Abstract. We give examples of infinite type surfaces with end spaces that are not self-similar, but a unique maximal type of end, either a singleton or Cantor set.

1. Introduction

The paper [6] introduced the notion of self-similar end spaces for infinite type surfaces, and proved that a self-similar end space necessarily contains a unique maximal type of end, with the set of ends of this type either a singleton or a Cantor set. See Section 2 for the definition of the partial order on ends types. This notion has turned out to be a useful one and it has appeared many times in the literature, see for instance [1, 2, 3, 7]. A partial converse to this statement was proved in [6, Prop. 4.8]. Namely, under the additional hypothesis that an infinite type surface Σ contains no nondisplaceable subsurfaces, it was shown that the end space of Σ is self-similar if and only if the set of maximal ends is either a singleton or a Cantor set of points of the same type. However, the necessity of this extra hypothesis (no nondisplaceable subsurfaces) was not discussed there, raising the question of whether it could be eliminated. This appeared as Question 1.4 in [4], and Remark 6.2 in [5]. Here we answer the question, showing the strict converse (without extra assumptions) to [6, Prop. 4.8] is false:

Theorem 1.1. There exist examples of surfaces that have non self-similar end spaces with a unique maximal type of end and set of maximal ends homeomorphic to a singleton or to a Cantor set.

To prove this, we first provide details on a local construction of non-comparable points (an idea sketched loosely in [6]), and then use this to build examples first in the case where the maximal end is a singleton, followed by the Cantor set case.

2. Toolkit: Non-comparable points

For a surface Σ, we denote the space of ends of Σ by $E(\Sigma)$ or simply $E$. We define an equivalence relation on the end space by saying points $x, x' \in E$ are the same type if there exists some clopen neighborhood of $x$ in $E$ that is homeomorphic to some clopen neighborhood of $x'$. This is equivalent to saying that there is a homeomorphism of $\Sigma$ such the induced map on the end space sends $x$ to $x'$. For an end $x$, we let $E(x) \subset E$ denote the set of points of type $x$ and let Accu($x$) denote the set of accumulation points of $E(x)$.

In [6, Section 4], we defined a partial order on types of ends of a given end space. For $x, y \in E$, we say $y \preceq x$ if $x$ is an accumulation point of $E(y)$, that is, if $y \in \text{Accu}(x)$. While this is not necessarily a partial order on $E$, it is a partial order on the types of ends. We say $x$ is a maximal type if $x \preceq y$ implies $y \preceq x$. We denote the set of maximal points in $E$
by $\mathcal{M}(E)$. We say $x, y \in E$ are non-comparable if neither $x \lesssim y$ nor $y \lesssim x$ holds. See [6] for more details and discussion.

The first building block in our construction is a sequence of surfaces $D_n$ indexed by $n \in \mathbb{N}$, each with one boundary component, such that $D_n$ contains a unique maximal end $z_n$ and for all $i \neq j$ the ends $z_i$ and $z_j$ are non-comparable (the reader should picture $D_i$ and $D_j$ as disjoint subsurfaces of $\Sigma$).

Note that a construction such as this is not possible when the surface is planar and has a countable number of ends. A classical result of Mazurkiewicz and Sierpinski states that, for any surface with countable ends, there exists a countable ordinal $\alpha$ such that the end space $E$ is homeomorphic to the ordinal $\omega^\alpha \cdot m + 1$ where $m$ is a positive integer. The assumption that $E$ has one maximal point implies that $m = 1$. Now assume $D, D'$ are two genus zero surfaces with one boundary and a countable end space ($E$ and $E'$) such that that each end space has one maximal end ($x \in E$ and $x' \in E'$). Then their end spaces are respectively homeomorphic to $\omega^\alpha + 1$ and $\omega^{\alpha'} + 1$ for some countable ordinals $\alpha$ and $\alpha'$. Now if $\alpha \leq \alpha$ then $x \preceq x'$, which means $x$ and $x'$ are comparable.

We carry out the construction in both remaining cases, namely, when the set of ends is uncountable and the surface is planar (the proof easily generalizes to non-planar surfaces), and when the set of ends is countable and the surface is non-planar.

**Uncountable planar case.** Let $D$ be a disc, let $C_n = Q_n \cup C \subset D$ be the union of a countable set $Q_n$ and a Cantor set $C$, with Cantor-Bendixson rank $n$ such that, for each derived set of $C_n$ that has isolated points, the accumulation set of the isolated points contains the Cantor set. For example, one may take the $C$ to be the standard middle-thirds Cantor set, and insert in each missing interval a set homeomorphic to $\omega^n + 1$ to form $Q_n$. Now for each $C_n$, select a single point $z_n$ and let $C'_n$ be another Cantor set contained in $D$ so that $C_n \cap C'_n = \{z_n\}$. Puncturing $D$ along $C_n \cup C'_n$ gives a surface $D_n$ with one boundary component such that $z_n$ is the unique maximal end. By construction, $z_i$ and $z_j$ are non-comparable when $i \neq j$.

**Countable non-planar case.** Let $D$ be a disk and let $\alpha$ and $\beta$ be two countable ordinals with $\beta < \alpha$. Let $E_\alpha$ be a subset of $D$ homeomorphic to $\omega^\alpha + 1$ and denote its (unique) maximal point by $z_{\alpha,\beta}$. Now, consider a closed subset $E_\beta \subset E_\alpha$ homeomorphic to $\omega^\beta + 1$ where $z_{\alpha,\beta}$ is again the maximal point of $E_\beta$. For every isolated point $y$ of $E_\beta$ remove a disk around $y$ (keeping these disks pairwise disjoint) and glue back in a one-ended, infinite genus surface with one boundary component. We also puncture $D$ along the remaining points of $E_\alpha$ to obtain a surface $D_{\alpha,\beta}$. The point $z_{\alpha,\beta}$ is the unique maximal end of this surface. Moreover, for two pairs or countable ordinals $(\alpha, \beta)$ and $(\alpha', \beta')$ (satisfying $\beta < \alpha$ and $\beta' < \alpha'$) if $\alpha \geq \alpha'$ and $\beta < \beta'$ then $D_{\alpha,\beta}$ and $D_{\alpha',\beta'}$ are non-comparable. In fact, no end of $D_{\alpha,\beta}$ is of the same type as $z_{\alpha',\beta'}$ and vice versa. Hence we can, for example, fix $\alpha$ and vary $\beta$ to get a countable family of surfaces with one boundary where the maximal points are non-comparable.

**Uncountably many non-comparable points.** It is also possible for a surface to contain uncountably many non-comparable points. For example, let $\Sigma$ be a sphere minus a Cantor set. Visualize $\Sigma$ as a union of pairs of pants. Enumerate the pairs of pants, remove a disk from each pair of pants, and glue back in a copy of $D_n$ to the $n$-th pair of pants. Call the resulting surface $\Sigma'$. Then all the ends of $\Sigma'$ coming from $\Sigma$ are non-comparable since small enough neighborhoods of any two such ends contain non-comparable points.
3. Proof of the main theorem

Now we construct the surface that will furnish the examples needed for Theorem 1.1. We give the construction first for the case where $M(E)$ is a singleton. We then modify the construction to produce examples where $M(E)$ is a Cantor set.

Start with a flute surface, meaning a sphere punctured along a sequence of points $p_1, p_2, \ldots$ accumulating at an end $p_\infty$. For each $i \neq \infty$, replace a neighborhood of the puncture $p_i$ with a Cantor tree $T_i$. We think of $T_i$ as a union of pants surfaces, indexed by finite binary strings, so that the pants indexed by a string $s_1 \ldots s_n$ has cuffs glued to the pants indexed by $s_1 \ldots s_{n-1}$, $s_1 \ldots s_n 0$, and $s_1 \ldots s_n 1$, and the first pair of pants $P_\emptyset$ is glued on where the original puncture was removed. Now for each $i$, we will replace a countable set of discs in $T_i$ with discs homeomorphic to copies of the previously constructed discs $D_n$ (from either construction in the previous section), according to the following recipe.

For tree $T_i$, place copies of $D_1, D_2, \ldots D_i$ on the first pants surface, $P_\emptyset$, and place a copy of $D_k$ on each pants indexed by a word of length $k - i$. Thus, for each $k \geq i$ there are $2^{k-i}$ copies of $D_k$ on $T_i$. Call the resulting punctured surface $S$. An illustration is given in Figure 1.

Note that all of the ends of each of the trees $T_i$ are pairwise locally homeomorphic. The end $p_\infty$ of our surface $S$ is the unique accumulation point of these tree ends, so it is the unique maximal end. We will now show that the end space of $S$ is not self-similar. Let $E_i$ denote the end space of the tree $T_i$.

Consider the decomposition of the end space $E_1 \sqcup (E - E_1)$. Since $E_1$ does not contain a maximal end, to show the end space of the surface is not self-similar, it suffices to show that its complement contains no homeomorphic copy of $E_1$. Suppose for contradiction that we could find such. Note the the sets $U_i := \bigcup_{n=1}^{\infty} E_n$ form a neighborhood basis of $p_\infty$ in the end space. Since $p_\infty$ is the unique maximal end and $E_1$ is closed, any homeomorphic copy of $E_1$ must avoid some neighborhood of $p_\infty$ so is contained in a finite union $E_2 \cup E_3 \cup \ldots \cup E_N$.

By construction, $E_1$ contains $2^N$ locally homeomorphic copies of the end $z_{N+1}$. But $E_2 \cup E_3 \cup \ldots \cup E_N$ contains $\sum_{i=1}^{N-1} 2^i < 2^N$ copies of $z_{N+1}$. A contradiction. Thus, $E_1$ cannot be mapped into its complement, so the end space is not self-similar.

![Figure 1. Construction of the surface with unique maximal end.](image-url)
Cantor set case. A variation on the construction above can be used to produce a non self-similar surface with a unique maximal type and a Cantor set of maximal ends. First, following a similar procedure to the construction of the punctured trees $T_i$, for each $i \in \mathbb{N}$ we can build a Cantor tree $T'_i$ with a single boundary component that contains one copy of each of the discs $D_1, D_2, ... D_i$, and for each $k > i$ contains $2^{(2^k-i)}$ copies of $D_k$, with each end of the tree locally homeomorphic.

Now instead of starting with the flute, start with a Cantor tree constructed of pairs of pants indexed by binary strings, with the first pair of pants $P_\emptyset$ capped off by a disc on one of its boundary components. From each pair of pants indexed by a string of length $i$, remove a disc and glue in a copy of $T'_{i+1}$ to it along its boundary. In particular, $T'_1$ is glued to the first pair of pants indexed by the empty string. In the resulting surface, the ends of the original Cantor tree are precisely the maximal ends, forming a Cantor set of maximal ends of a single type. We claim again that this is not self-similar. To see this, let $E_1$ denote the end space of $T'_1$ and consider the decomposition of its end space into $E_1 \sqcup (E - E_1)$. Suppose for contradiction that $E - E_1$ contained a homeomorphic copy of $E_1$. As before, since $E_1$ and the set of maximal ends are both closed, the homeomorphic image of $E_1$ avoids some neighborhood of the maximal ends, so is contained in a union of end spaces of trees homeomorphic to $T'_i$ for a bounded set of indices $i$. We consider the maximal such index $N$, and again count copies of ends of type $z_{N+1}$. Without loss of generality, we may take $N \geq 4$. The set $E_1$ contains $2^{(2^{N+1})}$ copies of $z_{N+1}$. Since our surface is constructed using $2^k$ copies of each tree $T'_k$, the number of copies of $z_{N+1}$ in the union of all trees $T'_k$ for $2 \leq k \leq N$ is equal to

$$2 \cdot 2^{(2^{N-1})} + 2^2 \cdot 2^{(2^{N-2})} + \ldots + 2^{N-1} \cdot 2^{(2^{N-N+1})}$$

Set $k = 2^{N-1} + 1$. Then this sum is bounded above by

$$2^k + 2^{k-1} + \ldots + 2^{k-N+1} < 2^{k+1} < 2^{(2^N)}$$

which gives the desired contradiction.

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