RELATIVE EQUILIBRIA AT SINGULAR POINTS OF THE MOMENT MAP

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Abstract. We prove a criterion for stability of relative equilibria in symmetric Hamiltonian systems at singular points of the momentum map. This generalizes a theorem of G.W. Patrick. The method of the proof is also useful in studying the bifurcation of relative equilibria.

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1. INTRODUCTION

Let $(M,\omega)$ be a symplectic manifold, $G$ a Lie group acting properly on $M$ and preserving the symplectic form. Assume further that there is an equivariant moment map $\Phi : M \to \mathfrak{g}^*$ corresponding to the action. Let $h$ be a $G$ invariant smooth function on $M$. Denote its Hamiltonian vector field by $X_h$ and the flow of the vector field by $e^{tX_h}$. Since the function $h$ and the form $\omega$ are invariant, the flow $e^{tX_h}$ is $G$ equivariant. Hence it descends to a flow on the quotient space $M/G$ (throughout the paper we will assume for simplicity that all flows exist for all times).

Recall that a point $m$ in $M$ is a relative equilibrium of the Hamiltonian $h$ (really of the flow $e^{tX_h}$) if the orbit $G \cdot m$, thought of as a point in $M/G$, is fixed by the induced flow on $M/G$. Equivalently, $m$ is a relative equilibrium iff

- the vector $X_h(m)$ is tangent to the orbit $G \cdot m$ iff
- there is $\xi \in \mathfrak{g}$, the Lie algebra of $G$, such that the corresponding vector field $\xi M$ on $M$ satisfies $\xi M(m) = X_h(m)$ iff
- $m$ is a critical point of $h - \langle \Phi, \xi \rangle$ for some $\xi \in \mathfrak{g}$.

Remark 1.1. It is standard that $\xi M(m) = X_h(m)$ implies that $e^{t\xi} \cdot m = e^{tX_h}(m)$ for all $t$. Since the moment map $\Phi$ is constant along the flow of $X_h$ and is equivariant, it follows that $\Phi(m) = \Phi(e^{tX_h}(m)) = \Phi(e^{t\xi} \cdot m) = e^{t\xi} \cdot \Phi(m)$. Hence $\xi$ is in the isotropy Lie algebra of $\Phi(m)$.

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1 “·” denotes the action of $G$ on $M$
Since we are dealing with a Hamiltonian system, it does not make sense to talk about asymptotic stability. The best one can hope for is something along the lines of:

**Definition 1.2.** Let \((M, \omega, G, \Phi, h)\) be as above — a symplectic manifold \((M, \omega)\) with a symmetry group \(G\) acting properly on \(M\) in a Hamiltonian fashion with a moment map \(\Phi\) and with a \(G\)-invariant Hamiltonian \(h\). Let \(H\) be a subgroup of \(G\). A relative equilibrium \(m\) of \(h\) is **H-stable** if for any \(H\) invariant neighborhood \(V\) of the orbit \(H \cdot m\) there is an \(H\) invariant neighborhood \(U \subset V\) such that \(\exp(tX_h)(U) \subset V\) for all times \(t\).

In his thesis [P], G. Patrick proved the following result (see also [P2]):

**Theorem [Patrick]** Given a Hamiltonian system \((M, \omega, G, \Phi, h)\) as above, assume additionally that

1. A relative equilibrium \(m\) is a regular point of the moment map \(\Phi\),
2. the isotropy group \(H\) of \(\mu = \Phi(m)\) acts properly on \(M\),
3. there exists an \(\text{Ad}(H)\) invariant inner product on the Lie algebra \(g\) of \(G\),
4. for an element \(\xi \in g\) with \(\xi_M(m) = X_h(m)\) the Hessian of \(h - \langle \Phi, \xi \rangle\) is definite when restricted to a complement of the tangent space \(T_m(H \cdot m)\) to the orbit \(H \cdot m\) in the tangent space to the level set \(T_m(\Phi^{-1}(m))\).

Then the relative equilibrium \(m\) is **H stable**.

**Remark 1.3 (Hessians).** Let \(N\) be a manifold, \(f \in C^\infty(N)\) a function and \(x \in N\) a point.
The differential \(df(x)\) of \(f\) at \(x\) is a well-defined functional on the tangent space \(T_xN\) which behaves well under pull-backs: if \(\psi : Z \to N\) is a smooth map of manifolds with \(\psi(z) = x\), then \(d(\psi^*f)(z) = \psi^*(df(x))\).

On the other hand, unless \(df(x) = 0\), the Hessian \(d^2f(x)\) is not a well-defined quadratic form on \(T_xN\). Of course, for every choice of coordinates on \(N\) near \(x\) we get a symmetric matrix of partial derivatives of \(f\). But this matrix behaves badly under a change of coordinates and, more generally, under pull-backs by smooth maps.

If \(df(x) = 0\) then the Hessian \(d^2f(x)\) is well-defined\(^2\) and behaves well under pull-backs and restrictions: if \(\psi : Z \to N\) is a smooth map of manifolds with \(\psi(z) = x\), then \(d^2(\psi^*f)(z) = \psi^*(d^2f(x))\). In particular, if \(Z\) is a submanifold of \(N\), then

\[
(1.1) \quad d^2f(x)|_{T_zZ} = d^2(f|_Z)(x).
\]

**Remark 1.4.** Note that that since \(m\) is assumed to be a regular point of the moment map, the tangent space to the level set of \(\Phi\) at \(m\), \(T_m\Phi^{-1}(m)\), is \(\ker(df(m))\). Further, a complement to \(T_m(H \cdot m)\) in \(T_m\Phi^{-1}(m)\) is a symplectic subspace isomorphic to the tangent space of the reduced space \(\Phi^{-1}(\mu)/H\) at the point \(H \cdot m\). Hence assumption \(\text{[P]}\) in the theorem above amounts to saying that the Hessian of the reduced Hamiltonian is positive definite at the point \(H \cdot m\). In particular the equilibrium point \(H \cdot m\) is stable in the reduced system.

A simple example below shows that assumption of stability in the reduced system is not enough to guarantee that the corresponding relative equilibrium is relatively stable.

In this paper we extend Patrick’s result to the case where the relative equilibrium \(m\) in question is not necessarily a regular point of the moment map. Patrick’s theorem follows as a

\(^2\) Recall that \(df(x)\) is the class of \(f - f(x)\) in \(\mathfrak{M}/\mathfrak{M}^2\) where \(\mathfrak{M}\) is the ideal of functions which vanish at \(x\). If \(df(x) = 0\), then \(f - f(x) \in \mathfrak{M}^2\) and defines \(d^2f(x)\) to be the class of \(f - f(x)\) in \(\mathfrak{M}^2/\mathfrak{M}^3\).
special case. The difficulty with extending Patrick’s result lies in the fact that in general there is no reason for the reduced space \( \Phi^{-1}(\mu)/H \) to be an orbifold in a neighborhood of the point \( H \cdot m \in \Phi^{-1}(\mu)/H \) (We use the notation of Patrick’s Theorem). So while the Hamiltonian dynamics on the reduced space and the notion of stability still make sense (SL, BL, ACC), the notions of the tangent space and of the Hessian of the reduced Hamiltonian do not. The tangent space at \( m \) to \( \Phi^{-1}(\mu) \) does not make sense either (except in the sense of Zariski).

On the other hand the kernel of the differential \( d\Phi_m \) is still a vector space and it still contains \( T_m(H \cdot m) \) as a subspace. So it makes sense to replace the assumption (4) of Patrick’s theorem ("for an element \( \xi \in \mathfrak{g} \) with \( \xi_M(m) = X_h(m) \) the Hessian of \( h - \langle \Phi, \xi \rangle \) is definite when restricted to a complement of the tangent space \( T_m(H \cdot m) \) to the orbit \( H \cdot m \) in the tangent space to the level set \( T_m(\Phi^{-1}(m)) \)”)

for a well-chosen element \( \xi \in \mathfrak{h} \) with \( \xi_M(m) = X_h(m) \) the restriction of the Hessian of \( h - \langle \Phi, \xi \rangle \) to \( \ker(d\Phi_m) \) is semi-definite and the kernel of the restriction is precisely \( T_m(H \cdot m) \), the tangent space to the orbit \( H \cdot m \).

Since for any \( \xi \in \mathfrak{h} \) the restrictions \( \langle \Phi, \xi \rangle |_{H \cdot m} \) is constant and since \( h |_{H \cdot m} \) is constant as well, we have that the restriction \( d^2(h - \langle \Phi, \xi \rangle)(m)|_{T_{H \cdot m}(H \cdot m)} \) of the Hessian of \( h - \langle \Phi, \xi \rangle \) to the tangent space to the \( H \)-orbit through the relative equilibrium is always zero. Thus \( d^2(h - \langle \Phi, \xi \rangle)(m)|_{\ker d\Phi_m} \) descends to a well-defined quadratic form on \( V := \ker(d\Phi_m)/T_m(H \cdot m) \).

**Remark 1.5** (symplectic slice). Note that since \( \ker(d\Phi_m) \) is the symplectic perpendicular \( T_m(G \cdot m)^\omega \) and since \( T_m(H \cdot m) = T_m(G \cdot m)^\omega \cap T_m(G \cdot m) \), it follows that \( V := \ker(d\Phi_m)/T_m(H \cdot m) \) is also a symplectic vector space. It is called the symplectic slice to the orbit \( G \cdot m \). If \( m \) is a regular point of the moment map, then, up to an action of finite group, the symplectic slice is symplectomorphic to a neighborhood of the point \( H \cdot m \) the corresponding reduced space.

Thus the main result of this paper reads:

**Theorem 1.6.** Let \((M, \omega)\) be a symplectic manifold, \( G \) a Lie group acting on \( M \) and preserving the symplectic form. Assume further that there is an equivariant moment map \( \Phi : M \to \mathfrak{g}^* \) corresponding to the action. Let \( h \) be a \( G \)-invariant smooth function on \( M \). Suppose \( m \) is a relative equilibrium of \( h \). Suppose

1. the isotropy group \( H \) of \( \mu = \Phi(m) \) acts properly on \( M \),
2. there exists an \( \text{Ad}(H) \)-invariant inner product on the Lie algebra \( \mathfrak{g} \) of \( G \),
3. the restriction of the Hessian of \( h - \langle \Phi, \xi \rangle \) to \( \ker(d\Phi_m) \) descends to a definite form on the symplectic slice at \( m \), where \( \xi \in \mathfrak{h} \) is orthogonal (with respect to the \( \text{Ad}(H) \)-invariant inner product) to the isotropy Lie algebra \( \mathfrak{g}_m \) of \( m \) and satisfies \( \xi_M(m) = X_h(m) \).

Then \( m \) is \( H \)-stable.

**Remark 1.7.** Since Theorem 1.6 is local, one can relax the assumption on the action of \( H \). Namely, it is enough to assume that \( H \) acts properly on a neighborhood of the orbit \( H \cdot m \).

We end the section with an example illustrating that a relative equilibrium may be relatively stable within a level set, and unstable in the whole phase space.

**Example 1.8.** Consider the standard action of \( SO(2) \) on \( \mathbb{R}^2 \). It lifts to a Hamiltonian action on \( T^*\mathbb{R}^2 = \{(q_1, q_2, p_1, p_2)\} \) with a moment map \( f(q, p) = q_1 p_2 - q_2 p_1 \). Now consider a Hamiltonian
system on the product \( M = T^*\mathbb{R}^2 \times T^*S^1 = \{(q_1, q_2, p_1, p_2, \theta, \theta_0)\} \) (standard symplectic form) with the Hamiltonian
\[
h = (q_1p_2 - q_2p_1) + p_\theta(p_1^2 + p_2^2 - q_1^2 - q_2^2).
\]
The Hamiltonian \( h \) Poisson commutes with \( p_\theta \), which is a moment map for the action of \( S^1 \) on the second factor.

The reduced spaces \( M_\mu \) are all symplectomorphic to \( T^*\mathbb{R}^2 \) and the reduced Hamiltonians \( h_\mu \) are given by
\[
h_\mu = (q_1p_2 - q_2p_1) + \mu(p_1^2 + p_2^2 - q_1^2 - q_2^2).
\]
Note that the origin \((0,0,0,0) \in T^*\mathbb{R}^2\) is stable for \( h_0 = q_1p_2 - q_2p_1 \) and that the Hessian of \( h_0 \) is not positive definite at the origin. For \( \mu \neq 0 \), the reduced Hamiltonian is of the form \( h_\mu = f + \mu g \) with \( \{f, g\} = 0 \). It is easy to see that the origin is unstable for the flow of \( g \), hence is unstable for \( h_\mu \) for all \( \mu \neq 0 \). Therefore the relative equilibria of the form \((q_1, q_2, p_1, p_2, \theta, 0)\) are all unstable in \( M \), even though the corresponding fixed point is stable in the reduced space \( M_0 \).

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2. Reduction to the case where the group orbit through the relative equilibrium is isotropic

We begin the proof of Theorem 1.6 by reducing it to the case where the image \( \mu = \Phi(m) \) of the relative equilibrium is fixed by the coadjoint action of \( G \), equivalently, when the orbit \( G \cdot m \) is isotropic.

Notation 2.1. If \( P \) is a principal \( L \)-bundle and \( N \) an \( L \)-manifold, we can form the quotient \( P \times_L N := (P \times N)/L \), the corresponding associated bundle. We denote the equivalence class in \( P \times_L N \) of a point \((p,n) \in P \times N \) by \([p,n]\). If \( N \) is a product \( N_1 \times N_2 \), the points of \( P \times_L (N_1 \times N_2) \) will be written \([p,n_1,n_2]\).

Suppose a Lie group \( L \) acts properly on a manifold \( N \), i.e., the map \( L \times N \to N \times N \), \((a,x) \mapsto (a \cdot x,x)\) is proper. A slice for this action at \( n \in N \) is a submanifold \( S \) of \( N \), \( n \in S \subset N \), such that

1. \( S \) is invariant under the action of the isotropy group \( L_n \) of \( n \),
2. \( L \cdot S \) is open in \( N \),
3. the map \( L \times_{L_n} S \to L \cdot S \), \([l,s] \mapsto l \cdot s\), is an \( L \)-equivariant diffeomorphism.

Hence if \( S \) is a slice, we have \( T_n N = T_n S \oplus T_n(L \cdot n) \) and \( L_n \) orbits in \( S \) parameterize \( L \) orbits near \( L \cdot n \).

Let \( G \) be a Lie group. A point \( \alpha \in \mathfrak{g}^* \) is split if its isotropy Lie algebra \( \mathfrak{g}_\alpha \) has a \( G_\alpha \) invariant complement \( \mathfrak{n} \) in \( \mathfrak{g} \), \( \mathfrak{g} = \mathfrak{g}_\alpha \oplus \mathfrak{n} \) (\( G_\alpha \) equivariant). Note that if a \( G_\alpha \) invariant inner product exists on \( \mathfrak{g} \) (or, equivalently, on \( \mathfrak{g}^* \)), then \( \alpha \) is split: we can take \( \mathfrak{n} \) to be the perpendicular to \( \mathfrak{g}_\alpha \) under the inner product.
Assume that $\alpha \in \mathfrak{g}^*$ is split. Since the tangent space to the coadjoint orbit $G \cdot \alpha$ is naturally isomorphic to the annihilator $\mathfrak{g}_\alpha^*$ of $\mathfrak{g}_\alpha$ in $\mathfrak{g}^*$, one can show that a $G_\alpha$ invariant neighborhood $B$ of $\alpha$ in the affine subspace $\alpha + \mathfrak{n}^\circ$ is a slice at $\alpha$ for the coadjoint action of $G$. Now if $\Phi : M \to \mathfrak{g}^*$ is an equivariant moment map, then the equivariance of $\Phi$ implies that $\Phi$ is transversal to $B$. Hence $R := \Phi^{-1}(B)$ is a submanifold of $M$.

**Theorem 2.2** (cf. [GS2], Theorem 26.7 and [GLS], Corollary 2.3.6). Let $(M,\omega)$ be a Hamiltonian $G$ space with momentum map $F : M \to \mathfrak{g}^*$. Suppose $\alpha \in \mathfrak{g}^*$ is split, and $\mathfrak{g} = \mathfrak{g}_\alpha \oplus \mathfrak{n}$ is a corresponding $G_\alpha$-invariant splitting.

Then for a small enough $G_\alpha$ invariant neighborhood $B$ of $\alpha$ in $\alpha + \mathfrak{n}^\circ$, the preimage $R = F^{-1}(B)$ is a symplectic submanifold of $M$. Moreover, the action of $G_\alpha$ on $R$ is Hamiltonian with momentum map $F_R : R \to \mathfrak{g}_\alpha^*$ being the restriction of $F$ to $R$ followed by the projection of $\mathfrak{g}^*$ onto $\mathfrak{g}_\alpha^*$.

**Remark 2.3.** The submanifold $R = F^{-1}(B)$ is called a symplectic cross-section. It has the property that for $m \in F^{-1}(\alpha)$ the $G_\alpha$ orbit is isotropic in $R$. Also the cross-section $R$ is the smallest symplectic submanifold of $M$ containing the fiber $F^{-1}(\alpha)$.

Let $(M,\omega)$ be a symplectic manifold, $G$ a Lie group acting on $M$ and preserving the symplectic form. Assume further that there is an equivariant moment map $\Phi : M \to \mathfrak{g}^*$ corresponding to the action. Let $h$ be a $G$ invariant smooth function on $M$ and suppose $m \in M$ is a relative equilibrium of the Hamiltonian $h$. Let $\mu = \Phi(m)$. Assume that there exists an $H$-invariant inner product on $\mathfrak{g}$, where $H$ denotes the isotropy group of $\mu$. Then $\mu$ is split. Let $\mathfrak{n}$ be the $H$ invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$ (orthogonal to $\mathfrak{h}$ with respect to the inner product), $B$ be a $H$ invariant neighborhood of $\mu$ in $\mu + \mathfrak{n}^\circ$ which is a slice for the action of $G$ and let $R$ be a corresponding cross-section passing through $m$. Let $\Phi_R : R \to \mathfrak{h}^*$ be a corresponding moment map. Since $\mu$ is fixed by $H$, $\pi(\mu) \in \mathfrak{h}^*$ is fixed by the coadjoint action of $H$ (here $\pi : \mathfrak{g}^* \to \mathfrak{h}^*$ is the projection).

Let $h_R$ be the restriction of our Hamiltonian $h \in C^\infty(M)^G$ to the cross-section $R$. Since the flow of the Hamiltonian vector field $X_h$ of $h$ preserves the fibers of the moment map and the cross-section is a union of fibers, the vector field $X_h$ is tangent to the cross-section. It follows that the Hamiltonian vector field of $h_R$ on the symplectic manifold $(R,\omega|_R)$ is the restriction of $X_h$ to $R$.

**Lemma 2.4.** Under the hypothesis of the previous two paragraphs, if $m \in R$ is an $H$ stable equilibrium of $h_R$ then $m \in M$ is an $H$ stable equilibrium of $h$.

**Proof.** Since $B$ is a slice for the coadjoint action of $G$, $G \cdot B$ is a neighborhood of $\mu$ in $\mathfrak{g}^*$ diffeomorphic to the associated bundle $G \times_H B$. By the equivariance of the moment map, $G \cdot R$ is an open $G$ invariant subset of the manifold $M$ diffeomorphic to the associated bundle $G \times_H R$. The lemma now follows from the proposition below. \qed

**Proposition 2.5.** Let $G$ be a group, $H \subseteq G$ a compact subgroup. Let $\pi : P = G \times_H F \to G/H$ be a $G$ homogeneous fiber bundle for some $H$-manifold $F$. Suppose $X$ is a $G$ invariant vector field on $P$ which is tangent to the fibers. Suppose further that $p \in F := \pi^{-1}(1H)$ is $H$ stable for the vector field $X$ on $F$ ($1$ denotes the identity in $G$). The $p$ is $H$ stable for the vector field $X$ on the whole space $P$, where $H$ acts on $P$ as a subgroup of $G$. 
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$\Phi$ preserves fibers, it induces a flow $\phi_t$ on $F$ by $[1, \phi_t(f)] = \Phi_t([1, f])$. Therefore

\begin{equation}
\Phi_t([a, f]) = a \cdot \Phi_t([1, f]).
\end{equation}

The subgroup $H$ acts on $G$ by conjugation, $a \cdot g = aga^{-1}$ for all $a \in H$, $g \in G$. It also acts on $G/H$ by $a \cdot gH = agH$. The projection $\varpi : G \to G/H$ is equivariant with respect to these actions. Since $H$ is compact, there exists a local equivariant section $s$ defined on some $H$ invariant neighborhood $U$ of $1H$ in $G/H$. By definition $s(a \cdot x) = as(x)a^{-1}$. The section defines a diffeomorphism $\psi : U \times F \to \pi^{-1}(U) \subset F$, $\psi(x, f) = [s(x), f]$.

Since $H$ acts on $F$ and on $U$, it acts on $U \times F$: $a \cdot (x, f) = (a \cdot x, a \cdot f)$. We have a flow on $U \times F$: $\Phi_t(x, f) = (x, \phi_t(f))$. It is not hard to see that $\psi$ is $H$ equivariant and that it intertwines the flow $\Phi_t$ on $U \times F$ with the flow $\Phi_t$ on $\pi^{-1}(U)$. Indeed,

$\psi([a \cdot x, a \cdot f]) = [s(a \cdot x), a \cdot f] = [as(x)a^{-1}, a \cdot f] = [as(x), f] = a \cdot [a, f] = a\psi(x, f)$;

and

$\psi([x, \phi_t(f)]) = [s(x), \phi_t(f)] = \Phi_t([s(x), f]) = \Phi_t(\psi(x, f))$.

Consequently it is enough to prove that $(1H, p)$ is $H$ relatively stable in $U \times F$ for the flow $\Phi_t$.

Let $V$ be an $H$ invariant neighborhood of $(1H, p)$ in $U \times F$. Without loss of generality we may assume that $V$ is of the form $V_1 \times V_2$, where both $V_1 \subset U$ and $V_2 \subset F$ are $H$ invariant. Since $p$ is $H$ stable for the flow $\phi_t$, there is a neighborhood $U_2 \subset V_2$ of $p$ such that $\phi_t(U_2) \subset V_2$. Consequently $\Phi_t(V_1 \times U_2) \subset V_1 \times V_2$.

To complete the reduction to the case where the orbit through the relative equilibrium is isotropic, it remains to prove one more lemma.

**Lemma 2.6.** Let $(M, \omega)$ be a symplectic manifold, $G$ a Lie group acting on $M$ and preserving the symplectic form and $\Phi : M \to \mathfrak{g}^*$ an equivariant moment map corresponding to the action. Let $m \in M$ be a point, $R \subset M$ a symplectic cross-section passing through $m$. Then

$\ker d\Phi(m) = \ker d(\Phi|_R)(m)$.

Hence, if $h \in C^\infty(M)^G$ is an invariant Hamiltonian such that $d(h - \langle \Phi, \xi \rangle)(m) = 0$ for some $\xi$ in the isotropy Lie algebra of $\Phi(m)$ (cf. Remark 2.2), then

$d^2(h - \langle \Phi, \xi \rangle)(m)|_{\ker d\Phi(m)} = d^2(h|_R - \langle \Phi|_R, \xi \rangle)(m)|_{\ker d(\Phi|_R)(m)}$

(cf. Remark 2.3).

**Proof.** The set $G \cdot R$ is a neighborhood of the point $m$ in $M$ and is diffeomorphic to the associated bundle $G \times_H R$, where $H$ is the isotropy group of $\mu = \Phi(m)$. The diffeomorphism $G \times_H R \to G \cdot R$ is given by $[g, r] \mapsto g \cdot r$. Similarly, the set $G \cdot B$ is a neighborhood of the point $\mu$ in $\mathfrak{g}^*$ and is diffeomorphic to the associated bundle $G \times_H B$, where $B = \Phi(R)$ (cf. Theorem 2.3). If $s : G/H \supset U \to G$ is a local section, it simultaneously trivializes the associated bundles $G \times_H R \to G/H$ and $G \times_H B \to G/H$. With respect to these trivializations, the moment map $\Phi : M \supset G \times_H R \supset U \times R \to U \times B \subset G \times_H B \subset \mathfrak{g}^*$ takes the form

$\Phi(u, r) = (u, \Phi(r))$, 

Remark 3.3. The form $\omega_Y$ on $Y$ has the further property that orbit $H \cdot [1,0,0]$ is isotropic and that the symplectic slice at $[1,0,0]$ is $V$. 

3. Relative equilibria lying on isotropic orbits

We now consider the following situation. Let $(M, \omega)$ be a symplectic manifold, let $G$ be a Lie group acting properly on $(M, \omega)$ in Hamiltonian fashion and let $\Phi : M \to g^*$ be a corresponding equivariant moment map. Let $h$ be a $G$ invariant smooth function on $M$. Suppose that $m \in M$ is a relative equilibrium of the Hamiltonian $h$. Suppose further that $\Phi(m)$ is fixed by the coadjoint action of $G$. Then it is no loss of generality to assume that $\Phi(m) = 0$. We may also assume that $h(m) = 0$.

Since the point 0 is fixed by the action of $G$, the $G$-orbit through the point $m$ is isotropic in $(M, \omega)$. Since the orbit $G \cdot m$ is isotropic, the tangent space $T_m(G \cdot m)$ is contained in its symplectic perpendicular $T_m(G \cdot m)^\omega$. Hence the quotient $V = T_m(G \cdot m)^\omega / T_m(G \cdot m)$ is the symplectic slice at $m$ (cf. Remark 1.3). Note that $V$ is a natural symplectic representation of the isotropy group $K$ of $m$. Note also that since the action of $G$ is proper, the isotropy group $K$ is compact.

Theorem 3.1 (Local normal form for a neighborhood of an isotropic orbit [GS1], [Ma]). Let $\Psi : N \to \mathfrak{h}^*$ be a moment map associated to a Hamiltonian proper action of a Lie group $H$ on a symplectic manifold $(N, \sigma)$. Suppose the orbit $H \cdot x$ is isotropic in $N$. Let $H_x$ denote the stabilizer of $x$ in $H$, let $\mathfrak{h}_x^0$ denote the annihilator of its Lie algebra in $\mathfrak{h}^*$, and let $H_x \to \text{Sp}(V)$ denote the symplectic slice representation.

Given an $H_x$-equivariant embedding, $i : \mathfrak{h}_x^* \to \mathfrak{h}^*$, there exists an $H$-invariant symplectic two-form, $\omega_Y$, on the manifold $Y = H \times_{H_x} (\mathfrak{h}_x^0 \times V)$, such that

1. the form $\omega_Y$ is nondegenerate near the zero section of the bundle $Y \to H/H_x$,
2. there exists a neighborhood $U_x$ of the orbit of $x$ in $N$ and an equivariantly diffeomorphism
   $\psi$ from a neighborhood $U_0$ of the zero section in $Y$ to $U_x$ such that $\psi^* \omega = \omega_Y$, and
3. the action of $H$ on $(Y, \omega_Y)$ is Hamiltonian with a moment map $\Phi_Y : Y \to \mathfrak{h}_x^*$ given by
   $$\Phi_Y([g, \eta, v]) = g \cdot (\eta + i(\Phi_V(v)))$$

where $\cdot$ denotes the coadjoint action, and $\Phi_V : V \to \mathfrak{h}_x^*$ is the moment map for the slice representation of $H_x$.

Consequently, the equivariant symplectic embedding $\psi : U_0 \hookrightarrow M$ provided by (2) above intertwines the two moment maps, up to translation: $\psi^* \Psi = \Phi_Y + \psi(x)$.

Remark 3.2. Since the action of $H$ is assumed to be proper, the isotropy group $H_x$ is compact. Therefore an $H_x$ equivariant embedding $i : \mathfrak{h}_x^* \to \mathfrak{h}^*$ always exists: Choose an $H_x$ invariant inner product on $\mathfrak{h}$. The orthogonal projection $\mathfrak{h} \to \mathfrak{h}_x$ with respect to the inner product is $H_x$ equivariant. Its transpose $i : \mathfrak{h}_x^* \to \mathfrak{h}^*$ is an $H_x$ equivariant embedding. Moreover, the perpendicular $\mathfrak{h}_x^\perp$ of $\mathfrak{h}_x$ (with respect to the inner product) is the annihilator of $i(\mathfrak{h}_x^*)$ in $\mathfrak{h}^*$.

Remark 3.3. The form $\omega_Y$ on $Y$ has the further property that orbit $H \cdot [1,0,0]$ is isotropic and that the symplectic slice at $[1,0,0]$ is $V$. 

(cf. [L], p. 814). Hence for any point $r$ in the cross-section, we have $\ker d\Phi(r) = \ker d(\Phi|_R)(r)$. 

\[\square\]
Theorem 3.1 is essentially an equivariant version of Weinstein’s isotropic embedding theorem. The proof in the case that the group $H$ is compact can be found in [GS1] and in [GS2]. The case of proper actions is discussed in [31]. See also [Ma].

Let us now go back to studying the relative equilibrium $m \in M$ lying on the zero level set of the moment map $\Phi$. By Theorem 3.1 there exists a $G$-invariant neighborhood $U$ of $m$, a $G$ invariant neighborhood $U_0$ of $[1, 0, 0]$ in $Y := G \times K$ and an equivariant diffeomorphism $\psi : U_0 \to U$ such that $\psi([1, 0, 0]) = m$, $\psi^*\omega = \omega_Y$ and $\psi^*\Phi = \Phi_Y$ (since $\Phi_Y([1, 0, 0]) = 0 = \Phi(m)$). Let $h_Y = \psi^*h$. Then

$$d\psi(\ker d\Phi([1, 0, 0])) = \ker d\Phi(m);$$

$$d(h - \langle \Phi, \xi \rangle)(m) = 0 \quad \text{if and only if} \quad d(h_Y - \langle \Phi_Y, \xi \rangle)([1, 0, 0]) = 0;$$

and consequently

$$\psi^*(d^2(h - \langle \Phi, \xi \rangle)(m)) = d^2(h_Y - \langle \Phi_Y, \xi \rangle)([1, 0, 0]).$$

(cf. Remark 3.3). Moreover if $d^2(h - \langle \Phi, \xi \rangle)(m)$ descends to a definite quadratic form on $V$ then $d^2(h_Y - \langle \Phi_Y, \xi \rangle)([1, 0, 0])$ descends to a definite quadratic form on $V$ as well ($V$ is embedded in $Y$ as the subset $\{[1, 0, v] \mid v \in V\}$). Therefore, it is no loss of generality to assume that $(M, \omega) = (Y, \omega_Y)$ and $m = [1, 0, 0]$.

Now, if $\xi \in \mathfrak{t}^\perp \subset \mathfrak{g}$, then $\langle \Phi, \xi \rangle([1, 0, v]) = i(\Phi_Y(v), \xi) = 0$ by Remark 3.2. Therefore

$$0 = d(h - \langle \Phi, \xi \rangle)(m)|_V = d(h - \langle \Phi, \xi \rangle)(m)|_V = d(h)(m) \quad \text{and} \quad d^2(h - \langle \Phi, \xi \rangle)(m)|_V = d^2(h)(m).$$

Next, observe that the spaces $C^\infty(Y)^G$ and $C^\infty(\mathfrak{t}^0 \times V)^K$ are isomorphic: the isomorphism sends a function $f \in C^\infty(Y)^G$ to the function $\tilde{f} \in C^\infty(\mathfrak{t}^0 \times V)^K$ defined by $\tilde{f}(\eta, v) = f([1, \eta, v])$. Therefore the conditions on our Hamiltonian $h \in C^\infty(Y)^G$ translate into the following two conditions on the corresponding function $\tilde{h} \in C^\infty(\mathfrak{t}^0 \times V)^K$:

1. $d_v\tilde{h}(0, 0) = 0$ and
2. $d^2_v\tilde{h}(0, 0)$ is definite.

Here $d_v$ denotes the differential of a function on $\mathfrak{t}^0 \times V$ in the $V$ variables and $d_v^2$ has similar meaning. It is no loss of generality to assume that the quadratic form $d^2_v\tilde{h}(0, 0)$ on $V$ is positive definite.

Note that the norm squared of the moment map $|\Phi|^2$ is $G$-invariant and satisfies

$$|\Phi|^2([a, \eta, v]) = |\Phi|^2([1, \eta, v]) = |\eta + i(\Phi_Y(v))|^2 = |\eta|^2 + |i(\Phi_Y(v))|^2.$$

Therefore, if $\gamma(t) = [a(t), \eta(t), v(t)]$ is an integral curve of the Hamiltonian vector field of $h$ we have

$$|\Phi|^2(\gamma(0)) = |\Phi|^2(\gamma(t)) \geq |\eta(t)|^2$$

and

$$h(\gamma(0)) = h(\gamma(t))$$

for all time $t$. Thus the proof of stability reduces to the proof of the proposition below.

**Proposition 3.4.** Let $W$ and $V$ be normed finite dimensional vector spaces, $h \in C^\infty(W \times V)$ a function with $d_\lambda h(0, 0) = 0$ and $d_\lambda^2 h(0, 0) > 0$, and $\varphi \in C^\infty(W \times V)$ a function with $\varphi(\lambda, v) \geq |\lambda|^2$.

Then for any $\epsilon > 0$ there is $\delta > 0$ such that for any curve $\gamma(t) = (\lambda(t), v(t))$ in $W \times V$ satisfying $|\gamma(0)|^2 < \delta$, $h(\gamma(t)) = h(\gamma(0))$ and $\varphi(\gamma(t)) = \varphi(\gamma(0))$ for all $t$, we have $|\gamma(t)|^2 < \epsilon$ for all $t$. 
Proof. We first apply the Morse lemma (see, for example [3], Lemma 1.2.2). Since \((0,0)\) is a critical point of \(h\) and since the Hessian \(d^2h(0,0)\) is nondegenerate, there is, by the implicit function theorem, a function \(\sigma(\lambda)\) defined on a neighborhood of 0 in \(W\) such that \(\partial_v h(\lambda, \sigma(\lambda)) = 0\).

Moreover, there exist neighborhoods \(U_1\) of 0 in \(W\) and \(U_2\) of 0 in \(V\) and a map \(\tau: U_1 \times U_2 \to W \times V\), \(\tau(\lambda, v) = (\lambda, y(\lambda, v))\), \(\tau(0,0) = (0,0)\) such that \(\tau(U_1 \times U_2)\) is open \(\tau: U_1 \times U_2 \to \tau(U_1 \times U_2)\) is a diffeomorphism and

\[
(3.3) \quad h(\lambda, v) = h(\lambda, \sigma(\lambda)) + \frac{1}{2} d^2_v h(\lambda, \sigma(\lambda))(y(\lambda, v), y(\lambda, v)).
\]

Since \(d^2_v h(0,0)\) is positive definite there is \(c > 0\) such that \(d^2_v h(0,0)(v,v) > c|v|^2\) for all \(v \in V\). By continuity, there is \(\delta_3 > 0\) such that if \(|\lambda| < \delta_3\) then \(d^2_v h(\lambda, \sigma(\lambda))(y, y) > \frac{1}{2} c|y|^2\) for all \(y \in V\).

Since \(\tau\) is a diffeomorphism onto its image and \(\tau(0,0) = (0,0)\), \(\tau^{-1}\) is continuous near \((0,0)\). Hence for any \(\epsilon > 0\) there are \(\delta_1, \delta_2 > 0\) with

\[
(3.4) \quad |\lambda|^2 < \delta_1, |y(\lambda, v)|^2 < \delta_2 \quad \text{implies} \quad |\lambda|^2 + |y(\lambda, v)|^2 < \epsilon.
\]

Choose \(\epsilon_1 > 0\) such that \(8\epsilon_1/c < \delta_2\). Since \(\lambda \mapsto h(\lambda, \sigma(\lambda))\) is continuous, there is \(\epsilon_2 > 0\) such that

\[
|\lambda|^2 < \epsilon_2 \quad \text{implies} \quad |h(\lambda, \sigma(\lambda))| < \epsilon_1.
\]

Let \(\epsilon_4 = \min(\epsilon_3, \epsilon_2, \delta_1)\). Since \(h\) and \(\varphi\) are continuous, there is \(\delta > 0\) such that

\[
|\gamma(0)| < \delta \quad \text{implies} \quad \begin{cases} h(\gamma(0)) < \epsilon_2 \\ \varphi(\gamma(0)) < \epsilon_4 \end{cases}.
\]

Since \(h\) is constant along \(\gamma\) and \(\varphi(\gamma(0)) = \varphi(\gamma(t)) \geq |\lambda(t)|^2\),

\[
|\gamma(0)| < \delta \quad \text{implies} \quad \begin{cases} h(\lambda(t), \sigma(\lambda(t))) < \epsilon_2 \\ |\lambda(t)|^2 < \epsilon_4 \end{cases}.
\]

Since

\[
|\lambda(t)|^2 < \epsilon_4 \quad \text{implies} \quad \begin{cases} d^2_v h(\lambda, \sigma(\lambda))(y, y) \geq \frac{1}{2} c|y|^2 \\ h(\lambda(t), \sigma(t)) < \epsilon_1 \end{cases},
\]

we have

\[
|y(\gamma(t))|^2 \leq \frac{2}{c} d^2_v h(\lambda(t), \sigma(\lambda(t))(y(\gamma(t)), y(\gamma(t))) \\
\leq \frac{4}{c} (|h((\lambda(t), v(t))| + |h((\lambda(t), \sigma(\lambda(t))))|) \quad \text{by (3.3)}
\leq \frac{4}{c} (\epsilon_1 + \epsilon_1) < \delta_2.
\]

Since we arranged \(|\lambda(t)| < \delta_1\) we have, by (3.4),

\[
|\gamma(t)|^2 \leq |\lambda(t)|^2 + |v(t)|^2 < \epsilon \quad \text{for all } t.
\]

\[\square\]
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