A STRICT NON-STANDARD INEQUALITY .999\ldots < 1

KARIN USADI KATZ AND MIKHAIL G. KATZ

Abstract. Is .999\ldots equal to 1? A. Lightstone’s decimal expansions yield an infinity of numbers in [0, 1] whose expansion starts with an unbounded number of repeated digits “9”. We present some non-standard thoughts on the ambiguity of the ellipsis, modeling the cognitive concept of generic limit of B. Cornu and D. Tall. A choice of a non-standard hyperinteger \( H \) specifies an \( H \)-infinite extended decimal string of 9s, corresponding to an infinitesimally diminished hyperreal value \( 1.5 \). In our model, the student resistance to the unital evaluation of .999\ldots is directed against an unspoken and unacknowledged application of the standard part function, namely the stripping away of a ghost of an infinitesimal, to echo George Berkeley. So long as the number system has not been specified, the students’ hunch that .999\ldots can fall infinitesimally short of 1, can be justified in a mathematically rigorous fashion.

Contents

1. The problem of unital evaluation
2. A geometric sum
3. Arguing by “I told you so”
4. Coming clean
5. Squaring .999\ldots < 1 with reality
6. Hyperreals under magnifying glass
7. Zooming in on slope of tangent line
8. Hypercalculator returns .999\ldots
9. Generic limit and precise meaning of infinity
10. Limits, generic limits, and Flatland
11. A non-standard glossary

Date: February 24, 2009.

2000 Mathematics Subject Classification. Primary 26E35; Secondary 97A20, 97C30.

Key words and phrases. decimal representation, generic limit, hyperinteger, infinitesimal, Lightstone’s semicolon, non-standard calculus, unital evaluation.

*Supported by the Israel Science Foundation (grants no. 84/03 and 1294/06) and the BSF (grant 2006393).
Student resistance to the evaluation of \( .999\ldots \) as the real number 1 (henceforth referred to as the unital evaluation) has been widely discussed in the mathematics education literature. It has been suggested that the source of such resistance lies in a psychological predisposition in favor of thinking of \( .999\ldots \) as a process, or iterated procedure, rather than the final outcome, see for instance D. Tall’s papers [28, p. 6], [25, p. 221], [23] (see also [27] for another approach).

We propose an alternative model to explain such resistance, in the framework of non-standard analysis. From this point of view, the resistance is directed against an unspoken and unacknowledged application of the standard part function “\( st \)” (see Section 11 item 11.3), namely the stripping away of a ghost of an infinitesimal, to echo George Berkeley [3], implicit in unital evaluation:

\[
\text{st} \left( 0.999999999999999\ldots \right) = 1.
\]

**Figure 1.1.** Taking standard part of wikiartist’s conception of an infinity of 9s
A STRICT NON-STANDARD INEQUALITY \( .999 \ldots < 1 \)

What is the significance of such fine (indeed, infinitesimal) distinctions between \( .999 \ldots \) and 1? As this text is addressed to a wide audience, it may be worth recalling that dividing by the difference \( 1 - .999 \ldots \) will only be possible if the latter is nonzero, a matter of immediate import for studying rates of change and calculating derivatives (see Section 7).

The hyperreal approach to a certain extent vindicates the student resistance to education professionals’ toeing the standard line on unital evaluation: so long as the number system has not been specified, the students’ hunch that \( .999 \ldots \) can fall infinitesimally short of 1, can be justified in a mathematically rigorous fashion.

In Section 2, based on a formula for a geometric sum, we summarize the standard argument in favor of unital evaluation. In Section 3, we summarize the standard resistance to the latter. In Section 4, we construct a hyperreal decimal in \([0, 1)\), represented by a string of \( H\)-infinitely many repeated 9s, and identify its Lightstone representation. In Section 5, we square the strict inequality with standard reality, by means of the standard part function. In Section 6, we represent the hyperreal graphically by means of an infinite-magnification microscope, already exploited for pedagogical purposes by Keisler [15] and Tall [23]. In Section 7, we exploit the hyperreal to calculate \( f'(1) \).

In Section 8, we develop an applied-mathematical model of a hypercalculator so as to explain a familiar phenomenon of a calculator returning a string of 9s in place of an integer. In Section 9, we examine the cognitive concept of generic limit of Cornu and Tall in mathematics education, in relation to a hyperreal approach to limits. In Section 10, we set the situation in perspective, with E. A. Abbott.

Section 11 is a technical appendix containing basic material on non-standard calculus. The historical Section 12 contains an examination of the views of Courant, Lakatos, and E. Bishop. Section 13 contains a 10-step proposal concerning the problem of unital evaluation.

2. A geometric sum

Evaluating the formula

\[
1 + r + r^2 + \ldots + r^{n-1} = \frac{1 - r^n}{1 - r}
\]

at \( r = \frac{1}{10} \), we obtain

\[
1 + \frac{1}{10} + \frac{1}{100} + \ldots + \frac{1}{10^{n-1}} = \frac{1 - \frac{1}{10^n}}{1 - \frac{1}{10}},
\]

or alternatively
Multiplying by \( \frac{9}{10} \), we obtain
\[
\overline{.999 \ldots 9} = \frac{9}{10} \left( \frac{1 - \frac{1}{10}}{1 - \frac{1}{10}} \right) = 1 - \frac{1}{10^n}
\]
for every \( n \in \mathbb{N} \). As \( n \) increases without bound, the formula
\[
\overline{.999 \ldots 9} = 1 - \frac{1}{10^n}
\]
becomes
\[
.999 \ldots = 1.
\]
Or does it?

3. Arguing by “I Told You So”

When I tried this one on my teenage daughter, she remained unconvinced. She felt that the number \( .999 \ldots \) is smaller than 1. After all, just look at it! There is something missing before you reach 1. I then proceeded to give a number of arguments. Apologetic mumbo-jumbo about the alleged “non-unicity of decimal representation” fell on deaf ears. The one that seemed to work best was the following variety of the old-fashioned “because I told you so” argument: factor out a 3:
\[
3(\overline{.333 \ldots}) = 1
\]
to obtain
\[
\overline{.333 \ldots} = \frac{1}{3}
\]
and “everybody knows” that the number \( .333 \ldots \) is exactly “a third” on the nose. Q.E.D. This worked for a few minutes, but then the validity of (3.1) was called into question, as well.

4. Coming Clean

Then I finally broke down. In Abraham Robinson’s theory of hyperreal analysis [20], there is a notion of an infinite hyperinteger (see Section 11 item 11.8). H. Jerome Keisler [15] took to denoting such an entity by the symbol

\( H \),
possibly because its inverse is an infinitesimal \( h \) typically appearing in
the denominator of the familiar definition of derivative (it most decid-
edly does not stand for “Howard”). Taking infinitely many terms in
formula (2.1) is interpreted as replacing \( n \in \mathbb{N} \) by an infinite hyperin-
teger
\[
H \in \mathbb{N}^* \setminus \mathbb{N}.
\]
The transfer principle (see Section 11, item 11.1) applied to (2.1) then
yields
\[
\underbrace{.999 \ldots}_{H} = 1 - \frac{1}{10^H}
\]
where the infinitesimal quantity \( \frac{1}{10^H} \) is nonzero:
\[
\frac{1}{10^H} > 0.
\]
Therefore we obtain the strict nonstandard inequality
\[
\underbrace{.999 \ldots}_{H} < 1 \quad (4.1)
\]
and my teenager was right all along. Note that hyperreal extended
decimal expansions were treated by A. Lightstone in [17, pp. 245–247].
In his notation, the hyperreal appearing in the left-hand-side of (4.1)
appears as the extended decimal
\[
.999 \ldots; \ldots .99^\hat{9}
\]
with the hat “\(^\hat{\phantom{0}}\)" indicating the \( H \)-th decimal place, where the last
repeated digit 9 occurs. We have employed the underbrace notation as
in (4.1) rather than Lightstone’s semicolon, as it parallels the finite case
more closely, and seems more intuitive. An alternative construction of
a strict inequality \( .999 \ldots < 1 \) may be found in [18], however in a
number system which is not a field.

5. Squaring \( .999 \ldots < 1 \) with reality

To obtain a real number in place of the hyperreal \( \underbrace{.999 \ldots}_{H} \), we apply
the standard part function “st” (see Section 11, item 11.3):
\[
\text{st} \left( \underbrace{.999 \ldots}_{H} \right) = \text{st} \left( 1 - \frac{1}{10^H} \right) = 1 - \text{st} \left( \frac{1}{10^H} \right) = 1.
\]
To elaborate further, one could make the following remark. Even in
standard analysis, the expression \( .999 \ldots \) is only shorthand for the
limit of the finite expression (2.1) when the standard integer \( n \) increases without bound. From the hyperreal viewpoint, “taking the limit” is accomplished in two steps:

1. evaluating the expression at an infinite hyperinteger \( H \), and then
2. taking the standard part.

The two steps form the non-standard definition of limit (see Section 11 item 11.10). Now the first step (evaluating at \( H \)) produces a hyperreal number dependent on \( H \) (in all cases it will be strictly less than 1). The second step will strip away the infinitesimal part and produce the standard real number 1 in the cluster (see Section 11 item 11.4) of points infinitely close to it.

**Remark 5.1 (Multiple infinities).** The existence of more than one infinite hyperreal is not only a requirement to have a field, but is actually extremely useful. For example, using the natural hyperreal extension \( f^* \) of \( f \) (see Section 11 item 11.1), it is possible to write down a pointwise definition of uniform continuity of a function \( f \) (see below). Such a definition considerably reduces the quantifier complexity of the standard definition.

To elaborate, note that the standard definition of uniform continuity of a real function \( f \) can be said to be global rather than local (i.e. pointwise), in the sense that, unlike ordinary continuity, uniform continuity cannot be defined as a pointwise property of \( f \). Meanwhile, in the framework of Robinson’s theory, it is possible to give a definition of uniform continuity of the real function \( f \) in terms of its natural hyperreal extension, denoted

\[
f^*,
\]

in such a way that the definition is local, in the sense of depending only on each pointwise cluster (see Section 11 item 11.4) in the domain of \( f^* \). Thus, \( f \) is uniformly continuous on \( \mathbb{R} \) if the following condition is satisfied:

\[
\forall x \in \mathbb{R}^* \quad (y \approx x \implies f^*(y) \approx f^*(x)).
\]

Here \( \approx \) stands for the relation of being infinitely close. The condition must be satisfied at the infinite hyperreals \( x \), in addition to the finite ones. This addition is what distinguishes uniform continuity from ordinary continuity.
6. Hyperreals under magnifying glass

The symbol “∞” is employed in standard real analysis to define a formal completion of the real line $\mathbb{R}$, namely

$$\mathbb{R} \cup \{\infty\}$$

(sometimes a formal point “$-\infty$” is added, as well).

Such a formal device is helpful in simplifying the statements of certain theorems (which would otherwise have a number of subcases). The symbol is used in a different sense in topology and projective geometry, where the Thom compactification $\mathbb{R} \cup \{\infty\}$ of $\mathbb{R}$ is a circle:

$$\mathbb{R} \cup \{\infty\} \approx S^1.$$  

(6.2)

We have refrained from using the symbol “∞” to denote an infinite hyperreal, as in

$$\overbrace{.999\ldots}^{\infty},$$

even though the symbol “∞” does convey the idea of the infinite more effectively than the symbol “$H$” that we have used. The reason is so as
to avoid the risk of creating a false impression of the uniqueness of an
infinite point (as in (6.1) or (6.2) above), in the field $\mathbb{R}^*$ (see Section 11, item 11.2).

We represent the hyperreal

\[ \underbrace{.999\ldots}_H \]  \hspace{1cm} (6.3)

visually by means of an infinite-resolution microscope already exploited for pedagogical purposes by Keisler [15]. The hyperreal $\underbrace{.999\ldots}_H$ appearing in the diagram of Figure 6.1 illustrates graphically the strict hyperreal inequality

\[ \underbrace{.999\ldots}_H < 1, \]

where, as we already mentioned, the symbol $H$ is exploited to denote a fixed infinite Robinson hyperinteger (see Section 11, item 11.8).

7. Zooming in on slope of tangent line

To calculate the slope of the tangent line to the curve $y = x^2$ at $x = 1$, we first compute the ratio

\[ \frac{\Delta y}{\Delta x} = \frac{(.9\ldots)^2 - 1^2}{.9\ldots - 1} = \frac{(.9\ldots - 1)(.9\ldots + 1)}{.9\ldots - 1} = .9\ldots + 1, \]

where we have deleted the underbrace $\underbrace{.999\ldots}_H$ and also shortened the symbol $\underbrace{.999\ldots}_H$ to $.9\ldots$, so as to lighten the notation.

Therefore the slope is

\[ \frac{dy}{dx} = \text{st} \left( \frac{\Delta y}{\Delta x} \right) = \text{st}(.9\ldots + 1) = \text{st}(.9\ldots) + 1 = 1 + 1 = 2. \]

Note that

\[ \Delta x = .9\ldots - 1 = -.0\ldots01 \]

in Lightstone’s notation, where the digit “1” appears in the $H^{th}$-decimal place, as illustrated in Figure 7.1.

8. Hypercalculator returns .999 . . .

Everyone who has ever held an electronic calculator is familiar with the curious phenomenon of it sometimes returning the value

.999999
A Strict Non-Standard Inequality .999\ldots < 1

Figure 7.1. Zooming in on the slope of tangent line to the curve $y = x^2$ at $x = 1$

in place of the expected 1.000000. For instance, a calculator programmed to apply Newton’s method to find the zero of a function, may return the .999999 value as the unique zero of the function $\log x$.

Developing a model to account for such a phenomenon is complicated by the variety of the degree of precision displayed, as well as the greater precision typically available internally than that displayed on the LCD. To simplify matters, we will consider an idealized model, called a hypercalculator, of a theoretical calculator that applies Newton’s method precisely $H$ times, where $H$ is a fixed infinite hyperinteger, as discussed in the previous section.

**Theorem 8.1.** Let $f$ be a concave increasing differentiable function with domain an open interval $(1 - \epsilon, 1 + \epsilon)$ and vanishing at its midpoint. Then the hypercalculator applied to $f$ will return a hyperreal decimal .999\ldots with an initial segment consisting of an unbounded number of repeated 9’s.

**Proof.** Assume for simplicity that $f(x_0) < 0$. We have

$$x_1 = x_0 + \frac{|f(x_0)|}{f'(x_0)}.$$
By the mean value theorem, there is a point \( c \) such that \( x_0 < c < 1 \) where \( f'(c) = \frac{|f(x_0)|}{1-x_0} \), or
\[
\frac{|f(x_0)|}{f'(c)} = 1 - x_0.
\]
Since \( f \) is concave, its derivative \( f' \) is decreasing, hence
\[
x_1 = x_0 + \frac{|f(x_0)|}{f'(x_0)} < x_0 + 1 - x_0 = 1.
\]
Thus \( x_1 < 1 \). Inductively, the point \( x_{n+1} = x_n + \frac{|f(x_n)|}{f'(x_n)} \) satisfies \( x_n < 1 \) for all \( n \). By the transfer principle (see Section 11, item 11.1), the hyperreal \( x_H \) satisfies a strict inequality
\[
x_H < 1,
\]
as well (cf. (4.1)). Hence the hypercalculator returns a value strictly smaller than 1 yet infinitely close to 1, proving the theorem. \( \square \)

9. **Generic limit and precise meaning of infinity**

The precise meaning of the finite expression
\[
.999...9, \ n \text{ times}
\]
is that the repeated digit 9 occurs precisely \( n \) times. The standard non-terminating decimal
\[
.999...,\ 
\]
as it is traditionally written, is said to have an unbounded number of repeated digits 9, but the expression “infinitely many 9’s” is only a figure of speech, as “infinity” is not a number in standard analysis, in the sense that, whenever a precise meaning is attributed to the phrase “infinitely many 9’s”, it is almost invariably in terms of limits. D. Tall writes in \[26\] as follows:

[...] the infinite decimal 0.999... is intended to signal the limit of the sequence 0.9, 0.99, 0.999, ... which is 1, but in practice it is often imagined as a limiting process which never quite reaches 1.

Tall \[24, 26\] describes a concept in cognitive theory he calls a *generic limit concept* in the following terms:

[...] if a quantity repeatedly gets smaller and smaller and smaller without ever being zero, then the limiting object is naturally conceptualised as an extremely small
A STRICT NON-STANDARD INEQUALITY \( .999 \ldots < 1 \)

quantity that is not zero (Cornu [7]). Infinitesimal quantities are natural products of the human imagination derived through combining potentially infinite repetition and the recognition of its repeating properties. (see [11] for the related notion of procept). A technical realisation of the cognitive concept of generic limit is achieved in the hyperreal line (see Section 11, item 11.2) as follows. Given an infinite hyperinteger \( H \), consider the hyperreal repeated decimal where the repeated digit 9 occurs precisely \( H \) times, in the sense routinely used in non-standard calculus, for example, when one partitions a compact interval into \( H \) parts in the proof of the extreme value theorem (see Section 11, item 11.9). Such a number can be denoted suggestively by \( \underbrace{.999 \ldots}_{H} \) with an underbrace indicating that the digit 9 occurs \( H \) times, resulting in a strict inequality \( \underbrace{.999 \ldots}_{H} < 1 \) (with the underbrace indicating that we are not referring to the standard real). In Lightstone’s notation, this hyperreal would be expressed by the hyperdecimal

\[
.999_{H} \ldots_{H} \ldots_{H} 9,
\]

the last digit 9 occurring in the \( H \)-th position. Such a hyperreal appears to be the mathematical counterpart of the cognitive concept of generic limit.

10. LIMITS, GENERIC LIMITS, AND FLATLAND

As far as standard limits are concerned, given the sequence \( u_1 = .9, u_2 = .99, u_3 = .999, \) etc., from the hyperreal viewpoint we have

\[
\lim_{n \to \infty} u_n = \text{st}(u_H),
\]

where “st” is the standard part function which “strips away” the infinitesimal part, and outputs the standard real in the cluster of the hyperreal \( u_H \) (see Section 11, item 11.4). What may be bothering the students is this unacknowledged application of the standard part function, resulting in a loss of an infinitesimal.

We have, in fact, been looking at the problem “from above”, in the context of non-standard analysis. Perhaps a helpful parallel is provided by the famous animated film Flatland (cf. [11]), where the two-dimensional creatures are unable to conceive of what we think of as the sphere in 3-space, due to their dimension limitation. Similarly, one can conceive of the difficulty in the understanding of the unital evaluation of \( .999 \ldots, \) as due to the limitation of the standard real vision.
The notion of an infinitesimal appeals to intuition (see R. Courant’s comment in Section 12) and would not go away inspite of what is, by now, over a century of $(\varepsilon, \delta)$-ideology (see E. Bishop’s comment in Section 12). Highschool students are exposed to the thorny $.999\ldots$ issue before they are exposed to any rigorous notion of a real number. They are not aware of fine differences between rational numbers, algebraic numbers, real numbers, hyperreal numbers. A related point is make by Keisler in his textbook [15], when he points out that “we have no way of knowing what the line in physical space looks like”. Most students (perhaps all) initially believe that the mysterious number with “infinitely many” repeated digits $9$ falls short of the value $1$. If an education professional claims that the students are making a mistake, might he in fact be making a pedagogical error?

11. A non-standard glossary

In this section we present some illustrative terms and facts from non-standard calculus [15]. The relation of being infinitely close is denoted by the symbol $\approx$. Thus, $x \approx y$ if and only if $x - y$ is infinitesimal.

11.1. Natural hyperreal extension $f^\ast$. The extension principle of non-standard calculus states that every real function $f$ has a hyperreal extension, denoted $f^\ast$ and called the natural extension of $f$. The transfer principle of non-standard calculus asserts that every real statement true for $f$, is true also for $f^\ast$. For example, if $f(x) > 0$ for every real $x$ in its domain $I$, then $f^\ast(x) > 0$ for every hyperreal $x$ in its domain $I^\ast$. Note that if the interval $I$ is unbounded, then $I^\ast$ necessarily contains infinite hyperreals. We will typically drop the star $\ast$ so as not to overburden the notation.

11.2. Internal set. Internal set is the key tool in formulating the transfer principle, which concerns the logical relation between the properties of the real numbers $\mathbb{R}$, and the properties of a larger field denoted $\mathbb{R}^\ast$ called the hyperreal line. The field $\mathbb{R}^\ast$ includes, in particular, infinitesimal (“infinitely small”) numbers, providing a rigorous mathematical realisation of a project initiated by Leibniz. Roughly speaking, the idea is to express analysis over $\mathbb{R}$ in a suitable language of mathematical logic, and then point out that this language applies equally well to $\mathbb{R}^\ast$. This turns out to be possible because at the set-theoretic level, the propositions in such a language are interpreted to apply only to internal sets rather than to all sets. Note that the term “language” is used in a loose sense in the above. A more precise term is theory.
A STRICT NON-STANDARD INEQUALITY \(.999 \ldots < 1\)  

in first-order logic. Internal sets include natural extension of standard sets.

11.3. **Standard part function.** The standard part function “st” is the key ingredient in Abraham Robinson’s resolution of the paradox of Leibniz’s definition of the derivative as the ratio of two infinitesimals \(\frac{dy}{dx}\).

The standard part function associates to a finite hyperreal number \(x\), the standard real \(x_0\) infinitely close to it, so that we can write

\[
\text{st}(x) = x_0.
\]

In other words, “st” strips away the infinitesimal part to produce the standard real in the cluster. The standard part function “st” is not defined by an internal set (see item 11.2 above) in Robinson’s theory.

11.4. **Cluster.** Each standard real is accompanied by a cluster of hyperreals infinitely close to it. The standard part function collapses the entire cluster back to the standard real contained in it. The cluster of the real number 0 consists precisely of all the infinitesimals. Every infinite hyperreal decomposes as a triple sum

\[
H + r + \epsilon,
\]

where \(H\) is a hyperinteger, \(r\) is a real number in \([0, 1)\), and \(\epsilon\) is infinitesimal. Varying \(\epsilon\) over all infinitesimals, one obtains the cluster of \(H + r\).

11.5. **Derivative.** To define the real derivative of a real function \(f\) in this approach, one no longer needs an infinite limiting process as in standard calculus. Instead, one sets

\[
f'(x) = \text{st} \left( \frac{f(x + \epsilon) - f(x)}{\epsilon} \right), \tag{11.1}
\]

where \(\epsilon\) is infinitesimal, yielding the standard real number in the cluster of the hyperreal argument of “st” (the derivative exists if and only if the value \((11.1)\) is independent of the choice of the infinitesimal).

The addition of “st” to the formula resolves the centuries-old paradox famously criticized by George Berkeley [3] (in terms of the *Ghosts of departed quantities*, cf. [21] Chapter 6), and provides a rigorous basis for the calculus.

11.6. **Continuity.** A function \(f\) is continuous at \(x\) if the following condition is satisfied: \(y \approx x\) implies \(f(y) \approx f(x)\).
11.7. **Uniform continuity.** A function \( f \) is uniformly continuous on \( I \) if the following condition is satisfied:

- **standard:** for every \( \epsilon > 0 \) there exists a \( \delta > 0 \) such that for all \( x \in I \) and for all \( y \in I \), if \( |x - y| < \delta \) then \( |f(x) - f(y)| < \epsilon \).
- **non-standard:** for all \( x \in I^* \), if \( x \approx y \) then \( f(x) \approx f(y) \).

See Remark [5.1](#) for a more detailed discussion.

11.8. **Hyperinteger.** A hyperreal number \( H \) equal to its own integer part \( H = \lfloor H \rfloor \) is called a hyperinteger (here the integer part function is the natural extension of the real one). The elements of the complement \( \mathbb{Z}^* \setminus \mathbb{Z} \) are called infinite hyperintegers.

11.9. **Proof of extreme value theorem.** Let \( H \) be an infinite hyperinteger. The interval \([0, 1]\) has a natural hyperreal extension. Consider its partition into \( H \) subintervals of equal length \( \frac{1}{H} \), with partition points \( x_i = i/H \) as \( i \) runs from 0 to \( H \). Note that in the standard setting, with \( n \) in place of \( H \), a point with the maximal value of \( f \) can always be chosen among the \( n + 1 \) partition points \( x_i \), by induction. Hence, by the transfer principle, there is a hyperinteger \( i_0 \) such that \( 0 \leq i_0 \leq H \) and

\[
 f(x_{i_0}) \geq f(x_i) \quad \forall i = 0, \ldots, H. \tag{11.2}
\]

Consider the real point

\[ c = \text{st}(x_{i_0}). \]

An arbitrary real point \( x \) lies in a suitable sub-interval of the partition, namely \( x \in [x_{i-1}, x_i] \), so that \( \text{st}(x_i) = x \). Applying “\( \text{st} \)” to the inequality (11.2), we obtain by continuity of \( f \) that \( f(c) \geq f(x) \), for all real \( x \), proving \( c \) to be a maximum of \( f \) (see [15, p. 164]).

11.10. **Limit.** We have \( \lim_{x \to a} f(x) = L \) if and only if whenever the difference \( x - a \) is infinitesimal, the difference \( f(x) - L \) is infinitesimal, as well, or in formulas: if \( \text{st}(x) = a \) then \( \text{st}(f(x)) = L \).

Given a sequence of real numbers \( \{x_n|n \in \mathbb{N}\} \), if \( L \in \mathbb{R} \) we say \( L \) is the limit of the sequence and write \( L = \lim_{n \to \infty} x_n \) if the following condition is satisfied:

\[
 \text{st}(x_H) = L \quad \text{for all infinite } H \tag{11.3}
\]

(here the extension principle is used to define \( x_n \) for every infinite value of the index). This definition has no quantifier alternations. The standard \((\epsilon, \delta)\)-definition of limit, on the other hand, does have quantifier
A STRICT NON-ST ANDARD INEQUALITY .999 \ldots < 1

alternations:

\[ L = \lim_{n \to \infty} x_n \iff \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N} : n > N \implies d(x_n, L) < \varepsilon. \] (11.4)

11.11. Non-terminating decimals. Given a real decimal

\[ u = .d_1d_2d_3 \ldots, \]

consider the sequence \( u_1 = .d_1, \ u_2 = .d_1d_2, \ u_3 = .d_1d_2d_3, \) etc. Then by definition,

\[ u = \lim_{n \to \infty} u_n = \text{st}(u_H) \]

for every infinite \( H. \) Now if \( u \) is a non-terminating decimal, then one obtains a strict inequality \( u_H < u \) by transfer from \( u_n < u. \) In particular,

\[ .999 \ldots \hat{9} = \underbrace{.999 \ldots}_{H} = 1 - \frac{1}{10^H} < 1, \] (11.5)

where the hat \( \hat{\ } \) indicates the \( H \)-th Lightstone decimal place. The standard interpretation of the symbol .999 \ldots as 1 is necessitated by notational uniformity: the symbol .a_1a_2a_3 \ldots in every case corresponds to the limit of the sequence of terminating decimals .a_1 \ldots a_n. Alternatively, the ellipsis in .999 \ldots could be interpreted as alluding to an infinity of nonzero digits specified by a choice of an infinite hyperinteger \( H \in \mathbb{N}^* \setminus \mathbb{N}. \) The resulting \( H \)-infinite extended decimal string of 9s corresponds to an infinitesimally diminished hyperreal value (11.5). Such an interpretation is perhaps more in line with the naive initial intuition persistently reported by teachers.

12. Courant, Lakatos, and Bishop

Prior to Robinson, mathematicians thought of infinitesimals in terms of “naive befogging” and “vague mystical ideas”. Thus, Richard Courant [8, p. 81] wrote as follows:

We must, however, guard ourselves against thinking of \( dx \) as an “infinitely small quantity” or “infinitesimal”, or of the integral as the “sum of an infinite number of infinitesimally small quantities”. Such a conception would be devoid of any clear meaning; it is only a naive befogging of what we have previously carried out with precision.

and again on page 101:
We have no right to suppose that first $\Delta x$ goes through something like a limiting process and reaches a value which is infinitesimally small but still not 0, so that $\Delta x$ and $\Delta y$ are replaced by “infinitely small quantities” or “infinitesimals” $dx$ and $dy$, and that the quotient of these quantities is then formed. Such a conception of the derivative is incompatible with the clarity of ideas demanded in mathematics; in fact, it is entirely meaningless. For a great many simple-minded people it undoubtedly has a certain charm, the charm of mystery which is always associated with the word “infinite”; and in the early days of the differential calculus even Leibnitz[sic] himself was capable of combining these vague mystical ideas with a thoroughly clear understanding of the limiting process. It is true that this fog which hung round the foundations of the new science did not prevent Leibnitz[sic] or his great successors from finding the right path. But this does not release us from the duty of avoiding every such hazy idea [emphasis added–MK] in our building-up of the differential and integral calculus.

Needless to say, Courant’s criticism was not without merit at the time of its writing (a quarter century prior to Robinson’s discovery). I. Lakatos [16, p. 44] wrote in ’66 as follows:

Robinson’s work... offers a rational reconstruction of the discredited infinitesimal theory which satisfies modern requirements of rigour and which is no weaker than Weierstrass’s theory. This reconstruction makes infinitesimal theory an almost respectable ancestor of a fully fledged, powerful modern theory, lifts it from the status of pre-scientific gibberish, and renews interest in its partly forgotten, partly falsified history.

Not everyone was persuaded. A decade later, Courant’s duty of avoiding every such hazy idea was taken up under a constructivist banner by E. Bishop. In his essay [5] cast in the form of an imaginary dialog between Brouwer and Hilbert, E. Bishop anchors his foundational stance in a species of mathematical constructivism. Thus, Bishop’s opposition to Robinson’s infinitesimals, expressed in a bristling review [6] of Keisler’s textbook [15], was to be expected (and in fact was anticipated by editor Halmos). In a memorable parenthetical remark, Bishop writes [6]:
Although it seems to be futile, I always tell my calculus
students that mathematics is not esoteric: It is common
sense. (Even the notorious \((\epsilon, \delta)\)-definition of limit is
**common sense** [emphasis added–MK], and moreover it is central to the important practical problems of approxima-
tion and estimation.)

Bishop is referring, of course, to the type of *common-sense* definition
reproduced in (11.4), which he favors over Keisler’s definition (11.3) in
terms of Robinson’s hyperreals.

Bishop expressed his views about non-standard analysis and its use
in teaching in a brief paragraph toward the end of his essay “Crisis in con-
temporary mathematics” [5, p. 513-514]. After discussing Hilbert’s
formalist program he writes:

> A more recent attempt at mathematics by formal finesse
> is non-standard analysis. I gather that it has met with
> some degree of success, whether at the expense of giv-
> ing significantly less meaningful proofs I do not know.
> My interest in non-standard analysis is that attempts
> are being made to introduce it into calculus courses. It
> is difficult to believe that **debasement of meaning**
> [emphasis added–MK] could be carried so far.

Bishop’s view of the introduction of non-standard analysis in the class-
room as no less than a “debasement of meaning”, was duly noted by
J. Dauben [9].

To illustrate how Bishop anchors his foundational stance in a species
of mathematical constructivism, note that in [5, pp. 507-508], he writes:

> To my mind, it is a major defect of our profession that
we refuse to distinguish [...] between integers that are
computable and those that are not [...] the distinction
between computable and non-computable, or construc-
tive and non-constructive is the source of the most fa-
mous disputes in the philosophy of mathematics...

On page 511, Bishop defines a principle (LPO) as the statement that
it is possible to search “a sequence of integers to see
whether they all vanish”,

and goes on to characterize the dependence on the LPO as a procedure
both Brouwer and Bishop himself reject. S. Feferman [10] explains the
principle as follows:

> Bishop criticized both non-constructive classical math-
ematics and intuitionism. He called non-constructive
mathematics “a scandal”, particularly because of its “deficiency in numerical meaning”. What he simply meant was that if you say something exists you ought to be able to produce it, and if you say there is a function which does something on the natural numbers then you ought to be able to produce a machine which calculates it out at each number.

The constructivist objections to LPO are similar to objections to the law of excluded middle, and to proofs by contradiction (for example, the traditional argument for the irrationality of $\sqrt{2}$ is a proof by contradiction, but using classical continued fraction estimates, it is easy to rewrite it in a numerically meaningful fashion, acceptable from the constructivist viewpoint).

Given that a typical construction of Robinson’s infinitesimals (see Keisler [15, p. 911]) certainly does rely on LPO, Bishop’s opposition to such infinitesimals, expressed in a bristling review [6] of Keisler’s textbook, was to be expected.

Non-standard calculus in the classroom was analyzed in the Chicago study by K. Sullivan [22]. Sullivan showed that students following the non-standard calculus course were better able to interpret the sense of the mathematical formalism of calculus than a control group following a standard syllabus. Such a conclusion was also noted by M. Artigue [2, p. 172].

13. A 10-STEP PROPOSAL

In the matter of teaching decimal notation, we would like to obtain some reaction from educators to the following proposal concerning the problem of the unital evaluation of .999.... A student may ask:

What does the teacher mean to happen exactly after nine, nine, nine when he writes dot, dot, dot?

How is a teacher to handle such a question? Experience shows that toeing the standard line on the unital evaluation of .999... possesses a high-frustration factor in the classroom. Rather than baffling the student with such a categorical claim, a teacher could proceed by presenting the following ten points, based on the material outlined in Section [11]:

1. the reals are not, as the rationals are not, the maximal number system;
2. there exist larger number systems, containing infinitesimals;
3. in such larger systems, the interval [0, 1] contains many numbers infinitely close to 1;
A STRICT NON-STANDARD INEQUALITY .999... < 1

(4) in a particular larger system called the hyperreal numbers, there is a generalized notion of decimal expansion for such numbers, starting in each case with an unbounded number of digits “9”;
(5) all such numbers therefore have an arguable claim to the notation “.999...” which is patently ambiguous (the meaning of the ellipsis “...” requires disambiguation);
(6) all but one of them are strictly smaller than 1;
(7) the convention adopted by most professional mathematicians is to interpret the symbol “.999...” as referring to the largest such number, namely 1 itself;
(8) thus, the students’ intuition that .999... falls just short of 1 can be justified in a mathematically rigorous fashion;
(9) the said extended number system is mostly relevant in infinitesimal calculus (also known as differential and integral calculus);
(10) if you would like to learn more about the hyperreals, go to your teacher so he can give you further references.

14. EPILOGUE

A goal of our, admittedly non-standard, analysis is both to educate and to heal. The latter part involves placing balm upon the bewilderment of myriad students of decimal notation, frustrated by the reluctance of their education professionals to yield as much as an infinitesimal iota in their evaluation of the symbol .999..., or to acknowledge the ambiguity of an ellipsis.

ACKNOWLEDGMENTS

We are grateful to David Ebin and David Tall for a careful reading of an earlier version of the manuscript, and for making numerous helpful suggestions.

REFERENCES

[1] Abbott, Edwin A.: The annotated Flatland. A romance of many dimensions. With an introduction and notes by Ian Stewart. Reprint of the 2002 original [Perseus Publ., Cambridge]. Basic Books, New York, 2008.
[2] Artigue, Michèle: Analysis, in Advanced Mathematical Thinking (ed. David Tall), Springer-Verlag (1994), p. 172 (“The non-standard analysis revival and its weak impact on education”).
[3] Berkeley, George: The Analyst, a Discourse Addressed to an Infidel Mathematician (1734).
[4] Bernstein, Allen R.; Robinson, Abraham: Solution of an invariant subspace problem of K. T. Smith and P. R. Halmos, Pacific Journal of Mathematics 16 (1966), 421-431.
[5] Bishop, E.: The crisis in contemporary mathematics, *Historia Math.* 2 ('75), no. 4, 507-517.

[6] Bishop, E.: Review: H. Jerome Keisler, Elementary calculus, *Bull. Amer. Math. Soc.* 83 ('77), 205-208.

[7] Cornu, B.: Limits, pp. 153-166, in Advanced mathematical thinking. Edited by David Tall. Mathematics Education Library, 11. Kluwer Academic Publishers Group, Dordrecht, 1991.

[8] Courant, R.: Differential and integral calculus. Vol. I. Translated from the German by E. J. McShane. Reprint of the second edition (1937). Wiley Classics Library. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1988.

[9] Dauben, J.: Appendix (1992): revolutions revisited. pp. 72–82 in Revolutions in mathematics. Edited and with an introduction by Donald Gillies. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1992.

[10] Feferman, Solomon: Relationships between constructive, predicative and classical systems of analysis. Proof theory (Roskilde, 1997), 221–236, Synthese Lib., 292, Kluwer Acad. Publ., Dordrecht, 2000.

[11] Gray, E.; Tall, D.: Duality, Ambiguity, and Flexibility: A “Proceptual” View of Simple Arithmetic. Journal for Research in Mathematics Education 25(2) (1994), 116-140.

[12] Halmos, Paul R.: I want to be a mathematician. An automathography. Springer-Verlag, New York, 1985.

[13] Halmos, Paul R.: Invariant subspaces of polynomially compact operators, *Pacific Journal of Mathematics* 16 (1966), 433-437.

[14] Kanovei, V.; Shelah, S.: A definable nonstandard model of the reals. *J. Symbolic Logic* 69 (2004), no. 1, 159–164.

[15] Keisler, H. Jerome: Elementary Calculus: An Infinitesimal Approach. Second Edition. Prindle, Weber & Schmidt, Boston, '86.

[16] Lakatos, Imre: Cauchy and the continuum: the significance of nonstandard analysis for the history and philosophy of mathematics. Math. Intelligencer 1 (1978), no. 3, 151–161 (originally published in ’66).

[17] Lightstone, A. H.: Infinitesimals. *Amer. Math. Monthly* 79 (1972), 242–251.

[18] Richman, F.: Is 0.999... = 1? *Math. Mag.* 72 (1999), no. 5, 396–400.

[19] Robinson, Abraham: Non-standard analysis. North-Holland Publishing Co., Amsterdam 1966.

[20] Robinson, Abraham: Non-standard analysis. Reprint of the second (1974) edition. With a foreword by Wilhelmus A. J. Luxemburg. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1996.

[21] Stewart, I.: From here to infinity. A retitled and revised edition of The problems of mathematics [Oxford Univ. Press, New York, 1992]. With a foreword by James Joseph Sylvester. The Clarendon Press, Oxford University Press, New York, 1996.

[22] Sullivan, Kathleen: The Teaching of Elementary Calculus Using the Nonstandard Analysis Approach, *Amer. Math. Monthly* 83 (1976), 370-375.

[23] Tall, D.: Mathematical intuition, with special reference to limiting processes, Proceedings of the Fourth International Congress on Mathematical Education, Berkeley, 170-176 (1980).
[24] Tall, D.: The psychology of advanced mathematical thinking, in Advanced mathematical thinking. Edited by David Tall. Mathematics Education Library, 11. Kluwer Academic Publishers Group, Dordrecht, 1991.

[25] Tall, D.: “Cognitive Development In Advanced Mathematics Using Technology”. Mathematics Education Research Journal 12 (3) (2000), 210-230.

[26] Tall, D.: Dynamic mathematics and the blending of knowledge structures in the calculus, pp. 1-11 in Transforming Mathematics Education through the use of Dynamic Mathematics, ZDM (june ’09).

[27] Tall, D.: How humans learn to think mathematically (forthcoming). 370 pages.

[28] Tall, D; Schwarzenberger, R.: Conflicts in the Learning of Real Numbers and Limits. Mathematics Teaching 82 (1978), 44-49.

DEPARTMENT OF MATHEMATICS, BAR ILAN UNIVERSITY, RAMAT GAN 52900 ISRAEL

E-mail address: {katzmik}@macs.biu.ac.il (remove curly braces)