Solution of the task of planar geometric optics with a non-stationary source

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Abstract. In the present paper a geometric model of acquisition of receiver curve in the tasks of planar geometric optics with a non-stationary emitter is proposed. The model is based on cyclographic mapping of Euclidean space on a plane and significantly differs in terms of simplicity from the existing mathematical models applied in solution of similar tasks. Furthermore, the proposed model differs in applying uniform solution algorithms for direct and inverse planar geometric optics tasks. The variety of such tasks is dictated by variety of geometric forms of the sought element of the triad “emitter-reflector-receiver” as well as the initial conditions, where a single element of the triad is unknown and sought. The results of the present paper can find application in the area of metrology for optical profile control of machined cylindrical workpieces constituting reflector of the triad.

1. Introduction

Ray focusing of given electromagnetic radiation into a certain line or on certain surface is essential in various areas of industry. If wavelength of such radiation matches wavelength of optical radiation (light), i.e. lies between $10^{-3} \text{µm}$ and $10^{3} \text{µm}$, then the laws of geometric optics are applicable to the problem [1]. The mentioned wavelength band includes the ultraviolet, visible and infrared parts of the spectrum. The triad of curves and surfaces “emitter – reflector – receiver” known in geometric optics is widely applied in the mentioned wavelength band. Manipulation of a triad, in general, boils down to optical transformation of one bundle of straight lines (rays) into another, e.g. in the task of acquiring optimal reflector geometry given emitter and receiver. Similar tasks are relatively common: ultraviolet drying of paintwork materials, uniform illuminance drying of printed sheets in polygraphy [2], antenna design [3,4], solar concentrator formation for eco-friendly energy generation [5], optical inspection of workpiece geometry, etc.

Optical inspection of geometry of various workpieces finds more and more industrial application, e.g. in paper [6] an algorithm of profile generation and revolution surface contour representation on the basis of optical triangulation with use of mirrored emission is considered. The algorithm is based on numerical solution to a system of differential equations, which complicates computational process of workpiece profile acquisition. The authors of the present paper propose an approach to optical triangulation modeling based on cyclographic mapping of Euclidean surface on a plane. The approach utilizes the laws of geometric optics and considers the source (emitter) as a non-stationary object.

2. Problem definition

The existing triangulation method for finding profiles and contour pictures of rotation surfaces based on specularly reflected emanation uses a mathematical model characterized by computational complexity based on numerical methods. The task is to build a simpler geometric model and the
corresponding algorithm for determining the receiver with a fixed reflector and a moving source. The model is based on a cyclographic mapping of Euclidean space onto a plane.

3. Theory

3.1. Cyclographic mapping in the tasks of planar geometric optics.

A mapping of points of space as cycles (directed circles) of plane \((xy)\) is called cyclographic mapping. It was first introduced and further developed by German geometers W. Fiedler (1882), E. Müller and J. Krames (1929). The modern level of cyclographic mapping and its numerous theoretical realizations and practical applications are considered in papers [7-10]. The cyclographic method puts a point of space in correspondence with a certain projecting cone with vertex matching the point and angle at the vertex between its generatrix and axis equal to 45°. In classic cyclography such cone is called an “\(\alpha\)-cone”. The intersection of an \(\alpha\)-cone with plane of projection \(\Pi_1(xy)\) results in a circle, which is further transformed into a cycle. The direction of the cycle is determined by the position of the initial point in space: if the point is above the plane of projection, the cycle is directed counter clockwise, otherwise the cycle is directed clockwise.

Solutions to the tasks of planar geometric optics with stationary elements of the triad “emitter – reflector – receiver” are considered in papers [11-13]. In these papers the solutions to the tasks of direct and inverse problems of geometric optics for various geometric schemes, different in form of stationary elements of the triad and their positional relationship, are considered. In these solutions each element of the triad is put into correspondence with its spatial cyclographic image. A central bundle of rays is put into correspondence with an \(\alpha\)-cone, orthogonal projection of its vertex determined by correspondent central bundle coordinates. A parallel bundle of rays defined by a straight line on a plane is put into correspondence with a plane inclined on angle equal to 45° with respect to projection plane. A diffuse bundle defined by a certain curve on projection plane \(\Pi_1(xy)\) is put into correspondence with a linear surface (\(\alpha\)-surface), its generatrices inclined on angle equal to 45° with respect to projection plane. A cylindrical surface is constructed for a line representing reflector, for which this line serves as directrix [11-14].

The article deals with the task of receiver reconstruction given a source (emitter) moving along a certain trajectory and a stationary reflector is considered.

3.2 Algorithm of solution of the task of geometric optics with non-stationary emitter.

Let us consider reflector and emitter trajectory both given in the form of two arbitrary curves lying in the same plane, \(\bar{x}\) and \(\bar{a}\) correspondingly (figure 1). It is required to acquire the emitter. Let us consider the emitter fixed on a certain point of its trajectory and put the emitter and the receiver in correspondence with cyclographic images: an \(\alpha\)-cone \(\mathcal{P}_i\) and a cylindrical surface \(\Sigma\) correspondingly. In this case, the task in question boils down to the direct task of geometric optics, i.e. acquiring the receiver. In the approach considered in papers [11-13], the curve of intersection \(\bar{P}\) of two \(\alpha\)-surfaces is acquired first, its cyclographic projection is acquired subsequently on the bases of formulas [9]

\[
\begin{align*}
x_{P(1,2)} &= x + z \frac{-x'z + y' \sqrt{(x')^2 + (y')^2 - (z')^2}}{(x')^2 + (y')^2}, \\
y_{P(1,2)} &= y + z \frac{-y'z + x' \sqrt{(x')^2 + (y')^2 - (z')^2}}{(x')^2 + (y')^2}.
\end{align*}
\]

(1)

where \(x', y', z'\) represent derivatives of functions of coordinates \(x(t), y(t), z(t)\) describing the curve of intersection, with respect to parameter \(t\).
The acquired cyclographic projection in general consists of two branches: one representing the sought receiver and the other matching the base of the cone modeling the emitter. Since the emitter is a non-stationary object, let us construct more $\alpha$-cones of projection $\Psi_1, \Psi_{i+1}, \ldots, \Psi_n$ with vertices $K_1, K_{i+1}, \ldots, K_n$ correspondingly, orthogonal projections $K_1^l, K_{i+1}^l, \ldots, K_n^l$ of which lie on emitter trajectory $\vec{a}$ (figure 2). The intersection between the generated $\alpha$-cones and the cylindrical reflector surface $\Sigma$ occurs in curves $\vec{P}_1, \vec{P}_{i+1}, \ldots, \vec{P}_n$, each forming a corresponding cyclographic projection in projection plane $\Pi_{i}(xy)$ consisting of two branches included in multitudes $\vec{P}_{1,i}, \vec{P}_{i+1,i}, \ldots, \vec{P}_{n,i}$ and $\vec{P}_{2,i}, \vec{P}_{2,i+1}, \ldots, \vec{P}_{2,n}$ by one. Only one of the resultant multitudes, in this case $\vec{P}_{1,i}, \vec{P}_{1,i+1}, \ldots, \vec{P}_{1,n}$, consists of actual receivers; the other one consists of imaginary receivers. The subsequent solution of the considered task involves only the multitude of actual receivers, so the subsequent actions consider only the curves of multitude $\{\vec{P}_{1,i}\}_{i\in N}$ (figure 2).

It is obvious that the sought receiver constitutes an envelope of multitude of cyclographic projections $\{\vec{P}_{1,i}\}_{i\in N}$. Let us acquire this envelope. A one-parameter set of $\alpha$-cones moving along emitter trajectory defines two linear surfaces. As follows from the definition of cyclographic mapping, an $\alpha$-cone has equal height and radius, therefore, it is possible to acquire two curves constituting directrices of the sought $\alpha$-surfaces. The first curve $\vec{m}(x_m, y_m, z_m)$ is planar and connects the vertices of all $\alpha$-cones, while the second curve $\vec{n}$ constitutes an envelope of $\alpha$-cone bases, consisting of two branches $\vec{n}_1$ and $\vec{n}_2$. The curve $\vec{n}$ can be acquired through the equations of equidistant with respect to curve $\vec{a}$, as, inherently, due to constant radius of $\alpha$-cone bases, the envelopes constitute equidistant curves of one-parameter multitude of circles of bases of the $\alpha$-cones. The equations of equidistant curves are known and of the following form:

$$
\begin{align*}
x_{ni} &= x_n \pm \frac{h \cdot y_n}{\sqrt{(x_n)^2 + (y_n)^2}}, \\
y_{ni} &= y_n \pm \frac{h \cdot x_n}{\sqrt{(x_n)^2 + (y_n)^2}},
\end{align*}
$$

where $h$ represents distance from the initial curve, on which equidistant curve is introduced (in this case equal to $\alpha$-cone radius), $i = 1, 2$. 

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**Figure 1.** The initial data for the task of acquiring the emitter.
Figure 2. Curves of intersection \( \{ P^\alpha \}_i \) of various \( \alpha \)-cones \( \{ \Psi^\alpha \}_i \) with cylindrical reflector surface \( \Sigma \) and cyclographic projections \( \{ \overline{P}^\alpha \}_i \).

The \( \alpha \)-surfaces in question can be constructed utilizing the formulas for linear surfaces acquisition known in differential geometry [15]:

\[
\begin{align*}
X_i(t, l) &= x_m(t) + l\left[ x_{n,j}(t) - x_m(t) \right], \\
Y_i(t, l) &= y_m(t) + l\left[ y_{n,j}(t) - y_m(t) \right], \\
Z_i(t, l) &= z_m(t) + l\left[ z_{n,j}(t) - z_m(t) \right].
\end{align*}
\]

(2)

As a result, making use of the formulas (2), we acquire two linear surfaces \( \Phi_1 \) and \( \Phi_2 \) (figure 3) intersecting the cylindrical surface \( \Sigma \). The intersection between the surface \( \Phi_1 \) and the cylindrical surface \( \Sigma \) occurs along a certain curve \( \overline{S} \), which, in turn, constitutes an envelope of a one-parameter multitude of spatial curves \( \{ \overline{B}^\alpha \}_i \), \( n \in N \) of intersection between earlier constructed \( \alpha \)-cones and the reflector surface \( \Sigma \). After a cyclographic projection of curve \( \overline{S} \) is constructed according to the formulas (1), two branches of envelope \( \overline{S}^1_i \) and \( \overline{S}^2_i \) are acquired, one of them being the sought envelope of multitude of curves \( \{ \overline{B}^\alpha \}_i \) and, therefore, the sought receiver curve (figure 3, 4).

Supposing a cross section of a certain workpiece of cylindrical form is accepted as a reflector, the acquired curve \( \overline{S}^1_i \) will constitute an image of cross section of the initial workpiece. Comparing the acquired image with the given reference cross section, it is possible to study form deviation of the machined workpiece in the considered section.

4. Results of experiments

In order to validate the mentioned algorithm, the calculations were performed for the case mimicking cross section of a workpiece. In order to facilitate the calculations, a curve of second order was accepted as section contour, while emitter moved along a circular trajectory. Formulas of the mentioned curves are of the following form:
ellipse modeling workpiece cross section: \( x_1 = 4 \cdot \cos(t); y_1 = 3 \cdot \sin(t) \), where \( 0 \leq t \leq 2\pi \);
emitter trajectory: \( x_2 = \sqrt{29} \cdot \cos(\phi); y_2 = \sqrt{29} \cdot \sin(\phi) \), where \( 0 \leq \phi \leq 2\pi \).

The acquired receiver (curve \( \overline{S}_1 \)) is presented on figure 5. The equation of the acquired curve \( \overline{S}_1 \) is of the following form:

\[
x_{ni} = 4 \cos(t) + \frac{1}{A} \left( (7 - \sqrt{29} + \sqrt{A}) \cdot (\frac{-28 \sin(t)^2 \cdot \cos(t)}{\sqrt{A}} + 3 \cos(t) \sqrt{A - \frac{49 \cos(t)^2 \cdot \sin(t)}{A}}) \right);
\]
\[
y_{ni} = 3 \sin(t) + \frac{1}{A} \left( (7 - \sqrt{29} + \sqrt{A}) \cdot (\frac{-21 \cos(t)^2 \cdot \sin(t)}{\sqrt{A}} + 4 \sin(t) \sqrt{A - \frac{49 \cos(t)^2 \cdot \sin(t)}{A}}) \right),
\]

where \( A = 16 \sin(t)^2 + 9 \cos(t)^2 \), \( 0 \leq t \leq 2\pi \).

Figure 3. The resultant solution of the task of finding receiver with non-stationary emitter

Figure 4. The final result of solution of the task of finding receiver with non-stationary emitter on a plane
5. Consideration of results

Considering the above, we can draw a conclusion that the method of cyclographic mapping allows us to acquire receiver curve of virtually any form given a non-stationary emitter and a stationary reflector. Analysis of results of computational experiments shows us that the considered task is computationally simple, while its practical application in cylindrical workpiece optical form inspection systems can significantly facilitate the mathematical model as well as the corresponding computational tools and processing coordinates of physical points of a workpiece with subsequent reconstruction in CAD and similar specific software.

![Figure 5](image)

**Figure 5.** The result of construction of the envelope curve of the receiver $\overline{S}_i$ given a reflector in the form of ellipse and non-stationary emitter moving along circular trajectory.

6. Conclusion

The paper considers a geometric model of acquisition of receiver curve given a reflector curve and a non-stationary emitter. The model allows us to perform calculations based on acquiring a cyclographic projection of a curve and its optical property. Application of cyclographic projection of a curve in solution of such tasks allows us to avoid solving simultaneous differential equations by numerical methods, which is specific to the existing solutions. The results of the present paper can find application in the area of metrology for optical control of machined cylindrical workpieces.

7. References

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