0. Introduction

0.1. The Weil representation. In his celebrated 1964 Acta paper [W] André Weil constructed a certain (projective) unitary representation of a symplectic group over local fields. This representation has many fascinating properties which have gradually been brought to light over the last few decades. It now appears that this representation is a central object in mathematics, bridging between various topics in mathematics and physics, including classical invariant theory, the theory of theta functions and automorphic forms, harmonic analysis, and last (but probably not least) quantum mechanics. Although it holds such a fundamental status, it is satisfying to observe that the Weil representation already appears in "real life" situations. Given a linear space $L$, there exists an associated (split) symplectic vector space $V = L \times L^*$. The Weil representation of the group $Sp = Sp(V, \omega)$ can be realized on the Hilbert space

$$\mathcal{H} = L^2(L, \mathbb{C}).$$

Interestingly, elements of the group $Sp$ act by certain kinds of generalized Fourier transforms. In particular, there exists a specific element $w \in Sp$ (called the Weyl element) whose action is given, up to a normalization, by the standard Fourier transform. From this perspective, the classical theory of harmonic analysis seems to be devoted to the study of a particular (and not very special) operator in the Weil representation.

In this paper we will be concerned only with the case of the Weil representations of symplectic groups over finite fields of odd characteristic. Let us note that already in this setting there is no simple way to obtain the Weil representation. A possible approach to its construction is through the use of an auxiliary two-step unipotent group called the Heisenberg group. In this
approach, the Weil representation appears as a collection of intertwining operators of a special irreducible representation of the Heisenberg group. It takes some additional work to realize that in fact, in the finite field case, it can be linearized into an honest representation.

0.2. Geometric approach and invariant presentation. The Weil representation over a finite field is an object of algebraic geometry! More specifically, in the case of finite fields, all groups involved are finite, yet of a very special kind, consisting of rational points of corresponding algebraic groups. In this setting it is an ideology, due to Grothendieck, that any (meaningful) set-theoretic object is governed by a more fundamental algebro-geometric one. The procedure by which one lifts to the setting of algebraic geometry is called geometrization. In this procedure, sets are replaced by algebraic varieties (defined over the finite field) and functions are replaced by corresponding sheaf-theoretic objects (i.e., objects of the $\ell$-adic derived category). The procedure translating between the two settings is called sheaf-to-functions correspondence. This ideology has already proved to be extremely powerful, not just as a technique for proving theorems, but much more importantly, it supplies an appropriate framework without which certain deep mathematical ideas cannot even be stated.

It is reasonable to suspect that the Weil representation is meaningful enough so that it can be geometrized. Indeed, this was carried out in a letter written by Deligne to Kazhdan in 1982 [D1]. In his letter Deligne proposed a geometric analogue of the Weil representation. More precisely, he proposed to construct an $\ell$-adic Weil perverse sheaf on the algebraic group $\text{Sp}$ (in this paper we use boldface letters to denote algebraic varieties).

Although the main ideas already appear in that letter, the content is slightly problematic for several reasons. First, it was never published; and this is probably a good enough reason for writing this text. Second, in his letter, Deligne chooses a specific realization [Ge], which is equivalent to choosing a splitting $V = L \times L'$. As a result he obtains a sheaf

$$K_{L,L'}$$

on the variety $\text{Sp} \times L' \times L'$. The sheaf $K_{L,L'}$ is given in terms of explicit formulas which of course depend on the choice of $L$ and $L'$. These arbitrary choices make the formulas quite unpleasant (see [GH] for the $SL_2$ case) and, moreover, help to hide some delicate geometric phenomena underlying the Weil representation.

In this paper, we present a construction of the geometric Weil representation in all dimensions, yet from a different perspective. In our construction we do not use any specific realization. Instead, we invoke the idea of invariant presentation of an operator and as a consequence we obtain an invariant construction of the Weil representation sheaf

$$\mathcal{K}$$

on the variety $\text{Sp} \times V$. 
This avoids most of the unpleasant computations, and moreover, brings to the forefront some of the geometry behind this representation. As a by-product of working in the geometric setting, we obtain a new proof of the linearity of the Weil representation over finite fields. The upshot is that it is enough to prove the multiplicativity property on an open subvariety of the symplectic group. It is possible to choose such a variety on which verifying multiplicativity becomes simple. This kind of argument has no analogue in the set-theoretic setting and it is our hope that it exemplifies some of the powerful elegance of the geometric method.

Although it was first proposed some 25 years ago, we would like to remark that recently a striking application was found for the geometric Weil representation in the area of quantum chaos and the associated theory of exponential sums [GH]. We believe that this is just the tip of the iceberg and the future promises great things for this fundamental object. This belief is our motivation to write this paper.

0.3. Results.

(1) The main result of this paper is the new invariant construction of the geometric Weil representation sheaf in all dimensions. This is the content of Theorem 3.2.2.1 which claims the existence and some of the properties of this sheaf.

(2) We present a new construction of the set-theoretic Weil representation in all dimensions, which uses the notion of invariant presentation of an operator. In addition, algebro-geometric techniques (such as perverse extension) are used to obtain a new proof of the multiplicativity (i.e., linearity) property. Finally, explicit formulas are supplied.

(3) We present a character formula for the Weil representation. This is the content of Theorem 2.2.1.

0.4. Structure of the paper. The paper is divided into several sections.

• Section 1 In this section we present some preliminaries. In the first part of this section, the Heisenberg representation is presented and the notion of invariant presentation of an operator is discussed. In the second part of this section the classical construction of the Weil representation is given.

• Section 2 The Weil representation and its character are obtained using the idea of invariant presentation of an operator.

• Section 3 The geometric Weil representation sheaf is constructed. The construction is obtained by means of Grothendieck’s geometrization procedure concluding with Theorem 3.2.2.1 which claims the existence of a sheaf-theoretic analogue of the Weil representation. Some preliminaries from algebraic geometry and ℓ-adic sheaves are
also discussed. In particular, we recall Grothendieck’s sheaf-to-function correspondence procedure which is the main connection between Sections 3 and 2.

- Appendix A. The relation between our invariant construction of the Weil representation sheaf and the sheaf that was proposed by Deligne \[ D1 \] is discussed. The upshot is that they are related by a certain kind of \( \ell \)-adic Fourier transform.

- Appendix B. A proof of the main theorem of the paper, i.e., Theorem 3.2.2.1 is given.

- Appendix C. Certain additional proofs are given.

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1. Preliminaries

In this paper we denote by \( k = \mathbb{F}_q \) a finite field of \( q \) elements, where \( q \) is odd, and by \( \overline{k} \) an algebraic closure.

1.1. The Heisenberg representation. Let \( (V, \omega) \) be a \( 2N \)-dimensional symplectic vector space over the finite field \( k \). There exists a two-step nilpotent group \( H = H(V, \omega) \) associated to the symplectic vector space \( (V, \omega) \). The group \( H \) is called the Heisenberg group. It can be realized as the set \( H = V \times k \) equipped with the following multiplication rule

\[
(v, z) \cdot (v', z') = (v + v', z + z' + \frac{1}{2} \omega(v, v')).
\]  

(1.1.1)

The center of \( H \) is \( Z = Z(H) = \{(0, z) : z \in k\} \). Fix a non-trivial additive character \( \psi : Z \rightarrow \mathbb{C}^\times \). We have the following fundamental theorem

**Theorem 1.1.2** (Stone-Von Neumann). *There exists a unique (up to isomorphism) irreducible representation \((\pi, H, \mathcal{H})\) with central character \( \psi \), i.e., \( \pi(z) = \psi(z) \text{Id}_H \) for every \( z \in Z \).*

We call the representation \( \pi \), appearing in Theorem 1.1.2 the Heisenberg representation associated with the central character \( \psi \).
Remark 1.1.3. Although the representation $\pi$ is unique, it admits a multitude of different models (realizations). In fact this is one of its most interesting and powerful attributes. In particular, for any Lagrangian splitting $V = L \oplus L'$, there exists the Schrödinger model $(\pi_{L,L'}, H, S(L'))$, where $S(L')$ denotes the space of complex-valued functions on $L'$. In this model we have the following actions

- $[\pi_{L,L'}(l) \triangleright f](x) = \psi(\omega(x,l)) f(x)$;
- $[\pi_{L,L'}(l') \triangleright f](x) = f(x + l')$;
- $[\pi_{L,L'}(z) \triangleright f](x) = \psi(z) f(x)$,

where $l \in L$, $x, l' \in L'$, and $z \in Z$.

1.2. Invariant presentation of operators. Let $A : \mathcal{H} \rightarrow \mathcal{H}$ be a linear operator. Choosing an identification $\beta$ of $\mathcal{H}$ with the space $S(X)$ of functions on a finite set $X$ (which is equivalent to choosing a basis of $\mathcal{H}$), we can realize $A$ as a function $A_\beta \in S(X \times X)$ (which is equivalent to writing $A$ as a matrix). The function $A_\beta$ acts by convolution on functions in $S(X)$. Choosing a different identification $\beta$, of course, changes the realization $A_\beta$. This is all very trivial, nevertheless, we would like to pose the following question:

**Is there a correct realization?**

Representation theory suggests an answer to this seemingly strange question. We will realize operators as certain functions on the group $H$. In more detail, associated to a central character $\psi : Z \rightarrow \mathbb{C}^\times$ we denote by $S(H, \psi)$ the space of functions on the Heisenberg group which behave $\psi^{-1}$-equivariantly with respect to left (right) multiplication of the center, i.e., functions $f \in S(H)$ satisfying

$$f(z \cdot h) = \psi^{-1}(z) f(h),$$

(1.2.1)

for every $z \in Z$.

Recall that $\mathcal{H}$ constitutes an irreducible representation space of the Heisenberg group (with central character $\psi$). This implies that given an operator $A$ there exists a unique function $\hat{A} \in S(H, \psi)$ such that $\pi(\hat{A}) = A$, where $\pi(\hat{A}) = \frac{1}{|Z(H)|} \sum_{h \in H} \hat{A}(h) \pi(h)$. The relation between $A$ and $\hat{A}$ is expressed in the following formula

$$\hat{A}(h) = \frac{1}{\dim \mathcal{H}} Tr(A \pi(h^{-1})).$$

(1.2.2)

The transform $A \mapsto \hat{A}$ is well known in the literature and is usually referred to as the Weyl transform [We2]. It behaves well with respect to composition of operators: Given two operators $A, B \in \text{End}(\mathcal{H})$, it is a direct calculation to verify that $\hat{A} \hat{B} = \hat{A} * \hat{B}$, where $*$ denotes the normalized convolution product of functions on the group

$$\hat{A} * \hat{B}(h) = \frac{1}{|Z(H)|} \sum_{h_1, h_2 = h} \hat{A}(h_1) \hat{B}(h_2).$$

(1.2.3)
Concluding, using the Weyl transform we are able to establish an invariant presentation of operators.

**Theorem 1.2.4** (Invariant presentation of operators [H2]). Let \((\pi, H, \mathcal{H})\) be the Heisenberg representation. The Weyl transform \(\hat{\cdot} : \text{End}(\mathcal{H}) \to S(H, \psi)\) is an isomorphism of algebras, with the inverse given by the extended action \(\pi : S(H, \psi) \to \text{End}(\mathcal{H})\),

\[
f \mapsto \pi(f) = \frac{1}{|Z(H)|} \sum_{h \in H} f(h)\pi(h).
\]

Because of the equivariance property (1.2.1), functions in \(S(H, \psi)\) are determined by their restriction to the vector space \(V \subset H\), hence we can (and will) identify \(S(H, \psi)\) with \(S(V)\). The Convolution (1.2.3) is realized on \(S(V)\) as follows, for two functions \(f, g \in S(V)\)

\[
(f * g)(v) = \sum_{v_1 + v_2 = v} \psi(-\frac{1}{2}\omega(v_1, v_2)) f(v_1)g(v_2).
\]

We will refer to the induced isomorphism \(\hat{\cdot} : \text{End}(\mathcal{H}) \to S(V)\) also as the Weyl transform.

### 1.3. The Weil representation .

Let \(G = Sp(V, \omega)\) be the group of symplectic linear automorphisms of \(V\). The group \(G\) acts by group automorphism on the Heisenberg group through its tautological action on the vector space \(V\). This induces an action of \(G\) on the category \(\text{Rep}(H)\) of representations of \(H\), i.e., given a representation \(\pi \in \text{Rep}(H)\) and an element \(g \in G\) one obtains a new representation \(\pi^g\) (realized on the same Hilbert space) defined by \(\pi^g(h) = \pi(g \cdot h)\). It is clear that this action does not affect the central character and sends an irreducible representation to an irreducible one. Let \(\pi\) be the Heisenberg representation associated with a central character \(\psi\). Invoking Theorem 1.1.2, we conclude that for every element \(g \in G\) we have

\[
\pi^g \simeq \pi.
\]

Denote by \(\rho(g) : \mathcal{H} \to \mathcal{H}\) an intertwiner which realizes this isomorphism. Equivalently, this means that \(\rho(g)\) satisfies and in fact is determined up to scalar by the following equation

\[
\rho(g)\pi(h)\rho(g)^{-1} = \pi(g \cdot h), \quad (1.3.1)
\]

for every \(g \in G\).

The above equation are sometimes referred to in the literature as the Egorov identity. Having that all \(\pi^g, g \in G\), are irreducibles, and using Schur’s lemma, we conclude that the collection \(\{\rho(g); g \in G\}\) forms a projective representation \(\rho : G \to PGL(\mathcal{H})\). It is a non-trivial argument that the projective representation \(\rho\) can be linearized into an honest representation which we also denote by \(\rho : G \to GL(\mathcal{H})\). This representation is called the *Weil representation*. Let us summarize this in the following theorem
Theorem 1.3.2. There exists a canonical\footnote{Unique unless \( q = 3 \) and \( N = 1 \) (see Remark 3.3.2.2 for the canonical choice in the latter case).} representation
\[ \rho : G \to GL(\mathcal{H}), \]
satisfying the equation (1.3.1).

Remark 1.3.3. The fact that the projective representation \( \rho \) can be linearized is a peculiar phenomenon of the finite field situation. If one deals with the analogue constructions, for example over infinite Archimedian fields, it is a deep fact that \( \rho \) can be de-projectivized up to a \( \pm \) sign, which is called the metaplectic sign. Hence, in this case \( \rho \) can be corrected to a representation of a double cover
\[ 1 \to \mathbb{Z}_2 \to \tilde{Sp} \to Sp \to 1, \]
which is called the metaplectic cover. This fact is responsible for several fundamental phenomena in quantum mechanics, such as the nonzero energy of the vacuum state of the harmonic oscillator.

2. Invariant presentation of the Weil representation

The Weil representation was obtained classically by a very implicit construction (Section 1.3). This is already true before the linearization procedure (Theorem 1.3.2). What we would like to do next is to obtain a concrete realization (formulas) for the representation \( \rho \). In order to carry this out, it seems that one would be obliged to choose a basis of the Hilbert space \( \mathcal{H} \) and then write every element \( \rho(g) \) as an appropriate matrix. This procedure, in principle, could be carried out and, in fact, in most references we know this is the way \( \rho \) is presented. We would like to suggest a different approach based on the idea of invariant presentation discussed in Section 1.2. Namely, every operator \( \rho(g) \) will be presented by a function (kernel) \( K_g = \widehat{\rho(g)} \in S(V) \) satisfying the multiplicative rule \( K_{gh} = K_g \ast K_h \). The collection of kernels \( \{K_g\}_{g \in G} \) can be thought of as a single function
\[ K \in S(G \times V), \quad (2.1) \]
satisfying the following convolution property
\[ (m \times Id)^* K = p_1^* K \ast p_2^* K, \quad (2.2) \]
where the map \( m \times Id : G \times G \times V \to G \times V \) is given by \( (g_1, g_2, v) \mapsto (g_1 \cdot g_2, v) \), and the maps \( p_i(g_1, g_2, v) = (g_i, v) \) are the projectors on the first and second \( G \)-coordinate respectively. The right-hand side of (2.2) means \( p_1^* K \ast p_2^* K(g_1, g_2, v) = K_{g_1} \ast K_{g_2}(v) \). In the sequel we will sometime suppress the \( V \)-coordinate, writing \( m : G \times G \to G \), and \( p_i : G \times G \to G \), we will also suppress the projections \( p_i \) in (2.2) obtaining a much cleaner convolution formula
\[ m^* K = K \ast K. \quad (2.3) \]
2.1. Ansatz. We would like to establish an ansatz for the kernels \( K_g \) for elements \( g \) in some appropriate subset of \( G \). Let \( O \) be the subset of all elements \( g \in \text{End}(V) \) such that \( g - I \) is invertible. We define the following map \( \kappa : O \rightarrow \text{End}(V) \)

\[
\kappa(g) = \frac{g + I}{g - I}.
\]

It is a direct verification that \( \kappa \) is injective, identifying the set \( U = G \cap O \) with a subset of the Lie algebra \( sp(V) \). The map \( \kappa \) is well known in the literature \cite{We} and referred to as the Cayley transform. It establishes an interesting algebraic relation between the group \( G \) and its Lie algebra.

We summarize some basic properties of the Cayley transform

- \( \kappa(g^{-1}) = -\kappa(g) \).
- \( \kappa^2 = \text{Id} \).
- \( \omega(\kappa(g)v, u) = -\omega(v, \kappa(g)u) \) for every \( g \in U \) (this is just the statement that \( \kappa(g) \in sp(V) \)).
- The following two equivalent identities hold

\[
\begin{align*}
\kappa(gh) &= [I + \kappa(g)][\kappa(g) + \kappa(h)]^{-1}[I - \kappa(g)] + \kappa(g), \\
\kappa(gh) &= [I + \kappa(h)][\kappa(g) + \kappa(h)]^{-1}[I + \kappa(g)] - I,
\end{align*}
\]

for every pair \( g, h \in U \) such that \( gh \in U \). We note that for such a pair \( \kappa(g) + \kappa(h) \) is invertible, therefore, the formulas indeed make sense.

Let \( \sigma : k^\times \rightarrow \mathbb{C}^\times \) be the unique quadratic character of the multiplicative group (also called the Legendre character). Denote by \( \epsilon \) the value of the one-dimensional Gauss sum \( \epsilon = \sum_{z \in \mathbb{Z}} \psi(z^2) \).

**Ansatz:** Define

\[
K_g(v) = \nu(g)\psi\left(\frac{1}{4}\omega(\kappa(g)v, v)\right), \quad g \in U,
\]

where \( \nu(g) = \frac{\epsilon^N}{q^N}\sigma(\det(\kappa(g) + I)) \).

**Remark 2.1.4.** The only non-trivial part of the ansatz (2.1.3) is the normalization factor \( \nu(g) \). The expression which involves the \( \psi \) can be easily obtained \cite{Pl}. In more detail, the operator \( \rho(g) \) is determined up to a scalar by the Egorov identity (1.3.1). Hence, starting with the identity operator \( \text{Id} : \mathcal{H} \rightarrow \mathcal{H} \) and taking the average

\[
A = \sum_{v \in V} \pi(-g \cdot v)\text{Id}\pi(v) = \pi(\hat{A}) \in \text{Hom}\mathcal{H}(\pi, \pi^g),
\]

we recover the operator \( \rho(g) \) up to a scalar. It is a direct calculation, using formula (1.2.2), to verify that \( \hat{A}(v) = \psi\left(\frac{1}{4}\omega(\kappa(g)v, v)\right) \).

**Theorem 2.1.5 (Extension).** There exists a unique multiplicative extension of (2.1.3) to \( G \).
The uniqueness is clear since we have $U \cdot U = G$, the existence of this extension will be proved in Section 3 using algebraic geometry.

**Remark 2.1.6.** In the case where $q = 3$ and $N = 1$ the Weil representation is not unique, the extension established in Theorem 2.1.5 gives, in a sense that will be explained later (see Remark 3.3.2.2), a natural choice.

2.2. Application to characters. Denote by $\tau = \rho \ltimes \pi$ the representation of the semi-direct product $J = G \ltimes H$ on the Hilbert space $\mathcal{H}$. The group $J$ is sometime referred to as the Jacobi group. We will call the representation $(\tau, J, \mathcal{H})$ the Heisenberg-Weil representation. As a result of the explicit expression (2.1.3) we obtain, using formula (1.2.2), a new (cf. [Ge, H1, H2]) explicit formulas for the character of the Weil and Heisenberg-Weil representations when restricted to the subsets $U$ and $U \times H$, respectively.

**Theorem 2.2.1** (Character formulas). The character $ch_\tau$ of the Heisenberg-Weil representation $\tau$ when restricted to the subset $U \times H$ is given by

$$ch_\tau(g, v, z) = \frac{2^N}{q} \sigma(\det(\kappa(g) + I)) \psi\left(\frac{1}{4} \omega(\kappa(g)v, v) + z\right),$$

and the character $ch_\rho$ of the Weil representation $\rho$ when restricted to the subset $U$ is given by

$$ch_\rho(g) = \frac{2^N}{q} \sigma(\det(\kappa(g) + I)).$$

3. Geometric Weil representation

In this section we are going to construct a geometric counterpart to the set-theoretic Weil representation. This will be achieved using geometrization. This is a formal procedure, invented by Grothendieck and his school, by which sets are replaced by algebraic varieties (over the finite field) and functions are replaced by certain sheaf-theoretic objects.

3.1. Preliminaries from algebraic geometry. Next we have to use some space to recall notions and notations from algebraic geometry and the theory of $\ell$-adic sheaves.

3.1.1. Varieties. In the sequel, we are going to translate back and forth between algebraic varieties defined over the finite field $k$ and their corresponding sets of rational points. In order to prevent confusion between the two, we use bold-face letters to denote a variety $X$ and normal letters $X$ to denote its corresponding set of rational points $X = X(k)$. For us, a variety $X$ over the finite field is a quasi-projective algebraic variety, such that the defining equations are given by homogeneous polynomials with coefficients in the finite field $k$. In this situation, there exists a (geometric) Frobenius endomorphism $Fr : X \to X$, which is a morphism of algebraic varieties. We denote by $X$ the set of points fixed by $Fr$, i.e., $X = X(k) = X^{Fr} = \{x \in X : Fr(x) = x\}$. The category of algebraic varieties over $k$ will be denoted by $\text{Var}_k$. 


3.1.2. **Sheaves.** Let $\mathcal{D}^b(\mathcal{X})$ denote the bounded derived category of constructible $\ell$-adic sheaves on $\mathcal{X}$ [BBD]. We denote by $\text{Perv}(\mathcal{X})$ the Abelian category of perverse sheaves on the variety $\mathcal{X}$, that is the heart with respect to the autodual perverse t-structure in $\mathcal{D}^b(\mathcal{X})$. An object $F \in \mathcal{D}^b(\mathcal{X})$ is called $n$-perverse if $F[n] \in \text{Perv}(\mathcal{X})$. Finally, we recall the notion of a Weil structure (Frobenius structure) [D2]. A Weil structure associated to an object $F \in \mathcal{D}^b(\mathcal{X})$ is an isomorphism $\theta : Fr^* F \sim F$.

A pair $(F, \theta)$ is called a Weil object. By an abuse of notation we often denote $\theta$ also by $Fr$. We choose once an identification $\mathbb{Q}_{\ell} \simeq \mathbb{C}$, hence all sheaves are considered over the complex numbers.

**Remark 3.1.2.1.** All the results in this section make perfect sense over the field $\mathbb{Q}_{\ell}$, in this respect the identification of $\mathbb{Q}_{\ell}$ with $\mathbb{C}$ is redundant. The reason it is specified is in order to relate our results with the standard constructions.

Given a Weil object $(F, Fr^* F \simeq F)$ one can associate to it a function $f^F : \mathcal{X} \to \mathbb{C}$ to $F$ as follows

$$f^F(x) = \sum_i (-1)^i \text{Tr}(Fr_{|H^i(F)}).$$

This procedure is called Grothendieck's sheaf-to-function correspondence. Another common notation for the function $f^F$ is $\chi_{Fr}(F)$, which is called the Euler characteristic of the sheaf $F$.

### 3.2. Geometric Weil representation.

We shall now start the geometrization of the Weil representation.

#### 3.2.1. Replacing sets by varieties.

The first step we take is to replace all sets involved by their geometric counterparts, i.e., algebraic varieties. The symplectic space $(V, \omega)$ is naturally identified as the set $V = V(k)$, where $V$ is a $2N$-dimensional symplectic vector space in $\text{Var}_k$. The Heisenberg group $H$ is naturally identified as the set $H = H(k)$, where $H = V \times \mathbb{A}^1$ is the group variety equipped with the same multiplication formulas [1.1.1]. The subvariety $Z = Z(E) = \{(0, \lambda) : \lambda \in \mathbb{A}^1\}$ is the center of $H$. Finally, the group $G$ is naturally identified as $G = G(k)$, where $G = \text{Sp}(V, \omega)$.

#### 3.2.2. Replacing functions by sheaves.

The second step is to replace functions by their sheaf-theoretic counterparts [Ga]. The central character $\psi : Z \to \mathbb{C}^\times$ is associated via the sheaf-to-function correspondence to the Artin-Schreier sheaf $\mathcal{L}_\psi$ living on $Z$, i.e., we have $f^{\mathcal{L}_\psi} = \psi$. The Legendre character $\sigma$ on $k^\times \simeq \mathbb{G}_m(k)$ is associated to the Kummer sheaf $\mathcal{L}_\sigma$ on $\mathbb{G}_m$. Looking back at formula [2.1.3], the constant $\epsilon$ is replaced by the following Weil object

$$\mathcal{E} = \int \mathcal{L}_{\psi(z^2)} \in \mathcal{D}^b(pt),$$

$$\simeq Z.$$
where, for the rest of this paper, \( \int = \int_{\text{c}} \) denotes integration with compact support \([BBD]\). Grothendieck’s Lefschetz trace formula \([Gr]\) implies that, indeed, \( f^E = e \). In fact, there exists a quasi-isomorphism \( E \overset{q.i.}{\rightarrow} H^1(E)[-1] \) and \( \dim H^1(E) = 1 \), hence, \( E \) can be thought of as a one-dimensional vector space, equipped with a Frobenius operator, sitting at cohomological degree 1.

Finally, given an element \( a \) in the Lie algebra \( sp(V) \), we will use the notation

\[
\mathcal{G}_a = \int_{v \in V} \mathcal{L}_{\psi(\tfrac{1}{4}\omega(a_v,v))},
\]

for the symplectic gauss integral (noting that \( \omega(a\cdot,\cdot) \) is a symmetric bilinear form).

Our main objective, in this section, is to construct a multiplicative extension \( K : G \times V \rightarrow \mathbb{C} \) of the anzats (2.1.3). The extension appears as a direct consequence of the following geometrization theorem (see Appendix B for a proof).

**Theorem 3.2.2.1** (Geometric Weil representation). There exists a geometrically irreducible \( \dim(G) \)-perverse Weil sheaf \( K \) on \( G \times V \) of pure weight \( w(K) = 0 \), satisfying the following properties

1. Convolution property: There exist an isomorphism \( m^*K \simeq K \ast K \).

2. Function property: We have \( f^K(g,v) = K\big|_g(v) \) for every \( g \in U \).

**Remark 3.2.2.3.** Note that, as promised in Theorem 2.1.5, taking \( K = f^K \), we obtain a multiplicative extension of the anzats! The nice thing about this construction is that it uses geometry and, in particular (see Appendix B.1), the notion of perverse extension which has no counterpart in the set-function theoretic setting.

### 3.3. Additional properties

We describe two additional properties of the sheaf \( K \).

**3.3.1. Restriction property.** We consider an irreducible subvariety \( M \subset G \) and the associated open subvariety \( U_M = U \cap M \), where \( U = \{ g \in G; g-I \text{ is invertible} \} \).

**Definition 3.3.1.1.** We say that \( M \) is multiplicative if \( M \cdot M \subset M \). A multiplicative subvariety \( M \subset G \) is called \underline{openly generated} if the induced morphism

\[
m : U_M \times U_M \rightarrow M,
\]

is smooth, surjective and with connected fibers.

**Theorem 3.3.1.3.** Let \( M \subset G \) be a multiplicative, openly generated subvariety, then the restriction \( K\big|_{M \times V} \) is geometrically irreducible \( \dim(M) \)-perverse Weil sheaf of pure weight zero.
For a proof see Appendix C.1

3.3.2. Product property. Consider two symplectic vector spaces \((V_1, \omega_1)\) and \((V_2, \omega_2)\) in \(\text{Var}_k\). Denote by \((V, \omega)\) the product space with \(V = V_1 \times V_2\) and \(\omega = \omega_1 + \omega_2\). Let \(G_1, G_2\) and \(G\) be the corresponding symplectic groups. We have the natural inclusion \(i : G_1 \times V_1 \times G_2 \times V_2 \hookrightarrow G \times V\). Let us denote by \(K_1, K_2\) and \(K\) the corresponding Weil representation sheaves.

**Theorem 3.3.2.1.** There exist an isomorphism \(i^*K \cong K_1 \boxtimes K_2\), where \(\boxtimes\) denotes exterior tensor product.

For a proof see Appendix C.2

**Remark 3.3.2.2 (\(\text{[G]}\)).** The product property confirms that the set theoretic Weil representation that we constructed coincides with the one in \(\text{[Ge]}\). This is the unique family \((\rho_V, Sp(V), H_V)\), where \(V\) runs in the category of symplectic vector spaces, which satisfies

\[\rho_{V_1 \times V_2} \cong \rho_{V_1} \boxtimes \rho_{V_2},\]

for every pair of symplectic vector spaces \(V_1\) and \(V_2\).

### Appendix A. Relation to Deligne’s sheaf

In his letter \(\text{[D1]}\) Deligne proposed a sheaf-theoretic analogue of the Weil representation of \(G = Sp(V, \omega)\), where \((V, \omega)\) is a \(2N\)-dimensional symplectic vector space over the finite field \(k = \mathbb{F}_q\) with \(q\) odd. His proposed construction was given in terms of specific formulas. In this short section we would like to elaborate on the relation between the Weil representation sheaf \(K\) which we constructed in the previous subsection, and Deligne’s sheaf (see \(\text{[CH]}\) for the formal construction of Deligne’s sheaf in the \(SL_2\) case).

Deligne’s sheaf was given in terms of a specific realization of the Weil representation. Let us elaborate on his construction. Fixing a Lagrangian splitting \(V = L \oplus L'\), we consider the Schrödinger realization (Remark 1.1.3) \((\pi_{L,L'}, H, H_{L,L'})\) of the Heisenberg representation, where \(H_{L,L'} = S(L')\). In this realization every operator \(\rho(g), g \in G\), is given by a kernel \(K_{L,L'}(g) : L' \times L' \to \mathbb{C}\). The collection of kernels \(\{K_{L,L'}(g)\}_{g \in G}\) is equivalent to a single function \(K_{L,L'}\) on \(G \times L' \times L'\) which satisfies the convolution property

\[m^*K_{L,L'} = K_{L,L'} \ast K_{L,L'} . \quad (A.1)\]

In his letter Deligne constructed a perverse Weil sheaf, which we will denote by \(K_{L,L'}\), on the variety \(G \times L' \times L'\) satisfying the following two properties

- **Function property:** The function \(K_{L,L'} = f^{K_{L,L'}}\).
- **Convolution property:** There exists an isomorphism

\[m^*K_{L,L'} \simeq K_{L,L'} \ast K_{L,L'} . \]
Let us now explain the relation between our sheaf $K$ and Deligne’s sheaf $K_{L,L'}$. On the level of functions, $K$ and $K_{L,L'}$ are related as follows

$$K_{L,L'}(g,x,y) = \langle \delta_x | \rho(g) \delta_y \rangle = \sum_{l' \in L', l \in L} K(g,l'+l) \langle \delta_x | \psi(\omega(\cdot,l)L_l \delta_y) \psi(\frac{1}{2} \omega(l,l')) \rangle \psi\left(\omega(\frac{1}{2} \omega(l,l') - \frac{1}{2} \omega(l,y-x))\right),$$

where the first equality is the definition of $K_{L,L'}(g,x,y)$ and the second equality follows from substituting $\pi_{L,L'}(l',l) = \psi\left(\frac{1}{2} \omega(l,l')\right)$.

It will be convenient to rewrite (A.2) in a more functorial manner. Let $\alpha: G \times L' \times L \to G \times L \times L'$ be the map given by $\alpha(g,x,y) = (g,y-x,x+y)$. Formula (A.2) is equivalent to

$$K_{L,L'} = \alpha^* \text{Four}_L(K),$$

where $\text{Four}_L$ denotes the (non-normalized) fiber-wise Fourier transform from functions on the vector bundle $G \times L' \times L \to G \times L$ to functions on the dual vector bundle $G \times L' \times L' \to G \times L'$; the duality is manifested by the symplectic form $\omega$ which induces a non-degenerate pairing between $L$ and $L'$.

Formula (A.3) can easily be translated into the geometric setting. We define Deligne’s sheaf associated with the Schrödinger realization $\pi_{L,L'}$ to be

$$K_{L,L'} = \alpha^* \text{Four}_L(K),$$

where now $\text{Four}_L$ denotes the $\ell$-adic Fourier transform $[KL]$. Since $\text{Four}_L$ shifts the perversity degree by $N [KL]$ it follows that $K_{L,L'}$ is an irreducible $\dim(G) + N$- perverse Weil sheaf.

**Appendix B. Proof of the geometric Weil representation theorem**

Appendix B is devoted to the proof of Theorem 3.2.2.1, i.e., to the construction of the Weil representation sheaf $K$.

**B.1. Construction.** The construction of the sheaf $K$ is based on the ansatz (2.1.3).

Let $U \subset G$ be the open subvariety consisting of elements $g \in G$ such that $g - I$ is invertible. The Cayley transform is given by an algebraic function, hence it makes sense also in the geometric setting, we denote it also by $\kappa$.

The construction proceeds in several steps
• **Non-normalized kernels:** On the variety \( \text{End}(V) \times V \) define the sheaf

\[
\tilde{K}(a,v) = L_{\psi \left( \frac{1}{4} \omega(au,v) \right)}.
\]

Denote by \( \tilde{K}_U \) the pull back \( \tilde{K}_U (g, v) = \tilde{K}(\kappa(g), v) \) defined on \( U \times V \).

• **Normalization coefficients:** On the open subvariety \( U \times V \) define the sheaf

\[
C = E \otimes_{N} L_{\sigma(\det(\kappa(g) + I))}[4N](2N).
\]

We note that for \( g \in U \) we have \( \kappa(g) + I = \frac{2g}{g - I} \), thus it is invertible and consequently \( L_{\sigma(\det(\kappa(g) + I))} \) is smooth on \( U \).

• **Normalized kernels:** On the open subvariety \( U \times V \) define the sheaf

\[
K_U = C \otimes \tilde{K}_U.
\]

Finally, take

\[
K = j_! K_U,
\]

where \( j : U \hookrightarrow G \) is the open imbedding, and \( j_* \) is the functor of perverse extension [BBD]. It follows directly from the construction that the sheaf \( K \) is irreducible \( \text{dim}(G) \)-perverse of pure weight 0.

Section B.2 is devoted to proving that the kernel sheaf \( K \) (B.1.2) satisfies the conditions of Theorem 3.2.2.1.

**B.2. Proof of the properties.** The function property is clear from the construction. We are left to show the convolution property.

We need to show that

\[
m^* K \simeq K \ast K,
\]

where \( m : G \times G \rightarrow G \) is the multiplication morphism.

**Proof.** The morphism \( m \) forms a principal \( G \)-bundle, hence it is smooth and surjective. We denote by \( m_U : U \times U \rightarrow G \) the restriction of \( m \) to the open subvariety \( U \times U \hookrightarrow G \times G \). It is easy to verify that \( m_U \) is smooth and surjective. The following technical lemma (see Appendix [B.3] for a proof) plays a central role in the proof.

**Lemma B.2.2.** There exists an isomorphism \( m^*_U K \simeq K_U \ast K_U \).

Lemma B.2.2 implies that the sheaves \( m^* K \) and \( K \ast K \) are isomorphic when restricted to the open subvariety \( U \times U \). The sheaf \( m^* K \) is irreducible \( \text{dim}(G \times G) \)-perverse as a pull-back by a smooth morphism of an irreducible \( \text{dim}(G) \)-perverse sheaf on \( G \times V \). Hence, it is enough to show that the sheaf \( K \ast K \) is also irreducible \( \text{dim}(G \times G) \)-perverse. Consider the map \( (m_U, Id) : U \times U \times G \rightarrow G \times G \). It is clearly smooth and surjective. It is enough to show that the pull-back \((m_U, Id)^* K \ast K \) is irreducible \( \text{dim}(U \times U \times G) \)-perverse. Again, using Lemma B.2.2 and also invoking some direct diagram chasing one obtains

\[
(m, Id)^* K \ast K \simeq K_U \ast K_U \ast K.
\]

(B.2.3)
The right-hand side of (B.2.3) is principally a subsequent application of a properly normalized, symplectic Fourier transforms (see Formula (B.3.1) below) on $\mathcal{K}$, hence by the Katz-Laumon theorem [KL] it is irreducible dim$(U \times U \times \mathbf{G})$-perverse.

Let us summarize. We showed that both sheaves $m^* \mathcal{K}$ and $\mathcal{K} \ast \mathcal{K}$ are irreducible dim$(\mathbf{G} \times \mathbf{G})$-perverse and are isomorphic on an open subvariety. This implies that they must be isomorphic. This concludes the proof of Property (B.2.1). □

B.3. Proof of Lemma [B.2.2] The proof will proceed in several steps.

- Step 1. We show that the sheaf $\mathcal{K}_U \ast \mathcal{K}_U$ is irreducible dim$(U \times U)$-perverse. This is done by a direct computation

$$\mathcal{K}_U \ast \mathcal{K}_U(g, h, v) \simeq C_g \otimes C_h \otimes \mathcal{K}_U \ast \mathcal{K}_U(g, h, v).$$

Writing the convolution more explicitly

$$\mathcal{K}_U \ast \mathcal{K}_U(g, h, v) \simeq \int_{u \in \mathbf{V}} \mathcal{L} \psi(\frac{1}{2} \omega(v, [\kappa(g) - I]u) \otimes \mathcal{K}(\kappa(g) + \kappa(h), u).$$

(B.3.1)

Hence, we see that $\mathcal{K}_U \ast \mathcal{K}_U$ is principally an application of Fourier transform, which implies that $\mathcal{K}_U \ast \mathcal{K}_U$ is irreducible dim$(U \times U)$-perverse.

- Step 2. It is enough to show that the sheaves $m^* \mathcal{K}$ and $\mathcal{K}_U \ast \mathcal{K}_U$ are isomorphic on an open subvariety. Let $W \subset U \times U$ be the open subvariety consisting of pairs $(g, h) \in U \times U$ such that $gh \in U$. We will show that the sheaves $m^* \mathcal{K}$ and $\mathcal{K}_U \ast \mathcal{K}_U$ are isomorphic on $W$. Direct computation of the integral in (B.3.1), using completion to squares, reveals that

$$\mathcal{K}_U \ast \mathcal{K}_U(g, h, v) \simeq \mathcal{K}(\kappa(g) + [I + \kappa(g)] [\kappa(g) + \kappa(h)]^{-1} [I - \kappa(g)], v) \otimes \mathcal{G}_{\kappa(g) + \kappa(h)},$$

for every $(g, h) \in W$.

Invoking identity (2.1.1), we obtain that

$$\mathcal{K}_U \ast \mathcal{K}_U(g, h, v) \simeq \mathcal{K}(gh, v) \otimes \mathcal{G}_{\kappa(g) + \kappa(h)}.$$

Consequently, it is enough to show

**Proposition B.3.2.** There exists an isomorphism

$$C_g \otimes C_h \otimes \mathcal{G}_{\kappa(g) + \kappa(h)} \simeq C_{gh}.$$ 

**Proof.** Denote by $T$ the Tate vector space $T = H^2(\int \mathbb{P}^1 \mathbb{P}^1)[-2]$. Recall that we defined $C_g = E^\otimes 2N \otimes \mathcal{L}_{\sigma(\det[\kappa(g) + I])}[4N] (2N)$ and $C_h = E^\otimes 2N \otimes \mathcal{L}_{\sigma(\det[\kappa(h) + I])}[4N] (2N)$.
Using the character property of the sheaf $L_\sigma$ we obtain

$$C_g \otimes C_h \simeq E \otimes L_{\sigma[\det[\kappa(g)+I]\det[\kappa(h)+I]]}[4N](2N). \quad (B.3.3)$$

In addition, short analysis of the symplectic Gauss integral reveals

$$G_{\kappa(g)+\kappa(h)} \simeq E \otimes L_{\sigma[\det[\kappa(g)+\kappa(h)]]}. \quad (B.3.5)$$

Combining formulas (B.3.3), (B.3.5), and identity (2.1.2), we obtain

$$C_g \otimes C_h \otimes G_{\kappa(g)+\kappa(h)} \simeq \mathcal{E} \otimes L_{\sigma[\det[\kappa(gh)+I]]}[4N](2N) = C_{gh}. \quad \Box$$

**Appendix C. Additional Proofs**

**C.1. Proof of Theorem 3.3.1.3.**

**Proof.** Denote $F = \mathcal{K}|_{M \times V}$ and $F_U = \mathcal{K}|_{U_M \times V}$.

Since restriction does not increase weight, it is enough to prove that $F$ is geometrically irreducible $\dim(M)$- perverse. Due to the assumption that $m$ is smooth and surjective, it is enough to show that $m^*F$ (where $m$ is the morphism (3.3.1.2)) is a geometrically irreducible $\dim(U_M \times U_M)$- perverse. We have $m^*F \simeq F_U \ast F_U$, implying that $m^*F$ is an application of a properly normalized $\ell$-adic Fourier transform to an irreducible smooth sheaf (See formula (B.3.1)), consequently, the assertion follows by Katz-Laumon theorem [KL]. \quad \Box

**C.2. Proof of Theorem 3.3.2.1.**

**Proof.**

**Step 1.** The sheaves $i^*\mathcal{K}$ and $\mathcal{K}_1 \boxtimes \mathcal{K}_2$ are geometrically irreducible $[\dim(G_1 \times G_2)]$- perverse Weil sheaves on $G_1 \times G_2$. The statement is evident for $\mathcal{K} \boxtimes \mathcal{K}$, and for $i^*\mathcal{K}$ it follows from the restriction property (Theorem 3.3.1.3) of the sheaf $\mathcal{K}$ with respect to the multiplicative subvariety $M = G_1 \times G_2 \subset G$.

**Step 2.** By perversity it is enough to show that $i^*\mathcal{K}$ and $\mathcal{K}_1 \boxtimes \mathcal{K}_2$ are isomorphic when restricted to the open subvariety $U_1 \times V_2 \times U_2 \times V_2$, where $U_i = \{g \in G_i; \det(g-I) \neq 0\}$. This is a simple verification using formula (B.1.1). \quad \Box
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