Weighted Bergman-Dirichlet and Bargmann-Dirichlet spaces of order $m$
Explicit formulae for reproducing kernels and asymptotic

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Abstract

We introduce new functional spaces that generalize the weighted Bergman and Dirichlet spaces on the disk $D(0, R)$
in the complex plane and the Bargmann-Fock spaces on the whole complex plane. We give a complete description of
the considered spaces. Mainly, we are interested in giving explicit formulas for their reproducing kernel functions and
their asymptotic behavior as $R$ goes to infinity.

Keywords: Weighted Bergman-Dirichlet spaces, Weighted Bargmann-Dirichlet spaces, Reproducing kernel
function, Hypergeometric function, Asymptotic behavior

1. Introduction and main results

Let $D$ be the unit disk in the complex plane $\mathbb{C}$ and denote by $D_\gamma$; $\gamma \in \mathbb{R}$, the functional space of all analytic
functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $D$ such that its norm $\|f\|_\gamma$ is finite, where

$$\|f\|_\gamma^2 = \sum_{n=0}^{\infty} (n+1)^\gamma |a_n|^2.$$  

Thus for special values $\gamma = -1$, $\gamma = 0$ and $\gamma = 1$ we have the Bergman, Hardy and Dirichlet space, respectively. More general, $D_\gamma$ is a weighted Bergman space when $\gamma < 0$ and a weighted Dirichlet space when $\gamma > 0$. Such spaces play important roles in function theory and operator theory, as well as in modern analysis, probability and statistical analysis. For a nice introduction and surveys of these spaces in the context function and operator theories, see \cite{4,5,7,1} and the references therein.

Added to the sequential characterization, the weighted Bergman space can be described differently. It can be realized as the $(1-|z|^2)^\alpha d\lambda$ square integrable functions on $D$ that are holomorphic on $D$

$$\mathcal{A}^{2,\alpha}(D) := L^2(D; (1-|z|^2)^\alpha d\lambda) \cap \mathcal{H}^o(D),$$

where $d\lambda(z) = dx dy = \frac{1}{2} dz \wedge d\bar{z}$ with $z = x + iy$; $x, y \in \mathbb{R}$, is the two dimensional Lebesgue area measure. The corresponding reproducing kernel is known to be given through

$$K^\alpha(z, w) = \left(\frac{\alpha + 1}{\pi} \right) \left(\frac{1}{1 - z\bar{w}}\right)\alpha.$$
Moreover, a function \( f \) is finite. A more convenient norm to use on the classical Dirichlet space is the following semi-norm defined by the Dirichlet integral

\[
D(f) := \int_D |f'(z)|^2 \, d\Lambda(z) = \sum_{n=0}^{\infty} |a_n|^2
\]

is finite. A more convenient norm to use on the classical Dirichlet space is the following

\[
\|f\|_{1,0}^2 := |f(0)|^2 + \frac{1}{\pi} \int_D |f'(z)|^2 \, d\Lambda(z).
\]

The reproducing kernel of the classical Dirichlet space with respect to this norm is known to be given by \([1]\)

\[
K(z; w) = \frac{1}{\pi} \left( 1 + \log \left( \frac{1}{|z w|} \right) \right): \quad z, w \in \mathbb{D}.
\]

Theorem 1.1. The space \( \mathcal{A}_{R,m}^\alpha(\mathbb{D}_R) \) is non trivial if and only if \( \alpha > -1 \). In this case \( \mathcal{A}_{R,m}^\alpha(\mathbb{D}_R) \) is a reproducing kernel Hilbert space. Its reproducing kernel is given explicitly in terms of the \( _2F_1 \)-hypergeometric function as

\[
K_{R,m}^\alpha(z, w) = \frac{(\alpha + 1)}{\pi R^2} \left\{ \sum_{n=0}^{\infty} \frac{\Gamma(\alpha + 2 + n)}{\Gamma(n+1)\Gamma(\alpha+2)} \left( \frac{w}{R^2} \right)^n + \frac{(\alpha \Gamma(\alpha+1) R^{2\alpha-m+2})}{(m!)^2} \Gamma(\alpha+1) \Gamma(\alpha+\alpha+2) \right\}.
\]

Moreover, a function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) on \( \mathbb{D}_R \) belongs to \( \mathcal{A}_{R,m}^\alpha(\mathbb{D}_R) \) if and only if

\[
\|f\|_{R,m}^2 = \sum_{n=0}^{\infty} \left( \frac{n \Gamma(\alpha+1) R^{2\alpha-n+2}}{\Gamma(n+\alpha+2)} \right) |a_n|^2 + \sum_{n=m}^{\infty} \left( \frac{(n!)^2 \Gamma(\alpha+1) R^{2(\alpha-m)+2}}{(n-m)! \Gamma(\alpha+\alpha+2)} \right) |a_n|^2 < +\infty.
\]

Remark 1.2. 1. The special case of \( R = 1 \) and \( m = 0 \) leads to the weighted Bergman space \([1]\). In this case, the expression of the reproducing kernel in \([4]\) reduces further to the Bergman reproducing kernel \([2]\).

2. For \( R = 1, \alpha = 0 \) and \( m = 1 \), the corresponding space is the classical Dirichlet space. In this case the expression of the reproducing kernel \([4]\) reduces further to the reproducing kernel given through \([3]\) of the classical Dirichlet space.

What we do in the construction of \( \mathcal{A}_{R,m}^\alpha(\mathbb{D}_R) \) works mutatis mutandis to introduce and study their analogues on the whole complex plane \( \mathbb{C} \), the Bargmann-Dirichlet spaces \( \mathcal{B}_{m}^\alpha(\mathbb{C}) \) of order \( m \) (see Section 3). The following is the analog of Theorem 1.1 for these spaces.

Theorem 1.3. The space \( \mathcal{B}_{m}^\alpha(\mathbb{C}) \) is a reproducing kernel Hilbert space. Its reproducing kernel function is given in terms of the \( _2F_1 \)-hypergeometric function as

\[
K_m^\alpha(z, w) = \frac{1}{\pi} \sum_{k=0}^{m-1} \left( \frac{v^k w}{k!} \right)^2 + \frac{(\alpha \Gamma(\alpha+1) R^{2\alpha-m+2})}{(m!)^2} \Gamma(\alpha+1) \Gamma(\alpha+\alpha+2) \right\}.
\]

Moreover, a function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) belongs to \( \mathcal{B}_{m}^\alpha(\mathbb{C}) \) if and only if

\[
\|f\|_{\alpha,m}^2 = \pi \left\{ \sum_{n=0}^{m-1} \left( \frac{n!}{\sqrt{n+1}} \right) |a_n|^2 + \sum_{n=m}^{\infty} \left( \frac{(n!)^2}{(n-m)!} \right) |a_n|^2 \right\} < +\infty.
\]
Remark 1.4. For \( m = 0 \), the space \( \mathcal{B}_{m}^{2,\alpha}(\mathbb{C}) \) is the Bargmann-Fock space, consisting of holomorphic functions on \( \mathbb{C} \) that are \( e^{|z|^2} \) \( d\lambda \)-square integrable. Its reproducing kernel function is known to be given by

\[
K'(z, w) = \left( \frac{\lambda}{\pi} \right)^{\alpha} e^{-\frac{|z|^2}{\lambda}}.
\]

Motivated by the fact that the flat hermitian geometry on \( \mathbb{C} \) can be approximated by the complex hyperbolic geometry of the disks \( \mathbb{D}_R \) of radius \( R > 0 \) associated to an appropriate scaled Bergman Kähler metric [2] (see Section 4), we show that the spaces \( \mathcal{B}_{m}^{2,\alpha}(\mathbb{C}) \), with \( \alpha = vR^2 \), can be seen as the limit of the spaces \( \mathcal{A}_{R,m}^{2,\alpha}(\mathbb{D}_R) \) as \( R \) goes to infinity, in the sense that we have

Theorem 1.5. For every fixed nonnegative integer \( m \), the reproducing kernel \( K_{R,m}^{2,\alpha}(z, w) \) of the weighted Bergman-Dirichlet space \( \mathcal{A}_{R,m}^{2,\alpha}(\mathbb{D}_R) \) converges pointwise and uniformly on compact sets of \( \mathbb{C} \times \mathbb{C} \) to the reproducing kernel function of weighted Bargmann-Dirichlet space \( \mathcal{B}_{m}^{2,\alpha}(\mathbb{C}) \).

The paper is organized as follows. In the succeeding sections (Sections 2 and 3), we discuss the proofs of our main results, Theorems 1.1 and 1.3, stated in this introductory section. Moreover, we give a complete description of the considered Hilbert spaces \( \mathcal{A}_{R,m}^{2,\alpha}(\mathbb{D}_R) \) and \( \mathcal{B}_{m}^{2,\alpha}(\mathbb{C}) \), including the explicit formulae for their reproducing kernel functions. In Section 4, we show that the \( L^2 \)-eigenprojector kernel of \( \mathcal{A}_{R,m}^{2,\alpha}(\mathbb{D}_R) \) on \( \mathbb{D}_R \) gives rise to its analogue of \( \mathcal{B}_{m}^{2,\alpha}(\mathbb{C}) \) on \( \mathbb{C} \) by letting \( R \) tends to infinity.

2. Weighted Bergman-Dirichlet spaces of order \( m \) on the disk \( \mathbb{D}_R \)

Denote by \( \mathbb{D}_R = \{ z \in \mathbb{C} : |z| < R \} \) the disk of radius \( R > 0 \) in the complex plane \( \mathbb{C} \). For given \( \alpha \in \mathbb{R} \), let \( L^2(\mathbb{D}_R; d\mu_{\alpha,R}) := L^2(\mathbb{D}_R; d\mu_{\alpha,R}) \) be the space of complex valued functions on \( \mathbb{D}_R \) that are square-integrable with respect to the density measure

\[
d\mu_{\alpha,R}(z) = \left( 1 - \frac{|z|^2}{R^2} \right)^\alpha d\lambda(z),
\]

\( d\lambda \) being the two dimensional Lebesgue area measure on \( \mathbb{C} \). The space \( L^2(\mathbb{D}_R; d\mu_{\alpha,R}) \) is a Hilbert space in the norm

\[
\| f \|_{\mu_{\alpha,R}}^2 := \int_{\mathbb{D}_R} |f(z)|^2 d\mu_{\alpha,R}(z)
\]

(8)

corresponding to the hermitian scaler product

\[
\langle f, g \rangle_{\mu_{\alpha,R}} := \int_{\mathbb{D}_R} f(z)\overline{g(z)}d\mu_{\alpha,R}(z).
\]

(9)

By \( \mathcal{H}ol(\mathbb{D}_R) \), we denote the vector space of all convergent entire series \( f(z) = \sum_{n=0}^{+\infty} a_n z^n \) on \( \mathbb{D}_R \). Note that, for a given arbitrary nonnegative integer \( m = 0, 1, 2, \ldots \), we can split any \( f \in \mathcal{H}ol(\mathbb{D}_R) \) as

\[
f(z) = f_{1,m}(z) + f_{2,m}(z),
\]

(10)

where

\[
f_{1,m}(z) := \sum_{n=0}^{m-1} a_n z^n \quad \text{and} \quad f_{2,m}(z) := \sum_{n=m}^{+\infty} a_n z^n = f(z) - f_{1,m}(z)
\]

so that \( f^{(m)} = f_{2,m}^{(m)} \), with the convention that \( f_{1,0}(z) = 0 \) when \( m = 0 \). Thus for any fixed nonnegative integer \( m \), we consider the functional space \( \mathcal{A}_{R,m}^{2,\alpha}(\mathbb{D}_R) \) of all \( f \in \mathcal{H}ol(\mathbb{D}_R) \) such that

\[
\| f \|_{\mathcal{A}_{R,m}^{2,\alpha}(\mathbb{D}_R)}^2 := \| f_{1,m} \|_{\mu_{\alpha,R}}^2 + \| f_{2,m}^{(m)} \|_{\mu_{\alpha,R}}^2 < +\infty.
\]

(11)
We denote by \( \langle \cdot, \cdot \rangle_{a,m,R} \) the associated hermitian scalar product defined by
\[
\langle f, g \rangle_{a,m,R} := \langle f_{1,m}, g_{1,m} \rangle_{a,R} + \left( f_{2,m} - g_{2,m} \right)_{a,R}
\] (12)
for given \( f, g \in A^2_{R,0}(\mathbb{D}) \).

The aim of this section is to give a concrete description of \( A^2_{R,0}(\mathbb{D}) \) and prove Theorem 1. We begin with the following

**Lemma 2.1.** Keep notations as above.

(i) The monomials \( e_n(z) = z^n \) are pairwise orthogonal with respect to the hermitian scalar product \( \langle \cdot, \cdot \rangle_{a,m,R} \) in (12).

(ii) The monomials \( e_n(z) = z^n \) belong to \( A^2_{R,0}(\mathbb{D}) \) if and only if \( \alpha > -1 \).

(iii) For \( \alpha > -1 \), we have
\[
\|e_n\|^2_{a,m,R} = \pi \begin{cases} \frac{1}{R^{2n+2}} \frac{\Gamma((n+1)\alpha)}{\Gamma(n+2)} & \text{for } n < m \\ \frac{1}{R^{2(n-m)+2}} \frac{\Gamma((n-m+1)\alpha)}{\Gamma(n-m+2)} & \text{for } n \geq m \end{cases}
\] (13)

**Proof.** For (i), we distinguish three cases. Indeed, we have
\[
\langle e_n, e_k \rangle_{a,m,R} = \begin{cases} \langle e_n, e_k \rangle_{a,R} & \text{for } n, k < m \\ 0 & \text{for } n < m, k \geq m \\ \frac{n!}{(n-m)!} \langle e_{n-m}, e_k \rangle_{a,R} & \text{for } n, k \geq m \end{cases}
\] (14)

This reduces further to the computation of \( \langle e_n, e_k \rangle_{a,R} \), which can be handled using polar coordinates \( z = r e^{i\theta} \) with \( r \in [0, 1] \) and \( \theta \in [0, 2\pi] \). Thus, we have
\[
\langle e_n, e_k \rangle_{a,R} = \int_{\mathbb{D}} z^n \overline{z}^k \left( 1 - \frac{|z|^2}{R} \right)^\alpha dA(z)
\]
\[
= \int_{[0,1] \times [0,2\pi]} (re^{i\theta})^n(e^{-i\theta})^k(1 - r^2)^\alpha r^2 dr d\theta.
\]

By means of Fubini-Tonelli theorem, we get
\[
\langle e_n, e_k \rangle_{a,R} = R^{n+k+2} \int_0^1 r^{n+k+1} (1 - r^2)^\alpha \left( \int_0^{2\pi} e^{in\theta} d\theta \right) dr
\]
\[
= 2\pi R^{n+k+2} \int_0^1 r^{n+k+1} (1 - r^2)^\alpha dr \delta_{n,k}.
\] (15)

Whence in view of (14), we conclude that \( \langle e_n, e_k \rangle_{a,m,R} = 0 \) for \( n \neq k \).

The proof of (ii) follows by taking \( k = n \) in (15) and next making use of the change \( t = r^2 \). Indeed, we obtain
\[
\|e_n\|^2_{a,m,R} = \pi R^{2n+2} \int_0^1 t^n (1 - t)^{\alpha} dt.
\]
The involved integral is then a special case of the well known Euler Beta function \[\text{B}(x, y) := \int_0^1 t^{x-1} (1 - t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \text{ p. 18}\]

provided that \( \Re(x) > 0 \) and \( \Re(y) > 0 \). Therefore, the norm \( \|e_n\|^2_{a,m,R} \) is finite if and only if \( \alpha > -1 \). In this case, we have
\[
\|e_n\|^2_{a,m,R} = \frac{\pi R^{2n+2} \Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}.
\]
By substituting this in (14), it follows
\[
\|e_n\|^2_{\mathcal{H}_{a,m,R}} = \begin{cases} 
\pi R^{2n+2} \frac{n! \Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} & n < m \\
\pi R^{2(n-m)+2} \frac{(n-1)! \Gamma(\alpha+1)}{\Gamma(n-m+\alpha+2)} & n \geq m
\end{cases}
\]
Thus the proof is completed. \( \square \)

The first main result of this section is the following

**Lemma 2.2.** The space \( \mathcal{H}_{R,m}^{a,\alpha}(\mathbb{D}) \) is nontrivial if and only if \( \alpha > -1 \). In this case, a function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) belongs to \( \mathcal{H}_{R,m}^{a,\alpha}(\mathbb{D}) \) if and only if \( (a_n)_n \) satisfies the growth condition \( \sum_{n=0}^{\infty} |a_n|^2 \|e_n\|^2_{\mathcal{H}_{a,m,R}} < +\infty \), which reads explicitly as,

\[
\|f\|^2_{\mathcal{H}_{a,m,R}} = \pi \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n-1} \frac{n! \Gamma(\alpha+1) R^{2k+2}}{\Gamma(n+k+\alpha+2)} \right) |a_n|^2 + \sum_{n=0}^{\infty} \left( \sum_{k=1}^{n} \frac{(n)! \Gamma(\alpha+1) R^{2(n-k)+2}}{k! \Gamma(n-k+\alpha+2)} \right) |a_n|^2 < +\infty.
\] (16)

**Proof.** Lemma 2.1 shows that the monomials \( z^n \) belong to \( \mathcal{H}_{R,m}^{a,\alpha}(\mathbb{D}) \) under the assumption \( \alpha > -1 \). For the converse, assume that \( \mathcal{H}_{R,m}^{a,\alpha}(\mathbb{D}) \) is nontrivial and pick a nonzero function \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) such that \( \|f\|^2_{\mathcal{H}_{a,m,R}} < +\infty \). Therefore, according to (i) of Lemma 2.1, we get

\[
\|f\|^2_{\mathcal{H}_{a,m,R}} = \sum_{n=0}^{\infty} |a_n|^2 \|e_n\|^2_{\mathcal{H}_{a,m,R}} < +\infty.
\]

This implies that \( |a_n|^2 \|e_n\|^2_{\mathcal{H}_{a,m,R}} < +\infty \) for every \( n \) and in particular for certain \( n_0 \) for which \( a_{n_0} \neq 0 \). Thus from (ii) of Lemma 2.1 we deduce that \( \alpha > -1 \). \( \square \)

**Remark 2.3.** For \( R = 1 \) and \( \alpha = 0 \), the spaces \( \mathcal{H}_{R,m}^{a,\alpha}(\mathbb{D}) \) corresponding to \( m = 0 \) and \( m = 1 \) can be identified respectively to

\[
\left\{ (a_n)_{n \geq 0} \subset \mathbb{C}; \sum_{n=0}^{\infty} \frac{|a_n|^2}{n+1} < +\infty \right\} \quad \text{and} \quad \left\{ (a_n)_{n \geq 1} \subset \mathbb{C}; \sum_{n=1}^{\infty} n|a_n|^2 < +\infty \right\}
\]

which are respectively the sequential characterization of the classical Bergman and Dirichlet spaces.

**Definition 2.4.** We will call \( \mathcal{H}_{R,m}^{a,\alpha}(\mathbb{D}) \), when \( \alpha > -1 \), the weighted Bergman-Dirichlet space of order \( m \).

From now on we assume that \( \alpha > -1 \).

**Lemma 2.5.** For every fixed \( m \), the space \( \mathcal{H}_{R,m+1}^{a,\alpha}(\mathbb{D}) \), \( \alpha > -1 \), is continuously embedded in \( \mathcal{H}_{R,m}^{a,\alpha}(\mathbb{D}) \), in the sense that

\[
\|f\|^2_{\mathcal{H}_{a,m,R}} \leq \frac{1}{C_{a,R,m}} \|f\|^2_{\mathcal{H}_{a,m+1,R}}.
\] (17)

for every \( f \in \mathcal{H}_{R,m}^{a,\alpha}(\mathbb{D}) \), where

\[
C_{a,R,m} = \min \left( 1, \frac{R^{2m+2}\Gamma(\alpha+2)}{m!\Gamma(m+\alpha+2)} \frac{\alpha}{R^2} \right).
\]

In particular, \( \mathcal{H}_{R,m}^{a,\alpha}(\mathbb{D}) \) is continuously embedded in the weighted Bergman space \( \mathcal{H}_{R}^{a,\alpha}(\mathbb{D}) = \mathcal{H}_{R,m}^{a,\alpha}(\mathbb{D}) \).
Proof. According to Lemma 2.2, any \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) in \( \mathcal{A}_{R,m+1}^{2,0}(\mathbb{D}_R) \) satisfies

\[
\|f\|_{\alpha,m+1,R}^2 = \sum_{n=0}^{\infty} |a_n|^2 \|e_n\|_{\alpha,m+1,R}^2 < +\infty; \ e_n(z) = z^n.
\]

Now by means of (3), we get

\[
\|e_n\|_{\alpha,m+1,R}^2 = \begin{cases} \|e_n\|_{\alpha,m,R}^2 & \text{for } n < m < m + 1 \\ \frac{R^2m\Gamma(\alpha + 2)}{m!\Gamma(m + \alpha + 2)} \|e_m\|_{\alpha,m,R}^2 & \text{for } n = m < m + 1 \\ \frac{(n-\alpha)(n-m+1)}{R^2} \|e_n\|_{\alpha,m,R}^2 & \text{for } n \geq m + 1 \geq m \end{cases}
\]

and therefore

\[
\|e_n\|_{\alpha,m+1,R}^2 \geq \min\left(1, \frac{R^2m\Gamma(\alpha + 2)}{m!\Gamma(m + \alpha + 2)} \frac{\alpha}{R^2}\right) \|e_n\|_{\alpha,m,R}^2 \geq C_{\alpha,R,m} \|e_n\|_{\alpha,m,R}^2,
\]

where the constant \( C_{\alpha,R,m} \) is independent of \( n \). It is given by

\[
C_{\alpha,R,m} = \min\left(1, \frac{R^2m\Gamma(\alpha + 2)}{m!\Gamma(m + \alpha + 2)} \frac{\alpha}{R^2}\right).
\]

Thus, it follows

\[
\|f\|_{\alpha,m+1,R}^2 \geq C_{\alpha,R,m} \sum_{n=0}^{\infty} |a_n|^2 \|e_n\|_{\alpha,m,R}^2 \geq C_{\alpha,R,m} \|f\|_{\alpha,m,R}^2.
\]

This implies in particular that \( \|f\|_{\alpha,m,R}^2 \) is finite, so that \( f \in \mathcal{A}_{R,m}^{2,0}(\mathbb{D}_R) \) and the embedding mapping is continuous from \( \mathcal{A}_{R,m+1}^{2,0}(\mathbb{D}_R) \) into \( \mathcal{A}_{R,m}^{2,0}(\mathbb{D}_R) \). This completes the proof. \( \square \)

An other basic property for the spaces \( \mathcal{A}_{R,m}^{2,0}(\mathbb{D}_R) \) is the following

**Proposition 2.6.** The space \( \mathcal{A}_{R,m}^{2,0}(\mathbb{D}_R) \) is a Hilbert space and the monomials \( e_n(z) = z^n \); \( n \geq 0 \), constitute an orthogonal basis of it.

**Proof.** Since \( \mathcal{A}_{R,m}^{2,0}(\mathbb{D}_R) \) is continuously embedded in the weighted Bergman space \( \mathcal{A}_{R,m}^{2,0}(\mathbb{D}_R) \), it is not difficult to see that \( \mathcal{A}_{R,m}^{2,0}(\mathbb{D}_R) \) is a Hilbert space. What is needed, to show that \( \{e_n\}_n \) is a basis of \( \mathcal{A}_{R,m}^{2,0}(\mathbb{D}_R) \), is completeness. Indeed, let \( f(z) = \sum_{n=0}^{\infty} a_n z^n \) be in the orthogonal of the linear span of \( \{e_n\}_n \) in \( (\mathcal{A}_{R,m}^{2,0}(\mathbb{D}_R), \langle \cdot, \cdot \rangle_{\alpha,m,R}) \),

\[
f(z) = \sum_{n=0}^{\infty} a_n z^n \in \left(S \text{pan}\{e_n; \ n \geq 0\}\right)^{\perp_{\alpha,m,R}}.
\]

Thus, we have \( \langle f, e_n \rangle_{\alpha,m,R} = 0 \) for every \( n \). Now, since

\[
\langle f, e_n \rangle_{\alpha,m,R} = \begin{cases} \|e_n\|_{\alpha,R}^2 & \text{for } n < m \\ (m!)^2 \|e_{m-n}\|_{\alpha,R}^2 & \text{for } n \geq m \end{cases},
\]

it follows that \( a_n = 0 \) for all \( n \geq 0 \). This proves

\[
|0| = (S \text{pan}\{e_{m+n}; \ n \geq 0\}^{\perp_{\alpha,m,R}} = (S \text{pan}\{e_n; \ n \geq 0\}^{\perp_{\alpha,m,R}}.
\]
and therefore

\[ S \text{pan}\{e_n; \ n \geq 0\}^{\|.,_{\alpha,R}} = K^{2,\alpha}_{R,R}(\mathbb{D}_R). \]

In order to prove that \( K^{2,\alpha}_{R,R}(\mathbb{D}_R) \) is a reproducing kernel Hilbert space, we need to show the following

**Lemma 2.7.** The point evaluation in \( \mathbb{D}_R \) is a bounded operation, namely for any fixed \( z \in \mathbb{D}_R \), there exists a constant \( C_z \) such that

\[ |f(z)| \leq C_z \|f\|_{\alpha,R} \]

for every \( f \in K^{2,\alpha}_{R,R}(\mathbb{D}_R) \). Moreover, the mapping \( z \mapsto C_z \) is continuous.

**Proof.** For every \( f = \sum_{n=0}^{+\infty} a_n z^n \in K^{2,\alpha}_{R,R}(\mathbb{D}_R) \), we have

\[ |f(z)| \leq \sum_{n=0}^{+\infty} |a_n| |e_n(z)| = \sum_{n=0}^{+\infty} \left( \frac{|a_n|}{\|e_n\|_{\alpha,R}} \right) \left( \|a_n\| \|e_n\|_{\alpha,R} \right). \]

By the Cauchy-Schwartz inequality, we get

\[ |f(z)| \leq \left( \sum_{n=0}^{+\infty} \frac{|a_n|^2}{\|e_n\|^2_{\alpha,R}} \right)^{\frac{1}{2}} \left( \sum_{n=0}^{+\infty} |a_n|^2 \right)^{\frac{1}{2}} \|e_n\|^2_{\alpha,R}. \]

Thence, \( f \) satisfies the pointwise estimate \( |f(z)| \leq C_z \|f\|_{\alpha,R} \), where \( C_z \) stands for

\[ C_z := \left( \sum_{n=0}^{+\infty} \frac{|a_n|^2}{\|e_n\|^2_{\alpha,R}} \right)^{\frac{1}{2}}. \]

Thus, the evaluation mapping \( \delta_z : f \mapsto f(z) \) is a continuous linear form on \( K^{2,\alpha}_{R,R}(\mathbb{D}_R) \). \( \square \)

Therefore \( K^{2,\alpha}_{R,R}(\mathbb{D}_R) \) is a reproducing kernel Hilbert space by Riesz representation theorem, whose reproducing kernel function is given explicitly in terms of the \( 3F_2 \)-hypergeometric function [3, Chapter 5],

\[ 3F_2 \left( \begin{array}{c} a, b, c \\ \alpha', \beta' \end{array} \right | x \right) = \sum_{k=0}^{+\infty} \frac{(a)_k (b)_k (c)_k x^k}{(\alpha')_k (\beta')_k k!} \left( \frac{z}{z'} \right)^k, \quad |x| < 1, \]

where \( (a)_k = a(a+1) \cdots (a+k-1) \) is the Pochhammer symbol. Namely, we have the following.

**Proposition 2.8.** The reproducing kernel of \( K^{2,\alpha}_{R,R}(\mathbb{D}_R) \) is given by

\[ K^{\alpha}_{R,R}(z,w) = \frac{(\alpha+1)}{\pi R^2} \left( \sum_{n=0}^{+\infty} \frac{(\alpha + 2 + n)(\frac{z^\alpha}{R})^n}{n! R^{2n}} + \frac{(\alpha + 2 + n)(\frac{z^\alpha}{R})^n}{n! R^{2n}} \right) \cdot \left( 3F_2 \left( \begin{array}{c} 1, 1, \alpha + 2 \\ m + 1, m + 1 \end{array} \right | \frac{z}{R} \right). \quad (18) \]

**Proof.** Recall from above that \( (z^\alpha)_{n=0}^{+\infty} \) is an orthogonal basis of the reproducing kernel Hilbert space \( K^{2,\alpha}_{R,R}(\mathbb{D}_R) \). Therefore, the reproducing kernel function \( K^{\alpha}_{R,R}(z,w) ; z, w \in \mathbb{D}_R \) of \( K^{2,\alpha}_{R,R}(\mathbb{D}_R) \) can be computed by evaluating the sum

\[ K^{\alpha}_{R,R}(z,w) = \sum_{n=0}^{+\infty} \frac{\|e_n\|_{\alpha,R}^2}{\|e_n\|_{\alpha,R}^2}. \]

More explicitly, we have

\[ K^{\alpha}_{R,R}(z,w) = \frac{1}{\pi R^2} \left( \sum_{n=0}^{+\infty} \frac{\Gamma(\alpha + 2 + n)(\frac{z^\alpha}{R})^n}{\Gamma(n+1) \Gamma(\alpha+1) R^{2n}} + \sum_{n=0}^{+\infty} \frac{\frac{\Gamma(n-m+1) \Gamma(\alpha+2+n-m)}{\Gamma(n+1) \Gamma(\alpha+1)}}{R^{2(n-m)}} \right). \]
By means of \((\alpha + 1)\Gamma(\alpha + 1) = \Gamma(\alpha + 2)\) and the change of index \(n - m = p\), we get
\[
K^\alpha_{R,m}(z, w) = \frac{(\alpha + 1)}{\pi R^2} \left( \sum_{n=0}^{m-1} \frac{\Gamma(\alpha + 2 + n)}{n!} \left( \frac{\overline{w}}{R^2} \right)^n + \left( \frac{\overline{w}}{R^2} \right)^m \sum_{p=0}^{\infty} \frac{\Gamma^2(p + 1)\Gamma(\alpha + 2 + p)}{(m + 1)p + 1} \left( \frac{\overline{w}}{R^2} \right)^p \right).
\]

Finally, since \(\Gamma(p + 1) = \Gamma(p + m + 1) = m!(m + 1)p\) and \(\Gamma(\alpha + 2 + p) = (\alpha + 2)_p\), it follows
\[
K^\alpha_{R,m}(z, w) = \frac{(\alpha + 1)}{\pi R^2} \left( \sum_{n=0}^{m-1} \frac{(\alpha + 2)_n}{n!} \left( \frac{\overline{w}}{R^2} \right)^n + \left( \frac{\overline{w}}{R^2} \right)^m \sum_{p=0}^{\infty} \frac{(1)_p(\alpha + 2)_p}{(m + 1)p + 1} \left( \frac{\overline{w}}{R^2} \right)^p \right).
\]

This completes the proof.

\[\Box\]

**Remark 2.9.** Making use of the fact \( \left(\begin{array}{c} a, a, c \\ a, a \end{array}\right) \right) = (1 - x)^{-c}\), we see that for the special case of \(m = 0\), the formula (18) reads simply
\[
K^\alpha_{R,0}(z, w) = \frac{(\alpha + 1)}{\pi R^2} \left( \frac{R^2}{R^2 - \overline{w}} \right)^{\alpha + 2},
\]
which corresponds to the reproducing kernel of the weighted Bergman space \(\mathcal{A}^2(\mathbb{D}_R)\) on \(\mathbb{D}_R ((\overline{\mathbb{D}}))\).

**Remark 2.10.** For \(R = 1\), \(\alpha = 0\) and \(m = 1\), the corresponding reproducing kernel reduces further to the classical reproducing kernel of the logarithmic Dirichlet space,
\[
K^\alpha_{R,1}(z, w) = \frac{1}{\pi} \left( 1 + \overline{w} \right) \left( \begin{array}{c} 1, 1 \\ \frac{1}{2} \end{array}\right) \left( \frac{1}{\overline{w}} \right) = \frac{1}{\pi} \left( 1 + \ln \left( \frac{1}{1 - \overline{w}} \right) \right)
\]
thanks to the transformation \([\overline{a}, p. 109]\)
\[
x_2F_1 \left( \frac{1, 1}{2}, 1 \right) = \ln \left( \frac{1}{1 - x} \right).
\]

We conclude this section by noting that the proof of Theorem 1.1, stated in the introductory section, is contained in the previous established results (essentially Lemmas 2.7 and 2.8 and Propositions 2.4 and 2.6).

3. **Weighted Bargmann-Dirichlet spaces of order \(m\) on the complex plane**

Fix a real number \(\nu > 0\) and denote by \(L^2(\mathbb{C}; e^{-\nu|z|^2} \, d\lambda)\) the usual Hilbert space of all square-integrable functions on \(\mathbb{C}\) with respect to the Gaussian measure \(d\mu(z) = e^{-\nu|z|^2} \, d\lambda(z)\). The hermitian inner product is defined by
\[
(f, g)_\nu := \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-\nu|z|^2} \, d\lambda(z),
\]
and the associated norm by
\[
\|f\|_\nu^2 := \int_{\mathbb{C}} |f(z)|^2 d\mu(z).
\]

For fixed nonnegative integer \(m = 0, 1, 2, \cdots\), any \(f(z) = \sum_{n=0}^{\infty} a_n z^n\) in the vector space \(\mathcal{H}^{\nu}(\mathbb{C})\) of all convergent entire series on \(\mathbb{C}\), can be written as \(f(z) = f_{1,m}(z) + f_{2,m}(z)\), where
\[
f_{1,m}(z) := \sum_{n=0}^{m-1} a_n z^n \quad \text{and} \quad f_{2,m}(z) := \sum_{n=m}^{\infty} a_n z^n.
\]
Then, one can perform the functional space \( B^{2,\nu}_m(\mathbb{C}) \) of all entire functions \( f \in \mathcal{H}(\mathbb{C}) \) such that

\[
\|f\|_{r,m}^2 := \|f_1,m\|_{\nu}^2 + \|f_{2,m}\|_{\nu}^2 < +\infty.
\]

Notice that the hermitian inner product \( \langle \cdot, \cdot \rangle_{r,m} \) associated to the norm \( \|\cdot\|_{r,m} \) is given through

\[
\langle f, g \rangle_{r,m} := \langle f_{1,m}, g_{1,m} \rangle_{\nu} + \langle f_{2,m}, g_{2,m} \rangle_{\nu}.
\]

**Definition 3.3.** Precisely, there exists a constant \( C \) which can be handled using polar coordinates

\[
\int_{\mathbb{C}} |z|^2 e^{-|z|^2} d\lambda = \pi n! \delta_{n,k}.
\]

which can be handled using polar coordinates \( z = re^{i\theta} \) and the change \( t = vr^2 \), combined with the known facts

\[
\int_0^{2\pi} e^{i(n-k)\theta} d\theta = 2\pi \delta_{n,k} \quad \text{and} \quad \int_0^{+\infty} t^\nu e^{-rt} dt = n!.
\]

Finally, (23) follows by orthogonality of the monomials in \( B^{2,\nu}_m(\mathbb{C}) \) keeping in mind the explicit expression of \( \|e_n\|_{r,m}^2 \) given through (22) and the fact that the series \( f \) belongs to \( B^{2,\nu}_m(\mathbb{C}) \) if and only if \( \|f\|_{r,m}^2 \) is finite.

**Remark 3.2.** For \( m = 0 \), the considered space is to the classical Bargmann-Fock Hilbert space \( B^{2,\nu}(\mathbb{C}) := L^2(\mathbb{C}; e^{-|z|^2} d\lambda) \cap \mathcal{H}(\mathbb{C}) \). While for \( m = 1 \), it reads simply

\[
B^{2,\nu}_1(\mathbb{C}) := \left\{ f \in \mathcal{H}(\mathbb{C}), \quad \frac{\pi}{\nu} |f(0)|^2 + \int_{\mathbb{C}} |f'(z)|^2 e^{-|z|^2} d\lambda(z) < +\infty \right\}.
\]

**Definition 3.3.** We call \( B^{2,\nu}_m(\mathbb{C}) \) weighted Bargmann-Dirichlet spaces of order \( m \).

**Lemma 3.4.** For every fixed nonnegative integer \( m \), the space \( B^{2,\nu}_{m+1}(\mathbb{C}) \) is continuously embedded in \( B^{2,\nu}_m(\mathbb{C}) \). More precisely, there exists a constant \( C_{r,m} \) depending only in \( \nu \) and \( m \) such that

\[
\|f\|_{r,m}^2 \leq C_{r,m} \|f\|_{r,m+1}^2.
\]

In particular, the weighted Bargmann-Dirichlet space \( B^{2,\nu}_m(\mathbb{C}) \) is continuously embedded in the classical Bargmann-Dirichlet space.

**Proof.** It is similar to the one given for Lemma 2.5. Thanks to the previous obtained results, one can proceed exactly as in the proof of Proposition 2.6 and Lemma 2.7 to show the following
Proposition 3.5. $L^2_m(C)$ is a Hilbert space and the monomials $e_n(z) = z^n, n \geq 0$ constitute an orthogonal basis of it. Moreover, the evaluation map $\delta_z : f \mapsto f(z)$, for fixed $z \in C$, is a continuous linear form on $L^2_m(C)$ and satisfies

$$|f(z)| \leq C_z \|f\|_{m}$$

for every $f \in L^2_m(C)$, where

$$C_z = \left( \sum_{n=0}^{\infty} \frac{|z^n|^2}{\|e_n\|^2_{L^2_m}} \right)^{\frac{1}{2}}.$$

This shows that $L^2_m(C)$ is a reproducing kernel Hilbert space. Its reproducing kernel function is given explicitly in terms of the $2F_2$-hypergeometric function [3, Chapter 5],

$$2F_2 \left( \begin{array}{c} a, b, \bar{a}, \bar{b} \end{array} \right) = \sum_{k=0}^{+\infty} \frac{(a)_k(b)_k}{(a')_k(b')_k} \frac{x^k}{k!}.$$

More precisely, we assert

Proposition 3.6. The reproducing kernel of space $L^2_m(C)$ is given by

$$K_m^*(z, w) = \frac{v}{\pi} \left( \sum_{n=0}^{m-1} \frac{y^n(z \bar{w})^n}{n!} + \frac{(z \bar{w})^m}{(m!)^2} 2F_2 \left( \begin{array}{c} 1, 1 \end{array} \right) \Gamma(m+1) \Gamma(2m+n+1) \left( \frac{v(z \bar{w})^m}{(m!)} \right) \right).$$

Proof. We have

$$K_m^*(z, w) = \sum_{n=0}^{+\infty} \frac{r^n(z \bar{w})^n}{\|e_n\|^2_{L^2_m}}$$

$$= \frac{v}{\pi} \left( \sum_{n=0}^{m-1} \frac{y^n(z \bar{w})^n}{(m!)^2} \Gamma(m+1) \Gamma(2m+n+1) \left( \frac{v(z \bar{w})^m}{(m!)} \right) \right).$$

This completes the proof.

Remark 3.7. For $m = 0$, we recover the reproducing kernel function of the Bargmann-Fock space which is known to be given by

$$K^*(z, w) = \left( \frac{v}{\pi} \right)^m e^{\bar{w}z}.$$

4. Weighted Bargmann-Dirichlet spaces as limit of weighted Bergman-Dirichlet spaces

The complex space $C$ endowed with the flat metric $ds^2_\infty = dz \otimes d\bar{z}$ can be seen as a Kählerian manifold. It is shown in [3] that the flat hermitian geometry on $C$ can be approximated by the complex hyperbolic geometry of the disks $D_R$ of radius $R > 0$ associated to the scaled Bergman Kähler metric

$$ds^2_R = \frac{R^4}{(R^2 - |z|^2)^2} dz \otimes d\bar{z}.$$
one associated to the metric $ds^2_R$. It converges to the volume measure associated to $ds^2_\infty$, when $R$ goes to $+\infty$, being indeed

$$\lim_{R \to +\infty} \left(1 - \frac{|z|}{R}\right)^{2R^2} d\lambda = e^{-|z|^2} d\lambda.$$ 

Thus we have instead of general $\alpha > -1$, we consider the particular case of $\alpha = \nu R^2$ with $\nu > 0$, so that

$$\lim_{R \to +\infty} d\mu_{\alpha,R} = d\mu_{\nu}.$$

The main result of this section concerns the pointwise convergence of the reproducing kernel functions.

**Theorem 4.1.** Let $K_{R,m}^\alpha$ with $\alpha = \nu R^2$ (resp. $K_{m}^\nu$) be the reproducing kernel of the weighted Bergman-Dirichlet (resp. Bargmann-Dirichlet) space of order $m$. Then, we have

$$\lim_{R \to +\infty} K_{R,m}^{\nu R^2}(z, w) = K_{m}^{\nu}(z, w)$$

for every fixed $(z, w) \in \mathbb{C} \times \mathbb{C}$.

The proof of this theorem, follows by making use of the explicit expression of the reproducing kernels $K_{R,m}^{\nu R^2}$ and $K_{m}^{\nu}$ as given by (2.8) and (24), respectively, combined with the following lemma

**Lemma 4.2.** For every fixed $\xi \in \mathbb{C}$, we have

$$\lim_{\rho \to +\infty} \rho^{p+1} F_p\left(\begin{array}{c}
a_1, \cdots, a_p, c + \rho \xi \\ a_1', \cdots, a_p'
\end{array}\right) = \rho F_p\left(\begin{array}{c}
a_1, \cdots, a_p \\ a_1', \cdots, a_p'
\end{array}\right),$$

where $a_j, a_j'; j = 1, \cdots, p$, are complex numbers. Moreover, the convergence is uniform on compact sets of $\mathbb{C}^n$.

**Proof.** It can be checked easily in a formal way. For a rigorous proof, one can proceed exactly as in [2] for $p = 1$. □

**Remark 4.3.** According to Lemma 4.2, the convergence in Theorem 4.1 of the reproducing kernel functions is uniform in $z, w$ for $z, w$ in any compact set of $\mathbb{C} \times \mathbb{C}$.

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