Twist-2 Light-Ray Operators: 
Anomalous Dimensions and Evolution Equations

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Abstract

The non–singlet and singlet anomalous dimensions of the twist–2 light–ray operators for unpolarized and polarized deep inelastic scattering are calculated in $O(\alpha_s)$. We apply these results for the derivation of evolution equations for partition functions, structure functions, and wave functions which are defined as Fourier transforms of the matrix elements of the light-ray operators. Special cases are the Altarelli–Parisi and Brodsky–Lepage kernels. Finally we extend Radyushkin’s solution from the non–singlet to the singlet case.

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1 Introduction

The study of the Compton amplitude for scattering a virtual photon off a hadron is one of the basic tools in QCD to understand the short-distance behavior of the theory. In the general case of non-forward scattering the Compton amplitude is given by

\[ T_{\mu\nu}(p_+, p_-, Q) = i \int d^4x e^{iqx} \langle p_2 | T(J_\mu(x/2)J_\nu(-x/2)) | p_1 \rangle, \]  

where \( p_\pm = p_2 \pm p_1, Q = (q_1 + q_2)/2 \), and \( p_1 + q_1 = p_2 + q_2 \).

The time-ordered product in eq. (1) can be represented in terms of the non-local operator product expansion \([4]\) :

\[ T(J_\mu(x/2)J_\nu(-x/2)) \approx \int_{-\infty}^{+\infty} d\kappa_- \int_{-\infty}^{+\infty} d\kappa_+ \left[ C_a(x^2, \kappa_-, \kappa_+, \mu^2) S_{\mu\nu}^{\rho\sigma} \bar{x}_\rho O^\rho_{\sigma}(\kappa_+ \bar{x}, \kappa_- \bar{x}, \mu^2) + C_{a,5}(x^2, \kappa_-, \kappa_+, \mu^2) \varepsilon_{\mu\nu\rho\sigma} \bar{x}_\rho O^\rho_{\sigma,5}(\kappa_+ \bar{x}, \kappa_- \bar{x}, \mu^2) \right] \]  

with \( S_{\mu\nu\lambda\tau} = g_{\mu\nu} g_{\lambda\sigma} - g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\lambda\nu} \) and \( \varepsilon_{\mu\nu\rho\sigma} \) denoting the Levi–Civita symbol. The light–like vector

\[ \bar{x} = x - \eta(x.\eta/\eta.\eta) + \frac{x.\eta}{\eta.\eta} \sqrt{1 - \frac{x.x\eta.\eta}{(x.\eta)^2}} \]  

is related to \( x \) and a constant subsidiary four–vector \( \eta \) which drops out in leading order. Unlike the local operator product expansion \([4]\), which is usually applied in the case of forward scattering, eq. (2) straightforwardly leads to compact expressions for the coefficient functions \( C_a(x^2, \kappa_-, \kappa_+, \mu^2) \) and light–ray operators \( O^\rho_{\sigma}(\kappa_+ \bar{x}, \kappa_- \bar{x}, \mu^2) \). Indeed this expansion is a summed–up local light–cone expansion \([4, 5]\). In lowest order it contains only quark operators with two external legs. These operators are to be decomposed into operators of different twist. Here, we consider the contribution of twist–2 operators only\(^2\) for which we calculate the anomalous dimensions entering the corresponding renormalization group equation \([4]\).

Different processes, such as deep inelastic (forward) scattering \([3]\), deeply virtual Compton scattering \([5,7–12]\), or more generally non–forward scattering processes, cf. e. g. \([13]\), and photoproduction of mesons \([13–16]\) contain as an essential input the same light–ray operators of twist–2. All interesting non–perturbative partition functions are Fourier transforms of corresponding matrix elements of these operators. Therefore, all evolution equations result from the renormalization group equation of the light–ray operators involved. The evolution kernels can be determined from the anomalous dimensions of these operators. This procedure will be carried out here. The Altarelli–Parisi kernels as well as the Brodsky-Lepage/Efremov-Radyushkin kernels appear as special cases.

A second interesting feature of this approach is that non–forward processes contain more variables than forward processes. In the generalized Bjorken region this implies that instead of only one distribution parameter in the forward case, the Bjorken variable \( x \), two distribution parameters appear in the present case. One of them is related to the sum and the other to the difference of the external momenta. The explicit solution of the evolution equation in the non–singlet case was firstly found by Radyushkin \([17]\). Here, we extend his method to the singlet case. On the other hand, by imposing specific conditions on the momenta also one-parameter equations may be obtained which are equivalent to the general case treating a parameter which is fixed for special cases as a free second variable.

\(^2\)The notion of twist is not quite unambiguous in this context. It is rather used to label the type of operators being concerned, for which in the limit \( p_2 \to p_1 \) the notion applies \([4]\).
In this paper we present the results of a calculation of the non–singlet and singlet twist-2 anomalous dimensions for the general case both for unpolarized and polarized scattering, cf. [3]. We derive general evolution equations both for the operators and matrix elements in the non–forward case. Furthermore we derive a series of special cases previously discussed in the literature.

2 Twist-2 Light-Ray Operators and Different Choices of Partition Functions

We consider the following twist-2 flavor singlet operators\(^3\) which are known to mix under renormalization

\[
O^q(\kappa_1, \kappa_2) = \frac{i}{2} \left[ \psi_r(\kappa_1 \bar{x}) \gamma_\mu \bar{x}^\mu \psi_r(\kappa_2 \bar{x}) - \overline{\psi}_r(\kappa_2 \bar{x}) \gamma_\mu \bar{x}^\mu \psi_r(\kappa_1 \bar{x}) \right],
\]

\[
O^G(\kappa_1, \kappa_2) = \bar{x}^\mu \bar{x}^{\mu'} F^a_{\mu\nu}(\kappa_1 \bar{x}) F^a_{\mu'\nu}(\kappa_2 \bar{x}),
\]

\[
O_5^i(\kappa_1, \kappa_2) = \frac{i}{2} \left[ \psi_r(\kappa_1 \bar{x}) \gamma_5 \gamma_\mu \bar{x}^\mu \psi_r(\kappa_2 \bar{x}) + \overline{\psi}_r(\kappa_2 \bar{x}) \gamma_5 \gamma_\mu \bar{x}^\mu \psi_r(\kappa_1 \bar{x}) \right],
\]

\[
O_5^G(\kappa_1, \kappa_2) = \frac{1}{2} \bar{x}^\mu \bar{x}^{\mu'} \left[ F^a_{\mu\nu}(\kappa_1 \bar{x}) \tilde{F}^a_{\mu'\nu}(\kappa_2 \bar{x}) - F^a_{a\nu}(\kappa_2 \bar{x}) \tilde{F}^a_{\mu\nu}(\kappa_1 \bar{x}) \right].
\]

Here \(\psi_r\) denotes the quark field, \(F^a_{\mu\nu}\) and \(\tilde{F}^a_{\mu\nu}\) the gluon field strength and its dual, respectively, and furthermore we use

\[
\kappa_1 = \kappa_+ - \kappa_-, \quad \kappa_2 = \kappa_+ + \kappa_- .
\]

To simplify the considerations we apply the axial gauge, \(\bar{x}_\mu A^\mu = 0\), by which the phase factor connecting gauge dependent fields equals to unity.

The operators (5) and (7) appear only in the next order in \(\alpha_s\) in the Compton scattering amplitude. However, due to the mixing of operators of the same dimension and twist they emerge in the renormalization group equations. Whereas for the determination of the anomalous dimensions it is sufficient to investigate the operators (4 5 7), by considering the evolution equations for observables it is necessary to introduce generalized partition functions related to the matrix elements of the operators \(O^i, i = q, G\); (the index 5 will be suppressed in the following). The expectation values may be represented by either

\[
\langle p_1 | O^i | p_2 \rangle \frac{h_i}{(i\bar{x}p_+)^{h_i}} = e^{-i\kappa_+ \bar{x}_-} \int_{-1}^{+1} dz_+ dz_- e^{-i\kappa_- (\bar{x}_p z_+ + \bar{x}_- z_-)} F_i(z_+, z_-)
\]

or

\[
\langle p_1 | O^i | p_2 \rangle \cdot (\kappa_-)^{h_i} = e^{-i\kappa_+ \bar{x}_-} \int_{-1}^{+1} dz_+ dz_- e^{-i\kappa_- (\bar{x}_p z_+ + \bar{x}_- z_-)} G_i(z_+, z_-) .
\]

Here \(h_i\) denotes the degree of homogeneity, \(h_i\), with respect to a rescaling of the \(\kappa\)-variables of the quarkonic and gluonic operators (cf. 3) with

\[
h_q = 1, \quad h_G = 2 .
\]

\(^3\)In leading order the non–singlet operator is related to eq. (4). The corresponding results which are derived below for this operator apply literally to \(O_{NS}(\kappa_1, \kappa_2)\) in this order.
One way to express the partition function, eq. (9), is connected with the states and has the property, that in the forward limit, $|p_1\rangle \rightarrow |p_2\rangle$, these amplitudes lead to the well known structure functions appearing in the Altarelli–Parisi-equations. The second expression, eq. (10), which is related to the light–like spread of the operators leads to simpler two–variable evolution equations the solution of which is more straightforwardly obtained, cf. section 7.

3 Anomalous Dimensions of Twist 2 Operators

The renormalization group equations for the singlet operators read:

$$
\frac{\mu^2}{d\mu^2}O^i(\kappa_1, \kappa_2) = \frac{\alpha_s(\mu^2)}{2\pi} \int D\alpha K^{ij}(\alpha_1, \alpha_2, \kappa_\perp)O_j(\kappa'_{1}, \kappa'_{2})
$$

(12)

with

$$
D\alpha = d\alpha_1 d\alpha_2 \theta(1 - \alpha_1 - \alpha_2) \theta(\alpha_1) \theta(\alpha_2),
$$

(13)

$$
\alpha_s = g^2/(4\pi)
$$

the strong coupling constant, $\mu$ the renormalization scale, and

$$
\kappa'_1 = (1 - \alpha_1)\kappa_1 + \alpha_1\kappa_2, \quad \kappa'_2 = \alpha_2\kappa_1 + (1 - \alpha_2)\kappa_2.
$$

By $K = (K^{ij})$ we denote the matrix of singlet anomalous dimensions in the unpolarized case. The singlet evolution equations for the polarized case are obtained by replacing the operators $O^{q,G}$ by $O^{q,G}_p$ and the matrix $K$ by $\Delta K$. As far as relations are concerned which are valid both for the unpolarized and the polarized case under this replacement, we will, for brevity, only give that for the unpolarized case in the following. The matrices of the singlet anomalous dimensions are

$$
K = \begin{pmatrix}
K^{qq} & K^{qG} \\
K^{Gq} & K^{GG}
\end{pmatrix}
$$

(14)

and

$$
\Delta K = \begin{pmatrix}
\Delta K^{qq} & \Delta K^{qG} \\
\Delta K^{Gq} & \Delta K^{GG}
\end{pmatrix},
$$

respectively. In leading order, the non–singlet anomalous dimensions obey $K^{NS} = \Delta K^{NS} = K^{qq} = \Delta K^{qq}$.

For the unpolarized case the anomalous dimensions are given by

$$
K^{qq}(\alpha_1, \alpha_2) = C_F \left\{ 1 - \delta(\alpha_1) - \delta(\alpha_2) + \delta(\alpha_1) \left[ \frac{1}{\alpha_2} \right] + \delta(\alpha_2) \left[ \frac{1}{\alpha_1} \right] + \frac{3}{2} \delta(\alpha_1)\delta(\alpha_2) \right\},
$$

(15)

$$
K^{qG}(\alpha_1, \alpha_2) = -2N_f T_R \kappa_\perp \left\{ 1 - \alpha_1 - \alpha_2 + 4\alpha_1\alpha_2 \right\},
$$

(16)

$$
K^{Gq}(\alpha_1, \alpha_2) = -C_F \frac{1}{\kappa_\perp} \left\{ \delta(\alpha_1)\delta(\alpha_2) + 2 \right\},
$$

(17)

$$
K^{GG}(\alpha_1, \alpha_2) = C_A \left\{ 4(1 - \alpha_1 - \alpha_2) + 12\alpha_1\alpha_2 \\
+ \delta(\alpha_1) \left[ \frac{1}{\alpha_2} \right] - 2 + \alpha_2 \right\} + \delta(\alpha_2) \left[ \frac{1}{\alpha_1} \right] - 2 + \alpha_1 \right\} + \frac{\beta_0}{2} \delta(\alpha_1)\delta(\alpha_2),
$$

(18)

in $O(\alpha_s)$ where $C_F = (N_c^2 - 1)/2N_c \equiv 4/3, T_R = 1/2, C_A = N_c \equiv 3, \beta_0 = (11C_A - 4T_R N_f)/3$. The $[\ ]_+$-prescription is defined by

$$
\int_0^1 dx [f(x, y)]_+ \varphi(x) = \int_0^1 dx f(x, y) \left[ \varphi(x) - \varphi(y) \right],
$$

(19)

if the singularity of $f$ is of the type $\sim 1/(x - y)$. 

4
Correspondingly the anomalous dimensions for the polarized case are

\[
\begin{align*}
\Delta K^{q}(\alpha_1, \alpha_2) & = K^{q}(\alpha_1, \alpha_2), \\
\Delta K^{qG}(\alpha_1, \alpha_2) & = -2N_f T_R \kappa_- \{1 - \alpha_1 - \alpha_2\}, \\
\Delta K^{Gq}(\alpha_1, \alpha_2) & = -C_F \frac{1}{\kappa_-} \{\delta(\alpha_1)\delta(\alpha_2) - 2\}, \\
\Delta K^{GG}(\alpha_1, \alpha_2) & = K^{GG}(\alpha_1, \alpha_2) - 12C_A \alpha_1 \alpha_2.
\end{align*}
\]

Whereas the anomalous dimensions for the polarized case are derived for the first time in [3, 4], those for the unpolarized case were found several years ago [7, 8]. Recent calculations have also been performed in [9, 10]. The matrices \(K^{ij}\) and \(\Delta K^{ij}\) determine the evolution of the operators \(O^{NS}_{{\alpha}_i} O^{q}_j\), \(O^q_{{\alpha}_i}\), and \(O^G_{{\alpha}_i}\), respectively, in \(O(\alpha_s)\).

The determination of these anomalous dimensions is straightforward. In lowest order one has to calculate the one-particle irreducible one-loop Feynman diagrams containing the operators considered as first vertex. The corresponding Feynman rules and the calculation in the covariant gauge for one example are presented in ref. [7]. Here we have performed the calculation in the axial gauge which leads to essential simplifications. We used the Leibbrandt-Mandelstam prescription for the axial poles [13].

### 4 Evolution Equations of Partition Functions

The non–perturbative partition functions defined in section 2 have to be determined experimentally at an input scale \(Q^2\). Their values at higher scales \(Q^2\) can be calculated perturbatively solving the corresponding evolution equations. Here we derive only the equations which emerge from our two choices eqs. (9, 10). These equations follow immediately from the renormalization group equations of the operators, eq. (12).

For simplicity we introduce different variables which are most convenient, cf. [7], for the subsequent discussion

\[
\begin{align*}
w_1 & = \alpha_1 - \alpha_2 = \frac{\kappa_+^\prime - \kappa_-}{\kappa_-}, \quad w_2 = 1 - \alpha_1 - \alpha_2 = \frac{\kappa_-^\prime}{\kappa_-}, \\
D\alpha & = \frac{1}{2} D\kappa = \frac{1}{2} dw_1 dw_2 \Theta(1 - w_1 - w_2) \Theta(1 + w_1 - w_2) \Theta(w_2).
\end{align*}
\]

In these variables the RG-equation (12) reads

\[
\mu^2 \frac{d}{d\mu^2} O^j(\kappa_+, \kappa_-) = \frac{\alpha_s(\mu^2)}{2\pi} \int D\kappa D\kappa^\prime O^j(\kappa_1 + w_1 \kappa_-, w_2 \kappa_-).
\]

By Fourier transform we get an evolution equation for the distribution functions \(G^i\), inserting the definition (10) also into the right hand side

\[
\begin{align*}
\mu^2 \frac{d}{d\mu^2} G^i(z_+, z_-) & = \int_{-\infty}^{+\infty} \frac{d\kappa_- \bar{x}p_+}{2\pi} \int_{-\infty}^{+\infty} \frac{d\kappa_+ \bar{x}p_-}{2\pi} e^{i(\kappa_+ \bar{x}p_+ - \kappa_- \bar{x}p_- - \epsilon \bar{x}p_- - \epsilon \bar{x}p_+ z_+ \bar{z}_+)} G^i(\epsilon z_+ \bar{z}_+), \\
\frac{\alpha_s(\mu^2)}{4\pi} \int D\kappa D\kappa^\prime O^j(\kappa_1 + w_1 \kappa_-, w_2 \kappa_-) & \int_{-\infty}^{+\infty} \frac{d\bar{z}_+ \bar{z}_+}{2\pi} \int_{-\infty}^{+\infty} \frac{dz_+^\prime dz_+^\prime}{2\pi} \epsilon^{-i \bar{z}_+ \bar{z}_+} e^{i \epsilon \bar{z}_+ \bar{z}_+ \bar{z}_+ \bar{z}_+} G^i(\epsilon z_+ \bar{z}_+).
\end{align*}
\]

\footnote{A different, but related, quantity was studied in [18].}
The $\kappa_-$-dependence of the evolution kernel is $K^{ij}(w_1, w_2, \kappa_-) = \tilde{K}^{ij}(w_1, w_2)(\kappa_-)^{h_j - h_i}$. Carrying out the integrations with respect to $d\kappa_-$ we get

$$
\mu^2 \frac{d^2}{d\mu^2} G^i(z_+, z_-) = \frac{\alpha_s(\mu^2)}{4\pi} \int Dw \tilde{K}^{ij}(w_1, w_2) w_2^{-h_j} \int_{-1}^{+1} dz_{-} \int_{-1}^{+1} dz_{-}' 
\delta(z_- - w_2 z_-' - w_1) \delta(z_+ - w_2 z_+') G^j(z_+ z_-')
$$

leading to

$$
\mu^2 \frac{d^2}{d\mu^2} G^i(z_+, z_-) = \frac{\alpha_s(\mu^2)}{4\pi} \int_{-1}^{+1} \int_{-1}^{+1} \frac{dz_{-}' dz_{-}'}{|z_{+}'|} \tilde{K}^{ij}(z_- - z_{+}' z_{-}' z_+' z_+') \left( \frac{z_{+}'}{z_+} \right)^{h_j} G^j(z_{+}' z_{-}') .
$$

The derivation of the evolution equations for the partition functions (4) follows the same line. The difference is that the integration with respect to $d\kappa_- \tilde{x}_p$ leads not directly to a $\delta$-function so that one obtains

$$
\mu^2 \frac{d^2}{d\mu^2} F^i(z_+, z_-) = \frac{\alpha_s(\mu^2)}{2\pi} \int_{-1}^{+1} \int_{-1}^{+1} d z_{-}' d z_{-}' \Gamma^{ij}(z_+, z_-; z_{+}', z_{-}') F^j(z_{+}', z_{-}') ,
$$

where

$$
\Gamma^{ij}(z_+, z_-; z_{+}', z_{-}') = \int \frac{dw_2}{2} \int_{-\infty}^{+\infty} \frac{d\kappa_- \tilde{x}_p}{4\pi} e^{i\kappa_- \tilde{x}_p(z_+ - w_2 z_{-})} \tilde{K}^{ij}(z_- - w_2 z_{-}', w_2) ,
$$

with $h_{ij} = h_i - h_j$. Whether the $d\kappa_- \tilde{x}_p$-integration leads to a $\delta$-function, its derivative, or an integration depends on the values of $i$ and $j$. In the case of an integration a $\Theta$-function appears, for which the integration constant has to be determined separately.

## 5 One-Variable Evolution Equations

The equations given above cover the most general case depending on two partition variables $z_+$ and $z_-$. To reveal the relation of the general case to more special cases we introduce a kinematic condition by $\tilde{x}_p = \tau \tilde{x}_p$. This condition appears natural having in mind scale invariance. For special cases being discussed below the parameter $\tau$ will be fixed. On the other hand, one may consider $\tau$ as a second general parameter aside $\tilde{x}_p$ resembling the general two-variable case. Instead of the variables $z_-$ and $z_+$ we use $t$ and $\tau$ as new variables defining

$$
\Phi_i(t, \tau) = \int_{-\infty}^{+\infty} dz_- F_i(t - \tau z_-, z_-) ,
$$

$$
F_i(z_+, z_-) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} d\tau \frac{|z_-|}{\tau^2} e^{iz_- (z_+ - \tau)/\tau} \Phi_i(t, \tau) .
$$

These partition functions are related to the expectation values of the operators $Q^q$ and $O^G$ by

$$
\langle p_1 \mid O^q(-\kappa_- \tilde{x}, \kappa_- \tilde{x}) \mid p_2 \rangle \bigg|_{\tilde{x}_p = \tau \tilde{x}_p} = \int_{-\infty}^{+\infty} dte^{-in_- \tilde{x}_p t} \Phi_q(t, \tau) ,
$$

$$
\langle p_1 \mid O^G(-\kappa_- \tilde{x}, \kappa_- \tilde{x}) \mid p_2 \rangle \bigg|_{(i\tilde{x}_p)^2} = \int_{-\infty}^{+\infty} dte^{-in_- \tilde{x}_p t} t \Phi_G(t, \tau) .
$$
The evolution equations are:
\[
\mu^2 \frac{d}{d\mu^2} \Phi^i(t, \tau) = \frac{\alpha_s(\mu^2)}{2\pi} \int_{-\infty}^{+\infty} dt' V_{ext}^{ij}(t, t', \tau) \Phi^j(t', \tau).
\] (34)

The corresponding extended kernels read
\[
V_{ext}^{ij}(t, t', \tau) = \int [d\alpha] K^{ij}(\alpha_1, \alpha_2, \kappa_\tau) \frac{1}{2\pi} \exp\left\{ ip_\pm \kappa_\tau \left[ t - (1 - \alpha_1 - \alpha_2)t' + \tau(\alpha_1 - \alpha_2) \right] \right\}
\]

and obey the scaling relation
\[
V_{ext}^{ij}(t, t', \tau) = \frac{1}{\tau} V_{ext}^{ij}(t, t', 1).
\] (36)

For convenience we write the general expressions for the evolution kernels in the variables
\[
x = \frac{1}{2} \left(1 + \frac{t}{\tau}\right), \quad \bar{x} = \frac{1}{2} \left(1 - \frac{t}{\tau}\right), \quad y = \frac{1}{2} \left(1 + \frac{t'}{\tau}\right), \quad \bar{y} = \frac{1}{2} \left(1 - \frac{t'}{\tau}\right).
\] (37)

They are given by:
\[
V_{ext}^{qq}(t, t', \tau) = \frac{1}{2\tau} \left\{ V^{qq}(x, y)\rho(x, y) + V^{qq}(\bar{x}, \bar{y})\rho(\bar{x}, \bar{y}) + \frac{1}{2} C_F \delta(x - y) \right\}
\] (38)
\[
V_{ext}^{qG}(t, t', \tau) = \frac{1}{2\tau} \left\{ V^{qG}(x, y)\rho(x, y) - V^{qG}(\bar{x}, \bar{y})\rho(\bar{x}, \bar{y}) \right\} \left(\frac{2y - 1}{2}\right)
\] (39)
\[
V_{ext}^{Gq}(t, t', \tau) = \frac{1}{2\tau} \left\{ V^{Gq}(x, y)\rho(x, y) - V^{Gq}(\bar{x}, \bar{y})\rho(\bar{x}, \bar{y}) \right\} \left(\frac{2}{2x - 1}\right)
\] (40)
\[
V_{ext}^{GG}(t, t', \tau) = \frac{1}{2\tau} \left\{ V^{GG}(x, y)\rho(x, y) + V^{GG}(\bar{x}, \bar{y})\rho(\bar{x}, \bar{y}) \right\} \left(\frac{2y - 1}{2x - 1}\right)
\] (41)
\[
+ \frac{1}{4\tau} \beta_0 \delta(x - y)
\]
\[
\Delta V_{ext}^{qq}(t, t', \tau) = V_{ext}^{qq}(t, t', \tau)
\] (42)
\[
\Delta V_{ext}^{qG}(t, t', \tau) = \frac{1}{2\tau} \left\{ \Delta V^{qG}(x, y)\rho(x, y) - \Delta V^{qG}(\bar{x}, \bar{y})\rho(\bar{x}, \bar{y}) \right\} \left(\frac{2y - 1}{2}\right)
\] (43)
\[
\Delta V_{ext}^{Gq}(t, t', \tau) = \frac{1}{2\tau} \left\{ \Delta V^{Gq}(x, y)\rho(x, y) - \Delta V^{Gq}(\bar{x}, \bar{y})\rho(\bar{x}, \bar{y}) \right\} \left(\frac{2}{2x - 1}\right)
\] (44)
\[
\Delta V_{ext}^{GG}(t, t', \tau) = \frac{1}{2\tau} \left\{ \Delta V^{GG}(x, y)\rho(x, y) + \Delta V^{GG}(\bar{x}, \bar{y})\rho(\bar{x}, \bar{y}) \right\} \left(\frac{2y - 1}{2x - 1}\right)
\] (45)
\[
+ \frac{1}{4\tau} \beta_0 \delta(x - y),
\]
with
\[
\rho(x, y) = \theta \left(1 - \frac{x}{y}\right) \theta \left(\frac{x}{y}\right) \operatorname{sign}(y),
\] (46)
and
\[
V^{qq}(x, y) = C_F \left[ \frac{x}{y} - \frac{1}{y} + \frac{1}{(y - x)_+} \right]
\] (47)
\[ V^{qG}(x, y) = -2N_f T_R \frac{x}{y} \left[ 4(1 - x) + \frac{1 - 2x}{y} \right] \] (48)

\[ V^{Gq}(x, y) = C_F \left[ 1 - \frac{x^2}{y} \right] \] (49)

\[ V^{GG}(x, y) = C_A \left[ 2 \frac{x^2}{y} \left( 3 - 2x + \frac{1 - x}{y} \right) + \frac{1}{(y - x)_+} - \frac{y + x}{y^2} \right] \] (50)

\[ \Delta V^{qq}(x, y) = V^{qq}(x, y) \] (51)

\[ \Delta V^{qG}(x, y) = -2N_f T_R \frac{x}{y^2} \] (52)

\[ \Delta V^{Gq}(x, y) = C_F \left[ \frac{x^2}{y} \right] \] (53)

\[ \Delta V^{GG}(x, y) = C_A \left[ 2 \frac{x^2}{y^2} + \frac{1}{(y - x)_+} - \frac{y + x}{y^2} \right] \] (54)

Note that the kernels given in eqs. (38–54) apply to the full range of the variables, i.e. they represent the kernels completely. The function \( V_{ext}^{qq}(t, t', \tau) \) was already derived in refs. [20, 13], the complete treatment of both singlet cases are given in [5]. For the calculation one may fix the parameter \( \tau \), apply the scaling properties and take into account the necessary general structure in the \( t, t' \)-plane derived in [13]. Note, that our kernels are represented in a very compact form. Equivalent results obtained by other authors [9–11] consist at least of two separately calculated expressions. For \( \tau = 1 \), our expressions correspond to \( \zeta = 1 \) in [9]. In the notation of ref. [11] our parameter \( \tau \) equals to \( -\xi/2 \). Note also refs. [12].

6 Special Cases

The evolution kernels given above cover limiting cases which were studied before. These are characterized by special kinematic conditions for the matrix elements. This concerns the case of forward scattering \( \langle p_2 \rangle \rightarrow \langle p_1 \rangle \equiv \langle p \rangle \) and the transition from the vacuum state \( \langle 0 \rangle \) to a hadron state \( \langle p \rangle \) wave functions.

6.1 The Brodsky–Lepage Limit

For \( \tau = \pm 1 \) the equations (38–54) correspond to the limit \( \langle p_2 \rangle \rightarrow \langle p \rangle, \langle p_1 \rangle \rightarrow \langle 0 \rangle \), which is known as the Brodsky–Lepage [14] and Efremov–Radyushkin [13] case. This limit \( p_1 \rightarrow 0 \) may be performed formally, i.e. irrespective of any other quantum numbers, leading to correct results, cf. [20]. The corresponding evolution equations are given by eq. (37) using as variables \( x \) and \( y \), given in eq. (57). As an example we consider the simplest case in the range \( x, y < 1, x \neq y \)

\[ V^{qq}(x, y) = C_F \left\{ \Theta(y - x) \left[ \frac{x}{y} - \frac{1}{y} + \frac{1}{(y - x)} \right] + \Theta(x - y) \left[ \frac{1 - x}{1 - y} - \frac{1}{1 - y} + \frac{1}{(x - y)} \right] \right\}. \] (55)

6.2 The Near Forward Representation

This representation has been introduced in [13] and also used in ref. [3]. Essentially it consists of the part for \( t > \tau \) and \( t' > \tau \) of the general kernel. It contains the forward scattering evolution kernels case as limiting cases. Especially, for a correct application of the evolution equation to
the near forward matrix elements this representation is needed (for \( t > \tau \)) but additionally another representation for \( t < \tau \) has to be taken into account. In an example we show that this representation follows from our general kernel, i.e. that the general structure of \( V^{q q}_{\text{ext}} \) covers also the case \( t > \tau, t' > \tau \). For this range \( \text{sign}(\overline{y}) = -\text{sign}(y) \) and \( \Theta(1 - \frac{x}{y}) = \Theta(y - x) \) holds. Using these changes and dropping, for simplicity, the +-prescriptions we obtain

\[
V^{q q}(x, y) = C_F \Theta(y - x) \left\{ \frac{x}{y} \left[ 1 + \frac{1}{y - x} \right] - \frac{1 - x}{1 - y} \left[ 1 + \frac{1}{x - y} \right] \right\}
\]

\[
= C_F \Theta(y - x) \left[ \frac{1}{y - x} \left[ 1 + \frac{x y}{y y} \right] \right],
\]

(56)

where again \( x \) and \( y \) are taken from eq. (37).

6.3 The Altarelli–Parisi Limit

The Altarelli-Parisi kernels can be directly determined from the anomalous dimensions and suitably defined quark and gluon distribution functions

\[
f^q(z, \mu) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d(2p_\perp \vec{\kappa}) \langle p | O^q | p \rangle \Theta(\kappa_-, \mu) \frac{e^{2ip_\perp \vec{\kappa} / 2}}{2ip_\perp \vec{\kappa}},
\]

(57)

\[
z f^G(z, \mu) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d(2p_\perp \vec{\kappa}) \langle p | O^G | p \rangle \Theta(\kappa_-, \mu) \frac{e^{2ip_\perp \vec{\kappa} / (2ip_\perp \vec{\kappa})^2}}{2ip_\perp \vec{\kappa}}.
\]

(58)

The respective polarized parton densities are obtained by replacing \( f^q, G \) by \( \Delta f^q, G \) and \( O^q, G \) by \( O^q, G_5 \). In ref. [5] this procedure has been described and leads to the known results, [6]. On the other hand we should remark, that our general expressions for the one–variable evolution equations and the corresponding kernels contain the Altarelli-Parisi case in the limit \( \tau \to 0 \). The determination of this limit is nontrivial because cancellations of divergent terms occur. As an example we formulate the basic steps for the kernel \( V^{Gq}_{\text{ext}} \).

\[
(2x - 1)V^{Gq}_{\text{ext}}(x, y) = C_F \{ \Theta(\frac{x}{y}) \Theta(1 - \frac{x}{y}) \text{sign}(y) (1 - \frac{x^2}{y})
\]

\[
- \Theta(\frac{x}{y}) \Theta(1 - \frac{x}{y}) \text{sign}(\overline{y}) (1 - \frac{x^2}{\overline{y}}) \} \frac{1}{\tau}.
\]

(59)

We re-express the variables \( x, y \) again by the \( t, t' \)-variables. Finally the limit \( \tau \to 0 \) is performed yielding

\[
\lim_{\tau \to 0} V^{Gq}_{\text{ext}}(t, t') = \frac{1}{t'} C_F \left( 1 + \frac{(1 - z)^2}{z} \right), \quad z = \frac{t}{t'}.
\]

(60)

This is the corresponding Altarelli-Parisi splitting function. Another possibility consists in deriving first the near-forward representation and performing then the limit to the forward case, cf. [4].

7 Solutions

Here we look for solutions of the two variable equations in the singlet case. We extend the method of ref. [17] to this case. The idea follows the original solution of the Brodsky-Lepage
In this way a first diagonalization is obtained. The kernels \( \Gamma \) equation given by Efremov and Radyushkin [15]. The method consists in forming moments of the partition functions such that the resulting evolution equations for these moments can be solved. We start form eq. (27) and look for equations for the moments of the partition functions \( n \), solved. We start form eq. (27) and look for equations for the moments of the partition functions \( n \), solved.

One obtains from eq. (27)

\[
\mu^2 \frac{d}{du^2} \int_{-1}^{+1} dz_+ z^+_n G^i(z_+, z_-) = \frac{1}{2} \int_{-1}^{+1} dz' \int_{-1}^{+1} \frac{dz'_+}{|z'_+|} \left( \frac{z'_+}{z_+} \right)^{h_j} \hat{K}_{ij} \left( z_--\frac{z_+ z'_-}{z'_+}, \frac{z_+}{z'_+} \right) G^j(z'_+, z'_-) \]

yielding

\[
\mu^2 \frac{d}{du^2} G^i_n(z_-) = \frac{1}{2} \frac{\alpha_s(\mu^2)}{2\pi} \int_{-1}^{+1} dz'_- \Gamma^{ij}_n(z_-, z'_-) G^j_n(z'_-),
\]

with

\[
\Gamma^{ij}_n(z_-, z'_-) = \int_{-1}^{+1} dz'_+ \left( \frac{z'_+}{z'_+} \right)^{n-h_j} \hat{K}_{ij} \left( z_--\frac{z_+ z'_-}{z'_+}, \frac{z_+}{z'_+} \right).
\]

In this way a first diagonalization is obtained. The kernels \( \Gamma^{ij}_n(z_-, z'_-) \) for the unpolarized case read

\[
\Gamma^{qq}_n(z, z') = C_F \left\{ \Theta(z-z') [\left( \frac{1-z}{1-z'} \right)^n \left( \frac{1}{n} + \frac{2}{(z-z')_+} \right) ] + \Theta(z'-z) [\left( \frac{1+z}{1+z'} \right)^n \left( \frac{1}{n} + \frac{2}{(z'-z)_+} \right) ] + 3\delta(z-z') \right\},
\]

\[
\Gamma^{qg}_n(z, z') = -2N_f T_R \left\{ \Theta(z-z') [\left( \frac{1-z}{1-z'} \right)^n \left( \frac{n^2 + 2n(z-z') - (2zz' - 1)}{(n+1)n(n-1)} \right) ] + \Theta(z'-z) [\left( \frac{1+z}{1+z'} \right)^n \left( \frac{n^2 + 2n(z'-z) - (2zz' - 1)}{(n+1)n(n-1)} \right) ] \right\},
\]

\[
\Gamma^{gq}_n(z, z') = -C_F \left\{ \Theta(z-z') [\left( \frac{1-z}{1-z'} \right)^n \frac{1}{n} ] + \Theta(z'-z) [\left( \frac{1+z}{1+z'} \right)^n \frac{1}{n} ] + \delta(z-z') \right\},
\]

\[
\Gamma^{gg}_n(z, z') = C_A \left\{ \Theta(z-z') [\left( \frac{1-z}{1-z'} \right)^n \left( \frac{3(1+z)(1-z)}{n-1} \right) + \frac{6zz'}{n} \left( 3(1+z')(1-z) - \frac{2}{n} + \frac{2}{(z-z')_+} \right) ] + \Theta(z'-z) [\left( \frac{1+z}{1+z'} \right)^n \left( \frac{3(1-z)(1+z')}{n-1} \right) + \frac{6zz'}{n} \left( 3(1-z')(1+z) - \frac{2}{n} + \frac{2}{(z'-z)_+} \right) ] \right\} + \beta_0 \delta(z-z').
\]
Similarly one obtains for the polarized kernels

\[ \Delta \Gamma^q_{n}(z, z') = C_F \{ \Theta(z - z')[\left( \frac{1 - z}{1 - z'} \right)^n \left( \frac{1}{n} + \frac{2}{(z - z')^+} \right) + \Theta(z' - z)[\left( \frac{1 + z}{1 + z'} \right)^n \left( \frac{1}{n} + \frac{2}{(z' - z)^+} \right) + 3\delta(z - z')}, \quad (70) \]

\[ \Delta \Gamma^G_{n}(z, z') = -2N_f T_R \{ \Theta(z - z')[\left( \frac{1 - z}{1 - z'} \right)^n \frac{1}{n} + \Theta(z' - z)[\left( \frac{1 + z}{1 + z'} \right)^n \frac{1}{n} \}, \quad (71) \]

\[ \Delta \Gamma^{Gq}_{n}(z, z') = +C_F \{ \Theta(z - z')[\left( \frac{1 - z}{1 - z'} \right)^n \frac{1}{n} + \Theta(z' - z)[\left( \frac{1 + z}{1 + z'} \right)^n \frac{1}{n} - \delta(z - z')}, \quad (72) \]

\[ \Delta \Gamma^{GG}_{n}(z, z') = C_A \{ \Theta(z - z')[\left( \frac{1 - z}{1 - z'} \right]^n \frac{4}{n} + \frac{2}{(z - z')^+} + \Theta(z' - z)[\left( \frac{1 + z}{1 + z'} \right)^n \frac{4}{n} + \frac{2}{(z' - z)^+} \}] + \beta_0 \delta(z - z'). \quad (73) \]

As a next step we have to find a solution of eq. (77). One finds that the following symmetry–relations hold for kernels \( \Gamma^{GG}_{n}(z, z') \) and \( \Delta \Gamma^{GG}_{n}(z, z') \):

\[ (1 - z'^2)^n \Gamma^{ij}_{n}(z, z') = \Gamma^{ij}_{n}(z', z)(1 - z^2)^n, \quad (74) \]

\[ (1 - z'^2)^n \Delta \Gamma^{ij}_{n}(z, z') = \Delta \Gamma^{ij}_{n}(z', z)(1 - z^2)^n. \quad (75) \]

This is expected because of similar problems studied previously \[17, 15\]. The final diagonalization of eq. (74), referring to the partition functions eq. (10) for which the quarkonic and gluonic operators are dealt with equally, can thus be performed using Gegenbauer polynomials.

\[ \Gamma^{ij}_{n}(z, z') = \sum \Gamma^{ij}_{n,m} (1 - z^2)^n C_m^{n+1/2}(z) C_m^{n+1/2}(z') N_{n,m}, \quad (76) \]

\[ \Delta \Gamma^{ij}_{n}(z, z') = \sum \Delta \Gamma^{ij}_{n,m} (1 - z^2)^n C_m^{n+1/2}(z) C_m^{n+1/2}(z') N_{n,m}. \quad (77) \]

The coefficients \( N_{n,m} \) are the normalization factors of the Gegenbauer polynomials

\[ \int_{-1}^{+1} dz (1 - z^2)^n C_l^{n+1/2}(z) C_k^{n+1/2}(z) = \delta_{lk} N_{n,k} = \delta_{lk} 2^{2n+1} \Gamma^2 \left( n + \frac{1}{2} \right) \frac{(k + n + 1/2) \cdot k!}{2\pi \Gamma(2n + 1 + k)} \]. \quad (78) \]

\( \Gamma^{ij}_{n,m} \) and \( \Delta \Gamma^{ij}_{n,m} \) are the respective expansion coefficients which can be easily calculated. As well-known \[21\] the various possible representations of the solution of evolution equations using expansions in orthogonal polynomials are badly convergent \[22, 23\] requesting high-precision representations \[23\] which are difficult to handle in practical applications.

Another method of solution consists in a forming also Mellin moments in the variable \( z_\ast \),

\[ \int_{-1}^{+1} dz_\ast z_\ast \cdot C_i(z_\ast) = G^i_{nw}. \]

Since the kernels \( \Gamma^i_{n}(z_\ast, z'\ast) \) and \( \Delta \Gamma^i_{n}(z_\ast, z'\ast) \) obey the representation

\[ \Gamma^i_{n}(z_\ast, z'\ast) = \int_{0}^{1} dw_2 w_2^{n-h_i} G^i_{n}(z_\ast - w_2 z'_\ast, w_2), \quad (79) \]

the evolution equations may be written as (see also \[9\])

\[ \mu^2 \frac{d}{d\mu^2} G^i_{n} = \frac{\alpha_s(\mu^2)}{2\pi} \sum_{j,l} \Gamma^i_{n,k} C_{nl}^{ij}. \quad (80) \]
with the transformed kernels given by

\[
\Gamma_{n,kl}^{i,j} = \frac{1}{2} \int Dw \ w_{2}^{n+l-h_{j},l-h_{j}} w_{1}^{k-l} \hat{K}_{i,j}(w_{1}, w_{2}) \left( \begin{array}{c} k \\ l \end{array} \right).
\]

These evolution equations are not diagonal with respect to the indices \((k, l)\). The explicit expressions for \(\Gamma_{n,kl}^{i,j}\) and \(\Delta \Gamma_{n,kl}^{i,j}\) are given in ref. [7,5b]. For fixed \(n\) they form triangular matrices. The eigenvalues are the diagonal elements \(k = l\)

\[
\Gamma_{n,kk}^{i,j} = \frac{1}{2} \int Dw \ w_{2}^{n+k-h_{j},k-h_{j}} \hat{K}_{i,j}(w_{1}, w_{2}) = \gamma_{n+k-h_{j}}^{i,j}.
\]

The coefficients \(\gamma_{n+k-h_{j}}^{i,j}\) are the anomalous dimensions of the forward case with a shifted Mellin index.

8 Conclusions

The evolution kernels for the twist 2 light-ray operators both for the case of unpolarized and polarized deep inelastic non-forward scattering were derived for the flavor non-singlet and singlet cases. In general the partition functions depend on two distribution variables. One may study as well specialized evolution equations in one distribution variable implying external constraints, covering the case of evolution equations for non-forward parton densities. In this way, among various others, also the well-known evolution equations as the Brodsky–Lepage or Altarelli–Parisi equations can be obtained. The solution of the two-variable evolution equations in the non-singlet and singlet cases can be either performed applying a single Mellin transform and using Gegenbauer polynomials or by a two-fold Mellin transform.

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