ABSTRACT: We solve for the retarded Greens function for linearized gravity in a background with a negative cosmological constant, anti de Sitter space. In this background, it is possible for a signal to reach spatial infinity in a finite time. Therefore the form of the Greens function depends on a choice of boundary condition at spatial infinity. We take as our condition that a signal which reaches infinity should be lost, not reflected back. We calculate the Greens function associated with this condition, and show that it reproduces the correct classical solution for a point mass at the origin, the anti de Sitter-Schwarzschild solution.
INTRODUCTION

The physics of gravitation at energies small compared to the Planck scale is well described by Einstein’s theory of general relativity. Unfortunately, the nonrenormalizibility of this theory means that we cannot use perturbation theory to extract predictions at higher energies. This means that in this case, either perturbation theory or general relativity is inapplicable. It is often believed that to learn anything interesting about quantum gravity, we must first find a formulation of it which allows us to make predictions at all orders. Unfortunately, it seems that we are a long way away from accomplishing this goal.

An alternative approach is to consider quantum gravity at low energies, and see what might be learned. At low energies there is no doubt that physics is correctly described by the action functional of Einstein’s gravity, plus an infinite series of counterterms which diverge in the ultraviolet, but go to zero in the infrared limit [1]. This approach tells us, for example, that the gravitational anomaly of the standard model must cancel, which gives an observationally verified restriction on the particle content of the model [2].

This low-energy gravity approach has recently been used by Tsamis and Woodard to suggest a mechanism for suppression of the cosmological constant [3,4]. They consider quantum gravity in a de Sitter space background, the maximally symmetric background for gravity with a positive cosmological constant. They develop a formalism for doing low-energy perturbation theory for gravity in this background [5,6,7], and argue that there is a natural relaxation of the effective cosmological constant over time [1].

In this paper we take the first step toward making a similar analysis for the case of a negative cosmological constant. The maximally symmetric background for gravity with a negative cosmological constant is anti de Sitter space. In this paper, we determine the appropriate retarded Greens function for the linearized graviton kinetic operator on a background which is anti de Sitter everywhere and at all times. The retarded Greens function $G(x, x')$ allows us to determine the linearized response of the metric to a distribution of
energy-momentum \( T(x) \), as

\[
\text{Response} = \text{Initial Value Term} + \int d^4x' \ G(x, x') \ T(x') \quad (1)
\]

Furthermore, the average of the retarded and advanced Greens functions gives the imaginary part of the free propagator for the theory:

\[
\text{Im} \left\{ i \left[ \rho \sigma \Delta^{\alpha \beta} \right] (x, x') \right\} = \frac{1}{2} \left\{ \left[ \rho \sigma G_{\text{ret}}^{\alpha \beta} \right] (x, x') + \left[ \rho \sigma G_{\text{adv}}^{\alpha \beta} \right] (x, x') \right\} \quad (2)
\]

From this, the free Feynman propagator can be deduced, up to terms which are dependent on the vacuum. This of course is the necessary ingredient in formulating a theory of Feynman diagrams for anti de Sitter space gravity.

In the first section of this paper (after this introduction), we discuss the geometry of the anti de Sitter space background, and define coordinate systems which we will need to use. We do this for an arbitrary number \( D \) of spacetime dimensions. We show that this background has the peculiar feature that it is possible for an observer to travel from finite to in coordinate values (or vice-versa) in a finite time. Therefore, in solving for (1) one needs to specify what happens to a signal which reaches infinity [8]. In the next section we write and solve the gauge fixed equations of motion for gravity in the anti de Sitter background. In the section after that one, we solve explicitly for the retarded Greens function of anti de Sitter gravity in four spacetime dimensions, using the boundary condition that any signal which reaches infinity should be lost. In the penultimate section we show that our Greens function gives the right classical limit for a certain source; specifically that it gives the anti de Sitter-Schwarzchild solution, up to a coordinate transformation, when the source \( T(x) \) is a single point mass. Finally we will conclude by discussing where to proceed from here, in particular how anti de Sitter space might fit into realistic models of the history of the universe, and how the analysis in this paper might be modified to accomodate this.
Anti de Sitter Space

Anti de Sitter space (AdS) in \( D \) dimensions can be described in terms of \( D + 1 \) coordinates* \( X^0, X^i, X^D \), in terms of which the metric is

\[
ds^2 = -(dX^0)^2 + dX^i dX^i - (dX^D)^2
\]  

(3)

We obtain Anti de Sitter space by restricting these coordinates to obey

\[
-(X^0)^2 + X^i X^i - X^D X^D = -1/h^2
\]

(4)

for some constant \( h \). In this paper we will set \( h = 1 \) for convenience.

The space AdS contains closed timelike curves (e.g. the circle \((X^0)^2 + (X^D)^2 = 1\)), along which an observer could travel and return to his own past. This undesirable feature can be removed by extending de Sitter space to its universal covering space (CAdS). This just means that we introduce an integer \( N \) (winding number) which we increment by 1 every time we go around the loop, and we consider different values of \( N \) for the same \( X \) to be different spacetime locations.

It is of course possible to solve (4) in terms of a set of \( D \) coordinates \( x^\mu \). These coordinates will then have a non-flat metric, given by

\[
g_{\mu\nu} = -\frac{\partial X^0}{\partial x^\mu} \frac{\partial X^0}{\partial x^\nu} + \frac{\partial X^i}{\partial x^\mu} \frac{\partial X^i}{\partial x^\nu} - \frac{\partial X^D}{\partial x^\mu} \frac{\partial X^D}{\partial x^\nu}
\]

(5)

One such solution (given here for \( D = 4 \), but easily generalized to any \( D \)) is

\[
X^0 = \sqrt{r^2 + 1} \sin \tau
\]

(6a)

\[
X^1 = r \sin \theta \cos \phi
\]

(6b)

\[
X^2 = r \sin \theta \sin \phi
\]

(6c)

* Notation: Latin indices \( i, j \), etc. are spatial indices which go from 1 to \( D - 1 \). Greek indices will run from 0 to \( D - 1 \).
\[ X^3 = r \cos \theta \quad (6d) \]
\[ X^4 = \sqrt{r^2 + 1} \cos \tau \quad (6e) \]

The metric in these coordinates can be calculated using (5). It is the static and isotropic metric
\[ g_{\tau \tau} = -1 - r^2 \quad (7a) \]
\[ g_{rr} = \frac{1}{1 + r^2} \quad (7b) \]
\[ g_{\theta \theta} = r^2 \quad (7c) \]
\[ g_{\phi \phi} = r^2 \sin^2 \theta \quad (7d) \]

To obtain de Sitter space, we should take the range of \( \tau \) to be 0 to \( 2\pi \), and identify all observables at these two boundaries. The covering space CaDS can be obtained simply by letting \( \tau \) take any real value; it is not necessary to introduce winding numbers when these coordinates are used.

From the metric (7) we can show that the Riemann tensor for anti de Sitter space
\[ R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\gamma,\delta} - \Gamma^\alpha_{\beta\delta,\gamma} + \Gamma^\alpha_{\gamma\rho} \Gamma^\rho_{\beta\delta} - \Gamma^\alpha_{\delta\rho} \Gamma^\rho_{\beta\gamma} \quad (8) \]

is
\[ R^\alpha_{\beta\gamma\delta} = -\delta^\alpha_\gamma g_{\beta\delta} + \delta^\alpha_\delta g_{\beta\gamma} \quad (9) \]

From this, we can see that the anti de Sitter metric obeys the gravitational equations of motion
\[ R_{\alpha\beta} = \Lambda g_{\alpha\beta} \quad (10) \]

for \( \Lambda = -(D - 1) \).

The equation of motion of a massless signal is obtained by setting \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \) to zero. For a radially moving signal, this gives \( r = \tan(\tau - \tau_0) \). This shows that a massless signal can get from zero to infinity or vice-versa in a finite time, namely \( \frac{\pi}{2} \). This means that
in CAdS a local event will causally influence all spatial locations after a certain finite time. The consequence of this is that the CAdS Greens function depends on the choice of an extra boundary condition, at \( r = \infty \) [8]. We will argue that the condition which works is that any signal which reaches \( r = \infty \) should be lost.

The static coordinate system is not the most convenient for the calculations which are to follow. From (10) we can show that the conformal or Weyl tensor

\[
C_{\alpha\beta\gamma\delta} \equiv R_{\alpha\beta\gamma\delta} - \frac{1}{D-2} \left( g_{\alpha\gamma} R_{\beta\delta} + g_{\beta\delta} R_{\alpha\gamma} - g_{\alpha\delta} R_{\beta\gamma} - g_{\beta\gamma} R_{\alpha\delta} \right)
\]

\[
+ \frac{1}{(D-1)(D-2)} \left( g_{\alpha\gamma} g_{\beta\delta} - g_{\alpha\delta} g_{\beta\gamma} \right) R
\]

(11)
is identically zero for anti de Sitter space. This fact implies that there exists conformal coordinate systems in which the anti de Sitter metric is proportional to the Minkowski metric. One such coordinate system is* \( t, y, x^i \), in terms of which

\[
X^0 = \frac{t}{y}
\]

(12a)

\[
X^i = \frac{x^i}{y}, \quad i = 1, \ldots, D - 2
\]

(12b)

\[
X^{D-1} = \frac{1 + t^2 - x^i x^i - y^2}{2y}
\]

(12c)

\[
X^D = \frac{1 - t^2 + x^i x^i + y^2}{2y}
\]

(12d)

We then find

\[
g_{\mu\nu} = \frac{1}{y^2} \eta_{\mu\nu}
\]

(13)

These conformal coordinates do not naturally extend to CAdS; winding numbers \( N \) need to be introduced. The two regions \( y > 0 \) and \( y < 0 \) actually correspond to different coordinate patches. It is not possible to go from positive \( y \) to negative \( y \) by passing through

* Barred Latin indices such as \( \bar{i} \) take values from 1 to \( D - 2 \). This convention and others will be fully introduced in the next section.
$y = 0$, but it is possible to do so by going through $y = \infty$. $y = 0$ is not really part of the space since no finite values of the $X$ coordinates make $y$ zero. To emphasize the separation between positive and negative $y$ regions, we will take the convention that regions of $y > 0$ are given integer winding numbers, while regions with $y < 0$ are given half integer numbers.

Now consider a massless signal originating at $x'\mu$ on the patch with winding number $N$. By setting $ds^2 = 0$, we see that the signal will follow the usual path of a lightlike beam, the path $x$ such that $(x - x')^2 = 0$. If the signal is initially headed toward $y = 0$, then it will hit $y = 0$. If it is initially headed away from $y = 0$, then it will go into the far future of sheet $N$, and reappear in the far past on the sheet $N + \frac{1}{2}$. On this sheet it will be heading toward $y = 0$. Our boundary conditions will be that any signal which hits $y = 0$ is lost. Therefore a massless signal from sheet $N$ can at most get to sheet $N + \frac{1}{2}$ before it is lost. Massive (timelike) signals, on the other hand, can remain present for all $N$. These facts will be important in the construction of the Greens function.

We see that an observer at $x\mu$ on sheet $N$ can causally observe any event $x'\mu$ on sheet $N' = N$ for $t' < t - \sqrt{(x - x')^2 + (y - y')^2}$; any event on sheet $N' = N - \frac{1}{2}$ for $t' < t + \sqrt{(x - x')^2 + (y - y')^2}$; and any event on sheets $N' < N - \frac{1}{2}$ no matter where or when. This is the same effect we saw using static coordinates. Although in conformal coordinates it takes an infinite amount of time $t$ to cross to another $N$ level, it is still possible for a massless signal to cross the entire spatial extent of the manifold without taking the entire temporal lifetime of the manifold to do it.

The condition (4), and hence the anti de Sitter metric, is invariant under the anti de Sitter group, the $D(D + 1)/2$ parameter group of Lorentz rotations of the $X$ coordinates. In static coordinates, $\tau$ translation and spatial rotations which shift $\theta, \phi$ are some of the anti de Sitter transformations; the others are non-linearly realized in these coordinates. In conformal coordinates, the anti de Sitter symmetries are realized as translational invariance in the $D-1$ dimensional flat subspace, Lorentz invariance in this subspace, invariance under
simultaneous dilatation of all coordinates, and $D - 1$ nonlinear symmetries. This is exactly analogous to the de Sitter case [9].

An important invariant under the anti de Sitter group is the distance function

$$1 - z(X, X') \equiv -\frac{1}{4}(X - X')^2$$ (14)

In the static coordinates this is

$$1 - z(x, x') = \frac{1}{2} + \frac{1}{2} \sqrt{r^2 + 1} \sqrt{r'^2 + 1} \cos(\tau - \tau') + \frac{r r'}{2} (\sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta')$$ (15)

In conformal coordinates it is

$$1 - z(x, x') = -\frac{1}{2yy} (x - x')^2$$ (16)

Setting $1 - z = 0$ in either coordinate system is an alternate way to find the path of a massless signal. Since our Greens function should be invariant under the anti de Sitter group (at least up to gauge transformations), it will naturally be a function of $1 - z$.

**Gauge Fixing and Classical Solutions**

We now expand the gravitational Lagrangian

$$L = \frac{1}{\kappa^2} \sqrt{-g} (R - (D - 2)\Lambda)$$ (17)

about the anti de Sitter background

$$g_{\mu\nu} = \frac{1}{y^2} (\eta_{\mu\nu} + \kappa \psi_{\mu\nu})$$ (18)

The resulting Lagrangian for $\psi$ is invariant under the gauge transformation

$$\delta \psi_{\mu\nu} = -2 \epsilon_{(\mu, \nu)} + \frac{2}{y} \eta_{\mu\nu} e_y$$ (19)

provided that the functions $e_\mu(x)$ are well-behaved enough at infinity to allow integration by parts.
We borrow the following notation from [5] (some of which is standard): Indices are raised and lowered by the (spacelike) Minkowski metric $\eta_{\mu\nu}$ (since we have given up general covariance by working exclusively in the conformal coordinate system). Greek indices go from 0 to $D-1$, while Latin indices take only spacelike values 1 to $D-1$. Indices in parentheses are symmetrized ($a_{(\mu\nu)} \equiv \frac{1}{2}(a_{\mu\nu} + a_{\nu\mu})$). A bar over a $\delta$ or $\eta$ tensor indicates the suppression of its non-flat (in this case $y$) components; e.g. $\overline{\eta}_{\mu\nu} = \eta_{\mu\nu} - \delta_\mu^y \delta_\nu^y$. We also introduce the new notation that barred indices only run over flat coordinates. Thus a tensor with a barred index such as $A^{\overline{\mu}}$ must have zero $y$ component.

Following [5], we expand out the quadratic part of the Lagrangian, and add the gauge fixing term $-\frac{1}{2} F_{\mu} F_{\nu} \eta_{\mu\nu}$ where

$$F_\mu \equiv y^{1-D/2} \left( \psi_{\mu,\nu}^{\rho} - \frac{1}{2} \psi_{,\mu}^{\rho} - \frac{D-2}{y^D} \psi_{\mu y}^{\rho} \right)$$

Then the gauge fixed quadratic Lagrangian is

$$\mathcal{L}_{GF}^{(2)} = \frac{1}{2} \psi^{\mu\nu} D_{\mu\nu}^{\rho\sigma} \psi_{\rho\sigma}$$

where

$$D_{\mu\nu}^{\rho\sigma} \equiv \left( \frac{1}{2} \delta_\mu^{(\rho} \delta_\nu^{\sigma)} - \eta_{\mu\nu} \eta^{\rho\sigma} \right) D_0 + \frac{D-2}{y^D} \delta_\nu^{(\rho} \delta_\mu^{\sigma)}$$

with

$$D_0 \equiv y^{2-D} \left( \partial^2 - \frac{D-2}{y} \partial y \right)$$

We will now investigate solutions of the homogeneous equations of motion. The gauge-invariant equations are

$$D_{\mu\nu}^{\rho\sigma} \psi_{\rho\sigma} + \frac{1}{y} F_{(\mu,\nu)}^{\rho\sigma} + \frac{1}{2y} \eta_{\mu\nu} F_{,\rho}^{\rho\sigma} + \frac{D-2}{2y^2} \delta_\nu^{(\rho} \delta_\mu^{\sigma)} F_{,\nu}^{\rho\sigma} + \frac{D-2}{4y^2} \eta_{\mu\nu} F_{y}^{\rho\sigma} = 0$$

If we demand $F = 0$, then solutions of the gauge-fixed equations $(D\psi)_{\mu\nu} = 0$ are also solutions of the invariant equations. Let us look for such solutions. Following the analogous treatment of the $\Lambda > 0$ case [6], it is convenient to reexpress $\psi$:

$$\psi_{\rho\sigma} = \left( \delta_\rho^{(\mu} \delta_\sigma^{\overline{\nu})} - \frac{1}{D-3} \eta_{\rho\sigma} \eta^{\mu\nu} \right) \psi_{A}^{\mu\nu} + 2 \delta_\nu^{(\rho} \delta_\sigma^{\overline{\nu})} \psi_{B}^{\mu} + \left( \frac{1}{D-3} \eta_{\rho\sigma} - \delta_\rho^y \delta_\sigma^y \right) \psi_{C}$$
where $\psi^A$ is symmetric. We then calculate

\[
D_{\mu\nu} \psi^\rho = \frac{1}{2} \left( \delta(\mu^2) \delta(\nu^2) + \frac{1}{D-3} \delta(\mu) \delta(\nu) \frac{\pi^2}{\pi^2} \right) D_0 \psi_{\alpha^2}^A + \frac{\pi^2}{\pi^2} D_{D-2} \psi_{\alpha^2}^B - \frac{1}{2} \left( \frac{D-2}{D-3} \right) \delta(\mu) \delta(\nu) D_2(D-3) \psi_C
\]

where

\[
D_n \equiv D_0 + \frac{n}{yD}
\]

Thus the equations of motion for the $A, B, C$ components are

\[
D_0 \psi^A_{\mu\nu} = 0 \quad (28a)
\]

\[
D_{D-2} \psi^B_{\mu} = 0 \quad (28b)
\]

\[
D_{2(D-3)} \psi^C = 0 \quad (28c)
\]

The general solutions to these equations are

\[
\psi^A_{\mu\nu} = \int \frac{dD-2k}{(2\pi)^{D-1}} \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} e^{ikx^\mu x^\nu - i\sqrt{\alpha^2 + k^2}t} \frac{\alpha y^{D/2} h_{D-2}(\alpha y) A_{\mu\nu}(\alpha, k^2) + c.c.}{(2\pi)^{D-1}} \]

\[
\psi^B_{\mu} = \int \frac{dD-2k}{(2\pi)^{D-1}} \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} e^{ikx^\mu - i\sqrt{\alpha^2 + k^2}t} \frac{\alpha y^{D/2} h_{D-1}(\alpha y) B_{\mu}(\alpha, k^2) + c.c.}{(2\pi)^{D-1}} \]

\[
\psi^C = \int \frac{dD-2k}{(2\pi)^{D-1}} \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} e^{ikx^\mu - i\sqrt{\alpha^2 + k^2}t} \frac{\alpha y^{D/2} h_{D-6}(\alpha y) C(\alpha, k^2) + c.c.}{(2\pi)^{D-1}} \]

where $h_l$ are spherical Hankel functions

\[
h_l(x) \equiv (-x)^l \left( \frac{1}{x} \frac{d}{dx} \right)^l \left( \frac{e^{ix}}{ix} \right)
\]

These functions satisfy recursion relations

\[
\frac{d}{dx} h_l(x) - \frac{l}{x} h_l(x) = -h_{l+1}(x)
\]

* Specifically, they are spherical Hankel functions of the first kind. Their complex conjugates $h_l^*$ are Hankel functions of the second kind.

** Note that for $D = 4$, (29c) involves $h_{-1}$. In this case we define $h_{-1} \equiv ih_0$, so that these relations work for $l = -1$ and $l = 1$ respectively.
\[
\frac{d}{dx} h_l(x) + \frac{i+1}{x} h_l(x) = h_{l-1}(x) \quad (31b)
\]

We then need to enforce the gauge condition \( F_\mu = 0 \). In terms of the polarization coefficients \( A, B, C \) this condition becomes

\[
iki^j A_{\bar{\mu} j} + i\sqrt{k^2 + \alpha^2} A_{\bar{\mu} j} - \alpha B_{\bar{\mu}} = 0 \quad (32a)
\]

and

\[
\frac{1}{D-3} \alpha A^\bar{\mu}_{\bar{\mu}} + i k^j B_{\bar{\mu}} + i\sqrt{k^2 + \alpha^2} B_{\bar{\mu}} + \frac{D-2}{D-3} \alpha C = 0 \quad (32b)
\]

The space of solutions still has a residual gauge invariance under transformations which preserve \( F_\mu = 0 \). We could use this residual invariance to solve (32) and eliminate some parameters. There are several ways to do this. One would be to set all the \( B \)’s and \( C \) to zero. Then we also need \( A \) to be traceless, and to satisfy \( k^j A_{\bar{\mu} j} + \sqrt{k^2 + \alpha^2} A_{\bar{\mu} j} = 0 \).

Alternatively we could set all timelike components zero: \( A_{\bar{\mu} j} = B_j = 0 \). Then we need to have \( B_{\bar{\mu}} = i k^j A_{\bar{\mu} j} \) and \( C = \frac{1}{D-2} \left( -A_{\bar{\mu}} - i\alpha(D-3) k^j B_{\bar{\mu}} \right) \). This last solution gives a Fock space of manifestly nonnegative norm.

These are the solutions for \( F = 0 \). We can also ask if there are any solutions with \( F \) nonzero. In order that such solutions satisfy both the invariant and fixed equations of motion, \( F \) must satisfy

\[
F_{(\mu,\nu)} - \frac{1}{2} \eta_{\mu\nu} F^\rho_{\rho,\nu} + \frac{D-2}{2y} \delta_{(\mu}^y F_{\nu)} + \frac{D-2}{4y} \eta_{\mu\nu} F^y = 0 \quad (33)
\]

The only solutions to this are

\[
F_\mu = y^{1-D/2} \delta_{\mu}^\bar{\mu} \left( c_{\bar{\mu}}^{\bar{\nu}} x_{\bar{\nu}} + d_{\bar{\mu}} \right) \quad (34)
\]

where \( c \) is antisymmetric. There are not any \( \psi \) solutions of the free equations which are bounded for \( |x^{\bar{\nu}}| \rightarrow \infty \) and give such results for \( F \).
The retarded Green’s function for our theory must satisfy

\[ D_{\mu \nu} \rho \sigma \left[ \rho \sigma G^{\alpha \beta}_{\text{ret}}(x, x') \right] = \delta_{\mu}^{(\alpha} \delta_{\nu}^{\beta)} \delta^{D}(x - x') \]  

(35)

It must also obey retarded boundary conditions. As we have discussed previously, we need to extend AdS to the covering space CAdS. The time coordinates \( t, t' \) determine which is the earlier point only in the case that the winding numbers \( N, N' \) are equal. If \( N < N' \) then \( x \) corresponds to an earlier event than \( x' \), and vice versa, regardless of \( t \) and \( t' \). So the correct retarded boundary condition is

\[ G(x, x') = 0 \text{ if } t < t' \text{ and } N = N' \]

and

\[ G(x, x') = 0 \text{ if } N < N' \]

We will proceed to solve this explicitly shortly, but first we discuss in general whether the solution to the gauge-fixed equations of motion generated by

\[ \psi_{\alpha \beta}(x) = \int d^D x' \left[ \alpha \beta G^{\mu \nu}_{\text{ret}}(x, x') \right] T_{\mu \nu}(x') \]  

(36)

must also solve the gauge-invariant equations of motion

\[ \frac{\delta S_{\text{Inv}}}{\delta \psi_{\alpha \beta}} = T_{\alpha \beta} \]  

(37)

when the source \( T \) satisfies the appropriate conservation law. This law is the requirement that

\[ \int d^D x T^{\alpha \beta}(x) \delta e_{\psi}(x) = 0 \]  

(38)

for any gauge parameters \( e_{\mu}(x) \). If \( T \) does not satisfy (38) then no solutions exist to (37).

Now consider the variation of the gauge-fixed action

\[ S_{\text{Fixed}} = S_{\text{Inv}} - \frac{1}{2} \int d^D x F_{\mu}(x) F^{\mu}(x) \]  

(39)
under a gauge transformation:

\[
\int d^D x \left( \delta e^\alpha \psi^\beta (x) \right) \frac{\delta S_{\text{Fixed}}}{\delta \psi^\alpha \psi^\beta (x)} = 0 - \int d^D x F_\mu (x) \delta e F^\mu (x)
\]  

(40)

Now evaluate this with \( \psi \) set equal to the solution (36). Then \( \frac{\delta S_{\text{Fixed}}}{\delta \psi^\alpha \psi^\beta (x)} \) is \( T^\alpha \beta (x) \) by the gauge-fixed equations of motion. Then the left side of (40) must be zero because \( T \) is a conserved current. Then so too must the right side, and integrating this by parts we see that we must have

\[
\left( -\partial^2 + \frac{D(D-2)}{4y^2} \right) F_{\mu} = 0
\]

(41a)

\[
\left( -\partial^2 + \frac{(D-2)(D-4)}{4y^2} \right) F_y = 0
\]

(41b)

It is easy to see that (33) implies (41), but not the other way around. If our solution also satisfies (33), then it will obey the invariant as well as gauge fixed equations of motion.

We now move on to the explicit determination of \( G \). After some analysis which exactly parallels that of the de Sitter case [5], we determine the tensor structure of \( G \):

\[
\left[ \rho \sigma G^\alpha \beta_{\text{ret}} \right] (x, x') = a(x, x') \left[ \rho \sigma T_A^\alpha \beta \right] + b(x, x') \left[ \rho \sigma T_B^\alpha \beta \right] + c(x, x') \left[ \rho \sigma T_C^\alpha \beta \right]
\]

(42)

where

\[
\left[ \rho \sigma T_A^\alpha \beta \right] \equiv 2 \delta^\alpha (\rho^\beta \sigma) - \frac{2}{D-3} \eta^\alpha \beta \eta_{\rho \sigma}
\]

(43a)

\[
\left[ \rho \sigma T_B^\alpha \beta \right] \equiv 4 \delta^\beta (\rho^\sigma \delta^\alpha \partial \partial y)
\]

(43b)

\[
\left[ \rho \sigma T_C^\alpha \beta \right] \equiv \frac{1}{D-2} \left[ \frac{2}{D-3} \eta_{\rho \sigma} \eta^\alpha \beta - 2 \delta^\delta \rho^\sigma \delta^\alpha \eta^\beta - 2 \eta_{\rho \sigma} \delta^\alpha \delta^\beta + 2(D-3) \delta^\delta \rho^\sigma \delta^\alpha \delta^\beta \right]
\]

(43c)

and \( a, b, c \) obey the equations

\[
D_0 a(x, x') = \delta^D (x - x')
\]

(44a)

\[
D_{D-2} b(x, x') = \delta^D (x - x')
\]

(44b)

\[
D_{2(D-3)} c(x, x') = \delta^D (x - x')
\]

(44c)
Because of the delta functions, we can write these equations in the symmetric form
\[(yy')^{2-D} \left( \frac{\partial^2 - \frac{D}{y} \frac{\partial}{\partial y} + \frac{n}{y^2}}{} \right) G_n = \delta^D(x - x') \] (45)

To solve these equations explicitly, we expand the delta functions:
\[
\delta^D(x - x') = \delta^{D-2}(\tilde{x} - \tilde{x}') \delta(y - y') \delta(t - t') \delta_{N,N'} \] (46a)
\[
\delta^{D-2}(\tilde{x} - \tilde{x}') = \left( \frac{1}{2\pi} \right)^{D-2} \int d^{D-2} k \, e^{ik \cdot (x - x')^2} \] (46b)
\[
\delta(y - y') = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \, \frac{\alpha^2 \, (yy')^{D/2}}{\sqrt{k^2 + \alpha^2}} h_l(\alpha y) h_l^*(\alpha y') \] (46c)
The expansion (46c) works for any \(l\), but we choose \(l = \sqrt{\frac{1}{4}(D - 1)^2 - n}\) to diagonalize \(D_n\).

We then similarly expand \(G_n\) as
\[
G_n(x, x') = \int \frac{d^{D-2} k}{(2\pi)^{D-2}} \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \, e^{ik \cdot (x - x')^2} \alpha^2 \, (yy')^{D/2} \frac{\sqrt{k^2 + \alpha^2}}{\alpha^2} h_l(\alpha y) h_l^*(\alpha y') \, g_n(t, t', \alpha, k^2) \] (47)

Then we find
\[
\left( -\frac{\partial^2}{\partial t^2} - k^2 - \alpha^2 \right) g_n = \delta(t - t') \delta_{N,N'} \] (48)

We will first solve this for \(N = N'\) and later generalize. This solution is
\[
g_n = -\frac{\sin(\sqrt{k^2 + \alpha^2}(t - t'))}{\sqrt{k^2 + \alpha^2}} \theta(t - t') \] (49)
so
\[
G_n = -\int \frac{d^{D-2} k \, d\alpha}{(2\pi)^{D-1}} \, e^{ik \cdot (x - x')^2} \alpha^2 \, (yy')^{D/2} \, \frac{\sin(\sqrt{k^2 + \alpha^2}(t - t'))}{\sqrt{k^2 + \alpha^2}} h_l(\alpha y) h_l^*(\alpha y') \, \theta(t - t') \] (50)
The evaluation of this integral depends strongly on the value of \(D\). We will now evaluate it for the physical case of \(D = 4\). To do so, we notice that our theory is Lorentz invariant in the flat subspace \(t, x^i\). If \((t - t')^2 - (x - x')^2 < 0\) then we can, without changing the background metric, go to a frame where \(t - t' = 0\), in which case we see that \(G_n = 0\). Otherwise, we can go to a frame where \(x^i - x'^i = 0\). Then
\[
G_n = -\int \frac{d^2 k \, d\alpha}{(2\pi)^3} \, \alpha^2 (yy')^2 \, \frac{\sin(\sqrt{k^2 + \alpha^2}(t - t'))}{\sqrt{k^2 + \alpha^2}} h_l(\alpha y) h_l^*(\alpha y') \, \theta(t - t') \] (51)
We write the $k$ integral in polar coordinates and substitute $u = \sqrt{k^2 + \alpha^2}$ to get

$$G_n = -\frac{(yy')^2}{t-t'} \theta(t-t') \int \frac{d\alpha}{(2\pi)^2} \alpha^2 \cos \alpha (t-t') \, h_l(\alpha y) h_l^*(\alpha y')$$

(52)

Now we look at this for the $n$ values of interest. First $n = 0$ gives $l = 1$. We get

$$G_0 = -\frac{yy'}{2(t-t')} \theta(t-t') \int \frac{d\alpha}{(2\pi)^2} \left( 1 + \frac{i(y'-y)}{yy'\alpha} + \frac{1}{\alpha^2} \right) \left( e^{i\alpha(t-t'+y-y')} + e^{i\alpha(-t+t'+y-y')} \right)$$

(53)

The first term in (53) just gives delta functions. The others can be evaluated by contour integration, and we get

$$G_0 = \frac{1}{4\pi} \theta(t-t') \left[ \theta(t-t' - |y-y'|) - \frac{yy'}{t-t'} (\delta(t-t'+y-y') + \delta(t-t'-y+y')) \right]$$

(54)

Now we need to generalize this result to $x^i - x'^i \neq 0$ and $N \neq N'$. The straightforward invariant and causal generalization is

$$G_0(x, x') = \frac{1}{4\pi} \left\{ \left[ \theta(t-t') \delta_{N,N'} + \theta(t'-t) \delta_{N,N'+1/2} \right] \left[ \theta(1 - z(x, x')) - \frac{1}{2} \delta(1 - z(x, x')) \right] ight. $$

$$+ \theta(t-t') \delta_{N,N'+1/2} + \theta(N - N' - \frac{3}{4}) \}$$

(55)

where we recall that $1 - z(x, x')$ is given by (16). (55) is constructed following the discussion at the end of the second section. The delta function term is a massless signal so it only propagates to the next half level. The theta function term is a timelike signal so its influence is felt on the next half level $N = N' + \frac{1}{2}$ at all times such that $t > t' - \sqrt{(x-x')^2(x-x')^2}$, and thereafter at all locations for $N \geq N' + 1$ (hence the final term with no theta function).

If we had used some sort of reflective boundary conditions then the delta function would need to be continued to all $N > N'$ (possibly with an alternating sign, depending on which type of boundary conditions were used).

So much for the case of $n = 0$. The other two cases are the same ($n = 2$) in $D = 4$. For $N = N'$ and $x^i = x'^i$ they evaluate to

$$G_2 = -\frac{yy'}{4\pi(t-t')} \theta(t-t') \left[ \delta(t-t'+y-y') + \delta(-t+t'+y-y') \right]$$

(56)
which generalizes to

\[ G_2(x, x') = -\frac{1}{8\pi} \left[ \theta(t - t')\delta_{N,N'} + \theta(t' - t)\delta_{N,N'+1/2} \right] \delta(1 - z(x, x')) \] (57)

Putting it all together with the tensor factors, the final answer for our Greens function is

\[ \left[ \rho\sigma G^{\alpha\beta} \right](x, x') = \frac{1}{4\pi} \left\{ \left[ \theta(t - t')\delta_{N,N'} + \theta(t' - t)\delta_{N,N'+1/2} \right] \left[ \theta(1 - z(x, x')) \left( 2\delta^{(\alpha}_{\rho} \delta^{\beta)}_{\sigma} - 2\pi\rho\sigma\eta^{\alpha\beta} \right) \right. \right. \\
\left. \left. - \frac{1}{2} \delta(1 - z(x, x')) \left( 2\delta^{(\alpha}_{\rho} \delta^{\beta)}_{\sigma} - \eta_{\rho\sigma}\eta^{\alpha\beta} \right) \right] \right. \\
\left. + \left[ \theta(t - t')\delta_{N,N'+1/2} + \theta(N - N' - \frac{3}{4}) \right] \left( 2\delta^{(\alpha}_{\rho} \delta^{\beta)}_{\sigma} - 2\pi\rho\sigma\eta^{\alpha\beta} \right) \right\} \] (58)

In the next section we show that this Greens function gives the correct anti de Sitter-Schwarzchild solution as its response to a single point mass.
Response to a Point Mass

Consider a point mass which is at rest at \( r = 0 \) in the static coordinates (7). These coordinates are related to our conformal coordinates as follows:

\[
y = \left( r \cos \theta + \sqrt{1 + r^2 \cos \tau} \right)^{-1} \tag{59a}
\]

\[
t = \sqrt{1 + r^2} \sin \tau \left( r \cos \theta + \sqrt{1 + r^2 \cos \tau} \right)^{-1} \tag{59b}
\]

\[
x^1 = r \sin \theta \cos \phi \left( r \cos \theta + \sqrt{1 + r^2 \cos \tau} \right)^{-1} \tag{59c}
\]

\[
x^2 = r \sin \theta \sin \phi \left( r \cos \theta + \sqrt{1 + r^2 \cos \tau} \right)^{-1} \tag{59d}
\]

We see that a point mass at \( r = 0 \) has \( x^1 = x^2 = 0 \), and \( y^2 = t^2 + 1 \). So the path is \( y = \pm \sqrt{t^2 + 1} \), the sign depending on whether we are on an \( N = \) integer level \( (y > 0) \) or \( N = \) half-integer level \( (y < 0) \). The action for a point particle of mass \( M \) which follows a spacetime path \( q^\mu(\tau) \) is

\[
S = -M \int d\tau \sqrt{-g_{\mu\nu} \dot{q}^\mu \dot{q}^\nu} \tag{60}
\]

The linearized energy-momentum tensor for the particle is then

\[
T^{\alpha\beta}(x) = -\frac{\delta S}{\delta \psi_{\alpha\beta}(x)} \bigg|_{\psi=0} \tag{61}
\]

Calculating this for our particle, we find the nonzero components of \( T \) are

\[
T^{tt}(x) = \mp \frac{1}{2} \kappa M \left( x^t \right)^2 \delta (y \mp \sqrt{t^2 + 1}) \tag{62a}
\]

\[
T^{ty}(x) = -\frac{1}{2} \kappa M \frac{t}{\sqrt{t^2 + 1}} \left( x^t \right)^2 \delta (y \mp \sqrt{t^2 + 1}) \tag{62b}
\]

\[
T^{yy}(x) = \mp \frac{1}{2} \kappa M \frac{t^2}{t^2 + 1} \left( x^t \right)^2 \delta (y \mp \sqrt{t^2 + 1}) \tag{62c}
\]

This \( T \) must satisfy the conservation law which we get from (38):

\[
T^{\mu\nu,\nu} + \frac{1}{y} \delta^\mu y T^{\nu}_\nu = 0 \tag{63}
\]

It is easily verified that (63) is in fact obeyed by (62).
We now wish to explicitly compute the response to the source $T(x)$:

$$\kappa \psi_{\rho \sigma}(x) = \kappa \int d^4 x' \left[ \rho \sigma G^{\alpha \beta} \right](x, x') T_{\alpha \beta}(x') \quad (64)$$

We can tell by the forms of (58) and (62) that this response will have the form

$$\kappa \psi_{\alpha \beta}(x) = \eta_{\alpha \beta} A(x) + \delta^t_{\alpha} \delta^t_{\beta} B(x) + \delta^y_{\alpha} \delta^y_{\beta} C(x) + \left( \delta^t_{\alpha} \delta^y_{\beta} + \delta^y_{\alpha} \delta^t_{\beta} \right) D(x) \quad (65)$$

Now we need to evaluate the functions $A$ through $D$. For definiteness we will take both $y$ and $t$ to be positive. We need to look at each term in (58) separately: First,

$$\left[ \rho \sigma G^1_{\alpha \beta} \right](x, x') = \frac{1}{4\pi} \left[ \theta(t - t') \delta_{N, N'} + \theta(t' - t) \delta_{N, N' + 1/2} \right] \theta(1 - z(x, x')) \left( 2 \delta^\rho_{\alpha} \delta^\sigma_{\beta} - 2 \bar{\eta}_{\rho \sigma} \bar{\eta}_{\alpha \beta} \right) \quad (66)$$

$G_1$ annihilates all components of $T$ except $T^{tt}$. Thus its contribution will be of the form $\bar{\eta}_{\rho \sigma} + \delta^t_{\rho} \delta^t_{\sigma}$. So $A_1 = B_1 = -C_1$, and $D_1 = 0$. Then we calculate

$$A_1 = -\frac{1}{4\pi} \kappa^2 M \int dt' \left[ \theta(t - t') \theta(2y \sqrt{t'^2 + 1 - 2tt' - l}) - \theta(-t + t') \theta(-2y \sqrt{t'^2 + 1 + 2tt' + l}) \right]$$

where

$$l \equiv -t^2 + x^i x^i + y^2 + 1 \quad (68)$$

(67) evaluates to

$$A_1 = -4MG \left( \frac{lt - ys}{2(y^2 - t^2)} + t \right) \quad (69)$$

with

$$s \equiv \sqrt{l^2 + 4(t^2 - y^2)} \quad (70)$$

Note that (70) gives a real $s$ for any $t$ and $y$. Now for the second part of $G$, which is:

$$\left[ \rho \sigma G^2_{\alpha \beta} \right](x, x') = -\frac{1}{8\pi} \left[ \theta(t - t') \delta_{N, N'} + \theta(t' - t) \delta_{N, N' + 1/2} \right] \delta(1 - z(x, x')) \left( 2 \delta^\rho_{\alpha} \delta^\sigma_{\beta} - \eta_{\rho \sigma} \eta_{\alpha \beta} \right) \quad (71)$$

When we contract this with (62), we get
\[
\frac{M\kappa^2}{2\pi y} \int dt' \sum_{\pm} \theta(\pm(t - t')) \delta(\pm 2y\sqrt{t'^2 + 1} - 2tt' - l) \\
\times \left[ \pm \left( \delta^t \delta^t + \frac{1}{2} \eta_{\rho\sigma} \right) \sqrt{t'^2 + 1} \pm \left( \delta^t \delta^t - \frac{1}{2} \eta_{\rho\sigma} \right) \frac{t'^2}{\sqrt{t'^2 + 1}} - \left( \delta^t \delta^t + \delta^y \delta^t \right) \right]
\] (72)

The delta function for the minus sign is not zero for any \( t' \) values for which \( \theta(-t + t') \neq 0 \).

For the plus sign, there is one such value, which is

\[
t'_0 = \frac{lt - ys}{2(y^2 - t^2)}
\] (73)

Then we use the identity

\[
\delta(f(t')) = \left| \frac{df}{dt'} \right|_{t'=t'_0}^{-1} \delta(t')
\] (74)

to find the results

\[
A_2 = 4MG \frac{y}{s}
\] (75a)

\[
B_2 = 4MG \frac{y(y - ts)^2}{2s(y^2 - t^2)^2}
\] (75b)

\[
C_2 = -2A_2 + B_2
\] (75c)

\[
D_2 = -4MG \frac{y(lt - ys)(ly - ts)}{2s(y^2 - t^2)^2}
\] (75d)

Finally, there is the third term in the Greens function

\[
\left[ \rho\sigma G_3^{\alpha\beta} \right](x, x') = \frac{1}{4\pi} \left[ \theta(t - t')\delta_{NN'+1/2} + \theta(N - N' - \frac{3}{4}) \right] \left( 2\delta_\rho^\alpha \delta_\sigma^\beta - 2\eta_{\rho\sigma}\eta^{\alpha\beta} \right)
\] (76)

This gives a contribution of the form

\[
-4mG \left( \eta_{\rho\sigma} + \delta_\rho^t \delta_\sigma^t \right) (T - t)
\] (77)

The constant \( T \) is formally divergent, but this divergence does not affect any physical observables. This is because it can be removed by a simple rescaling of the flat spatial coordinates:

\[ x^i \rightarrow (1 + 2\kappa^2 T)x^i \]

This divergence is a result of the unphysical assumption that our space
was anti de Sitter since infinitely long ago. If we had used reflective boundary conditions, there would have been additional divergent terms whose forms would be similar to (75). These do not seem to be removable by coordinate transformations.

For the form (65), we calculate
\[ \kappa F_\mu = \frac{1}{y} \left( - \frac{1}{y} B - \frac{1}{2} C \right)_\mu + \delta^t_\mu \left( - \frac{2}{y} A + C \right)_t - \frac{2}{y} C - D_t + \frac{2}{y} D \] (78)
which evaluates to
\[ \kappa F_\mu = \frac{4 m G}{y} \delta^t_\mu \] (79)
We can see that this \( F \), while not zero, satisfies the equation (33). Thus our solution will obey the gauge invariant equations of motion. One can check this directly by computing the first-order Riemann tensor for the solution (65):
\[ R^t_{yt} = - \frac{1}{y^2} y_y + \frac{1}{2y^2} \left( - y (B + C) . y + y^2 (A + B) . y + 2 y D_t - 2 y^2 D_y + y^2 (A + C) . t \right) \] (80a)
\[ R^t_{yi} = 0 \] (80b)
\[ R^t_{yt} = \frac{1}{2y} \left( - A + C + y (A - B) . y + y D_t \right) \] (80c)
\[ R^t_{yy} = \frac{1}{2y} \left( - D - y D_y + y (A + C) . t \right) \] (80d)
\[ R^t_{yi} = - \frac{1}{y^2} \delta^t_{ij} + \frac{1}{2} \left( - A + B \right) . j + \frac{1}{2y^2} \left( 2 C + 2 y A . y - y B . y + 2 y D_t + y^2 A . t \right) \] (80e)
\[ R^t_{iy} = \frac{1}{2} D . j + \frac{1}{2y} \left( A . t + C . t + y A . t \right) \] (80f)
\[ R^t_{ij} = \frac{1}{2} \left( - y D + A . t \right) \] (80g)
\[ R^y_{iy} = - \frac{1}{y^2} \delta^y_{ij} - \frac{1}{2} \left( A + C \right) . j + \frac{1}{2y^2} \left( 2 C - y C . y - y^2 A . y \right) \] (80h)
\[ R^y_{ij} = \frac{1}{2} \left( \frac{1}{y} A + \frac{1}{y} C + A . y \right) \] (80i)
\[ R^t_{jkl} = \frac{1}{y^2} \left( - 1 + C - \frac{y^2}{2} A . m m + y A . y \right) \] (80j)
where, as one might expect, \( \tau_{12} = -\tau_{21} = 1, \tau_{11} = \tau_{22} = 0 \). From (80), we obtain the Ricci tensor

\[
R_{tt} = \frac{3}{y^2} - \frac{3}{y^2}(B+C) + \frac{1}{2} \partial^2(A-B) + \frac{1}{2y}(-4A+C+3B),y - \frac{3}{y}D,t + D,ty - \frac{1}{2}(2A+B+C),tt
\]

(81a)

\[
R_{yt} = -\frac{3}{y}D - \frac{1}{2}\bar{\partial}_j D_{,ij} - \frac{1}{y}(A+C),t - A,t_y
\]

(81b)

\[
R_{yy} = -\frac{3}{y^2} - \frac{1}{2} \partial^2(A+C) - \frac{1}{2y}(3C + B),y + \frac{1}{y}(B + C),yy + \frac{1}{y}D,t - D,ty
\]

(81c)

\[
R_{ti} = \left( -\frac{1}{y}D + \frac{1}{2}D,y - A,t - \frac{1}{2}C,t \right) \bar{\partial}_i
\]

(81d)

\[
R_{yi} = \left( -\frac{1}{y}A - \frac{1}{y}C - A,y + \frac{1}{2}B,y - \frac{1}{2}D,t \right) \bar{\partial}_i
\]

(81e)

\[
R_{ij} = -\frac{3}{y^2} \bar{\partial}_{ij} + \left( -A - \frac{1}{2}C + \frac{1}{2}B \right) \bar{\partial}_{ij} + \left( -\frac{1}{2} \partial^2 A + \frac{3}{2y}C + \frac{1}{2y}(4A - B - C),y + \frac{1}{y}D,t \right) \bar{\partial}_{ij}
\]

(81f)

Substituting the explicit values for the coefficient functions, we can show

\[
R_{\mu\nu} = -\frac{3}{y^2} (\eta_{\mu\nu} + \kappa \psi_{\mu\nu})
\]

(82)

to linearized order.

This shows that the point mass response solves Einstein’s equations. Now we need to determine whether the solution generated is equivalent to the anti de Sitter-Schwarzchild solution. In static spherical coordinates \( \tau, r, \theta, \phi \), the anti de Sitter-Schwarzchild metric is

\[
g_{\tau\tau} = -1 - r^2 + \frac{2MG}{r}
\]

(83a)

\[
g_{rr} = \left( 1 + r^2 - \frac{2MG}{r} \right)^{-1} = \frac{1}{1 + r^2} + \frac{2MG}{r(1 + r^2)^2} + O(G^2)
\]

(83b)

\[
g_{\theta\theta} = r^2
\]

(83c)

\[
g_{\phi\phi} = r^2 \sin^2 \theta
\]

(83d)

with all off-diagonal components of \( g \) zero. Does a coordinate transformation exists from \( \tau, r, \theta, \phi \) to \( t, x^1, x^2, y \) such that (83) is transformed into our point mass solution? If
so, then it is given by (59) to lowest order, but there will will be order $MG$ corrections to (59) which we don’t know. This ambiguity in the order $MG$ coordinate transformations is related to the fact that we had a choice of gauge in solving for our Greens function. Making order $MG$ coordinate transformations on the point mass solution will give the same solution in a different gauge. It would be quite difficult to find the exact transformation which would give the solution in our gauge.

It is easier to show indirectly that our solution is equivalent to (83). We can do this by comparing the Weyl tensors for both solutions. This is given in general by (11); for a metric such as ours which obeys Einstein’s equations with $\Lambda = -3$, this is

$$C^\lambda_{\mu\nu\kappa} = R^\lambda_{\mu\nu\kappa} + \delta^\lambda_{\nu} g_{\mu\kappa} - \delta^\lambda_{\kappa} g_{\mu\nu} \quad (84)$$

Evaluating this for our solution, we find

$$C^t_{y12} = 0 \quad (85a)$$

$$C^t_{1t2} = 96MG \frac{x_1 x_2 y (t^2 + y^2)}{s^5} \quad (85b)$$

$$C^t_{1y2} = -192MG \frac{x_1 x_2 y^2}{s^5} \quad (85c)$$

$$C^t_{112} = -48MG \frac{x_2 y (1 + t^2 + x_1^2 + x_2^2 - y^2)}{s^5} \quad (85d)$$

$$C^y_{112} = -48MG \frac{x_2 y^2 (1 + t^2 + x_1^2 + x_2^2 - y^2)}{s^5} \quad (85e)$$

$$C^y_{1y1} = -8MGy \frac{s^2 + 12t^2 x_1^2 - 12y^2 x_2^2}{s^5} \quad (85f)$$

These are the independent components of $C$; the other components can be obtained from these by switching $x_1 \leftrightarrow x_2$, using the standard permutation symmetries of $C$, or using the fact that contracting any two indices of $C$ with the metric gives zero. Compare this with the first-order Weyl tensor obtained from the anti de Sitter-Schwarzschild solution:

$$C^{tttt} = -\frac{2MG}{r^3} \quad (86a)$$
\[ C^{\theta \tau \theta} = \frac{MG}{(r^2 + 1)r^5} \]  
\[ C^{\theta r \theta} = -\frac{MG(r^2 + 1)}{r^5} \]  
\[ C^{\phi r \phi} = -\frac{MG(r^2 + 1)}{r^5 \sin^2 \theta} \]  
\[ C^{\phi \tau \phi} = \frac{MG}{r^5(r^2 + 1) \sin^2 \theta} \]  
\[ C^{\phi \theta \phi} = \frac{2MG}{r^7 \sin^2 \theta} \]  

We have written $C$ with all indices up, to facilitate transforming it to conformal coordinates. The transformed $C$ is $C'$, where

\[ C'^{\alpha \beta \gamma \delta} = \frac{\partial x'^{\alpha}}{\partial x^\mu} \frac{\partial x'^{\beta}}{\partial x^\nu} \frac{\partial x'^{\gamma}}{\partial x^\rho} \frac{\partial x'^{\delta}}{\partial x^\sigma} C^{\mu \nu \rho \sigma} \]  

where $x^\mu = (\tau, r, \theta, \phi)$, and $x'^{\alpha} = (t, x_1, x_2, y)$. Because $C$ is zero for the background anti de Sitter metric, we need only use the zero-order transformations (59). The results of this are

\[ C^{t y 1 2} = 0 \]  
\[ C^{t 1 1 2} = -\frac{3MG \cos \phi \sin \phi \sin^2 \theta (1 + \sin^2 \tau + r^2 \sin^2 \tau)}{r^3 \left( r \cos \theta + \sqrt{1 + r^2 \cos \tau} \right)^6} \]  
\[ C^{t 1 2 2} = -\frac{6MG \sin \phi \cos \phi \sin^2 \theta \sin \tau \sqrt{1 + r^2}}{r^3 \left( r \cos \theta + \sqrt{1 + r^2 \cos \tau} \right)^6} \]  
\[ C^{t 1 1 2} = -\frac{3MG \sin \phi \sin \tau \sin \theta (r \sqrt{1 + r^2} + \cos \tau \cos \theta (r^2 + 1))}{r^3 \left( r \cos \theta + \sqrt{1 + r^2 \cos \tau} \right)^6} \]  
\[ C^{y 1 1 2} = -\frac{3MG \sin \phi \sin \theta (r + \cos \tau \cos \theta \sqrt{r^2 + 1})}{r^3 \left( r \cos \theta + \sqrt{1 + r^2 \cos \tau} \right)^6} \]  
\[ C^{y 1 y 1} = -\frac{MG}{r^3} \left[ \frac{1}{(r \cos \theta + \sqrt{1 + r^2 \cos \tau})^3} + 3 \sin^2 \theta \sin^2 \phi + (r^2 + 1) \sin^2 \tau \cos^2 \phi \right] \]  

(86b) (86c) (86d) (86e) (86f) (88a) (88b) (88c) (88d) (88e) (88f)
We then invert (59) to find

\[ r = \frac{s}{2|y|} \]  
(89a)

\[ \tau = \tan^{-1} \frac{2t}{l} \]  
(89b)

\[ \theta = \tan^{-1} \frac{\sqrt{x_1^2 + x_2^2}}{2 - l} \]  
(89c)

\[ \phi = \tan^{-1} \frac{x_2}{x_1} \]  
(89d)

Substituting these expressions into (88), and using the background metric \( y^{-2}\eta_{\alpha\beta} \) to lower the last three indices, we find that (88) is equal to (85). Thus, our solution is just the anti de Sitter-Schwarzchild solution in different coordinates.
Conclusions

We have determined the retarded Greens function for quantum gravity in the idealized case of a background which is anti de Sitter at all locations and all times. To do so, we needed to establish a boundary condition at spatial infinity (which corresponds to $y = 0$ in our conformal coordinate system). We mandated the boundary condition that any signal which reaches spatial infinity should be lost, not reflected back. We showed that this boundary condition gives the correct classical solution for the case of a point mass at the origin, up to of course a change of coordinates.

It is easy to go from the Greens function (58) to the Feynman propagator for gravity in anti de Sitter space for the case $N = N'$, up to real analytic terms which are annihilated by the kinetic operator. This will be

$$i \left[ \rho \sigma \Delta^{\alpha \beta} \right] (x, x') = \frac{1}{16\pi^2} \left\{ \frac{1}{1 - z(x, x') + i\epsilon} - \ln \left[ (x - x')^2 + i\epsilon \right] \right\}$$

(90)

As in the de Sitter case [5,9], the propagator cannot be made invariant under the anti de Sitter symmetry.

For $N \neq N'$ it is not clear how to obtain the propagator, because of the other terms in the Greens function. Loss of quantum coherence may be a problem here, since information which reaches spatial infinity ($y = 0$ in conformal coordinates) will be lost. We still advocate the boundary conditions we have used here rather than reflective ones, because as we have shown, our boundary conditions give the correct classical solution (at least for one specific case). This loss of coherence will not be present when we go to physically realistic situations. If it occurs in the case of infinite eternal anti de Sitter space, then it just serves to underscore the unphysicality of this mathematical model.

To proceed further, we would need to understand what modifications of this work would be necessary so that it might apply to a physically realistic situation. A negative cosmological constant might have existed in the physical universe as the result of a field theoretic phase
transition associated with a symmetry breaking. Such a phase transition alters the effective vacuum energy by redefining the vacuum. There are two differences between this realistic situation and our mathematical model. First, by causality the phase transition cannot occur throughout the whole universe at one time. Therefore a region of anti de Sitter space would be surrounded by normal space. Secondly, the universe would not have been anti de Sitter for all time, as our model space in this paper was. These features will likely make dealing with a realistic anti de Sitter phase more difficult than the simple mathematical model which we have solved in this paper.

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References

1) N. C. Tsamis and R. P. Woodard, *Phys. Lett.* **B301** (1993) 351; “Strong Infrared Effects in Quantum Gravity”, UFIFT-HEP-92-24/CRETE-92-17.

2) J. Minahan, P. Ramond, and R. Warner, *Phys. Rev. D* **41** (1990) 715.

3) L. Abbott, *Sci. Am.* **258** (1988) 106.

4) S. Weinberg, *Rev. Mod. Phys.* **61** (1989) 1.

5) N. C. Tsamis and R. P. Woodard, “The Structure of Perturbative Quantum Gravity on a De Sitter Background”, Florida preprint UFIFT-92-14, to appear in *Comm. Math. Phys.*
6) N. C. Tsamis and R. P. Woodard, *Phys. Lett.* **B292** (1992) 269.

7) N. C. Tsamis and R. P. Woodard, UFIFT-93-17/CRETE-93-11.

8) S. J. Avis, C. J. Isham, and D. Storey, *Phys. Rev. D* **18** (1978) 3565.

9) G. Kleppe, *Phys. Lett.* **B317** (1993) 305.