Symmetry nonintegrability for extended $K(m,n,p)$ equation

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Abstract
In the present paper we study symmetries of extended $K(m,n,p)$ equations and prove that the equations from this class have no generalized symmetries of order greater than five and hence are not symmetry integrable.

1 Introduction
Consider the equation, cf. [2, 14] and references therein:

$$u_t = a(u^p)_{xxxx} + b(u^n)_{xxx} + c(u^m)_x + f(u),$$

(1)

to which we shall refer to as the extended $K(m,n,p)$ equation, as for $f = 0$ equation (1) boils down to the $K(m,n,p)$ equation from [2]:

$$u_t = a(u^p)_{xxxx} + b(u^n)_{xxx} + c(u^m)_x.$$  

(2)

Here $a, b, c$ are real constants and $m, n, p$ are integers.

Equation (2), as well as its special case for $a = 0$, the well-known $K(m,n)$ equation, see e.g. [8, 14] and references therein, are the subject of intense research since they admit compacton solutions [2, 8, 9]. Such solutions are of significant interest since they share many features with the famous solitons but, unlike the latter, have compact support [8, 9].

It is well known that integrable systems have rich symmetry algebras, cf. e.g. [6, 7, 12] and references therein. A simple observation that the computation of symmetries is, to a large extent, an algorithmic procedure has, in a series of nontrivial developments, lead to symmetry-based integrability tests and the related notion of symmetry integrability [6, 7, 4, 5, 11, 13]. For a recent survey on the latter, see [3], and for integrability in general see e.g. [12, 1, 5, 3] and references therein.

Recall that integrable systems enjoy many remarkable properties like infinite hierarchies of conservation laws or large sets of explicit exact solutions, cf. e.g. [4, 5, 7, 11, 12] and references therein, which makes clear the importance of establishing integrability of a given equation.

In the remaining part of the present paper we show that for $a \neq 0$ and $p \neq 1, -4$ equation (1), and hence the original $K(m,n,p)$ equation (2) as well, is not symmetry integrable as it has no generalized symmetries of order greater than five, see Theorem 1 below for details.

2 Main result
For partial derivatives in $x$ we employ the usual notation $u_j = \partial^j u / \partial x^j$, so $u_0 \equiv u$, $\partial u / \partial x = u_1 \equiv u_x$, $\partial^2 u / \partial x^2 \equiv u_{xx} = u_2$, etc.
Let \( \mathcal{A} \) be an algebra of smooth functions of \( x, t, u^{1/5}, u^{-1/5} \) and finitely many \( u_j \) (we stress that of course different functions from \( \mathcal{A} \) may depend on a different, but always finite, number of \( u_j \)). In spirit of [7] the elements of \( \mathcal{A} \) will be referred to as \textit{differential functions}, and it will be tacitly assumed from now on that all functions encountered below are differential functions unless otherwise explicitly stated.

**Theorem 1.** For any \( f(u) \) and arbitrary real constants \( a, b, c \) and arbitrary integers \( m, n, p \) such that \( a \neq 0 \) and \( p \neq 1, -4 \), the extended \( K(m, n, p) \) equation

\[
    u_t = a(u^p)_xxxxx + b(u^n)xxx + c(u^m)x + f(u)
\]

has no generalized symmetries of order greater than 5.

3 Preliminaries

Here we briefly recall a few known results following mostly [4, 5, 7, 10, 13].

Consider for now an evolution equation in one dependent and two independent variables of the form

\[
    u_t = F(x, u, u_1, \ldots, u_n), \quad n \geq 2.
\]

The operators of total derivatives \( D_x \) and \( D_t \) then take the form [4, 5]

\[
    D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} u_{i+1} \frac{\partial}{\partial u_i}, \quad D_t = \frac{\partial}{\partial t} + \sum_{i=0}^{\infty} D_i^x(F) \frac{\partial}{\partial u_i}
\]

An evolutionary vector field \( v_Q = Q \partial / \partial u \) is called [7] a \textit{generalized symmetry} for \( (4) \) if its characteristic \( Q \) satisfies

\[
    D_t(Q) = D_F(Q).
\]

Here for an \( F = F(x, t, u, u_1, \ldots, u_k) \) we denote [7] by \( D_F \) its (formal) Fréchet derivative

\[
    D_F = \sum_{j=0}^{k} \frac{\partial F}{\partial u_j} D_j^x.
\]

4 Proof of the main result

To simplify writing, denote now by \( F \) the right-hand side of (3) and let \( n = \text{ord } F \), so in our case \( n = 5 \).

It follows from well-known general results, cf. e.g. [7, 10], that if \( Q \) is a characteristic of generalized symmetry for \( (3) \) and \( k = \text{ord } Q > 1 \) then we have

\[
    \frac{\partial Q}{\partial u_k} = c_k(t) \left( \frac{\partial F}{\partial u_n} \right)^{k/n}
\]

for some function \( c_k(t) \).

Moreover, Theorem 1 from [10] then implies that if furthermore \( k > n \) then we have

\[
    D_x \left( \frac{\partial Q}{\partial u_{k-n+1}} (\rho_{-1})^{n-k-1} \right) = D_x(h_{k-n+1}) + \frac{k}{n^2} c_k(t) D_t(\rho_{-1}) + \frac{1}{n} \dot{c}_k(t) \rho_{-1},
\]

for some differential function \( h_{k-n+1} \).
Here $\rho_{-1}$ stands for the so-called minus first canonical density [4]:

$$\rho_{-1} = \left(\frac{\partial F}{\partial u_n}\right)^{-1/n}.$$

It is immediate that for (7) to hold, the expression

$$\frac{k}{n^2} \hat{c}_k(t) D_t \left( \left(\frac{\partial F}{\partial u_n}\right)^{-1/n} \right) + \frac{1}{n} \hat{c}_k(t) \left(\frac{\partial F}{\partial u_n}\right)^{-1/n}$$

should belong to the image of $D_x$.

In turn, a well-known, see e.g. [7], necessary condition for the latter is

$$\frac{\delta}{\delta u} \left( \frac{k}{n^2} \hat{c}_k(t) D_t \left( \left(\frac{\partial F}{\partial u_n}\right)^{-1/n} \right) + \frac{1}{n} \hat{c}_k(t) \left(\frac{\partial F}{\partial u_n}\right)^{-1/n} \right) = 0,$$

where by $\frac{\delta}{\delta u}$ we mean the operator of variational derivative, cf. e.g. [7, 13],

$$\frac{\delta}{\delta u} = \sum_{i=0}^{\infty} (-1)^i D^i_x \circ \frac{\partial}{\partial u_i}.$$

Denote the left hand-side of (8) by $C$ and observe that

$$\frac{\partial C}{\partial u_4} = \frac{2k}{625} (ap)^{4/5} c_k u^{(4/5)p-19/5}(p-1)(p+4)(3p+2).$$

Clearly, for $C$ to vanish it is necessary that $\partial C/\partial u_4$ vanishes as well.

It is obvious from (9) that under the assumptions of our theorem $\partial C/\partial u_4$ cannot vanish unless $c_k = 0$, and thus under the said assumptions equation (3) cannot have a generalized symmetry of order greater than 5.

5 Conclusion

Thus, for $a \neq 0$ and $p \neq 1$, equation (3) has no generalized symmetries of order greater than five and cannot have an infinite hierarchy of generalized symmetries of arbitrarily high order, so (3) for $a \neq 0$ and $p \neq 1$, $-4$ is not symmetry integrable. This result paves the way to finding all generalized symmetries for (3), or its special case (2), because it suffices now to find the generalized symmetries with the characteristics of order up to 5, which can be done for instance using computer algebra software like [1]. However, in doing so one runs into plenty of various special cases, treating all of which in detail would go beyond the scope of the present paper.

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