AREA-EXPANDING EMBEDDINGS OF RECTANGLES

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Abstract. We estimate whether there is a k-expanding embedding from one n-dimensional rectangle into another. Our estimates are accurate up to a constant factor $C(n)$.

Suppose that $U, V \subset \mathbb{R}^n$ are open sets. An embedding $I : V \to U$ is called $k$-expanding if, for every $k$-dimensional surface $\Sigma \subset V$, the volume of $I(\Sigma)$ is at least the volume of $\Sigma$. Our theorem describes when there is a $k$-expanding embedding from one $n$-dimensional rectangle into another. It is sharp up to a constant factor in each dimension.

**Theorem 1.** For each dimension $n$, there is a constant $c(n) > 0$ so that the following holds. Let $R$ be an $n$-dimensional rectangle with dimensions $R_1 \leq \ldots \leq R_n$, and let $S$ be an $n$-dimensional rectangle with dimensions $S_1 \leq \ldots \leq S_n$.

If there is a $k$-expanding embedding from $S$ into $R$, then, for all integers $j, l$ in the ranges $0 \leq j < k \leq l \leq n$,

$$(R_1 \ldots R_j) \to R_{j+1} \ldots R_l \geq c(n)(S_1 \ldots S_j) \to S_{j+1} \ldots S_l.$$  

($\ast$)

**Theorem 2.** Conversely, for each dimension $n$ there is a constant $C(n) > 0$ so that the following holds. If, for all integers $j, l$ in the ranges $0 \leq j < k \leq l \leq n$,

$$(R_1 \ldots R_j) \to R_{j+1} \ldots R_l \geq C(n)(S_1 \ldots S_j) \to S_{j+1} \ldots S_l,$$  

(\ast\ast)

then there is a $k$-expanding embedding from $S$ into $R$.

Note that the necessary conditions ($\ast$) and the sufficient conditions (\ast\ast) are identical except that the constant $c(n)$ is replaced by the larger constant $C(n)$.

Some special cases of Theorem 1 were proven in [3]. The main contribution of this paper is to prove Theorem 1 in the remaining harder cases. In order to put the new methods in context, we give an overview of the problem, starting with the simplest cases.

**Overview of area-expanding embeddings**

We begin by discussing the two easy cases $k = n$ and $k = 1$. If $k = n$, then ($\ast$) reduces to the one inequality $R_1 \ldots R_n \geq S_1 \ldots S_n$, which says that the volume of $R$ is bigger than the volume of $S$. This one condition is sufficient for finding an $n$-expanding embedding from $S$ into $R$. For example, one can find a linear $n$-expanding embedding.

If $k = 1$, then ($\ast$) says $R_1 \ldots R_l \geq S_1 \ldots S_l$ for each $1 \leq l \leq n$. These inequalities say that the smallest $l$-dimensional cross-section of $S$ has smaller volume than the smallest $l$-dimensional cross-section of $R$. We’ll say more about the proof of this inequality a little lower in the introduction. In order to prove Theorem 2, we need to use nonlinear maps. For example, suppose that $R$ is the unit square and that $S$
is a long thin rectangle with dimensions \((1/2)\epsilon \times (1/2)\epsilon^{-1}\), \(\epsilon < 1/10\). There is no linear 1-expanding embedding from \(S\) into \(R\), but there is a non-linear 1-expanding embedding that folds \(S\) into \(R\), as shown in the following figure.

Using these folding maps repeatedly, it’s not hard to prove Theorem 2 for \(k = 1\).

With that background, we turn to the main case \(2 \leq k \leq n - 1\). To construct \(k\)-expanding embeddings, we use the two methods above. We use \(k\)-expanding linear maps, and we also use simple folding maps like the one in Figure 1. Composing these two kinds of maps, we construct enough embeddings to prove Theorem 2.

We now return to the proof of Theorem 1, which is the main subject of the paper. For general \(k\), the inequalities in \((*)\) can be divided into two kinds. First, we have the inequalities \(R_1 \cdots R_l \gtrsim S_1 \cdots S_l\) for each \(k \leq l \leq n\). We have already seen this kind of inequality about cross-sectional volumes in the case \(k = 1\). Second, we have more complicated inequalities with \(j > 0\). These more complicated inequalities appear only when \(k\) is in the range \(2 \leq k \leq n - 1\). For example, if \(k = 2\), we have the inequality \(R_1^2 R_2 R_3 \gtrsim S_1^2 S_2 S_3\).

The proof of the first inequalities \(R_1 \cdots R_l \gtrsim S_1 \cdots S_l\) follows from a sweepout estimate as follows. The rectangle \(R\) may be sliced into parallel \(l\)-dimensional rectangles with dimensions \(R_1 \times \cdots \times R_l\). If we take the pullback of these surfaces in \(S\), then we get a family of surfaces sweeping out the rectangle \(S\). We refer to these surfaces as slices of \(S\). This construction is illustrated in the figure below.

Now \(R_1 \cdots R_l\) is the volume of each rectangular slice of \(R\). It follows from linear algebra that if \(l \geq k\), then a \(k\)-expanding map is also \(l\)-expanding. (The linear algebra is described in Appendix 1.) Therefore, each slice of \(S\) has volume at most \(R_1 \cdots R_l\). Next we apply the sweepout estimate of Almgren and Gromov.
**Sweepout Estimate.** (Almgren, Gromov [7, 3]) A family of $l$-dimensional surfaces sweeping out $S$ contains a surface of volume at least $c(n)S_1...S_l$.

Each slice of $S$ has volume at most $R_1...R_l$, but one slice of $S$ has volume at least $c(n)S_1...S_l$, and so we conclude that $R_1...R_l \gtrsim S_1...S_l$, proving (*) in the case $j = 0$.

If $j > 0$, then the algebra in (*) is complicated looking. We can think of (*) as a statement about the $j$-dimensional width and the $l$-dimensional width of $R$ and $S$. This point of view becomes clearer if we rewrite (*) in the following equivalent way.

\[ R_1...R_l \gtrsim \left[ (S_1...S_j)/(R_1...R_j) \right]^{\frac{l-k}{k-j}} S_1...S_l. \]

If there is a $k$-expanding embedding from $S$ into $R$, we already know that $R_1...R_l \gtrsim S_1...S_l$. If $R_1...R_j \geq S_1...S_j$, then (*) follows automatically. So we only need to consider the case that $R_1...R_j$ is much smaller than $S_1...S_j$. In this case, (*) says that the $l$-dimensional width of $R$ must be substantially larger than the $l$-dimensional width of $S$: larger by a factor $\sim \left[ (S_1...S_j)/(R_1...R_j) \right]^{\frac{l-k}{k-j}}$. In other words, it is possible to squeeze $S$ into a rectangle $R$ with much smaller $j$-dimensional width only if $R$ has much larger $l$-dimensional width.

**The tightening construction**

Now we describe the new technique in this paper. As in Figure 2, we look at the preimages in $S$ of parallel $l$-dimensional rectangles in $R$. We will give a proof by contradiction, so we assume that (*) is violated. If $j = 0$, we saw above that the slices of $S$ do not have enough volume to sweep out $S$. If $j > 0$, then the slices of $S$ have enough volume to sweep out $S$, but in a subtler way, we will show that they are still not big enough to sweep out $S$. The rough idea is that since $[0, R_1] \times ... \times [0, R_l]$ is shaped very differently from $[0, S_1] \times ... \times [0, S_l]$, the slices in $S$ have to “scrunched up”.

![Figure 3. Two "scrunched up" slices in the rectangle S](image)

The two curves in Figure 3 are long enough to stretch from the bottom of $S$ to the top of $S$, but they are too scrunched up to do so. If a family of curves sweeps out the rectangle $S$, then they cannot all be as scrunched up as these.
In order to prove that the slices are “scrunched up”, and in order to exploit this scrunching, we proceed as follows. We subdivide the rectangle \([0, R_1] \times \ldots \times [0, R_l]\) into subrectangles at a well-chosen scale. Each slice of \(S\) is thus subdivided into pieces given by the inverse images of the subrectangles. A subdivision of the slices is shown in Figure 4.

Figure 4. Each slice is divided into three pieces

Figure 4 shows a magnified view of the curves from Figure 3. Each slice has been subdivided into three pieces. The large dots mark the endpoints of the pieces.

Now, we use an isoperimetric inequality to “tighten” each piece of each slice. This is the key step in the proof. It involves a new variant of the isoperimetric inequality. We describe it in more detail below. Continuing informally, we show the new tightened slices in Figure 5.

Figure 5. Each piece of each slice has been pulled tight

The dots in Figure 5 are in the same locations as the dots in Figure 4, but instead of connecting them with scrunched up curves we have connected them with straight lines. In the body of the paper, the tightened pieces are not completely flat but are constructed by a Federer-Fleming type argument. This tightening reduces the volume of the slices, and if (*) is violated, then the tightened slices do not have enough volume to sweep out \(S\). This finishes our cartoon outline of the proof of Theorem 1.

The key step of tightening the pieces is done with the help of an isoperimetric inequality. In order to do this, we need to choose a piece of the slice so that the
piece itself has a large volume but the boundary of the piece has a small volume. These pieces exist because $R_1 \ldots R_j$ is much smaller than $S_1 \ldots S_j$.

Let’s give a more precise description in a simple example. Suppose that $j = 1$, $k = 2$, and $l = 3$. Furthermore, suppose that $R_1 = 1$ and that $R_2$ and $R_3$ are much bigger than 1. Now we divide the rectangle $[0, R_1] \times [0, R_2] \times [0, R_3]$ into subrectangles of dimensions $1 \times L \times L$ for a large number $L < R_2$. One of these subrectangles has volume $L^2$. Its relative boundary has area $4L$. (The relative boundary of the subrectangle consists of four faces with dimensions $1 \times L$. The absolute boundary also contains two large faces with dimensions $L \times L$, but these large faces lie in the boundary of $R$.) For comparison, notice that a 2-cycle $z$ in $\mathbb{R}^n$ with area $4L$ must bound a 3-chain with volume $\lesssim L^{3/2}$, which is much smaller than $L^2$. So the subrectangle is a large 3-chain with a small relative boundary.

The preimage of this subrectangle in $S$ is a relative 3-chain with a small relative boundary. In order to tighten it, we need to prove an isoperimetric inequality for relative cycles in the rectangle $S$. In particular, we will prove and use the following estimate for relative integral cycles in $S$.

**Isoperimetric Lemma.** If $z$ is a $p$-dimensional relative cycle in $S$ with volume $c(n)S_1 \ldots S_j A^{p-1}$ for some $A$ in the range $S_1 < A < S_{j+1}$, then $z$ bounds a $(p+1)$-chain with volume at most $C(n)S_1 \ldots S_j A^{p-j+1}$.

The Isoperimetric Lemma is a modification of the Federer-Fleming isoperimetric inequality. For reference, we recall their inequality.

**Federer-Fleming Isoperimetric Inequality.** If $z$ is a $q$-dimensional cycle in $\mathbb{R}^n$ with volume $A^q$, then $z$ bounds a $(q+1)$-dimensional chain with volume at most $C(n)A^{q+1}$.

The inequality in the Isoperimetric Lemma depends on the dimensions of $S$. This is necessary: there is no isoperimetric inequality for relative cycles that holds uniformly for all rectangles. Instead, there is a different isoperimetric profile for each rectangle, and we have to estimate how the profile depends on the dimensions of the rectangle. In the paper, we give a fairly precise description of this isoperimetric profile, and the Isoperimetric Lemma above is a special case.

The algebra in the Isoperimetric Lemma is somewhat complicated. To understand it, it helps me to consider the special case that $z$ has the form $[0, S_1] \times \ldots \times [0, S_j] \times z'$, where $z'$ is an absolute $(p-j)$-dimensional cycle in $[0, S_{j+1}] \times \ldots \times [0, S_n]$. The cycle $z'$ would have volume $c(n)A^{p-j}$, and the Federer-Fleming inequality implies that $z'$ bounds a chain $y$ of volume at most $C(n)A^{p-j+1}$. Hence $z$ bounds $[0, S_1] \times \ldots \times [0, S_j] \times y$, which has volume at most $C(n)S_1 \ldots S_j A^{p-j+1}$. The Isoperimetric Lemma says that the same estimate holds for a general cycle $z$ as long as the volume of $z$ lies in an appropriate range. We prove it by using the construction of Federer-Fleming at a sequence of different scales.

We are now ready to fill in all the details in the cartoon outline above. In order to keep the algebra simple, we again focus on the special case $j = 1$, $k = 2$, $l = 3$. In this case, condition $(\ast)$ reads $R_1^2R_2R_3 \gtrsim S_1^2S_2S_3$. We already know that $R_1R_2R_3 \gtrsim S_1S_2S_3$, so we only need to prove $(\ast)$ in the case that $R_1$ is much smaller than $S_1$. We consider a 3-dimensional rectangle in $R$ with dimensions $[0, R_1] \times [0, R_2] \times [0, R_3]$, parallel to the smallest 3-face of $R$. We let $z$ denote the inverse image of this rectangle in $S$. The relative cycle $z$ is one of the scrunched up slices in Figure 3.
Next we divide $z$ into pieces. First we subdivide the rectangle $[0, R_1] \times [0, R_2] \times [0, R_3]$ into subrectangles of dimensions $R_1 \times L \times L$, for a number $L > R_1$, which we choose later. We let $C_i$ be the inverse images of these subrectangles in $S$. The chains $C_i$ are the pieces of the slices in Figure 4. We have $z = \sum C_i$, and we know that each chain $C_i$ has volume at most $R_1L^2$.

Now we look at the boundaries of the chains $C_i$. Each of our 3-dimensional subrectangles of dimension $R_1 \times L \times L$ has a relative boundary with area at most $4R_1L$. Since the map $I$ is 2-expanding, the relative boundary of each chain $C_i$ has area at most $4R_1L$. Now we apply the Isoperimetric Lemma to the boundary of $C_i$. To make the proof work, we have to choose $L$ so that $R_1L$ is between $S_2$ and $S_1S_2S_3$. Then the Isoperimetric Lemma guarantees that $\partial C_i$ bounds some 3-chain $C'_i$ with volume at most $\sim (R_1/S_1)R_1L^2$. In other words, our bound for the volume of $C'_i$ is better than the bound for the volume of $C_i$ by a factor $\sim (R_1/S_1)$. To tighten the slice $z$, we replace each chain $C_i$ with the chain $C'_i$. The chains $C'_i$ are the segments in Figure 5. We define a relative cycle $z' = \sum C'_i$. The relative cycle $z'$ is one of the tightened slices in Figure 5. The total volume of $z'$ is at most $\sim (R_1/S_1)R_1R_2R_3$.

We perform the same tightening operation on every slice. Each tightened slice has volume at most $\sim (R_1/S_1)R_1R_2R_3$. Because of the sweepout lemma, one of the tightened slices must have volume at least $c(n)S_1S_2S_3$. Hence $(R_1/S_1)R_1R_2R_3 \gtrsim S_1S_2S_3$, and rearranging we get $R_1^2R_2R_3 \gtrsim S_1^2S_2S_3$ as desired.

In general the tightening procedure is a little bit more involved. We use our control of the $k$-skeleton of the slice to tighten the $(k+1)$-skeleton. Then we use our improved control of the $(k+1)$-skeleton to tighten the $(k+2)$-skeleton, and so on until we get to the $1$-skeleton of the slice.

**Complexes of cycles**

Lastly, I want to say a word about the language we use in the proof. We outlined our argument informally in terms of families of surfaces, but families of surfaces are not a convenient language. For one problem, the tightening construction we just described does not depend continuously on the surface. Instead, we use a discrete analogue of a family of cycles that we call a complex of cycles. Over several papers, I have found complexes of cycles to be a simple, convenient language for arguments about area-contracting maps.

A complex of cycles $C$ in $S$ is parametrized by a polyhedral complex $X$. For each $p$-face $F$ of $S$, the complex $C$ associates a $p$-dimensional relative chain $C(F)$, and these chains are required to fit together in a coherent way. Figure 6 shows an example of a complex of cycles, illustrating the way the chains should fit together.

![Figure 6. A complex of cycles parametrized by a triangle](image)
In this example, the polyhedral complex $X$ is a triangle. Each side of the triangle corresponds to an oriented relative 1-chain in $S$. The solid line in the triangle corresponds to the two solid curves in $S$, and so on.

Complexes of cycles were introduced by Almgren in his thesis [1] on the homotopy groups of spaces of cycles. He begins with a continuous family of cycles, but the first step in his argument is to replace the continuous family by a complex of cycles that approximates it. Complexes of cycles were then used by Gromov in his proof of the Sweepout Estimate [7]. The first step in Gromov’s proof is also to replace the continuous family by a complex of cycles approximating it. Almgren and Gromov did not name the object that they use. The name complex of cycles comes from [6].

Here is an outline of the paper. In Section 1, we prove estimates for the isoperimetric profile of a rectangle. In Section 2, we state a generalization of Theorem 1. In Section 3, we define complexes of cycles. In Section 4 we prove a version of the sweepout lemma for complexes of cycles. With this lemma, we prove Theorem 1 in the easy case $j = 0$. In Section 5, we give some algebraic preliminaries which reduce the general case of Theorem 1 to a slightly more special case. In Section 6, we explain the tightening construction and prove Theorem 1. This section is the heart of the paper. In Section 7, we construct area-expanding embeddings of rectangles, proving Theorem 2. The paper ends with two appendices. The first appendix covers the linear algebra related to area-expanding or area-contracting maps. The second appendix covers generalizations of our results to shapes other than rectangles.

**Acknowledgements.** This paper is a simplified version of the main result of my thesis [5]. The proof in my thesis was very convoluted. I am grateful to my thesis advisor, Tom Mrowka, for his support and encouragement.

1. The isoperimetric profile of a rectangle

Let $R$ denote the $n$-dimensional rectangle $[0, R_1] \times \ldots \times [0, R_n]$, where the dimensions are ordered so that $R_1 \leq \ldots \leq R_n$. In this section, we estimate the isoperimetric profile for relative integral cycles in $R$. Our goal is to understand the way that the isoperimetric profile depends on the dimensions $R_i$.

If $z$ is a relative integral $k$-cycle in $R$, the filling volume of $z$ is the smallest volume of any relative $(k+1)$-chain $y$ with $\partial y = z$. Let $I^R_k(V)$ denote the largest filling volume of any $k$-dimensional relative integral cycle in $R$ with volume at most $V$.

Remark: We use the following definition for volume. If a chain $C$ is given by $\sum c_i f_i$ where $c_i \in \mathbb{Z}$ and $f_i$ is a Lipschitz map from the standard $k$-simplex to $S$, then the volume of the chain $C$ is defined to be $\sum |c_i| Vol(f_i^* Euc)$, where $Euc$ denotes the Euclidean metric on $R$. This quantity is also called the mass of $C$. We denote the volume of $C$ by $|C|$.

The following theorem estimates the isoperimetric profile $I^R_k$ for the rectangle $R$.

**Theorem 3.** There are constants $c(n) > 0, C(n)$ so that the following holds.

If $V \leq c(n) R_1 \ldots R_k$, then write $V = c(n) R_1 \ldots R_j \rho^{k-j}$ for some $0 \leq j \leq k - 1$ and some $\rho$ in the range $R_j \leq \rho \leq R_{j+1}$. (These conditions determine $j$ and $\rho$ uniquely.)
Then $I_R^k(V) \leq C(n) R_1 \ldots R_j \rho^{k-j+1}.$

In any case, $I_R^k(V) \leq C(n) R_{k+1} V.$

Before we prove the theorem, we consider two examples of relative cycles in $R.$ These examples show that our upper bounds for $I_R^k$ are fairly sharp. They also help me to remember the formulas.

Pick an integer $j$ in the range $0 \leq j \leq k - 1.$ Then consider the cycle $[0, R_1] \times \ldots \times [0, R_j] \times S^{k-j}(\rho)$ for $R_j \leq \rho \leq (1/10) R_{j+1}.$ In this equation, $S^{k-j}(\rho)$ denotes a sphere of dimension $k - j$ and radius $\rho$ contained in $[0, R_{j+1}] \times \ldots \times [0, R_n],$ with center at the center of the rectangle, $(R_{j+1}/2, \ldots, R_n).$ This cycle has volume $V \sim R_1 \ldots R_j \rho^{k-j}.$ The best filling of the cycle is just $[0, R_1] \times \ldots \times [0, R_j] \times B^{k-j+1}(\rho).$

To clarify the notation, $B^{k-j+1}$ is a Euclidean ball of dimension $k - j + 1$ with boundary $S^{k-j}(\rho).$ This filling has volume $V \sim R_1 \ldots R_j \rho^{k-j+1}.$

Second, consider the relative cycle $[0, R_1] \times \ldots \times [0, R_k] \times \{p\}$ with multiplicity $M,$ where $p$ is the center of the rectangle $[0, R_{k+1}] \times \ldots \times [0, R_n].$ (Alternatively, consider $M$ nearby parallel rectangles.) The volume of this cycle is $V = MR_1 \ldots R_k.$

This cycle has filling volume $V \sim MR_1 \ldots R_{k+1} = R_{k+1} V.$

Remarks: These examples give lower bounds for $I_R^k(V).$ The lower bounds match the upper bounds in the theorem up to a constant factor except in the delicate range $c(n) R_1 \ldots R_k \leq V \leq R_1 \ldots R_k.$ If $R_{k+1} \gg R_k,$ then the function $I_R^k(V)$ grows very rapidly over the course of this range. It appears plausible that $I_R^k(V)$ is discontinuous, perhaps at the value $V = R_1 \ldots R_k.$

Now we turn to the proof of the theorem.

Proof. We begin by using the deformation theorem of Federer and Fleming, which we record as a lemma.

Lemma 1.1. Suppose that $z$ is a relative $k$-cycle in $R.$ Consider a rectangular lattice inside $R$ with each side-length roughly equal to $L$ (up to a factor of 2), for some $L \leq R_1,$ and suppose that the boundary of $R$ lies in the $(n-1)$-skeleton of the lattice. Then there is another relative cycle $z'$ in $R$ contained in the $k$-skeleton of the lattice and obeying the following inequalities.

1. The volume of $z'$ is at most $C(n)|z|.$
2. The filling volume of $z' - z$ is at most $C(n) L |z|.$

Remark: Morally, we are using a cubical lattice. We allow a slightly non-cubical lattice so that we can arrange for the boundary of $R$ to lie in the $(n-1)$-skeleton of the lattice.

Proof. (sketch) We sketch the proof of Federer and Fleming. For more details, see [5]. We begin with a relative cycle $z$ with boundary $\partial z$ contained in $\partial R.$ We build a sequence of homologies $z = z_n \sim z_{n-1} \sim \ldots \sim z_k = z',$ where each $z_p$ has the same boundary as $z$ and $z_p$ lies in the union of the $p$-skeleton of our lattice and $\partial R.$ (If we think of $z_p$ as relative chains, then they are all cycles and $z_p$ lies in the $p$-skeleton of our lattice.)

The homology from $z_p$ to $z_{p-1}$ is constructed as follows. For each interior $p$-face of our lattice, we push $z_p \cap F$ into $\partial F$ while keeping $z_p \cap \partial F$ fixed. To do this, we pick a random point $x$ in $F$ and push $F - \{x\}$ radially into the boundary of $F.$ For a random point $x,$ this operation stretches volume by at most a constant.
$C(n)$. Therefore, the volume of $z_{p-1}$ is at most $C(n) |z_p|$. Similarly, the volume of the homology from $z_p$ to $z_{p-1}$ is at most $C(n) L |z_p|$. \hfill \Box

Using this lemma, we prove the isoperimetric inequality by induction on $k$. When $k = 0$, $z$ is just a weighted sum of points $\sum c_i p(i)$, where $c_i \in \mathbb{Z}$ and $p(i)$ is a point in $R$. The volume of $z$ is defined to be $\sum |c_i|$. A point $p$ with coordinates $(p_1, ..., p_n)$ bounds a segment $[0, p_1] \times \{p_2\} \times ... \times \{p_n\}$. Applying this operation to each point $p(i)$ with multiplicity $c_i$, we get a filling of $z$ with volume at most $R_1 V \text{ol}(z)$. This argument gives the base for our induction.

Now we come to the inductive step. Suppose that $z$ is a $k$-cycle with volume $V$. We proceed in two cases. If $V \leq c(n) R_1^k$, then we select $L = C(n) V^{1/k} \leq R_1$ and pick a rectangular lattice with sidelengths roughly $L$ and with $\partial R$ in the $(n-1)$-skeleton of the lattice. Then we use Lemma 1.1 to move $z$ to a new relative cycle $z'$ with volume at most $C(n) V$ lying in the $k$-skeleton of our lattice. Since $C(n) V \leq L^k$, the new cycle $z'$ is simply 0. Lemma 1.1 also guarantees us a homology from $z$ to $z'$ with volume at most $C(n) LV$, which is at most $C(n) V^{k+1}/k+1$. This upper bound is the one we needed to prove.

In the second case, we suppose that $V \geq c(n) R_1^k$. In this case, we select $L = R_1$ and pick a rectangular lattice with sidelengths roughly $L$ and with $\partial R$ in the $(n-1)$-skeleton of the lattice. We pick the lattice so that each lattice point has $x_1$ coordinate either 0 or $R_1$. Then we use Lemma 1.1 to move $z$ to a new relative cycle $z'$ with volume at most $C(n) V$ lying in the $k$-skeleton of our lattice. The homology from $z$ to $z'$ has volume at most $C(n) R_1 V$. The cycle $z'$ need not be 0, but it is a union of interior $k$-faces of our lattice. Each interior $k$-face has the form $[0, R_1] \times ..., and so the cycle $z'$ has the special form $z' = [0, R_1] \times z_1$ for some relative cycle $z_1$ in the $(n-1)$-dimensional rectangle $[0, R_2] \times ... \times [0, R_n]$. The cycle $z_1$ has volume at most $C(n) V/R_1$.

By induction, we can assume that our theorem holds for $z_1$. Therefore, $z_1$ bounds a relative chain $C_1$ with a certain volume bound that we calculate below. Then $z'$ bounds $[0, R_1] \times C_1$. We will calculate that the volume of this filling obeys the inequality stated in the theorem.

If the volume of $z$ is at most $c(n) R_1 ... R_k$, then the volume of $z_1$ is at most $c(n-1) R_2 ... R_k$. If the volume of $z$ is equal to $c(n) R_1 ... R_j \rho^{k-j}$ for some $\rho$ in the range $R_j \leq \rho \leq R_{j+1}$, then the volume of $z_1$ is roughly $R_2 ... R_j \rho^{(k-1)-(j-1)}$ for the same $\rho$. By induction, $z_1$ bounds a chain $C_1$ with volume at most $C(n-1) R_2 ... R_j \rho^{k-j+1}$, and so $[0, R_1] \times C_1$ has volume at most $C(n-1) R_1 ... R_j \rho^{k-j+1}$. Also, the homology from $z$ to $z'$ has volume at most $C(n) R_1 (R_1 ... R_j) \rho^{k-j} \leq C(n) R_1 ... R_j \rho^{k-j}$. Therefore, the filling volume of $z$ is at most $C(n) R_1 ... R_j \rho^{k-j+1}$.

In any case, $z_1$ bounds a $k$-chain $C_1$ of volume at most $C(n-1) R_{k+1} V/R_1$. Hence $z' = [0, R_1] \times z_1$ bounds a $(k+1)$-chain of volume at most $C(n-1) R_{k+1} V$. The homology from $z$ to $z'$ has volume at most $C(n) R_1 V$. Therefore, the filling volume of $z$ is at most $C(n) R_{k+1} V$. \hfill \Box

The algebra above is a bit complicated. In the sequel, we only use the following special case, which is easier to remember.

If $z$ is a relative $p$-cycle in $R$ with volume $V$ at most $c(n) R_1 ... R_j \rho^{p-j}$, then it bounds a relative $(p+1)$-chain $y$ in $R$ with volume at most $C(n) R_j V$. 
2. Statement of the main inequalities

In the paper, we will prove an estimate which is a little more general than Theorem 1. We now formulate it in terms of k-dilation. Recall that the k-dilation of a smooth map $\Phi$ is defined to be $\|\Lambda^k d\Phi\|_{L^\infty}$. The k-dilation measures by what factor the map $\Phi$ stretches k-dimensional areas. The k-dilation of $\Phi$ is at most $\Lambda$ if and only if $\Phi$ maps every k-dimensional surface of volume $V$ to an image of volume at most $\Lambda V$.

Recall that $R$ is an $n$-dimensional rectangle with dimensions $R_1 \leq \ldots \leq R_n$ and $S$ is an $n$-dimensional rectangle with dimensions $S_1 \leq \ldots \leq S_n$. We let $Q_i$ denote the quotient $S_i/R_i$. We now state the main estimates of the paper.

**Estimate 1.** Suppose that $U$ is an open set in $R$ and that $\Phi$ is a map of pairs $(U, \partial U) \to (S, \partial S)$ of degree $D > 0$. Suppose that $j$ and $l$ lie in the ranges $0 \leq j < k \leq l \leq n$. Then the k-dilation of $\Phi$ is bounded below by the following inequality.

$$\text{dil}_k(\Phi) \geq c(n)Q_1\ldots Q_j(Q_{j+1}\ldots Q_l)^{k-j}.$$ 

For example, if $I$ is a k-expanding embedding from $S$ into $R$, then we take $U$ to be the image of $S$, and we take $\Phi$ to be the inverse of $I$. The map $\Phi$ has k-dilation at most 1, and it has degree 1, and so Estimate 1 implies Theorem 1. Estimate 1 is slightly more general because $\Phi$ need not be a diffeomorphism.

If the degree $D$ is large, then we can strengthen some of the lower bounds in Estimate 1 as follows.

**Estimate 2.** With the same assumptions as above, for any $0 \leq j < k$, the k-dilation of $\Phi$ is bounded below by the following inequality.

$$\text{dil}_k(\Phi) \geq c(n)D^{k-j}Q_1\ldots Q_j(Q_{j+1}\ldots Q_n)^{k-j}.$$ 

In the paper [3], I proved Estimate 1 if either $j = 0$ or $l = n$. We will prove all the cases of Estimate 1 in this paper. The proof of the case $j = 0$ is essentially the same as the one in [3], but this paper gives a new proof for the case $l = n$.

3. Complexes of cycles

We introduce some vocabulary that we will use in our proof.

A complex of cycles in a rectangle $S$ is a collection of chains of different dimensions that fit together in a coherent way. It consists of the following data. There is a polyhedron $X$ which is like a parameter space for the complex. Then there is a map $C$ which assigns to each d-dimensional face $F^d$ of $X$ a d-dimensional relative chain in $S$. These chains have to fit together so that if the boundary of $F^d$ is equal to $\sum_{i=1}^N F_i^{d-1}$, then the boundary of the chain $C(F)$ should be $\sum_{i=1}^N C(F_i)$. In this paper, we work with complexes of cycles over $\mathbb{Z}$, and so all the faces and chains in the above discussion are oriented.

More formally, the map $C$ is a chain map between two complexes. The first complex is generated by the faces of $X$ with integral multiplicities and the natural boundary operations. The second complex is the complex of integral relative Lipschitz cycles in $S$, which we denote $I_{rel}(S)$.

This definition is due to Almgren. Almgren introduced it in his paper on the topology of the space of cycles [1]. For more explanation of the definition, see Section 1 of [4].
We remark that the complex $X$ may have dimension bigger than $n$. Even if $d > n$, the definition of Lipschitz d-chain in $S$ makes sense.

We give an example of a complex of cycles. If $U \subset R$ is an open set and $\Phi$ is a map from $(U, \partial U)$ to $(S, \partial S)$, then we can define a complex of cycles by noticing where $\Phi$ maps various chains. Let us fix a polyhedral structure $P$ on $R$. For each face $F$ of this structure, we define $C_p(F)$ to be $\Phi(F \cap U)$. The complex $C_p$ sends each face $F$ contained in the boundary of $R$ to zero, and so we can say that $C_p$ is parametrized by $(R, \partial R)$.

Since $C$ is a chain map, it induces a map on homology from $H_*(X, \mathbb{Z})$ to $H_*(S, \partial S, \mathbb{Z})$. In particular, if $C$ is a complex of cycles parametrized by $(R, \partial R)$, then it induces a map from $H_*(R, \partial R, \mathbb{Z})$ to $H_*(S, \partial S, \mathbb{Z})$. We define the degree of $C$ to be the degree of this map on $H_n$. The degree of $C_p$ is the same as the degree of $\Phi$.

A homotopy of complexes of cycles is a complex $C$ parametrized by $X \times [0,1]$. If the restriction of $C$ to $X \times \{0\}$ is a complex $C_0$ and the restriction of $C$ to $X \times \{1\}$ is $C_1$, then we say that $C$ is a homotopy from $C_0$ to $C_1$. If $C_0$ and $C_1$ are homotopic, then the induced maps on homology $H_*(X, \mathbb{Z}) \rightarrow H_*(S, \partial S, \mathbb{Z})$ are the same.

4. The Sweepout Lemma

We now prove a lemma that says that if all the chains in a complex are small enough then the complex is null-homotopic. The lemma and proof are based on an argument of Gromov from page 134 of [7].

**Lemma 4.1.** There is a constant $c(n) > 0$ so that the following estimate holds.

Suppose that $C_0$ is a complex of cycles in $S$ parametrized by $X$. Suppose that for each vertex $v$ of $X$, $C_0(v)$ is equal to 0. Suppose that for each $p$-face $F^p$ in $X$, $C_0(F^p)$ has volume at most $c(n)S_1...S_p$. Then $C_0$ is null-homotopic.

Lemma 4.1 is closely related to the Sweepout Estimate stated in the introduction. Gromov used this argument to prove the sweepout estimate on page 134 of [7].

**Proof.** We let $C_1$ denote the zero map. We have to prove that $C_0$ is homotopic to $C_1$ by constructing a homotopy $C$ between them. The homotopy $C$ needs to be defined on $X \times [0,1]$, and it is already defined on $X \times \{0\}$ and on $X \times \{1\}$. We define $C$ one skeleton at a time.

We will prove inductively that we can extend $C$ to the p-skeleton of $X \times [0,1]$ while preserving the inequality $|C(F^p)| \leq c(n)S_1...S_p$ for all $p \leq n$. To start the induction, we define $C$ on the 1-skeleton by setting $C(v \times [0,1])$ equal to zero for each vertex $v$ of $X$. Since $C_0(v) = 0 = C_1(v)$, this choice is allowed and it clearly obeys our volume estimate. By induction, we may assume that we have done the extension to the (p-1)-skeleton of $X \times [0,1]$. When we extend to the p-skeleton, we have to define $C(F^p)$ for each p-face so that $\partial C(F^p) = C(\partial F^p)$. By induction, $C(\partial F^p)$ is a (p-1)-cycle in $S$ with volume at most $c(n)S_1...S_{p-1}$. According to Theorem 3, we can fill this cycle with volume at most $c(n)S_1...S_{p-1}S_{p-1} \leq c(n)S_1...S_p$.

Next we have to extend $C$ to the (n+1)-skeleton of $X$. We have already defined $C$ on the n-skeleton. In particular, $C(\partial F^{n+1})$ is a relative n-cycle in $S$ with volume at most $c(n)S_1...S_n < S_1...S_n$. Therefore this n-cycle is exact. We define $C(F^{n+1})$ to be any (n+1)-chain with the given boundary. Finally we extend to the higher skeletons. There is no obstruction to finding an extension to the higher skeletons because $H_p(S, \partial S) = 0$ for $p \leq n + 1$. □
(The same proof works for a complex of cycles parametrized by \( (R, \partial R) \). In this case, we get a homotopy parametrized by \( (R \times [0, 1], \partial R \times [0, 1]) \).)

Using this lemma, we can prove the easiest cases of Estimates 1 and 2. These cases were first proven in [3], but we include them here for completeness.

**Proposition 4.1.** If \( U \) is an open set in \( R \) and if \( \Phi : (U, \partial U) \to (S, \partial S) \) is a map of degree \( D \neq 0 \), then the \( k \)-dilation of \( \Phi \) is at least \( c(n)Q_1...Q_k \).

**Proof.** By scaling, we may assume that \( \Phi \) is \( k \)-contracting and it then suffices to prove that \( R_1...R_k \geq c(n)S_1...S_k \). We assume that \( R_1...R_k < c(n)S_1...S_k \) and proceed to a contradiction.

We cut \( R \) into rectangular blocks which are each congruent to \([0, R_1] \times ... \times [0, R_k] \times [0, \epsilon]^{n-k}\) for some small number \( \epsilon > 0 \). All the rectangular blocks are parallel, and they form a grid of dimension \( 1 \times ... \times 1 \times (R_{k+1}/\epsilon) \times ... \times (R_n/\epsilon) \). Now we look at the complex \( C_{\Phi} \) corresponding to this decomposition.

If \( p < k \), then each \( p \)-face of our decomposition lies on the boundary of \( R \) and so is mapped to 0. Each \( k \)-face of our decomposition has volume at most \( R_1...R_k \). For each \( k \)-face \( F^k \), \( C_{\Phi}(F^k) \) has volume less than \( c(n)S_1...S_k \), since \( \Phi \) is \( k \)-contracting. Similarly, for \( p > k \), each \( p \)-face \( F^p \) has volume at most \( R_1...R_k \epsilon^{p-k} \). In Appendix 1, we prove that if \( \Phi \) is \( k \)-contracting then it is also \( l \)-contracting for each \( l \geq k \).

So \( C_{\Phi}(F^p) \) has volume less than \( R_1...R_k \epsilon^{p-k} \). If we choose \( \epsilon \) small enough, then Lemma 4.1 implies that \( C_{\Phi} \) is null-homotopic. In particular \( C_{\Phi} \) has degree zero. But we have already seen that \( C_{\Phi} \) has degree \( D \) which we assumed non-zero. \( \square \)

Since the map \( \Phi \) is also \( l \)-contracting for all \( l \geq k \), we get the following more general proposition.

**Proposition 4.2.** If \( l \geq k \), if \( U \) is an open set in \( R \), and if \( \Phi : (U, \partial U) \to (S, \partial S) \) is a map of degree \( D \neq 0 \), then the \( k \)-dilation of \( \Phi \) is at least \( c(n)(Q_1...Q_l)^{k/l} \). Also, the \( k \)-dilation of \( \Phi \) is at least \( (|D|Q_1...Q_n)^{k/n} \).

**Proof.** By the last proposition, the \( l \)-dilation of \( \Phi \) is at least \( c(n)Q_1...Q_l \). Also, the \( n \)-dilation of any degree \( D \) map is at least \( |D|Q_1...Q_n \). Therefore, the \( k \)-dilation of \( \Phi \) is at least \( c(n)(Q_1...Q_l)^{k/l} \) and at least \( (|D|Q_1...Q_n)^{k/n} \). \( \square \)

Proposition 4.2 proves Estimates 1 and 2 in the case \( j = 0 \).

5. Algebraic preliminaries

We rewrite the remaining cases of our estimates.

**Estimate 1. (Non-trivial cases)** There is a constant \( c(n) > 0 \) so that the following holds. Let \( R, S \) be \( n \)-dimensional rectangles. Suppose \( U \subset R \) is an open set. Suppose that \( \Phi \) is a \( k \)-contracting map from \( U \) to \( S \) of degree \( D \neq 0 \). Suppose \( 0 < j < k < l \).

Then
\[
[(R_1...R_j)/(S_1...S_j)] \xrightarrow{\epsilon} R_1...R_l \geq c(n)S_1...S_l.
\]

**Estimate 2. (Non-trivial cases)** In the same situation as above, the following inequality holds.

\[
[(R_1...R_j)/(S_1...S_j)] \xrightarrow{\epsilon} R_1...R_n \geq c(n)|D|S_1...S_n.
\]
Fix $j$. We define $L$ by the equation $R_1\ldots R_jL^{k-j} = \delta(n)S_1\ldots S_jS_j^{k-j}$, where $\delta(n) > 0$ is a small dimensional constant.

In the next section, we will prove Estimates 1 and 2 under the assumption that $L \leq R_{j+1}$.

We now check that it suffices to prove the estimates in this special case. This checking just takes a little algebra - the geometric part of our proof is in the next section.

We can rewrite our inequalities as follows.

$$\frac{\|R_1\ldots R_j\|}{\|S_1\ldots S_j\|} \geq c(n)\frac{\|S_1\ldots S_i\|}{\|R_1\ldots R_i\|}, \quad (1')$$

$$\frac{\|R_1\ldots R_j\|}{\|S_1\ldots S_j\|} \leq c(n)\frac{\|D\|}{\|R_1\ldots R_n\|}, \quad (2')$$

The right-hand sides of both equations is independent of $j$. So it suffices to pick the one value of $j$ that minimizes the left-hand side and to prove our theorem for this one value of $j$. Now for this value of $j$, we will prove that $L \leq R_{j+1}$.

We see this inequality in two cases. In the first case, it may happen that $j = k - 1$. In this case, $L = \delta(n)S_1\ldots S_{k-1}/(R_1\ldots R_{k-1}) \leq \delta(n)S_1\ldots S_k/(R_1\ldots R_{k-1})$. But by Proposition 4.1, $R_1\ldots R_k \geq c(n)S_1\ldots S_k$. Therefore, $\delta(n)S_1\ldots S_k/(R_1\ldots R_{k-1}) \leq \delta(n)c(n)^{-1}R_k$. If we choose $\delta(n)$ small enough, then $\delta(n)c(n)^{-1}R_k \leq R_k = R_{j+1}$.

In the second case $j < k - 1$. In this case, $j + 1$ was a legal competitor for $j$, and so we conclude that

$$\frac{\|R_1\ldots R_j\|}{\|S_1\ldots S_j\|} \leq \frac{\|R_1\ldots R_{j+1}\|}{\|S_1\ldots S_{j+1}\|}. \quad (3')$$

We raise each side of the equation to the power $(k-j)(k-j-1)$ and then move all the powers of $R$ to the righthand side.

$$S_1\ldots S_jS_j^{k-j} \leq R_1\ldots R_jR_j^{k-j}. \quad (4')$$

A fortiori, $S_1\ldots S_jS_j^{k-j} \leq R_1\ldots R_jR_j^{k-j}$. On the other hand, $S_1\ldots S_jS_j^{k-j} \geq R_1\ldots R_jL^{k-j}$. Therefore, $L \leq R_{j+1}$.

6. Tightening a complex of cycles

In this section we prove our main estimates by cutting the rectangle $R$ into pieces, mapping the pieces into $S$, and then pulling them tight with the isoperimetric inequality. To begin, we cut $R$ into subrectangles of a carefully chosen size.

We define $L$ by the equation $R_1\ldots R_jL^{k-j} = \delta(n)S_1\ldots S_jS_j^{k-j}$, where $\delta(n) > 0$ is a small constant that we can choose later. In this section we will make the mild assumption that $L \leq R_{j+1}$. In Section 5, we explained how the general case follows from this special case by high-school algebra. We pick a polyhedral structure on $R$ by cutting it into rectangular blocks of dimensions $R_1 \times \ldots \times R_j \times L \times \ldots \times L$. (By making a mild change in the dimensions of $R$, we may also assume that $L$ divides $R_i$ for each $i \geq j + 1$.)

We let $B$ be the chain complex generated by the interior faces of this decomposition. The homology of $B$ is $H_*(R, \partial R, \mathbb{Z})$. We let $C_0$ be the chain map $B \to I_{rel}(S)$ associated to $\Phi$. In other words, if $F$ is a face of $B$, then $C_0(F)$ is $\Phi(F \cap U)$. The degree of $C_0$ is $\delta$, the degree of $\Phi$.

By repeatedly using the isoperimetric inequality, we will “tighten” $C_0$ to a new complex of cycles $C_1$. 

The complex $C_1$ agrees with $C_0$ for faces of dimension at most $k$. For faces of higher dimension, $C_1$ is different from $C_0$. We define $C_1$ by induction on the dimension.

First we define $C_1(F^{k+1})$. We have already defined $C_1(\partial F^{k+1})$. Each face of $\partial F^{k+1}$ has $k$-volume at most $\delta(n)S_1...S_jS_j^{k-j}$, and so $C_1(\partial F^{k+1})$ has volume at most $\delta(n)2^nS_1...S_jS_j^{k-j}$. If we pick $\delta$ small enough, the isoperimetric inequality tells us that $C_1(\partial F^{k+1})$ bounds a $(k+1)$-chain with volume at most $\delta(n)CS_1...S_jS_j^{k-j+1}$.

We define $C_1(F^{k+1})$ to be a $(k+1)$-chain with this volume bound. We repeat this construction for every $(k+1)$-face in our decomposition of $R$.

Then we proceed inductively, defining $C_1$ one skeleton at a time so that at each stage it obeys the inequality $|C_1(F^p)| < \delta CS_1...S_jS_j^{p-j}$. Suppose we have defined $C_1$ on the $p$-skeleton and that $F^{p+1}$ is a (p+1)-face. We have already defined $C_1(\partial F^{p+1})$ and it has volume at most $\delta CS_1...S_jS_j^{p-j}$. Assuming $\delta$ is sufficiently small, we can apply the isoperimetric inequality to fill $C_1(\partial F^{p+1})$ by a (p+1)-chain of volume at most $\delta CS_1...S_jS_j^{p+1-j}$. We define $C_1(F^{p+1})$ to be this chain.

A key point in the proof is that the new complex $C_1$ has the same degree as the original complex $C_0$.

**Key Lemma.** The degree of $C_1$ is equal to the degree of $\Phi$.

This point is the subtlest part of our argument, and so we defer the proof until the end of the section.

### Gluing a complex of cycles

In place of $B$, we now consider a coarser decomposition of the rectangle $R$. This time we divide $R$ into blocks with dimensions $R_1 \times ... \times R_l \times L \times ... \times L$. Each new n-dimensional block is a union of $N = (R_{j+1}/L)...(R_l/L)$ blocks from the old decomposition. More generally, each interior p-face of the new decomposition is a union of $N$ p-faces of the old decomposition. We let $B^+$ be the complex generated by the interior faces of this coarser decomposition. Note that each interior face of $B^+$ has dimension $p \geq l$ and dimensions $R_1 \times ... \times R_l \times L \times ... \times L$. (There are $p-l$ factors of $L$ in this formula.)

Any complex of cycles $C : B \to I_{rel}(S)$ can easily be glued together to form a new complex of cycles $C^+ : B^+ \to I_{rel}(S)$. Suppose that $F$ is a p-face of $B^+$. As we observed above, $F$ is a union of p-faces from $B$: $F = \sum_{i=1}^{N} F_i$, where $F_i$ is a face of $B$. Now we just define $C^+(F) = \sum_{i=1}^{N} C(F_i)$. The degree of $C$ and the degree of $C^+$ are always the same.

In particular, $C_1^+$ is the glued-together version of $C_1$. The volume of $C_1^+(F^p)$ is at most $C(n)\delta NS_1...S_jS_j^{p-j}$. Plugging in the value of $N$, we see that the volume of $C_1^+(F^p)$ is at most $C(n)\delta R_{j+1}...R_lL^{l-j}S_1...S_jS_j^{p-j}$. Finally, plugging in the value of $L$, we see that the volume of $C_1^+(F^p)$ is at most

$$C(n, \delta)|\frac{R_1...R_j}{S_1...S_j}|^{l-j}R_1...R_lS_j^{p-j}. \quad (V)$$

Using the volume bound $(V)$ and the key lemma, we can now prove estimates (1) and (2). To prove inequality (2), we set $l = n$. In this case $B^+$ consists of only one n-face, which is the whole rectangle $R$. According to the formula above, $C_1^+(R)$ has volume at most $C(n)|\frac{R_1...R_n}{S_1...S_j}|^{n-k}R_1...R_n$. On the other hand, by the
Key Lemma. $C^+_1$ has degree $D$, and so $C^+_1(R)$ must have volume at least $|D|S_1\ldots S_n$. We conclude the following inequality.

$$(R_1\ldots R_j)^{i-b}R_1\ldots R_n \geq c(n)|D|(S_1\ldots S_j)^{i-b}S_{1\ldots n}.$$ 

This inequality is equivalent to (2).

Next we prove inequality (1) using Lemma 4.1. Recall that each interior face of $B^+$ has dimension $p \geq k$. Since $C^+_1$ has degree $D \neq 0$, Lemma 4.1 guarantees that for some dimension $p$, we can find an interior face $F^p$ so that $C^+_1(F^p)$ has volume at least $c(n)S_1\ldots S_p$. On the other hand, this same volume is bounded above by (V). Combining these equations, we conclude the following.

$$C(n)\left[\frac{R_1\ldots R_j}{S_1\ldots S_j}\right]^{i-b}R_1\ldots R_lS_{j}^{p-l} \geq c(n)S_1\ldots S_p.$$ 

Rearranging this inequality, we get the following.

$$\left[\frac{R_1\ldots R_j}{S_1\ldots S_j}\right]^{i-b}R_1\ldots R_l \geq c(n)S_1\ldots S_pS_{j}^{-(p-l)} \geq c(n)S_1\ldots S_l.$$ 

This proves inequality (1).

We have now finished proving our main estimates except for the proof of the key lemma which tells us that the degree of $C_1$ is equal to $D$.

**Gradually tightening chains**

**Key Lemma.** The degree of $C_1$ is equal to the degree of $\Phi$.

To prove the lemma, we will need to construct some homotopies between chain maps. We use the following lemma, which generalizes Lemma 4.1.

**Lemma 6.1.** There is a constant $c(n) > 0$ so that the following holds. Suppose that $C_0$ and $C_1$ are two chain maps $X \to I_{rel}(S)$. Suppose that $C_0$ and $C_1$ agree on the $k$-skeleton of $X$. Suppose that for each $p$-face $F^p$ in $X$ of dimension $p \geq k+1$, the volumes $|C_0(F^p)|$ and $|C_1(F^p)|$ are at most $c(n)S_1\ldots S_p$. Then $C_0$ and $C_1$ are homotopic.

**Proof.** We have to build a chain map $C : X \times [0,1] \to I_{rel}(S)$, extending $C_0$ and $C_1$. If $p \leq k$, we define $C(F^p \times [0,1])$ to be $0$.

We will prove inductively that we can extend $C$ to the $(n)$-skeleton of $X \times [0,1]$ while preserving the inequality $|C(F^p \times [0,1])| \leq c(n)S_1\ldots S_{p+1}$ for $p \leq n-1$.

When we extend to the $(p+1)$-skeleton, we have to define $C(F^p \times [0,1])$ for each $p$-face so that $\partial C(F^p \times [0,1]) = C((\partial F^p) \times [0,1]) + C_1(F^p) - C_0(F^p)$. By induction, the right-hand side is a $p$-cycle in $S$ with volume at most $c(n)S_1\ldots S_p$. According to our isoperimetric inequality, we can fill this cycle with volume at most $c(n)S_1\ldots S_pS_p \leq c(n)S_1\ldots S_{p+1}$.

Next we extend $C$ to the $(n+1)$-skeleton. We have to define $C(F^n \times [0,1])$. We have already defined $C$ on $\partial(F^n \times [0,1])$; it is an $n$-cycle with volume less than $S_1\ldots S_n$. Hence it is an exact $n$-cycle, and we can choose a filling for it. We can then extend to the higher-dimensional faces because $H_q(S,\partial S) = 0$ for all $q \geq n+1$. $\Box$

At first we might hope to apply this lemma to build a homotopy from $C_0$ to $C_1$. (Recall that $C_0$ and $C_1$ agree on the $k$-skeleton of $B$.) In general, this does not work, because the volumes $|C_0(F^p)|$ may be too large. Morally, the problem is that in building $C_1$ we have suddenly tightened the chains into a quite different
position. To build a homotopy, we want to gradually tighten the chains so that at each step they move only slightly. Then we can use the lemma above to build a homotopy between the small steps.

**Proof of key lemma**

Let \( B_s \) be the division of \( R \) into rectangular blocks with dimensions \( R_1 \times \ldots \times R_j \times 2^{-s} L \times \ldots \times 2^{-s} L \). The division \( B_0 \) is just \( B \), and the other \( B_s \) are finer subdivisions of \( B \).

Next we define chain maps \( \Gamma_s : B_s \rightarrow I_{rel}(S) \) as follows. For each face \( F^p \) in \( B_s \) of dimension \( p \leq k \), we define \( \Gamma_s(F) \) to be \( \Phi(F \cap U) \). Then we extend \( \Gamma_s \) to faces of dimension \( p \geq k + 1 \) inductively, using the isoperimetric inequality for rectangles at each step as in the construction of \( C_1 \). Because the constructions agree exactly, we may take \( \Gamma_0 \) to be equal to \( C_1 \).

First we check that \( \Gamma_{s+1} \) and \( \Gamma_s \) have the same degree. We let \( \Gamma^+_s : B_s \rightarrow I_{rel}(S) \) be the glued version of \( \Gamma_{s+1} \). As in the previous gluing construction, \( \Gamma^+_s \) and \( \Gamma_{s+1} \) have the same degree. We will use Lemma 6.1 to show that \( \Gamma^+_s \) and \( \Gamma_s \) are homotopic. By construction, they have the same restriction to the k-skeleton of \( B_s \). By the same argument that we used for \( C_1 \), \( \Gamma_s(F^p) \) has volume at most \( \delta(n) S_1 \ldots S_j S^j_{p-j} \leq \delta(n) S_1 \ldots S_p \). The same holds true for \( \Gamma^+_s \) and hence for \( \Gamma^{s+1} \).

Applying Lemma 6.1, we see that \( \Gamma_s \) and \( \Gamma_{s+1} \) have the same degree.

Let \( \beta_s : B_s \rightarrow I_{rel}(S) \) be the chain map sending a face \( F^p \) to \( \Phi(F^p \cap U) \) for every \( p \). The map \( \beta_s \) is analogous to \( C_0 \), and so it has degree \( D \) for every \( s \).

If \( p \geq k + 1 \), then the volume of \( \beta_s(F^p) \) is at most \( |F^p| \), which is at most \( R_1 \ldots R_j L^{p-j} 2^{-p-j} \). Since \( j < k \), we may choose \( s \) sufficiently large so that for each \( p \geq k + 1 \), this volume is at most \( c(n) S_1 \ldots S_p \). We now fix \( s \) to be this sufficiently large value. The two chain maps \( \beta_s \) and \( \Gamma_s \) agree on the k-skeleton of \( B_s \). Because of our choice of \( s \), the volume of \( \beta_s(F^p) \) is at most \( c(n) S_1 \ldots S_p \) for each \( p \geq k + 1 \). We checked above that the same inequality holds for \( \Gamma_s \). According to Lemma 6.1, \( \beta_s \) and \( \Gamma_s \) are homotopic and so have the same degree.

To summarize, the degree of \( C_1 \) is equal to the degree of \( \Gamma_0 \), which is equal to the degree of \( \Gamma_s \), which is equal to the degree of \( \beta_s \), which is equal to \( D \).

This concludes the proof of Estimates 1 and 2, and hence the proof of Theorem 1.

7. Constructing area-expanding embeddings

In this section, we prove Theorem 2.

**Theorem 2.** There is a constant \( C(n) \) so that the following holds.

Suppose that the dimensions of \( R \) and \( S \) obey the following inequalities for all \( 0 \leq j < k \leq l \leq n \).

\[
(R_1 \ldots R_j) \xrightarrow{\Delta} R_{j+1} \ldots R_l \geq C(n)(S_1 \ldots S_j) \xrightarrow{\Delta} S_{j+1} \ldots S_l.
\]

Then there is a \( k \)-expanding embedding from \( S \) into \( R \).

We will construct our embedding by composing a \( k \)-expanding linear map and a simple folding map analogous to the one in Figure 1.

If \( R \) and \( S \) are 2-dimensional rectangles with \( R_1 > 3S_1 \) and \( R_1R_2 > 9S_1S_2 \), then there is a 1-expanding embedding of \( S \) into \( R \). This embedding is illustrated in Figure 1.
Next, let \( a < b \) be integers between 1 and \( n \). If \( R_i = S_i \) except when \( i \) is equal to \( a \) or \( b \) and \( R_a > 3S_a \) and \( R_b > 9S_b \), then there is a 1-expanding embedding of \( S \) into \( R \). This embedding is the direct product of the folding map for the coordinates \( a \) and \( b \) and the identity in the other coordinates.

Composing these folding embeddings proves the following lemma.

**Lemma 7.1.** There is a constant \( C(n) \) so that the following holds. If \( R_1 \ldots R_p > C(n)S_1 \ldots S_p \) for each \( p \) between 1 and \( n \), then there is a 1-expanding embedding from \( S \) into \( R \).

The rest of the proof is just algebra, although it’s rather tedious. We put it in the form of a lemma.

**Lemma 7.2.** Suppose that the dimensions of \( R \) and \( S \) obey the following inequalities for all \( 0 \leq j < k \leq l \leq n \):

\[
R_1 \ldots R_j (R_{j+1} \ldots R_l)^{\frac{k-j}{l-j}} \geq S_1 \ldots S_j (S_{j+1} \ldots S_l)^{\frac{k-j}{l-j}}. \tag{I n}
\]

Then there is a \( k \)-contracting linear diffeomorphism from \( R \) to a rectangle \( T \) with \( T_1 \ldots T_p \geq S_1 \ldots S_p \) for all \( p \).

Given these lemmas, we finish the proof of Theorem 2. Under the hypothesis of the theorem, Lemma 7.2 tells us that we can find a \( k \)-contracting linear diffeomorphism from \( R \) to \( T \) where \( T_1 \ldots T_p \geq C(n)S_1 \ldots S_p \) for all \( p \). Then we use Lemma 7.1 to construct a 1-expanding embedding of \( S \) into \( T \). Now we turn to the proof of Lemma 7.2.

**Proof.** If \( S_1 \ldots S_p \leq R_1 \ldots R_p \) for every \( p \), then we take \( T = R \) and we are done. Let \( b \) be the smallest integer so that \( S_1 \ldots S_b > R_1 \ldots R_b \). Because of all the inequalities in the hypothesis of the lemma, we know that \( b < k \).

We will define a sequence of linear diffeomorphisms \( R = R(0) \rightarrow R(1) \rightarrow \ldots \rightarrow R(c) \), for some integer \( c \) between 1 and \( k-1 \). The diffeomorphism to \( R(q) \) is called \( L_q \). When \( q \) is less than \( c \), the rectangle \( R(q) \) has \( R(q)_1 = \ldots = R(q)_{q+1} = \ldots = R(q)_q+1 \). The linear map \( L_q \) increases each \( R(q-1)_i \), for \( i \) between 1 and \( q \) by a factor of \( \lambda_q \) and decreases every other \( R(q-1)_i \) by a factor of \( \lambda_q^{-q/(k-q)} \), for some number \( \lambda_q > 1 \). From the last sentence, it follows that each \( L_q \) is \( k \)-contracting. If \( c \) is not bigger than \( b \), then \( R(c)_1 \ldots R(c)_b = S_1 \ldots S_b \). If \( c \) is bigger than \( b \), then \( R(c)_1 \ldots R(c)_c = S_1 \ldots S_c \).

Now we define the maps \( L_q \). It suffices to define \( \lambda_q \). There is a maximum value of \( \lambda_q \) which increases \( R(q-1)_q \) and decreases \( R(q-1)_{q+1} \) until they meet. If there is a lesser value of \( \lambda_q \) which makes \( R(q)_1 \ldots R(q)_m = S_1 \ldots S_m \), where \( m \) is the maximum of \( b \) and \( q \), then use that value and take \( c = j \). If not, use the maximal value. As we increase \( q \), \( R(q)_1 \ldots R(q)_b \) increases. If \( R(b)_1 \ldots R(b)_b < S_1 \ldots S_b \), then \( R(b)_1 \ldots R(b)_{b+1} < S_1 \ldots S_{b+1} \). More generally, for \( q \) at least \( b \), if \( R(q)_1 \ldots R(q)_q < S_1 \ldots S_q \), then \( R(q)_1 \ldots R(q)_{q+1} < S_1 \ldots S_{q+1} \).

From the formula for the map \( L_q \), it follows that \( R(q)_1 \ldots R(q)_k = R_1 \ldots R_k \) for every \( q \), and by hypothesis \( R_1 \ldots R_k \geq S_1 \ldots S_k \). Therefore, the above construction terminates with \( c \) less than or equal to \( k-1 \).

Recall that \( m \) is the maximum of \( b \) and \( c \). We have proven above that \( R(c)_1 \ldots R(c)_m = S_1 \ldots S_m \). Moreover, for every \( p \) less than \( m \), \( R(c)_1 \ldots R(c)_p \geq S_1 \ldots S_p \). If \( b \) is greater than or equal to \( c \), this follows because \( R_1 \ldots R_p \geq S_1 \ldots S_p \), and the definition of \( L_j \) shows that \( R(q)_1 \ldots R(q)_p \geq R_1 \ldots R_p \) for every \( p \) less than \( k \). If \( c \) is greater than \( b \),
already know that \( R(c)_{1} = R(c)_{m} \) and \( R(c)_{1} R(c)_{m} = S_{1} S_{m} \). In either case it is true.

The maps \( L_{q} \) preserve many of the inequalities in \((In)\). In particular, if \( j \geq q \), then the following equality holds.

\[
R(q)_{1} R(q)_{j} R(q)_{j+1} ... R(q)_{l} (k-j)/(l-j) = R_{1} R_{j} R_{j+1} ... R_{l} (k-j)/(l-j).
\]

Therefore, if \( j \geq m \) then

\[
R(c)_{1} R(c)_{j} R(c)_{j+1} ... R(c)_{l} (k-j)/(l-j) \geq S_{1} S_{j} S_{j+1} ... S_{l} (k-j)/(l-j).
\]

Since \( R(c)_{1} R(c)_{m} = S_{1} S_{m} \), we can divide the above inequality on both sides, leaving the following inequality for all \( j \geq m \).

\[
R(c)_{m+1} ... R(c)_{j} R(c)_{j+1} ... R(c)_{l} (k-j)/(l-j) \geq S_{m+1} S_{j} S_{j+1} ... S_{l} (k-j)/(l-j). \quad (*)
\]

At this point, we employ induction on the dimension of the rectangles.

We define \( R' \) to be the \((n-m)\)-directional rectangle with dimensions \( R(c)_{m+1} \times \ldots \times R_{n}(n) \), so that \( R(c) = [0, R(c)_{1}] \times \ldots \times [0, R(c)_{m}] \times R' \). We define \( S' = S_{m+1} \times \ldots \times S_{n} \).

We can rewrite \((*)\) in terms of \( R' \) and \( S' \). To do this, let \( k' = k - m, j' = j - m \) and \( l' = l - m \). Then \((*)\) tells us that for any \( j', l' \) in the ranges \( 0 \leq j' < k' \leq l' \leq n - m \), we have the following inequalities.

\[
R'_{1} R'_{j'} R'_{j'+1} ... R'_{l'} (k'-j')/(l'-j') \geq S'_{1} S'_{j'} S'_{j'+1} ... S'_{l'} (k'-j')/(l'-j'). \quad (+)'
\]

By induction on the dimension \( n \), we can assume that there is a \( k' \)-contracting linear diffeomorphism from \( R' \) to some rectangle \( T' \) so that \( T'_{1} ... T'_{p} \geq S'_{1} ... S'_{p} \) for any \( 1 \leq p \leq n - m \).

We finally define \( T \) to be the rectangle with dimensions \( R(c)_{1} \times \ldots \times R(c)_{m} \times T'_{1} \times \ldots \times T'_{n-m} \). The direct product of the \((k-m)\)-contracting linear map from \( R' \) to \( T' \) with the identity map is a \( k \)-contracting linear diffeomorphism from \( R(c) \) to \( T \).

Since we already have a \( k \)-contracting linear map from \( R \) to \( R(c) \), we can compose the two maps to get a \( k \)-contracting linear diffeomorphism from \( R \) to \( T \). Also, we already know that \( T_{1} ... T_{p} = R(c)_{1} ... R(c)_{p} \geq S_{1} ... S_{p} \) when \( p \) is less than or equal to \( m \). But for larger \( p \), \( T_{1} ... T_{p} = T_{1} ... T_{m} T_{m} ... T_{p-m} \geq S_{1} ... S_{m} S_{1} ... S_{p-m} = S_{1} ... S_{p} \).

\[\square\]

8. Appendix: k-dilation and linear algebra

In this section we record some basic facts about k-dilation that follow from linear algebra.

If \( L \) is a linear map from \( \mathbb{R}^{M} \) to \( \mathbb{R}^{N} \), then we can write \( L \) in the form \( 0_{1} DO_{2} \).

In this equation, \( O_{2} \) is an \( M \times M \) orthogonal matrix, \( O_{1} \) is an \( N \times N \) orthogonal matrix, and \( D \) is an \( M \times N \) matrix which vanishes off the diagonal and with all diagonal entries at least 0. If we let \( n \) be the minimum of \( M \) and \( N \), then the diagonal entries of \( D \) are \( 0 \leq s_{1} \leq \ldots \leq s_{n} \). The numbers \( s_{1}, ..., s_{n} \) are called the singular values of \( L \). The Lipschitz constant of \( L \) is the largest singular value \( s_{n} \). The k-dilation of \( L \) is the product of the \( k \) largest singular values: \( s_{n-k+1} \ldots s_{n} \). Using this fact, we prove some basic inequalities about k-dilation.
Lemma 8.1. Suppose that \( l > k \). Then the following inequality holds between the \( l \)-dilation and the \( k \)-dilation.

\[
|\Lambda^l L|^{k/l} \leq |\Lambda^k L|.
\]

Proof. Let \( s_i \) denote the singular values of \( L \). Then the left-hand side is \((s_{n-l+1}...s_n)^{k/l}\). This expression is less than \((s_{n-k+1}...s_n)\), which is the right-hand side. \( \Box \)

Corollary 8.1. Suppose that \( \Phi \) is a map with \( k \)-dilation \( D(k) \) and \( l \)-dilation \( D(l) \) with \( l \geq k \). Then \( D(l)^{k/l} \leq D(k) \).

Proof. Recall that \( D(l) \) is the supremum of \(|\Lambda^l d\Phi|\). For each point \( x \), \(|\Lambda^l d\Phi(x)|^{k/l} \leq |\Lambda^k d\Phi(x)|\). Passing to the supremum proves the corollary. \( \Box \)

We include one other piece of linear algebra related to \( k \)-dilation. We don’t use this result in our paper, but I think it’s worth knowing for context. If \( j < k \), a linear map with \( k \)-dilation equal to 1 may have arbitrarily large \( j \)-dilation, but it must pay for a large \( j \)-dilation by having a small \( l \)-dilation for each \( l > k \). This tradeoff is described by the following lemma.

Lemma 8.2. Suppose that \( j \leq k \leq l \) and that \( L \) is a linear map.

Then \(|\Lambda^j L|^{l-k} |\Lambda^l L|^{k-j} \leq |\Lambda^k L|\).

Proof. The idea is to rewrite everything in terms of singular values.

\[
|\Lambda^j L|^{l-k} |\Lambda^l L|^{k-j} = (s_{n-j+1}...s_n)^{l-k} (s_{n-l+1}...s_n)^{k-j} \\
= (s_{n-l+1}...s_{n-j})^{k-j} (s_{n-j+1}...s_n)^{l-j} \leq (s_{n-k+1}...s_{n-j})^{l-j} (s_{n-j+1}...s_n)^{l-j} \\
= |\Lambda^k L|^{l-j}.
\]

Taking \((l-j)^{th}\) roots of both sides finishes the proof. \( \Box \)

I call this lemma the expansion/contraction inequality: a \( k \)-contracting linear map may expand in some directions and has to pay for it by contracting in others. Unlike the last lemma, this one has no direct analogue for non-linear maps. A \( k \)-contracting map \( \Phi \) may have large \( j \)-dilation and \( l \)-dilation equal to 1. Suppose at one point \( x \) that \(|\Lambda^l d\Phi_x| = 10^6\). It follows that the \( j \)-dilation of \( \Phi \) is at least \( 10^6 \). It also follows that the \( l \)-dilation of \( \Phi \) at the point \( x \) is small. But the \( l \)-dilation of \( \Phi \) globally may still be 1 because at some other point \( y \), we may have \( d\Phi_y \) equal to the identity.

Nevertheless, the results in this paper can be viewed as an analogue of the expansion/contraction inequality for nonlinear maps. For example, suppose that \( \Phi \) is a degree 1 \( k \)-contracting map from \( R \) to \( S \). Suppose that \( R_1...R_j << S_1...S_j \). Because of the sweepout lemma, the \( j \)-dilation of \( \Phi \) must be at least \( c(n)S_1...S_j/R_1...R_j \). If \( \Phi \) were linear, its \( l \)-dilation would then be bounded by a small number coming from the expansion/contraction inequality. The actual \( l \)-dilation of \( \Phi \) may be 1, but the map \( \Phi \) can be in some sense approximated by the complex of chains \( C_1 \) (defined in Section 6). Up to a constant factor \( C(n) \), the volumes of the chains in \( C_1 \) obey the same bounds that would follow if the \( l \)-dilation of \( \Phi \) obeyed the expansion/contraction inequality.
9. Appendix 2: Minor Generalizations

In this section, we discuss how far our results generalize to shapes that are not rectangles.

First we briefly consider replacing $S$ by another shape. We note that all our arguments depended only on knowing the isoperimetric profile of $S$. Therefore, our methods should adapt to give some estimates for any target where we can estimate the isoperimetric profile.

Second we consider replacing $R$ by a more general shape. Our arguments apply to products of the form: $X^{i} \times Y^{i} \times Z^{n-i}$, where $X$ and $Z$ may be any Riemannian manifolds, but the middle factor $Y$ is still a rectangle. In this case, our estimate survives, reading as follows.

**Proposition 9.1.** Suppose that $U$ is an open set in $X \times Y \times Z$, where $X^{i}$ and $Z^{n-i}$ are Riemannian manifolds and $Y^{i-1}$ is a rectangle. Suppose that $\Phi$ is a $k$-contracting degree non-zero map from $U$ to an $n$-dimensional rectangle $S$. Then the volumes of $X$ and $Y$ are bounded below by the following inequalities.

$$|X|^{|\Phi|} |Y| \geq c(n)(S_{1} \ldots S_{j})^{|\Phi|} S_{j+1} \ldots S_{l}.$$  

If $X$ and $Z$ are oriented, $\mu = n$, and the degree of the map is large, we also get an analogue of Estimate 2: $|X|^{|\Phi|} |Y| \geq c(n) |D| (S_{1} \ldots S_{j})^{|\Phi|} S_{j+1} \ldots S_{n}$.

**Proof.** (sketch) Use the argument of the paper, cutting the domain into pieces each a product of the form $X$ times a cube in $Y$ with side-length $L$ times a tiny simplex in $Z$. If the domain is not orientable, use mod 2 chains instead of integral chains. □

The statement of the proposition would still make sense if we allowed the middle factor $Y$ to be any manifold, but the rectangular structure is used crucially in the proof, mostly when we cut $Y$ into cubes. I strongly believe that the estimate above does not generalize to all Riemannian products $X \times Y \times Z$.

The product structure can also be relaxed a little. Suppose our domain $A^n$ admits a map $\pi$ onto $Y^{i-1} \times Z^{n-i}$, where as above $Y$ is a rectangle and $Z$ is any Riemannian manifold. Suppose that for any $p$-chain $C$ in $Y \times Z$, the $(p+j)$-dimensional volume of $\pi^{-1}(C)$ is at most $V|C|$. Suppose that $U$ is an open set in $A$ admitting a k-contracting map of non-zero degree to the $n$-dimensional rectangle $S$. Then our inequality again survives in the form $V^{|\Phi|} |Y| \geq c(n)(S_{1} \ldots S_{j})^{|\Phi|} S_{j+1} \ldots S_{l}$. (And the analogue of Estimate 2 holds also.)

For example, we can replace $R$ by an ellipsoidal metric on the $n$-sphere. Define $E^n$ by the equation $\sum_{i=0}^{n}(x_i/E_i)^{2} = 1$. Here $E$ is an ellipsoid with principal axes $E_0 \leq \ldots \leq E_n$. The manifold $E$ is $C(n)$-bilipschitz to the double of the rectangle $[0,E_1] \times \ldots \times [0,E_n]$. So for any $j \geq 0$, there is a map $\pi$ from $E$ to $[0,E_{j+1}] \times \ldots \times [0,E_n]$ which obeys the conditions of the last paragraph with $V \sim E_{1} \ldots E_{j}$. Applying our generalized version of Estimates 1 and 2, we get the following corollary.

**Corollary.** Suppose that $E$ and $E'$ are $n$-dimensional ellipsoids with principal axes $E_0 \leq \ldots \leq E_n$ and $E'_0 \leq \ldots \leq E'_n$, and quotients $Q_i = E'_i/E_i$. Suppose that $\Phi$ is a map from $E$ to $E'$ with degree $D \neq 0$. Then the $k$-dilation of $\Phi$ is bounded below by the following formulas. First, if $0 \leq j < k \leq l \leq n$,
\[ \text{dil}_k(\Phi) \geq c(n)Q_1...Q_j(Q_{j+1}...Q_l)^{\frac{k-j}{n-j}}. \]

Second, if \(0 \leq j < k\),

\[ \text{dil}_k(\Phi) \geq c(n)|D|^{\frac{k-j}{n-j}}Q_1...Q_j(Q_{j+1}...Q_n)^{\frac{k-j}{n-j}}. \]

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