Abstract

In high-dimensional data setting, the sample covariance matrix is singular. In order to get a numerical stable and positive definite modification of the sample covariance matrix in the high-dimensional data setting, we in this paper consider the condition number constrained covariance matrix approximation problem and present its explicit solution with respect to Frobenius norm. The condition number constraint guarantees the numerical stability and positive definiteness of the approximation form simultaneously. By exploiting the special structure of the data matrix in high-dimensional data setting, we also propose some new algorithms based on efficient matrix decomposition techniques. Numerical experiments are also given to show the computational efficiency of the proposed algorithms.

Keywords: covariance matrix, condition number constraint, singular value decomposition, high-dimensional data

MSC[2010]: 65F35, 15A12, 15A60

*The work is supported by a project of Shandong Province Higher Educational Science and Technology Program (Grant No. J17KA160) and the author is partly supported by the Natural Science Foundation of Shandong Province (Grant No. ZR2019MA002).

†Corresponding author. Email addresses: shxwang@qfnu.edu.cn or shwangmy@163.com
1 Introduction

In multivariate statistical analysis, the covariance matrix is an fundamental component of many statistical models, such as linear (quadratic) discriminant analysis, principal component analysis and canonical correlation analysis [1]. The covariance matrix also finds its popularity in many other applied disciplines. Examples includes classification of gene expression [20], machine learning [7], portfolio management [25, 22] and so on. However, in the most of practical applications, the underlying true covariance matrix is never to be known and usually estimated from the given data. Thus many covariance matrix estimation procedures have been developed under various data settings, and the interested readers should be referred to [7, 22, 35, 24, 34] and the references therein.

The Bag Data era brings much more complex data settings and challenges for covariance matrix approximation. Especially, the high-dimensional data analysis, where the number of variables is much larger than the observations, is a very active topic in many scientific researches. In such data setting, the sample covariance matrix can not be a “good” substitute of the covariance matrix. To show this, let $x_1, \cdots, x_n$ be $n$ observations of the random vector $x \in \mathbb{R}^p$ from a multivariate normal distribution $\mathcal{N}_p(0, \Sigma)$. Though the sample covariance matrix

$$S_n = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T = \frac{1}{n} X_n X_n^T \quad \text{with} \quad X_n = [x_1, \cdots, x_n]$$

(1.1)

maximizes the likelihood function and is an asymptotically unbiased estimate of $\Sigma$, the dispersion property makes it have poor performance in approximating the eigenstructure of $\Sigma$, especially when $p$ is close to or larger than $n$ [22, 4]. We generate 100 groups of data from $\mathcal{N}(0, I_p)$, and plot the mean of the $i$-th largest eigenvalues of $S_n$ ($i = 1, \cdots, p$) in Figure 1. Since we set $\Sigma = I_p$, the true eigenvalues of $\Sigma$ are equal to 1 and $\Sigma$ is also well-conditioned. Figure 1 shows that the larger sample eigenvalues are biased upwards, whereas the smaller ones downwards, which makes $S_n$ ill-conditioned when $p \gtrsim n$, and singular when $p > n$.

To get a suitable approximation or estimation of $\Sigma$ from such data setting, a natural idea is to remove or decrease the influence of the dispersion property.

To approximate $\Sigma$ in high-dimensional data setting, numerous statistical or numerical methods have been proposed with respect to different criterions (cf.[13, 19]). In statistics,
Figure 1: The distribution of eigenvalues of $S_n = \frac{1}{n} \sum_{i=1}^{n} x_i x_i^T$ with $x_i \sim \mathcal{N}(0, I_p)$ and $n = 100$.

the main research interest on high-dimensional covariance matrix estimation is to establish the statistical consistency, which relies on the assumption of covariance structure and the distribution of random variables [6, 5, 21, 27]. However, the computational efficient thresholding techniques may lead the estimates to be indefinite [6, 27]. Therefore, some shrinkage and constraint methods are given to ensure the positive definiteness of the estimates [35, 24, 22, 23, 19]. Considering numerical stability, Won et al. [34] proposed the condition number constrained maximum likelihood estimator of covariance matrix, which enjoys the statistical consistency property under some strong assumptions [16]. The idea was extended to the maximum likelihood estimation of structured covariance matrix [3] and precision matrix [16]. The introduction of condition number constraint can not only guarantee the positive definiteness of covariance matrix but also its numerical stability.

The covariance structure and distribution assumptions may be restrictive in applications. Under different loss functions or transformations, some optimal matrix approximation techniques are proposed [18, 11, 10, 13] to obtain positive semidefinite covariance matrix approximations. Considering positive definiteness and numerical stability, Tanaka and Nakata [28] considered how to obtain a positive definite covariance matrix from a given symmetric data matrix, which depends on the full eigenvalue decomposition of the data matrix and may be time-consuming. In this paper, we will reconsider the condition number constrained covariance matrix approximation problem, but limited to the high-dimension data setting. In Section 2, a reformulation of condition number constrained covariance matrix approximation problem will be given, and we also give a comparison with the existing works. Our main
results are given in Section 3, in which we present the explicit expression of the solution to condition number constrained covariance matrix approximation problem and propose some new and efficient algorithms. To show the efficiency of our results, the numerical experiments are performed in Section 4. Section 5 contains the concluding remark of the whole paper.

2 Problem reformulation

The main idea to overcome the dispersion property of sample eigenvalue is to pull the extreme sample eigenvalues back to some target, so some shrinkage methods are proposed [22, 23, 19]. However, we note that when \( p \) is much larger than \( n \), the linear shrinkage techniques may give unreliable approximation of covariance matrix [32, 22, 23]. To remove the influence of extreme sample eigenvalues, a more straightforward method is to directly bound the eigenvalues or condition number of covariance matrix, which has been widely used in high-dimensional covariance matrix estimation with respect to different loss functions [35, 34, 24, 16].

In this work, we also confine ourself to the high-dimensional data setting, but we will reconsider the following condition number constrained covariance matrix approximation (C3MA) problem that can be stated as follows

\[
\hat{\Sigma} = \arg\min_{\Sigma \in \mathbb{P}_+^p} \| \Sigma - S_n \|_F , \quad \text{subject to} \quad \kappa(\Sigma) \leq \kappa_n ,
\]

(2.1)

where \( \| \cdot \|_F \) denotes the Frobenius norm, \( \kappa(\Sigma) = \lambda_1(\Sigma)/\lambda_p(\Sigma) \) is the condition number of \( \Sigma \) with \( \lambda_1 \) and \( \lambda_p \) being the maximum and minimum eigenvalues of \( \Sigma \), and \( \kappa_n \) is a finite positive real number larger than 1. Since the constraint guarantees the numerical stability and positive definiteness of \( \hat{\Sigma} \) simultaneously and the rank of \( S_n \) is no larger than \( n \) according to (1.1), \( S_n \) is singular and can not be the feasible solution to (2.1) in high-dimensional data setting. Moreover, it is easy to check that the C3MA problem (2.1) is convex and its proof can be similarly derived as [29].

2.1 Related works and useful result

The idea of using condition number constraint to ensure the numerical stability of covariance matrix may be first used in maximum likelihood estimation of covariance matrix with appli-
cations in portfolio management and radar signal processing [33, 34, 3]. Without considering the distribution of random variable, some scholars considered condition number constrained covariance matrix approximation problem. For example, Tanaka and Nakata considered the positive definite matrix approximation with condition number constraint [28]. Then the result was extended to condition number constrained non-square matrix approximation problem arising from communication systems [30]. There are also some other related works on various matrix estimation problem with condition number constraint, and interested readers are referred to [16, 29].

We need to point out that the most relevant work is given by Tanaka and Nakata [28], in which model (2.1) was discussed with the norm to be unitary invariant and $S_n$ being only symmetric. The following result was given in [28], which will be useful to characterize the solution of C3MA problem (2.1).

**Lemma 2.1.** Let the norm in (2.1) be unitary invariant and $S_n$ a symmetric matrix. Given the eigenvalue decomposition $S_n = U\hat{\Lambda}U^T$ with $\hat{\Lambda} = \text{Diag}(\hat{\lambda}_1, \cdots, \hat{\lambda}_p)$ and $\hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_p$, if $\Sigma = 0$ is not the feasible solution, then for any optimal solution $\Lambda^*$ to the following problem

$$\Lambda^* = \arg\min_{\Lambda = \text{Diag}(\lambda_1, \cdots, \lambda_p)} \|\Lambda - \hat{\Lambda}\|, \quad \text{subject to } \min \lambda_i \geq 0 \text{ and } \max \lambda_i \leq \kappa_n \min \lambda_i,$$

(2.2)

$\hat{\Sigma} = U\Lambda^*U^T$ is the optimal solution to problem (2.1).

**Remark 2.1.** With Lemma 2.1, Tanaka and Nakata showed that when the Ky Fan $p-k$ norm [26, Ch. 3] is used, the solution to (2.2) can be further simplified, and they also presented its analytical solutions with respect to spectral and trace norms [28]. Tanaka and Nakata suggested to use the full eigenvalue decomposition of $S_n$ to solve (2.1). But, this can be infeasible in high-dimensional data setting due to its heavy computational burden. We also find that in high-dimensional data setting the solution to (2.1) is more depending on $\kappa_n$, actually the inequality constraint becomes the equality constraint. The high-dimensional data setting impart $S_n$ a very spiked rank structure and it has at most $n$ nonzero eigenvalues that much smaller than $p$. Thus it should be of interest to design some efficient numerical methods to solve (2.1) by extensively exploiting the special structure of $S_n$.  

5
3 Main results

3.1 The solution of the C3MA problem

In the high-dimensional data setting, $p$ is much larger than $n$, which makes $S_n$ can not be the feasible solution of the C3MA problem (2.1). Taking this special structure of $S_n$ into consideration, we present the solution to (2.1) in the following Theorem 3.1 and its proof is given in Appendix. Without loss of generality, $X_n$ is assumed to be full-column rank and thus $\text{rank}(S_n) = n$.

**Theorem 3.1.** Let $S_n = U\hat{\Lambda}U^T$ be the eigenvalue decomposition of $S_n$ with $\hat{\Lambda} = \text{Diag}(\hat{\lambda}_1, \cdots, \hat{\lambda}_p)$ and $\hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_n > \hat{\lambda}_{n+1} = \cdots = \hat{\lambda}_p = 0$. Then the solution to (2.1) is given by

$$\hat{\Sigma} = U\Lambda^*U^T,$$

where $\Lambda^*$ is given as follows: **Case 1.** when $\hat{\lambda}_1/\hat{\lambda}_n \leq \kappa_n$,

$$\Lambda^* = \text{Diag}\left(\tilde{\Lambda}_1^*, \tilde{\Lambda}_2^*\right),$$

with $\tilde{\Lambda}_1^* = \text{Diag}(\hat{\lambda}_1, \cdots, \hat{\lambda}_n)$, $\tilde{\Lambda}_2^* = \mu^*I_{p-n}$ and $\mu^* = \lambda_1/\kappa_n$; **Case 2.** when $\hat{\lambda}_1/\hat{\lambda}_n > \kappa_n$,

$$\Lambda^*(\mu^*) = \text{Diag}\left(\tilde{\Lambda}_1^*(\mu^*), \tilde{\Lambda}_2^*(\mu^*)\right),$$

where $\tilde{\Lambda}_1^*(\mu^*) = \text{Diag}(\kappa_n\mu^*, \cdots, \kappa_n\mu^*, \hat{\lambda}_{\alpha^*+1}, \cdots, \hat{\lambda}_{\beta^*+1}, \mu^*, \cdots, \mu^*)$, $\tilde{\Lambda}_2^*(\mu) = \mu^*I_{p-n}$,

$$\mu^* = \frac{\kappa_n \sum_{i=1}^{\alpha^*} \hat{\lambda}_i + \sum_{i=\beta^*}^{n} \hat{\lambda}_i}{\alpha^*\kappa_n^2 + p - \beta^* + 1},$$

and the optimal $\alpha^*$ and $\beta^*$ can be determined with about $O(n)$ flops.

**Remark 3.1.** In [28], the authors only gave the analytical solution of (2.2) with respect to spectral norm and trace norm. Here, with respect to Frobenius norm, we present the solution of C3MA problem (2.1) in high-dimensional data setting. When $n \geq p$, $S_n$ may have $p$ nonzero eigenvalues and we denote it by $\hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_p > 0$. It can be deduced that, in this case, the $\Lambda_2^*$ in Theorem 3.1 will be removed, and the solution to (2.1) is given by

$$\hat{\Sigma} = U\Lambda^*U^T,$$
where $\Lambda^*$ is given as follows: 

Case 1. when $\hat{\lambda}_1 / \hat{\lambda}_p \leq \kappa_n$, $\Lambda^* = \tilde{\Lambda}^*$, where $\tilde{\Lambda}^* = \text{Diag}(\hat{\lambda}_1, \ldots, \hat{\lambda}_p)$;

Case 2. when $\hat{\lambda}_1 / \hat{\lambda}_p > \kappa_n$, $\Lambda^*(\mu^*) = \tilde{\Lambda}^*(\mu^*)$, where $\mu^* = \frac{\kappa_n \sum_{i=1}^{n} \hat{\lambda}_i + \sum_{i=m}^{p} \hat{\lambda}_i}{\alpha \kappa_n + p - \beta^* + 1}$, and the optimal $\alpha^*$ and $\beta^*$ can be determined with about $O(p)$ flops.

A similar result can also be derived from [28, Theorem 2] by substituting the Ky Fan $p-k$ norm with the Frobenius norm. We can note that when $n \geq p$ the condition number of $\hat{\Sigma}$, i.e. $\kappa(\hat{\Sigma})$, can be smaller than $\kappa_n$, whereas in high-dimensional data setting $\kappa(\hat{\Sigma})$ is always equal to $\kappa_n$.

### 3.2 Algorithm

According to (1.1), the sample covariance matrix $S_n$ is positive semidefinite in high-dimensional data setting. Suppose that $\text{rank}(S_n) = n$. Then the eigenvalue decomposition of $S_n$ is given by

$$S_n = U \begin{bmatrix} \hat{\Lambda}_1 & 0 \\ 0 & 0 \end{bmatrix} U^T,$$

where $\hat{\Lambda}_1 = \text{Diag}(\hat{\lambda}_1, \hat{\lambda}_2, \ldots, \hat{\lambda}_n)$ with $\hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_n > 0$, $U \in \mathbb{R}^{p \times p}$, and $UU^T = I_p$.

Since $\text{rank}(S_n) = \text{rank}(X_n)$, the singular value decomposition (SVD) of $1/\sqrt{n}X_n$ can be written as follows

$$\frac{1}{\sqrt{n}}X_n = U \hat{\Delta} V^T,$$

where $V \in \mathbb{R}^{n \times n}$, $VV^T = I_n$, and $\hat{\Delta} = \begin{bmatrix} \hat{\Delta}_1 & 0 \\ 0 & 0 \end{bmatrix}$ with $\hat{\Delta}_1 = \text{Diag}(\hat{\delta}_1, \hat{\delta}_2, \ldots, \hat{\delta}_n) \in \mathbb{R}^{n \times n}$ and $\hat{\delta}_1 \geq \hat{\delta}_2 \geq \cdots \geq \hat{\delta}_n > 0$ [15, Ch. 8]. Then

$$S_n = \frac{1}{n}X_nX_n^T = U \begin{bmatrix} \hat{\Delta}_1 & 0 \\ 0 & 0 \end{bmatrix} V^T V \begin{bmatrix} \hat{\Delta}_1 & 0 \\ 0 & 0 \end{bmatrix} U^T = U \begin{bmatrix} \hat{\Delta}_1^2 & 0 \\ 0 & 0 \end{bmatrix} U^T,$$

which is exactly the eigenvalue decomposition (3.1) with $\hat{\Lambda}_1 = \hat{\Delta}_1^2$.

We note that in [28] the full eigenvalue decomposition was suggested to find the optimal solution of the C3MA problem (2.1). However, in high-dimensional data setting the computation of full eigenvalue decomposition of $S_n$ may be very time-consuming, which requires
about $O(9p^3)$ flops when the symmetric QR algorithm is used. The preceding paragraph shows that if the left singular vectors $U$ and singular values $\hat{\Delta}_1$ of $1/\sqrt{n}X_n$ are available, the optimal solution of (2.1) can also be derived. Besides, if the SVD is employed, its computational complexity is about $O(4p^2n - 8pn^2)$ when $U$ and $\hat{\Delta}$ are needed and computed by Golub-Reinsch SVD (GR-SVD) algorithm [15]. When $p$ is much larger than $n$, Chan showed that the computational efficiency of GR-SVD can be further improved by combining the QR factorization and GR-SVD algorithm, which was called the MOD-SVD [8]. Thus, with respect to computational complexity, using SVD can lead to a significant reduction of computational burden compared with the full eigenvalue decomposition in high-dimensional data setting.

Therefore, we employ the MOD-SVD to compute the left singular vectors $U$ and singular values $\hat{\Delta}_1$ of $1/\sqrt{n}X_n$. Specifically, we first compute the QR factorization of $1/\sqrt{n}X_n$ with Householder transformation

$$\frac{1}{\sqrt{n}}X_n = Q \begin{bmatrix} R \\ 0 \end{bmatrix},$$

where $Q \in \mathbb{R}^{p \times p}$, $QQ^T = I_p$ and $R \in \mathbb{R}^{n \times n}$ is an upper triangular matrix, and then the SVD of $R$ is computed by GR-SVD algorithm

$$R = U_1 \hat{\Delta}_1 V^T,$$

where $U_1 \in \mathbb{R}^{n \times n}$ and $U_1 U_1^T = I_n$. Finally, the SVD of $1/\sqrt{n}X_n$ is given by

$$\frac{1}{\sqrt{n}}X_n = Q \begin{bmatrix} U_1 & 0 \\ 0 & I_{p-n} \end{bmatrix} \begin{bmatrix} \hat{\Delta}_1 \\ 0 \end{bmatrix} V^T.$$

A detailed comparison of computational complexity and storage requirement between GR-SVD and MOD-SVD can be found in [8]. We would like to apply these two different algorithms to compute the SVD of the factor $1/\sqrt{n}X_n$ in the present paper, and a numerical comparison will also be given in the numerical experiment part.

Summarizing the above discussion, we present the following three algorithms for solving the C3MA problem (2.1). The first one is based on full eigenvalue (FU-EIG) decomposition of $S_n$, which was suggested in [28]. The other two are based on the GR-SVD and MOD-SVD, respectively.
Algorithm 1 FU-EIG based solver for the C3MA problem

**Input:** upper bound of the condition number $\kappa_n$, and data matrix $X_n$.

**Output:** optimal approximation $\hat{\Sigma}$.

1. compute $S_n = \frac{1}{n} X_n X_n^T$.

2. compute the full eigenvalue decomposition $S_n$,

   $$S_n = U \hat{\Lambda} U^T,$$

   and output $U$ and $\hat{\Lambda}$.

3. find the optimal $u^*$ with $\hat{\Lambda}$, and give the optimal $\Lambda^*$ similar to Remark 3.1.

4. construct $\hat{\Sigma}$ by

   $$\hat{\Sigma} = U \Lambda^* U^T.$$

Algorithm 2 GR-SVD based solver for the C3MA problem

**Input:** upper bound of the condition number $\kappa_n$, and data matrix $X_n$.

**Output:** optimal approximation $\hat{\Sigma}$.

1. compute the SVD of $1/\sqrt{n} X_n$,

   $$\frac{1}{\sqrt{n}} X_n = U \begin{bmatrix} \hat{\Delta}_1 \\ 0 \end{bmatrix} V^T,$$

   and only output $U$ and $\hat{\Delta}_1$.

2. find the optimal $u^*$ with $\hat{\Delta}_1^2$, and give the optimal $\Lambda^*$ by Theorem 3.1.

3. construct $\hat{\Sigma}$ by

   $$\hat{\Sigma} = U \Lambda^* U^T.$$
Algorithm 3 MOD-SVD based solver for the C3MA problem

**Input:** upper bound of the condition number $\kappa_n$, and data matrix $X_n$.

**Output:** optimal approximation $\hat{\Sigma}$.

1. compute the QR factorization of $1/\sqrt{n}X_n$,

$$\frac{1}{\sqrt{n}}X_n = Q \begin{bmatrix} R \\ 0 \end{bmatrix},$$

and output $Q$ and $R$.

2. compute the SVD of $R$,

$$R = U_1 \hat{\Delta}_1 V^T,$$

and only output $U_1$ and $\hat{\Delta}_1$.

3. find the optimal $u^*$ with $\hat{\Delta}_1^2$, and give the optimal $\Lambda^*$ by Theorem 3.1.

4. construct $\hat{\Sigma}$ by

$$\hat{\Sigma} = Q \begin{bmatrix} U_1 & 0 \\ 0 & I_{p-n} \end{bmatrix} \Lambda^* \begin{bmatrix} U_1^T & 0 \\ 0 & I_{p-n} \end{bmatrix} Q^T.$$
Remark 3.2. The special structure of $S_n$ makes it possible to use SVD to reduce the computational burden of solving C3MA problem (2.1). The idea can not be directly applied to the situation that $S_n$ is only symmetric, which was considered in [28]. However, if some priori information on the low-rank structure of $S_n$ is available, we may use fast sampling techniques to construct its low-rank approximations and then use the proposed method. The low-rank and symmetric $S_n$ often arises in high-dimensional robust statistics, we leave its efficient approximation as the near future work. For the computation of SVD, Chan [8] showed that when $p \gtrsim 2n$ the MOD-SVD can achieve as much as 50 percent savings compared with GR-SVD algorithm, and this has been incorporated into LINPACK [2]. Based on the LINPACK routines, the MATLAB function `svd()` was built to compute the SVD of a matrix. However, as an interesting finding (presented in Section 4), we note that when the data matrix is large, the MOD-SVD should be introduced earlier, that is, when $p/n > 1$ the MOD-SVD should also be used, not necessarily until $p/n \approx 2$. We will show this in our numerical experiment.

3.3 Further discussions on $\kappa_n$

Note that in Theorem 3.1 $\mu$ and $\kappa_n$ are used to truncate the extreme eigenvalues of $S_n$, and $\mu$ is a function of $\kappa_n$. It may be of interest to study how $\kappa_n$ influences the truncation process, which was originally discussed on condition number constrained maximum likelihood estimation of covariance matrix [34]. An interesting finding is that under our loss function (2.1) the monotone property given in [34, Proposition 1] will not hold. Specifically, if we define the univariate functions $\mu = \mu(\kappa_n)$ and $\nu = \nu(\kappa_n) = \kappa_n \mu$, then both $\mu$ and $\nu$ are not the monotone functions of $\kappa_n$ any more as $\kappa_n$ increases from 1 to infinity. To illustrate this, we consider the following special case. For a given $\kappa_{n_0}$, let $\alpha$ be the largest index such that $\hat{\lambda}_\alpha > \nu(\kappa_{n_0})$, $\beta$ the smallest index such that $\mu(\kappa_{n_0}) > \hat{\lambda}_\beta$, and the following inequalities hold

$$\hat{\lambda}_\alpha > \nu*(\kappa_{n_0}) > \hat{\lambda}_{\alpha+1} \quad \text{and} \quad \hat{\lambda}_{\beta-1} > \mu*(\kappa_{n_0}) > \hat{\lambda}_\beta.$$ 

Then we can find some $\kappa_n$ in a small neighbourhood of $\kappa_{n_0}$ such that

$$\hat{\lambda}_\alpha > \nu*(\kappa_n) > \hat{\lambda}_{\alpha+1} \quad \text{and} \quad \hat{\lambda}_{\beta-1} > \mu*(\kappa_n) > \hat{\lambda}_\beta.$$
With basic calculus techniques, it can be check that both
\[
\mu(\kappa_n) = \frac{\kappa_n \sum_{i=1}^{\alpha} \hat{\lambda}_i + \sum_{i=\beta}^{n} \hat{\lambda}_i}{\alpha \kappa_n^2 + p - \beta + 1}
\]
and \(\nu(\kappa_n) = \frac{\kappa_n^2 \sum_{i=1}^{\alpha} \hat{\lambda}_i + \kappa_n \sum_{i=\beta}^{n} \hat{\lambda}_i}{\alpha \kappa_n^2 + p - \beta + 1}\)
are not the monotone functions of \(\kappa_n \in [1, \infty)\), and its maximizers are given by
\[
\kappa_\mu = \max \left\{ \sqrt{\left( \frac{\sum_{i=\beta}^{n} \hat{\lambda}_i}{\sum_{i=1}^{\alpha} \hat{\lambda}_i} \right)^2 + \frac{p - \beta + 1}{\alpha}} - \frac{\sum_{i=\beta}^{n} \hat{\lambda}_i}{\sum_{i=1}^{\alpha} \hat{\lambda}_i}, 1 \right\},
\]
and
\[
\kappa_\nu = \frac{p - \beta + 1}{\alpha} \frac{\sum_{i=1}^{\alpha} \hat{\lambda}_i}{\sum_{i=\beta}^{n} \hat{\lambda}_i} + \sqrt{\left( \frac{p - \beta + 1}{\alpha} \frac{\sum_{i=1}^{\alpha} \hat{\lambda}_i}{\sum_{i=\beta}^{n} \hat{\lambda}_i} \right)^2 + \frac{p - \beta + 1}{\alpha}},
\]
respectively. Thus we can not determine the monotone relationship between \(\mu(\kappa_n)\) and \(\nu(\kappa_n)\), and also \(\nu(\kappa_n)\) and \(\nu(\kappa_n)\). However, we note that if \(\gamma_1 = \sum_{i=1}^{\alpha} \hat{\lambda}_i/\sum_{i=\beta}^{n} \hat{\lambda}_i\) is large and also much larger than \(\gamma_2 = (p - \beta + 1)/\alpha\), which often empirically holds in high-dimensional data setting according to our numerical experiments, then we have \(\kappa_\mu = O(\sqrt{\gamma_2})\) and \(\kappa_\nu = O(\gamma_1 \gamma_2)\), and for the \(\kappa_n \geq \kappa_{n_0}\) in \([\kappa_\mu, \kappa_\nu]\) the following inequalities hold
\[
\mu(\kappa_{n_0}) \geq \mu(\kappa_n) \quad \text{and} \quad \nu(\kappa_{n_0}) \leq \nu(\kappa_n).
\]
The above inequalities show that the monotone properties corresponding to [34, Proposition 1] may be established in a limited interval. But, strictly speaking, a theoretical justification of the assumptions on the truncated eigenvalues of \(S_n\) is much more difficult, and the above discussion can only be taken as an illustration to show the possibility of establishing monotone properties of \(\mu\) and \(\nu\).

Comparing Theorem 3.1 with Remark 1, we can find that in high-dimensional data setting the condition number of \(\hat{\Sigma}\) actually equals to \(\kappa_n\), whereas \(\kappa(\hat{\Sigma})\) may smaller than \(\kappa_n\) when \(p \leq n\). Thus it is more crucial to determine \(\kappa_n\) in high-dimensional data setting. The determination of \(\kappa_n\) usually stems from the following two aspects. Firstly, the main purpose of introducing condition number constraint is to guarantee the numerical stability, and this can be achieved by giving a user-chosen \(\kappa_n\). For example, consider the following classical Markowitz mean-variance model
\[
\min_{x \in \mathbb{R}^p} \frac{1}{2} x^T \Sigma x - u^T x \quad \text{subject to} \ 1^T x = 1,
\]
and its rigorous or first order relative perturbation bounds can be unified as the following form

$$\frac{\|\Delta x\|_2}{\|x\|_2} \leq \kappa(\Sigma) f(\|\Delta \Sigma\|, \|\Delta u\|),$$

where $f$ is a function of perturbations $\Delta \Sigma$ and $\Delta u$ to covariance matrix $\Sigma$ and mean vector $u$, respectively [32, Section 3]. The perturbation bounds show that given the amount of perturbations $\|\Delta \Sigma\|$ and $\|\Delta u\|$ a small relative solution error may be obtained by controlling $\kappa_n$, and thus $\kappa_n$ can be determined by perturbation analysis of the applied models. Secondly, the data-driven methods seem to be more natural to select $\kappa_n$. The high-dimensional data setting and dispersion property makes the extreme eigenvalues of $S_n$ can not be used to determine $\kappa_n$ for $\hat{\lambda}_p = 0$. Inspired by Section 3.2, a natural question is whether we can use the extreme singular values of $1/\sqrt{n}X_n$ to determine $\kappa_n$? The answer is still pessimistic, when $p$ is much larger than $n$. Let the elements of $X_n$ be independent standard normal random variables, then the expectation of the smallest singular value $\hat{\delta}_n$ of $1/\sqrt{n}X_n$ satisfies $E\hat{\delta}_n \geq \sqrt{p/n} - 1$ [9], which shows that it is the ratio $p/n$ that controls the smallest singular value of $1/\sqrt{n}X_n$, even for well-conditioned underlying covariance matrix. Some statistical methodologies were proposed to estimate the spectrum of random matrices, but these methods can not give reliable estimates of extreme eigenvalues in high-dimensional data setting [12, 17]. Thus some theoretical guaranteed data-driven procedures for selecting $\kappa_n$ in the high-dimensional data setting are still in need and should be treated as future research work.

**Remark 3.3.** The discussion on monotone properties of truncation function is merely for a comparison with the results given in [34], and should not be taken as rigorous proof of establishing monotone properties of truncation functions on limited interval. The user-chosen $\kappa_n$ may look like too artificial, but this is vary common in high-dimensional covariance matrix estimation. For example, to ensure the positive definiteness of covariance matrix, a common technique is to force its minimum eigenvalue larger than some given constant [35, 24]. However, the authors found that bounding the minimum eigenvalue may lead to ill-conditioned covariance matrix estimation, when $p$ is much larger than $n$ [32]. Thus considering numerical stability, it is more appropriate to bound the condition number, which guarantees both the positive definiteness and numerical stability of covariance matrix estimates. In practical applications, $\kappa_n$ may be chosen from the interval $[10^4, 10^6]$ , if the priori information on $\kappa_n$
is unavailable [28].

4 Numerical experiment

In this part, the numerical experiments will be given to check the results presented in this paper. All the computations are performed in MATLAB R2014a on PC with 4 GB RAM and Intel Core i5-6600 CPU running at 3.30 GHz.

Example 4.1. In this example, we will give a comparison of the three algorithms given in Section 3.2 with respect to different settings. The data is generated from multivariate normal distribution $\mathcal{N}(0, I_p)$ with sample size $n$, and we set $\kappa_n = 10^3$ in all experiments for simplicity. For each pair of $p$ and $n$, we repeat the numerical experiment 100 times and report the mean values of the CPU time in seconds. We first give a comparison of Algorithms 2 and 3 to check the superiority of MOD-SVD. The numerical results are reported in Table 1, from which we can see that as the ratio $p/n$ increases the MOD-SVD based method becomes more and more efficient. This coincides with the conclusion given in [8].

| $n = 100$ | $p = 150$ | $p = 200$ | $p = 250$ | $p = 300$ | $p = 350$ |
|-----------|-----------|-----------|-----------|-----------|-----------|
| GR-SVD    | 0.0316    | 0.0473    | 0.0826    | 0.1292    | 0.2061    |
| MOD-SVD   | 0.0215    | 0.0214    | 0.0224    | 0.0246    | 0.0259    |

In practical applications, the use of built-in function provided by data analysis software will substantially improve the performance of the proposed algorithm. To compare the three algorithms given in Section 3.2, we employ the MATLAB functions `eig()`, `qr()`, and `svd()` in the following computations, and the numerical results are reported in Table 2. According to Table 2, in high-dimensional data setting the SVD based methods outperform the full eigenvalue decomposition based method. However, when $p/n \gtrsim 2$, the first part of Table 2 shows that the GR-SVD based algorithm outperforms our MOD-SVD based algorithm that directly combines the function `qr()` and `svd()`. This is because the QR factorization step has been incorporated into the well programmed `svd()` function, which has been discussed in Remark 3.2.
Table 2: CPU time comparison of the three algorithms.

|       | n = 500 | p = 500 | p = 1000 | p = 2000 | p = 3000 | p = 4000 |
|-------|---------|---------|----------|----------|----------|----------|
| FU-EIG | 0.0694  | 0.4319  | 4.9537   | 17.2565  | 37.3999  |
| GR-SVD | 0.0692  | 0.1654  | 0.5851   | 1.5105   | 2.8391   |
| MOD-SVD| 0.0710  | 0.1614  | 0.7262   | 2.0560   | 3.9470   |

|       | n = 2000 | p = 2500 | p = 3000 | p = 3500 | p = 4000 | p = 4500 |
|-------|----------|----------|----------|----------|----------|----------|
| FU-EIG | 10.1859  | 16.6683  | 24.7184  | 37.5496  | 49.3076  |
| GR-SVD | 8.3104   | 10.4214  | 7.8683   | 9.4778   | 9.8960   |
| MOD-SVD| 6.3723   | 7.0242   | 7.4115   | 9.2190   | 10.1207  |

As an interesting finding, we note that, when \( n = 2000 \) and \( p/n < 2 \), the MOD-SVD based algorithm outperforms the GR-SVD based one. For example, when \( n = 2000 \) and \( p = 3000 \) the MOD-SVD based algorithm can achieve even more than 30 percent savings compared with the GR-SVD based one. Therefore, we suggest that when the data set is large, even if \( p \) is only slightly larger than \( n \) the QR factorization step should also be incorporated into the \texttt{svd()} function.

Example 4.2. In this example, we mainly consider the truncation effect of \( \kappa_n \), which should be treated as numerical complementary support to the discussions given in Section 3. The data is generated from the multivariate normal distribution \( \mathcal{N}(0, \Sigma) \). The covariance matrix \( \Sigma \) is a random matrix and its eigenvalues are equally distributed from \( 10^i \) to \( 10^{-i} \), which leads to \( \kappa(\Sigma) = 10^{2i} \).

To investigate the truncation process of \( \kappa_n \), we set \( i = 3 \) that equivalent to \( \kappa(\Sigma) = 10^6 \). By varying \( \kappa_n \), we show how the eigenvalues of \( S_n \) are truncated. The numerical results are presented in Figure 2, and for better presentation we take logarithm of the eigenvalues in the second and third rows of Figure 2. From Figure 2, we note that when \( \kappa_n = 10 \) both the larger and smaller eigenvalues of \( S_n \) are truncated. However, as \( \kappa_n \) increases, the C3MA model (2.1) tends to preserve the larger eigenvalues and the smaller ones are more likely to be truncated. When \( \kappa_n = 10^4 \), only the smaller eigenvalues are truncated. This can be used to give some explanation for why some high-dimensional covariance matrix estimation procedures would like to bound the eigenvalue from below \([35, 24]\).
Figure 2: The truncation effect of C3MA model (2.1) with $\kappa(\Sigma) = 10^6$. 
We note from Figure 2 that, as $\kappa_n$ increases, the number of truncated eigenvalues decreases, and the truncation locations $\alpha$ and $\beta$ move to the corresponding endpoints. For a further exploration, here despite the truncation locations $\alpha$ and $\beta$, we also check that under different regularization levels whether $\kappa_n$ is contained in the interval $[\kappa_\mu, \kappa_\nu]$. For each setting, we set $\kappa(\Sigma) = 10^4$ and repeat the numerical experiment 1000 times. For $\alpha$ and $\beta$, we report its smallest and largest values, and the percentage of $\kappa_n$ contained in $[\kappa_\mu, \kappa_\nu]$ is also given. The results are reported in Table 3, from which we note that although as $\kappa_n$ increases the average range between $\alpha$ and $\beta$ becomes wider, the smaller eigenvalues always tend to be truncated compared with the larger ones. Moreover, in our simulation all $\kappa_n$s are contained in the interval $[\kappa_\mu, \kappa_\nu]$, and this may numerically verified the discussion given in Section 3.3.

When $p$ is much larger than $n$ and the regularization level is not very high, the leading $n$ eigenvalues of $S_n$ are all reserved, which to some extent may explain that why we cannot directly use the singular values of $1/\sqrt{n}X_n$ to determine $\kappa_n$ in high-dimensional data setting.

### Table 3: The distribution of truncated eigenvalues

| $n = p = 400$ | $\kappa_n = 10$ | $\kappa_n = 10^2$ | $\kappa_n = 10^3$ | $\kappa_n = 10^4$ |
|----------------|-----------------|-----------------|-----------------|-----------------|
| $\alpha$       | [10, 14]        | [1, 2]          | [1, 1]          | [1, 1]          |
| $\beta$        | [218, 224]      | [322, 328]      | [370, 375]      | [389, 393]      |
| IN             | 100%            | 100%            | 100%            | 100%            |

| $n = 200, p = 400$ | $\alpha$       | [2, 6]        | [1, 1]          | [1, 1]          |
|-------------------|-----------------|-----------------|-----------------|-----------------|
| $\beta$           | [150, 157]      | [200, 200]      | [200, 200]      | [200, 200]      |
| IN                | 100%            | 100%            | 100%            | 100%            |

5 Concluding Remark

In this paper, we investigated the C3MA problem (2.1) in the high-dimensional data setting and presented its explicit solution with respect Frobenius norm, which has not been considered in the current literatures. By exploring the special structure of the data matrix, efficient SVDs based numerical algorithms were proposed to solve the C3MA problem. Our numerical experiments showed that the proposed algorithms are quite efficient. It should be noted
that it is the spiked rank structure of the data matrix that leads us to use SVDs to improve the performance of our algorithms. If the factor structure of the sample covariance matrix is not available, the proposed method in this paper will not be applicable. Note that the big data matrix are usually well approximated by low rank matrices \([31]\). Thus when the given data matrix \(S_n\) is only symmetric and low rank, it should be treated as a future work to study how to apply the proposed method to such problem.

**Appendix**

*Proof of Theorem 3.1.* Let \(\lambda_1, \lambda_2, \ldots, \lambda_p\) be the eigenvalues of \(\Sigma\) and satisfy \(\lambda_1 \geq \cdots \geq \lambda_p > 0\). With Lemma 2.1, \((2.2)\) amounts to

\[
\Lambda^* = \arg\min_{\exists \mu > 0, 0 < \mu \leq \lambda_p \leq \cdots \leq \lambda_1 \leq \kappa_n \mu} \sum_{i=1}^{p} (\lambda_i - \hat{\lambda}_i)^2.
\]

(5.1)

In high-dimensional data setting, \(S_n\) has at most \(n\) nonsingular eigenvalues, thus \((5.1)\) is equivalent to

\[
\Lambda^* = \arg\min_{\exists \mu > 0, 0 < \mu \leq \lambda_p \leq \cdots \leq \lambda_1 \leq \kappa_n \mu} \left\{ \sum_{i=1}^{n} (\lambda_i - \hat{\lambda}_i)^2 + \sum_{i=n+1}^{p} \lambda_i^2 \right\}
\]

(5.2)

which gives \(\lambda_{n+1}^* = \cdots = \lambda_p^* = \mu\), and the rest we need to do is to determine the auxiliary variable \(\mu\) and the first \(n\) eigenvalues of \(\Sigma\). Note that the zero eigenvalues of \(S_n\) give little essential influence on the solution of \((5.1)\). Without loss of generality, we assume that \(\hat{\lambda}_n > 0\), and then instead of \((5.2)\) we only need to consider the following problem

\[
\tilde{\Lambda}_1 = \arg\min_{\exists \mu > 0, 0 < \mu \leq \lambda_n \leq \cdots \leq \lambda_1 \leq \kappa_n \mu} \sum_{i=1}^{n} (\lambda_i - \hat{\lambda}_i)^2.
\]

(5.3)

To characterize the optimal solution of \((5.3)\), the derivation should be separated into the following two cases.

**Case 1.** When \(\hat{\lambda}_1/\hat{\lambda}_n \leq \kappa_n\), the constraint in \((5.3)\) is inactive and we get the optimal solution

\[
\tilde{\Lambda}_1 = \text{Diag}(\hat{\lambda}_1, \hat{\lambda}_2, \cdots, \hat{\lambda}_n).
\]
Combined with (5.2) and considering the constraint, the solution to (5.1) is given by

$$\Lambda^* = \text{Diag}(\tilde{\Lambda}_1^*, \tilde{\Lambda}_2^*)$$

with $\tilde{\Lambda}_2^* = \mu^* I_{p-n}$ and $\mu^* = \hat{\lambda}_1/\kappa_n$.

**Case 2.** When $\hat{\lambda}_1/\hat{\lambda}_n > \kappa_n$, we set $f_i(\lambda_i, \hat{\lambda}_i) = (\lambda_i - \hat{\lambda}_i)^2$ with $i = 1, \ldots, n$. For a fixed $\mu$ the minimizer of $f_i(\lambda_i, \hat{\lambda}_i)$ is given by

$$\lambda_i^*(\mu) = \arg\min_{\exists \mu > 0, 0 < \mu \leq \kappa_n} f_i(\lambda_i, \hat{\lambda}_i) = \min \left\{ \max(\mu, \hat{\lambda}_i), \kappa_n \mu \right\}, \quad (5.4)$$

and satisfies $\mu \leq \lambda_1^*(\mu) \leq \cdots \leq \lambda_n^*(\mu) \leq \kappa_n \mu$. In this case, the solution to (5.1) is given by

$$\Lambda^*(\mu^*) = \text{Diag} \left( \lambda_1^*(\mu^*), \cdots, \lambda_n^*(\mu^*) \right),$$

where $\lambda_i^*(\mu^*) = \text{Diag} \left( \lambda_1^*(\mu^*), \cdots, \lambda_n^*(\mu^*) \right)$, $\lambda_i^*(\mu^*) = \mu^* I_{p-n}$, and $\mu^*$ minimizes the following univariate function

$$u^* = \arg\min_{\mu > 0} \sum_{i=1}^{n} (\lambda_i^*(u) - \hat{\lambda}_i)^2 + (p-n) u^2. \quad (5.6)$$

To determine the optimal $\mu^*$, with (5.4) and (5.6) we define the following univariate function

$$f(\mu) = \sum_{i=1}^{n} f_i(\lambda_i^*(\mu), \hat{\lambda}_i) + (p-n) u^2$$

$$= \sum_{i: \lambda_i < \mu} f_i(\lambda_i^*(\mu), \hat{\lambda}_i) + \sum_{i: \mu \leq \lambda_i \leq \kappa_n \mu} f_i(\lambda_i^*(\mu), \hat{\lambda}_i) + \sum_{i: \kappa_n \mu < \lambda_i} f_i(\kappa_n \mu, \hat{\lambda}_i) + (p-n) u^2.$$

It can be easily checked that $f(\mu)$ is convex and continuously differentiable for $\mu \in (0, \infty)$. For $\alpha \in \{1, \cdots, n-1\}$ and $\beta \in \{2, \cdots, n\}$ with $\beta - 1 \geq \alpha$, we define

$$R_{\alpha, \beta} = \{ \mu : \hat{\lambda}_{\alpha} > \kappa_n \mu \geq \hat{\lambda}_{\alpha+1} \text{ and } \hat{\lambda}_{\beta-1} \geq \mu > \hat{\lambda}_{\beta} \},$$

$$f_{\alpha, \beta}(\mu) = \sum_{i=1}^{\alpha} f_i(\kappa_n \mu, \hat{\lambda}_i) + \sum_{i=\alpha+1}^{\beta-1} f_i(\hat{\lambda}_i, \hat{\lambda}_i) + \sum_{i=\beta}^{n} f_i(\mu, \hat{\lambda}_i) + (p-n) u^2$$

$$= \sum_{i=1}^{\alpha} (\kappa_n \mu - \hat{\lambda}_i)^2 + \sum_{i=\beta}^{n} (\mu - \hat{\lambda}_i)^2 + (p-n) u^2,$$

and then $f(\mu) = f_{\alpha, \beta}(\mu)$ for $\mu \in R_{\alpha, \beta}$. Since $f_{\alpha, \beta}''(\mu) = 2\kappa_n^2 \alpha + 2(p-\beta + 1) > 0$ for $\mu \in R_{\alpha, \beta}$, $f_{\alpha, \beta}'(\mu)$ is strictly increasing on $(\hat{\lambda}_n, \hat{\lambda}_1/\kappa_n)$. Similar to [16, 34], we can check that the unique minimizer of $f(\mu)$ is in $(\hat{\lambda}_n, \hat{\lambda}_1/\kappa_n)$ and satisfies $f'(\mu) = 0$. The solution to $f_{\alpha, \beta}'(\mu) = 0$ for $\mu \in (0, \infty)$ is

$$\mu_{\alpha, \beta} = \frac{\kappa_n \sum_{i=1}^{\alpha} \hat{\lambda}_i + \sum_{i=\beta}^{n} \hat{\lambda}_i}{\alpha \kappa_n^2 + p - \beta + 1},$$

thus $\mu_{\alpha, \beta}$ is also the solution to $f'(\mu) = 0$ if and only if $\mu_{\alpha, \beta} \in R_{\alpha, \beta}$, and we have $\mu^* = \mu_{\alpha, \beta}$.

The optimal $\mu^*$ can be found in $O(n)$ flops, which can be similarly derived as [34, 16].
References

[1] T.W. Anderson, An Introduction to Multivariate Statistical Analysis, third ed., John Wiley & Sons, New Jersey, 2003.

[2] E. Anderson, Z. Bai, C. Bischof, S. Blackford, J. Demmel, J. Dongarra, J. Du Croz, A. Greenbaum, S. Hammarling, A. McKenney, D. Sorensen, LAPACK Users’ Guide, third ed, SIAM, Philadelphia, 1999.

[3] A. Aubry, A. De Maio, L. Pallotta, A. Farina, Maximum likelihood estimation of a structured covariance matrix with a condition number constraint, IEEE Trans. Signal Process. 60(6)(2012) 3004-3021.

[4] Z.D. Bai, Methodologies in spectral analysis of large dimensional random matrices, a review, in: Z.H. Chen, J.T. Zhang, F.F. Hu (Eds.), Advances in Statistics, Word Scientific, 2008, pp. 174-240.

[5] J. Bien, R.J. Tibshirani, Sparse estimation of a covariance matrix, Biometrika 98(4)(2011) 807-820.

[6] P.J. Bickel, E. Levina, Covariance regularization by thresholding, Ann. Statist. 36(6)(2008) 2577-2604.

[7] G. Cao, C. Bouman, Covariance estimation for high dimensional data vectors using the sparse matrix transform, in: D. Koller, D. Schuurmans, Y. Bengio, L. Bottou (Eds.), Advances in Neural Information Processing Systems, 2009, pp. 225-232.

[8] T.F. Chan, An improved algorithm for computing the singular value decomposition, ACM Trans. Math. Software 8(1)(1982) 72-83.

[9] K.R. Davidson, S.J. Szarek, Local operator theory, random matrices and Banach spaces, in: W.B. Johnson, J. Lindenstrauss (Eds.), Handbook of the Geometry of Banach Spaces, Elsevier, Amsterdam, 2001, pp. 317-366.
[10] X. Duan, J. Bai, M. Zhang, X. Zhang, On the generalized low rank approximation of the correlation matrices arising in the asset portfolio, Linear Algebra Appl. 461(2014) 1-17.

[11] X.F. Duan, J.C. Bai, J.F. Li, J.J. Peng, On the low rank solution of the Q-weighted nearest correlation matrix problem, Numer. Linear Algebra Appl. 23(2)(2016) 340-355.

[12] N. El Karoui, Spectrum estimation for large dimensional covariance matrices using random matrix theory, Ann. Statist. 36(6)(2008) 2757-2790.

[13] J. Fan, Y. Liao, H. Liu, An overview of the estimation of large covariance and precision matrices. Econom. J. 19(1)(2016) C1-C32.

[14] D.I. Georgescu, N.J. Higham, G.W. Peters, Explicit solutions to correlation matrix completion problems, with an application to risk management and insurance. R. Soc. open sci. 5(3)(2018) 172348.

[15] G.H. Golub, C.F. Van Loan, Matrix Computations, fourth Ed., Johns Hopkins University Press, Maryland, 2013.

[16] X. Guo, C. Zhang, The effect of L1 penalization on condition number constrained estimation of precision matrix, Stat Sinica. 27( 2017)1299-1317.

[17] A. Hendrikse, L. Spreeuwers, R. Veldhuis, A bootstrap approach to eigenvalue correction. In 2009 Ninth IEEE International Conference on Data Mining, 2009, pp. 818-823.

[18] N.J. Higham, Computing the nearest correlation matrix $\hat{\rho}$ problem from finance, IMA J. Numer. Anal. 22(3)(2002) 329-343.

[19] N.J. Higham, N. Strabic, V. Sego, Restoring definiteness via shrinking, with an application to correlation matrices with a fixed block, SIAM Rev. 58(2)(2016) 245-263.

[20] J. Khan et al., Classification and diagnostic prediction of cancers using gene expression profiling and artificial neural networks. Nature Medicine, 7(6)(2001) 673.

[21] C. Lam, J. Fan, Sparsistency and rates of convergence in large covariance matrix estimation, Ann. Statist. 37(2009) 4254-4278.
[22] O. Ledoit, M. Wolf, A well-conditioned estimator for large-dimensional covariance matrices, J. Multivariate Anal. 88(2)(2004) 365-411.

[23] O. Ledoit, M. Wolf, Nonlinear shrinkage estimation of large-dimensional covariance matrices, Ann. Statist. 40(2)(2012) 1024-1060.

[24] H. Liu, L. Wang, T. Zhao, Sparse covariance matrix estimation with eigenvalue constraints, J. Comput. Graph. Statist. 23(2)(2014) 439-459.

[25] H. Markowitz, Portfolio selection, J. Finance, 7(1)(1952) 77-91.

[26] H. Roger, R.J. Charles, Topics in Matrix Analysis, Cambridge University Press, Cambridge, 1994.

[27] A.J. Rothman, E. Levina, J. Zhu, Generalized thresholding of large covariance matrices, J. Amer. Statist. Assoc. 104(485)(2009) 177-186.

[28] M. Tanaka, K. Nakata, Positive definite matrix approximation with condition number constraint, Optim. Lett. 8(3)(2014) 939-947.

[29] M. Tanaka, K. Nakata, Successive projection method for well-conditioned matrix approximation problems, IEEE Signal Process. Lett. 21(4)(2014) 418-422.

[30] J. Tong, Q. Guo, S. Tong, J. Xi, Y. Yu, Condition number-constrained matrix approximation with applications to signal estimation in communication systems, IEEE Signal Process. Lett. 21(8)(2014) 990-993.

[31] M. Udell, A. Townsend, Why are big data matrices approximately low rank? SIAM Journal on Mathematics of Data Science, 1(1)(2019) 144-160.

[32] S. Wang, H. Yang, Conditioning theory of the equality constrained quadratic programming and its applications, Linear Multilinear Algebra https://doi.org/10.1080/03081087.2019.1623858

[33] J.H. Won, S.J. Kim, Maximum likelihood covariance estimation with a condition number constraint, in: M.B. Matthews (Eds.), Proc. 40th Asilomar Conf. Signals, Systems and Computers, New York, 2006, pp. 1445-1449.
[34] J.H. Won, J. Lim, S.J. Kim, B. Rajaratnam, Condition-number-regularized covariance estimation, J. R. Stat. Soc. Ser. B Stat. Methodol. 75(3)(2013) 427-450.

[35] L. Xue, S. Ma, H. Zou, Positive-definite $l^1$-penalized estimation of large covariance matrices, J. Amer. Statist. Assoc. 107(500)(2012) 1480-1491.

[36] J. Yao, S. Zheng, Z.D. Bai, Sample Covariance Matrices and High-dimensional Data Analysis, Cambridge University Press, Cambridge, 2015.