Covariants of \( SL_4(\mathbb{C}) \) and incidence geometry of \( \mathbb{P}^3(\mathbb{C}) \)

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Abstract. We determine a minimal generating set for the algebra of covariants of an arbitrary number of points, lines and planes in three-dimensional projective space. Moreover, we discuss the corresponding minimal set of geometric incidence relations.

1. Introduction

For linear subspaces of the complex projective space \( \mathbb{P}^3(\mathbb{C}) \), one can formulate incidence relations such as 'point \( x \) lies on line \( l' \) or 'plane \( \mathcal{E} \) and line \( l \) intersect in one point' et cetera. We can translate these relations into the vanishing (or nonvanishing) of covariants of \( SL_4(\mathbb{C}) \) such as 'point \( x \) nonzero if and only if \( dim(\mathcal{E}) \). For linear subspaces of the complex projective space \( \mathbb{P}^3(\mathbb{C}) \)

Consider the exterior algebra \( \Lambda(\mathbb{C}^4) \) and let us denote the exterior product inside this algebra by \( \bigwedge \) instead of \( \wedge \) for geometrical reasons. Then a \( k \)-dimensional linear subspace \( A \) of \( \mathbb{C}^4 \) with basis \( x_1, \ldots, x_k \) is identified - by slight abuse of notation - with \( A := x_1 \bigwedge \ldots \bigwedge x_k \in \Lambda^k(\mathbb{C}^4) \). We call such elements of \( \Lambda(\mathbb{C}^4) \) extensors (of step \( k \)). Moreover, we call the exterior product \( A \bigwedge B \) of two extensors the join, as it is an extensor being nonzero if and only if \( dim(A \cap B) = 0 \) and in this case it corresponds to the span of \( A \) and \( B \). We identify extensors of step four with scalars and denote them by the familiar bracket notation \( [x_1 \cdots x_4] := x_1 \bigwedge \ldots \bigwedge x_4 \) of the determinant. To represent the intersection of subspaces \( A = a_1 \bigwedge \ldots \bigwedge a_j \) and \( B = b_1 \bigwedge \ldots \bigwedge b_k \), we define the operation \( A \bigwedge B \), called the meet, by

\[
A \bigwedge B := \sum_{(a',a'') \vdash (a_1 \cdots a_j)} \text{sgn}(\vdash) [a'_1 \cdots a'_{4-k} b_1 \cdots b_k] \cdot a''_1 \bigwedge \ldots \bigwedge a''_{j+k-4},
\]

where \( (a',a'') \vdash (a_1 \cdots a_j) \) means that we sum over all decompositions of \( a_1 \cdots a_j \) in two ordered subwords \( a',a'' \) and \( \text{sgn}(\vdash) \) is the sign of the underlying permutation \( a \mapsto a'a'' \). Then \( A \bigwedge B \) is nonzero if and only if \( A \) and \( B \) span \( \mathbb{C}^4 \) and in this case it corresponds to the intersection of the subspaces \( A \) and \( B \). We call \( \Lambda(\mathbb{C}^4) \) the Grassmann-Cayley-Algebra. For more details we refer to [1, 2]. The Grassmann-Cayley-Algebra is a fundament for quantum logic, see [3, 4, 5].

Now identify \( k \)-dimensional linear subspaces of \( \mathbb{C}^4 \) with \( (k-1) \)-dimensional linear-projective subspaces of \( \mathbb{P}^3(\mathbb{C}) \). The incidence relations from above stay untouched under the natural action of \( SL_4 \) on \( \mathbb{C}^4 \), they are invariant. This is to say, they must be given by polynomials in their homogeneous coordinates - invariant under the action of \( SL_4 \). The invariants are polynomial expressions in the brackets from above, but there are invariant functions, the covariants, that are not scalar-valued but tensor valued. One can see them as 'incomplete invariants' (like the meet \( A \bigwedge B \) from above) that can be completed to bracket expressions by generic subspaces of
\( \mathbb{P}^3(\mathbb{C}) \). An example of an invariant is the determinant \([x_1 x_2 x_3 x_4]\) that is zero if and only if the points \(x_1, \ldots, x_4\) lie in a common plane. An example of a covariant would be \([x_1 x_2 * *]\) := \(x_1 \lor x_2\), which is zero if and only if \(x_1 = x_2\), being equivalent to the condition that for a generic ‘dummy’ line \(l\) the dimension of the span \(x_1 \lor x_2 \lor l\) is smaller than three, i.e. the bracket \([x_1 x_2]\) vanishes.

Incidence relations and covariants of \(\text{SL}_n\) are on the one hand important in classical mechanics, rigidity of joint-bar-body-hinge-frameworks \([6, 7, 8]\), with applications in such fields as computer aided geometric design \([9]\), machine learning \([10]\), robotics \([11]\) and biology \([12, 13]\). On the other hand, they as well come into play in quantum mechanics, entanglement and quantum information theory \([5, 14]\).

From the mathematical perspective, all geometric incidence relations can be formulated by means of covariants \([1, \text{Eq. 3.3.11}]\) and these form - due to Hilbert’s famous work \([15]\) - a finitely generated algebra. Thus the classical invariant theoretic task of finding a minimal generating set for the algebra of invariants or covariants becomes important from the geometric point of view as it equals a minimal set of geometric relations that suffices to describe all geometric relations. The determination of such minimal generating set has been done by the author in \([16]\) for the algebra of invariants of arbitrary points, lines and planes in \(\mathbb{P}^3(\mathbb{C})\) and for the algebra of invariants of arbitrary points, lines, planes and hyperplanes in \(\mathbb{P}^4(\mathbb{C})\). The work \([16]\) uses a hypergraph approach relying on the bracket notation to denote the invariants. Loosely speaking, to each bracket we associate a vertex of degree \(n + 1\) (where \(n = 3, 4\) for the above cases respectively) and to each extensor of step \(k\) a \(k\)-edge - a hyperedge with \(k\) ‘arms’ - that is connected to a vertex if and only if a basis element of the corresponding projective subspace appears in the respective bracket. The Plücker relations that hold between brackets become relations between the graphs and can together with graph theory be used to identify minimal generating sets. See \([17]\) for a somewhat dual approach for invariants of binary forms.

In Section 2 of the present work, we derive from these invariants a minimal generating set for the algebra of covariants in the case \(\mathbb{P}^3(\mathbb{C})\), which due to \([18]\) corresponds to a set of subgraphs of the invariant graphs. To formalize this, in analogy to the dummy line \(l\) from above we introduce dummy edges (of gray color) corresponding to generic extensors so that each vertex of a subgraph by adding dummy edges is still of degree four. The minimal generating set of covariants is given in Table 1.

In Section 3, finally, we give geometric relations corresponding to the vanishing of covariants. It is no problem to study Table 1 with the understanding from this introduction and proceed directly to Section 3 for the geometric meaning of the covariants.

2. Covariants of \(\text{SL}_4\)

The purpose of this section is to provide a rigorous foundation for the following theorem, giving a minimal generating set of the algebra of covariants of points, lines and planes in \(\mathbb{P}^3(\mathbb{C})\). Nevertheless, the rough definition of the graphs from the introduction is perfectly enough to understand their meaning. We give a slightly more formal way of thinking about the graphs before stating the theorem: let \(e_1^\star, \ldots, e_4^\star\) be the standard basis of \((\mathbb{C}^4)^\ast\). To each vertex associate a tensor \(e_1^\star \lor \cdots \lor e_4^\star \in \Lambda^4(\mathbb{C}^4)\) while for each \(k\)-edge choose an extensor of step \(k\). Take the tensor product of all these tensors and for each connection of an edge to a vertex connect a corresponding pair of indices. The result is the value of the covariant on the respective extensors.

**Theorem 2.1.** A minimal generating set for the covariants of points, lines and planes in \(\mathbb{P}^3(\mathbb{C})\) is given in Table 1.

**Remark 2.2.** We will refer to the entry in the \(i\)-th row and \(j\)-th column as \(I_{ij}\). For an actual configuration of \(n_1\) points, \(n_2\) lines and \(n_3\) planes, one can extract a minimal generating set from Table 1 by taking all combinations of points, lines and planes for the 1-, 2- and 3-edges with respect to the restriction that \(I_{ij}\) is antisymmetric in \(k\)-edges - for all but \(I_{22}\), which is symmetric.
Table 1. Covariant graphs of SL$_4$.

| 1-edges | 2-edges | 3-edges | (1, 2)-edges | (2, 3)-edges | (1, 3)-edges | (1, 2, 3)-edges |
|---------|---------|---------|--------------|--------------|--------------|-----------------|
| ![Graphs](image.png) | ![Graphs](image.png) | ![Graphs](image.png) | ![Graphs](image.png) | ![Graphs](image.png) | ![Graphs](image.png) | ![Graphs](image.png) |

in 2-edges, $I_{64}$, which is symmetric in 1-edges, and $I_{65}$, which is symmetric in 3-edges. That is to say, if for example we consider three points, five lines and four planes, there will be no covariants $I_{41}$ and $I_{62}$ in a minimal generating set and exactly $\binom{5}{2}$ of type $I_{22}$, five of type $I_{32}$, $\binom{5}{3} \cdot 6$ of type $I_{64}$ et cetera.

Now we give the formal background for proving Theorem 2.1. This is based on the invariant case from [16], which we refer to for a more detailed exposition. The foundation is the symbolic method for antisymmetric tensors from [19]. We adopt from there a slightly different definition of the bracket than the one from the introduction, suited for invariants of antisymmetric tensors. Let $n_1, n_2, n_3 \in \mathbb{N}$ as well as $V := \mathbb{C}^4$ and $V_{k,j} := \Lambda^k(V)$. Consider the SL$_4$-module

$$W_{(n_1,n_2,n_3)} := \bigoplus_{j=1}^{n_1} V_{1,j} \oplus \bigoplus_{j=1}^{n_2} V_{2,j} \oplus \bigoplus_{j=1}^{n_3} V_{3,j},$$

where the action of SL$_4$ is induced by the standard action on $\mathbb{C}^4$. The notation $V_{k,j}$ is used to distinguish different copies of $\Lambda^k(V)$ (or different extensors of the same step $k$). To each $V_{k,j}$ associate an infinite supply of letters $a_{k,j,i}$. Choose a total order on the set of these letters.

A (skew) bracket, which we denote by $[\ast \ast \ast \ast]$, contains four of these letters. A bracket monomial is a product of brackets, such that if a letter $a_{k,j,i}$ appears in the bracket monomial, it does so $k$ times, maybe in different brackets. A bracket polynomial is a linear combination of bracket monomials. Now we impose the following identities of bracket monomials: first, a
bracket is symmetric in its entries, i.e.

$$[abcd] = [\sigma(a)\sigma(b)\sigma(c)\sigma(d)]$$

for each permutation $\sigma$ of $\{a, b, c, d\}$. Second, we have the exchange identity from [19, p. 60]:

$$\sum_{(u',w'')=u} [u'v][u''w] = (-1)^{d-[w]} \sum_{(v',v'')=v} [u'v][v''w],$$

where $u, v, w$ are words in the letters $a_{k,j,i}$ and all summands with brackets containing more than four letters are set to zero - which is the case for example if $|u| \geq 5$, where the right side will be zero. We will refer to this relation as the Plücker relation.

Observe that there is no 'sgn(−)' so at least for letters $a_{1,j,i}$, the relation is kind of an 'unsigned version' of the original Plücker relation. The correct signs appear when we assign polynomial invariants of the action of $\text{SL}_4$ on $W_{(n_1,n_2,n_3)}$ to the bracket monomials. We do this in the following way: inside each bracket, arrange the letters according to the total order from above. Consider an element $\sum t_{k,j}$ of $W_{(n_1,n_2,n_3)}$, where $t_{k,j} \in V_{j-1}$. Take the tensor product of tensors $e_1^* \vee \ldots \vee e_4^*$ for each bracket in the monomial and $t_{k,j}$ for each letter $a_{k,j,i}$ (not for each appearance of the letter) in the monomial. The image of $\sum t_{k,j}$ in $\mathbb{C}$ by the polynomial map associated to the bracket monomial is the contraction of this tensor, where the $m$-th index of $t_{k,j}$ is contracted with the $n$-th index of some $e_j^* \vee \ldots \vee e_i^*$ if and only if the $m$-th appearance of the associated letter $a_{k,j,i}$ in the bracket monomial is at the $n$-th position of the corresponding bracket.

The resulting linear map $U$ (called the umbral operator in [19]) from the bracket algebra to $\mathbb{C}[W]^*\text{SL}_4$ is surjective. So a minimal generating set of the bracket algebra modulo ker($U$) is in one-to-one-correspondence to a minimal generating set of $\mathbb{C}[W]^*\text{SL}_4$.

Now what we did in [16] was replacing the bracket algebra by an algebra of hypergraphs as sketched in the introduction. The exact way is the following: identify bracket monomials with graphs so that for each bracket the graph has a vertex and for each letter $a_{k,j,i}$ it has a $k$-edge of color $j$ and shading $i$ connected to a vertex if and only if $a_{k,j,i}$ appears in the corresponding bracket. So for fixed $k$, the different colors $j$ of $k$-edges allow to distinguish between different appearances of $V^k$ and for fixed $k$ and $j$, the shadings allow to distinguish between different appearances of $V^k$ in the tensor product to be contracted. Since it only affects signs, we will omit the shading from now on, compare [16]. Now extend this identification linearly to bracket polynomials.

When we speak of applying the Plücker relation to some specific edges, we mean the relation between graphs coming from the Plücker relation between brackets with the word $u$ corresponding to these specific edges. If it is not clear which edges correspond to $v$ and $w$ respectively, we will say we pull the edges to the vertex $V$, where $V$ corresponds to the bracket containing the word $v$. So on the one hand, we have Plücker relations between graphs, on the other hand we have graph theoretical methods available now. The following example shows how these fit together and can be used to determine sets of generators.

**Example 2.3.** Consider $W = W_{(0,n_2,0)}$, i.e. $W$ only consists of copies of $\Lambda^2V$. The corresponding hypergraphs only have 2-edges, so are true graphs. We want to find a minimal generating set of the graph algebra. Let $\Gamma$ be an arbitrary such graph. If $\Gamma$ has a vertex with two looping edges - which corresponds to a bracket $[aabb]$ - then either this is the only vertex of $\Gamma$ or $\Gamma$ is disconnected and equals the product of two distinct graphs. So we can assume $\Gamma$ has no such vertex. If now there is a vertex with no looping edge at all, then the Plücker relation applied to any edge connected to this vertex results in $\Gamma$ being equal to a sum of graphs with a looping edge at this vertex and no looping edges of other vertices affected. Applying this procedure to every vertex, we can assume that each vertex has a looping edge. The subgraph with these looping edges removed must be a cycle, thus $\Gamma$ equals a sum of cycles of the form
Now one can show by applying the Plücker relation to a looped and 3-edges interchanged differ from \( \Gamma \) by a disconnected graph. Thus when considering a generating set, one can assume that these cycles are antisymmetric in the edges. We actually found a generating set, the same as the one in [20], where brackets have been used, no graphs.

Which brings us to the point that this generating set is by no means minimal. In fact, a minimal generating set only consists of the vertices with two looping edges and the cycles with three vertices, compare \( I \) and \( \Lambda^3V \equiv V^\ast \), see Example 2.3, the cycle \( I_{63} \) can be assumed antisymmetric in the edges, see [23] to complete Shmelkin’s classification from [22] of representations of \( \text{SL}_n \).

We now have all we need to prove Theorem 2.1. But let us first formalize the statement what we mean by covariant graphs of points, lines and planes in \( \mathbb{P}^3(\mathbb{C}) \): that is, a set of general graphs, from which for a fixed arbitrary number of \( n_1 \) points, \( n_2 \) lines and \( n_3 \) planes by suitably coloring 1-, 2- and 3-edges with colors \( 1, \ldots, n_1, 1, \ldots, n_2, 1, \ldots, n_3 \) respectively, one can extract a minimal generating set of \( \text{Cov}(W_{(n_1,n_2,n_3)})^{\text{SL}_4} \). Such set is often called a set of typical covariants, see [24].

**Proof of Theorem 2.1.** A minimal generating set of the algebra of invariants is given in [16, Thm. 1.1]. These are exactly the graphs from Table 1 without dummy edges. Moreover, all graphs with one dummy edge and one vertex must be included in a minimal generating set. By duality of \( V \) and \( \Lambda^3V \equiv V^\ast \), we have to include subgraphs \( I_{15} \) - which is dual to \( I_{14} \) - as well.

Now to extract a minimal generating set from all possible remaining subgraphs, we use the already mentioned fact that we can evaluate the graph at \( e_1 \lor \ldots \lor e_k \) for each dummy \( k \)-edge. The resulting polynomials are exactly the algebra of invariants \( \mathbb{C}[W]^U \) for a maximal unipotent subgroup \( U \) of \( \text{SL}_n \), see [21].
one or two dummy 1-edges - three in total. We have the analogous statement for subgraphs of the dual graph $I_{65}$.

Only subgraphs of $I_{27}$ and $I_{47}$ remain. These graphs are antisymmetric in the 2-edges, so we have to include subgraphs with one dummy 1-, 2-, or 3-edge and those with a dummy 1- and a dummy 3-edge. The resulting set of covariants is generating due to the arguments above. It is minimal since the set of invariants from [16, Thm. 1.1] is minimal: if one of the covariant graphs could be expressed as a polynomial in the others by Plücker relations, the same would hold for the respective invariant graphs with dummy edges replaced by real edges.

3. Geometric relations given by covariants

In this section, we give geometric interpretations of vanishing of the covariants from Table 1. We show how Grassmann-Cayley-Algebra expressions can be translated directly into the (skew) brackets - with $k$ identic letters standing for an extensor of step $k$ - and graphs from Section 2 and how this is the more immediate way compared to e.g. the approach from [1] using $k$ different letters corresponding to a basis of a $k$-extensor. To illustrate this, we begin with the following example:

Example 3.1. Consider the case of lines in $\mathbb{P}^2(\mathbb{C})$. In [1, Ex. 3.3.3], the Grassmann-Cayley-Algebra expression for the condition that three pairwise different lines $l_1, l_2, l_3$ in $\mathbb{P}^2(\mathbb{C})$ meet in one point, which is

$$(l_1 \land l_2) \land l_3 = 0,$$

is translated in a bracket algebra expression in the following way: assume each $l_i$ is spanned by two points $x_i, y_i$, then the left hand side of the above condition becomes

$$((x_1 \lor y_1) \land (x_2 \lor y_2)) \land (x_3 \lor y_3).$$

By the definition of join and meet, this can be translated into the bracket expression

$$[x_1y_1x_2][y_2x_3y_3] - [x_1y_1y_2][x_2x_3y_3].$$

This is at least unsatisfying as far as we had to choose a basis for the lines. But we can do better. Associate letters $a_i$ to the $l_i$ as proposed in Section 2. Now using the definition of the skew bracket, the first expression in the $l_i$ is translated into $[a_1a_1a_2][a_2a_3a_3]$ or equivalently and even more self-evidently into the graph

\[\begin{array}{c}
1 \\
\circ \\
3 \\
\circ \\
2
\end{array}\]

This graph as well as the bracket exactly corresponds to the determinant in the coordinates of the $l_i$, which is exactly what we want and in fact reflects the underlying geometry: the lines $l_i$ meet in one point if and only if the corresponding planes in $\mathbb{C}^3$ meet in one line. This is the case if and only if the normal vectors of these planes, which are nothing else than the coordinate vectors of the extensors $l_i$, are linearly dependent, i.e. if their determinant vanishes.

We move to $\mathbb{P}^3(\mathbb{C})$ again. The translation from Grassmann-Cayley-Algebra expressions into bracket expressions and graphs can be done by the definitions of meet and join, adapted to the definitions of skew brackets from Section 2. Let $C$ be an expression in the Grassmann-Cayley-Algebra involving only $\lor$ and $\land$, no addition.

Then for $A \lor B$ in $C$, where $A$ and $B$ are extensors of step $k_A$ and $k_B$, either $k_A + k_B > 4$, which means that the expression is zero by definition, or replace $A \lor B$ by the bracket $[a_A \cdots a_A a_B \cdots a_B * \cdots *]$ containing $k_A$ and $k_B$ times the letters $a_A$ and $a_B$ respectively. On
the other hand, for $A \wedge B$, either $k_A + k_B < 4$ - then the expression is zero by definition - or replace it by

$$
\left( \frac{k_B}{4 - k_A} \right) [a_A \cdots a_A a_B \cdots a_B] [a_B \cdots a_B * \cdots *],
$$

with $k_A$ and $4 - k_A$ times the letters $a_A$ and $a_B$ in the first and $k_A + k_B - 4$ times the letter $a_B$ in the second bracket. The prefactor here is due to the 'implicit shuffles' of the letters $a_B$ with $A$

means that the respective covariant vanishes if and only if the extensor is equal to zero, i.e. the light gray transparent planes are auxiliary planes, while all other points, lines and planes represent one of the edges of the covariant graph. The dashing in the fist three entries of Table 2, thus our condition equals $\left[ w_1 \cdots w_n \right][v_1] \cdots [v_m][wv* \cdots *].$

In the first case, either $|w| + |v| = 4$, then again the expression is equal to zero (a special case of this is one of $|w|$ and $|v|$ being equal to four, i.e. one of the bracket monomials is not a covariant but an invariant), or we can replace it by

$$
\left[ w_1 \cdots w_n \right][v_1] \cdots [v_m][wv* \cdots *].$
$$

In the second case, either $|w| + |v| < 4$ - the expression is equal to zero - or replace it by

$$
\sum_{(v',v'') \vdash v} \text{shuff}_{v',v''} \cdot \left[ w_1 \cdots w_n \right][v_1] \cdots [v_m][wv''][v''* \cdots *],
$$

where the prefactor $\text{shuff}_{v',v''}$ counts the number of shuffles resulting in the same decomposition $(v', v'')$ of $v$. Thus if $v$ only consists of one letter, it becomes the binomial prefactor from above.

Iterating this procedure leads to a bracket polynomial and from this to a graph. The reverse procedure is more involved and there is no general algorithm available, see [1, Sec. 3.5], but in most cases of the covariants from Table 1, the corresponding Grassmann-Cayley-Algebra expressions are obvious. We now give an example of how generic subspaces come into play. From now on we let letters in brackets corresponding to extensors be the same as the naming letter of the respective subspace, but in standard math-italic font.

**Example 3.2.** We want to find a Grassmann-Cayley-Algebra expression, a bracket polynomial and a graph that detect if two lines $l_1, l_2$ in $\mathbb{P}^3(\mathbb{C})$ are identical. This property is equivalent to saying their span has dimension one. But we have no immediate criterion for this in the Grassmann-Cayley-Algebra. The operations therein more or less can only 'detect' if two lines intersect or not. But we have a criterion for a point $x$ lying on a line $l$, namely $x \lor l = 0$ or $[xll*] = 0$ equivalently. So we can reformulate the condition ‘$l_1$ and $l_2$ are identical’ by ‘a generic point of $l_1$ lies on $l_2$’. We can create generic points on $l_1$ by the meet of $l_1$ with a generic plane $\mathcal{C}$, thus our condition equals

$$
0 = (\mathcal{C} \land l_1) \lor l_2 = [EELL_1][l_1l_2l_2*]
$$

If we now replace the star in the second bracket by a generic point $x$ on $\mathcal{C}$, we still have the same condition and the corresponding graph is entry $I_{32}$ from Table 1:

$$
\begin{array}{c}
\begin{array}{c}
\text{2}
\end{array}
\end{array}
\text{\bigcirc}
\text{\bigoplus}
$$

In Table 2, we list configurations for which the covariants from the first four rows of Table 1 vanish. The light gray transparent planes are auxiliary planes, while all other points, lines and planes represent one of the edges of the covariant graph. The dashing in the fist three entries means that the respective covariant vanishes if and only if the extensor is equal to zero, i.e.
Table 2. Configurations of subspaces for vanishing covariants.

| points | lines | planes | pts. & lns. | lns. & pns. | pts. & pns. | pts. & lns. & pns. |
|--------|-------|--------|-------------|-------------|-------------|-------------------|
| ![Configuration](image1.png) | ![Configuration](image2.png) | ![Configuration](image3.png) | ![Configuration](image4.png) | ![Configuration](image5.png) | ![Configuration](image6.png) | ![Configuration](image7.png) |

One sees that naturally the more edges the graphs have, the more complicated the configurations become. For example in the case of $I_{4,7}$ we have a point $x$, four lines $l_1, \ldots, l_4$ and a plane $E$. The configuration for vanishing $I_{4,7}$ described in words would be that (1) the intersection point of the line $l_4$ and the plane $E$, (2) the intersection point of the line $l_2$ and the plane through the line $l_1$ and the point $x$, and (3) the line $l_3$ - all three lie on a common plane.

For this reason, we do not list configurations for all covariants from Table 1, but in all cases but for the cycles $I_{7,2}$ and $I_{8,2}$, it is possible to derive a Grassmann-Cayley-Algebra expression by starting at an 'end-vertex' and reversing the process from above, i.e. replacing subsets of brackets with elementary $\lor$- and $\land$-operations. In the case of $I_{8,2}$, the reference [25] provides a geometric interpretation: the invariant $I_{8,2}$ vanishes, if the six lines form a line complex, which means they either have a common transversal or are reciprocal to a common screw. Then $I_{7,2}$ vanishes if the five lines and a generic sixth line either all have a common transversal (which only happens when they intersect in one point) or are reciprocal to a common screw.

Let us finish with a comment on [1, Ex. 3.4.5, 3.4.6]. In these examples, bracket expressions for four and five lines having a common transversal are derived. These expressions are invariants, and since the only generators for the algebra of invariants for five or less lines are those of type $I_{2,2}$ - with one invariant of this type for each pair of lines - they can be described only by means of $I_{2,2}$. This can be seen directly in the formula from [1, Thm. 3.4.7] but is not so obvious in the case of the formula for four lines on page 106.

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