$T'$ and the Cabibbo Angle

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The use of the binary tetrahedral group ($T'$) as flavor symmetry is discussed. I emphasize the CKM quark and PMNS neutrino mixings. I present a novel formula for the Cabibbo angle.

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1. Introduction on renormalizability

In particle theory phenomenology, model building fashions vary with time and because the present lack of data (soon to be compensated by the Large Hadron Collider) does not allow discrimination between models some fashions develop a life of their own. In the present talk we take the apparently retrogressive step of imposing the requirement of renormalizability, as holds for quantum electrodynamics (QED), quantum chromodynamics (QCD) and the standard electroweak model, to show that non-abelian flavor symmetry becomes then much more restrictive and predictive. In a specific model we show that a normal neutrino mass hierarchy is strongly favored over an inverted hierarchy.

For several years now there has been keen interest in the use of $A_4$ as a finite flavor symmetry in the lepton sector, especially neutrino mixing. In particular, the empirically approximate tribimaximal mixing of the three neutrinos can be predicted. It is usually stated that either normal or inverted neutrino mass spectrum can be predicted.

We revisit these two questions in a minimal $A_4$ framework with only one $A_4$-3 of Higgs doublets coupling to neutrinos and permitting only renormalizable couplings. For such a minimal model there is more predictivity regarding neutrino masses.

Although the standard model was originally discovered using the criterion of renormalizability, it is sometimes espoused that renormalizability is not prerequisite in an effective lagrangian. Nevertheless, imposing renormalizability in the present case is more sensible because it does render the model far more predictive by avoiding the many additional parameters associated with higher-order irrelevant operators. Our choice of Higgs sector also minimizes the number of free parameters.

It is sufficiently important to emphasize the concept that every result mentioned in this talk would be impossible without imposing renormalizability.
Although it has been fruitful in low-energy QCD, heavy-quark effective theory and technicolor this idea is inappropriate to fundamental model building in particle phenomenology.

2. $A_4$ symmetry

The group $A_4$ is the order $g=12$ symmetry of a regular tetrahedron $T$ and is a subgroup of the rotation group $SO(3)$. $A_4$ has irreducible representations which are three singlets $1_1, 1_2, 1_3$ and a triplet $3$. In the embedding $A_4 \subset SO(3)$ the $3$ of $A_4$ is identified with the adjoint $3$ of $SO(3)$.

Since the only Higgs doublets coupling to neutrinos in our model are in a $3$ of $A_4$, it is very useful to understand geometrically the three components of a $3$.

A regular tetrahedron has four vertices, four faces and six edges. Straight lines joining the midpoints of opposite edges pass through the centroid and form a set of three orthogonal axes. Regarding the regular tetrahedron as the result of cutting off the four odd corners from a cube, these axes are parallel to the sides of the cube. With respect to the regular tetrahedron, a vacuum expectation value (VEV) of the $3$ such as $< 3 > = v(1,1,-2)$, as will be used, clearly breaks $SO(3)$ to $U(1)$ and correspondingly $A_4$ to $Z_2$, since it requires a rotation by $\pi$ about the $3$-axis to restore the tetrahedron.

At the same time, we can understand the appearance of tribimaximal mixing with matrix

$$U_{TBM} = \begin{pmatrix} \frac{\sqrt{1}}{\sqrt{6}} & \frac{\sqrt{1}}{\sqrt{6}} & \frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{1}}{\sqrt{6}} & \frac{\sqrt{1}}{\sqrt{6}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ \frac{\sqrt{1}}{\sqrt{2}} & -\frac{\sqrt{1}}{\sqrt{2}} & 0 \end{pmatrix},$$

and our definitions are such that the ordering $\nu_{1,2,3}$ and $\nu_{\tau,\mu,e}$ satisfy

$$\begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} = U_{TBM} \begin{pmatrix} \nu_{\tau} \\ \nu_{\mu} \\ \nu_{e} \end{pmatrix}$$

Assuming no CP violation, the Majorana matrix $M_{\nu}$ is real and symmetric and therefore of the form

$$M_{\nu} = \begin{pmatrix} A & B & C \\ B & D & F \\ C & F & E \end{pmatrix}$$

and is related to the diagonalized form by
\[ M_{\text{diag}} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} = U_{TBM} M_\nu U^T_{TBM}. \] (4)

Substituting Eq. (1) into Eq. (4) shows that \( M_\nu \) must be of the general form in terms of real parameters \( A, B, C \):

\[ M_\nu = \begin{pmatrix} A & B & C \\ B & A & C \\ C & C & A + B - C \end{pmatrix}, \] (5)

which has eigenvalues

\[ m_1 = (A + B - 2C) \]
\[ m_2 = (A + B + C) \]
\[ m_3 = (A - B). \] (6)

The observed mass spectrum corresponds approximately to \(|m_1| = |m_2|\) which requires either \( C = 0 \) or \( C = 2(A + B) \). For a normal hierarchy, \( A + B = 0 \) and \( C = 0 \). For an inverted hierarchy \( A = B \) and \( C = 0 \) or \( C = 4A \).

Now we study our minimal \( A_4 \) model to examine the occurrence of the Majorana matrix Eq. (5) and the eigenvalues Eq. (6).

3. Minimal \( A_4 \) model

We assign the leptons to \((A_4, Z_2)\) irreps as follows

\[ \begin{pmatrix} \nu_e \\ \tau^- \\ \nu_\mu \\ \mu^- \\ \nu_e^- \end{pmatrix}_L \begin{pmatrix} \tau^-_R (1, -1) \\ \mu^+_R (1, -1) \\ e^+_R (1, -1) \end{pmatrix} \begin{pmatrix} N^{(1)}_R (1, +1) \\ N^{(2)}_R (1_2, +1) \\ N^{(3)}_R (1_3, +1) \end{pmatrix}. \] (7)

The lepton lagrangian is

\[ \mathcal{L}_Y^{(\text{leptons})} = \frac{1}{2} M_1 N^{(1)}_R N^{(1)}_R + M_{23} N^{(2)}_R N^{(3)}_R + \left[ Y_1 \left( L_L N^{(1)}_R H_3 \right) + Y_2 \left( L_L N^{(2)}_R H_3 \right) \right] \\
+ Y_3 \left( L_L N^{(3)}_R H_3 \right) + Y_\tau \left( L_L \tau_R H'_3 \right) + Y_\mu \left( L_L \mu_R H'_3 \right) \\
+ Y_e \left( L_L e_R H'_3 \right) + \text{h.c.} \] (8)
where $SU(2)$-doublet Higgs scalars are in $H_3(3, +1)$ and $H'_3(3, -1)$. The charged lepton masses originate from $< H'_3 > = (m_\tau/Y_\tau, m_\mu/Y_{\mu}, m_e/Y_e)$ and are, to leading order, disconnected from the neutrino masses if we choose a flavor basis where the charged leptons are mass eigenstates. The $N_R^L$ masses break the $L_\tau \times L_\mu \times L_e$ symmetry but change the charged lepton masses only by very small amounts $\propto Y^2 m_i/M_N$ at one-loop level. The right-handed neutrinos have mass matrix

$$M_N = \begin{pmatrix} M_1 & 0 & 0 \\ 0 & 0 & M_{23} \\ 0 & M_{23} & 0 \end{pmatrix}. \quad (9)$$

We take the VEV of the scalar $H_3$ to be

$$< H_3 > = (V_1, V_2, V_3), \quad (10)$$

whereupon the Dirac matrix is

$$M_D = \begin{pmatrix} Y_1 V_1 & Y_2 V_3 & Y_3 V_2 \\ Y_1 V_3 & Y_2 V_2 & Y_3 V_1 \\ Y_1 V_2 & Y_2 V_1 & Y_3 V_3 \end{pmatrix}. \quad (11)$$

The Majorana mass matrix $M_\nu$ is given by

$$M_\nu = M_D M_N^{-1} M_D^T. \quad (12)$$

Technical details are provided in arXiv:0806.1707

Our conclusion is that the $A_4$ model in a minimal form does favor the normal hierarchy. We have considered a more restrictive model based on $A_4$ than previously considered. The theory has been required to be renormalizable and the Higgs scalar content is the minimum possible. We have required that the neutrino mixing matrix be of the tribimaximal form. We then find that the masses for the neutrinos are highly constrained and can be in a normal, not inverted hierarchy.

Most, if not all, previous $A_4$ models in the literature permit higher-order irrelevant non-renormalizable operators and their concomitant proliferation of parameters and hence allow a wide variety if possibilities for the neutrino masses. We believe the renormalizability condition is sensible for these flavor symmetries because of the higher predictivity.

The next step which is the subject of the rest of this talk is whether the present renormalizable $A_4$ model can be extended to a renormalizable $T'$ model. It is necessary but not sufficient condition for this that a successful renormalizable $A_4$ model, as presented here, exists.
4. \( T' \) symmetry.

The first use of the binary tetrahedral group \( T' \) in particle physics was\(^1\) by Case, Karplus and Yang in 1956 who were motivated to consider gauging a finite \( T' \) subgroup of \( SU(2) \) in Yang-Mills theory. This led Fairbairn, Fulton and Klink (FFK) in 1964 to make an analysis\(^2\) of \( T' \) Clebsch-Gordan coefficients. As a flavor symmetry, \( T' \) first appeared\(^3\) in 1994 motivated by the idea of representing the three quark families with the third treated differently from the first two. Since \( T' \) is the double cover of \( A_4 \), it was natural to suggest that \( T' \) be employed to accommodate quarks and simultaneously the established \( A_4 \) model building for tribimaximal neutrino mixing.

We shall discuss such a \( T' \) model with simplifications to emphasize the largest quark mixing, the Cabibbo angle, for which we shall derive an entirely new formula\(^4,5\) as an exact angle.

Recall that charged lepton masses arise from the vacuum expectation value

\[
<H_3^\prime> = \left( \frac{m_\tau}{Y_\tau}, \frac{m_\mu}{Y_\mu}, \frac{m_e}{Y_e} \right) = (M_\tau, M_\mu, M_e) \tag{13}
\]

where \( M_i \equiv m_i / Y_i \) (\( i = \tau, \mu, e \)). Neutrino masses and mixings come from the see-saw mechanism and the VEV

\[
<H_3> = V(1, -2, 1) \tag{14}
\]

We shall now promote \( A_4 \) to \( T' \) keeping renormalizability and including quarks.

5. Minimal \( T' \) model

The left-handed quark doublets \((t, b)_L, (c, d)_L, (u, d)_L\) are assigned under \( (T' \times Z_2) \) to

\[
\begin{pmatrix}
  t \\
  b
\end{pmatrix}_L \quad (1_L, +1)
\]

\[
\begin{pmatrix}
  c \\
  s \\
  u \\
  d
\end{pmatrix}_L \quad (2_L, +1)
\tag{15}
\]

and the six right-handed quarks as
We add only two new scalars $H_{11}(1_1, +1)$ and $H_{13}(1_3, +1)$ whose VEVs

$$< H_{11} > = m_t/Y_t \quad < H_{13} > = m_b/Y_b$$

provide the $(t, b)$ masses. In particular, no $T'$ doublet $(2_1, 2_2, 2_3)$ scalars have been added. This allows a non-zero value only for $\Theta_{12}$. The other angles vanish making the third family stable.\footnote{As we shall discuss non-vanishing $\Theta_{23}$ and $\Theta_{13}$ are related to $(d, s)$ masses.}

The allowed quark Yukawa and mass terms are

$$\mathcal{L}^{\text{quarks}} = Y_t(\{Q_L\}_{11}\{t_R\}_{11} H_{11}) + Y_b(\{Q_L\}_{11}\{b_R\}_{12} H_{13}) + Y_C(\{Q_L\}_{21}\{C_R\}_{23} H_{3}) + Y_S(\{Q_L\}_{21}\{S_R\}_{22} H_{3}) + \text{h.c.}$$

The use of $T'$ singlets and doublets\footnote{It is discrete anomaly free. We thank the UF-Gainesville group for discussions.} for quark families in Eqs.\ref{eq:15},\ref{eq:16} permits the third family to differ from the first two and thus make plausible the mass hierarchies $m_t \gg m_b, m_b > m_{c,u}$ and $m_b > m_{s,d}$.

6. The Cabibbo angle

The nontrivial $(2 \times 2)$ quark mass matrices $(c, u)$ and $(s, d)$ will be respectively denoted by $U'$ and $D'$ and calculated using the $T'$ Clebsch-Gordan coefficients of Fairbairn, Fulton and Klink. Dividing out $Y_C$ and $Y_S$ in Eq.\ref{eq:18} gives $U$ and $D$ matrices ($\omega = e^{i\pi/3}$)

$$U \equiv \left( \frac{1}{Y_C} \right) U' = \left( \begin{array}{cc} \sqrt{\frac{2}{3}} \omega^2 M_c & \frac{1}{\sqrt{3}} M_e \\ -\frac{1}{\sqrt{3}} \omega^2 M_e & \sqrt{\frac{2}{3}} M\mu \end{array} \right)$$
Let us first consider $U$ of Eq. (19). Noting that $m_\tau > m_\mu > m_e$ we may simplify $U$ by setting the electron mass to zero, $M_e = 0$. This renders $U$ diagonal leaving free the $c$, $u$, $\tau$ and $\mu$ masses. This leaves only the matrix $D$ in Eq. (20) which predicts both $\Theta_{12}$ and $(m_d^2/m_s^2)$. The hermitian square $D \equiv DD^\dagger$ is

$$D \equiv \left(\frac{1}{\sqrt{3}}\right) \begin{pmatrix} \frac{1}{\sqrt{3}} & -2\sqrt{\frac{2}{3}} \omega \\ \sqrt{\frac{2}{3}} & \frac{\sqrt{3}}{\omega} \end{pmatrix}$$

(20)

which leads by diagonalization to a formula for the Cabibbo angle

$$\tan 2\Theta_{12} = \left(\frac{\sqrt{2}}{3}\right)$$

(22)

or equivalently $\sin \Theta_{12} = 0.218..$ close to the experimental value $\sin \Theta_{12} \simeq 0.227$. Our result of an exact angle for $\Theta_{12}$ can be regarded as on a footing with the tribimaximal values for neutrino angles $\theta_{ij}$.

Note that the tribimaximal $\theta_{12}$ presently agrees with experiment within one standard deviation (1$\sigma$). On the other hand, our analogous exact angle for $\Theta_{12}$ differs from experiment already by 9$\sigma$ which is probably a reflection of the fact that the experimental accuracy for $\Theta_{12}$ is $\sim 0.5\%$ while that for $\theta_{12} = \sim 6\%$.

It is thus very important to acquire better experimental data on $\theta_{12}$, $\theta_{23}$ and $\theta_{13}$ to detect their similar deviation from the exact angles predicted by TBM. Our result for $(m_d^2/m_s^2)$ from Eq. (21) is exactly 0.288.. compared to the central experimental value $\simeq 0.003$ in a simplified model whose generalization to an extended scalar sector including $T'$ doublets can avoid $\Theta_{23} = \Theta_{13} = 0$ and thereby change $(m_d^2/m_s^2)$ due to mixing of $(d, s)$ with the $b$ quark.

This $T' \times Z_2$ extension of the standard model is an first step to tying the quark and lepton sectors together, providing calculability, and at the same time reducing the number of standard model parameters. The ultimate goal would be to understand the origin of this discrete symmetry. Since gauge symmetries can break to discrete symmetries, and gauge symmetries arise naturally from strings, perhaps there is a clever construction of our model with its fundamental origin in string theory.

7. Summary

Renormalizability and simplification of $(A_4 \times Z_2)$ then $(T' \times Z_2)$ models lead to: Cabibbo angle formula
\tan 2\Theta_{12} = \left( \frac{\sqrt{2}}{3} \right)

References

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