The Extension of TOPSIS Method for Multi-Attribute Decision-Making With q-Rung Orthopair Hesitant Fuzzy Sets

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ABSTRACT As a combination of q-rung orthopair fuzzy sets (q-ROFSs) and hesitant fuzzy sets (HFSs), q-rung orthopair hesitant fuzzy sets (q-ROHFSs) are more effective, powerful, and meaningful in solving the complexity, ambiguity, and expert hesitancy of membership and non-membership in multi-attribute decision-making (MADM) problems. And so, based on the advantages of q-ROHFSs, we herein propose an improved TOPSIS model in the q-rung orthopair hesitant fuzzy environment. This model can provide more accuracy in expressing fuzzy and ambiguous information. At first, we propose the distance and similarity measures of q-ROHFSs and the properties related to the distance and similarity measures of q-ROHFSs, and secondly, the axiomatized definition and formula for the entropy of q-ROHFSs. Then, for the case where the attribute weights are totally unknown, a combination of subjective and objective attribute weighting model is proposed. This model not only considers the expert’s decision preference, but also the objective situation of the attributes. In addition to the above-mentioned outcomes, this paper also improves the relative closeness formula, increases the preference coefficient, and considers the risk-preference of decision makers. Finally, the proposed model is compared with other methods and used to evaluate the effectiveness of military aircraft overhaul. The method is verified to be scientific, reliable and effective for solving MADM problems.

INDEX TERMS q-Rung orthopair hesitant fuzzy sets, distance measure, entropy, multi-attribute decision-making, TOPSIS method.

I. INTRODUCTION
Since there are many ambiguities, uncertainties and other phenomena in real life, it is not possible to simply use a single value to describe multiple decision-making information. Relatively speaking, as the number of practical applications keeps increasing, the multi-attribute decision-making (MADM) model needs to be considered. The multi-attribute decision-making model has been extensively studied in recent years [1]–[4] for obtaining relevant guidance for practical decision-making. Zadeh [5] proposed the concept of fuzzy sets. The fuzzy set theory provides a quantitative analysis method for solving uncertain problems in complex systems. Ever since the introduction of fuzzy sets by Zadeh [5] in 1965, the fuzzy set theory has attracted the attention of many scholars and has also been rapidly developed, and is presently being widely used in all aspects of social production and life [6]–[10]. With the continuous development of social economy over the past two decades, the objective world has become increasingly complicated. In order to express and explain the problems of the real world more accurately, several scholars have developed and expanded the form of fuzzy sets, and many of them have come up with L-fuzzy sets [11], type 2 fuzzy sets [12], interval fuzzy sets [13], intuitionistic fuzzy sets (IFSs) [14], interval intuitionistic fuzzy sets [15], fuzzy multiple sets [16], and hesitant fuzzy sets [17].

As an effective extension of the fuzzy set [5], since the intuitionistic fuzzy set (IFS) [14] can simultaneously consider the membership as well as non-subordination of elements belonging to the set, it can describe the fuzzy nature of the objective world from three crucial aspects: support, opposition and neutrality. For that reason, it has been widely
Concerned by decision-making researchers and has delivered fruitful results [18]–[27], involving various operations of IFS [18], [19], the intuitionistic fuzzy information integration operator [20]–[24], the intuition Fuzzy clustering [25], and the correlation coefficients of IFS [26], [27].

Compared with fuzzy sets, IFSs consider both the membership and non-subordination of elements to the set, and are typically more objective and authentic. But, an intuitionistic fuzzy set also has certain limitations in the application of MADM. For instance, it can only describe those fuzzy phenomena where the sum of membership and non-membership exceeds 1 and if it does not, then IFS is practically useless. Therefore, Yager proposed the Pythagorean fuzzy set (PFS) [28], [29], which solved the above-mentioned problem. Based on Yager’s research, Pythagorean fuzzy sets (PFSs) have been comprehensively conducted and the valuable results were obtained [30]–[34]. Among them, Wang and Li [33] proposed the Pythagorean fuzzy interaction power Bonferroni mean operator and weighted Pythagorean fuzzy interaction power Bonferroni mean operator and Firoozja et al. [34] studied a new similarity measure for Pythagorean fuzzy sets (PFSs) by using triangular conorms. But PFSs has a definite requirement that the sum of squares of membership and non-membership does not exceed 1, which limits the development of PFSs to some extent. To this end, Yager [35] proposed q-rung orthopair fuzzy sets (q-ROFSs) represented as:

\[ Q = \{ x, (a(x), b(x)), q \} | x \in U \]  

where the \( a(x) : U \rightarrow [0, 1] \) denotes the membership degree, the \( b(x) : U \rightarrow [0, 1] \) denotes the non-membership degree. For any element \( x \in U \), it satisfies:

\[ a(x)^q + b(x)^q \leq 1, \quad q \geq 1 \]

It is worth pointing out that q-ROFSs are the generalizations of IFS and PFSs. When \( q = 1 \), q-ROFSs degenerate to IFS; when \( q = 2 \), q-ROFSs degenerate to PFSs. If \( q_2 \geq q_1 \), Yager [35] further proves that \( q_2 \)-rung fuzzy set is essentially \( q_2 \)-rung fuzzy set. Liu and Liu [36] utilized the Bonferroni average operator to study the decision-making problem in q-rung fuzzy environment. Yager and Alajlan [37] studied the application of q-ROFS in approximate reasoning while Liu and Wang [38] studied its application in decision-making based on Archimedean norm. In actual decision-making problems, decision-makers (DMs) often hesitate between numerous possible solutions when making the decisions, and the number of possible solutions provided by different DMs is usually different. To solve this type of problem, Torra [17] proposed a hesitant fuzzy set (HFS). Hesitant fuzzy sets (HFSs) are a generalization of fuzzy sets, which generalize the membership of elements from a number in the interval \([0, 1]\). As a novel tool for dealing with uncertain decision-making problems, the proposition of HFSs aroused great interest among scholars worldwide, and the theory and its application have been rapidly developed thereafter [39]–[43]. Zhu et al. [41] defined the hesitant fuzzy preference relations to solve hesitant fuzzy MADM problems with preference relations. Zhang and Yang [42] defined the degree of inclusion and similarity of HFSs, which are used to solve hesitant fuzzy multi-attribute group decision-making (MAGDM) problems. Li and Wei [43] proposed a gray correlation method to solve the hesitant fuzzy multi-attribute group decision-making problems with completely determined attribute weights.

Because the fuzzy set considers only the membership degree of elements, the non-membership degree of elements is ignored. Therefore, Zhu et al. [44] put forward the dual hesitant fuzzy set (DHFS) combining HFS and IFS, which can describe the decision-maker’s indecision completely, and is more suitable for the actual situation. Then, Zang et al. [45] put forward the gray relation decision-making method. Singh [46] defined distance measure and similarity measure of dual hesitant fuzzy sets (DHFSs). Wang et al. [47] came up with the dual hesitant fuzzy integral aggregation operator for solving the dual hesitant fuzzy multi-attribute group decision-making problem. Liang and Xu [48] put forward the Pythagorean hesitant fuzzy set (PHFS) and the technique for order preference by similarity to ideal solution (TOPSIS) to solve the problem of Pythagorean hesitant fuzzy multiple attribute decision.

Since q-rung orthopair fuzzy set (q-ROFS) is a generalization of PFS, its membership degree \( a(x) \) and the non-membership degree \( b(x) \) can take multiple values. Therefore, combining HFS and q-ROFS is of great significance to the expansion and practical application of fuzzy set theory. Liu et al. [49] put forward the concept of q-rung orthopair hesitant fuzzy set (q-ROHFS), investigated the algorithm and operation properties of q-ROHFS, defined the score function and accurate function of q-ROHFS, and gave the ranking method between q-ROHFSs. Additionally, they also studied the distance measure of q-ROHFS, established the TOPSIS model based on the distance of q-ROHFS, and verified the feasibility and effectiveness of the model through an example.

Combination of q-rung fuzzy set and HFS, which incorporates their common advantages, can not only describe the fuzzy phenomena but also reflect the decision-maker’s indecision in membership and non-membership. Accordingly, it has more advantages in MADM.

With the purpose of describing the uncertainty in the q-rung orthopair hesitant fuzzy information, the q-ROHFS information measures are introduced into the application of MADM. The q-ROHFS information measures include distance, similarity and entropy. The similarity measure of q-rung orthopair hesitant fuzzy sets is used predominantly for identifying and evaluating different hesitant fuzzy information. The entropy of q-rung orthopair hesitant fuzzy sets (q-ROHFSs) is typically used to measure the uncertainty of hesitant fuzzy linguistic information.

Taking into account, the extremely important position of information measure in MADM, this paper studies the distance measure and entropy of q-ROHFSs. To this end, the research content is organized as follows: Part 1 reviews
the concepts of IFS, q-ROFSs, q-ROHFSs and so on. Part 2 defines the distance and similarity measures of q-ROHFSs. Their specific formulas are given along with the discussion on their properties. Part 3 defines the entropy of the q-ROHFS and gives the specific formula for its entropy. Part 4 presents the construction of the TOPSIS model based on the q-runghorthopair hesitant fuzzy environment, to solve MADM problems. Part 5 introduces a decision-making example of military aircraft overhaul effectiveness evaluation and the use of the proposed TOPSIS model to get the final decision result and the relevant analysis. The influence of change of each parameter on the decision result is compared with other literature methods to illustrate the rationality and effectiveness of the TOPSIS method presented in this paper. Part 6 summarizes the research content.

II. PRELIMINARIES

A. THE q-RUNG ORTHOPAIR FUZZY SET

The only requirement that IFSs have, is that the sum of membership degree and non-membership degree must not exceed 1, which limits its development to a certain extent. In order to improve upon this drawback, Yager [35] proposed the q-rung orthopair fuzzy sets (q-ROFSs). Some fundamental theorems regarding the q-ROFSs are presented as follows.

Definition 1 [35]: Let $U$ be a fixed set. Then, the q-ROFSs take the following form:

$$Q = \{[(x, (a(x), b(x)), q)] | x \in U\}$$

where the $a(x) : U \rightarrow [0, 1]$ denotes the membership degree, the $b(x) : U \rightarrow [0, 1]$ denotes the non-membership degree. For any element $x \in U$, it satisfies

$$[a(x)]^q + [b(x)]^q \leq 1, \quad q \geq 1$$

The degree of indeterminacy can be expressed as:

$$\pi(x) = \left[1 - \left(a(x)^q + b(x)^q\right)^{\frac{1}{q}}\right]$$

For convenience, we consider $B = (a(x), b(x), q)$ a q-rung orthopair fuzzy number (q-ROFN).

Definition 2 [36]: Let $B = (a(x), b(x), q)$, a score function can be expressed by the formula:

$$S(B) = \frac{1}{2} \left(1 + a(x)^q - b(x)^q\right), \quad S(B) \in [0, 1]$$

Definition 3 [36]: Let $B = (a(x), b(x), q)$, an accuracy function can be expressed by the formula:

$$H(B) = a(x)^q + b(x)^q, \quad H(B) \in [0, 1]$$

B. q-RUNgh ORTHOPAIR HESITANT FUZZY SETS

Definition 4 [49]: Based on the HFSs and q-ROFSs, Liu et al. [49] proposed the q-ROHFS. Let $E$ be a fixed set. The q-ROHFS $E$ defined on $X$ is denoted as:

$$E = \{< x, \Gamma_E(x), \Psi_E(x) > \} | x \in X \}$$

where $\Gamma_E(x)$ and $\Psi_E(x)$ are the two non-empty finite subsets in $[0,1]$, and respectively denote the membership and non-membership degrees of the element $x \in X$. For any element $x \in U$, these satisfy $\forall x \in X, \forall \mu_E(x) \in \Gamma_E, \forall \nu_E(x) \in \Psi_E$ respectively, with the following restrictions:

$$\mu_E^q(x) + \nu_E^q(x) \leq 1, \quad 0 \leq \mu_E \leq 1, \quad 0 \leq \nu_E(x) \leq 1$$

The degree of indeterminacy can be expressed as:

$$\pi_E(x) = \left[1 - (\Gamma_E(x)^q + \Psi_E(x)^q)^{\frac{1}{q}}\right]$$

We consider the $< \Gamma_E(x), \Psi_E(x) >$ as the q-rung orthopair hesitant fuzzy number (q-ROHFN). For convenience, the q-ROHFN is recorded as $h = < \Gamma_h, \Psi_h >$.

Furthermore, if $|\Gamma_h| = |\Psi_h| = 1$, the q-ROHFS degenerates into q-rung orthopair fuzzy set and when $q = 1$, the q-ROHFS degenerates to the DHFS.

Definition 5 [49]: Let $h = < \Gamma_h, \Psi_h >$ as a q-ROHFN. Then, the score function of the q-ROHFS can be expressed by the following formula:

$$S_h = \frac{1}{2} \left(1 + \frac{1}{|\Gamma_h|} \sum_{\mu_E(x) \in \Gamma_h} \mu_E^q(x) - \frac{1}{|\Psi_h|} \sum_{\nu_E(x) \in \Psi_h} \nu_E^q(x)\right), \quad S_h \in [0, 1]$$

Definition 6 [49]: Let $h = < \Gamma_h, \Psi_h >$ as a q-ROHFN. An accuracy function of the q-ROHFS can be expressed by the following formula:

$$D_h = \frac{1}{2} \left(1 - \frac{1}{|\Gamma_h|} \sum_{\mu_E(x) \in \Gamma_h} \mu_E^q(x) + \frac{1}{|\Psi_h|} \sum_{\nu_E(x) \in \Psi_h} \nu_E^q(x)\right), \quad D_h \in [0, 1]$$

Let $h_1 = < \Gamma_{h_1}, \Psi_{h_1} >, h_2 = < \Gamma_{h_2}, \Psi_{h_2} >$ be two q-ROHFNs. Then according to the Definition 5 and Definition 6, $h_1$ and $h_2$ can be compared using the above-mentioned methods. If $S_{h_1} > S_{h_2}$, then $h_1 > h_2$; if $S_{h_1} = S_{h_2}$, then (1) if $D_{h_1} = D_{h_2}$, then $h_1 = h_2$; (2) if $D_{h_1} < D_{h_2}$, then $h_1 < h_2$; (3) if $D_{h_1} > D_{h_2}$, then $h_1 > h_2$.

Definition 7 [49]: Let $h_1 = < \Gamma_{h_1}, \Psi_{h_1} >, h_2 = < \Gamma_{h_2}, \Psi_{h_2} >$ be two q-ROHFNs. Then, the following operation theory can be obtained.

$$h_1 \vee h_2 = \left(\bigcup_{\mu_{E_1} \in \Gamma_1} \{\min\{\mu_{E_1}, \mu_{E_2}\}\}, \bigcup_{\nu_{E_1} \in \Psi_1} \{\min\{\nu_{E_1}, \nu_{E_2}\}\}\right)$$

$$h_1 \wedge h_2 = \left(\bigcup_{\mu_{E_1} \in \Gamma_1} \{\mu_{E_1}, \mu_{E_2}\}, \bigcup_{\nu_{E_1} \in \Psi_1} \{\max\{\nu_{E_1}, \nu_{E_2}\}\}\right)$$

$$\exists \lambda \lambda h_1 = \left(\bigcup_{\mu_{E_1} \in \Gamma_1} \{\lambda \mu_{E_1}\}, \bigcup_{\nu_{E_1} \in \Psi_1} \{(1 - \lambda) \nu_{E_1}\}\right)$$

$$\forall \lambda \lambda h_1 = \left(\bigcup_{\mu_{E_1} \in \Gamma_1} \{\lambda \mu_{E_1}\}, \bigcup_{\nu_{E_1} \in \Psi_1} \{(1 - \lambda) \nu_{E_1}\}\right)$$

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III. DISTANCE AND SIMILARITY MEASURES OF q-RUNG ORTHOPAIR HESITANT FUZZY SETS

A. DISTANCE MEASURE OF q-RUNG ORTHOPAIR HESITANT FUZZY NUMBERS

Definition 8: Let \( h_1 = \langle \Gamma_{h_1}, \Psi_{h_1} >_q \), \( h_2 = \langle \Gamma_{h_2}, \Psi_{h_2} >_q \), \( h_3 = \langle \Gamma_{h_3}, \Psi_{h_3} >_q \) be three q-ROHFNs. A real function \( D : q \rightarrow \text{RHFN}(X) \times q \rightarrow \text{RHFN}(X) \rightarrow [0, 1] \) satisfies the following properties:

1. \( 0 \leq D(h_1, h_2) \leq 1 \);
2. \( D(h_1, h_2) = D(h_2, h_1) \);
3. \( D(h_1, h_2) = 0 \iff h_1 = h_2 \);
4. \( D(h_1, h_3) + D(h_2, h_3) \geq D(h_1, h_2) \).

So we consider \( D(h_1, h_2) \) as the distance measure between the q-ROHFNs \( h_1 \) and \( h_2 \).

We need to make the following assumptions in order to calculate the distance between, and the entropies of the q-ROHFNs.

(A1) The elements in \( \Gamma_{h_1} \) and \( \Psi_{h_1} \) of q-ROHFN-\( h_1(x) \) are sorted in the ascending order. At the same time, \( \mu_{h_1}^{(i)} \) and \( \upsilon_{h_1}^{(i)} \) respectively represent the \( i^{th} \) element values from small to large in \( \Gamma_{h_1} \) and \( \Psi_{h_1} \).

(A2) Let \( h_1 = \langle \Gamma_{h_1}, \Psi_{h_1} >_q \), \( h_2 = \langle \Gamma_{h_2}, \Psi_{h_2} >_q \) be two q-ROHFNs. If \( |\Gamma_{h_1}| \neq |\Gamma_{h_2}| \), then, let \( |\Gamma_{h_1}| = \max(|\Gamma_{h_1}|, |\Gamma_{h_2}|) \) and \( |\Gamma_{h_2}| \) be the number of elements in \( \Gamma_{h_1} \) and \( \Gamma_{h_2} \) respectively. If \( |\Gamma_{h_1}| > |\Gamma_{h_2}| \), then the elements in \( |\Gamma_{h_1}| \) are added until \( |\Gamma_{h_1}| = |\Gamma_{h_2}| \). If \( |\Gamma_{h_1}| < |\Gamma_{h_2}| \), then the elements in \( |\Gamma_{h_1}| \) are added until \( |\Gamma_{h_1}| = |\Gamma_{h_2}| \). In this case, for \( \Psi_{h_1} \) and \( \Psi_{h_2} \), the same process is repeated.

(A3) Let \( h = \langle \Gamma_{h}, \Psi_{h} >_q \) be a q-ROHFN where satisfies \( |\Gamma_{h}| = |\Psi_{h}| \).

For the principle of adding elements, we referred to the following principles:

Definition 9: Let \( h(x) = \{\gamma_i | i = 1, 2, \cdots, l_i\}, h^+ \) and \( h^- \) respectively represent the maximum and minimum values in \( h(x) \), i.e., \( h^+ = \max \{\gamma_i | i = 1, 2, \cdots, l_i\} \) and \( h^- = \min \{\gamma_i | i = 1, 2, \cdots, l_i\} \). Moreover, the element is added to \( h(x) \) according to the following formula:

\[
h(x) = \alpha h^+ + (1 - \alpha) h^- \quad 0 \leq \alpha \leq 1
\]

Example 1: Let \( X = \{x_1, x_2\} \),

\[
H_1 = \begin{cases} < x_1, [0.5, 0.6, 0.7], [0.4, 0.5] >, \\
< x_2, [0.6, 0.8], [0.4, 0.5] > \end{cases}
\]

\[
H_2 = \begin{cases} < x_1, [0.6, 0.8], [0.4, 0.55] >, \\
< x_2, [0.7, 0.8], [0.3, 0.4] > \end{cases}
\]

be two q-ROHFNs on \( X \), then \( H_1 \) and \( H_2 \) can be extended as follows:

\[
H_1' = \begin{cases} < x_1, [0.5, 0.6, 0.7], [0.4, 0.5, 0.5] >, \\
< x_2, [0.6, 0.8], [0.4, 0.5] > \end{cases}
\]

\[
H_2' = \begin{cases} < x_1, [0.6, 0.8, 0.8], [0.4, 0.55, 0.55] >, \\
< x_2, [0.7, 0.8], [0.3, 0.4] > \end{cases}
\]

Definition 10: Let \( h_1 = \langle \Gamma_{h_1}, \Psi_{h_1} >_q \), \( h_2 = \langle \Gamma_{h_2}, \Psi_{h_2} >_q \) be two q-ROHFNs. We propose the three distance formulas between \( h_1 \) and \( h_2 \), as follows:

1. The q-rung orthopair hesitant fuzzy Hamming distance (q-ROHFHD):

\[
D_1(h_1, h_2) = \frac{1}{2l} \sum_{i=1}^{l} (|\mu_{h_1}^{(i)}(x) - \mu_{h_2}^{(i)}(x)| + |\upsilon_{h_1}^{(i)}(x) - \upsilon_{h_2}^{(i)}(x)|)
\]

2. The q-rung orthopair hesitant fuzzy Euclidean distance (q-ROHFED):

\[
D_2(h_1, h_2) = \left\{ \frac{1}{2l} \sum_{i=1}^{l} (|\mu_{h_1}^{(i)}(x) - \mu_{h_2}^{(i)}(x)|^2 + |\upsilon_{h_1}^{(i)}(x) - \upsilon_{h_2}^{(i)}(x)|^2) \right\}^{\frac{1}{2}}
\]

3. The q-rung orthopair hesitant fuzzy Generalized distance (q-ROHGED):

\[
D_3(h_1, h_2) = \left\{ \frac{1}{2l} \sum_{i=1}^{l} (|\mu_{h_1}^{(i)}(x) - \mu_{h_2}^{(i)}(x)|^p + |\upsilon_{h_1}^{(i)}(x) - \upsilon_{h_2}^{(i)}(x)|^p)^{\frac{1}{p}} \right\}^{\frac{1}{p}} \quad p > 0
\]

B. DISTANCE MEASURE OF q-RUNG ORTHOPAIR HESITANT FUZZY SETS

Definition 11: Let \( H_1 \) and \( H_2 \) be two q-ROHFSs, and let \( h_1 = \langle \Gamma_{h_1}, \Psi_{h_1} >_q \), \( h_2 = \langle \Gamma_{h_2}, \Psi_{h_2} >_q \) be two q-ROHFNs of \( H_1 \) and \( H_2 \). We propose the distance measure between \( H_1 \) and \( H_2 \) as follows:

1. The Hamming distance

\[
D'_1(H_1, H_2) = \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{2l} \sum_{i=1}^{l} (|\mu_{h_1}^{(j)}(x_j) - \mu_{h_2}^{(j)}(x_j)| + |\upsilon_{h_1}^{(j)}(x_j) - \upsilon_{h_2}^{(j)}(x_j)|) \right)
\]

2. The Euclidean distance

\[
D'_2(H_1, H_2) = \left\{ \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{2l} \sum_{i=1}^{l} (|\mu_{h_1}^{(j)}(x_j) - \mu_{h_2}^{(j)}(x_j)|^2 + |\upsilon_{h_1}^{(j)}(x_j) - \upsilon_{h_2}^{(j)}(x_j)|^2) \right)^{\frac{1}{2}} \right\}^{\frac{1}{2}}
\]

3. The generalized Euclidean distance

\[
D'_3(H_1, H_2) = \left\{ \frac{1}{n} \sum_{j=1}^{n} \left( \frac{1}{2l} \sum_{i=1}^{l} (|\mu_{h_1}^{(j)}(x_j) - \mu_{h_2}^{(j)}(x_j)|^p + |\upsilon_{h_1}^{(j)}(x_j) - \upsilon_{h_2}^{(j)}(x_j)|^p) \right)^{\frac{1}{p}} \right\}^{\frac{1}{p}} \quad p > 0
\]
(4) The fourth distance

\[ D_3(H_1, H_2) = \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{1}{2} \max \{ |\mu_1^q(x_j) - \mu_2^q(x_j)| \} \right. \]

\[ + |\nu_1^q(x_j) - \nu_2^q(x_j)| + |\pi_1^q(x_j) - \pi_2^q(x_j)| \right] \]

(18)

(5) The mixing distance

\[ D_5(H_1, H_2) = \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{1}{4j} \sum_{i=1}^{lb} \left( |\mu_1^q(x_j) - \mu_2^q(x_j)| \right. \right. \]

\[ + |\nu_1^q(x_j) - \nu_2^q(x_j)| + |\pi_1^q(x_j) - \pi_2^q(x_j)| \]

\[ + \frac{1}{4} \max \{ |\mu_1^q(x_j) - \mu_2^q(x_j)| \} \]

\[ + |\nu_1^q(x_j) - \nu_2^q(x_j)| + |\pi_1^q(x_j) - \pi_2^q(x_j)| \right] \]

(19)

Definition 12: Let \( H_1 \) and \( H_2 \) be two \( q \)-ROHFSs, and let \( h_1 =< \Gamma_{h_1}, \Psi_{h_1} >_q, h_2 =< \Gamma_{h_2}, \Psi_{h_2} >_q \) be two \( q \)-ROHFSs of \( H_1 \) and \( H_2 \). We propose the weighted generalized distance between \( H_1 \) and \( H_2 \) as follows:

\[ D_w^1(H_1, H_2) = \left( \sum_{j=1}^{n} \omega_j \left[ \frac{1}{2j} \sum_{i=1}^{lb} \left( |\mu_1^q(x_j) - \mu_2^q(x_j)| \right. \right. \right. \]

\[ + |\nu_1^q(x_j) - \nu_2^q(x_j)| + |\pi_1^q(x_j) - \pi_2^q(x_j)| \right] \right)^{\frac{1}{2}} \]

(20)

where \( p \geq 1, p \) is a constant and \( p > 0 \).

If \( p = 1 \), the weighted generalized distance is reduced to the weighted Hamming distance

\[ D_w^2(H_1, H_2) = \sum_{j=1}^{n} \omega_j \left[ \frac{1}{2j} \sum_{i=1}^{lb} \left( |\mu_1^q(x_j) - \mu_2^q(x_j)| \right. \right. \right. \]

\[ + |\nu_1^q(x_j) - \nu_2^q(x_j)| + |\pi_1^q(x_j) - \pi_2^q(x_j)| \right] \]

(21)

If \( p = 2 \), the weighted generalized distance is reduced to the weighted Euclidean distance

\[ D_w^3(H_1, H_2) = \left( \sum_{j=1}^{n} \omega_j \left[ \frac{1}{2j} \sum_{i=1}^{lb} \left( |\mu_1^q(x_j) - \mu_2^q(x_j)|^2 \right. \right. \right. \]

\[ + |\nu_1^q(x_j) - \nu_2^q(x_j)|^2 + |\pi_1^q(x_j) - \pi_2^q(x_j)|^2 \right] \right)^{\frac{1}{2}} \]

(22)

Example 2: Let \( X = \{ x_1, x_2 \} \),

\( H_1 = \{ x_1 < 0.5, 0.6, 0.7, 0.3, 0.4, 0.5 \} \]

\( H_2 = \{ x_1 < 0.6, 0.65, 0.8, 0.4, 0.5, 0.55 \} \),

and be two \( q \)-ROHFSs on \( X \), and \( \omega = (0.3, 0.7), q = 3 \), then \( D_w^2(H_1, H_2) = 0.1024, D_w^3(H_1, H_2) = 0.0952 \).

Theorem 1: Let \( H_1 \) and \( H_2 \) be two \( q \)-ROHFSs, and let \( h_1 =< \Gamma_{h_1}, \Psi_{h_1} >_q, h_2 =< \Gamma_{h_2}, \Psi_{h_2} >_q \) be two \( q \)-ROHFSs of \( H_1 \) and \( H_2 \). The weighted generalized distance between \( H_1 \) and \( H_2 \) can be given as follows:

\[ (1) \ 0 \leq D_w(h_1, H_2) \leq 1; \]

\[ (2) D_w(h_1, H_2) = D_w(H_2, H_1); \]

\[ (3) D_w(h_1, H_2) = 0 \Leftrightarrow H_1 = H_2. \]

Proof:

(1) Since \( |\mu_1^q(x_j) - \mu_2^q(x_j)|^p \leq 1 \), we have \( 0 \leq |\mu_1^q(x_j) - \mu_2^q(x_j)| \) \leq 1, \( 0 \leq |\pi_1^q(x_j) - \pi_2^q(x_j)|^p \leq 1 \).

Then

\[ 0 \leq D_w(h_1, H_2) \leq \left( \sum_{j=1}^{n} \omega_j \left[ \frac{1}{2j} \sum_{i=1}^{lb} \left( |\mu_1^q(x_j) - \mu_2^q(x_j)|^p \right. \right. \right. \]

\[ + |\nu_1^q(x_j) - \nu_2^q(x_j)|^p + |\pi_1^q(x_j) - \pi_2^q(x_j)|^p \right] \right)^{\frac{1}{p}} = 1. \]

(2)\[ D_w(h_1, H_2) = \left( \sum_{j=1}^{n} \omega_j \left[ \frac{1}{2j} \sum_{i=1}^{lb} \left( |\mu_1^q(x_j) - \mu_2^q(x_j)|^p \right. \right. \right. \]

\[ + |\nu_1^q(x_j) - \nu_2^q(x_j)|^p + |\pi_1^q(x_j) - \pi_2^q(x_j)|^p \right] \right)^{\frac{1}{p}} = D_w(H_2, H_1). \]

(3) \( D_w(h_1, H_2) = 0 \Leftrightarrow |\mu_1^q(x_j) - \mu_2^q(x_j)|^p = 0 \]

\[ |\mu_1^q(x_j) - \mu_2^q(x_j)|^p = 0, |\pi_1^q(x_j) - \pi_2^q(x_j)|^p = 0. \]

\[ \Leftrightarrow \{ \mu_1^q(x_j) = \mu_2^q(x_j), \mu_1^q(x_j) = \nu_2^q(x_j) \text{ for all } i \} \]

\[ \Leftrightarrow H_1 = H_2. \]

C. SIMILARITY MEASURE OF q-ROHFSs

Definition 13: Let \( h_1 =< \Gamma_{h_1}, \Psi_{h_1} >_q, \)

\( h_2 =< \Gamma_{h_2}, \Psi_{h_2} >_q \) be two \( q \)-ROHFSs. And a real function \( S: q-ROHFNs \times q-ROHFNs \rightarrow [0, 1] \) satisfies the following properties:

\[ (1) \ 0 \leq S(h_1, h_2) \leq 1; \]

\[ (2) S(h_1, h_2) = S(h_2, h_1); \]

\[ (3) S(h_1, h_2) = 1 \Leftrightarrow h_1 = h_2; \]

Then \( S(h_1, h_2) \) is called the similarity measure of the \( q \)-ROHFSNs \( h_1 \) and \( h_2 \).

Definition 14: Let \( H_1 \) and \( H_2 \) be two \( q \)-ROHFSs, and let \( h_1 =< \Gamma_{h_1}, \Psi_{h_1} >_q, h_2 =< \Gamma_{h_2}, \Psi_{h_2} >_q \) be two \( q \)-ROHFSs of \( H_1 \) and \( H_2 \). We propose the similarity measures between \( H_1 \) and \( H_2 \) as follows:

\[ (1) S_1(H_1, H_2) = 1 - \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{1}{2j} \sum_{i=1}^{lb} \left( |\mu_1^q(x_j) - \mu_2^q(x_j)| \right. \right. \right. \]

\[ + |\nu_1^q(x_j) - \nu_2^q(x_j)| + |\pi_1^q(x_j) - \pi_2^q(x_j)| \right] \]

(23)
\[ S_2(H_1, H_2) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \frac{1}{2j} \sum_{i=1}^{l_j} \left( |\mu_1^{(i)}(x_j) - \mu_2^{(i)}(x_j)|^2 + |\nu_1^{(i)}(x_j) - \nu_2^{(i)}(x_j)|^2 + |\pi_1^{(i)}(x_j) - \pi_2^{(i)}(x_j)|^2 \right) \right)^{\frac{1}{2}} \]  

(24) 

\[ S_3(H_1, H_2) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \frac{1}{2j} \sum_{i=1}^{l_j} \left( |\mu_1^{(i)}(x_j) - \mu_2^{(i)}(x_j)|^p + |\nu_1^{(i)}(x_j) - \nu_2^{(i)}(x_j)|^p + |\pi_1^{(i)}(x_j) - \pi_2^{(i)}(x_j)|^p \right) \right)^{\frac{1}{p}} \]  

(25) 

\[ S_4(H_1, H_2) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \frac{1}{2j} \sum_{i=1}^{l_j} \left( \left| \left( \mu_1^{(i)}(x_j) \right)^2 - \left( \mu_2^{(i)}(x_j) \right)^2 \right| + \left| \left( \nu_1^{(i)}(x_j) \right)^2 - \left( \nu_2^{(i)}(x_j) \right)^2 \right| + \left| \left( \pi_1^{(i)}(x_j) \right)^2 - \left( \pi_2^{(i)}(x_j) \right)^2 \right| \right) \right)^{\frac{1}{2}} \]  

(26) 

\[ S_5(H_1, H_2) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \frac{1}{2j} \sum_{i=1}^{l_j} \left( \left| \max_{1 \leq l \leq j} \left( |\mu_1^{(i)}(x_j) - \mu_2^{(i)}(x_j)| \right) \right| + \left| \nu_1^{(i)}(x_j) - \nu_2^{(i)}(x_j) \right| + \left| \pi_1^{(i)}(x_j) - \pi_2^{(i)}(x_j) \right| \right) \right)^{\frac{1}{2}} \]  

(27) 

\[ S_6(H_1, H_2) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \frac{1}{2j} \sum_{i=1}^{l_j} \left( \left| \max_{1 \leq l \leq j} \left( |\mu_1^{(i)}(x_j) - \mu_2^{(i)}(x_j)| \right) \right| + \left| \nu_1^{(i)}(x_j) - \nu_2^{(i)}(x_j) \right| + \left| \pi_1^{(i)}(x_j) - \pi_2^{(i)}(x_j) \right| \right) \right)^{\frac{1}{2}} \]  

(28) 

\[ S_7(H_1, H_2) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \frac{1}{2j} \sum_{i=1}^{l_j} \left( \left| \mu_1^{(i)}(x_j) - \mu_2^{(i)}(x_j) \right| + \left| \nu_1^{(i)}(x_j) - \nu_2^{(i)}(x_j) \right| + \left| \pi_1^{(i)}(x_j) - \pi_2^{(i)}(x_j) \right| \right) \right)^{\frac{1}{2}} \]  

(29) 

\[ S_{\omega}(H_1, H_2) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \frac{1}{2j} \sum_{i=1}^{l_j} \left( |\mu_1^{(i)}(x_j) - \mu_2^{(i)}(x_j)|^p + |\nu_1^{(i)}(x_j) - \nu_2^{(i)}(x_j)|^p + |\pi_1^{(i)}(x_j) - \pi_2^{(i)}(x_j)|^p \right) \right)^{\frac{1}{p}} \]  

(30) 

\[ S_{\omega}^{2}(H_1, H_2) = 1 - \sum_{j=1}^{n} \omega_j \left( \frac{1}{2j} \sum_{i=1}^{l_j} \left( |\mu_1^{(i)}(x_j) - \mu_2^{(i)}(x_j)|^p + |\nu_1^{(i)}(x_j) - \nu_2^{(i)}(x_j)|^p + |\pi_1^{(i)}(x_j) - \pi_2^{(i)}(x_j)|^p \right) \right)^{\frac{1}{p}} \]  

(31) 

If \( p = 2 \), the weighted generalized similarity is reduced to \( S_{\omega}^{2}(H_1, H_2) \) 

\[ S_{\omega}^{2}(H_1, H_2) = 1 - \frac{1}{n} \sum_{j=1}^{n} \frac{1}{2j} \sum_{i=1}^{l_j} \left( |\mu_1^{(i)}(x_j) - \mu_2^{(i)}(x_j)|^2 + |\nu_1^{(i)}(x_j) - \nu_2^{(i)}(x_j)|^2 + |\pi_1^{(i)}(x_j) - \pi_2^{(i)}(x_j)|^2 \right) \]  

(32) 

Example 3: Let \( X = \{ x_1, x_2 \} \), 

\[ H_1 = \{ x_1, [0.5, 0.6, 0.7], [0.4, 0.5] \} \]  

and 

\[ H_2 = \{ x_2, [0.6, 0.65], [0.3, 0.5] \} \] 

be two q-ROHFNs on X, and \( \omega = (0.3, 0.7) \), \( q = 3 \). Then \( H_1 \) and \( H_2 \) can be extended as follows: 

\[ H_1' = \{ x_1, [0.5, 0.6, 0.7], [0.4, 0.5, 0.5] \} \]  

\[ H_2' = \{ x_2, [0.6, 0.65, 0.65], [0.3, 0.5, 0.5] \} \]  

and then \( S_{\omega}^{2}(H_1, H_2) = 0.8699, S_{\omega}^{3}(H_1, H_2) = 0.8767 \). 

IV. ENTROPY OF q-RUNG ORTHOPAIR HESITANT FUZZY SETS 

A. AXIOMATIC DEFINITION OF ENTROPY OF q-ROHFSs 

Definition 16: Let A be a q-ROHFS and let \( h_1 = \langle \Gamma_1, \Psi_1 \rangle \), \( h_2 = \langle \Gamma_2, \Psi_2 \rangle \) be two q-ROHFNs of \( H_1 \) and \( H_2 \). We propose the weighted generalized similarity between \( H_1 \) and \( H_2 \) as follows: 

\[ S_{\omega}^{1}(H_1, H_2) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \frac{1}{2j} \sum_{i=1}^{l_j} \left( |\mu_1^{(i)}(x_j) - \mu_2^{(i)}(x_j)| + |\nu_1^{(i)}(x_j) - \nu_2^{(i)}(x_j)| + |\pi_1^{(i)}(x_j) - \pi_2^{(i)}(x_j)| \right) \right)^{\frac{1}{q}} \]  

(33) 

where \( q \geq 1 \), \( p \) is a constant and \( p > 0 \). 

If \( p = 1 \), the weighted generalized similarity is reduced to 

\[ S_{\omega}^{2}(H_1, H_2) = 1 - \sum_{j=1}^{n} \omega_j \left[ \frac{1}{2j} \sum_{i=1}^{l_j} \left( |\mu_1^{(i)}(x_j) - \mu_2^{(i)}(x_j)| \right) \right] \]  

(34) 

where \( H_0 = \{ x \mid x \in X \} \). Then \( E(A) \) is called the entropy of the q-ROHFS A. 

B. ENTROPY OF q-RUNG ORTHOPAIR HESITANT FUZZY SETS 

Theorem 2: Let A be the q-ROHFS on the universe X = \{x_1, x_2, \ldots, x_n\}, \( h_A(x_j) \) be the q-ROHFN of the element \( x_j \) corresponding to \( X \) of A, and \( E : HFE_q(X) \rightarrow [0, 1] \), then 

A \mapsto E(A) = 1 - D(A, H^0) 

where \( H^0 = \{ x \mid x \in X \} \). Then, \( E(A) \) is the entropy of q-ROHFS A. 

Proof: According to the Definition 16, we have 

(1) When A is a distinct set, i.e., A = \{ \{1\}, \{0\} \} or A = \{ \{0\}, \{1\} \}, we have \( D(A, H^0) = 1 \), and therefore \( E(A) = 0 \).
(2) When \( \forall x \in X, \Gamma_A(x) = \Psi_A(x) = \{0\} \), i.e. \( A = H^0 \); then \( D(A, H^0) = 0 \), and therefore \( E(A) = 0 \).

(3) Then due to \( D(A, H^0) = D(A^c, H^0) \), we have \( 1 - D(A, H^0) = 1 - D(A^c, H^0) \), which proves \( E(A) = E(A^c) \).

(4) Further, if \( D(A, H^0) \leq D(B, H^0) \), where \( H^0 = \{ x \in X \mid \{0\} \} \), then we have \( 1 - D(A, H^0) \geq 1 - D(B, H^0) \), which proves \( E(A) \geq E(B) \). Hence, the theorem is proved.

Moreover, based on the axiomatic definition of entropy of q-ROHFSs, the entropy formula of the q-ROHFS is given as:

\[
E(A) = S(A, H^0)
\]

where \( H^0 = \{ x \in X \mid \{0\} \} \), then \( E(A) \) is the entropy of q-ROHFS \( A \).

\textbf{Example 4:} Let \( X = \{x_1, x_2\} \),

\[
H_1 = \{ x_1, \{0.5, 0.6, 0.7\}, \{0.4, 0.5\} >, \langle x_2, \{0.5, 0.7\}, \{0.4, 0.5\} > \}
\]

and

\[
H_2 = \{ x_1, \{0.6, 0.65\}, \{0.3, 0.5\} >, \langle x_2, \{0.7, 0.8\}, \{0.3, 0.4\} > \}
\]

be two q-ROHFNs on \( X \), and \( \omega = (0.3, 0.7), q = 3 \), then \( H_1 \) and \( H_2 \) can be extended as follows:

\[
H_1' = \{ x_1, \{0.5, 0.6, 0.7\}, \{0.4, 0.5, 0.5\} >, \langle x_2, \{0.5, 0.7\}, \{0.4, 0.5\} > \};
\]

\[
H_2' = \{ x_1, \{0.6, 0.65, 0.65\}, \{0.3, 0.5, 0.5\} >, \langle x_2, \{0.7, 0.8\}, \{0.3, 0.4\} > \};
\]

then \( E_1(H_1) = 0.4064, E_1(H_2) = 0.4520, E_2(H_1) = 0.3759, E_2(H_2) = 0.4194 \).

\vspace{0.5cm}

\textbf{V. COMBINED WEIGHTING TOPSIS MADM BASED ON q-ROHFS}

\textbf{Phase 1: Attribute weight solution model based on combined weighting}

The attribute weight solution model based on combined weighting can be specifically divided into the following stages.

\textbf{Step 1: Constructing a q-rung hesitant fuzzy decision matrix}

Let \( Y = (y_1, y_2, \cdots, y_i) \) be a set of alternatives, and \( C = \{c_1, c_2, \cdots, c_j\} \) be the attribute set of the evaluation target element, scheme \( Y \) be the evaluation value of the q-ROHFN of the attribute \( C \), and is represented as \( a_{ij}, a_{ij} = \langle \Gamma_h, \Psi_h \rangle \).

So we can obtain the decision matrix \( A = (a_{ij})_{m \times n} \), where \( i = 1, 2, \cdots, m \) and \( j = 1, 2, \cdots, n \).

Owing to the calculation needs of the proposed q-rung orthopair hesitant fuzzy distance formula, we need to expand the decision matrix \( A = (a_{ij})_{m \times n} \) into a new decision matrix \( A' = (a_{ij}')_{m \times n} \) according to the assumptions “A1, A2, and A3” made in Definition 8.

\[\begin{array}{c}
165157
\end{array}\]
iterative process is performed. The specific flow chart for the FA is shown in Fig. 1, and the weight index solved by FA is represented by \( \omega_i \).

---

**Flow chart of firefly algorithm.**

![Flow chart of firefly algorithm.](image)

In keeping with the judgment matrix satisfying the condition of complete consistency, the objective function in FA can be established as follows:

\[
\begin{align*}
\min \text{CIF} (n) &= \frac{1}{n} \sum_{i=1}^{n} \left| \sum_{k=1}^{n} (a_{ik}\omega_k) - n\omega_i \right| \\
\text{s.t.} \quad \omega_k &> 0, \quad k = 1, 2, \ldots, n; \quad \sum_{k=1}^{n} \omega_k = 1
\end{align*}
\]  

(45)

where \( \text{CIF} (n) \) is the consistency index function and \( \omega_k \) is the optimization variable. Examination of the consistency of judgment matrix is equivalent to optimizing and solving the formula (45), which makes the judgment matrix more accurate.

**Step 5:** Entropy weight method objectively seek weight

In this paper, the axiom definition of entropy is proposed. Based on this, an objective formula for solving index attribute weights is herein proposed. The solution of objective weight \( \omega_j \) needs to combine the decision matrix \( A \). The formula for solving the weight \( \omega_j \) is as follows:

\[
\omega_j = \frac{1 - E(c_j)}{n - \sum_{j=1}^{n} E(c_j)}
\]  

(46)

where

\[
E(c) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{1}{2l_j} \sum_{i=1}^{l_j} (|\mu_A^i(x_j)| + |\nu_A^i(x_j)| + |\pi_A^i(x_j) - 1|) \right] \right)^{\frac{1}{p}}
\]

In this paper, the axiom definition of entropy is proposed. Based on this, an objective formula for solving index attribute weights is herein proposed. The solution of objective weight \( \omega_j \) needs to combine the decision matrix \( A \). The formula for solving the weight \( \omega_j \) is as follows:

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\]

(45)

where

\[
E(c) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{1}{2l_j} \sum_{i=1}^{l_j} (|\mu_A^i(x_j)| + |\nu_A^i(x_j)| + |\pi_A^i(x_j) - 1|) \right] \right)^{\frac{1}{p}}
\]

In this paper, the axiom definition of entropy is proposed. Based on this, an objective formula for solving index attribute weights is herein proposed. The solution of objective weight \( \omega_j \) needs to combine the decision matrix \( A \). The formula for solving the weight \( \omega_j \) is as follows:

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E(c) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{1}{2l_j} \sum_{i=1}^{l_j} (|\mu_A^i(x_j)| + |\nu_A^i(x_j)| + |\pi_A^i(x_j) - 1|) \right] \right)^{\frac{1}{p}}
\]

(46)

where

\[
E(c) = 1 - \left( \frac{1}{n} \sum_{j=1}^{n} \left[ \frac{1}{2l_j} \sum_{i=1}^{l_j} (|\mu_A^i(x_j)| + |\nu_A^i(x_j)| + |\pi_A^i(x_j) - 1|) \right] \right)^{\frac{1}{p}}
\]
of subjective and objective weights. The combination weighting formula is as follows:

$$\omega_z = s \omega_j + (1 - s) \omega_i$$  \hspace{1cm} (47)$$

where: $0 \leq s \leq 1$. The new extension of TOPSIS method for multiple attribute decision making with q-ROHFSs

This paper proposes an improved TOPSIS Method for Multiple Attribute Decision Making with q-ROHFSs. A generalized weighted distance formula of the q-ROHFS is proposed herein, and based on this, the distance between each scheme and the positive and negative ideal points is calculated. The formula for solving the closeness is improved, and the risk preference coefficient $\theta$ is added. The influence of risk preference coefficient $\theta$ on the ranking of alternatives is also discussed herein.

**Step 7:** Determination of positive and negative ideal solutions

According to step 1, a new q-rung orthopair hesitant fuzzy decision matrix $A'$ is established to determine the positive ideal solution $A'^+ = (a_{ij}^+, a_{2j}^+, \ldots, a_{mj}^+)$ along with the negative ideal solution $A'^- = (a_{ij}^-, a_{2j}^-, \ldots, a_{mj}^-)$ for the index attributes. The positive and negative ideal solutions serve the purpose of classifying the index attributes that need to be referenced. As a general rule, the index attributes of the evaluated objects are divided into benefit types and cost types. The ideal solution is specifically determined by referring to the following formula:

$$a_{ij}^+ = \begin{cases} \max_{i} a_{ij}', j \in U_1 \\ \min_{i} a_{ij}', j \in U_2 \end{cases}; \hspace{1cm} (48)$$

$$a_{ij}^- = \begin{cases} \min_{i} a_{ij}', j \in U_1 \\ \max_{i} a_{ij}', j \in U_2 \end{cases} \hspace{1cm} (49)$$

where $U_1$ and $U_2$ represent the benefit-type attribute set and the cost-type attribute set, respectively.

**Step 8:** Calculating the mixed weighted distance between each alternative and the positive and negative ideal points

Combined with the integrated weight vector $\omega_z$, we use the weighted distance formula to calculate the positive mixed weighted distance measure $D_i^+$ between the evaluation value $a_{ij}$ of each option $y_i$ under each attribute and the positive ideal point $a_{ij}^+$. And calculate the negative mixed weighted distance measure $D_i^-$ between the evaluation value $a_{ij}$ and the negative ideal point $a_{ij}^-$.\n
$$D_i^+(H_1, H_2) = \left[ \sum_{j=1}^{l} \omega_j \left( \frac{1}{2l} \sum_{i=1}^{m} (|\mu_1(i)(x_j) - \mu_2(i)(x_j)|^p + |\nu_1(i)(x_j) - \nu_2(i)(x_j)|^p) \right)^{\frac{1}{p}} \right]^{\frac{1}{q}} \hspace{1cm} (50)$$

$$D_i^- = \sum_{z=1}^{m} \omega_z D_i^{+} \hspace{1cm} (51)$$

**Step 9:** Calculating the relative closeness $RC_i$ of each alternative.

This paper presents an improved formula for calculating the relative closeness. Compared with Ref [30], this paper introduces the risk preference coefficient in the closeness formula. The formula is as follows:

$$RC_i = \frac{(1 - \theta)D_i^{-}}{(1 - \theta)D_i^{+} + \theta D_i^{+}} \hspace{1cm} (53)$$

where $\theta$ is the risk preference coefficient, $0 \leq \theta \leq 1$. $\theta > 0.5$ implies that the decision-maker (DM) is risk-acceptor; when $\theta < 0.5$, the DM is risk-averse; when $\theta = 0.5$ implies that the DM is risk-balanced.

**Step 10:** Ranking the alternatives

According to the relative closeness $RC_i$, the alternative $y_i$ is sorted. The larger the $RC_i$ value, better the alternative $y_i$ will be.

VI. EXAMPLE APPLICATION

In the present section, we use the new extension of TOPSIS Method for Multiple Attribute Decision Making under q-rung orthopair hesitant fuzzy environment for analysis of an example to verify the applicability of this method. In addition, a sensitivity analysis is carried out to discuss the changes in various parameters to sort out the program the impact of the results, and comparison with other q-rung orthopair hesitant fuzzy environment methods is required for verifying the superiority, reliability, and scientific correctness of the method.

A. NUMERICAL EXAMPLE

With the renewal and iteration of the aviation equipment, the process of military aircraft overhaul presents the characteristics of “many types, long cycle, complicated process”, etc. Scientific and reasonable assessment of the effectiveness of military aircraft overhaul is conducive for deepening the repair potential, improving overhaul capabilities, rationally allocating resources, shortening the overhaul cycle, easing the backlog problem, and ensuring normal execution of combat training tasks. In the present work, selecting a better aircraft overhaul plan to ensure the duty rate of aircraft missions is of primary importance. For effective evaluation of aircraft overhaul efficiency, the values of this parameter were taken from four overhaul plants ($y_1, y_2, y_3, y_4$). Values from repairing of the same type of aircraft were selected for evaluation.

This paper analyzes the establishment of an index system for military aircraft repairs and also presents the analysis of related repair-work processes, consults a large number of overhaul-related literature for both domestic and foreign aircrafts, and combines the field survey of the overall workflow of the overhaul plant, screens and summarizes 4 indicators that affect the efficiency of overhaul: equipment support $c_1,$
repair quality $c_2$, repair tasks $c_3$, and repair resources $c_4$, all of which are shown in Table 1.

| Attribute index | Indicator description |
|-----------------|-----------------------|
| equipment support $c_1$ | The equipment support mainly reflects the efficiency of the overhaul in terms of the indicators such as the punctuality of spare parts and the turnover rate of key equipment inventory. |
| repair quality $c_2$ | Maintenance quality is reflected in the rate of rework incidents, the number of quality problems arising during the whole machine repair and the successful pass-rate of flight test. |
| repair tasks $c_3$ | Maintenance tasks are reflected in terms of repair volume, capacity utilization rate, repair cycle, and in-plant cycle. |
| repair resources $c_4$ | Matching rate for spare parts variety, production equipment utilization rate, working time utilization rate, personnel technical skill level, machine location circulation rate, etc. can reflect the impact of repair resources. |

\[ \text{Step 1: Constructing a q-rung orthopair hesitant fuzzy decision matrix.} \]

According to Table 1, let $Y = (y_1, y_2, y_3, y_4)$ be a set of alternative military aircraft overhaul plants, and $C = (c_1, c_2, c_3, c_4)$ be the attribute set of the evaluation target element under the q-rung orthopair hesitant fuzzy environment. So, we can obtain the q-rung orthopair hesitant fuzzy decision matrix as $A = (a_{ij})_{m \times n}$, as shown in Table 2.

Owing to the calculation needs of the proposed q-rung orthopair hesitant fuzzy distance formula, we need to expand the decision matrix $A = (a_{ij})_{m \times n}$ into a new decision matrix $A' = (a_{ij}^2)_{m \times n}$. This paper assumes that the decision-maker is optimistic and let, $\theta = 1$. Brining this assumption into formula 1 and combining it with the assumption in definition 8, the new decision matrix $A'$ can be obtained, as shown in Table 3.

It is easy to verify that each q-rung hesitant fuzzy element in Table 2 satisfies $q \geq 3$. Without loss of generality, this article takes $q = 3$.

Next, the attribute weight determination method mentioned in the above section is used to calculate the attribute weight. The detailed steps are as follows:

\[ \text{Step 2: Initial judgment matrix establishment.} \]

Experts are invited to score the factors affecting the aircraft overhaul efficiency in Table 1. The initial judgment matrix is as follows:

\[
C = \begin{bmatrix}
C & c_1 & c_2 & c_3 & c_4 \\
1 & 1/4 & 1/3 & 2 \\
4 & 1 & 2 & 4 \\
3 & 1/2 & 1 & 3 \\
1/2 & 1/4 & 1/3 & 1
\end{bmatrix}
\]

\[ \text{Step 3: Optimization of the initial judgment matrix $C''$.} \]

With the aim of reducing the influence of expert subjective factors, the original matrix $C''$ is optimized by using formulas 40-44 to obtain the optimized matrix $P$.

\[
P = \begin{bmatrix}
1 & 0.3926 & 0.6148 & 1.1839 \\
2.5474 & 1 & 1.5660 & 3.0158 \\
1.6267 & 0.6386 & 1 & 1.9258 \\
0.8447 & 0.3316 & 0.5193 & 1
\end{bmatrix}
\]

\[ \text{Step 4: Determination of the weight of indicators based on FA.} \]

In the initial setting of FA algorithm in this paper, the number of fireflies considered, $m = 20$, the maximum number of iterations $G = 200$, the disturbance factor $\alpha = 0.2$, $\beta = 0.97$, and the light attraction factor $\lambda = 1$. Referring to the basic flow of the firefly algorithm, and bringing the matrix $P$ from the above-mentioned step 3 into the formula 45, the attribute weight $\omega_j$ is obtained as follows:

\[
\omega_j = (0.1664, 0.4232, 0.2703, 0.1401)
\]

\[ \text{FIGURE 2. Matrix P iterative optimization graph.} \]

Fig. 2 shows the simulated solution of FA in MATLAB. The optimization process in the weighting process is done by the firefly algorithm, and the optimal result is the best choice for the consistency of the judgment matrix.

\[ \text{Step 5: Entropy weight method for solving objective weights.} \]

This paper also proposes an objective method for solving attribute weights. The objective weight $\omega_j$ can be obtained by bringing the elements from matrix $A'$ into the formula 46. At first, we can obtain the entropy value $E_i$ for each attribute as:

\[
E_1 = 0.4352, E_2 = 0.3335, E_3 = 0.3648, E_4 = 0.4902
\]

then the values for $\omega_j$ are:

\[
\omega_j^1 = 0.2377, \omega_j^2 = 0.2805, \omega_j^3 = 0.2673, \omega_j^4 = 0.2155
\]

Consequently, the subjective and objective weights of the attributes can be obtained as shown in Table 4.

\[ \text{Step 6: Obtaining the combined weight $\omega_{ij}$ of the attributes} \]

This model not only considers the experts’ preference for the evaluation attribute indexes of military aircraft overhaul effectiveness, but also considers each attribute index. Objective information on the effectiveness of military aircraft overhauls can successfully reduce the subjective impact and make...
TABLE 2. The q-rung orthopair hesitant fuzzy decision matrix $A$.

|   | $c_1$       | $c_2$       | $c_3$       | $c_4$       |
|---|-------------|-------------|-------------|-------------|
| $y_1$ | $\{0.5,0.6\}$, $\{0.4\}$ | $\{0.7,0.9\}$, $\{0.1,0.3\}$ | $\{0.7,0.8\}$, $\{0.3\}$ | $\{0.3,0.5,0.6\}$, $\{0.4,0.5\}$ |
| $y_2$ | $\{0.4,0.5\}$, $\{0.5,0.6\}$ | $\{0.7,0.9\}$, $\{0.1,0.2\}$ | $\{0.6,0.8\}$, $\{0.2,0.4\}$ | $\{0.2,0.5\}$, $\{0.5,0.7\}$ |
| $y_3$ | $\{0.3,0.5,0.7\}$, $\{0.6,0.8\}$ | $\{0.7,0.8\}$, $\{0.2\}$ | $\{0.7\}$, $\{0.3,0.4\}$ | $\{0.3,0.5\}$, $\{0.3,0.4,0.6\}$ |
| $y_4$ | $\{0.4,0.6\}$, $\{0.3,0.5\}$ | $\{0.7,0.8\}$, $\{0.2,0.3\}$ | $\{0.8\}$, $\{0.1,0.4\}$ | $\{0.4,0.6\}$, $\{0.5\}$ |

TABLE 3. New decision matrix $A'$.

|   | $c_1$       | $c_2$       | $c_3$       | $c_4$       |
|---|-------------|-------------|-------------|-------------|
| $y_1$ | $\{0.5,0.6\}$, $\{0.4,0.4\}$ | $\{0.7,0.9\}$, $\{0.1,0.3\}$ | $\{0.7,0.8\}$, $\{0.3,0.3\}$ | $\{0.3,0.5,0.6\}$, $\{0.4,0.5,0.5\}$ |
| $y_2$ | $\{0.4,0.5\}$, $\{0.5,0.6\}$ | $\{0.7,0.9\}$, $\{0.1,0.2\}$ | $\{0.6,0.8\}$, $\{0.2,0.4\}$ | $\{0.2,0.5\}$, $\{0.5,0.7\}$ |
| $y_3$ | $\{0.3,0.5,0.7\}$, $\{0.6,0.8,0.8\}$ | $\{0.7,0.8\}$, $\{0.2,0.2\}$ | $\{0.7,0.7\}$, $\{0.3,0.4\}$ | $\{0.3,0.5,0.5\}$, $\{0.3,0.4,0.6\}$ |
| $y_4$ | $\{0.4,0.6\}$, $\{0.3,0.5\}$ | $\{0.7,0.8\}$, $\{0.2,0.3\}$ | $\{0.8,0.8\}$, $\{0.1,0.4\}$ | $\{0.4,0.6\}$, $\{0.5,0.5\}$ |

TABLE 4. Subjective and objective weights of each attribute.

|   | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|------|------|------|------|
| $\omega_1$ | 0.2377 | 0.2805 | 0.2673 | 0.2155 |
| $\omega_2$ | 0.1309 | 0.4814 | 0.2954 | 0.0923 |

TABLE 5. The distance between each solution and the positive and negative ideal solutions.

|   | $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|---|------|------|------|------|
| $D^+_j$ | 0.1843 | 0.2549 | 0.2911 | 0.2056 |
| $D^-_j$ | 0.2654 | 0.2044 | 0.2238 | 0.2786 |

the assessment results more scientific and reliable. Assuming $s = 0.5$ and bringing it into formula 47, the combined weight $\omega_c$ of the attributes is found to be:

$$\omega_c = (0.1950, 0.3603, 0.2785, 0.1662)$$

**Step 7**: Determination of positive ideal solutions $A'^+$ and negative ideal solutions $A'^-$.

Since the four attributes that affect the efficiency of aircraft overhaul are all benefit-type attributes, therefore, according to the decision matrix $A'$ and formulas 48 and 49, the positive and negative ideal points of the attribute can be found as:

$$A'^+ = (<0.7>, [0.3], <0.9>, [0.1], >$$
$$<0.8>, [0.1], >, <0.6>, [0.3])$$

$$A'^- = (<0.3>, [0.8], >, <0.7>, [0.3], >$$
$$<0.6>, [0.4], >, <0.2>, [0.7])$$

**Step 8**: Calculation of the mixed weighted distances $D^+_j$ and $D^-_j$.

The attribute weight value $\omega_c$ is brought into the generalized Euclidean weighted distance formulas 51 and 52 to obtain the mixed weighted distances $D^+_j$ and $D^-_j$ as shown in Table 5.

**Step 9**: Calculation of the relative closeness $RC_i$ of each alternative

The present paper improves the calculation formula of relative closeness, and increases the risk preference of DMs. Assuming $\theta = 0.5$, $D^+_j$ and $D^-_j$ from step 8 are brought into the relative closeness formula 52 to get the $RC_i$ value for each scheme as follows:

$$RC_1 = 0.5902, RC_2 = 0.4451, RC_3 = 0.4347, RC_4 = 0.5754$$

**Step 10**: Sorting the alternatives

Based on the results obtained from the closeness of each program, the programs are ranked to get $y_1 \succ y_4 \succ y_2 \succ y_3$. Therefore, it becomes clear that the optimal solution is $y_1$. That is, the military aircraft overhaul efficiency of the overhaul plant $y_1$ is the best.

**B. SENSITIVITY ANALYSIS OF PARAMETERS**

With the aim of analyzing the influence of different parameters on the ranking of the schemes, the present work conducted a sensitivity analysis on the parameters $p$, $s$, and $\theta$.

**Phase 1**: Influence of parameter $p$ on the result of scheme selection.

Values of the parameter $s$ are given below. Different values of distance are chosen for the entropy of the q-ROHFS and the parameter $p$, followed by calculation of the closeness between each scheme along with the positive and negative ideal solutions, and sensitization of the decision result analysis.

Taking $s = 0$, $s = 0.5$, $s = 1$ respectively, and using the entropy formula $E_3$ of the q-ROHFS and the weighted distance formula $D_{i\omega}$, the closeness between each scheme and
FIGURE 3. The influence of parameter $p$ on the decision result.

FIGURE 4. The influence of parameter $s$ on the decision result.
the ideal solution is calculated. The change of the decision result with the parameter \( p \) is shown in Fig. 3.

From the analysis shown in Fig. 3, we can see that as the parameter \( p \) continues to change, the closeness of each program also changes incessantly. When the value of the parameter \( p \) increases to a certain value, the closeness of each program stabilizes gradually, and the final decision result is \( y_4 \succ y_1 \succ y_3 \succ y_2 \). Moreover, when different parameters \( s \) are taken, the ranking of the closeness results for each scheme tends to be stable. When \( s = 1 \), the ranking result for the closeness of each scheme tends to stabilize fastest; when \( s = 0.5 \), the ranking result for the closeness of each scheme tends to be stable; when \( s = 0 \), each sorting result for the closeness of the schemes tends to be stable at the slowest speed.

Phase 2: Influence of parameter \( s \) on the result of scheme selection

Values of the parameter \( p \) are given below. The parameter \( s \) in the combination weighting method selects different values, calculates the closeness between each scheme as well as the positive and negative ideal solutions, in addition to performing a sensitivity analysis on the decision result.

Taking \( p = 1, p = 4, p = 7, p = 15 \), respectively, and using the entropy formula \( E_3 \) and weighted distance formula \( D_{0} \) of the q-ROHFS, the closeness between each scheme and the ideal solution is calculated. The variation of the decision result with the parameter \( s \) is shown in Fig. 4.

From the analysis in Fig. 4, it can be seen that as the parameter \( s \) changes continuously, the closeness of each scheme also changes continuously, but the ranking results for the closeness of each scheme are \( y_1 \succ y_4 \succ y_2 \succ y_3 \). In addition, when different values of the parameter \( p \) are taken, the closeness of each scheme is different.

Phase 3: The influence of parameter \( \theta \) on the result of scheme selection

It can be seen from Fig. 5 that despite the continuous change in the risk preference coefficient \( \theta \), the ranking result for the closeness of each scheme always remains \( y_1 \succ y_4 \succ y_2 \succ y_3 \), which is consistent with the above-mentioned decision result. It also shows that the ranking result for the closeness of each scheme is not sensitive to the change in risk preference coefficient \( \theta \). Among these, as the risk preference coefficient \( \theta \) is close to 0.5, and the distinction of the closeness for each scheme is more obvious; especially when \( \theta = 0.5 \), the ranking for the closeness of each scheme is the best.

C. COMPARATIVE ANALYSIS

A comparative analysis is carried out to illustrate the effectiveness and superiority of the proposed method over the method of HFSs proposed by Xu and Zhang [52], the method of q-ROFSs developed by Liu et al. [4], the method of PHFs proposed by Liang and Xu [48] and PHFSs Khan et al. [53], and other methods using the same illustrative example.

1) COMPARATIVE ANALYSIS BETWEEN THE PROPOSED METHOD AND THE EXISTING MADM METHOD WITH HFSs

HFSs can be regarded as a special case of q-ROHS under the condition of \( q = 1 \), in which only the membership degrees considered. For comparison, q-ROHS can be translated to HFSs by restricting \( q = 1 \) and only considering membership degrees. The hesitant fuzzy information is shown in Table 6.

| \( y_1 \) | \( y_2 \) | \( y_3 \) | \( y_4 \) |
|---|---|---|---|
| \([0.5,0.6]\) | \([0.4,0.5]\) | \([0.3,0.5,0.7]\) | \([0.4,0.6]\) |

Firstly, the hesitant fuzzy TOPSIS method introduced by Xu and Zhang [52] is used to calculate the ranking of alternatives. The closeness of each scheme is calculated to be \( RC_1 = 0.6975, RC_2 = 0.5587, RC_3 = 0.5662, RC_4 = 0.6023 \).

According to the closeness of each scheme, the ranking is \( y_1 \succ y_4 \succ y_2 \succ y_3 \), which shows the best scheme is \( y_1 \).

Obviously, the ranking of alternatives calculated by the hesitant fuzzy TOPSIS method is different from the proposed method, but the two methods obtain the same best choice of \( y_1 \). The main reason is that HFSs TOPSIS method only considers the membership degree, while ignores the non-membership degree, which causes information distortion and fails to meet the actual decision-making situation. This further demonstrates the effectiveness of the proposed method.

2) COMPARATIVE ANALYSIS WITH THE EXISTING MADM METHOD WITH q-ROHS

Q-ROFSs can be regarded as a special case of q-ROHS when there is only one membership degree and one non-membership degree. Here, for comparison, q-ROHSs can be converted to q-ROFSs by calculating the average value of the membership degrees and non-membership degrees. The q-rung orthopair hesitant fuzzy information is shown in Table 7.

| \( y_1 \) | \( y_2 \) | \( y_3 \) | \( y_4 \) |
|---|---|---|---|
| \([0.55,0.4]\) | \([0.8,0.2]\) | \([0.75,0.3]\) | \([0.47,0.45]\) |
| \([0.45,0.55]\) | \([0.8,0.15]\) | \([0.7,0.3]\) | \([0.35,0.6]\) |
| \([0.5,0.7]\) | \([0.75,0.2]\) | \([0.7,0.35]\) | \([0.4,0.43]\) |
| \([0.5,0.4]\) | \([0.75,0.25]\) | \([0.8,0.25]\) | \([0.5,0.5]\) |

Secondly, the MADM method based on q-rung fuzzy orthopair sets introduced by Liu et al. [4] is utilized to
determine the ranking of alternatives. The closeness of each scheme is calculated to be $RC_1 = 0.7686$, $RC_2 = 0.5281$, $RC_3 = 0.3415$, $RC_4 = 0.6959$.

Thirdly, the MADM method based on Pythagorean hesitant fuzzy sets introduced by Liang and Xu [48] is used to calculate the ranking of alternatives. The relative closeness value are $RC_1 = 0.6602$, $RC_2 = 0.5797$, $RC_3 = 0.5611$, $RC_4 = 0.6270$. The ranking of all alternatives is $y_1 > y_4 > y_2 > y_3$, which indicates $y_1$ is the best selection. Moreover, the MADM method based on Pythagorean hesitant fuzzy sets introduced by Khan et al. [53] is used to calculate the ranking of alternatives. The relative closeness values are $RC_1 = 0.6458$, $RC_2 = 0.5824$, $RC_3 = 0.5621$, $RC_4 = 0.6329$. The ranking of all alternatives is $y_1 > y_4 > y_2 > y_3$, which indicates $y_1$ is the best selection.

Obviously, the rankings of alternatives calculated by the PHFSs MADM methods are same with the proposed method. However, the sum of the squares of membership and non-membership degrees due to the complexity and uncertainty of the evaluation environment. In this case, PHFSs is not applicable while q-ROHFSs can overcome these limitations.

4) COMPARATIVE ANALYSIS WITH OTHER MADM METHODS

With the aim of illustrating the reliability of the decision result, the TOPSIS decision method of the q-ROHFS proposed in [49] is used to calculate the relative closeness between each scheme and the ideal solution. The optimal scheme is determined according to the closeness ranking results.

According to the method in [49], the decision matrix $A$ is expanded to a new decision matrix $A'^\prime$.

$$a'_{11} = \{0.5, 0.5, 0.5, 0.6, 0.6, 0.6\},$$
$$a'_{12} = \{0.4, 0.4, 0.4, 0.4, 0.4, 0.4\}$$
$$a'_{13} = \{0.7, 0.7, 0.7, 0.9, 0.9, 0.9\},$$
$$a'_{14} = \{0.1, 0.1, 0.3, 0.3, 0.3, 0.3\}$$
$$a'_{21} = \{0.3, 0.3, 0.5, 0.5, 0.5, 0.5\},$$
$$a'_{22} = \{0.4, 0.4, 0.4, 0.5, 0.5, 0.5\}$$

$$a'_{31} = \{0.5, 0.6, 0.6, 0.6, 0.6, 0.6\}$$
$$a'_{32} = \{0.6, 0.6, 0.6, 0.6, 0.6, 0.6\}$$
$$a'_{33} = \{0.7, 0.7, 0.7, 0.7, 0.7, 0.7\}$$
$$a'_{34} = \{0.2, 0.2, 0.2, 0.2, 0.2, 0.2\}$$
$$a'_{41} = \{0.2, 0.2, 0.2, 0.2, 0.2, 0.2\}$$
$$a'_{42} = \{0.5, 0.5, 0.5, 0.5, 0.5, 0.5\}$$

The data in decision matrix $A'^\prime$ is substituted into the TOPSIS method taken from [49]. The positive mixed weighted distance measure $D_j^+$ and the negative mixed weighted distance measure between the negative ideal point $D_j^-$ are calculated, as presented in Table 6.

### TABLE 8. Pythagorean hesitant fuzzy decision matrix.

| $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|-------|-------|-------|-------|
| $c_1$ | $[0.5, 0.5, 0.6, 0.4]$ | $[0.6, 0.6, 0.7, 0.5]$ | $[0.7, 0.7, 0.8, 0.5]$ | $[0.8, 0.8, 0.9, 0.5]$ |
| $c_2$ | $[0.6, 0.6, 0.7, 0.5]$ | $[0.7, 0.7, 0.8, 0.5]$ | $[0.8, 0.8, 0.9, 0.5]$ | $[0.9, 0.9, 1.0, 0.6]$ |
| $c_3$ | $[0.7, 0.7, 0.8, 0.5]$ | $[0.8, 0.8, 0.9, 0.5]$ | $[0.9, 0.9, 1.0, 0.6]$ | $[1.0, 1.0, 1.0, 0.7]$ |
| $c_4$ | $[0.8, 0.8, 0.9, 0.5]$ | $[0.9, 0.9, 1.0, 0.6]$ | $[1.0, 1.0, 1.0, 0.7]$ | $[1.1, 1.1, 1.1, 0.8]$ |
The closeness of each scheme is calculated to be $RC_1 = 0.5592$, $RC_2 = 0.5231$, $RC_3 = 0.3969$, $RC_4 = 0.5298$. According to the closeness of each scheme, the ranking is $y_1 \succ y_4 \succ y_2 \succ y_3$, which indicates the best scheme is $y_1$.

Lastly, we calculate the comprehensive evaluation values using the q-Rung dual hesitant fuzzy weighted heronian mean ($q-\text{RDHFHWHM}$) operator Xu et al. [54]. The score values are $S(y_1) = 0.6267$, $S(y_2) = 0.5604$, $S(y_3) = 0.5527$, $S(y_4) = 0.6160$. The ranking of all alternatives is $y_1 \succ y_4 \succ y_2 \succ y_3$, which indicates $y_1$ is the best selection.

Obviously, the ranking of alternatives calculated by Liu et al. [49] and q-\text{RDHFHWHM} Xu et al. [54] is same with the proposed method. However, the MADM method by Liu et.al. [49] possesses the following shortcomings: (1) the distance formula does not consider the degree of hesitation, which causes loss of information and affects the real decision-making results; (2) the attribute weight is not objective enough and the calculation method of attribute weight is not given; (3) the relative closeness formula does not consider the risk preference of the decision maker. Besides, the q-\text{RDHFHWHM} developed by Xu et al. [54] does not formulate the expressions for calculating the weight, making the result not objective enough.

The ranking values of the above discussion is given in Table 10.

*TABLE 9.* The distance between each solution and the positive and negative ideal solutions.

| $y_1$ | $y_2$ | $y_3$ | $y_4$ |
|------|------|------|------|
| $D^+_1$ | 0.2246 | 0.2322 | 0.2723 | 0.2316 |
| $D^-_1$ | 0.2849 | 0.2547 | 0.1792 | 0.2610 |

Based on the above comparison analyses, the advantages of the proposed methods are summarized as the follows.

The q-rung orthopair hesitant fuzzy sets (q-ROHFSs) are very suitable for illustrating uncertain or fuzzy information in MCDM problems because the membership degrees and non-membership degrees can be two sets of several possible values, which cannot be achieved by q-ROFSs, HFSs and PHFSs.

Besides, the proposed method of combined weighting is used to describe the weight of q-rung orthopair hesitant fuzzy decision information, in which the two aspects of subjectivity and objectivity are incorporated. Therefore, the evaluation results are more scientific and effective. Moreover, the proposed method comprehensively considers the membership degree, non-membership degree and hesitation degree of q-rung orthopair hesitant fuzzy information, which matches the real complex decision-making environment. Nevertheless, this effect cannot be achieved by the methods proposed by Liu et al. [49] and Xu et al. [54]. Furthermore, the effect of relevant parameters on the decision results is discussed, which makes the decision results more reliable and flexible. In addition, this paper also considers the decision-maker’s risk preference in the decision-making process. A sensitivity analysis of the decision result to the risk preference coefficient is conducted, which makes the decision result more reasonable and accurate. All in all, the proposed method is effective and reliable.

*VII. CONCLUSION*

In this paper, we define the distance and similarity measures of q-rung orthopair hesitant fuzzy sets, provide their specific formulas, and discuss their properties. Based on them, the entropy of q-rung orthopair hesitant fuzzy sets is defined. The relationship between the entropy and the distance of q-rung orthopair hesitant fuzzy sets is studied. The theorem for constructing the entropy based on the distance of q-rung orthopair hesitant fuzzy sets is proposed and the specific formulas for the entropy of the q-rung orthopair hesitant fuzzy sets are given. Based on these results, we further construct a TOPSIS multi-attribute decision-making model under the q-rung orthopair hesitant fuzzy environment. In addition, we introduce the decision-making example of military aircraft overhaul effectiveness evaluation, use the proposed TOPSIS multi-attribute decision-making model to derive the final decision result, and analyze the effect of various parameters on the result. Compared with the existing decision methods, i.e., q-ROFSs, HFSs, PHFSs and q-ROHFSs, it proves that the proposed TOPSIS attribute decision-making method is reasonable and effective.

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