THE IN-MEDIUM FEW-BODY PROBLEM

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Abstract. We are concerned with few-particle correlations in a fermionic system at finite temperature and density. Within the many-body Green functions formalism the description of correlations is provided by the Dyson equation approach that leads to effective few-body equations. They contain the dominant medium effects, which are self energy corrections and the Pauli blocking. Hence the effective two-body interactions between quasiparticles are momentum/energy-dependent and therefore they can be used in the medium modified, momentum space, integral AGS equations for three- and four-body systems. To investigate correlations and clusters beyond four-body, we employ, instead, the configuration space two-variable integro-differential equations (IDEA) for A-body bound systems which are based on Hyperspherical Harmonics and the Faddeev decomposition of the wave function in two-body amplitudes. This requires the transformation of the energy dependent two-body interactions to equivalent local, energy independent, ones. To achieve this we use inverse scattering techniques the resulting interactions being, on- and (to all practical purposes) off-shell equivalent to the energy dependent potentials. In this way we obtain binding energy results for the 2-, 3-, 4-, and 16-particle in a medium at a finite temperature and various densities. Several aspects of the problem are discussed and the behavior of the potential surfaces obtained in the extreme adiabatic approximation, below and above the Mott transition, is investigated.
1. Introduction

The properties of nuclear matter can be studied using various methods such as path integral methods, computer simulations, or perturbation theory using many-body Green functions [1]. In the latter, which will be followed here, the equations of state may be expressed in terms of many-body Green functions and the inclusion of correlations is provided by the Dyson equation approach developed for this purpose in [2, 3, 4, 5, 6, 7], and reviewed in [8]. Within the Dyson equation approach the familiar in-medium two-body equations, known as Galitskii-Feynman or Bethe-Goldstone equations (depending on details), can be derived [1]. Recently, these equations have been extended to investigate correlations and clusters of more than two particles and applied to various aspects of fragmentation in a heavy ion collision at intermediate energies [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. For three- and four-body correlations this approach leads to modified momentum space integral equations of the Alt-Grassberger-Sandhas (AGS) [19, 20] type.

For correlations with more particles ($A > 4$), relevant e.g. in fragmentation of heavy ion collisions at intermediate energies [21], the momentum space integral equations are not appropriate and thus one has to resort to effective two-body interactions for the clusters involved and consider the system under consideration as a few-body system. For instance, the $^{16}$O may be considered as a $4\alpha$ system interacting via the $\alpha-\alpha$ interaction. In such a formalism, however, one faces the problem of how to construct the inter-cluster interactions which is a difficult task to implement. Apart from this, most of the existing and well established theories for the description of the medium-light nuclei, $5 \leq A \leq 20$ say, are in configuration space and are based on energy-independent two-body potentials. Therefore, the energy dependent effective interactions can not be directly employed and thus one should resort to some localization procedure which transforms the $V(E, r)$ for each chemical potential $\mu$ and temperature $T$, to an equivalent local potential (ELP) $V_L(r)$ which is energy-independent. The construction of the $V_L(r)$ is thus desirable for two reasons. Firstly, to investigate how the characteristics of the in-medium two-body interaction are manifested in the presence of the Fermi function $f(E)$. Secondly, to enable us to employ a configuration space formalism describing the $A$-body system embedded in nuclear matter.

In this respect, the two-variable integro-differential equation formalism (IDEA) in configuration space [22, 23, 24], valid for three- and many-body systems is employed. The IDEA is based on the Faddeev formalism, where the $A$-body wave function can be written as a sum of two-body amplitudes, and on the Hyperspherical Harmonics. The reduced equations give bind-
ing energy results for the A-body which are in excellent agreement with variational, Green Function Monte Carlo, and other competing methods. In what follows, we shall describe first in Sect. 2 our formalism starting from a brief description of the effective in-medium equations, the localization procedure, and the IDEA method. Our results are discussed in Sect. 3 and our conclusions are summarized in Sect. 4.

2. Formalism

We use the techniques of many-body Green functions. Reference to textbook treatments is given in [1]. The Dyson equation approach to correlations along with several approximation schemes has been developed in [2, 3, 4, 5, 6, 7], reviewed in [8] and will be briefly summarized in the next section. In recent years we have systematically applied this approach up to four-nucleon clusters [9, 10, 11, 12, 13, 14, 15, 16, 17, 18] and also generalized it to the light front to describe quark correlations in hot and dense relativistic nuclear matter [25, 26, 27, 28].

2.1. THE DYSON EQUATION APPROACH

Let the many-particle Hamiltonian be given by

$$H = \sum_{11'} H_0(1,1') a_1 a_{1'} + \frac{1}{2} \sum_{121'2'} V_2(12,1'2') a_1 a_{1'} a_2 a_{2'}$$

where $H_0$ is the kinetic energy, $V_2$ a generic two-body potential, and $a, a'$ are Fermionic destruction and creation operators. We define a chronological Green function for a given number of particles as follows

$$G_{\alpha\beta}^{t-t'} = -i\langle TA_{\alpha}(t) A_{\beta}^\dagger(t')\rangle$$

$$= -i \left( \theta(t-t') \langle A_{\beta}^\dagger(t') A_{\alpha}(t) \rangle \mp \theta(t'-t) \langle A_{\beta}^\dagger(t') A_{\alpha}(t) \rangle \right).$$

The average $\langle \cdots \rangle$ is taken over the exact ground state and the upper (lower) sign is for fermions (bosons). The operators $A_{\alpha}(t)$ could be build out of any number of destruction and/or creation operators. For $n$-particle correlations $A_{\alpha} = a_1 a_2 a_3 \ldots a_n$. The time dependence of the operators is given in the Heisenberg picture by $A(t) = e^{iHt} A e^{-iHt}$. Note that the multi-particle (or cluster) Green function is defined at equal or global time.

In finite temperature formalism the above definition can be generalized. For a grand canonical ensemble, relevant here, the Heisenberg picture assumes the form $A(t) = e^{(H-\mu N)t} A e^{-(H-\mu N)t}$, which includes the chemical potential $\mu$ and the number operator $N$. For $\tau = it$ (imaginary time) the original Heisenberg picture is formally recovered. The average is now taken over the
(equilibrium) grand canonical statistical operator \( \rho \), viz. \( \langle \cdots \rangle = \text{tr}\{\cdots\} \). To accommodate the convention of Fetter and Walecka [1] the imaginary time Green function is given by

\[
G_{\alpha\beta}^{\tau-\tau'} = -\langle T_\tau A_\alpha(\tau)A_\beta^{\dagger}(\tau') \rangle,
\]

where the time ordering is according to the value of \( \tau \), with the smallest at the right and the imaginary unit is dropped compared to (2).

Dyson equations can be established for both forms [8, 10],

\[
i \frac{\partial}{\partial \tau} G_{\alpha\beta}^{\tau-\tau'} = \delta(\tau - \tau') \langle [A_\alpha, A_\beta^{\dagger}] \rangle \pm \sum_\gamma \int d\bar{\tau} \mathcal{M}_{\alpha\gamma}^{\tau-\bar{\tau}} G_{\gamma\beta}^{\bar{\tau}-\tau'}.
\]

The mass matrix that appears in (4) is given by

\[
\mathcal{M}_{\alpha\beta}^{\tau-\tau'} = \delta(\tau - \tau') \mathcal{M}_{0,\alpha\beta}^{\tau-\tau'} + \mathcal{M}_{r,\alpha\beta}^{\tau-\tau'}
\]

(5)

\[
(\mathcal{M}_0\mathcal{N})^{\tau-\tau'}_{\alpha\beta} = \langle [A_\alpha, H](\tau), [A_\beta^{\dagger}(\tau)] \rangle
\]

(6)

\[
(\mathcal{M}_r\mathcal{N})^{\tau-\tau'}_{\alpha\beta} = \sum_\gamma \langle T_\tau [A_\alpha, H](\tau), [A_\beta^{\dagger}, H](\tau') \rangle \text{irreducible}
\]

(7)

where we have used \( \mathcal{N}_{\alpha\beta}^{\tau} = \langle [A_\alpha, A_\beta^{\dagger}] \rangle \). The first term in (5) is instantaneous and related to the mean field approximation, the second term is the retardation term. This form suggests to first solve the mean field problem (neglecting the retardation term) and evaluate higher order contributions involving \( \mathcal{M}_{r,\alpha\beta}^{\tau-\tau'} \) in the mean field basis. Hence (4) may be written in the following form

\[
i \frac{\partial}{\partial \tau} G_{0,\alpha\beta}^{\tau-\tau'} = \delta(\tau - \tau') \mathcal{N}_{0,\alpha\beta}^{\tau-\tau'} + \sum_\gamma \mathcal{M}_{0,\alpha\gamma}^{\tau-\tau'} G_{0,\gamma\beta}^{\tau-\tau'},
\]

(8)

\[
G_{\alpha\beta}^{\tau-\tau'} = G_{0,\alpha\beta}^{\tau-\tau'} + \sum_\gamma \int \int d\bar{\tau} d\bar{\tau}' G_{0,\alpha\gamma}^{\tau-\bar{\tau}} (\mathcal{N}\mathcal{M})^{\tau-\bar{\tau}}_{\gamma\delta} G_{\delta\beta}^{\bar{\tau}'-\tau'}.
\]

(9)

As a first approximation, we neglect the retardation part and hence solve only the mean field equation.

It is convenient to use the Matsubara-Fourier representation [1] of the Dyson equations. This leads to the Matsubara frequency instead of the “time” derivative in (8)

\[
z_\lambda = \frac{\lambda \pi}{-i\beta} + \mu
\]

(10)

where \( \lambda = 2n \) for bosons and \( \lambda = 2n + 1 \) for fermions.

For a one body operator, (8) leads to the standard mean field approximation (Hartree-Fock approximation) and an effective Hamiltonian that
describes uncorrelated quasiparticles. To be more specific, for \( A_\alpha = a_1 \) the one-particle Green function, as follows from (8), the use of (6) and the of Matsubara-Fourier transformation, is
\[
G(z) = (z - \varepsilon_1)^{-1}
\] (11)
where the Matsubara frequency has been analytically continued \( z_\lambda \rightarrow z \).

The quasiparticle self energy is
\[
\varepsilon_1 = \frac{k_1^2}{2m_1} + \sum V_2(12, \mathbf{i}2)f_2 = \frac{k_1^2}{2m_1} + \Delta_{HF} \] (12)
where \( \mathbf{i}2 \) denotes proper antisymmetrization of particles 1 and 2. The Fermi function \( f_i \equiv f(\varepsilon_i) \) for the \( i \)-th particle in an uncorrelated medium is given by
\[
\langle a^\dagger a \rangle = \text{tr}(\rho a^\dagger a) = f(\varepsilon_i) = \frac{1}{e^{\beta(\varepsilon_i - \mu)} + 1}.
\] (13)

The density operator of an uncorrelated medium is characterized only by single particle operators, viz.
\[
\rho_0 = \frac{e^{-\beta K}}{\text{tr}(e^{-\beta K})}, \quad K = \sum_i (\varepsilon_i - \mu)a_i^\dagger a_i.
\] (14)

For a general operator \( A_\alpha \) other than one body, (8) is called cluster mean field approximation. Using the density operator for an uncorrelated medium to evaluate the traces, the equation hierarchy decouples. This is a consistent way to arrive at equations for each cluster that can be solved with few-body techniques.

The resolvent \( G_0 \) for \( n \) noninteracting quasiparticles is
\[
G_0(z) = (z - H_0)^{-1}N \equiv R_0(z)N, \quad H_0 = \sum_{i=1}^n \varepsilon_i
\] (15)
where \( G_0, H_0, \) and \( N \) are formally matrices in \( n \) particle space. The Pauli-blocking for \( n \)-particles is
\[
N = \tilde{f}_1\tilde{f}_2\ldots\tilde{f}_n \pm f_1f_2\ldots f_n, \quad \tilde{f} = 1 - f
\] (16)
Note that \( NR_0 = R_0N \). Straightforward but tedious evaluation of (8) using \( A_\alpha = a_1a_2a_3\ldots a_n \) (4) and the Wick theorem, leads also to the full resolvent \( G(z) \). For each number of particles in the cluster the resolvents have the same formal structure and after Matsubara-Fourier transformation can be written in a convenient way as in the isolated system, viz.
\[
G(z) = (z - H_0 - V)^{-1}N \equiv R(z)N, \quad V \equiv \sum_{\text{pairs } \alpha} N_\alpha^2V_\alpha^\alpha.
\] (17)
Note that $V^\dagger \neq V$ and $R(z)N \neq NR(z)$. To be specific, for an interaction in pair $\alpha = (12)$ the effective potential reads

$$
\langle 123 \ldots | N_2^{(12)} V_2^{(12)} | 1'2'3' \ldots \rangle = (\tilde{f}_1 f_2 - f_1 f_2) V_2(12, 1'2') \delta_{33'} \ldots \tag{18}
$$

For further use in the Alt-Grassberger-Sandhas (AGS) equations [19, 20], we give also the channel resolvent

$$
G_\alpha(z) = (z - H_0 - N_\alpha^2 V_\alpha^2)^{-1} N \equiv R_\alpha(z) N. \tag{19}
$$

For the two-body case as well as for a two-body subsystem embedded in the $n$-body cluster the standard definition of the $t$ matrix leads to the familiar $t$ matrix equation for finite temperature and densities [1],

$$
T_2^\alpha(z) = V_2^\alpha + V_2^\alpha N_2^\alpha R_0(z) T_2^\alpha(z). \tag{20}
$$

This equation, given in the quasiparticle approximation, is used as a driving kernel for the few-particle equations. Going beyond the quasiparticle approximation would extend the scope of the present work. Note, however, that improvements can be systematically achieved by including the retardation part of the effective mass matrix, although technically this is quite a difficult task. If multi-particle correlations are neglected (binary collision approximation) it is possible to solve the $t$ matrix in a self consistent way, see e.g. [29].

### 2.2. EFFECTIVE IN-MEDIUM EQUATIONS

The in-medium Schrödinger type equation equivalent to (20) is given by

$$
(H_0 - z) \Psi(12) + \sum_{1'2'} (1 - f_1 - f_2) V_2(12, 1'2') \Psi(1'2') = 0. \tag{21}
$$

In the present work we are interested in a region of rather low density in the vicinity of the Mott transition ($\sim \rho_0/10$, where $\rho_0 = 0.16 \text{ fm}^{-3}$ is normal nuclear matter density). We therefore use an effective mass approximation for the quasi-particle energy $\varepsilon$; i.e. after evaluation of Eq. (12) we fit the effective mass $m_{\text{eff}}$ via

$$
\varepsilon = \frac{k^2}{2m} + \Delta_{\text{HF}}(k) \simeq \frac{k^2}{2m_{\text{eff}}} + \Delta_{0}^{\text{HF}}. \tag{22}
$$

The constant shift $\Delta_{0}^{\text{HF}}$ can be absorbed in a redefinition of the chemical potential $\mu_{\text{eff}} = \mu - \Delta_{0}^{\text{HF}}$. To simplify our analysis further, we assume that the two-body system is at rest in the medium. It is well known that the influence of the medium fades away for larger relative momentum respective
to the medium [30]. Upon introducing relative and c.m. coordinates for the two-particle system \((p \text{ and } P = 0)\), the kinetic energy term reads
\[
\frac{k_1^2}{2m_{\text{eff}}} = \frac{k_2^2}{2m_{\text{eff}}} = \frac{p^2}{2m_{\text{eff}}} = \frac{1}{2} E
\]
and \(E = z - E_{\text{cont}}\) is the two-body binding/scattering energy in the c.m. system. In the effective mass approximation and for \(P = 0\), \(E_{\text{cont}}\) is given by \(E_{\text{cont}} = 2\Delta^{\text{HF}}_{\text{HF}}[30]\). Using a local bare nucleon nucleon potential \(V_2(r)\), where \(r\) is the conjugate of \(p\), the effective potential becomes energy dependent
\[
V(E, r) = (1 - 2f(E)) V_2(r),
\]
where the Fermi function is now explicitly given by
\[
f(E) = \frac{1}{e^{\beta(E/2 - \mu_{\text{eff}})} + 1}.
\]
The effective in-medium Schrödinger-type equation for scattering states then reads
\[
\left( \frac{p^2}{m_{\text{eff}}} + V(E, r) - E \right) \psi(E, r) = 0.
\]
Note that this equation holds for \(E > 0\), since \(E = p^2/m_{\text{eff}}\) for the eigenvalues.

The potential \(V(E, r)\) can be employed to study its effects and how is manifested in \(A\)-body systems. For example, for \(A = 2\), one may study its analytical behavior in the complex \(k\)-plane as the nuclear density \(\rho\) changes [31] or study the influence it has on clusters embedded in the medium. Due to its energy-dependence, it can be employed in momentum space integral equations for three– [13] and four–body [16] systems embedded in the medium.

2.3. ENERGY-INDEPENDENT INTERACTIONS

Equivalence between two potentials \(V_1\) and \(V_2\), irrespective of their nature and structure, means that the scattering wave functions for the two-potentials are related by [32]
\[
u_1(r) = f(r)u_2(r)
\]
with
\[
f(r) \xrightarrow{r \to \infty} 1
\]
i.e the two wave functions must be asymptotically the same and thus the phase shifts are also the same. In our case, however, we have to obtain
an equivalent potential to \( V(E, r) \) which should not only provide the same asymptotic scattering wave function but it should generate differences in the interior region which are minimal \( (f(r) \sim 1 \forall r) \) while at the same time the corresponding binding energy \( E_b \) obtained by the \( E \)-independent interaction should be the same as the one obtained from \( V(E_b, r) \).

From all methods employed to construct \( E \)-independent ELP’s, the inverse scattering technique is the most natural one. The two-body potential is, in this case, directly obtained from the available bound and scattering information in a unique way and without an \emph{a priori} assumption about the shape and range of the interaction. This not only guarantees the recovery of the on-shell behavior of the wave function but of the bound states -when present- as well. The off-shell differences between the original wave function and the one generated by the ELP potential obtained by inversion, described by the function \( f(r) \), Eq. (28), are also minimal. From all variant inverse scattering methods, the fixed-\( \ell \) inversion of Marchenko [33, 34, 35] is perhaps the most suitable to employ in order to construct an \( E \)-independent ELP to \( V(E, r) \) as defined by (24). The required input is the \( S \)-matrix \( S_\ell(k) \), defined by

\[
S_\ell(k) = \frac{f_\ell(-k)}{f_\ell(k)} = e^{2i\delta_\ell(k)},
\]

where \( f_\ell(k) \) is the Jost-function the analytical properties of which describe the scattering, bound, and resonances of the system and \( \delta_\ell(k) \) are the scattering phase shifts at the energy \( E = \frac{\hbar^2 k^2}{2\mu} \) which are used to obtain the Fourier transform \( F_\ell(r, r') \) of the \( S \)-matrix \( S_\ell(k) \),

\[
F_\ell(r, r') = \frac{1}{2\pi} \int_{-\infty}^{\infty} h_\ell^{(+)}(kr) [1 - S_\ell(k)] h_\ell^{(+)}(kr') dk + \sum_{n=1}^{N_b} A_{n\ell} h_\ell^{(+)}(b_n r) h_\ell^{(+)}(b_n r'),
\]

where \( h_\ell^{(+)}(z) \) is the Riccati-Hankel function, \( N_b \) is the number of bound states, and \( A_{n\ell} \) are the corresponding asymptotic normalization constants [35]. The \( F_\ell \) in turns serve as an input to the Marchenko integral equation,

\[
K_\ell(r, r') + \mathcal{F}_\ell(r, r') + \int_r^{\infty} K_\ell(r, s) \mathcal{F}_\ell(s, r') ds = 0 .
\]

Once the \( K_\ell(r, r') \) is obtained from the solution of (31), the potential \( V_\ell(r) \) for the partial wave \( \ell \) is calculated from the relation

\[
V_\ell(r) = -2 \frac{dK_\ell(r, r)}{dr}.
\]
It is clear that while in the Schrödinger equation one starts from the knowledge of the potential to obtain the wave function which provide us with the physical information, in the inverse scattering procedure one uses instead experimental scattering data (or data obtained via a theoretical approach) to construct the underlying unknown potential first. The practical implementation of the method has been described in [31, 36).

2.4. INTEGRO-DIFFERENTIAL EQUATIONS APPROACH

The method we employ to study the dynamics of the A-boson$^1$ like system is based on the Faddeev decomposition of the wave function and on Hyperspherical Harmonics. The $A$-body Schrödinger equation reads

$$\left[ T + \sum_{i<j \leq A} V(r_{ij}) - E \right] \Psi(x) = 0 \quad (33)$$

where $x$ stands for all the coordinates of all particles in the center-of-mass frame, $V(r_{ij})$ is a central two-body potential, and $r_{ij} = |x_i - x_j|$. Assuming that the wave function is written as a sum of amplitudes,

$$\Psi(x) = \sum_{i<j \leq A} \Phi(r_{ij}, x) = H_{[L_m]}(x) \sum_{i<j \leq A} F(r_{ij}, r) \quad (34)$$

where $H_{[L_m]}(x)$ is a harmonic polynomial of minimal degree $L_m$ and $r$ is the hyperradius $r = (\sum_{i<j \leq A} r_{ij}^2)/A$, one obtains the Faddeev-type equation for the amplitude $F(r_{ij}, r)$

$$[T - E] H_{[L_m]}(x) F(r_{ij}, r) = -V(r_{ij}) H_{[L_m]}(x) \sum_{k<l \leq A} F(r_{kl}, r) \quad (35)$$

This equation may be modified by introducing in both sides the average of the potential over the unit hypersphere i.e the so-called hypercentral potential $V_0(r)$ defined by [22]

$$V_0(r) = \int_{-1}^{+1} V(r \sqrt{(1 + z)/2}) W(z) dz \quad (36)$$

where $z = 2r_{ij}^2/r^2 - 1$ and where the weight function $W(z)$ is given by

$$W(z) = (1 - z)\alpha (1 + z)\beta \quad (37)$$

$^1$Presently we neglect spin-isospin dependence, which means that the effective few-body equations derived are for bose-like particles. This restriction seems not severe as it might be interpreted as a spin averaging. This kind of averaging procedures (e.g. angular momentum averaging of Pauli factors) are widely used for quasiparticles and lead for example to the possibility of angular momentum expansion. This does not affect the Fermi statistics in use.
with \( \alpha = (D - 5)/2, \beta = 1/2, \) and \( D = 3(A - 1) \). Then instead of (35) one has the modified Faddeev equation

\[
T + \frac{A(A - 1)}{2}V_0(r) - E \right] H_{[L_m]}(x) \phi(r_{ij}, r) = - \left[ V(r_{ij}) - V_0(r) \right] H_{[L_m]}(x) \sum_{k < t \leq A} \phi(r_{kt}, r) \tag{38}
\]

It is noted that by summing aver all pairs \((ij)\) one recovers the Schrödinger equation. Letting \( \phi(r_{ij}, r) = P(z, r) / r^{(D-1)/2} \) for the modified Faddeev component, multiplying from left by \( H_{[L_m]}^*(x) \), and integrating over the \((N - 1)\)-hyperangles \(d\Omega_{N-1}\) (except those of the \((i, j)\)-pair) we obtain the integro-differential equation (IDEA) for the amplitude \(P(z, r)\) \cite{22, 23}

\[
\left\{ \frac{\hbar^2}{m} \left[ D_r + \frac{4}{r^2} D_z \right] + \frac{A(A - 1)}{2} V_0(r) - E \right\} P(z, r) = - \left[ V(r \sqrt{1 + z}/2) - V_0(r) \right] \Pi(z, r) \tag{39}
\]

where we use the abbreviations

\[
D_r \equiv - \frac{\partial^2}{\partial r^2} + \frac{L(L + 1)}{r^2}, \quad D_z = - \frac{1}{W(z)} \frac{\partial}{\partial z} (1 - z^2) W(z) \frac{\partial}{\partial z} \tag{40}
\]

and

\[
\Pi(z, r) = P(z, r) + \int_{-1}^{+1} f(z, z') P(z', r) \, dz'. \tag{41}
\]

In the above, \( L = (D - 3)/2 \) and the function \( f(z, z') \) is given by

\[
f(z, z') = W(z') \sum_K (f_K^2 - 1) P_K(z) P_K(z') / h_K \tag{42}
\]

where \( P_K(z) \equiv P_K^{\alpha, \beta}(z) \) are Jacobi polynomials associated to the weight function \( W(z) \). The constant \( f_K \) is given by

\[
f_K^2 = 1 + \left[ 2(A - 2) P_K(-1/2) + \frac{(A - 2)(A - 3)}{2} P_K(-1) \right] / P_K(1) \tag{43}
\]

while \( h_K \), a normalization constant, by

\[
h_K = \int_{-1}^{+1} [P_K(z)]^2 W(z) \, dz \tag{44}
\]

Details on the IDEA formalism can be found in Refs. \cite{22, 23} and will not be discussed here.
In the adiabatic approximation one assumes that the radial motion (coordinate $r$) and the orbital motion (coordinate $z$) are nearly decoupled i.e. we may write $P(z,r) \approx \Pi(z,r)$ and that $\Pi(z,r)$ is varied slowly with $r$ and thus one can omit its derivatives with respect to $r$. Then Eq. (39) can be split into two equations ($\hbar^2/m = 1$)

$$\left[ -\frac{4}{r^2}D_z + U_\lambda(r) \right] P_\lambda(z,r) = \left[ V(r\sqrt{(1+z)/2}) - V_0(r) \right] \Pi_\lambda(z,r)$$

and

$$\left[ D_r + \frac{A(A-1)}{2} V_0(r) + U_\lambda(r) \right] u_\lambda(r) = 0$$

The $U_\lambda(r)$ is the so-called eigen-potential sometimes referred to as potential surface. It is a characteristic for each system and it can be used to calculate reliably the $A$-boson like bound states via (46). The solution thus obtained provide us a lower bound for the binding energy $E_{\lambda}^{\text{eaa}}$ [22, 23, 24].

3. Results

In our investigations we employ the Volkov potential [37] which is widely used in model nuclear structure calculations,

$$V(r) = -83.3400196 \exp(-(r/1.6)^2) + 144.843409 \exp(-(r/0.82)^2)$$

the two-body binding energy being $E_2 = 0.5462$ MeV. The medium-modified energy dependent potentials $V(E,r)$ are constructed using Eq. (24). Employing $V(E,r)$ one can obtain phase shifts at any energy $E$ corresponding to a specific $\mu_{\text{eff}}$ and effective mass $m_{\text{eff}}$. In the present work we use two nuclear densities namely $\rho = 0.003$ fm$^{-3}$ and $\rho = 0.034$ fm$^{-3}$ and $T = 10$. The low energy behavior of these phase shifts are shown in Fig. 1. The distortion of the potential for $\rho = 0.003$ fm$^{-3}$ results in different phase shifts but the potential still sustains a bound state at $E_2 = 0.0282$ MeV. In contrast the phase shift for $\rho = 0.034$ fm$^{-3}$ start from zero and thus, according to Levinson’s theorem, no bound exists in this case.

![Figure 1](image-url)
Figure 2. Comparison of the scattering wave functions ψ(r) at E = 25 MeV obtained with the E-independent Marchenko potential V_L(r) (—) with those of V(E, r) (---) for ρ = 0.003 fm$^{-3}$ (Fig. a) and ρ = 0.034 fm$^{-3}$ (Fig. b) at T = 10 MeV.

Using the above phase shifts we constructed the E-independent potentials by means of the Marchenko inverse scattering method, the quality of which can be seen in Fig. 2 where the original and the resulted wave functions at E = 25 MeV are drawn. It is seen that, to all practical purposes, the two wave functions are the same i.e the potential V(E, r) is not only on–shell equivalent but (nearly) off–shell equivalent to the one obtained by inversion, V_L(r), as well. Similar results are obtained at all energies.

Once the quality of the interactions is checked, they are employed in the IDEA equations (45) and (46) to obtain the binding energies for the three–, four– and sixteen–body systems in the adiabatic approximation. The binding energy results for the three densities used are given in table 1. It is seen that with the bare interaction used, the three-body is the first to 'dissolve' in the medium for density ρ = 0.034 fm$^{-3}$. The four-body results show similar trends to the three-body system. However, the binding energy for the 16–body system, is not as drastically reduced and for ρ = 0.034 fm$^{-3}$ which is quite higher than the Mott density, the binding energy is still significant.

In Fig. 3 the ELP’s for the medium modified 2-body interaction is plotted together with the potential surfaces obtained from the solution of (45). It is

| ρ (fm$^{-3}$) | E_2 (MeV) | E_3 (MeV) | E_4 (MeV) | E_16 (MeV) |
|-------------|-----------|-----------|-----------|------------|
| 0           | 0.5462    | 8.6704    | 30.7250   | 1630.1462  |
| 0.003       | 0.0282    | 4.7202    | 22.2798   | 1554.0695  |
| 0.034       | -         | -         | 5.4093    | 1410.5321  |

TABLE 1. Binding energy results for 2–, 3–, 4–, and 16–body systems.
seen that the essential features of the medium modified two-body interaction with the Volkov (bare) potential are similar to those obtained Ref. [31] with the Malfliet-Tjon potential: As the density of the medium increases the attractive well becomes narrower and eventually for densities above the Mott one, the two nucleons become unbound. It is interesting to note that at these densities the repulsion in the interaction region appears. The analytical properties for the corresponding Jost function of the potentials have been studied in Ref. [31] where it was shown that the bound state, instead of being continued off the imaginary axis to the resonances region, moves along the negative imaginary $k$-axis, i.e. it becomes an anti-bound state – at least for the densities used where the repulsive hump in the interaction region is weak. The potential surfaces exhibit similar behavior and in addition they become shallower. The eigen-potentials for the 16–body system tend, in addition, to shift the repulsion in the interior region in a more pronounced manner.

Figure 3. The two-body potentials and the corresponding potential energy surfaces (in the extreme adiabatic approximation) for 3-, 4- and 16-body systems for zero distortion (---), for $\rho = 0.003$ fm$^{-3}$ (---), and $\rho = 0.034$ fm$^{-3}$ (···).
4. Conclusions

We propose a method to investigate the A-body system embedded in the medium at a finite temperature and for densities below and above the Mott density. Unlike previous investigations which were carried out in momentum space, the present one is in configuration space and thus advantageous in visualizing the effects of the medium on the system under consideration.

To achieve this, we firstly apply inverse scattering techniques to transform the medium modified two-body interactions which are energy dependent, into an energy independent potentials. This allow us to employ the potential directly in configuration space where various formalisms exist. One such formalism, is the IDEA method for few- and many-body systems which, apart from its simplicity, gives results which are in excellent agreement with those of other methods and therefore they can be safely employed in our investigations on the in-medium few-body problem.

We demonstrate our approach by calculating the binding energies for the 2–, 3–, 4–, and 16–boson like systems at finite temperatures and densities. The behavior of the adiabatic approximation potential surfaces, below and above the Mott density, is also investigated. We found that the adiabatic potentials exhibit similar features to those found in Ref. [31], namely, that the attractive well moves towards the interior region while at the same time becomes narrower as the densities become larger. The analytical properties of the corresponding Jost function in the complex $k$-plane for these potentials were investigated in [31] where it was shown that the bound state $E_b = -\hbar^2/2\mu b^2$ moves, downwards on the imaginary $k$-axis and eventually become an anti-bound state instead of a resonance (as implied by the appearance of a small repulsive hump in the interaction region). This reflects a vanishing imaginary part of the respective two-body spectral function. Hence, the two-body cluster (even for $b < 0$) retains the character of a quasiparticle, i.e. the correlation does not decay. In a more general approach the self-consistency requirement [29] as well as three-body collisions [11] lead to imaginary parts of the two-body spectral function, and therefore to a finite width (life time) of the two-particle cluster irrespective of $b$ smaller or larger than zero.

It is interesting to note that the change of the eigen-potential for the 16-body shows a lesser sensitivity to the density changes and the binding energies are still quite deep. Further investigations with other bosonic systems and two-body forces are needed to elucidate the behavior and phase transition of large clusters in the medium.
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