Quantum kinetics derivation as generalization of the quantum hydrodynamics method

Pavel A. Andreev

M. V. Lomonosov Moscow State University, Moscow, Russia.

(Dated: May 5, 2014)

Abstract

We present a new way of quantum kinetic equation derivation. This method appears as a natural generalization of the many-particle quantum hydrodynamic method. Kinetic equations are derived for different system of particles. First of all we consider quantum plasma and pay special attention to the spin evolution. We show that we need a set of two kinetic equations for description of spinning particles. One of these equations is the equation for distribution function, however this equation contains new function, even in the self-consistent field approximation. This is a spin-distribution function introduced in the paper. Therefore we have to derive kinetic equation for spin distribution function evolution, which is presented here and used to construct a closed set of kinetic equations. We also present kinetic equation for system of neutral particles with a short-range interaction in the first order by the interaction radius approximation. We derive a set of kinetic equations for particles having electric dipole moment, this set analogous to the equations set for spinning particles, but it has some differences. As a special topic we find kinetic equations for graphene carriers in the vicinity of the Dirac points. Derived equations, in general case, contain two-particle distribution functions, which take into account contribution of the quantum correlations including the exchange interaction, but we restrict ourself by the self-consistent field approximation to obtain closed kinetic description, in the system of particles with the short-range interaction.

PACS numbers:
I. INTRODUCTION

Quantum plasma studying requires developments of new theoretical methods. In classical regime it has been very fruitful to use hydrodynamical and kinetic methods. So, we can expect that their quantum generalization will be as profound. In last decade the method of quantum hydrodynamics has been developed by many teams \[1\]-\[13\]. The semi-relativistic interactions, the spin-spin \[2\], the spin-current \[4\], the spin-orbit \[9\], the current-current \[12\] interactions, and the Darwin term and the semi-relativistic amendment to the kinetic energy \[12\], have been included in the quantum hydrodynamics scheme along with the Coulomb interaction. Wigner kinetic approximation has been used for studying of the quantum plasma \[14\]-\[22\]. A physical kinetics, free of any special assumptions on the space-time geometry, was developed in Ref. \[23\]. A quantum distribution function such that calculating statistical averages leads to the same local values of the number of particles, the momentum, and the energy as those in quantum mechanics was found in Ref. \[24\]. This distribution function coincides with the Wigner function only for spatially homogeneous systems. It was used for derivation of an quantum kinetic equation \[24\]-\[27\] and consideration of some physical problems \[25\]-\[29\]. Kinetic equation for charged spinning particles defining the distribution function of ten variables the coordinate \(r\), the momentum \(p\), the spin \(s\), and the time \(t\) was considered in Ref. \[14\]. Integration of this distribution function \(f(r, p, s, t)\) on the spin and momentum gives concentration of particles \(n(r, t)\). Integrating the product of the spin vector and the distribution function on the same variables gives spin density. Quantum kinetic equation for \(f(r, p, s, t)\) with the semi-relativistic effects caused by interaction was suggested in Ref. \[18\]. Obtaining of the kinetic equation by averaging of the one-particle hydrodynamic equations on the Maxwell or the Fermi distribution function was considered in Ref. \[30\]. Set of equations consisting of the Vlasov equation, including ponderomotive force acting of the magnetic moments, for particle motion and Bargman-Michel-Telegdi (BMT) equation, describing magnetic moment evolution, was used in Ref. \[31\] to consider dispersion of elementary excitations for the tensor gyromagnetic ratio. Review of the recent achievements in the quantum kinetics can be found in Ref.s \[32\], \[33\].

Microscopic density of particles in classical physics can be presented as the sum of Dirac
delta functions [34]-[36]

\[ n(r, t) = \sum_{n=1}^{N} \delta(r - r_n(t)), \]

where \(N\) is the total number of particles in the system. This definition of the particles concentration was used for construction of the quantum particles concentration and derivation of the quantum hydrodynamic equations [1], [8], [9]. Corresponding microscopic distribution function can be written as

\[ f = \sum_n \delta(r - r_n(t))\delta(p - p_n(t)). \]

Averaging of this function leads to macroscopic distribution function allowing to derive a chain of relativistic kinetic equations [36]. General form of quantum distribution function might be presented in the following form [24]

\[ \hat{f}(r, p) = \sum_n \hat{n}_n(p)\hat{n}_n(r), \]

and it was expressed via the one-particle statistical operator. However, we in this paper consider the quantum mechanical averaging [37]

\[ \langle L \rangle = \int \psi^* \hat{L} \psi dR \]

of \( \hat{f} \) defined as

\[ \hat{f} = \sum_n \delta(r - \hat{r}_n)\delta(p - \hat{p}_n), \]

where \(\hat{p}_n\) is the momentum operator for \(n\)-th particle, and \(p\) is the numerical vector function which arithmetizes the momentum space, as the coordinate \(r\) arithmetizes coordinate space.

We show that evolution of the quantum mechanical averaging of the operator [1] leads to arising of the second function, which is the quantum mechanical averaging of the following operator

\[ \hat{S}^\alpha = \sum_n \hat{\sigma}^\alpha_n(r - \hat{r}_n)\delta(p - \hat{p}_n). \]

Thus, we should use a set of two kinetic equations for description of spinning particles evolution.

We use this method for derivation of kinetic equations for different physical systems. These are the spinning quantum plasma briefly discussed above, charged or neutral particles having electric dipole moment, neutral particles with the short-range interaction, graphene
carriers and graphene excitons. Corresponding quantum hydrodynamic equation have been obtained previously. At both the graphene carriers and the graphene excitons description we have deal with the quasi-spin. Consequently, we have to obtain a couple kinetic equations as for spinning quantum plasma. At description of the electrically polarized particles lead to appearing of a new function, analogous to (2), but we should put the electric dipole moment operator instead of the spin operator.

This paper is organized as follows. In Sec. II we start our microscopic derivation of quantum kinetic equation. This derivation is presented for the simples case of charged particles in an external electric field described by scalar potential with the Coulomb interaction. The self-consistent field approximation considered for obtained kinetic equation. In Sec. III we introduce the distribution function in a magnetic field and derive kinetic equation for charged spinless particles in an external electromagnetic field. In Sec. IV we consider kinetics of spinning particles, we derive a couple of kinetic equations for the distribution function and spin distribution function. In Sec. V we obtain kinetic equations for particles having electric dipole moment. In Sec. VI we study kinetic of neutral particles with the short range interaction. In Sec. VII we find a set of kinetic equation for carriers in the graphene. In Sec. VIII a set of kinetic equations for the graphene excitons is obtained. In Sec. IX we present the brief summary of our results. In Appendix we critically examined some general method derivation of kinetic equation used as for quantum and for classic systems.

II. CONSTRUCTION OF MACROSCOPIC EQUATIONS

A. Kinetic equation for spinless particles: General form

The equation of quantum kinetics is derived from the non-stationary Schrodinger equation for system of N particles:

\[ i\hbar \partial_t \psi(R, t) = \left( \sum_n \left( \frac{1}{2m_n} \hat{D}_n^2 + e_n \varphi_{n, ext} \right) + \frac{1}{2} \sum_{n,k \neq k} e_n e_k G_{nk} \right) \psi(R, t). \]  

(3)

The following designations are used in the equation (3): \( \hat{D}_n = -i\hbar \partial_n^\alpha - e_n A_{n, ext}^\alpha / c, \varphi_{n, ext}, A_{n, ext}^\alpha \) are the potentials of the external electromagnetic field, \( \partial_n^\alpha = \nabla_n^\alpha \) is the derivative on
the space variables of \( n \)-th particle, and \( G_{nk} = 1/r_{nk} \) is the Green functions of the Coulomb interaction, \( \psi(R, t) \) is the psi function of N particle system, \( R = (r_1, ..., r_N) \), \( e_n, m_n \) are the charge and the mass of particle, \( \hbar \) is the Planck constant and \( c \) is the speed of light.

The first step in derivation of the kinetic equation is a definition of distribution function of particles. We determine the distribution function of particles as the quantum-mechanical average of the operator of distribution function:

\[
\hat{f} = \sum_n \delta(r - r_n)\delta(p - \hat{p}_n).
\]

This function is the microscopic distribution function in classic physics, \( \delta(r) \)-is the Dirac’s \( \delta \)-function.

In that way, the distribution function has form:

\[
f(r, p, t) = \frac{1}{2} \int \psi^*(R, t) \sum_n \left( \delta(r - r_n)\delta(p - \hat{p}_n) + \delta(p - \hat{p}_n)\delta(r - r_n) \right) \psi(R, t) dR, \quad (4)
\]

where \( dR = \prod_{n=1}^N dr_n \). This definition symmetric relatively to operators \( \delta(r - r_n) \) and \( \delta(p - \hat{p}_n) \), but it still is not fully symmetric. To get the fully symmetric definition of the distribution function we need to add complex conjugated quantity. Thus we have following definition

\[
f(r, p, t) = \frac{1}{4} \int \left( \psi^*(R, t) \sum_n \left( \delta(r - r_n)\delta(p - \hat{p}_n) + \delta(p - \hat{p}_n)\delta(r - r_n) \right) \psi(R, t) + c.c. \right) dR, \quad (5)
\]

where c.c. stands for the complex conjugation.

For the first step we will consider set of the charged particles in external quasi-static electric field including the Coulomb interaction between particles, this means that we do not include the vector potential. Next, we will consider electromagnetic external field, and we will also consider spinning particles with spin-spin interaction.

Integrating of the distribution function over momentum we have concentration

\[
n(r, t) = \int f(r, p, t) dp \quad (6)
\]

where the concentration has familiar from the quantum hydrodynamics form \( f_8, f_1 \)

\[
n(r, t) = \int dR \sum_n \delta(r - r_n)\psi^*(R, t)\psi(R, t), \quad (7)
\]
Integrating of the product of momentum on the distribution function over momentum we find particles current

\[ j(r, t) = \int p f(r, p, t) dp, \tag{8} \]

details of calculations you can find in appendix.

In the same way we have the kinetic energy

\[ \varepsilon_{\text{kin}}(r, t) = \int \frac{p^2}{2m} f(r, p, t) dp. \tag{9} \]

In the absence of the inter-particle interaction we find

\[ \partial_t f + \frac{1}{m} p \partial_r f + e \frac{i}{\hbar} \varphi(r, t) \sin(\nabla_r \nabla_p) f = 0, \tag{10} \]

where

\[ \sin(\nabla_r \nabla_p) = \sum_{l=1}^{\infty} \frac{(i\hbar)^{2l+1}}{(2l+1)!} (\nabla_r \nabla_p)^{2l+1}. \]

In equation (10) we have used designation \( \varphi(r, t) \nabla_r \), it means that the gradient operator \( \nabla \) on spatial variables acts on the left-hand side, instead of usual acting of operators on function standing on the right-hand side. We do not write the Plank constant \( \hbar \) in the argument of \( \sin \) to make this notation more handy.

The last term appears here due to the commutation of the electric scalar potential with the \( \delta(p - \hat{p}_n) \). At \( l \geq 2 \) terms are proportional to the plank constant \( \sim \hbar^{l-1} \). They disappear in the classical limit \( \hbar \to 0 \).

In the quasi-classical limit we have to include one term of the sum in the last term only, so we have

\[ \partial_t f + \frac{1}{m} p \partial_r f + e E \nabla_p f = 0, \tag{11} \]

where we have written \( E = -\partial_r \varphi \).

We also admit that distribution function \( f \) satisfy to the following equation

\[ \partial_t f + \frac{1}{m} p \partial_r f + e \frac{i}{\hbar} \varphi(r, t) \sum_{l=1}^{\infty} \frac{(i\hbar)^l}{l!} (\nabla_r \nabla_p)^l f = 0. \]

At derivation of kinetic equation the distribution current arises in the second terms of equations (10) and (19)

\[ J^\alpha(r, p, t) = \frac{1}{8} \int \left( \hat{p}^\alpha \psi^*(R, t) \right) \sum_n \left( \delta(r - r_n) \delta(p - \hat{p}_n) + \delta(p - \hat{p}_n) \delta(r - r_n) \right) \psi(R, t) \]
\[ + \psi^*(R,t) \sum_n \left( \delta(r - r_n) \delta(p - \hat{p}_n) + \delta(p - \hat{p}_n) \delta(r - r_n) \right) \hat{p}_n^\alpha \psi(R,t) + c.c. \] \) \) (12)

which we approximately present in the following

\[ J^\alpha(r,p,t) = p^\alpha f(r,p,t), \] \) \) (13)

which corresponds to the traditional structure of kinetic equation. The last formula has been already used in formulas \([10]\) and \([19]\).

Integrating equation \((10)\) and other form of kinetic equation obtained in this paper we can find that they give us the equations of the many-particle quantum hydrodynamics obtained in Refs. \([1]\), \([8]\), \([9]\).

Now we are going to include the Coulomb inter-particle interaction

\[ \partial_t f + \frac{1}{m} p \partial_r f + e^2 \phi \sin(\vec{\nabla}_r \vec{\nabla}_p) f \]
\[ + \frac{1}{4} \frac{\hbar}{m} \int \sum_{n,k} (\varepsilon_n \varepsilon_k G_{nk}) \sin(\vec{\nabla}_{r,n} \vec{\nabla}_p) \times \]
\[ \times \left( \psi^*(R,t) \left( \delta(r - r_n) \delta(p - \hat{p}_n) + \delta(p - \hat{p}_n) \delta(r - r_n) \right) \psi(R,t) + c.c. \right) dR = 0. \] \) \) (14)

In equation \((14)\) we find one more term in comparison with equation \((10)\). This new term is the last term in equation \((14)\), it is caused by the Coulomb interaction.

Equation \((14)\) can be rewritten using two-particle function

\[ \partial_t f + \frac{1}{m} p \partial_r f + e^2 \phi \sin(\vec{\nabla}_r \vec{\nabla}_p) f \]
\[ + e^2 \frac{\hbar}{m} \int G(r - r') \sin(\vec{\nabla}_r \vec{\nabla}_p) f_2(r,p,r',p',t) dr' dp' = 0, \] \) \) (15)

where

\[ f_2(r,p,r',p',t) = \frac{1}{4} \int \left( \psi^*(R,t) \sum_{n,k} \delta(r' - r_k) \delta(p' - \hat{p}_k) \times \right. \]
\[ \left. \times \left( \delta(r - r_n) \delta(p - \hat{p}_n) + \delta(p - \hat{p}_n) \delta(r - r_n) \right) \psi(R,t) + c.c. \right) dR, \] \) \) (16)

is the two-particle distribution function.

In the quasi-classical limit we have to include one term of the sum in the last term only, so we have

\[ \partial_t f + \frac{1}{m} p \partial_r f + eE \vec{\nabla}_p f - e^2 \int \nabla_r G(r - r') \cdot \nabla_p f_2(r,p,r',p',t) dr' dp' = 0. \] \) \) (17)
Equation (17) looks like classical kinetic equation. However we should admit that two-particle distribution function $f_2$ is defined via wave function $\psi$. Therefore, we see, analogously to the quantum hydrodynamics [1], [8], this equation is a quantum kinetic equation containing information about quantum effects, in particular, the exchange interaction. Some methods of obtaining of the closed quantum kinetic description was discussed and developed in Ref. [38].

**B. The self-consistent field approximation**

Considering many-particle quantum kinetics we find that equation of distribution function $f$ evolution contains two-particle distribution function $f_2$ that corresponds to the classical kinetics. For particles with a long-range interaction, in our case it is the Coulomb interaction, we can introduce, well-known in classical physics and quantum hydrodynamics, the self-consistent field approximation. In this approximation we separate two-particle distribution function $f_2$ in the product of two one-particle distribution functions $f$

$$f_2(r, p, r', p', t) = f(r, p, t)f(r', p', t).$$

This approximation gives us closed mathematical apparatus in the form of one nonlinear integral equation, which in the quasi-classic limit to be

$$\partial_t f + \frac{1}{m} p \partial_r f + eE \nabla_p f - e^2 \int \nabla_r G(r - r') f(r', p', t) dr' dp' \cdot \nabla_p f(r, p, t) = 0.$$  \hfill (19)

In general case, from equation (20), in the self-consistent field approximation, we find

$$\partial_t f + \frac{1}{m} p \partial_r f + e\frac{i}{\hbar} \varphi \sin(\nabla_r \nabla_p f)$$

$$+ e^2 \frac{i}{\hbar} \int G(r - r') \sin(\nabla_r \nabla_p f(r, p, t)f(r', p', t)dr' dp' = 0,$$  \hfill (20)

Introducing electric field for inter-particle interaction we come to the set of kinetic equation and field equations (Maxwell equations) instead of integro-differential equation (19). This set appears as

$$\partial_t f + \frac{1}{m} p \partial_r f + eE \nabla_p f = 0,$$  \hfill (21)

$$\nabla \times E = 0, \ \ \ \nabla E = 4\pi e \int f(r, p, t) dp.$$  \hfill (22)

In the result we see that derived equations corresponds to the Vlasov equation. It also coincides with the results obtained in Ref.s [15]-[21], where was used the Wigner distribution function.
III. CHARGED SPINLESS PARTICLES IN MAGNETIC FIELD

Studying charged particles in an magnetic field we have long derivative
\[ D_n = -i \hbar \nabla_n - e A_n / c \] instead of the short one \(-i \hbar \nabla_n\). So, we have to include it at definition of the
distribution function \( f(\mathbf{r}, \mathbf{p}, t) \) and in this case we have

\[ f(\mathbf{r}, \mathbf{p}, t) = \frac{1}{4} \int \left( \psi^*(R, t) \sum_n \left( \delta(\mathbf{r} - \mathbf{r}_n) \delta(\mathbf{p} - \hat{D}_n) + \delta(\mathbf{p} - \hat{D}_n) \delta(\mathbf{r} - \mathbf{r}_n) \right) \psi(R, t) + \text{c.c.} \right) dR, \tag{23} \]

In the absence of the inter-particle interaction we find

\[ \partial_t f + \frac{1}{m} \mathbf{p} \partial_\mathbf{r} f - \frac{e}{mc} \partial^\alpha A^{\alpha} \partial^\beta \mathbf{p} \left( p^\alpha f \right) - \frac{e}{c} (\partial_t A) \nabla_\mathbf{p} f \]

\[ - \frac{e}{mc} \frac{i}{\hbar} A^\alpha \sin(\nabla_\mathbf{r} \nabla_\mathbf{p})(p^\alpha f) + \frac{e}{\hbar} \phi \sin(\nabla_\mathbf{r} \nabla_\mathbf{p}) f = 0. \tag{24} \]

Including in consideration of the vector potential of the external electromagnetic field leads to appearing of three additional terms in the kinetic equation \([24]\), which are the third-fifth terms. The fourth term gives contribution in the force acting on the charge of the external electric field. The third and fifth terms is the magnetic part of the quantum Lorentz force. In the quasi-classical limit it has well-known form presented below.

In the quasi-classical limit we have to include one term of the sum in the last term only, so we have

\[ \partial_t f + \frac{\mathbf{p}}{m} \partial_\mathbf{r} f + e \left( \mathbf{E} + \frac{1}{mc} \mathbf{p} \times \mathbf{B} \right) \nabla_\mathbf{p} f = 0, \tag{25} \]

where we have written \( \mathbf{E} = -\partial_\mathbf{r} \phi - \partial_t A / c \).

We should write the Lorentz force in equation \([25]\) via velocity \( \mathbf{p} / m = \mathbf{v} \), but distribution function \( f = f(\mathbf{r}, \mathbf{p}, t) \) depends on momentum, and we write all coefficients via momentum \( \mathbf{p} \).

IV. KINETIC EVOLUTION OF SPINNING PARTICLES

In section III we present the Schrodinger equation contained only the Coulomb interaction. Now we going to consider kinetics of spinning charged particles. In this section we are particularly interested in the spin-spin interactions. The whole Hamiltonian, used in this case reads
$$H = \sum_n \left( \frac{1}{2m_n} \hat{D}_n^\alpha \hat{D}_n^\alpha + e_n \mathcal{F}_n^\text{ext} - \gamma_n \hat{\sigma}_n^\alpha \mathcal{B}_n^\alpha \right) + \frac{1}{2} \sum_{k,n \neq p} (e_k e_n G_{kn} - \gamma_k \gamma_n \mathcal{G}_{kn}^\alpha \hat{\sigma}_k^\alpha \hat{\sigma}_n^\beta),$$

where

$$\hat{D}_n^\alpha = -i \hbar \partial_n^\alpha - \frac{e_n}{c} A_{n,\text{ext}}^\alpha.$$

In these formulas we have used following designations. The Green’s functions of the Coulomb, the spin-spin and the spin-current interactions has the following form

$$G_{pn} = \frac{1}{r_{pn}}, \quad G_{\alpha\beta pn} = 4\pi \delta_{\alpha\beta} \delta(p_{pn}) + \frac{\partial_\alpha p}{r_{pn}} \partial_\beta p \left( \frac{1}{r_{pn}} \right).$$

The quantities $\varphi_{pn}^\text{ext} = \varphi(r_p, t), A_{p(\text{ext})}^\alpha = A^\alpha(r_p, t)$ are the scalar and the vector potentials of the external electromagnetic field:

$$B_{\alpha(\text{ext})}(r_p, t) = \varepsilon_{\alpha\beta\gamma} \partial_\beta p A_{\gamma(\text{ext})}(r_p, t),$$

$$E_{\alpha(\text{ext})}(r_p, t) = -\nabla_\alpha \varphi_{\text{ext}}(r_p, t) - \frac{1}{c} \partial_t A_{\alpha(\text{ext})}(r_p, t).$$

$\hat{\sigma}_p^\alpha$ is the Pauli matrix, a commutation relations for them is

$$[\hat{\sigma}_p^\alpha, \hat{\sigma}_n^\beta] = 2i \delta_{pn} \varepsilon^{\alpha\beta\gamma} \hat{\sigma}_p^\gamma.$$  

First we present kinetic equation for spinning particles in the absence of the inter-particle interaction

$$\partial_t f + \frac{1}{m} p \partial_r f - \frac{e}{mc} \partial_\alpha A_\beta \partial_\beta p^\alpha f - \frac{e}{c} (\partial_t A_\alpha) \nabla_p f$$

$$- \frac{e}{mc} \hat{\varphi}_{\text{ext}}(\hat{\nabla}_r \nabla_p)(p^\alpha f) + \frac{e}{\hbar} \varphi_{\text{ext}}(\hat{\nabla}_r \nabla_p) f$$

$$- \mu \frac{q}{\hbar} B^\alpha \sin(\hat{\nabla}_r \nabla_p) S^\alpha(r, p, t) = 0$$

In the quasi-classical limit we have to include one term of the sum in the last term only, so we have

$$\partial_t f + \frac{P}{m} \partial_r f + e \left( \mathbf{E} + \frac{1}{mc} \mathbf{p} \times \mathbf{B} \right) \nabla_p f + \partial_\alpha B_\beta(r, t) \partial_\alpha p \partial_\beta S^\beta(r, p, t) = 0,$$

where we have new quantity in these equations, we can call it spin-distribution function, it’s explicit form is

$$S^\alpha(r, p, t) = \frac{1}{4} \int \left( \psi^\dagger(R, t) \sum_n \left( \delta(r_\text{ext} - r_n) \delta(p_{n, \text{ext}} - p_{\hat{D}_n}) + \delta(p_{n, \text{ext}} - p) \delta(r_\text{ext} - r_n) \right) \sigma_{n, \text{ext}}^\alpha \psi(R, t) + h.c. \right) dR,$$
where h.c. stands for the Hermitian conjugation. $S^\alpha(r, p, t)$ is kinetic analog of the spin density, which arises in the quantum hydrodynamics and has form

$$S^\alpha(r, t) = \int dR \sum_n \delta(r - r_n) \psi^*(R, t) \hat{\sigma}_n^\alpha \psi(R, t),$$

(30)

we have used same letter for designation of the spin density and the spin-distribution function, but they differ by set of arguments.

It can be shown that

$$S^\alpha(r, t) = \int S^\alpha(r, p, t) dp.$$  

(31)

Magnetization $M^\alpha(r, t)$ usually used in the quantum hydrodynamics and has simple connection with the spin density $M^\alpha(r, t) = \gamma S^\alpha(r, t)$, where $\gamma$ is the gyromagnetic ratio for considering species of particles.

To get complete description of systems of spinning particles we have to derive an equation for the spin-distribution function $S^\alpha(r, p, t)$.

A. Including of interaction

Introducing of interaction makes the kinetic equation more larger, but we present it’s explicit form

$$\partial_t f + \frac{1}{m} p \partial_r f - \frac{e}{mc} \partial^\alpha A^\beta \partial_\mu \left( p^\alpha f \right) - \frac{e}{c} \partial_t A f$$

$$- \frac{e}{mc} \frac{i}{\hbar} A^\alpha \sin(\nabla_r \nabla_p) (p^\alpha f) + e \frac{i}{\hbar} \varphi \sin(\nabla_r \nabla_p)$$

$$- \mu \frac{i}{\hbar} B^\alpha \sin(\nabla_r \nabla_p) S^\alpha (r, p, t)$$

$$+ \frac{1}{4} \frac{i}{\hbar} \int \sum_{n,k \neq n} (e_n e_k G_{nk}) \sin(\nabla_r \nabla_p) \times$$

$$\times \left( \psi^*(R, t) \left( \delta(r - r_n) \psi(R, t) + h.c. \right) dR \right)$$

$$+ \frac{1}{4} \frac{i}{\hbar} \int \sum_{n,k \neq n} (e_n e_k G_{nk}^{\mu \nu}) \sin(\nabla_r \nabla_p) \times$$

$$\times \left( \psi^*(R, t) \left( \delta(r - r_n) \psi(R, t) + h.c. \right) dR = 0. \right)$$

(32)
Let’s describe meaning of terms in this equation. The first seven terms coincide with the same terms in the kinetic equation for charged spinless particles \(^{(24)}\). They are time evolution of distribution function presented by the first term, convective part of distribution function evolution, presented by the second and third terms, correspondingly. Terms four, and six give quantum-kinetic generalization of the Lorentz force describing action of the external magnetic field on particle charges. The fifth and seventh terms show the force acting on charges exerted by the external electric field. The eighth term describes action of the external magnetic field on spin of particles entering equation via the spin-distribution function \(S^\alpha(r, p, t)\). The last two terms present inter-particle interaction, the Coulomb and the spin-spin interactions correspondingly, and it’s influence on evolution of the distribution function.

Introducing two-particle spin-distribution function as

\[
S^{\alpha\beta}_2(r, p, r', p', t) = \frac{1}{4} \int \left( \psi^*(R, t) \sum_{n,k \neq n} \delta(r' - r_k)\delta(p' - \hat{D}_k) \times \right.
\]

\[
\times \left( \delta(r - r_n)\delta(p - \hat{D}_n) + \delta(p - \hat{D}_n)\delta(r - r_n) \right) \sigma^\alpha_n \sigma^\beta_k \psi(R, t) + h.c. \right) dR
\]

we can rewrite previous equation in the following form

\[
\frac{\partial f}{\partial t} + \frac{1}{m} p \frac{\partial f}{\partial r} - \frac{e}{mc} A^\alpha \frac{\partial f}{\partial p} - \frac{e}{c} (\partial_t A) \nabla_p f
\]

\[
- \frac{e}{mc} \frac{\hbar^2}{\hbar} A^\alpha \sin(\bar{\nabla}_r \nabla_p) (p^\alpha f) + e \frac{\hbar}{\hbar} \phi \sin(\bar{\nabla}_r \nabla_p) f
\]

\[
- \mu \frac{\hbar}{\hbar} B^\alpha \sin(\bar{\nabla}_r \nabla_p) S^\alpha(r, p, t)
\]

\[
+ e^2 \frac{\hbar}{\hbar} \int G(r - r') \sin(\bar{\nabla}_r \nabla_p) f_2(r, p, r', p', t) dr' dp'
\]

\[
+ \gamma \frac{\hbar}{\hbar} \int G^{\mu\nu}(r - r') \sin(\bar{\nabla}_r \nabla_p) S^{\mu\nu}_2(r, p, r', p', t) dr' dp' = 0
\]

Using of the two-particle functions we have the two last terms in equation \((34)\) written shortly. It also gives a form of kinetic equation analogous to the BBGKY equations, that allows us use well-known and useful ideas to get closed set of equations, such as introduction of the self-consistent field approximation, which we discuss for spinning below.
B. spin-distribution function evolution

In kinetic equation for spinning particles appears the spin distribution function. Therefore, for construction of the closed set of equation describing spinning particles we have to find equation evolution of the spin distribution function. For this goal we differentiate spin distribution function with respect to time, after some calculations we find the kinetic equation for spin distribution function evolution

$$\partial_t S^\alpha (\mathbf{r}, \mathbf{p}, t) + \frac{1}{m} \mathbf{p} \partial_r S^\alpha - \frac{e}{mc} \partial^\gamma A^\beta \partial_p \left( p^\gamma S^\alpha \right) - \frac{e}{c} (\partial_t \mathbf{A}) \nabla_p S^\alpha$$

$$- \frac{e}{mc} \frac{\hbar}{2} A^\beta \sin \left( \frac{\hbar}{\mathbf{r}} \nabla_p \right) \left( p^\beta S^\alpha \right) + \frac{e}{c} \varphi \sin \left( \frac{\hbar}{\mathbf{r}} \nabla_p \right) S^\alpha$$

$$- \mu \frac{\hbar}{2} B^\alpha \sin \left( \frac{\hbar}{\mathbf{r}} \nabla_p \right) f (\mathbf{r}, \mathbf{p}, t)$$

$$+ e^2 \frac{\hbar}{2} \int G (\mathbf{r} - \mathbf{r}') \sin \left( \frac{\hbar}{\mathbf{r}} \nabla_p \right) M_2 (\mathbf{r}, \mathbf{p}, \mathbf{r}', \mathbf{p}', t) d\mathbf{r} d\mathbf{p}'$$

$$+ \gamma^2 \frac{\hbar}{2} \int G^\alpha \beta (\mathbf{r} - \mathbf{r}') \sin \left( \frac{\hbar}{\mathbf{r}} \nabla_p \right) N_2^\beta (\mathbf{r}, \mathbf{p}, \mathbf{r}', \mathbf{p}', t) d\mathbf{r} d\mathbf{p}'$$

$$- \frac{2\gamma \hbar}{\hbar} \varepsilon^\alpha \beta \gamma \left( B^\beta S^\gamma + \gamma \int G_2^\alpha \beta (\mathbf{r}, \mathbf{r}') S_2^\beta (\mathbf{r}, \mathbf{p}, \mathbf{r}', \mathbf{p}', t) d\mathbf{r} d\mathbf{p}' \right).$$ (35)

There is no high order derivatives on space variables in the last term. In the second term of equation (35) we have used following presentation for the kinetic spin current

$$J^\alpha \beta (\mathbf{r}, \mathbf{p}, t) = p^\beta S^\alpha (\mathbf{r}, \mathbf{p}, t),$$ (36)

which has following explicit form

$$J^\alpha \beta (\mathbf{r}, \mathbf{p}, t) = \frac{1}{8} \int \left( \hat{D}^\beta_n \psi^* (\mathbf{R}, t) \right) \sum_n \left( \delta (\mathbf{r} - \mathbf{r}_n) \delta (\mathbf{p} - \hat{\mathbf{D}}_n) + \delta (\mathbf{p} - \hat{\mathbf{D}}_n) \delta (\mathbf{r} - \mathbf{r}_n) \right) \hat{\sigma}^\alpha_n \psi (\mathbf{R}, t)$$

$$+ \psi^* (\mathbf{R}, t) \sum_n \left( \delta (\mathbf{r} - \mathbf{r}_n) \delta (\mathbf{p} - \hat{\mathbf{D}}_n) + \delta (\mathbf{p} - \hat{\mathbf{D}}_n) \delta (\mathbf{r} - \mathbf{r}_n) \right) \hat{D}^\beta \hat{\sigma}^\alpha_n \psi (\mathbf{R}, t) + h.c. \right) d\mathbf{R}. \quad (37)$$

We have used a couple of new two-particle functions

$$M_2^\alpha (\mathbf{r}, \mathbf{p}, \mathbf{r}', \mathbf{p}', t) = \frac{1}{4} \int \left( \psi^* (\mathbf{R}, t) \sum_{n,k \neq n} \delta (\mathbf{r}' - \mathbf{r}_k) \delta (\mathbf{p}' - \hat{\mathbf{D}}_k) \timesight.$$  

$$\times \left( \delta (\mathbf{r} - \mathbf{r}_n) \delta (\mathbf{p} - \hat{\mathbf{D}}_n) + \delta (\mathbf{p} - \hat{\mathbf{D}}_n) \delta (\mathbf{r} - \mathbf{r}_n) \right) \sigma_n^\alpha \psi (\mathbf{R}, t) + h.c. \right) d\mathbf{R}, \quad (38)$$
and

\[ N_2^\alpha(r, p, r', p', t) = \frac{1}{4} \int \left( \psi^*(R, t) \sum_{n, k \neq n} \delta(r' - r_k) \delta(p' - \hat{D}_k) \times \right. \]

\[ \times \left( \delta(r - r_n) \delta(p - \hat{D}_n) + \delta(p - \hat{D}_n) \delta(r - r_n) \right) \sigma_k^\alpha \psi(R, t) + h.c. \) dR. \tag{39} \]

These two two-particle distribution function describe correlations between spin dynamic of one particle with the space motion (concentration evolution) other particle. Consequently, in the self-consistent field approximation this function represents as

\[ M_2^\alpha(r, p, r', p', t) = S^\alpha(r, p, t) f(r', p', t), \tag{40} \]

and

\[ N_2^\alpha(r, p, r', p', t) = f(r, p, t) S^\alpha(r', p', t). \tag{41} \]

Using equation (35) we can find hydrodynamical equation for spin evolution

\[ S^\alpha(r, t) = S^\alpha(r, p, t). \tag{35} \]

Terms containing \( \nabla_p f(r, p, t) \) give no contribution in the hydrodynamical spin evolution equation. Consequently, the First, second, and last terms of equation (35) give contribution in the hydrodynamical spin evolution equation, and we have

\[ \partial_t S^\alpha(r, t) + \frac{1}{m} \int \partial_r(\mathbf{p} S^\alpha(r, p, t)) dp \]

\[ - \frac{2 \gamma}{\hbar} \varepsilon^{\alpha \beta \gamma} \left( B^\beta S^\gamma(r, t) + \gamma \int G^{\beta \delta}(r, r') S_2^\gamma(r, r', t) dr' \right) = 0, \tag{42} \]

which corresponds to equations presented in Ref.s [4], [9]. We admit that we have not introduced velocity field in equation (42). The second term in equation (42) is the divergence of the spin current and the last term describes interactions. Hydrodynamical definition of spin-current we briefly discuss in the Appendix, where we show that presented here definition corresponds to the hydrodynamical one.

The second term in equation (42) is the divergence of the spin-current, and the last term in a big brackets describes interaction. Hydrodynamical spin-current definition we briefly discuss in Appendix.

**C. selfconsistent field approximation for spinning particles**

In previous sections we have considered the self-consistent field approximation for charged spinless particles. After derivation of the general set of kinetic equations for spinning particles we should consider self-consistent field approximation of these equations, to be short,
we present equations in the quasi-classic limit. We have presented approximate form for $M^2_\alpha$ and $N^2_\alpha$, but we have one more two-particle function $S^\alpha_2$, which is represented as

$$S^\alpha_2(r, p, r', p', t) = S^\alpha(r, p, t)S^\beta(r', p', t)$$

(43)

In the result we have next set of equations

$$\partial_t f + \frac{p}{m} \partial_r f + e \left( E + \frac{1}{mc} p \times B \right) \nabla_p f + \partial_\alpha B^\beta(r, t) \partial_p \alpha S^\beta(r, p, t) = 0,$$

(44)

and

$$\partial_t S^\alpha + \frac{p}{m} \partial_r S^\alpha + e \left( E + \frac{1}{mc} p \times B \right) \nabla_p S^\alpha + \partial_\gamma B^\alpha(r, t) \partial_p \gamma f(r, p, t) - \frac{2\gamma}{\hbar} \varepsilon^\alpha\beta\gamma B^\beta S^\gamma = 0,$$

(45)

Due to the fact that we have considered spin-spin interaction and we have not included spin-current and current-current interaction we have magnetic field satisfying to the following equation $\nabla \times B = 4\pi \nabla \times \int S(r, p, t) dp$. Using additivity of the electromagnetic fields we can include magnetic field caused by the electric currents, therefore we have

$$\nabla \times E = 0, \quad \nabla E = 4\pi e \int f(r, p, t) dp$$

(46)

$$\nabla \times B = 4\pi \nabla \times \int S(r, p, t) dp + \frac{4\pi e}{mc} \int pf(r, p, t) dp, \quad \nabla B = 0.$$ 

(47)

Equations (44) and (45) are the set of kinetic equations for the quantum plasma of spinning particles. Kinetic equations for quantum spin plasma have been derived in other papers [14], [15]-[22]. However, obtained in this paper equations have several differences with attained before, and main difference is the fact that we get set of two kinetic equation for distribution function $f$ and the spin distribution function $S^\alpha$, instead of one equation for the distribution function $f$ only.

V. KINETICS OF PARTICLES HAVING ELECTRIC DIPOLE MOMENT

The electric dipole moment is the property of neutral and charged particles which is, analogously to the spin, leads to anisotropic inter-particles interaction. Analogously to the spin kinetics we have to have two kinetic equations, one for usual distribution function $f$ and another one we can called the dipole distribution function, which is the average of the following operator

$$\hat{P}^\alpha = \sum_n d_n^\alpha \delta(r - \hat{r}_n)\delta(p - \hat{p}_n),$$

(48)
where $d_n$ is the electric dipole moment of particle, usually molecules happen to be polar particle, i.e. having electric dipole moment. Quantum hydrodynamics of particles having electric dipole moment was considered in Ref. [8]. Boltzmann equation has been used in Ref.s [39], [40]. As an example of infinite bubble diagram expansion method using for evolution of particles having electric dipole moment see Ref. [41]. To reveal main features of the electric dipole moment kinetics we present kinetic equations for neutral particles having electric dipole moment. We start our derivation from the many-particle Schrodinger equation with the Hamiltonian

$$\hat{H} = \sum_i \left( \frac{1}{2m_i} \vec{p}_i^2 - d_\alpha^i E_\alpha^{i,ext} + V_{\text{trap}}(\mathbf{r}_i, t) \right) - \frac{1}{2} \sum_{i,j \neq i} \left( d_\alpha^i d_\beta^j G_{\alpha\beta}^{ij} \right),$$ \hspace{1cm} (49)

where the Hamiltonian of the electric dipole interaction is

$$H_{dd} = -\partial^\alpha \partial^\beta \frac{1}{r} \cdot d_1^\alpha d_2^\beta,$$ \hspace{1cm} (50)

which corresponds to the Maxwell’s equation, as it was shown in Ref. [42].

Using well-known identity

$$-\partial^\alpha \partial^\beta \frac{1}{r} = \frac{\delta^{\alpha\beta} - 3r^\alpha r^\beta / r^2}{r^3} + \frac{4\pi}{3} \delta^{\alpha\beta} \delta(r),$$ \hspace{1cm} (51)

we can see that the Hamiltonian \(50\) differs from usually used one

$$H_{dd} = \frac{\delta^{\alpha\beta} - 3r^\alpha r^\beta / r^2}{r^3} d_1^\alpha d_2^\beta.$$ \hspace{1cm} (52)

Necessity to consider the Hamiltonian of electric dipole interaction in the form \(50\) caused by the fact that it must accord to the Maxwell’s equation. This connection exists since Maxwell’s equations describe electric field and it’s connection with the sources. In our case source is the density of electric polarization. Action of the electric field on the density of polarization come in to equations of motion via the force field. As the result it describes interaction of polarization (electric dipole moments). To be short we present them in the self-consistent field approximation

$$\partial_t f + \frac{P}{m} \partial_r f + e \left( \mathbf{E} + \frac{1}{mc} \mathbf{p} \times \mathbf{B} \right) \nabla_p f + \partial_\alpha B^\beta(r, t) \partial_{p\alpha} P^\beta(r, p, t) = 0,$$ \hspace{1cm} (53)

and

$$\partial_t P^\alpha + \frac{P}{m} \partial_r P^\alpha + e \left( \mathbf{E} + \frac{1}{mc} \mathbf{p} \times \mathbf{B} \right) \nabla_p P^\alpha + \partial_\gamma E^\beta(r, t) \partial_{p\gamma} R^{\alpha\beta}(r, p, t) = 0,$$ \hspace{1cm} (54)
where
\[ R^{\alpha \beta}(r, p, t) = \frac{1}{4} \int \left( \psi^* \sum_n d_n^\alpha d_n^\beta (\delta(r - \hat{r}_n)\delta(p - \hat{p}_n) + \delta(p - \hat{p}_n)\delta(r - \hat{r}_n)) + c.c. \right) dR \]

is the one particle distribution function depending on product of the two operators of the electric dipole moment of one particle, and the electric field caused inter-particle interaction satisfy to the Maxwell equations
\[ \nabla \times E = 0, \quad \nabla E = -4\pi \nabla \int P(r, p, t) dp. \tag{55} \]

At development of the quantum hydrodynamical description of systems of particles having electric dipole moment \([8]\) has been suggested a method of closing of the QHD set. Analogous approximation can be used here.

VI. KINETICS OF NEUTRAL PARTICLES WITH SHORT RANGE INTERACTION

Presented derivation can be used for systems of particles with the short-range interaction. The quantum hydrodynamics of neutral particles with the short-range interaction was considered in Ref. \([43]\), where ultracold quantum gases were considered particularly. Following ideas developed in this paper and in Ref. \([43]\) we can derive kinetic equation for described systems. To be short we present corresponding kinetic equation in the quasi-classical limit
\[ \partial_t f + \frac{p}{m} \partial_r f - \nabla V_{ext} \nabla_p f - \int \nabla r U(r - r') \cdot \nabla_p f_2(r, p, r', p', t) dr dp' = 0, \tag{56} \]

using the fact that we consider the short-range interaction we can rewrite general equation \((56)\) in the following form
\[ \partial_t f + \frac{p}{m} \partial_r f - \nabla V_{ext} \nabla_p f + \frac{1}{2} \Upsilon \partial_r \partial_p \int f_2(r, p, r', p', t) dp' = 0, \tag{57} \]

where
\[ \Upsilon = \frac{4\pi}{3} \int dr (r)^2 \frac{\partial U(r)}{\partial r}, \tag{58} \]

and we have made additional approximation, we have considered the short-range interaction in the first order by the interaction radius \([43]\). In general case the short-range interaction leads to the expansion of the term describing interaction in the series, and we have...
included only first term of this expansion. This scheme was developed in quantum hydro-
dynamics \cite{43}, contribution of the next nonzero term in this expansion, in hydrodynamical
approximation, can be found in Ref.s \cite{43}, \cite{44}, \cite{45}, \cite{46}.

Integrating the last term in equation (57) gives us trace of hydrodynamical two-particle
concentration at great length considered in Ref.s \cite{43}, \cite{44}

\[ n_2(r, r, t) \equiv Trn_2(r, r', t) = \int f_2(r, p, r, p', t)dpdp'. \]

Trace of the two-particle concentration was calculated in Ref. \cite{43}, where was shown, in
the first order by the interaction radius, that

\[ n_2(r, r, t) = n^2(r, t), \]

for system of particles being in the Bose-Einstein condensate state.

Methods developed in Ref. \cite{43} can be used for further studying of the structure of the
quantum kinetic equation for quantum gases.

\section{VII. GRAPHENE KINETICS}

This section is dedicated to development of the quantum kinetic theory for graphene
carriers. It is important to mention several interesting papers have been recently published,
where possible applications of graphene for future devises is discussed \cite{47}, \cite{48} and history
of graphene studies is presented \cite{49}.

Methods of physical kinetics are usually used for studying of collective properties of
many-particle systems, such as dispersion dependencies of collective excitations \textit{and} their
linear and nonlinear evolution. Properties of collective excitations in graphene, including
spectrum of linear plasmons, were considered in Ref.s \cite{50}, \cite{51}, \cite{52}, \cite{53} by means of the
random phase approximation (RPA).

Working with the graphene we have deal with the system of charged spinning particles
in the presence, in general case, of an external magnetic field. Therefore we have to use the
distribution function defined by formula (23). Before derivation of the kinetic equation for
graphene we will describe basic equation which drives graphene electrons in the vicinity of
the Dirac points.
We use the many-particle spinor massless Dirac equation \[54\], \[55\]

\[
\imath \hbar \partial_t \psi = \left( \sum_i \left( v_F \sigma^\alpha \hat{D}_i^\alpha + e_i \varphi_{i,ext} \right) + \frac{1}{2} \sum_{i,j \neq i} e_i e_j G_{ij} \right) \psi.
\] (59)

The following designations are used in the Hamiltonian (59): \(D_i^\alpha = -\imath \hbar \partial_t^\alpha - e_i A_i^\alpha,\), \(A_i^\alpha,\) - is the potentials of the external electromagnetic field, \(\mathbf{B}_{i,ext} = \text{curl} \mathbf{A}_{i,ext}\) is the magnetic field, quantities \(e_i, m_i\)-are the charge and mass of particles, \(\hbar\)-is the Planck constant, and \(G_{ij} = 1/r_{ij}\) is the Green functions of the Coulomb interaction. In equation (59) the spinor wave function \(\psi = \psi(R, t)\) depend on 2N coordinates \(R = [r_1, ..., r_N]\) and time, where \(r_i = [x_i, y_i]\) is the 2D coordinates of each particle. Potentials \(\varphi_{i,ext} = \varphi_{ext}(r_i, t), A_i^\alpha = A_i^\alpha(r_i, t)\) also depend on 2D variables. This fact has deep consequences. Potential part of electric field connected with the scalar potential via space derivative: \(E_i = -\nabla_i \varphi_i.\) Consequently in equation (59) there is no contribution of external electric field directed perpendicular to the graphene plane \(E_z\) (Contribution of \(E_z\) might appear via \(\partial_t A_z\)). Physically, there is no limitation on attendance of \(z\) projection of electric field and it’s action on graphene electrons. The magnetic field vector to be

\[
\mathbf{B} = \text{curl} \mathbf{A} = e_x (\partial_y A_z - \partial_z A_y) +
+ e_y (\partial_z A_x - \partial_x A_z) + e_z (\partial_x A_y - \partial_y A_x),
\]

two component of the vector potential of the magnetic field \(A_x, A_y\) are presented in Hamiltonian (59), and they does not depend on coordinate \(z.\) Therefore equation (59) contain \(z\) component of the magnetic field only. In this paper we interested in action of the external magnetic field directed at angle of graphene plane. Therefore, we get graphene kinetic equations including whole vector of magnetic field \(B \cdot e_z \rightarrow \mathbf{B} = [B_x, B_y, B_z].\)

Quantum hydrodynamics corresponding to described microscopic theory was developed in Ref. \[56\], \[57\]. Kinetic equation for graphene was used in Ref. \[58\] for derivation of the hydrodynamic equations.

It should be admitted that the spin matrix vector \(\sigma^\alpha\) describe not spin, but so called quasi-spin. This notion appears in graphene due to the fact that that Wigner-Seitz cell lattice contains two carbon atoms. Thus, graphene lattice consists of two sub-lattices, which are usually called lattice A and lattice B. Therefore, in his motion the graphene \(p_Z\) electron moves from atom A (B) to atom B (A), and transitions of the graphene electron between sub-lattices A and B is modelled by means quasi-spin.
Kinematic term in equation (59) is the helicity of particles $\sigma^\alpha p^\alpha$, so we have here that the Hamiltonian is proportional to the helicity of particles, and we consider dynamics governed by the spin 1/2 two dimensional helicity as the kinetic energy of particle. In Ref. [59] we have considered the QHD formulation of the particle dynamics governed by the spin-1 two dimensional helicity. It has been suggested that such equations might describe evolution of excitons in the graphene, existing in two layer graphene. In the next section we will consider kinetic formulation of this model. Here we continue development of the kinetic theory for electron in monolayer graphene.

Differentiating distribution function (23) with respect to time and using equation (59) for time derivatives of the wave function we find the graphene kinetic equation. In general case it has form

$$\partial_t f + v_F \nabla S - \frac{e}{c} \partial_t A \nabla_p f - v_F \frac{e}{c} \partial^\alpha A^\beta \partial^p S^\alpha - v_F \frac{\varphi}{\hbar} \sin(\hat{\nabla}_r \nabla_p) S^\alpha + e \frac{\varphi}{\hbar} \sin(\hat{\nabla}_r \nabla_p) f$$

$$+ e^2 \int \partial_\alpha G(r,r') \sin(\hat{\nabla}_r \nabla_p) f_2(r,p,r',p',t) dr' dp' = 0. \quad (60)$$

This equation contains the spin distribution function $S^\alpha (r,p,t)$, so we will derive equation for the spin distribution function $S^\alpha (r,p,t)$. As well, we rewrite equation (62) in the classical-like limit

$$\partial_t f + v_F \nabla S + e E \nabla_p f - v_F \frac{e}{c} \alpha^\beta \gamma^\gamma B^\gamma \partial^p S^\alpha$$

$$- e^2 \int \partial^\alpha G(r,r') \partial^p f_2(r,p,r',p',t) dr' dp' = 0. \quad (61)$$

As it discussed in Ref. [50], graphene properties have no classic limit, and the notion "classical-like limit" is used here to show that we consider the first term of operator $\sin(\hat{\nabla}_r \nabla_p)$ only. Obtained equation (61) is quantum equation, but it accounts leading on $\hbar$ terms only.

We have to admit that vectors $r$ and $p$ are two-dimensional vectors, however vectors $E(r,t)$ and $B(r,t)$, and even pseudo-spin vector $S(r,t)$ can be considered as three dimensional vectors. Belonging of the graphene electrons to the lattice A and B is modelled by increasing of amplitude of $\psi_A$ or $\psi_B$, where $\psi^* = (\psi_A, \psi_B)$ spinor function of electron, and appearing of the $S_z$ during derivative at using of the commutation relation for spin matrices is quite reasonable. Since belonging of the electron to lattice A or B describes by the struc-
ture of the Pauli matrices, and we can easily use some combination of the Pauli matrices as they appear at derivation.

Kinetic equation for the spin distribution function appears as

$$\partial_t S^\alpha + v_F \partial_r f + \frac{2v_F}{\hbar} \varepsilon^{\alpha \beta \gamma} p^\gamma S^\beta - \frac{e}{c} \partial_t A^\beta \partial_p^\beta S^\alpha$$

$$-v_F \frac{e}{c} \partial^\alpha A^\beta \partial^\beta f + v_F \frac{e}{c} \frac{i}{\hbar} A^\alpha \sin(\varphi r \nabla p) f + \frac{e}{\hbar} \varphi \sin(\varphi r \nabla p) S^\alpha$$

$$+ e^2 \frac{i}{\hbar} \int G(r, r') \sin(\varphi r \nabla p) M_2^\alpha(r, p, r', p', t) dr' dp' = 0,$$  \hspace{1cm} (62)

and in the classical like limit it has form

$$\partial_t S^\alpha + v_F \partial_r f + \frac{2v_F}{\hbar} \varepsilon^{\alpha \beta \gamma} p^\gamma S^\beta + eE^\beta \partial_p^\beta S^\alpha - v_F \frac{e}{c} \varepsilon^{\alpha \beta \gamma} B^\gamma \partial_p^\beta f$$

$$- e^2 \int \nabla r G(r, r') \nabla p M_2^\alpha(r, p, r', p', t) dr' dp' = 0.$$  \hspace{1cm} (63)

In the absence of inter-particle interaction we can rewrite the couple of graphene kinetic equations

$$\partial_t f + v_F \nabla S + eE \nabla_p f - v_F \frac{e}{c} \varepsilon^{\alpha \beta \gamma} B^\gamma \partial_p^\beta S^\alpha = 0,$$  \hspace{1cm} (64)

and

$$\partial_t S^\alpha + v_F \partial_r f + \frac{2v_F}{\hbar} \varepsilon^{\alpha \beta \gamma} p^\gamma S^\beta + eE \nabla_p S^\alpha - v_F \frac{e}{c} \varepsilon^{\alpha \beta \gamma} B^\gamma \partial_p^\beta f = 0.$$  \hspace{1cm} (65)

In the self-consistent field approximation the graphene kinetic equation (61), (63) transforms in

$$\partial_t f + v_F \nabla S + eE \nabla_p f - v_F \frac{e}{c} \varepsilon^{\alpha \beta \gamma} B^\gamma \partial_p^\beta S^\alpha$$

$$- e^2 \int \partial^\alpha G(r, r') f(r', p', t) dr' dp' \cdot \partial_p^\alpha f(r, p, t) = 0,$$  \hspace{1cm} (66)

and

$$\partial_t S^\alpha + v_F \partial_r f + \frac{2v_F}{\hbar} \varepsilon^{\alpha \beta \gamma} p^\gamma S^\beta + eE \nabla_p S^\alpha - v_F \frac{e}{c} \varepsilon^{\alpha \beta \gamma} B^\gamma \partial_p^\beta f$$

$$- e^2 \int \nabla r G(r, r') f(r', p', t) dr' dp' \cdot \nabla_p S^\alpha(r, p, t) = 0.$$  \hspace{1cm} (67)

Due to two-dimensionality of the system we can not introduce the field of interaction, or if we actually would be forced to do it we get \(\delta\) function in the right-hand side of the field equation \(\nabla E = 4\pi \rho = 4\pi e \delta(z) \int f(r, p, t) d^2 p\).

In the result we have the closed set of quantum kinetic equations for graphene, where evolution of the two distribution functions obeys to the two kinetic equations, see for example (66 and 67).
VIII. GRAPHENE EXCITON KINETICS

In the previous section we have considered kinetic equation for carriers in graphene, which can be considered as a spin-1/2 helicity governed system of charged particles. In this section we going to study kinetic properties of the spin-1 helicity governed particles motion. This model has been suggested in Ref. [59] for obtaining of the quantum hydrodynamical descriptions of the graphene excitons and dispersion dependence of collective excitations of the Bose-Einstein condensate of the graphene excitons. Here we present corresponding kinetic equations.

We present a basic equation here for evolution description of excitons in graphene

\[ i\hbar \partial_t \psi = \left( \sum_i \left( v_F \hat{s}_i^a p_i^a + V_{i,ext} \right) + \frac{1}{2} \sum_{i,j \neq i} U_{ij} \right) \psi. \quad (68) \]

Using this equation we derive and present below the equations for description of excitons collective motion. Equation (68) differs from (59) by the form of spin matrix and the form of interaction. Equation (68) contains following quantities: wave function \( \psi = \psi(R,t) \), \( R \) is the whole particles coordinates \( R = [r_1, \ldots, r_i, \ldots, r_N] \), \( r_i = [x_i, y_i] \), \( \hat{s}_i^a \) are the spin-1 matrixes for \( i \)-th particle, \( p_i^a = -\hbar \nabla \) is the momentum operator, \( V_{i,ext} \) is the potential of external field, \( U_{ij} \) is the short-range interaction potential describing the interaction between excitons in graphene. We notice that at the same time some particle might interact with several particles, by means short-range interaction potential \( U_{ij} \). For this statement illustration we refer to the liquid where molecules is neutral and interacts with the several neighbor molecules. We consider quasi spin-1 particles and spin operators are 3 \( \times \) 3 matrixes

\[ \hat{s}_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \hat{s}_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \hat{s}_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \]

and the commutation relation for spin-1 matrixes is

\[ [\hat{s}_i^a, \hat{s}_j^b] = i\delta_{ij} \varepsilon^{abc} \hat{s}_i^c. \quad (69) \]

Differentiating the distribution function, defined by formula (23), but without vector potential, with respect to time we find following kinetic equation

\[ \partial_t f = v_F \nabla S - \nabla V_{ext} \nabla_p f + \frac{1}{2} \chi_{2D} \partial_\alpha p^\alpha \int f_2(r, p, r, p', t) dp' = 0. \quad (70) \]
which contains quasi-spin distribution function \((29)\), where we have operators \(\hat{s}_n^\alpha\) instead of the Pauli matrices \(\hat{\sigma}_n^\alpha\), and, in the same way, we obtain kinetic equation for the spin-distribution function

\[
\partial_t S^\alpha + v_F \partial_r^\alpha C^\alpha = \frac{v_F}{\hbar} \varepsilon^{\alpha\beta\gamma} p^\gamma S^\beta - \nabla^\beta V_{ext} \partial_p^\beta S^\alpha + \frac{1}{2} \Upsilon_{2D} \nabla_r \nabla_p \int M_2^\alpha (r, p, r', t) dp' = 0. \tag{71}
\]

where

\[
C^\alpha = \frac{1}{4} \int \left( \psi^* (R, t) \sum_n \left( \delta(r - r_n) \delta(p - \hat{D}_n) + \delta(p - \hat{D}_n) \delta(r - r_n) \right) \times \sigma_n^\alpha \sigma_n^\beta \psi (R, t) + h.c. \right) dR, \tag{72}
\]

and

\[
\Upsilon_{2D} = \pi \int dr(r) r \frac{\partial U(r)}{\partial r}. \tag{73}
\]

\(\Upsilon_{2D}\) is the interaction constant for short-range interaction in two-dimensional system. We have obtained additional one particle function \(C^\alpha\), which unlike spin-1/2 case does not represents via \(f\) and \(S^\alpha\). As for electrically polarized particles, in section (V), appearing of additional function make truncation more difficult. We have suggested some approximation for hydrodynamical description \([8, 42, 59]\), where we have considered collective wave dispersion. We can suggest analogous approximations, but developing method allows us to perform truncation using approximate form of the wave function.

IX. CONCLUSION

Methods of studying of the quantum plasma have been developing last decade. The task of the interparticle interaction account reveals as a nontrivial problem. However we can conclude that this problem has been successfully solve in nonrelativistic in semi-relativistic quantum hydrodynamics. In classical physics kinetics method is more popular and profound method of the plasma studying. Thus it is worthwhile to derive the quantum kinetic theory, and it is also important to trace the interparticle interaction from microscopic level, as it was made in quantum hydrodynamics. First of all the kinetics allows to consider thermal effects. In most cases, when we have deal with the quantum plasma, quantum properties
might reveal at low temperatures. Thus using of the hydrodynamics in quantum case is more useful than in classical one. Nevertheless a some thermal spreading might leads to new effects in quantum plasma, and we have to have quantum kinetic theory to get correct description.

This program has been realized in the paper. We have obtained quantum kinetic description of different physical systems and presented corresponding kinetic equations in various approximations.

We have suggested new definition for the quantum distribution function which gives us possibility to derive a chain of the kinetics equations directly from many-particle Schrodinger equation including explicitly inter-particle interaction. This derivation is the direct generalization of the many-particle quantum hydrodynamic method. This definition changes at account of the vector potential of the external electromagnetic field by evident explicit replacement of the momentum operator $\hat{p}_n = -i\hbar \nabla_n$ on $\hat{D}_n = -i\hbar \nabla_n - e_n A_n/c$. In the quasi-classical limit it gives the Lorentz force in the usual form, and it also corresponds to the quantum hydrodynamics of spinning particles. Developed description corresponds to the Vlasov equation in the classical limit.

We have derived kinetic equations for charged spinning particles (quantum plasma), neutral particles with short-range interaction, electrons in graphene, electrically polarized molecules, and graphene excitons. We have shown that in the system of spinning particles we have to use set of two kinetic equations, one for usual distribution function and the another for spin-distribution function which arise in the equation of distribution function evolution. We have obtained closed set of kinetic equation which allow to use them for studying of particular problems.

Appendix A

Here we will proof following formula (8), which we rewrite here

$$j(r, t) = \int p f(r, p, t) dp,$$

we will make including presence of an external magnetic field, that corresponds to the fact that we should use general definition of the distribution function (23). Distribution function $f$ defined by formula (23), or in the absence of the magnetic field and the rotational electric
field the distribution function has more simple form (5). Hydrodynamical current \( j(\mathbf{r},t) \) has form [1], [2], [4], [8], [9]

\[
j(\mathbf{r},t) = \int \sum_{n=1}^{N} \frac{1}{2} \delta(\mathbf{r} - \mathbf{r}_n) \left( (\hat{p}\psi)^* \psi + \psi^* \hat{p}\psi \right) dR. \tag{A2}
\]

Now we are ready to demonstration of the formula (8) or (A3).

\[
\int p f(\mathbf{r},p,t) dp = \frac{1}{4} \int p \sum_{n=1}^{N} \left( \psi^* \left( \delta(\mathbf{r} - \mathbf{r}_n) \delta(\mathbf{p} - \hat{p}_n) + \delta(\mathbf{p} - \hat{p}_n) \delta(\mathbf{r} - \mathbf{r}_n) \right) \psi + c.c. \right) dp
\]

\[
= \frac{1}{4} \int \sum_{n=1}^{N} \left( \psi^* \left( \delta(\mathbf{r} - \mathbf{r}_n) \delta(\mathbf{p} - \hat{p}_n) + \delta(\mathbf{p} - \hat{p}_n) \delta(\mathbf{r} - \mathbf{r}_n) \right) \psi 
+ \left( \delta(\mathbf{r} - \mathbf{r}_n) \delta(\mathbf{p} - \hat{p}_n) + \delta(\mathbf{p} - \hat{p}_n) \delta(\mathbf{r} - \mathbf{r}_n) \right) \psi^* \cdot \psi \right) dp
\]

\[
= \frac{1}{4} \int \int \sum_{n=1}^{N} \left( \psi^* \left( \delta(\mathbf{r} - \mathbf{r}_n) \delta(\mathbf{p} - \hat{p}_n) + \delta(\mathbf{p} - \hat{p}_n) \delta(\mathbf{r} - \mathbf{r}_n) \right) \psi \right. 
+ \left( \delta(\mathbf{r} - \mathbf{r}_n) \delta(\mathbf{p} - \hat{p}_n) + \delta(\mathbf{p} - \hat{p}_n) \delta(\mathbf{r} - \mathbf{r}_n) \right) (\mathbf{p}\psi)^* \cdot \psi \right) dp
\]

Integrating on momentum \( p \) we replace \( p \) on \( \hat{p}_n \) due to the delta functions \( \delta(\mathbf{p} - \hat{p}_n) \) and find

\[
\int p f(\mathbf{r},p,t) dp = \frac{1}{2} \int \sum_{n=1}^{N} \frac{1}{2} \left( (\hat{p}_n\psi)^* \left( \delta(\mathbf{r} - \mathbf{r}_n) + \delta(\mathbf{r} - \mathbf{r}_n) \right) \psi 
+ \psi^* \left( \delta(\mathbf{r} - \mathbf{r}_n) + \delta(\mathbf{r} - \mathbf{r}_n) \right) \hat{p}_n\psi \right) dp = j(\mathbf{r},t)
\]

Analogously, in the presence of the external magnetic field, we can show that

\[
j(\mathbf{r},t) = \int p f(\mathbf{r},p,t) dp,
\tag{A3}
\]

where, from hydrodynamic point of view

\[
j(\mathbf{r},t) = \int \sum_{n=1}^{N} \frac{1}{2} \delta(\mathbf{r} - \mathbf{r}_n) \left( (\hat{D}\psi)^* \psi + \psi^* \hat{D}\psi \right) dR. \tag{A4}
\]

In formula (5) the hydrodynamic kinetic energy was mentioned, it’s explicit form is

\[
\varepsilon(\mathbf{r},t) = \int \sum_{n=1}^{N} \frac{1}{4m_n} \delta(\mathbf{r} - \mathbf{r}_n) \left( (\hat{D}^2\psi)^* \psi + \psi^* \hat{D}^2\psi \right) dR. \tag{A5}
\]
Formula (9) can be proved analogously to the proving of formula (8) or (A3).

The hydrodynamical spin-current appears in the formula (42), where the second term we can rewrite as

\[
\frac{1}{m} \int \partial_\beta (p^\beta S^\alpha (r, p, t)) dp = \partial_\beta \frac{1}{m} \int p^\beta S^\alpha (r, p, t) dp \equiv \partial_\beta J^{\alpha\beta} (r, t),
\]

where

\[
J^{\alpha\beta} (r, t) = \int \sum_n \frac{1}{2} \delta (r - r_n) \left( (\hat{D}_\beta \psi)^* \sigma^\alpha \psi + \psi^* \hat{D}_\beta \sigma^\alpha \psi \right) dR.
\]

It can be proved that formula (A7) corresponds to the QHD definition of the spin-current [2], [4], [60].

1. Discussion of some methods derivation of kinetic equations

In nineteen century, Boltzmann suggested his famous kinetic equation to describe an evolution of a gas of interacting particles. To describe interaction he introduced the collision integral. Since then the idea of using collision integral become widely used. It has been used even for plasma description, along with self-consistent field. Fundamental meaning of the collision integral we can understand by reading famous Landau’s course of theoretical physics [61]. In last decade a Boltzmann type equation has been used to description of ultracold quantum gases to describe a part ”existing in excited states” that corresponds to the part of particles at non-zero temperature [62]-[65]. A derivation of kinetic equation from NLSE was made in Ref. [66] using Wigner distribution function.

Let’s pay attention to the ideas presented in Landau’s course of theoretical physics [61]. At description of non-equilibrium states of a Fermi- or Bose-liquid one uses quasi-particle distribution function \( \hat{n} \). This function depends on momentum, coordinate and time, and satisfies a transport (kinetic) equation. One takes the transport equation in the form of

\[
\frac{d\hat{n}(r, p, t)}{dt} = I(\hat{n}),
\]

where \( I(n) \) is the collision integral and \( d/dt \) is an operator which has following form:

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + \hat{r} \frac{\partial}{\partial r} + \hat{p} \frac{\partial}{\partial p},
\]

where \( \hat{r} \) and \( \hat{p} \) should be find from Hamilton’s equations.
Using of such representation to find an evident form of a transport equation is not correct, because transport equation is specified in six dimensional space, where \( r, p \) and \( t \) are independent variables and quantities \( \dot{r} \) and \( \dot{p} \) are not specified and have no sense.

Consideration of transport equation as a continuity equation in six dimensional phase space was very useful, it was a way of understanding of meaning of transport equation by many physicists. But kinetics chains have structure different from structure of hydrodynamics one.

We have made criticism of the way of obtaining the left-hand side of the transport equation, but one of the crucial points of this paper that we must not use form (A8) for transport (kinetic) equation. Moreover, we should not expect existence of collision integral in kinetic equation for description of short range interaction of neutral particles. An example of kinetic equation for neutral particles without collision term is presented in Ref. [66].

New kinetic equation gives us another method for study of particle dynamics. This equation is derived for a system of neutral interacting particles. One does not contain collisions integral. It differs from Boltzmann equation. Equation contains interaction via term containing the trace of two-particle concentration \( f_2 \) and interaction constant. The equation contains field variables only. In this case we can call it field kinetic equation. We obtain equation for the case of particles with short-range interaction. We do not use conditions of that interaction is weak or concentration is small. We found equation as for Bose as for Fermi particles, difference between Bose and Fermi particles reveals at further calculations of \( f_2 \). It is interesting that we do not need to know cross-section for particles collisions as it is in Boltzmann equation. In this way we are free to calculate various properties of particles system without direct knowledge inter-particle potential of interaction. Equation (57) contains information about inter-particle potential of interaction via interaction constant. The last one is the integral of inter-particle potential of interaction. The interaction constant can be found from measuring of dispersion dependence calculated from our equation. We do not need to know information about scattering length or cross-section before solving of the equation.
Acknowledgments

The author thanks Professor L. S. Kuz’menkov for fruitful discussions.

[1] L. S. Kuz’menkov and S. G. Maksimov, Teor. i Mat. Fiz., 118 287 (1999) [Theoretical and Mathematical Physics 118 227 (1999)].
[2] L. S. Kuz’menkov, S. G. Maksimov, and V. V. Fedoseev, Theor. Math. Fiz. 126 136 (2001) [Theoretical and Mathematical Physics, 126 110 (2001)].
[3] F. Haas, G. Manfredi, M. Feix, Phys. Rev. E 62, 2763(2000).
[4] P. A. Andreev and L. S. Kuz’menkov, Russian Phys. Jour. 50, 1251 (2007).
[5] Mattias Marklund and Gert Brodin, Phys. Rev. Lett. 98, 025001 (2007).
[6] G. Brodin and M. Marklund, New J. Phys. 9, 277 (2007).
[7] G. Brodin, M. Marklund, and G. Manfredi, Phys. Rev. Lett. 100, 175001 (2008).
[8] P. A. Andreev, L. S. Kuz’menkov, M. I. Trukhanova, Phys. Rev. B 84, 245401 (2011).
[9] P. A. Andreev, L.S. Kuz’menkov, Int. J. Mod. Phys. B 1250186 (2012).
[10] S. M. Mahajan and F. A. Asenjo, Phys. Rev. Lett. 107, 195003 (2011).
[11] F. A. Asenjo, V. Munoz, J. A. Valdivia, and S. M. Mahajan, Phys. Plasmas 18, 012107 (2011).
[12] A. Yu. Ivanov, P. A. Andreev, and L. S. Kuzmenkov, arXiv:1209.6124
[13] F. Haas, B. Eliasson, P. K. Shukla, Phys. Rev. E 85, 056411 (2012).
[14] Gert Brodin, Mattias Marklund, Jens Zamanian, Asa Ericsson, and Piero L. Mana, Phys. Rev. Lett. 101, 245002 (2008).
[15] M. Marklund, J. Zamanian, G. Brodin, Transport Theory and Statistical Physics, 39, 502, (2011).
[16] G. Brodin, M. Marklund, J. Zamanian and M. Stefan, Plasma Phys. Control. Fusion 53, 074013 (2011).
[17] M. Stefan, J. Zamanian, G. Brodin, A. Prasad Misra, and M. Marklund, Phys. Rev. E 83, 036410 (2011).
[18] F. A. Asenjo, J. Zamanian, M. Marklund, G. Brodin, and P. Johansson, New J. Phys. 14, 073042 (2012).
[19] F. Haas, M. Marklund, G. Brodin, J. Zamanian, Physics Letters A 374, 481 (2010).
[20] J. Zamanian, M. Stefan, M. Marklund, and G. Brodin, Physics of Plasmas **17**, 102109 (2010).
[21] J. Zamanian, M. Marklund and G. Brodin, New Journal of Physics **12**, 043019 (2010).
[22] F. Haas, J. Zamanian, M. Marklund and G. Brodin, New Journal of Physics **12**, 073027 (2010).
[23] M.V. Altaisky, Physics Letters A, **374**, 522 (2010).
[24] L. S. Kuz'menkov and S. G. Maksimov, Teor. i Mat. Fiz., **131** 231 (2002) [Theoretical and Mathematical Physics **131** 641 (2002)].
[25] S. G. Maximov, L. S. Kuzmenkov, J. L. Guardado Zavala, International Journal of Quantum Chemistry, **100**, 311, (2004).
[26] S. G. Maximov and L. S. Kuzmenkov, Int. J. Theor. Phys. **11**, 1 (2004).
[27] L. S. Kuz’menkov. and S. G. Maksimov, Theoretical and Mathematical Physics, **143**, 821 (2005).
[28] S. Maximov, L. S. Kuzmenkov, Chaos, Solitons and Fractals, **37**, 369 (2008).
[29] S. Maximov, Physica D: Nonlinear Phenomena, **238**, 1937 (2009).
[30] M. V. Kuzelev, A. A. Rukhadze, Usp. Fiz. Nauk, **169** 687 (1999) [Phys. Usp. **42**, 603 (1999)].
[31] D. V. Vagin, P. A. Polyakov, and N. E. Rusakova, Moscow University Physics Bulletin, **64**, 133 (2009).
[32] P. K. Shukla, B. Eliasson, Phys. Usp. **53** 51 (2010) [Uspehi Fizhceskih Nauk **180**, 55 (2010)].
[33] P. K. Shukla, B. Eliasson, Rev. Mod. Phys. **83**, 885 (2011).
[34] Yu. L. Klimontovich, *Statistical Physics* [in Russian], Nauka, Moscow (1982); English transl., Harwood, New York (1986).
[35] S. Weinberg, *Gravitation and Cosmology* (John Wiley and Sons, Inc., New York, 1972).
[36] L. S. Kuz’menkov, Theoretical and Mathematical Physics **86**, 159 (1991).
[37] L. D. Landau, E. M. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory*. Vol. 3 (3rd ed.). Pergamon Press, (1977).
[38] O. Petrus, Physica **82A**, 273 (1976).
[39] C.-K. Chan, C.-J. Wu, W.-C. Lee, and S. Das Sarma, Phys. Rev. A **81**, 023602 (2010).
[40] S. Ronen and J. Bohn, Phys. Rev. A **81**, 033601 (2010).
[41] Qiuzi Li, E. H. Hwang, and S. Das Sarma, Phys. Rev. B **82**, 235126 (2011).
[42] P. A. Andreev and L. S. Kuz’menkov, arXiv: 1201.2440.
[43] P. A. Andreev, L. S. Kuz’menkov, Phys. Rev. A **78**, 053624 (2008).
[44] P. A. Andreev, L. S. Kuz’menkov, Russian Physics Journal **52**, 912 (2009).
[45] P. A. Andreev, L. S. Kuz’menkov, Mod. Phys. Lett. B 26, 1250152 (2012).
[46] K. V. Zezyulin, P. A. Andreev, and L. S. Kuz’menkov, arXiv:1205.1161.
[47] P. Pasanen, M. Voutilainen, M. Helle, X. Song and P. J Hakonen, Phys. Scr. T146, 014025 (2012).
[48] S. Bae, S. Jin Kim, D. Shin, Jong-Hyun Ahn, and B. Hee Hong, Phys. Scr. T146, 014024 (2012).
[49] A. K. Geim, Phys. Scr. T146, 014003 (2012).
[50] S. Das Sarma and E. H. Hwang, Phys. Rev. Lett. 102, 206412 (2009).
[51] M. Jablan, H. Buljan, M. Soljacic, Phys. Rev. B 80, 245435 (2009).
[52] L. A. Falkovsky, A. A. Varlamov, Eur. Phys. J. B 56, 281 (2007).
[53] A. A. Dubinov, V. Ya. Aleshkin, V. Mitin, T. Otsuji, V. Ryzhii, J. Phys.: Condens. Matter 23, 145302 (2011).
[54] D. E. Sheehy and J. Schmalian, Phys. Rev. Lett. 99, 226803 (2007).
[55] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, A. A. Firsov, Nature 438, 197 (2005).
[56] P. A. Andreev, arXiv: 1201.0779.
[57] P. A. Andreev, PIERS Proceedings, p.154, August 19-23, Moscow, Russia (2012).
[58] D. Svintsov, V. Vyurkov, S. Yurchenko, T. Otsuji, and V. Ryzhii, arXiv:1201.0592.
[59] P. A. Andreev, arXiv: 1201.6553.
[60] P. A. Andreev and L. S. Kuz’menkov, arXiv: 1210.1090.
[61] L. D. Landau and E. M. Lifshitz, Statistical Physics, Part 2: Theory of the Condensed State, Course of theoretical Physics Vol. 9 (Pergamon Press, London, 1987).
[62] T. Nikuni and A. Griffin, Phys. Rev. A, 65, 011601(R) (2001).
[63] T. Nikuni and A. Griffin, Phys. Rev. A, 69, 023604 (2004).
[64] J. E. Williams and A. Griffin, Phys. Rev. A, 63, 023612 (2001).
[65] T. Nikuni and A. Griffin, Phys. Rev. A, 63, 033608 (2001).
[66] M. Marklund and P. K. Shukla, Eur. Phys. J. B 48, 71 (2005).