Diffusive heat waves in random conformal field theory

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We propose and study a conformal field theory (CFT) model with random position-dependent velocity that, as we argue, naturally emerges as an effective description of heat transport in certain one-dimensional quantum many-body systems with static random impurities. We present exact analytic results for this model that elucidate how purely ballistic heat waves in standard CFT can acquire normal and anomalous diffusive contributions due to static impurities. Our results include impurity-averaged Green’s functions describing the time evolution of the energy density and the heat current, and an explicit formula for the thermal conductivity that, in addition to a universal Drude peak, has a non-trivial real regular contribution depending sensitively on the impurity correlations.

Keywords: Heat transport; CFT; Waves in random media; Normal and anomalous diffusion

Introduction. Heat transport has been modeled successfully by the diffusion equation since the time of Fourier. Still, the mathematical derivation of diffusion from microscopic models has remained an outstanding challenge [1]. Recent concerted efforts addressing this problem in one spatial dimension (1D) have led to important progress [2], including derivations in [3, 4] of diffusive effects within hydrodynamical descriptions of integrable quantum many-body systems [5, 6]. Meanwhile, exact results from conformal field theory (CFT), routinely used to effectively describe universal properties of 1D quantum many-body systems, show that standard (pure) CFT only supports purely ballistic transport, see, e.g., [7–10]. This points to the importance of randomness and impurities. Such a route was recently explored in [11] with positive results for CFT extended by a complex dynamical environment that varies in time and space. This, however, does not shed light on the question if and how static impurities in CFT can lead to diffusive behavior.

In this paper we propose and study CFT with random position-dependent velocity \( v(x) \) that, as we argue, emerges naturally as an effective description of certain 1D quantum many-body systems with random static impurities. Defining this velocity in terms of a Gaussian random function and using recent generalizations of CFT to inhomogeneous situations [10, 12–15], we obtain exact analytic results which elucidate how ballistic transport in standard CFT can acquire normal and anomalous diffusive contributions due to static impurities.

As a motivating example, consider a generalized XXZ spin chain with uniformly varying couplings \( J_i^x = J_i^y = J_i \) and \( J_i^z = J_i \Delta \) between spins on adjacent sites \( i \) and \( i+1 \), with constant \( \Delta \). It was argued in [14, 15] that, if \( J_i \) varies on length scales much larger than the lattice spacing and if \( -1 < \Delta < 1 \), then a generalized Luttinger model with position-dependent velocity \( v(x) \) provides an effective description of this system. Such a Luttinger model is an example of inhomogeneous CFT with central charge \( c = 1 \). We propose to use inhomogeneous CFT with random velocity \( v(x) \) to effectively describe, e.g., generalized XXZ spin chains with couplings \( J_i \) modeling static impurities that vary on mesoscopic scales, see Figure 1. We stress that there are many more examples to which our results apply, see [10].

![Figure 1](image-url)

Figure 1. Illustration of a velocity \( v(x) \) effectively describing a lattice system with impurities varying on a mesoscopic scale \( \alpha_{\text{imp}} \) much larger than the lattice spacing \( a \). For the XXZ spin chain described in the main text, \( x = i a \), and the color and size of the dots indicate the magnitude of the couplings \( J_i \).

We study heat transport in the random CFT model described above in two ways: (i) By deriving and exactly solving effective equations for heat transport. (ii) By computing the linear-response thermal conductivity as a function of frequency and deriving an explicit formula for its real regular part. Both approaches are non-perturbative but widely different in nature. For instance, (i) is beyond linear response. By averaging over the impurities, we show that the resulting heat waves deform diffusively, and we obtain exact results for the thermal diffusivity \( \alpha_{\text{th}} \) at long times and the zero-frequency limit \( L_{\text{th}} \) of the real regular thermal conductivity. We find that \( L_{\text{th}} \) and \( \alpha_{\text{th}} \) are proportional to each other, which establishes a link between the two approaches. The results obtained by both approaches show that, in general, there are normal and anomalous diffusive contributions to heat transport on top of a ballistic one.

The derivations of our results make use of established mathematical tools known from wave propagation in random media used in a wide range of fields, including optics, acoustics, polymer physics, geophysics, and electrical engineering [16]. While such tools are known in condensed matter physics, they have not been used in the context of non-equilibrium quantum statistical mechanics, as far as we know. Our results show that these tools are useful in this context. Finally, we remark that the exact computation of \( L_{\text{th}} \) was recently advocated as a means to classify integrable models into purely ballistic or diffusive [17].
Random CFT. By inhomogeneous CFT we mean a quantum field theory model with Hamiltonian

\[ H = \int dx \, v(x) [T_x(x) + T_-(x)], \tag{1} \]

where \( v(x) > 0 \) is a velocity that varies smoothly in space and \( T_x(x) \) are operators satisfying the commutation relations well-known from standard CFT [18],

\[ [T_x(x), T_y(y)] = \mp 2i \delta'(x - y) T_x(y) \]

\[ \pm i \delta(x - y) T'_x(y) = \frac{c}{24\pi} i \delta'''(x - y) \tag{2} \]

and \( [T_x(x), T_x(x')] = 0 \), with \( c > 0 \) the central charge. The time evolution of observables \( \mathcal{O} \) is determined by the Heisenberg equation \( \partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)] \) (we set \( \hbar = k_B = 1 \)). We note that, in specific models, \( T_x(x) \) (and other observables) are represented by operators on some Fock space; there are many examples of interest, including the Luttinger model already mentioned, see [10] and references therein. Clearly, the special case \( v(x) = v \) (constant) corresponds to standard CFT.

We define random CFT as inhomogeneous CFT with random velocity

\[ v(x) = v/[1 - \xi(x)] \tag{3} \]

with \( \xi(x) \) a Gaussian random function [19] specified by \( \mathbb{E}[\xi(x)] = 0 \) and the covariance

\[ \Gamma(x - x') = \mathbb{E}[\xi(x)\xi(x')], \tag{4} \]

where \( \mathbb{E}[\cdot] \) denotes the average over impurities. We assume that \( \Gamma(x) \) is even, has non-negative Fourier transform, and is such that \( \Gamma_0 = \int \Gamma(x) dx \) is finite. Note that this parameter \( \Gamma_0 \) is non-negative and has the dimension of length. We find it convenient to introduce another length parameter \( a_0 > 0 \) to write \( \Gamma(x) = (\Gamma_0/a_0) f(|x|/a_0) \), with \( f(u) \) some suitable function of the dimensionless variable \( u = |x|/a_0 \). Four examples (a)–(d) of such functions \( f(u) \), which we use to illustrate our results, are given in Table I. Note that standard CFT can be recovered by setting \( \Gamma_0 = 0 \).

For later reference, we recall the following well-known property of Gaussian random functions:

\[ \mathbb{E}[e^{-\lambda f_u} \, du \, \delta^{(d)}(\xi(x))] = e^{-\lambda^2 \lambda(x-y)^2/2} \tag{5} \]

for real \( \lambda \) with \( \Lambda(x-y) = \int_0^{\infty} d\xi \int_y^x dx_2 \Gamma(x_1 - x_2) \). Note that \( \Lambda(x) \) is even, \( \geq 0 \), and \( \Lambda(x) = \Gamma_0 a_0 F(|x|/a_0) \) with \( F(u) = \int_0^u dv_1 \int_0^{u} dv_2 f(|v_1 - v_2|) \). As will be seen, this function \( F(u) \) is important since it determines the nature of diffusion in the model, see Table I for our examples (a)–(d).

We note that the model with fixed impurities \( \xi(x) \) can be made mathematically precise using Minkowskian CFT on a circle. As explained in [10], this model can be solved by straightening out the position-dependent velocity using conformal transformations and taking the thermodynamic limit. However, we will mostly use simpler arguments to derive our results in the present paper.

\[ \begin{array}{|c|c|c|c|}
\hline
f(u) & F(u) & F'(u) \\
\hline
(a) & \delta(u) & u & 1 \\
(b) & \frac{1}{2} e^{-u} & u + e^{-u} - 1 & 1 - e^{-u} \\
(c) & \frac{1}{2\pi} e^{-\sqrt\pi} & u + 2e^{-\sqrt\pi}(\sqrt{u} + 1) - 2 & 1 - e^{-\sqrt\pi} \\
(d) & \frac{1}{2\pi} e^{-\sqrt\pi} & u - \arctan(u) & \frac{u^2}{1 + u^2} \\
\hline
\end{array} \]

Table I. Examples (a)–(d) of functions \( f(u) \) for \( u \geq 0 \) defining covariance functions as \( \Gamma(x) = (\Gamma_0/a_0) f(|x|/a_0) \). The functions \( F(u) \) and \( F'(u) \) are discussed in the text.

Equations of motion. We observe that the energy density operator given by the Hamiltonian in (1) is \( E(x) = v(x)[T_x(x) + T_-(x)] \). The corresponding heat current operator is \( J(x) = v(x)^2 [T_x(x) - T_-(x)] \). Indeed, using the Heisenberg equation and (2), one verifies that the continuity equation \( \partial_t E(x, t) + \partial_x J(x, t) = 0 \) holds true. In a similar manner, one finds the corresponding equation of motion \( \partial_t J(x, t) + v(x) \partial_x [v(x) E(x, t) + S(x)] = 0 \), where \( S(x) = -\frac{c}{16\pi} [v(x) v'(x) - \frac{1}{2} v'(x)^2] \) is an anomaly originating from the Schwinger term in (2). The continuity equation guarantees conservation of total energy as in standard CFT. However, the current is more complicated: the second equation above implies that the total current \( \int dx J(x) \) is not conserved in general. Similarly, one can check that, in general, the momentum operator \( \int dx [T_x(x) - T_-(x)] \) is not conserved either.

We define \( E(x, t) = \langle E(x, t) \rangle \) and \( J(x, t) = \langle J(x, t) \rangle \) obtained by taking the expectation value \( \langle \cdot \rangle \) of the corresponding operators in some arbitrary state in the thermodynamic limit. These expectations satisfy the same equations of motion as the operators. They have the following static solutions: \( E_{\text{stat}}(x) = [C_1 - S(x)]/v(x) \) and \( J_{\text{stat}}(x) = C_2 \) with real constants \( C_1 \) and \( C_2 \). These solutions describe an equilibrium state if \( C_2 = 0 \) and a non-equilibrium steady state if \( C_2 \neq 0 \). We consider the deviations \( E(x, t) = \tilde{E}(x, t) - E_{\text{stat}}(x) \) and \( J(x, t) \) from equilibrium, assuming the initial conditions \( \tilde{E}(x, 0) = c_0(x) \) and \( J(x, 0) = 0 \) independent of impurities, where \( c_0(x) \) is some function describing a local energy density injected into the equilibrium state at time \( t = 0 \).

We are interested in the impurity averages \( e(x, t) = \mathbb{E}[\tilde{E}(x, t)] \) and \( j(x, t) = \mathbb{E}[J(x, t)] \) of the energy density and heat current deviations. To compute these we use the following PDEs that follow from our discussion above:

\[ \partial_t \tilde{E}(x, t) + \partial_x J(x, t) = 0, \tag{6a} \]

\[ \partial_t J(x, t) + v(x) \partial_x [v(x) \tilde{E}(x, t)] = 0. \tag{6b} \]

These equations are reminiscent of effective equations of motion for local expectations in the framework of generalized hydrodynamics recently proposed in [5, 6]. We recall that the latter is an extension of ordinary hydrodynamics in that it takes into account the (possibly) infinite
number of conserved quantities in integrable systems, see [20] for a related study in the case of a non-integrable perturbation to CFT. Importantly, even if the PDEs are simple, they are non-trivial due to the presence of the random velocity.

Heat waves in random media. Inspired by [21, 22], we use tools known from wave propagation in random media to compute $e(x,t)$ and $j(x,t)$. To this end, we write $\dot{E}(x,t) = \left[u_+(x,t) + u_-(x,t)\right]/v(x)$ and $J(x,t) = u_+(x,t) - u_-(x,t)$, with functions $u_+(x,t)$ satisfying $\partial_t u_+(x,t) + v(x) \partial_x u_+(x,t) = 0$. Our initial conditions $E(x,0) = e_0(x)$ and $J(x,0) = 0$ translate into $u_+(x,0) = v(x)e_0(x)/2$.

Using standard PDE methods we find the exact solution of the initial value problem for $u_+(x,t)$ above,

$$u_+(x,t) = \int dy \frac{\theta(\pm(x-y))}{2} \int \frac{d\omega}{2\pi} e^{i\omega (x-y)/v} G_{\pm}(x-y,t) e_0(y)$$

with the Heaviside function $\theta(x)$. Inserting (3), using (5) to compute the impurity average, and using a standard Gaussian integral, we obtain

$$\mathbb{E}[u_+(x,t)] = \int dy \frac{v}{2} G_{\pm}(x-y,t) e_0(y)$$

with the impurity-averaged Green’s functions

$$G_{\pm}(x,t) = \theta(\pm x) \frac{e^{-(x\pm vt)^2/4\Lambda(x)}}{\sqrt{2\pi\Lambda(x)}}. \quad (9)$$

In a similar manner we compute $\mathbb{E}[u_+(x,t)/v(x)]$. The results can be summarized as follows:

$$e(x,t) = \int dy \left[G^E_+(x,y,t) + G^E_-(x,y,t)\right] e_0(y), \quad (10a)$$

$$j(x,t) = \int dy \left[G^J_+(x,y,t) + G^J_-(x,y,t)\right] e_0(y) \quad (10b)$$

with

$$G^E_+(x,t) = \frac{1}{2} \left[1 - \frac{(x \mp vt)^2}{4\Lambda(x)}\right] G_{\pm}(x,t), \quad (11a)$$

$$G^J_+(x,t) = \pm \frac{v}{2} G_{\pm}(x,t). \quad (11b)$$

The functions in (9) are Gaussian distributions with variance $\Lambda(x)$. A few remarks are in order: 1) Since $G_{\pm}(x,t)$ becomes $\delta(x)$ as $t \to 0^+$, the initial conditions are satisfied. 2) Since $G_{\pm}(x,t)$ becomes $\theta(\pm x)\delta(x \mp vt)$ as $t \to 0^-$, the standard CFT results of [10, 23] are recovered. 3) One can verify that the total energy is conserved in (10a) [24].

Diffusion equation. The functions $G_{\pm}(x,t)$ in (9) provide an explicit description of how heat spreads in our system. Clearly, $G_{\pm}(x,t)$ describes a wave moving to the right ($+$) or left ($-$) with speed $v$. However, different from standard CFT, this heat wave is not purely ballistic: in general, as it moves, there is a gradual increase of its width, which is a clear indication of diffusion.

To characterize this diffusive behavior we note that $G_{\pm}(x,t)$ solves the propagation-diffusion equation [25]

$$\left[v^{-1} \partial_t \pm \partial_x - \gamma(x) \partial_x^2\right] G_{\pm}(x,t) = 0 \quad (12)$$

for $x > 0$ and $t > 0$, with $\gamma(x) = \pm \Lambda'(x)/2v = \left(\Gamma_0/2v^2\right) F'(v|\alpha_0) > 0$ becoming constant for large $|x|$. Phenomena described by a PDE of the form in (12) are referred to as temporal diffusion in [25], with $\gamma(x)$ a temporal diffusion coefficient [26].

This is similar to the usual notion of diffusion, the difference being that space and time have switched roles. It is important to note that one can also interpret this as standard diffusion in a frame of reference moving with the wave. To see this, change variables to $\tilde{x} = x \mp vt$, $\tilde{t} = |x|/v$ and define $\tilde{G}_{\pm}(\tilde{x}, \tilde{t}) = G_{\pm}(x,t)$. This is a natural choice of variables: $\tilde{x}$ is the coordinate of the observer moving with the wave, and $\tilde{t}$ is her time measured by the position of the wave. Eq. (12) then becomes

$$\left[\partial_{\tilde{t}} - \alpha_{\text{th}}(\tilde{t})\partial_{\tilde{x}}^2\right] \tilde{G}_{\pm}(\tilde{x}, \tilde{t}) = 0 \quad (13)$$

for $\tilde{t} > 0$ and $\tilde{x} > -vt$, with the thermal diffusivity $\alpha_{\text{th}}(\tilde{t}) = (\Gamma_0\nu/2) F'(v|\alpha_0)$. In general, $\alpha_{\text{th}}(\tilde{t})$ is time-dependent, see Table I for $F'(u)$ in our examples. The exception is Example (a), in which the thermal diffusivity is equal to the constant

$$\alpha_{\text{th}} = \frac{\Gamma_0\nu^2}{2}, \quad (14)$$

while it converges to this value for large $vt/\alpha_0$ in Examples (b)- (d).

Eq. (13) is a diffusion equation in a moving frame (the underlying ballistic motion) with heat waves changing according to a diffusion process given by $\alpha_{\text{th}}(\tilde{t})$. Equivalently, the variance of this process is $\Lambda(x)$, which in the new coordinates equals $2 \int_{0}^{\tilde{t}} dt' \alpha_{\text{th}}(t') = \Gamma_0 \alpha_0 F(v|\alpha_0)$ and thus goes as $\Gamma_0 vt$ plus a non-linear correction term, see Table I. This indicates that there are both normal and anomalous diffusive contributions [27] on top of a ballistic part. The normal diffusion is determined by the leading term $u$ of the function $F(u)$ and is in this sense universal: it is independent of model details. The anomalous diffusive part is determined by the sub-leading term $F(u) - u$ and can thus be qualitatively different in different cases.

Linear response theory. In what follows, we consider the thermal conductivity $\kappa_{\text{th}, \xi}(\omega)$ as a function of frequency $\omega$ at fixed impurity configuration, with the subscript $\xi$ emphasizing the dependence on the Gaussian random function. To be clear, we define $\kappa_{\text{th}, \xi}(\omega)$ as the response function related to the total heat current obtained by perturbing the equilibrium state at temperature $\beta^{-1}$ with a unit pulse perturbation $V = -(\delta / \beta) \int dx W(x) \xi(x)$ at time zero, where $W(x)$ is an asymptotically flat smooth function equal to $1/2 (-1/2)$ to the far left (right), but otherwise arbitrary, cf. [10, 23].
Using standard linear-response theory [28], one derives the following Green-Kubo formula:

$$\kappa_{\text{th},\xi}(\omega) = \beta \int_0^\infty dx \int_0^\infty dt e^{i\omega t} \int dx \int dx' \partial_x'[-W(x')] \langle \mathcal{J}(x,t)\mathcal{J}(x',i\tau) \rangle^\xi \beta,$$

(15)

where $$\langle \mathcal{J}(x,t)\mathcal{J}(x',i\tau) \rangle^\xi$$ is the connected part of the current-current correlation function in thermal equilibrium with respect to $$H$$ in (1). Since translational invariance is broken, we cannot change variables to perform the integral over $$x'$$.

In general, the real part of the thermal conductivity can be partitioned as $$\Re \kappa_{\text{th}}(\omega) = D_{\text{th}} \pi \delta(\omega) + \Re \kappa_{\text{reg}}^{\text{th}}(\omega)$$, where $$D_{\text{th}}$$ is the thermal Drude weight and $$\kappa_{\text{reg}}^{\text{th}}(\omega)$$ is the regular part, see, e.g., [10, 17]. A non-zero $$D_{\text{th}}$$ corresponds to a ballistic contribution, while a non-zero $$\Re \kappa_{\text{th}}^{\text{reg}}(\omega)$$ for $$\omega = 0 (\neq 0)$$ corresponds to a normal (anomalous) diffusive contribution [27]. In inhomogeneous CFT, by deriving an explicit formula for the current-current correlation function using standard CFT tools, one can show that (15) yields the same universal $$D_{\text{th}} = \pi v c/(3\beta)$$ as in standard CFT, and

$$\Re \kappa_{\text{th},\xi}(\omega) = \frac{\pi c}{6\beta} \left[1 + \left(\frac{\omega \beta}{2\pi}\right)^2\right] \int dx \int dx' \partial_x'[-W(x')] \left(1 - \frac{v}{\nu(x)}\right) \cos\left(\omega \int_{x'}^{x} \frac{dx}{\nu(x)}\right).$$

(16)

We obtained this formula by straightforward but tedious computations using CFT results developed in [10] (we plan to give details elsewhere). Clearly, (16) vanishes for standard CFT.

To average over impurities, we write the cosine as sum of exponentials, insert (3), and use (5). After averaging, translation invariance is recovered, which allows us to do the $$x'$$-integral and obtain a result independent of $$W(x)$$. We thus find that $$\Re \kappa_{\text{th}}^{\text{reg}}(\omega) = \mathbb{E}[\Re \kappa_{\text{th},\xi}(\omega)]$$ is given by the following explicit integral:

$$\Re \kappa_{\text{th}}^{\text{reg}}(\omega) = \frac{\pi c}{6\beta} \left[1 + \left(\frac{\omega \beta}{2\pi}\right)^2\right] \int dx e^{-\frac{1}{2}(\omega/v)^2 \Lambda(x)} \cos\left(\frac{\omega x}{v}\right).$$

(17)

if $$\Gamma_0 > 0$$ and zero otherwise. Eq. (17) can be easily evaluated for different examples of $$\Lambda(x)$$, and in Figure 2 we plot it for our examples (a)–(d) in Table I.

For $$\Lambda(x) = \Gamma_0 |x|$$ (Example (a) in Table I), one can compute the integral in (17) analytically to obtain

$$\Re \kappa_{\text{th}}^{\text{reg}}(\omega) = \frac{\pi c}{6\beta} \left[1 + \left(\frac{\omega \beta}{2\pi}\right)^2\right] \Gamma_0 / \left[1 + \left(\frac{\omega \beta}{2v}\right)^2\right],$$

which implies

$$L_{\text{th}} = \lim_{\omega \to 0} \Re \kappa_{\text{th}}^{\text{reg}}(\omega) = \frac{\pi c}{6\beta} \Gamma_0.$$

(18)

This result is actually true in any example. To see this, change the integration variable in (17) to $$y = \omega x/v$$ and note that the function in the exponential becomes

$$\frac{-1}{2} \Gamma_0 a_0 (\omega/v)^2 F(v|g|/\omega a_0),$$

which is equal to $$\frac{-1}{2} \Gamma_0 |g|/v$$ up to sub-leading terms not contributing to the integral in the limit $$\omega \to 0$$. Since the parameter $$L_{\text{th}}$$ characterizes normal diffusion, this confirms our conclusion above that the normal diffusion in our model is universal. Moreover, (14) and (18) imply $$L_{\text{th}} = \pi c \alpha_{\text{th}}$$, which provides a direct link between the two approaches.

However, the full frequency dependency of $$\Re \kappa_{\text{th}}^{\text{reg}}(\omega)$$ depends on model details, as can be seen from Figure 2. In particular, the asymptotic behavior for large $$\omega$$ is qualitatively different in different cases: for our examples in Table I, while $$\Re \kappa_{\text{th}}^{\text{reg}}(\omega)/L_{\text{th}}$$ becomes constant for large $$\omega$$ in (a), it grows linearly in (b) and sub-linearly in (c). The large-$\omega$ result in Example (d) also tends to a constant but in a way that seems to depend more sensitively on the choice of parameters compared to the other examples. The asymptotic behavior can be understood by analyzing the integral in (17) for large values of $$\omega$$, similarly as above. One finds that, for large $$\omega$$, the integral is dominated by the small-$u$ behavior of $$F(u)$$, which is different in the different examples. It would be interesting to explore this dependence on model details more systematically, but this is beyond the scope of this paper.

![Figure 2](image-url)

**Figure 2.** $$\Re \kappa_{\text{th}}^{\text{reg}}(\omega)$$ in (17) for Examples (a)–(d) in Table I and representative parameter values. In all plots, $$\omega_0 = v/\Gamma_0$$, $$\Gamma_0/a_0 = 0.6$$ is fixed, and the parameter varied is $$\beta \omega_0$$ equal to 1.8 (blue solid line), 1.2 (red dotted), and 0.6 (yellow dashed).

**Conclusions.** We proposed and studied a CFT model with random position-dependent velocity amenable to an exact analytical treatment. Such models, we argued, naturally emerge as effective descriptions of certain 1D quantum many-body systems with static random impurities. We presented two exact results for heat transport in this model which prove, in complimentary ways, that impurities lead to diffusive contributions on top of the well-known ballistic one of standard CFT. In particular, we found that the diffusive behavior is encoded in a func-
tion $F(u)$ that has a universal leading term $u$, describing normal diffusion, and a sub-leading term $F(u) - u$ corresponding to a non-universal anomalous diffusive contribution. It would be interesting to study the connection between our two results from the general point of view of hydrodynamics [3, 29].

We finally note that models similar to the generalized XXZ spin chain discussed in the introduction have recently received a lot of attention in the context of many-body localization [30]. It would be interesting to investigate if potential signatures of many-body localization can be established in random CFT, similarly as in [11]. We hope to come back to this question in future work.

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