Forces Induced by Non-Equilibrium Fluctuations: The Soret-Casimir Effect

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The notion of fluctuation-induced forces is generalized to the cases where the fluctuations have nonequilibrium origin. It is shown that a net force is exerted on a single flat plate that restricts scale-free fluctuations of a scalar field in a temperature gradient. This force tends to push the object to the colder regions, which is a manifestation of thermophoresis or the Soret effect. In the classic two-plate geometry, it is shown that the Casimir forces exerted on the two plates differ from each other, and thus the Newton’s third law is violated.

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I. INTRODUCTION

In his pioneering work in 1948, Casimir introduced the notion of fluctuation-induced forces when he showed that quantum fluctuations of the electromagnetic fields can generate measurable long-ranged forces between conducting plates [1]. The idea, however, has been subsequently generalized to various cases where: (i) the fluctuations are of classical thermal origin, (ii) the fluctuating media are complex fluids, and (iii) the boundaries are moving [2–5].

In light of the extensive development of the original idea of fluctuation-induced interactions, it may seem quite natural to ask what happens if the fluctuations that cause the interaction are of nonequilibrium nature. A first attempt towards answering this question was made by G.I. Taylor already in 1951, when he was studying the swimming mechanism of flagella (tails of spermatozoa) and the theoretically challenging question of the possibility of “swimming at low Reynolds number” [6]. In this classic work, Taylor has shown that two flagella undergoing wavelike harmonic deformations have a tendency to attract each other, because their undulations introduce (nonequilibrium) hydrodynamic fluctuations in the surrounding viscous fluid that could induce an effective interaction between them. More recently, another generalization has been introduced where localized sources drive the fluctuating medium out of equilibrium [7]. We also note that the idea of depletion forces [8], which are also fluctuation–induced in origin, has been generalized to systems driven out of equilibrium [9].

A particularly interesting way of driving a thermally fluctuating system out of equilibrium is by imposing spatial or temporal variations in the temperature profile. Many interesting phenomena are known to appear under such conditions, ranging from anomaly in diffusion to the old problem of thermophoresis or Soret effect [10,11]. In this effect, whose microscopic mechanism is still subject to debate [12], solutions of colloidal particles [13] or polymers [14] that are placed in a temperature gradient experience separation, which signals the appearance of a net driving force on particles as a result of the temperature gradient. It has also been recently shown that inhomogeneity in the temperature profile could lead to deviation of the self-diffusion coefficient of a tracer undergoing Brownian motion from the simplistic local Einstein’s relation [15].

![Diagram](image1.png)

**FIG. 1.** (a) A single plate immersed in a fluctuating medium. (b) Two parallel plates separated by a distance $H$ in a same medium. In both cases $\Pi^\pm$ represent the stresses exerted on a plate from the right- or the left-hand side of it.

Here, we attempt to generalize the notion of Casimir forces for nonequilibrium systems in which temperature profile is not uniform [16]. We consider a near-equilibrium fluctuating system in which we can define local and instantaneous temperature. We study the dynamics of a classical scalar field governed by a Langevin equation, in which the strength of the noise is proportional to the local and instantaneous temperature in the
system. We consider external objects that restrict the fluctuations of the scalar field and calculate the resulting fluctuation–induced stress on these objects. We concentrate on the examples of a single flat plate as well as two parallel plates, as depicted in Fig. 1, immersed in various temperature profiles. For the case of a single plate, we find that a net force is exerted on the plate due to the imbalance of the fluctuation–induced forces from the two sides, which has a tendency to push it to the colder regions. We also calculate the Casimir force between two parallel plates in nonuniform temperature profiles and find that the two plates experience different forces in violation of the third law of Newton.

The rest of the paper is organized as follows: In Secs. II and III, we introduce the Langevin dynamics of the scalar field, which we expect to capture the most physically relevant consequences of nonequilibrium temperature profiles, and define the quantities of interest that are to be calculated. Section IV is devoted to the discussion of the example of a single plate placed in various temperature profiles, which is followed by the corresponding discussions for the two-plate geometry in Sec. V. Finally Sec. VI concludes the paper, while an Appendix outlines some details of the calculations.

II. THE MODEL

Let us consider an equilibrium system described by a scalar field $\phi(R, t)$ that undergoes thermal fluctuations in a space that is bounded by a number of external objects. The field could represent a component of the electromagnetic field (e.g. the electric potential) in a dielectric medium or vacuum [1–3], an order parameter field for a critical binary mixture or a magnetic system [4], a massless Goldstone mode arising from a continuous symmetry breaking such as nematic liquid crystals or superfluid helium, an elastic deformation field for fluctuating membranes and surfaces, or the electrostatic potential in charged fluids at very low salt concentrations [5]. In all of the above systems, it is possible to write down a Hamiltonian for the fluctuations around the equilibrium state, which in Gaussian approximation reads

$$\mathcal{H} = \frac{K}{2} \int d^3 R \left[ \nabla \phi(R) \right]^2, \quad (1)$$

where $K$ is an elastic modulus describing the stiffness of the system for the fluctuations around equilibrium state.

We consider the situation where temperature has a slowly varying profile such that thermal equilibrium can be achieved locally. This means that we have many heat reservoirs locally in contact with our system, which have different temperatures. The temperature difference between neighboring reservoirs should be sufficiently small so that the condition of local equilibrium is fulfilled. Similarly, we can consider a temperature profile with temporal variations that are sufficiently slow such that instantaneous equilibrium state can be defined. We would like to drive the system described by the Hamiltonian in Eq. (1) out of equilibrium by imposing such temperature profiles and examine the corresponding fluctuation–induced forces exerted on external boundaries in the medium.

Our specific choices of boundaries are sketched in Fig. 1. We consider both the case of a single plate of area $A$, as well as two such parallel plates separated by a distance $H$. We choose the $z$ axis to be the normal direction to the plates, such that the three dimensional position vector can be represented as $R = (r, z)$. We assume the Dirichlet boundary condition, which means that the fluctuating field $\phi$ is restricted to vanish on the plates.

The physical quantity that we would like to calculate for such systems is the pressure or the normal force per unit area exerted on each plate from the fluctuating medium. In the prescribed geometry, this will be $\Pi_{zz}$—the $zz$ component of the stress tensor. Using the Hamiltonian in Eq. (1), one can calculate the components of the stress tensor in terms of the two point correlation functions of the field. For example, the local pressure exerted from the right side on the plate that is located at $z = 0$ is given as

$$\Pi_{zz}^+(r, t) = -\frac{K}{2} \partial_z \partial_{z'} \langle \phi(r, z, t) \phi(r, z', t) \rangle \big|_{z, z' \to 0^+,} \quad (2)$$

where the minus sign denotes that this pressure tends to push the plate to the $-z$ direction. (Note that the normal to plate at this point is in $+z$ direction.) A similar definition and expression can be given for the pressure from the left side $\Pi_{zz}^-$, and the total normal force per unit area

$$\Pi_{\text{tot}} = \Pi_{zz}^- - \Pi_{zz}^+, \quad (3)$$

exerted on the plate in the positive $z$ direction. Note that due to the translational symmetry of the infinite plates in the parallel directions, the lateral forces are all zero, even for the case where temperature is not uniform.

III. DYNAMICAL RELAXATION TO LOCAL EQUILIBRIUM

For calculating the correlation functions in the local and instantaneous near-equilibrium states resulted from a prescribed temperature profile, we use the fact that the such states could be reachable through an equilibrium dynamical relaxation. We assume a dynamical Langevin equation as

$$\gamma \partial_t \phi(R, t) = K \nabla^2 \phi(R, t) + \eta(R, t), \quad (4)$$

with the boundary conditions $\phi(R, 0, t) = \phi(R, H, t) = 0$. Here $\gamma$ is the friction coefficient of the system, and the random force $\eta(R, t)$ is a Langevin noise whose spectrum is given by the local and instantaneous fluctuation—dissipation theorem:
\[ \langle \eta(R, t) \eta(R', t') \rangle = 2\gamma k_B T(R, t) \delta^3(R - R') \delta(t - t'), \]  
where \( T(R, t) \) is the temperature profile in the system.

For solving the above Langevin equation, we define the appropriate Green’s function as the solution of the following equation

\[ (\gamma \partial_t - K \nabla^2) G(R, t; R', t') = \delta^3(R - R') \delta(t - t'), \]

with the boundary conditions

\[ G(R, t; R', t') |_{R \text{ on boundary}} = 0, \]

for each value of \( R' \). The solution of the Langevin equation can then be written as

\[ \phi(R, t) = \int d^3 R' dt' G(R, t; R', t') \eta(R', t'), \]

which yields the two point correlation function of the field as

\[ \langle \phi(R, t) \phi(R', t') \rangle = 2\gamma k_B \int d^3 \mathbf{R} \int dt_1 T(\mathbf{R}_1, t_1) \times G(R, t; \mathbf{R}_1, t_1) G(\mathbf{R}', t'; \mathbf{R}_1, t_1). \]

Note that the fluctuations in the \( \phi \)-field are affected by the entire temperature profile through the scale-free relaxation dynamics governed by Eq. (4).

To obtain the form of the Green’s function for the specific geometries discussed above, we proceed by direct solution of the differential equation. We focus only on the two-plate case, since the single-plate Green’s function can be deduced from it by simply setting \( H = 0 \). Because of the translational symmetry in time, as well as the space direction parallel to the plates, we introduce the Fourier transformation

\[ G_{q, \omega}(z, z') = \int d^3 r dt e^{iq \cdot (r - r') - i\omega(t - t')} G(r, z, t; r', z', t'), \]

which satisfies the following equation:

\[ (Q_-^2 - \partial_z^2) G_{q, \omega}(z, z') = \frac{1}{K} \delta(z - z'), \]

where \( Q_\pm = \sqrt{q^2 \pm i\omega / K} \). We can divide the space into three distinct subspaces: left (L), middle (M), and right (R), as shown in Fig. 1(b), and label the Green’s function in each subspace with the corresponding indices: L, M, or R. The boundary conditions can be summarized as

\[ G_{q, \omega}^{L,M}(0, z') = G_{q, \omega}^{M,R}(L, z') = 0, \]
\[ G_{q, \omega}^{L}(\infty, z') = G_{q, \omega}^{R}(\infty, z') = 0, \]
\[ G_{q, \omega}^{L,M,R}(z^-, z') = G_{q, \omega}^{L,M,R}(z^+, z'), \]
\[ \partial_z G_{q, \omega}^{L,M,R}(z^-, z') - \partial_z G_{q, \omega}^{L,M,R}(z^+, z') = \frac{1}{K}. \]

Using the above boundary conditions we can solve Eq.(11) and directly obtain the closed form for Green’s function in the different parts of the space as

\[ G_{q, \omega}^{L}(z, z') = \begin{cases} - \sinh[Q_- - z'] \frac{\exp[-Q_-z']}{KQ_-} & z > z', \\ - \sinh[Q_- - z'] \frac{\exp[-Q_-z']}{KQ_-} & z < z', \end{cases} \]
\[ G_{q, \omega}^{M}(z, z') = \begin{cases} - \sinh[Q_- - z'] \sinh[Q_- - (z' - H)] \frac{\exp[-Q_-z']}{KQ_-} & z > z', \\ - \sinh[Q_- - z'] \sinh[Q_- - (z' - H)] \frac{\exp[-Q_-z']}{KQ_-} & z < z', \end{cases} \]
\[ G_{q, \omega}^{R}(z, z') = \begin{cases} \sinh[Q_- - (z' - H)] \frac{\exp[-Q_-z']}{KQ_-} & z > z', \\ \sinh[Q_- - (z - H)] \frac{\exp[-Q_-z']}{KQ_-} & z < z', \end{cases} \]

The above expressions for the Green’s function can be used to calculate the normal stress exerted on the plates for arbitrarily given temperature profiles. While the equilibrium fluctuation-induced forces are universal, it appears that forces induced by nonequilibrium fluctuations are not universal and do depend on the specific choice of dynamics as well as the microscopic details, as will be shown in the next two sections.

**IV. SINGLE-PLATE GEOMETRY**

In this section we focus on the example of a single plate that is embedded in a fluctuating background with local temperature. When the temperature profile is uniform, the stress resulted from the field configurations on the left side cancels completely with the corresponding counterpart coming from the right side, and the plate experiences no net force. However, when temperature is not uniform the strength of the fluctuations are also not uniform, and we expect a force imbalance due to the asymmetry caused by the temperature gradient. To further simplify the problem, we assume that temperature only depends on the \( z \) coordinate, i.e. it is independent of time and the parallel coordinates. The two contributions to the normal stress at the position of the plate can be written as

\[ \Pi^- = -\frac{\gamma k_B}{K} \int \frac{d^2 q d\omega}{(2\pi)^3} \int_{-\infty}^{0} dz \ T(z) \ e^{(Q_+ + Q_-)z}, \]

and

\[ \Pi^+ = -\frac{\gamma k_B}{K} \int \frac{d^2 q d\omega}{(2\pi)^3} \int_{0}^{\infty} dz \ T(z) \ e^{-(Q_+ + Q_-)z}. \]

Due to the nonuniformity of the temperature profile, the nonuniversal contributions to the stress on the two sides do not cancel out. Therefore, the forces mediated by nonequilibrium fluctuations are not universal and they
also depend on the specific imposed temperature profile. Examining the behavior of Eqs. (14) and (15) above, we can distinguish between three different categories: (i) the case where temperature is continuous on the plate and its asymptotic values are equal at \( z = -\infty \) and \( z = +\infty \), (ii) the case where temperature is continuous on the plate but its asymptotic values at the two infinities are not the same, and (iii) the case where the temperature is not continuous on the plate. Each category is examined separately below.

A. Continuous on plate and \( T(-\infty) \neq T(\infty) \)

We assume a specific profile for the temperature as [see Fig. 2(a)]

\[
T(z) = \begin{cases} 
T_L & ; z \leq -\xi, \\
-\frac{1}{2}(T_L - T_R) - \frac{1}{2}(T_L + T_R) & ; -\xi \leq z \leq \xi, \\
T_R & ; \xi \leq z,
\end{cases}
\]

which exemplifies the general class where the two half-spaces are kept at different temperatures \( T_L \) and \( T_R \) and connected by a crossover domain of size \( \xi \). Using the one-plate Green’s function from Eq. (13) above, we can calculate the total stress on the plate as

\[
\Pi_{\text{tot}} = c_1 \frac{k_B(T_L - T_R)}{\xi^2 a},
\]

where \( a \) is a microscopic length scale, and \( c_1 \) is a nonuniversal numerical constant of order unity. The above result can be rewritten as

\[
\Pi_{\text{tot}} \sim -\frac{k_B}{\xi a} \nabla T,
\]

where \( \nabla T = \frac{T_R - T_L}{2\xi} \), leading to a Soret coefficient \[17\]

\[
S_T \sim A \frac{a}{\xi} \left( \frac{T_L + T_R}{2} \right)^{-1}.
\]

Note that the length scale \( \xi \) appears in an anomalous way, in that it is not represented entirely in the form of \( \nabla T \).

B. Continuous on the plate and \( T(-\infty) = T(\infty) \)

As another example, which represents the case when the temperature is continuous on the plate and has equal asymptotic values on the far left and right sides of the plate, we choose [see Fig. 2(b)]

\[
T(z) = \begin{cases} 
T_0 - \delta T_0(e^{-z/\xi} - e^{-mz/\xi}) & ; z > 0, \\
T_0 + \delta T_0(e^{z/\xi} - e^{mz/\xi}) & ; z < 0,
\end{cases}
\]

for any arbitrary value of \( m \). The above temperature profile yields

\[
\Pi_{\text{tot}} = c_2(m-1) \frac{k_B\delta T_0}{\xi a^2},
\]

where \( c_2 \) is a numerical prefactor of order one. The above result can be recast in the form

\[
\Pi_{\text{tot}} \sim -\frac{k_B}{\xi a^2} \nabla T,
\]

where \( \nabla T = (1-m)\delta T_0/\xi \), from which a Soret coefficient can be deduced as

\[
S_T \sim A \frac{a}{\xi} T_0.
\]

Note that in this case the length scale \( \xi \) appears only through the combination \( \nabla T \).

C. discontinuous on the plate

Finally, we consider a most general class of temperature profiles as shown in Figs. 2(c) and (d), using the following example

\[
T(z) = \begin{cases} 
T_R + \delta T e^{-z/\xi} & ; z > 0, \\
T_L + \delta T - e^{z/\xi} & ; z < 0,
\end{cases}
\]
Due to the discontinuity of temperature at the boundary, the plate experiences the strongest form of force that reads

$$\Pi_{\text{tot}} \sim -\frac{k_B \delta T}{a^3}, \quad (25)$$

to the leading order, where $\delta T = (T_R + \delta T_+) - (T_L + \delta T_-)$ is the local temperature difference across the plate. Note that due to the singular form of the temperature profile, it is not possible to define a Soret coefficient in this case.

V. TWO-PLATE GEOMETRY

Here we consider the case of two parallel plates immersed in a nonuniform temperature profile. We commence by the equilibrium Casimir problem and then go ahead to generalize it to the nonequilibrium cases. We focus on two different types of behaviors corresponding to where the temperature profile is uniform or nonuniform in the exterior regions.

A. Equilibrium Casimir Force

Here we derive the Fluctuation–induced force on the plates for the case when temperature profile is uniform everywhere, namely $T(r, t) = T_0$ as shown in Fig. 3. The stress on the plate located at $z = 0$ has two contributions coming from right and left sides. The contribution from the right is given as

$$\Pi^+ = -k_B T_0 \int \frac{d^2 q dq \omega}{(2\pi)^3} \left[ \frac{\gamma}{K(Q_+ + Q_-)} - \frac{Q_+[\coth(Q_+H) - 1] - Q_-[\coth(Q_-H) - 1]}{2i \omega} \right], \quad (26)$$

The above stress has two terms; the first term is divergent and the second one is finite. The contribution from the left side reads

$$\Pi^- = -\frac{\gamma k_B T_0}{K} \int \frac{d^2 q dq \omega}{(2\pi)^3} \frac{1}{Q_+ + Q_-}, \quad (27)$$

which exactly cancels the divergent part of Eq. (26), so that the total pressure on this plate is rendered finite as

$$\Pi_{\text{tot}} = \frac{\zeta(3) k_B T_0}{8\pi} \frac{1}{H^3}, \quad (28)$$

Note that the force is universal, and attractive (as it tends to push the plate at $z = 0$ to the right). We can easily check that the other plate experiences the same amount of pressure in the reversed direction, and hence, the Newton’s third law applies. As we will see below, this is a direct consequence of the fact that this force is mediated by equilibrium fluctuations.

FIG. 3. Cross section of the temperature profile for the uniform temperature case. The two plates are located at $z = 0$ and $z = H$.

FIG. 4. Cross section of the temperature profile for the non-uniform temperature case. The two plates are located at $z = 0$ and $z = H$. (a) Temperature is uniform in the exterior. (b) Temperature is nonuniform in the exterior. In both cases the $z$-dependence of the temperature profile the interior is arbitrary.

The cancellation of the divergent parts in Eqs. (26) and (27) is directly related to the assumption that temperature is continuous on the plates. If we considered the case where temperature is uniform but its value is, say, $T_0$ in the region M and $T_0'$ in the region L, then in addition to the above Casimir force we would obtain a divergent part proportional to $(T_0 - T_0')$, similar to the single-plate forces discussed in Sec. IV C above.
B. Temperature profile independent of \( z \) in the exterior

To extend the above calculations to the nonequilibrium situations, we first consider the case where temperature is independent of \( z \) in the exterior regions (R and L) but it is \( z \)-dependent in the interior region (M), as shown in Fig. 4(a). Moreover, we also maintain the possibility of the dependence of temperature on time, as well as the in-plane coordinates \( r \), and assume that the temperature is continuous on the plates to avoid the prescribed singularities.

We can calculate the two contributions to the local stress on the plate located at \( z = 0 \) as

\[
\Pi^+(r, t) = -\frac{\gamma k_B}{8K} \int \frac{d^2q d\omega}{(2\pi)^3} \int_0^H dz' T(q, L - z', \omega) \\
\times \int \frac{d^2q' d\omega'}{(2\pi)^3} \sinh(P_+ z') \sinh(P_- z') \frac{1}{P_+ + P_-} e^{i(q \cdot r - \omega t)},
\]

(29)

and

\[
\Pi^- = -\frac{\gamma k_B}{8K} \int \frac{d^2q d\omega}{(2\pi)^3} T(q, 0, \omega) \\
\times \int \frac{d^2q' d\omega'}{(2\pi)^3} \frac{1}{P_+ + P_-} e^{i(q \cdot r - \omega t)},
\]

(30)

where \( P_{\pm} = \sqrt{\frac{1}{2}(q + q')^2 + i \frac{\gamma}{2\pi}(\omega + \omega')}, \) and we have used the Fourier transform of temperature in time and in the in-plane coordinates. Due to the continuity of temperature, the divergent parts of the two contributions cancel each other as expected, and the net stress on the plate can be written as (see the Appendix)

\[
\Pi(r, t) = \zeta(3) \frac{k_B T(r, 0, t)}{8\pi H^3} - \frac{k_B}{8\pi H^3} \int \frac{d^2q d\omega}{(2\pi)^3} T(q, 0, \omega) \\
\times f \left( \frac{1}{4} q^2 H^2 + i \frac{\gamma}{2K} \omega H^2 \right) e^{i(q \cdot r - \omega t)},
\]

(31)

where

\[
f(u) = \zeta(3) - \frac{4}{3} u^{3/2} + 2u \ln(1 - e^{2\sqrt{u}}) \\
+ 2\sqrt{u} \text{Li}_2(e^{2\sqrt{u}}) - \text{Li}_3(e^{2\sqrt{u}}),
\]

(32)

with \( \text{Li}_n(z) = \sum_{k=1}^{\infty} z^k/k^n \) being the polylogarithm function. The function \( f(u) \) is analytic in the upper-half complex plane is asymptotes to \( \zeta(3) \) everywhere on the semicircle at infinity. For small values of \( u \), however, it has a power series expansion as: \( f(u) \approx u - \frac{2u^{3/2}}{3} + \frac{1}{3} u^2 + O(u^{5/2}) \). An interesting feature of the above result is that the stress on the plate is not sensitive to the \( z \)-dependence of the temperature profile in the interior region.

In the case of uniform temperature Eq. (31) recovers the equilibrium Casimir force [Eq. (28)] because temperature will only have the zero frequency and wavevector component in Fourier space, and the correction term will vanish due to the factor of \( f(0) = 0 \). When temperature is modulated in time or in-plane space directions, the plate will experience local and instantaneous Casimir-like pressure as given by the first term in Eq. (31), while the second term yields corrections as

\[
\Pi(r, t) = \zeta(3) \frac{k_B T(r, 0, t)}{8\pi H^3} \\
+ \frac{k_B}{16\pi H} \left( \frac{1}{2} q^2 + \frac{\gamma}{K} \right) T(r, 0, t),
\]

(33)

to the leading order. Note that the above result is an expansion in powers of the ratio \( H/K \), where \( \lambda \) is the typical wavelength set by the variations of the temperature in space or time, and is valid in the limit \( H/K \ll 1 \).

In the opposite limit of \( H/K \ll 1 \), the fluctuation–induced interactions are screened by the length scale \( \lambda \). This is reminiscent of the screening of the electric field for periodic charge distributions, where the screening length is also set by the period of the charge distribution. The screening manifests itself in the result of Eq. (31), where for large values of the wavevector or frequency the function \( f(u) \) tends to \( \zeta(3) \), thus leading to a cancellation of the first term.

The above calculations can be repeated for the second plate to yield similar expressions for the stress, except that the value of the temperature will be replaced by its local values in the neighborhood of the second plate. In the general case when the two values are different, the two plates will experience different stresses and thus the third law of Newton will be violated. This effect is due to the nonequilibrium nature of the fluctuations, and has also been reported recently in the theory of depletion forces [9].

When temperature varies only with \( z \), we can simplify the expressions for the stress on the left plate as

\[
\Pi_{\text{tot}}(0) = \frac{\zeta(3) k_B T(0)}{8\pi H^3},
\]

(34)

and correspondingly for the right plate as

\[
\Pi_{\text{tot}}(H) = -\frac{\zeta(3) k_B T(H)}{8\pi H^3}.
\]

(35)

Considering the fact that Newton’s third law is violated, one can extract the pressure exerted on the system of the two plates as

\[
\Pi_{\text{tot}}^{2p} = -\frac{\zeta(3) k_B [T(H) - T(0)]}{8\pi H^3}.
\]

(36)

Note that the above result is reminiscent of Eq. (25), provided that we replace the microscopic length scale \( a \) by \( H \), which is the effective thickness of the compound object.
C. Nonuniform temperature profile in the exterior

The fact that the fluctuation–induced interactions are independent of the interior temperature inhomogeneity in the \( z \) direction is a general result and does not depend on whether or not the exterior temperature is \( z \)-dependent. However, the form of the force that is acting on each plate depends on the neighboring exterior temperature profile.

To study this class of temperature profiles, we consider the simple case where the temperature profile is independent of time and the in-plane space coordinates. We assume the following profile for the temperature:

\[
T(z) = \begin{cases} 
T_L + (T - T_L) e^{z/\xi} & ; \ z \leq 0, \\
T_R + (T - T_R) e^{-(z-H)/\xi} & ; \ z \geq H,
\end{cases}
\]

as a typical example for the class of profiles depicted in Fig. 4(b). Note that we also assume that temperature is continuous on each plate. We can then calculate the stress on the left plate as

\[
\Pi_{\text{tot}}(0) = \frac{\zeta(3) k_B T}{8\pi H^3} + c_3 \frac{k_B (T_L - T)}{\xi a^2},
\]

and correspondingly on the right plate as

\[
\Pi_{\text{tot}}(H) = -\frac{\zeta(3) k_B T}{8\pi H^3} + c_3 \frac{k_B (T - T_R)}{\xi a^2}.
\]

We can extract the pressure exerted on the system of the two plates as

\[
\Pi_{\text{tot}}^{2\mu} = c_3 \frac{k_B (T_L - T_R)}{\xi a^2},
\]

which is reminiscent of Eq. (25), and thus yields a Soret coefficient of the form given in Eq. (23) above.

VI. CONCLUDING REMARKS

In conclusion, we have shown that external objects located in a nonuniform temperature profile experience nonuniversal fluctuation–induced forces that tend to move them towards the colder region, which is a manifestation of the so-called Soret effect. This effect is examined in various temperature profiles, and it is shown that the behavior of the fluctuation–induced forces strongly depends on: (1) whether the temperature is continuous across the object, (2) the asymptotic values of the temperature profile, and (3) temporal or spatial variations of temperature. In the case of two external objects immersed in a medium undergoing nonequilibrium thermal fluctuations, it is shown that the third law of Newton is violated in that the two objects experience different forces depending on where they are located.

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APPENDIX A: INTEGRATION OVER A KERNEL

In this Appendix, we outline some details of the calculations involved in Eqs. (29) and (30) above. We are dealing with the following integral:

\[
U(q, q', \omega, z') = \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\sinh(P_+ z') \sinh(P_- H)}{\sinh(P_+ H) \sinh(P_- H)}
\]

with \( P_+ \) being defined below Eq. (30). The integrand has two branch points, and their corresponding branch cuts, in the complex \( \omega' \) plane. Now, by closing the integration contour around these cuts, we can directly check that the integral vanishes for \( z' < H \), while it is divergent for the limiting case of \( z' = H \). This behavior is similar to a delta-function centered at \( z = H \). We can then make the assignment:

\[
U(q, q', \omega, z') = u(q, q', \omega) \delta(z' - H)
\]

and try to find the function \( u \). Note that if \( u \) is found to be a well-behaved finite function, our assignment will become rigorous. Integrating both sides of the above equation with respect to \( z' \) yields:

\[
u(q, q', \omega) = \frac{2iK}{\gamma} \int \frac{d\omega'}{2\pi} \frac{\coth(P'H)}{\omega' - iK(q, q')} \].

The integrand of Eq. (A2) has a pole and a branch cut as shown in Fig. 5. By closing the integration contour in the upper-half plane we obtain

\[
u(q, q', \omega) = -\frac{2K}{\gamma} \sqrt{\frac{1}{4} (q^2 + q'^2) + i \frac{\gamma}{2K} \omega} \times \coth \sqrt{\frac{1}{4} (q^2 + q'^2) + i \frac{\gamma}{2K} \omega},
\]

which completes the frequency integration.

![Complex \( \omega' \) Plane](image_url)

FIG. 5. Complex \( \omega' \) plane and the contour that is used in the integration. The point \( P = iK(q, q')/\gamma \) is the pole and \( BP = \omega - \frac{1}{2} \gamma (q - q')^2 \) is the branch point corresponding to the branch cut.
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[17] The Soret coefficient is defined as follows: the flux of particles $J$ is related to the temperature gradient via an Onsager relation $J = -c D_T \nabla T$, where $c$ is the (particle) concentration and $D_T$ is the so-called thermal diffusion coefficient [11]. The Soret coefficient is then defined as $S_T = D_T / D$ where $D$ is the self-diffusion coefficient. Using the relation $J = c \mu F$ where $\mu$ is the mobility of the plate and $F = \Pi A$ is the overall force exerted on it, one can extract a relation between the stress on the plate and the temperature gradient in terms of the Soret coefficient as: $\Pi = -\frac{k_B T}{A} S_T \nabla T$, in which the Einstein’s relation $D = \mu k_B T$ is used.