Flat Lorentz 3-manifolds and cocompact Fuchsian groups

William M. Goldman and Gregory A. Margulis

1. Introduction

Consider Minkowski 2+1-space $\mathbb{E}$ and let $G \subset \text{SO}(2,1)^0$ be a discrete subgroup. Suppose that a group of affine isometries of $\mathbb{E}$ with linear part $G$ acts properly and freely on $\mathbb{E}$. In a remarkable preprint [20], Geoffrey Mess proved the following theorem:

**Theorem.** $G$ is not cocompact in $\text{SO}(2,1)^0$.

Mess deduces this result as part of a general theory of domains of dependence in constant curvature Lorentzian 3-manifolds. We give an alternate proof, using an invariant introduced by Margulis [18, 19] and Teichmüller theory.

We thank Scott Wolpert for helpful conversations concerning Teichmüller theory. We also wish to thank Paul Igodt and the Algebra Research Group at the Katholieke Universiteit Leuven at Kortrijk, Belgium for their hospitality at the “Workshop on Crystallographic Groups and their Generalizations II”, where these results were obtained.

2. Background

Let $\mathbb{R}^{2,1}$ be a 3-dimensional real vector space with inner product

$$\mathbb{B}(x, y) = x_1y_1 + x_2y_2 - x_3y_3.$$ 

The group of linear isometries of $\mathbb{R}^{2,1}$ will be denoted by $\text{SO}(2,1)$. Let $\text{Isom}(\mathbb{R}^{2,1})$ denote the group of affine isometries, that is, the group of all transformations of the form

$$h : \mathbb{R}^{2,1} \longrightarrow \mathbb{R}^{2,1}$$

$$x \longmapsto g(x) + u$$

where $g \in \text{O}(2,1)$ and $u \in \mathbb{R}^{2,1}$. We write $g = \mathbb{L}(h)$ and $h = (g, u)$. Evidently $\text{Isom}(\mathbb{R}^{2,1})$ is isomorphic to the semidirect product $\text{O}(2,1) \rtimes \mathbb{R}^{2,1}$ where $\mathbb{R}^{2,1}$ denotes the vector group of translations of $\mathbb{E}$.

Both authors gratefully acknowledge partial support from NSF grants.
Let \( G \subset O(2,1) \) be a subgroup. An affine deformation of \( G \) is a homomorphism \( \phi : G \to \text{Isom}(\mathbb{R}^{2,1}) \) such that \( L(\phi(g)) = g \). An affine deformation \( \phi \) is proper if the resulting action of \( G \) by affine transformations on \( \mathbb{R}^{2,1} \) is a proper action. Write \( \phi(g) = (g, u(g)) \).

The condition that \( \phi \) be a homomorphism is that the map \( u = u_\phi : G \to \mathbb{R}^{2,1} \) satisfy the cocycle condition
\[
(1) \quad u_\phi(g_1 g_2) = u_\phi(g_1) + g_1 u_\phi(g_2).
\]
A map \( u : G \to \mathbb{R}^{2,1} \) satisfying (1) is called a cocycle and the vector space of cocycles is denoted by \( Z^1(G, \mathbb{R}^{2,1}) \).

If \( \phi_1, \phi_2 \) are affine deformations of \( G \) which are conjugate by translation by \( v \in \mathbb{R}^{2,1} \), then the difference \( u_{\phi_1} - u_{\phi_2} \) is the cocycle
\[
\delta v : g \mapsto v - g(v).
\]
Such a cocycle is called a coboundary. The subspace of coboundaries is denoted by \( B^1(G, \mathbb{R}^{2,1}) \). We say that \( \phi_1, \phi_2 \) are translationally conjugate. Translational conjugacy classes of affine deformations of \( G \) correspond to elements in the cohomology group
\[
H^1(G, \mathbb{R}^{2,1}) = Z^1(G, \mathbb{R}^{2,1})/B^1(G, \mathbb{R}^{2,1}).
\]

Suppose that \( \phi : G \to \text{Isom}(\mathbb{R}^{2,1}) \) is a proper affine deformation. By Fried-Goldman [11], the group \( G \) is solvable or the linear part
\[
L \circ \phi : G \to O(2,1)
\]
is an isomorphism onto a discrete subgroup of \( O(2,1) \). (Indeed, this conclusion is obtained for any proper affine action on \( \mathbb{R}^3 \).) The solvable groups are easily classified by embedding them as lattices in Lie subgroups which themselves act properly. When \( G \) is not solvable, then interesting examples do exist (Margulis [18, 19]). Furthermore every torsionfree non-cocompact discrete subgroup \( G \subset O(2,1) \) for which \( H^1(G; \mathbb{R}^{2,1}) \neq 0 \) admits proper affine deformations (Drumm [8]).

Recall that an element of \( O(2,1) \) is hyperbolic if it has three distinct real eigenvalues. A subgroup \( G \subset O(2,1) \) is purely hyperbolic if every element is hyperbolic. A cocompact discrete subgroup contains a purely hyperbolic subgroup of finite index.

3. An invariant of affine isometries

In [18, 19], Margulis defines an invariant \( \alpha_\phi : G \to \mathbb{R} \) of an affine deformation \( \phi \) of a purely hyperbolic subgroup \( G \subset O(2,1) \) as follows. We assume that \( G \subset SO(2,1)^0 \). Choose a component \( N_+ \) of the complement of 0 in the lightcone. Since any element \( g \) of \( G \) is hyperbolic its three eigenvalues are distinct positive real numbers
\[
\lambda(g) < 1 < \lambda(g)^{-1}.
\]
Choose an eigenvector $x^-(g) \in \mathcal{N}_+$ for $\lambda(g)$ and an eigenvector $x^+(g) \in \mathcal{N}_+$ for $\lambda(g)^{-1}$, respectively. Then there exists a unique eigenvector $x^0(g)$ for $g$ with eigenvalue 1 such that:

- $\mathbb{B}(x^0(g), x^0(g)) = 1$;
- $(x^-(g), x^+(g), x^0(g))$ is a positively oriented basis.

Notice that $x^0(g^{-1}) = -x^0(g)$.

If $\phi$ is an affine deformation corresponding to a cocycle $u$, then $\alpha_\phi$ is defined as:

$$\alpha_\phi : G \to \mathbb{R}$$

$$g \mapsto \mathbb{B}(x^0(g), u(g)).$$

More generally, $\alpha_\phi(g) = \mathbb{B}(x^0(g), \phi(g)(x) - x)$ for any $x \in E$. Furthermore $\alpha_\phi$ is a class function on $G$ and recently Drumm-Goldman [10] have proved that the mapping

$$H^1(G, \mathbb{R}^{2,1}) \to \mathbb{R}^G$$

$$[u] \mapsto \alpha_\phi$$

is injective, that is, $\alpha$ is a complete invariant of the conjugacy class of the affine deformation.

In [13], Margulis proved the following theorem (see also Drumm [7]):

**Theorem 1 (Margulis).** Suppose that $G \subset \text{SO}(2,1)^0$ is purely hyperbolic and let $\phi : G \to \text{Isom}(\mathbb{R}^{2,1})$ be an affine deformation. If there exist $g_1, g_2 \in G$ such that $\alpha_\phi(g_1) > 0 > \alpha_\phi(g_2)$, then $\phi$ is not proper.

Affine deformations defining free actions correspond to cocycles for which $\alpha(g) \neq 0$ for $g \neq I$. We shall say that a cocycle $u$ is positive (respectively negative) if $\alpha(g) > 0$ (respectively $\alpha(g) < 0$) whenever $I \neq g \in G$. Clearly $u$ is positive if and only if $-u$ is negative. We conjecture a converse to Theorem 1: an affine deformation is proper if and only if its cocycle is positive or negative.

### 4. Deformation-theoretic interpretation of $\alpha$

We reduce the proof of Mess’s theorem to facts about deformations of hyperbolic Riemann surfaces. Let $M$ be a surface with a complete hyperbolic structure and $\pi = \pi_1(M)$ its fundamental group. A representation $\phi : \pi \to \text{SO}(2,1)^0$ is *Fuchsian* if it is an embedding onto a discrete subgroup of $\text{SO}(2,1)^0$. When $M$ is a closed surface, the space of conjugacy classes of Fuchsian representations $\phi : \pi \to \text{SO}(2,1)^0$ is an open subset of the space of conjugacy classes of all representations, which identifies with the *Teichmüller space* $T(M)$ of $M$. (See Weil [26, 27, 28], §VI of Raghunathan [22] for the general theory and Goldman [12, 13] for the case of surface groups.) Its tangent space identifies with the cohomology group $H^1(G, \mathbb{R}^{2,1})$ where $G = \phi(\pi)$.

Since the classical theory of Fuchsian groups is usually phrased in terms of $\text{SL}(2, \mathbb{R})$ (rather than $\text{SO}(2,1)$), and since $2 \times 2$ matrices are more tractable than $3 \times 3$ matrices, we work with $\text{SL}(2, \mathbb{R})$. The Lie groups $\text{SL}(2, \mathbb{R})$ and $\text{SO}(2,1)$ are locally isomorphic, but not *globally* isomorphic. One model for the local isomorphism is the adjoint representation, as follows. The trace form of any nontrivial representation (for example the Killing form) provides the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ with a Lorentzian inner product invariant under the adjoint representation. Thus $\mathfrak{sl}(2, \mathbb{R})$ is isometric...
to \( \mathbb{R}^{2,1} \); we give an explicit orthogonal basis. In this way the adjoint representation 
\[ \text{Ad} : \text{SL}(2, \mathbb{R}) \rightarrow \text{Isom}(\mathfrak{sl}(2, \mathbb{R})) \]
defines a local isomorphism \( \rho : \text{SL}(2, \mathbb{R}) \rightarrow \text{SO}(2, 1) \) of Lie groups.

The local isomorphism \( \rho : \text{SL}(2, \mathbb{R}) \rightarrow \text{O}(2, 1) \) is not injective — its kernel consists of the center \( \{ \pm I \} \) of \( \text{SL}(2, \mathbb{R}) \). Nor is \( \rho \) surjective — its image is the identity component \( \text{SO}^0(2, 1) \) of \( \text{O}(2, 1) \). Neither issue is problematic here, since purely hyperbolic discrete subgroups of \( \text{SO}(2, 1) \) lift to subgroups of \( \text{SL}(2, \mathbb{R}) \) (Abikoff [1], Culler [6], Kra [17]). Let \( G \) be a purely hyperbolic subgroup of \( \text{SO}(2, 1) \), with inclusion \( i : G \hookrightarrow \text{SL}(2, \mathbb{R}) \). Then there exists a representation \( \tilde{i} : G \rightarrow \text{SL}(2, \mathbb{R}) \) such that \( i = \rho \circ \tilde{i} \). Furthermore composition with the local isomorphism \( \rho \) induces a covering space
\[ \text{Hom}(G, \text{SL}(2, \mathbb{R})) \rightarrow \text{Hom}(G, \text{Isom}^0(\mathbb{R}^{2,1})). \]

Thus smooth paths in \( \text{Hom}(G, \text{Isom}^0(\mathbb{R}^{2,1})) \) lift to \( \text{Hom}(G, \text{SL}(2, \mathbb{R})) \). Henceforth we suppress \( \tilde{i} \) (identifying \( G \) with its image \( i(G) \) in \( \text{SL}(2, \mathbb{R}) \)) and consider paths in \( \text{Hom}(G, \text{SL}(2, \mathbb{R})) \).

5. \( \mathfrak{sl}(2, \mathbb{R}) \) and \( \mathbb{R}^{2,1} \)

For the calculations later, we now give a detailed description of the local isomorphism \( \rho \) derived from the adjoint representation.

For convenience, consider the Lie algebra \( \mathfrak{sl}(2, \mathbb{R}) \) with inner product
\[ (3) \quad B(X, Y) := \frac{1}{2} \text{tr}(XY). \]

The basis
\[ e_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \]
is orthogonal with respect to \( B \) and satisfies
\[ B(e_1, e_1) = B(e_2, e_2) = 1, \quad B(e_3, e_3) = -1. \]

This provides an isometry of Lorentzian vector spaces
\[ \psi : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathbb{R}^{2,1} \]
\[ \begin{pmatrix} v_1 & v_2 \\ v_3 & -v_1 \end{pmatrix} \mapsto \begin{pmatrix} v_1 \\ (v_2 + v_3)/2 \\ (-v_2 + v_3)/2 \end{pmatrix}. \]

With respect to this isometry the adjoint representation defines a local isomorphism \( \rho : \text{SL}(2, \mathbb{R}) \rightarrow \text{O}(2, 1) \) satisfying:
\[ \psi(\text{Ad}(g)v) = \rho(g)\psi(v) \]
whenever \( g \in \text{SL}(2, \mathbb{R}) \) and \( v \in \mathfrak{sl}(2, \mathbb{R}) \). (In other words, \( \psi : \mathfrak{sl}(2, \mathbb{R})\text{Ad} \rightarrow \mathbb{R}^{2,1} \) is \( \rho \)-equivariant.) Explicitly,
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 1 + 2bc & -ac + bd \\ -ab + cd & (a^2 - b^2 - c^2 + d^2)/2 \\ ab + cd & (a^2 + b^2 - c^2 + d^2)/2 \end{pmatrix} \begin{pmatrix} ac + bd \\ (a^2 - b^2 + c^2 + d^2)/2 \\ (a^2 + b^2 + c^2 + d^2)/2 \end{pmatrix}
\]
(where $ad - bc = 1$). Differentiation at $I \in \text{SL}(2, \mathbb{R})$ (that is, at $a = d = 1$, $b = c = 0$) gives the Lie algebra isomorphism

$$\mathfrak{sl}(2, \mathbb{R}) \longrightarrow \mathfrak{o}(2, 1)$$

$$\begin{bmatrix} v_1 & v_2 \\ v_3 & -v_1 \end{bmatrix} \mapsto \begin{bmatrix} 0 & v_3 - v_2 & v_2 + v_3 \\ v_2 - v_3 & 0 & 2v_1 \\ v_2 + v_3 & -2v_1 & 0 \end{bmatrix}.$$ 

An element $g \in \text{SL}(2, \mathbb{R})$ is hyperbolic if it has two real distinct eigenvalues, which are necessarily reciprocal. If $g$ has eigenvalues $\mu, \mu^{-1}$ with $|\mu| < 1$, then $\rho(g)$ has eigenvalues $\lambda = \mu^2, 1, \mu^{-2}$. In particular $g \in \text{SL}(2, \mathbb{R})$ is hyperbolic if and only if $\rho(g)$ is hyperbolic. There exists $f \in \text{SL}(2, \mathbb{R})$ such that

$$fgf^{-1} = g_0$$

where

$$g_0 = \pm \begin{bmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{bmatrix}$$

and

$$0 < \mu < 1 < \mu^{-1}.$$ 

The eigenvectors of $g_0 = \rho(g_0)$ are:

$$x^-(g_0) = \psi \left( \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$x^+(g_0) = \psi \left( \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$x^0(g_0) = \psi \left( \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$ 

The eigenvectors for $g$ are the images of the eigenvectors of $g_0$ under $f$.

Now we derive a formula for $\alpha(g)$ for an affine deformation $\phi$ which is of the form $h = (\rho(g), \psi(v)(g))$ where $g \in G \subset \text{SL}(2, \mathbb{R})$ and $v \in \mathfrak{sl}(2, \mathbb{R})$. Suppose that $g \in \text{SL}(2, \mathbb{R})$ is hyperbolic. We use the embedding $\text{SL}(2, \mathbb{R}) \hookrightarrow \mathfrak{gl}(2, \mathbb{R})$. Orthogonal projection

$$\mathfrak{gl}(2, \mathbb{R}) \longrightarrow \mathfrak{sl}(2, \mathbb{R})$$

$$g \mapsto g - \frac{\text{tr}(g)}{2}.$$ 

maps $g_0$ to a diagonal matrix of trace zero. Dividing $\Pi(g_0)$ by

$$\text{sgn}(|\text{tr}(g)|) \sqrt{\det(\Pi(g_0))}$$

gives the diagonal matrix corresponding to $x^0(g_0) \in \mathbb{R}^{2,1}$ (where $\text{sgn}(x)$ denotes the sign of a nonzero real number $x$). Since $\text{tr}(g_0) = \pm(\mu + \mu^{-1})$,

$$\det(\Pi(g_0)) = -(\mu - \mu^{-1})^2 = -\left( \text{tr}(g_0)^2 - 4 \right)/4.$$
so

\[
\text{sgn}(\text{tr}(g_0)) \Pi(g_0) / \sqrt{-\det(\Pi(g_0))} = \text{sgn}(\text{tr}(g_0)) \left( g_0 - \frac{\text{tr}(g_0)}{2} \mathbb{I} \right) / \left( \frac{\sqrt{\text{tr}(g_0)^2 - 4}}{2} \right)
\]

\[
= \left[ -1 \ 0 \\
0 \ 1 \right]
\]

corresponds to \( x^0(g)\). Conjugation by \( f \) gives the general formula

\[
(4) \quad \psi : \text{sgn}(\text{tr}(g)) \left( g - \frac{\text{tr}(g)}{2} \mathbb{I} \right) / \left( \frac{\sqrt{\text{tr}(g)^2 - 4}}{2} \right) \mapsto x^0(g)
\]

From (4) follows a formula for \( \alpha(g) \) in terms of traces. Suppose that \( G \subset \text{SL}(2, \mathbb{R}) \) is purely hyperbolic and \( u \in Z^1(G, \mathfrak{sl}(2, \mathbb{R})) \cong Z^1(G, \mathbb{R}^{2,1}) \). Taking the trace of the product of (4) with \( u(g) \), and applying (2) and (3) yields:

\[
(5) \quad \alpha(g) = \text{sgn}(\text{tr}(g)) \frac{\text{tr}(u(g)g)}{\sqrt{\text{tr}(g)^2 - 4}}
\]

6. Trace and displacement length

Let Hyp denote the subset of \( \text{SL}(2, \mathbb{R}) \) consisting of hyperbolic elements. The image of the trace function \( \text{tr} : \text{Hyp} \rightarrow \mathbb{R} \) consists of the disjoint two intervals \((-\infty, -2)\) and \((2, \infty)\). Furthermore hyperbolic elements \( g \in \text{Hyp} \) are determined up to conjugacy by their trace. In terms of hyperbolic geometry, \( \text{tr}(g) \) relates to the displacement length \( \ell(g) \), that is, the minimum distance \( g \) moves a point \( x \in \mathbb{H}^2_\mathbb{R} \). This minimum is realized when \( x \) lies in the \( g \)-invariant geodesic, which is necessarily unique. Equivalently \( \ell(g) \) is the length of the shortest homotopically nontrivial closed curve in the quotient \( \mathbb{H}^2_\mathbb{R} / \langle g \rangle \). Such a shortest curve is necessarily a simple closed geodesic. Let \( \tilde{g} \in \text{SL}(2, \mathbb{R}) \) be a lift of \( g \in \text{Isom}(\mathbb{R}^{2,1}) \) to \( \text{SL}(2, \mathbb{R}) \), that is, \( g = \rho(\tilde{g}) \). Displacement length of \( g \) relates to \( \text{tr}(\tilde{g}) \) and the eigenvalue \( 0 < \mu < 1 \) by:

\[
\ell(g) = -2 \log \mu \\
| \text{tr}(\tilde{g}) | = 2 \cosh(\ell(g)/2)
\]

(the sign of \( \text{tr}(\tilde{g}) \) is ambiguous since \( \ker(\rho) = \{ \pm \mathbb{I} \} \)). Since

\[
(6) \quad \frac{d| \text{tr} |}{d\ell} = \sinh(\ell/2) > 0
\]

trace depends monotonically on displacement length.

Associated to a cocycle \( u \in Z^1(G, \mathbb{R}^{2,1}) \) are real analytic paths \( \tilde{i}_t \) in \( \text{Hom}(G, \text{SL}(2, \mathbb{R})) \) of the form

\[
\tilde{i}_t(g) = g \exp \left( tu(g) + O(t^2) \right)
\]

where \( t \) is defined in an open interval \( I_g \) containing zero. (In general \( I_g \) may depend on \( g \).) We say that the cocycle \( u \) is tangent to the path \( \tilde{i}_t \).
Given a path $\tilde{\iota}_t \in \text{Hom}(G, \text{SL}(2, \mathbb{R}))$ where $\tilde{\iota}_t(G) \subset \text{Hyp}$, consider the two functions

$$\tau_g : I_g \longrightarrow \mathbb{R},$$

$$t \longmapsto |\text{tr}(\tilde{\iota}_t(g))|$$

and

$$L_g : I_g \longrightarrow \mathbb{R},$$

$$t \longmapsto \ell(\tilde{\iota}_t(g)).$$

When $\tilde{\iota}_t$ corresponds to a path $\mu(t)$ in $\mathcal{T}(M)$, then $L_g = \ell_g \circ \mu$ where $\ell_g : \mathcal{T}(M) \rightarrow \mathbb{R}$ is the geodesic length function associated to $g$.

**Lemma 2.** Let $\phi$ be an affine deformation of $G$ corresponding to the cocycle $u \in Z^1(G, \mathbb{R}^{2,1})$ and let $g \in G$. Suppose that $\mu(t)$ is a path in $\mathcal{T}(M)$ tangent to $u$. Then

$$\alpha_{\phi}(g) = L'_g(0).$$

Furthermore $\alpha_{\phi}(g)$ and $\tau'_g(0)$ have the same sign.

**Proof.** Let $\tilde{\iota}_t : G \longrightarrow \text{SL}(2, \mathbb{R})$ be a smooth path of representations starting at the inclusion $\iota$ corresponding to $\mu(t)$.

$$\tau'_g(0) = \left. \frac{d}{dt} \right|_{t=0} |\text{tr}(\tilde{\iota}_t(g))|$$

$$= \pm \left. \frac{d}{dt} \right|_{t=0} \text{tr}(g(\exp(tu(g) + O(t^2))))$$

$$= \pm \left. \frac{d}{dt} \right|_{t=0} \text{tr}(g(1 + tu(g) + O(t^2)))$$

$$= \pm \text{tr}(gu(g))$$

where the sign equals $\text{sgn}(|\text{tr}(\tilde{\iota}_t(g))|) = \text{sgn}(|\text{tr}(\tilde{\iota}_0(g))|)$. Applying (8) to the last expression gives

$$\tau'_g(0) = \sqrt{\text{tr}(g)^2 - 4} \alpha(g).$$

Thus $\tau'_g(0)$ has the same sign as $\alpha(g)$ as claimed.

To prove (7), apply (8) and the chain rule to obtain:

$$
\tau'_g(0) = \sinh \left( \frac{L_g(0)}{2} \right) L'_g(0).
$$

Since

$$\sinh \left( \frac{L_g(0)}{2} \right) = \frac{\sqrt{\text{tr}(g)^2 - 4}}{2},$$

(8) follows from (8) and (9). Thus a cocycle is positive (respectively negative) in the sense of Theorem 1 if and only if the corresponding deformation in $\mathcal{T}(M)$ increases (respectively decreases) lengths of closed curves, to first order.
7. Reduction to Teichmüller theory

Suppose that $G \subset \text{SL}(2, \mathbb{R})$ and $\phi : G \to \text{Isom}(\mathbb{R}^{2,1})$ is a proper affine deformation. By Theorem 1, the corresponding cocycle $u \in Z^1(G, \mathbb{R}^{2,1})$ is either positive or negative; by replacing $u$ by $-u$ if necessary, we assume that $u$ is positive.

By Fried-Goldman [11], $G$ is necessarily discrete and is isomorphic to its image in the group of affine isometries. Suppose that $G$ is cocompact. By passing to a subgroup of finite index, we may assume that $G$ is torsion free. Then $G$ acts freely on the real hyperbolic plane $H^2_0$ and since $G$ is discrete and cocompact, $H^2_0/G$ is a closed hyperbolic surface $M$. Furthermore $G$ is isomorphic to the fundamental group $\pi_1(M)$. The representation $\tilde{\iota}$ corresponds to a point $O$ in the Teichmüller space $\mathcal{T}(M)$ and the cohomology class $[u] \in H^1(G, \mathbb{R}^{2,1})$ corresponds to a tangent vector $v$ to $\mathcal{T}(M)$ at $O$.

**Lemma 3.** There exists a path $\mu(t)$ in $\mathcal{T}(M)$, defined for all $0 \leq t < \infty$ starting at $O \in \mathcal{T}(M)$ with tangent vector $v \in T_O \mathcal{T}(M)$:

\begin{align}
\mu(0) &= O \\
\mu'(0) &= v
\end{align}

such that, for each $g \in G$, the geodesic length function $\ell_g$ is convex along $\mu(t)$.

Assuming Lemma 3 and that $u$ is positive, we obtain a contradiction. Since $\alpha(g) > 0$, the directional derivative

$$\mu'(0) \ell_g = v \ell_g = L'_g(0) > 0$$

by Lemma 4. Convexity implies that $\mu'(t) \ell_g$ cannot decrease as $t \to +\infty$. Thus

$$(\ell_g \circ \mu)'(t) = \mu'(t) \ell_g \geq \mu'(0) \ell_g = \alpha(g) > 0$$

for all $t \geq 0$. In particular $\ell_g \circ \mu$ is monotone. Furthermore

$$(11) \quad \ell_g(\mu(t)) \to +\infty \text{ as } t \to +\infty,$$

that is, each closed geodesic on the hyperbolic surface $\mu_t$ lengthens as $t \to +\infty$.

Such a path $\mu$ cannot exist for closed hyperbolic surfaces. Let $N > 0$. Then for only finitely many conjugacy classes $F = \{[g_1], \ldots, [g_m]\}$ in $G \cong \pi_1(M)$, the corresponding closed geodesics in $M$ have length $< N$. (Here $[g]$ denotes the conjugacy class of $g \in G$.) For any $g \in G$ with $[g] \notin F$, the length function $L_g(t) > L_g(0) \geq N$. Now consider $[g_i] \in F$. Let

$$\alpha_0 = \min_{1 \leq i \leq m} \alpha(g_i) > 0.$$ 

Convexity, together with (9) implies that

$$L_{g_i}(t) \geq L_{g_i}(0) + t\alpha(g_i) \geq t\alpha_0.$$ 

Hence, for $t > N/\alpha_0$,

$$L_{g_i}(t) = \ell_{g_i}(\mu_t) > N$$

for all $g \in G - \{\mathbb{I}\}$.

However, for any closed hyperbolic surface $M$ there exists a simple closed geodesic of length at most $2 \log(2 - 2\chi(M))$ (Lemma 5.2.1 of Buser [2]). Taking $N > 2 \log(2 - 2\chi(M))$, we obtain the desired contradiction. \qed
Proof of Lemma 3. Here are two constructions for \( \mu \), the first based on the Riemannian geometry of \( T(M) \) with the Weil-Petersson metric and the second based on Thurston’s earthquake flows.

Let \( \mu(t) \) be the Weil-Petersson geodesic satisfying (10). By Corollary 4.7 of Wolpert [30], the geodesic length function \( \ell_g \) is strictly convex along \( \mu(t) \) and the directional derivative \( \nu \ell_g > 0 \), for any \( g \in G - \{1\} \). Therefore \( \ell_g \circ \mu(t) \) is monotonically increasing for \( t > 0 \).

However, in general the Weil-Petersson metric is geodesically incomplete (Chu [5], Wolpert [31]), so that \( \mu(t) \) is only defined for \( t_1 < t < t_2 \) where \( t_1 < 0 < t_2 \). We show this is impossible under our assumptions on \( \mu'(0) = \nu \).

By Mumford’s compactness theorem (Mumford [21], Harvey [14], 2.5.1 or Buser [2], 6.6.5), the subspace of moduli space consisting of hyperbolic surfaces whose injectivity radius is larger than any positive constant is compact. An incomplete geodesic on a Riemannian manifold must leave every compact set. Therefore, if the Weil-Petersson geodesic \( \mu(t) \) cannot be extended to \( t_2 < \infty \), then
\[
\lim_{t \to t_2} \inf_{g \in G - \{1\}} \ell_g(\mu(t)) = 0,
\]
contradicting monotonicity of \( \ell_g \).

Hence \( \mu(t) \) is defined for all \( t < \infty \). As above, convexity implies (11).

Alternatively, take \( \mu \) to be the earthquake path introduced by Thurston (see Kerckhoff [15, 16] and Thurston [24]). For the given tangent vector \( \nu \), there exists a unique measured geodesic lamination \( \lambda \) such that the corresponding earthquake path \( \mu(t) = E_\lambda(t) \) satisfies (10) (Kerckhoff [16], Proposition 2.6). By Kerckhoff [15] (see also Wolpert [29]), each length function \( \ell_g \) is convex along the earthquake path \( E_\lambda \), implying (11). Indeed, \( \ell_g \) is strictly convex along \( \mu \) since the lamination \( \lambda \) fills up \( M \) — that is, every nonperipheral simple closed curve \( \sigma \) intersects \( \lambda \). For otherwise \( \ell_\sigma \) would be constant along \( \mu \), contradicting
\[
\frac{d}{dt} \bigg|_{t=0} \ell_\sigma \circ \mu(t) > 0.
\]

Remark. Another proof, closer in spirit to the proof in [20], involves the density of simple closed curves in the projective measured lamination space. Let \( S \) denote the set of isotopy classes of simple closed curves on \( M \) and let \( P \mathcal{L}(M) \) denote Thurston’s space of projective equivalence classes of measured geodesic laminations on \( M \). Since
\[
\mathcal{ML}(M) \longrightarrow T_0 T(M)
\]
\[
\lambda \mapsto E'_\lambda(0)
\]
is a homeomorphism (Proposition 2.6 of [16]), there exist \( \lambda \in \mathcal{ML}(M) \) satisfying \( E'_\lambda(0) = \nu \neq 0 \). Theorem 5.1 of [25] implies
\[
P \mathcal{L}(M) \longrightarrow T_0 T(M)
\]
\[
[\lambda] \mapsto d \log \ell_\lambda
\]
is an embedding onto a convex sphere in \( T_0 T(M) \) (where \( \ell_\lambda(N) \) denotes the length of the lamination \( \lambda \) as measured in \( N \)). Since \( S \) is dense in \( P \mathcal{L}(M) \), there exist
\[ \gamma_1, \gamma_2 \in S \text{ such that} \]
\[ (d \log \ell_{\gamma_1})(\lambda) > 0 \]
\[ (d \log \ell_{\gamma_2})(\lambda) < 0. \]

Let \( g_1, g_2 \in \pi_1(M) \) correspond to \( \gamma_1, \gamma_2 \) respectively. Then
\[ \nu L_{g_1} > 0, \nu L_{g_2} < 0, \]
contradicting Theorem 1 and Lemma 2.

Remark. Mess’s original proof uses Lorentzian geometry, and in particular
the theory of domains of dependence in constant curvature Lorentzian space forms
developed in [20] and Scannell [23]. As part of his general theory, Mess shows that
any affine deformation sufficiently near the holonomy of a complete flat Lorentz 3-
manifold is the holonomy of a complete flat Lorentz 3-manifold, that is, the nearby
action is also proper and free. The cocycle \( u \) corresponds to the velocity vector to an
earthquake path \( E_\lambda \) along a measured geodesic lamination \( \lambda \), and \( \lambda \) is approximated
by a finite measured geodesic lamination, that is, a disjoint union of simple closed
geosics. However for a finite lamination, the corresponding group action is not
free (elements of \( G \) corresponding to curves disjoint from \( \lambda \) have fixed points), a
contradiction.

References
1. Abikoff, W., Appel, K. and Schupp, P., Lifting surface groups to \( SL(2, \mathbb{C}) \), in “Kleinian groups
and related topics (Oaxtepec, 1981)”, Lecture Notes in Math. 971, Springer, Berlin-New York
(1983) 1–5.
2. Buser, P., “Geometry and Spectra of Compact Riemann Surfaces,” Progress in Mathematics
106, Birkhäuser Boston (1992)
3. Charette, V., Drumm, T., Goldman, W. and Morrill, M., Flat Lorentz Manifolds : A Survey,
Proc. A. Besse Round Table on Global Pseudo-Riemannian Geometry, (to appear).
4. Charette,V. and Goldman, W., Affine Schottky groups and crooked tilings, these proceedings
5. Chu, T., The Weil-Petersson metric in moduli space, Chinese J. Math. 4 (1976), 29–51.
6. Culler, M., Lifting representations to covering groups, Adv. in Math. 59 (1986), (1), 64–70.
7. Drumm, T., Examples of nonproper affine actions, Mich. Math. J. 39 (1992), 435–442
8. ______, Linear holonomy of Margulis space-times, J.Diff.Geo. 38 (1993), 679–691
9. ______, Translations and the holonomy of complete affine flat manifolds, Math. Res. Letters
1, 757–764 (1994).
10. Drumm, T. and Goldman, W., On Margulis’s invariant of affine actions, (in preparation)
11. Fried, D. and Goldman, W., Three-dimensional affine crystallographic groups, Adv. Math.
47 (1983), 1–49.
12. Goldman, W., The symplectic nature of fundamental groups of surfaces, Adv. Math. 54
(1984), 200–225.
13. ______, Invariant functions on Lie groups and Hamiltonian flows of surface group represen-
tations, Inv. Math. 85 (1986), 1–40.
14. Harvey, W., Spaces of Discrete Groups, in “Discrete Groups and Automorphic Functions,”
(1977), 295–348 Academic Press, London New York San Francisco.
15. Kerckhoff, S., The Nielsen realization problem, Ann. Math. 117 (1983), 235–265.
16. ______, Earthquakes are analytic, Comm. Math. Helv. 60 (1985), no. 1, 17–30.
17. Kra, I., On lifting Kleinian groups to \( SL_2(\mathbb{C}) \), in “Differential geometry and complex analy-
sis,” Springer, Berlin-New York (1985), 181–193.
18. Margulis, G., Free properly discontinuous groups of affine transformations, Dokl. Akad. Nauk
SSSR 272 (1983), 937–940;
19. ______, Complete affine locally flat manifolds with a free fundamental group, J. Soviet Math.
134 (1987), 129–134
20. Mess, G., Lorentz spacetimes of constant curvature, I.H.E.S. preprint (1990).
21. Mumford, D., A remark on Mahler’s compactness theorem, Proc. A.M.S. 29 (1971), 289–294.
22. Raghunathan, M. S., “Discrete subgroups of Lie groups,” Ergebnisse der Mathematik Und ihrer Grenzgebiete 68 (1972).
23. Scannell, K., Flat conformal structures and the classification of de Sitter manifolds, Comm. Anal. Geom. 7 (1999), (2), 325–345.
24. Thurston, W., Earthquakes in two-dimensional hyperbolic geometry, in “Low-dimensional Topology and Kleinian Groups,” London Math. Soc. Lecture Notes Series 112, Cambridge University Press (1986), 91–112.
25. Minimal stretch maps between hyperbolic surfaces, preprint.
26. Weil, A., Discrete subgroups of Lie groups I, Ann. Math. 72 (1960), 369–384.
27. Discrete subgroups of Lie groups II, Ann. Math. 75 (1962), 578–602.
28. Remarks on the cohomology of groups, Ann. Math. 80 (1964), 149–157.
29. Wolpert, S., On the symplectic geometry of deformations of a hyperbolic surface, Ann. Math. (2) 117 (2), (1983), 207–234.
30. Geodesic length functions and the Nielsen problem, J. Diff. Geo. 25 (1987), 275–296.
31. Non-completeness of the Weil-Petersson metric for Teichmüller space, Pac. J. Math. 61 (1975), 573–577.