1 Introduction

In recent years, we have been pursuing a QCD based model\cite{1–5} using the linear cum Coulomb Cornell potential and reported the results of various static and dynamic properties of heavy flavour mesons with considerable theoretical success. The model uses the perturbation method of quantum mechanics\cite{6} where either of the two pieces of potential (linear or coulomb) is taken as parent and perturbation successively and chosen the better option by comparing with data. However such approach is fraught with inherent limitation: there are intermediate range of inter quark separation where both short range Coulomb and the long range linear parts of the potential are equally importance and preference of one above the other as parent or perturbation makes no sense. It is therefore instructive to go back to the traditional variational method\cite{7–10} and see if a proper trial wave-function can effectively generate the Coulomb and Linear effect without assuming such divide between the long range and the short range effects.

Another shortcoming of the previous approach with perturbation method is the proper incorporation of relativistic effect in the motion of the light quark in Heavy-Light mesons. In this context, earlier, a Dirac modification factor ($\alpha_0^{-\epsilon}$)\cite{2,4} was introduced as overall relativistic correction, in analogy with QED.\cite{11} However, as will be discussed below, such effect does not conform to the the necessity of positivity of mass of heavy-light mesons. We therefore abandon this modification factor, instead we modify the bare quark mass in meson Hamiltonian by introducing a term $p^2/2m$, consistent with the Hamiltonian used recently by Hwang, Kim, and Namgung.\cite{7}

Application of variational method in Heavy quark physics was first started by Hwang \textit{et al}.\cite{7} using the linear cum coulomb potential, which was later successfully applied by Rai \textit{et al}.\cite{10} to find masses and decay constants of heavy flavour mesons using a power law potential ($\sim -A/r + br^\alpha$). While the former calculated only masses and decay constants of pseudo-scalar mesons and the later\cite{10} calculated also the mass difference between pseudo-scalar and vector mesons using the method. More recently, the variational method is applied to heavy flavour physics by Vega and Flores using super-symmetric potential.\cite{8}

The above discussion suggests the relevance of variational method in the present day quark dynamics. This has motivated us to study the QCD based potential model afresh using variational method.

The manuscript is arranged as below:

In Sec. 2 we provide the formalism where we use variational method with Gaussian, Coulomb and Airy trial function. In the same section, we also discuss the relativistic correction for the light quark/anti-quark present in heavy-light meson. In Sec. 3 we provide the results of mass, decay constant, mass difference of pseudo-scalar and vector mesons, ratios of decay constants. Using the masses and decay constants we then calculate the oscillation frequencies of neutral B mesons and branching ratios of leptonic decays of charged mesons. A comparison is also made with QCD sum rules, lattice QCD, and experimental results. In Sec. 4 we summarize the conclusion.
2 Formalism

2.1 Variational Approach with Gaussian Trial Wave-Function

We apply a variational method which is similar to Ref. [7], considering the trial wave-function to be of Gaussian form as,
\[
\psi(r) = \left( \frac{\alpha}{\sqrt{\pi}} \right)^{3/2} e^{-\alpha^2 r^2 / 2},
\]
where, \( \alpha \) is the variational parameter. We choose the \( \bar{Q}Q \) potential as
\[
V(r) = - \frac{4\alpha_s}{3r} + br,
\]
where \( \alpha_s \) is the strong coupling constant and \( b \) is the standard confinement parameter \((b \sim 0.183 \text{ GeV}^2) \). This is the general form of Cornell potential, which considers both the properties of quark interaction-asymptotic freedom and confinement, following variational scheme, the ground state energy is given by
\[
E(\alpha) = \langle \psi | H | \psi \rangle.
\]

Now, using the trial wave function we obtain the expectation values of each term in the Hamiltonian,
\[
\left\langle -\frac{\nabla^2}{2\mu} \right\rangle = -\frac{3\mu^2 \alpha^2}{4},
\]
\[
\left\langle -4\alpha_s \right\rangle = -\frac{\mu^3 \alpha A}{\sqrt{\pi}},
\]
\[
\langle br \rangle = \frac{\mu^3 b}{\sqrt{\pi} \alpha}.
\]
Therefore, adding all the three equations above we get
\[
E(\alpha) = -\frac{3\mu^2 \alpha^2}{4} - \frac{\mu^3 \alpha A}{\sqrt{\pi}} + \frac{\mu^3 b}{\sqrt{\pi} \alpha},
\]
where, \( A = 4\alpha_s/3 \). Now, by minimising \( E(\alpha) \) with respect to \( \alpha \) we can find the variational parameter \( \alpha \) for different heavy flavoured mesons. The minimization condition to find the expectation value of Hamiltonian as
\[
\frac{dE(\alpha)}{d\alpha} = 0,
\]
at \( \alpha = \alpha' \).

This equation is solved by using Mathematica 7 and we find the variational parameter for different Heavy Flavour mesons which is shown in Table 1.

2.2 Variational Approach with Coulombic Trial Wave-Function

Let us now consider the trial wave-function to be
\[
\psi(r) = \frac{(\mu \alpha')^{3/2}}{\sqrt{\pi}} e^{-\mu \alpha' r},
\]
where, \( \alpha' \) is the variational parameter. Now, with the potential as given in Eq. (2) we calculate the expectation value of the hamiltonian,
\[
E(\alpha') = \langle \psi | H | \psi \rangle = \frac{1}{2} \mu \alpha'^2 - A \mu \alpha' + \frac{3b}{2\mu \alpha'},
\]
where, \( A = 4\alpha_s/3 \), and \( \alpha_s \) is the strong coupling constant. Now, minimising, \( dE/d\alpha' = 0 \), we get,
\[
\alpha'^2 - A \alpha'^2 - \frac{3b}{2\mu^2} = 0.
\]
This equation is solved by using Mathematica 7 and we find the variational parameter for different Heavy Flavour mesons which is shown in Table 2.

2.3 Variational Approach with Airy Trial Wave-Function

It is well known that the solution of Schrodinger equation with linear potential gives a wave-function that contains Airy function. We therefore consider the trial wave-function as Airy function,
\[
\psi(r) = \frac{N}{2 \sqrt{\pi} r} A_i[(2\mu b')^{1/3} + \varrho_0 n].
\]
Here, \( b' \) is the variational parameter and \( \varrho_0 \) are the zeroes of Airy function such that \( A_i[\varrho_0 n] = 0 \), and is given as \([14]\)
\[
\varrho_0 n = -\left[ \frac{3\pi (4n - 1)}{8} \right]^{2/3}.
\]
For different S states few zeroes of the Airy function is listed below.

### Table 1

| Heavy Flavour Mesons | Variational parameter (\( \alpha' \)) |
|---------------------|-------------------------------------|
| \( D(\bar{c}u/\bar{d}b) \) | 0.3087 |
| \( D(c\bar{s}) \) | 0.310 |
| \( B(ab/\bar{d}b) \) | 0.301 |
| \( B_s(ab) \) | 0.537 |

### Table 2

| Mesons | \( \alpha' \) |
|--------|----------------|
| \( D(c\bar{u}/\bar{d}b) \) | 1.7285 |
| \( D(c\bar{s}) \) | 1.4642 |
| \( B(ab/\bar{d}b) \) | 1.5164 |
| \( B_s(ab) \) | 1.230 |
| \( B(bc) \) | 0.6978 |

### Table 3

| States | \( \varrho_0 n \) |
|--------|------------------|
| \( 1s \) (\( n = 1, l = 0 \)) | -2.3194 |
| \( 2s \) (\( n = 2, l = 0 \)) | -4.083 |
| \( 3s \) (\( n = 3, l = 0 \)) | -5.5182 |
| \( 4s \) (\( n = 4, l = 0 \)) | -6.782 |
which was based on QED analogy. The modification is, the wave-function at the origin with suitable a cut-off

\( r \)

condition develops a singularity. In the previous works, for a very short distance. When in Sakurai

\( r/a \)

tic correction to the wave-function was made by intro-

\[ A_i[\rho] = a_0 \left[ 1 + \frac{\rho^3}{3!} + \frac{\rho}{6!} + \frac{\rho^7}{9!} + \cdots \right] \]

\( b_0 \left[ \rho + \frac{\rho^4}{4!} + \frac{\rho^7}{7!} + \frac{\rho^{10}}{10!} + \cdots \right], \quad (14) \)

with, \( a_0 = 0.355 \ 028 \ 1 \) and \( b_0 = 0.258 \ 819 \ 4 \). Now, the normalization condition is

\[ \int_0^\infty 4\pi r^2 \psi^* \psi \, dr = 1. \]

(15)

Substituting the wave-function from Eq. (12) we get

\[ N = \frac{1}{\left( \int_0^\infty A_i^2[(2\mu dr)^{1/3}] - 2.3194 \right)^{1/2}}. \]

(16)

This can be easily calculated in Mathematica 7. The corresponding energies are\(^{[13-14]} \)

\[ E_n = -\left[ \frac{b^2}{2\mu} \right]^{1/3} \rho_0 n. \]

(17)

While dealing with the Airy function as the trial wave-function of variational method with the Cornell potential, the main problem is that the wave-function has got a singularity at \( r = 0 \). The presence of singularity in a wave-function is not new and in QED also such singularities appear.\(^{[11,15-16]} \) Therefore, to calculate the wave-function at the origin, we follow a method valid for S-wave as Ref. [17]. In this method, the wave-function at the origin is found from the condition, \( |\psi(0)|^2 = \mu/2\pi \langle dV/dr \rangle. \)

(17)

We find the variational parameter \( b' \) as Table 4.

| Mesons       | \( b' \) |
|--------------|---------|
| \( D(\bar{c}u/\bar{d}d) \) | 2.050   |
| \( D(\bar{c}s) \)      | 1.597   |
| \( B(\bar{c}b/\bar{d}b) \) | 1.7269  |
| \( B_s(\bar{c}b) \)       | 1.2709  |
| \( B(\bar{b}c) \)         | 0.585   |

It is worthwhile to mention that Airy function is an infinite series in itself, given as:

\[ A_i[\rho] = a_0 \left[ 1 + \frac{\rho^3}{3!} + \frac{\rho}{6!} + \frac{\rho^7}{9!} + \cdots \right] \]

\( -b_0 \left[ \rho + \frac{\rho^4}{4!} + \frac{\rho^7}{7!} + \frac{\rho^{10}}{10!} + \cdots \right], \quad (14) \)

To study Heavy-light meson in any potential model one needs to incorporate the relativistic effect in the light quark. In some of our previous works\(^{[1-5]} \) the relativistic correction to the wave-function was made by introducing a Dirac modification term \( (r/a_0)^{-\epsilon} \), where \( \epsilon = 1 - \sqrt{1 - (4\alpha_s/3)^2} \) is the Dirac modification factor and \( a_0 = 3/4\mu \alpha_s \). But, the problem with such regularization is that, for pseudo-scalar mesons, the masses loses positivity. The mass formula for pseudo-scalar mesons,

\[ M_P = m_Q + m - \frac{8\pi \alpha_s}{3m_Q m_Q} |\psi(0)|^2. \]

(20)

\[ D(\bar{c}u/\bar{d}d) \]

\( 0.089 \)

\( 0.0073 \)

\( D(\bar{c}s) \)

\( 0.037 \)

\( 0.0055 \)

\( B(\bar{c}b/\bar{d}b) \)

\( 0.0035 \)

\( 1.452 \times 10^{-9} \)

\( B_s(\bar{c}b) \)

\( 0.0033 \)

\( 1.638 \times 10^{-9} \)

\( B(\bar{b}c) \)

\( 0.002 \)

\( 3.872 \times 10^{-9} \)

Therefrom the positivity of mass we obtain the corresponding lower bound on the cut-off parameter \( r_0 \) as follows

\[ m_Q + m \geq \frac{8\pi \alpha_s}{3m_Q m_Q} |\psi(0)|^2. \]

(21)

From this inequality, with suitable cut-off \( r_0 \) in the airy trial wave-function, we calculate the lower bound on \( r_0 \) from positivity of mass, which are shown below for various Heavy Flavour mesons. As an illustration, experimental value of mass of the \( D \) meson (1.869 GeV) will yield a value of \( r_0 \sim 19 \text{ GeV}^{-1} \), which exceeds the size of meson itself. From the Table 4 it is clear that the regularization of wave-function with QED analogy as depicted in Refs. [2, 4] fails from the prospective of positivity of mass. The positivity of mass of pseudo-scalar mesons yields a regularization length which exceeds that given by QED analogy of H-atom. This is a distinctive feature of Heavy flavour mesons of QCD compared to H-atom of QED.

Therefore, the relativistic effect in the light quark is introduced as in Ref. [6]. In the works\(^{[7,10]} \) the light quark is considered relativistically with the Hamiltonian as \( H = M + p^2/2m + \sqrt{p^2 + m^2 + V(r)} \), where, \( M \) is the mass of Heavy quark and \( m \) is the mass of light quark. In the present work, we just use a simplified variant of Ref. [7] and take only first order corrections of the relativistic effect as \( E = m + p^2/2m \) in the light quark.

\[ M_P = M + m + \frac{p^2}{2m} + \Delta E. \]

(22)

\[ M_P = M + m + \frac{p^2}{2m} + \Delta E. \]

(22)

Here, \( M \) and \( m \) are the masses of Heavy quark/anti-quark and light quark/anti-quark respectively and we have considered the 1st order relativistic correction to the light quark/anti-quark. The energy shift of mass splitting due to spin interaction in the perturbation theory.
reads\textsuperscript{[1–2,6,16]} 
\[
\Delta E = \frac{32\pi\alpha_s}{9Mm} S_Q \cdot S_Q |\psi(0)|^2.
\] (23)

For pseudo-scalar mesons, \( S_Q \cdot S_Q = -3/4\), so pseudo-scalar meson masses can be expressed as
\[
M_P = M + m + \left( -\frac{\nabla^2}{2m} - \frac{8\pi\alpha_s}{3Mm} |\psi(0)|^2 \right).
\] (24)

Similarly, for vector mesons \( S_Q \cdot S_Q = 1/4\), so
\[
M_V = M + m + \left( -\frac{\nabla^2}{2m} + \frac{8\pi\alpha_s}{9Mm} |\psi(0)|^2 \right).
\] (25)

This particular aspect was overlooked in Refs.\textsuperscript{[2–5,13].}

2.6 Decay Constants of Heavy Flavoured Mesons

For pseudo-scalar mesons, the decay constant \( f_p \) is related to the ground state wave function at the origin \( \psi(0) \) according to the Van-Royen-Weisskopf formula,\textsuperscript{[19]} in the non-relativistic limit as,
\[
f_p = \sqrt{\frac{12|\psi(0)|^2}{M_p} C^2} .
\] (26)

where, \( M_p \) is the mass of pseudo-scalar meson. Now with QCD correction factor\textsuperscript{[2] it can be written as,
\[
f_p = \sqrt{\frac{12|\psi(0)|^2}{M_p} \tilde{C}^2} .
\] (27)

With,
\[
\tilde{C}^2 = 1 - \frac{\alpha_s}{\pi} \left[ 2 - \frac{m_Q - m_Q \ln m_Q}{m_Q} \right] .
\] (28)

Again, the ratios of pseudo-scalar decay constants e.g. for \( B_s \) and \( D_s \) meson can be expressed as
\[
\frac{f_{B_s}}{f_{D_s}} = \sqrt{\frac{M_{D_s}^2 \psi_{B_s}(0)}{M_{B_s}^2 \psi_{D_s}(0)}} .
\] (29)

2.7 Mass Difference of Vector and Pseudo-Scalar Mesons

The mass difference between the Pseudo-scalar and vector meson is given by\textsuperscript{[10,20]}
\[
M_{(QQ)}^* - M_{(QQ)} = \frac{8\pi A}{3m_Q m_Q} |\psi_{QQ}(0)|^2 ,
\] (30)

where \( m_Q \) is the mass of heavy quark and \( m_Q \) is the mass of antiquark. This is attributed to the hyperfine interaction and \( A = 4\alpha_s/3 \) where \( \alpha_s \) is the strong coupling constant.

2.8 Oscillation Frequency

It has been well established that the \( B_D \) and \( B_S \) meson mix with their antiparticles by means of Box diagram and involves exchange of \( W \) bosons and \( u, c, t \) quarks, which leads to oscillation between mass eigenstates.\textsuperscript{[21–23] The oscillation is parametrized by mixing mass parameter \( \Delta m \) given by,
\[
\Delta m_B = \frac{G_F^2 m_B^2 M_{BS} f_{BS}^2 g(x_t) \eta_t |V_{tb}^* V_{ts}|^2 B}{8\pi} ,
\] (31)

where, \( \eta_t \) is the gluonic correction to oscillation (= 0.55\textsuperscript{[24]} and \( B \) is the bag parameter (= 1.34\textsuperscript{[24]} and the parameter \( g(x_t) \) is given as,\textsuperscript{[25]}
\[
g(x_t) = \frac{1}{4} + \frac{9}{4(1-x_t)} - \frac{3}{2(1-x_t)^2} - \frac{3x_t^2}{2(1-x_t)^3} ,
\] (32)

and, \( x_t = m_t^2/M_B^2 \). From data of Particle data group,\textsuperscript{[26]}
\( m_t = 174 \text{ GeV}, M_{B_s} = 80.403 \text{ GeV}, |V_{tb}| = 1, |V_{td}| = 0.0074, |V_{ts}| = 0.04 \).

2.9 Leptonic Decay Width and Branching Ratio

It is also well known that charged mesons (\( \pi^\pm, K^\pm, D^\pm, D_s^\pm, B^\pm \)) can decay to a charged lepton pair, when they annihilate via a virtual \( W^\pm \) boson. Purely leptonic decays are rare, but there are clear experimental signatures because of the presence of highly energetic lepton in the final state. Absence of hadrons in the final state indicates that the theoretical predictions are very clean.

The partial decay width for the process is given by,\textsuperscript{[22]}
\[
\Gamma(P \rightarrow l\nu) = \frac{G_F^2 f_P^2 M_P m_l^2}{8\pi} \left( 1 - \frac{m_l^2}{M_P^2} \right)^2 |V_{ls}|^2 .
\] (33)

With the computed masses and decay constants, the leptonic decay widths for separate lepton channel \( m_l = \mu, \tau, e \) can be easily calculated. Here, \( V_{ls} \) is the CKM matrix element for quark flavours \( f \) and \( g \), also, \( G_F, P, f_P, M_P, m_l \) denote the Fermi constant, generic pseudo-scalar (PS) meson, PS-meson weak-decay constant, PS-meson mass, and lepton mass respectively. The branching ratio of heavy flavour mesons is calculated by using relation
\[
B = \tau_P \Gamma(P \rightarrow l\nu) .
\] (34)

Here, \( \tau_P \) is the lifetime of pseudo-scalar mesons. For calculation we take the world average values reported by particle data group\textsuperscript{[20]} as, \( \tau_D = 1.04 \text{ ps}, \tau_{D_s} = 0.5 \text{ ps}, \tau_B = 1.63 \text{ ps} \) and \( |V_{td}| = 0.230, |V_{ts}| = 1.023, |V_{tb}| = 3.89 \times 10^{-3} \).

3 Results

3.1 Masses

With the formalism developed in Sec. 2, we calculate the masses of some Heavy Flavour mesons, which are shown in Table 6. The input parameters are \( m_u/d = 0.336 \text{ GeV}, m_b = 4.95 \text{ GeV}, m_c = 1.55 \text{ GeV}, m_s = 0.483 \text{ GeV}, \) and \( b = 0.183 \text{ GeV}\textsuperscript{2,29} also we take \( \alpha_s = 0.39 \) for C-scale and \( \alpha_s = 0.22 \) for b-scale.\textsuperscript{[4] We make a comprehensive comparison of our results with lattice results,\textsuperscript{[27]} QCD sum rule,\textsuperscript{[28]} other models\textsuperscript{[4,9]} and present experimental results. Our result agrees well with the present results of lattice QCD, QCD sum rules and experimental data. The difference of the predictions of our
model with the more advanced approaches such as lattice QCD\cite{27} and QCD sum rules\cite{28} are insignificant. As an illustration, from Table 6, the predicted mass of D and B meson with Gaussian trial wave-function are (1.94 GeV and 5.35 GeV), which are quite close to the lattice results (1.885 GeV and 5.283 GeV) and QCD sum rule results (1.87 GeV and 5.283 GeV). The pattern is similar to other mesons. Similarly, the mass difference between the pseudo-scalar and the vector mesons are given in Table 9 and compared with lattice results. Here too, our results agree with lattice results.

### 3.2 Decay Constants

The decay constants of Heavy flavoured mesons are calculated using Eq. (27) and are shown in Table 7, where also comparison is made with the results of lattice QCD, QCD sum rules and experimental data. Here too, the agreement of the predictions of our model with a Gaussian trial wave-function with that of lattice QCD\cite{29–31} and QCD sum rules\cite{32} are good. As an illustration, the decay constant of B meson is (0.198 GeV), which is close to lattice result (0.218 GeV) and QCD sum rule result (0.193 GeV). Again in non relativistic case, \( f_B / f_D \approx \sqrt{M_D / M_B} = 0.59 \), but using relativistic correction to the light quark/anti-quark we get, \( f_B / f_D = 0.70 = 0.59 \times 1.18 \), i.e. \( f_B / f_D \) is enhanced by a factor 1.18. This is to be compared with Hwang’s work, where the factor is, 1.13, 1.31\cite{33} (\( f_B / f_D = 0.67 \) or 0.77). In Table 8 we show the ratios of decay constants and compared with the lattice results. Also the ratio \( \xi = (\sqrt{B_S} f_{B_S} / \sqrt{B_D} f_{B_D} \approx f_{B_S} / f_{B_D} = 1.045 \) which is well agreement with Ref. \cite{22}.

#### 3.3 Oscillation Frequencies

The mixing mass parameter \( \Delta m \), which is connected to the oscillation of neutral mesons is estimated and compared with the results of QCD sum rules, lattice QCD and experimental data, which is shown in Table 10. Here too, our predictions with a Gaussian trial wave-function is in good agreement with lattice results,\cite{33} QCD sum rule result\cite{34} and the experimental data.\cite{35–36}

#### 3.4 Branching Ratios

The Branching ratio of different decay channels in the leptonic decays of Heavy Flavour mesons are calculated and shown in Table 11 (Gaussian), Table 12 (Coulomb) and Table 13 (Airy). Comparison is also done with other model,\cite{37} and the experimental results.

### Table 6 Masses of Heavy Flavoured mesons (in GeV).

| Mesons | \( M_P \) (Gaussian) | \( M_P \) (Coulomb) | \( M_P \) (Airy) | Ref. \cite{16} | Ref. \cite{4} | Lattice\cite{27} | Q. sum rule\cite{28} | Exp. Mass |
|--------|---------------------|---------------------|-----------------|----------------|----------------|-------------------|------------------|----------|
| \( D(c\bar{u} /cd) \) | 1.94 | 1.606 | 1.28 | 1.972 | 2.378 | 1.885 | 1.87 | 1.869 ± 0.0016 |
| \( D(c\bar{s}) \) | 2.032 | 1.739 | 1.69 | 2.154 | 2.076 | 1.969 | 1.97 | 1.968 ± 0.0033 |
| \( B(\bar{u} /\bar{d}) \) | 5.35 | 5.11 | 5.14 | 5.314 | 5.798 | 5.283 | 5.28 | 5.279 ± 0.0017 |
| \( B_s(\bar{u} /\bar{d}) \) | 5.48 | 5.40 | 5.372 | 5.6 | 5.331 | 5.366 | 5.37 | 5.366 ± 0.0024 |
| \( B(\bar{b} /\bar{c}) \) | 6.4 | 6.38 | 6.5 | 6.8 | 6.278 | 6.277 ± 0.006 |

### Table 7 Decay constants of Heavy Flavoured mesons (GeV).

| Mesons | \( f_P \) (Gaussian) | \( f_P \) (Coulomb) | \( f_P \) (Airy) | QCD Sum rules\cite{32} | Lattice | Exp. value |
|--------|---------------------|---------------------|-----------------|---------------------|---------|------------|
| \( D(c\bar{u} /cd) \) | 0.282 | 0.377 | 0.801 | 0.206 ± 0.002 | 0.220 ± 0.003\cite{18} | 0.205 ± 0.85 ± 0.022\cite{35,38} |
| \( D(c\bar{s}) \) | 0.336 | 0.431 | 0.683 | 0.245 ± 0.015 | 0.258 ± 0.001\cite{31} | 0.254 ± 0.053\cite{35,38} |
| \( B(\bar{u} /\bar{d}) \) | 0.198 | 0.264 | 0.764 | 0.193 ± 0.012 | 0.218 ± 0.005\cite{18} | 0.198 ± 0.014\cite{39} |
| \( B_s(\bar{u} /\bar{d}) \) | 0.207 | 0.238 | 0.627 | 0.232 ± 0.018 | 0.228 ± 0.010\cite{30} | 0.237 ± 0.017\cite{39} |
| \( B(\bar{b} /\bar{c}) \) | 0.563 | 0.59 | 0.333 | 0.562\cite{24} |

### Table 8 Ratios of pseudoscalar decay constants.

| Ratios | \( f_{PE} / f_{T} \) (Gaussian) | \( f_{PE} / f_{T} \) (Coulomb) | \( f_{PE} / f_{T} \) (Airy) | \( f_{DE} / f_{T} \) (Gaussian) | \( f_{DE} / f_{T} \) (Coulomb) | \( f_{DE} / f_{T} \) (Airy) |
|--------|---------------------|---------------------|-----------------|---------------------|---------------------|-----------------|
| Our model | 1.045 | 0.90 | 0.821 | 1.19 | 1.14 | 0.857 |
| Lattice result | 1.16 ± 0.06\cite{40} | 1.188\cite{37} |
| QCD Sum rule | 1.20 ± 0.03\cite{41} | 1.17 ± 0.03\cite{41} |

### Table 9 Mass difference of pseudo-scalar and vector mesons (GeV).

| Mesons | \( M_{PS} – M_P \) (Gaussian) | \( M_{PS} – M_P \) (Coulomb) | \( M_{PS} – M_P \) (Airy) | \( M_{PS} – M_P \) (Lattice)\cite{38} |
|--------|---------------------|---------------------|-----------------|---------------------|
| \( D(c\bar{u} /cd) \) | 0.44 | 0.81 | 0.66 | 0.067 |
| \( D(c\bar{s}) \) | 0.312 | 0.28 | 0.46 | 0.066 |
| \( B(\bar{u} /\bar{d}) \) | 0.038 | 0.088 | 0.11 | 0.034 |
| \( B_s(\bar{u} /\bar{d}) \) | 0.026 | 0.16 | 0.081 | 0.027 |
| \( B(\bar{b} /\bar{c}) \) | 0.0305 | 0.0305 | 0.025 | |
Table 10 Mixing mass parameter (ps$^{-1}$).

| Meson | $\Delta m_R$ (Gaussian) | $\Delta m_R$ (Coulomb) | $\Delta m_R$ (Airy) | Sum rule | Lattice | Exp.value |
|-------|-------------------------|------------------------|---------------------|----------|---------|-----------|
| $B_D$ | 0.45                    | 0.78                   | 0.27                | 0.48$^{[34]}$ | 0.63$^{[33]}$ | 0.5$^{[36]}$ |
| $B_S$ | 16                      | 60                     | 9.3                 | > 14.6$^{[34]}$ | 19.6$^{[33]}$ | 17.76$^{[35]}$ |

Table 11 Branching ratio of Heavy Flavour mesons (with Gaussian).

| Mesons | $BR_r \times 10^{-4}$ | $BR_p \times 10^{-4}$ | $BR_{\pi} \times 10^{-8}$ |
|--------|------------------------|------------------------|--------------------------|
| $D$    | 2.08 (this work)       | 7.6 (this work)        | 1.82 (this work)         |
| Ref. $[37]$ | 0.9               | 6.6                   | 1.5                      |
| Expt.$^{[24]}$ | < 2.1         | 4.4 ± 0.7             | < 8.8                    |

| Mesons | $BR_r \times 10^{-2}$ | $BR_p \times 10^{-3}$ | $BR_{\pi} \times 10^{-7}$ |
|--------|------------------------|------------------------|--------------------------|
| $D$    | 6.94                   | 10                     | 0.0025                   |
| Ref. $[37]$ | 8.4               | 7.7                   | 1.8                      |
| Expt.$^{[24]}$ | 6.6 ± 0.6         | 6.2 ± 0.6             | < 1.2                    |

| Mesons | $BR_r \times 10^{-4}$ | $BR_p \times 10^{-6}$ | $BR_{\pi} \times 10^{-6}$ |
|--------|------------------------|------------------------|--------------------------|
| $B$    | 1.14                   | 0.489                  | 0.00014                  |
| Expt.$^{[26]}$ | 1.8               | < 1                   | < 1.9                    |

Table 12 Branching ratio of Heavy Flavour mesons (with Coulomb).

| Mesons | $BR_r \times 10^{-3}$ | $BR_p \times 10^{-4}$ | $BR_{\pi} \times 10^{-8}$ |
|--------|------------------------|------------------------|--------------------------|
| $D$    | 2.5                    | 7.6                    | 1.82                     |
| Ref. $[37]$ | 0.9               | 6.6                   | 1.5                      |
| Expt.$^{[24]}$ | < 2.1         | 4.4 ± 0.7             | < 8.8                    |

| Mesons | $BR_r \times 10^{-2}$ | $BR_p \times 10^{-3}$ | $BR_{\pi} \times 10^{-7}$ |
|--------|------------------------|------------------------|--------------------------|
| $D$    | 6.94                   | 10                     | 0.0025                   |
| Ref. $[37]$ | 8.4               | 7.7                   | 1.8                      |
| Expt.$^{[24]}$ | 6.6 ± 0.6         | 6.2 ± 0.6             | < 1.2                    |

| Mesons | $BR_r \times 10^{-4}$ | $BR_p \times 10^{-6}$ | $BR_{\pi} \times 10^{-6}$ |
|--------|------------------------|------------------------|--------------------------|
| $B$    | 1.14                   | 0.489                  | 0.00014                  |
| Expt.$^{[26]}$ | 1.8               | < 1                   | < 1.9                    |

Table 13 Branching ratio of Heavy Flavour mesons (with Airy).

| Mesons | $BR_r \times 10^{-3}$ | $BR_p \times 10^{-4}$ | $BR_{\pi} \times 10^{-8}$ |
|--------|------------------------|------------------------|--------------------------|
| $D$    | 7.7                    | 0.27                   | 61                       |
| Ref. $[37]$ | 0.9               | 6.6                   | 1.5                      |
| Expt.$^{[24]}$ | < 2.1         | 4.4 ± 0.7             | < 8.8                    |

| Mesons | $BR_r \times 10^{-2}$ | $BR_p \times 10^{-3}$ | $BR_{\pi} \times 10^{-7}$ |
|--------|------------------------|------------------------|--------------------------|
| $D$    | 70                     | 2.4                    | 0.55                     |
| Ref. $[37]$ | 8.4               | 7.7                   | 1.8                      |
| Expt.$^{[24]}$ | 6.6 ± 0.6         | 6.2 ± 0.6             | < 1.2                    |

| Mesons | $BR_r \times 10^{-4}$ | $BR_p \times 10^{-6}$ | $BR_{\pi} \times 10^{-6}$ |
|--------|------------------------|------------------------|--------------------------|
| $B$    | 3.26                   | 0.11                   | 0.0026                   |
| Expt.$^{[26]}$ | 1.8               | < 1                   | < 1.9                    |

From all the tables, it can be easily inferred that while exploring static properties of Heavy Flavour mesons variational approach with Gaussian trial wave-function is phenomenologically preferable.

4 Conclusion

We have investigated the static properties of heavy flavour mesons using variational method with Cornell potential in co-ordinate space with three different trial wave-functions. Specifically, we consider the trial wave-functions viz. Gaussian, Coulomb, and Airy function. Also, since the light quark of a Heavy-Light meson is relativistic, we have incorporated first order relativistic correction in a minimal way. The Gaussian wave-function appears to be better choice compared with Coulomb and Airy trial wave-functions. We have also compared our results with those of lattice QCD and the QCD sum rules. Our results with a Gaussian trial wave-function conform with the results of both of them. In a sense, it is an improvement over our previous perturbative approaches$^{[1−5,11,22]}$ where an arbitrary choice between the linear and the Coulomb part of Cornell potential as parent/perturbation is necessary.

To conclude, the variational method with a Gaussian trial wave-function provides a simple method to study the static and dynamic properties of pseudo-scalar mesons which are close to the corresponding results of the lattice QCD and QCD sum rules. Such effective H.O. wave function can presumably be generated by a QQ potential, which is polynomial in $r$, $V(r) = \sum_{n=0}^{\infty} a_n r^n$, with, $a_2 \gg a_{l,l'/2}$, a feature noticed by Godfrey and Isgur$^{[42]}$ as early as in 1980’s. The present analysis, therefore seems to indicate the relevance of such a simple model based on the Schrödinger equation as far as phenomenology is concerned, in spite of advanced tools like lattice QCD and QCD sum rules available in the current literature.

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