0 \) due to equation (2.12)). We express the expansion (3.42) as
\[ e^{-2k_1 |\varepsilon|} e^{i\kappa_2 \xi_2} W_{SS}(\xi_2 - \varepsilon, \xi_2) \sim |\kappa_1| \left( \cosh(2k_2 \xi_2^\pm) + (k_2^2 + \kappa_2^2)^{1/2} \right), \quad \varepsilon \to \pm \infty \] (3.43)

where
\[ \xi_2^\pm = \xi_2 \mp x_2, \quad x_2 = \frac{\ln(\Delta \Gamma)}{2k_2} < 0 \] (3.44)

is a shifted moving coordinate. Then we write the expansion (3.41) in terms of this coordinate, and use the relations (2.27)–(2.28), giving
\[ e^{-2k_1 |\varepsilon|} e^{i\kappa_2 \xi_2} V_{SS}(\xi_2 - \varepsilon, \xi_2) \sim k_2 (k_2^2 + \kappa_2^2)^{1/4} |\kappa_2|^{1/2} |\kappa_1| e^{i\kappa_2 \xi_2} e^{i(i(\gamma_1 + \frac{1}{2} \delta_1 - \kappa_2 x_2))} e^{i\kappa_2 \xi_2^\pm} \cosh(k_2 \xi_2^\pm + \frac{1}{2} i \lambda_2), \quad \varepsilon \to \pm \infty. \] (3.45)

Hence we have
\[ e^{i\kappa_2 \xi_2} V_{SS}(\xi_2 - \varepsilon, \xi_2) / W_{SS}(\xi_2 - \varepsilon, \xi_2) = \tilde{f}_{SS}(\xi_2 - \varepsilon, \xi_2) \sim e^{i(i(\gamma_1 + \frac{1}{2} \delta_1 - \kappa_2 x_2))} e^{i\kappa_2 \xi_2^\pm} \mathcal{U}_{SS}(\xi_2^\pm), \quad \varepsilon \to \pm \infty. \] (3.46)

The phase factor \( e^{i(i(\gamma_1 + \frac{1}{2} \delta_1 - \kappa_2 x_2))} \) can be combined with \( e^{i\phi_2} \) to get a shifted phase angle
\[ \phi_2^\pm = \phi_2 \pm \eta_2, \quad \eta_2 = -\gamma_1 - \frac{1}{2} \delta_1 + \kappa_2 x_2. \] (3.47)

Similarly, the expansion (3.40) gives
\[ e^{-2k_1 |\varepsilon|} e^{i\kappa_2 \xi_2} V_{SS}(\xi_2 - \varepsilon, \xi_2) \sim 0, \quad \varepsilon \to \pm \infty \] (3.48)

whence
\[ e^{i\kappa_2 \xi_2} V_{SS}(\xi_2 - \varepsilon, \xi_2) / W_{SS}(\xi_2 - \varepsilon, \xi_2) = \tilde{f}_{SS}(\xi_2 - \varepsilon, \xi_2) \sim 0, \quad \varepsilon \to \pm \infty. \] (3.49)
Combining equations (3.46) and (3.49) with equation (2.15), we see that the asymptotic expansion of the Sasa-Satsuma oscillatory 2-soliton solution with respect to its rightmost coordinate is given by
\[ u(t, x) \sim e^{i\phi_2} e^{i\nu_2 t} e^{i\xi_2^2} U_{SS}(\xi_2^2) = u_2^+(t, x), \quad \xi_2 = \text{const.}, \quad \varepsilon = \xi_2 - \xi_1 \to \pm \infty. \quad (3.50) \]

3.3. **Asymptotic expansion for large time.** The precise correspondence between the moving coordinate expansion given by \( \varepsilon \to \pm \infty \) and an asymptotic expansion \( t \to \pm \infty \) will now be explained. In particular, through equations (3.3) and (3.4), we will determine how large \( |t| \) must so that the expansions (3.16) and (3.28) derived for the Hirota oscillatory 2-soliton and the expansions (3.39) and (3.50) derived for the Sasa-Satsuma oscillatory 2-soliton are approximately valid over some interval in \( x \) at a finite time \(-\infty < t < \infty\).

From equations (3.6)–(3.8) and equations (3.29)–(3.31), we see that the expansions (3.16) and (3.39) remain approximately valid if \( \pm k_2 \xi_2 \gg 1 \) holds, with \( k_1 |\xi_1^\pm| = O(1) \). These two conditions can be expressed explicitly as conditions on \( t, x \) after we use equations (3.3), (3.4), (3.10) and (3.33) to get
\[ \xi_2^\pm = \xi_1^\pm + x_1 + (c_1 - c_2) t. \]
Then the condition \( \pm k_2 \xi_2 \gg 1 \) gives
\[ \pm (\xi_1^\pm + (c_1 - c_2) t) \gg \frac{1}{k_2} - x_1 \quad (3.51) \]
while the other condition \( k_1 |\xi_1^\pm| = O(1) \) implies
\[ \pm \xi_1^\pm \gtrsim -\frac{1}{k_1}. \quad (3.52) \]
We now combine these two inequalities (3.51) and (3.52), yielding
\[ \pm t \gg \frac{1}{k_2} + \frac{1}{k_1} - x_1 \quad (3.53) \]
which determines the minimum size of \( t \). Finally, from inequality (3.52), we have
\[ c_1 t \pm x_1 - \frac{1}{k_1} \lesssim x \lesssim c_1 t \pm x_1 + \frac{1}{k_1} \quad (3.54) \]
which determines the interval in which \( x \) lies. These are the conditions on \( t, x \) under which the expansions (3.16) and (3.39) approximately hold, giving
\[ u(t, x) \sim e^{i\phi_1^\pm} e^{i\nu_1 t} \tilde{f}(x - c_1 t \mp x_1) = u_1^\pm(t, x). \quad (3.55) \]

There are similar conditions for the expansions (3.16) and (3.39) to hold approximately from equations (3.18)–(3.20) and equations (3.40)–(3.42), yielding
\[ u(t, x) \sim e^{i\phi_2^\pm} e^{i\nu_2 t} \tilde{f}(x - c_2 t \mp x_2) = u_2^\pm(t, x). \quad (3.56) \]
We see that these equations remain approximately valid if \( \mp k_1 \xi_1 \gg 1 \) holds, with \( k_2 |\xi_2^\pm| = O(1) \). By using equations (3.3) and (3.4), we get \( \xi_1 = \xi_2^\pm - x_2 - (c_1 - c_2) t \). The condition \( \mp k_1 \xi_1 \gg 1 \) thereby gives
\[ \mp (\xi_2^\pm - (c_1 - c_2) t) \gg \frac{1}{k_1} + x_2 \quad (3.57) \]
while the other condition \( k_2 |\xi_2^\pm| = O(1) \) implies
\[ \mp \xi_2^\pm \gtrsim -\frac{1}{k_2}. \quad (3.58) \]
Then, combining these two inequalities (3.57) and (3.58), we obtain

$$\pm t \gg \frac{1}{k_2} + \frac{1}{k_1} + x_2$$

(3.59)

which determines the minimum size of \( t \). Then from inequality (3.58), we have

$$c_2 t \pm x_2 - \frac{1}{k_2} \lesssim x \lesssim c_2 t \pm x_2 + \frac{1}{k_2}$$

(3.60)

which determines the interval in which \( x \) lies.

An important observation now is that the approximate expansions (3.55) and (3.56) will hold simultaneously if \( t \) satisfies both conditions (3.53) and (3.59). Since equations (3.10), (3.23), (3.33), (3.44) show that \( x_1 > 0 \) and \( x_2 < 0 \) in all cases, a simple sufficient condition on \( t \) is given by

$$|t| \gg \frac{k_1 + k_2}{k_1 k_2 (c_1 - c_2)}.$$  

(3.61)

Another useful observation is that the previous analysis holds independently of the signs of \( c_1 \) and \( c_2 \), including cases when one of \( c_1 \) or \( c_2 \) is zero. Hence, we have established the following results.

**Lemma 1.** For \( t \) satisfying the condition (3.61), the Hirota oscillatory 2-soliton solution (2.14), (2.16)–(2.18) with parameters \( \phi_1, \phi_2, \nu_1, \nu_2, c_1 > c_2 \) has the form of an asymptotic superposition \( u \simeq u_1^\pm + u_2^\pm \) in which \( u_1^\pm \) and \( u_2^\pm \) are distinct oscillatory waves having respective speeds \( c_1 \) and \( c_2 \), temporal frequencies \( \nu_1 \) and \( \nu_2 \), phase angles \( \phi_1^\pm \) and \( \phi_2^\pm \) given by expressions (3.13) and (3.25), and having positions that are determined by the respective moving coordinates (3.10) and (3.23).

**Lemma 2.** For \( t \) satisfying the condition (3.61), the Sasa-Satsuma oscillatory 2-soliton solution (2.14), (2.19)–(2.21) with parameters \( \phi_1, \phi_2, \nu_1, \nu_2, c_1 > c_2 \) has the form of an asymptotic superposition \( u \simeq u_1^\pm + u_2^\pm \) in which \( u_1^\pm \) and \( u_2^\pm \) are distinct oscillatory waves having respective speeds \( c_1 \) and \( c_2 \), temporal frequencies \( \nu_1 \) and \( \nu_2 \), phase angles \( \phi_1^\pm \) and \( \phi_2^\pm \) given by expressions (3.36) and (3.47), and having positions that are determined by the respective moving coordinates (3.33) and (3.44).

When these oscillatory 2-soliton solutions for the Hirota equation and the Sasa-Satsuma equation have either \( c_1 = 0 \) or \( c_2 = 0 \), then the respective asymptotic wave \( u_1^\pm \) or \( u_2^\pm \) as \( t \to \pm \infty \) is a standing wave.

4. **Constants of Motion**

For the Hirota equation (2.1) and the Sasa-Satsuma equation (2.2), we recall that the conserved integrals defining momentum, energy, and Galilean energy are given by [9] (up to
arbitrary normalization factors)

\[ \mathcal{P} = \int_{-\infty}^{+\infty} |u|^2 \, dx \]  \hspace{1cm} (4.1)

\[ \mathcal{E} = \int_{-\infty}^{+\infty} 3(|u_x|^2 - 4|u|^4) \, dx \]  \hspace{1cm} (4.2)

\[ \mathcal{C} = \int_{-\infty}^{+\infty} 3t(|u_x|^2 - 4|u|^4) - x|u|^2 \, dx \]  \hspace{1cm} (4.3)

which yield constants of motion for all smooth solutions \( u(t, x) \) with sufficiently rapid decay \( u \to 0 \) as \( x \to \pm \infty \). These integrals are related to the center of momentum defined by

\[ \mathcal{X}(t) = \frac{1}{\mathcal{P}} \int_{-\infty}^{+\infty} x|u|^2 \, dx = \mathcal{X}(0) + \frac{\mathcal{E}}{\mathcal{P}} t \]  \hspace{1cm} (4.4)

where

\[ \mathcal{C} = t\mathcal{E} - \mathcal{P}\mathcal{X}(t) = \mathcal{C}(0) = -\mathcal{P}\mathcal{X}(0). \]  \hspace{1cm} (4.5)

This is the same relation that holds for the corresponding constants of motion of the mKdV equation [7].

The Hirota equation admits an additional conserved integral given by the angular twist [9] (up to an arbitrary normalization factor)

\[ \mathcal{W} = \int_{-\infty}^{+\infty} \text{Re} \left( iu\bar{u}_x \right) \, dx = -i \int_{-\infty}^{+\infty} |u|^2 \arg(u)_x \, dx \]  \hspace{1cm} (4.6)

holding for all smooth solutions \( u(t, x) \) with sufficiently rapid decay \( u \to 0 \) as \( x \to \pm \infty \). This integral is not conserved for the Sasa-Satsuma equation.

It is straightforward to evaluate these constants of motion explicitly for the oscillatory 1-soliton solutions from Proposition 1 for the Hirota equation and the Sasa-Satsuma equation. For notional convenience we will denote

\[ \beta_\pm = 3\sqrt{\frac{c}{3}} \pm \frac{\nu}{2} \pm \nu/2. \]  \hspace{1cm} (4.7)

**Theorem 1.** The Hirota oscillatory 1-soliton (2.5), (2.7) has angular twist, momentum, energy, and Galilean energy given by

\[ \mathcal{W} = \frac{1}{2}kk = \sqrt{3}(\beta_-^2 - \alpha_+^2)/8 \]  \hspace{1cm} (4.8)

\[ \mathcal{P} = \frac{1}{2}k = \sqrt{3}(\beta_- + \alpha_+)/4 \]  \hspace{1cm} (4.9)

\[ \mathcal{E} = \frac{1}{2}k(k^2 - 3\kappa^2) = \sqrt{3}(\beta_- + \alpha_+)c/4 \]  \hspace{1cm} (4.10)

\[ \mathcal{C} = 0 \]  \hspace{1cm} (4.11)

The Sasa-Satsuma oscillatory 1-soliton (2.5), (2.8) has momentum, energy, and Galilean energy given by

\[ \mathcal{P} = k = \sqrt{3}(\beta_- + \alpha_+)/2 \]  \hspace{1cm} (4.12)

\[ \mathcal{E} = k(k^2 - 3\kappa^2) = \sqrt{3}(\beta_- + \alpha_+)c/2 \]  \hspace{1cm} (4.13)

\[ \mathcal{C} = 0 \]  \hspace{1cm} (4.14)

In both cases, the center of momentum is \( \mathcal{X}(t) = ct \) with \( c = \mathcal{E}/\mathcal{P} \).
We note that the center of momentum of these oscillatory 1-solitons can be shifted arbitrarily by means of a space translation \( x \rightarrow x - x_0 \) applied to the moving coordinate \( \xi = x - ct \) in equation (2.5), which leads to

\[
\mathcal{X}(t) = x_0 + ct.
\]

(4.15)

This changes the Galilean energy

\[
\mathcal{C} = -x_0 \mathcal{P}
\]

(4.16)

while the momentum and energy are unchanged.

From the previous expressions, we can evaluate the momentum, energy, and Galilean energy of the oscillatory 2-soliton solutions from Proposition 2 for the Hirota equation and the Sasa-Satsuma equation. In particular, we know from Lemmas 1 and 2 that each solution \( u \sim u_1^\pm + u_2^\pm \) is asymptotically a superposition of two waves \( u_1^\pm \) and \( u_2^\pm \) as \( t \rightarrow \pm \infty \). Hence the conserved integrals (4.1), (4.2), (4.3) are respectively given by a sum of the momenta \( \mathcal{P}_1, \mathcal{P}_2 \), the energies \( \mathcal{E}_1, \mathcal{E}_2 \), and the Galilean energies \( \mathcal{C}_1, \mathcal{C}_2 \) associated with each individual wave. This yields the following result, using the notation (2.11).

**Theorem 2.** The Hirota oscillatory 2-soliton (2.14)–(2.15), (2.16)–(2.18) has angular twist, momentum, energy, and Galilean energy given by

\[
\mathcal{W} = \mathcal{W}_1 + \mathcal{W}_2 = \frac{1}{2} \kappa_1 k_1 + \frac{1}{2} \kappa_2 k_2 = \sqrt{3}(\beta_{1-}^2 - \beta_{1+}^2 + \beta_{2-}^2 - \beta_{2+}^2)/8
\]

(4.17)

\[
\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 = \frac{1}{2} k_1 + \frac{1}{2} k_2 = \sqrt{3}(\beta_{1-} + \beta_{1+} + \beta_{2-} + \beta_{2+})/4
\]

(4.18)

\[
\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = \frac{1}{2} k_1(k_1^2 - 3\kappa_1^2) + \frac{1}{2} k_2(k_2^2 - 3\kappa_2^2) = \sqrt{3}((\beta_{1-} + \beta_{1+})c_1 + (\beta_{2-} + \beta_{2+})c_2)/4
\]

(4.19)

\[
\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 = \mp(\frac{1}{2} k_1 x_1 + \frac{1}{2} k_2 x_2) = 0
\]

(4.20)

where \( x_1 \) and \( x_2 \) are given by equations (3.10) and (3.23). The Sasa-Satsuma oscillatory 2-soliton (2.14)–(2.15), (2.19)–(2.21) has momentum, energy, and Galilean energy given by

\[
\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2 = k_1 + k_2 = \sqrt{3}(\beta_{1-} + \beta_{1+} + \beta_{2-} + \beta_{2+})/2
\]

(4.21)

\[
\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = k_1(k_1^2 - 3\kappa_1^2) + k_2(k_2^2 - 3\kappa_2^2) = \sqrt{3}((\beta_{1-} + \beta_{1+})c_1 + (\beta_{2-} + \beta_{2+})c_2)/2
\]

(4.22)

\[
\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 = \mp(k_1 x_1 + k_2 x_2) = 0
\]

(4.23)

where \( x_1 \) and \( x_2 \) are given by equations (3.33) and (3.44). In both cases,

\[
\mathcal{X}(t) = \frac{\mathcal{E}}{\mathcal{P}} t
\]

(4.24)

is the center of momentum, which moves at constant speed

\[
c = \frac{\mathcal{E}}{\mathcal{P}} = \frac{\mathcal{P}_1 c_1 + \mathcal{P}_2 c_2}{\mathcal{P}_1 + \mathcal{P}_2} = \frac{(\beta_{1-} + \beta_{1+})c_1 + (\beta_{2-} + \beta_{2+})c_2}{\beta_{1-} + \beta_{1+} + \beta_{2-} + \beta_{2+}}.
\]

(4.25)

5. **Position shifts and phase shifts**

In the asymptotic past \( t \rightarrow -\infty \) and future \( t \rightarrow \infty \), the Hirota and Sasa-Satsuma oscillatory 2-soliton solutions given in Proposition 2 reduce to a superposition \( u \sim u_1^\pm + u_2^\pm \) of oscillatory 1-solitons \( u_1^\pm \) and \( u_2^\pm \) having speeds \( c_1, c_2 \), temporal frequencies \( \nu_1, \nu_2 \), phase angles \( \phi_1^\pm, \phi_2^\pm \), and having centers of momentum \( \chi_1^\pm(t) = c_1 t \pm x_1, \chi_2^\pm(t) = c_2 t \pm x_2 \), with
\( c_1 \neq c_2 \). Without loss of generality, we will assume \( c_1 > c_2 \) hereafter, since \( u \) is symmetric under simultaneously interchanging \( c_1 \leftrightarrow c_2, \nu_1 \leftrightarrow \nu_2, \phi_1 \leftrightarrow \phi_2 \).

We begin by examining some properties of the asymptotic oscillatory 1-solitons

\[
\begin{align*}
    u_1^\pm &= \exp(i\phi_1^\pm)\exp(i\nu_1 t)\exp(i\kappa_1 \xi_1^\pm)U_1(\xi_1^\pm) \\
    u_2^\pm &= \exp(i\phi_2^\pm)\exp(i\nu_2 t)\exp(i\kappa_2 \xi_2^\pm)U_2(\xi_2^\pm)
\end{align*}
\]

where

\[
\begin{align*}
    \xi_1^\pm &= x - c_1 t \mp x_1, \quad \xi_2^\pm &= x - c_2 t \mp x_2
\end{align*}
\]

are shifted moving coordinates, and where both \( U_1 \) and \( U_2 \) are given by the envelope function (2.7) in the Hirota case and (2.8) in the Sasa-Satsuma case. First, the functions are shifted moving coordinates, and where both momentum are symmetric around \( \kappa_1 \) in the Hirota case \( (2.8) \) in the Sasa-Satsuma case. First, the functions for the two asymptotic oscillatory waves. Second, at these positions, the phase of both has exponential decay

\[
|u_1^\pm| = |U_1| \sim O(\exp(-k_1|\xi_1^\pm|)), \quad |\xi_1^\pm| \gg 1/k_1
\]

while their phase has linear behaviour

\[
\begin{align*}
    \arg(u_1^\pm) &= \phi_1^\pm + \nu_1 t + \kappa_1 \xi_1^\pm + \arg(U_1) \sim \psi_1 + \phi_1^\pm + \nu_1 t + \kappa_1 \xi_1^\pm, \quad |\xi_1^\pm| \gg 1/k_1 \\
    \arg(u_2^\pm) &= \phi_2^\pm + \nu_2 t + \kappa_2 \xi_2^\pm + \arg(U_2) \sim \psi_2 + \phi_2^\pm + \nu_2 t + \kappa_2 \xi_2^\pm, \quad |\xi_2^\pm| \gg 1/k_2
\end{align*}
\]

where \( \psi_1 = \psi_2 = 0 \) in the Hirota case, and \( \psi_1 = \text{sgn}(\xi_1^\pm)\arg(\kappa_1(\kappa_1 + ik_1)), \psi_2 = \text{sgn}(\xi_2^\pm)\arg(\kappa_2(\kappa_2 + ik_2)) \) in the Sasa-Satsuma case. For graphical and analytical purposes, it will be more useful to work with the envelope phase of the two waves (5.1).

We recall that the envelope phase of an oscillatory wave \( u = \exp(i\phi)\exp(i\nu t)\exp(i\kappa \xi)U(\xi) \) expressed in terms of a moving coordinate \( \xi = x - ct - \chi_0 \), with phase angle \( \phi \), temporal frequency \( \nu \), speed \( c \), and center of momentum \( \chi(t) = ct + \chi_0 \), is defined by \[ \varphi(u) = \arg(u) - \kappa x - (\nu - \kappa c)t = \phi - \kappa \chi_0 + \arg(U) \]. Applied to the asymptotic oscillatory waves (5.1), this yields the envelope phases

\[
\begin{align*}
    \varphi(u_1^\pm) &= \arg(U_1) + \phi_1^\pm \mp \kappa_1 x_1 \sim \psi_1 + \phi_1^\pm \mp \kappa_1 x_1, \quad |\xi_1^\pm| \gg 1/k_1 \\
    \varphi(u_2^\pm) &= \arg(U_2) + \phi_2^\pm \mp \kappa_2 x_2 \sim \psi_2 + \phi_2^\pm \mp \kappa_2 x_2, \quad |\xi_2^\pm| \gg 1/k_2
\end{align*}
\]

which approach constant values away from the center of momentum positions (5.3). Exactly at the centers of momentum, the envelope phases are simpler, due to the phase property (5.4) which yields

\[
\begin{align*}
    \varphi(u_1^\pm)|_{\xi_1^\pm=0} &= \phi_1^\pm \mp \kappa_1 x_1 = \varphi_1^\pm, \quad \varphi(u_2^\pm)|_{\xi_2^\pm=0} = \phi_2^\pm \mp \kappa_2 x_2 = \varphi_2^\pm.
\end{align*}
\]
As \( t \to \pm \infty \), the asymptotic positions (5.3) of the two oscillatory waves (5.1) lie on straight lines in the \((t,x)\)-plane, with the lines \( x = \chi_1^-(t) \) and \( x = \chi_2^+(t) \) being each shifted relative to the lines \( x = \chi_1^+(t) \) and \( x = \chi_2^-(t) \) by a constant value

\[
\Delta x_1 = \chi_1^+(t) - \chi_1^-(t) = 2x_1, \quad \Delta x_2 = \chi_2^+(t) - \chi_2^-(t) = 2x_2. \tag{5.9}
\]

Likewise, the asymptotic phase angles of the two oscillatory waves are each shifted by a constant value

\[
\Delta \phi_1 = \phi_1^+ - \phi_1^-, \quad \Delta \phi_2 = \phi_2^+ - \phi_2^- . \tag{5.10}
\]

Hence the envelope phases also undergo shifts

\[
\Delta \varphi_1 = \varphi(u_1^+) - \varphi(u_1^-) = \Delta \phi_1 - \kappa_1 \Delta x_1 = \phi_1^+ - \phi_1^- \\
\Delta \varphi_2 = \varphi(u_2^+) - \varphi(u_2^-) = \Delta \phi_2 - \kappa_2 \Delta x_2 = \phi_2^+ - \phi_2^- \tag{5.11}
\]

which are determined entirely by the asymptotic shifts (5.9) and (5.10).

From Lemmas 1 and 2, we have the following expressions for the shifts (5.9) and (5.11).

**Theorem 3.** For \( t \to \pm \infty \) in the Hirota oscillatory 2-soliton (2.14), (2.16)–(2.18), with \( c_1 > c_2 \), the asymptotic soliton with speed \( c_1 \) and temporal frequency \( \nu_1 \) undergoes a shift in position and envelope phase given by

\[
\Delta x_1 = \frac{1}{k_1} \ln \left( \frac{(k_1 + k_2)^2 + (\kappa_1 - \kappa_2)^2}{(k_1 - k_2)^2 + (\kappa_1 - \kappa_2)^2} \right) > 0 \tag{5.12}
\]

\[
\Delta \varphi_1 = -2 \arg \left( (k_1 + k_2)(k_1 - k_2) + (\kappa_1 - \kappa_2)^2 + i2k_2(\kappa_1 - \kappa_2) \right) \tag{5.13}
\]

while the asymptotic soliton with speed \( c_2 \) and temporal frequency \( \nu_2 \) undergoes a shift in position and envelope phase given by

\[
\Delta x_2 = -\frac{1}{k_2} \ln \left( \frac{(k_1 + k_2)^2 + (\kappa_1 - \kappa_2)^2}{(k_1 - k_2)^2 + (\kappa_1 - \kappa_2)^2} \right) < 0 \tag{5.14}
\]

\[
\Delta \varphi_2 = -2 \arg \left( (k_1 + k_2)(k_1 - k_2) - (\kappa_1 - \kappa_2)^2 + i2k_1(\kappa_1 - \kappa_2) \right) \tag{5.15}
\]

where \( k_1, k_2, \kappa_1, \kappa_2 \) are given in terms of \( c_1, c_2, \nu_1, \nu_2 \) by equations (2.9)–(2.11).

**Theorem 4.** For \( t \to \pm \infty \) in the Sasa-Satsuma oscillatory 2-soliton (2.14), (2.19)–(2.21), with \( c_1 > c_2 \), \( \nu_1 \neq 0 \) and \( \nu_2 \neq 0 \), the asymptotic soliton with speed \( c_1 \) and temporal frequency \( \nu_1 \) undergoes a shift in position and envelope phase given by

\[
\Delta x_1 = \frac{1}{k_1} \ln \left( \frac{(k_1 + k_2)^2 + (\kappa_1 - \kappa_2)^2}{(k_1 - k_2)^2 + (\kappa_1 - \kappa_2)^2} \right) \sqrt{\frac{(k_1 + k_2)^2 + (\kappa_1 + \kappa_2)^2}{(k_1 - k_2)^2 + (\kappa_1 + \kappa_2)^2}} > 0 \tag{5.16}
\]

\[
\Delta \varphi_1 = -2 \arg \left( (k_1 + k_2)(k_1 - k_2) + (\kappa_1 - \kappa_2)^2 + i2k_2(\kappa_1 - \kappa_2) \right) \tag{5.17}
\]

\[+ \arg \left( (k_1 + k_2)(k_1 - k_2) + (\kappa_1 + \kappa_2)^2 + i2k_2(\kappa_1 + \kappa_2) \right) \]

while the asymptotic soliton with speed \( c_2 \) and temporal frequency \( \nu_2 \) undergoes a shift in position and envelope phase given by

\[
\Delta x_2 = -\frac{1}{k_2} \ln \left( \frac{(k_1 + k_2)^2 + (\kappa_1 - \kappa_2)^2}{(k_1 - k_2)^2 + (\kappa_1 - \kappa_2)^2} \right) \sqrt{\frac{(k_1 + k_2)^2 + (\kappa_1 + \kappa_2)^2}{(k_1 - k_2)^2 + (\kappa_1 + \kappa_2)^2}} < 0 \tag{5.18}
\]

\[
\Delta \varphi_2 = -2 \arg \left( (k_1 + k_2)(k_1 - k_2) - (\kappa_1 - \kappa_2)^2 + i2k_1(\kappa_1 - \kappa_2) \right) \tag{5.19}
\]

\[+ \arg \left( (k_1 + k_2)(k_1 - k_2) - (\kappa_1 + \kappa_2)^2 - i2k_1(\kappa_1 + \kappa_2) \right) \]
where \( k_1, k_2, \kappa_1, \kappa_2 \) are given in terms of \( c_1, c_2, \nu_1, \nu_2 \) by equations (2.9)–(2.11).

For both the Hirota and Sasa-Satsuma oscillatory 2-solitons, as \( t \to \pm \infty \) the centers of momentum and the phase angles of the two asymptotic oscillatory waves (5.1) are given by

\[
\chi_1^\pm(t) = c_1 t \pm \frac{1}{2} \Delta x_1, \quad \chi_2^\pm(t) = c_2 t \pm \frac{1}{2} \Delta x_2
\]

and

\[
\phi_1^\pm = \phi_1 \pm \frac{1}{2} \Delta \phi_1, \quad \phi_2^\pm = \phi_2 \pm \frac{1}{2} \Delta \phi_2.
\]

5.1. Oscillatory wave collisions. The Hirota and Sasa-Satsuma oscillatory 2-soliton solutions (2.14)–(2.15) describe a collision between two asymptotic oscillatory waves with speeds \( c_1 > c_2 \) (or \( c_1 < c_2 \)). The collision is a right-overtake if \( c_1 > c_2 \geq 0 \) (or \( c_2 > c_1 \geq 0 \)), a left-overtake if \( 0 \geq c_1 > c_2 \) (or \( 0 \geq c_2 > c_1 \)), and a head-on if \( c_1 > 0 \geq c_2 \) (or \( c_2 > 0 \geq c_1 \)). As will be now illustrated, in all cases the net effect of the collision is only to shift the asymptotic position and asymptotic phase angle of each wave, where these shifts are given in Theorems 3 and 4.

The positions shifts are seen graphically in the asymptotic amplitude of the 2-soliton solution, since for large \(|t|\) we have

\[
|u| \sim \begin{cases} |u_1^\pm| = |U_1(\xi_1^\pm)|, \quad x \simeq \chi_1^\pm(t) \\ |u_2^\pm| = |U_2(\xi_2^\pm)|, \quad x \simeq \chi_2^\pm(t) \end{cases}
\]

from Lemmas 1 and 2, where \( \xi_1^\pm \) and \( \xi_2^\pm \) are the shifted moving coordinates (5.2) which determine the positions (5.3) of the two asymptotic oscillatory waves, and where \( U_1 \) and \( U_2 \) are the envelope functions for these waves, given in Proposition 1. Similarly, we have

\[
\arg(u) \sim \begin{cases} \arg(u_1^\pm) = \varphi(u_1^\pm) + \kappa_1 x + (\nu_1 - c_1 \kappa_1)t, \quad x \simeq \chi_1^\pm(t) \\ \arg(u_2^\pm) = \varphi(u_2^\pm) + \kappa_2 x + (\nu_2 - c_2 \kappa_2)t, \quad x \simeq \chi_2^\pm(t) \end{cases}
\]

yielding the asymptotic phase of the 2-soliton solution.

To see the phase shifts graphically, it is useful to remove the asymptotic linear part of \( \arg(u) \) by defining an envelope phase for the 2-soliton solution in analogy to the definition for oscillatory waves [8]. Consider, for a 2-soliton solution \( u \),

\[
\varphi(u) = \arg(u) - (\kappa_1 x + (\nu_1 - c_1 \kappa_1)t)\theta_1 + (\kappa_2 x + (\nu_2 - c_2 \kappa_2)t)\theta_2
\]

where

\[
\theta_1 = |V_1|/(|V_1| + |V_2|), \quad \theta_2 = |V_2|/(|V_1| + |V_2|)
\]

are interpolation functions which satisfy the properties

\[
\theta_1 + \theta_2 = 1
\]

\[
\theta_1 \sim \begin{cases} 1, & x \simeq \chi_1^\pm(t) \\ 0, & x \simeq \chi_2^\pm(t) \end{cases}
\]

\[
\theta_2 \sim \begin{cases} 1, & x \simeq \chi_2^\pm(t) \\ 0, & x \simeq \chi_1^\pm(t) \end{cases}
\]
with $V_1$ and $V_2$ given by the envelope functions in Proposition 2. From these properties, we then have

$$
\varphi(u) \sim \begin{cases} 
\varphi(u_1^+), & x \simeq \chi_1^+(t) \\
\varphi(u_2^+), & x \simeq \chi_2^+(t) 
\end{cases} \quad (5.29)
$$

whence the envelope phase of $u$ matches the envelope phase (5.8) of each asymptotic oscillatory wave. We remark that there is a lot of freedom in the choice of interpolation functions (5.25) (for example, replacing $|V_1|$ and $|V_2|$ by some power like $|V_1|^2$ and $|V_2|^2$), however, the qualitative features of the envelope phase can be expected to remain the same.

The amplitude and envelope phase of the oscillatory 2-soliton solutions (2.14)–(2.15) are illustrated in Fig. 1–Fig. 6 for the Hirota case, and in Fig. 7–Fig. 12 for the Sasa-Satsuma case.

![Figure 1](image1)

**Figure 1.** Hirota oscillatory 2-soliton right-overtake with $c_1 = 4$, $c_2 = 2$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$

![Figure 2](image2)

**Figure 2.** Hirota oscillatory 2-soliton right-overtake (in black) and oscillatory 1-solitons (in red and blue) with $c_1 = 4$, $c_2 = 2$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$
Figure 3. Hirota oscillatory 2-soliton left-overtake with $c_1 = -2$, $c_2 = -4$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$

Figure 4. Hirota oscillatory 2-soliton left-overtake (in black) and oscillatory 1-solitons (in red and blue) with $c_1 = -2$, $c_2 = -4$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$

Figure 5. Hirota oscillatory 2-soliton head-on with $c_1 = 4$, $c_2 = -2$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$
Figure 6. Hirota oscillatory 2-soliton head-on (in black) and oscillatory 1-solitons (in red and blue) with $c_1 = 4$, $c_2 = -2$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$

Figure 7. Sasa-Satsuma oscillatory 2-soliton right-overtake with $c_1 = 4$, $c_2 = 2$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$

Figure 8. Sasa-Satsuma oscillatory 2-soliton right-overtake (in black) and oscillatory 1-solitons (in red and blue) with $c_1 = 4$, $c_2 = 2$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$
Figure 9. Sasa-Satsuma oscillatory 2-soliton left-overtake with $c_1 = -2$, $c_2 = -4$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$

Figure 10. Sasa-Satsuma oscillatory 2-soliton left-overtake (in black) and oscillatory 1-solitons (in red and blue) with $c_1 = -2$, $c_2 = -4$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$

Figure 11. Sasa-Satsuma oscillatory 2-soliton head-on with $c_1 = 4$, $c_2 = -2$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$
5.2. **Position shifts in oscillatory wave collisions.** In a right-overtake collision with \( c_1 > c_2 > 0 \), the asymptotic solitons \( u_1^\pm \) and \( u_2^\pm \) are oscillatory waves that each move to the right, where \( u_1^\pm \) is the faster wave and \( u_2^\pm \) is the slower wave. The effect of the collision on the asymptotic positions of these waves is to shift the fast wave forward (i.e. to the right, since \( \Delta x_1 > 0 \)) and the slow wave backward (i.e. to the left, since \( \Delta x_2 < 0 \)). Similarly, in a left-overtake collision with \( 0 > c_1 > c_2 \), the asymptotic solitons \( u_1^\pm \) and \( u_2^\pm \) are oscillatory waves that each move to the left, where now \( u_1^\pm \) is the slower wave and \( u_2^\pm \) is the faster wave. The collision affects the asymptotic positions of the two waves by shifting the fast wave forward (i.e. to the left, since \( \Delta x_2 < 0 \)) and the slow wave backward (i.e. to the right, since \( \Delta x_1 > 0 \)). In contrast, in a head-on collision with \( c_1 > 0 > c_2 \), the asymptotic soliton \( u_1^\pm \) is a right-moving oscillatory wave while the other asymptotic soliton \( u_2^\pm \) is a left-moving oscillatory wave. The collision has the effect that the asymptotic positions of both waves are shifted forward relative to their directions of motion, since the right-moving wave undergoes a shift to the right (due to \( \Delta x_1 > 0 \)) and the left-moving wave undergoes a shift to the left (due to \( \Delta x_2 < 0 \)).

In all cases, the asymptotic positions shifts \( \Delta x_1 \) and \( \Delta x_2 \) satisfy the algebraic relation

\[
 k_1 \Delta x_1 + k_2 \Delta x_2 = 0
\]

which holds as a direct consequence of the oscillatory 2-soliton solution having a center of momentum that moves at a constant speed, as shown by equation (4.24).

5.3. **Standing waves.** An oscillatory 1-soliton (1.5) with \( c = 0 \) and \( \nu \neq 0 \) is a time-periodic standing wave. The standing wave solutions for the Hirota equation (2.1) and the Sasa-Satsuma equation (2.2) are presented in oscillatory form in Proposition 1.

Collisions of an oscillatory wave with a standing wave are described by the oscillatory 2-soliton (2.14) when \( c_1 = 0 \), \( c_2 \neq 0 \), \( \nu_1 \neq 0 \), or when \( c_2 = 0 \), \( c_1 \neq 0 \), \( \nu_2 \neq 0 \). These collision solutions for the Hirota and Sasa-Satsuma equations are special cases of the solutions presented in Proposition 2 and are shown respectively in Fig. 13–Fig. 14 and Fig. 15–Fig. 16. They have not previously appeared in the literature.
Figure 13. Hirota 2-soliton collision of a left-moving oscillatory wave and a standing wave with $c_1 = 0$, $c_2 = -4$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$

Figure 14. Hirota 2-soliton collision of a right-moving oscillatory wave and a standing wave with $c_1 = 4$, $c_2 = 0$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$

Figure 15. Sasa-Satsuma collision of a left-moving oscillatory soliton and a standing wave with $c_1 = 0$, $c_2 = -4$, $\nu_1 = 2$, $\nu_2 = 5$, $\varphi_1^- = 0$, $\varphi_2^- = \pi/2$
We remark that Theorems 3 and 4 hold for collisions of an oscillatory wave and a standing wave. Thus, in a right-overtake collision with $c_1 > c_2 = 0$, the effect of the collision on the asymptotic positions of the waves is to shift the right-moving asymptotic oscillatory wave $u_1^\pm$ in a forward direction (i.e. to the right, since $\Delta x_1 > 0$) while the asymptotic standing wave $u_2^\pm$ is displaced in the opposite direction (i.e. to the left, since $\Delta x_2 < 0$). Similarly, in a left-overtake collision with $0 = c_1 > c_2$, the collision affects the asymptotic positions of the two waves by shifting the left-moving asymptotic oscillatory wave $u_2^\pm$ in a forward direction (i.e. to the left, since $\Delta x_2 < 0$) while the asymptotic standing wave $u_1^\pm$ is displayed in the opposite direction (i.e. to the right, since $\Delta x_1 > 0$). In both cases, the asymptotic positions and the asymptotic phase angles of the waves are given by equations (5.20)–(5.21). The position and phase shifts are shown in Fig. 17–Fig. 18 for the Hirota equation, and Fig. 19–Fig. 20 for the Sasa-Satsuma equation.
In previous work [7], collisions of ordinary solitary waves (1.4) (i.e. with no temporal harmonic modulation) have been studied for the Hirota equation (2.1) and the Sasa-Satsuma equation (2.2). A collision in this case consists of a right-moving faster solitary wave with
speed $c_1$ and phase $\phi_1$ overtaking a right-moving slower solitary wave with speed $c_2$ and phase $\phi_2$. The corresponding ordinary 2-soliton solutions exhibit several distinguishing properties. First, at a particular time $t = t_0$ the amplitude displays invariance $|u(t_0, x - \chi(t_0))| = |u(t_0, \chi(t_0) - x)|$ under spatial reflection around the center of momentum $x = \chi(t)$ of the two solitary waves. This time $t = t_0$ can be understood to represent the moment of greatest nonlinear interaction of the waves during the collision. Second, the amplitude is always non-zero, $|u(t, x)| \neq 0$, throughout the collision. As a consequence of these two properties, the interaction of the two waves can be characterized primarily by the convexity of $|u(t_0, x)|$ at the center of momentum $x = \chi(t_0)$ at time $t = t_0$. The case of negative convexity describes a collision such that the waves undergo a merge-split interaction in which $|u(t_0, x)|$ has a single peak with an exponentially decreasing tail, while the case of positive convexity describes a collision such that the waves exhibit either a bounce-exchange interaction in which $|u(t_0, x)|$ has a double peak with an exponentially decreasing tail, or an absorb-emit interaction in which $|u(t_0, x)|$ has a pair of side peaks around a central peak and an exponentially decreasing tail, depending on the speed ratio and relative phase angle of the two waves, as explained in Ref. [7].

In contrast, collisions of oscillatory waves (1.5) described by the 2-soliton solutions from Proposition 2 have very different features (animations can be seen at http://lie.math.brocku.ca/~sanco/solitons/oscillatory.php):

1. the amplitude $|u|$ exhibits invariance under spatial reflections only in special cases;
2. the amplitude $|u|$ vanishes at certain positions $x$ and times $t$ (i.e. $u$ has nodes);
3. the phase arg($u$) exhibits rapid spatial change at certain positions $x$ and times $t$ (i.e. $u$ has phase coils with large spatial winding);
4. the phase gradient arg($u_x$) changes sign at certain positions $x$ and times $t$ (i.e. $u$ has spatial reversals of phase winding).

A detailed study of the interactions of oscillatory waves for these equations will be presented in a sequel paper. Our work in the present paper has two immediate extensions. First, the Hirota equation (2.1) and the Sasa-Satsuma equation (2.2) are known to be gauge-equivalent to third-order NLS equations [13]

$$q_{\tilde{t}} \pm i\sqrt{v/3}(3q_{\tilde{x}\tilde{x}} + \alpha|q|^2 q) + \alpha|q|^2 q_{\tilde{x}} + \beta(|q|^2)_{\tilde{x}} q + q_{\tilde{x}\tilde{x}\tilde{x}} = 0$$

(6.1)

through the Galilean-phase transformation

$$\tilde{t} = t, \quad \tilde{x} = x + vt, \quad u(t, x) = q(\tilde{t}, \tilde{x}) \exp \left( \pm i\sqrt{v/3}(\tilde{x} - (2v/3)\tilde{t}) \right)$$

(6.2)

where $v > 0$ is a speed parameter, with $\alpha = 24$, $\beta = 0$ in the Hirota case and $\alpha = 12$, $\beta = 6$ in the Sasa-Satsuma case. Under this transformation, the oscillatory 1-solitons (2.5)–(2.8) shown in Proposition 1 for the Hirota and Sasa-Satsuma equations correspond to NLS solitons of the same form

$$q(\tilde{t}, \tilde{x}) = \exp(i\phi) \exp(i\tilde{v}\tilde{t}) \tilde{q}(\tilde{\xi}), \quad \tilde{\xi} = \tilde{x} - \tilde{c}\tilde{t}$$

(6.3)

parameterized by a speed $\tilde{c}$, a temporal frequency $\tilde{v}$, and a phase $\phi$, where

$$\tilde{c} = c + v, \quad \tilde{v} = v \pm \sqrt{v/3}(c + 4v/3), \quad \tilde{q}(\tilde{\xi}) = \exp(\mp i\sqrt{v/3} \tilde{\xi}) \tilde{f}(\tilde{\xi})$$

(6.4)

Consequently, we obtain NLS oscillatory 2-solitons

$$q(\tilde{t}, \tilde{x}) = \exp(i\phi_1) \exp(i\tilde{v}_1\tilde{t}) \tilde{q}_1(\tilde{\xi}_1, \tilde{\xi}_2) + \exp(i\phi_2) \exp(i\tilde{v}_2\tilde{t}) \tilde{q}_2(\tilde{\xi}_1, \tilde{\xi}_2)$$

(6.5)
with
\[ \tilde{\xi}_1 = \tilde{x} - \tilde{c}_1 \tilde{t}, \quad \tilde{\xi}_2 = \tilde{x} - \tilde{c}_2 \tilde{t}, \]
\[ \tilde{c}_1 = c_1 + v, \quad \tilde{c}_2 = c_2 + v, \]
\[ \tilde{\nu}_1 = \nu_1 + \sqrt{v/3}(c_1 + 4v/3), \quad \tilde{\nu}_2 = \nu_2 + \sqrt{v/3}(c_2 + 4v/3). \]

where the functions \( \tilde{f}_1(\tilde{\xi}_1, \tilde{\xi}_2) \) and \( \tilde{f}_2(\tilde{\xi}_1, \tilde{\xi}_2) \) are given in Proposition 2 for the oscillatory 2-solitons (2.14)–(2.21) of the Hirota and Sasa-Satsuma equations. In addition, we obtain NLS oscillatory breathers in the special case \( c_1 = c_2 \neq 0 \), discussed in Ref. [7].

The main results stated in Theorems 3 and 4 on the properties of collisions described by oscillatory 2-solitons, carry over directly to the third-order NLS equation (6.1). In particular, the net effect of a collision is to shift the asymptotic positions and phases of the individual oscillatory waves while the speed and the temporal frequency of each wave remains unchanged, such that the center of momentum of the waves is preserved in the collision.

Second, the Hirota equation (2.1) has two natural multi-component generalizations given by \( U(N) \)-invariant integrable mKdV equations [14]
\[ \tilde{u}_t + 12(\tilde{u} \tilde{u}_x + (\tilde{u}_x \cdot \tilde{u}) \tilde{u}) + \tilde{u}_{xxx} = 0 \]
(6.10)
and
\[ \tilde{u}_t + 24(\tilde{u} \tilde{u}_x + (\tilde{u}_x \cdot \tilde{u}) \tilde{u}) - (\tilde{u}_x \cdot \tilde{u}) \tilde{u} + \tilde{u}_{xxx} = 0 \]
(6.11)
where \( \tilde{u}(t, x) \) is a \( N \)-component complex vector variable. For all \( N \geq 2 \), these two vector equations admit vector oscillatory wave solutions of the form
\[ \tilde{u}(t, x) = \exp(\mp i\sqrt{v/3} \tilde{\xi}_1) \tilde{f}_1(\tilde{\xi}_1, \tilde{\xi}_2), \]
\[ \tilde{u}(t, x) = \exp(\mp i\sqrt{v/3} \tilde{\xi}_2) \tilde{f}_2(\tilde{\xi}_1, \tilde{\xi}_2), \]
(6.9)
with wave speed \( c \) and temporal frequency \( \nu \), satisfying the kinematic relation (3.1), where \( \psi \) is an arbitrary constant complex unit vector and \( \tilde{f}_H \) is the complex envelope function (2.7) for the oscillatory 1-soliton solution of the scalar Hirota equation (2.1). In forthcoming work, we plan to generalize the results in the present paper to study the vector oscillatory 2-soliton solutions and vector oscillatory breather solutions of both equations (6.10) and (6.11).

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