Abstract. Graph associahedra are natural generalizations of the classical associahedra. They provide polytopal realizations of the nested complex of a graph $G$, defined as the simplicial complex whose vertices are the tubes (i.e. connected induced subgraphs) of $G$ and whose faces are the tubings (i.e. collections of pairwise nested or non-adjacent tubes) of $G$. The constructions of M. Carr and S. Devadoss, of A. Postnikov, and of A. Zelevinsky for graph associahedra are all based on the nested fan which coarsens the normal fan of the permutahedron. In view of the combinatorial and geometric variety of simplicial fan realizations of the classical associahedra, it is tempting to search for alternative fans realizing graphical nested complexes.

Motivated by the analogy between finite type cluster complexes and graphical nested complexes, we transpose in this paper S. Fomin and A. Zelevinsky’s construction of compatibility fans from the former to the latter setting. For this, we define a compatibility degree between two tubes of a graph $G$. Our main result asserts that the compatibility vectors of all tubes of $G$ with respect to an arbitrary maximal tubing on $G$ support a complete simplicial fan realizing the nested complex of $G$. In particular, when the graph $G$ is reduced to a path, our compatibility degree lies in $\{-1, 0, 1\}$ and we recover F. Santos’ Catalan many simplicial fan realizations of the associahedron.

Keywords. Graph associahedra · finite type cluster algebras · compatibility degrees · compatibility fans.

1. Introduction

Associahedra. The $n$-dimensional associahedron is a simple polytope whose $\frac{1}{n+2} \binom{2n+2}{n+1}$ vertices correspond to Catalan objects (triangulations of an $(n+3)$-gon, binary trees on $n+1$ nodes, ...) and whose edges correspond to mutations between them (diagonal flips, edge rotations, ...). Its combinatorial structure appeared in early works of D. Tamari [Tam51] and J. Stasheff [Sta63], and was first realized as a convex polytope by M. Haiman [Hai84] and C. Lee [Lee89]. Since then, the associahedron has motivated a flourishing research trend with rich connections to combinatorics, geometry and algebra: polytopal constructions [Lod04, HL07, CSZ11, LP13], Tamari and Cambrian lattices [MHPS12, Rea04, Rea06], diameter and Hamiltonicity [STT88, Deh10, Pon14, HN99], geometric properties [BHLT09, HLR10, PS13], combinatorial Hopf algebra [LR98, HNT05, Cha00, CP14], to cite a few. The associahedron was also generalized in several directions, in particular to secondary and fiber polytopes [GKZ08, BFS90], graph associahedra and nestohedra [CD06, Dev09, Pos09, FS05, Zel06, Pil13], pseudotriangulation polytopes [RS03], cluster complexes and generalized associahedra [FZ03b, CFZ02,HLT11, Ste13, Hoh12], and brick polytopes [PS12, PS15].

Graph associahedra. This paper deals with graph associahedra, which were defined by M. Carr and S. Devadoss [CD06] in connection to C. De Concini and C. Procesi’s wonderful arrangements [DCP95]. Given a simple graph $G$ with $\kappa$ connected components and $n + \kappa$ vertices, the $G$-associahedron $\text{Asso}(G)$ is an $n$-dimensional simple polytope whose combinatorial structure encodes the connected subgraphs of $G$ and their nested structure. More precisely, the face lattice of the polar of the $G$-associahedron is isomorphic to the nested complex $\text{Nest}(G)$ on $G$, defined as the simplicial complex of all collections of tubes (connected induced subgraphs) of $G$ which are pairwise compatible (either nested, or disjoint and non-adjacent). As illustrated in Figures 1, 2 and 3, the graph associahedra of certain special families of graphs happen to coincide with well-known families of polytopes: classical associahedra are path associahedra, cyclohedra are cycle...
associahedra, and permutahedra are complete graph associahedra. The graph associahedra were extended to the nestohedra, which are simple polytopes realizing the nested complex of arbitrary building sets [Pos09, FS05]. Graph associahedra and nestohedra have been geometrically realized in different ways: by successive truncations of faces of the standard simplex [CD06], as Minkowski sums of faces of the standard simplex [Pos09, FS05], or from their normal fans by exhibiting explicit inequality descriptions [Dev09, Ze06]. The resulting polytopes all have the same normal fan which coarsens the type A Coxeter arrangement: its rays are the characteristic vectors of the tubes, and its cones are generated by characteristic vectors of compatible tubes. Alternative realizations of graph associahedra with different normal fans are obtained by successive truncations of faces of the cube in [Vol10, DFRS15]. The objective of this paper is to provide a new unrelated family of complete simplicial fans realizing the graphical nested complex Nest(G) for any graph G.

Cluster algebras and cluster fans. Our construction is directly inspired from combinatorial and geometric properties of finite type cluster algebras and generalized associahedra introduced by S. Fomin and A. Zelevinsky in [FZ02, FZ03a, FZ03b]. Note that A. Zelevinsky [Ze06] already underlined the closed connection between nested complexes, nested fans, and nestohedra on one hand and cluster complexes, cluster fans and generalized associahedra on the other hand. Our paper contributes to strengthen this connection: we use ideas from cluster algebras to obtain results on graphical nested complexes, which in turn translate to relevant properties of the geometry of finite type cluster algebras.

Cluster algebras are commutative rings generated by a set of cluster variables grouped into overlapping clusters. The clusters are obtained from an initial cluster $X^0$ by a mutation process. Each mutation exchanges a single variable in a cluster according to a formula controlled by a combinatorial object. We refer to [FZ02] since precise details on this mutation are not needed here. Two cluster variables are exchangeable if they belong to a same cluster and compatible if they are not compatible and belong to two clusters connected by a mutation. The cluster complex of a cluster algebra $A$ is the pure simplicial complex whose vertices are cluster variables of $A$ and whose facets are the clusters of $A$. A cluster algebra is of finite type if it has finitely many cluster variables, and thus if its cluster complex is finite. Finite type cluster algebras were classified in [FZ03a]: up to isomorphism, there is one finite type cluster algebra for each finite crystallographic root system.

The Laurent Phenomenon [FZ02] asserts that each cluster variable $x$ can be expressed as a Laurent polynomial in terms of the cluster variables $x_1, \ldots, x_n$ of the initial cluster $X^0$. The d-vector of $x$ with respect to $X^0$ is the vector $d(X^0, x)$ whose $i$th coordinate is the exponent of the initial variable $x_i$ in the denominator of $x$. In finite type, this exponent was also interpreted in [FZ03b, CP15] as the compatibility degree $(x_i \parallel x)$ between the cluster variables $x_i$ and $x$. The compatibility degree $(\cdot \parallel \cdot)$ has the following properties: for any distinct cluster variables $x$ and $x'$, we have $(x \parallel x') \geq 0$ with equality if and only if $x$ and $x'$ are compatible, and $(x \parallel x') = 1 = (x' \parallel x)$ if and only if $x$ and $x'$ are exchangeable. The d-vectors can be used to construct a simplicial fan realization of the cluster complex, called d-vector fan: it is known for certain initial clusters in finite type cluster algebras, that the cones generated by the d-vectors of all collections of compatible cluster variables form a complete simplicial fan realizing the cluster complex. This is proved by S. Fomin and A. Zelevinsky [FZ03a] for the bipartite initial cluster, by S. Stella [Ste13] for all acyclic initial clusters, and by F. Santos [CSZ11, Section 5] for any initial cluster in type A. In fact, we expect this property to hold for any initial cluster, acyclic or not, of any finite type cluster algebra: this paper proves it for types A, B, and C (our general proof provides as a particular case a new proof in type A, similar to that of [CSZ11, Section 5]), and the other finite types are investigated in a current project of the authors in collaboration with C. Ceballos.

There is another complete simplicial fan realizing the cluster complex, whose rays are now given by the g-vectors of the cluster variables, defined as the multi-degrees of the cluster variables expressed in the cluster algebra with principal coefficients [FZ07]. The fact that the cones generated by the g-vectors of all collections of compatible cluster variables form a complete simplicial fan is a consequence of [Rea14]. When the initial cluster is acyclic, the resulting g-vector fan is the Cambrian fan of N. Reading and D. Speyer [RS09] and it coarsens the Coxeter fan.
Figure 1. The classical associahedron (left) is the path associahedron (right).

Figure 2. The cyclohedron (left) is the cycle associahedron (right).

Figure 3. The permutahedron (left) is the complete graph associahedron (right).
Motivated by the combinatorial and geometric richness of the \(d\)- and \(g\)-vector fans described above for finite type cluster algebras, we want to construct various simplicial fan realizations of the nested complex. As it coarsens the Coxeter fan, the normal fan of the graph-associahedra and nestohedra of [CD06, Dev09, Pos09, FS05, Zel06] should be considered as an analogue of the \(g\)-vector fan. A tentative approach to construct alternative \(g\)-vector fans for tree associahedra can be found in [Pil13]. Other approaches are still in progress.

In this paper, we construct an analogue of the \(d\)-vector fan for graphical nested complexes. We define the compatibility degree \((t \parallel t')\) between two tubes \(t, t'\) of a graph \(G\) to be \((t \parallel t') = -1\) if \(t = t'\), \((t \parallel t') = 0\) if \(t\) and \(t'\) are compatible, and \((t \parallel t') = |\{neighbors of t in t' \setminus t\}|\) otherwise. Similar to the compatibility degree for cluster algebras, it satisfies \((t \parallel t') \geq 0\) for any two distinct tubes \(t, t'\) of \(G\), with equality if and only if \(t\) and \(t'\) are compatible, and \((t \parallel t') = 1 = (t' \parallel t)\) if and only if \(t\) and \(t'\) are exchangeable. We define the compatibility vector \(d(T^o, t) := [(t \parallel t_1), \ldots, (t \parallel t_n)]\) of a tube \(t\) with respect to an initial maximal tubing \(T^o := \{t_1, \ldots, t_n\}\), and the compatibility matrix \(d(T^o, T) := [(t \parallel t_j)]_{i \in [n], j \in [m]}\) of a tubing \(T := \{t_1, \ldots, t_m\}\) of \(G\) with respect to the initial tubing \(T^o\). We write \(\mathbb{R}_{\geq 0} d(T^o, T)\) to denote the polyhedral cone generated by the compatibility vectors of the tubes of \(T\) with respect to \(T^o\). We also define the dual compatibility vector \(d^*(T^o, t) := [(t' \parallel t)]_{i \in [n]}\) of \(t\) with respect to \(T^o\) and the dual compatibility matrix \(d^*(T^o, T) := [(t' \parallel t_j)]_{i \in [n], j \in [m]}\). Note that \(d(T^o, T)\) and \(d^*(T^o, T)\) are transpose to each other.

Although there is no denominators involved anymore, we still use the letter \(d\) to stand for compatibility degree vector, and to match with the cluster algebra notations. Indeed, our compatibility degrees for paths and cycles coincide with the compatibility degree of [FZ03b] in types \(A, B,\) and \(C\). Compatibility degrees on type \(A\) cluster variables correspond to compatibility (and dual compatibility) degrees on tubes of paths while compatibility degrees in type \(C\) (resp. \(B\)) cluster variables correspond to compatibility (resp. dual compatibility) degrees on tubes of cycles.

Our main result is the following analogue of the compatibility fan for path associahedra constructed by F. Santos in [CSZ11, Section 5].

**Theorem 1.** For any graph \(G\), the compatibility vectors (resp. dual compatibility vectors) of all tubes of \(G\) with respect to any initial maximal tubing \(T^o\) on \(G\) support a complete simplicial fan realizing the nested complex on \(G\). More precisely, both collections of cones

\[
\mathcal{D}(G, T^o) := \{\mathbb{R}_{\geq 0} d(T^o, T) \ | \ T\ tubing on G\} \quad \text{and} \quad \mathcal{D}^*(G, T^o) := \{\mathbb{R}_{\geq 0} d^*(T^o, T) \ | \ T\ tubing on G\}
\]

are complete simplicial fans.

We then study the number of distinct compatibility fans. As in [CSZ11], we consider that two compatibility fans \(\mathcal{D}(G, T^o)\) and \(\mathcal{D}(G', T'^o)\) are equivalent if they differ by a linear isomorphism. Such a linear isomorphism induces an isomorphism between the nested complexes \(\mathcal{N}(G)\) and \(\mathcal{N}(G')\). Besides those induced by graph isomorphisms between \(G\) and \(G'\), there are non-trivial nested complex isomorphisms: for example, the complementation \(t \rightarrow V \setminus t\) on the complete graph, or the map on tubes of the path corresponding to the rotation of diagonals in the polygon. Extending these two examples, we exhibit a non-trivial nested complex isomorphism on any spider (a set of paths attached by one endpoint to a clique). Our next statement shows that these are essentially the only non-trivial nested complex isomorphisms.

**Theorem 2.** All nested complex isomorphisms \(\mathcal{N}(G) \rightarrow \mathcal{N}(G')\) are induced by graph isomorphisms \(G \rightarrow G'\), except if one of the connected components of \(G\) is a spider.

**Corollary 3.** Except if one of the connected components of \(G\) is a spider, the number of linear isomorphism classes of compatibility fans of \(G\) is the number of orbits of maximal tubings on \(G\) under graph automorphisms of \(G\).

The next step would be to realize all these complete simplicial fans as normal fans of convex polytopes. This question remains open, except for some particular graphs: besides all graphs with at most 4 vertices, we settle the case of paths and cycles following a similar proof as [CSZ11].

**Theorem 4.** All compatibility and dual compatibility fans of paths and cycles are polytopal.
Overview. The paper is organized as follows. We first recall in Section 2 definitions and basic notions on polyhedral fans and graphical nested complexes. In particular, we state in Proposition 5 a crucial sufficient condition for a set of vectors indexed by a set $X$ to support a simplicial fan realization of a simplicial complex on $X$. We also briefly survey the classical constructions of graph associahedra of [CD06, Pos09, Zel06].

In Section 3, we define the compatibility degree between two tubes of a graph, review its combinatorial properties, and state our main geometric results on compatibility and dual compatibility fans.

We study various examples in Section 4. After an exhaustive description of the compatibility fans of all graphs with at most 4 vertices, we study four families of graphs: paths, cycles, complete graphs, and stars. The first two families connect our construction to S. Fomin and A. Zelevinsky’s $d$-vector fans for type $A$, $B$, and $C$ cluster complexes.

Section 5 discusses some further topics. We first study the behavior of the compatibility fans with respect to products and links. We then describe all nested complex isomorphisms in order to show that most compatibility fans are not linearly isomorphic. We also discuss the question of the realization of our compatibility fans as normal fans of convex polytopes.

Finally, we have chosen to gather all proofs of our results in Section 6 with the hope that the properties and examples of compatibility fans treated in Sections 4 and 5 help the reader’s intuition.

2. Preliminaries

In this section, we briefly review classical material to recall basic definitions and fix notations. The reader familiar with polyhedral geometry and graph associahedra can skip these preliminaries and go directly to Section 3.

2.1. Polyhedral geometry and fans. We first recall classical definitions from polyhedral geometry. More details can be found in the textbooks [Zie95, Lecture 1] and [DRS10, Section 2.1.1].

A closed polyhedral cone is a subset of $\mathbb{R}^n$ defined equivalently as the positive span of finitely many vectors or as the intersection of finitely many closed linear halfspaces. The dimension of a cone is the dimension of its linear span. The faces of a cone $C$ are the intersections of $C$ with the supporting hyperplanes of $C$. Faces of polyhedral cones are polyhedral cones. The faces of dimension 1 (resp. codimension 1) are called rays (resp. facets). We will only consider pointed cones, which contain no entire line of $\mathbb{R}^n$. Therefore, $C$ is the positive span of its rays and the intersection of the halfspaces defined by its facets and containing it. We say that $C$ is simplicial if its rays form a linear basis of its linear span.

A polyhedral fan is a collection $\mathcal{F}$ of polyhedral cones of $\mathbb{R}^n$ closed under faces and which intersect properly, i.e.

1. if $C \in \mathcal{F}$ and $F$ is a face of $C$, then $F \in \mathcal{F}$;
2. the intersection of any two cones of $\mathcal{F}$ is a face of both.

A polyhedral fan is simplicial if all its cones are, and complete if the union of its cones covers the entire space $\mathbb{R}^n$. We will use the following characterization of complete simplicial fans, whose formal proof can be found e.g. in [DRS10, Corollary 4.5.20].

**Proposition 5.** For a simplicial sphere $\Delta$ with vertex set $X$ and a set of vectors $V := (v_x)_{x \in X}$ of $\mathbb{R}^n$, the collection of cones $\{R_{\geq 0}v_\Delta \mid \Delta \in \Delta \}$, where $R_{\geq 0}v_\Delta$ denotes the positive span of $V_\Delta := \{v_x \mid x \in \Delta \}$, forms a complete simplicial fan if and only if

1. there exists a facet $\Delta$ of $\Delta$ such that $V_\Delta$ is a basis of $\mathbb{R}^n$ and such that the open cones $R_{>0}v_\Delta$ and $R_{>0}v_{\Delta'}$ are disjoint for any facet $\Delta'$ of $\Delta$ distinct from $\Delta$;
2. for any two adjacent facets $\Delta, \Delta'$ of $\Delta$ with $\Delta \setminus \{x\} = \Delta' \setminus \{x'\}$, the coefficients $\alpha, \alpha'$ in the unique (up to rescaling) linear dependence

$$\alpha v_x + \alpha' v_{x'} + \sum_{y \in \Delta \cup \Delta'} \beta_y v_y = 0$$

on $V_{\Delta \cup \Delta'}$ have the same sign (different from 0).
A polytope is a subset $P$ in $\mathbb{R}^n$ defined equivalently as the convex hull of finitely many points in $\mathbb{R}^n$ or as a bounded intersection of finitely many closed half-spaces of $\mathbb{R}^n$. The faces of $P$ are the intersections of $P$ with its supporting hyperplanes. The (outer) normal cone of a face $F$ of $P$ is the cone generated by the outer normal vectors of the facets (codimension 1 faces) of $P$ containing $F$. Finally, the (outer) normal fan of $P$ is the collection of the (outer) normal cones of all its faces.

2.2. Graphical nested complexes. We now review the definitions and basic properties of graphical nested complexes. We refer to [CD06, Dev09] for the original construction of graph associahedra. This construction extends to nested complexes on arbitrary building sets, see [Pos09, FS05, Zel06]. Although we remain in the graphical situation, our presentation borrows results from these papers.

Fix a graph $G$ with vertex set $V$. Let $\kappa(G)$ denote the set of connected components of $G$ and define $n := |V| - |\kappa(G)|$. A tube of $G$ is a non-empty subset $t$ of vertices of $G$ inducing a connected subgraph $G[t]$ of $G$. The inclusion maximal tubes of $G$ are its connected components $\kappa(G)$; all other tubes are called proper. Two tubes $t, t'$ of $G$ are compatible if they are either nested (i.e. $t \subseteq t'$ or $t' \subseteq t$), or disjoint and non-adjacent (i.e. $t \cup t'$ is not a tube of $G$). A tubing on $G$ is a set $T$ of pairwise compatible proper tubes of $G$. The collection of all tubings on $G$ is a simplicial complex, called nested complex of $G$ and denoted by $N(G)$.

Example 6. To illustrate the content of the paper, we will follow a toy example, presented in Figure 4. We have represented a graph $G_{ex}$ on the left with a tube $t_{ex} = \{a, b, d, f, g, h, i, k, l\}$, and a maximal tubing $T_{ex}^0$ on $G_{ex}$ on the right.

For a tubing $T$ on $G$ and a tube $t$ of $T \cup \kappa(G)$, we define $\lambda(t, T) := t \setminus \bigcup_{t' \subseteq t} t'$. The sets $\lambda(t, T)$ for $t \in T \cup \kappa(G)$ form a partition of the vertex set of $G$. When $T$ is a maximal tubing, each set $\lambda(t, T)$ contains a unique vertex of $G$ that we call the root of $t$ in $T$.

The nested complex $N(G)$ is an $(n - 1)$-dimensional simplicial sphere, and we denote by $F(G)$ the dual graph of this complex: its vertices are maximal tubings on $G$ and its edges are flips between them, i.e. pairs of distinct maximal tubings $T, T'$ on $G$ such that $T \setminus \{t\} = T' \setminus \{t'\}$ for some tubes $t \in T$ and $t' \in T'$. Since $N(G)$ is a sphere, any tube of any maximal tubing can be flipped, and the resulting tube is described in the following proposition, whose proof is left to the reader. Remember that we denote by $G[U]$ the subgraph of $G$ induced by a subset $U \subseteq V$.

Proposition 7. Let $t$ be a tube in a maximal tubing $T$ on $G$, and let $\overline{t}$ be the inclusion minimal tube of $T \cup \kappa(G)$ which strictly contains $t$. Then the unique tube $t'$ such that $T' = T \triangle \{t, t'\}$ is again a maximal tubing on $G$ is the connected component of $G[T \setminus \lambda(t, T)]$ containing $\lambda(\overline{t}, T)$.

We say that two distinct tubes $t, t'$ of $G$ are exchangeable if there exists two adjacent maximal tubings $T, T'$ on $G$ such that $T \setminus \{t\} = T' \setminus \{t'\}$. Note that several such pairs $\{T, T'\}$ are possible, but they all contain certain tubes. We call forced tubes of the exchangeable pair $\{t, t'\}$ any tube which belongs to any adjacent maximal tubings $T, T'$ such that $T \setminus \{t\} = T' \setminus \{t'\}$. These tubes are easy to describe: they are precisely the tube $\overline{t} := t \cup t'$ and the connected components of $\overline{t} \setminus \{r, r'\}$.
Example 8. Figure 5 illustrates the flip between two maximal tubings $T_{ex}$ and $T'_{ex}$ on $G_{ex}$. The flipped tubes $t_{ex} = \{a, b, c, d, f, g, h, k, l, m\}$ (with root $g$) and $t'_{ex} = \{c, d, e, h, i, m\}$ (with root $i$) are colored green, while the forced tubes of the exchangeable pair $\{t_{ex}, t'_{ex}\}$ are colored red.

2.3. Graphical nested fans and graph associahedra. Although we will not use them in the remaining of this paper, we survey some constructions of the nested fan and graph associahedron to illustrate the previous definitions. The nested fan is constructed implicitly in [CD06], and explicitly in [Pos09, FS05, Zel06] in the more general situation of nested complexes on arbitrary building sets.

Let $(e_v)_{v \in V}$ be the canonical basis of $\mathbb{R}^V$, let $\mathbb{H} = \{x \in \mathbb{R}^V \mid \sum_{w \in W} x_w = 0 \text{ for all } W \in \kappa(G)\}$ and $\pi : \mathbb{R}^V \to \mathbb{H}$ denote the orthogonal projection on $\mathbb{H}$. Let $g(t) = \pi(\sum_{v \in t} e_v)$ denote the projection of the characteristic vector of a tube $t$ of $G$, and define $g(T) = \{g(t) \mid t \in T\}$ for a tubing $T$ on $G$. These vectors support a complete simplicial fan realization of the nested complex:

**Theorem 9 ([CD06, Pos09, FS05, Zel06]).** For any graph $G$, the collection of cones

$$G(G) := \{R_{\geq 0} g(T) \mid T \text{ tubing on } G\}$$

is a complete simplicial fan of $\mathbb{H}$, called nested fan of $G$, which realizes the nested complex $N(G)$.

**Remark 10.** The cones of this fan can be encoded as follows. Fix a tubing $T$ on $G$. The spine of $T$ is the forest $S$ given by the Hasse diagram of the inclusion poset on $T \cup \kappa(G)$ where the vertex corresponding to a tube $t$ is labeled by $\lambda(t, T)$. Then the cone $R_{\geq 0} g(T)$ is the braid cone of $S$ and is polar to the incidence cone of $S$:

$$R_{\geq 0} g(T) = \{x \in \mathbb{H} \mid x_v \leq x_w \text{ for all } v \to w \in S\} = (R_{\geq 0} \{e_v - e_w \mid v \to w \in S\})^\circ$$

where $v \to w \in S$ means that there is a directed path in $S$ from the node containing $v$ to the node of $S$ containing $w$. Spines are called $B$-trees in [Pos09].

It is proved in [CD06, Dev09, Pos09, FS05, Zel06] that the nested fan comes from a polytope.

**Theorem 11 ([CD06, Dev09, Pos09, FS05, Zel06]).** For any graph $G$, the nested fan $G(G)$ is the normal fan of the graph associahedron $Asso(G)$.

In fact, these different papers define different constructions and realizations for the graph associahedron $Asso(G)$. Originally, M. Carr and S. Devadoss constructed $Asso(G)$ by iterative truncations of faces of the standard simplex. S. Devadoss then gave explicit integer coordinates for the facets in [Dev09]. In a different context, A. Postnikov [Pos09] and independently E. M. Feichtner and B. Sturmfels [FS05] constructed graph associahedra (more generally nestohedra) by Minkowski sums of faces of the standard simplex. Finally, A. Zelevinsky [Zel06] discussed which polytopes can realize the nested fan using a characterization of all possible facet inequality descriptions.
3. Compatibility degrees, vectors, and fans

In this section, we define our compatibility degree on tubes and state our main results. To go straight to results and examples, the proofs are delayed to Section 6. We refer to Section 4 for examples of compatibility fans of relevant graphs in connection to the geometry of type $A, B$ and $C$ cluster complexes, and to Section 5 for further properties of compatibility fans.

3.1. Compatibility degree. Motivated by the compatibility degrees in finite type cluster algebras, we introduce an analogous notion on tubes of graphical nested complexes.

**Definition 12.** For two tubes $t, t'$ of $G$, the compatibility degree of $t$ and $t'$ is

$$
(t \parallel t') = \begin{cases}
-1 & \text{if } t = t', \\
\frac{|\{\text{neighbors of } t \text{ in } t' \setminus t\}|}{|\{\text{neighbors of } t' \text{ in } t \setminus t'\}|} & \text{if } t \not\subseteq t', \\
0 & \text{otherwise}.
\end{cases}
$$

**Example 13.** On the graph $G_{ex}$ of Examples 6 and 8, the compatibility degrees of the green tubes $t_{ex}, t_{ex}'$ and of the red tube $t_{ex} = t_{ex} \cup t_{ex}'$ of Figure 5 with the tube $t_{ex}^2$ of Figure 4 (left) are given by

$$(t_{ex} \parallel t_{ex}') = |\{i\}| = 1, \quad (t_{ex} \parallel t_{ex}^2) = |\{g\}| = 1, \quad (t_{ex} \parallel t_{ex}') = 0.$$  

We will see in Sections 4.2 and 4.3 that our compatibility degree on tubes of paths (resp. cycles) corresponds to the compatibility degree on cluster variables in type $A$ (resp. $B/C$) cluster algebras defined in [FZ03b]. The compatibility degree in cluster algebras encodes compatibility and exchangeability between cluster variables. The analogous result for the graphical compatibility degree is given by the following proposition, proved in Section 6.1.

**Proposition 14.** For any two tubes $t, t'$ of $G$,

- $(t \parallel t') < 0 \iff (t' \parallel t) < 0 \iff t = t'$,
- $(t \parallel t') = 0 \iff (t' \parallel t) = 0 \iff t$ and $t'$ are compatible, and
- $(t \parallel t') = 1 \iff (t' \parallel t) \iff t$ and $t'$ are exchangeable.

**Remark 15.** It can happen that $(t \parallel t') = 1$ while $(t' \parallel t) \neq 1$, in which case $t$ and $t'$ are not exchangeable. This situation appears as soon as $G$ contains a cycle or a trivalent vertex. See e.g. Example 13.

Proposition 14 should be understood informally as follows: the compatibility degree between two tubes measures how much they are incompatible. It is natural to use this measure to construct fan realizations of the nested complex: intuitively, pairs of tubes with low compatibility degrees should correspond to rays closed to each other. We make this idea precise in the next section.

3.2. Compatibility fans. Similar to the $d$-vector fan defined by S. Fomin and A. Zelevinsky [FZ03a], we consider the compatibility vectors with respect to an arbitrary initial maximal tubing $T^\circ$. Remember that any maximal tubing on $G$ has precisely $n := |V| - |\alpha(G)|$ tubes.

**Definition 16.** Let $T^\circ := \{t_1^\circ, ..., t_n^\circ\}$ be an arbitrary initial maximal tubing on $G$. The compatibility vector of a tube $t$ of $G$ with respect to $T^\circ$ is the integer vector $d(T^\circ, t) := [(t_1^\circ \parallel t), \ldots, (t_n^\circ \parallel t)]$. The compatibility matrix of a tubing $T = \{t_1, \ldots, t_m\}$ on $G$ with respect to $T^\circ$ is the matrix $d(T^\circ, T) := [(t_i^\circ \parallel t_j)]_{i \in [m], j \in [n]}$.

Remember that we denote by $\mathbb{R}_{\geq 0}$ the polyhedral cone generated by the column vectors of a matrix $M$. Note that the compatibility vectors of the initial tubes are given by the negative of the basis vectors, while all other compatibility vectors lie in the positive orthant: $d(T^\circ, T^\circ) = -I_n$ and $d(T^\circ, t) \in \mathbb{R}_{\geq 0}I_n$ for $t \not\subseteq T^\circ$. Our main result asserts that these compatibility vectors support a complete simplicial fan realization of the graphical nested complex.
Theorem 17. For any graph \( G \) and any maximal tubing \( T^o \) on \( G \), the collection of cones
\[
D(G, T^o) := \{ \mathbb{R}_{\geq 0} d(T^o, t) \mid t \text{ tubing on } G \}
\]
is a complete simplicial fan which realizes the nested complex \( \mathcal{N}(G) \). We call it the compatibility fan of \( G \).

We prove this statement in Section 6.3. The proof relies on the characterization of complete simplicial fans presented in Proposition 5. Unfortunately, we are not able to compute the linear dependence between the compatibility vectors involved in an arbitrary flip. To illustrate the difficulty, we show in the following example that these linear dependences may be complicated. In particular, they do not always involve only the forced tubes of the flip.

Example 18. Consider the initial maximal tubing \( T^o_{ex} \) of the graph \( G_{ex} \) of Figure 4 (right) and the flip \( T_{ex} \setminus \{ t_{ex} \} = T^o_{ex} \setminus \{ t'_{ex} \} \) illustrated in Figure 5. The linear dependence between the compatibility vectors of the tubes of \( T_{ex} \setminus T^o_{ex} \) with respect to \( T^o_{ex} \) is
\[
2d(T^o_{ex}, t_{ex}) + d(T^o_{ex}, t'_{ex}) - d(T^o_{ex}, \{ d \}) - d(T^o_{ex}, \{ e \}) - 3d(T^o_{ex}, \{ m \}) + 4d(T^o_{ex}, \{ k, l \}) - 3d(T^o_{ex}, \{ c, d, h \}) = 0.
\]
Observe that the tube \( \{ d \} \) is involved in this linear dependence although it is not a forced tube of the exchangeable pair \( \{ t_{ex}, t'_{ex} \} \).

Theorem 17 has the following consequences, whose direct proof would require some work.

Corollary 19. For any initial tubing \( T^o \) on \( G \),
- the compatibility vector map \( t \mapsto d(T^o, t) \) is injective: \( d(T^o, t) = d(T^o, t') \Rightarrow t = t' \).
- the compatibility matrix \( d(T^o, T) \) of any maximal tubing \( T \) on \( G \) has full rank.

3.3. Dual compatibility fan. It is also interesting to consider the following dual notion of compatibility vectors, where the roles of \( t \) and \( t^n_1, \ldots, t^n_n \) are reversed. The results are similar, and the motivation for this dual definition will become clear in Section 4.3.

Definition 20. Let \( T^o := \{ t^n_1, \ldots, t^n_n \} \) be an arbitrary initial maximal tubing on \( G \). The dual compatibility vector of a tube \( t \) of \( G \) with respect to \( T^o \) is the vector \( d^*(t, T^o) := [t \parallel t^n_1, \ldots, t \parallel t^n_n] \). The dual compatibility matrix of a tubing \( T := \{ t_1, \ldots, t_n \} \) on \( G \) with respect to \( T^o \) is the matrix \( d^*(T, T^o) := [t_i \parallel t^n_j]_{i,j \in [n]} \).

The following statement is the analogue of Theorem 17.

Theorem 21. For any graph \( G \) and any maximal tubing \( T^o \) on \( G \), the collection of cones
\[
D^*(G, T^o) := \{ \mathbb{R}_{\geq 0} d^*(T, T^o) \mid T \text{ tubing on } G \}
\]
is a complete simplicial fan which realizes the nested complex \( \mathcal{N}(G) \). We call it the dual compatibility fan of \( G \).

The proof of this statement appears in Section 6.4. It is a direct application of Theorem 17 using duality between compatibility and dual compatibility matrices.

Example 22. Consider the initial maximal tubing \( T^o_{ex} \) on the graph \( G_{ex} \) of Figure 4 (right) and the flip \( T_{ex} \setminus \{ t_{ex} \} = T^o_{ex} \setminus \{ t'_{ex} \} \) illustrated in Figure 5. The linear dependence between the dual compatibility vectors of the tubes of \( T_{ex} \setminus T^o_{ex} \) with respect to \( T^o_{ex} \) is
\[
2d^*(t_{ex}, T^o_{ex}) + d^*(t'_{ex}, T^o_{ex}) - d^*(\{ e \}, T^o_{ex}) - d^*(\{ c, d, h \}, T^o_{ex}) = 0.
\]

4. Examples for specific graphs

In this section, we provide examples of compatibility fans for particular families of graphs. These examples will help the intuition for the proofs in Section 6. We start with graphs with few vertices to illustrate the variety of compatibility fans. We then describe compatibility fans for paths, cycles, complete graphs and stars. For paths and cycles, we give an explicit connection to the compatibility degree in cluster algebras of types \( A, B, \) and \( C \).
4.1. **Graphs with few vertices.** In view of Proposition 37 below, we restrict to connected graphs. The only connected graphs with 3 vertices are the 3-path and the triangle, whose compatibility fans are represented in Figure 6. The other possible choices for the initial tubing in these pictures would produce the same fans: it is clear for the triangle as all maximal tubings are obtained from one another by graph isomorphisms; for the path, it is an illustration of the non-trivial isomorphisms between compatibility fans studied in Section 5.2.

The first interesting compatibility fans appear in dimension 3 for connected graphs on 4 vertices. All possibilities up to linear transformations are represented in Figure 8. Instead of representing cones in the 3-dimensional space, we intersect the compatibility vectors with the unit sphere, make a stereographic projection of the resulting points on the sphere (the pole of the projection is the point of the sphere in direction $-e_1 - e_2 - e_3$), and draw the cones on the resulting planar points. Under this projection, the three external vertices correspond to the tubes of the initial tubing, and the external face corresponds to the initial tubing. For the sake of readability, we do not label the remaining vertices of the projection. Their labels can be reconstructed from the initial tubes by flips. For example, the tubes corresponding to the vertices of the top pictures of Figure 8 are given in Figure 7.

The pictures become more complicated in dimension 4. To illustrate them, we have represented in Figure 9 the stereographic projection of the compatibility fan for an arbitrary initial maximal tubing on the path, cycle, complete graph, and star on 5 vertices.
Figure 8. All possible compatibility fans up to linear isomorphism, for all connected graphs on 4 vertices (see also the end of the picture on page 12 for the two remaining graphs). Instead of representing the cones in the 3-dimensional space, we intersect the compatibility vectors with the unit sphere, make a stereographic projection of the resulting points on the sphere (the pole of the projection is the point of the sphere in direction $-e_1 - e_2 - e_3$), and draw the cones on the resulting planar points.
Complete graph minus one edge

Complete graph minus two incident edges
4.2. Paths. We now consider the nested complex $\mathcal{N}(P_{n+1})$ and the compatibility fan $\mathcal{D}(P_{n+1}, T^\circ)$ for the path $P_{n+1}$ on $n+1$ vertices. As already mentioned in the introduction, the nested complex $\mathcal{N}(P_{n+1})$ is isomorphic to the $n$-dimensional simplicial associahedron, i.e. the simplicial complex of sets of pairwise non-crossing diagonals of an $(n+3)$-gon. It is convenient to present the correspondence as follows. Consider an $(n+3)$-gon $Q_{n+3}$ with vertices labeled from left to right by $0, 1, \ldots, n+2$ and such that all vertices $1, \ldots, n+1$ are located strictly below the boundary edge $[0, n+2]$. We can therefore identify the path $P_{n+1}$ with the path $1, \ldots, n+1$ on the boundary of $Q_{n+3}$. We then associate to a diagonal $\delta$ of $Q_{n+3}$ the tube $t_{\delta}$ of $P_{n+1}$ whose vertices are located strictly below $\delta$, see Figure 10. Finally, we associate to a set $\Delta$ of pairwise non-crossing internal diagonals of $Q_{n+3}$ the set of tubes $T_\Delta = \{t_{\delta} \mid \delta \in \Delta\}$, see Figure 1. The reader can check that the map $\Delta \mapsto T_\Delta$ defines an isomorphism between the simplicial associahedron and the nested complex $\mathcal{N}(P_{n+1})$: two diagonals $\delta, \delta'$ of $Q_{n+3}$ are non-crossing if and only if the corresponding tubes $t_{\delta}, t_{\delta'}$ of $P_{n+1}$ are compatible.

It follows by classical results on the associahedron that the path $P_{n+1}$ has:

- $n(n+3)$ proper tubes\footnote{OEIS, A000096} (internal diagonals of the $(n+3)$-gon),
- $\frac{1}{2(n+2)} \binom{2n+2}{n+1}$ maximal tubings\footnote{OEIS, A000108} (triangulations of the $(n+3)$-gon),
- $\frac{1}{k+1} \binom{n+k+2}{k}$ tubings with $k$ tubes\footnote{OEIS, A033282} (dissections of the $(n+3)$-gon into $k$ parts).
The following statement, whose proof is left to the reader, describes the behavior of the map \( \delta \mapsto t_\delta \) with respect to compatibility degrees.

**Proposition 23.** For any two diagonals \( \delta, \delta' \) of \( Q_{n+3} \), the compatibility degree of the corresponding tubes \( t_\delta, t_{\delta'} \) of \( P_{n+1} \) is given by

\[
(t_\delta \parallel t_{\delta'}) = \begin{cases} 
-1 & \text{if } \delta = \delta', \\
0 & \text{if } \delta \neq \delta' \text{ do not cross}, \\
1 & \text{if } \delta \neq \delta' \text{ cross}.
\end{cases}
\]

In other words, our compatibility degree between tubes of \( P_{n+1} \) coincides with the compatibility degree between type \( A \) cluster variables defined by S. Fomin and A. Zelevinsky in [FZ03b], and our graphical compatibility fan coincides with the type \( A \) compatibility fan defined for an acyclic initial cluster in [FZ03a] and for any initial cluster in [CSZ11, Section 5]. We thus obtain an alternative proof of F. Santos’ result [CSZ11, Section 5].

**Corollary 24.** For type \( A \) cluster algebras, the denominator vectors (or compatibility vectors) of all cluster variables with respect to any initial cluster support a complete simplicial fan which realizes the cluster complex.

**Remark 25 (Dual compatibility fan).** The compatibility fan \( D(P_{n+1}, T^o) \) and the dual compatibility fan \( D^*(P_{n+1}, T^o) \) coincide since the compatibility degree is symmetric for tubes of \( P_{n+1} \).

**Remark 26 (Linear dependences).** In the case of the path \( P_{n+1} \), the linear dependences are explicitly described in [CSZ11]. They are derived from the case of the octagon by edge contraction in the interpretation in terms of triangulations. They can only involve the two flipped tubes and the forced tubes, and the coefficients are either 1 or 2 for the flipped tubes and \(-1\) or 0 for the forced tubes. See Section 6.6 for more details.

**Remark 27.** The compatibility degree for tubes of a path takes values in \( \{-1, 0, 1\} \). It is tempting to construct compatibility fans for graphical nestohedra using the naive compatibility degree defined by \( (t \parallel t') = -1 \) if \( t = t' \), \( (t \parallel t') = 0 \) if \( t \neq t' \) are compatible, and \( (t \parallel t') = 1 \) if \( t \neq t' \) are incompatible. This naive approach works for the paths but fails for any other connected graph since two distinct tubes would get the same compatibility vectors. See Figure 11 for examples on the triangle and on the tripod.

**Figure 10.** Isomorphism between the simplicial associahedron and the nested complex of a path: diagonals are sent to tubes (left), preserving the compatibility (middle) and incompatibility (right). See also Figure 1.

**Figure 11.** Counter-examples to the naive definition of compatibility degrees: both on the triangle and on the tripod, all tubes of the initial maximal tubing on the left are incompatible with the two distinct tubes on the right.
4.3. Cycles. We now consider the nested complex $N(O_{n+1})$ and the compatibility fan $D(O_{n+1}, T^e)$ for the cycle $O_{n+1}$ on $n + 1$ vertices. As already mentioned in the introduction, the nested complex $N(O_{n+1})$ is isomorphic to the $n$-dimensional simplicial cyclohedron, i.e., the simplicial complex of sets of pairwise non-crossing pairs of centrally symmetric internal diagonals (including duplicated long diagonals) of a regular $(2n + 2)$-gon $R_{2n+2}$. The explicit correspondence works as follows. We label the vertices of $R_{2n+2}$ cyclically with two copies of $[n+1]$. We then associate

- to a duplicated long diagonal $\delta$ with vertices labeled by $i$ the tube $t_\delta := [n+1] \setminus \{i\}$ of $O_{n+1}$,
- to a pair of centrally symmetric diagonals $\{\delta, \bar{\delta}\}$ the tube $t_\delta$ of $O_{n+1}$ which consists of the labels of the vertices of $R_{2n+2}$ separated from the center of $R_{2n+2}$ by $\delta$ and $\bar{\delta}$.

Finally, we associate to a set $\Delta$ of pairwise non-crossing pairs of centrally symmetric internal diagonals of $R_{2n+2}$ the set of tubes $T_\Delta := \{t_\delta | \delta \in \Delta\}$. See Figures 12 and 2. The reader can check that the map $\Delta \mapsto T_\Delta$ defines an isomorphism between the simplicial cyclohedron and the nested complex $N(O_{n+1})$: two pairs of centrally symmetric diagonals (or duplicated long diagonals) $\{\delta, \bar{\delta}\}$ and $\{\delta', \bar{\delta}'\}$ of $R_{2n+2}$ are non-crossing if and only if the corresponding tubes $t_\delta$ and $t_{\delta'}$ of $O_{n+1}$ are compatible.

![Figure 12. Isomorphism between the simplicial cyclohedron and the nested complex of a cycle: centrally symmetric pairs of diagonals are sent to tubes, preserving the compatibility and incompatibility. See also Figure 2.](image)

It follows by classical results on the cyclohedron that the cycle $O_{n+1}$ has:

- $n(n + 1)$ proper tubes [OEIS, A002378] (centrally symmetric pairs of diagonals),
- $\binom{2n}{n}$ maximal tubings [OEIS, A000984] (centrally symmetric triangulations),
- $\binom{n}{k} \binom{n+k}{k}$ tubings with $k$ tubes [OEIS, A063007] (centrally symmetric dissections).

The following statement, whose proof is left to the reader, describes the behavior of the map $\{\delta, \bar{\delta}\} \mapsto t_\delta$ with respect to compatibility degrees.

**Proposition 28.** For any two pairs of centrally symmetric diagonals (or duplicated long diagonals) $\{\delta, \bar{\delta}\}$ and $\{\delta', \bar{\delta}'\}$ of $R_{2n+2}$, the compatibility degree $(t_\delta \parallel t_{\delta'})$ of the corresponding tubes $t_\delta$ and $t_{\delta'}$ of $O_{n+1}$ is the number of crossings between the two diagonals $\delta$ and $\bar{\delta}$ and the diagonal $\delta'$.

In other words, our compatibility degree (resp. dual compatibility degree) between tubes of $O_{n+1}$ coincides with the compatibility degree between type $C$ (resp. type $B$) cluster variables defined by S. Fomin and A. Zelevinsky in [FZ03b]. Moreover, our graphical compatibility fan (resp. dual compatibility fan) coincides with the type $C$ (resp. type $B$) compatibility fan defined for an acyclic initial cluster in [FZ03a]. This extends for any arbitrary initial cluster to the following corollary.

**Corollary 29.** For type $B$ and $C$ cluster algebras, the denominator vectors (or compatibility vectors) of all cluster variables with respect to any initial cluster support a complete simplicial fan which realizes the cluster complex.
Remark 30 (Dual compatibility fan). Since the compatibility degree is not symmetric for tubes of $O_{n+1}$, the compatibility fan $\mathcal{D}(O_{n+1}, T^\circ)$ and the dual compatibility fan $\mathcal{D}^*(O_{n+1}, T^\circ)$ do not coincide. Figures 13 and 14 show both fans for different initial tubings on the cycles $O_3$ and $O_4$.

Figure 13. Compatibility (left) and dual compatibility (right) fans for the triangle.

Figure 14. Compatibility (top) and dual compatibility (bottom) fans for the cycle on 4 vertices with respect to different initial tubings.
Remark 31 (Linear dependences). As for paths, only finitely many linear dependences occur for all cycles $O_{n+1}$, both on compatibility vectors as on dual compatibility vectors. Indeed, with the interpretation of the maximal tubings in terms of centrally symmetric triangulations, the same kind of arguments as in [CSZ11] ensure that all these dependences can be inferred by checking the cycle $O_8$ on 8 vertices. As for the path, these linear dependences only involve flipped and forced tubes, and the coefficients of the flipped tubes may only be 1 or 2 and these of the forces tubes may only be 0, −1 or −2. See Section 6.6 for more details.

It is easy to find an example of a maximal tubing on the tripod such that one of the linear dependences obtained with respect to this maximal tubing does not only involve forced tubes. It implies in particular that the paths and cycles are the only graphs that have this property. It is then tempting to ask whether it is a coincidence that these graphs also are the only ones whose corresponding associahedra also are generalized associahedra.

4.4. Complete graphs. We now consider the nested complex $\mathcal{N}(K_{n+1})$ and the compatibility fan $\mathcal{D}(K_{n+1}, T^0)$ for the complete graph $K_{n+1}$ on $n + 1$ vertices. As already mentioned in the introduction, the nested complex $\mathcal{N}(K_{n+1})$ is isomorphic to the $n$-dimensional simplicial permutohedron, i.e. the simplicial complex of collections of pairwise nested subsets of $[n+1]$. See Figure 3.

It follows by classical results on the permutahedron that the complete graph $K_{n+1}$ has:

- $2^n - 2$ proper tubes [OEIS, A000918] (proper subsets of $[n]$),
- $n!$ maximal tubings [OEIS, A000142] (permutations of $[n]$).
- $k! S(n,k)$ tubings with $k$ tubes, where $S(n,k)$ is the Stirling number of second kind (i.e. the number of ways to partition a set of $m$ elements into $p$ non-empty subsets) [OEIS, A008277].

The compatibility degree between two tubes $t, t'$ of $K_{n+1}$ is $(t \parallel t') = -1$ if $t = t'$, $(t \parallel t') = 0$ if $t$ and $t'$ are distinct and nested, and $(t \parallel t') = |t' \setminus t|$ otherwise. This connects the compatibility degree $d(T, t)$ to an alternative combinatorial model for the permutahedron in terms of lattice paths. Since all maximal tubings are equivalent, we can assume w.l.o.g. that $T^0 = \{[i] \mid i \in [n]\}$.

For any tube $t$ of $K_{n+1}$, we consider the lattice paths $\phi(t)$ and $\psi(t)$ whose horizontal steps above abscissa $[i, i+1]$ lie at height $|t \setminus [i]|$ and $([i] || t)$ respectively. The following properties are illustrated in Figure 15, where $\phi(t)$ is the plain path while $\psi(t)$ is dotted until it meets $\phi(t)$.

Proposition 32. (i) For any tube $t$ of $K_{n+1}$, the lattice path $\phi(t)$ is decreasing from $(0, |t|)$ to $(n + 1, 0)$ with vertical steps of height 0 or 1.

(ii) $\phi$ is surjective on the decreasing paths ending at $(n+1, 0)$ with vertical steps of height 0 or 1.

(iii) For any tubes $t, t'$ of $K_{n+1}$, we have $t \subseteq t'$ if and only if $\phi(t')$ decreases when $\phi(t)$ decreases. In particular, the paths $\phi(t)$ and $\phi(t')$ are then non-crossing.

(iv) For a tubing $T$ on $K_{n+1}$, the map $\sigma(T) : i \mapsto | \{ t \in T \mid \phi(t) \text{ has a descent at abscissa } i \} | + 1$ is a surjection from $[n+1]$ to $|T| + 1$, and therefore $\pi(T) = \bigcup_{i \in [T]} \sigma^{-1}(i)$ is an ordered partition of $[n+1]$ into $|T| + 1$ parts. The map $T \mapsto \pi(T)$ defines an isomorphism form the nested complex $\mathcal{N}(K_{n+1})$ to the refinement poset of ordered partitions.

(v) For a tube $t$ of $K_{n+1}$ not in $T^0$, the path $\psi(t)$ is obtained from the path $\phi(t)$ by replacing the initial down stairs by an horizontal path at height 0. See Figure 15, where $\psi(t)$ is dotted until it meets $\phi(t)$.

Remark 33 (Dual compatibility fan). As discussed in Section 5.2 below, the complementation $t \mapsto V \setminus t$ defines an automorphism of the nested complex $\mathcal{N}(K_{n+1})$, which dualizes the compatibility degree: $(t \| t') = (V \setminus t' || V \setminus t)$ for any tubes $t, t'$ of $K_{n+1}$. Therefore, the dual compatibility fans are compatibility fans: for any tubing $T^0$ on $K_{n+1}$,

$$D^*(K_{n+1}, T^0) = D(K_{n+1}, \{V \setminus t^0 \mid t^0 \in T^0\}).$$

Remark 34 (Linear dependences). For the complete graph, the linear dependences between compatibility vectors of tubes involved in a flip can already be complicated. However, the coefficients $(\alpha, \alpha')$ of the flipped tubes in these dependences can only take the following values:

$(k, k)$ with $k > 0$, or $(k, kp)$ with $k, p > 0$, or $((k+1) p, kp)$ with $k, p > 0$. 

4.5. Stars. We finally consider the nested complex $\mathcal{N}(X_{n+1})$ and the compatibility fan $\mathcal{D}(X_{n+1}, T^\circ)$ for the star $X_{n+1}$ with $n + 1$ vertices, i.e. the tree with $n$ leaves $\ell_1, \ldots, \ell_n$ all connected to a central vertex denoted $\ast$. The graph associahedron $\text{Asso}(X_{n+1})$ is called stellohedron. We have represented in Figure 16 two realizations of the 3-dimensional stellohedron.

One easily checks that the star $X_{n+1}$ has:

- $2^n + n - 1$ proper tubes [OEIS, A052944] (distinguish tubes containing $\ast$ or not),
- $n! \sum_{i=0}^{n} \frac{1}{i!}$ maximal tubings [OEIS, A000522] (consider the minimal tube containing $\ast$),
- $\sum_{i=0}^{k} \binom{n}{k-i} (i-1)! (i S(n-k+i, i) + S(n-k+i, i-1))$ tubings with $k$ tubes, where $S(m, p)$ denotes the Stirling number of second kind (i.e. the number of ways to partition a set of $m$ elements into $p$ non-empty subsets) [OEIS, A008277] (to see it, sum over the number $i$ of tubes containing $\ast$), and
- $4^n! \prod_{n_i=n} \frac{1}{n_i} - 1 = \sum_{i \geq 1} (i+1)^n/2^i$ tubings in total (including the empty tubing). This is the number of chains in the boolean lattice on an $n$-element set [OEIS, A007047] (an immediate bijection is given by the spines of the tubings).

**Figure 15.** The tubing $\{146, 12468, 123468\}$ corresponds to three non-crossing decreasing lattice paths, and to the ordered partition $57|3|28|146$.

**Figure 16.** Two polytopal realizations of the 3-dimensional stellohedron: their normal fans are the nested fan (left) and a compatibility fan (right).
the cones of three different tubings are represented in Figure 18 (right).

Thus always equal 1 while those of the forced tubes (not in the initial tubing as in the beginning of the proof of Theorem 17 in Section 6.3. The coefficients of the flipped tubes

Remark 36 (Linear dependences) In the special case discussed in this section, all linear dependences between compatibility vectors of tubes involved in a flip are inclusion-exclusion dependences as in the beginning of the proof of Theorem 17 in Section 6.3. The coefficients of the flipped tubes thus always equal 1 while those of the forced tubes (not in the initial tubing $T^\circ$) always equal $-1$.

5. Further topics

In this section, we first study the behavior of the compatibility fans with respect to products and links. We then explain that most compatibility fans are not linearly isomorphic, which requires a description of all nested complex isomorphisms. Finally, we discuss the question of the realization of our compatibility fans as normal fans of convex polytopes. The results in this section are only stated for compatibility fans, but similar results hold for dual compatibility fans.

5.1. Products and restrictions. In all examples that we discussed earlier, we only considered connected graphs. Compatibility fans for disconnected graphs can be reconstructed from those for connected graphs by the following statement, whose proof is left to the reader.

Proposition 37. If $G$ has connected components $G_1, \ldots, G_k$, then the nested complex $\mathcal{N}(G)$ is the join of the nested complexes $\mathcal{N}(G_1), \ldots, \mathcal{N}(G_k)$. Moreover, for any maximal tubings $T^\circ_1, \ldots, T^\circ_k$ on $G_1, \ldots, G_k$ respectively, the compatibility fan $\mathcal{D}(G, T^\circ)$ with respect to the maximal tubing $T^\circ := T^\circ_1 \cup \cdots \cup T^\circ_k$ on $G$ is the product of the compatibility fans $\mathcal{D}(G_1, T^\circ_1), \ldots, \mathcal{D}(G_k, T^\circ_k)$:

$$\mathcal{D}(G, T^\circ) = \mathcal{D}(G_1, T^\circ_1) \times \cdots \times \mathcal{D}(G_k, T^\circ_k) = \{C_1 \times \cdots \times C_k | C_i \in \mathcal{D}(G_i, T^\circ_i) \text{ for all } i \in [k]\}.$$ 

Figure 18 (right) illustrates Proposition 37 with the compatibility fan of a graph formed by two paths. Compatibility fans of paths are discussed in Section 4.2. Besides all compatibility vectors, the cones of different tubings are represented in Figure 18 (right).
As observed in [CD06], all links of graphical nested complexes are joins of graphical nested complexes. The following statement asserts that the compatibility fans reflect this property on coordinate hyperplanes. To be more precise, for a tube \( t^o \) of \( G \), we denote by \( G[t^o] \) the restriction of \( G \) to \( t^o \) and by \( G^*t^o \) the reconnected complement of \( t^o \) in \( G \), i.e. the graph with vertex set \( V \setminus t^o \) and edge set \( \{ e \in {\binom{V \setminus t^o}{2}} \mid e \cup t^o \text{ is connected in } G \} \). A maximal tubing \( T^o \) on \( G \) containing \( t^o \) induces maximal tubings \( T^o[t^o] := \{ t \mid t \in T^o, t \subseteq t^o \} \) on the restriction \( G[t^o] \) and \( T^o* t^o := \{ t \setminus t^o \mid t \in T^o, t \not\subset t^o \} \) on the reconnected complement \( G^*t^o \). See Figure 19 for an example.

**Proposition 38.** The link of a tube \( t^o \) in the nested complex \( \mathcal{N}(G) \) is isomorphic to the join of the nested complexes \( \mathcal{N}(G[t^o]) \) and \( \mathcal{N}(G^*t^o) \). Moreover, for an initial maximal tubing \( T^o \) containing \( t^o \), the intersection of the compatibility fan \( \mathcal{D}(G, T^o) \) with the coordinate hyperplane orthogonal to \( e_{t^o} \) is the product of the compatibility fans \( \mathcal{D}(G[t^o], T^o[t^o]) \) and \( \mathcal{D}(G^*t^o, T^o* t^o) \).

This statement follows from Lemmas 55 and 56 and Theorem 17, which are all proved in Section 6.
5.2. Many compatibility fans. In this section, we show that we obtained many distinct compatibility fans. Following [CSZ11], we classify compatibility fans up to linear isomorphisms: two fans \( F, F' \) of \( \mathbb{R}^n \) are linearly isomorphic if there exists an invertible linear map which sends the cones of \( F \) to the cones of \( F' \). We immediately observe that:

- The only linear isomorphisms between compatibility fans are generated by permutations of coordinates, positive affinities orthogonal to coordinate hyperplanes, or reflections with respect to coordinate hyperplanes containing all but two compatibility vectors. Indeed, any nested fan contains precisely \( n \) pairs of opposite compatibility vectors, given by the vectors \( e_i \) of the canonical basis and their opposite \(-e_i\). These pairs of opposite compatibility vectors are preserved (up to rescaling) by linear isomorphisms, thus the coordinate vectors are sent to (dilates of) coordinate vectors or their opposite. Finally, the coordinate hyperplane \( H_i \), orthogonal to \( e_i \), separates \(-e_i\) from all other compatibility vectors not contained in \( H_i \). Thus, the negative coordinate vector \(-e_i\) has to be sent to a (dilate of a) negative coordinate vector, except if the only two compatibility vectors not contained in \( H_i \) are \( e_i \) and \(-e_i\) (in which case the underlying graph has a connected component reduced to a segment).

- If two compatibility fans \( D(G, T) \) and \( D(G', T') \) are linearly isomorphic, then the two nested complexes \( \mathcal{N}(G) \) and \( \mathcal{N}(G') \) are (combinatorially) isomorphic, meaning that there is a bijection \( \Phi \) from the tubes of \( G \) to the tubes of \( G' \) which preserves the compatibility. Note that the converse statement need not be true: not all isomorphisms between the nested complexes \( \mathcal{N}(G) \) and \( \mathcal{N}(G') \) induce linear isomorphisms between the compatibility fans \( D(G, T) \) and \( D(G', T') \). However, understanding the nested complex isomorphism group yields a lower bound on the number of distinct compatibility fans (Corollary 45).

In the sequel, we describe all nested complex isomorphisms. Observe first that an isomorphism \( \Phi \) between two graphs \( G \) and \( G' \) automatically induces an isomorphism \( \Phi \) between the nested complexes \( \mathcal{N}(G) \) and \( \mathcal{N}(G') \) defined by \( \Phi(t) = \{ \phi(v) \mid v \in t \} \) for all tubes \( t \) on \( G \). We say that such a nested complex isomorphism \( \Phi \) is trivial. Trivial isomorphisms clearly preserve compatibility degrees: \( (\Phi(t) \parallel \Phi(t')) = (t \parallel t') \) for any tubes \( t, t' \) on \( G \). We are interested in non-trivial nested complex isomorphisms. We first want to underline two relevant examples:

(i) The complementation \( t \mapsto V \setminus t \) is a non-trivial automorphism of the nested complex \( \mathcal{N}(K_{n+1}) \) of the complete graph \( K_{n+1} \). It dualizes the compatibility degree: \( (V \setminus t \parallel V \setminus t') = (t \parallel t') \) for any tubes \( t, t' \) of \( K_{n+1} \).

(ii) The map \( \circ \) defined for \( 1 \leq j \leq k \leq n+1 \) by

\[
\circ [j, k] = \begin{cases} [k + 1, n + 1] & \text{if } j = 1, \\
[j - 1, k - 1] & \text{if } j > 1,
\end{cases}
\]

is a non-trivial automorphism of the nested complex \( \mathcal{N}(P_{n+1}) \) of the path \( P_{n+1} \). Indeed, \( \circ \) is the conjugate of the (combinatorial) 1-vertex clockwise rotation of the \((n + 3)\)-gon \( Q_{n+3} \) by the bijection \( \delta \mapsto t_5 \) of Section 4.2. See Figure 20.

\[ \text{Figure 20. The 1-vertex clockwise rotation of the } (n + 3)\text{-gon induces a non-trivial automorphism } \circ \text{ of the nested complex } \mathcal{N}(P_{n+1}). \]
Remark 39. \( \Omega \) generalizes both non-trivial nested complex automorphisms described earlier:

(i) When all legs of the spider are empty, the automorphism \( \Omega \) of \( \mathcal{N}(X_{(0)}^{n+1}) \) clearly specializes to the complementation \( t \mapsto V \setminus t \) on \( \mathcal{N}(K_{n+1}) \).
(ii) For the path, different spiders are possible. The path $P_{n+1}$ coincides with the spider $X^I$ with body $\{1\}$ and the single leg $[2, n+1]$, and the automorphism $\Omega$ of $N(X^I)$ is the composition of the rotation automorphism $\circ$ with the vertical reflection automorphism $\leftrightarrow$. Similarly, for any $2 \leq p \leq n+1$, the spider $X^p := X_{\{p-2, n-p+1\}}$ coincides with $P_{n+1}$, and the automorphism $\Omega$ of $N(X^p)$ is the composition of $\circ^p$ with $\leftrightarrow$. Finally, for the spider $X^{n+2}$ with body $\{n+1\}$ and the single leg $[n]$, the automorphism $\Omega$ of $N(X^{n+2})$ is the composition of $\circ^{n+2}$ with $\leftrightarrow$.

This actually suggests an alternative description of $\Omega$ on arbitrary spiders $X_n$. Namely, $\Omega$ is equivalently described by the following steps: shift all leg tubes towards the body, complement all body tubes, delete all edges $\{v_i^0, v_i^0\}$ for $i \neq i' \in [f]$ of the body, replace them by the clique $\{v_i^0, v_i'\}$ for $i \neq i' \in [f]$ on the feet of the spider, and finally apply the trivial isomorphism from the resulting spider back to the initial spider. Our original presentation of $\Omega$ will nevertheless be easier to handle in the proofs.

The following statement is proved in Section 6.5.

**Proposition 40.** The map $\Omega$ is a non-trivial automorphism of the nested complex $N(X_n)$ which dualizes the compatibility degree: $(\Omega(t) \parallel \Omega(t')) = (t' \parallel t)$.

**Remark 41.** It follows from Proposition 40 that all dual compatibility fans of a spider $X_n$ are also compatibility fans of $X_n$ for any tubing $T^0$ on $X_n$,

$$D^\ast(X_n, T^0) = D(X_n, \Omega(T^0)).$$

Note that we already used this observation for complete graphs in Remark 33.

In fact, these non-trivial automorphisms of the nested complexes of the spiders are essentially the only non-trivial nested complex isomorphisms. The following statements are proved in Section 6.5.

**Proposition 42.** A nested complex isomorphism $\Phi : N(G) \rightarrow N(G')$ restricts to nested complex isomorphisms $N(H) \rightarrow N(H')$ between maximal connected subgraphs $H$ of $G$ and $H'$ of $G'$.

**Theorem 43.** Let $G$ and $G'$ be two connected graphs and $\Phi : N(G) \rightarrow N(G')$ be a non-trivial nested complex isomorphism. Then $G$ and $G'$ are spiders and there exists a graph isomorphism $\psi : G \rightarrow G'$ which induces a nested complex isomorphism $\Psi : N(G) \rightarrow N(G')$ (defined by $\Psi(t) = \{\psi(v) \mid v \in t\}$) such that the composition $\Psi^{-1} \circ \Phi$ coincides with the non-trivial nested complex automorphism $\Omega$ on $N(G)$.

**Corollary 44.** For connected graphs $G, G'$, any nested complex isomorphism $\Phi : N(G) \rightarrow N(G')$ either preserves or dualizes the compatibility degree: either $(\Phi(t) \parallel \Phi(t')) = (t' \parallel t)$ for all $t, t'$ of $G$, or $(\Phi(t) \parallel \Phi(t')) = (t' \parallel t)$ for all $t, t'$ of $G$.

**Corollary 45.** If a connected graph $G$ is not a spider, the number of linear isomorphism classes of compatibility fans of $G$ is the number of orbits of maximal tubings on $G$ under automorphisms of $G$.

**Remark 46.** Not only we obtain many non-isomorphic complete simplicial fan realizations for graphical nested complexes (as stated in Corollary 45), but these realizations cannot be derived from the existing geometric constructions for graph associahedra. Indeed, all previous polytopal realizations of graph associahedra can be obtained by successive face truncations of a simplex [CD06] or of a cube [Vol10, DFRS15]. Not all compatibility fans can be constructed in this way. For example, the leftmost compatibility fan of Figure 7 is not linearly isomorphic to the normal fan of a polytope obtained by face truncations of the 3-dimensional simplex or cube.

5.3. **Polytopality.** To conclude, we briefly discuss the polytopality of our compatibility fans for graphical nested complexes. A complete polyhedral fan is said to be polytopal (or regular) if it is the normal fan of a polytope. It is well known that not all complete polyhedral fans (even simplicial) are polytopal. Examples are easily constructed from non-regular triangulations, see e.g. the discussion in [DRS10, Chapter 2].

**Polytopality of cluster fans.** The polytopality of cluster fans has been studied since the foundations of finite type cluster algebras. For the compatibility fans, polytopality was shown for
particular initial clusters by F. Chapoton, S. Fomin and A. Zelevinsky \cite{CFZ02} and in type $A$ by F. Santos \cite[Section 5]{CSZ11}.

**Theorem 47** (\cite{CFZ02, CSZ11}). The $d$-vector fan (or compatibility fan) is polytopal for
- any initial cluster in any type $A$ cluster algebra \cite[Section 5]{CSZ11}, and
- the bipartite initial cluster in any finite type cluster algebra \cite{CFZ02}.

The polytopality of the $g$-vector fan was studied by C. Hohlweg, C. Lange and H. Thomas \cite{HLT11}. See also recent alternative proofs by S. Stella \cite{Ste13} and V. Pilaud and C. Stump \cite{PS15}.

**Theorem 48** (\cite{HLT11, Ste13, PS15}). The $g$-vector fan (or Cambrian fan) is polytopal for any acyclic initial cluster in any finite type cluster algebra.

To our knowledge, the polytopality of the $d$- and $g$-vector fans remains open in all other cases. Interestingly, all these results rely on the following characterization of polytopality for complete simplicial fans, closely connected to the regular triangulations of vector configurations and the theory of secondary polytopes of \cite{GKZ08}, see also \cite{DRS10}. Equivalent formulations of this characterization appear \emph{e.g.} in \cite[Lemma 2.1]{CFZ02}, \cite[Proposition 6.3]{Zel06}, \cite[Theorem 4.1]{HLT11}, or \cite[Lemma 5.4]{CSZ11}. Here, we follow the presentation of the first two which suits perfectly our previous notations.

**Proposition 49.** Let $F$ be a complete simplicial fan in $\mathbb{R}^n$ and let $R$ denote a set of vectors generating its rays (1-dimensional cones). Then the following are equivalent:

1. $F$ is the normal fan of a simple polytope in $(\mathbb{R}^n)^*$:
2. There exists a map $\omega : R \to \mathbb{R}_{>0}$ such that for any two maximal adjacent cones $\mathbb{R}_{\geq 0}S$ and $\mathbb{R}_{\geq 0}S'$ of $F$ with $S, S' \subseteq R$ and $S \setminus \{s\} = S' \setminus \{s'\}$, we have
   \[ \alpha \omega(s) + \alpha' \omega(s') + \sum_{r \in S \cap S'} \beta_r \omega(r) > 0, \]
   where
   \[ \alpha s + \alpha' s' + \sum_{r \in S \cap S'} \beta_r r = 0 \]
   is the unique (up to rescaling) linear dependence with $\alpha, \alpha' > 0$ between the rays of $S \cup S'$.

Under these conditions, $F$ is the normal fan of the polytope defined by
\[ \{ \phi \in (\mathbb{R}^n)^* \mid \langle \phi | r \rangle \leq \omega(r) \text{ for all } r \in R \}. \]

**Polytopality of compatibility fans.** We have seen in Section 2.3 that the nested fan is the normal fan of the graph associahedron of \cite{CD06, Dev09, Pos09, Zel06}. For the compatibility fan, the question of the polytopality remains open:

**Conjecture 50.** All compatibility and dual compatibility fans for graphical nested complexes are polytopal.

To settle this conjecture, the hope would be to apply the characterization of polytopality for complete simplicial fans presented in Proposition 49. Besides finding an explicit function $\omega$ on the compatibility vectors of the tubes of a graph, our main issue is that we do not control the details of the linear dependence between the compatibility vectors of the tubes involved in a flip. See the proof of Theorem 17 in Section 6.3.

To support Conjecture 50, we have studied the polytopality of the compatibility fans of the specific families of graphs discussed in Section 4. We show in Section 6.6 that Conjecture 50 holds for paths and cycles.

**Theorem 51.** All compatibility and dual compatibility fans of paths and cycles are polytopal.

Note that the case of paths is covered by the results of \cite[Section 5]{CSZ11} presented in Theorem 47. For cycles, the result was unknown except for the bipartite initial tubing by the results of \cite{CFZ02} on type $B$ and $C$ cluster algebras. Via the correspondences given in Propositions 23 and 28, Theorem 51 translates to the following relevant property of $d$-vector fans.
Corollary 52. In types $A$, $B$ and $C$ cluster algebras, the $d$-vector fan with respect to any initial cluster (acyclic or not) is polytopal.

We were not able to settle Conjecture 50 for complete graphs. We believe that this question is worth investigating. As already mentioned, it requires a better understanding of all linear dependences between the compatibility vectors of the tubes involved in a flip.

In another direction, we checked empirically that all 3-dimensional compatibility and dual compatibility fans of Section 4.1 and Figure 8 are polytopal. Using the characterization given in Proposition 49, it boils down to check the feasibility of (many) linear programs.

Finally, as a curiosity and to conclude on a recreative note, we provide a polytopal realization of the compatibility fan for the star $X_{n+1}$ with respect to the initial tubing $T^o := \{\{\ell_1\}, \ldots, \{\ell_n\}\}$ whose tubes are the $n$ leaves of $X_{n+1}$. We first observe that this fan is linearly isomorphic to the fan $\mathcal{G}(G)$ of Theorem 9. Therefore, it can be realized by an affine transformation of the graph associahedra constructed in Theorem 11.

Here, we prefer to give a direct construction with integer coordinates. We provide both the vertex and the facet descriptions of this realization. On the one hand, for each maximal tubing $T$ on $X_{n+1}$ we define a point $x(T) \in \mathbb{R}^n$ whose $i$th coordinate is the cardinality of the inclusion minimal tube of $T \cup \{\ell_i\}$ containing the leaf $\ell_i$ minus 1. The set $(x(T) \mid T$ maximal tubing on $X_{n+1})$ is the orbit of permutation coordinates of the set $\{\sum_{i=k}^n i \cdot e_i \mid 0 \leq k \leq n\}$. On the other hand, for a tube $t$ of $X_{n+1}$ containing the central vertex $\ast$, we observed earlier that the compatibility vector of $t$ with respect to $T^o$ is the characteristic vector of the leaves of $X_{n+1}$ not contained in $t$. Let $f(k) := \sum_{j=k}^n j = \frac{1}{2}(n+k)(n+1-k)$ and define a half-space $H^2(t) \subset \mathbb{R}^n$ by

$$H^2(t) := \left\{ x \in \mathbb{R}^n \mid \langle d(T^o, t) \mid x \rangle \leq f(|t|) \right\} = \left\{ x \in \mathbb{R}^n \mid \sum_{\ell_i \in t} x_i \leq f(|t|) \right\}.$$  

Finally, for the tubes of the initial tubing $T^0$, we define

$$H^2(\{\ell_i\}) := \left\{ x \in \mathbb{R}^n \mid \langle d(T^o, \{\ell_i\}) \mid x \rangle \leq 0 \right\} = \left\{ x \in \mathbb{R}^n \mid x_i \geq 0 \right\}.$$

Proposition 53. The compatibility fan $\mathcal{D}(X_{n+1}, T^o)$ for the initial tubing $T^o := \{\{\ell_1\}, \ldots, \{\ell_n\}\}$ whose tubes are the leaves of $X_{n+1}$ is the normal fan of the $n$-dimensional simple polytope defined equivalently as

- the convex hull of the points $x(T)$ for all maximal tubings $T$ on $X_{n+1}$, or
- the intersection of the half-spaces $H^2(t)$ for all tubes $t$ of $X_{n+1}$.

The proof of this statement is given in Section 6.6. As an illustration, the 3-dimensional stellohedron defined in Proposition 53 is represented in Figure 16 (right). Figure 22 represents two Schlegel diagrams (see [Zie95, Lecture 5] for definition) for the 4-dimensional stellohedron defined in Proposition 53. In both pictures, we have distinguished two particular facets:

- The red facet corresponds to all tubings containing the tube $\{\ast\}$. Since the reconnected complement of $\{\ast\}$ in $X_5$ is the complete graph $K_4$, it has the combinatorics of the permutohedron. In fact, by definition of our polytopal realization, this facet is the classical permutohedron, obtained as the convex hull of the orbit of $\sum_{i\in\{\ast\}} i \cdot e_i$ under permutation of the coordinates.
- The blue facet corresponds to all tubings containing the tube $\{\ell\}$, where $\ell$ is the bottom left leaf of $X_5$. This facet contains the initial tubing $T^o$ at the back. Note that there are 4 isometric facets to this blue facet, corresponding to the four leaves of $X_5$. This is visible in Figure 22 (right).

The blue (resp. red) facet is the projection facet on the left (resp. right) picture.

Remark 54. To conclude, observe that we could have replaced the function $f$ in the definition of the half-spaces $H^2(t)$ by any concave function. This follows from Proposition 49 since the linear
Figure 22. Two Schlegel diagrams for the 4-dimensional stellohedron defined in Proposition 53. The red facet corresponds to all tubings containing the tube \{∗\} while the blue facet corresponds to all tubings containing the tube \{ℓ\}, where ℓ is the bottom left leaf of X₅.

dependence between the compatibility vectors of the tubes of \(T \cup T'\) is given for any adjacent maximal tubings \(T, T'\) on \(X_{n+1}\) distinct from \(T^o\) such that \(T \setminus \{t\} = T' \setminus \{t'\}\) by
\[
d(T^o, t) + d(T^o, t') = d(T^o, t) + d(T^o, t')
\]
where \(t := t \cap t'\) and \(t' := t \cap t'\) (which are tubes of \(X_{n+1}\)). Details are left to the reader.

6. Proofs

This section contains all proofs of our results. Some of them require additional technical steps, which motivated us to separate them from the rest of the paper. We also hope that the many examples treated in Section 4 help the reader’s intuition throughout these proofs.

6.1. Compatibility degree (Proposition 14). We start with the proof that our graphical compatibility degree encodes compatibility and exchangeability between tubes. We show the three points of Proposition 14:

\( (t \parallel t') < 0 \iff (t' \parallel t) < 0 \iff t = t' \).

This is immediate from the definition since the compatibility degree of two distinct tubes is either a cardinal or 0.

\( (t \parallel t') = 0 \iff (t' \parallel t) = 0 \iff t \text{ and } t' \text{ are compatible.} \)

Consider two distinct tubes t, t' of G. If they are compatible, then either t ⊆ t', or t' ⊆ t, or t and t' are non-adjacent. In the first case, \( (t \parallel t') = 0 \) by the last line of Definition 12 of the compatibility degree. In the last two cases, \( (t \parallel t') = \left|\{ \text{neighbors of } t \text{ in } t' \setminus t \}\right| = |\emptyset| = 0. \)

Conversely, if \( (t \parallel t') = 0 \), then either t has no neighbor in t', or t' ⊆ t, or t ⊆ t', so that the two tubes are compatible.

\( (t \parallel t') = 1 \iff (t' \parallel t) \iff t \text{ and } t' \text{ are exchangeable.} \)

The \( \iff \) part follows from the explicit flip description in Proposition 7. Indeed, assume that t and t' are exchangeable, let \( t := t \cup t' \), and let T, T' be two adjacent maximal tubings on G such

\( (t \parallel t') = 1 \iff (t' \parallel t) \iff t \text{ and } t' \text{ are exchangeable.} \)

The \( \iff \) part follows from the explicit flip description in Proposition 7. Indeed, assume that t and t' are exchangeable, let \( t := t \cup t' \), and let T, T' be two adjacent maximal tubings on G such
that \( T \prec \{t\} = T' \prec \{t'\} \). Since \( t' \) is the connected component of \( G[\xi \setminus \lambda(t, T)] \) containing \( \lambda(t, T) \), the root \( \lambda(t, T) \) is the unique neighbor of \( t \) in \( t' \prec t \). Therefore, \( (t \parallel t') = 1 \) and \( (t' \parallel t) = 1 \) by symmetry.

Assume conversely that \( (t \parallel t') = 1 = (t' \parallel t) \). Since \( (t' \parallel t) = 1 \), there exists a unique neighbor \( r \) of \( t' \) in \( t \prec t' \). Similarly, there exists a unique neighbor \( r' \) of \( t \) in \( t' \prec t \). We want to find two adjacent maximal tubings \( T, T' \) on \( G \) such that \( T \prec \{t\} = T' \prec \{t'\} \). We start with the forced tubes (see the end of Section 2.2): we define \( \xi := t \cup t' \) and we let \( s_1, \ldots, s_\ell \) be the connected components of \( \xi \setminus \{r, r'\} \). We choose an arbitrary maximal tubing \( S_i \) on \( G[s_i] \) for each \( i \), and an arbitrary maximal tubing \( S \) on \( G \) containing \( \xi \). The set of tubes

\[
R := \{\xi, s_1, \ldots, s_\ell\} \cup S_1 \cup \cdots \cup S_\ell \cup \{s \mid s \in S, s \notin \xi\}.
\]

is clearly a tubing, and is compatible with both \( t \) and \( t' \). We now compute the cardinality of \( R \). Observe first that \( |S| = |V| - |\kappa(G)| \) so that \( |\{s \mid s \in S, s \notin \xi\}| = |V| - |\kappa(G)| - |\xi| \) since \( \xi \) is a tube of \( G \). Moreover, \( |S_i| = |s_i| - 1 \) since \( s_i \) is a tube, and \( \sum_i |s_i| = |\bigcup_i s_i| = |t \setminus \{r, r'\}| = |\xi| - 2 \). We conclude that

\[
|R| = (1 + \ell) + (|\xi| - 2 - \ell) + (|V| - |\kappa(G)| - |\xi|) = |V| - |\kappa(G)| - 1
\]

Therefore, \( R \) is a ridge of the nested complex \( \mathcal{N}(G) \), so that \( \mathcal{T} := R \cup \{t\} \) and \( \mathcal{T}' := R \cup \{t'\} \) are maximal tubings related by the flip of \( t \) into \( t' \).

\[
6.2. \textbf{Restriction on coordinate hyperplanes (Proposition 38).} \textbf{We now state two lemmas needed in the proofs of Theorem 17. They have essentially the same content as Proposition 38, except that they focus on compatibility vectors (rays) and not on the other cones of the compatibility fan since Theorem 17 is not proved yet. In particular, they will imply Proposition 38 once Theorem 17 will be established.}

\[
\text{Remember that for a tube} \ t^o \text{ of } G, \text{ we denote by } G[t^o] \text{ the restriction of } G \text{ to } t^o \text{ and by } G^*t^o \text{ the reconnected complement of } t^o \text{ in } G, \text{ defined as the graph with vertex set } V \setminus t^o \text{ and edge set } \{e \in (V \setminus t^o) \mid e \text{ or } e \cup t^o \text{ is connected in } G\}.
\]

\[
\textbf{Lemma 55 ([CD06]).} \textbf{For a tube } t^o \text{ of } G, \text{ the map}
\]

\[
s \mapsto \overline{s} := \begin{cases} s & \text{if } s \subseteq t^o \\
\emptyset & \text{if } s \supseteq t^o \text{ or } s \cap t^o = \emptyset \end{cases}
\]

\[
\text{between the tubes of } G \text{ compatible with } t^o \text{ and the tubes of } G := G[t^o] \cup G^*t^o \text{ defines an isomorphism between the link of } t^o \text{ in the nested complex } \mathcal{N}(G) \text{ and the nested complex } \mathcal{N}(G). \]

\[
\text{We denote by } \overline{T} := \{\overline{t} \mid t \in T\} \text{ the image of a tubing } T \text{ on } G. \text{ This map actually preserves the compatibility degrees between tubes and the compatibility vectors.}
\]

\[
\textbf{Lemma 56.} \text{Let } t^o \text{ be a tube of } G. \text{ The map } s \mapsto \overline{s} \text{ between the link of } t^o \text{ in the nested complex } \mathcal{N}(G) \text{ and the nested complex } \mathcal{N}(G) \text{ of the graph } \overline{G} := G[t^o] \cup G^*t^o \text{ defined in Lemma 55 preserves the compatibility degree: } (t \parallel t') = (\overline{t} \parallel \overline{t}') \text{ for any tubes } t, t' \text{ of } G \text{ compatible with } t^o. \text{ Therefore, for any maximal tubing } T^o \text{ on } G \text{ containing } t^o \text{ and any tube } t \text{ of } G \text{ compatible with } t^o, \text{ the compatibility vector } d(T^o, t) \text{ is obtained from the compatibility vector } d(T^o, t) \text{ by deletion of its vanishing } t^o\text{-coordinate.}
\]

\[
\textbf{Proof.} \text{If } t \text{ and } t' \text{ are compatible, so are } \overline{t} \text{ and } \overline{t}' \text{ by Lemma 55, thus the result follows from Proposition 14. We can therefore assume that } t \text{ and } t' \text{ are incompatible, so } t \text{ and } \overline{t}'. \text{ Therefore, the compatibility degrees } (t \parallel t') \text{ and } (t' \parallel \overline{t}') \text{ actually count neighbors. However, it follows immediately from the definitions of the graph } \overline{G} \text{ and of the map } s \mapsto \overline{s} \text{ that the neighbors of } t \text{ in } t' \setminus t \text{ are precisely the neighbors of } \overline{t} \text{ in } \overline{t}' \setminus \overline{t}. \text{ This proves the equality between the compatibility degrees. The equality between the compatibility vectors follows coordinate by coordinate.}
\]
6.3. Compatibility fan (Theorem 17). In order to show that the cones of the compatibility matrices of all tubings on \( G \) form a complete simplicial fan, we need the following refinement.

**Theorem 57.** For any graph \( G \) and any maximal tubing \( T^o \) on \( G \), the compatibility vectors with respect to \( T^o \) have the following properties.

- **Span Property:** For any tube \( u \) of \( G \), the span of \( \{ d(T^o, s) \mid s \in T, s \subseteq u \} \), for a maximal tubing \( T \) on \( G \) containing \( u \), is independent of \( T \).
- **Flip Property:** For any two adjacent maximal tubings \( T, T' \) on \( G \) with \( T \setminus \{ t \} = T' \setminus \{ t' \} \), there exists a linear dependence

\[
\alpha d(T^o, t) + \alpha' d(T^o, t') + \sum_{s \in T \cap T'} \beta_s d(T^o, s) = 0
\]

between the compatibility vectors of \( T \cup T' \) with respect to \( T^o \) which is:

- **Separating:** the hyperplane spanned by \( \{ d(T^o, s) \mid s \in T \cap T' \} \) separates \( d(T^o, t) \) and \( d(T^o, t') \), i.e. the coefficients \( \alpha \) and \( \alpha' \) have the same sign different from \( 0 \).
- **Local:** the dependence is supported by tubes included in \( T \), i.e. \( \beta_s = 0 \) for all \( s \not\subseteq T \).

Theorem 17 follows from the **Separating Flip Property** and the characterization of complete simplicial fans in Proposition 5. The **Span Property** and **Local Flip Property** are not required to get Theorem 17 but we use them to obtain the proof of the **Separating Flip Property**. Observe also that we do not need to prove that the linear dependence between the compatibility vectors of \( T \cup T' \) is unique: it is a consequence of Proposition 5 once we know the **Separating Flip Property**.

Before entering details, let us sketch the general idea of the proof of Theorem 57. We seek for a linear dependence between the compatibility vectors of the tubes of \( T \cup T' \) with respect to the initial tubing \( T^o \), that is, for a linear relation satisfied by their compatibility degrees with respect to any tube of \( T^o \). There are simple combinatorial relations between the compatibility degrees of the tubes of \( T \cup T' \) with respect to all tubes of \( T^o \) not contained in \( T \). Our strategy is to start from such a relation and adapt it iteratively such that it holds for the other tubes of \( T^o \) as well. This transformation is done in two steps:

- We first deal with the tubes of \( T^o \) contained in \( T \) and maximal for this property. They determine the coefficients of the linear dependence on the forced tubes (\( T \) and the connected components of \( T \setminus \{ r, r' \} \)).
- For the remaining tubes of \( T^o \), we need to make successive corrections to the linear dependence. We first get an explicit linear dependence assuming that \( T \cap T' \) contains certain suitable tubes included in \( T \). We then use inductively the **Span Property** and the **Local Flip Property** to get an implicit linear dependence in general.

The key of the proof is that our transformation increases the set of tubes of \( T^o \) for which the relation between the compatibility degrees of the tubes of \( T \cup T' \) is valid.

We now start the formal proof. We proceed by induction on the dimension of the nested complex \( \mathcal{N}(G) \). It is immediate when this dimension is \( 0 \). We now consider an arbitrary graph \( G \) and assume that we have shown Theorem 57 and thus Corollary 19 for any graph \( H \) such that \( \dim(\mathcal{N}(H)) < \dim(\mathcal{N}(G)) \). Given a exchangeable pair of tubes \( t, t' \) of \( G \), our first objective is to exhibit **separating** and **local** linear dependences for some adjacent maximal tubings \( T, T' \) on \( G \) such that \( T \setminus \{ t \} = T' \setminus \{ t' \} \). We will show later the **Span Property** and use it to prove that the linear dependence is **separating** and **local** for all adjacent maximal tubings \( T, T' \) on \( G \) such that \( T \setminus \{ t \} = T' \setminus \{ t' \} \).

**Lemma 58.** For any two exchangeable tubes \( t, t' \) of \( G \), there exists adjacent maximal tubings \( T, T' \) on \( G \) such that \( T \setminus \{ t \} = T' \setminus \{ t' \} \) and a linear dependence between the compatibility vectors of the tubes of \( T \cup T' \) which is both **separating** and **local**.
Proof. We fix some notations for the forced tubes of the exchangeable pair \( \{t, t'\} \). We set \( \mathfrak{t} = t \cup t' \), and we denote by \( T_1, \ldots, T_k \) the connected components of \( G[t \cap t'] \), by \( a_1, \ldots, a_r \) the connected components of \( G[t \setminus (t' \cup \{r\})] \), and by \( a_1', \ldots, a_p \), the connected components of \( G[t' \setminus (t \cup \{r\})] \). Although it is not a tube, we set \( \mathfrak{t} = \bigcup_{i \in [k]} T_i \) and we abuse notation to write \( (t^\circ \parallel \mathfrak{t}) \) for \( \sum_{i \in [k]} (T_i^\circ \parallel \mathfrak{t}) \) and similarly \( d(T^\circ, \mathfrak{t}) \) for \( \sum_{i \in [k]} d(T^\circ, T_i) \). We will use in the same way the notations \( a \) and \( a' \).

We need to distinguish different cases, for which the linear dependences are slightly different, while the proofs are essentially identical. To simplify the discussion, we assume in Cases (A), (B) and (C) below that \( t, t' \notin \mathcal{T}^o \) and that no tube of \( \mathcal{T}^o \) is compatible with both \( t \) and \( t' \). At the end of the proof, Cases (D) and (E) show how to restrict to these hypotheses.

(A) A first relation. Consider a tube \( t^o \) of \( \mathcal{T}^o \) not contained in \( \mathfrak{t} \). We claim that

\[
(t^\circ \parallel t) + (t^\circ \parallel t') = (t^o \parallel \mathfrak{t}) + (t^\circ \parallel \mathfrak{t}).
\]

Indeed, since \( t^o \notin \mathfrak{t} \), these four compatibility degrees actually count neighbors of \( t^o \). The formula thus follows from inclusions-exclusion principle since \( \mathfrak{t} = t \cup t' \) and \( \mathfrak{t} = t \cap t' \).

There are other tubes of \( \mathcal{T}^o \) satisfying this relation. Indeed, consider a tube \( t^o \) of \( \mathcal{T}^o \) included in \( \mathfrak{t} \) which contains \( r \) but does not contain nor is adjacent to \( r' \). Then \( t^o \subseteq t \subseteq \mathfrak{t} \) so that \( (t^o \parallel t) = (t^o \parallel \mathfrak{t}) = 0 \). Moreover, \( t^o \) is incompatible with \( t' \) and all \( T_1, \ldots, T_k \). Since \( r' \) is not adjacent to \( t^o \), all neighbors of \( t^o \) in \( t^o \setminus t^o \) are in \( t \setminus t' \). Therefore \( (t^o \parallel t') = (t^o \parallel \mathfrak{t}) \). The relation follows. Similarly, the relation follows if \( t^o \) contains \( r' \) and does not contain nor is adjacent to \( r \).

If all tubes of \( \mathcal{T}^o \) included in \( \mathfrak{t} \) satisfy the previous conditions, we have obtained a separating and local linear dependence:

\[
d(T^o, t) + d(T^o, t') = d(T^o, \mathfrak{t}) + d(T^o, \overline{\mathfrak{t}}).
\]

This linear dependence would be valid for any adjacent maximal tubings \( \mathcal{T}, \overline{\mathcal{T}} \) on \( G \) such that \( \mathcal{T} \setminus \{t\} = \overline{\mathcal{T}} \setminus \{t'\} \) since the tube \( \mathfrak{t} \) and the tubes \( t \) are forced in any such pair. Unfortunately, these conditions do not always hold for all tubes of \( \mathcal{T}^o \). In this case, we will therefore adapt the linear dependence (1) to cover all possible configurations for the tubes of \( \mathcal{T}^o \).

(B) No tube of \( \mathcal{T}^o \) contained in \( \mathfrak{t} \) contains both \( r \) and \( r' \). Except if the linear dependence (1) is valid, there must exist w.l.o.g. a tube \( t^o \in \mathcal{T}^o \) contained in \( \mathfrak{t} \), containing \( r \) and adjacent to \( r' \). Choose \( t^o \) maximal for these properties. Since we have assumed that no tube of \( \mathcal{T}^o \) is compatible with both \( t \) and \( t' \), all tubes of \( \mathcal{T}^o \) included in \( t^o \) contain \( r \). These tubes thus form a nested chain \( t^o = T_0^o \supseteq T_1^o \supseteq \cdots \supseteq T_p^o = \{r\} \). For \( i \in [p] \), define \( T_i^o \) to be the connected component of \( G[t^o_i \setminus \{r\}] \) containing the singleton \( t^o_{i-1} \setminus t^o_i \).

Set

\[
\alpha = (t^o \parallel a) + (t^o \parallel t') \quad \text{and} \quad \alpha' = (t^o \parallel \mathfrak{t}) + (t^o \parallel a),
\]

and define inductively \( \beta_1, \ldots, \beta_p \) by

\[
\beta_i = \alpha' (T_i^o \parallel t') - \alpha (T_i^o \parallel \mathfrak{t}) - (\alpha - \alpha') (T_i^o \parallel a) - \sum_{j \in [i-1]} \beta_j (T_j^o \parallel T_i^o).
\]

We claim that

\[
\alpha d(T^o, t) + \alpha' d(T^o, t') = \alpha d(T^o, \mathfrak{t}) + \alpha' d(T^o, \overline{\mathfrak{t}}) + (\alpha - \alpha') d(T^o, a) + \sum_{i \in [p]} \beta_i d(T^o, T_i^o).
\]

To prove it, we check this linear dependence coordinate by coordinate.

Observe first that \( (T_i^o \parallel t) = (T_i^o \parallel \mathfrak{t}) = 0 \) for all \( 0 \leq i \leq p \) since \( T_i^o \supseteq t \subseteq \mathfrak{t} \). Moreover, for all \( i < j \), we have by definition \( T_j^o \subseteq T_i^o \), so that \( (T_i^o \parallel T_j^o) = 0 \). Finally, we have \( (T_i^o \parallel T_j^o) = 1 \) since \( T_i^o \) and \( T_j^o \) are incompatible and the only neighbor of \( T_j^o \) in \( T_i^o \setminus T_j^o \) is the singleton \( t^o_{i-1} \setminus t^o_i \). Therefore, Relation (2) holds for \( t^o \) by definition of \( \alpha \) and \( \alpha' \) and for \( T_i^o \) by definition of \( \beta_i \).

Consider now a tube \( s^o \) of \( \mathcal{T}^o \) not included in \( t^o \). Suppose that \( s^o \) is included in \( \mathfrak{t} \). Then \( s^o \) contains precisely one of \( r \) and \( r' \) (it cannot contain both by assumption (B), and it cannot avoid both as it would be compatible with both \( t \) and \( t' \)). If \( s^o \) contains \( r \), it contains \( t^o \) and therefore equals \( t^o \) by maximality of the latter. Otherwise, \( s^o \) contains \( r' \), thus is adjacent to \( t^o \) thus contains it (by compatibility), and thus contains \( r \), a contradiction. We therefore obtained that \( s^o \) is not included in \( \mathfrak{t} \), so that all compatibility degrees \( (s^o \parallel t), (s^o \parallel \mathfrak{t}), (s^o \parallel \mathfrak{t}) \) and \( (s^o \parallel a) \) actually
count neighbors of \( s^* \). Observe now that \( r \) cannot be adjacent to \( s^\circ \) (except if it belongs to \( s^\circ \)), since \( r \) belongs to \( t^\circ \) which is compatible with \( s^\circ \). Therefore, we have 

\[
(s^\circ \parallel t) = (s^\circ \parallel \emptyset) + (s^\circ \parallel a),
\]

since \( t = \emptyset \cup a \cup \{r\} \) and \( t' = \emptyset \setminus a \setminus \{r\} \). Finally, since any \( t^*_i \) is contained in \( t^\circ \), it is compatible with \( s^\circ \), so that \( (s^\circ \parallel t^*_i) = 0 \) by Proposition 14. Combining these equalities, we obtain that Relation (2) holds for any \( s^\circ \in T^\circ \).

We found a linear dependence between the compatibility vectors of \( \{t, t', \emptyset\} \cup \emptyset \cup a \cup \{t^*_i \mid i \in [p]\} \) with respect to the initial tubing \( T^\circ \). Any two of these tubes except \( t, t' \) are compatible:

- the forced tubes are pairwise compatible;
- each \( t^*_i \) is a connected component of \( t^\circ \setminus \{r\} \), thus is contained in \( \emptyset \setminus \{r, r'\} \), and thus is compatible with the connected components of \( \emptyset \setminus \{r, r'\} \) (i.e. all forced tubes);
- for \( t^*_i \supseteq t^*_j \), any connected component of \( t^*_i \setminus \{r\} \) is contained in a connected component of \( t^*_j \setminus \{r\} \) and thus is compatible with all connected components of \( t^*_j \setminus \{r\} \). In particular \( t^*_i \) and \( t^*_j \) are compatible.

Therefore, there exists adjacent maximal tubings \( T, T' \) on \( G \) such that \( T \setminus \{t\} = T' \setminus \{t'\} \) and \( T \cup T' \supseteq \{t, t', \emptyset\} \cup \emptyset \cup a \cup \{t^*_i \mid i \in [p]\} \). For this choice of \( T, T' \), we thus obtained a separating and local linear dependence (2) between the compatibility vectors of the tubes of \( T \cup T' \).

(C) A tube of \( T^\circ \) contained in \( \emptyset \) contains both \( r \) and \( r' \). We distinguish again two cases.

(C.1) No tube of \( T^\circ \) contained in \( \emptyset \) contains \( r \) and is adjacent to \( r' \) or conversely. There must exist a tube \( t^\circ \in T^\circ \) included in \( t \) and containing \( r \). Choose \( t^\circ \) maximal for these properties.

With the same arguments as in Case (B), the tubes of \( T^\circ \) included in \( t^\circ \) form a nested chain \( t^\circ = t^\circ_0 \supseteq t^\circ_1 \supseteq \cdots \supseteq t^\circ_p = \{r\} \). For \( i \in [p] \), define \( t^*_i \) to be the connected component of \( G[t^\circ_{i-1} \setminus \{r\}] \) containing the singleton \( t^\circ_{i-1} \setminus t^\circ_i \). We define the tube \( t^\circ_i \), the chain \( t^\circ = t^\circ_0 \supseteq t^\circ_1 \supseteq \cdots \supseteq t^\circ_p = \{r\} \) and the tubes \( t^*_i \) for \( i \in [p] \) similarly.

Consider now the inclusion minimal tube \( \emptyset \) of \( T^\circ \) contained in \( \emptyset \) and containing both \( r \) and \( r' \). Since we assumed that no tube of \( T^\circ \) is compatible with both \( t \) and \( t' \), we have \( \emptyset = t^\circ \cup t^\circ \cup \{r\} \) where \( \emptyset \in \emptyset \). Let \( \emptyset \) be the connected component of \( G[\emptyset \setminus \{r, r'\}] \) containing \( \emptyset \).

Set 

\[
\alpha := (t^\circ \parallel \emptyset) + (t^\circ \parallel a') \quad \text{and} \quad \alpha' := (t^\circ \parallel \emptyset) + (t^\circ \parallel a),
\]

and define inductively \( \beta_1, \ldots, \beta_p \) and \( \beta'_1, \ldots, \beta'_p \) by

\[
\beta_i = \alpha' (t^\circ_i \parallel t) - (\alpha + \alpha') (t^\circ_i \parallel \emptyset) - \alpha (t^\circ_i \parallel a) + \alpha' (t^\circ_i \parallel \emptyset) - \sum_{j \in [i-1]} \beta_j (t^\circ_i \parallel t^*_j),
\]

\[
\beta'_i = \alpha (t^\circ_i \parallel t) - (\alpha + \alpha') (t^\circ_i \parallel \emptyset) - \alpha' (t^\circ_i \parallel a') + \alpha (t^\circ_i \parallel \emptyset) - \sum_{j \in [i-1]} \beta'_j (t^\circ_i \parallel t^*_j).
\]

We claim that

\[
\alpha \mathbf{d}(T^\circ, t) + \alpha' \mathbf{d}(T^\circ, t') = (\alpha + \alpha') \mathbf{d}(T^\circ, \emptyset) + \alpha \mathbf{d}(T^\circ, a) + \alpha' \mathbf{d}(T^\circ, a') - \alpha \alpha' \mathbf{d}(T^\circ, \emptyset)
\]

\[
+ \sum_{i \in [p]} \beta_i \mathbf{d}(T^\circ, t^*_i) + \sum_{i \in [p]} \beta'_i \mathbf{d}(T^\circ, t^*_i).
\]

To prove it, we check this linear dependence coordinate by coordinate.

We start with \( t^\circ \) and \( t^\circ \). We have \( (t^\circ \parallel t) = 0 \) since \( t^\circ \subsetneq t \). Moreover, \( (t^\circ \parallel \emptyset) = [\{\emptyset\}] = 1 \). Finally, we have by definition \( t^*_i \subsetneq t^\circ \) so that \( (t^\circ \parallel t^*_i) = 0 \) for all \( j \in [p] \). Combining these equalities, Relation (3) follows for \( t^\circ \) from the definition of \( \alpha \) and \( \alpha' \). The argument is identical for \( t^\circ \).

We now consider the tubes \( t^*_i \) and \( t^\circ_i \). Observe first that \( (t^*_i \parallel t) = 0 \) for all \( 0 \leq i \leq p \) since \( t^*_i \subsetneq t \). Moreover, for all \( i < j \), we have by definition \( t^*_i \subsetneq t^*_j \), so that \( (t^*_i \parallel t^*_j) = 0 \). In addition, we have \( (t^*_i \parallel t^*_j) = 0 \) for all \( i \in [p] \) and \( j \in [p] \) since \( t^*_i \) and \( t^*_j \) are compatible. Finally, we have \( (t^*_i \parallel t^*_j) = 1 \) since \( t^*_i \) and \( t^*_j \) are incompatible and the only neighbor of \( t^*_i \) in \( t^*_i \setminus t^*_j \) is the singleton \( t^*_i \setminus t^*_j \). Therefore, Relation (3) holds for \( t^\circ_i \) by definition of \( \beta_i \). The argument is identical for \( t^\circ_i \).
Finally, we consider a tube $s^o$ of $T^o$ not strictly contained in $\overline{t}$. With similar arguments as in Case (B), the compatibility degrees $(s^o \parallel t)$, $(s^o \parallel t^i)$, $(s^o \parallel \overline{t})$, $(s^o \parallel a)$ and $(s^o \parallel a^i)$ actually count neighbors of $s^o$, and (by assumption $C$) satisfy

$$(s^o \parallel t) = (s^o \parallel \overline{t}) + (s^o \parallel a) \quad \text{and} \quad (s^o \parallel t^i) = (s^o \parallel \overline{t}) + (s^o \parallel a^i).$$

Moreover, since all $t^i$, $t^i*$ and $t^*$ are contained in $\overline{t}$, they are all compatible with $s^o$, so that $(s^o \parallel t^i) = (s^o \parallel t^*) = (s^o \parallel \overline{t}) = 0$ by Proposition 14. Combining these equalities, we obtain that Relation (3) holds for any $s^o \in T^o$.

With the same arguments as in Case (B), there exists adjacent maximal tubings $T, T'$ on $G$ such that $T \setminus \{t\} = T' \setminus \{t^i\}$ and $T \cup T' \supseteq \{t, t', \overline{t}\} \cup \overline{t} \cup a \cup a' \cup \{t^i | i \in [p]\} \cup \{t^i* | i \in [p]\}$. For this choice of $T, T'$, we thus obtained a separating and local linear dependence (3) between the compatibility vectors of the tubes of $T \cup T'$.

(C.2) A tube of $T^o$ contained in $\overline{t}$ contains $r$ and is adjacent to $r'$ or conversely. W.l.o.g., consider an inclusion maximal tube $t^o \in T^o$ contained in $\overline{t}$, containing $r$ and adjacent to $r'$. Since we have assumed that no tube of $T^o$ is compatible with both $t$ and $t'$, all tubes of $T^o$ included in $t^o$ contain $r$. These tubes thus form a nested chain $t^o = t^o_1 \supseteq t^o_2 \supseteq \cdots \supseteq t^o_p = \{r\}$. For $i \in [p]$, define $t^i_*$ to be the connected component of $G[t^i_{i-1} \setminus \{r\}]$ containing the singleton $t^i_{i-1} \setminus t^i_0$.

Set

$$\alpha := (t^o \parallel t') - (t^o_0 \parallel \overline{t}) = |\{r'\}| = 1 \quad \text{and} \quad \alpha' := (t^o \parallel t) + (t^o \parallel a),$$

and define inductively $\beta_1, \ldots, \beta_p$ by

$$\beta_i = \alpha' (t^i_0 \parallel t') - (1 + \alpha') (t^i_0 \parallel \overline{t}) - (t^i_0 \parallel a) - \sum_{j \in [i-1]} \beta_j (t^i_j \parallel t^i_*) .$$

We claim that

$$(4) \quad d(T^o, t) + \alpha' d(T^o, t') = (1 + \alpha) d(T^o, \overline{t}) + d(T^o, a) + \alpha' d(T^o, a') + \sum_{i \in [p]} \beta_i d(T^o, t^i_*) .$$

To prove it, we check this linear dependence coordinate by coordinate.

Observe first that $(t^i_0 \parallel t) = 0$ for all $0 \leq i \leq p$ since $t^i_0 \subset t$. Moreover, for all $i < j$, we have by definition $t^i_0 \subset t^j_0$, so that $(t^i_0 \parallel t^j_0) = 0$. Finally, we have $(t^i_0 \parallel t^j_0) = 1$ since $t^i_0$ and $t^j_0$ are incompatible and the only neighbor of $t^i_0$ in $t^i_0 \setminus t^i_0$ is the singleton $t^i_{i-1} \setminus t^i_0$. Therefore, Relation (2) holds for $t^o$ by definition of $\alpha$ and $\alpha'$ and for $t^o_0$ by definition of $\beta_i$.

Consider now a tube $s^o$ of $T^o$ not strictly contained in $t^o$. With similar arguments as in Case (B), the compatibility degrees $(s^o \parallel t)$, $(s^o \parallel t')$, $(s^o \parallel \overline{t})$, $(s^o \parallel a)$ and $(s^o \parallel a')$ actually count neighbors of $s^o$, and (by assumption $C$) satisfy

$$(s^o \parallel t) = (s^o \parallel \overline{t}) + (s^o \parallel a) \quad \text{and} \quad (s^o \parallel t') = (s^o \parallel \overline{t}) + (s^o \parallel a').$$

Moreover, since all $t^i_0$ are contained in $t^o$, they are all compatible with $s^o$, so that $(s^o \parallel t^i_0) = 0$ by Proposition 14. Combining these equalities, we obtain that Relation (3) holds for any $s^o \in T^o$.

With the same arguments as in Case (B), there exists adjacent maximal tubings $T, T'$ on $G$ such that $T \setminus \{t\} = T' \setminus \{t^i\}$ and $T \cup T' \supseteq \{t, t', \overline{t}\} \cup \overline{t} \cup a \cup a' \cup \{t^i | i \in [p]\}$. For this choice of $T, T'$, we thus obtained a separating and local linear dependence (4) between the compatibility vectors of the tubes of $T \cup T'$.

We assumed in Cases (A), (B) and (C) above that $t, t' \notin T^o$ and that no tube of $T^o$ is compatible with both $t$ and $t'$. The remaining two cases show how to force this assumption.

(D) A tube $t^o$ of $T^o$ is compatible with both $t$ and $t'$. We treat this case by induction on the number of tubes of $T^o$ compatible with both $t$ and $t'$. By Lemma 56, the compatibility vectors with respect to $T^o$ of the tubes of $G$ compatible with $t^o$ correspond to the compatibility vectors of the tubes of $G := G[t^o] \cup G*t^o$ with respect to the maximal tubing $\overline{T^o}$. Since there are strictly less tubes of $\overline{T^o}$ compatible with both $t$ and $t'$, the induction hypothesis ensures that there exists adjacent maximal tubings $\overline{T}, \overline{T}'$ on $G$ such that $\overline{T} \setminus \{t\} = \overline{T}' \setminus \{t^i\}$ and a separating and local linear dependence between the compatibility vectors of $\overline{T} \cup \overline{T}'$ with respect to $\overline{T^o}$. By Lemma 55, the sets $\overline{T} := \{t^o\} \cup \{s | s \in \overline{t}\}$ and
T′ := \{t^2\} \cup \{s′ \mid s′ \in \overline{T′}\} are tubings on G (since t^2 is compatible with all preimages of tubes of \overline{G}) and they are maximal by cardinality. Moreover, Lemma 56 ensures that the linear dependence between the compatibility vectors of the tubes of T ∪ T′ with respect to T° coincides with the linear dependence between the compatibility vectors of the tubes of \overline{T} ∪ \overline{T′} with respect to \overline{T°}.

The linear dependence clearly remains separating and local, which concludes when there is a tube t′ \in T° compatible with both t and t′.

(E) t or t′ belongs to T°. We can assume w.l.o.g. that t = t° belongs to T°. Consider any two adjacent maximal tubings T, T′ on G such that T \setminus \{t\} = T′ \setminus \{t′\}. Since any tube s in T ∩ T′ is compatible with t = t°, the t°-coordinate of the compatibility vector d(T°, s) vanishes. The same happens for the vector v := d(T°, t) + d(T°, t′) since (t° \parallel t) = −1 (as t = t°) while (t° \parallel t′) = 1 (by Proposition 14). The set of vectors \{v\} ∪ \{d(T°, s) \mid s \in T ∩ T′\} has cardinality |T°| but is contained in the hyperplane of \mathbb{R}^{|T°|} orthogonal to e_0. Therefore, there is a linear dependence between these vectors, which translates into a linear dependence between the compatibility vectors of the tubes of T ∪ T′ with the same coefficient on d(T°, t) and d(T°, t′). This coefficient cannot vanish: otherwise, we would have a linear dependence on the compatibility vectors of \overline{T} with respect to \overline{T°}. Since \overline{T} is a maximal tubing on \overline{G}, and dim(Λ(\overline{G})) = dim(Λ(G)) − 1, this would contradict Corollary 19 and thus the induction hypothesis for \overline{G}. This linear dependence is therefore separating. It is automatically local if \overline{v} = V since all tubes are then subsets of \overline{v}. Otherwise, we prove that it is local by restriction. We distinguish two cases.

(E.1) A tube \overline{t} of T° contains t ∪ \{t′\} and is contained in \overline{t}. Consider the restricted graph \overline{G} = G[t ∪ \{t′\}]. Any tube s of G included in t is also a tube of \overline{G}. Therefore, the set \overline{T} := \{s \mid s \in T, s \subseteq t\} is a tubing on \overline{G} for any tubing T containing t. Define also the tube t′ := t′ \setminus a′ of \overline{G}. The existence of \overline{t} implies that (s° \parallel t′) = (s° \parallel a′) for any s° \in T° not included in t. It follows that the compatibility vector d(\overline{T°}, \overline{t′}) is the restriction of d(T°, t′) − d(T°, a′) to the coordinates indexed by \overline{T°}. Similarly, for any tube s ∈ T contained in t, the compatibility vector d(\overline{T°}, s) is the restriction of d(T°, s) to the coordinates indexed by \overline{T°}. This shows that the linear dependence on \{d(\overline{T°}, s) \mid s \in \overline{T} ∪ \overline{T′}\} provides a linear dependence on \{d(T°, s) \mid s \in T ∪ T′, s \subseteq t\} ∪ \{d(T°, t′) − d(T°, a′)\}. The resulting linear dependence on \{d(T°, s) \mid s \in T ∪ T′\} is local.

(E.2) No tube of T° contains t ∪ \{t′\} and is contained in \overline{t}. The proof is identical to Case (E.1), replacing d(T°, t′) − d(T°, a′) by d(T°, t′) − d(T°, \overline{t′}). Details are left to the reader. □

We can now prove the Span Property.

Lemma 59. For any tube u of G, the span of \{d(T°, s) \mid s \in T, s \subseteq u\}, for a maximal tubing T on G containing u, is independent of T.

Proof. We proceed by induction on the size of u. The result is immediate if u is a singleton. Consider now two adjacent maximal tubings T, T′ on G containing u such that T \setminus \{t\} = T′ \setminus \{t′\}. Assume first that t and t′ are contained in u. By Lemma 58, there exists adjacent maximal tubings S, S′ on G containing u such that S \setminus \{t\} = S′ \setminus \{t′\} and a linear dependence between the compatibility vectors of the tubes of S ∪ S’ with respect to T° which is both separating and local. By definition, this implies that there exists α > 0 and α’ > 0 such that the vector α d(T°, t) + α’ d(T°, t′) belongs to vect(\{d(T°, s) \mid s \in S ∪ S′, s \subseteq t\}). However,

\{s \in S ∪ S′ \mid s \subseteq \overline{t}\} = \{t\} ∪ \{s \in S ∪ S′ \mid s \subseteq t\} ∪ \{s \in S ∪ S′ \mid s \subseteq a\} ∪ \{s \in S ∪ S′ \mid s \subseteq a′\}.

By induction hypothesis applied to each tube of t, we have

\text{vect}(\{d(T°, s) \mid s \in S ∪ S′, s \subseteq \overline{t}\}) = \text{vect}(\{d(T°, s) \mid s \in S ∪ S′, s \subseteq t\}) \quad \text{(as S ∪ S′ = t \not\subseteq \overline{t})}

\text{vect}(\{d(T°, s) \mid s \in T ∩ S′, s \subseteq \overline{t}\}) \quad \text{induction hypothesis)

\text{vect}(\{d(T°, s) \mid s \in T ∩ S', s \subseteq \overline{t}\}) \quad \text{(as T ∩ S' = t \not\subseteq \overline{t})

and similarly replacing t by a or a’. It follows that the vector α d(T°, t) + α’ d(T°, t′) also belongs to vect(\{d(T°, s) \mid s \in T ∪ T′, s \subseteq u\}). Since α \neq 0, this implies that d(T°, t) belongs
to $\text{vect} (\{d(T^0, s) \mid s \in T, s \subseteq u\})$. Similarly, $d(T^0, t')$ belongs to $\text{vect} (\{d(T^0, s) \mid s \in T, s \subseteq u\})$.

We therefore obtained that:

$$\text{vect} (\{d(T^0, s) \mid s \in T, s \subseteq u\}) = \text{vect} (\{d(T^0, s) \mid s \in T', s \subseteq u\}).$$

This also clearly holds when $t$ and $t'$ are not contained in $u$. This concludes the proof since the graph of flips on the maximal tubings on $G$ containing $u$ is connected.

We can finally conclude the proof of Theorem 57 using both Lemmas 58 and 59.

**Proof of Theorem 57.** The **Span Property** is proved in Lemma 59. It only remains to show the **Flip Property** for arbitrary adjacent maximal tubings. Consider two adjacent maximal tubings $T, T'$ on $G$ with $T \setminus \{t\} = T' \setminus \{t'\}$. By Lemma 58, there exists adjacent maximal tubings $S, S'$ on $G$ such that $S \setminus \{t\} = S' \setminus \{t'\}$ and a linear dependence between the compatibility vectors of the tubes of $S \cup S'$ with respect to $T^0$ which is both separating and local. By definition, this implies that there exists $\alpha > 0$ and $\alpha' > 0$ such that the vector $\alpha d(T^0, t) + \alpha' d(T^0, t')$ belongs to $\text{vect} (\{d(T^0, s) \mid s \in S \cap S', s \subseteq t\})$. Lemma 59 applied to $t$ ensures that

$$\text{vect} (\{d(T^0, s) \mid s \in T \cap T', s \subseteq t\}) = \text{vect} (\{d(T^0, s) \mid s \in S \cap S', s \subseteq t\})$$

and similarly replacing $S$ by $a$ or $a'$. We thus conclude that

$$\text{vect} (\{d(T^0, s) \mid s \in T \cap T', s \subseteq t\}) = \text{vect} (\{d(T^0, s) \mid s \in S \cap S', s \subseteq t\})$$

contains the vector $\alpha d(T^0, t) + \alpha' d(T^0, t')$ with $\alpha > 0$ and $\alpha' > 0$. In other words, we obtained a separating and local linear dependence on $\{d(T^0, s) \mid s \in C \cup T'\}$.

### 6.4. Dual compatibility fan (Theorem 21)

The fact that dual compatibility vectors support a complete simplicial fan realizing the nested complex is a direct consequence of Theorem 17, using that the dual compatibility matrix $d^*(T, T^0)$ is the transpose of the compatibility matrix $d(T^0, T)$.

Observe indeed that the **Separation Flip Property** for compatibility vectors is equivalent to the condition $\det (d(T^0, T)) \cdot \det (d(T^0, T')) < 0$ for any adjacent maximal tubings $T, T'$ on $G$. Since the transposition preserves the determinant, this implies that $\det (d^*(T, T^0)) \cdot \det (d^*(T', T^0)) < 0$ for any adjacent maximal tubings $T, T'$ on $G$, and thus the **Separation Flip Property** for dual compatibility vectors. We conclude again by Proposition 5.

**Remark 60.** Observe that this proof does not provide us with any explicit linear dependence between dual compatibility vectors. In particular, we do not know whether the following analogue properties of Theorem 57 hold:

**Dual Span Property:** For any tube $u$ of $G$, the span of $\{d(t, T^0) \mid t \in T, t \not\subset u\}$, for a maximal tubing $T$ on $G$ containing $u$, is independent of $T$.

**Dual Local Flip Property:** For any two maximal tubings $T, T'$ with $T \setminus \{t\} = T' \setminus \{t'\}$, the unique linear dependence between the dual compatibility vectors of $T \cup T'$ with respect to $T^0$ is supported by tubes not strictly included in a connected component of $z = t \cap t'$.

### 6.5. Nested complex isomorphisms (Proposition 40, Proposition 42 and Theorem 43)

We now prove various results on nested complex isomorphisms presented in Section 5.2. We first show that the map $\Omega$ on the tubes of a spider $X_\omega$ defines a nested complex automorphism that dualizes the compatibility degree.

**Proof of Proposition 40.** First, $\Omega$ clearly sends tubes of $X_\omega$ to tubes of $X_\omega$. We just have to show that $(\Omega(t) \mid \Omega(t')) = (t' \mid t)$ for any two tubes $t, t'$ of $X_\omega$, and Proposition 14 will imply that $\Omega$ is a nested complex automorphism. This follows from the definition of $\Omega$ and the fact that

$$\left[ \left[ \left[ v_j, v_{k'} \right] \ | \ \left[ v_j, v_{k'} \right] \right] \right] = \delta_{i=i'} \cdot (\delta_{j<j'<k+1<k'+1} + \delta_{j'<j<k'\leq k+1}),$$

$$\left[ \left[ \left[ v_j, v_{k} \right] \ | \ \left[ \cup_{h \in \ell} \left[ h_{0}, h_{k_0} \right] \right] \right] \right] = \delta_{j<k+1\leq k},$$

and

$$\left( \left( \bigcup_{i \in \ell} \left[ v_i, v_{k_i} \right] \bigcup_{i \in \ell} \left[ v_i, v_{k_i} \right] \right) = \left\{ i \in [\ell] \mid k_i < k'_i \right\} \cdot \delta_{i \in [\ell], k_i < k'_i}. \quad \Box$$
Our objective is to show that these maps $\Omega$ on spiders are essentially the only non-trivial nested complex isomorphisms. We fix an isomorphism $\Phi$ between two nested complexes $\mathcal{N}(G)$ and $\mathcal{N}(G')$.

We first show that $\Phi$ preserves connected components in the following sense.

**Lemma 61.** Two tubes $t$ and $t'$ of $G$ belong to the same connected component of $G$ if and only if their images $\Phi(t)$ and $\Phi(t')$ belong to the same connected component of $G'$.

**Proof.** Observe first that two tubes from distinct connected components are automatically incompatible. Assume now that $t$ and $t'$ are in the same connected component of $G$. If $t$ and $t'$ are incompatible, then $\Phi(t)$ and $\Phi(t')$ are also incompatible and therefore in the same connected component of $G'$. If $t$ and $t'$ are compatible, then there exists a tube $t''$ incompatible with both $t$ and $t'$:

- if $t \cap t' = \emptyset$, consider a path connecting a neighbor of $t$ to a neighbor of $t'$ in $G[V \setminus \{t \cup t'\}]$;
- if $t \subseteq t'$, consider a path connecting a neighbor of $t$ to a neighbor of $t'$ in $G[V \setminus t]$.

We obtain that $\Phi(t'')$ is incompatible with both $\Phi(t)$ and $\Phi(t')$, so that they all belong to the same connected component of $G'$. This proves one direction. For the other direction, consider $\Phi^{-1}$. □

Consequently, the nested complex $\mathcal{N}(G)$ records the sizes of the connected components of the graph $G$. We call connected size partition of $G$ the partition $\lambda(G) := |V_1|, |V_2|, \ldots, |V_\kappa|$ of $|V|$, where $V_1, \ldots, V_\kappa$ are the connected components of $G$ ordered such that $|V_i| \geq |V_{i+1}|$.

**Corollary 62.** Two graphs whose nested complexes are isomorphic have the same connected size partitions: $\mathcal{N}(G) \simeq \mathcal{N}(G') \implies \lambda(G) = \lambda(G')$. In particular, they have the same number of vertices.

**Proof.** Consider a maximal tubing $T$ on $G$ and decompose it into subtubings $T_1, \ldots, T_\kappa$ on the connected components $V_1, \ldots, V_\kappa$ of $G$. Their images $\Phi(T_1), \ldots, \Phi(T_\kappa)$ decompose the maximal tubing $\Phi(T)$. Moreover, Lemma 61 ensures that two tubes $\Phi(t) \in \Phi(T_i)$ and $\Phi(t') \in \Phi(T_j)$ belong to the same connected component of $G'$ if and only if $i = j$. We therefore obtain that $\lambda(G) = \{|V_1|, \ldots, |V_\kappa|\} = \{|T_1| + 1, \ldots, |T_\kappa| + 1\} = \{\Phi(T_1) + 1, \ldots, \Phi(T_\kappa)\} = \lambda(G')$. □

Proposition 42 is another immediate consequence of Lemma 61: since it sends all tubes in a connected component $H$ of $G$ to tubes in the same connected component $H'$ of $G'$, the map $\Phi$ induces a nested complex isomorphism between $\mathcal{N}(H)$ and $\mathcal{N}(H')$. From now on, we assume without loss of generality that $G$ is connected. Our next step is a crucial structural property of $\Phi$.

**Lemma 63.** For any nested complex isomorphism $\Phi : \mathcal{N}(G) \to \mathcal{N}(G')$ and any tube $t$ of $G$, either $|\Phi(t)| = |t|$ or $|\Phi(t)| = |V| - |t|$.

**Proof.** By Lemma 55, the join of a tube $t$ in $\mathcal{N}(G)$ is isomorphic to the nested complex $\mathcal{N}(G[t] \cup G^*t)$ of the union of the restriction $G[t]$ with the reconnected complement $G^*t$. The former has $|t|$ vertices while the latter has $|V| - |t|$ vertices. Since $\Phi$ induces an isomorphism from the join of $t$ in $\mathcal{N}(G)$ to the join of $\Phi(t)$ in $\mathcal{N}(G')$, the result follows from Corollary 62. □

We say that $\Phi$ **maintains** the tube $t$ if $|\Phi(t)| = |t|$ and that $\Phi$ **swaps** the tube $t$ if $|\Phi(t)| = |V| - |t|$.

**Proposition 64.** If it maintains all tubes of $G$, then $\Phi$ is the trivial nested complex isomorphism induced by the graph isomorphism $\psi : G \to G'$ defined by $\Phi(\{v\}) = \{\psi(v)\}$.

**Proof.** Two vertices $v$ and $w$ of $G$ are adjacent if and only if the two tubes $\{v\}$ and $\{w\}$ are incompatible. Since $\Phi$ preserves the compatibility relation, this shows that $v$ and $w$ are adjacent if and only if $\psi(v)$ and $\psi(w)$ are, i.e. that $\psi$ defines a graph isomorphism. Let $\Psi$ denote the nested complex isomorphism induced by $\psi$, i.e. defined by $\Psi(t) = \{\psi(v) \mid v \in t\}$.

We prove by induction on $|t|$ that $\Phi(t) = \Psi(t)$ for any tube $t$ on $G$. It holds for singletons. For the induction step, consider an arbitrary tube $t$ of $G$. Let $v \in V \setminus t$ be a neighbor of $t$. Since $\{v\}$ and $t$ are incompatible, so are $\Phi(\{v\}) = \{\psi(v)\}$ and $\Phi(t)$, and thus $\psi(v)$ is a neighbor of $\Phi(t)$. Let $w \in V$ be such that $\psi(w)$ is a neighbor of $\psi(v)$ in $\Phi(t)$. If $w \notin t$ then it is incompatible with $t \cup \{v\}$, and thus $\Phi(\{w\}) = \{\psi(w)\}$ is incompatible with $\Phi(t \cup \{v\})$. Therefore, $\Phi(t \cup \{v\})$ is adjacent to and does not contain $\psi(w)$ which is in $\Phi(t)$. Since $|\Phi(t \cup \{v\})| = |t| + 1 = |\Phi(t)| + 1$, this implies that $\Phi(t \cup \{v\})$ is incompatible with $\Phi(t)$, a contradiction. Therefore, we know...
that \(w \in t\). Let \(t_1, \ldots, t_k\) denote the connected components of \(G[t \setminus \{w\}]\). By induction hypothesis, \(\Phi(t_i) = \Psi(t_i)\) for all \(i \in [k]\). Moreover, since \(\Phi(t_i)\) is compatible with \(\Phi(t)\) and adjacent to \(\psi(w) \in \Phi(t)\), it is included in \(\Phi(t)\). We thus obtain \(\Psi(t) = \{\psi(w)\} \cup \Psi(t_1) \cup \cdots \cup \Psi(t_k) \subseteq \Phi(t)\) and thus \(\Phi(t) = \Psi(t)\) since \(|\Phi(t)| = |t| = |\Psi(t)|\).

\[\square\]

**Lemma 65.** If \(\Phi\) does not maintain a tube \(t\) of \(G\) (i.e., \(|\Phi(t)| \neq |t|\)), then

(i) \(\Phi\) swaps any tube of \(G\) containing \(t\),

(ii) \(\Phi\) maintains any tube of \(G\) disjoint from and non-adjacent to \(t\), and

(iii) \(\Phi\) swaps at least one singleton included in \(t\).

**Proof.** Consider a tube \(s\) of \(G\) strictly containing \(t\). The link of \([s, t]\) in \(\mathcal{N}(G)\) is isomorphic to the nested complex of the union of the graphs \(G[s]\) and \(G\) with \(|V| - |s|\) and \(|V| - |s|\) vertices respectively. Therefore, Corollary 62 ensures that the link of \(\{\Phi(s), \Phi(t)\}\) in \(\mathcal{N}(G')\) is isomorphic to the nested complex of a graph with three connected components with \(|s| - |t|\) and \(|V| - |s|\) vertices respectively. If \(\Phi(s)\) is not contained in \(\Phi(t)\), then the link of \(\{\Phi(s), \Phi(t)\}\) in \(\mathcal{N}(G')\) would be isomorphic to the nested complex of a graph with one connected component \(G'(|\Phi(t)|)\) having \(|\Phi(t)|\) vertices. We reach a contradiction as \(|\Phi(t)| = |V| - |t|\) is neither \(|t|\) (by assumption on \(t\)), nor \(|V| - |s|\) (since \(|s| > |t|\)), nor \(|s| - |t|\) (since \(|s| < |V|\)). Therefore, \(\Phi(s)\) is contained in \(\Phi(t)\) and the link of \(\{\Phi(s), \Phi(t)\}\) in \(\mathcal{N}(G')\) is isomorphic to the union of the graphs \(G[\Phi(s)]\), \((G'\Phi(s))\Phi(t)\setminus\Phi(s)\), and \(G'\Phi(t)\) with \(|\Phi(s)|\), \(|\Phi(t)| - |\Phi(s)|\) and \(|V| - |\Phi(t)|\) vertices respectively. If \(|\Phi(s)| \neq |V| - |s|\), then it forces \(|\Phi(s)| = |s| - |t| = |s|\), a contradiction. This proves (i).

Consider now a tube \(s\) of \(G\) disjoint from and non-adjacent to \(t\). Note that \(|s| + |t| < |V|\) as there is at least a vertex separating them. The link of \([s, t]\) in \(\mathcal{N}(G)\) is isomorphic to the nested complex of the union of the graphs \(G[s]\), \(G\) with \(|s|, |t|\) and \(|V| - |s| - |t|\) vertices respectively. Therefore, Corollary 62 ensures that the link of \(\{\Phi(s), \Phi(t)\}\) in \(\mathcal{N}(G')\) is isomorphic to the nested complex of a graph with three connected components with \(|s|, |t|\) and \(|V| - |s| - |t|\) vertices respectively. If \(\Phi(s)\) is not contained in \(\Phi(t)\), then the link of \(\{\Phi(s), \Phi(t)\}\) in \(\mathcal{N}(G')\) would be isomorphic to the nested complex of a graph with one connected component \(G'[\Phi(t)]\) having \(|\Phi(t)|\) vertices. We reach a contradiction as \(|\Phi(t)| = |V| - |t|\) is neither \(|t|\) (by assumption on \(t\)), nor \(|s|\) (since \(|s| + |t| < |V|\)), nor \(|V| - |s| - |t|\) (since \(|s| > 0\)). Therefore, \(\Phi(s)\) is contained in \(\Phi(t)\) and the link of \(\{\Phi(s), \Phi(t)\}\) in \(\mathcal{N}(G')\) is isomorphic to the union of the graphs \(G[\Phi(s)]\), \((G'\Phi(s))\Phi(t)\setminus\Phi(s)\), and \(G'\Phi(t)\) with \(|\Phi(s)|\), \(|\Phi(t)| - |\Phi(s)|\) and \(|V| - |\Phi(t)|\) vertices respectively. If \(|\Phi(s)| \neq |s|\), then it forces \(|\Phi(s)| = |V| - |s| - |t| = |V| - |s|\), a contradiction. This proves (ii).

Finally, to prove (iii) we can assume that \(t\) is not a singleton. Thus \(\Phi(t)\) is not an inclusion maximal tube. Let \(t'\) be a tube of \(G\) such that \(\Phi(t')\) is a maximal tube of \(G'\) containing \(\Phi(t)\). Since \(\Phi^{-1}\) swaps \(\Phi(t)\) and \(\Phi(t')\) contains \(\Phi(t)\), \(\Phi^{-1}\) also swaps \(\Phi(t')\) by (i). Thus, \(t'\) is a singleton swapped by \(\Phi\) and contained in \(t\).

\[\square\]

**Lemma 66.** Denote by \(M := \{v \in V \mid |\Phi\{\{v\}\}| = 1\}\) the set of vertices maintained by \(\Phi\) and by \(S := \{v \in V \mid |\Phi\{\{v\}\}| = |V| - 1\}\) the set of points swapped by \(\Phi\). Then

(i) \(S\) forms a clique of \(G\),

(ii) any vertex in \(M\) has at most one neighbor in \(S\), and

(iii) any vertex in \(S\) has at most one neighbor in \(M\).

**Proof.** Let \(s, s' \in S\). Since \(|\Phi\{\{s\}\}| = |\Phi\{\{s'\}\}| = |V| - 1\), the tubes \(\Phi\{\{s\}\}\) and \(\Phi\{\{s'\}\}\) are incompatible. Therefore \(\{s\}\) and \(\{s'\}\) are incompatible, so that \(s\) and \(s'\) are neighbors. The set \(S\) thus forms a clique.

To prove (ii) and (iii), assume that some vertices \(m \in M\) and \(s \in S\) are neighbors. The tubes \(\{m\}\) and \(\{s\}\) are thus incompatible, so that \(\Phi\{\{m\}\}\) and \(\Phi\{\{s\}\}\) are also incompatible. Since \(|\Phi\{\{m\}\}| = 1\) while \(|\Phi\{\{s\}\}| = |V| - 1\), this implies \(\Phi\{\{s\}\}\) and \(\Phi\{\{m\}\}\) are incompatible. It follows that \(m\) cannot have another neighbor swapped by \(\Phi\) and \(s\) cannot have another neighbor maintained by \(\Phi\).

We are now ready to prove that any non-trivial nested complex isomorphism coincides, up to composition with a trivial nested complex isomorphism, with the isomorphism \(\Omega\) on a spider.
Proof of Theorem 43. The proof works by induction on the number \(|V|\) of vertices of \(G\). It is clear when \(|V| \leq 2\). For the induction step, assume that the result holds for all graphs on less than \(|V|\) vertices and consider a non-trivial nested complex isomorphism \(\Phi : \mathcal{N}(G) \to \mathcal{N}(G')\). Then \(\Phi\) does not maintain all tubes of \(G\) by Proposition 64, and thus swaps at least one singleton \(\{s\}\) by Lemma 65 (iii). Let \(s'\) denote the vertex of \(G'\) such that \(\Phi(\{s\}) = V' \setminus \{s'\}\).

The map \(\Phi\) induces a nested complex isomorphism between the link of \(\{s\}\) in \(\mathcal{N}(G)\) and the link of \(\Phi(\{s\})\) in \(\mathcal{N}(G')\). The former is isomorphic to the nested complex of the connected complement \(\tilde{G} := G'\Phi(\{s\})\) while the latter is isomorphic to the nested complex of the restriction \(\mathcal{N}(G)\Phi(\{s\})\). Let \(\tilde{\Phi} : \tilde{t} \mapsto \Phi(t)\) denote the resulting nested complex isomorphism between \(\mathcal{N}(G)\) and \(\mathcal{N}(G')\). This isomorphism \(\tilde{\Phi}\) is non-trivial: by Lemma 65, \(\Phi\) swaps any tube \(t\) containing \(\{s\}\), so that \(\tilde{\Phi}\) swaps the tube \(t \setminus \{s\}\). It follows by induction hypothesis that \(\tilde{G}\) and \(G'\) are spiders and that there exists a graph isomorphism \(\tilde{\psi} : \tilde{G} \to G'\) inducing a trivial nested complex isomorphism \(\Psi : \mathcal{N}(G) \to \mathcal{N}(G')\) such that \(\Psi = \Phi = \tilde{\Phi}\) is the automorphism of \(\mathcal{N}(G)\) described in Section 5.2. In other words, we can label by \(v^i_j\) the vertices of the spider \(\tilde{G}\) and by \(v^i_j\) the vertices of the spider \(G'\), with \(i \in [\tilde{\ell}]\) and \(0 \leq j \leq \tilde{n}_i\), such that

\[
\tilde{\Phi}(v^i_{\tilde{\ell}}, v^i_{\tilde{\ell}}) = [v^i_{\tilde{n}_i+1-k}, v^i_{\tilde{n}_i+1-j}] \quad \text{and} \quad \tilde{\Phi} \left( \bigcup_{i \in [\tilde{\ell}]} [v^0_i, v^i_{\tilde{n}_i}] \right) = \bigcup_{i \in [\tilde{\ell}]} [v^0_i, v^i_{\tilde{n}_i-1-k_i}].
\]

We now claim that \(G\) and \(G'\) are both spiders. To prove it, we distinguish two cases:

**Body case:** all neighbors of \(s\) in \(G\) are swapped by \(\Phi\). Then they form a clique in \(G\) (by Lemma 66 (i)), so that \(G\) is the spider \(\tilde{G}\) where we add one more body vertex \(s\) with no attached leg. Moreover, \(s'\) is necessarily swapped by \(\Phi^{-1}\) (otherwise \(\Phi^{-1}(\{s'\})\) would be a neighbor of \(s\) maintained by \(\Phi\)). We conclude by symmetry that \(G'\) is the spider \(\tilde{G}'\) where we add one more body vertex \(s'\) with no attached leg.

**Leg case:** \(s\) has a neighbor \(m\) maintained by \(\Phi\). It is unique by Lemma 66 (iii) and not connected to any other vertex swapped by \(\Phi\) by Lemma 66 (ii). Therefore, \(G\) is the spider \(\tilde{G}\) where we replace the edges connecting \(m\) to all other body vertices of \(G\) by a new body vertex \(s\) with an edge to \(m\). Moreover, \(s'\) is necessarily maintained by \(\Phi^{-1}\) (otherwise \(\Phi^{-1}(\{s'\})\) should be \(V \setminus \{s\}\) which is not connected). We conclude that \(G'\) is the spider \(\tilde{G}'\) where we add one additional leg vertex \(s'\) to the free endpoint of a leg.

We now label by \(v^i_j\) the vertices of \(G\) according to the labels \(v^i_j\) of \(\tilde{G}\) and by \(v^i_j\) the vertices of \(G'\) according to the labels \(v^i_j\) of \(\tilde{G}'\). We follow the two cases above:

**Body case:** We set \(\ell := \tilde{\ell} + 1, n_i := \tilde{n}_i\) for \(i \in [\tilde{\ell}]\) and \(n_\tilde{\ell} = 0\). For any \(i \in [\tilde{\ell}]\) and \(0 \leq j \leq \tilde{n}_i\), we label by \(v^i_j\) the vertex of \(G\) corresponding to the vertex labeled by \(\tilde{v}^i_j\) in \(\tilde{G}\), and similarly we label by \(v^i_j\) the vertex of \(G'\) corresponding to the vertex labeled by \(\tilde{v}^i_j\) in \(\tilde{G}'\). Finally, we label \(s\) by \(v^0_0\) and \(s'\) by \(v^0_{\tilde{n}_i}\).

**Leg case:** Assume that the neighbor of \(s\) maintained by \(\Phi\) corresponds to the vertex labeled by \(\tilde{v}^a_{\tilde{n}_a}\) in \(\tilde{G}'\). We set \(\ell := \tilde{\ell}, n_a := \tilde{n}_a + 1\) for \(i \in [\tilde{\ell}]\) and \(n_a := \tilde{n}_a + 1\). For any \(i \in [\tilde{\ell}]\) and \(0 \leq j \leq \tilde{n}_a\), we label by \(v^i_j\) if \(i \neq a\) and \(v^i_j+1\) if \(i = a\) the vertex of \(G\) corresponding to the vertex labeled by \(\tilde{v}^i_j\) in \(\tilde{G}\), and by \(v^i_j\) the vertex of \(G'\) corresponding to the vertex labeled by \(\tilde{v}^i_j\) in \(G'\). Finally, we label \(s\) by \(v^0_0\) and \(s'\) by \(v^0_{\tilde{n}_a}\).

By our previous description of the graphs \(G\) and \(G'\), these labelings are indeed valid labelings of spiders, meaning that the edges of \(G\) are indeed given by \(\{v^i_j, v^i_{j+1}\} \mid i \in [\tilde{\ell}], j \in [n_i]\} \cup \{v^0_i, v^i_j\} \mid i \neq i' \in [\tilde{\ell}]\), and similarly for \(G'\). We moreover claim that \(\Phi\) is given by

\[
\Phi([v^i_j, v^i_k]) = [v^i_{n_i+1-k}, v^i_{n_i+1-j}] \quad \text{and} \quad \Phi \left( \bigcup_{i \in [\tilde{\ell}]} [v^0_i, v^i_{n_i}] \right) = \bigcup_{i \in [\tilde{\ell}]} [v^0_i, v^i_{n_i-1-k_i}].
\]

It is immediate for all tubes compatible with \(\{s\} = \{v^0_0\}\) as it is easily transported from (5). Therefore, we only have to check it for the tubes of \(G\) adjacent to \(s = v^0_0\) and not containing it. Observe
first that $\Phi([v^a_1, v^a_k])$ is a tube with $k$ vertices (by Lemma 65 (iii)), and that it contains $s' = v^a_{n_a}$ since it has to be incompatible with $\Phi(s) = V' \setminus \{s'\}$. Therefore, $\Phi([v^a_1, v^a_k]) = [v^a_{n_a+1-k}, v^a_{n_a}]$. Consider now a tube $t = \bigcup_{i \in [k]} [v^a_0, v^a_k]$ not containing $s = v^a_0$ (i.e., with $k_a = -1$). Since the nested tubes $t$ and $t \cup \{s\}$ are both swapped, we have $\Phi(t) = \Phi(t \cup \{s\}) \cup \{s'\}$. Since $\Phi(t \cup \{s\})$ is given by Equation (6), so is $\Phi(t)$. This concludes the proof that $\Phi$ is given by Equation (6), so that it coincides with $\Omega$ up to the graph automorphism defined by $v_j \mapsto v^a_j$. \hfill $\square$

6.6. Polytopality of compatibility fans (Theorem 51 and Proposition 53). This section provides the proof of the polytopality results presented in Section 5.3. Using a similar method as [CSZ11, Section 5] based on Proposition 49, we first prove that all compatibility and dual compatibility fans of paths and cycles are polytopal.

Proof of Theorem 51. We use the characterization of polytopality of complete simplicial fans given in Proposition 49. For this, we need to understand better the linear dependences on compatibility vectors for paths and cycles.

Consider first the case of the path. When $(T \cup T') \cap T^\circ = \emptyset$, the linear dependences can only be of the form

$$d(T^\circ, t) + d(T^\circ, t') = d(T^\circ, t) + d(T^\circ, t'),$$
$$d(T^\circ, t) + d(T^\circ, t') = d(T^\circ, a) + d(T^\circ, a'),$$
$$2d(T^\circ, t) + d(T^\circ, t') = d(T^\circ, a) + d(T^\circ, a'),$$
$$2d(T^\circ, t) + d(T^\circ, t') = d(T^\circ, a) + d(T^\circ, a'),$$

up to exchanging simultaneously $t$ with $t'$ and $a$ with $a'$. If $T \cap T'$ contains a tube $t^o \in T^o$, then the compatibility degree of all tubes of $(T \cup T') \setminus \{t^o\}$ with $t^o$ vanishes, so that the tube $t^o$ cannot appear in the linear dependence. When $t, t' \not\in T^o$ but $(T \cap T') \cap T^o \neq \emptyset$, the relations are thus obtained from the ones above by deleting terms in their right hand sides. The dependences when $t$ or $t'$ belong to $T^o$ will be treated separately.

We now define a height function $\omega$ on tubes on $P_{n+1}$ by

$$\omega(t) = \begin{cases} f(|t|) & \text{if } t \not\in T^o, \\ \Omega & \text{otherwise,} \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}_{>0}$ is any strictly concave increasing positive function and $\Omega \in \mathbb{R}$ is a large enough constant. When $(T \cup T') \cap T^o = \emptyset$, we obtain by definition of $T$, $t$, $a$ and $a'$, and using that $f$ is concave and increasing, that

$$\omega(t) + \omega(t') > \omega(T) + \omega(t),$$
$$\omega(t) + \omega(t') > \omega(a) + \omega(a'),$$
$$2\omega(t) + \omega(t') > \omega(T) + \omega(a),$$
$$2\omega(t) + \omega(t') > \omega(T) + \omega(a').$$

Moreover, the inequalities still hold when we delete terms in their right hand sides since $\omega$ is positive. Therefore, $\omega$ satisfies Condition (2) of Proposition 49 when we do not flip an initial tube. Finally, initial tubes only appear in the left hand sides of linear dependences, so choosing $\omega(t^o) = \Omega$ large enough ensures that $\omega$ satisfies Condition (2) of Proposition 49 for any flip. Observe that this is essentially the same proof as in [CSZ11, Section 5].

We now adapt this proof for the cycle $O_{n+1}$. Clearly, the dependences described above for the path also appear for the cycle (as cycles contain paths). Beside those, when $(T \cup T') \cap T^o = \emptyset$, a straightforward case analysis shows that the linear dependences can only be of the form

$$d(T^o, t) + d(T^o, t') = 2d(T^o, t_1), \quad \text{where } t_1 \in t.$$
above by deleting terms in their right hand sides. The dependences when \( t \) or \( t' \) belong to \( T^\circ \) will again be treated separately.

We choose the same height function \( \omega \) as before. For the same reasons, the linear dependences for the path are again transformed to strict inequalities on \( \omega \). Moreover, as \( t_1 \subseteq t \cap t' \) and \( f \) is increasing, we have

\[
\omega(t) + \omega(t') > 2\omega(t_1).
\]

We conclude as before by choosing \( \Omega \) large enough that \( \omega \) satisfies the Condition (2) of Proposition 49 for any flip.

Finally, for dual compatibility vectors, a straightforward case analysis shows that the linear dependences are all of the form

\[
d^*(t, T^\circ) + d^*(t', T^\circ) = d^*(\bar{t}, T^\circ) + d^*(\bar{t}', T^\circ),
\]

\[
d^*(t, T^\circ) + d^*(t', T^\circ) = d^*(a, T^\circ) + d^*(a', T^\circ),
\]

\[
2d^*(t, T^\circ) + d^*(t', T^\circ) = d^*(\bar{t}, T^\circ) + d^*(a, T^\circ),
\]

\[
2d^*(t, T^\circ) + d^*(t', T^\circ) = d^*(\bar{t}, T^\circ) + d^*(a', T^\circ),
\]

when none of \( t, t' \) and \( \bar{t} \) have \( n \) vertices. We can also have the linear dependences

\[
d^*(t, T^\circ) + d^*(t', T^\circ) = 2d^*(\bar{t}, T^\circ) + d^*(\bar{t}', T^\circ),
\]

\[
2d^*(t, T^\circ) + d^*(t', T^\circ) = 2d^*(\bar{t}, T^\circ) + d^*(a, T^\circ),
\]

when \( |\bar{t}| = n \) and

\[
d^*(t, T^\circ) + d^*(t', T^\circ) = d^*(\bar{t}_1, T^\circ), \quad \text{where } \bar{t}_1 \in \bar{t}.
\]

when \( |t| = |t'| = n \). Again, no tube of \( T^\circ \) can appear in the right hand sides of the linear dependences. Therefore, when \( t, t' \notin T^\circ \) but \( (T \cap T') \cap T^\circ \neq \emptyset \), the linear dependences are obtained from the generic ones above by deleting terms in their right hand sides.

We now define a height function \( \omega \) on tubes on \( O_{n+1} \) by

\[
\omega(t) = \begin{cases} 
\frac{f(|t|)}{2} & \text{if } t \notin T^\circ \text{ and } |t| \neq n, \\
\Omega & \text{otherwise,}
\end{cases}
\]

where \( f : \mathbb{R} \to \mathbb{R}_{>0} \) is any strictly concave increasing positive function and \( \Omega \in \mathbb{R} \) is a large enough constant. By definition of \( \bar{t}, \bar{t}_1, a \) and \( a' \), and using that \( f \) is concave and increasing, we obtain that \( \omega \) satisfies a strict inequality for each linear dependence above. We conclude as before by choosing \( \Omega \) large enough that \( \omega \) satisfies the Condition (2) of Proposition 49 for any flip.

Our last proof concerns the polytopality of the compatibility fan for the star, for which we have presented a candidate in Section 5.3.

Proof of Proposition 53. We thus just have to show that for any tube \( t \) and any maximal tubing \( T \) on \( X_{n+1} \), the point \( x(T) \) belongs to the half-space \( H^\geq(t) \) and to the boundary of this half-space if and only if \( t \in T \).

Consider first a tube \( t \) not in \( T^\circ \). Let \( \bar{t} \) denote the inclusion minimal tube of \( T \cup V \) containing the central vertex \( \ast \). Then the other tubes of \( T \) are all leaves of \( X_{n+1} \) contained in \( \bar{t} \) and a nested chain of tubes \( t = t_{|t|} \subseteq t_{|t|+1} \subseteq \cdots \subseteq t_{n+1} = V \) of \( X_{n+1} \). Therefore, we have \( x(T)_i = 0 \) if \( \{\ell_i\} \subseteq \bar{t} \) and \( x(T)_i = j - 1 \) if \( \{\ell_i\} \vdash t_j \setminus t_{j-1} \). We conclude that

\[
\langle d(T^\circ, t) \mid x(T) \rangle = \sum_{i \in [n]} x(t)_i = \sum_{\substack{|t| \leq j \leq n+1 \\bar{t}_j \vdash t_{j-1} \in \bar{t}}} \sum_{j=|t|}^{n} (j - 1) \leq \sum_{j=|t|}^{n} j = f(|t|),
\]

with equality if and only if \( t_j \setminus t_{j-1} \not\subseteq t \) for all \( |t| \leq j \leq n+1 \), i.e. if and only if \( t = t_{|t|} \).

Finally, for any \( i \in [n] \), we have \( \langle d(T^\circ, \{\ell_i\}) \mid x(T) \rangle = -x(T)_i \leq 0 \), with equality if and only if the inclusion minimal tube of \( T \cup V \) containing \( i \) is \( \{\ell_i\} \), i.e. if and only if \( \{\ell_i\} \in T \).
References

[BFS90] Louis J. Billera, Paul Filliman, and Bernd Sturmfels. Constructions and complexity of secondary polytopes. Adv. Math., 83(2):155–179, 1990.

[BHLT99] N. Bergeron, C. Hohlweg, C. Lange, and H. Thomas. Isometry classes of generalized associahedra. Sémin. Lothar. Combin., 61A:Art. B61Aa, 15, 2009.

[CD06] Michael P. Carr and Satyan L. Devadoss. Coxeter complexes and graph-associahedra. Topology Appl., 153(12):2155–2168, 2006.

[CFZ02] Frédéric Chapoton, Sergey Fomin, and Andrei Zelevinsky. Cluster algebras. I. Foundations. J. Amer. Math. Soc., 15(2):409–455, 2002.

[CFZ03a] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. II. Finite type classification. Invent. Math., 154(1):63–121, 2003.

[CFZ03b] Sergey Fomin and Andrei Zelevinsky. Cluster algebras. III. Cluster complexes. Compos. Math., 133(1):108–160, 2002.

[CP14] Grégory Chatel and Vincent Pilaud. Cambrian Hopf Algebras. Preprint, arXiv:1411.3704, 2014.

[CDP95] C. De Concini and C. Procesi. Wonderful models of subspace arrangements. Selecta Math. (N.S.), 2(1):249–286, 1995.

[Deh10] Patrick Dehornoy. On the rotation distance between binary trees. Adv. Math., 223(4):1316–1355, 2010.

[Dev09] Satyan L. Devadoss. A realization of graph associahedra. Pacific J. Math., 238(2):423–442, 2009.

[DRS10] Jesus A. De Loera, Jörg Rambau, and Francisco Santos. Triangulations: Structures for Algorithms and Applications, volume 25 of Algorithms and Computation in Mathematics. Springer Verlag, 2010.

[Hai84] Mark Haiman. Constructing the associahedron. Unpublished manuscript, 11 pages, available at http://www.math.berkeley.edu/~mhaiman/ftp/assoc/manuscript.pdf, 1984.

[HAI97] Christophe Hohlweg and Carsten Lange. Realizations of the associahedron and cyclohedron. Discrete Comput. Geom., 37(4):517–543, 2007.

[HLLR10] Christophe Hohlweg, Jonathan Lortie, and Annie Raymond. The centers of gravity of the associahedron and of the permutohedron are the same. Electron. J. Combin., 17(1):Research Paper 72, 14, 2010.

[HLT11] Christophe Hohlweg, Carsten Lange, and Hugh Thomas. Permutohedra and generalized associahedra. Adv. Math., 226(1):608–640, 2011.

[HNT05] Ferran Hurtado and Marc Noy. Graph of triangulations of a convex polygon and tree of triangulations. Comput. Geom., 33(1):19–33, 2005.

[HNT06] Florent Hivert, Jean-Christophe Novelli, and Jean-Yves Thibon. The algebra of binary search trees. Theoret. Comput. Sci., 359(1):129–157, 2006.

[Hoh12] Christophe Hohlweg. Permutohedra and associahedra. In Folkert Müller-Hoissen, Jean Pallo, and Jim Stasheff, editors, Associahedra, Tamari Lattices and Related Structures − Tamari Memorial Festschrift, volume 299 of Progress in Mathematics, pages 129–159. Birkhäuser, 2012.

[Lee89] Carl W. Lee. The associahedron and triangulations of the n-gon. European J. Combin., 10(6):541–560, 1989.

[Lod04] Jean-Louis Loday. Realization of the Stasheff polytope. Arch. Math. (Basel), 83(3):267–278, 2004.

[LP13] Carsten Lange and Vincent Pilaud. Using spines to revisit a construction of the associahedron. Preprint, arXiv:1307.4391, 2013.

[LR98] Jean-Louis Loday and María O. Ronco. Hopf algebra of the planar binary trees. Adv. Math., 139(2):293–309, 1998.

[MHPS12] Folkert Müller-Hoissen, Jean Marcel Pallo, and Jim Stasheff, editors, Associahedra, Tamari Lattices and Related Structures. Tamari Memorial Festschrift, volume 299 of Progress in Mathematics. Birkhäuser, Basel, 2012.

[OEIS] The On-Line Encyclopedia of Integer Sequences. Published electronically at http://oeis.org, 2010.

[Pil13] Vincent Pilaud. Signed tree associahedra. Preprint, arXiv:1309.5222, 2013.
[Pos09] Alexander Postnikov. Permutohedra, associahedra, and beyond. *Int. Math. Res. Not.* IMRN, (6):1026–1106, 2009.

[Pou14] Lionel Pournin. The diameter of associahedra. *Adv. Math.*, 259:13–42, 2014.

[PS12] Vincent Pilaud and Francisco Santos. The brick polytope of a sorting network. *European J. Combin.*, 33(4):632–662, 2012.

[PS13] Vincent Pilaud and Christian Stump. Vertex barycenter of generalized associahedra. To appear in *Proc. Amer. Math. Soc.*, arXiv:1210.3314, 2013.

[PS15] Vincent Pilaud and Christian Stump. Brick polytopes of spherical subword complexes and generalized associahedra. To appear in *Adv. Math.*, arXiv:1111.3349, 2015.

[Rea04] Nathan Reading. Lattice congruences of the weak order. *Order*, 21(4):315–344 (2005), 2004.

[Rea06] Nathan Reading. Cambrian lattices. *Adv. Math.*, 205(2):313–353, 2006.

[Rea14] Nathan Reading. Universal geometric cluster algebras. *Math. Z.*, 277(1-2):499–547, 2014.

[RS09] Nathan Reading and David E. Speyer. Cambrian fans. *J. Eur. Math. Soc. (JEMS)*, 11(2):407–447, 2009.

[RSS03] Günter Rote, Francisco Santos, and Ileana Streinu. Expansive motions and the polytope of pointed pseudo-triangulations. In *Discrete and computational geometry*, volume 25 of *Algorithms Combin.*, pages 699–736. Springer, Berlin, 2003.

[Sta63] Jim Stasheff. Homotopy associativity of H-spaces I, II. *Trans. Amer. Math. Soc.*, 108(2):293–312, 1963.

[Ste13] Salvatore Stella. Polyhedral models for generalized associahedra via Coxeter elements. *J. Algebraic Combin.*, 38(1):121–158, 2013.

[STT88] Daniel D. Sleator, Robert E. Tarjan, and William P. Thurston. Rotation distance, triangulations, and hyperbolic geometry. *J. Amer. Math. Soc.*, 1(3):647–681, 1988.

[Tam51] Dov Tamari. *Mono¨ıdes pr´eordonn´es et chaˆınes de Malcev*. PhD thesis, Universit´e Paris Sorbonne, 1951.

[Vol10] V. D. Volodin. Cubical realizations of flag nestohedra and a proof of Gal’s conjecture for them. *Uspekhi Mat. Nauk*, 65(1(391)):183–184, 2010.

[Zel06] Andrei Zelevinsky. Nested complexes and their polyhedral realizations. *Pure Appl. Math. Q.*, 2(3):655–671, 2006.

[Zie95] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.