On quasi-linearization of correlations in quadratic elasticity theory for isotropic media

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Abstract. This paper exposes the strain potential of the quadratic theory of elasticity for isotropic media. The strain potential is represented as a sum of the quadratic and the cubic part. A system of experiments, carried out to define five constants included in the potential, is exposed. Different variants of building quasi-linear correlations are discussed. The rationality of linearizing constitutive correlations by reducing the degree of the potential’s cubic part by division by the quadratic convolution module of stress tensor components is doubted because the correlations thus found are lengthier than the correlations from the quadratic elasticity theory.

1. Introduction
There are a lot of works in continuum strain mechanics on building constitutive elastic theory correlations for media sensitive to the kind of stressed state. For a detailed overview of these papers see articles [1,2] and monographs [3].

The approach chosen from various proposals on building constitutive correlations for such media is based on using the potential of the quadratic elasticity theory.

The invariants chosen in works [4, 5] as basic are related to the octahedral plane recorded as

$$\sigma = (\sigma_1 + \sigma_2 + \sigma_3)/3, \quad \tau_0 = \sqrt{\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{3}},$$

$$\tau_0^3 \cos 3 \varphi = \sqrt{2} (\sigma_1 - \sigma) (\sigma_2 - \sigma) (\sigma_3 - \sigma). \quad (1.1)$$

Where $\sigma$ is the normal plane stress, $\tau_0$ is the tangent stress, $\varphi$ is the angle determining the direction of the tangent stress in the octahedral plane.

The quadratic potential recorded via physical invariants (1.1) is

$$W = A\sigma^2 + B\tau_0^2 + C\tau_0^3 \cos 3 \varphi + D\sigma^3 + E\sigma \tau_0^2. \quad (1.2)$$

The strain potential of the most primitive variant of the varied-module elasticity theory is set in work [4] as

$$W = A\sigma^2 + (B + C \cos 3 \varphi)\tau_0^2. \quad (1.3)$$
If to take in potential (1.1) that $D = E = 0$ and use $\tau_o^2 \cos 3 \varphi$ instead of $\tau_o^3 \cos 3 \varphi$, the result will be potential (1.2). Thus, according to L A Tolokonnikov’s suggestion, the power of $\tau_o^3 \cos 3 \varphi$ was lowered by dividing that function by its integral stress state characteristic $\tau_o$. This procedure was called quasi-linearization.

If to use the invariants recorded as $I_1 = \sigma_1 + \sigma_2 + \sigma_3$, $I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1$, $I_3 = \sigma_1 \sigma_2 \sigma_3$, (1.4)

the quadratic potential will be

$$W = AI_1^2 + BI_2^2 + CI_3^3 + DI_1 I_2 + EI_1.$$ (1.5)

If to quasi-linearize the cubic part of the potential by dividing by the module of the stress vector in the 3D space of main stresses, the result stemming from (1.4) will be the quasi-quadratic potential recorded as

$$W = AI_1^2 + BI_2^2 + (CI_3 + DI_1 I_2 + EI_1) / \Sigma,$$ (1.6)

where $\Sigma = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2}$. Potential (1.5) was extensively used in [3] to solve applied problems.

V P Myasnikov and his coauthors in [6] used the idea of quasi-linearization to record the quasi-quadratic stress potential based on convolutions of main strains and represented as

$$W = a I_1^2 + b I_2^2 + (c I_1^4 + d I_1 I_2 + e I_3) / \sqrt{I_1} + (f I_1^4 + g I_1 I_3) / I_2 + ...,$$ (1.7)

where $I_1 = e_1 + e_2 + e_3$, $I_2 = e_1^2 + e_2^2 + e_3^2$, $I_3 = e_1^3 + e_2^3 + e_3^3$.

Thus the quadratic elasticity theory potential is found as the sum of the quadratic and the cubic part, whereas the quasi-quadratic potential is found by lowering the power of the cubic part, which is done by dividing the cubic part by a certain integral characteristic of stress or strain state in [3-6].

The tangential octahedral stress was used as an integral characteristic of the stress in [4, 5]. The module of the main stress vector was used in [3]. Quadratic convolution of the strain tensor was used in [6].

2. Determinative correlation of quadratic elasticity theory

Let us consider the straining of an isotropic medium relegated to the orthogonal frame of reference $x_i (i = 1, 2, 3)$.

Experimental straining tests of many isotropic materials [7-11, 14-17] reveal a nonlinear dependence between the components of strain tensor and the components of stress tensor. An option recommended for describing the stress-strain relation in the nonlinear elasticity theory is a polynomial basis from [18] recorded as

$$W(\sigma) = W_1 + W_2 + W_3 + ...,$$

where $W_n$ are the $n + 1$ homogeneous polynomials by stresses.

The properties of symmetry determine the dependence of the strain potential on the stress tensor invariant [18, 19].

Let us build the correlations of the theory of quadratic elasticity of isotropic media by choosing as the stress tensor invariants the convolutions of the components of the stress tensor from [20, 21]

$$\Sigma_1 = \sigma_{ij} \delta_{ij} = \sigma_1 + \sigma_2 + \sigma_3, \Sigma_2 = \sigma_{ik} \sigma_{kj} = \sigma_1^2 + \sigma_2^2 + \sigma_3^2,$$

$$\Sigma_3 = \sigma_{ik} \sigma_{im} \sigma_{mj} = \sigma_1^3 + \sigma_2^3 + \sigma_3^3.$$ (2.1)
It is more preferable to use invariants (2.1) instead of (1.4) because the integral stress state characteristic of the stress vector module is already included in the list of invariants (2.1); plus, unlike the value of \( I_3 \), invariant \( \Sigma_3 \) in the uniaxial compression and extension test differs from zero.

The potential of the quadratic elasticity theory for an isotropic moderately nonlinear medium is the polynomial basis of stress tensor convolutions

\[
W(\Sigma_1, \Sigma_2, \Sigma_3) = W_1(\Sigma_1, \Sigma_2, \Sigma_3) + W_2(\Sigma_1, \Sigma_2, \Sigma_3),
\]

plus

\[
W_1 = A_4 \Sigma_1^2 + A_2 \Sigma_2 = A_1 (\sigma_1 + \sigma_2 + \sigma_3)^2 + A_2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2),
\]

\[
W_2 = A_3 \Sigma_1^3 + A_4 \Sigma_1 \Sigma_2 + A_5 \Sigma_3 = A_3 (\sigma_1 + \sigma_2 + \sigma_3)^3 + A_4 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2) + A_5 (\sigma_1^3 + \sigma_2^3 + \sigma_3^3),
\]

where \( A_1, \ldots, A_5 \) are the constants characterizing the mechanical properties of a moderately nonlinear isotropic medium.

Quadratic potential (2.1.3) determines the linear stress-strain relation and contains two constants.

Cubic potential (2.1.4) contains three constants and characterizes a moderate nonlinear stress-strain relation.

The constitutive correlations stemming from potentials (2.1.2) and (2.1.4) are

\[
e_1 = 2A_3 \Sigma_1 + 2A_2 \sigma_1 + 3A_4 \Sigma_1^2 + A_4 (2\Sigma_1 \sigma_1 + \Sigma_2) + A_5 \sigma_1^2,
\]

(2.5)

The symbol (123) means that the remaining relations are obtained by circular permutation of the indices principal stresses.

Let us show that, to find the constants included in correlations (2.5), it will be enough to conduct three basic tests at various values of the Lode-Nadai coefficient from [22]:

\[
\mu = \frac{2\sigma_2 - \sigma_1 - \sigma_3}{\sigma_1 - \sigma_3}, \quad \sigma_1 \geq \sigma_2 \geq \sigma_3.
\]

Here \( \mu = 0 \) corresponds to the normal pure shear test and \( \mu = -1 \) and \( \mu = 1 \) – to the uniaxial compression and extension, respectively.

The main maximal and minimal stress determined in the normal pure shear test at \( \mu = 0 \) are \( \sigma_1 = -\sigma_3 = \tau \), whereas intermediate main stress \( \sigma_2 \) is zero.

The dependences for approximating the test data are stemmed from (2.5) and recorded as

\[
e_1' = 2A_4 \tau + (2A_4 + 3A_4)\tau^2, \quad e_2' = 2A_4 \tau^2, \quad e_3' = -A_3 \tau + (2A_4 + 3A_4)\tau^2.
\]

(2.6)

Here, the \( e_i' \) symbols stand for the main strain values in the normal pure shear test.

It follows from (2.6) that

\[
A_2 = \frac{e_1' - e_3'}{4\tau}, \quad A_4 = \frac{e_2'}{2\tau^2} = \Delta, \quad A_5 = \frac{1}{3} \left( \frac{e_1' + e_3'}{2\tau^2} - \Delta \right),
\]

where \( G_\tau \) is the pure shear modulus.

Mind that the measurement of strain in the direction of intermediate main stress \( \sigma_2' \), made in the normal pure shear test is essential because the finding of constants \( A_4 \) and \( A_5 \) lies in the beginning of the string of calculating constants characterizing the mechanical properties of an isotropic continuous medium at the fair representation of quadratic potential (2.2) by the sum of (2.3, 2.4) begins with finding constants \( A_4 \) and \( A_5 \).
Let us also consider the uniaxial extension test at $\mu = -1$ and the uniaxial compression test at $\mu = 1$. The stress state at uniaxial extension is characterized by the combination of stresses $\sigma_1 = \sigma_+ = 0$. The dependences found from (2.5) for approximating the test data are

$$e_1^+ = 2(A_1 + A_2)\sigma_+ + 3(A_3 + A_4 + A_5)\sigma_+^2,$$

$$e_2^+ = e_3^+ = 2A_1\sigma_+ + (3A_3 + A_4)\sigma_+^2. \quad (2.7)$$

Here $e^+$ stands for the strain values measured in the uniaxial sample extension test.

Uniaxial compression is determined by the system of stresses $\sigma_1 = \sigma_2 = 0$, $\sigma_3 = -\sigma_-$. The constitutive correlations are recorded as

$$e_1^- = e_2^- = -2A_1\sigma_- + (3A_3 + A_4)\sigma_-^2,$$

$$e_3^- = -2(A_1 + A_2)\sigma_- + 3(A_3 + A_4 + A_5)\sigma_-^2. \quad (2.8)$$

Here, $e_i^-$ stand for the strain values measured in the uniaxial sample compression test.

Since constants $A_2$, $A_4$ and $A_5$ are known, the regression of the uniaxial extension and compression test data by dependences (2.7), (2.8) allows calculating constants $A_1$ and $A_3$.

3. Quasi-quadratic potential

Let us consider two variants of quasi-linearizing the correlations from the quadratic elasticity theory.

In works [5] Tolokonnikov proposed to replace for moderately nonlinear media the cubic part of the quadratic elasticity theory potential recorded via physical invariants of the stress tensor with the quasi-quadratic equation derived by dividing the cubic part of the potential by octahedral stress.

Work [3] propose to apply a similar procedure to potential (2.2) of the quadratic elasticity theory by replacing cubic part (2.4) with a quasi-quadratic function recorded as

$$W^* = B_1\Sigma_2^2 + B_2\Sigma_3^2 + (B_1\Sigma_1^3 + B_3\Sigma_1\Sigma_2 + B_3\Sigma_3)\Sigma^{-1},$$

where $\Sigma = \sqrt{\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2}$.

In this case the constitutive quasi-linear correlations are

$$e_1 = 2B_1\Sigma_1 + 2B_2\sigma_1 + B_3\chi(3\Sigma_1 - \chi^3\sigma_1) + B_4(\Sigma + \chi\sigma_1) + B_5(3\chi - \lambda)\sigma_1, \quad (123),$$

where $\chi = \Sigma_1 / \Sigma$, $\lambda = \Sigma_3 / \Sigma^3$.

To find the constants included in correlations (3.1), let us use the data from the three basic tests. Let us consider the normal pure shear test: $\sigma_1 = -\sigma_3 = \tau$, $\sigma_2 = 0$ ($\chi = \lambda = 0$).

Correlations (3.1) are recorded as

$$e_1^* = (2B_2 + \sqrt{2}B_4)\tau, \quad e_2^* \sqrt{2}B_4\tau, \quad e_3^* = -(2B_2 - \sqrt{2}B_4)\tau. \quad (3.2)$$

The two constants found by approximating the test data by dependences (3.2) are

$$B_2 = 1/4G_\tau, \quad B_4 = \Delta_\tau / \sqrt{2},$$

where $G_\tau$ is the shear module found in the pure shear test, $\Delta_\tau = e_2^* / \tau$ is the compliance coefficient at the pure shear in the direction of intermediate main stress $\sigma_2$. 

4
At the uniaxial compression recorded as \( \sigma_1 = \sigma_+ \), \( \sigma_2 = \sigma_3 = 0 \) \((\chi = \lambda = 1)\) the test data are exposed to a regression analysis using the following dependences:

\[
e_1^* = 2(B_1 + B_2 + B_3 + B_4 + B_5)\sigma_+, \quad e_2^* = e_3^* = (2B_1 + 2B_3 + B_4)\sigma_+^2.
\]

The plus sign indicates that the strains are from the uniaxial extension test. Therefore,

\[
\frac{1}{E_+} = 2(B_1 + B_2 + B_3 + B_4 + B_5), \quad \frac{\nu_+}{E_+} = -(2B_1 + 3B_3 + B_4),
\]

where \( E_+ \) and \( \nu_+ \) are, respectively, Young’s module and Poisson’s ratio at uniaxial extension.

The stress state determined in the uniaxial compression test is \( \sigma_1 = \sigma_+ = 0 \), \( \sigma_3 = -\sigma_- \) \((\chi = \lambda = -1)\).

The dependences used to approximate the test data are

\[
e_1^- = e_2^- = (-2B_1 + 3B_3 + B_4)\sigma_- , \quad e_3^- = 2(-B_1 - B_2 + B_3 + B_4)\sigma_- .
\]

The minus sign indicates that the strains are from the uniaxial compression test. Therefore,

\[
\frac{1}{E_-} = 2(B_1 + B_2 - B_3 - B_4 - B_5), \quad \frac{\nu_-}{E_-} = -2B_1 + 3B_3 + B_4,
\]

where \( E_- \) and \( \nu_- \) are, respectively, Young’s module and Poisson’s ratio at uniaxial compression.

Constants \( B_1, B_2, B_3, B_4 \) and \( B_5 \) are found by jointly solving equations (3.3) and (3.4) as

\[
B_1 = -\frac{1}{4}\left(\frac{\nu_-}{E_-} + \frac{\nu_+}{E_+}\right), \quad B_3 = \frac{1}{3}\left(\frac{\nu_-}{E_-} - \frac{\nu_+}{E_+}\right)B_4, \quad B_4 = \Delta_e / \sqrt{2} ,
\]
\[
B_2 = \frac{1}{4}\left(1+\frac{\nu_-}{E_-} + 1+\frac{\nu_+}{E_+}\right), \quad B_5 = \frac{1}{4}\left(1 - \frac{\nu_-}{E_-} + \frac{1}{6}\left(\frac{\nu_-}{E_-} - \frac{\nu_+}{E_+}\right) - \frac{2}{3}B_4 .
\]

Work [3] illustrate the advantage of approximating the test data by dependences (3.1) on VPP and ARP graphite [7, 8, 11, 25], concrete [9, 15-17], and SCH15-32 grey cast iron [10] in comparison with the best known models of the varied-module elasticity theory [3, 4, 12, 13, 23, 24].

Unfortunately, those test data were not approximated by quadratic elasticity theory correlations (3.1).

The second variant of deriving quasi-linear correlations of the quadratic elasticity theory consists in directly quasi-linearizing correlations (3.1), for which purpose the powers of the summands with square stresses are lowered by dividing them by the module vector of main tensions in the 3D vector space of main stresses.

After this procedure the quasi-linear correlations of the quadratic elasticity theory are recorded as

\[
e_1 = 2A_1\sigma_1 + 2A_2\sigma_1^2 + 3A_3\sigma_1\chi + A_4(2\chi\sigma_1 + \Sigma) + A_5\sigma_1^3 . \quad (123)
\]

where \( \alpha_i = \sigma_i / \Sigma \) are the normalized main stresses; plus, \( \chi = \alpha_1 + \alpha_2 + \alpha_3 \).

Dimensionless stresses \( \alpha_i \) have a clear geometric meaning as the guiding lines of the cosine of the vector of main stresses in the 3D space of main stresses.

The method extensively used in solving multiple applied problems from by means of the quasi-linear correlations of the quadratic elasticity theory from was step-by-step approach.
The applied problem solutions were compared with the similar solutions obtained at the averaged mechanical characteristics of an isotropic medium and using the constitutive correlations proposed by other authors [3, 4, 12, 13, 23, 24].

4. Conclusion
1. Since a correlation of the quadratic elasticity theory is derived using the third stress tensor invariant, the suggested correlations allow taking account of the influence of the kind of stress state at the strain of an isotropic continuous medium.
2. The basic normal pure shear and uniaxial extension and compression tests allow calculating five constants included in the constitutive correlations of the quadratic elasticity theory.
3. It is more complicated to solve applied problems using quasi-linear correlations of the quadratic elasticity theory than by means of quadratic correlations because the nonlinear part of quadratic correlations contains quadratic products of stresses and the quasi-linear part of quasi-linear correlations contains correlations between the quadratic products of stresses and the square root of the quadratic convolution of stresses.
4. The errors introduced by quasi-linearizing the potential of the quadratic elasticity theory are yet to be evaluated.
5. The absence of any applied problem solutions using correlations of the quadratic elasticity theory does not allow evaluating for fidelity the solutions obtained using quasi-linear stress-strain correlations.

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