RELATIVE DEHN FUNCTIONS OF AMALGAMATED PRODUCTS AND HNN–EXTENSIONS.

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Abstract. We obtain an upper bound for relative Dehn functions of amalgamated products and HNN–extensions with respect to certain collections of subgroups. Our main results generalize the combination theorems for relatively hyperbolic groups proved by Dahmani.

1. Introduction

Relative Dehn functions were introduced in [14] (a particular case was also considered in [3]) in order to obtain an isoperimetric characterization of relatively hyperbolic groups. In this paper we prove two theorems concerning relative Dehn functions of amalgamated products and HNN–extensions. In particular, our results generalize the Combination Theorem for relatively hyperbolic groups proved by Dahmani [6]. Results of similar type for ordinary Dehn functions can be found in [2, 4, 5, 9, 11].

We begin with definitions. Let $G$ be a group, $\{H_\lambda\}_{\lambda \in \Lambda}$ a collection of subgroups of $G$, $X$ a subset of $G$. We say that $X$ is a relative generating set of $G$ with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$ if $G$ is generated by $X$ together with the union of all $H_\lambda$. (We always assume $X$ to be symmetrized, i.e. $X^{-1} = X$.) In this situation the group $G$ can be regarded as a quotient group of the free product

$$F = \ast_{\lambda \in \Lambda} H_\lambda \ast F(X),$$

where $F(X)$ is the free group with the basis $X$. Let $N$ denote the kernel of the natural homomorphism $F \to G$. If $N$ is a normal closure of a subset $R \subseteq N$ in the group $F$, we say that $G$ has relative presentation

$$\langle X, H_\lambda, \lambda \in \Lambda \mid R \rangle.$$

If $\#X < \infty$ and $\#R < \infty$, the relative presentation (2) is said to be finite and the group $G$ is said to be finitely presented relative to the collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$.

Let

$$\mathcal{H} = \bigsqcup_{\lambda \in \Lambda} (H_\lambda \setminus \{1\}).$$

2000 Mathematics Subject Classification. Primary 20F65; Secondary 20F67, 20E06.

Key words and phrases. Relative Dehn function, relatively hyperbolic group, HNN–extension, free product with amalgamation.

This work has been supported by the RFBR Grant № 02-01-00892.
Given a word $W$ in the alphabet $X \cup H$ such that $W$ represents 1 in $G$, there exists an expression

$$W = F \prod_{i=1}^{k} f_i^{-1} R_i \pm 1,$$

with the equality in the group $F$, where $R_i \in \mathcal{R}$ and $f_i \in F$ for $i = 1, \ldots, k$. The smallest possible number $k$ in a representation of the form (4) is called the relative area of $W$ and is denoted by $\text{Area}_{rel}(W)$.

**Definition 1.1.** We say that a function $f : \mathbb{N} \to \mathbb{N}$ is a relative isoperimetric function of $G$ with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$ (associated to the relative presentation (2)) if for any $n \in \mathbb{N}$ and any word $W$ in $X \cup H$ of length at most $n$ representing the identity in the group $G$, we have $\text{Area}_{rel}(W) \leq f(n)$. The smallest relative isoperimetric function of (2) is called the relative Dehn function of $G$ with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$ and is denoted by $\delta_{rel}(f, H_\lambda)_{\lambda \in \Lambda}$ (or simply by $\delta_{rel}$ when the group $G$ and the collection of subgroups are fixed).

Observe that $\delta_{rel}(n)$ is not always well–defined, since the number of words of bounded relative length can be infinite. Indeed consider the group

$$G = \langle a, b \mid [a, b] = 1 \rangle \cong \mathbb{Z} \times \mathbb{Z}$$

and the cyclic subgroup $H$ generated by $a$. Clearly $X = \{b^{\pm 1}\}$ is a relative generating set of $G$ with respect to $H$. It is easy to see that the word $W_n = [a^n, b]$ has length 4 as a word over $X \cup (H \setminus \{1\})$ for every $n$ since $a^n$ can be regarded as a letter from the alphabet $H \setminus \{1\}$. On the other hand $\text{Area}_{rel}(W_n)$ grows linearly as $n \to \infty$. Thus we do not have any bound on $\text{Area}_{rel}(W_n)$ in terms of $\|W_n\|$ in this case.

However if $G$ is finitely presented relative to $\{H_\lambda\}_{\lambda \in \Lambda}$ and $\delta_{rel}$ is well-defined, it is independent of the choice of the finite relative presentation up to the following equivalence relation [14, Theorem 2.32]. For two functions $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ if there are positive constants $A, B, C$ such that $f(n) \leq Ag(Bn) + Cn$. One says that $f$ and $g$ are equivalent if $f \preceq g$ and $g \preceq f$.

Recall that a function $f : \mathbb{N} \to \mathbb{N}$ is said to be superadditive if

$$f(a + b) \geq f(a) + f(b)$$

for any $a, b \in \mathbb{N}$. Given an arbitrary function $f : \mathbb{N} \to \mathbb{N}$, the superadditive closure of $f$ is defined to be

$$\overline{f}(n) = \max_{i=1, \ldots, n} \left( \max_{a_i + \cdots + a_i = n, a_i \in \mathbb{N}} (f(a_1) + \cdots + f(a_i)) \right)$$

In fact, $\overline{f}$ is the smallest superadditive function such that $\overline{f}(n) \geq f(n)$ for all $n$. The main results of our paper are the following two theorems.

**Theorem 1.2.** Suppose that a group $H$ is finitely presented with respect to a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda} \cup \{K\}$ and the corresponding relative Dehn function $\gamma$ is well–defined. Assume also that $K$ is finitely generated and for some $\nu \in \Lambda$, there exists a monomorphism $\iota : K \to H_\nu$. Then the HNN–extension

$$G = \langle H, t \mid t^{-1}kt = \iota(k), k \in K \rangle$$
is finitely presented relative to \{H_\lambda\}_{\lambda \in \Lambda}. Moreover, the relative Dehn function \(\delta\) of \(G\) with respect to \{H_\lambda\}_{\lambda \in \Lambda} is well–defined and satisfies the inequality
\[
\delta \leq \gamma \circ \gamma.
\]

Similarly for amalgamated products, we have

**Theorem 1.3.** Suppose that a group \(A\) (respectively \(B\)) is finitely presented relative to a collection of subgroups \(\{A_\mu\}_{\mu \in M} \cup \{K\}\) (respectively \(\{B_\nu\}_{\nu \in N}\)) and the corresponding relative Dehn function \(\gamma_1\) (respectively \(\gamma_2\)) is well–defined. Assume in addition that \(K\) is finitely generated and for some \(\eta \in N\), there is a monomorphism \(\xi : K \to B_\eta\). Then the amalgamated product \(A \ast_{K=\xi(K)} B\) is finitely presented relative to the collection \(\{A_\mu\}_{\mu \in M} \cup \{B_\nu\}_{\nu \in N}\). Moreover, the relative Dehn function \(\delta\) of \(A \ast_{K=\xi(K)} B\) with respect to the collection \(\{A_\mu\}_{\mu \in M} \cup \{B_\nu\}_{\nu \in N}\) is well–defined and satisfies the inequality
\[
\delta \leq \gamma \circ \gamma,
\]
where \(\gamma\) is defined by
\[
(7) \quad \gamma(n) = \max\{\gamma_1(n), \gamma_2(n)\}
\]
for all \(n \in \mathbb{N}\).

In fact Theorem 1.3 can be derived from Theorem 1.2 via an easy observation concerning the behavior of relative Dehn functions under taking retracts (see Section 3).

Recall that a group \(G\) is hyperbolic relative to a collection of subgroups \(\{H_\lambda\}_{\lambda \in \Lambda}\) if the relative Dehn function of \(G\) with respect to \(\{H_\lambda\}_{\lambda \in \Lambda}\) is linear [14]. Thus we obtain the following corollaries of Theorem 1.2 and Theorem 1.3. For finitely generated groups, these results were obtained by Dahmani in [6] and used to prove that limits groups introduced by Sela [16] are hyperbolic relative to certain collections of maximal abelian subgroups.

**Corollary 1.4.** In the notation of Theorem 1.2, if \(H\) is hyperbolic relative to \(\{H_\lambda\}_{\lambda \in \Lambda} \cup \{K\}\), then \(G\) is hyperbolic relative to \(\{H_\lambda\}_{\lambda \in \Lambda}\).

**Corollary 1.5.** In the notation of Theorem 1.3, if \(A\) and \(B\) are hyperbolic relative to \(\{A_\mu\}_{\mu \in M} \cup \{K\}\) and \(\{B_\nu\}_{\nu \in N}\) respectively, then \(A \ast_{K=\xi(K)} B\) is hyperbolic relative to \(\{A_\mu\}_{\mu \in M} \cup \{B_\nu\}_{\nu \in N}\).

**Example 1.6.** Recall that for any quasi–convex subgroups \(Q_1, Q_2\) of an ordinary hyperbolic group \(H\), \(H\) is hyperbolic relative to \(\{Q_1, Q_2\}\) whenever \(|Q_1^h \cap Q_1| < \infty\) for any \(i = 1, 2\) and \(h \notin Q_i\), and \(|Q_1^g \cap Q_2| < \infty\) for any \(g \in H\) [3, 13]. Suppose that there is a monomorphism \(\iota : Q_1 \to Q_2\). Then the corresponding HNN–extension \(H_{\ast \iota}\) is hyperbolic relative to \(Q_2\) by Corollary 1.4. As \(Q_2\) is hyperbolic itself (see [3]), \(H_{\ast \iota}\) is a hyperbolic group [7].

Similarly suppose that \(Q_1\) and \(Q_2\) are quasi–convex subgroups of hyperbolic groups \(H_1\) and \(H_2\) respectively, \(|Q_1^g \cap Q_1| < \infty\) for any \(g \notin Q_1\), and \(|Q_2^g \cap Q_2| < \infty\) for any \(h \notin Q_2\). Assume that there is a monomorphism \(\xi : Q_1 \to Q_2\). Then the amalgamated product \(H_1 \ast_{Q_1=\xi(Q_1)} H_2\) is hyperbolic relative to \(Q_2\). In particular, \(H_1 \ast_{Q_1=\xi(Q_1)} H_2\) is hyperbolic. These results are well known and were proved independently by many authors (see [2, 9, 11]).
2. Preliminaries

Word metrics and Cayley graphs. Let $G$ be a group generated by a symmetrized set $A$. Recall that the Cayley graph $\Gamma(G, A)$ of a group $G$ with respect to the set of generators $A$ is an oriented labelled 1–complex with the vertex set $V(\Gamma(G, A)) = G$ and the edge set $E(\Gamma(G, A)) = G \times A$. An edge $e = (g, a)$ goes from the vertex $g$ to the vertex $ga$ and has the label $\phi(e) = a$. As usual, we denote the origin and the terminus of the edge $e$, i.e., the vertices $g$ and $ga$, by $e_-$ and $e_+$ respectively. Given a combinatorial path $p = e_1 e_2 \ldots e_k$ in the Cayley graph $\Gamma(G, A)$, where $e_1, e_2, \ldots, e_k \in E(\Gamma(G, A))$, we denote by $\phi(p)$ its label. By definition, $\phi(p) = \phi(e_1) \phi(e_2) \ldots \phi(e_k)$. We also denote by $p_- = (e_1)_-$ and $p_+ = (e_k)_+$ the origin and the terminus of $p$ respectively. The length $l(p)$ of $p$ is, by definition, the number of edges of $p$.

Associated to $A$ is the so–called word metric on $G$. More precisely, the length $|g|_A$ of an element $g \in G$ is defined to be the length of a shortest word in $A^\pm 1$ representing $g$ in $G$. This defines a metric on $G$ by $\text{dist}_A(f, g) = |f^{-1}g|_A$. We also denote by $\text{dist}_A$ the natural extension of the word metric to the Cayley graph $\Gamma(G, A)$.

Van Kampen Diagrams. Recall that a van Kampen diagram $\Delta$ over a presentation

$$G = \langle A \mid O \rangle$$

is a finite oriented connected simply–connected 2–complex endowed with a labelling function $\phi : E(\Delta) \rightarrow A$, where $E(\Delta)$ denotes the set of oriented edges of $\Delta$, such that $\phi(e^{-1}) = (\phi(e))^{-1}$. Labels and lengths of paths are defined as in the case of Cayley graphs. Given a cell $\Pi$ of $\Delta$, we denote by $\partial \Pi$ the boundary of $\Pi$; similarly, $\partial \Delta$ denotes the boundary of $\Delta$. The labels of $\partial \Pi$ and $\partial \Delta$ are defined up to a cyclic permutation. An additional requirement is that for any cell $\Pi$ of $\Delta$, the boundary label $\phi(\partial \Pi)$ is equal to a cyclic permutation of a word $P^\pm 1$, where $P \in O$. Sometimes it is convenient to use the notion of 0–refinement in order to assume diagrams to be homeomorphic to a disc. We do not explain here this notion and refer the interested reader to [13, Ch. 4].

The van Kampen lemma states that a word $W$ over the alphabet $A$ represents the identity in the group given by (8) if and only if there exists a simply–connected planar diagram $\Delta$ over $A$ such that with boundary label $W$ [13, Ch. 5, Theorem 1.1].

$H_\lambda$–components. For a group $G$ and a collection of subgroups $\{H_\lambda\}_{\lambda \in \Lambda}$ of $G$, we denote by $\Gamma(G, X \cup \mathcal{H})$ the Cayley graph of $G$ with respect to the generating set $A = X \cup \mathcal{H}$, where $\mathcal{H}$ is defined by (2). We also fix a relative presentation (2) of $G$ with respect to $\{H_\lambda\}_{\lambda \in \Lambda}$.

Let $q$ be a path in $\Gamma(G, X \cup \mathcal{H})$. A subpath $p$ of $q$ is called an $H_\lambda$–component for some $\lambda \in \Lambda$ (or simply a component) of $q$, if the label of $p$ is a word in the alphabet $H_\lambda \setminus \{1\}$ and $p$ is not contained in a bigger subpath of $q$ with this property. Two $H_\lambda$–components $p_1, p_2$ of a path $q$ in $\Gamma(G, X \cup \mathcal{H})$ are called connected if there exists a path $c$ in $\Gamma(G, X \cup \mathcal{H})$ that connects some vertex of $p_1$ to some vertex of $p_2$ and $\phi(c)$ is a word consisting of letters from $H_\lambda \setminus \{1\}$. In algebraic terms this means that all vertices of $p_1$ and $p_2$ belong to the same coset $gH_\lambda$ for a certain $g \in G$. 


Note that we can always assume $c$ to have length at most 1, as every nontrivial element of $H_\lambda$ is included in the set of generators. An $H_\lambda$–component $p$ of a path $q$ is called isolated if no distinct $H_\lambda$–component of $q$ is connected to $p$.

The next lemma is a simplification of Lemma 2.27 from [14].

Lemma 2.1. Suppose that the relative presentation (2) is finite and the corresponding relative Dehn function $\delta$ is well-defined. Then for each subgroup $H_\lambda$ there exists a constant $C > 0$ and a finite subset $\Omega_{H_\lambda} \subseteq H_\lambda$, such that the following condition holds. Let $q$ be a cycle in $\Gamma$, $p_1, \ldots, p_k$ a set of isolated $H_\lambda$–components of $q$, $g_1, \ldots, g_k$ the elements of $G$ represented by the labels of $p_1, \ldots, p_k$ respectively. Then for any $i = 1, \ldots, k$, $g_i$ belongs to the subgroup $\langle \Omega_{H_\lambda} \rangle \leq G$ and the lengths of $g_i$ with respect to $\Omega_{H_\lambda}$ satisfy the inequality

$$\sum_{i=1}^{k} |g_i|_{\Omega_{H_\lambda}} \leq C\delta(l(q)).$$

Conventions and notation. Given a word $W$ in an alphabet $A$, we denote by $\|W\|$ its length. We also write $W \equiv V$ to express letter–for–letter equality of words $W$ and $V$. Finally for elements $g, t$ of a group $G$, $gt$ denotes the element $t^{-1}gt$. Similarly $Ht$ denotes $t^{-1}Ht$ for a subgroup $H$ of $G$.

3. Diagrams over HNN–extensions

To deal with van Kampen diagrams over HNN–extensions, we need certain standard technical tools. The first one is a $t$-band (other people call them corridors or strips). This notion goes back to the paper [12]; here we describe it shortly. Suppose that one has a finite presentation of the form

$$(9) \langle A, t \mid T \rangle,$$

where $t \notin A$ and the only relators involving $t$ are of the form

$$(10) t^{-1}x_i t y_i,$$

where $x_i, y_i$ are some words over $A$. If the boundary label $\partial \pi$ of a cell $\pi$ of a van Kampen diagram over (9) is of the form (10), we call $\pi$ a $t$-cell. The subpath of $\partial \pi$ having the label $x_i$ (respectively $y_i$) is called a bottom (respectively top) of $\pi$. A $t$-band is a sequence of pairwise distinct $t$-cells $\tau = (\pi_1, \ldots, \pi_n)$ in a van Kampen diagram over (9) such that each two consecutive cells in this sequence have a common edge labelled $t$. The path formed by tops (respectively bottoms) of $\pi_1, \ldots, \pi_n$ is called a top (respectively bottom) of the $t$-band $\tau$ and is denoted by $\text{top}(\tau)$ (respectively $\text{bot}(\tau)$). The length of a $t$-band is the number of its 2-cells. A $t$-band $\pi_1, \ldots, \pi_n$ is maximal if it is not contained in another $t$-band of greater length. In what follows we always assume all $t$–bands under consideration to be maximal. Finally a (maximal) $t$-band $\tau = (\pi_1, \ldots, \pi_n)$ is called a $t$-annulus if it has no common edges with $\partial \Delta$ labelled $t$.

We will use the following obvious information about $t$-bands.

- Distinct $t$-bands have no common 2-cells.
- The labels of the top and the bottom of any $t$-band contain no $t^{\pm 1}$.

Furthermore, $\partial \Delta$ and non–annular $t$-bands bound domains of $\Delta$, which are maximal connected disjoint subdiagrams of $\Delta$ containing no $t$–cells except for those from
Figure 1. Structure of diagrams over HNN–extensions; bottoms of $t$–bands are marked in bold.

t–annuli. The set of all domains of $\Delta$ is denoted by $\mathcal{D}(\Delta)$. By $T(D)$ we denote the set of all non–annular $t$–bands $\tau$ such that $\text{bot}(\tau)$ lies on the boundary of a domain $D$. For example, $T(D) = \{\tau, \sigma\}$ in Fig. 1.

In the rest of the paper we use the notation of Theorem 1.2. Let us fix a finite presentation

$$H = \langle X, K, H_\lambda, \lambda \in \Lambda \mid R \rangle$$

of $H$ with respect to $\{H_\lambda\}_{\lambda \in \Lambda} \cup \{K\}$. We denote by $Y$ an arbitrary symmetrized finite generating set of $K$. For a word $R \in \mathcal{R}$, let us denote by $R'$ the word obtained from $R$ by replacing each letter $k \in K$ with a shortest word in $Y$ representing the element $k$ in $H$. Set $\mathcal{R}' = \{R', R \in \mathcal{R}\}$, $T = \{t^{-1}yt = \iota(y), y \in Y\}$ (here $\iota(y) \in H_\nu$ is regarded as a letter), and $\widetilde{\mathcal{R}} = \mathcal{R}' \cup T$. Obviously the HNN–extension (6) has the finite relative presentation

$$G = \langle X \cup Y \cup \{t\}, H_\lambda, \lambda \in \Lambda \mid \widetilde{\mathcal{R}} \rangle.$$

Associated to the relative presentations (11) and (12) are ordinary (non–relative) presentations

$$H = \langle X \cup \mathcal{H} \cup (K \setminus \{1\}) \mid \mathcal{R} \cup \mathcal{S} \cup \mathcal{L} \rangle$$

and

$$G = \langle X \cup Y \cup \{t\} \cup \mathcal{H} \mid \widetilde{\mathcal{R}} \cup \mathcal{S} \rangle,$$

where $\mathcal{H}$ is defined by (3), $\mathcal{S} = \bigcup_{\lambda \in \Lambda} S_\lambda$ and $S_\lambda$ (respectively $\mathcal{L}$) is the set of all words in the alphabet $H_\lambda \setminus \{1\}$ (respectively $K \setminus \{1\}$) representing 1 in $H$.

**Definition 3.1.** Consider two $t$–bands $\tau_1, \tau_2 \in T(D)$, where $D$ is a domain of a diagram $\Delta$ over (11). The boundary of $D$ can be decomposed as

$$\partial D = \text{bot}(\tau_1)a\text{bot}(\tau_2)b$$

for certain subpaths $a, b$ of $\partial D$. We say that $\tau_1$ and $\tau_2$ are $K$–connected if $\phi(a)$ (or, equivalently, $\phi(b)$) represents an element of $K$ in $G$. 
The following lemma plays the key role in the proof of Theorem 1.2.

**Lemma 3.2.** Suppose that $W$ is a word in the alphabet $X \cup Y \cup \{t\} \cup H$ such that $W = 1$ in $G$. Then there is a van Kampen diagram $\Delta$ over $(14)$ with boundary label $W$ such that the following conditions hold.

1. $\Delta$ contains no $K$–connected $t$–bands.
2. Any non–annular $t$–band $\tau$ in $\Delta$ has length $|k(\tau)|_Y$, where $k(\tau)$ denotes the element of $K$ represented by $\phi(\text{bot}(\tau))$.

**Proof.** To prove the first assertion of the lemma we proceed by induction on the number $n$ of appearances of the letters $t^{\pm 1}$ in $W$. If $n = 0$, the lemma is obvious, so we assume $n > 0$. Let $\Delta$ be an arbitrary diagram over $(14)$ with boundary label $W$. Suppose that for some $D \in D(\Delta)$, there are two $K$–connected $t$–bands $\tau_1, \tau_2 \in T(D)$. Then the boundary of $\Delta$ is decomposed as

$$\partial \Delta = u_1e_1v_1e_2u_2e_3v_2e_4,$$

where $e_2, e_4$ (respectively $e_1, e_4$) are common edges of $\partial \Delta$ and $\tau_1$ (respectively $\tau_2$) labelled $t$. Let $U_i, V_i$ denote the labels of $u_i$ and $v_i$, $i = 1, 2$ (see Fig.2). We are going to rebuild the diagram $\Delta$ as follows.

Since $\tau_1$ and $\tau_2$ are $K$–connected, $V_i$ represents an element of $K$ in $G$ for $i = 1, 2$. Let $B_i$ be a word in $Y$ such that $B_i = V_i$ in $G$. Further let $T_i$ be the word in the alphabet $H_\nu \setminus \{1\}$ obtained from $B_i$ by replacing each letter $y \in Y$ with the letter from $H_\nu \setminus \{1\}$ representing the element $y^t$ in $G$. For $i = 1, 2$, we denote by $\Delta_i$ the van Kampen diagram over $(14)$ with boundary label

$$\phi(\Delta_i) \equiv t^{-1}V_itT_i^{-1}$$

that is obtained by gluing the $t$–band with the bottom labelled $B_i$ to a diagram over $(14)$ corresponding to the equality $V_i = B_i$ in $G$. Clearly

$$U_1T_1U_2T_2 = U_1t^{-1}V_1tU_2t^{-1}V_2t = 1$$

**Figure 2. Rebuilding of diagrams containing $K$–connected $t$–bands.**
in $G$. Denote by $\Delta_0$ a diagram over $[14]$ with boundary label

$$\phi(\Delta_0) \equiv U_1 T_1 U_2 T_2.$$ 

We now glue the diagrams $\Delta_0$, $\Delta_1$, $\Delta_2$ in the obvious way (see Fig. 2) and denote the obtained diagram by $\Xi$. It is clear that any domain $D$ of $\Xi$ is a domain in $\Delta_j$ for a certain $j \in \{0, 1, 2\}$ and two $t$–bands $\tau_1, \tau_2 \in T(D)$ are $K$–connected in $\Xi$ if and only if they are $K$–connected in $\Delta_j$. Since the number of appearances of $t^{\pm 1}$ in the boundary labels of each of the diagrams $\Delta_0$, $\Delta_1$, $\Delta_2$ is at most $(n - 2)$, we may assume that $\Delta_0$, $\Delta_1$, and $\Delta_2$ contain no $K$–connected $t$–bands. Therefore no $t$–bands of $\Xi$ are $K$–connected.

Let us prove the second assertion. Let $\Delta$ be an arbitrary diagram over $[14]$ with boundary label $W$, $\tau$ a non–annular $t$–band in $\Delta$, $T \equiv \phi(\text{top}(\tau))$, $B \equiv \phi(\text{bot}(\tau))$. Let also $A$ be a shortest word in $Y$ representing the same element of $K$ as $B$. Given these data, we proceed as follows. First we remove $\tau$ from $\Delta$ thus cutting the rest of $\Delta$ in two parts denoted by $\Delta'$ and $\Delta''$. Further we take the $t$–band $\sigma$ with $\phi(\text{bot}(\sigma)) \equiv A$. Note that

$$\phi(\text{top}(\sigma)) = t^{-1} A t = t^{-1} B t = T$$

in $G$. Let $\Sigma_1$, $\Sigma_2$ be diagrams over $[14]$ having boundary labels $(\phi(\text{top}(\sigma)))^{-1}T$ and $AB^{-1}$ respectively. Gluing $\Delta'$, $\Sigma_1$, $\sigma$, $\Sigma_2$, and $\Delta''$ in the obvious way (see Fig. 3), we obtain a new diagram with boundary label $W$, where

$$l(\sigma) = ||A|| = |k(\sigma)|_Y.$$ 

Obviously this procedure does not violate condition 1) from the statement of the lemma. Doing this for all non–annular $t$–bands, we get what we need. □
4. Proofs of the main results

By technical reasons, it is convenient to introduce an auxiliary presentation
\[ G = \langle X \cup Y \cup \{t\} \cup \mathcal{H} \mid \tilde{R} \cup Q \cup S \rangle, \]
where \( Q \) is the set of all words in \( Y \) representing 1 in \( K \), and \( S, \tilde{R} \) are defined as in (13) and (14). Speaking about van Kampen diagrams over presentations (13), (14), and (15), we call a cell \( \Pi \) an \( R \)-cell (respectively \( \tilde{R} \), \( R' \), \( \mathcal{L} \), \( \mathcal{S} \), \( \mathcal{S}_\lambda \), \( Q \)-cell) if the boundary of \( \Pi \) is labelled by a word from \( R \) (respectively \( \tilde{R} \), \( R' \), \( \mathcal{L} \), \( \mathcal{S} \), \( \mathcal{S}_\lambda \), \( Q \)).

For a diagram \( \Delta \), the set of all \( R \)-, \( \tilde{R} \)-, etc., cells (respectively the number of such cells) is denoted by \( \mathcal{R}(\Delta) \), \( \tilde{\mathcal{R}}(\Delta) \), etc. (respectively \( N_{\mathcal{R}}(\Delta) \), \( N_{\tilde{\mathcal{R}}}(\Delta) \), etc.) Further we set \[ M = \max_{R \in \mathcal{R}} \|R\|. \]

Let also \( \gamma \) denote the relative Dehn function of \( H \) with respect to \( \{H_\lambda\}_{\lambda \in \Lambda} \cup \{K\} \) associated to (11). In terms of van Kampen diagrams this means that for any word \( W \) in \( X \cup \mathcal{H} \cup (K \setminus \{1\}) \) representing 1 in \( H \), there exists a diagram \( \Delta \) over (13) such that \( N_{\mathcal{R}}(\Delta) \leq \gamma(n) \).

**Lemma 4.1.** Let \( W \) be a word in the alphabet \( X \cup Y \cup \mathcal{H} \) of length at most \( n \) such that \( W = 1 \) in \( G \). Then there exists a van Kampen diagram \( \Delta \) over (14) with boundary label \( W \) and the number of \( \tilde{R} \)-cells
\[ N_{\tilde{R}}(\Delta) \leq n + (M + 1)\gamma(n). \]

**Proof.** Note that we can regard \( W \) as a word in \( X \cup \mathcal{H} \cup (K \setminus \{1\}) \) as \( Y \subseteq K \). Obviously \( W \) represents 1 in \( H \). Let \( \Delta_0 \) be a diagram over (13) such that:

a) \( \partial \Delta_0 = W \);

b) The number of \( R \)-cells in \( \Delta_0 \) is \( N_{\mathcal{R}}(\Delta_0) \leq \gamma(n) \).

c) \( \Delta_0 \) has minimal total number of \( L \)- and \( S \)-cells among all diagrams over (13) satisfying the first two conditions.

Note that any internal edge of \( \Delta_0 \) (i.e., a common edge of two cells) belong to the boundary of some \( R \)-cell. Indeed, if two \( S_\lambda \)-cells \( \Pi_1 \) and \( \Pi_2 \) have a common edge \( e \), we can erase \( e \) replacing \( \Pi_1 \) and \( \Pi_2 \) with one cell since \( S_\lambda \) contains all words in \( H_\lambda \setminus \{1\} \) representing 1 in \( H \). However this contradicts c). The same argument can be applied if two \( L \)-cells of \( \Delta_0 \) have a common edge.

For an edge \( e \) in \( \Delta_0 \) labelled by a letter \( k \in K \setminus \{1\} \), we denote by \( V(e) \) a shortest word in \( Y \) representing \( k \) in \( G \). It is clear that replacing \( e \) with \( V(e) \) for all such edges \( e \) of \( \Delta_0 \), we obtain a diagram \( \Delta_1 \) over (13). Note that \( l(\partial \Delta_1) = l(\partial \Delta_0) \) as any letter from \( K \setminus \{1\} \) labelling an edge on \( \partial \Delta_0 \) belongs to \( Y \). To obtain a diagram over (14) it remains to get rid of \( Q \)-cells. For every \( Q \)-cell \( \Pi \) of \( \Delta_1 \), we construct a diagram \( \Sigma_{\Pi} \) over (14) as follows. Consider the \( t \)-annulus \( \sigma \) whose outer contour is \( \text{bot}(\sigma) \) and \( \phi(\text{bot}(\sigma)) \equiv \phi(\partial \Pi) \). Obviously \( \phi(\text{top}(\sigma)) \) represents \( t^{-1}\phi(\Pi)t = 1 \) in \( G \) and is a word in the alphabet \( H_\nu \setminus \{1\} \). Therefore, we can glue an \( S_\nu \)-cell to the inner contour of \( \sigma \). The resulting disk diagram is denoted by \( \Sigma_{\Pi} \).

Now replacing every \( Q \)-cell in \( \Delta_1 \) with \( \Sigma_{\Pi} \), we obtain a diagram \( \Delta \) over (14). Evidently the number of \( \tilde{R} \)-cells of \( \Delta \) satisfies
\[ N_{\tilde{R}}(\Delta) \leq N_{\tilde{R}}(\Delta_0) + \sum_{\Pi \in Q(\Delta_1)} l(\partial \Pi). \]
As any internal edge of $\Delta_0$ belongs to the boundary of some $\mathcal{R}$–cell, any internal edge of $\Delta_1$ belongs to the boundary of some $\mathcal{R}'$–cell. Hence we have
\begin{equation}
\sum_{\Pi \in \mathcal{Q}(\Delta_1)} l(\partial \Pi) \leq l(\partial \Delta_1) + \sum_{\Pi \in \mathcal{R}'(\Delta_1)} l(\partial \Pi) \leq \|W\| + MN_\mathcal{R}(\Delta_0) \leq n + M\gamma(n).
\end{equation}
Combining (17) and (18) yields (16). \qed

**Proof of Theorem 1.2.** Let $W$ be a word in the alphabet $X \cup Y \cup \{ t \} \cup \mathcal{H}$ such that $\|W\| \leq n$ and $W = 1$ in $G$. We consider a van Kampen diagram $\Delta$ that satisfies conditions 1), 2) from Lemma 3.2. To prove the theorem we have to bound the number of $\mathcal{R}$ cells in $\Delta$.

Let $D$ be a domain of $\Delta$. Since the boundary label of $D$ contains no letters $t^\pm$, we can think of $\phi(\partial D)$ as a word in $X \cup \mathcal{H} \cup (K \setminus \{ 1 \})$. Denote by $q_D$ an arbitrary cycle in $\Gamma(H, X \cup \mathcal{H} \cup (K \setminus \{ 1 \}))$ having label $\phi(\partial D)$. For each $\tau \in T(D)$, $\text{bot}(\tau)$ gives rise to a $K$–component of $q_D$. Since no $t$–bands of $\Delta$ are $K$–connected, these components are isolated in $q_D$.

Recall that for a $t$–band $\tau$, $k(\tau)$ denotes the element of $K$ represented by $\phi(\text{bot}(\tau))$. Let $c_D$ denote the cycle in $\Gamma(H, X \cup \mathcal{H} \cup (K \setminus \{ 1 \}))$ obtained from $q_D$ by replacing all components with single edges. We call the edges of $c_D$ corresponding to bottoms of $t$–bands from $T(D)$ distinguished. Note that distinguished edges are isolated $K$–components of $c_D$ and for a $t$–band $\tau \in T(D)$, label of the distinguished edge corresponding to $\tau$ represents the element $k(\tau)$. Without loss of generality we may assume that $Y$ contains the set $\Omega_K$ that is provided by Lemma 2.1 applied to the group $H$ and the collection of subgroups $\{ H_\lambda \}_{\lambda \in \Delta} \cup \{ K \}$. Applying the second assertion of Lemma 3.2 and Lemma 2.1 we obtain
\begin{equation}
\sum_{\tau \in T(D)} l(\tau) = \sum_{\tau \in T(D)} |k(\tau)|_Y \leq C\gamma(l(c_D)).
\end{equation}

By $N_{\text{top}}(D)$ and $L_{\partial \Delta}(D)$ we denote the number of $t$–bands $\tau$ of $\Delta$ such that $\text{top}(\tau) \in \partial D$ and the number of common edges of $\partial D$ and $\partial \Delta$ respectively. Clearly
\begin{equation}
l(c_D) \leq 2(N_{\text{top}}(D) + L_{\partial \Delta}(D)).
\end{equation}
Denote by $T(\Delta)$ the set of all non–annular $t$–bands in $\Delta$. Obviously we have
\begin{equation}
\sum_{D \in \mathcal{D}(\Delta)} l(c_D) \leq \sum_{D \in \mathcal{D}(\Delta)} 2(N_{\text{top}}(D) + L_{\partial \Delta}(D)) \leq 2(\text{card}(T(\Delta)) + n) \leq 3n.
\end{equation}
Summing (19) over all domains of $\Delta$ and taking into account (20), we obtain the following bound on the total length of non–annular $t$–bands in $\Delta$
\begin{equation}
\sum_{\tau \in T(\Delta)} l(\tau) = \sum_{D \in \mathcal{D}(\Delta)} \sum_{\tau \in T(D)} l(\tau) \leq \sum_{D \in \mathcal{D}(\Delta)} C\gamma(l(c_D)) \leq C\gamma(3n).
\end{equation}

Furthermore, we have
\begin{equation}
\sum_{D \in \mathcal{D}(\Delta)} l(\partial D) \leq l(\partial \Delta) + 2 \sum_{\tau \in T(\Delta)} l(\tau) \leq n + 2C\gamma(3n).
\end{equation}
By Lemma 3.1 and inequality (22), we may assume that the total number of all $\mathcal{R}$–cells in all domains of $\Delta$ is at most
\begin{equation}
\sum_{D \in \mathcal{D}(\Delta)} \left[ l(\partial D) + (M + 1)\gamma(l(\partial D)) \right] \leq n + 2C\gamma(3n) + (M + 1)\gamma(n + 2C\gamma(3n)).
\end{equation}
Finally summing (21) and (23) and taking into account that \( n \preceq f(n) \) for any superadditive function \( f \neq 0 \), we obtain \( N_R(\Delta) \preceq \gamma(n) \).

The next lemma is a relative analogue of a well–known property of ordinary Dehn functions (see, for example, [1]). The proof is straightforward (and the same as in the non–relative case), so we leave it to the reader.

**Lemma 4.2.** Let \( U \) be a group that is finitely presented relative to a collection of subgroups \( \{U_\lambda\}_{\lambda \in \Lambda} \), and let \( U_1 \) be a retract of \( U \) that contains all subgroups from the set \( \{U_\lambda\}_{\lambda \in \Lambda} \). Suppose that the relative Dehn function \( \delta \) of \( U \) with respect to \( \{U_\lambda\}_{\lambda \in \Lambda} \) is well–defined. Then \( U_1 \) is finitely presented relative to \( \{U_\lambda\}_{\lambda \in \Lambda} \), the relative Dehn function \( \delta_1 \) of \( U_1 \) with respect to \( \{U_\lambda\}_{\lambda \in \Lambda} \) is well–defined and satisfies the inequality \( \delta_1 \preceq \delta \).

**Proof of Theorem 1.3.** Recall that the amalgamated product \( A *_{K=\xi(K)} B \) is a retract of the HNN–extension of the free product \( A * B \) with the associated subgroups \( K \) and \( \xi(K) \) [10]. It remains to note that the relative Dehn function \( \gamma \) of \( A * B \) with respect to \( \{A_\mu\}_{\mu \in M} \cup \{B_\nu\}_{\nu \in N} \) satisfies (4). This is well–known for ordinary Dehn functions; the proof in the relative case is obvious and actually the same, so we leave it as an exercise to the reader. Now applying subsequently Theorem 1.2 and Lemma 4.2 we get Theorem 1.3. □

**References**

[1] G. Baumslag, C.F. Miller III, H. Short, *Isoperimetric inequalities and the homology of groups*, Invent. Math. **113** (1993), no. 3, 531–560.

[2] M. Bestvina, M. Feighn, *A combination theorem for negatively curved groups*, J. Differential Geom. **35** (1992), no. 1, 85–101.

[3] B.H. Bowditch, *Relatively hyperbolic groups*, prep., 1999.

[4] S.G. Brick, J.M. Corson, *On Dehn functions of amalgamations and strongly undistorted subgroups*, IJAC, **10** (2000), 5, 665–681.

[5] S.G. Brick, J.M. Corson, *Dehn functions and complexes of groups*, Glasgow Math. J. **40** (1998), no. 1, 33–46.

[6] F. Dahmani, *Combination of convergence groups*, Geom. Topol. **7** (2003), 933–963.

[7] B. Farb, *Relatively hyperbolic groups*, GAFA, **8** (1998), 810–840.

[8] M. Gromov, *Hyperbolic groups*, Essays in Group Theory, MSRI Series, Vol.8, (S.M. Gersten, ed.), Springer, 1987, 75–263.

[9] O. Kharlampovich, A.G. Myasnikov, *Hyperbolic groups and free constructions*, Trans. Amer. Math. Soc. **350** (1998), no. 2, 571–613.

[10] R.C. Lyndon, P.E. Shupp, *Combinatorial Group Theory*, Springer–Verlag, 1977.

[11] K.V. Mikhajlovskii, A. Yu. Olshanskii, *Some constructions relating to hyperbolic groups*, Geometry and cohomology in group theory (Durham, 1994), 263–290, London Math. Soc. Lecture Note Ser., 252, Cambridge Univ. Press, Cambridge, 1998.

[12] Ch.F. Miller III, P.E. Schupp, *The geometry of Higman–Neumann–Neumann extensions*, Comm. Pure and Appl. Math. XXVI (1973), 787–802.

[13] A.Yu. Ol’shanskii, Geometry of defining relations in groups, Kluwer Academic Publisher, 1991.

[14] D.V. Osin, *Relatively hyperbolic groups: Intrinsic geometry, algebraic properties, and algorithmic problems*, Memoirs Amer. Math. Soc., to appear.

[15] D.V. Osin, *Elementary subgroups of hyperbolic groups and bounded generation*, Internat. J. Alg. Comp., to appear.

[16] Z. Sela, *Diophantine geometry over groups I: Makanin – Razborov diagrams*, IHES Publ. Math., **93** (2001), 31–105.
