Fail-safe optimization of viscous dampers for seismic retrofitting

Nicolò Pollini

Department of Wind Energy, Technical University of Denmark, Roskilde, 4000, Denmark

Correspondence
Nicolò Pollini, DTU Wind Energy, Fredriksborgvej 399, 4000 Roskilde, Denmark.
Email: nipol@dtu.dk

Summary
This paper presents a new optimization approach for designing minimum-cost fail-safe distributions of fluid viscous dampers for seismic retrofitting. Failure is modeled as either complete damage of the dampers or partial degradation of the dampers’ properties. In general, this leads to optimization problems with large number of constraints. This may result in high computational costs if all the constraints are simultaneously considered during the optimization analysis. Thus, to reduce the computational effort, the use of a working-set optimization algorithm is proposed in this paper. The main idea is to solve a sequence of relaxed optimization subproblems with a small subset of all constraints. The algorithm terminates once a solution of a subproblem is found that satisfies all the constraints of the problem. The retrofitting cost is minimized with constraints on the interstory drifts at the peripheries of frame structures. The structures considered are subjected to a realistic ensemble of ground motions, and their response is evaluated with time-history analyses. The transient optimization problem is efficiently solved with a gradient-based sequential linear programming algorithm. The gradients of the response functions are calculated with a consistent adjoint sensitivity analysis procedure. Promising results attained for 3-D irregular frames are presented and discussed. The numerical results highlight the fact that the optimized layout and size of the dampers can change significantly even for moderate levels of damage.

KEYWORDS
adjoint sensitivity analysis, fail-safe design, passive control, seismic retrofitting, transient optimization, viscous dampers

1 | INTRODUCTION

Fluid viscous dampers are a technology initially developed for military applications, but with the end of the Cold War in 1990, their use was allowed also in civil engineering applications. Because of their proven robustness and reliability during decades of Cold War applications, their use in commercial structures took place quickly.1 In particular, the use of viscous dampers in earthquake engineering applications was first validated between the years 1990 and 1993, when their benefit for wind and other types of transient excitation was also shown.2,3 Design procedures for buildings with added damping systems have also been included in design standards. For example, Chapter 18 of ASCE 7-164 refers to new structures and Chapter 15 of ASCE 41-175 to the retrofitting of existing structures, and they originated from the MCEER report 00-0010 of Ramirez et al.6–10 Fluid viscous dampers help to reduce the deformation demand on the structure by dissipating part of the
energy coming from an earthquake. As a consequence, if fluid viscous dampers are properly sized and placed, inter-story drifts, floor accelerations, and story shear forces can be significantly reduced.\textsuperscript{11} The use of passive energy dissipation devices has gained much attention in academia and practice, and the reader is referred to the comprehensive textbooks for more details.\textsuperscript{12-14}

Two aspects strongly influence the structural performance of an added damping system made of fluid viscous dampers. The first is the size of the dampers, which is typically expressed in terms of their damping coefficient. The second is the distribution of the dampers in the structure that needs to be retrofitted. At the same time, these aspects affect not only the structural performance but also the associated retrofitting cost, which can be crucial in promoting the use of fluid viscous dampers over other seismic retrofitting techniques. These aspects led to the development of several approaches for the sizing and placement of fluid viscous dampers assisted by optimization, as recently reviewed by De Domenico et al.\textsuperscript{15} The available methodologies can be grouped based on the formulation used in the optimization problem. A first group consists of those approaches that rely on continuous design variables (i.e., damping coefficients).\textsuperscript{16-31} This leads to optimization problems that are computationally efficient and applicable also to large-scale problems. The optimized designs attained may consist of a wide variety of different damper sizes. To reduce some of the costs, the number of different size groups (i.e., dampers with the same mechanical properties) should be reduced. Thus, another group consists of methodologies that make use of discrete design variables to represent the damping coefficients and that lead to optimized designs of dampers characterized by a limited number of size groups.\textsuperscript{32-36} These approaches rely on predefined parameters such as the available dampers’ sizes or the number of dampers, and this may have a considerable restraining effect on the optimized solutions that can be attained. Moreover, in some cases, the resulting optimization problems are relatively difficult to solve, due to their combinatorial nature. Recently, an attempt was made to develop methodologies that combine the good aspects of the two set of approaches mentioned above: practical (i.e., near discrete) distributions of dampers with a reasonable computational cost.\textsuperscript{37-40} This is achieved using a gradient-based continuous optimization approach for the placement and sizing of linear viscous dampers coupled with material interpolation techniques, typically applied in topology optimization.\textsuperscript{41} In recent work, material interpolation functions have been used also for the optimization-based seismic design of steel moment-resisting frames.\textsuperscript{42}

The optimization approaches discussed previously identify optimized dampers’ distributions under a given set of conditions. These conditions are typically defined by the objective function to be minimized (e.g., the cost), the optimization constraints (e.g., the structural displacements or accelerations), and the type of modeling of the structural system considered. However, the viscous dampers may experience damage during their service lifetime or an earthquake, and therefore, they may not perform as anticipated.\textsuperscript{43} This can potentially cause tragic human and economic losses. To prevent an unexpected and undesired catastrophic failure of a structure retrofitted with viscous dampers, during the design phase, fail-safe structural optimization methodologies should be applied. The basic idea of the fail-safe structural design philosophy is that a structure should be designed to survive normal design loading conditions when damage occurs. The resulting design is safe even if certain predefined types of damage conditions apply. The damage can be in the form of complete or partial failure of a structural member. For the seismic retrofitting with viscous dampers, this can be translated into complete or partial failure of one or several dampers. An additional problem could be represented by the uncertainties in the dampers’ constitutive behavior and the inability of simplified constitutive models to reliably capture their behavior. In general, the fail-safe optimization subject falls into the category of design involving uncertainties, known generally as reliability-based design optimization (RBDO).\textsuperscript{44,45} However, reliability-based optimization approaches rely on statistics and probabilistic analyses, whereas fail-safe approaches rely on the definition of worst case locations of failure scenarios and their associated optimization constraints.\textsuperscript{46,47}

In the context of more traditional structural design applications, the fail-safe optimization for static loading has received an increasing attention by several researchers. One of the first contributions is from 1976 by Sun et al.,\textsuperscript{48} in the field of fail-safe truss structural optimization. The problems considered by Sun et al. minimize the structural weight with constraints on stresses, nodal displacements, and natural period of the structure. They define a priori the number and locations of damaged members. In Achtziger and Bendsoe,\textsuperscript{49} the structural topology optimization of degraded trusses is discussed. The fail-safe structural topology optimization of continuum structures is first presented by Jansen et al.\textsuperscript{50} in 2014, where the local failure of the structure is modeled by removing material in predefined portions of the 2-D design domain. Zhou and Fleury\textsuperscript{46} in 2016 generalized the work of Jensen et al.\textsuperscript{50} to 3-D continuum structures. Kanno\textsuperscript{51} in 2017 proposed an approach for fail-safe structural optimization of trusses based on the worst case scenario of structural degradation. Lüdeker and Kriegesmann\textsuperscript{47} in 2019 discussed the fail-safe optimization of beam structures. They define the failure scenarios as complete removal of one beam at a time. In a recent work, Stolpe\textsuperscript{52} discussed the fail-safe optimization of truss structures. Both partial and full failures of the truss elements are considered, and the optimization problem
is formulated as a convex conic programming problem. All the damage scenarios are considered, and to reduce the computational burden, a working-set approach is used. It should be noted that the references listed above focused on the fail-safe structural optimization with static loading conditions.

Therefore, even though the attention towards fail-safe approaches for the structural optimization of static load-bearing structures seems to be growing in recent years and to gain momentum, to the best of the author's knowledge the fail-safe optimization of structural dynamics problems has not been discussed yet. This is also true in the context of seismic retrofitting with viscous dampers. The use of a fail-safe approach would lead to optimized dampers’ distributions that are more robust and can guarantee a desired performance even when certain failure scenarios occur, something much needed in the earthquake engineering community. The purpose of this paper is to fulfill this need.

Thus, this paper presents a novel optimization approach for the minimum-cost fail-safe optimization of fluid viscous dampers for seismic retrofitting. The dampers’ placement and size is simultaneously optimized. Failure is modeled as either complete or partial damage of the dampers. The damage is expressed through degradation of the mechanical properties of the dampers, that is, their damping coefficient. In general, this results in an optimization problem with a large number of constraints, which requires a high computational effort during the optimization process. The computational cost is reduced using a working-set strategy, similarly to Verbart and Stolpe 2018: a sequence of relaxed subproblems each defined by an expanding subset of the constraints is solved cyclically. In every cycle, new constraints that are active in correspondence of the current optimized solution are added, and the updated optimization subproblem is solved again. The procedure stops once a solution that satisfies simultaneously all of the constraints of the original problem is found. The dampers’ retrofitting cost is minimized with constraints on the inter-story drifts of the retrofitted structure. The response of the retrofitted structures is evaluated with time-history analyses considering an ensemble of realistic ground motions. The problem at hand is formulated through continuous design variables (the damping coefficients of the dampers), and it is solved with a computationally efficient gradient-based algorithm, based on sequential linear programming (SLP).

The remainder of the article is organized as follows. Section 2 presents the modeling of the damage scenarios. Section 3 describes the optimization problem formulation with details on the objective cost function, the governing equations, and the response constraints. Section 4 focuses on the proposed working-set strategy used to reduce the computational cost during the fail-safe optimization process. The details of the adjoint sensitivity analysis and other computational considerations are given in Section 5. Numerical results are presented and discussed in Section 6, followed by concluding remarks in Section 7.

2 | FAILURE SCENAROIS

The problem considered herein consists in sizing and placing linear fluid viscous dampers in given frame structures subjected to realistic ground motions. The designs are identified by minimizing the dampers’ cost while fulfilling expected performance criteria of the retrofitted structure. Moreover, the dampers’ layouts are identified while also considering possible failure or damage scenarios for the dampers, thus obtaining safer optimized designs. In regard to structures equipped with viscous dampers, studies have been conducted on how to achieve an acceptable reliability level, expressed typically by a total probability of collapse during the lifetime of the structure or, in more simple terms, a probability of collapse for the maximum expected earthquake. This work, however, considers failure scenarios associated only to the dampers, which are also consistent with the linear elastic structural behavior considered. Hence, this section starts the discussion by presenting the scenarios considered for complete or partial damage of the viscous dampers.

2.1 | Complete failure of dampers

We consider linear fluid viscous dampers. Hence, their force velocity behavior is formulated as follows:

\[ f_d = c_d \dot{d}, \]

where \( d \) is the derivative in time of the relative displacement between the damper’s ends; \( c_d \) is the damping coefficient of the damper; and \( f_d \) is the damper’s output resisting force. The design variables of the problem are the damping coefficients of the dampers. If we imagine that we have the possibility of placing a maximum of \( N_d \) dampers in a structure for retrofitting purposes, then \( I = \{1, \ldots, N_d\} \) is the set of all the dampers’ indices. Thus, the resulting damping matrix due to the added damping if calculated as follows:

\[ C_d = \sum_{i \in I} T_i^T c_{d,i} T_i, \]
where \( c_{d,i} \) is the damping coefficient of the \( i \)th damper; \( T_i \) is a transformation matrix of the \( i \)th damper from global coordinates to the local coordinates of the dampers (damper elongation); and \( C_d \) is the added damping matrix. We consider \( n_c \) complete damage scenarios where in each case, \( m_c \) dampers are completely damaged. The set of completely damaged dampers’ indices for the damage scenario \( \alpha \) is \( J^\alpha \subseteq I \). The damaged added damping matrix is formulated as follows:

\[
C^\alpha_d = \sum_{i \in T \setminus J^\alpha} T_i^T c_{d,i} T_i.
\] (3)

### 2.2 Partial failure of dampers

Similar to the case of complete damage, we consider \( n_p \) partial damage scenarios where in each case, \( m_p \) dampers are partially damaged. The set of partially damaged dampers’ indices for the damage scenario \( \alpha \) is \( J^\alpha \subseteq I \). The damaged added damping matrix is formulated as follows:

\[
C^\alpha_d = \sum_{i \in T \setminus J^\alpha} T_i^T c_{d,i} T_i + \sum_{j \in J^\alpha} T_j^T \nu_j c_{d,j} T_j,
\] (4)

where \( \nu_j \) is a damage coefficient that reduces the damping capacity of the damper \( j \) and satisfies \( 0 \leq \nu_j \leq 1 \) (e.g., \( \nu_j = 0.5 \forall j \)).

### 3 Optimization Problem Formulation

This section provides important details regarding the cost function minimized in the optimization analysis, the governing equations of motion, and the structural response constraints considered. This section is concluded presenting the final optimization problem formulation.

#### 3.1 Objective cost function and design variables

The overall aim of this work is to propose an effective and efficient fail-safe optimization approach for minimizing the cost due to the seismic retrofitting with fluid viscous dampers, with constraints on the structural performance. In a recent work, a realistic cost function for the retrofitting with viscous dampers has been proposed.\(^{39}\) The cost function includes the costs associated to the number of locations in the structure taken by dampers, the dampers’ manufacturing cost, and the cost associated to the dampers’ prototype testing. In this work, the focus is on the novel fail-safe optimization approach for seismic retrofitting with viscous dampers proposed, and we will consider only the cost associated to the manufacturing of the dampers. The cost of a single fluid viscous damper depends on its displacement capacity (stroke), peak force, performance and testing requirements, number of devices ordered, delivery schedule, and other aspects related to a competitive market environment. In this work, a simplified manufacturing cost formulation is considered because the focus is directed to the novel fail-safe optimization approach. To simplify the manufacturing cost of a damper, we first observe that the peak stroke experienced by a damper is strongly correlated with the peak interstory drift, which is constrained in our problem formulation. For this reason, the damper stroke is not explicitly considered in the cost formulation here. Assuming a dominant mode behavior, the velocity in the damper in location \( i \) can be expressed as \( \omega_1 d_i \), where \( \omega_1 \) is the dominant frequency and \( d_i \) is the envelope peak drift at the location \( i \). Experience shows that, usually, dampers are located where the drifts reach their allowable values, which are known values.\(^{23}\) Thus, an estimate of the maximum velocities in the dampers is known in advance, and based on Equation (1), the minimization of the dampers’ peak forces is here replaced by the minimization of the damping coefficients. Based on these considerations, the retrofitting cost function minimized is

\[
J(c_d) = \sum_{i=1}^{N_d} c_{d,i},
\] (5)

where \( c_d \) is a vector that collects the damping coefficients \( c_{d,i} \) of the dampers. The damping coefficient of each damper is formulated as follows:

\[
c_{d,i} = \tilde{c}_d x_i, \quad \text{with } 0 \leq x_i \leq 1, \quad i = 1, \ldots, N_d.
\] (6)
In Equation (6), \( \bar{c}_d \) is the maximum available damping coefficient considered in the optimization analysis and it is defined a priori (e.g., \( \bar{c}_d = 150000 \, \text{ksi} \cdot \text{in} / \text{sec} \)). It defines the available design domain \([0, \bar{c}_d]\) for each damper, and it should be set high enough in order to ensure the feasibility of the optimization problem at hand. Moreover, it allows to define the actual optimization design variables \( x_i \) as continuous numbers that can vary between 0 and 1, which is beneficial from a numerical point of view for the optimizer. The values \( x_i \) are collected in the vector \( x \). Hence, the objective function can be normalized by the parameter \( \bar{c}_d \) leading to

\[
J(x) = \sum_{i=1}^{N_d} x_i. 
\]

Thus, the objective cost function \( J(x) \) of Equation (7) is considered in the final optimization problem formulation.

### 3.2 Equations of motion

In this work, linear structures equipped with linear fluid viscous dampers are considered. When a structure is retrofitted using viscous damper, a linear behavior of the retrofitted structure may be desired. With the methodology proposed herein (as it is further discussed in Section 3.3), this could be achieved by limiting the interstory drifts to allowable limits that enforce a linear structural behavior, if feasible. Moreover, we consider linear fluid viscous dampers because of their out-of-phase effect.\(^2\) If needed, the optimized distributions of linear dampers can be translated into equivalent nonlinear ones by equating the energy dissipated per cycle by linear and nonlinear dampers as described by Lin and Chopra.\(^{57}\) Alternatively, in a recent work, De Domenico and Ricciardi\(^{58}\) discussed a stochastic linearization method for establishing an equivalence between the damping coefficients of linear and nonlinear dampers suitable for linear methods of analysis (e.g., the response spectrum method), or linear random vibration theory. Thus, the equations of motion for 3-D irregular structures considered herein are the following:

\[
\begin{align*}
M \ddot{u}(t) + \begin{bmatrix} C_s & C_{d}(x) \end{bmatrix} \dot{u}(t) + K u(t) &= -a_g(t) Me \\
\dot{u}(0) &= u_0, \quad \ddot{u}(0) = \dot{u}_0,
\end{align*}
\]

where \( M, C_s, \) and \( K \) are the mass, inherent damping, and stiffness matrices of the structure, respectively; \( C_d \) is the added damping matrix that depends on the design variables \( x \); \( u(t), \dot{u}(t), \) and \( \ddot{u}(t) \) are the displacements, velocities, and accelerations at time \( t \) of the degrees of freedom relatively to the ground; and \( e \) is the influence vector and it represents the displacements of the masses resulting from static application of a unit ground displacement. Basically, the vector \( e \) assigns the acceleration to the degrees of freedom of the structure affected by the ground motion. \( a_g(t) \) is the ground acceleration record as a function of time. In general, the local coordinates of the dampers are different from the global ones. Thus, a transformation of coordinates is performed during the assembly of the added damping matrix \( C_d \), as it is shown in Equation (2).

The equations of motion in Equation (8) are discretized and solved in time using Newmark’s time stepping method. In particular, we rely on the average acceleration method because it is stable for any choice of time step. The specific approach adopted for evaluating the structural response directly affects the adjoint sensitivity analysis (which will be discussed in Section 5) where the gradients of the response constraints are calculated. Therefore, for the sake of clarity, the details of the time stepping scheme adopted in this work are provided in Appendix A.

### 3.3 Structural response constraints

The retrofitting designs are obtained by solving an optimization problem with nonlinear constraints imposed on the structural performance of the structure equipped with fluid viscous dampers. In principle, there are several local responses of interest associated to the structural behavior and damage, such as interstory drifts and total story accelerations, to name a few. Fluid viscous dampers have proven to be effective in reducing interstory drifts, story shear forces, and floor accelerations.\(^{11}\) Several authors used the interstory drifts as performance measures for the optimization-based seismic retrofitting with viscous dampers.\(^{20,36,37,39}\) In the following, interstory drifts are the structural performance index explicitly constrained in the optimization problem, because they have been shown to be also a good measure of both structural and nonstructural damage.\(^{59}\) It should be noted that, in principle, the proposed approach could be deployed also considering
other structural performance constraints in the optimization problem, as described by Lavan.\textsuperscript{60} In the following, the peak interstory drifts normalized by a predefined allowed value are chosen as the local performance indices constrained in the optimization problem at hand:

\[
d_{ci} = \max_i (d_i(x, t)/d_{allow}) \leq 1, \forall i = 1, \ldots, N_{drifts},
\]

where \(d_i(t)\) is the \(i\)th interstory drift constrained at time \(t\); \(d_{allow}\) is the maximum allowed value of interstory drift (e.g., 3.5 cm) and it is a predefined input parameter; and \(N_{drifts}\) is the number of interstory drifts constrained.

The optimization approach adopted in this work relies on a gradient-based algorithm (more details are provided in Section 5). Hence, all the objective and constraint functions involved in the optimization problem formulation have to be differentiable. The constraint formulation of Equation (9) relies on the max function, which is not differentiable. We use a \(p\)-norm function that approximates \(\max_i (d_i(x, t)/d_{allow})\)\textsuperscript{38} and that is differentiable:

\[
\tilde{d}_c(u, x) = \left[ \frac{1}{t_f - t_0} \int_{t_0}^{t_f} (D^{-1}(d_{allow}) D(Hu(x, t)))^p dt \right]^{1/p} \cdot 1 \leq 1,
\]

where \(t_0\) and \(t_f\) are the initial and final time of the time-history analysis; \(p\) is a large even number; \(D()\) is an operator that transforms a vector into a diagonal matrix with the vector on the main diagonal; and \(H\) is a matrix that transforms displacements \(u\) into interstory drifts \(d\). Additionally, in order to reduce the number of gradients of constraint functions from \(N_{drifts}\) to one, we aggregate the constraints (10) into a single constraint:

\[
g(u, x) = \frac{1^T D(\tilde{d}_c(u, x))^q + 1}{1^T D(\tilde{d}_c(u, x))^q} - 1 \leq 0.
\]

Equation (11) is differentiable, and for increasing values of \(q\), Equation (11) approximates with increasing accuracy the maximum value of \(\tilde{d}_c(u, x)\), that is, \(\max_i (\tilde{d}_c(u, x))\) for \(i = 1, \ldots, N_{drifts}\). The details of the numerical values assigned to \(q\) are given in Section 6.

In this work, we consider \(n_c\) scenarios of total failure of \(n_c\) dampers and \(n_p\) scenarios of partial failure of \(n_p\) dampers. We consider also the case without any failure of the dampers. Thus, in total, for a given design layout of dampers, we evaluate the structural response for \(N_{FS} = 1 + n_c + n_p\) different scenarios. Section 4 presents a procedure used to reduce the number of failure scenarios actually considered during the optimization analysis to only a subset. However, in principle, the number of response constraints considered is \(N_{FS}\) (i.e., one for each failure scenario):

\[
g_{a}(u, x) = \frac{1^T D(\tilde{d}_c^a(u, x))^q + 1}{1^T D(\tilde{d}_c^a(u, x))^q} - 1 \leq 0, \quad \text{for} \quad a = 1, \ldots, N_{FS}.
\]

Equation (12) represents a list of \(N_{FS}\) nonlinear constraints considered in the optimization analysis. It should be noted that, in principle, \(N_{FS}\) can be a large number, resulting in a lot of response and sensitivity analyses that need to be performed in every optimization iteration.

### 3.4 Final optimization problem

We present now the final optimization problem formulation for the fail-safe design of fluid viscous dampers for seismic retrofitting. The optimization problem is stated as follows:

\[
\begin{aligned}
\min_{x \in \mathbb{R}^{N_d}} & \quad J(x) \\
\text{subject to} & \quad g_{a}(u, x) \leq 0, \quad \forall a \in S_{FS} \\
& \quad 0 \leq x_i \leq 1, \quad \text{for} \quad i = 1, \ldots, N_d \\
& \text{with:} \quad \ddot{u}(t) + [C_s + C_d(x)] \dot{u}(t) + Ku(t) = -a_g(t)Me \quad \forall a_g \in E \\
& \quad u(0) = u_0, \quad \dot{u}(0) = \ddot{u}_0,
\end{aligned}
\]

\(P_{FS}\)
where $\mathcal{E}$ is the ensemble of ground motions considered. $S_{FS}$ is the set of all indices that identify the failure scenarios considered, and its cardinality is $|S_{FS}| = N_{FS}$. The optimization problem 13 is solved with a SLP approach. With this approach, in every optimization iteration, the problem is linearized and solved locally. Hence, the gradients of the objective and constraints functions need to be calculated. The objective function $J$ is formulated explicitly in terms of the design variables of the problem. Thus, its gradient can be calculated directly. The gradients of the constraint functions need to be calculated with a dedicated adjoint sensitivity analysis instead. More details about the sensitivity analyses performed in this work are given in Section 5, but from a computational point of view, each sensitivity analysis weighs as much as a linear response time history analysis. In principle, having $N_{FS}$ fail-safe scenarios means that in each optimization iteration, $N_{FS}$ adjoint sensitivity analyses need to be performed in order to calculate the gradients of the $g_a(u, x)$ constraint functions, for $\alpha = 1, \ldots, N_{FS}$. Thus, in order to reduce the number of sensitivity analyses performed in each optimization iteration, and hence the overall computational cost, in Section 4, the use of a working-set strategy is proposed. The goal is to consider a subset of constraints during the optimization and to reduce in this way the required computational cost and time.

4 WORKING-SET STRATEGY

In the fail-safe optimization approach discussed herein, we consider $N_{FS}$ failure scenarios. In every optimization iteration, the structural response should be evaluated for each of these scenarios. Moreover, the gradient of the aggregated drift constraint (12) is calculated for each failure scenario in each optimization iteration as well, at the cost of an additional time-history analysis per constraint (more details are given in Section 5). Thus, the solution of (13) requires in every iteration the evaluation of $2 \times N_{FS}$ time-history analyses, and this may result in a very high computational cost. In this section, we discuss the working-set strategy adopted in order to reduce the number of failure scenarios (hence constraints) actually considered in the optimization analysis and, as a consequence, the overall computational cost. This strategy is inspired by the one presented by Verbart and Stolpe. In particular, we present the relaxed formulation of the optimization subproblems considered, the strategy for updating the set of constraints considered between consecutive subproblems $k$ and $k + 1$, and the stopping criterion for the optimization process.

Instead of solving the full optimization problem 13, we solve a sequence of relaxed subproblems that consider a subset of failure scenarios:

$$
\minimize \ J(x) \quad \text{subject to} \quad \begin{array}{lcl}
g_a(u, x) & \leq & 0, \quad \forall \alpha \in S_{WS}^k \\
x_i & \leq & 1, \quad \text{for} \quad i = 1, \ldots, N_d
\end{array}
$$

($P_{WS}^k$)

where $S_{WS}^k \subset S_{FS}$ is the subset of indices that identify the failure scenarios actually considered in the $k$th subproblem. Its cardinality is $|S_{WS}^k| = N_{WS}^k$, with $N_{WS}^k << N_{FS}$. Once the subproblem $P_{WS}^k$ has been solved, the failure scenarios (i.e., aggregated drift constraints) that are active and dominant in correspondence of the optimized solution and that were not considered in the solution of $P_{WS}^k$ are added to the following subproblem $P_{WS}^{k+1}$.

A working-set $S_{WS}^k$ is a set of failure scenarios’ indices considered while solving the optimization subproblem $P_{WS}^k$. Each failure scenario with index $a$ corresponds to an aggregated inter-story drift constraint $g_a(u, x)$ in the optimization problem. The first working-set considered, $S_{WS}^k$ with $k = 0$, does not include any failure scenarios. It contains only one aggregated drift constraint, as in Equation (11), associated to the case without any failure. This leads to a dampers’ design that does not account for any failure, similar to the approach of Lavan and Levy. We denote this solution as $x_0$. Then, this solution is checked with all the failure scenarios defined, and the most critical ones are included in the next working-set $S_{WS}^{k+1}$. To this end, we first build the following temporary set:

$$
T^k = \{ i \mid (g_{max}^k - g_i^k)/g_{max}^k \leq \epsilon \land g_i^k > 0 \} \quad \text{for} \quad i \in S_{FS}.
$$

(13)
where \( g^k_i = g_i(\mathbf{u}_k, \mathbf{x}_k) \), \( T^k \) is a temporary set that contains the indices of failure scenarios not included in \( S^k_{WS} \), and \( g^k_{\text{max}} \) is the maximum constraint value:

\[
g^k_{\text{max}} = \max_i \left( g^k_i \right) \quad \text{for} \quad i \in S_{FS}.
\]

The parameter \( \epsilon \) determines the number of constraints considered as critical and hence included in the working-set \( T^k_{WS} \) (e.g., \( \epsilon = 5\% \)). The new working-set \( S^{k+1}_{WS} \) is then defined as

\[
S^{k+1}_{WS} = T^k \cup S^k_{WS},
\]

and it is further updated after solving the optimization subproblem \( P^{k+1}_{WS} \) based on the solution \( \mathbf{x}_{k+1} \). The proposed strategy is built in a way that ensures that critical failure scenarios can only be added, and hence, \( S^k_{WS} \subset S^{k+1}_{WS} \). As a result, it allows for significant computational savings both in terms of response analyses and adjoint sensitivity analyses.

The sequential solution of optimization subproblems \( \{ P^k_{WS}; P^{k+1}_{WS}; \ldots \} \) is terminated once an optimized solution is found such that in correspondence of this solution, the set of constraints in \( S_{FS} \) that are violated, namely, \( Y^k \), is empty:

\[
Y^k = \{ i \in S_{FS} \mid g_i(\mathbf{u}_k, \mathbf{x}_k) > 0 \}.
\]

The pseudo code of the working-set strategy adopted herein is provided in Algorithm 1.

**Algorithm 1 Working-set strategy as a sequence of optimization subproblems**

Set: \( k = 0; \left| S^0_{WS} \right| = 1 \) (nonfailure scenario only); \( f\text{lag} = 1 \)

while \( f\text{lag} = 1 \) do

Solve \( P^k_{WS} \)
Evaluate \( Y^k \) as in Equation (16)
if \( Y^k = \emptyset \) then
\( f\text{lag} = 0 \) (i.e., stop)
end if
Define \( S^{k+1}_{WS} \) based on Equation (15)
\( k = k + 1 \)
end while

### 5 | SENSITIVITY ANALYSIS AND COMPUTATIONAL CONSIDERATIONS

As it has been already anticipated in Section 3.4, the optimization subproblems \( P^k_{WS} \) are solved with a modified SLP approach inspired by the cutting planes method.\(^61\) This is an iterative gradient-based optimization approach, where in every iteration, the problem is linearized and the resulting linear programming problem is solved locally. For more information and details regarding SLP and nonlinear constrained optimization, the interested reader is referred to the relevant literature.\(^62,63\) Hence, the gradients of the objective and constraint functions need to be calculated. The objective function \( J \) is formulated explicitly in terms of the design variables of the problem. Thus, its gradient \( \nabla J \) can be calculated directly: \( \nabla J = \mathbf{1} \), where \( \mathbf{1} \) is a vector with all entries equal to one and dimensions \( [N_d \times 1] \). The gradient of each aggregated constraint (i.e., \( \nabla g_{\text{agg}} \)), on the other hand, requires a sensitivity analysis. Because we assume that the number of design variables is larger than the number of constraints considered in each subproblem, we rely on an adjoint sensitivity analysis.\(^64\) This is ensured by the working-set strategy adopted, which has been described in Section 4. In the case of a number of design variables smaller than the number of constraints, it would have been recommended to adopt the direct differentiation method for calculating the constraints’ gradients.\(^53\) Moreover, to ensure the consistency of the sensitivity calculated, we rely on the so called discretize-then-differentiate adjoint variable method.\(^65-67\) According to this method, the discrete version of the governing equilibrium (Equation A1) is considered in the gradient calculation.
5.1 Adjoint sensitivity analysis

The goal is to calculate the gradient of the constraint function defined in Equation (12). If we consider the discrete structural response in time $u_i$ at time step $t_i$ obtained from the solution of Equation (A1), we have

$$
\nabla g = \sum_{i=1}^{N} \frac{du_i}{dx} \frac{dg}{du_i},
$$

(17)

where for simplicity, we have dropped the subscript $\alpha$. In order to calculate the gradient $\nabla g$ without having to calculate the implicit derivatives $du_i/dx$, the adjoint sensitivity analysis described in this section is used. First, we define an augmented function $\hat{g}$, which is obtained by adding zero terms to the definition of $g$. These terms are the residuals of the discrete dynamic equilibrium equations defined in Equation (A1):

$$
\hat{g}(u, x) = g(u, x) + \sum_{i=1}^{N} \lambda_{u,i}^T R_{u,i} + \sum_{i=1}^{N} \lambda_{v,i}^T R_{v,i} + \sum_{i=1}^{N} \lambda_{a,i}^T R_{a,i},
$$

(18)

where $N$ is the number of time steps; $\lambda_{u,i}$, $\lambda_{v,i}$, and $\lambda_{a,i}$ are vectors that collect the adjoint variables; and

$$
R_{u,i} = M\dot{u}_i + [C_s + C_d(x)] \ddot{u}_i + K u_i + M a_i,
$$

$$
R_{v,i} = -\dot{\ddot{u}}_i + \frac{\gamma}{\beta \Delta t} (u_i - u_{i-1}) + \left(1 + \frac{\gamma}{\beta} \right) \dot{u}_{i-1} + \Delta t \left(1 - \frac{\gamma}{2\beta} \right) \ddot{u}_{i-1}
$$

(19)

$$
R_{a,i} = -\ddot{u}_i + \frac{1}{\beta \Delta t^2} (u_i - u_{i-1}) - \frac{1}{\beta \Delta t} \dot{u}_{i-1} - \left(\frac{1}{2\beta} - 1\right) \dddot{u}_{i-1}.
$$

When the equilibrium is satisfied in every time step, we have that $\hat{g}(u, x) = g(u, x)$, and hence, $\nabla \hat{g} = \nabla g$. The gradient of the augmented function is then calculated as follows:

$$
\nabla \hat{g} = \sum_{i=1}^{N} \left( \frac{du_i}{dx} \frac{dg}{du_i} + \frac{du_i}{dx} \frac{dg}{du_i} + \frac{du_i}{dx} \frac{dg}{du_i} + \frac{du_i}{dx} \frac{dg}{du_i} \right) \lambda_{u,i}
$$

$$
+ \sum_{i=1}^{N} \left( \frac{du_i}{dx} \frac{dR_{u,i}}{du_i} + \frac{du_i}{dx} \frac{dR_{u,i}}{du_i} + \frac{du_i}{dx} \frac{dR_{u,i}}{du_i} + \frac{dR_{a,i}}{du_i} \right) \lambda_{v,i}
$$

$$
+ \sum_{i=1}^{N} \left( \frac{du_i}{dx} \frac{dR_{v,i}}{du_i} + \frac{du_i}{dx} \frac{dR_{v,i}}{du_i} + \frac{du_i}{dx} \frac{dR_{v,i}}{du_i} + \frac{dR_{a,i}}{du_i} \right) \lambda_{a,i}
$$

(20)

To avoid the calculation of the implicit derivatives of the state variables with respect to the design variables (i.e., $\frac{da_i}{dx}$, $\frac{d\dot{u}_i}{dx}$, and $\frac{d\ddot{u}_i}{dx}$), once (20) is differentiated, we collect all the terms multiplying these derivatives and we equate them to zero. Therefore, in each time step $i = 1, \ldots, N-1$, we have

$$
M^T \lambda_{u,i} + \lambda_{a,i} - \Delta t \left(1 - \frac{\gamma}{2\beta} \right) \lambda_{v,i+1} + \left(\frac{1}{2\beta} - 1\right) \lambda_{a,i+1} = 0
$$

$$
C^T \lambda_{u,i} + \lambda_{v,i} - \left(1 + \frac{\gamma}{\beta} \right) \lambda_{v,i+1} + \frac{1}{\beta \Delta t} \lambda_{a,i+1} = 0
$$

(21)

$$
K^T \lambda_{u,i} - \frac{\gamma}{\beta \Delta t} (\lambda_{v,i} - \lambda_{v,i+1}) - \frac{\gamma}{\beta \Delta t^2} (\lambda_{a,i} - \lambda_{a,i+1}) + \frac{d\hat{g}}{du_i} = 0.
$$
where \( C = C_s + C_d(\mathbf{x}) \), and we also included the fact that \( \frac{dg}{du} = 0 \) and \( \frac{dg}{da} = 0 \). In matrix form, the system of Equation (21) can be written as \( \mathbf{A} \mathbf{\xi}_i = \mathbf{b}_i \), where

\[
\begin{bmatrix}
\mathbf{M}^T & 0 & 1 \\
\mathbf{C}^T & 1 & 0 \\
\mathbf{K}^T & -\frac{\gamma}{\beta \Delta t} \mathbf{I} & -\frac{\gamma}{\beta \Delta t^2} \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
\lambda_{u,i} \\
\lambda_{v,i} \\
\lambda_{a,i}
\end{bmatrix}
= 
\begin{bmatrix}
\Delta t \left( 1 - 2\beta \right) \lambda_{v,i+1} - \frac{1}{\beta} \lambda_{a,i+1} \\
\left( 1 - \beta \right) \lambda_{v,i+1} - \frac{1}{\beta \Delta t} \lambda_{a,i+1} \\
-\frac{\gamma}{\beta \Delta t^2} \lambda_{v,i+1} - \frac{\gamma}{\beta \Delta t} \lambda_{a,i+1} - \frac{dg}{du}
\end{bmatrix},
\tag{22}
\]

where \( \mathbf{I} \) is the identity matrix with dimensions \([N_{dof} \times N_{dof}]\); \( N_{dof} \) is the number of structural degrees of freedom; and \( \mathbf{A} \) has dimensions \([3N_{dof} \times 3N_{dof}]\), \( \mathbf{\xi}_i \) and \( \mathbf{b}_i \) \([3N_{dof} \times 1]\). Numerically, Equation (10) is calculated in discrete form as follows:

\[
\dot{\mathbf{d}}_c(\mathbf{u}, \mathbf{x}) = \left( \frac{1}{t_f - t_0} \sum_{i=1}^{N} w_i (D^{-1}(\mathbf{d}_{allow})D(\mathbf{H}_i(\mathbf{x})))^p \right)^{1/p} \cdot \mathbf{1},
\tag{23}
\]

where \( w_i \) is a weight used for numerical integration (e.g., \( w_i = \Delta t \) for \( i \neq \{0, N\} \) and \( w_i = \Delta t / 2 \) for \( i = \{0, N\} \)). Thus, in Equation (22), \( \frac{dg}{du} \) is explicitly calculated as follows:

\[
\frac{dg}{du} = -\mathbf{H}^T D^{-1}(\mathbf{d}_{allow}) \left( \frac{1}{t_f - t_0} \sum_{i=1}^{N} w_i (D^{-1}(\mathbf{d}_{allow})D(\mathbf{H}_i(\mathbf{x})))^p \right)^{\frac{i-1}{p}} \left( \frac{1}{t_f - t_0} w_i (D^{-1}(\mathbf{d}_{allow})D(\mathbf{H}_i(\mathbf{x})))^p \right)^{p-1}
\]

\[
\frac{1}{(den)^2} \left( den \ (q + 1) D(\tilde{\mathbf{d}}_c)^{q+1} \mathbf{1} - num \ q \ D(\tilde{\mathbf{d}}_c)^{q+1} \mathbf{1} \right),
\tag{24}
\]

where

\[
num = \mathbf{1}^T D(\tilde{\mathbf{d}}_c)^{q+1} \mathbf{1}; \ den = \mathbf{1}^T D(\tilde{\mathbf{d}}_c)^q \mathbf{1}.
\tag{25}
\]

Essentially, the adjoint sensitivity analysis consists in solving for each time step \( i \rightarrow i - 1 \) the linear system of equations (Equation 22). It results in a linear time-history analysis solved backwards in time and with known final conditions:

\[
\mathbf{A} \mathbf{\xi}_N = \mathbf{b}_N \quad \text{with} \quad \mathbf{b}_N = \left[ \begin{array}{c} \mathbf{0} \ -\frac{dg}{du} \end{array} \right]^T.
\tag{26}
\]

It should be noted that the system of equations of the adjoint sensitivity analysis (22) has dimensions \([3N_{dof} \times 3N_{dof}]\). The dimensions of the system of equations for the evaluation of the structural response (A1) are \([N_{dof} \times N_{dof}]\), once the Newmark’s method is implemented. However, also the dimensions of the system of equations (22) can be reduced to \([N_{dof} \times N_{dof}] \) if, for example, \( \lambda_{u,i} \) and \( \lambda_{a,i} \) are expressed in terms of \( \lambda_{a,i} \).

Once the adjoint variables \( \mathbf{\xi}_i \) are known in each time step \( i \), the constraint gradient is calculated as follows:

\[
\nabla g = \sum_{i=1}^{N} \mathbf{u}_i^T \frac{dC_{\mathbf{u}_i}}{d\mathbf{x}} \lambda_{a,i},
\tag{27}
\]

The pseudo code for the adjoint sensitivity analysis is provided in Algorithm 2.
## 5.2 Computational considerations

In order to successfully adopt existing algorithms for the nonlinear and nonconvex optimization problems at hand, that is (14), some practical and conservative measures need to be taken in the optimization algorithm implementation. These include the selection of a dominant ground motion from the ensemble of records considered; the management of the linearized drift constraints; a continuation scheme for the control of certain parameters; and convergence criteria.

### 5.2.1 Ground motion selection

In general, the full set of ground motions in the ensemble should be considered when solving (14). However, this may further increase the computational cost. Thus, this work follows the procedures suggested by Lavan and Levy \[19\] where the dominant “active” ground motion is selected (i.e., the one with largest spectral displacements in correspondence to the structure’s natural period). Once the optimization with the single ground motion is terminated, the optimized design is tested with all the acceleration records from the ensemble. If the inter-story drift constraint is violated with any of the records not considered for optimization, a new optimization analysis is performed considering the initial record plus all the records for which a violation of the constraints was encountered. The process terminates when a design is found for which the drift constraint is not violated with all the ground motions from the ensemble considered.

### 5.2.2 Managing the linearized constraints

As has been already mentioned, we apply a modified SLP approach inspired by the cutting planes method to solve (14). In every iteration of standard SLP, a linear subproblem is solved. In the algorithm used herein, the subproblems grow in dimension, because in each iteration, a new linearized approximation of the aggregated constraint (one for each \( \alpha \in S_{WS}^k \)) is added to the set of constraints considered. Because the problem at hand is nonconvex, it may happen that a constraint is active even though the current solution strictly falls into the feasible domain. In other words, it may happen that a constraint cuts the feasible domain directing the algorithm towards a very conservative solution. This is clearly shown in Figure 2 of Levy and Lavan \[20\]. In the SLP algorithm used in this work, these undesired constraints are disregarded and their effect is nullified in the following optimization iterations.

### 5.2.3 Continuation scheme for parameter control

The optimization problem (14) includes several highly nonlinear components, namely, the differential equivalents of the max functions in the aggregated constraint. Therefore, difficulties to converge smoothly towards a good optimized solution are expected. A common approach for promoting a smooth convergence of the optimization process is to gradually increase the parameters that control the degree of nonlinearity. This applies to the parameters \( p \) and \( q \) in Equations (10) and (11). Furthermore, a conservative move limit strategy is applied in the solution of the subproblems, meaning that in each optimization iteration \( i \), the updates of \( x \) are searched in a close neighborhood of the solution corresponding to the previous iteration \( i - 1 \): \( x_{i-1} - ml \leq x_i \leq x_{i-1} + ml \). Specific details regarding the values of these parameters are given in the numerical examples of Section 6.
Convergence criteria

The methodology is assumed to have reached the final solution in a $k$th optimization subproblem (14) after a minimum of $i_{\text{min}}$ iterations, and once we have that, $\Delta x < \delta$, with $\Delta x = \|x_i - x_{i-1}\|$ and $\delta = 0.10 \, ml \sqrt{N_d}$. The value of $\delta$ and $i_{\text{min}}$ considered is given in Section 6. The overall optimization approach halts once the final solution $x^{*,k}$ of the current subproblem considered (14) satisfies the drift constraints in all the failure scenarios identified for the problem. The solution $x^{*,k}$ is hence considered the final solution of (13).

6 | NUMERICAL EXAMPLES

In the following section, several numerical results are presented and discussed. They are obtained by optimizing two realistic structures. As already mentioned in Section 5, the optimization problem formulation (14) is solved with a modified SLP approach inspired by the cutting planes method, which has been implemented in Python 2.7 by the author. All the numerical analyses were performed on a Linux machine with 8 Gb of RAM and a dual core Intel i7 CPU at 2.00 GHz.

We consider two examples of asymmetric frames made of reinforced concrete, as introduced in Tso and Yao\(^6\) in 1994. These two test cases were also considered by Lavan and Levy\(^2\) in 2006 where an optimal continuous damping was found and in Lavan and Amir\(^3\) in 2014 but yielding a discrete damping distribution. The same examples were also considered by Pollini et al.\(^3\) in 2016, where a realistic retrofitting cost function was minimized. In both examples, the column sizes are $0.5 \times 0.5$ m in frames 1 and 2 and $0.7 \times 0.7$ m in frames 3 and 4 (see Figure 1). The beam sizes are $0.4 \times 0.6$ m and the floor mass is uniformly distributed with a weight of $0.75$ ton/m$^2$. The selection of a specific ensemble of acceleration records depends on the requirements of each specific retrofitting problem. Here, to demonstrate the applicability of the approach proposed, we considered the ensemble of ground acceleration records LA 10% in 50 years.\(^6\) Out of the ensemble considered, LA16 has the largest maximal displacement for reasonable values of the periods of the structures in both examples. Hence, LA16 was the ground motion considered first in both examples, acting in the $y$ direction.\(^2\) In the present work, 5% of critical damping for the first two modes is considered in order to build the Rayleigh damping matrix of the structures.

In each example, 16 dampers can potentially be sized and placed in the structures considered. Three groups of failure scenarios are considered at the same time.

1) In the first, no failure is considered. The dampers are optimized without considering any damage scenario.
2) In the second, complete failure of one damper at a time is considered. This is equivalent to groups of 16 dampers ($n_c = 16$) into groups of 1 damper ($k_c = 1$) per group. Thus, the number of distinct complete failure scenarios is

$$\frac{n_c!}{k_c!(n_c - k_c)!} \quad \text{with} \quad n_c = 16, \quad k_c = 1.$$  

Hence,

$$\frac{16!}{1!(15)!} = 16.$$  

\[(28)\]
Numerically, the complete damage of an \(i\)th damper in the failure scenario \(\alpha\) is enforced by multiplying the corresponding damping coefficient by 0: \(c_{d,i}^\alpha = 0 \times c_{d,i}\) with \(\alpha = 1, 2, \ldots, 16;\)  
3) In the third, partial failure of two dampers at a time is considered. This is equivalent to groups of 16 dampers \((n_p = 16)\) into groups of 2 dampers \((k_p = 2)\) per group. Thus, the number of distinct partial failure scenarios is

\[
\frac{n_p!}{k_p!(n_p-k_p)!} \quad \text{with} \quad n_p = 16, \ k_p = 2.
\]

Hence,
\[
\frac{16!}{2!(14)!} = 120.
\]

Numerically, the partial damage of an \(i\)th damper in the failure scenario \(\alpha\) is enforced by multiplying the corresponding damping coefficient by 0.5 (50\% of damage): \(c_{d,i}^\alpha = 0.5 \times c_{d,i}\) with \(\alpha = 1, 2, \ldots, 120.\)

Therefore, the total number of failure scenarios considered in the following numerical examples is \(N_{\text{FS}} = 1 + 16 + 120 = 137.\) In principle, many more failure scenarios can be identified, by varying the number of dampers simultaneously damaged and the level of damage. However, as it will be shown in the numerical examples, only few of the scenarios previously identified will be actually governing the design.

In regard to the parameters that define the approach discussed herein, the following settings were selected for the numerical examples: the maximum damping coefficient available from Equation (6) is \(c_d = 150000 \text{kN/m};\) the maximum allowed value of interstory drift in Equation (9) is set here to \(d_{\text{allow}} = 3.5 \text{ cm},\) which corresponds to 1\% of story height; the parameters \(p\) and \(q\) introduced in Equations (10) and (11) are set to 100 and increased by steps of 500 up to \(10^6;\) the parameter \(\epsilon\) used in Equation (13) is set to 0.05; the moving limit considered is \(m_{l} = 0.02;\) the value of \(\delta\) considered for the convergence criteria is 0.008; and \(l_{\text{min}} = 50\) iterations.

6.1 | Example 1: Eight-story three bay by three bay asymmetric structure

A 3-D view of the first frame considered is displayed in Figure 1A. The same structure was considered by Lavan and Levy\(^{22}\) in a similar problem, for the optimization of viscous dampers but without considering any failure scenario. They showed that, for a seismic excitation acting in the \(y\) direction, the optimization algorithm placed dampers only at the peripheries in the \(y\) direction, even though locations at internal frames and in the \(x\) direction were also allowed. The results of Lavan and Levy confirm the intuition: dampers in the peripheral frames in the \(y\) direction are more effective in controlling the interstory drifts originating from the translation of the structure in the \(y\) direction and from the rotation of the floors. Thus, here, 16 locations where dampers can be placed are selected at the exterior frames in the \(y\) direction. If needed, more locations in the structure could be allowed for the placement of dampers. To each additional location would correspond an additional design variable (i.e., damping coefficient). As a result of the working-set strategy adopted during the optimization process, four optimization analyses were performed, consisting of subproblems with 1, 2, 3, and 4 failure scenarios. The optimization analyses run for 82, 105, 91, and 75 iterations for a total computational time of 15 min and 8 s over 353 iterations. The final optimized solution is shown in Table 1. For comparison, the results obtained without considering any failure scenarios are also included in the table. They are referred to as “basic design” in contrast to the “fail-safe design.” It can be observed that the two solutions are significantly different in terms of dampers’ number and size. The basic design has 26\% of the total added damping of the fail-safe design. Moreover, the fail-safe design relies on more dampers and of larger size. In particular, the fail-safe design has 13 dampers and the final value of the objective function is \(J = 615.875 \text{kN/m}\). The basic design has nine dampers and the final value of the objective function is \(J = 161.925 \text{kN/m}\).

The structure has been tested with both the fail-safe and the basic dampers’ designs for all 137 failure scenarios, to compare the performances of the two solutions. Figure 2 shows a plot of the maximum value of drift constraint (Equation (12)) for all failure cases. It is possible to observe that with the fail-safe design in correspondence of the optimized solution, few failure scenarios are actually governing the design because their associated normalized peak drift is equal to, or close to, one. With the basic design, instead, a significant constraint violation is observed for several failure scenarios. This highlights the robustness of the layout of dampers obtained with the fail-safe approach discussed in this paper. Figure 3 shows the time history of all the interstory drifts of the structure retrofitted with the fail-safe dampers’ design for all failure scenarios defined. None of the interstory drifts exceeds the maximum allowed value.

The optimized fail-safe design was evaluated with the other 19 ground motions in the ensemble, and no other constraint violations were encountered for all the failure scenarios.
TABLE 1  Optimized damping values for the asymmetric eight-story frame of Section 6.1

| Location | Basic design (kNs/m) | Fail-safe design (working-set, kNs/m) | Fail-safe design (full set, kNs/m) |
|----------|----------------------|-------------------------------------|----------------------------------|
| 1        | 1682                 | 126469                              | 126470                           |
| 2        | 32 585               | 61 554                              | 61 389                           |
| 3        | 23 454               | 96 247                              | 96 367                           |
| 4        | 19 285               | 51 637                              | 51 839                           |
| 5        | 13 054               | 28 122                              | 28 432                           |
| 6        | 0                    | 24 871                              | 24 706                           |
| 7        | 0                    | 20 444                              | 20 001                           |
| 8        | 0                    | 0                                   | 0                                |
| 9        | 24 584               | 25 045                              | 12 908                           |
| 10       | 29 129               | 32 975                              | 32 828                           |
| 11       | 17 490               | 46 308                              | 46 509                           |
| 12       | 662                  | 13 264                              | 13 335                           |
| 13       | 0                    | 21 144                              | 20 979                           |
| 14       | 0                    | 0                                   | 0                                |
| 15–16    | 0                    | 0                                   | 0                                |
| Final cost J | 161 925           | 615 875                            | 615 864                           |

Note. The results obtained with the working-set strategy, with the full set of failure scenarios, and without considering any damage scenarios (i.e., the basic design) are shown for comparison. All results have been obtained considering the record LA16.

FIGURE 2  Maximum value of the drift constraints $g_\alpha$ of the retrofitted structure of Example 1 (Section 6.1) for all failure scenarios. Results for the fail-safe design (blue) and basic design (red). The record considered for optimization is LA16. The dashed line marks the maximum value allowed of normalized interstory drift, that is, 1.0. The basic design significantly violates the interstory drift constraint in several failure scenarios.

FIGURE 3  Values in time of the interstory drifts $d(t) = Hu(t)$ of the structure equipped with the fail-safe optimized dampers’ layout for all fail-safe scenarios, in Example 1 (Section 6.1). The record considered for optimization is LA16. The dashed lines mark the maximum allowed interstory drift $d_{\text{allow}} = 35$ mm.

To verify the extent of the computational saving achieved using the working-set strategy, an additional optimization analysis was performed considering at once all the failure scenarios. The purpose of this test was to compare the computational cost required for the solution of the full problem (13) to the one required for the solution of the subproblems (14), as explained in Section 4. The optimization converged after 71 iterations, taking 11 h, 36 min, and 19 s.

The final results are presented in Table 1. In Table 2, we compare the computational time and effort of the working-set approach with the one required for the full problem. The working-set approach requires the solution of multiple subproblems, which are essentially a relaxation of the original full problem. However, every subproblem considers a significantly smaller number of failure scenarios. This results in a smaller number of constraints to be considered in every subproblem and hence less time history and adjoint sensitivity analyses. As a result, the computational time is reduced by 97.8%, and only approximately a tenth of the function evaluations (i.e., time history and sensitivity analyses) is required.
| Approach    | Iterations | Number of failure scenarios | Time (min) | Number of function evaluations |
|------------|------------|-----------------------------|------------|-------------------------------|
| Working set| \{82, 105, 91, 75\} | \{1, 2, 3, 4\} | 15.14 | 1730 |
| Full set   | \{71\}     | \{137\}         | 696.31     | 19454 |

Note. The number of function evaluations counts the number of time history and adjoint sensitivity analyses performed. The working-set approach requires a number of function evaluations 10 times smaller.

### TABLE 3 Optimization results of the asymmetric eight-story setback frame of Section 6.2

| Location | Basic design (LA16, kNs/m) | Fail-safe design (LA16, kNs/m) | Fail-safe design (LA14, LA16, LA18; kNs/m) |
|----------|-----------------------------|-------------------------------|---------------------------------------------|
| 1        | 0                           | 18304                         | 19890                                      |
| 2        | 14996                       | 8963                          | 9370                                      |
| 3        | 0                           | 10965                         | 13726                                      |
| 4        | 0                           | 0                             | 126                                       |
| 5        | 0                           | 7613                          | 5237                                      |
| 6        | 0                           | 5661                          | 6236                                      |
| 7–8      | 0                           | 0                             | 0                                          |
| 9        | 0                           | 15508                         | 13572                                     |
| 10       | 22198                       | 20947                         | 21726                                     |
| 11       | 26552                       | 34011                         | 36235                                     |
| 12       | 2506                        | 25812                         | 26569                                     |
| 13       | 0                           | 4129                          | 4741                                      |
| 14       | 0                           | 1225                          | 0                                         |
| 15–16    | 0                           | 0                             | 0                                          |
| Final cost J | 66 252 | 153 138           | 156 428                                   |

Note. The results obtained with the working-set strategy considering only the record LA16, and the records LA14, LA16, and LA18 at the same time are both listed. The fail-safe results and the results obtained without considering any damage scenarios (i.e., the basic design) are shown for comparison.

### 6.2 Example 2: Eight-story three bay by three bay setback frame structure

A 3-D view of the second frame considered is shown in Figure 1B. Also, this structure was considered by Lavan and Levy, and they observed also in this case the tendency of the optimization algorithm to place dampers at the peripheries in the y direction. As a result of the working-set strategy adopted, five optimization analyses were performed. Each consisted of a subproblem with 1, 2, 3, 4, and 9 failure scenarios. The optimization analyses run for 68, 60, 72, 74, and 67 iterations for a total computational time of 21 min and 35 s over 341 iterations. The final optimized solution is shown in Table 3. Also in this case, the dampers optimized without considering any failure scenarios are included in the table for comparison (i.e., basic design). They are referred to as “basic design” in contrast to the “fail-safe design.” It can be observed that also in this case, the two solutions are significantly different in terms of dampers’ number and size. The basic design has 43% of the total added damping of the fail-safe design. In fact, the fail-safe design relies on more dampers and of larger size. In particular, the fail-safe design has 11 dampers and the final value of the objective function is \( J = 153 138 \text{kNs/m} \). The basic design has four dampers and the final value of the objective function is \( J = 66 252 \text{kNs/m} \).

Also in this case, the structure has been tested with both the fail-safe and the basic dampers’ designs for all 137 failure scenarios, to compare the performances of the two solutions. Looking at Figures 4 and 5, the same conclusions as in the previous example with respect to Figures 2 and 3 can be drawn.

The optimized fail-safe design was tested with the other 19 ground motions in the ensemble. In two cases, a significant constraint violation was observed: 7% with LA14 and 4% with LA18. Thus, another optimization analysis was performed considering three records at the same time, namely, LA14, LA16, and LA18. The results are shown in Table 3. The optimization analysis run, in the past through three subproblems for 60, 69, and 74 iterations, for a total of 203 iterations and a computational time of 1 h, 47 min, and 51 s. The computational time increased because three acceleration records were considered in the optimization analysis. Hence, in each optimization iteration, three time history analyses and three sensitivity analyses were performed for each failure scenario considered. Moreover, each subproblem consisted of 3, 4, and 5 failure scenarios. The obtained fail-safe design was then tested with all the records from the ensemble considered, and no constraint violation was encountered for all failure scenarios. This is shown in Figure 6.
The numerical results from Sections 6.1 and 6.2 show that significantly different optimized solutions are obtained if failure scenarios are considered in the design phase. This is clearly shown in Tables 1 and 3. The solutions significantly differ from the solutions obtained without considering any damage in the dampers in terms of number of dampers allocated in the structure and dampers’ size. Moreover, the dampers’ designs obtained with the fail-safe approach are more robust compared with the designs optimized without considering any damage in the dampers. This can be observed in Figures 2 and 4. In the first example, we also showed that by using a working-set strategy, it is possible to significantly reduce to 10 times the computational cost required for the optimization process. This is a crucial aspect, because it allows the approach discussed herein to be implemented on standard desktop computers and hence to assist in their activity the engineers and practitioners that have access only to modest computational resources. In the second example, the fail-safe design obtained considering a single dominant acceleration record (i.e., LA16) did not fulfill the interstory drift requirements with two of the records from the LA 10% in 50 years ensemble. Thus, an additional optimization analysis was performed considering simultaneously three records: LA14, LA16, and LA18. In this way, it was shown that the methodology handles realistic ensembles of ground motions, and it can thus be used in practical performance-based design applications of 3-D irregular structures. Moreover, thanks to the working-set strategy adopted, the computational effort proved reasonable also when three acceleration records were considered at the same time. The computational cost would have been most likely prohibitive without the working-set strategy.

7 | FINAL CONSIDERATIONS

This paper presents a new optimization approach for designing minimum-cost fail-safe distributions of fluid viscous dampers for seismic retrofitting. Failure is modeled as either complete damage of the dampers or degradation of the
dampers’ properties. The dampers’ cost function is minimized, with constraints on the interstory drifts at the peripheries of irregular 3-D structures. These are computed with time history analyses considering an ensemble of realistic ground motions. Therefore, the proposed methodology can be used for the fail-safe performance-based seismic retrofitting of 3-D irregular structures.

The novelty of the proposed approach lies in its optimization problem formulation, which leads to optimized added damping systems that are more robust compared with those obtained with the approaches currently available in the literature. The computational cost is significantly reduced by means of a working-set strategy: only few dominant failure scenarios are actually considered during the optimization, and the final designs fulfill all the performance constraints associated to all the failure scenarios initially identified.

The numerical results show how the proposed methodology successfully handled realistic design cases. A large number of failure scenarios was considered successfully, solving a sequence of relaxed subproblems instead of the original full problem. Each subproblem considered only a working-set of constraints associated to the most critical failure scenarios. The methodology discussed herein ensures that the working-set is updated after every subproblem is solved and that the failure scenarios previously considered are contained in the new working-set. In other words, the working-sets expand until the overall optimization process terminates. The numerical results also show that the optimized dampers’ layout and size can be significantly different when failure scenarios are considered in the design phase: the algorithm tends to place more dampers and of bigger size. This is in good agreement with the engineering intuition according to which if failure scenarios are considered in the design, the level of redundancy of the designed system increases. Moreover, the working-set strategy significantly reduces the computational effort required for optimization. This means that the methodology can be used by engineers in practice relying on standard computational resources. Because a linear structural behavior is considered during the optimization, the obtained layout of dampers must be checked with a high fidelity structural model during a post-processing phase in order to verify the compliance of the retrofitted structure with design standards and codes and, if needed, to fine tune the dampers’ properties. The methodology discussed in this paper is expected to promote the fail-safe optimization-based design of dampers even for large-scale structures, where the number of design variables may become very large and other optimization approaches (e.g., genetic algorithms) would require prohibitive computational efforts and resources.

ACKNOWLEDGEMENTS
The author would like to thank the editor and two anonymous reviewers for their knowledgeable comments and suggestions that helped to improve the manuscript. The author wishes to thank also Oren Lavan from Technion—Israel Institute of Technology, and Mathias Stolpe from Technical University of Denmark for their valuable feedback.

ORCID
Nicolò Pollini https://orcid.org/0000-0001-8504-6005

REFERENCES
1. Berquist M, De Pasquale R, Frye S, et al. Fluid Viscous Dampers—General Guidelines for Engineers Including a Brief History: Taylor Devices Inc; 2019.
2. Constantinou MC, Symans MD. Experimental and Analytical Investigation of Seismic Response of Structures with Supplemental Fluid Viscous Dampers: National Center for Earthquake Engineering Research Buffalo, NY; 1992.
3. McNamara RJ, Taylor DP. Fluid viscous dampers for high-rise buildings. Struct Des Tall Build. 2003;12(2):145-154.
4. ASCE/Structural Engineering Institute. Minimum Design Loads for Buildings and Other Structures: ASCE/SEI 7-16; 2016.
5. ASCE/Structural Engineering Institute. Seismic Evaluation and Retrofit of Existing Buildings: ASCE/SEI 41-17; 2017.
6. Ramirez OM, Constantinou MC, Kircher CA, et al. Development and Evaluation of Simplified Procedures for Analysis and Design of Buildings with Passive Energy Dissipation Systems, Rep. No: MCEER-00-0010: Multidisciplinary Center for Earthquake Engineering Research, University at Buffalo, State University of New York, Buffalo, NY; 2000.
7. Ramirez OM, Constantinou MC, Whittaker AS, Kircher CA, Chrysostomou CZ. Elastic and inelastic seismic response of buildings with damping systems. Earthquake Spectra. 2002;18(3):531-547.
8. Ramirez OM, Constantinou MC, Gomez JD, Whittaker AS, Chrysostomou CZ. Evaluation of simplified methods of analysis of yielding structures with damping systems. Earthquake Spectra. 2002;18(3):501-530.
9. Whittaker AS, Constantinou MC, Ramirez OM, Johnson MW, Chrysostomou CZ. Equivalent lateral force and modal analysis procedures of the 2000 NEHRP provisions for buildings with damping systems. Earthquake Spectra. 2003;19(4):959-980.
10. Ramirez OM, Constantinou MC, Whittaker AS, Kircher CA, Johnson MW, Chrysostomou CZ. Validation of the 2000 NEHRP provisions' equivalent lateral force and modal analysis procedures for buildings with damping systems. *Earthquake Spectra.* 2003;19(4):981-999.

11. Constantinou MC, Symans MD. Experimental study of seismic response of buildings with supplemental fluid dampers. *Struct Des Tall Build.* 1992;2(2):93-132.

12. Soong TT, Dargush GF. *Passive energy dissipation systems in structural engineering.* New York: John Wiley & Sons. 1997:282.

13. Takewaki I. *Building Control with Passive Dampers: Optimal Performance-Based Design for Earthquakes.* John Wiley & Sons; 2011.

14. Filippou F, Christopoulos C. *Principles of Passive Supplemental Damping and Seismic Isolation.* IUS Press, Pavia, Italy; 2006.

15. De Domenico D, Ricciardi G, Takewaki I. Design strategies of viscous dampers for seismic protection of building structures: a review. *Soil Dyn Earthq Eng.* 2019;118:144-165.

16. Gluck N, Reinhorn AM, Gluck J, Levy R. Design of supplemental dampers for control of structures. *J Struct Eng.* 1996;122(12):1394-1399.

17. Wu B, Ou J-P, Soong TT. Optimal placement of energy dissipation devices for three-dimensional structures. *Eng Struct.* 1997;19(2):113-125.

18. Takewaki I. Optimal damper placement for minimum transfer functions. *Earthq Eng Struct Dyn.* 1997;26(11):1113-1124.

19. Lavan O, Levy R. Optimal design of supplemental viscous dampers for irregular shear-frames in the presence of yielding. *Earthq Eng Struct Dyn.* 2005;34(8):889-907.

20. Lavan O, Levy R. Optimal design of supplemental viscous dampers for linear framed structures. *Earthq Eng Struct Dyn.* 2006;35(3):337-356.

21. Lavan O, Levy R. Optimal peripheral drift control of 3d irregular framed structures using supplemental viscous dampers. *J Earthq Eng.* 2006;10(06):903-923.

22. Lavan O, Levy R. Optimal peripheral drift control of 3d irregular framed structures using supplemental viscous dampers. *J Earthq Eng.* 2006;10(06):903-923.

23. Levy R, Lavan O. Fully stressed design of passive controllers in framed structures for seismic loadings. *Struct Multidiscip Optim.* 2006;32(2):485-498.

24. Silvestri S, Trombetti T. Physical and numerical approaches for the optimal insertion of seismic viscous dampers in shear-type structures. *J Earthq Eng.* 2007;11(5):787-828.

25. Lavan O, Cimellaro GP, Reinhorn AM. Noniterative optimization procedure for seismic weakening and damping of inelastic structures. *J Struct Eng.* 2008;134(10):1638-1648.

26. Cimellaro GP, Soong TT, Reinhorn AM. Integrated design of controlled linear structural systems. *J Struct Eng.* 2009;135(7):853-862.

27. Almazán JL, de la Llera JC. Torsional balance as new design criterion for asymmetric structures with energy dissipation devices. *Earthq Eng Struct Dyn.* 2009;38(12):1421-1440.

28. Lavan O. A methodology for the integrated seismic design of nonlinear buildings with supplemental damping. *Struct Control Health Monit.* 2015;22(3):484-499.

29. Altiere D, Tubaldi E, De Angelis M, Patelli E, DallÀÂZZAsta A. Reliability-based optimal design of nonlinear viscous dampers for the seismic protection of structural systems. *Bull Earthq Eng.* 2018;16(2):963-982.

30. Del Gobbo GM, Williams MS, Blakeborough A. Comparing fluid viscous damper placement methods considering total-building seismic performance. *Earthq Eng Struct Dyn.* 2018;47(14):2864-2886.

31. De Domenico D, Ricciardi G. Earthquake protection of structures with nonlinear viscous dampers optimized through an energy-based stochastic approach. *Eng Struct.* 2019;179:523-539.

32. Zhang R-H, Soong TT. Seismic design of viscoelastic dampers for structural applications. *J Struct Eng.* 1992;118(5):1375-1392.

33. Agrawal AK, Yang JN. Optimal placement of passive dampers on seismic and wind-excited buildings using combinatorial optimization. *J Intell Mater Syst Struct.* 1999;10(12):997-1014.

34. Dargush GF, Sant RS. Evolutionary aseismic design and retrofit of structures with passive energy dissipation. *Earthq Eng Struct Dyn.* 2005;34(13):1601-1626.

35. Lavan O, Dargush GF. Multi-objective evolutionary seismic design with passive energy dissipation systems. *J Earthq Eng.* 2009;13(6):758-790.

36. Kanno Y. Damper placement optimization in a shear building model with discrete design variables: a mixed-integer second-order cone programming approach. *Earthq Eng Struct Dyn.* 2013;42(11):1657-1676.

37. Lavan O, Amir O. Simultaneous topology and sizing optimization of viscous dampers in seismic retrofitting of 3d irregular frame structures. *Earthq Eng Struct Dyn.* 2014;43(9):1325-1342.

38. Pollini N, Lavan O, Amir O. Towards realistic minimum-cost optimization of viscous fluid dampers for seismic retrofitting. *Bull Earthq Eng.* 2016;14(3):971-998.

39. Pollini N, Lavan O, Amir O. Minimum-cost optimization of nonlinear fluid viscous dampers and their supporting members for seismic retrofitting. *Earthq Eng Struct Dyn.* 2017;46(12):1941-1961.

40. Pollini N, Lavan O, Amir O. Optimization-based minimum-cost seismic retrofitting of hysteretic frames with nonlinear fluid viscous dampers. *Earthq Eng Struct Dyn.* 2018;47(15):2985-3005.

41. Bendsoe MP, Sigmund O. *Topology Optimization: Theory, Methods, and Applications.* Springer; 2013.

42. Idelso O, Lavan O. Performance based formal optimized seismic design of steel moment resisting frames. *Comput Struct.* 2020;235:106269.

43. Miyamoto HK, Gilani ASJ, Wada A, Arikartana C. Limit states and failure mechanisms of viscous dampers and the implications for large earthquakes. *Earthq Eng Struct Dyn.* 2010;39(11):1279-1297.

44. Rozvany GIN, Maute K. Analytical and numerical solutions for a reliability-based benchmark example. *Struct Multidiscip Optim.* 2011;43(6):745-753.
45. Lazarov BS, Schevenels M, Sigmund O. Topology optimization considering material and geometric uncertainties using stochastic collocation methods. *Struct Multidiscip Optim*. 2012;46(4):597-612.

46. Zhou M, Fleury R. Fail-safe topology optimization. *Struct Multidiscip Optim*. 2016;54(5):1225-1243.

47. Lüdeker JK, Kriegesmann B. Fail-safe optimization of beam structures. *J Comput Des Eng*. 2019;6(3):260-268.

48. Sun P-F, Arora JS, Haug Jr EJ. Fail-safe optimal design of structures. *Eng Optim*. 1976;2(1):43-53.

49. Achtziger W, Bendsøe MP. Optimal topology design of discrete structures resisting degradation effects. *Struct Optim*. 1999;17(1):74-78.

50. Jansen M, Lombaert G, Schevenels M, Sigmund O. Topology optimization of fail-safe structures using a simplified local damage model. *Structural and Multidisciplinary Optimization*. 2014;49(4):657-666.

51. Kanno Y. Redundancy optimization of finite-dimensional structures: concept and derivative-free algorithm. *Journal of Structural Engineering*. 2017;143(1):04016151.

52. Stolpe M. Fail-safe truss topology optimization. *Structural and Multidisciplinary Optimization*. 2019;1-14.

53. Verbart A, Stolpe M. A working-set approach for sizing optimization of frame-structures subjected to time-dependent constraints. *Structural and Multidisciplinary Optimization*. 2018;58(4):1367-1382.

54. Miyamoto HK, Gilani ASI, Wada A, Ariyaratana C. Identifying the collapse hazard of steel special moment-frame buildings with viscous dampers using the FEMA P695 methodology. *Earthquake Spectra*. 2011;27(4):1147-1168.

55. Dimopoulos AI, Tzimas AS, Karavasilis TL, Vamvatsikos D. Probabilistic economic seismic loss estimation in steel buildings using post-tensioned moment-resisting frames and viscous dampers. *Earthquake Engineering & Structural Dynamics*. 2016;45(11):1725-1741.

56. Kitayama S, Constantinou MC. Seismic performance of buildings with viscous damping systems designed by the procedures of ASCE/SEI 7-16. *Journal of Structural Engineering*. 2018;144(6).

57. Lin WH, Chopra AK. Asymmetric one-storey elastic systems with non-linear viscous and viscoelastic dampers: earthquake response. *Earthquake Engineering & Structural Dynamics*. 2003;32(4):555-577.

58. De Domenico D, Ricciardi G. Improved stochastic linearization technique for structures with nonviscous dampers. *Soil Dynamics and Earthquake Engineering*. 2018;113:415-419.

59. Ch BMP DC, Komodromos P, Phocas MC. Optimized earthquake response of multi-storey buildings with seismic isolation at various elevations. *Earthquake Engineering & Structural Dynamics*. 2012;41(15):2289-2310.

60. Lavan O. Optimal design of viscous dampers and their supporting members for the seismic retrofitting of 3d irregular frame structures. *Journal of Structural Engineering*. 2015;141(11):04015026.

61. Kelley JE. The cutting-plane method for solving convex programs. *Journal of the Society for Industrial and Applied Mathematics*. 1960;8(4):703-712.

62. Christensen PW, Klarbring A. *An Introduction to Structural Optimization*: Springer; 2008.

63. Nocedal J, Wright S. *Numerical Optimization*: Springer; 2006.

64. Michaleris P, Tortorelli DA, Vidal CA. Tangent operators and design sensitivity formulations for transient non-linear coupled problems with applications to elastoplasticity. *International Journal for Numerical Methods in Engineering*. 1994;37(14):2471-2499.

65. Jensen JS, Nakshatrala PB, Tortorelli DA. On the consistency of adjoint sensitivity analysis for structural optimization of linear dynamic problems. *Structural and Multidisciplinary Optimization*. 2014;49(5):831-837.

66. Pollini N, Lavan O, Amir O. Adjoint sensitivity analysis and optimization of hysteretic dynamic systems with nonlinear viscous dampers. *Structural and Multidisciplinary Optimization*. 2018;57(6):2273-2289.

67. Lavan O. Adjoint sensitivity analysis and optimization of transient problems using the mixed Lagrangian formalism as a time integration scheme. *Structural and Multidisciplinary Optimization*. 2019;1-16.

68. Tso WK, Yao S. Seismic load distribution in buildings with eccentric setback. *Canadian Journal of Civil Engineering*. 1994;21(1):50-62.

69. Somerville P, Smith N, Punnamurthula S, Sun J. Development of ground motion time histories for phase 2 of the FEMA/SAC steel project. *Technical Report SAC/BD-97-04*. 1997.

70. Chopra AK. *Dynamics of Structures. Theory and Applications to Earthquake Engineering*: Prentice Hall; 2012.

**How to cite this article:** Pollini N. Fail-safe optimization of viscous dampers for seismic retrofitting. *Earthquake Engng Struct Dyn*. 2020;49:1599–1618. [https://doi.org/10.1002/eqe.3319](https://doi.org/10.1002/eqe.3319)
APPENDIX A: NEWMARK’S TIME STEPPING SCHEME

The equations of motion are discretized in time, and accelerations and velocities are expressed in terms of displacements:\(^70\)

\[
\mathbf{M}\ddot{\mathbf{u}}_{i+1} + [\mathbf{C}_s + \mathbf{C}_d(\mathbf{x})]\dot{\mathbf{u}}_{i+1} + \mathbf{K}_s\mathbf{u}_{i+1} = -a_{g,i+1}\mathbf{M}
\]

\[
\begin{align*}
\dot{\mathbf{u}}_{i+1} &= \frac{1}{\beta\Delta t^2}(\mathbf{u}_{i+1} - \mathbf{u}_i) - \frac{1}{\beta\Delta t}\ddot{\mathbf{u}}_i - \left(\frac{1}{2\beta} - 1\right)\dot{\mathbf{u}}_i \\
\dot{\mathbf{u}}_{i+1} &= \frac{\gamma}{\beta\Delta t}(\mathbf{u}_{i+1} - \mathbf{u}_i) + \left(1 - \frac{\gamma}{\beta}\right)\dot{\mathbf{u}}_i + \Delta t\left(1 - \frac{\gamma}{2\beta}\right)\dddot{\mathbf{u}}_i
\end{align*}
\]  \(\text{(A1)}\)

where \(\beta = 1/4\) and \(\gamma = 1/2\) are used in the average acceleration method and \(\beta = 1/6\) and \(\gamma = 1/2\) are used in the linear acceleration method. Here, we rely on the average acceleration method because it is stable for any choice of \(\Delta t = t_{i+1} - t_i\). It should be noted that in each time step \(i+1\), Equation (A1) is solved for \(\mathbf{u}_{i+1}\), and then, \(\dot{\mathbf{u}}_{i+1}\) and \(\dddot{\mathbf{u}}_{i+1}\) are calculated.