CONGRUENCE PROPERTIES FOR THE TRINOMIAL COEFFICIENTS

Moa Apagodu
Department of Mathematics, Statistics and Computer Science, St. Olaf College, Northfield, Minnesota
apagod1@stolaf.edu

Ji-Cai Liu
Department of Mathematics, Wenzhou University, Wenzhou, People’s Republic of China
jcliu2016@gmail.com

Received: 8/28/19, Revised: 12/26/19, Accepted: 4/29/20, Published: 5/8/20

Abstract
In this paper, we state and prove some congruence properties for the trinomial coefficients, one of which is similar to Wolstenholme’s theorem.

1. Introduction
In 1819, Babbage [4] showed for any odd prime $p$,
\[
\binom{2p-1}{p-1} \equiv 1 \pmod{p^2}.
\] (1)

In 1862, Wolstenholme [21] proved that the above congruence holds modulo $p^3$ for any prime $p \geq 5$, i.e.,
\[
\binom{2p-1}{p-1} \equiv 1 \pmod{p^3},
\] (2)

which is known as Wolstenholme’s theorem. It is well-known that Wolstenholme’s theorem is a fundamental congruence in combinatorial number theory. We refer to [14] for various extensions of Wolstenholme’s theorem.

In the past few years, $(q)$-congruences for sums of binomial coefficients have attracted the attention of many researchers (see, for instance, [2, 3, 6, 7, 8, 9, 10, 11, 19, 20]). In 2011, Sun and Tauraso [20] proved that for any prime $p \geq 5$,
\[
\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \binom{p}{3} \pmod{p^2},
\] (3)
\[
\sum_{k=0}^{p-1} \binom{2k}{k} \frac{1}{k+1} = \frac{3}{2} \binom{p}{3} - \frac{1}{2} \pmod{p^2},
\]
\quad (4)
where \( \binom{r}{p} \) denotes the Legendre symbol. Note that \( \binom{2k}{k} \frac{1}{k+1} \) is the \( n \)-th Catalan number \( C_n \), which plays an important role in various counting problems. Extensions of (3) and (4) have been established in [3, 10].

In 2018, the first author [2] conjectured two congruences on sums of the super Catalan numbers (named by Gessel [5]):
\[
\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} \binom{2i}{i} \binom{2j}{j} \binom{-(i+j)}{i} \equiv \binom{p}{3} \pmod{p},
\]
\[
\sum_{i=0}^{p-1} \sum_{j=0}^{p-1} (3i + 3j + 1) \binom{2i}{i} \binom{2j}{j} \binom{-(i+j)}{i} \equiv -7 \binom{p}{3} \pmod{p},
\]
which were confirmed by the second author [11].

In this paper, we will study the congruence properties for the trinomial coefficients. Here we consider the coefficients of the trinomial
\[
(1 + x + x^{-1})^n = \sum_{j=-n}^{n} \binom{n}{j} x^j.
\]

Two immediate consequences of this definition are
\[
\binom{n}{j} = \binom{n}{-j},
\]
and
\[
\binom{n}{j} = \binom{n-1}{j-1} + \binom{n-1}{j} + \binom{n-1}{j+1}.
\]
We have the following multinomial theorem (see [18, page 17]):
\[
(x + y + z)^n = \sum_{a+b+c=n} \frac{n!}{a!b!c!} x^a y^b z^c.
\]
Letting \( y = 1 \) and \( z = 1/x \) in (5), we get
\[
\binom{n}{j} = \sum_{a+b+c=n} \frac{n!}{a!b!c!} = \sum_{c=0}^{n} \binom{n}{c} \binom{n-c}{c+j}.
\]
We first prove a congruence for the trinomial coefficients, which is similar to Wolstenholme’s theorem.
Theorem 1. For any prime \( p \geq 5 \), we have

\[
\binom{2p}{p} \equiv 2 + \frac{2p^2}{3} \binom{p}{3} B_{p-2} \left( \frac{1}{3} \right) \pmod{p^3},
\]

(7)

where \( B_n(x) \) is the Bernoulli polynomial of order \( n \).

The second result consists of the following two congruences on single sums of trinomial coefficients.

Theorem 2. For any prime \( p \geq 5 \), we have

\[
\sum_{j=0}^{p} \binom{p}{j} \equiv \frac{1}{2} (1 + 3^p) \pmod{p^2},
\]

(8)

\[
\sum_{j=0}^{p-1} \binom{p-1}{j} \equiv \frac{1}{2} \left( 1 + \binom{p}{3} \right) \pmod{p}.
\]

(9)

The third aim of the paper is to establish a congruence on double sums of trinomial coefficients.

Theorem 3. For any prime \( p \geq 5 \) and integer \( j \) with \( 0 < j < p \), we have

\[
\sum_{k=0}^{p-1} \binom{k}{j} \equiv \frac{(-1)^j + 1}{2} \cdot (-1)^{\frac{p-j-1}{2}} \pmod{p},
\]

(10)

\[
\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \binom{m}{n} \equiv \frac{1}{2} \left( (-1)^{\frac{p-1}{2}} + 1 \right) \pmod{p}.
\]

(11)

The rest of this paper is organized as follows. We shall prove Theorems 1, 2 and 3 in Sections 2, 3 and 4, respectively. An open problem on \( q \)-congruence is proposed in the last section for further research.

2. Proof of Theorem 1

By (2) and (6), we have

\[
\binom{2p}{p} = \sum_{k=0}^{p-1} \binom{2p}{k} \binom{2p-k}{k+p}
\]

\[
\equiv 2 + \sum_{k=1}^{p-1} \binom{2p}{k} \binom{2p-k}{k+p} \pmod{p^3}.
\]

(12)
Let $H_k$ denote the $k$-th harmonic number:

$$H_k = \sum_{j=1}^{k} \frac{1}{j}.$$ 

For any integer $s$ and $0 \leq k \leq p - 1$, we have

$$\binom{sp - 1}{k} \equiv (-1)^k (1 - spH_k) \pmod{p^2}, \quad (13)$$

and

$$\binom{sp + k}{k} \equiv 1 + spH_k \pmod{p^2}. \quad (14)$$

It follows from (1), (13) and (14) that for $1 \leq k \leq \frac{p-1}{2}$,

$$\binom{2p}{k} = \frac{2p}{k} \binom{2p - 1}{k - 1} \equiv \frac{2p(-1)^{k-1}}{k} (1 - 2pH_{k-1}) \pmod{p^3}, \quad (15)$$

and

$$\binom{2p - k}{k + p} = \frac{(2p)^2}{4} \cdot \frac{\binom{p-1}{k-1}}{(2p-1)} \cdot \frac{(p+k)}{k}$$

$$\equiv (-1)^{k-1} \binom{2k}{k} \cdot \frac{pH_{2k-1} - 1}{2(1 - 2pH_{k-1})(1 + pH_k)} \pmod{p^2}. \quad (16)$$

Substituting (15) and (16) into (12) gives

$$\binom{\binom{2p}{p}}{p} \equiv 2 + p \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} (pH_{2k-1} - 1) \pmod{p^3} \quad (17)$$

Note that

$$\frac{pH_{2k-1} - 1}{1 + pH_k} \equiv -1 + (H_k + H_{2k-1})p \pmod{p^2}. \quad (18)$$

Combining (17) and (18), we arrive at

$$\binom{\binom{2p}{p}}{p} \equiv 2 - p \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} H_{2k} + p^2 \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} (H_k + H_{2k-1}) \pmod{p^3}$$

$$= 2 + p \left( \frac{p}{2} \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} H_{2k} - \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} \right) + p^2 \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k} H_k - \frac{p^2}{2} \sum_{k=1}^{\frac{p-1}{2}} \frac{(2k)}{k^2}.$$
Using the following congruence [12, (2.8)]:
\[
p \sum_{k=1}^{n-1} \frac{(2k)}{k} H_{2k} - \sum_{k=1}^{n-1} \frac{(2k)}{k^2} = \frac{5p}{2} \sum_{k=1}^{n-1} \frac{(2k)}{k^2} - 2p \sum_{k=1}^{n-1} \frac{(2k)}{k} H_k \quad (\text{mod } p^2),
\]
we have
\[
\left( \binom{2p}{p} \right) \equiv 2 + 2p^2 \sum_{k=1}^{n-1} \frac{(2k)}{k^2} - p^2 \sum_{k=1}^{n-1} \frac{(2k)}{k} H_k \quad (\text{mod } p^3). \quad (19)
\]

We have the following two congruences (see [12, (1.1)] and [13, page 156]):
\[
\sum_{k=1}^{n-1} \frac{(2k)}{k} H_k \equiv \frac{1}{3} \left( \binom{p}{3} \right) B_{p-2} \left( \frac{1}{3} \right) \quad (\text{mod } p), \quad (20)
\]
and
\[
\sum_{k=1}^{n-1} \frac{(2k)}{k^2} \equiv \frac{1}{3} \binom{p}{3} B_{p-2} \left( \frac{1}{3} \right) \quad (\text{mod } p). \quad (21)
\]
Finally, combining (19)–(21), we complete the proof of (7).

3. Proof of Theorem 2

Proof of (8). We begin with the following identity, which is A027914 of the Online Encyclopedia of Integer Sequences [17]:
\[
\sum_{j=0}^{n} \left( \binom{n}{j} \right) = \frac{1}{2} \left( 3^n + \sum_{k=0}^{n} \binom{n}{k} \binom{n-k}{k} \right). \quad (22)
\]
Letting \( n = p \) in the above gives
\[
\sum_{j=0}^{p} \binom{p}{j} = \frac{1}{2} \left( 3^p + \sum_{k=0}^{p-1} \binom{p}{k} \binom{p-k}{k} \right).
\]
Note that for \( 1 \leq k \leq \frac{p-1}{2} \),
\[
\binom{p}{k} \binom{p-k}{k} = \frac{p(p-1) \cdots (p-k)(p-k+1)(p-k-1) \cdots (p-2k+1)}{k!^2} \equiv -\frac{p}{2k} \binom{2k}{k} \quad (\text{mod } p^2).
\]
Thus,
\[
\sum_{j=0}^{p} \left( \binom{p}{j} \right) = \frac{1}{2} \left( 3^p + 1 - \frac{p}{2} \sum_{k=1}^{p-1} \frac{(2k)}{k} \right) \pmod{p^2}.
\]

We have the following congruence [15, (1.6)]:
\[
\sum_{k=1}^{p-1} \frac{(2k)}{k} \equiv 0 \pmod{p}.
\]

It follows that
\[
\sum_{j=0}^{p} \left( \binom{p}{j} \right) = \frac{1}{2} (3^p + 1) \pmod{p^2},
\]

as desired.

Proof of (9). Letting \( n = p - 1 \) in (22), we obtain
\[
\sum_{j=0}^{p-1} \left( \binom{p-1}{j} \right) = \frac{1}{2} \left( 3^{p-1} + \frac{p-1}{0} \sum_{k=0}^{p-2} \binom{p-1}{k} \binom{p-1-k}{k} \right).
\]

Note that for \( 0 \leq k \leq \frac{p-1}{2} \),
\[
\binom{p-1}{k} \equiv (-1)^k \pmod{p},
\]

and
\[
\binom{p-1-k}{k} = \frac{(p-1-k)(p-2-k)\cdots(p-2k)}{k!} \equiv (-1)^k \binom{2k}{k} \pmod{p}.
\]

By the above two congruences and Fermat’s little theorem, we have
\[
\sum_{j=0}^{p-1} \left( \binom{p-1}{j} \right) = \frac{1}{2} \left( 1 + \sum_{k=0}^{p-1} \frac{(2k)}{k} \right) \pmod{p}. \tag{23}
\]

Then the proof of (9) follows from (3) and (23).
4. Proof of Theorem 3

Proof of (10). Exchanging the summation order, we get

\[
\sum_{k=0}^{p-1} \binom{k}{j} = \sum_{k=0}^{p-1} \sum_{i=0}^{k} \binom{k}{i} \binom{k-i}{i+j} = \sum_{i=0}^{p-1} \sum_{k=i}^{p-1} \binom{k}{i} \binom{k-i}{i+j}.
\]

Since

\[
\binom{k}{i} \binom{k-i}{i+j} = \binom{2i+j}{i} \binom{k}{2i+j},
\]

we have

\[
\sum_{k=0}^{p-1} \binom{k}{j} = \sum_{i=0}^{p-1} \binom{2i+j}{i} \sum_{k=i}^{p-1} \binom{k}{2i+j} = \sum_{i=0}^{p-1} \binom{2i+j}{i} \binom{p}{2i+j+1},
\]

where we have utilized the identity (proved by induction):

\[
\sum_{k=0}^{n} \binom{k}{m} = \binom{n+1}{m+1}.
\]

Note that for \(1 \leq k \leq p-1\),

\[
\binom{p}{k} \equiv 0 \pmod{p}.
\]

If \(j\) is odd, then

\[
\sum_{k=0}^{p-1} \binom{k}{j} \equiv 0 \pmod{p}.
\]

If \(j\) is even, then

\[
\sum_{k=0}^{p-1} \binom{k}{j} \equiv \binom{p-1}{j-1} \equiv (-1)^{\frac{p-j-1}{2}} \pmod{p}.
\]

This completes the proof of (10).
Proof of (11). By (10), we have

\[
\sum_{m=0}^{p-1} \sum_{n=0}^{p-1} \binom{m}{n} \equiv \sum_{n=0}^{p-1} \frac{(-1)^n + 1}{2} \cdot (-1)^{\frac{p-2n-1}{2}} \quad \text{(mod } p) \\
= \sum_{n=0}^{p-1} \frac{(-1)^{\frac{p-2n-1}{2}}}{2} \\
= \frac{1}{2} \left( (-1)^{\frac{p-1}{2}} + 1 \right),
\]

as claimed. \(\square\)

Remark. Theorems 2 and 3 can also be established by using the method of the first author and Zeilberger [3].

5. Concluding Remarks

We have three \(q\)-anlogs corresponding to the trinomial coefficients as given in [16], namely,

\[
T_1(n, j, q) := \sum_{k=0}^{n} q^{k+j} \binom{n}{k} q^{n-k} q^{j},
\]

\[
T_2(n, j, q) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} q^{2n - 2k} q^{n-k-j},
\]

\[
T_3(n, j, q) := \sum_{k=0}^{n} (-q)^k \binom{n}{k} q^{2n - 2k} q^{n-k-j},
\]

where the \(q\)-binomial coefficients \(\binom{n}{k}_q\) are defined as

\[
\binom{n}{k}_q = \begin{cases} 
(1-q^n)(1-q^{n-1})\ldots(1-q^{n-k+1})/(1-q)(1-q^2)\ldots(1-q^k), & \text{if } 0 \leq k \leq n, \\
0, & \text{otherwise.}
\end{cases}
\]

It is not hard to prove the following \(q\)-congruences.

Proposition 1. For any odd prime \(p\) and integer \(1 \leq s \leq 3\), we have

\[
\sum_{j=0}^{p-1} T_s(p, j, q) \equiv 1 \quad \text{(mod } [p]_q),
\]

where the \(q\)-integers are given by \([n]_q = (1-q^n)/(1-q)\).
The proof of Proposition 1 is trivial and left to the interested reader.

In 1999, Andrews [1] established an interesting $q$-analog of Babbage’s congruence (1):

$$
\left( \binom{2p-1}{p-1} \right)_q = q^{\frac{n(n+1)}{2}} \pmod{\lfloor p \rfloor^2},
$$

for any odd prime $p$. It is natural to ask whether the congruence (7) possesses a $q$-analog. For convenience sake, let

$$
\left( \binom{n}{j} \right)_q = T_1(n, j, q).
$$

Numerical calculation suggests the following $q$-congruence, and we propose this conjecture for further research.

**Conjecture 1.** For any prime $p \geq 5$, we have

$$
\left( \binom{2p}{p} \right)_q = 2 \left( \frac{p + 3}{6} \right) + p \left( q^p - 1 \right) + 2 \pmod{\lfloor p \rfloor^2},
$$

where $\lfloor x \rfloor$ denotes the integral part of real $x$.

**Acknowledgments.** The first author would like to thank Professor Dennis Stanton of the School of Mathematics, University of Minnesota for indoctrinating him into the $q$-series during his visit in the fall 2019. The second author was supported by the National Natural Science Foundation of China (grant 11801417). The authors are also grateful to the referee for valuable comments and interesting suggestions which have improved the quality of the paper.

**References**

[1] G. E. Andrews, $q$-Analogs of the binomial coefficient congruences of Babbage, Wolstenholme and Glaisher, *Discrete Math.* **204** (1999), 15–25.

[2] M. Apagodu, Elementary proof of congruences involving sum of binomial coefficients, *Int. J. Number Theory* **14** (2018), 1547–1557.

[3] M. Apagodu and D. Zeilberger, Using the “freshman’s dream” to prove combinatorial congruences, *Amer. Math. Monthly* **124** (2017), 597–608.

[4] C. Babbage, Demonstration of a theorem relating to prime numbers, *Edinburgh Philosophical J.* **1** (1819), 46–49.

[5] I. Gessel, Super ballot numbers, *J. Symbolic Comput.* **14** (1992), 179–194.

[6] V. J. W. Guo, Proof of a $q$-congruence conjectured by Tauraso, *Int. J. Number Theory* **15** (2019), 37–41.

[7] V. J. W. Guo and M. J. Schlosser, Some new $q$-congruences for truncated basic hypergeometric series, *Symmetry* **11(2)** (2019), Art. 268.
Some congruences involving central $q$-binomial coefficients, *Adv. in Appl. Math.* **45** (2010), 303–316.

V. J. W. Guo and W. Zudilin, A $q$-microscope for supercongruences, *Adv. Math.* **346** (2019), 329–358.

J.-C. Liu, On two conjectural supercongruences of Apagodu and Zeilberger, *J. Difference Equ. Appl.* **22** (2016), 1791–1799.

J.-C. Liu, Congruences on sums of super Catalan numbers, *Results Math.* **73** (2018), Art. 140.

G.-S. Mao and Z.-W. Sun, Two congruences involving harmonic numbers with applications, *Int. J. Number Theory* **12** (2016), 527–539.

S. Mattarei and R. Tauraso, Congruences for central binomial sums and finite polylogarithms, *J. Number Theory* **133** (2013), 131–157.

R. Meštrović, Wolstenholme’s theorem: Its Generalizations and Extensions in the last hundred and fifty years (1862–2012), preprint (2011), arXiv:1111.3057.

H. Pan and Z.-W. Sun, A combinatorial identity with application to Catalan numbers, *Discrete Math.* **306** (2006), 1921–1940.

A. V. Sills, *An Invitation to the Rogers-Ramanujan Identities*, CRC Press, Boca Raton, 2018.

N. J. A. Sloane, Sequence A027914 in OEIS (On-Line Encyclopedia of Integer Sequences), http://oeis.org/A027914.

R. P. Stanley, *Enumerative Combinatorics*, vol. 1, Cambridge Univ. Press, Cambridge, 1986.

Z.-W. Sun and R. Tauraso, New congruences for central binomial coefficients, *Adv. in Appl. Math.* **45** (2010), 125–148.

Z.-W. Sun and R. Tauraso, On some new congruences for binomial coefficients, *Int. J. Number Theory* **7** (2011), 645–662.

J. Wolstenholme, On certain properties of prime numbers, *Quart. J. Pure Appl. Math.* **5** (1862), 35–39.