FAITHFUL ACTIONS OF THE ABSOLUTE GALOIS GROUP ON CONNECTED COMPONENTS OF MODULI SPACES

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Abstract. We give a canonical procedure associating to an algebraic number $a$ first a hyperelliptic curve $C_a$, and then a triangle curve $(D_a, G_a)$ obtained through the normal closure of an associated Belyi function.

In this way we show that the absolute Galois group Gal($\bar{\mathbb{Q}}/\mathbb{Q}$) acts faithfully on the set of isomorphism classes of marked triangle curves, and on the set of connected components of marked moduli spaces of surfaces isogenous to a higher product (these are the free quotients of a product $C_1 \times C_2$ of curves of respective genera $g_1, g_2 \geq 2$ by the action of a finite group $G$). We show then, using again the surfaces isogenous to a product, first that it acts faithfully on the set of connected components of moduli spaces of surfaces of general type (amending an incorrect proof in a previous ArXiv version of the paper); and then, as a consequence, we obtain that for every element $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, not in the conjugacy class of complex conjugation, there exists a surface of general type $X$ such that $X$ and the Galois conjugate surface $X^{\sigma}$ have nonisomorphic fundamental groups.

Using polynomials with only two critical values, we can moreover exhibit infinitely many explicit examples of such a situation.

Introduction

In the 60’s J. P. Serre showed in [Ser64] that there exists a field automorphism $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, and a variety $X$ defined over $\bar{\mathbb{Q}}$ such that $X$ and the Galois conjugate variety $X^{\sigma}$ have non isomorphic fundamental groups, in particular they are not homeomorphic.

In this note we give new examples of this phenomenon, using the so-called ‘surfaces isogenous to a product’ whose weak rigidity was proven in [Cat00] (see also [Cat03]) and which by definition are quotients of a product of curves $(C_1 \times C_2)$ of respective genera at least 2 by the free action of a finite group $G$.

One of our main results is a strong sharpening of the phenomenon discovered by Serre: observe in this respect that, if $\mathfrak{c}$ denotes complex conjugation, then $X$ and $X^{\mathfrak{c}}$ are diffeomorphic.

Theorem 0.1. If $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ is not in the conjugacy class of $\mathfrak{c}$, then there exists a surface isogenous to a product $X$ such that $X$ and the Galois conjugate variety $X^{\sigma}$ have non isomorphic fundamental groups.

Moreover, we give some faithful actions of the absolute Galois group Gal($\bar{\mathbb{Q}}/\mathbb{Q}$), related among them.

The following results are based on the concept of a (symmetry-) marked variety. A marked variety is a triple $(X, G, \eta)$ where $X$ is a projective variety,
and $\eta: G \to \text{Aut}(X)$ is an injective homomorphism (one says also that we have an effective action of the group $G$ on $X$): here two such triples $(X, G, \eta)$, $(X', G', \eta')$ are isomorphic iff there are isomorphisms $f: X \to X'$, and $\psi: G \to G'$ such that $f$ carries the first action $\eta$ to the second one $\eta'$ (i.e., such that $\eta' \circ \psi = \text{Ad}(f) \circ \eta$, where $\text{Ad}(f)(\phi) := f\phi f^{-1}$). A particular case of marking is the one where $G \subset \text{Aut}(X)$ and $\eta$ is the inclusion: in this case we may denote a marked variety simply by the pair $(X, G)$.

**Theorem 0.2.** To any algebraic number $a \notin \mathbb{Z}$ there corresponds, through a canonical procedure (depending on an integer $g \geq 3$), a marked triangle curve $(D_a, G_a)$. This correspondence yields a faithful action of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of isomorphism classes of marked triangle curves.

**Theorem 0.3.** The absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of connected components of the (coarse) moduli spaces of étale marked surfaces of general type.

With a rather elaborate strategy we can then show the stronger result:

**Theorem 0.4.** The absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of connected components of the (coarse) moduli space of surfaces of general type.

Our method is closely related to the so-called theory of ‘dessins d’ enfants’ (see [Gro97]). Dessins d’ enfants are, in view of Riemann’s existence theorem (generalized by Grauert and Remmert in [GR58]), a combinatorial way to look at the monodromies of algebraic functions with only three branch points. We emphasize once more how we make here an essential use of Belyi functions ([Belyi79]) and of their functoriality. Our point of view is however more related to the normal closure of Belyi functions, i.e., pairs $(C, G)$ with $G \subset \text{Aut}(C)$ such that the quotient $C/G \cong \mathbb{P}^1$ and the quotient map is branched exactly in three points.

In the first section we describe a simple but canonical construction which, for each choice of an integer $g \geq 3$, associates to a complex number $a \in \mathbb{C} \setminus \mathbb{Q}$ a hyperelliptic curve $C_a$ of genus $g$, and in such a way that $C_a \cong C_b$ iff $a = b$.

In the later sections we construct the associated triangle curves $(D_a, G_a)$ and prove the above theorems.

It would be interesting to obtain similar types of results, for instance forgetting about the markings in the case of triangle curves, or even using only Beauville surfaces (these are the surfaces isogenous to a product which are rigid: see [Cat00] for the definition of Beauville surfaces and [Cat03], [BCG05], [BCG06] for further properties of these).

Theorem (0.4) was announced by the second author at the Alghero Conference ‘Topology of algebraic varieties’ in september 2006, and asserted with an incorrect proof in the previous ArXiv version of the paper ([BCG07]). The survey article [Cat09] then transformed some of the theorems of [BCG07] into conjectures. The present article then takes up again some conjectures made previously in [BCG07] and repeated in [Cat09].
The main new input of the present paper is the systematic use of twists of the second component of an action on a product $C_1 \times C_2$ by an automorphism of the group $G$ and the discovery that this leads to an injective homomorphism of the Kernel $\mathfrak{K}$ of the action (on the set of connected components $\pi_0(\mathfrak{K})$ of the moduli space of surfaces of general type) into some Abelian group of the form $\circledast_G(Z(\text{Out}(G)))$, $Z$ denoting the centre of a group. Then we use a known result (cf. [F-J08]) that $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ does not contain any nontrivial normal abelian subgroup.

Observe that Robert Easton and Ravi Vakil ([E-V07]), with a completely different type of examples, showed that the Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates faithfully on the set of irreducible components of the moduli spaces of surfaces of general type.

In the last section we use Beauville surfaces and polynomials with two critical values in order to produce infinitely many explicit and simple examples of pairs of surfaces of general type with nonisomorphic fundamental groups which are conjugate under the absolute Galois group (observe in particular that the two fundamental groups have then isomorphic profinite completions).

1. Very special hyperelliptic curves

Fix a positive integer $g \in \mathbb{N}$, $g \geq 3$, and define, for any complex number $a \in \mathbb{C} \setminus \{-2g, 0, 1, \ldots, 2g - 1\}$, $C_a$ as the hyperelliptic curve of genus $g$

\[ w^2 = (z - a)(z + 2g)\prod_{i=0}^{2g-1} (z - i) \]

branched over $\{-2g, 0, 1, \ldots, 2g - 1, a\} \subset \mathbb{P}^1_\mathbb{C}$.

**Proposition 1.1.**

1. Consider two complex numbers $a, b$ such that $a \notin \mathbb{Q}$: then $C_a \cong C_b$ if and only if $a = b$.

2. Assume now that $g \geq 6$ and let $a, b \in \mathbb{C} \setminus \{-2g, 0, 1, \ldots, 2g - 1\}$ be two complex numbers. Then $C_a \cong C_b$ if and only if $a = b$.

**Proof.** One direction being obvious, assume that $C_a \cong C_b$.

1) Then the two sets with $2g + 2$ elements $B_a := \{-2g, 0, 1, \ldots, 2g - 1, a\}$ and $B_b := \{-2g, 0, 1, \ldots, 2g - 1\}$ are projectively equivalent over $\mathbb{C}$ (the latter set $B_b$ has also cardinality $2g + 2$ since $C_a \cong C_b$ and $C_a$ smooth implies that also $C_b$ is smooth).

In fact, this projectivity $\varphi$ is defined over $\mathbb{Q}$, since there are three rational numbers which are carried into three rational numbers (because $g \geq 2$).

Since $a \notin \mathbb{Q}$ it follows that $\varphi(a) \notin \mathbb{Q}$ hence $\varphi(a) = b \notin \mathbb{Q}$ and $\varphi$ maps $B := \{-2g, 0, 1, \ldots, 2g - 1\}$ to $B$, and in particular $\varphi$ has finite order. Since $\varphi$ yields an automorphism of $\mathbb{P}^1_\mathbb{K}$, it either leaves the cyclical order of $\{-2g, 0, 1, \ldots, 2g - 1\}$ invariant or reverses it, and since $g \geq 3$ we see that there are 3 consecutive integers such that $\varphi$ maps them to 3 consecutive integers. Therefore $\varphi$ is either an integer translation, or an affine symmetry of the form $x \mapsto -x + 2n$, where $2n \in \mathbb{Z}$. In the former case $\varphi = id$, since it has finite order, and it follows in particular that $a = b$. In the latter case it must be $2g + 2n = \varphi(-2g) = 2g - 1$ and $2n = \varphi(0) = 2g - 2$, and we derive the contradiction $-1 = 2n = 2g - 2$. 
2) The case where \(a, b \in \mathbb{Q}\) is similar to the previous one: \(\varphi\) preserves or reverses the cyclical order of the two sets, and we are done as before if \(\varphi(a) = b\).

Observe that the set \(B_a := \{-2g, 0, 1, \ldots, 2g - 1, a\}\) admits a parabolic transformation \(\psi(x) := x + 1\), with \(\infty\) as fixed point, with the properties that \(|\psi'(B_a) \cap B_a| \geq 2g - 1\), and that there is an element \(0 \in B_a\) such that \(\psi(0), \ldots, \psi^{2g-1}(0) \in B_a\). If \(B_b\) is projectively equivalent to \(B_a\), then also \(B_b\) inherits such a parabolic transformation \(\tau\) with this property.

Assume that the fixed point of \(\tau\) is \(\infty\): then \(\varphi(\infty) = \infty\), \(\varphi\) is affine and, since \(\varphi \circ \psi = \tau \circ \varphi\), \(\tau\) is of the form \(\tau(x) = x + m\), for some \(m \in \mathbb{Q}\), and moreover \(\varphi(x) = mx + r\), for some \(r \in \mathbb{Q}\). In fact, if we set \(\varphi(x) = ax + r\), \(\varphi \circ \psi = \tau \circ \varphi \iff a(x + 1) + r = ax + r + m \iff m = a\).

Since \(\tau(x) = x + m\), the above property implies that \(m = \pm 1\), hence also \(\varphi(x) = \pm x + r\).

**Claim:** \(\varphi(x) = \pm x + r \implies a = b\).

**First proof of the claim:**

If \(\varphi(x) = x + r\), then \(B_b\) contains \(\{r, \ldots, (2g - 1) + r\}\) and either

1. \(r = 0\) and \(\varphi = id\) or
2. \(r = 1\), \(B_b\) contains \(2g\) and \(-2g + 1\), a contradiction, or
3. \(r = -1\), \(B_b\) contains \(-1\) and \(-2g - 1\), a contradiction.

Similarly, if \(\varphi(x) = -x + r\), \(B_b\) contains \(\{r - (2g - 1), \ldots, r\}\) and either

1. \(r = 2g - 1\) and \(a = b = 4g - 1\) or
2. \(r = 2g\), \(B_b\) contains \(2g\) and \(4g\), a contradiction, or
3. \(r = 2g - 2\), \(B_b\) contains \(-1\) and \(-4g - 2\), a contradiction.

\(\square\)

Let \(w\) be the fixed point of \(\tau\): then we may assume that \(w \in \mathbb{Q}\) and we must exclude this case. Observe that in the set \(B_b\) each consecutive triple of points is a triple of consecutive integers, if no element in the triple is \(-2g\) or \(b\). This excludes at most six triples. Keep in mind that \(a \in B_a\) and consider all the consecutive triples of integers in the set \(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}\): at most two such triples are not a consecutive triple of points of \(B_a\). We conclude that there is a triple of consecutive integers in the set \(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}\) mapping to a triple of consecutive integers under \(\varphi\). Then either \(\varphi\) is an integer translation \(x \mapsto x + n\), or it is a symmetry \(x \mapsto -x + 2n\) with \(2n \in \mathbb{Z}\).

**Second proof of the claim:**

In both cases the intervals equal to the respective convex spans of the sets \(B_a, B_b\) are sent to each other by \(\varphi\), in particular the length is preserved and the extremal points are permuted. If \(a \in [-2g, 2g - 1]\) also \(b \in [-2g, 2g - 1]\) and in the translation case \(n = 0\), so that \(\varphi(x) = x\) and \(a = b\). We see right away that \(\varphi\) cannot be a symmetry, because only two points belong to the left half of the interval.

If \(a < -2g\) the interval has length \(2g - 1 - a\), if \(a > 2g - 1\) the interval has length \(2g + a\). Hence, if both \(a, b < -2g\), since the length is preserved, we find that \(a = b\); similarly if \(a, b > 2g - 1\).
By symmetry of the situation, we only need to exclude the case $a < -2g$, $b > 2g - 1$: here we must have $2g - 1 - a = 2g + b$, i.e., $a = -b - 1$. We have already treated the case where $b = \varphi(a)$, hence we have a translation $\varphi(x) = x + n$ and since $\varphi(2g - 1) = b$, $\varphi(2g - 2) = 2g - 1$, it follows that $n = 1$, and then $\varphi(-2g) = -2g + 1$ gives a contradiction.

We shall assume from now on that $a, b \in \bar{Q} \setminus Q$ and that there is a field automorphism $\sigma \in \text{Gal}(\bar{Q}/Q)$ such that $\sigma(a) = b$. (Obviously, for any $\sigma \in \text{Gal}(\bar{Q}/Q)$ different from the identity, there are $a, b \in \bar{Q} \setminus Q$ with $\sigma(a) = b$ and $a \neq b$.)

The following is a special case of Belyi’s celebrated theorem asserting that an algebraic curve $C$ can be defined over $\bar{Q}$ if and only if it admits a Belyi function, i.e., a holomorphic function $f : C \to \mathbb{P}^1$ whose only critical values are in the set $\{0, 1, \infty\}$. The main assertion concerns the functoriality of a certain Belyi function.

**Proposition 1.2.** Let $P \in Q[x]$ be the minimal polynomial of $a \in \bar{Q}$ and consider the field $L := Q[x]/(P)$. Let $C_x$ be the hyperelliptic curve over $L$

$$w^2 = (z - x)(z + 2g)\prod_{i=0}^{2g-1}(z - i).$$

Then there is a rational function $F_x : C_x \to \mathbb{P}^1_L$ such that for each $a \in C$ with $P(a) = 0$ it holds that the rational function $F_a$ (obtained under the specialization $x \mapsto a$) is a Belyi function for $C_a$.

**Proof.** Let $f_x : C_x \to \mathbb{P}^1_L$ be the hyperelliptic involution, branched in $\{-2g, 0, 1, \ldots, 2g - 1, x\}$. Then $P \circ f_x$ has as critical values:

- the images of the critical values of $f_x$ under $P$, which are $\in Q$,
- the critical values $y$ of $P$, i.e. the zeroes of the discriminant $h_1(y)$ of $P(z) - y$ with respect to the variable $z$.

$h_1$ has degree $\deg(P) - 1$, whence, inductively as in [Belyi79], we obtain $\tilde{f}_x := h \circ P \circ f$ whose critical values are all contained in $Q \cup \{\infty\}$ (see [Wo01] for more details). If we take any root $a$ of $P$, then obviously $f_a$ has the same critical values.

Let now $r_1, \ldots, r_n \in Q$ be the (pairwise distinct) finite critical values of $\tilde{f}_x$. We set:

$$y_i := \frac{1}{\prod_{j \neq i}(r_i - r_j)}.$$

Let $N \in \mathbb{N}$ be a positive integer such that $m_i := Ny_i \in \mathbb{Z}$. Then we have that the rational function

$$g(t) := \prod_i(t - r_i)^{m_i} \in \mathbb{Q}(t)$$

is ramified at most at $\infty$ and $r_1, \ldots, r_n$. In fact, $g'(t)$ vanishes at most when $g(t) = 0$ or at the points where the logarithmic derivative $G(t) := \frac{g'(t)}{g(t)} = \sum_i m_i \frac{1}{t - r_i}$ has a zero. However, $G(t)$ has simple poles at the $n$ points $r_1, \ldots, r_n$ and by the choice made we claim that it has a zero of order $n$ at $\infty$. 

...
In fact, consider the polynomial \( \frac{1}{N} G(t) \Pi_i (t - r_i) \), which has degree \( \leq n - 1 \) and equals
\[
\sum_i y_i \Pi_{j \neq i}(t - r_j) = \sum_i \Pi_{j \neq i}(t - r_j) \frac{1}{\Pi_{j \neq i}(r_i - r_j)}.
\]
It takes value 1 in each of the points \( r_1, \ldots, r_n \), hence it equals the constant 1.

It follows that the critical values of \( g \circ \tilde{f}_x \) are at most 0, \( \infty \), \( g(\infty) \).

We set \( F_x := \Phi \circ g \circ \tilde{f}_x \) where \( \Phi \) is the affine map \( z \mapsto g(\infty)^{-1} z \), so that the critical values of \( F_x \) are equal to \( \{0, 1, \infty\} \). It is obvious by our construction that for any root \( a \) of \( P \), \( F_a \) has the same critical values as \( F_x \), in particular, \( F_a \) is a Belyi function for \( C_a \).

Since in the sequel we shall consider the normal closure (we prefer here, to avoid confusion, not to use the term ‘Galois closure’ for the geometric setting) \( \psi_a : D_a \rightarrow \mathbb{P}^1_C \) of each of the functions \( F_a : C_a \rightarrow \mathbb{P}^1_C \), we recall in the next section the ‘scheme theoretic’ construction of the normal closure.

## 2. Effective construction of normal closures

In this section we consider algebraic varieties over the complex numbers, endowed with their Hausdorff topology, and, more generally, ‘good’ covering spaces (i.e., between topological spaces which are locally arcwise connected and semilocally simply connected).

**Lemma 2.1.** Let \( \pi : X \rightarrow Y \) be a finite ‘good’ unramified covering space of degree \( d \) between connected spaces \( X \) and \( Y \).

Then the normal closure \( Z \) of \( \pi : X \rightarrow Y \) (i.e., the minimal unramified covering of \( Y \) factoring through \( \pi \), and such that there exists an action of a finite group \( G \) with \( Y = Z/G \)) is isomorphic to any connected component of
\[
W := W_\pi := (X \times_Y \ldots \times_Y X) \setminus \Delta \subset X^d \setminus \Delta,
\]
where \( \Delta := \{(x_1, \ldots, x_d) \in X^d | \exists i \neq j \quad x_i = x_j\} \) is the big diagonal.

**Proof.** Choose base points \( x_0 \in X \), \( y_0 \in Y \) such that \( \pi(x_0) = y_0 \) and denote by \( F_0 \) the fibre over \( y_0 \), \( F_0 := \pi^{-1}(\{y_0\}) \).

We consider the monodromy \( \mu : \pi_1(Y, y_0) \rightarrow \mathfrak{S}_d = \mathfrak{S}(F_0) \) of the unramified covering \( \pi \). The monodromy of \( \phi : W \rightarrow Y \) is induced by the diagonal product monodromy \( \mu^d : \pi_1(Y, y_0) \rightarrow \mathfrak{S}(F_0^d) \), such that, for \( (x_1, \ldots, x_d) \in F_0^d \), we have \( \mu^d(\gamma)(x_1, \ldots, x_d) = (\mu(\gamma)(x_1), \ldots, \mu(\gamma)(x_d)) \).

It follows that the monodromy of \( \phi : W \rightarrow Y \), \( \mu_W : \pi_1(Y, y_0) \rightarrow \mathfrak{S}(\mathfrak{S}_d) \) is given by left translation \( \mu_W(\gamma)(\tau) = \mu(\gamma) \circ (\tau) \).

If we denote by \( G := \mu(\pi_1(Y, y_0)) \subset \mathfrak{S}_d \) the monodromy group, it follows right away that the components of \( W \) correspond to the cosets \( G\tau \) of \( G \). Thus all the components yield isomorphic covering spaces.

\( \square \)
The theorem of Grauert and Remmert ([GR58]) allows to extend the above construction to yield normal closures of morphisms between normal algebraic varieties.

**Corollary 2.2.** Let \( \pi: X \to Y \) be a finite morphism between normal projective varieties, let \( B \subset Y \) be the branch locus of \( \pi \) and set \( X^0 := X \setminus \pi^{-1}(B) \), \( Y^0 := Y \setminus B \).

If \( X \) is connected, then the normal closure \( Z \) of \( \pi \) is isomorphic to any connected component of the closure of \( W^0 := (X^0 \times_{Y^0} \ldots \times_{Y^0} X^0) \setminus \Delta \) in the normalization \( W^n \) of \( W := (X \times_Y \ldots \times_Y X) \setminus \Delta \).

**Proof.** The irreducible components of \( W \) correspond to the connected components of \( W^0 \), as well as to the connected components \( Z \) of \( W^n \). So, our component \( Z \) is the closure of a connected component \( Z^0 \) of \( W^0 \). We know that the monodromy group \( G \) acts on \( Z^0 \) as a group of covering transformations and simply transitively on the fibre of \( Z^0 \) over \( y_0 \): by normality the action extends biholomorphically to \( Z \), and clearly \( Z/G \cong Y \).

\[ \square \]

3. Faithful action of the absolute Galois group on the set of marked triangle curves (associated to very special hyperelliptic curves)

Let \( a \) be an algebraic number, \( g \geq 3 \), and consider as in section 1 the hyperelliptic curve \( C_a \) of genus \( g \) defined by the equation

\[
w^2 = (z - a)(z + 2g)\prod_{i=0}^{2g-1}(z - i).
\]

Let \( F_a: C_a \to \mathbb{P}^1 \) be the Belyi function constructed in proposition 1.2 and denote by \( \psi_a: D_a \to \mathbb{P}^1 \) the normal closure of \( C_a \) as in corollary 2.2.

**Remark 3.1.**

1. We denote by \( G_a \) the monodromy group of \( D_a \) and observe that there is a subgroup \( H_a \subset G_a \) acting on \( D_a \) such that \( D_a/H_a \cong C_a \).
2. Observe moreover that the degree \( d \) of the Belyi function \( F_a \) depends not only on the degree of the field extension \( [\mathbb{Q}(a) : \mathbb{Q}] \), but much more on the height of the algebraic number \( a \); one may give an upper bound for the order of the group \( G_a \) in terms of these.

The pair \((D_a,G_a)\) that we get is a so called triangle curve, according to the following definition (see [Cat00]):

**Definition 3.2.**

1. A marked variety is a triple \((X,G,\eta)\) where \( X \) is a projective variety and \( \eta: G \to \text{Aut}(X) \) is an injective homomorphism.
2. Equivalently, a marked variety is a triple \((X,G,\alpha)\) where \( \alpha: X \times G \to X \) is an effective action of the group \( G \) on \( X \).
3. Two marked varieties \((X,G,\alpha),(X',G',\alpha')\) are said to be isomorphic if there are isomorphisms \( f: X \to X' \), and \( \psi: G \to G' \) transporting the action \( \alpha: X \times G \to X \) into the action \( \alpha': X' \times G' \to X' \), i.e., such that

\[
f \circ \alpha = \alpha' \circ (f \times \psi) \iff \eta' \circ \psi = \text{Ad}(f) \circ \eta, \quad \text{Ad}(f)(\phi) := f \phi f^{-1}.
\]
(4) If $G$ defined as a subset of $\text{Aut}(X)$, then the natural marked variety is the triple $(X, G, i)$, where $i: G \to \text{Aut}(X)$ is the inclusion map, and shall sometimes be denoted simply by the pair $(X, G)$.

(5) A marked curve $(D, G, \eta)$ consisting of a smooth projective curve of genus $g$ and an effective action of the group $G$ on $D$ is said to be a marked triangle curve of genus $g$ if $D/G \cong \mathbb{P}^1$ and the quotient morphism $p: D \to D/G \cong \mathbb{P}^1$ is branched in three points.

Remark 3.3. Observe that:

1) we have a natural action of $\text{Aut}(G)$ on marked varieties, namely
\[
\psi(X, G, \eta) := (X, G, \eta \circ \psi^{-1}).
\]

2) the action of the group $\text{Inn}(G)$ of inner automorphisms does not change the isomorphism class of $(X, G, \eta)$ since, for $\gamma \in G$, we may set $f := (\eta(\gamma))$, $\psi := \text{Ad}(\gamma)$, and then $\eta \circ \psi = \text{Ad}(f) \circ \eta$, since $\eta(\psi(g)) = \eta(\gamma g \gamma^{-1}) = \eta(\gamma) \eta(g) \eta(\gamma^{-1}) = \text{Ad}(f)(\eta(g))$.

3) In the case where $G = \text{Aut}(X)$, we see that $\text{Out}(G)$ acts simply transitively on the isomorphism classes of the $\text{Aut}(G)$-orbit of $(X, G, \eta)$.

Consider now our triangle curve $D_a$: without loss of generality we may assume that the three branch points in $\mathbb{P}^1$ are $\{0, 1, \infty\}$ and we may choose a monodromy representation
\[
\mu: \pi_1(\mathbb{P}^1 \setminus \{0, 1, \infty\}) \to G_a
\]

corresponding to the normal ramified covering $\psi_a: D_a \to \mathbb{P}^1$. Denote further by $\tau_0, \tau_1, \tau_\infty$ the images of geometric loops around $0, 1, \infty$. Then we have that $G_a$ is generated by $\tau_0, \tau_1, \tau_\infty$ and $\tau_0 \cdot \tau_1 \cdot \tau_\infty = 1$. By Riemann’s existence theorem the datum of these three generators of the group $G_a$ determines a marked triangle curve (see [Cat00], [BCG05]).

We can phrase our previous considerations in a theorem, after preliminarily observing:

Remark 3.4. 1) $\sigma \in \text{Aut}(\mathbb{C})$ acts on $\mathbb{C}[z_0, \ldots, z_n]$, by sending $P(z) = \sum_{I=(i_0, \ldots, i_n)} a_I z^I \mapsto \sigma(P)(z) := \sum_{I=(i_0, \ldots, i_n)} \sigma(a_I) z^I$.

2) Let $X$ be a projective variety
\[
X \subset \mathbb{P}^n_{\mathbb{C}}, X := \{z | f_i(z) = 0 \ \forall i\}.
\]
The action of $\sigma$ extends coordinatewise to $\mathbb{P}^n_{\mathbb{C}}$, and carries $X$ to the set $\sigma(X)$ which is another variety, denoted $X^\sigma$, and called the conjugate variety. In fact, since $f_i(z) = 0$ implies $\sigma(f_i)(\sigma(z)) = 0$, one has that
\[
X^\sigma = \{w | \sigma(f_i)(w) = 0 \ \forall i\}.
\]

3) Likewise, if $f: X \to Y$ is a morphism, its graph $\Gamma_f$ is a subscheme of $X \times Y$, hence we get a conjugate morphism $f^\sigma: X^\sigma \to Y^\sigma$.

4) Similarly, if $G \subset \text{Aut}(X)$, and $i: G \to \text{Aut}(X)$ is the inclusion, then $\sigma$ determines another marked variety $(X^\sigma, G, \text{Ad}(\sigma) \circ i)$, image of $(X, G, i)$.

In other words, we have $G^\sigma \subset \text{Aut}(X^\sigma)$ in such a way that, if we identify $G$ with $G^\sigma$ via $\text{Ad}(\sigma)$, then $(X/G)^\sigma \cong X^\sigma/G$. 
Theorem 3.5. To any algebraic number \( a \notin \mathbb{Z} \) there corresponds, through a canonical procedure (depending on an integer \( g \geq 3 \)), a marked triangle curve \((D_a, G_a)\).

This correspondence yields a faithful action of the absolute Galois group \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) on the set of isomorphism classes of marked triangle curves.

Proof. Let \((D, G)\) be a marked triangle curve, and \( \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \): extend \( \sigma \) to \( \sigma \in \text{Gal}(\mathbb{C}/\mathbb{Q}) \) and take the transformed curve \( D^\sigma \) and the transformed graph of the action, a subset of \( D^\sigma \times D^\sigma \times G \).

Since there is only a finite number of isomorphism classes of such pairs \((D, G)\) of a fixed genus \( g \) and with fixed group \( G \), it follows that \( D \) is defined over \( \bar{\mathbb{Q}} \) and the chosen extension of \( \sigma \) does not really matter.

Finally, apply the action of \( \sigma \) to the triangle curve \((D_a, G_a)\) and assume that the isomorphism class of \((D_a, G_a)\) is fixed by the action. This means then, setting \( b := \sigma(a) \), that there is an isomorphism \( f: D_a \rightarrow D_b = D^\sigma_a \) such that \( \text{Ad}(f) = \text{Ad}(\sigma) \).

In other words, \( \sigma \) identifies \( G_a \) with \( G_b \) by our assumption and the two actions of \( G_a \) on \( D_a \) and \( D_b \) are transported to each other by \( f \).

It suffices to show that under the isomorphism \( f \) the subgroup \( H_a \) corresponds to the subgroup \( H_b \) (i.e., \( \text{Ad}(f)(H_a) = H_b \)).

Because then we conclude that, since \( C_a = D_a/H_a \), \( C_b = D_b/H_b \), \( f \) induces an isomorphism of \( C_a \) with \( C_b \).

And then by proposition 1.2 we conclude that \( a = b \).

We use now that \( \text{Ad}(f) = \text{Ad}(\sigma) \), so it suffices to show the following

Lemma 3.6. \( \text{Ad}(\sigma)(H_a) = H_b \).

Proof of the Lemma.

Let \( K \) be the Galois closure of the field \( L (= \text{splitting field of the field extension } \mathbb{Q} \subset L) \), and view \( L \) as embedded in \( \mathbb{C} \) under the isomorphism sending \( x \mapsto a \).

Consider the curve \( \hat{C}_x \) obtained from \( C_x \) by scalar extension \( \hat{C}_x := C_x \otimes L K \). Let also \( \hat{F}_x := F_x \otimes L K \) the corresponding Belyi function with values in \( \mathbb{P}^1_K \).

Apply now the effective construction of the normal closure of section 2; hence, taking a connected component of \((\hat{C}_x \times_{\mathbb{P}^1_K} \cdots \times_{\mathbb{P}^1_K} \hat{C}_x) \setminus \Delta \) we obtain a curve \( D_x \) defined over \( K \).

Note that \( D_x \) is not geometrically irreducible, but, once we tensor with \( \mathbb{C} \), it splits into several components which are Galois conjugate and which are isomorphic to the conjugates of \( D_a \).

Apply now the Galois automorphism \( \sigma \) to the triple \( D_a \rightarrow C_a \rightarrow \mathbb{P}^1 \). Since the triple is induced by the triple \( D_x \rightarrow C_x \rightarrow \mathbb{P}^1_K \) by taking a tensor product \( \otimes_K \mathbb{C} \) via the embedding sending \( x \mapsto a \), and the morphisms are induced by the composition of the inclusion \( D_x \subset (C_x)^d \) with the coordinate projections,
Remark 3.7. Assume that the two triangle curves $D_a \to C_a \to \mathbb{P}^1$ isomorphic through a complex isomorphism $\sigma \in \Sigma$. Essentially, there are the following exact sequences for the Grothendieck étale fundamental group, we define $C_0^0 := F_{x}^{-1}(\mathbb{P}^1 \setminus \{0, 1, \infty\})$, and accordingly $C_0^0$ and $D_x^0$.

There are the following exact sequences for the Grothendieck étale fundamental group (compare Theorem 6.1 of [SGA1]):

\[
1 \to \pi_1^{alg}(D_a^0) \to \pi_1^{alg}(D_x^0) \to \text{Gal}(\overline{\mathbb{Q}}/K) \to 1
\]

\[
1 \to \pi_1^{alg}(C_a^0) \to \pi_1^{alg}(C_x^0) \to \text{Gal}(\overline{\mathbb{Q}}/K) \to 1
\]

\[
1 \to \pi_1^{alg}(\mathbb{P}^1_C \setminus \{0, 1, \infty\}) \to \pi_1^{alg}(\mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\}) \to \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1
\]

where $H_a$ and $G_a$ are the respective factor groups for the (vertical) inclusions of the left hand sides, corresponding to the first and second sequence, respectively to the first and third sequence.

On the other hand, we also have the exact sequence

\[
1 \to \pi_1^{alg}(\mathbb{P}^1_C \setminus \{0, 1, \infty\}) \to \pi_1^{alg}(\mathbb{P}^1_\mathbb{Q} \setminus \{0, 1, \infty\}) \to \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to 1.
\]

The finite quotient $G_a$ of $\pi_1^{alg}(\mathbb{P}^1_\mathbb{C} \setminus \{0, 1, \infty\})$ (defined over $K$) is sent by $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to another quotient, corresponding to $D_{\sigma(a)}$, and the subgroup $H_a$, yielding the quotient $C_a$, is sent to the subgroup $H_{\sigma(a)}$.

**Remark 3.7.** Assume that the two triangle curves $D_a$ and $D_b = D_{\sigma(a)}$ are isomorphic through a complex isomorphism $f: D_a \to D_b$ (but without that necessarily $(f, \text{Ad}(\sigma))$ yields an isomorphism of marked triangle curves $(D_a, G_a)$, $(D_b, G_b)$).

We define $\psi: G_a \to G_a$ to be equal to $\psi := \text{Ad}(\sigma^{-1} \circ f)$.

Then $\text{Ad}(f) = \text{Ad}(\sigma) \circ \psi$ and applying to $y \in D_b, y = f(x)$ we get

\[
\text{Ad}(f)(g)(y) = (\text{Ad}(\sigma) \circ \psi)(g)(y) \iff f(g(x)) = (\text{Ad}(\sigma) \circ \psi)(f(x)).
\]

Identifying $G_a$ with $G_b$ under $\text{Ad}(\sigma)$, one can interpret the above formula as asserting that $f$ is only ‘twisted’ equivariant $(f(g(x)))^\sigma = \psi(g)(f(x))$.

**Proposition 3.8.** Assume that the two triangle curves $D_a$ and $D_b = D_{\sigma(a)}$ are isomorphic under a complex isomorphism $\psi \in \text{Aut}(G)$ such that $f(g(x)) = (\text{Ad}(\sigma) \circ \psi)(f(x))$ is inner.

Then $C_a \cong C_b$, hence $a = b$.

**Proof.** If $\psi$ is inner, then the marked triangle curves $(D_a, G_a) = (D_a, G_a, i_a)$ ($i_a$ being the inclusion map of $G_a \subset \text{Aut}(D_a)$), and its transform by $\sigma$, $(D_b, G_b) = (D_b, G_a, \text{Ad}(\sigma) \circ i_a)$ are isomorphic.

Then the argument of theorem [3.5] implies that $C_a \cong C_b$, hence $a = b$. □
We pose here the following conjecture, which is a strengthening of the previous theorem 3.5.

**Conjecture 3.9.** (Conjecture 2.13 in [Cat09]) The absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of isomorphism classes of (unmarked) triangle curves.

The following definition will be useful in the proof of theorem 0.4.

**Definition 3.10.** Let $a \in \bar{\mathbb{Q}}$ be an algebraic number and let $(D_a, G_a)$ be the associated marked triangle curve obtained by the canonical procedure above (depending on an integer $g \geq 3$). Then $(D_a, \text{Aut}(D_a))$ is called the fully marked triangle curve associated to $a$.

**Remark 3.11.** If we consider instead of $(D_a, G_a)$ the fully marked triangle curve $(D_a, \text{Aut}(D_a))$ we have also the subgroup $H_a \leq \text{Aut}(D_a)$ such that $D_a/H_a = C_a$, where $C_a$ is the very special hyperelliptic curve associated to the algebraic number $a$.

The same proof as the proof of theorem 3.5 gives

**Theorem 3.12.** To any algebraic number $a \notin \mathbb{Z}$ there corresponds, through a canonical procedure (depending on an integer $g \geq 3$), a fully marked triangle curve $(D_a, \text{Aut}(D_a))$.

This correspondence yields a faithful action of the absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the set of isomorphism classes of fully marked triangle curves.

4. Connected components of moduli spaces associated to very special hyperelliptic curves

Fix now an integer $g \geq 3$, and another integer $g' \geq 2$.

Consider now all the algebraic numbers $a \notin \mathbb{Q}$ and all the possible smooth complex curves $C'$ of genus $g'$, observing that the fundamental group of $C'$ is isomorphic to the standard group

$$\pi_{g'} := \langle \alpha_1, \beta_1, \ldots, \alpha_{g'}, \beta_{g'} | \prod_{i=1}^{g'} [\alpha_i, \beta_i] = 1 \rangle.$$  

Since $g' \geq 2$ and $G_a$ is 2-generated there are plenty of epimorphisms (surjective homomorphisms) $\mu: \pi_{g'} \to G_a$. For instance it suffices to consider the epimorphism $\theta: \pi_{g'} \to F_{g'}$ from $\pi_{g'}$ to the free group $F_{g'} := \langle \lambda_1, \ldots, \lambda_{g'} \rangle$ in $g'$ letters given by $\theta(\alpha_i) = \theta(\beta_i) = \lambda_i$, $\forall$ $1 \leq i \leq g'$, and to compose $\theta$ with the surjection $\phi: F_{g'} \to G_a$, given by $\phi(\lambda_1) = \tau_0$, $\phi(\lambda_2) = \tau_1$, and $\phi(\lambda_i) = 1$ for $3 \leq i \leq g'$.

Consider all the possible epimorphisms $\mu: \pi_{g'} \to G_a$. Each such $\mu$ gives a normal unramified covering $D' \to C'$ with monodromy group $G_a$.

Let us recall now the basic definitions underlying our next construction: the theory of surfaces isogenous to a product, introduced in [Cat00](see also [Cat03]), and which holds more generally for varieties isogenous to a product.

**Definition 4.1.** (1) A surface isogenous to a (higher) product is a compact complex projective surface $S$ which is a quotient $S = (C_1 \times C_2)/G$
of a product of curves of resp. genera $g_1, g_2 \geq 2$ by the free action of a finite group $G$. It is said to be unmixed if the embedding $i: G \to \Aut(C_1 \times C_2)$ takes values in the subgroup (of index at most two) $\Aut(C_1) \times \Aut(C_2)$.

(2) A Beauville surface is a surface isogenous to a (higher) product which is rigid, i.e., it has no nontrivial deformation. This amounts to the condition, in the unmixed case, that $(C_i, G)$ is a triangle curve.

(3) An étale marked surface is a triple $(S', G, \eta)$ such that the action of $G$ is fixpoint free. An étale marked surface can also be defined as a quintuple $(S, S', G, \eta, F)$ where $\eta: G \to \Aut(S')$ is an effective free action, and $F: S \to S'/G$ is an isomorphism.

Observe that a surjection of the fundamental group $r: \pi_1(S, y) \to G$ determines an étale marked surface. Once a base point $y$ is fixed, then the marking provides the desired surjection $r$. Moving the base point $y$ around amounts to replacing $r$ with the composition $r \circ \Ad(\gamma)$, for $\gamma \in \pi_1(S, y)$. However,

$$(r \circ \Ad(\gamma))(\delta) = r(\gamma \delta \gamma^{-1}) = (\Ad(r(\gamma)) \circ r)(\delta).$$

Therefore we see that we have to divide by the group $\Inn(G)$ acting on the left.

In this case the associated subgroup of the covering is a normal subgroup, hence uniquely determined, independently of the choice of a base point above $y$; however, the corresponding isomorphism of the quotient group with $G$ changes, and as a result the epimorphism $r$ is modified by an inner automorphism of $G$. On the other hand the action of nontrivial elements in $\Out(G) := \Aut(G)/\Inn(G)$ may transform the marking into a non isomorphic one.

**Remark 4.2.** Consider the coarse moduli space $\mathcal{M}_{x,y}$ of canonical models of surfaces of general type $X$ with $\chi(\mathcal{O}_X) = x, K_X^2 = y$. Gieseker ([Ge77]) proved that $\mathcal{M}_{x,y}$ is a quasi-projective variety.

We denote by $\mathcal{M}$ the disjoint union $\bigcup_{x,y \geq 1} \mathcal{M}_{x,y}$, and we call it the moduli space of surfaces of general type.

Fix a finite group $G$ and consider the moduli space $\mathcal{M}_{x,y}^G$ for étale marked surfaces $(X, X', G, \eta, F)$, where the isomorphism class $[X] \in \mathcal{M}_{x,y}$.

This moduli space $\mathcal{M}_{x,y}^G$ is empty if there is no surjection $r: \pi_1(S, y) \to G$, otherwise we obtain that $\mathcal{M}_{x,y}^G$ is a finite étale covering space of $\mathcal{M}_{x,y}$ with fibre over $X$ equal to the quotient set $\Epi(\pi_1(X, y), G)/\Inn(G)$.

By the theorem of Grauert and Remmert ([GR58]) $\mathcal{M}_{x,y}^G$ is a quasi-projective variety.

Recall the following result concerning surfaces isogenous to a product ([Cat00], [Cat03]):

**Theorem 4.3.** Let $S = (C_1 \times C_2)/G$ be a surface isogenous to a product. Then any surface $X$ with the same topological Euler number and the same fundamental group as $S$ is diffeomorphic to $S$. The corresponding subset of the moduli
space \( \mathcal{M}_S^{top} = \mathcal{M}_S^{diff} \), corresponding to surfaces homeomorphic, resp, diffeomorphic to \( S \), is either irreducible and connected or it contains two connected components which are exchanged by complex conjugation.

If \( S \) is a Beauville surface (i.e., \( S \) is rigid) this implies: \( X \cong S \) or \( X \cong \overline{S} \).

It follows also that a Beauville surface is defined over \( \overline{\mathbb{Q}} \), whence the Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) acts on the discrete subset of the moduli space \( \mathcal{M} \) of surfaces of general type corresponding to Beauville surfaces.

It is tempting to make the following

**Conjecture 4.4.** *(Conjecture 2.11 in [Cat09])* The absolute Galois group \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) acts faithfully on the discrete subset of the moduli space \( \mathcal{M} \) of surfaces of general type corresponding to Beauville surfaces.

**Definition 4.5.** Let \( \mathcal{N}_a \) be the subset of the moduli space of surfaces of general type given by surfaces isogenous to a product of unmixed type \( S \cong (D_a \times D')/G_a \), where \( D_a, D' \) are as above (and the group \( G_a \) acts by the diagonal action).

From [Cat00] and especially Theorem 3.3 of [Cat03] it follows:

**Proposition 4.6.** For each \( a \in \overline{\mathbb{Q}} \), \( \mathcal{N}_a \) is a union of connected components of the moduli spaces of surfaces of general type.

Moreover, for \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \), \( \sigma(\mathcal{N}_a) = \mathcal{N}_{\sigma(a)} \).

**Proof.** Since \( D_a \) is a triangle curve, the pair \((D_a, G_a)\) is rigid, whereas, varying \( C' \) and \( \mu \), we obtain the full union of the moduli spaces for the pairs \((D', G_a)\), corresponding to the possible free topological actions of the group \( G_a \) on a curve \( D' \) of genus \(|G_a|(g' - 1) + 1\).

Thus, the surfaces \( S \cong (D_a \times D')/G_a \) give, according to the cited theorem 3.3 of [Cat03], a union of connected components of the moduli space \( \mathcal{M} \) of surfaces of general type.

Choose now a surface \( S \) as above (thus, \([S] \in \mathcal{N}_a\)) and apply the field automorphism \( \sigma \in \text{Aut}(\mathbb{C}) \) to a point of the Hilbert scheme corresponding to the 5-canonical image of \( S \) (which is isomorphic to \( S \), since the canonical divisor of \( S \) is ample). We obtain a surface which we denote by \( S^\sigma \).

By taking the fibre product of \( \sigma \) with \( D_a \times D' \rightarrow S \) it follows that \( S^\sigma \) has an étale covering with group \( G_a \) which is the product \((D_a)^\sigma \times (D')^\sigma\).

Recall that \((C_a)^\sigma = C_{\sigma(a)} \) (since \( \sigma(a) \) corresponds to another embedding of the field \( L \) into \( \mathbb{C} \)), and recall the established equality for Belyi maps \((F_a)^\sigma = F_{\sigma(a)} \), which implies \((D_a)^\sigma = D_{\sigma(a)} \).

On the other hand, the quotient of \((D')^\sigma \) by the action of the group \( G_a \) has genus equal to the dimension of the space of invariants \( \dim(H^0(\Omega^1_{D'} G_a)) \), but this dimension is the same as \( g' = \dim(H^0(\Omega^1_{D'} G_a)) \). Hence the action of \( G_a \) on \((D')^\sigma \) is also free (by Hurwitz’ formula), and we have shown that \( S^\sigma \) is a surface whose moduli point is in \( \mathcal{N}_{\sigma(a)} \).

Finally, since the subscheme of the Hilbert scheme corresponding to these points is defined over \( \overline{\mathbb{Q}} \), it follows that the action on the set of connected
components of this subscheme, which is the set of connected components of the moduli space, depends only on the image $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Let us explain the rough idea for our strategy: for each $\sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ which is nontrivial we would like to find $a$ such that, setting $b := \sigma(a)$:

$$a \neq b \text{ and } \mathcal{N}_a \text{ and } \mathcal{N}_b \text{ do not intersect.}$$

If (***) holds, then we can easily conclude that $\sigma$ acts nontrivially on the set $\pi_0(\mathcal{M})$ of connected components of $\mathcal{M}$. If (***') does not hold for each $a$, the strategy must be changed and becomes a little bit more complicated.

Observe that the condition that $\mathcal{N}_a$ and $\mathcal{N}_b$ intersect implies, by the structure theorem for surfaces isogenous to a product, only the weaker statement that the two triangle curves $D_a$ and $D_b$ are isomorphic.

In order to resort to the result established in the previous section we first of all consider some connected components of moduli spaces of étale marked surfaces, specifically of étale marked surfaces isogenous to a product.

Let therefore $S = (C_1 \times C_2)/G$ be a surface isogenous to a product, of unmixed type. Then the conjugate surface $S^\sigma$ has an étale cover with group $G$, and we see that $\sigma$ acts on the étale marked surface $(S, C_1 \times C_2, G, \eta, F)$ carrying it to $(S^\sigma, C_1^\sigma \times C_2^\sigma, G^\sigma, \eta^\sigma, F^\sigma)$. In particular, $\text{Ad}(\sigma)$ identifies the group $G$ acting on $C_1 \times C_2$ with the one acting on $C_1^\sigma \times C_2^\sigma$.

Now, $S^\sigma$ belongs to the same connected component of $S$, or to its complex conjugate, if and only if there exists an isomorphism $\Phi: \pi_1(S^\sigma) \to \pi_1(S)$.

Identifying $G$ and $G^\sigma$ via $\text{Ad}(\sigma)$, the surjection $r: \pi_1(S) \to G$, whose kernel is $\pi_1(C_1 \times C_2)$, yields a second surjection $r \circ \Phi: \pi_1(S^\sigma) \to G$, which has kernel $\pi_1(C_1^\sigma \times C_2^\sigma)$, by the unicity of the minimal realization of a surface isogenous to a product ([Cat00], prop. 3.15).

Hence $r \circ \Phi$ differs from the surjection $r^\sigma$ via an automorphism $\psi \in \text{Aut}(G)$ such that

$$r \circ \Phi = \psi \circ r^\sigma.$$

The condition that moreover we get the same étale marked surface is that the automorphism $\psi$ is inner. The above argument yields now the following

**Theorem 4.7.** The absolute Galois group $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of connected components of the (coarse) moduli space of étale marked surfaces isogenous to a higher product.

**Proof.** Given $a \in \bar{\mathbb{Q}}$, consider a connected component $\hat{\mathcal{M}}_x^{G_a}$ of the space of $G$-marked surfaces of general type $\mathcal{M}_{x,y}^{G_a}$ corresponding to a given homomorphism

$$\mu: \pi_{g'} \to G := G_a.$$

This homomorphism has a kernel isomorphic to $\pi_{g_2}$, and conjugation by elements of $\pi_{g'}$ determines a homomorphism

$$\rho: G \to \text{Out}^+(\pi_{g_2}) = \text{Map}_{g_2}.$$
which is well defined, up to conjugation in the mapping class group $\text{Map}_{g_2}$ ($\rho$ is the topological type of the action of $G$).

Our theorem follows now from the following

**Main Claim:** if $\hat{\mathcal{N}}_a^\sigma = \sigma(\hat{\mathcal{N}}_a^\sigma)$, then necessarily $a = \sigma(a)$.

Our assumption says that there are two curves $C, C'$ of genus $g'$, and two respective covering curves $C_2, C'_2$, with group $G_a$ and monodromy type $\mu$ (equivalently, with topological type $\rho$ of the action of $G_a$), such that there exists an isomorphism

$$f : D_a^\sigma \times C_2^\sigma \rightarrow D_a \times C'_2$$

commuting with the action of $G_a$ on both surfaces.

By the rigidity lemma 3.8 of [Cat00], $f$ is of product type, and since one action is not free while the other is free, we obtain that $f = f_1 \times f_2$, where $f_1 : D_a^\sigma \rightarrow D_a$ commutes with the $G_a$ action.

Therefore the marked triangle curves $(D_a, G_a, i_a)$ and $(D_a^\sigma, G_a, \text{Ad}(\sigma)i_a)$ are isomorphic and by Theorem 3.5 we get $a = \sigma(a)$.

□

4.1. What happens, if we forget the marking?

Assume now that $\sigma$ acts as the identity on the subset $\pi_0(\mathcal{N}_a)$ of $\pi_0(\mathcal{M})$, whose points correspond to the connected components $\mathcal{N}_a^\sigma$ (image of $\mathcal{N}_a^\sigma$ in the moduli space $\mathcal{M}$ of surfaces of general type).

Then we use the following trick: let $\lambda \in \text{Aut}(G)$, and consider now the epimorphism $\lambda \circ \mu$, to which corresponds the connected component $\mathcal{N}_a^{\sigma_{\lambda^{-1}}}$ of the moduli space.

The component contains a surface $S_\lambda$ which is the quotient of the same product of curves $C_1 \times C_2$, but where the action of $G$ is different, since we divide by another subgroup, the subgroup

$$G(\lambda) : = \{ (g, \lambda(g)) \subset G \times G \}.$$  

Therefore the Galois action on $C_1 \times C_2$ is always the same, and we get, by the above assumption, that there is an automorphism $\psi_\lambda$ of $G$, induced by an isomorphism of $\pi_1(S_\lambda)$ with $\pi_1(S_\lambda^\sigma)$.

Indeed, this isomorphism of fundamental groups is induced by an isomorphism of $S_\lambda$ with the conjugate $(S_\lambda^\sigma)^\sigma$ of another surface $S_\lambda^\sigma$ in the connected component; this isomorphism lifts to an isomorphism of product type

$$f_1 \times f_2 : C_1 \times C_2 \cong C_1^\sigma \times (C_2^\sigma)^\sigma.$$  

Notice that, for each $\lambda$, $(S_\lambda^\sigma)^\sigma$ is a quotient of $C_1^\sigma \times (C_2^\sigma)^\sigma$.

Identifying these two surfaces to $S_\lambda = (C_1 \times C_2)/G$ via this isomorphism, we get that the Galois automorphism $\sigma$ acts on $G \times G$ by a product automorphism $\psi_1 \times \psi_2$, and the automorphism $\psi_\lambda$ of $G$ is induced by the identification of $G \cong G(\lambda)$ given by the first projection.
Note that, while $\psi_1$ is unique, $\psi_2$ is only defined up to an inner automorphism, corresponding to an automorphism of $C_2$ contained in $G$.

We must now have that
\[(\psi_1 \times \psi_2)(G(\lambda)) = G(\lambda) \iff (\psi_1(g), \psi_2(\lambda(g))) \in G(\lambda) \iff \psi_2(\lambda(g)) = \lambda(\psi_1(g)) \forall g \in G.\]

By setting $\lambda = Id$, we obtain $\psi_1 = \psi_2$, and using $\psi_2 \circ \lambda = \lambda \circ \psi_1$ we reach the following conclusion

**Proposition 4.8.** $\psi_1$ lies in the centre $Z(Aut(G))$ of $Aut(G)$, in particular the class $[\psi_1] \in Out(G)$ lies in the centre $Z(Out(G))$.

Clearly, if this class is trivial, then the triangle curves $(D_a, G)$ and $(D_b, G)$ differ by an inner automorphism of $G$ and we conclude by proposition 3.8 that $C_a \cong C_b$, hence $a = b$, a contradiction.

Hence we may assume that the class $[\psi_1] \in Z(Out(G))$ is nontrivial.

We are now ready for the proof of

**Theorem 4.9.** The absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ acts faithfully on the set of connected components of the (coarse) moduli space of surfaces of general type.

There are two main intermediate results, which obviously together imply theorem 0.4.

For the first we need a new definition.

**Definition 4.10.** Let $a \in \overline{\mathbb{Q}} \setminus \mathbb{Q}$ and define $\tilde{G}_a := Aut(D_a)$.

Given a surjective homomorphism
\[\tilde{\mu} : \pi_{g'} \to \tilde{G}_a,\]
with topological type $\tilde{\rho}$, consider all the étale covering spaces $C_2 \to C_2/\tilde{G}_a = C'$ of curves $C'$ of genus $g'$ with this given topological type.

Consider then the connected component $\tilde{\mathcal{N}}^a$ of the moduli space of surfaces of general type $\mathcal{M}$ corresponding to surfaces isogenous to a product of the type $S = (D_a \times C_2)/\tilde{G}_a$.

**Proposition 4.11.** Let $\mathfrak{K}$ be the kernel of the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\pi_0(\mathcal{M})$. Then $\mathfrak{K}$ is an abelian subgroup.

**Proof.** We want to embed the kernel $\mathfrak{K}$ in an abelian group, e.g. a direct product of groups of the form $Z(Out(G))$, using proposition 4.8.

Assume that $\sigma$ lies in the kernel $\mathfrak{K}$. Then, for each algebraic number $a$, and every $\tilde{\rho}$ as above, $\sigma$ stabilizes the connected component $\tilde{\mathcal{N}}^a$.

Let us denote here for simplicity $G := \tilde{G}_a$.

Hence to $\sigma$ we associate an element $[\psi_1] \in Z(Out(G))$, which is nontrivial if and only if $\sigma(a) \neq a$. 
Therefore it suffices to show that, for a fixed \( a \in \bar{\mathbb{Q}} \), and \( \tilde{\rho} : G \to \text{Map}_{g_2} = \text{Out}(\pi_{g_2}) \), the function explained before stating proposition 4.8
\[
\sigma \mapsto [\psi_1] \in Z(\text{Out}(G))
\]
is a homomorphism.

Observe that in fact, there is a dependence of \( \psi_1 \) on \( \sigma \) and on the algebraic number \( a \). To stress these dependences, we change the notation and denote the isomorphism \( \psi_1 \) corresponding to \( \sigma \) and \( a \), by \( \psi_{\sigma,a} \), i.e.,
\[
\psi_{\sigma,a}(g) = \Phi_1^{-1} \circ g^\sigma \circ \Phi_1 = \Phi_1^{-1} \circ \sigma g \sigma^{-1} \circ \Phi_1,
\]
where \( \Phi_1 : D_a \to D_{\sigma(a)} \) is the isomorphism induced by the fact that \( \sigma \) stabilizes the component \( \tilde{\mathfrak{N}}_{\tilde{\rho}_a} \).

Since \( D_a \) is fully marked, whenever we take another isomorphism \( \Phi : D_a \to D_{\tau(a)} \) we have that \( (\Phi)^{-1} \circ \Phi_1 \in \text{Aut}(D_a) = G \). Therefore, since we work in \( \text{Out}(G_a) \), \( \psi_{\sigma,a} \) does not depend on the chosen isomorphism \( \Phi : D_a \to D_{\sigma(a)} \).

Let now \( \sigma, \tau \) be elements of \( \mathfrak{K} \). We have then (working always up to inner automorphisms of \( G \)):

- \( \psi_{\sigma,a} = \text{Ad}(\varphi^{-1}\sigma) \), for any isomorphism \( \varphi : D_a \to D_{\sigma(a)} \), and any algebraic number \( a \);
- \( \psi_{\tau,a} = \text{Ad}(\Phi^{-1}\tau) \), for any isomorphism \( \Phi : D_a \to D_{\tau(a)} \), and any algebraic number \( a \);
- \( \psi_{\tau\sigma,a} = \text{Ad}(\Psi^{-1}\tau\sigma) \), for any isomorphism \( \Psi : D_a \to D_{\tau\sigma(a)} \).

We can choose \( \Psi := \varphi^\tau \circ \Phi \), and then we see immediately that
\[
(1) \quad \psi_{\tau\sigma,a} = \text{Ad}((\varphi^\tau \circ \Phi)^{-1}\tau\sigma) = \text{Ad}(\Phi^{-1}(\varphi^\tau)^{-1}\tau\sigma) = \text{Ad}(\Phi^{-1}\tau\varphi^{-1}\tau^{-1}\tau\sigma) = \text{Ad}(\Phi^{-1}\tau) \text{Ad}(\varphi^{-1}\sigma) = \psi_{\sigma,a} \circ \psi_{\tau,a}.
\]

This shows that the injective map
\[
\mathfrak{K} \to \prod_{a \in \bar{\mathbb{Q}}} \prod_{\tilde{\rho}} Z(\text{Out}(\tilde{G}_a)),
\]
is in fact a group homomorphism. Therefore \( \mathfrak{K} \) is abelian (as subgroup of an abelian group).

\[\square\]

**Proposition 4.12.** Any abelian normal subgroup \( \mathfrak{K} \) of \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) is trivial.

**Proof.** Let \( N \subset \bar{\mathbb{Q}} \) be the fixed subfield for the subgroup \( \mathfrak{K} \) of \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \).

\( N \) is a Galois extension of the Hilbertian field \( \mathbb{Q} \) and \( N \), if \( \mathfrak{K} \) is not trivial, is not separably closed. Hence, by proposition 16.11.6 of [F-J08] then \( \text{Gal}(N) \) is not prosolvable, in particular, \( \mathfrak{K} = \text{Gal}(N) \) is not abelian, a contradiction.

\[\square\]

Theorem 4.4 has the following consequence:

**Theorem 4.13.** If \( \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) is not in the conjugacy class of complex conjugation \( c \), then there exists a surface isogenous to a product \( X \) such that \( X \) and the Galois conjugate surface \( X^\sigma \) have non isomorphic fundamental groups.
Proof. By a theorem of Artin (see cor. 9.3 in [Lang]) we know that any \( \sigma \) which is not in the conjugacy class of \( c \) has infinite order.

By theorem 0.4 the orbits of \( \sigma \) on the subset of \( \pi_0(\mathcal{M}) \) corresponding to the union of the \( N_a \)'s have unbounded cardinality, otherwise there is a power of \( \sigma \) acting trivially, contradicting the statement of 0.4.

Take now an orbit with three elements at least: then we get surface \( S_0, S_1 := S_0^\sigma, S_2 := S_1^\sigma \), which belong to three different components. Since we have at most two different connected components where the fundamental group is the same, we conclude that either \( \pi_1(S_0) \neq \pi_1(S_1) \) or \( \pi_1(S_1) \neq \pi_1(S_2) \).

\[ \square \]

Observe that \( X_a \) and \( (X_a)^\sigma \) have isomorphic Grothendieck étale fundamental groups. In particular, the profinite completion of \( \pi_1(X_a) \) and \( \pi_1((X_a)^\sigma) \) are isomorphic. In the last section we shall give explicit examples where the actual fundamental groups are not isomorphic.

Another interesting consequence is the following. Observe that the absolute Galois group \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) acts on the set of connected components of the (coarse) moduli spaces of minimal surfaces of general type. Theorem 4.13 has as a consequence that this action of \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) does not induce an action on the set of isomorphism classes of fundamental groups of surfaces of general type.

**Corollary 4.14.** \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) does not act on the set of isomorphism classes of fundamental groups of surfaces of general type.

*Proof.* In fact, complex conjugation does not change the isomorphism class of the fundamental group \( (X \text{ and } \bar{X} \text{ are diffeomorphic}). \) Now, if we had an action on the set of isomorphism classes of fundamental groups, then the whole normal closure \( \mathcal{H} \) of the \( \mathbb{Z}/2 \) generated by complex conjugation (the set of automorphisms of finite order, by the cited theorem of E. Artin, see corollary 9.3 in [Lang]) would act trivially.

By Theorem 4.13 the subgroup \( \mathcal{H} \) would then be equal to the union of these elements of order 2 in \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \). But a group where each element has order \( \leq 2 \) is abelian, and again we would have a normal abelian subgroup, \( \mathcal{H} \), of \( \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \), contradicting 4.12.

\[ \square \]

The above arguments show that the set of elements \( \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \) such that for each surface of general type \( S \) and \( S^\sigma \) have isomorphic fundamental groups is indeed a subgroup where all elements of order two, in particular it is an abelian group of exponent 2.

**Question 4.15.** *(Conjecture 2.5 in [Cat09]*) Is it true that for each \( \sigma \in \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \), different from the identity and from complex conjugation, there exists a surface of general type \( S \) such that \( S \) and \( S^\sigma \) have non isomorphic fundamental groups?

It is almost impossible to calculate explicitly the fundamental groups of the surfaces constructed above, since one has to explicitly calculate the monodromy of the Belyi function of the very special hyperelliptic curves \( C_a \).
Therefore we give in the next section explicit examples of pairs of rigid surfaces with non isomorphic fundamental groups which are Galois conjugate.

5. Explicit examples

In this section we provide, as we already mentioned, explicit examples of pairs of surfaces with non isomorphic fundamental groups which are conjugate under the absolute Galois group. Hence they have non isomorphic fundamental groups with isomorphic profinite completions (recall that the completion of a group \( G \) is the inverse limit 

\[ \hat{G} = \lim_{\rightarrow} G/K, \]

of the factors \( G/K, K \) being a normal subgroup of finite index in \( G \)).

The surfaces in our examples are rigid.

In fact, we can prove the following

**Theorem 5.1.** Beauville surfaces yield explicit examples of Galois conjugate surfaces with non-isomorphic fundamental groups (whose profinite completions are isomorphic).

We consider (see \[BCG06\] for an elementary treatment of what follows) polynomials with only two critical values: \( \{0, 1\} \).

Let \( P \in \mathbb{C}[z] \) be a polynomial with critical values \( \{0, 1\} \).

In order not to have infinitely many polynomials with the same branching behaviour, one considers normalized polynomials \( P(z) := z^n + a_{n-2}z^{n-2} + \ldots + a_0 \).

The condition that \( P \) has only \( \{0, 1\} \) as critical values, implies, as we shall briefly recall, that \( P \) has coefficients in \( \overline{\mathbb{Q}} \). Denote by \( K \) the number field generated by the coefficients of \( P \).

Fix the types \((m_1, \ldots, m_r)\) and \((n_1, \ldots, n_s)\) of the cycle decompositions of the respective local monodromies around 0 and 1: we can then write our polynomial \( P \) in two ways, namely as:

\[ P(z) = \prod_{i=1}^{r} (z - \beta_i)^{m_i}, \]

and

\[ P(z) = 1 + \prod_{k=1}^{s} (z - \gamma_k)^{n_k}. \]

We have the equations \( F_1 = \sum m_i \beta_i = 0 \) and \( F_2 = \sum n_k \gamma_k = 0 \) (since \( P \) is normalized). Moreover, \( m_1 + \ldots + m_r = n_1 + \ldots + n_s = n = \text{deg}P \) and therefore, since \( \sum_j (m_j - 1) + \sum_i (n_i - 1) = n - 1 \), we get \( r + s = n + 1 \).

Since we have \( \prod_{i=1}^{r} (z - \beta_i)^{m_i} = 1 + \prod_{k=1}^{s} (z - \gamma_k)^{n_k} \), comparing coefficients we obtain further \( n - 1 \) polynomial equations with integer coefficients in the variables \( \beta_i, \gamma_k \), which we denote by \( F_3 = 0, \ldots, F_{n+1} = 0 \). Let \( \mathbb{V}(n; (m_1, \ldots, m_n), (n_1, \ldots, n_s)) \) be the algebraic set in affine \((n+1)\)-space defined by the equations \( F_1 = 0, \ldots, F_{n+1} = 0 \). Mapping a point of this algebraic
set to the vector \((a_0, \ldots, a_{n-2})\) of coefficients of the corresponding polynomial
\(P\) we obtain a set
\[
\mathbb{W}(n; (m_1, \ldots, m_n), (n_1, \ldots, n_s))
\]
(by elimination of variables) in affine \((n - 1)\) space. Both these are finite
algebraic sets defined over \(\mathbb{Q}\) since by Riemann’s existence theorem they are
either empty or have dimension 0.

Observe also that the equivalence classes of monodromies \(\mu: \pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}) \to \mathfrak{S}_n\) correspond to the orbits of the group of \(n\)-th roots of 1 (we refer to [BCG06] for more details).

**Lemma 5.2.**
\[
\mathbb{W} := \mathbb{W}(7; (2, 2, 1, 1); (3, 2, 2))
\]
is irreducible over \(\mathbb{Q}\) and splits into two components over \(\mathbb{C}\).

**Proof.** This can easily be calculated by a MAGMA routine. \(\square\)

The above lemma implies that \(\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})\) acts transitively on \(\mathbb{W}\). Looking
at the possible monodromies, one sees that there are exactly two real non
equivalent polynomials.

In both cases, which will be explicitly described later on, the two permutations,
of types \((2, 2)\) and \((3, 2, 2)\), are seen to generate \(\mathfrak{A}_7\) and the respective
normal closures of the two polynomial maps are easily seen to give (we use
here the fact that the automorphism group of \(\mathfrak{A}_7\) is \(\mathfrak{S}_7\)) nonequivalent triangle
curves \(D_1, D_2\).

By Hurwitz’s formula, we see that
\[
g(D_i) = \frac{|\mathfrak{A}_7|}{2}(1 - \frac{1}{2} - \frac{1}{6} - \frac{1}{7}) + 1 = 241.
\]

**Definition 5.3.** Let \((a_1, a_2, a_3)\) and \((b_1, b_2, b_3)\) be two spherical
systems of generators of a finite group \(G\) of the same signature, i.e.,
\(\{\text{ord}(a_1), \text{ord}(a_2), \text{ord}(a_3)\} = \{\text{ord}(b_1), \text{ord}(b_2), \text{ord}(b_3)\}\). Then \((a_1, a_2, a_3)\) and
\((b_1, b_2, b_3)\) are called Hurwitz equivalent iff they are equivalent under the equivalence
relation generated by
\[
(a_1, a_2, a_3) \equiv (a_2, a_2^{-1}a_1a_2, a_3),
\]
\[
(a_1, a_2, a_3) \equiv (a_1, a_3, a_3^{-1}a_2a_3).
\]

It is well known that two such triangle curves are isomorphic, compatibly
with the action of the group \(G\), if and only if the two spherical systems of
generators are Hurwitz equivalent.

**Lemma 5.4.** There is exactly one Hurwitz equivalence class of triangle curves
given by a spherical system of generators of signature \((5, 5, 5)\) of \(\mathfrak{A}_7\).

**Proof.** This is shown by an easy MAGMA routine. \(\square\)

**Remark 5.5.** In other words, if \(D_1\) and \(D_2\) are two triangle curves given by
spherical systems of generators of signature \((5, 5, 5)\) of \(\mathfrak{A}_7\), then \(D_1\) and \(D_2\) are
not only isomorphic as algebraic curves, but they have the same action of \(G\).
Let $D$ be the triangle curve given by a(ny) spherical systems of generators of signature $(5, 5, 5)$ of $\mathfrak{A}_7$. Then by Hurwitz' formula $D$ has genus 505.

Consider the two triangle curves $D_1$ and $D_2$ as in example 5.2. Clearly $\mathfrak{A}_7$ acts freely on $D_1 \times D$ as well as on $D_2 \times D$ and we obtain two non isomorphic Beauville surfaces $S_1 := (D_1 \times D)/G$, $S_2 := (D_2 \times D)/G$.

**Proposition 5.6.** 1) $S_1$ and $S_2$ have nonisomorphic fundamental groups.

2) There is a field automorphism $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $S_2 = (S_1)^\sigma$. In particular, the profinite completions of $\pi_1(S_1)$ and $\pi_1(S_2)$ are isomorphic.

**Remark 5.7.** The above proposition proves theorem 5.1.

**Proof.** 1) Obviously, the two surfaces $S_1$ and $S_2$ have the same topological Euler characteristic. If they had isomorphic fundamental groups, by theorem 3.3 of Cat03, $S_2$ would be the complex conjugate surface of $S_1$. In particular, $C_1$ would be the complex conjugate triangle curve of $C_2$: but this is absurd since we shall show that both $C_1$ and $C_2$ are real triangle curves.

2) We know that $(S_1)^\sigma = ((C_1)^\sigma \times (C)^\sigma)/G$. Since there is only one Hurwitz class of triangle curves given by a spherical system of generators of signature $(5, 5, 5)$ of $\mathfrak{A}_7$, we have $(C)^\sigma \cong C$ (with the same action of $G$).

We determine now explicitly the respective fundamental groups of $S_1$ and $S_2$.

In general, let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_m)$ be two sets of spherical generators of a finite group $G$ of respective order signatures $r := (r_1, \ldots, r_n)$, $s := (s_1, \ldots, s_m)$. We denote the corresponding ‘polygonal’ curves by $D_1$, resp. $D_2$.

Assume now that the diagonal action of $G$ on $D_1 \times D_2$ is free. We get then the smooth surface $S := (D_1 \times D_2)/G$, isogenous to a product.

Denote by $T_r := T(r_1, \ldots, r_n)$ the polygonal group

$$\langle x_1, \ldots, x_{n-1} | x_1^{r_1} = \ldots = x_n^{r_n} = 1 \rangle.$$

We have the exact sequence (cf. Cat00 cor. 4.7)

$$1 \to \pi_1 \times \pi_2 \to T_r \times T_s \to G \times G \to 1,$$

where $\pi_i := \pi_1(D_i)$.

Let $\Delta_G$ be the diagonal in $G \times G$ and let $H$ be the inverse image of $\Delta_G$ under $\Phi : T_r \times T_s \to G \times G$. We get the exact sequence

$$1 \to \pi_1 \times \pi_2 \to H \to G \cong \Delta_G \to 1.$$

**Remark 5.8.** $\pi_1(S) \cong H$ (cf. Cat00 cor. 4.7).

We choose now an arbitrary spherical system of generators of signature $(5, 5, 5)$ of $\mathfrak{A}_7$, for instance $((1, 7, 6, 5, 4), (1, 3, 2, 6, 7), (2, 3, 4, 5, 6))$. Note that we use here MAGMA's notation, where permutations act on the right (i.e., $ab$ sends $x$ to $(xa)b$).

A MAGMA routine shows that

1. $\langle (1, 2)(3, 4), (1, 5, 7)(2, 3)(4, 6), (1, 7, 5, 2, 4, 6, 3) \rangle$

...
and

\[(3) \quad ((1, 2)(3, 4), (1, 7, 4)(2, 5)(3, 6), (1, 3, 6, 4, 7, 2, 5))\]

are two representatives of spherical generators of signature \((2, 6, 7)\) yielding two non isomorphic triangle curves \(C_1\) and \(C_2\), each of which is isomorphic to its complex conjugate. In fact, an alternative direct argument is as follows. First of all, \(C_i\) is isomorphic to its complex conjugate triangle curve since, for an appropriate choice of the real base point, complex conjugation sends \(a \mapsto a^{-1}, b \mapsto b^{-1}\) and one sees that the two corresponding monodromies are permutation equivalent (see Figure 1 and Figure 2).

Moreover, since \(\text{Aut}(\mathfrak{A}_7) = S_7\), if the two triangle curves were isomorphic, then the two monodromies were conjugate in \(S_7\). That this is not the case is seen again by the following pictures.

**Figure 1. Monodromy corresponding to (1)**

![Figure 1](image1)

**Figure 2. Monodromy corresponding to (2)**

![Figure 2](image2)

The two corresponding homomorphisms \(\Phi_1: T_{(2, 6, 7)} \times T_{(5, 5, 5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7\) and \(\Phi_2: T_{(2, 6, 7)} \times T_{(5, 5, 5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7\) give two exact sequences

\[1 \to \pi_1(C_1) \times \pi_1(C) \to T_{(2, 6, 7)} \times T_{(5, 5, 5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7 \to 1,\]

and

\[1 \to \pi_1(C_2) \times \pi_1(C) \to T_{(2, 6, 7)} \times T_{(5, 5, 5)} \to \mathfrak{A}_7 \times \mathfrak{A}_7 \to 1,\]

yielding two non isomorphic fundamental groups \(\pi_1(S_1) = \Phi_1^{-1}(\Delta_{3_7})\) and \(\pi_1(S_2) = \Phi_2^{-1}(\Delta_{3_7})\) fitting both in an exact sequence of type

\[1 \to \pi_{241} \times \pi_{505} \to \pi_1(S_j) \to \Delta_{3_7} \cong \mathfrak{A}_7 \to 1,\]

where \(\pi_{241} \cong \pi_1(C_1) \cong \pi_1(C_2), \pi_{505} = \pi_1(C)\).

Using the same trick that we used for our main theorems, namely, using a surjection of a group \(\Pi_g \to \mathfrak{A}_7, g \geq 2\), we get infinitely many examples of pairs of fundamental groups which are nonisomorphic, but which have isomorphic profinite completions.

This implies the following
Theorem 5.9. There is an infinite sequence \( g_1 < g_2 < \ldots < g_i < \ldots \) and for each \( g_i \) there is a pair of surfaces \( S_1(g_i) \) and \( S_2(g_i) \) isogenous to a product, such that

- the corresponding connected components \( \mathcal{N}(S_1(g_i)) \) and \( \mathcal{N}(S_2(g_i)) \) are disjoint,
- there is a \( \sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) such that \( \sigma(\mathcal{N}(S_1(g_i))) = \mathcal{N}(S_2(g_i)) \),
- \( \pi_1(S_1(g_i)) \) is non isomorphic to \( \pi_1(S_2(g_i)) \), but they have isomorphic profinite completions,
- the fundamental groups fit into an exact sequence

\[
1 \to \Pi_{241} \times \Pi_{g_i'} \to \pi_1(S_j(g_i)) \to \mathfrak{A}_7 \to 1, \ j = 1, 2.
\]

Remark 5.10. 1) Many more explicit examples as the one above (but with cokernel group different from \( \mathfrak{A}_7 \)) can be obtained using polynomials with two critical values.

2) A construction of polynomials with two critical values having a very large Galois orbit was proposed to us by D. van Straten.

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