STANLEY DECOMPOSITIONS AND PARTITIONABLE SIMPLICIAL COMPLEXES

JÜRGEN HERZOG, ALI SOLEYMAN JAHAN AND SIAMAK YASSEMI

Dedicated to Takayuki Hibi on the occasion of his fiftieth birthday

ABSTRACT. We study Stanley decompositions and show that Stanley’s conjecture on Stanley decompositions implies his conjecture on partitionable Cohen-Macaulay simplicial complexes. We also prove these conjectures for all Cohen-Macaulay monomial ideals of codimension 2 and all Gorenstein monomial ideals of codimension 3.

INTRODUCTION

In this paper we discuss a conjecture of Stanley [St2] concerning a combinatorial upper bound for the depth of a \( \mathbb{Z}^n \)-graded module. Here we consider his conjecture only for \( S/I \), where \( I \) is a monomial ideal.

Let \( K \) be a field, \( S = K[x_1, \ldots, x_n] \) the polynomial ring in \( n \) variables. Let \( u \in S \) be a monomial and \( Z \) a subset of \( \{x_1, \ldots, x_n\} \). We denote by \( uK[Z] \) the \( K \)-subspace of \( S \) whose basis consists of all monomials \( uv \) where \( v \) is a monomial in \( K[Z] \). The \( K \)-subspace \( uK[Z] \subset S \) is called a Stanley space of dimension \( |Z| \).

Let \( I \subset S \) be a monomial ideal, and denote by \( I^c \subset S \) the \( K \)-linear subspace of \( S \) spanned by all monomials which do not belong to \( I \). Then \( S = I^c \oplus I \) as a \( K \)-vector space, and the residues of the monomials in \( I^c \) form a \( K \)-basis of \( S/I \).

A decomposition \( \mathcal{D} \) of \( I^c \) as a finite direct sum of Stanley spaces is called a Stanley decomposition of \( S/I \). Identifying \( I^c \) with \( S \) through the residues, a Stanley decomposition yields a decomposition of \( S/I \) as well. The minimal dimension of a Stanley space in the decomposition \( \mathcal{D} \) is called the Stanley depth of \( \mathcal{D} \), denoted \( \text{sdepth}(\mathcal{D}) \).

We set \( \text{sdepth}(S/I) = \max \{ \text{sdepth}(\mathcal{D}) : \mathcal{D} \text{ is a Stanley decomposition of } S/I \} \), and call this number the Stanley depth of \( S/I \).

In [St Conjecture 5.1] Stanley conjectured the inequality \( \text{sdepth}(S/I) \geq \text{depth}(S/I) \). We say \( I \) is a Stanley ideal, if Stanley’s conjecture holds for \( I \).

Not many classes of Stanley ideals are known. Apel [Ap2 Corollary 3] showed that all monomial ideals \( I \) with \( \dim S/I \leq 1 \) are Stanley ideals. He also showed [Ap2 Theorem 3 & Theorem 5] that all generic monomial ideals and all cogenic Cohen-Macaulay monomial ideals are Stanley ideals, and Soleyman Jahan [So Proposition 2.1] proved that all monomial ideals in a polynomial ring in \( n \) variables of codimension less than or equal 1 are Stanley ideals. This implies in particular a result of Apel which says that all monomial ideals in the polynomial ring in three variables are Stanley ideals.

In [HePo] the authors attach to each monomial ideal a multi-complex and introduce the concept of shellable multi-complexes. In case \( I \) is a squarefree monomial ideal, this concept of shellability coincides with non-pure shellability introduced by Björner and Wachs [BjWa]. It is shown in [HePo Theorem 10.5] that if \( I \) is pretty clean (see the definition in Section 2), then the multi-complex attached to \( I \) is shellable and \( I \) is a Stanley ideal. The concept of pretty clean modules is a generalization of clean modules introduced by Dress [Dr]. He showed that a simplicial complex is shellable if and only if its Stanley-Reisner ideal is clean.
We use these results to prove that any Cohen-Macaulay monomial ideal of codimension 2 and that any Gorenstein monomial ideal of codimension 3 is a Stanley ideal, see Proposition 1.4 and Theorem 2.1. For the proof of Proposition 1.4 we observe that the polarization of a perfect codimension 2 ideal is shellable, and show this by using Alexander duality and result of [HCHIZ] in which it is proved that any monomial ideal with 2-linear resolution has linear quotients. The proof of Theorem 2.1 is based on a structure theorem for Gorenstein monomial ideals given in [BrHe1]. It also uses the result, proved in Proposition 2.3, that a pretty clean monomial ideal remains pretty clean after applying a substitution replacing the variables by a regular sequence of monomials.

In the last section of this paper we introduce squarefree Stanley spaces and show in Proposition 3.2 that for a squarefree monomial ideal $I$, the Stanley decompositions of $S/I$ into squarefree Stanley spaces correspond bijectively to partitions into intervals of the simplicial complex whose Stanley-Reisner ideal is the ideal $I$. Stanley calls a simplicial complex $\Delta$ partitionable if there exists a partition $\Delta = \bigcup_{i=1}^{r} [F_i, G_i]$ of $\Delta$ such that for all intervals $[F_i, G_i] = \{ F \in \Delta : F_i \subset F \subset G_i \}$ one has that $G_i$ is a facet of $\Delta$. We show in Corollary 3.5 that the Stanley-Reisner ideal $I_\Delta$ of a Cohen-Macaulay simplicial complex $\Delta$ is a Stanley ideal if and only if $\Delta$ is partitionable. In other words, Stanley’s conjecture on Stanley decompositions implies his conjecture on partitionable simplicial complexes.

1. **STANLEY DECOMPOSITIONS**

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring and $I \subset S$ a monomial ideal. Note that $I$ and $I^c$ as well as all Stanley spaces are $K$-linear subspaces of $S$ with a basis which is a subset of monomials of $S$. For any $K$-linear subspace $U \subset S$ which is generated by monomials, we denote by $\text{Mon}(U)$ the set of elements in the monomial basis of $U$. It is then clear that if $u_i K[Z_i]$, $i = 1, \ldots, r$ are Stanley spaces, then $I^c = \bigoplus_{i=1}^{r} u_i K[Z_i]$ if and only if $\text{Mon}(I^c)$ is the disjoint union of the sets $\text{Mon}(u_i K[Z_i])$.

Usually one has infinitely many different Stanley decompositions of $S/I$. For example if $S = K[x_1, x_2]$ and $I = (x_1x_2)$, then for each integer $k \geq 1$ one has the Stanley decomposition

$$\mathcal{D}_k: S/I = K[x_2] \oplus \bigoplus_{j=1}^{k} x_1^j K \oplus x_1^{k+1} K[x_1]$$

deom S/I. Each of these Stanley decompositions of $S/I$ has Stanley depth 0, while the Stanley decomposition $K[x_2] \oplus x_1 K[x_1]$ of $S/I$ has Stanley depth 1.

Even though $S/I$ may have infinitely many different Stanley decompositions, all these decompositions have one property in common, as noted in [So, Section 2]. Indeed, if $\mathcal{D}$ is a Stanley decomposition of $S/I$ with $s = \text{dim} S/I$. Then the number of Stanley sets of dimension $s$ in $\mathcal{D}$ is equal to the multiplicity $e(S/I)$ of $S/I$.

There is also an upper bound for $\text{depth}(S/I)$ known, namely

$$\text{sdepth}(S/I) \leq \min \{ \text{dim} S/P : P \in \text{Ass}(S/I) \}.$$ 

see [Ap2, Section 3]. Note that for $\text{depth}(S/I)$ the same upper bound is valid. As a consequence of these observations one has

**Corollary 1.1.** Let $I \subset S$ be a monomial ideal such that $S/I$ is Cohen-Macaulay. Then the following conditions are equivalent:

(a) $I$ is a Stanley ideal.
(b) There exists a Stanley decomposition $\mathcal{D}$ of $S/I$ such that each Stanley space in $\mathcal{D}$ has dimension $d = \text{dim} S/I$.
(c) There exists a Stanley decomposition $\mathcal{D}$ of $S/I$ which has $e(S/I)$ summands.

The following result will be needed later in Section 2.
Proposition 1.2. Let $I \subset S$ be a monomial complete intersection ideal. Then $S/I$ is clean. In particular, $I$ is a Stanley ideal.

Proof. Let $u \in S$ be a monomial. We call $\text{supp}(u) = \{x_i : x_i \text{ divides } u\}$ the support of $u$. Now let $G(I) = \{u_1, \ldots, u_m\}$ be the unique minimal set of monomial generators of $I$. By our assumption, $u_1, \ldots, u_m$ is a regular sequence. This implies that $\text{supp}(u_i) \cap \text{supp}(u_j) = \emptyset$ for all $i \neq j$.

It follows from the definition of the polarization of a monomial ideal (see for example [So]), that for the polarized ideal $I^p = (u_1^p, \ldots, u_m^p)$ one again has $\text{supp}(u_i^p) \cap \text{supp}(u_j^p) = \emptyset$ for all $i \neq j$.

Thus $J = I^p$ is a squarefree monomial ideal generated by the regular sequence of monomials $v_1, \ldots, v_m$ with $v_i = u_i^p$ for all $i$.

Let $\Delta$ be the simplicial complex whose Stanley-Reisner ideal $I_{\Delta}$ is equal to $J$. The Alexander dual $\Delta^\vee$ of $\Delta$ is defined to be the simplicial complex whose faces are $\{[n] \setminus F : F \notin \Delta\}$. The Stanley-Reisner ideal of $\Delta^\vee$ is minimally generated by all monomials $x_i \cdots x_k$ where $(x_i, \ldots, x_k)$ is a minimal prime ideal of $I_{\Delta}$.

In our case it follows that $I_{\Delta^\vee}$ is minimally generated by the monomial of the form $x_i \cdots x_m$ where $x_j \in \text{supp}(v_j)$ for $j = 1, \ldots, m$. Thus we see that $I_{\Delta^\vee}$ is the matroidal ideal of the transversal matroid attached to the sets $\text{supp}(v_1), \ldots, \text{supp}(v_m)$, see [CoHe, Section 5]. In [HeTa, Lemma 1.3] and [CoHe, Section 5] it is shown that any polymatroidal ideal has linear quotients, and this implies that $\Delta$ is a shellable simplicial complex, see for example [HeHiZh1, Theorem 1.4]. Hence by the theorem of Dress quoted in the next section, $S/I_{\Delta}$ is clean. Now we use the result in [So, Theorem 3.10] which says that a monomial ideal is pretty clean (see the definition in Section 2) if and only if its polarization is clean. Therefore we conclude that $S/I$ is pretty clean. Since all prime ideals in a pretty clean filtration are associated prime ideals of $S/I$ (see [HePo, Corollary 3.4]) and since $S/I$ is Cohen-Macaulay, the prime ideals in the filtration are minimal. Hence $S/I$ is clean. Thus we conclude from [HePo, Theorem 6.5] that $I$ is Stanley ideal.

Corollary 1.3. Let $I \subset S$ be a monomial ideal with depth $S/I \geq n - 1$. Then $I$ is a Stanley ideal.

Proof. The assumption implies that $I$ is a principal ideal. Thus the assertion follows from Proposition 1.2. □

With the same techniques as in the proof of Proposition 1.2, we can show

Proposition 1.4. Let $I \subset S$ be a monomial ideal which is perfect and of codimension 2. Then $S/I$ is clean. In particular, $I$ is a Stanley ideal.

Proof. We will show that the polarized ideal $I^p$ defines a shellable simplicial complex. Then, as in the proof of Proposition 1.2, it follows that $S/I$ is clean. Note that $I^p$ is a perfect squarefree monomial ideal of codimension 2. Let $\Delta$ be the simplicial complex defined by $I^p$. By the Eagon–Reiner theorem [EaRe] and a result of Terai [T], the ideal $I_{\Delta^\vee}$ has a 2-linear resolution. Now we use the fact, proved in [HeHiZh, Theorem 3.2], that an ideal with 2-linear resolution has linear quotients which in turn implies that $\Delta$ is shellable, as desired. □

Combining the preceding results with Apel’s result according to which all monomial ideals with $\dim S/I \leq 1$ are Stanley ideals we obtain

Corollary 1.5. Let $I \subset S$ be a monomial ideal. If $n \leq 4$ and $S/I$ is Cohen-Macaulay, then $I$ is a Stanley ideal.

2. GORENSTEIN MONOMIAL IDEALS OF CODIMENSION 3

As the main result of this section we will show

Theorem 2.1. Each Gorenstein monomial ideal of codimension 3 is a Stanley ideal.
Proposition 2.3. Obviously, any clean ideal is pretty clean. In [HePo, Theorem 6.5] it is shown that if the monomial generators of the Reisner ideal. Furthermore, let $u$ be a pretty clean filtration $\mathcal{F}$ of $T/I$ with $I_k/I_{k+1} = T/P_k$ for all $k$.

Observe that the $K$-homomorphism $\phi : T \to S$ is flat, since $u_1, \ldots, u_r$ is a regular sequence. Hence if we set $J_k = \phi(I_k)S$ for $k = 1, \ldots, m$, then we obtain the filtration $\phi(I)S = J_0 \subset J_1 \subset \cdots \subset J_m = S$ with $J_k/J_{k+1} \cong S/\phi(P_k)S$.

Suppose $P_k = (y_{i_1}, \ldots, y_{i_k})$, then $\phi(P_k)S = (u_{i_1}, \ldots, u_{i_k})$. In other words, $\phi(P_k)S$ is a monomial complete intersection, and hence by Proposition 1.2, we have that $S/\phi(P_k)S$ is clean. Therefore there exists a prime filtration $J_k = J_{k_1} \subset J_{k_2} \subset \cdots \subset J_{k_n} = J_{k+1}$ such that $J_k/J_{k+1} \cong S/P_k$ where $P_k$ is a minimal prime ideal of $\phi(P_k)S$. Since $\phi(P_k)S = (u_{i_1}, \ldots, u_{i_k})S$ is a complete intersection, all minimal prime ideals of $\phi(P_k)$ have height $t_k$.

Composing the prime filtrations of the $J_k/J_{k+1}$, we obtain a prime filtration of $S/\phi(I)S$. We claim that this prime filtration is (pretty) clean. In fact, let $P_{k_i}$ and $P_{j_i}$ be two prime ideals in the support of this filtration. We have to show: if $P_{k_i} \subset P_{j_i}$ for $k < \ell$, or $P_{k_i} \subset P_{j_i}$ for $k = \ell$ and $i < j$, then $P_{k_i} = P_{j_i}$. In case $k = \ell$, we have height($P_{k_i}$) = height($P_{j_i}$) = $t_k$, and the assertion follows. In case $k < \ell$, by using the fact that $\mathcal{F}$ is a pretty clean filtration, we have that $P_k = P_\ell$ or $P_k \not\subset P_\ell$. In the first case, the prime ideals $P_k$ and $P_\ell$ have the same height, and the assertion follows. In the second case there exists a variable $y_\ell \in P_k \setminus P_\ell$. Then the monomial $u_\ell$ belongs to $\phi(P_\ell)S$ but not to $\phi(P_\ell)S$. This implies that $P_k$ contains a variable which belongs to the support of $u_\ell$. However this variable cannot be a generator of $P_{j_\ell}$, because the support of $u_\ell$ is disjoint of the support of all the monomial generators of $\phi(P_k)S$. This shows that $P_{k_i} \not\subset P_{j_i}$. 

Corollary 2.4. Let $\Delta$ be a shellable simplicial complex and $I_\Delta \subset T = K[y_1, \ldots, y_r]$ its Stanley-Reisner ideal. Furthermore, let $u_1, \ldots, u_r \subset S = K[x_1, \ldots, x_n]$ be a regular sequence of monomials, and let $\phi(y_i) = u_i$ for $i = 1, \ldots, r$. Then $\phi(I_\Delta)S$ is a Stanley ideal.
Proof. By the theorem of Dress, the ring $T/I_{\Delta}$ is clean. Therefore, $S/\varphi(I_{\Delta})S$ is again clean, by Proposition 2.3. In particular, $S/\varphi(I_{\Delta})S$ is pretty clean which according to [HePo] Theorem 6.5 implies that $\varphi(I_{\Delta})S$ is a Stanley ideal. □

Proof of Theorem 2.7 Let $\Delta$ be the simplicial complex whose Stanley-Reisner ideal

$$I_{\Delta} \subset T = K[y_1, \ldots, y_{2m+1}]$$

is generated by the monomials $y_iy_{i+1} \cdots y_{i+m-1}$, $i = 1, \ldots, 2m+1$, where $y_i = y_{i-2m-1}$ whenever $i > 2m+1$, and let $u_1, \ldots, u_{2m+1} \subset S = K[x_1, \ldots, x_n]$ be the regular sequence given in Theorem 2.1.

Then we have $I = \varphi(I_{\Delta})S$ where $\varphi(y_j) = u_j$ for all $j$. Therefore, by Corollary 2.4 it suffices to show that $\Delta$ is shellable.

Identifying the vertex set of $\Delta$ with $[2m+1] = \{1, \ldots, 2m+1\}$ and observing that $I_{\Delta}$ is of codimension 3, it is easy to see that $F \subset [2m+1]$ is a facet of $\Delta$ if and only if $F = [2m+1] \setminus \{a_1, a_2, a_3\}$ with

$$a_2 - a_1 < m + 1, \quad a_3 - a_2 < m + 1, \quad a_3 - a_1 > m.$$ 

We denote the facet $[2m+1] \setminus \{a_1, a_2, a_3\}$ by $F(a_1, a_2, a_3)$.

We will show that $\Delta$ is shellable with respect to the lexicographic order. Note that $F(a_1, a_2, a_3) < F(b_1, b_2, b_3)$ in the lexicographic order, if and only if either $b_1 < a_1$, or $b_1 = a_1$ and $b_2 < a_2$, or $a_1 = b_1$, $a_2 = b_2$ and $a_3 < b_3$.

In order to prove that $\Delta$ is shellable we have to show: if $F = F(a_1, a_2, a_3)$ and $G = F(b_1, b_2, b_3)$ with $F < G$, then there exists $c \in G \setminus F$ and some facet $H$ such that $H < G$ and $G \setminus H = \{c\}$.

We know that $|G \setminus F| \leq 3$. If $|G \setminus F| = 1$, then there is nothing to prove. In the following we discuss the cases $|G \setminus F| = 2$ and $|G \setminus F| = 3$. The discussion of these cases is somewhat tedious but elementary. For the convenience of the reader we list all the possible cases.

Case 1: $|G \setminus F| = 2$.

(i) If $b_1 = a_1 < b_2 < a_2$, then we choose $H = (G \setminus \{a_2\}) \cup \{b_2\}$.

(ii) If $b_1 < b_2 = a_1$ or $b_1 < b_2 < a_1 < a_2 = b_3 < a_3$, then we choose $H = (G \setminus \{a_3\}) \cup \{b_1\}$.

(iii) If $b_1 < a_1 < b_2 < a_2 = b_3 < a_3$, we consider the following two subcases:

- for $a_3 - b_2 < m + 1$, we choose $H = (G \setminus \{a_3\}) \cup \{b_3\}$.
- for $a_3 - b_2 \geq m + 1$, we choose $H = (G \setminus \{a_3\}) \cup \{b_1\}$.

(iv) If $b_1 < a_1 < a_2 = b_2 < a_3$, then we choose $H = (G \setminus \{a_3\}) \cup \{b_3\}$.

(v) If $b_1 < a_1 < a_2 = b_2 < a_3 < b_3$ or $b_1 < a_1 < a_2 < a_3 = b_2 < b_3$, then we choose $H = (G \setminus \{a_1\}) \cup \{b_1\}$.

Case 2: $|G \setminus F| = 3$.

(i) If $b_1 < a_1 < a_2 < a_3 < b_3$, then we choose $H = (G \setminus \{a_1\}) \cup \{b_1\}$.

(ii) If $b_1 < b_2 < b_3 < a_1 < a_2 = a_3 < b_3$ or $b_1 < b_2 < a_1 < a_2 < a_3$ and $a_1 < b_3$, then we choose $H = (G \setminus \{a_1\}) \cup \{b_2\}$.

(iii) If $b_1 < a_1 < b_2 < a_3 < a_2 < a_3$, then we choose $H = (G \setminus \{a_2\}) \cup \{b_3\}$.

(iv) If $b_1 < a_1 < a_2 < b_2 < a_3$, we consider the following two subcases:

- for $a_3 - b_2 < m + 1$, we choose $H = (G \setminus \{a_3\}) \cup \{b_3\}$.
- for $a_3 - b_2 \geq m + 1$, we choose $H = (G \setminus \{a_3\}) \cup \{b_1\}$.

(v) If $b_1 < a_1 < a_2 < b_2 < b_3 < a_3$, then we choose $H = (G \setminus \{a_3\}) \cup \{b_3\}$. □

Combining the result of Theorem 2.7 with the result of Apel [Ap2, Corollary 3] we obtain

Corollary 2.5. Let $I \subset S$ be monomial ideal. If $n \leq 5$ and $S/I$ is Gorenstein, then $I$ is a Stanley ideal.
3. **Squarefree Stanley decompositions and partitions of simplicial complexes**

A Stanley space $uK[Z]$ is called a *squarefree Stanley space*, if $u$ is a squarefree monomial and $\supp(u) \subseteq \mathbb{Z}$. We shall use the following notation: for $F \subseteq [n]$ we set $x_F = \prod_{i \in F} x_i$ and $Z_F = \{ x_i : i \in F \}$. Then a Stanley space is squarefree if and only if it is of the form $x_F K[Z_G]$ with $F \subseteq G \subseteq [n]$.

A Stanley decomposition of $S/I$ is called a *squarefree Stanley decomposition* of $S/I$, if all Stanley spaces in the decomposition are squarefree.

**Lemma 3.1.** Let $I \subseteq S$ be a monomial ideal. The following conditions are equivalent:

(a) $I$ is a squarefree monomial ideal.

(b) $S/I$ has a squarefree Stanley decomposition.

**Proof.** (a) $\Rightarrow$ (b): We may view $I$ as the Stanley-Reisner ideal of some simplicial complex $\Delta$. With each $F \in \Delta$ we associate the squarefree Stanley space $x_F K[Z_F]$. We claim that $\bigoplus_{F \in \Delta} x_F K[Z_F]$ is a (squarefree) Stanley decomposition of $S/I$. Indeed, a monomial $u \in S$ belongs to $I^c$ if and only if $\supp(u) \subseteq \Delta$, and these monomial form a $K$-basis for $I^c$. On the other hand, a monomial $u \in S$ belongs to $x_F K[Z_F]$ if and only if $\supp(u) = F$. This shows that $I^c = \bigoplus_{F \in \Delta} x_F K[Z_F]$.

(b) $\Rightarrow$ (a): Let $\bigoplus_{i \in F} K[Z_i]$ be a squarefree Stanley decomposition of $S/I$. Assume that $I$ is not a squarefree monomial ideal. Then there exists $u \in G(I)$ which is not squarefree and we may assume that $x_{i_1}|u$. Then $u' = u/x_{i_1} \in F$, and hence there exists $i$ such that $u' \in u_i K[Z_i]$. Since $x_1|u'$ it follows that $x_1 \in Z_i$. Therefore $u \in u_i K[Z_i] \subset I^c$, a contradiction.

Let $\Delta$ be a simplicial complex of dimension $d-1$ on the vertex set $V = \{ x_1, \ldots, x_n \}$. A subset $\mathcal{I} \subset \Delta$ is called an *interval*, if there exits faces $F, G \in \Delta$ such that $\mathcal{I} = \{ H \in \Delta : F \subseteq H \subseteq G \}$. We denote this interval given by $F$ and $G$ also by $[F, G]$ and call $\dim G - \dim F$ the *rank* of the interval. A partition $\mathcal{P}$ of $\Delta$ is a presentation of $\Delta$ as a disjoint union of intervals. The $r$-vector of $\mathcal{P}$ is the integer vector $r = (r_0, r_1, \ldots, r_d)$ where $r_i$ is the number of intervals of rank $i$.

**Proposition 3.2.** Let $\mathcal{P} : \Delta = \bigcup_{i=1}^d [F_i, G_i]$ be a partition of $\Delta$. Then

(a) $D(\mathcal{P}) = \bigoplus_{i=1}^d x_{F_i} K[Z_{G_i}]$ is squarefree Stanley decomposition of $S/I$.

(b) The map $\mathcal{P} \mapsto D(\mathcal{P})$ establishes a bijection between partitions of $\Delta$ and squarefree Stanley decompositions of $S/I$.

**Proof.** (a) Since each $x_{F_i} K[Z_{G_i}]$ is a squarefree Stanley space it suffices to show that $I^c$ is indeed the direct sum of the Stanley spaces $x_{F_i} K[Z_{G_i}]$. Let $u \in \Mon(I^c)$; then $H = \supp(u) \in \Delta$. Since $\mathcal{P}$ is a partition of $\Delta$ it follows that $H \in [F_i, G_i]$ for some $i$. Therefore, $u = x_{F_i} u'$ for some monomial $u' \in K[Z_{G_i}]$. This implies that $u \in x_{F_i} K[Z_{G_i}]$. This shows that $\Mon(I^c)$ is the union of sets $\Mon(x_{F_i} K[Z_{G_i}])$. Suppose there exists a monomial $u \in x_{F_i} K[Z_{G_i}] \cap x_{F_j} K[Z_{G_j}]$. Then $\supp(u) \in [F_i, G_i] \cap [F_j, G_j]$. This is only possible if $i = j$, since $\mathcal{P}$ is partition of $\Delta$.

(b) Let $[F_i, G_i]$ and $[F_j, G_j]$ be two intervals. Then $x_{F_i} K[Z_{G_i}] = x_{F_i} K[Z_{G_j}]$ if and only if $[F_i, G_i] = [F_j, G_j]$. Indeed, if $x_{F_i} K[Z_{G_i}] = x_{F_j} K[Z_{G_j}]$, then $x_{F_i} \in x_{F_j} K[Z_{G_j}]$, and hence $x_{F_i} | x_{F_j}$. By symmetry we also have $x_{F_j} | x_{F_i}$. In other words, $F_i = F_j$, and it also follows that $K[Z_{G_i}] = K[Z_{G_j}]$. This implies $G_i = G_j$. These considerations show that $\mathcal{P} \mapsto D(\mathcal{P})$ is injective.

On the other hand, let $\mathcal{P} : S/I = \bigoplus_{i=1}^d x_{F_i} K[Z_{G_i}]$ be an arbitrary squarefree Stanley decomposition of $S/I$. By the definition of a squarefree Stanley set we have $F_i \subseteq G_i$, and since $x_{F_i} K[Z_{G_i}] \subset I^c$, it follows that $G_i \in \Delta$. Hence $[F_i, G_i]$ is an interval of $\Delta$, and a squarefree monomial $x_F$ belongs to $x_{F_i} K[Z_{G_i}]$ if and only if $F \in [F_i, G_i]$.

Let $F \subset \Delta$ be an arbitrary face. Then $x_F \in \Mon(I^c) = \bigcup_{i=1}^d \Mon(x_{F_i} K[Z_{G_i}])$. Hence the squarefree monomial $x_F$ belongs to $x_{F_i} K[Z_{G_i}]$ for some $i$, and hence $F \in [F_i, G_i]$. This shows that
\[ \bigcup_{i=1}^{d}[F_i, G_i] = \Delta. \] Suppose \( F \in [F_i, G_i] \cap [F_j, G_j] \). Then \( x_F \in x_{F_i}K[Z_{G_i}] \cap x_{F_j}K[Z_{G_j}] \), a contradiction. Hence we see that \( \mathcal{P}: \Delta = \bigcup_{i=1}^{d}[F_i, G_i] \) is a partition of \( \Delta \) with \( D(\mathcal{P}) = \mathcal{P} \). \( \square \)

Now let \( I \subset S \) be a squarefree monomial ideal. Then we set
\[
s\text{depth}(S/I) = \max\{s\text{depth}(\mathcal{P}): \mathcal{P} \text{ is a squarefree Stanley decomposition of } S/I\},
\]
and call this number the \textit{squarefree Stanley depth} of \( S/I \).

As the main result of this section we have

**Theorem 3.3.** Let \( I \subset S \) be a squarefree monomial ideal. Then \( s\text{depth}(S/I) = \text{depth}(S/I) \).

**Proof.** Let \( \mathcal{P} \) be any Stanley decomposition of \( S/I \), and let \( \Delta \) be the simplicial complex with \( I = I_\Delta \). For each \( F \in \Delta \) we have \( x_F \in I' \). Hence there exists a summand \( uK[Z] \) with \( x_F \in uK[Z] \).

Since \( x_F \) is squarefree it follows that \( u = x_G \) is squarefree and \( F \subseteq G \cup Z \). Let \( \mathcal{P}' \) the sum of those Stanley spaces \( uK[Z] \) in \( \mathcal{P} \) for which \( u \) is a squarefree monomial. Then this sum is direct. Therefore the intervals \([G, G \cup Z]\) corresponding to the summands in \( \mathcal{P}' \) are pairwise disjoint. On the other hand these intervals cover \( \Delta \), as we have seen before, and hence form a partition of \( \mathcal{P} \) of \( \Delta \). It follows from the construction of \( \mathcal{P} \) that \( s\text{depth}(D(\mathcal{P})) \geq \text{depth}(\mathcal{P}) \). This shows that \( s\text{depth}(S/I) \geq \text{depth}(S/I) \). The other inequality \( s\text{depth}(S/I) \leq \text{depth}(S/I) \) is obvious. \( \square \)

**Corollary 3.4.** Let \( \Delta \) be a simplicial complex. Then the following conditions are equivalent:

(a) \( I_\Delta \) is a Stanley ideal.

(b) There exists a partition \( \Delta = \bigcup_{i=1}^{d}[F_i, G_i] \) with \( |G_i| \geq \text{depth}(K[\Delta]) \) for all \( i \).

Let \( \Delta \) be a simplicial complex and \( \mathcal{F}(\Delta) \) its set of facets. Stanley calls a simplicial complex \( \Delta \) \textit{partitionable} if there exists a partition \( \Delta = \bigcup_{i=1}^{d}[F_i, G_i] \) with \( \mathcal{F}(\Delta) = \{G_1, \ldots, G_d\} \). We call a partition with this property a \textit{nice partition}. Stanley conjectures [St1] Conjecture 2.7 (see also [St2] Problem 6) that each Cohen-Macaulay simplicial complex is partitionable. In view of Corollary 1.1 it follows that the conjecture of Stanley decompositions implies the conjecture on partitionable simplicial complexes. More precisely we have

**Corollary 3.5.** Let \( \Delta \) be a Cohen-Macaulay simplicial complex with \( h \)-vector \((h_0, h_1, \ldots, h_d)\). Then the following conditions are equivalent:

(a) \( I_\Delta \) is a Stanley ideal.

(b) \( \Delta \) is partitionable.

(c) \( \Delta \) admits a partition whose \( r \)-vector satisfies \( r_i = h_{d-i} \) for \( i = 0, \ldots, d \).

(d) \( \Delta \) admits a partition into \( e(K[\Delta]) \) intervals.

Moreover, any nice partition of \( \Delta \) satisfies the conditions (c) and (d).

**Proof.** (a) \( \iff \) (b) follows from Corollary 3.4. In order to prove the implication (b) \( \Rightarrow \) (c), consider a nice partition \( \Delta = \bigcup_{i=1}^{d}[F_i, G_i] \) of \( \Delta \). From this decomposition the \( f \)-vector of \( \Delta \) can be computed by the following formula
\[ \sum_{i=0}^{d} f_{i-1}t^i = \sum_{i=0}^{d} r_it^{d-i}(1 + t)^i. \]

On the other hand one has
\[ \sum_{i=0}^{d} f_{i-1}t^i = \sum_{i=0}^{d} h_it^{d-i}(1 + t)^i, \]
see [BrHe] p. 213]. Comparing coefficients the assertion follows.

The implication (c) \( \Rightarrow \) (d) follows from the fact that \( e(K[\Delta]) = \sum_{i=0}^{d} h_i \), see [BrHe]. Proposition 4.1.9. Finally (d) \( \Rightarrow \) (a) follows from Corollary 1.1. \( \square \)
We conclude this section with some explicit examples. Recall that constructibility, a general-
ization of shellability, is defined recursively as follows: (i) a simplex is constructible, (ii) if \( \Delta_1 \)
and \( \Delta_2 \) are \( d \)-dimensional constructible complexes and their intersection is a \((d-1)\)-dimensional
constructible complex, then their union is constructible. In this definition, if in the recursion we
restrict \( \Delta_2 \) always to be a simplex, then the definition becomes equivalent to that of (pure) shella-
bility. The notion of constructibility for simplicial complexes appears in [St3]. It is known and
easy to see that

\[
\text{Shellable} \Rightarrow \text{constructible} \Rightarrow \text{Cohen-Macaulay}.
\]

Since any shellable simplicial complex is partitionable (see [St1, p. 79]), it is natural to ask
whether any constructible complex is partitionable? This question is a special case of Stanley’s
conjecture that says that Cohen-Macaulay simplicial complexes are partitionable. We do not know
the answer yet! In the following we present some examples where the complexes are not shellable
or are not Cohen-Macaulay but the ideals related to these simplicial complexes are Stanley ideals.

**Example 3.6.** The following example of a simplicial complex is due to Masahiro Hachimori
[Ha]. The simplicial complex \( \Delta \) described by the next figure is 2-dimensional, non shellable but
constructible. It is constructible, because if we divide the simplicial complex by the bold line,
we obtain two shellable complexes, and their intersection is a shellable 1-dimensional simplicial
complex.

![Diagram of a 2-dimensional simplicial complex](image)

Indeed we can write \( \Delta = \Delta_1 \cup \Delta_2 \) where the shelling order of the facets of \( \Delta_1 \) is given by:

\[148, 149, 140, 150, 189, 348, 349, 378, 340, 390, 590, 569, 689, 678,\]

and that of \( \Delta_2 \) is given by:

\[125, 126, 127, 167, 235, 236, 237, 356.\]

We use the following principle to construct a partition of \( \Delta \): suppose that \( \Delta_1 \) and \( \Delta_2 \) are \( d \)-dimensional partitionable simplicial complexes, and that \( \Gamma = \Delta_1 \cap \Delta_2 \) is \((d-1)\)-dimensional pure simplicial complex. Let \( \Delta_1 = \bigcup_{i=1}^{s} [K_i, L_i] \) be a nice partition of \( \Delta_1 \), and \( \Delta_2 = \bigcup_{j=1}^{t} [F_i, G_i] \) a nice
partition of \( \Delta_2 \). Suppose that for each \( i \), the set \([F_i, G_i] \setminus \Gamma\) has a unique minimal element \( H_i \). Then
\( \Delta_1 \cup \Delta_2 = \bigcup_{i=1}^{s} [K_i, L_i] \cup \bigcup_{j=1}^{t} [H_i, G_i] \) is a nice partition of \( \Delta_1 \cup \Delta_2 \). Notice that \([F_i, G_i] \setminus \Gamma\) has a
unique minimal element if and only if for all \( F \in [F_i, G_i] \cap \Gamma \) there exists a facet \( G \) of \( \Gamma \) with
\( F \subseteq G \subseteq G_i \).

Suppose that \( \Delta_2 \) is shellable with shelling \( G_1, \ldots, G_s \). Let \( F_i \) be the unique minimal subface of
\( G_i \) which is not a subface of any \( G_j \) with \( j < i \). Then \( \Delta_2 = \bigcup_{i=1}^{s} [F_i, G_i] \) is the nice partition induced
by this shelling. The above discussions then show that \( \Delta_1 \cup \Delta_2 \) is partitionable, if for all \( i \) and all
\( F \in \Gamma \) such that \( F \subseteq G_i \) and \( F \not\subseteq G_j \) for \( j < i \), there exists a facet \( G \in \Gamma \) with \( F \subseteq G \subseteq G_i \).
In our particular case the shelling of $\Delta_1$ induces the following partition of $\Delta_1$:

$$\emptyset, 148, [9, 149], [0, 140], [5, 150], [89, 189], [3, 348], [39, 349], [7, 378], [30, 340], [90, 390], [59, 590], [6, 569], [68, 689], [67, 678],$$

and the shelling of $\Delta_2$ induces the following partition of $\Delta_2$:

$$\emptyset, 125, [6, 126], [7, 127], [67, 167], [3, 235], [36, 236], [37, 237], [56, 356].$$

The facets of $\Gamma = \Delta_1 \cap \Delta_2$ are: 15, 56, 67, 73.

The restriction of the intervals of this partition of $\Delta_2$ to the complement of $\Gamma$ do not all give intervals. For example we have $[6, 126] \setminus \Gamma = \{16, 26, 126\}$. This set has two minimal elements, and hence is not an interval. On the other hand, the following partition of $\Delta_2$ (which is not induced from a shelling)

$$\emptyset, 237, [1, 125], [5, 356], [6, 167], [17, 127], [25, 235], [26, 126], [36, 236]$$

restricted to the complement of $\Gamma$ yields the following intervals

$$[2, 237], [12, 125], [35, 356], [16, 167], [17, 127], [25, 235], [26, 126], [36, 236],$$

which together with the intervals of the partition of $\Delta_1$ give us a partition of $\Delta$.

**Example 3.7.** (The Dunce hat) The Dunce hat is the topological space obtained from the solid triangle $abc$ by identifying the oriented edges $\overrightarrow{ab}$, $\overrightarrow{bc}$ and $\overrightarrow{ac}$. The following is a triangulation of the Dunce hat using 8 vertices.

![Dunce hat diagram](image)

The facets arising from this triangulation are

$$124, 125, 145, 234, 348, 458, 568, 256, 236, 138, 128, 278, 678, 237, 137, 167, 136.$$

It is known that the simplicial complex corresponding to this triangulation is not shellable (not even constructible), but it is Cohen-Macaulay, see [H2], and it has the following partition:

$$\emptyset, 124, [3, 234], [5, 145], [6, 236], [7, 137], [8, 348], [13, 138], [16, 136], [18, 128], [25, 125], [27, 237], [28, 278], [56, 256], [67, 167], [68, 568], [78, 678], [58, 458].$$

Therefore we have again depth($\Delta$) = dim($\Delta$) = sdepth($\Delta$) = 3.

**Example 3.8.** (The Cylinder) The ideal $I = (x_1x_4, x_2x_5, x_3x_6, x_1x_3x_5, x_2x_4x_6) \subset K[x_1, \ldots, x_6]$ is the Stanley-Reisner ideal of the triangulation of the cylinder shown in the next figure. The corresponding simplicial complex $\Delta$ is Buchsbaum but not Cohen-Macaulay.

![Cylinder diagram](image)
The facets of $\Delta$ are $123, 126, 156, 234, 345, 456$, and it has the following partition:

$$[0, 123], [4, 234], [5, 345], [6, 456], [15, 156], [16, 126], [26, 26].$$

Therefore we have $\text{depth}(\Delta) = \text{sdepth}(\Delta) = 2 < 3 = \dim(\Delta)$. Although $\Delta$ is not partitionable, $I_\Delta$ is a Stanley ideal.

ACKNOWLEDGMENTS

This paper was prepared during the third author’s visit of the Universität Duisburg-Essen, where he was on sabbatical leave from the University of Tehran. He would like to thank Deutscher Akademischer Austausch Dienst (DAAD) for the partially support. He also thanks the authorities of the Universität Duisburg-Essen for their hospitality during his stay there.

REFERENCES

[Ap1] J. Apel, On a conjecture of R. P. Stanley; Part I-Monomial Ideals, J. of Alg. Comb. 17, (2003), 36–59.
[Ap2] J. Apel, On a conjecture of R. P. Stanley; Part II-Quotients Modulo Monomial Ideals, J. of Alg. Comb. 17, (2003), 57–74.
[BiWa] A. Björner, M. Wachs, Shellable nonpure complexes and posets. I, Trans. Amer. Math. Soc. 349 (1997), 3945–3975.
[BrHe1] W. Bruns, J. Herzog, On multigraded resolutions, Math. Proc. Cambridge Phil. Soc. 118, (1995), 234-251.
[BrHe] W. Bruns, J. Herzog, Cohen Macaulay rings, Revised Edition, Cambridge, 1996.
[Dr] A. Dress, A new algebraic criterion for shellability, Beitrage zur Alg. und Geom., 34(1), (1993), 45–55.
[EaRe] J. Eagon and V. Reiner, Resolutions of Stanley–Reisner rings and Alexander duality, J. Pure Appl. Algebra 130 (1998), 265–275.
[CoHe] A. Conca, J. Herzog, Castelnuovo-Mumford regularity of products of ideals. Collect. Math. 54 (2003), 137-152.
[Ha] M. Hachimori, Decompositions of two-dimensional simplicial complexes. To appear in Discrete Mathematics.
[HeHiZh] J. Herzog, T. Hibi, X. Zheng, Monomial ideals whose powers have a linear resolution. Math. Scand. 95 (2004), no. 1, 23–32
[HeHiZh1] J. Herzog, T. Hibi, X. Zheng, Dirac’s theorem on chordal graphs and Alexander duality. European J. Combin. 25 (2004), no. 7, 949–960.
[HePo] J. Herzog, D. Popescu, Finite filtrations of modules and shellable multicompleses, manuscripta math. 121 (2006), 385–410.
[HeTa] J. Herzog and Y. Takayama, Resolutions by mapping cones, in: The Roos Festschrift volume Nr.2(2), Homology, Homotopy and Applications 4, (2002), 277 – 294.
[So] A. Soleyman Jahan, Prime filtrations of monomial ideals and polarizations, to appear in J. Algebra.
[St1] R. P. Stanley, Linear Diophantine equations and local cohomology, Invent. Math. 68, (1982), 175–193.
[St1] R. P. Stanley, Combinatorics and Commutative Algebra, Birkhäuser, 1983.
[St2] R. P. Stanley, Positivity Problems and Conjectures in Algebraic Combinatorics, In Mathematics: Frontiers and Perspectives (V. Arnold, M. Atiyah, P. Lax, and B. Mazur, eds.), American Mathematical Society, Providence, RI, 2000, pp. 295-319.
[St3] R. P. Stanley, Cohen-Macaulay rings and constructible polytopes, Bull. Amer. Math. Soc. 81, (1975), 133–135
[T] N. Terai, Generalization of Eagon–Reiner theorem and $h$-vectors of graded rings, Preprint 2000.