Constructing multi-loop scattering amplitudes with manifest singularity structure

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The infrared exponentiation properties of dimensionally-regularized multi-loop scattering amplitudes are typically hidden at the level of the integrand, materializing only after integral evaluation. We address this long-standing problem by introducing an appropriate integral basis which is simultaneously finite and uniform weight. As an example, we cast the integrand for the QCD corrections to the two-loop massless quark electromagnetic form factor into a form where it is manifest that the $\epsilon^{-4}$ and $\epsilon^{-3}$ pole terms arise from the $\epsilon$ expansion of the square of our one-loop master integral.

INTRODUCTION

Our understanding of the infrared structure of dimensionally-regularized gauge theory scattering amplitudes involving massless quanta is still not complete, despite decades of research [1–44]. One curious feature of multi-loop hard scattering amplitudes is that, at least in all realistic models, they do not exhibit any physically-meaningful pole structure when written in integral form with respect to any known Feynman integral basis. This is surprising because one would naively think that the higher $\epsilon$ pole terms should have their origin in factorizable topologies; Catani [16] explained long ago how to predict the $\epsilon^{-4}$ and $\epsilon^{-3}$ pole terms of quite general on-shell two-loop scattering amplitudes from the corresponding one-loop results and appropriate universal one-loop quantities. We show in this Letter that it is possible to make manifest at least this aspect of gauge theory infrared exponentiation at the integrand level, simply by making a more appropriate choice of integral basis.

The key idea is to combine two extant Feynman integral basis paradigms to construct a new integral basis which is simultaneously finite [45, 46] and uniform weight [47, 48]. Note that extensions of the original uniform weight basis construction intended to be applicable to general scattering processes (i.e. beyond the realm of multiple polylogarithms) have been proposed in the literature [49–60]. In this Letter, we study processes where the integral basis is polylogarithmic in nature, though it seems likely to us that our construction will ultimately generalize to processes with richer analytic structures, such as the production of a top-antitop pair in proton-proton collisions (see e.g. [61, 62]). A fully general construction would be of independent interest as a bookkeeping device for scattering amplitude computations because it would deliver Feynman integrals with near-optimal properties, perfect for both analytical and numerical explorations (see e.g. [63] to gain perspective on this point). We will call integrals which are both finite and uniform weight uniformly finite below.

In the next section, we motivate our construction by studying the master integrals for the unrenormalized one-loop quark form factor and one-loop four-point gluon scattering amplitudes of massless QCD. We then show how not all uniformly-finite integral bases are equally interesting and explain how to construct a good uniformly-finite basis for the unrenormalized two-loop quark form factor of massless QCD. Finally, we discuss our result, highlighting the ways in which it is superior to previously-known representations. We conclude with an outlook and some remarks about the potential of our new integral basis paradigm.

MOTIVATION

In this section, we suggest how one could first discover non-trivial uniformly-finite integrals. First, note that it is trivial to build a uniformly-finite Feynman integral out of any finite integral which evaluates to a single power product of Gamma functions [64, 65]. Consider the one-loop form factor master integral from [46]:

$$\left. e^{\gamma_E} \epsilon^{3-\epsilon} \int \frac{d^6k}{((p_1 + k)^2(\epsilon - (p_2 - k)^2)^2)\Gamma(1 + \epsilon)} \right|_{q^2 = -1}$$

$$= e^{\gamma_E} \epsilon^{3-\epsilon} \frac{\Gamma^2(1 - \epsilon)\Gamma(1 + \epsilon)}{\Gamma(2 - \epsilon)}$$

$$= 1 + 2\epsilon + \left(4 - \frac{1}{2}\epsilon^2\right)\epsilon^2 + O(\epsilon^3). \quad (1)$$

In the above, we have set the virtuality $q^2 = (p_1 + p_2)^2$ to $-1$. To construct our preferred uniformly-finite master integral from Eq. (1), we simply dress it with a factor of $1 - 2\epsilon$. Let us remind the reader that it is straightforward to identify finite Feynman integrals belonging to a given integral topology [45] and that there exists a Reduze 2 [66, 67] job for exactly this purpose.

The one-loop fully-massless box integral provides a more non-trivial example because its all-orders-in-$\epsilon$ evaluation involves a sum of $_2F_1$ functions [71]. Using a Tarasov dimension shift [72] together with integration by parts [72, 73], one can derive the identity

$$\left. \frac{1}{\epsilon^2} \left( s(1-2\epsilon) \right) \left( t(1-2\epsilon) \right) \right|_{q^2 = -1}$$

$$= \frac{2}{s} \left( (s + t)(1 - 2\epsilon) \right) \left( \left. \frac{1}{\epsilon^2} \left( s(1-2\epsilon) \right) \left( t(1-2\epsilon) \right) \right|_{q^2 = -1} \right) \quad (2)$$
From Eq. 11 and the discussion of $6 - 2\epsilon$ boxes in [74], we know that the dressed bubble integrals in [9] are uniformly finite and that the box integral on the right-hand side is finite. Now, due to the fact that the $4 - 2\epsilon$ massless box is well-known to be uniform weight with leading singularity $s t [72]$, we can immediately conclude from Eq. [2] that the $6 - 2\epsilon$ massless box dressed with $(s + t)(1 - 2\epsilon)$ is a uniformly-finite integral.

The form of the result for the uniformly-finite massless box is incredibly simple and we arrived at the result without studying any explicit $\epsilon$ expansions. Instead, we were able to leverage our knowledge of the usual uniform weight integral in $4 - 2\epsilon$. As we shall soon see, it is harder to produce satisfactory, non-factorizable, uniformly-finite integral candidates at the multi-loop level.

\[
\mathcal{F}_2^\epsilon (s) = C_F^2 \left( \frac{1}{\epsilon^4} \right) \left[ f_1 \left( (1 - 2\epsilon)^2 \right) + f_2 \left( (8 - 2\epsilon) \right) \right] + \frac{32(1 - 2\epsilon)(3 - 2\epsilon)(1 - 3\epsilon)(2 - 3\epsilon)}{(1 - \epsilon)(1 + 2\epsilon)} f_3 \epsilon \left( (8 - 2\epsilon) \right) + C_F C_A \left( \frac{1}{\epsilon^4} \right) \left[ f_5 \left( -8(1 - 3\epsilon)(2 - 3\epsilon) \right) + f_6 \left( \frac{32(1 - 2\epsilon)(3 - 2\epsilon)(1 - 3\epsilon)(2 - 3\epsilon)}{(1 - \epsilon)(1 + 2\epsilon)} \right) \right] + C_F N_f \left( \frac{32(1 - 2\epsilon)(3 - 2\epsilon)(1 - 3\epsilon)(2 - 3\epsilon)}{(1 - \epsilon)(1 + 2\epsilon)} \right) .
\]

In Eq. 8, uniformly-finite products are enclosed with braces and the $f_i$ are both non-singular and non-vanishing in the $\epsilon \to 0$ limit. If desired, explicit expressions for the $f_i$ may be derived in a straightforward manner using the Reduze 2 implementation of Laporta’s algorithm [12] and the Tarasov dimension shift [71, 77], together with Eq. (10) of [73]. Clearly, the expression for $\mathcal{F}_2^\epsilon (s)$ above sheds no light on its infrared structure, despite the fact that we have exhibited uniformly-finite integral candidates for all non-factorizable subtopologies.

In fact, insisting upon the conventional master integral topologies is the problem here; a much more fruitful approach is to allow for integral topologies which are reducible with respect to integration by parts reduction. One can then perform a detailed scan of the available finite master integrals across all integral topologies using Reduze 2. Two reducible topologies which stand out are

\[ \text{and} \]

**UNIFORMLY-FINITE MASTER INTEGRALS FOR THE TWO-LOOP QUARK FORM FACTOR**

Our goal in this section is to produce an explicit uniformly-finite integral basis which makes manifest the IR exponentiation properties of the two-loop quark form factor of massless QCD. Unfortunately, both the two-loop one-external-mass sunrise and the two-loop one-external-mass bubble-triangle admit trivial uniformly-finite integral candidates which contribute to the $\epsilon^{-4}$ and $\epsilon^{-3}$ pole terms of the unrenormalized two-loop form factor. Working with the finite two-loop non-planar form factor integral employed in [10] for now, we find

\[
\begin{align*}
&\frac{1}{6} (1 - 2\epsilon) \left( \frac{4 - 2\epsilon}{6} \right) = \zeta_3 + \frac{3}{5} \zeta_2^2 \epsilon + \left( -\zeta_2 \zeta_3 + 7 \zeta_5 \right) \epsilon^2 \\
&\quad + \left( \frac{99}{35} \zeta_3^2 - \frac{23}{3} \zeta_2^2 \right) \epsilon^3 + \left( -\frac{17}{2} \zeta_2^2 \zeta_3 - 7 \zeta_2 \zeta_5 + 49 \zeta_7 \right) \epsilon^4 \\
&\quad + \left( \frac{3777}{350} \zeta_4^2 + \frac{23}{3} \zeta_2 \zeta_3^2 - \frac{1306}{15} \zeta_3 \zeta_5 \right) \epsilon^5 + O(\epsilon^6)
\end{align*}
\]
and

\[- (1 - 2\epsilon)(1 - 3\epsilon) \quad \begin{array}{c}
\hline
\hline
\end{array}
\]
\[= \zeta_2 + \zeta_3 + \frac{4}{5} \zeta_2^2 \epsilon^2
\]
\[+ \left( -\frac{29}{3} \zeta_2 \zeta_3 + 3\zeta_5 \right) \epsilon^3 + \left( -\frac{281}{70} \zeta_2^2 - \frac{29}{3} \zeta_2^3 \right) \epsilon^4
\]
\[+ \left( -\frac{379}{30} \zeta_2^2 \zeta_3 - \frac{257}{5} \zeta_2 \zeta_5 + 9\zeta_7 \right) \epsilon^5
\]
\[+ \left( -\frac{4502}{175} \zeta_2^2 + \frac{425}{9} \zeta_2 \zeta_3^2 - \frac{402}{5} \zeta_3 \zeta_5 \right) \epsilon^6 + O(\epsilon^7). \quad (5)
\]

To fix the dressing factors in the above, we found it natural to compute virtual-two-propagator cuts. As desired, the improved basis integrals above do not contribute to the $\epsilon^{-4}$ or $\epsilon^{-3}$ pole terms of the amplitude.

One might think that such simple results arise because the integrals have an underlying representation in terms of Gamma functions. In contrast, the basic scalar two-loop non-planar form factor integral was shown in [78] to contain $\bar{3}F_2$ and $4F_3$ functions in its all-orders-in-$\epsilon$ evaluation. It is therefore of some importance to show that we can readily construct a uniformly-finite integral candidate for the non-planar topology.

The non-planar topology’s basic scalar integral in $4 - 2\epsilon$ is well-known to be uniform weight (see e.g. [59] for a review). In general, however, the identification of uniform weight integrals in topologies which admit them is a non-trivial task and several programs which help one to accomplish this task have been published in the last few years [90, 91]. For form factors in $4 - 2\epsilon$, a dedicated method has been developed [57, 44]. If the result was not already known, the above-cited method would have allowed us to discover that the basic scalar integral in $4 - 2\epsilon$ is uniform weight with very little effort.

The non-planar topology first admits finite integrals in $6 - 2\epsilon$ and there are just two independent candidates,

\[\begin{array}{c}
\hline
\hline
\end{array}
\]
\[= (6 - 2\epsilon)
\]
\[\text{and}
\]
\[\begin{array}{c}
\hline
\hline
\end{array}
\]
\[= (6 - 2\epsilon)
\]

For this topology, a maximal cut analysis [95, 97] of the finite integrals is not enough by itself, but it imposes strong constraints on the structure of the result. Eq. (2.13) of [57] implies

\[\begin{align*}
\left( \begin{array}{c}
\hline
\hline
\end{array} \right) &= \frac{4i\pi^3 e^{-i\pi\epsilon} e^{2g\gamma\epsilon} \Gamma(1 - 2\epsilon)}{(1 - \epsilon)(1 - 4\epsilon) \Gamma(1 - 4\epsilon)} (q^2)^{1 + 2\epsilon} \\
\left( \begin{array}{c}
\hline
\hline
\end{array} \right) &= \frac{16i\pi^3 e^{-i\pi\epsilon} e^{2g\gamma\epsilon} \Gamma(1 - 2\epsilon)}{(1 - \epsilon)(1 - 4\epsilon) \Gamma(1 - 4\epsilon)} (q^2)^{2 + 2\epsilon}
\end{align*}
\]
\[\quad \text{(6)}
\]
\[\quad \text{(7)}
\]

and we immediately see that the simplest linear combination of finite integrals which could turn out to simultan-
We see from Eq. (10) that the connection between the one-loop form factor and the $\epsilon^{-4}$ and $\epsilon^{-3}$ poles of the two-loop form factor now originate from a term proportional to the square of the one-loop master integral, we have furthermore eliminated a hidden zero that Eq. (3) had in the $\epsilon^{-4}$ term of its $C_A C_F$ color structure. Even better, the impact of the non-Abelian exponentiation theorem [2, 3] on the $C_F^2$ color structure is manifest now as well: we see from Eq. (10) that the square of the dressed one-loop bubble generates even the $\epsilon^{-2}$ divergence of $C_F^2$.

In this Letter, we introduced a new Feynman integral basis which unifies the well-known finite and uniform weight integral basis paradigms. We have seen that, by making judicious choices for the basis elements, a finite and uniform weight basis can expose structure in the integrands of phenomenologically-relevant scattering amplitudes which has until now remained completely hidden from view. Furthermore, our explicit construction was shown to require little effort beyond that which is needed for the setup of an ordinary uniform weight basis.

Of course, many interesting questions remain. As mentioned in the introduction, it is natural to wonder whether a similar construction could work in the presence of massive quarks. A first test would be to rerun the analysis of this Letter for the two-loop heavy quark form factor in QCD [99, 104]. It is also important to continue studying scattering amplitudes in massless QCD, both with more legs and more loops. In this direction, we have already obtained a number of non-trivial results for one-loop five-gluon and two-loop four-gluon uniformly-finite basis integrals. The three-loop quark form factor of massless QCD [106, 111] would be a natural next step to shed additional light on our proposal for the manifest exponentiation of the higher poles in $\epsilon$.

Naively, it would seem that we already have a problem, due to the fact that our one-loop form factor master lives in $6-2\epsilon$, but we chose the $4-2\epsilon$ kite integral as one of our two-loop form factor basis integrals. Fortunately, it turns out that one can lift the $4-2\epsilon$ kite integral to a dressed sum of two $6-2\epsilon$ planar ladder integrals [112]. Anyway, it would certainly be useful to better understand which integral topologies fit well into our framework.

DISCUSSION AND OUTLOOK

Adopting the uniformly-finite integral basis constructed in the previous section, our final result for the unrenormalized two-loop massless QCD quark form factor at virtuality $q^2 = -1$ has the schematic form

\[
\mathcal{F}_2^2(\epsilon) = C_F^2 \left( \frac{c_1}{\epsilon^4} \right) (6-2\epsilon) + \frac{c_2}{\epsilon} (1-2\epsilon) + \frac{c_3}{\epsilon} (1-2\epsilon)(1-3\epsilon) + \cdots
\]

where the $c_i$ are both non-singular and non-vanishing in the $\epsilon \to 0$ limit.

Eq. (10) makes manifest the connection between the one-loop form factor and the $\epsilon^{-4}$ and $\epsilon^{-3}$ poles of the two-loop form factor. Not only do these divergences now originate from a term proportional to the square of the one-loop master integral, we have furthermore eliminated a hidden zero that Eq. (3) had in the $\epsilon^{-4}$ term of its $C_A C_F$ color structure. Even better, the impact of the non-Abelian exponentiation theorem [2, 3] on the $C_F^2$ color structure is manifest now as well: we see from Eq. (10) that the square of the dressed one-loop bubble generates even the $\epsilon^{-2}$ divergence of $C_F^2$.
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