ON GENERALIZED QUASI-EINSTEIN GRW SPACE-TIMES

UDAY CHAND DE AND SAMEH SHENAWY

Abstract. Recently, it is proven that generalized Robertson-Walker space-times in all orthogonal subspaces of Gray’s decomposition but one (unrestricted) are perfect fluid space-times. GRW space-times in the unrestricted subspace are identified by having constant scalar curvature. Generalized quasi-Einstein GRW space-times have a constant scalar curvature. It is shown that generalized quasi-Einstein GRW space-times reduce to Einstein space-times or perfect fluid space-times.

1. Introduction

The warped product $M = I \times f M^*$ of an open connected interval $(I, -dt^2)$ of $\mathbb{R}$ and a Riemannian manifold $M^*$ with warping function $f : I \to \mathbb{R}^+$ is called a generalized Robertson-Walker space-time (or GRW space-times) [12, 15]. This family of Lorentzian space-times broadly extends the classical Robertson-Walker space-times, Friedmann cosmological models, Einstein-de Sitter space-times and many others [2, 15]. The classical Robertson-Walker spacetime is regarded as cosmological models since it is spatially homogeneous and spatially isotropic whereas GRW space-times serve as inhomogeneous extension of Robertson-Walker space-times that admit an isotropic radiation [2] (see also [4, 15]). A Lorentzian manifold is called a perfect fluid space-time if the Ricci tensor $\text{Ric}$ takes the form

$$\text{Ric} (X, Y) = a g(X, Y) + b A(X) A(Y)$$

where $a, b$ are scalars and $A$ is a 1-form metrically equivalent to a unit time-like vector field [13, 14]. Perfect fluid space-times in the language of differential geometry are called quasi-Einstein spaces where $A$ is metrically equivalent to a unit space-like vector field. Recently, in [14], it is proven that a perfect fluid space-time with divergence-free conformal curvature tensor is a GRW space-time with Einstein fibers given that the scalar curvature is constant. Many sufficient conditions on perfect fluid space-times to be a GRW space-time are derived.

Gray presented an invariant orthogonal decomposition of the covariant derivative of the Ricci tensor [5] (see also [10]). Recently, Carlo Mantica et al proved that the Ricci tensor of a generalized Robertson-Walker space-time in all classes of Gray’s decomposition but $A \oplus B$ is either Einstein or takes the form of a perfect fluid whereas $A \oplus B$ is not restricted [13]. The class $A \oplus B$ is characterized by $\nabla R = 0$ i.e. the scalar curvature is constant. Now, the following question arises.

Does the Ricci tensor of all GRW space-times in $A \oplus B$ reduce to be Einstein or take the form of a perfect fluid?

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In this work, we get a partial positive answer. A (pseudo-)Riemannian manifold \((M, g)\) is called a generalized quasi-Einstein manifold if its Ricci curvature satisfies
\[
\text{Ric}(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma [A(X)B(Y) + A(Y)B(X)],
\]
where \(\alpha, \beta\) and \(\gamma\) are non-zero constants, \(A\) and \(B\) are 1-forms corresponding to two orthonormal vector fields \([1,3,6–9]\). If \(\gamma = 0\), then \(M\) reduces to a quasi-Einstein manifold. It is clear that generalized quasi-Einstein space-times are generally imperfect fluid space-times with constant scalar curvature \(R = n\alpha + \beta\). However, we prove that generalized quasi-Einstein GRW space-times are either Einstein or perfect fluid space-times.

**Remark 1.** It is noted that any vector field orthogonal to a time-like vector field is space-like. Thus the generators couldn’t be time-like. Now, one may assume that one of the generators is time-like and the other is space-like. Generally, the results of this article still hold in this case with minor changes.

2. **Notes on generalized quasi-Einstein manifolds**

Let \(M\) be a generalized quasi-Einstein (pseudo-)Riemannian manifold i.e.
\[
R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma (A_i B_j + A_j B_i),
\]
where \(A_i A^i = B_j B^j = 1\) and \(A_i B^i = 0\). The trace of this equations gives
\[
R = n\alpha + \beta.
\]
It is noted that
\[
A^i A^j R_{ij} = \alpha + \beta, \\
A^i B^j R_{ij} = \gamma, \\
B^i B^j R_{ij} = \alpha,
\]
and consequently \(\beta = (A^i A^j - B^i B^j) R_{ij}\). Hence we can state the following result.

**Proposition 1.** Let \(M\) be a generalized quasi-Einstein manifold with generators \(A\) and \(B\). Then the scalar curvature is
\[
R = (n - 1) B^i B^j R_{ij} + A^i A^j R_{ij}.
\]

Now assume that \(A\) is an eigenvector of the Ricci tensor with eigenvalue \(\xi\) i.e. \(A^i R_{ij} = \xi A_j\). A contraction of Equation (2.1) with \(A^i\) yields
\[
A^i R_{ij} = \alpha a_i g_{ij} + \beta A_i A_j + \gamma (A_i A_j B_i + A_j A_i B_j)
\]
implies \((\xi - \alpha - \beta) A_j = \gamma B_j\). Thus \(\gamma = 0\) and \(\xi = \alpha + \beta\). If \(B\) is an eigenvector of the Ricci tensor with eigenvalue \(\phi\), then
\[
B^i R_{ij} = \alpha B_i g_{ij} + \beta B_i A_j A_j + \gamma (B_i A_j B_j + B_j A_i B_i)
\]
infers \((\phi - \alpha) B_j = \gamma A_j\).

Consequently, \(\phi = \alpha\) and \(\gamma = 0\). Conversely, assume that \(M\) is a quasi-Einstein manifold with generator \(A\). Then
\[
R_{ij} = \alpha g_{ij} + \beta A_i A_j
\]
yields
\[
A^i R_{ij} = \alpha A_i g_{ij} + \beta A_i A_j = (\alpha + \beta) A_j.
\]
Thus, for a Codazzi Ricci tensor, the generators are both closed. In this case,

$$B^i R_{ij} = \alpha B^i g_{ij} + \beta B^i A_i A_j = \alpha A_j.$$ 

This leads to the following.

**Theorem 1.** Let $M$ be a generalized quasi-Einstein manifold. Then, $M$ reduces to a quasi-Einstein manifold if and only if one of the generators is an eigenvector of the Ricci tensor.

The covariant derivative of the Ricci tensor of a generalized quasi-Einstein manifold is given by

$$\nabla_k R_{ij} = \beta (\nabla_k A_i) A_j + \beta A_i (\nabla_k A_j)$$

$$+ \gamma ([\nabla_k A_i] B_j + A_i (\nabla_k B_j) + (\nabla_k A_j) B_i + A_j (\nabla_k B_i))$$

and hence

$$\nabla_i R_{kj} = \beta (\nabla_i A_k) A_j + \beta A_k (\nabla_i A_j)$$

$$+ \gamma ([\nabla_i A_k] B_j + A_k (\nabla_i B_j) + (\nabla_i A_j) B_k + A_j (\nabla_i B_k))$$

Thus, the Codazzi deviation tensor $D$ is

$$D_{kij} = \nabla_k R_{ij} - \nabla_i R_{kj}$$

$$= \beta (\nabla_k A_i) A_j + \beta A_i (\nabla_k A_j) - \beta (\nabla_i A_k) A_j - \beta A_k (\nabla_i A_j)$$

$$+ \gamma ([\nabla_k A_i] B_j + A_i (\nabla_k B_j) + (\nabla_k A_j) B_i + A_j (\nabla_k B_i))$$

$$- \gamma ([\nabla_i A_k] B_j + A_k (\nabla_i B_j) + (\nabla_i A_j) B_k + A_j (\nabla_i B_k))$$

Now, we have the following cases

$$A^i D_{kij} = \beta (\nabla_k A_i - \nabla_i A_k) + \gamma (\nabla_k B_i - \gamma \nabla_i B_k)$$

$$B^i D_{kij} = \nabla_k A_i - \nabla_i A_k$$

Thus, for a Codazzi Ricci tensor, the generators are both closed. In this case,

$$0 = D_{kij}$$

$$= \beta A_i (\nabla_k A_j) - \beta A_k (\nabla_i A_j) + \gamma A_i (\nabla_k B_j)$$

$$+ \gamma (\nabla_k A_i) B_j - \gamma A_k (\nabla_i B_j) - \gamma (\nabla_i A_j) B_k$$

$$= \nabla_j [\beta A_i A_k + \gamma A_i B_k + \gamma A_k B_i]$$

$$- 2 \beta A_k (\nabla_j A_i) - 2 \gamma (\nabla_j A_i) B_k - 2 \gamma A_k (\nabla_j B_i)$$

$$= \nabla_j R_{ik} - 2 (\beta A_k + \gamma B_k) \nabla_j A_i - 2 \gamma A_k \nabla_j B_i.$$

Thus we have the following.

**Proposition 2.** Let $M$ be a generalized quasi-Einstein manifold. Assume that $M$ is Einstein-like of class $\mathcal{B}$ (i.e. the Ricci tensor is a Codazzi tensor). Then $A$ and $B$ are closed. Moreover, since

$$\nabla_j R_{ik} = 2 (\beta A_k + \gamma B_k) \nabla_j A_i + 2 \gamma A_k \nabla_j B_i.$$ 

A contraction of $D_{kij}$ by $g^{ij}$ and then by the generators $A^k$ and $B^k$ infers

$$0 = (\beta A^i + \gamma B^i) \nabla_i A_k + (\beta A_k + \gamma B_k) (\nabla_i A^i) + \gamma A_k (\nabla_i B^i) + \gamma A^i \nabla_i B_k,$$

$$0 = \beta (\nabla_i A^i) + \gamma \nabla_i B^i,$$

$$0 = \gamma \nabla_i A^i.$$

Thus $\nabla_i A^i = \nabla_i B^i = 0.$
Assume that $M$ is Einstein-like of class $\mathcal{P}$ (that is, the Ricci tensor is a parallel, $\nabla_k R_{ij} = 0$). Then,
\[
0 = \beta (\nabla_i A_k) A_j + \beta A_k (\nabla_i A_j) + \gamma (\nabla_i A_k) B_j + \gamma A_k (\nabla_i B_j)
\]
Contraction by $A^k$ and $B^k$ imply
\[
0 = \beta \nabla_i A_j + \gamma \nabla_i B_j,
\]
\[
0 = \gamma \nabla_i A_j.
\]
Assume that $A$ is not parallel, then $\beta = \gamma = 0$. Thus we conclude.

**Theorem 2.** Let $M$ be a Ricci-symmetric generalized quasi-Einstein manifold. Then, $M$ is Einstein if the generator $A$ is not covariantly constant.

3. Generalized quasi-Einstein GRW space-times

A Lorentzian manifold $M$ is a GRW space-time if and only if $M$ has a unit time-like vector field $u_i$ such that
\[
\nabla_k u_j = \varphi (g_{kj} + u_k u_j),
\]
which is also an eigenvector of the Ricci tensor i.e. $R_{ij} u^i = \xi u_j$ for some scalar functions $\varphi$ and $\xi$ [11,13]. We say that $u$ is a nontrivial torse-forming vector field if $\varphi \neq 0$. This characterization is an alternative of Chen’s theorem in [2]. If $M$ is a generalized quasi-Einstein manifold, then
\[
R_{ij} = \alpha g_{ij} + \beta A_i A_j + \gamma (A_i B_j + A_j B_i).
\]
A contraction by $u^i$ yields
\[
u^i R_{ij} = \alpha u^i g_{ij} + \beta u^i A_i A_j + \gamma (u^i A_i B_j + A_j u^i B_i)
\]
which implies
\[
(\xi - \alpha) u_j = \beta (u^i A_i) A_j + \gamma (u^i A_i) B_j + \gamma A_j (u^i B_i)
\]
and hence
\[
(\xi - \alpha) u_j = (\beta (u^i A_i) + \gamma (u^i B_i)) A_j + \gamma (u^i A_i) B_j
\]
Two different contractions by the generators give
\[
(\xi - \alpha - \beta) (u^i A_i) - \gamma (u^i B_i) = 0
\]
and
\[
(\xi - \alpha) (u^i B_i) - \gamma (u^i A_i) = 0.
\]
Thus
\[
(\xi - \alpha) u_j = (\beta (u^i A_i) + (\xi - \alpha - \beta) (u^i A_i)) A_j + (\xi - \alpha) (u^i B_i) B_j
\]
\[
= (\xi - \alpha) (u^i A_i) A_j + (\xi - \alpha) (u^i B_i) B_j
\]
and hence
\[
(\xi - \alpha) [u_j - (u^i A_i) A_j - (u^i B_i) B_j] = 0.
\]
It is clear that $u_j$ is not a linear combination of $A_j$ and $B_j$ only since $u^i$ is time-like whereas $A^i$ and $B^i$ are orthonormal space-like fields so $\xi = \alpha$. Therefore,
\[
\beta (u^i A_i) + \gamma (u^i B_i) = 0
\]
\[
\gamma (u^i A_i) = 0
\]
It is noted that $\gamma = \beta = 0$ if $(u^i A_i)$ is not zero. Suppose that $(u^i B_i)$ does not vanish. Then Equation (3.3) implies that either $\gamma = 0$ or $(u^i A_i) = 0$. The later case with Equation (3.2) yield $\gamma = 0$ i.e. $M$ is quasi-Einstein if $(u^i B_i) \neq 0$. Now, assume that $u^i$ is orthogonal to both the generators i.e. $(u^i A_i) = (u^i B_i) = 0$. The Ricci tensor of a GQE manifold is

$$R_{ij} = \xi g_{ij} + \beta A_i A_j + \gamma (A_i B_j + A_j B_i)$$

and so

$$\nabla_k R_{ij} = \beta A_j \nabla_k A_i + \beta A_i \nabla_k A_j + \gamma (B_j \nabla_k A_i + A_i \nabla_k B_j + A_j \nabla_k B_i + B_i \nabla_k A_j)$$

A contraction by $u^i$ implies

$$u^i \nabla_k R_{ij} = 0$$
$$\nabla_k (u^i R_{ij}) - R_{ij} \nabla_k u^i = 0$$

It is noted that $u^i$ is an eigenvector of the Ricci tensor (i.e. $u^i R_{ij} = \xi u_{ij}$) and $\nabla_k u^i = \varphi (\delta^i_k + u_k u^i)$. Thus

$$\nabla_k (\xi u_{ij}) - R_{ij} \varphi (\delta^i_k + u_k u^i) = 0$$
$$\xi \nabla_k u_j - \varphi \delta^i_k R_{ij} - \varphi u_k u^i R_{ij} = 0$$
$$\xi \varphi (g_{kj} + u_k u_j) - \varphi R_{kj} - \varphi u_k u_j = 0$$
$$\varphi (\xi g_{kj} - R_{kj}) = 0$$

So $M$ is Einstein if $u$ is a nontrivial torse-forming vector field.

**Theorem 3.** Let $M$ be a generalized quasi-Einstein GRW space-time. Then $u^i R_{ij} = \alpha u_{ij}$ i.e. $\alpha$ is the eigenvalue of the eigenvector $u^i$ and

1. $M$ reduces to be Einstein space-time if $u^i$ is orthogonal to both the generators provided $\varphi \neq 0$.
2. $M$ reduces to be Einstein space-time if $u^i$ is not orthogonal to first generator.
3. $M$ reduces to be perfect fluid space-time if $u^i$ is not orthogonal to the second generator.

**Corollary 1.** Let $M$ be a generalized quasi-Einstein Lorentzian manifold admitting a unit time-like non-trivial torse-forming vector field. Then $M$ reduces to an Einstein GRW space-time or a perfect fluid GRW space-time.

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(U. C. De) Department of Pure Mathematics, University of Calcutta, 35, Ballygaunge Circular Road, Kolkata 700019, West Bengal, India, E-mail address: ucde@yahoo.com

(S. Shenawy) Basic Science Department, Modern Academy for Engineering and Technology, Maadi, Egypt, E-mail address: drshenawy@mail.com