On White Noise Space and Lévy’s Brownian Motion on the Circle

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Abstract: In this article, we show that the Brownian motion on the circle constructed in Lévy (1959) is a regular Euclidean Brownian motion on the half-circle with its own mirror image on the other half-circle, and is degenerated in the sense of Minlos (1959). This raises the question of what the white noise is on the circle. We then formally define the white noise space and its associated Brownian bridge.

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1. Introduction.

The study of Brownian motions leads to many interesting and important branches in probability (Wiener [24], Lévy [15], Gross [6], Hida [7]). Most of these developments are in Euclidean spaces. On the circle, Lévy [16] studied and constructed Brownian motion. In Lévy’s construction, a white noise process is assumed to exist first, and the Brownian motion is introduced as the direct integration of this white noise process along the half-circle. As we discuss in more detail in Section 3, this Brownian motion on the circle can be shown to be a regular Euclidean Brownian motion on the half-circle with its own mirror image on the rest half-circle, and is deemed degenerated in the sense of Minlos [19]. This leads to the more fundamental question of the formal treatment of the white noise and Brownian type processes on the circle. Note that the white noise and other random processes on the circle have wide applications in probability and statistics, for example, the positive definite functions (Wood [25], Pinkus [20], Menegatto et al. [18], Sun [23], Huang et al. [10], Gneiting [5]), the study of the intrinsic random functions (Huang et al. [11]), the shape modeling (Hobolth and Jensen [9], Aletti and Ruffini [3]), and stochastic differential equations (Lindgren et al. [17]).

A white noise is a mathematical idealization of a stationary random process with a constant density, for example, see Yaglom [26]. It is not an ordinary random process, but a generalized random process (Itô [12], Gel’fand and Vilenkin [4]). In this manuscript, we first formally define the white noise space on the circle following the approach in Hida [7], Hida et al. [8], Kuo [14], Si [22]. The
associated Brownian bridge is obtained in Section 3 with the discussion of the issues with Lévy’s construction of Brownian motion on the circle.

2. White noise space on the circle

The study of the white noise space can be widely found in Hida [7], Hida et al. [8], Kuo [14] and Si [22] based on the Bochner-Minlos Theorem. Minlos (1959, [19]) extended Bochner Theorem to infinite dimensional space with Kolmogorov [13]’s refinement in proof. A comprehensive discussion can be found in Hida [7]. Most of these developments are in Euclidean spaces. In this Section, we adapt this approach and formally define the white noise on the circle.

We start with a Gel’fand triple for functions on the unit circle $S$:

$E \subset L^2(S) \subset E'$,

where the space $E$ and its dual space $E'$ will be introduced shortly, and $L^2(S)$ is the collection of functions $f(t), t \in S$ such that

$\int_S |f(t)|^2 dt < \infty$.

This $L^2(S)$ is a Hilbert space with the inner product

$\langle f, g \rangle = \int_S f(t)\overline{g(t)} dt$,

the associated norm

$\|f\| = \left( \int_S |f(t)|^2 dt \right)^{1/2}$,

and the orthonormal basis

$\{ \psi_n(t) = e^{i2\pi nt}, t \in S \}, \quad n \in \mathbb{Z}$.

For the space $E$, we consider the Sobolev space of order 1 on the circle

$H_1(S) = \{ f \in L^2(S) : \sum_n |(f, \psi_n)|^2 n^2 < \infty \}$.

Clearly $H_1(S) \subset L^2(S)$. Note that the Sobolev space of negative index is introduced as the dual space of the Sobolev space of the corresponding positive index with additional constraints (See Adams [1, Chap. 3]). For Sobolev spaces of negative index for periodic functions, Robinson [21, Chap. 5] defines $H_{-1}(S)$ as the dual space of $H_{1,0}(S)$, where

$H_{1,0}(S) = \{ f \in H_1(S) : (f, \psi_0) = 0 \}$,

then,

$H_{-1}(S) = (H_{1,0}(S))'$.
Note that \( \psi_0(t) \equiv 1 \), then, the extra condition is:

\[
(f, \psi_0) = \int_S f(t) dt,
\]

that is, the functions in \( H_{1,0}(S) \) has the Fourier coefficient for \( \psi_0 \) to be zero. We now have the Gel’fand triple

\[
H_{1,0}(S) \subset L^2(S) \subset H_{-1}(S),
\]

with a natural pair

\[
(\xi, x), \quad \xi \in H_{1,0}(S), \quad x \in H_{-1}(S).
\]

**Lemma 1.** The injection \( H_{1,0}(S) \mapsto L^2(S) \) is of Hilbert-Schmidt type, so is the injection \( L^2(S) \mapsto H_{-1}(S) \).

To show this, note that the norms in both \( H_{1,0}(S) \) and \( L^2(S) \) are

\[
\|f\|_{H_{1,0}(S)}^2 = \sum_n n^2 |(f, \psi_n)|^2, \quad \|f\|^2 = \sum |f(\cdot, \psi_n)|^2.
\]

Let \( \xi_n = \frac{1}{n} \psi_n, n \neq 0 \), then \( \{\xi_n\} \) is an orthonormal basis for \( H_{1,0}(S) \), and

\[
\sum \|\xi_n\|^2 = \sum \frac{1}{n^2} < \infty.
\]

So the injection \( H_{1,0}(S) \mapsto L^2(S) \) is of Hilbert-Schmidt type. Similarly, the injection \( L^2(S) \mapsto H_{-1}(S) \) is also of Hilbert-Schmidt type. Parallel results can be found in Kuo [14], Si [22].

Consider the following functional \( C(\xi), \xi \in H_{1,0}(S) \),

\[
C(\xi) = \exp \left( -\frac{1}{2} \|\xi\|^2 \right), \quad \xi \in H_{1,0}(S),
\]

where \( \|\cdot\| \) is the \( L^2(S) \) norm \( \|\xi\|^2 = (\xi, \xi) = \int_S |\xi(t)|^2 dt \).

**Lemma 2.** \( C(\xi) \) is positive definite for \( \xi \in H_{1,0}(S) \).

To show that \( C(\xi) \) is positive definite, i.e., for any \( z_j \in \mathcal{C} \) and \( \xi_j \in H_{1,0}(S), j = 1, \ldots, n \),

\[
\sum_{j,k=1}^n z_j C(\xi_j - \xi_k) \overline{z_k} \geq 0.
\]

For \( \xi_1, \ldots, \xi_n \in H_{1,0}(S) \), let \( V \) be the (finite-dimensional) subspace of \( H_{1,0}(S) \) spanned by \( \xi_1, \ldots, \xi_n \) with the \( L^2(R) \) norm. Let \( \mu_V \) be the standard Gaussian measure on \( V \), since \( V \) is finite-dimensional, one can directly apply Bochner’s Theorem. Then for any \( \xi \in V \),

\[
\int_V e^{i(\xi, y)} d\mu_V(y) = e^{-\frac{1}{2} \|\xi\|^2}.
\]
Therefore,
\[
\sum_{j,k=1}^{n} z_j C(\xi_j - \xi_k) \sigma_k = \sum_{j,k=1}^{n} \int_{V} z_j e^{i(\xi_j - \xi_k, y)} d\mu_V(y) \sigma_k
\]
\[
= \int_{V} \left| \sum_{j=1}^{n} z_j e^{i(\xi_j, y)} \right|^2 d\mu_V(y) \geq 0.
\]

The function \( C(\xi) \) is obviously continuous on \( H_{1,0}(S) \) and \( C(0) = 1 \). That is, \( C(\xi) \) is a characteristic functional of \( \xi \in H_{1,0}(S) \).

It is obvious that \( C(\xi) \) is continuous on \( H_{1,0}(S) \) and \( C(0) = 1 \). That is, \( C(\xi) \) is a characteristic functional of \( \xi \in H_{1,0}(S) \). Therefore, by Minlos’ Theorem (Minlos [19], Kolmogorov [13]), we have the following theorem.

**Theorem 3.** The functional \( C(\xi), \xi \in H_{1,0}(S) \) in (1) determines the probability measure \( \mu \) on \( (H_{-1}(S), \mathcal{B}) \) satisfying
\[
\int_{H_{-1}(S)} e^{i(x, \xi)} d\mu(x) = C(\xi),
\]
where \( \mathcal{B} \) is the Borel \( \sigma \)-algebra on \( H_{-1}(S) \), i.e., the \( \sigma \)-algebra generated by the cylinder subsets of \( H_{-1}(S) \). \( C(\xi) \) is the characteristic functional of the measure \( \mu \).

**Definition 4.** The measure space \( (H_{-1}(S), \mathcal{B}, \mu) \) determined by the characteristic functional \( C(\xi) \) in (1) is called a white noise space on the circle. The measure \( \mu \) is called the standard Gaussian measure on \( H_{1,0}(S) \).

Notation wise, we denote the natural dual space pair \( (\xi, x), \xi \in H_{1,0}(S), x \in H_{-1}(S) \) as \( \xi(x) \), which is a random variable in this white noise space and has the following probability properties.

**Proposition 5.** The random variable \( \xi(x) \sim N(0, \|\xi\|^2) \), and the random variables \( \xi_1(x) \) and \( \xi_2(x) \) are independent when \( (\xi_1, \xi_2) = 0 \).

To show this, we first show that \( (\xi_1(x), \ldots, \xi_n(x)) \) follow a multivariate normal distribution through the following characteristic function derivation.
\[
\mathbb{E} \exp \{ it_1 \xi_1(x) + it_2 \xi_2(x) + \ldots + it_n \xi_n(x) \}
\]
\[
= \int_{H_{-1}} \exp \{ i(t_1 \xi_1 + t_2 \xi_2 + \ldots + t_n \xi_n, x) \} d\mu(x)
\]
\[
= \exp \left( -\frac{1}{2} \left\| t_1 \xi_1 + t_2 \xi_2 + \ldots + t_n \xi_n \right\|^2 \right).
\]

Therefore, \( (\xi_1(x), \ldots, \xi_n(x)) \) forms a Gaussian system. In particular,
\[
\xi(x) \sim N(0, \|\xi\|^2),
\]
that is, \( \xi(x) \) is a Gaussian random variable with mean zero, and variance \( \| \xi \|^2 \).
And \( \xi_1(x), \xi_2(x) \) are jointly bivariate Gaussian random vector with characteristic function
\[
E \exp\{it_1\xi_1(x) + it_2\xi_2(x)\} = \exp\left( -\frac{1}{2}(t_1^2\|\xi_1\|^2 + t_2^2\|\xi_2\|^2 + 2t_1t_2\langle \xi_1, \xi_2 \rangle) \right).
\]
When \( \langle \xi_1, \xi_2 \rangle = 0 \), \( \xi_1(x) \) and \( \xi_2(x) \) are independent.

This Proposition indicates that
\[
C(\xi_1 + \xi_2) = C(\xi_1)C(\xi_2),
\]
whenever \( \text{supp}(\xi_1) \cap \text{supp}(\xi_2) = \emptyset \), where \( \text{supp}(\xi) \) stands for the support of \( \xi \). Based on the discussion in Hida \[7, Sect. 4.5\], this process has independent values at every moment, or equivalently, \( \xi_1(x) \) and \( \xi_2(x) \) are independent random variables if \( \text{supp}(\xi_1) \cap \text{supp}(\xi_2) = \emptyset \). This is the so-called white noise property, see also in Alder and Taylor \[2\], Lindgren et al. \[17\].

3. Brownian bridge on the circle.

In this Section, we formally introduce Brownian bridge and discuss the issues with Lévy’s Brownian motion construction in \[16\].

Given the white noise space on the circle \((H_{-1}(S), S, \mu)\) with the characteristic functional \(C(\xi)\) in \(1\). We have the following extension theorem.

Lemma 6. The process \( \xi(x), \xi \in H_{1,0}(S) \) has an extension to \( L^2_0(S) \), where \( L^2_0(S) = \{ f \in L^2(S), (f, \psi_0) = 0 \} \), additionally,
\[
\int_{H_{-1}(S)} (f, x)(g, x)d\mu(x) = (f, g)
\]
With \( H_{1,0}(S) \) dense in \( L^2_0(S) \), suppose \( f \in L^2_0(S) \), take a sequence \( \{ \xi_n \} \) in \( H_{1,0}(S) \) such that \( \xi_n \to f \) in \( L^2_0(S) \). Then, the sequence \( \{ \xi_n(x) \} \) of random variables is Cauchy in \( L^2(H_{-1}(S), B, \mu) \) since
\[
\int_{H_{-1}(S)} |\xi_k(x) - \xi_n(x)|^2d\mu(x) = \|\xi_k - \xi_n\|^2.
\]
Denote
\[
f(x) = \lim_{n \to \infty} \xi_n(x), \text{ in } L^2(H_{-1}(S), B, \mu).
\]
This limit is independent of the sequence. Moreover, the random variable \( f(x) \) is normally distributed with mean 0 and variance \( \|f\|^2 \), and for two \( f, g \in L^2_0(S) \),
the covariance of \( f(x) \) and \( g(x) \) is their \( L_2^0(S) \) inner product by the continuity of extension, namely,

\[
\int_{H_{-1}(S)} f(x)g(x)d\mu(x) = (f,g).
\]

In Euclidean space \( R \), a Brownian motion can be introduced through \( 1_{[0,t)} \) as

\[
(1_{[0,t)},1_{[0,s)}) = \min(s,t),
\]

which directly lead to a Brownian motion in \( R \). However, \( 1_{[0,t)} \notin L_2^0(S) \) as it has Fourier coefficient \((1_{[0,t)},\psi_0) = t \neq 0\), and cannot be extended from \( H_{1,0}(S) \). Therefore, instead of \( 1_{[0,t)} \), we have

\[
1_{[0,t)} - t \in L_2^0(S),
\]

which yields

\[
(1_{[0,t)} - t,1_{[0,s)} - s) = \min(s,t) - st.
\]

This is the covariance function for the Brownian bridge. Therefore, we have the following result.

**Theorem 7.** Given the white noise space \((H_{-1}(S), B, \mu)\) on the circle, the generalized random process

\[
(1_{[0,t)} - t,x), \quad x \in H_{-1}(S),
\]

(2)

is a Brownian bridge.

It is clear that the additional condition of \((f,\psi_0) = 1\) of the test function space \( H_{1,0}(S) \) introduced in Section 2 plays a direct role in obtaining this Brownian bridge.

Next, we will revisit the Brownian motion on the circle constructed by Lévy [16]. Lévy constructed a Brownian motion \( B(t), t \in S \) on the circle with the covariance

\[
\text{cov}(B(t), B(s)) = \frac{1}{2}(r(o,t) + r(o,s) - r(s,t)),
\]

(3)

where \( r(s,t) \) is the circular distance between two points on the circle, \( o \) is an arbitrary pre-chosen origin point. This mimics the Brownian motion in Euclidean space. In his construction, Levy assumes the existence of the white noise on the circle first, and obtains the process \( B(t) \) at \( t \in S \) to be the integration of the white noise process along the half-circle centered at \( t \) (Lévy [16, Equation 5]). Direct computation can show that this resulting process possesses the covariance function of (3). Additionally, it can be shown that such process has an equality (Lévy [16, Equation 8]):

\[
B(t) + B(t') = B(s) + B(s') \quad a.s.,
\]

(4)

where \( t', s' \) are the opposites of \( t, s \) on the circumference. Similar relationship is revealed on the sphere in [10].
Note that equation (3) implies that if given the process on the half-circle \( \{B(t), t \in [0, \frac{1}{2}]\} \), then for any point \( s \in (\frac{1}{2}, 1) \), it can be obtained directly by (4),

\[
B(s) = B(0) + B\left(\frac{1}{2}\right) - B(s - \frac{1}{2}), \quad a.s.
\]

In other words, this Brownian motion is actually a regular one-dimensional Euclidean Brownian motion on \([0, \frac{1}{2}]\), and the mirror image on \((\frac{1}{2}, 1)\).

Given our discussion of the white noise space on the circle, one can obtain that there is a test function \( \eta_t(u), u \in S \):

\[
\eta_t(u) = \frac{1}{\sqrt{2\pi}} \sum_{k \text{ is odd}} e^{i2\pi kt} \frac{1}{|k|} e^{i2\pi ku}.
\]

This test function has the property that

\[
(\eta_t, \eta_s) = \min(s, t).
\]

It is clear that \( \eta_t(u) \in L^2(S) \) and \( (\eta_t, \psi_0) = 0 \). Lemma 6 can apply and the generalized random process

\[
\eta_t(x), \quad x \in H_{-1}(S)
\]

shares the same covariance with the Brownian motion in Lévy [16]. It is also obvious from this development that the function \( \eta_t(u) \) only has odd terms, and the process becomes degenerated in the sense of Minlos [19, Section 3]. The covariance in (3) can be shown to be not strictly positive definite on the circle (Pinkus [20]).

**Remark.** In Euclidean spaces, a white noise, sometimes, can be thought of as the generalized derivative of a Brownian motion. With our discussion, Lévy’s Brownian motion on the circle is not proper for this consideration. However, the Brownian bridge in (2) is well-defined and its generalized derivative can be viewed as the white noise on the circle.

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