On Mass Spectrum in SQCD. Unequal quark masses

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Abstract

$N = 1$ SQCD with $N_c$ colors and two types of light quarks, $N_l$ flavors with smaller masses $m_l$ and $N_h = N_F - N_l$ flavors with larger masses $m_h$, $N_c < N_F < 3N_c$, $0 < m_l \leq m_h \ll \Lambda_Q$, is considered within the dynamical scenario in which quarks can form the coherent colorless diquark-condensate. There are several phase states at different values of parameters $r = m_l/m_h$, $N_l$, and $N_F$. Properties of these phases and the mass spectra therein are described.

1. Introduction.

In the present paper the results obtained in [1] for equal quark masses are generalized to the case of unequal masses. We do not consider here the most general case of arbitrary quark masses. Only one specific (but sufficiently representative) choice of unequal masses is considered: there are $N_l \neq N_c$ flavors with the smaller masses $m_l$ and $N_h = N_F - N_l$ flavors with the larger masses $m_h \geq m_l > 0$, $N_c < N_F < 3N_c$. Some abbreviations used below: DC means the diquark-condensate, HQ stands for a heavy quark, the $l$-quarks means the quarks with the smaller masses $m_l$, the $h$-quarks are those with the larger masses $m_h$. The masses $m_l$ and $m_h$ are the running current quark masses normalized at $\mu = \Lambda_Q$, and $\mathcal{M}_l$ or $\mathcal{M}_h$ are the chiral diquark condensates of the $l$ or $h$-quarks also normalized at $\mu = \Lambda_Q$, $\Lambda_Q$ (independent of quark masses) is the scale parameter of the gauge coupling constant. All quark masses are small, $0 < m_l \leq m_h \ll \Lambda_Q$.

So, the whole theory can be considered as being defined by three numbers $N_c$, $N_F$, $N_l$ and by three dimensional parameters $\Lambda_Q$, $m_l$, $m_h$ (i.e. all dimensional observables will be expressed through these three).

As will be shown below, within the dynamical scenario used, there are different phase states in this theory at different values of parameters $r = m_l/m_h \leq 1$, $N_l$, and $N_F$:

a) the DC$_l$ – DC$_h$ phase appears at $m_h^{\mathrm{pole}} \ll \mathcal{M}_h \ll \mathcal{M}_l \ll \Lambda_Q$ in both cases $N_l > N_c$ and $N_l < N_c$ ($m_h^{\mathrm{pole}}$ is the perturbative pole mass of the $h$-quarks);

b) the DC$_l$ – HQ$_h$ phase appears at $\mathcal{M}_h \ll \mathcal{M}_l \ll m_h^{\mathrm{pole}} \ll \Lambda_Q$, and at $N_l > N_c$ only;
c) the another regime of the DC_l – HQ_h phase appears at $\mathcal{M}_\text{ch}^h \ll m_h^{\text{pole}} \ll \mathcal{M}_\text{ch}^l \ll \Lambda_Q$, and in both cases $N_l > N_c$ and $N_l < N_c$;
d) the Higgs_l – DC_h or Higgs_l – HQ_h- phases appear at $\mathcal{M}_\text{ch}^l \gg \Lambda_Q$, and at $N_l < N_c$ only.

It is implied that a reader is familiar with the previous paper [1], because all the results from [1] are used essentially in this paper.

The paper is organized as follows. The properties of the DC_l – DC_h phase are considered in section 2. The phase DC_l – HQ_h (in two regimes) is considered in sections 3 and 4. The phases Higgs_l – DC_h and Higgs_l – HQ_h with higgsed l-quarks are considered in section 5. The section 6 contains a short conclusion.

2. The DC_l – DC_h phase.

Let us recall first the effective Lagrangian for equal mass quarks and just below the physical threshold at $\mu < \mu_H = \mathcal{M}_\text{ch}$, after the evolution of all quark degrees of freedom has been finished [1] (b_0 = 3N_c - N_F, $\overline{N_c} = N_F - N_c$, see also the footnote 5 in [1]):

$$L = \int d^2 \theta d^2 \overline{\theta} \left\{ \text{Tr} \sqrt{\Pi^\dagger \Pi} + Z_Q \text{Tr} \left( Q^\dagger e^V Q + \overline{Q}^\dagger e^{-V} \overline{Q} \right) \right\} + \int d^2 \theta \left\{ \frac{2\pi}{\alpha(\mu)} S + W_Q \right\},$$

$$W_Q = \left( \frac{\text{det} \Pi}{\Lambda_{b_0}^{\text{ch}}} \right)^{1/N_c} \left\{ \text{Tr} \left( \overline{Q}^\dagger \Pi^{-1} Q \right) - N_F \right\} + \text{Tr} \left( m_Q \Pi \right), \quad Z_Q = \frac{m_Q}{\mathcal{M}_\text{ch}}.$$ (1)

Here: $\langle m_Q \rangle_i^T \equiv m_Q(\mu = \Lambda_Q)_i^T$, where $m_Q(\mu)_i^T$ are the running quark masses, and $\langle \Pi^i_j \rangle = \langle (\overline{Q} Q^i)_\mu = \Lambda_Q \rangle \equiv (\mathcal{M}_\text{ch}^2)^{\frac{i}{2}}_j$. For equal quark masses: $\langle m_Q \rangle = m_Q \delta^i_j$, $\langle \mathcal{M}_\text{ch}^2 \rangle = \mathcal{M}_\text{ch}^2 \delta^i_j$, $m_C \mathcal{M}_\text{ch}^2 = \langle S \rangle = (\Lambda_{b_0}^{\text{ch}} \text{det} m_Q)^{1/N_c}$; (here and everywhere below for DC-phases: as for concrete forms of the pion Kahler terms , see the footnote 5 in [1]).

Well above the highest physical threshold, $\mu_H \ll \mu \ll \Lambda_Q$, the quark fields $\overline{Q}, Q$ describe the original quarks with the small running current masses $m_Q(\mu)$, while below the threshold they become the fields of heavy quarks with the large constituent masses $\mu_C = \mathcal{M}_\text{ch}$.  

The fields $\Pi$ are defined as ”the light part of $\overline{Q} Q$”. i.e. well above the threshold, when the large constituent mass of quarks is not yet formed, $\Pi$ and $\overline{Q} Q$ are both the same living diquark operator of light quarks, so that $\overline{Q} \Pi^{-1} Q$ is a unit c-number matrix, and the projector $P = \text{Tr} \left( \overline{Q} \Pi^{-1} Q \right) - N_F = 0$. Moreover, the term $(\text{det} \Pi/\Lambda_{b_0}^{\text{ch}})^{1/N_c}$ is dominated by contributions of light quantum quark fields, and represents not a constant mass, but a living interaction. But below the threshold, at $\mu < \mathcal{M}_\text{ch}$, after the large constituent mass $\mathcal{M}_\text{ch}$ appeared, the light $\Pi$ and heavy $\overline{Q} Q$ become quite different, so that $P$ becomes a non-trivial

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1 The Konishi anomaly [2] for the canonically normalized constituent quark fields $C = Q/Z_Q^{1/2}$ and $\overline{C} = \overline{Q}/Z_Q^{-1/2}$ looks as: $\langle \overline{C} C \rangle = \langle S \rangle / \mu_C$. But the form of its explicit realization is a matter of convention. One convention is that it is realized directly through the one-loop triangle diagram with the heavy constituent quarks forming a loop and emitting two gluinos. Second convention is that the one loop constituent quark contributions into the vacuum polarization are transferred to the gluon kinetic term at a first stage, and then there appears a term $S \ln \mu_C$, while the quark term in $W_Q$ in (1) has to be used now for calculations with the valence constituent quarks only. The Konishi anomaly originates then from this vacuum polarization term and looks now as: $\langle \overline{C} C \rangle = \langle \partial/\partial \mu_C (S \ln \mu_C) \rangle = \langle S \rangle / \mu_C.$
nonzero term. Besides, below the threshold all $N_F^2$ fields $\Pi$ become "frozen", in a sense that all of them contain the large $c$-number vacuum part $\mathcal{M}_{ch}^{\Pi}$ and the light quantum pion fields $\pi$ with the small masses $m_Q$, whose contributions to amplitudes are smaller, $|\pi| \leq \mu < \mathcal{M}_{ch}$. As a result, the whole term $(\det \Pi / \Lambda_{b_Q}^{(0)})^{1/\mathcal{N}_c}(Z_Q \Pi)^{-1}$ in $W_Q$ is dominated now by the $c$-number vacuum part which becomes the large constituent mass $\mathcal{M}_{ch}$ of the quark fields $Q$ and $\overline{Q}$.

Let us start with $m_l = m_h$ and begin to make $m_h > m_l$, so that a gap appears between $\mathcal{M}_{ch}^l > \mathcal{M}_{ch}^h$.

\[
\left( \mathcal{M}_{ch}^l \right)^2 = \frac{1}{m_l} \left( \Lambda_{Q}^{b_Q} \det m \right)^{1/\mathcal{N}_c} = \Lambda_{Q}^{b_Q/\mathcal{N}_c} \frac{m_1}{m_l}(N_1-N_0)/\mathcal{N}_c m_h^{(N_F-N_0)/N_c},
\]

\[
\left( \mathcal{M}_{ch}^h \right)^2 = \frac{1}{m_h} \left( \Lambda_{Q}^{b_Q} \det m \right)^{1/\mathcal{N}_c} = \Lambda_{Q}^{b_Q/\mathcal{N}_c} \frac{m_1}{m_h} \frac{N_0}{N_c} m_h^{(N_F-N_0)/N_c} = \frac{m_l}{m_h} \left( \mathcal{M}_{ch}^l \right)^2.
\]

Clearly, at scales $\mu \gg \mathcal{M}_{ch}^l$ the large constituent masses $\mu_C = \mathcal{M}_{ch}^l$ and $\mu_C = \mathcal{M}_{ch}^h$ are not yet formed, and all quarks behave as perturbative massless particles. So, the fields $\Pi$ are not yet frozen, the factor $(\det \Pi)^{1/\mathcal{N}_c}$ in (1) is really $(\det \overline{Q}Q)^{1/\mathcal{N}_c}$ and it is still a living interaction, not a mass. As a result, there is still no difference between the (light at lower scales) fields $\Pi$ and (heavy at lower scales) fields $\overline{Q}Q$. Therefore, the projector $\mathcal{P}$ in curly brackets in (1) is still zero:

\[
\mathcal{P} = \text{Tr} \left( \overline{Q}Q^{-1}Q \right) - \mathcal{N}_F = 0, \quad \mu > \mathcal{M}_{ch}^l > \mathcal{M}_{ch}^h.
\]

Now, the main point is that the projector $\mathcal{P}$ begins to be nonzero only after the decreasing scale $\mu$ crosses the physical threshold at $\mu \sim \mathcal{M}_{ch}^h < \mathcal{M}_{ch}^l$ (and not before, at $\mu \sim \mathcal{M}_{ch}^l > \mathcal{M}_{ch}^h$), where the mass gap between the heavy constituent quarks $(\overline{Q}_h, Q_h)^{(const)}$ with the masses $\mathcal{M}_{ch}^h$ and the light pions $\Pi_h^{\Pi} = (\overline{Q}_h, Q_h)^{(light)}$ with the masses $\sim m_h$ appears and "begins to work", so that the fields $Q_h, \overline{Q}_h$ become frozen. Before this, at $\mu > \mathcal{M}_{ch}^h$, the constituent mass $\mathcal{M}_{ch}^h$ is not yet formed and the operator $\Pi_h^{\Pi}$ is not yet frozen and represents rather two still living light quarks $\overline{Q}_h, Q_h$, whose quantum part still dominates over its $c$-number vacuum part. So, in the first term in $W_Q$ the common factor $(\det \Pi)^{1/\mathcal{N}_c}$ is not yet frozen completely, and still describes some interaction, not a mass. Therefore, the constituent masses are not yet formed not only for the $\overline{Q}_h, Q_h$ quarks, but also for the $\overline{Q}_l, Q_l$ quarks. This shows that the very presence of still living perturbative light quarks $\overline{Q}_h, Q_h$ at $\mathcal{M}_{ch}^h < \mu < \mathcal{M}_{ch}^l$ prevents also the quarks $\overline{Q}_l, Q_l$ from acquiring the large constituent mass $\mathcal{M}_{ch}^l$. So, nothing happens yet at $\mu \sim \mathcal{M}_{ch}^l$ and the perturbative regime does not stop here, but continues down to $\mu \sim \mathcal{M}_{ch}^h$. This is the real physical threshold $\mu_H$, and the nonzero non-perturbative contributions to the quark superpotential appear only after crossing this region, and they appear simultaneously for all flavors.\(^3\)

\(^2\) But, to remain in the same DC - phase for all flavors, there will be a restriction on the values of $m_l$ and $m_h$, such that $r = m_l/m_h$ can’t be too small. The explicit form of this restriction will be presented below.

\(^3\) In a sense, the constituent quarks can be thought of as the extended solitons. And this shows also that the characteristic size of the heavier constituent quarks $\overline{Q}_h, Q_h$ is not $R_l \sim 1/\mathcal{M}_{ch}^l$, but larger: $R_l \sim 1/\mathcal{M}_{ch}^h > 1/\mathcal{M}_{ch}^l$, this is typical for a soft soliton, whose size is much larger than its Compton wavelength, $R_{sol}^{(soft)} > 1/M_{sol}^{(soft)}$. I.e., the size $R_l$ is the same as the size $R_h$ of the lighter constituent quarks $\overline{Q}_h, Q_h$ : $R_h \sim 1/\mathcal{M}_{ch}^h$, which are, in this sense, the hard solitons.
So, at \( \mu < \mathcal{M}_{ch}^{h} \), instead of (1), the effective Lagrangian takes now the form:

\[
L = \int d^2 \theta \, d^2 \overline{\theta} \left\{ \text{Tr} \sqrt{\Pi^\dagger \Pi} + Z_l \text{Tr}_l \left( Q^l e^V Q \right) + Z_h \text{Tr}_h \left( Q^h e^{-V} Q \right) + \left( Q \rightarrow \overline{Q} \right) \right\} + 
+ \int d^2 \theta \left\{ -\frac{2\pi}{\alpha(\mu)} S + W_Q \right\}, \quad W_Q = \left( \frac{\text{det} \Pi}{\Lambda_Q^{b_0}} \right)^{1/N_c} \left\{ \text{Tr} \left( \overline{Q} \Pi^{-1} Q \right) - N_F \right\} + \text{Tr}(m \Pi),
\]

\[
Z_l = \frac{m_l}{\mathcal{M}_{ch}^{l}}, \quad Z_h = \frac{m_h}{\mathcal{M}_{ch}^{h}}.
\]

(5)

Here: \( \Pi \) is the total \( N_F \times N_F \) matrix of all pions, and \( \overline{Q}, Q \) with \( l \) or \( h \) - flavors are the constituent quarks with the masses \( \mathcal{M}_{ch}^{l} \) or \( \mathcal{M}_{ch}^{h} \), respectively.

After integrating out all heavy constituent quarks (this leaves behind a large number of hadrons made of constituent quarks which are weakly confined, the string tension \( \sqrt{\sigma} \sim \Lambda_{YM} \ll \mathcal{M}_{ch}^{h} < \mathcal{M}_{ch}^{l} \)) and proceeding in the same way as in \([1]\), one obtains the same form as in \([1]\):

\[
L = \int d^2 \theta \, d^2 \overline{\theta} \left\{ \text{Tr} \sqrt{\Pi^\dagger \Pi} \right\} + \int d^2 \theta \left\{ -\frac{2\pi}{\alpha_{YM}(\mu, \Lambda_L)} S - N_F \left( \frac{\text{det} \Pi}{\Lambda_Q^{b_0}} \right)^{1/N_c} + \text{Tr}(m \Pi) \right\},
\]

\[
\Lambda_L^3 = \left( \frac{\text{det} \Pi}{\Lambda_Q^{b_0}} \right)^{1/N_c}, \quad \langle \Lambda_L \rangle = \Lambda_{YM} \ll \mu \ll \mathcal{M}_{ch}^{h}.
\]

(6)

So, the only difference with the case of the equal quark masses is that now the masses entering \( \text{Tr}(m \Pi) \) are not equal.

Proceeding further as in \([1]\) and going through the Veneziano-Yankielowicz (VY) procedure for gluons \([3]\), one obtains that there is a large number of gluonia with masses \( M_{gl} \sim \Lambda_{YM} \ll \mathcal{M}_{ch}^{l} \), and the lightest particles are the pions with the Lagrangian:

\[
L_\pi = \int d^2 \theta \, d^2 \overline{\theta} \left\{ \text{Tr} \sqrt{\Pi^\dagger \Pi} \right\} + \int d^2 \theta \left\{ -N_c \left( \frac{\text{det} \Pi}{\Lambda_Q^{b_0}} \right)^{1/N_c} + \text{Tr}(m \Pi) \right\}, \quad \mu \ll \Lambda_{YM}.
\]

(7)

The pion masses are proportional to a sum of their two quark masses, i.e.: \( M_\pi^{(ll)} = c_o 2m_l, \, M_\pi^{(hh)} = c_o 2m_h, \, M_\pi^{(lh)} = c_o (m_l + m_h) \), where \( c_o \) is a constant \( O(1) \).

Clearly, when the quark masses become equal, \( m_h \rightarrow m_l \), the Lagrangian (7) matches smoothly those in (1) (and vice versa). This is as it should be, until both types of quarks remain in the same DC - phase.

On the whole, it is seen that starting with the case of the equal quark masses and splitting them smoothly, one obtains very similar results. The only essential restriction is that theory has to stay in the DC_\( l - DC_\chi \) phase. And the only new non-trivial point is that there

\[^4\text{It is only worth noting that because there is only one common threshold } \mu_H = \mathcal{M}_{ch}^{h} \text{ for all flavors, the renormalization factors } Z_\sigma \text{ of all } N_F^2 \text{ pions are the same: } Z_\sigma = z_Q^{-1}(\Lambda_Q, \, \mathcal{M}_{ch}^{h}) \equiv z_Q^{-1}, \text{ where } z_Q \ll 1 \text{ is the perturbative renormalization factor of the massless quark, see \([1]\).}\]
is only one common physical threshold $\mu_H$ where the non-perturbative effects turn on and change the form of the Lagrangian, and this threshold is determined by the smallest diquark condensate $\mu_H = M_{\text{ch}}^h < M_{\text{ch}}^t$, see (2),(3).

Let us write finally the conditions for theory to be in the DC$_t$–DC$_h$ phase. When going down from $\mu \sim \Lambda_Q$, the massless perturbative evolution will be stopped either at $\mu_H = M_{\text{ch}}^h$ if $M_{\text{ch}}^h > m_h^{\text{pole}}$, or at $\mu_H = m_h^{\text{pole}}$ if $m_h^{\text{pole}} > M_{\text{ch}}^h$. So, the h-quarks will be in the DC$_h$ phase at $M_{\text{ch}}^h > m_h^{\text{pole}}$ and in the HQ$_h$ phase at $M_{\text{ch}}^h < m_h^{\text{pole}}$. The phase transition occurs at $M_{\text{ch}}^h \sim m_h^{\text{pole}}$. Using (3), one obtains that the DC$_h$ phase persists from $r = (m_t/m_h) = 1$ down to $r > r_1$:

$$M_{\text{ch}}^h = m_h^{\text{pole}} = m_h \left( \frac{\Lambda_Q}{m_h} \right)^{\gamma_+} \to r = \frac{m_t}{m_h} = r_1 \equiv \left( \frac{m_h}{\Lambda_Q} \right)^{\sigma} \ll 1, \quad \sigma = \frac{1}{N_t} \left[ \frac{2N_c}{1 + \gamma_+} - (N_F - N_c) \right],$$

$$3N_c/2 < N_F < 3N_c : \quad \gamma_+ = \frac{2N_c - N_F}{N_F - N_c} \quad \to \quad \sigma = \frac{N_F - N_c}{N_t}. \quad (8)$$

In (8) : $\gamma_+$ is the quark anomalous dimension. It is known in the conformal window while, to have definite answers, the value $\gamma_+ = (2N_c - N_F)/(N_F - N_c)$ used in [1] for $N_c < N_F < 3N_c/2$ is used also here and below in the text.

However, (8) is not the only condition, as if $r = m_t/m_h$ will be too small at $N_t < N_c$, then $M_{\text{ch}}^t$ will become larger than $\Lambda_Q$ and the l-quarks will be higgsed. This happens, see (2), at :

$$N_t < N_c : \quad M_{\text{ch}}^t = \Lambda_Q \quad \to \quad r = \frac{m_t}{m_h} = r_2 \equiv \left( \frac{m_h}{\Lambda_Q} \right)^{\frac{N_F - N_c}{N_c - N_t}} \ll 1. \quad (9)$$

On the whole, the theory is in the DC$_t$–DC$_h$ phase at :

- $a)$ $(m_t/m_h) > r_1$ for $N_t > N_c$;
- $b)$ $(m_t/m_h) > \max (r_1, r_2)$ for $N_t < N_c$;
- $r_2 > r_1$ at $N_t < N_o$, $N_o = \left\{ \begin{array}{ll} N_c b_0/2N_F & \text{for} \quad 3N_c/2 < N_F < 3N_c \text{ for } N_c < N_F < 3N_c/2. \end{array} \right.$

$$r_2 > r_1 \quad \text{at} \quad N_t < N_o, \quad N_o = \left\{ \begin{array}{ll} N_c b_0/2N_F & \text{for} \quad 3N_c/2 < N_F < 3N_c \text{ for } N_c < N_F < 3N_c/2. \end{array} \right. \quad (10)$$

3. The DC$_t$–HQ$_h$ phase : $M_{\text{ch}}^h \ll m_h^{\text{pole}}$, $N_t > N_c$.

a) $3N_c/2 < N_F < 3N_c$, $3N_c/2 < N_t < N_F$.

This is the different phase when the lighter l-quarks $Q^l$, $\overline{Q}_l$ are in the DC phase, while the heavier h-quarks $Q^h$, $\overline{Q}_h$ are in the HQ phase.

For definiteness, let us agree to use below the following procedure. Theory is defined at $\mu = \Lambda_Q$ by the values of quark masses : $m_l \equiv m_l (\mu = \Lambda_Q) \leq m_h \equiv m_h (\mu = \Lambda_Q) \ll \Lambda_Q$. Starting with $m_l = m_h$, the unequal quark masses will be obtained with $m_h$ staying intact, while $m_l$ will become smaller, $m_l \ll m_h \ll \Lambda_Q$.

5
At \( r \equiv m_l/m_h = 1 \) theory is in the DC\(_l\) – DC\(_h\) phase, with the highest physical scale \( \mu_H \) given by \( \mu_H = M_{ch}^h = M_{ch}^l \ll \Lambda_Q \). As was explained in section 2, the constituent masses of \( Q^l, \bar{Q}^l \) quarks can’t be formed alone, but only after all flavors will be frozen. So, as \( r \) begins to decrease, the highest physical scale \( \mu_H \) is determined by a competition between \( \mathcal{M}_{ch}^h < \mathcal{M}_{ch}^l \) and the pole mass \( m_{h}^\text{pole} \) of \( Q^h, \bar{Q}_h^h \) quarks, \( m_{h}^\text{pole} = m_h(\mu = m_h^\text{pole}) \).

The DC\(_l\) – DC\(_h\) phase persists until \( \mathcal{M}_{ch}^h > m_h^\text{pole} \), while at \( \mathcal{M}_{ch}^h < m_h^\text{pole} \) the coherent condensate of \( Q^h, \bar{Q}_h^h \) quarks can’t be maintained any more, so that there is a phase transition from the DC\(_l\) – DC\(_h\) phase to the DC\(_l\) – HQ\(_h\) one. This happens at \( r \sim r_1 \ll 1 \), see (8).

Although at \( r < r_1 \) theory is in the DC\(_l\) – HQ\(_h\) phase, there are two different regimes (see section 4 below), depending on whether \( r < r_1^* \ll r_1 \), or \( r_1^* < r < r_1 \), with \( r_1^* \) determined by

\[
\mathcal{M}_{ch}^l = m_h^\text{pole} \rightarrow r = r_1^* = \left( \frac{m_h}{\Lambda_Q} \right)^\rho \ll r_1 \ll 1, \quad \rho = \frac{1}{N_l - N_c} \left[ \frac{2N_c}{1 + \gamma_+} - (N_F - N_c) \right]. \tag{11}
\]

The regime at \( r < r_1^* \ll r_1 \) is much simpler and is considered first in this section. So, let us take \( r \ll r_1^* \) and consider the properties of this DC\(_l\) – HQ\(_h\) phase. Here, the highest physical scale \( \mu_H \) is given by the pole mass \( m_h^\text{pole} = \Lambda_Q(m_h/\Lambda_Q)^{1/(1+\gamma_+)} \ll \Lambda_Q \) of the heavier quarks \( Q^h, \bar{Q}_h^h \), see (8).

The condition \( r \ll r_1^* \), i.e. \( \mathcal{M}_{ch}^l \ll m_h^\text{pole} \), see (11), means that even if \( Q^l, \bar{Q}_l^l \) quarks were trying to freeze in the threshold region around \( \mu \sim m_h^\text{pole} \) by forming the largest possible constituent mass \( m_C^l = \mathcal{M}_{ch}^l \), this is impossible if \( \mathcal{M}_{ch}^l \ll m_h^\text{pole} \), as even this mass will be too small for freezing. Therefore, no non-perturbative effects turn on at \( \mu \sim m_h^\text{pole} \) in this case, and the region \( \mu \sim m_h^\text{pole} \) is crossed in the pure perturbative regime.

At \( \mu < m_h^\text{pole} \) the heavy \( h \)-quarks decouple from the lower energy theory and can be integrated out. What remains, is the lower energy theory with \( N_c \) colors and \( N_l > 3N_c/2 \) light flavors \( Q^l, \bar{Q}_l^l \), which at \( \mu_H' < \mu \ll \mu_H = m_h^\text{pole} \) will be also in the conformal regime (\( \mu_H' \) is the new highest physical scale of this lower energy theory). Let us denote by \( \hat{\Lambda}_Q \) the scale parameter of the new gauge coupling. Its value can be found from the following considerations. At \( m_h^\text{pole} < \mu \ll \hat{\Lambda}_Q \) the original coupling \( \alpha(\mu) \) is already frozen at the value \( \alpha_1^* = O(1) \). At \( \mu \ll m_h^\text{pole} \) the new coupling will be also frozen at a new value \( \alpha_2^* = O(1), \alpha_2^* > \alpha_1^* \). So, going from \( \mu \ll m_h^\text{pole} \) up to \( \mu \sim m_h^\text{pole} \), the coupling of the lower energy theory becomes living in the interval \( \delta \mu \sim m_h^\text{pole} \) around \( \mu = m_h^\text{pole} \), where it decreases significantly from \( \alpha_2^* \) to \( \alpha_1^* \). This is only possible if the scale factor \( \hat{\Lambda}_Q \) of the lower energy theory is \( \hat{\Lambda}_Q \sim m_h^\text{pole} \).

So, at \( \mu < \hat{\Lambda}_Q \) we remain with \( N_c \) colors, \( N_l > 3N_c/2 \) light flavors with the small current masses \( \hat{m}_l \equiv m_l(\mu = \hat{\Lambda}_Q) = \hat{z}_Q^{-1}(\Lambda_Q, m_h^\text{pole}) m_l \ll \hat{\Lambda}_Q \), and the coupling \( \alpha(\mu) \) with the scale parameter \( \hat{\Lambda}_Q = m_h^\text{pole} \). Moreover, the value of the diquark condensate of \( l \)-flavors is, see (3), (8):

\[
\langle (\bar{Q}lQ^l)_{\mu = \hat{\Lambda}_Q} \rangle = \hat{z}_Q^{4} (\hat{\mathcal{M}}_{ch}^l)_{\mu = \hat{\Lambda}_Q}^2, \quad \hat{\mathcal{M}}_{ch}^l = z_Q^{1/2} (\Lambda_Q, m_h^\text{pole}) \mathcal{M}_{ch}^l \ll \hat{\Lambda}_Q = m_h^\text{pole}, \tag{12}
\]

\[
z_Q(\Lambda_Q, m_h^\text{pole}) = \left( \frac{m_h^\text{pole}}{\Lambda_Q} \right)^{\gamma_+} = \left( \frac{m_h}{\Lambda_Q} \right)^{\gamma_+}, \quad \gamma_+^{\text{conf}} = b_o/N_F.
\]

The properties of such a theory have been described in [II] - it is in the DC\(_l\) phase. Its highest physical scale is \( \mu_H' = \hat{\mathcal{M}}_{ch}^l \ll \hat{\Lambda}_Q \), so that at \( \hat{\mathcal{M}}_{ch}^l < \mu \ll \hat{\Lambda}_Q \) it is in the conformal
regime, while below the threshold at $\mu \sim \tilde{\mathcal{M}}_{\text{ch}}^{l}$ the quarks $Q^l$, $\bar{Q}_l$ acquire the constituent masses $\mu_{Q}^l = \mathcal{M}_{\text{ch}}^{l}$ and there appear $N_c^2$ light pions. The low energy Lagrangian of these pions at $\mu \ll \Lambda_{YM} = (\Lambda_{\text{QCD}}^3 N_c - N_l) \det \hat{m}_l)^{1/3N_c} = (\Lambda_Q^3 N_c - N_F \det m)^{1/3N_c}$ is:

$$L_\pi = \int d^2\theta d^2\bar{\theta} \sqrt{\text{Tr} \Pi_l^\dagger \Pi_l} + \int d^2\theta \left\{ -(N_l - N_c) \left( \frac{\det \Pi_l}{\Lambda_{\text{QCD}}^{3N_c - N_l}} \right)^{1/(N_l - N_c)} + \hat{m}_l \text{Tr} \hat{\Pi}_l \right\}. \quad (13)$$

The normalization of the pion fields $\hat{\Pi}_l \equiv \langle \bar{Q}_l Q^l \rangle_{\mu = \Lambda_Q}$, $\langle \Pi_l \rangle = (\Lambda_{\text{QCD}}^{3N_c - N_l} m_l^{\dagger N_l - N_c})^{1/N_c}$ is the most natural one, from the viewpoint of the lower energy theory. But it is also useful to rewrite (13) with the "old normalization" of fields at $\mu = \Lambda_Q$, $\Pi_l \equiv \langle \bar{Q}_l Q^l \rangle_{\mu = \Lambda_Q}$, $\langle \Pi_l \rangle = \langle S \rangle / m_l \equiv \Lambda_{YM}^{3N_c} m_l = m_l^{-1}(\Lambda_Q^3 \det m)^{1/N_c}$. Then it looks as :

$$L_\pi = \int d^2\theta d^2\bar{\theta} \left\{ z_Q(\Lambda_Q, m_{ch}^{\text{pole}}) \sqrt{\text{Tr} \Pi_l^\dagger \Pi_l} \right\} + \int d^2\theta \left\{ -(N_l - N_c) \left( \frac{\det \Pi_l}{\Lambda_{\text{QCD}}^{3N_c - N_l}} \right)^{1/(N_l - N_c)} + m_l \text{Tr} \Pi_l \right\}, \quad z_Q(\Lambda_Q, m_{ch}^{\text{pole}}) = \left( \frac{m_{ch}^{\text{pole}}}{\Lambda_Q} \right)^{b_0/3N_c} \ll 1. \quad (14)$$

On the whole, the mass spectrum includes in this case : a) a large number of heaviest hh-hadrons with their mass scale $\sim m_{ch}^{\text{pole}}$, b) a large number of $ll$- mesons with masses $\sim \mathcal{M}_{\text{ch}}^{(l)}$ made of non-relativistic quarks $Q^l$, $\bar{Q}_l$ with the constituent masses $\mathcal{M}_{\text{ch}}^{(l)} \ll m_{ch}^{\text{pole}}$, c) a large number of hybrid hl-mesons made of above constituents (all quarks are weakly confined, the string tension is $\sqrt{\sigma} \sim \Lambda_{YM} \ll \mathcal{M}_{\text{ch}}^{(l)} \ll m_{ch}^{\text{pole}}$), d) a large number of gluonia with masses $\sim \Lambda_{YM} = (\Lambda_{\text{QCD}}^{3N_c - N_F} \det m)^{1/3N_c} \ll \mathcal{M}_{\text{ch}}^{(l)}$, det $m \equiv m_l^{N_l} m_{ch}^{N_F - N_l}$, e) and finally, $N_c^2$ lightest l-pions $\hat{\Pi}_l$ with masses $M_\pi^l \sim \hat{m}_l = \hat{z}_Q^l(\Lambda_Q, m_{ch}^{\text{pole}}) m_l \ll \Lambda_{YM}$.

b) $3N_c/2 < N_F < 3N_c$, $N_c < N_l < 3N_c/2$.

The difference with the case "a" above is that at $\mu < \mu_H = m_{ch}^{\text{pole}}$ and after the heaviest quarks $Q^h$, $\bar{Q}_h$ have been integrated out, the lower energy theory is not in the conformal regime at $\mu_H < \mu < m_{ch}^{\text{pole}}$, but in the strong coupling one, see [1]. I.e., its new coupling grows in a power-like fashion at $\mu \ll \mu_H = m_{ch}^{\text{pole}}$. This allows to determine its new scale parameter $\Lambda'$, from matching of couplings at $\mu = \mu_H = m_{ch}^{\text{pole}}$, where both are $O(1)$. This is only possible with $\Lambda' = \mu_H = m_{ch}^{\text{pole}} \equiv \hat{\Lambda}_Q$. Therefore, at $\mu < \hat{\Lambda}_Q$ we remain with $N_c$ colors, $N_c < N_l < 3N_c/2$ light flavors with the current masses $\hat{m}_l \equiv m_l(\mu = \hat{\Lambda}_Q) \ll \hat{\Lambda}_Q$, and the coupling with the scale parameter $\hat{\Lambda}_Q$. All this is exactly as it was in the case "a" above, only the value of $N_l$ is smaller now.

As was explained in [1], only the perturbative behavior in the interval of scales $\mu_H = \mathcal{M}_{\text{ch}}^{l} < \mu < \hat{\Lambda}_Q$ differs in this case from the conformal behavior in the case "a" above, while at $\mu < \mathcal{M}_{\text{ch}}^{l}$ all properties and mass spectra are the same. In particular, the lowest energy pion Lagrangian is the same as in (13), (14), etc.

c) $N_c < N_F < 3N_c/2$, $N_c < N_l < N_F$.

In this case the original theory (at $\mu_H = m_{ch}^{\text{pole}} < \mu < \Lambda_Q$) and the lower energy one (at $\mu < \mu_H$) are both in the strong coupling regime. Their couplings $\alpha_{\pm}(\mu)$ will be matched at
\[ \mu = \mu_H = m_h^{\text{pole}}, \text{where} \ m_h^{\text{pole}} \text{is the pole mass of} \ Q^h, \ \overline{Q}^h \text{quarks:} \ m_h^{\text{pole}} = m_h(\mu = m_h^{\text{pole}}). \]  

Because \( m_h^{\text{pole}} \ll \Lambda_Q \), the upper (i.e. original) coupling \( \alpha_+(\mu = m_h^{\text{pole}}) \) is parametrically large, and so will be \( \alpha_-^{(\mu = m_h^{\text{pole}})}. \) Therefore, it is clear that its scale parameter \( \Lambda' \gg m_h^{\text{pole}}. \)

To obtain definite expressions, let us make a (sufficiently weak) assumption that at \( N_c < N_F < 3N_c/2 \) the quark perturbative factor \( z_Q^+(\Lambda, \mu) \) and the coupling \( a_+ = N_c\alpha_+/(2\pi) \) of the higher energy theory, and \( z_Q^-(\Lambda', \mu), \ a_- = N_c\alpha_-/(2\pi) \) of the lower energy one behave as

\[ z_Q^+(\Lambda, \mu) = \left( \frac{\mu}{\Lambda_Q} \right)^\gamma_+ \ll 1, \quad a_+(\mu) = \left( \frac{\Lambda_Q}{\mu} \right)^\nu_+ \gg 1, \quad \nu_+ = \frac{N_F\gamma_+ - b_0}{N_c} > 0, \quad \mu \ll \Lambda_Q, \]

\[ z_Q^-(\Lambda', \mu) = \left( \frac{\mu}{\Lambda'} \right)^\gamma_- \ll 1, \quad a_-(\mu) = \left( \frac{\Lambda'}{\mu} \right)^\nu_- \gg 1, \quad \nu_- = \frac{N_F\gamma_- - b_0'}{N_c} > \nu_+, \quad \mu \ll \Lambda', \]

\[ b_0 = (3N_c - N_F), \quad b_0' = (3N_c - N_l), \quad \Lambda' \ll \Lambda_Q. \quad (15) \]

Because \( \nu_- > \nu_+ > 0 \), it is seen from (16) that \( m_h^{\text{pole}} \ll \Lambda' \ll \Lambda_Q. \quad (16) \]

Therefore, after the heaviest quarks \( Q^I, \overline{Q}^I \) have been integrated out at \( \mu < m_h^{\text{pole}} \), we have now \( N_c \) colors, \( N_c < N_l < 3N_c/2 \) flavors and the gauge coupling with the scale parameter \( \Lambda', \ m_h^{\text{pole}} \ll \Lambda' \ll \Lambda_Q \), determined from (16). The value of the current mass \( m_l' \) of \( Q^I, \overline{Q}^I \) quarks and the pion fields \((\Pi_l')^I = (\overline{Q}^I Q^I)^{(\text{light})}_{\mu = \Lambda'}\) normalized at \( \mu = \Lambda' \) look as:

\[ m_l' \equiv m_l(\mu = \Lambda') = z_Q^-(\Lambda', m_h^{\text{pole}}) \hat{m}_l, \quad z_Q^-(\Lambda', m_h^{\text{pole}}) = \left( \frac{m_h^{\text{pole}}}{\Lambda'} \right)^\gamma_+ \ll 1, \quad \hat{m}_l = m_l \left( \frac{\Lambda_Q}{\Lambda'} \right)^{\frac{\gamma_+}{1+\gamma_+}}, \]

\[ (\Pi_l')^I_j = \delta^I_\beta \Pi_j^l, \quad \hat{\Pi}_l = \frac{1}{z_Q^-(\Lambda', m_h^{\text{pole}})} \hat{\Pi}_l, \quad \hat{\Pi}_l = \Pi_l z_Q^+(\Lambda_Q, m_h^{\text{pole}}) = \Pi_l \left( \frac{m_h}{\Lambda_Q} \right)^{\frac{\gamma_+}{1+\gamma_+}}. \quad (17) \]

Therefore, the low energy pion Lagrangian will have the form (13), with the replacements: \( \Pi_l \to \Pi_l', \hat{m}_l \to m_l', \Lambda \to \Lambda'. \) The pion mass is now \( m_l' \). Being expressed through the pion fields \( \Pi_l \) normalized at \( \mu = \Lambda_Q \), its superpotential has the universal form (14), and only the \( Z^l_\pi \)-factor multiplying the Kahler term of \( l \)-pions is different, now it is:

\[ Z^l_\pi = \left( \frac{m_h}{\Lambda_Q} \right)^\Delta, \quad \Delta = \frac{\delta}{1+\gamma_+}, \quad \delta = \left[ \gamma_+ - \gamma_-(\frac{\nu_+}{\nu_-}) \right], \quad m_l' = m_l/Z^l_\pi. \quad (18) \]

So, in the case considered, there are in the mass spectrum: a) the heaviest \( h \)-hadrons with their mass scale \( \sim m_h^{\text{pole}} \) given by (16); b) the \( ll \)-mesons made of the non-relativistic

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5 As a concrete example, one can use the values from \( \Pi \): \( \gamma_+ = (2N_c - N_F)/(N_F - N_c), \quad \gamma_- = (2N_c - N_l)/(N_l - N_c), \quad \nu_+ = (3N_c - 2N_F)/(N_F - N_c), \quad \nu_- = (3N_c - 2N_l)/(N_l - N_c). \) The value of \( \Delta \) in (18) is \( 0 < \Delta = (N_F - N_l)/(3N_c - 2N_l) < 1/2 \) in this case.

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quarks $Q^l$, $\overline{Q}^l$ with the constituent masses $\mu'_c = (\Pi_l)^{1/2}$ (17); c) the hybrid $h\ell$-mesons made of the above constituents; d) the gluonia with the universal mass scale $\Lambda_{YM}$; d) and $N_l^2$ lightest $l$-pions with the masses $m'_l \ll \Lambda_{YM}$, see (18).

On the whole for this regime of the DC$_l$ – HQ$_h$ phase and $N_l > N_c$, the hierarchy of scales in the mass spectrum is always the same:
a) the largest masses are the pole masses $m_h^\text{pole} \ll \Lambda_Q$ of $Q^h$, $\overline{Q}^h$ quarks ; b) the next ones are the constituent masses $\mu'_C$ of $Q^l$, $\overline{Q}^l$ quarks, they are almost much smaller than $m_h^\text{pole}$, but their concrete values depend on the case considered ; c) the next one is the universal mass scale of gauge particles, it is always $\Lambda_{YM} = (\Lambda_Q^{\text{ho}} \det m)^{1/3N_c}$; d) the lightest are $N_l^2$ $l$-pions, their low energy Lagrangian has the universal form (14), but the value of the $Z_l^\prime$-factor in front of the Kahler term (and so their masses $M_n^l$) depends on the case considered.

4. The DC$_l$ – HQ$_h$ phase: $\mathcal{M}_{\text{ch}}^h \ll m_h^\text{pole} \ll M_{\text{ch}}^l$.

Let us consider now the most difficult regime with $r'_l \ll r \ll r_1$, i.e. $\mathcal{M}_{\text{ch}}^h \ll m_h^\text{pole} \ll M_{\text{ch}}^l$.

Let us trace the RG-flow when the running scale $\mu$ starts with $\mu = \Lambda_Q$ and decreases. As was argued in section 2, even the large value of the running coherent condensate $\mathcal{M}_{\text{ch}}^l(\mu)$ does not mean, by itself, that the large constituent mass $\mathcal{M}_{\text{ch}}^l(\mu)$ of $Q^l$, $\overline{Q}^l$ quarks is really already formed, as the projector $\mathcal{P}$ in (4) becomes nonzero only after decreasing $\mu$ reaches such a value $\mu_2$ that both flavors, $l$ and $h$, entering $\det(\overline{Q}Q)$ acquire masses larger than $\mu_2$ and become frozen. Therefore, the first point where this can happen in the DC$_l$ – HQ$_h$ phase with $\mathcal{M}_{\text{ch}}^h \ll m_h^\text{pole} \ll M_{\text{ch}}^l$ is the pole mass $m_h^\text{pole}$. So, there is the narrow threshold region $\mu_2 = m_h^\text{pole} / (\text{several}) < \mu < \mu_1 = (\text{several}) m_h^\text{pole}$ around $m_h^\text{pole}$, where the non-perturbative effects turn on at $\mu_1$ and saturate at $\mu_2$. In a sense, what will be going on in this transition region is qualitatively similar to those described in section 2 for the DC$_l$ – DC$_h$ phase, only the role played by the coherent condensate $\mathcal{M}_{\text{ch}}^h$ of the $Q^h$, $\overline{Q}^h$ quarks plays here their perturbative pole mass $m_h^\text{pole}$. Therefore, all flavors become frozen in the threshold region $\mu_2 < \mu < \mu_1$ around $m_h^\text{pole}$. The $Q^h$, $\overline{Q}^h$ quarks - because their evolution is stopped by their pole mass $m_h^\text{pole}$, while the $Q^l$, $\overline{Q}^l$ quarks - because their large constituent mass $\mathcal{M}_{\text{ch}}^l \gg m_h^\text{pole}$ is formed in this threshold region.

So, what form will the superpotential take at $\mu < m_h^\text{pole}$, after the non-perturbative RG-flow has finished, and all quark masses become frozen? (The heaviest are the constituent $Q^l$, $\overline{Q}^l$ quarks with masses $\mathcal{M}_{\text{ch}}^l$, the next ones are the $Q^h$, $\overline{Q}^h$ quarks with masses $m_h^\text{pole}$, and the lightest are the pions $\Pi_l$ with the masses $m_l$, plus all gluons which are still massless). Let us look at the superpotential in (1) or (5). Because there are only $\Pi_l$ - pions, while the $h$ - quarks are in the HQ phase and there is no difference between $\det(\overline{Q}^h Q^h)$ and $\Pi_h^h$, the $h$-quark contributions cancel in the projector $\mathcal{P} = \text{Tr} (\overline{Q} \Pi^{-1} Q) - N_F$, and it will have the form: $\mathcal{P} = \text{Tr} (\overline{Q} \Pi^{-1} Q^l) - N_l$. Now, what form can det $\Pi$ in (1) or (5) take at $\mu < m_h^\text{pole}$, after the evolution of all quark degrees of freedom has been finished and the quarks $Q^h$, $\overline{Q}^h$ have been integrated out? In other words, what their fields $\Pi^h_l = (\overline{Q}^h Q^h)$ will be substituted by in det $\Pi$? The only possible form is.

$$\Pi^h_l = (\overline{Q}^h Q^h) \rightarrow \left( m^{-1} \right)^h \left( \frac{\det \Pi_l}{\Lambda_Q^{\text{ho}} \det m_h} \right)^{1/(N_l-N_c)}, \quad (19)$$
In (21) the specific properties for the case considered are: a) the fields entering (1), (5) and (20) are normalized at \( \mu = \Lambda_Q \):

\[
L = \int d^2 \theta d^2 \bar{\theta} \left\{ \text{Tr} \sqrt{\Pi_l^2} \Pi_l + Z_l \text{Tr}_l \left( Q_l^\dagger e^{V} Q_l + Z_h \text{Tr}_h \left( Q_h^\dagger e^{V} Q_h \right) + \left( Q \to Q_h \right) \right) \right\} +
\int d^2 \theta \left\{ -\frac{2\pi}{\alpha(\mu)} S + W_Q \right\}, \quad Z_l = \frac{m_l}{M_{\text{ch}}}, \quad Z_h = \frac{m_h}{m_{h,\text{pole}}},
\]

\[
m_{h,\text{pole}} = \frac{m_h}{z_Q^+ (\Lambda_Q, m_{h,\text{pole}})} = m_h \left( \frac{\Lambda_Q}{m_h} \right)^{\frac{\gamma_+}{1+\gamma_+}} \gg m_h,
\]

\[
W_Q = \left( \frac{\det \Pi_l}{\Lambda_{Q_l}^{b_0} \det m_h} \right)^{1/(N_l-N_c)} \left\{ \text{Tr}_l \left( Q_l^{-1} \right) - N_l \right\} + m_h \text{Tr}_h (Q_h) + m_l \text{Tr} (\Pi_l).
\]

The meaning of (20) is the same as before in (1) or (5). All terms with the quark fields are retained only to recall the values of their masses and, besides, it is implied that they can be used, for instance, for some calculations where these quarks appear as valence ones. If one is not interested in all this at \( \mu < m_{h,\text{pole}} \), all quark terms in (20) can be omitted.

Now, let us write the explicit form of the inverse Wilsonian coupling \[4\] \( 2\pi/\alpha_W(\mu) \) in (20). The simplest way to obtain it is to write out the result of the overall RG-flow from \( \mu = \Lambda_Q \) down to \( \mu = m_{h,\text{pole}} / (\text{several}) \) (because the RG is a group). So, one obtains:

\[
\frac{2\pi}{\alpha_W(\mu)} = N_c \ln \left( \frac{\mu_2^3}{\Lambda_Q^3} \right) - \ln \left( \frac{\det \mu_l^3}{\Lambda_Q^{N_l}} \right) - N_h \ln \left( \frac{m_{h,\text{pole}}^3}{\Lambda_Q^{N_h}} \right) + N_l \ln \left( \frac{1}{Z_l} \right) + N_h \ln \left( \frac{1}{Z_h} \right).
\]

In (21) the specific properties for the case considered are: a) the \( Z_h \)-factor of the \( Q_h, \bar{Q}_h \) quarks is \( Z_h = m_h / m_{h,\text{pole}} \), because their mass \( m_h(\mu) \) started with the value \( m_h \) at \( \mu = \Lambda_Q \) and finished with the value \( m_{h,\text{pole}} \) at \( \mu = \mu_2 \), b) the constituent mass \( \mu_C^l \) of the \( Q_l, \bar{Q}_l \) quarks in (21) has the form, see (20):

\[
\left( \frac{\mu_C^l}{\Lambda_{Q_l}^{b_0} \det m_h} \right)^{1/(N_l-N_c)} \left( \Pi_l^{-1} \right)_i^j.
\]

\( \left( \Pi_l^{-1} \right)_i^j \) is determined uniquely by symmetries: a) the flavor symmetry \( SU(N_l)_L \times SU(N_l)_R \times SU(N_h)_L \times SU(N_h)_R \), b) by the R-charges of the higher energy theory: \( R(Q_l^l) = R(\bar{Q}_l) = R(Q_h^h) = R(\bar{Q}_h) = R(\Pi_l)/2 = (N_F - N_c)/N_F \), \( R(m_l) = R(m_h) = 2N_c/N_F \), c) by the R'-charges of the lower energy theory: \( R'(Q_l^l) = R'(\bar{Q}_l) = R'(\Pi_l)/2 = (N_l - N_c)/N_l \), \( R'(Q_h^h) = R'(\bar{Q}_h) = 1 \), \( R'(m_l) = 2N_c/N_l \), \( R'(m_h) = 0 \), d) the overall normalization in (19) is determined by the Konishi anomaly: \( m_h(\bar{Q}_h Q_h) = \langle S \rangle \), see also (2).
So, the coupling in (20) at $\Lambda_{YM} \ll \mu < \mu_2 = m_{h^{\text{pole}}}/(\text{several})$ is weak and has the form (see also section 2 in [1]):

$$\frac{2\pi}{\alpha_W(\mu, \Lambda_L)} = \frac{2\pi}{\alpha(\mu, \Lambda_L)} - N_c \ln \frac{1}{\theta_2(\mu, \langle \Lambda_L \rangle)} = 3N_c \ln \left( \frac{\mu}{\Lambda_L} \right),$$

and the Lagrangian at $\mu < \mu_2$ looks as:

$$L = \int d^2 \theta d^2 \bar{\sigma} \left\{ \text{Tr} \sqrt{\Pi^2} \Pi_I \right\} +$$

$$\int d^2 \theta \left\{ -\frac{2\pi}{\alpha(\mu, \Lambda_L)} S - N_I \left( \frac{\det \Pi_I}{\Lambda_{Q}^{\text{det}} m_h} \right)^{1/(N_I - N_C)} + \text{Tr}\left( m_i \Pi_I \right) \right\}.$$  (24)

It describes gluonia with the universal mass scale $M_{gl} \sim \Lambda_{YM}$ (interacting with the pions $\Pi_I$), and after integrating them out through the VY-procedure [3], one obtains finally the lowest energy Lagrangian of pions:

$$L = \int d^2 \theta \left\{ \text{Tr} \sqrt{\Pi_I^2} \Pi_I \right\} + \int d^2 \theta \left\{ -(N_I - N_C) \left( \frac{\det \Pi_I}{\Lambda_{Q}^{\text{det}} m_h} \right)^{1/(N_I - N_C)} + \text{Tr}\left( m_i \Pi_I \right) \right\}.$$  (25)

It describes $N_l^2$ $l$-pions $\Pi_I$ with the masses $\sim m_l$, and their superpotential has the standard universal form for the DC$_l -$ HQ$_h$ phase, see (14).

On the whole, the mass spectrum includes in this case: a) the $ll$-hadrons made of the $Q^i$, $\bar{Q}_i$-quarks with the constituent masses $\mu_C^i = M_{ch}^i \ll \Lambda_Q$, b) the $hh$-hadrons made of the non-relativistic $Q^h$, $\bar{Q}_h$-quarks with the pole masses $m_{h^{\text{pole}}} \ll M_{ch}^i$, c) the hybrid $hl$-hadrons made of the above constituents (all quarks are weakly confined, the string tension is $\sqrt{\sigma} \sim \Lambda_{YM} \ll m_{h^{\text{pole}}} \ll M_{ch}^i$), d) the gluonia with their universal mass scale $M_{gl} \sim \Lambda_{YM} \ll m_{h^{\text{pole}}}$, e) and finally, $N_l^2$ lightest $l$-pions with the masses $\sim m_l \ll \Lambda_{YM}$, and the superpotential (25) which is universal one for the DC$_l -$ HQ$_h$ phase. In a sense, this mass spectrum is similar to those described in section 3, the main difference is that the hierarchy $m_{h^{\text{pole}}} \gg \mu_C^i$ from section 3 is reversed here.

Finally, let us look how the mass spectrum changes on both sides around the phase transition at $r \sim r_1$, with $M_{ch}^i \sim m_{h^{\text{pole}}} \ll \Lambda_{ch}^i$, see (8) and section 2.

a) The h-flavors. In the DC$_l -$ DC$_h$ phase at $r > r_1$, there are many heavy $h$-hadrons made of the non-relativistic $Q^h$, $\bar{Q}_h$-quarks with the constituent masses $M_{ch}^h$, see (3), and $N_h^2$ light $h$-pions with the masses $\sim m_h$. As $r$ crosses $r_1$, the coherent condensate of the h-flavors breaks down and theory appears in the DC$_l -$ HQ$_h$ phase. The above $N_h^2$ light $h$-pions with the masses $\sim m_h$ disappear from the mass spectrum. At the same time, because $M_{ch}^h = m_{h^{\text{pole}}}$, the constituent masses $M_{ch}^h$ of $Q^h$, $\bar{Q}_h$ quarks are substituted smoothly by
their perturbative pole masses \( m_h^{\text{pole}} \), so that the mass spectrum of the heavy \( h \)-hadrons made now of the non-relativistic current quarks \( Q^h \), \( \overline{Q}^h \) changes smoothly.

b) The \( l \)-flavors. In the DC\(_l\) – DC\(_h\) phase at \( r > r_1 \), there are many heavy \( l \)-hadrons made of the \( Q^l \), \( \overline{Q}^l \) - quarks with the constituent masses \( M_{\text{ch}}^l \gg M_{\text{ch}}^h \), see (2), and \( N_l^2 \) lightest \( l \)-pions with the masses \( \sim m_l \ll m_h \). In the DC\(_l\) – HQ\(_h\) phase at \( r < r_1 \), all these \( l \)-hadrons and \( N_l^2 \) \( l \)-pions are still present in the spectrum and their masses remain the same.

c) The hybrid \( hl \)-flavors. In the DC\(_l\) – DC\(_h\) phase at \( r > r_1 \), there are many heavy \( hl \)-mesons with the masses \( (M_{\text{ch}}^h + M_{\text{ch}}^l) \) and corresponding \( hl \)-pions with the small masses \( (m_h + m_l) \). In the DC\(_l\) – HQ\(_h\) phase at \( r < r_1 \), these light hybrid pions are absent. As for the heavy hybrid mesons, their masses change smoothly from \( (M_{\text{ch}}^h + M_{\text{ch}}^l) \) to \( (m_h^{\text{pole}} + M_{\text{ch}}^l) \).

d) Finally, all gluons remain massless down to the scale \( \mu \sim \Lambda_{YM} \), and there is a large number of gluonia with the same masses \( M_{gl} \sim \Lambda_{YM} \) in both phases.

5. The Higgs\(_l\) – DC\(_l\) and Higgs\(_l\) – HQ\(_h\) phases. \( M_{\text{ch}}^l > \Lambda_Q \), \( N_l < N_c - 1 \).

There are only two different phases at \( N_l > N_c \) (because always \( M_{\text{ch}}^l < \Lambda_Q \), and the lighter quarks are never higgsed): a) DC\(_l\) – DC\(_h\), and b) DC\(_l\) – HQ\(_h\).

At \( N_l < N_c \), in addition to the above two phases, two new phases appear at \( M_{\text{ch}}^l > \Lambda_Q \), when the lighter \( l \)-quarks are higgsed, \( \langle Q^l \rangle = \langle \overline{Q}^l \rangle \neq 0 \), while the heavier quarks are either in the DC phase, or in the HQ phase.

So, let us take \( r \ll r_2 \) (see (9), but not too small, see below) and look for a mass spectrum in this phase. One can proceed in a close analogy with the case of the Higgs phase for \( N_F < N_c - 1 \) in \([1]\), the only difference is that not all flavors are higgsed now, only \( Q^l \), \( \overline{Q}^l \).

So, one begins with the scale of the large gluon mass, \( \mu = \mu_{gl} = g_{gl} M_{\text{ch}}^l \gg \Lambda_Q \), \( \delta_{gl}^l = 4\pi\alpha(\mu = \mu_{gl}) \ll 1 \). \( \langle Q^l \rangle_{\mu = \mu_{gl}} = \delta_l^l M_{\text{ch}}^l \), \( \langle \overline{Q}^l \rangle_{\mu = \mu_{gl}} = \delta_l^l M_{\text{ch}}^l \). The gauge symmetry \( SU(N_c) \) is broken down to \( SU(N_c - N_l) \) at this scale \( \mu = \mu_{gl} \) and \( (2N_l N_c - N_l^2) \) gluons become massive. The same number of degrees of freedom of \( Q^l \), \( \overline{Q}^l \) quarks acquire the same masses and become the superpartners of these massive gluons (in a sense, they can be considered as the heavy "constituent quarks"), and there remain \( N_l^2 \) light complex pion fields \( \tilde{\Pi}_l = \langle Q^l \overline{Q}^l \rangle_{\mu = \mu_{gl}} \), \( \langle \tilde{\Pi}_{\overline{l}} \rangle = (M_{\text{ch}}^l)^2 \). The value of \( m_l(\mu) \) at this scale is \( \tilde{m}_l \equiv m_l(\mu = \mu_{gl}) \) (this will be the \( l \)-pion mass), and similarly, the mass of \( h \)-quarks at this scale is \( \tilde{m}_h \equiv m_h(\mu = \mu_{gl}) \).

Besides, let us denote as \( \tilde{\Pi}_{hl} \) and \( \tilde{\Pi}_{lh} \) the hybrids (in essence, these are the \( h \)-quark fields \( Q^h_{\alpha l} \), \( \overline{Q}^h_{\alpha l} \) with broken colors \( a = 1...N_l \) ), while \( Q^h \), \( \overline{Q}^h \) will be now the \( h \)-quark fields with unbroken colors.

Consider first the case \( N_l < b_o/2 \), i.e. \( b'_o = (b_o - 2N_l) > 0 \). After integrating out all heaviest particles with masses \( \sim \mu_{gl} \) and proceeding in the same way as in \([1]\), one obtains the lower energy Lagrangian at the scale \( \mu \lesssim \mu_{gl} \):

\[
L = \int d^3 \theta \int d^2 \theta' \left\{ \text{Tr} \sqrt{\tilde{\Pi}^\dagger_\overline{l} \tilde{\Pi}_l} + \text{Tr}_h \left( \hat{Q}^l e^{\frac{\overline{Q}^l}{\mu}} \hat{Q} + (\hat{Q} \to \overline{Q}) \right) + \text{Tr} \left( \hat{\Pi}_{hl} \hat{\Pi}_{hl} \right) + \cdots \right\} + \int d^2 \theta \left\{ -\frac{2\pi}{\alpha(\mu, \Lambda)} \dot{S} + \tilde{m}_l \text{Tr} \hat{\Pi}_l + \tilde{m}_h \text{Tr}_h \left( \overline{Q} \dot{Q} \right) + \hat{m}_h \text{Tr} \left( \hat{\Pi}_{hl} \hat{\Pi}_{hl} \right) \right\},
\]

\[
\Delta b'_o = \frac{b_o}{z_{Q_o}^{N_l}} \text{det} \hat{\Pi}_l \left( \frac{z_{Q_o}^l}{z_{Q_o}} \right)^{N_l - N_l}, \quad b'_o = (b_o - 2N_l) > 0,
\] (26)
\[ z_Q = z_Q(\mu_{gl}, \Lambda_Q \mid N_c, N_F) = \frac{\hat{m}_l}{m_l}, \quad z_Q' = z_Q(\mu_{gl}, \langle \hat{\Lambda} \rangle \mid N_c - N_l, N_F - N_l) = \frac{\hat{m}_h}{m'_h}. \] (27)

Here: \( \hat{S} = \hat{W}_\alpha^2/32\pi^2 \), \( \hat{W}_\alpha \) are the gauge field strengths of \((N_c - N_l)^2 - 1\) remaining massless gluon fields, \( \alpha(\mu, \hat{\Lambda}) \) is the gauge coupling of this lower energy theory and \( \hat{\Lambda} \) is its scale parameter, \( z_Q \ll 1 \) is the massless quark renormalization factor from \( \mu = \mu_{gl} \) down to \( \mu = \Lambda_Q \) in the original theory with \( N_c \) colors and \( N_F \) flavors, \( z_Q' \ll 1 \) is the analogous renormalization factor from \( \mu = \mu_{gl} \) down to \( \mu = \langle \hat{\Lambda} \rangle \) in the lower energy theory with \( N_c - N_l \) colors and \( N_l \) remained active h-flavors \( Q^h, \bar{Q}^h \). \( m'_h \ll \langle \hat{\Lambda} \rangle \) is the current mass of \( Q^h, \bar{Q}^h \)-quarks in this lower energy theory at \( \mu = \langle \hat{\Lambda} \rangle \). All fields in (26) are normalized at \( \mu = \mu_{gl} \). Finally, the dots in (26) denote residual D-term interactions. It is supposed that these play no significant role for the case considered in this section and will be neglected in what follows.

Therefore, the hybrids \( \hat{\Pi}_{hl} \) and \( \hat{\Pi}_{lh} \) will appear in the spectrum as (weakly interacting) particles with the masses \( \hat{m}_h \). These hybrids will not be written explicitly below (but implied).

The lower energy theory with \( N'_c = N_c - N_l \) colors, \( N'_F = N_h \) flavors of \( Q^h, \bar{Q}^h \)-quarks, with \( b'_\alpha > 0 \) and \( m'_h \ll \langle \hat{\Lambda} \rangle \) will be in the \( DC_h \) - phase \[^1\]. The constituent mass \( \mu_{C}^h = (z_Q')^{1/2} \hat{M}_{C}^h \ll \langle \hat{\Lambda} \rangle \) is formed in the threshold region \( \mu \sim \mu_C^h \) and there appear \( N'_h \) h-pions \( \Pi'_h \), \( \langle \Pi'_h \rangle_h = \delta^{3/2}(\mu_C^h)^2 \), with masses \( m'_h \). After integrating out these constituent h-quarks, one remains with the Yang-Mills theory with \( N'_c = N_c - N_l \) colors and with the new scale factor \( \Lambda_L \) of the gauge coupling, and with \( N'_h \) h-pions \( \Pi'_h \):

\[ L = \int d^2\theta \, d^2\bar{\theta} \left\{ 2 \text{ Tr} \sqrt{\hat{\Pi}_l^1 \hat{\Pi}_l + \text{ Tr} (\Pi'_h)^{i_1} \Pi'_h} \right\} \] (28)

\[ + \int d^2\theta \left\{ -\frac{2\pi}{\alpha(\mu, \Lambda_L)} \hat{S} - N'_F \left( \frac{\text{det} \Pi'_h}{\Lambda'_h} \right)^{1/(N'_F - N'_c)} + \hat{m}_l \text{ Tr} \hat{\Pi}_l + m'_h \text{ Tr} \Pi'_h \right\}, \]

\[ \Lambda_L^3 = \left( \frac{\text{det} \Pi'_h}{\Lambda'_h} \right)^{1/(N'_F - N'_c)}, \quad \langle \Lambda_L \rangle = \Lambda_{YM} = \left( \Lambda_Q^{3N_c} \text{ det } m \right)^{1/3N_c}, \quad \text{det } m = m_l N_l m_h^{N_F - N_l}. \]

Proceeding through the VY-procedure, one obtains the lowest energy pion Lagrangian:

\[ L = \int d^2\theta \, d^2\bar{\theta} \left\{ 2 \text{ Tr} \sqrt{\hat{\Pi}_l^1 \hat{\Pi}_l + \text{ Tr} (\Pi'_h)^{i_1} \Pi'_h} \right\} + \]

\[ + \int d^2\theta \left\{ -(N'_F - N'_c) \left( \frac{\text{det} \Pi'_h}{\Lambda'_h} \right)^{1/(N'_F - N'_c)} + \hat{m}_l \text{ Tr} \hat{\Pi}_l + m'_h \text{ Tr} \Pi'_h \right\}. \] (29)

Substituting \( \hat{\Lambda} \) from (26), this takes the form:

\[ L = \int d^2\theta \, d^2\bar{\theta} \left\{ 2 \text{ Tr} \sqrt{\hat{\Pi}_l^1 \hat{\Pi}_l + \text{ Tr} (\Pi'_h)^{i_1} \Pi'_h} \right\} + \]

\[ + \int d^2\theta \left\{ -\hat{S} \left( \frac{\text{det} \Pi'_h}{\Lambda'_h} \right)^{1/(N'_F - N'_c)} + \hat{m}_l \text{ Tr} \hat{\Pi}_l + m'_h \text{ Tr} \Pi'_h \right\}. \]

\[^7\text{Both } z_Q \text{ and } z_Q' \text{ are only logarithmic in the case considered.}\]
\[ + \int d^2 \theta \left\{ -(N_F - N_c) \left( \frac{\zeta_{Q}^{N_F} \det \Pi_{I} \det \Pi_{h}^{'}}{\Lambda_{Q}^{b_0}} \right)^{1/(N_F - N_c)} + \bar{m}_{l} \text{Tr} \Pi_{I} + \bar{m}_{h} \text{Tr} \Pi_{h}^{'} \right\}. \] (30)

The Lagrangian (30) (with the hybrid pions \( \Pi_{hl} \) and \( \Pi_{ih} \) reinstated), being expressed through the fields \( \Pi_{I} \), \( \Pi_{h} \), \( \Pi_{hl} \), \( \Pi_{ih} \) and the masses \( \bar{m}_{l} \), \( \bar{m}_{h} \) normalized at the "old scale" \( \mu = \Lambda_{Q} \), takes the form:

\[ L = \int d^2 \theta d^2 \bar{\theta} \left\{ \frac{2}{\zeta_{Q}} \text{Tr} \sqrt{\Pi_{I}^{2} \Pi_{I}} + \frac{\zeta_{Q}^{2}}{\zeta_{Q}} \text{Tr} \sqrt{\Pi_{h}^{2} \Pi_{h}} + \frac{1}{\zeta_{Q}} \text{Tr} \left( \Pi_{hl}^{2} \Pi_{hl} + \Pi_{hl}^{2} \Pi_{ih} \right) + \ldots \right\} + \]

\[ + \int d^2 \theta \left\{ -(N_F - N_c) \left( \frac{\det \Pi_{I} \det \Pi_{h}}{\Lambda_{Q}^{b_0}} \right)^{1/(N_F - N_c)} + \bar{m}_{l} \text{Tr} \Pi_{I} + \bar{m}_{h} \text{Tr} \left( \Pi_{h} + \Pi_{hl} \Pi_{ih} \right) \right\}, \]

\[ \Pi_{h}^{'} = \frac{\zeta_{Q}}{\zeta_{Q}} \Pi_{h}, \quad \Pi_{I} = \frac{1}{\zeta_{Q}} \Pi_{I}, \quad \bar{m}_{l} = \frac{\bar{m}_{l}}{\zeta_{Q}}, \quad \bar{m}_{h} = \frac{\bar{m}_{h}}{\zeta_{Q}} = \frac{\zeta_{Q}}{\zeta_{Q}} \bar{m}_{h}. \] (31)

On the whole for this case when theory is deeply in the Higgs - DC phase (i.e. \( \mathcal{M}_{\text{ch}}^{l} \gg \Lambda_{Q} \)), the mass spectrum looks as follows. There is: a) \((2N_{l}N_{c} - N_{l}^{2})\) massive gluons and the same number of their superpartners - the "constituent l-quarks" with heaviest masses \( \mu_{gl} \gg \Lambda_{Q} \), b) a large number of hadrons made of non-relativistic constituent \( Q^{h}, \overline{Q}_{h}^{l} \)-quarks with masses \( \sim \mu_{C}^{l} \gg \Lambda_{Q} \ll \mu_{gl} \), c) a large number of strongly coupled gluonia with the mass scale \( M_{gl} \sim \Lambda_{YM} \ll \mu_{C}^{h} \), d) \( N_{h}^{2} \) h-pions with masses \( \sim m_{h}^{l} \ll \Lambda_{YM} \), e) the hybrid pions \( \Pi_{hl} \) and \( \Pi_{ih} \) (these are \( Q^{h} \) and \( \overline{Q}_{h}^{l} \) - quarks with higgsed colors) with masses \( \bar{m}_{h} \ll \bar{m}_{l}^{l} \), f) \( N_{l}^{2} \) lightest l-pions with masses \( \bar{m}_{l}^{l} \ll \bar{m}_{h} \).

At \( N_{l} < N_{c} - 1 \), starting with \( r \equiv m_{l}^{l}/m_{h} \) = 1 when all quarks are in the DC phase and diminishing \( r \), there always will be a number of phase transitions. The DC\(_{l} \) - DC\(_{h} \) phase is maintained until (10) is fulfilled.

Let us take \( N_{l} < N_{o} \), see (10). Then, as \( r \) approaches \( r_{2} \) from above, \( \mathcal{M}_{\text{ch}}^{l} \) approaches \( \Lambda_{Q} \) from below, with all quarks in the DC\(_{l} \) - DC\(_{h} \) phase. When \( \mathcal{M}_{\text{ch}}^{l} \) overshoots \( \Lambda_{Q} \), there is a phase transition as the l-quarks become higgsed. The crucial parameter here (i.e. at \( \mathcal{M}_{\text{ch}}^{l} \gg \Lambda_{Q} \), but not too large, see below) is \( b_{o}^{l} = (3N_{c} - N_{l}') = (b_{o} - 2N_{l}) \). The \( Q^{h}, \overline{Q}_{h}^{l} \)-quarks will be in the DC\(_{h} \) phase at \( b_{o}^{l} > 0 \), and in the HQ\(_{h} \) phase at \( b_{o}^{l} < 0 \). If \( N_{l} < N_{o} \), then \( N_{l} < b_{o}/2 \) also, so that as \( \mathcal{M}_{\text{ch}}^{l} \) overshoots \( \Lambda_{Q} \) and the l-quarks are higgsed, the DC\(_{h} \)-phase of h-quarks is maintained.

Let us trace how the mass spectrum changes on both sides around this phase transition between the DC\(_{l} \) - DC\(_{h} \) and Higgs\(_{l} \) - DC\(_{h} \) phases, at \( r \sim r_{2} \ll 1 \) (see section 2).

a) The gluons. In the DC\(_{l} \) - DC\(_{h} \) phase at \( r < r_{2} \), all \( (N_{l}^{2} - 1) \) gluons can be thought of as having the small masses \( M_{gl} \sim \Lambda_{YM} \). In the Higgs\(_{l} \) - DC\(_{h} \) phase at \( r > r_{2} \), the SU\(_{l}(N_{c}) \) gauge symmetry is broken down to the non-Abelian SU\(_{l}(N_{c} - N_{l}) \) one, \( N_{l} < N_{c} - 1 \). So, \( (2N_{l}N_{c} - N_{l}^{2}) \) gluons acquire the large masses \( M_{gl} \sim \Lambda_{Q} \gg \Lambda_{YM} \), while \( (N_{c} - N_{l})^{2} - 1 \) gluons remain with the same small masses \( \sim \Lambda_{YM} \).

b) The l\(_{l}\)-flavors. In the DC\(_{l} \) - DC\(_{h} \) phase at \( r < r_{2} \), the confined \( Q^{l}, \overline{Q}_{h}^{l} \) quarks have large constituent masses \( \mu_{C}^{l} = \mathcal{M}_{\text{ch}}^{l} \sim \Lambda_{Q} \) and there are \( N_{l}^{2} \) light l\(_{l}\)-pions with small masses \( M_{pi}^{l} \sim m_{l} \). In the Higgs\(_{l} \) - DC\(_{h} \) phase at \( r > r_{2} \), there is \( (2N_{l}N_{c} - N_{l}^{2}) \) massive quarks which
are the superpartners of massive gluons and so have the same masses \( \sim \Lambda_Q \). In a sense, these quarks can be thought of as remnants of the previous constituent \( l \)-quarks and their masses match smoothly across \( r \sim r_2 \). As for the \( ll \)-pions, their number and masses also match smoothly across the phase transition.

c) The \( hh \)-flavors. Nothing happens also with the confined constituent \( Q^h, \bar{Q}^h \)-quarks (i.e. those with unbroken colors) with the masses \( \mu_{Q}^h = M_{ch}^h \ll \Lambda_Q \), and with \( N_{b}^2 \) hh-pions with masses \( \sim m_h \gg m_l \). But in the Higgs\(_l\) - DC\(_h\) phase at \( r > r_2 \), the \( Q^h, \bar{Q}^h \)-quarks with broken colors appear now individually in the spectrum as light particles with the masses \( \sim (m_h - m_l) \approx m_h \). They can be thought of as remnants of the previous hybrid \( \Pi_{hl} \) and \( \Pi_{lh} \)-pions with the masses \( (m_h + m_l) \approx m_h \), which were present in the spectrum in the DC\(_l\) - DC\(_h\) phase at \( r < r_2 \).

Let us take now \( N_l > N_o \). The theory is in the DC\(_l\) - DC\(_h\) phase at \( r = 1 \). As \( r \) decreases, there is first a phase transition to the DC\(_l\) - HQ\(_h\) phase at \( r \sim r_1 \gg r_2 \), which persists until \( r \) approaches \( r_2 \) from above. If \( N_l > b_o/2 \), as \( M_{ch}^l \) overshoots \( \Lambda_Q \) and the \( l \)-quarks are higgsed, the HQ\(_h\)-phase of the \( Q^h, \bar{Q}^h \)-quarks is maintained.

But there are such values of \( N_l < N_F < 3N_c \) and \( N_l < N_c - 1 \) that \( b_o < N_l < b_o/2 \). In this case, the theory stays in the DC\(_l\) - HQ\(_h\) phase as \( M_{ch}^l \) approaches \( \Lambda_Q \) from below, while as \( M_{ch}^l \) overshoots \( \Lambda_Q \) the \( h \)-quarks condense and there appear \( \Pi_{hl} \)-pions, and theory will be in the Higgs\(_l\) - DC\(_h\) phase. So, not only the \( l \)-quarks change their phase, but the \( Q^h, \bar{Q}^h \)-quarks also. The reason for this is the following. At \( M_{ch}^l \) slightly above \( \Lambda_Q \) when the \( l \)-quarks are already higgsed, the remaining lower energy theory has \( \langle \Lambda \rangle \sim \Lambda_Q \), \( N_l' = N_c - N_l \) colors, \( N_F' = N_F - N_l \) flavors, and with \( m_h \ll \Lambda_Q \) and \( M_{ch}^h \) staying intact because \( \langle \Lambda \rangle \sim \Lambda_Q \). But the pole mass \( m_pole^h \) of the \( Q^h, \bar{Q}^h \)-quarks is smaller now in this new theory than it was before higgsing, \( m_pole^h \ll m_pole^h \), because the quark anomalous dimension diminished. So, while the hierarchy was \( m_pole^h \gg M_{ch}^l \) before higgsing, it is reversed now after higgsing, \( m_pole^h \ll M_{ch}^l \), and the \( Q^h, \bar{Q}^h \)-quarks also change their phase simultaneously with the \( Q^l, \bar{Q}^l \) ones.

This is not the end of the story with \( b_o' > 0 \) however, because to stay in the Higgs\(_l\) - DC\(_h\) phase the condition \( m_{sl}^l = m_{isl} (\mu = \chi) \ll \langle \Lambda \rangle \) is necessary, so that \( r = m_l/m_{isl} \) has not to be too small. As \( r \) decreases at \( N_l < N_c - 1 \) and \( b_o' > 0 \), \( \langle \Lambda \rangle \) in (26) decreases in a power-like fashion because \( M_{ch}^l \) grows \( \sim (1/r)^\omega \), \( \omega = (N_c - N_l)/2N_c \), see (2), \( (M_{ch}^l \sim M_{ch}^h \) up to a logarithmic factor), while \( m_{isl}^l \) changes only logarithmically. So, as decreasing \( r \) crosses the smaller value \( r_3 \ll r_2 \) where decreasing \( \langle \Lambda \rangle \) becomes \( \langle \Lambda \rangle < m_{isl}^l \), there is the phase transition from the Higgs\(_l\) - DC\(_h\) phase to the Higgs\(_l\) - HQ\(_h\) one. The coherent condensate of the \( Q^h, \bar{Q}^l \)-quarks breaks down, the \( \Pi_{hl} \)-pions disappear, and at \( r \ll r_3 \) the heavy \( Q^h, \bar{Q}^h \)-quarks will be in the perturbative weak coupling regime, like \( h \)-quarks with higgsed colors (but weakly confined, the string tension is small: \( \sqrt{\sigma} \sim \Lambda_{YM} \)). I.e., the lower energy theory at \( \mu \ll \mu_{sl} \) contains: the unbroken non-Abelian gauge group \( SU(N'_c) \), \( N'_c = N_c - N_l \), with the scale factor \( \Lambda \) (26) of its gauge coupling, and \( N_F' = N_F - N_l \) (\( N'_c < N'_F < 3N'_c \)) flavors of the heavy non-relativistic quarks \( Q^h, \bar{Q}^h \) with their pole masses \( m_{pole}^h \gg \langle \Lambda \rangle \), plus the \( l \)-pions entering \( \Lambda \), see (26), and plus the hybrid pions \( \Pi_{hl}, \Pi_{lh} \) (these are the light \( Q^h, \bar{Q}^h \) quarks with higgsed colors, weakly interacting through residual D-terms interactions). So, we don’t give here further detail, because this is a simple regime and it is evident how to deal with this case. The mass spectrum in this Higgs\(_l\) - HQ\(_h\) phase at \( r < r_3 \) looks as follows. There is: \ a) \( (2N_l N_c - N_F^2) \) of massive gluons and the same number of their superpartners- “the constituent \( l \)-quarks” with the heaviest masses \( \mu_{sl} \gg \Lambda_Q \), \ b) \ a large number of hadrons made of
non-relativistic $Q^h$, $Q_{\overline{h}}$-quarks with the perturbative pole masses $m_{h}^{\text{pole}}$ (the hierarchies look here as: $\langle \hat{\Lambda} \rangle \ll \Lambda_{YM} \ll m_{h}^{\text{pole}} \ll \Lambda_{Q}$, with $\hat{\Lambda}$ from (26), while $\Lambda_{YM}$ is the gauge coupling scale arising after the $Q^h$, $Q_{\overline{h}}$-quarks were integrated out), c) the hybrids $\Pi_{hi}, \Pi_{\overline{h}i}$ (these are $Q^h$, $Q_{\overline{h}}$ quarks with higgsed colors) with the masses $\Lambda_{YM} \ll \bar{m}_{h} = m_{h}(\mu = \mu_{gl}) \ll m_{h}^{\text{pole}}$, d) a large number of strongly coupled gluonia with the mass scale $M_{gl} \sim \Lambda_{YM}$, e) $N_{l}^{2}$ lightest $l$-pions with the masses $2\bar{m}_{l} = 2m_{l}(\mu = \mu_{gl}) = 2z_{Q} m_{l}$. The lowest energy Lagrangian of these $l$-pions will have the same Kahler term as in (31), and the same universal superpotential as in (14).

Finally, let us consider the case $\mathcal{M}_{\text{ch}}^{t} > \Lambda_{Q}, N_{l} < N_{c} - 1$, $b'_{0} = (b_{0} - 2N_{l}) < 0$. As was pointed out above, as $\mathcal{M}_{\text{ch}}^{t}$ approaches $\Lambda_{Q}$ from below, the theory is already in the DC$_{1}$–HQ$_{h}$ phase, and as $\mathcal{M}_{\text{ch}}^{t}$ overshoots $\Lambda_{Q}$ and the $l$-quarks become higgsed, the confined $Q^h$, $Q_{\overline{h}}$-quarks remain in the HQ$_{h}$ phase. So, on the whole, this is the Higgs$_{i}$ – HQ$_{h}$ phase. But now, at $b'_{0} = (b_{0} - 2N_{l}) < 0$ and $\mathcal{M}_{\text{ch}}^{t} \gg \Lambda_{Q}, \langle \hat{\Lambda} \rangle \gg \mathcal{M}_{\text{ch}}^{t}$, see (26), and in the interval of scales $\Lambda_{YM} \ll \mu \ll \mu_{gl}$ the remained non-Abelian $SU(N_{c} - N_{l})$ gauge theory with $N_{F} - N_{l}$ of confined $Q^h$, $Q_{\overline{h}}$ quarks is in the weak coupling logarithmic regime. On the whole, this is also a very simple case (see sections 2 and 8 in [1]), and it is clear what will be the mass spectrum. Qualitatively, it is similar to those described in the preceding paragraph with $b'_{0} > 0$ and in the same Higgs$_{i}$ – HQ$_{h}$ phase at $r < r_{3}$ (and the lowest energy Lagrangian of the lightest $l$-pions will be the same), so that we are not going into further detail with this case.

6. Conclusions.

As was described above within the dynamical scenario considered in this paper, $\mathcal{N} = 1$ SQCD with $N_{c}$ colors (with the scale factor $\Lambda_{Q}$ of their gauge coupling), $N_{c} < N_{F} < 3N_{c}$ flavors of light quarks, with $N_{l}$ lighter flavors with masses $m_{l}$ and $N_{h} = N_{F} - N_{l}$ heavier ones with masses $m_{h}$, $0 < m_{l} < m_{h} \ll \Lambda_{Q}$, will be in the different phase states, depending on the values of the above parameters. Besides, the mass spectra are also highly sensitive to the values of these parameters.

The lighter $Q^{t},Q_{\overline{t}}$-quarks may be in two different phases: either in the DC (diquark condensate) phase at $\mathcal{M}_{\text{ch}}^{t} \ll \Lambda_{Q}$ (both at $N_{l} < N_{c}$ and $N_{l} > N_{c}$), or in the Higgs phase at $\mathcal{M}_{\text{ch}}^{t} \gg \Lambda_{Q}$ (at $N_{l} < N_{c}$ only). The heavier $Q^{h},Q_{\overline{h}}$-quarks may be also in two different phases: either in the DC phase at $\mathcal{M}_{\text{ch}}^{h} \gg m_{h}^{\text{pole}}$, or in the HQ (heavy quark) phase at $\mathcal{M}_{\text{ch}}^{h} \ll m_{h}^{\text{pole}}$. So on the whole, four different phases are realized in this theory. For each of them, the mass spectra and the corresponding interaction Lagrangians were described above in the text.\footnote{The $N_{c}$-dependence of various quantities (e.g. $\langle S \rangle \sim N_{c}^{0}$, etc.) used everywhere above in the text is the same as in the main text in [1]. And as in [1], the correct $N_{c}$ dependence ($\langle S \rangle \sim N_{c}$, etc.) can be easily reinstated (see section 9 in [1]).}

We did not consider in this paper the Seiberg dual theories [3][10] with unequal quark masses. As was argued in [1], the direct and dual theories are not equivalent even in a simpler case of equal quark masses. There are no chances that the situation will be better for unequal quark masses.

This work is supported in part by the RFBR grant 07-02-00361-a.
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