Finding Principal Null Directions
for Numerical Relativists

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abstract

We present a new method for finding principal null directions (PNDs). Because our method assumes as input the intrinsic metric and extrinsic curvature of a spacelike hypersurface, it should be particularly useful to numerical relativists. We illustrate our method by finding the PNDs of the Kastor-Traschen spacetimes, which contain arbitrarily many $Q = M$ black holes in a de Sitter background.

Key Words: General Relativity, Asymptotic structure, Exact solutions, Numerical Relativity, Gravitational waves: theory.
1 Introduction

According to General Relativity, light propagating from a spherical source to a distant observer through spacetime curved by the mass-energy of intervening bodies conveys to that observer a distorted (i.e., non-circular) image of the source. The amount of observed distortion depends on both the direction and the distance to the source, and of course vanishes as that distance approaches zero. Interestingly, the distortion growth rate also vanishes in this limit, independently of the direction from which the observer approaches the source. We may sum all this up by writing \( D = \frac{1}{2} C(\epsilon, \theta, \phi) \epsilon^2 \), where \( D \) is a measure of the image distortion, \( \epsilon \) is a measure of the distance to the source, and the coefficient \( C(\epsilon, \theta, \phi) \) depends smoothly on \( \epsilon \) (even at \( \epsilon = 0 \)). Remarkably, for certain special directions \((\theta_i, \phi_i)\), it happens that \( C(0, \theta, \phi) = 0 \). In these directions, of which there can be at most four, the distortion vanishes unusually fast as the observer approaches the source. These approximately distortion-free directions are called principal null directions (PNDs) [1].

PNDs provide a great deal of gauge-invariant information about solutions to Einstein’s equation. Indeed, in vacuum, they provide nearly as much information as the entire Riemann curvature. In the early 1960s, many researchers exploited this fact to obtain a large class of exact vacuum solutions with fewer than four distinct principal null directions at each point [2]. Some of these so-called algebraically special solutions were easy to interpret. Those with only one PND at each point, for example, clearly represented gravitational waves. But the significance of algebraically special solutions as a class remained somewhat obscure until Sachs proved his celebrated ‘peeling theorem’ [3]. This theorem asserts that the PNDs determined by the gravitational field of a bounded source gradually coalesce as distance from the source increases. The greater the distance, the more accurately we may approximate the field by ever more ‘special’ algebraically special solutions.

Sachs’s peeling theorem was only the first of many results linking the behavior of a spacetime’s PNDs to the physical situation that the spacetime models. More recently, for example, Arianrhod et. al. have shown how to infer some interesting physical characteristics of the static, cylindrically symmetric Curzon spacetime from the (admittedly somewhat complex) behavior of its PNDs [4].

Despite their proven usefulness in constructing and interpreting exact solutions to
Einstein’s equation, PNDs have played almost no role in work on numerical solutions. A language barrier seems at least partly responsible for this unfortunate state of affairs. Spinors and null tetrads have traditionally been the tools of choice in work on PNDs, and neither of these tools fits naturally into the ‘3+1’ framework \[6\] that is the mainstay of numerical relativity.

In this paper, we present a new method for calculating PNDs tailored to the needs of numerical relativists. Our method assumes as input the induced metric and extrinsic curvature of a spacelike hypersurface, and produces as output the projections into that hypersurface of the PNDs.

In the next section, we present the details of our method, and in section 3 we apply it to obtain the PNDs of the Kastor-Traschen solutions. The result is a series of pictures analogous to those in Arianrhod et. al. showing how the PNDs of various Kastor-Traschen solutions vary from point to point within the spacelike hypersurfaces of a natural slicing.

2 A ‘3+1’ Method for computing PNDs

2.1 Our strategy

Here, we present our method as it applies to vacuum spacetimes with cosmological constant $\Lambda$. We begin by fixing a triple $(\Sigma, h_{ab}, p_{ab})$, where $\Sigma$ is a smooth 3-manifold, $h_{ab}$ is a Riemannian metric on $\Sigma$, and $p_{ab}$ is the extrinsic curvature of $\Sigma$. The constraints on $h_{ab}$ and $p_{ab}$ are

\[
R - p_{ab}p^{ab} + p^2 = 2\Lambda, \quad (1)
\]
\[
D_a(p^{ab} - ph_{ab}) = 0. \quad (2)
\]

Here $p = p_{ab}h^{ab}$, $D_a$ is the (unique) torsion-free derivative operator compatible with $h_{ab}$, and $R = R_{ab}h^{ab}$, where $R_{ab}v^b = -2D_a[D_mv^m]$ for all smooth $v^a$ on $\Sigma$.

From $(h_{ab}, p_{ab})$ we construct two further tensor fields $E_{ab}, B_{ab}$ as follows:

\[
E_{ab} = R_{ab} - p_a^m p_{bm} + pp_{ab} - \frac{2}{3}\Lambda h_{ab}, \quad (3)
\]
\[
B_{ab} = \varepsilon_a^{mn} D_m p_{nb}, \quad (4)
\]

where the tensor field $\varepsilon_{abc} = \varepsilon_{[abc]}$ satisfies $\varepsilon_{abc}\varepsilon^{abc} = 3!$ It follows from (1) and (2) that the fields $E_{ab}, B_{ab}$ are both trace-free and symmetric.
The next step is to choose a unit vector field \( \hat{z}^a \) on \( \Sigma \), and to decompose \( E_{ab}, B_{ab} \) into components along and perpendicular to \( \hat{z}^a \). We set

\[
\begin{align*}
e & \quad = E_{ab} \hat{z}^a \hat{z}^b, \\
e_a & \quad = E_{bc} \hat{z}^b (\delta_c^a - \hat{z}_a \hat{z}^c), \\
e_{ab} & \quad = E_{cd} (\delta_c^a - \hat{z}_a \hat{z}^c) (\delta_d^b - \hat{z}_b \hat{z}^d) + \frac{1}{2} e_s_{ab}, \\
b & \quad = B_{ab} \hat{z}^a \hat{z}^b, \\
b_a & \quad = B_{bc} \hat{z}^b (\delta_c^a - \hat{z}_a \hat{z}^c), \\
b_{ab} & \quad = B_{cd} (\delta_c^a - \hat{z}_a \hat{z}^c) (\delta_d^b - \hat{z}_b \hat{z}^d) + \frac{1}{2} e_s_{ab},
\end{align*}
\]

where \( s_{ab} = h_{ab} - \hat{z}_a \hat{z}_b \). Finally, we set

\[
\begin{align*}
\Psi_0 & = (-e_{ab} + J_a^c b_{bc}) m^a m^b, \\
\Psi_1 & = \frac{1}{\sqrt{2}} (e_a - J_a^c b_{bc}) m^a, \\
\Psi_2 & = \frac{1}{2} (e - ib), \\
\Psi_3 & = \frac{1}{\sqrt{2}} (e_a + J_a^c b_{bc}) \bar{m}^a, \\
\Psi_4 & = (-e_{ab} - J_a^c b_{bc}) \bar{m}^a \bar{m}^b.
\end{align*}
\]

where \( J_a^b \equiv \varepsilon_{ac} b_c \hat{z}_a \) is a rotation by 90 degrees in the plane orthogonal to \( \hat{z}^a \), and \( m^a = \frac{1}{\sqrt{2}} (\hat{x}^a - i \hat{y}^a) \) for some pair of orthogonal unit vector fields that span that plane.

Given \( \Psi_0, \ldots, \Psi_4 \), we need only solve the equation

\[
\Psi_4 z^4 + 4 \Psi_3 z^3 + 6 \Psi_2 z^2 + 4 \Psi_1 z + \Psi_0 = 0.
\]

to get our final answer: for each root \( z_i = \tan \frac{\theta_i}{2} e^{-i \phi_i} \) \( (i = 1, \ldots, 4) \), the unit vector

\[
P^{(i)}_a = \cos \theta_i \hat{z}^a + \sin \theta_i \cos \phi_i \hat{x}^a + \sin \theta_i \sin \phi_i \hat{y}^a
\]

determines a principal null direction. By this we mean the following: at points of \( \Sigma \) in the 4-dimensional maximal evolution \( \mathcal{M} \) of \( (\Sigma, h_{ab}, p_{ab}) \), \( t^a + P^{(i)}_a \) is a principal null vector for each \( i \), where \( t^a \) is the unit normal to \( \Sigma \) in \( \mathcal{M} \).

This completes the description of our method. In the following section, we recall the steps required to solve (16).
2.2 Procedure for finding PNDs

We follow the d’Inverno-Russel-Clark method [5] to find the solutions $z_i$ of (16). Upon setting $y = \Psi_4 z + \Psi_3$, (16) becomes

$$y^4 + 6Hy^2 + 4Gy + K = 0,$$

(18)

where $H, G$ and $K$ stand for the following combinations of the complex numbers $\Psi_0, \cdots, \Psi_4$:

$$I \equiv \Psi_4 \Psi_0 - 4\Psi_1 \Psi_3 + 3\Psi_2^2,$$

(19)

$$J \equiv \det \begin{vmatrix} \Psi_4 & \Psi_3 & \Psi_2 \\ \Psi_3 & \Psi_2 & \Psi_1 \\ \Psi_2 & \Psi_1 & \Psi_0 \end{vmatrix},$$

(20)

$$K \equiv \Psi_4^2I - 3H^2,$$

(21)

$$H \equiv \Psi_4 \Psi_2 - \Psi_3^2,$$

(22)

and $G \equiv \Psi_4^2 \Psi_1 - 3\Psi_4 \Psi_3 \Psi_2 + 2\Psi_3^3.$

(23)

The solutions of (18) are simply expressed in terms of the solutions of

$$\lambda^3 - I\lambda + 2J = 0,$$

(24)

which are

$$\lambda_1 = -(P + \frac{I}{3P}),$$

(25)

$$\lambda_2 = -(e^{i2\pi/3} P + e^{i4\pi/3} \frac{L}{3P}),$$

(26)

$$\lambda_3 = -(e^{i4\pi/3} P + e^{i2\pi/3} \frac{L}{3P})$$

(27)

where $P = \{J + \sqrt{J^2 - (I/3)^3}\}^{1/3}$. From the $\lambda_i \ (i = 1, 2, 3)$ we now determine three further complex numbers, $\alpha, \beta, \gamma$, using the following equations:

$$\alpha^2 = 2\Psi_4 \lambda_1 - 4H,$$

(28)

$$\beta^2 = 2\Psi_4 \lambda_2 - 4H,$$

(29)

$$\gamma^2 = \alpha^2 + \beta^2 + 4H,$$

(30)

and $\alpha\beta\gamma = 4G$

(31)

(28) - (30) determine $\alpha, \beta, \gamma$ up to sign, and (31) determines the signs. Then the solutions of (18) are given as $\frac{1}{2}(\alpha + \beta + \gamma), \frac{1}{2}(\alpha - \beta - \gamma), \frac{1}{2}(-\alpha + \beta - \gamma)$ and $\frac{1}{2}(-\alpha - \beta + \gamma)$. Finally,
from $\alpha, \beta, \gamma$, we obtain the following four complex numbers:

\[
\begin{align*}
  z_1 &= -\{\Psi_3 + \frac{1}{2}(\alpha + \beta + \gamma)\}/\Psi_4, \\
  z_2 &= -\{\Psi_3 + \frac{1}{2}(\alpha - \beta - \gamma)\}/\Psi_4, \\
  z_3 &= -\{\Psi_3 + \frac{1}{2}(-\alpha + \beta - \gamma)\}/\Psi_4, \\
  z_4 &= -\{\Psi_3 + \frac{1}{2}(-\alpha - \beta + \gamma)\}/\Psi_4.
\end{align*}
\]

(32)\hspace{1cm}(33)\hspace{1cm}(34)\hspace{1cm}(35)

These numbers are the solutions of (16).

3  PNDs of the Kastor-Traschen solutions

In this section, we illustrate our method by finding the PNDs of the Kastor-Traschen (KT) solutions [7]. Because these solutions admit no timelike Killing vector field and contain black holes that undergo rapid relative motion, we would expect their PNDs to exhibit a variety of interesting and instructive behaviors.

3.1 Kastor-Traschen solutions

The KT solutions to Einstein's equation with cosmological constant contain arbitrary many $Q = M$ black holes that participate in an overall de Sitter expansion or contraction. In the $\Lambda \to 0$ limit, the KT solutions reduce to the Majumdar-Papapetrou solutions, in which the balance between gravitational attraction and electrostatic repulsion among the black holes causes each to maintain its position relative to the others eternally. To write the KT metric, we first choose $(x, y, z) \in \mathbb{R}^3$, $i = 1, 2, \ldots, N$, and set $r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}$. Then

\[
ds^2 = -\frac{1}{\Omega^2} dt^2 + a(t)^2 \Omega^2 (dx^2 + dy^2 + dz^2),
\]

(36)

where $\Omega = 1 + \sum_{i=1}^{N} \frac{M_i}{ar_i}$, $a = e^{Ht}$ and $H = \pm \sqrt{\frac{\Lambda}{3}}$.

Naively, we interpret $M_i$ as the mass of the $i$th black hole, although we have neither an asymptotically flat region nor event horizons available to convert this naive interpretation into a rigorous one.
3.2 PNDs of Kastor-Traschen solutions

To apply the method of section 2 to the KT solutions, we must first choose spacelike hypersurfaces $\Sigma$ and orthonormal triads $\hat{x}^{a}$, $\hat{y}^{a}$, $\hat{z}^{a}$ tangent to these $\Sigma$. The natural choice would seem to be the $t = \text{const.}$ surfaces for $\Sigma$, and

$$\hat{x}^{a} = \frac{1}{a\Omega} \left( \frac{\partial}{\partial x} \right)^{a}, \quad \hat{y}^{a} = \frac{1}{a\Omega} \left( \frac{\partial}{\partial y} \right)^{a}, \quad \hat{z}^{a} = \frac{1}{a\Omega} \left( \frac{\partial}{\partial z} \right)^{a}$$

(37)

for the triad. We keep this choice in force throughout in what follows.

We now select various values of the free parameters $(x, y, z)$ and $M$ appearing in the KT metric (36), and plot the PNDs our method generates. For simplicity, we take all $z = 0$ — that is, we confine all black holes to the $x - y$ coordinate plane — and plot the projections of the PNDs into that plane.

For the case of one black hole, the KT solution is just the Reissner-Nordstr"{o}m-de Sitter solution in cosmological coordinates, and its horizons are located at $r_{\pm} = (1 - 2M|H| \pm \sqrt{1 - 4M|H|})/2a|H|$, where $r_{-}$ and $r_{+}$ denote the outer black hole horizon and de Sitter horizon, respectively.

Figure 2 shows the PNDs of a KT solution containing two black holes of equal mass ($M_{1} = M_{2} = 1.0$) separated by a coordinate distance $d = 5$. In this case, $r_{+} = 7.87M$ and $r_{-} = 0.127M$, so that the size of each black hole is the same as black points indicating its location. Four PNDs at each point are presented using arrows, but because these four PNDs coincide in pairs, with the coincident pairs pointing in opposite directions, we get a picture that looks like the electric field lines of equal charges. This indicates that the spacetime is Petrov type D at all points.

As we decrease the separation between two black holes, we expect that at a certain critical value of that separation, their horizons will coalesce. When two equal-mass black holes inhabit a contracting ($H < 0$) KT spacetime (total mass $M$), the numerical results of [8] suggest a value of $d \sim 0.45M$ for this critical separation, which leads us to believe that the two black holes in Figure 2 have distinct horizons.

In Figure 3, we show the case of two black holes with different masses ($M_{1} : M_{2} = 1 : 10$) separated by (a) $d = 5$, (b) $d = 2$ and (c) $d = 0.5$ in a contracting background ($H < 0$). Since the black holes move along the natural trajectories in that background, we may interpret these figures as snapshots of coalescing black holes. These cases have
none of the symmetry of Figure 2, and as a result, the field of PNDs is algebraically general (Petrov type I). However, we also see that the spacetime near each black hole is almost Petrov type D, as is a single black hole, and that at great distances from the holes the PNDs exhibit the same almost type-D behavior. This is just as we would expect. Moreover, we see that the PNDs on the axis connecting the two black holes exhibit Petrov type D behavior. Although we have no value for the critical separation in the case of two black holes with different masses, Figure 3(c) suggests that as the separation tends to zero, the spacetime’s Petrov type approaches that of a single black hole.

We can also draw the pattern of PNDs for KT solutions containing many black holes. As an example, we show in Figure 4 the PNDs of three black holes of equal mass located at the vertices of an equilateral triangle.

4 Conclusions and discussion

The results of the previous section illustrate the advantages of plotting PNDs over other methods of assessing how nearly algebraically special a spacetime is. Particularly when the spacetime is known only approximately, other methods are hard to apply, for they require us to determine whether certain scalar combinations of curvature components (e.g., $I^3 - 6J^2$) are exactly zero or not.

We hope the method presented in this paper will help numerical relativists to deal with at least two subtle issues. First, there is the problem of how to set initial data representing bounded sources free of incoming radiation. Because Sachs’s peeling theorem assumes no incoming radiation, any failure of the PNDs computed from $(\Sigma, h_{ab}, p_{ab})$ to exhibit the proper peeling behavior shows that $(\Sigma, h_{ab}, p_{ab})$ is an unsatisfactory model of the instantaneous state of an isolated system. Our method thus allows us to recognize and reject initial data cluttered with spurious incoming radiation. This should be of some help. Second, there is the problem of detecting spurious gravitational waves in numerically generated spacetimes that result from physically inappropriate boundary conditions. We hope that such waves will manifest themselves in a pathological time dependence of the PNDs near numerical boundaries.

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Figure Captions

Figure 1:
The solutions $z_{k}$ (32)-(35) for each $k = 1, \ldots, 4$ at each point determine the PNDs. If we write $z_{k}$ as $z_{k} = r_{k}e^{-i\phi_{k}}$, and set $r_{k} = \tan \frac{\theta_{k}}{2}$, then each PND $P_{(k)}^{a}$ points in the direction $(\theta_{k}, \phi_{k})$ indicated by the bold arrow in the figure, which we write as in (17).

Figure 2:
PNDs of the Kastor-Traschen solution for the case of two black holes of equal mass ($M_{1} = M_{2} = 1$) separated by a coordinate distance $d = 5$. Black holes are located at $(x, y) = (3.0, 5.5)$ and $(8.0, 5.5)$, and the arrows indicate PNDs at each point.

Figure 3:
PNDs of a KT solution containing two black holes with different masses ($M_{1} : M_{2} = 1 : 10$) separated by (a) $d = 5$, (b) $d = 2$ and (c) $d = 0.5$. In these cases, the symmetry of Figure 2 is absent, we see that the field is algebraically general (Petrov type I). We also see, however, that the spacetime near each black hole and along the axis joining the two black holes is almost Petrov type D, and that in regions far from both holes, the spacetime is also almost Petrov type D, like a single black hole.

Figure 4:
PNDs of a KT solution with three black holes of equal mass at the vertices of an equilateral triangle. Black holes are located at $(x, y) = (2.5, 3.0)$, $(8.5, 3.0)$ and $(5.5, 8.2)$.
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