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To cite this version:
Mohammed Safi, Lucie Baudouin, Alexandre Seuret. Refined exponential stability analysis of a coupled system. IFAC World Congress, Jul 2017, Toulouse, France. hal-01496136

HAL Id: hal-01496136
https://laas.hal.science/hal-01496136v1
Submitted on 27 Mar 2017

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Refined exponential stability analysis
of a coupled system

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Abstract: The objective of this contribution is to improve recent stability results for a system
coupling ordinary differential equations to a vectorial transport partial differential equation by
proposing a new structure of Lyapunov functional. Following the same process of most of the
investigations in literature, that are based on an a priori selection of Lyapunov functionals and
use the usual integral inequalities (Jensen, Wirtinger, Bessel...), we will present an efficient
method to estimate the exponential decay rate of this coupled system leading to a tractable
test expressed in terms of linear matrix inequalities. These LMI conditions stem from the new
design of a candidate Lyapunov functional, but also the inherent properties of the Legendre
polynomials, that are used to build a projection of the infinite dimensional part of the state of
the system. Based on these polynomials and using the appropriate Bessel-Legendre inequality,
we will prove an exponential stability result and in the end, we will show the efficiency of our
approach on academic example.

Keywords: Transport equation, Lyapunov stability, integral inequalities.

1. INTRODUCTION

When modeling a control problem phenomenon using a
state space formulation, the trade off between capturing a
certain level of complexity and obtaining a model on which
the tools we have can be applied is inevitable. Considering
systems modeled by ordinary differential equations (ODE)
ensures huge literature and quantity of very well developed
tool control tools whereas the choice of a partial differential
equation (PDE) model brings in a very different set of
approaches and references. Our work lies in the more
narrow domain of the stability study of coupled ODE-
PDE system and should be seen as a first step in the
understanding of a possible way to get a simple common
ground to consider at the same time both of the two states
of such a heterogeneous system.

PDE systems stand out as having important applications
in the modeling and control of physical networks: hydraulic
(Coron et al. (2008)), gas pipeline networks (Gugat et al.
(2011)) and road traffic (Coclite et al. (2005)) for instance.
Systems coupling PDEs to ODEs have attracted some
more attention in the last decade as in Krstic (2009),
Hasan et al. (2016), Stinner et al. (2014) and Friedmann
(2015). Stability analysis and stabilization of such systems
have also appeared recently in e.g. Susto Gian and Krstic
(2010), Prieur et al. (2008) and Tang et al. (2015).

To control these coupled systems, the backstepping ap-
proach can be considered. It was originally developed for
parabolic and second-order hyperbolic PDEs, as well as
for several challenging physical problems such as turbulent
flows magnetohydrodynamics Vazquez and Krstic (2008).
In many recent works, it is used to stabilize ODE-PDE
systems. For example, a first order hyperbolic PDE cou-
pled to an ODE has been stabilized by this approach in
Krstic and Smyshlyaev (2008).

In this article, we are considering a vectorial transport
equation, whose state is of infinite dimension, coupled with
a classical system of ODEs. The stability of this specific
kind of coupled systems has already been investigated
recently in Castillo et al. (2015), Baudouin et al. (2016)
and Saifi et al. (2016) and one should mention that it can
also be considered as specific formulation of a Time-Delay
System (TDS). Actually, there is a very large literature
in TDS and among many others, we can refer to Xu and
Sallet (2002), Mondie et al. (2005) and Xu et al. (2006).
In our paper, following the classical Lyapunov method for
stability study (see e.g. Gu et al. (2003), Fridman (2014),
Gyurkovics and Takacs (2016) and Seuret and Gouaisbaut
(2015)), we will provide an efficient approach for assessing
stability of this first ODE-PDE system.

Most of the contributions on stability in the TDS frame-
work are based on the good selection of a Lyapunov-
Krasovskii functional (LKF) leading to sufficient stability
conditions (see Gu et al. (2003)), and a polynomial ap-
proximation to estimate the infinite dimensional state of
the system, which is not a new idea (see Papachristodoulou
and Peet (2006), Peet (2014) and Ahmadi et al. (2014)).
But in this work, we aim at showing that a better design
of the LKF may improve stability studies of such coupled
systems and gives interesting results for the convergence
rate of this system, getting closer to what an appropriate
frequency analysis can give.

Notations: N is the set of positive integers, $\mathbb{R}^n$ is the n-
dimensional Euclidean space with vector norm $| \cdot |_n$. We
denote $\mathbb{R}^{n \times m}$ the set of real matrices of dimension $n \times m$.
$I_n \in \mathbb{R}^{n \times n}$ is the identity matrix, $0_{n,m}$ the null matrix,
and $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ replaces the symmetric matrix $\begin{bmatrix} A & B \\ B & C \end{bmatrix}$. We

* The paper was partially supported by the ANR projects LimICoS
and SCIDIS. Corresponding author . Fax +33-561336411.
We consider the following system governed by the coupling of a transport PDE and a finite dimensional system of ODE:
\[
\begin{align*}
\dot{X}(t) &= AX(t) + Bz(1,t), & t > 0, \\
\dot{z}(x,t) + \rho \partial_x z(x,t) = 0, & x \in (0,1), t > 0, \quad (1) \\
z(0,t) &= CX(t) + Dz(1,t), & t > 0.
\end{align*}
\]

The state of this coupled system is composed not only of the complete state of the system, and in this one, the first quadratic term is dedicated to the vector state \(X(t)\) and the last term to the infinite dimensional state \(z(x,t)\). The two terms that remain in between are formed through the functions \(Q(x)\) and \(T(x_1,x_2)\), and have the specificity of building a link between those finite and infinite dimensional states, both constitutive of the ODE-PDE coupling system. The challenge in the precise design of this candidate Lyapunov functional (2) relies on the choice of the functions \(Q\) and \(T\) that depend on the integration parameters \(x, x_1\) and \(x_2\). In our approach, we will use polynomial functions of a given degree (a basis of Legendre polynomials) to construct a truncated decomposition of those functions as follows:
\[
\begin{align*}
Q(x) &= \sum_{k=0}^{N} Q(k) L_k(x), \\
T(x_1, x_2) &= \sum_{i,j=0}^{N} T(i,j) L_i(x_1) L_j(x_2),
\end{align*}
\]
where \(N \in \mathbb{N}\), and where \(L_k\), for \(k \in \mathbb{N}\), denote the shifted Legendre polynomials of degree \(k\) considered over the interval \([0,1]\). These polynomials and their properties will be detailed in the next section. Using the decomposition (3) of the functions \(Q\) and \(T\), the Lyapunov functional becomes:
\[
V_N(X(t), z(t)) = \left[ \begin{array}{c} X(t) \\ Z_N(t) \end{array} \right]^T \begin{bmatrix} P & Q_N \\ Q_N^T & T_N \end{bmatrix} \left[ \begin{array}{c} X(t) \\ Z_N(t) \end{array} \right] + \int_0^1 e^{-\frac{2\pi^2}{4}\int_0^1 z(t,S) \dot{z}(t,S) dx} dt,
\]
(4)
where
\[
Q_N = \left[ Q(0), \ldots, Q(N) \right] \in \mathbb{R}^{n,m(N+1)}
\]
and
\[
T_N = \left[ T(i,j) \right]_{i,j=0..N} \in \mathbb{R}^{m(N+1),m(N+1)}
\]
and
\[
Z_N(t) = \left[ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right] \in \mathbb{R}^{m(N+1)}
\]
(5)
is the projection of the \(m\) components of the infinite dimensional state \(z(x,t)\) over the \(N+1\) first Legendre polynomials.

In this article, we aim specifically at showing that taking the last integral term in (4) with an exponential weight \(e^{-\frac{2\pi^2}{4}\int_0^1 z(t,S) \dot{z}(t,S) dx} dt\), allows to derive a better estimate of the decay rate of the system compared to Baudouin et al. (2016), which uses a different method. In addition to the contribution of Baudouin et al. (2016), an additional term (i.e. \(Dz(1,t)\)) has been included in the boundary condition of system (1) so that a larger class of systems can be covered, as, for instance, systems with commensurate delays.

3. PRELIMINARIES

3.1 Legendre polynomials

The shifted Legendre polynomials (see for instance Courant and Hilbert (1953)) we will use are denoted \(\{L_k\}_{k \in \mathbb{N}}\) and
Moreover, the derivative of those polynomials is given by
\[ V^\prime(\overline{z}) \] will use the shorthand notation
\[ V, L \]
act over \([0,1]\). The family \( \{L_k\}_{k \in \mathbb{N}} \) forms an orthogonal basis of \( L^2([0,1]; \mathbb{R}) \) and we have precisely
\[
\int_0^1 L_j(x)L_k(x) \, dx = \frac{1}{2k+1} \delta_{jk},
\]
where \( \delta_{jk} \) represents Kronecker's coefficient, equal to 1 if \( j = k \) and 0 otherwise. The boundary values are given by:
\[
L_k(0) = (-1)^k, \quad L_k(1) = 1. \tag{6}
\]
Moreover, the derivative of those polynomials is given by
\[
\frac{d}{dx} L_k(x) = \begin{cases} 0, & k = 0, \\ \sum_{j=0}^{k-1} \ell_{kj} L_j(x), & k \geq 1. \end{cases} \tag{7}
\]
with
\[
\ell_{kj} = \begin{cases} (2j+1)(1-(-1)^{k+j}), & \text{if } j \leq k - 1, \\ 0, & \text{if } j \geq k. \end{cases} \tag{8}
\]

3.2 Bessel-Legendre inequality

The following lemma gives a Bessel-type inequality that compares an \( L^2([0,1]; \mathbb{R}^n) \) and \( R \in S^m_\mathbb{R} \). The following integral inequality holds for all \( N \in \mathbb{N} \):
\[
\int_0^1 z^T(x)Rz(x) \, dx \geq Z_N^T R^N Z_N, \tag{9}
\]
with
\[
R^N = \text{diag}(R, 3R, \ldots, (2N+1)R). \tag{10}
\]

Proof : The proof is easily conducted, as we show in Baudouin et al. (2016), by considering the difference between the state \( z \) and its projections over the \( N+1 \) first Legendre polynomials. Indeed, denoting \( y_N(x,t) = z(x,t) - \sum_{k=0}^N(2k+1)L_k(x) \int_0^1 z(\xi,t)L_k(\xi) \, d\xi \), the orthogonality of the Legendre polynomials and the Bessel inequality allows to obtain (9) from the positive definiteness and the expansion of \( \int_0^1 z_N^T(x,t)y_N(x,t) \, dx \), as in e.g. Seuret and Gouaisbaut (2015). \( \square \)

4. STABILITY RESULTS

4.1 Exponential stability

To assess the exponential stability of system (1), we will show that the Lyapunov functional (4) satisfies the following inequalities:
\[
\varepsilon_1 E(t) \leq V_N(X(t), z(t)) \leq \varepsilon_2 E(t), \tag{11}
\]
\[
\dot{V}_N(X(t), z(t)) + 2\delta V_N(X(t), z(t)) \leq -\varepsilon_3 E(t) \tag{12}
\]
for some positive scalars \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \). From now on, we will use the shorthand notation \( V_N(t) \). Since \( V_N \) partly depends on a projection \( Z_N \) of the state \( z \), as written in (4), then in order to compute the time derivative of \( V_N(t) \) in (12) we will need the time derivative of \( Z_N(t) \). The following property thus provides a simple expression.

Property 1. Consider \( z \in C(\mathbb{R}_+; L^2([0,1]; \mathbb{R}^m)) \) satisfying the transport equation in system (1). The time derivative of the projection vector \( Z_N(t) \) is given by:
\[
\dot{Z}_N(t) = \rho L_N Z_N(t) + (\Gamma^*_N D - \Gamma_N) z(1,t) + \rho L_N^2 C X(t), \tag{13}
\]
where we used the notations
\[
\begin{aligned}
\Gamma_N &= [I_m \cdots I_m]_+^T \in \mathbb{R}^{m(N+1),m}, \\
\Gamma^*_N &= [I_m - I_m \cdots (-1)^N I_m]_+^T \in \mathbb{R}^{m(N+1),m}, \\
L_N &= [k_k I_m]_{k=0,N} \in \mathbb{R}^{m(N+1),m(N+1)},
\end{aligned} \tag{14}
\]
the \( \ell_{kj} \) being defined in (8).

Proof : First, let us compute the time derivative of the projection of the infinite dimensional state \( z(x,t) \) over the \( k^\text{th} \) Legendre polynomial \( L_k \). Using the transport equation in (1), integration by parts and properties (6) and (7) of the Legendre polynomials, we obtain the following expression:
\[
\frac{d}{dt} \int_0^1 z(x,t)L_k(x) \, dx = -\int_0^1 \rho \partial_x z(x,t)L_k(x) \, dx \\
= -[\rho z(x,t)L_k(x)]_0^1 + \int_0^1 \rho \partial_t z(x,t) \frac{d}{dx} L_k(x) \, dx \\
= -\rho(z(1,t) + (-1)^k \rho z(0,t)) + \sum_{j=0}^{\max[0,k-1]} \ell_{kj} \rho \int_0^1 z(x,t)L_j(x) \, dx.
\]
Consequently, using the notations recently introduced and omitting the time variable \( t \), we have
\[
\frac{d}{dt} Z_N(t) = \rho L_N Z_N(t) - \rho \Gamma_N z(1,t) + \rho \Gamma_N z(0,t).
\]
The proof is concluded by injecting the boundary condition \( z(0,t) = CX(t) + D z(1,t) \) in the previous expression. \( \square \)

Remark 1. As mentioned before, the vector \( Z_N \) corresponds to the projections of the state \( z \) of the PDE dynamics over a set of polynomials of limited degree in \( L^2([0,1]; \mathbb{R}^m) \). We can note in (13) that the components of \( Z_N \) are computed by several integration of a combination of \( z(1,t) \) and \( z(0,t) \), since the matrices \( L_N \) are strictly lower triangular for any integer \( N \). Therefore, the augmented variable \( Z_N \) cannot be exponentially stable. However, in this work we are interested in stability of the global coupled system and not only the augmented system given in Property 1.

Based on the previous discussions, the following theorem is stated.

Theorem 2. Consider system (1) with a given transport speed \( \rho > 0 \). Recall that the matrices \( L_N, \Gamma_N, \Gamma^*_N \) are defined in (14), the matrix \( R^N \) is given by (10) and define the following \( \mathbb{R}^{m(N+1),m(N+1)} \) matrices
\[
S^N = \text{diag}(S, 3S, \ldots, (2N+1)S), \\
\Gamma^*_N = \text{diag}(I_m, 3I_m, \ldots, (2N+1)I_m). \tag{15}
\]
If there exists an integer \( N > 0 \) such that there exists \( \delta > 0, P \in \mathbb{S}^n_+, Q_N \in \mathbb{R}^{n(N+1)m}, T_N \in \mathbb{S}^{(N+1)m}, S \) and \( R \in \mathbb{S}^{n+1}_+ \) satisfying the following LMIs
\[
\Phi_N = \begin{bmatrix} P & Q_N \\ Q_N^T T_N + e^{-\delta t} S^N \end{bmatrix} > 0, \tag{16}
\]

Proof :
\[ \Psi_N(\rho, \delta) = \begin{bmatrix} \Psi_{11} & \Psi_{12} & \Psi_{13} \\ * & \Psi_{22} & \Psi_{23} \\ * & * & \Psi_{33} \end{bmatrix} < 0, \] (17)

where

\[
\begin{align*}
\Psi_{11} &= H(PA + \rho Q_N I_N^T C + 2\delta P + \rho C^T(S + R)C), \\
\Psi_{12} &= PB + \rho Q_N (I_N^TD - I_N) + \rho C^T(S + R)D, \\
\Psi_{13} &= A^TQ_N + \rho C^T I_N^T T_N - \rho Q_N L_N + 2\delta Q_N, \\
\Psi_{22} &= -\rho e^{-\frac{2\delta}{\rho}} S + \rho D^T(S + R)D, \\
\Psi_{23} &= B^TQ_N + \rho (I_N^TD - I_N)^T T_N, \\
\Psi_{33} &= \rho He(T_N L_N) + 2\delta T_N - \rho e^{-\frac{2\delta}{\rho}} R^N,
\end{align*}
\]

then system (1) is exponentially stable. Indeed, there exists a constant \( K > 0 \) and a guaranteed decay rate \( \delta^* > \delta \) such that the energy of the system verifies, for all \( t > 0 \),

\[ E(t) \leq Ke^{-2\delta^* t} \left( |X|^2 + |z|^2 \right) \]

(18).

Proof: To prove this stability result, we have to show that the Lyapunov functional \( V_N \) given in (4) verifies the inequalities (11) and (12) for some positive scalars \( \varepsilon_1, \varepsilon_2 \) and \( \varepsilon_3 \). The proof falls then into four steps.

Exponential stability: As soon as we will obtain that the Lyapunov functional \( V_N \) satisfies (11) and (12), we can prove the exponential stability of system (1), since we get easily

\[ V_N(t) + \left( 2\delta + \frac{\varepsilon_3}{\varepsilon_2} \right) V_N(t) \leq 0. \]

Indeed, integrating on the interval \([0, t] \) and using \( 2\delta^* = 2\delta + \frac{\varepsilon_3}{\varepsilon_2} \), inequality \( V_N(t) \leq V_N(0)e^{-2\delta^* t} \) holds for all \( t > 0 \) and using (11) once again, we get

\[ \varepsilon_1 E(t) \leq V_N(t) \leq V_N(0)e^{-2\delta^* t} \leq \varepsilon_2 E(0)e^{-2\delta^* t}, \]

which yields (18).

Existence of \( \varepsilon_1 \): On the one hand, since \( \rho > 0 \) and \( \Phi_N > 0 \), there exists a sufficiently small \( \varepsilon_1 > 0 \) such that

\[ S > \varepsilon_1 e^{-\frac{2\delta}{\rho}} I_m, \quad \Phi_N = P \begin{bmatrix} Q_N & \Phi_N \\ * & T_N + e^{-\frac{2\delta}{\rho}} S \end{bmatrix} > \varepsilon_1 \begin{bmatrix} I_n & 0 \\ 0 & T_N \end{bmatrix}. \]

On the other hand, \( V_N \) defined by (4) satisfies, \( \forall t \geq 0 \),

\[ V_N(t) \geq \int_{\frac{1}{\rho}} \left( X(t)^T Z_N(t) \right)^T \Phi_N \left( X(t)^T Z_N(t) \right) + e^{-\frac{2\delta}{\rho}} \int_0^t z^T(x,t)S z(x,t)dx. \]

Replacing \( \Phi_N \) by its lower bound depending on \( \varepsilon_1 \) and introducing \( \varepsilon_1 \) in the last integral term, we have

\[ V_N(t) \geq \varepsilon_1 |X(t)|^2 + \varepsilon_1 \int_0^t z^T(x,t)z(x,t)dx - Z_N(t)(e^{-\frac{2\delta}{\rho}} S - \varepsilon_1 I_m)Z_N(t) + \int_0^t z^T(x,t)(e^{-\frac{2\delta}{\rho}} S - \varepsilon_1 I_m)z(x,t)dx. \]

By noting that \( S - \varepsilon_1 e^{-\frac{2\delta}{\rho}} I_m > 0 \), Lemma 1 ensures that the sum of the two last terms is positive and we thus obtain that there exists \( \varepsilon_1 > 0 \) such that \( V_N(t) \geq \varepsilon_1 E(t) \).

Existence of \( \varepsilon_2 \): There exists a sufficiently large scalar \( \beta > 0 \) that allows

\[ \begin{bmatrix} P & Q_N \\ Q_N^T & T_N \end{bmatrix} \preceq \beta \begin{bmatrix} I_n & 0 \\ 0 & I_N \end{bmatrix}, \]

such that, under the assumptions \( S > 0 \) and \( R > 0 \), we get

\[ V_N(t) \leq \beta |X(t)|^2 + \beta Z_N(t)I_N Z_N(t) \]

\[ + \int_0^t e^{-\frac{2\delta}{\rho}} z^T(x,t)(S + (1 - x)R)z(x,t)dx \leq \beta |X(t)|^2 + \beta Z_N(t)I_N Z_N(t) \]

\[ + \int_0^t z^T(x,t)(S + R)z(x,t)dx. \]

Applying Lemma 1 to the second term of the right-hand side gives

\[ V_N(t) \leq \beta |X(t)|^2 + \int_0^1 z^T(x,t)(\beta I_m + S + R)z(x,t)dx \leq \beta |X(t)|^2 + \varepsilon_2 \|z\|_{2,0}^2 \leq \varepsilon_2 E(t), \]

where \( \varepsilon_2 = \beta + \lambda_{\text{max}}(S) + \lambda_{\text{max}}(R) \). Therefore, the proof of (11) is complete.

Existence of \( \varepsilon_3 \): Defining a kind of finite dimensional augmented state vector, of size \( n + (N + 2)m \) given by

\[ \xi_N(t) = \begin{bmatrix} X^T(t) z^T(t) \end{bmatrix}, \]

and using Property 1 and the definition of \( \Psi_N(\rho, \delta) \), several calculations, based on (1), lead to the following expression of the time derivative of \( V_N \), we obtain

\[ \dot{V}_N(t) + 2\delta V_N(t) \leq \varepsilon_3 \xi_N(t) \Psi_N(\rho, \delta) \xi_N(t) \]

\[ + \rho e^{-\frac{2\delta}{\rho}} R^N Z_N(t) - \varepsilon_3 E(t), \]

(19)

These calculations are omitted because of space limitations. Following the same procedure as for the existence of \( \varepsilon_1 \), the LMI (17) ensures that there exists a sufficiently small \( \varepsilon_3 > 0 \) such that

\[ R > 0 \varepsilon_3 e^{-\frac{2\delta}{\rho}} I_m, \quad \Psi_N(\rho, \delta) < -\varepsilon_3 \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} \]

(17).

Hence, using these two LMIs in (19) yields

\[ \dot{V}_N(t) + 2\delta V_N(t) \leq -\varepsilon_3 \left( |X(t)|^2 + \int_0^1 |z(x,t)|^2 dx \right) + \rho e^{-\frac{2\delta}{\rho}} R^N z_N(t) - \varepsilon_3 E(t), \]

\[ - \varepsilon_3 \int_0^1 z^T(x,t)(\rho e^{-\frac{2\delta}{\rho}} R - \varepsilon_3 I_m)z(x,t)dx. \]

Since \( R - \frac{1}{\rho} \varepsilon_3 e^{-\frac{2\delta}{\rho}} I_m > 0 \), Lemma 1 ensures that the sum of the two last terms of the previous equation is negative. Thus the Lyapunov functional \( V_N \) satisfies \( \dot{V}_N(t) + 2\delta V_N(t) \leq -\varepsilon_3 E(t) \), which concludes on the exponential stability of system (1).

\( \square \)

5. NUMERICAL EXAMPLE

To test our approach, we consider the following academic time-delay system

\[ \dot{X}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} X(t) + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} X(t - h), \]

(20)

which is given under the form of system (1) by \( \rho = \frac{1}{h} \) and

\[ A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}. \]
\[ C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}. \]

This example was studied in Baudouin et al. (2016) using a different Lyapunov functional. Figure 1 gives the maximum decay rate \( \delta \) for each value of the transport speed \( \rho \) and for several values of \( N \) using the LF (4). We can note that accelerating the system by a refine value of the transport speed \( \rho \) may improve its decay rate. The value of \( \delta \) reaches a maximum value \( \delta_{\text{max}} > 2.5 \) at the order \( N = 1 \), which corresponds to the particular case Wirtinger-based inequality (see Seuret and Gouaisbaut (2013)), rather than \( \delta_{\text{max}} \simeq 2.5 \) at the order \( N = 4 \) in Baudouin et al. (2016).

To explain the large difference between the two studies, we can note in Figure 2 that the analysis in Baudouin et al. (2016) was limited by the constraint \( (\rho - 2\delta)R - 2\delta S > 0 \) which is presented by the dashed line \( \delta = \frac{\rho}{2} \). In fact, we have \(-2\delta S < 0\) and to satisfy this constraint and obtain a positive term in \( (\rho - 2\delta)R \), the decay rate \( \delta \) has to be strictly lower than the half of the transport speed \( \rho \). The new analysis with Lyapunov functional (4) allow us to avoid this constraint and reduce the conservatism of our study and which is illustrated with the blue curve in Figure 2.

The most important result to illustrate our approach appears in Figure 3 which shows that we obtain the same values as frequency analysis in Breda et al. (2015), for the exponential decay rate \( \delta \) of the energy at the order \( N = 4 \). We remark also that we can improve the convergence rate by limiting the transport speed in system (1) since we have a maximum decay rate \( \delta_{\text{max}} = 4.3562 \) for the transport speed \( \rho = 3.5010 \).

Fig. 1. Evolution of the decay rate \( \delta \) obtained with Theorem 2 with respect to \( \rho \) for several value of \( N \).

Fig. 2. Comparison between the decay rate \( \delta \) obtained in Theorem 2 and in (Baudouin et al., 2016) with respect to \( \rho \) and for \( N=4 \).

Now, to test a more general case, we take the same dynamic matrix \( A \) with \( D \neq 0 \) by considering the following system

\[
\dot{X}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix} X(t) + \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix} (0.05X(t - \frac{h}{2}) + 0.95X(t - h)),
\]

which takes the shape of (1) for \( \rho = \frac{2}{\pi} \) and

\[
A = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 & -1 & 0 \\ -1 & -1 & -1 & -1 \end{bmatrix},
\]

\[
C = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.05 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 19 & 0 & 0 & 0 \end{bmatrix}.
\]

Using Theorem 2, the maximum transport speed for stability is estimated as follows

| Variables | \( N=0 \) | \( N=1 \) | \( N=2 \) | \( N=3 \) |
|-----------|-----------|-----------|-----------|-----------|
| \( \delta = 0 \) | \( \rho_{\text{min}} = 0.253 \) | 0.233 | 0.232 | 0.232 |
| \( \delta = 0.01 \) | \( \rho_{\text{min}} = 0.391 \) | 0.367 | 0.366 | 0.365 |

Table 1. Minimal allowable transport speed

When considering this example as a time delay system, the following table shows that the results are very close to those found in Gu et al. (2003), and to the analytical limit, even for small values of \( N \).

| \( \rho_{\text{max}} \) | \( N=4 \) | \( N=5 \) |
|---------------------|-----------|-----------|
| \( h_{\text{max}} \) | 8.594   | 8.596   |

Table 2. Comparison in term of time delay with Gu et al. (2003)
The variables $N_{d1}$ and $N_{d2}$ in Gu et al. (2003) present the discretization degrees, respectively for the delay terms $h_1 = \frac{1}{2}$ and $h_2 = h$.

6. CONCLUSION

In this document, we show as a first result that the structure of the proposed Lyapunov functional may improve the estimation of the convergence rate of system (1) and gives better assessment stability results. By changing the design of the LF, we give a novel approach for the stability analysis of coupled ODE - transport PDE systems issued from recent developments on time-delay systems.

Indeed, we provide an efficient result of stability of a coupled ODE-transport PDE system in terms of tractable LMIs depending on the transport speed $\rho$ and the order $N$ of a polynomial approximation of the infinite dimensional part of the state.

Moreover, the estimation of the decay rate that we generate with this approach is very close, in a definite order $N$, to the frequency results which are more precise, and we show that the maximum of this estimation is greater than the one found in (Baudouin et al., 2016).

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