PERFECT POWERS IN CATALAN AND NARAYANA NUMBERS

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Abstract. When a Catalan number or a Narayana number is a (non-trivial) perfect power? For Catalan numbers, we show that the answer is “never”. However, we prove that for every $b$, the Narayana number $N(a, b)$ is a (non-trivial) perfect square for infinitely many values of $a$, and we show how to compute all of them. We also conjecture that $N(a, b)$ is never a (non-trivial) perfect $k$-th power for $k \geq 3$ and we prove some cases of this conjecture.

Introduction

Given two natural numbers $a$ and $b$, the Narayana number $N(a, b)$ is defined by the formula

$$N(a, b) := \frac{1}{a} \binom{a}{b} \left( \frac{a}{b-1} \right).$$

These numbers are well known in discrete mathematics, since they count several families of mathematical objects (see [8] for a classical reference, or [11, 17, 18, 22, 23, 31, 32, 34] for some more recent occurrence), e.g. $N(m + n - 1, m)$ is the number of parallelogram polyominoes in a rectangular $m \times n$ box (cf. [6]).

There is a natural link with the famous Catalan numbers

$$C_n := \frac{1}{n+1} \binom{2n}{n};$$

given by the identity $\sum_k N(n, k) = C_n$.

The Catalan numbers are ubiquitous in mathematics: see [29, 30, Exercise 6.19] for about 200 families of mathematical objects counted by these numbers.

In algebraic combinatorics $q, t$-analogues of Catalan numbers have been studied in connection with the so called $n!$-conjecture (now $n!$-theorem of Haiman) about the renown diagonal harmonics (see [12, 15, 16]). More recently, a $q, t$-analogue of the Narayana numbers has been shown to be intimately related to the decade old shuffle conjecture about the Frobenius characteristic of the diagonal harmonics (see [4, 5, 14]), renewing the interest for these numbers.

From a number theoretic point of view, it is natural to ask about divisibility properties of these numbers, which are of course related to divisibility properties of the binomial coefficients.

For the Catalan numbers, such properties have been studied by several authors. In particular their parity was studied in [2], while more generally their congruence modulo a power of 2 has been recently investigated in [21, 33]. Their divisibility by prime powers was completely determined in [3] via arithmetic techniques. In [9], among

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other results, the 2-adic valuation of Catalan numbers has been studied by means of
certain group actions. Similar results for generalizations of Catalan numbers have been
studied in \[19, 25\].

Lately, also the Narayana numbers have received more attention in this direction.
In particular, in \[7\], using a theorem of Kummer on the \(p\)-adic valuation of binomial
coefficients, the authors study the divisibility of \(N(a, b)\) by primes, in relation with the
description of \(N(a, b)\) in base \(p\).

In this work we study the following number theoretic question:

**Question.** When a Catalan number or a Narayana number is a (non-trivial) perfect
power?

For us an integer is a (non-trivial) perfect power if it is of the form \(m^k\) where \(m\) and
\(k\) are both integers \(\geq 2\) (so 1 is not a perfect power). For \(k = 2\), we call this integer a
perfect square.

The question for Catalan numbers has a negative answer: it follows easily from
a classical theorem of Ramanujan on the distribution of primes in intervals that the
sequence of Catalan numbers does not contain perfect powers. We show this in Section 1.

Interestingly enough, the situation for the Narayana numbers is very different: there
are a lot of them which are perfect squares.

We study this case in Section 3. We start by exhibiting infinitely many pairs \((a, b)\)
such that \(N(a, b)\) is a perfect square, see Proposition 3.1. Then, in Theorem 3.2 we
give an effective algorithm to compute all such pairs, proving in particular the stronger
result that for any given \(b > 1\), there are infinitely many \(a > b\) such that \(N(a, b)\) is a
perfect square. It turns out that this problem can be reduced to the study of certain
generalized Pell’s equations: we recall the facts about these equations that we need in
Section 2.

In Section 4 we show how the algorithm works with an explicit example.

We conclude, in Section 5 by studying the seemingly more complicated case of
higher powers. We make the following conjecture.

**Conjecture 1.** \(N(a, b)\) is never a non-trivial perfect \(k\)-th power for \(k \geq 3\).

Conjecture 1 seems to be quite hard. For example, for \(b = 3\) (the case \(b = 2\) follows
from some known results), it is related to a generalization of the Catalan’s conjecture
by Pillai. However we are able to provide some evidence for our conjecture, by showing
that it holds when \(b\) is “not too small” (see Theorems 5.3 and 5.4).

1. **Catalan numbers are not perfect powers**

The famous Catalan numbers are defined, for \(n \geq 1\), by the formula

\[ C_n := \frac{1}{n + 1} \binom{2n}{n} . \]

In this section we answer the question:

**Question.** Are there perfect powers in the Catalan sequence \(\{C_n\}_{n \geq 1}\)?
Though Catalan numbers have been extensively studied, to the best of our knowledge this is the first investigation of this kind.

The negative answer to our question follows easily from the following classical theorem, which is due to Ramanujan (see [27, Section 9.3B] for a proof).

**Theorem 1.1** (Ramanujan). For $n \geq 6$ there are at least two primes between $n$ and $2n$.

Here is a complete answer to our question.

**Theorem 1.2.** For all $n$, the $n$-th Catalan number $C_n$ is never a perfect power.

**Proof.** We write

\[
C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)(2n-1) \cdots (n+2)}{n!}.
\]

By Ramanujan’s Theorem, for $n \geq 6$ there are at least two primes between $n$ and $2n$. Since they cannot be both $n$ and $n+1$, this implies that there is at least one prime between $n+2$ and $2n$. Hence this prime divides exactly $C_n$ (since it divides the numerator in (1.1), but not the denominator), showing that it cannot be a perfect power.

Since $C_1 = 1, C_2 = 2, C_3 = 5, C_4 = 14$ and $C_5 = 42$ are not perfect powers, this completes the proof. □

2. **The generalized Pell’s equation** $n^2 - dm^2 = z^2$

Before considering the problem of when a Narayana number is a non-trivial perfect power, we recall some results on Pell’s equation which will be used in the following section.

Let $z$ and $d$ be positive integers, with $d$ squarefree. In this section we want to describe all the positive integral solutions $(n, m)$ with $m$ even of the generalized Pell’s equation

\[
n^2 - dm^2 = z^2.
\]

We start recalling the following classical well known results. We suggest [35, Chapter 1] as general reference.

2.1. **Solving the generalized Pell’s equations** $n^2 - dm^2 = z^2$. From [35, Proposition 1.5], all positive integral solutions $(n, m)$ of the generalized Pell’s equation

\[
n^2 - dm^2 = z^2
\]

are obtained as

\[
n \pm m\sqrt{d} = (n' + m'\sqrt{d})(n_1 + m_1\sqrt{d})^k
\]

where $k \in \mathbb{Z}$, $(n_1, m_1)$ is the fundamental solution of the corresponding Pell’s equation

\[
n^2 - dm^2 = 1,
\]
and \((n', m')\) is a particular positive solution of \((2.2)\), belonging to a finite set effectively computable only in terms of \(n_1, m_1, d\) and \(z\). In particular \((n', m')\) can be chosen so that

\[|n'| < z\sqrt{n_1 + m_1\sqrt{d}} \quad \text{and} \quad |m'| < z\sqrt{n_1 + m_1\sqrt{d}}.\]

The fundamental solution \((n_1, m_1)\) of \((2.3)\) is easily computable: \(n_1/m_1\) is in fact the truncation of the continued fraction expansion of \(\sqrt{d}\) to the end of its first period, if this period has even length, or to the end of its second period, if this period has odd length. Moreover, all positive solutions of \((2.3)\) are of the form \((n, m)\) with

\[n_k + m_k\sqrt{d} := (n_1 + m_1\sqrt{d})^k \quad \text{for} \quad k \in \mathbb{N}.

2.2. Finding integral solutions \((n, m)\) with \(m\) even. We go back to the original purpose of this section, that is to find integral solutions of \((2.2)\) with \(m\) even.

Observe that if \(d\) is even, then \(n\) and \(z\) must have the same parity. We have

\[dm^2 = n^2 - z^2 = (n - z)(n + z),\]

hence in this case \(dm^2\) is divisible by \(2^2 = 4\); since \(d\) is squarefree, we must have that \(2\) divides \(m^2\), so \(m\) is even.

Therefore, we assume from now on that \(d\) is odd.

Consider the general product

\[x_2 + y_2\sqrt{d} := (x_1 + y_1\sqrt{d})(x_0 + y_0\sqrt{d}),\]

where all \(x_i\)'s and \(y_i\)'s are integers. We have

\[(x_1 + y_1\sqrt{d})(x_0 + y_0\sqrt{d}) = (x_1x_0 + y_1y_0d) + \sqrt{d}(x_1y_0 + x_0y_1),\]

hence

\[x_2 = x_1x_0 + y_1y_0d \quad \text{and} \quad y_2 = x_1y_0 + x_0y_1.\]

It is now clear that:

- if \(x_0\) and \(x_1\) are both even, and \(y_0\) and \(y_1\) are both odd, then \(x_2\) is odd while \(y_2\) is even;
- similarly, if \(y_0\) and \(y_1\) are both even, and \(x_0\) and \(x_1\) are both odd, then \(x_2\) is odd while \(y_2\) is even.

On the other hand

- if \(x_0\) and \(y_1\) are both even, and \(y_0\) and \(x_1\) are both odd, then \(x_2\) is even while \(y_2\) is odd.

Moreover

- if only one of \(x_0, y_0, x_1, y_1\) is odd, then both \(x_2\) and \(y_2\) are even;
- if only one of \(x_0, y_0, x_1, y_1\) is even, then both \(x_2\) and \(y_2\) are odd.

Now, since \((n_k, m_k)\) are all solutions of the Pell’s equation \((2.3)\), necessarily \(n_k\) and \(m_k\) have different parities for all \(k\) (we are assuming that \(d\) is odd!).

In fact, from what we observed, we easily deduce that if \(n_1\) is odd, then \(n_k\) is odd (and hence \(m_k\) is even) for all \(k \in \mathbb{Z}\). Similarly, if \(n_1\) is even, then \(n_{2k+1}\) is even (and hence \(m_{2k+1}\) is odd) for all \(k \in \mathbb{Z}\), while \(n_{2k}\) is odd (and hence \(m_{2k}\) is even) for all \(k \in \mathbb{Z}\).
Now all positive solutions \((n, m)\) of (2.2) are obtained as
\[
n \pm m\sqrt{d} = (n' + m'\sqrt{d})(n_k + m_k\sqrt{d}) \quad \text{for } k \in \mathbb{Z},
\]
from which \(m = |n'm_k + m'n_k|\). So the parity of \(m\) depends on the parities of \(n'\) and \(m'\). From the discussion above, we easily deduce the following lemma.

**Lemma 2.1.** In the notation above, all positive solutions \((n, m)\) of (2.2) with \(m\) even are obtained as
\[
n \pm m\sqrt{d} = (n' + m'\sqrt{d})(n_{2k} + m_{2k}\sqrt{d}) \quad \text{for } k \in \mathbb{Z},
\]
if \(d\) is even or if both \(n'\) and \(m'\) are even; \(n \pm m\sqrt{d} = (n' + m'\sqrt{d})(n_{2k+1} + m_{2k+1}\sqrt{d}) \quad \text{for } k \in \mathbb{Z},
\]
if both \(m'\) and \(d\) are odd, while \(n'\) is even.

### 3. The case of the squares

Given two natural numbers \(a\) and \(b\), the Narayana number \(N(a, b)\) is defined by the formula
\[
N(a, b) := \frac{1}{a} \binom{a}{b} \left( \frac{a}{b-1} \right).
\]
In this section we study when \(N(a, b)\) is a perfect square.

Since given two natural numbers \(a\) and \(b\), we have \(N(a, a) = N(a, 1) = 1\), while \(N(a, b) = 0\) for \(a < b\), we will always assume in what follows that \(a > b > 1\).

It is not hard to see that there are infinitely many pairs \((a, b)\) for which \(N(a, b)\) is a square. We can prove a little more by providing explicit families of such pairs.

**Proposition 3.1.** There are infinitely many pairs \((a, b)\) such that \(N(a, b)\) is a square. More precisely:

1. if \(n\) is odd, then \(N\left(n^2, \frac{n^2+1}{2}\right)\) is a square;
2. if \(n\) is even, then \(N\left(n^2 - 2, \frac{n^2-2}{2}\right)\) is a square;
3. for all \(n\), \(N(n^2(n^2+1), n^2+1)\) is a square.

**Proof.** We start with the following simple, but quite useful, manipulation:
\[
N(a, b) = \frac{1}{a} \binom{a}{b} \left( \frac{a}{b-1} \right) = \frac{b}{a(a-b+1)} \left( \frac{a}{b} \right)^2.
\]

Hence to check that \(N(a, b)\) is a square it is enough to check that \(b/a(a-b+1)\) is.

For \(n \in \mathbb{N}, n \geq 1\) odd, \((n^2 + 1)/2\) is a positive integer, so, letting \(a := n^2\) and \(b := (n^2 + 1)/2\) we compute
\[
\frac{a(a-b+1)}{b} = \frac{2}{(n^2+1)} \left( n^2 \left( n^2 - \frac{(n^2+1)}{2} + 1 \right) \right) = n^2.
\]
This shows that, for $n > 1$ odd $N\left(\frac{n^2 - 1}{2}, \frac{n^2 + 1}{2}\right)$ is always a square, proving (1).

Similarly, for $n \in \mathbb{N}$, $n > 1$ even, $(n^2 - 2)/2$ is a positive integer. Setting $a := n^2 - 2$ and $b := (n^2 - 2)/2$, we compute
\[
\frac{a(a - b + 1)}{b} = \frac{2}{(n^2 - 2)} \left( (n^2 - 2) \left( \frac{n^2 - 2}{2} + 1 \right) \right) = 2 \left( \frac{n^2 - 2}{2} + 1 \right) = n^2.
\]
So for $n > 2$ even, $N\left(\frac{n^2 - 2}{2}, \frac{n^2 + 2}{2}\right)$ is always a square, establishing (2).

Finally, for any integer $n \in \mathbb{N}$, letting $a := n^2(n^2 + 1)$ and $b := n^2 + 1$ gives
\[
\frac{a(a - b + 1)}{b} = \frac{n^2(n^2 + 1) (n^2(n^2 + 1) - (n^2 + 1) + 1)}{(n^2 + 1)} = n^6,
\]
so $N(n^2(n^2 + 1), n^2 + 1)$ is a square too, proving (3). □

Remark 3.1. Notice that Proposition 3.1 does not cover all the pairs $(a, b)$ such that $N(a, b)$ is a square. For instance $N(1728, 28)$ and $N(63, 28)$ are both squares (as we will see in the next section) but they are not in the families appearing in Proposition 3.1.

In fact we can do much better: for any given $b$, we can produce all the $a$’s for which $N(a, b)$ is a square. It turns out that there are infinitely many of them for every $b$.

We will show that the problem of finding all pairs $(a, b)$ such that $N(a, b)$ is a square reduces to finding solutions of a general Pell’s equation.

The following theorem is the main result of this section. We remark here that its proof gives an algorithm to compute all the pairs $(a, b)$ for which $N(a, b)$ is a perfect square. Some explicit computations will be made in Section 4.

Theorem 3.2. For every fixed integer $b > 1$, $N(a, b)$ is a perfect square for infinitely many and effectively computable integers $a$.

Proof. Let $b$ be a positive integer, with $b = ds^2$ and $d$ is square-free.

We want to find all integers $a$’s such that $N(a, b) = c^2$ for some integer $c$. Using (3.3), this is equivalent to
\[
N(a, b) = \frac{b}{a(a - b + 1)} \left( \frac{a}{b} \right)^2 = c^2,
\]
so $N(a, b)$ is a square if and only if $ab(a - b + 1)$ is a square.

We now show that this problem is equivalent to finding the integral solutions $(n, m)$ of the Pell’s equation
\[
n^2 - dm^2 = (b - 1)^2
\]
such that $m$ is even.

In fact, assume that $a$ is an integer such that
\[
ab(a - b + 1) = c^2
\]
for some integer \( c' \). Notice that from (3.3), \( b \) divides \((c')^2\), hence \( ds \) divides \( c' \). It is now easy to check that the pair

\[
(n, m) = \left(2a + 1 - b, 2s \frac{c'}{b}\right)
\]

is an integral solution of (3.2) with \( m \) even.

On the other hand, suppose to be given a solution \((n, m)\) of (3.2) with \( m \) even. Notice that this implies that \( n \) and \( b - 1 \) have the same parity. So

\[
a = \frac{n + b - 1}{2}
\]

is an integer and one can easily check that

\[
N(a, b) = \left(\frac{2s}{m} \frac{a}{b}\right)^2.
\]

Now Lemma 2.1 shows how to compute all the positive solutions \((n, m)\) of equation (3.2) with \( m \) even.

There are always infinitely many, since for example, in the notation of the lemma, we can always choose \((n', m') = (b - 1, 0)\). This completes the proof. \(\square\)

4. SOME EXPLICIT COMPUTATION

We show how the proof of Theorem 3.2 is effective by computing an explicit example.

Let \( b = 28 \). We want all the \( a \)'s such that \( N(a, 28) \) is a square greater than 1. The algorithm is the following.

Keeping the above notation, we write \( b = 7 \cdot 2^2 \), so \( d = 7, s = 2 \) and equation (3.2) becomes

\[
n^2 - 7m^2 = 27^2.
\]

To solve it, from the discussion of Section 2, first we have to find the fundamental solution \((n_1, m_1)\) of the equation

\[
n^2 - 7m^2 = 1.
\]

The continued fraction expansion of \( \sqrt{7} \) is

\[
\sqrt{7} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ldots}}}}.
\]

or better, in the standard notation, \( \sqrt{7} = [2, 1, 1, 4, 1, 1, 1, 4, \ldots] \). So it has period of length 4, which is even, hence truncating the expansion in (4.3) at the end of the first period we get \( \frac{8}{3} \). Therefore \((n_1, m_1) = (8, 3)\) is the fundamental solution we sought for.

Now all positive solutions \((n, m)\) of (3.1) can be found as

\[
n \pm \sqrt{7}m = (n' + m' \sqrt{7})(8 + 3\sqrt{7})^k,
\]

or
where \( k \in \mathbb{Z} \) and \((n', m')\) is any solution of (4.1) such that
\[
|n'| < 27 \sqrt{8 + 3\sqrt{7}} < 27 \cdot 4 \quad \text{and} \quad |m'| < 27 \sqrt{\frac{8 + 3\sqrt{7}}{7}} < 27 \cdot 2.
\]
So it sufficient to compute \((n')^2 - 7(m')^2\) for all such values of \(n'\) and \(m'\) and see which one satisfies (4.1). In this case all the solutions \((n', m')\) of (4.1) in the range (4.4) are
\[(4.5) \quad (27, 0), \ (29, 4), \ (36, 9), \ (48, 15), \ (69, 24), \ (99, 36).
\]
We denote by \((n_k, m_k)\) the solution of (4.2) obtained as
\[
n_k + m_k\sqrt{7} = (8 + 3\sqrt{7})^k
\]
with \( k \in \mathbb{Z} \).

All the positive solutions \((n, m)\) of (4.1) are then of the form
\[
n \pm m\sqrt{d} = (n' + m'\sqrt{d})(n_k + m_k\sqrt{d}) \quad \text{for} \ k \in \mathbb{Z}
\]
where \((n', m')\) is in our list (4.5).

**Remark 4.1.** In fact notice that
\[
(36 + 9\sqrt{7})(n_{-1} + m_{-1}\sqrt{7}) = (36 + 9\sqrt{7})(8 - 3\sqrt{7}) = 99 - 36\sqrt{7},
\]
and consequently
\[
(99 + 36\sqrt{7})(n_{-1} + m_{-1}\sqrt{7}) = (99 + 36\sqrt{7})(8 - 3\sqrt{7}) = 36 - 9\sqrt{7}.
\]
Similarly,
\[
(48 + 15\sqrt{7})(n_{-1} + m_{-1}\sqrt{7}) = (48 + 15\sqrt{7})(8 - 3\sqrt{7}) = 69 - 24\sqrt{7},
\]
and consequently
\[
(69 + 24\sqrt{7})(n_{-1} + m_{-1}\sqrt{7}) = (69 + 24\sqrt{7})(8 - 3\sqrt{7}) = 48 - 15\sqrt{7}.
\]
So we can restrict our list (4.5) to
\[(4.6) \quad (27, 0), \ (29, 4), \ (36, 9), \ (48, 15).
\]
Remember that we are looking for integral solutions of (4.1) with \(m\) even.

Since \(d = 7\) is odd and \(n_1 = 8\) is even, applying Lemma 2.1 we have that all the positive solutions \((n, m)\) of (4.1) with \(m\) even are of the form
\[
n \pm m\sqrt{7} = \begin{cases} 
27(n_{2k} + m_{2k}\sqrt{7}) \\
(29 + 4\sqrt{7})(n_{2k} + m_{2k}\sqrt{7}) \\
(36 + 9\sqrt{7})(n_{2k+1} + m_{2k+1}\sqrt{7}) \\
(48 + 15\sqrt{7})(n_{2k+1} + m_{2k+1}\sqrt{7})
\end{cases} \quad \text{with} \ k \in \mathbb{Z}.
\]

Now for all such solutions \((n, m)\), the proof of Theorem 3.2 shows that
\[
a := \frac{n + b - 1}{2} = \frac{n + 27}{2}
\]
is an integer such that \(N(a, b) = N(a, 28)\) is a perfect square whenever \(a > 28 = b\).

In this way we can effectively construct, by a finite search in an explicit bounded interval, all the pairs \((a, 28)\) for which \(N(a, 28)\) is a square.

We now list some explicit computations.
Example 4.1. Let us take \((n', m') = (27, 0)\).

For \(k = 0\), \((n_0, m_0) = (1, 0)\) and we get \(a = (1 + 27)/2 = 14 < b\), which we disregard.

For \(k = 1\) we get \((n_{2k}, m_{2k}) = (n_2, m_2) = (127, 48)\), so \((n, m) = (27 \cdot 127, 27 \cdot 48) = (3429, 1296)\) and
\[
a = \frac{3429 + 27}{2} = 1728
\]
for which
\[
N(1728, 28) = 36393925811128600489003879513323005869574641433293468096956^2
\]
\[
= \left( \frac{2 \cdot 2}{1296} \right)^2 \left( \frac{1728}{28} \right)^2
\]

For \(k = 2\) we have \((n_{2k}, m_{2k}) = (n_4, m_4) = (32257, 12192)\) and \((n, m) = (27 \cdot 32257, 27 \cdot 12192) = (870939, 329184)\), which gives \(a = (870939 + 27)/2 = 435483\). Indeed it is possible to check that
\[
N(435483, 28) = \left( \frac{2 \cdot 2}{329184} \left( \frac{435483}{28} \right) \right)^2
\]

Example 4.2. As another example, take \((n', m') = (36, 9)\) from the list.

For \(k = -1\) we have
\[
(36 + 9\sqrt{7})(n_{2k+1} + m_{2k+1}\sqrt{7}) = (36 + 9\sqrt{7})(n_{-1} + m_{-1}\sqrt{7})
\]
\[
= (36 + 9\sqrt{7})(8 - 3\sqrt{7}) = 99 - 36\sqrt{7},
\]
So \((n, m) = (99, 36)\), hence \(a = (99 + 27)/2 = 63\) and indeed
\[
N(63, 28) = 69923143311577493^2 = \left( \frac{2 \cdot 2}{36} \left( \frac{63}{28} \right) \right)^2
\]

Finally, for \(k = 0\) we have
\[
(36 + 9\sqrt{7})(n_{2k+1} + m_{2k+1}\sqrt{7}) = (36 + 9\sqrt{7})(n_1 + m_1\sqrt{7})
\]
\[
= (36 + 9\sqrt{7})(8 + 3\sqrt{7}) = 477 + 180\sqrt{7},
\]
which gives \((n, m) = (477, 180)\). Therefore \(a = (477 + 27)/2 = 252\), and
\[
N(252, 28) = 266280675495914347757098255444196475^2 = \left( \frac{2 \cdot 2}{180} \left( \frac{252}{28} \right) \right)^2
\]

It is amusing to see how the pairs \((a, b)\) for which \(N(a, b)\) is a square distribute. We plotted in Figure 1 such pairs for \(a \leq 2000\), but only for the values \(b \leq a/2\), because of the symmetry \(N(a, b) = N(a, a - b + 1)\).
5. Higher powers

In this section we investigate when a Narayana number is a perfect power $m^k$ of some integer $m$ with $k > 2$. Compared to the case of squares, here things become more complicated.

We are concerned here with Conjecture 1 from the introduction, i.e. that no Narayana number is a perfect $k$-th power of an integer for $k \geq 3$.

While the full conjecture seems to be out of reach, we provide here evidences by presenting some partial results.

Consider the equation

$$N(a, b) = m^k$$

for integers $m \geq 2$ and $k \geq 1$.

From this and (3.1) we get the two equations

\[(5.1) \quad (a - b + 1) \left( \frac{a}{b - 1} \right)^2 = ab m^k\]

and

\[(5.2) \quad b \left( \frac{a}{b} \right)^2 = a(a - b + 1) m^k.\]

We start with the following proposition.

**Proposition 5.1.** Let $a, b$ be positive integers with $b \leq a/2$. Suppose that $N(a, b) = m^k$ for some positive integers $m$ and $k$. Then:

1. if $a = p$ is a prime, then $k = 1$;
2. if $a = p^2$ is the square of a prime, then $k \leq 2$.

**Proof.** Case 1: Observe that clearly $p$ does not divide both $b$ and $p - b + 1$; moreover $p$ divides $\left( \frac{p}{b} \right)$ exactly once. Hence (5.2) with $a = p$ implies that $p$ divides $m^k$ exactly once, therefore we must have $k = 1$. 

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Case [2]: We start by recalling the following formula. For a prime \( p \), we denote by \( v_p \) the \( p \)-adic valuation, i.e. for \( n \in \mathbb{N} \), \( v_p(n) \) is the greatest nonnegative integer \( h \) such that \( p^h \) divides \( n \). Then it is well known and easy to show that
\[
v_p\left(\frac{p^n}{t}\right) = n - v_p(t)
\]
for all positive integers \( n, t \).

So
\[
v_p(N(p^2, b)) = -2 + (2 - v_p(b)) + (2 - v_p(b - 1)).
\]

Notice that \( p \) cannot divide both \( b \) and \( b - 1 \). Moreover \( v_p(b) \) and \( v_p(b - 1) \) are either 0 or 1, since \( a = p^2 > b > b - 1 \). Summing up, this tells us that \( v_p(N(p^2, b)) \in \{1, 2\} \) and so \( N(p^2, b) \) cannot be a perfect \( k \)-th power with \( k > 2 \).

**Remark 5.1.** Notice that for \( a = p^r \) with \( r \geq 3 \), and for general \( b \), the same argument only shows that \( k \leq r \). So in this case we need another strategy.

We are going to use the following result, well known as Bertrand’s postulate, and first proved by Tchebysev.

**Theorem 5.2 (Tchebyshev).** For all \( n > 0 \) there is a prime \( p \) such that \( n < p \leq 2n \).

The following theorem is the main result of this section.

**Theorem 5.3.** Let \( a \) be a positive integer and let \( p \) be the biggest prime such that \( p < a \). Suppose that \( a/2 \geq b > a - p + 1 \). Then \( N(a, b) = m^k \) for some \( m \in \mathbb{N} \), only for \( k \leq 2 \).

**Proof.** By Tchebysev’s Theorem, we can find a prime \( p \) such that \( \lfloor (a + 1)/2 \rfloor < p < a + 1 \). Because of Lemma 5.1 we can assume that \( p < a \). Observe that \( p \) cannot divide \( a \).

By assumption \( a/2 \geq b \), so \( p > b \). Hence \( p \) does not divide \( b \).

We look first at the case where \( p \neq a - b + 1 \).

We set \( c := a - p \), so that \( a - b + 1 = p + c - b + 1 \). Now \( p \) does not divide \( a - b + 1 \), since it does not divide \( c - b + 1 \): indeed \( b - c - 1 > 0 \), thus \( b - c - 1 \leq b - 2 < p \). Now (5.2) with \( a = p + c \) becomes
\[
b \left( \frac{p + c}{b} \right)^2 = (p + c)(p + c - b + 1)m^k.
\]

By hypothesis \( b > c + 1 \), hence \( p \) divides \( \left( \frac{p + c}{b} \right) \) exactly once. Since \( p \) does not divide \( b \), \( p + c = a \) and \( p + c - b + 1 = a - b + 1 \), it must divide \( m \). But \( p \) divides the left hand side exactly twice, so it must divide \( m^k \) exactly twice. In particular we must have \( k \leq 2 \).

It remains to check the case where \( p \) is equal to \( a - b + 1 \). If we set \( a - b + 1 = p \) in the equation (5.1), we get
\[
p \left( \frac{a}{p} \right)^2 = abm^k.
\]
Now, since \( |(a + 1)/2| < p < a \), \( p \) does not divide \( \binom{a}{p} \). Since \( p \) does not divide both \( a \) and \( b \), it must divide exactly once \( m^k \), which implies \( k = 1 \).

This completes the proof of the theorem. \( \square \)

For \( c \in \mathbb{N} \), let us call \( P(c) \) the greatest prime \( p \) that divides \( c \). It can be shown \( 28 \) (see also \( 20 \)) that for \( n \geq 2k > 0 \)

\[
P\left( \binom{n}{k} \right) > 1.95k.
\]

**Theorem 5.4.** If \( N(a, b) = m^k \) and \( a/2 \geq b \geq \sqrt{a}/1.95 \), then \( k \leq 2 \).

**Proof.** Using \( 5.2 \), we can rewrite the condition \( N(a, b) = m^k \) as

\[
 b \left( \frac{a}{b} \right)^2 = a(a - b + 1)m^k.
\]

Let

\[
p := P\left( \binom{a}{b} \right).
\]

From what we observed before this theorem, we know that \( p > 1.95b \geq \sqrt{a} \).

By a theorem of Mignotte \( 21 \) (see also \( 20 \)), if we have the prime factorization

\[
\binom{n}{k} = \frac{n(n-1) \cdots (n-k+1)}{k!} = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j},
\]

then each prime power \( p_i^{\alpha_i} \) must divide one of the factors of the numerator.

Notice that \( p^2 > a \), so, using Mignotte’s theorem, \( p \) divides \( \binom{a}{b} \) exactly once.

Now \( p \) does not divide \( b \) since \( p > 1.95b > b \), so \( p \) divides the left hand side of \( 5.3 \) exactly twice.

But \( p \) cannot divide both \( a \) and \( a - b + 1 \), so it divides \( m^k \) one or two times. This implies that \( k \leq 2 \), as we wanted. \( \square \)

**Remark 5.2.** Notice that neither of the two theorems of this section implies the other. For example, for \( a = 1362 \), the greatest prime \( p \) which is smaller than \( a \) is 1361, hence Theorem \( 5.3 \) shows that \( N(1362, b) = m^k \) implies \( k \leq 2 \) for \( b > 1362 - 1361 + 1 = 2 \), while Theorem \( 5.4 \) gives the result only for \( b > \sqrt{1362}/1.95 \approx 18.93 \).

On the other hand, for \( a = 1360 \), the greatest prime \( p \) which is smaller than \( a \) is 1327, hence Theorem \( 5.3 \) shows that \( N(1360, b) = m^k \) implies \( k \leq 2 \) for \( b > 1360 - 1327 + 1 = 34 \), while Theorem \( 5.4 \) gives the result for \( b > \sqrt{1360}/1.95 \approx 18.91 \), which is a better bound.

In fact for \( a \) big enough (say \( a > 5000 \)) the bound on \( b \) of Theorem \( 5.3 \) seems to be always better than the one of Theorem \( 5.4 \) as Figure 2 suggests.
5.1. **Conjecture** for small values of $b$. Notice that both theorems cover only cases in which $b$ is “not too small”. For small $b$’s things can become complicated, though something can be said.

Consider for instance the case $b = 2$. Then

$$N(a, 2) = \frac{a(a - 1)}{2} = \binom{a}{2}$$

is just a binomial.

The fact that $N(a, 2) = m^k$ implies $k \leq 2$ is then proved in [13].

Consider now the case $b = 3$. Then the equation

$$N(a, 3) = \frac{a(a - 1)^2(a - 2)}{12} = m^k$$

is equivalent to

$$(a - 1)^4 - (a - 1)^2 = 12m^k$$

or

$$(2(a - 1)^2 - 1)^2 - 48m^k = 1.$$  

Now it would follow from Pillai’s generalization of Catalan’s conjecture that there are at most finitely many exceptions to Conjecture [14] in this case.
Conjecture 2 (Pillai’s conjecture). For any triple of positive integers $a, b, c$, the equation $ax^n - by^m = c$ has only finitely many solutions $(x, y, m, n)$ with $(m, n) \neq (2, 2)$.

Pillai’s conjecture is still open (it holds conditionally assuming the $abc$-conjecture).

In conclusion, other than some numerical evidence and the cases covered in this work, Conjecture 1 remains open.

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