Greybody factors of black holes in dRGT massive gravity coupled with nonlinear electrodynamics

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Abstract

In the context of the dRGT massive gravity coupled with nonlinear electrodynamics, we present new dRGT black hole solutions. Together with the thermodynamical properties of the solutions, we study the greybody factor of the corresponding black hole solution. To this end, we compute the rigorous bound on the greybody factor for the obtained dRGT black holes. The obtained results are graphically represented for different values of the theory’s physical parameters. Our analysis shows that the charged dRGT black holes of nonlinear electrodynamics evaporate quicker than the charged dRGT black holes originated from linear electrodynamics.
I. INTRODUCTION

The recent observations in the LIGO experiment suggest that the graviton is massive \[^{1}\]. It is also believed that the massive graviton of the Hubble scale may be responsible for the accelerated expansion of the universe \[^{2,3}\]. In fact, an experimental detection of graviton is a three-pipe problem, however theories of massive gravity have a number of pathologies \[^{4,5}\]. Cosmologically, the effects of the massive graviton should be considered in the large scales, at least larger than our solar system in which the GR runs properly. The classical tests of GR such as the perihelion precession of Mercury’s orbit, the deflection of light by the Sun and the gravitational redshift of light are the evidences that Einstein’s GR theory is accurate in astronomical small scales. The idea has been known ever since 1939 from the works of Fierz and Pauli (FP) \[^{6,7}\]. They constructed a linear theory of non-interactive massive graviton in a flat background spacetime which was ghost-free \[^{9}\]. In 1972, Boulware and Deser \[^{8}\] introduced the non-linear generalization of the theory in a curved background which contains the so-called van Dam-Veltman-Zakharov discontinuity \[^{10,11}\] as well as the ghost instabilities. There were attempts to fix the problem such as using the
Vainshtein mechanism [12]. But finally, de Rham, Gabadadze, and Tolley (dRGT) [13–15] have introduced a ghost free non-linear generalization of the FP theory [16, 17], which is valid in any dimensions. In this new massive gravity, adding mass to the graviton does not change significantly the physics on a small scale from the GR, as it was expected [4]. There are numerous works published in dRGT theory which lead a growing interest in this theory [18]. For instance, some cosmological and black hole solutions obtained in the dRGT theory can be seen in Refs. [43–50] and [51–64], respectively. In particular, the first non-trivial black hole solution of the 3 + 1-dimensional dRGT gravity with cosmological constant was found by Vegh in Ref. [65]. Regarding to the definition of the Hawking radiation [68] and greybody factor, there are different methods to evaluate the transmission probability and greybody factor, such as the WKB approximation, matching method [69–75], and rigorous bound method [76]. Studies about greybody factors have been increasingly gaining attention in the literature due to its observational evidence potential (see for example [19–42] and references therein). In the present study, we first introduce the 3 + 1-dimensional black hole solutions in the dRGT massive gravity coupled with nonlinear electrodynamics and then analyze their greybody factors with the method of rigorous bound [77].

The remainder of this paper is organized as follows: Section II lays out the 3 + 1-dimensional black hole solutions in the dRGT massive gravity coupled with nonlinear electrodynamics. In Sec. III, we consider the massless scalar perturbations in the geometry of the 3 + 1-dimensional black hole solutions in the dRGT massive gravity coupled with nonlinear electrodynamics. Section IV is devoted to the computation of the greybody factors with the method of rigorous bound. Finally, we discuss our results and conclude in Sec. V. We follow the metric signature (− + ++) and use the geometrized units, where \( G = c = 1 \).

II. 3+1-DIMENSIONAL BLACK HOLE SOLUTION IN DRGT MASSIVE GRAVITY COUPLED WITH NONLINEAR ELECTRODYNAMICS

In this section, we first consider the action of the dRGT massive gravity without matter source and cosmological constant in 3 + 1−dimensions, which is given by [13, 15, 78, 80]

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \left[ R(g) + m_g^2 (\mathcal{U}_2 + \alpha_3 \mathcal{U}_3 + \alpha_4 \mathcal{U}_4) + \mathcal{L} \right],
\]

(1)
in which

\[ U_2 = Tr \left(K^2\right) - Tr \left(K^2\right), \tag{2} \]
\[ U_3 = Tr \left(K^3\right) - 3Tr \left(K\right) Tr \left(K^2\right) + 2Tr \left(K^3\right), \tag{3} \]
and

\[ U_4 = Tr \left(K^4\right) - 6Tr \left(K^2\right) Tr \left(K^2\right) + 8Tr \left(K^3\right) Tr \left(K\right) + 3Tr \left(K^2\right)^2 - 6Tr \left(K^4\right). \tag{4} \]

Herein, \( m_g \) is the mass of the graviton, \( \alpha_3 \) and \( \alpha_4 \) are constants of the theory, and \( K \) represents a \( 4 \times 4 \) matrix defined by

\[ K_{\mu}^{\nu} = \delta_{\mu}^{\nu} - \sqrt{g^{\alpha\gamma} f_{\gamma\beta}}. \tag{5} \]

In latter equation, \( g^{\alpha\gamma} \) is the inverse of the metric tensor and \( f_{\gamma\beta} \) is a symmetric tensor which is called reference (or fiducial) metric. The nonlinear electrodynamics Lagrangian \( \mathcal{L} \) is defined by \[ 67 \]

\[ \mathcal{L} = \frac{-F}{1 - \frac{b}{\sqrt{8}} \sqrt{-F}}, \tag{6} \]

where \( b \) is a positive parameter and \( F = F_{\alpha\beta} F^{\alpha\beta} \) is nothing but the Maxwell invariant with a pure electric field

\[ F = E(r) dt \wedge dr. \tag{7} \]

Variation of the action with respect to electric potential admits the following Maxwell non-linear equation

\[ d \left( \tilde{F} \frac{d\mathcal{L}}{dF} \right) = 0, \tag{8} \]

where \( \tilde{F} \) is the dual of \( F \). The variation of the metric with respect to the metric tensor yields the following field equations

\[ G_{\mu}^{\nu} + m^2 g_{\mu\nu} X^{\mu} = T_{\mu}^{\nu}, \tag{9} \]

in which

\[ X_{\mu\nu} = K_{\mu\nu} - K g_{\mu\nu} - \alpha \left(K^2_{\mu\nu} - K K_{\mu\nu} + \frac{U_2}{2} g_{\mu\nu}\right) + 3 \beta \left(K^3_{\mu\nu} - K K^2_{\mu\nu} + \frac{U_2}{2} K_{\mu\nu} - \frac{U_3}{6} g_{\mu\nu}\right), \tag{10} \]

with

\[ \alpha = 1 + 3 \alpha_3. \tag{11} \]
and
\[ \beta = \alpha_3 + 4\alpha_4. \]  

Furthermore, \( T'_{\mu} \) denotes the nonlinear electrodynamics energy momentum tensor, which is given by
\[ T'_{\mu} = \frac{1}{2} (\mathcal{L}\delta_{\mu} - 4\mathcal{L}\mathcal{F}_{\mu\lambda}\mathcal{F}^{\nu\lambda}). \]  

In spherically symmetric spacetime, we consider a line-element of the form
\[ ds^2 = -n(r) dt^2 + \frac{dr^2}{f(r)} + L(r)^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right), \]  
where \( n(r) \), \( f(r) \), and \( L(r) \) are to be obtained. The reference metric tensor can be chosen as
\[ f_{\gamma\beta} = \text{diag} [0, 0, h(r)^2, h(r)^2 \sin^2 \theta], \]  
in which \( h(r)^2 \) is a coupling function. Having considered the actual metric \([5]\) and the reference metric \([6]\), one finds
\[ \mathcal{K}_{\alpha\beta} = \text{diag} \left[0, 0, 1 - \frac{h(r)}{L(r)}, 1 - \frac{h(r)}{L(r)}\right], \]  
and consequently
\[ X^t_t = X^r_r = -\frac{3L - 2h}{L} - \alpha \frac{(3L - h)(L - h)}{L^2} - \beta \frac{3(L - h)^2}{L^2}, \]  
and
\[ X^\theta_\theta = X^\phi_\phi = -\frac{3L - h}{L} - \alpha \frac{(3L - 2h)}{L} - \beta \frac{3(L - h)^2}{L}. \]  

Also, the nonlinear Maxwell equation admits an electric field of the form \([66, 67]\)
\[ E(r) = \frac{2}{b} \left(1 - \frac{1}{\sqrt{1 + \frac{\phi}{T^2}}} \right), \]  
and thus the energy momentum tensor components are explicitly found to be
\[ T^t_t = T^r_r = \frac{-E^2}{\left(1 - \frac{bE}{T}\right)^2}, \]  
and
\[ T^\theta_\theta = T^\phi_\phi = \frac{E^2}{1 - \frac{bE}{T}}. \]
As $X^t_t = X^r_r$, one should impose $G^t_t = G^r_r$ which implies that

$$
\frac{d}{dr} \left( \frac{n}{f} L^2 \right) = 0,
\tag{22}
$$

where a prime denotes the derivative of a function with respect to its argument. One can easily check that for the case of $n(r) = f(r)$ and $L(r) = r$, Eq. (22) is satisfied. A substitution into the $tt$ or $rr$ components of the Einstein’s equations yield

$$
f(r) = 1 - \frac{2M}{r} + \frac{8r^2}{3b^2} \left(1 + \frac{qb}{r^2}\right)^{3/2} - \frac{4q}{b} \left(1 + \frac{2r^2}{3qb}\right) +
\frac{m_g^2}{r} \int dr \left(3 (1 + \alpha + \beta) r^2 - 2(1 + 2\alpha + 3\beta) hr + h^2 (\alpha + 3\beta)\right),
\tag{23}
$$
in which $M$ is an integration constant. Finally, $\theta\theta$ or $\phi\phi$ components of the Einstein’s equation admit a trivial solution for the function $h(r) = h_0$ where $h_0$ is a constant parameter. Setting $L(r) = r$ and using the $rr$ component of the Einstein’s equation, after some algebra, we obtain the following analytical metric function for the charged dGRT black hole in the nonlinear electrodynamics

$$
f(r) = 1 - \frac{2M}{r} + \frac{8r^2}{3b^2} \left(1 + \frac{qb}{r^2}\right)^{3/2} - \frac{4q}{b} \left(1 + \frac{2r^2}{3qb}\right) +
\frac{m_g^2}{r} \left((1 + \alpha + \beta) r^2 - (1 + 2\alpha + 3\beta) h_0 r + h_0^2 (\alpha + 3\beta)\right),
\tag{24}
$$
in which $M$ is an integration constant. On the other hand, it is also possible to obtain a second set of solutions by considering $h(r) = \frac{3\beta + 2\alpha + 1}{\alpha + 3\beta} r$. After making straightforward calculations, one gets the following black hole solution

$$
f(r) = 1 - \frac{2M}{r} - \left(m_g^2 \frac{1 + \alpha^2 + \alpha - 3\beta}{3(\alpha + 3\beta)} + \frac{8r^2}{3b^2}\right) \frac{1 + \frac{qb}{r^2}}{\frac{4q}{b} + \frac{8r^2}{3b^2}} \left(1 + \frac{qb}{r^2}\right)^{3/2}.
\tag{25}
$$

### III. CHARGED SCALAR PERTURBATIONS FOR CHARGED DRGT MASSIVE GRAVITY BLACK HOLES IN NONLINEAR ELECTRODYNAMICS

In this section, we shall study the thermal radiation of the charged dRGT massive gravity (coupled with nonlinear electrodynamics) black holes. To this end, we first consider the massless charged Klein-Gordon equation

$$
\frac{1}{\sqrt{-g}} D_\mu \left[ \sqrt{-g} g^{\mu\nu} D_\nu \right] \Psi = 0,
\tag{26}
$$
where
\[ D_\mu = \partial_\mu - iqA_\mu, \] (27)
in which the electromagnetic potential is defined as
\[ A_t = -\frac{2}{b} \left( r - \sqrt{r^2 + q^2 b} \right), \quad A_r = A_\theta = A_\varphi = 0. \] (28)

Plugging the line-element (14) of the charged dRGT massive gravity black hole in the Klein-Gordon equation (26), we get
\[
\left[ -\frac{1}{f(r)} \partial_t^2 \Psi + \frac{1}{f(r)} q^2 A_t^2 \Psi + \frac{2i q A_t}{f(r)} \partial_t \Psi + \frac{2f}{r} \partial_r \Psi + 
\right.
\left. f'(r) \partial_r \Psi + f(r) \partial_r^2 \Psi + \frac{\cos \theta}{r^2 \sin \theta} \partial_\theta \Psi + \frac{1}{r^2} \partial^2_\theta \Psi + \frac{1}{r^2 \sin^2 \theta} \partial^2_\varphi \Psi \right] = 0. \] (29)

We use the following ansatz for the wave function
\[ \Psi(t, r, \Omega) = e^{i\omega t} \frac{\varphi(r)}{r} Y_{lm}(\Omega), \] (30)
in which \( e^{i\omega t} \) is the oscillating function and \( Y_{lm}(\Omega) \) are spherical harmonics, which satisfy the following angular equation
\[ \frac{1}{\sin^2 \theta} \partial^2 Y_{\varphi} + \frac{1}{\sin \theta} \left[ \partial \left( \sin \theta \frac{\partial Y_{\varphi}}{\partial \theta} \right) \right] = -\lambda Y_{\varphi}, \] (31)
where \( \lambda = l(l + 1) \) is the eigenvalue having orbital quantum number \( l \). Thus, the radial equation reads
\[ \frac{f}{\varphi} \frac{d}{dr} \left[ r^2 f \frac{d}{dr} \left( \frac{\varphi}{r} \right) \right] + (\omega - qA_t)^2 - \frac{\lambda f}{r^2} = 0. \] (32)

The tortoise coordinate is defined by \( \frac{dr}{dr^*} = \frac{1}{f(r)} \), which helps us to permute the radial equation to the form of one-dimensional Schrödinger equation
\[ \frac{d^2 \varphi(r)}{dr^2} + \left[ \omega^2 - V_{eff} \right] \varphi(r) = 0, \] (33)
where the effective potential in general form for dRGT massive gravity black holes with nonlinear electrodynamics is defined as
\[ V_{eff} = 2\omega qA_t - q^2 A_t^2 + \frac{\lambda f}{r^2} + \frac{f}{r} f', \] (34)
in which \( f' = \frac{df}{dr} \). Hereafter we split our calculations to the first and second solution and clarify them by indexes 1 and 2. Let’s rearrange the Eq. (24) as
\[ f_1(r) = 1 - \frac{2M}{r} + \frac{8r^2}{3b^2} \left( 1 + \frac{qb}{r} \right)^{3/2} - \frac{4q}{b} \left( 1 + \frac{2r^2}{3qb} \right) + (Ar^2 - Br + C), \quad (35) \]

where

\[
A = m_g^2 (1 + \alpha + \beta),
B = m_g^2 (1 + 2\alpha + 3\beta) h_0,
C = m_g^2 (\alpha + 3\beta) h_0^2. \quad (36)
\]

By substituting Eq. (35) and Eq. (28) in the general formula (34), then the effective potential for the first solution can be obtained as

\[
V_{\text{eff}(1)} = 2\omega q \left( -\frac{2}{b} \left( r - \sqrt{r^2 + qb} \right) \right) - \left( -\frac{2q}{b} \left( r - \sqrt{r^2 + qb} \right) \right)^2 + \frac{\lambda}{r^2} \left( 1 - \frac{2M}{r} + \frac{8r^2}{3b^2} \left( 1 + \frac{qb}{r} \right)^{3/2} - \frac{4q}{b} \left( 1 + \frac{2r^2}{3qb} \right) + (Ar^2 - Br + C) \right) + \frac{1}{r} \left( 1 - \frac{2M}{r} + \frac{8r^2}{3b^2} \left( 1 + \frac{qb}{r} \right)^{3/2} - \frac{4q}{b} \left( 1 + \frac{2r^2}{3qb} \right) + (Ar^2 - Br + C) \right) \times \left( \frac{2M}{r^2} + \sqrt{1 + \frac{qb}{r} \left( \frac{16r}{3b^2} + \frac{4q}{3b} \right) - \frac{16qr}{3qb^2} + 2Ar - B} \right). \quad (37)
\]

Following the approach of above to derive the effective potential of dRGT massive gravity with nonlinear electrodynamics for second solution. The metric function has been introduced by Eq. (25), which we can rewrite it as

\[ f_2(r) = 1 - \frac{2M}{r} + \left( D + \frac{8}{3b^2} \right) r^2 - \frac{4q}{b} + \frac{8r^2}{3b^2} \left( 1 + \frac{qb}{r^2} \right)^{3/2}, \quad (38) \]

where

\[
D = m_g^2 \frac{(1 + \alpha^2 + \alpha - 3\beta)}{3 (\alpha + 3\beta)}. \quad (39)
\]

The effective potential for the second solution is given by
\[ V_{eff(2)} = 2\omega q \left( -\frac{2}{b} \left( r - \sqrt{r^2 + qb} \right) \right) - \left( -\frac{2q}{b} \left( r - \sqrt{r^2 + qb} \right)^2 \right) + \]
\[ \frac{\lambda}{r^2} \left( 1 - \frac{2M}{r} - \left( D + \frac{8}{3b^2} \right) r^2 - \frac{4q}{b} + \frac{8r^2}{3b^2} \left( 1 + \frac{qb}{r^2} \right)^{3/2} \right) + \]
\[ \frac{1}{r} \left( 1 - \frac{2M}{r} - \left( D + \frac{8}{3b^2} \right) r^2 - \frac{4q}{b} + \frac{8r^2}{3b^2} \left( 1 + \frac{qb}{r^2} \right)^{3/2} \right) \times \]
\[ \left( \frac{2M}{r^2} - 2r \left( D + \frac{8}{3b^2} \right) + \sqrt{1 + \frac{qb}{r^2}} \left( \frac{16r}{3b^2} - \frac{8q}{3br} \right) \right). \] (40)

The behavior of dRGT effective potential for both solutions i.e., Eqs. (37) and (40) are depicted in Figs. (1) and (2) by varying the controlling parameter of \( \omega \) which is appeared in the effective potential by coupling of nonlinear electrodynamics. The parameters \( B \) and \( C \) are chosen to be zero and \( A = -1 \). It can be seen from both figures that \( V_{eff} \), which vanishes at the horizon, peaks right after the horizon and then quickly dampens towards the asymptotic region, this procedure happened for the second solution in a smaller amount rather than the first. Moreover for both, by increasing the frequency the potential peak increase as well. On the other hand, when the energy of the scalar waves increases, the peak value of the potential barrier near the event horizon also increases, which may lead to the caged of the waves. As being stated in Refs. [77, 79, 81, 82], since the main contribution to the transmission amplitude comes from the \( l = 0 \) mode (i.e., s-wave case [83]), it is adequate to qualitatively analyze the potential (37) for s-waves. In a general comparison, we can see the behavior of the potential for second solution is smoother in the same period than first solution, in this case the role of constant parameter \( b \) is significant.

IV. RIGOROUS BOUNDS ON THE GREYBODY FACTOR

A. First Solution

In this section, we shall apply the rigorous bounds [76, 77] on the greybody factor to the 3+1-dimensional black hole in dRGT massive gravity coupled with nonlinear electrodynamics. To this end, we first recall the formulation of the greybody factor (\( T \)) [81, 82]

\[ T \geq \sec h^{2} \left( \int_{-\infty}^{+\infty} \vartheta dr_{*} \right), \] (41)
FIG. 1: Plots of $V_{eff}$ versus $r$ for the metric function (24). The plot is governed by Eq. (37). The physical parameters are chosen as; $M = 1, b = 50, q = 8$, and $\lambda = 0$.

in which

$$\vartheta = \sqrt{\left[h'(r_*)\right]^2 + \left[\omega^2 - V(r_*) - h^2(r_*)\right]^2 \over 2h(r_*)},$$

(42)

where $h(r_*) > 0$, which should satisfy $h(-\infty) = h(+\infty) = \omega$. Therefore, one can set $h = \omega$ and hence Eq. (42) simplifies to

$$T \geq \sec h^2 \left( \frac{1}{2\omega} \int_{-\infty}^{+\infty} Vdr_* \right).$$

(43)

By using the tortoise coordinate and the effective potential of first solution (Eq. (37)), then the greybody factor equation (43) can be written as
FIG. 2: Plots of $V_{\text{eff}}$ versus $r$ for the metric function (25). The plot is governed by Eq. (40). The physical parameters are chosen as; $M = 1, b = 10, q = 3$, and $\lambda = 0$.

$T_1 \geq \sec^2 \frac{1}{2\omega} \left\{ \int_{r_h}^{R_h} \left( \frac{\lambda}{r^2} + \frac{2M}{r^3} - \frac{2q^2}{r^4} + 2A - \frac{B}{r} \right) dr + \right.$

$\int_{r_h}^{R_h} \frac{2\omega q^2 r}{Ar^4 - Br^3 + (1 + c) r^2 - 2Mr + q^2} dr -$

$\int_{r_h}^{R_h} \frac{\omega q^3 b}{2(Ar^5 - Br^4 + (1 + c) r^3 - 2Mr^2 + q^2 r)} dr$

$- \int_{r_h}^{R_h} \frac{q^4}{Ar^4 - Br^3 + (1 + c) r^2 - 2Mr + q^2} \right\},$ \hspace{1cm} (44)

the result is an awkward formula in point of integration view so to prevail over this issue we use the Taylor expansion, which accomplish the greybody factor as

$T_1 \geq \sec^2 \frac{1}{2\omega} \left\{ -\frac{\lambda}{R_h - r_h} - \frac{M}{R_h^2 - r_h^2} + \frac{2q^2}{3(R^3_h - r^3_h)} - (B + \frac{1}{2}\omega q b) \ln (R_h - r_h) + W_1 (R_h - r_h) + X_1 (R^2_h - r^2_h)$

$+ Y_1 (R^3_h - r^3_h) + Z_1 (R^4_h - r^4_h) - P_1 (R^5_h - r^5_h) \right\},$ \hspace{1cm} (45)

where

$W_1 = 2A - \frac{\omega b M}{q} - q^2,$ \hspace{1cm} (46)
\[ X_1 = \left( \omega + \frac{\omega_b}{4q^2} \left( q (1 + c) - \frac{4M^2}{q} \right) \right) - M, \]  
(47)

\[ Y_1 = -\frac{\omega_b}{6q^2} \left( qB - \frac{4M (q^2 (1 + c) - 2M^2)}{q^3} \right) + \frac{q^2 (1 + c) - 4M^2 + 4\omega M}{3q^2}, \]  
(48)

and

\[ Z_1 = \frac{\omega}{2q^2} \left[ (1 + c) + \frac{4M^2}{q^2} \right] - \frac{1}{8q^3} \left( -\omega qbA + \frac{\omega b (1 + c)^2}{q} + \frac{4\omega bM (Bq^4 - 3Mq^2 (1 + c) + 4M^3)}{q^5} \right) - \frac{1}{4q^2} \left( q^2 B + \frac{4M (-q^2 (1 + c) + 2M^2)}{q^2} \right), \]  
(49)

and

\[ P_1 = \frac{\omega b}{5q^5} \left( (1 + c) + 6M^2 \right) B - 2MAq^2 \]  
\[ - \frac{2\omega bM(1 + c)}{5q^4} (-q^2 (1 + c) + 2M^2) - \frac{\omega bM}{5q^4} (-q^4 (1 + c^2) + 12M^2q^2 (1 + c) - 2cq^4 - 16M^4) + \]  
\[ \frac{1}{5q^2} \left( -q^2 A + 4BM + (1 + c)^2 - \frac{4M^2 (3q^2 (1 + c) - 4M^2)}{q^4} \right) - \frac{2}{q^2} \left( \omega B + \frac{4\omega M (-q^2 (1 + c) + 2M^2)}{q^4} \right), \]  
(50)

two parameters \( R_h \) and \( r_h \), are upper and lower rigorous bound respectively, which they can obtained by [31].

\[ R_h = \frac{2}{(-2A)^{1/3}} \left[ \sqrt{\frac{2\sqrt{3}}{\beta}} + 4 \cos \left( \frac{1}{3} \sec^{-1} \left( -\sqrt{\frac{\sqrt{3}}{\beta}} + 2 \left( 2\sqrt{2}\beta + \sqrt{6} \right) \right) \right) \right] - 1, \]  
(51)

and the lower one reads

\[ r_h = \frac{-2}{(-2A)^{1/3}} \left[ \sqrt{\frac{2\sqrt{3}}{\beta}} + 4 \cos \left( \frac{1}{3} \sec^{-1} \left( -\sqrt{\frac{\sqrt{3}}{\beta}} + 2 \left( 2\sqrt{2}\beta + \sqrt{6} \right) \right) + \frac{\pi}{3} \right) \right] + 1. \]  
(52)

We demystify our results obtained, by illustrating the greybody factors for different charge values, in this case to approach in ideal form of figure we got a significantly smaller amount...
of \(b\) (around 0.1) than its value \((b = 50)\) in effective potential case. The remarkable point in Fig. (3) is that greybody factor for \(h_0 = 0\) behaves as in the case of the AdS/dS black string ([31, 79]). From this figure, one can see that the greybody factor increase by only increasing a small range of charge but after it has an inverse behaviour.

**B. Second Solution**

Based on previous part, let us substitute the effective potential of second solution Eq. (40) in Eq. (43) to get,

\[
T_2 \geq \sec^2 \frac{1}{2\omega} \left\{ \int_{r_h}^{R_h} \left( \frac{\lambda}{r^2} + \frac{2M}{r^3} - \frac{2q^2}{r^4} - 2D \right) dr + \int_{r_h}^{R_h} \frac{2\omega q^2 r}{-Dr^4 + r^2 - 2Mr + q^2} dr - \int_{r_h}^{R_h} \frac{\omega q^2 b}{2(-Dr^5 + r^3 - 2Mr^2 + q^2r)} dr - \right. \\
\left. - \int_{r_h}^{R_h} \frac{q^4}{-Dr^4 + r^2 - 2Mr + q^2} dr \right\},
\]

then after integration and using the Taylor expansion, the greybody equation is defined as

\[
T_2 \geq \sec^2 \frac{1}{2\omega} \left\{ \frac{-\lambda}{R_h - r_h} - \frac{M}{R_h^2 - r_h^2} + \frac{2q^2}{3(R_h^3 - r_h^3)} - \frac{1}{2}\omega b \ln(R_h - r_h) - W_2(R_h - r_h) + X_2(R_h^2 - r_h^2) + \right. \\
\left. Y_2(R_h^3 - r_h^3) + Z_2(R_h^4 - r_h^4) + P_2(R_h^5 - r_h^5) \right\},
\]

where

\[
W_2 = 2D + q^2 + \frac{\omega b M}{q},
\]

\[
X_2 = \omega - \left( -\frac{\omega b}{4q} + \frac{\omega b M^2}{q^3} \right) - M,
\]

\[
Y_2 = \frac{q^2 - 4M^2 + 4\omega M}{3q^2} + \frac{2\omega b M (q^2 - 2M^2)}{3q^5}
\]

\[
Z_2 = \frac{\omega}{2q^2} \left( \frac{4M^2}{q^2} - 1 \right) - \frac{\omega b}{8q^2} \left( -qD + \frac{(q^2 - 12M^2)}{q^3} + \frac{16M^4}{q^5} \right),
\]

and
\[ P_2 = -\frac{1}{10q^3} \left( 2\omega b M \left( 2D + \frac{3}{q^2} - \frac{16M^2}{q^4} + \frac{16M^4}{q^6} \right) \right) - \frac{1}{5q^2} \left( 1 - q^2 D + \frac{4M}{q^4} \left( M \left( 4M^2 - 3q^2 \right) + 2\omega \left( -q^2 + 2M^2 \right) \right) \right). \]  

(59)

FIG. 3: Plots of \( T \) versus \( \omega \) for the metric function \( f_1 \). The plot is governed by Eq. \((54)\). The physical parameters are chosen as; \( M = 1, b = 0.1, \lambda = 0, A = -1, \) and \( B = C = 0 \).

From the Eq. \((54)\), one can see the rigorous bounds on the greybody factors for the second solution of dRGT massive gravity coupled with nonlinear electrodynamics and its plotted as shown in Fig. \((4)\), the constant parameter \( b \) is chosen to be small in comparison with the potential case, for both solution of greybody factor. We can see that by increasing the charge parameter gradually, the greybody factor approach to its maximum value then it starts to dwindle, this alteration happen after \( q = 4.5 \) for both solutions. Therefore we can conclude that, having a monotonous behaviour in existence of charge for greybody factor is far from expectation.
The physical parameters are chosen as; $M = 1$, $b = 0.1$, $\lambda = 0$, and $D = 0.8$.

V. CONCLUSIONS

In this study, we have first sought for the dRGT black holes in nonlinear electrodynamics. We have shown that there exists two possible class of $3+1$-dimensional black solutions in the dRGT massive gravity coupled with nonlinear electrodynamics. The obtained spacetimes admit static and spherically symmetric metric (14). However, each class of the charged dRGT black holes has different metric functions (23) and (24) depending on the considered coupling functions $h(r) = h_0$ and $h(r) = \frac{3q+2b+1}{a+3b}r$, which are emerged from the Einstein’s field equations, respectively. We have then derived the effective potential through the decoupled set of radial and angular equations resulting from the massless charged Klein-Gordon equation. The behaviors of the effective potential for both solutions have been depicted in Figs. (1) and (2) for the metric functions $f_1$ and $f_2$, respectively. The impression of this utility method in greybody radiation is illustrated in the Figs. (3) and (4). In fact, the greybody factors in dRGT massive gravity with linear electrodynamics was studied in
Thus, we have revealed the influence of other parameters, which are consequences of coupling with nonlinear electrodynamics, on the potential and greybody factor for the two different black holes solutions.

The greybody factors that we are interested in are just the transmission probabilities for scalar wave modes propagating through the effective potential. We have managed to obtain several rigorous bounds that are placed on the greybody factors of the charged dRGT black holes. In particular, we have seen that the structure of the effective potential is deterministic for the rigorous bound on the greybody factor. Furthermore, we have depicted the greybody factors, which are derived from the rigorous bound. Based upon our analysis, we have seen that charged dRGT black holes of nonlinear electrodynamics evaporate quickly as compared to the charged dRGT black holes originated from linear electrodynamics [37]. Namely, nonlinear electrodynamics gives rise to the dRGT black holes radiate more thermal flux of quantum particles. For this reason, they will disappear in a shorter time than the charged one belonging to the linear electrodynamics.

Since the rotating black hole solutions in modified gravity theories are significant as they offer an arena to test these theories through astrophysical observations, in the near future, we plan to obtain the rotating dGRT black holes having charge in nonlinear electrodynamics by using the standard Newman-Janis algorithm [84] and reveal the effect of the rotation on their evaporation.

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