Holographic three-point functions:
one step beyond the tradition

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Abstract

Within the program of holographic renormalization, we discuss the computation of three-point correlation functions along RG flows. We illustrate the procedure in two simple cases. In an RG flow to the Coulomb branch of \(\mathcal{N} = 4\) SYM theory we derive a compact and finite expression for the three-point function of lowest CPO’s dual to inert scalars. In the GPPZ flow, that captures some features of \(\mathcal{N} = 1\) SYM theory, we compute the three-point function with insertion of two inert scalars and one active scalar that mixes with the stress tensor. By amputating the external legs at the mass poles we extract the trilinear coupling of the corresponding superglueballs. Finally we outline the procedure for computing three-point functions with insertions of the stress tensor as well as of (broken) R-symmetry currents.

1 Introduction

Holographic renormalization [1, 2] is a powerful tool that produces finite correlation functions and allows explicit checks of (anomalous) Ward identities along RG flows dual to asymptotically AdS domain walls. One of the
main achievement of the program is an efficient and consistent algorithm for computing finite one-point functions in the presence of external sources. By sources we mean the coefficient functions, two per each bulk field component, which are not fixed by the near-boundary analysis [3, 4]. One of those corresponds to an operator deformation of the fixed point theory, while the other corresponds to giving a VEV to an operator [5].

The non-local and non-linear relation between VEV and source which amounts to requiring regularity of the classical solution of the non-polynomial bulk field equations is hard to find in general. Luckily in order to compute two-point correlation functions it is enough to know this relation at the linear level, e.g. by solving the linearized fluctuation equations around a given domain wall solution. Disentangling the mixing between longitudinal/trace modes of the stress tensor and the active scalar(s) proved subtler [1, 6], than initially expected [7, 8]. Similar difficulties appearing in the mixing of the broken symmetry currents with Goldstone bosons were disposed of in [2, 9, 10]. Transverse modes are easier to analyze since their fluctuations can be expressed in terms of an auxiliary massless scalar [11, 12]. In particular, after getting rid of non-canonical kinetic terms, (transverse) vector fields display a universal effective mass $M^2 = -2A''$ [11, 13, 14] that is manifestly positive thanks to the convexity of the scale factor $A(r)$ which is at the heart of the holographic c-theorem [15, 16]. Interesting mass spectra emerge from momentum analysis of holographic two-point functions but so far tradition has prevented the study of even the simplest interactions.

There is little hope of computing higher-point correlation functions except possibly for three-point functions$^3$. Higher point functions require the knowledge of the bulk-to-bulk propagator that appears in exchange diagrams. For three-point functions, the relevant Witten diagrams at tree-level only involve bulk-to-boundary propagators that are known in many cases, e.g. solving the linearized fluctuation equations with prescribed boundary conditions. Bulk-to-bulk propagators would only appear in loop diagrams that are suppressed by the 5-D Newton constant i.e. by inverse powers of $N$ in the spirit of the AdS/CFT correspondence [17, 18, 19]$^4$. Thus restricting to the planar limit captured, at strong 't Hooft coupling, by the supergravity approximation one

$^3$Following the procedure of holographic renormalization, three-point functions at the conformal point dual to pure AdS have been computed by D. Freedman, U. Gürsoy and K. Skenderis as reported in [23].

$^4$For recent reviews see e.g. [20, 21, 22, 23] and references therein.
can go a step beyond the tradition and address the problem of computing three-point functions. By ‘computing’ we actually mean writing down closed-form expressions that allow one to extract interesting information about the interactions in particular regimes. The only ingredient that is missing from previous analyses is the exact form of some bulk cubic couplings along the flows and of some additional counterterms. In this note we fill this gap and compute simple three-point functions of both active and inert scalars in the two most studied holographic RG flows: the GPPZ flow [24] and the CB flow [29, 12].

In order to fix the notation we briefly review the procedure of holographic renormalization in Sect. 2. We then describe the general strategy for computing three-point functions in Sect. 3 and collect some basic formulae for the two flows in Sect. 4. The only real novelty that deserves a special mention is the identification of additional logarithmically divergent counterterms in the GPPZ flow when the bulk field dual to the gluino bilinear comes into play.

In Sect. 5 we specialize our analysis to the CB flow. With little effort we write down a compact expression for the three-point function of inert scalars belonging to the \((3, 3)\) representation of the \(SO(4)\) subgroup of \(SO(6)\) preserved along the flow. We check that our result is finite for generic momenta and try to convince the reader that it displays the correct UV behavior for large momenta. For comparison and in order to give a flavor of the complexity of the problem, we compute the Fourier transform of the conformal three-point functions of scalar primary operators which is not easy to find in the existing literature [25]. We thus report the relevant formula for \(\Delta = 2\) in an appendix.

In Sect. 6 we analyze the three-point function with one insertion of the active scalar that interpolates for the trace of the stress tensor and two insertions of the inert scalars, which is a “caricature” of the gluino bilinear in the GPPZ flow. As expected the resolution of the mixing with the stress tensor at the required order in the fluctuations turns out to be quite laborious but straightforward. The naive result is logarithmically divergent in the contact terms that are cancelled by the additional counterterms previously identified. By amputating the external legs we extract the finite and rather explicit trilinear coupling of the relevant superglueball states.

Ideally one would like to compute three-point correlation functions with insertions of the stress tensor and/or of (broken) R-symmetry currents. We
plan to return to this problem soon. In the concluding Section we outline the procedure for computing such three-point functions and speculate how similar computations, following or extending when needed the approach of holographic renormalization along the lines of [14], should be feasible for theories which are dual to full-fledged string solutions possibly with logarithmic corrections even in the UV [26, 27, 28].

2 Holographic Renormalization strikes back

In this section we fix our notation. Since there only minor changes with respect to [1, 2, 23], readers familiar with holographic renormalization may skip directly to Section 3.

To be specific we focus on asymptotically AdS domain walls which are dual to RG flows in \( d = 4 \). The set of fields driving the flow together with some ‘spectator’ fields represents a consistent truncation of some gauged supergravity in \( D = 5 \), which in turn is the low-energy effective theory of some superstring compactifications on an Einstein manifold with stabilizing fluxes. As usual, we consider Poincaré invariant solutions of the form [15, 29, 16, 24, 12]

\[
ds^2 = e^{2A(r)} dx \cdot dx + dr^2 \quad \Phi = \Phi(r)
\]

where \( \Phi \) denote one or more canonical scalar fields that drive the flow. The AdS boundary (field theory UV) is at \( r \to +\infty \). The (IR) interior presents a naked singularity unless the flow terminates at another (super)conformal fixed point. Preserving some supersymmetry reduces the problem to solving first order ‘gradient flow’ equations

\[
A'(r) = -\frac{g}{3} W(\Phi) \quad \Phi'(r) = \frac{g}{2} \partial_\Phi W
\]

where \( g = 2/L \) is the bulk gauge coupling and \( L \) is the characteristic AdS scale. The scalar potential reads

\[
V = \frac{g^2}{4} \left[ \frac{1}{2} \left( \frac{\partial W}{\partial \Phi} \right)^2 - \frac{2}{3} W^2 \right]
\]

To extract some physics from the domain wall solution one may not forgo a near-boundary analysis of the non-linear field equations [4]. To this end it is
customary to introduce a different radial variable $z$ such that $dr = -Ldz/z$, i.e. $z = \exp(-r/L)$. The scale factor behaves as $A(r) \approx r/L \approx -\log z$ near the UV boundary at $z = 0$. When $\exp(2A) \approx \exp(+2r/\bar{L})$ as $r \to -\infty$ one has an AdS ‘horizon’ with $\bar{L} < L$ that describes a different fixed point at the endpoint of the flow [29, 30]. Barring string corrections or uplift to $D = 10$ [31, 32], many interesting solutions display a naked singularity at some finite value of the radial variable. Depending on the behavior of the active scalar field(s) near the boundary one has flows along which (super)conformal invariance is either spontaneously broken, by the presence of operator VEV’s, or explicitly broken, by a relevant deformation of the fixed point action.

For scalar fields dual to operators with UV dimension $\Delta$, i.e. with AdS mass $(ML)^2 = \Delta(\Delta - 4)$, the near boundary expansion, that reads

$$\Phi(z, x) = z^{4-\Delta}(\phi(0)(x) + z\phi(1)(x) + ...) + z^\Delta(\tilde{\phi}(0)(x) + z\tilde{\phi}(1)(x) + ... ) \quad (2.4)$$

displays two a priori independent asymptotic behaviors. The iterative solution of the field equations allows one to express the higher order coefficient functions $\phi(n)$ and $\tilde{\phi}(n)$ in terms of $\phi(0)$ and $\tilde{\phi}(0)$. For $\Delta$ an integer, only coefficient functions with ‘even’ indices appear and it is convenient to change radial variable to $\rho = z^2$. Moreover the two dominant behaviors differ by an integer and additional logarithmic terms are needed in order to satisfy the field equations$^5$. Logarithmic behaviors can be associated to conformal anomalies that, being local, can be written in terms of $\phi(0)$, playing the role of an operator source, and a finite number of derivatives thereof. It is not difficult to see that $\tilde{\phi}(0)$ is related to the VEV of the corresponding operator [5, 4].

Naive extension of the AdS/CFT correspondence to holographic RG flows suggests one to compute the on-shell action as a functional of the boundary data and then take functional derivatives in order to compute correlation functions. However the unregulated on-shell action is infinite due to the infinite volume of the bulk. A convenient regularization consists in a radial cutoff $\rho > \epsilon$ that allows one to easily isolate a finite number of terms diverging as $\epsilon \to 0$. For the case of scalars coupled to gravity the on-shell action takes

$^5$In the very special case $\Delta = 2$, even the dominant term contains a logarithm $\Phi(\rho, x) = \rho \log \rho \phi(0)(x) + \rho \tilde{\phi}(0)(x) + ...$. 
the schematic form

\[
S_{\text{reg}}[\phi(0), g(0); \epsilon] = \int_{\rho=\epsilon} d^4 x \sqrt{g(0)} \left[ \frac{a(0)}{\epsilon} + \frac{a(2)}{\epsilon^2 - 1} + \ldots - \log \epsilon a_{(2\nu)} + \mathcal{O}(\epsilon^0) \right]
\]  

(2.5)

where \( \nu = 2 \) and \( a_{(2k)} \) are local functions of the sources, \( \phi(0) \) and \( g(0) \), that appear in the expansions of \( \Phi \) and of the metric

\[
g(0)_{ij} = g(0)_{ij} + \rho g(2)_{ij} + \rho^2 g(4)_{ij} + \rho^2 \log \rho h(4)_{ij} + \ldots
\]  

(2.6)

The counterterm action \( S_{\text{ct}}[\Phi, \gamma; \epsilon] \), chosen in such a way as to cancel the divergent terms in \( S_{\text{reg}}[\phi(0), g(0); \epsilon] \), is to be expressed in terms of the fields \( \Phi(x, \epsilon) \) ‘living’ at the regulating surface \( \rho = \epsilon \) and of the induced metric, \( \gamma_{ij} = g_{ij}(x, \epsilon)/\epsilon \). This is required for covariance and follows from a straightforward but often laborious “inversion” of the expansions (2.6),(2.1). The counterterms are universal in that they make the on-shell action finite for any solution of the bulk field equations with given boundary data. The counterterms are however different for different (consistent) truncations, e.g. for different potentials \( V(\Phi) \). So far we have described a “minimal” scheme in which only the divergences of \( S_{\text{reg}} \) are subtracted. As in standard quantum field theory, one has the freedom to add finite invariant counterterms. These correspond to a change of scheme aimed for instance to restoring some (super)symmetry.

For later purposes, it proves convenient to define a subtracted action at the cutoff

\[
S_{\text{sub}}[\Phi, \gamma; \epsilon] = S_{\text{reg}}[\phi(0), g(0); \epsilon] + S_{\text{ct}}[\Phi, \gamma; \epsilon].
\]  

(2.7)

The subtracted action has a finite limit as \( \epsilon \to 0 \), and the renormalized action is a functional of the sources defined by this limit, i.e.

\[
S_{\text{ren}}[\phi(0), g(0)] = \lim_{\epsilon \to 0} S_{\text{sub}}[\Phi, \gamma; \epsilon]
\]  

(2.8)

The distinction between \( S_{\text{sub}} \) and \( S_{\text{ren}} \) is needed because the variations required to obtain correlation functions are performed before the limit \( \epsilon \to 0 \) is taken. In particular, the expectation value of a scalar operator in the presence of sources, defined by

\[
\langle O_\phi \rangle = \frac{1}{\sqrt{g(0)}} \frac{\delta S_{\text{ren}}}{\delta \phi(0)}
\]  

(2.9)
where $g(0) = \det(g(0)_{ij})$ can be computed by rewriting it in terms of the fields living at the regulating surface\(^6\)

$$\langle O_{\phi} \rangle = \lim_{\epsilon \to 0} \left( \frac{1}{\epsilon^{2\Delta/2}} \frac{1}{\sqrt{\gamma}} \frac{\delta S_{\text{sub}}}{\delta \Phi(x, \epsilon)} \right)$$

(2.10)

where $\gamma = \det(\gamma_{ij})$. In general one can prove that

$$\langle O_{\phi} \rangle = (2\Delta - 4) \tilde{\phi}(0) + \text{local}$$

(2.11)

where the local terms are completely fixed by the choice of $\phi(0)$.

The hard problem is to determine the non-local relation between the operator VEV $\tilde{\phi}(0)$ and the operator source $\phi(0)$ that guarantees regularity of the solution in the deep interior (IR regime) where the supergravity solution may develop a singularity. Traditionally only two-point function have been explicitly computed along holographic RG flows [1, 2, 12, 9, 6, 10, 30]. These only require the non-local relation at linear order

$$\tilde{\phi}(0)(x) = \frac{1}{2\Delta - 4} \int d^{4}x' C_{2}(x - x') \phi(0)(x')$$

(2.12)

or better, in momentum space,

$$\tilde{\phi}(0)(p) = \frac{1}{2\Delta - 4} C_{2}(p) \phi(0)(p)$$

(2.13)

that is established by solving the linearized fluctuation equations with prescribed boundary conditions, i.e. regularity in the interior. Upon differentiation one gets the two-point correlation function

$$\langle O_{\phi}(x) O_{\phi}(x') \rangle = C_{2}(x - x') + \text{contact terms}$$

(2.14)

### 3 Three-point functions: general strategy

Our aim in this section is to sketch the further necessary steps that one should take in order to compute three-point functions. For simplicity we detail the procedure for inert scalar fields with cubic couplings, a case that

\(^6\)For scalars of UV dimension $\Delta = 2$, an additional $\log \epsilon$ is needed in this formula.
finds application in Section 5. Mixing with metric fluctuations or with vector fields is not discussed in this section. We will address this issue in Section 6.

For three-point functions one needs to take into account quadratic terms in the field equations. To this end consider the Euclidean action for a real inert scalar around a domain wall

\[ S = \frac{N^2}{2\pi^2} \int d^4x d\rho \sqrt{G} \left[ \frac{1}{2} Z(\rho)(\partial\Phi)^2 + \frac{1}{2} M^2(\rho)\Phi^2 + \frac{1}{3} T(\rho)\Phi^3 + ... \right] \] (3.15)

The effective mass \( M(\rho) \) and the cubic coupling \( T(\rho) \) may depend on the radial variable \( \rho \) via their functional dependence on the active scalar(s). The kinetic term \( Z(\rho) \) may also depend on \( \rho \). In the cases we explicitly discuss here scalar fields are canonical. In order to keep our formulae as compact as possible we thus assume \( Z = 1 \) henceforth.

The non-linear field equation is of the form

\[ (\nabla_G^2 - M^2)\Phi = T\Phi^2 + ... \] (3.16)

At the classical level the solution can be found by summing trees. Loops are suppressed by powers of \( 1/N^2 \) and we neglect them since they can only be consistently taken into account after embedding the model in string theory. For a given boundary source \( j(x) \) we find

\[ \Phi(x, \rho) = \int d^4x'K(x, \rho; x')j(x') + \]

\[ \int d\rho'd^4x'\sqrt{G}D(x, \rho; x', \rho')T(\rho') \left[ \int d^4x''K(x', \rho'; x'')j(x'') \right]^2 + ... \]

where \( D(x, \rho; x', \rho') \) is the bulk-to-bulk propagator, \( K(x, \rho; x') \) is the bulk-to-boundary propagator and ... denote higher-order terms in \( j(x) \).

The linearized approximation, which is enough to compute two-point functions, yields

\[ \Phi_{(1)}(\rho, p) = K(\rho, p)j(p) \] (3.18)

in momentum space. The near boundary analysis then gives\(^8\)

\[ \phi_{(0|1)}(p) = j(p) \] (3.19)

\(^7\)The overall constant is determined by careful reduction from \( D = 10 \) [19].

\(^8\)The second index denotes the order in \( j \).
and
\[ \tilde{\phi}_{(0|1)}(p) = \frac{1}{2\Delta - 4} C_2(p) j(p) \]  

(3.20)

To quadratic order in \( j \)
\[ \Phi_{(2)}(\rho, p) = \int d\rho' \sqrt{G} D(p, \rho; \rho') T(\rho') \int d^4 q K(q, \rho') j(q) K(p - q, \rho') j(p - q) \]  

(3.21)

Expanding both sides in powers of \( \rho \) yields
\[ \phi_{(0|2)}(p) = 0 \]  

(3.22)

and
\[ \tilde{\phi}_{(0|2)}(p) = \frac{1}{2\Delta - 4} \int d\rho \sqrt{G} K(p, \rho) T(\rho) \int d^4 q K(q, \rho) j(q) K(q - p, \rho) j(p - q) \]  

(3.23)

since near the AdS boundary, where \( \rho \approx 0 \),
\[ D(x, \rho; x', \rho') \approx \frac{\rho^{\Delta/2}}{2\Delta - 4} K(x, x', \rho') \]  

(3.24)

Plugging this back into the on-shell action or better into the one-point function given above, differentiating w.r.t. \( j \) and then setting \( j \) to zero, one finally gets
\[ \langle O_\phi(p_1) O_\phi(p_2) O_\phi(p_3) \rangle = (2\pi)^4 \delta(\sum_i p_i) \int_{\rho > \epsilon} d\rho \sqrt{G(\rho)} T(\rho) \prod_i K(p_i, \rho) \]  

(3.25)

modulo contact terms that are polynomial in the momenta and cancel potential divergences arising when the regulator of the radial integration is removed.

In all cases we consider, the linearized fluctuation equations reduce to hypergeometric equations in a suitable radial variable \( w = w(\rho) \). The solution which is regular in the deep IR interior (which is a naked singularity of the ‘good’ kind [33]) is of the form
\[ \phi_p(w, x) = w^a (1 - w)^b F(a, b; c; w) \exp(ipx) \]  

(3.26)

Analytically continuing the hypergeometric function from \( w = 0 \), that represents the deep IR interior (singularity), to \( w = 1 \), that represents the UV boundary, determines the bulk to boundary propagators. The reader may find some useful formulae in appendix B.
4 Our favorite holographic RG flows

For completeness we gather in this section some relevant formulae about two particularly simple yet interesting asymptotically AdS domain wall solutions: the CB flow [29, 12] and the GPPZ flow [24]. The former is dual to an RG flow from $\mathcal{N} = 4$ SYM at its maximally superconformal point to a locus in the Coulomb branch where superconformal symmetry is broken spontaneously. The latter is dual to an RG flow to a ’confining’ $\mathcal{N} = 1$ SYM theory where superconformal invariance is broken explicitly by turning on relevant (mass) operators for the three chiral multiplets.

4.1 Coulomb branch flow

The first case we consider is the CB flow with $n = 2$, in the notation of [29]. The supersymmetric domain wall solution describes a continuous distribution of D3-branes on a disk. The relevant superpotential is given by

$$W(\Phi) = -e^{-2\Phi/\sqrt{6}} - \frac{1}{2}e^{4\Phi/\sqrt{6}}. \quad (4.27)$$

Using (2.3), one can easily compute the potential. Near $\Phi = 0$ it can be expanded as

$$V(\Phi) = -\frac{3}{L^2} - \frac{2}{L^2} \Phi^2 + \frac{4}{3\sqrt{6}L^2} \Phi^3 + \mathcal{O}(\Phi^4) \quad (4.28)$$

which exhibits a tachyonic mass $M^2 L^2 = -4$, as appropriate for a field dual to a chiral primary operator (CPO) of UV dimension $\Delta = 2$.

It is very convenient to express the domain-wall solution in terms of a new radial variable $v$. The solution and the relation between $v$ and $r$ are

$$\Phi = \frac{1}{\sqrt{6}} \log v, \quad e^{2A} = \frac{\ell^2}{L^2} \frac{v^{2/3}}{1 - v}, \quad \frac{dv}{dr} = \frac{2}{L} v^{2/3} (1 - v) \quad (4.29)$$

The boundary is at $v = 1$ and the solution has a curvature singularity at $v = 0$. The parameter $\ell$ is the radius of the disk of branes [29, 30] in the 10-dimensional “lift” of this solution and sets the scale of a mass gap in the spectrum of excitations to $m_o = \ell/L^2$. 

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The scalar field vanishes at the boundary at the rate
\[ \Phi \approx -\frac{1}{\sqrt{6}}(1 - v) = -\frac{1}{\sqrt{6}} \exp(-2r/L) \] (4.30)
that corresponds to the dual operator of UV dimension \( \Delta = 2 \) taking a VEV. The endpoint of the flow describes \( \mathcal{N} = 4 \) SYM theory at a locus in the Coulomb branch where the \( SO(6) \) R-symmetry is broken to \( SO(2) \times SO(4) \). For any finite \( N \), the \( SO(2) \) symmetry is actually broken \([2, 9]\).

The scalar that gets a VEV is the neutral singlet component of the \( \text{CPO } Q_{ij}^{20'} = \text{Tr}(X^i X^j)|_{20'} \) that decomposes according to \( 20' \rightarrow (1, 1)_0 \oplus (1, 1)_{+2} \oplus (1, 1)_{-2} \oplus (2, 2)_{+1} \oplus (2, 2)_{-1} \oplus (3, 3)_0 \) under \( SO(4) \times SO(2) \).

### 4.2 GPPZ flow

The GPPZ flow \([24]\) corresponds to adding an operator of dimension \( \Delta = 3 \) to the Lagrangian, namely the top component of the superpotential \( \Delta \mathcal{W} = \sum_{I} Z_{I}^{2} \), that gives a common mass to the three \( \mathcal{N} = 1 \) chiral multiplets \( Z_{I} \) appearing in the decomposition of the \( \mathcal{N} = 4 \) vector multiplet. The solution was proposed as the holographic dual of pure \( \mathcal{N} = 1 \) SYM theory \([24]\). Although it does not capture all of the expected properties of the field theory, it is particularly simple and still displays some interesting features including a discrete spectrum of superglueballs.

The active scalar is a singlet under an \( SO(3) \) subgroup of \( SO(6) \). A consistent truncation to the \( SO(3) \) singlets yields \( \mathcal{N} = 2 \) gauged supergravity coupled to two hypermultiplets describing a \( G_{2(2)}/SO(4) \) coset \([31, 11]\). After lengthy calculation one gets the 5d superpotential that reads
\[ W(\Phi, \Sigma) = -\frac{3}{4} \left[ \cosh \left( \frac{2\Phi}{\sqrt{3}} \right) + \cosh \left( 2\Sigma \right) \right] = -\frac{3}{2} - \frac{1}{2}(\Phi^2 + \Sigma^2) + \ldots \] (4.31)

Near \( \Phi = \Sigma = 0 \), the potential admits the expansion
\[ V(\Phi, \Sigma) = -\frac{3}{L^2} - \frac{3}{2L^2}(\Phi^2 + \Sigma^2) - \frac{1}{3L^2}(\Phi^4 - 3\Sigma^4 + 6\Phi^2\Sigma^2) + \ldots \] (4.32)

The mass of both \( \Phi \) and \( \Sigma \) is \( M^2 L^2 = -3 \), indicating that the dual scalar operators have UV dimension \( \Delta = 3 \). For simplicity we only consider the
domain-wall solution with $\Sigma = 0$, and

$$\Phi = \frac{\sqrt{3}}{2} \log \frac{1 + \sqrt{1 - u}}{1 - \sqrt{1 - u}} \quad e^{2A} = \frac{u}{1 - u},$$

(4.33)

where $u = 1 - \exp(-2r/L)$. The boundary is at $u = 1$ and the solution has a

naked singularity of ‘good’ type at $u = 0$. Since $\Phi \approx \sqrt{3} \exp(-r/L)$ near the

boundary, we are dealing with an operator deformation rather than a VEV.

Along the flow

$$W(u) = -\frac{3}{2u} \quad W_\phi(u) = -\sqrt{3} \frac{\sqrt{1 - u}}{u}$$

(4.34)

$$W_{\phi\phi}(u) = -\frac{2 - u}{u} \quad W_{\phi\phi\phi}(u) = -4 \frac{\sqrt{1 - u}}{u}$$

(4.35)

For our later purposes, we need the effective mass of the field $\Sigma$ along the

GPPZ flow

$$M_\sigma^2(u) = V_{\sigma\sigma} = -\frac{3}{L^2} \left[ 2 \cosh \left( \frac{2\Phi}{\sqrt{3}} \right) - 1 \right] = -\frac{3}{L^2} \frac{4 - 3u}{u}$$

(4.36)

and its cubic coupling to the active scalar $\Phi$

$$\partial_\phi M_\sigma^2(u) = V_{\phi\sigma\sigma} = -\frac{4\sqrt{3}}{L^2} \sinh \left( \frac{2\Phi}{\sqrt{3}} \right) = -\frac{8\sqrt{3}}{L^2} \frac{\sqrt{1 - u}}{u}$$

(4.37)

also along the flow.

5 Three-point functions in CB flow

In order to simplify our lives we only consider inert scalars of UV dimension

$\Delta = 2$, i.e. those among the 20 tachyonic scalars which are not singlet under

the $SO(4) \times SO(2)$ preserved in the CB flow. In particular we concentrate

our attention on the $(3, 3)_0$ components $Q^{l_1l_2}$ and aim to compute

$$G_{CB}^{l_1l_2j_1j_2k_1k_2}(x_1, x_2, x_3) = \langle Q^{l_1l_2}(x_1)Q^{j_1j_2}(x_2)Q^{k_1k_2}(x_3) \rangle$$

(5.38)

More general solutions with a non-trivial profile for the bulk field $\Sigma$ dual to the gaugino

bilinear have been considered by GPPZ.
The rationale behind this choice is that there is a unique $SO(4) \times SO(2)$ invariant trilinear coupling among the $Q$’s. It is proportional to $\varepsilon_{iLjLkL}\varepsilon_{iRjRkR}$. In order to avoid additional derivative couplings, it is necessary to identify three components of $Q^{iLjR}$ with canonical kinetic term around the CB domain wall solution. It is well known that the icosaplet parametrizes a coset $SL(6)/SO(6)$. Taking as coset representative a symmetric matrix $S$ with $\det(S) = 1$, the scalar potential reads

$$V = -\frac{g^2}{32}[(tr(S))^2 - 2tr(S^2)] \quad (5.39)$$

$S$ can be diagonalized by an orthogonal $SO(6)$ transformation

$$S = R(\theta)\tilde{S}(\beta)R^T(\theta) \quad (5.40)$$

where $\tilde{S}(\beta) = diag(\exp(2\beta_1), ..., \exp(2\beta_6))$ with $\sum_i \beta_i = 0$. The six $\beta$’s should then be expressed in terms of five independent fields. A convenient choice that leads to canonically normalized scalars is

$$\beta_1 = \frac{\sqrt{2}}{\sqrt{3}}[-\Phi + \sqrt{3}(\lambda + \sigma + \omega)]$$
$$\beta_2 = \frac{\sqrt{2}}{\sqrt{3}}[-\Phi + \sqrt{3}(\lambda - \sigma - \omega)]$$
$$\beta_3 = \frac{\sqrt{2}}{\sqrt{3}}[-\Phi + \sqrt{3}(-\lambda + \sigma - \omega)]$$
$$\beta_4 = \frac{\sqrt{2}}{\sqrt{3}}[-\Phi + \sqrt{3}(-\lambda - \sigma + \omega)]$$
$$\beta_5 = \frac{\sqrt{2}}{\sqrt{3}}[2\Phi + \sqrt{6}\nu]$$
$$\beta_6 = \frac{\sqrt{2}}{\sqrt{3}}[2\Phi - \sqrt{6}\nu] \quad (5.41)$$

Taking $\Phi$ as active scalar, one can convince oneself that $\nu$ transforms under the lower $SO(2)$ subgroup while $\lambda, \sigma, \omega$ only transform under the upper $SO(4)$. The former is part of the charged singlet $(1, 1)_\pm$. The latter three are part of the neutral ennaplet $Q^{iLjR} \in (3, 3)_0$. 

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We can proceed by switching off all the other inert scalars, setting $\Phi$ to its background value and keeping only terms up to cubic order in $\lambda, \sigma, \omega$. We find

$$V(\Phi, \lambda, \sigma, \omega, \ldots) = V(\Phi) + \frac{1}{2} M^2(\Phi)[\lambda^2 + \sigma^2 + \omega^2] + T(\Phi)\lambda\sigma\omega + \ldots \quad (5.42)$$

As expected $\lambda, \sigma, \omega$ have the same effective mass

$$M^2(\Phi) = -\frac{4}{L^2} e^{2\Phi/\sqrt{6}} \quad (5.43)$$

and a unique trilinear coupling

$$T(\Phi) = \frac{1}{\sqrt{2}L^2} [2e^{-4\Phi/\sqrt{6}} - e^{2\Phi/\sqrt{6}}] \quad (5.44)$$

Along the CB flow

$$M^2(v) = -\frac{4}{L^2} v^{1/3} \quad T(v) = \frac{1}{\sqrt{2}L^2} v^{-2/3}(2 - v) \quad (5.45)$$

The relevant wave equation for the inert scalars $Q(x, v) = e^{ipx}Q(v)$ reads

$$Q'' + \left(\frac{2}{v} + \frac{1}{1 - v}\right)Q' + \left[\frac{p^2}{4L^2} v^2(1 - v) + \frac{1}{v(1 - v)^2}\right] Q = 0 \quad (5.46)$$

with $Q = \lambda, \sigma, \omega$. The solution that vanishes at the singularity $v = 0$ is

$$Q(v) = v^a(1 - v)F(a, a; 2a; v) \quad (5.47)$$

where

$$a(p) = -\frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{p^2}{m_o^2}} \quad (5.48)$$

The inclusion of the fields $\lambda, \sigma$ and $\omega$ does not modify in any significant way the near boundary analysis performed in [1, 2]. In particular the renormalized one-point function of the operator dual to any one of them (say $\lambda$) is simply given by

$$\langle O_\lambda(x) \rangle = 2\lambda(0)(x) \quad (5.49)$$

\footnote{Using $P^{(0)}_\nu(z) = F(-\nu, \nu + 1; 1; (1 - z)/2), F(a, a; 2a; v)$ can be expressed in terms of associated Legendre functions, but little is gained in this rewriting.}
Analytic continuation of $F(a,a;2a;v)$ reveals the non-local relation between $\tilde{\lambda}(0)$ and $\lambda(0)$ at the linearized level and yields the two-point function

$$\langle O_\lambda(p)O_\lambda(q) \rangle = 4\frac{N^2}{2\pi^2}(2\pi)^4\delta(p+q)[\psi(1)-\psi(a)] \quad (5.50)$$

where the correct normalization of the action has been taken into account. The spectrum is continuous above a mass gap $m_o = \ell/L^2$. Similar analysis leads to the bulk-to-boundary propagator (in momentum space)

$$K_p(v) = -\frac{\Gamma(a)}{\Gamma(2a)}v^a(1-v)F(a,a;2a;v) \quad (5.51)$$

Taking the mixed derivative of (5.49) w.r.t. the sources of the remaining two fields ($\sigma(0)$ and $\omega(0)$) gives the desired three-point function

$$\langle O_\lambda(p_1)O_\sigma(p_2)O_\omega(p_3) \rangle = (2\pi)^4\delta(\sum_p p_i) \int_0^1 dv \sqrt{G(v)}T(v)\prod_i K_{p_i}(v) \quad (5.52)$$

Plugging in the relevant expressions yields

$$G_{CB}(p_1,p_2,p_3) = \kappa_{CB}\delta(\sum_p p_i)\prod_i \frac{\Gamma(a_i)^2}{\Gamma(2a_i)} \int_0^1 dv (2-v)\prod_i v^{a_i}F(a_i,a_i;2a_i;v) \quad (5.53)$$

where $\kappa_{CB} = -(2N^2/\sqrt{3})(2\pi)^2(\ell^4/L^2)$. Since the $SO(4) = SU(2)_L \times SU(2)_R$ tensor structure is unique, Wigner-Eckart theorem gives

$$\langle Q^{iLJR}(p_1)Q^{jLR}(p_2)Q^{kLR}(p_3) \rangle = \varepsilon^{iL,LJ,LK}_L \varepsilon^{iR,JR,KR}_R G_{CB}(p_1,p_2,p_3) \quad (5.54)$$

Although the integral cannot be performed elementarily, it is easy to see that $G_{CB}(p_1,p_2,p_3)$ is finite for generic momenta, as expected for operators of UV dimension $\Delta = 2$, and vanishes at the threshold $p_i^2 = -m_o^2$ where $a_i = -1/2$. Series integration yields a compact expression

$$G_{CB}(p_1,p_2,p_3) = \kappa_{CB}\sum_{n_i} \prod_{i} \frac{\Gamma(a_i+n_i)^2}{\Gamma(2a_i+n_i)n_i!} \left[ \frac{2}{1+\sum_i(a_i+n_i)} - \frac{1}{2+\sum_i(a_i+n_i)} \right] \quad (5.55)$$

where $\delta(\sum_i p_i)$ is understood.
Direct comparison with the conformal three-point function of scalar primary operators of dimension $\Delta = 2$ computed in appendix A seems hopeless. However using the AdS/CFT correspondence for pure AdS we know that

$$G_{\text{CFT}}(x_1, x_2, x_3) = \kappa_{\text{CFT}} T(0) \int \frac{dz d^3x}{z^5} \prod_i \frac{z^2}{[z^2 + (x - x_i)^2]^2}$$  \hfill (5.56)$$
can be written in momentum space, up to constants, as

$$G_{\text{CFT}}(p_1, p_2, p_3) = \tilde{\kappa}_{\text{CFT}} \delta(\sum_i p_i) \int \frac{dz}{z^5} T(0) \prod_i z^2 \mathcal{K}_0(zp_i)$$  \hfill (5.57)$$
where

$$\mathcal{K}_0(u) = \sum_{n=0}^{\infty} \frac{u^{2n}}{2^{2n} (n!)^2} \left[ \log(u/2) - \psi(n + 1) \right]$$  \hfill (5.58)$$
is a modified Bessel function. Changing radial variable according to

$$1 - v = \frac{\ell^2}{L^4} z^2 - \frac{2}{3} \frac{\ell^4}{L^8} z^4 + \mathcal{O}(z^6)$$  \hfill (5.59)$$
it is easy to convince oneself that the UV limit $\ell \to 0$ ‘explodes’ the boundary region $v \approx 1$ where indeed the asymptotics of $K(p,v)$ coincides with the expansion of $z^2 \mathcal{K}_0(zp_i)$ and the trilinear coupling goes to its fixed point value $T(0)$. This fixes the UV behavior including the overall normalization.

6 Three-point function in the GPPZ flow

Although the exact form of the fully non-polynomial potential of the $\mathcal{N} = 2$ gauged supergravity that governs the GPPZ flow is known\textsuperscript{12}, it is not easy to isolate trilinear couplings that only involve inert scalars. Partly for this reason and partly to push our analysis another step forward, we are lead to consider the still relatively simple 3-point function

$$G_{\text{GPPZ}}(x_1, x_2, x_3) = \langle O_\sigma(x_1) O_\sigma(x_2) O_\phi(x_3) \rangle$$  \hfill (6.60)$$
\textsuperscript{11}The exact coordinate transformation with $\rho = z^2/L^2$ can be found in [2].
\textsuperscript{12}We thank G. Dall’Agata for disclosing its secret form to us.
where $\mathcal{O}_\phi$ is the operator dual to the active scalar $\Phi$ and $\mathcal{O}_\sigma$ is the operator dual to the inert scalar $\Sigma$. $G_{PPPZ}$ vanishes at the superconformal fixed point in the UV by $SO(6)$ R-symmetry arguments.

The situation is subtler than in the previous Section. On the one hand $\Phi$ mixes with the longitudinal and trace components of the metric. On the other hand a naive analysis inspired by the “dynamical scalar gauge” shows that the above 3-point function is logarithmically divergent as $\epsilon \to 0$ if no new counterterm is generated when $\Sigma$ is included. However, slightly at variance to what happens when only $\Phi$ is present, we find two new logarithmically divergent counterterms. One is quartic in $\Sigma$ the other is biquadratic in $\Phi$ and $\Sigma$.

In order to derive the precise form of these counterterms one has to perform a detailed near boundary analysis of the bulk field equations. The results we need are

$$g_{ij}^{(2)} = \frac{1}{2} (R_{ij}^{(0)} - \frac{1}{6} R^{(0)} g^{ij} ) - \frac{1}{3} (\phi_{(0)}^2 + \sigma_{(0)}^2 ) g_{ij}^{(0)}$$  \hspace{1cm} (6.61)

$$h_{(4)} = -2(\phi_{(0)} \tilde{\phi}_{(2)} + \sigma_{(0)} \tilde{\sigma}_{(2)})$$  \hspace{1cm} (6.62)

$$g_{(4)} = \frac{1}{4} (g_{(2)} g_{(2)} - 2(\phi_{(0)} \phi_{(2)} + \sigma_{(0)} \sigma_{(2)}) + 4(\phi_{(0)} \tilde{\phi}_{(2)} + \sigma_{(0)} \tilde{\sigma}_{(2)})
+ \frac{1}{9} (\phi_{(0)}^4 - 3\sigma_{(0)}^4 + 6\phi_{(0)}^2 \sigma_{(0)}^2$$  \hspace{1cm} (6.63)

$$\tilde{\phi}_{(2)} = -\frac{1}{4} (\nabla^2_{(0)} \phi_{(0)} - (g_{(2)} \phi_{(0)}) - \frac{1}{3} \phi_{(0)}^3 - \phi_{(0)} \sigma_{(0)}^2$$  \hspace{1cm} (6.64)

$$\tilde{\sigma}_{(2)} = -\frac{1}{4} (\nabla^2_{(0)} \sigma_{(0)} - (g_{(2)} \sigma_{(0)}) + \sigma_{(0)}^3 - \phi_{(0)}^2 \sigma_{(0)}$$  \hspace{1cm} (6.65)

where $R_{ij}^{(0)} = g_{ij}^{(0)} R_{(0)ij}$, $g_{(2)} = g_{ij}^{(2)} g_{(2)ij}$, $g_{(4)} = g_{ij}^{(4)} g_{(4)ij}$, $h_{(4)} = g_{ij}^{(4)} h_{(4)ij}$ and $(g_{(2)} g_{(2)}) = g_{ij}^{(2)} g_{(2)ij} g_{kl}^{(2)} g_{(2)kl}$.

Using Einstein equations in the convenient form

$$R_{\mu\nu} = -2(\partial_{\mu} \phi \partial_{\nu} \Phi + \partial_{\mu} \Sigma \partial_{\nu} \Sigma + \frac{2}{3} VG_{\mu\nu})$$  \hspace{1cm} (6.67)

and including the Gibbons-Hawking boundary term, it is easy to see that the (regulated) on-shell action is simply given by

$$S_{reg} = -\frac{2}{3} \int_\rho \sqrt{d^4x} \sqrt{G} V + \epsilon \partial_{\epsilon} \int \frac{d^4x}{\epsilon^2} \sqrt{g(x, \epsilon)}$$  \hspace{1cm} (6.68)
Inserting the data arising from the near boundary analysis gives

\[ S_{\text{reg}}[g(0), \phi(0), \sigma(0); \epsilon] = \int d^4x \sqrt{g(0)} \left\{ -\frac{3}{2\epsilon^2} + \frac{1}{2\epsilon}(\phi^2(0) + \sigma^2(0)) \right\} \]

\[ + \log \epsilon \left[ \frac{1}{32} \left( R_{(0)ij}R_{(0)}^{ij} - \frac{1}{3} R_{(0)}^{2} \right) + \frac{1}{2} (\phi(0) \nabla^2_{(0)} \phi(0) + \sigma(0) \nabla^2_{(0)} \sigma(0)) \right. \]

\[ + \frac{1}{12} R(0)(\phi^2(0) + \sigma^2(0)) - 2\sigma^4(0) + 2\phi^2(0)\sigma^2(0) \right\} + O(\epsilon^0) \]  

(6.69)

Once expressed in terms of the regulated fields, the counterterms read

\[ S_{\text{ct}}[\gamma, \Phi, \Sigma; \epsilon] = \int d^4x \sqrt{\gamma} \left\{ \frac{3}{2} - \frac{1}{8} R[\gamma] + \frac{1}{2}(\Phi^2 + \Sigma^2) \right\} \]

\[ - \log \epsilon \left[ \frac{1}{4} \left( \Phi \nabla_{[\gamma} \Phi + \frac{1}{6} R[\gamma] \Phi^2 \right) + \frac{1}{4} \left( \Sigma \nabla_{[\gamma} \Sigma + \frac{1}{6} R[\gamma] \Sigma^2 \right) \right. \]

\[ + 2\Sigma^4 - 2\Sigma^2 \Phi^2 + \frac{1}{32} \left( R_{[\gamma]} R_{ij}^{ij}[\gamma] - \frac{1}{3} R[\gamma] \right) \]  

(6.70)

(6.71)

(6.72)

As anticipated the symmetry between \( \Phi \) and \( \Sigma \) is spoiled by the logarithmically divergent quartic terms at the beginning of the last line.

The one-point function of \( \mathcal{O}_\phi \) in the presence of sources is

\[ \langle \mathcal{O}_\phi \rangle = \lim_{\epsilon \to 0} \left\{ 2(\phi(2) + \tilde{\phi}(2)) - \frac{2}{9} \phi^3(0) - 4 \log \epsilon \phi(0) \sigma^2(0) \right\} \]

(6.73)

Despite appearance \( \langle \mathcal{O}_\phi \rangle \) is finite as we will see.

In order to compute the desired three-point function we need to determine the (non-local) dependence of \( \tilde{\phi}(2) \) on \( \sigma(0) \). Because of the mixing of the fluctuation \( \varphi \) of the active scalar \( \Phi \) with the longitudinal \( (H) \), radial \( (h_{rr}) \) and trace \( (h = g_{(0)ij} h_{ij}) \) components of the metric one has to study the coupled field equations to linear order in \( \varphi, h_{rr}, h_{ij} \) and \( H \) and to quadratic order in \( \sigma \). Following the steps detailed by \[7, 8\], but keeping \( h_{rr} \), one gets

\[ h' + \frac{16}{3} \Phi' \varphi - 4A'h_{rr} = J_C \]

(6.74)

\[ H'' + 4A'H' - \frac{1}{2} e^{-2A}(h + 2h_{rr}) = J_H \]

(6.75)

\[ 3A'(h' - \partial^2 H) + \frac{3}{4} e^{-2A} \partial^2 h + 4V h_{rr} - 4\Phi' \varphi' - 4V_{\phi} \varphi = J_C \]  

(6.76)
where $\partial^2 = \delta^{ij}\partial_i\partial_j$ and

\[
J_C = \frac{16}{3\partial^2} \partial^i (\sigma' \partial_i \sigma) = \frac{16}{3\partial^2} \partial^i T^{(2\sigma)}_{\nu} (6.77)
\]

\[
J_H = \frac{4e^{-2A}}{3\partial^2} \left( \frac{4 \partial^i \partial^j - \delta^{ij}}{\partial^2} \right) \partial_i \sigma \partial_j \sigma = \frac{4e^{-2A}}{3\partial^2} \left( \frac{4 \partial^i \partial^j - \delta^{ij}}{\partial^2} \right) T^{(2\sigma)}_{\nu} (6.78)
\]

\[
J_G = 2(\sigma')^2 - 2e^{-2A}\partial^i \sigma \partial_i \sigma - 2M^2 \sigma^2 = 4T^{(2\sigma)}_{\nu} (6.79)
\]

Introducing the ‘gauge invariant’ combinations

\[
P = h + \frac{16W}{3W_\phi} \varphi \quad R = h_{rr} + 2\partial_r \left( \frac{\varphi}{W_\phi} \right) \quad Q = H' - 2e^{-2A} \frac{\varphi}{W_\phi} (6.80)
\]

one gets

\[
P' - 4A'R = -J_C (6.81)
\]

\[
Q' + 4A'Q - \frac{1}{2} e^{-2A}P - e^{-2A}R = J_H (6.82)
\]

\[
3A'P' - 3A'\partial^2 Q + \frac{3}{4} e^{-2A} \partial^2 P + 4VR = J_G (6.83)
\]

Manipulating these equations as in [7, 1] eventually boils down to

\[
R'' + (2W_\phi - 4W)R' + (2W_\phi W_\phi - 4W^2 + \frac{32}{9} W^2 - \frac{8}{3} W W_\phi - e^{-2A}P^2)R = J_R (6.84)
\]

where the $\sigma$-dependent source

\[
J_R = -4\sigma'\sigma' - \frac{4}{3} M^2 \sigma^2 - \partial_r \left( \frac{\sigma^2 \partial M^2}{W_\phi \partial_\sigma} \right) - 2A' \left( \frac{\sigma^2 \partial M^2}{W_\phi \partial_\sigma} \right) (6.85)
\]

turns out to be relatively simple thanks to the identity

\[
J'_G + 4A'J_G + \frac{3}{4} \partial^2 e^{-2A} J_C - 3A'J_H + 3A'J'_C + 12(A')^2J_C = -2\partial_r(M_{\sigma}^2)\sigma^2 (6.86)
\]

that follows from

\[
\nabla^\mu B^\mu T^{(2\sigma)}_{\nu} = -\frac{1}{2} \partial_\nu (M_{\sigma}^2)\sigma^2 (6.87)
\]

where $\nabla^\mu B$ is the background covariant derivative.
Switching to the standard variable \( u \) and to momentum space yields

\[
R'' + \frac{1}{u(1-u)} R' + \frac{1}{u(1-u)} \left( \frac{2u - 1}{u(1-u)} - \frac{p^2 L^2}{4} \right) R = J_R(u) \tag{6.88}
\]

The solution of the homogeneous equation is

\[
R_p^{(0)} = u(1-u) F(a_+^{(\phi)}, a_-^{(\phi)}; 3; u) = \frac{u(1-u) \Gamma(3)}{\Gamma(a_+^{(\phi)} \Gamma(a_-^{(\phi)})} \sum_{n=0}^{\infty} \frac{(a_+^{(\phi)})_n (a_-^{(\phi)})_n}{(n!)^2} \times
\]

\[
\times (1-u)^n [2 \psi(n+1) - \psi(n + a_+^{(\phi)}) - \psi(n + a_-^{(\phi)}) - \log(1-u)] \tag{6.89}
\]

where

\[
a_\pm^{(\phi)} = \frac{3}{2} \pm \frac{1}{2} \sqrt{1 - p^2 L^2} \tag{6.90}
\]

For \( u \approx 1 \), \( R(u) \) admits an expansion of the form

\[
R = R_{(0)} (1-u) \log(1-u) + \tilde{R}_{(0)} (1-u) + \ldots \tag{6.91}
\]

In the axial gauge \( h_{rr} = 0 \) one finds

\[
\tilde{R}_{(0)} = -\frac{4}{\sqrt{3}} (\tilde{\phi}_2 + \tilde{\phi}_2 - \phi_{(0)}) \quad R_{(0)} = -\frac{4}{\sqrt{3}} \tilde{\phi}_{(2)} = \frac{1}{\sqrt{3}} \nabla^2_{(0)} \phi_{(0)} + \ldots \tag{6.92}
\]

To linear order around the GPPZ flow, where \( \phi_{(0)} = \sqrt{3} \) and \( \tilde{\phi}_{(0)} = 1/\sqrt{3} \), while \( \phi_2 = \sigma_{(0)} = 0 \),

\[
\langle O_{\phi} \rangle = -\frac{\sqrt{3}}{2} \tilde{R}_{(0)} \tag{6.93}
\]

Differentiating \( \tilde{R}_{(0)} \) w.r.t. \( \phi_{(0)} \), brings in an overall factor of \( p^2 \) that shows up in the two-point function

\[
\langle O_{\phi}(p) O_{\phi}(-p) \rangle = \frac{N^2}{4 \pi^2} p^2 \left[ \psi(a_+^{(\phi)}(p)) + \psi(a_-^{(\phi)}(p)) - 2 \psi(1) \right] \tag{6.94}
\]

A similar though simpler analysis [7, 8] leads to the other two-point function

\[
\langle O_{\sigma}(p) O_{\sigma}(-p) \rangle = \frac{N^2}{4 \pi^2} \left( p^2 - \frac{8}{L^2} \right) \left[ \psi(a_+^{(\sigma)}(p)) + \psi(a_-^{(\sigma)}(p)) \right] \tag{6.95}
\]
where
\[ a_\pm^{(\sigma)}(p) = \frac{3}{2} \pm \frac{1}{2} \sqrt{9 - p^2L^2} \] (6.96)

Using
\[ \psi(z) = -\gamma + \sum_{n=0}^{\infty} \frac{z - 1}{(n + 1)(z + n)} \] (6.97)

it is easy to identify the mass poles \([11]\) and compute their positive (!) residues.

For \( O_\phi \), that belongs to the \( \mathcal{N} = 1 \) `anomaly' multiplet \( \mathcal{A} = \sum_i \text{Tr}(Z_i^2) \) dual to the active hypermultiplet, one has
\[ (m_\phi L)^2 = 4(k + 1)(k + 2) \quad k = 0, 1, 2, \ldots \] (6.98)

For \( O_\sigma \), that belongs to the \( \mathcal{N} = 1 \) `Lagrangian' multiplet \( \mathcal{S} = \text{Tr}(W^2 + ...) \) dual to the dilaton hypermultiplet,
\[ (m_\sigma L)^2 = 4k(k + 3) \quad k = 0, 1, 2, \ldots , \] (6.99)

including a zero-mass pole.

It is amusing to observe that the superglueball decay constants arising from the residues at the \( k^{th} \) mass pole are the same for \( \phi \) and \( \sigma \)
\[ |f_\phi(k)|^2 = \frac{N^2 m_\phi^2(k)(2k + 3)}{\pi^2 L^2} = |f_\sigma(k)|^2 \] ! (6.100)

We are ready to compute the three-point function. To quadratic order in \( \sigma \)'s
\[ \langle O_\phi(p) \rangle_\epsilon = -\frac{\sqrt{3}}{2} \tilde{R}(0)(p) = -\frac{\sqrt{3}}{2} \int_0^{1-\epsilon} du \sqrt{G(u)}K_R(u,p)J_R(u,p) \] (6.101)

where, setting the active scalar to its background,
\[ J_R(u) = -16(\sigma')^2 - 32(1 - u)\sigma'\sigma - 12\sigma^2 \] (6.102)

Differentiating (6.101) twice w.r.t. \( \sigma(0) \) one gets the logarithmically divergent integral
\[ G_{GPPZ}(p_1, p_2, p_3) = \kappa_{GPPZ} \delta(\sum_i p_i) \prod_i \Gamma(a_+(p_i))\Gamma(a_-(p_i)) \int_0^{1-\epsilon} du \ u^3(1 - u)^2 \times \]
\[ F_R(p_3) \ [F_\sigma'(p_1)F_\sigma'(p_2) + F_\sigma(p_1)F_\sigma'(p_2)] - 2(1 - u)F_\sigma'(p_1)F_\sigma'(p_2) \] (6.103)
where \( \kappa_{GP\, P\, Z} = (N^2 \sqrt{3}/2\pi^2 L)(2\pi)^4 \) and
\[
F_R(p) = F(a_+^{(\phi)}(p), a_-^{(\phi)}(p); 3; u) \quad (6.104)
\]
and
\[
F_\sigma(p) = F(a_+^{(\sigma)}(p), a_-^{(\sigma)}(p); 3; u) \quad (6.105)
\]
The logarithmic divergence of \( \tilde{R}(0) \) is a contact term independent of the momenta. Its coefficient \( \eta = 4\sqrt{3} \) is exactly cancelled by the last contact term in \( \langle O_\phi \rangle \) when \( \phi(0) \) is put to its background value of \( \sqrt{3} \). So that \( \langle O_\phi \rangle \) as well as \( G_{GP\, P\, Z} \) are finite as they should. It is relatively easy to check that \( G_{GP\, P\, Z} \) vanish in the UV, i.e. at large momenta as expected on \( SO(6) \) R-symmetry grounds.

Although the above integral cannot be expressed in terms of elementary functions, it is not difficult to integrate the products of hypergeometric functions by series. Moreover the expressions drastically simplify when one goes on-shell i.e. \( p_i^2 = -m_i^2 \) and amputates the external legs thus effectively computing the irreducible vertex that enters the 3-body scattering amplitude or, equivalently, the superglueball decay amplitude.

To this end observe that near the mass-shell where
\[
p_i^2 L^2 = -m(k_i)^2 L^2 + \epsilon_i \quad (6.106)
\]
\[
a_-(p_i) \approx -k_i - \frac{\epsilon_i}{4(2k_i + 3)} \quad \text{and} \quad a_+(p_i) \approx 3 + k_i + \frac{\epsilon_i}{4(2k_i + 3)} \quad (6.107)
\]
This is true both for \( \sigma \) and \( \phi \). Moreover, at the mass-shell the hypergeometric functions truncate and become degree \( k \) (Gegenbauer or Jacobi) polynomials.

The mass poles of the connected 3-point function \( G_{GP\, P\, Z}(p_1, p_2, p_3) \) are exposed by the \( \Gamma \)'s in front of the integral. Indeed
\[
\Gamma(a_-(p_i)) \approx \Gamma \left( -k_i - \frac{\epsilon_i}{4(2k_i + 3)} \right) \approx \frac{(-)^{k_i+1} 4(2k_i + 3)}{k_i! \epsilon_i} \quad (6.108)
\]
while
\[
\Gamma(a_+(p_i)) \approx \Gamma \left( 3 + k_i + \frac{\epsilon_i}{4(2k_i + 3)} \right) \approx (k_i + 2)! \quad (6.109)
\]
The relevant integrals are of the form
\[
\int_0^1 du (1 - u)^2 u^{\Sigma n + 1} [2n_1 n_2 (1 - u) - (n_1 + n_2) u] = \quad (6.110)
\]
\[
\frac{3! 2n_1 n_2}{(\Sigma n + 2)(\Sigma n + 3)(\Sigma n + 4)(\Sigma n + 5)} - \frac{2! (n_1 + n_2)}{(\Sigma n + 3)(\Sigma n + 4)(\Sigma n + 5)}
\]
replacing the Pochhammer symbols in the hypergeometric functions with their explicit expressions

\[ (-k_i)_{n_i} = \frac{(-)^{n_i} k_i!}{(k_i - n_i)!} \quad (3 + k_i)_{n_i} = \frac{(k_i + n_i + 2)!}{(k_i + 2)!} \]  

(6.111)

and putting everything together yields

\[ A_{GPPZ}(k_1, k_2, k_3) = \frac{(2\pi)^5 \sqrt{3}}{N L} \prod_{i=1}^{3} \sqrt{(k_i + 1)(k_i + 2)(2k_i + 3)} \times \]  

(6.112)

\[ \sum_{n_i=0}^{k_i} \left\{ \frac{6n_1n_2 - (n_1 + n_2)(\sum n + 2)}{(n_3 + 2) \prod_{s=2}^{5} \prod_{n+s}^{n+s} \prod_{i=1}^{3} \left[ \frac{(-)^{k_i+n_i} (k_i + n_i + 2)!}{n_i!(n_i + 1)! (k_i - n_i)!} \right] } \right\} \]

which has the expected mass dimension, \([m]^1\) and is correctly suppressed by the \(1/N\) factor at large \(N\). Comparison with field theory results seems prohibitive for the time being since it would require a non-perturbative understanding of the dynamics of \(\mathcal{N} = 1\) SYM which is well beyond reach. One may take the holographic result as a very precise prediction for the amplitude at strong 't Hooft coupling.

### 7 Concluding remarks and speculations

We have studied three-point functions of scalar operators along two holographic RG flows. For the CB flow we have considered inert scalars and derived a compact expression that is finite and behaves correctly in the UV. In the GPPZ flow, though the initial situation was subtler because of the mixing of \(\Phi\) with the metric, we have been able to extract the irreducible vertices of three superglueballs. Thanks to the operator identity

\[ T^i_i = \beta \mathcal{O}_\phi \]  

(7.113)

with \(\beta = -\sqrt{3}\), the very same modes couple to the trace of the stress tensor \([1, 6]\). This should elucidate the evolution with scale according to the holographic renormalization group \([34]\). Extending the analysis to the transverse modes of the stress tensor or of the (broken) currents is straightforward in principle since the relevant trilinear bulk couplings are completely fixed by
gauge invariance. The expected form of these correlators should be

\[ \langle T_{ij}(p_1)O_\sigma(p_2)O_\sigma(p_3) \rangle_t = \kappa_t \delta(\sum_i p_i) \int dw \sqrt{G(w)} K_{ij,kl}(p_1, w) \times (7.114) \]

\[ \times [e^{-2A(w)}(p_k^i p_3^j - p_k^j p_3^i) + 4\delta^{kl} M^2_\sigma(w)] K_\sigma(p_2, w) K_\sigma(p_3, w) + \text{contact} \]

and

\[ \langle J_i(p_1)O_\sigma(p_2)O_\tilde{\sigma}(p_3) \rangle_t = \kappa_j \delta(\sum_i p_i) \int dw \sqrt{G(w)} K_{i,k}(p_1, w) \times (7.115) \]

\[ (p_k^i - p_3^i) K_\sigma(p_2, w) K_\tilde{\sigma}(p_3, w) + \text{contact} \]

As expected, the integrals diverge but the contact terms suggested by holographic renormalization cancel these divergences. Similar considerations apply to three-point functions with more insertions of the stress tensor or of broken currents. As usual, bulk gauge invariance translates into Ward identities of the boundary quantum field theory. The holographically computed one-point functions in the presence of sources do satisfy these Ward identities including anomalies and this carries over to higher point functions.

It would be nice to address similar issues in full-fledged string solutions that display confinement and logarithmic corrections to the pure AdS behavior in the UV [26, 27, 28]. The results obtained by Krasnitz for the case of two-point functions in the KT flow are very encouraging in this respect [14].

As far as higher than three point correlation functions are concerned, if the miracle that in AdS transforms exchange diagrams into contact ones does not take place in non-trivial holographic RG flows then three-point functions will set the standard of a new holographic tradition, at least for a while.

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Appendix: Conformal 3-point function in momentum space

In order to have a glimpse of the result of the integral for the holographic 3-point correlation function in the CB flow and to eventually compare with not-so-standard field theory results it is convenient to compute the Fourier transform of the 3-point function of scalar primary operators of dimension $\Delta_i, i = 1, 2, 3$.

$$G_{\text{CFT}}(x_1, x_2, x_3) = \langle O(x_1)O(x_2)O(x_3) \rangle = \kappa_{\text{CFT}}(x_{12}^2)^{-\Delta_{12}}(x_{23}^2)^{-\Delta_{23}}(x_{31}^2)^{-\Delta_{31}}$$

(7.116)

where $x_{ij}^2 = (x_i - x_j)^2$ and $\Delta_{ij} = \Delta_i + \Delta_j - \Sigma$ with $\Sigma = (\Delta_1 + \Delta_2 + \Delta_3)/2$.

For simplicity we set the constant $\kappa_{\text{CFT}} = 1$ henceforth. By definition

$$G_{\text{CFT}}(p_1, p_2, p_3) = \int d^4x_1 d^4x_2 d^4x_3 e^{-i(p_1x_1 + p_2x_2 + p_3x_3)} G_{\text{CFT}}(x_1, x_2, x_3)$$

(7.117)

In $d$ dimensions, Feynman parametrization leads to the integral

$$G_{\text{CFT}}(p_1, p_2, p_3) = \frac{\pi^{(\Sigma+d)/2}\Gamma(d/2 - \Delta_{23})\Gamma(d - \Sigma)}{2^\Sigma\Gamma(d/2)\Gamma(\Delta_{12})\Gamma(\Delta_{13})} \times \int_0^1 d\sigma F(\Sigma - \frac{d}{2}, \Delta_{23}; \frac{d}{2}; 1 - \xi(\sigma)) \frac{(1 - \sigma)^{d/2 - \Delta_{13} - 1}\sigma^{d/2 - \Delta_{12} - 1}}{[(1 - \sigma)(p_3 + p_2)^2 + \sigma p_2^2)^d - \Sigma}$$

where

$$\xi(\sigma) = \frac{(1 - \sigma)\sigma p_3^2}{(1 - \sigma)(p_3 + p_2)^2 + \sigma p_2^2}$$

(7.118)

We have checked that this expression correctly factorizes in the extremal case $\Delta_1 = \Delta_2 + \Delta_3$.

Let us specialize to the case $\Delta_i = 2$ in $d = 4$, so that $\Sigma = 3$ and $\Delta_{ij} = 1$.

Putting

$$\mu_2 \equiv p_2^2/p_1^2 \quad \mu_3 \equiv p_3^2/p_1^2$$

(7.119)

one gets

$$G_{\text{CFT}}(p_1, p_2, p_3) = \frac{\pi^{7/2}}{8p_1^2} \delta(\sum_i p_i) A(\mu_2, \mu_3)$$

(7.120)

where

$$A(\mu_2, \mu_3) \equiv \int_0^1 d\sigma \frac{\log[(1 - \sigma)\sigma] - \log[\sigma\mu_2 + (1 - \sigma)\mu_3]}{(1 - \sigma)\sigma + \sigma\mu_2 + (1 - \sigma)\mu_3}$$

(7.121)

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is strikingly reminiscent of the box integral $B(r, s)$ computed \( e.g. \) in [35]. Indeed $A(\mu_2, \mu_3)$ is tightly related to $B(r, s)$.

It is easy to check the symmetry $A(\mu_2, \mu_3) = A(\mu_3, \mu_2)$. It is slightly more laborious to express it in terms of logs and dilogs. To this end it is convenient to factorize the denominator as

\[
\sigma(1 - \sigma) + \mu_2\sigma + (1 - \sigma)\mu_3 = -(\sigma - \sigma_+)(\sigma - \sigma_-) \tag{7.122}
\]

where

\[
\sigma_\pm = \frac{1}{2}[1 + \mu_2 - \mu_3 \pm \sqrt{(1 + \mu_2 - \mu_3)^2 + 4\mu_3}] \tag{7.123}
\]

Repeatedly and carefully (since they are valid for $0 \leq x \leq 1$) using

\[
Li_2(x) + Li_2(1 - x) = -\log(x)\log(1 - x) + \frac{\pi^2}{6} \tag{7.124}
\]

and

\[
Li_2(x) + Li_2\left(\frac{x - 1}{x}\right) = -\frac{1}{2}\log^2(x) \tag{7.125}
\]

where\(^{13}\)

\[
Li_2(z) = -\int_0^1 \frac{d\sigma}{\sigma} \log(1 - z\sigma) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \tag{7.126}
\]

one finally gets the desired result

\[
A(\mu_2, \mu_3) = \frac{1}{\nu} \left(2Li_2\left(\frac{1 + \mu_2 - \mu_3 - \sqrt{\nu}}{2\mu_2}\right) + 2Li_2\left(\frac{1 + \mu_2 - \mu_3 - \sqrt{\nu}}{2\mu_2}\right) \right. \\
\left. - \log(\mu_2)\log(\mu_3) - \left[\log\left(\frac{\mu_2 + \mu_3 - 1 - \sqrt{\nu}}{2}\right)\right]^2 \right) \tag{7.127}
\]

where

\[
\nu = 1 + \mu_2^2 + \mu_3^2 - 2\mu_1 - 2\mu_2 - 2\mu_2\mu_3 \tag{7.128}
\]

$A(\mu_2, \mu_3)$ is manifestly symmetric and real for $\nu \geq 0$. Up to an overall constant, $A(\mu_2, \mu_3)$ coincides with $B(r, s)$ or better $g(x_1, x_2, x_3)$ in [35] after replacing $\mu_2$ and $\mu_3$ with $r$ and $s$, \textit{i.e.} $p_1 \rightarrow x_{23}, p_2 \rightarrow x_{31}, p_3 \rightarrow x_{12}$.

\(^{13}\)In the notation of Abramowitz-Stegun [36] (pag.1004) $Li_2(z) = f(1 - z)$.
Appendix B: Other useful formulae

For the RG flows under consideration, homogeneous solution of the fluctuation equations may be expressed in terms of hypergeometric functions

\[ \phi_p(w, x) = w^\alpha (1 - w)^\beta F(a, b; c; w) \exp(ipx) \quad (7.129) \]

We assume that \( z = 0 \) represents the deep IR interior (singularity) and \( w = 1 - \epsilon \) the (regulated) UV boundary. In order to fix the overall normalization of the (regulated) bulk-to-boundary propagators \( K_\epsilon(p, z) \) we need the analytic continuation of \( F(a, b, c; w) \) to \( w \approx 1 \).

For \( c = a + b \)

\[ F(a, b; a + b; w) = -\frac{\Gamma(a + b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(n!)^2} (1 - w)^n \times (7.130) \]

\[ \times [\log(1 - w) - 2\psi(n + 1) + \psi(a + n) + \psi(b + n)] \]

For \( c = a + b + m \)

\[ F(a, b; a + b + m; w) = \frac{\Gamma(m)}{\Gamma(a + b + m)\Gamma(a + m)\Gamma(b + m)} \sum_{n=0}^{m-1} \frac{(a - m)_n(b - m)_n}{n!(1 - m)_n} \times \]

\[ \times (1 - w)^{n-m} \quad (7.131) \]

\[ \times [\log(1 - w) - \psi(n + 1) - \psi(n + m + 1) + \psi(a + n) + \psi(b + n)] \]

Correspondingly, the (regulated) bulk to boundary propagators read

\[ K_\epsilon(p, w) = -\frac{w^\alpha (1 - w)^\beta \Gamma(a)\Gamma(b)}{\epsilon^\beta \log \epsilon \Gamma(a + b)} F(a, b; a + b; w) \quad (7.133) \]
\[ K_\epsilon(p, w) = \frac{w^\alpha(1-w)^\beta \Gamma(a)\Gamma(b)}{\epsilon^\beta \Gamma(m)\Gamma(a+b-m)} F(a, b; a+b-m; w) \quad (7.134) \]

\[ K_\epsilon(p, w) = \frac{w^\alpha(1-w)^\beta \Gamma(a+m)\Gamma(b+m)}{\epsilon^\beta \Gamma(m)\Gamma(a+b+m)} F(a, b; a+b+m; \epsilon) \quad (7.135) \]

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