Lie superbialgebra structures on the centerless twisted $N = 2$ superconformal algebra

Huanxia Fa, Junbo Li†

Department of Mathematics, Changshu Institute of Technology, Changshu 215500, China

Abstract. In this paper, Lie superbialgebra structures on the centerless twisted $N = 2$ superconformal algebra $\mathcal{L}$ are considered which are proved to be coboundary triangular.

Key words: Lie superbialgebras, Yang-Baxter equation, the twisted $N = 2$ superconformal algebra.

Mathematics Subject Classification (2000): 17B05, 17B37, 17B62, 17B66.

§1. Preliminaries

The superconformal algebras were constructed by Kac (see [8]) and by Ademollo et al. [10] originally but independently, which are closely related the conformal field theory and the string theory, and act important roles in both mathematics and physics supplying the underlying symmetries of string theory. It is well-known that the $N = 2$ superconformal algebras fall into four sectors: the Neveu-Schwarz sector, the Ramond sector, the topological sector and the twisted sector. A series of good results have been obtained on these algebras (see [1, 4, 5, 9, 15, 19] and the reference cited therein). All sectors are closely related to the Virasoro algebra and the super-Virasoro algebra which play great roles in the two-dimensional conformal filed.

The notion of Lie bialgebras was introduced in 1983 by Drinfeld (see [2, 3]) during the process of investigating quantum groups. Then there appeared several papers on Lie bialgebras and Lie superbialgebras (e.g., [10, 11, 13, 18, 19]). In [10]–[13], the Lie bialgebra structures on Witt and Virasoro algebras were investigated, which are shown to be triangular coboundary. Moreover, the Lie bialgebra structures on the one-sided Witt algebra were completely classified. In [18, 19], the Lie superbialgebra structures on the generalized super-Virasoro algebra and Ramond $N = 2$ superconformal algebra were investigated. In this paper, we shall study the Lie super-bialgebra structures on the centerless twisted $N = 2$ superconformal algebra, which is proved to be coboundary triangular.

Firstly, let us recall some related definitions. Let $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ be a vector space over the complex number field $\mathbb{C}$. If $x \in \mathcal{L}_{[x]}$, then we say that $x$ is homogeneous of degree $[x]$ and we write $\text{deg} x = [x]$. Denote by $\tau$ the super-twist map of $\mathcal{L} \otimes \mathcal{L}$, i.e.,

$$\tau(x \otimes y) = (-1)^{[x][y]} y \otimes x, \quad \forall x, y \in \mathcal{L}.$$ 

For any $n \in \mathbb{N}$, denote by $\mathcal{L} \otimes^n$ the tensor product of $n$ copies of $\mathcal{L}$ and $\xi$ the super-cyclic map cyclically permuting the coordinates of $\mathcal{L} \otimes^3$, i.e.,

$$\xi = (1 \otimes \tau) \cdot (\tau \otimes 1) : x_1 \otimes x_2 \otimes x_3 \mapsto (-1)^{[x_1][x_2]+[x_3]} x_2 \otimes x_3 \otimes x_1, \quad \forall x_i \in \mathcal{L}, \ i = 1, 2, 3,$$
where $\mathbb{1}$ is the identity map of $\mathcal{L}$. Then the definition of a Lie superalgebra can be described in
the following way: A Lie superalgebra is a pair $(\mathcal{L}, \varphi)$ consisting of a vector space $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$
and a bilinear map $\varphi: \mathcal{L} \otimes \mathcal{L} \to \mathcal{L}$ satisfying:

$$\varphi(L_i, L_j) \subset L_{i+j},$$

$$\text{Ker}(1 \otimes 1 - \tau) \subset \text{Ker} \varphi,$$

$$\varphi \cdot (1 \otimes \varphi) \cdot (1 \otimes 1 + \xi + \xi^2) = 0.$$  \hspace{1cm} (1.1)

Meanwhile, the definition of a Lie super-coalgebra can be described in the following way: A Lie super-coalgebra is a pair $(\mathcal{L}, \Delta)$ consisting of a vector space $\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1$ and a linear map $\Delta: \mathcal{L} \to \mathcal{L} \otimes \mathcal{L}$ satisfying:

$$\Delta(L_i) \subset \sum_{j \in \mathbb{Z}_2} L_j \otimes L_{i-j},$$

$$\text{Im} \Delta \subset \text{Im}(1 \otimes 1 - \tau),$$

$$\left(1 \otimes 1 \otimes 1 + \xi + \xi^2\right) \cdot (1 \otimes \Delta) \cdot \Delta = 0.$$ \hspace{1cm} (1.2)

Now one can give the definition of a Lie super-bialgebra, which is a triple $(\mathcal{L}, \varphi, \Delta)$ satisfying:

(i) $(\mathcal{L}, \varphi)$ is a Lie superalgebra,

(ii) $(\mathcal{L}, \Delta)$ is a Lie super-coalgebra,

(iii) $\Delta \varphi(x \otimes y) = x \ast \Delta y - (-1)^{|x||y|} y \ast \Delta x \ \forall \ x, y \in \mathcal{L},$

where the symbol “$\ast$” means the adjoint diagonal action

$$x \ast (\sum a_i \otimes b_i) = \sum (\langle x, a_i \rangle \otimes b_i + (-1)^{|x||a_i|} a_i \otimes \langle x, b_i \rangle), \ \forall \ x, a_i, b_i \in \mathcal{L},$$ \hspace{1cm} (1.3)

and in general $[x, y] = \varphi(x \otimes y)$ for $x, y \in \mathcal{L}$.

Denote by $\mathcal{U}(\mathcal{L})$ the universal enveloping algebra of $\mathcal{L}$ and $A \setminus B = \{x \mid x \in A, x \notin B\}$ for any two sets $A$ and $B$. If $r = \sum a_i \otimes b_i \in \mathcal{L} \otimes \mathcal{L}$, then the following elements are in $\mathcal{U}(\mathcal{L}) \otimes \mathcal{U}(\mathcal{L}) \otimes \mathcal{U}(\mathcal{L})$

$$r^{12} = \sum a_i \otimes b_i \otimes 1 = r \otimes 1, \ r^{23} = \sum 1 \otimes a_i \otimes b_i = 1 \otimes r,$$

$$r^{13} = \sum a_i \otimes 1 \otimes b_i = (1 \otimes \tau)(r \otimes 1) = (\tau \otimes 1)(1 \otimes r),$$

while the following elements are in $\mathcal{L} \otimes \mathcal{L} \otimes \mathcal{L}$

$$[r^{12}, r^{23}] = \sum_{i,j} a_i \otimes [b_i, a_j] \otimes b_j,$$

$$[r^{12}, r^{13}] = \sum_{i,j} (-1)^{|a_j||b_i|} [a_i, a_j] \otimes b_i \otimes b_j,$$

$$[r^{13}, r^{23}] = \sum_{i,j} (-1)^{|a_j||b_i|} a_i \otimes a_j \otimes [b_i, b_j].$$
**Definition 1.1.** (i) A coboundary super-bialgebra is a quadruple \((\mathcal{L}, \varphi, \Delta, r)\), where \((\mathcal{L}, \varphi, \Delta)\) is a Lie super-bialgebra and \(r \in \text{Im}(1 \otimes 1 - \tau) \subset \mathcal{L} \otimes \mathcal{L}\) such that \(\Delta = \Delta_r\) is a coboundary of \(r\), i.e.,

\[
\Delta_r(x) = (-1)^{|[x]|r}x \ast r, \quad \forall \ x \in \mathcal{L}.
\] (1.4)

(ii) A coboundary Lie super-bialgebra \((\mathcal{L}, \varphi, \Delta, r)\) is called triangular if it satisfies the following classical Yang-Baxter Equation

\[
c(\cdot) := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0. \quad \text{(CYBE)}
\] (1.5)

Let \( V = V_0 \oplus V_1 \) be an \( \mathcal{L} \)-module where \( \mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_1 \). A \( \mathbb{Z}_2 \)-homogenous linear map \( d : \mathcal{L} \rightarrow V \) is called a homogenous derivation of degree \([d] \in \mathbb{Z}_2\), if \( d(\mathcal{L}_i) \subset V_{i+[d]} \) (\( \forall \ i \in \mathbb{Z}_2\)),

\[
d([x, y]) = (-1)^{|d|}[x] \ast d(y) - (-1)^{|y|}[d][x]y \ast d(x), \quad \forall \ x, y \in \mathcal{L}.
\] (1.6)

Denote by \( \text{Der}_i(\mathcal{L}, V) \) \((i = 0, 1)\) the set of all homogenous derivations of degree \(i\). Then the set of all derivations from \( \mathcal{L} \) to \( V \), \( \text{Der}(\mathcal{L}, V) = \text{Der}_0(\mathcal{L}, V) \oplus \text{Der}_1(\mathcal{L}, V) \). Denote by \( \text{Inn}_i(\mathcal{L}, V) \) \((i = 0, 1)\) the set of homogenous inner derivations of degree \(i\), consisting of \(a_{\text{inn}}\), \(a \in V_i\), defined by

\[
a_{\text{inn}} : x \mapsto (-1)^{|a|[x]}x \ast a, \quad \forall \ x \in \mathcal{L}.
\] (1.7)

Then the set of inner derivations \( \text{Inn}(\mathcal{L}, V) = \text{Inn}_0(\mathcal{L}, V) \oplus \text{Inn}_1(\mathcal{L}, V) \).

Denote by \( H^1(\mathcal{L}, V) \) the first cohomology group of \( \mathcal{L} \) with coefficients in \( V \). Then

\[
H^1(\mathcal{L}, V) \cong \text{Der}(\mathcal{L}, V) / \text{Inn}(\mathcal{L}, V).
\]

An element \( r \) in a superalgebra \( \mathcal{L} \) is said to satisfy the modified Yang-Baxter equation if

\[
x \ast c(\cdot) = 0, \quad \forall \ x \in \mathcal{L}. \quad \text{(MYBE)}
\] (1.8)

The centerless twisted \( N = 2 \) superconformal algebra \( \mathcal{L} \) consists of the Virasoro algebra generators \( L_m, m \in \mathbb{Z} \), corresponding to the stress-energy tensor, a Heisenberg algebra \( T_r \), with half-integral \( r \in \frac{1}{2} + \mathbb{Z} \), corresponding to the \( U(1) \) current, and the fermionic generators \( G_p, p \in \frac{1}{2} \mathbb{Z} \), which are the modes of the two spin-\( \frac{3}{2} \) fermionic fields with the following commutation relations (see, e.g., [4]),

\[
[L_m, L_n] = (m - n)L_{m+n},
\]

\[
[T_r, T_s] = 0,
\]

\[
[L_m, T_r] = -rT_{r+m},
\]

\[
[L_m, G_p] = (\frac{m}{2} - p)G_{p+m},
\]

\[
[G_p, G_q] = \begin{cases} (-1)^{2p}2L_{p+q} & \text{if } p + q \in \mathbb{Z}, \\ (-1)^{2p+1}(p - q)T_{p+q} & \text{if } p + q \in \frac{1}{2} + \mathbb{Z}. \end{cases}
\] (1.9)
for $m,n \in \mathbb{Z}$, $r,s \in \frac{1}{2} + \mathbb{Z}$, $p,q \in \frac{1}{2}\mathbb{Z}$. Obviously, $\mathcal{L}$ is $\mathbb{Z}_2$-graded: $\mathcal{L} = \mathcal{L}_{\pi} \oplus \mathcal{L}_{T}$, with

$$
\mathcal{L}_{\pi} = \text{span}_\mathbb{C}\{L_m, T_r \mid m, r \in \frac{1}{2} + \mathbb{Z}\}, \quad \mathcal{L}_{T} = \text{span}_\mathbb{C}\{G_p \mid p \in \frac{1}{2}\mathbb{Z}\}.
$$

(1.10)

The Cartan subalgebra of $\mathcal{L}$ is $\mathcal{H} = \mathbb{C}L_0$ and $\mathcal{W} = \text{span}_\mathbb{C}\{L_m \mid m \in \mathbb{Z}\}$ is the well-known centerless Virasoro algebra.

The main result of this paper can be formulated as follows.

**Theorem 1.2.** Every Lie super-bialgebra structure on the centerless twisted $N = 2$ super-conformal algebra $\mathcal{L}$ defined in (1.9) is triangular coboundary.

§2. Proof of the main result

The following result for the non-super case can be found in [13] while its super case can be found in [19].

**Lemma 2.1.** Let $\mathcal{L}$ be a Lie superalgebra, $r \in \text{Im}(1 \otimes 1 - \tau) \subset \mathcal{L} \otimes \mathcal{L}$ with $[r] = 0$. Then

$$(1 + \xi + \xi^2) \cdot (1 \otimes \Delta_r) \cdot \Delta_r(x) = x \ast c(r), \quad \forall \ x \in \mathcal{L}. \quad (2.1)$$

Thus $(\mathcal{L}, [\cdot, \cdot], \Delta_r)$ is a Lie super-bialgebra if and only if $r$ satisfies (MYBE) (see(1.8)).

The following lemma can be obtained by using the similar techniques of [16, Lemma 2.2].

**Lemma 2.2.** Regarding $\mathcal{L}^\otimes n$ as an $\mathcal{L}$-module under the adjoint diagonal action of $\mathcal{L}$, if $r \in \mathcal{L}^\otimes n$ such that $x \ast r = 0$, $\forall \ x \in \mathcal{L}$, then one has $r = 0$.

As a conclusion of Lemma 2.2, one immediately obtains

**Corollary 2.3.** An element $r \in \text{Im}(1 \otimes 1 - \tau) \subset \mathcal{L} \otimes \mathcal{L}$ satisfies CYBE in (1.5) if and only if it satisfies MYBE in (1.8).

**Proposition 2.4.** $\text{Der}(\mathcal{L}, \mathcal{V}) = \text{Inn}(\mathcal{L}, \mathcal{V})$, where $\mathcal{V} = \mathcal{L} \otimes \mathcal{L}$, equivalently, $H^1(\mathcal{L}, \mathcal{V}) = 0$.

**Proof.** Note that $\mathcal{V} = \bigoplus_{i \in \frac{1}{2}\mathbb{Z}} \mathcal{V}_i$ is also $\frac{1}{2}\mathbb{Z}$-graded with $\mathcal{V}_i = \sum_{j+k=i} \mathcal{L}_j \otimes \mathcal{L}_k$, where $i,j,k \in \frac{1}{2}\mathbb{Z}$. We say a derivation $d \in \text{Der}(\mathcal{L}, \mathcal{V})$ is homogeneous of degree $i \in \frac{1}{2}\mathbb{Z}$ if $d(\mathcal{V}_j) \subset \mathcal{V}_{i+j}$ for all $j \in \frac{1}{2}\mathbb{Z}$. Set $\text{Der}(\mathcal{L}, \mathcal{V})_i = \{d \in \text{Der}(\mathcal{L}, \mathcal{V}) \mid \text{deg} \ d = i\}$ for $i \in \frac{1}{2}\mathbb{Z}$.

For any $d \in \text{Der}(\mathcal{L}, \mathcal{V})$, $i \in \frac{1}{2}\mathbb{Z}$, $u \in \mathcal{L}_j$ with $j \in \frac{1}{2}\mathbb{Z}$, we can write $d(u) = \sum_{k \in \mathbb{Z}} v_k \in \mathcal{V}$ with $v_k \in \mathcal{V}_k$, then we set $d_i(u) = v_{i+j}$. Then $d_i \in \text{Der}(\mathcal{L}, \mathcal{V})_i$ and

$$
d = \sum_{i \in \frac{1}{2}\mathbb{Z}} d_i \quad \text{where} \ d_i \in \text{Der}(\mathcal{L}, \mathcal{V})_i, \quad (2.2)
$$

which holds in the sense that for every $u \in \mathcal{L}$ only finitely many $d_i(u) \neq 0$, and $d(u) = \sum_{i \in \mathbb{Z}} d_i(u)$ (we call such a sum in (2.2) summable).
Claim 1. If $i \in \frac{1}{2}\mathbb{Z}\setminus\{0\}$, then $d_i \in \text{Inn}(\mathcal{L}, \mathcal{V})$.

Denote $u = -\frac{1}{2}d_i(L_0) \in \mathcal{V}_i$. For any $x_j \in \mathcal{L}_j$ and $j \in \frac{1}{2}\mathbb{Z}$, applying $d_i$ to $[L_0, x_j] = -jx_j$, using $d_i(x_j) \in \mathcal{V}_{i+j}$ and the action of $L_0$ on $\mathcal{V}_{i+j}$ is the scalar $L_0|_{\mathcal{V}_{i+j}} = -(i+j)$, one has

$$-(i+j)d_i(x_j) - (-1)^{|d_i|}x_j \cdot d_i(L_0) = -jd_i(x_j),$$

i.e., $d_i(x_j) = u_{\text{inn}}(x_j)$, which implies $d_i$ is inner.

Claim 2. $d_0(L_0) = 0$.

Using (2.3) with $i = 0$, we obtain $x \ast d_0(L_0) = 0$, $\forall x \in \mathcal{L}_j$, $j \in \frac{1}{2}\mathbb{Z}$, which together with Lemma 2.2 gives $d_0(L_0) = 0$.

Claim 3. For any $d_0 \in \text{Der}_0(\mathcal{L}, \mathcal{V})$, replacing $d_0$ by $d_0 - u_{\text{inn}}$ for some $u \in \mathcal{V}_0$, one can suppose $d_0(\mathcal{L}) = 0$.

For any $n \in \mathbb{Z}^+$, one can write $d_0(L_n)$ as

$$\sum_{i \in \mathbb{Z}} a_{n,i}L_{i+n} \otimes L_{-i} + \sum_{j \in \mathbb{Z}} (d_{n,j}L_{j+n} \otimes G_{-j} + e_{n,j}G_{j+n} \otimes L_{-j}) + \sum_{p \in \frac{1}{2}\mathbb{Z}} b_{n,p}G_{p+n} \otimes G_{-p}$$

$$+ \sum_{r \in \frac{1}{4}+\mathbb{Z}} c_{n,r}T_{r+n} \otimes T_{-r} + \sum_{r \in \frac{1}{4}+\mathbb{Z}} (f_{n,r}G_{r+n} \otimes T_{-r} + g_{n,r}T_{r+n} \otimes G_{-r}),$$

for some $a_{n,i}, b_{n,p}, c_{n,r}, d_{n,j}, e_{n,j}, f_{n,r}, g_{n,r} \in \mathbb{C}$, where the sums are all finite. Noticing that for any $i, j \in \mathbb{Z}$, $r \in \frac{1}{2} + \mathbb{Z}$, $p \in \frac{1}{2}\mathbb{Z}$, one has

$$L_1 \ast (T_r \otimes T_{-r}) = rT_r \otimes T_{1-r} - rT_{1+r} \otimes T_{-r},$$

$$L_1 \ast (T_r \otimes G_{-r}) = (0.5 + r)T_r \otimes G_{1-r} - rT_{1+r} \otimes G_{-r},$$

$$L_1 \ast (G_r \otimes T_{-r}) = rG_r \otimes T_{1-r} - (r - 0.5)G_{1+r} \otimes T_{-r},$$

$$L_1 \ast (L_i \otimes L_{-i}) = (i+1)L_i \otimes L_{1-i} - (i-1)L_{1+i} \otimes L_{-i},$$

and

$$L_1 \ast (L_j \otimes G_{-j}) = (j + 0.5)L_j \otimes G_{1-j} - (j - 1)1_{1+j} \otimes G_{-j},$$

$$L_1 \ast (G_j \otimes L_{-j}) = (j+1)G_j \otimes L_{1-j} - (j-0.5)G_{1+j} \otimes L_{-j},$$

$$L_1 \ast (G_p \otimes G_{-p}) = (p+0.5)G_p \otimes G_{1-p} - (p-0.5)G_{1+p} \otimes G_{-p}.$$

Denote

$$M_{1,1} = \max\{|i| \mid a_{1,i} \cdot b_{1,i} \cdot d_{1,i} \cdot e_{1,i} \neq 0\}, \quad M_{1,2} = \max\{|r| \mid b_{1,r} \cdot c_{1,r} \cdot f_{1,r} \cdot g_{1,r} \neq 0\}.$$

Using the induction on $M_{1,1} + M_{1,2}$, and replacing $d_0$ by $d_0 - u_{\text{inn}}$, where $u$ is a combination of some $L_i \otimes L_{-i}$, $G_p \otimes G_{-p}$, $T_r \otimes T_{-r}$, $G_r \otimes T_{-r}$, $T_r \otimes G_{-r}$, $L_j \otimes G_{-j}$ and $G_j \otimes L_{-j}$, one can suppose

$$a_{1,i} = b_{1,p} = c_{1,r} = d_{1,j} = e_{1,k} = f_{1,r} = g_{1,r_2} = 0,$$
for any $i \in \mathbb{Z}\backslash\{-2,1\}$, $j \in \mathbb{Z}\backslash\{1\}$, $k \in \mathbb{Z}\backslash\{-2\}$, $p \in \frac{1}{2}\mathbb{Z}\backslash\{-\frac{3}{2}, \frac{1}{2}\}$, $r \in \frac{1}{2} + \mathbb{Z}$, $r_1 \in \frac{1}{2} + \mathbb{Z}\backslash\{\frac{3}{2}\}$, $r_2 \in \frac{1}{2} + \mathbb{Z}\backslash\{\frac{3}{2}\}$, using which one can rewrite $d_0(L_1)$ as

\begin{align*}
  d_0(L_1) &= a_{1,-2}L_{-1} \otimes L_2 + a_{1,1}L_2 \otimes L_{-1} + b_{1,0}G_{-\frac{1}{2}} \otimes G_{\frac{3}{2}} + b_{1,\frac{1}{2}}G_{\frac{1}{2}} \otimes G_{-\frac{1}{2}} \\
  &\quad + d_{1,1}L_2 \otimes G_{-1} + e_{1,-2}G_{-1} \otimes L_2 + f_{1,\frac{1}{2}}G_{\frac{1}{2}} \otimes T_{-\frac{1}{2}} + g_{1,-\frac{1}{2}}T_{-\frac{1}{2}} \otimes G_{\frac{1}{2}}. \quad (2.5)
\end{align*}

Then one can see that $2L_{-1} \ast d_0(L_1)$ is equal to

\begin{align*}
  -6a_{1,-2}L_{-1} \otimes L_1 - 6a_{1,1}L_1 \otimes L_{-1} - 6d_{1,1}L_1 \otimes G_{-1} - 6e_{1,-2}G_{-1} \otimes L_1 \\
  + d_{1,1}L_2 \otimes G_{-2} + e_{1,-2}G_{-2} \otimes L_2 - 4b_{1,0}G_{-\frac{1}{2}} \otimes G_{\frac{1}{2}} - 4b_{1,\frac{1}{2}}G_{\frac{1}{2}} \otimes G_{-\frac{1}{2}} \\
  - 4f_{1,\frac{1}{2}}G_{\frac{1}{2}} \otimes T_{-\frac{1}{2}} + f_{1,\frac{1}{2}}G_{\frac{1}{2}} \otimes T_{-\frac{1}{2}} + g_{1,-\frac{1}{2}}T_{-\frac{1}{2}} \otimes G_{\frac{1}{2}} - 4g_{1,-\frac{1}{2}}T_{-\frac{1}{2}} \otimes G_{\frac{1}{2}},
\end{align*}

while $L_1 \ast d_0(L_{-1}) - \sum_{p \in \frac{1}{2}\mathbb{Z}} (\left(\frac{3}{2} - p\right)b_{-1,p} + (\frac{3}{2} + p)b_{-1,p+1})G_p \otimes G_{-p}$ is equal to

\begin{align*}
  \sum_{i \in \mathbb{Z}} (\left((2 - i)a_{-1,i} + (2 + i)a_{1,i+1}\right)L_i \otimes L_{-i} + \sum_{r \in \frac{1}{2} + \mathbb{Z}} \left((1 - r)c_{-1,r} + (1 + r)c_{-1,r+1}\right)T_r \otimes T_{-r} \\
  + \sum_{j \in \mathbb{Z}} \left((2 - j)d_{-1,j} + (2 + j)d_{1,j+1}\right)L_j \otimes G_{-j} + \left((3 - j)e_{-1,j} + (2 + j)e_{1,j+1}\right)G_j \otimes L_{-j} \\
  + \sum_{r \in \frac{1}{2} + \mathbb{Z}} \left((\left(\frac{3}{2} - r\right)f_{-1,r} + (2 + 1)f_{-1,r+1}\right)G_r \otimes T_{-r} + \left((1 - r)g_{-1,r} + (2 + r)g_{1,r+1}\right)T_r \otimes G_{-r}).
\end{align*}

Applying $d_0$ to $[L_1, L_{-1}] = 2L_0$ and using Claim 2, we obtain

\begin{equation}
  L_{-1} \ast d_0(L_1) = L_1 \ast d_0(L_{-1}). \quad (2.6)
\end{equation}

Comparing the coefficients of $T_r \otimes T_{-r}$, $L_i \otimes L_{-i}$ and $G_p \otimes G_{-p}$ in (2.6), one has

\begin{align*}
  (r - 1)c_{-1,r} &= (r + 1)c_{-1,r+1}, \\
  3a_{1,-1} + a_{-1,0} + 3a_{1,-2} &= a_{-1,1} + 3a_{-1,2} + 3a_{1,1} = (i - 2)a_{-1,i} - (i + 2)a_{1,i+1} = 0, \\
  2b_{1,0} + b_{-1,1} + 2b_{1,-\frac{3}{2}} = b_{-1,\frac{1}{2}} + 2b_{-1,\frac{1}{2}} + 2b_{1,\frac{1}{2}} = (p - \frac{3}{2})b_{-1,p} - (p + \frac{3}{2})b_{-1,p+1} = 0,
\end{align*}

for any $i \in \mathbb{Z}\backslash\{\pm 1\}$, $r \in \frac{1}{2} + \mathbb{Z}$ and $p \in \frac{1}{2}\mathbb{Z}\backslash\{\pm \frac{1}{2}\}$, which together with our purpose that all the sets $\{a_{-1,i} | i \in \mathbb{Z}\}$, $\{b_{-1,p} | p \in \frac{1}{2}\mathbb{Z}\}$, $\{c_{-1,r} | r \in \frac{1}{2} + \mathbb{Z}\}$ are of finite rank, imply

\begin{align*}
  c_{-1,r} = b_{-1,p} = 2b_{-1,0} = b_{-1,\frac{1}{2}} + 2b_{-1,\frac{1}{2}} + 2b_{1,\frac{1}{2}} = (p - \frac{3}{2})b_{-1,p} - (p + \frac{3}{2})b_{-1,p+1} = 0, \\
  a_{-1,i} = 3a_{-1,1} + a_{-1,0} + 3a_{1,-2} = a_{-1,1} + 3a_{-1,2} + 3a_{1,1} = a_{-1,0} + a_{1,1} = 0,
\end{align*}

for any $i \in \mathbb{Z}\backslash\{\pm 1, 0, 2\}$, $r \in \frac{1}{2} + \mathbb{Z}$, and $p \in \frac{1}{2}\mathbb{Z}\backslash\{\pm \frac{1}{2}, \frac{3}{2}\}$. Comparing the coefficients of $L_j \otimes G_{-j}$ and $G_k \otimes L_{-k}$ in (2.6), one has

\begin{align*}
  (j - 1)d_{-1,j} &= (j + 3)\frac{3}{2}d_{-1,j+1}, \quad d_{-1,1} + \frac{5}{2}d_{-1,2} + 3d_{1,1} = \frac{7}{2}d_{-1,3} - \frac{1}{2}d_{1,1} = 0, \\
  (k - \frac{3}{2})e_{-1,k} &= (k + 2)e_{-1,k+1}, \quad \frac{5}{2}e_{-1,1} + e_{-1,0} + 3e_{1,-2} = \frac{7}{2}e_{-1,2} - \frac{1}{2}e_{1,-2} = 0,
\end{align*}

for any $j \in \mathbb{Z}\backslash\{\pm 1\}$, $k \in \mathbb{Z}\backslash\{-2\}$, $p \in \frac{1}{2}\mathbb{Z}\backslash\{-\frac{3}{2}, \frac{1}{2}\}$, $r \in \frac{1}{2} + \mathbb{Z}$, and $r_1 \in \frac{1}{2} + \mathbb{Z}\backslash\{\frac{3}{2}\}$, $r_2 \in \frac{1}{2} + \mathbb{Z}\backslash\{\frac{3}{2}\}$.
for any \( j \in \mathbb{Z} \setminus \{1, 2\} \) and \( k \in \mathbb{Z} \setminus \{-1, -2\} \), which together with our suppose that all the sets \( \{d_{-1,j} \mid j \in \mathbb{Z}\} \) and \( \{e_{-1,k} \mid k \in \mathbb{Z}\} \) are of finite rank, imply

\[
d_{1,1} = e_{1,-2} = d_{-1,j} = e_{-1,j} = 0, \quad \forall \ j \in \mathbb{Z}.
\]

Comparing the coefficients of \( G_r \otimes T_{-r} \) and \( T_r \otimes G_{-r} \) in (2.6), one has

\[
(r - \frac{3}{2}) f_{-1,r} = (1 + r) f_{-1,r+1}, \quad f_{-1,\frac{1}{2}} + \frac{3}{2} f_{-1,\frac{3}{2}} + 2 f_{1,\frac{1}{2}} = \frac{5}{2} f_{-1,\frac{5}{2}} - \frac{1}{2} f_{1,\frac{1}{2}} = 0,
\]

\[
(s - 1) g_{-1,s} = \left( \frac{3}{2} + s \right) g_{-1,s+1}, \quad \frac{3}{2} g_{-1,-\frac{1}{2}} + g_{-1,\frac{1}{2}} + 2 g_{1,-\frac{3}{2}} = \frac{5}{2} g_{-1,-\frac{5}{2}} - \frac{1}{2} g_{1,-\frac{3}{2}} = 0,
\]

for any \( r \in \frac{1}{2} + \mathbb{Z} \setminus \{\frac{1}{2}, \frac{3}{2}\} \) and \( s \in \frac{1}{2} + \mathbb{Z} \setminus \{-\frac{1}{2}, -\frac{3}{2}\} \), which together with our suppose that all the sets \( \{f_{-1,r} \mid r \in \frac{1}{2} + \mathbb{Z}\} \) and \( \{g_{-1,r} \mid r \in \frac{1}{2} + \mathbb{Z}\} \) are of finite rank, give

\[
f_{1,\frac{1}{2}} = g_{1,-\frac{3}{2}} = 0 = f_{-1,r} = g_{-1,r}, \quad \forall \ r \in \frac{1}{2} + \mathbb{Z}.
\]

Then one can rewrite \( d_0(L_{-1}) \) (see (2.4)) as

\[
d_0(L_{-1}) = 3(a_{-1,2} + a_{1,1})(L_0 \otimes L_0 - L_0 \otimes L_{-1}) + a_{-1,2} L_1 \otimes L_2 - (a_{-1,2} + a_{1,1} + a_{1,-2}) L_2 \otimes L_1 + (b_{-1,\frac{1}{2}} + b_{1,\frac{1}{2}} - b_{1,-\frac{3}{2}}) G_{-\frac{1}{2}} \otimes G_{\frac{1}{2}} - 2(b_{-1,\frac{1}{2}} + b_{1,\frac{1}{2}}) G_{-\frac{1}{2}} \otimes G_{\frac{1}{2}} - b_{-1,\frac{3}{2}} G_{\frac{1}{2}} \otimes G_{-\frac{3}{2}}.
\]

Applying \( d_0 \) to \([L_2, L_{-1}] = 3L_1\), we obtain

\[
L_2 \ast d_0(L_{-1}) - L_{-1} \ast d_0(L_2) = 3d_0(L_1), \tag{2.7}
\]

where

\[
L_2 \ast d_0(L_{-1}) = -4(a_{-1,2} + a_{1,1} + a_{1,-2}) L_0 \otimes L_1 - (a_{-1,2} + a_{1,1} + a_{1,-2}) L_2 \otimes L_3 + 3(a_{-1,2} + a_{1,1})(3L_1 \otimes L_0 - 2L_2 \otimes L_{-1}) + 3(a_{-1,2} + a_{1,1})(2L_{-1} \otimes L_2 - 3L_0 \otimes L_1) + a_{-1,2} L_3 \otimes L_2 + 4a_{-1,2} L_1 \otimes L_0 + (b_{-1,\frac{1}{2}} + b_{1,\frac{1}{2}} - b_{1,-\frac{3}{2}})(\frac{5}{2} G_{\frac{1}{2}} \otimes G_{\frac{1}{2}} + \frac{1}{2} G_{-\frac{1}{2}} \otimes G_{\frac{3}{2}}) - 3(b_{-1,\frac{1}{2}} + b_{1,\frac{1}{2}}) (G_{\frac{3}{2}} \otimes G_{-\frac{1}{2}} + G_{-\frac{1}{2}} \otimes G_{\frac{3}{2}}) + \frac{1}{2} b_{-1,\frac{3}{2}} G_{\frac{3}{2}} \otimes G_{-\frac{3}{2}} + \frac{5}{2} b_{-1,\frac{3}{2}} G_{\frac{3}{2}} \otimes G_{\frac{1}{2}},
\]

and \( L_{-1} \ast d_0(L_2) \) can be rewritten as

\[
\sum_j ((j - \frac{3}{2}) d_{2,j} \ast L_{j+1} \otimes G_{-j} + \sum_k ((k - 2) e_{2,k} \ast L_{k+1} \otimes L_{-k}) + \sum_r ((r - \frac{3}{2}) g_{2,r} \ast L_{r+1} \otimes G_{-r} + \sum_k ((r - 1) f_{2,k} \ast L_{k+1} \otimes L_{-k}) + \sum_i ((i - 2) a_{2,i} \ast L_{i+1} \otimes L_{-i} + \sum_p ((p - \frac{3}{2}) b_{2,p} \ast L_{p+1} \otimes L_{-p}) + \sum_r ((r - 1) c_{2,r} \ast L_{r+1} \otimes L_{-r}) + \sum_r ((r - 2) d_{2,r} \ast L_{r+1} \otimes L_{-r}).
\]
Comparing the coefficients of $G_{p+1} \otimes G_{-p}$ and $T_{r+1} \otimes T_{-r}$ in (2.7) with $p \in \mathbb{Z}$, $r \in \frac{1}{2} + \mathbb{Z}$, we obtain
\[
(p - \frac{3}{2})b_{2,p-1} = (p + \frac{5}{2})b_{2,p}, \quad (r - 1)c_{2,r-1} = (r + 2)c_{2,r},
\]
which together with our suppose that all the sets $\{b_{2,p} \mid p \in \mathbb{Z}\}$ and $\{c_{2,r} \mid r \in \frac{1}{2} + \mathbb{Z}\}$ are of finite rank, give
\[
b_{2,p} = c_{2,r} = 0, \quad \forall \ p \in \mathbb{Z}, \ r \in \frac{1}{2} + \mathbb{Z}.
\]
Comparing the coefficients of $L_{i+1} \otimes L_{-i}$ in (2.7) with $i \in \mathbb{Z}$, one has
\[
\sum_{i \in \mathbb{Z}} ((i - 2)a_{2,i-1} - (i + 3)a_{2,i})L_{i+1} \otimes L_{-i} + 3(2a_{-1,2} + 3a_{1,1})L_{2} \otimes L_{-1}
= -(a_{-1,2} + a_{1,1} + a_{-1,2})L_{2} \otimes L_{3} + 3(2a_{-1,2} + 2a_{1,1} - a_{-1,2})L_{-1} \otimes L_{2}
- (13a_{-1,2} + 13a_{1,1} + 4a_{1,2})L_{0} \otimes L_{1} + (13a_{-1,2} + 9a_{1,1})L_{1} \otimes L_{0} + a_{-1,2}L_{3} \otimes L_{-2},
\]
which together with our suppose that the set $\{a_{2,i} \mid i \in \mathbb{Z}\}$ is of finite rank, give
\[
a_{1,-2} + a_{1,1} = a_{-1,2} = a_{2,i} = 0, \quad 4a_{2,-3} = -15a_{1,1} - a_{2,0},
a_{2,-2} = 6a_{1,1} + a_{2,0}, \quad 2a_{2,-1} = -9a_{1,1} - 3a_{2,0}, \quad 4a_{2,1} = 9a_{1,1} - a_{2,0},
\]
for any $i \in \mathbb{Z} \setminus \{-3, \cdots, 1\}$. Comparing the coefficients of $G_{p+1} \otimes G_{-p}$ in (2.7) with $p \in \frac{1}{2} + \mathbb{Z}$, one has
\[
\sum_{p} ((p - \frac{3}{2})b_{2,p-1} - (p + \frac{5}{2})b_{2,p})G_{p+1} \otimes G_{-p} - \frac{1}{2}b_{-1,\frac{3}{2}}G_{\frac{1}{2}} \otimes G_{-\frac{1}{2}}
= \frac{1}{2}(b_{-1,\frac{3}{2}} + b_{1,\frac{1}{2}} - b_{-1,\frac{1}{2}})G_{-\frac{1}{2}} \otimes G_{\frac{1}{2}} - 3(b_{-1,\frac{3}{2}} + b_{1,\frac{1}{2}} + b_{-1,\frac{1}{2}})G_{-\frac{1}{2}} \otimes G_{\frac{1}{2}}
+ \frac{5}{2}(2b_{-1,\frac{1}{2}} + b_{1,\frac{1}{2}} - b_{-1,\frac{1}{2}})G_{\frac{1}{2}} \otimes G_{-\frac{1}{2}} - 3(b_{-1,\frac{1}{2}} + 2b_{1,\frac{1}{2}})G_{\frac{1}{2}} \otimes G_{-\frac{1}{2}},
\]
which together with our suppose that the set $\{b_{2,p} \mid p \in \frac{1}{2} + \mathbb{Z}\}$ is of finite rank, give
\[
b_{2,-\frac{3}{2}} = -6b_{1,\frac{1}{2}} + 3b_{2,\frac{1}{2}} = -b_{2,-\frac{1}{2}}, \quad b_{1,-\frac{3}{2}} = b_{1,\frac{1}{2}}, \quad b_{2,-\frac{1}{2}} = 4b_{1,\frac{1}{2}} - b_{2,\frac{1}{2}}, \quad b_{-1,\frac{3}{2}} = b_{2,p} = 0,
\]
for any $p \in \frac{1}{2} + \mathbb{Z} \setminus \{-\frac{5}{2}, \cdots, \frac{1}{2}\}$. Comparing the coefficients of $L_{j+1} \otimes G_{-j}$ in (2.7) with $j \in \mathbb{Z}$, one has
\[
\sum_{j \in \mathbb{Z}} ((j - \frac{3}{2})d_{2,j-1} - (j + 3)d_{2,j})L_{j+1} \otimes G_{-j} = 0,
\]
which together with our suppose that the set $\{d_{2,j} \mid j \in \mathbb{Z}\}$ is of finite rank, give
\[
d_{2,j} = 0, \quad \forall \ j \in \mathbb{Z}.
\]
Comparing the coefficients of $G_{k+1} \otimes L_{-k}$ in (2.7) with $k \in \mathbb{Z}$, one has

$$\sum_{k \in \mathbb{Z}} ((k - 2)e_{2,k-1} - (k + \frac{5}{2})e_{2,k})G_{k+1} \otimes L_{-k} = 0,$$

which together with our suppose that the set $\{e_{2,k} \mid k \in \mathbb{Z}\}$ is of finite rank, give $e_{2,k} = 0, \ \forall \ k \in \mathbb{Z}$.

Comparing the coefficients of $G_{r+1} \otimes T_{-r}$ in (2.7) with $r \in \frac{1}{2} + \mathbb{Z}$, one has

$$\sum_{r \in \frac{1}{2} + \mathbb{Z}} ((r - 1)f_{2,r-1} - (r + \frac{5}{2})f_{2,r})G_{r+1} \otimes T_{-r} = 0,$$

which together with our suppose that the set $\{f_{2,r} \mid r \in \frac{1}{2} + \mathbb{Z}\}$ is of finite rank, give $f_{2,r} = 0, \ \forall \ r \in \frac{1}{2} + \mathbb{Z}$.

Comparing the coefficients of $T_{r+1} \otimes G_{-r}$ in (2.7) with $r \in \frac{1}{2} + \mathbb{Z}$, one has

$$\sum_{r \in \frac{1}{2} + \mathbb{Z}} ((r - \frac{3}{2})g_{2,r-1} - (r + 2)g_{2,r})T_{r+1} \otimes G_{-r} = 0,$$

which together with our suppose that the set $\{g_{2,r} \mid r \in \frac{1}{2} + \mathbb{Z}\}$ is of finite rank, give $g_{2,r} = 0, \ \forall \ r \in \frac{1}{2} + \mathbb{Z}$.

Then one can rewrite $d_0(L_1)$, $d_0(L_{-1})$ and $d_0(L_2)$ as

$$d_0(L_1) = a_{1,1}(L_2 \otimes L_{-1} - L_{-1} \otimes L_2) + b_{1,\frac{1}{2}}(G_{-\frac{1}{2}} \otimes G_{\frac{1}{2}} + G_{\frac{1}{2}} \otimes G_{-\frac{1}{2}}),$$

$$d_0(L_{-1}) = 3a_{1,1}(L_{-1} \otimes L_0 - L_0 \otimes L_{-1}) - 2b_{1,\frac{1}{2}}G_{-\frac{1}{2}} \otimes G_{-\frac{1}{2}},$$

$$d_0(L_2) = -\frac{1}{4}(15a_{1,1} + a_{2,0})L_{-1} \otimes L_3 + (6a_{1,1} + a_{2,0})L_0 \otimes L_2 - \frac{3}{2}(3a_{1,1} + a_{2,0})L_1 \otimes L_1 + a_{2,0}L_2 \otimes L_0 + \frac{1}{4}(9a_{1,1} - a_{2,0})L_3 \otimes L_{-1} + (4b_{1,\frac{1}{2}} - b_{2,\frac{1}{2}})G_{-\frac{1}{2}} \otimes G_{\frac{1}{2}} + b_{2,\frac{1}{2}}G_{\frac{1}{2}} \otimes G_{-\frac{1}{2}} + 3(2b_{1,\frac{1}{2}} - b_{2,\frac{1}{2}})(G_{\frac{1}{2}} \otimes G_{\frac{1}{2}} - G_{\frac{1}{2}} \otimes G_{\frac{1}{2}}).$$

Applying $d_0$ to $[L_1, L_{-2}] = 3L_{-1}$, one has

$$L_1 \ast d_0(L_{-2}) - L_{-2} \ast d_0(L_1) = 3d_0(L_{-1}),$$

while $L_1 \ast d_0(L_{-2})$ can be written as

$$\sum_{j}((3 - j)d_{-2,j} + (j + \frac{3}{2})d_{-2,j+1})L_{j-1} \otimes G_{j-1} + \sum_{k}((\frac{5}{2} - k)e_{-2,k} + (k + 2)e_{-2,k+1})G_{k-1} \otimes L_{-k} + \sum_{r}((\frac{5}{2} - r)f_{-2,r} + (r + 1)f_{-2,r+1})G_{r-1} \otimes T_{-r} + \sum_{r}((2 - r)g_{-2,r} + (r + \frac{3}{2})g_{-2,r+1})T_{r-1} \otimes G_{-r},$$

$$+ \sum_{i}((3 - i)a_{-2,i} + (i + 2)a_{-2,i+1})L_{i-1} \otimes L_{-i} + \sum_{p}((\frac{5}{2} - p)b_{-2,p} + (p + \frac{3}{2})b_{-2,p+1})G_{p-1} \otimes G_{-p} + \sum_{r}((2 - r)c_{-2,r} + (r + 1)c_{-2,r+1})T_{r-1} \otimes T_{-r},$$
and
\[ L_{-2} * d_0(L_1) = a_{1,1}(L_{-3} \otimes L_2 - L_2 \otimes L_{-3}) + 4a_{1,1}(L_{-1} \otimes L_0 - L_0 \otimes L_{-1}) - \frac{1}{2}(b_{1,\frac{1}{2}}G_{-\frac{3}{2}} \otimes G_{\frac{3}{2}} + b_{1,\frac{1}{2}}G_{\frac{3}{2}} \otimes G_{-\frac{3}{2}}) - 5b_{1,\frac{1}{2}}G_{-\frac{3}{2}} \otimes G_{-\frac{3}{2}}. \]

Comparing the coefficients of \( L_{i-1} \otimes L_{-i} \) in (2.8), one has
\[ \sum_{i \in \mathbb{Z}} ((3 - i)a_{-2,i} + (i + 2)a_{-2,i+1})L_{i-1} \otimes L_{-i} = a_{1,1}L_{-3} \otimes L_2 + 13a_{1,1}L_{-1} \otimes L_0 - 13a_{1,1}L_0 \otimes L_{-1} - a_{1,1}L_2 \otimes L_{-3}, \]
which forces
\[ a_{1,1} = a_{-2,i} = 0, \quad \forall i \in \mathbb{Z}. \]

Comparing the coefficients of \( G_{p-1} \otimes G_{-p} \) in (2.8), one has
\[ \sum_{p} ((\frac{5}{2} - p)b_{-2,p} + (p + \frac{3}{2})b_{-2,p+1})G_{p-1} \otimes G_{-p} = -\frac{1}{2}b_{1,\frac{1}{2}}G_{-\frac{5}{2}} \otimes G_{\frac{5}{2}} - 11b_{1,\frac{1}{2}}G_{-\frac{1}{2}} \otimes G_{\frac{1}{2}} - \frac{1}{2}b_{1,\frac{1}{2}}G_{\frac{1}{2}} \otimes G_{-\frac{1}{2}}, \]
which forces
\[ b_{1,\frac{1}{2}} = b_{-2,p} = 0, \quad \forall p \in \frac{1}{2}\mathbb{Z}. \]

Comparing the coefficients of \( T_{r-1} \otimes T_{-r}, L_{j-1} \otimes G_{-j}, G_{k-1} \otimes L_{-k}, G_{r-1} \otimes T_{-r} \) and \( T_{r-1} \otimes G_{-r} \) in (2.8), one has
\[ (r - 2)c_{-2,r} = (r + 1)c_{-2,r+1}, \quad (j - 3)d_{-2,j} = (j + \frac{3}{2})d_{-2,j+1}, \quad (k - \frac{5}{2})e_{-2,k} = (k + 2)e_{-2,k+1}, \]
\[ (r - \frac{5}{2})f_{-2,r} = (r + 1)f_{-2,r+1}, \quad (r - 2)g_{-2,r} = (r + \frac{3}{2})g_{-2,r+1}, \]
which force
\[ c_{-2,r} = d_{-2,j} = e_{-2,k} = f_{-2,r} = g_{-2,r} = 0, \quad \forall j, k \in \mathbb{Z}, ~ r \in \frac{1}{2} + \mathbb{Z}. \]

Then one can rewrite \( d_0(L_{\pm 1}) \) and \( d_0(L_{\pm 2}) \) as
\[ d_0(L_1) = d_0(L_{-1}) = d_0(L_{-2}) = 0, \]
\[ d_0(L_2) = -\frac{1}{4}a_{2,0}L_{-1} \otimes L_3 + a_{2,0}L_3 \otimes L_{-1} + a_{2,0}(L_0 \otimes L_2 + L_2 \otimes L_0) - \frac{3}{2}a_{2,0}L_1 \otimes L_1 + b_{2,\frac{1}{2}}(G_{\frac{3}{2}} \otimes G_{-\frac{3}{2}} - G_{-\frac{3}{2}} \otimes G_{\frac{3}{2}}) - 3b_{2,\frac{1}{2}}(G_{\frac{1}{2}} \otimes G_{-\frac{1}{2}} - G_{-\frac{1}{2}} \otimes G_{\frac{1}{2}}). \]

Applying \( d_0 \) to \([L_2, L_{-2}] = 4L_0\), one has \( L_{-2} * d_0(L_2) = 0 \), which implies
\[ a_{2,0} = b_{2,\frac{1}{2}} = 0. \]
Then one can claim \( d_0(L_2) \) is equal to zero. Hence \( d_0(L_i) = 0, \forall i \in \mathbb{Z} \).

One can write

\[
d_0(G_{\frac{1}{2}}) = \sum_{i \in \mathbb{Z}} (a_i L_i \otimes T_{\frac{1}{2} - i} + b_i T_{\frac{1}{2} - i} \otimes L_i + c_i L_i \otimes G_{\frac{1}{2} - i} + d_i G_{\frac{1}{2} - i} \otimes L_i) \]

where the sums are all finite. Applying \( d_0 \) to \([L_1, G_{\frac{1}{2}}] = 0\) and comparing the coefficients, one can deduce \( d_0(G_{\frac{1}{2}}) \) must be zero, which together with \([L_m, G_{\frac{1}{2}}] = \frac{m-1}{2} G_{\frac{1}{2}+m}\) (\( \forall m \in \mathbb{Z} \)) and \([L_{-1}, G_{\frac{1}{2}}] = -3G_{\frac{1}{2}}\) forces

\[
d_0(G_r) = 0, \quad \forall r \in \frac{1}{2} + \mathbb{Z}.
\] (2.9)

Write \( d_0(G_0) \) as

\[
\sum_{i \in \mathbb{Z}} a_i L_i \otimes L_{-i} + \sum_{p \in \frac{1}{2} + \mathbb{Z}} b_p G_p \otimes G_{-p} + \sum_{r \in \frac{1}{2} + \mathbb{Z}} c_r T_r \otimes T_{-r} + \sum_{j \in \mathbb{Z}} d_j L_j \otimes G_{-j}
\]

\[
+ \sum_{k \in \mathbb{Z}} e_k G_k \otimes L_{-k} + \sum_{r \in \frac{1}{2} + \mathbb{Z}} f_r G_r \otimes T_{-r} + \sum_{r \in \frac{1}{2} + \mathbb{Z}} g_r T_r \otimes G_{-r},
\]

where the sums are all finite. Applying \( d_0 \) to \([G_0, G_0] = 2L_0\) and comparing the coefficients, one can deduce \((\forall r \in \frac{1}{2} + \mathbb{Z}, \ i \in \mathbb{Z})\)

\[
rb_r = c_r, \quad f_r = g_r, \quad ia_i = -4b_i, \quad d_i = -e_i.
\]

Then \( d_0(G_0) \) can be rewritten as

\[
d_0(G_0) = \sum_{i \in \mathbb{Z}} (a_i L_i \otimes L_{-i} - \frac{ia_i}{4} G_i \otimes G_{-i} + d_i L_i \otimes G_{-i} - d_i G_i \otimes L_{-i})
\]

\[
+ \sum_{r \in 1/2 + \mathbb{Z}} (b_r G_r \otimes G_{-r} + rb_r T_r \otimes T_{-r} + f_r G_r \otimes T_{-r} + f_r T_r \otimes G_{-r}).
\]

Applying \( d_0 \) to \([L_{-1}, [L_1, G_0]] = -\frac{3}{4}G_0\), we obtain

\[
L_{-1} \ast L_1 \ast d_0(G_0) = -\frac{3}{4}d_0(G_0).
\] (2.10)

Comparing the coefficients of \( L_i \otimes L_{-i}, L_i \otimes G_{-i}, G_r \otimes G_{-r}, G_r \otimes T_{-r} \) and \( T_r \otimes T_{-r} \), one has

\[
(8i^2 - 13)a_i = 4(i - 2)^2 a_{i-1} + 4(i + 2)^2 a_{i+1},
\]

\[
(4i^2 - 4)d_i = (i - 2)(2i - 3)d_{i-1} + (i + 2)(2i + 3)d_{i+1},
\]

\[
2r^2 f_r + (r - 1)f_{r-1} \frac{3}{2} - r) + (r + 1)(-r - \frac{3}{2})f_{r+1} = 0,
\]

\[
(r - \frac{3}{4})b_r + (\frac{3}{2} - r)(r - \frac{3}{2}) b_{r-1} + (\frac{3}{2} + r)(-r - \frac{3}{2})b_{r+1} = 0,
\]

which together with our suppose the set \( \{ i \mid i \in \mathbb{Z}, a_i \cdot b_{i+1} \cdot d_i \neq 0 \} \) is of finite rank, imply

\[
a_i = 0, \quad b_r = f_r = 0, \quad d_j = 0, \quad d_{-1} = -d_1,
\]
for any $i \in \mathbb{Z}$, $r \in \frac{1}{2} + \mathbb{Z}$ and $j \in \mathbb{Z}\setminus\{\pm 1\}$. Then $d_0(G_0)$ can be rewritten as

$$d_0(G_0) = -d_1 L_{-1} \otimes G_1 + d_1 L_1 \otimes G_{-1} + d_1 G_{-1} \otimes L_1 - d_1 G_1 \otimes L_{-1}. \quad (2.11)$$

One can write $d_0(T_{\frac{1}{2}})$ as

$$d_0(T_{\frac{1}{2}}) = \sum_{i \in \mathbb{Z}} a_i L_i \otimes T_{\frac{1}{2} - i} + \sum_{i \in \mathbb{Z}} b_i T_{\frac{1}{2} - i} \otimes L_i + \sum_{i \in \mathbb{Z}} c_i L_i \otimes G_{\frac{1}{2} - i} + \sum_{i \in \mathbb{Z}} d_i G_{\frac{1}{2} - i} \otimes L_i$$

$$+ \sum_{i \in \mathbb{Z}} e_i G_i \otimes T_{\frac{1}{2} - i} + \sum_{i \in \mathbb{Z}} f_i T_{\frac{1}{2} - i} \otimes G_i + \sum_{i \in \mathbb{Z}} g_i G_i \otimes G_{\frac{1}{2} - i} + \sum_{i \in \mathbb{Z}} h_i G_{\frac{1}{2} - i} \otimes G_i,$$

where the sums are all finite. Applying $d_0$ to $[L_{-1}, [L_1, T_{\frac{1}{2}}]] = \frac{3}{4} T_{\frac{1}{2}}$, we obtain

$$L_{-1} * L_1 * d_0(T_{\frac{1}{2}}) = \frac{3}{4} d_0(T_{\frac{1}{2}}). \quad (2.12)$$

Comparing the coefficients of $L_i \otimes T_{\frac{1}{2} - i}$ and $T_{\frac{1}{2} - i} \otimes L_i$ in the both sides of (2.12), we obtain

$$(2i^2 - i - 2)x_i + (2 - i)(i - \frac{3}{2})x_{i-1} - (i + \frac{1}{2})(i + 2)x_{i+1} = 0 \quad \text{for } x = a \text{ or } b,$$

which imply

$$x_i = 0, \quad \forall \; i \in \mathbb{Z}\setminus\{0, \pm 1\} \quad \text{while} \quad x_0 = -2x_1 = -2x_{-1} \quad \text{for } x = a \text{ or } b.$$

Comparing the coefficients of $L_i \otimes G_{\frac{1}{2} - i}$ and $G_{\frac{1}{2} - i} \otimes L_i$ in the both sides of (2.12), we obtain

$$(2i^2 - i - \frac{11}{4})y_i - (i - 2)^2 y_{i-1} - (i + 1)(i + 2)y_{i+1} = 0 \quad \text{for } y = c \text{ or } d,$$

which imply

$$y_i = 0 \quad \text{for } y = c \text{ or } d, \quad \forall \; i \in \mathbb{Z}.$$

Comparing the coefficients of $G_i \otimes T_{\frac{1}{2} - i}$ and $T_{\frac{1}{2} - i} \otimes G_i$ in the both sides of (2.12), we obtain

$$(2i^2 - i - \frac{3}{4})z_i - (\frac{3}{2} - i)^2 z_{i-1} - (i + \frac{1}{2})(i + \frac{3}{2})z_{i+1} = 0 \quad \text{for } z = e \text{ or } f,$$

which imply

$$z_i = 0 \quad \text{for } z = e \text{ or } f, \quad \forall \; i \in \mathbb{Z}.$$

Comparing the coefficients of $G_i \otimes G_{\frac{1}{2} - i}$ and $G_{\frac{1}{2} - i} \otimes G_i$ in the both sides of (2.12), we obtain

$$(2i^2 - i - \frac{3}{2})w_i + (\frac{3}{2} - i)(i - 2)w_{i-1} - (i + 1)(i + \frac{3}{2})w_{i+1} = 0 \quad \text{for } w = g \text{ or } h,$$

which imply

$$w_i = 0, \quad \forall \; i \in \mathbb{Z}\setminus\{0, 1\} \quad \text{while} \quad w_0 = -w_1 \quad \text{for } w = g \text{ or } h.$$
Then $d_0(T_{\frac{r}{2}})$ can be rewritten as

$$d_0(T_{\frac{r}{2}}) = a_1(L_{-1} \otimes T_{\frac{r}{2}} - 2L_0 \otimes T_{\frac{r}{2}} + L_1 \otimes T_{-\frac{r}{2}}) + g_1(G_1 \otimes G_{-\frac{r}{2}} - G_0 \otimes G_{\frac{r}{2}}) + b_1(T_{\frac{r}{2}} \otimes L_{-1} - 2T_{\frac{r}{2}} \otimes L_0 + T_{-\frac{r}{2}} \otimes L_1) + h_1(G_{-\frac{r}{2}} \otimes G_1 - G_{\frac{r}{2}} \otimes G_0). \quad (2.13)$$

Applying $d_0$ to $[G_0, G_{\frac{r}{2}}] = \frac{1}{2}T_{\frac{r}{2}}$, one has

$$2G_0 \cdot d_0(G_{\frac{r}{2}}) + 2G_{\frac{r}{2}} \cdot d_0(G_0) = d_0(T_{\frac{r}{2}}).$$

Using (2.9), (2.11) and (2.13) and comparing the coefficients of all the products in the above identity, we obtain

$$d_1 = a_1 = g_1 = b_1 = h_1 = 0,$$

which implies

$$d_0(G_0) = d_0(T_{\frac{r}{2}}) = 0.$$

By now, we have proved

$$d_0(L_i) = d_0(G_r) = d_0(G_0) = d_0(T_{\frac{r}{2}}) = 0, \quad \forall i \in \mathbb{Z}, \ r \in \frac{1}{2} + \mathbb{Z},$$

which together with (1.9), implies $d_0(\mathcal{L}) = 0$ for the case $d_0 \in \text{Der}_0(\mathcal{L}, \mathcal{V})$. Thus the claim follows.

**Claim 4.** Suppose $d_0 \in \text{Der}_1(\mathcal{L}, \mathcal{V})$ is odd. By replacing $d_0$ by $d_0 - u_{\text{inn}}$ for some $u \in \mathcal{V}_0$, we can suppose $d_0(\mathcal{L}) = 0$.

Employing the similar techniques used in Claim 3, one can see the claim holds.

**Claim 5.** The sum in (2.2) is finite.

For any $i \in \mathbb{Z}$, suppose $d_i = (v_i)_{\text{inn}}$ for some $v_i \in \mathcal{V}_i$. If $|\{i \mid v_i \neq 0\}|$ is infinite, then $d(L_0) = \sum_{i \in \mathbb{Z}} L_0 \ast v_i = -\sum_{i \in \mathbb{Z}} iv_i$ is an infinite sum, which contradicts $d \in \text{Der}(\mathcal{L}, \mathcal{V})$. Thus the claim and proposition follow.

**Lemma 2.5.** If $r \in \mathcal{V}$ satisfies $x \ast r \in \text{Im}(1 \otimes 1 - \tau) (\forall x \in \mathcal{L})$, then $r \in \text{Im}(1 \otimes 1 - \tau)$.

**Proof.** Note $\mathcal{L} \ast \text{Im}(1 \otimes 1 - \tau) \subset \text{Im}(1 \otimes 1 - \tau)$. Write $r = \sum_{i \in \frac{1}{2}\mathbb{Z}} r_i$ with $r_i \in \mathcal{V}_i$. Obviously, $r \in \text{Im}(1 \otimes 1 - \tau)$ if and only if $r_i \in \text{Im}(1 \otimes 1 - \tau)$ for all $i \in \frac{1}{2}\mathbb{Z}$. Thus without loss of generality, one can suppose $r = r_i$ is homogeneous.

If $i \in \frac{1}{2}\mathbb{Z}^*$, then $r_i = -\frac{1}{i} L_0 \ast r_i \in \text{Im}(1 \otimes 1 - \tau)$. For the case $i = 0$, one can write

$$r_0 = \sum_{i \in \mathbb{Z}} a_i L_i \otimes L_{-i} + \sum_{p \in \frac{1}{2}\mathbb{Z}} b_p G_p \otimes G_{-p} + \sum_{r \in \frac{1}{2} + \mathbb{Z}} c_r T_r \otimes T_{-r} + \sum_{j \in \mathbb{Z}} d_j L_j \otimes G_{-j} + \sum_{k \in \mathbb{Z}} e_k G_k \otimes L_{-k} + \sum_{r \in \frac{1}{2} + \mathbb{Z}} f_r G_r \otimes T_{-r} + \sum_{r \in \frac{1}{2} + \mathbb{Z}} g_r T_r \otimes G_{-r},$$

where $a_i, b_p, c_r, d_j, e_k, f_r, g_r \in \mathbb{C}$.
where the sum are all finite. Since the elements of the form \( u_{1,i} := L_i \otimes L_{-i} - L_{-i} \otimes L_i \), 
\( u_{2,p} := G_p \otimes G_{-p} - G_{-p} \otimes G_p \), \( u_{3,r} := T_r \otimes T_{-r} - T_{-r} \otimes T_r \), \( v_i := L_i \otimes G_{-i} - G_{-i} \otimes L_i \) and 
\( w_r := G_r \otimes T_{-r} - T_{-r} \otimes G_r \) are all in \( \text{Im}(1 \otimes 1 - \tau) \), replacing \( v \) by \( v - u \), where \( u \) is a combination of some \( u_{1,i}, u_{2,p}, u_{3,r}, v_i \) and \( w_r \), one can suppose

\[
\begin{align*}
   a_i &\neq 0 \implies i \in \mathbb{Z}_+; \\
   b_p &\neq 0 \implies p \in \frac{1}{2} \mathbb{Z}_+; \\
   c_r &\neq 0 \implies r \in \frac{1}{2} + \mathbb{Z}_+; \\
   e_i & = g_i = 0, \quad \forall i \in \mathbb{Z}, \quad r \in \frac{1}{2} + \mathbb{Z}.
\end{align*}
\]  

(2.14)

(2.15)

(2.16)

(2.17)

Then \( r_0 \) can be rewritten as

\[
   r_0 = \sum_{i \in \mathbb{Z}_+} a_i L_i \otimes L_{-i} + \sum_{p \in \frac{1}{2} \mathbb{Z}_+} b_p G_p \otimes G_{-p} + \sum_{r \in \frac{1}{2} + \mathbb{Z}_+} c_r T_r \otimes T_{-r} + \sum_{j \in \mathbb{Z}} d_j L_j \otimes G_{-j} + \sum_{r \in \frac{1}{2} + \mathbb{Z}} f_r G_r \otimes T_{-r}.
\]

First assume that \( a_i \neq 0 \) for some \( i > 0 \). Choose \( j < 0 \) such that \( i + j < 0 \). Then we see that the term \( L_i \otimes L_{j-i} \) appears in \( L_j \cdot r_0 \), but (2.14) implies that the term \( L_{i+j} \otimes L_{-i} \) does not appear in \( L_j \cdot r_0 \), a contradiction with the fact that \( L_j \cdot r_0 \in \text{Im}(1 \otimes 1 - \tau) \). Then one further can suppose \( a_i = 0, \quad \forall i \in \mathbb{Z}^* \). Similarly, one also can suppose \( b_p = c_r = 0 \) for all \( p \in \frac{1}{2} \mathbb{Z}^* \), \( r \in \frac{1}{2} + \mathbb{Z} \). Therefore, \( r_0 \) can be rewritten as

\[
   r_0 = \sum_{j \in \mathbb{Z}} d_j L_j \otimes G_{-j} + \sum_{r \in \frac{1}{2} + \mathbb{Z}} f_r G_r \otimes T_{-r} + a_0 L_0 \otimes L_0 + b_0 G_0 \otimes G_0.
\]

(2.18)

Finally, we mainly use the fact \( \text{Im}(1 \otimes 1 - \tau) \subset \text{Ker}(1 \otimes 1 + \tau) \) and the assumption that \( L \cdot r_0 \subset \text{Im}(1 \otimes 1 - \tau) \) to deduce \( a_0 = d_0 = d_j = f_r = 0 \) for all \( j \in \mathbb{Z}, \quad r \in \frac{1}{2} + \mathbb{Z} \). One has

\[
   0 = (1 \otimes 1 + \tau)(L_1 \cdot r_0)
\]

\[
   = 2a_0(L_0 \otimes L_0 + L_1 \otimes L_0) + b_0(G_0 \otimes G_0 + G_0 \otimes G_0)
\]

\[
   + \sum_{r \in \frac{1}{2} + \mathbb{Z}} ((3/2 - r)f_{r-1} + rf_r)(G_r \otimes T_{1-r} + T_{1-r} \otimes G_r)
\]

\[
   + \sum_{j \in \mathbb{Z}} ((2 - j)d_{j-1} + (1/2 + j)d_j)(L_j \otimes G_{1-j} + G_{1-j} \otimes L_j).
\]

Then noticing both the sets \( \{ j \mid d_j \neq 0 \} \) and \( \{ r \mid f_r \neq 0 \} \) of finite rank and comparing the coefficients of the tensor products, one immediately gets

\[
a_0 = b_0 = d_j = f_r = 0, \quad \forall p \in \mathbb{Z}, \quad r \in \frac{1}{2} + \mathbb{Z}.
\]

Thus the lemma follows. □

**Proof of Theorem 1.2.** Let \((\mathcal{L}, [\cdot, \cdot], \Delta)\) be a Lie super-bialgebra structure on \( \mathcal{L} \). Then \( \Delta = \Delta_r \) is defined by (1.4) for some \( r \in Y_0 \). By (1.2), \( \text{Im} \Delta \subset \text{Im}(1 \otimes 1 - \tau) \). Thus by Lemma 2.5, \( r \in \text{Im}(1 \otimes 1 - \tau) \). Then (1.2), (2.1) and Corollary 2.3 show that \( c(r) = 0 \). Thus \((\mathcal{L}, [\cdot, \cdot], \Delta)\) is triangular coboundary. □
References

[1] S. L. Cheng, V.G. Kac, A new $N = 6$ superconformal algebra, Commun. Math. Phys., 186 (1997), 219.

[2] V.G. Drinfeld, Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of classical Yang-Baxter equations, Dokl. Akad. Nauk, 268 (1983), 285–287.

[3] V.G. Drinfeld, Quantum groups, Proceeding of the International Congress of Mathematicians, 1, 2 (1986), 798–820.

[4] Matthias Dörrazapf, Beatriz Gato-Rivera, Singular Dimensions of the $N = 2$ Superconformal Algebras II: The Twisted $N = 2$ Algebra, Comm. Math. Phys., 220 (2001), 263–292.

[5] W. Eholzer, M.R. Gaberdiel, Unitarity of rational $N = 2$ superconformal theories, Commun. Math. Phys., 186 (1997), 61–85.

[6] P. J. Hilton, U. Stammbach, A Course in Homological Algebra., 2nd ed. New York: Springer-Verlag, (1997).

[7] E. Kiritsis, Character formula and the structure of the representations of the $N = 1, N = 2$ superconformal algebras, J. Mod. Phys. A 3 (1988), 1871–1906.

[8] V.G. Kac, Lie superalgebras, Adv. Math., 26 (1977), 8–97.

[9] V.G. Kac, J.W. van de Leur, On classification of superconformal algebras. Strings 88, Singapore: World Scientific, (1988).

[10] W. Michaelis, A class of infinite-dimensional Lie bialgebras containing the Virasoro algebras, Adv. Math., 107 (1994), 365–392.

[11] W.D. Nichols, The structure of the dual Lie coalgebra of the Witt algebra, J. Pure Appl. Alg., 68 (1990), 395–364.

[12] A. Neveu, J.H. Schwarz, Factorizable dual model of pions, Nucl. Phys. B, 31 (1971), 86–112.

[13] S.H. Ng, E.J. Taft, Classification of the Lie bialgebra structures on the Witt and Virasoro algebras, J. Pure Appl. Alg., 151 (2000), 67–88.

[14] P. Ramond, Dual theory of free fermions, Phys. Rev. D, 3 (1971), 2451–2418.

[15] A. Schwimmer, N. Seiberg, Comments on the $N = 2, 3, 4$ superconformal algebras in two dimensions, Phys. Lett.B, 184 (1986), 191–196.

[16] G. Song, Y. Su, Lie bialgebras of generalized Witt type, Sci China Ser A 49 (2006), 533–44.

[17] E.J. Taft, Witt and Virasoro algebras as Lie bialgebras, J. Pure Appl. Alg., 87 (1993), 301–312.

[18] H. Yang, Y. Su, Lie super-bialgebra structures on the generalized super-Virasoro algebras, Acta Mathematica Sinica, English Series, accepted.

[19] H. Yang, Y. Su, Lie bialgebras over the Ramond $N = 2$ super-Virasoro algebras, Chaos, Solutions & Fractals, (2008), in press.