Crystalline realizations of 1-motives

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Abstract

We consider the crystalline realization of Deligne’s 1-motives in positive characteristics and prove a comparison theorem with the De Rham realization of (formal) liftings to zero characteristic.

We then show that one dimensional crystalline cohomology of an algebraic variety, defined by forcing universal cohomological descent via de Jong’s alterations, coincides with the crystalline realization of the Picard 1-motive, over perfect fields of characteristic $> 2$.

Grothendieck’s vision of crystals is well explained in [11] (see also [16] and the subsequent papers by Berthelot and Messing [3]). Grothendieck pointed out that one can recover the first De Rham cohomology of an abelian scheme in characteristic zero via the Lie algebra of the universal $G_a$-extension of the dual. Moreover, in positive characteristics, this universal $G_a$-extension and the Poincaré biextension are crystalline in nature and depend only on the $p$-divisible group associated to the abelian scheme. Recall the following classical results.

If $G$ is any group scheme over a base scheme $S$ such that $\text{Hom}(G, G_a) = 0$ and $\text{Ext}(G, G_a)$ is a locally free $\mathcal{O}_S$-module of finite rank, the universal $G_a$-extension $E(G)$ exists: it is an extension of $G$ by the vector group $\text{Ext}(G, G_a)^\vee$.

Theorem A ([11, Prop. I.4.1.4 & § II.1.5]): Let $A_0$ be an abelian scheme over a perfect field $k$ of characteristic $p > 0$. Let $A$ be a (formal) lifting to the Witt vectors $\mathbf{W}(k)$ and let $A^\vee$ be the dual. Let $T_{\text{crys}}(A_0)$ be the contravariant Dieudonné module associated to the Barsotti-Tate group $A_0[p^\infty]$. Then

$$T_{\text{crys}}(A_0) \cong \text{Lie} E(A^\vee) \cong H^1_{\text{DR}}(A/\mathbf{W}(k))$$

Providing a good definition of crystalline topology (cf. [4]) one can recover one dimensional crystalline cohomology from the above. We then also have:

Theorem B ([11 II.3.11.2]): Let $X$ be smooth and proper over a perfect field $k$ of characteristic $p > 0$. Let $\text{Pic}^{0,\text{red}}(X)$ be the abelian Picard scheme. Let $T_{\text{crys}}(-)$ denote the covariant Dieudonné module. Then

$$T_{\text{crys}}(\text{Pic}^{0,\text{red}}(X)) \cong H^1_{\text{crys}}(X/\mathbf{W}(k))$$

A geometric way to prove it is by applying the former to the universal extension of a lifting, considering the Albanese variety $\text{Alb}(X) = \text{Pic}^{0,\text{red}}(X)^\vee$. Let $\text{Pic}_{\text{crys}}^{0,0}(X)$ be the sheaf on the (small) fppf site on $\mathbf{W}(k)$ given by the functor associating to $T$ the group of isomorphism
classes of crystals of invertible $\mathcal{O}^{\text{crys}}_{X \times \mathbf{W}(k)/T}$-modules (which are algebraically equivalent to 0 on the Zariski site on $X \times \mathbf{W}(k)/T$). By construction we have that $\text{Lie Pic}^{\text{crys}, 0} = H^1_{\text{crys}}$, inducing a canonical isomorphism

$$\text{Lie Pic}^{\text{crys}, 0}(\text{Alb}(X)) \xrightarrow{\sim} \text{Lie Pic}^{\text{crys}, 0}(X)$$

by the Albanese mapping. Furthermore, $\text{Lie Pic}^{\text{crys}, 0}(\text{Alb}(X))$ can be identified to the Lie algebra of the universal extension of a (formal) lifting of $\text{Pic}^{0, \text{red}}(X)$ to the Witt vectors. Note that $\text{Pic}^{\text{crys}, 0}$ is the natural substitute of the usual functor $\text{Pic}^{0, \sigma}$ given by invertible sheaves with an integrable connection, and recall that, in general, in positive characteristics, $H^1_{\text{DR}}(X/k)$ cannot be recovered from the Picard scheme, as illustrated by Oda [19].

Another application of Grothendieck’s vision is provided by Deligne’s definition of the De Rham realization of a 1-motive (see [7] and [2]). In fact, the universal $\mathbf{G}_a$-extension $\mathcal{E}(\mathbb{M})$ of a 1-motive $\mathbb{M}$ over any base scheme exists and the De Rham realization is naturally defined via the Lie algebra (see [7] 10.1.10-11, cf. Section 2 below).

The results

The purpose of this paper, after providing a 1-motive of the crystalline realization, is to draw the picture above in the motivic world, over a perfect field $k$.

The target category of our realizations is the category of filtered $F$-crystals of level 1 over $k$, for simplicity referred to as filtered $F$-crystals in the sequel. Actually, we consider the category of filtered $F$-$\mathbf{W}(k)$-modules consisting of finitely generated $\mathbf{W}(k)$-modules endowed with an increasing filtration and a $\sigma$-linear operator, the Frobenius $F$, respecting the filtration (here $\sigma$ is the Frobenius on $\mathbf{W}(k)$). Such category is, in an obvious way, a tensor category. Filtered $F$-crystals are the objects whose underlying $\mathbf{W}(k)$-modules are free and there exists a $\sigma^{-1}$-linear operator, the Verschiebung $V$, such that $V \circ F = F \circ V = p$. We let $\mathbf{W}(k)(1)$ be the filtered $F$-crystal $\mathbf{W}(k)$, with filtration $W_n = \mathbf{W}(k)$ if $n \geq -2$ and $W_n = 0$ for $n < -2$ and with the $\sigma$-linear operator $F$ given by $1 \mapsto 1$ and the $\sigma^{-1}$-linear operator $V$ defined by $1 \mapsto p$.

For a 1-motive $\mathbb{M}$ over a perfect field define $\mathbb{M}[p^n]$ as $H^{-1}(\mathbb{M}/p^n\mathbb{M})$, where $\mathbb{M}/p^n\mathbb{M}$ is the cone of multiplication by $p^n$ on $\mathbb{M}$. Then get a Barsotti-Tate group $\mathbb{M}[p^\infty]$ taking the direct limit. Passing to its contravariant Dieudonné module we get a filtered $F$-crystal over $k$ which we denote $\mathbf{T}^{\text{crys}}(\mathbb{M})$ (see Section 1 for details). We note that such a realization is an instance (the good reduction case) of a more general theory of $p$-adic realizations of 1-motives over a local field due to J.-M. Fontaine (to appear in [3]).

On the other hand, following Deligne, define $\mathbf{T}^{\text{DR}}(\mathbb{M})$ by $\text{Lie } \mathcal{E}(\mathbb{M}^\vee)$ where $\mathbb{M}^\vee$ is the Cartier dual (see Section 2, cf. [12] for duals and [21] for extensions). We show that for $p > 2$ the universal extension and the Poincaré biextension are crystalline (see Section 3). Furthermore, $\mathbf{T}^{\text{DR}}(-)$ yields a filtered crystal (no restriction on $p$), providing the Dieudonné crystal of the 1-motive (in the terminology of [10], see [5,3]). We then have the following comparison theorem.

**Theorem A’**: Let $\mathbb{M}_0$ be a 1-motive over a perfect field $k$ of positive characteristic $p > 0$. Choose a (formal) 1-motive $\mathbb{M}$ over $\mathbf{W}(k)$ lifting $\mathbb{M}_0$. Then, there is a canonical isomorphism

$$\mathbf{T}^{\text{crys}}(\mathbb{M}_0) \cong \mathbf{T}^{\text{DR}}(\mathbb{M})$$

of filtered $F$-crystals. Let $\mathbf{T}^{\text{crys}}(\mathbb{M}_0)$ be the crystal given by $\mathbf{T}^{\text{crys}}(\mathbb{M}_0^\vee)$. There is a bilinear perfect pairing of filtered $\mathbf{W}(k)$-modules

$$\mathbf{T}^{\text{crys}}(\mathbb{M}_0) \otimes \mathbf{W}(k) \rightarrow \mathbf{W}(k)(1).$$
See \[2,1\] for the definition of the filtration on \(T^{\text{DR}}(M)\) and \[1,3\] for the definition of the filtration on \(T^{\text{crys}}(M_0)\). The proof of Theorem A’ is given in Section 4 (see [1,3] for a list of properties of \(T^{\text{crys}}\)) and it ‘mostly’ adorns Grothendieck’s original proof for abelian schemes, cf. [16]. We stress that the description of \(T^{\text{crys}}(M_0)\) in terms of universal extensions is essential to prove Theorem B’ below.

Now let \(V\) be an algebraic variety over a perfect field \(k\). Applying de Jong’s method [6] we obtain a pair \((X, Y)\) where \(X\) is a smooth proper simplicial scheme, \(Y\) is a normal crossing divisor in \(X\), and \(V := X - Y\) is a smooth hypercovering of \(V\). Note that we may and will assume that \(V_0 \to V\) is generically étale by [6] Thm. 4.1. Let \(\text{Pic}^+(V) := \text{Pic}^+(X, Y)\) be the Picard 1-motive of \(V\) (see Section 5 and Appendix [A] cf. [2] and [22]). In the same way, as de Jong suggested in [6] p. 51-52, we set \(H^*_\text{crys}(V/W(k)) := H^*_\text{logcrys}(X, Y)\) where \((X, Y)\) here denotes the simplicial logarithmic structure on \(X\), determined by \(Y\). (see Section 6). We show in [6,6] that \(H^*_\text{logcrys}(X, Y)\) is naturally a free filtered \(F\)-\(W(k)\)-module. We have the following link.

**Theorem B’:** Let \(V\) be an algebraic variety over a perfect field \(k\) of characteristic \(p \geq 3\). There is a functorial isomorphism of filtered \(F\)-\(W(k)\)-modules

\[
T^{\text{crys}}(\text{Pic}^+(V)) \cong H^1_{\text{crys}}(V/W(k))(1).
\]

The proof of Theorem B’ is the full Section 7 and, very roughly speaking, goes as follows. By the comparison Theorem A’, we are left to compute the universal extension crystal of the 1-motive \(\text{Pic}^+(V)\). Thus, one gets to deal with a simplicial version of the functor of invertible \(\text{Pic}^{\text{logcrys}}\), showing that its Lie algebra coincides with \(H^1_{\text{logcrys}}\) (see 7.3–7.5).

It is shown in the Appendix [A] that \(\text{Pic}^+(V)\) is independent of the choices made, i.e., of the pair \((X, Y)\), and in particular functorial in \(V\). Thus via Theorem B’, see Corollary 7.5.4

\(H^1_{\text{crys}}(V/W(k))\) is independent of the chosen hypercovering \(V \to V\) so that one can define

\[H^1_{\text{crys}}(V/W(k)) := H^1_{\text{crys}}(V/W(k))\]

forcing descent for crystalline cohomology. This answers a question raised in [6] p. 52 concerning the independence of \(H^i_{\text{crys}}(V/W(k)) \otimes \mathbb{Q}\) from the choice of hypercoverings, at least for \(i = 1\) and \(p \geq 3\).

For the sake of exposition we omit the case \(p = 2\) deserving an \textit{ad hoc} explanation. Actually, in this case, the independence of \(\text{Pic}^+(V)\) from the choices made and \(H^1_{\text{crys}}(V/W(k))\) are still in place. Some technical difficulties in proving Theorem B’ arise from the fact that the standard divided power structures on the ideal \(2W(k)\) are not topologically nilpotent; see [6,4]. We will treat these matters elsewhere.

Note that the abelian quotient of \(\text{Pic}^+(V)\) is the (reduced) identity component of the kernel of the canonical map \(\text{Pic}^{0,\text{red}}(X_0) \to \text{Pic}^{0,\text{red}}(X_1)\). Moreover, for \(V\) a normal proper scheme \(\text{Pic}^+(V) = \text{Pic}^{0,\text{red}}(V)\) is abelian, e.g., for every prime \(p\) we obtain from Theorem B’

\[H^1_{\text{crys}}(V/W(k)) \cong \text{Ker}(H^1_{\text{crys}}(X_0) \to H^1_{\text{crys}}(X_1))\]

if \(V\) is a normal projective variety.

Recall that the Cartier dual of \(\text{Pic}^+(V)\) is the (homological) Albanese 1-motive \(\text{Alb}^{-}(V)\) see [2] (cf. [22]), e.g., for \(V\) a normal proper scheme is the cokernel of the map \(\text{Alb}(X_1) \to \text{Alb}(X_0)\).
Corollary: There is a functorial $\mathbf{W}(k)$-linear isomorphism

$$T_{\text{crys}}(\text{Alb}^-(V)) \xrightarrow{\simeq} \text{Hom}(H^i_{\text{crys}}(V/\mathbf{W}(k))(1), \mathbf{W}(k)(1)).$$

This suggests that one can define the first crystalline homology group (modulo torsion!) as $H^i_{\text{crys}}(V/\mathbf{W}(k)) := \text{Hom}(H^i_{\text{crys}}(V/\mathbf{W}(k))(1), \mathbf{W}(k)(1))$. Finally note that Theorem B' can be reformulated in terms of the first rigid cohomology group of $V$, i.e.,

$$T_{\text{crys}}(\text{Pic}^+(V)) \otimes \mathbb{Q} \xrightarrow{\simeq} H^1_{\text{crys}}(V/\mathbf{W}(k))(1) \otimes \mathbb{Q} \xrightarrow{\simeq} H^1_{\text{rig}}(V/\text{Frac}(\mathbf{W}(k)))(1).$$

The last isomorphism follows using the recent proof of cohomological descent of rigid cohomology for proper coverings due to N. Tsuzuki [27] and the comparison between log-crystalline cohomology and rigid cohomology (for proper and smooth schemes with strict normal crossing divisors) due to A. Shiho [24]. This suggests, as remarked by B. Chiarellotto and K. Joshi, that one could approach Theorem B' using rigid cohomology. On the other hand, our approach allows to prove the version of Theorem B' with integral coefficients.

In the same spirit, the Corollary can be rephrased for rigid homology. However, if we compare to [21], we find that $H^*_{\text{crys}}(V/\text{Frac}(\mathbf{W}(k))) \neq H^*_{\text{rig}}(V/\text{Frac}(\mathbf{W}(k)))$ in general, since the rigid homology defined in [21, §2] is the Borel-Moore variation of crystalline homology, e.g., they coincide for proper schemes.

Open problems

A natural question is to extend our crystalline realization functor to mixed motives. Recall that the category of 1-motives (up to isogeny) is embedded in Voevodsky’s triangulated category of motives. However, for 1-motives over a perfect field, the more striking question is to prove the following claim.

Let $(X, Y)$ be a pair such that $Y$ is a closed subvariety of an algebraic variety $X$ over a perfect field. In [1], over an algebraically closed field of characteristic zero, effective 1-motives with torsion $\text{Pic}^+(X, Y; i) (= M_{i+1}(X, Y)$ in [Π]) are constructed showing Deligne’s conjecture [7, 10.4.1], e.g., showing the algebraicity of the maximal mixed Hodge structure of type $(0, 0), (0, -1), (-1, 0), (-1, -1)$ contained in $H^{i+1}(X, Y; \mathbb{Q}(1))$ if $k = \mathbb{C}$. By using an appropriate resolution it is easy to modify the construction yielding $\text{Pic}^+(X, Y; i)$ over a perfect field, such that $\text{Pic}^+(X, 0; 0) = \text{Pic}^+(X)$ above (however, it is not clear, for $i > 0$, if $\text{Pic}^+(X, Y; i)$ is integrally well defined!). Furthermore, it is easy to obtain a Barsotti-Tate crystal of an effective 1-motive, using Fontaine’s theory [8]. Note that now we deal with finite groups and therefore we can lift to the Witt vectors only the free part of an effective 1-motive. However, we may define $T_{\text{crys}}(-)$ by means of the covariant Dieudonné module, i.e., given by the Dieudonné module of the dual of the associated formal $p$-group.

Conjecture C: Provide a weight filtration $W_*$ on the crystalline cohomology $H^*_\text{crys}((X, Y)/\mathbf{W}(k))$ of the pair $(X, Y)$. Let $H^i_{\text{crys},(1)}((X, Y)/\mathbf{W}(k))$ denote the submodule of $W_i H^*_\text{crys}((X, Y)/\mathbf{W}(k))$ whose image in $\text{Gr}^W_2$ is generated by the image of the discrete part of $\text{Pic}^+(X, Y; i)$ under a suitable cycle map. Then there is a canonical isomorphism (eventually up to $p$-power isogenies)

$$T_{\text{crys}}(\text{Pic}^+(X, Y; i)) \xrightarrow{\simeq} H^{i+1}_{\text{crys},(1)}((X, Y)/\mathbf{W}(k))(1)$$

of filtered $F\cdot \mathbf{W}(k)$-modules.
The corresponding statement for De Rham cohomology over a field of characteristic zero is Theorem 3.5 in [1]. Theorem B’ above corresponds to this statement for \( i = 0 \) and \( Y = \emptyset \). Note that only the free part of \( \text{Pic}^+(X, Y; i) \) is independent of the hypercoverings in char. 0 (cf. [1] 2.5 and [2] 4.4.4). Here \( H^*_{\text{sys}}((X, Y)/W(k)) \) is defined following [3]. However, by dealing with rigid cohomology Conjecture C, can be rephrased switching crystalline to rigid. Note that a cycle map for rigid cohomology is fully described in [21].

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Notation

We let our base schemes be locally noetherian. For a perfect field \( k \) we let \( W(k) \) (resp. \( W_n(k) \) with \( n \in \mathbb{N} \)) denote the ring of (truncated) Witt vectors with the standard divided power structure on its maximal ideal. Note that for \( p = 2 \) the standard divided power structure on the maximal ideal of \( W_n(k) \) is not nilpotent. For \( S_0 \) a base scheme such that \( p \) is locally nilpotent we let \( S_0 \to S \) be a thickening defined by an ideal \( I \) with nilpotent divided powers.

Let \( G \) denote a group scheme over \( S \). We consider \( G \) as a sheaf for the fppf topology on \( S \). We sometimes denote \( g \in G \) an \( S \)-point. Denote \( G_a \) and \( G_m \) the usual additive and multiplicative structure \( S \)-group schemes.

If \( G \) is a \( p \)-divisible group over a perfect field \( k \) of characteristic \( p \), we denote by \( D(G) \) the contravariant Dieudonné module of \( G \) defined as the module over the Dieudonné ring \( D_k := W(k)[F, V]/(FV = VF = p) \) of homomorphisms from \( G \) to the group of Witt covectors over \( k \); see [8, §III.1.2]. By [8, §III.6.1] such functor defines an antiequivalence from the category of \( p \)-divisible groups over \( k \) to the category of \( D_k \)-modules which are finite and free as \( W(k) \)-modules. By the comparison Theorem [10, Thm. 15.3] \( D(G) \) coincides with the Lie algebra of the universal extension of a lifting of the dual \( p \)-divisible group \( G^\vee \) to \( W(k) \). For the latter approach to contravariant Dieudonné theory we refer the reader to [17, IV.2.4.3] or [16, §9.2].

For a simplicial \( S \)-scheme \( X \), we denote \( d^j_i \) the faces \( X_j \to X_{j-1} \), e.g., \( d^2_0, d^2_1, d^2_2 : X_2 \to X_1 \), over \( S \), and sometimes we omit the \( j \)-index when it is clear from the context.

1  Barsotti-Tate crystal of a 1-motive

First recall some definitions and constructions. We refer to [7, §10] and [2, §1] for more details on 1-motives.

1.1 Deligne’s 1-motives

Recall that algebraic 1-motives are originally defined in [7, Définition 10.1.2] over an algebraically closed field. However, in [7, Variante 10.1.10] (cf. [2]), the definition of 1-motif lisse is taken over arbitrary base schemes, as follows. Let \( S \) be a scheme. A 1-motive \( \mathbb{M} \) over \( S \), is

a) a group scheme \( \mathbb{X} \) over \( S \) which étale locally on \( S \) is constant, free and of finite type as \( \mathbb{Z} \)-module;
b) a semi-abelian scheme $G$ over $S$, extension of an abelian scheme $A$ over $S$ by a torus $T$ over $S$;

c) a homomorphism of $S$-group schemes $u: X \rightarrow G$.

As customary, we view the category of commutative $S$-group schemes as a full sub-category of the derived category $D^b_{fppf}(S)$, identifying an $S$-group scheme with the complex having its underlying fppf sheaf concentrated in degree $0$. Analogously, we identify a 1-motive $M$ with the complex in $C^b_{fppf}(S)$

$$\cdots \rightarrow 0 \rightarrow X \xrightarrow{u} G \rightarrow 0 \rightarrow \cdots$$

of fppf sheaves over $S$ concentrated in degree $-1$ and $0$. One sees (cf. [2]) that

$$\text{Hom}_{C^b_{fppf}(S)}(M_1, M_2) = \text{Hom}_{D^b_{fppf}(S)}(M_1, M_2).$$

By [23, Prop. 2.3.1] the morphisms of 1-motives, from $[X_1 \xrightarrow{u_1} G_1]$ to $[X_2 \xrightarrow{u_2} G_2]$, also as objects of $D^b_{fppf}(S)$, are morphisms of complexes, i.e., given by pairs of homomorphisms $X_1 \rightarrow X_2$ and $G_1 \rightarrow G_2$ of $S$-group schemes making the following diagram commute. Given a 1-motive $M$ we define a weight filtration $W_0(M) := M$, $W_{-1}(M) := [0 \rightarrow G]$, $W_{-2}(M) := [0 \rightarrow T]$ and $W_{n< -2}(M) = 0$.

### 1.2 Cartier duality

Cartier duality is naturally extended to 1-motives over a field (see [7, 10.2.11] and cf. [2, §1.5]). Over a base scheme it is further extended as follows. Let $S$ be a locally noetherian base scheme. Let $A$ be an abelian scheme over $S$. Let $A^\vee := \text{Pic}^0_{A/S}$ be the abelian scheme dual to $A$. Let $P \rightarrow A \times S A^\vee$ be the Poincaré $G_m$-bundle rigidified along $\{0\} \times_S A^\vee$ and $A \times_S \{0\}$: it defines a biextension of $A$ and $A^\vee$ by $G_{m,S}$ [11, VIII.3.2]. For every $S$-scheme $T$, an extension of $A \times S T$ by $G_{m,T}$ defines a $G_{m,T}$-bundle over $A \times S T$ and, hence, a $T$-valued point of $A^\vee$. The map associating to $x \in A^\vee(T)$ the $G_{m,T}$-bundle $P_{A \times T}(y)$ rigidified over $0 \in A(T)$ defines a homomorphism form $A^\vee(T)$ to $\text{Ext}^1_T(A \times_S T, G_{m,T})$. These two maps are inverse one of the other and define an isomorphism $A^\vee \xrightarrow{\sim} \text{Ext}^1(A, G_{m,S})$.

1) let $T$ be a torus over $S$ and let $\mathbb{Y}$ be its character group. To give an extension $G$ of $A$ by $T$ is equivalent to give a homomorphism $u_{A^\vee}: \mathbb{Y} \rightarrow A^\vee$ over $S$.

If $y \in \mathbb{Y}$, then the image of $-y$ (note the minus sign!) in $A^\vee$, identified with $\text{Ext}^1(A, G_{m,S})$, is the unique rigidified extension of $A$ by $G_{m,S}$ obtained as the push-out of

$$0 \rightarrow Y \rightarrow G \rightarrow A \rightarrow 0$$

by $y$;

2) to give a homomorphism $u: X \rightarrow G$ is equivalent to give
• a homomorphism $u_A : X \to A$

• a trivialization, as biextension, of the pull back of $P$ by the homomorphism

$$u_A \times u_A^\vee : X \times_S Y \to A \times_S A^\vee.$$ 

In particular, let $M := [X \overset{u}{\to} G]$ be a 1-motive. Define the dual 1-motive

$$M^\vee := [X^\vee \overset{u^\vee}{\to} G^\vee]$$

as follows:

• the group scheme $X^\vee$ is the character group $Y$ of $T$;

• the semiabelian scheme $G^\vee$ is the extension of $A^\vee$ by the torus $X^\vee \otimes Z \mathbb{G}_{m,S}$ defined by the composite homomorphism $u_A : X \to G \to A = (A^\vee)^\vee$;

• the homomorphism $u^\vee$ is defined by the trivialization of the biextension $(u_A \times u_A^\vee)^* (P)$ (cf. [7, 10.2.11]).

1.3 The functors $T^\text{crys}$ and $T^\text{crys}$

Let $p$ be a prime number. Let $n \in \mathbb{N}$. Define the group scheme $M[p^n]$ as $H^{-1}(M/p^nM)$, where $M/p^nM$ is the cone of multiplication by $p^n$ on $M$. More explicitly,

$$M[p^n] := \frac{\ker (u + p^n : X \times_S G \to G)}{\text{im}((p^n, -u) : X \to X \times_S G)}.$$ 

It is a finite and flat group scheme over $S$ and it sits in the exact sequence

$$0 \to G[p^n] \to M[p^n] \to \left(\frac{X}{p^nX}\right) \to 0. \tag{1}$$

Note also that we have the exact sequence

$$0 \to T[p^n] \to G[p^n] \to A[p^n] \to 0.$$ 

Define

$$M[p^\infty] := \lim_{n \to \infty} M[p^n];$$

the direct limit being taken using the natural inclusions $M[p^n] \subset M[p^m]$ for $m \geq n$. Then, $M[p^\infty]$ defines a Barsotti-Tate group in the sense of [17, Def I.2.1]. It sits in the exact sequence

$$0 \to G[p^\infty] \to M[p^\infty] \to X[p^\infty] \to 0, \tag{2}$$

where $X[p^\infty] := X \otimes_{\mathbb{Z}} \left(\mathbb{Q}_p/\mathbb{Z}_p\right)$. We also get the exact sequence

$$0 \to T[p^\infty] \to G[p^\infty] \to A[p^\infty] \to 0. \tag{3}$$

Note that we clearly get a splitting of (2) over a suitable faithfully flat extension of $S$. In fact, choose a faithfully flat extension $S' \to S$ and a compatible set of homomorphisms

$$\{u_n : \frac{1}{p^n} (X \times_S S') \to G \times_S S'\}_{n \in \mathbb{N}}.$$
such that $u_0 = u \times_S S'$. Such a choice determines a splitting of (1) over $S'$, for any $n$, defining
\[ \left( \mathbb{A} / p^n \mathbb{A} \right) \times_S S' \to \mathcal{M}[p^n] \times_S S' \text{ by } \mathfrak{p} \mapsto (x, -u_n(p^{-n}x)) \]. Hence, we get a splitting of (2) over $S'$.

Let $S$ be a scheme where $p$ is locally nilpotent. Denote by
\[ D(\mathcal{M}[p^\infty]) \]
the contravariant Dieudonné crystal, on the crystalline site of $S$, associated to the Barsotti-Tate group $\mathcal{M}[p^\infty]$ (cf. [17, IV.2.4.3]).

Let $k$ be a perfect field of characteristic $p$ and let $\mathbf{W}(k)$ be the Witt vectors of $k$. Suppose that $\mathcal{M}$ is defined over $k$.

**Definition 1.3.1** We call crystalline realizations of $\mathcal{M}$ the following $\mathbf{W}(k)$–modules
\[ T_{\text{crys}}(\mathcal{M}) := \lim_{\rightarrow \infty} D(\mathcal{M}[p^\infty]) \left( \text{Spec } (k) \hookrightarrow \text{Spec } (\mathbf{W}(k)) \right) \]
and
\[ T_{\text{crys}}(\mathcal{M}) := \lim_{\rightarrow \infty} D(\mathcal{M}[p^\infty]) \left( \text{Spec } (k) \hookrightarrow \text{Spec } (\mathbf{W}(k)) \right). \]

Call $T_{\text{crys}}(\mathcal{M})$ the *Barsotti-Tate crystal* of the 1-motive $\mathcal{M}$. The functor associating to a 1-motive its $p$-divisible group is exact and covariant. The Dieudonné functor is exact and contravariant. It follows from (2) and (3) that $T_{\text{crys}}(\mathcal{M})$ admits Frobenius and Verschiebung operators and a filtration (respected by Frobenius and Verschiebung): $W_{\geq 0}(T_{\text{crys}}(\mathcal{M})) := T_{\text{crys}}(\mathcal{M})$, $W_1(T_{\text{crys}}(\mathcal{M})) := T_{\text{crys}}(G)$, $W_2(T_{\text{crys}}(\mathcal{M})) := T_{\text{crys}}(T)$ and $W_{\leq -3}(T_{\text{crys}}(\mathcal{M})) := 0$. Hence, $T_{\text{crys}}$ (resp. $T_{\text{crys}}$) defines a contravariant (resp. covariant) functor from the category of 1-motives over $k$ to the category of filtered $F$-crystals.

We remark that the Poincaré biextension in (2) yields a perfect pairing (cf. [7, 10.2.5])
\[ \mathcal{M}[p^n] \otimes \mathcal{M}^\vee[p^n] \to \mathbb{Z}/p^n(1) \]
identifying $\mathcal{M}^\vee[p^n]$ with the Cartier dual $\mathcal{M}^\vee[p^n]$ of $\mathcal{M}[p^n]$. We thus get a perfect pairing of Barsotti-Tate groups
\[ \mathcal{M}[p^\infty] \otimes \mathcal{M}^\vee[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p(1) \]
We therefore obtain a perfect pairing of filtered $\mathbf{W}(k)$-modules
\[ T_{\text{crys}}(\mathcal{M}) \otimes \mathbf{W}(k) \to \mathbf{W}(k)(1). \]
We then may regard the Barsotti-Tate crystal of the motivic Cartier dual as the Dieudonné module of the usual Cartier dual.

**2 Universal vector extension of a 1-motive**

We give some generalities on vector extensions of 1-motives; see [7, 10.1.7]. As in [17] and [16] this concept will be essential to define the Dieudonné module of a 1-motive directly in terms of the 1-motive, without using the associated Barsotti-Tate group.
2.1 Vector extensions of 1-motives

A vector group scheme over $S$ is a group scheme endowed with an action of the group ring $A^1$ and isomorphic to $G_{a,S}$ (compatibly with the $A^1$-action), for some $r \in \mathbb{N}$, locally on $S$. If $E$ is a (fixed, Zariski) locally free $O_S$-module, denote by $W$ the vector group scheme over $S$ whose sections over an $S$-scheme $T$ are

$$W(T) := \Gamma(T, O_T \otimes_{O_S} E),$$

e.g., if $E = O_S$ then $W = G_{a,S}$.

A vector extension

$$0 \rightarrow W \rightarrow E \rightarrow M \rightarrow 0$$

of a 1-motive $M = [X \rightarrow G]$ is an extension of the complex given by $M$ (here $X$ is in degree $-1$ and $G$ in degree 0) by a complex consisting of a vector group scheme $W$ over $S$ concentrated in degree 0.

To give a vector extension $E$ of $M$ is equivalent to give a complex $[X \rightarrow W E G]$ where

i) $E G$ is a vector extension

$$0 \rightarrow W \rightarrow E \rightarrow G \rightarrow 0$$

of $G$ by a vector group scheme $W$ over $S$;

ii) $u_G : X \rightarrow E G$ is a group homomorphism so that the map $X \rightarrow W E G$ coincides with $u$.

Let $M = [X \rightarrow G]$ be a 1-motive over $S$. A vector extension

$$0 \rightarrow W(M) \rightarrow E(M) \rightarrow M \rightarrow 0 \quad (6)$$

of $M$ is called universal if, for any vector extension $0 \rightarrow W \rightarrow E \rightarrow M \rightarrow 0$, there exists a unique homomorphism of $S$-vector group schemes $\phi : W(M) \rightarrow W$ such that $E$ is the push-out of $E(M)$ by $\phi$.

In the following Sections 2.2 and 2.3 we will show (cf. [7, 10.1.7]) that there exists a universal vector extension $E(M)$ over $S$. Moreover, we clearly have (cf. [2, 1.4]) an exact sequence:

$$0 \rightarrow E(G) \rightarrow E(M)_G \rightarrow E([X \rightarrow 0]) \rightarrow 0. \quad (7)$$

Following Deligne’s notation (see [7, 10.1.11]) we let

$$T_{\text{DR}}(M) := \text{Lie} \left( E(M)_G \right).$$

The weight filtration on $M$, defined in [1.1] induces a weight filtration on $T_{\text{DR}}$: $W_{\geq 0}(T_{\text{DR}}(M)) := T_{\text{DR}}(M)$, $W_{-1}(T_{\text{DR}}(M)) := T_{\text{DR}}(G)$, $W_{-2}(T_{\text{DR}}(M)) := T_{\text{DR}}(T)$ and $W_{\leq -3}(T_{\text{DR}}(M)) := 0$.

The fact that $W_n(T_{\text{DR}}(M))$, as defined above, are submodules of $T_{\text{DR}}(M)$ follows from 2.2 and 2.3 below. The short exact sequence (7) yields a corresponding short exact sequence of De Rham realizations (cf. [2, 1.4]). Denote $T_{\text{DR}}(M) := T_{\text{DR}}(M^0)$. 
2.2 Construction of $E(A)$, $E(G)$ and $E([X \to 0])$

By [17] or [16] the abelian scheme $A$ admits a universal vector extension $E(A)$ over $S$. Let $A^\vee$ be the abelian scheme dual to $A$. Let $\text{Pic}^{0,0}$ be the functor of invertible sheaves in $\text{Pic}^0$ endowed with an integrable $S$-connection (see [16]). The map which forgets the connection gives to this functor $\text{Pic}^{0,0}_{A^\vee/S}$ the structure of a functor over $A = \text{Pic}^0_{A^\vee/S}$. One proves (see [16, I 2.6 and 3.2.3]) that this functor is representable and

$$E(A) \cong \text{Pic}^{0,0}_{A^\vee/S}.$$ 

In particular, the kernel of the forgetful map

$$\text{Pic}^{0,0}_{A^\vee/S} \to \text{Pic}^0_{A^\vee/S}$$

classifies all possible connections on the structure sheaf of $A^\vee$. We conclude that, if $\omega_{A^\vee/S}$ is the $O_S$-module of invariant differentials on $A^\vee$ dual to $A$, then

$$\mathcal{W}(A) \to \mathcal{W}(\omega_{A^\vee/S})$$

in the notation of 2.1.

**Lemma 2.2.1** The universal extension $E(G)$ exists and is isomorphic to $G \times_A E(A)$.

**Proof:** Applying the functor $\text{Hom}_{\text{fppf}}(\cdot, G_{a,S})$ to the exact sequence

$$0 \to T \to G \to A \to 0$$

we get the long exact sequence

$$0 \to \text{Hom}(A, G_{a,S}) \to \text{Hom}(G, G_{a,S}) \to \text{Hom}(T, G_{a,S}) \to \text{Ext}^1(A, G_{a,S}) \to \text{Ext}^1(G, G_{a,S}) \to \text{Ext}^1(T, G_{a,S}).$$

Note that

i) $\text{Hom}(A, G_{a,S}) = \{0\}$, since $f: A \to S$ is proper, smooth and geometrically irreducible so that $f_*(O_A) = O_S$;

ii) $\text{Hom}(T, G_{a,S}) = \{0\}$, since $\text{Hom}(G_{m,S}, G_{a,S}) = \{0\}$ by a direct computation comparing the comultiplications on the associated Hopf algebras;

iii) $\text{Ext}^1(T, G_{a,S}) = \{0\}$ (cf. [7, 10.1.7.b]).

We conclude that $\text{Ext}^1(A, G_{a,S}) \to \text{Ext}^1(G, G_{a,S})$. The map is defined by sending an extension $E$ of $A$ by $G_{a,S}$ to $G \times_A E$. Hence, $G$ admits a universal vector extension $E(G)$ over $S$ defined by $G \times_A E(A)$.

**Lemma 2.2.2** (i) The sheaf $X \otimes_Z G_{a,S}$ is represented by a vector group scheme over $S$.

(ii) The universal extension $E([X \to 0])$ is defined by the homomorphism

$$X \to X \otimes_Z G_{a,S}$$

which sends $x \mapsto x \otimes 1$. 
Proof: Let \( S' \) be an étale cover of \( S \) such that \( X \times_S S' \) is split. Then, \( (X \otimes \mathbb{Z} G_{a,S}) \times_S S' \) is the spectrum of the symmetric algebra \( \text{Sym}(I') \) with \( I' = X \times \mathbb{Z} \mathcal{O}_{S'} \). By descent theory the latter descends to a locally free \( \mathcal{O}_S \)-module \( I \). Hence, \( X \otimes \mathbb{Z} G_{a,S} \) is represented by the vector group scheme associated to \( \text{Hom}(I, \mathcal{O}_S) \). This proves part (i).

The homomorphism in (ii) is well defined over \( S \). Due to the uniqueness of the universal extensions and descent theory for morphisms, it suffices to prove (ii) over \( S' \). Hence, we may assume that \( X = \mathbb{Z} e_1 \oplus \cdots \oplus \mathbb{Z} e_n \).

By the discussion in 2.1 the homomorphism \( u : X \to \mathbb{G}_m \) defines a vector extension of \( \mathbb{M} \). We leave to the reader the proof that it is (canonically isomorphic to) the universal vector extension of \( \mathbb{M} \) (cf. [2, 1.4]). By construction we then get the claimed exact sequence (7).

\[ 2.3 \quad \text{Construction of } \mathbb{E}(\mathbb{M}) \]

See [2, §1.4] for a construction of \( \mathbb{E}(\mathbb{M}) \). For our purposes the following less canonical construction will be useful. Let \( S' \) be an étale cover of \( S \) such that \( X \times_S S' \cong \mathbb{Z}^n \), for some \( n \in \mathbb{N} \). Assume that a universal vector extension \( \mathbb{E}(\mathbb{M} \times_S S') \) exists over \( S' \). The universal property of \( \mathbb{E}(\mathbb{M} \times_S S') \) gives the necessary descent data to get \( \mathbb{E}(\mathbb{M}) \) descending \( \mathbb{E}(\mathbb{M} \times_S S') \) from \( S' \) to \( S \). Since \( \mathbb{M} \) is defined over \( S \), the descent is effective. Hence, we may assume that \( X = \mathbb{Z} e_1 \oplus \cdots \oplus \mathbb{Z} e_n \). In the notation of 2.1, let

\[ \mathbb{E}(\mathbb{M})_G := \mathbb{E}(\mathbb{G}) \times_S \left(X \otimes \mathbb{Z} G_{a,S}\right). \]

It is naturally endowed with the structure of commutative group scheme over \( S \).

Note that

\[ \mathbb{E}(\mathbb{M})_G / \mathbb{E}(\mathbb{G}) \xrightarrow{\sim} X \otimes \mathbb{Z} G_{a,S} \]

and \( \mathbb{E}(\mathbb{M})_G \to \mathbb{G} \) is a vector extension of \( \mathbb{G} \). Let

\[ \psi : X \to \mathbb{E}(\mathbb{G}) \]

be a homomorphism lifting \( u \); by our assumption on \( X \) and possibly replacing \( S \) with an étale cover, it exists. Let

\[ u_{\mathbb{E}(\mathbb{M})} : X \to \mathbb{E}(\mathbb{M})_G \]

be the homomorphism

\[ x = \sum_i x_i e_i \mapsto \left( \sum_i x_i \cdot \psi(e_i), \sum_i x_i e_i \otimes 1 \right). \]

It is a homomorphism of group schemes and the composition with \( \mathbb{E}(\mathbb{M})_G \to \mathbb{G} \) coincides with \( u \). By the discussion in 2.1 the homomorphism \( u_{\mathbb{E}(\mathbb{M})} \) defines a vector extension of \( \mathbb{M} \). We leave to the reader the proof that it is (canonically isomorphic to) the universal vector extension of \( \mathbb{M} \) (cf. [2, 1.4]). By construction we then get the claimed exact sequence (7).

\[ 2.4 \quad \mathbb{E}(\mathbb{M}) \text{ via } \mathbb{E}(\mathbb{X} \to \mathbb{A}) \]

Let \( \mathbb{M} \) be a 1-motive and let \( u_{\mathbb{A}} : \mathbb{X} \to \mathbb{A} \) be the induced 1-motive such that

\[ 0 \to \mathbb{T} \to \mathbb{M} \to [\mathbb{X} \to \mathbb{A}] \to 0 \]

is exact. Taking \( \text{Ext}^1(-, \mathbb{G}_{a,S}) \) we see that

\[ \text{Ext}^1([\mathbb{X} \to \mathbb{A}], \mathbb{G}_{a,S}) \xrightarrow{\sim} \text{Ext}^1(\mathbb{M}, \mathbb{G}_{a,S}) \]

in such a way that \( \mathbb{E}([\mathbb{X} \to \mathbb{A}]) \) pulls back to \( \mathbb{E}(\mathbb{M}) \).
Let \( M_1 := \left[ X_1 \overset{\text{u}_1}{\rightarrow} A_1 \right] \) and \( M_2 := \left[ X_2 \overset{\text{u}_2}{\rightarrow} A_2 \right] \) be 1-motives over \( S \) with \( A_1 \) and \( A_2 \) abelian schemes over \( S \). Assume we are given

1) vector extensions \( E_1 \) and \( E_2 \) of \( M_1 \) and \( M_2 \) respectively;

2) a \( G_{m,S} \)-bienstension \( P \) of \( E_1, A_1 \times_S E_2, A_2 \);

3) a map \( \alpha: X_1 \times_S X_2 \rightarrow P \) lifting \( (u_1, u_2) \) and inducing a trivialization, as a bienstension, of \( (u_1, u_2)^*(P) \).

For every \( x_2 \in X_2 \), the element \( \alpha(0, x_2) \) defines a rigidification of \( P|_{E_1 \times_S \{x_2\}} \) over the 0 element of \( E_1 \cong E_1 \times_S \{x_2\} \). Because of the bienstension structure of \( P \), we get that \( P|_{E_1 \times_S \{x_2\}} \) is endowed with the structure of a commutative group scheme, having \( \alpha(0, x_2) \) as 0 element and sitting in an exact sequence

\[
0 \rightarrow G_{m,S} \rightarrow P|_{E_1 \times_S \{x_2\}} \rightarrow E_1 \rightarrow 0.
\]

Moreover, the map \( X_1 \rightarrow P|_{E_1 \times_S \{x_2\}} \) defined by \( x_1 \mapsto \alpha(x_1, x_2) \) is a group homomorphism. Hence, the data 1) – 3) above define an extension of \( E \) via the exact sequence

\[
\begin{array}{c}
0 \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow \\
\rightarrow 0
\end{array}
\]

applying the functor \( \text{Hom}_{\text{fppf}}(\_ , G_{a,S}) \). Then,

\[
\text{Hom}(G[p^N], G_{a,S}) \longrightarrow \text{Ext}^1(G, G_{a,S})
\]

is an isomorphism. Moreover, by [17] Prop IV.1.3,\( \text{Hom}(G[p^N], G_{a,S}) \cong \text{Hom}(\omega_{G[p^N]}^\vee, G_{a,S}) \),

where \( G[p^N]^\vee \) is the Cartier dual to \( G[p^N] \). The universal map \( G[p^N] \rightarrow G[p^N]^\vee \) is defined by

\[
G[p^N] \overset{\sim}{\rightarrow} \text{Hom}(G[p^N]^\vee, G_{m,S}) \rightarrow \text{Hom}(\text{Inf}^1(G[p^N]^\vee), G_{m,S}) \cong G[p^N]^\vee,
\]

2.5 Universal vector extension of a Barsotti-Tate group

In [17] and [16] it is proven that, if \( p \) is locally nilpotent on \( S \) and \( G \) is a Barsotti-Tate group, there exists a universal vector extension \( E(G) \) of \( G \).

Suppose that \( p^N S = 0 \). As in [17] Prop IV.1.10, we may classify the extensions of \( G \) by \( G_{a,S} \) via the exact sequence

\[
0 \rightarrow G[p^N] \rightarrow G \rightarrow G \rightarrow 0
\]

applying the functor \( \text{Hom}_{\text{fppf}}(\_ , G_{a,S}) \). Then,

\[
\text{Hom}(G[p^N], G_{a,S}) \longrightarrow \text{Ext}^1(G, G_{a,S})
\]

is an isomorphism. Moreover, by [17] Prop IV.1.3,\( \text{Hom}(G[p^N], G_{a,S}) \cong \text{Hom}(\omega_{G[p^N]}^\vee, G_{a,S}) \),

where \( G[p^N]^\vee \) is the Cartier dual to \( G[p^N] \). The universal map \( G[p^N] \rightarrow G[p^N]^\vee \) is defined by

\[
G[p^N] \overset{\sim}{\rightarrow} \text{Hom}(G[p^N]^\vee, G_{m,S}) \rightarrow \text{Hom}(\text{Inf}^1(G[p^N]^\vee), G_{m,S}) \cong G[p^N]^\vee,
\]
where \( \text{Inf}^1(\mathcal{G}[p^N]) \hookrightarrow \mathcal{G}[p^N] \) is the first infinitesimal neighborhood of the identity.

Let \( S_0 \) be a scheme on which \( p \) is locally nilpotent and let \( \mathcal{G}_0 \) be a Barsotti-Tate group on \( S_0 \). The construction above can be extended to define a crystal in fppf sheaves on the nilpotent crystalline site of \( S_0 \) as follows. Let \( S_0 \hookrightarrow S \) be a locally nilpotent thickening with nilpotent divided powers and let \( \mathcal{G} \) be a (any) lifting of \( \mathcal{G}_0 \) to \( S \). Define

\[
\mathcal{E}(\mathcal{G}_0) \left( S_0 \hookrightarrow S \right) := \mathcal{E}(\mathcal{G}).
\]

(8)

It is proven in [17, Thm IV.2.2] that indeed this defines a crystal and is functorial in \( \mathcal{G}_0 \).

Recall that the Dieudonné crystal \( D(\mathcal{G}_0[p^\infty]) \), on the crystalline site of \( S_0 \), associated to the Barsotti-Tate group \( \mathcal{G}_0[p^\infty] \) in [17, IV.2.4.3], is by definition the Lie algebra of the crystal \( \mathcal{E}(\mathcal{G}_0) \).

2.6 Another construction of \( \mathcal{E}(\mathcal{G}) \) for \( p \) nilpotent

Let \( \mathcal{G} \) be a semiabelian scheme over \( S \). One can construct the universal vector extension of \( \mathcal{G} \) exactly as in 2.5 substituting \( \mathcal{G} \) to \( \mathcal{G} \).

Assume that \( p^N S = 0 \). The composite map

\[
\text{Hom}(\mathcal{G}[p^N], \mathcal{G}_{a,S}) \longrightarrow \text{Ext}^1(\mathcal{G}, \mathcal{G}_{a,S}) \longrightarrow \text{Ext}^1(\mathcal{G}[p^\infty], \mathcal{G}_{a,S})
\]

is the isomorphism in 2.5. Since all these cohomology groups are represented by locally free \( \mathcal{O}_S \)-modules of the same rank, all the maps are isomorphisms. Let \( \Delta \) be a splitting of \( \mathcal{E}(\mathcal{G}) \rightarrow \mathcal{G} \) Zariski locally on \( \mathcal{G} \). Then, \( \rho := p^N \cdot \Delta : \mathcal{G} \rightarrow \mathcal{E}(\mathcal{G}) \) is a well defined homomorphism not depending on \( \Delta \). Hence, we have a diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{G}[p^N] & \longrightarrow & \mathcal{G} & \overset{p^N}{\longrightarrow} & \mathcal{G} & \longrightarrow & 0 \\
\downarrow & & \downarrow \rho & & \downarrow \iota & & \downarrow \\
0 & \longrightarrow & \mathcal{W}(\mathcal{G}) & \longrightarrow & \mathcal{E}(\mathcal{G}) & \longrightarrow & \mathcal{G} & \longrightarrow & 0.
\end{array}
\]

Let \( \mathcal{G}[p^\infty] \) be the Barsotti-Tate group associated to \( \mathcal{G} \). Restricting this diagram to \( \mathcal{G}[p^\infty] \) and using 2.5 we get:

Lemma 2.6.1 ([17, Thm V.2.1]) The map \( \mathcal{G}[p^N] \longrightarrow \mathcal{W}(\mathcal{G}) \) is universal for vector extensions of \( \mathcal{G} \) over \( S \). Moreover, the canonical map

\[
\mathcal{E}(\mathcal{G}[p^\infty]) \longrightarrow \mathcal{E}(\mathcal{G}) \times_{\mathcal{G}} \mathcal{G}[p^\infty]
\]

is an isomorphism.

2.7 Another construction of \( \mathcal{E}(\mathbb{M}) \) for \( p \) nilpotent

Let \( \mathbb{M} \) be a 1-motive over a scheme \( S \) on which \( p^N = 0 \). Let \( \rho \) be as in 2.6. By construction we have \( \rho \circ u = p^N u_{\mathcal{M}} \). Let

\[
q : X \times_S \mathbb{G} \rightarrow \mathbb{G}
\]

be the map \((x, g) \mapsto u(x) + p^N g\). Define

\[
\Phi : X \times_S \mathbb{G} \longrightarrow \mathcal{E}(\mathbb{M})_G
\]
by $\Phi(x, g) := u_{E(M)}(x) + \rho(g)$. Then,

$$
\Phi(p^N x, -u(x)) = p^N u_{E(M)}(x) - \rho(u(x)) = 0.
$$

The composition of $\Phi$ with the projection $\pi: E(M)_G \to G$ is $(x, g) \mapsto u(x) + p^N x$. Hence, $\pi \circ \Phi = q$ and

$$
\ker (\pi \circ \Phi) = \ker (q) = \{(x, g) \in G \times_S X | u(x) = -p^N g\}.
$$

By Lemma 1.3 we get a surjective map $\ker (q) \to M[p^N]$. Summarizing we get a diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & \ker (q) & \longrightarrow & X \times_S G & \longrightarrow & G & \longrightarrow & 0 \\
& & & & & \Phi \downarrow & & \downarrow & \\
0 & \longrightarrow & W(M) & \longrightarrow & E(M)_G & \longrightarrow & G & \longrightarrow & 0.
\end{array}
$$

Moreover, the induced map $\ker (q) \to W(M)$ factors as

$$
\ker (q) \longrightarrow M[p^N] \longrightarrow W(M).
$$

This map is functorial in $M$. Taking $G \to M$ and in virtue of Lemma 2.6 the composition of

$$
G[p^N] \longrightarrow M[p^N] \longrightarrow W(M)
$$

factors via $W(G)$ and is the universal map for vector extensions. Replacing $S$ with a suitable étale cover, we may assume that $X = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$. By Lemma 2.2.2 and functoriality of $\Phi$, the induced map

$$
M[p^N] \longrightarrow X/p^N X \longrightarrow E([X \to 0]) = X \otimes_{\mathbb{Z}} G_{a, S}
$$

is defined by $x \mapsto x \otimes 1$. Hence, it coincides with the universal map

$$
X/p^N X \longrightarrow \omega_{X/p^N X}^\vee = \omega_{X \otimes_{\mathbb{Z}_p} \mathbb{Z}_p} = X \otimes_{\mathbb{Z}} \omega_{p^N}
$$

of Lemma 2.5 (the computation is left to the reader). By Lemma 2.5 the map $M[p^N] \to W(M)$ induced by $\Phi$ factors via a unique map $\omega_{M[p^N]} \to W(M)$. Putting together all the remarks given so far, we conclude that this map is an isomorphism and, hence, it is the universal one.

### 2.8 Comparison between $E(M)$ and $E(M[p^\infty])$ for $p$ nilpotent

Fix $n \in \mathbb{N}$. By Proposition 1.3 the group scheme $\tilde{M}[p^n]$ is the quotient of the subgroup

$$
\tilde{M}[p^n] := \{(x, g) \in X \times_S G | u(x) = -p^n g\} \hookrightarrow X \times_S G
$$

by $(p^n, -u)(X)$. The inclusion $M[p^n] \hookrightarrow \tilde{M}[p^{n+m}]$ is induced by the inclusion $\tilde{M}[p^n] \hookrightarrow \tilde{M}[p^{n+m}]$ defined by $(x, g) \mapsto (p^m x, g)$. Multiplication by $p^n: \tilde{M}[p^{n+m}] \to \tilde{M}[p^n]$ is induced by the map $\tilde{M}[p^{n+m}] \to \tilde{M}[p^n]$ defined by $(x, g) \mapsto (x, p^n g)$. Let

$$
\tilde{M}[p^\infty] := \lim_{n \to \infty} \tilde{M}[p^n].
$$

Then, $M[p^\infty]$ is the quotient of $\tilde{M}[p^\infty]$ by the image of $X$. Note that $\tilde{M}[p^n] = \ker (q)$. Let $M[p^\infty]'$ be the fiber product of $p^N: M[p^\infty] \to M[p^\infty]$ and the inclusion $G[p^\infty] \subset M[p^\infty]$ over $M[p^\infty]$. 
Note that \( M[p^\infty]' \) surjects onto \( G[p^\infty] \) with kernel \( M[p^N] \). Let \( \widetilde{M[p^\infty]}' \) be the fiber product of \( M[p^\infty] \to \widetilde{M[p^\infty]} \to M[p^\infty] \) (the first map being multiplication by \( p^N \)) and \( G[p^\infty] \subset M[p^\infty] \) over \( M[p^\infty] \). By construction we have a surjective map \( q': \widetilde{M[p^\infty]}' \to G[p^\infty] \) with kernel \( M[p^N] \).

If \((x, g) \in \widetilde{M[p^\infty]}' \cap \widetilde{M[p^n]} \) with \( n \geq N \), then \( x \left\langle \frac{1}{p^{n-N}} \right\rangle \) lies in \( X \) and \( q'((x, g)) = u \left( \frac{x}{p^{n-N}} \right) + p^N g = q \left( \frac{x}{p^{n-N}}, g \right) \). Thus, we obtain a commutative diagram:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Ker}(q) & \longrightarrow & \widetilde{M[p^\infty]}' & \longrightarrow & G[p^\infty] & \longrightarrow & 0 \\
\| & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Ker}(q) & \longrightarrow & X \times_S G & \longrightarrow & G & \longrightarrow & 0.
\end{array}
\]

Note that \( M[p^\infty]' \) is the quotient of \( \widetilde{M[p^\infty]}' \) by the image of \( X \). By \( \text{[.6]} \) the push-out of \( M[p^\infty]' \) along \( \omega_{M[p^N]'} \) is \( \omega_{M[p^N]'} \) is the pull-back of \( E(M[p^\infty]) \) via \( G[p^\infty] \to M[p^\infty] \). Hence, the push-out of the extensions in the diagram above via the composite of \( \text{Ker}(q) \to M[p^N] \to \omega_{M[p^N]'} \) yields, by \( \text{[.7]} \) the following:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \omega_{M[p^N]'} & \longrightarrow & E(M[p^\infty]) \times_{M[p^\infty]} G[p^\infty] & \longrightarrow & G[p^\infty] & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{W}(M) & \longrightarrow & E(M)_G & \longrightarrow & G & \longrightarrow & 0.
\end{array}
\]

**Proposition 2.8.1** Let \( S \) be a scheme on which \( p \) is locally nilpotent. Let \( M \) be a 1-motive over \( S \). There is a canonical and functorial isomorphism

\[
E(M[p^\infty]) \times_{M[p^\infty]} G[p^\infty] \cong E(M)_G \times_G G[p^\infty].
\]

In particular, it induces an isomorphism on the level of formal groups of \( E(M[p^\infty]) \) and of \( E(M) \) and of Lie algebras

\[
\text{Lie} \left( E(M[p^\infty]) \right) \longrightarrow \text{Lie} \left( E(M)_G \right) = T_{\text{DR}}(M).
\]

### 3 Universal extension crystal of a 1-motive

Let \( S_0 \) be a scheme such that \( p \) is locally nilpotent. Let \( M_0 := [u_0: X_0 \to G_0] \) be a 1-motive over \( S_0 \). Let \( S_0 \subseteq S \) be a locally nilpotent pd thickening of \( S_0 \): it is a closed immersion defined by an ideal sheaf \( \mathcal{I} \) endowed with locally nilpotent divided powers structure \( \{ \gamma_n: \mathcal{I} \to \mathcal{I} \}_n \). Let \( \mathcal{M} := [u: X \to G] \) and \( \mathcal{M'} := [u': X' \to G'] \) be two 1-motives over \( S \) lifting \( M_0 \). We will show that there is a canonical isomorphism \( E(M) \cong E(M') \).

#### 3.1 Exp and Log

Let \( \text{Spec}(D) \) be an affine flat scheme over \( S \). The divided power structure on \( \mathcal{I} \) extends uniquely to a divided power structure on \( ISD \) by the flatness of \( D \); see \( \text{[4]} \) Cor 3.22. We get a homomorphism

\[
\exp: \left\{ a \in D | a \equiv 0 \mod ISD \right\} \to \left\{ m \in D | m \equiv 1 \mod ISD \right\}
\]

defined by \( a \mapsto \sum_n \gamma_n(a) \). The inverse exists and is the logarithm \( m \mapsto \log(m) := \sum_n (n - 1)! \gamma_n(m) \). Thus, \( \exp \) and \( \log \) are isomorphisms.
3.2 The crystalline nature of $\mathbb{E} ([X \to A])$

The category of étale group schemes over $S$ is equivalent to the category of étale group schemes over $S_0$. Hence, we get canonical isomorphisms

$$\rho: X \cong X',$$  \hfill (9)

since their base change to $S_0$ are isomorphic to $X_0$. Moreover, there is a canonical isomorphism of $S$-group schemes $\sigma: \mathbb{E} (A) \iso \mathbb{E} (A')$. This follows from [17] and [18]: in loc. cit. a crystal, over the crystalline site of $S_0$, is constructed whose value at $S_0 \hookrightarrow S$ is canonically $\mathbb{E} (A)$ (equivalently $\mathbb{E} (A')$).

**Lemma 3.2.1** There is a unique isomorphism of complexes of $S$-group schemes

$$\bar{\xi}: \mathbb{E} ([X \to A]) \iso \mathbb{E} ([X' \to A']).$$

making the following diagram commute:

$$\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{E} (A) & \longrightarrow & \mathbb{E} ([X \to A]) & \longrightarrow & \mathbb{E} ([X \to 0]) & \longrightarrow & 0 \\
\downarrow{\sigma} & & \downarrow{\xi} & & \downarrow{\rho} & & \downarrow{\rho} & & \\
0 & \longrightarrow & \mathbb{E} (A') & \longrightarrow & \mathbb{E} ([X' \to A']) & \longrightarrow & \mathbb{E} ([X' \to 0]) & \longrightarrow & 0,
\end{array}$$

where the vertical map $\sigma$ on the left is the isomorphism defined in [20] and the vertical map on the right is the isomorphism

$$\mathbb{E} ([X \to 0]) \cong \mathbb{E} ([X' \to 0])$$
deduced from [20].

**Proof:** By the asserted uniqueness and by descent theory we may assume that $X_0 \cong \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_n$ and $\mathcal{I}$ is nilpotent. Let $\pi': \mathbb{E} (A') \to A'$ be the projection. We may assume that $\text{Ker} (\pi')$ is isomorphic to $G_{a,S}^{\dim (A')}$ Choose homomorphisms $\psi: X \to \mathbb{E} (A)$ and $\psi': X' \to \mathbb{E} (A')$, lifting $u$ and $u'$ respectively, such that $\psi \times_S S_0 = \psi' \times_S S_0$. By [20] we have

$$\mathbb{E} ([X \to A])_A = \mathbb{E} (A) \times_S (X \otimes_{\mathbb{Z}} G_{a,S})$$

and

$$\mathbb{E} ([X' \to A'])_{A'} = \mathbb{E} (A') \times_S \left( X' \otimes_{\mathbb{Z}} G_{a,S} \right).$$

Let $\xi := \sigma \times \rho$. With the notation of [20] let $v := u_{\mathbb{E} (X \to A)}$ and $v' := u_{\mathbb{E} (X' \to A')}$. Consider $d := \xi \circ v - v' \circ \rho: X \to \mathbb{E} ([X' \to A'])_{A'}$. Its projection to $X' \otimes_{\mathbb{Z}} G_{a,S}$ is 0 by construction.

Let $i = 1, \ldots, n$. Notice that $(\pi \circ d)(e_i) \in \text{Ker} (A(S) \to A'(S_0))$. Hence, $\pi(d(e_i))$ is a point defined in the formal group of $A'$. Since $\mathcal{I}$ is nilpotent, such point factors through a finite flat subgroup scheme $A'[p^N]$ for suitable $N$. Let $S'[e]$ be the dual numbers over $S$. Let $A'[p^N]^\vee := \text{Spec} (\mathcal{D})$, let $\Delta: \mathcal{D} \to \mathcal{D} \otimes_{\mathcal{O}_S} \mathcal{D}$ be the comultiplication and let $cu: \mathcal{D} \to \mathcal{O}_S$ be the counit. We have

$$A'[p^N](S) = \text{Hom}_S (A'[p^N]^\vee, G_{m,S})$$

$$= \{ d \in \mathcal{D} | \Delta (x) = x \otimes x, cu (x) = 1 \}$$

$$=: \text{Cospec} (\mathcal{D}).$$
Note that
\[
\text{Lie}\left(\mathbb{A}'[p^N]\right) = \left\{ \tau : \mathbb{A}'[p^N] \times_S S[\varepsilon] \to \mathbb{G}_{m,S}[\varepsilon]|\tau \times_S S[\varepsilon] S = 1 \right\}
= \{d \in D|\Delta(x) = x \otimes 1 + 1 \otimes x, \text{cu}(x) = 0\}
=: \text{Prim}(D).
\]
Since \(D\) is a flat \(\mathcal{O}_S\)-module, as explained in 3.1 the exponential and the logarithm define an isomorphism
\[
\exp : \text{Prim}(D) \cap \mathcal{I} D \xrightarrow{\sim} \text{Ker}\left(\text{Cospec}(D) \to \text{Cospec}(D/\mathcal{I} D)\right);
\]
see [17, Rmk III.2.2.6]. Hence, the exponential defines an isomorphism
\[
\exp : \text{Lie}\left(\mathbb{A}'[p^N]\right) \cap \mathcal{I} D \xrightarrow{\sim} \text{Ker}\left(\mathbb{A}'[p^N](S) \to \mathbb{A}'[p^N](S_0)\right).
\]
Let \(y_i := \exp^{-1}\left((\pi \circ d)(e_i)\right)\). Let \(G_{a,S} \to \text{Lie}\left(\mathbb{A}'[p^N]\right)\) be the homomorphism of group schemes sending 1 to \(y_i\). The composition with \(\exp\) defines the unique map \(G_{a,S} \to \mathbb{A}'[p^N](S_0)\). Since \(E(\mathbb{A}')\) is the extension of \(\mathbb{A}'\) by \(G_{a,S}^{\dim(\mathbb{A}')}\), there exists a unique map from \(G_{a,S}\) to \(E(\mathbb{A}')\) sending 1 to \(d(e_i)\). Hence, there exists a unique map \(t : X \otimes_Z G_{a,S} \to E(\mathbb{A}')\) such that, if
\[
\xi := \begin{pmatrix} \sigma & i \\ 0 & \rho \end{pmatrix},
\]
then \(\xi \circ \nu = \nu' \circ \rho\).

### 3.3 Deformations of biextensions

Let \(S_0\) be a scheme. Let \(S_0 \hookrightarrow S\) be a thickening defined by an ideal \(\mathcal{I}\) with nilpotent divided powers.

**Lemma 3.3.1** Let \(X\) be a flat and separated group scheme over \(S\). The group of isomorphism classes of extensions of \(X\) by \(G_{m,S}\), endowed with a trivialization over \(X \times_S S_0\), is naturally isomorphic to the group of isomorphism classes of extensions of \(X\) by \(G_{a,S}\), endowed with a trivialization over \(X \times_S S_0\).

Given representatives of two corresponding isomorphism classes \(P_m\) and \(P_a\), there is a natural isomorphism between the group of automorphisms of \(P_m\), preserving the trivialization over \(X \times_S S_0\), and the group of automorphisms of \(P_a\), preserving the trivialization over \(X \times_S S_0\).

**Proof:** Let \(G = G_{m,S}\) or \(G_{a,S}\). Let \(P\) be an extension of \(X\) by \(G\) and let
\[
\alpha : P \times_S S_0 \xrightarrow{\sim} X \times_S G \times_S S_0
\]
be a trivialization of the extension over \(X \times_S S_0\). Choose an open affine covering \(\{U_i\}_i\) of \(X\) and, for each \(i\), a trivialization
\[
\beta_i : P \times_S U_i \xrightarrow{\sim} G \times_S U_i
\]
compatible with \(\alpha\). For \(i \neq j\), let \(\text{Spec}(\mathcal{D}_{i,j}) := U_{i,j}\). The trivializations \(\beta_i\) and \(\beta_j\) restricted to \(U_{i,j} := U_i \times_X U_j\) differ by...
i) a multiplicative cocycle $m_{i,j} \in \mathcal{D}_{i,j}^*$ such that $m_{i,j}$ is 1 mod $\mathcal{I}$ if $G = G_{m,S}$.

ii) an additive cocycle $a_{i,j} \in \mathcal{D}_{i,j}$ such that $a_{i,j}$ is 0 mod $\mathcal{I}$ if $G = G_{a,S}$.

In case (i) define the extension

$$0 \longrightarrow G_{a,S} \longrightarrow P_a \longrightarrow X \longrightarrow 0$$

by the additive cocycle $a_{i,j} := \log(m_{i,j}) \in \mathcal{D}_{i,j}$; see §3.1 for log. In case (ii) define the extension

$$0 \longrightarrow G_{m,S} \longrightarrow P_m \longrightarrow X \longrightarrow 0$$

by the multiplicative cocycle $m_{i,j} := \exp(a_{i,j}) \in \mathcal{D}_{i,j}$. It is easily checked that these maps define an equivalence between $G_{m,S}$-torsors and $G_{a,S}$-torsors with a trivialization over $S_0$.

Let $m$, $p_1$ and $p_2$ be the maps form $X \times_S X$ to $X$ defined by the multiplication, the first and the second projection respectively. Let $P$ be a $G$-torsor over $X$. To give a multiplication law on $P$, compatible with the one on $X$ and with the action of $G$ and inducing the standard group law on $P \times_S S_0$, is equivalent to give a map

$$X \times_S X \longrightarrow m^*(P)p_1^*(P)^{-1}p_2^*(P)^{-1}$$

reducing to the identity after base change to $S_0$. Proceeding as before, using the logarithm and the exponential, one passes from the case $G = G_{m,S}$ to the case $G = G_{a,S}$ and viceversa. Clearly the commutativity and the associativity of the multiplication is preserved. Hence, the conclusion.

The proof of the second part of the Lemma is left as an exercise for the reader. ⊓⊔

**Corollary 3.3.2** Let $A$ and $B$ be flat and separated group schemes over $S$. There is a natural isomorphism between the group of isomorphism classes of biextensions [14, Ex. VII] of $A$ and $B$ by $G_{m,S}$, endowed with a trivialization over $(A \times_S S_0) \times_{S_0} (B \times_S S_0)$, and the group of isomorphism classes of biextensions [14, Ex. VII] of $A$ and $B$ by $G_{a,S}$, endowed with a trivialization over $(A \times_S S_0) \times_{S_0} (B \times_S S_0)$.

Given representatives of two corresponding isomorphism classes $P_m$ and $P_a$ in this two groups, there is a natural isomorphism between the group of automorphisms of $P_m$, preserving the trivialization over $(A \times_S S_0) \times_{S_0} (B \times_S S_0)$, and the group of automorphisms of $P_a$, preserving the trivialization over $(A \times_S S_0) \times_{S_0} (B \times_S S_0)$. Such group is isomorphic to

$$\text{Ker} \left( \text{Bil} \left( A \times_S B, G_{a,S} \right) \to \text{Bil} \left( A_0 \times_{S_0} B_0, G_{a,S_0} \right) \right),$$

where Bil stands for the bilinear homomorphisms.

**Proposition 3.3.3** Let $A$ be an abelian scheme over $S$. Let $X$ be a flat and separated group scheme over $S$. Let $G$ be $G_{a,S}$ or $G_{m,S}$. Then, the group of biextensions of $X$ and $\mathbb{E}(A)$ by $G$, endowed with a trivialization over $(X \times \mathbb{E}(A)) \times_S S_0$, is identified with a subgroup of the homomorphisms from $X$ to the sheaf of isomorphism classes of extensions of $\mathbb{W}(A)$ by $G$ endowed with a trivialization over $S_0$. Such identification is compatible with the isomorphisms in Lemma 3.3.1 and Corollary 3.3.2.
Proof: Let \( \mathcal{B} \) be a biextension of \( X \) and \( \mathbb{E}(\mathbb{A}) \) by \( G \). For every \( S \)-scheme \( T \) and any \( T \)-valued point \( x \) of \( X \) the fiber \( \mathcal{B}|_{\{x\} \times \mathbb{E}(\mathbb{A})} \) defines an extension \( \mathcal{E}_x \) of \( \mathbb{E}(\mathbb{A}) \times_T T \) by \( G \times_T T \). This defines a map \( \vartheta: \text{Biext}^1\left(X, \mathbb{E}(\mathbb{A}); G\right) \to \text{Hom}\left(X, \mathbf{Ext}^1(\mathbb{E}(\mathbb{A}), G)\right). \) By [13] Ex. VIII, 1.1.4 the kernel of this map is \( \text{Ext}^1\left(X, \text{Hom}\left(\mathbb{E}(\mathbb{A}), G\right)\right). \)

A trivialization of \( \mathcal{B} \) over \( (X \times \mathbb{E}(\mathbb{A})) \times T S_0 \) induces a trivialization of \( \mathcal{E}_x \) over \( \mathbb{E}(\mathbb{A})_0 \). This construction is compatible with the isomorphisms of 3.3.1 and 3.3.2 Hence, it suffices to consider the case \( G = G_{a,S} \). The exact sequence (6) induces a long exact sequence

\[
0 \to \text{Hom}(A, G_{a,S}) \to \text{Hom}(\mathbb{E}(A), G_{a,S}) \to \text{Hom}(\mathbb{W}(A), G_{a,S}) \to \text{Ext}^1(A, G_{a,S}) \to \text{Ext}^1(\mathbb{E}(A), G_{a,S}) \to \text{Ext}^1(\mathbb{W}(A), G_{a,S}).
\]

Note that

- \( \text{Hom}(A, G_{a,S}) = \{0\} \) since \( A \to S \) is proper;
- the map \( \delta \) is defined by push forward of (6) and is an isomorphism by definition of universal extension.

We conclude that \( \text{Hom}(\mathbb{E}(A), G_{a,S}) = \{0\} \) so that \( \vartheta \) is injective for \( G = G_a \). Furthermore, we get an injective map \( \text{Ext}^1(\mathbb{E}(A), G_{a,S}) \to \text{Ext}^1(\mathbb{W}(A), G_{a,S}). \) Since the homomorphisms from \( \mathbb{E}(A_0) \) to \( G_{a,S_0} \) are trivial, such map is injective also considering extensions of \( \mathbb{E}(A) \) (resp. \( \mathbb{W}(A) \)) by \( G_{a,S} \) endowed with trivialization over \( S_0 \).

Proposition 3.3.4 Let \( M := [X \to A] \) and \( N := [Y \to B] \) be 1-motives over \( S \) with trivial toric part. Then, kernel of the base change map

\[
\text{Biext}^0(\mathbb{E}(N)_B, \mathbb{E}(M)_A; G_{m,S}) \to \text{Biext}^0(\mathbb{E}(N)_B, \mathbb{E}(M)_A; G_{m,S_0})
\]

is isomorphic to the kernel of

\[
\text{Hom}(Y \otimes X, G_{a,S}) \to \text{Hom}(Y \otimes X_0, G_{a,S_0}).
\]

Proof: By Corollary 3.3.2 it suffices to prove the proposition for the kernel of the homomorphism \( \text{Biext}^0(\mathbb{E}(N)_B, \mathbb{E}(M)_A; G_{a,S}) \to \text{Biext}^0(\mathbb{E}(N)_B, \mathbb{E}(M)_A; G_{a,S_0}). \) Note that \( \text{Biext}^0(\mathbb{E}(N)_B \times_S \mathbb{E}(M)_A; G_{a,S}) \cong \text{Hom}(\mathbb{E}(N)_B, \text{Hom}(\mathbb{E}(M)_A, G_{a,S})). \) Proceeding as in the proof of Proposition 3.3.3 get that \( \text{Hom}(\mathbb{E}(M)_A, G_{a,S}) \cong \text{Hom}(\mathbb{E}(X), G_{a,S}) \cong \text{Hom}(X, G_{a,S}). \) Analogously, since \( \mathbb{E}(N)_B \) is the extension of \( \mathbb{E}(B) \) by \( Y \otimes G_{a,S}, \) we get that \( \text{Hom}(\mathbb{E}(N)_B, X^* \otimes G_{a,S}) \) is isomorphic to \( \text{Hom}(Y \otimes X^* \otimes G_{a,S}). \) The latter is \( \text{Hom}(Y \otimes X, G_{a,S}). \)

3.4 The crystalline nature of the Poincaré biextension

Let \( M_0 := [X_0 \to A_0] \) be a 1-motive over \( S_0 \). Let \( \mathcal{P}_0 \) be the pull back to the universal extension \( \mathbb{E}\left([X_0 \times_{S_0} X_0^\vee \to A_0 \times_{S_0} A_0^\vee]\right)_{A_0 \times_{S_0} A_0^\vee} \) of the Poincaré biextension on \( A_0 \times S_0 A_0^\vee \). Let

\[
\alpha_0 : X_0 \times_{S_0} X_0^\vee \to \mathcal{P}_0
\]
be as in 2.4. Let $M := [X \to G]$ and $M' := [X' \to G']$ be 1-motives over $S$ lifting $M_0$. Let
\[\sigma : E \left( [X \times_S X' \to A \times_S A'] \right) \sim \to E \left( [X' \times_S X'' \to A' \times_S A''] \right)\]
be the isomorphism defined in Lemma 3.2.1. Let
\[\mathcal{P} \longrightarrow E \left( [X \times_S X' \to A \times_S A'] \right)_{A \times_S A'}\]
and
\[\mathcal{P}' \longrightarrow E \left( [X' \times_S X'' \to A' \times_S A''] \right)_{A' \times_S A''}\]
be the pull back of the Poincaré biextension on $A \times_S A'$ (resp. on $A' \times_S A''$). Let
\[\alpha : X \times_S X' \longrightarrow \mathcal{P}\quad \text{and} \quad \alpha_2 : X' \times_S X'' \longrightarrow \mathcal{P}'\]
be the maps defining $E(M)$ (resp. $E(M')$) as in 2) of 2.3. Then,

**Proposition 3.4.1** There is a unique isomorphism
\[\mathcal{P} \sim \to \sigma^*(\mathcal{P}'),\]
as biextensions of $E \left( [X \times_S X' \to A \times_S A'] \right)_{A \times_S A'}$, compatible with the trivializations $\alpha$ and $\alpha'$ via the identification $X \times_S X' \cong X' \times_S X''$ of (7).

**Proof:** By the claimed uniqueness and using descent, we may assume that $X_0 \cong \mathbb{Z}^r$ and that the toric part of $G_0$ is split. This is equivalent to require that $X$, $X'$, $X''$ and $X'$ are constant group schemes. In particular, the map $E \left( [X \times_S X' \to A \times_S A'] \right)_{A \times_S A'} \longrightarrow A \times_S A'$ can be factored via
\[E \left( [X \times_S X' \to A \times_S A'] \right)_{A \times_S A'} \longrightarrow \pi \longrightarrow A \times_S A'.\]
We conclude that $\mathcal{P}$ is also the pull back via $p$ of the pull back $Q$ of the Poincaré biextension on $A \times_S A'$ via $\pi$ and, hence, does not depend on the choice of $p$. Denote by $Q'$ the pull back to $E(A' \times_S A'')$ of the Poincaré biextension on $A' \times_S A''$. Consider the isomorphism
\[\varrho : E \left( A \times S A' \right) \sim \to E(A) \times_S E(A') \longrightarrow E(A') \times_S E(A'') \sim \to E \left( A' \times_S A'' \right)\]
defined in 3.2. Let $G = G_{m,S}$. Consider the element $\varrho^*(Q')Q^{-1}$: it is a biextension of $E(A)$ and $E(A')$ by $G_{m,S}$ whose base change to $E(A_0) \times E(A''_0)$ is canonically trivialized. By Proposition 3.3.3 such biextensions form a subgroup of the group of homomorphisms from $E(A)$ to the sheaf of isomorphism classes of extensions of $\mathbb{W}(A')$ by $G_{a,S}$ endowed with a trivialization over $S_0$. Since $A$ is a divisible sheaf and $\mathbb{W}(A')$ and $G_{a,S}$ are torsion sheaves, such homomorphisms are a subgroup of the group $\mathcal{K}$ of homomorphisms from $\mathbb{W}(A)$ to the sheaf of isomorphism classes of extensions of $\mathbb{W}(A')$ by $G_{a,S}$ (equivalently $G_{m,S}$) endowed with a trivialization over $S_0$. Note that $Q$ defines the trivial $G_{m,S}$-biextension of $\mathbb{W}(A)$ and $\mathbb{W}(A')$ since it is pulled-back from $A \times_S A'$. Thus, the image of $\varrho^*(Q')Q^{-1}$ in $\mathcal{K}$ coincides with the pull-back of $P'$ via the map $\mathbb{W}(A) \times_S \mathbb{W}(A') \rightarrow A' \times_S A''$ induced by $\varrho$. Hence, the image of $\varrho^*(Q')$ in $\mathcal{K}$ lies in the image of homomorphisms from $A'$ to the sheaf of isomorphism classes of extensions of $\mathbb{W}(A')$. 


by \( G_{a,S} \) endowed with a trivialization over \( S_0 \), which is trivial. Therefore, \( \varphi^*(Q') \) is isomorphic to \( Q \) as biextension. We conclude that \( P \) is isomorphic to \( \sigma^*(P') \) as biextensions lifting \( P_0 \).

By Proposition 3.3.4 the automorphism group of a biextension over \( E\bigl( [X \times_S X' \to A \times_S A'] \bigr) \), reducing to the identity after base change to \( S_0 \), is isomorphic to the kernel of

\[
\text{Hom}_{fppf}(X \otimes X^\vee, G_{a,S}) \rightarrow \text{Hom}_{fppf}(X_0 \otimes X_0^\vee, G_{a,S_0}).
\]

The set of maps \( \alpha: X \times_S X^\vee \rightarrow P \), as in 3) of 2.4, deforming the map \( \alpha_0 \) defining \( M_0 \), is a principal homogenous space under the kernel of

\[
\text{Hom}_{fppf}(X \otimes X^\vee, G_{a,S}) \rightarrow \text{Hom}_{fppf}(X_0 \otimes X_0^\vee, G_{a,S_0}).
\]

Hence, the conclusion. ⊓⊔

Corollary 3.4.2 There is a unique isomorphism of \( S \)-group schemes

\[
\zeta: E(M) \rightarrow E(M')
\]

making the following diagram commute

\[
\begin{array}{ccccccc}
0 & \longrightarrow & T & \longrightarrow & E(M) & \longrightarrow & E([X \to A]) & \longrightarrow & 0 \\
& & \tau \downarrow & & \zeta \downarrow & & \xi \downarrow & & \xi \downarrow & & 0,
\end{array}
\]

where \( \xi \) is defined in Lemma 3.2.1 and \( \tau \) is the canonical isomorphism between the tori \( T \) and \( T' \) as deformations of \( T_0 \).

Proof: By 2.4 the universal extension of \( M \) is defined by the universal extension \( E\bigl( [X \times_S X' \to A \times_S A'] \bigr) \) and a trivialization \( \alpha \) of the pull-back \( P \) to \( X \times X' \). Analogous description exists for \( M' \). By Proposition 3.4.1 we get an isomorphism \( E(M) \rightarrow E(M') \) with the claimed properties. Since the homomorphisms from a vector group scheme to a torus and from an abelian scheme to a torus are trivial, we have \( \text{Hom}(E([X \to A]), T') = 0 \). This implies the claimed uniqueness. ⊓⊔

### 3.5 Crystals via universal extensions

Let \( S_0 \) be a scheme such that \( p \) is locally nilpotent. Let \( M_0 \) be a 1-motive over \( S_0 \). Define the crystal of group schemes \( E(M_0) \) on the nilpotent crystalline site of \( S_0 \). Let \( S_0 \rightarrow S \) be a locally nilpotent pd thickening of \( S_0 \). Let \( M \) be a 1-motive lifting \( M_0 \) to \( S \). Then,

\[
E(M_0)(S_0 \leftarrow S) := E(M),
\]

Here \( M_0 \rightarrow S_0 \) can be lifted locally and by virtue of Corollary 3.4.2 we get indeed a crystal which we call the universal extension crystal of the 1-motive.

Define the contravariant functor associating to a 1-motive \( M_0 \) over \( S_0 \) the crystal \( D(M_0) \) on the nilpotent crystalline site of \( S_0 \) as follows:

\[
D(M_0)(S_0 \leftarrow S) := T^{DR}(M).
\]
4 PROOF OF THEOREM A'

See 2.1 for the notation $\mathbf{T}^{\mathrm{DR}}(\mathbb{M})$. Furthermore, let $\mathcal{P}_0$ be the pull back to $E\left(\mathbb{X}_0 \times_{S_0} \mathbb{X}_0^\vee \to \mathbb{A}_0 \times_{S_0} \mathbb{A}_0^\vee\right)$ of the Poincaré biextension on $\mathbb{A}_0 \times_{S_0} \mathbb{A}_0^\vee$. Define the Poincaré crystal of biextensions $\mathcal{P}_0$ over the crystal $E(M_0)$ on the crystalline site of $S_0$ as follows. Let $S_0 \hookrightarrow S$ be as above and let $\mathbb{M}$ be a 1-motive lifting $M_0$ to $S$. Let $\mathcal{P}$ be the pull back to $E\left(\mathbb{X} \times_S \mathbb{X}^\vee \to \mathbb{A} \times_S \mathbb{A}^\vee\right)$ of the Poincaré biextension on $\mathbb{A} \times_S \mathbb{A}^\vee$. Then,

$$\mathcal{P}_0(S_0 \hookrightarrow S) := \mathcal{P}(S).$$

Due to Proposition 3.4.1 this defines a crystal. Define the $\mathcal{O}_{S_0}^{\text{crys}}$-bilinear pairing of crystals

$$\Phi : \mathbf{D}(M_0) \times \mathbf{D}(M_0^\vee) \longrightarrow \mathbf{D}(G_{m,S_0}^\vee)$$

as follows. By [7, Prop 10.2.7.4] there is a unique $\mathbb{Z}$-structure on the biextension $\mathcal{P}$. See [7, 10.2.7.2] for this notion. Its curvature defines an $\mathcal{O}_S$-bilinear map [7, 10.2.7.3]

$$\Phi : \operatorname{Lie} \left( E(M^\vee) \right) \otimes_{\mathcal{O}_S} \operatorname{Lie} \left( E(M) \right) \longrightarrow \operatorname{Lie} \left( G_{m,S} \right)$$

where the Lie algebras are applied to the degree zero component of the complexes, i.e., to connected group schemes, yielding a pairing of filtered vector bundles over $S$.

Remark 3.5.1 If $S_0 \subset S$ is a pd thickening defined by a locally nilpotent ideal, but the pd structure is not locally nilpotent, then the universal extension of a lifting of $M_0$ to $S$ need not be crystalline. This holds, for example, if $p = 2$ and $S$ is a scheme over $W(k)$ with pd structure compatible with the one on the maximal ideal of $W(k)$. On the other hand, thanks to Proposition 2.8.1 and [10, §11], its formal completion at the origin is crystalline. Thus, the crystal $\mathbf{D}(M_0)$ is defined on the full crystalline site of $S_0$.

4 Proof of Theorem A'

Let $M_0$ be a 1-motive defined over a perfect field $S_0 = \text{Spec}(k)$ of positive characteristic $p > 0$. Let $\mathbb{M}$ denote a lifting to $S = \text{Spec}(W_n(k))$. We first show that the crystal $\left[1\right]$ is a filtered $F$-crystal, i.e., we show how to get Frobenius and Verschiebung compatibly with the weight filtration.

4.1 Frobenius and Verschiebung

Let $\sigma : \text{Spec}(k) \rightarrow \text{Spec}(k)$ be the Frobenius map defined on $k$ by $x \mapsto x^p$. By abuse of notation we denote by $\sigma : \text{Spec}(W_n(k)) \rightarrow \text{Spec}(W_n(k))$ also the map associated to the Frobenius map on Witt vectors. Let $M_0^{(p)}$ be the pull-back of $M_0$ via $\sigma$ (as complex of group schemes, i.e., $M_0^{(p)} := \left[ X_0^{(p)} \to G_0^{(p)} \right]$). The Frobenius map on $M_0$ defines a morphism of 1-motives $F : M_0 \rightarrow M_0^{(p)}$ over $k$. Given the lifting $\mathbb{M}$ of $M_0$ to $W_n(k)$, then $\sigma^*(\mathbb{M})$ is a lift of $M_0^{(p)}$ to the truncated Witt vectors $W_n(k)$. Hence, we obtain a map, called Verschiebung,

$$V : \operatorname{Lie} \left( E(M) \right) \longrightarrow \operatorname{Lie} \left( \sigma^*(E(M)) \right) \cong \operatorname{Lie} \left( E(M) \right) \otimes_{\sigma} W_n(k).$$

The Frobenius map on $M_0^{(p)}$ defines by duality a morphism of 1-motives $V : M_0^{(p)} \rightarrow M_0^{(p)}$ over $k$. Note that $F \circ V = p$ and $V \circ F = p$: this is clear on the lattice $X_0$, on the torus $T_0$ and it is
classical on the the abelian part $A_0$. The Verschiebung on $M_0$ defines a homomorphism, called Frobenius,

$$F : \text{Lie} \left( \mathbb{E}(M) \right) \otimes_\sigma W_n(k) \rightarrow \text{Lie} \left( \mathbb{E}(M) \right).$$

Then, $F \circ V = p$ and $V \circ F = p$. Analogously, Frobenius on $M_0^\vee$ defines $V$ on $\text{Lie} \left( \mathbb{E}(M_0^\vee) \right)$ and Frobenius on $M_0$ defines, by duality, Verschiebung on $M_0^\vee$ and Frobenius on $\text{Lie} \left( \mathbb{E}(M_0^\vee) \right)$. Clearly, $F$ and $V$ preserve the weight filtrations. Passing to the limit over $n$ we then obtain $F$ and $V$ on $\lim_n D(\mathbb{M}_0) \left( \text{Spec}(k) \subset \text{Spec}(W_n(k)) \right)$ and $\lim_n D(\mathbb{M}_0^\vee) \left( \text{Spec}(k) \subset \text{Spec}(W_n(k)) \right)$ over $W(k)$.

Note that, for $M := G_m$ we get $T_{\text{crys}}(M) = W(k)$ as $W(k)$-module, with $F$ sending $1 \mapsto 1$ and $V$ sending $1 \mapsto p$, i.e., $T_{\text{crys}}([0 \rightarrow G_m]) = W(k)(1)$ as $F$-crystals according with the notation adopted above.

Finally, we remark that Frobenius defines a map from the Poincaré biextension $P_0$ on $\mathbb{E} \left( [X_0 \times_{S_0} X_0^\vee \rightarrow A_0 \times_{S_0} A_0^\vee] \right)$ to the Poincaré biextension $P_0^{(p)}$ on $\mathbb{E} \left( [X_0^{(p)} \times_{S_0} (X_0^\vee)^{(p)} \rightarrow A_0^{(p)} \times_{S_0} (A_0^\vee)^{(p)}] \right)$. It follows that, for $p \geq 3$, the pairing (13) satisfies $V_{G_m} \circ \Phi_M = p \sigma^{-1} \circ \Phi_M = \Phi_{M^{(p)}} \circ (V_M \times V_M^\vee)$ and, thus, $p F_{G_m} \circ \Phi_M = \Phi_M \circ (F_M \times F_M)$.

### 4.2 The comparison

We then show that $T_{\text{crys}}(M_0) = T_{\text{DR}}(M)$. By [16] §15 the contravariant Dieudonné module of a $p$-divisible group over $k$ is canonically isomorphic to the Lie algebra of the universal extension of a lifting to $W(k)$ of the dual $p$-divisible group. It follows from 2.8.1 applied to $S = \text{Spec}(W_n(k))$ that $D(M_0) (S_0 \subset S) = T_{\text{DR}}(M)$, defined in [11], and $D(M_0[p^\infty]) (S_0 \subset S)$, defined in [13], are canonically isomorphic as filtered $W_n(k)$-modules. By definition of $T_{\text{crys}}$ and $T_{\text{crys}}$, see Definition 1.3.1 we get canonical isomorphisms $T_{\text{crys}}(M_0) \cong \lim_n D(M_0) \left( \text{Spec}(k) \subset \text{Spec}(W_n(k)) \right)$ and, since $M_0[p^\infty]^\vee = M_0[p^\infty]$, $T_{\text{crys}}(M_0) \cong \lim_n D(M_0^\vee) \left( \text{Spec}(k) \subset \text{Spec}(W_n(k)) \right)$ as filtered $W(k)$-modules. By the functoriality claimed in Proposition 2.8.1 they are compatible with Frobenius and Verschiebung. This proves the first claim of Theorem A’.

The second claim of Theorem A’ follows from the above and 4.1 at least for $p \geq 3$, or from [5] for any $p$. Note that, for $p \geq 3$, both the pairings (13) and (5) are induced by the Poincaré extension on a formal lifting of $M[p^\infty]$ and $M$, respectively, to $W(k)$. One can prove that, indeed, the two pairings are the same. Since we will not use this compatibility in the sequel, we omit the proof and leave it to the reader.

We also have obtained the following result.

**Corollary 4.2.1** Let $R$ be a complete discrete valuation ring with perfect residue field $k$ and with ramification index smaller than $p - 1$. Let $M$ be a $1$-motive over $\text{Spec}(R)$. Then, there is a canonical isomorphism

$$T_{\text{crys}}(M_k) \otimes_{W(k)} R \cong T_{\text{DR}}(M).$$

**Proof:** The hypothesis on the ramification implies that the maximal ideal of $R$ is endowed with canonical divided powers structure compatible with that on the maximal ideal of $W(k)$. The $1$-motive $M_n$ obtained base changing $M$ to $R/(p^n)$ is a lifting of $M_k$. Let $M'_n$ be a lifting...
of $M_k$ to $W_n(k)$. It follows from \[2.8.1\] that we have canonical isomorphisms $\text{Lie} \left( E(M_n[p^\infty]) \right) \cong \text{Lie} \left( E(M_n[p^\infty]) \otimes W(k) \right) \cong D(M_k[p^\infty]) \left( \text{Spec}(k) \subset \text{Spec}(W_n(k)) \right) \otimes W(k) R$. ☐

### 4.3 Properties of $T_{\text{crys}}(M)$

We explicitly state some of the properties of $T_{\text{crys}}(M)$ (similarly for $T^{\text{crys}}(M)$), which can be deduced using either its definition as the Dieudonné module of $M[p^\infty]^\vee$ or via $D(M)$:

1) it is a free $W(k)$-module of rank equal to $\dim(G) + \text{rk}(\mathcal{X})$;

2) it admits a weight filtration $W$

   2.a) $W_{\geq 0} \left( T_{\text{crys}}(M) \right) = T_{\text{crys}}(M)$;

   2.b) $W_{-1} \left( T_{\text{crys}}(M) \right) = T_{\text{crys}}(G)$. It is a free $W(k)$-module of rank $\dim(G)$;

   2.c) $W_{-2} \left( T_{\text{crys}}(M) \right) = T_{\text{crys}}(T)$. It is a free $W(k)$-module of rank $\dim(T)$;

   2.d) $W_{\leq -3} \left( T_{\text{crys}}(M) \right) = 0$;

3) the graded pieces of the weight filtration satisfy

   3.a) $\text{Gr}^W_2 \left( T_{\text{crys}}(M) \right) := T_{\text{crys}}([0 \to T])$. It is a free $W(k)$-module of rank $\dim(T)$;

   3.b) $\text{Gr}^W_{-1} \left( T_{\text{crys}}(M) \right) := T_{\text{crys}}([0 \to A])$. It is a free $W(k)$-module of rank $\dim(A)$;

   3.c) $\text{Gr}^W_0 \left( T_{\text{crys}}(M) \right) := T_{\text{crys}}([\mathcal{X} \to 0])$. It is a free $W(k)$-module of rank $\text{rk}(\mathcal{X})$.

4) Let $\sigma$ be the Frobenius homomorphism on $W(k)$. Then $T_{\text{crys}}(M)$ is endowed with a $\sigma$-linear morphism $F$ and a $\sigma^{-1}$-linear morphism $V$ such that

   4.a) they respect the weight filtration;

   4.b) $F \circ V = V \circ F = p$;

   4.c) $V$ is an isomorphism on $\text{Gr}^W_0 \left( T_{\text{crys}}(M) \right)$;

   4.d) $F$ is an isomorphism on $\text{Gr}^W_2 \left( T_{\text{crys}}(M) \right)$;

5) there is a perfect bilinear pairing of filtered $W(k)$-modules

   $\langle -, - \rangle : T_{\text{crys}}(M) \otimes W(k) T^{\text{crys}}(M) \rightarrow W(k)(1). \quad (14)$

   satisfying $\langle -, - \rangle \sigma^{-1} = (V(-), V(-))$ and $p\langle -, - \rangle \sigma = \langle F(-), F(-) \rangle$.

### 5 Picard 1-motives and $\zeta$-structures

Let $S_0 = \text{Spec}(k)$ where $k$ is any field. Let $\pi : X_\bullet : S_0$ be a simplicial $S_0$-scheme. Note that the simplicial structure provide a complex of $S_0$-schemes in the sense of [1] \[2\]. Let $\pi_i : X_i : S_0$ denote the structure morphisms. We say that $\pi$ is proper, smooth, etc. if each $\pi_i$ is proper, smooth, etc. Assume $X_\bullet$ proper over $S_0$ so that, for each component $X_i$, the usual Picard
fppf-sheaf $\text{Pic}_{X_i/S_0} := R^1(\pi_i)_* \mathbb{G}_{m,X_i}$ is representable by a (commutative) group scheme, locally of finite type over $S_0$. We refer to [2] for the general framework of Picard functors.

If, in addition $S_0 = \text{Spec}(k)$ is a perfect field, and $X_i$ is smooth over $S_0$ then the reduced connected component of the identity $\text{Pic}_{X_i/S_0}^{0,\text{red}}$ is an abelian scheme over $S_0$, i.e., this is the case if $\pi$ is proper and smooth over $S_0$ (see [2]).

5.1 $\text{Pic}^+$

For any simplicial $S_0$-scheme $X$, let $\text{Pic}(X_\cdot)$ be the group of isomorphism classes of simplicial line bundles on $X_\cdot$ (we refer to [2, §4.1 & A.3] for the basic properties of the simplicial Picard functor). By descent we see that

$$\text{Pic}(X_\cdot) \cong H^1_{\text{ét}}(\mathcal{F}, \mathbb{G}_{m,X_\cdot}) \cong H^1_{\text{fppf}}(\mathcal{F}, \mathbb{G}_{m,X_\cdot}).$$

Denote $T \mapsto \text{Pic}_{X_\cdot/S_0}(T)$ the Picard fppf-sheaf obtained by sheafifying the functor

$$T \mapsto \text{Pic}(X_\cdot \times_{S_0} T)$$

with respect to the fppf-topology on $S_0$, i.e., if $\pi : X_\cdot \times_{S_0} T \to T$, then

$$\text{Pic}_{X_\cdot/S_0}(T) \cong H^0_{\text{fppf}}(T, R^1\pi_* \mathbb{G}_{m,X_\cdot \times_{S_0} T}).$$

Considering the canonical spectral sequence (see [7, 5.2.7.1])

$$E^{p,q}_1 = R^q(\pi_p)_* \mathbb{G}_{m,X_p} \Rightarrow R^{p+q}\pi_* \mathbb{G}_{m,X},$$

we obtain an exact sequence of fppf-sheaves:

$$0 \to \ker((\pi_1)_* \mathbb{G}_{m,X_1} \to (\pi_2)_* \mathbb{G}_{m,X_2}) \to \text{Pic}_{X_\cdot/S_0} \to \text{Ker} (\text{Pic}_{X_0/S_0} \to \text{Pic}_{X_1/S_0})$$

We then have the following result.

**Lemma 5.1.1** Let $X_\cdot$ be a smooth proper simplicial $S_0$-scheme, $S_0 = \text{Spec}(k)$. Then the Picard fppf-sheaf $\text{Pic}_{X_\cdot/S_0}^{0,\text{red}}$ is representable by a group scheme, locally of finite type over $S_0$. Denote by $\text{Pic}_{X_\cdot/S_0}^{0,\text{red}}$ the connected component of the identity of the simplicial Picard scheme of $X_\cdot$ over a perfect field $S_0 = \text{Spec}(k)$, endowed with its reduced structure: it is a semi-abelian scheme, extension of the abelian scheme

$$\text{Ker}^{0,\text{red}} (\text{Pic}_{X_0/S_0}^{0,\text{red}} \to \text{Pic}_{X_1/S_0}^{0,\text{red}})$$

by a torus $T$ (here $\text{Ker}^{0,\text{red}}$ denote the identity component of the kernel endowed with its reduced structure).

**Proof:** It follows from [2, Lemma 4.1.2] and the proof of [2, Prop. 4.1.3] (see also [22, 3.2-3.5]). Actually, using (15) it follows from (16) and the representability of $\text{Pic}_{X_i/S_0}$, since the fppf-sheaf $(\pi_i)_* \mathbb{G}_{m,X_i}$ is representable by a torus and the resulting extensions are representable, e.g., by [20, Prop. 17.4].
Remark 5.1.2 We describe the torus $T$ appearing in Lemma 5.1.1. Consider the homology of the following complex of tori (cf. 15) and 16)

$$(\pi_0)_*\mathbb{G}_m, X_0 \to (\pi_1)_*\mathbb{G}_m, X_1 \to (\pi_2)_*\mathbb{G}_m, X_2.$$ 

The torus $T$ is given by the quotient of $T_2 := \text{Ker}^{0,\text{red}}((\pi_1)_*\mathbb{G}_m, X_1 \to (\pi_2)_*\mathbb{G}_m, X_2)$ by $T_1 := \text{Im}((\pi_0)_*\mathbb{G}_m, X_0 \to (\pi_1)_*\mathbb{G}_m, X_1)$. We also describe the cocharacter group of $T$ as follows. Assume that $S_0 = \text{Spec} (k)$ and $k$ is algebraically closed. Let $C_2, C_1$ and $C_0$ be the free abelian groups generated by the connected (= irreducible) components of $X_2, X_1$ and $X_0$ respectively. Let

$$C_2 \xrightarrow{d_2} C_1 \xrightarrow{d_1} C_0$$

be the group homomorphisms defined by the alternating sums of the faces maps of the simplicial scheme. Let

$$C^0 \xrightarrow{d^1} C^1 \xrightarrow{d^2} C^2$$

be the dual complex, actually given by the cocharacters of the above complex of tori. The map $C_1 \to C_0$ associates to a component $X_{1,j}$ of $X_1$ the element $X_{0,h} - X_{0,i}$ where $d^1_0(X_{1,j}) \subset X_{0,h}$ and $d^1_1(X_{1,j}) \subset X_{0,i}$. In particular, the image of $C_1 \to C_0$ is a direct summand of $C_0$. This implies that the kernel of $(\pi_0)_*\mathbb{G}_m, X_0 \to (\pi_1)_*\mathbb{G}_m, X_1$ is a torus. In particular, the group of cocharacters of $T_1$ coincides with $\text{Im} d^1$. Since $\text{Ker} d^2$ is the cocharacter group of $T_2$, we conclude that the cocharacter group of $T$ is the free group $\text{Ker} d^2/\text{Im} d^1$.

Let $(X, Y)$ be a simplicial pair over $S_0$ such that $X_0$ is as above (cf. Lemma 5.1.1) and $Y_0 \subseteq X_0$ is a reduced simplicial $S_0$-divisor with strict normal crossings. Here we have that every $Y_i$ is a divisor with strict normal crossings in $X_i$, i.e., $Y_i = \cup Y_{ij}$ such that $Y_{ij}$ is smooth over $S_0$ and $Y_{ij}$ has pure codimension 1 in $X_i$. For example, by de Jong’s theory of alterations, every algebraic variety $V$ over a perfect field $S_0 = \text{Spec} (k)$ admits a proper hypercovering $V \to V'$ such that $V' = X \times Y$ for some $(X, Y)$ as above, and moreover $V_0 \to V$ is generically étale. Consider the fppf-sheaf $\text{Pic}_{[X, Y]/S_0}$ associated to the presheaf (cf. [7, 5.1.13.1])

$$T \mapsto \mathbb{H}^1_{Y_0, S_0} T (X_0, X_0, T_0, \mathbb{G}_m).$$

We obtain a canonical map (forgetting the support, cf. [2, §4.2])

$$\tilde{u} : \text{Pic}_{[X, Y]/S_0} \to \text{Pic}_{X, Y}/S_0$$

Since $S_0 = \text{Spec} (k)$ and $k$ is a perfect field, we then describe $u$ as follows. Let $\overline{S_0} = \text{Spec} (\overline{k})$ be some algebraic closure of $k$. Let $\text{Gal}(\overline{k}/k)$ be the Galois group. Let $\overline{X}_0$ and $\overline{Y}_0$ denote the base change to $\overline{S_0}$. Denote

$$\text{Div}_{Y_0, \overline{X}_0} (\overline{X}_0)$$

the group of those Weil divisors $D$ on $X_0 \times_{\overline{S_0}} \overline{S_0}$, supported on $Y_0 \times_{\overline{S_0}} \overline{S_0}$, such that $d^0_0(D) = d^1_0(D)$ as divisors on $X_1 \times_{\overline{S_0}} \overline{S_0}$ (note that on $X_0$ and $X_1$ Cartier and Weil divisors coincide).

The group $\text{Div}_{Y_0, \overline{X}_0} (\overline{X}_0)$ is naturally a $\text{Gal}(\overline{k}/k)$-module with respect to the Galois action on Weil divisors. Moreover, we have a natural identification of the group of $\overline{S_0}$-points of $\text{Pic}_{[\overline{X}_0, \overline{Y}_0]/\overline{S_0}}$ with $\text{Div}_{Y_0, \overline{X}_0} (\overline{X}_0)$. In fact, the analogous spectral sequence 17 for the cohomology with supports
degenerates over $\overline{S_0}$ and yields the claimed group isomorphism (see [2] (33 p. 56]). We then obtain canonical Gal $(\overline{k}/k)$-equivariant maps

$$
\begin{align*}
Div_Y(\overline{X_r}) & \xrightarrow{u'} \text{Pic}(\overline{X_r}) \\
\downarrow & \downarrow \\
\text{Pic}_{[X_r,Y_r]/S_0}(\overline{S_0}) & \longrightarrow \text{Pic}_{X_r/S_0}(\overline{S_0})
\end{align*}
$$

(18)

where $u'$ is the canonical map sending a Weil divisor $D$ on $\overline{X_0}$ to the corresponding line bundle $O_{\overline{X_0}}(D)$. Let $\text{Pic}_{[X_r,Y_r]/S_0}^{0,\text{red}}$ be the inverse image of $\text{Pic}_{X_r/S_0}^{0,\text{red}} \subset \text{Pic}_{X_r/S_0}$ under the map $u$. Remark that $\text{Pic}_{X_r/S_0}^{0,\text{red}}$ is stable under the Gal $(\overline{k}/k)$-action on $\text{Pic}_{X_r/S_0} \times_{S_0} \overline{S_0}$.

The following definition is equivalent to [2, Def. 4.2.1] and [22, Def. 4.1] but the proof of the independence of the choices made, for positive characteristics, appears in our Appendix A only.

**Definition 5.1.3** Let $(X_r,Y_r)$ be a pair as above. Define

$$
\text{Pic}^+(X_r,Y_r) := [\text{Pic}_{[X_r,Y_r]/S_0}^{0,\text{red}} u \rightarrow \text{Pic}_{X_r/S_0}^{0,\text{red}}].
$$

If $V_r = X_r - Y_r$ is an hypercovering of an algebraic variety $V$ over a perfect field (such that $V_0 \rightarrow V$ is generically étale) we let

$$
\text{Pic}^+(V) := \text{Pic}^+(X_r,Y_r)
$$

denote the cohomological Picard 1-motive of $V$ (see the Appendix A for the fact that $\text{Pic}^+$ is well defined and contravariantly functorial).

Conversely, we remark that the 1-motive $\text{Pic}^+(X_r,Y_r)$ over $S_0$ is determined by the Gal $(\overline{k}/k)$-equivariant map

$$
\begin{align*}
Div_Y(\overline{X_r}) & \xrightarrow{u'} \text{Pic}(\overline{X_r}) \cong \text{Pic}_{X_r/S_0}(\overline{S_0}) \\
\text{Div}_{Y}^{0}(\overline{X_r}) & \text{Pic}^+(V)_{\overline{k}} \equiv \text{Pic}^+(V_{\overline{k}})
\end{align*}
$$

over $\overline{S_0}$, in particular $\text{Pic}^+(V)_{\overline{k}} = \text{Pic}^+(V_{\overline{k}})$. We may and will denote $\text{Div}_{Y}^{0}(\overline{X_r})$ the subgroup of those divisors in $\text{Div}_{Y}^{0}(\overline{X_r})$ mapping to zero in $\text{NS}(\overline{X_r}) := \pi_0(\text{Pic}_{X_r/S_0})$.

Now let $S_0 \hookrightarrow S_n$ be a thickening defined by an ideal with nilpotent divided powers. As above, we are mainly interested in the case that the thickening $S_n = \text{Spec}(W_{n+1}(k))$ is the affine scheme defined by the Witt vectors of length $n + 1$ (or equivalently by $W(k)/p^{n+1}$). We then always get a lifting $\text{Pic}^+(V)_n$ of the 1-motive $\text{Pic}^+(V)$ to $S_n$.

Remark that if $(X_r,Y_r)$ itself lift (this is not the case in general!) to a similar pair $(X_n,Y_n)_n$ over $S_n$ and $\text{Pic}$ is representable over $S_n$, we may set

$$
\text{Pic}^+(V)_n = \text{Pic}^+((X_r,Y_r)_n).
$$

Nevertheless, a formal lifting of the 1-motive $\text{Pic}^+(V)$ always exists.
5.2 $\mathbf{Pic}^\natural$

Recall (see [2, §4.5]) the construction of the functor $\mathbf{Pic}^\natural_{X_+/S_0}$. Here $S_0$ is the spectrum of a perfect field and $X_+$ is smooth and proper over $S_0$.

The group $\mathbf{Pic}^\natural(X_+)$ is given by isomorphism classes of pairs $(\mathcal{L}_+, \nabla_+)$, where $\mathcal{L}_+$ is a simplicial line bundle and $\nabla_+$ is a simplicial integrable connection

$$\nabla_+: \mathcal{L}_+ \to \mathcal{L}_+ \otimes_{\mathcal{O}_{X_+}} \Omega^1_{X_+/S_0}.$$ 

There is a natural map

$$\mathbf{Pic}^\natural(X_+) \to \mathbb{H}^1(X_+, \mathcal{O}_{X_+}^\times \xrightarrow{\text{dlog}} \Omega^1_{X_+/S_0}).$$

In characteristic zero this map is an isomorphism and provides the group of $k$-points of the universal extension of $\mathbf{Pic}^0$, whose Lie algebra is the first De Rham cohomology (see [2, 4.5]), i.e., the simplicial $\natural$-Picard functor $\mathbf{Pic}^\natural_{X_+/S_0}$, obtained by sheafifying the functor $T \mapsto \mathbf{Pic}^\natural(X_+ \times_{S_0} T)$

with respect to the fppf-topology on $S_0$, yields an exact sequence of Zariski (or étale) sheaves

$$0 \to \pi_* \Omega^1_{X_+/S_0} \to \mathbf{Pic}^\natural_{X_+/S_0} \to \mathbf{Pic}_{X_+/S_0} \to R^1\pi_* \Omega^1_{X_+/S_0} \quad (19)$$

and

$$\text{Lie } \mathbf{Pic}^\natural_{X_+/S_0} \cong H^1_{\text{DR}}(X_+/S_0) \quad (20)$$

over $S_0$ of characteristic zero.

In positive characteristics one dimensional De Rham cohomology cannot be recovered as above, e.g., we only have an injection

$$H^1_{\text{DR}}(X_+/S_0) \subseteq \mathbb{H}^1(X_+, \mathcal{O}_{X_+} \to \Omega^1_{X_+/S_0})$$

determined by the Oda subspace $H^0(X_0, \Omega^1_{X_0/S_0})_{d=0}$ (cf. [19, p. 121]).

However, for curves, $\text{Pic}^+$ coincides with Deligne’s definition of motivic cohomology and its universal extension is given by the $\natural$-Picard functor (see [7, 10.3.13] and cf. [19, 5.2]). A similar case is given by $X_+$ such that each $X_i$ has connected components which are abelian schemes (cf. [19, 5.1]).

Actually, whenever the Hodge-De Rham spectral sequence degenerates we get De Rham cohomology as above.

**Lemma 5.2.1** The $\natural$-Picard sheaf $\mathbf{Pic}^\natural_{X_+/S_0}$ is representable, by a group scheme locally of finite type over $S_0$. We have an extension

$$0 \to \pi_* \Omega^1_{X_+/S_0,d=0} \to \mathbf{Pic}^\natural_{X_+/S_0} \to \mathbf{Pic}^0_{X_+/S_0} \to 0$$

where we set $\Omega^*_{X_+/S_0,d=0}$ according to Oda [19, Def. 5.5].

If we further assume that the Hodge-De Rham spectral sequence of the components $X_i$ degenerates then [19] and [2] hold over $S_0$. 


Proof: Representability follows by a similar argument to the proof of Lemma 5.1.1, since Pic$^\natural_{X_i/S_0}$ is representable, e.g., by using [20, Prop. 17.4].

By applying the arguments in [16, 4.1.2] and [2, p. 62] to our setting we can see that the indeterminacy in putting an integrable connection on the trivial simplicial line bundle is the space of closed simplicial 1-forms. Moreover, the obstruction map in putting an integrable connection on a simplicial line bundle defines a homomorphism from Pic$^\natural_{X_i/S_0}$ to the vector group $R^1\pi_*\Omega^1_{X_i/S_0,d=0}$. Thus, when restricted to the semiabelian scheme Pic$^0_{X_i/S_0}$ it is trivial, proving the exactness on the right of the sequence in the Lemma. Note that here Pic$^\natural,0_{X_i/S_0}$ is just the pullback of Pic$^0_{0,red}X_i/S_0$ along Pic$^\natural_{X_i/S_0} \to$ Pic$^\natural X_i/S_0$.

For the components $X_i$ such that all global 1-forms are closed the arguments provided by [2, 4.5.1] apply; an easy spectral sequence and Lie algebra computations yields the result (cf. [16, §4]).

Note that if we consider the component $X_i$ of any smooth proper simplicial scheme $X$, and let $A_i$ denote Alb$_{X_i/S_0} := (\text{Pic}^0_{X_i/S_0})^\vee$, the Albanese (abelian) scheme, we have that the following extension

$$0 \to (\pi_0)_*\Omega^1_{A_i/S_0} \to \text{Pic}^{0,0}_{A_i/S_0} \to \text{Pic}^{0,red}_{X_i/S_0} \to 0$$

is the universal extension of the abelian scheme Pic$^{0,red}_{X_i/S_0}$.

6 Log-crystalline cohomology

We refer the reader to [15, §5] for the basics on log-crystalline theory.

6.1 Logarithmic structures

Let $k$ be a perfect field of characteristic $p$. For every $n \in \mathbb{N}$, let $(S_n, L_n, \gamma_n)$ be the triple consisting of

1) $S_n := \text{Spec} (W_{n+1}(k))$ the spectrum of the Witt vectors of length $n + 1$ over $k$;

2) $L_n$ the trivial logarithmic structure on $S_n$ defined by the map (of multiplicative monoids) $\mathbb{N} \to W_{n+1}(k)$ given by $1 \mapsto 0$;

3) $\gamma_n$ the standard divided powers structure on the ideal $\mathcal{I}_n := (p)$ of $W_{n}(k)$.

Assume these data to be compatible for varying $n$. As usual we view the category of schemes over $S_n$ as the full subcategory of log-schemes over $S_n$ with logarithmic structure induced by $L_n$.

Remark 6.1.1 (cf. [16, §1.5]) Let $X$ be a smooth projective scheme over $S_0$. Let $Y \subset X$ be a divisor with normal crossings. Let $M$ be the logarithmic structure defined by $Y$ on $X$ as follows. We can cover $X$ with affine schemes $\{U_i = \text{Spec} (A_i)\}_i$, étale over $X$, such that for each $i$ there exists irreducible elements $\{\pi_{i,\alpha}\}_\alpha \subset A_i$ such that $Y \cap U_i$ is defined by the ideal $\prod_\alpha \pi_{i,\alpha}$. Then, $M_{U_i} = \left\{ g \in A_i[\pi_{i,\alpha}^{\pm 1}]_\alpha \right\}$ and coincides with the logarithmic structure associated to the pre-logarithmic structure

$$\prod_\alpha \mathbb{N} \to A_i, \quad (a_\alpha)_\alpha \mapsto \prod_\alpha \pi_{i,\alpha}^a.$$
6.1.2 Log structures and crystals

We continue with the discussion in 6.1.1. Let \((U_i, T_i), M_{T_i}, \iota, \delta)\) be an open set for the logarithmic crystalline site of \((X, M)\) over \((S_n, L_n, \gamma_n)\) defined in 6.1. Then,

\[
\iota: (U_i, M_{U_i}) \hookrightarrow (T_i, M_{T_i})
\]

is an exact closed immersion of logarithmic schemes defined by an ideal \(I_i\) with divided powers \(\delta; [15, \S 5.2]\). In particular, \(I_i\) is a nilpotent ideal and the log-structure on \(U_i\) is the one induced from \(T_i\). Therefore, for every \(\alpha\) there exists \(m_{i,\alpha} \in \Gamma(T_i, \mathcal{O}_{T_i})\) such that \(\iota^*(m_{i,\alpha}) \in A_i^*: \pi_{i,\alpha}\). Any two elements \(m_{i,\alpha}\) and \(m'_{i,\alpha}\), having the same property, satisfy

\[
m_{i,\alpha}m'^{-1}_{i,\alpha} \in \Gamma(T_i, \mathcal{O}_{T_i}).
\]

Hence, the ideal \((m_{i,\alpha})\) defines a Cartier divisor on \(T_i\) relative to \(S_0\) lifting the Cartier divisor \((\pi_{i,\alpha})\) on \(U_i\). Hence, for every element \(\pi\) in the group \(M_{\log}^\text{gp}\) associated to \(M\) we get a well defined invertible sheaf \(\mathcal{O}_{T_i}[\pi^{-1}]\) on \(T_i\) lifting the invertible sheaf on \(U_i\) defined by \(\pi\).

Let \((X_i, Y_i)\) be a simplicial pair over \(S_0\) such that \(X_i\) is a projective and smooth simplicial scheme over \(S_0\) and \(Y_i \subseteq X_i\) is a simplicial divisor with normal crossings relative to \(S_0\). Identify the pair \((X_i, Y_i)\) with the scheme \(X_i\) with the fine logarithmic structure defined by \(Y_i\). Then, \((X_i, Y_i)\) lies over \((S_0, L_0)\) in the sense of log schemes and it is log-smooth over \((S_0, L_0)\) [15 Ex 3.7].

**Definition 6.1.3** A simplicial logarithmic pair over \((S_0, L_0)\) is a simplicial pair \((X_i, Y_i)\) over \(S_0\) as above where \((X_i, Y_i)\) is regarded as a scheme \(X_i\) with the fine logarithmic structure defined by \(Y_i\).

6.2 Čech coverings

Let \((X, M)\) be a logarithmic scheme log-smooth over \((S_0, L_0)\). Assume that \(X\) is projective and smooth (in the classical sense) over \(S_0\). Let \(\{U_i\}_i\) be a covering family of \(X\) by affine schemes étale over \(X\). We choose and fix a total order on the set of indices \(\{i\}\). Each \(U_i\) lifts to an open \(U_i \hookrightarrow V_i\) in the logarithmic site of \((X, M)\) over \((S_n, L_n)\) so that \(V_i\) is smooth over \(S_n\). In particular, \(V_i\) is flat over \(S_n\) and \(\mathcal{I}_n V_i\) has a unique PD structure extending the one on \(\mathcal{I}_n\). Then, \(\{U_i \hookrightarrow V_i\}_i\) is a covering family in the site \(\mathcal{(X, M)/(S_n, L_n)}\)\(\log\). For any \(i < j\) let \(U_{i,j} := U_i \cap U_j\). Since \(X\) is separated, \(U_{i,j}\) is an affine scheme. Let \(V^i_j\) (resp. \(V^j_i\)) be the open subscheme of \(V_i\) (resp. \(V_j\)) defined by \(U_{i,j}\). They are isomorphic. Define \(V_{i,j} := V^i_j\). Then \(V_{i,j}\) is the logarithmic divided power envelope of the thickening \(U_{i,j} \subseteq V_{i,j}\) [15 Def 5.4]. Continuing in this fashion one defines

\[
U_{i_\bullet} \hookrightarrow V_{i_\bullet}
\]

for every \(n + 1\)-uple of indices \(i_\bullet = (i_0, \ldots, i_n)\). One gets an hypercovering

\[
\cdots \amalg i_0 < i_1 U_{(i_0, i_1)} \Rightarrow \amalg U_i
\]

and a Leray spectral sequence

\[
E_2^{s,t} = H^s_{\log}(\amalg i_0 < \cdots < i_s \amalg U_{(i_0, \ldots, i_s)}/(S_n, L_n)) \Rightarrow H^{s+t}_{\log}(\mathcal{(X, M)/(S_n, L_n)}).
\]
By [15, Thm 6.4] we have
\[
H^t_{\log \cryst}(U_{i_0,\ldots,i_s}/(S_n,L_n)) \cong H^t(V_{i_0,\ldots,i_s},\omega^*_V(S_n,L_n));
\]
where \(\omega^*,\log\) is the de Rham complex of differentials defined in the logarithmic sense [15, §1.7].

**Lemma 6.2.1** The homology of the total complex of the bicomplex \(\oplus_{i_0<\cdots<i_s}\omega^{*,\log}_{V_{i_0,\ldots,i_s}}/(S_n,L_n)\) is
\[
H^{s+t}_{\log \cryst}((X,M)/(S_n,L_n)).
\]

### 6.3 Crystalline versus log-crystalline

The notation is as in [62]. Let \((X,L_0)\) be the scheme \(X\) with logarithmic structure defined by pull back of \(L_0\). The morphism \(f: (X, M) \to (X, L_0)\) induces a natural morphism of topoi [15, §5.9]
\[
f^*_{\log \cryst}: ((X,M)/(S_n,L_n))^\log \cryst \longrightarrow ((X,L_0)/(S_n,L_n))^\log \cryst = (X/S_n)^{\cryst}.
\]
Let \(I\) be a sheaf on \((X/S_n)^{\cryst}\). Let \((U,M_U) \hookrightarrow (T,M_T)\) be an open on \((X,M)^{\log \cryst}\) defined by a PD ideal, then \(f^*_{\log \cryst}(I)((U,M_U)) \subset (T,M_T)) = I(U \subset T)\). This implies that \(f^*_{\log \cryst}\) is exact and commutes with taking global sections. In particular, let \(\mathcal{O}_X^{\cryst} \to I^*\) and \(\mathcal{O}_{(X,M)}^{\log \cryst} \to J^*\) be injective resolutions. By the exactness of \(f^*_{\log \cryst}\) we get a commutative diagram with exact rows
\[
\begin{array}{ccc}
0 & \longrightarrow & f^*_{\log \cryst}(\mathcal{O}_X^{\cryst}) \longrightarrow f^*_{\log \cryst}(I^*) \longrightarrow \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{O}_{(X,M)}^{\log \cryst} \longrightarrow J^*.
\end{array}
\]

Taking global sections we obtain for every \(i \in \mathbb{N}\) homomorphisms
\[
H^i_{\cryst}(X/S_n) \longrightarrow H^i_{\log \cryst}((X,M)/(S_n,L_n)).
\]
We wish to determine kernel and cokernel of the homomorphism for \(i = 0\) and 1. To do that we use the spectral sequence of the 'Čech cohomology' for the log-crystalline as in Lemma 6.2.1. Since we are considering and comparing two logarithmic structures on \(X\), we write a superscript log whenever we consider \((X,M)\) and we use no superscript whenever we consider \((X,L_0)\). Fix indices \(i_0 < \cdots < i_s\). Let \(\alpha: M_{V_{i_0,\ldots,i_s}} \to \mathcal{O}_{V_{i_0,\ldots,i_s}}\) be the logarithmic structure on \(V_{i_0,\ldots,i_s}\). For every \(r\) and \(n\) let
\[
\mathcal{Q}^r_{i_0,\ldots,i_s} := \omega^{r,\log}_{V_{i_0,\ldots,i_s}}/(S_n,L_n)/\omega^{r}_{i_0,\ldots,i_s}/S_n
\]
be the quotient of the \(r\)-th wedge product of the logarithmic differentials of \((V_{i_0,\ldots,i_s},M_{V_{i_0,\ldots,i_s}})\) relative to \((S_n,L_n)\) by the \(r\)-th wedge product of the usual Kähler differentials of \(V_{i_0,\ldots,i_s}\) relative to \(S_n\). By possibly shrinking the open subschemes \(U_i\) and by the assumptions of logarithmic smoothness, we may assume that there exists \(\{v_1,\ldots,v_w\}\) global sections of \(\mathcal{O}_{V_{i_0,\ldots,i_s}}\) and \(\{m_1,\ldots,m_z\}\) global sections of \(M_{V_{i_0,\ldots,i_s}}\) such that
\[
\omega^{1,\log}_{V_{i_0,\ldots,i_s}} = \mathcal{O}_{V_{i_0,\ldots,i_s}}dv_1 + \cdots + \mathcal{O}_{V_{i_0,\ldots,i_s}}dv_w + \mathcal{O}_{V_{i_0,\ldots,i_s}}d\log(m_1) + \cdots + d\log(m_z).
\]
In particular,
\[
\mathcal{Q}^1_{i_0,\ldots,i_s} \cong \mathcal{O}_{V_{i_0,\ldots,i_s}}/m_1\mathcal{O}_{V_{i_0,\ldots,i_s}} + \cdots + \mathcal{O}_{V_{i_0,\ldots,i_s}}/m_z\mathcal{O}_{V_{i_0,\ldots,i_s}}.
\]
Consider the following diagram

\[
\begin{array}{cccccccc}
0 & \to & \oplus_i \mathcal{O}_{V_i} & \to & \oplus_{i,j} \mathcal{O}_{V_{i,j}} & \oplus_i \omega^1_{V_i/S_n} & \to & \oplus_{i,j,k} \mathcal{O}_{V_{i,j,k}} & \oplus_i \omega^2_{V_i/S_n} & \oplus_i j \omega^1_{V_i/S_n} & \oplus_i j \omega^1_{V_i/S_n} \\
| & & & & & & & & & & \\
0 & \to & \oplus_i \mathcal{O}_{V_i} & \to & \oplus_{i,j} \mathcal{O}_{V_{i,j}} & \oplus_i \omega^1_{V_i/(S_n,L_n)} & \to & \oplus_{i,j,k} \mathcal{O}_{V_{i,j,k}} & \oplus_i j \omega^2_{V_i/(S_n,L_n)} & \oplus_i j \omega^1_{V_i/(S_n,L_n)} & \\
0 & \to & \oplus_i Q^1_i & \to & \oplus_i j Q^1_{i,j} & \oplus_i Q^2_i & \\
\end{array}
\]

By Lemma 6.2.11, the homology of the first row computes \( H^*_\text{crys}(X/S_n) \), while the homology of the second row computes \( H^*_\text{logcrys}((X,M)/(S_n,L_n)) \).

**Corollary 6.3.1** The group \( H^0_{\text{logcrys}}((X,M)/(S_n,L_n)) \) is equal to \( H^0_{\text{crys}}(X/S_n) \) and is a locally free \( \mathcal{O}_{S_n} \)-module of rank equal to the number of the geometric irreducible components of \( X \to S_0 \).

**Proof:** By 6.3, the formation of \( H^0_{\text{crys}}(X/S_n) \) and of \( H^0_{\text{logcrys}}((X,M)/(S_n,L_n)) \) commute with field extensions of \( S_0 \). Hence, we may assume that \( X \to S_0 \) is geometrically irreducible. Furthermore, both groups injects in

\[
K_n := \text{Ker} \left( \prod_i \Gamma(V_i, \mathcal{O}_{V_i}) \to \prod_{i<j} \Gamma(V_{i,j}, \mathcal{O}_{V_{i,j}}) \right)
\]

and we have inclusions \( \mathcal{O}_{S_n} \hookrightarrow H^0_{\text{crys}}(X/S_n) \hookrightarrow H^0_{\text{logcrys}}((X,M)/S_n) \). It suffices to prove that the composite \( \mathcal{O}_{S_n} \hookrightarrow K_n \) is an isomorphism. Proceeding by induction on \( n \) and since for every \( n \) we have the exact sequence \( 0 \to K_n \xrightarrow{L^n} K_{n+1} \to K_n \), one concludes.

**Proposition 6.3.2** Let \( M \) denote the log-structure defined by a strict normal crossing divisor \( Y \) in \( X \) smooth and projective over \( S_0 \). We have an exact sequence

\[
0 \to H^1_{\text{crys}}(X) \to H^1_{\text{logcrys}}(X,M) \to \text{Div}_Y(X) \otimes_{\mathbb{Z}} W(k) \to H^2_{\text{crys}}(X).
\]

The map \( \text{Div}_Y(X) \otimes_{\mathbb{Z}} W(k) \to H^2_{\text{crys}}(X) \) is defined via the crystalline first Chern class and its kernel coincides with \( \text{Div}_Y^1(X) \otimes_{\mathbb{Z}} W(k) \).

**Proof:** We let \( Q^n_r \) denote the complex \( 0 \to \oplus_i Q^n_{i} \to \oplus_{i,j} Q^n_{i,j} \to \cdots \). By the arguments in 6.3 for every \( n \in \mathbb{N} \), we have an exact sequence

\[
0 \to H^1_{\text{crys}}(X/S_n) \to H^1_{\text{logcrys}}((X,M)/(S_n,L_n)) \to \text{Ker} \left( H^0(Q^1_1) \to H^0(Q^2_2) \right) \to H^2_{\text{crys}}(X/S_n).
\]

The claimed exact sequence follows inspecting such exact sequence and taking limits over \( n \in \mathbb{N} \). Note that all groups appearing above are finitely generated \( W_{n+1}(k) \)-modules and, thus, satisfy the Mittag-Leffler condition. We further may assume that \( k = \overline{k} \) is algebraically closed. For \( j = 1, \ldots, z \) the image of \( d(\log(m_j)) \) via the derivation \( \omega^1_{V_i/(S_n,L_n)} \to \omega^2_{V_i/(S_n,L_n)} \) is \( 0 \). Hence, the map \( H^0(Q^1_1) \to \oplus_i Q^2_i \) is zero and \( \text{Ker} \left( H^0(Q^1_1) \to H^0(Q^2_2) \right) = H^0(Q^1_1) \). Since the log-structure \( M \) is defined by a strict normal crossing divisor \( Y \) of \( X \), if \( \{ Y_j \} \) is the set of
irreducible components of $Y$, it follows from Corollary 6.3.1 that $H^0(Q_1^1) = \prod_j H^0_{\text{crys}}(Y_j/S_n) = \text{Div}_Y(X) \otimes_{\mathbb{Z}} W_{n+1}(k)$. The first Chern class map
\[
c_1^{\text{crys}} : H^1(X,\mathcal{O}_X^*) \to H^2_{\text{crys}}(X/S_n) \quad \text{resp. } c_1^{\text{logcrys}} : H^1(X,\mathcal{O}_X^*) \to H^2_{\logcrys}(X/(S_n, L_n))
\]
is defined taking the long exact sequence of cohomology groups associated to the short exact sequence $0 \to 1 + p\mathcal{O}_{X/S_n}^* \to \mathcal{O}_{X/S_n}^* \to \mathcal{O}_{X/S_0}^* \to 0$ and the map $\log : 1 + p\mathcal{O}_{X/S_n}^* \to \mathcal{O}_{X/S_n}^*$ (analogously for $c_1^{\text{logcrys}}$). Comparing $c_1^{\text{crys}}$ and $c_1^{\text{logcrys}}$ via the long exact sequence relating the crystalline and the log–crystalline cohomology groups, one sees that the composite of the projection $H^1_{\logcrys}(X/S_n,\mathcal{O}_{X/S_n}^{\logcrys*}) \to H^1(X,\mathcal{O}_X^*)$ and $c_1^{\text{crys}}$ factors via the connecting homomorphism $H^0(Q_1^1) \to H^2_{\text{crys}}(X/S_n)$ of (21). As remarked in 6.1.2, $\text{Div}_Y(X) \to H^1(X,\mathcal{O}_X^*)$ factors via $H^1_{\logcrys}(X/S_n,\mathcal{O}_{X/S_n}^{\logcrys*})$. Thus, $c_1^{\text{crys}}$ restricted to $\text{Div}_Y(X)$ factors as $\text{Div}_Y(X) \to H^0(Q_1^1) \to H^2_{\text{crys}}(X/S_n)$.

By [10] II.6.8.0] and the comparison between crystalline and de Rham-Witt cohomology for proper and smooth schemes over perfect fields, the Chern class $c_1^{\text{crys}}$ induces an injection $\text{NS}(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p \hookrightarrow H^2_{\text{crys}}(X)$. By [10] II.5.10.1 and [10] II.6.8.4] the map $\text{NS}(X) \otimes_{\mathbb{Z}} W(k) \to H^2_{\text{crys}}(X)$ is injective as well. Since $\text{Div}_{\text{crys}}^0(X)$ is the subgroup of $\text{Div}_Y(X)$ of divisors algebraically equivalent to zero, the conclusion follows.

\section{Crystalline via simplicial log-crystalline}

Let $(X_i, Y_i)$ be a simplicial pair and $n \in \mathbb{N}$. For every sheaf $\mathcal{F}_i$ of abelian groups over the simplicial logarithmic crystalline site $(X_i, Y_i)/(S_n, L_n, \gamma_n)$ define
\[
\mathbb{H}_{\logcrys}^i((X_i, Y_i)/(S_n, L_n, \gamma_n), \mathcal{F}_i)
\]
as the right derived functor of the following left exact functor
\[
\mathcal{F}_i \mapsto \ker \left( \Gamma\left( (X_0, Y_0)/(S_n, L_n, \gamma_n), \mathcal{F}_0 \right) \xrightarrow{(d_0)^*} \Gamma\left( (X_1, Y_1)/(S_n, L_n, \gamma_n), \mathcal{F}_1 \right) \right).
\]
Note that each $\mathcal{F}_i$ is a sheaf on the scheme $X_i$ with the logarithmic structure. Define
\[
\mathbb{H}_{\logcrys}^i((X_i, Y_i)/(S_n, L_n, \gamma_n))
\]
as $\mathbb{H}_{\logcrys}$ of the structure sheaf $\mathcal{O}_{X_i}$. We write $\mathbb{H}_{\logcrys}^i((X_i, Y_i)/S_n)$ whenever the logarithmic structure and the divided powers structures $(L_n, \gamma_n)$ are the ones fixed in 6.1 Define
\[
\mathbb{H}_{\logcrys}^i(X_i, Y_i) := \lim_{\mathcal{H}_{\logcrys}^i((X_i, Y_i)/S_n)}.
\]
Denote $\mathbb{H}_{\text{crys}}^i(V_i/W(k)) := \mathbb{H}_{\logcrys}^i(X_i, Y_i)$ where $V_i = X_i - Y_i$ is the corresponding smooth simplicial scheme.
6.5 A spectral sequence, \( H^1_{\text{logcrys}} \) and \( H^0_{\text{logcrys}} \)

From the spectral sequence

\[
E^{p,q}_1 = H^q_{\text{logcrys}}((X_p, Y_p)/S_n) \Rightarrow \mathbb{H}^{p+q}_{\text{logcrys}}((X, Y)/S_n).
\]

we deduce exact sequences describing \( H^1_{\text{crys}} \) and \( H^0_{\text{crys}} \):

\[
0 \to \ker \left( H^0_{\text{logcrys}}((X_1, Y_1)/S_n) \to H^0_{\text{logcrys}}((X_2, Y_2)/S_n) \right) \to H^1_{\text{logcrys}}((X, Y)/S_n) \to \ker \left( H^1_{\text{logcrys}}((X_0, Y_0)/S_n) \to H^1_{\text{logcrys}}((X_1, Y_1)/S_n) \right) \to \ker \left( H^0_{\text{logcrys}}((X_2, Y_2)/S_n) \to H^0_{\text{logcrys}}((X_3, Y_3)/S_n) \right) \to \ker \left( H^0_{\text{logcrys}}((X_1, Y_1)/S_n) \to H^0_{\text{logcrys}}((X_2, Y_2)/S_n) \right).
\]

The arrows are defined by alternating sums of pull-backs along the faces of the simplicial structure (cf. [2] §4). By Corollary 6.3.1 we may replace \( H^0_{\text{logcrys}} \) with \( H^0_{\text{crys}} \) everywhere. Analogously, we have a spectral sequence giving us \( H^0_{\text{logcrys}} \) as

\[
\mathbb{H}^0_{\text{logcrys}}(X, S_n) = \ker \left( H^0_{\text{logcrys}}(X_0/S_n) \to H^0_{\text{logcrys}}(X_1/S_n) \right).
\]

In particular, \( \mathbb{H}^0_{\text{logcrys}}((X, Y)/S_n) = \mathbb{H}^0_{\text{crys}}(X, S_n) \). Taking limits over \( n \in \mathbb{N} \) one concludes that

\[
\frac{\ker \left( H^0_{\text{logcrys}}(X_1, Y_1) \to H^0_{\text{logcrys}}(X_2, Y_2) \right)}{\ker \left( H^0_{\text{logcrys}}(X_0, Y_0) \to H^0_{\text{logcrys}}(X_1, Y_1) \right)}
\]

is a free \( W(k) \)-module. In fact, in the notation of Remark 5.1.2 the latter is identified with \( (\ker d^2/\text{Im} d^1) \otimes W(k) \) and \( d^2/\text{Im} d^1 \) is a free abelian group.

6.6 Properties of \( H^1_{\text{logcrys}}(X, Y) \)

The \( W_{n+1}(k) \)-modules appearing in 6.6.5 are finitely generated and, thus, they satisfy the Mittag-Leffler condition. In particular, we get a canonical (decreasing) weight filtration \( W \) on our crystalline cohomology \( H^1_{\text{crys}}(V/W(k)) := H^1_{\text{logcrys}}(X, Y) \) as follows:

- \( W_{\geq 2} := H^1_{\text{logcrys}}(X, Y) \);
- \( W_1 := H^1_{\text{crys}}(X) \) which is a \( W(k) \)-submodule of \( H^1_{\text{logcrys}}(X, Y) \) by 6.3.1 and 24;
- \( W_0 := \frac{\ker \left( H^0_{\text{crys}}(X_1) \to H^0_{\text{crys}}(X_2) \right)}{\text{Im} \left( H^0_{\text{crys}}(X_0) \to H^0_{\text{crys}}(X_1) \right)} \) which is a free \( W(k) \)-submodule of \( H^1_{\text{crys}}(X) \) by 23 and the discussion in 6.5.
- \( W_{<0} := 0 \).

Moreover we have the following description of the graded pieces \( \text{Gr}^W_i \)
7 PROOF OF THEOREM B'

- $\text{Gr}_2^W$ over $k = \overline{k}$, by 7.3.1 and 7.3.2, is contained in the $W(k)$-module $\text{Div}_X^1(T, X_{\dagger}) \otimes W(k)$: here $\text{Div}_X^1(T, X_{\dagger})$ are the divisors defined in 5.1. Therefore, $\text{Gr}_2^W$ is a finitely generated free $W(k)$-module;

- $\text{Gr}_1^W$ is contained in $\ker \left( H^1_{\text{crys}}(X_0) \rightarrow H^1_{\text{crys}}(X_1) \right)$. In particular, $\text{Gr}_1^W$ is a finitely generated, free $W(k)$-module by [11 Prop. II.3.11].

For $p > 2$ it follows from the proof of Theorem B', see 7.3.2 and 7.3.6, that in fact

$$\text{Gr}_1^W = \ker \left( H^1_{\text{crys}}(X_0) \rightarrow H^1_{\text{crys}}(X_1) \right)$$

and, by 7.5.3, also $\text{Gr}_2^W = \text{Div}_X^1(T, X_{\dagger}) \otimes W(k)$. Anyway, we can already conclude that $H^1_{\text{crys}}(V, W(k))$ is a finitely generated free $W(k)$-module.

Next, we show that $H^1_{\text{crys}}(V, W(k))$ is naturally endowed with the structure of a filtered $W(k)$-module. The filtration has been discussed above. We define Frobenius. Let $(X_{\dagger}^{(p)}, Y_{\dagger}^{(p)})$ be the simplicial scheme obtained pulling back $(X_{\dagger}, Y_{\dagger})$ via the Frobenius map $\text{Spec}(k) \rightarrow \text{Spec}(k)$. Then, Frobenius on $(X_{\dagger}, Y_{\dagger})$ defines a morphism of simplicial $k$-schemes $(X_{\dagger}, Y_{\dagger}) \rightarrow (X_{\dagger}^{(p)}, Y_{\dagger}^{(p)})$ and, hence, a $W(k)$-linear map

$$H^1_{\text{logcrys}}(X_{\dagger}^{(p)}, Y_{\dagger}^{(p)}) \cong H^1_{\text{logcrys}}(X_{\dagger}, Y_{\dagger}) \otimes W(k) \rightarrow H^1_{\text{logcrys}}(X_{\dagger}, Y_{\dagger})$$

($\sigma =$Frobenius on $W(k)$), which respects the weights filtration by functoriality. In conclusion, we obtain a $\sigma$-linear map

$$F: H^1_{\text{logcrys}}(X_{\dagger}, Y_{\dagger}) \rightarrow H^1_{\text{logcrys}}(X_{\dagger}, Y_{\dagger}).$$

7 Proof of Theorem B'

Our main task here is to compute, given a variety $V$ over a perfect field $k$ of characteristic $p \geq 3$, the Lie algebra of the universal vector extension of a lifting of $\Pic^+ (V)$ to the Witt vector. See 24 for some remarks on the characteristic 2 case.

7.1 Intermediate functors

First construct some simplicial variations on themes from 16 and log-crystalline variations on themes from [2].

7.1.1 The functor $\Pic^{\text{logcrys}, 0}$

Assume that $p \geq 3$. Let $(X_{\dagger}, Y_{\dagger})$ be a simplicial logarithmic pair, see 6.1. Thus $X_{\dagger}$ is a simplicial log-scheme over $S_0$. Let

$$\Pic^{\text{logcrys}, 0}_{(X_{\dagger}, Y_{\dagger})/S_0} \subset \Pic^{\text{logcrys}}_{(X_{\dagger}, Y_{\dagger})/S_n}$$

be the sheaves over the fppf site of $S_n$ associated to the following presheaves. Let $T$ be a scheme flat and of finite presentation over $S_n$. Consider the logarithmic and divided powers structure on $T$ induced from those on $S_n$. Let $\Pic^{\text{logcrys}}((X_{\dagger}, Y_{\dagger})/T)$ be the group of isomorphism
classes of simplicial crystals of invertible $\mathcal{O}_{X_1 \times S_n T/T}^{\logcrys}$-modules for the nilpotent crystalline site of $X_1 \times S_n T$ relative to $T$. There is a canonical pull back to the Zariski site $(X_1 \times S_n T)_{zar}$. Let $\text{Pic}^{\logcrys,0}((X_1, Y)/T)$ denote the subgroup of sections of $\text{Pic}^{\logcrys}((X_1, Y)/T)$ which land in $\text{Pic}^{0,\red}_{X_1/S_0}(T \times S_n S_0)$ (cf. 5.1). We might describe an element of $\text{Pic}^{\logcrys,0}((X_1, Y)/T)$ as the giving of a pair $(L, \alpha)$

\begin{enumerate}
  \item $L$ yields an element of $\text{Pic}^{\logcrys,0}((X_0, Y_0)/T)$;
  \item and the isomorphism $\alpha: (d_0^1)^* (L) \sim (d_1^1)^*(L)$ as elements of $\text{Pic}^{\logcrys,0}((X_1, Y_1)/T)$ satisfying the cocycle condition
    $$((d_1^2)^*(\alpha))^{-1} \circ ((d_2^2)^*(\alpha)) \circ ((d_0^2)^*(\alpha)) = \text{Id}.$$\end{enumerate}

Analogous description holds for $\text{Pic}^{\logcrys}((X_0, Y)/T)$. For the sake of notation, in the following, we sometimes omit reference to the pair $(X_1, Y_1)$ if it is clear from the context.

**Lemma 7.1.2** We have an exact sequence of functors, respecting the group laws

\[
\begin{array}{c}
0 \rightarrow \text{Ker}^\dagger \left( \pi_{1,*}^{\logcrys} (G_{m, X_1}) \rightarrow \pi_{2,*}^{\logcrys} (G_{m, X_2}) \right) \\
\text{Im} \left( \pi_{0,*}^{\logcrys} (G_{m, X_0}) \rightarrow \pi_{1,*}^{\logcrys} (G_{m, X_1}) \right) \rightarrow \text{Pic}^{\logcrys,0}_{(X_1, Y_1)/S_n} \\
\rightarrow \text{Ker} \left( \text{Pic}^{\logcrys,0}_{(X_0, Y_0)/S_n} \rightarrow \text{Pic}^{\logcrys,0}_{(X_1, Y_1)/S_n} \right)
\end{array}
\]

where $\text{Ker}^\dagger (-)$ denotes the subsheaf of $\text{Ker} (-)$ of those elements which land in $\text{Ker}^{0,\red}(-)$ over $S_0$.

**Proof:** Here, for every scheme $T$ flat over $S_n$ and any element $(L, \alpha)$ of $\text{Pic}^{\logcrys,0}((X_1, Y)/T)$, the canonical map is $(L, \alpha) \mapsto L$. The kernel consists of the automorphisms of $(d_0^1)^*(\mathcal{O}_{X_0 \times T}^{\logcrys}) \sim (d_1^1)^*(\mathcal{O}_{X_0 \times T}^{\logcrys})$ satisfying the usual cocycle condition modulo the automorphisms of $\mathcal{O}_{X_0 \times T}^{\logcrys}$. This description proves the lemma with the correction $\text{Ker}^\dagger (-)$ according to the description of the toric part of $\text{Pic}^{0,\red}_{X_1/S_0}$ in Remark 5.1.2.

**7.1.3 Dual numbers**

Let $Z$ be a log-scheme and $(I_Z, \gamma)$ a nilpotent ideal with divided powers structure. Let $Z[\varepsilon] := \text{Spec} (\mathcal{O}_Z[\varepsilon]/(\varepsilon^2))$ be the scheme of dual numbers over $Z$. Then, $I_Z \mathcal{O}_Z[\varepsilon]/(\varepsilon^2)$ is endowed with a unique divided powers structure extending the one on $I_Z$; cf. [4] Cor. 3.22. Let $L_n$ be the unique fine logarithmic structure extending the one on $Z$. Let

\[
i_Z: Z \rightarrow Z[\varepsilon] \quad \text{and} \quad j_Z: Z[\varepsilon] \rightarrow Z
\]

be the closed immersion defined by $\varepsilon = 0$ and, respectively, the natural projection map. These maps are compatible with logarithmic and divided powers structures.
7.1.4 The functor $\text{InfDef}$

Let

$$\text{InfDef} \left( O_{X_i/S_n}^{\text{logcrys}} \right)$$

be the group of infinitesimal deformations of the structural sheaf $O_{X_i/S_n}^{\text{logcrys}}$ or, equivalently, the group of isomorphism classes of pairs $(\mathcal{M}, \tau_{\mathcal{M}})$ where

i) $\mathcal{M}$ is a crystal of invertible $O_{X_i \times S_n S_n[\varepsilon]/S_n[\varepsilon]}^{\text{logcrys}}$ modules,

ii) $\tau_{\mathcal{M}}: \ i_{X_i,\text{logcrys}}^* (\mathcal{M}) \sim O_{X_i/S_n}^{\text{logcrys}}$ is an isomorphism of $O_{X_i/S_n}^{\text{logcrys}}$-modules.

The group structure is defined by the tensor product as $O_{X_i \times S_n S_n[\varepsilon]/S_n[\varepsilon]}^{\text{logcrys}}$ modules. The identity element is equal to $O_{X_i \times S_n S_n[\varepsilon]/S_n[\varepsilon]}^{\text{logcrys}}$. Multiplication by elements of $1 + \varepsilon W_n(\kappa)$ induces the structure of $W_n(\kappa)$-module. We might describe an element of $\text{InfDef} \left( O_{X_i/S_n}^{\text{logcrys}} \right)$ as the giving of a pair $(L, \alpha)$

i) $L$ yields an element of $\text{InfDef} \left( O_{X_0/S_n}^{\text{logcrys}} \right)$;

ii) $\alpha: (d_0^\dagger)^* (L) \sim (d_1^\dagger)^* (L)$ as elements of $\text{InfDef} \left( O_{X_i/S_n}^{\text{logcrys}} \right)$ (i.e., as sheaves of $O_{X_i/S_n[\varepsilon]}^{\text{logcrys}}$-modules that $i_{X_i,\text{logcrys}}^* (\alpha) = \text{Id}$) satisfying the cocycle condition

$$( (d_1^\dagger)^* (\alpha) )^{-1} \circ ( (d_2^\dagger)^* (\alpha) ) \circ ( (d_0^\dagger)^* (\alpha) ) = \text{Id}.$$ 

Note that the functor $\text{InfDef}$ is functorial in $X_i$ and, in particular, the Frobenius $X_i \to X_i^{(p)}$ defines a $\sigma$-linear homomorphism on $\text{InfDef} \left( O_{X_i/S_n}^{\text{logcrys}} \right)$.

We link $\text{InfDef}$ to $\text{Pic}^{\text{logcrys}}$ as follows. For every $i$ let $f_i: (X_i[\varepsilon]/S_n[\varepsilon])^{\text{logcrys}} \to (X_i/S_n)^{\text{logcrys}}$ be the standard morphism of topos: for $(U \subset T, \delta, L)$ an object of the crystalline site of $X_i$ relative to $S_n$ we let $f_i^{-1}(U \subset T, \delta, L)$ be $U[\varepsilon] \subset T[\varepsilon]$. As in [16, II.1.5] we get an exact sequence

$0 \to O_{X_i/S_n} \to (O_{X_i/S_n}^{\text{logcrys}})^{\dagger} \to (O_{X_i/S_n}^{\text{logcrys}})^{\dagger} \to 0$ compatibly with the simplicial structure.

Let $\text{Lie} \left( \text{Pic}^{\text{logcrys}}_{(X, Y_i)/S_n} \right)$ be the group of $S_n[\varepsilon]$-valued points of $\text{Pic}^{\text{logcrys}}_{(X, Y_i)/S_n}$ reducing to the identity modulo $\varepsilon$. We conclude that

$$\text{Lie} \left( \text{Pic}^{\text{logcrys}}_{(X, Y_i)/S_n} \right) \sim \text{InfDef} \left( O_{X_i/S_n}^{\text{logcrys}} \right).$$

Analogously, let $\text{Lie} \left( \text{Pic}^{\text{logcrys},0}_{(X, Y_i)/S_n} \right)$ be the group of $S_n[\varepsilon]$-valued points of $\text{Pic}^{\text{logcrys},0}_{(X, Y_i)/S_n}$ reducing to the identity modulo $\varepsilon$. Then, we have a canonical, Frobenius equivariant homomorphism

$$\text{Lie} \left( \text{Pic}^{\text{logcrys},0}_{(X, Y_i)/S_n} \right) \to \text{Lie} \left( \text{Pic}^{\text{logcrys}}_{(X, Y_i)/S_n} \right) \sim \text{InfDef} \left( O_{X_i/S_n}^{\text{logcrys}} \right).$$
Lemma 7.1.5 We have a canonical isomorphism
\[ \text{InfDef} \left( O^\text{logcrys}_{X_*/S_n} \right) \cong H^1_{\text{logcrys}} \left( (X_*, Y_*/S_n) \right) \]
compatibly with Frobenius and the exact sequences:

\[ 0 \to \text{Ker} \left( \text{Aut} \left( O^\text{logcrys}_{X_1/S_n[\varepsilon]} \right) \to \text{Aut} \left( O^\text{logcrys}_{X_2/S_n[\varepsilon]} \right) \right) \to \text{InfDef} \left( O^\text{logcrys}_{X_*/S_n} \right) \]
\[ \to \text{Ker} \left( \text{InfDef} \left( O^\text{logcrys}_{X_0/S_n} \right) \to \text{InfDef} \left( O^\text{logcrys}_{X_1/S_n} \right) \right) \]

and

\[ 0 \to \text{Ker} \left( H^0_{\text{logcrys}} \left( X_1/S_n \right) \to H^0_{\text{logcrys}} \left( X_2/S_n \right) \right) \to H^1_{\text{logcrys}} \left( (X_*, Y_*/S_n) \right) \]
\[ \to \text{Ker} \left( H^1_{\text{logcrys}} \left( (X_0, Y_0)/S_n \right) \to H^1_{\text{logcrys}} \left( (X_1, Y_1)/S_n \right) \right) \]

where \( \text{Aut} \left( O^\text{logcrys}_{X_1/S_n[\varepsilon]} \right) \) consists of the automorphisms reducing to the identity via \( i^*_{X_1, \text{logcrys}} \).

Proof: Using the identification \( \text{Lie} \left( \text{Pic}^\text{logcrys} \left( (X_*, Y_*/T) \right) \right) \cong \text{InfDef} \left( O^\text{logcrys}_{X_*/S_n} \right) \), the argument is a simplicial variant of [16, §II.1.5]. Note that Frobenius on \( (O^\text{logcrys}_{X_i[\varepsilon]/S_n[\varepsilon]})^* \) induces the Frobenius on \( O^\text{logcrys}_{X_i/S_n[\varepsilon]} \).

Lemma 7.1.6 Let \( X \) and \( Y \) be projective and smooth schemes over \( k \). Then,
\[ \text{Pic}^\text{logcrys,0} \left( X \times_k Y/S_n \right) = \text{Pic}^\text{logcrys,0} \left( X/S_n \right) \times \text{Pic}^\text{logcrys,0} \left( Y/S_n \right). \]

Proof: We proceed by induction on \( n \). If \( n = 0 \), then \( S_0 = \text{Spec} \left( k \right) \) and for every logarithmic scheme \( Z \) smooth and log-smooth over \( k \) the group \( \text{Pic}^\text{logcrys,0} \left( X \times_k Z/k \right) \) is the group of isomorphism classes of invertible sheaves on \( Z \times_k \mathbb{F} \) algebraically equivalent to \( 0 \) and endowed with a logarithmic integrable connection. Hence, denoting by \( \omega^1_{Z/k} \) the sheaf of logarithmic differentials, we have an exact sequence
\[ 0 \to H^0 \left( Z, \omega^1_{Z/k} \right)_{d=0} \to \text{Pic}^\text{logcrys,0} \left( Z/k \right) \to \text{Pic}^\text{logcrys,0} \left( Z/k \right). \]

Note that \( \text{Pic}^\text{0,red} \left( X \times_k Y/k \right) \cong \text{Pic}^\text{0,red} \left( X/k \right) \times_k \text{Pic}^\text{0,red} \left( Y/k \right) \) (see [5]). Let \( \pi_1 \) and \( \pi_2 \) be the two projections from \( X \times_k Y \) to \( X \) and \( Y \) respectively. Then, \( \omega^1_{X \times_k Y/k} = \pi_1^* (\omega^1_{X/k}) \oplus \pi_2^* (\omega^1_{Y/k}) \). Since \( X \) and \( Y \) are projective we have, \( \pi_1_* (\pi_1^* (\omega^1_{X/k})) = \omega^1_{X/k} \) and \( \pi_2_* (\pi_2^* (\omega^1_{Y/k})) = \omega^1_{Y/k} \). Hence
\[ H^0 \left( X \times_k Y, \omega^1_{X \times_k Y/k} \right)_{d=0} \cong H^0 \left( X, \omega^1_{X/k} \right)_{d=0} \oplus H^0 \left( Y, \omega^1_{Y/k} \right)_{d=0}. \]
We conclude that the lemma holds for \( n = 0 \). Suppose that the lemma is proven for \( n \). Pulling back via the projections \( X \times_k Y \to X \) and \( X \times_k Y \to Y \) we obtain the following commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
0 \\
\downarrow \\
H^1_{\logcrys}(X/k) \oplus H^1_{\logcrys}(Y/k) \\
\downarrow \phi^n \\
Pic^{\logcrys,0}_{X/S_{n+1}} \times Pic^{\logcrys,0}_{Y/S_{n+1}} \\
\downarrow \\
Pic^{\logcrys,0}_{X \times_k Y/S_{n+1}} \\
\end{array}
\end{array} \xrightarrow{\sim} \begin{array}{c}
\begin{array}{c}
0 \\
\downarrow \\
H^1_{\logcrys}(X \times_k Y/k) \\
\downarrow \phi^n \\
Pic^{\logcrys,0}_{X \times_k Y/S_{n+1}} \\
\downarrow \\
Pic^{\logcrys,0}_{X \times_k Y/S_n}.
\end{array}
\end{array}
\]

Indeed, the kernel of the bottom vertical right hand side map is isomorphic to the crystals of invertible modules on the crystalline site of \( X \times_k Y \) relative to \( S_{n+1} \) reducing to the trivial crystal \( \mathcal{O}^{\logcrys}_{X \times_k Y/S_n} \). Using ‘Cech cohomology’ by 6.2, it can be identified with the cohomology group \( H^1_{\logcrys}(X \times_k Y/k) \otimes_k (p^n \mathbf{W}_{n+1}(k)) \). One gets an analogous description of the kernel of the bottom vertical left hand side map. The bottom horizontal isomorphism exists by inductive hypothesis, while the top horizontal isomorphism follows from the equality \( H^1_{\logcrys}(X \times_k Y/k) = \text{Lie} \left( Pic^{\logcrys,0}_{X \times_k Y/k} \right) \) and the \( n = 0 \) case.

7.2 The case of abelian varieties

This is Grothendieck’s remark [11], cf. [16]. Let \( B_0 \) be an abelian variety over \( S_0 \). Since the deformation functor of abelian varieties is unobstructed, there exists an abelian scheme \( B_n \) over \( S_n \) lifting \( B_0 \). Then, \( B_n[\varepsilon] \) is a lifting of \( B_0 \), smooth over \( S_n[\varepsilon] \). Hence, the category of crystals of invertible \( \mathcal{O}^{\text{crys}}_{B_0 \times S_0, S_n[\varepsilon]/S_n[\varepsilon]} \)-modules over the nilpotent crystalline site of \( B_0 \times S_0 \) \( S_n[\varepsilon]/S_n[\varepsilon] \) relative to \( S_n[\varepsilon] \) is equivalent to the category of line bundles over \( B_n[\varepsilon] \) with integrable connection. Let \( \mathbb{E}(B_n^\vee) \) be the universal extension of the abelian scheme dual to \( B_n \). It actually classifies isomorphism classes of line bundles over \( B_n \) algebraically equivalent to 0 and endowed with integrable connection. Hence, we have an equivalence of functors over the fppf site of \( S_n \), compatible with group structures,

\[
\mathbb{E}(B_n^\vee) \cong \text{Pic}^{\logcrys,0}_{B_n/S_n} \cong \text{Pic}^{\logcrys,0}_{B_0/S_n}.
\]

Taking \( \text{Lie} \) we get a natural isomorphism of \( \mathcal{O}_{S_n} \)-modules

\[
\text{Pic}^{\logcrys,0}_{B_0/S_n} \cong \text{Pic}^{\logcrys,0}_{B_0/S_n} \cong \text{InfDef} \left( \mathcal{O}^{\logcrys}_{B_0/S_n} \right).
\]

7.3 The compact case

Here we assume that \( Y = \emptyset \), thus all logarithmic structures are trivial. We then write \( \text{Pic}^{\text{crys,0}} \) for the functor \( \text{Pic}^{\logcrys,0} \) defined in the previous sections. We also assume that each irreducible component of \( X_0 \), \( X_1 \) and \( X_2 \) is geometrically irreducible. For the general case, since \( k \) is assumed to be perfect we pass to a separable closure \( k^{\text{sep}} \) of \( k \). Then, one easily verifies that the morphisms \( \phi_i \) in 7.3.1 and consequently \( \phi^{\ast}_{\text{crys}} \) in 7.3.6 are \( \text{Gal}(k^{\text{sep}}/k) \)-equivariant.
7.3.1 Comparison via the Albanese

Let $i = 0, 1$ or $2$. Let $X_i := I\!I X_i^j$ be the irreducible components of $X_i$. Define

$$X_i^{[2]} := \Pi_j X_i^j \times_k X_i^j, \quad \text{and} \quad A(X_i) := \Pi_j \text{Alb} (X_i^j/k).$$

For every $j$ there exists a unique morphism $\phi^j_i : X_i^j \times_k X_i^j \to \text{Alb} (X_i^j/k)$ such that for every $x_i^j \in X_i^j(k)$ the map $\phi^j_i (x_i^j, \cdot) : X_i^j \to \text{Alb} (X_i^j/k)$ is the Albanese map sending $x_i^j \mapsto 0$. Let

$$\Pi_j \phi^j_i =: \phi_i : X_i^{[2]} \longrightarrow A(X_i)$$

be the induced map. By the functoriality of the Albanese construction it is easily checked that the maps $d_0^j, d_1^j : X_i^j \to X_0^j$ and $d_0^j, d_1^j, d_2^j : X_2^j \to X_1^j$ and maps $d_0^j, d_1^j : A(X_1) \to A(X_0)$ and $d_0^j, d_1^j, d_2^j : A(X_2) \to A(X_1)$. Then, $(X_i^{[2]}, d^j_i)$ and $(A(X_i), d^j_i)$ are simplicial schemes over $k$ and the maps $\{\phi_i : X_i^{[2]} \to A(X_i)\}_{i=0,1,2}$ define a map of 2-truncated simplicial schemes. By functoriality of $\text{Pic}^{\text{crys},0}$ we have the map

$$\phi^* : \text{Ker} \left( \text{Pic}^{\text{crys},0}_{\text{A}(X_0)/S_n} \to \text{Pic}^{\text{crys},0}_{\text{A}(X_1)/S_n} \right) \longrightarrow \text{Ker} \left( \text{Pic}^{\text{crys},0}_{X_0^{[2]}/S_n} \to \text{Pic}^{\text{crys},0}_{X_1^{[2]}/S_n} \right).$$

Let $A_0$ be the reduced kernel of $(d_0^j)^* : \text{Pic}^{0,\text{red}}_{X_0/k} \to \text{Pic}^{0,\text{red}}_{X_1/k}$. Let $B_{0,n}$ and $B_{1,n}$ be abelian schemes over $S_n$ lifting $\left(\text{Pic}^{0,\text{red}}_{X_0/S_0}\right)^\vee$ and $\left(\text{Pic}^{0,\text{red}}_{X_1/S_0}\right)^\vee$. Let $A_n$ be an abelian scheme over $S_n$ lifting $A_0$. The sequence

$$(\text{Pic}^{0,\text{red}}_{X_0/S_0})^\vee \longrightarrow (\text{Pic}^{0,\text{red}}_{X_0/S_0})^\vee \longrightarrow \overline{A_0^\vee} \longrightarrow 0$$

is exact. Hence, we get from (7.2) a commutative diagram of sheaves over the fppf site of $S_n$:

$$\begin{array}{ccc}
\text{Pic}^{\text{crys},0}_{A_0^\vee/S_n} & \longrightarrow & \text{Pic}^{\text{crys},0}_{\left(\text{Pic}^{0,\text{red}}_{X_0/S_0}\right)^\vee/S_n} \\
\downarrow & & \downarrow \\
\text{E}(A_0^\vee) & \longrightarrow & \text{E}(B_{0,n})
\end{array} \quad \text{(24)}$$

Since $A_0 \subset \text{Pic}^{0,\text{red}}_{X_0/S_0}$ is a closed immersion, $a_0$ is a closed immersion. Since $a_n \times S_n S_0$ is $a_0$, then also $a_n$ is a closed immersion. In particular, $a_n$ is injective. Using Lemma 7.1.6 to get the second equality we have

$$\text{Pic}^{\text{crys},0}_{A(X_i)/S_n} = \prod_j \text{Pic}^{\text{crys},0}_{\text{Alb}(X_i^j/k)/S_n} = \text{Pic}^{\text{crys},0}_{A(X_i)/S_n}.$$

Note that $\left(\text{Pic}^{0}_{X_i/S_0}\right)^\vee/S_n = \text{Pic}^{\text{crys},0}_{A(X_i)/S_n}$. In conclusion, we obtain

$$\text{Pic}^{\text{crys},0}_{A_0^\vee/S_n} \hookrightarrow \text{Ker} \left( \text{Pic}^{\text{crys},0}_{A(X_0)/S_n} \to \text{Pic}^{\text{crys},0}_{A(X_1)/S_n} \right).$$

For every $i$ and $j$ the composite of the diagonal embedding $X_i^j \to X_i^j \times_k X_i^j$ and $\phi_i^j : X_i^j \times_k X_i^j \to \text{Alb} (X_i/k)$ factors via the identity of $\text{Alb} (X_i/k)$. Hence, the homomorphism

$$\begin{array}{ccc}
\text{Pic}^{\text{crys},0}_{\text{Alb}(X_i^j/k)/S_n} & \longrightarrow & \text{Pic}^{\text{crys},0}_{X_i^j \times_k X_i^j/S_n} \\
\downarrow & & \downarrow \\
\text{Pic}^{\text{crys},0}_{X_i^j/S_n}
\end{array}$$
Let $G$ be the zero map. By Lemma 7.1.6 we have $\text{Pic}^{\text{crys},0}_{X'_i/\kappa} = \text{Pic}^{\text{crys},0}_{X'_i/S_n} \times \text{Pic}^{\text{crys},0}_{X'_i/S_n}$ and the map to $\text{Pic}^{\text{crys},0}_{X'_i/S_n}$ is the sum. Hence, the map of fppf sheaves on $S_n$

$$\phi^*_i: \text{Pic}^{\text{crys},0}_{\text{Alb}(X'_i/k)/S_n} \rightarrow \text{Pic}^{\text{crys},0}_{X'_i/S_n}$$

factors via $(\text{Id}, -\text{Id}) : \text{Pic}^{\text{crys},0}_{X'_i/\kappa} \rightarrow \text{Pic}^{\text{crys},0}_{X'_i/\kappa} \times \text{Pic}^{\text{crys},0}_{X'_i/\kappa}$. Hence, the map $\phi^*$ induces a homomorphism

$$\phi^*: \text{Pic}^{\text{crys},0}_{\kappa'/\kappa} \rightarrow \text{Ker}\left(\text{Pic}^{\text{crys},0}_{X_0/\kappa} \rightarrow \text{Pic}^{\text{crys},0}_{X_1/\kappa}\right).$$

Taking $\text{Lie}$ of these functors we get a map

$$\phi^*_i: \mathcal{H}^{\text{crys}}_1(A_0^\vee/S_n) \rightarrow \text{Ker}\left(\mathcal{H}^{\text{crys}}_1(X_0/S_n) \rightarrow \mathcal{H}^{\text{crys}}_1(X_1/S_n)\right)$$

**Proposition 7.3.2** The induced map

$$T^{\text{crys}}_0(A_0) \xrightarrow{=\text{Ker}} \text{Ker}\left(\mathcal{H}^{\text{crys}}_1(X_0) \rightarrow \mathcal{H}^{\text{crys}}_1(X_1)\right)$$

is an isomorphism.

**Proof:** Using Illusie’s theory [10] II.1 we can compute crystalline cohomology of the smooth proper schemes $X_1$ as the hypercohomology of De Rham Witt complexes. In particular, by [10] II.3.11.2 the groups $\mathcal{H}^{\text{crys}}_1(X_i)$ and $\mathcal{H}^{\text{crys}}_1(\text{Alb}(X_i/k))$ coincide. Since the sequence $(\text{Pic}^{0,\text{red}}_{X_1/\kappa})^\vee \rightarrow (\text{Pic}^{0,\text{red}}_{X_0/\kappa} \times \kappa) \rightarrow A^\vee \rightarrow 0$ is exact and the contravariant Dieudonné functor defines a fully faithful functor from the category of abelian varieties to the category of $F$-crystals, the associated sequence

$$0 \rightarrow \mathcal{H}^{\text{crys}}_1(A_0^\vee) \rightarrow \mathcal{H}^{\text{crys}}_1((\text{Pic}^{0,\text{red}}_{X_0/\kappa})^\vee) \rightarrow \mathcal{H}^{\text{crys}}_1((\text{Pic}^{0,\text{red}}_{X_1/\kappa})^\vee)$$

is exact. Since $\mathcal{H}^{\text{crys}}_1(A_0^\vee) = T^{\text{crys}}_0(A_0)$, the conclusion follows. 

### 7.3.3 Souping up

Let $G$ be the semiabelian scheme $\text{Pic}^{0,\text{red}}_{X_i/\kappa}$. Let $A_0$ be the abelian part of $G$. Fix $n \in \mathbb{N}$. Let $G_n \rightarrow S_n$ be a lift of $G_0 \rightarrow S_0$ as a semiabelian scheme. Let $A_n$ be the abelian part of $G_n$. By the discussion above, see 7.3.1 and the identification $E(A_n) \cong \text{Pic}^{\text{crys},0}_{\kappa'/\kappa}$, see 7.2 we have a homomorphism of sheaves of abelian groups on the fppf site of $S_n$

$$\phi^*: E(A_n) \rightarrow \text{Ker}\left(\text{Pic}^{\text{crys},0}_{X_0/S_n} \rightarrow \text{Pic}^{\text{crys},0}_{X_1/S_n}\right).$$

Define $E'_n$ as the fibre product

$$E'_n \xrightarrow{\phi^*} \text{Pic}^{\text{crys},0}_{X_i/\kappa}$$

$$\downarrow \quad \downarrow$$

$$E(A_n) \xrightarrow{\phi^*} \text{Ker}\left(\text{Pic}^{\text{crys},0}_{X_0/S_n} \rightarrow \text{Pic}^{\text{crys},0}_{X_1/S_n}\right)$$

By 7.3.2 the kernel of the right vertical map is represented by a torus $T_n$ over $S_n$. 

Lemma 7.3.4 We have $E'_n \simeq E_G$ as extensions of $E(\mathbb{A}_n)$ by $T_n$.

Proof: By the discussion above we have a sequence

$$0 \rightarrow T_n \rightarrow E'_n \rightarrow E(\mathbb{A}_n) \rightarrow 0.$$ 

We claim that it is exact on the right and, hence, exact. We have a homomorphism of abelian sheaves

$$\text{Ker} \left( \text{Pic}^{\text{crys},0}_{X_0/S_n} \rightarrow \text{Pic}^{\text{crys},0}_{X_1/S_n} \right) \longrightarrow \left( \pi_{3, *}^{\text{crys}}(G_{m,X_3})/\text{Im} \left( \pi_{2,*}^{\text{crys}}(G_{m,X_2}) \right) \right)$$

associating to any crystal $L$ of invertible modules on $(X_0/S_n)$, such that there exists an isomorphism $\alpha: (d_1)^*(L) \cong (d_1)^*(L)$, the class of $((d_2)^* (\alpha))^{-1} \circ ((d_3)^* (\alpha)) \circ ((d_3)^* (\alpha))$. Note that such a class is trivial if and only if there exists $\alpha$ satisfying the cocycle condition, i.e., if and only if $\alpha$ lifts to an element of $\text{Pic}^{\text{crys}}_{X_n/S_n}$. Thus, to prove the exactness it suffices to prove that the composite $\Upsilon_n : E(\mathbb{A}_n) \rightarrow E'_n := \pi_{3,*}^{\text{crys}}(G_{m,X_3})/\text{Im} \left( \pi_{2,*}^{\text{crys}}(G_{m,X_2}) \right)$ is trivial. We proceed by induction on $n$. For $n = 0$ the triviality follows from $\text{Hom}(G_{n,k}, G_{m,k}) = 0$ and $\text{Hom}(\mathbb{A}_0, G_{m,k}) = 0$. If $\Upsilon_n$ is trivial, then $\Upsilon_{n+1}$ factors as $E(\mathbb{A}_n) \rightarrow \text{Ker}(\mathbb{A}_{n+1} \rightarrow E'_n)$. The group on the right hand side is a sum of $G_{n,k}$'s. Since $E(\mathbb{A}_0)$ is the universal $G_{n,k}$-extension of $\mathbb{A}_0$, we have $\text{Hom}(E(\mathbb{A}_n), G_{n,k}) = \{0\}$. This concludes the inductive step and proves the claimed exactness.

Interpreting $\text{Pic}^{\text{crys},0}_{X_0/S_n}$ as isomorphism classes of invertible sheaves endowed with integrable connection, $E'_0$ is the fibre product of $\text{Pic}^{\text{crys}}_{X_0/k}$ and $\text{Pic}^{\text{crys}}_{\mathbb{A}_0/k}$ over the kernel of $\text{Pic}^{\text{crys}}_{X_0/k} \rightarrow \text{Pic}^{\text{crys}}_{X_1/k}$. It follows from the definition of $G_0$ that $E'_0 \cong E(G_0)$.

Since $E'_n$ is an extension of representable sheaves over the fppf site of $S_n$, [20 Prop. 17.4] guarantees that $E'_n$ is itself representable. Moreover, the base change of $E'_n$ to $S_0$ is isomorphic to $E'_0$. It follows from the crystalline nature of $E(G_n)$ (see 3.4.2) that there exists a unique isomorphism of extensions $E(G_n) \cong E'_n$ lifting the isomorphism $E_n \times_{S_n} S_0 \cong E'_0 \cong E(G_0)$ over $S_0$.

7.3.5 The comparison

In conclusion, given a lift $G_n \rightarrow S_n$ of $G_0 \rightarrow S_0$, we get a canonical homomorphism

$$E(G_n) \longrightarrow \text{Pic}^{\text{crys},0}_{X_n/S_n}.$$ 

Taking $\text{Lie}$ and using [4.1.4] and [4.1.5] we get functorial homomorphisms of $\mathcal{O}_{S_n}$-modules

$$\text{Lie} \left( E(G_n) \right) \longrightarrow \text{InfDef} \left( \mathcal{O}^{\text{crys}}_{X_n/S_n} \right)$$

$$\downarrow \quad i \quad \downarrow i$$

$$\mathbb{T}^{\text{crys}}(G) \otimes W(k) W_n(k) \quad \mathbb{H}^1_{\text{crys}}(X_n/S_n).$$

Proposition 7.3.6 For $p \geq 3$ the inverse limit of the above maps defines an isomorphism

$$\phi^{\text{crys}} : \mathbb{T}^{\text{crys}}(G_0) \cong \mathbb{H}^1_{\text{crys}}(X_0/S_0).$$

respecting the weight filtrations and the actions of Frobenius in 4.3 and in 6.6.
Proof: By \[4.3\] and \[6.6\] the two \(W(k)\)-modules admit a weight filtration, preserved by \(\hat{o}^\text{crys}_n\). Hence, it suffices to prove that it is an isomorphism on the various graded pieces. For \(\text{Gr}_1\) use \[7.3.2\] and for \(\text{Gr}_0\) use \[6.5\] (cf. \[5.1.2\]).

7.4 On the case \(p = 2\)

For \(p = 2\) there is a problem in defining the functor \(\text{Pic}^\text{logcrys,0}\) in \[7.4\] since the divided power structure we consider on \(pW_{n+1}(k)\) is not nilpotent. Following \[10\] we may replace \(\text{Pic}^\text{logcrys,0}\) with the sheaf

\[
\text{Pic}^\text{logcrys,0}(X, Y_\nu)/S_n
\]

associated to the following presheaf over the fppf site of \(S_n\). Let \(T\) be a scheme over \(S_n\) flat and of finite presentation. Consider the logarithmic and divided power structure on \(T\) induced from those on \(S_n\). Consider the group of isomorphism classes of simplicial crystals of invertible \(O^\text{logcrys}_{X, S_n T/T}\)-modules for the full crystalline site of \(X, S_n T\) relative to \(T\) endowed with a trivialization over \(T \times S S_0\); compare with \[16, \S 11.1\].

By \[4, \text{Ex. 4.14} \& \text{Cor. 6.8}\] the category of crystals of invertible \(O^\text{logcrys}_{X, S_n T/T}\)-modules for the full crystalline site of \(X, S_n T\) relative to \(T\) is equivalent to the category of invertible simplicial sheaves on \(X\) with an integrable, quasi-nilpotent connection. In particular, if \(A_0\) is an abelian variety, \(\text{Pic}^\text{logcrys,0}_{A_0}/S_n\) is the formal group \(E(A_\nu)\) of the universal extension of the dual of a lifting \(A_n\) of \(A_0\) to \(S_n\). Note that \(\text{Pic}^\text{logcrys,0}_{A_0}/S_n\) and \(E(A_\nu)\) have the same Lie algebras.

For \(p \geq 3\), if we consider the nilpotent crystalline site, the category of crystals of invertible \(O^\text{logcrys}_{X, S_n T/T}\)-modules is equivalent to the category of invertible simplicial sheaves on \(X\) with an integrable connection (no nilpotence assumption). Thus, we get a natural inclusion \(\overline{\text{Pic}^\text{logcrys,0}} \subset \text{Pic}^\text{logcrys,0}\) inducing an isomorphism at the level of tangent spaces \(\text{Lie}\). By \[4, \text{Ex. 4.14}\] \(\overline{\text{Pic}^\text{logcrys,0}}_{X_\nu}/S_n\) is the fiber product of \(\text{Pic}^\text{logcrys,0}_{X_\nu}/S_n\) and \(\text{Pic}^\text{logcrys,0}_{X_\nu}/S_0\) over \(\text{Pic}^\text{logcrys,0}_{X_\nu}/S_0\).

Unfortunately, this is not enough to get Theorem B’ for \(p = 2\) since \[7.3.2\] is lacking.

7.5 The general case

See \[5.1\] for the general definition of the Picard 1-motive \(\mathbb{M}_0 = \text{Pic}^+(V)\) associated to the algebraic \(k\)-variety \(V\). Let \(\text{Div}^0_{Y_\nu}(X_\nu)\) be the group defined in \[11\] over \(\overline{k}\) along with a Galois action; that is an étale sheaf on \(S_0 = \text{Spec}(k)\) and identified with the discrete part of \(\text{Pic}^+(V)\).

Definition 7.5.1 Let \(n \in \mathbb{N}\). Define

\[
v_n : X \longrightarrow \text{Pic}^\text{logcrys,0}((X_\nu, Y_\nu)/S_n)
\]

to be the homomorphism associating to a local section \(D\) of \(X\) the crystal \(O^\text{logcrys}_{X_\nu/S_n}[D^{-1}]\) of invertible \(O^\text{logcrys}_{X_\nu/S_n}\)-modules via the procedure in \[7.3.2\].
Proposition 7.5.2 For $n \in \mathbb{N}$ there is a unique homomorphism
\[
\psi_n: E(M_n)_{G_n} \longrightarrow \text{Pic}^{\log \text{crys}, 0}(X, Y)/S_n
\]
such that

a) for varying $n$ the $\psi_n$ are compatible, and

b) $\psi_n$ makes the following diagram commute
\[
\begin{array}{ccc}
E(G_n) & \longrightarrow & \text{Pic}^{\text{crys}, 0}_{X_n/S_n} \\
\downarrow & & \downarrow \\
E(M_n)_{G_n} & \xrightarrow{\psi_n} & \text{Pic}^{\log \text{crys}, 0}_{(X_n, Y_n)/S_n} \\
u_n & & v_n \\
X & \xrightarrow{=} & X,
\end{array}
\]
where the morphism of top line is defined in 7.3.5.

Proof: The $\psi_n$ are clearly obtained after an étale covering of $S_0$. We may then assume that $X$ is a constant group scheme. Note that $M_n = [X \rightarrow G_n]$. Proceed by induction on $n$. If $n = 0$, the claim is granted by 7.3.5 and the following commutative diagram
\[
\begin{array}{ccc}
E(M_0)_{G_0} & \rightarrow & \text{Pic}^{\log \text{crys}, 0}_{(X, Y)/S_0} \\
p_0 & & q_0 \\
G_0 & \rightarrow & \text{Pic}^{0, \text{red}}_{X/S_0},
\end{array}
\]
Note that $G_0$ is the semiabelian part of the universal extension of $M_0$, the kernel of the map $p_0$ is $\text{Ext}(M_0, G_a)\gamma$, and the diagram is provided by universality as soon as $q_0$ is a vector group extension of $M_0$. To see this, note that
\[
\text{Pic}^{z, 0}_{X/S_0} \hookrightarrow \text{Pic}^{z-\log, 0}_{(X, Y)/S_0} = \text{Pic}^{\log \text{crys}, 0}_{(X, Y)/S_0}
\]
and the kernel of the surjection $q_0$ is the group scheme representing connections on the structure sheaf of the simplicial scheme $X$ with logarithmic poles along the simplicial divisor $Y$. Hence, it is isomorphic to the vector group scheme
\[
\mathbb{H}^0(X, \Omega^1_{X, S_0}(\log(Y)))_{d=0}
\]
by the arguments in the proof of Lemma 5.2.1. The universal property of $E(M_0)$ yields the claimed $\psi_0$ uniquely.

Suppose that $\psi_n$ has been defined for $n < N$. Since we have a non-canonical splitting
\[
E(M_N)_{G_N} \overset{\sim}{\longrightarrow} E(G_N) \times S_N E[X \rightarrow 0]
\]
we certainly can lift $\psi_{N-1}$ to a homomorphism
\[
\psi'_N: E(M_N)_{G_N} \longrightarrow \text{Pic}^{\log \text{crys}, 0}_{(X, Y)/S_N}
\]
such that the upper square in (b) commutes. The possible liftings \( \psi'_N \) are a principal homogeneous space under

\[
\text{Hom} \left( X, \text{Lie} \left( \text{Pic}_{\log \text{crys},0}(X,Y)/S_N \right) \right).
\]

Hence, there is a unique \( \psi'_N \) such that the lower square in (b) commutes.

\[\therefore\]

### 7.5.3 Concluding

By taking \( \text{Lie} \) of the map \( \psi_n \) in Proposition 7.5.2, we get a \( W_n(k) \)-linear homomorphism

\[
\mathbb{T}_\text{crys}(M_0) \otimes W(k) \rightarrow \mathbb{H}^1_{\log \text{crys}} \left( (X,Y)/S_n \right)(1)
\]

as claimed in Theorem B', compatible for varying \( n \), preserving the weight filtrations \( W \) and the action of Frobenius. Due to Proposition 7.3.6 in order to conclude the proof of Theorem B', it suffices to check that the above map is an isomorphism after taking the inverse limit over \( n \in \mathbb{N} \) on the \( \text{Gr}_0^W \) parts. Thus we are left to deal with the case of \( k = \overline{k} \) as all maps are clearly Galois equivariants. By 6.6 we have \( \iota: \text{Gr}_2^W \left( \mathbb{H}^1_{\log \text{crys}}(X,Y) \right) \subseteq \text{Div}^0_{Y'}(X_\ast) \otimes_{Z} W(k) \).

Furthermore \( \text{Div}^0_{Y'}(X_\ast) \otimes_{Z} W(k) = \text{Gr}_0^W \left( \mathbb{T}_\text{crys}(M_0) \right) \) by definition of \( M_0 = \text{Pic}^+(V) \), see 5.1.

By Proposition 7.5.2 we obtain a map

\[
\eta: \text{Gr}_0^W \left( \mathbb{T}_\text{crys}(M_0) \right) \rightarrow \text{Gr}_0^W \left( \mathbb{H}^1_{\log \text{crys}}(X,Y)/(1) \right).
\]

We now show that \( \iota \circ \eta \) is the identity modulo \( p \). This suffices to conclude the proof of Theorem B'. Using Section 6.3 for the case \( S_n = S_0 = \text{Spec}(k) \) we get, by construction, the following commutative diagram

\[
\begin{array}{ccc}
\text{Gr}_0^W \left( \mathbb{H}^1_{\log \text{crys}}(X,Y)/(1) \right) & \xrightarrow{\iota} & \text{Div}^0_{Y'}(X_\ast) \otimes_{Z} W(k) \\
\downarrow & & \downarrow \\
\mathbb{H}^0(X_\ast, \Omega^1_{X_\ast}/S_0(\log(Y))_{d=0}/\mathbb{H}^0(X_\ast, \Omega^1_{X_\ast}/S_0)_{d=0} & \xrightarrow{\text{Res}} & \text{Div}^0_{Y'}(X_\ast) \otimes_{Z} k,
\end{array}
\]

where \( \text{Res} \) is the map defined by taking residues and \( \iota \) is the inclusion above. By the proof of Proposition 7.5.2 for every \( D \in \text{Div}^0_{Y'}(X_\ast) = X \) we have \( \psi_0(u_0(D)) = (O_{X_\ast}(-D), d) \), where \( d \) is the canonical connection on \( O_{X_\ast}(-D) \). Thus, the map \( \overline{\psi_0}: \mathbb{E}(M_0)\mathbb{G}_a/\mathbb{E}(G_0) = X \otimes \mathbb{G}_a \rightarrow \text{Pic}_{(X,Y)/S_0}/\text{Pic}_{X_\ast}/S_0 \) induced by \( \psi_0 \) is the unique \( A^1 \)-linear map sending \( D \otimes 1 \) to the class of \( (O_{X_\ast}(-D), d) \) and the induced map on Lie algebras yields the lower horizontal arrow in the following commutative diagram

\[
\begin{array}{ccc}
\text{Gr}_0^W \left( \mathbb{T}_\text{crys}(M_0) \right) & \xrightarrow{\eta} & \text{Gr}_0^W \left( \mathbb{H}^1_{\log \text{crys}}(X,Y)/(1) \right) \\
\downarrow & & \downarrow \\
\text{Gr}_0^W \left( \mathbb{T}_\text{crys}(M_0) \right) \otimes \mathbb{Z}/p\mathbb{Z} & \xrightarrow{\eta k} & \mathbb{H}^0(X_\ast, \Omega^1_{X_\ast}/S_0(\log(Y))_{d=0}/\mathbb{H}^0(X_\ast, \Omega^1_{X_\ast}/S_0)_{d=0}.
\end{array}
\]

Furthermore, using the description of \( \overline{\psi_0} \) given above, one verifies that \( \text{Res} \circ \eta k \) is simply the inclusion \( \text{Div}^0_{Y'}(X_\ast) \otimes_{Z} k \rightarrow \text{Div}^0_{Y'}(X_\ast) \otimes_{Z} k \). Patching the two diagrams we conclude that \( \iota \circ \eta \) is the identity modulo \( p \) as claimed.
Corollary 7.5.4 Let $V$ be an algebraic variety over a perfect field $k$ of characteristic $p > 2$. The filtered $F\cdot \mathcal{W}(k)$-module $H^1_{\text{crys}}(V/\mathcal{W}(k)) := H^1_{\text{crys}}(V, \mathcal{W}(k))$ is independent of the choice of the hypercovering $V \to V$ (subject to the condition that $V_0 \to V$ is generically étale).

Proof: Let $V, \to V$ and $V' \to V$ be two such hypercoverings. We may assume that there is a map $\varphi : V' \to V$, of $V$-simplicial schemes compatibly with the normal crossing boundaries, i.e., which is the restriction of a map $X' \to X$. (cf. Appendix A). By Theorem B', the Appendix A and functoriality we get an induced isomorphism $T_{\text{crys}}(\text{Pic}^+(V)) \cong H^1_{\text{crys}}(V/\mathcal{W}(k)) \cong T_{\text{crys}}(\text{Pic}^+(V'))$ of filtered $F\cdot \mathcal{W}(k)$-modules.

\section{Appendix}

We provide details showing that the cohomological Picard 1-motive $\text{Pic}^+(V)$ is independent of the choices made, i.e., it is independent of choices of hypercoverings and compactifications, over perfect fields. In characteristic zero this is provided by \cite{4} Prop. 2.5, see also \cite{5} Remark 4.4.4]. However, the argument for positive characteristics is slightly more involved.

Suppose we are given two such smooth proper hypercoverings $f : V, \to V$ and $f' : V' \to V$ which admit smooth compactifications with normal crossing boundaries, $V, = X, - Y,$ and $V' = X', - Y'$ and assume that $V_0 \to V$ and $V'_0 \to V$ are generically étale. We refer to De Jong's theory of alterations over perfect fields for their existence, see \cite{5} Thm. 4.1]. Note that, as usual (cf. \cite{5} §6.2], we can always find a third one mapping to both (cf. 5.1.7 and 5.2.4 in Exposé V bis of [\cite{13}]]. We then can assume that there is a map $\varphi : V' \to V$, of $V$-simplicial schemes compatibly with the normal crossing boundaries, i.e., it is the restriction of a proper map $\varphi : X' \to X$.

Then, by pulling back (see \cite{5} 6.2], we get a map of 1-motives $\varphi^* : \text{Pic}^+(X, Y) \to \text{Pic}^+(X', Y')$ and thus, equivalently, we have a $\text{Gal}(\bar{k}/k)$-equivariant map of complexes

$$\varphi^* : [\text{Div}_{\varphi^*}(\mathcal{X},) \to \text{Pic}^0(\mathcal{X},)] \to [\text{Div}^0_{\varphi^*}(\mathcal{X},) \to \text{Pic}^0(\mathcal{X},)].$$

We are then reduced to deal with $k = \bar{k}$ since our constructions are natural enough to be automatically compatible with $\text{Gal}(\bar{k}/k)$-actions. The proof is divided in two steps. First of all, in Section A.1 we show that, for $\ell \neq p = \text{char}(k)$, the map $\varphi^*$ induces an isomorphism of the $\ell$-adic realization of the above 1-motives. This implies that $\varphi^*$ is an isogeny of $p$-th power order. The argument follows closely \cite{5} Thm. 4.4.3]. Then, in Section A.2 we prove that $\varphi^*$ is an isomorphism: here, the techniques used are specific to characteristic $p$.

\subsection{\ell-adic realizations}

Let $p = \text{char}(k)$ and $k = \bar{k}$. Recall \cite{5} 10.1.10] for the definition of $\ell$-adic realization $T_{\ell}(\mathbb{M})$ of a 1-motive $\mathbb{M}$ over $k$ for $\ell \neq p$; this is given by taking the inverse limit over $\nu$ of the inverse system $\mathbb{M}[\ell^\nu]$ as defined in our section \cite{13}. From \cite{4] it yields an exact sequence (cf. \cite{5} §1.3)

$$0 \to \prod_{\ell \neq p} T_{\ell}(\mathbb{G}) \to \prod_{\ell \neq p} T_{\ell}(\mathbb{M}) \to \prod_{\ell \neq p} T_{\ell}(\mathbb{X}) \to 0 \tag{28}$$
for \( M = [X \to G] \). Note that for a complex of abelian groups \( C = [F \to G] \) such that \( \ell \cdot F = 0 \) and \( G/\ell^\nu = 0 \) (for all \( \nu > 0 \)) \( T_\ell(C) \) can be defined along with an exact sequence

\[
0 \to \lim_{\nu} \ell \cdot G \to T_\ell(C) \to \lim_{\nu} F/\ell^\nu \to \lim_{\nu} \ell \cdot G
\]

explaining (28). The following is a modification of [2, Prop. 1.3.1].

**Lemma A.1.1** Let \( p = \text{char}(k) \) and \( k = \overline{k} \). The functor \( \prod_{\ell \neq p} T_\ell \) from the category of 1-motives to abelian groups is faithful. Moreover, if \( M \to M' \) is a map of 1-motives such that

\[
\prod_{\ell \neq p} T_\ell(M) \cong \prod_{\ell \neq p} T_\ell(M')
\]

then \( M \to M' \) is an isogeny of \( p \)-power order.

**Proof:** Consider \( M = [X \to G], M' = [X' \to G'] \) and \( f: M \to M' \). By making use of the exact sequence (28) we can see that it is enough to check faithfulness separately for maps of semi-abelian schemes or lattices. Since torsion points coprime to \( p \) are Zariski dense in a semi-abelian scheme over \( k = \overline{k} \), \( \prod_{\ell \neq p} T_\ell(f) = 0 \) implies \( f = 0 \) for morphisms \( f \) between semi-abelian schemes. Moreover, \( \prod_{\ell \neq p} T_\ell(X[1]) = \prod_{\ell \neq p} X \otimes \mathbb{Z}_\ell \) is clearly faithful.

If \( M \to M' \) induces an isomorphism \( \prod_{\ell \neq p} T_\ell(M) \cong \prod_{\ell \neq p} T_\ell(M') \) then by (28) we have that \( \prod_{\ell \neq p} T_\ell(G) \) injects into \( \prod_{\ell \neq p} T_\ell(G') \) and \( \prod_{\ell \neq p} T_\ell(X[1]) \) surjects onto \( \prod_{\ell \neq p} T_\ell(X'[1]) \), therefore we have an exact sequence

\[
0 \to X'' \to X \to X' \to A \to 0
\]

where \( A \) is finite and killed by a power of \( p \). Moreover by the snake lemma applied to the resulting diagram given by (28) we get that

\[
\prod_{\ell \neq p} T_\ell(X''[1]) \cong \prod_{\ell \neq p} T_\ell(G') / T_\ell(G).
\]

Since \( \prod_{\ell \neq p} T_\ell(G) \) injects into \( \prod_{\ell \neq p} T_\ell(G') \) we have that \( B = \text{Ker}(G \to G') \) is a finite group. Let \( B\{p\} \) be the sub-group of \( p^n \)-torsion elements for some \( n \gg 0 \) then

\[
B/B\{p\} \cong \prod_{\ell \neq p} T_\ell(G/B) / T_\ell(G) \cong \prod_{\ell \neq p} T_\ell(G') / T_\ell(G).
\]

Thus \( p^n B = 0 \), for some \( n \gg 0 \), since \( B/B\{p\} \) injects into \( \prod_{\ell \neq p} T_\ell(X''[1]) \) which is torsion free. If we let \( G'' \) denote the cokernel of the map \( G \to G' \), we then get the following exact sequence of complexes

\[
0 \to [X'' \to B] \to [X \to G] \to [X' \to G'] \to [A \to G''] \to 0.
\]

Applying \( \prod_{\ell \neq p} T_\ell \) we have

\[
\prod_{\ell \neq p} T_\ell([X'' \to B]) \to \prod_{\ell \neq p} T_\ell(M) \cong \prod_{\ell \neq p} T_\ell(M') \to \prod_{\ell \neq p} T_\ell([A \to G''])
\]

where \( T_\ell([X'' \to B]) = T_\ell([X''[1]]) \) and \( T_\ell([A \to G'']) = T_\ell([G'']) \) since \( \nu B = 0 \) and \( A/\ell^\nu = 0 \) (for all \( \nu > 0 \) and \( \ell \neq p \)) respectively. Therefore the composition of the induced maps here-above
is the zero map as well as an isomorphism. Thus $\prod_{\ell \neq p} T_\ell(X''[1]) = \prod_{\ell \neq p} T_\ell(G'') = 0$ whence $X'' = G'' = 0$, i.e., $G \to G'$ is an isogeny with kernel the finite group $B$, $X \to X'$ is injective with cokernel the finite group $A$ and we can find a positive integer $\tau$ such that $p^\tau A = p^\tau B = 0$.

Now just apply Lemma A.1.1 to our map $\varphi^*$ and obtain an induced isomorphism

$$\prod_{\ell \neq p} T_\ell(\varphi^*) : \prod_{\ell \neq p} T_\ell(\text{Pic}^+(X', Y')) \xrightarrow{\cong} \prod_{\ell \neq p} T_\ell(\text{Pic}^+(X', Y'))$$

by cohomological descent. In fact, the same arguments in the proof of Theorem 4.4.3 in [2] applies here (see also [2, 7.2] for compatibility with Galois actions) provided that $\ell \neq p$, and therefore

$$T_\ell(\text{Pic}^+(X', Y')) = H^1_{\text{et}}(V, \mathbb{Z}_\ell(1)) = T_\ell(\text{Pic}^+(X', Y')).$$

We then have that $\varphi^*$ is an isogeny of $p$-power order.

**Corollary A.1.2** If $M \to M'$ is a map of 1-motives such that

$$\prod_{\ell \neq p} T_\ell(M) \cong \prod_{\ell \neq p} T_\ell(M')$$

and $T_{\text{crys}}(M) \cong T_{\text{crys}}(M')$, then $M \to M'$ is an isomorphism.

**Proof:** By Lemma A.1.1 we know that $M \to M'$ is a $p$-power isogeny. In particular, $G \to G'$ is a $p$-power isogeny and the map $X \to X'$ is injective with $p$-power cokernel. By definition the crystalline realization $T_{\text{crys}}(M)$ of $M$ is the covariant Dieudonné module of the $p$-divisible group $M[p^\infty]$. Since the Dieudonné functor is fully faithful, we deduce that $M[p^\infty] \to M'[p^\infty]$ is an isomorphism. Note that we have an exact sequence $0 \to G[p^\infty] \to M[p^\infty] \to X[p^\infty] \to 0$. Since the map $X[p^\infty] = X \otimes \mathbb{Q}_p/\mathbb{Z}_p \to X'[p^\infty] = X' \otimes \mathbb{Q}_p/\mathbb{Z}_p$ is injective, we deduce from the above exact sequence that it is also surjective and that $G[p^\infty] \to G'[p^\infty]$ is an isomorphism. The conclusion follows.

### A.2 $p$-adic realization

Let $M := [X \to G] := \text{Pic}^+(X', Y')$ and $M' := [X' \to G'] := \text{Pic}^+(X', Y')$. Let $G$ be an extension of the abelian variety $A$ by the torus $T$. Let $G'$ be an extension of the abelian variety $A'$ by the torus $T'$. All cohomology groups in the sequel are considered for the fppf-topology.

**Lemma A.2.1** Let $\pi : X_\cdot \to \text{Spec}(k)$ be the structural morphism. Let $F$ be a finite commutative group scheme over $k$. Let $F' := \text{Hom}(F, G_m)$ be the Cartier dual of $F$. The natural map of fppf-sheaves over $k$

$$R^1\pi_* (F_{X_\cdot}) \to \text{Hom}(F', \text{Pic}_{X_{/k}}),$$

defined by push-forward of simplicial $F$-torsors via elements of $F'$, is an isomorphism.

**Proof:** By [13] Prop. III.4.16] the maps $R^1(\pi_i)_* (F_{X_i}) \to \text{Hom}(F', \text{Pic}_{X_{i/k}})$ are isomorphisms for every $i$. Thus, the map

$$\text{Ker} (R^1(\pi_0)_*(F_{X_0}) \to R^1(\pi_1)_*(F_{X_1})) \to \text{Hom} (F', \text{Ker} (\text{Pic}_{X_{0/k}} \to \text{Pic}_{X_{1/k}}))$$
is an isomorphism. Furthermore, \((\pi_1)_*(\mathcal{F}_X) = \text{Hom}(\mathcal{F}_\wedge, (\pi_1)_*(G_m))\). By \cite[Lemma III.4.17]{13} we have \(\mathcal{E}xt^1(\mathcal{F}_\wedge, G_m) = 0\) for every \(i\). Then, \(\text{Hom}(\mathcal{F}_\wedge, \_\) preserves exact sequences of tori. Using the spectral sequences for \(R^1\pi_*(\mathcal{F}_X)\) and for \(R^1\pi_*(G_{m,X})\) the lemma follows. \(\square\)

**Corollary A.2.2** We have \(H^1(X, \mathbb{Z}_p) \cong \text{Hom}(\mu_{p^\infty}, \mathbb{G})\).

**Proof:** By Lemma A.2.1 the map \(H^1(X, \mathbb{Z}_p) \rightarrow \text{Hom}(\mu_{p^\infty}, \text{Pic}_{X, k})\) is an isomorphism. If \(\mathcal{F}\) is a finite \(k\)-group scheme or a discrete group scheme, we have \(\text{Hom}(\mu_{p^\infty}, \mathcal{F}) = 0\). Let \(K := \text{Ker}(\text{Pic}_{X_0/k} \rightarrow \text{Pic}_{X_1/k})\) and let \(\mathcal{F} := K/K^{0, \text{red}}\). Thus, \(\text{Hom}(\mu_{p^\infty}, K^{0, \text{red}}) \rightarrow \text{Hom}(\mu_{p^\infty}, K)\) is an isomorphism. Hence, the map \(\text{Hom}(\mu_{p^\infty}, \mathcal{G}) \rightarrow \text{Hom}(\mu_{p^\infty}, \text{Pic}_{X, /k})\) is an isomorphism, as claimed. \(\square\)

**Lemma A.2.3** Let \(X\) be a normal \(k\)-scheme and let \(U \subset X\) be an open dense subscheme. The map \(H^1(X, \mathbb{Z}/p^n\mathbb{Z}) \rightarrow H^1(U, \mathbb{Z}/p^n\mathbb{Z})\) is injective.

**Proof:** Let \(Y \rightarrow X\) be a \(\mathbb{Z}/p^n\mathbb{Z}\)-torsor. Since the map \(Y \rightarrow X\) is étale and \(X\) is normal, \(Y\) is normal. Suppose that \(Y|_U \rightarrow U\) is trivial, \(i.e., Y|_U = \Pi_{i=1}^{n} U\). In particular, the normalization \(Y' := \Pi_{i=1}^{n} X\) of \(X\) in \(Y'|_U\) is the trivial \(\mathbb{Z}/p^n\mathbb{Z}\)-torsor. The normality of \(Y\) implies that \(Y \cong Y'\) as \(\mathbb{Z}/p^n\mathbb{Z}\)-torsors over \(X\). Thus, \(Y\) is the trivial torsor as claimed. \(\square\)

**Lemma A.2.4** The group
\[
\lim_{\infty \leftarrow n} \left( H^1(V_i, \mathbb{Z}/p^n\mathbb{Z})/H^1(X_i, \mathbb{Z}/p^n\mathbb{Z}) \right)
\]
is torsion free as \(\mathbb{Z}_p\)-module. The map
\[
\alpha_i: \lim_{\infty \leftarrow n} \left( H^1(V_i, \mathbb{Z}/p^n\mathbb{Z})/H^1(X_i, \mathbb{Z}/p^n\mathbb{Z}) \right) \rightarrow \lim_{\infty \leftarrow n} \left( H^1(V'_i, \mathbb{Z}/p^n\mathbb{Z})/H^1(X'_i, \mathbb{Z}/p^n\mathbb{Z}) \right)
\]
is injective.

**Proof:** If \(Z \subset X_i\) is an irreducible divisor, we let \(R_Z\) be a complete dvr which is an extension of the completed local ring \(\hat{O}_{X_i, Z}\) of \(X_i\) at the generic point of \(Z\) with residue field equal to an algebraic closure \(k_Z\) of the fraction field \(k(Z)\) of \(Z\) and with maximal ideal generated by that of \(\hat{O}_{X_i, Z}\). Let \(K_Z\) be the fraction field of \(R_Z\) and let \(m_Z\) be the maximal ideal of \(R_Z\). For every \(r \in \mathbb{N}\) let \(U_{Z,(r)}\) be the group of units of \(R_Z\) congruent to 1 modulo \(m_Z^r\). Let \(W_{Z,(r)} := U_{Z,(1)}/U_{Z,(r)}\) and \(W := \lim_{\infty \leftarrow r} W_{Z,(r)}\). For every integer \(1 \leq i \leq r - 1\) prime to \(p\) let \(r_i\) be the smallest integer such that \(p^{r_i} \geq r/i\). By \cite[§V.9 Prop. 9]{25}\] the group \(W_{Z,(r)}\) is isomorphic to the product of truncated Witt vectors \(W_{r_i}(k_Z)\) over the integers \(1 \leq i \leq r - 1\) prime to \(p\). By purity of the branch locus \cite[X.3.4]{12}\] a \(\mathbb{Z}/p^n\mathbb{Z}\)-torsor over \(V_i\) which is unramified at the generic points of \(Y_i\) is unramified. Thus, \(H^1(V_i, \mathbb{Z}/p^n\mathbb{Z})/H^1(X_i, \mathbb{Z}/p^n\mathbb{Z})\) is contained in \(\oplus_j H^1(K_{Y_{ij}}, \mathbb{Z}/p^n\mathbb{Z})/H^1(R_{Y_{ij}}, \mathbb{Z}/p^n\mathbb{Z})\). By class field theory, cf. \cite[§XV.2]{25}\], the latter is isomorphic to \(\oplus_j \text{Hom}(W_{Y_{ij}}, \mathbb{Z}_p)\). In particular, the map
\[
\Psi_{X_i}: \lim_{\infty \leftarrow n} \left( H^1(V_i, \mathbb{Z}/p^n\mathbb{Z})/H^1(X_i, \mathbb{Z}/p^n\mathbb{Z}) \right) \rightarrow \oplus_j \text{Hom}(W_{Y_{ij}}, \mathbb{Z}_p)
\]
is injective. Let $Y_i'$ be an irreducible component of $Y_i'$ dominating $Y_{ij}$. Let $\text{Norm}_{j\ell}$ be the homomorphism $W_{Y_i'/h} \to W_{Y_{ij}}$ defined by the norm map from $K_{Y_i'/h}$ to $K_{Y_{ij}}$. Then, the map from $H^1(V_i, \mathbb{Z}/p^n\mathbb{Z})/H^1(X_i, \mathbb{Z}/p^n\mathbb{Z})$ to $H^1(V_{ij}', \mathbb{Z}/p^n\mathbb{Z})/H^1(X_{ij}', \mathbb{Z}/p^n\mathbb{Z})$ is compatible with the map obtained applying $\text{Hom}(-, \mathbb{Z}/p^n\mathbb{Z})$ to $\prod_{j,h} \text{Norm}_{j\ell}$. By class field theory the quotient of $W_{Y_i'/h} \to W_{Y_{ij}}$ is isomorphic to the wild inertia of the abelianized Galois group of the extension $K_{Y_i'/h}/K_{Y_{ij}}$; see [26, Cor. XV.2.3]. In particular, it is a finite $p$-group. Applying $\text{Hom}(-, \mathbb{Z}_p)$ we conclude that the map $\alpha_i$ is injective as claimed.

**Corollary A.2.5** The map $\text{Hom}(\mu_{p\infty}, G) \to \text{Hom}(\mu_{p\infty}, G')$ is an isomorphism.

**Proof:** Since $V_i$ and $V_{ij}'$ are hypercoverings of $V_i$, we have cohomological descent for the étale cohomology of constant sheaves. In particular, $\mathbb{H}^i(V_i, \mathbb{Z}/p^n\mathbb{Z}) \cong H_i^i(V, \mathbb{Z}/p^n\mathbb{Z}) \cong \mathbb{H}^i(V_i', \mathbb{Z}/p^n\mathbb{Z})$ for every $n$. Hence, the induced map $\mathbb{H}^i(V_i, \mathbb{Z}/p^n\mathbb{Z}) \to \mathbb{H}^i(V_{ij}', \mathbb{Z}/p^n\mathbb{Z})$ is an isomorphism. Taking inverse limits over $n \in \mathbb{N}$ we have a commutative diagram with exact rows (using Lemma A.2.3)

\[
\begin{array}{cccc}
0 & \longrightarrow & \mathbb{H}^1(X_i, \mathbb{Z}_p) & \longrightarrow & \mathbb{H}^1(V_i, \mathbb{Z}_p) & \longrightarrow & H^1(V_0, \mathbb{Z}_p)/H^1(X_0, \mathbb{Z}_p) \\
\downarrow & & \downarrow i & & \downarrow \alpha \circ & & \downarrow \alpha_0 \\
0 & \longrightarrow & \mathbb{H}^1(X_{ij}', \mathbb{Z}_p) & \longrightarrow & \mathbb{H}^1(V_{ij}', \mathbb{Z}_p) & \longrightarrow & H^1(V_0, \mathbb{Z}_p)/H^1(X_0, \mathbb{Z}_p)
\end{array}
\]

By Lemma A.2.4 the map $\alpha_0$ is injective. Thus, the left vertical map is an isomorphism. By Corollary A.2.2 such map coincides with the map $\text{Hom}(\mu_{p\infty}, G) \to \text{Hom}(\mu_{p\infty}, G')$. This proves the claim.

**Lemma A.2.6** The maps $\text{Hom}(\alpha_p, \text{Pic}_{X_0/k}) \to \text{Hom}(\alpha_p, \text{Pic}_{X_0'/k})$ and $\text{Hom}(\mathbb{Z}/p\mathbb{Z}, \text{Pic}_{X_0/k}) \to \text{Hom}(\mathbb{Z}/p\mathbb{Z}, \text{Pic}_{X_0'/k})$ are injective. Thus $\text{Hom}(\alpha_p, G) \to \text{Hom}(\alpha_p, G')$ and $\text{Hom}(\mathbb{Z}/p\mathbb{Z}, G) \to \text{Hom}(\mathbb{Z}/p\mathbb{Z}, G')$ are injective.

**Proof:** For any finite commutative $k$-group scheme $\mathcal{F}$, let $\mathcal{F}^\vee$ be the Cartier dual of $\mathcal{F}$. By [13, Prop. III.4.16] we have $\text{Hom}(\mathcal{F}, \text{Pic}_{X_0/k}) \cong H^0(X_0, \mathbb{O}^\vee_{X_0/k})$ (and analogously for $X_0'$). For $\mathcal{F} = \alpha_p$ or $\mathcal{F} = \mathbb{Z}/p\mathbb{Z}$, the latter group is identified with a subgroup of $H^0(X_0, \mathbb{O}^\vee_{X_0/k})$ (resp. $H^0(X_0', \mathbb{O}^\vee_{X_0'/k})$); cf. [13, Prop. III.4.14]. Note that $H^0(X_0, \mathbb{O}^\vee_{X_0/k}) \subset \mathbb{O}^\vee_{X_0/k}$ and that $H^0(X_0', \mathbb{O}^\vee_{X_0'/k}) \subset \mathbb{O}^\vee_{X_0'/k}$. Furthermore, the map $X_0' \to X_0$ is generically separable by construction; thus, $\mathbb{O}^\vee_{X_0'/k} \subset \mathbb{O}^\vee_{X_0/k}$. This proves the first claim. Since any homomorphism from $\alpha_p$ or $\mathbb{Z}/p\mathbb{Z}$ to a torus is trivial, it suffices to prove the second assertion for $\mathbb{A}$ in place of $G$ and $\mathbb{A}'$ in place of $G'$. By construction $\mathbb{A}$ is the reduced kernel of the map $\text{Pic}_{X_0/k}^0 \to \text{Pic}_{X_0'/k}^0$ and $\mathbb{A}'$ is the reduced kernel of the map $\text{Pic}_{X_0'/k}^0 \to \text{Pic}_{X_0'/k}^0$. Thus, it suffices to prove the Lemma for $\text{Pic}_{X_0/k}^0$ instead of $G$ and $\text{Pic}_{X_0'/k}^0$ instead of $G'$. The conclusion follows.

**Corollary A.2.7** The homomorphism $G \to G'$ is an isomorphism.

**Proof:** By Section A.1 for every $\ell \neq p$ the homomorphism of Tate modules $T_\ell(G) \to T_\ell(G')$ is an isomorphism. This implies that the map $G \to G'$ is an isogeny with kernel $\mathcal{F}$ of $p$-power
order. By Corollary A.2.5 and Lemma A.2.6, $\mathcal{F}$ does not contain any subgroup isomorphic to $\mu_p$ or $\alpha_p$ or $\mathbb{Z}/p\mathbb{Z}$. Hence, $\mathcal{F} = 0$.

Proposition A.2.8 The homomorphism $\varphi^*: \mathbb{M} \to \mathbb{M}'$ is an isomorphism.

Proof: By Section A.1 the map $\varphi^*$ is an isogeny of $p$-power order. In particular, the lattices $\mathbb{X}$ and $\mathbb{X}'$ have the same rank and $\mathbb{X}'/\mathbb{X}$ is killed by a power of $p$. By Corollary A.2.7 the map induced by $\varphi^*$ on the semiablian parts is an isomorphism. It then suffices to prove that the map $\mathbb{X}/p\mathbb{X} \to \mathbb{X}'/p\mathbb{X}'$ is injective. Let $\mathcal{C} := [\text{Div}_{Y_1}(X_1) \to \text{Pic}(X_1)]$ as a complex of abelian groups. As in A.1 denote

$$\mathbf{T}_{\mathbb{Z}/p\mathbb{Z}}(\mathcal{C}) := \left\{ (D, \mathcal{L}, \eta) \in \text{Div}_{Y_1}(X_1) \times \text{Pic}(X_1), \eta : \mathcal{L}^p \cong \mathcal{O}_X(-D) \right\}.$$ 

It sits in an exact sequence

$$0 \to \mathbf{T}_{\mathbb{Z}/p\mathbb{Z}}(\text{Pic}(X_1)) \to \mathbf{T}_{\mathbb{Z}/p\mathbb{Z}}(\mathcal{C}) \to \text{Div}_{Y_1}(X_1)/p\text{Div}_{Y_1}(X_1) \to 0.$$ 

Let $\rho_p : \mathbf{T}_{\mathbb{Z}/p\mathbb{Z}}(\mathcal{C}) \to \text{Pic}(X_1)$ be the map $\rho_p((D, \mathcal{L}, \eta)) := (\mathcal{L}, \eta)|_{Y_1}$ defined via simplicial Kummer theory, cf. [2] §4.4. As in loc. cit. the induced map $\mathbf{T}_{\mathbb{Z}/p\mathbb{Z}}(\text{Pic}(X_1)) \to \text{Pic}(X_1)$ is an isomorphism. Since $\mu_p(X_1) = 0$, the map $\mathbf{T}_{\mathbb{Z}/p\mathbb{Z}}(\mathcal{C}) \to \text{Div}_{Y_1}(X_1)/p\text{Div}_{Y_1}(X_1)$ is injective. Note that $\rho_p((D, \mathcal{L}, \eta))$ is a simplicial $\mu_p$-covering of $X$, and the associated covering of $X_0$ is ramified exactly over the support of $D$. Thus, the inverse image of $\text{Div}_{Y_1}(X_1)/p\text{Div}_{Y_1}(X_1)$ is isomorphic to $\text{Div}_{Y_1}(X_1)/p\text{Div}_{Y_1}(X_1)$ and the induced map $\text{Div}_{Y_1}(X_1)/p\text{Div}_{Y_1}(X_1) \to \text{Pic}(X_1)$ is injective. Hence, in the following commutative diagram

$$\begin{array}{cccccc}
0 & \longrightarrow & \mathbf{T}_{\mathbb{Z}/p\mathbb{Z}}(\text{Pic}(X_1)) & \longrightarrow & \mathbf{T}_{\mathbb{Z}/p\mathbb{Z}}(\mathcal{C}) & \longrightarrow & \text{Div}_{Y_1}(X_1)/p\text{Div}_{Y_1}(X_1) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \text{Div}_{Y_1}(X_1) & \longrightarrow & \text{Div}_{Y_1}(X_1)/p\text{Div}_{Y_1}(X_1) & \longrightarrow & \text{Pic}(X_1) & \longrightarrow & 0
\end{array}$$

the left and right vertical arrows are injective. Therefore, we conclude that

$$\mathbf{T}_{\mathbb{Z}/p\mathbb{Z}}(\mathcal{C}) \hookrightarrow \text{Div}_{Y_1}(X_1).$$ (29)

is an injective map.

Note that we assumed that $V_0 \to V_0'$ is a dominant map of normal schemes over $k$ which is generically finite and separable, see [3] Thm. 4.1]. Let $k(V_0)$ and $k(V_0')$ be the function fields of $V_0$ and $V_0'$ respectively. Consider the diagram

$$\begin{array}{cccccc}
\text{H}^1(V_0, \mu_p) & \longrightarrow & \text{H}^1(k(V_0), \mu_p) & \longrightarrow & k(V_0)^*/(k(V_0)^*)^p \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{H}^1(V_0', \mu_p) & \longrightarrow & \text{H}^1(k(V_0'), \mu_p) & \longrightarrow & k(V_0')^*/(k(V_0')^*)^p
\end{array}$$

By normality the horizontal arrows are injective. By the separability assumption the right vertical arrow is injective. Thus, the induced map $\text{H}^1(V_0, \mu_p) \to \text{H}^1(V_0', \mu_p)$ is injective. Let $\mathcal{C}' := [\text{Div}_{Y_1}(X_1') \to \text{Pic}(X_1')]$. 


Using (29) for $C$ and $C'$, we obtain that
\[ T_{Z/pZ}(C) \rightarrow T_{Z/pZ}(C') \] (30)
is injective.

Let $T_{Z/pZ}(\mathbb{M})$ be (as in Section 1.3) the group of $k$-valued $p$-torsion points of $\mathbb{M}$: it sits in the exact sequence (1). By construction it maps to $T_{Z/pZ}(\mathbb{C})$. Let $\text{NS}(X) := \pi_0(\text{Pic} X_{/k})$. It is a discrete group and it coincides with $\text{Pic}_{X_{/k}}(k)/\text{Pic}_{X_{/k}}^{0,\text{red}}(k)$. The map $T_{Z/pZ}(\mathbb{G}) \rightarrow T_{Z/pZ}(\text{Pic}(X))$ is injective. Since $X$ is the fiber product of $\text{Div}_{Y_{/k}}(X) \rightarrow \text{Pic}(X)$ and $\text{Pic}_{0,\text{red}}^{X_{/k}}(X_{/k})$ the map $\text{Pic}_{0,\text{red}}^{X_{/k}}(X_{/k}) \rightarrow \text{Pic}_{0,\text{red}}^{X_{/k}}(X_{/k})$ is an isomorphism. The $p$-power torsion of $\text{Pic}(X)$ injects into $\text{Pic}(X_0)$, as $\mu_p(X_0) = 0$ (as well as for $X'$). Then, by Lemma A.2.6 the map $\text{Pic}(X) \rightarrow \text{Pic}(X')$ is injective on the $p$-power torsion. Thus, the map $\text{NS}(X) \rightarrow \text{NS}(X')$ is injective on $p$-power torsion: in particular, $\mathcal{F}$ injects into $\mathcal{F}'$. Therefore, the kernel of $T_{Z/pZ}(\mathbb{M}) \rightarrow T_{Z/pZ}(\mathbb{C})$ is contained in the kernel of $T_{Z/pZ}(\mathbb{M}') \rightarrow T_{Z/pZ}(\mathbb{C}')$. By (30) we then get that the map $T_{Z/pZ}(\mathbb{G}) \rightarrow T_{Z/pZ}(\mathbb{G}')$ is injective. Since by Corollary A.2.7 the map $\text{Pic}_{0,\text{red}}^{X_{/k}}(X_{/k}) \rightarrow \text{Pic}_{0,\text{red}}^{X_{/k}}(X_{/k})$ is an isomorphism, we conclude that the map $X/pX \rightarrow X'/pX'$ is injective as claimed.

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