The Minimum Hartree Value for the Quantum Entanglement Problem

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Abstract. A general n-partite state $|\Psi\rangle$ of a composite quantum system can be regarded as an element in a Hilbert tensor product space $H = \otimes_{k=1}^{n} H_k$, where the dimension of $H_k$ is $d_k$ for $k = 1, \cdots, n$. Without loss of generality we may assume that $d_1 \leq \cdots \leq d_n$. A separable (Hartree) n-partite state $|\phi\rangle$ can be described by $|\phi\rangle = \otimes_{k=1}^{n} |\phi^{(k)}\rangle$ with $|\phi^{(k)}\rangle \in H_k$. We show that $\sigma := \min \{ \langle \Psi | \phi \Psi \rangle : |\Psi\rangle \in H, \langle \Psi | \Psi \rangle = 1 \}$ is a positive number, where $|\phi\Psi\rangle$ is the nearest separable state to $|\Psi\rangle$. We call $\sigma$ the minimum Hartree value of $H$. We further show that $\sigma \geq 1/\sqrt{d_1 \cdots d_{n-1}}$. Thus, the geometric measure of the entanglement content of $\Psi$, $\| |\Psi\rangle - |\phi\Psi\rangle | \leq \sqrt{2 - 2\sigma} \leq \sqrt{2 - 2 \left( 1/\sqrt{d_1 \cdots d_{n-1}} \right)}$.

Key Words. quantum entanglement, Hilbert tensor product space, separable (Hartree) state.

1 Introduction

The quantum entanglement problem is now regarded as a central problem in quantum information processing [2, 3, 6]. A general n-partite state $|\Psi\rangle$ of a composite quantum system can be regarded as an element in a Hilbert tensor product space $H = \otimes_{k=1}^{n} H_k$, where the dimension of $H_k$ is $d_k$ for $k = 1, \cdots, n$. Without loss of generality we may assume that $d_1 \leq \cdots \leq d_n$. A natural geometrical measure of the entanglement content of $|\Psi\rangle$ is the distance from its nearest separable (Hartree) state [6].

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farther away from the set of separable states, the more entangled a state is \[1, 6\]. A separable (Hartree) \(n\)-partite state \(|\phi\rangle\) can be described by \(|\phi\rangle = \otimes_{k=1}^{n} |\phi^{(k)}\rangle\) with \(|\phi^{(k)}\rangle \in \mathcal{H}_k\). In the next section, we show that \(\sigma := \min \{ \langle \Psi | \phi_\Psi \rangle : |\Psi\rangle \in \mathcal{H}, \langle \Psi | \Psi \rangle = 1 \}\) is a positive number, where \(|\phi_\Psi\rangle\) is the nearest separable state to \(|\Psi\rangle\). We call \(\sigma\) the minimum Hartree value of \(\mathcal{H}\). In Section 3, for \(n = 2\), we show that \(\sigma = 1/\sqrt{d_1}\). We further show in Section 4 that \(\sigma \geq 1/\sqrt{d_1 \cdots d_{n-1}}\) when \(n \geq 3\). Thus, the geometric measure of the entanglement content of \(\Psi\), \(\|\Psi\rangle - |\phi_\Psi\rangle\| \leq \sqrt{2} - 2\sigma \leq \sqrt{2} - 2\left(1/\sqrt{d_1 \cdots d_{n-1}}\right)\). Some final remarks are given in Section 5.

### 2 The Minimum Hartree Value

Let \(|\Psi\rangle\) be a general \(n\)-partite pure state of a composite quantum system. Then we may denote \(|\Psi\rangle \in \mathcal{H}\), where \(\mathcal{H}\) is a Hilbert tensor product space \(\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n\), and the dimension of \(\mathcal{H}_k\) is \(d_k\) for \(k = 1, \ldots, n\). We have \(\langle \Psi | \Psi \rangle = 1\). An arbitrary separable \(n\)-partite state \(|\phi\rangle \in \mathcal{H}\) can be described by \(|\phi\rangle = \otimes_{k=1}^{n} |\phi^{(k)}\rangle\), where \(|\phi^{(k)}\rangle \in \mathcal{H}_k\) and \(\| |\phi^{(k)}\rangle\| = 1\) for \(k = 1, \ldots, n\). Denote the set of all separable states in \(\mathcal{H}\) as \(\text{Separ}(\mathcal{H})\).

For a general \(n\)-partite state \(|\Psi\rangle \in \mathcal{H}\), a geometric measure of its entanglement content can be defined as \([6]\)

\[
d = \| |\Psi\rangle - |\phi_\Psi\rangle\| = \min \{ \| |\Psi\rangle - |\phi\rangle\| : |\phi\rangle \in \text{Separ}(\mathcal{H}) \}, \tag{2.1}
\]

where \(|\phi_\Psi\rangle \in \text{Separ}(\mathcal{H})\) is the nearest separable state of \(|\Psi\rangle\). Since the minimization in (2.1) was taken with a continuous function on a compact set in a finite dimensional space, the nearest separable state to \(|\Psi\rangle\) always exists.

For convenience, as in \([6]\), instead of studying (2.1), we may study

\[
d^2 = \| |\Psi\rangle - |\phi_\Psi\rangle\|^2 = \min \{ \| |\Psi\rangle - |\phi\rangle\|^2 : |\phi\rangle \in \text{Separ}(\mathcal{H}) \}. \tag{2.2}
\]

Note that

\[
\| |\Psi\rangle - |\phi\rangle\|^2 = \langle \Psi | \Psi \rangle + \langle \phi | \phi \rangle - \langle \Psi | \phi \rangle - \langle \phi | \Psi \rangle = 2 - \langle \Psi | \phi \rangle - \langle \phi | \Psi \rangle. \tag{2.3}
\]
Thus the minimization problem in (2.2) is equivalent to the following maximization problem:
\[
\max \left\{ \langle \Psi | \left( \bigotimes_{k=1}^{n} | \phi^{(k)} \rangle \right) | \Psi \rangle : \langle \phi^{(k)} | \phi^{(k)} \rangle = 1, k = 1, \cdots, n \right\}.
\]
(2.4)

Introducing Lagrange multipliers \( \lambda_k \) for \( k = 1, \cdots, n \), we have
\[
\langle \Psi | \left( \bigotimes_{j \neq k}^{n} | \phi^{(j)} \rangle \right) \rangle = \lambda_k \langle \phi^{(k)} | \phi^{(k)} \rangle \]
(2.5)
and
\[
\left( \bigotimes_{j \neq k}^{n} | \phi^{(j)} \rangle \right) | \Psi \rangle = \lambda_k | \phi^{(k)} \rangle.
\]
(2.6)

We see that
\[
\lambda \equiv \lambda_k = \langle \Psi | \phi \rangle = \langle \phi | \Psi \rangle
\]
is a real number in \([-1, 1]\) [6]. Then (2.5) and (2.6) become
\[
\langle \Psi | \left( \bigotimes_{j \neq k}^{n} | \phi^{(j)} \rangle \right) \rangle = \lambda \langle \phi^{(k)} | \phi^{(k)} \rangle
\]
(2.7)
and
\[
\left( \bigotimes_{j \neq k}^{n} | \phi^{(j)} \rangle \right) | \Psi \rangle = \lambda | \phi^{(k)} \rangle.
\]
(2.8)

Then the largest entanglement eigenvalue \( \lambda_* \) corresponds the nearest separable state \( | \phi_{\Psi} \rangle \), and is equal to the maximal overlap [6]:
\[
\lambda_* = \langle \Psi | \phi_{\Psi} \rangle = \max \{|\langle \Psi | \phi \rangle| : \phi \in \text{Separ}(\mathcal{H})\}.
\]
(2.9)

We now consider the function defined by the maximum function in (2.9):
\[
g(|z\rangle) := \max \{|\langle z | \phi \rangle| : \phi \in \text{Separ}(\mathcal{H})\},
\]
for \(|z\rangle \in \mathcal{H}\). We see that \( g(|z\rangle) \geq 0 \) and \( g(|z\rangle) = 0 \) if and only if \(|z\rangle = 0\).
Furthermore, for \(|z\rangle, |w\rangle \in \mathcal{H}\), we have \( g(|z\rangle + |w\rangle) \leq g(|z\rangle) + g(|w\rangle)\).
Hence, \( g \) defines a norm in the finite dimensional space \( \mathcal{H} \). Note that \( h(z) = \sqrt{\langle z | z \rangle} \) also defines a norm in \( \mathcal{H} \). According to the norm equivalence theorem in the finite dimensional space [4], \( \sigma := \min \{ \langle \Psi | \phi_{\Psi} \rangle : | \Psi \rangle \in \mathcal{H}, \langle \Psi | \Psi \rangle = 1 \} \) is a positive number. We call \( \sigma \) the minimum Hartree value of \( \mathcal{H} \). Thus, the geometric measure of the entanglement content of \(| \Psi \rangle\), \( \| | \Psi \rangle - | \phi_{\Psi} \rangle \| \leq \sqrt{2 - 2\sigma} \).

We now summarize this result in the following theorem:
Theorem 2.1. Let the minimum Hartree value of $H$ be defined as $\sigma := \min \{\langle \Psi | \phi \rangle : |\Psi\rangle \in H, \langle \Psi | \Psi \rangle = 1\}$, where $|\phi\rangle$ is the nearest separable state to $|\Psi\rangle$. Then $\sigma > 0$, and for any $|\Psi\rangle \in H$, we have $\langle \Psi | \phi \rangle \geq \sigma$.

Furthermore, the geometric measure of the entanglement content of $|\Psi\rangle$, $\|\|\Psi\rangle - |\phi\rangle\| \leq \sqrt{2 - 2\sigma}$.

3 The Minimum Hartree Value when $n = 2$

We assume that $n = 2$. Let $|e_i\rangle$ for $i = 1, \cdots, d_1$ be an orthonormal basis for $H_1$, and $|s_j\rangle$ for $j = 1, \cdots, d_2$ be an orthonormal basis for $H_2$. Write

$$|\Psi\rangle = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} a_{ij} |e_i\rangle |s_j\rangle,$$

$$|\phi^{(1)}\rangle = \sum_{i=1}^{d_1} u_i |e_i\rangle$$

and

$$|\phi^{(2)}\rangle = \sum_{j=1}^{d_2} v_j |s_j\rangle,$$

where the overbar denotes conjugation. Then (2.7) has the form

$$a_{ij} v_j = \lambda u_i$$

and

$$a_{ij} u_i = \lambda v_j.$$

Then $A = (a_{ij})$ is a $d_1 \times d_2$ matrix, $u = (u_i)$ is a $d_1$-dimensional vector and $v = (v_j)$ is a $d_2$-dimensional vector. We have [2]

$$Au = \lambda u \quad \text{and} \quad A^\dagger u = \lambda v,$$

where the dagger denotes the Hermitian conjugate. Then as in [2], we see that $\lambda$ is a singular value of $A$, and $\lambda_*$ is the largest singular value of $A$.

On the other hand, since $\langle \Psi | \Psi \rangle = 1$, we see that

$$\sum_{i=1}^{d_1} \sum_{j=1}^{d_2} |a_{ij}|^2 = 1.$$

By linear algebra, $\sigma$, the minimum value of $\lambda_*$ for all $|\Psi\rangle \in H$ with $\langle \Psi | \Psi \rangle = 1$, is $1/\sqrt{d_1}$. Thus, we have the following theorem:

Theorem 3.1. For $n = 2$, the minimum Hartree value $\sigma = 1/\sqrt{d_1}$.

Let $a_{jj} = 1/\sqrt{d_1}$ and $a_{ij} = 0$ if $i \neq j$. Then we see that $|\Psi\rangle$ is a pure state and $\lambda_* = 1/\sqrt{d_1} = \sigma$, i.e., the value $\sigma = 1/\sqrt{d_1}$ is attainable.
4 A Lower Bound for the Minimum Hartree Value when \( n \geq 3 \)

In general, let \( |e_{i_k}^{(k)}\rangle \) for \( i_k = 1, \ldots, d_k \) be an orthonormal basis for \( \mathcal{H}_k \), \( k = 1, \ldots, n \). Write

\[
|\Psi\rangle = \sum_{i_1, \ldots, i_n} a_{i_1 \cdots i_n} |e_{i_1}^{(1)}\rangle \cdots |e_{i_n}^{(n)}\rangle, \\
|\varphi^{(k)}\rangle = \sum_{i_k=1}^{d_k} u_{i_k}^{(k)} |e_{i_k}^{(k)}\rangle.
\]

Let \( A \) be a hypermatrix defined by \( A = (a_{i_1 \cdots i_n}) \). By (2.9), we have

\[
\lambda_* \equiv \sigma(A) := \max \left\{ \left| \sum_{i_1 \cdots i_n} a_{i_1 \cdots i_n} u_{i_1}^{(1)} \cdots u_{i_n}^{(n)} \right| : \|u^{(k)}\| = 1, \text{ for } k = 1, \ldots, n \right\}.
\]

Define matrix \( A_{i_1 \cdots i_{n-2}} = (A_{i_1 \cdots i_{n-2}ij}) \) by \( A_{i_1 \cdots i_{n-2}ij} := a_{i_1 \cdots i_{n-2}ij} \). By (4.1),

\[
\sigma(A_{i_1 \cdots i_{n-2}}) \leq \sigma(A).
\]

Let \( \|A\| \) and \( \|A_{i_1 \cdots i_{n-2}}\| \) be the Frobenius norm of \( A \) and \( A_{i_1 \cdots i_{n-2}} \) respectively, i.e.,

\[
\|A\|^2 = \sum_{i_1, \ldots, i_n} |a_{i_1 \cdots i_n}|^2,
\]

and

\[
\|A_{i_1 \cdots i_{n-2}}\|^2 = \sum_{i_{n-1}, i_n} |a_{i_1 \cdots i_n}|^2.
\]

By linear algebra, we have

\[
\|A_{i_1 \cdots i_{n-2}}\|^2 \leq d_{n-1} \sigma(A_{i_1 \cdots i_{n-2}}).
\]

Since \( \langle \Psi | \Psi \rangle = 1 \), we have \( \|A\| = 1 \). Putting all of these together, we have

\[
1 = \|A\|^2 = \sum_{i_1, \ldots, i_{n-2}} \|A_{i_1 \cdots i_{n-2}}\|^2 \leq \sum_{i_1, \ldots, i_{n-2}} d_{n-1} \sigma(A_{i_1 \cdots i_{n-2}})^2 \leq \sum_{i_1, \ldots, i_{n-2}} d_{n-1} \sigma(A)^2 = d_1 \cdots d_{n-1} \lambda_*^2.
\]
By this and the definition of $\sigma$, we have
$$\sigma \geq 1/\sqrt{d_1 \cdots d_{n-1}}.$$ 

We now have the following theorem:

**Theorem 4.1.** In general, the minimum Hartree value $\sigma \geq 1/\sqrt{d_1 \cdots d_{n-1}}$. 

By (2.3), the geometric measure of the entanglement content of $|\Psi\rangle$, $\|\Psi\rangle - |\phi_\Psi\rangle\| \leq \sqrt{2 - 2\sigma} \leq \sqrt{2 - 2 \left(1/\sqrt{d_1 \cdots d_{n-1}}\right)}$.

5 Final Remarks

The discussion here follows the spirit of the discussion of the best rank-one approximation ratio in [5]. The best rank-one approximation ratio discussion in [5] only deals with real values vectors and hypermatrices. Also, the minimum Hartree value here has explicit physical meanings. These are the differences. This also stimulates further research to find the exact value of $\sigma$ when $n \geq 3$.

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