SMALL DATA GLOBAL REGULARITY FOR NAVIER-STOKES-SCHRÖDINGER MAP SYSTEM IN THREE DIMENSIONS

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Abstract. In this article, we prove global regularity and scattering for Navier-Stokes-
Schrödinger map system with small and localized data in three dimensions. We use Coulomb
gauge to rewrite the system, and then use Fourier analysis and vector fields method to prove
global solution and scattering.

1. Introduction

In this paper, we consider the Navier-Stokes-Schrodinger map initial-value problem
\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla P &= \Delta u - \text{div}(\nabla \phi \odot \nabla \phi), \\
\text{div} u &= 0, \\
\partial_t \phi + u \cdot \nabla \phi &= \phi \times \Delta \phi, \\
(u, \phi)|_{t=0} &= (u_0, \phi_0).
\end{aligned}
\]
(1.1)

Here \(d = 2, 3\), \(u : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d\) represents the velocity field of the flow, \(P\) is the pressure
function, and \(\phi : \mathbb{R}^d \times [0, T] \to \mathbb{S}^2 \subset \mathbb{R}^3\) denotes the magnetization field. The notation \(\times\) is
the cross product for vectors in \(\mathbb{R}^3\), and the term \(\nabla \phi \odot \nabla \phi\) denotes the \(d \times d\) matrix whose
\((i, j)\)-th entry is given by \(\partial_i \phi \cdot \partial_j \phi\) \((1 \leq i, j \leq d)\), i.e.
\[
(\nabla \phi \odot \nabla \phi)_{ij} = \sum_{k=1}^d \partial_i \phi_k \partial_j \phi_k,
\]
and
\[
(\text{div}(\nabla \phi \odot \nabla \phi))_j = \sum_{i=1}^d \sum_{k=1}^d \partial_i (\partial_t \phi_k \partial_j \phi_k) = \sum_{k=1}^d \Delta \phi_k \partial_j \phi_k + \frac{1}{2} \partial_j |\nabla \phi_k|^2.
\]

This model is a coupled system of the incompressible Navier-Stokes equations and Schrödinger
map flow which can be used to describe the dispersive theory of magnetization of ferromagnets
with quantum effects. The system (1.1) also arise in the simulation of classical fluids. In
\([3, 4, 5]\) they formulated and simulated classical fluids using a \(C^2\)-valued Schrödinger equation
subject to an incompressibility constraint. Such a fluid flow was called an incompressible
Schrödinger flow, which is used to simulated classical fluids with particular advantage in its
simplicity and its ability of capturing thin vortex dynamics. They further considered the
incompressible Schrödinger flow with Berger metric, and derived the Euler-Schrödinger map
...
system, meanwhile simulating viscous fluid remains an open problem. If \( u \equiv 0 \), the model (1.1) is reduced to the Schrödinger flow of maps from \( \mathbb{R}^d \) into \( S^2 \), which is an interesting equation known as the ferromagnetic chain system, and has been intensely studied in the last decades. We refer to [13] for the detailed review.

Different from the standard Schrödinger Map equation, the presence of material derivative \( \partial_t + u \cdot \nabla \) makes the \( \phi \)-equation become a quadratic Schrödinger equation. There are many works devoted to small data global regularity for quadratic Schrödinger equations of the form

\[
(1.2) \quad i \partial_t u + \Delta u = N(u, \bar{u}, \nabla u, \nabla \bar{u}), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.
\]

To shorten the discussion, we focus here on recent developments corresponding to nonlinearities involving one \( \nabla u \) or \( \nabla \bar{u} \). On one hand, the nonlinearity with null structure makes estimates easier. On the other hand, the loss of derivatives makes estimates more complicated. In dimension 3, Hayashi-Miao-Naumkin [8] and Hayashi-Naumkin [9] were able to prove global existence and scattering for small data and for any quadratic nonlinearity involving at least one derivative. In dimension 2, Delort [6] proved global existence for a nonlinearity of the form \( u \nabla u \) or \( \bar{u} \nabla \bar{u} \) by the vector fields method, a normal form transform and microlocal analysis. Finally, Germain-Masmoudi-Shatah [7] used the space-time resonance to prove global regularity and scattering for a nonlinearity \( N = Q(u, u) + Q(\bar{u}, \bar{u}) \), where \( Q \) is like a derivative for low frequencies, and the identity for high frequencies.

The first author [10] proved the local existence and uniqueness of (1.1) for large data by parabolic approximation and parallel transport. In this article, we establish the global regularity and scattering of (1.1) with small and localized data in three dimensions by Coulomb gauge and vector fields method.

1.1. The main theorem. Here we start with two conservation laws for solutions of the system (1.1). One of the conservation law is its energy defined by

\[
E(u, \phi) = \frac{1}{2} \| u \|_{L^2}^2 + \int_0^t \| \nabla u \|_{L^2}^2 ds + \frac{1}{2} \| \nabla \phi \|_{L^2}^2.
\]

And (1.1) has the scaling invariance property:

\[
u(t, x) \to \lambda u(\lambda^2 t, \lambda x), \quad \phi(t, x) \to \phi(\lambda^2 t, \lambda x).
\]

Thus this scaling would suggest the critical Sobolev space for \( (u, \phi) \) to be \( \dot{H}^{\frac{d}{2}-1} \times \dot{H}^{\frac{d}{2}} \), and \( d = 2 \) is the energy critical case. Also the mass is conserved in the (1.1)

\[
(1.3) \quad M(\phi) = \frac{1}{2} \| \phi - Q \|_{L^2}^2, \quad \text{if} \quad \| \phi_0 - Q \|_{L^2} < \infty, \quad \text{for some} \quad Q \in S^2.
\]

For \( k \geq 0 \) let \( H^k(\mathbb{R}^d) \) denote the usual Sobolev spaces of complex valued functions on \( \mathbb{R}^d \). Given a point \( Q \in S^2 \), we define the extrinsic Sobolev space \( H^k_Q \) by

\[
H^k_Q := \{ u : \mathbb{R}^d \to \mathbb{R}^3 : |u(x)| = 1 \text{ and } u - Q \in H^k \},
\]

which is equipped with the metric \( d_Q(f, g) = \| f - g \|_{H^k} \). We also define the metric spaces

\[
H^\infty := \bigcap_{k=1}^{\infty} H^k, \quad H^\infty_Q := \bigcap_{k=1}^{\infty} H^k_Q.
\]
Since \( \phi \) actually satisfies a quadratic Schrödinger equation, we would use vector fields to define its weighted energy. Precisely, we define the perturbed angular momentum operators \( \tilde{\Omega} \) as

\[
\tilde{\Omega}_i = \Omega_i + A_i,
\]

where \( \Omega = x \wedge \nabla \) is the usual rotation vector field and

\[
A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

We define the scaling vector-field \( S \) by

\[
S = 2t\partial_t + x_i\partial_{x_i}.
\]

For simplicity of notations, we set

\[
\Gamma \in \{ \partial_t, \partial_1, \partial_2, \partial_3, \tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3 \}, \quad Z^a = S^{a1}\Gamma^{a'},
\]

where \( a = (a_1, a_2, \cdots, a_8) \in \mathbb{N}^8 \) and \( \Gamma^{a'} = \Gamma^{a_2} \cdots \Gamma^{a_8} \). We will explain the choice of the above vector fields in the next subsection 1.2.

Then, our main result is as follows:

**Theorem 1.1** (Small data global regularity and scattering). Given \( Q \in \mathbb{S}^2 \). Assume that the initial data \( (u_0, \phi_0) \in H^\infty \times H^\infty_Q \) satisfies

\[
\|u_0\|_{H^3} + \|\phi_0\|_{H^4} + \|Zu_0\|_{H^1} + \|\nabla Z\phi_0\|_{H^1} \leq \epsilon_0,
\]

Then (1.1) admits a unique global solution \( (u, \phi) \) with the energy bounds

\[
\sup_{|a| \leq 1} \{ \|Z^a u(t)\|_{H^N(a)} + \|\nabla Z^a u\|_{L^2([0,t];H^N(a))} \} + \|\phi(t)\|_{H^N_Q(0;H^1)} + \|\nabla Z\phi(t)\|_{H^N(1)} \lesssim \epsilon_0,
\]

for any \( 0 \leq t \leq T \in [0, \infty) \), where \( N(0) = 3, N(1) = 1 \). Moreover, \( \nabla \phi \), expressed in a Coulomb frame, scatters to the free solution of a suitable linear Schrödinger equation, and

\[
\lim_{t \to \infty} (\|u(t)\|_{L^\infty} + \|\phi(t) - Q\|_{L^\infty}) = 0.
\]

**1.2. Main ideas.** The main strategy to prove global regularity for (1.1) relies on an interplay between the control of high order energies and decay estimates, which is based on the Fourier analysis and vector-field method. The main ingredients include decay estimates, energy and weighted energy estimates, and \( L^2 \) weighted bounds on the profile \( \Psi = e^{-it\Delta}\psi \) associated with differentiated fields \( \psi \). However, there are still some difficulties to overcome.

In the prior works in \cite{7,9} and so on for quadratic Schrödinger equations, the vector field \( G = x + 2it\nabla \) was applied to prove the small data global regularity, which commutes with the Schrödinger operator \( i\partial_t + \Delta \). However, the operator \( G \) does not commute with the heat operator \( \partial_t - \Delta \). Furthermore, it works well only when the nonlinearities \( N(u) \) is self-conjugate, namely, \( N(e^{it}u) = e^{it}N(u) \) for all \( \theta \in \mathbb{R} \). Thus these facts do not allow us to use the vector field \( G \) to define the weighted energy functional. In view of the symmetries of Navier-Stokes equation and Schrödinger equation, as a system we would only choose the following vector fields to define its weighted energy

\[
\{ \partial_t, \partial_1, \partial_2, \partial_3, \tilde{\Omega}_1, \tilde{\Omega}_2, \tilde{\Omega}_3, S \}.
\]
However, the dispersive estimates could not connect the decay and weighted energy directly without the vector field \( G = x + 2it \nabla \). Inspired by [8] we can consider the operator \( G \cdot \nabla \) as a connection between decay estimates and weighted energy, and thus give the decay of differentiated field \( \psi \). Precisely, the time decay estimates of the derivative of the solution \( \psi \) are obtained by a priori estimate of the norm \( \| G \nabla \psi \|_{L^2} \). Then, we apply the inequality
\[
\| G \nabla f \|_{L^2} \lesssim \| (x \cdot \nabla + 2it\Delta) f \|_{L^2} + \| \Omega f \|_{L^2},
\]
which is valid for any smooth function \( f \). Note that the operator \( x \cdot \nabla + 2it\Delta \) can be replaced by the scaling vector field \( S = 2t \partial_t + x \cdot \nabla \) via the identity \( x \cdot \nabla + 2it\Delta = S + 2it(i\partial_t + \Delta) \). Then it suffices to prove the decay estimates of the nonlinear term \( (i\partial_t + \Delta) \psi \) in \( L^2 \) for the solution \( \psi \) of the nonlinear Schrödinger equation. This decay estimate is achieved by Fourier analysis and bootstrap assumptions, and hence the decay of the solution \( \psi \) can be obtained.

1.3. Notations. The notation \( A \lesssim B \) means there exists some universal constant \( C > 0 \) such that \( A \leq CB \). We denote \( \mathcal{R} \) as the Riesz transformation \( \mathcal{R} = \frac{\nabla}{|\nabla|} \). Let \( \mathbb{P} \) be the Leray projection
\[
\mathbb{P} = I_d + \nabla(\Delta)^{-1}\nabla.
\]
The usual Sobolev spaces \( W^{s,p} \) are defined by
\[
\| f \|_{W^{s,p}} = \sum_{k=0}^{s} \| f \|_{W^{k,p}} = \sum_{k=0}^{s} \sum_{|\alpha| = k} \| \partial^\alpha f \|_{L^p}, \quad \text{for } \alpha \in \mathbb{N}^d, \ s \in \mathbb{N}.
\]
When \( p = 2 \), the Sobolev spaces \( H^s \) for any \( s \in \mathbb{R} \) are also defined by
\[
\| f \|_{H^s} = \| \langle \xi \rangle^s \hat{f}(\xi) \|_{L^2},
\]
where \( \langle \xi \rangle = \sqrt{1 + |\xi|^2} \) and \( \hat{f} \) is the Fourier transform of \( f \),
\[
\hat{f}(\xi) = \mathcal{F}(f)(\xi) := \int_{\mathbb{R}^d} f(x)e^{-ix\cdot\xi} \, dx.
\]

1.4. Outline. In section 2 we rewrite the \( \phi \)-equation for its differentiated fields. In section 3 we fix notation and state the main bootstrap proposition. We also state several lemmas. In section 4, we begin with the linear decay estimate, which is the key lemma in deriving the decay estimate of \( \psi \). Then we use the bootstrap assumptions to derive various decay estimates of velocity \( u \), connection coefficients \( A \) and the nonlinearities in \( \psi \)-equation. In section 5, we use these decays and Fourier analysis to prove the energy and weighted energy estimates, and also give the scattering of \( \nabla\phi \) in Coulomb gauge. Finally, we use local existence and bootstrap Proposition 3.4 to prove Theorem 1.1.

2. The differentiated equations and main theorem

In this section we start with a smooth solution \( \phi \) to the Schrödinger map equation and a smooth orthonormal frame \( (v, w) \) in \( T_{\phi}S^2 \). Then we construct the fields \( \psi_m \) and the connection coefficients \( A_m \), and derive the differentiated Schrödinger map equations satisfied by these functions under Coulomb gauge.
2.1. The differentiated equations. We begin with a smooth function \( \phi : \mathbb{R}^d \times (-T, T) \rightarrow S^2 \). Instead of working directly on equation (1.1) for the function \( \phi \), it is convenient to study the equations satisfied by its derivatives \( \partial_m \phi \) for \( m = 1, d + 1 \), where \( \partial_{d+1} = \partial_t \). These are tangent vectors to the sphere at \( \phi(x, t) \).

**Proposition 2.1** (Coulomb gauge, [1], Proposition 2.3). Assume \( T \in [0, 1] \), \( Q \in S^2 \), and

\[
\begin{align*}
\phi & \in C([-T, T] : H^\infty_Q); \\
\partial_t \phi & \in C([-T, T] : H^\infty).
\end{align*}
\]

Then there are continuous functions \( v, w : \mathbb{R}^d \times [-T, T] \rightarrow S^2 \), \( \phi \cdot v = 0 \), \( w = \phi \times v \), such that

\[
\partial_m v, \partial_m w \in C([-T, T] : H^\infty) \quad \text{for} \quad m = 1, \cdots, d, d + 1
\]

where \( \partial_{d+1} = \partial_t \). In addition,

\[
(2.1) \quad \text{if} \quad A_m = (\partial_m v) \cdot w \quad \text{for} \quad m = 1, \cdots, d, \quad \text{then} \quad \sum_{j=1}^d \partial_m A_m = 0.
\]

Assume that \( \phi, v, w \) are as in Proposition 2.1. In addition to the connection coefficients \( A_m \), we can define the differentiated variables \( \psi_m \) for \( m = 1, \cdots, d, d + 1 \) and \( A_{d+1} \)

\[
(2.2) \quad \psi_m = \partial_m \phi \cdot v + i \partial_m \phi \cdot w, \quad A_{d+1} = (\partial_t v) \cdot w.
\]

These allow us to express \( \partial_m \phi \), \( \partial_m v \) and \( \partial_m w \) in the frame \((\phi, v, w)\) as

\[
(2.3) \quad \begin{cases}
\partial_m \phi = v \Re \psi_m + w \Im \psi_m, \\
\partial_m v = -\phi \Re \bar{\psi}_m + w A_m, \\
\partial_m w = -\phi \Im \bar{\psi}_m - v A_m.
\end{cases}
\]

Denote the covariant derivative as

\[
D_t = \partial_t + i A_t.
\]

We then obtain the curl type relations between the variables \( \psi_m \)

\[
(2.4) \quad D_t \psi_m = D_m \psi_t
\]

and the curvature of the connection

\[
(2.5) \quad D_t D_m - D_m D_t = i(\partial_t A_m - \partial_m A_t) = i \Im (\psi_t \bar{\psi}_m).
\]

Assume that the smooth function \( \phi \) satisfies the Schrödinger map equation in (1.1). Then we derive the Schrödinger equations for the function \( \psi_m \). By (1.1), \( \phi \times v = w \) and \( w \times \phi = v \), we have

\[
(2.6) \quad \psi_{d+1} = -u \cdot \psi + i D_t \psi_t.
\]

Applying \( D_m \) to this, by (2.4) and (2.5) we obtain

\[
D_{d+1} \psi_m = -\partial_m u \cdot \psi - u \cdot D \psi_m + \Im (\psi_t \bar{\psi}_m) \psi_t + i D_t D_t \psi_m,
\]

which is equivalent to

\[
(2.7) \quad i(\partial_t + u \cdot \nabla) \psi_m + \Delta \psi_m = -2 i A \cdot \nabla \psi_m + (A_{d+1} + (u + A) \cdot A - i \nabla \cdot A) \psi_m \\
- i \partial_m u \cdot \psi + i \Im (\psi_t \bar{\psi}_m) \psi_t.
\]
Consider the system of equations which consists of (2.7), (2.4) and (2.5). The solution \( \psi_m \) for the above system can’t be uniquely determined as it depends on the choice of the orthonormal frame \((v, w)\). Precisely, it is invariant with respect to the gauge transformation 
\[
\psi_m \rightarrow e^{i\theta} \psi_m, \quad A_m \rightarrow A_m + \partial_m \theta.
\]
In order to obtain a well-posed system one needs to make a choice which uniquely determines the gauge. Here we choose to use the Coulomb gauge (2.1), which in view of (2.5) leads to (2.8)
\[
\Delta A_m = \partial_t \text{Im}(\psi \overline{\psi}_m).
\]
Similarly, by (2.5) and (2.6) we also have the compatibility condition
\[
\partial_t A_m - \partial_m A_{d+1} = \text{Im}(\psi_{d+1} \overline{\psi}_m) = \text{Im}[(u \cdot \psi + i D_i \psi_l) \overline{\psi}_m].
\]
This combined with the Coulomb gauge implies (2.9)
\[
\Delta A_{d+1} = \partial_m \text{Im}[(u \cdot \psi - i D_i \psi_l) \overline{\psi}_m].
\]
In conclusion, under the Coulomb gauge \(\nabla \cdot A = 0\) by (2.7), (2.8) and (2.9) we obtain the system for velocity \(u\) and differentiated fields \(\psi_m\)
\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla P &= \Delta u - \partial_j \text{Re}(\psi \overline{\psi}_j), \\
\text{div} u &= 0, \\
i(\partial_t + u \cdot \nabla) \psi_m + \Delta \psi_m &= -2iA \cdot \nabla \psi_m + (A_{d+1} + (u + A) \cdot A) \psi_m \\
&\quad - i \partial_m u \cdot \psi + i \text{Im}(\psi_l \overline{\psi}_m) \psi_l, \\
(u, \psi) \big|_{t=0} &= (u_0, \psi_0),
\end{align*}
\]
where connection coefficients \(A\) and \(A_{d+1}\) are determined at fixed time in an elliptic fashion via the following equations
(2.10)
\[
\begin{align*}
\Delta A_m &= \partial_t \text{Im}(\psi_l \overline{\psi}_m), \\
\Delta A_{d+1} &= \partial_m \text{Im}[(u \cdot \psi - i D_i \psi_l) \overline{\psi}_m].
\end{align*}
\]
we can assume that the following conditions hold at infinity in an averaged sense:
\[
A_m(\infty) = 0; \quad A_{d+1}(\infty) = 0.
\]
These are needed to insure the unique solvability of the above elliptic equations in a suitable class of functions.

Finally, we prove quantitative estimates for the differentiated fields \(\psi\) with respect to \(\phi\).

Lemma 2.2. With the notation in Proposition [2.7], if the map \(\phi\) has the additional property 
\[
\|\phi\|_{H^{k+1}} + \|\nabla Z \phi\|_{H^{k-2}} \leq \epsilon \text{ for any } k \geq 3,
\]
then for its differentiated fields \(\psi_m\) with \(1 \leq m \leq d\) we have the bounds
(2.11)
\[
\|\psi\|_{H^k} + \|Z \psi\|_{H^{k-2}} \lesssim \|\phi\|_{H^{k+1}} + \|\nabla Z \phi\|_{H^{k-2}}.
\]

Proof. We prove this bound by induction. First, we bound the first term \(\psi\). By \(|v| = |w| = 1\), (2.2) and Sobolev embedding, we have
\[
\|\psi\|_{L^2 \cap L^\infty} \leq \|\nabla \phi\|_{L^2 \cap L^\infty} (\|v\|_{L^\infty} + \|w\|_{L^\infty}) \leq \|\nabla \phi\|_{H^k},
\]
\[
\|A\|_{L^2} \lesssim \|P_{\leq 0}(\psi^2)\|_{L^1} + \|P_{>0}(\psi^2)\|_{L^2} \lesssim \|\psi\|_{L^2} \|\psi\|_{L^2 \cap L^\infty},
\]
and
\[ \|\nabla v\|_{L^2} + \|\nabla w\|_{L^2} \lesssim \|\psi\|_{L^2} + \|A\|_{L^2} \lesssim \|\psi\|_{L^2}. \]

To prove (2.12) for \( \psi \) in \( H^k \), we assume that
\[ \|\psi\|_{H^n} + \|A\|_{H^n} + \|\nabla v\|_{H^n} + \|\nabla w\|_{H^n} \lesssim \|\phi\|_{H_Q^{k+1}}, \quad \text{for any } n < l \leq k. \]

By this assumption, (2.2) and \( |\phi| = |v| = |w| = 1 \), we have
\[ \|\nabla^{l_1} \psi\|_{L^2} \lesssim \|\nabla^{l_1+1} \phi\|_{L^2} + \sum_{l_1 + l_2 = l, 0 < l_2 < l} \|\nabla^{l_1+1} \phi\|_{L^4} \|\nabla^{l_2} (v, w)\|_{L^4} \]
\[ + \|\nabla \phi\|_{L^\infty} \|\nabla^l (v, w)\|_{L^2} \lesssim \|\phi\|_{H_Q^{k+1}} (1 + \|\nabla (v, w)\|_{H^{l-1}}) \lesssim \|\phi\|_{H_Q^{k+1}} \]
and
\[ \|\nabla^l A\|_{L^2} \lesssim \|\nabla^{l-1} (\psi^2)\|_{L^2} \lesssim \|\psi\|_{H^{l-1}} \|\psi\|_{L^\infty} \lesssim \|\phi\|_{H_Q^{k+1}}^2. \]

Similarly, by (2.3) we also have
\[ \|\nabla^{l_1+1} v\|_{L^2} \lesssim \|\nabla^{l_1} v\|_{L^2} + \sum_{l_1 + l_2 > 0} \|\nabla^{l_1} v\|_{L^4} \|\nabla^{l_2} \phi\|_{L^4} + \|A\|_{H^l} \|\nabla w\|_{H^{l-1}} \]
\[ \lesssim \|\phi\|_{H_Q^{k+1}} + \|\psi\|_{H^l} \|\nabla w\|_{H^{l-1}} \lesssim \|\phi\|_{H_Q^{k+1}}, \]
and
\[ \|\nabla^{l_1+1} w\|_{L^2} \lesssim \|\phi\|_{H_Q^{k+1}}. \]

Next, we bound the second term \( Z\psi \). By (2.2) we have
\[ \|Z\psi\|_{L^2} \lesssim \|Z \nabla \phi\|_{L^2} + \|\nabla \phi\|_{L^3} \|(Z v, Z w)\|_{L^6} \]
\[ \lesssim \|\nabla Z \phi\|_{L^2} + \epsilon \|\nabla Z (v, w)\|_{L^2}, \]
\[ \|Z A\|_{L^2} \lesssim \|\nabla^{-1} (Z \psi \cdot \psi)\|_{L^2} \lesssim \|Z \psi\|_{L^2} \|\psi\|_{L^2 \cap L^\infty}, \]
and
\[ \|\nabla Z (v, w)\|_{L^2} \lesssim \|Z (\psi, A)\|_{L^2} + \|(\psi, A)\|_{L^3} \|Z (\phi, v, w)\|_{L^6} \]
\[ \lesssim \|Z \psi\|_{L^2} + \epsilon \|\nabla Z (\phi, v, w)\|_{L^2}. \]

These imply
\[ \|Z \psi\|_{L^2} + \|Z A\|_{L^2} + \|\nabla Z (v, w)\|_{L^2} \lesssim \|\nabla Z \phi\|_{L^2} \lesssim \epsilon. \]

Hence, we assume that
\[ \|Z \psi\|_{H^n} + \|Z A\|_{H^n} + \|\nabla Z (v, w)\|_{H^n} \lesssim \epsilon, \quad \text{for any } n < l \leq k - 2. \]
By (2.2), (2.3) and Sobolev embedding we have
\[
\|\nabla^l Z\psi\|_{L^2} \lesssim \sum_{l_1+l_2=l} \|\nabla^{l_1} Z\phi\|_{L^2} \|\nabla^{l_2} (v, w)\|_{L^\infty}
\]
\[
+ \sum_{l_1+l_2=l, l_2>0} \|\nabla^{l_1+1}\phi\|_{L^\infty} \|\nabla^l Z(v, w)\|_{L^2} + \|\nabla^{l_1+1}\phi\|_{L^3} \|Z(v, w)\|_{L^6}
\]
\[
\lesssim \epsilon + \|\nabla^l \phi\|_{H^{l+1}} \|\nabla Z(v, w)\|_{L^2} \lesssim \epsilon,
\]
\[
\|\nabla^lZA\|_{L^2} \lesssim \|\nabla^{l-1}(Z\psi \cdot \psi)\|_{L^2} \lesssim \|Z\psi\|_{H^{l-1}} \|\psi\|_{H^{l+1}} \lesssim \epsilon^2,
\]
and
\[
\|\nabla^{l+1}Z(v, w)\|_{L^2} \lesssim \|\nabla^{l+1}(v, w)\|_{L^2} + \|\nabla^{l} Z(\psi_0 + A(v, w))\|_{L^2}
\]
\[
\lesssim \epsilon + \sum_{l_1+l_2=l} \|\nabla^{l_1} Z(\psi, A)\|_{L^2} \|\nabla^{l_2} (\phi, v, w)\|_{L^\infty}
\]
\[
+ \sum_{l_1+l_2=l, l_2>0} \|\nabla^{l_1} (\psi, A)\|_{L^3} \|\nabla^{l_2} Z(\phi, v, w)\|_{L^2}
\]
\[
+ \|\nabla^l (\psi, A)\|_{L^3} \|Z(\phi, v, w)\|_{L^6}
\]
\[
\lesssim \epsilon + \epsilon \|Z\psi\|_{H^l} \lesssim \epsilon.
\]
This completes the proof of the lemma. \qed

3. Preliminaries and the main propositions

In this section, we start by summarizing our main definitions and notation.

3.1. Some analysis tools. For a function \(u(t, x)\) or \(u(x)\), let \(\hat{u} = \mathcal{F}u\) denote the Fourier transform in the spatial variable \(x\). Fix a smooth radial function \(\phi : \mathbb{R}^d \to [0, 1]\) supported in \([-2, 2]\) and equal to 1 in \([-1, 1]\), and for any \(i \in \mathbb{Z}\), let
\[
\varphi_i(x) := \varphi(x/2^i) - \varphi(x/2^{i-1}).
\]
We then have the spatial Littlewood-Paley decomposition,
\[
\sum_{i=-\infty}^{\infty} P_i(D) = 1,
\]
where \(P_i\) localizes to frequency \(2^i\) for \(i \in \mathbb{Z}\), i.e,
\[
\mathcal{F}(P_i u) = \varphi_i(\xi) \hat{u}(\xi).
\]
For simplicity of notation, we set
\[
u_j = P_j u, \quad \nu_{\leq j} = \sum_{i=-\infty}^{j} P_i u, \quad \nu_{\geq j} = \sum_{i=j}^{\infty} P_i u.
\]
Next, we state some basic lemmas.
Lemma 3.1 (Decay estimates of heat operator). For any Schwartz function $f \in \mathcal{S}(\mathbb{R}^3)$, we have

$$
\|e^{\Delta t}\nabla^j f\|_{W^{N,q}} \lesssim t^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{q}) - \frac{1}{2}} \|f\|_{W^{N,p}}, \quad 1 \leq p \leq q \leq \infty.
$$

In the proof of energy estimates, one often needs to analyze the symbols. We define a class of symbol as follows

$$
\mathcal{S}^\infty := \{ m : \mathbb{R}^6 \to \mathbb{C}, m \text{ is continuous and } \| \mathcal{F}^{-1} m \|_{L^1} < \infty \},
$$

whose associated norms are defined as

$$
\| m \|_{S^\infty} := \| \mathcal{F}^{-1} m \|_{L^1},
$$

and

$$
\| m \|_{S^\infty_{k_1,k_2}} := \| m(\xi,\eta)\varphi_k(\xi)\varphi_{k_1}(\xi - \eta)\varphi_{k_2}(\eta)\|_{S^\infty}.
$$

Then we have

Lemma 3.2 (Bilinear estimate, [12]). Given $m \in \mathcal{S}^\infty$ and two well-defined functions $f_1$, $f_2$, then the following estimate holds:

$$
\| \mathcal{F}^{-1} \left( \int_{\mathbb{R}^3} m(\xi,\eta)\widehat{f_1}(\xi - \eta)\widehat{f_2}(\eta) d\eta \right)(x) \|_{L^r} \lesssim \| m \|_{S^\infty} \| f_1 \|_{L^p} \| f_2 \|_{L^q}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}.
$$

We also need the following lemma to connect the weighted $L^2$-norm of profile to weighted energy.

Lemma 3.3 (Lemma 2.7, [11]). For any $N \geq 0$ and Schwartz function $f \in \mathcal{S}(\mathbb{R}^3)$, we have

$$
\| \mathcal{F}^{-1}(\| \xi \| \nabla \widehat{f})\|_{H^N} \lesssim \| x \cdot \nabla_x f \|_{H^N} + \| \Omega f \|_{H^N} + \| f \|_{H^N}.
$$

3.2. The main bootstrap proposition. Applying the vector fields $S$ and $\Gamma$ to (2.10) and (2.11), we obtain

$$
\begin{cases}
(\partial_t - \Delta)(S - 1)^{a_1}\Gamma^{c'} u = - \sum_{b+c=a} C^b_a Z^b u \cdot \nabla Z^c u - \nabla Z^a P - \sum_{b+c=a} \partial_j (Z^b \psi_j Z^c \psi), \\
\text{div } Z^a u = 0, \\
i \partial_t Z^a \psi + \Delta Z^a \psi = - i \sum_{b+c=a} C^b_a Z^b (u + 2A) \cdot \nabla (S + 1)^c \Gamma^{c'} \psi \\
+ \sum_{b+c=a} C^b_a Z^b A_{d+1}(S + 2)^c \Gamma^{c'} \psi \\
+ \sum_{b+c=e+a} C^b_a Z^b (A + u) \cdot Z^c A(S + 2)^c \Gamma^{c'} \psi \\
- i \sum_{b+c=a} C^b_a \nabla Z^b u \cdot (S + 1)^c \Gamma^{c'} \psi \\
+ i \sum_{b+c=e+a} C^b_a \text{Im}(Z^b \psi_j Z^c \psi)(S + 2)^c \Gamma^{c'} \psi_l,
\end{cases}
$$

(3.4)
with connection coefficients $A$ and $A_{d+1}$ satisfy
\[
\Delta Z^a A = \sum_{b+c=a} C^b_a \partial_t \text{Im}(Z^b \psi_l(S + 1) \Gamma^c \bar{\psi}),
\]
\[
\Delta Z^a A_{d+1} = -\sum_{b+c=a} C^b_a \text{Re} \partial_m(\partial_t Z^b \psi_l Z^c \bar{\psi}_m) + \sum_{b+c+c=a} C^b_{a,b} \text{Im} \partial_m(Z^b(u + A) \cdot Z^c \psi Z^c \bar{\psi}_m),
\]
where the coefficients $C^b_a$ and $C^b_{a,b}$ are
\[
C^b_a = \frac{a!}{b!(a-b)!}, \quad C^b_{a,b} = \frac{a!}{b!c!(a-b-c)!}.
\]
We define $u^{(a)}$, $\psi^{(a)}$ and the profile of $\psi$
\[
u^{(a)} = Z^a u, \quad \psi^{(a)} = Z^a \psi, \quad \Psi = e^{-it\Delta} \psi.
\]
Our main result is the following proposition:

**Proposition 3.4** (Bootstrap proposition). Assume that $(u, \psi)$ is a solution to (2.10) on some time interval $[0, T]$, $T \geq 1$ with initial data satisfying the assumption
\[
\sup_{|a| \leq 1} \{ \| u^{(a)} \|_{H_N(a)} + \| \psi^{(a)} \|_{H_N(a)} \} \leq \epsilon_0.
\]
Assume also that the solution satisfies the bootstrap hypothesis
\[
\sup_{|a| \leq 1, t \in [0, T]} \{ \| u^{(a)} \|_{H_N(a)} + \| \nabla u^{(a)} \|_{L^2([0, t]; H_N(a))} + \| \psi^{(a)} \|_{H_N(a)} \} \leq \epsilon_1,
\]
\[
\sup_{t \in [0, T]} \| x \cdot \nabla \Psi \|_{H_N^{(1)}} \leq \epsilon_1,
\]
where $\epsilon_1 = \epsilon_0^{2/3}$, $N(0) = 3$, $N(1) = 1$. Then the following improved bounds hold
\[
\sup_{|a| \leq 1, t \in [0, T]} \{ \| u^{(a)} \|_{H_N(a)} + \| \nabla u^{(a)} \|_{L^2([0, t]; H_N(a))} + \| \psi^{(a)} \|_{H_N(a)} \} \lesssim \epsilon_0,
\]
\[
\sup_{t \in [0, T]} \| x \cdot \nabla \Psi \|_{H_N^{(1)}} \lesssim \epsilon_0.
\]

In the next sections, we will concern on the proof of Proposition 3.4 and hence Theorem 1.1.

4. Decay of velocity field and differentiated field

In this section, we give the various decay estimates of $u$ and $\psi$, which will be useful in the energy estimates in the next sections. Here we start with the basic decay estimates.

**Lemma 4.1** ([1], Lemma 2.4). For any Schwartz function $f$, $d = 3$ and $t > 1$, we have
\[
\| \nabla f \|_{L^5} \lesssim t^{-1/2} \| \Theta f \|_{L^2},
\]
\[
\| f \|_{L^\infty} \lesssim t^{-3/2} \| \Theta f \|_{L^2}^{3/2} \| f \|_{L^2}^{1/2},
\]
where $\Theta = (x \cdot \nabla + 2it\Delta, \Omega)$.

We then use these two estimates to give the decay of $\psi$. 

**Lemma 4.2** (Decay of fields $\psi$). With the notations and hypothesis in Proposition 3.4, for any $t \in [0, T]$ we have

(4.3) \[ \| \nabla \psi(t) \|_{W^{N(1),6}} \lesssim \epsilon_1(t)^{-1}, \]
(4.4) \[ \| \psi(t) \|_{W^{N(1),\frac{1}{4}+\infty}} \lesssim \epsilon_1(t)^{-3/4}. \]

Proof. By (4.1) and the equality

\[ (x \cdot \nabla + 2it\Delta)e^{it\Delta} = e^{it\Delta}(x \cdot \nabla), \]
we have for $t > 1$

\[ \| \nabla e^{it\Delta} \psi_k \|_{L^6} \lesssim t^{-1} \left( \| (x \cdot \nabla + 2it\Delta)e^{it\Delta} \psi_k \|_{L^2} + \| \Omega \psi_k \|_{L^2} \right) \lesssim t^{-1} \left( \| x \cdot \nabla \psi_k \|_{L^2} + \epsilon_1 \right), \]

Then the bound (4.3) follows by (3.7).

The bound (4.4) can be proved similarly by (4.2), and hence this completes the proof of the lemma. \(\square\)

Next, we use Lemma 4.2 and Duhamel’s formula to prove the decays of velocity $u$.

**Lemma 4.3.** With the notations and hypothesis in Proposition 3.4, for any $t \in [0, T]$, we have

(4.5) \[ \| u \|_{W^{N(0)-7/4,\infty}} \lesssim \epsilon_1(t)^{-3/4}. \]

Proof. By Duhamel’s formula and (3.1), it suffices to prove

(4.6) \[ \int_0^t \| e^{(t-s)\Delta} \mathbb{P}(u \cdot \nabla u + \nabla(\psi^2)) \|_{W^{N(0)-7/4,\infty}} \lesssim \epsilon_1^2(t)^{-3/4} + \epsilon_1 \sup_{s \in [t/2, t]} \| u(s) \|_{L^\infty}. \]

By (3.1), $\text{div} u = 0$ and Sobolev embedding we bound the first term by

\[ \int_0^t \| e^{(t-s)\Delta} \mathbb{P}(u \cdot \nabla u) \|_{W^{N(0)-7/4,\infty}} \lesssim \int_0^{t/2} (t-s)^{-2} \| u^2 \|_{W^{N(0)-7/4,1}} ds \]
\[ + \int_{t/2}^t (t-s)^{-1+\delta} (t-s)^{-1/4-\delta} \| u^2 \|_{H^{N(0)-5/4+2\delta}} ds \]
\[ \lesssim \epsilon_1^2(t)^{-1} + \epsilon_1 \sup_{s \in [t/2, t]} \| u(s) \|_{L^\infty}. \]

Similarly, by (4.4) we easily have

\[ \int_0^t \| e^{(t-s)\Delta} \nabla(\psi^2) \|_{W^{N(0)-7/4,\infty}} \lesssim \epsilon_1^2(t)^{-1} + \epsilon_1 \sup_{s \in [t/2, t]} \| \psi(s) \|_{L^\infty} \lesssim \epsilon_1^2(t)^{-3/4}. \]

These give the bound (4.6), and conclude the decay estimate (4.5). \(\square\)

We then use this decay estimate to prove the following more important decay estimate for $\nabla u$.

**Lemma 4.4.** With the notations and hypothesis in Proposition 3.4. For any $t \in [0, T]$, we have

(4.7) \[ \| \nabla u \|_{W^{N(1),\infty}} \lesssim \epsilon_1(t)^{-5/4}. \]
Proof. By Duhamel’s formula and (3.1), it suffices to prove

\begin{equation}
\int_0^t \|e^{(t-s)\Delta}P\nabla(u \cdot \nabla u + \nabla (\psi^2))\|_{W^N(1,\infty)} \lesssim \epsilon_1^2(t)^{-5/4}.
\end{equation}

By (3.1), \(\text{div} u = 0\) and Sobolev embedding we bound the first term by

\begin{align*}
\int_0^t \|e^{(t-s)\Delta}P\nabla(u \cdot \nabla u)\|_{W^N(1,\infty)} &\lesssim \int_0^t/2 (t-s)^{-5/2}\|u^2\|_{W^N(1,1)} ds \\
&\quad + \int_{t/2}^t (t-s)^{-1+\delta}\|u^2\|_{W^N(1)+25,\infty} ds \\
&\lesssim \epsilon_1^2(t)^{-3/2} + \langle t \rangle^\delta \sup_{s \in [t/2,t]} \|u(s)\|^2_{W^N(1)+25,\infty} \\
&\lesssim \epsilon_1^2(t)^{-3/2+\delta}.
\end{align*}

Similar, by (4.4) we easily have

\begin{align*}
\int_0^t \|e^{(t-s)\Delta}P\nabla^2(\psi^2)\|_{W^N(1,\infty)} &\lesssim \epsilon_1^2(t)^{-3/2} + \langle t \rangle^\delta \sup_{s \in [t/2,t]} \|\psi(s)\|^2_{W^N(1)+25,\infty} \\
&\lesssim \epsilon_1^2(t)^{-3/2+\delta}.
\end{align*}

These give the bound (4.8), and conclude the decay estimate (4.7). \(\square\)

By the decays of \(\psi\) and \(u\) we obtain the decays for \(A\) and \(A_{d+1}\).

**Corollary 4.5** (Decays of \(A\) and \(A_{d+1}\)). With the notations and hypothesis in Proposition 3.4, for any \(t \in [0,T]\) we have

\begin{align}
\|\nabla A^{(a)}\|_{H^N(\alpha)} &\lesssim \epsilon_1^2(t)^{-1/2}, \quad |a| \leq 1, \\
\|\nabla A\|_{W^N(1,\infty)} &\lesssim \epsilon_1^2(t)^{-3/2+3\delta/4}.
\end{align}

\begin{align}
\|A_{d+1}^{(a)}\|_{H^N(\alpha)} &\lesssim \epsilon_1^2(t)^{-1/2}, \quad |a| \leq 1, \\
\|A_{d+1}\|_{W^N(1,\infty)} &\lesssim \epsilon_1^2(t)^{-5/4+3\delta}.
\end{align}

**Proof.** We start with the first estimate (4.9). By (2.8) and (4.4) we have for \(|a| \leq 1\)

\begin{align*}
\|\nabla A^{(a)}\|_{H^N(\alpha)} &\lesssim \|\nabla^{-1/2}(\psi^{(a)} \psi)\|_{L^2} + \|\psi^{(a)} \psi\|_{H^N(\alpha)} \\
&\lesssim \|\psi^{(a)} \psi\|_{L^2} + \|\psi^{(a)} \psi\|_{H^N(\alpha)} \\
&\lesssim \|\psi^{(a)}\|_{L^2} \|\psi\|_{L^2} + \|\psi\|_{L^\infty}^2 + \epsilon_1^2(t)^{-3/4} \\
&\lesssim \epsilon_1^2(t)^{-1/2}.
\end{align*}

We use Sobolev embedding and (4.4) to give the second estimate (4.10)

\begin{align*}
\|\nabla A\|_{W^N(1,\infty)} &\lesssim \|R(\psi^2)\|_{W^N(1)+35,2+\delta} \\
&\lesssim \|\psi\|_{W^N(1)+35,\infty} \|\psi\|_{W^N(1)+35,2+\delta} \\
&\lesssim \epsilon_1^2(t)^{-3/2+3\delta/4}.
\end{align*}
For the third bound (4.11), when \( |a| = 0 \), by (2.9), Sobolev embedding, (4.3) and (4.4) we have

\[
\|A_{d+1}\|_{H^{N(0)}} \lesssim \|\nabla^{-1}(\nabla \psi \psi + (u + A)\psi^2)\|_{H^{N(0)}} \\
\lesssim \|\nabla \psi \psi + (u + A)\psi^2\|_{L^{6/5}} + \|\nabla \psi \psi + (u + A)\psi^2\|_{L^{6/5}}^0 \|H^{N(0)-1}} \\
\lesssim \|\nabla \psi\|_{L^3} \|\psi\|_{L^2} + \|u, A\|\|\psi\|_{L^4} + \|\nabla \psi\|_{L^3} \|\psi\|_{L^6} + \|\psi\|_{W^{1,\infty}} \|\psi\|_{H^{N(0)}} \\
+ \|u, A\|\|\psi\|_{H^{N(0)}} \|\psi\|_{L^6} + \|\nabla \psi\|_{H^{N(0)}} \|u, A\|\|\psi\|_{L^6} \|\psi\|_{L^6} \\
\lesssim \epsilon_1^2(t)^{-1/2} + \epsilon_1^2(t)^{-1} + \epsilon_1^2(t)^{-3/4} + \epsilon_1^3(t)^{-3/2} \lesssim \epsilon_1^2(t)^{-1/2}.
\]

When \( |a| = 1 \), by (2.9), Sobolev embedding, (4.3) and (4.4) we have

\[
\|A_{d+1}^{(a)}\|_{H^{N(1)}} \lesssim \|\nabla^{-1}\left( \sum_{|b+c|=1} \nabla \psi^{(b)} \psi^{(c)} + \sum_{|b+c+e|=1} (u + A)^{(b)} \psi^{(c)} \psi^{(e)} \right)\|_{H^{N(1)}}.
\]

By (4.3) and (4.4), the first term in the right-hand side can be bounded by

\[
\|\nabla^{-1}(\nabla \psi^{(1)} \psi + \nabla \psi \psi^{(1)})\|_{H^{N(1)}} \\
\lesssim \|\mathcal{R}(\psi^{(1)} \psi) + \nabla^{-1}(\nabla \psi \psi^{(1)})\|_{H^{N(1)}} \\
\lesssim \|\psi^{(1)}\|_{H^{N(1)}} \|\psi\|_{W^{1,\infty}} + \|\nabla \psi \psi^{(1)}\|_{L^{6/5}} + \|\nabla \psi \psi^{(1)}\|_{H^{N(1)-1}} \\
\lesssim \epsilon_1^2(t)^{-3/4} + \|\nabla \psi\|_{L^3} \|\psi\|_{L^6} + \|\nabla \psi\|_{W^{1,\infty}} \|\psi\|_{H^{N(1)}} \\
\lesssim \epsilon_1^2(t)^{-3/4} + \epsilon_1^2(t)^{-1/2} + \epsilon_1^3(t)^{-1} \lesssim \epsilon_1^2(t)^{-1/2}.
\]

We use (4.4), (4.5) and (2.8) to bound the second term by

\[
\|\nabla^{-1}\left( \sum_{|b+c|=1} (u + A)^{(b)} \psi^{(c)} \psi^{(e)} \right)\|_{H^{N(1)}} \\
\lesssim \|\mathcal{R}(\psi^{(b)} \psi^{(c)} \psi^{(e)})\|_{L^{6/5}} \|\psi\|_{L^6} + \|\nabla \psi\|_{L^3} \|\psi\|_{L^6} \|\psi\|_{L^8} \\
+ \|\psi\|_{W^{1,\infty}} \|\psi\|_{W^{1,\infty}} \|\psi\|_{W^{1,\infty}} \|\psi\|_{W^{1,\infty}} \|\psi\|_{W^{1,\infty}} \|\psi\|_{W^{1,\infty}} \|\psi\|_{W^{1,\infty}} \\
\lesssim \epsilon_1^2(t)^{-1} + \epsilon_1^3(t)^{-3/2} \lesssim \epsilon_1^2(t)^{-1}.
\]

Finally, we prove the last bound (4.12). By (4.3) and (4.4) we have

\[
\|A_{d+1}\|_{W^{N(1),\infty}} \lesssim \|\nabla^{-1}\mathcal{R}(\nabla \psi \psi + (u + A)\psi^2)\|_{W^{N(1),\infty}} \\
\lesssim \|\nabla \psi \psi + (u + A)\psi^2\|_{L^{1/38}} \|\nabla \psi \psi + (u + A)\psi^2\|_{W^{N(1),\frac{3}{13}}} \\
\lesssim \|\nabla \psi\|_{L^{1/38}} \|\psi\|_{W^{N(1),\frac{3}{13}}} \|\psi\|_{L^\infty} \|\psi\|_{L^\infty} \\
+ \|\nabla \psi\|_{W^{N(1),\frac{3}{13}}} \|\psi\|_{W^{N(1),\frac{3}{13}}} \|\psi\|_{W^{N(1),\infty}} \|\psi\|_{W^{N(1),\infty}} \\
\lesssim \epsilon_1^2(t)^{-5/4+3\delta} + \epsilon_1^3(t)^{-3/2} \lesssim \epsilon_1^2(t)^{-5/4+3\delta}.
\]
This completes the proof of the lemma.

We then obtain the decay estimates for the nonlinearities
\[ \mathcal{N} := (A_{d+1} + (A_i^2 + u \cdot A))\psi_{m} - i\partial_{m} u \cdot \psi + i \text{Im}(\psi_{m} \bar{\psi}_{m})\psi_{t}, \]
\[ \mathcal{N}^{(1)} := \sum_{|b+c| \leq 1} (A_{d+1}^{(b)}\psi^{(c)} + \partial_{m} u^{(b)} \cdot \psi) + \sum_{|b+c+e| \leq 1} [(A + u)^{(b)} \cdot A^{(c)} \psi^{(e)} + \psi^{(b)} \psi^{(e)} \psi^{(e)}]. \]

**Corollary 4.6 (Decay of \( \mathcal{N}^{(a)} \)).** With the notations and hypothesis in Proposition 3.4, for any \(|a| \leq 1\) and \(t \in [0, T]\) we have
\[ (4.13) \quad \| \mathcal{N}^{(a)} \|_{H^{N(a)}} \lesssim \epsilon_{1}^{2}(t)^{-5/4+3\delta} + \epsilon_{1}(t)^{-3/4} \| \nabla u^{(a)} \|_{H^{N(a)}}. \]

**Proof.** By (3.5), (4.7), (4.4) and Lemma 3.5 we bound \( \mathcal{N} \) by
\[ \| \mathcal{N} \|_{H^{N(0)}} \lesssim \| (A_{d+1}, \nabla u) \|_{H^{N(0)}} + \| (u, A) \|_{H^{N(0)}} \| (u, A) \|_{L^{\infty}} \| \psi \|_{L^{\infty}} \]
\[ + \| (A_{d+1}, \nabla u) \|_{L^{\infty}} + \| (u, A) \|_{L_{\infty}^{2}} \| \psi \|_{H^{N(0)}} + \| \psi \|_{H^{N(0)}} \| \psi \|_{L_{\infty}^{2}} \]
\[ \lesssim \epsilon_{1}^{2}(t)^{-1/2+3\delta} + \| \nabla u \|_{H^{N(0)}} \epsilon_{1}(t)^{-3/4} + \epsilon_{1}^{2}(t)^{-5/4+3\delta} \]
\[ \lesssim \epsilon_{1}^{2}(t)^{-5/4+3\delta} + \epsilon_{1}(t)^{-3/4} \| \nabla u \|_{H^{N(0)}}. \]

We could also bound \( \mathcal{N}^{(1)} \) by
\[ \| \mathcal{N}^{(1)} \|_{H^{N(1)}} \lesssim \| (A_{d+1}^{(1)}, \nabla u^{(1)}) \|_{H^{N(0)}} + \| (u^{(1)} + A^{(1)}) \|_{H^{N(0)}} \| (u + A) \|_{W^{N(1),\infty}} \| \psi \|_{W^{N(1),\infty}} \]
\[ + \| (A_{d+1}, \nabla u) \|_{W^{N(1),\infty}} + \| (u + A) \|_{W^{N(1),\infty}} \| \psi^{(1)} \|_{H^{N(1)}} \]
\[ + \| \psi^{(1)} \|_{H^{N(1)}} \| \psi \|_{W^{N(1),\infty}}^{2} \]
\[ \lesssim \epsilon_{1}^{2}(t)^{-1/2+3\delta} + \| \nabla u^{(1)} \|_{H^{N(0)}} \epsilon_{1}(t)^{-3/4} + \epsilon_{1}^{2}(t)^{-5/4+3\delta} \]
\[ \lesssim \epsilon_{1}^{2}(t)^{-5/4+3\delta} + \epsilon_{1}(t)^{-3/4} \| \nabla u^{(1)} \|_{H^{N(1)}}. \]

This concludes the proof of the Corollary.

**Lemma 4.7.** With the notations and hypothesis in Proposition 3.4, for any \(t \in [0, T]\), we have
\[ (4.14) \quad \| \nabla^{2} u \|_{H^{N(1)}} \lesssim \epsilon_{0}(t)^{-1} \]
\[ (4.15) \quad \| \mathcal{F}^{-1}(\mathcal{L} \nabla \tilde{u}(\xi)) \|_{H^{N(1)}} \lesssim \epsilon_{1}. \]

**Proof.** We prove the first bound. By Duhamel’s formula, (3.1) and (1.6), it suffices to prove
\[ (4.16) \quad \int_{0}^{t} \| e^{(t-s)\Delta} \nabla^{2} \mathbb{P}(u \cdot \nabla u + \nabla (\psi^{2})) \|_{H^{N(1)}} \, ds \lesssim \epsilon_{1}^{2}(t)^{-5/4+3\delta} + \epsilon_{1} \sup_{s \in [t/2, t]} \| \nabla^{2} u \|_{H^{N(1)}}. \]

By (3.1) and (3.6) we have
\[ \int_{0}^{t/2} \| e^{(t-s)\Delta} \nabla^{2} \mathbb{P}(u \cdot \nabla u + \nabla (\psi^{2})) \|_{H^{N(1)}} \, ds \]
\[ \lesssim \int_{0}^{t/2} (t-s)^{-9/4} \| u \|_{H^{N(1)}}^{2} + \| \psi \|_{H^{N(1)}}^{2} \, ds \lesssim \epsilon_{1}^{2}(t)^{-5/4}. \]
We use (3.1), (4.5), (4.7) and Hölder inequality to bound the remainder integral by

\[\int_{t/2}^{t} \| e^{(t-s)\Delta} \nabla^2 (u \cdot \nabla u) \|_{H^1_N} ds \]

\[\lesssim \int_{t/2}^{t} (t-s)^{-1/2} (\| \nabla u \|_{H^1_N} \| \nabla u \|_{L^\infty} + \| u \|_{W^N, \infty} \| \nabla^2 u \|_{H^1_N}) ds \]

\[\lesssim \int_{t/2}^{t} (t-s)^{-1/2} (\epsilon_1^{3/2} \langle s \rangle^{-5/4} \| \nabla^2 u(s) \|_{H^1_N}^2 + \epsilon_1 \langle s \rangle^{-3/4} \| \nabla^2 u(s) \|_{H^1_N}) ds \]

\[\lesssim \epsilon_1^2 (t)^{-3/2} + \epsilon_1 \sup_{s \in [t/2, t]} \| \nabla^2 u \|_{H^1_N} \]

\[\int_{t/2}^{t} \| e^{(t-s)\Delta} \nabla^2 \mathbb{P}(\psi^2) \|_{H^1_N} ds \lesssim \int_{t/2}^{t} (t-s)^{-1+\delta/2} \| \nabla^\delta (\nabla \psi \psi) \|_{H^1_N} ds, \]

By Littlewood-Paley decomposition, (4.3) and (4.4) we bound the integrand by

\[2^{(N(1)+\delta)k^+} \| P_k (\nabla \psi \psi) \|_{L^2} \lesssim 2^{(N(1)+\delta)k^+} (\| \nabla \psi_k \|_{L^{6(1+\delta)}} \| \psi_k \|_{L^3} + \sum_{k_1 > k} \| \nabla \psi_{k_1} \|_{L^6} \| \psi_{k_1} \|_{L^3}) \]

\[\lesssim 2^{(N(1)+\delta)k^+} (\| \nabla \psi_k \|_{L^6}^{1-3\delta} \| \nabla \psi_k \|_{L^2}^{3\delta} \| \psi_k \|_{L^\infty}^{1+6\delta} \| \psi_k \|_{L^2}^{2-6\delta}) \]

\[+ \epsilon_1 \langle s \rangle^{-1} \| \psi_k \|_{L^6}^{1/3} \| \psi_k \|_{L^2}^{2/3} + \sum_{k_1 > k} \epsilon_1 \langle s \rangle^{-1} \| \psi_{k_1} \|_{L^6}^{1/3} \| \psi_{k_1} \|_{L^2}^{2/3} \]

\[\lesssim 2^{-(N(1)+2N(0))k/3} \epsilon_1^2 \langle s \rangle^{-5/4} + \sum_{k_1 > k} 2^{-(N(1)+2N(0))k_1/3} \epsilon_1^2 \langle s \rangle^{-5/4} \]

\[\lesssim 2^{3\delta (N(1)-N(0)+\delta)k^+ + \delta k^+} \epsilon_1^2 \langle s \rangle^{-5/4} + \sum_{k_1 > k} 2^{3\delta (N(1)-N(0)+\delta)k^+ + \delta k^+} \epsilon_1^2 \langle s \rangle^{-5/4} \]

This gives

\[\| P_{\geq 0} \nabla \psi \psi \|_{H^N_{(1)+\delta}} \lesssim \epsilon_1^2 \langle s \rangle^{-5/4+3\delta/2}. \]

For the low-frequency part, we easily have

\[\| P_{< 0} \nabla \psi \psi \|_{H^N_{(1)+\delta}} \lesssim \| \nabla \psi \|_{L^6} \| \psi \|_{L^3} \lesssim \epsilon_1^2 \langle s \rangle^{-5/4}. \]

From these two estimates, we have

\[\int_{t/2}^{t} \| e^{(t-s)\Delta} \nabla^3 \mathbb{P}(\psi^2) \|_{H^1_N} ds \lesssim \epsilon_1^2 (t)^{-5/4+2\delta}. \]

This concludes the bound (4.16), and hence gives the decay estimate (4.14).
Next, we prove the second bound \((4.15)\). By \(u\)-equation and \((4.14)\) we have
\[
\|\partial_t u\|_{H^N(\Omega)} \lesssim \|\nabla^2 u\|_{H^N(\Omega)} + \|u \nabla u + \psi \nabla \psi\|_{H^N(\Omega)} \\
\lesssim \epsilon_1(t)^{-1} + \|u\|_{W^{N,\infty}(\Omega)} \|\nabla u\|_{H^N(\Omega)} + \|\psi\|_{W^{N,\infty}(\Omega)} \|\nabla \psi\|_{W^N(\Omega)} \\
\lesssim \epsilon_1(t)^{-1} + \epsilon_2(t)^{-5/4} \lesssim \epsilon_1(t)^{-1}.
\]
This combined with \((3.3)\) and \((3.6)\) to yields
\[
\|\mathcal{F}^{-1}(|\xi| \nabla \hat{u}(\xi))\|_{H^N(\Omega)} \lesssim \|u^{(1)}\|_{H^N(\Omega)} + t \|\partial_t u\|_{H^N(\Omega)} + \|u\|_{H^N(\Omega)} \lesssim \epsilon_1.
\]
We complete the proof of the lemma. \(\square\)

Finally, We state the following useful lemma.

**Lemma 4.8.** With the hypothesis in Proposition 3.4, for any \(t \in [0, T]\), we have
\[(4.17) \quad \left(\sum_k 2^{2N(1)k^+} \|\xi| \nabla \hat{\psi}_k\|_{L^2}^2\right)^{1/2} \lesssim \epsilon_1.\]

**Proof.** By \((3.3)\), \((3.6)\) and \((3.7)\) we bound this by
\[
\left(\sum_k 2^{2N(1)k^+} \|\xi| \nabla \hat{\psi}_k\|_{L^2}^2\right)^{1/2} \lesssim \|\mathcal{F}^{-1}(|\xi| \nabla \hat{\psi})\|_{H^N(\Omega)} + \|\psi\|_{H^N(\Omega)} \\
\lesssim \|x \cdot \nabla_x \Psi\|_{H^N(\Omega)} + \|\Omega \Psi\|_{H^N(\Omega)} + \|\psi\|_{H^N(\Omega)} \lesssim \epsilon_1.
\]
\(\square\)

### 5. Proof of Proposition 3.4 and Theorem 1.1

In this section we prove the bootstrap proposition 3.4 and then prove the main theorem 1.1. Here we start with the energy estimates \((3.8)\).

**Proposition 5.1.** With the notation and hypothesis in Proposition 3.4, for any \(t \in [0, T]\), we have
\[
\sum_{|n| \leq 1} \|u^{(a)}\|_{H^{N(\omega)}}^2 + \int_0^t \|\nabla u^{(a)}\|_{H^{N(\omega)}}^2 \|u^{(a)}\|_{H^{N(\omega)}}^2 + \|\psi^{(a)}\|_{H^{N(\omega)}}^2 \lesssim \epsilon_0^2.
\]

**Proof.** We start with the energy estimates of velocity \(u^{(a)}\). We define the energy functional by
\[
E_u^{(a)}(t) := \frac{1}{2} \|(S - 1)^{a_1} \Gamma^{a'} u\|_{H^{N(\omega)}} + \int_0^t \|\nabla (S - 1)^{a_1} \Gamma^{a'} u\|_{H^{N(\omega)}}^2 \|u^{(a)}\|_{H^{N(\omega)}} + \|\psi^{(a)}\|_{H^{N(\omega)}} = \mathcal{E}_u(t).
\]

Then by \(u\)-equation in \((3.4)\) we have
\[
\frac{d}{dt} E_u^{(a)}(t) \lesssim \sum_{|n| \in \{0, N(\omega)\}} \langle \partial^n \hat{Z}^a u, \partial^n (u^{(b)} \nabla u^{(c)} + \partial_j (\psi^{(b)} \psi^{(c)})) \rangle \\
\lesssim \|\nabla \hat{Z}^a u\|_{H^{N(\omega)}} \left( \sum_{b+c=a} \|\nabla u^{(b)}\|_{H^{N(\omega)}} \|u^{(c)}\|_{H^{N(\omega)}} + \sum_{|b| \leq 1} \|\psi^{(b)}\|_{H^{N(\omega)}} \|\langle \nabla \rangle \|_{L^\infty} \right),
\]
where \(\tilde{Z} = (S - 1)^{a_1} \Gamma^{a'}\). By the assumption \((3.6)\), \((3.3)\) and H"older inequality we obtain
\[
E_u^{(a)}(t) \lesssim E_u^{(a)}(0) + \int_0^t \partial_s E_u^{(a)}(s) \|u^{(a)}\|_{H^{N(\omega)}} \|u^{(a)}\|_{H^{N(\omega)}} \lesssim \epsilon_0^2.
\]
Next, we prove the energy estimate of differentiated field $\psi$. Let $\mathcal{N}$ be the nonlinearities in $\psi$-equation

$$\mathcal{N} := (A_{d+1} + (A_t^2 + u \cdot A))\psi_m - i\partial_m u \cdot \psi + i \text{Im}(\psi_t \overline{\psi}_m)\psi_t.$$  

By $\psi$-equation and $\text{div}\ u = \text{div}\ A = 0$ we have

$$\frac{d}{dt}\left\|\psi\right\|_{H^N(0)}^2 = \sum_{|a| \leq 0, N(0)} \left[ - \text{Re}\left\{\partial^a\psi, \partial^a((u + 2A) \cdot \nabla \psi)\right\} + \text{Re}\left\{\partial^a\psi, \partial^a\mathcal{N}\right\} \right]$$

$$\lesssim \|\psi\|_{H^N(0)}\left(\|\nabla (u + A) \cdot \psi\|_{H^N(0)} + \|\mathcal{N}\|_{H^N(0)}\right).$$

We use (4.7) and (4.3) to bound the second term by

$$\|\nabla (u + A) \cdot \psi\|_{H^N(0)} \lesssim \|\nabla (u + A)\|_{L^\infty} \|\psi\|_{H^N(0)} + \|\nabla (u + A)\|_{H^N(0)} \|\psi\|_{L^\infty} \lesssim \epsilon_1^3 t^{-5/4}.$$  

By this estimate, (4.13), (3.6) and Hölder inequality we obtain

$$\|\psi\|_{H^N(0)}^2 \lesssim \|\psi(0)\|_{H^N(0)}^2 + \int_0^t \epsilon_1 (\epsilon_1 (s)^{-5/4 + 3\delta} + \epsilon_1 (s)^{-3/4}) \|\nabla u\|_{H^N(0)} ds$$

$$\lesssim \epsilon_0^2 + \epsilon_1^3 \lesssim \epsilon_0^2.$$

Finally, we prove the energy estimate for $\psi^{(a)}$ with $|a| = 1$. By $\psi$-equation in (3.4) and $\text{div}\ u = \text{div}\ A = 0$ we have

$$\frac{d}{dt}\|\psi^{(a)}\|_{H^N(1)}^2 = \sum_{|a| = 1} \left[ - \text{Re}\left\{\partial^a\psi^{(a)}, \sum_{b+c=a} \partial^b((u^{(b)} + 2A^{(b)}) \cdot \nabla \psi^{(c)})\right\} + \text{Re}\left\{\partial^a\psi^{(a)}, \partial^a\mathcal{N}^{(a)}\right\} \right]$$

$$\lesssim \sum_{|a| = 1} \|\psi^{(a)}\|_{H^N(1)} \left(\|\nabla \frac{1}{2}(u^{(a)}, A^{(a)})\|_{H^N(1)} \|\nabla \psi\|_{W^N,6} + \|\nabla (u, A)\|_{W^N,\infty} \|\psi^{(a)}\|_{H^N(1)} + \|\mathcal{N}^{(a)}\|_{H^N(1)}\right).$$

Then this combined with Lemma (4.5) and (4.13) to give

$$\|\psi^{(a)}\|_{H^N(1)}^2 \lesssim \epsilon_0^2 + \int_0^t \epsilon_1 (\epsilon_1 (s)^{-5/4 + 3\delta} + \epsilon_1 (s)^{-1}) \|\nabla^{1/2} u^{(a)}\|_{H^N(1)} + \epsilon_1 (s)^{-3/4} \|\nabla u\|_{H^N(0)} ds$$

$$\lesssim \epsilon_0^2 + \epsilon_1^3 \lesssim \epsilon_0^2.$$  

The energy estimate for $\psi^{(a)}$ follows, and hence we complete the proof of the proposition.

Next, we prove the bound (3.9).

**Proposition 5.2.** With the hypothesis in Proposition 3.4, for any $t \in [0, T]$, we have

$$\|x \cdot \nabla \Psi\|_{H^N(1)} \lesssim \epsilon_0.$$

**Proof.** By (3.3), the fact that $[e^{it\Delta}, S] = 0$ and (3.8), we have

$$\|x \cdot \nabla \Psi\|_{H^N(1)} \lesssim \|S\Psi\|_{H^N(1)} + \|t\partial_t \Psi\|_{H^N(1)}$$

$$\lesssim \|S\Psi\|_{H^N(1)} + \|t\partial_t \Psi\|_{H^N(1)} \lesssim \epsilon_0 + t\|\partial_t \Psi\|_{H^N(1)}.$$

(5.2)
Then it suffices to estimate the last term in the right-hand side of (5.2). By the $\psi$-equation in (3.3) and (3.4), it suffices to prove

(5.3) \[ \|u \cdot \nabla \psi\|_{H^{N(1)}} + \|A \cdot \nabla \psi\|_{H^{N(1)}} + \|N\|_{H^{N(1)}} \lesssim \epsilon_1^2(t)^{-5/4+3\delta}. \]

First, we estimate the term $u \cdot \nabla \psi$. We use (3.3) and integration by parts to rewrite this by

\[
\mathcal{F} P_k(u_k \cdot \nabla \psi_k)(\xi) = \varphi_k(\xi) \int e^{-i|\eta|^2} \overline{u_k}(\xi - \eta) \cdot i\eta \Psi_k(\eta) d\eta \\
= (2t)^{-1} \varphi_k(\xi) \int \nabla_\eta \overline{u_k}(\xi - \eta) \hat{\psi}_k(\eta) + \overline{u_k}(\xi - \eta) e^{-i|\eta|^2} \nabla_\eta \hat{\Psi}_k(\eta) d\eta.
\]

For the low-high interaction, i.e. $k_1 \leq k - 5$, $k_2 = k + O(1)$. By (3.2), Sobolev embedding and interpolation inequality we have

\[
\|P_k(u_{<k-5} \cdot \nabla \psi_k)\|_{L^2} \lesssim t^{-1} \left( \|\mathcal{F}^{-1}(\nabla_\eta \overline{u_{<k-5}}(\eta))\|_{L^6} \|\psi_k\|_{L^3} + \|u_{<k-5}\|_{L^6} \|\mathcal{F}^{-1}(\nabla_\eta \hat{\Psi}_k(\eta))\|_{L^6} \right) \\
\lesssim t^{-1} \left( \|\mathcal{F}^{-1}(|\eta|\nabla_\eta \overline{u_{<k-5}}(\eta))\|_{L^2} \|\psi_k\|_{L^6}^{1/3} \|\psi_k\|_{L^2}^{2/3} \\
+ \|u\|_{L^6} \|u\|_{L^2} \|\mathcal{F}^{-1}(|\eta|\nabla_\eta \hat{\Psi}(\eta))\|_{L^2} \right).
\]

We then use (4.5), (4.15), (4.4) and (4.17) to bound this by

\[
\left( \sum_k 2^{2N(1)k^+} \|P_k(u_{<k-5} \cdot \nabla \psi_k)\|^2_{L^2} \right)^{1/2} \lesssim \epsilon_1^2(t)^{-5/4}.
\]

For the high-low interaction, i.e. $k-5 \leq k_1 \leq k+5$, $k_2 < k+6$. By (3.2), Sobolev embedding and interpolation inequality we have

\[
\sum_{k_2 < k+6} \|P_k(u_k \cdot \nabla \psi_{k_2})\|_{L^2} \\
\lesssim \sum_{k_2 < k+6} t^{-1} \left( \|\mathcal{F}^{-1}(\nabla_\eta \hat{u}_k(\eta))\|_{L^6} \|\psi_{k_2}\|_{L^6} \right) \\
+ \|u_k\|_{L^{6/3}} \|\mathcal{F}^{-1}(\nabla_\eta \hat{\Psi}_{k_2}(\eta))\|_{L^{6/3}} \\
\lesssim t^{-1} \left( \|\mathcal{F}^{-1}(|\eta|\nabla_\eta \hat{u}_k(\eta))\|_{L^2} \|\psi\|_{W^{1,\frac{6}{7+3\delta}}} \\
+ \|u_k\|_{L^{6/3}} \|u_k\|_{L^2} \left( ||\mathcal{F}^{-1}(|\eta|\nabla_\eta \hat{\Psi}(\eta))\|_{H^1} + \|\psi\|_{H^1} \right) \right).
\]

We use interpolation inequality, (4.4), (4.15), (4.3) and (4.17) to bound this by

\[
\left( \sum_k 2^{2N(1)k^+} \|P_k(u_{k+O(1)} \cdot \nabla \psi_{<k+6})\|^2_{L^2} \right)^{1/2} \lesssim \epsilon_1^2(t)^{-\frac{5}{4}+2\delta}
\]

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For the high-high interaction, i.e. $k_1 > k + 5$, $k_2 = k + O(1)$, by Bernstein’s inequality and (3.2) we have
\[
\|P_k(u_{k_1} \cdot \nabla \psi_{k_2})\|_{L^2} \lesssim t^{-1} 2^{3(\delta k - N(1)k^+)} \epsilon(t)^{-1/36} \left( \|\mathcal{F}^{-1}(|\eta| \nabla \hat{u}_{k_1}(\eta))\|_{L^2} + \|\mathcal{F}^{-1}(|\eta| \nabla \hat{\Psi}(\eta))\|_{L^2} \right).
\]
We then use (4.4) and (4.5) to bound this by
\[
\sum_{k_1 \sim k_2 > k + O(1)} \|P_k(u_{k_1} \cdot \nabla \psi_{k_2})\|_{L^2} \lesssim t^{-1} 2^{3(\delta k - N(1)k^+)} \epsilon(t)^{-1/36} \left( \|\mathcal{F}^{-1}(|\eta| \nabla \hat{u}_{k_1}(\eta))\|_{L^2} + \|\mathcal{F}^{-1}(|\eta| \nabla \hat{\Psi}(\eta))\|_{L^2} \right).
\]
By (3.6), (4.15) and (4.17), this implies
\[
\left[ \sum_k 2^{N(1)k^+} \sum_{k_1 \sim k_2 > k + O(1)} \|P_k(u_{k_1} \cdot \nabla \psi_{k_2})\|_{L^2} \right]^{1/2} \lesssim \epsilon(t)^{-1/36} \left( \|\mathcal{F}^{-1}(|\eta| \nabla \hat{u}(\eta))\|_{H^1} + \|u\|_{H^1} + \|\mathcal{F}^{-1}(|\eta| \nabla \hat{\Psi}(\eta))\|_{H^1} + \|\psi\|_{H^1} \right) \lesssim \epsilon(t)^2 \epsilon(t)^{-1/36}.
\]
Hence, the bound (5.3) for $u \cdot \nabla \psi$ follows.

By (4.9) and (4.3) we bound the second term in (5.3) by
\[
\|A \cdot \nabla \psi\|_{H^{N(1)}} \lesssim \|\nabla^{1/2} A\|_{H^{N(1)}} \|\nabla \psi\|_{W^{N(1),6}} \lesssim \epsilon(t)^{-3/4 + 3(\delta)}.
\]
The last term in (5.3) can be proved using the similar argument to that of (4.13) with regularity index $N(1)$. We have
\[
\|\mathcal{N}\|_{H^{N(1)}} \lesssim \epsilon(t)^{-5/4 + 3(\delta)} + \epsilon(t)^{-3/4} \|\nabla u\|_{H^{N(1)}} \lesssim \epsilon(t)^{-5/4 + 3(\delta)}.
\]
This completes the proof of the lemma. \(\square\)

From the above estimate of $\partial_t \Psi$ we obtain the scattering.

**Proposition 5.3** (Scattering). With the hypothesis in Proposition 3.4, for any $t \in [0, T]$, we have
\[
\lim_{t \to \infty} \|\psi - e^{it\Delta} \Psi_\infty\|_{H^{N(1)}} = 0.
\]

**Proof.** By the definition of $\Psi$ we have
\[
\Psi(t) = \Psi(0) + \int_0^t \partial_s \Psi(s) ds = \Psi(0) - \int_0^t e^{-is\Delta} (u + 2A) \cdot \nabla \psi + \mathcal{N}(s) ds.
\]
By (5.3) we obtain
\[ \| \Psi(t_1) - \Psi(t_2) \|_{H^N(1)} \lesssim \int_{t_1}^{t_2} \| ((u + 2A) \cdot \nabla \psi + N)(s) \|_{H^N(1)} ds \]

\[ \lesssim \int_{t_1}^{t_2} \epsilon_1^2(s)^{-5/4 + 3\delta} ds \rightarrow 0, \quad \text{as } t_1, t_2 \rightarrow \infty. \]

This motivates us to define the function \( \Psi_\infty \) as
\[ \Psi_\infty := \Psi(0) - \int_0^\infty e^{-is\Delta} ((u + 2A) \cdot \nabla \psi + N)(s) ds. \]

Then we have
\[ \| \psi - e^{it\Delta} \Psi_\infty \|_{H^N(1)} = \| e^{-it\Delta} \psi - \Psi_\infty \|_{H^N(1)} \]
\[ \leq \int_t^\infty \| ((u + 2A) \cdot \nabla \psi + N)(s) \|_{H^N(1)} ds \]
\[ \lesssim \epsilon_1^2(t)^{-1/4 + 3\delta}, \quad \text{as } t \rightarrow \infty. \]

This implies that \( \psi \) scatters to the linear solution \( e^{it\Delta} \Psi_\infty \) in the lower regularity Sobolev space \( H^N(1) \).

In order to prove the Theorem 1.1, we need the following local existence result and an useful lemma.

**Proposition 5.4** (Local solution). The Cauchy problem (1.1) with \( (u_0, \phi_0) \in H^k \times H^k_{Q} \), for any integer \( k \geq \left[ \frac{d}{2} \right] + 1 \), admits a unique local solution \( (u, \phi) \) satisfying
\[ \| u \|_{H^k} + \| \nabla u \|_{L^2([0,t];H^k)} + \| \phi \|_{H^k_{Q}} \leq C(k, \| u_0 \|_{H^k}, \| \phi_0 \|_{H^k_{Q}}), \]
for any \( t \in [0,T] \), where \( T = T(\| u_0 \|_{H^2}, \| \nabla \phi_0 \|_{H^2_{Q}}). \)

**Lemma 5.5**. With the notation in Proposition 2.1, if the differentiated fields \( \psi \) has the additional property
\[ \sup_{|a| \leq 1, t \in [0,T]} \| \psi^{(a)}(t) \|_{H^N(0)} \leq \epsilon. \]

Then we have the bound
\[ \| \nabla \phi \|_{H^N(0)} + \| \nabla Z \phi \|_{H^N(1)} \lesssim \epsilon. \]

**Proof.** In fact, we prove
\[ \| \nabla (\phi, v, w) \|_{H^N(0)} + \| \nabla Z (\phi, v, w) \|_{H^N(1)} \lesssim \epsilon. \]

Recall the identity (2.3),
\[
\begin{align*}
\partial_m \phi &= v \Re \psi_m + w \Im \psi_m, \\
\partial_m v &= -\phi \Re \psi_m + w A_m, \\
\partial_m w &= -\phi \Im \psi_m - v A_m.
\end{align*}
\]

Since \( |\phi| = |v| = |w| = 1 \), by (2.8) and Sobolev embedding we have
\[ \| \nabla (\phi, v, w) \|_{L^2} \lesssim \| \psi \|_{L^2} + \| A \|_{L^2} \lesssim \epsilon. \]
We then prove the bound the first term in (5.5) by induction. Precisely, assume that
\[ \| \nabla \phi \|_{H^0} + \| \nabla v \|_{H^0} + \| \nabla w \|_{H^0} \lesssim \epsilon, \]
for any \( n < l \leq N(0) \).

By Sobolev embedding and the above inductive assumption, we have
\[ \| \nabla^{l+1} \phi \|_{L^2} \lesssim \| \nabla \phi \|_{L^2} + \sum_{l_1 + l_2 = l, 0 < l_2 < l} \| \nabla^{l_1} \psi \|_{L^4} \| \nabla^{l_2} (v, w) \|_{L^4} + \| \psi \|_{L^\infty} \| \nabla^l (v, w) \|_{L^2} \]
\[ \lesssim \| \psi \|_{H^{N(0)}} (1 + \| \nabla (v, w) \|_{H^{l-1}}) \]
\[ \lesssim \| \psi \|_{H^{N(0)}} \lesssim \epsilon. \]

Similarly, we also have
\[ \| \nabla^{l+1} v \|_{L^2} + \| \nabla^{l+1} w \|_{L^2} \lesssim (\| \psi \|_{H^{N(0)}} + \| A \|_{H^{N(0)}}) (1 + \| \nabla (\phi, v, w) \|_{H^{l-1}}) \]
\[ \lesssim \| \psi \|_{H^{N(0)}} \lesssim \epsilon. \]

We continue to bound the second term \( \nabla Z (\phi, v, w) \). By (2.3) we have
\[ \| \nabla Z (\phi, v, w) \|_{L^2} \lesssim \| \nabla (\phi, v, w) \|_{L^2} + \| Z (\psi, A) \|_{L^2} \| (\phi, v, w) \|_{L^\infty} \]
\[ + \| (\psi, A) \|_{L^2} \| Z (\phi, v, w) \|_{L^6} \]
\[ \lesssim \epsilon + \epsilon \| \nabla Z (\phi, v, w) \|_{L^2}. \]

Similarly, we also have
\[ \| \nabla^2 Z (\phi, v, w) \|_{L^2} \lesssim \| \nabla^2 (\phi, v, w) \|_{L^2} + \| Z (\psi, A) \|_{H^1} \| (\phi, v, w) \|_{W^{1,\infty}} \]
\[ + \| \nabla (\psi, A) \|_{L^2} \| Z (\phi, v, w) \|_{L^6} + \| (\psi, A) \|_{L^\infty} \| \nabla Z (\phi, v, w) \|_{L^2} \]
\[ \lesssim \epsilon. \]

These conclude the bound (5.5), and complete the proof of the lemma. \( \Box \)

Finally, from the above Proposition 5.4, the bootstrap Proposition 3.4 and Lemma 5.5, we prove our main result: Proof of Theorem 1.1.

First, we use continuity method to prove global existence. Since initial data \((u_0, \phi_0)\) satisfies (1.6), by Proposition 5.4 we assume that there exists maximal lifespan \( T > 1 \) such that for any \( 0 \leq t \leq T \)
\[ \sup_{|a| \leq 1} \{ \| Z^a u(t) \|_{H^{N(a)}} + \| \nabla Z^a u \|_{L^2([0,t];H^{N(a)})} \} + \| \phi(t) \|_{H^{N(0)+1}} + \| \nabla Z \phi(t) \|_{H^{N(0)}} \lesssim \epsilon_1, \]
and in the Coulomb gauge
\[ \| \nabla \psi \|_{H^{N(1)}} \lesssim \epsilon_1, \]
where \( \epsilon_1 = \epsilon_0^{2/3} \), \( \Psi = e^{-u \Delta} \psi \) and \( \psi = \partial \phi \cdot v + i \partial \phi \cdot w \) as in section 2. Then by Lemma 2.2 we obtain that \((u, \psi)\) is a solution of system (2.10) with initial data \((u_0, \psi_0)\) satisfying
\[ \sup_{|a| \leq 1} \{ \| u_0^{(a)} \|_{H^{N(a)}} + \| \psi_0^{(a)} \|_{H^{N(a)}} \} \lesssim \epsilon_0, \]
and also satisfies the bound
\[ \sup_{|a| \leq 1, t \in [0, T]} \{ \| u^{(a)}(t) \|_{H^{N(a)}} + \| \nabla u^{(a)} \|_{L^2([0,t];H^{N(a)})} + \| \psi^{(a)} \|_{H^{N(a)}} \} \lesssim \epsilon_1. \]
From Proposition 3.4 we obtain that the solution \((u, \psi)\) have the improved bound
\[
(5.6) \sup_{|a| \leq 1, t \in [0, T]} \|u^{(a)}\|_{H^N(a)} + \|\nabla u^{(a)}\|_{L^2(\mathbb{R}^d; H^N(a))} + \|\psi^{(a)}\|_{H^N(a)} \lesssim \epsilon_0.
\]

By mass conservation (1.3) and Lemma 5.5, this gives the improved bound (1.7). Hence, We can extend the solution \((u, \phi)\) by Proposition 5.4 and obtain the global solution.

Next, by Bernstein’s inequality we have
\[
\|\phi - Q\|_{L^\infty} \lesssim \sum_k 2^{dk/p} \|P_k(\phi - Q)\|_{L^2}^{2/p} \|P_k(\phi - Q)\|_{L^\infty}^{1-2/p} \lesssim \sum_k 2^{(d+2-\delta)k} \|P_k(\phi - Q)\|_{L^2}^{2/p} \|\nabla P_k(\phi - Q)\|_{L^\infty}^{1-2/p} \lesssim \|\phi - Q\|_{H^1}^{2/p} \|\nabla \phi\|_{L^\infty}^{1-2/p},
\]
where we choose \(p = d + 2 - \delta\) with \(\delta > 0\) small. Then by (2.3) and (4.4) we obtain
\[
\|\phi - Q\|_{L^\infty} \lesssim \epsilon_0 \langle t \rangle^{-\frac{3}{2} + \frac{d}{2p}} \to 0, \quad \text{as} \ t \to \infty.
\]
The scattering of \(\nabla \phi\) in Coulomb gauge is given by Proposition 5.3, and the decay of \(u\) is obtained by (4.5). Hence we complete the proof of Theorem 1.1. □

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