Membrane’s Discrete Spectrum and BRST Residual Symmetry in the Conformal Light Cone Gauge.

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Abstract

In this work we present a proof of the discretness of the spectrum for bosonic membrane, in a flat minkowski space. This may be useful to show the quantum mechanical consistence of others bosonics extended models. This proof includes the BRST residual symmetry and was directly performed over the discretized membrane model. The BRST residual invariant effective action is explicitly contructed.

1 Introduction

Recently there has been a renewed interest in the theories of Super-p-branes, specially due to the duality relations between some Super-p-branes, Dirichlet Branes and Superstring in several dimension [1]. These duality relation are usually stablished at different coupling limits (strong-weak) and different compactification limits (large-short radios) of low energy phenomenological actions [2]. There are also remarkable results for the BPS spectra, and calculations of entropy for these new membrane theories [3].

Due to the new membrane theories, the problem of the supermembrane spectrum comes out as a matter of intense research. Some years ago it was probed that the spectrum of the supermembrane was continuous without gap from zero in flat space time [4]. More recently it has been claim [5] that by compactifying on a torus some of the spatial dimensions the spectrum renders to be discrete, but others authors [6] disagrees and obtains still obtain a continuous spectrum. In a recent paper [7] it is shown that both answers are correct depending on the irredicucibility number of the wrapping on the torus.

In contrast with the extense work devoted to the Supermembranes, there exist relatively few works about the quantization of the bosonic membrane
most of them are semiclassical approximations and indicate a discrete spectrum. It is widely accepted that the membrane spectrum is discrete, but in some cases, the authors claim that for physical reasons (that involves the confinement of wave fronts due to the uncertainty principle) the spectrum turns out to be discrete. So the objective of this paper is to present a proof of the discretness of the bosonic membrane spectrum, from a BRST functional integral point of view.

The organization of this paper is the following: in section 2 we review the classical membrane theory, its invariances, constraints and residual gauge symmetry in the conformal light cone gauge fixing. In section 3 we explain the discretized membrane model and explicitly construct the invariant residual BRST effective action. In section 4 we present a proof of the discretness of the bosonic membrane spectrum, taking into account the residual BRST gauge invariance.

2 Gauge Fixing and Residual Gauge Group

We will start from the usual second order action

$$S = -\frac{1}{8\pi^2\beta} \int_B \sqrt{-g}(g^{ab}x_a^\mu x_{\mu,b} - 1)d^3\sigma$$

which is equivalent to the Nambu-Goto-Dirac action over the field equations.

The classical hamiltonian and constraints can be obtained from (1), following Dirac’s procedure and using the ADM parametrization:

$$\mathcal{H} = \frac{N}{2\sqrt{\gamma}}(p^2 + \gamma) + N^i(px,i)$$

where $\frac{N}{2\sqrt{\gamma}}$ and $N^i$ are the Lagrange multipliers associated with the 3-d diffeomorphism generating first class constraints:

$$\phi = \frac{1}{2}(p^2 + \gamma)$$
$$\phi_i = px,i$$
The conformal gauge fixing for this theory is defined as in \[12\]

\[
\begin{align*}
g_{0j} &= 0 \\
g_{00} + \gamma &= 0
\end{align*}
\]

(3)

It was shown some time ago \[13\] that even though this gauge fixes the Lagrange multipliers, there is still a residual gauge group that allows us to fix the light cone gauge (LCG). This residual gauge group has as parameters the solutions to the homogeneous equations that arise from the gauge invariance of the gauge fixing conditions, i.e. the solutions to:

\[
\begin{align*}
\delta (g_{00} + \gamma) &= 2 [-\partial_o \zeta^o_{res} + \partial_i \zeta^i_{res}] = 0 \\
\delta (g_{0i}) &= \gamma_{ij} \partial_o \zeta^o_{res} + \gamma \partial_i \zeta^i_{res} = 0
\end{align*}
\]

(4)

(5)

The temporal evolution of \(x^+\) is given by the field equation: \(\partial_{00} x^+ = 0\), that could be directly integrated as: \(x^+ = c^+ \tau + \kappa(\sigma^1, \sigma^2)\) where the function \(\kappa(\sigma^1, \sigma^2)\) could be determined using the residual gauge parameters \(\zeta^a_{res}\). Note that \(\delta x^+ = c^+ \tau\) fixes the residual gauge parameter \(\zeta^o_{res}\), so:

\[
\partial_o \zeta^i_{res} = 0 \\
\Rightarrow \quad \zeta^i_{res} = \epsilon^{ij} \partial_j f(\sigma^1, \sigma^2)
\]

\[
\partial_i \zeta^i_{res} = 0
\]

The LCG fixing allows us to determine the \(d-2\) transverse part, explicitly solving the constraints we get:

\[
\partial_i x^- = \frac{1}{c^+} \partial_i \vec{x}.\vec{p}
\]

(6)

\[
p^+ = c^+
\]

(7)

\[
p^- = \partial_0 x^-
\]

(8)

These equations do not exhaust the content of the constraints \[2\]. Indeed, if we take the 2\(d\) curl of \(\phi_1\), we get the residual constraint \[14\]

\[
T = \epsilon^{ij} \vec{p}_{i,j} \vec{x}_{ij} = \vec{p}_{1,1} \vec{x}_{2,2} - \vec{p}_{2,2} \vec{x}_{1,1}
\]

(9)

3
that generates the residual group. This group is the subgroup of 2 dimensional diffeomorphisms that preserve areas, and its generator has a closed first class algebra, namely:

\[
\{T(\sigma), T(\bar{\sigma})\} = \epsilon^{ij} \partial_j T(\sigma) \partial_j \delta(\sigma - \bar{\sigma})
\]  

(10)

The action of this group on the canonical variables is given as follows:

\[
\delta_T x = \{x(\sigma), \int d^2 \sigma \lambda T(\bar{\sigma}) = \xi^i \partial_i x(\sigma)\}
\]  

(11)

\[
\delta_T p = \{p(\sigma), \int d^2 \sigma \lambda T(\bar{\sigma}) = \partial_j (\xi^j p(\sigma))\}
\]  

(12)

where \(\xi^i \equiv \epsilon^{ij} \partial_j \lambda\) may be identified with \(\zeta^i_{res}\). According to the above formulas, the coordinates \(x\) transform as a scalars while their corresponding momenta \(p\) as scalar densities in two dimensions as expected.

Performing variations of the action (1) with respect to the variables \(g_{ab}\), and using (3) we get

\[
\partial_a x^- = -\partial_0 x^I \partial_a x^I
\]  

(13)

\[
\partial_0 x^- = -\frac{1}{2}(\partial_0 x^I)^2 - \frac{1}{2} det(\partial_a x^I \partial_b x^I)
\]  

(14)

where we denoted by \(x^I\) the Light cone transverse part of \(x^m\). These equations allow us to solve the minus sector \(x^-\).

The equations for the d-2 transverse sector, may be obtained the following effective transverse action

\[
L = \frac{1}{2}(\partial_0 x^I)^2 - \frac{1}{4} det(\partial_a x^I \partial_b x^I)
\]  

(15)

this could be rewritten as

\[
L = \frac{1}{2}(\partial_0 x^I)^2 - \frac{1}{4} \{x^I, x^J\}_{LB} \{x^I, x^J\}_{LB}
\]  

(16)

where the Lie bracket is definied by
\{A, B\}_{LB} \equiv \epsilon^{ij} \partial_i A \partial_j B \tag{17}

The action (13) is invariant under gauge transformations generated by the constraints (3) but with the parameters \(\xi^i = \epsilon^{ij} \partial_j \lambda\), the transformation of \(x\) by the Lie brackets follows from

\[\delta x(\sigma) = \{x(\sigma), \lambda(\sigma)\}_{LB}\] \tag{18}

that is equivalent to the residual gauge invariance generated by \(T\) through

\[\delta_T x = \{x(\sigma), \int d^2 \bar{\sigma} \lambda(\bar{\sigma}) T(\bar{\sigma})\}_{PB}\]

\[= \{x(\sigma), \lambda(\sigma)\}_{LB} = \zeta^j \partial_j x(\sigma)\] \tag{19}

where: \(\zeta^j \equiv \epsilon^{ij} \partial_j \lambda(\sigma)\) \tag{20}

The transverse action (16) has just the structure of the action for a Yang Mills theory compactified to one dimension, in the Coulomb gauge \([4]\). This equivalence is a particular characteristic of the 2-brane and could not be easily extended to other p-branes. Altough we still have to discuss the residual gauge symmetry of the membrane generated by the residual constraint (9) that in principle is absent from a Yang Mills theory.

3 Discretized membrane model.

Introducing a base of functions over the section of B at constant time \([4]\)

\[x^I(\tau, \sigma^i) = x^I_o(\tau, \sigma^i) + \sum_A x^{IA}(\tau) Y_A(\sigma^1, \sigma^2),\] \tag{21}

\[p^I(\tau, \sigma^i) = p^I_o(\tau, \sigma^i) + \sum_A p^{IA}(\tau) Y_A(\sigma^1, \sigma^2)\] \tag{22}
the Hamiltonian is obtained from (17) using $D_\alpha x^I = \partial_\alpha x^I = p^I$

$$H = \frac{1}{2c}[p^Ip^I + p^{IA}p^{IA}] + \frac{1}{4}[f_{ABC}x^{IA}x^{IB}]^2$$  \hspace{1cm} (23)

$$f_{ABC} = \int d^2 \sigma Y_A \{Y_B, Y_C\}$$  \hspace{1cm} (24)

where $f_{ABC}$ are Lie algebra structure functions analogs.

To obtain a correct theory of discretized membranes we must impose the residual constraint (9) of the membrane, this implies a set of constraints over our discretized membrane model that are the 2-brane analogous of Virasoro constraints.

$$T = \{P_I, x^I\} = (x^{IA}p^B)f^{C}_{AB}Y_C = 0$$

$$\Rightarrow L_A = f_{ABC}x^{IB}p^C_I = 0$$

We may define a first quantization theory for the discretized membrane, where the Hilbert space consists of the scalar wave functions valued over the infinite set of coefficients $x^{IA}(t)(A = 1, ..., \infty \ I = 1, ..., d - 2)$ instead of $x^{IA}(\tau, \sigma^1, \sigma^2)$.

$$\Phi(x^{IA}) : \mathbb{R}^{N(d-2)} \to \mathbb{C}$$  \hspace{1cm} (25)

The operators position and momentum are defined in the Schrödinger representation as

$$X^{IA}\Phi >= x^{IA}\Phi > \quad \text{and} \quad P_{IA}\Phi >= -i\frac{\partial}{\partial x^{IA}}\Phi >$$  \hspace{1cm} (26)

Eliminating the zero mode from (23) we get the Schrödinger equation

$$\left[-\frac{1}{2}(\frac{\partial^2}{\partial x^0 A^2} + \frac{1}{2}f_{ABC}x^{IB}x^{JC})^2\right]\Phi >= E\Phi >$$  \hspace{1cm} (27)

that jointly with the residual constraints $L_A$ are the equations for the wave functions.
\[-if_{ABC}x^{IA}\left(\frac{\partial}{\partial x^{IA}}\right)|\Phi| > 0\] (28)

From these equations it is evident that the theory is not uniquely defined. For example consider a membrane with periodic boundary conditions \(x(\sigma^1) = x(\sigma^1 + 2\pi k/m)\) and \(x(\sigma^2) = x(\sigma^2 + 2\pi k/n)\).

A complete set of functions is

\[Y_{mn} = \exp(im\sigma_1 + in\sigma_2)\] (29)

and the structure constants are

\[f_{ABC} = f_{mn,pq,rs} = (A \times B)\delta^{A+B}_C\] (30)

where \(A=(m,n)\); \(B=(p,q)\) and \(C=(r,s)\) \(\in \mathbb{Z}^2\) that coincides with the structure constants of a \(N\times N\) matrix realization of \(SU(N)\) in the \(N \to \infty\) limit [1].

It is easy to see that coefficients \(L_A\) of the constraint \(T\) satisfy the same Poisson algebra, that the base (29) in term of Lie Bracket. In fact

\[\{L_A, L_D\}_P = [f_{ABC}f_{DCF} - f_{DBC}f_{DCF}]x^BP^F\]

but \(T_A = f_{A(BC)}\) correspond to the adjunt representation , that satisfy

\([f_{A(BC)}, f_{D(CF)}] = f_{ADE}f_{E(BF)}\) then \(\{L_A, L_D\}_P = f_{ADF}L_F\) (31)

The BRST generating charge for a closed constraint algebra [13] is

\[\Omega = c^AL_A - \frac{1}{2}c^Ac^Bf_{ABC}^C\] (32)

Following a modified BFV [16] approach The Functional Integral including the BRST invariant terms is given as:
\[
I = \int Dz \, e^{\int dt \, p \dot{x} - \mu A \dot{v}^A - H_o(\text{brst}) - \delta(\lambda^A \mu_A) + \delta \underline{\Theta} \chi^A)}
\]

where \( p, x \) and \( \mu, c \) are canonical variables, while the others variables are not canonical.

The BRST invariant Hamiltonian is given by

\[
H_o(\text{brst}) = H_o + \mu_a (1) V^A_B c^B
\]

where \( V^A_B \) are the coefficients of the commutator

\[
\{L_B(\sigma), H_o(\sigma)\} = (1) V^A_B L_A
\]

we obtain that this coefficients are null in virtue of

\[
\{Y^F L_F, H_o\} = \{T, H_o\} = \epsilon^{ij} \partial_i \{\phi_j(\sigma), \phi_3(\sigma)\} = 0
\]

this implies that \( V^A_B = 0 \forall A \text{ and } B \). So we get

\[
H_o(\text{brst}) = H_o = \frac{1}{2} p^I p^I A + C_{IAB} f^C E x^A x^B x^E x^D
\]

The transformation laws of a object depending on canonical variables are given by

\[
\delta F(p, x, \mu, c) = \{F, \Theta\}
\]

while the non canonical variables the transformation laws are given by

\[
\delta \underline{\Theta} = B_A \quad \delta B_A = 0 \\
\delta \lambda^A = \Theta^A \quad \delta \Theta^A = 0
\]

Using the transformation laws and the Hamiltonian \((37)\) into \((33)\) we get
\[ I = \int Dz \, e^{\int dt \, p^A \dot{x}^A - H_{o(brst)} - \lambda^A (L_A - e^B f_{AB}^C \mu_C) + B_A X^A - \xi_A \delta X^A} \]  

(40)

\[ Dz = Dp \, Dx \, D\mu \, Dc \, D\xi \, D\lambda \, DB \, D\Theta \]

Note that \( H_{o(brst)} \) is the BRST invariant Hamiltonian and not the BRST effective Hamiltonian, the former which be deduced from the effective action in the functional integral after fixing the gauge.

First we will integrate in the \( \lambda \) variables and get functional deltas over the BRST extended constraints

\[ L_{A(brst)} = L_A - e^B f_{AB}^C \mu_C \]  

(41)

that generates the same constraint algebra (31) than \( L_A \)

\[ \{L_{A(brst)}, L_{D(brst)}\} = f_{ADF} L_{F(brst)} \]  

(42)

We will now fix the residual gauge freedom generated by the constraints taking

\[ \chi^C = \lambda^C - \kappa^C \]  

(43)

where \( \kappa^C \) is a suitable collection of constants so \( \delta \kappa^C = \theta_C \). Integrating (40) in the auxiliary variables \( B \) and \( x^I \) we obtain

\[ I = \int Dz \, \delta(L_{A(brst)}) \delta(\lambda^C - \kappa^C) \delta(\xi_A - \mu_A) e^{\int dt \, p^A \dot{x}^A - H_{o(brst)}} \]  

(44)

\[ Dz = Dp \, Dx \, D\mu \, Dc \]

the last term in the exponential is the effective Hamiltonian

\[ H_{eff} = H_{o(brst)} = \frac{1}{2} p^A \cdot p^A + \frac{1}{4} f_{AB}^C \int CEDx^I A x^J B x^I E x^J D \]  

(45)

that only in this gauge choice coincides with the BRST invariant Hamiltonian, but submitted to the restrictions implied by the deltas in the functional integral, they are:

\[ \xi_A = \mu_A \quad \lambda^A = \kappa^A \quad \text{and} \quad L_A - e^B f_{AB}^C \mu_C = 0 \]  

(46)
4 Discrete Spectrum of the Membrane.

In this section we will probe that the spectrum of the membranes is discrete, performed directly on the discretized membrane model. We also take into account the local constraints \( L_A \) and the residual BRST invariance they generate in a fix gauge choice, but the BFV theorem \[15] garantized that our result is gauge independent and our proof have the advantage that we never take the limit \( N \to \infty \) of the group \( SU(N) \).

We will use a corollary due to B. Simon \[17\] of a beautiful theorem due to Fefferman and Phong \[18\] about the spectral dimension of the quantum Hamiltonian. This Corollary establishes that the number of eigenvalues (counting multiplicities) of the Hamiltonian is finite for every finite total system energy value, if the Hamiltonian operator for the quantum system is

\[
\mathcal{H} = -\nabla^2 + V(X), \quad x \in \mathcal{R}^M \text{ and } V(x) \geq 0 \quad (47)
\]

and the potential \( V(x) \) can be written as a sum of homogeneous polynomials of degree 2

\[
V(x) = \sum_{j=1,\ldots,m} Q_j^2, \quad \text{that satisfies } \sum_{j=1,\ldots,m; \alpha=a,\ldots,n} \left( \frac{\partial Q_j}{\partial x^\alpha} \right)^2 \geq 0 \quad (48)
\]

So we are going to probe that the effective BRST Hamiltonian accomplishes all these conditions.

From (45) it is evident that the potential could be written as

\[
V(x^{IB}) = \sum_{A,j} (Q^A_{ij})^2 \quad (49)
\]

where \( Q^C_{ji} = f_{ABC}x^A_ix^B_j \) are homogeneous polynomials of degree 2. It is also evident that \( V(x) \geq 0 \).

The left hand side of (49) is given by

\[
\sum_{A,I,E} x^{IA} f_{ABC} f_{EBC} x^E_I = (x^I, x_I)_K \quad (50)
\]
as \( T_A = f_{A(BC)} \) correspond to the adjoint representation of (31) then \( x^I = x^I A f_{A(BC)} \) are forms valued on the adjoint representation and the product in (30) correspond to the usual definition of the Killing product.

We only have to probe that this Killing product is not negative when \( x^I \neq 0 \)

\[
(x^I, x_I)_K = x^I A x^E K_{AE} \tag{51}
\]

where the Killing metric is diagonal \( K_{AE} = (AxB)(ExB) \delta_{A+B,E+B} \) which implies that \( K = tr(T_A T_E) = \sum_A \sum_B (AxB)^2 > 0 \) then the Killing product (51) is positive definite when \( x^I \neq 0 \)

\[
(x^I, x_I)_K = K \eta_{IJ} x^I A x^J A > 0 \tag{52}
\]

because the light cone metric \( \eta_{IJ} \) is positive and \( (x^I, x^I)_K \) is a sum of positive terms.

So this discrete membrane model accomplishes all the above conditions then we conclude that the spectrum is discrete for every finite amount of energy.

5 Conclusions.

In this paper we obtain the BRST effective Hamiltonian for the membrane in a gauge fixing, namely the conformal light cone gauge plus the residual gauge fixing conditions. We conclude that the effective Hamiltonian satisfies the conditions of the Simon and Fefferman and Phong theorems, this means that for finite energy the spectrum of the membrane is discrete and finite. Although this result was obtained in a particular gauge fixing, due to the BFV theorem the result must be valid in all gauges.

In Physical terms, this is an example of a potential that is non confinant for a system of classical particles but that is quantically confinant for wave functions due to the uncertainty principle.

The way in which Supersymmetry breaks this result must be study carefully to include the residual Symmetry, allowed gauge fixing conditions, and global constraints.
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