W-entropy formulas on super Ricci flows and Langevin deformation on Wasserstein space over Riemannian manifolds

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In this survey paper, we give an overview of our recent works on the study of the W-entropy for the heat equation associated with the Witten Laplacian on super-Ricci flows and the Langevin deformation on Wasserstein space over Riemannian manifolds. Inspired by Perelman’s seminal work on the entropy formula for the Ricci flow, we prove the W-entropy formula for the heat equation associated with the Witten Laplacian on n-dimensional complete Riemannian manifolds with the CD(K, m)-condition, and the W-entropy formula for the heat equation associated with the time dependent Witten Laplacian on n-dimensional compact manifolds equipped with a (K, m)-super Ricci flow, where K ∈ R and m ∈ [n, ∞]. Furthermore, we prove an analogue of the W-entropy formula for the geodesic flow on the Wasserstein space over Riemannian manifolds. Our result recaptures an important result due to Lott and Villani on the displacement convexity of the Boltzmann-Shannon entropy on Riemannian manifolds with non-negative Ricci curvature. To better understand the similarity between above two W-entropy formulas, we introduce the Langevin deformation of geometric flows on the cotangent bundle over the Wasserstein space and prove an extension of the W-entropy formula for the Langevin deformation. Finally, we make a discussion on the W-entropy for the Ricci flow from the point of view of statistical mechanics and probability theory.

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1 Introduction

Entropy was introduced by R. Clausius [10] in 1865 in the study of thermodynamics. In 1872, L. Boltzmann [8] introduced the H-entropy and formally derived the H-theorem for the evolution equation of the probability distribution of ideal gas (now called the Boltzmann equation). The statistical interpretation of the H-entropy was given by Boltzmann [9] in 1877. In 1948, C. Shannon [45] introduced the Shannon entropy in the theory of communication and transformation of information. In 1958, J. Nash [38] used the Boltzmann entropy to study the continuity of solutions of parabolic and elliptic equations.

Now entropy has been an important tool in many areas of mathematics. For example, the Kolmogorov-Sinai entropy plays an important role in the study of dynamical systems and ergodic theory, the exponential decay of the Boltzmann entropy is closely related to the logarithmic Sobolev inequalities, the rate function in Sanov’s theorem in the theory of

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large deviation is the relative Boltzmann entropy with respect to the reference measure. More recently, the displacement convexity of the Boltzmann-Shanon entropy or the Rényi entropy is a key tool in Lott, Villani and Sturm’s works \[34, 33, 52, 53, 46, 47, 48\] to develop analysis and geometry on metric measure spaces.

In 1982, R. Hamilton \[13\] introduced the Ricci flow and initiated the program to prove the Poincaré conjecture and the Thurston geometrization conjecture using the Ricci flow (see also \[14\]). In a seminal paper \[44\], Perelman gave a gradient flow reformulation for the Ricci flow and proved the $W$-entropy formula along the conjugate heat equation of the Ricci flow. More precisely, let $M$ be an $n$-dimensional compact manifold and define

$$\mathcal{F}(g, f) = \int_M (R + |\nabla f|^2)e^{-f}dv,$$

where $g \in \mathcal{M} = \{\text{Riemannian metric on } M\}$, $f \in C^\infty(M)$, $R$ denotes the scalar curvature on $(M, g)$, and $dv$ denotes the volume measure on $(M, g)$. Under the constraint condition that the weighted volume measure $d\mu = e^{-f}dv$ is fixed, Perelman \[44\] proved that the gradient flow of $\mathcal{F}$ with respect to the standard $L^2$-metric on $M \times C^\infty(M)$ is given by the following modified Ricci flow for $g$ together with the conjugate heat equation for $f$, i.e.,

$$\begin{align*}
\partial_t g &= -2(Ric + \nabla^2 f), \\
\partial_t f &= -\Delta f - R.
\end{align*}$$

Moreover, Perelman \[44\] introduced the $W$-entropy as follows

$$W(g, f, \tau) = \int_M \left[\tau(R + |\nabla f|^2) + f - n\right] \frac{e^{-f}}{(4\pi\tau)^{n/2}}dv,$$

(1)

where $\tau > 0$, and $f \in C^\infty(M)$ such that

$$\int_M (4\pi\tau)^{-n/2}e^{-f}dv = 1,$$

and proved that if $(g(t), f(t), \tau(t))$ satisfies the evolution equations

$$\begin{align*}
\partial_t g &= -2Ric, \\
\partial_t f &= -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau}, \\
\partial_t \tau &= -1,
\end{align*}$$

(2) (3) (4)

then the following remarkable $W$-entropy formula holds

$$\frac{d}{dt}W(g, f, \tau) = 2\int_M \tau \left|Ric + \nabla^2 f - \frac{g}{2\tau}\right|^2 \frac{e^{-f}}{(4\pi\tau)^{n/2}}dv.$$

(4)

In particular, the $W$-entropy is monotonic increasing in $t$ and the monotonicity is strict except that $(M, g(\tau), f(\tau))$ is a shrinking Ricci soliton, i.e.,

$$Ric + \nabla^2 f = \frac{g}{2\tau}.$$

As an application, Perelman \[44\] proved the no local collapsing theorem, which “removes the major stumbling block in Hamilton’s approach to geometrization” and plays an important role in the final resolution of the Poincaré conjecture and Thurston’s geometrization conjecture.
It is natural and interesting to ask the problems what is the hidden idea for Perelman to introduce the mysterious $W$-entropy, what is the reason for him to call the quantity in (1) the $W$-entropy, and whether there is some essential link between the $W$-entropy and the Boltzmann entropy in statistical mechanics and probability theory.

Inspired by Perelman [44] and related works [39, 40], the second author of this paper proved in [20] the $W$-entropy formula for the heat equation of the Witten Laplacian on compact Riemannian manifolds with the $CD(0, m)$-condition and gave a probabilistic interpretation of the $W$-entropy for the Ricci flow. Later, the $W$-entropy formula and a rigidity theorem for the $W$-entropy were proved in [22, 24] for the fundamental solution to the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $CD(0, m)$-condition, and the $W$-entropy formula was proved in [21] for the Fokker-Planck equation of the Witten Laplacian on complete Riemannian manifolds with the $CD(0, m)$-condition. The relationship between Perelman’s $W$-entropy formula for the Ricci flow and the Boltzmann H-theorem for the Boltzmann equation was discussed in [23] from the point of view of the statistical mechanics. In [26, 27, 29, 30, 31], we extended the $W$-entropy formula to the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $CD(K, m)$-condition and on compact Riemannian manifolds equipped with $(K, m)$-super Ricci flows. Moreover, we proved in [28, 31] an analogue of the $W$-entropy formula for the geodesic flow on the Wasserstein space over Riemannian manifolds with the $CD(0, m)$-condition, which recaptures an important result due to Lott and Villani [34, 33] on the displacement convexity of the Boltzmann-Shannon entropy on the Wasserstein space over Riemannian manifolds with non-negative Ricci curvature. To better understand the similarity between the $W$-entropy formula for the heat equation of the Witten Laplacian and the $W$-entropy formula for the geodesic flow on the Wasserstein space over Riemannian manifolds, we introduced in [28, 31] the Langevin deformation of geometric flows on the Wasserstein space over Riemannian manifolds, which interpolates the backward gradient flow of the Boltzmann-Shannon entropy and the geodesic flow on the Wasserstein space, and proved an extension of the $W$-entropy formula for the Langevin deformation. The rigidity models are also proposed for the Langevin deformation of flows. In particular, two rigidity theorems were proved in [22, 24, 28, 31] for the gradient flow of the Boltzmann-Shannon entropy and the geodesic flow on the Wasserstein space over complete Riemannian manifolds with the $CD(0, m)$-condition.

The purpose of this survey paper is to give an overview of our works in [20, 22, 23, 24, 26, 27, 29, 30, 31] and to make a discussion on the $W$-entropy for the Ricci flow from the point of view of statistical mechanics and probability theory.

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2 $W$-entropy formulas for Witten Laplacian on Riemannian manifolds

Since Perelman’s preprint [44] was published on Arxiv in 2002, many people have studied the $W$-like entropy for other geometric flows on Riemannian manifolds [39, 40, 11, 35, 15]. Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold with a fixed metric (and
with bounded geometry condition), and let

\[ u = \frac{e^{-f}}{(4\pi t)^{n/2}} \]

be a positive solution to the linear heat equation

\[ \partial_t u = \Delta u \]

with \( \int_M u(x, 0)dv(x) = 1 \). In \([39, 40]\), Ni introduced the \( W \)-entropy for the linear heat equation \( (5) \) by

\[ W(f, t) = \int_M \left[ t|\nabla f|^2 + f - n \right] \frac{e^{-f}}{(4\pi t)^{n/2}} dv, \]

and proved the following \( W \)-entropy formula

\[ \frac{d}{dt} W(f, t) = -2 \int_M t \left( |\nabla^2 f - \frac{g}{2t}|^2 + \text{Ric}(\nabla f, \nabla f) \right) \frac{e^{-f}}{(4\pi t)^{n/2}} dv. \]

In particular, the \( W \)-entropy for the linear heat equation \( (5) \) is decreasing on complete Riemannian manifolds with non-negative Ricci curvature. In \([18]\), Li and Xu extended Ni’s \( W \)-entropy formula \( (6) \) to the heat equation \( \partial_t u = \Delta u \) on complete Riemannian manifolds with fixed metric satisfying \( \text{Ric} \geq -Kg \), where \( K \geq 0 \) is a constant.

Let \((M, g)\) be a complete Riemannian manifold, \( \phi \in C^2(M) \). Let \( d\mu = e^{-\phi} dv \), where \( dv \) is the Riemannian volume measure on \((M, g)\). The Witten Laplacian, called also the weighted Laplacian,

\[ L = \Delta - \nabla \phi \cdot \nabla \]

is a self-adjoint and non-negative operator on \( L^2(M, \mu) \). By Itô’s calculus, one can construct the symmetric diffusion process \( X_t \) associated to the Witten Laplacian by solving the SDE

\[ dX_t = \sqrt{2}dW_t - \nabla \phi(X_t) dt, \]

where \( W_t \) is the Brownian motion on \( M \). Moreover, it is well known that the transition probability density of the diffusion process \( X_t \) is the fundamental solution to the heat equation of \( L \), i.e., the heat kernel of the Witten Laplacian \( L \). In view of this, it is a fundamental problem to study the heat equation and related properties for the Witten Laplacian on Riemannian manifolds with various geometric conditions.

To develop the study of the \( W \)-entropy formula for the heat equation of the Witten Laplacian, we need to introduce some notations. Let \( n = \text{dim} \, M \), and \( m \geq n \) a constant. Following Bakry and Emery \([1]\), we introduce

\[ \text{Ric}_{m,n}(L) = \text{Ric} + \nabla^2 \phi - \frac{\nabla \phi \otimes \nabla \phi}{m-n}, \]

and call it the \( m \)-dimensional Bakry-Emery Ricci curvature associated with the Witten Laplacian \( L \) on \((M, g, \phi)\). We make the convention that \( m = n \) if and only if \( L = \Delta \) and \( \phi \) is a constant. In this case, \( \text{Ric}_{m,n}(L) = \text{Ric} \). When \( m = \infty \), we introduce

\[ \text{Ric}(L) = \text{Ric} + \nabla^2 \phi. \]

Following Bakry and Emery \([1]\), we say that the \( CD(K, m) \) condition holds for the Witten Laplacian \( L = \Delta - \nabla \phi \cdot \nabla \) on \((M, g, \phi)\) if and only if \( \text{Ric}_{m,n}(L) \geq K \), where \( K \in \mathbb{R} \) and \( m \in [n, \infty] \).
2.1 The case of $CD(0, m)$-condition

In \cite{20 22 24}, inspired by Perelman’s work on the $W$-entropy formula for Ricci flow, the second author of this paper proved the $W$-entropy formula for the heat equation associated with the Witten-Laplacian on complete Riemannian manifolds with the $CD(0, m)$-condition, which extends the above mentioned result due to Ni \cite{39 40}. More precisely, we have

**Theorem 2.1** (\cite{20 22 24}) Let $(M, g)$ be a compact Riemannian manifold, or a complete Riemannian manifold with bounded geometry condition\footnote{Here we say that $(M, g)$ satisfies the bounded geometry condition if the Riemannian curvature tensor $\text{Riem}$ and its covariant derivatives $\nabla^k \text{Riem}$ are uniformly bounded on $M$, $k = 1, 2, 3$.}, and $\phi \in C^4(M)$ with $\nabla \phi \in C^3_b(M)$. Let $m \in [n, \infty)$, and $u = \frac{e^{-f}}{(4\pi t)^{m/2}}$ be a positive solution of the heat equation $\partial_t u = Lu$ when $M$ is compact, or the fundamental solution associated with the Witten Laplacian, i.e., the heat kernel to the heat equation $\partial_t u = Lu$, when $M$ is complete non-compact. Let

\[ H_m(u, t) = -\int_M u \log u \, d\mu + \frac{m}{2} (1 + \log(4\pi t)). \]

Define the $W$-entropy for the Witten-Laplacian by

\[ W_m(u, t) = \frac{d}{dt}(tH_m(u)). \]

Then

\[ W_m(u, t) = \int_M \left[ t|\nabla f|^2 + f - m \right] \frac{e^{-f}}{(4\pi t)^{m/2}} \, d\mu, \]

and

\[ \frac{d}{dt}W_m(u, t) = -2\int_M t \left( \nabla^2 f - \frac{g}{2t} \right)^2 + Ric_{m,n}(L)(\nabla f, \nabla f) \right) \, d\mu \]
\[ - \frac{2}{m-n} \int_M t \left( \nabla \phi \cdot \nabla f + \frac{m-n}{2t} \right)^2 \, d\mu. \]

By calculation and integration by part, we have

\[ \frac{d}{dt}H_m(u, t) = -\int_M \left( L \log u + \frac{m}{2t} \right) \, d\mu. \]

By \cite{19 22}, if $Ric_{m,n}(L) \geq 0$, the generalized Li-Yau Harnack inequality (\cite{17}) holds

\[ L \log u + \frac{m}{2t} \geq 0, \quad \forall t > 0. \]

Therefore, $H_m(u, t)$ is non-increasing in time $t$ for the heat equation $\partial_t u = Lu$ on complete Riemannian manifolds with the $CD(0, m)$-condition, i.e., $Ric_{m,n}(L) \geq 0$.

As a corollary of Theorem 2.1 if $(M, g, \phi)$ is complete Riemannian manifold with the bounded geometry condition and $Ric_{m,n}(L) \geq 0$, then the $W$-entropy for the heat equation $\partial_t u = Lu$ is decreasing in time $t$, i.e.,

\[ \frac{d}{dt}W_m(u, t) \leq 0, \quad \forall t \geq 0. \]

Moreover, under the condition $Ric_{m,n}(L) \geq 0$, it was proved in \cite{22} that $W_m(u, t)$ attains its minimum at some point $t = t_0 > 0$, i.e.,

\[ \frac{d}{dt}W_m(u, t) = 0 \quad \text{at some} \quad t = t_0 > 0, \]
if and only if $(M, g)$ is isometric to Euclidean space $\mathbb{R}^n$, $m = n$, $\phi \equiv C$ for a constant $C \in \mathbb{R}$, and

$$u(x, t) = e^{-\frac{\|x\|^2}{4\pi t}} \frac{m}{(4\pi t)^{n/2}}, \quad \forall x \in \mathbb{R}^n, t > 0.$$ 

In other words, the Euclidean space $\mathbb{R}^n$ is the unique equilibrium state for the $W$-entropy of the Witten-Laplacian in the statistical ensemble of complete Riemannian manifolds $(M, g, \phi)$ with bounded geometry condition and with the $CD(0, m)$-condition.

In [26], we gave a new proof of Theorem 2.1 by using the warped product approach. Let $m \in \mathbb{N}$, $m \geq n$. Let $\tilde{M} = M \times N$, where $(N, g_N)$ is a compact Riemannian manifold with dimension $m - n$. Let $\phi \in C^2(M)$. We consider the following warped product metric on $\tilde{M}$:

$$\tilde{g} = g_M \bigoplus e^{-\frac{\|x\|^2}{4t}} g_N.$$ 

Let $\nu_N$ be the normalized volume measure on $N$. Then the volume measure on $(\tilde{M}, \tilde{g})$ is

$$d\text{vol}_{\tilde{M}} = d\mu \otimes d\nu_N.$$ 

Let $\tilde{\nabla}$ be the Levi-Civita connection on $(\tilde{M}, \tilde{g})$, $\tilde{\nabla}^2$ and $\tilde{\Delta}$ the Hessian and the Laplace-Beltrami operator on $(\tilde{M}, \tilde{g})$. By direct calculation, we have

$$\left|\tilde{\nabla}^2 f - \frac{\tilde{g}}{2t}\right|^2 = \left|\nabla^2 f - \frac{g}{2t}\right|^2 + \frac{1}{m-n} \left(\nabla \phi \cdot \nabla f + \frac{m-n}{2t}\right)^2,$$

and

$$\tilde{\Delta} = L + e^{-\frac{2\phi}{m-n}} \Delta_N.$$ 

**A new proof of Theorem 2.1 ([26]).** To avoid technical issue, we only consider the case of compact manifolds. Let $u = \frac{-\nabla f}{(4\pi t)^{n/2}}$ be a positive solution to the heat equation $\partial_t u = Lu$. Then it satisfies the following heat equation on $(\tilde{M}, \tilde{g})$

$$\partial_t u = \tilde{\Delta} u.$$ 

Since $f$ depends only on the variable in the $M$-direction, we have $\tilde{\nabla} f = \nabla f$. Therefore the $W$-entropy $W_m(u, t)$ defined by (3) coincides with the $W$-entropy $\tilde{W}_m(u, t)$ defined on $(\tilde{M}, \tilde{g})$ as follows

$$\tilde{W}_m(u, t) = \int_{\tilde{M}} \left[t|\tilde{\nabla} f|^2 + f - m\right] e^{-f} \frac{m}{(4\pi t)^{n/2}} d\text{vol}_{\tilde{M}}.$$ 

Applying Ni’s $W$-entropy formula (4) to $(\tilde{M}, \tilde{g})$, we have

$$\frac{d}{dt} \tilde{W}_m(u, t) = -2 \int_{\tilde{M}} t \left(\left|\tilde{\nabla}^2 f - \frac{\tilde{g}}{2t}\right|^2 + \tilde{Ric}(\tilde{\nabla} f, \tilde{\nabla} f)\right) u \mu d\nu_N.$$ 

By (10) and $\tilde{Ric}(\tilde{\nabla} f, \tilde{\nabla} f) = Ric_{m,n}(L)(\nabla f, \nabla f)$, we derive (9) from (11). \qed

**Remark 2.2.** One of the advantages of the above proof is that: when $m \in \mathbb{N}$ and $m > n$, the quantity $\frac{m-n}{m-n} \left(\nabla \phi \cdot \nabla f + \frac{m-n}{2t}\right)^2$ appeared in the $W$-entropy formula in Theorem 2.1 has a natural geometric interpretation. It corresponds to the vertical component of the quantity $\left|\tilde{\nabla}^2 f - \frac{\tilde{g}}{2t}\right|^2$ on $\tilde{M} = M \times N$ equipped with the warped product metric (10).
2.2 The case of CD\((K, m)\)-condition

Theorem 2.1 can be viewed as the W-entropy formula for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the CD\((0, m)\)-condition. It is natural to raise the question whether we can extend Theorem 2.1 to the heat equation of the Witten Laplacian on complete Riemannian manifolds with the CD\((K, m)\)-condition for general \(K \in \mathbb{R}\) and \(m \in [n, \infty)\).

In \([26]\), we extended Theorem 2.1 to the Witten Laplacian on complete Riemannian manifolds with the CD\((K, m)\)-condition for \(K \in \mathbb{R}\) and \(m \in [n, \infty)\).

**Proposition 2.3**\([26]\)  Let \((M, g)\) be a complete Riemannian manifold with bounded geometry condition, and \(\phi \in C^4(M)\) satisfying the condition in Theorem 2.1. Let \(u\) be a positive solution to the heat equation
\[
\partial_t u = Lu.
\]
Then, under the CD\((-K, m)\)-condition, i.e., \(\text{Ric}_{m,n}(L) \geq -K\), where \(K \in \mathbb{R}\) and \(m \in [n, \infty)\), the following Harnack inequality holds
\[
\frac{|\nabla u|^2}{u^2} - \left(1 + \frac{2}{3}Kt\right) \frac{\partial u}{u} \leq \frac{m}{2t} + \frac{mK}{2} \left(1 + \frac{Kt}{3}\right), \quad \forall t > 0.
\]

**Theorem 2.4**\([22, 27]\)  Let \(u = e^{-\frac{1}{4\pi t}m/2}\) be the fundamental solution to the heat equation \(\partial_t u = Lu\). Under the same assumption as in Theorem 2.1, define
\[
H_{m,K}(u, t) = -\int_M u \log u \, d\mu - \frac{m}{2} \left(1 + \log(4\pi t)\right) - \frac{m}{2}Kt \left(1 + \frac{1}{6}Kt\right), \quad (12)
\]
and the W-entropy by the Boltzmann formula
\[
W_{m,K}(u, t) = \frac{d}{dt}(tH_{m,K}(u)). \quad (13)
\]
Then
\[
W_{m,K}(u, t) = \int_M \left(t|\nabla f|^2 + f - m \left(1 + \frac{1}{2}Kt\right)^2\right) u \, d\mu,
\]
and
\[
\frac{d}{dt}W_{m,K}(u, t) + 2t\int_M \left(|\nabla^2 f - \left(\frac{1}{2t} + \frac{K}{2}\right) g|^2\right) u \, d\mu
+ 2t \frac{m-n}{m-n} \int_M \left(\nabla \phi \cdot \nabla f + (m-n) \left(\frac{1}{2t} + \frac{K}{2}\right)^2\right) u \, d\mu
= 2t \int_M (\text{Ric}_{m,n}(L) + Kg)(\nabla f, \nabla f) u \, d\mu. \quad (14)
\]

In particular, if the CD\((-K, m)\)-condition holds, i.e., \(\text{Ric}_{m,n}(L) \geq -K\), then
\[
\frac{d}{dt}H_{m,K}(u, t) \leq 0,
\]
and
\[
\frac{d}{dt}W_{m,K}(u, t) \leq 0.
\]
Moreover, under the CD\((-K, m)\)-condition, the left hand side in (13) equals to zero at some \(t = t_0 > 0\) if and only if \((M, g, \phi)\) is a \((-K, m)\)-Ricci soliton, i.e.,
\[
\text{Ric}_{m,n}(L) = -Kg.
2.3 The case of $CD(K, \infty)$-condition

When $m = \infty$, we cannot use the above definition formulas \eqref{12} and \eqref{13} to introduce $H_{K, \infty}(u, t)$ and to define the $W$-entropy for the Witten Laplacian on Riemannian manifolds with the $CD(K, \infty)$-condition. Based on the reversal logarithmic Sobolev inequality due to Bakry and Ledoux [2], we proved the following result.

**Theorem 2.5** (\[27, 30, 31\]) Let $M$ be a complete Riemannian manifold with bounded geometry condition, $\phi \in C^4(M)$ with $\nabla \phi \in C^3_b(M)$. Suppose that $\text{Ric} + \nabla^2 \phi \geq K$, where $K \in \mathbb{R}$ is a constant. Let $u(\cdot, t) = P_t f$ be a positive solution to the heat equation $\partial_t u = Lu$ with $u(\cdot, 0) = f$, where $f$ is a positive and measurable function on $M$. Let

$$H_K(f, t) = D_K(t) \int_M (f \log f - P_t f \log P_t f) d\mu,$$

where $D_0(t) = \frac{1}{t}$ and $D_K(t) = \frac{2K}{1 - e^{-2Kt}}$ for $K \neq 0$. Then for all $t > 0$

$$\frac{d}{dt}H_K(f, t) \leq 0,$$

and for all $t > 0$, we have

$$\frac{d^2}{dt^2}H_K(t) + 2K \coth(Kt) \frac{d}{dt}H_K(t) \leq -2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu.$$

**Theorem 2.5** suggests us a new way to introduce the $W$-entropy for the Witten Laplacian on Riemannian manifolds with the $CD(K, \infty)$-condition

$$W_K(f, t) = H_K(f, t) + \frac{\sinh(2Kt)}{2K} \frac{d}{dt}H_K(f, t).$$

In this way, for all $t > 0$, we prove that \eqref{27}

$$\frac{d}{dt}W_K(f, t) + (e^{2Kt} + 1) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu$$

$$= -(e^{2Kt} + 1) \int_M (\text{Ric}(L) - Kg)(\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu. \quad (15)$$

In particular, for all $t > 0$, we have

$$\frac{d}{dt}W_K(f, t) \leq 0.$$

Moreover, under the $CD(K, \infty)$-condition, the left hand side in \eqref{15} equals to zero at some $t = t_0 > 0$ if and only if $(M, g, \phi)$ is a $K$-Ricci soliton, i.e.,

$$\text{Ric} + \nabla^2 \phi = Kg.$$

3 W-entropy formulas for Witten Laplacian on $(K, m)$-super Ricci flows

In Section 2, we extend the $W$-entropy formula to the heat equation of the Witten Laplacian on complete Riemannian manifolds with the $CD(K, \infty)$-condition. It is an interesting question whether we can further extend the $W$-entropy formula to the heat equation associated with the time dependent Witten Laplacian on compact or complete Riemannian manifolds with time dependent metrics and potentials.
We now introduce the notion of the \((K, m)\)-super Ricci flow on manifolds with time dependent metrics and potentials. By definition, we call \((M, g(t), \phi(t), t \in [0, T])\) a \((K, m)\)-super Ricci flow if
\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L) \geq Kg, \quad \forall \ t \in [0, T],
\]
where \(K \in \mathbb{R}\) and \(m \in [n, \infty]\) are two constants. Note that a Riemannian manifold equipped with a stationary \((K, m)\)-super Ricci flow (i.e., \((g(t), \phi(t))\) is independent of time) if and only if the \(CD(K, m)\)-condition holds, i.e.,
\[
\text{Ric}_{m,n}(L) \geq Kg.
\]
In the case \(m = n\), the notion of the \((K, n)\)-super Ricci flow is indeed the \(K\)-super Ricci flow in geometric analysis
\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric} \geq Kg, \quad \forall \ t \in [0, T],
\]
and in the case \(m = \infty\), the \((K, \infty)\)-super Ricci flow equation reads
\[
\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) \geq Kg, \quad \forall \ t \in [0, T].
\]
In view of this, the Perelman Ricci flow is indeed the \((0, \infty)\)-Ricci flow together with the conjugate heat equation
\[
\frac{\partial g}{\partial t} = -2\text{Ric}(L), \quad \frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \left( \frac{\partial g}{\partial t} \right).
\]

Let \((M, g(t), \phi(t), t \in [0, T])\) be a complete Riemannian manifold with a family of time dependent metrics \(g(t)\) and potentials \(\phi(t)\). Let
\[
L = \Delta_{g(t)} - \nabla_{g(t)} \phi(t) \cdot \nabla_{g(t)}
\]
be the time dependent Witten Laplacian on \((M, g(t), \phi(t))\). Let
\[
d\mu(t) = e^{-\phi(t)} d\text{vol}_{g(t)}.
\]
Suppose that
\[
\frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \left( \frac{\partial g}{\partial t} \right).
\]
Then \(\mu(t)\) is independent of \(t \in [0, T]\), i.e.,
\[
\frac{\partial d\mu(t)}{\partial t} = 0, \quad t \in [0, T].
\]

We now state the main results of this section, which extend Theorems 2.1, 2.4 and 2.5 to the heat equation associated with the time dependent Witten Laplacian on compact manifolds with a \((K, m)\)-super Ricci flow, where \(K \in \mathbb{R}\) and \(m \in [n, \infty]\).
3.1 The case of \((0, m)\)-super Ricci flow

In [26], we proved the \(W\)-entropy formula to the heat equation associated with the time dependent Witten Laplacian on compact manifolds equipped with a \((0, m)\)-super Ricci flow, which can be regarded as the \(m\)-dimensional analogue of Perelman’s \(W\)-entropy formula for the Ricci flow.

**Theorem 3.1** ([26]) Let \((M, g(t), \phi(t), t \in [0, T])\) be a compact manifold with family of time dependent metrics and \(C^2\)-potentials. Suppose that \(g(t)\) and \(\phi(t)\) satisfy the conjugate equation (16). Let \(u = \frac{e^{-f}}{(4\pi t)^{m/2}}\) be a positive solution of the heat equation

\[
\partial_t u = L u
\]

with initial data \(u(0)\) satisfying \(\int_M u(0) d\mu(0) = 1\). Let

\[
H_m(u, t) = - \int_M u \log u d\mu - \frac{m}{2} \left(1 + \log(4\pi t)\right).
\]

Define

\[
W_m(u, t) = \frac{d}{dt}(tH_m(u)).
\]

Then

\[
W_m(u, t) = \int_M \left[t|\nabla f|^2 + f - m\right] u d\mu,
\]

and

\[
\frac{d}{dt} W_m(u, t) + 2t \int_M \left|\nabla^2 f - \frac{g}{2t}\right|^2 u d\mu + \frac{2t}{m-n} \int_M \left(\nabla\phi \cdot \nabla f + \frac{m-n}{2t}\right)^2 u d\mu
\]

\[
= -2t \int_M \left(\frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L)\right)(\nabla f, \nabla f) u d\mu. \tag{17}
\]

In particular, if \(\{g(t), \phi(t), t \in (0, T]\}\) is a \((0, m)\)-super Ricci flow and satisfies the conjugate equation (16), then \(W_m(u, t)\) is decreasing in \(t \in (0, T]\), i.e.,

\[
\frac{d}{dt} W_m(u, t) \leq 0, \quad \forall t \in (0, T].
\]

Moreover, the left hand side in (17) identically equals to zero on \((0, T]\) if and only if \((M, g(t), \phi(t), t \in (0, T]\)) is a \((0, m)\)-Ricci flow in the sense that

\[
\frac{\partial g}{\partial t} = -2\text{Ric}_{m,n}(L), \quad \frac{\partial \phi}{\partial t} = \frac{1}{2} \text{Tr} \left(\frac{\partial g}{\partial t}\right).
\]

3.2 The case of \((K, m)\)-super Ricci flow

In general we have the following result which extends Theorem 2.4 to \((K, m)\)-super Ricci flow for general \(K \in \mathbb{R}\) and \(m \in [n, \infty)\).

**Theorem 3.2** ([26, 27]) Under the same notation as in Theorem 3.1, define

\[
H_{m,K}(u, t) = - \int_M u \log u d\mu - \frac{m}{2} \left(1 + \log(4\pi t)\right) - \frac{m}{2} K t \left(1 + \frac{1}{6} K t\right), \tag{18}
\]

with
and
\[ W_{m,K}(u,t) = \frac{d}{dt}(tH_{m,K}(u)). \] (19)

Then
\[ W_{m,K}(u,t) = \int_M \left[ t|\nabla f|^2 + f - m \left( 1 + \frac{1}{2}Kt \right)^2 \right] u d\mu, \]

and
\[
\begin{align*}
\frac{d}{dt}W_{m,K}(u,t) &+ 2t \int_M \left| \nabla^2 f - \left( \frac{1}{2t} + \frac{K}{2} \right) g \right|^2 u d\mu \\
&+ \frac{2t}{m-n} \int_M \left( \nabla \phi \cdot \nabla f + (m-n) \left( \frac{1}{2t} + \frac{K}{2} \right) \right)^2 u d\mu \\
&= -2t \int_M \left( \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}_{m,n}(L) + Kg \right) (\nabla f, \nabla f) u d\mu.
\end{align*}
\] (20)

In particular, if \((M, g(t), \phi(t), t \in (0, T])\) is a \((-K, m)\)-super Ricci flow and satisfies the conjugate equation \((16)\), then \(W_{m,K}(u,t)\) is decreasing in \(t \in (0, T)\), i.e.,
\[ \frac{d}{dt}W_{m,K}(u,t) \leq 0, \quad \forall t \in (0, T]. \]

Moreover, the left hand side in \((20)\) identically equals to zero on \((0, T]\) if and only if \((M, g(t), \phi(t), t \in (0, T])\) is a \((-K, m)\)-Ricci flow in the sense that
\[
\begin{align*}
\frac{\partial g}{\partial t} &= -2(\text{Ric}_{m,n}(L) + Kg), \\
\frac{\partial \phi}{\partial t} &= \frac{1}{2} \text{Tr} \left( \frac{\partial g}{\partial t} \right).
\end{align*}
\]

### 3.3 The case of \((K, \infty)\)-super Ricci flow

In \([27, 30, 31]\), we proved the equivalence between the \((K, \infty)\)-super Ricci flow and two families of logarithmic Sobolev inequalities for the time dependent Witten Laplacian on Riemannian manifolds with time dependent metrics and potentials. Based on this result, we have the following \(W\)-entropy formula for the time dependent Witten Laplacian on compact Riemannian manifolds with \((K, \infty)\)-super Ricci flow, which can be viewed as the natural extension of the \(W\)-entropy formula for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the \(CD(K, \infty)\)-condition.

**Theorem 3.3** \([27, 30, 31]\) Let \((M, g(t), \phi(t), t \in [0, T])\) be a compact \((K, \infty)\)-super Ricci flow satisfying the conjugate heat equation \((16)\). Let \(u(\cdot, t) = P_t f\) be a positive solution to the heat equation \(\partial_t u = Lu\) with \(u(\cdot, 0) = f\), where \(f\) is a positive and measurable function on \(M\). Define
\[ H_K(f, t) = D_K(t) \int_M \left( f \log f - P_t f \log P_t f \right) d\mu, \]
where \(D_0(t) = \frac{1}{t}\) and \(D_K(t) = \frac{2e^K}{t^2}\) for \(K \neq 0\). Then for all \(t \in [0, T]\)
\[ \frac{d}{dt}H_K(f, t) \leq 0, \]
and for all $t \in (0, T]$, we have
\[
\frac{d^2}{dt^2} H_K(t) + 2K \coth(Kt) \frac{d}{dt} H_K(t) \leq -2D_K(t) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu.
\]

Define the $W$-entropy by the revised Boltzmann entropy formula
\[
W_K(f, t) = H_K(f, t) + \frac{\sinh(2Kt)}{2K} \frac{d}{dt} H_K(f, t).
\]

Then for all $t \in (0, T]$, we have
\[
\frac{d}{dt} W_K(f, t) + (e^{2Kt} + 1) \int_M |\nabla^2 \log P_t f|^2 P_t f d\mu
= - (e^{2Kt} + 1) \int_M \left( \frac{1}{2} \frac{\partial g}{\partial t} + \text{Ric}(L) - Kg \right) (\nabla \log P_t f, \nabla \log P_t f) P_t f d\mu. \quad (21)
\]

In particular, for all $t \in (0, T]$, we have
\[
\frac{d}{dt} W_K(f, t) \leq 0.
\]

Moreover, the left hand side of (21) identically equals to zero on $(0, T]$ if and only if $(M, g(t), \phi(t))$ is the $(K, \infty)$-Ricci flow satisfying the conjugate equation (16), i.e., for all $t \in (0, T]$,
\[
\frac{\partial g}{\partial t} = -2(\text{Ric} + \nabla^2 \phi - Kg),
\]
\[
\frac{\partial \phi}{\partial t} = -R - \Delta \phi + nK.
\]

4 \quad W$-entropy formula for geodesic flow on Wasserstein space

Starting from Brenier’s work [6, 7] on the Monge-Kantorovich optimal transport problem with quadratic cost function, Otto, Lott, McCann, Villani and Sturm [11, 12, 34, 33, 52, 53, 46, 47, 48] among others have developed the optimal transport theory. In particular, they developed an infinite dimensional Riemannian geometry and the theory of the gradient flow on the Wasserstein space over Euclidean space, compact Riemannian manifolds and metric measure spaces. The displacement convexity of the Boltzmann-Shannon entropy or the Renyi entropy along geodesics on the Wasserstein space has been a key tool in [34, 33, 52, 53, 46, 17, 18] to introduce the notions of the upper bound of the dimension and the lower bound of the Ricci curvature on metric measure spaces. In [37], McCann and Topping proved the contraction property of the $L^2$-Wasserstein distance between solutions of the backward heat equation on closed manifolds equipped with the Ricci flow, which extends previous results for the Fokker-Planck equation on Euclidean space (due to Otto [11]) and on complete Riemannian manifolds with suitable Bakry-Emery curvature condition (due to Sturm and von Renesse [49]). See also [50, 51]. In [33], Lott further proved two convexity results of the Boltzmann-Shannon type entropy along the geodesics on the Wasserstein space over closed manifolds equipped with the backward Ricci flow, which are closely related to Perelman’s result on the monotonicity of the $W$-entropy for the Ricci flow. In [25], we extended Lott’s convexity results to the Wasserstein space on compact Riemannian manifolds equipped with the backward Perelman Ricci flow.
Let \((M, g)\) be a complete Riemannian manifold equipped with a weighted volume measure \(d\mu = e^{-\phi} dv\), where \(\phi \in C^2(M)\) and \(dv\) denotes the volume measure on \((M, g)\). The Boltzmann-Shannon entropy of the probability measure \(\rho d\mu\) with respect to the reference measure \(\mu\) is defined by

\[
\text{Ent}(\rho) := \int_M \rho \log \rho \, d\mu.
\]

Let \(P_2(M, \mu)\) (resp. \(P_2^\infty(M, \mu)\)) be the Wasserstein space (reps. the smooth Wasserstein space) of all probability measures \(\rho(x) \, d\mu(x)\) with density function \(\rho\) on \(M\) such that \(\int_M d^2(o, x) \rho(x) \, d\mu(x) < \infty\), where \(d(o, \cdot)\) denotes the distance function from a fixed point \(o \in M\). Following Otto \cite{Otto01} and Lott \cite{Lott10, Lott11}, the tangent space \(T_{\rho \, d\mu}P_2^\infty(M, \mu)\) is identified as follows

\[
T_{\rho \, d\mu}P_2^\infty(M, \mu) = \left\{ s = \nabla_\mu^* (\rho \nabla f) : f \in C^\infty(M), \int_M |\nabla f|^2 \, d\mu < \infty \right\},
\]

where \(\nabla_\mu^*\) denotes the \(L^2\)-adjoint of the Riemannian gradient \(\nabla\) with respect to the weighted volume measure \(d\mu\) on \((M, g)\). For \(s_i = \nabla_\mu^* (\rho \nabla f_i) \in T_{\rho \, d\mu}P_2^\infty(M, \mu)\), we introduce Otto’s infinite dimensional Riemannian metric on \(P_2^\infty(M, \mu)\) as follows

\[
\langle (s_1, s_2) \rangle := \int_M \nabla f_1 \cdot \nabla f_2 \, d\mu,
\]

provided that

\[
||s_i||^2 := \int_M |\nabla f_i|^2 \, d\mu < \infty, \quad i = 1, 2.
\]

Let \(T_{\rho \, d\mu}P_2(M, \mu)\) be the completion of \(T_{\rho \, d\mu}P_2^\infty(M, \mu)\) equipped with Otto’s infinite dimensional Riemannian metric. Then \(P_2(M, \mu)\) is an infinite dimensional Riemannian manifold.

By Benamou and Brenier \cite{Benamou00}, for any given \(\mu_i = \rho_i d\mu \in P_2(M, \mu)\), \(i = 0, 1\), the \(L^2\)-Wasserstein distance between \(\mu_0\) and \(\mu_1\) coincides with the geodesic distance between \(\mu_0\) and \(\mu_1\) in \(P_2(M, \mu)\) equipped with Otto’s infinite dimensional Riemannian metric, i.e.,

\[
W_2^2(\mu_0, \mu_1) = \inf \left\{ \frac{1}{2} \int_0^1 |\nabla f(t, x)|^2 \rho(x, t) \, d\mu(x) : \partial_t \rho = \nabla_\mu^* (\rho \nabla f), \rho(0) = \rho_0, \rho(1) = \rho_1 \right\}.
\]

By \cite{Otto99}, given \(\mu_0 = \rho(\cdot, 0) \mu\), \(\mu_1 = \rho(\cdot, 1) \mu \in P_2^\infty(M, \mu)\), it is known that there is a unique minimizing Wasserstein geodesic \(\{\mu(t), \mu(t) \in [0, 1]\}\) of the form \(\mu(t) = (F_t)_* \mu_0\) joining \(\mu_0\) and \(\mu_1\) in \(P_2(M, \mu)\), where \(F_t \in \text{Diff}(M)\) is given by \(F_t(x) = \exp_x (-t \nabla f(\cdot, 0))\) for an appropriate Lipschitz function \(f(\cdot, t)\). See also \cite{Lott10, Lott11}. If the Wasserstein geodesic in \(P_2(M, \mu)\) belongs entirely to \(P_2^\infty(M, \mu)\), then the geodesic flow \((\rho, f) \in T^*P_2^\infty(M, \mu)\) satisfies the transport equation and the Hamilton-Jacobi equation

\[
\begin{align*}
\partial_t \rho - \nabla_\mu^* (\rho \nabla f) &= 0, \quad (22) \\
\partial_t f + \frac{1}{2} |\nabla f|^2 &= 0, \quad (23)
\end{align*}
\]

with the boundary condition \(\rho(0) = \rho_0\) and \(\rho(1) = \rho_1\). When \(\rho_0, f_0 \in C^\infty(M)\), defining \(f(\cdot, t) \in C^\infty(M)\) by the Hopf-Lax solution

\[
f(x, t) = \inf_{y \in M} \left( f_0(y) + \frac{d^2(x, y)}{2t} \right),
\]

and solving the transport equation \(22\) by the characteristic method, it is known that \((\rho, f)\) satisfies \(22\) and \(23\) with \(\rho(0) = \rho_0\) and \(f(0) = f_0\). See \cite{Lott10} Sect. 5.4.7. See also \cite{Lott10, Lott11}. In
view of this, the transport equation (22) and the Hamilton-Jacobi equation (23) describe the geodesic flow on the cotangent bundle $T^*P_\infty^2(M, \mu)$ over the Wasserstein space $P_2(M, \mu)$. Note that the Hamilton-Jacobi equation (23) is also called the eikonal equation in geometric optics.

The main result of this section is the following $W$-entropy formula for the geodesic flow on the Wasserstein space $P_\infty^2(M, \mu)$.

**Theorem 4.1** ([28, 31]) Let $(M, g)$ be a compact Riemannian manifold, $\phi \in C^2(M)$, $d\mu = e^{-\phi}dv$. Let $\rho : M \times [0, T] \to \mathbb{R}^+$ and $f : M \times [0, T] \to \mathbb{R}$ be smooth solutions to the transport equation (22) and the Hamilton-Jacobi equation (23). For any $m \geq n$, define the $H_m$-entropy and $W_m$-entropy for the geodesic flow $(\rho, f)$ on $T^*P_\infty^2(M, \mu)$ as follows

$$H_m(\rho, t) = -\text{Ent}(\rho(t)) - \frac{m}{2} \left(1 + \log(4\pi t^2)\right),$$

and

$$W_m(\rho, t) = \frac{d}{dt}(tH_m(\rho, t)).$$

Then for all $t > 0$, we have

$$\frac{d}{dt}W_m(\rho, t) = -t \int_M \left[\left|\nabla^2 f - \frac{g}{t}\right|^2 + \text{Ric}_{m,n}(L)(\nabla f, \nabla f)\right] \rho d\mu$$

$$- \frac{t}{m-n} \int_M \left|\nabla\phi \cdot \nabla f + \frac{m-n}{t}\right|^2 \rho d\mu. \quad (24)$$

In particular, if $\text{Ric}_{m,n}(L) \geq 0$, then $W_m(\rho, t)$ is decreasing in time $t$ along the geodesic flow on $T^*P_\infty^2(M, \mu)$.

As a corollary of Theorem 4.1, we can recapture the following beautiful result due to Lott and Villani [34, 33].

**Corollary 4.2** ([24, 32]) Let $(M, g, \phi)$ be a compact Riemannian manifold with $\text{Ric}_{m,n}(L) \geq 0$. Then $t\text{Ent}(\rho(t)) + mt \log t$ is convex in time $t$ along the geodesic on $P_2(M, \mu)$.

5 **Comparison between Theorem 2.1 and Theorem 4.1**

In this section, we compare the $W$-entropy formula (9) in Theorem 2.1 and the $W$-entropy formula (24) in Theorem 4.1.

- The $W$-entropy formula (9) for the heat equation of the Witten Laplacian in Theorem 2.1 and the $W$-entropy formula (24) for the geodesic flow on the Wasserstein space in Theorem 4.1 have similar expressions. Moreover, similarly to Corollary 4.2 from Theorem 4.1 we can derive the following

**Corollary 5.1** Let $(M, g, \phi)$ be a compact Riemannian manifold with $\text{Ric}_{m,n}(L) \geq 0$. Then $t\text{Ent}(u(t)) + \frac{m}{2}t \log t$ is convex in time $t$ along the heat equation $\partial_t u = Lu$ on $M$.

- By [22, 24], Theorem 2.1 and a rigidity theorem hold on complete Riemannian manifolds with bounded geometric condition and with the $CD(0, m)$-condition: $W_m(u, t)$ achieves its minimum at some $t = t_0 > 0$ if and only if $M = \mathbb{R}^n$, $m = n$, and $u(x, t) = u_m(x, t) = \frac{1}{(4\pi t^2)^n} e^{-\frac{|x|^2}{4t}}$ is the heat kernel of the heat equation $\partial_t u = \Delta u$ on...
\[ \text{Ent}(u_m(t)) = -\frac{m}{2}(1 + \log(4\pi t)). \]

Thus the \( H_m \)-entropy for the heat equation of the Witten Laplacian is given by
\[ H_m(u(t)) = \text{Ent}(u_m(t)) - \text{Ent}(u(t)), \]
and the \( W_m \)-entropy for the heat equation of the Witten Laplacian is given by the Boltzmann entropy formula
\[ W_m(u, t) := \frac{d}{dt} (t[\text{Ent}(u_m(t)) - \text{Ent}(u(t))]). \] (25)

This gives a natural probabilistic interpretation of the \( W \)-entropy for the heat equation of the Witten Laplacian on Riemannian manifolds. See also Section 6 for the probabilistic interpretation of the Perelman \( W \)-entropy for the Ricci flow.

• On the other hand, when \( m \in \mathbb{N} \), we can check that the following \((\rho_m, f_m)\)
\[ \rho_m(x, t) = \frac{1}{(4\pi t^2)^{m/2}} e^{-\frac{\|x\|^2}{4t}}, \]
\[ f_m(x, t) = \frac{\|x\|^2}{2t}, \]
where \( t > 0, x \in \mathbb{R}^m \), is a solution to the transport equation (22) and the Hamilton-Jacobi equation (23) on \( \mathbb{R}^m \) equipped with the standard Lebesgue measure, i.e.,
\[ \partial_t \rho + \nabla \cdot (\rho \nabla f) = 0, \]
\[ \partial_t f + \frac{1}{2} |
\[ \nabla f|^2 = 0, \]
respectively. Moreover, the Boltzmann-Shannon entropy of the probability measure \( \rho_m(t, x)dx \) (which equals to \( u_m(t^2, x)dx \)) is given by
\[ \text{Ent}(\rho_m(t)) = -\frac{m}{2}(1 + \log(4\pi t^2)). \]

Thus we can reformulate the \( H_m \)-entropy and the \( W_m \)-entropy for the geodesic flow on the Wasserstein space \( P_2(M, \mu) \) as follows
\[ H_m(\rho(t)) = \text{Ent}(\rho_m(t)) - \text{Ent}(\rho(t)), \] (26)
and
\[ W_m(\rho, t) := \frac{d}{dt} (t[\text{Ent}(\rho_m(t)) - \text{Ent}(\rho(t))]). \] (27)

• The relative entropy \( H_m(\rho(t)) \) defined by (26) is the difference between the Boltzmann-Shannon entropy of the probability measure \( \rho(t)d\mu \) on \( (M, \mu) \) and the Boltzmann-Shannon entropy of the reference model \( \rho_m(t)dx \) on \( (\mathbb{R}^m, dx) \), and \( W_m(\rho, t) \) defined by (27) is the time derivative of \( tH_m(\rho(t)) \). In [28, 31], similarly to the case of Theorem 2.1, we extended the \( W \)-entropy formula (24) in Theorem 4.1 to complete Riemannian manifolds with bounded geometry condition. In view of this, we proved that the rigidity model for the \( W \)-entropy for the geodesic flow on the Wasserstein space \( P^\infty_2(M, \mu) \) over complete Riemannian manifolds with the CD(0, m)-condition is \( M = \mathbb{R}^n \), \( m = n \), \( \rho = \rho_m \) and \( f = f_m \).

\footnote{Following Villani [29, 53], we call \( H_m(u(t)) \) the \textit{relative entropy} even though it is slightly different from the classical definition of the relative entropy in probability theory.}
6 Langevin deformation of geometric flows on Wasserstein space

We can raise a natural question how to understand the similarity between the W-entropy formulas in Theorem 2.1 and Theorem 4.1. Can we pass through one of them to another one? One possible approach to answer this question is to use the vanishing viscosity limit method from the heat equation to the Hamilton-Jacobi equation. However, it seems that one cannot easily use this approach to pass through the $W$-entropy formula for the heat equation of the Witten Laplacian to the $W$-entropy formula for the geodesic flow on the Wasserstein space.

Inspired by J.-M. Bismut’s works (see [3, 4]) on the deformation of hypoelliptic Laplacians on the cotangent bundle over Riemannian manifolds, which interpolates the usual Laplacian on the underlying Riemannian manifold $M$ and the Hamiltonian vector field which generates the geodesic flow on the cotangent bundle over $M$, we introduced in [28, 31] the Langevin deformation of geometric flows on the cotangent bundle of the Wasserstein space over compact Riemannian manifolds.

More precisely, for $c \in (0, \infty)$, let $(\rho, f)$ be smooth solution to the following equations:

\begin{equation}
\partial_t \rho - \nabla \mu(\rho \nabla f) = 0, \tag{28}
\end{equation}

\begin{equation}
c^2 \left( \partial_t f + \frac{1}{2} |\nabla f|^2 \right) = -f + V'(\rho). \tag{29}
\end{equation}

where $V \in C^\infty((0, \infty), \mathbb{R})$. Eq. (28) is indeed the transport equation [24], while Eq. (29) can be viewed as the Langevin equation on $T^*P^2(M, \mu)$. When $c \to \infty$, Eq. (29) implies that $f$ should satisfy the Hamilton-Jacobi equation [24]. In this case, $(\rho, f)$ is indeed the geodesic flow on $T^*P_2(M, \mu)$. On the other hand, when $c = 0$, Eq. (29) implies that $f = V'(\rho)$. In this case, $\rho$ is the backward gradient flow of $U(\rho) = \int_M V(\rho) d\mu$ on $P^2(M, \mu)$ equipped with Otto’s infinite dimensional Riemannian metric

\begin{equation}
\partial_t \rho = \nabla \mu(\rho \nabla V'(\rho)). \tag{30}
\end{equation}

In [28, 31], we proved the existence and uniqueness of the Langevin deformation between the backward gradient flow of the Boltzmann-Shannon entropy $\text{Ent}(\rho) = \int_M \rho \log \rho d\mu$ (respectively, the Renyi entropy $U(\rho) = \frac{1}{m-1} \int_M \rho^m d\mu$ for $m > 1$), which is the backward heat equation $\partial_t \rho = -L_\rho$ (respectively, the backward porous medium equation $\partial_t \rho = -L \rho^m$) of the Witten Laplacian on $M$, and the geodesic flow on the cotangent bundle of the smooth Wasserstein space $P^2_M(M, \mu)$ over compact Riemannian manifolds. Moreover, we proved an extension of the $W$-entropy formula along the Langevin deformation of geometric flows. The rigidity models are also proposed for the Langevin deformation. Due to the limit of the length of the paper, we refer the reader to [28, 31] for the details of these results.

7 The $W$-entropy, statistical mechanics and probability theory

In [43], Perelman gave a heuristic interpretation for the $W$-entropy using statistical mechanics. Recall that the partition function for the canonical ensemble at temperature $\beta^{-1}$ is given by $Z_\beta = \int_E e^{-\beta E} d\omega(E)$, where $d\omega(E)$ denotes the “density of states” measure, whose physical meaning is the number of microstates with energy levels in the range $[E, E + dE]$. The average of the energy with respect to the Gibbs measure $dP(E) = \frac{e^{-\beta E}}{Z_\beta} d\omega(E)$ is

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z_\beta,$$
and the Boltzmann entropy $S$ satisfies the Boltzmann entropy formula

$$S = \log Z_\beta - \beta \frac{\partial}{\partial \beta} \log Z_\beta.$$  

Equivalently, letting $\tau = \beta^{-1}$, then

$$S = \frac{\partial}{\partial \tau} (\tau \log Z_\beta).$$

(31)

The fluctuation of the energy is given by

$$\sigma := \langle (E - \langle E \rangle)^2 \rangle = \frac{\partial^2}{\partial \beta^2} \log Z_\beta,$$

and the derivative of the Boltzmann entropy with respect to $\beta$ satisfies

$$\frac{\partial S}{\partial \beta} = -\beta \sigma.$$  

Let $(M, g(\tau))$ be a family of closed Riemannian manifolds, $dm(\tau) = (4\pi \tau)^{-n/2} e^{-f(\tau)} dv_{g(\tau)}$ a probability measure on $(M, g(\tau))$, where $g(\tau)$ satisfies the backward Ricci flow equation

$$\partial_\tau g = 2\text{Ric},$$

$f(\tau)$ satisfies the heat equation $\partial_\tau f = \Delta f - |\nabla f|^2 + R - \frac{n}{2\tau}$ and $\tau = T - t$. Assume that there is a canonical ensemble with a “density of state measure” $d\omega(E)$ such that the partition function $Z_\beta = \int_{\mathbb{R}} e^{-\beta E} d\omega(E)$ is given by

$$\log Z_\beta = \int_M \left( -f + \frac{n}{2} \right) dm,$$

(32)

where $\beta = \frac{1}{\tau}$, and the backward time $\tau = T - t$ is regarded as the temperature. Then, using the above formulas in statistical mechanics, Perelman formally derived that

$$\langle E \rangle = -\tau^2 \int_M \left( R + |\nabla f|^2 - \frac{n}{2\tau} \right) dm,$$

$$S = -\int_M \left( \tau (R + |\nabla f|^2) + f - n \right) dm,$$

$$\sigma = 2\tau^4 \int_M \left| \text{Ric} + \nabla^2 f - \frac{g}{2\tau} \right|^2 dm.$$

This yields

$$W(g, f, \tau) = -S,$$

and

$$\frac{d}{dt} W(g, f, \tau) = 2 \int_M \tau \left| \text{Ric} + \nabla^2 f - \frac{g}{2\tau} \right|^2 dm.$$

This gives an interpretation of the $W$-entropy for the backward Ricci flow by Boltzmann’s entropy formula. However, the problem whether there is a canonical ensemble with a “density of states” measure $d\omega(E)$ such that the partition function $Z_\beta = \int_{\mathbb{R}} e^{-\beta E} d\omega(E)$ satisfies Perelman’s requirement remains open. See [23] for further discussion on this issue.

In [22, 23], the second author of this paper gave a probabilistic interpretation of Perelman’s $W$-entropy for the Ricci flow. Observing that

$$\log Z_\beta = \text{Ent}(u(\tau)) + \frac{n}{2} \left( 1 + \log(4\pi \tau) \right),$$
where
\[
\text{Ent}(u(\tau)) = \int_M u \log udv = -\int_M f + \frac{n}{2} \log(4\pi\tau) \frac{e^{-f}}{(4\pi\tau)^{n/2}} dv
\]
is the Boltzmann-Shannon entropy of the heat kernel measure \( dm = u(\tau)dv_{g(\tau)} \) with respect to the volume measure \( dv_{g(\tau)} \) on \((M, g(\tau))\), where \( u = \frac{e^{-f}}{(4\pi\tau)^{n/2}} \). On the other hand, let
\[
u_n(x, \tau) = \frac{e^{-\frac{|x|^2}{2\tau}}}{(4\pi\tau)^{n/2}}, \quad \forall x \in \mathbb{R}^n, \tau > 0
\]
be the Gaussian heat kernel on \( \mathbb{R}^n \). Then it is well-known that the Boltzmann-Shannon entropy of the Gaussian measure \( d\gamma_n(\tau, x) = u_n(\tau, x)dx \) with respect to the Lebesgue measure is given by
\[
\text{Ent}(u_n) = -\frac{n}{2}(1 + \log(4\pi\tau)).
\]
Hence
\[
\log Z_\beta = \text{Ent}(u(\tau)) - \text{Ent}(u_n(\tau))
\]
is the difference of the Boltzmann-Shannon entropy of the heat kernel measure \( dm = u(\tau)dv_{g(\tau)} \) on \((M, g(\tau))\) and the Boltzmann-Shannon entropy of the Gaussian measure \( \gamma_n \) on \( \mathbb{R}^n \). In view of this, we have the following probabilistic interpretation of the \( W \)-entropy for the Ricci flow
\[
W(g, f, \tau) := \frac{d}{d\tau}(\tau[\text{Ent}(u_n(\tau)) - \text{Ent}(u(\tau))]). \tag{33}
\]
Similarly to (33), we can give the probabilistic interpretation of the \( W \)-entropy for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the \( CD(0, m) \)-condition. See [22] in Section 5. By [20] and [21], we have the similar probabilistic interpretation of the \( W \)-entropy for the geodesic flow on the Wasserstein space over Riemannian manifolds with the \( CD(0, m) \)-condition. See [22].

It is natural and interesting to ask the question whether we can give a probabilistic interpretation of the \( W \)-entropy for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the \( CD(K, m) \) and \( CD(K, \infty) \)-conditions. This question is closely related to the question whether there exist the rigidity models of the \( W \)-entropy for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the \( CD(K, m) \) and \( CD(K, \infty) \)-conditions.

In our recent paper [29], we gave a probabilistic interpretation of the \( W_{m,K} \)-entropy for the heat equation of the Witten Laplacian on complete Riemannian manifolds with the \( CD(K, m) \)-condition. More precisely, let \( m \in \mathbb{N}, M = \mathbb{R}^m, g_0 \) the Euclidean metric, \( \phi_K(x) = -\frac{K|x|^2}{2} \) and \( d\mu_K(x) = e^{-\frac{K|x|^2}{2}} dx \), where \( K \in \mathbb{R} \). Then \( \nabla \phi_K(x) = -Kx \), and \( \nabla^2 \phi_K = -K\text{Id}_{\mathbb{R}^m} \). We consider the Ornstein-Ulenbeck operator on \( \mathbb{R}^m \) given by
\[
L = \Delta + Kx \cdot \nabla.
\]
Note that \((\mathbb{R}^m, g_0, \phi_K)\) is a complete shrinking Ricci soliton, i.e., \( \text{Ric}(L) = -Kg_0 \). The Ornstein-Ulenbeck diffusion process on \( \mathbb{R}^m \) is the solution to the Langevin SDE
\[
dX_t = \sqrt{2}dW_t + KX_t dt, \quad X_0 = x.
\]
It is well-known that the law of \( X_t \) is Gaussian \( N\left( e^{Kt}x, \frac{e^{2Kt} - 1}{K} \text{Id} \right) \), and the heat kernel of \( X_t \) with respect to the Lebesgue measure on \( \mathbb{R}^m \) is given by
\[
u_{m,K}(x, y, t) = \left( \frac{K}{2\pi(e^{2Kt} - 1)} \right)^{m/2} \exp \left( \frac{K|y - e^{Kt}x|^2}{2(e^{2Kt} - 1)} \right).
\]
By direct calculation, the relative Boltzmann-Shannon entropy of the law of $X_t$ with respect to the Lebesgue measure on $\mathbb{R}^m$ is given by

$$\text{Ent}(u_{m,K}(x,y,t)|dy) = -\frac{m}{2} \left( 1 + \log(4\pi\sigma^2_K(t)) \right),$$

where $\sigma^2_K(t) = \frac{e^{2Kt} - 1}{2K}$. When $t \to 0$, we have

$$\text{Ent}(u_{m,K}(x,y,t)|dy) = -\frac{m}{2} \left( 1 + \log(4\pi t) + Kt + \frac{K^2 t^2}{6} \right) + O(t^4).$$

Thus, when $t \to 0^+$, the second term in the definition formula (18) of the $H_{m,K}$-entropy is asymptotically equivalent (at the order $O(t^4)$) to the Boltzmann-Shannon entropy of the heat kernel at time $t$ of the Ornstein-Uhlenbeck operator on $\mathbb{R}^m$ with respect to the Lebesgue measure on $\mathbb{R}^m$. That is to say, when $t \to 0^+$, we have

$$H_{m,K}(u(t)) = \text{Ent}(u_{m,K}(t)|dy) - \text{Ent}(u(t)|\mu) + O(t^4),$$

and

$$W_{m,K}(u(t)) = \frac{d}{dt} \left( t H_{m,K}(u(t)) \right).$$

To end this paper, let us mention that, in a forthcoming paper [16], Kuwada and the second author of this paper prove an analogue of the $W$-entropy monotonicity theorem on metric measure spaces with the so-called $RCD(0,N)$-condition.

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