Abstract

We generalize the Umbral Calculus of G-C. Rota [40] by studying not only sequences of polynomials and inverse power series, or even the logarithms studied in [24], but instead we study sequences of formal expressions involving the iterated logarithms and $x$ to an arbitrary real power.

Using a theory of formal power series with real exponents, and a more general definition of factorial, binomial coefficient, and Stirling numbers to all the real numbers, we define the Iterated Logarithmic Algebra $\mathcal{I}$. Its elements are the formal representations of the asymptotic expansions of a large class of real functions, and we define the harmonic logarithm basis of $\mathcal{I}$ which will be interpreted as a generalization of the powers $x^n$ since it behaves nicely with respect to the derivative.

We classify all operators over $\mathcal{I}$ which commute with the derivative (classically these are known as shift-invariant operators), and formulate several equivalent definitions of a sequence of binomial type. We then derive many formulas useful towards the calculation of these sequences including the Recurrence Formula, the Transfer Formula, and the Lagrange Inversion Formula. Finally, we study Sheffer sequences, and give many examples.

L’algèbre des Logarithmes Iterés

On généralise ici le calcul ombral de G-C Rota en étudiant non seulement les suites de polynômes, de séries à exposants entier négatif, ou encore de logarithmes [24], mais d’une façon plus générale on s’intéresse aux suites formelles des logarithms itérés de $x$ et de $x$ à un exposant réel quelconque.

On s’appuie sur la théorie des séries de puissances formelles à exposants réels, sur une définition générale des factorielles, des coefficients du binôme, et des nombres de Stirling à tous les nombres réel pour définir l’Algebre de Logarithmes Itérés $\mathcal{I}$. Cette algèbre a comme éléments les représentations asymptotiques de beaucoup de fonctions réelles par rapport de l’échelle des monômes dans les Logarithmes Itérés. On définit alors une base de $\mathcal{I}$ de logarithms harmoniques, lesquels pourront être considérés comme généralisant les puissances $x^n$, puisqu’ils se comportent comme $x^n$ lors de la dérivation.

On caractérise tous les opérations sur $\mathcal{I}$ qui commutent avec l’opérateur de dérivation (ce sont usuellement les “opérateurs invariants par translation”), ce qui nous permet de formuler plusieurs définition équivalentes des suite de séries logarithmiques de type binomial. On en déduit de nombreuses formules utiles au calcul de ces séries, en particulier la formule de récurrence, la formule de transfert, et la formule d’inversion de Lagrange. Enfin, on étudie les suites du Sheffer, ainsi que de nombreux exemples.
Dedicated to
Professor Gian-Carlo Rota
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Chapter 1

Introduction

Over the years, mathematicians have studied many special sequences of functions especially polynomials. These sequences were found to have many important similarities; however, it was not until G-C. Rota’s Umbral Calculus [40] that this notion was formalized. Only could one study under a single theory most of the important sequences of polynomials—for example, $x^n$, Abel, lower factorial, upper factorial. These sequences are all sequences of binomial type.

Nevertheless, yet more work needed to be done, for the theory only applied to polynomials. For example, the theory of “factor sequences” was developed specifically to handle inverse formal power series. However, this theory was completely separate from the Umbral calculus of polynomials. An early version of this article was published under the title Formal Power Series of Logarithmic Type [24]. Not only did this paper allow us to simultaneously consider polynomials and inverse formal power series, but it also allowed us to consider expressions involving the logarithm $\log x$.

This article differs from [24] principally in that it considers two further generalizations to the logarithmic algebra $\mathcal{L}$. By introducing the Iterated logarithms $\log\log x$, $\log\log\log x$, and so on to the Logarithmic algebra, we have the discrete theory. Then by including $x$ to any real power, we derive the continuous theory. Thus, the logarithmic algebra $\mathcal{L}$ is the set of all complex functions on the real numbers with asymptotic expansions in a neighborhood of $+\infty$ with respect to the ladder of comparison $x^a(\log x)^b \cdots$. With only a handful of obvious exceptions, all of the results of Formal Power Series of Logarithmic Type carry over into these new contexts with only minor changes. For example, we conclude that every sequence of polynomials binomial type can be uniquely extended to a pseudobasis of $\mathcal{L}$ called a Roman graded sequence (after Prof. Steve Roman). In fact, this sequence is usually easier to calculate than the sequence of binomial type itself using classical techniques. Thus, even when one is only interested in polynomials it still pays to introduce logarithms.
1.1 Discrete and Continuous

In this paper, all results and sections regarding the discrete iterated logarithmic algebra will be denoted “Discrete” and the results regarding the more general continuous iterated logarithmic algebra will be denoted “Continuous.” Readers interested in only one of these theories may safely omit all material pertaining to the other. Sections and results numbered with a D and paragraphs starting with Discrete are relevant only to the discrete theory whereas sections and results numbered with a C and paragraphs starting with Continuous are relevant only to the continuous theory. Otherwise, the remainder of this article may be interpreted discretely by supposing that all variables $a, b, c, \ldots$ are integers, and that $\alpha, \beta$ are vectors of integers, or it may be interpreted continuously by supposing that all variables $a, b, c, \ldots$ are real numbers and that $\alpha, \beta$ are vectors of real numbers.

Digressions and all sections marked with the word “Appendix” or the letter “A” are independent of all later material, and are included for their own sake.
Chapter 2

D-Invariant Operators

2.1 The Operator Topology

It is a classical result that the algebra of formal differential operators \( \sum_{n \geq 0} a_n D^n \) acts on the vector space of polynomials. In view of the fact that the derivative \( D \) is invertible in \( I^\alpha \) for \( \alpha \neq (0) \) (Proposition ??), we can define on \( I^+ \) the action of a more general class of differential operators, called Artinian operators. These are linear operators which commute with all powers of the derivative, and act on each level in a similar fashion; this notion will be made more precise later.

The primary goal of this chapter is to classify all Artinian operators. We see (Theorem 2.4.1) that they are merely Artinian series in the derivative, and that the operator topology—which we are shortly to define—corresponds to the Artinian topology. This implies (see §2.4.4A) that all linear differential equations have a unique canonical solution. Finally, along the way we find the opportunity (in §2.3) to apply this theory to Real Analysis.

We begin by defining a topology on the ring of continuous linear operators acting on formal power series of logarithmic type.

Let \( \mathcal{J} \) be a subspace of \( \mathcal{I} \). We say that a sequence \( (\theta_n)_{n \geq 0} \) of continuous linear operators of \( \mathcal{J} \) into itself converges in the operator topology on \( \mathcal{J} \) when for every \( p(x) \in \mathcal{J} \) the sequence \( (\theta_n p(x))_{n \geq 0} \) converges in \( \mathcal{J} \).
**Proposition 2.1.1** Let \( J = I, I^+, \) or \( I^a. \) Then the set of continuous linear operators in the operator topology of \( J \) is a complete topological \( K \)-algebra whose operations are given by:

\[
(\theta \phi)p(x) = \theta(\phi p(x))
\]

\[
(a\theta)p(x) = a(\theta p(x))
\]

\[
(\theta + \phi)p(x) = (\theta p(x)) + (\phi p(x)).
\]

**Proof:** Let \( (\theta_n)_{n \geq 0} \) and \( (\phi_n)_{n \geq 0} \) be convergent sequences of continuous linear operators on \( J. \) Thus, for any \( p(x) \in J, \) the sequences \( (\theta_n p(x))_{n \geq 0} \) and \( (\phi_n p(x))_{n \geq 0} \) are Cauchy. In particular, \( (\theta_n \phi_k p(x))_{n \geq 0} \) is Cauchy for any \( k \geq 0, \) and since \( \theta_k \) is continuous, \( (\theta_k \phi_n p(x))_{n \geq 0} \) is also Cauchy for any \( k \geq 0. \) Hence, \( (\theta_n \phi_n p(x))_{n \geq 0} \) is Cauchy, and \( (\theta_n \phi_n)_{n \geq 0} \) converges.

\[
((\theta_n + \phi_n)p(x))_{n \geq 0}
\]

converges since \( J \) is a topological space. Hence, \( (\theta_n + \phi_n)_{n \geq 0} \) converges.

The ring is complete since \( \theta p(x) = \lim_{n \to +\infty} \theta_n p(x) \) is the limit of a Cauchy sequence.

We shall write infinite series \( \sum_{k \geq d} \phi_k \) of operators, which are understood to denote the limits of their partial sums.

We list below some notable operators and comment briefly on them:

**Example 2.1.1 (Derivative)** The derivative \( D^a \) is defined via Definition ?? on all of \( I \) when \( a \) is a nonnegative integer. Otherwise, it is defined on only the positive logarithmic algebra \( I^+ \) via

**Discrete** Proposition ??.

**Continuous** Definition ??.

Note that the map \( a \mapsto D^a \) is continuous in the operator topology.

\[
D^a \lambda_b^\alpha(x) = \frac{|b|!}{|b-a|!} \lambda_{b-a}^\alpha(x).
\]

**Example 2.1.2 (Shift Operator)** For all complex numbers \( z, \) the shift operator, \( E^z : I \to I \) is given by the sum

\[
E^z = \sum_{n \geq 0} z^n D^n / n!.
\]

We see that this is a convergent sum, so \( E^z \) is well defined. In fact, we show that \( E^z \) is a field isomorphism.

Note that this is the first use we make of the fact that the field of complex numbers \( \mathbb{C} \) has characteristic zero.
2.1. THE OPERATOR TOPOLOGY

Example 2.1.3 (Elementary $D$-Invariant Operator) For any pair of vectors $\alpha$ and $\beta$ the elementary $D$-invariant operator from $\alpha$ to $\beta$—denoted $E_{\beta\alpha}$—is the linear map on the logarithmic algebra defined for all $\alpha$ and $\gamma$ by

$$E_{\beta\alpha} \lambda^\alpha_\gamma(x) = \begin{cases} \lambda^\beta_\alpha(x) & \text{if } \alpha = \gamma, \text{ and} \\ 0 & \text{if } \alpha \neq \gamma \end{cases}$$

where $a$ is not a negative integer if $\alpha = (0)$. In the discrete case, the elementary $D$-invariant operators are continuous.

Note that $\operatorname{proj}_\alpha = E_{\alpha\alpha}$ is the projection map $\mathcal{I} \to \mathcal{I}^\alpha$. In other words,

$$\operatorname{proj}_{\beta}(\lambda^\alpha_\alpha(x)) = \begin{cases} \lambda^\alpha_\alpha(x) & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta \end{cases}$$

These projections commute with $D^a$. Note however that not all continuous, linear projections which commute with $D^a$ are expressible in terms of these projections.

Continuous A subspace $\mathcal{J}$ of $\mathcal{I}^+$ is said to be D-invariant if it is invariant under the fractional derivative $D^a$ for all real numbers $a$. An operator $\theta$ on a D-invariant subspace $\mathcal{J}$ is said to be D-invariant when $D^a \theta = \theta D^a$ for all real $a$.

Discrete A subspace $\mathcal{J}$ of $\mathcal{I}$ is said to be D-invariant or shift-invariant if it is invariant under the derivative $D$, or equivalently if it is invariant under $E^z$ for all complex numbers $z$. An operator $\theta$ on a D-invariant subspace $\mathcal{J}$ is said to be D-invariant or shift-invariant when $D \theta = \theta D$, or equivalently if $DE^z = E^z D$ for all complex numbers $z$.

For example, all the operators mentioned above are D-invariant.

A continuous linear operator $\theta$ on $\mathcal{I}$ or $\mathcal{I}^+$ is said to be a regular operator if it commutes with every elementary D-invariant operator $E_{\alpha\beta}$ (except possibly when $\beta = 0$), that is, such that

$$\theta E_{\alpha\beta} = E_{\alpha\beta} \theta$$

for $\beta \neq 0$. For example, $D^a$ and $E^z$ are regular operators.

A regular D-invariant operator on $\mathcal{I}^+$ is called a Artinian operator, and one on all of $\mathcal{I}$ is called a differential operator. The set of Artinian operators is denoted by $\Lambda^+$, and the set of differential operators by $\Lambda$. Beware that this notation is inconsistent with the notation for the corresponding concept in [24]. The notation used here was chosen because it is more logical.

Clearly every differential operator restricts to an Artinian operator. Thus, $\Lambda^+ \subseteq \Lambda$

Proposition 2.1.2 1. The set of Artinian operators $\Lambda^+$ is a complete topological ring in the operator topology of $\mathcal{I}^+$. 
2. The set of differential operators $\Lambda$ is a complete topological ring in the operator topology of $\mathcal{I}$.

Proof: $\Lambda^+$ and $\Lambda$ clearly are $K$-algebras, so it suffices to show that the limit of any Cauchy sequence of Artinian (resp. differential) operators is again an Artinian (resp. differential) operator.

Let $(\theta_n)_{n \geq 0}$ be such a Cauchy sequence, and let $\theta$ be its limit. Now,

$$\theta E^z p(x) = \lim_{n \to +\infty} \theta_n E^z p(x) = \lim_{n \to +\infty} E^z \theta_n p(x) = E^z \lim_{n \to +\infty} \theta_n p(x)$$

since $E^z$ is a continuous operator. This in turn equals $E^z \theta p(x)$, so $\theta$ is D-invariant. Mutatis mutandis, we have $E_{st} \theta = \theta E_{st}$, so $\theta$ is regular.

Our objective is to obtain structural characterizations of Artinian and differential operators.

### 2.2 Taylor’s Formula

We shall derive analogs of Taylor’s formula in the Logarithmic algebra $\mathcal{I}$. We begin by giving the following alternate definition of the shift operator:

**Proposition 2.2.1** For all complex numbers $z$, $E^z$ is a well defined continuous field isomorphism of $\mathcal{I}$ which fixes all constants.

Proof: Need only check that $E^z(p(x)q(x)) = (E^z p(x))(E^z q(x))$ for all $p(x), q(x) \in \mathcal{I}$.

$$E^z(p(x)q(x)) = \sum_{k \geq 0} \frac{z^k}{k!} D^k(p(x)q(x))$$

$$= \sum_{k \geq 0} \frac{z^k}{k!} \left( \sum_{n+m=k} \binom{k}{n} (D^n p(x))(D^m q(x)) \right)$$

$$= \sum_{n,m \geq 0} \frac{z^{n+m}}{n!m!} (D^n p(x))(D^m q(x))$$

$$= (E^z p(x))(E^z q(x)).$$
Proposition 2.2.1 shows that $E^z$ satisfies the conditions of the characterization of Artinian composition in [25]. Any continuous field isomorphism of $I$ which fixes all constants is determined by its values on the iterated logarithms $\ell_k$. Thus, $E^z$ is the 0-composition associated with the substitution of $x + z$ for $x$, and $\log x + \sum_{j>0}(-1)^{j+1}z^j/jx^j$, and so on. In general,

$$E^z\ell_k = \sum_{n \geq 0} \frac{z^n}{n!} D^n \ell_k$$

$$= \sum_{n \geq 0} \frac{z^n}{[-n]!n!} \lambda^{\langle 0, \ldots, 0, 1, 0, \ldots \rangle}(x)$$

$$= \ell_k - \sum_{n \geq 0} \sum_{\ell(\rho)=k, \rho_k=1} \frac{(-z)^n s(-n, \rho_1)}{[n]!n} \left[ \prod_{j=1}^{k-2} e_{\rho_j-\rho_{j+1}}(-1, \ldots, -\rho_j + 1) \right] \ell^{(-n),\rho^*}$$

$$= \ell_k + \sum_{\ell(\rho)=k, \rho_k=1} (-1)^{\rho_1+|\rho|} \left[ \prod_{j=1}^{k-2} e_{\rho_j-\rho_{j+1}}(-1, \ldots, -\rho_j + 1) \right] \sum_{n \geq 0} \frac{z^n(n - 1)! s(-n, \rho_1)}{n} \ell^{(-n),\rho^*}$$

$$= \ell_k + \sum_{\ell(\rho)=k, \rho_k=1} (-1)^{\rho_{k-1}+|\rho|} \left[ \prod_{j=1}^{k-2} e_{\rho_j-\rho_{j+1}}(1, \ldots, \rho_j - 1) \right] \sum_{n \geq 0} \frac{z^n(n - 1)! s(-n, \rho_1)}{n} \ell^{(-n),\rho^*}$$

where $\rho^* = (\rho_1, \ldots, \rho_{k-1})$. End of Digression.

Observe that $E^z_1 E^z_2 = e^{z_1} D e^{z_2} = e^{(z_1 + z_2)} D = E^{z_1 + z_2}$.

**Discrete** By applying the shift operator to the harmonic logarithms of order $(1)$ and degree $n$ a nonnegative integer, $\lambda_n^{(1)}(x)$, we obtain the following identity:

$$(1 + a)^n \left( \log(1 + a) - 1 - \frac{1}{2} - \cdots - \frac{1}{n} \right)$$

$$= \left[ (x + a)^n \left( \log(x + a) - 1 - \frac{1}{2} - \cdots - \frac{1}{n} \right) \right]_{x=1}$$

$$= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} x^i \left( \log x - 1 - \frac{1}{2} - \cdots - \frac{1}{i} \right) + \sum_{i>n} \binom{n}{i} a^i x^{n-i}$$

$$= \sum_{i=0}^{n} \binom{n}{i} a^{n-i} \left( 1 + \frac{1}{2} + \cdots + \frac{1}{i} \right) + \sum_{i>n} \binom{n}{i} a^i,$$
and therefore:

\[(1 + a)^n \log(1 + a) = -((1 + a)^n - 1) \frac{s(-n, 1)}{|-n|!} + \sum_{i=0}^{n-1} \binom{n}{i} a^{n-i} \frac{s(-i, 1)}{|-i|!} + \sum_{i>n} \binom{n}{i} a^i \]

\[= ((1 + a)^n - 1) \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right) - na \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) - \frac{n}{2} a^{n-2} - na^{n-1} \]

\[+ a^{n+1} \left(\frac{1}{1} + a^{n+2} \left[\frac{1}{-2}\right] + \cdots.\right)\]

Contrast with [35, Corollary 5.6].

### 2.3 Real Analysis

Proposition 2.2.1 can be summarized as stating that

\[E^z p(x) = e^{zD} p(x) = p(x + z)\]

for all logarithmic series \(p(x) \in \mathcal{I}\). Note that this identity is tautological, since we cannot “evaluate” the variable \(x\). However, in the case of complex numbers, we can, and we have

**Proposition 2.3.1** All formal power series of logarithmic type \(p(x) \in \mathcal{I}\) represent asymptotic expansions of a complex valued function \(\tilde{p}(x)\) in a neighborhood of infinity. Moreover, for all real numbers \(c\), \(E^c p(x)\) represents an asymptotic expansion of \(\tilde{p}(x + c)\).

Conversely, any asymptotic expansion of a function in a neighborhood of infinity relative to the ladder of comparison given by \(\lambda_a^n(x)\) is a formal power series of logarithmic type.

**Proof:** The asymptotic expansions follow from Proposition 2.2.1. We conclude by observing

\[E^c p(x) |_{x=z} = e^{zD}\tilde{p}(x) |_{x=z}\]

\[= \sum_{k \geq 0} \frac{c^k}{k!} D^k \tilde{p}(x) |_{x=z}\]

\[= \sum_{k \geq 0} \frac{x^k}{k!} [D^k p(x)]_{x=c}.\]
2.4 Characterization of Various Classes of Operators

2.4.1 Artinian Operators

Every Artinian operator \( \theta \) maps \( I_\alpha \) into itself for every nonzero vector \( \alpha \). We denote by \( \theta \) its restriction to \( I_\alpha \), by an abuse of notation, and we say that \( \theta \) is an Artinian operator of \( I_\alpha \) into itself.

As planned, we can now characterize the algebra of Artinian operators and its topology. It is isomorphic to an algebra (mentioned in [25]) we denote \( \mathbb{C} < x > \); it is in a sense the dual of the Noetherian algebra of Definition ??.

**Discrete** In the discrete case, \( K < x > \) is the algebra of Laurent series in the variable \( x \) with complex coefficients.

**Continuous** In the continuous case, \( K < x > \) is the algebra of Artinian series in the variable \( x \) with complex coefficients.

\[
K < x > = \left\{ \sum_{a \in \mathbb{R}} c_a x^a : c_a \in K \text{ and for all } a \in \mathbb{R}, \text{ there exists finitely many} \ b \geq 0 \text{ with } c_a \neq 0 \right\}.
\]

Its operations and topology are similar to that of the Noetherian algebra.

**Theorem 2.4.1** The algebra of Artinian operators \( \Lambda^+ \) is naturally isomorphic to the Artinian algebra \( \mathbb{C} < x > \) as a topological algebra \( \Lambda^+ = \mathbb{C}(D)^{R_0} \).

**Proof:** \( (\mathbb{C}(D)^{R_0} \subseteq \Lambda^+) \) Note that \( D^a D^b = D^b D^a \), and \( D^a E_{\alpha \beta} = E_{\alpha \beta} D^a \).

\( (\Lambda^+ \subseteq \mathbb{C}(D)^{R_0}) \) Let \( \theta \in \Lambda^+ \) be an Artinian operator. By regularity, \( \theta \) is determined by its action on the harmonic logarithms of order \( \alpha \) for any particular \( \alpha \neq (0) \).

\[
\theta \lambda^\alpha_b(x) = \sum_b e_{ab} \lambda^\alpha_b(x).
\]

Notice, that for any \( a \) and \( b \), \( e_{ab} \neq 0 \) only for finitely many \( b' \geq b \). Next,

\[
\theta D^a \lambda^\alpha_b(x) = \frac{[b]!}{[b-a]!} \lambda^\alpha_{b-a}(x)
\]

\[
= \frac{[b]!}{[b-a]!} \sum_c e_{a-b,c} \lambda^\alpha_c(x)
\]

\[
D^a \theta \lambda^\alpha_b(x) = D^a \sum_c e_{bc} \lambda^\alpha_c(x)
\]

\[
= \sum_c e_{bc} \frac{[c]!}{[c-a]!} \lambda^\alpha_{c-a}(x).
\]
Thus, $\left\lfloor \frac{b}{a} \right\rfloor e_{a-b,c} = \left\lfloor \frac{c+a}{c} \right\rfloor e_{b,c+b}$. In particular, setting $a = b$,

$$e_{0,c} = \left\lfloor \frac{c}{b} \right\rfloor e_{b,c} + b.$$ 

Hence, $\theta$ is determined by the $e_{0,c}$, and therefore equals the Artinian series $\sum e_{c,0}D^c / |c|!$.

**Continuity** It remains only to show that Artinian series in the derivative are continuous in the operator topology (§2.1), and that Artinian operators are continuous in the Artinian topology. However, $c_aD^{a_n}$ converges whenever $a_n$ and $c_n$ converge, and $c_nD^{a_n}$ converges to zero whenever $a_n$ increases without bound. Moreover, all Cauchy sequences of operators are linear combinations of the above.

If $f(D)$ is an Artinian operator which is represented by a delta series (that is, a series of degree one), then $f(D)$ is called a *delta operator*.

**Discrete** Recall the important fact that if $f(D)$ is a delta operator, then the sequence of powers $(f(D)^n)_{n \in \mathbb{Z}}$ is a pseudobasis for the field of Artinian operators $\Lambda^+$. That is, for every Artinian operator $g(D) \in \Lambda^+$ of degree $d$ there is a sequence $(c_k)_{k \leq d}$ indexed by integers $k \leq d$ such that $g(D) = \sum_{k \leq d} c_k f(D)^k$. This sequence can actually be calculated via Theorem 3.3.3. The notation $f(D)^{a,n}$ comes from the continuous case (see below); in the context of the discrete theory, it should be read simply as $f(D)^a$.

**Continuous** Recall the important fact from [25] that if $f(D) = \sum_{a \in \mathbb{R}} c_a x^a$ is an Artinian operator of degree $d \neq 0$ (for example, a delta operator), and $n$ is an integer, then the set $\{ f(D)^{a,n} : a \in \mathbb{R} \}$ is a pseudobasis for the field of Artinian operators $\Lambda^+$ where

$$g(x)^{t,n} = m^t e^{it(\theta + 2\pi n)} c_d^{t,n} x^a \sum_{M} \left( \prod_{a \in M} \frac{c_{a+d}}{c_d} x^a \right)$$

and $m = |c_d|$ and $\theta = \arg(c_d)$ are the modulus and argument of the leading coefficient of $f(D)$.

That is, for every Artinian operator $g(D) \in \Lambda^+$ there is a sequence $(c_a)_{a \in \mathbb{R}}$ such that $\sum_{a \in \mathbb{R}} c_a f(D)^{a,n}$ converges to $g(D)$. When $f(D)$ is a delta operator, this sequence can be calculated via Theorem 3.3.3 or the methods of [20].

Similarly, the series of nonnegative integers powers of any delta operator $f(D)$ is a pseudobasis for $\Lambda$.

### 2.4.2 Differential Operators

In analogy with the preceding result for Artinian operators, we obtain the following structure theorem for differential operators:

**Corollary 2.4.2** The topological algebra of differential operators is naturally isomorphic to the topological algebra of formal power series in the derivative as a topological algebra: $\Lambda = \mathbb{C}[[D]]$.

**Proof:** Such series are the only members of $\Lambda^+$ which are well defined on $I^{(0)}$.

As opposed to Artinian operators which are invertible if nonzero, a differential operator is invertible in $\Lambda$ if and only if it is of degree 0.
2.4. CHARACTERIZATION OF VARIOUS CLASSES OF OPERATORS

2.4.3A D-invariant Operators

The rings of continuous, linear, D-invariant operators (which are not necessarily regular) over \( \mathcal{I} \) and \( \mathcal{I}^+ \) are structured as follows:

**Proposition 2.4.3** Let \( \mathcal{R} \) and \( \mathcal{R}_0 \) be the rings of D-invariant linear operators on \( \mathcal{I}^+ \) and \( \mathcal{I} \) respectively which are continuous on each \( \mathcal{I}^\alpha \). Then:

**Discrete**  
1. \( \mathcal{R} \) is the closure \( D \), in the operator topology, of the span of the operators \( D^n E_{\alpha\beta} \) where \( \alpha \) and \( \beta \) are nonzero vectors with finite support of integers, and \( n \) is an integer.
2. \( \mathcal{R}_0 \) is the closure \( D_0 \), in the operator topology, of the span of the operators \( D^n E_{\alpha\beta} \) where
   (a) \( \alpha \) and \( \beta \) are vectors with finite support of integers,
   (b) \( n \) is an integers,
   (c) \( \alpha \neq (0) \) unless \( \beta = (0) \), and
   (d) \( \beta \neq (0) \) unless \( n \) is a nonnegative integer.

**Continuous**  
1. \( \mathcal{R} \) is the closure \( D \), in the operator topology, of the span of the operators \( D^n E_{\alpha\beta} \) where \( \alpha \) and \( \beta \) are nonzero vectors with finite support of real numbers, and \( a \) is a real number.
2. \( \mathcal{R}_0 \) is the closure \( D_0 \), in the operator topology, of the span of the operators \( D^n E_{\alpha\beta} \) where
   (a) \( \alpha \) and \( \beta \) are vectors with finite support of real numbers,
   (b) \( a \) is a real number,
   (c) \( \alpha \neq (0) \) unless \( \beta = (0) \), and
   (d) \( \beta \neq (0) \) unless \( a \) is a nonnegative integer.

**Proof:** (\( D \subseteq \mathcal{R} \) and \( D_0 \subseteq \mathcal{R}_0 \)) Elementary D-invariant operators commute with the derivative. Thus, they commute with all Artinian and differential operators. Hence, they are in fact D-invariant.

Observe that \( E_{\alpha\beta} D^a \) is continuous, linear and D-invariant for \( \alpha, \beta \neq (0) \). We conclude that every operator in \( \mathcal{D} \) and \( \mathcal{D}_0 \) is continuous, linear, and D-invariant.

(\( \mathcal{R} \subseteq \mathcal{D} \)) Let \( \theta \) be a continuous, linear, D-invariant operator. For each pair of vectors \( \alpha, \beta \neq (0) \), define \( \theta_{\alpha\beta} = \text{proj}_{\alpha} \theta \text{proj}_{\beta} \). Obviously, \( \theta = \sum_{\alpha, \beta \neq (0)} \theta_{\alpha\beta} \). It suffices to show that for all nonzero vectors \( \alpha \) and \( \beta \), there is an Artinian operator \( f_{\alpha\beta}(D) \in \Lambda^+ \) such that \( \theta_{\alpha\beta} = f(D)E_{\beta\alpha} \).

However, \( E_{\beta\alpha} \theta_{\alpha\beta} \) is a continuous, linear, D-invariant operator on \( \mathcal{I}^\alpha \), so \( E_{\beta\alpha} \theta_{\alpha\beta} = f(D)\text{proj}_{\beta} \) for some Artinian operator \( f(D) \). Hence, \( \theta_{\beta\alpha} = E_{\beta\alpha} f(D)E_{\alpha\beta} = f(D)E_{\beta\alpha} \) as desired.

(\( \mathcal{R}_0 \subseteq \mathcal{D}_0 \)) Similarly, it suffices to show that \( \theta_{\alpha, (0)} \) (as defined above) is equal to zero for \( \alpha \neq (0) \). Assume not towards contradiction. By the reasoning above, \( \theta_{\alpha, (0)} = f(D)E_{\alpha, (0)} \) for some nonzero differential operator.
CHAPTER 2. D-INVARIANT OPERATORS

Let \( f(D) \) be the inverse of \( f(D) \). Since \( g(D) \) is a D-invariant operator, the product \( g(D) \theta \) is also D-invariant. Hence, without loss of generality, \( \theta_{\alpha,(0)} = E_{\alpha,(0)} \). We calculate that \( E_{\alpha,(0)} D1 = E_{\alpha,(0)}^0 \). However, we also know that \( DE_{\alpha,(0)} 1 = E_{\ell(0), \alpha} \neq 0 \). Contradiction.

We omit a discussion of non-D-invariant continuous, linear, regular operators and their expansions in terms of \( D \) and \( \sigma \) (to be defined later), since this would be an unnecessary digression.

However, there are other interested related questions left unsolved.

**Open Problem 2.4.4** Is there a simple characterization of non-linear, continuous, shift-invariant operators on polynomials? Or of the logarithmic algebra? For example, \( p(x) \mapsto p(x)^2 \).

### 2.4.4A Differential Equations

We next make some general remarks about solutions of differential equations of infinite order (and thus, in particular, of difference equations). We have seen (Theorem 2.4.1) that the algebra of Artinian operators is isomorphic to the Artinian algebra in the “variable” \( D \) as a topological algebra over the complex numbers. In particular, every Artinian operator is invertible. Hence, every differential equation of the form

\[
f(D)p(x) = q(x)
\]

where \( f(D) \in \Lambda^+ \) is an Artinian operator and \( q(x) \in \mathcal{I}^+ \) has a unique solution \( p(x) \) in \( \mathcal{I}^+ \). For example, \( q(x) \) may be any rational function of \( x \) whose numerator is of smaller degree than the denominator.

If \( q(x) \in \mathcal{I} \) (for example, if \( q(x) \) is an arbitrary rational function) the solution may not be unique. We can nevertheless define the natural solution of equation (2.1) as follows. Let \( \pi \) be any bijection between the equipotent sets \( A \) and \( A - \{0\} \) where \( A \) is the set of all vectors \( \alpha \) with finite support of reals (resp. integers). Define \( P_\pi \) to be the linear map (continuous on each \( \mathcal{I}^\alpha \)) defined by

\[
P_\pi = \sum_\alpha E_{\pi,\alpha,\alpha}
\]

or equivalently

\[
P_\pi \lambda_\alpha^\sigma(x) = \lambda_\alpha^{\pi,\sigma}(x)
\]

For example, we might have

\[
P_\pi = \sum_{n \geq 0} E_{(n)(n+1)} + \sum_{\alpha \neq (n)} E_{\alpha\alpha}
\]

where the second sum in each equation is over vectors which do not consist of a single nonnegative integer. Thus, \( P_\pi : \mathcal{I} \to \mathcal{I}^+ \) and its inverse \( P_\pi^{-1} : \mathcal{I}^+ \to \mathcal{I} \) is given by \( P_\pi^{-1} \lambda_\alpha^\sigma(x) = \lambda_\alpha^{\pi^{-1},\sigma}(x) \). Now, let \( s(x) = P_\pi q(x) \), and consider the differential equation

\[
f(D)r(x) = s(x).
\]
Since \( s(x) \in \mathcal{I}^+ \), this differential equation has a unique solution \( r(x) \in \mathcal{I}^+ \). Now, set \( p(x) = P^{-1}_\pi r(x) \) to obtain a solution of equation (2.1). The present definition agrees with (and is simpler than) all other definitions given of a natural solution over the complex numbers. Moreover, since \( p(x) \) does not depend on the choice of \( \pi \), the solution has been chosen naturally.

A notable example is the difference equation
\[
\Delta p(x) = 1/x.
\]
By the above remarks, it has a unique solution in \( \mathcal{I}^+ \), which turns out to be the \( \psi \)-function \( \psi(x) \), the logarithmic derivative of the gamma function. Thus, the theory of the \( \psi \)-function can be developed purely formally. (See chapter 5.1.1.)

So far, we have only consider linear differential equations. This leads us to the following open problem.

**Open Problem 2.4.5** In general, what differential equations have solutions? And when do they have canonical solutions?

The logarithmic algebra is not a *differentially closed field* since \( Dp(x) = p(x) \) only has one solution—\( p(x) = 0 \). However, we can redefine the degree of a differential operator so that we are consistent with [25] by insisting that its degree is the lowest (rather than highest) exponent of \( D \) with a nonzero coefficient. Under these circumstances, \( Dp(x) = p(x) \) is no longer a counterexample.

**Open Problem 2.4.6C** Under this definition of degree, is the logarithmic algebra differentially closed? And if so, can we modify the logarithmic algebra so that it is differentially closed under the usual definition?

Recall that differentially closed fields are known to exist in every character; however, no example of a differentially closed field of character zero has been found.

### 2.5 Augmentation

Although it is impossible to evaluate at zero any expression involving logarithms, the following definition nevertheless serves as the logarithmic analog of evaluation at zero.

**Definition 2.5.1** (Augmentation) For each vector \( \alpha \), we define the augmentation of order \( \alpha \) to be the linear functional \( \langle \rangle_\alpha \) (continuous on each \( \mathcal{I}^\alpha \)) from the logarithmic algebra to the complex numbers such that
\[
\langle \lambda_\alpha^0(x) \rangle_\alpha = \delta_{\alpha,0} \delta_{0,0},
\]
or equivalently using the notation introduced after Corollary 2.20,
\[
\langle p(x) \rangle_\alpha = [\lambda_\alpha^0(x)] p(x).
\]
We digress to indicate the algebraic significance of augmentation in the discrete case. First, note that in this case \( \langle \rangle_0 \) is continuous.

When \( \langle \rangle_0 \) is restricted to \( I(0) \), the augmentation reduces the evaluation of a polynomial at \( x = 0 \). That is, \( \langle p(x) \rangle_0 = p(0) \) for \( p(x) \in I(0) \). An augmentation of nonzero order can be viewed as a generalization of evaluation at \( x = 0 \); it is closely related to the residue of complex variable theory. (Recall that the residue of a Laurent series is its coefficient of \( x^{-1} \).) For instance, for \( p(x) \in I(1) \),

\[
\langle p(x) \rangle_1 = \text{Res}(Dp(x)).
\]

In fact, for \( \alpha \) a vector of \( j \) nonnegative integers and \( p(x) \in I^\alpha \),

\[
\langle p(x) \rangle_\alpha = \text{Res}
\left(D \left( \prod_{k=1}^{j} \frac{f^{(0)}(x)}{\alpha_k!} \right) p(x) \right).
\]  

(2.2)

Note that equation (2.2) also holds for all \( p(x) \in \bigoplus_{\beta \leq \alpha} I^\beta \). End of Digression.

We derive formulas relating the augmentation to the derivative.

**Proposition 2.5.2**

1. For all \( a \) and \( b \), and all \( \alpha, \beta \neq (0) \)

\[
\langle D^b \lambda_\alpha^a(x) \rangle_\beta = |a|! \delta_{ab} \delta_{\alpha \beta}.
\]

2. For all \( a \) and all nonnegative integers \( n \),

\[
\langle D^n \lambda_\alpha^{(0)}(x) \rangle_0 = n! \delta_{an}.
\]

3. If \( f(D) \) is an Artinian operator with coefficients \( [D^a]f(D) = c_a \), and \( p(x) \) is a formal power series of logarithmic type with coefficients \( b_a = [\lambda_\alpha^a(x)]p(x) \), then the augmentation \( \langle f(D)p(x) \rangle_\alpha \) is given by the finite sum

\[
\langle f(D)p(x) \rangle_\alpha = \sum_a |a|! c_a b_\alpha^a.
\]

4. If \( f(D) \) is a differential operator with coefficients \( c_k = [D^k]f(D) \), and \( p(x) \) is a formal power series of logarithmic type with coefficients \( b_a = [\lambda_\alpha^a(x)]p(x) \), then the augmentation \( \langle f(D)p(x) \rangle_\alpha \) is given by the finite sum

\[
\langle f(D)p(x) \rangle_\alpha = \sum_{k \geq 0} k! c_k b_\alpha^k.
\]
Some special cases are of interest. The augmentation of the derivative of a formal power series of logarithmic type can be described by \( \langle D^{a}p(x) \rangle_{a} = [a]!b_{a}^{\alpha} \) where \( p(x) \) is as in Proposition 2.5.2. Similarly, the augmentation of the action of a differential operator on a harmonic logarithm is \( \langle f(D)\lambda_{a}^{\alpha}(x) \rangle_{a} = [a]!c_{n} \) where \( f(D) \) is as in Proposition 2.5.2.

The augmentation leads us to a version of Taylor’s formula for formal power series of logarithmic type:

**Theorem 2.5.3 (Taylor’s Formula)** Let \( p(x) \in \mathcal{I}^{+} \). Then we have the following convergent expansion in \( \mathcal{I}^{\alpha} \),

\[
p(x) = \sum_{\alpha \neq (0)} \sum_{a} \frac{\langle D^{a}p(x) \rangle_{a}}{[a]!} \lambda_{a}^{\alpha}(x).
\]

**Proof:** By linearity and continuity, it suffices to consider the case \( p(x) = \lambda_{a}^{\alpha}(x) \). This special case was treated above (Proposition 2.5.2).

Note that the series of the right hand side is convergent.

**Discrete** Observe that in the discrete case, any formal power series of logarithmic type is determined by the augmentations of its derivative.

**Continuous** Nevertheless, in the continuous case, members of \( \mathcal{I}^{(0)} \) can not be recovered from the augmentations of their derivatives. For example, \( \langle f(D)\sqrt{x} \rangle_{(0)} = 0 \) for all differential operators \( f(D) \).

On the other hand, in either case, an Artinian operator can be recovered from the augmentations of its action on the harmonic logarithms of any particular order \( \alpha \neq (0) \) and a differential operator can be recovered from the augmentations of the harmonic logarithms of any particular order as is demonstrated by the following theorem.

**Theorem 2.5.4 (Expansion Theorem)**

1. Let \( f(D) \) be an Artinian operator, and let \( \alpha \neq (0) \), then we have the following convergent expansion

\[
f(D) = \sum_{a} \frac{\langle f(D)\lambda_{a}^{\alpha}(x) \rangle_{a}}{[a]!} D^{a}.
\]

2. Similarly, if \( f(D) \) is a differential operator, and \( \alpha \) is a vector, then we have the following convergent series

\[
f(D) = \sum_{n \geq 0} \frac{\langle f(D)\lambda_{n}^{\alpha}(x) \rangle_{a}}{[n]!} D^{n}.
\]
The following argument is used repeatedly in the next two sections—often implicitly.

**Proposition 2.5.5** (Spanning Argument)

1. Let \( p(x) \in I^+ \). If \( \langle f(D)p(x) \rangle_{\alpha} = 0 \) for all vectors \( \alpha \neq (0) \) and all Artinian operators \( f(D) \), then \( p(x) = 0 \).

2. Let \( \alpha \neq (0) \) be a vector, and let \( f(D) \) be an Artinian operator. If \( \langle f(D)p(x) \rangle_{\alpha} = 0 \) for all \( p(x) \in I^\alpha \), then \( f(D) = 0 \).

3. **Discrete** Similarly, in the discrete case, for \( I^{(0)} \), let \( p(x) \in I^{(0)} \). If \( \langle f(D)p(x) \rangle_{(0)} = 0 \), for all differential operators \( f(D) \in \Lambda \), then \( p(x) = 0 \).

4. Let \( \alpha \) be a vector, and let \( f(D) \) be a differential operator. If for all \( p(x) \in I^\alpha \), \( \langle f(D)p(x) \rangle_{\alpha} = 0 \), then \( f(D) = 0 \).

**Open Problem 2.5.6** Is there a simple formula expressing any monomial \( \ell^\alpha \) or product of harmonic logarithms \( \lambda^\alpha_a(x)\lambda^\beta_b(x) \) in terms of harmonic logarithms? That is, is there a simple way to calculate the coefficients of \( \ell^\alpha \) or \( \lambda^\alpha_a(x)\lambda^\beta_b(x) \) given by **Theorem 2.5.3**?

**Discrete** Note that the expansion of \( \lambda^{(k)}_n(x)\lambda^{(k)}_m(x) \) is known.
Chapter 3

Roman Graded Sequences

3.1 Graded Sequences

This chapter is devoted to the study of the logarithmic analog of sequences of polynomials of binomial type. However, before one can walk; one must crawl. We must first define the logarithmic analog of a sequence of polynomials. Classically, a sequence of polynomials \((p_n(x))_{n \geq 0}\) is subject to the requirements that \(\text{deg}(p_n(x)) = n\) for all \(n\). Here the requirements are slightly more complicated, yet the idea remains the same.

**Definition 3.1.1 (Graded Sequences of Logarithmic Series)** The sequence \(\{p^\alpha_a(x) : a \text{ a real (resp. an integer) and } \alpha \text{ a vector of reals (resp. integers)}\}\) is called a continuous (resp. discrete) graded sequence of formal power series of logarithmic type if the following six (resp. five) conditions hold. (Part of a typical such sequence is illustrated by Figure 3.1)

1. For all \(a\) and \(\alpha \neq (0)\), \(p^\alpha_a(x)\) is a homogeneous formal power series of logarithmic type of order \(\alpha\). For example, the series in the \(k\)th row of Figure 3.1 are homogeneous of order \(k\).

2. For \(n\) a negative integer, \(p^{(0)}_n(x) = 0\). That is, the left side of the middle row of Figure 3.1 is filled with zeroes.
3. For a not a negative integer, \( p^{(0)}_a(x) \) is a logarithmic series of degree \( a \). In other words, in the discrete case for \( n \) a nonnegative integer \( p^{(0)}_n(x) \) is a polynomial of degree \( n \). That is, the right side of the middle row of Figure 3.1 is filled with polynomials in order of degree.

4. For \( \alpha \neq (0) \), \( p^\alpha_a(x) \) is of degree \( a \). For example, the \( n \)th column of Figure 3.1 consists of series of degree \( n \).

5. (Regularity) For all \( a \) and \( \beta \), and all \( \alpha \neq (0) \), \( E_{\alpha \beta} p^\beta_a(x) = p^\alpha_a(x) \). For example, the operator \( E_{(j)(k)} \) sends the \( k \)th row of Figure 3.1 to the \( j \)th row.

6. Continuous (Continuity) In the continuous case only, the map from the reals to \( \mathcal{I}^\alpha \) defined by

\[
a \mapsto \begin{cases} 
p^\alpha_a(x) & \text{a not a negative integer} 
p^{\alpha+1}_a(x) & \text{a is a negative integer} \end{cases}
\]

must be continuous for all \( \alpha \).

When a sequence \( p^\alpha_a(x) \) has the property that its leading coefficients are all positive real numbers, then it is said to be standard.

The logarithmic series \( p^{(1)}_{-1}(x) \) is called the residual series of the graded sequence \( p^\alpha_a(x) \). (Indicated by a box in Figure 3.1.)

The principle subsequence of \( p^\alpha_a(x) \) is the subsequence \( (\tilde{p}_a(x))_{n \in \mathbb{R}} \) defined by

\[
\tilde{p}_a(x) = \begin{cases} 
p^{(0)}_a(x) & \text{for a not a negative integer, and} 
p^{(1)}_a(x) & \text{for a a negative integer.}
\end{cases}
\]

For example,
Proposition 3.1.2 \( \lambda^n(x) \) is a standard graded sequence of formal power series of logarithmic type. Its residual series is \( 1/x \), and its principal subsequence is the sequence of powers of \( x \), \( (x^n)_{n \in \mathbb{Z}} \).

Proof: The only thing worth checking is that the sequence is continuous. However, since the Gamma function is infinitely differentiable at all points other than the nonnegative integers, the map \( a \mapsto s(a, k) \) is continuous. Hence, the map defined by equation (3.1) is continuous.

The nonzero elements of any graded sequence form a pseudobasis for \( I \), and the restriction of a graded sequence to any particular \( \alpha \) forms a pseudobasis for \( I^\alpha \). Thus, for any pair of graded sequences \( p^n_\alpha(x) \) and \( q^n_\alpha(x) \), there is a unique continuous linear operator \( \theta \) such that \( \theta p^n_\alpha(x) = q^n_\alpha(x) \). We study such operators in detail in chapter 4.

Theorem 3.1.3 Let \( p^n_\alpha(x) \) be a graded sequence. Then every logarithmic series \( g(x) \in I \) can be uniquely written as an expression

\[
g(x) = \sum_{\alpha, \alpha} b^n_\alpha p^n_\alpha(x)
\]

where \( g^\alpha(x) \in I^\alpha \). This is a convergent expansion in the topology of Noetherian series, and an asymptotic expansion in the topology of the complex numbers as \( x \) tends towards infinity.

We are able to actually calculate the constants \( b^n_\alpha \) in many cases using such results as Theorems 3.3.4 and the corresponding result in [22].

3.2 Roman Graded Sequences

In this section, we introduce the central concept of this work. It is known that the operational calculus of formal differential operators is intimately associated with sequences of polynomials of binomial type, that is, with sequences of polynomials \( p_n(x) \) satisfying the binomial identity (Definition 3.6.1)

\[
p_n(x + a) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)p_{n-k}(a)
\]

A good many sequences of polynomials occurring in combinatorics and in the theory of special functions turn out to be of binomial type. For example, the powers \( x^n \), the lower factorial \( (x)_n \), the upper factorial \( (x)^n \), the Abel polynomials \( A_n(x) \), the LaGuerre polynomials \( L_n(x) \), and the inverse-Abel polynomials \( \mu_n(x) \) are all sequences of polynomials of binomial type. We give here the logarithmic generalization of this notion; such graded sequences of formal power series of logarithmic type are called Roman graded sequences. We derive five equivalent characterizations of such graded sequences. We anticipate the fact (Theorem 3.6.4) that the five notions introduced below coincide:

1. Roman graded sequence (Definition 3.2.2),
2. associated graded sequence (Definition 3.3.2),
3. basic graded sequence (Definition 3.4.1), and
4. conjugate graded sequence (Definition 3.5.1),
5. graded sequence of logarithmic binomial type. (Definition 3.6.2)

We motivate the definition of a Roman graded sequence, by deriving a formula for the action of a product of two Artinian operators on the harmonic logarithm.

**Proposition 3.2.1** 1. Let \( f(D) \) and \( g(D) \) be Artinian operators. Then for all \( a \) and for all \( \alpha \neq (0) \) we have the following finite sum:

\[
\langle f(D)g(D)\lambda_\alpha^a(x)\rangle_\alpha = \sum_b \left\lfloor \frac{a}{b} \right\rfloor \langle f(D)\lambda_\alpha^b(x)\rangle_\alpha \langle g(D)\lambda_\alpha^{a-b}(x)\rangle_\alpha. (3.2)
\]

2. Similarly, let \( f(D) \) and \( g(D) \) be differential operators. Then equation (3.2) holds for all vectors \( \alpha \).

**Proof:** Let \( c_a = [D^a]f(D) \), and \( d_a = [D^a]g(D) \). The product of the two series is given by the sum

\[ f(D)g(D) = \sum_b \left( \sum_a c_a d_{a-b} \right) D^b. \]

Hence, by Theorem 2.5.2,

\[
\langle f(D)g(D)\lambda_\alpha^a(x)\rangle_\alpha = [b]! \sum_a c_a d_{b-a} \\
= \sum_a \frac{[b]!}{[a]![b-a]!} ([a]! x_a)([b-a]! d_{b-a}) \\
= \sum_a \left\lfloor \frac{b}{a} \right\rfloor \langle f(D)\lambda_\alpha^a(x)\rangle_\alpha \langle g(D)\lambda_\alpha^{a-b}(x)\rangle_\alpha. \]

The extension to products of more than two operators follows easily by induction.

We introduce Roman graded sequences by the following definition. It will shortly be seen that simpler alternate definitions can be given.

**Definition 3.2.2** (Roman Graded Sequences) Let \( p_\alpha^a(x) \) be a graded sequence of formal power series of logarithmic type. The graded sequence is a Roman graded sequence if for all \( a \) and \( \alpha \), we have the following finite sum

\[
\langle f(D)g(D)p_\alpha^a(x)\rangle_\alpha = \sum_b \left\lfloor \frac{a}{b} \right\rfloor \langle f(D)p_\alpha^b(x)\rangle_\alpha \langle g(D)p_\alpha^{a-b}(x)\rangle_\alpha. (3.3)
\]

where \( f(D) \) and \( g(D) \) are Artinian operators if \( \alpha \neq (0) \), and are differential operators if \( \alpha = (0) \).
3.3. ASSOCIATED GRADED SEQUENCES

For example, \( \lambda^a_0(x) \) is a Roman graded sequence.

**Proposition 3.2.3** A graded sequence is Roman if and only if equation (3.3) holds when \( f(D) = D^a \) and \( g(D) = D^b \).

*Proof:* Linearity and continuity. □

**Proposition 3.2.4** A graded sequence \( p^a_\alpha(x) \) with coefficients \( d_{ab} = [\lambda^a_0(x)]p^a_\alpha(x) \) is Roman if and only if for all \( a, b, \) and \( c \):

\[
\left[ \begin{array}{c} a + b \\ a \end{array} \right] d_{c,a+b} = \sum_c \left[ \begin{array}{c} c \\ e \end{array} \right] d_{e,c}d_{c-a,b}.
\]

(3.4)

*Proof:* We demonstrate the “only if;” the reasoning for the other implication is similar.

Note that in the case \( \alpha = (0) \), we need only consider nonnegative integers \( a \) and \( b \), so the same argument applies. □

3.3. ASSOCIATED GRADED SEQUENCES

We proceed to derive an altogether different characteristic property of Roman graded sequences, which uses delta operators. Such differential operators may be viewed as playing the role of the derivative—much like the forward difference operator \( \Delta \) (Definition 5.1.1) acts on the polynomial sequence of lower factorials \( (x)_n = x(x-1)\cdots(x-n+1) \).

**Proposition 3.3.1** Let \( f(D) \) (and \( n \) an integer) be a delta operator in \( \Lambda^\pm \). Then there is a unique graded sequence of formal power series of logarithmic type, \( p^a_\alpha(x) \), such that for all \( a, b \) and \( \alpha \)

\[
\begin{cases}
\text{Discrete} & \langle f(D)^a p^a_\alpha(x) \rangle_\alpha = [a]! \delta_{ab} \\
\text{Continuous} & \langle f(D)^a p^a_\alpha(x) \rangle_\alpha = [a]! \delta_{ab}.
\end{cases}
\]

(3.5)
Proof: (Uniqueness) Spanning argument (Proposition 2.5.5.)

(Existence) We need only consider the continuous case. Let \( \alpha^\delta = [D^b] f(D)^{a,n} \), and let \( c_b = c_b^{(1)} = [D^b] f(D) \). Thus, \( c_a^{(a)} \neq 0 \) for all real numbers \( a \). Similarly, let \( d^{\alpha}_{ab} = [\lambda^\alpha_a(x)] p^a_n(x) \) be the coefficients of \( p^a_n(x) \). In this notation, equation (3.5) is equivalent to the equation

\[
\sum_b [b]! c_b^{(a)} d^{\alpha}_{eb} = [e]! \delta_{eb}.
\]

Hence, the coefficients \( d^{\alpha}_{ab} \) can be computed recursively as:

\[
d^{\alpha}_{ab} = \frac{1}{[b]! a^{(b)}_b} \left( [a]! \delta_{ab} - \sum_{e>b} [e]! c_e^{(b)} d^{\alpha}_{eb} \right).
\]

The recursion is well defined since \( d^{\alpha}_{ab} \) is a Noetherian sequence in \( b \). Finally, notice that \( p^a_n(x) \) is regular by symmetry, and that \( \deg(p^a_n(x)) = n \) since \( b^a_{nn} = [n]! / [n]! a^{(n)}_n \neq 0 \). Finally, note that the map defined by equation (3.1) is continuous since \( a \mapsto f(D)^{a,n} \) is continuous.

**Definition 3.3.2 (Associated Graded Sequence)** Let \( f(D) \) be a delta operator (and \( n \) an integer). The unique graded sequence mentioned in Proposition 3.3.1 is called the \( n \)-th associated graded sequence of the delta operator \( f(D) \). We also say that the sequence is \( n \)-associated with \( f(D) \). (The term \( n \)-th applies only in the continuous case.)

For example, by Proposition 2.5.2, \( \lambda^\alpha_a(x) \) is the standard graded sequence associated with the delta operator \( D \).

We can now generalize Theorem 2.5.4 to explicitly determine the coefficients in the expansion of an arbitrary Artinian operator in terms of the powers of a delta operator:

**Theorem 3.3.3 (Expansion Theorem)** Let the graded sequence \( p^a_n(x) \) be \( n \)-associated with the delta operator \( f(D) \). Then for all Artinian operators \( g(D) \) and vectors \( \alpha \neq (0) \) we have the following convergent sum.

\[
\begin{align*}
\text{Discrete} & \quad g(D) = \sum_k \frac{(g(D)p^a_n(x))\alpha_k}{[k]!} f(D)^k \\
\text{Continuous} & \quad g(D) = \sum_a \frac{(g(D)p^a_n(x))\alpha_a}{[a]!} f(D)^{a,n}.
\end{align*}
\]

(3.6)

When \( \alpha = (0) \), equation (3.6) holds for all differential operators \( g(D) \).
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Proof: By Definition 2.5.1 we have,
\[ \sum_a \frac{(g(D)p_a^\alpha(x))_\alpha}{[a]!} f(D)^{a;n} p_b^\alpha(x) \] = \sum_a \frac{(g(D)p_a^\alpha(x))_\alpha}{[a]!} (f(D)^{a;n} p_b^\alpha(x))_\alpha
= \langle g(D)p_a^\alpha(x) \rangle_\alpha.

The conclusion follows by the spanning argument.

Dually, we obtain the explicit form of the expansion of an arbitrary formal series of logarithmic type as a linear combination of elements of a Roman graded sequence. This gives a useful generalization of Theorem 2.5.3.

Theorem 3.3.4 (Logarithmic Taylor’s Theorem) Let \( p_a^\alpha(x) \) be the \( (n^{th}) \) graded sequence associated with the delta operator \( f(D) \). Then for every formal power series of logarithmic type \( p(x) \in \mathcal{I}^+ \) we have the following convergent sum

\[
\begin{align*}
\text{Discrete} & \quad p(x) = \sum_{\alpha \neq (0)} \sum_k \frac{\langle f(D)^k p(x) \rangle_\alpha}{[k]!} p_k^\alpha(x) \\
\text{Continuous} & \quad p(x) = \sum_{\alpha \neq (0)} \sum_a \frac{\langle f(D)^{a;n} p(x) \rangle_\alpha}{[a]!} p_a^\alpha(x).
\end{align*}
\]

Proof: We need only consider the continuous case, but first a simple lemma. For all \( p(x), q(x) \in \mathcal{I}^+ \), by Part (1) of the spanning argument (Proposition 2.5.5), it is clear that \( p(x) = q(x) \) if and only if for all \( a \) and \( \alpha \neq (0) \), \( \langle D^a p(x) \rangle_\alpha = \langle D^a q(x) \rangle_\alpha \).

When we apply the Expansion Theorem where \( g(D) \) is set equal to \( D^a \), we find that
\[ D^a = \sum_b \frac{\langle D^a p_b^\alpha(x) \rangle_\alpha}{[b]!} f(D)^{b;n}. \]

Thus,
\[ \langle D^a p(x) \rangle_\alpha = \sum_b \frac{\langle D^a p_b^\alpha(x) \rangle_\alpha}{[b]!} \langle f(D)^{b;n} p(x) \rangle_\alpha = \langle D^a \sum_b p_b^\alpha(x) \frac{\langle f(D)^{b;n} p(x) \rangle_\alpha}{[b]!} \rangle_\alpha. \]

Therefore, the above remarks, we conclude that
\[ p(x) = \sum_{\alpha \neq (0)} \sum_a \frac{\langle f(D)^a p(x) \rangle_\alpha}{[a]!} p_a^\alpha(x). \]

In the continuous case, where this is relevant, we may now classify standard associated sequences (Definition 3.1.1).
Proposition 3.3.5C The \( n \)th associated sequence of a delta operator \( f(D) \) is standard if and only if the leading coefficient of \( f(D) \) is a positive real number and \( n = 0 \).

Such delta operators are be called standard operators.

3.4 Basic Graded Sequences

Definition 3.4.1 (Basic Graded Sequence) Let \( f(D) \) be a delta operator. A graded sequence of formal power series of logarithmic type \( p_\alpha^a(x) \) is called the \((n)\) basic graded sequence for \( f(D) \) if for all \( \alpha \),

1. \( \langle p_\alpha^a(x) \rangle_\alpha = 1 \);  
2. For all \( a > 0 \) (and thus all \( a \neq 0 \)), \( \langle p_\alpha^a(x) \rangle_\alpha = 0 \); and 
3. Discrete For all integers \( n \), and all vectors \( \alpha \) of integers with finite support, \( f(D)p_\alpha^a(x) = [n]p_\alpha^{a-1}(x) \).  
   Continuous For all real numbers \( a \) and \( b \), and all nonzero vectors \( \alpha \) of real numbers with finite support, 
   \[
   f(D)^b\cdot p_\alpha^a(x) = \frac{|a|!}{|a-b|!} p_\alpha^{|a-b|}(x). 
   \] (3.7)

Theorem 3.4.2 1. Let \( p_\alpha^a(x) \) be a logarithmic graded sequence. Such a graded sequence is the \((n)\) basic graded sequence for the delta operator \( f(D) \) if and only if it is the \((n)\) associated graded sequence for the delta operator \( f(D) \).

2. Every delta operator has a unique \((n)\) basic sequence.

3. Every basic sequence is basic for a unique delta operator (and integer \( n \)).

Proof: (1: If) Properties 1 and 2 of Definition 3.4.1 follow from Definition 3.3.2. Property 3 follows from the following series of equalities, and the spanning argument as shown here in the continuous case:

\[
\langle f(D)^{a\cdot n} (f(D)^{b\cdot n} p_\epsilon^\alpha(x)) \rangle_\alpha = \langle f(D)^{a+b\cdot n} p_\epsilon^\alpha(x) \rangle_\alpha = \frac{|c|!}{|c-b|!} \delta_{c,b} (f(D)^{a-b\cdot n} p_\epsilon^{a-b}(x))_\alpha
\]

(1: Only if) We proceed differently in the continuous and discrete cases:
Discrete  By induction we have, \( f(D)^k p_n^\alpha(x) = [n]! p_{n-k}^\alpha(x)/[n-k]! \). Hence,

\[
\langle f(D)^k p_n^\alpha(x) \rangle_\alpha = \left\langle \frac{[n]!}{[n-k]!} p_{n-k}^\alpha(x) \right\rangle_\alpha = [n]! \delta_{n-k,0} = [n]! \delta_{n,k}.
\]

Hence, \( p_n^\alpha(x) \) is the associated graded sequence of \( f(D) \) by definition.

Continuous  By equation (3.7), \( \langle f(D)^b p_n^\alpha(x) \rangle_\alpha = \frac{|a|!}{|a-b|!} \delta_{a-b,0} = |a|! \delta_{ab} \).

(2 and 3) Immediate from 1.  

The simplest example of a basic graded sequence is the graded sequence of harmonic logarithms \( \lambda_n^\alpha(x) \); by Theorem 77, it is the standard basic graded sequence for the delta operator \( D \). It can be viewed as the natural logarithmic extension of the sequence of powers of \( x \). More generally, every sequence of binomial type (and every factor sequence) has a natural logarithmic extension into a basic graded sequence related to the same delta operator.

\( D \) is invertible on \( I^+ \). Hence, just as in the continuous case (equation (3.7)), we have:

Porism 3.4.3D Let \( p_n^\alpha(x) \) be the associated graded sequence of formal power series of logarithmic type of the delta operator \( f(D) \), then for all integers \( n \) and \( k \), and all nonzero vectors \( \alpha \) with finite support of integers.

\[
f(D)^k p_n^\alpha(x) = \frac{[n]!}{[n-k]!} p_{n-k}^\alpha(x)
\]

Moreover, when \( k \) is a nonnegative integer, equation (3.8) holds for all \( \alpha \).  

We may now connect the notion of an associated graded sequence with that of a Roman graded sequence.

Proposition 3.4.4 Let \( p_n^\alpha(x) \) be an associated graded sequence, and let \( f(D) \in \Lambda^+ \) be an Artinian operator. Then we have the following convergent sum

\[
f(D)p_n^\alpha(x) = \sum_b \left\lfloor \frac{a}{b} \right\rfloor \langle f(D)p_b^\alpha(x) \rangle_\alpha p_{n-b}^\alpha(x)
\]

for all \( \alpha \neq (0) \), and also for \( \alpha = (0) \) if \( f(D) \) is in fact a differential operator.
Proof: We need only consider the continuous case, suppose \( p_\alpha^a(x) \) is \( n \)-associated with the delta operator \( g(D) \). Then for all \( b \)

\[
\begin{align*}
g(D)^{b:n} p_\alpha^a(x) &= \frac{[a]!}{[a-b]!} p_{a-b}^\alpha(x) \\
&= \sum_c \binom{a}{c} \langle g(D)^{b:n} p_c^\alpha(x) \rangle_\alpha p_{a-c}^\alpha(x).
\end{align*}
\]

Since \( \{ g(D)^{b:n} : b \in \mathbb{R} \} \) is a pseudobasis, we can use continuity and linearity to replace \( g(D)^{b:n} \) by \( f(D) \).

The following identity is classically proven [40] by introducing a new variable. However, we have a much simpler proof.

**Corollary 3.4.5** Let \( p_\alpha^a(x) \) be an associated graded sequence with coefficients

\[ c_{ab} = [\lambda_b^a(x)] p_\alpha^a(x). \]

Then for all \( a, b, \) and \( \alpha \) we have

\[
D^b p_\alpha^a(x) = \sum_{d \geq b} \frac{[a]! c_{ab}}{[a-d]!} p_{a-d}^\alpha(x).
\]

**Proof:** Proposition 3.4.4.

As another consequence of Proposition 3.4.4 we obtain:

**Proposition 3.4.6** A graded sequence of formal power series of logarithmic type \( p_\alpha^a(x) \) is an associated graded sequence if and only if it is a Roman graded sequence.

**Proof:** (Only if) Let \( f(D) \) and \( g(D) \) be Artinian operators. By Proposition 3.4.4,

\[
f(D) p_\alpha^a(x) = \sum_b \binom{a}{b} \langle f(D) p_b^\alpha(x) \rangle_\alpha p_{a-b}^\alpha(x).
\]

Thus,

\[
\langle f(D) g(D) p_\alpha^a(x) \rangle_\alpha = \left\langle g(D) \left( \sum_b \binom{a}{b} \langle f(D) p_b^\alpha(x) \rangle_\alpha p_{a-b}^\alpha(x) \right) \right\rangle_\alpha
\]

\[
= \sum_b \binom{a}{b} \langle f(D) p_b^\alpha(x) \rangle_\alpha \langle g(D) p_{a-b}^\alpha(x) \rangle_\alpha.
\]
3.5. CONJUGATE GRADED SEQUENCES

(If) Conversely, let \( p_\alpha(x) \) be a Roman graded sequence. We define a sequence of Artinian operators \( f_b(D) \) by the relation
\[
\left< f_b(D)p_\alpha^{(1)}(x) \right>_{(1)} = [a]\delta_{ab}.
\]
It suffices to show that
\[
\begin{cases}
\text{Discrete} & f_b(D) = f(D)^{b,n} \\
\text{Continuous} & f_b(D) = f(D)^{b}
\end{cases}
\]
for some delta operator \( f(D) \) (and some integer \( n \)).

By the spanning argument, \( f_b(D) \) is well defined. Now,
\[
\left< f_b(D)\lambda_a^{(1)}(x) \right>_{(1)} = 0
\]
for \( a < b \), and
\[
\left< f_b(D)\lambda_b^{(1)}(x) \right>_{(1)} \neq 0,
\]
hence \( \text{deg}(f_b(D)) = b \). In particular, \( f_1(D) \) is a delta operator. Since \( p_\alpha^{(1)}(x) \) is a Roman graded sequence, we infer that
\[
\left< f_b(D)f_c(D)p_\alpha^{(1)}(x) \right>_{(1)} = \sum_d \left< f_b(D)p_d^{(1)}(x) \right>_{(1)} \left< f_c(D)p_{a-d}^{(1)}(x) \right>_{(1)}
\]
\[
= [a]!\delta_{a,b+c}
\]
\[
= \left< f_{b+c}(D)p_a^{(1)}(x) \right>_{(1)}.
\]
By the spanning argument, \( f_i(D)f_j(D) = f_{i+j}(D) \). Hence,

\textbf{Discrete} By induction, \( f_n(D) = f_1(D)^n \).

\textbf{Continuous} By the characterization of exponentiation in [25], there is an integer \( n \) such that \( f_b(D) = f_1(D)^{b,n} \) for all \( n \).

Thus, every Roman graded sequence is associated with a unique delta operator (and integer) and \textit{visa versa}.

3.5 Conjugate Graded Sequences

Each delta operator (and integer) has another graded sequence associated with it: its conjugate graded sequence. We see that these sequences are Roman; however, they are not associated with the same delta operator for which they are conjugate.
CHAPTER 3. ROMAN GRADED SEQUENCES

**Definition 3.5.1** (Conjugate Graded Sequence) Let \( f(D) \in \Lambda^+ \) be a delta operator. Its \((n^{th})\) conjugate graded sequence \( q^a_\alpha(x) \) is defined as

\[
\begin{align*}
\text{Discrete} & \quad q^a_\alpha(x) = \sum_{k \leq n} \frac{\langle f(D)^k \lambda^a_\alpha(x) \rangle}{[k]!} \lambda^k_\alpha(x) \\
\text{Continuous} & \quad q^a_\alpha(x) = \sum_{b \leq a} \frac{\langle f(D)^b \lambda^a_\alpha(x) \rangle}{[b]!} \lambda^b_\alpha(x)
\end{align*}
\]

for all \( a \) and \( \alpha \).

Indeed, for all delta operators \( f(D) \) (and integers \( n \)), the graded sequence \( q^a_\alpha(x) \) as defined by equation (3.9) automatically meets the conditions of Definition 3.1.1.

The canonical example of a conjugate graded sequence is the graded sequence of harmonic logarithms; it is the standard conjugate graded sequence of the delta operator D, since by Theorem 2.5.3,

\[
\lambda^a_\alpha(x) = \sum_{b} \langle D^b \lambda^a_\alpha(x) \rangle \lambda^b_\alpha(x)/[b]!.
\]

In the above example, the \( n^{th} \) conjugate graded sequence of a delta operator is a Roman graded sequence. This fact is true in general:

**Proposition 3.5.2** A graded sequence of formal power series of logarithmic type \( q^a_\alpha(x) \) is Roman if and only if it is the \((n^{th})\) conjugate graded sequence of a delta operator. Moreover, the delta operator (and integer) are unique.

**Proof:** (*If*) We need only consider the continuous case. Let \( q^a_\alpha(x) \) be the \( n^{th} \) conjugate graded sequence of the delta operator \( f(D) \). Now, by equation (3.9) the coefficients \( c_{ab} \) of \( p^a_\alpha(x) \) are given by

\[
c_{ab} = \left[ \lambda^a_b(x) \right] p^a_\alpha(x) = \frac{\langle f(D)^b \lambda^a_\alpha(x) \rangle}{[b]!}.
\]

It suffices to show that the \( c_{ab} \) satisfy equation (3.4). For any \( b_1, b_2 \),

\[
\sum_d \binom{a}{d} c_{db_2} c_{a-d,b_1} = \sum_d \binom{a}{d} \frac{\langle f(D)^{b_2} \lambda^a_\alpha(x) \rangle}{[b_2]!} \frac{\langle f(D)^{b_1} \lambda^{a-d}_\alpha(x) \rangle}{[b_1]!}
\]

\[
= \frac{\langle f(D)^{b_1+b_2} \lambda^a_\alpha(x) \rangle}{[b_1]! [b_2]!}
\]

since \( \lambda^a_\alpha(x) \) is a Roman graded sequence. The last expression equals \( \left[ \frac{b_1+b_2}{b_1} \right] c_{a,b_1+b_2} \). Hence, equation (3.4) holds.
3.5. CONJUGATE GRADED SEQUENCES

(Only if) Conversely, suppose that \( q^\alpha_a(x) \) is a Roman graded sequence. Let \( c_{ab} \) denote the coefficients of \( q^\alpha_a(x) \)
\[
c_{ab} = [\lambda^\alpha_b(x)] q^\alpha_a(x).
\]
Define the Artinian operator \( f_d(D) \) by
\[
\langle f_d(D) \lambda^\alpha_a(x) \rangle_\alpha = [d]! c_{ad}.
\]
By the spanning argument, this condition defines \( f_d(D) \). As in the proof of Proposition 3.4.6 which employs a similar technique, it suffices to observe that \( f_d(D) = f(D)^d \) (resp. \( f(D)^{d,n} \)) for some delta operator \( f(D) \) (and integer \( n \)).

In fact, we have:
\[
f_1(D) = \sum_a \frac{c_{a,1}}{a!} D^a.
\]
Thus, \( f_1(D) \) is a delta operator. Now,
\[
\langle f_{b_1+b_2}(D) \lambda^\alpha_a(x) \rangle_\alpha = [b_1 + b_2]! b_{a,b_1+b_2}
\]
by definition, and that expression equals
\[
\sum_d \left[ \begin{array}{c} a \\ d \end{array} \right] \langle f_{b_1}(D) \lambda^\alpha_d(x) \rangle_\alpha \langle f_{b_2}(D) \lambda^\alpha_{a-d}(x) \rangle_\alpha
\]
by equation (3.4). This in turn equals
\[
\langle f_{b_1}(D) f_{b_2}(D) \lambda^\alpha_a(x) \rangle_\alpha
\]
since \( \lambda^\alpha_a(x) \) is a Roman graded sequence. In other words,
\[
\langle f_{b_1+b_2}(D) \lambda^\alpha_a(x) \rangle_\alpha = \langle f_{b_1}(D) f_{b_2}(D) \lambda^\alpha_a(x) \rangle_\alpha,
\]
so by the spanning argument \( f_{i+j}(D) = f_i(D) f_j(D) \).

**Discrete** Hence, by induction, \( f_n(D) = f_1(D)^n \).

**Continuous** Notice first that \( d \mapsto f_d(D) \) is a continuous map. Hence, by the characterization of exponentiation in [25], there is an integer \( n \) such that \( f_b(D) = f_1(D)^{b:n} \) for all \( n \).

In other words, \( q^\alpha_a(x) \) is the \( (n^{th}) \) conjugate graded sequence of the delta operator \( f_1(D) \) (and no other).

In the continuous case where this is relevant, we may now classify standard conjugate sequences.

**Proposition 3.5.3C** The \( n^{th} \) conjugate sequence of a delta operator \( f(D) \) is standard if and only if \( f(D) \) is a standard operator and \( n = 0 \).
3.6 Graded Sequences of Logarithmic Binomial Type

We motivate our final reformulation of the Roman condition by briefly recalling some facts about sequences of polynomials of binomial type.

**Definition 3.6.1 (Polynomial Sequence of Binomial Type)** A sequence \((p_n(x))_{n \geq 0}\) of polynomials is of binomial type if each polynomial \(p_n(x)\) is of degree \(n\), and for all field elements \(a\), and all nonnegative integers \(n\),
\[
p_n(x + a) = \sum_{k=0}^{n} \binom{n}{k} p_k(a)p_{n-k}(x).
\]

It is a basic result of Umbral calculus (see for example [40]) that every sequence of binomial type is associated with a unique delta operator and vice versa. \((p_n(x))_{n \geq 0}\) is the associated sequence of the delta operator \(f(D)\) if and only if \(f(D)p_n(x) = np_{n-1}(x)\) for positive \(n\), and \(p_n(0) = \delta_{n,0}\) for nonnegative \(n\).

In the present theory, the logarithmic analogs of sequences of binomial type are the graded sequences of binomial type defined below. As before, this definition turns out to be equivalent to that of a Roman graded sequence.

**Definition 3.6.2 (Graded Sequence of Logarithmic Binomial Type)** A graded sequence of formal power series of logarithmic type \(p_\alpha^\alpha(x)\) is of logarithmic binomial type if for all \(a\) and \(\alpha\), and all complex numbers \(z\),
\[
\sum_b \binom{a}{b} \left( E^z p_b^{(0)}(x) \right)_{(0)} p_a^{-\alpha}(x) \text{ is a convergent sum which equals } E^z p_\alpha^\alpha(x).
\]

**Discrete** Note that in the discrete case \(\left( E^z p_b^{(0)}(x) \right)_{(0)}\) is merely \(p_b^{(0)}(z)\). Thus, for \(\alpha = (0)\), we obtain simply the definition of a sequence of polynomials of binomial type. For \(\alpha = (1)\) and \(a\) a negative integer, we obtain factor sequences. [39]

Thus, the notion of a graded sequence of logarithmic binomial type subsumes both the notion of a sequence of polynomials of binomial type, and the notion of a factor sequence. In view of the present theory, these older notions can be seen as obsolete.

We next prove that, as promised, every graded sequence of logarithmic binomial type is a Roman graded sequence, and conversely.

**Theorem 3.6.3** A logarithmic graded sequence \(p_\alpha^\alpha(x)\) is the \((n^{th})\) basic graded sequence for some delta operator \(f(D)\) (and integer \(n\)) if and only if it is a graded sequence of logarithmic binomial type.
Proof: (Only if) Proposition 3.4.4.

(If) To prove that a graded sequence is basic, we must demonstrate each of the three properties enumerated in Definition 3.4.1.

(1) By Definition 3.6.2, 
\[ p_0^{(0)}(x) = p_0^{(0)}(0)p_0^{(0)}(x), \]
and \( I \) is a field, so we have 
\[ p_0^{(0)}(x) = 1. \]

(2) Whereas for \( a \) positive, we have
\[ 0 = p_a^{(0)}(x) - p_0^{(0)}(x) = \sum_{b > 0} \left\lfloor \frac{a}{b} \right\rfloor \left\lfloor \frac{a-b}{b} \right\rfloor p_a^{(0)}(x), \]
and the \( p_a^{(0)}(x) \) form a pseudobasis for \( I^{(0)} \), so \( \left\langle p_b^{(0)}(x) \right\rangle_{(0)} = 0 \) for \( b > 0 \). Thus, by regularity,
\[ \langle p_a^{(0)}(x) \rangle_{\alpha} = \delta_{n,0} \]
for all \( a \) and \( \alpha \).

(3) Define a sequence of continuous, linear operators \( Q^b \) by the identity
\[ Q^b p_a^\alpha(x) = \frac{|a|}{|a-b|} p_{a-b}^\alpha(x). \]
Clearly, \( Q^c Q^b = Q^{b+c} \), and the map \( b \mapsto Q^b \) is continuous in the operator topology, so by the characterization of exponentiation in [25], it remains only to show that \( Q = Q^1 \) is a delta operator.

By inspection \( Q \) is regular, and lowers the degree of any logarithmic series by one. We now demonstrate via the following string of identities that \( Q \) is D-invariant:
\[
QE^\alpha x p_a^\alpha(x) = Q \sum_b \left[ \frac{a}{b} \right] \left\langle E^\alpha x p_b^{(0)}(x) \right\rangle_{(0)} p_{a-b}^\alpha(x) \\
= \sum_b \left[ \frac{a}{b} \right] [a-b] \left\langle E^\alpha x p_b^{(0)}(x) \right\rangle_{(0)} p_{a-b-1}^\alpha(x) \\
= \sum_b \left[ \frac{a-1}{b} \right] [a] \left\langle E^\alpha x p_b^{(0)}(x) \right\rangle_{(0)} p_{a-b-1}^\alpha(x) \\
= E^\alpha Q p_a^\alpha(x). \]

We summarize the results of the preceding sections in the following theorem:
Theorem 3.6.4 For any graded sequence of formal power series of logarithmic type $p_\alpha^n(x)$, the following statements are equivalent:

1. $p_\alpha^n(x)$ is a Roman graded sequence. (Definition 3.2.2)
2. $p_\alpha^n(x)$ is the ($n^{th}$) associated graded sequence for some unique delta operator $f(D)$. (Definition 3.3.2)
3. $p_\alpha^n(x)$ is the ($m^{th}$) conjugate graded sequence for some unique delta operator $g(D)$. (Definition 3.5.1)
4. $p_\alpha^n(x)$ is the ($n^{th}$) basic graded sequence for some unique delta operator $f(D)$. (Definition 3.4.1)
5. $p_\alpha^n(x)$ is a graded sequence of logarithmic binomial type. (Definition 3.6.2)
6. $p_\alpha^n(x)$ is a graded sequence of formal power series of logarithmic type whose coefficients

\[ p_\alpha^n(x) = \sum_{k \leq n} b_{nk} \lambda_k^n(x) \]

satisfy equation (3.4).

Furthermore, in this case, $p_\alpha^n(x)$ is associated with and basic for the same delta operator $f(D)$ (and integer $n$).

Proof: Condition 1 is equivalent to Condition 6 by Porism 3.2.3, it is equivalent to Condition 3 by Proposition 3.5.2, and it is equivalent to Condition 2 by Proposition 3.4.6. Next, Condition 2 is equivalent to Condition 4 by Theorem 3.4.2. Finally, Condition 4 is equivalent to Condition 5 by Theorem 3.6.3. □
Chapter 4

Relations Among Roman Graded Sequences

Continuous In this chapter, we focus our attention primarily upon standard operators, and their corresponding 0th associated and conjugate sequences which we have seen (Propositions 3.3.5C and 3.5.3C) are standard sequences.

Discrete In this chapter, we maintain the same level of generality as before. However, our results are still less general than the corresponding results in the continuous case.

4.1 Transfer Operators

Let $p_\alpha(x)$ be the standard associated graded sequence for the delta operator $f(D)$. In view of Propositions 3.5.2, and 3.4.6, the standard conjugate graded sequence for the operator $f(D)$ is in general some other Roman graded sequence, say $q_\alpha(x)$. By Proposition 3.3.1, this graded sequence is in turn the standard associated graded sequence for another delta operator $g(D)$. What is the relationship between $f(D)$ and $g(D)$? And between $p_\alpha(x)$ and $g(D)$?

We shall prove (Corollary 4.2.9) the remarkable fact that the formal power series $f(D)$ and $g(D)$ are inverses to each other in the sense of LaGrange Inversion [25], and $p_\alpha(x)$ is the conjugate graded sequence for $g(D)$. Actually, the results we shall obtain are more sweeping, and lead to a powerful technique for establishing identities among formal power series of logarithmic type.

We shall occasionally use the boldface $p$ to denote a logarithmic graded sequence $p_\alpha(x)$. Thus,
DEFINITION 4.1.1 (Umbral Composition) Let $p$ and $q$ be two graded sequences, and let $c_{ab}$ denote the coefficients of the $p$ sequence $c_{ab} = |\lambda^a_b(x)|p^a_b(x)$. We define the umbral composition of the $q$ graded sequence with the $p$ graded sequence to be the graded sequence defined by the following convergent summation

$$q^a_p(x) = \sum_b c_{ab}p^a_b(x).$$

PROPOSITION 4.1.2 Given two graded sequences $p^a_\alpha(x)$ and $q^a_\alpha(x)$. Their composition $p^a_\alpha(q)$ is a well defined graded sequence.

Proof: We need only consider the continuous case. By the Noetherian condition, their composition is well defined. The conditions on its degree and order follow immediately. Continuity follows since the composition of two continuous functions is itself continuous.

By [25], D is the two-sided identity for the group of delta operators under (0-)composition, by definition the graded sequence of harmonic logarithms is the two-sided identity for the semigroup formed by the operation of composition of standard Roman graded sequences. We shall show that this semigroup is actually a group, and that the two groups are naturally isomorphic. The isomorphism is given by the function which associates each delta operator with its associated graded sequence. The crucial role in obtaining these results is played by the notion of a transfer operator which we proceed to define:

DEFINITION 4.1.3 (Transfer Operator) Let $p^a_\alpha(x)$ be a Roman graded sequence. The transfer operator associated with the graded sequence is the continuous linear operator $\tau_p: I \to I$ defined as

$$\tau_p \lambda^a_\alpha(x) = p^a_\alpha(x)$$

for all $a$ and $\alpha$.

Thus, $\tau_p q^a_\alpha(x) = q^a_p(x)$.

Discrete If $p^a_\alpha(x)$ is the associated graded sequence for the delta operator $f(D)$, we frequently write $\tau_f$ for the transfer operator associated with $p^a_\alpha(x)$.

Continuous If $p^a_\alpha(x)$ is the $n^{th}$ associated graded sequence for the delta operator $f(D)$, we frequently write $\tau_{f,n}$ for the transfer operator associated with $p^a_\alpha(x)$. We also write $\tau_f$ for $\tau_{f,0}$.

PROPOSITION 4.1.4 The transfer operator is a well defined regular operator.

Proof: Proposition 4.1.2.
4.2 Adjoints

We must now define the adjoint of a regular operator. This definition is equivalent to the usual adjoint with respect to the inner product defined in §4.3.1A.

**Definition 4.2.1** (Adjoint) If $\theta$ is a continuous linear operator on $I^\alpha$ for some $\alpha \neq (0)$; then the adjoint of $\theta$ is defined to be the linear operator $\text{adj}(\theta)$ on Artinian operators defined by

$$\langle [\text{adj}(\theta)] f(D) p(x) \rangle_\alpha = \langle f(D) \theta p(x) \rangle_\alpha \quad (4.1)$$

for all formal power series of logarithmic type $p(x) \in I^\alpha$, and for all Artinian operators $f(D) \in \Lambda^+$.

**Discrete** In the discrete case, if $\theta$ is a continuous, linear operator on $I^{(0)}$, then the adjoint of $\theta$ is defined to be the linear operator $\text{adj}(\theta) : \Lambda \rightarrow \Lambda$ such that equation (4.1) holds for all formal power series of logarithmic type $p(x) \in I^{(0)}$, and for all differential operators $f(D) \in \Lambda$.

**Proposition 4.2.2** Let $\theta$ be as in Definition 4.2.1. Then $\text{adj}(\theta)$ is a well defined continuous linear operator.

**Proof:** (Well Defined) Spanning argument (Proposition 2.5.5).

(Linear) Consider the following string of equalities

$$\langle \text{adj}(a_1 \theta_1 + a_2 \theta_2) f(D) p(x) \rangle_\alpha = \langle f(D)(a_1 \theta_1 + a_2 \theta_2)p(x) \rangle_\alpha = a_1 \langle f(D)\theta_1 p(x) \rangle_\alpha + a_2 \langle f(D)\theta_2 p(x) \rangle_\alpha = a_1 \langle \text{adj}(\theta_1) f(D)p(x) \rangle_\alpha + a_2 \langle \text{adj}(\theta_2) f(D)p(x) \rangle_\alpha = \langle (a_1 \text{adj}(\theta_1) + a_2 \text{adj}(\theta_2)) f(D)p(x) \rangle_\alpha.$$  

(Continuity) Note that the expression $\langle f(D) \theta p(x) \rangle_\alpha = \langle \text{adj}(\theta) f(D)p(x) \rangle_\alpha$ is continuous in $f(D)$ and $p(x)$.

Let $\theta$ be a continuous linear operator on $I$ or $I^+$ which maps each $I^\alpha$ to itself. The adjoint of the restriction of $\theta$ to $I^\alpha$ is denoted by $\text{adj}(\theta)^\alpha$. If $\theta$ is a regular, continuous, linear operator on $I^+$ or all of $I$, the adjoint of $\theta$ is the unique operator which coincides with $\text{adj}(\theta)^\alpha$ for $\alpha \neq (0)$.

The adjoint of nonregular operators on $I$ or $I^+$ can be similarly defined; however, we omit the discussion, since it would be an unnecessary digression.

Let us consider several important operators, and as an exercise compute their adjoints.
CHAPTER 4. RELATIONS AMONG ROMAN GRADED SEQUENCES

Example 4.2.1 Let $f(D)$ be an Artinian operator. Then

$$[\text{adj}(f(D))](g(D)) = f(D)g(D).$$

In other words, the adjoint of an Artinian operator $f(D)$ is the operator of multiplication by $f(D)$.

The Roman shift is an operator which is crucial to our work in the remainder of this chapter; moreover, it is not a $D$-invariant operator, so its adjoint is of particular interest.

Example 4.2.2 We define a regular, linear operator $\sigma$ on $I^{\alpha}$ continuous on each $I^{\alpha}$ by requiring that for all $a$,

$$\sigma \lambda_a^\alpha(x) = \begin{cases} 
\lambda_{a+1}^\alpha(x) & \text{if } a \neq -1, \text{ and} \\
0 & \text{if } a = -1
\end{cases}$$

We call the operator $\sigma$ the standard Roman shift. It is not a $D$-invariant operator. For example, $D\sigma - \sigma D = I \neq 0$. The standard Roman shift has as its adjoint the Pincherle derivative operator $\text{adj}(\sigma)(D^a) = aD^{a-1}$ for all integers $a$, since

$$\langle bD^{b-1}\lambda_a^\alpha(x) \rangle_{\alpha} = [b]!(1 - \delta_{0,b})\delta_{a+1,b} = \langle D^b \sigma \lambda_a^\alpha(x) \rangle_{\alpha}.$$ 

Continuous In the continuous case, $\sigma$ is not continuous over all of $I^{\alpha}$ since the limit of $\sigma p(x)$ as $p(x)$ approaches $1/x$ is not well defined.

The standard Roman shift is a logarithmic generalization of the operator $x$ of multiplication by $x$, since $\sigma x^a = x^{a+1}$ for $a \neq -1$. See Theorem 4.3.7 for an amazing property obeyed by the Roman shift.

The Pincherle derivative has several equivalent definitions.

Definition 4.2.3 (Pincherle Derivative) Let $f(D)$ be an Artinian operator. We can define its Pincherle derivative $f'(D)$ by any of the following equivalent formulations:

1. The Pincherle derivative is the continuous, linear map

$$' : \Lambda^+ \to \Lambda^+$$

$$D^a \mapsto aD^{a-1}.$$

2. $f'(D) = f(D)\sigma - \sigma f(D)$.

3. The Pincherle derivative is the adjoint of the standard Roman shift $\sigma$. In other words, $f'(D) = \text{adj}(\sigma)f(D)$. 

4. The Pincherle derivative of an Artinian operator is its derivative as an Artinian series. (See the section on derivatives in [25], and Theorem 2.4.1 here.)

Before considering the adjoint of an arbitrary transfer operator. Let us consider the following transfer operator which illustrates an interesting quirk of the continuous theory; it clarifies the meaning of the “; n” in the definition of the exponentiation of Artinian series, and their composition.

Example 4.2.3C Let $\psi_n$ be the transfer operator $\tau_{D,n}$. Thus, $\psi_n$ is the continuous linear operator

$$\psi_n \lambda_a(x) = e^{2\pi i n} \lambda_a(x).$$

$\psi_n$ is $D$-invariant; however, $\psi_n$ is not $D$-invariant for $n \neq 0$. For example, $\psi_1 D^{1/2} = -D^{1/2} \psi_1$. On the other hand, $\psi_n \psi_m = \psi_{n+m} = \psi_m \psi_n$.

Now, $\text{adj}(\psi_n) f(D) = f(D; n)$ where for $f(x) = \sum_a c_a x^a$, $f(g; n) = \sum_a c_a g(x)^{a/n}$.

Finally, we consider the adjoint of an arbitrary transfer operator.

Proposition 4.2.4 If $\tau$ is a transfer operator, then its adjoint $\text{adj}(\tau)$ is an automorphism of $\Lambda^+$. Discrete Moreover, $\text{adj}(\tau)^{(0)}$ is an automorphism of $\Lambda$.

Proof: We show that $\text{adj}(\tau)$ is an monomorphism and that it acts on delta operators. This implies that $\text{adj}(\tau)$ preserves degree, and thus, that $\text{adj}(\tau)$ is an automorphism.

Let $\tau$ be associated with the Roman graded sequence $p_a(x)$.

(Injectivity) Assume that $\text{adj}(\tau) f(D) = \text{adj}(\tau) g(D)$ for some pair of Laurent operators $f(D)$ and $g(D)$. Thus, for all $p(x) \in \mathcal{I}^{(1)}$, $
\langle f(D)p(x) \rangle_{(1)} = \langle g(D)p(x) \rangle_{(1)},$

so by the spanning argument (Proposition 2.5.5), we infer $f(D) = g(D)$.

(Morphism) By Proposition 4.2.2, $\text{adj}(\tau)$ is continuous and linear, so we need only confirm that $\text{adj}(\tau)$ preserves multiplication. Let $f(D)$ and $g(D)$ be Artinian operators. Then we have for all $a$,

$$\langle \text{adj}(\tau)(f(D)g(D)) \lambda_a^{(1)}(x) \rangle_{(1)} = \langle f(D)g(D)\tau \lambda_a^{(1)}(x) \rangle_{(1)}$$

$$= \langle f(D)g(D)p_a^{(1)}(x) \rangle_{(1)}$$

$$= \sum_b \left\lfloor a \right\rfloor \left\lceil b \right\rceil \langle f(D)p_b^{(1)}(x) \rangle_{(1)} \langle g(D)p_{a-b}^{(1)}(x) \rangle_{(1)}$$

$$= \sum_b \left\lfloor a \right\rceil \left\lceil b \right\rceil \langle f(D)\tau \lambda_b^{(1)}(x) \rangle_{(1)} \langle g(D)\tau \lambda_{a-b}^{(1)}(x) \rangle_{(1)}$$

$$= \sum_b \left\lfloor a \right\rceil \left\lceil b \right\rceil \langle \text{adj}(\tau)f(D) \rangle_{(1)} \langle \text{adj}(\tau)g(D) \rangle_{(1)} \langle \lambda_b^{(1)}(x) \rangle_{(1)} \langle \lambda_{a-b}^{(1)}(x) \rangle_{(1)}$$

$$= \langle \text{adj}(\tau)f(D) \rangle_{(1)} \langle \text{adj}(\tau)g(D) \rangle_{(1)} \langle \lambda_a^{(1)}(x) \rangle_{(1)}$$
Thus, $\text{adj}(\tau)f(D) = (\text{adj}(\tau)f(D))(\text{adj}(\tau)g(D))$ by the spanning argument.

**Degre** We need only consider the continuous case. Suppose $p_a^n(x)$ is the $(n^{th})$ associated graded sequence for the delta operator $f(D)$, then for all $a$,

$$\langle \text{adj}(\tau)f(D)^{k\cdot n} \lambda_a(1)(x) \rangle_{(1)} = \langle f(D)^{k\cdot n} p_a^{(1)}(x) \rangle_{(1)} = [a]! \delta_{ab} = \langle D^k \lambda_a(1)(x) \rangle_{(1)}$$

and by the spanning argument (Proposition 2.5.5), $\text{adj}(\tau)f(D) = D$. The conclusion now follows by the Expansion Theorem 2.5.4.

The most important properties of transfer operators are stated in the following proposition.

**Proposition 4.2.5**

1. A transfer operator maps Roman graded sequences to Roman graded sequences.

2. If $\tau : p^\alpha_a(x) \mapsto q^\alpha_a(x)$ is a continuous linear operator, where the $p^\alpha_a(x)$ and $q^\alpha_a(x)$ are the $(n^{th}$ and $m^{th})$ associated graded sequences for the delta operators $f(D)$ and $g(D)$ respectively, then we have $\text{adj}(\tau)g(D) = f(D)$.

3. Continuous Moreover, $\text{adj}(\tau)g(D)^{a\cdot n} = f(D)^{b\cdot n}$ for all $a$, and if $p^\alpha_a(x)$ and $q^\alpha_a(x)$ are standard then the operator $\tau$ above is a transfer operator.

**Discrete** The operator $\tau$ above is a transfer operator.

**Proof:** We need only consider the continuous case.

(1) Let $\tau : \lambda^\alpha_a(x) \mapsto p^\alpha_a(x)$ be a transfer operator. By Theorem 4.2.4, $\text{adj}(\tau)$ is an isomorphism of $\Lambda^+$. Let $q^\alpha_a(x)$ be the $m^{th}$ associated graded sequence for the delta operator $g(D)$. Then,

$$\langle \text{adj}(\tau)^{-1} g(D)^{b\cdot m} q^\alpha_a(x) \rangle_{\alpha} = \langle g(D)^{b\cdot m} q^\alpha_a(x) \rangle_{\alpha} = [a]! \delta_{ab}.$$  

By 25 and Proposition 4.2.4, $\text{adj}(\tau)^{-1} g(D)^{b\cdot m} = (\text{adj}(\tau)^{-1} g(D))^{b\cdot n}$ for some $n$.

Hence, $\tau q^\alpha_a(x)$ is an associated graded sequence for the delta operator $\text{adj}(\tau)^{-1} g(D)$.

(2) We have the following sequence of equalities:

$$\langle \text{adj}(\tau)g(D)p_a^{(1)}(x) \rangle_{(1)} = \langle g(D)\tau p_a^{(1)}(x) \rangle_{(1)} = \langle g(D)q_a^{(1)}(x) \rangle_{(1)} = \delta_{a,1} = \langle f(D)p_a^{(1)}(x) \rangle_{(1)}.$$
4.2. ADJOINTS

Hence, by the spanning argument, \( \text{adj}(\tau)g(D) = f(D) \).

\( (3) \) Similar to \( (2) \).

\( (4) \) More generally, for an arbitrary Artinian operator \( \sum b_c g(D)^{b_c} \), we have by Proposition 4.2.4 and [25], \( \text{adj}(\tau) \sum b_c g(D)^{b_c} = \sum c_b g(D)^{c_b} \) for some \( m \). Specialize to the case of \( \sum b_c D^b = f^{(-1)}(D)^{b_c} \) to get

\[
\text{adj}(\tau) \left( f^{(-1)}(g) \right)^{b_c} = e^{2\pi ibm} D^b,
\]

and hence

\[
\left\langle \left( f^{(-1)}(g) \right)^{b_c} \tau \lambda_n^{\alpha}(x) \right\rangle_{\alpha} = \langle e^{2\pi i an} D^b \lambda_n^{\alpha}(x) \rangle_{\alpha}.
\]

In other words, \( \tau e^{2\pi i a m} \lambda_n^{\alpha}(x) \) is associated with the delta operator \( f^{(-1)}(g) \). Finally, note that \( m = 0 \) when all of the sequences in question are standard.

The following results illustrate the applications of the preceding proposition.

**Proposition 4.2.6** If \( p_n^{\alpha}(x) \) and \( q_n^{\alpha}(x) \) are the standard associated graded sequences of the delta operators \( g(D) \) and \( f(D) \) respectively, then the composition \( g(f) \) (resp. \( g(f;0) \)) is the delta operator with standard associated graded sequence \( q_n^{\alpha}(p_n^{\alpha}) \).

*Proof:* If \( \tau : \lambda_n^{\alpha}(x) \rightarrow p_n^{\alpha}(x) \) is a transfer operator, then as in the proof of Part (1) of Proposition 4.2.5, \( \tau q_n^{\alpha}(x) \) is the standard associated graded sequence for \( (\text{adj}(\tau))^{-1} g(D) \). However, \( \tau q_n^{\alpha}(x) = q_n^{\alpha}(p_n^{\alpha}) \) by Definition 4.1.1. Moreover, Part (2) of Proposition 4.2.5 asserts that \( \text{adj}(\tau) f(D) = D \), so \( \text{adj}(\tau)^{-1} D = f(D) \), and, thus, \( \text{adj}(\tau)^{-1} g(D) = g(f) \) (resp. \( g(f;0) \)). The conclusion follows.

**Corollary 4.2.7** The set of Roman graded sequences is closed under umbral composition. Similarly, the set of standard Roman graded sequences is closed under umbral composition.

**Corollary 4.2.8** (Inverses) Let \( p_n^{\alpha}(x) \), and \( q_n^{\alpha}(x) \) be (standard) Roman graded sequences associated with the (standard) delta operators \( f(D) \), and \( g(D) \) respectively. Then the following statements are equivalent:

1. \( f(g) = D \) (resp. \( f(g;0) = D \)),
2. \( g(f) = D \) (resp. \( g(f;0) = D \)),
3. \( q_n^{\alpha}(p) = \lambda_n^{\alpha}(x) \), and
4. \( p_n^{\alpha}(q) = \lambda_n^{\alpha}(x) \).

**Corollary 4.2.9** The standard associated Roman graded sequence for the standard operator \( f(D) \) is the standard conjugate Roman graded sequence for the delta operator \( f^{(-1;0)}(D) \), and visa versa.

*Proof:* Theorem 3.3.4 and Corollary 4.2.8.
4.3 The Roman Shift

The following definition generalizes to all Roman graded sequences the notion of the standard Roman shift (Example 4.2.2).

**Definition 4.3.1 (Roman Shift)** If \( p_\alpha^a(x) \) is a Roman graded sequence, the roman shift relative to \( p_\alpha^a(x) \) is the linear operator \( \sigma_p : \mathcal{I} \rightarrow \mathcal{I} \) defined by

\[
\sigma_p p_\alpha^a(x) = \begin{cases} 
p_\alpha^{a+1}(x) & \text{if } a \neq -1, \\
0 & \text{if } a = -1.
\end{cases}
\]

(4.2)

for all \( a \) and \( \alpha \) where \( \sigma_p \) is continuous over each \( \mathcal{I}^\alpha \) but not continuous simultaneously over all of \( \mathcal{I} \).

As with Transfer operators, if \( p_\alpha^a(x) \) is (\( n \))-associated with the delta operator \( f(D) \), we also write \( \sigma_f \) (or \( \sigma_{f,n} \)) instead of \( \sigma_p \). When no graded sequence of delta operator has been specified, we assume \( \sigma = \sigma_{D,0} = \sigma_\lambda \), that is, that \( \sigma \) is the standard Roman shift as previously defined.

**Definition 4.3.2 (Artinian Derivation)** Let \( Q \) be an operator on the Artinian (resp. Noetherian) algebra such that

\[
Q f(x)^{a-n} g(x)^{b+m} = a(Q f(x))(f(x)^{a-1:n} g(x)^{b:m} + b(Q g(x)) f(x)^{a:n} g(x)^{b-1:m}).
\]

Note that an Artinian (resp. Noetherian) derivation is automatically a derivation, and thus is linear.

**Lemma 4.3.3** The derivative is an Artinian (resp. Noetherian) derivation.

**Proposition 4.3.4**

1. A regular, continuous, linear operator \( \theta \) defined on \( \mathcal{I}^+ \) is a Roman shift if and only if \( \text{adj}(\theta) \) is a continuous, everywhere defined (Artinian) derivation of the algebra of Artinian operators \( \Lambda^+ \) for which \( \text{adj}(\theta f(D)) = 1 \) for some delta operator \( f(D) \).

2. A regular, continuous, linear operator \( \theta \) defined on the logarithmic algebra \( \mathcal{I} \) is a Roman shift if and only if \( \text{adj}(\theta) \) is a continuous, everywhere defined, (Artinian) derivation of the algebra of differential operators \( \Lambda \) for which \( \text{adj}(\theta f(D)) = 1 \) for some delta operator \( f(D) \).
4.3. THE ROMAN SHIFT

Proof: We need only consider the continuous case.

(Only if) Let \( p^\alpha_a(x) \) be a Roman graded sequence of formal power series of logarithmic type. By Proposition 3.5.2, \( p^\alpha_a(x) \) is \((n-)\)-associated with a delta operator \( f(D) \). Now, we have

\[
\langle \text{adj}(\sigma_f) f(D)^{b,n} p^1_a(x) \rangle_{(1)} = \langle f(D)^{b,n} \sigma_f p^1_a(x) \rangle_{(1)} = \delta_{a,-1} \langle f(D)^{b,n+1} p^1_a(x) \rangle_{(1)} = (a+1) [a]! \delta_{a+1,b} = b [a]! \delta_{a,b-1} = \langle b f(D)^{b-1,n} p^1_a(x) \rangle_{(1)}
\]

Hence, by the spanning argument, \( \langle \text{adj}(\sigma_f) f(D)^{b,n} = b f(D)^{b-1,n} \rangle \). Also, \( \text{adj}(\sigma_f) f(D) = 1 \). By the continuity of \( \text{adj}(\sigma_f) \), the result follows.

(If) Conversely, suppose \( \text{adj}(\theta) \) is a continuous, everywhere defined derivation of \( \Lambda^+ \) with \( \text{adj}(\theta) f(D) = 1 \). Let \( \sigma_p \) be the Roman shift associated with \( p^\alpha_a(x) \), the \( n \)th associated graded sequence for \( f(D) \). We shall show that \( \theta = \sigma_p \).

Thus, by the spanning argument, \( \text{adj}(\sigma_f) = (\text{adj}(\sigma_f) g(D)) \text{adj}(\sigma_g) \). ■

Next, we derive the chain rule for Roman shifts.

**Proposition 4.3.5** (Chain Rule) Suppose \( \sigma_f \), and \( \sigma_g \) are Roman shift operators. Then

\[
\text{adj}(\sigma_f) = (\text{adj}(\sigma_f) g(D)) \text{adj}(\sigma_g).
\]

**Proof:** We need only consider the continuous case.

For any Artinian operator \( h(D) = \sum_b c_b g(D)^{b,n} \),

\[
\text{adj}(\sigma_f) h(D) = \sum_b b c_b g(D)^{b-1,n} \text{adj}(\sigma_f) g(D) = [\text{adj}(\sigma_f) g(D)] [\text{adj}(\sigma_g) h(D)],
\]

so

\[
\text{adj}(\sigma_f) = (\text{adj}(\sigma_f) g(D)) \text{adj}(\sigma_g). \quad \blacksquare
\]

The following proposition allows us to relate two Roman shift operators.
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Proposition 4.3.6 If $\sigma_f$, and $\sigma_g$ are regular shift operators, then

$$\sigma_f = \sigma_g \text{adj}(\sigma_f)g(D).$$

Proof: For any Artinian series $h(D)$ and logarithmic series $p(x)$, we have

$$h(D)\sigma_f p(x) = \text{adj}(\sigma_f)h(D)p(x) = (\text{adj}(\sigma_g)h(D))(\text{adj}(\sigma_f)g(D))p(x)$$

by the chain rule. This in turn equals $h(D)\sigma_g(\text{adj}(\sigma_f)g(D))p(x)$.

4.3.1A Orthogonal Functions

We can now define a symmetric non-degenerate bilinear form associated with any Roman graded sequence $p^*_a(x)$. Consider the principal subsequence $\tilde{p}_a(x)$ (Definition 3.1.1). Define an inner product

$$\langle \tilde{p}_a(x)|\tilde{p}_b(x) \rangle = \delta_{ab} |a|!.$$ 

One verifies that relative to this bilinear form, the operator $f(D)$ is adjoint to the Roman shift $\sigma_f$. For example, for the harmonic logarithm one has $\tilde{\lambda}_a(x) = x^a$ for all $a$, and

$$\langle D^a|x^b \rangle = \langle x^a|\sigma x^b \rangle$$

where $\sigma$ is the standard Roman shift, which restricts to the operator $x$ of multiplication by $x$. The principal sequence $\tilde{p}_a(x)$ is the set of eigenfunctions of the operator $\sigma_f f(D)$ with eigenvalue $n$. For example, the $x^a$ form the set of eigenvalues of the operator $xD$.

The bilinear form is not definite, since $\langle 1 + x^{-2}|1 + x^{-2} \rangle = 0$.

**Discrete** However, in the discrete case, we can define a Hermitian form

$$\langle p(x)|q(x) \rangle_c = - \overline{\langle p(x)|q(x) \rangle},$$

so we have

$$\langle (ix)^n(ix)^n \rangle_c = \langle (ix)^n(ix)^n \rangle$$

$$= \begin{cases} -\langle x^n|x^n \rangle & \text{for } n \text{ even, and} \\ \langle x^n|x^n \rangle & \text{for } n \text{ odd,} \end{cases}$$

and thus for $n$ negative

$$\langle (ix)^n(ix)^n \rangle_c = 1/(-n - 1)! > 0.$$
Extending by linearity, we obtain a Hermitian inner product which is positive definite, where the sequence \( h_n(x) \) defined as
\[
h_n(x) = \begin{cases} 
x^n & \text{for } n \geq 0, \\
(ix)^n & \text{for } n < 0
\end{cases}
\]
is a complete orthogonal sequence in the Hilbert space obtained by completion relative to this Hermitian inner product. We can obtain this sort of sequence using the Knuth coefficients [19] with \( \epsilon = -i \) in place of the Roman coefficients. In this manner,
\[
[n]! = \begin{cases} 
n! & \text{if } n \geq 0, \\
(-1)^n i/(-n - 1)! & \text{if } n < 0.
\end{cases}
\]
One can therefore develop a spectral theory of the operator \( xD \) in this Hilbert space (rather than with the smaller one including only positive powers of \( x \), as is done classically).

### 4.3.2A Rodrigues’ Formula

Note the following amazing fact.

**Theorem 4.3.7** Let \( p(x) \) be a finite linear combination of harmonic logarithms \( \sum_{i=1}^{k} c_i \lambda_0^{(i)}(x) \). Then for any complex number \( z \), we have the following formal identity
\[
(e^{-z \sigma} D e^{z \sigma})p(x) = (D - zI)p(x).
\]

By \( e^{-a \sigma} D e^{a \sigma} p(x) \) we merely mean to indicate the expression
\[
\sum_{j \geq 0 \atop k \geq 0} \frac{(-1)^j a^{j+k}}{j! k!} \sigma^j D \sigma^k p(x)
\]
which we assert does in fact converge to the indicated value. The “operator” \( e^{z \sigma} \) can not be extended to all of \( \mathbb{I} \), since the series \( \sum_{k \geq 0} z^k \sigma^k / k! \).

The corresponding classical identity
\[
e^{-x} D e^{x} = D + I
\]
is associated with the classical identity
\[
D x - x D = I
\]
which corresponds to the logarithmic identity
\[
D \sigma - \sigma D = D' = I.
\]
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Proof: By linearity, it suffices to consider the case \( p(x) = \lambda_\alpha^a(x) \). There are three cases to consider.

(When \( a \) is not a nonpositive integer) Classical proof can be applied \textit{mutatis mutandis}.

(When \( a = 0 \)) We calculate:

\[
e^{-z\sigma}D e^{z\sigma} \lambda_0^\alpha(x) = \sum_{j \geq 0} \sum_{k \geq 0} \frac{(-1)^j z^{j+k}}{j!k!} \sigma^j D \sigma^k \lambda_0^\alpha(x)
\]

\[
= \lambda_1^\alpha(x) - z \sum_{j \geq 0} \sum_{k \geq 0} \frac{(-1)^j z^{j+k}}{j!(k+1)!} \sigma^j \lambda_0^\alpha(x)
\]

\[
= \lambda_1^\alpha(x) - z \sum_{j \geq 0} \sum_{k \geq 0} \frac{(-1)^j z^{j+k}}{j!} \sigma^j \lambda_0^\alpha(x)
\]

\[
= \lambda_1^\alpha(x) - z \left( \sum_{k \geq 0} k \sum_{j=0}^k (-1)^j \binom{k}{j} \lambda_0^\alpha(x) \right)
\]

\[
= \lambda_1^\alpha(x) - z \lambda_0^\alpha(x)
\]

(When \( a \) is a negative integer) We similarly compute

\[
e^{-z\sigma}D e^{z\sigma} \lambda_a^\alpha(x) = \sum_{j \geq 0} \sum_{k \geq 0} \frac{(-1)^j a^{j+k}}{j!k!} \sigma^j D \sigma^k \lambda_a^\alpha(x)
\]

\[
= \sum_{k \geq 0} \sum_{j=0}^k \frac{(-1)^j a^k}{j!(k-j)!} \sigma^j \lambda_a^\alpha(x)
\]

\[
= -a^{-1} \sum_{k=0}^j \sum_{j=0}^{a-1} \frac{(-1)^j a^k(a + k - j)}{j!(k-j)!} \lambda_{a+k-1}^\alpha(x)
\]

\[
= a \lambda_{a-1}^\alpha(x) + z \lambda_a^\alpha(x)
\]

\[
= (D - zI) \lambda_a^\alpha(x).
\]

4.4 Explicit Formulas for Roman Graded Sequences

We are now ready to derive the following recurrence formula for Roman graded sequences:

**Theorem 4.4.1** (Recurrence Formula) If \( p_a^\alpha(x) \) is an associated graded sequence of delta operator \( f(D) \), then for all \( a \neq -1 \) and for all \( \alpha \)

\[
p_{a+1}^\alpha(x) = \sigma(f'(D))^{-1} p_a^\alpha(x)
\]

where \( \sigma \) is the standard Roman shift.
4.4. EXPLICIT FORMULAS FOR ROMAN GRADED SEQUENCES

Proof: Let $a \neq -1$, then by Proposition 4.3.6.

\[
p_{n+1}^a(x) = \sigma f p_n^a(x) = \sigma (\text{adj}(\sigma f(D)))^{-1} p_n^a(x) = \sigma f'(D)^{-1} p_n^a(x).
\]

Next, we give an explicit formula for the associated graded sequence of a delta operator in terms of the residual series of the graded sequence of harmonic logarithms.

**Proposition 4.4.2** If $p_n^a(x)$ is the $(n^{th})$ associated graded sequence of formal power series of logarithmic type for the delta operator $f(D)$, then for all $a$ and all $\alpha \neq (0)$:

\[
\begin{align*}
\text{Discrete} & \quad p_n^a(x) = [a]! f'(D) f(D)^{-1-a} \lambda_1^\alpha(x) \\
\text{Continuous} & \quad p_n^a(x) = [a]! f'(D) f(D)^{-1-a} \lambda_1^\alpha(x)
\end{align*}
\]

Proof: We need consider only the continuous case. Let $q_n^a(x)$ be the graded sequence defined by:

\[
q_n^a(x) = [a]! f'(D) f(D)^{-1-a} \lambda_1^\alpha(x).
\]

It suffices to verify that $q_n^a(x)$ is the $n^{th}$ basic graded sequence for $f(D)$. $q_n^a(x)$ is indeed is a graded sequence since $\deg (f'(D) f(D)^{-1-a}) = -1 - a$ and all the operators in question are continuous. Then, if $a \neq 0$, we have

\[
\langle q_n^a(x) \rangle_\alpha = \langle [a]! f'(D) f(D)^{-1-a} \lambda_1^\alpha(x) \rangle_\alpha = [a]! \langle (f(D)^{-1-a})' \lambda_1^\alpha(x) \rangle_\alpha.
\]

Now, $\langle f(D)^{-a} \lambda_1^\alpha(x) \rangle_\alpha = [D^{-1}] (f(D)^{-a})'$.

The crucial observation in any theorem related to Lagrange inversion is that for all Artinian operators $g(D)$, the coefficient $[D^{-1}] g'(D)$ is zero. Hence, $\langle q_n^a(x) \rangle_\alpha = 0$. Thus, Property 2 of Definition 3.4.1 is satisfied.

Now, consider the case. $a = 0$, and $\alpha \neq (0)$. Then we have:

\[
\langle q_0^a(x) \rangle_\alpha = \langle f'(D) f(D)^{-1} \lambda_1^\alpha(x) \rangle_\alpha.
\]

Say that $c_0 = [D^0 f(D)]$. The coefficient $[D^{-1}] f(D)^{-1} = c_1^{-1}$, and the coefficient $[D^0] f'(D) = c_1$. Neither operator has any terms of lower degree. Hence, the coefficient $[D^{-1}] (f'(D) f(D)^{-1}) = 1$, and there are no terms of lower degree. Hence, $\langle q_0^a(x) \rangle_\alpha = 1$. Thus, Property 1 of Definition 3.4.1 is satisfied.
Finally,

\[ f(D)^{b,n} q^a_0(x) = [a]! f'(D) f(D)^{-a} \lambda^a_{-1} (x) = [n] q^a_{a-1} (x) \]

so Property 3 is also satisfied. Therefore, \( p^a_\alpha (x) = q^a_\alpha (x) \). 

Proposition 4.4.2 is unusual in that it does not readily generalize to \( I^{(0)} \) because for \( \alpha = (0) \), \( \lambda^\alpha_{-1}(x) \) equals 0. Moreover, for \( a \geq 0 \), \( f(D)^{-1-a} \) (resp. \( f(D)^{-1-a}; n \)) has negative degree, and thus is not a member of \( \Lambda \).

Nonetheless, its consequence—the transfer formula—still holds for \( I^{(0)} \):

**Theorem 4.4.3 (Transfer Formula)** Let \( f(D) \) be a differential operator of degree one, and let \( p^a_\alpha (x) \) be its \((n)\) associated graded sequence. Then for all \( a \) and \( \alpha \),

\[
\begin{align*}
\text{Discrete} & \quad p^a_\alpha (x) = f'(D) \left( \frac{D}{f(D)} \right)^{a+1} \lambda^a_\alpha (x) \\
\text{Continuous} & \quad p^a_\alpha (x) = f'(D) \left( \frac{D}{f(D)} \right)^{a+1:n} \lambda^a_\alpha (x)
\end{align*}
\]

In particular, its residual series is given by

\[ p^{(1)}_{-1}(x) = f'(D)x^{\frac{1}{2}}. \]

**Proof:** (When \( \alpha \neq (0) \)) The conclusion is immediate from the preceding proposition.

(When \( \alpha = (0) \)) We need only consider the continuous case. By regularity, we have the following string of equalities for all \( a \):

\[
\begin{align*}
p^{(0)}_a(x) & = E_{(0),(1)} p^{(1)}_a(x) \\
& = E_{(0),(1)} f'(D) \left( \frac{D}{f(D)} \right)^{a+1:n} \lambda^{(1)}_a(x) \\
& = f'(D) \left( \frac{D}{f(D)} \right)^{a+1:n} E_{(0),(1)} \lambda^{(1)}_a(x) \\
& = f'(D) \left( \frac{D}{f(D)} \right)^{a+1:n} \lambda^{(0)}_a(x).
\end{align*}
\]

The following variants of the transfer formula are often useful:
4.4. EXPLICIT FORMULAS FOR ROMAN GRADED SEQUENCES

Corollary 4.4.4 Let \( p_\alpha^a(x) \) and \( q_\alpha^a(x) \) be the \((n)th\) and \((m)th\) associated sequences for the delta operators \( f(D) \) and \( g(D) \) respectively, then for all \( a \) and \( \alpha \),

\[
\begin{align*}
\text{Discrete} \quad p_\alpha^a(x) &= \frac{f'(D)g'(D)^{-1}(D)^{a+1}}{f(D)^{a+1}} q_\alpha^a(x) \\
\text{Continuous} \quad p_\alpha^a(x) &= \frac{f'(D)g'(D)^{-1}(D)^{a+1;m}}{f(D)^{a+1;n}} q_\alpha^a(x).
\end{align*}
\]

Proposition 4.4.5 In the notation of Proposition 4.4.2,

\[
\begin{align*}
\text{Discrete} \quad p_\alpha^a(x) &= g(D)^{-a} \lambda_\alpha^a(x) - (g(D)^{-a})' \lambda_{a-1}^a(x) \\
\text{Continuous} \quad p_\alpha^a(x) &= g(D)^{-a;n} \lambda_\alpha^a(x) - (g(D)^{-a;n})' \lambda_{a-1}^a(x)
\end{align*}
\]

for \( a \neq 0 \), where \( g(D) = f(D)/D \).

Proof: We need only consider the continuous case. By Proposition 4.4.2,

\[ p_\alpha^a(x) = f'(D)g(D)^{-1-a;n} \lambda_\alpha^a(x). \]

However,

\[
\begin{align*}
f'(D)g(D)^{-1-a} &= (Dg(D))' g(D)^{-a-1;n} \\
&= (D'g(D) + Dg'(D)) g(D)^{-a-1;n} \\
&= g(D)^{-a} + Dg'(D)g(D)^{-a-1;n} \\
&= g(D)^{-a} + (g(D)^{-a;n})' D/a
\end{align*}
\]

so that

\[ f'(D)g(D)^{-1-a;n} \lambda_\alpha^a(x) = g(D)^{-a;n} \lambda_\alpha^a(x) - (g(D)^{-a;n})' \lambda_{a-1}^a(x). \]

Note that Proposition 4.4.5 does not in general hold for \( a = 0 \).

Corollary 4.4.6 In the notation of Proposition 4.4.5,

\[
\begin{align*}
\text{Discrete} \quad p_\alpha^a(x) &= \sigma g(D)^{-a} \lambda_{a-1}^a(x) \\
\text{Continuous} \quad p_\alpha^a(x) &= \sigma g(D)^{-a;n} \lambda_{a-1}^a(x)
\end{align*}
\]

for \( a \neq 0,1 \), where \( \sigma \) is the standard Roman shift.
Proof: We need only consider the continuous case. By Proposition 4.4.5,
\[ p_\alpha^n(x) = g(D)^{-a:n} \lambda_\alpha^n(x) - (g(D)^{-a:n})' \lambda_\alpha^{n-1}(x) \]
\[ = g(D)^{-a:n} \lambda_\alpha^n(x) - g(D)^{-a:n} \sigma \lambda_\alpha^{n-1}(x) + \sigma g(D)^{-a:n} \lambda_\alpha^{n-1}(x). \]

We conclude with an important remark about graded sequences of formal power series of logarithmic type. Let \( p_\alpha^n(x) \) be a Roman graded sequence with coefficients \( d_{ab}^{\alpha\beta} = \frac{\lambda_\beta^b(x)}{\lambda_\alpha^a(x)} \). By regularity,

1. For \( \alpha, \beta \neq (0) \) and for all \( a \) and \( b \), \( d_{ab}^{\alpha\alpha} = d_{ab}^{\beta\beta} \).
2. For \( a \) and \( b \) not a negative integer, \( d_{ab}^{(0)(0)} = d_{ab}^{(0)(0)} \), and
3. For \( \alpha \neq \beta \), and for all \( a \) and \( b \), \( d_{ab}^{\alpha\beta} = 0 \).

In view of this, we see that in computations with Roman graded sequences it suffices for most purposes to compute in the subspace \( I^{(1)} \). In other words, even in computations with polynomials it is preferable to deal with logarithms first! A quick survey of the examples chapter 5 demonstrates the utility of \( I^{(1)} \) as compared the algebra of polynomials.

### 4.5 Composition of Formal Series

Let \( f(D) \) be a delta operator. All of the formulas for composition and inversions of series (in particular, the various versions of the LaGrange inversion formula) are consequences of the following theorem. See also [25] and [22] for other results concerning the composition of series.

**Proposition 4.5.1** 1. If \( f(D) \) is the delta operator with \((n^{th})\) associated graded sequence \( p_\alpha^n(x) \), then for every Artinian operator \( g(D) \) we have the following convergent series:

\[
\begin{align*}
\text{Discrete} & \quad g(f^{-1}) = \sum_b \frac{\langle g(D)p_\alpha^n(x) \rangle_{ab}}{b!} D^b \\
\text{Continuous} & \quad g(f^{(-1;n)}) = \sum_b \frac{\langle g(D)p_\alpha^n(x) \rangle_{ab}}{b!} D^b
\end{align*}
\]

(4.3)

where \( \alpha \neq (0) \).

2. Equation \((4.3)\) holds for \( \alpha = (0) \) as well whenever \( g(D) \) is a differential operator.
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Proof: We need only consider the continuous case. By the expansion theorem (Theorem 3.3.3),
\[ g(D) = \sum_b \frac{\langle g(D)p_n^a(x) \rangle_b}{[b]!} f(D)^{b,n}. \]

Hence, substituting \( f^{(-1;n)}(D) \) for \( D \) (by the characterization of composition in [25]),
\[
\begin{align*}
g(f^{(-1;n)}) &= \sum_b \frac{\langle g(D)p_n^a(x) \rangle_b}{[b]!} f(f^{(-1;n)})^{b,n} \\
&= \sum_b \frac{\langle g(D)p_n^a(x) \rangle_b}{[b]!} D^b.
\end{align*}
\]

Theorem 4.5.1 has many versions and many corollaries. We first deduce from the Expansion Theorem the following convergent expansion
\[
\begin{cases}
\text{Discrete} & \sum_n \frac{\langle g(D)p_n^a(x) \rangle_a}{[n]!} D^n = g(f^{(-1)}) = \sum_n \frac{\langle g(f^{(-1)})\lambda_n^a(x) \rangle_a}{[n]!} D^n \\
\text{Continuous} & \sum_b \frac{\langle g(D)p_n^a(x) \rangle_b}{[b]!} D^b = g(f^{(-1;n)}) = \sum_b \frac{\langle g(f^{(-1;n)})\lambda_n^a(x) \rangle_a}{[b]!} D^b
\end{cases}
\]

Hence,
\[
\begin{cases}
\text{Discrete} & \langle g(D)p_n^a(x) \rangle_a = \langle g(f^{(-1)})\lambda_n^a(x) \rangle_a \\
\text{Continuous} & \langle g(D)p_n^a(x) \rangle_a = \langle g(f^{(-1;n)})\lambda_n^a(x) \rangle_a
\end{cases}
\]

. An application of the Transfer Formula (Theorem 4.4.3) gives (in, for example, the continuous case)
\[
[a]! \langle g(D)f'(D)f(D)^{-1-a;n}\lambda_a^{(1)}(x) \rangle_{(1)} = \langle g(D)p_a^{(1)}(x) \rangle_{(1)} = \langle g(f^{(-1;n)})\lambda_a^{(1)}(x) \rangle_{(1)}.
\]

By the spanning argument (Proposition 2.5.5), we have the following corollary.

**Corollary 4.5.2** Let \( f(D) \) be a delta operator, and let \( g(D) \) be any Artinian operator. Then we have the following convergent sum:
\[
\begin{cases}
\text{Discrete} & g(f^{(-1)}) = \sum_k \langle g(D)f'(D)f(D)^{-1-k}\lambda_{-1}^{(1)}(x) \rangle_{(1)} D^k \\
\text{Continuous} & g(f^{(-1;n)}) = \sum_a \langle g(D)f'(D)f(D)^{-1-a;n}\lambda_{-1}^{(1)}(x) \rangle_{(1)} D^a.
\end{cases}
\]

By taking \( g(D) = D^n \), we obtain powers of \( f^{(-1;n)}(D) \) (resp. \( f^{(-1)}(D) \)).
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**Corollary 4.5.3** If $f(D)$ is a delta operator with $(n^{th})$ associated graded sequence $p^n_0(x)$, then we have the following convergent expansion

\[
\begin{align*}
\text{Discrete} \quad f^{(-1)}(D)^a &= \sum_k \frac{\langle D^k p_0^{(1)}(x) \rangle_{(1)}}{[k]!} D^k \\
\text{Continuous} \quad f^{(-1; n)}(D)^a &= \sum_b \frac{\langle D^n p_b^{(1)}(x) \rangle_{(1)}}{[b]!} D^b.
\end{align*}
\]
Chapter 5

Examples

5.1 Roman Graded Sequences

We prefix some general considerations about the computation of the Roman graded sequence $p^n_\alpha(x)$ (n-) associated with a delta operator $f(D)$. The crucial step is the computation of the residual series $p^{(1)}_{-1}(x)$. This is given by the simple formula

$$p^{(1)}_{-1}(x) = f'(D)(1/x).$$

Once the residual series is known, any of the series $p^n_\alpha(x)$ can be obtained from the residual series by applying a suitable power of $f(D)$; that is, by the Transfer Formula (Theorem 4.4.3),

$$\begin{align*}
\text{Discrete} & \quad p^n_\alpha(x) = f'(D) \left( \frac{f(D)}{D} \right)^{-k-1} x^k \\
\text{Continuous} & \quad p^n_\alpha(x) = f'(D) \left( \frac{f(D)}{D} \right)^{-a-1}; n x^a
\end{align*}$$

when $a$ is a negative integer, and

$$\begin{align*}
\text{Discrete} & \quad p^n_\alpha(x) = f'(D) \left( \frac{f(D)}{D} \right)^{-k-1} x^k \left( \log x - 1 - \frac{1}{2} - \ldots - \frac{1}{k} \right) \\
\text{Continuous} & \quad p^n_\alpha(x) = f'(D) \left( \frac{f(D)}{D} \right)^{-a-1}; n x^a \left( \log x - \frac{s(-a, 1)}{[-a]!} \right)
\end{align*}$$

otherwise.
If we denote the coefficients of $p^{(1)}_a(x)$ by $c_{ab} = [\lambda^{(1)}_b(x)]p^{(1)}_a(x)$, then it follows from regularity that

$$p^\alpha_a(x) = \sum_b c_{ab} \lambda^\alpha_b(x)$$

for all $\alpha$.

Note that for $\alpha = (0)$ we obtain

$$\begin{cases}
\text{Discrete} & p^{(0)}_k(x) = \sum_{j=0}^b c_{kj}x^j \\
\text{Continuous} & p^{(0)}_{aa}(x) = \sum_{b \in \mathbb{R} - N} c_{ab}x^b.
\end{cases}$$

In the discrete case, this is the sequence of polynomials of binomial type associated with the delta operator $f(D)$; thus we see that even in the case of polynomials, it may be speedier to compute via the logarithmic graded sequence.

### Table 5.1: Examples of Roman Graded Sequences

| Delta Operator | Associated Graded Sequence | Inverse Operator | Conjugate Graded Sequence |
|----------------|---------------------------|-----------------|--------------------------|
| $\Delta = E - I$ | $\lambda^\alpha_a(x)$ | $D$ | $\lambda^\alpha_a(x)$ |
| $\nabla = 1 - E^{-1}$ | $\lambda^\alpha_a(x)$ | $-\log(1 - D)$ | $\phi^\alpha_a(x)$ |
| $A_a = DE^a$ | $\lambda^\alpha_a(x)$ | $\log(1 + D)$ | $\phi^\alpha_a(x)$ |
| $E^z(E - 1)$ | $\lambda^\alpha_a(x)$ | $-\log(1 - D)$ | $\phi^\alpha_a(x)$ |
| $K = (D - I)$ | $L^\alpha_a(x)$ | $K = D/(D - I)$ | $L^\alpha_a(x)$ |

### 5.1.1 Lower Factorial

Other than the Harmonic logarithms (which we have already discussed at great length), our first example of a Roman graded sequence is the logarithmic lower factorial graded sequence.

**Definition 5.1.1** (Forward Difference Operator) Define the forward difference operator $\Delta = E - I = e^D - 1$. Let $(x)^\alpha_a$ denote its standard associated graded sequence; it is called the logarithmic lower factorial graded sequence. Let $\phi^\alpha_a(x)$ denote its standard conjugate graded sequence; it is called the logarithmic exponential graded sequence.
5.1. ROMAN GRADED SEQUENCES

Table 5.2: Lower Factorials \((x)_n^\alpha\)

| \((x)_0^0\) | \((x)_0^1\) | \((x)_0^2\) | \((x)_0^3\) | \((x)_0^4\) |
|---|---|---|---|---|
| \(x(x-1)\) | \(\lambda_2(x) - \lambda_1(x) + B_{3,3}/3x - B_{4,3}/12x^2 + \cdots\) | \(1/(x+1)\) | \(1/(x+1)(x+2)\) |
| \(x\) | \(x \log(x) - x + B_{2,2}/2x - B_{3,2}/6x^2 + \cdots\) | \(\log(x+1) + B_{1,1}/(1 + x) - B_{2,1}/2(1 + x)^2 + B_{3,1}/3(1 + x)^3 - \cdots\) |
| \(1\) | \(\log(x+1) + B_{1,1}/(1 + x) - B_{2,1}/2(1 + x)^2 + B_{3,1}/3(1 + x)^3 - \cdots\) |
| \(0\) | \(1\) |

Now, \(\Delta' = E\), so by Theorem 4.4.3, we calculate the residual series

\[
(x)_{-1}^{(1)} = Ex^{-1} = \frac{1}{x+1}. \tag{5.1}
\]

In general,

**Proposition 5.1.2** For \(n\) a negative integer,

\[
(x)_n^{(1)} = \frac{1}{(x+1) \cdots (x-n)} = (x)_n.
\]

**Proof:** We proceed by induction. The case \(n = 1\) amounts to equation (5.1) which we verified above. Now, suppose proposition holds for \(n\) then

\[
(x)_{-1}^{(1)} = \frac{(-n-2)(x)_n^{(1)}}{(x+1) \cdots (x+n)} = \frac{1}{(x+1) \cdots (x+n+1)}.
\]

Similarly, we have the continuous analog of [39, p. 133]:

**Proposition 5.1.3** For \(a\) not a negative integer, \((x)_a^{(0)} = (x)_a\).

**Proof:** Proposition 3.3.1

Thus,

**Corollary 5.1.4** For all \(a\), \(\widetilde{(x)}_a = (x)_a\).
Now, we determine the series \((x)_0^{(1)}\). By Theorem 4.4.3, we have
\[
(x)_0^{(1)} = E \frac{D}{\Delta} \log x,
\]
or equivalently
\[
\int_x^{x+1} (t)_0^{(1)} \, dt = \log(x + 1).
\]
Moreover, since by the Euler-MacLaurin formula
\[
\frac{D}{\Delta} = \sum_{k \geq 0} B_k D^k / k!,
\]
we have
\[
(x)_0^{(1)} = \sum_{k \geq 0} \frac{B_k}{k!} D^k \log(x + 1)
\]
\[
= \log(x + 1) + \frac{B_1}{1 + x} - \frac{B_2}{2(1 + x)^2} + \frac{B_3}{3(1 + x)^3} - \cdots
\]
where the \(B_k\) are the Bernoulli numbers. Hence, we discover that \((x)_0^{(1)} = \psi(x + 1)\) coincides with the classical \(\psi\)-function (the logarithmic derivative of the gamma function) introduced by Gauss. Similarly, one finds that \((x)_1^{(1)}\) and \((x)_2^{(1)}\) coincide with the digamma and trigamma functions.

The classical expansion
\[
\begin{align*}
\text{Discrete} & \quad (D/\Delta)^n = \sum_{k \geq 0} B_{kn} D^k / k! \\
\text{Continuous} & \quad (D/\Delta)^{a;0} = \sum_{k \geq 0} B_{ka} D^k / k!
\end{align*}
\]
defines the graded sequence \(B_{ab}\) of Bernoulli numbers of order \(b\) and degree \(a\). In terms of these higher order Bernoulli numbers, we obtain for all \(a\):
\[
(x)_a^a = (D/\Delta)^{1+a;0} E \lambda_a^a(x)
\]
\[
= \sum_{k \geq 0} B_{k,a+1} D^k / k! \lambda_a^a(x + 1)
\]
\[
= \sum_{k \geq 0} B_{k,a+1} \binom{a}{k} \lambda_a^{a-k}(x + 1)
\]

**Discrete** We now give an explicit calculation of \((x)_n^{(2)}\) for \(n\) a negative integer. To begin with, by Theorem 4.4.3, we can calculate the residual series of order \((2)\).
\[
(x)_n^{(2)} = E \left(2x^{-1} \log x\right)
\]
\[
= \frac{2 \log(x + 1)}{x + 1}.
\]
Continuing in this way,

\[(x)_{-2}^{(2)} = -\Delta (x)_{-1}^{(2)}
\]

\[= (x)_{-1}^{(2)} - (x + 1)_{-1}^{(2)}
\]

\[= 2 \left( \frac{\log(x + 1)}{x + 1} - \frac{\log(x + 2)}{x + 2} \right)
\]

\[= 2 \left( \frac{\log(x + 1) - (x + 1) \log\left(\frac{x + 2}{x + 1}\right)}{(x + 1)(x + 2)} \right),
\]

and

\[(x)_{-3}^{(2)} = -\frac{1}{2} \Delta (x)_{-2}^{(2)}
\]

\[= \log(x + 1) - (x + 1) \log\left(\frac{x + 2}{x + 1}\right) - \log(x + 2) - (x + 2) \log\left(\frac{x + 3}{x + 2}\right)
\]

\[= \frac{2 \log(x + 1)}{(x + 1)(x + 2)(x + 3)} + \frac{\log\left(\frac{x + 3}{x + 1}\right)}{(x + 3)}.
\]

In general for \(n\) a positive integer,

\[(x)_{-n}^{(2)} = \frac{2 \log(x + 1)}{(x + 1)(x + 2)\cdots(x + n)} + \frac{2 \log\left(\frac{x + n}{x + 1}\right)}{(n - 1)!(x + n)}.
\]

Similarly, we calculate \((x)_{n}^{(0,1)}\). The residual series of order \((0,1)\) is:

\[(x)_{-1}^{(0,1)} = E(1/x \log x)
\]

\[= 1/(x + 1) \log(x + 1).
\]

Next,

\[(x)_{-2}^{(0,1)} = -\Delta (x)_{-1}^{(0,1)}
\]

\[= \frac{(x + 1) \log\left(\frac{x + 2}{x + 1}\right) + \log(x + 2)}{(x + 1)(x + 2) \log(x + 1) \log(x + 2)},
\]

and so on.

The identity

\[(x + a)_{0}^{(1)} = \sum_{k \geq 0} \left[ \begin{array}{c} 0 \\ k \end{array} \right] (a)_k (x)_{-k}^{(1)}
\] (5.4)
gives a classical identity satisfied by the $\psi$-function, that is:

$$\psi(x + a + 1) = \psi(x + 1) + \sum_{k \geq 0} \frac{(-1)^{k+1}a(a-1)\cdots(a-k+1)}{k(x+1)(x+2)\cdots(x+k)}.$$  

Similar identities can be obtained for the digamma and trigamma functions.

The logarithmic Taylor’s theorem (Theorem 3.3.4) gives the following generalization of Newton’s expansion:

**Proposition 5.1.5** Every logarithmic series $p(x)$ can be uniquely expanded as a convergent series

$$p(x) = \sum_{a,\alpha} d^a_\alpha (x)_a / \lfloor a \rfloor!$$

where the coefficients $d^a_\alpha$ are given by

$$d^a_\alpha = \langle \Delta^a p(x) \rangle_\alpha.$$  

For example,

$$\frac{1}{x} = \sum_{k \geq 0} \frac{1}{k!(x+1)\cdots(x+k+1)}.$$  (5.5)

We *digress* to indicate the meaning of such equalities. Formally, we merely mean that when both sides of the equality are expanded in terms of harmonic logarithms $\lambda^a_\alpha(x)$ the resulting coefficients are identical. However, because of such results as Theorem 3.1.3, we are allowed to make computations in the real or complex numbers, and thus obtain asymptotic expansions. In equation (5.5), the right side and left side are both approximately 0.01 for $x = 100$; in fact, the error is about 0.0000015 when you compute 20 or more terms of the summation. Similarly for $x = 69$, when one computes the first 14 terms of the summation, one finds that the left side is 0.14488 and the right side is 0.14493. *End of Digression.*

**Open Problem 5.1.6** How are the Stirling numbers $s(a, b)$ which are the coefficients of the formal power series $(y)_a$ related to the coefficients of the logarithmic series $(x)_a$?
5.1. ROMAN GRADED SEQUENCES

| Table 5.3: Upper Factorials $\langle x \rangle_n^a$ |
|--------------------------------------------------|
| $\langle x \rangle_0^{(0)}$ = $x(x + 1)$ | $\langle x \rangle_2^{(1)}$ = $\lambda_2^{(1)}(x) + \lambda_1^{(1)}(x) - B_{3,3}/3x - B_{4,3}/12x^2 - \cdots$ |
| $\langle x \rangle_0^{(1)}$ = $x$ | $\langle x \rangle_1^{(1)}$ = $-x \log(x) + x - B_2/2x - B_3/6x^2 - \cdots$ |
| $\langle x \rangle_0^{(0)}$ = 1 | $\langle x \rangle_0^{(1)}$ = $\log(x - 1) - B_1/(x - 1) + B_2/2(x - 1)^2 - B_3/3(x - 1)^3 + \cdots$ |
| $\langle x \rangle_{-1}$ | $1/(x - 1)$ |
| $\langle x \rangle_{-2}$ | $1/(x - 1)(x - 2)$ |

5.1.2 Upper Factorial

**Definition 5.1.7** (Backward Difference Operator) Define the backward difference operator $\nabla = I - E^{-1} = I - e^{-D}$. Let $\langle x \rangle_n^a$ denote its standard associated graded sequence; it is called the logarithmic upper factorial graded sequence.

As before, the residual series is given by

$$\langle x \rangle_{-1}^{(1)} = E^{-1}x^{-1} = \frac{1}{x - 1},$$

and for $n$ a negative integer

$$\langle x \rangle_n^{(1)} = 1/(x - 1) \cdots (x + n).$$

Similarly, for $a$ not a negative integer, we have:

$$\langle x \rangle_a^{(0)} = \Gamma(x + a)/\Gamma(x).$$

Thus, for all $a$,

$$\tilde{\langle x \rangle}_a = \Gamma(x + a)/\Gamma(x)$$

For $a = 0$ we have:

**Proposition 5.1.8** $\langle x \rangle_0^{(1)} = \log(x - 1) - B_1/x - 1 + B_2/2(x - 1)^2 - B_3/3(x - 1)^3 + \cdots$.

**Proof:** By Theorem 4.4.3, we have

$$\langle x \rangle_0^{(1)} = E^{-1}D \log x.$$
From the Euler-MacLaurin formula, namely from
\[
\frac{D}{\nabla} = \frac{-D}{e^{-D} - 1} = \frac{D}{\Delta}(-D; 0) = \sum_{k \geq 0} \frac{B_k}{k!}(-D)^k = \sum_{k \geq 0} (-1)^k \frac{B_k}{k!} D^k.
\]
we infer
\[
\langle x \rangle_0^{(1)} = \sum_{k \geq 0} (-1)^k \frac{B_k}{k!} D^k \log(x + 1) = \log(x - 1) - \frac{B_1}{x - 1} + \frac{B_2}{2(x - 1)^2} - \frac{B_3}{3(x - 1)^3} + \cdots.
\]

We have, in terms of the Bernoulli numbers of higher order:
\[
\left( \frac{D}{\nabla} \right)^{a;0} = \sum_{k \geq 0} (-1)^k B_{k,a} D^k/k!.
\]
Hence, for all \(a\),
\[
\langle x \rangle_a^\alpha = \left( \frac{D}{\nabla} \right)^{1+a;0} E^{-1} \lambda_a^\alpha(x) = \sum_{k \geq 0} (-1)^k B_{k,a+1} D^k \lambda_a^\alpha(x-1)/k! = \sum_{k \geq 0} (-1)^k B_{k,a+1} \left[ \binom{a}{k} \right] \lambda_a^{\alpha-k}(x-1).
\]

**Discrete** As with the upper factorial graded sequence, we compute \(\langle x \rangle_a^{(2)}\) for \(n\) a negative integer as follows starting with the residual series of order (2).
\[
\langle x \rangle_{-1}^{(2)} = E^{-1} \left( 2 \frac{x^{-1} \log x}{x} \right) = 2 \log(x - 1)
\]
\[
\langle x \rangle_{-2}^{(2)} = -\nabla \langle x \rangle_{-1}^{(2)} = 2 \left( \frac{\log(x-2)}{x-2} - \frac{\log(x-1)}{x-1} \right) = \frac{\log(x-2) - (x-2) \log \left( \frac{x-1}{x-2} \right)}{(x-1)(x-2)},
\]

and in general
\[
\langle x \rangle^{(2)}_n = \frac{2 \log(x - n)}{(x - 1) \cdots (x - n)} + \frac{2 \log \left( \frac{x - 1}{x - n} \right)}{(n - 1)! (x - n)}.
\]

5.1.3 Abel

The logarithmic extension of the Abel polynomials turns out to be surprisingly pleasing.

**Definition 5.1.9 (Abel Operator)** Define the Abel operator \( A_z = D E^z \). Its standard associated graded sequence \( A_n(x) \) is called the logarithmic Abel graded sequence. Its standard conjugate graded sequence \( \mu_n(x) \) is called the logarithmic inverse-Abel graded sequence.

| Table 5.4: Abel Graded Sequence \( A_n(x) \) |
|---------------------------------------------|
| \( A_0(x) = x(x - 2z) \) & \( A_1(x) = \sigma \lambda_1^1(x - 2z) \) |
| \( A_0(x) = x \) & \( A_1(x) = \sigma \log(x - z) \) |
| \( A_0(x) = 1 \) & \( A_1(x) = \log(x - z)/x \) |
| \( A_0(x) = x(z + x)^{-2} \) & \( A_1(x) = x(z + x)^{-3} \) |

By Corollary 4.4.6, for \( a \neq 0, 1 \), we obtain
\[
A_n^a(x) = \sigma E^{-az} \lambda^a_{n-1}(x) \\
= \sigma \sum_{k \geq 0} \binom{a - 1}{k} (-az)^k \lambda^a_{n-k-1}(x) \\
= \sum_{\substack{k \geq 0 \\ k \neq a}} \binom{a - 1}{k} (-az)^k \lambda^a_{n-k}(x).
\]

In particular, when \( a \) is not a negative integer, we obtain a simple generalization of the classical Abel polynomials.

\[
\begin{align*}
\text{Discrete} & \quad A_n^0(x) = x(x - nz)^{n-1} \\
\text{Continuous} & \quad A_n^a(x) = x(x - az)^{a-1},
\end{align*}
\]

and for \( a \) a negative integer, we have
\[
A_n^{(1)}(x) = x(x - az)^{a-1}.
\]
Thus, its residual series is
\[ A_1^{(1)} = \frac{x}{(x+z)^2}, \]
and its principal sequence is
\[
\begin{align*}
\text{Discrete} \quad \tilde{A}(x) &= \frac{x}{(x-az)^{a+1}} \\
\text{Continuous} \quad \tilde{A}(x) &= \frac{x}{(x-az)^a}.
\end{align*}
\]
\[ A_0^{(1)}(x) \] can be calculated via Theorem 4.4.3. It is expressed very simply.
\[
A_0^{(1)}(x) = A_1 E^{-z} \log x = (1 + zD) \log x = \log x + z/x,
\]
so
\[ A_0^a = \lambda_0^a(x) + z\lambda_{-1}^a(x). \]

From the logarithmic binomial identity, we have the sum
\[
A_0^{(1)}(x + b) = \sum_{k \geq 0} \binom{0}{k} A_k^{(0)}(b) A_{-k}^{(1)}(x)
\]
over all nonnegative integers \( k \). Thus, we infer the remarkable identity
\[
\frac{a}{x+b} + \log(x+b) = \frac{a}{x} + \log x + \sum_{k \geq 1} (-1)^{k+1} \frac{b(1+k)(b-ak)^{k-1}x}{k(x+ak)^{k+1}}
\]
\[ (5.9) \]
For example, we can substitute here the values \( a = 1, b = 2, \) and \( x = 5 \). If we compute the first 12 terms of
the series, the left hand and right hand sides of equation \( (5.9) \) are both approximately 2.0887673.

In general, by Theorem 4.4.3
\[
A_n^a(x) = E^{-az}(1 + zD)\lambda_n^a(x) = \lambda_n^a(x - az) + z [a] \lambda_{a-1}^a(x - az).
\]
For example,
\[ A_1^{(1)}(x) = x \log(x - z) + z - x. \]
Again, by Theorem 3.3.4, every formal power series of logarithmic type can be expanded in terms of Abel series
\[
p(x) = \sum_{a,\alpha} \frac{d_{\alpha}^a}{[a]!} A_n^a(x)
\]
where
\[ d^a_\alpha = \langle E^{a}D^{\alpha}p(x) \rangle_\alpha. \]

For example, (See equation (5.13))
\[
\log x = \sum_{k \leq 0} (ka)^{-k} \left[ \begin{array}{c} 0 \\ k \end{array} \right] A_k^{(1)}(x) = A_0^{(1)}(x) + zA_{-1}^{(1)}(x) - 2z^2 A_{-2}^{(1)}(x) + 9z^3 A_{-3}^{(1)}(x) - \ldots
\]
\[
= \log x - \frac{z}{x} \sum_{k > 0} \left( \frac{k}{x + kz} \right)^{k+1} z^k x.
\]

That is,
\[
x^{-2} = \sum_{k > 0} \left( \frac{ka}{x + ka} \right)^{k+1}.
\]

5.1.4 Gould

**Definition 5.1.10** (Logarithmic Gould Graded Sequence) We define the logarithmic Gould graded sequence \(G^n_{\alpha}(x)\) to be the standard graded sequence associated with the delta operator \(E^z \Delta = E^{z+1} - E^z\).

| Table 5.5: Gould Graded Sequence \(G^n_{\alpha}(x)\) |
|--------------------------------------------------|
| \(G_0^{(0)}(x)\) | \(x(x - 2a - 1)\) |
| \(G_1^{(0)}(x)\) | \(x\) |
| \(G_n^{(0)}(x)\) | 1 |
| \(G_2^{(1)}(x)\) | \(\langle x \rangle_2^{(1)} - a(x)_1^{(1)} + \sum_{k \geq 2} \frac{(-1)^{n+k} \binom{kn}{k} [n - k]}{(n-1)(n-2)\ldots(n-k)}\) |
| \(G_1^{(1)}(x)\) | \(x \log(x) - x + \sum_{k \geq 2} \left[ B_k^{(2)} \left( \frac{2}{k} \right) x^{1-k} + \frac{\binom{k}{k+1}(x-k-1)}{(k+1)x-k-1)} \right]\) |
| \(G_0^{(1)}(x)\) | \(\log(x) + B_1/(x + 1) - B_2/(x + 1)(x + 2) + a/2!(x + 1)(x + 2) + \cdots\) |
| \(G_{-1}^{(1)}(x)\) | \(x/(x + a)(x + a + 1)\) |
| \(G_{-2}^{(1)}(x)\) | \(x/(x + 2a)(x + 2a + 1)(x + 2a + 2)\) |
The Pincherle derivative of $E^z \Delta$ is $(z + 1)E^{z+1} - aE^z$, so the residual series is given by
\[
G^{(1)}_{-1}(x) = \frac{(z + 1)E^{z+1} - aE^z}{x} \left( \frac{1}{x} \right)
\]
\[
= \frac{z + 1}{x + z + 1} - \frac{z}{x + z}
\]
\[
= \frac{x}{(x + z)(x + z + 1)}.
\]
Since Roman graded sequences are basic, we have
\[
G^{(1)}_{-2}(x) = -E^z \Delta G^{(1)}_{-1}(x)
\]
\[
= E^z \left( \frac{x}{(x + z)(x + z + 1)} - \frac{x + 1}{(x + z + 1)(x + z + 2)} \right)
\]
\[
= \frac{x}{(x + 2z)(x + 2z + 1)(x + 2z + 2)}
\]
\[
= \frac{x}{x + 2z}(x + 2z - 2).
\]
Similarly, by induction we have for $n$ positive
\[
G^{(1)}_{-n}(x) = x(x + nz - 1)_{-n-1},
\]
and, by induction for $n$ nonnegative
\[
G^{(0)}_n(x) = x(x - nz - 1)_{n-1}.
\]
In general for all $a$,
\[
\bar{G}_a(x) = x(x - az - 1)_{a-1}.
\]
See §5.2.6 for an explicit computation of $G^a_n(x)$.

### 5.1.5 Laguerre

Our final example of a Roman graded sequence is the logarithmic Laguerre graded sequence.

**Definition 5.1.11 (Laguerre Operator)** Define the Laguerre operator to be
\[
K = D/(D - 1).
\]
Define the logarithmic Laguerre graded sequence to be its (0th) associated graded sequence denoted $L^a_0(x)$. 
Table 5.6: Laguerre Graded Sequence $L^\alpha_n(x)$

| $L^0_0(x)$ | $x^2 - 2x$ |
|-------------|-------------|
| $L^1_0(x)$ | $-x$ |
| $L^0_0(x)$ | $1$ |
| $L^1_1(x)$ | $\frac{1}{2}(-x^2 + 2x)$ |
| $L^0_1(x)$ | $\log(x) - \sum_{k \geq 0} (-1)^k k! x^{k-1}$ |
| $L^1_2(x)$ | $-\sum_{k \geq 2} (-1)^k (k-1) k! x^{k-2}$ |

**Continuous** Note that the Laguerre operator $K$ and the Laguerre graded sequence $L^\alpha_n(x)$ are not standard, since the leading of $K$ is $-1$. However, $-K$ and $K(-D;0)$ are both standard operators we discuss later ($\S 5.2.1$ and $\S 5.2.2$), and if $r^\alpha_a(x)$ and $p^\alpha_a(x)$ are their standard associated sequences, then

$$L^\alpha_a(x) = \psi_{-1/2} r^\alpha_a(x) \quad (5.11)$$

where $\psi_a$ is as defined in Example 4.2.3C, and

$$L^\alpha_a(x) = (-1)^{\pi a} p^\alpha_a(x). \quad (5.12)$$

Now, the Pincherle derivative of the Laguerre operator is given by $K' = -(D-I)^{-2}$, so by Theorem 4.4.3,

$$L^\alpha_a(x) = -(D-I)^{-2}(D-I)^{a+1;0} \lambda^\alpha_a(x)$$

$$= -(D-I)^{a-1;0} \lambda^\alpha_a(x).$$

Hence, for all $a$,

$$L^\alpha_a(x) = e^{\pi i a} \left( \sum_{k \geq 0} (-1)^k \binom{a}{k} D^k \right) \lambda^\alpha_a(x)$$

$$= e^{\pi i a} \sum_{k \geq 0} (-1)^k \binom{a}{k} D^k \lambda^\alpha_a(x)$$

$$= e^{\pi i a} \sum_{k \geq 0} (-1)^k \binom{a}{k} \frac{|a|}{|a-k|} \lambda^\alpha_a(x).$$

Thus,

$$L^\alpha_1(x) = -\lambda^\alpha_1(x).$$

Similarly, we have derived the following expansion

$$L^{(1)}_0(x) = \log(x) + \frac{1}{x} - \frac{1}{x^2} + \frac{2}{x^3} - \frac{6}{x^4} + \cdots.$$
CHAPTER 5. EXAMPLES

**Discrete** Thus, $L_n^\alpha(x)$ does not contain any terms of negative degree when $n$ is a positive integer. This is true only for Roman sequences associated with $D/(aD + bI)$ where $a$ and $b$ are nonzero complex numbers. Note that the series of degree 0 contains no negative terms only if the operator in question is $aD$ where $a$ is a nonzero complex number.

**Continuous** In the continuous case, $L_n^\alpha(x)$ contains terms of negative degree for all $n$. The only Roman sequences which do not always contain terms of negative degree are those associated with delta operators of the form $zD$ where $z$ is a nonzero complex number.

From Theorem 4.3.7, we derive a logarithmic extension of the classical Rodrigues’ formula for Laguerre polynomials.

$$L_n^\alpha(x) = -e^{\sigma D}e^{(D^{-1}e^{-\sigma})\lambda_n^\alpha(x)}.$$ 

### 5.2 Connection Constants

Given two graded sequences $p_n^\alpha(x)$ and $q_n^\alpha(x)$; we would like to express one in terms of the other.

$$p_n^\alpha(x) = \sum_b d_{ab} q_b^\alpha(x).$$

The coefficients $d_{ab}$ are called the connection constants from the graded sequence $q_n^\alpha(x)$ to the graded sequence $p_n^\alpha(x)$, and are denoted $[q_n^\alpha(x)]p_n^\alpha(x)$.

If $p_n^\alpha(x)$ and $q_n^\alpha(x)$ are the standard Roman graded sequences associated with the delta operators $f(D)$ and $g(D)$ respectively, then by Proposition 4.2.6

$$r_n^\alpha(x) = \sum_b c_{ab} \lambda_n^\alpha(x)$$

is the standard Roman graded sequence associated with $g(f^{-1};0)$. Thus, to determine the connection constants it suffices merely to calculate $r_n^\alpha(x)$. This easy device for the computation of connection constants is the most effective application of the present theory.

#### 5.2.1 Upper Factorial to Lower Factorial

To express $(x)_n^\alpha$ in terms of $(x)_a^\alpha$ we first calculate

$$\Delta(\nabla^{(-1)};0) = e^{-\log(I-D)} - I$$

$$= \frac{I}{I - D} - I$$

$$= \frac{D}{I - D}$$

$$= -K(D)$$
where \( K(D) \) is the Laguerre operator (Definition 5.1.11). Thus, \( r_\alpha^a(x) \) is the standard graded sequence related to the logarithmic Laguerre graded sequence defined by equation (5.11),
\[
r_\alpha^a(x) = \Psi_{1/2} L_\alpha^a(x)
\]
where \( \Psi_{1/2} \) is as defined in Example 4.2.3C.

\[\psi_{1/\lambda}^a(x) = e^{\pi i a} L_\lambda^a(x).\]

In a sense, \( \psi_{1/\lambda}^a(x) \) is the logarithmic analog of substitution of \(-x\) for \(x\).

We now apply the results from §5.1.5. Thus, for all \( a \) and \( \alpha \)

| Table 5.7: Lower Factorial in Terms of Upper Factorial |
| --- |
| \((x)_a^{(0)} = (x)_0^{(0)} + 2(x)_1^{(0)}\) | \((x)_a^{(1)} = (x)_1^{(1)}\) |
| \((x)_a^{(1)} = (x)_0^{(1)} + 2(x)_1^{(1)}\) | \((x)_a^{(1)} = (x)_1^{(1)} - \sum_{k \geq 0} k! (x)^{(1)}_{k-1} \) |
| \((x)_a^{(1)} = (x)_0^{(1)} - \sum_{k \geq 1} k! (x)^{(1)}_{k-1} \) | \((x)_a^{(1)} = \sum_{k \geq 1} (k-1)k! (x)^{(1)}_{k-1} / 2 \) |

\[(x)_a^a = \sum_{k \geq 0} \left( a - 1 \right)_k \frac{|a|!}{|a-k|!} (x)_a^{a-k}(x).\]

From Theorem 4.3.7, we have the identity
\[
\begin{cases}
\text{Discrete} & (x)_a^a = -e^{s \psi} \nabla^{a-1} e^{-s \psi} (x)_a^a. \\
\text{Continuous} & (x)_a^a = -e^{s \psi} \nabla^{a-10} e^{-s \psi} (x)_a^a.
\end{cases}
\]

### 5.2.2 Lower Factorial to Upper Factorial

Conversely, to compute \(\langle x \rangle_a^a\) in terms of \((x)_a^a\), we need the delta operator
\[
\nabla(\Delta^{-1:0}) = \frac{D}{D + 1} = K(-D).
\]

Hence, the connection constants from \((x)_a^a\) to \(\langle x \rangle_a^a\) are given by the coefficients of \(e^{-\pi i a} L_\alpha^a(x)\) (equation (5.12)). Hence, for all \(a\) and \(\alpha\),
\[
\langle x \rangle_a^a = \sum_{k \geq 0} (-1)^k \left( a - 1 \right)_k \frac{|a|!}{|a-k|!} \lambda_{a-k}(x).
\]
Table 5.8: Upper Factorial in Terms of Lower Factorial

\[
\begin{array}{c|c}
\langle x \rangle^{(0)}_0 &= (x)^{(0)}_0 \\
\langle x \rangle^{(0)}_1 &= (x)^{(0)}_1 \\
\langle x \rangle^{(1)}_0 &= (x)^{(1)}_0 \\
\langle x \rangle^{(1)}_1 &= (x)^{(1)}_1 \\
\langle x \rangle^{(1)}_2 &= (x)^{(1)}_2 \\
\end{array}
\]

5.2.3 Laguerre to Harmonic

If \( p^\alpha_a(x) \) is the standard Roman graded sequence associated with the delta operator \( f(D) \), then finding the connection constants from \( p^\alpha_a(x) \) to \( \lambda^\alpha_a(x) \) is tantamount to finding the standard graded sequence associated with \( f(-1)^{\alpha-1}D \); that is, the standard conjugate graded sequence for \( f(D) \).

Note that since \( x/(x-1) \) is the 0-compositional inverse of itself, the logarithmic Laguerre graded sequence is self-conjugate, so we have for all \( a \) and \( \alpha \)

\[
\lambda^\alpha_a(x) = \sum_{k \geq 0} (-1)^k \binom{a-1}{k} \frac{|a|!}{|a-k|!} L^\alpha_{a-k}(x)
\]

and from Theorem 4.3.7,

\[
\lambda^\alpha_a(x) = L^\alpha_a(L) = -e^{\sigma L} K^a e^{-\sigma L} L^\alpha_a(x).
\]

5.2.4 Lower Factorial to Harmonic

Similarly, the connection constants from \( (x)^\alpha_a \) to \( \lambda^\alpha_a(x) \) are given by \( \phi^\alpha_a(x) \)—the logarithmic exponential graded sequence—which we now compute.

The relevant delta operator is

\[
\log(1 + D) = \sum_{k > 0} (-1)^{k+1} D^k / k.
\]

Thus, by Theorem 4.4.3, we can calculate the residual series.

\[
\phi^\alpha_{a-1}(x) = \left( \frac{1}{1 + D} \right) \lambda^\alpha_{a-1}(x)
= \sum_{k \geq 0} (-1)^k D^k \lambda^\alpha_{a-1}(x)
= \sum_{k \geq 0} \frac{(-1)^k}{|k-1|!} \lambda^\alpha_{a-1-k}(x)
= \sum_{k \geq 0} k! \lambda^\alpha_{a-1-k}(x).
\]
Finally, by Theorems 4.4.1 and 4.3.7, for $a \neq -1$, we obtain the recursion formula

$$\phi_{a+1}^\alpha(x) = \sigma(I + D)\phi_{a}^\alpha(x) = \sigma e^{-\sigma}D e^{\sigma}\phi_{a}^\alpha(x).$$

### 5.2.5 Abel to Harmonic

The compositional inverse of the Abel operator is not easily calculated. Nevertheless, we may calculate the logarithmic inverse-Abel graded sequence, using the theory of conjugate graded sequences.

Thus, by Definition 3.5.1

$$\mu_{\alpha}^\alpha(x) = \sum_{b} \frac{\langle D^b E_{b\pi} \lambda_{\alpha}^\alpha(x) \rangle}{[b]!} \lambda_{\alpha}^\alpha(x).$$

Now,

$$D^b E_{b\pi} \lambda_{\alpha}^\alpha(x) = D^b \sum_{j \geq 0} \begin{pmatrix} a \\ j \end{pmatrix} (bz)^j \lambda_{a-j}^\alpha(x) = \sum_{j \geq 0} (bz)^j \begin{pmatrix} a \\ j \end{pmatrix} \frac{[a]!}{j! [a-b-j]!} \lambda_{a-b-j}^\alpha(x).$$

Thus,

$$\mu_{\alpha}^\alpha(x) = \sum_{k \geq 0} ((a - k)z)^k \frac{[a]!}{k!0!} \lambda_{a-k}^\alpha(x) / [a - k]! = \sum_{k \geq 0} ((a - k)z)^k \begin{pmatrix} a \\ k \end{pmatrix} \lambda_{a-k}^\alpha(x).$$

Hence, the remarkable identity

$$\lambda_{\alpha}^\alpha(x) = \sum_{k \geq 0} ((a - k)z)^k \begin{pmatrix} a \\ k \end{pmatrix} A_{a-k}^\alpha(x).$$

(5.13)
5.2.6 Upper Factorial to Gould

Assume that Gould parameter \( z \) is a real number which we denote by \( t \). The relevant operator is \( f(D) = -D(I-D)^{-t,0} \) whose associated graded sequence is:

\[
          r^a_\alpha(x) = f'(D) \left( \frac{f(D)}{D} \right)^{-a-1,0} \lambda^a_\alpha(x) \\
          = e^{2\pi ia}((t-1)D + I)(I-D)^{at-1,0} \lambda^a_\alpha(x). \tag{5.14}
\]

As expected, for \( t = 1 \) we rederive the Laguerre graded sequence \( L^a_\alpha(x) \), and for \( t = 0 \) we get a variant on the harmonic graded sequence \( \psi_{\lambda^n} \).

For \( a \neq 0,1 \), instead of equation (5.14), we may use Corollary 4.4.6.

\[
          r^a_0(x) = e^{\pi ia}(I - D)^{at} \lambda^a_{a-1}(x) \\
          = e^{\pi ia} \left( \sum_{k \geq 0} (-1)^k \binom{at}{k} D^k \right) \lambda^a_{a-1}(x) \\
          = \sum_{k \geq 0 \atop k \neq n} e^{\pi i(a+k)} \binom{at}{k} \frac{[a-1]!}{[a-k-1]!} \lambda^a_{a-k}(x).
\]

Thus, for \( a \neq 0,1 \), we obtain the following remarkable identity relating the Gould graded sequence to the lower factorial graded sequence.

\[
          G^a_\alpha(x) = \sum_{k \geq 0 \atop k \neq n} e^{\pi i(a+k)} \binom{at}{k} \frac{[a-1]!}{[a-k-1]!} \langle x \rangle^a_{a-k}.
\]

For \( a = 0 \), equation (5.14) reduces to

\[
          r^a_0(x) = ((t-1)D + I)(I-D)^{-1} \lambda^a_0(x) \\
          = (1 + tD + tD^2 + tD^3 + \cdots) \lambda^a_0(x) \\
          = \lambda^a_0(x) + t \sum_{k>0} (-1)^{k+1}(k-1)! \lambda^a_{-k}(x).
\]

Thus, the Gould series of degree zero are given by the elegant expression

\[
          G^a_0(x) = \langle x \rangle^a_0 + t \sum_{k>0} (-1)^{k+1}(k-1)! \langle x \rangle^a_{-k}.
\]

Thus,

\[
          G^{(1)}_0(x) = \log(x+1) + \frac{B_1}{x+1} - \frac{2!B_2}{(1+x)^2} + \cdots \\
          + \frac{t}{x+1} - \frac{t}{(x+1)(x+2)} + \frac{2!t}{(x+1)(x+2)(x+3)} + \cdots.
\]
Finally, from Proposition 4.4.5, we obtain
\[
\begin{align*}
 r_\alpha^n(x) &= -(I - D)^t \lambda_\alpha^n(x) - t(I - D)^{t-1} \lambda_\alpha^0(x) \\
 &= \lambda_\alpha^n(x) + \sum_{k \geq 2} \left( \binom{t}{k} + t \left( \binom{t-1}{k-1} \right) \right) \lambda_{1-k}^\alpha(x) / |1-k|!
\end{align*}
\]
\[
\begin{align*}
&= \lambda_\alpha^n(x) + \sum_{k \geq 2} \left( \binom{t}{k} + t \left( \binom{t-1}{k-1} \right) \right) (-1)^k (k-2)! \lambda_{1-k}^\alpha(x),
\end{align*}
\]
so
\[
\begin{align*}
 G_\alpha^n(x) &= (x)_1^\alpha + \sum_{k \geq 2} \left( \binom{t}{k} + t \left( \binom{t-1}{k-1} \right) \right) (-1)^k (k-2)! (x)_{1-k}^\alpha \\
 G_1^{(1)}(x) &= x \log(x) - x + \sum_{k \geq 2} B_k^{(2)} \left[ \frac{2}{k} x^{1-k} + \sum_{k \geq 2} \frac{(-1)^k \left( \binom{t}{k} + t \left( \binom{t-1}{k-1} \right) \right) (k-2)!}{(x-1) \cdots (x-k+1)} \right]
\end{align*}
\]

### 5.2.7 Lower Factorial to Gould

Similarly, here we are interested in the operator \( f(D) = D(I + D)^t \) where \( t \) is real. For \( t = 0 \), we rederive the harmonic logarithms \( \lambda_\alpha^n(x) \), and for \( t = -1 \), we get a variant of the Laguerre graded sequence \((-1)^{\pi a} L_\alpha^n(x)\). (See equation (5.12) and §5.2.2.)

The standard associated graded sequence for \( f(D) \) is
\[
\begin{align*}
 r_\alpha^a(x) &= f'(D) \left( \frac{f(D)}{D} \right)^{a-1;0} \lambda_\alpha^a(x) \\
 &= ((t+1)D + I)(I + D)^{-at-1;0} \lambda_\alpha^0(x).
\end{align*}
\]

For \( a \neq 0,1 \),
\[
\begin{align*}
 r_\alpha^a(x) &= \sigma(I + D)^{-at;0} \lambda_{a-1}^\alpha(x) \\
 &= \sum_{\substack{k \geq 0 \cr k \neq a}} \left( \frac{-at}{k} \right) \left[ \frac{|a-1|!}{|a-k-1|!} \right] \lambda_k^\alpha(x).
\end{align*}
\]
Thus, for \( a \neq 0,1 \),
\[
\begin{align*}
 G_\alpha^a(x) = \sum_{\substack{k \geq 0 \cr k \neq n}} \left( \frac{-at}{k} \right) \left[ \frac{|a-1|!}{|a-k-1|!} \right] (x)_{a-k}^\alpha.
\end{align*}
\]
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Now, for \( a = 0 \),

\[
    r_0^a(x) = (t + 1)D + I)(I + D)^{-1} \lambda_0^a(x) \\
    = (1 + tD - tD^2 + tD^3 - \cdots) \lambda_0^a(x) \\
    = \lambda_0^a(x) + t \sum_{k>0} (k-1)! \lambda_k^a(x),
\]

so that again

\[
    G_0^a(x) = (x)_0^a(x) + t \sum_{k>0} (k-1)! \langle x \rangle_k^a.
\]

For \( n = 1 \),

\[
    r_1^a(x) = (I + D)^{-t} \lambda_1^a(x) - t(I + D)^{-t-1} \lambda_0^a(x) \\
    = \lambda_1^a(x) + t \sum_{k\geq2} \left( \binom{-t}{k} + t(k-1) \binom{-t-1}{k-1} \right)(-1)^k k! \lambda_{k-1}^a(x),
\]

hence another remarkable expansion for a residual series:

\[
    G_{-1}^a(x) = \sum_{k\geq0} \left( \binom{t}{k} + t \binom{t-1}{k-1} \right)(-1)^k k! \langle x \rangle_{-1-k}^a.
\]

As noted in [20], nearly any sequence may be used as a “factorial” on which to base on umbral calculus. In particular, we could have chosen the Gaussian coefficients as our factorials. If so, then would have derived the \( q \)-analog of our theory.

**Open Problem 5.2.1** *What are the \( q \)-analogs of these examples?*

We hope the preceding examples display the utility of the theory of formal power series of logarithmic type.
Bibliography

[1] P. Appell, and J. Kampé de Fériet, “Fonctions Hypergéométriques et Hypersphériques, Polynomes d’Hermite,” Gauthier-Villars, Paris, 1926.

[2] R. Askey, “Orthogonal Polynomials and Special Functions,” Regional Conference Series in Applied Mathematics, SIAM (1975).

[3] M. Barnabei, A. Brini, and G. Nicoletti, Polynomial Sequences of Integral Type, Journal of Mathematical Analysis and Its Applications, 78 (1980), 598–617.

[4] M. Barnabei, A. Brini, and G. Nicoletti, Recursive Methods and the Umbral Calculus, Journal of Algebra, 75 (1982), 546–573.

[5] N. Bourbaki, “Fonctions d’une Reele Variable.”

[6] J. Cigler, Some Remarks on Rota’s Umbral Calculus, Indagationes Mathematicae, 40 (1978), 27–42.

[7] Bieren De Haan, “Nouvelles tables d’integral definies.”

[8] A. Di Bucchianico, On Rota’s Theory of Polynomials of Binomial Type, Mathematics Department, University of Groninger, Technical Report TW-20.

[9] A. Erdélyi, “Asymptotic Expansions,” Dover Publications, 1956.

[10] J. M. Freeman, and F. Hoffman, A Semigroup and Gaussian Polynomials, Discrete Mathematics, 36 (1981), 247–260.

[11] J. M. Freeman, Transforms of Operators on K[x][[t]], Congressus Numerantium, 48 (1985), 125–132.

[12] A. Garsia, An exposé of the Mullin-Rota Theory of Polynomials of Binomial Type, Linear and Multi-linear Algebra, 1 (1973), 47–65.

[13] A. Garsia, and S. A. Joni, A New Expression for Umbral Operators and Power Series Inversion, Proceedings of the American Mathematical Society, 64 (1977), 179–185.

[14] A. J. Goldstein, “A Residue Operator in Formal Power Series,” Computing Science Technical Report # 26, Bell Telephone Laboratories, 1975.
[15] HARDY, “Integration of a Function of a Single Variable.”
[16] HARDY, “Orders of Infinity.”
[17] S. A. JONI, Lagrange Inversion in Higher Dimensions and Umbral Operators, 6 (1978), 111–121.
[18] K. KNOPP, “Theory and Application of Infinite Series,” Hafner Publishing Company, New York.
[19] D. LOEB, A Generalization of the Binomial Coefficients, To appear.
[20] D. LOEB, A Generalization of the Stirling Numbers, To appear.
[21] D. LOEB, The Iterated Logarithmic Algebra, MIT Department of Mathematics Thesis (1989).
[22] D. LOEB, The Iterated Logarithmic Algebra II: Sheffer Sequences, To appear.
[23] D. LOEB, Sequences of Symmetric Functions of Binomial Type, To appear.
[24] D. LOEB AND G.-C. ROTA, Formal Power Series of Logarithmic Type, Advances in Mathematics, 75 (1989), 1–118.
[25] D. LOEB, Series with General Exponents, To appear ???.
[26] I. G. MACDONALD, “Symmetric Functions and Hall Polynomials,” Oxford Mathematical Monographs, Claredon Press, Oxford, 1979.
[27] L. M. MILNE-THOMSON, “The Calculus of Finite Differences,” MacMillan, London, 1951.
[28] R. A. MORRIS, Frobenius Endomorphisms in the Umbral Calculus, Studies in Applied Mathematics 58 (1978), 95–117.
[29] R. MULLIN AND G.-C. ROTA, On the Foundations of Combinatorial Theory: III. Theory of Binomial Enumeration, Graph Theory and Its Applications, (1970) 168–211.
[30] H. POINCARE, Acta Math., 8 (1886) 295–344.
[31] D. L. REINER, The Combinatorics of Polynomial Sequences, Studies in Applied Mathematics, 58 (1978), 95–117.
[32] D. L. REINER, Multivariate Sequences of Binomial Type, Studies in Applied Mathematics, 57 (1977), 119–133.
[33] S. ROMAN, The Algebra of Formal Series, Advances in Mathematics 31 (1979) 309–329.
[34] S. ROMAN, The Algebra of Formal Series II: Sheffer Sequences, Journal of Mathematical Analysis and Applications 74 (1980), 120–143.
[35] S. ROMAN, A Generalization of the Binomial Coefficients, To Appear.
[36] S. ROMAN, The Theory of the Umbral Calculus: I, Journal of Mathematical Analysis and Its Applications, 87 (1982), 58–115.
[37] S. Roman, *The Theory of the Umbral Calculus: II*, Journal of Mathematical Analysis and Its Applications, 89 (1982), 290–314.

[38] S. Roman, *The Theory of the Umbral Calculus: III*, Journal of Mathematical Analysis and Its Applications, 95 (1983), 528–563.

[39] S. Roman, “The Umbral Calculus,” Academic Press, 1984.

[40] S. Roman and G.-C. Rota, *The Umbral Calculus*, Advances in Mathematics, 27 (1978) 95–188.

[41] G.-C. Rota, “Finite Operator Calculus,” Academic Press, 1975.

[42] G.-C. Rota, D. Kahaner, and A. Odlyzko, *Finite Operator Calculus*, Journal of Mathematical Analysis and Applications, 42 (1973).

[43] A. J. Stam, *Two Identities in the Theory of Polynomials of Binomial Type*, Journal of Mathematical Analysis and Its Applications, 122 (1987), 439–443.

[44] K. Ueno, *Umbral Calculus and Special Functions*, Advances in Mathematics, 67 (1988), 174–229.

[45] A. J. van der Poorten, *A Generalization of Turán’s Main Theorems to Binomials and Logarithms*, Bulletin of the Australian Mathematical Society, 2 (1970), 183–195.

[46] L. Verde-Star, *Dual Operators and Lagrange Inversion in Several Variables*, Advances in Mathematics, 58 (1985), 89–108.

[47] W. Wasow, “Asymptotic Expansions for Ordinary Differential Equations,” Interscience Publishers, 1965.

[48] T. Watanabe, *On a Dual Relation for Addition Formulas of Additive Groups: I*, Nagoya Mathematical Journal, 94 (1984), 171–191.

[49] T. Watanabe, *On a Dual Relation for Addition Formulas of Additive Groups: II*, Nagoya Mathematical Journal, 97 (1985), 95–135.

[50] K.-W. Yang, *Integration in the Umbral Calculus*, Journal of Mathematical Analysis and Its Applications, 74 (1980), 200–211.

[51] D. Zeilberger, *Some Comments on Rota’s Umbral Calculus*, Mathematical Analysis and Its Applications, 74 (1980), 456–463.