A quantitative sharpening of Moriwaki’s arithmetic Bogomolov inequality

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A. Moriwaki proved the following arithmetic analogue of the Bogomolov unstability theorem. If a torsion-free hermitian coherent sheaf on an arithmetic surface has negative discriminant then it admits an arithmetically destabilising subsheaf. In the geometric situation it is known that such a subsheaf can be found subject to an additional numerical constraint and here we prove the arithmetic analogue. We then apply this result to slightly simplify a part of C. Soulé’s proof of a vanishing theorem on arithmetic surfaces.

1 Introduction and statement of result

Let $K$ be a number field with ring of integers $\mathcal{O}_K$ and $X/\text{Spec}(\mathcal{O}_K)$ an arithmetic surface, i.e. a regular, integral, purely two-dimensional scheme, proper and flat over $\text{Spec}(\mathcal{O}_K)$ and with smooth and geometrically connected generic fibre. Attached to a hermitian coherent sheaf on $X$ are the usual characteristic classes with values in the arithmetic Chow-groups $\widehat{CH}^i(X)$ (cf. [GS1], 2.5), and in particular the discriminant of $E$

$$\Delta(E) := (1 - r)\hat{c}_1(E)^2 + 2r\hat{c}_2(E) \in \widehat{CH}^2(X)$$

where $r := \text{rk}(E)$. The arithmetic degree map

$$\hat{\deg} : \widehat{CH}^2(X)_\mathbb{R} \rightarrow \mathbb{R}$$
is an isomorphism \([GS2]\) and we will use the same symbol to denote an element in \(\widetilde{CH}^2(X)_{\mathbb{R}}\) and its arithmetic degree in \(\mathbb{R}\), see \([GS2]\), 1.1 for the definition of arithmetic Chow-groups with real coefficients \(\widetilde{CH}^2(X)_{\mathbb{R}}\). Following \([Mo2]\) we define the positive cone of \(X\) to be

\[
\hat{C}^+(X) := \{ x \in \widetilde{CH}^1(X)_{\mathbb{R}} \mid x^2 > 0 \text{ and } \deg_K(x) > 0 \}.
\]

Given a torsion-free hermitian coherent sheaf \(\mathcal{E}\) of rank \(r \geq 1\) on \(X\) and a subsheaf \(\mathcal{E}' \subseteq \mathcal{E}\) we endow \(\mathcal{E}'\) with the metric induced from \(\mathcal{E}\) and consider the difference of slopes

\[
\xi_{\mathcal{E}', \mathcal{E}} := \frac{\hat{c}_1(\mathcal{E}')}{\text{rk}(\mathcal{E}')} - \frac{\hat{c}_1(\mathcal{E})}{r} \in \widetilde{CH}^1(X)_{\mathbb{R}}.
\]

Recall that a subsheaf \(\mathcal{E}' \subseteq \mathcal{E}\) is saturated if the quotient \(\mathcal{E}/\mathcal{E}'\) is torsion-free. Our main result is the following.

**Theorem 1** Let \(\mathcal{E}\) be a torsion-free hermitian coherent sheaf of rank \(r \geq 2\) on the arithmetic surface \(X\), satisfying

\[
\Delta(\mathcal{E}) < 0.
\]

Then there is a non-zero saturated subsheaf \(\mathcal{E}' \subseteq \mathcal{E}\) such that \(\xi_{\mathcal{E}', \mathcal{E}} \in \hat{C}^+(X)\) and

\[
\xi^2_{\mathcal{E}', \mathcal{E}} \geq -\frac{\Delta}{r^2(r - 1)}.
\]

**Remark 2** The existence of an \(\mathcal{E}' \subseteq \mathcal{E}\) with \(\xi_{\mathcal{E}', \mathcal{E}} \in \hat{C}^+(X)\) is the main result of \([Mo2]\) and means that \(\mathcal{E}' \subseteq \mathcal{E}\) is arithmetically destabilising with respect to any polarisation of \(X\), c.f. loc. cit. for more details on this. The new contribution here is the inequality \(\xi^2\) which is the exact arithmetic analogue of a known geometric result, c.f. for example \([HL]\), Theorem 7.3.4.

**Remark 3** A special case of Theorem \(\xi^2\) appears in disguised form in the proof of \([So]\), Theorem 2: Given a sufficiently positive hermitian line bundle \(L\)
on the arithmetic surface $X$ and some non-torsion element $e \in H^1(X, L^{-1}) \simeq \text{Ext}^1(L, \mathcal{O}_X)$, C. Soulé establishes a lower bound for

$$||e||^2 := \sup_{\sigma: \mathbb{K} \hookrightarrow \mathbb{C}} ||\sigma(e)||^2_{L^2}$$

by considering the extension determined by $e$

$$E : 0 \to \mathcal{O}_X \to E \to L \to 0$$

and suitably metrised as to have $\hat{c}_1(E) = \mathcal{L}$ and $2\hat{c}_2(E) = \sum_{\sigma} ||\sigma(e)||^2_{L^2}$, hence $\Delta(E) = -\mathcal{L}^2 + 2\sum_{\sigma} ||\sigma(e)||^2_{L^2}$ (where we write $\mathcal{L} = \hat{c}_1(L)$ following the notation of loc. cit.).

If $E_{\mathbb{Q}}$ is semi-stable the arithmetic Bogomolov inequality concludes the proof. Otherwise, the main point is to show the existence of an arithmetic divisor $D$ satisfying

(2) $\deg_K(D) \leq \deg_K(L)/2$ and

(3) $2(\mathcal{L} - D)D \leq [K : \mathbb{Q}] \cdot ||e||^2$,

c.f. (28) and (32) of loc. cit. where these inequalities are established by some direct argument. We wish to point out that the existence of some $D$ satisfying (2) and (3) is a special case of Theorem 1. In fact, let $E' \subseteq E$ be as in Theorem 1 and define $D := L - \hat{c}_1(E')$. We then compute

$$\xi_{E', E} = \mathcal{L}^2/2 - D$$

and $\xi_{E', E} \in \mathcal{C}^{++}(X)$ implies (2). Furthermore, the inequality (1) in the present case reads

$$\xi^2_{E', E} = \mathcal{L}^2/4 + D^2 - \mathcal{L}D \geq -\Delta/4 = \mathcal{L}^2/4 - 1/2 \sum_{\sigma} ||\sigma(e)||^2_{L^2}$$

i.e.

$$2(\mathcal{L} - D)D \leq \sum_{\sigma} ||\sigma(e)||^2_{L^2},$$

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hence the trivial estimate \([K : \mathbb{Q}] : ||e||^2 \geq \sum |\sigma(e)|_2^2\) gives \([\mathfrak{M}]\).

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2 Proof of Theorem 1

We collect some lemmas first. We call a short exact sequence
\[ \mathcal{E} : 0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0 \]
of hermitian coherent sheaves on \(X\) isometric if the metrics on \(\mathcal{E}'\) and \(\mathcal{E}''\) are induced from the one on \(\mathcal{E}\). This implies that \(\hat{c}_1(\mathcal{E}) = \hat{c}_1(\mathcal{E}') + \hat{c}_1(\mathcal{E}')(\text{i.e. } \tilde{c}_1(\mathcal{E}) = 0)\). We also have
\[ \hat{c}_2(\mathcal{E}) = \hat{c}_2(\mathcal{E}' \oplus \mathcal{E}'') - a(\tilde{c}_2(\mathcal{E})) \text{ in } \overline{CH}^2(X), \]
where
\[ a : \tilde{A}^{1,1}(X_{\mathbb{R}}) \rightarrow \overline{CH}^2(X) \]
is the usual map \([SABK]\), chapter III.

**Lemma 4** If
\[ \mathcal{E} : 0 \rightarrow \mathcal{E}' \rightarrow \mathcal{E} \rightarrow \mathcal{E}'' \rightarrow 0 \]
is an isometric short exact sequence of hermitian coherent sheaves on \(X\) with ranks \(r', r, r'' \geq 1\) and discriminants \(\Delta', \Delta, \Delta''\), then
\[ \frac{\Delta'}{r'} + \frac{\Delta''}{r''} - \frac{\Delta}{r} = \frac{rr'}{r''} \xi^2_{\mathcal{E}' \mathcal{E}} + 2a(\tilde{c}_2(\mathcal{E})) \text{ in } \overline{CH}^2(X)_{\mathbb{R}}. \]

**Proof** We omit the computation using the formulas for \(\tilde{c}_i(\mathcal{E})\) recalled above which shows that the left hand side of the stated equality equals
\[ \hat{c}_1(\mathcal{E})^2 \left( \frac{r - 1}{r} + \frac{1 - r'}{r'} \right) + \hat{c}_1(\mathcal{E}'')^2 \left( \frac{r - 1}{r} + \frac{1 - r''}{r''} \right) + \]
Similarly one writes \( \xi^2_{E,E'} \) as a rational linear combination of \( \hat{c}_1(E)^2, \hat{c}_1(E'')^2 \) and \( \hat{c}_1(E') \hat{c}_1(E'') \) and comparing the results, the stated formula drops out. □

**Lemma 5** For \( E \) as in Lemma 4 and \( G'' \subseteq E'' \) a saturated subsheaf of rank \( s \geq 1 \) carrying the induced metric, put

\[
\overline{G} := \ker(E \to E'' \to E''/G'') \subseteq \overline{E}
\]

with the induced metric. Then

\[
\xi_{G,E} = \frac{r'(r''-s)}{(r'+s)r''} \xi_{E,E'} + \frac{s}{r'+s} \xi_{G'',E'} \quad \text{in } \overline{CH^1(X)}_{\mathbb{R}}.
\]

Observe that the coefficients in the last expression are non-negative rational numbers.

**Proof** We have a commutative diagram with exact rows and columns

Here, we have endowed \( E/G, E''/G'' \) and \( H \) with the metrics induced from \( \overline{E}, \overline{E''} \) and \( \overline{G} \), hence all rows and columns are isometric by definition. A
minor point to note is that with this choice of metrics the two indicated isomorphisms are isometric, indeed this only means that taking sub- (resp. quotient-)metrics is transitive. One has

\[ \xi_{E'E} = \frac{r'' - r' c_1(E')}{{r'} r} \]

and analogously for any isometric exact sequence in place of \( E \). Using this and the diagram one writes both sides of the stated equality as a \( \mathbb{Q} \)-linear combination of \( c_1(E'), c_1(G'') \) and \( c_1(E''/G'') \) to obtain the same result, namely

\[ \frac{r'' - s}{{(r' + s)} r} c_1(E') + \frac{r'' - s}{{(r' + s)} r} c_1(G') - \frac{1}{r} c_1(E''/G''). \]

Finally, we will need the following observation about the intersection theory on \( X \) where, for \( x \in \hat{C}_{++}(X) \), we write \( |x| := (x^2)^{1/2} \in \mathbb{R}^+ \).

**Lemma 6** The subset \( \hat{C}_{++}(X) \subseteq \hat{CH}^1(X)_{\mathbb{R}} \) is an open cone, i.e. \( x, y \in \hat{C}_{++}(X) \) and \( \lambda \in \mathbb{R}^+ \) implies that \( x + y, \lambda x \in \hat{C}_{++}(X) \). For \( x, y \in \hat{C}_{++} \) we have \( |x + y| \geq |x| + |y| \).

**Proof** This is [Mo2], (1.1.2.2) except for the final assertion which is obvious if \( x \in \mathbb{R} y \) and we can thus assume that \( V := \mathbb{R} x + \mathbb{R} y \subseteq \hat{CH}^1(X)_{\mathbb{R}} \) is two-dimensional. We claim that the restriction of the intersection-pairing makes \( V \) a real quadratic space of type \((1, -1)\). As we have \( x \in V \) and \( x^2 > 0 \) we only have to exhibit some \( v \in V \) with \( v^2 < 0 \). To achieve this let \( h \in \hat{CH}^1(X)_{\mathbb{R}} \) be the first arithmetic Chern class of some sufficiently positive hermitian line bundle on \( X \) such that the arithmetic Hodge index theorem holds for the Lefschetz operator defined by \( h \), c.f. [GS2], Theorem 2.1, ii). Then \( a := x h \) (resp. \( b := y h \)) are non-zero real numbers for otherwise we would have \( x^2 < 0 \) (resp. \( y^2 < 0 \)). Thus \( v := \frac{x}{a} - \frac{b}{h} \in V \) satisfies \( v \neq 0 \) and \( vh = 0 \), hence \( v^2 < 0 \).

Fix a basis \( e, f \in V \) with \( e^2 = 1, f^2 = -1 \) and write

\[ x = \alpha e + \beta f \]
\[ y = \gamma e + \delta f. \]

To show that \(|x + y| \geq |x| + |y|\) we can assume, changing both the signs of \(x\) and \(y\) if necessary, that \(\alpha > 0\). We then claim that \(\gamma > 0\). For otherwise there would be \(\lambda_1, \lambda_2 \in \mathbb{R}_+\) such that \(v := \lambda_1 x + \lambda_2 y\) would have \(e\)-coordinate equal to zero, hence \(v^2 \leq 0\) contradicting the fact that either \(-v\) or \(v\) lies in \(\hat{C}_+\) (depending on whether or not we changed the signs of \(x\) and \(y\) above).

From \(x^2 = \alpha^2 - \beta^2, \ y^2 = \gamma^2 - \delta^2 > 0\) we obtain \(\alpha = |\alpha| \geq |\beta|\) and \(\gamma = |\gamma| \geq |\delta|\) and then \(\alpha \gamma \geq |\beta \delta| \geq \beta \delta\), i.e.

\[ xy = \alpha \gamma - \beta \delta \geq 0. \]  

(4)

To conclude, we use the following chain of equivalent statements

\[
\begin{align*}
|x + y| & \geq |x| + |y| \iff \\
(x + y)^2 - (|x| + |y|)^2 & \geq 0 \iff \\
2xy - 2|x||y| & \geq 0 \iff \\
xy & \geq |x||y| \iff \\
(x^2)^2 & \geq |x|^2 |y|^2 \iff \\
(\alpha \gamma - \beta \delta)^2 & \geq (\alpha^2 - \beta^2)(\gamma^2 - \delta^2) \iff \\
\alpha^2 \gamma^2 + \beta^2 \delta^2 - 2\alpha \beta \gamma \delta & \geq \alpha^2 \gamma^2 - \alpha^2 \delta^2 - \beta^2 \gamma^2 + \beta^2 \delta^2 \iff \\
2\alpha \beta \gamma \delta & \leq \alpha^2 \delta^2 + \beta^2 \gamma^2 \iff \\
0 & \leq (\alpha \delta - \beta \gamma)^2.
\end{align*}
\]

\[ \square \]

**Proof of Theorem**

We first remark that for a torsion-free hermitian coherent sheaf \(\mathcal{F}\) of rank one on \(X\) we always have \(\Delta(\mathcal{F}) \geq 0\). In fact,

\[ F \simeq \mathcal{L} \otimes \mathcal{I}_Z \]
for some line-bundle $L$ and $I_Z$ the ideal sheaf of some closed subscheme $Z \subseteq X$ of codimension 2. This becomes an isometry for the trivial metric on $I_Z$ and a suitable metric on $L$ (since $I_Z$ is trivial on the generic fibre of $X$). Then

$$\Delta(F) = 2\hat{c}_2(T \otimes I_Z) = 2\hat{c}_2(I_Z) = 2\text{length}(Z) \geq 0.$$ 

By the main result of [Mo2], there is $0 \neq E' \subseteq E$ saturated such that $\xi_{E',E} \in \hat{C}_{++}(X)$. We can assume that, as $E'$ varies through these subsheaves, the real numbers $\xi_{E',E}^2$ remain bounded for otherwise there is nothing to prove. So we can choose $0 \neq E' \subseteq E$ saturated with $\xi_{E',E'} \in \hat{C}_{++}(X)$ and $\xi_{E',E}^2$ maximal subject to these conditions. Put $E'' := E/E'$ and consider the isometric exact sequence

$$E : 0 \to E' \to E \to E'' \to 0$$

with discriminants $\Delta', \Delta, \Delta''$ and ranks $r', r, r''$. We claim that $\Delta' \geq 0$. This is clear in case $r = 2$ from the remark made at the beginning of the proof. In case $r \geq 3$ we assume that $\Delta' < 0$ and we let $G \subseteq E'$ be a saturated subsheaf with $\xi_{G,E'} \in \hat{C}_{++}$. Then $G \subseteq E$ is saturated and using lemma 6 we get

$$|\xi_{G,E'}| = |\xi_{G,E} + \xi_{E',E'}| \geq |\xi_{G,E'}| + |\xi_{E',E'}| > |\xi_{E',E'}|$$

contradicting the maximality of $|\xi_{E',E'}|$. So we have indeed $\Delta' \geq 0$. Assume now, contrary to our assertion, that

$$\frac{\Delta}{r} < -r(r-1)\xi_{E',E}^2.$$ 

Then from Lemma 1 $\Delta' \geq 0$, $\xi_{E',E} \leq 0$ ([Mo1], 7.2) we get

$$\frac{\Delta''}{r''} \leq \frac{\Delta}{r} + \frac{r'r''}{r''} \xi_{E',E}^2 < \left(-r(r-1) + \frac{r'r''}{r''}\right) \xi_{E',E}^2$$

$$= -r^2 \frac{r'' - 1}{r''} \xi_{E',E} ^2 \leq 0,$$

hence $\Delta'' < 0$. By induction, there is $0 \neq G'' \subseteq E''$ saturated with $\xi_{G'',E''} \in \hat{C}_{++}(X)$ and

$$\xi_{G'',E''}^2 \geq \frac{-\Delta''}{r''(r'' - 1)} > \frac{r^2}{r''^2} \xi_{E',E}^2.$$ 

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Clearly $\overline{G} := \ker(E \to E''/G'') \subset \overline{E}$ is saturated and from Lemma 5 the positivity of the coefficients appearing there and lemma 6 we get

$$|\xi_{E',E}| \geq \frac{r'(r'' - s)}{(r' + s)r''} |\xi_{E',E}^*| + \frac{s}{r'} |\xi_{E',E}^*|$$

$$> \frac{r'(r'' - s)}{(r' + s)r''} |\xi_{E',E}^*| + \frac{s}{r'} \frac{r}{r'' + s} |\xi_{E',E}^*|$$

$$= \left( \frac{r'(r'' - s) + rs}{r''(r' + s)} \right) |\xi_{E',E}^*| = |\xi_{E',E}^*|.$$  

This again contradicts the maximality of $|\xi_{E',E}|$ and concludes the proof.

References

[GS1] H. Gillet, C. Soulé, An arithmetic Riemann-Roch theorem, Invent. Math. 110 (1992), no. 3, 473–543.

[GS2] H. Gillet, C. Soulé, Arithmetic analogs of the standard conjectures, Motives (Seattle, WA, 1991), 129–140, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[HL] D. Huybrechts, M. Lehn, The geometry of moduli spaces of sheaves, Aspects of Mathematics, E31, Friedr. Vieweg and Sohn, Braunschweig, 1997.

[Mo1] A. Moriwaki, Inequality of Bogomolov-Gieseker type on arithmetic surfaces, Duke Math. J. 74 (1994), no. 3, 713–761.

[Mo2] A. Moriwaki, Bogomolov unstability on arithmetic surfaces, Math. Res. Lett. 1 (1994), no. 5, 601–611.

[So] C. Soulé, A vanishing theorem on arithmetic surfaces, Invent. Math. 116 (1994), no. 1-3, 577–599.

[SABK] C. Soulé, Lectures on Arakelov geometry, with the collaboration of D. Abramovich, J.-F. Burnol and J. Kramer, Cambridge Studies in Advanced Mathematics, 33, Cambridge University Press, Cambridge, 1992.