Quantum conformal mechanics

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Abstract

The quantum mechanics of one degree of freedom exhibiting the exact conformal $SL(2, \mathbb{R})$ symmetry is presented. The starting point is the classification of the unitary irreducible representations of the $SL(2, \mathbb{R})$ group (or, to some extent, its universal covering). The coordinate representation is defined as the basis diagonalizing the special conformal generator $\hat{K}$. It is indicated how the resulting theory emerges from the canonical/geometric quantization of the Hamiltonian dynamics on the relevant coadjoint orbits.

1 Introduction

The $SL(2, \mathbb{R})$ group provides the prototype of conformal groups. It describes conformal transformations of $(1 + 0)$-dimensional space-time; the generators of $SL(2, \mathbb{R})$ corresponds to time translations, dilatations and special conformal transformations. The dynamical model exhibiting such a symmetry, the so called conformal mechanics, has been introduced in Refs. [1] and [2] (for their supersymmetric extensions see [3, 4]). Since then there appeared number of papers devoted to the detailed study of the conformal mechanics [3]-[37], both on classical and quantum levels.

The basic quantities entering the description of conformal mechanics are the Hamiltonian $H$, the dilatation generator $D$ and the generator of conformal transformations $K$:

$$H = \frac{p^2}{2m} + \frac{g^2}{2x^2},$$

$$D = -\frac{1}{2}xp,$$

$$K = \frac{m}{2}x^2,$$
where $x$ and $p$ are standard canonical variables, $m$ – the mass of a particle (which we will put equal to one in what follows) and the coupling constant $g$. They obey $SL(2, \mathbb{R})$ commutations rules with respect to the standard Poisson brackets

$$\{D, H\} = -H, \quad \{D, K\} = K, \quad \{K, H\} = -2D.$$  (1.4)

This structure survives on the quantum level provided $D$ is ordered appropriately. The value of the Casimir operator can be expressed in terms of the coupling constant as follows

$$C = HK - D^2 = \frac{g^2}{4},$$  (1.5)

or on the quantum level ($\hbar = 1$),

$$\hat{C} = \frac{1}{2}(\hat{H}\hat{K} + \hat{K}\hat{H}) - \hat{D}^2 = \frac{g^2}{4} - \frac{3}{16}.$$  (1.6)

The main restriction usually imposed on the Hamiltonian $H$ is that the potential is repulsive, $g^2 > 0$. Then the phase space $P$ is a half-plane $\{(x, p)|x \in \mathbb{R}_+, p \in \mathbb{R}\}$. This phase space is complete in the sense that, given any initial point, $(x(0), p(0)) \in P$, the whole trajectory $(x(t), p(t)) \in P$, $-\infty < t < \infty$, belongs to $P$. This is no longer the case for the non-positive coupling constant. If $g = 0$ we have the free motion so the natural candidate for the phase space is the whole plane. The situation gets even more complicated when $g$ becomes negative. The singularity of the potential at $x = 0$ defines the boundary of the phase space which again becomes the half-plane $x > 0, p \in \mathbb{R}$. However, it is now no longer complete due to the ”falling on the center” phenomena: the particle can reach the boundary of phase space at finite time.

This trouble has its counterpart on the quantum level. Upon quantization the Hamiltonian becomes the differential operator of the second order. Actually, it is a formal differential expression which becomes meaningful provided it defines a self-adjoint operator acting in the Hilbert space of states. To this end one has to impose the appropriate boundary conditions yielding such an operator \[38\]-\[41\]. It appears that in many cases (including the one considered here) such a procedure is not unique. In fact, for the conformal Hamiltonian (1.1) one arrives at the following classification \[42\]:

- for $g^2 \geq \frac{3}{4}$ there exists only one self-adjoint Hamiltonian (1.1); the energy spectrum is purely continuous and extends over the whole nonnegative semiaxis $E \geq 0$;

- for $\frac{3}{4} > g^2 > -\frac{1}{4}$ there exists a one-parameter family of self-adjoint Hamiltonians; the physical interpretation of this fact in terms of regularized cut-off potential is discussed in the classical Landau and Lifschitz book \[43\]; however, such an interpretation does not cover all possibilities \[42\]. In the case under consideration the continuous part of the energy spectrum again extends over the whole nonnegative semiaxis. However, for certain self-adjoint extensions there exists an additional single bound state of nonnegative energy (surprisingly enough, this can happen even in the region $\frac{3}{4} > g^2 \geq 0$, i.e., for the repulsive potential);
For $g^2 = -\frac{1}{4}$ again there exists a one-parameter family of self-adjoint Hamiltonians. The spectrum consists of continuous part $E \geq 0$ together with a single bound state of negative energy.

- For $g^2 < -\frac{1}{4}$ there exists also a one parameter family. The continuous spectrum consists of all nonnegative energies; there exists an infinite family of negative energy bound states concentrating exponentially to zero and going exponentially to $-\infty$.

In all cases, except the first one, the self-adjoint Hamiltonian is defined by dimensionful parameter which sets the scale of the bound state(s). Therefore, the scale symmetry must be broken as well as the conformal one. Let us note that it is not a priori clear whether their generators can be defined as genuine self-adjoint operators obeying $sl(2, \mathbb{R})$ algebra except in formal sense and whether the resulting algebra can be integrated to yield the representation of the (covering) of the $SL(2, \mathbb{R})$ group. The question arises whether the exists a quantum mechanical model carrying genuine (i.e., unbroken) $SL(2, \mathbb{R})$ symmetry. The answer seems to be positive for strong enough repelling potentials. In this case both on the classical and quantum level we are dealing with conformal invariant systems. However, in other cases the situation becomes more complicated; classically, the accessible phase space becomes incomplete in the sense described above while quantum mechanically trouble arises with scale invariant definitions of self-adjoint group generators.

In the present paper we construct fully conformally invariant quantum mechanics for all values of the coupling constant. The starting point is the construction of conformal mechanics as described in Refs. [32] and [36] where the general and elegant method of coadjoint orbits, which allows to define Hamiltonian dynamics invariant under a transitive action of a given symmetry group, has been applied. The main conclusion following from the construction presented there is that, at least on the classical level, the troubles arising in the case of nonrepelling potential are the artifacts arising due to the nontrivial topology of the phase space. Once this is properly recognized the motion becomes completely regular and the singularity at $x = 0$ appears to be completely spurious as resulting from the fact that the phase space cannot be covered by a single map. So, on the classical level we are dealing with perfectly regular (in fact, rather simple) dynamical system. The only inconvenience is that, in general, there exists no globally defined system of Darboux coordinates.

Our aim in the present paper is to quantize the classical dynamics defined with the help of the orbits method. We show that for all values of the coupling constant one can find the relevant quantum mechanical system exhibiting exact $SL(2, \mathbb{R})$ conformal symmetry. Its Hilbert space of states spans an irreducible unitary representation of the $SL(2, \mathbb{R})$ group (or its universal covering). As it is known [44]-[47] the irreducible unitary representations of $SL(2, \mathbb{R})$ can be classified as follows. First, there exists the continuous serie characterized by the pairs $(\rho, \epsilon)$, where $\epsilon = 0, 1; \rho \in \mathbb{R}$ for $\epsilon = 0$ while $\rho \in \mathbb{R} \setminus \{0\}$ for $\epsilon = 1$; two such representations $(\rho, \epsilon), (\rho', \epsilon')$ are equivalent if $\epsilon = \epsilon'$ and $\rho = \pm \rho'$. Second, there are discrete series characterized by the integers $m \geq 1$; they are all inequivalent. Further, there is the supplementary serie indexed by $\rho \in (-1, 1)$,
\(\rho \neq 0\); again the representations corresponding to \(\rho\) and \(-\rho\) are unitary equivalent. Finally, there are two mock representations which can be viewed as the formal limits \(m \to 0^+\) of the discrete ones or as two irreducible components of the representation \((0,1)\) of the continuous serie.

All these representations are used as the building blocks in our construction. Due to the equation (1.6) one can relate the values of coupling constants and Casimir operators which allows to identify the representations arising for particular values of coupling. As a result the following picture emerges:

![Figure 1](image)

In all cases the full conformal symmetry is preserved.

The paper is organized as follows. In Section 2 we remind briefly the results obtained in Refs. [32] and [36] concerning the classical \(SL(2,\mathbb{R})\) dynamics. Section 3 deals with discrete series. The continuous series are considered in Section 4. Section 5, 6 and 7 are devoted to the intermediate interval \(-\frac{1}{4} \leq g^2 \leq \frac{3}{4}\) of the coupling constant values as well as some special cases. Finally, Section 8 contains some conclusions.

## 2 Classical \(SL(2,\mathbb{R})\)-invariant systems

The \(SL(2,\mathbb{R})\) group is locally isomorphic to \(SO(2,1)\). This is easily seen by defining

\[
K = M_0 + M_1, \quad H = M_0 - M_1, \quad D = M_2. \tag{2.1}
\]

Then the \(so(2,1)\) algebra

\[
[M_\mu, M_\nu] = -i\varepsilon_{\mu\nu}^\alpha M_\alpha, \tag{2.2}
\]

(we adopt the conventions \(\varepsilon_{012} = 1, g_{\mu\nu} = \text{diag}(+,-,-)\)) becomes

\[
[D, H] = -iH, \quad [D, K] = iK, \quad [K, H] = -2iD. \tag{2.3}
\]

Let \(\xi_\alpha\) be the coordinates in the dual space to \(so(2,1)\); the coadjoint action of \(SL(2,\mathbb{R})\) (faithful action of \(SO(2,1)\)) reads

\[
\xi'_\alpha = (\Lambda^{-1})^\beta_\alpha \xi_\beta. \tag{2.4}
\]

The invariant (degenerate) Poisson structure reads

\[
\{\xi_\alpha, \xi_\beta\} = -\varepsilon_{\alpha\beta}^\gamma \xi_\gamma. \tag{2.5}
\]

There are three families of coadjoint orbits:
(i) upper ($\xi_0 > 0$) and lower ($\xi_0 < 0$) sheets of two-sheeted hyperboloids

$$\xi^\alpha \xi_\alpha = \lambda^2 > 0,$$

(ii) one-sheeted hyperboloids

$$\xi^\alpha \xi_\alpha = -\lambda^2 < 0,$$

(iii) forward ($\xi_0 > 0$) and backward ($\xi_0 < 0$) cones

$$\xi^\alpha \xi_\alpha = 0.$$

According to the general theory [48]-[52] the Poisson structure (2.5), when restricted to a coadjoint orbit, becomes nondegenerate yielding the symplectic manifold with invariant action of the $SL(2, \mathbb{R})$ group providing thus an invariant Hamiltonian formalism.

According to the eqs. (2.1) the generators of conformal algebra are represented by the following functions

$$H = \xi_0 - \xi_1, \quad K = \xi_0 + \xi_1, \quad D = \xi_2.$$  

The Hamiltonian equations of motion take the form

$$\dot{\xi}_\alpha = \{\xi_\alpha, H\},$$

and yield

$$\dot{\xi}_0 = -\xi_2, \quad \dot{\xi}_1 = -\xi_2, \quad \dot{\xi}_2 = \xi_1 - \xi_0.$$  

One arrives at regular dynamics which remains regular when restricted to the orbit.

In order to make contact with standard form of the Hamiltonian mechanics one looks for the Darboux coordinate. Let us first consider the case (i) (we take the upper sheet $\xi_0 > 0$ for convenience). The following transformation

$$\xi_0 = \frac{p^2}{4} + \frac{\lambda^2}{x^2} + \frac{x^2}{4},$$

$$\xi_1 = -\frac{p^2}{4} - \frac{\lambda^2}{x^2} + \frac{x^2}{4},$$

$$\xi_2 = -\frac{1}{2} xp,$$

maps the upper sheet onto the half plane $0 < x < \infty$, $-\infty < p < \infty$. The mapping is smooth, one-to-one and $(x, p)$ become the Darboux coordinates, $\{x, p\} = 1$. By comparing eqs. (1.1), (2.9) and (2.12) we find

$$g^2 = 4\lambda^2.$$  

Therefore, one arrives at the standard form of the conformal mechanics with repelling potential.

The case (ii) is more complicated and more interesting. One has to choose at least two maps to cover the phase space manifold. Both provide the local Darboux
coordinates. These details are given in Ref. [32]. However, in order to make contact with the standard form of the conformal mechanics we consider the intersection of our hyperboloid with the plane \( \xi_0 + \xi_1 = 0 \). It consists of two straight lines \( \xi_0 + \xi_1 = 0 \), \( \xi_2 = \pm \lambda \). Consider two submanifolds

\[ M_\pm = \{ \xi_\alpha \xi^\alpha = -\lambda^2 \mid 0 \leq \xi_0 + \xi_1 \}. \tag{2.14} \]

Together with two lines defined above they cover the whole hyperboloid (see Figure 1 in Refs. [32, 36]). Equations

\[
\begin{align*}
\xi_0 &= \frac{p^2}{4} - \frac{\lambda^2}{x^2} + \frac{x^2}{4}, \\
\xi_1 &= -\frac{p^2}{4} + \frac{\lambda^2}{x^2} + \frac{x^2}{4}, \\
\xi_2 &= -\frac{1}{2} xp,
\end{align*}
\tag{2.15}
\]

provide the smooth one-to-one mapping of \( M_+ \) onto the half-plane \( x > 0, -\infty < p < \infty \). Analogously,

\[
\begin{align*}
\xi_0 &= -\frac{p^2}{4} + \frac{\lambda^2}{x^2} - \frac{x^2}{4}, \\
\xi_1 &= \frac{p^2}{4} - \frac{\lambda^2}{x^2} - \frac{x^2}{4}, \\
\xi_2 &= -\frac{1}{2} xp,
\end{align*}
\tag{2.16}
\]

define the smooth one-to-one mapping of \( M_- \) onto the half-plane \( x < 0, -\infty < p < \infty \). For both mappings \((x, p)\) are Darboux coordinates. We see that the image of \( M_+ \) yields the standard form of the conformal mechanics with

\[ g^2 = -4\lambda^2 < 0. \tag{2.17} \]

However, we conclude that the description of the dynamics in terms of positive values of \( x \) coordinate is incomplete. The singularity related to the effect of the falling on the center in finite time is spurious. It is an artefact of the choice of coordinates in the symplectic manifold. The situation is somewhat similar to that encountered in general relativity. For example, the only real singularity in Schwarzschild solution resides at the center while with the standard choice of the coordinates the metric diverges at the horizon.

Once the necessity of adjoining the submanifold \( M_- \) is clearly recognized the apparent singularity at the origin \((x = 0)\) disappears. Let us stress again that due to the fact that \( M_+ \) and \( M_- \) do not cover the whole phase manifold the above singularity is still present in the explicit formulae given above. However, the rules relating the dynamics for \( x > 0 \) and \( x < 0 \) are uniquely defined (also on the Hamiltonian level). Alternatively one could work with genuine covering of the phase manifold with no (even apparent) singularities. Finally, the case of light cones corresponds to the free motion. Let
us consider, for example, the forward cone $\xi_0 > 0$. Define the canonical (Darboux) variables by

\[
\xi_1 = -\frac{p^2}{4} + \frac{x^2}{4}, \quad (2.18)
\]

\[
\xi_2 = -\frac{1}{2}xp. \quad (2.19)
\]

Defining further

\[
\xi = \xi_1 + i\xi_2, \quad u = \frac{1}{2}(x - ip), \quad (2.20)
\]

one can rewrite eqs. (2.18) and (2.19) as

\[
\xi = u^2. \quad (2.21)
\]

Therefore, the Darboux variables parametrize the Riemann surface of $\sqrt{\xi}$.

We conclude that, in the classical case, the coadjoint orbits method allows us to completely regular dynamics invariant under the action of conformal $SL(2, \mathbb{R})$ group. In the following section we construct the quantum counterparts of such dynamical systems exhibiting the unbroken conformal symmetry.

### 3 Quantum conformal mechanics: discrete series

There exist two discrete series $D^+_m$, $m = 1, 2, 3, \ldots$, of the inequivalent unitary irreducible representations of the $SL(2, \mathbb{R})$ [44]-[47]. The representation $D^+_m$ acts in the Hilbert space of functions analytic in the upper half-plane ($z = x + iy$, $\text{Im}z > 0$) equipped with the scalar product:

\[
(f, g) = \frac{1}{\Gamma(m)} \int_{y > 0} y^{m-1} \overline{f(z)} g(z) dx dy. \quad (3.1)
\]

The action of $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in SL(2, \mathbb{R})$ is given by

\[
(D^+_m(g)f)(z) = (g_{12}z + g_{22})^{-m-1} f \left( \frac{g_{11}z + g_{21}}{g_{12}z + g_{22}} \right). \quad (3.2)
\]

Analogously, $D^-_m$ acts in the Hilbert space of functions analytic in the lower half-plane equipped with the scalar product

\[
(f, g) = \frac{1}{\Gamma(m)} \int_{y < 0} |y|^{m-1} \overline{f(z)} g(z) dx dy. \quad (3.3)
\]

Let us consider the $D^+_m$ serie. In order to find the relevant generators representing $\hat{H}$, $\hat{K}$, and $\hat{D}$ we adopt the following form of the generators in the defining representation of $SL(2, \mathbb{R})$.

\[
H = i\sigma_+ , \quad K = -i\sigma_-, \quad D = -\frac{i}{2} \sigma_3. \quad (3.4)
\]
Consider first the Hamiltonian $\hat{H}$. Due to

$$e^{i\lambda \hat{H}} = e^{-\lambda \sigma_+} = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix},$$  

we find

$$(e^{i\lambda \hat{H}} f)(z) = (1 - \lambda z)^{-m-1} f \left( \frac{m}{1 - \lambda z} \right).$$  

(3.6)

Expanding both sides in $\lambda$ and comparing the linear terms we find

$$\hat{H} = -i(m + 1)z - iz^2 \frac{d}{dz}. \quad (3.7)$$

Similarly,

$$\hat{K} = -i \frac{d}{dz},$$  

$$\hat{D} = -iz \frac{d}{dz} - i \frac{(m + 1)}{2}. \quad (3.8)$$

Computing the Casimir operator (1.6) yields

$$g^2 = m^2 - \frac{1}{4}, \quad m = 1, 2, \ldots, \quad (3.9)$$

Therefore the coupling constant takes the values in discrete set

$$g^2 = \frac{3}{4}, \frac{15}{4}, \ldots, \quad (3.10)$$

(for other values of $g^2 \geq \frac{3}{4}$, see Section 7). It is easy to find the eigenvectors of $\hat{H}$:

$$\hat{H} f_E(z) = E f_E(z). \quad (3.11)$$

The energy spectrum is purely continuous and positive. The eigenvectors read

$$f_E(z) = \frac{(2E)^{\frac{m}{2}}}{\sqrt{2\pi}} e^{-iE} z^{-\frac{(m+1)}{2}}. \quad (3.12)$$

They are orthonormal and form a complete set

$$\langle f_E, f_{E'} \rangle = \delta(E - E'),$$  

$$\int_0^\infty dE (g, f_E)(f_E, h) = (g, h). \quad (3.13)$$

In order to make contact with the picture sketched in the previous section we need also the eigenvectors of $\hat{K}$:

$$\hat{K} g_\lambda(z) = \lambda g_\lambda(z). \quad (3.14)$$
The spectrum is again purely continuous and positive. The eigenvectors read

\[ g_\lambda(z) = \frac{(2\lambda)^{\frac{m}{2}}}{\sqrt{2\pi}} e^{i\lambda z}, \quad (3.15) \]

and form the orthonormal and complete set. In order to define the coordinate representation we put

\[ \lambda = \frac{x^2}{2}, \quad x > 0; \quad (3.16) \]

and normalize \( g_\lambda(z) \) to \( \delta(x - x') \); so eq. (3.15) is replaced by

\[ g_x(z) = \frac{x^{m+\frac{1}{2}}}{\sqrt{2\pi}} e^{\frac{ix^2}{2}}. \quad (3.17) \]

Consequently, the generator of the special conformal transformations takes the form

\[ \hat{K} = \frac{x^2}{2}. \quad (3.18) \]

Now, one can compute the wave function of any vector \( f(z) \) in the coordinate representation \( (z = w + iv) \)

\[ \tilde{f}(x) = (g_x, f) = \frac{1}{\Gamma(m)} \int_{-\infty}^{\infty} dw \int_{0}^{\infty} dv v^{m-1} g_x(z) f(z) \quad (3.19) \]

Let us compute the wave functions of the Hamiltonian eigenvectors

\[ \tilde{f}_E(x) = (g_x, f_E) = \frac{x^{m+\frac{1}{2}}(2E)^{\frac{m}{2}}}{2\pi\Gamma(m)} \int_{-\infty}^{\infty} dw \int_{0}^{\infty} dv v^{m-1} (w + iv)^{-(m+1)} \]

\[ \cdot \exp \left( -i \left( \frac{x^2}{2} (w - iv) + \frac{E}{w + iv} \right) \right). \quad (3.20) \]

In polar coordinates (3.20) takes the form

\[ \tilde{f}_E(x) = \frac{x^{m+\frac{1}{2}}(2E)^{\frac{m}{2}}}{2\pi\Gamma(m)} \int_{0}^{\pi} d\theta r^{m-1} \sin^{m-1} \theta e^{-i(m+1)\theta} e^{-ive^{i\theta}(\frac{x^2}{2} + \frac{E}{r})}. \quad (3.21) \]

First, we do the \( r \)-integration (see [53])

\[ \int_{0}^{\infty} drr^{-1} \exp \left( -i \left( \frac{x^2 e^{-i\theta}}{2r} + \frac{Ee^{-i\theta}}{r} \right) \right) = 2K_0(x\sqrt{2E}ie^{-i\theta}), \quad (3.22) \]

so that eq. (3.21) takes the form

\[ \tilde{f}_E(x) = \frac{x^{m+\frac{1}{2}}(2E)^{\frac{m}{2}}}{\pi\Gamma(m)} \int_{0}^{\pi} d\theta \sin^{m-1} \theta e^{-i(m+1)\theta} K_0(x\sqrt{2E}ie^{-i\theta}). \quad (3.23) \]
Using (see again [53])

\[ K_0(z) = -\ln \left( \frac{z}{2} \right) I_0(z) + \sum_{k=0}^{\infty} \frac{\psi(k+1)}{2^{2k}(k!)^2} z^{2k}, \]  
\[ I_0(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} \left( \frac{z}{2} \right)^{2k}, \]  

one easily concludes that the second term on the right-hand side of eq. (3.24) does not contribute. The first one yields

\[ \tilde{f}_E(x) = i^{-(m+1)} \sqrt{x} J_m(\sqrt{2Ex}), \]  

which proves that

\[ \hat{H} = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{m^2 - \frac{1}{4}}{2x^2}. \]  

Operator \( \hat{D} \) can be recovered in a similar way.

Using the isomorphism between \( SL(2, \mathbb{R}) \) and \( SU(1, 1) \) defined by

\[ SU(1, 1) \ni \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \rightarrow \begin{pmatrix} \text{Re}(\alpha + \beta) & -\text{Im}(\alpha - \beta) \\ \text{Im}(\alpha + \beta) & \text{Re}(\alpha - \beta) \end{pmatrix} \in SL(2, \mathbb{R}), \]  

one can construct an alternative model of the unitary irreducible representations of the discrete series. To this end one considers the Hilbert space of the functions analytic in the unit disc \((w = x + iy, |w| < 1)\) and equipped with the scalar product

\[ (f, g) = \frac{1}{\Gamma(m)} \int_{|w|<1} f(w)g(w)(1-|w|^2)^{m-1}dxdy. \]  

The unitary representation \( \tilde{D}_m^+ \) of \( SU(1, 1) \) (and, consequently also \( SL(2, \mathbb{R}) \)) is given by

\[ \left( \tilde{D}_m^+(g)f \right)(w) = (\bar{\alpha} + \beta w)^{-(m+1)} f \left( \frac{\alpha w + \bar{\beta}}{\bar{\alpha} + \beta w} \right). \]  

The representation \( \tilde{D}_m^- \) is obtained by the formula \( \tilde{D}_m^-(g) = \tilde{D}_m^+(g) \). The relation between the representations expressed in terms of the analytic functions on the upper half-plane and unit disc reads

\[ w \equiv w(z) = \frac{z - i}{z + i}, \quad z \equiv z(w) = \frac{1 + w}{1 - w}, \]  

\[ \tilde{f}(w) = 2(1-w)^{-(m+1)} f(z(w)); \]  

here \( z \) belongs to the upper half-plane while \( w \) to the open unit disc. Eqs. (3.31) and (3.7)-(3.8) lead to the following form of generators

\[ \hat{\xi}_0 = \frac{1}{2}(\hat{K} + \hat{H}) = \frac{m+1}{2} + w \frac{d}{dw}, \]  
\[ \hat{\xi}_1 = \frac{1}{2}(\hat{K} - \hat{H}) = -\frac{m+1}{2} w - \frac{1}{2}(1+w^2) \frac{d}{dw}, \]  
\[ \hat{\xi}_2 = \hat{D} = i \frac{m+1}{2} w - \frac{i}{2}(1-w^2) \frac{d}{dw}. \]  

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It is not difficult to construct, using the above realization in terms of analytic functions on the unit disc, the conformal dynamics in terms of wave functions defined on phase space. To this end let us note there is one-to-one correspondence between the points on the unit disc and on the (upper) sheet of the unit hyperboloid

\[(\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 = 1.\]  

(3.33)

It reads

\[\xi^0 = \frac{1 + |w|^2}{1 - |w|^2}, \quad \xi^1 + i\xi^2 = \frac{2w}{1 - |w|^2}.\]  

(3.34)

Let us consider an arbitrary hyperboloid

\[(\xi^0)^2 - (\xi^1)^2 - (\xi^2)^2 = \lambda^2, \quad \lambda > 0.\]  

(3.35)

Upon rescaling \(\xi^\mu \rightarrow \xi^\mu / \lambda\) one converts the manifold (3.35) into the unit hyperboloid (3.33). Therefore, the former can be parametrized as follows

\[\xi^0 = \lambda \frac{1 + |w|^2}{1 - |w|^2}, \quad \xi^1 + i\xi^2 = \frac{2\lambda w}{1 - |w|^2}.\]  

(3.36)

It is not difficult to check that the \(SU(1, 1)\) action on unit disc is equivalent to the action of \(SO(2, 1)\) Lorentz group on \(\xi\) variables

\[w'(\xi) = w(\xi'), \quad \xi' = \Lambda(g)\xi, \quad g \in SU(1, 1).\]  

(3.37)

Let us define the function \(\hat{f}(\xi)\) by

\[\hat{f}(\xi) = \frac{1}{\sqrt{\Gamma(m)} (\xi^0 + \lambda)^{m+1/2}} \tilde{f}(w(\xi)).\]  

(3.38)

Then the scalar product (3.29) becomes

\[\int \overline{\hat{f}(\xi)} \tilde{g}(\xi) \theta(\xi^0) \delta(\xi^2 - \lambda^2) d^3\xi,\]  

(3.39)

while the action of \(SU(1, 1) \simeq SL(2, \mathbb{R})\) takes the form

\[\left(D_m^+(g) \hat{f}\right)(\xi) = \left(\frac{\Gamma(\xi^0 + \lambda) + \beta(\xi^1 + i\xi^2)}{\Gamma(\xi^0 + \lambda) + \beta(\xi^1 + i\xi^2)}\right)^{-(m+1)} \hat{f}(\Lambda(g)\xi).\]  

(3.40)

However, one should take into account that the functions \(f(w)\) are analytic. This imposes some constraints for \(\hat{f}(\xi)\). In terms of the functions defined on the unit disc it is simply the Cauchy-Riemann equation \(\partial f / \partial \bar{\xi} = 0\) which, due to (3.38) takes the form

\[\Delta \hat{f}(\xi) = 0,\]  

(3.41)

where \(\xi = \xi^1 + i\xi^2\) and

\[\Delta = \frac{m+1}{2} \xi + (\xi^0 + \lambda)^2 \frac{\partial}{\partial \xi} + \xi^2 \frac{\partial}{\partial \bar{\xi}}.\]  

(3.42)
The infinitesimal form of eq. (3.40) yields
\[
\hat{H} = i\xi^2 \frac{\partial}{\partial \xi^1} - i(\xi^0 + \xi^1) \frac{\xi}{\partial \xi^2} + \frac{m + 1}{2} \left(1 + \frac{\xi^1}{\xi^0 + \lambda}\right),
\]
\[
\hat{K} = i\xi^2 \frac{\partial}{\partial \xi^1} + i(\xi^0 - \xi^1) \frac{\xi}{\partial \xi^2} + \frac{m + 1}{2} \left(1 - \frac{\xi^1}{\xi^0 + \lambda}\right),
\]
\[
\hat{D} = -i\xi^0 \frac{\partial}{\partial \xi^1} - \frac{m + 1}{2} \frac{\xi^2}{\xi^0 + \lambda}.
\]

(3.43)

The invariance of the condition (3.41) follows from the commutation rules
\[
\frac{1}{2} [\hat{H} + \hat{K}, \Delta] = \Delta,
\]
\[
\frac{1}{2} [\hat{K} - \hat{H}, \Delta] = -\frac{i\xi}{\xi^0 + \lambda} \Delta,
\]
\[
[\hat{D}, \Delta] = \frac{-i\xi}{\xi^0 + \lambda} \Delta.
\]

(3.44)

Up to now the parameters \(\lambda\) and \(m\) stay unrelated; \(\lambda\) sets the scale of hyperboloid on which the wave functions are supported while \(m\) determines the value of the Casimir operator. To relate these quantities let us remind that in the classical theory the former parameter determines both the size of hyperboloid (the phase space) and the value of Casimir (and, simultaneously, that of the coupling constant). This suggests that the desired relation between \(\lambda\) and \(m\) is obtained by demanding that our operators (3.43) arise in the process of geometric quantization of the classical systems constructed with the help of the orbits method. Using the results obtained in Ref. [54] we find that the eqs. (3.43) (or, equivalently, eqs. (3.32)) are obtained by quantizing the classical \(SL(2, \mathbb{R})\)-invariant system provided
\[
\lambda = \frac{m + 1}{2}.
\]

(3.45)

Let us comment on the above formula. Consider the \(x-p\) representation of the \(sl(2, \mathbb{R})\) Lie algebra. The value of the relevant Casimir operator reads (cf. eq. (1.6))
\[
\hat{C} = \frac{g^2}{4} - \frac{3}{16}.
\]

(3.46)

The representation (3.43) of \(sl(2, \mathbb{R})\) yields
\[
\hat{C} = \frac{m^2 - 1}{4}.
\]

(3.47)

Comparing the above expressions for \(\hat{C}\) we get the relation between \(g\) nad \(m\)
\[
g^2 = m^2 - \frac{1}{4}.
\]

(3.48)
On the other hand $g$ is related to the size of hyperboloid by eq. (2.13):

$$g^2 = 4\lambda^2 = 4((\xi_0)^2 - (\xi_1)^2 - (\xi_2)^2). \quad (3.49)$$

Quantizing $\xi_\mu$, $\xi_\mu \rightarrow \hat{\xi}_\mu$ we find (see [54])

$$(\hat{\xi}_0)^2 - (\hat{\xi}_1)^2 - (\hat{\xi}_2)^2 = \lambda(\lambda - 1), \quad (3.50)$$

which, by (3.45), agrees with (3.47). The minimal value of $\lambda$ is 1; then $C = 0$ and $g^2 = \frac{3}{4}$. The "classical" phase space is then the (upper) sheet of the hyperboloid

$$(\xi_0)^2 - (\xi_1)^2 - (\xi_2)^2 = 1. \quad (3.51)$$

However, we should keep in mind that we are using the units with $\hbar = 1$. Reinserting $\hbar$ one concludes that the size of the classical phase space would be of order $\hbar$. The genuine classical limit is obtained by taking $\hbar \rightarrow 0$, $m \rightarrow \infty$ and $m\hbar = \text{const}$.

In a similar way one can analyse the case of lower sheet and/or the second discrete series $D_{m^-}$.

4 Quantum conformal mechanics: continuous series

The $SL(2, \mathbb{R}) \simeq SU(1, 1)$ group possesses the continuous series of the unitary irreducible representations [14]-[17]. They can be described as follows: the relevant Hilbert space consists of functions defined on the unit circle and square integrable with respect to the standard Lebesgue measure. The action of the $SU(1, 1)$ group is determined by two parameters, $\rho \in \mathbb{R}$ and $\epsilon = 0, 1$ and is given by

$$\left(\tilde{D}^{\rho,\epsilon}(g)f\right) (e^{i\psi}) = |\alpha + \beta e^{i\psi}|^{\rho-1-\epsilon}(\alpha + \beta e^{-i\psi})^\epsilon f \left(\frac{\alpha e^{i\psi} + \beta}{\alpha + \beta e^{i\psi}}\right). \quad (4.1)$$

All representations $\tilde{D}^{\rho,\epsilon}$ are irreducible expect $(\rho, \epsilon) = (0, 1)$ when we are dealing with the sum of two irreducible representations. Two representations $(\rho, \epsilon)$ and $(\rho', \epsilon')$ are equivalent if and only if $\rho = \rho'$, $\epsilon = \epsilon'$ or $\rho = -\rho'$, $\epsilon = \epsilon'$. In this section we will be dealing with the representations corresponding to $\rho \neq 0$. Therefore, we can assume $\rho > 0$, $\epsilon = 0, 1$. The corresponding generators are easily obtained via the isomorphism (3.28). Another equivalent and convenient form of the representation is given by the formula

$$\left(D^{\rho,\epsilon}(g)f\right) (z) = |g_{12}z + g_{22}|^{\rho-1-\epsilon}(g_{12}z + g_{22})^\epsilon f \left(\frac{g_{11}z + g_{21}}{g_{12}z + g_{22}}\right), \quad (4.2)$$

where $f \in L^2(\mathbb{R})$, again with respect to the standard Lebesgue measure.

As in the case of discrete series we used (3.4) to find the representation of the Lie algebra generators. However, to make contact with the classical theory presented in Section 2 it is more convenient to use the equivalent representation:

$$H = -i\sigma_+, \quad K = i\sigma_-, \quad D = -i\frac{1}{2}\sigma_3. \quad (4.3)$$
This choice leads to the following form of generators

\[
\hat{H} = -\frac{\rho + i}{2} \sin \psi - \frac{\epsilon}{2}(1 + \cos \psi) + i(1 + \cos \psi) \frac{d}{d\psi},
\]

\[
\hat{K} = \frac{\rho + i}{2} \sin \psi - \frac{\epsilon}{2}(1 - \cos \psi) + i(1 - \cos \psi) \frac{d}{d\psi},
\]

\[
\hat{D} = \frac{\rho + i}{2} \cos \psi - \frac{\epsilon}{2}\sin \psi + i\sin \psi \frac{d}{d\psi}.
\]

Computing the Casimir operator yields

\[
\hat{C} = -\frac{1}{4}(\rho^2 + 1) < -\frac{1}{4}.
\]

Our next step is to find the spectrum of the Hamiltonian \(\hat{H}\) and the generator \(\hat{K}\) of the special conformal transformation. The solutions to the eigenvalue equation

\[
\hat{H} f_E(\psi) = E f_E(\psi),
\]

reads

\[
f_E(\psi) = \frac{1}{\sqrt{2\pi}} e^{-i\psi} e^{-iE\tan(\frac{\psi}{2})}(1 + \cos \psi) \frac{i\rho - 1}{2} \left\{ \begin{array}{ll} 1, & 0 \leq \psi \leq \pi, \\ (-1') & \pi < \psi < 2\pi. \end{array} \right. \]

The eigenfunctions \(f_E(\psi)\) are properly normalized

\[
(f_E, f_{E'}) = \delta(E - E'),
\]

and obey the completeness relation

\[
\int_{-\infty}^{\infty} dE f_E(\psi)\overline{f_E(\psi')} = \delta(\psi - \psi').
\]

Similarly, one can look for the eigenvalue problem for the conformal generator \(\hat{K}\)

\[
\hat{K} g_\kappa(\psi) = \kappa g_\kappa(\psi).
\]

The solutions to eq. (4.10) read

\[
g_\kappa(\psi) = \frac{1}{\sqrt{2\pi}} e^{-i\psi} e^{i\kappa\cot(\frac{\psi}{2})}(1 - \cos \psi) \frac{i\kappa - 1}{2}.
\]

Again, they are normalized to \(\delta(\kappa - \kappa')\) and obey the completeness relation if the integration over \(\kappa\) extends over the whole real axis.

We conclude that both \(\hat{H}\) and \(\hat{K}\) have purely continuous spectrum extending from \(-\infty\) to \(\infty\). Comparing the energy spectrum with the properties of the classical motion we conclude that the continuous corresponds to the motion on one-sheeted hyperboloid.
In order to make contact with the classical parametrization (2.15) and (2.16) we parametrize its spectrum as follows

\[ \kappa = \begin{cases} \frac{x^2}{2}, & x > 0, \\ -\frac{x^2}{2}, & x < 0. \end{cases} \]  
(4.12)

The new representation with \( \hat{K} \) diagonal is defined as

\[ \tilde{f}(x) = |x|^{\frac{1}{2} + i \rho} (g_\kappa, f), \]  
(4.13)

with \( \kappa \) being related to \( x \) via formula (4.12). The prefactor \(|x|^{\frac{1}{2}}\) is introduced in order to obtain the Hilbert space of functions square integrable with respect to the standard Lebesque measure on \( \mathbb{R} \). An additional phase factor \(|x|^{i \rho}\) is added to provide the proper form of generators in \( x \)-representation.

Explicitly, we have

\[ \tilde{f}(x) = \frac{|x|^{\frac{1}{2} + i \rho}}{\sqrt{2\pi}} \begin{cases} 2\pi \int_0^{2\pi} d\psi e^{i\rho \frac{\psi}{2}} e^{-i\frac{x^2}{2} \cot(\frac{\psi}{2})} (1 - \cos \psi)^{-\frac{i \rho + 1}{2}} f(\psi), & x > 0, \\ 2\pi \int_0^{2\pi} d\psi e^{i\rho \frac{\psi}{2}} e^{-i\frac{x^2}{2} \cot(\frac{\psi}{2})} (1 - \cos \psi)^{-\frac{i \rho + 1}{2}} f(\psi), & x < 0. \end{cases} \]  
(4.14)

In particular, it is not difficult to find the energy eigenfunctions in the \( x \) representation. Using eq. (4.7) we find

\[ \tilde{f}_E(x) = |x|^{\frac{1}{2} + i \rho} (g_\kappa, f_E) = |x|^{\frac{1}{2} + i \rho} \int_0^{2\pi} g_\kappa(\psi) f_E(\psi) \]

\[ = \frac{|x|^{\frac{1}{2} + i \rho}}{2\pi} \int_0^\pi d\psi e^{-i\kappa \cot(\frac{\psi}{2}) - iE \tan(\frac{\psi}{2})} (1 + \cos \psi)^{-\frac{i \rho - 1}{2}} (1 - \cos \psi)^{-\frac{i \rho - 1}{2}} \]

\[ + (-1)^\epsilon |x|^{\frac{1}{2} + i \rho} \int_0^{\pi} d\psi e^{-i\kappa \cot(\frac{\psi}{2}) - iE \tan(\frac{\psi}{2})} (1 + \cos \psi)^{\frac{i \rho - 1}{2}} (1 - \cos \psi)^{-\frac{i \rho - 1}{2}}, \]  
(4.15)

where \( \kappa \) is related to \( x \) by eq. (4.12). Making the change of variables \( u = \tan(\frac{\psi}{2}) \), one finds after some manipulations

\[ \tilde{f}_E(x) = \begin{cases} \frac{|x|^{\frac{1}{2} + i \rho}}{\pi} \int_0^\infty du u^{-i \rho - 1} \cos(Eu + \frac{\kappa}{u}), & \epsilon = 0, \\ -i\frac{|x|^{\frac{1}{2} + i \rho}}{\pi} \int_0^\infty du u^{-i \rho - 1} \sin(Eu + \frac{\kappa}{u}), & \epsilon = 1. \end{cases} \]  
(4.16)

The above integrals can be easily taken (see [53]) to yield:
\[ \tilde{f}_E(x) = (2|E|)^\frac{i}{4}|x|^\frac{1}{2} (iJ_{-i\rho}(|x|\sqrt{2|E|}) \sinh\left(\frac{\pi\rho}{2}\right) - N_{-i\rho}(|x|\sqrt{2|E|}) \cosh\left(\frac{\pi\rho}{2}\right)) ; \] (4.17)

- for \( \epsilon = 0, \) \( Ex > 0 \)

\[ \tilde{f}_E(x) = \left(\frac{2}{\pi}\right)(|E|)^\frac{i}{4}|x|^\frac{1}{2} K_{-i\rho}(|x|\sqrt{2|E|}) \cosh\left(\frac{\pi\rho}{2}\right) ; \] (4.18)

- for \( \epsilon = 1, \) \( Ex > 0 \)

\[ \tilde{f}_E(x) = \text{sgn}(E)(|E|)^\frac{i}{4}|x|^\frac{1}{2} (iJ_{-i\rho}(|x|\sqrt{2|E|}) \cosh\left(\frac{\pi\rho}{2}\right) + N_{-i\rho}(|x|\sqrt{2|E|}) \sinh\left(\frac{\pi\rho}{2}\right)) ; \] (4.19)

- for \( \epsilon = 1, \) \( Ex < 0 \)

\[ \tilde{f}_E(x) = -\left(\frac{2\text{sgn}(E)}{\pi}\right)(|E|)^\frac{i}{4}|x|^\frac{1}{2} K_{-i\rho}(|x|\sqrt{2|E|}) \sinh\left(\frac{\pi\rho}{2}\right). \] (4.20)

We see that \( \tilde{f}_E(x) \) obey the eigenvalue equation(s):

\[ \left( -\frac{1}{2} \frac{d^2}{dx^2} - \frac{\rho^2 + \frac{1}{4}}{x^2} \right) \tilde{f}_E(x) = E\tilde{f}_E(x), \quad x > 0; \]
\[ \left( \frac{1}{2} \frac{d^2}{dx^2} + \frac{\rho^2 + \frac{1}{4}}{x^2} \right) \tilde{f}_E(x) = E\tilde{f}_E(x), \quad x < 0; \] (4.21)

The relation between the parameter \( \rho \) and the coupling constant \( g \) (cf. eq. (1.1)) reads

\[ g^2 = -\left(\rho^2 + \frac{1}{4}\right) = (i\rho)^2 - \frac{1}{4}. \] (4.22)

Eqs. (4.17)-(4.20) determine the way the solutions of the eigenvalue equations for \( x > 0 \) and \( x < 0 \) are glued together at \( x = 0 \) to yield the eigenvectors of the self-adjoint generator \( \hat{H} \). If we restrict ourselves to the \( x > 0 \) region for negative coupling constant then the conformal symmetry remains unbroken. This is not possible for eigenvalue problem on semiaxis where any boundary condition defining a self-adjoint extension of the symmetric operator given by the formal differential expression \( -\frac{1}{2} \frac{d^2}{dx^2} - \frac{g^2}{x^2} \) breaks the conformal symmetry [42].

It remains to relate the formalism presented above to the one emerging from canonical (geometric) quantization of the Hamiltonian dynamics defined on the one-sheeted hyperboloid. This hyperboloid is a submanifold of the three-dimensional space carrying the linear representation of \( SO(2, 1) \simeq SL(2, \mathbb{R})/\mathbb{Z}_2 \) (the adjoint representation \( SO(2, 1) \) or \( SL(2, \mathbb{R}) \)). On the other hand, as it is clearly seen from eq. (4.1), the irreducible representations under considerations are spanned by the function supported on
compact manifold \((S^1)\) on which \(SL(2, \mathbb{R})\) acts nonlinearity. This action is transitive so it results from nonlinear action of the group on compact coset manifold. Consider the compact subgroup generated by \(H + K\). The relevant group manifold can be viewed as the coset manifold \(SL(2, \mathbb{R})/G(D, K)\) where \(G(D, K) \subset SL(2, \mathbb{R})\) is the subgroup generated by \(D\) and \(K\). With our choice \((4.3)\)

\[ H + K = \sigma_2. \]  

(4.23)

The nonlinear action of \(g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \ ad - bc = 1\), is given by

\[ ge^{i\theta(H+K)} = e^{i\theta'(H+K)} e^{i\gamma(g, \theta)D} e^{i\delta(g, \theta)K}. \]  

(4.24)

Eq. (4.24) implies

\[ \tan \theta' = a \tan \theta + b \]
\[ c \tan \theta + d', \]
\[ e^\gamma = \frac{1 + \tan^2 \theta}{(a \tan \theta + b)^2 + (c \tan \theta + d)^2}, \]
\[ \delta = \frac{(b^2 + d^2 - a^2 - c^2) \tan \theta + (ab + cd)(\tan^2 \theta - 1)}{(a \tan \theta + b)^2 + (c \tan \theta + d)^2}. \]  

(4.25)

The action of \(SU(1, 1)\) on the variable \(\psi\) following from eq. (4.1) reads

\[ e^{i\psi'} = \frac{\alpha e^{i\psi} - \beta}{\alpha - \beta e^{i\psi}}. \]  

(4.26)

By comparing eqs. (4.25) and (4.26) and using the isomorphism (3.28) we conclude that one can identify

\[ \psi = 2\theta. \]  

(4.27)

In order to construct the variables transformation according to the linear representation of \(SL(2, \mathbb{R})\) we follow the method described in Refs. \([55, 56]\). First, we construct the three-dimensional representation \(D\) of \(SL(2, \mathbb{R})\) – the adjoint representation:

\[ D(e^{i\lambda D}) = \begin{pmatrix} \cosh \lambda & -\sinh \lambda & 0 \\ -\sinh \lambda & \cos \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]
\[ D(e^{i\lambda H}) = \begin{pmatrix} 1 + \frac{\lambda^2}{2} & -\frac{\lambda^2}{2} & -\lambda \\ \frac{\lambda^2}{2} & 1 - \frac{\lambda^2}{2} & -\lambda \\ -\lambda & \lambda & 1 \end{pmatrix}, \]
\[ D(e^{i\lambda K}) = \begin{pmatrix} 1 + \frac{\lambda^2}{2} & \frac{\lambda^2}{2} & \lambda \\ -\frac{\lambda^2}{2} & 1 - \frac{\lambda^2}{2} & -\lambda \\ \lambda & \lambda & 1 \end{pmatrix}, \]
\[ D(e^{i\theta(H+K)}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 2\theta & -\sin 2\theta \\ 0 & \sin 2\theta & \cos 2\theta \end{pmatrix}. \]  

(4.28)

(4.29)
The above representation, when restricted to the subgroup generated by $D$ and $K$ is not completely reducible. It has the two-dimensional invariant subspace spanned by the vectors

\[ e_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \] (4.30)

Any element of this subspace can be written as $\chi_1 e_1 + \chi_2 e_2$. The action of the subgroup generated by $K$ and $D$ reads

\[ e^{i\lambda K} : \quad \chi_1' = \chi_1 + \lambda \chi_2, \quad \chi_2' = \chi_2; \]
\[ e^{i\lambda D} : \quad \chi_1' = e^{-\lambda} \chi_1 + (e^{-\lambda} - 1) \chi_2, \quad \chi_2' = \chi_2; \] (4.31)

Now, according to the formalism described in Refs. [55, 56], the relation between the parameters $\theta, \chi_1$ and $\chi_2$ transforming nonlinearly under $SL(2, \mathbb{R})$ and the coordinates $\xi_0, \xi_1, \xi_2$, transforming linearly reads

\[ \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \end{pmatrix} = D(e^{i\theta(H+K)}) \begin{pmatrix} \chi_1 + \chi_2 \\ -\chi_1 - \chi_2 \\ \chi_2 \end{pmatrix}. \] (4.32)

Due to (4.29) we have the following relations

\[ \xi_0 = \chi_1 + \chi_2, \]
\[ \xi_1 = - (\chi_1 + \chi_2) \cos 2\theta - \chi_2 \sin 2\theta, \]
\[ \xi_2 = - (\chi_1 + \chi_2) \sin 2\theta + \chi_2 \cos 2\theta. \] (4.33)

Note that

\[ (\xi_0)^2 - (\xi_1)^2 - (\xi_2)^2 = -\chi_2^2, \] (4.34)

which is invariant under $SO(2, 1)$ as it follows from eqs. (4.31). Taking into account eq. (4.27) one can rewrite eq. (4.33) as

\[ \xi_1 = -\xi_0 \cos \psi - \chi_2 \sin \psi, \]
\[ \xi_2 = -\xi_0 \sin \psi + \chi_2 \cos \psi. \] (4.35)

By comparing eqs. (2.7) and (4.34) we conclude that the classical phase space may parametrized by $\xi_0$ and $\psi$ with $\chi_2 = \lambda$ kept fixed. The Poisson brackets (2.5) are equivalent to

\[ \{ \psi, \xi_0 \} = 1; \] (4.36)

The (geometric) quantization procedure is therefore, straightforward. On the quantum level

\[ [\hat{\psi}, \hat{\xi}_0] = i. \] (4.37)
Let us consider the following representation of the above commutation rule

\[ \hat{\psi} = \psi, \quad \hat{\xi}_0 = -i \frac{d}{d\psi} + \frac{\epsilon}{2}. \] (4.38)

With the appropriate ordering rule, \( \xi_0 f(\psi) \rightarrow \frac{1}{2}(\xi_0 f(\hat{\psi}) + f(\hat{\psi})\hat{\xi}_0) \) we find from eqs. (2.9) and (4.35)

\[ \tilde{H} = (\chi_2 + \frac{i}{2}) \sin \psi + \frac{\epsilon}{2} \frac{1 + \cos \psi}{d\psi}, \]

\[ \tilde{K} = -(\chi_2 + \frac{i}{2}) \sin \psi + \frac{\epsilon}{2} \frac{1 - \cos \psi}{d\psi}, \] (4.39)

\[ \tilde{D} = (\chi_2 + \frac{i}{2}) \cos \psi - \frac{\epsilon}{2} \sin \psi + \sin \psi \frac{d}{d\psi}. \]

Eqs. (4.39) coincide up to the Lie algebra isomorphism \( H \rightarrow -H, K \rightarrow -K, D \rightarrow D \) with eqs. (4.4) provided the identification \( \rho = 2\chi_2 \) has been made. Noting that \( \chi_2 = \lambda \) we find from eq. (4.22)

\[ g^2 = -4\lambda^2 - \frac{1}{4}, \] (4.40)

which should be compared with the classical relation (2.17). We see that these equations coincide up to the quantum correction. Let us note that, the parametrization (4.33) has been used by Plyushchay in his paper on quantization of \( SL(2,\mathbb{R}) \) symmetry [54].

The value of the Casimir operator is given by eq. (4.5).

\[ \hat{C} = -\frac{\rho^2}{4} - \frac{1}{4}. \] (4.41)

Therefore, due to the eq. (1.6) we find

\[ g^2 = -\rho^2 - \frac{1}{4}, \] (4.42)

so that \( g^2 < -\frac{1}{4} \). On the other hand the parametrization (4.33) yield, upon quantization, the proper value of the Casimir operator provided \( \chi_2 = \lambda = \rho/2 \). Due to eq. (1.41) the quantization of the one-sheeted hyperboloid \( (\xi_0)^2 - (\xi_1)^2 - (\xi_2)^2 = -\lambda^2 \) yields

\[ (\hat{\xi}_0)^2 - (\hat{\xi}_1)^2 - (\hat{\xi}_2)^2 = -\lambda^2 - \frac{1}{4}. \] (4.43)

The continuous series correspond to the quantization of the ”classical” theories described by the phase spaces in form of one-sheeted hyperboloids. Again, genuine classical limit is attained by \( \hbar \rightarrow 0, \rho \rightarrow \infty \) and \( \rho\hbar = \text{constant} \). The upper limit for the coupling constant is \(-\frac{1}{4}\).
5 Quantum conformal mechanics: supplementary serie

The representations considered up to now, i.e., the discrete and continuous series, cover the whole range of coupling constant except the interval $(-\frac{1}{4}, \frac{3}{4})$. In the present section we consider the supplementary serie which corresponds to the interval $(-\frac{1}{4}, \frac{3}{4})$. The special case $g^2 = -\frac{1}{4}$ will be dealt with in the next section.

The supplementary serie of the irreducible unitary representations is defined as follows [44]-[47]. For any $\rho$ such that $0 < |\rho| < 1$ we define the Hilbert space of functions of real variable running over the whole real axis $\mathbb{R}$ equipped with the scalar product

$$ (f, g) = \frac{1}{\Gamma(\rho)} \int \int_{\mathbb{R}^2} |y_1 - y_2|^{\rho - 1} f(y_1)g(y_2)dx_1dx_2. \quad (5.1) $$

The action of $SL(2, \mathbb{R})$ group is given by the formula

$$ (\Delta_{\rho}(g)f)(x) = |g_{12}y + g_{22}|^{-\rho - 1} f \left( \frac{g_{11}y + g_{21}}{g_{12}y + g_{22}} \right). \quad (5.2) $$

Alternatively, one can consider the equivalent representation of $SU(1, 1)$ acting in the space of functions defined on the unit circle equipped with the scalar product

$$ (f, g) = \frac{1}{\Gamma(\rho)} \int \int_{S^1 \times S^1} \left| \sin \left( \frac{\psi_1 - \psi_2}{2} \right) \right|^{\rho - 1} f(\psi_1)g(\psi_2)d\psi_1d\psi_2. \quad (5.3) $$

Then the group action reads

$$ (\tilde{\Delta}_{\rho}(g)f)(e^{i\psi}) = |\alpha e^{i\psi} + \beta e^{i\psi}|^{-\rho - 1} f \left( \frac{\alpha e^{i\psi} + \beta e^{i\psi}}{\alpha + \beta e^{i\psi}} \right). \quad (5.4) $$

The representation $\Delta_{\rho}$ and $\Delta_{-\rho}$ are unitary equivalent [46, 47]. Therefore, we may restrict ourselves to the parameters $\rho$ obeying $0 < \rho < 1$. In what follows we use the form of representations described by eqs. (5.1) and (5.2). As in the previous cases we start with determining the form of generators. They read

$$ \hat{H} = i(\rho + 1)y + iy^2 \frac{d}{dy}, $$

$$ \hat{K} = i \frac{d}{dy}, $$

$$ \hat{D} = -i(\rho + 1) - iy \frac{d}{dy}. \quad (5.5) $$

It is again easy to solve the spectral problems for $\hat{H}$ and $\hat{K}$. The general solution to the eigenvalue equation

$$ \hat{H}f_{E}(y) = Ef_{E}(y), \quad (5.6) $$
reads
\[ f_E(y) = \frac{|E|^\frac{\rho}{2}}{\sqrt{2 \cos(\frac{\rho\pi}{2})}} |y|^{-(\rho+1)} e^{\frac{iE}{y}}. \] (5.7)

The eigenfunctions \( f_E(y) \) are normalized to \( \delta(E - E') \) with respect to the scalar product (5.1). The completeness condition reads
\[ \int_{-\infty}^{\infty} dE \int_{\mathbb{R}^2} dy_1 dy_2 |z_1 - y_1|^{\rho-1} f_E(y_1) \overline{f_E(y_2)} |y_2 - z_2|^{\rho-1} = |z_1 - z_2|^{\rho-1}. \] (5.8)

With some effort one can verify the above equality for \( f_E(y) \) given by eq. (5.7) (to this end one can use the fact that the Fourier transform of the product equals the convolution of Fourier transforms of the factors as well as the form of Fourier transform of the distribution \( |y|^\lambda \), see [57]).

The eigenvalue problem for \( \hat{K} \),
\[ \hat{K} g_\kappa(y) = \kappa g_\kappa(y), \] (5.9)
can be also easily solved
\[ g_\kappa(y) = \frac{|\kappa|^\frac{\rho}{2}}{\sqrt{2 \cos(\frac{\rho\pi}{2})}} e^{-i\kappa y}. \] (5.10)

Again the eigenfunctions \( g_\kappa(y) \) are normalized to \( \delta(\kappa - \kappa') \) (with respect to the scalar product (5.1)) and obey the completeness condition
\[ \int_{-\infty}^{\infty} d\kappa \int_{\mathbb{R}^2} dy_1 dy_2 |z_1 - y_1|^{\rho-1} g_\kappa(y_1) \overline{g_\kappa(y_2)} |y_2 - z_2|^{\rho-1} = |z_1 - z_2|^{\rho-1}. \] (5.11)

We conclude that both \( \hat{H} \) and \( \hat{K} \) has purely continuous spectrum extending from \(-\infty\) to \(\infty\). In order to construct the "coordinate" representation we again label the eigenvalues of \( \hat{K} \) according to eq. (4.12). The wave functions in the coordinate representation are defined as
\[ \tilde{f}(x) = |x|^\frac{\rho}{2} (g_\kappa, f), \quad \kappa = \text{sgn}(x) \frac{x^2}{2}. \] (5.12)

In the coordinate representation \( \hat{K} \) takes the "standard" form
\[ \hat{K} = \begin{cases} \frac{x^2}{2}, & x > 0; \\ -\frac{x^2}{2}, & x < 0. \end{cases} \] (5.13)

The next step is to find the energy eigenvectors in coordinate representation. According to the definition (5.12) we have
\[ \tilde{f}_E(x) = \frac{|x|^\frac{\rho}{2}}{\Gamma(\rho)} \int_{\mathbb{R}^2} dy_1 dy_2 |y_1 - y_2|^{\rho-1} \left| \frac{x}{2} \right|^\frac{\rho}{2} e^{i\kappa y_1} \left| y_2 \right|^{-(\rho+1)} |E|^\frac{\rho}{2} e^{iE y_2}. \] (5.14)
Changing the variables $y_1 - y_2 \to y_1, y_2 \to y_2$ we find

$$\tilde{f}_E(x) = \frac{2^{-\frac{3}{2}} |x|^{\frac{1}{2} + \rho} |E|^{\frac{1}{2}}}{2\Gamma(\rho) \cos(\frac{\pi \rho}{2})} \int_{-\infty}^{\infty} dy_1 |y_1|^{\rho - 1} e^{iy_1} \int_{-\infty}^{\infty} dy_2 |y_2|^{-(\rho + 1)} e^{i(\kappa y_2 + \Phi)}, \quad (5.15)$$

The first integral on the right hand side is the Fourier transform of the generalized function $|y|^{\rho - 1}$ (see, [57]) while the second can be found in [53]. The final result reads

$$\tilde{f}_E(x) = 2\pi |x|^\frac{1}{2} \left( J_{-\rho}(|x|\sqrt{2|E|}) \sin(\frac{\pi \rho}{2}) - N_{-\rho}(|x|\sqrt{2|E|}) \cos(\frac{\pi \rho}{2}) \right), \quad (5.16)$$

for $E/x > 0$ and

$$\tilde{f}_E(x) = 4|x|^\frac{1}{2} K_{-\rho}(|x|\sqrt{2|E|}) \cos(\frac{\pi \rho}{2}), \quad (5.17)$$

for $E/x < 0$.

It is now easy to find the Hamiltonian in the coordinate representation. Eqs. (5.16) and (5.17) imply the following differential equations for $\tilde{f}_E(x)$:

$$(\frac{-1}{2} \frac{d^2}{dx^2} + \frac{\rho^2 - \frac{1}{4}}{2x^2}) \tilde{f}_E(x) = \begin{cases} E \tilde{f}_E(x), & x > 0, \\ -E \tilde{f}_E(x), & x < 0. \end{cases} \quad (5.18)$$

Therefore, the Hamiltonian in coordinate representation reads

$$\hat{H} = \begin{cases} -\frac{1}{2} \frac{d^2}{dx^2} + \frac{\rho^2 - \frac{1}{4}}{2x^2}, & x > 0, \\ \frac{1}{2} \frac{d^2}{dx^2} - \frac{\rho^2 - \frac{1}{4}}{2x^2}, & x < 0. \end{cases} \quad (5.19)$$

Let us note that the coupling constant variables are in the interval $\frac{1}{4} < g^2 < \frac{3}{4}$, i.e., it can attain both positive and negative values. In the classical case this would correspond both to the case of one- and two-sheeted hyperboloids (and a light cone) as the phase spaces. In spite of that in the quantum case the range of coordinate variable extends over the whole real axis for all $0 < \rho < 1$. The reason for that is quite simple and would be clearly seen if we introduced explicitly the Planck constant $\hbar$. The standard way of taking the classical limit is to keep the coupling fixed (or, at least, not infinitesimally small) when going $\hbar \to 0$. However, this implies the Casimir eigenvalue tends to infinity as $\hbar^{-2}$. This is here impossible due to the restriction $0 < \rho < 1$. In other words, our potential is proportional to $\hbar^2$ so we always dealing with purely quantum case. For example, for classical repelling potential we can argue that it is sufficient to restrict ourselves to the positive semiaxis. This is because the semiclassical tunneling factor $\exp(-\frac{1}{\hbar} \int x \sqrt{V(x) - E} dx)$ goes to zero as $x \to 0^+$. This is, however, not the case if $V(x)$ itself is proportional to $\hbar^2$; in fact, the semiclassical approximation makes here no sense in its standard form.

The most interesting case corresponds to $\rho = \frac{1}{2}$. As we see from eq. (5.19) the dynamics is then given, up to sign, by the free dynamics on each semiaxis. The generators of $SL(2, \mathbb{R})$ take the form

$$\hat{H} = \begin{cases} -\frac{1}{2} \frac{d^2}{dx^2}, & x > 0, \\ \frac{1}{2} \frac{d^2}{dx^2}, & x < 0; \end{cases} \quad (5.20)$$

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\[ K = \begin{cases} \frac{x^2}{2}, & x > 0, \\ -\frac{x^2}{2}, & x < 0; \end{cases} \] (5.21)

\[ \dot{D} = \frac{i}{2} x \frac{d}{dx} + \frac{i}{4}. \] (5.22)

It is not difficult to find the energy eigenfunctions in the coordinate representation. Following the same way as in the previous sections we find

\[ \tilde{f}_E(x) = \begin{cases} -2 \sqrt{\frac{2}{|2E|}} \sin(\sqrt{2|E|}x - \frac{\pi}{4}), & E > 0; \\ \sqrt{\frac{2}{|2E|}} e^{-\sqrt{2|E|}x}, & E < 0. \end{cases} \] (5.23)

It is not difficult to find the global action of \( \hat{K} \) and \( \hat{D} \). The global action of \( \hat{H} \) is nonlocal. Its kernel can be be explicitly found in terms of Fresnel integrals. It is, however, not very enlightening so we skip it here. It is important to note that the case under consideration does not correspond to the free motion. For example, the spectrum of \( \hat{H} \) as well as \( \hat{K} \) are not bounded from below as in the free case.

6 Quantum conformal mechanics: the exceptional cases

As far the unitary irreducible representations of \( SL(2, \mathbb{R}) \) are concerned we are left with the case \( \rho = 0, \epsilon = 0, 1 \). The case \( \rho = 0, \epsilon = 0 \) corresponds to a single irreducible representation. The case \( \epsilon = 1 \) is more interesting. The relevant representation is reducible, being direct sum of irreducible components. They can be formally viewed as the limits \( m \to 0^+ \) of the discrete series \( D^\pm_m \).

Let us start with the case \( \rho = 0, \epsilon = 0 \). Eq. (4.1) takes the form

\[ \left( \tilde{D}^{0,0}(g)f \right)(e^{i\psi}) = |\alpha + \beta e^{i\psi}|^{-1} f \left( \frac{\alpha e^{i\psi} + \beta}{\alpha - \beta e^{i\psi}} \right). \] (6.1)

The generators can be read off from eqs. (4.4):

\[ \hat{H} = -\frac{i}{2} \sin \psi + i(1 + \cos \psi) \frac{d}{d\psi}, \] (6.2)

\[ \hat{K} = \frac{i}{2} \sin \psi + i(1 - \cos \psi) \frac{d}{d\psi}, \] (6.3)

\[ \hat{D} = \frac{i}{2} \cos \psi + i \sin \psi \frac{d}{d\psi}, \] (6.4)

which leads to the energy eigenfunctions

\[ f_E(\psi) = \frac{1}{\sqrt{2\pi}} e^{-iE \tan(\frac{\psi}{2})} (1 + \cos \psi)^{-\frac{1}{2}}, \] (6.5)
and the eigenfunctions of $\hat{K}$

$$g_\kappa(\psi) = \frac{1}{\sqrt{2\pi}} e^{i\kappa \cot(\frac{\pi}{2})} (1 - \cos \psi)^{-\frac{1}{2}}. \quad (6.6)$$

We adopt eq. (4.13) as defining the coordinate representation

$$\tilde{f}(x) = |x|^\frac{1}{2} (g_\kappa, f), \quad (6.7)$$

which leads to

$$\tilde{f}_E(x) = \begin{cases} -|x|^\frac{1}{2} N_0(\sqrt{2|E||x|}), & E/x > 0; \\ -\frac{2}{\pi} |x|^\frac{1}{2} K_0(\sqrt{2|E||x|}), & E/x < 0. \end{cases} \quad (6.8)$$

Consequently, the Hamiltonian can be obtained by putting $\rho = 0$ in eqs. (4.21).

Let us now consider the case $\epsilon = 1$. The generators take the form

$$\hat{H} = -\frac{i}{2} \sin \psi - \frac{1}{2} (1 + \cos \psi) + i(1 + \cos \psi) \frac{d}{d\psi}, \quad (6.9)$$

$$\hat{K} = \frac{i}{2} \sin \psi - \frac{1}{2} (1 - \cos \psi) + i(1 - \cos \psi) \frac{d}{d\psi}, \quad (6.10)$$

$$\hat{D} = \frac{i}{2} \cos \psi - \frac{1}{2} \sin \psi + i \sin \psi \frac{d}{d\psi}. \quad (6.11)$$

Thus,

$$f_E(\psi) = \frac{1}{\sqrt{2\pi}} \text{sgn}(\pi - \psi) e^{-iE \tan(\frac{\pi}{2})} e^{-i\frac{\psi}{2}} (1 + \cos \psi)^{-\frac{1}{2}}. \quad (6.12)$$

Computing the coordinate representation of the energy eigenfunctions we find

$$\tilde{f}_E(x) = \begin{cases} -i \text{sgn}(E)|x|^\frac{1}{2} J_0(\sqrt{2|E||x|}), & E/x > 0; \\ 0, & E/x < 0. \end{cases} \quad (6.13)$$

The reducibility of representation is now clearly seen. The subspace of square integrable functions supported on positive semiaxis span the representation corresponding to the positive part of the spectrum of $\hat{H}$ and $\hat{K}$. In fact, this subspace is obviously invariant under the global action of $\hat{K}$ (multiplication by the $x$-dependent phase factor) and $\hat{D}$ (scaling of independent variable and multiplication by $x$-independent factor); the invariance under the action of $\hat{H}$ follows from the form of eigenfunctions. The subspace of functions supported on negative semiaxis carries the representation corresponding to negative eigenvalues of $\hat{H}$ and $\hat{K}$. Let us note that in accordance with general theory of $SL(2,\mathbb{R})$ representations, the wave functions (6.13) can be viewed as $m \to 0$ limit of the eigenfunctions spanning the representations belonging to the discrete series (cf. eq. (3.26)); the limiting representations are sometimes called the mock representations.
7 Representations of the universal covering

It follows from the previous analysis that the positive values of coupling constant are quantized. This restriction on the Casimir spectrum follows from the topology of $SL(2, \mathbb{R})$ group. For
\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \]
the defining condition $ad - bc = 1$ can be rewritten as
\[ x_1^2 + x_2^2 - x_3^2 - x_4^2 = 1, \]
with
\[ x_1 = \frac{1}{2}(a + d), \quad x_2 = \frac{1}{2}(b - c), \quad x_3 = \frac{1}{2}(a - d), \quad x_4 = \frac{1}{2}(b + c). \]
Therefore, the group manifold is a hyperboloid. It contains the unshrinkable circle
\[ x_1^2 + x_2^2 = 1, \quad x_3 = 0, \quad x_4 = 0, \]
which corresponds to the compact subgroup generated by $(H + K)$ (cf. eq. (4.3))
\[ g(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \]

It is not difficult to see that the homotopy group of $SL(2, \mathbb{R})$ is $\mathbb{Z}$ which is the homotopy group of a circle. Eq. (3.30) implies for $g$ given by eq. (7.5)
\[ \left( \hat{D}_m^+(g(\theta)) f \right)(w) = e^{-i(m+1)\theta} f(e^{-2i\theta} w), \]
which explains the quantization of $m$. On the contrary, eq. (4.1) describes the continuous series and does not impose any restriction on $\rho$ following from periodicity in $\theta$. The same concerns the supplementary series.

Admitting the representations of the universal covering $\widetilde{SL}(2, \mathbb{R})$ one can relax the quantization condition for the coupling constant in the "discrete" series. In the case of continuous series the parameter $\epsilon$ also ceases to be discrete. The most interesting case of appearance of the representations of the universal covering $\widetilde{SL}(2, \mathbb{R})$ is probably the case of free theory. Putting $g = 0$ in eq. (11.1) we find
\[ H = -\frac{1}{2} \frac{d^2}{dx^2}, \quad D = \frac{i x}{2} \frac{d}{dx} + \frac{i}{4}, \quad K = \frac{1}{2} x^2. \]
The above operators act in $L^2(\mathbb{R})$. Both $\hat{H}$ and $\hat{K}$ are positive so only discrete series enter the game. It follows from eq. (3.27) (or (3.48)) that $m = \pm \frac{1}{2}$ (for the universal covering of $SL(2, \mathbb{R})$ and the counterpart of the discrete series $m > -1$). Repeating the reasoning presented in Section 3 we find the coordinate representation for $m = \frac{1}{2}$
\[ \hat{f}_E(x) = i^{-\frac{1}{2}} \sqrt{x} J_{\frac{1}{2}}(\sqrt{2Ex}) = i^{-\frac{1}{2}} \sqrt{\frac{2}{\pi \sqrt{2E}}} \sin \sqrt{2E} x, \quad x > 0. \]
Putting \( m = -1/2 \) and choosing the parametrization of the eigenvalues of \( \hat{K} \) as

\[
\kappa = \frac{x^2}{2}, \quad x < 0,
\]

one easily finds

\[
\tilde{f}_E(x) = i^{-\frac{1}{2}} \sqrt{|x|} J_{-\frac{1}{2}}(|x|\sqrt{2E}) = i^{-\frac{1}{2}} \sqrt{\frac{2}{\pi\sqrt{2E}}} \cos \sqrt{2Ex}, \quad x < 0.
\]

(7.10)

So we are dealing with the representation acting in \( L^2(\mathbb{R}) \) which is reducible as a sum of representations corresponding to \( m = \frac{1}{2} \) and \( m = -\frac{1}{2} \) acting in the subspaces of functions having their supports in right and left semiaxes, respectively. Now, one can proceed as follows. The functions (7.8) can be defined on the whole real axis by antisymmetry while the functions (7.9) by symmetry. As a result the \( m = \frac{1}{2} \) and \( m = -\frac{1}{2} \) representations can be described as acting in the subspaces of \( L^2(\mathbb{R}) \) of odd or even functions, respectively. Our representation defined in \( L^2(\mathbb{R}) \), is obtained by decomposing any element \( f \in L^2(\mathbb{R}) \) into even and odd parts, \( f(x) = \frac{1}{2}((f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)) \) (cf. the results obtained in Ref. [45]). As in the previous cases one can start with the classical phase space. In the present case it is a (say, forward) light cone \( (\xi_0)^2 - (\xi_1)^2 - (\xi_2)^2 = 0 \). It is parametrized as follows (cf. eqs. (2.18) and (2.19))

\[
\begin{align*}
\xi_0 &= \frac{1}{4}(x^2 + p^2), \\
\xi_1 &= \frac{1}{4}(x^2 - p^2), \\
\xi_2 &= -\frac{1}{2}xp,
\end{align*}
\]

(7.11)

with \( x \) and \( p \) being the Darboux variables. On the quantum level

\[
\begin{align*}
\xi_0 &= \frac{1}{4}(x^2 - \frac{d^2}{dx^2}), \\
\xi_1 &= \frac{1}{4}(x^2 + \frac{d^2}{dx^2}), \\
\xi_2 &= \frac{i}{2}x \frac{d}{dx} + \frac{i}{4}.
\end{align*}
\]

(7.12) \quad (7.13) \quad (7.14)

Then one easily checks that

\[
(\xi_0)^2 - (\xi_1)^2 - (\xi_2)^2 = -\frac{3}{16},
\]

(7.15)

which perfectly agrees with eq. (3.46). Let us note that, with the line \( \xi_0 = -\xi_1, \xi_2 = 0 \) deleted, the cone can be mapped onto the half-plane \( x > 0 (x < 0), p \in \mathbb{R} \):

\[
x = \pm \sqrt{2(\xi_0 + \xi_1)}, \quad p = \frac{\mp 2\xi_2}{\sqrt{2(\xi_0 + \xi_1)}},
\]

(7.16)
However, if we consider the whole x-p plane, we are dealing with a double covering of the cone (cf. eq. (2.21)). This is a classical counterpart of the construction of quantum theory described above.

The full description of quantum systems related to the representations of the universal covering of $SL(2, \mathbb{R})$ will be given elsewhere.

8 Discussion

We have described the quantum version of the conformal mechanics of one degree of freedom from the point of view of the unitary representations of the $SL(2, \mathbb{R})$ group – conformal group in (1 + 0)-dimension. Let us emphasize the main points of the discussion.

According to the common wisdom the conformal theory is well-defined in the repelling case, i.e., for positive coupling constant $g^2 > 0$. The Hamiltonian can be then consistently defined as a self-adjoint operator generating unitary dynamics. On the contrary, the attractive case, $g^2 < 0$, is believed to be plagued by the ”falling on the center” phenomenon. However, even in this case one can define the self-adjoint operator starting from the formal differential expression (1.1). The point is that the resulting energy spectrum breaks the conformal symmetry. One obtains a dynamical system with the conformal symmetry broken ”quantum mechanically” (i.e., we are faced with kind of quantum anomalies). The source of the trouble may be traced back to the classical case. In the attractive case we are dealing with a kind of pathology: every trajectory hits in finite time (positive or negative) the boundary $x = 0$ of the phase space. This conclusion is, however, misleading. If we define an elementary conformal invariant system as the one described by the phase space on which the conformal group acts transitively we can classify such systems using the orbits method. It appears that the phase space for the attractive case is isomorphic to one-sheeted hyperboloid which has nontrivial topology. The conformal invariant dynamics is perfectly regular, with no singularities. The apparent singularity appears due to the fact that the phase space, being topologically nontrivial, cannot be covered by one map. When proper covering of phase space is constructed the dynamics becomes smooth. Once this fact is properly recognized both classical and quantum dynamics can be quite easily constructed. In the quantum case the starting point is the choice of a unitary representation of $SL(2, \mathbb{R})$; in this way the exact conformal symmetry is built into the theory from very beginning. It is then easy to find the generators as well-defined self-adjoint operators. It remains to define the coordinate representation. To this end we invoke eqs. (1.3), (2.12), (2.15) and (2.16) to define the x coordinate in terms of the spectrum of $\hat{K}$. The picture is closed by showing that the canonical (geometric) quantization of classical theory defined on the relevant coadjoint orbits yields the quantum picture we have started with.

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