Flavor Doubling and
the Nature of Asymptopia

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ABSTRACT

We consider the possibility that QCD with \( N \) flavors has a useful low-
energy description with \( 2N \) flavors. Specifically, we investigate a free the-
yory of \( 2N \) quarks. Although the free theory is \( U(N)_L \times U(N)_R \) invariant,
it admits a larger \( U(2N) \) invariance. However, when the axial anomaly
is accounted for in the effective theory by a ’t Hooft interac-
tion, only \( SU(N)_L \times SU(N)_R \times U(1)_B \subset U(2N) \) survives. There is however a residual
discrete symmetry that is not a symmetry of the QCD lagrangian. This \( S_2 \)
subgroup of \( U(2N) \) has many interesting properties. For instance, when
explicit chiral symmetry breaking effects are present, \( S_2 \) is broken unless
\( \bar{\theta} = 0 \) or \( \pi \). By expressing the free theory on the light-front, we show that
flavor doubling implies several superconvergence relations in pion-hadron
scattering. Implicit in the \( 2N \)-flavor effective theory is a Regge trajec-
tory with vacuum quantum numbers and unit intercept whose behavior
is constrained by \( S_2 \). In particular, \( S_2 \) implies that forward pion-hadron
scattering becomes purely elastic at high-energies, in good agreement with
experiment.

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1. Introduction

The hadron spectrum clearly exhibits regularity that is not explained by symmetries of the QCD lagrangian. For instance, to good approximation observed mesons and baryons fall on linear Regge trajectories with a universal slope parameter \([1]\). Furthermore, hadrons of different character exhibit universal mass-squared splittings with remarkable accuracy \([2]\). While QCD remains intractable at low energies one might hope to find an effective description which exhibits new regularity. One possibility investigated by many is that QCD is in some sense equivalent to a string theory. Presumably the world-sheet symmetries of an effective string description could explain features of hadron phenomenology not addressed by symmetries of the QCD lagrangian \([3]\). Although a consistent stringy description has not been found, there is one property of string models which certainly plays a fundamental and yet ill understood role in hadron physics: the pattern of asymptotic behavior of scattering amplitudes.

Asymptotic behavior in forward pion-hadron scattering is of special interest. The Goldstone nature of the pion allows a precise identification between Regge asymptotic behavior and algebraic constraints on the hadronic mass-squared matrix \([4]\). In particular, superconvergent sum rules in pion-hadron scattering have a one-to-one correspondence with symmetry properties of the hadronic mass-squared matrix. These algebraic constraints have always appeared mysterious, in part because they seem to have no place in the naive quark model. In this paper, we will show that the mass-squared matrix constraints and therefore the pattern of asymptotic behavior which they imply, can be understood very simply as consequences of flavor doubling.

One might object that such an underlying structure can have nothing to do with the real world since the number of flavors is known. We immediately reassure the reader that we are not suggesting a modification of QCD, which in this paper has \(N\) quark flavors. Although one can count the number of QCD flavors and colors in the low-energy theory through the effects of anomalies, it is important to realize that as a matter of principle it is not possible to measure the number of effective flavors inside a hadron at low energies, since quarks are not gauge invariant objects. In practice, one infers the number of effective flavors by observing global symmetries and their corresponding conserved currents in low-energy measurements. Therefore, in principle, there is no reason why QCD with \(N\) flavors cannot have an effective low-energy description with \(2N\) flavors. What it is important to ask is whether it is possible to double the number of flavors without introducing new conserved currents, since such currents would almost certainly be in violent disagreement with experiment. We will show that doubling the number of flavors introduces no new conserved currents when the axial anomaly is taken into account in the effective theory.
However, we will find that one cannot avoid a residual discrete symmetry which is certainly *not* a symmetry of the QCD lagrangian. Surprisingly, this discrete symmetry has distinct consequences that are in agreement with experiment. For instance, there can be no P and CP violation in the low-energy theory if this discrete symmetry is unbroken. Furthermore, a constraint on the hadronic mass-squared matrix, expressable as a superconvergent sum rule in pion-hadron scattering, is a direct consequence of the discrete symmetry. This is one of the algebraic sum rules alluded to above.

A further interesting property of the 2N-flavor effective theory is that it allows one to move from a current quark picture to a constituent picture with massless and massive constituent quarks and a conserved chiral current in the broken phase. Moreover, some of the results suggested by the constituent quark picture can be verified exactly using chiral symmetry and the new discrete symmetry in special Lorentz frames. Evidently doubling the number of flavors tells something about aspects of low-energy phenomenology which are mysterious from the point of view of the QCD lagrangian. At the end we will entertain some speculations on why this is the case.

This paper is organized as follows. In section 2 we review the algebraic sum rules and their relation to Regge asymptotic behavior. The main results of this section are summarized in table 1, which provides our motivation for doubling the number of flavors. In section 3 we introduce a simple free-field model with 2N flavors and discuss its symmetries. We investigate how this symmetry structure is modified by a ’t Hooft interaction. We then assume dynamical chiral symmetry breaking, and develop a simple constituent quark model. A conserved chiral current arises naturally in the broken phase, leading to the conjecture that Goldstone bosons are bound states of massless quarks. In section 4 we express the free theory of 2N flavors on the light-front, and show that the algebraic sum rules of table 1 are fundamental properties of the effective theory. In section 5 we use the symmetries of the 2N-flavor effective theory to find the chiral representations of the mesons, and in turn prove that Goldstone bosons must be bound states of massless constituent quarks as conjectured in section 3. The relation between the quark and meson mass-matrices is discussed in section 6. In section 7 we examine in some detail the phenomenology of the chiral representation involving the pion. We summarize our findings and discuss their possible meaning in section 8.

2. Mysterious Regularity in the Hadron Spectrum

Consider the process $\pi \alpha \rightarrow \pi \beta$, where $\alpha$ and $\beta$ are arbitrary single-hadron states, and $\pi$ is a massless Goldstone boson. Imagine writing down the most general chiral invariant lagrangian involving all possible operators that can contribute to this process. There are an infinite number of operators which couple the initial and final states to all possible
intermediate states in all channels, as well as an infinite number of contact interactions. Here we work in the approximation in which the continuum is left out (the tree graph approximation). The discussion is therefore exact for mesons in the large-$N_c$ limit.

The existence of low-energy theorems, and more generally of chiral perturbation theory, is a simple consequence of the fact that Goldstone bosons couple only through derivative interactions. It is less well known that this property of Goldstone bosons has additional interesting implications. Since the chiral lagrangian contains operators with arbitrary numbers of derivatives, S-matrix elements will have atrocious asymptotic behavior, unless there are intricate cancellations among the various momentum-dependent operators. One can obtain constraints based on the need for such cancellations by considering an expansion in inverse energy in Lorentz frames where all momenta are collinear. We will discuss the necessity of working in special Lorentz frames below. The coefficient in this expansion of zeroth-order in energy has special properties. This coefficient contains the term which is protected by chiral symmetry in the low-energy expansion, as well as contributions from non-Goldstone intermediate particle states, and yet it contains no unknown counterterms. All higher powers of energy contain unknown counterterms and are therefore unconstrained by chirality. As in Ref. 4, here we assume that these higher orders behave no worse at high energies than the zeroth order term. For $N = 2$, the zeroth-order coefficient takes the form

$$C^{(-)}_{\lambda b,\alpha a} \equiv \{i\epsilon_{abc}T_c - [X^{\lambda}_a, X^{\lambda}_b]\}_{\beta\alpha} \quad (1a)$$

$$C^{(+)}_{\lambda b,\alpha a} \equiv \frac{1}{2}\{[X^{\lambda}_b, [X^{\lambda}_a, M^2]] + [X^{\lambda}_a, [X^{\lambda}_b, M^2]]\}_{\beta\alpha} \quad (1b)$$

for the crossing-odd and crossing-even amplitudes, respectively. The roman subscripts are isospin indices, $T_a$ are the isospin matrices, and $M^2$ is the hadronic mass-squared matrix. The helicity, $\lambda$, is a conserved quantum number in the collinear frame, and $X^{\lambda}_a$ is an axial-vector coupling matrix, related to the matrix element of the process $\alpha(p, \lambda) \rightarrow \beta(p', \lambda') + \pi(q, a)$ in any frame in which the momenta are collinear:

$$\mathcal{M}_a(p'\lambda'\beta, p\alpha) = (4f_\pi)^{-1}(m^2_\alpha - m^2_\beta)[X^{\lambda}_a]_{\beta\alpha}\delta_{\lambda\lambda'}. \quad (2)$$

This identification holds both when $\alpha$ is at rest and moving at infinite momentum. The matrix $X^{\lambda}_a$ is independent of the reference frame. Below, in dealing with quarks, we will relate to this language by working in the light-cone frame.

It is important to realize that to this point we have assumed only chirality and the tree graph approximation. We will now assume that the coefficients of Eq. (1) behave no worse at high energies than the full amplitude is expected to behave on the basis of Regge
arguments [4]. Here by “high energies” we mean energies of order the characteristic scale
at which the momentum expansion fails. It is conventional to denote this scale \( \Lambda_\chi \sim m_\rho \).

The crossing-odd amplitude is pure \( I_t = 1 \), which Regge theory suggests is dominated by
the \( \rho \) trajectory, with intercept \( \alpha_1(0) \simeq 0.5 \). \( C^{(-)} \) vanishes if \( \alpha_1(0) < 1 \) and so one obtains
the generalized Adler-Weisberger (A-W) sum rule [4]

\[
[X^\lambda_a, X^\lambda_b]_{\beta\alpha} = i\epsilon_{abc}(T_c)_{\beta\alpha}.
\]  

Together with the defining relations, \([T_a, T_b] = i\epsilon_{abc}T_c \) and \([T_a, X^\lambda_b]_{\beta\alpha} = i\epsilon_{abc}(X^\lambda_c)_{\beta\alpha} \),
Eq. (3) closes the chiral algebra. It follows that for each helicity, \( \lambda \), hadrons fall into
representations of \( SU(2) \times SU(2) \), in spite of the fact that the group is spontaneously
broken [4]. However, \( X^\lambda_a \) is not a true symmetry generator since it does not commute
with the mass-squared matrix, \( \hat{M}^2 \). As we will see, in practice this means that the chiral
representations in the broken phase are reducible.

The crossing-even amplitude has both \( I_t = 0 \) and \( I_t = 2 \). Since no \( I = 2 \) states are
observed in nature, Regge pole theory suggests \( \alpha_2(0) < 0 \). The pomeron trajectory is
expected to dominate the \( I_t = 0 \) channel and so one further expects \( \alpha_0(0) = 1 \). Taken
together these Regge constraints imply that \( C_{ba}^{(0)} \) is proportional to \( \delta_{ba} \) (i.e. pure \( I_t = 0 \)).

It then follows that

\[
[X^\lambda_b, [M^2, X^\lambda_a]]_{\beta\alpha} \propto \delta_{ab}.
\]  

This sum rule implies that the hadronic mass-squared matrix is the sum of a chiral invariant
and the fourth component of a chiral four-vector [4]; that is,

\[
\hat{M}^2 = \hat{M}_0^2(\lambda) + \hat{M}_{(\bar{q}q)}^2(\lambda),
\]  

in an obvious notation. Note that although the mass-squared matrix is helicity independent, in principle the two parts of the mass-squared matrix can be separately helicity dependent. When \( \hat{M}_{(\bar{q}q)}^2(\lambda) \) vanishes, the mass matrix, \( \hat{M}^2 \), commutes with \( X^\lambda_a \), and hadrons of a given mass form complete chiral multiplets. One important consequence of
Eq. (3) and Eq. (4) is that hadrons which fall into an irreducible representation must be
degenerate [4,5]. A particularly striking consequence of these algebraic sum rules is the equation

\[
C^{(+)}_{\beta b,\alpha a} = -\delta_{ab}[M_{(\bar{q}q)}^2(\lambda)]_{\beta\alpha},
\]  

which relates the crossing-even forward amplitude at high energies to the symmetry breaking
part of the mass-squared matrix. This is a statement about diffraction: the constancy
Table 1: The equivalence of the first and second columns is exact in the tree-graph approximation (large-$N_c$ for pion-meson scattering). The first row implies that, for each helicity, mass eigenstates fill out generally reducible representations of $SU(2)_L \times SU(2)_R$.

| Hadrons | Regge in $\pi \alpha \to \pi \beta$ |
|---------|----------------------------------|
| $SU(2)_L \times SU(2)_R$ | $\alpha_1(0) < 1$ |
| $\hat{M}^2 = \hat{M}_0^2(\lambda) + \hat{M}_{(q\bar{q})}^2(\lambda)$ | $\alpha_2(0) < 0$, $\alpha_0(0) = 1$ |
| $[\hat{M}_0^2(\lambda), \hat{M}_{(q\bar{q})}^2(\lambda)] = 0$ | $\alpha_0(0) < 0$ $\alpha \neq \beta$ |

of cross sections at high energies. Here diffraction, or rather the existence of a pomeron, is equivalent to the existence of a non-vanishing chiral order parameter [4].

If $\hat{M}_{(q\bar{q})}^2(\lambda)$ is diagonal, scattering becomes purely elastic at high energies. In Regge language this constraint translates to $\alpha_0(0) < 0$ for $\alpha \neq \beta$ — no exchange of trajectories with vacuum quantum numbers when $\alpha \neq \beta$. Algebraically, this superconvergence relation takes the form

$$[\hat{M}_0^2(\lambda), \hat{M}_{(q\bar{q})}^2(\lambda)] = 0.$$  \hspace{1cm} (7)

This is perhaps the most puzzling of the sum rules since it implies a successful phenomenology and yet appears to be a statement completely unrelated to any QCD symmetry. The phenomenological status of this and the other sum rules is discussed in Ref. 4, Ref. 5, Ref. 6 and below. The algebraic sum rules have been generalized to arbitrary numbers of flavors in Ref. 5.

In table 1 we summarize the algebraic constraints and their equivalent statement in Regge theory. As we will see, the Adler-Weisberger sum rule, Eq. (3), is not so mysterious; it is a straightforward consequence of working in special Lorentz frames. On the other hand, the constraints on the mass-squared matrix are puzzling, particularly when viewed in the context of the quark model. We will discuss why this is the case. The primary purpose of this paper is to formulate an underlying description in which the algebraic sum rules are manifest.

3. The q-p Model

3.1 Symmetries

One way of constructing a field theory in which the algebraic sum rules are manifest is to double the usual number of flavors. In this paper, we will investigate this possibility in
Table 2: Symmetries of the $q$-$p$ model. The left column lists the familiar (classical) symmetries of the QCD lagrangian. The right column exhibits the new symmetries arising from flavor doubling. These symmetries continuously transform $q$ and $p$ into one another, as is made clear by the presence of the off-diagonal Pauli matrices, $\sigma_1$ and $\sigma_2$. The $P$ subscript implies permutation of $q$ and $p$.

| $\delta \psi/\psi$ | group          | $\delta \psi/\psi$ | group          |
|-------------------|---------------|-------------------|---------------|
| $-i\alpha$        | $U(1)_B$      | $-i\phi\sigma_1$ | $U(1)_P$      |
| $-i\vec{\alpha} \cdot \vec{T}$ | $SU(N)_V$ | $-i\vec{\phi} \cdot \vec{T}\sigma_1$ | $SU(N)_P$ |
| $-i\beta\sigma_3\gamma_5$ | $U(1)_A$ | $-i\omega\sigma_2\gamma_5$ | $U(1)_{P5}$ |
| $-i\vec{\beta} \cdot \vec{T}\sigma_3\gamma_5$ | $SU(N)_A$ | $-i\vec{\omega} \cdot \vec{T}\sigma_2\gamma_5$ | $SU(N)_{P5}$ |

The matter content of our effective theory consists of $2N$ Dirac fermions assembled into the vectors $q$ and $p$, each in the fundamental representation of $SU(N)$, which transform with respect to $SU(N)_L \times SU(N)_R$ as

$$(N, 1): \quad q_L \rightarrow Lq_L \quad p_R \rightarrow Lp_R$$

$$(1, N): \quad p_L \rightarrow Rp_L \quad q_R \rightarrow Rq_R.$$  

These $2N$ quarks are also assumed to carry a color charge, which will be suppressed. The most general $SU(N)_L \times SU(N)_R$ invariant free lagrangian one can build with $q$ and $p$ is

$$L_0 = \bar{q}i\vec{\phi}q + \bar{p}i\vec{\phi}p - M_0\bar{q}_Lp_R - M'_0\bar{q}_Rp_L + h.c.$$  

Parity conservation implies $M_0 = M'_0$ which gives

$$L_0 = \bar{q}i\vec{\phi}q + \bar{p}i\vec{\phi}p - M_0(\bar{q}p + \bar{p}q).$$  

This lagrangian clearly has symmetries beyond those assumed. In order to see the full symmetry structure it is convenient to define a new field,

$$\Psi = \frac{1}{\sqrt{2}} \left( \begin{array}{c} q + p \\ \gamma_5(q - p) \end{array} \right).$$  

In terms of this new field the lagrangian, $[Eq. (10)]$, takes the familiar form

$$L_0 = \bar{\Psi}i\vec{\phi}\Psi - M_0\bar{\Psi}\Psi.$$  

Since Ψ is a 2N-component vector, the full symmetry of the lagrangian is \( U(2N) \). Note that in terms of this new field, we need never mention chirality. For reasons that will become clear we will show how \( U(2N) \) arises in the original basis. We can assemble \( q \) and \( p \) into the 2N-component vector: \( \psi = (q \ p)^T \). The lagrangian, \( \text{Eq. (10)} \), then takes the form

\[
\mathcal{L}_0 = \bar{\psi}i\not\!\!\not{\partial}\psi - M_0\bar{\psi}\sigma_1\psi
\]

where \( \sigma_1 \) is a Pauli matrix acting in the \( q,p \) space. In this basis it is easy to classify symmetries. In table 2 we list the continuous global symmetries of \( \text{Eq. (13)} \). Note that these are generally non-commuting symmetries. There are four non-chiral symmetries given by \( \delta\psi = -i\Theta\psi \) with \( \Theta = \{\alpha, \bar{\alpha} \cdot \bar{T}, \phi \sigma_1, \bar{\phi} \cdot \bar{T}\sigma_1\} \) which we denote \( U(1)_B, SU(N)_V, U(1)_P \) and \( SU(N)_P \), respectively, and four chiral symmetries given by \( \delta\psi = -i\Theta_5\gamma_5\psi \) with \( \Theta_5 = \{\omega \sigma_2, \bar{\omega} \cdot \bar{T}\sigma_2, \beta \sigma_3, \bar{\beta} \cdot \bar{T}\sigma_3\} \) which we denote \( U(1)_P, SU(N)_P, U(1)_A \) and \( SU(N)_A \), respectively. \( T_a \) is an \( SU(N) \) generator and \( \alpha (\bar{\alpha}), \phi (\bar{\phi}), \omega (\bar{\omega}) \), and \( \beta (\bar{\beta}) \) are arbitrary parameters \((N^2 - 1 \text{ component vectors})\). Of special interest is the \( U(1)_P \) transformation

\[
\psi \rightarrow e^{-i\phi\sigma_1}\psi.
\]

\( U(1)_P \) is a subgroup of \( U(2N) \) which commutes with \( \sigma_1 \). Permutation of \( q \) and \( p \) generates the discrete subgroup of \( U(1)_P \) with \( \phi = \pi/2 \). This subgroup is the group \( S_2 \) of permutations of two objects\(^1\). Hence, \( S_2 \subset U(1)_P \subset U(2N) \).

It is straightforward to obtain the currents and conserved charges associated with the symmetries of table 2. We define the \( U(1) \) charges as \( Q^B, \bar{Q}^i \equiv Q^p, Q^2 \equiv Q^p \) and \( \bar{Q}^3 \equiv Q^A \), and the \( SU(N) \) charges as \( Q^a, G^1 = Q^p, G^2 = Q^p \) and \( G^3 = Q^A \). It is then easy to show that these eight charges satisfy the \( U(2N) \) algebra:

\[
\begin{align*}
[\bar{Q}^i, \bar{Q}^j] &= i\epsilon^{ijk}\bar{Q}^k, \quad [Q^a, Q^b] = i\delta_{ab}Q^c, \quad [\bar{Q}^i, Q^a] = 0 \\
[G^i_a, G^j_b] &= \frac{1}{2N}i\epsilon^{ijk}\delta_{ab}\bar{Q}^k + \frac{i}{2}\epsilon^{ijk}d_{abc}G^k_c + \frac{i}{4}\delta_{ij}f_{abc}Q^V_c \\
[\bar{Q}^i, G^a] &= i\epsilon^{ijk}G^k_a, \quad [Q^a, G^b] = i\delta_{ab}G^i_c \\
[Q^B, Q^B] &= [Q^B, \bar{Q}^i] = [Q^B, Q^a] = [Q^B, G^i_a] = 0,
\end{align*}
\]

as expected. Here \( \epsilon^{ijk} \) is the usual antisymmetric \( SU(2) \) tensor and \( f_{abc} \) and \( d_{abc} \) are generalized Gell-Mann coefficients defined by \([T_a, T_b] = if_{abc}T_c \) and \( \{T_a, T_b\} = \delta_{ab}/N + d_{abc}T_c \), where the \( T_a \) are \( SU(N) \) generators normalized such that \( tr(T_aT_b) = \delta_{ab}/2 \). The

\(^1\) \( S_2 \) is equivalent to \( \mathbb{Z}_2 \).
algebra of $U(2N)$, or $SU(2N) \times U(1)_B$, arises from the embedding $SU(2) \times SU(N)_V \to SU(2N)$. With $\bar{\sigma}_i \equiv \sigma_i/2$, the generators in the defining representation of $SU(2N)$ can be written as $\{\bar{\sigma}_1, \bar{\sigma}_2 \gamma_5, \bar{\sigma}_3 \gamma_5\} \otimes 1, 1 \otimes T_a$ and $\{\bar{\sigma}_1, \bar{\sigma}_2 \gamma_5, \bar{\sigma}_3 \gamma_5\} \otimes T_a$, which are in correspondence with the charges $Q^i, Q'_a$ and $G'_2$, respectively. This is similar to a spin-flavor symmetry. Of course here the $SU(2)$ —which mixes chiral and non-chiral symmetries— is a property of the special basis that we have chosen.

Table 2 makes clear the motivation for working in the $q$-$p$ basis. In this basis $U(2N)$ is decomposed into subgroups which do not mix $q$ and $p$ —identified with classical QCD symmetries— and subgroups which mix $q$ and $p$. Since we want to investigate the relevance of flavor doubling to QCD, we must include an explicit $SU(2)$ violating —$SU(N)_L \times SU(N)_R \times U(1)_B$ preserving— operator and therefore we must selectively break $U(2N)$. Below we will see that explicit $U(1)_A$ breaking effects also break $U(1)_P$, $SU(N)_P$ and $SU(N)_P$ completely, and break $U(1)_P$ to its $S_2$ subgroup.

### 3.2 The Effect of the Axial Anomaly

We now add a simple $U(1)_A$ violating quark interaction to take into account the effect of the axial anomaly. This is a sensible thing to do if we believe that the $2N$-flavor theory is a low-energy effective theory of QCD. Consider the $U(1)_A$ violating, $SU(N)_L \times SU(N)_R \times U(1)_B$ preserving 't Hooft interaction

$$
\mathcal{L}'(\bar{\theta}) = -\kappa \{e^{i\bar{\theta}} \det \bar{\psi}(1 - \sigma_3 \gamma_5)\psi + e^{-i\bar{\theta}} \det \bar{\psi}(1 + \sigma_3 \gamma_5)\psi\},
$$

(16)

where the determinant acts on $SU(N)$ matrices and $\kappa$ is a new parameter of mass dimension $4 - 3N$. We have included a $P$ and CP violating phase, $\bar{\theta}$. Note that $\bar{p}_R p_L + \bar{q}_L q_R = \frac{1}{2} \bar{\psi}(1 - \sigma_3 \gamma_5)\psi$ and $\bar{p}_R p_L + \bar{q}_R q_L = \frac{1}{2} \bar{\psi}(1 + \sigma_3 \gamma_5)\psi$. One can verify that the 't Hooft interaction also breaks $U(1)_P$, $SU(N)_P$ and $SU(N)_P$. $U(1)_P$ is also broken. However, a discrete subgroup survives; the $S_2$ transformation $\psi \to \pm i \sigma_1 \psi$ interchanges $\bar{\psi}(1 - \sigma_3 \gamma_5)\psi$ and $\bar{\psi}(1 + \sigma_3 \gamma_5)\psi$, which is equivalent to the transformation $L \leftrightarrow R$. That is,

$$
S_2 \mathcal{L}'(\bar{\theta}) S_2^{-1} = \mathcal{L}'(\bar{\theta}^{-1}).
$$

(17)

Of course in the absence of explicit chiral symmetry breaking effects we can perform the field redefinition, $\psi \to e^{i\bar{\theta} \sigma_3 \gamma_5/2N} \psi$, which removes $\bar{\theta}$ from the problem, and then $P$, CP and $S_2$ are manifest discrete symmetries of the effective theory.\footnote{Since $\psi \to \pm \sigma_1 \psi$ is also an invariance of the effective theory, strictly speaking the surviving discrete symmetry is $Z_4$ rather than $Z_2 = S_2$.}
If we include an $S_2$ invariant explicit chiral symmetry breaking term, $-m_q\bar{\psi}\psi$, we can again perform a field redefinition which removes $\bar{\theta}$ from the 't Hooft interaction. For small $\bar{\theta}$, $\bar{\psi}\psi \to \bar{\psi}\psi + i(\bar{\theta}/N)\bar{\psi}\sigma_3\gamma_5\psi$, which induces the P, CP and $S_2$ violating operator,

$$-\frac{m_q}{N} \bar{\theta} \bar{\psi}\sigma_3\gamma_5\psi. \quad (18)$$

Therefore, if $S_2$ is unbroken, the anomaly induces no P or CP violation in the $2N$-flavor effective theory. In general, P, CP and $S_2$ will be broken unless $\bar{\theta}=0$ or $\pi$. Of course, if $S_2$ is a global symmetry one does not expect it to be exact and even soft-breaking on the scale of strong interaction physics can grossly violate the experimental bound on $\bar{\theta}$. We will discuss this point below. In any case, given that $\bar{\theta}$ is an unconstrained parameter from the point of view of QCD, one might conclude that the $2N$-flavor effective theory cannot be describing the same low-energy physics as QCD. This might be the case. However, as we will see, $S_2$ makes other distinct predictions that agree well with experiment.

3.3 The Constituent $q$-$p$ Model

In summary, we have shown that because of the axial anomaly, doubling the number of quark flavors does not introduce new conserved currents. With the axial anomaly taken into account, our lagrangian takes the form

$$L = \bar{\psi}i\partial\psi - M_0\bar{\psi}\sigma_1\psi - \kappa\{\text{det} \bar{\psi}(1 - \sigma_3\gamma_5)\psi + \text{det} \bar{\psi}(1 + \sigma_3\gamma_5)\psi\} + \ldots \quad (19)$$

When $\kappa=0$, the lagrangian is $U(2N)$ invariant. When $\kappa \neq 0$, $U(2N)$ is broken explicitly to $SU(N)_L \times SU(N)_R \times U(1)_B$ by the anomaly. The lagrangian is also invariant with respect to the $S_2$ transformation $\psi \to \sigma_1\psi$. The dots refer to other invariant operators. Here we are working in the chiral limit and therefore P, CP and $S_2$ are exact symmetries of the theory.

We can now break $SU(N)_L \times SU(N)_R$ spontaneously to $SU(N)_V$ by assuming the non-vanishing condensates

$$\langle \bar{\psi}\psi \rangle = v_1 \quad \langle \bar{\psi}\sigma_3\psi \rangle = v_2. \quad (20)$$

Condensation of these quark bilinears is consistent with the QCD pattern of chiral symmetry breaking\(^3\). If $v_2 \neq 0$ then $S_2$ is spontaneously broken. We will keep an open mind regarding whether or not this is the case. How do we learn about the physical spectrum? In particular, if the chiral symmetry is truly spontaneously broken, where are the Goldstone bosons? In the $q$-$p$ basis the most general free lagrangian consistent with parity is

\(^3\) Since the 't Hooft interaction is a $2N$-quark operator, the four-flavor effective theory looks like an NJL model; presumably tuning $\kappa$ is one way of generating a condensate.
\[ \mathcal{L}_0 = \bar{q}i\not\!\!\!\!D_q + \bar{p}i\not\!\!\!\!D_p - M_0 (\bar{q}p + \bar{p}q) - M_1 \bar{q}q - M_2 \bar{p}p, \quad (21) \]

where \( M_1 \) and \( M_2 \) are new undetermined parameters. It is important to realize that \( q \) and \( p \) here are not the same as those above. The original \( q \) and \( p \) are current quarks, whereas these are constituent quarks. By inspection it is clear that if \( v_2 = 0 \), \( S_2 \) requires \( M_1 = M_2 \). It is useful to express \( S_2 \) invariance as a general constraint on the mass matrix. We can write the lagrangian, Eq. (21), as

\[ \mathcal{L}_0 = \bar{\psi}i\not\!\!\!\!D\psi - \bar{\psi} \hat{M} \psi, \quad (22) \]

where, in an obvious notation,

\[ \hat{M} = \hat{M}_0 + \hat{M}_{(\bar{q}q)}, \quad (23) \]

with \( \hat{M}_0 = \sigma_1 M_0 \) and \( \hat{M}_{(\bar{q}q)} = (M_1 + M_2)/2 + \sigma_3 (M_1 - M_2)/2 \). It is clear that \( \bar{\psi} \hat{M}_{(\bar{q}q)} \psi \) transforms like the condensates of Eq. (20). The action of \( S_2 \) on the lagrangian is such that

\[ S_2 \mathcal{L} S_2^{-1} - \mathcal{L} \propto \bar{\psi} [\hat{M}_0, \hat{M}_{(\bar{q}q)}] \sigma_1 \psi. \quad (24) \]

So in order that \( S_2 \) be preserved, it is sufficient that

\[ [\hat{M}_0, \hat{M}_{(\bar{q}q)}] = 0, \quad (25) \]

which clearly requires \( M_1 = M_2 \equiv M \). Note the similarity of Eq. (23) and Eq. (25) to the algebraic sum rules of table 1. Before addressing this issue in detail we can learn more about the spectrum in the constituent quark model. For simplicity we will assume throughout the rest of this section that \( v_2 = 0 \) and therefore \( S_2 \) is unbroken.

It is convenient to work in the diagonal basis:

\[ \phi_\pm \equiv \frac{1}{\sqrt{2}} (q \pm p). \quad (26) \]

These states transform as \((N, 1) \oplus (1, N)\), with respect to \( SU(N)_L \times SU(N)_R \) — as deduced from Eq. (8) — and as a doublet with respect to \( S_2 \). In this basis, the lagrangian, Eq. (21), takes the form

\[ \mathcal{L}_0 = \bar{\phi}_+ i\not\!\!\!\!D\phi_+ + \bar{\phi}_- i\not\!\!\!\!D\phi_- - (M + M_0) \bar{\phi}_+ \phi_+ - (M - M_0) \bar{\phi}_- \phi_- . \quad (27) \]

Are the Goldstone bosons implicit in this lagrangian? If so, we expect the presence of a conserved chiral current despite the fact that the chiral symmetry is spontaneously broken.
This is familiar from current algebra. This conserved current arises because it costs the Goldstone bosons no energy to move from one point on the vacuum manifold to another. Consider the axial vector current

\[ A_\mu^a = \bar{\phi} \gamma_\mu \gamma_5 T_a \phi, \]  

arising from the transformation \( \delta \phi / \phi = -i \gamma_5 \vec{\omega} \cdot \vec{T} \) where \( \vec{\omega} \) is an \( N^2 - 1 \) component vector and \( T_a \) is an \( SU(N) \) generator. This chiral transformation is not trivially related to the original chiral symmetry. One easily obtains

\[ \partial_\mu A_\mu^a = 2i(M - M_0) \bar{\phi} \gamma_5 T_a \phi. \]  

If we choose \( M = M_0 \), the \( \phi_- \) quarks are massless and we have a conserved chiral current in the broken phase. This suggests that \( M = M_0 \) is a consequence of Goldstone’s theorem and the Goldstone modes are bound states of massless \( \phi_- \) quarks with interpolating field \( \bar{\phi} \gamma_5 T_a \phi_- \). For now we will assume that this is the case. In a latter section we will use chiral symmetry and \( S_2 \) to prove that the Goldstone bosons must be \( \phi_- \) bound states. Note that the ’t Hooft operator contributes to the right side of Eq. (29). Therefore, there must exist another distinct operator which contributes with equal magnitude and opposite sign. Of course consistency with Goldstone’s theorem requires that the chiral current of Eq. (28) remain conserved in the presence of all interactions.

Finally, with \( M = M_0 \) we find

\[ L_0 = \bar{\phi} + i\partial_\phi \phi + \bar{\phi} - i\partial_\phi \phi - 2M_0 \bar{\phi} - \phi. \]  

In this constituent quark description, there are \( N \) massless and \( N \) massive quarks with a mass gap of \( 2M_0 \). Because of symmetry constraints the number of undetermined parameters has not changed.

It is straightforward to include explicit breaking effects. Consider the \( S_2 \) invariant current quark mass term

\[ L' = -m_q (\bar{\phi} + \phi + \bar{\phi} - \phi) = -m_q \bar{\psi} \psi. \]  

The constituent quark eigenvalues, including quark mass effects, become

\[ M_\pm \equiv M \pm M_0 + m_q. \]  

The constituent quark masses therefore contain a piece that comes from spontaneous chiral symmetry breaking, a piece which transforms as a chiral singlet and a piece that transforms like the current quark masses. We then have
\[ \partial_{\mu} A_{a}^{\mu} = 2im_{q}\bar{\phi}_{-}\gamma_{5}T_{a}\phi_{-}, \] 

(33)

when \( M = M_{0} \), as expected, and the free constituent quark lagrangian becomes

\[ \mathcal{L}_{0} + \mathcal{L}' = \bar{\phi}_{+}i\partial_{-}\phi_{+} + \bar{\phi}_{-}i\partial_{-}\phi_{-} - m_{q}\bar{\phi}_{-}\phi_{-} - (2M_{0} + m_{q})\bar{\phi}_{+}\phi_{+}. \] 

(34)

4. The q-p Model on the Light-Front

We saw in the last section that the properties of the mass matrix in the constituent quark model resemble the algebraic sum rules in pion-hadron scattering discussed in section 2. In this section we will show that the algebraic relations of table 1 are fundamental properties of the \( 2N \)-flavor effective theory. Since the algebraic relations are derived in collinear frames where helicity is conserved, in this section we will investigate the free effective theory in the light-cone frame where helicity is also conserved, in order to make a meaningful comparison. Although we will not discuss the \( \text{'t} \) Hooft interaction on the light-front, \( U(1)_{A} \) and all symmetries which continuously transform \( q \) and \( p \) into each other are assumed to be anomalous. We will see that on the light-front the effective theory with \( 2N \) flavors has special properties not shared by the naive quark model.

The generalized Adler-Weisberger sum rule expresses the fact that for each helicity hadron states fill out generally reducible representations of the full chiral group in the broken phase. The specialization to helicity conserving Lorentz frames can be understood as follows [7,8,9,10]. In the presence of the condensate the vacuum is teeming with quark-antiquark pairs. Therefore, at rest the axial charges do not connect single quark states but rather create quark-antiquark pairs. For this reason, at rest the chiral algebra is not useful for classification purposes; once the chiral symmetry is spontaneously broken there is no Lorentz invariant sense in which hadrons fill out representations of the full chiral group. On the other hand, if a system is moving past the vacuum at infinite momentum there is a sense in which the vacuum, and therefore the condensate, decouples from the system. Since helicity is conserved in the infinite momentum frame, it should not be surprising that—for each helicity— hadrons can be classified into representations of the full chiral group. Mathematically, one can show that on the light-front the axial charge operators conserve the number of quarks and antiquarks separately and count the helicity of all the quarks and antiquarks of a given state [8].

One can show that the \( 2N \)-flavor effective theory saturates the Adler-Weisberger sum rules by expressing the free theory on the light-front. The light-front Hamiltonian density corresponding to the free-field Lagrangian given in Eq. (22) can be written as
\[ \mathcal{H} = i \frac{\sqrt{2}}{4} \int dy^\perp \epsilon(x^\perp - y^\perp) \psi^\dagger_+(y) (\hat{M}^2 - \Delta_\perp) \psi_+(x), \]  

(35)

where \( \psi_+ \) is the dynamical component of \( \psi \) and \( \epsilon(x) \) is the sign function which satisfies \( \partial_x \epsilon(x) = 2 \delta(x) \). Here we assume that \([\hat{M}, T_a] = 0\). Since the light-front vector current, \( \tilde{J}_a^V \), is trivially conserved, the Hamiltonian is \( SU(N)_L \times SU(N)_R \) invariant if the light-front axial current

\[ \tilde{J}_a^A = J_a^A + i \frac{1}{4} \bar{\psi} \gamma_5 T_a \sigma_3 \{ \hat{M}, \sigma_3 \} \int dy^\perp \epsilon(x^\perp - y^\perp) \gamma^+ \psi_+(y) \]  

(36)

is conserved. It is straightforward to find

\[ \partial_\mu \tilde{J}_a^A = \epsilon \frac{1}{4} \bar{\psi} \gamma_5 [\hat{M}^2, \sigma_3] T_a \int dy^\perp \epsilon(x^\perp - y^\perp) \gamma^+ \psi_+(y). \]  

(37)

In the chiral invariant theory we have \( \hat{M} = M_0 \sigma_1 \) which gives \( \hat{M}^2 = M_0^2 \). If we express the Hamiltonian in the \( q-p \) variables we see that it is diagonal in \( q \) and \( p \).

Consider \( N = 2 \) QCD. We have the pattern of symmetry breaking \( SU(2)_L \times SU(2)_R \rightarrow SU(2)_V \). On the light-front this breaking manifests itself through explicit breaking operators in the Hamiltonian. In the broken phase, the axial charge operator is related to the amplitude for the absorption and emission of pions \([4,8,10]\). Focusing on the two flavor case, we can write

\[ \[ X_a^\lambda \]_{\beta \alpha} \delta_{\lambda \lambda} = \langle \beta, \lambda | \tilde{Q}_a^A | \alpha, \lambda' \rangle \]  

(38)

\[ (T_a)_{\beta \alpha} = \langle \beta, \lambda | \tilde{Q}_a^V | \alpha, \lambda' \rangle, \]

where \( \tilde{Q}_a^V \) and \( \tilde{Q}_a^A \) are the light-front vector and axial-vector charges, respectively, \( X_a^\lambda \) is defined in Eq. (2), and we have made use of helicity conservation. Taking matrix elements of the light-front charge algebra and inserting a complete set of states yields

\[ \[ X_a^\lambda, X_b^\lambda \]_{\beta \alpha} = i \epsilon_{abc} (T_c)_{\beta \alpha}, \]  

(39)

which is of course the generalized Adler-Weisberger sum rule. This result is not surprising since on the light-front the four-flavor effective theory is simply two copies of the free quark model, each of which satisfies the sum rules \([8]\). However, the manifestation of chiral symmetry breaking in the two pictures is fundamentally different.

In the naive quark model, quark mass effects do not break chiral symmetry on the light-front \([7,8]\). On the other hand, in the \( 2N \)-flavor effective theory, quark mass terms break chiral symmetry both at rest and on the light-front. This is easy to see by performing a simple spurion analysis. We can assign mass terms the \( SU(N)_L \times SU(N)_R \) transformation properties.
\[ M_0 \to M_0 \]
\[ \Gamma \to L\Gamma R^\dagger = R\Gamma L^\dagger, \]

where \( \Gamma = \{ m_0, M_1, M_2 \} \), and build all invariant light-front bilinears. Out of \( M_0 \) alone we can construct \( q_+^\dagger M_0^2 q_+ \) and \( p_+^\dagger M_0^2 p_+ \), which of course appear in Eq. (35) with \( \hat{M}^2 = M_0^2 \). If we allow a non-zero \( \Gamma \), we have the additional chiral invariant operators \( q_+^\dagger \Gamma^2 q_+ \) and \( p_+^\dagger \Gamma^2 p_+ \), as well as the operators \( q_+^\dagger \Gamma M_0 p_+ \) and \( p_+^\dagger \Gamma M_0 q_+ \) which break chiral symmetry explicitly. With \( N = 2 \) these operators transform like the fourth component of a chiral four-vector. If we include explicit breaking due to current quark masses, our mass-matrix becomes \( \hat{M} = M_0 \sigma_1 + m_q \) and so \( \hat{M}^2 = M_0^2 + m_q^2 + 2m_q M_0 \sigma_1 \). As is clear from the spurion analysis and Eq. (37), the piece proportional to \( \sigma_1 \) breaks chiral symmetry explicitly via the terms \( q_+^\dagger m_q M_0 p_+ \) and \( p_+^\dagger m_q M_0 q_+ \). Therefore in the \( 2N \)-flavor effective theory, quark masses break chiral symmetry on the light-front.

The way in which explicit chiral symmetry breaking enters in the quark model is peculiar. One can prove that on the light-front, one-body operators that break chiral symmetry like the fourth component of a chiral four-vector do not commute with the angular momentum operator [10]. Such operators therefore break chiral symmetry and spatial rotations. As pointed out in Ref. [10], one can get around this constraint by (i) introducing spin-orbit couplings, or (ii) adding mirror fermions. It is very interesting that several of the results we will derive below in the four-flavor effective theory can also be obtained in a two-flavor quark model by introducing operators with spin-orbit couplings [10].

In the light-front \( q-p \) basis we can express the mass-squared matrix as \( \hat{M}^2 = \hat{M}_0^2 + \hat{M}^2_{(qq)} \) where now

\[
\hat{M}_0^2 = 1 \{ M_0^2 + (M_1^2 + M_2^2)/2 \} + \sigma_3 (M_1^2 - M_2^2)/2
\]
\[
\hat{M}^2_{(qq)} = \sigma_1 M_0 (M_1^2 + M_2^2)
\]

with \( M'_i \equiv M_i + m_q \). Note that with \( \hat{M}, \hat{M}_0, \) and \( \hat{M}_{(qq)} \) defined in Eq. (23), \( \hat{M}^2 = (\hat{M})^2 \). However, \( \hat{M}_0^2 \neq (\hat{M}_0)^2 \) and \( \hat{M}^2_{(qq)} \neq (\hat{M}_{(qq)})^2 \). The chiral decomposition of \( \hat{M}^2 \) has been deduced from the spurion transformation properties.

The action of \( S_2 \) on the light-front Hamiltonian is such that

\[ S_2 \mathcal{H} S_2^{-1} - \mathcal{H} \propto \int dy^- \epsilon(x^- - y^-) \psi_+^\dagger(y) [\hat{M}_0^2, \hat{M}^2_{(qq)}] \sigma_1 \psi_+(x), \]

where we have used Eq. (41). Therefore, \( S_2 \) is preserved if \( [\hat{M}_0^2, \hat{M}^2_{(qq)}] = 0 \) which requires \( M_1 = M_2 \), as expected. In this case, the symmetries of the mass-squared matrix can be summarized in the equations.
\[ M^2 = \hat{M}_0^2 + \hat{M}_{(\bar{q}q)}^2 \]  
\[ [\hat{M}_0^2, \hat{M}_{(\bar{q}q)}^2] = 0. \]

Together with Eq. (39), these relations are familiar from table 1 as algebraic sum rules in forward pion-hadron scattering. Of course, here we have studied the symmetry properties of the quark mass-squared matrix whereas the sum rules of table 1 are for the hadronic mass-squared matrix. However, since here the algebraic sum rules are all statements of symmetry, they should hold also for the quark bound states. We will see that this is the case. As shown in section 2, Eq. (43a) implies the existence of a Regge trajectory with vacuum quantum numbers and unit intercept; i.e. a pomeron. Here \( \hat{M}_{(\bar{q}q)}^2 \) is independent of helicity. Therefore, Eq. (6) implies that pion-hadron scattering in the effective theory is helicity-independent at high-energies. Evidently, \( S_2 \) constrains the pomeron trajectory since Eq. (43b) implies that forward pion-hadron scattering becomes purely elastic at high energies.

We have now found two low-energy predictions of unbroken \( S_2 \) invariance: the absence of P and CP violation and the absence of inelastic diffraction, both of which are consistent with nature and yet neither of which is explained by QCD. We can now use \( S_2 \) to learn more about hadron spectra in the effective theory.

5. The Relevance of the Permutation Group

5.1 Symmetry Argument

In this section we will discuss what can be learned about meson states in the \( 2N \)-flavor effective theory purely from symmetry considerations. Our symmetry breaking pattern is \( G \to H \) where \( G = SU(N)_L \times SU(N)_R \) and \( H = SU(N)_V \). In the broken phase, the physical hadron states \( P_i \) fill out irreducible representations of \( H \). Like the constituents \( \phi_+ \) and \( \phi_- \)—for each helicity—the \( P_i \) fall into generally reducible representations of \( G \). That is, the \( P_i \) can be expressed as linear combinations of states \( B_i \) which are in irreducible representations of \( G \). A given reducible multiplet then takes the general form:

\[ |P_1\rangle = \sum_{k=1}^{n} u_{1k} |B_k\rangle \quad \cdots \quad |P_n\rangle = \sum_{k=1}^{n} u_{nk} |B_k\rangle \]  

(44)

where the \( u \)'s are mixing angles. Since the mixing angles are not fixed by \( G \) and \( n \) is arbitrary, \textit{a priori} these representations can be very complicated. Say we restrict ourselves to a subset of states within a given chiral multiplet that carry the same \( H \) charge. When the underlying theory is \( S_2 \) invariant, the constituents \( \phi_+ \) and \( \phi_- \) transform as an \( S_2 \) doublet.
Therefore, we can require that the physical states of a given $H$ charge be invariant with respect to arbitrary $S_2$ transformations:

$$
\begin{pmatrix}
B_i \\
B_j
\end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B_i \\
B_j
\end{pmatrix} \quad \forall \ i,j.
$$

(45)

Although this might seem ad hoc, we will see below that this transformation can be directly related to permutations of the composites. Invariant physical states must be either completely symmetric or completely antisymmetric with respect to this transformation. There is a single completely symmetric state for any $n$. However, there is a completely antisymmetric state only for $n=2$. There is therefore a single solution consistent with the permutation symmetry:

$$
|P_1\rangle = \frac{1}{\sqrt{2}} \{ |B_1\rangle + |B_2\rangle \} \quad |P_2\rangle = \frac{1}{\sqrt{2}} \{ |B_1\rangle - |B_2\rangle \}.
$$

(46)

This has been a naive way of finding that the only non-trivial representation of $S_2$ is a doublet consisting of a symmetric state and an antisymmetric state. Hence, in general, we expect that a physical hadron state will be either in an irreducible chiral representation or in a reducible representation of the form Eq. (46), where the states of definite chirality contribute with equal weight. This multiplet structure has an immediate consequence. Mass-squared splitting between the physical states comes about if the matrix element $\langle B_1 | \hat{M}^2 | B_2 \rangle$ is non-vanishing. This is so if $\hat{M}^2$ contains a piece which mixes different $G$ representations. We know that the full mass-squared matrix can be written $\hat{M}^2 = \hat{M}_0^2 + \hat{M}_{(qq)}^2$, where $\hat{M}_0^2$ transforms like a $G$ singlet, and $\hat{M}_{(qq)}^2$ breaks chiral symmetry and therefore mixes different $G$ representations. One can then readily check that Eq. (46) implies $\langle P_1 | \hat{M}_0^2 | P_2 \rangle = \langle P_1 | \hat{M}_{(qq)}^2 | P_2 \rangle = 0$, which immediately gives Eq. (43b), as expected.

5.2 \textit{N=2 Meson Multiplet Structure}

Now we apply our result to the two flavor case. We have the pattern of symmetry breaking $SU(2)_L \times SU(2)_R \rightarrow SU(2)_V$. Therefore, we expect physical hadron states to be states of definite isospin and— for each helicity— to fill out generally reducible representations of $SU(2)_L \times SU(2)_R$. All mass-squared splittings transform in the $(2, 2)$ (four-vector) representation. The quarks transform as $(1, 2)$ or $(2, 1)$ with respect to $SU(2)_L \times SU(2)_R$. In what follows, $\mu^2$ and $\delta$ represent generic elements of the mass-squared matrix which transform as $(1, 1)$ and $(2, 2)$, respectively; that is, $\mu^2 \in \hat{M}_0^2$ and $\delta \in \hat{M}_{(qq)}^2$. Mesons states carry isospin 0 and 1 and therefore transform as $(2, 2)$, $(3, 1)$, $(1, 3)$ and $(1, 1)$. One can easily check that the only products of these states that contain four-vectors are $(2, 2) \otimes (3, 1)$, $(2, 2) \otimes (1, 3)$, and $(2, 2) \otimes (1, 1)$. 
Physical meson states are states of definite parity and isospin and are therefore, without loss of generality, can be considered combinations of the isovectors $|2, 2\rangle_a, |1, 3\rangle_a - |3, 1\rangle_a \equiv \sqrt{2}|V\rangle_a$ and $|1, 3\rangle_a + |3, 1\rangle_a \equiv \sqrt{2}|A\rangle_a$, and the isoscalars $|2, 2\rangle_4$ and $|1, 1\rangle$. The subscripts are isospin indices. Mesons can fall into irreducible representations. However all states within an irreducible representation must be degenerate. Charge conjugation leaves $(2, 2)$ and $(1, 1)$ unchanged and interchanges $(1, 3)$ and $(3, 1)$. Therefore only $|V\rangle_a$ changes sign under charge conjugation. The two simplest solutions consistent with the multiplet structure implied by $S_2$ are:

**Case a:** $(2, 2) \otimes (3, 1)$ and $(2, 2) \otimes (1, 3)$

\begin{align*}
|I\rangle_a &= \frac{1}{\sqrt{2}}\{|2, 2\rangle_a - |A\rangle_a\} \quad M_I^2 = \mu^2 - \delta \\
|II\rangle_a &= \frac{1}{\sqrt{2}}\{|2, 2\rangle_a + |A\rangle_a\} \quad M_{II}^2 = \mu^2 + \delta \\
|III\rangle = |2, 2\rangle_4 & \quad |IV\rangle_a = |V\rangle_a \quad M_{III}^2 = M_{IV}^2 = \mu^2 \\
M_I^2 + M_{II}^2 &= 2M_{IV}^2 = 2M_{III}^2. 
\end{align*}

States $|I\rangle, |II\rangle$ and $|III\rangle$ have charge conjugation sign $\pm \epsilon$, $|IV\rangle$ has sign $\mp \epsilon$. States $|I\rangle$ and $|II\rangle$ form an $S_2$ doublet. In the right column we exhibit the mass relations implied by the representation content. The lowest lying member of this quartet must be an isovector.

**Case b:** $(2, 2) \otimes (1, 1)$

\begin{align*}
|I\rangle &= \frac{1}{\sqrt{2}}\{|2, 2\rangle_4 - |1, 1\rangle\} \quad M_I^2 = \mu^2 - \delta \\
|II\rangle &= \frac{1}{\sqrt{2}}\{|2, 2\rangle_4 + |1, 1\rangle\} \quad M_{II}^2 = \mu^2 + \delta \\
|III\rangle_a &= |2, 2\rangle_a \\
M_{III}^2 &= \mu^2 \\
M_I^2 + M_{II}^2 &= 2M_{III}^2. 
\end{align*}

These states have the same charge conjugation sign. Again states $|I\rangle$ and $|II\rangle$ form an $S_2$ doublet. The lowest lying member of this triplet must be an isoscalar.

We can also build other reducible representations which are consistent with the $S_2$ symmetry. However, if we assume that our meson states are bound states of two quarks, then the singlet $(1, 1)$ can only arise from the product $(1, 2) \otimes (1, 2) = (1, 1 \oplus 3)$, and therefore the singlet $(1, 1)$ is always grouped with the adjoint $(1, 3)$. This means that the singlet and the adjoint will never occur as distinct states within the same chiral multiplet. Cases (a) and (b) are then the only two possibilities. We will see how this comes about in more detail below when we build meson states out of constituent quarks. This pairing of singlet and adjoint can also be thought a consequence of the large-$N_c$ approximation. Of course in the large-$N_c$ limit one also has singlet-adjoint degeneracy.
This multiplet structure is not new. These results have been derived previously by assuming Regge behavior in pion-hadron scattering and working directly with the algebraic constraints of Table 1. In that derivation the algebraic nature of the sum rules was somewhat mysterious. In the effective theory, these results are consequences of chiral symmetry and the $S_2$ symmetry, with no need for further assumptions. This leads to a powerful statement about the ground state of the $2N$-flavor effective theory: *Since the Goldstone bosons are isovector, they must fall into a representation of type (a), and so must belong to a state of type $|I\rangle$.* We can now see if our identification of the Goldstone bosons as $\phi^-$ bound states is correct.

### 5.3 Direct Construction

Here we construct the meson states directly in the four-flavor effective theory and compare with what we found purely on the basis of symmetry. Consider products of the quark states defined in Eq. (26):

\[
\begin{align*}
\bar{\phi}^- \phi^- &\propto (\bar{q}_1 q_2 + \bar{p}_1 p_2) - (\bar{q}_1 p_2 + \bar{p}_1 q_2) \\
&= (2,2) \ (1, 1 \oplus 3) \oplus (1 \oplus 3, 1) \\
\bar{\phi}^+ \phi^+ &\propto (\bar{q}_1 q_2 + \bar{p}_1 p_2) + (\bar{q}_1 p_2 + \bar{p}_1 q_2) \\
&= (2,2) \ (1, 1 \oplus 3) \oplus (1 \oplus 3, 1) \\
\bar{\phi}^+ \phi^- &\propto (\bar{q}_1 q_2 - \bar{p}_1 p_2) + (\bar{q}_1 p_2 - \bar{q}_1 q_2) \\
&= (2,2) \ (1, 1 \oplus 3) \oplus (1 \oplus 3, 1) \\
\bar{\phi}^- \phi^+ &\propto (\bar{q}_1 q_2 - \bar{p}_1 p_2) - (\bar{q}_1 p_2 - \bar{q}_1 q_2) \\
&= (2,2) \ (1, 1 \oplus 3) \oplus (1 \oplus 3, 1).
\end{align*}
\]

The numerical scripts make the permutation properties clear, and we have used the chiral transformation properties of $q$ and $p$ given in Eq. (8). It is assumed that these are states of definite helicity, parity and isospin. Note that the insertion of additional gamma matrices can only change the parity of the state, or interchange the $(2,2)$ and $(1, 1 \oplus 3) \oplus (1 \oplus 3, 1)$ representations. Up to a phase, these “wavefunctions” are invariant with respect to the independent $S_2$ transformations

\[
\begin{align*}
q_i &\leftrightarrow p_i, \quad q_2 \leftrightarrow p_2 \quad \text{(50a)} \\
q_i &\leftrightarrow p_i, \quad q_i, p_i \text{ fixed } \ i \neq j. \quad \text{(50b)}
\end{align*}
\]

We see explicitly that the product of two $S_2$ doublets gives two $S_2$ doublets; i.e. $2 \otimes 2 = 2 \oplus 2$. However, invariance under charge conjugation necessarily unfolds one of the doublets since $\bar{\phi}^+ \phi^-$ and $\bar{\phi}^- \phi^+$ are not states of definite charge conjugation sign. The composite
wavefunctions of definite charge conjugation sign and their associated chiral representation content are:

\[ |I\rangle \sim \bar{\phi}_1 \phi_2 = \frac{1}{2} (\bar{q}_1 q_2 + \bar{p}_1 p_2) - \frac{1}{2} (\bar{q}_1 p_2 + \bar{p}_1 q_2) \]
\[ (2, 2) \quad (1, 1 \oplus 3) \oplus (1 \oplus 3, 1) \]

\[ |II\rangle \sim \bar{\phi}_-^1 \phi_2^2 = \frac{1}{2} (\bar{q}_1 q_2 + \bar{p}_1 p_2) + \frac{1}{2} (\bar{q}_1 p_2 + \bar{p}_1 q_2) \]
\[ (2, 2) \quad (1, 1 \oplus 3) \oplus (1 \oplus 3, 1) \]

\[ |III\rangle \sim \bar{\phi}_+^1 \phi_2^2 + \bar{\phi}_-^1 \phi_+^2 = \bar{q}_1 q_2 - \bar{p}_1 p_2 \]
\[ (2, 2) \]

\[ |IV\rangle \sim \bar{\phi}_+^1 \phi_2^2 - \bar{\phi}_-^1 \phi_+^2 = \bar{p}_1 q_2 - \bar{q}_1 p_2 \]
\[ (1, 1 \oplus 3) \oplus (1 \oplus 3, 1) \]

where we have used chiral symmetry to identify the wavefunctions with the physical states of the previous section. These states have charge conjugation sign: \( \pm \epsilon \) for \(|I\rangle\), \(|II\rangle\) and \(|III\rangle\), and \( \mp \epsilon \) for \(|IV\rangle\), and are invariant with respect to the permutation

\[ q_1 \leftrightarrow p_1 \quad q_2 \leftrightarrow p_2. \quad (52) \]

The permutation symmetry which we used to constrain the hadron wavefunctions (see Eq. (45)) implied equal weight of the \((2, 2)\) and \((1, 1 \oplus 3) \oplus (1 \oplus 3, 1)\) representations. From the point of view of the composites, this is a consequence of the permutation

\[ q_i \leftrightarrow p_i \quad q_j, p_j \text{ fixed } i \neq j, \quad (53) \]

which interchanges \((2, 2)\) and \((1, 1 \oplus 3) \oplus (1 \oplus 3, 1)\) representations and therefore leaves \(|I\rangle\) and \(|II\rangle\) invariant while interchanging \(|III\rangle\) and \(|IV\rangle\).

It is clear that the meson states contain multiplets (a) and (b) found above on the basis of symmetry arguments, as they must. Moreover, now we see that since a state that transforms like \(|I\rangle\) must be identified with a \( \bar{\phi}_- \phi_- \) state, Goldstone modes must be bound states of \( \phi_- \) quarks, as was conjectured based on the existence of a conserved chiral current in the broken phase.

6. Matching and the Mass Matrix

We can learn about the meson mass matrix by using a simple matching contraint. The leading explicit chiral symmetry breaking operator in chiral perturbation theory implies that \( M_\pi^2 \propto m_q \). Since in the effective theory the Goldstone bosons are bound states of quarks like other hadrons, and all mass terms are on the same footing (see Eq. (32)), we expect the constituent quark mass matrix to map to the hadronic mass-squared matrix. The matching constraint fixes the meson masses to be of the form:
\[ M_I^2 = B (M_- + M_-) = 2B (M - M_0 + m_q) \]
\[ M_{II}^2 = M_{II}'^2 = B (M_- + M_+) = 2B (M + m_q) \]
\[ M_{III}^2 = B (M_+ + M_+) = 2B (M + M_0 + m_q) \]

with associated meson quark content given in Eq. (51) and we have used Eq. (32). \( B \) is an undetermined constant, which in principle can be different for each chiral representation. This result maps to Eq. (47) and Eq. (48) only if \( B \) transforms like \((2, 2)\); that is \( B \rightarrow LBR^\dagger = RBL^\dagger \). This is in fact the case for the representation involving the pion; the pions are \( \phi_- \) bound states with \( M=M_0 \) which gives \( M^2_{\pi} = 2Bm_q \), where one identifies \( B = -(\bar{q}q)/2f^2_{\pi} \) in QCD at leading order in chiral perturbation theory. We then have the following identification:

\[ \mu^2 = 2B\{M + m_q\} \in \hat{M}_0^2 \]
\[ \delta = 2BM_0 \in \hat{M}^2_{(q)} \]

which is consistent with the spurion transformation properties of Eq. (40). Note that the explicit breaking effects due to current quark masses are contained in \( \hat{M}_0^2 \). Our results are consistent with Eq. (47) and Eq. (48) because the same symmetries are at work. In fact, Eq. (54) follows from assuming that the quarks are confined and that the matching constraint, \( M^2_{\pi} \propto m_q \), is satisfied. However, the matching constraint is correct only to leading order in chiral perturbation theory. For instance, it receives contributions of higher order in \( m_q \) from loop graphs involving pions. Therefore, consistent matching requires that the \( \phi_+ \) and \( \phi_- \) constituent quarks be weakly interacting in the same sense that low-energy pions are weakly interacting. This is precisely what one expects of constituent quarks.

7. Phenomenology of the Ground State

The lowest lying meson state must be a massless isovector, the pion, which must be in a representation of type (a). Since the pion is a Lorentz scalar, all states in this representation have zero-helicity. In the case of zero-helicity there is conservation of normality, \( \eta \equiv P(-1)^J \), where \( P \) is intrinsic parity and \( J \) is spin [4]. Since \( \pi \) has \( \eta=-1 \), only states of opposite normality communicate by single-pion emission and absorption. Here we consider a well-known grouping. The pion is joined by a scalar \( \epsilon \) (\( \eta=+1 \)), and the helicity-0 components of \( \rho \) (\( \eta=+1 \)) and \( a_1 \) (\( \eta=-1 \)). These are states with \( GP(-1)^J=+1 \) where \( G \) is \( G \)-parity. Following Ref. 4 we identify \( |I\rangle_a = |\pi\rangle_a \), \( |II\rangle_a = |a_1\rangle_a^{(0)} \), \( |III\rangle = |\epsilon\rangle \) and \( |IV\rangle_a = |\rho\rangle_a^{(0)} \). Here we review some consequences of this grouping as well as constraints on the decay constants that have not been considered previously using this method. It is instructive to introduce an arbitrary mixing angle, \( \phi \). The pion representation is then given by:
\(|\pi\rangle_a = -\cos\phi|2,2\rangle_a + \sin\phi|A\rangle_a\)
\(|a_1\rangle^{(0)}_a = \sin\phi|2,2\rangle_a + \cos\phi|A\rangle_a\)
\(|\epsilon\rangle = |2,2\rangle_A\quad |\rho\rangle^{(0)}_a = |V\rangle_a.\)

The superscript denotes helicity. In order to learn about the coupling of these states to the vector and axialvector currents in the collinear frame, we can define the decay constants

\[
\langle 0 | A_{a} | a_1 \rangle^{(0)}_b = \delta_{ab} f_1 \cos \phi \equiv \delta_{ab} f_{a_1}
\]
\[
\langle 0 | A_{a} | \pi \rangle_b = \delta_{ab} f_\pi \sin \phi \equiv \delta_{ab} f_{\pi}
\]
\[
\langle 0 | V_{a} | \rho \rangle^{(0)}_b = \delta_{ab} f \equiv \delta_{ab} f_\rho,
\]

where

\[
\langle 0 | V_{a} | V \rangle_b = \langle 0 | A_{a} | A \rangle_b \equiv \delta_{ab} f.
\]

The usual definitions of the decay constants are

\[
\langle 0 | A_{a\mu} | \pi \rangle_b = \delta_{ab} f_{\pi} p_{\mu}
\]
\[
\langle 0 | A_{a\mu} | a_1 \rangle^{(\lambda)}_b = \delta_{ab} f_{a_1} M_{a_1} \epsilon^{(\lambda)}_{\mu}
\]
\[
\langle 0 | V_{a\mu} | \rho \rangle^{(\lambda)}_b = \delta_{ab} f_\rho M_\rho \epsilon^{(\lambda)}_{\mu}
\]

where \(\epsilon^{(\lambda)}_{\mu}\) is the vector meson polarization vector. In the collinear frame with \(\lambda = 0\) we recover [Eq. (57)] from [Eq. (59)] if we identify

\[
(p_0^0 - p_3^0) \langle 0 | A_{a} | \alpha \rangle \equiv \langle 0 | A_{a}^0 - A_{a}^3 | \alpha \rangle
\]
\[
(p_0^0 - p_3^0) \langle 0 | V_{a} | \alpha \rangle \equiv \langle 0 | V_{a}^0 - V_{a}^3 | \alpha \rangle,
\]

where the 3-direction is the collinear direction of motion. It follows that \(f_\pi = f_\rho \sin \phi\) and \(f_{a_1} = f_\rho \cos \phi\). Setting \(M_\pi^2 = 0\) one obtains \(M_\rho^2 = \cos^2 \phi M_{a_1}^2\). By considering matrix elements of the pion transition operator, \(X_{a_1}^\lambda\), one also finds \(g_{\rho\pi}^2 f_\pi^2 = M_\rho^2 \sin^2 \phi\). One can then obtain combinations of masses and decay constants that are independent of the mixing angle. In particular it is clear that

\[
f_{a_1}^2 + f_\pi^2 = f_\rho^2
\]
\[
M_\rho^2 f_\rho^2 = M_{a_1}^2 f_{a_1}^2
\]

which is precisely the content of the first and second spectral function sum rules [11], respectively, evaluated in resonance saturation approximation. This is not so surprising since in both cases one can argue that chiral symmetry is the significant input. Note that the approach used here is completely distinct from the spectral function sum rule.
derivation which follows from constraints on off-shell asymptotic behavior [11]. Here we also find other relations that are independent of mixing angle such as $g_{\rho\pi\pi}f_{\pi}^2 = M_{\rho}^2$. Of course, understanding of why the chiral multiplet takes its specific form requires the $S_2$ permutation symmetry, or equivalently, the constraint Eq. (7).

The $S_2$ invariance implies that the irreducible chiral representations must enter with equal weight and so $\pi$ and $a_1^{(0)}$ form an $S_2$ doublet, $\cos \phi = \sin \phi = 1/\sqrt{2}$, and we obtain the familiar KSRF relations [12]

$$2g_{\rho\pi\pi}f_{\pi}^2 = M_{\rho}^2$$
$$2f_{\pi}g_{\rho\pi\pi} = M_{\rho}f_{\rho},$$

which are remarkably well satisfied experimentally [1,11]. Eq. (62) in turn implies $f_{a_1}^2 = f_{\pi}^2$ and $2M_{\rho}^2 = M_{a_1}^2$. We also obtain $M_{\epsilon}^2 = M_{\rho}^2$, and so the $\rho$ in the $S_2$ invariant four-flavor effective theory must be degenerate with a scalar. We emphasize that these relations, which have been derived previously on the basis of asymptotic constraints [4], are exact consequences of $S_2$ in the four-flavor effective theory.

Note that the matching constraints of the previous section give the mass-squared matrix of the pion “quartet” in terms of the fundamental parameters of the theory. So, for instance, besides $M_{\pi}^2 = 2Bm_q$, one also has $M_{\rho}^2 = 2B(M_0 + m_q)$. Therefore, in the four-flavor effective theory it is clear that as $B \to 0$, at least $\pi$, $\epsilon$ and $\rho$ and $a_1$ become massless. So in the event of a second-order phase transition, say at finite-temperature, one expects not the usual 4 of $O(4)$ sigma model scenario, but rather a new universality class based on the reducible “quartet” representation found above which in $O(4)$ notation corresponds to $4 \oplus 6$ [13].

Here we have concentrated on the ground state. The strong pion transitions of heavy mesons have also been studied using the algebraic sum rules [3]. There the sum rules have a great deal of predictive power because heavy quark symmetry provides additional constraints on the mass-squared matrix. We hope that flavor doubling will provide some understanding of the strong transitions and mass-squared splittings of the baryons, and of the manner in which the various helicities are related [14].

8. Summary and Speculations

In this paper we considered the possibility that QCD with $N$ flavors has a low-energy description with $2N$ flavors. We were motivated by regularity in the hadron spectrum that is not explained by QCD symmetries, particularly by asymptotic constraints in pion-hadron scattering that can be expressed in algebraic form. We constructed a free theory of $2N$ quark flavors arranged into a pair of vectors in the fundamental representation of $SU(N)$. The free theory has a $U(2N)$ invariance. However, we showed that a $U(1)_A$
violating 't Hooft operator explicitly breaks $U(2N)$ to $SU(N)_L \times SU(N)_R \times U(1)_B$. If $\bar{\theta} = 0$ or $\pi$, the 't Hooft operator leaves unbroken a discrete subgroup of $U(2N)$: the group $S_2$ of permutations of two objects. We then assumed the QCD pattern of chiral symmetry breaking and showed that in the $2N$-flavor effective theory it is possible to move from the free current quark theory to the free constituent quark model. In the constituent picture we identified a conserved chiral current in the presence of chiral symmetry breaking operators, and in turn conjectured that the Goldstone bosons in the theory are bound states of massless constituent quarks.

The $2N$-flavor effective theory was then studied on the light-front, where we showed that a generalized Adler-Weisberger sum rule and several superconvergence relations are manifest properties. We then showed that one can use $S_2$ and chiral symmetry to find the chiral representations filled out by mesons in the broken phase, obtained previously by assuming soft asymptotic behavior. Hadrons were then constructed directly out of constituent quarks and the conjecture that Goldstone bosons are bound states of massless quarks was proved. We considered the phenomenology of the chiral representation involving the pion and showed that predictions of $S_2$ give familiar results that are in good agreement with experiment.

Of course we have left several important questions unaddressed. Perhaps the most relevant question is whether it is possible that QCD has a low-energy effective description with a new symmetry which is not present in the QCD lagrangian.

One interesting possibility is that $S_2$ is a discrete gauge symmetry. If $S_2$ is a gauge symmetry, then the $2N$-flavor effective theory has the same global symmetries as $N$-flavor QCD. In this case, $S_2$ seems to solve the strong CP problem without conflicting with expectations that global symmetries are sacred and should therefore be shared by different descriptions of the same physics. This interpretation is consistent with the fact that $S_2$ only seems to have consequences related to asymptotic behavior of scattering amplitudes. Although gauge symmetries are redundancies, they do have algebraic consequences when married with asymptotic constraints on scattering amplitudes. In this sense gauge symmetries behave like spontaneously broken chiral symmetries. A nice example is that of the Drell-Hearn-Gerasimov sum rule which can be expressed algebraically as a statement of the (trivial) $U(1)$ algebra of electromagnetism (see the fourth entry in Ref. 4 and also Ref. 3).

The $S_2$ symmetry can also be interpreted as an accidental global symmetry which is relevant only at energy scales where the $2N$-flavor effective theory becomes a good description. There is some precedent for this sort of accidental symmetry. There exist supersymmetric gauge theories where the full global symmetry group is not visible at the
level of the perturbative definition of the theory, but only in the infrared, where there is an “accidentally” enhanced global symmetry [16].

A more ambitious interpretation of our results is that QCD with \( N \) flavors has a dual “magnetic” description with \( 2N \) flavors. This interpretation is suggested by the presence of the chiral invariant mass term, \( M_0 \). This mass scale could arise naturally from the condensation of scalar fields, which would transform in the adjoint representation of the gauge group. Since confinement is dual to the Higgs mechanism [17], one might then interpret the fields in the effective theory with \( 2N \) flavors as monopoles of the QCD degrees of freedom and condensation of the “magnetic” adjoint scalars as confinement of the QCD “electric” degrees of freedom. Chiral symmetry breaking would seem to require that the dual theory be in the Higgs phase since \( M_0 \) plays a fundamental role in establishing the existence of a conserved chiral current in the broken phase. Evidently the dual theory would have to be in the Higgs phase with confined quarks and the “magnetic” gauge theory would have the same number of colors as QCD in order that baryons in the two descriptions be constructed out of the same number of quarks. This duality conjecture is not strictly academic. As pointed out above, a very interesting example of matching of distinct descriptions exists on the light-front. A two-flavor quark model with spin-orbit couplings [10] gives results for the mesons identical to those found here in the four-flavor effective theory without spin-orbit couplings. In the two-flavor theory \( S_2 \) must be imposed by hand as a specific choice of symmetry breaking operators. It would be interesting to see if a similar mapping exists for the baryons [14].

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