Extensions of the linear bound in the Füredi–Hajnal conjecture

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Abstract

We present two extensions of the linear bound, due to Marcus and Tardos, on the number of 1’s in an \( n \times n \) 0-1 matrix avoiding a fixed permutation matrix. We first extend the linear bound to hypergraphs with ordered vertex sets and, using previous results of Klazar, we prove an exponential bound on the number of hypergraphs on \( n \) vertices which avoid a fixed permutation. This, in turn, solves various conjectures of Klazar as well as a conjecture of Brändén and Mansour. We then extend the original Füredi–Hajnal problem from ordinary matrices to \( d \)-dimensional matrices and show that the number of 1’s in a \( d \)-dimensional 0-1 matrix with side length \( n \) which avoids a \( d \)-dimensional permutation matrix is \( O(n^{d-1}) \).

1 Introduction

Füredi and Hajnal asked in [6] whether for every fixed 0-1 permutation matrix \( P \) the maximum number of 1’s in an \( n \times n \) 0-1 matrix \( M \) avoiding \( P \) is

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$O(n)$; the avoidance here means that $P$ cannot be obtained from $M$ by a series of row deletions, column deletions, and replacements of 1’s with 0’s (in particular, permuting rows or columns of $M$ is not allowed). The Füredi–Hajnal conjecture was settled by Marcus and Tardos in [10] where they proved that if $M$ avoids a $k \times k$ permutation matrix, then the number of 1’s in $M$ is at most $2k^4(k^2)^n$. In this paper we present extensions of their linear bound to more general structures.

The Marcus–Tardos bound can be reformulated in the language of graph theory, since matrices with entries 0 and 1 can be viewed as the incidence matrices of bipartite graphs. Thus if $P = ([2k], E(P))$ is a graph on the vertex set $[2k] = \{1, 2, \ldots, 2k\}$ with $k$ mutually disjoint edges, each of which connects the sets $[k]$ and $[k + 1, 2k] = \{k + 1, k + 2, \ldots, 2k\}$, and $M = ([2n], E(M))$ is a graph on $[2n]$ which only has edges connecting $[n]$ and $[n + 1, 2n]$ and does not contain $P$ as an ordered subgraph, then $M$ has only linearly many edges, i.e. $|E(M)| = O(n)$. It is easy to modify the proof in [10] so that it gives a linear bound for all $P$-avoiding graphs $G$ (not necessarily bipartite) on the vertex set $[2n]$ (and therefore $[n]$). In Section 2 we extend this bound further to hypergraphs with edges of arbitrary size. We also discuss exponential enumerative bounds which follow from the linear extremal bounds as corollaries.

In yet another light, 0-1 matrices can be viewed as the (characteristic matrices of) binary relations. In Section 3 we generalize the original proof of Marcus and Tardos to $d$ dimensions and show that every $d$-ary relation on $[n]$ which avoids a fixed $d$-dimensional permutation has at most $O(n^{d-1})$ elements.

### 2 Extensions to hypergraphs

For a graph $G' = ([k], E')$, we define $\text{gex}_<(n, G')$ to be the maximum number $|E|$ of edges in a graph $G = ([n], E)$ that does not contain $G'$ as an ordered subgraph. We represent a permutation $\pi = a_1a_2\ldots a_k$ of $[k]$ by the graph

$$P(\pi) = ([2k], \{\{i, k + a_i\} : i \in [k]\}).$$

As we mentioned in Section 1, it is easy to modify the proof in [10] to obtain the bound

$$\text{gex}_<(n, P(\pi)) = O(n) \quad (1)$$

where the constant in $O$ depends only on $\pi$. 

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For the hypergraph extension we need a few more definitions. A hypergraph is a finite collection $H = (E_i : i \in I)$ of finite nonempty edges $E_i$ which are subsets of $\mathbb{N} = \{1, 2, \ldots \}$. The vertex set is $V(H) = \bigcup_{i \in I} E_i$. For simplicity we do not allow (unlike in the graph case) isolated vertices; for our extremal problems this restriction is immaterial (isolated vertices in graphs can be represented by singleton edges in our extension). In general we will allow multiple edges, and will denote a hypergraph as simple if it has no multiple edges. We say that $H' = (E'_i : i \in I')$ is contained in $H = (E_i : i \in I)$, written $H' \prec H$, if there exists an increasing injection $f : V(H') \to V(H)$ and an injection $g : I' \to I$ such that $f(E'_i) \subset E_{g(i)}$ for every $i \in I'$; otherwise we say that $H$ avoids $H'$. To put it differently, $H' \prec H$ means that $H'$ can be obtained from $H$ by deleting some edges, deleting vertices from the remaining edges, and relabeling the vertices so that their ordering is preserved. This containment generalizes the ordered subgraph relation. Note that a simple hypergraph can contain a non-simple hypergraph.

The order $v(H)$ of $H$ is the number of vertices $v(H) = |V(H)|$, the size $e(H)$ is the number of edges $e(H) = |I|$, and the weight $i(H)$ is the number of incidences $i(H) = \sum_{i \in I} |E_i|$. We define two hypergraph extremal functions.

**Definition.** Let $F$ be any hypergraph. We associate with $F$ the functions $e(n, F), i(n, F) : \mathbb{N} \to \mathbb{N},$

$$e(n, F) = \max \{ e(H) : H \not\succ F \& H \text{ is simple} \& v(H) \leq n \}$$

$$i(n, F) = \max \{ i(H) : H \not\succ F \& H \text{ is simple} \& v(H) \leq n \}.$$

Obviously, $e(n, F) \leq i(n, F)$ for every $F$ and $n$. If $F$ has at least two edges and has no two separated edges (edges $E_1$ and $E_2$ satisfying $E_1 < E_2$), Klazar’s Theorem 2.3 in [9] gives an inequality in the opposite direction:

$$i(n, F) \leq (2v(F) - 1)(e(F) - 1)e(n, F).$$

So in particular, for every permutation $\pi$ of $[k],$

$$i(n, P(\pi)) \leq (4k - 1)(k - 1)e(n, P(\pi)). \quad (2)$$

Thus a linear bound on $i(n, P(\pi))$ follows directly from one on $e(n, P(\pi)).$

The latter bound can be derived using the techniques in [9] along with the graph bound in (1). To explain the reduction we need the notion of the blow-up of a graph. A graph $G'$ is an $m$-blow-up of a graph $G$ if for every edge
coloring of $G'$ by colors from $\mathbb{N}$ such that every color is used at most $m$ times, there exists a subgraph of $G'$ that is order-isomorphic to $G$ and no two of its edges have the same color. Let $G$ be a graph with $k$ vertices, $H$ be a $\left(\frac{k}{2}\right)$-blow-up of $G$, and $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $\ex(n, H) < nf(n)$ for every $n \in \mathbb{N}$. Then Theorem 3.1 in [9] states that, for every $n \in \mathbb{N}$,

$$\ex(n, G) < e(G) \cdot \ex(n, G) \cdot \ex(2f(n) + 1, G). \quad (3)$$

Combining the bounds in (1), (2), and (3) we obtain the following result:

**Theorem 2.1.** For every permutation $\pi$,

$$\ex(n, P(\pi)) = O(n).$$

**Proof.** For $m \in \mathbb{N}$ and a $k$-permutation $\pi$, consider a permutation graph $P(\pi')$ that arises from $P(\pi)$ by replacing every edge in $P(\pi)$ with a bundle of $k(m - 1) + 1$ edges so that the initial vertices of the edges in each bundle form an interval in $P(\pi')$ and the same holds for the final vertices. The positions of the intervals are as in $P(\pi)$, that is, for every selection of one edge from each bundle the resulting graph is order-isomorphic to $P(\pi)$. There are many such graphs $P(\pi')$ ($\pi'$ is a $(k^2(m - 1) + k)$-permutation) and each of them is, by the pigeonhole principle, an $m$-blow-up of $P(\pi)$.

We set $m = \left(\frac{2k}{2}\right)$. By the graph bound in (1), there are constants $c_\pi$ and $c'_{\pi'}$ such that

$$\ex(n, P(\pi)) < c_\pi n \quad \text{and} \quad \ex(n, P(\pi')) < c'_{\pi'} n$$

for every $n$. We set $H = P(\pi')$ and $f(n) = c_{\pi'}$ and apply the bound in (3) to get the linear bound

$$\ex(n, P(\pi)) < kc_\pi \cdot \ex(2c_{\pi'} + 1, P(\pi)) \cdot n.$$ 

By the bound in (2),

$$\ex(n, P(\pi)) < k(k - 1)(4k - 1)c_\pi \cdot \ex(2c_{\pi'} + 1, P(\pi)) \cdot n,$$ 

proving the claim. \qed

Klazar posed the following six extremal and enumerative conjectures in [8]:

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C1: The number of simple $H$ such that $v(H) = n$ and $H \not> P(\pi)$ is $< c_1^n$.

C2: The number of maximal simple $H$ with $v(H) = n$ and $H \not> P(\pi)$ is $< c_2^n$.

C3: For every simple $H$ with $v(H) = n$ and $H \not> P(\pi)$, we have $e(H) < c_3 n$.

C4: For every simple $H$ with $v(H) = n$ and $H \not> P(\pi)$, we have $i(H) < c_4 n$.

C5: The number of simple $H$ with $i(H) = n$ and $H \not> P(\pi)$ is $< c_5^n$.

C6: The number of $H$ with $i(H) = n$ and $H \not> P(\pi)$ is $< c_6^n$.

He showed that all six conjectures hold for a large class of permutations $\pi$ and that they hold for every $\pi$ in weaker forms: with almost linear and almost exponential bounds (respectively). Conjecture C4, however, is precisely the statement of Theorem 2.1, and it is easy to extend the proof given in this paper to affirm that all six conjectures hold for every permutation $\pi$.

We shall show how to amend the proofs in [8] to prove C1, and then note that C1 implies C2, C3, C5 and C6 via Lemma 2.1 of [8].

Corollary 2.2. For every permutation $\pi$ there exists a constant $c_1 > 1$ (depending on $\pi$) so that the number of simple hypergraphs on the vertex set $[n]$ avoiding $P(\pi)$ is $< c_1^n$.

Proof. Theorems 2.4 and 2.5 in [8] show that the number of hypergraphs with a given weight $i(H)$ that avoid $P(\pi)$ is at most $9^{(3^k + 2^k) i(H)}$. Thus by Theorem 2.1 we are done. □

The Stanley–Wilf conjecture (see Bóna [2]), proved by Marcus and Tardos in [10] as a corollary of their linear extremal bound, claimed that for every permutation $\pi$ there is a constant $c = c(\pi)$ such that the number of permutations $\sigma$ of $[n]$ avoiding $\pi$ is $< c^n$; the avoidance of permutations here means that $P(\sigma)$ is not an ordered subgraph of $P(\pi)$. In view of the reformulation from permutations to bipartite graphs mentioned in Section 1, Corollary 2.2 is an extension of the Stanley–Wilf conjecture. A related extension was proposed by Brändén and Mansour in Section 5 of [4]: they conjectured that the number of words over the ordered alphabet $[n]$ which have length $n$ and avoid $\pi$ is at most exponential in $n$. These words can be represented by simple graphs $G$ on $[2n]$ in which every edge connects $[n]$ and $[n + 1, 2n]$ and every $x \in [n]$ has degree exactly 1; the containment of ordered words is
then just the ordered subgraph relation. Hence this extension is subsumed in Corollary 2.2.

Corollary 2.2 subsumes yet another extension of the Stanley–Wilf conjecture to set partitions proposed by Klazar [7]. This extension is related to \( k \)-noncrossing and \( k \)-nonnesting set partitions whose exact enumeration was recently investigated by Chen et al. [5] and Bousquet-Mélou and Xin [3]. Consider, for a set partition \( H \) of \( [n] \), the graph \( G(H) = ([n], E) \) in which an edge connects two neighboring elements of a block (not separated by another element of the same block). Thus \( H \) is represented by increasing paths which are spanned by the blocks. \( H \) is a \( k \)-noncrossing (resp. \( k \)-nonnesting) partition iff \( P(12\ldots k) \) (resp. \( P(k(k - 1)\ldots 1) \)) is not an ordered subgraph of \( G(H) \). Thus Corollary 2.2 provides an exponential bound: for fixed \( k \), the numbers of \( k \)-noncrossing and \( k \)-nonnesting partitions of \( [n] \) grow at most exponentially.

### 3 An extension to \( d \)-dimensional matrices

We now generalize the original Füredi–Hajnal conjecture from ordinary 0-1 matrices to \( d \)-dimensional 0-1 matrices. As was mentioned in Section 1, these are just \( d \)-ary relations (or, as we will discuss later, \( d \)-uniform, \( d \)-partite hypergraphs). We keep the matrix terminology, however, both for the sake of consistency and to highlight the similarities with the original Marcus–Tardos proof in [10].

**Definition.** We will call a \((d + 1)\)-tuple \( M = (M; n_1, \ldots, n_d) \) where \( M \subset [n_1] \times \ldots \times [n_d] \) a \( d \)-dimensional (0-1) matrix, and will refer to the elements of \( M \) as edges.

If \( F = (F; k_1, \ldots, k_d) \) and \( M = (M; n_1, \ldots, n_d) \) are two \( d \)-dimensional matrices, we say that \( F \) is contained in \( M \), \( F \prec M \), if there exist \( d \) increasing injections \( f_i : [k_i] \to [n_i], \) \( i = 1, 2, \ldots, d \), such that for every \( (x_1, \ldots, x_d) \in F \) we have \((f_1(x_1), \ldots, f_d(x_d)) \in M\); otherwise we say that \( M \) avoids \( F \).

**Definition.** We set \( f(n, F, d) \) to be the maximum size \(|M|\) of a \( d \)-dimensional matrix \((M; n_1, \ldots, n_d)\) that avoids a \( d \)-dimensional matrix \( F \).

For \( i \in [d] \), we will denote the projection mapping from \([n_1] \times \ldots \times [n_d]\) to \([n_i]\) as \( \pi_i \). For \( t \in [d] \), we define the \( t \)-remainder of \( M = (M; n_1, \ldots, n_d) \) to be the \((d - 1)\)-dimensional matrix \( N = (N; n'_1, \ldots, n'_{d-1}) \) where \( n'_i =
Let $I_1 < I_2 < \cdots < I_r$ be a partition of $[n]$ into $r$ intervals and $M = (M; n, \ldots, n)$ a $d$-dimensional matrix. We define the contraction of $M$ (with respect to the intervals) to be the $d$-dimensional matrix $N = (N; r, \ldots, r)$ given by $(e_1, \ldots, e_{d-1}) \in N$ iff $M \cap (I_{e_1} \times \cdots \times I_{e_d}) \neq \emptyset$ (we could define the contraction operation for a general $d$-dimensional matrix and with distinct and general partitions in each coordinate but we will not need such generality).

We say that $P = (P; k, \ldots, k)$ is a $d$-dimensional permutation of $[k]$ if for every $i \in [d]$ and $x \in [k]$ there is exactly one edge $e \in P$ with $\pi_i(e) = x$. Note that $|P| = k$ and that there are exactly $(k!)^{d-1}$ $d$-dimensional permutations of $[k]$. For $d = 1$, the only 1-dimensional permutation $(P; k)$ is $[k]$, and for $d = 2$ the 2-dimensional permutations $P = (P; k, k)$ are precisely the $k \times k$ 0-1 permutation matrices. A $d$-dimensional permutation of $[k]$ can be thought of as a $d \times k$ matrix with the first row normalized to 1, 2, \ldots, $k$ and with each row being a permutation of 1, 2, \ldots, $k$. The columns would then give the coordinates of the $k$ edges in $P$.

It is also possible to view the structure $M = (M; n_1, \ldots, n_d)$ as an ordered, $d$-uniform, $d$-partite hypergraph with partitions $[n_i]$. In this interpretation, the image of $M$ by the projection $\pi_i$ is obtained by intersecting the edges with the $i$th partition, while the intersections with the union of all partitions except the $t$th one give the $t$-remainder of $M$ (in both cases we disregard multiplicity of edges). Furthermore, the set of $d$-dimensional permutations of $[k]$ would be the set of perfect matchings of the complete $d$-uniform, $d$-partite hypergraph on $kd$ vertices.

We will make use of two observations, analogous to those made in [10]:

1. For any $t \in [d]$, the $t$-remainder of a $d$-dimensional permutation of $[k]$ is a $(d-1)$-dimensional permutation of $[k]$. Furthermore, each edge of the resulting $t$-remainder can be completed (by adding the $t$-th coordinate) in a unique way to an edge of the original permutation.

2. If $M = (M; n, \ldots, n)$ avoids a $d$-dimensional permutation, then so does any contraction of $M$.

**Theorem 3.1.** For every fixed $d$-dimensional permutation $P$,

$$f(n, P, d) = \Theta(n^{d-1}).$$
On the other hand it is clear that for a $d$-dimensional permutation $P$ with $|P| > 1$ we have $f(n, P, d) \geq n^{d-1}$ ($f(n, P, d) = 0$ if $|P| = 1$). Thus, for a $d$-dimensional permutation $P$ with $|P| > 1$,

$$f(n, P, d) = \Theta(n^{d-1}).$$

This bound can be given an equivalent formulation. We say that a matrix $M = (M; n_1, \ldots, n_d)$ is a $d$-dimensional $k$-grid if each $[n_i]$ can be partitioned in $k$ intervals $I_{i,1} < I_{i,2} < \cdots < I_{i,k}$ so that $|M \cap (I_{1,j_1} \times I_{2,j_2} \times \cdots \times I_{k,j_k})| = 1$ for every $d$-tuple $(j_1, j_2, \ldots, j_d) \in [k]^d$ (thus, in particular, $|M| = k^d$). Let $g(n, k, d)$ be the maximum size of a $d$-dimensional $n \times n \times \cdots \times n$ matrix that contains no $d$-dimensional $k$-grid. Then

$$g(n, k, d) = \Theta(n^{d-1}).$$

It is clear that $g(n, k, d) \geq n^{d-1}$. The bound $g(n, k, d) = O(n^{d-1})$ implies $f(n, P, d) = O(n^{d-1})$ for every $P$ because every $d$-dimensional $k$-grid contains every $d$-dimensional permutation of $[k]$. In the other way, it is easy to see that there exist $d$-dimensional $k$-grids that are $d$-dimensional permutations of $[k^d]$. Thus $f(n, P, d) = O(n^{d-1})$ implies $g(n, k, d) = O(n^{d-1})$.

To prove Theorem 3.1, we will show that a $d$-dimensional matrix of big enough size must contain every $d$-dimensional permutation of $[k]$. We set

$$f(n, k, d) = \max_P f(n, P, d)$$

where $P$ runs through all $d$-dimensional permutations of $[k]$.

**Lemma 3.2.** Let $d \geq 2$, $m, n_0 \in \mathbb{N}$. Then

$$f(mn_0, k, d) \leq (k - 1)^d \cdot f(n_0, k, d) + dn_0m^d \binom{m}{k} \cdot f(n_0, k, d - 1).$$

**Proof.** Let $M = (M; mn_0, \ldots, mn_0)$ be a $d$-dimensional matrix that avoids $P$, a $d$-dimensional permutation of $[k]$. We aim to bound the size of $M$.

We split $[mn_0]$ into $n_0$ intervals $I_1 < I_2 < \cdots < I_{n_0}$, each of length $m$, and define, for $i_1, \ldots, i_d \in [n_0],$

$$S(i_1, \ldots, i_d) = \{ e \in M : \pi_j(e) \in I_{i_j} \text{ for } j = 1, \ldots, d \}.$$

Note that this partitions the set of edges of $M$ into $n_0^d$ pieces. We will call these sets of edges blocks and we define a cover of these blocks by a total of $dn_0 + 1$ sets $\{U_0\} \cup \{U(t, j) : t \in [d], j \in [n_0]\}$ as follows:
• $U(t, j) = \{S(i_1, \ldots, i_d) : i_t = j \text{ and } |\pi_t(S(i_1, \ldots, i_d))| \geq k\}$

• $U_0$ consists of the blocks which are not in any $U(t, j)$

Note that the total number of non-empty blocks is exactly the number of edges in the contraction of $M$ with respect to the partition $\{I_i\}$. Since $M$ does not contain $P$, the contraction of $M$ can not contain $P$, so the number of non-empty blocks is at most $f(n_0, k, d)$. Also note that any block $B$ in $U_0$ has at most $(k - 1)^d$ edges in it (because $B \subset X_1 \times \cdots \times X_d$ for some $X_i \subset [mn_0]$ with $|X_i| < k$). Hence

$$|\bigcup U_0| \leq (k - 1)^d \cdot f(n_0, k, d).$$

Now we fix $t \in [d]$ and $j \in [n_0]$. Clearly,

$$|\bigcup U(t, j)| \leq m^d|U(t, j)|.$$

We assume, for a contradiction, that $|U(t, j)| > \binom{m}{k} \cdot f(n_0, k, d - 1)$. By the definition of $U(t, j)$ and the pigeonhole principle, there are $k$ numbers $c_1 < c_2 < \cdots < c_k$ in $I_j$ and $r$ blocks $S_1, S_2, \ldots, S_r$ in $U(t, j)$ where $r > f(n_0, k, d - 1)$ such that for every $S_a$ and every $c_b$ there is an $e \in S_a$ with $\pi_t(e) = c_b$. Let $P'$ be the $t$-remainder of $P$ and $M' = (M'; n_0, \ldots, n_0)$ be the $(d - 1)$-dimensional matrix arising from contracting $(\bigcup_{i=1}^{r} S_i, n, \ldots, n)$ with respect to the intervals $I_i$ and then taking the $t$-remainder. Since $|M'| = r > f(n_0, k, d - 1)$, $M'$ contains $P'$. Thus among the blocks $S_1, S_2, \ldots, S_r$ there exist $k$ of them—call them $S_1, S_2, \ldots, S_k$—so that for any selection of $k$ edges $e_1 \in S_1, e_2 \in S_2, \ldots, e_k \in S_k$ their $t$-remainders form a copy of $P'$. Furthermore, due to the property of the blocks $S_i$, it is possible to select $e_1, \ldots, e_k$ so that their $t$-th coordinates agree with $P$. Then $e_1, \ldots, e_k$ form a copy of $P$, a contradiction. Therefore

$$|\bigcup U(t, j)| \leq m^d|U(t, j)| \leq m^d \binom{m}{k} \cdot f(n_0, k, d - 1)$$

and

$$|\bigcup_{t,j} U(t, j)| \leq dm^d \binom{m}{k} \cdot f(n_0, k, d - 1).$$

Combining this with the bound for $U_0$ gives the stated bound. $\square$

Theorem 3.1 will be a direct consequence of the following lemma:

**Lemma 3.3.** For $m = \lceil k^{d/(d-1)} \rceil$, $f(n, k, d) \leq k^d \left( dm \binom{m+1}{k} \right)^{d-1} n^{d-1}$. 

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Proof. We will proceed by induction on \( d + n \). For \( d = 1 \) this holds since \( f(n, k, 1) = k - 1 \) and for \( n < k^2 \), this holds trivially. Now given \( n \) and \( d \geq 2 \), assume that the hypothesis is true for all \( d', n' \) such that \( d' + n' < d + n \).

Let \( n_0 = \lfloor n/m \rfloor \) and

\[
c_d = k^d \left( dm \left( \frac{m+1}{k} \right) \right)^{d-1}.
\]

Using the inequality \( f(n, k, d) < f(mn_0, k, d) + dm n^{d-1} \), Lemma 3.2, the inductive hypotheses, and \( n_0 \leq n/m \), we get

\[
f(n, k, d) < \left( \frac{(k-1)^d}{m^{d-1}} c_d + dm \left( \binom{m}{k} c_{d-1} + 1 \right) \right) n^{d-1}.
\]

Since \( \frac{(k-1)^d}{m^{d-1}} \leq (1 - \frac{1}{k})^d \leq 1 - \frac{1}{k} \) and \( \binom{m}{k} c_{d-1} + 1 \leq \binom{m+1}{k} c_{d-1} \), it follows that \( f(n, k, d) < c_d n^{d-1} \) with the above defined \( c_d \).

4 Concluding remarks

We were informed recently that Balogh, Bollobás and Morris [1] derived Theorem 2.1 (their Theorem 2) and Corollary 2.2 (their Theorem 1) independently. The proofs in [1] are self-contained (not appealing to the results in [9]) and their approach is different from ours. In fact, they are able to prove stronger statements, which in turn imply Theorem 2.1 and Corollary 2.2 from this paper.

It should be noted that we make no effort to optimize any of the constants in Section 3, however it would be interesting to see if any of the constants could be drastically reduced. The constant achieved in this paper is double exponential in \( k \), whereas we conjecture the true constant is in fact much smaller.

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