Comment on ”Frame-dragging: meaning, myths, and misconceptions” by L. F. O. Costa and J. Natário

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I point out that the authors’ interpretation of their calculations differs from the standard interpretation, described in Sect. 84 of Landau-Lifshitz book. This casts doubt on the authors’ claim that Sagnac effect ”arises also in apparatuses which are fixed relative to the distant stars (i.e., to asymptotically inertial frames); in this case one talks about frame-dragging.”.

I. THE COMMENT.

In a recent work [1], Costa and Natário discussed the notion of frame-dragging in stationary (\(g_{\mu\nu}(x)\)) rotating (\(g_{\mu\nu} \neq 0\)) gravitational fields. The interval of such spacetime can be written in 1 + 3 block-diagonal form as follows:

\[ ds^2 = c^2 \left[ \sqrt{-g_{00}(dt - A_{i}dx^{i})} \right]^2 - dl^2, \]

\[ A_{i} \equiv -\frac{g_{0i}}{c\sqrt{-g_{00}}}, \quad dl^2 \equiv h_{ij}dx^{i}dx^{j}, \quad h_{ij} \equiv g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}, \quad e^{\Phi} \equiv \sqrt{-g_{00}}. \]  

A worldline \(x^{\mu} = (ct, x)\), where \(t \in \mathbb{R}\) while \(x = \text{const}\), is called an observer at rest (or laboratory observer) [1], see the discussion around Eq. (2) in [1].

In Sect. 2.1, after mentioning the Sagnac effect in flat spacetime (see Sect. 89 in [2]), the authors state: ”In a gravitational field, however, it (Sagnac effect) arises also in apparatuses which are fixed relative to the distant stars (i.e., to asymptotically inertial frames); in this case one talks about frame-dragging.”. To confirm the claim, Costa and Natário used the four dimensional geometry of general relativity [1], and considered a thought experiment, that implies the necessity of the following calculation: a photon was emitted by observer at rest at a spatial point \(O\), and arrived at the end to the point \(O\): \(x(0) = x(1)\). Knowing the closed loop \(x(\lambda)\), the task is to calculate the photons’ travel time, measured by \(O\).

To this aim, Costa and Natário separated the coordinate time \(dt\) from Eq. (1) with \(ds^2 = 0\), obtaining \(dt = A_{i}dx^{i} + \frac{dl}{c\sqrt{-g_{00}}}\). Identifying \(dt\) with the time measured by \(O\), they concluded that the travel time is

\[ t = \int_{C} A_{i}dx^{i} + \int_{C} \frac{dl}{c\sqrt{-g_{00}}}. \]

However, here the point is that the coordinate time \(dt\) in (1) does not represent the time measured by the laboratory of observer at rest \(O\). The latter should be found using the Landau-Lifshitz prescription, that associates the time interval and spatial distance to the infinitesimally closed events \(x^{\mu}\) and \(x^{\mu} + dx^{\mu}\). The ”true” time1 is

\[ dt_{p} = \sqrt{-g_{00}(dt - A_{i}dx^{i})}, \]

and is just the square root of first term on r.h.s. of Eq. (1). The spatial distance is equal to \(dl\), that is to the square root of second term. When the events \(x^{\mu}\) and \(x^{\mu} + dx^{\mu}\) represent the emission and absorption of a photon, these definitions together with the propagation law \(ds^2 = 0\) imply: \(v^{2} \equiv \frac{dl^{2}}{dt_{p}^{2}} = c^2\). That is the constancy of the speed of light in a vacuum is implicit in the Landau-Lifshitz prescription, see also [3].

Using (1) with \(ds^2 = 0\), the Landau-Lifshitz prescription (1) implies the travel time

\[ t_{p} = \frac{1}{c} \int_{C} dl, \]

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2 ”True” time is the direct translation of the terminology, used by Landau and Lifshitz in Russian version of [2].

See the discussion around unnumbered equation below the Eq. (84.5), and around Eq. (88.9) in [2]; or Sect. III A in [2].
which is different from [3].
All this casts doubt on the authors’ claim that the Sagnac effect “arises also in apparatuses which are fixed relative to the distant stars (i.e., to asymptotically inertial frames); in this case one talks about frame-dragging.”.

In conclusion, two comments are in order.
1. Even for non rotating metric \( (A_i = 0) \), the intervals \( dt \) and \( dt_p \) still different due to the conversion factor \( \sqrt{-g_{00}} \).

2. In discussing their equations (84.1)-(84.7), Landau and Lifshitz emphasize that their definitions of time and distance do not require any prior synchronization of clocks.

II. NOTE ADDED.

1. In their computations, authors of [1] used \( ds^2 = 0 \), and assumed that photon moves along a prescribed finite path \( C = x(\lambda) \). In other words, by the word ”photon/light signal” the authors mean any null worldline in spacetime [1].

As I understand, this was emphasized also in their Reply [7].

2. The quantity \( dt_{RE} = -Adx + e^{-\phi}dl \), introduced by authors on page 1 of [7], does not zero Eq. (1) of [7]. So, let me reconsider the two-way trip experiment to see, which equations will appear in the authors’ notation \( (x_R = x_E + dx) \).

Observer \( E \) emits a light flash at position \( x_E \), which is reflected by the mirror \( R \) at infinitesimally closed point \( x_R \), returning then to \( E \). Let me collect the “experimental dates” of two-way trip experiment as follows:

\[ t_{pE} = 0, \quad x_E, \quad x_R, \quad 2t_p > 0. \]

\( t_{pE} = 0 \) and \( 2t_p \) represent the proper time instants of emission and absorption of the photon at \( x_E \), measured by the clock of observer at rest at \( x_E \). The task is to restore the four-dimensional coordinates \( x_E^\mu, x_R^\mu \) and \( x_S^\mu \) of the events \( E, R \) and \( S \) \( (S \) is the event of arriving to the point \( x_E \) from the known three-dimensional quantities [6] measured in the laboratory.

In the stationary spacetime [1], we can identify \( x_E^0 = ct_{pE} = 0 \). Then \( x_S^0 = c \sqrt{-g_{00}} \), and we have the events

\[ x_E^\mu = (0, x_E), \quad x_S^\mu = (c \frac{2t_p}{\sqrt{-g_{00}}}, x_E), \quad x_R^\mu = (x_R^0, x_R), \]

and we need to restore the time coordinate \( x_R^0 \) of the event of reflection of the photon at \( x_R \). Each of one-way photons obeys the null-line condition, that according to [1] reads

\[ \left[ \sqrt{-g_{00}}(dx^0 - cA dx) \right]^2 = dl^2, \]

and should be specified for the corresponding differences of coordinates.

Consider the differences:

\[ dx_{ER}^\mu = x_R^\mu - x_E^\mu = (x_R^0, dx), \quad dx_{RS}^\mu = x_R^\mu - x_S^\mu = (x_R^0 - \frac{2ct_p}{\sqrt{-g_{00}}}, dx), \quad dx_{ES}^\mu = x_S^\mu - x_E^\mu = (c \frac{2t_p}{\sqrt{-g_{00}}}, 0). \]

The definition [6] implies the identity \( dx_{ES}^\mu = dx_{ER}^\mu - dx_{RS}^\mu \). In particular, the interval of coordinate time for the complete trip \( E \to R \to S \) is given by the difference

\[ dt_{ES} = dt_{ER} - dt_{RS} = 2e^{\phi} dl. \]

For the trip \( E \to R \), Eq. [5] reads

\[ \left[ \sqrt{-g_{00}}(x_R^0 - cA dx) \right]^2 = dl^2, \]

while for the trip \( R \to S \)

\[ \left[ \sqrt{-g_{00}}(x_R^0 - cA dx - \frac{2ct_p}{\sqrt{-g_{00}}}) \right]^2 = dl^2. \]

\[ ^3 \text{Null-line equation is invariant under the substitution } dx^0 \to -dx^0. \]
Comparing the two equations, we have
\[
[\sqrt{-g_{00}} (x^0_R - cA dx)]^2 = \left[\sqrt{-g_{00}} \left(x^0_R - cA dx - \frac{2ct_p}{\sqrt{-g_{00}}}\right)\right]^2.
\] (13)

Due to the condition \(2t_p > 0\), the only solution of this equation is (this is Eq. 11 in other notation)
\[
x^0_R = \frac{ct_p}{\sqrt{-g_{00}}} + cA dx.
\] (14)

Substitution of the obtained \(x^0_R\) back into the equation (11) or (12) gives
\[(ct_p)^2 = dl^2,
\] (15)
so the coordinate \(x^0_R\) can be presented also in the form
\[
x^0_R = \frac{dl}{\sqrt{-g_{00}}} + cA dx.
\] (16)

Using this expression to compute the difference \(dx^0_{ER} = x^0_R - x^0_E\) of coordinate times for the trip \(E \rightarrow R\), we obtain
\[
dx^0_{ER} = cA dx + \frac{dl}{\sqrt{-g_{00}}},
\] (17)
while \(dx^0_{RS} = x^0_R - x^0_S\) for the trip \(R \rightarrow S\) gives
\[
dx^0_{RS} = cA dx - \frac{dl}{\sqrt{-g_{00}}}.
\] (18)

They coincide with Eqs. (84.5) of the Landau-Lifshitz book. Observe that the term with \(dl\), but not \(cA dx\), changes sign with an inversion of direction in these notations.

Comments.
(I) \(t_p\) is one-half of the total proper time \(2t_p\) of the two-way trip. According to Landau-Lifshitz, from Eqs. (10) and (15) the observer \(E\) can interpret \(t_p\) as the time of one-way trip. Assuming this, the one-way photons passed equal distances at equal time \(t_p\), with the speed equal to the speed of light \(c\). By this way, with a pair of events separated by \((dx^0 = cdt, dx)\), they associated the time interval \(dt_p\), that should be computed according to Eq. (11).

(II) To give an interpretation of \(t_p\) in terms of the four-dimensional geometry, Landau and Lifshitz introduced the event \(x^\mu_I = (ct_p, \sqrt{-g_{00}}, x_E)\) on the worldline of observer \(E\), that he will consider as the event simultaneous with \(x^\mu_R\). This was used by Landau and Lifshitz in their book [2] to discuss the Sagnac effect in Minkowski space, see Sect. 89 entitled ”Rotation”.

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