CHEEGER’S CONSTANT AND THE FIRST EIGENVALUE OF A CLOSED FINSLER MANIFOLD

LIXIA YUAN AND WEI ZHAO

Abstract. In this paper, we consider Cheeger’s constant and the first eigenvalue of the nonlinear Laplacian on a closed Finsler manifold. A Cheeger type inequality and a Buser type inequality are established for closed Finsler manifolds. As an application, we obtain a Finslerian version of Yau’s lower estimate for the first eigenvalue.

1. Introduction

The study of the eigenvalues of Laplacian is a classical and important problem in Riemannian geometry, which highlights the interplay of the geometry-topology of the manifold with the analytic properties of functions. In order to bound below the first eigenvalue \( \lambda_1(M) \) of a closed Riemannian manifold \( (M^n, g) \), Cheeger [10] introduced Cheeger’s constant

\[
\mathfrak{h}(M) := \inf_{\Gamma} \frac{A(\Gamma)}{\min\{\text{Vol}(D_1), \text{Vol}(D_2)\}},
\]

where \( \Gamma \) varies over compact \((n-1)\)-dimensional submanifolds of \( M \) which divide \( M \) into disjoint open submanifolds \( D_1, D_2 \) of \( M \) with common boundary \( \partial D_1 = \partial D_2 = \Gamma \), and he proved that

\[
\lambda_1(M) \geq \frac{\mathfrak{h}^2(M)}{4}.
\]

Moreover, even if \( M \) has a boundary, Cheeger’s inequality still holds if \( \lambda_1(M) \) is subject to the Neumann boundary condition or the Dirichlet boundary condition [8, 17]. This inequality has found a number of applications, e.g., [6, 8, 15].

It is an important result due to Buser [7] that \( \mathfrak{h}(M) \) is actually equivalent to \( \lambda_1(M) \), with constants depending only on the dimension and the Ricci curvature of \( M \). More precisely, if \( \text{Ric} \geq -(n-1)\delta^2 \), then

\[
\lambda_1(M) \leq C(n)(\delta \mathfrak{h}(M) + \mathfrak{h}^2(M)).
\]

We refer to [10, 8, 13, 5] for more details of these two inequalities.

Finsler geometry is just Riemannian geometry without quadratic restriction. However, there are many Laplacians on a Finsler manifold, e.g., [3]. Among them, an important one was introduced by Shen [19], which is obtained by a canonical energy functional on the Sobolev space and is exactly the Laplace-Beltrami operator if the Finsler metric is Riemannian. This Laplacian has a close relationship with curvatures and plays a crucial role in establishing the comparison theorems for Finsler geometry.

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manifolds\cite{18,20,21,24}, but it is quasilinear and dependent on the measure. While the measure on a Finsler manifold can be defined in various ways and essentially different results may be obtained, e.g.,\cite{1,2}. In general, the eigenfunctions of this Laplacian are not smooth but $C^{1,\alpha}$\cite{14}. Hence, it seems indeed difficult to compute the first eigenvalue even for a the Euclidean sphere $S^n$ equipped with a Randers metric $F = \alpha + \beta$, where $\alpha$ is the canonical Riemannian metric and $\beta$ is a 1-form on $S^n$. The purpose of this paper is to investigate the relationship between Cheeger’s constant and the first eigenvalue of such Laplacian.

Let $(M, F, d\mu)$ be a closed Finsler $n$-manifold, where $d\mu$ be any measure on $M$. According to\cite{18,19}, the first nontrivial eigenvalue $\lambda_1(M)$ is defined as

$$\lambda_1(M) = \inf_{f \in \mathcal{H}_0(M) \setminus \{0\}} \frac{\int_M F^* (df) d\mu}{\int_M |f|^2 d\mu},$$

where $F^*$ is the dual metric of $F$ and $\mathcal{H}_0(M) := \{ f \in W^1_2(M) : \int_M f d\mu = 0 \}$. It follows from\cite{14,18,19} that $\lambda_1(M)$ is the smallest positive eigenvalue of Shen’s Laplacian. Inspired by\cite{11,18,19}, we define Cheeger’s constant of a closed Finsler manifold as

$$h(M) = \inf_{\Gamma} \frac{\min\{A_+(\Gamma)\}}{\min\{\mu(D_1), \mu(D_2)\}},$$

where $\Gamma$ varies over compact $(n-1)$-dimensional submanifolds of $M$ which divide $M$ into disjoint open submanifolds $D_1, D_2$ of $M$ with common boundary $\partial D_1 = \partial D_2 = \Gamma$ and $A_\pm(\Gamma)$ denote the areas of $\Gamma$ induced by the outward and inward normal vector fields $n_\pm$. In general, $A_+(\Gamma) \neq A_-(\Gamma)$. In fact, one can construct examples in which the ratio of these two areas can be arbitrarily large (see\cite{11}).

First, we have the following Cheeger type inequality.

**Theorem 1.1.** Let $(M, F, d\mu)$ be a closed Finsler manifold with the reversibility $\lambda_F \leq \lambda$. Then

$$\lambda_1(M) \geq \frac{h^2(M)}{4\lambda^2}. $$

It should be remarkable that if $M$ has a boundary, Chen\cite{11} proved the above inequality still holds if $\lambda_1$ subject to the Dirichlet boundary condition. Recall the uniform constant\cite{12} $\Lambda_F$ is defined as

$$\Lambda_F := \sup_{X,Y,Z \in TM \setminus \{0\}} \frac{g_z(X,X)}{g_z(Y,Y)}.$$ 

$\Lambda_F \geq 1$ with equality if and only if $F$ is Riemannian. As an application of Theorem\cite{11}, we obtain the following theorem, which is a Finslerian version of Yau’s lower estimate for the first eigenvalue\cite{22}.

**Theorem 1.2.** Let $(M, F, d\mu)$ be a closed Finsler manifold, where $d\mu$ is either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Then $\lambda_1(M)$ can be bounded from below in terms of the diameter, volume, uniform constant and a lower bound for the Ricci curvature.

Moreover, we also have a Buser type inequality for Finsler manifolds.
**Theorem 1.3.** Let \((M, F, d\mu)\) be a closed Finsler \(n\)-manifold with the Ricci curvature \(\text{Ric} \geq -(n-1)\delta^2\) and the uniform constant \(\Lambda_F \leq \Lambda\). Then
\[
\lambda_1(M) \leq C(n, \Lambda) \left(\delta h(M) + h^2(M)\right).
\]

In the Riemannian case, \(\lambda_F = \Lambda_F = 1\). Hence, Theorem 1.1 implies \((1.2)\) while Theorem 1.3 implies \((1.3)\). In particular, for a Randers metric \(F = \alpha + \beta\), the uniform constant \(\Lambda_F = (1+b)^2(1-b)^{-2}\), where \(b := \sup_{x \in M} \|\beta\|_\alpha(x)\) (see Corollary 6.3 below). For the Busemann-Hausdorff measure or the Holmes-Thompson measure, the S-curvature of a Berwald metric always vanishes (see Theorem 6.5 below). Then we have the following corollary.

**Corollary 1.4.** Let \((M, F, d\mu)\) be a closed Finsler \(n\)-manifold with the Ricci curvature \(\text{Ric} \geq -(n-1)k^2\).

1. If \(F = \alpha + \beta\), then \(\lambda_1(M) \leq C(n, b) \left(\delta h(M) + h^2(M)\right)\).
2. If \(F\) is a Berwald metric, then \(\lambda_1(M) \leq C(n, \lambda_F) \left(\delta h(M) + h^2(M)\right)\).

2. PRELIMINARIES

In this section, we recall some definitions and properties about Finsler manifolds. See \([14, 18]\) for more details.

Let \((M, F)\) be a (connected) Finsler \(n\)-manifold with Finsler metric \(F : TM \to [0, \infty)\). Let \((x, y) = (x', y')\) be local coordinates on \(TM\). Define
\[
g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j}, \quad G^i(y) := \frac{1}{4} y^i(y) \left(2 \frac{\partial g_{ij}}{\partial x^k}(y) - \frac{\partial g_{jk}}{\partial x^i}(y)\right) y^j y^k,
\]
where \(G^i\) is the geodesic coefficients. A smooth curve \(\gamma(t)\) in \(M\) is called a (constant speed) geodesic if it satisfies
\[
\frac{d^2 \gamma^i}{dt^2} + 2G^i \left(\frac{d\gamma}{dt}\right) = 0.
\]
Define the Ricci curvature by \(\text{Ric}(y) := \sum_{i=1}^n R^i_{ik}(y)\), where
\[
R^i_{ik}(y) := 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} + 2 G^j \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k} - \frac{\partial G^i}{\partial y^j} \frac{\partial G^j}{\partial y^k}.
\]
Set \(S_x M := \{y \in T_x M : F(x, y) = 1\}\) and \(SM := \cup_{x \in M} S_x M\). The reversibility \(\lambda_F\) and the uniformity constant \(\Lambda_F\) of \((M, F)\) are defined by
\[
\lambda_F := \sup_{y \in SM} F(-y), \quad \Lambda_F := \sup_{X, Y, Z \in SM} \frac{g_X(Y, Y)}{g_Z(Y, Y)}.
\]
Clearly, \(\Lambda_F \geq \lambda_F^2 \geq 1 \cdot \lambda_F = 1\) if and only if \(F\) is reversible, while \(\Lambda_F = 1\) if and only if \(F\) is Riemannian.

The dual Finsler metric \(F^*\) on \(M\) is defined by
\[
F^*(\eta) := \sup_{X \in T_x M \setminus \{0\}} \frac{\eta(X)}{F(X)}, \quad \forall \eta \in T_x^* M.
\]
The Legendre transformation \(\mathcal{L} : TM \to T^* M\) is defined by
\[
\mathcal{L}(X) := \begin{cases} \quad g_X(X, \cdot) & X \neq 0, \\ 0, & X = 0. \end{cases}
\]
In particular, $F^*(\mathcal{L}(X)) = F(X)$. Now let $f : M \rightarrow \mathbb{R}$ be a smooth function on $M$. The gradient of $f$ is defined by $\nabla f = \mathcal{L}^{-1}(df)$. Thus we have $d\nabla f = g_{ij}(\nabla f, X_i)$. Let $d\mu$ be a measure on $M$. In a local coordinate system $(x^i)$, express $d\mu = \sigma(x)dx^1 \wedge \cdots \wedge dx^n$. In particular, the Busemann-Hausdorff measure $d\mu_{BH}$ and the Holmes-Thompson measure $d\mu_{HT}$ are defined by

$$
\frac{\text{Vol}(\mathbb{B}^n)}{\text{Vol}(B_x M)} dx^1 \wedge \cdots \wedge dx^n,
$$

$$
\frac{1}{\text{Vol}(\mathbb{B}^n)} \int_{B_x M} \det g_{ij}(x,y)dy^1 \wedge \cdots \wedge dy^n dx^1 \wedge \cdots \wedge dx^n,
$$

where $B_x M := \{ y \in T_x M : F(x,y) < 1 \}$. For $y \in T_x M \setminus \{0\}$, define the distorsion of $(M,F,d\mu)$ as

$$
\tau(y) := \log \frac{\sqrt{\det g_{ij}(x,y)}}{\sigma(x)}.
$$

And S-curvature $S$ is defined by

$$
S(y) := \frac{d}{dt}[\tau(\gamma(t))]|_{t=0},
$$

where $\gamma(t)$ is the geodesic with $\dot{\gamma}(0) = y$.

### 3. A Cheeger type inequality for Finsler manifolds

**Definition 3.1 (14, 19).** Let $(M,F,d\mu)$ be a compact Finsler manifold. Denote $\mathcal{H}_0(M)$ by

$$
\mathcal{H}_0(M) := \left\{ \begin{array}{ll}
\{ f \in W^1_2(M) : \int_M f d\mu = 0 \}, & \partial M = \emptyset, \\
\{ f \in W^1_2(M) : f|_{\partial M} = 0 \}, & \partial M \neq \emptyset.
\end{array} \right.
$$

Define the canonical energy functional $E$ on $\mathcal{H}_0(M) \setminus \{0\}$ by

$$
E(u) := \frac{\int_M F^*(du)^2 d\mu}{\int_M u^2 d\mu}.
$$

$\lambda$ is an eigenvalue of $(M,F,d\mu)$ if there is a function $u \in \mathcal{H}_0(M) \setminus \{0\}$ such that $d_u E = 0$ with $\lambda = E(u)$. In this case, $u$ is called an eigenfunction corresponding to $\lambda$. The first eigenvalue of $(M,F,d\mu)$, $\lambda_1(M)$, is defined by

$$
\lambda_1(M) := \inf_{u \in \mathcal{H}_0(M) \setminus \{0\}} E(u),
$$

which is the smallest positive critical value of $E$.

**Remark 1.** It should be noticeable that $u$ is an eigenfunction corresponding to $\lambda$ if and only if

$$
\Delta u + \lambda u = 0 \text{ (in the weak sense)},
$$

where $\Delta := \text{div} \circ \nabla$ is the Laplacian induced by Shen[14, 18, 19].

Let $i : \Gamma \rightarrow M$ be a smooth hypersurface embedded in $M$. For each $x \in \Gamma$, there exist two 1-forms $\omega_\pm(x) \in T_x^* M$ satisfying $i^*(\omega_\pm(x)) = 0$ and $F^*(\omega_\pm(x)) = 1$. Then $n_\pm(x) := \mathcal{L}^{-1}(\omega_\pm(x))$ are two unit normal vectors on $\Gamma$. In general, $n_- \neq -n_+$ (see [11]). Let $d A_\pm$ denote the (area) measures induced by $n_\pm$, i.e., $d A_\pm = i^*(n_\pm) d\mu$. 
**Definition 3.2.** Let \((M, F, d\mu)\) be a closed Finsler manifold. Cheeger’s constant \(C(M)\) is defined by

\[
C(M) = \inf \frac{\min\{A\pm(\Gamma)\}}{\min\{\mu(D_1), \mu(D_2)\}},
\]

where \(\Gamma\) varies over compact \((n-1)\)-dimensional submanifolds of \(M\) which divide \(M\) into disjoint open submanifolds \(D_1, D_2\) of \(M\) with common boundary \(\partial D_1 = \partial D_2 = \Gamma\).

To prove Theorem 1.1, we need the following co-area formula.

**Theorem 3.3** ([18]). Let \((M, F, d\mu)\) be a Finsler manifold and \(\phi\) is piecewise smooth function with compact support. Then for any continuous function \(f\),

\[
\int_M f d\mu = \int_0^\infty \left[ \int_{\phi^{-1}(t)} f \frac{dA_n}{F(\nabla\phi)} \right] dt,
\]

where \(n := \nabla\phi / F(\nabla\phi)\).

Theorem 3.3 then yields the following lemma.

**Lemma 3.4.** For all positive function \(f \in C^\infty(M)\), we have

1. \(\int_0^\infty \mu(\Omega(t)) dt = \int_M f d\mu\),
2. \(C(M) \int_0^\infty \min\{\mu(\Omega(t)), \mu(M) - \mu(\Omega(t))\} dt \leq \int_M F^*(df) d\mu\),

where \(\Omega(t) := \{x \in M : f(x) \geq t\}\).

**Proof.** Without loss of generality, we assume that \(f\) is nonconstant. For almost every \(t \in [\min f, \max f]\), \(\Omega(t)\) is a domain in \(M\), with compact closure and smooth boundary. Note that \(n := \nabla\phi / F(\nabla\phi)\) is a unit normal vector field along \(\partial \Omega(t)\).

(1). It follows Theorem 3.3 that

\[
\frac{d}{dt} \mu(\Omega(t)) = - \int_{\phi^{-1}(t)} \frac{dA_n}{F(\nabla\phi)}.
\]

Thus, we have

\[
\int_0^\infty \mu(\Omega(t)) dt = - \int_0^\infty t d\mu(\Omega(t)) = \int_0^\infty t dt \int_{\phi^{-1}(t)} \frac{dA_n}{F(\nabla\phi)} = \int_M f d\mu.
\]

(2). Theorem 3.3 now yields

\[
\int_M F^*(df) d\mu = \int_0^\infty A_n(\partial \Omega(t)) dt \geq C(M) \int_0^\infty \min\{\mu(\Omega(t)), \mu(M) - \mu(\Omega(t))\} dt.
\]

**Proof of Theorem 1.1** Given a smooth function \(f\) on \(M\), let \(\alpha\) be a median of \(f\), i.e.,

\[
\mu(\{x : f(x) \geq \alpha\}) \geq \frac{1}{2} \mu(M), \mu(\{x : f(x) \leq \alpha\}) \geq \frac{1}{2} \mu(M).
\]

Set \(f_+ := \max\{f - \alpha, 0\}\) and \(f_- := \min\{f - \alpha, 0\}\). By the definition of median, one can check that for any \(t > 0\),

\[
\mu(\{x : f_+^2(x) \geq t\}) \leq \frac{1}{2} \mu(M), \mu(\{x : f_-^2(x) \geq t\}) \leq \frac{1}{2} \mu(M).
\]
Thus, the above inequalities together with Lemma 3.4 yield
\[
\int_M |f - \alpha|^2 d\mu = \int_M (f_+^2 + f_-^2) d\mu
\]
\[
= \int_M F^*(df_+^2) d\mu + \int_M F^*(df_-^2) d\mu = 2 \int_M f_+ F^*(df_+) + (-f_-) F^*(-df_-) d\mu
\]
\[
\leq 2\lambda \int_M |f - \alpha| F^*(df) d\mu \leq 2\lambda \left( \int_M |f - \alpha|^2 d\mu \right)^{1/2} \left( \int_M F^{*2}(df) d\mu \right)^{1/2}.
\]
Hence,
\[
\int_M F^{*2}(df) d\mu \geq \frac{\lambda^2}{4\lambda^2} \int_M |f - \alpha|^2 d\mu.
\]
Since \( \int_M f d\mu = 0 \),
\[
\inf_{\alpha \in \mathbb{R}} \int_M |f - \alpha|^2 d\mu \geq \int_M f^2 d\mu.
\]
\[\square\]

By a Croke type isoperimetric inequality, one has the following result. Also refer to Theorem 6.2, Proposition 6.4 for a reversible version of the isoperimetric inequality.

**Theorem 3.5** ([23, 24]). Let \((M, F, d\mu)\) be a closed Finsler manifold with \(\text{Ric} \geq (n-1)k\), where \(d\mu\) denotes either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Then
\[
\lambda_1(M) \geq \frac{(n-1)\mu(M)}{4 \text{Vol}(\mathbb{S}^{n-2}) \Lambda_{F}^{4n+4} \text{diam}(M) \int_0^{\text{diam}(M)} s_k^{n-1}(t) dt},
\]
where \(\text{diam}(M)\) denotes the diameter of \(M\).

Theorem 3.5 together with Theorem 1.1 now yields a Finslerian version of Yau’s lower estimate for the first eigenvalue [22].

**Theorem 3.6.** Let \((M, F, d\mu)\) be a closed Finsler manifold with \(\text{Ric} \geq (n-1)k\), where \(d\mu\) denotes either the Busemann-Hausdorff measure or the Holmes-Thompson measure. Then
\[
\lambda_1(M) \geq \left( \frac{(n-1)\mu(M)}{4 \text{Vol}(\mathbb{S}^{n-2}) \Lambda_{F}^{4n+4} \text{diam}(M) \int_0^{\text{diam}(M)} s_k^{n-1}(t) dt} \right)^2.
\]
That is, \(\lambda_1(M)\), of a closed Finsler manifold, can be bounded from below in terms of the diameter, volume, uniform constant and a lower bound for the Ricci curvature.

### 4. Volume Comparison

In this section, we will study the properties of the polar coordinate system of a Finsler manifold, which is useful to show Theorem 1.3. Refer to [20, 24] for more details.

Let \((M, F, d\mu)\) be a forward complete Finsler \(n\)-manifold. In the rest of this paper, we always assume that \(d\mu\) is either the Busemann-Hausdorff measure or the...
Homles-Thompson measure. Given \( p \in M \), denote by \((r,y) = (r,\theta^n), 1 \leq \alpha \leq n, \) the polar coordinates about \( p \). Express
\[
d\mu = \hat{\sigma}(r,y) dr \wedge dv_p(y),
\]
where \( dv_p \) is the measure on \( S_p M \) induced by \( F \).

**Lemma 4.1.** Let \((M, F, d\mu)\) be as above. If \( \text{Ric} \geq (n - 1)k \) \((k \leq 0)\) and \( \Lambda_F \leq \Lambda \), then

1. \( \hat{\sigma}(\min\{i_y, r\}, y) \geq \Lambda^{-2n} \frac{A_n,k(r)}{V_n,k(R)} \int_r^R \hat{\sigma}(\min\{i_y, t\}, y) dt, \forall 0 < r \leq R; \)
2. \( \int_{r_0}^{r_1} \hat{\sigma}(\min\{i_y, t\}, y) dt \geq \Lambda^{2n} \frac{V_n,k(r_1) - V_n,k(r_0)}{V_n,k(r_2) - V_n,k(r_1)} \int_{r_1}^{r_2} \hat{\sigma}(\min\{i_y, t\}, y) dt, \forall 0 < r_0 < r_1 < r_2; \)
3. \( \int_0^r \hat{\sigma}(\min\{i_y, t\}, y) dt \geq \Lambda^{-2n} \frac{V_n,k(r)}{V_n,k(R)} \int_0^R \hat{\sigma}(\min\{i_y, t\}, y) dt, \forall 0 < r \leq R. \)

Here, \( V_n,k(r) \) (resp. \( A_n,k(r) \)) is the volume (resp. area) of ball (resp. sphere) with radius \( r \) in the Riemannian space form of constant curvature \( k \), that is,
\[
A_n,k(r) = \text{Vol}(S^{n-1})g^{n-1}_k(r), \quad V_n,k(r) = \text{Vol}(S^n) \int_0^r g^{n-1}_k(t) dt.
\]

**Proof.** It is easy to check that \( \Lambda^{-n} \leq e^{-\tau(y)} \leq \Lambda^n \), for all \( y \in SM \). By \([20, 24]\), for each \( y \in S_p M \), we have
\[
\frac{\partial}{\partial r} \left( \frac{\hat{\sigma}_p(r,y)}{e^{-\tau(y)} g^{n-1}_k(r)} \right) \leq 0, \quad 0 < r < i_y,
\]
which implies
\[
\frac{\partial}{\partial r} \left( \frac{\hat{\sigma}_p(\min\{r, i_y\}, y)}{e^{-\tau(y)\min\{r, i_y\)} g^{n-1}_k(r)} \right) \leq 0, \text{ a.e. } r > 0.
\]

Hence,
\[
\frac{\hat{\sigma}_p(\min\{r, i_y\}, y)}{\hat{\sigma}_p(\min\{R, i_y\}, y)} \geq \frac{e^{-\tau(y)\min\{r, i_y\)} g^{n-1}_k(r)}{e^{-\tau(y)\min\{R, i_y\)} g^{n-1}_k(r)} \geq \Lambda^{-2n} \frac{g^{n-1}_k(r)}{g^{n-1}_k(R)}, \quad 0 < r \leq R.
\]

Then (1), (2) follows. And (3) follows from Gromov’s lemma\([9, \text{Lemma 3.1}]\). \(\square\)

Note that \( d\mathcal{A}_+(r,y) := \hat{\sigma}(r,y) dv_p(y) \) is the measure on \( S^+_g(r) \) induced by \( \nabla r \). Then we have the following result.

**Lemma 4.2.** Let \( i : \Gamma \hookrightarrow M \) be a hypersurface. If the reversibility \( \lambda_F \leq \lambda \), then
\[
dA_+(r,y) \geq \lambda^{-1} d\mathcal{A}_+(r,y), \text{ for any point } (r,y) = x \in \Gamma \ (r > 0).
\]

**Proof.** Let \( n \) denote a unit normal vector field on \( \Gamma \). Thus,
\[
d\mathcal{A}_+ = |i^*(\nabla r) d\mu| = |g_n(n, \nabla r)| dA_n \leq \lambda dA_n.
\]
\(\square\)
5. A Buser type isoperimetric inequality for starlike domains

In this section, we will extend Buser’s isoperimetric inequality[7, Lemma 5.1] to Finsler setting. However, the original method of Buser’s cannot be used directly, since the Finsler metrics considered here can be nonreversible. To overcome this difficulty, we introduce the “reverse” of a Finsler metric. The reverse of a Finsler metric $F$ is defined by $\tilde{F}(y) := F(-y)$.

A direct calculation yields the following two lemmas.

**Lemma 5.1.** For each $y \neq 0$, we have
\[ \tilde{G}^i(y) = G^i(-y), \quad \tilde{\text{Ric}}(y) = \text{Ric}(-y), \]
where $\tilde{G}^i$ (resp. $G^i$) is the spray of $\tilde{F}$ (resp. $F$) and $\tilde{\text{Ric}}$ (resp. $\text{Ric}$) is the Ricci curvature of $\tilde{F}$ (resp. $F$). Hence, if $\gamma$ is a geodesic of $F$, then the reverse of $\gamma$ is a geodesic of $\tilde{F}$.

**Lemma 5.2.** Let $(M, F)$ be a Finsler manifold. Then $d\bar{\mu} = d\mu$, where $d\bar{\mu}$ (resp. $d\mu$) denotes the Busemann-Hausdorff measure or the Holmes-Thompson measure of $\tilde{F}$ (resp. $F$).

**Corollary 5.3.** Let $(M, F, d\mu)$ be a Finsler manifold and $\Gamma$ be a smooth hypersurface embedded in $M$. Thus $\tilde{\Lambda}_\pm(\Gamma) = \Lambda_\mp(\Gamma)$, where $d\bar{\Lambda}$ (resp. $d\Lambda$) denote the induced measure on $\Gamma$ by $d\bar{\mu}$ (resp. $d\mu$).

**Proof.** Let $n_\pm$ (resp. $\tilde{n}_\pm$) be the unit normal vector along $\Gamma$ in $(M, F)$ (resp. $(M, \tilde{F})$). It is easy to check that $\tilde{n}_\pm = -n_\mp$. Then we are done by Lemma 5.2. $\square$

From above, we obtain the following key lemma.

**Lemma 5.4.** Let $D$ be a star-like domain (with respect to $p$) in $M$ with $B^+_p(r) \subset D \subset B^+_p(R)$. Let $\Gamma$ be a smooth hypersurface embedded in $D$ which divides $D$ into disjoint open sets $D_1, D_2$ in $D$ with common boundary $\partial D_1 = \partial D_2 = \Gamma$. Suppose $\text{Ric} \geq (n-1)(k < 0)$ and $\Lambda F \leq \Lambda$. If $\mu(D_1 \cap B^+_p(r/(2\sqrt{\Lambda}))) \leq \frac{1}{\alpha} \mu(B^+_p(r/(2\sqrt{\Lambda})))$, then
\[
\frac{\tilde{\Lambda}_\pm(\Gamma)}{\mu(D_1)} \geq \max_{0 < \beta < 2\sqrt{\Lambda}} \left\{ A_{n,k}(\beta) \left( \frac{-V_{n,k}(r)}{2\sqrt{\Lambda}} \right) - V_{n,k}(\beta) \right\},
\]

**Proof.** For convenience, set $B(\rho) := B^+_p(\rho)$, for any $\rho > 0$. Clearly, $\mu(D_1 \cap B(r/(2\sqrt{\Lambda}))) \leq \mu(D_2 \cap B(r/(2\sqrt{\Lambda})))$. Let $\alpha \in (0,1)$ be a constant which will be chosen later.

**Step 1:** Suppose $\mu(D_1 \cap B(r/(2\sqrt{\Lambda}))) \leq \alpha \mu(D_1)$.

For each $q \in D_1 - \text{Cut}_p$, set $q^*$ is the last point on the minimal geodesic segment $\gamma_{pq}$ from $p$ to $q$, where this ray intersects $\Gamma$. If the whole segment $\gamma_{pq}$ is contained in $D_1$, set $q^* := p$.

Fix a positive number $\beta \in (0, r/(2\sqrt{\Lambda}))$. Let $(t, y)$ denote the polar coordinate system about $p$. Given a point $q = (\rho, y) \in D_1 - \text{Cut}_p - B(r/(2\sqrt{\Lambda}))$, set
\[
\text{rod}(q) := \{(t, y) : \beta \leq t \leq \rho\}.
\]
Define
\[ D_1^1 := \{ q \in D_1 - \text{Cut}_p - B(\rho/(2\sqrt{A})) : q^* \notin B(\beta) \}; \]
\[ D_1^2 := \{ q \in D_1 - \text{Cut}_p - B(\rho/(2\sqrt{A})) : \text{rod}(q) \subset D_1 \}; \]
\[ D_1^3 := \{ q \in B(\rho/(2\sqrt{A})) - B(\beta) : \exists x \in D_1^2, \text{such that } q \in \text{rod}(x) \}. \]

By Lemma 4.1, we obtain that
\[ \frac{\mu(D_1^1)}{\mu(D_1^2)} \geq \Lambda^{-2n} \frac{V_{n,k}(\rho/(2\sqrt{A})) - V_{n,k}(\beta)}{V_{n,k}(\rho/(2\sqrt{A}))} =: \gamma^{-1}. \]

It follows from the assumption that
\[ (1 - \alpha)\mu(D_1) \leq \mu(D_1 - B(\rho/(2\sqrt{A}))), \quad \mu(D_1^1) \leq \mu(D_1 \cap B(\rho/(2\sqrt{A}))) \leq \alpha \mu(D_1). \]

Note that \( D_1 - B(\rho/(2\sqrt{A})) \subset D_1^1 \cup D_1^2 \). From above, we have
\[ (5.1) \quad (1 - \alpha)\mu(D_1) \leq \mu(D_1^2) + \mu(D_1^1) \leq \gamma \alpha \mu(D_1) + \mu(D_1^1). \]

Set \( \mathcal{C}_0 := \{ y \in S_p M : \exists t > 0, \text{such that } (t, y) \in D_1^1 \} \). Clearly,
\[ \mu(D_1^1) = \int_{\mathcal{C}_0} \: d\nu_p(y) \int_{r/(2\sqrt{A})}^{\min\{R, t_r\}} \chi_{D_1^1}(\exp_p(ty)) \cdot \hat{\sigma}(t, y)dt, \]
where
\[ \chi_{D_1^1}(x) = \begin{cases} 1, & x \in D_1^1, \\ 0, & x \notin D_1^1. \end{cases} \]

Given \( y \in \mathcal{C}_0 \), we can write
\[ \int_{r/(2\sqrt{A})}^{\min\{R, t_r\}} \chi_{D_1^1}(\exp_p(ty)) \cdot \hat{\sigma}(t, y)dt = \sum_{j_y} \int_{a_{j_y}}^{b_{j_y}} \hat{\sigma}(t, y)dt, \]
where \( \exp_p(b_{j_y}) \in \Gamma \) and \( \exp_p(a_{j_y}) \in \Gamma \) if \( a_{j_y} > r/(2\sqrt{A}) \).

Set
\[ c_{j_y} := \begin{cases} a_{j_y}, & a_{j_y} > r/(2\sqrt{A}); \\ F\left(\exp_p^{-1}\left(\left(\exp_p \frac{r}{2\sqrt{A}} y\right)^*\right)\right), & a_{j_y} = r/(2\sqrt{A}). \end{cases} \]

Thus, \( \beta \leq c_{j_y} \leq a_{j_y} \) and \( \exp_p(c_{j_y}) \in \Gamma \). Lemma 4.1 then yields
\[ \sum_{j_y} \int_{a_{j_y}}^{b_{j_y}} \hat{\sigma}(t, y)dt \leq \sum_{j_y} \int_{c_{j_y}}^{b_{j_y}} \hat{\sigma}(t, y)dt \leq \Lambda^{2n} \sum_{j_y} \frac{V_{n,k}(b_{j_y}) - V_{n,k}(c_{j_y})}{A_{n,k}(c_{j_y})} \hat{\sigma}(c_{j_y}, y) \]
\[ \leq \Lambda^{2n} \frac{V_{n,k}(R) - V_{n,k}(\beta)}{A_{n,k}(\beta)} \sum_{j_y} \hat{\sigma}(c_{j_y}, y). \]

The inequality above together with Lemma 4.2 yields
\[ \mu(D_1^1) \leq \Lambda^{2n} \frac{V_{n,k}(R) - V_{n,k}(\beta)}{A_{n,k}(\beta)} \int_{\mathcal{C}_0} \: d\nu_p(y) \int_{r/(2\sqrt{A})}^{\min\{R, t_r\}} \chi_{D_1^1}(\exp_p(ty)) \cdot \hat{\sigma}(t, y)dt \]
\[ \leq \Lambda^{2n+1} \frac{V_{n,k}(R) - V_{n,k}(\beta)}{A_{n,k}(\beta)} A_\pm(\Gamma). \]
Combining (5.1) and (5.2), we obtain
\[ \frac{A_n(\Gamma)}{\mu(D_1)} = \frac{A_n(\Gamma) \mu(D_1)}{\mu(D_1)} \geq \Lambda^{-\left(2n+\frac{1}{2}\right)}(1 - \alpha(1 + \gamma)) \frac{A_n,k(\beta)}{V_{n,k}(R) - V_{n,k}(\beta)}. \]

**Step 2:** Suppose \( \mu(D_1 \cap B(r/(2\sqrt{\Lambda}))) \geq \alpha \mu(D_1). \)

Then Fubini’s theorem together with [4, Lemma 8.5.4] yields that
\[ : \text{Suppose that} \]
\[ (q, w, \beta) \in \left( \frac{(q, w, \beta)}{w, \beta} \right) \text{and} \]
\[ \text{Hence, for each} \]
\[ (q, w, \beta) \in (W_1 \times W_2) \setminus N, \text{there exists a unique minimal geodesic} \]
\[ \gamma_{wp} \text{from} \]
\[ w \text{to} q \text{with the length} \]
\[ L_F(\gamma_{wp}) \leq r. \]

We claim that \( \gamma_{wp} \) is contained in \( B(r). \) In fact, if \( \gamma_{wp} \cap S_p^+(r) = \{ w_1, q_1 \} \) (which may coincide), then
\[ d(w_1, q_1) \geq d(p, w_1) - d(p, w) > \left( 1 - \frac{1}{2\sqrt{\Lambda}} \right) r, \]
\[ d(q_1, q) > \frac{1}{\sqrt{\Lambda}} \left( 1 - \frac{1}{2\sqrt{\Lambda}} \right) r. \]

Hence, \( L_F(\gamma_{wp}) \geq d(w_1, q_1) > r, \) which is a contradiction!

Since \( \gamma_{wp} \) is contained in \( B(r), \) it must intersect \( \Gamma. \) Denote by \( q^* \) the last point on \( \gamma_{wp} \) where \( \gamma_{wp} \) intersects \( \Gamma. \)

Define
\[ V_1 := \{ (q, w) \in W_1 \times W_2 : d(w, q^*) \geq d(q^*, q) \}, \]
\[ V_2 := \{ (q, w) \in W_1 \times W_2 : d(w, q^*) \leq d(q^*, q) \}, \]
where \( q^* \) is defined as above. Since \( \mu_\times(V_1 \cup V_2) = \mu_\times(W_1 \times W_2), \) we have
\[ \mu_\times(V_1) \geq \frac{1}{2} \mu_\times(W_1 \times W_2) \text{ or } \mu_\times(V_2) \geq \frac{1}{2} \mu_\times(W_1 \times W_2). \]

**Case I:** Suppose that \( \mu_\times(V_1) \geq \frac{1}{2} \mu_\times(W_1 \times W_2). \)

Note that
\[ \mu_\times(V_1) = \int_{w \in W_2} d\mu \int_{(q \in W_1: d(q^*, q) \geq d(q^*, q)) - \text{Cut}_w} d\mu. \]

Thus, there exist a point \( w_2 \in W_2 \) and a measurable set \( U_1 \subset W_1 \) such that
1. For each \( q \in U_1, \) \( d(w_2, q^*) \geq d(q^*, q) \) and \( (q, w_2) \notin N. \)
2. \( \mu(U_1) \geq \frac{1}{2} \mu(W_1): \)

Let \( (t, y) \) denote the polar coordinates about \( w_2. \) For \( q = (\rho, y) \in U_1, \) set \( q^* := (\rho^*, y). \) Since \( \rho^* = d(w, q^*) \geq d(q^*, q) = \rho - \rho^*, \rho^* \geq \rho/2. \) Set \( \rho^{**} := \sup \{ s : \exp_{w_2}(ty), t \in [\rho^*, s], \text{is contained in} \ U_1 - \text{Cut}_{w_2} \}. \) Then \( \tilde{q} := (\rho^{**}, y) \in B(r/(2\sqrt{\Lambda})), \) which implies
\[ \rho^{**} = d(w_2, \tilde{q}) \leq d(w_2, p) + d(p, \tilde{q}) < \frac{r}{2} + \frac{r}{2\sqrt{\Lambda}} \leq r. \]
Since \((\bar{q})^*=q^*, \rho^*\geq \rho^{**}/2\). Lemma 4.1 then yields
\[
\frac{\hat{\sigma}(\rho^*, y)}{\int_{\rho}^{\rho^*} \hat{\sigma}(t, y) dt} \geq \Lambda^{-2n} \frac{A_{n,k}(\rho^*)}{V_{n,k}(\rho^*) - V_{n,k}(\rho^*)} \geq \Lambda^{-2n} \frac{A_{n,k}(\rho^{**}/2)}{V_{n,k}(\rho^{**}) - V_{n,k}(\rho^{**}/2)}
\]

\[
\geq \Lambda^{-2n} \frac{A_{n,k}(r/2)}{V_{n,k}(r) - V_{n,k}(r/2)}
\]

Lemma 4.2 now yields that
\[
d A_{\pm}(\rho^*, y) \geq \Lambda^{-(2n+\frac{1}{2})} \frac{A_{n,k}(r/2)}{V_{n,k}(r) - V_{n,k}(r/2)} \left(\int_{\rho}^{\rho^*} \hat{\sigma}(t, y) dt\right) d\nu_{w^2}(y).
\]

Hence,
\[
A_{\pm}(\Gamma) \geq A_{\pm}(\Gamma \cap B(r/(2\sqrt{A}))) \geq \Lambda^{-(2n+\frac{1}{2})} \frac{A_{n,k}(r/2)}{V_{n,k}(r) - V_{n,k}(r/2)} \mu(U_1).
\]

By assumption, we have
\[
\alpha \mu(D_1) \leq \mu(D_1 \cap B(r/(2\sqrt{A}))) = \mu(W_1) \leq 2\mu(U_1),
\]
which implies
\[
\frac{A_{\pm}(\Gamma)}{\mu(D_1)} \geq \frac{\alpha}{2\Lambda^{2n+\frac{3}{2}}} \frac{A_{n,k}(r/2)}{V_{n,k}(r) - V_{n,k}(r/2)}.
\]

**Case II:** Suppose that \(\mu_x(V_2) \geq \frac{1}{2} \mu_x(W_1 \times W_2).

Then Fubini’s theorem yields that there exist a point \(q_1 \in W_1\) and a measurable set \(U_2 \subset W_2\) such that

(1) For each \(w \in U_2\), \(d(w, q_1^*) \leq d(q_1^*, q_1)\) and \((q_1, w) \notin N_2\).

(2) \(\mu(U_2) \geq \frac{1}{2} \mu(W_2)\);

It should be noticeable that \(q_1^*\) is dependent on the choice of \(w\). Let \(w^\sharp\) denote the first point on \(\gamma_{w,q_1}\) where the segment intersects \(\Gamma\). Thus, for each \(w \in U_2\),
\[
d(w, w^\sharp) \leq d(w, q_1^*) \leq d(q_1^*, q_1) \leq d(w^\sharp, q_1).
\]

Let \(\tilde{F}\) denote the reverse of \(F\). It follows from Lemma 5.1 that the reverse of the geodesic \(\tilde{\gamma}_{q_1, w}\) is a minimal geodesic from \(q_1\) to \(w\) in \((M, \tilde{F})\). Note that \(w^\sharp\) is the last point on \(\tilde{\gamma}_{q_1, w}\) where \(\tilde{\gamma}_{q_1, w}\) intersects \(\Gamma\). Let \(\tilde{N}\) be defined as \(N\) in \((M, \tilde{F})\). It is easy to see that \(\tilde{N} = N\). Denote by \(\tilde{d}\) the metric induced by \(\tilde{F}\). Thus, \(U_2 \subset W_2\) satisfies

(1) For each \(w \in U_2\), \(\tilde{d}(q_1, w^\sharp) \geq \tilde{d}(w^\sharp, w)\) and \((q_1, w) \notin \tilde{N}\).

(2) \(\tilde{\mu}(U_2) \geq \frac{1}{2} \tilde{\mu}(W_2)\);

Note that Lemma 5.1 also implies that \(\tilde{\text{Ric}} \geq (n-1)k\). A similar argument to the one in Case I together with Lemma 4.2 and Corollary 5.3 yields that
\[
A_{\pm}(\Gamma) \geq \Lambda^{-(2n+\frac{1}{2})} \frac{A_{n,k}(r/2)}{V_{n,k}(r) - V_{n,k}(r/2)} \mu(U_2).
\]

By assumption, we have
\[
\alpha \mu(D_1) \leq \mu(D_1 \cap B(r/(2\sqrt{A}))) \leq \mu(D_2 \cap B(r/(2\sqrt{A}))) = \mu(W_2) \leq 2\mu(U_2).
\]

Hence,
\[
\frac{A_{\pm}(\Gamma)}{\mu(D_1)} \geq \frac{\alpha}{2\Lambda^{2n+\frac{3}{2}}} \frac{A_{n,k}(r/2)}{V_{n,k}(r) - V_{n,k}(r/2)}.
\]
Step 3: From above, we obtain

\[
\frac{A_\pm(\Gamma)}{\mu(D_1)} \geq \begin{cases} 
\frac{1-\alpha(1+\Lambda^{2n}\mathcal{E})}{\Lambda^{2n+\frac{1}{2}}} \mathcal{A}, & \mu(D_1 \cap B(r/(2\sqrt{\Lambda}))) \leq \alpha \mu(D_1); \\
\frac{\alpha}{2\Lambda^{2n+\frac{1}{2}}} \mathcal{B}, & \mu(D_1 \cap B(r/(2\sqrt{\Lambda}))) \geq \alpha \mu(D_1), 
\end{cases}
\]

where

\[
\mathcal{A} := \frac{A_{n,k}(\beta)}{V_{n,k}(R) - V_{n,k}(\beta)}, \quad \mathcal{B} := \frac{A_{n,k}(r/2)}{V_{n,k}(r) - V_{n,k}(r/2)},
\]

To obtain the best possible bound, we set

\[
1-\alpha(1+\Lambda^{2n}\mathcal{E}) = \frac{\alpha^2}{2\Lambda^{2n+\frac{1}{2}}} \mathcal{B}.
\]

Thus,

\[
\alpha = \frac{2\mathcal{A}}{\mathcal{B} + 2\mathcal{A}(1+\Lambda^{2n}\mathcal{E})}.
\]

An easy calculation then yields

\[
\frac{A_\pm(\Gamma)}{\mu(D_1)} \geq \frac{A_{n,k}(\beta)}{2\Lambda^{4n+\frac{1}{2}}} \frac{V_{n,k}(\beta)}{V_{n,k}(r/2)} \frac{V_n}{V_{n,k}(R)}.
\]

Since \( \frac{A_\pm(\Gamma)}{\min(\mu(D_1), \mu(D_2))} \geq \frac{A_\pm(\Gamma)}{\mu(D_1)} \), we have the following Finslerian version of Buser’s isoperimetric inequality [9, Theorem 6.8].

**Corollary 5.5.** Let \( D \) be a star-like domain (with respect to \( p \)) in \( M \) with \( B_p^+(r) \subset D \subset B_p^+(R) \). If \( \text{Ric} \geq (n-1)k \) \( (k < 0) \) and \( \Lambda_F \leq \Lambda \), then

\[
h(D) := \inf_{\Gamma} \frac{\min(A_\pm(\Gamma))}{\min(\mu(D_1), \mu(D_2))} \geq \frac{\max_{0 < \beta < \frac{2}{\sqrt{\Lambda}}} \left\{ \frac{A_{n,k}(\beta)}{2\Lambda^{4n+\frac{1}{2}}} \frac{V_{n,k}(\beta)}{V_{n,k}(r/2)} \frac{V_n}{V_{n,k}(R)} \right\}}{\max(\mu(D_1), \mu(D_2))},
\]

where \( \Gamma \) varies over smooth hypersurfaces in \( D \) satisfying

1. \( \Gamma \) is embedded in \( \overline{D} \);
2. \( \Gamma \) divides \( D \) into disjoint open sets \( D_1, D_2 \) in \( D \) with common boundary \( \partial D_1 = \partial D_2 = \Gamma \).

6. **A Buser type inequality for Finsler manifolds**

Let \( (\phi, \varphi) := \int_M \phi \varphi d\mu \). Then we have the following minimax principle.

**Lemma 6.1.** Let \((M, F, d\mu)\) be a closed Finsler manifold with the reversibility \( \lambda_F \) and let \( D_1, D_2 \) be pairwise disjoint normal domains (i.e., with compact closures and nonempty piecewise \( C^\infty \) boundary) in \( M \). Then

\[
\lambda_1(M) \leq \lambda_F^2 \max\{\lambda_1(D_1), \lambda_1(D_2)\}.
\]
Proof. Suppose \( \lambda_1(D_1) \leq \lambda_1(D_2) \). And let \( \psi_i \) be the eigenfunction corresponding to \( \lambda_1(D_i) \), \( i = 1, 2 \). We extend \( \psi_i \) to \( M \) by letting \( \psi_i \equiv 0 \) on \( M \setminus D_i \).

There exists \( \alpha_1, \alpha_2 \), not all equal to zero, satisfying
\[
\alpha_1(\psi_1, \phi) + \alpha_2(\psi_2, \phi) = 0,
\]
where \( \phi \) is a nonzero constant function on \( M \). Set \( f := \alpha_1 \psi_1 + \alpha_2 \psi_2 \). Thus, \( (f, \phi) = 0 \) and \( f \in \mathcal{H}_0(M) \). Hence,
\[
\lambda_1(M) \leq \frac{\int_M F_s^2(df) \, d\mu}{\int_M f^2 \, d\mu} \leq \lambda_F^2 \frac{\int_{D_1} \alpha_1^2 F_s^2(d\psi_1) \, d\mu + \int_{D_2} \alpha_2^2 F_s^2(d\psi_2) \, d\mu}{\int_M (\alpha_1^2 \psi_1^2 + \alpha_2^2 \psi_2^2) \, d\mu} \leq \lambda_F^2 \lambda_1(D_2).
\]

The following lemma is clear.

**Lemma 6.2.** Let \((M, F, du)\) be a closed Finsler manifold. Given a positive constant \( C > 0 \), define a new Finsler metric \( \tilde{F} \) by \( \tilde{F}(y) := C^\frac{1}{2} F(y) \). Then
\[
\tilde{\text{Ric}}(y) = \frac{1}{C} \text{Ric}, \quad \tilde{\lambda}_1(M) = \frac{1}{C} \lambda_1(M), \quad \tilde{h}(M) = \frac{1}{\sqrt{C}} h(M).
\]

**Proof of Theorem 1.3** By Lemma 6.2 we can suppose that \( \text{Ric} \geq -(n-1) \), i.e., \( \delta = 1 \). Given any \( \epsilon > 0 \), let \( \Gamma, D_1 \) and \( D_2 \) be as in Definition 3.2 such that
\[
0 \leq \mathcal{F} - h(M) < \epsilon,
\]
where
\[
\mathcal{F} := \min\{A_\pm(\Gamma)\} / \min\{\mu(D_1), \mu(D_2)\}.
\]

**Step 1.** For \( n \geq 3 \), \( \Gamma \) may satisfy \( \max_{p \in M} d(p, \Gamma) < \rho \), for any \( \rho > 0 \) ("the problem of hair" [5]). Hence, we will find a new set \( \Gamma \) to replace \( \Gamma \).

Let \( \mathcal{P} := \{p_1, \ldots, p_k\} \) be a forward complete \( r \)-package in \( M \), that is
\begin{align*}
(1) & \quad d(p_i, p_j) \geq 2r, \text{ for } i \neq j; \\
(2) & \quad \cup_{1 \leq i \leq k} B^+(p_i, 2r\sqrt{\Lambda}) = M.
\end{align*}

For each \( p_i \in \mathcal{P} \), the Dirichlet region of \( p_i \) is defined by
\[
\mathcal{D}_i := \{q \in M : d(p_i, q) \leq d(p_j, q), \text{ for all } 1 \leq j \leq k\}.
\]

Property (1) and (2) imply that
\[
B_p(r/\sqrt{\Lambda}) \subset \mathcal{D}_i \subset B_p(2r\sqrt{\Lambda}).
\]

Lemma 4.1 yields
\[
\mu(\mathcal{D}_i) \leq \mu(B_{p_i}(2r\sqrt{\Lambda})) \leq \Lambda^{2n} \frac{V_{n-1}(2r\sqrt{\Lambda})}{V_{n-1}(r/(2\Lambda))} \mu(B_{p_i}(r/(2\Lambda))),(1)
\]

Now it follows from Corollary 5.5 that for \( 0 < r < \frac{1}{2\sqrt{\Lambda}} \),
\[
\mathcal{H}(\mathcal{D}_i) \geq \frac{A_{n-1}(\frac{r}{\sqrt{\Lambda}}) V_{n-1}(\frac{r}{2\sqrt{\Lambda}})}{4\Lambda^{4n+4} V_{n-1}(\frac{r}{\sqrt{\Lambda}}) V_{n-1}(2r\sqrt{\Lambda})} \geq \frac{1}{C_1(n, \Lambda) r} =: \mathcal{F}(r).
\]

Here, \( C_1(n, \Lambda) \) is a positive constant depending only on \( n \) and \( \Lambda \). One can easily check that
\[
\mathcal{F}(r) \leq \frac{A^{2n} V_{n-1}(4r\sqrt{\Lambda})}{V_{n-1}(r/(2\Lambda))} < \frac{1}{8}.
\]
for

$$0 < r < \min \left\{ \frac{1}{4\sqrt{\Lambda}}, \frac{1}{C_2(n, \Lambda)} \right\}.$$  

Claim: For $i \neq j$,

$$\mathcal{D}_i \cap \mathcal{D}_j = \{ q \in M : d(p_i, q) = d(p_j, q) \leq d(p_s, q), \text{ for all } 1 \leq s \leq k \}$$

has measure zero (with respect to $d\mu$).

Note that $d(p_i, \cdot)$ and $d(p_j, \cdot)$ are smooth on $\text{int}(\mathcal{D}_i \cap \mathcal{D}_j) - \text{Cut}_{p_i} \cup \text{Cut}_{p_j}$. If $\nabla(d(p_i, x) - d(p_j, x)) = 0$ for some $x \in \text{int}(\mathcal{D}_i \cap \mathcal{D}_j) - \text{Cut}_{p_i} \cup \text{Cut}_{p_j}$, then $d(d(p_i, x)) - d(d(p_j, x)) = 0$, that is, $d(d(p_i, x)) = d(d(p_j, x))$ and

$$\nabla d(p_i, x) = \nabla d(p_j, x).$$

This implies that the unique minimal geodesic from $p_i$ to $x$ overlaps the unique minimal geodesic from $p_j$ to $x$. Since $d(p_i, x) = d(p_j, x)$, we have $p_i = p_j$, which is a contradiction! Hence, $d(p_i, x) - d(p_j, x)$ is regular on $\text{int}(\mathcal{D}_i \cap \mathcal{D}_j) - \text{Cut}_{p_i} \cup \text{Cut}_{p_j}$, which implies $\dim(\text{int}(\mathcal{D}_i \cap \mathcal{D}_j) - \text{Cut}_{p_i} \cup \text{Cut}_{p_j}) \leq (n-1)$. Therefore, $\text{int}(\mathcal{D}_i \cap \mathcal{D}_j) = \emptyset$ and $\mathcal{D}_i \cap \mathcal{D}_j$ has measure zero. Thus, (6.4)

$$\mu(\mathcal{D}_i \cup \mathcal{D}_j) = \mu(\mathcal{D}_i) + \mu(\mathcal{D}_j).$$

Enumerate the collection $\{p_i\}_{i=1}^k$ in such way that

$$\mu(D_1 \cap B_{p_i}^+(r/2\Lambda)) \leq \frac{1}{2^i} \mu(B_{p_i}^+(r/2\Lambda)), \text{ for } i = 1, \cdots, m;$$

$$\mu(D_1 \cap B_{p_i}^+(r/2\Lambda)) > \frac{1}{2^i} \mu(B_{p_i}^+(r/2\Lambda)), \text{ for } i = m + 1, \cdots, k.$$

Lemma 5.4 implies that for $i = 1, \cdots, m$,

$$(6.5) \quad A_\pm(\Gamma \cap \text{int}(\mathcal{D}_i)) \geq \mathcal{F}(r) \mu(D_1 \cap \mathcal{D}_i).$$

From (6.1), (6.4) and (6.5), we obtain

$$\sum_{i=1}^m \mu(D_1 \cap \mathcal{D}_i) \leq \frac{1}{\mathcal{F}(r)} A_\pm(\Gamma),$$

which together with (6.3) yields that

$$\sum_{i=1}^m \mu(D_1 \cap \mathcal{D}_i) \leq \frac{1}{\mathcal{F}(r)} \min\{A_\pm(\Gamma)\}$$

which is a contradiction.
Likewise, there also exists a point $p_j \in \mathcal{P}$ such that
\[
\mu(D_2 \cap B_{p_j}^+(r/(2\Lambda))) > \frac{1}{2} \mu(B_{p_j}^+(r/(2\Lambda))).
\]

Thus, the following sets are not empty:
\[
\tilde{D}_1 := \left\{ q \in M : \mu(D_1 \cap B_q^+(r/(2\Lambda))) > \frac{1}{2} \mu(B_q^+(r/(2\Lambda))) \right\};
\]
\[
\tilde{D}_2 := \left\{ q \in M : \mu(D_2 \cap B_q^+(r/(2\Lambda))) > \frac{1}{2} \mu(B_q^+(r/(2\Lambda))) \right\}.
\]

Since the continuity of the map $q \mapsto \mu(D_1 \cap B_q^+(r/(2\Lambda))) - \mu(D_2 \cap B_q^+(r/(2\Lambda)))$, the open submanifolds $\tilde{D}_1$ and $\tilde{D}_2$ are separated by the closed subset
\[\tilde{\Gamma} := \{ q \in M : \mu(D_1 \cap B_q^+(r/(2\Lambda))) = \mu(D_2 \cap B_q^+(r/(2\Lambda))) \}.
\]

**Step 2.** Define
\[\tilde{\Gamma}^t := \{ q \in M : d(\tilde{\Gamma}, q) \leq t \}.
\]

Now choose a new forward complete $r$-package $\mathcal{Q} = \{q_1, \ldots, q_s\}$ in $M$ such that:
1. $q_1, \ldots, q_s \in \tilde{\Gamma}$ and $\tilde{\Gamma} \subset \bigcup_{1 \leq i \leq s} B_{q_i}^+(2r\sqrt{\Lambda})$;
2. $q_{s+1}, \ldots, q_m \in D_1$ and $q_{m+1}, \ldots, q_l \in D_2$.

Since $\tilde{\Gamma}^t \subset \bigcup_{1 \leq i \leq s} B_{q_i}^+(2r\sqrt{\Lambda} + t)$ and $q_1, \ldots, q_s \in \tilde{\Gamma}$, by (6.7) and Lemma 4.1, we have
\[
\mu(\tilde{\Gamma}^t) \leq \mu\left( \bigcup_{i=1}^s B_{q_i}^+(2r\sqrt{\Lambda} + t) \right) \leq \sum_{i=1}^s \mu(B_{q_i}^+(2r\sqrt{\Lambda} + t))
\]
\[
\leq \Lambda^2 n \frac{v_{n-1}(2r\sqrt{\Lambda} + t)}{V_{n-1}(r/(2\Lambda))} \sum_{i=1}^s \mu(B_{q_i}^+(r/(2\Lambda)))
\]
\[
= 2\Lambda^2 n \frac{v_{n-1}(2r\sqrt{\Lambda} + t)}{V_{n-1}(r/(2\Lambda))} \sum_{i=1}^s \mu(D_1 \cap B_{q_i}^+(r/(2\Lambda))).
\]

It follows from Corollary 5.5 that for $1 \leq i \leq s$,
\[
\frac{A_\pm(D_1 \cap B_{q_i}^+(r/(2\Lambda)))}{\mu(D_1 \cap B_{q_i}^+(r/(2\Lambda)))} \geq h(B_{q_i}^+(r/(2\Lambda))) \geq \mathcal{J}(r/(2\sqrt{\Lambda})) \geq \mathcal{J}(r).
\]

Since $d(q_i, q_j) \geq 2r$, $B_{q_i}^+(r/(2\Lambda)) \cap B_{q_j}^+(r/(2\Lambda)) = \emptyset$. Hence, we have
\[
\mu(\tilde{\Gamma}^t) \leq 2\Lambda^2 n \frac{v_{n-1}(2r\sqrt{\Lambda} + t)}{\mathcal{J}(r) V_{n-1}(r/(2\Lambda))} A_\pm(\Gamma),
\]
which implies
\[
\mu(\tilde{\Gamma}^t) \leq 2\Lambda^2 n \frac{v_{n-1}(2r\sqrt{\Lambda} + t)}{\mathcal{J}(r) V_{n-1}(r/(2\Lambda))} \min\{A_\pm(\Gamma)\}
\]
\[
= 2\Lambda^2 n \frac{\mathcal{J} V_{n-1}(2r\sqrt{\Lambda} + t)}{\mathcal{J}(r) V_{n-1}(r/(2\Lambda))} \min\{\mu(D_1), \mu(D_2)\}
\]
\[
\leq 2\Lambda^2 n \frac{\mathcal{J} V_{n-1}(2r\sqrt{\Lambda} + t)}{\mathcal{J}(r) V_{n-1}(r/(2\Lambda))} \mu(D_1).
\]

Now let $t = 2r\sqrt{\Lambda}$.
Claim: If \( q \in \tilde{D}_2 - \tilde{\Gamma}^{2r\sqrt{\Lambda}} \), then \( q \) must be contained in some Dirichlet region \( \mathcal{D}_i \), \( m + 1 \leq i \leq l \).

Choose a point \( x \in (\tilde{D}_1 \cup \Gamma) \). Suppose that the minimal geodesic from \( x \) to \( q \) intersects \( \Gamma \) at \( y \). Thus,

\[
d(x, q) \geq d(y, q) \geq d(\tilde{\Gamma}, q) > 2r\sqrt{\Lambda},
\]

which implies that \( d(q_i, q) > 2r\sqrt{\Lambda}, 1 \leq i \leq m \). Since \( \cup_{1 \leq i \leq l} B_{q_i}^+ (2r\sqrt{\Lambda}) \supset M \), there exists \( i \in \{m + 1, \ldots, l\} \) such that \( q \in B_{q_i}^+ (2r\sqrt{\Lambda}) \). Choose \( i_0 \in \{m + 1, \ldots, l\} \) such that

\[
d(q_{i_0}, q) = \min_{m+1 \leq i \leq l} d(q_i, q) < 2r\sqrt{\Lambda}.
\]

Then \( q \in \mathcal{D}_{i_0} \). The claim is true. Hence, \( \tilde{D}_2 - \tilde{\Gamma}^{2r\sqrt{\Lambda}} \subset \cup_{m+1 \leq i \leq l} \mathcal{D}_i \).

By (6.6), (6.8) and (6.3), we have

\[
\mu(\tilde{D}_1 - \tilde{\Gamma}^{2r\sqrt{\Lambda}}) \geq \mu(D_1 \cap (\tilde{D}_1 - \tilde{\Gamma}^{2r\sqrt{\Lambda}})) = \mu(D_1) - \mu(D_1 \cap (\tilde{D}_2 - \tilde{\Gamma}^{2r\sqrt{\Lambda}})) - \mu(D_1 \cap \tilde{\Gamma}^{2r\sqrt{\Lambda}}) \\
\geq \mu(D_1) - \sum_{i=m+1}^{l} \mu(D_1 \cap \mathcal{D}_i) - \mu(\tilde{\Gamma}^{2r\sqrt{\Lambda}}) \\
\geq \mu(D_1) - \frac{1}{4} \mu(D_1) - \frac{1}{4} \mu(D_1) = \frac{1}{2} \mu(D_1) > 0.
\]

Lemma 6.1 yields

\[
\lambda_1(M) \leq \Lambda \max \{ \lambda_1(\tilde{D}_1), \lambda(\tilde{D}_2) \}.
\]

Without loss of generality, we suppose that \( \lambda_1(\tilde{D}_1) \geq \lambda_1(\tilde{D}_2) \). Now, we estimate \( \lambda_1(\tilde{D}_1) \). Define a function on \( \tilde{D}_1 \) by

\[
f(q) := \begin{cases} 
\frac{d(\tilde{\Gamma}, q)}{2r\sqrt{\Lambda}}, & q \in \tilde{D}_1 \cap \tilde{\Gamma}^{2r\sqrt{\Lambda}} \\
1, & q \in \tilde{D}_1 - \tilde{\Gamma}^{2r\sqrt{\Lambda}}.
\end{cases}
\]

Clearly,

\[
F^{*2}(df(q)) = \begin{cases} 
\frac{1}{4r^{2\Lambda}}, & q \in \tilde{D}_1 \cap \tilde{\Gamma}^{2r\sqrt{\Lambda}} \\
0, & q \in \tilde{D}_1 - \tilde{\Gamma}^{2r\sqrt{\Lambda}}.
\end{cases}
\]

(6.8) together with (6.9) yields that

\[
\int_{\tilde{D}_1} F^{*2}(df) d\mu \leq \frac{1}{4r^{2\Lambda}} \mu(\tilde{\Gamma}^{2r\sqrt{\Lambda}}) \leq \Lambda^{2n-1} \frac{\mathscr{F} V_{n-1}(4r\sqrt{\Lambda})}{2r^{2} \mathscr{F}(r) V_{n-1}(r/(2\Lambda))} \mu(D_1),
\]

\[
\int_{\tilde{D}_1} f^2 d\mu \geq \mu(\tilde{\Gamma}^{2r\sqrt{\Lambda}}) \geq \frac{1}{2} \mu(D_1).
\]

Thus, by (6.2), we obtain

\[
\lambda_1(\tilde{D}_1) \leq \frac{\int_{\tilde{D}_1} F^{*2}(df) d\mu}{\int_{\tilde{D}_1} f^2 d\mu} \leq \Lambda^{2n-1} \frac{\mathscr{F} V_{n-1}(4r\sqrt{\Lambda})}{r^{2} \mathscr{F}(r) V_{n-1}(r/(2\Lambda))} \leq \frac{C_3(n, \Lambda) \mathcal{F}}{r},
\]
Choose and speed geodesic with \( \dot{\gamma}(0) = \gamma \).

**Proof.**

For the Holmes-Thompson measure, the S-curvature of a Berwald manifold always vanishes. Furthermore, we have the following theorem.

**Theorem 6.4.** Let \((M, \alpha + \beta)\) be a \(n\)-dimensional closed Randers manifold. Then

\[
\Lambda_F = \frac{(1 + b)^2}{(1 - b)^2} = \lambda^2_F,
\]

where \( b := \sup_{x \in M} \|\beta\|_x \). Hence, \( \lambda_1(M) \leq C(n,b) \left( \delta h(M) + h^2(M) \right) \).

**Proof.** Since \( M \) is closed, there exists a point \( x \in M \), such that \( \|\beta\|_x = b \). Choose \( y, X \in T_x M \) with \( \|X\|_x = \|y\|_x = 1 \). For convenience, we set \( y = s X + X^+_i \) and \( \beta = t X + X^+_i \). Here, we view \( \beta \) as a tangent vector in \((T_x M, \alpha)\) and \((X, X^+_i)\) is a basis of \( T_x M, \alpha \) where \( i = 1, 2 \). By \[18\] (1.6), one has

\[
g_y(X, X) = [1 + \beta(y)](1 - s^2) + (s + t)^2, \quad -1 \leq s \leq 1, \quad -b \leq t \leq b.
\]

Clearly, \( (1 - b)^2 \leq g_y(X, X) \), with equality if and only if \( y = \pm X \) and \( \beta = \mp b X \), and \( g_y(X, X) \leq (1 + b)^2 \), with equality if and only if \( y = \pm X \) and \( \beta = \pm b X \). Hence, \( \Lambda_F = (1 + b)^2/(1 - b)^2 \).

**Remark 2.** Given a Randers metric \( F = \alpha + \beta \), the Holmes-Thompson measure \( d\mu_{HT} = dV_\alpha \), where \( dV_\alpha \) is the Riemannian measure induced by \( \alpha \). By \[18\] Example 3.2.1, one can show

\[
\frac{\lambda_1(M, \alpha)}{(1 + b)^2} \leq \lambda_1(M, F) \leq \frac{\lambda_1(M, \alpha)}{(1 - b)^2},
\]

where \( \lambda_1(M, \alpha) \) (resp. \( \lambda_1(M, F) \)) is the first eigenvalue of \((M, \alpha)\) (resp. \((M, F, d\mu_{HT})\)).

By \[20\] \[24\], we can see that the upper bound for the uniform constant in Lemma \[41\] can be replaced by the lower bound for the S-curvature. Using the similar argument, one can show the following theorem.

**Theorem 6.5.** Let \((M, F, d\mu)\) be a closed Finsler \(n\)-manifold with the Ricci curvature \( \text{Ric} \geq -(n - 1)\delta^2 \), the S-curvature \( S \geq (n - 1)\eta \) and the reversibility \( \lambda_F \leq \lambda \). Then

\[
\lambda_1(M) \leq C(n, \lambda, \eta) \left( \delta h(M) + h^2(M) \right).
\]

It follows from \[18\] that for the Busemann-Hausdorff measure, the S-curvature of a Berwald manifold always vanishes. Furthermore, we have the following

**Theorem 6.6.** For the Holmes-Thompson measure, the S-curvature of a Berwald manifold also vanishes.

**Proof.** Let \((M, F)\) be a \(n\)-dimensional Berwald manifold and let \( \gamma_y(t) \) be a unit speed geodesic with \( \dot{\gamma}_y(0) = y \). Denote by \( P_t \) the parallel transportation along \( \gamma_y(t) \). Choose a basis \( \{e_i\} \) of \( T_{\gamma_y(t)} M \). Then \( E_i(t) := P_t e_i, \ 1 \leq i \leq n, \) is a basis of \( T_{\gamma_y(t)} M \). Let \( (y^i) \) (resp. \( (z^i) \)) denote the corresponding coordinate system in \( T_{\gamma_y(t)} M \) (resp. \( T_{\gamma_y(t)} M \)). Thus, \( z^i \circ P_t = y^i \).
For any $w \in S_{\gamma_0}(0)M$, we have
\[
\frac{d}{dt}g(\gamma_0(t), P_tw)(E_i(t), E_j(t)) = \frac{2}{F(P_tw)}A(\gamma_0(t), P_tw)(E_i(t), E_j(t), \nabla_{\gamma_0} P_tw) = 0.
\]
Note that $P_t(B_{\gamma_0}(0)M) = B_{\gamma_0}(t)M$, where $B_xM := \{y \in T_xM : F(x, y) < 1\}$. The equation above together with [18] Lemma 5.3.2 yields that
\[
\int_{w \in B_{\gamma_0}(0)M} \det g(\gamma_0(t), v)(E_i(t), E_j(t))dz^1 \wedge \cdots \wedge dz^n
\]
\[
= \int_{w \in B_{\gamma_0}(0)M} \det g(\gamma_0(t), P_tw)(P_t e_i, P_t e_j)P_t^* dz^1 \wedge \cdots \wedge P_t^* dz^n
\]
\[
= \int_{w \in B_{\gamma_0}(0)M} \det g(\gamma_0(0), w)(e_i, e_j)dy^1 \wedge \cdots \wedge dy^n.
\]
Thus, $\tau_{HT}(\gamma_0(t)) = \tau_{HT}(\gamma_0(0))$, which implies that $S_{HT} \equiv 0$. 

Theorem 6.4 together with Theorem 6.5 now yields the following

Corollary 6.6. Let $(M, F, d\mu)$ be a $n$-dimensional closed Berwald manifold with the Ricci curvature $\text{Ric} \geq -(n-1)\delta^2$ and the reversibility $\lambda_F \leq \lambda$. Then we have
\[
\lambda_1(M) \leq C(n, \lambda) \left( \delta h(M) + h^2(M) \right).
\]

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**Department of Mathematics, Fudan University, Shanghai, China**

*E-mail address: yuan_lixia@foxmail.com*

**Department of Mathematics, East China University of Science and Technology, Shanghai, China**

*E-mail address: szhao_we@yahoo.com*