ON HEREDITARY COREFLECTIVE SUBCATEGORIES OF Top

MARTIN SLEZIAK

Abstract. Let $A$ be a topological space which is not finitely generated and $\text{CH}(A)$ denote the coreflective hull of $A$ in $\text{Top}$. We construct a generator of the coreflective subcategory $\text{SCH}(A)$ consisting of all subspaces of spaces from $\text{CH}(A)$ which is a prime space and has the same cardinality as $A$. We also show that if $A$ and $B$ are coreflective subcategories of $\text{Top}$ such that the hereditary coreflective kernel of each of them is the subcategory $\text{FG}$ of all finitely generated spaces, then the hereditary coreflective kernel of their join $\text{CH}(A \cup B)$ is again $\text{FG}$.

Keywords: coreflective subcategory, hereditary coreflective subcategory, hereditary coreflective hull, hereditary coreflective kernel, prime space

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Introduction

Let $X$ be a topological space which is not finitely generated and $\text{SCH}(X)$ be the hereditary coreflective hull of $X$ in the category $\text{Top}$ of topological spaces. The aim of this paper is to construct a prime space $Y_X$ with the same cardinality as $X$ such that $\text{SCH}(X) = \text{CH}(Y_X)$ where $\text{CH}(Y_X)$ is the coreflective hull of $Y_X$. Obviously, if $X$ is finitely generated, then $\text{CH}(X) = \text{SCH}(X)$. If $X$ is not finitely generated, then, using the prime factors of $X$ we can easily construct a prime space $P_X$ such that $\text{SCH}(X) = \text{SCH}(P_X)$. Thus, it suffices to restrict our investigation to the case of prime spaces.

For the prime space $C(\omega_0)$ consisting of a convergent sequence and its limit point the problem was studied in [5], where a countable generator for the category $\text{SCH}(C(\omega_0))$ of subsequential spaces was produced.

Our procedure of constructing a generator $Y_A$ of the category $\text{SCH}(A)$ (where $A$ is a prime space that is not finitely generated) consists of two main steps. In the first step, using similar methods as in [5], we produce a set of special prime spaces which generates $\text{SCH}(A)$. Then, in the second step, we construct the generator $Y_A$ of $\text{SCH}(A)$ with the required properties.

This construction was inspired by the space $S_\omega$ from [2] and in the case $A = C(\omega_0)$ it gives a countable generator for the category of subsequential spaces different from that one presented in [5].

Finally, as an application of some above mentioned results we prove that if $A$ and $B$ are coreflective subcategories of $\text{Top}$ such that the hereditary coreflective kernel of $A$ as well as the hereditary coreflective kernel of $B$ is the category $\text{FG}$ of finitely generated spaces, then $\text{FG}$ is also the hereditary coreflective kernel of their join $\text{CH}(A \cup B)$. As a consequence of this result and some results of [9] we obtain that the collection of all those coreflective subcategories of $\text{Top}$ the hereditary coreflective kernel of which is $\text{FG}$ and the hereditary coreflective hull of which is
\textbf{Top} is closed under the formation of non-empty finite joins (in the lattice of all coreflective subcategories of \textbf{Top}) and arbitrary non-empty intersections.

1. Preliminaries

We recall some known facts about coreflective subcategories of the category \textbf{Top} of topological spaces (see [6]). All subcategories are supposed to be full and isomorphism-closed. The topological sum is denoted by \(\sqcup\).

Let \(A\) be a subcategory of \textbf{Top}. \(A\) is \textit{coreflective} if and only if it is closed under the formation of topological sums and quotient spaces. If \(A\) is a subcategory of \textbf{Top} or a class of topological spaces, then the \textit{coreflective hull} of \(A\) is the smallest coreflective subcategory of \textbf{Top} which contains \(A\) and we denote it by \(\text{CH}(A)\). \(\text{CH}(A)\) consists of all quotients of topological sums of spaces that belong to \(A\). If \(B = \text{CH}(A)\), then we say that \(A\) \textit{generates} \(B\) and the members of \(A\) are called \textit{generators} of \(B\). We use the notation \(B = \text{CH}(X)\) in this case.

Let \(A\) be a subcategory of \textbf{Top} and let \(SA\) denote the subcategory of \textbf{Top} consisting of all subspaces of spaces from \(A\). Then the following result is known (see [8, Remark 2.4.4(5)] or [3, Proposition 3.1]).

\textbf{Proposition 1.1.} If \(A\) is a coreflective subcategory of \textbf{Top}, then \(SA\) is also a coreflective subcategory of \textbf{Top}. (\(SA\) is the hereditary coreflective hull of \(A\).)

By \(FG\) we denote the category of all finitely generated spaces. It is well known (see e.g. [6]) that if \(X\) is not finitely generated, then \(FG \subseteq \text{CH}(X)\).

We say that a subcategory \(A\) of \textbf{Top} is \textit{hereditary} if with each topological space \(X\) it contains also all its subspaces. It is well known that the category of all finitely generated spaces and all its subcategories that are coreflective in \textbf{Top} are hereditary.

Some known hereditary coreflective subcategories of \textbf{Top} are \(\text{Gen}(\alpha)\) and \(\text{Top}(\alpha)\), where \(\alpha\) is an infinite cardinal. \(\text{Gen}(\alpha)\) is the subcategory of all spaces having tightness not exceeding \(\alpha\). \(\text{Top}(\alpha)\) is the category of all topological spaces such that the intersection of every family of open sets, which has cardinality less than \(\alpha\), is an open set.

Let \(A\) be a topological space. We say that \(A\) is a \textit{prime space} if it has precisely one accumulation point. The following assertion is obvious.

\textbf{Lemma 1.2.} Let \(X\) be a prime space with an accumulation point \(a\) and let \(Y\) be a subspace of \(X\) containing the point \(a\), then the map \(f : X \to Y\), defined by \(f(x) = x\) for \(x \in Y\) and \(f(x) = a\) for \(x \in X \setminus Y\), is a quotient map.

Given a topological space \(X\) and a point \(a \in X\), denote by \(X_a\) the space constructed by making each point, other than \(a\), isolated with \(a\) retaining its original neighborhoods. (I.e. a subset \(U \subseteq X\) is open in \(X_a\) if and only if \(a \notin U\) or there exists an open subset \(V\) of \(X\) such that \(a \in V \subseteq U\).) The topological space \(X_a\) is called \textit{prime factor of \(X\) at the point \(a\)}. It is clear that any prime factor is either a prime space or a discrete space.

\textbf{Proposition 1.3} ([3 Proposition 3.5]). If \(A\) is a hereditary coreflective subcategory of \textbf{Top} with \(FG \subseteq A\), then for each \(X \in A\) and each \(a \in X\) the prime factor \(X_a\) of \(X\) at \(a\) belongs to \(A\).
Let $A$ be a prime space with an accumulation point $a$. A subspace $B$ of $A$ is said to be a prime subspace of $A$ if $B$ is a prime space (i.e. $a \in B$ and $B \setminus \{a\} \ni a$).

**Lemma 1.4.** Let $(A_i; i \in I)$ be a family of prime spaces and let $a_i \in A_i$ be an accumulation point of $A_i$ for $i \in I$. A topological space $X$ belongs to $\text{CH}(\{A_i; i \in I\})$ if and only if for every non-closed subset $M$ of $X$ there exists $i \in I$, a prime subspace $B$ of $A_i$ and a continuous map $f: B \to X$ such that $f[B \setminus \{a_i\}] \subseteq M$ and $f(a_i) \notin M$.

**Proof.** Let $B \subseteq \text{Top}$ be the class of all topological spaces satisfying the given condition. First we show that $B$ is a coreflective subcategory of $\text{Top}$. It is evident that $B$ is closed under the formation of topological sums. Now let $X \in B$ and $q: X \to Y$ be a quotient map. Let $M$ be a non-closed subset of $Y$. Then $q^{-1}[M]$ is a non-closed subset of $X$, $X \in B$, so that there exists $i \in I$, a prime subspace $B$ of $A_i$ and a continuous map $g: B \to X$ such that $g[B \setminus \{a_i\}] \subseteq q^{-1}[M]$ and $g(a_i) \notin q^{-1}[M]$. Then for $f = q \circ g: B \to X$ we get $f[B \setminus \{a_i\}] \subseteq M$ and $f(a_i) \notin M$. Hence, $Y \in B$ and $B$ is a coreflective subcategory of $\text{Top}$.

Since evidently $A_i \in B$ for each $i \in I$, we have $\text{CH}(\{A_i; i \in I\}) \subseteq B$. To prove the reverse inclusion we construct a quotient map from a sum of subspaces of $A_i$ to arbitrary space $X \in B$. (Every subspace of $A_i$ belongs to $\text{CH}(A_i)$ by Lemma 1.2.)

Let $X \in B$. Let $f_j: B_j \to X$, $j \in J$, be the family of all continuous maps such that $B_j$ is a prime subspace of some $A_i$, $i \in I$. Let $D(X)$ be the discrete space on the set $X$ and $\text{id}_X: D(X) \to X$ be the identity map. It is easy to check that the map $f: D(X) \sqcup (\bigsqcup_{j \in J} B_j) \to X$ given by the maps $\text{id}_X$ and $f_j$, $j \in J$, is a quotient map.

Cardinals are initial ordinals where each ordinal is the (well-ordered) set of its predecessors. We denote the class of all ordinals by ON. If $\alpha$ is a cardinal, then by $\alpha^+$ we denote the cardinal which is a successor of $\alpha$. A net in a topological space defined on an ordinal $\alpha$ we call an $\alpha$-sequence.

From now on we assume that $A$ is a prime space with an accumulation point $a$ which is not finitely generated and the tightness of the space $A$ is $t(A) = \alpha$.

2. **Closure operator describing $\text{CH}(A)$**

The notion of sequential closure was used in [5] when studying sequential and subsequential spaces. Now we introduce a corresponding closure operator for the subcategory $\text{CH}(A)$.

Let $X$ be an arbitrary space and $M \subseteq X$. The set $M_1 = \{x \in X : \text{there exists a prime subspace } B \text{ of } A \text{ and a continuous map } f: B \to X \text{ such that } f[B \setminus \{a\}] \subseteq M \text{ and } f(a) = x\}$ is called the $A$-closure of $M$. Using transfinite induction we can define the set $M_\beta$ (the $\beta$-th $A$-closure of $M$) for each ordinal $\beta$ as follows. $M_0 = M$, $M_{\beta+1} = (M_\beta)_1$ for each ordinal $\beta$ and $M_\gamma = \bigcup_{\beta < \gamma} M_\beta$ for each limit ordinal $\gamma > 0$. Put $\widetilde{M} = \bigcup_{\beta \in \text{ON}} M_\beta$.

Evidently $(\widetilde{M})_1 = \widetilde{M}$, $\widetilde{M} \subseteq \overline{M}$. It is also clear that $M_\beta \subseteq M_\gamma$ holds for $\beta < \gamma$. If $A \subseteq B \subseteq X$, then $A_\beta \subseteq B_\beta$ for each ordinal $\beta$ and $A \subseteq B$. If $M_\beta = M_{\beta+1}$ for some ordinal $\beta$, then $\widetilde{M} = M_\beta$.

The following proposition characterizes the spaces belonging to $\text{CH}(A)$ using the closure operator $M \mapsto \widetilde{M}$. It is a special case of [8, Theorem 3.1.7] which includes more general cases of closure operators.
Proposition 2.1. A topological space $X$ belongs to $\text{CH}(A)$ if and only if $\overline{M} = \tilde{M}$ for every subset $M \subseteq X$.

Proof. Let $X \in \text{CH}(A)$ and $M \subseteq X$. Then $(\tilde{M})_1 \setminus \tilde{M} = \emptyset$, so that by Lemma 1.3 $\tilde{M}$ is closed and $\overline{M} = \tilde{M}$.

Conversely, if $\overline{M} = \tilde{M}$ for each $M \subseteq X$ and $M$ is non-closed, then $M_1 \setminus M \neq \emptyset$ and there exists a prime subspace $B$ of $A$ and a continuous map $f : B \to X$ such that $f[B \setminus \{a\}] \subseteq M$ and $f(a) \notin M$. Hence, according to Lemma 1.4 we conclude that $X \in \text{CH}(A)$.

Proposition 2.2. Let $A$ be a prime space with an accumulation point $a$, $X \in \text{CH}(A)$ and $a = t(A)$. Then for every subset $M \subseteq X$ it holds $M_{\alpha} = \overline{M}$.

Proof. If suffices to prove that $(M_{\alpha^+})_1 = M_{\alpha^+}$. Let $c \in (M_{\alpha^+})_1$. Then there exists a prime subspace $B$ of $A$ and a continuous map $f : B \to X$ with $f(a) = c$ and $f[B \setminus \{a\}] \subseteq M_{\alpha^+}$. Since $t(A) = \alpha$ and $a \in B \setminus \{a\}$, there exists $C \subseteq B \setminus \{a\}$ with $\text{card} C \leq \alpha$ such that $a \in \overline{C}$. The subspace $B_1 = C \cup \{a\}$ of $A$ is a prime subspace, $f|_{B_1} : B_1 \to X$ is continuous and $f|_{B_1}[C] \subseteq M_{\alpha^+}$.

For each $x \in C$ choose $\beta_x < \alpha^+$ such that $x \in M_{\beta_x}$ ($\alpha^+$ is a limit ordinal). Since $\text{card} C \leq \alpha < \alpha^+$ and $\alpha^+$ is a regular cardinal we obtain that $\gamma = \sup\{\beta_x, x \in C\} < \alpha^+$. Then $C \subseteq M_{\gamma}$ and, obviously, $f|_{B_1}(a) = f(a) = c \in M_{\gamma+1} \subseteq M_{\alpha^+}$. Thus, $(M_{\alpha^+})_1 \subseteq M_{\alpha^+}$.

3. $A$-sum

The notion of $A$-sum is a special case of the brush defined in $[8]$ and a generalization of the sequential sum introduced in [2]. The sequential sum was used in [5] for constructing the set of “canonical” prime spaces which generates the category of subsequential spaces. The notion of the $A$-sum will be used in a similar way to produce the set of special prime spaces that generates $\text{SCH}(A)$.

Definition 3.1. Let $A$ be a prime space with an accumulation point $a \in A$. Let us denote $B := A \setminus \{a\}$. Let for each $b \in B$ $X_b$ be a topological space and $x_b \in X_b$. Then the $A$-sum $\sum_A (X_b, x_b)$ is the topological space on the set $F = A \cup \left( \bigcup_{b \in B} (X_b \setminus \{x_b\}) \right)$ such that the map $\varphi : A \cup \left( \bigcup_{b \in B} X_b \right) \to F$ given by $\varphi(x) = x$ for $x \in A$, $\varphi(x) = (b, x)$ for $x \in X_b \setminus \{x_b\}$ and $\varphi(x_b) = b$ for every $b \in B$ is a quotient map. (We assume $A$ and all $\{b\} \times X_b$ to be disjoint.) The map $\varphi$ will be called the defining map of the $A$-sum.

Often it will be clear from the context what we mean under $A$ and we will abbreviate the notation of the $A$-sum to $\sum (X_b, x_b)$ or $\sum X_b$. The $A$-sum is obtained simply by identifying every $x_b \in X_b$ with the point $b \in A$. It is easy to see that the subspace $\varphi[X_b]$ is homeomorphic to $X_b$ and $A$ is also a subspace of the $A$-sum $\sum (X_b, x_b)$.

The $A$-sum is defined using topological sum and quotient map, thus if $A$ is a coreflective subcategory of $\text{Top}$ and $A$ contains $A$ and all $X_b$’s, then the $A$-sum $\sum X_b$ belongs to $A$.

The following lemma follows easily from the definition of the $A$-sum.

Lemma 3.2. A subset $U \subseteq \sum_A (X_b, x_b)$ is open (closed) if and only if $U \cap A$ is open (closed) in $A$ and $U \cap \varphi[X_b]$ is open (closed) in $\varphi[X_b]$ for every $b \in B$. 
Let for every $b \in B$ $X_b$ and $Y_b$ be topological spaces, $x_b \in X_b$, $y_b \in Y_b$ and let $f_b: X_b \to Y_b$ be a function with $f_b(x_b) = y_b$. Then we can define a map $f := \sum_{b} f_b: \sum_{b} X_b, x_b \to \sum_{b} Y_b, y_b$ by $f(b, x) = (b, f_b(x))$ for $x \in X_b \setminus \{x_b\}$ and $f(x) = x$ for $x \in A$. Let us note that $f \circ \varphi_1|_{X_b} = \varphi_2|_{Y_b} \circ f_b$ where $\varphi_1$ and $\varphi_2$ are the defining maps of the $A$-sums $\sum X_b$ and $\sum Y_b$ respectively.

We will need the following simple lemma:

**Lemma 3.3.** Let $f: X \to Y$ be a quotient map, $A \subseteq Y$ and let $f$ be one-to-one outside $A$. Then $f|_{f^{-1}[A]}: f^{-1}[A] \to A$ is a quotient map.

**Lemma 3.4.** Let $A$ be a prime space with an accumulation point $a$ and $B = A \setminus \{a\}$. Let for every $b \in B$ $f_b: X_b \to Y_b$ be a map between topological spaces, $x_b \in X_b$, $y_b \in Y_b$ and let $f_b(x_b) = y_b$.

(i) If all $f_b$’s are continuous, then $\sum f_b$ is continuous.

(ii) If all $f_b$’s are quotient maps, then $\sum f_b$ is a quotient map.

(iii) If all $f_b$’s are embeddings, then $\sum f_b$ is an embedding.

(iv) If all $f_b$’s are homeomorphisms, then $\sum f_b$ is a homeomorphism.

(v) Let $C$ be a prime subspace of $A$. Then $\sum_{C}(X_b, x_b)$ is a subspace of the space $\sum_{A}(X_b, x_b)$.

**Proof.** Put $f = \sum f_b$ and let $\varphi_1, \varphi_2$ be the defining maps of the $A$-sums $\sum_{b} X_b, x_b$, $\sum_{b} Y_b, y_b$ respectively. Let us denote $id_A \sqcup (\bigsqcup_{b \in B} f_b)$ by $h$. In this situation the following diagram commutes:

$$
\begin{array}{ccc}
A \sqcup (\bigsqcup_{b \in B} X_b) & \xrightarrow{h} & A \sqcup (\bigsqcup_{b \in B} Y_b) \\
\varphi_1 \downarrow & & \varphi_2 \downarrow \\
\sum_{b} (X_b, x_b) & \xrightarrow{f} & \sum_{b} (Y_b, y_b)
\end{array}
$$

The validity of (i) and (ii) follows easily from the fact that $\varphi_1$ and $\varphi_2$ are quotient maps.

(iii) Now, suppose that all $f_b$’s are embeddings. W.l.o.g. we can assume that $X_b \subseteq Y_b$ and $f_b$ is the inclusion of $X_b$ into $Y_b$ for every $b \in B$. Let $X'$ be the subspace of the space $\sum Y_b$ on the set $\sum X_b$. We have the following situation:

$$
\begin{array}{ccc}
A \sqcup (\bigsqcup_{b \in B} X_b) & \xrightarrow{h} & A \sqcup (\bigsqcup_{b \in B} Y_b) \\
\varphi_1 \downarrow & & \varphi_2 \downarrow \\
X' & \xrightarrow{f} & \sum Y_b
\end{array}
$$

We only need to prove that $X'$ has the quotient topology with respect to $\varphi_1$, because this implies that $X' = \sum X_b$ and $f$ is an embedding of $X' = \sum X_b$ to $\sum Y_b$. But $\varphi_2$ is one-to-one outside the set $A \sqcup (\bigsqcup X_b)$ and Lemma 3.3 implies that $\varphi_1$ is a quotient map.

(iv) It is an easy consequence of (ii) and (iii). (v) It follows easily from the definition of the $A$-sum. \[\square\]

**Corollary 3.5.** Let $A$ be a prime space with an accumulation point $a$ and let $C$ be a prime subspace of $A$. Let for every $b \in A \setminus \{a\} X_b$ be a topological space and
$x_b \in X_b$. Let for every $b \in C Y_b$ be a subspace of $X_b$ such that $x_b \in Y_b$. Then $\sum_{C}(Y_b, x_b)$ is a subspace of the space $\sum_{A}(X_b, x_b)$.

Let us note, that if for every $b \in A \setminus \{a\}$ $f_b$ is an embedding which maps isolated points of $X_b$ to isolated points of $Y_b$, then the embedding $\sum f_b$ has the same property.

4. The sets $T S_\gamma$, $T S S_\gamma$

In this section we construct the set of special prime spaces that generates $S C H(A)$ (where $A$ is a prime space which is not finitely generated and $t(A) = \alpha$). We start with defining the set $T S_\gamma$ of topological spaces for each ordinal $\gamma < \alpha^+$.

Let $T S_0 = \emptyset$ and $T S_1$ be the set of all prime subspaces of $A$.

If $\beta \geq 1$ is an ordinal, then $T S_{\beta+1}$ consists of all $B$-sums $\sum_{B}(X_b, x_b)$ where $B$ is a prime subspace of $A$, each $X_b \in T S_\beta$ and $x_b = a$.

If $\gamma > 0$ is a limit ordinal, then $T S_\gamma = \bigcup_{\beta < \gamma} T S_\beta$.

Sometimes, if we want to emphasize which prime space $A$ is used to construct this set, we use the notation $T S_\gamma(A)$.

Every space belonging to $T S_\gamma$ contains $B$ as a subspace and therefore it contains $a$. All spaces from $T S_\gamma$ are constructed from $A$ using $B$-sums, where $B \in C H(A)$, thus $T S_\gamma \subseteq C H(A)$ for each $\gamma$.

The following lemma is a generalization of [5, Lemma 6.2].

**Lemma 4.1.** Let $X$ be a topological space and $M \subseteq X$. If $p \in M_\beta \setminus M_\gamma$ for any $\gamma < \beta$, then there exists a space $S \in T S_\beta$ and a continuous map $f : S \to X$, which maps all isolated points of $S$ into $M$ and maps only the point $a$ to $p$.

**Proof.** For $\beta = 1$ the claim follows from the definition of $M_1$.

From the definition of $M_\beta$ it follows that $\beta$ is a non-limit ordinal. According to Proposition 2.2 $\beta < \alpha^+$. Suppose the assertion is true for any subset $K$ of $X$ and for any $\beta' < \beta$.

For a non-limit $\beta > 1$ there exists a prime subspace $B$ of $A$ and a continuous map $f : B \to X$ such that $f(a) = p$ and $f[B \setminus \{a\}] \subseteq M_{\beta-1}$.

If $\beta - 1$ is non-limit, we can moreover assume that $f[B \setminus \{a\}] \subseteq M_{\beta-1} \setminus M_{\beta-2}$.

(If necessary, we choose $B' = \{b \in B : f(b) \in M_{\beta-1} \setminus M_{\beta-2}\}$ and $f' = f|_{B'}$. $B'$ is a prime subspace of $A$, otherwise we get $x \in M_{\beta-1}$.)

If $\beta - 1$ is a limit ordinal, then for each point $x \in M_{\beta-1}$ there exists the smallest ordinal $\gamma < \beta - 1$ such that $x \in M_\gamma$. Obviously, $\gamma$ is a non-limit ordinal.

Thus for each $x \in f[B \setminus \{a\}]$ there exists a continuous map $f_x : S_x \to X$, where $S_x \in T S_{\beta-1}$, which sends all isolated points of $S_x$ into $M$ and $a$ to $x$.

Then $\sum_{B}(S_{f(b)}, a) \in T S_\beta$ and we can define a map $g : \sum_{B}(S_{f(b)}, a) \to X$ such that $g|_B = f$ and $g|(x) \times (S_x \setminus \{a\}) = f_x(y)$ for $y \in S_x \setminus \{a\}$. Clearly, $g$ maps isolated points into $M$. It remains only to show that $g$ is continuous.

The defining map $\varphi : B \sqcup (\bigsqcup_{b \in B \setminus \{a\}} S_{f(b)}) \to \sum_{B}(S_{f(b)}, a)$ is a quotient map. Thus, $g : \sum_{B}(S_{f(b)}, a) \to X$ is continuous if and only if $g \circ \varphi$ is continuous. But $g \circ \varphi|_B = f$ and $g \circ \varphi|_{S_x} = f_x$ are continuous, thus $g$ is continuous.

For any $S \in T S_\gamma$ we denote by $P(S)$ the subspace of the space $S$ which consists of all isolated points of $S$ and of the point $a$. Clearly, $P(S)$ is a prime space. We
denote by $TSS_\gamma$ the set of all spaces $P(S)$ where $S \in TS_\gamma$. The above lemma implies:

**Lemma 4.2.** If $p \in M_\beta$ and $p \notin M_\gamma$ for any $\gamma < \beta$, then there exists a space $T \in TSS_\beta$ and a continuous map $f : T \to X$, which maps all isolated points of the space $T$ into $M$ and such that $f(a) = p$.

**Proposition 4.3.** SCH($A$) is generated by the set $\bigcup_{\gamma < \alpha^+} TSS_\gamma$.

*Proof.* Let $X \in$ SCH($A$). According to Lemma 4.1 it suffices to prove that for any subset $M \subseteq X$ and any $x \in \overline{M} \setminus M$ there exists $T \in \bigcup_{\gamma < \alpha^+} TSS_\gamma$ and a continuous map $f : T \to X$ such that $f(a) = x$ and $f(T \setminus \{a\}) \subseteq M$.

Since $X \in$ SCH($A$) there exists $Y \in$ CH($A$) such that $X$ is a subspace of $Y$. Denote by $\overline{M}^Y$ the closure of $M$ in $Y$. Then $\overline{M} = \overline{M}^Y \cap X$ and $x \in \overline{M}^Y \setminus M$ in $Y$. By Proposition 2.2 $\overline{M}^Y = M_{\alpha^+} = \bigcup_{\beta < \alpha^+} M_\beta$. Let $\beta$ be the smallest ordinal with $x \in M_\beta$. Then $\beta > 0$ and for any $\gamma < \beta$ $x \notin M_\gamma$. By Lemma 4.1 there exists $S \in TS_\beta$ and a continuous map $f : S \to Y$ with $f(a) = x$ and $f(c) \in M$ for any isolated point of $S$. Then $P(S) \in TSS_\gamma$ and $f([P(S)]) \subseteq X$. Hence, $f|[P(S)] : P(S) \to X$ is a continuous map satisfying the required conditions. Consequently, $X \in$ CH($\bigcup_{\gamma < \alpha^+} TSS_\gamma$). □

**Remark 4.4.** It can be easily seen that if we define the sets $T'S_\gamma$, $\gamma < \alpha^+$, similarly as the sets $TS_\gamma$ but we use only the A-sums (and not all B-sums for prime subspaces $B$ of $A$) and then we put $T'S_\gamma = \{P(S) : S \in TSS_\gamma\}$ we obtain the set $\bigcup_{\gamma < \alpha^+} T'S_\gamma$ which also generates SCH($A$). This follows from the fact that any space from $\bigcup_{\gamma < \alpha^+} TSS_\gamma$ is a prime subspace of some space from $\bigcup_{\gamma < \alpha^+} T'S_\gamma$.

Similarly, if we put $T'S'_\gamma = \{S_a : S \in T'S_\gamma\}$ ($S_a$ is the prime factor of $S$ at $a$), then the set $\bigcup_{\gamma < \alpha^+} T'S'_\gamma$ generates SCH($A$) because $\bigcup_{\gamma < \alpha^+} T'S'_\gamma \subseteq$ SCH($A$) and for every $S \in \bigcup_{\gamma < \alpha^+} T'S_\gamma P(S)$ is a subspace of $S_a$.

5. The spaces $A_\omega$ and $(A_\omega)_a$

The space $A_\omega$ is defined similarly as $S_\omega$ in [2] using the A-sum and the space $A$ instead of the sequential sum and the space $C(\omega_0)$. We start with defining the space $A_n$ for each $n \in \mathbb{N}$ putting $A_1 = A$ and $A_{n+1} = \sum_{A} (A_n, a)$. Clearly, $A_1$ is a subspace of $A_2$ and if $A_{n-1}$ is a subspace of $A_n$, then, according to Lemma 3.4 $A_n = \sum_{A} (A_{n-1}, a)$ is a subspace of $A_{n+1} = \sum_{A} (A_n, a)$. Hence, $A_n$ is a subspace of $A_{n+1}$ for each $n \in \mathbb{N}$.

The Figure 1 represents the space $A_3$ for $A = C(\omega_0)$. (The space $C(\omega_0)$ is defined in Example 5.7)

The space $A_\omega$ is a topological space defined on the set $\bigcup_{n \in \mathbb{N}} A_n$ such that a subset $U$ of $\bigcup_{n \in \mathbb{N}} A_n$ is open in $A_\omega$ if and only if $U \cap A_n$ is open in $A_n$ for every $n \in \mathbb{N}$. It is obvious that for every $n \in \mathbb{N}$ the space $A_n$ is a subspace of $A_\omega$ and $A_\omega$ is a quotient space of the topological sum $\bigsqcup_{n \in \mathbb{N}} A_n$. Consequently, $A_\omega$ belongs to
CH(\(A\)). Observe that \(A_\omega\) can be considered as an inductive limit of its subspaces \(A_n\), \(n \in \mathbb{N}\).

Similarly as the space \(S_\omega\) in [2] the space \(A_\omega\) has the following important property.

**Proposition 5.1.** \(A_\omega = \sum_A (A_\omega, a)\)

**Proof.** Put \(X = \sum_A (A_\omega, a)\). For each \(n \in A\) the space \(A_n\) is a subspace of \(A_\omega\) and it follows that \(A_{n+1} = \sum_A (A_\omega, a)\) is a subspace of \(X\) (Lemma 3.4). Obviously, \(A = A_1\) is also a subspace of \(X\) and we obtain that for each \(n \in \mathbb{N}\) \(A_n\) is a subspace of \(X\).

Clearly, \(X = \bigcup A_n\). To finish the proof it suffices to check that if \(U\) is a subset of \(X\) and \(U \cap A_n\) is open in \(A_n\) for each \(n \in \mathbb{N}\), then \(U\) is open in \(X\).

Let us denote by \(A_n^b\) the subspace of \(X\) on the set \(\{b\} \cup (\{b\} \times (A_n \setminus \{a\}))\) and by \(A_n^b\) the subspace of \(X\) on the set \(\{b\} \cup (\{b\} \times (A_\omega \setminus \{a\}))\). Clearly, \(A_n^b\) is homeomorphic to \(A_n\) and \(A_n^b\) is homeomorphic to \(A_\omega\). \(A_n^b\) is a subspace of \(A_\omega\) and a subset \(V\) of \(A_n^b\) is open in \(A_n^b\) if and only if \(V \cap A_n^b\) is open in \(A_n^b\) for each \(n \in \mathbb{N}\).

If \(U \subseteq X\) and \(U \cap A_n\) is open in \(A_n\) for all \(n \in \mathbb{N}\), then \(U \cap A_n\) is open in \(A\) and \(U \cap A_{n+1}\) is open in \(A_{n+1} = \sum_A (A_n, a)\) for all \(n \in \mathbb{N}\). Then \(U \cap A_n^b\) is open in \(A_n^b\) for each \(n \in \mathbb{N}\) and \(b \in A \setminus \{a\}\) and it follows that \(U \cap A_n^b\) is open in \(A_n^b\) for each \(b \in A \setminus \{a\}\). Hence, \(U\) is open in \(X\).

The following lemma is evident.

**Lemma 5.2.** \(\text{card } A_\omega = \text{card } A\)

**Lemma 5.3.** For every ordinal \(\gamma\), \(1 \leq \gamma < \alpha^+\) and every space \(S \in \text{TS}_\gamma\) the space \(S\) is a subspace of \(A_\omega\). (Clearly, the point \(a\) of \(S\) coincides with the point \(a\) of \(A_\omega\)).

**Proof.** If \(\gamma = 1\), then \(S = B\) is a prime subspace of \(A = A_1\). Let \(\gamma\) be an ordinal, \(1 < \gamma < \alpha^+\) and the assertion hold for every ordinal \(\beta\), \(1 \leq \beta < \gamma\). If \(S = \sum_{B} X_b \in \text{TS}_\gamma\), then for each \(b \in B \setminus \{a\}\) \(X_b \in \text{TS}_{\beta_b}\) with \(1 \leq \beta_b < \gamma\). Hence, for each \(b \in B \setminus \{a\}\), \(X_b\) is a subspace of \(A_\omega\) and, according to Corollary 4.5 \(S\) is a subspace of \(A_\omega = \sum_A (A_\omega, a)\).

**Theorem 5.4.** Let \((A_\omega)_a\) be the prime factor of the space \(A_\omega\) at \(a\). Then \((A_\omega)_a\) is a prime space, \(\text{CH}((A_\omega)_a) = \text{SCH}(A)\) and \(\text{card} (A_\omega)_a = \text{card } A\).
Proof. Evidently, \((A_\omega)_a\) is a prime space and \(\text{card}(A_\omega)_a = \text{card} A\). Since \(A_\omega\) belongs to \(\text{CH}(A) \subseteq \text{SCH}(A)\), according to Proposition 5.4 \((A_\omega)_a\) belongs to \(\text{SCH}(A)\). Hence, it suffices to check that \(\bigcup_{\gamma < \alpha^+} TSS_\gamma \subseteq \text{CH}((A_\omega)_a)\).

Let \(T \in \bigcup_{\gamma < \alpha^+} TSS_\gamma\). Then there exists an ordinal \(\gamma, 1 \leq \gamma < \alpha^+, \) and \(S \in TSS_\gamma\) such that \(T = P(S)\). By Lemma 5.3 \(S\) is a subspace of \(A_\omega\) and, clearly, it follows that \(T = P(S)\) is a subspace of \((A_\omega)_a\). Consequently, there exists a quotient map \((A_\omega)_a \to T\) and we obtain that \(T\) belongs to \(\text{CH}((A_\omega)_a)\).

Finally, let \(X\) be an arbitrary topological space which is not finitely generated and \(\{X_c, c \in Y\}\) be the set of all prime factors of \(X\) that are not discrete spaces. Denote by \(A_X\) the quotient space of the topological sum \(\bigoplus_{c \in Y} (\{c\} \times X_c)\) obtained by collapsing all points of the subset \(\{(c, c), c \in Y\}\) of the space \(\bigoplus_{c \in Y} (\{c\} \times X_c)\) into one point \(a\).

The space \(A_X\) is a prime space which is not finitely generated, \(a\) is the accumulation point of \(A_X\), \(\text{card} A_X = \text{card} X\) and the following statement holds:

**Theorem 5.5.** \(\text{SCH}(X) = \text{CH}((A_X)_\omega)_a, \) \((A_X)_\omega)_a\) is a prime space and \(\text{card}((A_X)_\omega)_a = \text{card} X\).

*Proof.* Evidently, \(A_X \in \text{SCH}(X), X \in \text{CH}(A_X)\) and therefore \(\text{SCH}(X) = \text{SCH}(A_X)\). According to Theorem 5.4 \(\text{SCH}(A_X) = \text{CH}((A_X)_\omega)_a\), \((A_X)_\omega)_a\) is a prime space and \(\text{card}((A_X)_\omega)_a = \text{card} A_X = \text{card} X\). □

Recall, that a topological space \(X\) belongs to \(\text{Top}(\omega_1)\) if and only if every countable intersection of open subsets of \(X\) is open in \(X\) and \(\text{Top}(\omega_1)\) is a hereditary coreflective subcategory of \(\text{Top}\). If the space \(A\) belongs to \(\text{Top}(\omega_1)\), then \(\text{SCH}(A) \subseteq \text{Top}(\omega_1)\) and we can find smaller (and simpler) set of generators of \(\text{SCH}(A)\) than the set \(\bigcup_{\gamma < \alpha^+} TSS_\gamma(A)\) constructed in Proposition 5.4.

**Proposition 5.6.** If \(A \in \text{Top}(\omega_1)\), then \(\text{SCH}(A) = \text{CH}((P(A_n); 0 < n < \omega_0))\).

*Proof.* It suffices to show that \((A_\omega)_a \in \text{CH}((P(A_n); n < \omega_0))\). Clearly, each \(P(A_n)\) is a subspace of \((A_\omega)_a\). Denote by \(i_n: P(A_n) \to (A_\omega)_a\) the corresponding embedding and by \(f: \prod_{n \in \mathbb{N}} P(A_n) \to (A_\omega)_a\) the continuous map given by the maps \(i_n, n \in \mathbb{N}\). It is easy to see that this map is surjective. We claim that \(f\) is also a quotient map.

It suffices to show that if \(a \in U \subseteq (A_\omega)_a\) and \(U \cap P(A_n)\) is open in \(P(A_n)\) for each \(n, 0 < n < \omega_0\), then \(U\) is open in \((A_\omega)_a\). Since \(P(A_n)\) is a subspace of \(A_\omega\), there exist an open subset \(W_n\) of \(A_\omega\) such that \(W_n \cap P(A_n) = U \cap P(A_n)\). Put \(W = \bigcap_{0 < n < \omega_0} W_n\). The set \(W\) is open in \(A_\omega\) since \(A_\omega\) belongs to \(\text{Top}(\omega_1)\). We have \(W \cap P(A_n) \subseteq U \cap P(A_n)\) and \(\bigcup_{0 < n < \omega_0} P(A_n) = A_\omega\), therefore \(W \subseteq U\). Obviously, \(a \in W\). Hence, the set \(U\) is open in \((A_\omega)_a\).

Next we present some special cases of our construction.

**Example 5.7.** **Sequential spaces.** Recall that subspaces of sequential spaces are called subsequential. The category \textbf{Seq} of sequential spaces is the coreflective hull of the space \(C(\omega_0)\). The space \(C(\omega_0)\) is the topological space on the set \(\omega_0 + 1 = \omega_0 \cup \{\omega_0\}\) such that all points of \(\omega_0\) are isolated and a set containing \(\omega_0\) is
open if and only if its complement is finite. (Equivalently, the topology of $C(\omega_0)$ is the order topology given by the usual well-ordering of $\omega_0 + 1$.) The space $C(\omega_0)_{\omega}$ is homeomorphic to $S_\omega$ defined in [2]. Our results imply that the prime factor of the space $C(\omega_0)_{\omega}$ at $\omega_0$ is a generator of the category of subsequential spaces. Another countable generator of this category was constructed before in [5].

**Example 5.8.** The coreflective hull of the space $C(\alpha)$. Let $\alpha$ be a regular cardinal and $C(\alpha)$ be the topological space on the set $\alpha + 1 = \alpha \cup \{\alpha\}$ such that all points of $\alpha$ are isolated and a set containing $\alpha$ is open if and only if its complement has cardinality less than $\alpha$. It is well known that $X$ belongs to $\text{CH}(C(\alpha))$ if and only if a subset $V \subseteq X$ is closed in $X$ whenever for each $\alpha$-sequence of points from $V$ the set $V$ contains also all limits of this $\alpha$-sequence. The subcategories $\text{SCH}(C(\alpha))$ are minimal elements of the collection of all hereditary coreflective subcategories of $\text{Top}$ above $\text{FG}$. We use the subcategories $\text{SCH}(C(\alpha))$ in the next section. Our construction yields the generator $(C(\alpha)_{\omega})_\alpha$ of $\text{SCH}(C(\alpha))$ which has cardinality $\alpha$.

6. **Subcategories of $\text{Top}$ having $\text{FG}$ as their hereditary coreflective kernel**

Recall that a hereditary coreflective kernel of a subcategory $A$ of $\text{Top}$ is the largest hereditary coreflective subcategory of $\text{Top}$ contained in $A$. We denote it by $HCK(A)$. In this section we prove that if $A$ and $B$ are coreflective subcategories of $\text{Top}$ such that $HCK(A) = HCK(B) = \text{FG}$, then also $HCK(CH(A \cup B)) = \text{FG}$. The analogous result does not hold for infinite countable joins of coreflective subcategories. This problem is closely related to the subcategories $\text{SCH}(C(\alpha))$ because (see [3, Theorem 4.8]) $\text{FG}$ is the hereditary coreflective kernel of a coreflective subcategory $A$ of $\text{Top}$ if and only if $\text{FG} \subseteq A$ and for any regular cardinal $\alpha$ the category $\text{SCH}(C(\alpha))$ is not contained in $A$.

In [4, Problem 7] H. Herrlich and M. Hušek suggest to study coreflective subcategories of $\text{Top}$ such that their hereditary coreflective hull is the whole category $\text{Top}$ (i.e. $\text{SA} = \text{Top}$) and their hereditary coreflective kernel is the subcategory $\text{FG}$. In the paper [2] it is shown that there exists the smallest such subcategory of $\text{Top}$ and the collection of all such subcategories of $\text{Top}$ is closed under the formation of arbitrary non-empty intersections. In this section we prove that this collection is also closed under the formation of non-empty finite joins without being closed under the formation of infinite countable joins in the lattice of all coreflective subcategories of $\text{Top}$.

Throughout this section we will apply the results obtained in preceding sections to prime spaces $C(\alpha)$, $\alpha$ being a regular cardinal, defined in Example [5,8]. Note that $\alpha$ is an accumulation point of $C(\alpha)$ and $t(C(\alpha)) = \alpha$ for any regular cardinal $\alpha$. Since any prime subspace of $C(\alpha)$ is homeomorphic to $C(\alpha)$ it suffices to use only $C(\alpha)$-sums in the definition of $TS_\gamma$. For instance, if $n$ is a natural number, then $TS_n$ as well as $TSS_n$ contain precisely one space.

In order to prove the main result of this section, we first prove that if $\text{SCH}(C(\alpha)) \subseteq CH(A \cup B)$ for some coreflective subcategories $A$, $B$ of $\text{Top}$, then one of these subcategories contains $\text{SCH}(C(\alpha))$. We show it separately for the case $\alpha = \omega_0$ and $\alpha \geq \omega_1$.

We start with the case $\alpha = \omega_0$ where we can use some results presented in the paper [5]. As the sets $TSS_\gamma$, $\gamma < \omega_1$, introduced in [5] do not coincide with the sets $TSS_\gamma(C(\omega_0))$ defined in Section 4 we denote the sets used in [5] by $TSS_\gamma'(C(\omega_0))$. 
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The next lemma follows from [5, Theorem 7.1], resp. [5, Corollary 7.2].

**Lemma 6.1.** The category $\text{SSeq} = \text{SCH}(C(\omega_0))$ of subsequential spaces is the coreflective hull of the set $\bigcup_{n<\omega_1} TSS^n\beta(C(\omega_0))$.

As a consequence of [5, Theorem 7.1] and [5, Theorem 6.4] we obtain:

**Lemma 6.2.** If $\beta < \gamma < \omega_1$, then $TSS^n(\beta(C(\omega_0))) \subseteq \text{CH}(TSS^n\gamma(C(\omega_0)))$.

The following result concludes the part of this section concerning the subcategory $\text{SCH}(C(\omega_0))$.

**Proposition 6.3.** If $\text{SCH}(C(\omega_0)) \subseteq \text{CH}(\bigcup_{i \in I} A_i)$, $A_i$ is a coreflective subcategory of $\text{Top}$ for every $i \in I$ and $\text{card} I \leq \omega_0$, then there exists $i_0 \in I$ such that $\text{SCH}(C(\omega_0)) \subseteq A_{i_0}$.

**Proof.** Put $\beta_i = \sup \{\beta : TSS^n(\beta(C(\omega_0))) \subseteq A_i\}$ for $i \in I$. Since $\text{sup} \beta_i = \omega_1$ (Lemma 6.1) and $\omega_1$ is a regular cardinal, there exists $i_0 \in I$ such that $\beta_{i_0} = \omega_1$. By Lemma 6.1 and Lemma 6.2 we get that the coreflective subcategory $A_{i_0}$ contains the subcategory $\text{SSeq} = \text{SCH}(C(\omega_0))$. □

Next we want to prove a result analogous to Proposition 6.3 for the space $C(\alpha)$, where $\alpha \geq \omega_1$ is a regular cardinal. In the case $\alpha \geq \omega_1$ the desired result holds only for non-empty finite joins of coreflective subcategories of $\text{Top}$.

Recall that $C(\alpha)_1 = C(\alpha)$ and $C(\alpha)_{n+1} = \sum_{C(\alpha)} \langle C(\alpha)_n, \alpha \rangle$. According to Corollary 3.5 we obtain that $P(C(\alpha)_{n+1}) = P(\sum_{C(\alpha)} \langle P(C(\alpha)_n), \alpha \rangle)$ and it is easy to see that $\alpha^{\omega_1+1} \cup \{\alpha\}$ is the underlying set of the space $P(C(\alpha)_{n+1})$ and the subspace of $\sum P(C(\alpha)_n)$ on the set $\{\eta\} \cup (\{\eta\} \times \alpha^n)$ is homeomorphic to $P(C(\alpha)_{\eta})$ for each $\eta < \alpha$. To simplify the notation we will write $C(\alpha)_{\eta}$ instead of $P(C(\alpha)_{\eta})$.

The following result is a special case of Proposition 6.3.

**Proposition 6.4.** If $\alpha \geq \omega_1$ is a regular cardinal, then $\text{SCH}(C(\alpha)) = \text{CH}(\{C(\alpha)_{\eta} : 0 < n < \omega_0\})$.

**Lemma 6.5.** Let $\alpha \geq \omega_1$ be a regular cardinal. If $M$ is a subset of $C(\alpha)_n$ such that $\alpha \in \overline{M}$ and $M$ contains only isolated points of $C(\alpha)_n$, then there exists a subset $M' \subseteq M$ such that the subspace of the space $C(\alpha)_n$ on the set $\overline{M'}$ is homeomorphic to $C(\alpha)_n$.

**Proof.** The case $n = 1$ is clear. Let the assertion be true for $m$. Denote the subspace of $C(\alpha)_{m+1} = \sum_{C(\alpha)} C(\alpha)_m$ on the set $\{\eta\} \cup (\{\eta\} \times (C(\alpha)_m \setminus \{\alpha\})$, where $\eta < \alpha$, by $C(\alpha)_{\eta}^m$.

Put $B = \overline{M} \cap C(\alpha)$. Then $B$ is a prime subspace of $C(\alpha)$, for each $\eta \in B \setminus \{\alpha\}$ all points of the set $M_\eta = M \cap C(\alpha)_{\eta}^m$ are isolated in the space $C(\alpha)_{\eta}^m$ and $\eta \in \overline{M_\eta}$ in $C(\alpha)_{\eta}^m$ (observe that $\overline{M_\eta}$ in $C(\alpha)_{\eta}^m$ coincides with $\overline{M_\eta}$ in $C(\alpha)_{m+1}$ because $C(\alpha)_{\eta}^m$ is closed in $C(\alpha)_{m+1}$). Since $C(\alpha)_{\eta}^m$ is homeomorphic to $C(\alpha)_m$ by the induction assumption we obtain that there exists a subset $M'_\eta \subseteq M_\eta$ such that $\eta \in \overline{M'_\eta}$ and the subspace $\overline{M'_\eta}$ of $C(\alpha)_{\eta}^m$ is homeomorphic to some space $C(\alpha)_m$.

Let $B' = B \setminus \{\alpha\}$ and $M' = \bigcup_{\eta \in B'} M'_\eta$. Clearly, $M' \subseteq M$, $\overline{M'} = \bigcup_{\eta \in B'} \overline{M'_\eta} \cup \{\alpha\}$ in $S$ and $\overline{M'_\eta}$ is homeomorphic to $C(\alpha)_m$ for each $\eta \in B'$. □
The subspace $B$ of $C(\alpha)$ is homeomorphic to $C(\alpha)$ and it is easy to check that $\overline{M'}$ is homeomorphic to $\sum_{\alpha_{C}} C(\alpha)_{m} = C(\alpha)_{m+1}$.

**Corollary 6.6.** Let $\alpha \geq \omega_{1}$ be a regular cardinal, $0 < n < \omega_{0}$. Then every prime subspace $T$ of $C(\alpha)_{n}$ is homeomorphic to $C(\alpha)_{n}$.

**Proof.** Put $M = T \setminus \{\alpha\}$. Clearly, $\alpha \in \overline{M'}$. According to Lemma 6.5 there exists a subset $M'$ of $M$ such that the subspace $M' \cup \{\alpha\}$ of $C(\alpha)_{n}$ is homeomorphic to $C(\alpha)_{n}$. It follows from the proof of Lemma 6.5 that $M \setminus M'$ is a discrete clopen subspace of $C(\alpha)_{n}$ with cardinality at most $\alpha$. Hence, $T = M \cup \{\alpha\}$ is homeomorphic to $C(\alpha)_{n}$ as well.

**Proposition 6.7.** Let $\alpha \geq \omega_{1}$ be a regular cardinal and $0 < n < \omega_{0}$. If $C(\alpha)_{n}^{-} \subseteq \text{CH}(\bigcup_{i \in I} A_{i})$, where all $A_{i}$‘s are coreflective subcategories of $\text{Top}$, then there exists $i_{0} \in I$ such that $C(\alpha)_{n} \subseteq A_{i_{0}}$.

**Proof.** The space $C(\alpha)_{n}^{-}$ is a prime space with an accumulation point $\alpha$. If $C(\alpha)_{n}^{-} \subseteq \text{CH}(\bigcup_{i \in I} A_{i})$, then there exists a quotient map $f: \prod_{i \in I} B_{i} \to C(\alpha)_{n}^{-}$, where $B_{i}$ belongs to $A_{i}$ for each $i \in I$. Put $f_{i} = f|B_{i}$ and let $A_{i}$ be the space on the set $f_{i}[B_{i}]$ endowed with the quotient topology with respect to $f_{i}$ for each $i \in I$.

The topology of every space $A_{i}$ is finer than the topology of the corresponding subspace of $C(\alpha)_{n}^{-}$ and it follows that $A_{i}$ is either discrete or prime space. Clearly, a set $U \subseteq C(\alpha)_{n}^{-}$ is open in $C(\alpha)_{n}^{-}$ if and only if $U \cap A_{i}$ is open in $A_{i}$ for each $i \in I$ and $A_{i} \subseteq A_{i'}$. Obviously, there exists $i_{0} \in I$ such that $\alpha$ is an accumulation point of $A_{i_{0}}$ (otherwise $\alpha$ would be isolated in $C(\alpha)_{n}^{-}$).

We show that $C(\alpha)_{n}^{-} \subseteq \text{CH}(A_{i_{0}})$. Let $M$ be a non-closed subset of $C(\alpha)_{n}^{-}$. It suffices to find a continuous map $f: A_{i_{0}} \to C(\alpha)_{n}^{-}$ such that $f[A_{i_{0}} \setminus \{\alpha\}] \subseteq M$ and $f(\alpha) = \alpha$. According to Corollary 6.6 the subspace on the set $M \cup \{\alpha\}$ is homeomorphic to $C(\alpha)_{n}^{-}$. Let us denote the homeomorphism from $C(\alpha)_{n}^{-}$ to $M \cup \{\alpha\}$ by $g$. Moreover, there is a continuous map $i: A_{i_{0}} \to C(\alpha)_{n}^{-}$ defined by $i(x) = x$ for each $x \in A_{i_{0}}$. The desired continuous map is $f = g \circ i$.

If $X$ and $Y$ are prime spaces, then a continuous map $f: X \to Y$ is called a prime map if it maps only the accumulation point of $X$ to the accumulation point of $Y$.

**Lemma 6.8.** Let $\alpha \geq \omega_{1}$ be a regular cardinal and $0 < m < n < \omega_{0}$. There exists a quotient prime map $g: C(\alpha)_{m}^{-} \to C(\alpha)_{n}^{-}$.

**Proof.** Obviously, it suffices to prove the lemma for $n = m + 1$. In this case $C(\alpha)_{m+1} = P(\sum_{\alpha_{C}} C(\alpha)_{m})$ is a topological space on the set $\{\alpha\} \cup \alpha^{m+1}$ and $C(\alpha)_{m}^{-}$ is a topological space on the set $\{\alpha\} \cup \alpha^{m}$. We define a map $g: C(\alpha)_{m+1} \to C(\alpha)_{m}^{-}$ by $g(\alpha) = \alpha$ and $g((\eta, x)) = x$ for all $(\eta, x) \in C(\alpha)_{m+1} \setminus \{\alpha\}$. It is easy to check that the map $g$ is continuous and quotient.

**Corollary 6.9.** If $\alpha \geq \omega_{1}$ is a regular cardinal and $0 < m < n < \omega_{0}$, then $C(\alpha)_{m}^{-} \subseteq \text{CH}(C(\alpha)_{n}^{-})$.

**Proposition 6.10.** If $\alpha$ is a regular cardinal and $\text{SCH}(C(\alpha)) \subseteq \text{CH}(A \cup B)$, then $\text{SCH}(C(\alpha)) \subseteq \text{CH}(A)$ or $\text{SCH}(C(\alpha)) \subseteq \text{CH}(B)$.

**Proof.** Since the case $\alpha = \omega_{0}$ follows immediately from Proposition 6.9 we can assume that $\alpha \geq \omega_{1}$.
By Proposition 6.7 for each \( n, 0 < n < \omega_0 \), the space \( C(\alpha)_n^- \) belongs either to \( A \) or to \( B \). By Lemma 6.8 we have a quotient map \( f: C(\alpha)_n^- \to C(\alpha)_m^- \) for each \( n > m \). Hence, one of these two coreflective categories contains all spaces \( C(\alpha)_n^- \) and, consequently, it contains \( \text{SCH}(C(\alpha)) \).

Now we can state the main result of this section.

**Theorem 6.11.** If \( A, B \) are coreflective subcategories of the category \( \text{Top} \) and \( \text{HCK}(A) = \text{HCK}(B) = \text{FG} \), then \( \text{HCK}(\text{CH}(A \cup B)) = \text{FG} \).

**Proof.** Suppose the contrary. Then according to [3] Theorem 4.8 there exists a regular cardinal \( \alpha \) with \( \text{SCH}(C(\alpha)) \subseteq \text{CH}(A \cup B) \). Proposition 6.10 implies that \( \text{SCH}(C(\alpha)) \subseteq A \) or \( \text{SCH}(C(\alpha)) \subseteq B \), contradicting the assumption that the hereditary coreflective kernel of both these categories is \( \text{FG} \).

Let \( C \) be the conglomerate of all coreflective subcategories of \( \text{Top} \). It is well known that \( C \) partially ordered by \( \subseteq \) is a complete lattice. Denote by \( K \) the conglomerate of all coreflective subcategories \( A \) of \( \text{Top} \) with \( \text{HCK}(A) = \text{FG} \). The above theorem says that \( K \) is closed under the formation of non-empty finite joins in \( C \). We next show that \( K \) fails to be closed under the formation of infinite countable joins in \( C \). In the concrete we prove that if \( \alpha \geq \omega_1 \) is a regular cardinal, then all categories \( \text{CH}(C(\alpha)_n^-) \) belong to \( K \). According to Proposition 6.4 the category \( \text{SCH}(C(\alpha)) \) is the join of this family in \( C \) and, evidently, \( \text{SCH}(C(\alpha)) \notin K \). The proof is divided into three auxiliary lemmas.

**Lemma 6.12.** Let \( \alpha \geq \omega_1 \) be a regular cardinal and \( 2 \leq n < \omega_0 \). If there exists a prime map \( f: C(\alpha)_n^- \to C(\alpha)_{n+1}^- \), then there exists a prime map \( f': C(\alpha)_n^- \to C(\alpha)_{n+1}^- \) such that \( f'([\xi] \times \alpha^{n-1}) \cap (\bigcup_{\eta<\xi}\{\eta\} \times \alpha^n) = \emptyset \) for each \( \xi < \alpha \).

**Proof.** Let \( f: C(\alpha)_n^- \to C(\alpha)_{n+1}^- \) be a prime map. Denote by \( B_\xi \) the subspace of \( \sum C(\alpha)_{n-1}^- \) on the set \( \{\xi\} \cup (\{\xi\} \times \alpha^{n-1}) \) where \( \xi < \alpha \). The subspace \( B_\xi \) is homeomorphic to \( C(\alpha)_{n-1}^- \).

For each \( \xi < \alpha \) the set \( f^{-1}[\{\alpha\} \cup (\bigcup_{\eta \geq \xi}\{\eta\} \times \alpha^n)] \) is open in \( C(\alpha)_n^- \), therefore there exists an ordinal \( \gamma < \alpha \) such that for each \( \gamma' > \gamma \) the set \( \{\gamma'\} \cup (f^{-1}[\bigcup_{\eta > \xi}\{\eta\} \times \alpha^n] \cap B_{\gamma'} \) is open in \( B_\xi \). Hence, we can define an increasing sequence \( \{\gamma_\xi\}_{\xi<\alpha} \) such that \( C_\xi := \{\gamma_\xi\} \cup (f^{-1}[\bigcup_{\eta \geq \xi}\{\eta\} \times \alpha^n] \cap B_{\gamma_\xi}) \) is open in \( B_{\gamma_\xi} \). Clearly, \( f(C_\xi \setminus \{\gamma_\xi\}) \subseteq \bigcup_{\eta \geq \xi}\{\eta\} \times \alpha^n \).

According to Corollary 6.8 the subspace of \( B_{\gamma_\xi} \) on the set \( C_\xi \) is homeomorphic to \( C(\alpha)_{n-1}^- \). Hence, for each \( \xi < \alpha \) we can define an embedding \( h_\xi: C(\alpha)_{n-1}^- \hookrightarrow \sum C(\alpha)_{n-1}^- \) such that \( h_\xi[C(\alpha)_{n-1}^-] = C_\xi \). It is easy to see that the map \( h: \sum C(\alpha)_{n-1}^- \to \sum C(\alpha)_{n-1}^- \) given by \( h(\xi) = \gamma_\xi \) for each \( \xi < \alpha \), \( h(\alpha) = \alpha \) and \( h(\xi, x) = h_\xi(x) \) for each \( \xi < \alpha \) and \( x \in \alpha^{n-1} \) is also an embedding. Put \( A_\xi = \{\xi\} \times \alpha^{n-1} \). Then \( h[A_\xi] \subseteq h_\xi[C(\alpha)_{n-1}^-] = C_\xi \) and \( f[h[A_\xi]] \subseteq f[C_\xi \setminus \{\gamma_\xi\}] \subseteq \bigcup_{\eta \geq \xi}\{\eta\} \times \alpha^n \). Consequently, \( f \circ h \) is a prime map satisfying the required condition.

**Lemma 6.13.** Let \( \alpha \geq \omega_1 \) be a regular cardinal and \( 0 < n < \omega_0 \). Then there exists no prime map from \( C(\alpha)_n^- \to C(\alpha)_{n+1}^- \).
Lemma 6.14. Let
\[
\alpha \geq \omega_1 \text{ be a regular cardinal and } 0 < n < \omega_0. \quad \text{Then } \text{HCK}(\text{CH}(\alpha^{-n})) = \text{FG}.
\]

Proof. Recall (see [6]) that if \( \gamma > \delta \), then \( \text{Top}(\gamma) \cap \text{Gen}(\delta) = \text{FG} \). For \( \beta < \alpha \) we have \( \text{SCH}(\beta) \subseteq \text{Gen}(\beta) \) and \( \text{C}(\alpha^{-n}) \in \text{Top}(\alpha) \), hence \( \text{SCH}(\beta) \not\subseteq \text{Gen}(\beta) \).
CH(C(α)_n^-). Similarly if β > α, then SCH(C(β)) ⊆ Top(β) and C(α)_n^- ∈ Gen(α). Thus, SCH(C(β)) ⊈ CH(C(α)_n^-).

By Lemma 6.13 and Lemma 1.4, C(α)_n+1^- ⊈ CH(C(α)_n^-) (every prime subspace of C(α)_n^- is homeomorphic to C(α)_n^-) and C(α)_n+1^- ∈ SCH(C(α)), therefore SCH(C(α)) ⊈ CH(C(α)_n+1^-) as well. □

Denote by \( L \) the collection of all coreflective subcategories \( A \) of Top such that \( S_A = Top \) and HCK(\( A \)) = FG. In the paper [9] it is shown that \( L \) has the smallest element \( A_0 = CH(\{S^\alpha; \alpha \text{ is a cardinal}\}) \), where \( S \) is the Sierpiński doubleton, and \( L \) is closed under the formation of arbitrary non-empty intersections. This together with Theorem 6.11 yields:

**Theorem 6.15.** The collection \( L \) is closed under the formation of non-empty intersections, non-empty finite joins in \( \mathcal{C} \) and has the smallest element.

**Proposition 6.16.** There is no maximal coreflective subcategory \( A \) of Top such that HCK(\( A \)) = FG. Consequently, the collection \( L \) has no maximal element.

**Proof.** Suppose that \( A \) is maximal coreflective subcategory of Top with the property HCK(\( A \)) = FG. Let \( \alpha \geq \omega_1 \) be a regular cardinal. According to Lemma 6.14 and Theorem 6.11, HCK(CH(\( A \cup \{C(\alpha)_n^-\}) = FG \) for each \( n, 0 < n < \omega_0 \). Thus, we get \( C(\alpha)_n^- \in A \) for each \( n \) and by Proposition 6.4, SCH(C(\( A \)) ⊆ A, a contradiction.

The proof that \( L \) has no maximal elements is analogous. □

The family CH(\( A_0 \cup \{C(\alpha)_n^-\} \)), \( 0 < n < \omega_0 \), where \( \alpha \geq \omega_1 \) is a regular cardinal, is an example of a countable family of elements of \( L \) such that its join does not belong to \( L \).

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Department of Algebra and Number Theory, FMFI UK, Mlynská dolina, 842 48 Bratislava, Slovakia

E-mail address: sleziak@fmph.uniba.sk