Quantum Guessing via Deutsch-Jozsa

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Abstract

We examine the “Guessing Secrets” problem arising in internet routing, in which the goal is to discover two or more objects from a known finite set. We propose a quantum algorithm using $O(1)$ calls to an $O(\log N)$ oracle. This improves upon the best known classical result, which uses $O(\log N)$ questions and requires an additional $O(\log N^3)$ steps to produce the answer. In showing the possibilities of this algorithm, we extend the types of questions and function oracles that the Deutsch-Jozsa algorithm can be used to solve.
1 Introduction

Inspired by challenges in internet routing [10], Chung, Graham, and Leighton proposed the “Guessing Secrets” problem [1] in which a set of \( k \) objects is to be reconstructed based on the answers to a set of questions. It is a property of the answer set as a whole that gives us information, suggesting that perhaps a quantum algorithm could efficiently solve this problem. We propose using the Deutsch-Jozsa algorithm. Bernstein and Vazirani [4] showed a problem that requires \( \log N \) classical function calls but that is solved in one application of Deutsch-Jozsa; generalizations and other applications have followed [3, 5, 6, 11]. The present problem for two objects can be solved classically in \( O(\log N + (\log N)^3) \) steps [2]. In the quantum case, we can eliminate the \( \log N^3 \) term of the complexity.

The next section gives necessary background on the classical problem. It is followed by the presentation and analysis of the quantum algorithm in the case \( k = 2 \). Finally, we present some explorations in a related problem with more objects.

2 The problem for \( k = 2 \) and its classical approaches

The following summarizes the basic problem and approaches for \( k = 2 \) as presented by Chung, Graham, and Leighton [1]:

We are given a finite set of objects \( \Omega \), of size \( N \). An honest but possibly malicious adversary selects elements \( \{X_1, X_2\} \subset \Omega \). Our task is to deduce as much as possible about the \( X_i \)'s by asking yes/no questions. A question is simply a map from \( \Omega \) to \( \{0, 1\} \). For each question \( q \), the adversary must respond with either \( q(X_1) \) or \( q(X_2) \). Our goal is to construct a set of questions \( \{q_j\} \) and an algorithm to use the responses to determine as much as possible about \( \{X_1, X_2\} \).

Example: Suppose \( \Omega = \{1, 2, 3, 4\} \). The Adversary chooses \( \{X_1, X_2\} = \{1, 2\} \). If the question were “Is your object an odd number?” the Adversary could answer on behalf of 1 and say “Yes” or on behalf of 2 and say “No.” On the other hand, if the question were “Is your object less than three?” the Adversary must answer “Yes,” since that answer is true for both objects in the set.
There are inherent limits on what we can be assured of finding out. We can represent the elements of \( \Omega \) as vertices of a graph; each question eliminates some number of possible pairs of objects, which correspond to edges in the graph. If our graph contains two disjoint edges \((X_1, X_2)\) and \((X_3, X_4)\), we can ask a question \( q \) such that \( q(X_1) = q(X_2) \neq q(X_3) = q(X_4) \), the response to which will eliminate one of the edges. There are only two types of graphs that contain no disjoint pairs of edges, a star and a triangle, and once we are in one of these configurations, the Adversary can prevent us from gaining any new information. A star, a set of edges all sharing a common vertex, indicates that we know one of the objects we seek but have limited information about the other; a triangle means that our two objects are contained in a set of 3, but we don’t know which they are. The upshot is that the Adversary can limit our knowledge in this way no matter what questions we ask. So, our goal is reduced to arriving at a point of maximal knowledge as quickly as possible.

Chung, Graham, and Leighton noted that the minimal size of the question set is \( O(\log N) \) whether the questions depend on previous answers or not. As a result, we will use only non-adaptive algorithms, where the entire question set is submitted to the Adversary at once. It is then reasonable to define a vector \( A = (A_1, A_2, \ldots, A_m) \), where \( m \) is the number of questions asked and \( A_q \) is the adversary’s response to question \( q \).

Once we know the size of a minimal question set, the next goal is to create questions that allow efficient implementation and recovery of information about the \( \{X_i\} \). One productive idea is to represent \( \Omega \) as \( B^n \), a binary vector space for \( n = \lceil \log_2 N \rceil \), and also to identify the questions \( q \) as elements of \( B^n \), defining \( q(X) = q \cdot X \), the inner product (mod 2) of \( q \) and \( X \).

Given a set of questions that generates maximal information, efficiently extracting this information from the responses is not easy. Both [1] and [2] offer ways to do this, increasing the size of the question set by a constant factor to allow recovery in time polynomial in \( \log N \). In the quantum case, the recovery of information is fundamentally intertwined with the asking of the questions, so no distinction is made between them.

In the classical algorithm, the number of questions is equal to the number of calls to the adversary, and thus minimizing the size of the question set is paramount. In the quantum model, we can ask all the questions in superposition, so the number of calls to the adversary is the more important
measure of efficiency. We have chosen to use all of $B^n$ for our question set, which, though much larger than the minimum necessary, has two distinct advantages. First, it exhibits a symmetry that we will take advantage of. Second, we can efficiently generate a superposition of questions by applying a Hadamard gate to each individual qubit; no multi-qubit operations are needed. And while the set of questions is big, the number of calls to the Adversary will be small.

3 Quantum Procedure using Deutsch-Jozsa

3.1 Quantum Reformulations

In their paper, Chung, Graham, and Leighton describe their problem as a generalization of traditional “20 Questions”, a game in which there is one unknown object $X \in \Omega$:

20 Questions

Given an Adversary vector $A$ such that $\forall q, A_q = q(X)$ for some fixed $X$; find $X$.

We compare this to the parity problem of Bernstein and Vazirani [4], in which the quantum oracle inputs $q$ and returns the parity of the sum of a subset of the binary digits of $q$. This parity is exactly $q \cdot X \pmod{2}$ for some $X$, so we can write:

Parity Problem:

Given a quantum oracle for a function $f$, such that $\forall q, f(q) = q \cdot X$ for some fixed $X$; find $X$.

Since we have defined our questions in terms of the dot product, it is clear that the parity problem is the quantum analogue of 20 Questions. In a similar way, we can formulate a quantum version of Chung, Graham, and Leighton’s problem:

Quantum Guessing Secrets:

Given a quantum oracle for a function $f$, such that $\forall q, f(q) = q \cdot X_1$ or $f(q) = q \cdot X_2$ for some fixed $X_1, X_2$; find $X_1$ and $X_2$.

In this formulation, we define a quantum oracle based on the function $f(q) := A_q$, where $A$ is the vector of adversary responses described above. This oracle will be a unitary map that takes $|q\rangle|y\rangle$ to $|q\rangle|y \oplus A_q\rangle$ with addition
taken modulo 2. (Here, $|y\rangle$ represents a single qubit, while $|q\rangle$ is an n-qubit binary string $|q_1q_2\ldots q_n\rangle$ with $q = \sum_{i=1}^{n} q_i2^{n-i}$).

The Deutsch-Jozsa algorithm was designed to distinguish between constant and balanced functions. A more careful inspection of the measurement output shows that it can distinguish the $N$ different functions $f_j \in B^n$ defined by $f_j(X) = j \cdot X$ [8] and thus solves the parity problem [4]. We see then that the Deutsch-Jozsa algorithm solves “20 Questions” in only one call to the oracle. This inspires us to apply it to Guessing Secrets.

Much other work has been done to extend the classes of functions that this type of algorithm can distinguish [3, 5, 6]. In this paper we extend Deutsch-Jozsa in yet another direction by applying it to the guessing secrets problem. Terhal and Smolin [11] applied variations of Deutsch-Jozsa in order to recover an unknown object in a single query, but one oracle call will be insufficient for our purposes. There are $\binom{N}{3} + N$ triangles and maximal stars that our algorithm must distinguish, and Farhi, et al., have shown that such a quantum search must take at least 3 queries [7].

### 3.2 The Quantum Algorithm

**Proposition 1** With a single call to the Adversary oracle, the Deutsch-Jozsa algorithm outputs one of the desired objects with probability at least one half.

**Proof:** The algorithm is as follows:

- **Step 1:** Initialize $n$ bits to $|0\rangle$ plus an extra bit to $|1\rangle$.
- **Step 2:** Apply a Hamamard gate to each of the first $n$ bits, to get a uniform superposition of all questions $q$. Also apply a Hadamard gate to the final qubit to get an eigenstate of the NOT gate.
- **Step 3:** Apply the Adversary oracle.
- **Step 4:** Apply a Hadamard gate to each of the first $n$ bits.
- **Step 5:** Measure the first $n$ bits in the computational basis.
Formally, the system evolves like this:

\[ |0 \rangle^\otimes n |1 \rangle \xrightarrow{H^\otimes n+1} \frac{1}{\sqrt{2N}} \sum_{q=0}^{N-1} |q \rangle (|0 \rangle - |1 \rangle) \]

\[ \xrightarrow{\text{oracle}} \frac{1}{\sqrt{2N}} \sum_{q=0}^{N-1} (-1)^{f(q)} |q \rangle (|0 \rangle - |1 \rangle) \]

\[ \xrightarrow{H^\otimes n \otimes I} \frac{1}{\sqrt{2}} \sum_{j=0}^{N-1} C_j |j \rangle \frac{|0 \rangle - |1 \rangle}{\sqrt{2}} \]

The coefficient of \(|j \rangle\) is given by

\[ C_j = \sum_{q=0}^{N-1} (-1)^{j \cdot q + f(q)} = \#\{ q : j \cdot q = f(q) \} - \#\{ q : j \cdot q \neq f(q) \}. \tag{1} \]

What do we know about the coefficients of \(|X_1 \rangle\) and \(|X_2 \rangle\) (denoted \(C_{X_1}\) and \(C_{X_2}\))? For all \(q\), \(f(q) = q \cdot X_1 \) or \(f(q) = q \cdot X_2\), so there are only three types of questions \(q\):

- 1: \(q \cdot X_1 \neq q \cdot X_2\) and \(f(q) = q \cdot X_1\)
- 2: \(q \cdot X_1 \neq q \cdot X_2\) and \(f(q) = q \cdot X_2\)
- 3: \(q \cdot X_1 = q \cdot X_2 = f(q)\)

Let \(S_i\) be the number of questions of type \(i\), for \(i = 1, 2, 3\).

By the symmetry of \(B^n\), for any distinct objects \(X, Y \in \Omega\), \(q \cdot X = q \cdot Y\) for exactly half the possible values of \(q\). In particular, \(S_3 = N/2\). This allows us to calculate the probability that we measure \(X_1\) or \(X_2\):

\[
\begin{align*}
C_{X_1} &= S_3 + S_1 - S_2 \\
C_{X_2} &= S_3 - S_1 + S_2 \\
C_{X_1} + C_{X_2} &= 2S_3 = N \\
C_{X_1}^2 + C_{X_2}^2 &\geq \frac{N^2}{2} \\
\left(\frac{C_{X_1}}{N}\right)^2 + \left(\frac{C_{X_2}}{N}\right)^2 &\geq \frac{1}{2} \tag{2}
\end{align*}
\]

This means that the output of our measurement is one of the desired states with probability at least \(\frac{1}{2}\). \(\text{QED}\)
3.3 Finishing the Procedure

For given $\epsilon$, choose values for $m$ and $d$ such that

$$\text{Probability} \left( \text{Binomial} \left( m, \frac{1}{2} \right) \leq \frac{m}{2} - d \right) < \epsilon$$

with $d$ as small as possible for the selected $m$. Run the algorithm $m$ times and let $E$ be the number of outputs equal to either $X_1$ or $X_2$. Then

$$\text{Probability} \left( E \leq \frac{m}{2} - d \right) < \epsilon$$

Let $F(X) =$ the number of times $X$ appears as output in these $m$ runs, for all $X \in \Omega$. Then with probability $1 - \epsilon$,

$$(X_1, X_2) \in \left\{ (X', X'') : F(X') + F(X'') \geq \frac{m}{2} - d \right\}$$

(3)

There are two possible cases:

- **Case 1:** There is no $X \in \Omega$ with $F(X) \geq \frac{m}{2} - d$. Then with probability $1 - \epsilon$, $\{X_1, X_2\}$ is one of the edges $(X', X'')$. The corresponding graph is not too complex, and complexity bounds depend only on $\epsilon$. We can then ask few questions sequentially of the oracle to reduce the graph to a triangle or star.

- **Case 2:** There is a dominant $X'$ with $F(X') \geq \frac{m}{2} - d$. Then all that can be said with probability $1 - \epsilon$ is that the set of possible edges forms a star and that $X'$ is the center of it. This is entirely analogous to the classical case, in which our remaining uncertainty is determined by the number of points on the star. While the classical algorithm was a method of elimination, the quantum algorithm is a generator of outcomes; as such, it allows us to discover the center of the star quite quickly, even if it makes it harder to find all the possible second objects.

Let $X \in \Omega, X \notin \{X_1, X_2\}$. For every run of the algorithm, $C_{X_i} \geq |C_X|$ for $i = 1, 2$. (The requirement that half the answers for $X$ match those for $X_1$ and $X_2$ ensures that $C_X$ can be neither too big nor too small.) So the expected value $E[F(X_i)] \geq E[F(X)], i = 1, 2$. This fact will remain true even if the Adversary changes the oracle between calls and so will be useful in any further statistical analysis of the outcomes.
3.4 Examples

- Full star: The Adversary answers every question for the same object \( \forall q A_q = q \cdot X_1 \). The corresponding graph is a star centered at \( X_1 \) with \( N - 1 \) points. The quantum algorithm outputs \( X_1 \) with probability 1.

- Triangle: The Adversary chooses a third object \( X^* \) and answers for it whenever there is a choice. (If \( q \cdot X_1 \neq q \cdot X_2, A_q = q \cdot X^* \).) The corresponding graph is a triangle on \( X_1, X_2, X^* \). Our algorithm will output \( X_1, X_2, X^* \), or \( X_1 \oplus X_2 \oplus X^* \), each with probability \( \frac{1}{4} \). At most three additional questions will be needed to determine which 3 out of 4 are desirable.

- An intermediate case: For questions when \( q \cdot X_1 \neq q \cdot X_2 \), the Adversary chooses \( q \cdot X_1 \) three-fourths of the time. The corresponding graph is either a star centered at \( X_1 \) with only a few points or simply the edge \( (X_1, X_2) \). The quantum algorithm outputs \( X_1 \) with probability \( \frac{9}{16} \), \( X_2 \) with probability \( \frac{1}{16} \), and a handful of others, each with probability \( \leq \frac{1}{16} \).

4 Comparing Complexities of the Classical and Quantum Algorithms

The classical case for \( k = 2 \) requires \( O(\log N) \) questions that are answered sequentially. In the quantum case, each run of the algorithm requires only one call to the oracle. Since we are directly comparing the quantum performance to the classical, one must ask whether our large question set makes the oracle exponentially more complex. The answer is No. If the Adversary wishes to effect a triangle or an \((N - 1)\)-pointed star, it is straightforward to write an oracle algorithm in \( O(\log N) \) steps to assign appropriate values for \( f(q) \). (More generally, a quantum adversary can force the guesser into a graph \( G \) with an \( O(\log N) \) oracle if and only if a classical adversary could force the guesser into \( G \) with any \( O(\log N) \) questions from the set \( B^n \).)

Thus the questioning part of the algorithm takes \( O(\log N) \) steps in both cases. Classically, it is then necessary to extract information about \( \{X_1, X_2\} \) from the questions’ answers; the list decoding method in \( [2] \) accomplishes this in \( O((\log N)^3) \) steps. By contrast, the quantum algorithm simply runs \( O(1) \) times and the output is in an immediately useable form. The number
of runs (and calls to the oracle) is independent of the size of $\Omega$ and depends only on the desired level of certainty. It is the avoidance of a complicated decoding stage that allows the quantum algorithm to perform faster than its classical analogue.

5 What about $k > 2$?

In the original paper, Chung, Graham, and Leighton begin with a more general problem in which the guesser tries to determine a set of $k$ objects, with the promise that for all questions $q$, the Adversary’s response $A_q = q(X_i)$ for some $i \in \{1, 2, \ldots, k\}$. Both [1] and [2] observe that the situation gets much more complicated for $k > 2$, and the classical results in this case are far from complete or satisfying. For $k = 2$, there are 2 shapes for the final graph (star and triangle); for $k = 3$, there are 8 final hypergraph shapes, and any algorithm must be able to distinguish $O(N^7)$ of them [1]. According to [7], a quantum algorithm that can differentiate these requires at least 7 oracle calls. However, our present algorithm does not extend directly in the case of larger $k$.

To see this, let $k = 3$ and define the function by a “minority rule” (for fixed $X_1, X_2, X_3$, $f(q) := q \cdot X_1 \oplus q \cdot X_2 \oplus q \cdot X_3$); then $q \cdot X_i = f(q)$ for exactly half the questions and $C_{X_i} = 0$ for $i = 1, 2, 3$. The algorithm will never pick any of the correct objects! This sort of difficulty will persist for all higher values of $k$.

It is possible to generalize the original problem differently by placing stronger restrictions on the function $f$. For example, suppose that when asked a question, the Adversary must respond on behalf of the majority of the $X_1, X_2, \ldots, X_k$. This example is formalized below:

**Majority Problem**

Given a function $f : B^n \to \{0, 1\}$ such that $\forall q \in B^n, f(q) = q \cdot X_i$ for at least half of the $X_i \in \{X_1, X_2, \ldots, X_k\}$; determine $\{X_1, X_2, \ldots, X_k\}$.

Using the Deutsch-Jozsa algorithm with the function oracle does solve this problem. As before, we define a run of the algorithm as successful if its output is one of the $X_i$. 

9
Proposition 2  (i) If the $X_i$’s are linearly independent as vectors in $B^n$, the algorithm succeeds with probability $≥ p_k$, where $p_k > \frac{2}{π}$ if $k$ is odd; $p_k ≥ \frac{1}{2}$ if $k$ is even; and as $k → ∞$, $p_k → \frac{2}{π}$.

(ii) With no assumptions about the $X_i$’s and for $k$ odd, $P(\text{success}) ≥ \frac{1}{k}$, and the order of this bound is strict.

(iii) In the case $k = 3$, the Deutsch-Jozsa algorithm solves the Majority Problem, producing one of the desired objects with probability $≥ \frac{3}{4}$.

The issue of independence arises from the following:

Symmetry Principle:

For all linearly independent sets of vectors $\{X_1, X_2, \ldots, X_k\} ⊂ B^n$ and $v ∈ B^k$, the vector $(q · X_1, q · X_2, \ldots, q · X_k) = v$ for exactly $2^{n−k}$ questions $q ∈ B^n$.

It is obviously a disadvantage to require independence for our result. On the other hand, most sets of vectors $\{X_1, X_2, \ldots, X_k\}$ are linearly independent in the following sense: If $k$ vectors are chosen uniformly in $B^n$, a simple inductive argument shows that they are linearly independent with probability greater than $1 − 2^{k−n}$.

Combinatorial Proof of Proposition 2:

(i) For $M ⊂ \{1, 2, \ldots, k\}$, define

$$
\epsilon_{i,M} := \begin{cases} 1 & \text{if } i ∈ M \\ -1 & \text{if } i \notin M \end{cases}
$$

$$
S_M := \#\{q ∈ B^n : \{i : f(q) = q · X_i\} = M\}
$$

Thus $S_M$ counts the number of questions for which $M$ is the set of $X_i$ which agree with the function.

In Step 4 of the algorithm, the coefficient of $|X_i⟩$ is $\frac{C_{X_i}}{N}$, where

$$
C_{X_i} = \sum_{M ⊂ \{1,2,\ldots,k\}} \epsilon_{i,M} S_M.
$$

(4)

Observe that if $|M| < \frac{k}{2}$ then $S_M = 0$ by assumption, and if $|M| = \frac{k}{2}$ then $\sum_i \epsilon_{i,M} = 0$. This means that once we sum over $i$, we can restrict our
attention to subsets $M$ with $|M| > \frac{k}{2}$,

$$\sum_{i=1}^{k} C_{X_i} = \sum_{i,M} \epsilon_{i,M} S_M = \sum_{i:|M|>\frac{k}{2}} \epsilon_{i,M} S_M$$ (5)

If the $X_i$’s are linearly independent and $|M| > \frac{k}{2}$, then $S_M = 2\left(2^n-k\right) = \frac{N}{2^{k-1}}$ by the Symmetry Principle stated above, so

$$\sum_{i=1}^{k} C_{X_i} = \sum_{i:|M|>\frac{k}{2}} \epsilon_{i,M} S_M = \frac{N}{2^{k-1}} \sum_{i:|M|>\frac{k}{2}} \epsilon_{i,M}$$ (6)

For a given $j$, there are \( \binom{k}{j} \) subsets $M$ of size $j$. For a fixed $i$, we can then count how many $M$’s contain $i$ and how many do not. This is exactly the information encoded by $\epsilon_{i,M}$:

$$\sum_{|M|=j} \epsilon_{i,M} = \binom{k-1}{j-1} - \binom{k-1}{j}$$ (7)

Summing over all $j > \frac{k}{2}$

$$\sum_{i=1}^{k} C_{X_i} = \frac{N}{2^{k-1}} \sum_{i=1}^{k} \sum_{|M|>\frac{k}{2}} \epsilon_{i,M}$$

$$= \frac{N}{2^{k-1}} \sum_{i=1}^{k} \sum_{j=\lceil \frac{k+1}{2} \rceil}^{k} \left( \binom{k-1}{j-1} - \binom{k-1}{j} \right)$$

$$= \frac{N}{2^{k-1}} \sum_{i=1}^{k} \left( \frac{k-1}{\lceil \frac{k+1}{2} \rceil} - 1 \right)$$

$$= \frac{1}{2^{k-1}} \left( \frac{kN}{\lceil \frac{k+1}{2} \rceil} - 1 \right)$$ (8)

Dividing by $N$ yields

$$\frac{1}{N} \sum_{i=1}^{k} C_{X_i} = \frac{k}{2^{k-1}} \left( \frac{k-1}{\lceil \frac{k+1}{2} \rceil} \right)$$ (9)
The sum of the squares of the coefficients is minimized if they are all the same.

\[
P(\text{success}) = \sum_{i=1}^{k} \left( \frac{C_{X_i}}{N} \right)^2 \geq \frac{k}{2^{2(k-1)}} \left( \left\lfloor \frac{k-1}{2} \right\rfloor \right)^2 =: p_k \tag{10}
\]

It is easy to check that \(p_2 = \frac{1}{2}\) and \(p_3 = \frac{3}{4}\); and \(p_{2m}\) increases monotonically to \(\frac{2}{\pi}\) while \(p_{2m+1}\) decreases monotonically to \(\frac{2}{\pi}\).

(ii) The assumption that \(k\) is odd implies that for all \(M \subset \{1, 2, \ldots, k\}\), either \(|M| < \frac{k}{2}\) and \(S_M = 0\), or else \((2|M| - k) \geq 1\). So for all \(M\):

\[
(2|M| - k)S_M \geq S_M \tag{11}
\]

Also, for any \(M\),

\[
\sum_i \epsilon_{i,M} = |M| - (k - |M|) = 2|M| - k \tag{12}
\]

By definition, \(\sum_M S_M\) is the total number of questions, so

\[
\sum_{i=1}^{k} C_{X_i} = \sum_{i,M} \epsilon_{i,M} S_M = \sum_{M} (2|M| - k)S_M \geq \sum_{M} S_M = N \tag{13}
\]

Dividing by \(N\) gives

\[
\sum_{i=1}^{k} \frac{C_{X_i}}{N} = 1 \tag{14}
\]

The sum of the squares is minimized if each \(\frac{C_{X_i}}{N} = \frac{1}{k}\).

\[
P(\text{success}) = \sum_{i=1}^{k} \left( \frac{C_{X_i}}{N} \right)^2 \geq k \left( \frac{1}{k} \right)^2 = \frac{1}{k}. \tag{15}
\]

We cannot do significantly better than this bound, either. Let \(G\) be a subgroup of \(B^n\) and let \(X_1, X_2, \ldots, X_k\) be the nonzero elements of \(G\). (So \(k\) is one less than a power of two; in particular, \(k\) is odd.) Then \(f(q) = 0\) if and only if \(q \in G^\perp\), which implies that for all \(i \in \{1, 2, \ldots, k\}\):

\[
C_{X_i} = \left( \frac{N}{2} + |G^\perp| \right) - \left( \frac{N}{2} - |G^\perp| \right) = 2(|G^\perp|) = 2 \left( \frac{N}{k + 1} \right) \tag{16}
\]
Dividing by $N$ and summing the squares, we get

$$P(\text{success}) = \sum_{i=1}^{k} \left( \frac{C_{X_i}}{N} \right)^2 = k \left( \frac{2}{k+1} \right)^2 = \frac{4k}{(k+1)^2}$$

(17)

This is $O\left(\frac{1}{k}\right)$, so the bound above is more or less strict.

(iii) It should be noted that the Majority Problem for $k = 3$ is equivalent to the earlier example of the triangle (Section 3.4), where it was stated without proof that $X_1, X_2,$ and $X^*$ would each appear with probability $\frac{1}{4}$. A complete proof follows immediately from the results (i) and (ii):

Part (i) proved that $p_3 = \frac{3}{4}$ if the \{X_i\} are independent. If they are dependent and all nonzero, then they are the nonzero vectors of a subgroup $G$ as described in (ii) and $P(\text{success}) = \frac{4k}{(k+1)^2} = \frac{3}{4}$. Finally, if one of the vectors is zero, the vectors span a subgroup of dimension 2 and it is straightforward to show that each vector in this group will appear with probability $\frac{1}{4}$. Thus in any case, the algorithm succeeds will probability $\geq \frac{3}{4}$. QED

6 Conclusions

The Deutsch-Jozsa algorithm was written to show the power of a quantum computer in solving an artificially created problem. In this paper, we applied the same algorithm in a different way to an existing classical problem with active interest in combinatorics and computer science. This “Guessing Secrets” problem can solved classically in $O(\log N + (\log N)^3)$ steps while the quantum algorithm uses $O(1)$ calls to an $O(\log N)$ oracle. The quantum algorithm achieves this higher efficiency by producing output in an immediately useable form. It is certainly worth considering what other sorts of problems might be addressed in a similar fashion.

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