Quantum Diffusion with Drift and the Einstein Relation I

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Abstract: We study the dynamics of a quantum particle hopping on a simple cubic lattice and driven by a constant external force. It is coupled to an array of identical, independent thermal reservoirs consisting of free, massless Bose fields, one at each site of the lattice. When the particle visits a site \( x \) of the lattice it can emit or absorb field quanta of the reservoir at \( x \). Under the assumption that the coupling between the particle and the reservoirs and the driving force are sufficiently small, we establish the following results: The ergodic average over time of the state of the particle approaches a non-equilibrium steady state (NESS) describing a non-zero mean drift of the particle. Its motion around the mean drift is diffusive, and the diffusion constant and the drift velocity are related to one another by the Einstein relation.

KEY WORDS: diffusion, kinetic limit, quantum brownian motion

1. Introduction

In this paper and its companion [6], we study the quantum dynamics of a tracer particle driven by a constant external force field, \( F \), (e.g., a uniform gravitational field) and interacting with thermal reservoirs by emitting or absorbing gapless reservoir modes. The purpose of our analysis is to establish properties of the long-time effective dynamics of the particle and to justify some fluctuation-dissipation relations, notably the Einstein relations.

Among properties of the effective dynamics of the tracer particle coupled to thermal reservoirs at positive temperature, we expect the following ones to hold: A velocity-dependent friction force caused by scattering processes between the particle and the reservoir modes (emission of Cherenkov radiation) can be expected to counteract the external force driving the particle, in such a way that its mean velocity approaches a finite, non-zero limit, \( v(F) \), in the direction of the external force \( F \), as time \( t \) tends to infinity. Because of thermal fluctuations in the reservoir(s), the true motion of the particle is expected to be diffusive around its mean motion. One might guess that the mobility, \( \frac{\partial}{\partial F} \bigg|_{F=0} v(F) \), of the particle satisfies the ‘Einstein relation’

\[
\frac{\partial}{\partial F} \bigg|_{F=0} v(F) = \beta D(F = 0),
\]

where \( \beta \) is the inverse temperature of the gas and \( D(F) \equiv D(\beta, F) \) its diffusion constant. This would be the case if we modeled the dynamics of the particle by a Langevin equation. However, for a particle moving in physical space \( \mathbb{R}^3 \) and interacting with a single thermal reservoir filled with a massless, free Bose field (e.g., a...
weakly interacting Bose gas), this guess is almost certainly false! The reason is that, usually, the friction force caused by interactions of the particle with reservoir modes decreases, as the velocity of the particle grows. One therefore expects that a sufficiently large force $F$ will eventually overcome the friction force and cause a ‘run-away’ acceleration. At positive temperature, thermal fluctuations will, in the long run, always manage to kick the particle velocity into a region where ‘run-away’ occurs. In fact, some part of the intuition described here has recently been made precise in a (classical) Hamiltonian model describing the system in a limiting regime where the particle is very massive and the Bose gas in the reservoir is very dense (mean-field limit); see [22, 23]. In these papers, which only concern systems at zero-temperature, it is shown that the friction force, $F_{\text{fric}}(v)$, tends to 0, as $|v| \to \infty$, and (assuming rotational invariance) $|F_{\text{fric}}(v)| = |F_{\text{fric}}(|v|)|$ has a unique maximum, $F_{\text{max}}$, at some finite value of $|v|$. These properties imply that, for $|F| < F_{\text{max}}$, there are two stationary solutions of the equations of motion corresponding to two different values,

$$v_-, v_+, \quad \text{with} \quad v_+ > v_-,$$

of the speed of the particle. Particle motion with speed $v_+$ is likely to be unstable: One expects to find ‘run-away’ solutions accelerating to ever higher speed, for initial conditions close to the stationary solution corresponding to $v_+$.  

Thus, at positive temperatures, and for a particle moving in $\mathbb{R}^3$ interacting with a single thermal reservoir, we do not expect to ever observe an approach of the particle’s motion towards a uniform mean motion at a finite constant velocity determined by the external force $F$, as time $t$ tends to infinity, (with diffusive fluctuations around the mean motion). This type of motion – if observed – is a transient phenomenon that may be encountered at intermediate times, but will disappear at very large times. If the reasoning sketched here is correct the status of the Einstein relation (1.1) becomes rather questionable.

It actually seems to us that the effective dynamics of a tracer particle in the continuum driven by a constant external force and interacting with an infinitely extended thermal reservoir (corresponding, e.g., to an ideal or weakly interacting Bose gas at positive temperature), as described above, is too complicated to be treated with mathematical precision, at present. We therefore propose to study an idealized model for which the expected phenomenology is much simpler: We imagine that the tracer particle moves through a static crystalline background of ions modeled by a periodic potential. We assume that the lowest energy band in the given periodic potential is well separated from the higher lying bands. Assuming that the initial state of the particle is a superposition of states with energies belonging to the lowest band and following the particle’s motion only over a long, but finite interval of time, we may neglect transitions of the particle to states in higher-lying energy bands altogether and use a tight-binding approximation to describe the dynamics of the particle. This amounts to replacing physical space $\mathbb{R}^3$ by a lattice, e.g., the simple cubic lattice $\mathbb{Z}^3$, and to considering a quantum particle hopping on $\mathbb{Z}^3$. The particle is subject to a linear external potential, $-\mathbf{F} \cdot \mathbf{x}$, $\mathbf{x} \in \mathbb{Z}^3$, and interacts with a dispersive thermal reservoir at positive temperature. The crucial difference, compared to the continuum model described above, is that the speed of a quantum particle hopping on a lattice is uniformly bounded, and there are no ‘run-away solutions’ to the equations of motion.

Yet, the problem of analyzing the long-time motion of the particle in such a model is still rather challenging, and although a lot of attention was devoted to problems of this type, see e.g., [12, 23], we do not have any precise mathematical results, yet. The origin of the difficulties lies in memory effects within the reservoir: The probability for the tracer particle to re-absorb a reservoir mode it has emitted at another position in space, some time $\Delta t$ ago, does not decay to 0 sufficiently rapidly, as $\Delta t$ tends to infinity, to control the effective dynamics. The reason is that reservoir auto-correlation functions do not decay in time very fast, uniformly in space. (Assuming they decay in time integrably fast, uniformly in space, one may hope to face a problem that can be solved rigorously. However, a solution would still require a major effort; see [5, 21] for corresponding results when the external force $F$ vanishes.) We therefore propose to simplify the problem yet a little further: We assume that, at each site, $x$, of the lattice $\mathbb{Z}^3$, there is another thermal reservoir, $R_x$, and that reservoirs at different sites of the lattice are independent. Moreover, all these reservoirs are isomorphic to one another and are described by free quantum fields at some positive temperature $\beta^{-1}$. When the particle visits the site $x$ it only interacts with the reservoir $R_x$. Thus, for memory effects to occur, the particle has to return to a site it has visited previously. Such memory effects tend to decay exponentially fast in $\Delta t$. Models of this sort, but for a vanishing external force field $F$, have been introduced and studied in [20]. They are sufficiently simple to be analyzed rigorously. A somewhat similar model was treated in [2].

Assuming that, at this point, the reader has an idea of what the models are that will be analyzed in this paper (see Section 2 for precise definitions), we proceed to summarizing our main results; (for detailed, precise
We assume that the strength of interaction between the particle and the reservoirs is proportional to a coupling constant \( \lambda \) that will be chosen appropriately and that the external force field is given by \( F = \lambda^2 \chi \), where \( \chi \in \mathbb{R}^3 \) is a fixed vector. The initial state of the system is given by a product state, \( \rho_S \otimes \rho_{R,\beta} \), where \( \rho_S \) is a density matrix on the Hilbert space, \( \mathcal{H}_S = l^2(\mathbb{Z}^3) \), of the tracer particle localized around some site of the lattice, while \( \rho_{R,\beta} \) is the equilibrium state of the reservoirs at inverse temperature \( \beta \). The time evolution of the initial state in the Schrödinger picture is (formally) given by \( e^{-itH} \rho_S \otimes \rho_{R,\beta} e^{itH} \), where \( H \) is the Hamiltonian of the system; see Subsections 2.2, 2.3, 2.4. The precise definition of the Hamiltonian \( H \) involves the dispersion law, \( \epsilon \), of the particle and a form factor, \( \phi \), that appears in the interaction Hamiltonian coupling the particle to the reservoirs. In Subsection 3.1 two assumptions, on \( \epsilon \) and \( \phi \), are formulated that are sufficient for the results summarized below to hold and that shall not be described here.

We define the effective dynamics, \( Z_{[0,t]} \), of the tracer particle by

\[
\rho_S,t \equiv Z_{[0,t]} := \text{Tr}_R \left[ e^{-itH} (\rho_S \otimes \rho_{R,\beta}) e^{itH} \right],
\]

where \( \text{Tr}_R[\cdot] \) denotes the partial trace over the reservoir degrees of freedom. In the thermodynamic limit, the state \( \rho_{R,\beta} \) can no longer be represented by a density matrix and the above formula needs to be re-interpreted, but the left-hand side remains meaningful.

All the results summarized below are only known to hold, provided the coupling constant \( \lambda \) is sufficiently small.

Momentum space of the tracer particle is given by the torus \( \mathbb{T}^3 \). Let \( f \) be an arbitrary continuous function on momentum space. Our first result says that

\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \text{Tr}_S \left[ f(\cdot) \rho_{S,t} \right] = \langle f, \zeta^{\chi,\lambda} \rangle_{L^2(\mathbb{T}^3)},
\]

where \( \zeta^{\chi,\lambda} \) is a probability density on \( \mathbb{T}^3 \) describing a stationary state of the tracer particle corresponding to uniform motion at constant velocity \( v \neq 0 \). Here, \( \text{Tr}_S[\cdot] \) denotes the trace over the tracer particle Hilbert space \( \mathcal{H}_S \). The interpretation of this result is that the ergodic average over time of the states of the particle approaches a so-called ‘non-equilibrium steady state’ \( \text{NESS} \). If \( \chi = 0 \) then \( \zeta^{\chi,\lambda} \) is actually an equilibrium measure on \( \mathbb{T}^3 \) approximately equal to Maxwell’s velocity distribution, \( \propto e^{-3v_\perp} \), corresponding to the dispersion law \( \epsilon \).

Our second result says that, asymptotically, the average motion of the tracer particle is uniform, with asymptotic velocity

\[
v(\chi) = \lim_{t \to \infty} \frac{1}{t} \text{Tr}_S \left[ X \rho_{S,t} \right],
\]

given by \( v(\chi) = \langle \nabla \epsilon, \zeta^{\chi,\lambda} \rangle \). Assuming time-reversal invariance, one sees that \( v(\chi = 0) = 0 \), and one expects that \( \lim_{\chi \to \infty} v(\chi) = 0 \). To understand the latter, recall that the Hamiltonian of a particle hopping on \( \mathbb{Z}^3 \) and driven by a constant field \( F \), has discrete spectrum, corresponding to Bloch oscillations. The eigenvectors form the so-called Wannier-Stark ladder, they are localized and their localization length decreases as \( |F| \to \infty \). Viewed from this angle, it is quite remarkable that one gets transport upon coupling to the reservoirs.

The third result concerns the fact that the ‘true’ motion of the particle is diffusive. Because of thermal fluctuations in the reservoirs, the particle performs quantum Brownian motion around its uniform mean motion, with a diffusion tensor given by

\[
D^{ij}(\chi) = \lim_{T \to \infty} \frac{1}{T^2} \int_0^\infty dt e^{-t} \text{Tr}_S \left[ (X^i - v^i(\chi)t)(X^j - v^j(\chi)t) \rho_{S,t} \right].
\]

In the companion paper \([6]\), we will establish the ‘Einstein relation’

\[
\frac{\partial}{\partial \chi^i} \bigg|_{\chi = 0} v^i(\chi) = \lambda^2 \beta D^{ij}(\chi = 0).
\]

Next, we sketch some of the main steps that go into the proofs of these results. The key idea is to show that the effective dynamics, \( Z_{[0,t]} \), of the tracer particle defined in (1.2) is well approximated by its kinetic limit. Rescaling space and time as \( (x, t) = \lambda^{-2}(\xi, \tau) \), the kinetic limit is approached when \( \lambda \to 0 \), with \( (\xi, \tau) \) (arbitrary,
but) fixed. Let us consider the Wigner distribution, \( \nu(\xi, k) \), with \((\xi, k) \in \mathbb{R}^3 \times \mathbb{T}^3 \), corresponding to the state \( \rho_S \) of the particle at time \( t = 0 \). Then, in the kinetic limit, time evolution is given by a linear Boltzmann equation

\[
\frac{\partial}{\partial t} \nu_{\tau}(\xi, k) = (\nabla \epsilon)(k) \cdot \nabla \epsilon \nu_{\tau}(\xi, k) - \chi \cdot \nabla k \nu_{\tau}(\xi, k) + \int_{\mathbb{T}^3} dk' r(k', k) \nu_{\tau}(\xi, k') - r(k, k') \nu_{\tau}(\xi, k),
\]

where \( r(k, k') \) is the rate for a jump of the particle from momentum \( k \) to momentum \( k' \). The kernel \( r(k, k') \) can be expressed in terms of the reservoir auto-correlation function (see Subsections 3.1 and 4.2) and satisfies ‘detailed balance’. The second, but last term on the right-hand side of Equation (1.5) is a ‘gain term’, the last term is a ‘loss term’.

It is convenient to consider the Fourier transform in the variable \( \xi \) of Equation (1.5). We set

\[
\hat{\nu}_\tau^\kappa(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} d\xi \, e^{-i(k, \xi) \nu_{\tau}(\xi, k)},
\]

where the ‘pseudo-momentum’ \( \kappa \in \mathbb{R}^d \) is the variable dual to \( \xi \in \mathbb{R}^d \). Then \( \hat{\nu}_\tau^\kappa \) satisfies the equation

\[
\frac{\partial}{\partial \tau} \hat{\nu}_\tau^\kappa = M_{\kappa^* \kappa} \hat{\nu}_\tau^\kappa,
\]

where the operator \( M_{\kappa^* \kappa} \) is given by

\[
(M_{\kappa^* \kappa} g)(k) := i\kappa \cdot (\nabla \epsilon)(k) g(k) - \chi \cdot \nabla k g(k) + \int_{\mathbb{T}^3} dk' r(k', k) g(k') - \int_{\mathbb{T}^3} dk' r(k, k') g(k),
\]

for \( g \in L^2(\mathbb{T}^3) \). To understand our approach it is important to know that, for any force field \( \chi \) and arbitrary \( \tau \geq 0 \),

\[
(Z_{[0, \lambda - z \tau], t}) \lambda^2 \kappa \xrightarrow{\lambda \to 0} e^{\tau M_{\kappa^* \kappa}},
\]

strongly on \( L^2(\mathbb{T}^3) \); see Theorem 4.2 Equation (4.9). The properties of the dynamics generated by the operator \( M_{\kappa^* \kappa} \) are studied in detail in Section 5. It satisfies all the results summarized above: (approach to a NESS describing uniform motion at a finite velocity, diffusion around the average motion, Einstein relation, and vanishing of \( v(\chi) \), as \( \chi \to \infty \)).

Our goal is then to show that the true dynamics of the tracer particle, as described by the effective time evolution \( Z_{[0, \lambda - z \tau], t} \), on the fiber corresponding to pseudo-momentum \( \lambda^2 \kappa \) and with small, but non-zero \( \lambda \), is well approximated by the dynamics in the kinetic limit, as given by the propagator \( e^{\tau M_{\kappa^* \kappa}} \). Indeed, we will analyze the properties of \( Z_{[0, t]} \) on the fiber corresponding to pseudo-momentum \( \lambda^2 \kappa \), for arbitrarily large times \( t \), by viewing it as a perturbation of the propagator \( e^{\lambda^2 \tau M_{\kappa^* \kappa}} \). It will turn out to be convenient to develop the perturbation theory for the Laplace transforms of \( Z_{[0, \lambda - z \tau], t} \) and \( e^{\lambda^2 \tau M_{\kappa^* \kappa}} \). The Laplace transform of the latter is the resolvent of the operator \( \lambda^2 M_{\kappa^* \kappa} \), while the Laplace transform of \( Z_{[0, \lambda - z \tau], t} \) on the fiber with pseudo-momentum \( \lambda^2 \kappa \) is a ‘pseudo-resolvent’ that, in a suitable domain of the spectral parameter, can be viewed as a small perturbation of the resolvent of \( \lambda^2 M_{\kappa^* \kappa} \). This analysis is carried out in Section 7. The formalism in Section 7 relies on material gathered in 6 (standard Dyson expansion for \( Z_{[0, \lambda - z \tau], t} \) and for time-dependent correlation functions). The relevant results of 6 are summarized in Section 8 of the present paper. In Section 8.1, the system without external force, i.e., for \( \chi = 0 \), is analyzed in some detail, and, in Section 8.2, the proofs of the main results are completed.

Finally, let us also point out an interesting effect that we observe in our model and that forces us to state our results in the sense of ergodic averages (note indeed that a more natural expression in 1.4) would be \( D \chi(\lambda) = \lim_{z \to \infty} \frac{1}{z} \text{Tr}_{S_0} \{ (X^i - v^i(\chi) t)(X^j - v^j(\chi) t) \rho_{S,t} \} \). Consider matrix elements \( \rho_{S,t}(x, x') \) of the reduced density matrix. If \( |x - x'| \) is large compared to \( |F|^{-1} \), then the free Liouville equation (neglecting the reservoir) is dominated by the driving field and one can neglect the kinetic term, i.e., \( \rho_{S,t}(x, x') \approx e^{-i \int (x - x') \cdot F \rho_{S,0}(x, x') \} \). In that case, the dissipative effect of the reservoir vanishes. Indeed, recall that in many-body theory dissipation is related to the imaginary part of the self-energy, which emerges from virtual transitions. However, if we keep only the field term in \( H_S \) (as we just argued), then there are no such transitions because the coupling to the reservoirs is diagonal in the position basis. The conclusion is that these matrix elements \( \rho_{S,t}(x, x') \) do not decay with time, as one would naturally expect. In other words, decoherence is switched off by the field! The mathematical expression of this phenomenon is visible in Lemma 12 where we bound the operator \( \hat{M}^{\lambda^2 \kappa^* \kappa} \), which describes the lowest order (second order) effect of the particle-reservoir coupling. The point to note is that this operator is
not close to the operator $M^{\omega x}$, which describes the dynamics in lowest order in $\lambda$. The difference between these two operators comes from the fact that $\lambda$ also appears in the force field, since we set $F = \lambda^2 \chi$ and keep $\chi$ constant.

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2. Definition of the model

2.1. Notations and conventions.

2.1.1. Banach spaces. Given a Hilbert space $\mathcal{H}$, we use the standard notation

$$\mathcal{B}_p(\mathcal{H}) := \left\{ A \in \mathcal{B}(\mathcal{H}) : \text{Tr} \left[ (A^* A)^{p/2} \right] < \infty \right\}, \quad 1 \leq p \leq \infty,$$

with $\mathcal{B}_\infty(\mathcal{H}) = \mathcal{B}(\mathcal{H})$ the bounded operators on $\mathcal{H}$, and

$$\|A\|_p := \left( \text{Tr} \left[ (A^* A)^{p/2} \right] \right)^{1/p}, \quad \|A\| := \|A\|_\infty.$$

For operators acting on $\mathcal{B}_p(\mathcal{H})$, e.g., elements of $\mathcal{B}(\mathcal{B}_p(\mathcal{H}))$, we often use the calligraphic font: $V, W$ etc. An operator $A \in \mathcal{B}(\mathcal{H})$ determines bounded operators $Ad(A), ad(A), A_1, A_2$ on $\mathcal{B}_p(\mathcal{H})$ by

$$Ad(A)B := ABA^*, \quad ad(A)B := [A, B] = AB - BA$$

and

$$A_1B := AB, \quad A_2B := BA^*, \quad B \in \mathcal{B}_p(\mathcal{H}).$$

Note that $(A_1)(A_2)_r = (A_2)_r(A_1)_r$, as operators on $\mathcal{B}_p(\mathcal{H})$, $A_1, A_2 \in \mathcal{B}(\mathcal{H})$, i.e., the left- and right multiplications commute. The norm of operators in $\mathcal{B}(\mathcal{B}_p(\mathcal{H}))$ is defined by

$$\|W\| := \sup_{A \in \mathcal{B}_p(\mathcal{H})} \frac{\|W(A)\|}{\|A\|_p}.$$ 

In the following, we usually set $p = 1$ or $2$.

2.1.2. Scalar products. For vectors $\kappa \in \mathbb{C}^d$, we let $\text{Re} \kappa$ denote the vector $(\text{Re} \kappa^1, \ldots, \text{Re} \kappa^d)$, where $\text{Re}$ denotes the real part. Similar notation is used for the imaginary part, $\text{Im}$. The scalar product on $\mathbb{C}^d$ is written as $(\kappa_1, \kappa_2)$ or $\prod_1 \kappa_2$ and the norm as $|\kappa| := \sqrt{(\kappa, \kappa)}$. The scalar product on an infinite-dimensional Hilbert space $\mathcal{E}$ is written as $\langle \cdot, \cdot \rangle$, or, occasionally, as $\langle \cdot, \cdot \rangle_\mathcal{E}$. All scalar products are defined to be linear in the second argument and anti-linear in the first one.

2.1.3. Kernels. For $\mathcal{E} = L^2(\mathbb{Z}^d)$, we can represent $A \in \mathcal{B}_2(\mathcal{E})$ by its kernel $A(x, y)$, i.e., $(A)(x) = \sum_y A(x, y) f(y)$, $f \in \mathcal{E}$. Similarly, an operator, $A$, acting on $\mathcal{B}_2(\mathcal{E})$ can be represented by its kernel $A(x, y, x', y')$ satisfying $(\delta_\rho)(x, y) = \sum_{x', y'} A(x, y, x', y') \rho(x', y')$, $\rho \in \mathcal{B}_2(\mathcal{E})$. Occasionally, we use the notation $|x\rangle$ for $\delta_x \in \mathcal{E}$, defined by $\delta_x(x') = \delta_{x-x'}$, and $\langle x|$ for $(\delta_x, \cdot)$. In this notation $|x\rangle \langle y|$ stands for the rank-one operator $\delta_x \langle y, \cdot \rangle$. Similarly, for the choice $\mathcal{E} = L^2(\mathbb{T}^d)$, we often use the notation $|f\rangle$ for $f \in L^2(\mathbb{T}^d)$ and $\langle g|$ for $\langle g, \cdot \rangle$, $g \in L^2(\mathbb{T}^d)$. In this ‘Dirac notation’, $|f\rangle \langle g|$ stands for the rank-one operator $f(g, \cdot)$ on $L^2(\mathbb{T}^d)$.

2.2. The particle. Consider the hypercube $\Lambda = \Lambda_L = \mathbb{Z}^d \cap [-L/2, L/2]^d$, for some $L \in 2\mathbb{N}$. The particle Hilbert space is chosen as $\mathcal{H}_S = L^2(\Lambda)$ where the subscript $S$ refers to ‘system’.
To describe the hopping term (kinetic energy), we choose a real function \( \varepsilon : \mathbb{T}^d \to \mathbb{R} \) and we consider the self-adjoint operator \( T \equiv T^\Lambda \) on \( L^2(\Lambda) \) with symmetric kernel\(^1\)

\[
T(x, x') = \hat{\varepsilon}(x - x'),
\]
with \( \hat{\varepsilon} \) the Fourier transform of \( \varepsilon \). Since we will assume \( \varepsilon \) to be analytic, the hopping is short range.

A natural choice for the dispersion law is \( \varepsilon(k) = \sum_j 2(1 - \cos k_j) \), corresponding to \( T = -\Delta \), with \( \Delta \) the lattice Laplacian on \( L^2(\Lambda) \) with Dirichlet boundary conditions. This choice satisfies all our assumptions, to be stated in Section 3.1.

We define the particle Hamiltonian as

\[
H_\mathcal{S} := T - F \cdot X,
\]
where \( F \in \mathbb{R}^d \) is an external force field, e.g., an electric field, and \( X \equiv X^\Lambda \) denotes the position operator on \( \mathcal{H}_\mathcal{S} \), defined by \( Xf(x) = xf(x) \). In what follows we will write \( F = \lambda^2 \chi \), with \( \chi \) a rescaled field, (a notation to be motivated later).

### 2.3. The reservoir.

#### 2.3.1. Dynamics.

For each \( x \in \mathbb{Z}^d \), we define a reservoir Hilbert space at site \( x \) by

\[
\mathcal{H}_{\mathcal{R}_x} := \Gamma_x(L^2(\Lambda^\ast)),
\]
where \( \Lambda^\ast = \Lambda_L = \mathbb{R}^d \cap [-L/2, L/2]^d \) and \( \Gamma_x(\mathcal{E}) \) is the symmetric (bosonic) Fock space over the Hilbert space \( \mathcal{E} \).

We assume that the reader is familiar with basic concepts of second quantization, such as Fock space and creation/annihilation operators; (we refer to, e.g., [7] for definitions and background). The total reservoir Hilbert space is defined by

\[
\mathcal{H}_\mathcal{R} := \bigotimes_{x \in \Lambda} \mathcal{H}_{\mathcal{R}_x}.
\]

Note that for all \( x \), the spaces \( \mathcal{H}_{\mathcal{R}_x} \) are isomorphic to each other. We remark that there is no compelling reason to restrict the one-site reservoirs to the same region, \( [-L/2, L/2]^d \), as the particle system, but this simplifies our notation. The reservoir Hamiltonian is defined as

\[
H_\mathcal{R} := \sum_{x \in \Lambda} \sum_{q \in \Lambda^\ast} \omega(q) a_{x,q}^\ast a_{x,q},
\]
where \( \Lambda^\ast = \mathbb{Z}^d \) is the set of quasi-momenta for the reservoir at site \( x \), and the operators \( a_{x,q}^\ast \) are the canonical creation/annihilation operators satisfying the commutation relations

\[
[a_{x,q}, a_{x',q'}^\ast] = \delta_{x,x'} \delta_{q,q'}, \quad [a_{x,q}, a_{x',q'}] = [a_{x,q}^\ast, a_{x',q'}^\ast] = 0,
\]
and we choose the dispersion law \( \omega(q) = |q| + \delta_{q,0} \). Note that this dispersion law corresponds to photons or phonons, except for \( q = 0 \), where we have modified this dispersion law at \( q = 0 \) by adding an infrared regularization that does not affect any of our results; e.g., if we replace \( \delta_{0,q} \) by \( K \delta_{0,q} \), with \( K > 0 \), then all infinite-volume objects studied in this paper are independent of \( K \).

#### 2.3.2. Equilibrium state.

Next, we introduce the Gibbs state of the reservoir at inverse temperature \( \beta \), \( 0 < \beta < \infty \).

It is given by the density matrix

\[
\rho_{R,\beta} := \frac{1}{Z_{R,\beta}} e^{-\beta H_\mathcal{R}}, \quad \text{where } Z_{R,\beta} = \text{Tr}_{\mathcal{R}}[e^{-\beta H_\mathcal{R}}],
\]

where \( \text{Tr}_{\mathcal{R}}[\cdot] \) denotes the trace over \( \mathcal{H}_\mathcal{R} \).

An alternative way to describe this density matrix is to specify the expectation values of arbitrary observables, which we denote by \( \langle O \rangle_{\rho_{R,\beta}} := \text{Tr}_R [O \rho_{R,\beta}] \). For \( \varphi \in L^2(\Lambda^\ast) \), we write \( a_x(\varphi) = \sum_{q \in \Lambda^\ast} \varphi(q) a_{x,q} \), and we choose observables, \( O \), to be polynomials in the operators \( a_x(\varphi) \). One then finds that, for any \( x, x' \) and \( \varphi, \varphi' \in L^2(\Lambda^\ast) \):

\[
T^\Lambda(x, x') = \begin{cases} \hat{\varepsilon}(x - x'), & \text{if } x, x' \in \Lambda, \\ 0, & \text{else} \end{cases}
\]

\[1\) Later, we will consider \( T^\Lambda \) as an operator on \( L^2(\mathbb{Z}^d) \) by the natural embedding of \( L^2(\Lambda) \) into \( L^2(\mathbb{Z}^d) \). As such, it has the kernel

i.e., we impose Dirichlet boundary conditions.
2.4. The interaction. We define the Hilbert space of state vectors of the coupled system (particle and reservoirs) by
\[ \mathcal{H} := \mathcal{H}_S \otimes \mathcal{H}_R. \]
We pick a smooth ‘structure factor’ \( \phi \in L^2(\mathbb{R}^d) \) and we define its finite volume version \( \phi^\Lambda(q) := (2\pi/L)^{d/2}\phi(q) \), with the normalization chosen such that \( \|\phi\|_{L^2(\mathbb{R}^d)} = \lim_{L \to \infty} \|\phi^\Lambda\|_{L^2(\Lambda^*)} \). We will drop the superscript \( \Lambda \). The interaction between the particle and the reservoir at site \( x \) is given by
\[ H_{SR} := \sum_{x \in \Lambda} \mathbb{1}_x \otimes \Psi_x(\phi), \quad \text{where} \quad \Psi_x(\phi) = a_x(\phi) + a_x^\dagger(\phi) \]
is the field operator, and \( \mathbb{1}_x = |x\rangle \langle x| \) denotes the projection onto the lattice site \( x \). The interaction Hamiltonian is taken to be
\[ H := T \otimes 1 - \lambda^2 X \otimes 1 + 1 \otimes H_R + \lambda H_{SR}, \]
where \( \lambda \in \mathbb{R} \) is a coupling constant. The interaction term \( H_{SR} \) is relatively bounded w.r.t. \( H_S + H_R \) with arbitrarily small relative bound. It follows that \( H \) is essentially selfadjoint on the domain \( \mathcal{D}(H) \), \( (\text{Dom}(H_R) \subset \mathcal{H}_R) \)
denotes the domain of \( H_R \).

2.5. Effective dynamics. The time-evolution in the Schrödinger picture is given by
\[ \rho_t = e^{-itH} \rho e^{itH}, \quad \rho \in \mathcal{B}(\mathcal{H}). \]
We will usually choose an initial state \( \rho \) of the form \( \rho = \rho_S \otimes \rho_{R,0} \), with \( \rho_{R,0} \) as defined above. Of course, \( \rho_t \), with \( t > 0 \), will in general not be a simple tensor product, but we can always take the partial trace, \( \text{Tr}_R[\cdot] \), over \( \mathcal{H}_R \) to obtain the ‘reduced density matrix’ \( \rho_{S,t} \) of the system;
\[ \rho_{S,t} = \text{Tr}_R \left[ e^{-itH} (\rho_S \otimes \rho_{R,0}) e^{itH} \right] := Z_{[0,t]} \rho_S, \]
and we call \( Z_{[0,t]} : \mathcal{B}_1(\mathcal{H}_S) \to \mathcal{B}_1(\mathcal{H}_S) : \rho_S \mapsto \rho_{S,t} \) the reduced or effective dynamics. It is a trace-preserving and completely positive map.

In the present paper, we will mainly consider observables of the form \( O \otimes 1 \) with \( O \in \mathcal{B}(\mathcal{H}_S) \), in which case we can also write
\[ \langle O(t) \rangle_{\rho_S \otimes \rho_{R,0}} := \text{Tr} [O(t) \rho_S \otimes \rho_{R,0}] = \text{Tr}_S [O \rho_{S,t}], \]
where the trace \( \text{Tr}[\cdot] \) is over the Hilbert space \( \mathcal{H} \), the trace \( \text{Tr}_S[\cdot] \) is over the particle Hilbert space \( \mathcal{H}_S \) and \( O(t) \) is the Heisenberg picture time evolution of the observable \( O \otimes 1 \), i.e.,
\[ O(t) := e^{itH} (O \otimes 1) e^{-itH}. \]
Note that \( O(t) \) is, in general, not of the product form \( O' \otimes 1 \), for some \( O' \).

i. Gauge-invariance:
\[ \langle a_s^\dagger(\varphi) \rangle_{\rho_{S,t}} = \langle a_s(\varphi) \rangle_{\rho_{S,t}} = 0; \quad (2.5) \]

ii. Two-point correlations: Let \( g_\beta := (e^{i\omega} - 1)^{-1} \), with the one-particle dispersion law \( \omega(q) = |q| + \delta_{q,0} \), be the Bose-Einstein density (operator). Then
\[ \left( \begin{array}{cc} \langle a^\dagger_s(\varphi) a^\dagger_r(\varphi') \rangle_{\rho_{S,t}} & \langle a^\dagger_s(\varphi) a^\dagger_r(\varphi') \rangle_{\rho_{S,t}} \\ \langle a_s(\varphi) a_r(\varphi') \rangle_{\rho_{S,t}} & \langle a_s(\varphi) a_r(\varphi') \rangle_{\rho_{S,t}} \end{array} \right) = \delta_{s,r} \left( \begin{array}{cc} 0 & \langle \varphi', g_\beta \varphi \rangle \\ 0 & \langle \varphi, (1 + g_\beta) \varphi' \rangle \end{array} \right); \]

iii. Wick’s theorem:
\[ \langle a^\dagger_{x_1}(...a^\dagger_{x_n}(\varphi_{2n})...a_{x_1}(\varphi_1))_{\rho_{S,t}} \rangle = \sum_{\pi \in \text{Pair}(n)} \prod_{(r,s) \in \pi} \langle a^\dagger_{x_r}(\varphi_r) a^\dagger_{x_s}(\varphi_r) \rangle_{\rho_{S,t}}, \quad (2.6) \]
where \( \text{Pair}(n) \) denotes the set of partitions of \( \{1, \ldots, 2n\} \) into \( n \) pairs and the product is over these pairs \((r, s)\), with the convention that \( r < s \). Here, \( \# \) stands either for \( * \) or nothing.
2.6. Time-reversal. We define an anti-linear time-reversal operator \( \Theta = \Theta_S \otimes \Theta_R \), where \( \Theta_S \) is given by
\[
\Theta_S f(x) = f(x), \quad f \in \ell^2(\Lambda),
\]
and \( \Theta_R \) by \( \Theta_R := \Gamma_s(\theta_R) \), with the one-particle operator \( \theta_R \) given by
\[
\theta_R \varphi_x(q) = \varphi_x(-q), \quad \varphi_x \in \ell^2(\Lambda^*), \quad x \in \Lambda.
\]
If the dispersion law \( \varepsilon \) of the particle and the form factor \( \phi \) are invariant under time-reversal, i.e., \( \varepsilon(k) = \varepsilon(-k) \), \( \phi(q) = \phi(-q) \) (as will be assumed) then we have that
\[
\Theta H_{\chi=0} \Theta = H_{\chi=0},
\]
expressing time-reversal invariance of the model.

3. Assumptions and Results

3.1. Assumptions. The model introduced in the last section is parametrized by two functions: the dispersion law \( \varepsilon : \mathbb{T}^d \to \mathbb{R} \), and the form factor \( \phi : \mathbb{R}^d \to \mathbb{C} \). Here, we formulate our assumptions on these two functions.

The (multi-) strip \( V_\delta \) is defined by
\[
V_\delta := \{ z \in (\mathbb{T} + i\mathbb{T})^d : |\text{Im } z| \leq \delta \}. \quad (3.1)
\]

**Assumption A.** [Particle dispersion] The function \( \varepsilon \) extends to an analytic function in a region containing a strip \( V_\delta \), \( \delta > 0 \). In particular, the norm
\[
\| \varepsilon \|_{\infty, \delta} := \sup_{p \in V_\delta} |\varepsilon(p)|
\]
is finite, for some \( \delta > 0 \). Furthermore, there does not exist any \( v \in \mathbb{R}^d \) such that the function
\[
\mathbb{T}^d \ni k \mapsto (v, \nabla \varepsilon(k))
\]
vanishes identically.

This assumption allows us to estimate the free particle propagator \( e^{-itH_S} \) on the particle Hilbert space \( \mathcal{H}_S = \ell^2(\Lambda_L) \) as follows:
\[
|\langle e^{-itH_S}\rangle(x, x')| \leq C e^{-\nu|x-x'|} e^{t\|\text{Im } \varepsilon\|_{\infty, \nu}}. \quad (3.2)
\]

For \( L = \infty \), the bound \( (3.2) \) is the Combes-Thomas bound; for finite \( L \), it can be established in an analogous way. If we replace \( \mathbb{Z}^d \) by \( \mathbb{R}^d \), any physically acceptable dispersion law \( \varepsilon \) is unbounded, and there is no exponential decay in \( |x-x'| \). This is the main reason why the system studied in this paper is defined on a lattice.

The next assumption deals with the ‘time-dependent’ correlation function defined (in finite-volume) as
\[
\hat{\psi}^\Lambda(t) := \sum_{q \in \Lambda^*} |\psi^\Lambda(q)|^2 \left( \frac{e^{-it\omega(q)}}{\omega(q) - 1} + \frac{e^{it\omega(q)}}{1 - e^{-\beta\omega(q)}} \right), \quad (3.3)
\]
and in the thermodynamic limit as
\[
\hat{\psi}(t) := \int dq |\phi(q)|^2 \left( \frac{e^{-it|q|}}{|q| - 1} + \frac{e^{it|q|}}{1 - e^{-\beta|q|}} \right). \quad (3.4)
\]

Since the correlation function \( \hat{\psi} \) is determined by the form factor \( \phi \), the following assumption is in fact a constraint on the choice of \( \phi \). Define the strip \( \mathbb{H}_\beta \) by
\[
\mathbb{H}_\beta := \{ z \in \mathbb{C} : 0 \leq \text{Im } z \leq \beta \}. \quad (3.5)
\]
Assumption B. [Decay of reservoir correlation function] The form factor $\phi$ is a spherically symmetric function, i.e., $\phi(q) = \phi(|q|)$. The correlation functions $\hat{\psi}^\Lambda(z)$, $\hat{\psi}(z)$ are uniformly bounded in $\Lambda$ and $z \in \mathbb{H}_\beta$, and
\[
\lim_\Lambda \hat{\psi}^\Lambda(z) = \hat{\psi}(z)
\]
holds uniformly on compacts in $\mathbb{H}_\beta$, where $\lim_\Lambda$ stands for $\lim_{\Lambda \to \infty}$ (recall that $\Lambda \equiv \Lambda_L$). Furthermore, the number
\[
\sum_{q \in \Lambda^*} \omega(q)^{-1} |\phi^\Lambda(q)|^2
\]
is bounded uniformly in $\Lambda$. Most importantly, $\hat{\psi}(z)$ is continuous on $\mathbb{H}_\beta$ and
\[
|\hat{\psi}(z)| \leq C e^{-g_0|z|}, \quad z \in \mathbb{H}_\beta.
\]
This assumption mainly states that the reservoirs exhibit exponential loss of memory. This is a key ingredient for our analysis.

Often, one also considers the ‘spectral density’
\[
\hat{\psi}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dt \hat{\psi}(t) e^{it\omega}.
\]
It satisfies the so-called ‘detailed balance’ property $e^{i\omega}\hat{\psi}(\omega) = \hat{\psi}(-\omega)$, which expresses, physically, that the reservoir is in thermal equilibrium at inverse temperature $\beta$.

Assumptions [A] and [B] are henceforth required and will not be repeated.

3.2. Thermodynamic limit. Up to this point, we have considered a system in a finite volume (cube), $\Lambda$ or $\bar{\Lambda}$, characterized by its linear size $L$. However, if we wish to study dissipative effects, we must, of course, pass to the thermodynamic limit, in order to eliminate finite-volume effects such as Poincaré recurrence. This amounts to taking $\Lambda = \mathbb{Z}^d$, $\bar{\Lambda} = \mathbb{R}^d$ and is accomplished below.

In this section, we will explicitly put a label $\Lambda$ on all quantities referring to a system in a finite volume.

3.2.1. Observables of the system. We begin by defining some classes of infinite-volume system observables, (i.e., certain types of bounded operators on $\mathcal{H}_\beta$). We say that an operator $O \in \mathcal{B}(\mathcal{H}_\beta)$ is exponentially localized whenever
\[
|O(x, x')| \leq Ce^{-\nu(|x|+|x'|)}, \quad \text{for some } \nu > 0.
\]
An important rôle is played by the so-called quasi-diagonal operators. These are operators $O \in \mathcal{B}(\mathcal{H}_\beta)$ with the property that
\[
|O(x, x')| \leq Ce^{-\nu(|x-x'|)}, \quad \text{for some } \nu > 0.
\]
We denote by $\mathfrak{A}$ the class of quasi-diagonal operators and by $\mathfrak{A}$ its norm-closure.

An observable $O \in \mathcal{B}(\mathcal{H}_\beta)$ is said to be translation-invariant whenever $T_y O = O$, for arbitrary $y \in \mathbb{Z}^d$, where $T_y O(x, x') := O(x + y, x' + y)$. Translation-invariant operators on $\mathcal{H}_\beta$ form a commutative $C^*$-algebra denoted by $\mathfrak{C}_{\text{ti}}$. We also introduce the algebras
\[
\mathfrak{A}_{\text{ti}} := \mathfrak{C}_{\text{ti}} \cap \mathfrak{A}, \quad \mathfrak{A}_{\text{ti}} := \mathfrak{C}_{\text{ti}} \cap \mathfrak{A}.
\]

An operator $O \in \mathfrak{C}_{\text{ti}}/\mathfrak{A}_{\text{ti}}/\mathfrak{A}$ can be identified with a multiplication operator, $M_f$, on the Hilbert space $L^2(\mathbb{T}^d)$, i.e., $M_f g = fg$, $g \in L^2(\mathbb{T}^d)$, with $f : \mathbb{T}^d \to \mathbb{C}$ a bounded and measurable/real-analytic/continuous function. Physically, the variable in $\mathbb{T}^d$ is the momentum of the particle.

These classes of operators are introduced because certain expansions used in our analysis will apply to quasi-diagonal operators or translation-invariant quasi-diagonal operators, and they can be extended to the closures of these algebras by density.

In analyzing diffusion and in the proof of the Einstein relation we also need to consider certain observables that are unbounded operators: We introduce the $^*$-algebra $\mathfrak{X}$ that consists of polynomials in the components, $X^i$, $i = 1, \ldots, d$, of the particle-position operator $X$.

Given an infinite-volume observable $O \in \mathcal{B}(\mathcal{H}_\beta)$, $\mathcal{H}_\beta = L^2(\mathbb{Z}^d)$, or $O \in \mathfrak{X}$, we associate an observable $O^\Lambda = \mathbb{I}_\Lambda O \mathbb{I}_\Lambda$ on $\mathcal{H}_\beta^\Lambda = L^2(\Lambda)$ with it, where $\mathbb{I}_\Lambda$ is the orthogonal projection $L^2(\mathbb{Z}^d) \to L^2(\Lambda)$. 
3.2.2. Dynamics. We choose not to construct directly the time-evolution of infinite-volume observables and infinite-volume states, although this could be done by using the Araki-Woods representation of the system in the thermodynamic limit. Instead, we will analyze the infinite-volume dynamics of "reduced" states, i.e., of states restricted to particle observables and correlation (Green) functions of particle-observables by constructing these objects as thermodynamic limits of finite-volume expressions.

An infinite-volume density matrix of the particle system $\rho_S \in \mathcal{B}_1(\mathcal{H}_S)$ is called exponentially localized if

$$|\rho_S(x, x')| \leq Ce^{-\nu|x-x'|}, \quad \text{for some } \nu > 0. \quad (3.8)$$

Given such an infinite-volume density matrix $\rho_S$, we associate finite-volume density matrices

$$\rho^\Lambda := \frac{1}{Z^\Lambda} \mathbb{1}_\Lambda \rho_S \mathbb{1}_\Lambda \in \mathcal{B}_1(\mathcal{H}_S^\Lambda), \quad Z^\Lambda := \text{Tr}_\Lambda[\mathbb{1}_\Lambda \rho_S \mathbb{1}_\Lambda], \quad (3.9)$$

with it. Note that, due to the normalization by $Z^\Lambda$, $\rho^\Lambda$ is a density matrix on $\mathcal{H}_S^\Lambda$.

Recall the definition of the reduced dynamics, $Z^\Lambda_{[0,t]}$, introduced in Section 3.2.1 and set

$$Z_{[0,t]} := \lim_{\Lambda \to \infty} Z^\Lambda_{[0,t]} \rho^\Lambda_S. \quad (3.10)$$

The next lemma asserts that the thermodynamic limit (as $\Lambda$ and $\bar{\Lambda}$ increase to $\mathbb{Z}^d$, $\mathbb{R}^d$, respectively) in (3.10) exists, and that the resulting reduced dynamics $Z_{[0,t]}$ is translation-invariant.

**Lemma 3.1.** The limit on the right side of Equation (3.10) exists in $\mathcal{B}_1(\mathcal{H}_S)$, and this defines the map $Z_{[0,t]} : \mathcal{B}_1(\mathcal{H}_S) \to \mathcal{B}_1(\mathcal{H}_S)$. The map $Z_{[0,t]}$ preserves the trace, i.e., $\text{Tr}[Z_{[0,t]} \rho_S] = \text{Tr}[\rho_S]$, positivity and exponential localization of the state of the particle, i.e., if $\rho_S$ has any of these properties, then so does $Z_{[0,t]} \rho_S$. Moreover, $Z_{[0,t]}$ is translation-invariant; $\mathcal{T}_y Z_{[0,t]} \rho_S = Z_{[0,t]}$ for $y \in \mathbb{Z}^d$ with $\mathcal{T}_y$ as in Subsection 3.2.1. As a consequence of the above, for $O$ in $\mathfrak{A}$ or $\mathfrak{X}$, and for an exponentially localized state $\rho_S$, we can define

$$(O(t))_{\rho_S \otimes \rho_{\bar{\Lambda}}} := \text{Tr}_\Lambda(O Z_{[0,t]} \rho_S).$$

3.3. Results. Next, we summarize our main results. Throughout this section, it is understood that we consider the infinite-volume system; i.e., $\Lambda = \mathbb{Z}^d$, $\bar{\Lambda} = \mathbb{R}^d$.

Our first result describes the approach of the state of the system to a ‘non-equilibrium stationary state’ (NESS), in the limit of large times.

In the theorems below, we use the notation $O(t)$ for $O^\Lambda(t)$, even if $\chi \neq 0$. Recall also the multiplication operator $M_f$ on $L^2(\mathbb{T}^d)$ defined in Section 3.2.

**Theorem 3.2.** [Approach to NESS] There are constants $k_\Lambda, k_\chi, g > 0$, such that, for $0 < |\lambda| < k_\lambda$, $|\chi| < k_\chi$, there exists a real-analytic function $\zeta \equiv \zeta^{\chi, \lambda}$ on $\mathbb{T}^d$, satisfying $\zeta \geq 0$ and $\int_{\mathbb{T}^d} dk \zeta(k) = 1$, i.e., $\zeta$ is a probability density, such that the following statements hold for any exponentially localized density matrix, $\rho_S$, and continuous function $f : \mathbb{T}^d \to \mathbb{R}$:

i. For $\chi \neq 0$,

$$\frac{1}{T} \int_0^T dt \langle M_f(t) \rangle_{\rho_S \otimes \rho_{\bar{\Lambda}}} = \langle f, \zeta^{\chi, \lambda} \rangle_{L^2(\mathbb{T}^d)} + O(1/T), \quad \text{as } T \to \infty. \quad (3.11)$$

ii. For $\chi \equiv 0$,

$$\langle M_f(t) \rangle_{\rho_S \otimes \rho_{\bar{\Lambda}}} = \langle f, \zeta^{0, \lambda} \rangle_{L^2(\mathbb{T}^d)} + O(e^{-\lambda^2 gt}), \quad \text{as } t \to \infty, \quad (3.12)$$

and $\zeta^{0, \lambda}$ satisfies ‘time reversal invariance’: $\zeta^{0, \lambda}(k) = \zeta^{0, \lambda}(-k)$.

Our next result asserts that the motion of the particle is diffusive around an average uniform motion (i.e., a drift at a constant velocity).

**Theorem 3.3.** [Diffusion] Under the same assumptions as in Theorem 3.2

$$\lim_{t \to \infty} \frac{1}{t} \langle X(t) \rangle_{\rho_S \otimes \rho_{\bar{\Lambda}}} = v(\chi),$$

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where $v(\chi)$ is the ‘asymptotic velocity’ of the particle and is given by $v(\chi) = \langle \nabla \epsilon, \xi^{\chi, \lambda} \rangle$. For $\chi \neq 0$, we have $v(\chi) \neq 0$. The dynamics of the particle is diffusive, in the sense that the limits

$$D^j(\chi) := \lim_{T \to \infty} \frac{1}{T^2} \int_0^\infty dt \, e^{-t} \langle (X^j(t) - v^j(\chi)t)(X^j(t) - v^j(\chi)t) \rangle_{\rho_S \otimes \rho_{BR, \beta}}$$

exist, where the ‘diffusion tensor’ $D(\chi)$ is positive-definite, with $D(\chi) = \mathcal{O}(\lambda^{-2})$, as $\lambda \to 0$.

Note that the claim about the asymptotic velocity follows formally from Theorem 3.2 by defining the velocity operator as

$$V^j := i[H, X^j] = i[T, X^j] = M_{\mathcal{V}_{1, \epsilon}},$$

and writing $X(t) = X(0) + \int_0^t ds V(s)$. Although it is quite easy to make this reasoning precise, we warn the reader that, at this point, it is formal, because the Heisenberg-picture observables $X(t)$ and $V(t)$ have not been constructed as operators in the thermodynamic limit. They are formal objects appearing in correlation functions that are constructed as thermodynamic limits of finite-volume correlation functions.

3.4. Correlation functions and Einstein relation. In this section, we present some more results on our model, that are proven in [6]. We begin with introducing correlation functions. Let $O_1, O_2$ be two observables, i.e., $O_1, O_2$ belong to the algebras $\mathfrak{A}$ or $\mathfrak{X}$; see Section 3.2. For $\Lambda = \Lambda_L$, with $L \in 2\mathbb{N}$, we set $\rho_1^\Lambda = 1_{1, O_1} 1_{\Lambda}$. Similarly, given an exponentially localized density matrix $\rho_S \in \mathcal{B}_1(\ell^2(L^2))$, its finite-volume version $\rho_S^\Lambda \in \mathcal{B}_1(\ell^2(\Lambda))$ is defined in [3.3]. Let $t_1, t_2 \in \mathbb{R}$, then we define the (finite-volume) correlation function as

$$\langle O_2(t_2)O_1(t_1) \rangle_{\rho_S \otimes \rho_{BR, \beta}} := \text{Tr}[O_2(t_2)O_1(t_1)(\rho_S^\Lambda \otimes \rho_{BR, \beta})],$$

the trace being over the Hilbert space $\ell^2(\Lambda) \otimes \ell^2(\bar{\Lambda})$. The infinite-volume correlation function is defined as the limit

$$\langle O_2(t_2)O_1(t_1) \rangle_{\rho_S \otimes \rho_{BR, \beta}} := \lim_{\Lambda} \text{Tr}[O_2(t_2)O_1(t_1)(\rho_S^\Lambda \otimes \rho_{BR, \beta})],$$

and we claim that this limit exists for any exponentially localized $\rho_S$ and any $O_1, O_2$ in $\mathfrak{A}$ or $\mathfrak{X}$. We refer to [6] for a proof of this claim.

Apart from an initial state (density matrix) of the product form $\rho_S \otimes \rho_{BR, \beta}$, we also consider the Gibbs state of the coupled system when the external force field vanishes, $\chi = 0$. In finite volume, it is defined by

$$\rho_\beta^\Lambda := \frac{1}{Z_\beta} e^{-\beta H^{\chi=0}} \in \mathcal{B}_1(\ell^2(\Lambda) \otimes \ell^2(\bar{\Lambda})), \quad Z_\beta = \text{Tr}(e^{-\beta H^{\chi=0}}), \quad H^{\chi=0} = T \otimes 1 + 1 \otimes H_R + \lambda H_{SR},$$

and one easily checks that $\rho_\beta^\Lambda \in \mathcal{B}_1(\ell^2(\Lambda) \otimes \ell^2(\bar{\Lambda}))$. We then define, for $O_1, O_2$ as above,

$$\langle O_1(t_1) \rangle_{\rho_\beta^\Lambda} := \text{Tr}[O_1(t_1)\rho_\beta^\Lambda], \quad \langle O_2(t_2)O_1(t_1) \rangle_{\rho_\beta^\Lambda} := \text{Tr}[O_2(t_2)O_1(t_1)\rho_\beta^\Lambda].$$

One observes that, for $\chi = 0$, $\langle O_2(t_2+t)O_1(t_1) \rangle_{\rho_\beta^\Lambda} = \langle O_2(t_2)O_1(t_1) \rangle_{\rho_\beta^\Lambda}$, for any $t$, i.e., the correlation functions are time-translation invariant. More generally, one checks that, for $\chi = 0$, the correlation function [3.18] satisfies the KMS condition. In particular, we have that

$$\langle O_1(t_1)O_2(t_2) \rangle_{\rho_\beta^\Lambda} = \langle O_2(t_2)O_1(t_1 + i\beta) \rangle_{\rho_\beta^\Lambda}, \quad \langle O_2(t_2)O_1(t_1) \rangle_{\rho_\beta^\Lambda} := \lim_{\Lambda} \text{Tr}[O_2(t_2)O_1(t_1)\rho_\beta^\Lambda].$$

Note that we construct the thermodynamic limit of equilibrium correlation functions only for translation-invariant observables, since, pictorially, the particle is uniformly distributed in space and hence the expectation values of localized observables vanish. For more details we refer to [6].
Before we discuss the Einstein relation, let us mention that our model relaxes exponentially fast to equilibrium at vanishing external field; cf., Theorem 3.3. in [6]: For \( O_1, O_2 \in \mathbb{R}_{t_1}, t_1, t_2 \in \mathbb{R}_+ \), there is \( g > 0 \), such that
\[
\langle O_2(t_2)O_1(t_1) \rangle_{\rho_0 \otimes \rho_1, \beta} = \langle O_2(t_2)O_1(t_1) \rangle_{\rho_0} + O(e^{-\lambda^2gt_1}) \quad (t_2 > t_1),
\]
as \( t_1 \to \infty \), for \( \lambda \) sufficiently small and \( \chi = 0 \). Moreover, the equilibrium correlation functions exhibit the following ‘exponential cluster property’:
\[
\langle O_2(t_2)O_1(t_1) \rangle_{\rho_0} = \langle O_2(t_2) \rangle_{\rho_0} \langle O_1(t_1) \rangle_{\rho_0} + O(e^{-\lambda^2gt(t_2 - t_1)}) ,
\]
as \( t_2 - t_1 \to \infty \), for \( \lambda \) sufficiently small and \( \chi = 0 \). Finally, we mention that the equilibrium correlation function satisfies the KMS condition on the algebra \( \mathbb{A}_t \) of translation-invariant observables; cf., Lemma 3.2 in [6]. Of course, there is nothing special about the restriction to correlation functions with one or two observables and one can prove the statements above for any number of observables.

Our next result states that the equilibrium diffusion matrix \( D(\chi = 0) \) (which is in fact a multiple of the identity matrix) is related to the response of the particle’s motion to the field \( \chi \). The corresponding identity is known as the ‘Einstein relation’:

**Theorem 3.4.** [Einstein relation] *Under the same assumptions as in Theorem 3.3*
\[
\frac{\partial}{\partial \chi} \bigg|_{\chi = 0} v^j(\chi) = \lambda^2 \beta D^{ij}(\chi = 0),
\]
where \( D(\chi = 0) \) is defined in Equation (3.13) and it equals
\[
D^{ij}(\chi = 0) = \frac{1}{2} \int_\mathbb{R} (\langle V^i(t)V^j \rangle)_{\rho_0} dt .
\]

Note that, by the positivity and isotropy of the diffusion matrix, this theorem also shows that, for small but non-zero \( \chi \), \( v(\chi) \) does not vanish. The origin of the unfamiliar factor \( \lambda^2 \) on the right side of (3.23) is found in the fact that the driving force field in the Hamiltonian is \( \lambda^2 \chi \), rather than \( \chi \).

### 4. Strategy of Proofs and Discussion

Before we are able to present a comprehensible overview of the strategy of the proofs, we have to introduce some further notions and concepts, such as the fiber decomposition introduced next.

#### 4.1. Fiber decomposition
To start with, we note that \( \mathcal{H}_1(\mathcal{H}_2) \subset \mathcal{B}_2(\mathcal{H}_2) \subset L^2(2^d) \). Hence, we may view density matrices on \( \mathcal{H}_2 \) as elements of the space of Hilbert-Schmidt operators, \( \mathcal{B}_2(\mathcal{H}_2) \simeq L^2(\mathbb{T}^d \times \mathbb{T}^d, dk_1 dk_2) \).

We define
\[
\hat{O}(k_1, k_r) := \frac{1}{(2\pi)^d} \sum_{x_1, x_r \in \mathbb{Z}^d} O(x_1, x_r) e^{-ik_1 \cdot x_1 + ik_r \cdot x_r}, \quad O \in \mathcal{B}_2(\mathbb{Z}^d) .
\]

In what follows, we write \( O \) for \( \hat{O} \). To cope with the translation-invariance of our model, we make the following change of variables
\[
k := \frac{k_1 + k_r}{2}, \quad p := k_1 - k_r ,
\]
and, for a.a. \( p \in \mathbb{T}^d \), we obtain a well-defined function \( O_p \in L^2(\mathbb{T}^d) \) by putting
\[
(O_p)(k) := O(k + \frac{p}{2}, k - \frac{p}{2}).
\]

(4.1)

This follows from the fact that the Hilbert space \( \mathcal{B}_2(\mathcal{H}_2) \simeq L^2(\mathbb{T}^d \times \mathbb{T}^d, dk_1 dk_2) \) can be represented as a direct integral
\[
\mathcal{B}_2(\mathcal{H}_2) \simeq \int_{\mathbb{T}^d} \mathcal{H}_p, \quad O = \int_{\mathbb{T}^d} dp \mathcal{H}_p .
\]

(4.2)

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where each ‘fiber space’ \( \mathcal{H}_p \) can be identified with \( L^2(\mathbb{T}^d) \). Next, we define, for \( \theta = (\theta_1, \theta_2) \in \mathbb{C}^d \times \mathbb{C}^d \), operators \( J_\theta \) by
\[
J_\theta O := e^{-i(\theta_1, X)} O e^{-i(\theta_2, X)}, \quad O \in \mathcal{B}(\mathcal{H}_S).
\]
(4.3)

Note that \( J_\theta \) is unbounded if \( \theta \) has an imaginary part. Also note that a density matrix \( \rho_S \in \mathcal{B}_2(\mathcal{H}_S) \) is exponentially localized iff \( \|J_\theta \rho_S\|_2 < \infty \), for \( \theta = (\theta_1, \theta_2) \) in some complex neighborhood of \((0,0)\).

The following lemma captures some identities used later on. Recall the definition of the strip \( \mathcal{V}_\delta \) in (4.1).

**Lemma 4.1.** Let \( O \in \mathcal{B}_1(\mathcal{H}_S) \), then
\[
\text{Tr}_S[O e^{ipX}] = (1, O_p)_{L^2(\mathbb{T}^d)} = \int_{\mathbb{T}^d} dk \, O_p(k), \quad p \in \mathbb{T}^d.
\]
(4.4)

If there is a \( \delta > 0 \) such that \( \|J_{\theta/2} O\|_2 < \infty \), for \( |\theta| \leq \delta \), then \( p \mapsto O_p \) is analytic in the interior of the strip \( \mathcal{V}_\delta \).

(In the discussion above, for \( O \in \mathcal{B}_2(\mathcal{H}_S) \) the fiber operator \( O_p \) was defined for a.a. \( p \), but in the context of Lemma (4.1), \( O_p \) can be defined for any \( p \).) The first statement of the lemma follows from the singular-value decomposition for trace-class operators and standard properties of the Fourier transform. The second statement of Lemma (4.1) is the Paley-Wiener theorem, i.e., the relation between exponential decay of functions and analyticity of their Fourier transforms; see [19].

The fiber decomposition in Equation (4.2) is useful when one deals with operators \( A \) acting on \( \mathcal{B}_2(\mathcal{H}_S) \) that are translation invariant (TI), i.e., \( \mathcal{T}_Z AT_{-Z} = A \), with \( \mathcal{T}_Z \) defined as in Section 3.2. An important example of a TI operator \( A \) is the reduced time-evolution \( Z_{(0,t)} \); see Lemma 3.1. For TI operators \( A \), we find that \((AO)_p \) depends on \( O_p \) only, and hence it makes sense to write
\[
(AO)_p = A_p O_p, \quad A = \int_{\mathbb{T}^d} dp \, A_p.
\]
(4.5)

Similarly to Lemma (4.1) above, we find that, if \( J_{\theta/2} A J_{-\theta/2} \) is bounded for all \( \theta = (\theta_1, \theta_2) \), with \( |\theta| \leq \delta \), then the map \( p \mapsto A_p \) is analytic in a strip \( \mathcal{V}_\delta \). Or, in other words, the kernel of the operator \( A \), satisfies
\[
|A(x_1, x_r, x'_1, x'_r)| \leq C e^{-\nu |x_1 - x'_1| - \nu |x_r - x'_r|}, \quad \text{for} \nu < |\theta|/2, \quad (x_1, x_r, x'_1, x'_r) \in \mathbb{Z}^d.
\]
(4.6)

In particular, (4.3) means that \( A \) preserves the subspace of exponentially localized operators in \( \mathcal{B}_2(\mathcal{H}_S) \). We call such an \( A \) a quasi-diagonal operator on \( \mathcal{B}_2(\mathcal{H}_S) \).

### 4.2. Strategy of proofs of main results.

4.2.1. **Kinetic theory.** For small values of the coupling constant \( \lambda \), one can, at least heuristically, understand the model studied in this paper with the help of semiclassical kinetic theory. The reasoning proceeds as follows: If \( \lambda \) approaches zero one must wait a time of order \( \lambda^{-2} \) before one sees an effect of the particle-reservoir interactions. The effect is that the particle emits or absorbs a field quantum (i.e., a ‘photons’ or ‘phonon’) of one of the thermal reservoirs and thus changes its momentum. Since such emission/absorption processes are well separated in time, one can assume them to be independent, and this leads to a description of the particle motion in terms of a stochastic process. Since the maximal velocity of the particle is bounded (this is an effect of the lattice) and of order one, despite the presence of the driving field \( \chi \), the particle travels a distance of order \( \lambda^{-2} \) during a time of order \( \lambda^{-2} \). This motivates the introduction of the kinetic scale: We define ‘macroscopic’ variables, \((\xi, \tau)\), by setting \( \xi := \lambda^2 x \) and \( \tau := \lambda^2 t \), where the variables \((x, t)\) are the variables used in the definition of the model, henceforth called ‘microscopic’ variables. The fact that, for small enough \( \lambda \), our model is ‘well-described’ by kinetic theory can be expressed, impressionistically, as follows:

\[
\text{Hamiltonian evolution}_{\lambda}(\lambda^{-2} \xi, \lambda^{-2} \tau) \xrightarrow{\lambda \to 0} \text{Stochastic evolution } (\xi, \tau).
\]

The stochastic evolution appearing on the right side is discussed next.
4.2.2. Boltzmann equation. Consider a classical particle with position \( \xi \in \mathbb{R}^d \) and (quasi-) momentum \( k \in \mathbb{T}^d \). The momentum \( k \) evolves according to a Poisson process with rate \( r(k,k')dk' \) for a jump from momentum \( k \) to momentum \( k' \), where \( r(k,k') \) is given by
\[
r(k,k') := \psi'[k'] - \psi[k],
\]
and \( \psi \) is the spectral density given in Equation (4.7). Between two consecutive jumps, at times \( \tau \) and \( \tau + \Delta \tau \), the momentum grows linearly in time \( b(\tau + \Delta \tau) = b(\tau) + \chi \Delta \tau \) (where addition is defined on the torus \( \mathbb{T}^d \)). The change in position is governed by the (group-) velocity \( \nabla \xi(k) \):
\[
\xi(\tau + \Delta \tau) = \xi(\tau) + \int_{\tau}^{\tau+\Delta \tau} ds \nabla \xi(k(s)).
\]
From this, a Markov process on \( \mathbb{R}^d \times \mathbb{T}^d \) can be constructed using standard methods. We present here the associated Master Equation describing the time-evolution of the probability density, \( \nu_\tau(\xi,k) \geq 0 \), on phase space \( \mathbb{R}^d \times \mathbb{T}^d \) (with normalization \( \int d\xi \int dk \nu_\tau(\xi,k) = 1 \)):
\[
\frac{\partial}{\partial \tau} \nu_\tau(\xi,k) = (\nabla \xi(\xi,k)) \cdot \nabla \xi \nu_\tau(\xi,k) - \chi \cdot \nabla_\xi \nu_\tau(\xi,k) + \int_{\mathbb{T}^d} dk' \left[ r(k',k) \nu_\tau(\xi,k') - r(k,k') \nu_\tau(\xi,k) \right].
\]
For our purposes, it is convenient to consider the Fourier transform
\[
\hat{\nu}_\tau(\kappa,k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} d\xi e^{-i(\kappa,\xi)} \nu_\tau(\xi,k),
\]
where \( \kappa \in \mathbb{R}^d \) is the variable dual to \( \xi \in \mathbb{R}^d \). One verifies that \( \hat{\nu}_\tau(\kappa,k) \) satisfies the evolution equation
\[
\frac{\partial}{\partial \tau} \hat{\nu}_\tau(\kappa,k) = M^{\kappa,\chi} \hat{\nu}_\tau,
\]
where, for smooth functions \( g \) on \( \mathbb{T}^d \),
\[
(M^{\kappa,\chi} g)(k) := i\kappa \cdot (\nabla \xi(\xi,k)) g(k) - \chi \cdot \nabla_\xi g(k) + \int_{\mathbb{T}^d} dk' r(k',k) g(k') - \int_{\mathbb{T}^d} dk' r(k,k') g(k).
\]
One can easily check that \( M^{\kappa,\chi} \) generates a strongly continuous semigroup on \( L^2(\mathbb{T}^d) \). Its significance in understanding dynamical properties of our model stems from the fact that it describes the evolution \( Z_{[0,\lambda^{-2} \tau]} \) in the fiber indexed by \( \lambda^2 \kappa \), in the limit \( \lambda \to 0 \).

**Theorem 4.2.** [Kinetic limit] For any \( \chi \) and arbitrary \( \tau \geq 0 \),
\[
Z_{[0,\lambda^{-2} \tau]} \overset{\lambda^2 \kappa}{\longrightarrow}_{\lambda \to 0} e^{\tau M^{\kappa,\chi}}.
\]
strongly on \( L^2(\mathbb{T}^d) \).

The restriction to fibers of order \( \lambda^2 \) is equivalent to considering a macroscopic length scale \( \sim \lambda^{-2} \). One can also convince oneself that \( (\rho)_{\lambda^2 \kappa} \in L^2(\mathbb{T}^d) \) (the space which the operator on the left side of (4.9) acts on) is the rescaled Wigner transform of \( \rho \in H_2(\mathbb{R}^d) \), and one may check that the claim (4.9) is equivalent to the results in [11].

**4.2.3. Perturbation around the kinetic limit.** The strategy we follow to control the long-time behavior when \( \lambda \) is small but non-zero, is basically the same as in [20]. We represent the Laplace transform of the fibered dynamics \( Z_{[0,\lambda^{-2} \tau]} )_{\lambda^2 \kappa} \) as a small (in \( \lambda \)) perturbation of the resolvent of the Boltzmann generator \( M^{\kappa,\chi} \). This is accomplished by appropriately resumming diagrams, and this is the tedious part of our analysis, which is described in [6]. The ideas underlying this analysis are elementary, and our technique is actually a time-dependent counterpart of the use of ‘translation-analyticity in the spectral form factor’ first applied to the study of ‘confined’ open quantum systems by [14]. Later, these confined open quantum systems, where the particle does not have translational degrees of freedom, have been treated in greater generality; see [11] [8] [18].

A complication not present in [20], is that we need to keep track of the dependence of the poles in the Laplace transform on \( \lambda, \kappa, \chi \). However, the Hamiltonian at \( \chi = 0 \) is not relatively bounded w.r.t. the one at \( \chi = 0 \). This means that we need to develop some version of asymptotic (rather than analytic) perturbation theory, and this is done in Section [7].
4.2.4. Einstein relation. In this subsection, we derive the Einstein relation in finite volume (now dropping the superscript Λ, because all formulae of this subsection refer to a finite volume). We define the velocity operator as
\[ V^j := i[H, X^j] = i[T, X^j]. \] (4.10)
Note that, because of the finite lattice, this operator is not translation-invariant; however, its thermodynamic limit is. Using Duhamel’s principle we obtain
\[ \frac{\partial}{\partial \chi} \bigg|_{\chi=0} \langle V^j X(t) \rangle_{\rho_\beta} = -i\lambda^2 \int_0^t ds \langle [X^{i,0}(t-s), V^j] \rangle_{\rho_\beta}. \] (4.11)
For simplicity, we drop the spatial indices \( i, j \) in the following. Note that the right-hand side of (4.11) is independent of \( \chi \). By stationarity of the state \( \rho_\beta \), it can be written as \(-i\lambda^2 \int_0^t ds \langle [X^0(-s), V] \rangle_{\rho_\beta}\). In the remainder of this section, we always set \( \chi = 0 \) and drop this symbol from our notation. Using the KMS condition we find
\[ \int_0^t ds \langle [X(-s) \rangle_{\rho_\beta} = \int_0^t ds \langle XV(s) \rangle_{\rho_\beta} - \int_0^t ds \langle XV(i\beta + s) \rangle_{\rho_\beta} \]
\[ = i \int_0^\beta du \langle XV(iu) \rangle_{\rho_\beta} - i \int_0^\beta du \langle XV(iu + t) \rangle_{\rho_\beta} \]
\[ = i \int_0^\beta du \langle XV(iu) \rangle_{\rho_\beta} - i \int_0^\beta du \langle X(-t)V(iu) \rangle_{\rho_\beta}, \]
where, in the second line, we have used that the integral of the function \( z \mapsto \langle XV(z) \rangle_{\rho_\beta} \) vanishes along the contour \( 0, t, t + i\beta, i\beta, 0 \). The third line follows by time-translation invariance. Next, using \( X(-t) = \int_0^{-t} ds V(s) + X(0) \), by (3.14), we get
\[ \int_0^t ds \langle [X(-s) \rangle_{\rho_\beta} = \int_0^\beta du \int_0^t ds \langle V(s)V(iu) \rangle_{\rho_\beta} \]
\[ = i \int_0^\beta du \int_0^t ds \langle V(-s)V(i\beta - iu) \rangle_{\rho_\beta} \]
\[ = \frac{1}{2} \int_0^\beta du \int_{-t}^t ds \langle VV(s + iu) \rangle_{\rho_\beta} \]
\[ = \frac{\beta}{2} \int_{-t}^t ds \langle VV(s) \rangle_{\rho_\beta} + Q(t). \]
The second and third equality follow from time-reversal invariance and the KMS condition. To arrive at the last equality, we have used that the integral of the map \( z \mapsto \langle VV(z) \rangle_{\rho_\beta} \) vanishes along the contour \(-t, t, t + i\beta, -t + i\beta, -t\), and we have introduced the remainder term
\[ Q(t) := \frac{i}{2} \int_0^\beta du \int_0^\beta ds \langle VV(iu + t) \rangle_{\rho_\beta} - \frac{i}{2} \int_0^\beta du \int_0^\beta ds \langle VV(is - t) \rangle_{\rho_\beta}. \]
Recalling our starting point (4.11), we conclude that
\[ \frac{\partial}{\partial \chi} \bigg|_{\chi=0} \langle VX(t) \rangle_{\rho_\beta} = -\frac{i\lambda^2 \beta}{2} \int_{-t}^t ds \langle VV(s) \rangle_{\rho_\beta} + Q(t), \]
where the dynamics used on the right side is taken at \( \chi = 0 \). We now claim that, in the thermodynamic limit, \( Q(t) \to 0 \), as \( t \to \infty \); we refer to [6] for a proof. This proves the Einstein relation, which relates a non-equilibrium response (left-hand side) to an equilibrium correlation function (right-hand side).

5. Kinetic limit: Linear Boltzmann evolution

As announced in the previous section, we have to study the operator \( M \equiv M^{\kappa,\chi} \) introduced in Subsection 4.2.2 in order to unravel properties of the long-time dynamics of the particle. This operator is of the form
\[ M^{\kappa,\chi} = i\kappa \cdot (\nabla \epsilon) - \chi \cdot \nabla G + L, \] (5.1)
with the gain- and loss terms given by

$$(Gg)(k) := \int_{T^d} dk' r(k', k) g(k'), \quad (Lg)(k) := - \int_{T^d} dk' r(k', k) g(k).$$

The operator $M^{\kappa,\chi}$ is closable (as an operator defined on a domain dense in $L^2(T^d)$), because it is a bounded perturbation of the anti-selfadjoint operator $\chi \cdot \nabla$, a core being, e.g., $C^\infty_c(T^d)$. Our main aim is to control the spectrum of $M^{\kappa,\chi}$ and to establish good bounds on the resolvent $(z - M^{\kappa,\chi})^{-1}$. In our analysis, we will emphasize the usefulness of the $C_0$-semigroup generated by $M^{\kappa,\chi}$.

5.1. Concepts from the theory of $C_0$-semigroups. First, we recall some definitions and results from the theory of strongly continuous semigroups (hereafter $C_0$-semigroups). For a detailed discussion we refer to [10, 13]. For definiteness, we assume from the onset that the semigroup acts on $L^2(T^d)$. We say that $f \in L^2(T^d)$ is positive ($f \geq 0$), or strictly positive ($f > 0$) iff $f(k) \geq 0$, or $f(k) > 0$, respectively, for almost every $k \in T^d$.

A $C_0$-semigroup $(T_t)_{t \geq 0}$ is called

- positivity-preserving (or positive) if $0 \leq f$ implies $0 \leq T_tf$, for each $t \geq 0$;
- positivity-improving if $0 \leq f, f \neq 0$, implies $0 < T_tf$, for each $t > 0$.

For a $C_0$-semigroup $(T_t)_{t \geq 0}$, the growth bound, $\omega_0$, is defined by

$$\omega_0 := \inf \{ \omega \in \mathbb{R} : \exists K_\omega, \text{ with } 1 \leq K_\omega < \infty, \text{ such that } \|T_t\| \leq K_\omega \varepsilon^{\omega t}, \forall t \geq 0 \}.$$  

Here and in the following we use the symbol $\| \cdot \|$ for the norm on $L^2(T^d)$ and for the operator norm on $\mathcal{B}(L^2(T^d))$.

A $C_0$-semigroup $(T_t)_{t \geq 0}$ has a closed generator, $A$, and we use the standard notation $T_t = e^{tA}$. The spectral bound, $s(A)$, of $A$ is defined as

$$s(A) := \sup \{ \Re z : z \in \sigma(A) \},$$

where $\sigma(A)$ is the spectrum of $A$. For any $z \in \mathbb{C}$ with $\Re z > \omega_0$, $(z - A)^{-1} = \int_0^\infty dt e^{-zt} T_t$ exists, and we infer the bound $\| (z - A)^{-1} \| \leq \frac{K_\omega}{\Re(z - \omega_0)}$ and the inequality $s(A) \leq \omega_0$.

Let $O$ be a densely defined closed operator whose spectrum is not all of $\mathbb{C}$. We say (following [4], see also [15]) that $z \in \mathbb{C}$ belongs to the essential spectrum, $\sigma_{\text{ess}}(O)$, of $O$, iff $z \parallel O$ is not a Fredholm operator. We call $r_{\text{ess}}(O) = \sup \{ |z| : z \in \sigma_{\text{ess}}(O) \}$ the essential spectral radius. A $C_0$-semigroup $T_t$ is called quasi-compact if $r_{\text{ess}}(T_t) < 1$, for $t > 0$.

5.2. Spectral analysis of $M^{\kappa,\chi}$: Preliminaries. For now, we neglect the advection term $i\kappa \cdot (\nabla \varepsilon)$, i.e., we put $\kappa = 0$, and study $M^{0,\chi} = -\chi \cdot \nabla + G + L$. We therefore omit the superscript $\kappa$ everywhere in this section, and we simply write $M^\chi \equiv M^{0,\chi}$. Our main results are summarized in Lemma 5.2.

We say that a function $f$ on $T^d$ is real-analytic if it is analytic in a region containing a multistrip $\mathcal{D}$, for some $\delta > 0$. Starting from Assumptions A and B it is straightforward to verify that function $r(k, k')$, as defined in (4.5), is real-analytic in $k$ and $k'$, for some $\delta > 0$ determined by $g_\alpha$ and $\delta_\varepsilon$. Moreover, $r(k, k')$ is strictly positive almost everywhere for real arguments. Thus the functions $r(\cdot, k'), r(k, \cdot)$ can vanish only in isolated points. Therefore,

$$a_0 := \inf_{k \in T^d} \int_{T^d} dk' r(k, k') > 0.$$  

(5.2)

Setting $L(k) := - \int dk' r(k, k')$ (the loss operator $L$ being defined as multiplication by $L(k)$), we have that $L(k) \leq -a_0$, for all $k \in T^d$. The strict positivity of the rates $r(\cdot, \cdot)$ implies that the gain operator $G$ is positivity improving. Moreover, by the smoothness of $r(\cdot, \cdot)$, $G$ is a compact operator.

Lemma 5.1. The operator $-\chi \cdot \nabla + L$ generates a positivity-preserving $C_0$-semigroup $(S_t)_{t \geq 0}$ on $L^2(T^d)$ with growth bound $\omega_0 \leq -a_0$, and

$$\|(z + \chi \cdot \nabla - L)^{-1}\| \leq |\Re z + a_0|^{-1}, \quad \text{for } \Re z > -a_0.$$  

(5.3)

The operator $M^{0,\chi}$ generates a positivity-improving $C_0$-semigroup, $(T_t)_{t \geq 0}$, on $L^2(T^d)$. It has the constant function 1 as a left-eigenvector corresponding to the eigenvalue 1, i.e., $\langle 1, T_tf \rangle = \langle 1, f \rangle$, for all $f \in L^2(T^d)$ and all $t \geq 0$. 

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Proof. Define the $C_0$-semigroup $(S_t)_{t \geq 0}$ by

$$(S_t f)(k) := f(k - \chi t) e^{\int_0^t ds L(k + \chi(s-t))}, \quad t \geq 0, \quad f \in L^2(\mathbb{R}^d).$$  \hfill (5.4)

It is easy to check that $-\chi \cdot \nabla + L$ is the generator of $(S_t)_{t \geq 0}$, and the growth bound of $(S_t)_{t \geq 0}$ is smaller than $-a_0$. Since $G$ is bounded, the construction of the semigroup $(T_t)_{t \geq 0}$ is standard, e.g., by using the norm-convergent Dyson series

$$T_t := S_t + \sum_{n=1}^{\infty} \int_{0 \leq t_1 < \ldots < t_n \leq t} dt_1 \ldots dt_n S_{t-t_n} G S_{t_{n-1}-t_n} G \cdots S_{t_1}. \hfill (5.5)$$

Clearly, we have that $\|T_t\| \leq e^{t(-a_0 + \|G\|)}$. Observe that the semigroup $(S_t)_{t \geq 0}$ defined in Equation (5.4) has the property that $f > 0$ implies $S_t f > 0$, for any finite $t \geq 0$. Together with the fact that $G$ is positivity-improving, this implies that $(T_t)_{t \geq 0}$ is positivity-improving, for any $t > 0$. One easily checks that, for smooth $f$,

$$\frac{d}{dt} \langle 1, f_t \rangle = - \langle 1, \chi \cdot \nabla f_t \rangle + \langle 1, (G + L) f_t \rangle = 0, \quad f_t := T_t f,$$

(note that both terms vanish separately), and $\langle 1, T_t f \rangle = \langle 1, f \rangle$ holds for arbitrary $f \in L^2(\mathbb{R}^d)$, by a limiting argument.

From this lemma, we obtain information on the spectrum of the generator $M^\chi$.

Lemma 5.2. [Spectrum of $M^\chi$] All statements below hold for any $\chi \in \mathbb{R}^d$:

i. The essential spectrum of $M^\chi$ is contained in the region $\{ z \in \mathbb{C} : \text{Re} \, z \leq -a_0 \}$. Furthermore, the semigroup generated by $M^\chi$ is quasi-compact.

ii. Let $\chi \notin \sigma(M^\chi)$. If $f$ is real-analytic then $(z - M^\chi)^{-1} f$ is real-analytic, too.

iii. The spectrum of $M^\chi$ in the region $\{ z \in \mathbb{C} : \text{Re} \, z > -a_0 \}$ consists of isolated eigenvalues of finite (algebraic) multiplicity. The associated eigenfunctions are real-analytic functions.

iv. There is a constant $m > 0$ (independent of $\chi$) such that the region $\{ z \in \mathbb{C} : |\text{Im} \, z| \geq m, \text{Re} \, z > -a_0/2 \}$ does not contain any spectrum.

v. The only eigenvalue $\mu$ of $M^\chi$ with $\text{Re} \, \mu > 0$ is $\mu = 0$, and it is simple. The spectral projection associated with the eigenvalue $\mu = 0$, $P^{\mu, \chi} \equiv P^\chi$, is of the form $P^\chi = |\zeta^\chi\rangle \langle 1|$, where $\zeta^\chi$ is a strictly positive function, with $\langle 1, \zeta^\chi \rangle = 1$. Moreover, $\sup_{\chi} \|\zeta^\chi\| < \infty$.

Proof. i. By Lemma 4.1, we know that $s(-\chi \cdot \nabla + L) = -a_0$. Since $G$ is compact, Weyl’s theorem on the stability of the essential spectrum implies $\sigma_{\text{ess}}(-\chi \cdot \nabla + L) = \sigma_{\text{ess}}(-\chi \cdot \nabla + L + G)$. Moreover, compact perturbations of generators with strictly negative growth bound generate quasi-compact semigroups; see [10].

ii. By the analyticity of the function $r(k, k')$ in a strip, we have that

$$\|e^{\gamma \cdot \nabla} M^\chi e^{-\gamma \cdot \nabla} - M^\chi\| \leq O(\gamma), \hfill (5.6)$$

for sufficiently small $\gamma \in \mathbb{C}^d$. Hence, by standard perturbation theory, $e^{\gamma \cdot \nabla}(z - M^\chi)^{-1} e^{-\gamma \cdot \nabla}$ remains bounded for sufficiently small $\gamma$ (depending on $z$), for any $z \notin \sigma(M^\chi)$, and hence

$$\|e^{\gamma \cdot \nabla}(z - M^\chi)^{-1} f\| \leq C(z, \gamma)\|e^{\gamma \cdot \nabla} f\|.$$

The claim on analyticity follows then from the Paley-Wiener theorem.

iii. For any $z \in \mathbb{C}$ with $\text{Re} \, z > -a_0$, we write

$$z - (-\chi \cdot \nabla + L + G) = (z - (-\chi \cdot \nabla + L))(\mathbb{1} - \frac{1}{z + \chi \cdot \nabla - L} G).$$

It follows that $z - (-\chi \cdot \nabla + L + G)$ is invertible if and only if $(\mathbb{1} - \frac{1}{z + \chi \cdot \nabla - L} G)^{-1}$ is. Since $G$ is compact, the analytic Fredholm theorem implies that $(\mathbb{1} - \frac{1}{z + \chi \cdot \nabla - L} G)^{-1}$ is a meromorphic function with only finitely or countably many
isolated poles of finite (algebraic and geometric) multiplicity, the residues of which are finite rank operators. It follows that the spectrum of $M^\chi$ in the region $\{z \in \mathbb{C} : \text{Re}z > -a_0\}$ consists of isolated eigenvalues of finite multiplicity. Let $M^\chi f = \mu f$, $f \neq 0$, with $\text{Re} \mu > -a_0$. Since $\mu \notin \sigma(-\chi \cdot \nabla + L)$, we can rewrite this eigenvalue equation as

$$f = (\mu + \chi \cdot \nabla - L)^{-1} G f.$$  \hfill (5.7)

Consequently, for sufficiently small $\gamma \in \mathbb{C}^d$,

$$\|e^{\gamma \cdot \nabla} f\| \leq \| (\mu + \chi \cdot \nabla - e^{\gamma \cdot \nabla} Le^{-\gamma \cdot \nabla})^{-1} \| e^{\gamma \cdot \nabla} G \| \| f \|

\leq \frac{C}{|\text{Re} \mu - a_0| + O(\gamma)} \| f \|.$$  \hfill (5.8)

Indeed, the bound on the resolvent follows by Neumann series expansion, using (5.6), with $M^\chi$ replaced by $L$, and the resolvent bound on $(z + \chi \cdot \nabla - L)^{-1}$ from Lemma 5.1 whereas the bound $\|e^{\gamma \cdot \nabla} G\| < \infty$ follows from the analyticity of the kernel $r(\cdot, \cdot)$ of $G$. By the Paley-Wiener theorem, $f$ is real-analytic.

iv. Let $M^\chi f = \mu f$, with $\|f\| = 1$. Then, on one hand,

$$|\text{Im} \mu - \langle f, i\chi \cdot \nabla f \rangle| \leq |\text{Re} \mu| + \| (G + L) f \|.$$  \hfill (5.9)

On the other hand, by the functional calculus,

$$\|\langle f, i\chi \cdot \nabla f \rangle\| \leq \frac{1}{\nu} \langle f, e^{\nu \chi \cdot \nabla} f \rangle,$$  \hfill $\nu > 0$.

Since $\text{Re} \mu \geq -a_0/2$, the right-hand side of this equation can, for sufficient small $\nu$, be bounded independently of $\text{Im} \mu$. This follows from statement iii and Equation (5.8). Combining this $\text{Im} \mu$-independent bound with (5.9) yields the claim iv.

v. The claim that there is a unique simple eigenvalue with maximal real part (and strictly positive eigenvector $\zeta^\chi$) follows from a Perron-Frobenius-type argument; see, e.g., Chapter 6, Thm 3.5, in [10]. This theorem uses the quasi-compactness and the fact that, for sufficiently large real $z$, $(z - M^{0,\chi})^{-1}$ is positivity-improving, which in our case follows from the fact that $T_t$ is positivity-improving. The claim that this eigenvalue is zero, follows then immediately from the relation $(1, T_t \zeta^\chi) = (1, \zeta^\chi)$ and the spectral mapping theorem for generators of quasi-compact semigroups; see, e.g., Chapter 5, Theorem 4.7 in [10]. Finally, the uniformity in $\chi$ of the bound on $\|\zeta^\chi\|$ follows from (5.8), applied to $f = \zeta^\chi$, since

$$\|\zeta^\chi\| \leq \|(-\chi \cdot \nabla + L)^{-1}\| G \| \zeta^\chi\| \leq (1/a_0) \sup_k \| r(\cdot, k') \| \|\zeta^\chi\|_k,$$  \hfill where $\|\zeta^\chi\|_k := \int dk |\zeta^\chi| = (1, \zeta^\chi) = 1$, and we have used the explicit form of $G$.

\hfill \Box

5.3. Refined spectral analysis of $M^\chi = M^{0,\chi}$. In this subsection, we refine the conclusions of Lemma 5.2. The main difficulty we face is that the operator $M^\chi$ is not analytic (in any reasonable sense) in the parameter $\chi$, and hence, a priori, perturbation theory does not apply to the isolated eigenvalue at 0 and the corresponding eigenvector. This difficulty is overcome in the next lemma that shows that the resolvent and the spectrum of $M^\chi$ can be controlled in terms of $M^{0,\chi}$, for $\chi$ sufficiently small. Note that this would be obvious if $M^\chi$ were analytic in $\chi$. Afterwards, we also comment on the case of large $\chi$.

Lemma 5.3. There are constants $C, C'$ such that

$$\left| \frac{1}{z - M^\chi} \right| \leq \frac{C}{|\text{Re}z + g_M + O(\chi)|} + \frac{C'}{|z|},$$  \hfill for $\text{Re}z > -g_M + O(\chi)$,  \hfill (5.10)

as $\chi \to 0$, where $g_M := \text{dist}(i\mathbb{R}, \sigma(M^0)) \setminus \{0\}$.

In the proof of this lemma we make use of the transformation $B_\varepsilon := e^{\beta \varepsilon \cdot \nabla} B e^{-\beta \varepsilon \cdot \nabla}$, for any operator $B$ acting on $L^2(\mathbb{T}^d)$. Here $\varepsilon$ is the dispersion law of the particle. Since $\varepsilon$ is positive and bounded, it immediately follows that the spectra of $B_\varepsilon$ and $B$ coincide. In particular, we will consider

$$M^\chi_\varepsilon = G_\varepsilon + L + \frac{\beta}{2} (\chi \cdot \nabla \varepsilon) - \chi \cdot \nabla.$$  \hfill (5.11)
This transformation is useful, because the rates satisfy the identity
\[ r(k, k') = r(k', k) e^{-\beta(\zeta(k') - \zeta(k))}, \]
where \( \beta \) is the inverse temperature of the reservoirs. This identity is known as the detailed balance condition. It is a consequence of the KMS condition for the reservoirs and the time-reversal symmetry; (recall the discussion following Assumption B in Section 5.1). Equation (5.14) implies that \( G_z \) and \( G_z + L + \frac{\chi}{\zeta} (\chi \cdot \nabla \varepsilon) \) are selfadjoint. But note that \(-\chi \cdot \nabla\) is anti-selfadjoint. By \( P^x \) we denote the spectral projection associated to the eigenvalue \( \mu = 0 \) of \( M^x \), and we set \( P^x := I - P^x \). For \( \chi = 0 \), we infer from the detailed balance condition that \( P^0 = |\zeta^0(\chi)|I \), with \( \zeta^0 \) the ‘Gibbs state’ \( \zeta^0(k) = \frac{1}{(1 + e^{-\beta \varepsilon})} e^{-\beta \varepsilon} \). Finally, we note that \( P^x_\varepsilon \) and \( P^y_\varepsilon \) are orthogonal projections.

Proof of Lemma 5.3. We split
\[ (z - M^x)^{-1} = P^x(z - M^x)^{-1} P^x + \bar{P}^x(z - M^x)^{-1} \bar{P}^x, \]
and we remark that the first term is bounded by \( C \), since the eigenvalue \( \mu = 0 \) is simple and \( \|\zeta\| \) is bounded uniformly in \( \chi \), by Lemma 5.2 v. To deal with the second term, note that the left-eigenvector at \( \mu = 0 \) does not depend on \( \chi \), i.e., \( P^x = |\zeta^0\rangle \langle 1| \), which implies
\[ P^x P^x = P^x, \quad P^x \bar{P}^x = \bar{P}^x, \quad P^x \bar{P}^x = 0, \]
(5.13)
(however, \( \bar{P}^x P^x \neq 0 \), for \( \chi \neq \chi' \)). In particular, we have that
\[ M^x = \bar{P}^0 M^x \bar{P}^0 + \bar{P}^0 M^x P^0, \]
and, by straightforward algebra,
\[ (z - M^x)^{-1} = z^{-1} \bar{P}^0 + \bar{P}^0 (z \bar{P}^0 - \bar{P}^0 M^x \bar{P}^0)^{-1} \bar{P}^0 + \bar{P}^0 (z \bar{P}^0 - \bar{P}^0 M^x \bar{P}^0)^{-1} \bar{P}^0 M^x \bar{P}^0. \]
Using (5.13), this leads to
\[ \bar{P}^x(z - M^x)^{-1} \bar{P}^x = \bar{P}^x(\bar{P}^0(z - M^x) \bar{P}^0)^{-1} \bar{P}^x. \]
We recall the conjugation \( B \rightarrow B_\varepsilon \), introduced above, and write
\[ (\bar{P}^0_\varepsilon(z - M^x_\varepsilon) \bar{P}^0_\varepsilon)^{-1} = (R + iI)^{-1}, \]
where \( R \) is defined by \( R := \bar{P}^0(\text{Re} z - (G + L)_\varepsilon + \frac{\chi}{\zeta} (\chi \cdot \nabla \varepsilon)) \bar{P}^0_\varepsilon \) and \( I := \bar{P}^0(\text{Im} z + i \chi \cdot \nabla) \bar{P}^0_\varepsilon \). Note that \( R \) and \( I \) are selfadjoint operators. By the spectral calculus and the boundedness of \( \chi \cdot \nabla \varepsilon \),
\[ R \geq \text{Re} z - g_M + \mathcal{O}(\chi). \]
Hence, \( R > 0 \) (strictly positive in the sense that \( \inf \sigma(R) > 0 \), for \( \text{Re} z > g_M + \mathcal{O}(\chi) \), and we find that
\[ (R + iI)^{-1} = \frac{1}{\sqrt{R}} \left( 1 + \frac{i}{\sqrt{R}} R \right)^{-1} \frac{1}{\sqrt{R}}. \]
Using the selfadjointness of \( I \), the boundedness of \( e^{\beta \varepsilon} / 2 \) and \( P^x \) (uniform in \( \chi \)), we obtain the bound
\[ \| (z - M^x)^{-1} \bar{P}^x \| \leq C \| g_M + \mathcal{O}(\chi) + \text{Re} z \|^{-1}, \quad \text{Re} z > -g_M + \mathcal{O}(\chi). \]
\[ \square \]

As promised, we now turn to a discussion of the model for large \( \chi \). For simplicity, we restrict our attention to the one-dimensional case, \( d = 1 \), and comment on higher dimensions at the end of this discussion.

Lemma 5.4. Let \( d = 1 \). There are constants \( C, C' \) such that, for sufficiently large \( |\chi| \),
\[ \left\| \frac{1}{z - M^x} - \frac{1}{z} \frac{Q}{|\chi|} \right\| \leq \frac{C}{|\chi|}, \]
(5.14)
for \( z \in \mathbb{C} \), with \( |\text{Im} z| \leq C' \). Here \( Q \) is the orthogonal projection on the space of constant functions on \( T \).
Proof. In $d = 1$, the spectrum of the self-adjoint operator $\chi \cdot X$ is the lattice $\chi \mathbb{Z}$, and each of the eigenvalues corresponds to a one-dimensional eigenspace. The eigenspace corresponding to 0 is the space of constant functions. Hence, we may write

$$M^X = QM^0Q + (\chi \cdot X)Q + W,$$

with $W = QM^0Q + QM^0Q$, and $Q = \mathbb{1} - Q$. As argued previously, $Q(G + L) = 0$ (the constant function is a left eigenvector of $M^X$). Since $Q(\chi \cdot X)Q$ has spectrum in the region $\vert \text{Im } z \vert \geq \vert \chi \vert$ and $W$ is bounded, we obtain the claim of the lemma by a straightforward Neumann series expansion.

In higher dimensions, things are more subtle. If $\chi$ is a multiple of some element in $\mathbb{Z}^d$, then the above lemma is easily generalized by replacing $Q$ with the (infinite-dimensional) projection corresponding to the kernel of $\chi \cdot X$ and the term $\frac{1}{\mathbb{1} - Q} Q \frac{1}{\mathbb{1} - Q} Q$ by $\frac{1}{\mathbb{1} - Q} Q \frac{1}{\mathbb{1} - Q} Q$. In this situation, the spectrum of $\chi \cdot X$ is a lattice and the gap between 0 and the rest of the spectrum increases proportionally to $\vert \chi \vert$. Since the spectral analysis of $QM^0Q$ can be carried out similarly to that of $M^0$, this allows us to control the eigenvalue at 0 by perturbation theory, as $\chi \to \infty$. However, if the line $\chi \mathbb{R}$ does not hit any lattice point $\mathbb{Z}^d$, then the spectrum of $\chi \cdot X$ covers the whole real line and we need more subtle considerations to perform the limit $\chi \to \infty$. This is expected to be manageable; but we do not wish to address this point here.

5.4. Asymptotics of the semigroup $e^{tM^X}$. In this section, we discuss the large time asymptotics of the semigroup generated by $M^{\kappa \cdot X}$. In particular, we show that the diffusion tensor at vanishing external field is positive-definite.

Our interest is in the asymptotic behavior of solutions of the linear Boltzmann equation, i.e., of the probability density $\nu_t(x, k)$ satisfying the evolution equation (1.3). The distribution $\nu_t^0(k) := \int \text{d}x \nu_t(x, k)$ of the particle’s momentum evolves according to the semigroup $e^{tM^{\kappa \cdot X}}$. We recall that this semi-group is quasi-compact (and positivity improving) and that $\mu = 0$ is an isolated eigenvalue of $M^X$, by Lemma 5.2. Hence, we conclude (see, e.g., [10]) that, for any $\nu_{t=0} \in L^2(T^d)$,

$$\nu_t^0 = \hat{\zeta}^x + O(e^{-g_M(x)t}) , \quad t \to \infty ,$$

where $g_M(\chi) = \text{dist}(i\mathbb{R}, \sigma(M^{0 \cdot x}) \setminus \{0\})$. We know that $g_M(\chi) > 0$, for all $\chi$; but only for small $\chi$ we have established uniformity in $\chi$; see Lemma 5.3. Information on the distribution of the particle’s position is obtained from operators on fibers at non-zero $\kappa$, as explained in Section 4. Hence we have to determine the asymptotic behavior of $e^{tM^{\kappa \cdot X}}$, for small $\kappa$. Recall that $M^{\kappa \cdot X} = M^{0 \cdot X} + i\kappa \cdot \nabla_x$. Since $i\kappa \cdot \nabla_x$ is a bounded operator, analytic perturbation theory in $\kappa$ implies that the operator $M^{\kappa \cdot X}$ has an isolated, simple eigenvalue, $u_M(\kappa)$, close to 0, for small $\kappa \in \mathbb{C}^d$. Moreover, by the resolvent bound in Lemma 5.3 $g_M(\kappa, \chi) := \text{dist}(i\mathbb{R}, \sigma(M^{0 \cdot x}) \setminus \{u_M(\kappa)\})$ equals $g_M(\chi) + O(\kappa)$. For small $\kappa$ small enough, the semigroup generated by $M^{\kappa \cdot X}$ is quasi-compact, and we conclude, by similar reasoning as above, that

$$e^{tM^{\kappa \cdot X}} = P^{\kappa \cdot X}_M e^{tu_M(\kappa, \chi)} + O(e^{(u_M(\kappa, \chi) - g_M(\chi))t}) ,$$

with $P^{\kappa \cdot X}_M$ a rank-one operator that is a small perturbation of $\vert \hat{\zeta}^x \vert \chi$.

Following the discussion in Section 4, we know that the asymptotic velocity $v_M(\kappa)$ and diffusion constant $D_M(\chi)$ can be derived from $u_M$, with

$$v_M(\chi) = \frac{\partial}{\partial \kappa} \bigg|_{\kappa = 0} u_M(\kappa, \chi) , \quad D_M(\chi) = -\frac{1}{2} \frac{\partial^2}{\partial \kappa^2} \bigg|_{\kappa = 0} u_M(\kappa, \chi) .$$

These expressions can be computed using the standard Rayleigh-Schrödinger expansion of analytic perturbation theory:

$$u_M(\kappa, \chi) = i(1, \kappa, \nabla \epsilon) \psi^{0 \cdot x}(\kappa, \chi) - (1, \kappa, \nabla \epsilon) S^x(\kappa, \nabla \epsilon) \psi^{0 \cdot x}(\kappa, \chi) + O(\kappa^3) , \quad \kappa \to 0 ,$$

where by $S^x$ we denote the ‘reduced resolvent’ of $M^{0 \cdot X}$ at $z = 0$, i.e., $S^x = (\chi \cdot \nabla + G - L)^{-1}P^x$, with $P^x = 1 - P^X$, $P^X = \{\zeta \chi \} / (\vert \chi \vert)$. Note that $u_M(\kappa, \chi) = u_M(\kappa, \chi)$, so that $v_M(\chi)$ and $D_M(\chi)$ have real entries.

For $\chi = 0$, the detailed balance condition implies that $\chi \zeta^{0 \cdot x}(k) \propto e^{-\beta k}$, hence $v_M(\chi = 0) = 0$. To prove that the diffusion constant at vanishing external field is strictly positive, we use the transformation $B \mapsto B_e$ (as defined in Section 5.3) to find that, for any $a \in \mathbb{R}^d$,

$$a, D_M(\chi = 0)a = -\frac{1}{2} (1, \langle a, \nabla \epsilon \rangle \psi^0(a, \nabla \epsilon) \zeta^{0 \cdot x}) = -\frac{1}{2} \frac{1}{(1, e^{-\beta \kappa})} (e^{-\beta \kappa / 2}, \langle a, \nabla \epsilon \rangle \psi^0(a, \nabla \epsilon) e^{-\beta \kappa / 2}) .$$
By the spectral calculus and Lemma 5.2 we know that \(-S^0\) is strictly positive. Moreover, by Assumption A, 
\((a, \nabla \epsilon)\) does not vanish identically. It follows that \(D_M(x) = 0\) is a positive-definite matrix.

For \(\chi \neq 0\), one can establish smoothness (but not analyticity) of \(u_M(\kappa, \chi)\), \(v_M(\chi)\) and \(D_M(\chi)\) in \(\chi\), using asymptotic perturbation expansions in \(\chi\), for \(\chi\) small enough. This method is outlined in Lemma 6.1 of [6].

By the spectral calculus and using the ideas just presented, it is straightforward to prove that \((\chi, v_M(\chi)) \neq 0\), for \(\chi \neq 0\), and that \(D_M(\chi) > 0\), for small external forces \(\chi\). Moreover, one can also confirm the validity of the Einstein relation within the kinetic theory: \(\frac{\partial}{\partial t}\bigg|_{t=0} v_M(\chi) = \beta D_M(0)\).

Finally, let us turn to the large-\(\chi\) regime. In dimension \(d = 1\), Lemma 5.4 allows us to apply perturbation theory in the parameter \(1/|\chi|\), and we derive easily that both \(v(\chi)\) and \(D_M(\chi)\) vanish as \(1/|\chi|\), for \(|\chi| \to \infty\).

The main gap in our knowledge is for moderate \(|\chi|\): We are not able to prove that \(D_M(\chi) > 0\). The fact that this is difficult using spectral methods should not come as a surprise. In fact, modern approaches to the central limit theorem often use martingale techniques. However, a standard method we are aware of, which one may want to apply to prove the positivity of the diffusion constant and a central limit type theorem, the graded sector condition introduced in [10], does not appear to be applicable here.

6. Results from expansions

The aim of this section is to summarize properties of the effective dynamics \(Z_{[0,t]}\) that can be proven using expansion techniques. We only describe the main ideas and present formal arguments. Mathematically precise arguments are given in [6]. Sections 4 and 5, where elaborate expansion techniques are developed that can also be used to analyze correlation functions.

6.1. Survey of expansions. Here we sketch expansion techniques that are used to reduce the dynamics of the tracer particle. Let \(I \subset \mathbb{R}_+\) be a finite interval. We define the free particle dynamics, \(U^\Lambda_I\), on \(\mathcal{B}_2(\mathcal{H}_S)\), with \(\mathcal{H}_S = \ell^2(\Lambda)\), by

\[
U^\Lambda_I := e^{-i|I|\text{ad}(H_S)}, \quad H_S \equiv H^\Lambda_S = T^\Lambda - \lambda_2 \chi \cdot X^\Lambda,
\]

and the particle-reservoir interaction, \(H_{SR}(t) \equiv H^\Lambda_{SR}(t)\) in the interaction picture, which we may write as a sum over spatially localized terms, by

\[
H_{SR}(t) := e^{itH_A} H_{SR} e^{-itH_R} = \sum_{x \in \Lambda} \mathbb{1}_x \otimes e^{itH_A}(a_x(\phi) + a_x^*(\phi)) e^{itH_R}.
\]

Iterating Duhamel’s formula

\[
e^{it\text{ad}(H_R)} e^{-it\text{ad}(H)} = U_{[0,t]} - i\lambda \int_0^t ds U_{[s,t]} \text{ad}(H_{SR}(s)) e^{is\text{ad}(H_R)} e^{-is\text{ad}(H)},
\]

we find the (Lie-Schwinger-) Dyson series for \(Z_I\):

\[
Z^\Lambda_I(\epsilon) = \sum_{n \geq 0} (-i\lambda)^n \int_{t_n < t_{n-1} < \cdots < t_1 < t_0} dt_1 \cdots dt_n \text{Tr} \left[ U^\Lambda_{[t_0,t_1]} \cdots \text{ad}(H_{SR}(t_n)) \cdots \text{ad}(H_{SR}(t_1)) U^\Lambda_{[t_1,t_0]} \otimes \rho^\Lambda_{SR}[\mathcal{B}_2(\mathcal{H}_S)] \right],
\]

where \(t_0(\epsilon), t_1(\epsilon)\) denote the infimum and supremum of the interval \(I\), respectively. The trace over the reservoir Hilbert space can be evaluated using Wick’s theorem; see (2.47). For this purpose, we introduce the shorthand notations

\[
\mathbb{1}_{x,\zeta} := (\mathbb{1}_x)_\zeta, \quad \Psi_{x,\zeta}(t) := (-i\psi_x(t))_\zeta,
\]

for \(x \in \Lambda, \zeta \in \{1, r\}\) (the left- and right multiplications, \((.,.)_\zeta\), were introduced in (2.1)). We denote by Pair(n), the set of pairings of \(2n\) elements and denotes by \(\varphi_{r,\zeta}\) elements of \(\Lambda^{2n}, \{1, r\}^{2n}\), respectively. In this notation, the formal Dyson series for \(Z_I\) can, using Wick’s theorem, be written as

\[
Z^\Lambda_I = \sum_{n \geq 0} \int_{t_n < t_{n-1} < \cdots < t_2 < t_1} \left( \int_{t_1}^{t_2} dt \right)^n \sum_{x \in \Lambda} \sum_{\pi \in \text{Pair}(n)} \zeta^\Lambda((x, L, \zeta), \pi) \times U^\Lambda_{[t_2, t_1]} \mathbb{1}_{x_2, \zeta_2} U^\Lambda_{[t_1, \cdots, t_2]} \cdots U_{[t_1, x_1]} U^\Lambda_{[t_1, t_1]},
\]

(6.3)
where $\zeta^\Lambda$ denotes a reservoir correlation function given by

$$
\zeta^\Lambda((x,t),\zeta,\pi) := \prod_{(r,s) \in \pi} \lambda^2 h^\Lambda(t_s, t_r, \varsigma_s, \varsigma_r) \delta_{x_r, x_s},
$$

(6.4)

with

$$
h^\Lambda(u,v,\varsigma,\varsigma') :=
\begin{cases}
-\varphi^\Lambda(u-v), & \text{if } \varsigma = 1, \varsigma' = 1, \\
-\varphi^\Lambda(v-u), & \text{if } \varsigma = r, \varsigma' = r, \\
\varphi^\Lambda(v-u), & \text{if } \varsigma = r, \varsigma' = 1, \\
\varphi^\Lambda(u-v), & \text{if } \varsigma = 1, \varsigma' = r.
\end{cases}
$$

(6.5)

The reservoir two-point correlation function $\varphi^\Lambda$ has been defined in (3.3).

In the next step, we decompose the expansion (6.3) into a sum/integral over irreducible pairings: A pairing $\pi$ is irreducible whenever for any $m = 1, \ldots, 2n - 1$, there is a pair $(r, s) \in \pi$ such that $s \leq m < r$.

To that end, we define an operator $V^\Lambda$, by

$$
V^\Lambda := \sum_{n=0}^{\infty} \int_{t_n(I) = t_1, \ldots, t_{2n-1}(I) = t_n(I)} \left(2^n - 1\right) \sum_{\pi \in \text{Pair}(\Lambda)} \sum_{\tau \text{ irreducible}} \zeta^\Lambda((x,t),\zeta,\pi)
$$

$$
\times \mathbb{1}_{x_2n < x_1} \mathbb{1}_{t_2n - 1 < t_1} \cdots \mathbb{1}_{x_1, x_1} \mathbb{1}_{t_1, t_2} \mathbb{1}_{x_1, x_1},
$$

(6.6)

where the second sum is over irreducible pairings, and we only integrate over $2n - 2$ time coordinates, with $t_1 = t_{-1}(I)$ and $t_{2n} = t_{+1}(I)$ fixed. It is then easy to check that expression (6.3) can be rewritten as

$$
Z_I^\Lambda = \sum_{l \geq 0} \int_{t_{-1}(I) < t_l < \cdots < t_{l+1}(I)} \sum_{\mathbb{1}_{x_{2l+1} < x_{2l}}} \mathbb{1}_{t_{2l+1}(I)} \mathbb{1}_{x_{2l+1}, x_{2l+1}} \mathbb{1}_{t_{2l-1}(I)} \cdots \mathbb{1}_{x_1, x_1} \mathbb{1}_{t_1, t_2} \mathbb{1}_{x_1, x_1, t_{-1}(I), t_1}.
$$

(6.7)

Representation (6.7) is the starting point for the proof of Lemma 3.1. The details of our proof can be found in [5]: Section 5.1. Here we just indicate some of the main ideas underlying it: First, we recall the bound on $U_I$ from (3.2):

$$
|U^\Lambda_I(x, y, x', y')| \leq Ce^{\nu t} \text{ for all } \nu \geq 0.
$$

(6.8)

which holds uniformly in $\Lambda \subseteq \mathbb{Z}^d$. We observe that Assumption A ensures that $\lim_\Lambda U^\Lambda_I = U_I$, in the sense of convergence of kernels. Furthermore, the reservoir two-point correlation functions $\Psi^\Lambda(t)$ and $\Psi(t)$ are bounded uniformly in $t$ and $\Lambda$, and, by Assumption B, we have that $\lim_\Lambda \Psi^\Lambda(t) = \Psi(t)$, uniformly in $t$ on compact subsets of $\mathbb{R}$. One then proves that $V^\Lambda_I$ defines a bounded operator on $B_2(\mathscr{H}_S)$, for any $\Lambda$, including $\Lambda = \mathbb{Z}^d$, and that $\lim_\Lambda V^\Lambda_I = V_I$, in the sense of kernels. It is then straightforward to show that the expansion for $Z^\Lambda_I$ converges absolutely in norm as an operator on $B_2(\mathscr{H}_S)$, the bounds being uniform in $\Lambda$, and that $\lim_\Lambda Z^\Lambda_I$ has a limit $Z_I$, in the sense of convergence of kernels. Moreover, repeated use of (6.5) reveals that

$$
|Z_I^\Lambda(x, y, x', y')| \leq Ce^{-\nu|x-x'|-\nu|y-y'|},
$$

(6.9)

for some $\nu > 0$, where the constant $C$ is independent of $\Lambda \subseteq \mathbb{Z}^d$ and is uniform in $I$, for $I$ contained in compact subsets of $\mathbb{R}$. Thus, for any exponentially localized density matrix $\rho_S \in B_1(\mathscr{H}_S)$ and any finite time $t$, we can define

$$
\langle O(t) \rangle_{\rho_S \otimes \rho_{\mathbb{A}}} = \text{Tr}_S [O Z_{[0,t]} \rho_S] := \lim_\Lambda \text{Tr}_S [O^\Lambda Z^\Lambda_{[0,t]} \rho^\Lambda_S],
$$

(6.10)

where $O^\Lambda = \mathbb{1} \rho_{\mathbb{A}}$, with $O \in \mathbb{A}$ or $\mathbb{X}$. By the same reasoning, one also establishes that the infinite-volume objects are translation-invariant, i.e., $A = \mathcal{T}_- \mathcal{T}_{\mathcal{U} I}$ for $A = Z_{[0,t]}$, $U_I$, $V_I$ and $y \in \mathbb{Z}^d$. This becomes plausible if one recalls that translation-invariance in finite volume was broken only because of the Dirichlet boundary condition.

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The advantage of representation (6.7) over (6.3) is that, after Laplace transformation, it can be resummed: For \( z \in \mathbb{C} \), with \( \text{Re} \, z \) sufficiently large, we define

\[
\mathcal{R}(z) := \int_0^\infty dt \, e^{-zt} \mathcal{Z}_{[0,t]}, \tag{6.11}
\]

In order to identify the leading contributions to \( \mathcal{Z}_{[0,t]} \) and \( \mathcal{R} \), respectively, we define an operator

\[
\mathcal{V}^{(2)} := \sum_{n=2}^{\infty} \int \prod_{i=2}^{2n-1} dt_i \sum_{\lambda \in \text{Pair}(n)} \zeta((\mathcal{A}, \lambda), \pi) \times \prod_{x_{2n}, \varsigma \in \text{Pair}} \mathcal{U}[t_{2n-1}, t_{2n}] \mathcal{U}[t_1, t_2] \mathcal{U}[x_1, \varsigma], \tag{6.12}
\]

and the Laplace transforms

\[
\mathcal{M}(z) := \int_0^\infty dt \, (\mathcal{V}_{[0,t]} - \mathcal{V}^{(2)}_{[0,t]}), \quad \mathcal{R}_{ex}(z) := \int_0^\infty dt \, e^{-zt} \mathcal{V}^{(2)}_{[0,t]} . \tag{6.13}
\]

Recalling the definition of \( \zeta \) in (6.4) and of \( \mathcal{V} \) and \( \mathcal{V}^{(2)} \), we observe that, roughly speaking, \( \mathcal{M}(z) \) contains all contributions to second order in \( \lambda \) from the correlation functions \( \zeta \), but higher orders of \( \lambda \) enter \( \mathcal{M}(z) \) through the field term \( \lambda^2 \chi \cdot X \).

Recall the definition of the operator \( \mathcal{J}_0 \) in (5.13).

**Lemma 6.1.** The operator-valued function \( (z, \theta) \mapsto \mathcal{J}_0 \mathcal{A}(z) \mathcal{J}_0^{-1} \), with \( \mathcal{A} = \mathcal{M}, \mathcal{R}_{ex} \) is analytic in the region \( |\theta| < k_\theta, \text{Re} \, z > -k_z \), for some \( k_z, k_\theta > 0 \), and satisfies the bounds (as \( \lambda \to 0 \))

\[
\sup_{|\theta| < k_\theta, \text{Re} \, z > -k_z} \left\{ \begin{array}{l}
||\mathcal{J}_0 \mathcal{M}(z) \mathcal{J}_0^{-1}|| = O(\lambda^2), \\
||\mathcal{J}_0 \mathcal{R}_{ex}(z) \mathcal{J}_0^{-1}|| = O(\lambda^4).
\end{array} \right. \tag{6.14}
\]

Moreover, for \( \text{Re} \, z > 0 \),

\[
\mathcal{R}(z) = (z - \mathcal{L}_S - \mathcal{M}(z) - \mathcal{R}_{ex}(z))^{-1}, \tag{6.15}
\]

where \( \mathcal{L}_S = \text{ad}(H_S) \) is the Liouvillean of the particle system.

The proof of this Lemma is contained in [6], Section 5.3. Here we just sketch the main ideas. First, one observes that the time integrals in (6.7) are convolutions. Thus, when taking the Laplace transform, it suffices to consider the Laplace transforms of \( \mathcal{U}_{[0,t]} \) and \( \mathcal{V}^{(2)}_{[0,t]} \). The former being given by the resolvent of \( \mathcal{L}_S = \text{ad}(H_S) \), it suffices to consider the operators \( \mathcal{V}_{[0,t]} - \mathcal{V}^{(2)}_{[0,t]} \) and \( \mathcal{V}^{(2)}_{[0,t]} \), respectively. The Laplace transform of \( \mathcal{V}_{[0,t]} - \mathcal{V}^{(2)}_{[0,t]} \) can be computed explicitly (see below), and the claims concerning \( \mathcal{M} \) in Lemma 6.1 can be checked easily. It remains to analyze \( \mathcal{V}^{(2)}_{[0,t]} \), as defined in (6.12). The sum over the spatial coordinates \( x \) in (6.12) can be bounded using the Combesc-Thomason bound for the propagators \( \mathcal{U}_f \) (see (5.22)), at the price of a ‘mild’ exponential growth in time; (it is ‘mild’ because \( \nu \) on the right hand side of (6.2) can be chosen arbitrarily small, as long as it does not depend on \( \lambda \)). To bound the integrals over the time coordinates \( t \), we use the exponential decay in time of the correlation function \( \zeta \). We can cope with the ‘mild’ exponential growth coming from the sum over the spatial coordinates by slightly reducing the decay rate in the exponential decay coming from the correlations \( \zeta \). Then we are left with the problem of analyzing a one-dimensional ‘gas’ of pairings between points confined to an interval of the real line, with integrable (in fact, exponential) decay in the distance between points in each pair. It has been remarked repeatedly that, for such systems, one can integrate over all times and sum over all possible pairings. For more details, we refer to an earlier paper [20] and to the companion paper [6].

From here on, our analysis proceeds as follows: In the next subsection, we explicitly calculate the term \( \mathcal{M}(z) \) and show that, in some sense to be made precise, it is close to the generator \( \mathcal{M} \) of the linear Boltzmann equation discussed in Section 5. In a next step, carried out in Section 7, we show that \( \mathcal{R}(z) \) is comparable to the resolvent of \( \mathcal{M} \) when restricted to fibers corresponding to ‘small’ momenta.
6.2. Calculation of \( \mathcal{M}(z) \). We start with the calculation of \( \mathcal{M} \) defined in (6.13), i.e.,

\[
\mathcal{M}(z) = \lambda^2 \int_0^\infty dt e^{-zt} \sum_{x \in \mathbb{Z}^d} \sum_{s_1, s_2} h(t, s_1, s_2) \| 1_{x, s_1} e^{-it \Delta} 1_{x, s_1} \|.
\]  

(6.16)

We recall the fiber decomposition introduced in Section 4.1. Identifying the fiber spaces \( \mathcal{H}_p \) with \( L^2(\mathbb{T}^d) \), we interpret \( \mathcal{M}(z)_p \) as an operator acting on \( L^2(\mathbb{T}^d) \). The action of the unitary group \( e^{-it \Delta} \), with \( H_S = T - \lambda^2 \chi \cdot X \), is given by

\[
(e^{-it \Delta} f)(k) = f(k - \lambda^2 t \chi), \quad f \in L^2(\mathbb{T}^d),
\]

(6.17)

where \( \Phi(t) := \int_0^t ds \varepsilon(k - \lambda^2 \chi s) \). We split \( \mathcal{M}(z) \) into \( \mathcal{M}(z) = \sum_{\kappa, \chi} \mathcal{M}^{(1)}(z, \kappa, \chi) \), corresponding to the second sum in (6.15). A tedious but straightforward calculation, using (6.17), yields the fiber operators

\[
\begin{align*}
(\mathcal{M}^{(1)}(z)_p f)(k) &= -\lambda^2 \int_0^\infty dt \psi(t) \int_{\mathbb{T}^d} dk' e^{-zt - i\Phi_k + \frac{1}{2} i\Phi_k' - \frac{1}{2} i\Phi_k''}(t) f(k - \lambda^2 t \chi), \\
(\mathcal{M}^{(1)}(z)_p f)(k) &= -\lambda^2 \int_0^\infty dt \psi(-t) \int_{\mathbb{T}^d} dk' e^{-zt + i\Phi_k + \frac{1}{2} i\Phi_k' + \frac{1}{2} i\Phi_k''}(t) f(k - \lambda^2 t \chi), \\
(\mathcal{M}^{(1)}(z)_p f)(k) &= \lambda^2 \int_0^\infty dt \psi(t) \int_{\mathbb{T}^d} dk' e^{-zt + i\Phi_k + \frac{1}{2} i\Phi_k' - \frac{1}{2} i\Phi_k''}(t) f(k' - \lambda^2 t \chi), \\
(\mathcal{M}^{(1)}(z)_p f)(k) &= \lambda^2 \int_0^\infty dt \psi(t) \int_{\mathbb{T}^d} dk' e^{-zt - i\Phi_k + \frac{1}{2} i\Phi_k' + \frac{1}{2} i\Phi_k''}(t) f(k' - \lambda^2 t \chi),
\end{align*}
\]

where \( f \in L^2(\mathbb{T}^d) \).

6.3. Analysis of ladder diagrams. The idea of our analysis is to expand the restrictions to the fibers \( \mathcal{H}_\chi^x \) of the operators \( \mathcal{R}(z) \), i.e., \( (\mathcal{R}(z))_{\chi^x \kappa} \), for small \( z \), around the contributions of order \( \lambda^2 \). Note that, because of the small fiber momentum, \( \lambda^2 \chi \), there are no contributions of order 1. To capture these contributions, we define an operator \( \bar{M} := M + \delta M \) acting on \( L^2(\mathbb{T}^d) \approx \mathcal{H}_\chi^x \) that satisfies

\[
(\mathcal{L}_S + \mathcal{M}(0) + \mathcal{R}_\text{ex}(0))_{\chi^x \kappa} = \lambda^2 \bar{M} + \mathcal{O}(\lambda^4(1 + |\kappa|)),
\]

(6.18)

as \( \lambda \to 0, |\kappa| \to 0 \). Note that we have set the spectral parameter \( z \) in \( \mathcal{M} \) and \( \mathcal{R}_\text{ex} \) to zero. As the notation suggests, \( \bar{M} \) is closely related to \( \bar{M} := M_{\chi^x \kappa} \), the operator introduced and analyzed in Section 5 (note that, for \( \chi = 0, \bar{M} = M \)). The norms below refer to \( \mathcal{B}(L^2(\mathbb{T}^d)) \).

Lemma 6.2. Define \( \bar{M} := M + \delta M \) with

\[
\delta M = \delta M^{X, \lambda} := K(\chi) - K(0), \quad \text{where} \quad K(\chi) = (M(z = 0, \chi))_{\chi^x \kappa}.
\]

The operator \( \delta M \) is bounded, \( \| \delta M^{X, \lambda} \| \leq C \), and relatively bounded w.r.t. to \( \chi \cdot \nabla \) and \( M \), with a bound of order \( \lambda^2 \). More precisely,

\[
\| \delta M f \| \leq \lambda^2 C(\| f \| + \| \chi \cdot \nabla f \|),
\]

(6.19)

for any function \( f \) in the domain of \( \chi \cdot \nabla \). Furthermore, (6.18) holds.

Proof. By inspection of the expressions for \( \mathcal{M} \) in Section 6.2, we get

\[
\|K(\chi) f - K(0) f\| \leq C \int dt |\psi(t)| \left( \min(\|\lambda^2 t \chi\|, 1) \| f \| + \| f(\cdot + \lambda^2 t \chi) - f(\cdot) \| \right),
\]

and (6.19) follows from the exponential decay of \( \psi(t) \). To prove (6.18) we verify that

\[
\lambda^{-2} (\mathcal{L}_S)_{\chi^x \kappa} = i \chi \cdot \nabla e - \chi \cdot \nabla + \mathcal{O}(\lambda^2 \kappa),
\]

\[
\lambda^{-2} (M(z = 0, \chi = 0))_{\chi^x \kappa} = G + L + \mathcal{O}(\lambda^2 \kappa),
\]

by explicit computation. (The l-l and r-r terms in \( \mathcal{M} \) give rise to \( L \), the mixed ones to \( G \).)
We conclude with the remark that, presumably, Lemma 6.2 cannot be improved to
\[ \lim_{\chi \to 0} \| \delta M \| = 0. \] (6.20)
Such an estimate can easily be obtained for the l-r and r-l terms but not for l-l and r-r. To get a feeling for this, let us consider a constant dispersion law, \( \varepsilon(k) = \varepsilon(0) \) (which would actually violate our assumptions, but this should not matter here). Then \( \Phi_k(t) = t \varepsilon(0) \), and, by spectral calculus,
\[ (M^l(0)_p + M^r(0)_p) f(k) = -(2\pi)^d \lambda^2 \int_{-\infty}^{\infty} dt \hat{\psi}(t) f(k - \lambda^2 t \chi) = -(2\pi)^d \lambda^2 \psi(i \chi \cdot \nabla) f(k). \] (6.21)
Obviously \( \lim_{\chi \to 0} \| \psi(i \chi \cdot \nabla) - \psi(0) \| \to 0 \) holds only if the function \( \psi \) is constant.

6.4. Analysis of \( \tilde{M} \). In this subsection, we show that, in a small open neighborhood of the origin, \( \tilde{M} \) has an isolated simple eigenvalue \( \lambda^2 \)-close to that of \( M \). We recall the definition of the gap \( g_M(\chi) \) (see Section 5) and define \( B_r \) to be the disk \( B_r := \{ z \in \mathbb{C} : |z| \leq r \} \).

**Lemma 6.3.** There is a constant \( r > 0, r \propto g_M(0) \), such that, inside the ball \( B_r \), \( M \) and \( \tilde{M} = M + \delta M \) have unique simple eigenvalues \( u_M \equiv u_M(\kappa, \chi) \) and \( u_{\tilde{M}} \equiv u_{\tilde{M}}(\lambda, \kappa, \chi) \), respectively, with \( |u_{\tilde{M}} - u_M| = O(\lambda^2) \). Moreover, for \( z \in B_r \),
\[ \frac{1}{z - \lambda} = \frac{1}{z - u_{\tilde{M}}} P_{\tilde{M}} + O(z^0). \]

**Proof.** For \( M \), this has already been proven at \( \kappa = 0 \) in Section 5 and extended to \( \kappa \neq 0 \) by using perturbation theory of isolated eigenvalues (Section 5.4). Since \( \delta M \) is relatively bounded w.r.t. \( M \), we can again use perturbation theory to prove the claim for \( \tilde{M} \). Estimating the resolvent \( (z - \tilde{M})^{-1} \) by using a Neumann series expansion in \( \delta M \) and applying (6.19), we obtain that
\[ \| \delta M(z - M)^{-1} \| \leq \lambda^2 C \left( 1 + |z| + \|(z - M)^{-1}\| \right), \] (6.22)
which can be used to complete the proof of the lemma. (We refer the reader to [15] for details on the perturbation theory for isolated eigenvalues.)\( \square \)

Although this will not be used in our analysis, it is worthwhile pointing out an important difference between the spectral analysis of \( M \) and that of \( \tilde{M} \): Thanks to the resolvent bound (5.53), we know that the spectrum of \( M \), apart from the eigenvalue \( u \), is bounded away from the real axis. In the nomenclature of Section 5.4, \( g_M(\kappa, \chi) > 0 \). In fact, by Lemma 5.2, item iv, we have an explicit bound, uniform in \( \chi \), on the real part of eigenvalues with large imaginary part. For \( \tilde{M} \), analogous statements do not hold, because \( \delta M \) is not small in norm, but only relative to \( M \); see (6.22). To guarantee that \( z \notin \sigma(\tilde{M}) \), the right-hand side of (6.22) should be strictly smaller than one. Clearly, for any fixed \( \lambda \), this is not the case when \( \text{Im } z \to \pm \infty \).

7. Analysis of \( \mathcal{R}(z) \) Around \( z = 0 \)

In this section, we analytically continue the operator \( \mathcal{R}(z) \) (see (6.11)), a priori only defined for \( \text{Re } z > 0 \), to the region \( \{ z \in \mathbb{C} : |z| < \lambda^2 r \} \), for some \( r > 0 \) and \( \lambda \) sufficiently small. This is accomplished by applying perturbation theory to the fiber operators \( (\mathcal{R}(z))_{\lambda^2 \kappa} \). The guiding idea is that \( (\mathcal{R}(z))_{\lambda^2 \kappa} \) is a small perturbation of \( (z - \lambda^2 M^\kappa)^{-1} \), where \( M^\kappa \) has been analyzed in Section 5. However, it turns out to be more convenient to replace \( M \) by the operator \( \tilde{M} \) introduced in Section 6.3. In Section 7.1, we implement the perturbation theory developed on the basis of Lemma 6.1. The small parameters are the coupling constant \( \lambda \), the (scaled) fiber momentum \( \kappa \) and the field \( \chi \). All these three parameters must be assumed to be sufficiently small throughout our analysis, and we do not repeat this assumption in every step.

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7.1. Perturbation around the kinetic limit. Recall the definition of $\mathcal{R}(z)$ in equation (6.11).

$$\mathcal{R}(z) = \int_0^\infty dt \, e^{-zt} Z(0,t).$$

The main results of this subsection state that the operator $(\mathcal{R}(z))_{\lambda,\kappa}$ has a unique simple pole in a neighborhood of $z = 0$, whose residue, $P \equiv P^{\lambda,\kappa}$, is a rank-one operator (see Lemma 7.3) with the property that, in the fiber indexed by $\kappa = 0$,

$$P^{\lambda,\kappa=0,\chi} = |\zeta\rangle \langle 1|,$$

with $\|\zeta - \zeta_M\|_{L^2(\mathbb{T}^d)} = O(\lambda^2)$,

$$\text{(7.1)}$$

where $\zeta_M$ is the invariant state of the generator, $M$, of the linear Boltzmann evolution; see Section 5. This result is stated in Lemma 7.3. Moreover, we show that $P^{\lambda,\kappa}$ is an analytic function of $\kappa$ and a regular function of $\chi$; see Lemma 7.3. On an intuitive level, this means that the long-time dynamics of $(Z_{0,t})_{\lambda,\kappa}$, is dominated by the linear Boltzmann evolution $e^{tM}$. This statement is formalized in Section 5.1 for $\chi = 0$, and in Section 5.2, for $\chi \neq 0$.

To start with, we define an operator, $S$, acting on $L^2(\mathbb{T}^d)$ by

$$S \equiv S(z,\chi,\lambda,\kappa) := (\mathcal{L}_S + \mathcal{M}(z) + \mathcal{R}_{\text{ex}}(z))_{\lambda,\kappa}. \quad \text{(7.2)}$$

Note that $(\mathcal{R}(z))_{\lambda,\kappa} = (z - S)^{-1}$ (whenever both sides are well-defined), and that $S(z)$ is a closed operator on $L^2(\mathbb{T}^d)$. It is bounded except for the term $\chi \cdot \nabla$ that originates from $\mathcal{L}_S$.

For simplicity, we often abbreviate $S(z,\chi,\lambda,\kappa)$ by, for example, $S(z)$, when we consider the operator-valued function $z \mapsto S(z)$, with the other variables kept fixed. We use similar shorthand notation for $u_M \equiv u_M(\lambda,\kappa,\chi)$, $P \equiv P^{\lambda,\kappa}$, etc. in this and the remaining sections.

Let $\mathcal{D} \subset L^2(\mathbb{T}^d)$ be the dense subspace of real-analytic functions on $\mathbb{T}^d$; cf. Section 5. Recall the constant $k_z$ from Lemma 6.1.

**Lemma 7.1.**

i. $\mathcal{D}$ is a core for $S$ and $SD \subset S$. For all $z \in \mathbb{C}$ satisfying $\Re z \geq -k_z$ and such that $(z - S(z))^{-1}$ exists (i.e., as a bounded operator), we have that $(z - S(z))^{-1} \mathcal{D} \subset \mathcal{D}$.

ii. The differences $S(z) - S(0) = S(\kappa) - S(\kappa = 0)$ are bounded operators, and they are analytic in the variables $\kappa, z$ in the region $\Re z > -k_z$ and $|\kappa| < k_\theta$.

**Proof.** $\mathcal{D}$ is a core for $S$ because it is a core for $\chi \cdot \nabla$. Further, we first establish that, for $\gamma \in \mathbb{C}^d$ sufficiently small,

$$e^{\gamma \cdot \nabla} S e^{-\gamma \cdot \nabla} - S = O(\gamma). \quad \text{(7.3)}$$

Note that for $\theta \in \mathbb{C}^d$ sufficiently small, $J_\theta \mathcal{L}_S \mathcal{L}_S - \mathcal{A} = O(\theta)$, where $\mathcal{A} = \mathcal{L}_S + \mathcal{M}(z) + \mathcal{R}_{\text{ex}}(z)$. This follows from the analyticity of the dispersion law $\mathcal{L}_S$ and from Lemma 6.1 for $\mathcal{M}(z) + \mathcal{R}_{\text{ex}}(z)$. The bound (7.3) is then obtained by restricting to a fiber. A Neumann series expansion of $(z - S(z))^{-1}$, using (7.3), for some sufficiently small $\gamma$ (depending on $z$), yields boundedness of $e^{\gamma \cdot \nabla} (z - S(z))^{-1} e^{-\gamma \cdot \nabla}$. Together with (7.3), this implies part i, after an application of the Paley-Wiener theorem.

To prove part ii, it suffices to observe that the term $-\chi \cdot \nabla$ in $S$ is independent of $z$ and $\kappa$.

Next, we argue that the condition $z \in \sigma(S(z))$ has a unique solution $z^*$ in a neighborhood of $z = 0$:

**Lemma 7.2.** Fix some $r > 0$ sufficiently small, e.g., $r = g_M(0)/4$. Then there is a unique $z = z^*(\lambda,\kappa,\chi)$ in $B_{\lambda^2r}$ such that $z - S(z)$ is not invertible, i.e., such that $z \in \sigma(S(z))$. This unique $z^*$ is an isolated simple eigenvalue of $S(z^*)$.

**Proof.** We write

$$S(z) = \lambda^2 \widetilde{M} + \lambda^2 (\mathcal{M}(z) - \mathcal{M}(0))_{\lambda,\kappa} + (\mathcal{R}_{\text{ex}}(z))_{\lambda,\kappa} =: \lambda^2 \widetilde{M} + A(z, \lambda).$$

Recall that $\widetilde{M}$ has a unique, simple eigenvalue $u_{s,t}$ in $B_r$, for some $r > 0$. Since

$$\|A(z, \lambda)\| \leq C(\lambda^4 + \lambda^2 |z|) \leq C\lambda^4,$$

we have

$$\|A(z, \lambda)\| \leq C\lambda^4,$$
for \( z \in B_{\lambda r} \), an application of spectral perturbation theory shows that \( S(z) \) has a unique simple eigenvalue \( s(z) \) in (to be concrete) the disk \( B_{\lambda r/4} \). This eigenvalue is given by
\[
s(z) = \lambda^2 u_{\tilde{M}} + \text{Tr} \left[ P_{\tilde{M}} A(z, \lambda) \right] + O(\|A(z, \lambda)\|^2),
\]
where \( P_{\tilde{M}} \) is the spectral projection of \( \tilde{M} \) associated with the eigenvalue \( u_{\tilde{M}} \) and the trace is over \( L^2(\mathbb{T}^d) \).

Next, we show that there is a unique \( z \in B_r \) such that \( z \in \sigma(S(z)) \). First, we show uniqueness of \( z \): Assume that there are two solutions \( z_1, z_2 \) of \( z \in \sigma(S(z)) \). Then
\[
|z_1 - z_2| = |s(z_1) - s(z_2)| \leq C|z_1 - z_2| \sup_{z \in B_{\lambda r}} \left\| \frac{\partial}{\partial z} A(z, \lambda) \right\| \leq C\lambda^2 |z_1 - z_2|,
\]
which is a contradiction, for \( \lambda \) small enough. From \( (7.5) \), we also get \( |s(z) - \lambda^2 u_{\tilde{M}}| \leq C|\lambda|^4 \).

Second, we show that there is at least one \( z \in B_r \) such that \( z \in \sigma(S(z)) \): Assume there is no solution of \( z \in \sigma(S(z)) \). By taking \( \kappa \) sufficiently small, we can make sure that \( u_{\tilde{M}} \) lies in the ball \( B_{r/4} \). Then, denoting by \( C \) the positively oriented integration contour
\[
C := \{ z : |z| = \lambda^2 r/2 \}, \tag{7.5}
\]
we note that
\[
\sup_{z \in C} \left\| \frac{1}{z - \lambda^2 M} \right\| \leq C\lambda^{-2}, \quad \sup_{z \in C} \left\| \frac{1}{z - S(z)} \right\| \leq C\lambda^{-2}.
\]
The first bound is stated in Lemma 6.3. The second bound follows from the first one by Neumann series expansion, using the bound on \( A(z, \lambda) \). Next, we note the identity
\[
\frac{1}{z - S(z)} - \frac{1}{z - \lambda^2 M} = \frac{1}{z - S(z)} A(z, \lambda) \frac{1}{z - \lambda^2 M}
\]
and integrate both sides along the contour \( C \). The right-hand side is bounded in norm by \( C \), thus, after integration, it is bounded by \( C\lambda^2 \). On the left-hand side, the contour integral of \( \frac{1}{z - \lambda^2 M} \) yields the spectral projection \( P_{\tilde{M}} \).

We therefore arrive at a contradiction with the assumption that \( \frac{1}{z - S(z)} \) has no singular points. \qed

**Lemma 7.3.** The pole at \( z^* \) of \( z \mapsto (z - S(z))^{-1} \) is simple and its residue, \( P \), is a rank-one operator.

**Proof.** Simplicity of the pole has already been established in the above proof. To get hold of the residue, we expand \( S(z) \) in some neighborhood of \( z^* \):
\[
z - S(z) = z - z^* - (S(z^*) - z^*) - (z - z^*) S_1(z^*) - S_{>1}(z - z^*),
\]
where \( S_1(z^*) = \partial_z S(z^*) \) and \( S_{>1}(z - z^*) = \sum_{n \geq 2} \frac{z - z^*}{n!} (\partial_z)^n S(z^*) \). First, we rewrite
\[
\frac{1}{y - F_0 - yF_1} = \frac{1}{y - (1 - F_1)^{-1} F_0 - 1 - F_1}. \tag{7.6}
\]
From the considerations above, we know that \( F_0 \) has an isolated simple eigenvalue at 0. It follows that 0 is also an eigenvalue of \( (1 - F_1)^{-1} F_0 \). Since \( F_1 = O(\lambda^2) \), \( (1 - F_1)^{-1} - 1 \) is a relatively bounded perturbation of \( F_0 \) with small relative bound and hence perturbation theory ensures that this eigenvalue is again simple and isolated, and we call \( P_F \) the corresponding one-dimensional spectral projector. It follows that \( P_{\lambda} \frac{1}{1 - F_1} \) is the residue of the function \( \frac{1}{y - F_0 - yF_1} \) at \( y = 0 \). Then, we write
\[
\frac{1}{y - F_0 - yF_1 - F_2(y)} = \frac{1}{1 + (y - F_0 - yF_1)^{-1} F_2(y)} \frac{1}{y - F_0 - yF_1}. \tag{7.7}
\]
Since \( \|F_2(y)\| \leq C|y|^2 \) and \( \|(y - F_0 - yF_1)^{-1}\| \leq C|y|^{-1} \), the first factor on the right side of \( (7.7) \) is analytic in a neighborhood of \( y = 0 \) and it reduces to 1 at \( y = 0 \). It follows that the residue at \( y = 0 \) of the function \( \frac{1}{1 - F_1} \) is \( P_F \frac{1}{1 - F_1} \). Since \( P_F \) is one-dimensional, this is a rank-one operator. \qed
To continue, we denote the pole by \( u(\lambda, \kappa, \chi) = z^*(\lambda, \kappa, \chi) \).

In the statement of the next lemma, it is convenient to extend \( S \) to a function of \( \gamma \) defined in a complex neighborhood of \( \gamma = 0 \) by setting \( S(\gamma) := e^{\gamma \nabla} Se^{-\gamma \nabla} \). As argued in the proof of Lemma 7.1, \( \gamma \mapsto S(\gamma) \) is analytic, and the statements of the Lemmas 7.1 and 7.3 remain valid for sufficiently small \( \gamma \). We write
\[
(\mathcal{R}(z))_{\lambda, \kappa} = \frac{1}{z - S(z)} = \frac{1}{z - u} P^\kappa + R^\kappa(z),
\]
with \( z \mapsto R^\kappa(z) \) bounded and analytic in \( B_{\lambda r} \), for some \( r > 0 \). In the following, we often use the shorthand notations \( P \equiv P^\kappa \) and \( R(z) \equiv R^\kappa(z) \).

**Lemma 7.4.** The pole \( u \) and the operators \( P^\kappa, R^\kappa(z) \) are analytic in \( \kappa \) and \( \gamma \).

**Proof.** Residue and pole can be expressed as contour integrals, namely
\[
P = \frac{1}{2\pi i} \int_C dz \frac{1}{z - S(z)}, \quad uP = \frac{1}{2\pi i} \int_C dz \frac{z}{z - S(z)},
\]
where the contour \( C \) is defined in (7.5). Then the analyticity in \( \kappa \) and \( \gamma \) follows from the analyticity of \( S(z) \), as established in Lemma 7.1.

We now summarize our findings and derive some additional algebraic properties of the residue \( P \).

**Lemma 7.5.** For \( \kappa = 0 \), the residue, \( P \), can be written as \( P^{\kappa = 0} = |\zeta_{\lambda, \kappa = 0, \chi}\rangle \langle 1| \), with \( \zeta_{M} \equiv \zeta_{\lambda, \kappa, \chi} \in L^2(\mathbb{T}^d) \) a real-analytic function satisfying
\[
\| \zeta - \zeta_M \| = \mathcal{O}(\lambda^2),
\]
where \( \zeta_M \) is the invariant state of \( M \); see Section 4. For \( \kappa = 0 \), \( \zeta \) is a probability density on \( \mathbb{T}^d \). The function \( u \equiv u(\lambda, \kappa, \chi) \in \mathbb{C} \) satisfies
\[
u(\kappa) = u(-\kappa), \quad |u - \lambda^2 u_M| = \mathcal{O}(\lambda^2).
\]
Moreover, we have that
\[
u(\kappa = 0) = 0, \quad P^{\kappa = 0} R^{\kappa = 0}(z) = 0.
\]

**Proof.** For an arbitrary exponentially localized density matrix \( \rho_S \),
\[
\text{Tr}_S[e^{i\lambda \kappa \cdot \chi} Z_{[0,t]} \rho_S] = \text{Tr}_S[(e^{i\lambda \kappa \cdot \chi})^\dagger (Z_{[0,t]} \rho_S)^\dagger] = \text{Tr}_S[e^{-i\lambda \kappa \cdot X} Z_{[0,t]} \rho_S],
\]
where we have used that \( Z_{[0,t]} \) preserves positivity, in particular, hermiticity. Writing this in terms of fibers, taking the Laplace transforms and comparing the singular parts, we get \( u(-\kappa) = u(\kappa) \). For real \( z \), the operator \( \mathcal{R}(z) = \int dt e^{-itz} Z_t \) preserves positivity, hence \( \mathcal{R}(z) \rho_S \geq 0 \), for an arbitrary positive-definite \( \rho_S \), which implies the positivity of the function \( P^\kappa(\rho_S)_{00} \), where \( 0 \) refers to the zero fiber. Writing \( P^\kappa = |\zeta^\kappa\rangle \langle \zeta^\kappa| \), it then follows that \( \zeta^\kappa \) can be chosen to be positive. Since \( Z_{[0,t]} \) preserves the trace, it follows that \( \text{Tr}_S[\mathcal{R}(z) \rho_S] = 1/z \). Hence, from (7.3) (evaluated in the fiber indexed by \( \kappa = 0 \)),
\[
\frac{1}{z} = \frac{1}{z - u(\kappa = 0)} \langle 1, P^\kappa(\rho_S)_{00} \rangle + \langle 1, R^\kappa(\rho_S)_{00} \rangle = \frac{\lambda}{z} \langle 1, \zeta^\kappa \rangle \langle \zeta^\kappa| (\rho_S)_{00} \rangle + \langle 1, R^\kappa(\rho_S)_{00} \rangle.
\]
Since this identity has to hold for any density matrix \( \rho_S \), and because \( R^\kappa(z) \) is an analytic function, we conclude that \( u(\kappa = 0) = 0, P^\kappa R^\kappa(\rho_S)_{00} = 0 \) and \( \zeta^\kappa = 1 \). Choosing the normalization \( c = 1 \), it follows that \( \int_{\mathbb{T}^d} \zeta^\kappa = 1 \), i.e., \( \zeta^\kappa \) is a probability density. The analyticity of \( \gamma \mapsto e^{\gamma \nabla} P e^{-\gamma \nabla} \) implies boundedness of \( e^{\gamma \nabla} \zeta^\kappa, e^{\gamma \nabla} \zeta^\kappa \), and hence analyticity of \( k \mapsto \zeta^\kappa(k), \zeta^\kappa(k) \). The bounds by terms \( \mathcal{O}(\lambda^2) \) follow immediately from the perturbation theory outlined above.
8. Proof of main results

8.1. The equilibrium regime. If the field $\chi$ vanishes, our results can be strengthened, because $\delta M = 0$. For $\chi = 0$, the function $z \mapsto (\mathcal{R}(z))_{\chi^2}$ has only one pole, $u(\lambda, \kappa, \chi = 0)$, in the region $\Re z > -\lambda^2 g M(\chi, \chi = 0) + O(\lambda^4)$; (cf. the remark following Lemma 7.3). Then the pole $u(\lambda, \kappa, \chi = 0)$ determines the long-time properties of the dynamics. By applying an inverse Laplace transform, one then easily proves the following theorem.

**Theorem 8.1.** [Equilibrium asymptotics] We set $\chi = 0$. Then, for $0 < \lambda$ and $\kappa$ sufficiently small, there is a constant $g > 0$ such that

$$\| (Z_{\{0, t\}})_{\lambda^2} - e^{\lambda^2 t} P^0 \| = O(e^{-g \lambda^2 t}), \quad \text{as } t \to \infty,$$

as operators on $L^2(\mathbb{T}^d)$.

Note that $g$ can be chosen as $g = g M(0)/5$, for example. Also recall that $u(\kappa = 0) = 0$, by Lemma 7.5. For the proof, we refer to Theorem 4.5 of [20].

8.2. Proof of Theorems 3.2 and 3.3.

**Proof of Theorem 3.2.** We first set $\chi = 0$. Let $f$ be a continuous function on $\mathbb{T}^d$, (hence $M_f \in \mathcal{A}(\mathbb{T}^d)$). Then Lemma 4.1 and Theorem 8.1 yield

$$\text{Tr}_S[M_f Z_{\{0, t\}}] = \langle f, \zeta^0 \rangle + O(e^{-g \lambda^2 t}), \quad \text{as } t \to \infty,$$

proving (3.12).

To prove (3.11), we choose $\chi \neq 0$ and define $X := B(L^2(\mathbb{T}^d))$. Then the function $t \mapsto Z_{\{0, t\}}$ is in $L^\infty(\mathbb{R}_+, X)$. A standard Tauberian theorem, see e.g. [17] or [13], states the equivalence of

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \langle Z_{\{0, t\}} \rangle_0 = x, \quad \text{and} \quad \lim_{z \to 0, \Re z \geq 0} z \int_0^\infty dt e^{-zt} \langle Z_{\{0, t\}} \rangle_0 = x,$$

for some $x \in X$. Existence of the first limit yields Theorem 3.2. We show that the second limit exists: By Equation (7.8) we have that, for $|z|$ sufficiently small,

$$z \int_0^\infty dt e^{-zt} \langle Z_{\{0, t\}} \rangle_0 = \frac{z}{z - u(\kappa = 0)} P^0 + z R^0(z). \quad (8.1)$$

Since $u(0) = 0$ and $z \mapsto R^0(z)$ is analytic in a neighborhood of $z = 0$, the limit equals $P^0$. \hfill \Box

**Proof of Theorem 3.3.** We start from the identity

$$\langle X(t) \rangle_{\rho_S \otimes \rho_{\mathbb{R}^d}} = \int_0^t ds \langle V(s) \rangle_{\rho_S \otimes \rho_{\mathbb{R}^d}} + \langle X \rangle_{\rho_S \otimes \rho_{\mathbb{R}^d}},$$

which follows from the definition of the velocity operator (4.10) in finite-volume and can be easily justified, using Lemma 5.1 in the thermodynamic limit, for exponentially localized density matrices $\rho_S$; see Section 5.2 of [6] for details. Hence

$$\frac{1}{t} \langle X(t) \rangle_{\rho_S \otimes \rho_{\mathbb{R}^d}} = \frac{1}{t} \int_0^t ds \langle V(s) \rangle_{\rho_S \otimes \rho_{\mathbb{R}^d}} + \frac{1}{t} \langle X(0) \rangle_{\rho_S \otimes \rho_{\mathbb{R}^d}}$$

$$= \langle \nabla \epsilon, \zeta(\lambda) \rangle + O(t^{-1}), \quad (8.2)$$

as $t \to \infty$. Since $\rho_S$ is exponentially localized, the second term on the right-hand side of the last line vanishes as $t \to \infty$, and we conclude that $v(\chi) = \langle \nabla \epsilon, \zeta(\lambda) \rangle$. The statement that $v(\chi) \neq 0$, for $0 < |\chi|$ sufficiently small, follows from $v_M(\chi) \neq 0$, for $\lambda$ small enough.
We define the diffusion tensor $D \equiv D(\chi)$ by

$$D^{ij} := -\lambda^{-4} \frac{\partial^2}{\partial \kappa^2} \bigg|_{\kappa=0} u(\kappa),$$

where the factor $\lambda^{-4}$ is attributed to the fact that the fiber momentum is $\lambda^2 \kappa$, rather than $\kappa$. From $u(\kappa) = u(-\kappa)$, we conclude that $D$ has real entries. For $\chi \neq 0$ sufficiently small, positive-definiteness follows from the fact that $u(k)$ is a $C^\infty$ function in $\chi$ (see Lemma 6.1 in [2]), and the positive-definiteness of $D(\chi = 0)$. It remains to argue that the above definition in terms of the eigenvalue $u$ is equivalent to the one given in (3.13), namely,

$$D^{ij} = \lim_{T \to \infty} \frac{1}{T^2} \int_0^\infty dt \, e^{-zt} \langle (X^i(t) - v^i(\chi)t) (X^j(t) - v^j(\chi)t) \rangle_{\rho S \otimes \rho S}. $$

This is straightforward, and we omit details. But we do explain an analogous argument relating the expression for the asymptotic velocity $v(\chi)$ in terms of the eigenvalue to the one involving moments of $X$; i.e., we check that

$$v(\chi) = (\nabla \varepsilon, \zeta^{\chi, \lambda}) = i \frac{\partial}{\partial \kappa} \bigg|_{\kappa=0} u(\kappa).$$

It follows from our discussion in Section 4.4 that

$$\text{Tr}_{S}[X \rho_{S,t}] = i \frac{\partial}{\partial \kappa} \bigg|_{\kappa=0} \langle 1, (\rho_{S,t}) \chi \xi \rangle,$$

and we obtain, using $\text{Tr}_{S}[X \rho_{S,t}] = \mathcal{O}(t)$, that

$$z^2 \int_0^\infty dt \, e^{-zt} \text{Tr}_{S}[X \rho_{S,t}] = z^2 \int_0^\infty dt \, e^{-zt} \frac{\partial}{\partial \kappa} \bigg|_{\kappa=0} \langle 1, (\rho_{S,t}) \chi \xi \rangle$$

$$= z^2 \frac{\partial}{\partial \kappa} \bigg|_{\kappa=0} \int_0^\infty dt \, e^{-zt} \langle 1, (\rho_{S,t}) \chi \xi \rangle,$$

for $\text{Re} \, z > 0$. By straightforward manipulations, using (8.2), the limit $z \to 0$ of the left-hand side equals $v(\chi)$. We abbreviate $\frac{\partial}{\partial \kappa} \big|_{\kappa=0} f(\kappa)$ by $f'(0)$. Thanks to Lemma 7.3 it follows that

$$z^2 \int_0^\infty dt \, e^{-zt} \text{Tr}_{S}[X \rho_{S,t}] = iz^2 \langle 1, \frac{1}{z - u(0)} P^0(\rho S)_0 \rangle_t' + iz^2 \langle 1, R^0(z) (\rho S)_0 \rangle_t'$$

$$= \frac{iz^2}{(z - u(0))^2} \langle 1, P^0(\rho S)_0 \rangle + \frac{iz^2}{z - u(0)} \langle 1, P^0(\rho S)_0 \rangle t + \mathcal{O}(z^2)$$

$$= u'(0) + \mathcal{O}(z),$$

as $z \to 0$, where we have used that $P^\kappa, R^\kappa(z)$ are analytic in $z, \kappa$ and that $u(0) = 0, \langle 1, P^0(\rho S)_0 \rangle = 1$. Passing to the limit $z \to 0$, we confirm (8.4).

Note that, with our present methods, we cannot prove the existence of the limit

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T dt \, \frac{1}{T} \langle (X^i(t) - v^i(\chi)t)(X^j(t) - v^j(\chi)t) \rangle_{\rho S \otimes \rho S}. $$

The problem is that Tauberian theorems only hold if one can bound the integrand by a constant, whereas we only have a rough a priori bound, namely $\langle X^2(t) \rangle_{\rho S \otimes \rho S} \leq C|t|^2$. When $\chi \neq 0$ we therefore have to state our results in terms of Laplace transforms of time-dependent quantities in $t$.  

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