ON TEMPERED REPRESENTATIONS

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Abstract. Let $G$ be a unimodular locally compact group. We define a property of irreducible unitary $G$-representations $V$ which we call $c$-temperedness, and which for the trivial $V$ boils down to Følner’s condition (equivalent to the trivial $V$ being tempered, i.e. to $G$ being amenable). The property of $c$-temperedness is a-priori stronger than the property of temperedness.

We conjecture that for semisimple groups over local fields temperedness implies $c$-temperedness. We check the conjecture for a special class of tempered $V$’s, as well as for all tempered $V$’s in the cases of $G := SL_2(\mathbb{R})$ and of $G = PGL_2(\Omega)$ for a non-Archimedean local field $\Omega$ of characteristic 0 and residual characteristic not 2. We also establish a weaker form of the conjecture, involving only $K$-finite vectors.

In the non-Archimedean case, we give a formula expressing the character of a tempered $V$ as an appropriately-weighted conjugation-average of a matrix coefficient of $V$, generalizing a formula of Harish-Chandra from the case when $V$ is square-integrable.

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1. Introduction

1.1. Throughout the paper, we work with a unimodular second countable locally compact group $G$, and fix a Haar measure $dg$ on it. In the introduction, in §1.2 - §1.7 $G$ is assumed semisimple over a local field, while in §1.8 - §1.9 there is no such assumption. After the introduction, in §1.10 - §1.12 $G$ is assumed semisimple over a local field, while in §1.13 - §1.14 there is no such assumption. Unitary representations of $G$ are pairs $(V, \pi)$, but for lightness of notation we denote them by $V$, keeping $\pi$ implicit.
1.2. Assume that $G$ is a semisimple group over a local field\footnote{So, for example, $G$ can be taken $SL_n(\mathbb{R})$ or $SL_n(\mathbb{Q}_p)$.}. The characterization of temperedness of irreducible unitary $G$-representations in terms of the rate of decrease of $K$-finite matrix coefficients is well-studied (see for example \cite{Wa, CoHaHo, Ba}). Briefly, fixing a maximal compact subgroup $K \subset G$, an irreducible unitary $G$-representation $V$ is tempered if and only if for every two $K$-finite vectors $v_1, v_2 \in V$ there exists $C > 0$ such that
\[
|\langle gv_1, v_2 \rangle| \leq C \cdot \Xi_G(g)
\]
for all $g \in G$, where $\Xi_G : G \to \mathbb{R}_{>0}$ is Harish-Chandra’s $\Xi$-function (see \S 6.1 for a reminder on the definition of $\Xi_G$). When considering matrix coefficients of more general vectors, differentiating between tempered and non-tempered irreducible unitary $G$-representations becomes more problematic, as the following example shows.

Example 1.1 (see Claim \S 5.4). Let $G := \text{PGL}_2(\Omega)$, $\Omega$ a local field. Denote by $A \subset G$ the subgroup of diagonal matrices. Given a unitary $G$-representation $V$ let us denote
\[
\mathcal{M}_V(A) := \{ a \mapsto \langle av_1, v_2 \rangle \}_{v_1, v_2 \in V} \subset C(A),
\]
i.e. the set of matrix coefficients of $V$ restricted to $A$. Let us also denote
\[
\hat{L}^1(A) := \left\{ a \mapsto \int_A \chi(a) \cdot \phi(\chi) \cdot d\chi \right\}_{\phi \in L^1(A)} \subset C(A),
\]
i.e. the set of Fourier transforms of $L^1$-functions on $A$. Then for any non-trivial irreducible unitary $G$-representation $V$ we have
\[
\mathcal{M}_V(A) = \hat{L}^1(A).
\]

The remedy proposed in this paper is that, instead of analysing the pointwise growth of matrix coefficients, we arrange their "growth in average", i.e. the behaviour of integrals of norm-squared matrix coefficients over big balls.

1.3. We fix a norm $\| - \|$ on the vector space $\mathfrak{g} := \text{Lie}(G)$ and consider also the induced operator norm $\| - \|$ on $\text{End}(\mathfrak{g})$. We define the “radius” function $r : G \to \mathbb{R}_{\geq 0}$ by
\[
 r(g) := \log \left( \max \{ |\text{Ad}(g)|, |\text{Ad}(g^{-1})| \} \right)
\]
where $\text{Ad} : G \to \text{Aut}(\mathfrak{g})$ is the adjoint representation. We denote then by $G_{< r} \subset G$ the subset of elements $g$ for which $r(g) < r$.

Conjecture 1.2 (“asymptotic Schur orthogonality relations”). Let $V$ be a tempered irreducible unitary $G$-representation. There exist $d(V) \in \mathbb{Z}_{\geq 0}$ and $f(V) \in \mathbb{R}_{>0}$ such that for all $v_1, v_2, v_3, v_4 \in V$ we have
\[
\lim_{r \to +\infty} \int_{G_{< r}} \frac{\langle gv_1, v_2 \rangle \langle gv_3, v_4 \rangle}{r^d(V)} \cdot dg = \frac{1}{f(V)} \cdot \langle v_1, v_3 \rangle \langle v_2, v_4 \rangle.
\]

Remark 1.3 (see Claim \S 5.2). The validity of Conjecture 1.2 as well as the resulting invariants $d(V)$ and $f(V)$ (and of other similar results/conjectures below - see the formulation of Claim \S 5.2), do not depend on the choice of the norm $\| - \|$ on $\mathfrak{g}$ (used to construct the subsets $G_{< r}$).
Remark 1.4 (see Remark 1.10). An irreducible unitary $G$-representation $V$ for which the condition of Conjecture 1.2 is verified is tempered.

Remark 1.5. In the notation of Conjecture 1.2, $d(V) = 0$ if and only if $V$ is square-integrable. In that case, $f(V)$ is the well-known formal degree of $V$.

Remark 1.6 (following from Proposition 2.5). Let $V$ and $W$ be two tempered irreducible unitary $G$-representations for which Conjecture 1.2 holds, and which are non-isomorphic. Then for all $v_1, v_2 \in V$ and $w_1, w_2 \in W$ one has
\[
\lim_{r \to +\infty} \frac{\int_{G < r} \langle gv_1, v_2 \rangle \langle gw_1, w_2 \rangle \cdot dg}{r^{d(V) + d(W)/2}} = 0.
\]

1.4. We show the following statement, weaker than Conjecture 1.2:

Theorem 1.7 (see §1.4). Let $V$ be a tempered irreducible unitary $G$-representation and $K \subset G$ a maximal compact subgroup. There exists $d(V) \in \mathbb{Z}_{>0}$ such that:

(1) If $G$ is non-Archimedean, there exists $f(V) \in \mathbb{R}_{>0}$ such that for all $K$-finite $v_1, v_2, v_3, v_4 \in V$ we have
\[
\lim_{r \to +\infty} \frac{\int_{G < r} \langle gv_1, v_2 \rangle \langle gv_3, v_4 \rangle \cdot dg}{r^{d(V)}} = \frac{1}{f(V)} \langle v_1, v_3 \rangle \overline{\langle v_2, v_4 \rangle}.
\]

(2) If $G$ is Archimedean, for any given non-zero $K$-finite vectors $v_1, v_2 \in V$ there exists $C(v_1, v_2) > 0$ such that
\[
\lim_{r \to +\infty} \frac{\int_{G < r} |\langle gv_1, v_2 \rangle|^2 \cdot dg}{r^{d(V)}} = C(v_1, v_2).
\]

Remark 1.8. We expect that it should not be very difficult to establish the statement of item (1) of Theorem 1.7 also in the Archimedean case, instead of the weaker statement of item (2).

Concentrating on the non-Archimedean case for simplicity, Theorem 1.7 has as a corollary the following proposition, a generalization (from the square-integrable case to the tempered case) of a formula of Harish-Chandra (see [Ha2, Theorem 9]), expressing the character as a conjugation-average of a matrix coefficient.

Definition 1.9. Assume that $G$ is non-Archimedean. We denote by $C^\infty(G)$ the space of (complex-valued) smooth functions on $G$ and by $D_c^\infty(G)$ the space of smooth distributions on $G$ with compact support. We denote by $C^{-\infty}(G)$ the dual to $D_c^\infty(G)$, i.e. the space of generalized functions on $G$ (thus we have an embedding $C^\infty(G) \subset C^{-\infty}(G)$). Given an admissible unitary $G$-representation $V$, we denote by $\Theta_V \in C^{-\infty}(G)$ the character of $V$.

Proposition 1.10 (see §1.5). Let $V$ be a tempered irreducible unitary $G$-representation. Let $v_1, v_2 \in V$ be smooth vectors. Denote by $m_{v_1, v_2} \in C^\infty(G) \subset C^{-\infty}(G)$ the matrix coefficient $m_{v_1, v_2}(g) := \langle gv_1, v_2 \rangle$. Denoting $(g m)(x) := m(g^{-1} x g)$, the limit
\[
\lim_{r \to +\infty} \frac{\int_{G < r} g m_{v_1, v_2} \cdot dg}{r^{d(V)}}
\]

3When $G$ is non-Archimedean $K$-finite is the same as smooth (in particular does not depend on $K$).
exists in $C^{-\infty}(G)$, in the sense of weak convergence of generalized functions (i.e. convergence when paired against every element in $D_c^{\infty}(G)$), and is equal to

$$\langle v_1, v_2 \rangle_{f(V)} \cdot \Theta_V.$$

1.5. We are able to verify Conjecture 1.2 in some cases.

**Theorem 1.11** (see Theorem 6.2). Conjecture 1.2 is true for the principal series irreducible unitary representation of “slowest decrease”, i.e. the unitary parabolic induction of the trivial character via a minimal parabolic subgroup.

Here is the main result of the paper:

**Theorem 1.12** (see §7). Conjecture 1.2 is true for all tempered irreducible unitary representations of $G := \text{SL}_2(\mathbb{R})$ and of $G := \text{PGL}_2(\Omega)$, where $\Omega$ is a non-Archimedean field of characteristic 0 and of residual characteristic not equal to 2.

1.6. The proposition that follows shows that a seemingly weaker property implies that of Conjecture 1.2.

**Definition 1.13.** Given a unitary $G$-representation $V$ and vectors $v_1, v_2 \in V$ we define

$$M_{v_1,v_2}(r) := \int_{G < r} |\langle gv_1, v_2 \rangle|^2 \cdot dg.$$

**Proposition 1.14** (see §5.1). Let $V$ be an irreducible unitary $G$-representation. Let $v_0 \in V$ be a unit vector such that the following holds:

1. (1) For any vectors $v_1, v_2 \in V$ we have

$$\limsup_{r \to +\infty} \frac{M_{v_1,v_2}(r)}{M_{v_0,v_0}(r)} < +\infty.$$

2. (2) For any vectors $v_1, v_2 \in V$ and $r' > 0$ we have

$$\lim_{r \to +\infty} \frac{M_{v_1,v_2}(r + r') - M_{v_1,v_2}(r - r')}{M_{v_0,v_0}(r)} = 0.$$

Then Conjecture 1.2 holds for $V$.

**Question 1.15.** Does item (1) of Proposition 1.14 hold for arbitrary irreducible unitary $G$-representations?

**Remark 1.16** (see Proposition 5.3). An irreducible unitary $G$-representation for which there exists a unit vector $v_0 \in V$ such that conditions (1) and (2) of Proposition 1.14 are satisfied is tempered.

1.7. After finishing writing the current paper, we have found previous works [Mi] and [An]. Work [Mi] intends at giving an asymptotic Schur orthogonality relation for tempered irreducible unitary representations, but we could not understand its validity; on the first page the author defines a seminorm $||-||^2_p$ on $C^{\infty}(G)$ by a limit, but this limit clearly does not always exist. Work [An] (which deals with the more general setup of a symmetric space) provides an asymptotic Schur orthogonality relation for $K$-finite vectors in a tempered irreducible unitary $G$-representation, in the case when $G$ is real and under a regularity assumption on the central character. This work also seems to provide an interpretation of what we have denoted as $f(V)$ in terms of the Plancherel density (but it would be probably good to work this out in more detail).
1.8. Let now $G$ be an arbitrary unimodular second countable locally compact group. We formulate a property of irreducible unitary $G$-representations which we call c-temperedness (see Definition 2.1). The property of c-temperedness is, roughly speaking, an abstract version of properties (1) and (2) of Proposition 1.14. Here $G_r \subset G$ are replaced by a sequence $\{F_n\}_{n \geq 0}$ of subsets of $G$, which we call a Følner sequence, whose existence is part of the definition (so that we speak of a representation c-tempered with Følner sequence $\{F_n\}_{n \geq 0}$), while the condition replacing property (2) of Proposition 1.14 generalizes, in some sense, the Følner condition for a group to be amenable (i.e. for the trivial representation to be tempered).

We show in Corollary 3.16 that any c-tempered irreducible unitary $G$-representation is tempered and pose the question:

**Question 1.17.** For which groups $G$ every tempered irreducible unitary $G$-representation is c-tempered with some Følner sequence?

As before, c-tempered irreducible unitary $G$-representations enjoy a variant of asymptotic Schur orthogonality relations (see Proposition 2.3):

\[
\lim_{n \to +\infty} \frac{\int_{F_n} \langle gv_1, v_3 \rangle \overline{\langle gv_2, v_4 \rangle} \cdot dg}{\int_{F_n} |\langle gv_0, v_0 \rangle|^2 \cdot dg} = \frac{\langle v_1, v_2 \rangle}{\langle v_3, v_4 \rangle}
\]

for all $v_1, v_2, v_3, v_4 \in V$ and all unit vectors $v_0 \in V$. Also, we have a variant for a pair of non-isomorphic representations (see Proposition 2.5).

**Definition 1.18.** Let us say that two irreducible unitary $G$-representations are twins if their closures in $\hat{G}$ (w.r.t. the Fell topology) coincide.

**Question 1.19.** Let $V_1$ and $V_2$ be irreducible unitary $G$-representations and assume that $V_1$ and $V_2$ are twins. Suppose that $V_1$ is c-tempered with Følner sequence $\{F_n\}_{n \geq 0}$.

1. Is it true that $V_2$ is also c-tempered with Følner sequence $\{F_n\}_{n \geq 0}$?
2. If so, is it true that for unit vectors $v_1 \in V_1$ and $v_2 \in V_2$ we have

\[
\lim_{n \to +\infty} \frac{\int_{F_n} |\langle gv_1, v_1 \rangle|^2 \cdot dg}{\int_{F_n} |\langle gv_2, v_2 \rangle|^2 \cdot dg} = 1?
\]

1.9. For many groups there exist tempered representations with the slowest rate of decrease of matrix coefficients. For such representations it is often much easier to prove analogs of c-temperedness or of orthogonality relation (1.1) than for other representations - as exemplified by Theorem 1.11 above. See [BoGa] for hyperbolic groups.

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1.11. Throughout the paper, $G$ is a unimodular second countable locally compact group. We fix a Haar measure $dg$ on $G$, as well as Haar measures on the other unimodular groups we encounter ($dk$ on the group $K$, etc.). We denote by $\text{vol}_G(-)$ the volume with respect to $dg$.

All unitary $G$-representations are on separable Hilbert spaces.

Given a unitary $G$-representation $V$, vectors $v_1, v_2 \in V$ and a measurable subset $F \subset G$, we denote

$$M_{v_1, v_2}(F) := \int_F |\langle gv_1, v_2 \rangle|^2 \cdot dg.$$ 

So in the case of a semisimple group over a local field as above, we have set

$$M_{v_1, v_2}(r) := M_{v_1, v_2}(G_{<r}).$$

We write $L^2(G) := L^2(G, dg)$, considered as a unitary $G$-representation via the right regular action.

Given Hilbert spaces $V$ and $W$, we denote by $B(V; W)$ the space of bounded linear operators from $V$ to $W$, and write $B(V) := B(V; V)$.

We write $F_1 \setminus F_2$ for set differences and $F_1 \bigtriangleup F_2 := (F_1 \setminus F_2) \cup (F_2 \setminus F_1)$ for symmetric set differences.

2. Notion of c-temperedness

In this section, let $G$ be a unimodular second countable locally compact group. We introduce the notion of a c-tempered (with a given Følner sequence) irreducible unitary $G$-representation.

2.1. The following definition aims at a generalization of the hypotheses of Proposition 1.14, so as to make them suitable for a general group.

**Definition 2.1.** Let $V$ be an irreducible unitary $G$-representation. Let $F_0, F_1, \ldots \subset G$ be a sequence of measurable pre-compact subsets all containing a neighbourhood of 1. We say that $V$ is c-tempered with Følner sequence $F_0, F_1, \ldots$ if there exists a unit vector $v_0 \in V$ such that the following two conditions are satisfied:

1. For all $v_1, v_2 \in V$ we have

$$\limsup_{n \to +\infty} \frac{M_{v_1, v_2}(F_n)}{M_{v_0, v_0}(F_n)} < +\infty.$$ 

2. For all $v_1, v_2 \in V$ and all compact subsets $K \subset G$ we have

$$\lim_{n \to +\infty} \sup_{g_1, g_2 \in K} \frac{M_{v_1, v_2}(F_n \triangle g_2^{-1} F_n g_1)}{M_{v_0, v_0}(F_n)} = 0.$$ 

**Example 2.2.** The trivial unitary $G$-representation is c-tempered with Følner sequence $F_0, F_1, \ldots$ if for any compact $K \subset G$ we have

$$\lim_{n \to +\infty} \sup_{g_1, g_2 \in K} \frac{\text{vol}_G(F_n \triangle g_2^{-1} F_n g_1)}{\text{vol}_G(F_n)} = 0.$$ 

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4 “c” stands for “matrix coefficients”.
5 The notation $M_{\ldots, \ldots}(-)$ is introduced in [§1.11].
By Følner’s condition, the existence of such a sequence is equivalent to the trivial irreducible unitary $G$-representation being tempered, i.e. to $G$ being amenable.

2.2. Irreducible unitary $G$-representations which are $c$-tempered satisfy “asymptotic Schur orthogonality relations”:

**Proposition 2.3.** Let $V$ be an irreducible unitary $G$-representation. Assume that $V$ is $c$-tempered with Følner sequence $F_0, F_1, \ldots$ and let $v_0 \in V$ be a unit vector for which the conditions (1) and (2) of Definition 2.1 are satisfied. Then for all $v_1, v_2, v_3, v_4 \in V$ we have

$$\lim_{n \to +\infty} \frac{\int_{F_n} \langle gv_1, v_2 \rangle \langle gv_3, v_4 \rangle \cdot dg}{M_{v_0,v_0}(F_n)} = \langle v_1, v_3 \rangle \langle v_2, v_4 \rangle. \quad (2.2)$$

**Proof.** First, notice that in order to show that the limit in (2.2) holds, it is enough to show that for every sub-sequence there exists a further sub-sequence of it on which the limit holds. Replacing our sequence by the sub-sequence, it is therefore enough to show simply that there exists a sub-sequence on which the limit holds - which is what we will do.

Define bilinear maps $S_0, S_1, \ldots : V \times V \to L^2(G)$ by

$$S_n(v_1, v_2)(g) := \begin{cases} \frac{1}{\sqrt{M_{v_0,v_0}(F_n)}} \cdot \langle gv_1, v_2 \rangle, & g \in F_n \\ 0, & g \notin F_n \end{cases}. \quad (2.3)$$

Clearly those are bounded.

- The bilinear maps $S_n$ are jointly bounded, i.e. there exists $C > 0$ such that $||S_n||^2 \leq C$ for all $n$.

Indeed, by condition (1) of Definition 2.1 for any fixed $v_1, v_2 \in V$ there exists $C > 0$ such that $||S_n(v_1, v_2)||^2 \leq C$ for all $n$. By the Banach-Steinhaus theorem, there exists $C > 0$ such that $||S_n||^2 \leq C$ for all $n$.

Next, define quadlinear forms $\Phi_1, \Phi_2, \ldots : V \times V \times V \times V \to \mathbb{C}$ by

$$\Phi_n(v_1, v_2, v_3, v_4) := \langle S_n(v_1, v_2), S_n(v_3, v_4) \rangle$$

- The quadlinear forms $\Phi_n$ are jointly bounded, in fact $||\Phi_n|| \leq C$ for all $n$.

This follows immediately from the above finding $||S_n||^2 \leq C$ for all $n$.

- For all $g_1, g_2 \in G$ and $v_1, v_2, v_3, v_4 \in V$ we have

$$\lim_{n \to +\infty} (\Phi_n(g_1 v_1, g_2 v_2, g_1 v_3, g_2 v_4) - \Phi_n(v_1, v_2, v_3, v_4)) = 0. \quad (2.3)$$

---

6When stating Følner’s condition for the amenability of $G$ it is more usual to consider $g_2^{-1}F_n$ rather than $g_2^{-1}F_n g_1$ in (2.1), i.e. to shift only on one side. However, using, for example, [Gr, Theorem 4.1] applied to the action of $G \times G$ on $G$, we see that the above stronger “two-sided” condition also characterizes amenability.

7Recall that $L^2(G)$ denotes $L^2(G, dg)$, viewed as a unitary $G$-representation via the right regular action.
Indeed,

\[ |\Phi_n(g_1v_1, g_2v_2, g_1v_3, v_2v_4) - \Phi_n(v_1, v_2, v_3, v_4)| = \]

\[ = \left| \frac{\int_{F_n} \langle gg_1v_1, g_2v_2 \rangle \langle gg_1v_3, g_2v_4 \rangle \cdot dg - \int_{F_n} \langle gv_1, v_2 \rangle \langle gv_3, v_4 \rangle \cdot dg}{M_{v_0, v_0}(F_n)} \right| \leq \]

\[ \leq \frac{\int_{F_n} \Delta g_{g_1}^{-1} \langle gv_1, v_2 \rangle \cdot |\langle gv_3, v_4 \rangle| \cdot dg}{M_{v_0, v_0}(F_n)} \leq \]

\[ \leq \sqrt{\frac{M_{v_1, v_2}(F_n \Delta \Delta g_{g_1}^{-1} F_n g_1)}{M_{v_0, v_0}(F_n)}} \sqrt{\frac{M_{v_3, v_4}(F_n \Delta \Delta g_{g_1}^{-1} F_n g_1)}{M_{v_0, v_0}(F_n)}} \]

and the last expression tends to 0 as \( n \to +\infty \) by condition (2) of Definition 2.1.

- There exists a sub-sequence \( 0 \leq m_0 < m_1 < \ldots \) such that

\[ \lim_{n \to +\infty} \Phi_{m_n}(v_1, v_2, v_3, v_4) = \langle v_1, v_3 \rangle \langle v_2, v_4 \rangle \]

for all \( v_1, v_2, v_3, v_4 \in V \).

By the sequential Banach-Alaoglu theorem (which is applicable since \( V \) is separable), we can find a sub-sequence \( 0 \leq m_0 < m_1 < \ldots \) and a bounded quadlinear form

\[ \Phi : V \times V \times V \times V \to \mathbb{C} \]

such that \( \lim_{n \to +\infty} \Phi_{m_n} = \Phi \). Passing to the limit in equation (2.3) we obtain that for all \( g_1, g_2 \in G \) and all \( v_1, v_2, v_3, v_4 \in V \) we have

\[ \Phi(g_1v_1, g_2v_2, g_1v_3, g_2v_4) = \Phi(v_1, v_2, v_3, v_4). \]

Fixing \( v_2, v_4 \), we obtain a bounded bilinear form \( \Phi(\cdot, v_2, \cdot, v_4) : V \times V \to \mathbb{C} \) which is \( G \)-invariant, and hence by Schur’s lemma is a multiple of the form \( \langle \cdot, \cdot \rangle \), i.e. we have a uniquely defined \( c_{v_2,v_4} \in \mathbb{C} \) such that

\[ \Phi(v_1, v_2, v_3, v_4) = c_{v_2,v_4} \cdot \langle v_1, v_3 \rangle \]

for all \( v_1, v_3 \in V \). Similarly, fixing \( v_1, v_3 \) we see that we have a uniquely defined \( d_{v_1,v_3} \in \mathbb{C} \) such that

\[ \Phi(v_1, v_2, v_3, v_4) = d_{v_1,v_3} \cdot \langle v_2, v_4 \rangle \]

for all \( v_2, v_4 \in \mathbb{C} \). Since \( \Phi(v_0, v_0, v_0, v_0) = 1 \), plugging in \( (v_1, v_2, v_3, v_4) := (v_0, v_0, v_0, v_0) \) in the first equality we find \( c_{v_0,v_0} = 1 \). Then plugging in \( (v_1, v_2, v_3, v_4) := (v_1, v_0, v_3, v_0) \) in both equalities and comparing, we find \( d_{v_1,v_3} = \langle v_1, v_3 \rangle \). Hence we obtain

\[ \Phi(v_1, v_2, v_3, v_4) = \langle v_1, v_3 \rangle \langle v_2, v_4 \rangle \]

for all \( v_1, v_2, v_3, v_4 \in V \).

Now, writing explicitly \( \Phi_{m_n}(v_1, v_2, v_3, v_4) \), we see that the limit in (2.2) is valid on our sub-sequence, so we are done, as we explained in the beginning of the proof.
2.3. If one unit vector \( v_0 \) satisfies conditions (1) and (2) of Definition 2.1 then all unit vectors do:

Proposition 2.4. Let \( V \) be an irreducible unitary \( G \)-representation. Assume that \( V \) is c-tempered with Følner sequence \( F_0, F_1, \ldots \) and let \( v_0 \in V \) be a unit vector for which the conditions (1) and (2) of Definition 2.1 are satisfied. Then for any unit vector \( v' \in V \) the conditions (1) and (2) of Definition 2.1 are satisfied.

Proof. Let \( v' \in V \) be a unit vector. From (2.2) we get
\[
\lim_{n \to +\infty} \frac{M_{v'_0,v'_0}(F_n)}{M_{v_0,v_0}(F_n)} = 1.
\]
This makes the claim clear.

2.4. We also have the following version of “asymptotic Schur orthogonality relations” for a pair of non-isomorphic irreducible representations:

Proposition 2.5. Let \( V \) and \( W \) be irreducible unitary \( G \)-representations. Assume that \( V \) and \( W \) are c-tempered with the same Følner sequence \( F_0, F_1, \ldots \) and let \( v_0 \in V \) and \( w_0 \in W \) be unit vectors for which the conditions (1) and (2) of Definition 2.1 are satisfied. Then for all \( v_1, v_2 \in V \) and \( w_1, w_2 \in W \) we have
\[
\lim_{n \to +\infty} \int_{V \times V \times W \times W} \frac{\langle gv_1, v_2 \rangle \langle gw_1, w_2 \rangle \cdot dg}{\sqrt{M_{v_0,v_0}(F_n)} \sqrt{M_{w_0,w_0}(F_n)}} = 0.
\]

Proof. We proceed similarly to the proof of Proposition 2.3. Namely, again it is enough to find a sub-sequence on which the limit holds. We define quad linear forms
\[
\Phi_1, \Phi_2, \ldots : V \times V \times W \times W \to \mathbb{C}
\]
by
\[
\Phi_n(v_1, v_2, w_1, w_2) := \int_{V \times V \times W \times W} \frac{\langle gv_1, v_2 \rangle \langle gw_1, w_2 \rangle \cdot dg}{\sqrt{M_{v_0,v_0}(F_n)} \sqrt{M_{w_0,w_0}(F_n)}}.
\]
We see that these are jointly bounded, and that for all \( g_1, g_2 \in G \) and \( v_1, v_2 \in V \) and \( w_1, w_2 \in W \) we have
\[
\lim_{n \to +\infty} (\Phi_n(g_1 v_1, g_2 v_2, g_1 w_1, g_2 w_2) - \Phi_n(v_1, v_2, w_1, w_2)) = 0.
\]
We then find a bounded quad linear form
\[
\Phi : V \times V \times W \times W \to \mathbb{C}
\]
and a sub-sequence \( 0 \leq m_0 < m_1 < \ldots \) such that \( \lim_{n \to +\infty} \Phi_{m_n} = \Phi \). We get, for all \( g_1, g_2 \in G \) and \( v_1, v_2 \in V \) and \( w_1, w_2 \in W \):
\[
\Phi(g_1 v_1, g_2 v_2, g_1 w_1, g_2 w_2) = \Phi(v_1, v_2, w_1, w_2).
\]
By Schur’s lemma we obtain \( \Phi = 0 \), giving us the desired.

\( \square \)
2.5. It is easy to answer Question 1.17 in the case of square-integrable representations:

**Proposition 2.6.** Let $V$ be a square-integrable irreducible unitary $G$-representation. Then $V$ is c-tempered with Følner sequence any increasing sequence $F_0, F_1, \ldots$ of open pre-compact subsets in $G$, such that $1 \in F_0$ and $\cup_{n \geq 0} F_n = G$.

**Proof.** Recall, that matrix coefficients of a square-integrable irreducible representation are square integrable. Let $v_0 \in V$ be a unit vector. Let $F_0, F_1, \ldots$ be any increasing sequence of open pre-compact subsets in $G$ whose union is $G$ and with $1 \in F_0$. Let $v_1, v_2 \in V$. Condition (1) of Definition 2.1 holds because we have

$$M_{v_1, v_2}(F_n) \leq M_{v_1, v_2}(G) \leq \left( \frac{M_{v_1, v_2}(G)}{M_{v_0, v_0}(F_1)} \right) \cdot M_{v_0, v_0}(F_n).$$

As for condition (2) of Definition 2.1, let $\epsilon > 0$ and let $K \subset G$ be compact. There exists $n_0 \geq 0$ such that

$$M_{v_1, v_2}(G \setminus F_{n_0}) \leq \epsilon \cdot M_{v_0, v_0}(F_1).$$

There exists $n_1 \geq n_0$ such that $K F_{n_0} K^{-1} \subset F_{n_1}$. Let $n \geq n_1$ and let $g_1, g_2 \in K$. Notice that $(F_n \triangle g_2^{-1} F_n g_1) \cap F_{n_0} = \emptyset$. We have

$$M_{v_1, v_2}(F_n \triangle g_2^{-1} F_n g_1) \leq M_{v_1, v_2}(G \setminus F_{n_0}) \leq \epsilon \cdot M_{v_0, v_0}(F_1) \leq \epsilon \cdot M_{v_0, v_0}(F_n).$$

□

3. **c-Tempered irreps are tempered**

In this section, let $G$ be a unimodular second countable locally compact group. We introduce some intermediate concepts, with the goal of showing that c-tempered irreducible unitary $G$-representations are tempered (Corollary 3.16).

3.1. Let us recall some standard definitions and statements regarding weak containment.

**Definition 3.1.** Let $V$ and $W$ be unitary $G$-representations.

1. $V$ is **weakly contained** in $W$ if for every $v \in V$, compact $K \subset G$ and $\epsilon > 0$ there exist $w_1, \ldots, w_r \in W$ such that

$$|\langle gv, v \rangle - \sum_{1 \leq i \leq r} \langle gw_i, w_i \rangle| \leq \epsilon$$

for all $g \in K$.

2. $V$ is **Zimmer-weakly contained** in $W$ if for every $v_1, \ldots, v_r \in V$, compact $K \subset G$ and $\epsilon > 0$ there exist $w_1, \ldots, w_r \in W$ such that

$$|\langle gv_i, v_j \rangle - \langle gw_i, w_j \rangle| \leq \epsilon$$

for all $1 \leq i, j \leq r$ and $g \in K$.

To facilitate the formulation of the next lemma, let us also give the following intermediate definition:

---

8For “weakly contained in the sense of Zimmer”, following [12], Remark F.1.2.(ix)].
Definition 3.2. Let $V$ and $W$ be unitary $G$-representations. Let us say that $V$ is strongly-weakly contained in $W$ if for every $v \in V$, compact $K \subset G$ and $\epsilon > 0$ there exists $w \in W$ such that

$$|\langle gv, v \rangle - \langle gw, w \rangle| \leq \epsilon$$

for all $g \in K$.

Lemma 3.3. Let $V$ and $W$ be unitary $G$-representations.

1. If $V$ is Zimmer-weakly contained in $W$ then $V$ is strongly-weakly contained in $W$, and if $V$ is strongly-weakly contained in $W$ then $V$ is weakly contained in $W$.

2. If $V$ is weakly contained in $W$ then $V$ is strongly-weakly contained in $W^{\oplus \infty}$.

3. If $V$ is weakly contained in $W^{\oplus \infty}$ then $V$ is weakly contained in $W$.

4. If $V$ is irreducible and $V$ is weakly contained in $W$ then $V$ is strongly-weakly contained in $W$.

5. If $V$ is cyclic (in particular, if $V$ is irreducible) and $V$ is strongly-weakly contained in $W$ then $V$ is Zimmer-weakly contained in $W$.

6. If $V$ is strongly-weakly contained in $W$ then $V$ is Zimmer-weakly contained in $W^{\oplus \infty}$.

Proof. Statements (1), (2) and (3) are straightforward. For statement (4) see, for example, [BeHaVa, Proposition F.1.4]. For statement (5) see [Ke, proof of $(iii) \implies (iv)$ of Proposition 2.2]. For statement (6), again see [Ke, proof of $(iii) \implies (iv)$ of Proposition 2.2] (one writes $V$ as a Hilbert direct sum of countably many cyclic unitary $G$-representations, and uses item (5)). □

Corollary 3.4. Let $V$ and $W$ be unitary $G$-representations.

1. $V$ is weakly contained in $W$ if and only if $V$ is Zimmer-weakly contained in $W^{\oplus \infty}$.

2. If $V$ is irreducible, $V$ is weakly contained in $W$ if and only if $V$ is Zimmer-weakly contained in $W$.

The following definition of temperedness is classical:

Definition 3.5. A unitary $G$-representation $V$ is said to be tempered if $V$ is weakly contained in $L^2(G)$.

Remark 3.6. Notice that an irreducible unitary $G$-representation is tempered if and only if it is Zimmer-weakly contained in $L^2(G)$, by part (2) of Corollary 3.4.

3.2. The next definitions are related to the idea that one representation is weakly contained in another if there “almost” exists a $G$-intertwining isometric embedding from the one to the other.

Definition 3.7. Let $V$ and $W$ be unitary $G$-representations. A sequence $\{S_n\}_{n \geq 0} \subset B(V; W)$ is an asymptotic embedding if the following conditions are satisfied:

1. The operators $\{S_n\}_{n \geq 0}$ are jointly bounded, i.e. there exists $C > 0$ such that $\|S_n\|^2 \leq C$ for all $n \geq 0$.

---

9Here, $W^{\oplus \infty}$ stands for the Hilbert direct sum of countably many copies of $W$.

10Recall that $L^2(G)$ denotes $L^2(G, dg)$, viewed as a unitary $G$-representation via the right regular action.
Lemma 3.9. In the context of Definition 3.7, if conditions (1) and (2) of Definition 3.7 are satisfied then given compacts $L_1, L_2 \subset V$ and a compact $K \subset G$ we have
\[
\lim_{n \to +\infty} \sup_{v_1 \in L_1, v_2 \in L_2, g \in K} |\langle (S_n g - gS_n)v_1, S_n v_2 \rangle| = 0,
\]
and if conditions (1) and (3) of Definition 3.7 are satisfied then given compacts $L_1, L_2 \subset V$ we have
\[
\lim_{n \to +\infty} \sup_{v_1 \in L_1, v_2 \in L_2} |\langle S_n v_1, S_n v_2 \rangle - \langle v_1, v_2 \rangle| = 0.
\]

Proof. This follows from the well-known fact from functional analysis that pointwise convergence coincides with compact convergence on equi-continuous subsets, see [11] Proposition 32.5.

Lemma 3.10. In the context of Definition 3.7 assume that $V$ is irreducible. If conditions (1) and (2) of Definition 3.7 are satisfied then there exists a sub-sequence $0 \leq m_0 < m_1 < \ldots$ and $c \in \mathbb{R}_{\geq 0}$ such that for all $v_1, v_2 \in V$ we have
\[
\lim_{n \to +\infty} \langle S_{m_n} v_1, S_{m_n} v_2 \rangle = c \cdot \langle v_1, v_2 \rangle.
\]
In particular, if there exists $v \in V$ such that $\liminf_{n \to +\infty} ||S_n v||^2 > 0$ then there exists $d \in \mathbb{R}_{> 0}$ (in fact, $d^{-2} = \lim_{n \to +\infty} ||S_{m_n} v||^2/||v||^2$) such that $\{dS_{m_n} v\}_{n \geq 0}$ satisfies condition (3) of Definition 3.7 i.e. is an asymptotic embedding.

Proof. By the sequential Banach-Alaoglu theorem (applicable as $V$ is separable, and $\{S_n^* S_n\}_{n \geq 0}$ are jointly bounded by condition (1)), there exists a sub-sequence $1 \leq m_0 < m_1 < \ldots$ such that $\{S_{m_n}^* S_{m_n}\}_{n \geq 0}$ converges in the weak operator topology to some $S \in B(V)$.

Let us first check that $S$ is $G$-invariant. For $g \in G$ and $v_1, v_2 \in V$ we have
\[
|\langle S_n S_n g v_1, v_2 \rangle - \langle S_n^* S_n v_1, g^{-1} v_2 \rangle| = |\langle S_n g v_1, S_n v_2 \rangle - \langle S_n v_1, S_n g^{-1} v_2 \rangle| \leq |\langle (S_n g - g S_n) v_1, S_n v_2 \rangle| + |\langle S_n v_1, (g^{-1} S_n - S_n g^{-1}) v_2 \rangle|
\]
and both summands in the last expression converge to 0 as $n \to +\infty$ by condition (2). Therefore
\[
|\langle S g v_1, v_2 \rangle - \langle g S v_1, v_2 \rangle| = \lim_{n \to +\infty} |\langle S_{m_n}^* S_{m_n} g v_1, v_2 \rangle - \langle S_{m_n}^* S_{m_n} v_1, g^{-1} v_2 \rangle| = 0
\]
i.e. $\langle S g v_1, v_2 \rangle = \langle g S v_1, v_2 \rangle$. Thus, since $v_1$ and $v_2$ were arbitrary, $S g = g S$. This holds for all $g \in G$, i.e. $S$ is $G$-invariant.

\[\text{footnote}{"o" stands for “operator”} \]
By Schur’s lemma, we deduce $S = c \cdot \text{Id}_V$ for some $c \in \mathbb{C}$. This translates precisely to (3.11). The last claim is then straightforward.

**Remark 3.11.** Using Lemma 3.3 it is straight-forward that, assuming condition (1) of Definition 3.7, conditions (2) and (3) in Definition 3.7 are equivalent to the one condition that for $v_1, v_2 \in V$ and a compact $K \subset G$ one has

$$\lim_{n \to +\infty} \sup_{g \in K} |\langle gS_n v_1, S_n v_2 \rangle - \langle gv_1, v_2 \rangle| = 0.$$ 

Indeed, let us write

$$(3.2) \quad \langle gS_n v_1, S_n v_2 \rangle - \langle gv_1, v_2 \rangle = ((gS_n - S_n g)v_1, S_n v_2) + (\langle S_n g v_1, S_n v_2 \rangle - \langle gv_1, v_2 \rangle).$$

The current condition gives condition (3) by plugging in $g = 1$, and then (3.2) gives condition (2), using the uniformity provided by Lemma 3.9. Conversely, (3.2) shows immediately (again taking into consideration Lemma 3.9) that conditions (2) and (3) imply the current condition.

### 3.3. The concept of o-weak containment in fact coincides with that of Zimmer-weak containment:

**Proposition 3.12.** Let $V$ and $W$ be unitary $G$-representations. Then $V$ is o-weakly contained in $W$ if and only if $V$ is Zimmer-weakly contained in $W$.

**Proof.** Let $\{S_n\}_{n \geq 0} \subset B(V; W)$ be an asymptotic embedding. Given $v_1, \ldots, v_r \in V$, by Remark 3.11 given any compact $K \subset G$ we have

$$\lim_{n \to +\infty} \sup_{g \in K} |\langle gS_n v_i, S_n v_j \rangle - \langle gv_i, v_j \rangle| = 0$$

for all $1 \leq i, j \leq r$, and thus

$$\lim_{n \to +\infty} \sup_{g \in K} \sup_{1 \leq i, j \leq r} |\langle gS_n v_i, S_n v_j \rangle - \langle gv_i, v_j \rangle| = 0.$$ 

Thus by definition $V$ is Zimmer-weakly contained in $W$.

Conversely, suppose that $V$ is Zimmer-weakly contained in $W$. Let $\{e_i\}_{n \geq 0}$ be an orthonormal basis for $V$. Let $\{K_n\}_{n \geq 0}$ be an increasing sequence of compact subsets in $G$, with $1 \in K_0$ and with the property that for any compact subset $K \subset G$ there exists $n \geq 0$ such that $K \subset K_n$. As $V$ is Zimmer-weakly contained in $W$, given $n \geq 0$, let us find $w_n^0, \ldots, w_n^m \in W$ such that

$$\sup_{g \in K_n} |\langle ge_i, e_j \rangle - \langle gw_n^0, w_n^m \rangle| \leq \frac{1}{n + 1}$$

for all $0 \leq i, j \leq n$. Define $S_n : V \to W$ by

$$S_n \left( \sum_{i \geq 0} c_i \cdot e_i \right) := \sum_{0 \leq i \leq n} c_i \cdot w_n^i.$$ 

We want to check that $\{S_n\}_{n \geq 0}$ is an asymptotic embedding. As for condition (1), notice that

$$\left\| S_n \left( \sum_{i \geq 0} c_i e_i \right) \right\|^2 = \left\| \sum_{0 \leq i \leq n} c_i w_n^i \right\|^2 = \left\| \sum_{0 \leq i, j \leq n} c_i c_j \cdot \langle w_n^i, w_n^j \rangle \right\| \leq$$
\[
\sum_{0 \leq i, j \leq n} c_i c_j \cdot \langle e_i, e_j \rangle + \sum_{0 \leq i, j \leq n} c_i c_j \cdot (\langle w^n_i, w^n_j \rangle - \langle e_i, e_j \rangle) \leq \\
\sum_{0 \leq i \leq n} |c_i|^2 + \frac{1}{n+1} \left( \sum_{0 \leq i \leq n} |c_i|^2 \right)^2 \leq 2 \sum_{0 \leq i \leq n} |c_i|^2 \leq 2 \left( \sum_{i \geq 0} |c_i e_i|^2 \right)^2,
\]
showing that \( ||S_n||^2 \leq 2 \) for all \( n \geq 0 \). It is left to show the condition as in Remark 3.11. Let us thus fix a compact \( K \subset G \). Notice that it is straightforward to see that it is enough to check the condition for vectors in a subset of \( V \), the closure of whose linear span is equal to \( V \). So it is enough to check that
\[
\lim_{n \to +\infty} \sup_{g \in K} |\langle g S_n e_i, S_n e_j \rangle - \langle g e_i, e_j \rangle| = 0
\]
for any given \( i, j \geq 0 \). Taking \( n \) big enough so that \( K \subset K_n \) and \( n \geq \max\{i, j\} \), we have
\[
\sup_{g \in K} |\langle g S_n e_i, S_n e_j \rangle - \langle g e_i, e_j \rangle| = \sup_{g \in K} |\langle g w^n_i, w^n_j \rangle - \langle g e_i, e_j \rangle| \leq \frac{1}{n+1},
\]
giving the desired. \( \square \)

**Corollary 3.13.** An irreducible unitary \( G \)-representation is \( o \)-tempered if and only if it is tempered.

**Proof.** This is a special case of Proposition 3.12, taking into account Remark 3.6. \( \square \)

### 3.4
Here we give a weaker version of \( c \)-temperedness, which is technically convenient to relate to other concepts of this section.

**Definition 3.14.** Let \( V \) be an irreducible unitary \( G \)-representation. Let \( F_0, F_1, \ldots \subset G \) be a sequence of measurable pre-compact subsets all containing a neighbourhood of 1. We say that \( V \) is **right-c-tempered with Følner sequence** \( F_0, F_1, \ldots \) if there exists a unit vector \( v_0 \in V \) such that the following two conditions are satisfied:

1. For all \( v \in V \) we have
   \[
   \limsup_{n \to +\infty} \frac{M_{v, v_0}(F_n)}{M_{v_0, v_0}(F_n)} < +\infty.
   \]
2. For all \( v \in V \) and all compact subsets \( K \subset G \) we have
   \[
   \lim_{n \to +\infty} \sup_{g \in K} \frac{M_{v, v_0}(F_n \Delta F_n g)}{M_{v_0, v_0}(F_n)} = 0.
   \]

### 3.5
Finally, we can show that \( c \)-tempered irreducible unitary \( G \)-representations are tempered.

**Proposition 3.15.** Let \( V \) be an irreducible unitary \( G \)-representation. Assume that \( V \) is right-c-tempered (with some Følner sequence). Then \( V \) is \( o \)-tempered. More precisely, suppose that \( V \) is right-c-tempered with Følner sequence \( F_0, F_1, \ldots \) and let \( v_0 \in V \) be a unit vector for which the conditions (1) and (2) of Definition 3.14 are satisfied. Then the sequence of operators
\[
S_0, S_1, \ldots : V \to L^2(G)
\]
given by
\[ S_n(v)(x) := \begin{cases} \frac{1}{M_{v_0,v_0}(F_n)} \cdot \langle xv, v_0 \rangle, & x \in F_n \\ 0, & x \notin F_n \end{cases} \]

admits a sub-sequence which is an asymptotic embedding.

**Corollary 3.16.** Every c-tempered irreducible unitary $G$-representation (with some Følner sequence) is tempered.

**Proof.** It is clear that c-temperedness implies right-c-temperedness, Proposition 3.15 says that right-c-temperedness implies o-temperedness, and Corollary 3.13 says that o-temperedness is equivalent to temperedness. □

**Proof (of Proposition 3.15).** Clearly each $S_n$ is bounded. By condition (2) of Definition 3.14, for any fixed $v \in V$ there exists $C > 0$ such that $||S_n(v)||^2 \leq C$ for all $n$. By the Banach-Steinhaus theorem, this implies that the operators $S_0, S_1, \ldots$ are jointly bounded, thus condition (1) of Definition 3.7 is verified.

To verify condition (2) of Definition 3.14, fix $v \in V$ and a compact $K \subset G$. Given $g \in K$ and a function $f \in L^2(G)$ of $L^2$-norm one, we have
\[
|\langle S_n(gv) - gS_n(v), f \rangle| = \left| \frac{\int_{F_n} \langle xgv, v_0 \rangle f(x) \cdot dx - \int_{F_n} \langle xgv, v_0 \rangle f(g^{-1}x) \cdot dx}{\sqrt{M_{v_0,v_0}(F_n)}} \right| \leq \frac{\int_{F_n} \Delta(F_n) \langle xgv, v_0 \rangle f(x) \cdot dx}{\sqrt{M_{v_0,v_0}(F_n)}} \leq \sqrt{M_{v_0,v_0}(F_n)} \cdot \int_G |f(x)|^2 \cdot dx = \frac{M_{v_0,v_0}(F_n \Delta F_n)}{M_{v_0,v_0}(F_n)}.
\]

Since $f$ was arbitrary, this implies
\[
||S_n(gv) - gS_n(v)|| \leq \sqrt{\frac{M_{v_0,v_0}(F_n \Delta F_n)}{M_{v_0,v_0}(F_n)}}
\]

for $g \in K$. By condition (2) of Definition 3.14 this tends to 0 as $n \to +\infty$, uniformly in $g \in K$, and hence the desired.

Now, using Lemma 3.10 we see that some sub-sequence will satisfy condition (3) of Definition 3.7 once we notice that $||S_n v_0||^2 = 1$ for all $n$ by construction. □

4. **The case of $K$-finite vectors**

In this section $G$ is a semisimple group over a local field. We continue with notations from §1. The purpose of this section is to prove Theorem 1.7.
4.1. Let us first show that, when $G$ is non-Archimedean, it is enough to establish condition (2) of Theorem 1.7 and condition (1) will then follow. So we assume condition (2) and use the notation $C(v_1, v_2)$ therein.

Let us denote by $V \subset V$ the subspace of $K$-finite (i.e. smooth) vectors. By the polarization identity, it is clear that for all $v_1, v_2, v_3, v_4 \in V$ the limit
\[
\lim_{r \to +\infty} \int_{G_{< r}} \langle gv_1, v_2 \rangle \langle gv_3, v_4 \rangle \cdot dg \\
\]exists, let us denote it by $D(v_1, v_2, v_3, v_4)$, and $D$ is a quadlinear form $D : V \times V \times V \times V \to \mathbb{C}$.

Next, we claim that for all $v_1, v_2, v_3, v_4 \in V$ and all $g_1, g_2 \in G$ we have
\[
D(g_1 v_1, g_2 v_2, g_1 v_3, g_2 v_4) = D(v_1, v_2, v_3, v_4).
\]
Indeed, again by the polarization identity, it is enough to show that for all $v_1, v_2 \in V$ and all $g_1, g_2 \in G$ we have
\[
(4.1) \quad C(g_1 v_1, g_2 v_2) = C(v_1, v_2).
\]
There exists $r_0 \geq 0$ such that
\[
G_{< r_0} \subset g_2^{-1} G_{< r} g_1 \subset G_{< r + r_0}.
\]
We have:
\[
\int_{G_{< r}} |\langle gg_1 v_1, g_2 v_2 \rangle|^2 \cdot dg = \int_{g_2^{-1} G_{< r} g_1} |\langle gv_1, v_2 \rangle|^2 \cdot dg
\]
and therefore
\[
\int_{G_{< r - r_0}} |\langle gv_1, v_2 \rangle|^2 \cdot dg \leq \int_{G_{< r}} |\langle gg_1 v_1, g_2 v_2 \rangle|^2 \cdot dg \leq \int_{G_{< r + r_0}} |\langle gv_1, v_2 \rangle|^2 \cdot dg.
\]
Dividing by $r^d(V)$ and taking the limit $r \to +\infty$ we obtain (4.1).

Now, by Schur’s lemma (completely analogously to the reasoning with $\Phi$ in the proof of Proposition 2.3), we obtain that for some $C > 0$ we have
\[
D(v_1, v_2, v_3, v_4) = C \cdot \langle v_1, v_3 \rangle \langle v_2, v_4 \rangle
\]
for all $v_1, v_2, v_3, v_4 \in V$.

4.2. Thus, we aim at establishing condition (2) of Theorem 1.7 in either the non-Archimedean or the Archimedean cases. Since a complex group can be considered as a real group and the formulation of the desired theorem will not change, we assume that we are either in the real case or in the non-Archimedean case.

Also, notice that to show Theorem 1.7 for all maximal compact subgroups it is enough to show it for one maximal compact subgroup (in the non-Archimedean case because the resulting notion of $K$-finite vectors does not depend on the choice of $K$ and in the real case since all maximal compact subgroups are conjugate).
4.3. Let us fix some notation. We choose a maximal split torus $A \subset G$ and a minimal parabolic $P \subset G$ containing $A$. We denote

$$ a := \text{Hom}_\mathbb{Z}(X^*(A), \mathbb{R}). $$

We let $L \subset a$ to be $a$ itself in the real case and the lattice in $a$ corresponding to $X_*(A)$ in the non-Archimedean case. We let $\exp : L \to A$ be the exponential map constructed in the usual way:

- If $G$ is real, we let $\exp$ to be the composition $L = a \cong \text{Lie}(A) \to A$ where the last map is the exponential map from the Lie algebra to the Lie group, while the isomorphism is the identification resulting from the map $X^*(A) \to \text{Lie}(A)^*$ given by taking the differential at $1 \in A$.

- If $G$ is non-Archimedean, we let $\exp$ be the composition $L \cong X^*(A) \to A$ where the last map is given by sending $\chi$ to $\chi(\varpi^{-1})$, where $\varpi$ is a uniformizer.

We denote by

$$ \Delta \subset \tilde{\Delta} \subset X^*(A) \subset a^* $$

the set of simple roots $\Delta$ and the set of positive roots $\tilde{\Delta}$ (resulting from the choice of $P$). We identify $a$ with $\mathbb{R}^\Delta$ in the clear way. We set

$$ a^+ := \{ x \in a \mid \alpha(x) \geq 0 \ \forall \alpha \in \Delta \} $$

and $L^+ := L \cap a^+$.

Let us in the standard way choose a maximal compact subgroup $K \subset G$ “in good relative position” with $A$. In the real case this means $\text{Lie}(A)$ sitting in the $(-1)$-eigenspace of a Cartan involution whose 1-eigenspace is $\text{Lie}(K)$ and in the non-Archimedean case it is as in [Re, V.5.1., Théorème]. In the non-Archimedean case let us also, to simplify notation, assume that $G = K \exp(L^+)K$ (in general there is a finite subset $S \subset Z_G(A)$ such that $G = \bigsqcup_{s \in S} K \exp(L^+)sK$ and one proceeds with the obvious modifications).

Let us denote $\rho := \frac{1}{2} \sum_{\alpha \in \tilde{\Delta}} \mu_\alpha \cdot \alpha \in a^*$ where $\mu_\alpha \in \mathbb{Z}_{\geq 1}$ is the multiplicity of the root $\alpha$.

Fixing Haar measures, especially denoting by $dx$ a Haar measure on $L$, we have a uniquely defined continuous $\omega : L^+ \to \mathbb{R}_{\geq 0}$ such that the following integration formula holds:

$$ \int_G f(g) \cdot dg = \int_{K \times K} \left( \int_{L^+} \omega(x) f(k_1 \exp(x)k_2) \cdot dx \right) \cdot dk_1 dk_2. $$

Regarding the behaviour of $\omega(x)$, we can use [Ax1 around Lemma 1.1] as a reference. In the real case there exists $C > 0$ such that

$$ \frac{\omega(x)}{e^{2\rho(x)}} = C \cdot \prod_{\alpha} \left( 1 - e^{-2\alpha(x)} \right) $$

where $\alpha$ runs over $\tilde{\Delta}$ according to multiplicities $\mu_\alpha$. In the non-Archimedean case, for every $\Theta \subset \Delta$ there exists $C_{\Theta} > 0$ such that

$$ \frac{\omega(x)}{e^{2\rho(x)}} = C_{\Theta} $$

for all $x \in L^+$ satisfying $\alpha(x) = 0$ for all $\alpha \in \Theta$ and $\alpha(x) \neq 0$ for all $\alpha \in \Delta \setminus \Theta$. 


Since, by Claim 4.2 we are free in our choice of the norm $||-||$ on $g$, let us choose $||-||$ to be a supremum norm in coordinates gotten from an $A$-eigenbasis. Then
\[
\mathbf{r}(\exp(x)) = \log q \cdot \max_{\alpha \in \Delta} |\alpha(x)|
\]
where $q$ is the residual cardinality in the non-Archimedean case and $q := e$ in the real case. Let us denote
\[
a_{<r} := \{ x \in a \mid |\alpha(x)| < \frac{r}{\log q} \forall \alpha \in \Delta \}
\]
and $a_{<r}^{+} := a_{<r} \cap a^{+}$ and similarly $L_{<r} := L \cap a_{<r}$, $L_{<r}^{+} := L^{+} \cap L_{<r}$. Then $L_{<r} = \exp^{-1}(G_{<r})$. Hence there exists $r_{0} \geq 0$ such that
\[
(4.4) \quad K \exp(L_{<r}^{+} - r_{0})K \subset G_{<r} K \exp(L_{<r}^{+} + r_{0})K.
\]
4.4. Let now $V$ be a tempered irreducible unitary $G$-representation. Let us denote by $\mathcal{V} \subset V$ the subspace of $K$-finite vectors. Given $v_{1}, v_{2} \in V$, we will denote by $f_{v_{1}, v_{2}}$ the continuous function on $L^{+}$ given by
\[
f_{v_{1}, v_{2}}(x) := e^{\rho(x)} \langle \exp(x)v_{1}, v_{2} \rangle.
\]
We have
\[
M_{v_{1}, v_{2}}(r) = \int_{G_{<r}} |\langle gv_{1}, v_{2} \rangle|^{2} \cdot dg =
\]
\[
= \int_{K \times K} \left( \int_{L^{+} \cap \exp^{-1}(k_{2}G_{<r}k_{1}^{-1})} \omega(x) e^{2\rho(x)} |f_{k_{1}v_{1}, k_{2}v_{2}}(x)|^{2} \cdot dx \right) \cdot dk_{1}dk_{2}.
\]
In view of 1.4, in order to prove Theorem 1.7 it is enough to show:

Claim 4.1. There exists $d(V) \in \mathbb{Z}_{\geq 0}$ such that for every non-zero $v_{1}, v_{2} \in \mathcal{V}$ there exists $C(v_{1}, v_{2}) > 0$ such that
\[
\lim_{r \to +\infty} \int_{K \times K} \left( \int_{L^{+} \cap \exp^{-1}(k_{2}G_{<r}k_{1}^{-1})} \frac{\omega(x)}{e^{2\rho(x)}} |f_{k_{1}v_{1}, k_{2}v_{2}}(x)|^{2} \cdot dx \right) \cdot dk_{1}dk_{2} = C(v_{1}, v_{2}).
\]
4.5. We have the following:

Claim 4.2.

1. Given $v_{1}, v_{2} \in \mathcal{V}$, either $f_{v_{1}, v_{2}} = 0$ in which case we set $d(v_{1}, v_{2}) := -\infty$, or there exist $d(v_{1}, v_{2}) \in \mathbb{Z}_{>0}$ and $C(v_{1}, v_{2}) > 0$ such that
\[
\lim_{r \to +\infty} \frac{1}{d(v_{1}, v_{2})} \int_{L_{<r}^{+}} \frac{\omega(x)}{e^{2\rho(x)}} |f_{v_{1}, v_{2}}(x)|^{2} \cdot dx = C(v_{1}, v_{2}).
\]
2. In the real case, we have $d(v_{1}, Xv_{2}) \leq d(v_{1}, v_{2})$ for all $v_{1}, v_{2} \in \mathcal{V}$ and $X \in g$.
3. Denoting $d(V) := \sup_{v_{1}, v_{2} \in \mathcal{V}} d(v_{1}, v_{2})$, we have neither $d(V) = -\infty$ nor $d(V) = +\infty$ (i.e. $d(V) \in \mathbb{Z}_{\geq 0}$).

Let us establish Claim 4.1 given Claim 4.2.
Proof (of Claim 4.7 given Claim 4.3). Let us first handle the non-Archimedean case. Let us notice that we can replace \( v_1 \) and \( v_2 \) by \( g_1 v_1 \) and \( g_2 v_2 \) for any \( g_1, g_2 \in G \). Indeed, for some \( r_0 \geq 0 \) we have

\[
g_2^{-1} G_{< r-r_0} g_1 \subset G_{< r} \subset g_2^{-1} G_{< r+r_0} g_1
\]
and thus

\[
\int_{G_{< r-r_0}} |\langle gg_1 v_1, g_2 v_2 \rangle|^2 \cdot dg \leq \int_{G_{< r}} |\langle gv_1, v_2 \rangle|^2 \cdot dg \leq \int_{G_{< r+r_0}} |\langle gg_1 v_1, g_2 v_2 \rangle|^2 \cdot dg,
\]
from which the claim clearly follows. Since \( G \cdot v_1 \) spans \( V \) and \( G \cdot v_2 \) spans \( V \), we deduce that by replacing \( v_1 \) and \( v_2 \) we can assume that \( d(v_1, v_2) = d(V) \). Now, since the integral

\[
\int_{K \times K} \left( \int_{L_{>r}^+} \frac{\omega(x)}{e^{2\rho(x)}} |f_{k_1 v_1, k_2 v_2}(x)|^2 \cdot dx \right) \cdot dk_1 dk_2
\]
over \( K \times K \) is simply a finite linear combination the claim is clear.

Let us now handle the real case. First, we would like to see that for some \( k_1, k_2 \in K \) we have \( d(k_1 v_1, k_2 v_2) = d(V) \). To that end, let us denote by \( n \) and \( n^- \) the Lie algebras of \( N \) and \( N^- \) (the unipotent radicals of \( P \) and of \( P^- \), the opposite to \( P \) with respect to \( A \)) and identify \( a \) with the Lie algebra of \( A \) as before. Since \( \mathcal{U}(n^-) \mathcal{U}(a) K v_1 \) spans \( V \) and \( \mathcal{U}(n) \mathcal{U}(a) K v_2 \) spans \( V \), we can find \( k_1, k_2 \in K \) and some elements \( v'_1 \in \mathcal{U}(n^-) \mathcal{U}(a) k_1 v_1 \) and \( v'_2 \in \mathcal{U}(n) \mathcal{U}(a) k_2 v_2 \) such that \( d(v'_1, v'_2) = d(V) \). By Claim 4.3.2 this forces \( d(k_1 v_1, k_2 v_2) = d(V) \).

Next, given two continuous functions \( f_1, f_2 \) on \( a^+ \) and \( d \in \mathbb{Z}_{\geq 0} \) let us denote

\[
\langle f_1, f_2 \rangle_d := \lim_{r \to +\infty} \int_{a_{>r}^+} \frac{\omega(x)}{e^{2\rho(x)}} f_1(x) f_2(x) \cdot dx
\]
if the limit exists, and

\[
||f||^2_d := \langle f, f \rangle_d.
\]
We claim that the function \((k_1, k_2) \mapsto ||f_{k_1 v_1, k_2 v_2}||^2_d(V)\) on \( K \times K \) is continuous and that

\[
\lim_{r \to +\infty} \left( \int_{a_{>r}^+} \frac{\omega(x)}{e^{2\rho(x)}} |f_{k_1 v_1, k_2 v_2}(x)|^2 \cdot dx \right) \cdot dk_1 dk_2 = \int_{K \times K} ||f_{k_1 v_1, k_2 v_2}||^2_d(V) \cdot dk_1 dk_2.
\]
Then the right hand side is non-zero since we have seen that \( d(k_1 v_1, k_2 v_2) = d(V) \) for some \( k_1, k_2 \in V \), and we are done.

Let \((v^1_k)\) be a basis for the \( \mathbb{C} \)-span of \( \{kv_1\}_{k \in K} \) and let \((v^2_k)\) be a basis for the \( \mathbb{C} \)-span of \( \{kv_2\}_{k \in K} \). Let us write \( kv_1 = \sum_i c_i(k) v^1_i \) and \( kv_2 = \sum_j d_j(k) v^2_j \), so that \( c_i \) and \( d_j \) are continuous \( \mathbb{C} \)-valued functions of \( K \). Then

\[
\int_{a_{>r}^+} \frac{\omega(x)}{e^{2\rho(x)}} |f_{k_1 v_1, k_2 v_2}(x)|^2 \cdot dx = \sum_{i_1, i_2, j_1, j_2} c_{i_1}(k_1) c_{i_2}(k_1) d_{j_1}(k_2) d_{j_2}(k_2) \int_{a_{>r}^+} \frac{\omega(x)}{e^{2\rho(x)}} \cdot f_{v^1_{i_1}, v^2_{i_2}}(x) \cdot f_{v^1_{j_1}, v^2_{j_2}}(x) \cdot dx.
\]
Therefore

\[
||f_{k_1 v_1, k_2 v_2}||^2_d(V) = \sum_{i_1, i_2, j_1, j_2} c_{i_1}(k_1) c_{i_2}(k_1) d_{j_1}(k_2) d_{j_2}(k_2) \langle f_{v^1_{i_1}, v^2_{i_2}}, f_{v^1_{j_1}, v^2_{j_2}} \rangle_d(V).
\]
so \((k_1, k_2) \mapsto \|f_{v_1, v_2, k_2}\|^2_{d(V)}\) is indeed continuous. Also, it is now clear that we have
\[
\lim_{r \to +\infty} \frac{\int_{K \times K} \left( \int_{L_r} \omega(x) \|f_{k_1, k_2, 2}(x)\|^2 \cdot dx \right) \cdot dk_1 dk_2}{r d(V)} = 0.
\]

Let us now explain Claim 4.2 in the case when \(G\) is non-Archimedean. Let \(v_1, v_2 \in V\). Let us choose a positive integer \(k\) large enough so that \(k \cdot \mathbb{Z}_{\geq 0} \subset L\). By enlarging \(k\) even more if necessary, by [Ca, Theorem 4.3.3.] for every \(\Theta\) and every \(y \in (\mathbb{R}_{\geq 0} \times \mathbb{R}_{>0}^{\Delta \setminus \Theta}) \cap L^+\) the function
\[
f_{v_1, v_2, \Theta, y} : k \cdot \mathbb{Z}_{\geq 0} \to \mathbb{C}
\]
given (identifying \(\mathbb{R}^{\Delta \setminus \Theta}\) with a subspace of \(\mathbb{R}^2\) in the clear way) by \(x \mapsto f_{v_1, v_2}(y + x)\), can be written as
\[
\sum_{1 \leq i \leq p} c_i \cdot e^{\lambda_i(x \Delta \setminus \Theta)} q_i(x \Delta \setminus \Theta)
\]
where \(c_i \in \mathbb{C} \setminus \{0\}\), \(\lambda_i\) is a complex-valued functional on \(\mathbb{R}^{\Delta \setminus \Theta}\), and \(q_i\) is a monomial on \(\mathbb{R}^{\Delta \setminus \Theta}\). Here \(x \Delta \setminus \Theta\) is the image of \(x\) under the natural projection \(\mathbb{R}^2 \to \mathbb{R}^{\Delta \setminus \Theta}\).

We can assume that the couples in the collection \(\{(\lambda_i, q_i)\}_{1 \leq i \leq p}\) are pairwise different. Since \(V\) is tempered, by “Casselman’s criterion” we in addition have that for every \(1 \leq i \leq p\), \(\text{Re}(\lambda_i)\) is non-negative on \(\mathbb{R}_{\geq 0}^2\).

By Claim A.2, either \(p = 0\), equivalently \(f_{v_1, v_2, \Theta, y} = 0\) (in which case we set \(d_{v_1, v_2, \Theta, y} := -\infty\)), or there exists \(d_{v_1, v_2, \Theta, y} \in \mathbb{Z}_{\geq 0}\) such that the limit
\[
\lim_{r \to +\infty} \frac{1}{r d_{v_1, v_2, \Theta, y}} \sum_{x \in (k \cdot \mathbb{Z}_{\geq 0} \cap L^+)} |f_{v_1, v_2}(y + x)|^2
\]
extists and is strictly positive.

Now, given \(y \in L^+\) let us denote \(\Theta_y := \{\alpha \in \Delta \mid y_\alpha \leq k\}\) where by \(y_\alpha\) we denote the coordinate of \(y \in \mathbb{R}^\Delta\) at the \(\alpha\)-place. Let \(Y \subset L^+\) be the subset of \(y \in L^+\) for which \(y_\alpha \leq 2k\) for all \(\alpha \in \Delta\). Then \(Y\) is a finite set, and we have
\[
(4.5) \quad L^+ = \bigcap_{y \in Y} \left(y + k \cdot \mathbb{Z}_{\geq 0}^{\Delta \setminus \Theta_y}\right).
\]
Notice also that \(\omega(x)/\epsilon^{2\rho(x)}\) is a positive constant on each one of the subset of which we take union in (4.5). We set \(d(v_1, v_2) := \max_{y \in Y} d_{v_1, v_2, \Theta, y}\). We see that either \(f_{v_1, v_2} = 0\) (then \(d(v_1, v_2) = -\infty\)) or the limit
\[
\lim_{r \to +\infty} \frac{1}{r d(v_1, v_2)} \sum_{x \in L^+_r} \frac{\omega(x)}{\epsilon^{2\rho(x)}} |f_{v_1, v_2}(x)|^2
\]
extists and is strictly positive.
That $d(V)$ is finite follows from $d(v_1, v_2)$ being controlled by finitely many Jacquet modules, with the finite central actions on them.

4.7. Let us now explain Claim 1.12 in the case when $G$ is real. Using [CaMi] we know that, fixing $k > 0$, given $\Theta \subset \Delta$ the restriction of $\omega(x)^{1/2} f_{v_1, v_2}(x)$ to $\mathbb{R}^\Delta_{\geq k} \times [0,k]^\Theta$ can be written as

$$\sum_{1 \leq i \leq p} e^{\lambda_i(x_{\Delta \setminus \Theta})} q_i(x_{\Delta \setminus \Theta}) \phi_i(x)$$

where the notation is as follows. First, $\lambda_i$ is a complex-valued functional on $\mathbb{R}^{\Delta \setminus \Theta}$. Next, $q_i$ is a monomial on $\mathbb{R}^{\Delta \setminus \Theta}$. The couples $(\lambda_i, q_i)$, for $1 \leq i \leq p$, are pairwise distinct. The function $\phi_i$ is expressible as a composition

$$[0,k]^\Theta \times \mathbb{R}^\Delta_{\geq k} \xrightarrow{\text{id} \times e_i} [0,k]^\Theta \times (\mathbb{C}_{[1]})^{\Delta \setminus \Theta} \xrightarrow{\phi_i} \mathbb{C}$$

where $e_i$ is the coordinate-wise application of $x \mapsto e^{-x}$ and $\phi_i$ is a continuous function such that, for every $b \in [0,k]^\Theta$, the restriction of $\phi_i$ via $(\mathbb{C}_{[1]})^{\Delta \setminus \Theta} \xrightarrow{z \mapsto (b,z)} [0,k]^\Theta \times (\mathbb{C}_{[1]})^{\Delta \setminus \Theta}$ is holomorphic. Lastly, the function $b \mapsto \phi_i(b, \{0\}^{\Delta \setminus \Theta})$ on $[0,k]^\Theta$ is not identically zero. Since $V$ is tempered, by “Casselman’s criterion” we in addition have that for every $1 \leq i \leq p$, $\text{Re}(\lambda_i)$ is non-negative on $\mathbb{R}^\Delta_{\geq 0}$.

If $p = 0$, we set $d_{v_1, v_2, \Theta} := -\infty$. Otherwise, Claim 1.10 provides a number $d_{v_1, v_2, \Theta} \in \mathbb{Z}_{\geq 0}$, described concretely in terms of $\{(\lambda_i, q_i, \phi_i)\}_{1 \leq i \leq p}$, such that the limit

$$\lim_{r \to +\infty} \frac{1}{r^{d_{v_1, v_2, \Theta}}} \int_{\mathbb{R}^{\Delta \setminus \Theta} \cap \Delta_{> r}} \omega(x) e^{2\rho(x)} |f_{v_1, v_2}(x)|^2 \cdot dx$$

exists and is strictly positive. We set $d(v_1, v_2) := \max_{\Theta \subset \Delta} d_{v_1, v_2, \Theta}$. Then either $f_{v_1, v_2} = 0$ or $d(v_1, v_2) > 0$ and the limit

$$\lim_{r \to +\infty} \frac{1}{r^{d(v_1, v_2)}} \int_{\Delta_{> r}} \omega(x) e^{2\rho(x)} |f_{v_1, v_2}(x)|^2 \cdot dx$$

exists and is strictly positive. That $d(V)$ is finite follows from $d(v_1, v_2)$ being controlled by finitely many data, as in [CaMi]. Part (2) of Claim 1.12 follows easily from the concrete description of $d_{v_1, v_2, \Theta}$ in Claim 1.10.

5. Proofs for Remark 1.11, Remark 1.13, Proposition 1.10, Proposition 1.14 and Remark 1.16

In this section, $G$ is a semisimple group over a local field. We continue with notations from §1. We explain Remark 1.11 (in Claim 5.1), explain Remark 1.13 (in Claim 5.2), prove Proposition 1.10 (in §5.5), prove Proposition 1.14 (in §5.1) and explain Remark 1.16 (in Claim 5.3).

5.1. Lemma 5.1. Let $V$ be an irreducible unitary $G$-representation and suppose that there exists a unit vector $v_0 \in V$ satisfying properties (1) and (2) of Proposition 1.14. Let $0 < r_0 < r_1 < \ldots$ be a sequence such that $\lim_{n \to +\infty} r_n = +\infty$. Then $V$ is $c$-tempered with Følner sequence $G_{r_0} G_{r_1} \ldots$. 
Proof. Property (1) of Definition 2.1 is immediate from property (1) of Proposition 1.13. Let us check property (2) of Definition 2.1. Thus, let \( v_1, v_2 \in V \) and let \( K \subset G \) be a compact subset. Fix \( r' > 0 \) big enough so that \( K \subset G_{<r'} \) and \( K^{-1} \subset G_{<r'} \). We then have, for all \( r > 0 \) and all \( g_1, g_2 \in K \): 

\[
G_{<r} \Delta g_2^{-1} G_{<r} g_1 \subset G_{<r+2r'} \setminus G_{<r-2r'}.
\]

Therefore, using property (2) of Proposition 1.14,

\[
\limsup_{r \to +\infty} \sup_{g_1, g_2 \in K} \frac{M_{v_1, v_2}(G_{<r} \Delta g_2^{-1} G_{<r} g_1)}{M_{v_0, v_0}(G_{<r})} \leq \limsup_{r \to +\infty} \frac{M_{v_1, v_2}(G_{<r+2r'} \setminus G_{<r-2r'})}{M_{v_0, v_0}(G_{<r})} = 0
\]

and therefore also

\[
\lim_{n \to +\infty} \sup_{g_1, g_2 \in K} \frac{M_{v_1, v_2}(G_{<r} \Delta g_2^{-1} G_{<r} g_1)}{M_{v_0, v_0}(G_{<r_n})} = 0.
\]

\[\square\]

Proof (of Proposition 1.14). Let us fix a \( K \)-finite unit vector \( v'_0 \in V \), for some maximal compact subgroup \( K \subset G \). Let \( 0 < r_0 < r_1 < \ldots \) be a sequence such that \( \lim_{n \to +\infty} r_n = +\infty \). By Lemma 5.1 \( V \) is c-tempered with \( \text{Følner sequence} \ G_{<r_0}, G_{<r_1}, \ldots \) and hence by Proposition 2.3 we obtain

\[
\lim_{n \to +\infty} \frac{\int_{g \in G_{<r_n}} \langle g v_1, v_2 \rangle \langle g v_3, v_4 \rangle \cdot dg}{M_{v_0, v'_0}(v'_n)} = \langle v_1, v_3 \rangle \langle v_2, v_4 \rangle
\]

for all \( v_1, v_2, v_3, v_4 \in V \). Since this holds for any such sequence \( \{r_n\}_{n \geq 0} \), we obtain

\[
\lim_{r \to +\infty} \frac{\int_{g \in G_{<r}} \langle g v_1, v_2 \rangle \langle g v_3, v_4 \rangle \cdot dg}{M_{v_0, v'_0}(r)} = \langle v_1, v_3 \rangle \langle v_2, v_4 \rangle
\]

for all \( v_1, v_2, v_3, v_4 \in V \). By Theorem 1.7 we have

\[
\lim_{r \to +\infty} \frac{M_{v_0, v'_0}(r)}{r d(V)} = C
\]

for some \( C > 0 \). This enables to rewrite (5.1) as

\[
\lim_{r \to +\infty} \frac{\int_{g \in G_{<r}} \langle g v_1, v_2 \rangle \langle g v_3, v_4 \rangle \cdot dg}{r d(V)} = C \cdot \langle v_1, v_3 \rangle \langle v_2, v_4 \rangle
\]

for all \( v_1, v_2, v_3, v_4 \in V \), as desired.

\[\square\]

5.2. **Claim 5.2.** The validity of Conjecture 1.2 as well as the resulting invariants \( d(V) \) and \( f(V) \), of Theorem 1.7 as well as the resulting invariants \( d(V) \) and \( f(V) \) (the latter in the non-Archimedean case), and of Proposition 1.14 do not depend on the choice of the norm \( || - || \) on \( g \).

**Proof.** Let \( || - ||' \) be another norm on \( g \), let \( r' : G \to \mathbb{R}_{\geq 0} \) be the resulting function, and let \( G_{<r} \subset G \) be the resulting subsets. There exists \( r_0 \geq 0 \) such that

\[
e^{-r_0} \cdot ||X|| \leq ||X||' \leq e^{r_0} \cdot ||X||, \quad \forall X \in g
\]

and therefore

\[
e^{-2r_0} \cdot ||\text{Ad}(g)|| \leq ||\text{Ad}(g)||' \leq e^{2r_0} \cdot ||\text{Ad}(g)||, \quad \forall g \in G.
\]
Then
\[ G'_r < r \subset G_{r+2r_0}, \quad \forall r \geq 0 \]
and
\[ G_r < r \subset G'_{r+2r_0}, \quad \forall r \geq 0. \]
These “sandwich” relations readily imply the independence claims.

5.3.

**Claim 5.3.** An irreducible unitary $G$-representation for which there exists a unit vector $v_0 \in V$ such that conditions (1) and (2) of Proposition 1.14 are satisfied is tempered.

*Proof.* Clear from Lemma 5.1 coupled with Corollary 3.16.

5.4.

**Claim 5.4.** Let $G := \text{PGL}_2(\Omega)$, $\Omega$ a local field. Let $A \subset G$ be the subgroup of diagonal matrices. Then, for every non-trivial irreducible unitary $G$-representation $V$, the set of matrix coefficients of $V$ restricted to $A$ is equal to the set of function on $A$ of the form
\[ a \mapsto \int_{\hat{A}} \chi(a) \cdot \phi(\chi) \cdot d\chi \]
as $\phi$ runs over $L^1(\hat{A})$.

*Proof.* Denote by $B \subset G$ the subgroup of upper-triangular matrices and by $N \subset B$ its unipotent radical.

Let us recall that, by Mackey theory, there is a unique (up to isomorphism) infinite-dimensional irreducible unitary $B$-representation $W$, and the rest of irreducible unitary $B$-representations are killed by $N$. The restriction $\text{Res}^B_A W$ is isomorphic to the right regular unitary $A$-representation $L^2(A)$.

Let now $V$ be a non-trivial irreducible unitary $G$-representation. Recall that by the Howe-Moore theorem (or by a step in one of its usual proofs) $V$ does not contain non-zero $N$-invariant vectors. By decomposing the restriction $\text{Res}^G_B V$ into a direct integral of irreducible unitary $B$-representations, and using the fact that $V$ admits no non-zero $N$-invariant vectors, we see that $\text{Res}^G_B V$ is a multiple of $W$. Hence, we deduce that $\text{Res}^G_A A$ is a multiple of the right regular unitary $A$-representation $L^2(A)$.

Now, the matrix coefficients of a multiple of the right regular unitary $A$-representation $L^2(A)$ are easily seen to be the functions on $A$ of the form
\[ a \mapsto \int_{\hat{A}} \chi(a) \cdot \phi(\chi) \cdot d\chi \]
where $\phi \in L^1(\hat{A})$. 

5.5. Proof (of Proposition 1.10). Fix $d \in D_c^\infty(G)$. Let $K \subset G$ be an open compact subgroup such that $d$ is invariant under $K$ both on left and on right. Let us denote by $e_1, \ldots, e_n$ an orthonormal basis of $V^K$, and let us denote by $\pi_K : V \rightarrow V^K$ the orthonormal projection. Let us denote by $[-,-] : C^{-\infty}(G) \times D_c^\infty(G) \rightarrow \mathbb{C}$ the canonical pairing. We have

$$[g_m v_1, v_2, d] = [m v_1, v_2, d] = \langle m v_1, v_2 \rangle (de_i, e_j).$$

Hence

$$\int_{G_{<r}}[g_m v_1, v_2, d] \cdot \frac{dg}{d(V)} = \sum_{1 \leq i, j \leq n} \langle de_i, e_j \rangle \cdot \frac{\int_{G_{<r}} \langle m v_1, e_i \rangle \langle v_1, e_i \rangle \cdot \frac{dg}{d(V)}}{d(V)}$$

and therefore

$$\lim_{r \rightarrow +\infty} \int_{G_{<r}}[g_m v_1, v_2, d] \cdot \frac{dg}{d(V)} = \frac{1}{f(V)} \sum_{1 \leq i, j \leq n} \langle de_i, e_j \rangle \cdot \langle v_1, v_2 \rangle (de_i, e_j) = \frac{1}{f(V)} \sum_{1 \leq i \leq n} \langle de_i, e_i \rangle \cdot \langle v_1, v_2 \rangle = \frac{\langle v_1, v_2 \rangle}{f(V)} \Theta_V(d).$$

□

6. The case of the principal series representation $V_1$ of slowest decrease

In this section $G$ is a semisimple group over a local field. We continue with notations from §1. Our goal is to prove Theorem 1.11 (restated as Theorem 6.2 below).

6.1. We fix a minimal parabolic $P \subset G$ and a maximal compact subgroup $K \subset G$ such that $G = PK$. We consider the principal series unitary $G$-representation $V_1$ consisting of functions $f : G \rightarrow \mathbb{C}$ satisfying

$$f(pg) = \Delta_P(p)^{1/2} \cdot f(g) \quad \forall p \in P, g \in G$$

where $\Delta_P : P \rightarrow \mathbb{R}_{>0}$ is the modulus function of $P$. The $G$-invariant inner product on $V_1$ can be taken to be

$$\langle f_1, f_2 \rangle = \int_K f_1(k) \cdot \overline{f_2(k)} \cdot dk$$

(where we normalize the Haar measure on $K$ to have total mass 1). Recall that $V_1$ is irreducible. We denote by $f_0 \in V_1$ the spherical vector, determined by $f_0(k) = 1$ for all $k \in K$. We also write

$$\Xi_G(g) := \langle gf_0, f_0 \rangle.$$

Lemma 6.1. Given $r' \geq 0$ we have

$$\lim_{r \rightarrow +\infty} \frac{\int_{G_{<r}} \Xi_G(g)^2 \cdot dg}{\int_{G_{<r}} \Xi_G(g)^2 \cdot dg} = 0$$
and
\[ \lim_{r \to +\infty} \frac{\int_{G_{<r'}} \Xi_G(g)^2 \cdot dg}{\int_{G_{<r}} \Xi_G(g)^2 \cdot dg} = 1. \]

**Proof.** The second equality follows from the first, and the first is immediately implied by Theorem 1.7. \( \square \)

6.2. The main result of this section is:

**Theorem 6.2.** Let \( V \) be an irreducible tempered unitary \( G \)-representation. Suppose that there exist a unit vector \( v_0 \in V \) such that
\[ (6.1) \limsup_{r \to +\infty} \int_{G_{<r}} \Xi_G(g)^2 \cdot dg < +\infty. \]

Then Conjecture 1.2 holds for \( V \). In particular, Conjecture 1.2 holds for \( V_1 \).

6.3. We will prove Theorem 6.2 using the following result:

**Claim 6.3.** Let \( V \) be a tempered unitary \( G \)-representation. Then for all unit vectors \( v_1, v_2 \in V \) and all measurable \( K \)-biinvariant subsets \( S \subset G \) we have
\[ \int_S |\langle gv_1, v_2 \rangle|^2 \cdot dg \leq \int_S \Xi_G(g)^2 \cdot dg. \]

**Proof (of Theorem 6.2 given Claim 6.3).** To show that Conjecture 1.2 holds for \( V \) we will use Proposition 1.14, applied to our \( V \) and our \( v_0 \).

There exists \( r_0 \geq 0 \) such that \( KG_{<r} \subset G_{<r} + r_0 \) for all \( r \geq 0 \).

Let us verify condition (1) of Proposition 1.14. For unit vectors \( v_1, v_2 \in V \) we have
\[ \frac{M_{v_1,v_2}(r)}{M_{v_1,v_2}(r')} \leq \frac{\int_{G_{<r'+r_0}} \Xi_G(g)^2 \cdot dg}{\int_{G_{<r}} \Xi_G(g)^2 \cdot dg} \]
and therefore condition (1) of Proposition 1.14 follows from (6.1) and Lemma 6.1.

Let us now verify condition (2) of Proposition 1.14. For unit vectors \( v_1, v_2 \in V \) and \( r' \geq 0 \) we have
\[ \frac{M_{v_1,v_2}(r + r') - M_{v_1,v_2}(r - r')}{M_{v_1,v_2}(r')} \leq \frac{\int_{G_{<r'} \setminus G_{<r - r'}} \Xi_G(g)^2 \cdot dg}{\int_{G_{<r}} \Xi_G(g)^2 \cdot dg} \]
and therefore condition (2) of Proposition 1.14 follows from (6.1) and Lemma 6.1. \( \square \)

6.4. We will prove Claim 6.3 using the following result:

**Claim 6.4.** Let \( \phi \in L^2(G) \) be zero outside of a measurable \( K \)-biinvariant subset \( S \subset G \) of finite volume. Denote by \( T_\phi : L^2(G) \to L^2(G) \) the operator of convolution \( \psi \mapsto \phi * \psi \). Then
\[ \|T_\phi\| \leq \left( \int_S \Xi_G(g)^2 \cdot dg \right) \cdot \|\phi\|^2. \]

Here \( \|\phi\| \) stands for the \( L^2 \)-norm of \( \phi \).
Proof of Claim 6.3 given Claim 6.4. We can clearly assume that $S$ has finite volume. Let us denote
$$\phi(g) := \text{ch}_S(g) \cdot \langle gv_1, v_2 \rangle,$$
where ch$_S$ stands for the characteristic function of $S$. Let us denote by $S\phi: V \to V$ the operator
$$v \mapsto \int_G \phi(g) \cdot gv \cdot dg.$$
Since $V$ is tempered, we have $||S\phi|| \leq ||T\phi||$. Therefore
$$\int_S |\langle gv_1, v_2 \rangle|^2 \cdot dg = \int_G \phi(g) \cdot \langle gv_1, v_2 \rangle \cdot dg = \langle S\phi v_1, v_2 \rangle \leq ||S\phi|| \leq ||T\phi|| \leq$$
thus
$$\int_S |\langle gv_1, v_2 \rangle|^2 \cdot dg \leq \int_S \Xi_G(g)^2 \cdot dg$$
as desired. □

6.5. Finally, let us prove Claim 6.4 following [ChPiSa].

Proof of Claim 6.4. By [ChPiSa, Lemma 3.5] we can assume that $\phi$ is $K$-biinvariant and non-negative. By [ChPiSa, Proposition 4.3] we have
$$||T\phi|| = \int_G \Xi_G(g) \cdot \phi(g) \cdot dg.$$
Applying the Cauchy-Schwartz inequality, we obtain
$$||T\phi||^2 \leq \left( \int_S \Xi_G(g)^2 \cdot dg \right) \cdot ||\phi||^2,$$
as desired. □

7. The proof of Theorem 1.12

In this section we let $G$ be either $SL_2(\mathbb{R})$ or $PGL_2(\Omega)$, where $\Omega$ is a non-Archimedean local field of characteristic 0 and residual characteristic not equal to 2. We prove Theorem 1.12.

7.1. If $G = PGL_2(\Omega)$, we denote by $\varpi$ a uniformizer in $\Omega$, by $\mathcal{O}$ the ring of integers in $\Omega$, by $p$ the residual characteristic of $\Omega$ and $q := |\mathcal{O}/\varpi\mathcal{O}|$. 

\[14\] In the lemma we refer to it is assumed that $\phi$ is continuous but the arguments there apply to our $\phi$ without any modification.
7.2. We denote by \( A \subset G \) the subgroup of diagonal matrices and by \( U \subset G \) the subgroup of unipotent upper-triangular matrices. If \( G = \text{SL}_2(\mathbb{R}) \) we define the isomorphism
\[
a : \mathbb{R}^* \to A, \quad t \mapsto \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}
\]
and if \( G = \text{PGL}_2(\Omega) \) we define the isomorphism
\[
a : \Omega^* \to A, \quad t \mapsto \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}.
\]
We denote \( A^+ := \{ a \in A \mid |a^{-1}(a)| \geq 1 \} \).

If \( G = \text{SL}_2(\mathbb{R}) \) then we can (and will) take \( ||-|| \) on \( \mathfrak{g} \) to be such that
\[
\mathbf{r} \left( k_1 \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} k_2 \right) = \log \max \{|t|^2, |t|^{-2}\}
\]
where \( t \in \mathbb{R}^* \) and \( k_1, k_2 \in SO(2) \). If \( G = \text{PGL}_2(\Omega) \) then we can (and will) take \( ||-|| \) on \( \mathfrak{g} \) to be such that
\[
\mathbf{r} \left( k_1 \begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix} k_2 \right) = \log \max \{|t/s|, |s/t|\}
\]
where \( t, s \in \Omega^* \) and \( k_1, k_2 \in \text{PGL}_2(\mathcal{O}) \). Let us denote \( A^+_c := A^+ \cap G_c \).

If \( G = \text{SL}_2(\mathbb{R}) \) we set \( K := SO(2) \subset G \). If \( G = \text{PGL}_2(\Omega) \) we choose a non-square \( \zeta \in \mathcal{O}^* \) and set \( K \subset G \) to be the subgroup of elements of the form \( \begin{pmatrix} a & \zeta b \\ b & a \end{pmatrix} \), \( (a, b) \in \Omega^2 \setminus \{(0, 0)\} \) (so \( K \) is a closed compact subgroup in \( G \), but not open, and in particular not maximal).

We set \( \omega : A^+ \to \mathbb{R}_{\geq 0} \) to be given by \( \omega(a(t)) := |t^2 - t^{-2}| \) if \( G = \text{SL}_2(\mathbb{R}) \) and \( \omega(a(t)) := |t - t^{-1}| \) if \( G = \text{PGL}_2(\Omega) \). Then, taking the Haar measure on \( K \) to have total mass 1 and appropriately normalizing the Haar measure on \( A \), for all non-negative-valued measurable functions \( f \) on \( G \) we have
\[
\int_G f(g) \, dg = \int_{A^+} \omega(a) \left( \int_{K \times K} f(k_2ak_1) \cdot dk_1dk_2 \right) da.
\]

Given a unitary \( G \)-representation \( V \), vectors \( v_1, v_2 \in V \) and \( a \in A^+ \), we write
\[
M^o_{v_1, v_2}(a) := \int_{K \times K} |(k_2ak_1v_1, v_2)|^2 \cdot dk_1dk_2.
\]

We have
\[
(7.1) \quad M_{v_1, v_2}(r) := \int_{A^+_c} \omega(a) \cdot M^o_{v_1, v_2}(a) \cdot da
\]
(where \( M_{v_1, v_2}(r) \) was already defined in §1).

Given a unitary character \( \chi : A \to \text{U}(1) \) we consider the principal series unitary \( G \)-representation \( V_\chi \), consisting of functions \( f : G \to \mathbb{C} \) satisfying
\[
f(uag) = \chi(a) \cdot \Delta(a)^{1/2} \cdot f(g) \quad \forall a \in A, \, u \in U, \, g \in G,
\]
\footnote{\( \text{U}(1) \) denotes the subgroup of \( \mathbb{C}^* \) consisting of complex numbers with absolute value 1.}
where $\Delta(a) = |a^{-1}(a)|^2$ if $G = SL_2(\mathbb{R})$ and $\Delta(a) = |a^{-1}(a)|$ if $G = PGL_2(\Omega)$. Here $G$ acts by $(g'f)(g) := f(gg')$. The $G$-invariant inner product on $V_{\chi}$ can be expressed as

$$\langle f_1, f_2 \rangle = \int_K f_1(k) \cdot \overline{f_2(k)} \cdot dk.$$  

For $\theta \in \hat{K}$, let $h_{\theta}^k \in V_{\chi}$ denote the unique vector determined by $h_{\theta}^k(k) = \theta(k)$ for $k \in K$, if it exists, and write $\text{types}(V_{\chi}) \subset \hat{K}$ for the subset of $\theta$'s for which it exists. Thus $(h_{\theta}^k)_{\theta \in \text{types}(V_{\chi})}$ is a Hilbert basis for $V_{\chi}$.

7.3. Let us now give several preparatory remarks.

First, we do not establish Conjecture 1.2 directly but, rather, establish conditions (1) and (2) of Proposition 1.14 (which suffices by that proposition).

Second, for a square-integrable irreducible unitary $G$-representation $V$, establishing conditions (1) and (2) of Proposition 1.14 with any unit vector $v_0 \in V$ is straightforward (see the proof of Proposition 2.6 for a spelling-out). As is well-known, a tempered irreducible unitary $G$-representation which is not square-integrable is a direct summand in some $V_{\chi}$. Therefore, we establish conditions (1) and (2) of Proposition 1.14 for irreducible direct summands in $V_{\chi}$.

Third, if $\chi = 1$ when $G = SL_2(\mathbb{R})$ or if $\chi^2 = 1$ when $G = PGL_2(\Omega)$, $V_{\chi}$ satisfies Conjecture 1.2 by Theorem 6.2. So we assume throughout:

(7.2) \hfill $\chi \neq 1$ if $G = SL_2(\mathbb{R})$, \quad $\chi^2 \neq 1$ if $G = PGL_2(\Omega)$.

7.4. We reduce Conjecture 1.2 for an irreducible summand in $V_{\chi}$ to the following two claims.

Claim 7.1. Fix $\chi$ satisfying (7.2). Let $V$ be an irreducible direct summand in $V_{\chi}$. There exist $f \in V$, $r_0 \geq 0$ and $D > 0$ such that for all $r \geq r_0$ we have

$$M_{f,f}(r) \geq D \cdot r$$  

Claim 7.2. Fix $\chi$ satisfying (7.2). There exist $r_0 > 0$ and $C > 0$ (depending on $\chi$) such that for all $a \in A^+ \setminus A_{< r_0}$ we have

$$M^\circ_{f_1,f_2}(a) \leq C \cdot \omega(a)^{-1} \cdot ||f_1||^2 \cdot ||f_2||^2 \quad \forall f_1,f_2 \in V_{\chi}.$$  

Proof (of Conjecture 1.2 for summands in $V_{\chi}$ given Claim 7.1 and Claim 7.2 for $\chi$). Let $V$ be an irreducible direct summand in $V_{\chi}$. Let $f$, $r_0$, $D$ and $C$ be as in Claim 7.1 and as in Claim 7.2 (taking $r_0$ to be the maximum of the values from the two statements).

In order to verify Conjecture 1.2 for $V$, we will verify the conditions (1) and (2) of Proposition 1.14 where for $r_0$ we take our $f$.

Using (7.4) we obtain the existence of $E, E' > 0$ such that for all $r_0 \leq r_1 < r_2$ we have

$$M_{f_1,f_2}(r_2) - M_{f_1,f_2}(r_1) \leq E \cdot \text{vol}_A(A_{< r_2}^+ \setminus A_{< r_1}^+) \cdot ||f_1||^2 \cdot ||f_2||^2 \leq E' \cdot (1 + (r_2 - r_1)) \cdot ||f_1||^2 \cdot ||f_2||^2.$$  

From this and (7.3) the conditions (1) and (2) of Proposition 1.14 are immediate. \hfill $\square$
7.5. Let us prove Claim 7.1.

**Proof (of Claim 7.1).** Let \( V \) be an irreducible direct summand of \( V_\chi \).

Let us first treat the case \( G = PGL_2(\Omega) \). We use the (normalized) Jacquet \( A \)-module \( J(-) \) with respect to \( G \leftrightarrow AU \rightarrow A \). We denote by \( V \subset V \) the subspace of smooth vectors. \( J(V) \) is isomorphic to \( C_\chi \oplus C_{\chi^{-1}} \). We consider \( v \in V \) whose projection under the canonical \( V \to J(V) \) is non-zero and is an \( A \)-eigenvector with eigencharacter \( \chi \). By Casselman’s canonical pairing theory there exists a non-zero \( \alpha \in J(V)^* \) which is \( A \)-eigenvector with eigencharacter \( \chi^{-1} \) such that \( \langle av, v \rangle = |a|^{-1/2} \alpha(av) \) whenever \( a \in A^+ \setminus A_{<0}^0 \), for large enough \( r_0 \geq 0 \). Since we have \( \alpha(av) = \chi(a) \cdot \alpha(v) \) and \( \alpha(v) \neq 0 \), we deduce that for some \( C > 0 \) we have \( |\langle av, v \rangle|^2 = C \cdot |a|^{-1} \) for \( a \in A^+ \setminus A_{<0}^0 \). Let \( K_v \subset K \) be an open compact subgroup, small enough so that \( K_v \cdot v = v \). We have, again for \( a \in A^+ \setminus A_{<0}^0 \):

\[
M_{v,v}^o(a) = \int_{K \times K} |\langle k_2 \cdot a \cdot k_1 v, v \rangle|^2 \cdot dk_1 dk_2 \geq \]

\[
\geq \int_{K_v \times K_v} |\langle k_2 \cdot a \cdot k_1 v, v \rangle|^2 \cdot dk_1 dk_2 = C' \cdot |\langle av, v \rangle|^2
\]

for some \( C' > 0 \) and so \( M_{v,v}^o(a) \geq C'' \cdot |a|^{-1} \) for some \( C'' > 0 \). From this we obtain the desired.

Let us now treat the case \( G = SL_2(\mathbb{R}) \). Fix any \( \theta \in \text{types}(V_\chi) \). The leading asymptotic of \( K \)-finite vectors are well-known, and can be computed from explicit expressions in terms of the hypergeometric function (see [KIV] §6.5). In the case \( \chi|_{\text{a}(\mathbb{R}_{>0}^\times)} \neq 1 \), denoting by \( 0 \neq s \in \mathbb{R} \) the number for which \( \chi(\text{a}(t)) = t^{is} \) for all \( t \in \mathbb{R}_{>0}^\times \), we have

\[
\langle \text{a}(e^x) h_\theta^\chi, h_\theta^\chi \rangle \sim e^{-x} \cdot (E_1 \cdot e^{-isx} + E_2 \cdot e^{isx} + o(1)) \quad (x \to +\infty)
\]

for some non-zero \( E_1 \) and \( E_2 \) and so

\[
|\langle \text{a}(e^x) h_\theta^\chi, h_\theta^\chi \rangle|^2 \sim e^{-2x} \cdot (D + E_3 \cdot e^{-2isx} + E_4 \cdot e^{2isx} + o(1)) \quad (x \to +\infty)
\]

for some \( D > 0 \), \( E_3 \) and \( E_4 \). From this we obtain the desired. In the case \( \chi|_{\text{a}(\mathbb{R}_{>0}^\times)} = 1 \), and so \( \chi(\text{a}(-1)) = -1 \), we have

\[
\langle \text{a}(e^x) h_\theta^\chi, h_\theta^\chi \rangle \sim E \cdot e^{-x} \quad (x \to +\infty)
\]

for some non-zero \( E \) and so

\[
|\langle \text{a}(e^x) h_\theta^\chi, h_\theta^\chi \rangle|^2 \sim D \cdot e^{-2x} \quad (x \to +\infty)
\]

for some \( D > 0 \). From this we obtain the desired.

\( \square \)

7.6. We further reduce Claim 7.2.

**Claim 7.3.** Fix \( \chi \) satisfying (7.3). There exist \( r_0 > 0 \) and \( C > 0 \) (depending on \( \chi \)) such that for all \( \theta, \eta \in \text{types}(V_\chi) \) and all \( a \in A^+ \setminus A_{<0}^0 \) we have

\[
|\langle a h_\theta^\chi, h_\theta^\chi \rangle|^2 \leq C \cdot \omega(a)^{-1}. \tag{7.5}
\]
Proof (of Claim 7.2 for $\chi$ given Claim 7.3 for $\chi$). Let $f_1, f_2 \in V_\chi$ and write

$$f_1 = \sum_{\theta \in \text{types}(V_\chi)} c_\theta \cdot h_\theta^\chi, \quad f_2 = \sum_{\theta \in \text{types}(V_\chi)} d_\theta \cdot h_\theta^\chi$$

with $c_\theta, d_\theta \in \mathbb{C}$. Using Fourier expansion of the function $(k_1, k_2) \mapsto \langle ak_1 f_1, k_2 f_2 \rangle$ on $K \times K$ we have, for $a \in A^+ \setminus A_z^{\infty}r_o$:

$$M^\phi_{f_1,f_2}(a) = \int_{K \times K} |\langle ak_1 f_1, k_2 f_2 \rangle|^2 \cdot dk_1 dk_2 = \sum_{\theta, \eta \in \text{types}(V_\chi)} |c_\theta|^2 \cdot |d_\eta|^2 \cdot |\langle ah_\theta^\chi, h_\eta^\chi \rangle| \leq C \cdot \omega(a)^{-1} \cdot ||f_1||^2 \cdot ||f_2||^2.$$

\[\square\]

7.7. Let us now establish Claim 7.3 in the case $G = SL_2(\mathbb{R})$:

Proof (of Claim 7.3 in the case $G = SL_2(\mathbb{R})$). In the case $\chi_{|a(\mathbb{R}^{\infty})} \neq 1$, this is the contents of [BrCoNi1a] (theorem 2.1) (which contains a stronger claim, incorporating $\chi$ into the inequality). Let us therefore assume $\chi_{|a(\mathbb{R}^{\infty})} = 1$ and so $\chi(a(-1)) = -1$. For $n \in \mathbb{Z}$, let us denote by $\theta_n$ the character of $K$ given by

$$\begin{pmatrix} c & -s \\ s & c \end{pmatrix} \mapsto (c + is)^n.$$ We want to see that

$$cosh(x) \left| \langle a(c^x)h_{\theta_n}^\chi, h_{\theta_m}^\chi \rangle \right|$$

is bounded as we vary $x \in [0, +\infty)$ and $m, n \in 1 + 2\mathbb{Z}$. We have $V_\chi = V_\chi^- \oplus V_\chi^+$, where $V_\chi^-$ and $V_\chi^+$ are irreducible unitary $G$-representations, $\text{types}(V_\chi^-) = \{\theta_n : n \in -1 - 2\mathbb{Z}_{\geq 0}\}$ and $\text{types}(V_\chi^+) = \{\theta_n : n \in 1 + 2\mathbb{Z}_{\geq 0}\}$. Since the matrix coefficient in question vanishes when $m \in -1 - 2\mathbb{Z}_{\geq 0}$ and $n \in 1 + 2\mathbb{Z}_{\geq 0}$ or vice versa, and since the matrix coefficient in question does not change when we replace $(n, m)$ by $(-n, -m)$, we can assume $m, n \in 1 + 2\mathbb{Z}_{\geq 0}$. Furthermore, by conjugating by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

it is straight-forward to see that we can assume that $m \geq n$. We denote

$$k := \frac{m-n}{2}.$$ Let us use the well-known concrete expression of matrix coefficients in terms of the hypergeometric function (see [KI6] §6.5]):

$$cosh(x) \left| \langle a(c^x)h_{\theta_n}^\chi, h_{\theta_m}^\chi \rangle \right| = \left| \frac{1 + \frac{n-1}{2}k}{k!} \right|_{2F_1} \left( \frac{n-1}{2} + k + 1, \ -\frac{n-1}{2}, \ k + 1, \ tanh(x)^2 \right).$$

We want to show that this expression is bounded as we vary $x \in [0, +\infty)$, $n \in 1 + 2\mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$. Performing a change of variables $\frac{1}{\sqrt{2}} := \tanh(x)^2$, denoting $r := \frac{n-1}{2}$ and interpreting in terms of Jacobi polynomials, we can rewrite this last expression as:

$$Q^k_r(t) := \left( \frac{1 - \sqrt{r}}{2} \right)^{k/2} \cdot P_r^{(k,0)}(t).$$

We want to see that $Q^k_r(t)$ is bounded as we vary $t \in [-1, 1]$, $r \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$. But, it is known that under a suitable interpretation of the variable $t$, $Q^k_r(t)$ is equal to a matrix coefficient of unit vectors in an irreducible unitary representation of $SU(2)$, see, for example, [KI6] §6.3]. Therefore, we have $|Q^k_r(t)| \leq 1$ for all $t \in [-1, 1], r \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 0}$, as desired. See also [HaSc] Equation (20) for a direct proof of this last inequality.

\[\square\]
7.8. Finally, we want to establish Claim 7.3 in the case $G = \text{PGL}_2(\Omega)$. We will use the following proposition:

**Proposition 7.4.** There exists $C > 0$ such that the following holds. Let $\psi_1$ and $\psi_2$ be unitary characters of $\varpi \mathcal{O}$. Let $\alpha_1, \alpha_2 \in \mathcal{O}^\times x + x^2 \Omega[x] \subset \Omega[x]$ be power series. Let $\chi$ be a non-trivial unitary character of $\Omega^\times$. Denote by $c(\chi)$ the number $0$ if $|\chi|_{\mathcal{O}^\times} = 1$ and otherwise the smallest number $c \in \mathbb{Z}_{\geq 1}$ for which $|\chi|_{1+\varpi \mathcal{O}} = 1$. Also, denote by $d(\chi)$ the number $1/|1 - \chi(\varpi)|$ if $c(\chi) = 0$ and the number $0$ if $c(\chi) \neq 0$. Let $0 < m_1 \leq m_2 \leq n$ be integers. Then

$$\left| \int_{\varpi^{-m_1} \mathcal{O} \setminus \varpi^{-m_2} \mathcal{O}} \psi_1(\alpha_1(x)) \psi_2(\alpha_2(\varpi^m x^{-1})) \chi(x) \cdot \frac{dx}{|x|} \right| \leq C(c(\chi) + d(\chi) + 1).$$

**Proof (of Claim 7.3 in the case $G = \text{PGL}_2(\Omega)$ given Proposition 7.4).** Let us calculate more concretely the inner product appearing in Claim 7.3. We can normalize the inner product on $V_\chi$ so that for $f_1, f_2 \in V_\chi$ we have

$$\langle f_1, f_2 \rangle = \int_{\Omega} f_1 \left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) f_2 \left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) \cdot dx.$$

We then calculate

$$\left\langle t \begin{array}{cc} 0 & 1 \\ 0 & 1 \end{array} h_\theta, h_\mu^x \right\rangle = |t|^{-1/2} \chi(t) \int_{\Omega} h_\theta \left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) h_\mu^x \left( \begin{array}{cc} 1 & 0 \\ t^{-1}x & 1 \end{array} \right) \cdot dx$$

and thus we want to see that

$$I_{\theta, \mu}(t) := \int_{\Omega} h_\theta \left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) h_\mu^x \left( \begin{array}{cc} 1 & 0 \\ t^{-1}x & 1 \end{array} \right) \cdot dx$$

is bounded independently of $\theta, \mu \in \hat{K}$ and $t \in F$ satisfying $|t| \geq 1$. In general, for $\theta \in \hat{K}$ and $x \in \Omega$, we have

$$h_\theta \left( \begin{array}{cc} 1 & 0 \\ x & 1 \end{array} \right) = \frac{1}{|1 - \zeta x^2|^{1/2}} \chi(1 - \zeta x^2) \theta \left( \begin{array}{cc} 1 & \zeta x \\ x & 1 \end{array} \right).$$

And so we obtain

$$\begin{align*}
I_{\theta, \mu}(t) &= \int_{\Omega} \frac{1}{|1 - \zeta x^2|^{1/2}} \chi(1 - \zeta t^{-2}x^2) \theta \left( \begin{array}{cc} 1 & \zeta x \\ x & 1 \end{array} \right) \mu^{-1} \left( \begin{array}{cc} 1 & t^{-1}x \\ t^{-1}x & 1 \end{array} \right) \cdot dx.
\end{align*}$$

If for $D \subset F$ we denote by $I_{\theta, \mu}^D(t)$ the same expression as that for $I_{\theta, \mu}(t)$ in (7.6) but where integration is performed over $D$, we see

$$|I_{\theta, \mu}^\Omega(t)| \leq \int_{\varpi \mathcal{O}} \frac{dx}{1/q} = 1/q$$

and thus this is bounded. Furthermore,

$$|I_{\theta, \mu}^{\Omega \setminus \varpi \mathcal{O}}(t)| \leq \int_{\Omega \setminus \varpi \mathcal{O}} \frac{dx}{|t^{-1}x|} = 1/q$$

and thus this is bounded. Finally, for any specific $- \log_q |t| \leq m \leq 0$, we have

$$|I_{\theta, \mu}^{\mathcal{O}^\times}(t)| \leq \int_{\varpi^m \mathcal{O}^\times} \frac{dx}{|x|} = 1 - 1/q.$$
Therefore, denoting by $k \in \mathbb{Z}_{\geq 1}$ a number such that $\chi|_{\varpi^{k+1} \mathcal{O}} = 1$, it is enough to bound $I_{\theta, \mu}^{k \mathcal{O} \setminus \varpi^{-k} \mathcal{O}}(t)$. We have

$$I_{\theta, \mu}^{k \mathcal{O} \setminus \varpi^{-k} \mathcal{O}}(t) = \chi^{-1}(-\zeta) \theta \left( \begin{array}{cc} 0 & \zeta \\ 1 & 0 \end{array} \right) \int_{\varpi^{k+1} \mathcal{O} \setminus \varpi^{-k} \mathcal{O}} \chi^{-2}(x) \theta \left( \begin{array}{cc} 1 & x^{-1} \\ \zeta_1 x^{-1} & 1 \end{array} \right) \mu^{-1} \left( \begin{array}{cc} 1 & \zeta x^{-1} \\ t^{-1} x & 1 \end{array} \right) \frac{dx}{|x|}.$$ 

Let us denote by $K' \subset K$ the subgroup consisting of $\left( \begin{array}{cc} x & y \\ y & x \end{array} \right)$ for which $|y| \leq |x| \cdot |p|$. We have an isomorphism of topological groups $e : p\mathcal{O} \rightarrow K'$ given by

$$e(y) := \exp \left( \begin{array}{cc} 0 & \zeta y \\ y & 0 \end{array} \right).$$

Let us denote by $\alpha : p\mathcal{O} \rightarrow p\mathcal{O}$ the map given by $\alpha(y) := e^{-1} \left( \begin{array}{cc} 1 & \zeta y \\ y & 1 \end{array} \right)$. We have a power series expansion $\alpha(y) = y + \zeta y^3/3 + \zeta^2 y^5/5 + \ldots$. Let us now denote by $\tilde{\theta}$ the unitary character of $p\mathcal{O}$ satisfying $\theta|_{K'} \circ e = \tilde{\theta}$ and by $\tilde{\mu}$ the unitary character of $p\mathcal{O}$ satisfying $\mu|_{K'} \circ e = \tilde{\mu}$. Returning to our integral, we can take $k$ big enough so that $\varpi^n \in p\mathcal{O}$. Substituting $x^{-1}$ in place of $x$ in the integral we have, we see that we need to show that

$$\int_{\varpi^{k+1} \mathcal{O} \setminus \varpi^{-k+1-t} \mathcal{O}} \tilde{\theta}(\alpha(x)) \tilde{\mu}^{-1}(\alpha(x^{-1})) \cdot \chi^2(x) \cdot \frac{dx}{|x|}$$

is bounded independently of $\tilde{\theta}, \tilde{\mu} \in \widehat{p\mathcal{O}}$ and $t \in F$ satisfying $|t| \geq 1$. This is implied by Proposition 7.4.

**Remark 7.5.** We see from the proof that we have a more precise version of Claim 7.4. There exists $C > 0$ such that for $\chi$ satisfying (7.2), $\theta, \mu \in \text{types}(V_\chi)$ and $t \in \Omega^\times$ satisfying $|t| \geq 1$, we have

$$\left| \left( \begin{array}{cc} t & 0 \\ 0 & 1 \end{array} \right) h_\chi \cdot h_\mu \right| \leq C \cdot |t|^{-1/2} \cdot (c(\chi) + d(\chi) + 1).$$

Here $c(\chi)$ and $d(\chi)$ are as in the formulation of Proposition 7.4 (when we identify $\chi$ with a character of $\Omega^\times$ via the isomorphism $a : \Omega^\times \rightarrow A$).

Let us prove Proposition 7.4.

**Proof (of Proposition 7.4).** We can assume that the 1-th coefficients of $\alpha_1$ and $\alpha_2$ are equal to 1. Let us fix a unitary character $\psi$ of $\mathcal{O}$ satisfying $\psi|_{\varpi \mathcal{O}} = 1$ and $\psi|_{\varpi^{-1} \mathcal{O}} \neq 1$. For $i \in \{1, 2\}$, let $a_i \in \Omega$ be such that $\psi(a_i x) = \psi(x)$ for all $x \in \mathcal{O}$ ($a_i$ are defined up to addition of elements in $\varpi^{-1} \mathcal{O}$, so in particular we can assume that $a_i \neq 0$). Given $0 < m < n$ we define

$$J^m := \int_{\varpi^m \mathcal{O} \setminus \varpi^{m+1} \mathcal{O}} \psi \left( a_1 \alpha_1(x) + a_2 \alpha_2(\varpi^m x^{-1}) \right) \cdot \chi(x) \cdot \frac{dx}{|x|} =$$

$$= \chi(\varpi)^m \int_{\mathcal{O} \setminus \varpi^m \mathcal{O} \setminus \varpi^{m+1} \mathcal{O}} \psi \left( a_1 \alpha_1(x) + a_2 \alpha_2(\varpi^m x^{-1}) \right) \cdot \chi(x) \cdot dx,$$

so that the integral in question equals $\sum_{m_1 \leq m < m_2} J^m$. Let us abbreviate $b := (a_2 \varpi^{-m})/(a_1 \varpi^m)$. As we vary $m$, let us divide into cases.
Assume that \(|\varpi^m| < q^{-c(\chi)}\) and that \(|b| < q^{-c(\chi)}\). Set

\[
\beta(x) := \varpi^{-m}\alpha_1(\varpi^m x) + b \cdot \varpi^{-n}\alpha_2(\varpi^{n-m} x^{-1}).
\]

Then \(\beta\) gives a well-defined invertible analytic map \(O^* \rightarrow O^*\), whose derivative is everywhere a unit. Moreover, if \(c(\chi) > 0\), we have \(\beta^{-1}(x_0(1 + \varpi^{c(\chi)} O)) = x_0(1 + \varpi^{c(\chi)} O)\) for any \(x_0 \in O^*\). Hence

\[
J^m = \chi(\varpi)^m \int_{O^*} \psi(a_1 \varpi^m \beta(x)) \cdot \chi(x) \cdot dx = \chi(\varpi)^m \int_{O^*} \psi(a_1 \varpi^m x) \cdot \chi(x) \cdot dx.
\]

Therefore:

(a) Suppose that \(|a_1 \varpi^m| \leq 1\). Then \(J^m = \chi(\varpi)^m(1 - 1/q)\) if \(\chi\) is unramified and \(J^m = 0\) if \(\chi\) is ramified.

(b) Suppose that \(|a_1 \varpi^m \varpi^{c(\chi)}| > q^{-1}\). Then \(J^m = 0\) if \(\chi\) is unramified. If \(\chi\) is ramified, we write

\[
J^m = \chi(\varpi)^m \sum_{x_0 \in O^*/(1 + \varpi^{c(\chi)} O)} \psi(a_1 \varpi^m x_0) \chi(x_0) \int_{\varpi^{c(\chi)} O} \psi(a_1 \varpi^m x) \cdot dx.
\]

and each integral here is equal to 0, so that also in that case we obtain \(J^m = 0\).

(c) The case when neither of these two cases is satisfied corresponds to only finitely many values of \(m\), whose number is linearly bounded in terms of \(c(\chi) + 1\), and so we can be content with the crude estimate \(|J^m| \leq 1\) in this case.

Assume that \(|\varpi^{n-m}| < q^{-c(\chi)}\) and \(|b^{-1}| < q^{-c(\chi)}\). This case is dealt with analogously to the previous one; one denotes

\[
\beta(x) := \varpi^{-n-1}\alpha_2(\varpi^{n-m} x^{-1}) + b^{-1} \varpi^{-m}\alpha_1(\varpi^m x)
\]

and gets

\[
J^m = \chi(\varpi)^m \int_{O^*} \psi(a_2 \varpi^{n-m} x) \cdot \chi(x) \cdot dx.
\]

And thus:

(a) Suppose that \(|a_2 \varpi^{n-m}| \leq 1\). Then \(J^m = \chi(\varpi)^m(1 - 1/q)\) if \(\chi\) is unramified and \(J^m = 0\) if \(\chi\) is ramified.

(b) Suppose that \(|a_2 \varpi^{n-m} \varpi^{c(\chi)}| > q^{-1}\). Then \(J^m = 0\) if \(\chi\) is unramified. If \(\chi\) is ramified, we write

\[
J^m = \chi(\varpi)^m \sum_{x_0 \in O^*/(1 + \varpi^{c(\chi)} O)} \psi(a_2 \varpi^{n-m} x_0) \chi(x_0) \int_{\varpi^{c(\chi)} O} \psi(a_2 \varpi^{n-m} x) \cdot dx
\]

and each integral here is equal to 0, so that also in that case we obtain \(J^m = 0\).

(c) The case when neither of these two cases is satisfied corresponds to only finitely many values of \(m\), whose number is linearly bounded in terms of \(c(\chi) + 1\), and so we can be content with the crude estimate \(|J^m| \leq 1\) in this case.
The case when neither of these two cases is satisfied corresponds to only finitely many values of $m$, whose number is linearly bounded in terms of $c(\chi)$, and so we can be content with the crude estimate $|J^m| \leq 1$ in this case.

As our integral in question is equal to $\sum_{m_1 \leq m < m_2} J^m$, it is straightforward that the findings above give the boundedness as desired. \hfill $\square$

**Appendix A. Auxiliary claims regarding polynomial growth of exponential integrals and sums**

**A.1. Some notation.** We denote $[n] := \{1,2,\ldots,n\}$. We denote

$$\mathbb{C}_{<0} := \{ z \in \mathbb{C} \mid \Re(z) \leq 0 \}, \quad D := \{ z \in \mathbb{C} \mid |z| \leq 1 \}.$$  

Given $x = (x_1,\ldots,x_n) \in \mathbb{R}^n_0$ and $m = (m_1,\ldots,m_n) \in \mathbb{Z}_{\geq 0}^n$, we write $x^m := x_1^{m_1} \cdots x_n^{m_n}$. Given $\lambda \in \mathbb{C}_{\leq 0}^n$ we denote

$$J_{\lambda} := \{ 1 \leq j \leq n \mid \Re(\lambda_j) = 0 \}.$$  

Given $(\lambda,m) \in \mathbb{C}_{\leq 0}^n \times \mathbb{Z}_{\geq 0}^n$, we denote $d(\lambda,m) := \sum_{j \in J(\lambda)} (1 + m_j)$. Given $J \subset [n]$ and some set $X$, let us denote by $\text{res}_J : X^n \to X^J$ the natural restriction and by $\text{ext}^J : X^J \to X^n$ the natural extension by zero.

We fix a finite set $\mathcal{I} \subset \mathbb{R}^n_{\geq 0}$ with the property that given $j \in [n]$ there exists $v \in \mathcal{I}$ such that $\langle v,e_j \rangle \neq 0$, where $e_j$ the $j$-th standard basis vector. We denote

$$P_{cr} := \{ x \in \mathbb{R}^n_{\geq 0} \mid \langle v,x \rangle < r \forall v \in \mathcal{I} \}.$$  

Given $J \subset [n]$, we denote by $P_J \subset \mathbb{R}^J_{\geq 0}$ the convex pre-compact subset $\{ y \in \mathbb{R}^J_{\geq 0} \mid \text{ext}^J(y) \in P_{<1} \}$.

In §A3 we will also use the following notations. We consider a compact space $B$ equipped with a nowhere vanishing Radon measure $d\mu$. Let us say that a function $\phi : B \times \mathbb{R}^n_{\geq 0} \to \mathbb{C}$ is nice if it is expressible as

$$B \times \mathbb{R}^n_{\geq 0} \xrightarrow{\text{id}_B \times e^{-i}} B \times D^n \xrightarrow{\phi^o} \mathbb{C}$$

where $e^{-i}(x_1,\ldots,x_n) := (e^{-x_1},\ldots,e^{-x_n})$ and $\phi^o$ is continuous and holomorphic in the second variable (in the sense that when we fix the variable in $B$ it is the restriction of a holomorphic function on a neighbourhood of $D^n$). Given $J \subset [n]$ we denote by $\text{res}_J \phi : B \times \mathbb{R}^J_{\geq 0} \to \mathbb{C}$ the function given by $\text{res}_J \phi(b,y) := \phi^o(b,\text{ext}^J(\text{ei}(y)))$. We also write $\phi(b,+,\infty)$ for $\phi^o(b,0)$ etc.

**A.2. Growth - the case of summation over a lattice.**

**Lemma A.1.** Let $\lambda := (\lambda_1,\ldots,\lambda_n) \in \mathbb{C}_{\leq 0}^n$ and $m := (m_1,\ldots,m_n) \in \mathbb{Z}_{\geq 0}^n$. Let $K \subset \mathbb{R}^n_{\geq 0}$ be a compact subset. Assume that $\Re(\lambda) = 0$ and $\lambda \notin (2\pi i)\mathbb{Z}^n$. We have

$$\sup_{Q \subset K} \left| \frac{1}{r^n} \sum_{x \in \mathbb{Z}_{\geq 0}^n \cap Q} x^m e^{r \langle \lambda,x \rangle} \right| = O(r^{-1})$$

as $r \to +\infty$, where $Q$ denote convex subsets.
Proof. Let us re-order the variables, assuming that $\lambda \notin 2\pi i \mathbb{Z}$. Let us write $x = (x_1, x')$ where $x' = (x_2, \ldots, x_n)$ and analogously write $m'$ etcetera. Given a convex subset $Q \subset K$ and $x' \in \mathbb{R}_{>0}^{n-1}$ let us denote by $Q' \subset \mathbb{R}_{>0}$ the subset consisting of $x_1$ for which $(x_1, x') \in Q$ (it is an interval). Let us enlarge $K$ for convenience, writing it in the form $K = K_1 \times K'$ where $K_1 \subset \mathbb{R}_{>0}$ is a closed interval and $K' \subset \mathbb{R}_{>0}^{n-1}$ is the product of closed intervals.

We have

$$\sum_{x \in \mathbb{Z}_{>0} \cap Q} x^m e^{r(\lambda, x)} = \sum_{x' \in \mathbb{Z}_{>0}^{n-1} \cap K'} (x')^m e^{r(\lambda', x')} \left( \sum_{x_1 \in \mathbb{Z}_{>0} Q x'} x_1^{m_1} e^{r(\lambda_1, x_1)} \right).$$

We have $Q' \subset K'$ and it is elementary to see that

$$\sup_{R \subset K'} \left| \sum_{x_1 \in \mathbb{Z}_{>0} \cap R} x_1^{m_1} e^{r(\lambda_1, x_1)} \right| = O(1)$$

as $r \to +\infty$, where $R$ denote intervals. Therefore we obtain, for some $C > 0$ (not depending on $Q$) and all $r \geq 1$:

$$\left| \frac{1}{r^n} \sum_{x \in \mathbb{Z}_{>0} \cap Q} x^m e^{r(\lambda, x)} \right| \leq C \left( \frac{1}{r^{n-1}} \sum_{x' \in \mathbb{Z}_{>0}^{n-1} \cap K'} (x')^m \right) r^{-1}.$$

Since the expression in brackets is clearly bounded independently of $r$, we are done.

Lemma A.2. Let $(\lambda, m) \in \mathbb{C}_{\leq 0} \times \mathbb{Z}_{\geq 0}$. Then the limit

$$\lim_{r \to +\infty} \frac{1}{r^{d(\lambda, m)}} \sum_{x \in \mathbb{Z}_{>0} \cap P_{<r}} x^m e^{r(\lambda, x)} dx$$

exists, equal to 0 if $\text{res}_{J, \lambda} (\lambda) \notin 2\pi i \cdot \mathbb{Z}^{J, x}$ and otherwise equal to

$$\left( \int_{P_{\lambda}} y^{\text{res}_{J, \lambda} (m)} dy \right) \left( \sum_{z \in \mathbb{Z}_{\geq 0}} z^{\text{res}_{J, \lambda} (m)} e^{(\text{res}_{J, \lambda}(\lambda), z)} \right)$$

(the sum converging absolutely).

Proof. Let us abbreviate $J := J_\lambda$. Let us denote $\lambda' := \text{res}_{J}(\lambda)$ and $\lambda'' := \text{res}_{J'}(\lambda)$, and similarly for $m$. Given $x'' \in \mathbb{Z}_{>0}^J$ let us denote by $P_{x''} \subset \mathbb{R}_{>0}$ the subset consisting of $y'$ for which $\text{ext}^J (ry') + \text{ext}^{J'} (x'') \in P_{<r}$. We have

$$\sum_{x \in \mathbb{Z}_{>0} \cap P_{<r}} x^m e^{r(\lambda, x)} = r^{d(\lambda, m)-|J|} \sum_{x'' \in \mathbb{Z}_{>0}^J} (x'')^{m''} e^{r(\lambda'', x'')} \sum_{y' \in \mathbb{Z}_{>0} \cap P_{x''}} (y')^{m'} e^{r(\lambda', y')} := \Delta.$$
Let us assume first that $\lambda' \notin 2\pi i \cdot \mathbb{Z}^J$. Then by Lemma A.1 there exists $C > 0$ such that for all convex subsets $Q \subset P_J$ and all $r \geq 1$ we have

$$\left| \frac{1}{r^{|J|}} \sum_{y' \in \frac{1}{2}\mathbb{Z}_r^{<r} \cap Q} (y')^{m'} e^{r(\lambda',y')} \right| \leq C \cdot r^{-1}.$$

Therefore

$$|\Delta| \leq C r^{d(\lambda,m)-1} \sum_{x'' \in \mathbb{Z}_r^{<r}} (x'')^{m''} e^{(\Re(\lambda''),x'')} ,$$

and all

$$\left| \sum_{x'' \in \mathbb{Z}_r^{<r}} (x'')^{m''} e^{(\Re(\lambda''),x'')} \right| \leq C r^{-1} \sum_{x'' \in \mathbb{Z}_r^{<r}} (x'')^{m''}$$

giving the desired.

Now we assume $\lambda' \in 2\pi i \cdot \mathbb{Z}^J$. It is not hard to see that

$$\lim_{r \to +\infty} \frac{1}{r^d} \sum_{y' \in \frac{1}{2}\mathbb{Z}_r^{<r} \cap P_J} (y')^{m'} = \int_{P_J} (y')^{m'} \, dy'.$$

Hence we have (by dominated convergence)

$$\lim_{r \to +\infty} \frac{1}{r^d} \sum_{x'' \in \mathbb{Z}_r^{<r}} (x'')^{m''} e^{(\Re(\lambda''),x'')} \int_{P_J} (y')^{m'} \, dy'.$$

\[\square\]

**Claim A.3.** Let $p \geq 1$, let $\{(\lambda^{(\ell)},m^{(\ell)})\}_{\ell \in [p]} \subset \mathbb{C}^{n_0} \times \mathbb{Z}_{\leq 0}^n$ be a collection of pairwise different couples and let $\{e^{(\ell)}\}_{\ell \in [p]} \subset \mathbb{C} \setminus \{0\}$ be a collection of non-zero scalars. Denote $d := \max_{\ell \in [p]} d(2\Re(\lambda^{(\ell)}),2m^{(\ell)})$. The limit

$$\lim_{r \to +\infty} \frac{1}{r^d} \sum_{x \in \mathbb{Z}_r^{<r} \cap P_{<r}} \left| \sum_{\ell \in [p]} e^{(\ell)} x^{m^{(\ell)}} e^{(\Re(\lambda^{(\ell)}),x)} \right|^2$$

exists and is strictly positive.

**Proof.** Let us break the integrand into a sum following

$$\left| \sum_{\ell \in [p]} A_{\ell} \right|^2 = \sum_{\ell_1,\ell_2 \in [p]} A_{\ell_1} A_{\ell_2}.$$

Using Lemma A.2 we see the that resulting limit breaks down as a sum, over $(\ell_1,\ell_2) \in [p]^2$, of limits which exist, so the only thing to check is that the resulting limit is non-zero. It is easily seen that the limit at the $(\ell_1,\ell_2)$ place is zero unless $d(\lambda^{(\ell_1)},m^{(\ell_1)}) = d, d(\lambda^{(\ell_2)},m^{(\ell_2)}) = d, J_{\lambda^{(\ell_1)}} = J_{\lambda^{(\ell_2)}}, \text{ and } \text{res}_{J^{(\ell_1)}}(\lambda^{(\ell_2)}) - \text{res}_{J^{(\ell_1)}}(\lambda^{(\ell_1)}) \in 2\pi i \cdot \mathbb{Z}^J$. We thus can reduce to the case when, for a given $J \subset [n]$, we have $J_{\lambda^{(\ell)}} = J$ for all $\ell \in [p]$, we have $d(\lambda^{(\ell)},m^{(\ell)}) = d$ for all $\ell \in [p]$, and we have $\text{res}_{J}(\lambda^{(\ell_2)}) - \text{res}_{J}(\lambda^{(\ell_1)}) \in 2\pi i \cdot \mathbb{Z}^J$ for all $\ell_1,\ell_2 \in [p]$. We then obtain, using Lemma A.2 that our overall limit equals

$$\sum_{x \in \mathbb{Z}_r^{<r} \cap P_{<r}} \left| \int_{P_J} \left| \sum_{\ell \in [p]} e^{(\ell)} x^{\text{res}_{J}(m^{(\ell)})} z^{\text{res}_{J}(\lambda^{(\ell)})} e^{(\text{res}_{J}(\lambda^{(\ell)}),z)} \right|^2 \, dy. \right.$$
Proof. Let us re-order the variables, assuming that $\lambda \neq 0$. By the local linear independence of powers of $\lambda$, we can further assume that $\text{res}_J(m^{(\ell)})$ is independent of $\ell \in [p]$, and want to check that
\[
\sum_{\ell \in [p]} c^{(\ell)}_z \text{res}_J(m^{(\ell)}) \, \text{res}_J(\lambda^{(\ell)}), z),
\]
a function in $(z, y) \in \mathbb{Z}_{\geq 0} \times P_J$, is not identically zero. Notice that, by our assumptions, the elements in the collection \{($\text{res}_J(\lambda^{(\ell)}), \text{res}_J(m^{(\ell)}))_{\ell \in [p]}$\} are pairwise different. Thus the non-vanishing of our sum is clear (by linear algebra of generalized eigenvectors of shift operators on $\mathbb{Z}_+^r$).

A.3. Growth - the case of an integral.

**Lemma A.4.** Let $\lambda := (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}_+^n$ and $m := (m_1, \ldots, m_n) \in \mathbb{Z}_{\geq 0}$. Let $K \subset \mathbb{R}_{\geq 0}^n$ be a compact subset. Assume that $\text{Re}(\lambda) = 0$ and $\lambda \neq 0$. We have
\[
\sup_{Q \subset K} \left| \int_Q x^m e^{r(\lambda, x)} \, dx \right| = O(r^{-1})
\]
as $r \to +\infty$, where $Q$ denote convex subsets.

**Proof.** Let us re-order the variables, assuming that $\lambda_1 \neq 0$. Let us write $x = (x_1, x')$ where $x' = (x_2, \ldots, x_n)$ and analogously write $m'$ etcetera. Given a convex subset $Q \subset K$ and $x' \in \mathbb{R}_{\geq 0}^{n-1}$ let us denote by $Qx' \subset \mathbb{R}_{\geq 0}$ the subset consisting of $x_1$ for which $(x_1, x') \in Q$ (it is an interval). Let us enlarge $K$ for convenience, writing it in the form $K = K_1 \times K'$ where $K_1 \subset \mathbb{R}_{\geq 0}$ is a closed interval and $K' \subset \mathbb{R}_{\geq 0}^{n-1}$ is the product of closed intervals.

Using Fubini’s theorem
\[
\int_Q x^m e^{r(\lambda, x)} \, dx = \int_{K'} (x')^m e^{\lambda (x', x')} \left( \int_{Q \times \{x_1\}} x_1^m e^{r(x_1, x')} \, dx_1 \right) \, dx'.
\]
We have $Qx' \subset K'$ and it is elementary to see that
\[
\sup_{R \subset K'} \left| \int_R x_1^m e^{r(x_1, x')} \, dx_1 \right| = O(r^{-1})
\]
as $r \to +\infty$, where $R$ denote intervals. Therefore we obtain, for some $C > 0$ and all $r \geq 1$:
\[
\left| \int_Q x^m e^{r(\lambda, x)} \, dx \right| \leq C \left( \int_{K'} (x')^m \, dx' \right)^{r^{-1}},
\]
as desired.

**Lemma A.5.** Let $(\lambda, m) \in \mathbb{C}_+^n \times \mathbb{Z}_{\geq 0}^n$ and let $\phi : B \times \mathbb{R}_{\geq 0}^n \to \mathbb{C}$ be a nice function. Then the limit
\[
\lim_{r \to +\infty} \frac{1}{r^{d(\lambda, m)}} \int_B \int_{P \subset \mathbb{R}} x^m e^{r(\lambda, x)} \phi(b, x) \, dx \, db
\]
exists, equal to 0 if \( \text{res}_J \lambda \neq 0 \) and otherwise equal to
\[
\left( \int_{P \lambda} y^{\text{res}_J \lambda} (m) dy \right) \left( \int_{B} \int_{\mathbb{R}^J_{\geq 0}} z^{\text{res}_J \lambda} (m) e^{(\text{res}_J \lambda)z} \text{res}_J \phi (b,z) dz db \right)
\]
(the double integral converging absolutely).

Proof. Let us re-order the variables, assuming that \( J := J \lambda = [k] \). Let us write \( x = (x', x'') \) where \( x' \) consists of the first \( k \) components and \( x'' \) consists of the last \( k \) components. Let us write analogously \( m', \lambda' \) etc.

First, let us notice that if \( k \neq 0 \), we can write
\[
\phi (b,x) = e^{-x_1} \phi_0 (b,x) + \phi_1 (b,x)
\]
where \( \phi_0, \phi_1 : B \times \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{C} \) are nice functions and \( \phi_1 \) does not depend on \( x_1 \).

Dealing with \( e^{-x_1} \phi_0 (b,x) \) instead of \( \phi (b,x) \) makes us consider \( \lambda \) with smaller set \( J \) and thus \( (\lambda, m) \) with a smaller \( d(\lambda, m) \) and from this, reasoning inductively, we see that we can assume that \( \phi \) only depends on \( (b,x'') \). Let us write \( \phi'' := \text{res}_J \phi \).

Let us perform a change of variables \( y' := \frac{1}{r} x' \). Let \( P_{(r)} \subset \mathbb{R}_{>0}^n \) denote the transform of \( P_{<r} \) under these changes of variables (i.e. \( (x', x'') \in P_{<r} \) if and only if \( (y', x'') \in P_{(r)} \)). We obtain
\[
\int_{B} \int_{P \lambda} x'^n e^{\langle \lambda, x' \rangle} \phi (b,x) dx db = 
\]
\[
= \int_{B} \int_{P (r)} (y')^n e^{r \langle \lambda', y' \rangle} (x'')^n e^{\langle \lambda'', x'' \rangle} \phi'' (b,x'') dy' dx'' db =: \Delta.
\]

Given \( x'' \in \mathbb{R}_{>0}^J \), let us denote by \( P_{(r)}^{x''} \subset \mathbb{R}_{>0}^J \) the set consisting of \( y' \) for which \( (y', x'') \in P_{(r)} \). Notice that \( P_{(r_1)}^{x''} \subset P_{(r_2)}^{x''} \) for \( r_1 < r_2 \) and \( \cup_{r} P_{(r)}^{x''} = P_{J} \). Using Fubini’s theorem
\[
\Delta = \int_{B} \int_{\mathbb{R}_{>0}^J} (x'')^n e^{\langle \lambda'', x'' \rangle} \phi'' (b,x'') \left( \int_{P (r)} (y')^n e^{r \langle \lambda', y' \rangle} dy' \right) dx'' db.
\]

If \( \lambda' \neq 0 \), by Lemma \ref{lemma:A} there exists \( C > 0 \) such that for all convex subsets \( Q \subset P_{J} \) and all \( r \geq 1 \) we have
\[
\left| \int_{Q} (y')^n e^{r \langle \lambda', y' \rangle} dy' \right| \leq C \cdot r^{-1}.
\]
We have therefore
\[
|\Delta| \leq C \cdot r^{-1} \cdot \int_{B} \int_{\mathbb{R}_{>0}^J} (x'')^n e^{(\text{Re}(\lambda''), x'')} |\phi'' (b,x'')| dx'' db
\]
and thus indeed the desired limit is equal to 0.

Now we assume \( \lambda' = 0 \). Using Lebesgue’s dominated convergence theorem we have
\[
\lim_{r \to + \infty} \frac{1}{r} \Delta = \lim_{r \to + \infty} \int_{B} \int_{\mathbb{R}_{>0}^J} (x'')^n e^{\langle \lambda'', x'' \rangle} \phi'' (b,x'') \left( \int_{P (r)} (y')^n dy' \right) dx'' db =
\]
\[
= \int_{\mathbb{R}^\ell_{\geq 0}} (x''')^{m'''} e^{(x''',x''')^2} \left( \int_{P_J} (y')^{m'} dy' \right) dx'''' db
\]
as desired. \(\square\)

**Claim A.6.** Let \(\{ (\lambda^{(\ell)}, m^{(\ell)}) \}_{\ell \in [p]} \subseteq \mathbb{C} \times \mathbb{Z}_{\geq 0} \) be a collection of pairwise different couples. Let \(\{ \phi^{(\ell)} \}_{\ell \in [p]}\) be a collection of nice functions \(B \times \mathbb{R}^\ell_{\geq 0} \to \mathbb{C}\), such that for every \(\ell \in [p]\) the function \(b \mapsto \phi^{(\ell)}(b, +\infty)\) on \(B\) is not identically zero. Denote \(d := \max_{\ell \in [p]} d(2\text{Re}(\lambda^{(\ell)}), 2m^{(\ell)})\). The limit
\[
\lim_{r \to +\infty} \frac{1}{r^m} \int_{B} \int_{P_J} \left| \sum_{\ell \in [p]} x^{m^{(\ell)}} e^{(\lambda^{(\ell)}, x)^2} \phi^{(\ell)}(b, x) \right|^2 \ dx db
\]
exists and is strictly positive.

**Proof.** Let us break the integrand into a sum following
\[
\left| \sum_{\ell \in [p]} A_{\ell} \right|^2 = \sum_{\ell_1, \ell_2 \in [p]} A_{\ell_1} A_{\ell_2}.
\]
Using Lemma A.3, we see the that resulting limit breaks down as a sum, over \((\ell_1, \ell_2) \in [p]^2\), of limits which exist, so the only thing to check is that the resulting limit is non-zero. It is easily seen that the limit at the \((\ell_1, \ell_2)\) place is zero unless \(d(\lambda^{(\ell_1)}, m^{(\ell_1)}) = d, d(\lambda^{(\ell_2)}, m^{(\ell_2)}) = d, J_{\lambda^{(\ell_1)}} = J_{\lambda^{(\ell_2)}}\) and \(\text{res}_{J^{\ell_1}}(\lambda^{(\ell_1)}) = \text{res}_{J^{\ell_2}}(\lambda^{(\ell_2)})\). We thus can reduce to the case when, for a given \(J \subset [n]\), we have \(J_{\lambda^{(\ell)}} = J\) for all \(\ell \in [p]\), we have \(d(\lambda^{(\ell)}, m^{(\ell)}) = d\) for all \(\ell \in [p]\), and we have \(\text{res}_{J^{\ell}}(\lambda^{(\ell_1)}) = \text{res}_{J^{\ell}}(\lambda^{(\ell_2)})\) for all \(\ell_1, \ell_2 \in [p]\). We then obtain, using Lemma A.3 that our overall limit equals
\[
\int_{B} \int_{\mathbb{R}^\ell_{\geq 0}} \int_{P_J} \left| \sum_{\ell \in [p]} y^{\text{res}_{J^{\ell}}(m^{(\ell)})} z^{\text{res}_{J^{\ell}}(m^{(\ell)})} e^{(\text{res}_{J^{\ell}}(\lambda^{(\ell)}), z)} \phi^{(\ell)}(b, z) \right|^2 \ dy dz db.
\]
It is therefore enough to check that
\[
\sum_{\ell \in [p]} y^{\text{res}_{J^{\ell}}(m^{(\ell)})} z^{\text{res}_{J^{\ell}}(m^{(\ell)})} e^{(\text{res}_{J^{\ell}}(\lambda^{(\ell)}), z)} \phi^{(\ell)}(b, z),
\]
a function in \((b, z, y) \in B \times \mathbb{R}^\ell_{\geq 0} \times P_J\), is not identically zero. By the local linear independence of powers of \(y\), we can further assume that \(\text{res}_{J^{\ell}}(m^{(\ell)})\) is independent of \(\ell \in [p]\), and want to check that
\[
\sum_{\ell \in [p]} z^{\text{res}_{J^{\ell}}(m^{(\ell)})} e^{(\text{res}_{J^{\ell}}(\lambda^{(\ell)}), z)} \phi^{(\ell)}(b, z),
\]
a function in \((b, z) \in B \times \mathbb{R}^\ell_{\geq 0},\) is not identically zero. Notice that, by our assumptions, the elements in the collection \(\{ \text{res}_{J^{\ell}}(\lambda^{(\ell)}), \text{res}_{J^{\ell}}(m^{(\ell)}) \}_{\ell \in [p]}\) are pairwise different and for every \(\ell \in [p]\), the function \(b \mapsto \phi^{(\ell)}(b, \text{ext}^J(+\infty))\) on \(B\) is not identically zero. Considering the partial order on \(\mathbb{C}^J\) given by \(\mu_1 \leq \mu_2\) if \(\mu_2 - \mu_1 \in \mathbb{Z}_{\geq 0}\), we can pick \(\ell \in [p]\) for which \(\text{res}_{J^{\ell}}(\lambda^{(\ell)})\) is maximal among the \(\{ \text{res}_{J^{\ell}}(\lambda^{(\ell)}) \}_{\ell \in [p]}\). We can then pick \(b \in B\) such that \(\phi^{(\ell)}(b, \text{ext}^J(+\infty)) \neq 0\). We then boil down to Lemma A.7 that follows. \(\square\)
In the end of the proof of Claim A.6 we have used the following:

**Lemma A.7.** Let \( \{(\lambda^{(\ell)}, m^{(\ell)})\}_{\ell \in [p]} \subset \mathbb{C}^n \times \mathbb{Z}_{\geq 0}^n \) be a collection of pairwise different couples. Let \( \{\phi^{(\ell)}\}_{\ell \in [p]} \) be a collection of nice functions \( \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{C} \) (so here \( B = \{1\} \)). Suppose that \( \phi^{(\ell)}(\infty) \neq 0 \) for some \( \ell \in [p] \) for which \( \lambda^{(\ell)} \) is maximal among the \( \{\lambda^{(\ell')}\}_{\ell' \in [p]} \) with respect to the partial order \( \lambda_1 \leq \lambda_2 \) if \( \lambda_2 - \lambda_1 \in \mathbb{Z}_{\geq 0}^n \). Then the function

\[
x \mapsto \sum_{\ell \in [p]} x^{m^{(\ell)}} e^{(\lambda^{(\ell)} \cdot x)} \phi^{(\ell)}(x)
\]

on \( \mathbb{R}_{\geq 0}^n \) is not identically zero.

**Proof.** We omit the proof - one develops the \( \phi^{(\ell)} \) into power series in \( e^{-x_1}, \ldots, e^{-x_n} \) and uses separation by generalized eigenvalues of the partial differentiation operators \( \partial_{x_1}, \ldots, \partial_{x_n} \).

\[
\square
\]

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