GLOBAL ATTRACTORS FOR STRONGLY DAMPED WAVE EQUATIONS WITH DISPLACEMENT DEPENDENT DAMPING AND NONLINEAR SOURCE TERM OF CRITICAL EXPONENT

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Abstract. In this paper the long time behaviour of the solutions of the 3-D strongly damped wave equation is studied. It is shown that the semigroup generated by this equation possesses a global attractor in $H^1_0(\Omega) \times L^2(\Omega)$ and then it is proved that this is also a global attractor in $(H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$.

1. Introduction

We consider the following initial-boundary value problem for the strongly damped wave equation:

$$
w_{tt} - \Delta w_t + \sigma(w)w_t - \Delta w + f(w) = g(x) \quad \text{in} \ (0, \infty) \times \Omega, \quad (1.1)
$$

$$
w = 0 \quad \text{on} \ (0, \infty) \times \partial \Omega, \quad (1.2)
$$

$$
w(0, \cdot) = w_0, \quad w_t(0, \cdot) = w_1 \quad \text{in} \ \Omega, \quad (1.3)
$$

where $\Omega \subset R^3$ is a bounded domain with sufficiently smooth boundary and $g \in L^2(\Omega)$.

As shown in [6] and [13], equation (1.1) is related to the following reaction-diffusion equation with memory:

$$
w_t(t, x) = \int_{-\infty}^{t} K(t, s) \Delta w(s, x) ds - f(w(t, x)) + g(x). \quad (1.4)
$$

Namely, if $K(t, s) = \frac{1}{\lambda} \frac{1}{\lambda e^{-\lambda t} + 2\alpha \delta(t-s)}$ then (1.4) can be transformed into

$$
\lambda w_{tt} - \alpha \lambda \Delta w_t + (1 + \lambda f'(w))w_t - \Delta w + f(w) = g,
$$

where $\lambda > 0$, $\alpha \in [0, 1)$ and $\delta$ is a Dirac delta function. This equation is interesting from a physical viewpoint as a model describing the flow of viscoelastic fluids (see [6] and [13] for details).

When $\sigma(\cdot) \equiv 0$ the equation (1.1) becomes

$$
w_{tt} - \Delta w_t - \Delta w + f(w) = g. \quad (1.5)
$$

The long time behaviour (in terms of attractors) of solutions in this case has been studied by many authors (see [2], [5], [7], [14], [15], [19], [22] and references therein). In [14] the existence of a global attractor for (1.5) with critical source term (i.e. in the case when the growth of $f$ is of order 5) was proved. However, the regularity of the global attractor in that article was established only in the subcritical case. For
the critical case, the regularity of the global attractor of (1.5) was proved in [15], under the assumptions
\[ f \in C^1(R), \ |f'(s)| \leq c(1 + |s|^4), \ \forall s \in R \text{ and } \liminf_{|s| \to \infty} f'(s) > -\lambda_1 \] (1.6)
of
\[ f \in C^2(R), \ |f''(s)| \leq c(1 + |s|^3), \ \forall s \in R \text{ and } \liminf_{|s| \to \infty} \frac{f(s)}{s} > -\lambda_1, \] (1.7)
where \( \lambda_1 \) is a first eigenvalue of \(-\Delta\) with zero Dirichlet data. In that article the authors obtained a regular estimate for \( w_{tt} \) (when \( w(t, x) \) is a weak solution of (1.5)) and then proved the asymptotic regularity of the solution of the non-autonomous equation
\[-\Delta w_t - \Delta w + f(w) = g - w_{tt}.\]

In [5] and [19], the regularity of the global attractor of (1.5) was proved under the following weaker condition on the source term:
\[ f \in C(R), \ |f(u) - f(v)| \leq c(1 + |u|^4 + |v|^4)|u - v|, \ \forall u, v \in R \text{ and } \liminf_{|s| \to \infty} \frac{f(s)}{s} > -\lambda_1. \]

In [8], the authors investigated the weak attractor for the quasi-linear strongly damped equation
\[ w_{tt} - \Delta w_t - \Delta w + f(w) = \nabla \cdot \varphi'(\nabla w) + g \]
under the following conditions on the nonlinear functions \( f \) and \( \varphi \):
\[ f \in C^1(R), \ -C + a_1 |s|^q \leq f'(s) \leq C |s|^q, \ \forall s \in R, \]
\[ \varphi \in C^2(R^3, R), \ a_2 |\eta|^{p-1} |\xi|^2 \leq \sum_{i,j=1}^{3} \frac{\partial^2 \varphi(\eta)}{\partial \eta_i \partial \eta_j} \xi_i \xi_j \leq a_3 (1 + |\eta|^{p-1}) |\xi|^2, \ \forall \xi, \eta \in R^3, \]
for some \( a_i > 0, \ (i = 1, 2, 3), \ C > 0, \ q > 0 \) and \( p \in [1, 5) \). When \( \frac{\partial^2 \varphi}{\partial \eta_i \partial \eta_j} = 0, \ (i, j = 1, 2, 3) \), the strong attractor has also been studied. Recently, in [2], the authors have studied the global attractor for the strongly damped abstract equation
\[ w_{tt} + D(w, w_t) + Aw + F(w) = 0. \]

However, the approaches of the articles mentioned above, in general, do not seem to be applicable to (1.1). The difficulty is caused by the term \( \sigma(w)w_t \), when the function \( \sigma(\cdot) \) is not differentiable and the growth condition imposed on \( \sigma(\cdot) \) is critical. In this paper we prove the existence of the global attractors for (1.1)-(1.3) in \( H_0^1(\Omega) \times L_2(\Omega) \) and \( (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \). Then using the embedding \( H^{\frac{3}{4}+\varepsilon}(\Omega) \subset C(\bar{\Omega}) \) we show that these attractors coincide.

2. Well-posedness and the statement of the main result

We start with the conditions on nonlinear terms \( f \) and \( \sigma \).

- \( f \in C(R), \ |f(s) - f(t)| \leq c(1 + |s|^4 + |t|^4)|s - t|, \ \forall s, t \in R, \) (2.1)
- \( \liminf_{|s| \to \infty} \frac{f(s)}{s} > -\lambda_1, \) where \( \lambda_1 = \inf_{\varphi \in H_0^1(\Omega), \varphi \neq 0} \frac{||\nabla \varphi||^2_{L_2(\Omega)}}{||\varphi||^2_{L_2(\Omega)}}, \) (2.2)
- \( \sigma \in C(R), \ \sigma(s) \geq 0, \ |\sigma(s)| \leq c(1 + |s|^4), \ \forall s \in R. \) (2.3)
By the standard Galerkin’s method it is easy to prove the following existence theorem:

**Theorem 2.1.** Let conditions (2.1)-(2.3) hold. Then for every $T > 0$ and every $(\hat{w}_0, \hat{w}_1) \in \mathcal{H} := H_0^1(\Omega) \times L_2(\Omega)$, the problem (1.1)-(1.3) admits a weak solution  

$$w \in C([0, T]; H_0^1(\Omega)), \quad w_t \in C([0, T]; L_2(\Omega)) \cap L_2(0, T; H_0^1(\Omega)),$$

which satisfies the following energy equality

$$E(w(t)) + \int_0^t \| \nabla w_t(\tau) \|^2_{L_2(\Omega)} d\tau + \int_0^t \langle \sigma(w(\tau))w_t(\tau), w_t(\tau) \rangle d\tau + \langle F(w(t)), 1 \rangle - \langle g, w(t) \rangle = E(w(s)) + \langle F(w(s)), 1 \rangle - \langle g, w(s) \rangle, \quad 0 \leq s \leq t \leq T, \quad (2.4)$$

where $E(w(t)) = \frac{1}{2} \| \nabla w(t) \|^2_{L_2(\Omega)} + \| w_t(t) \|^2_{L_2(\Omega)}$, $\langle u, v \rangle = \int_\Omega u(x)v(x)dx$ and $F(w) = \int_0^w f(u)du$.

Now using the method of [10, Proposition 2.2] let us prove the following uniqueness theorem:

**Theorem 2.2.** Let conditions (2.1)-(2.3) hold. If $w(t, \cdot)$ and $\hat{w}(t, \cdot)$ are the weak solutions of (1.1)-(1.3), determined by Theorem 2.1, with initial data $(w_0, w_1)$ and $(\hat{w}_0, \hat{w}_1)$ respectively, then

$$\| w(T) - \hat{w}(T) \|^2_{H^1(\Omega)} + \| w_t(T) - \hat{w}_t(T) \|^2_{H^{-1}(\Omega)} \leq c(T, R) \left( \| w_0 - \hat{w}_0 \|_{H^1(\Omega)} + \| w_1 - \hat{w}_1 \|_{H^{-1}(\Omega)} \right)$$

where $c : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function with respect to each variable and $R = \max \{ \| (w_0, w_1) \|_\mathcal{H}, \| (\hat{w}_0, \hat{w}_1) \|_\mathcal{H} \}$.

**Proof.** By (2.1)-(2.4), it follows that

$$\| (w(t), w_t(t)) \|_\mathcal{H} + \| (\hat{w}(t), \hat{w}_t(t)) \|_\mathcal{H} \leq c_1(R), \quad \forall t \geq 0.$$

Denote $u(t, \cdot) = w(t, \cdot) - \hat{w}(t, \cdot)$ and $\tilde{u}(t, \cdot) = \int_0^t u(\tau, \cdot) d\tau$. Integrating (1.1) for $w(t, \cdot)$ and $\hat{w}(t, \cdot)$ on $[0, t]$ and taking the difference, we have

$$u_t - \Delta u + \Sigma(u) - \Sigma(\hat{w}) - \Delta \hat{u} + \int_0^t (f(w(\tau)) - f(\hat{w}(\tau))) d\tau =$$

$$\Sigma(w_0) - \Sigma(\hat{w}_0) - \Delta(w_0 - \hat{w}_0) + w_1 - \hat{w}_1, \quad \forall t \geq 0, \quad (2.5)$$

where $\Sigma(w) = \int_0^w \sigma(s)ds$. Testing (2.5) by $u$ and taking into account (2.1), (2.3), (2.4) and monotonicity of $\Sigma(\cdot)$, we find

$$\frac{d}{dt}E(\hat{u}(t)) + \frac{1}{2} \| \nabla u(t) \|^2_{L_2(\Omega)} \leq$$

$$\leq c_2(R) \left( \| \nabla (w_0 - \hat{w}_0) \|^2_{L_2(\Omega)} + \| w_1 - \hat{w}_1 \|^2_{H^{-1}(\Omega)} \right) +$$

$$+ c_2(R) \int_0^t \| \nabla u(\tau) \|^2_{L_2(\Omega)} d\tau, \quad \forall t \geq 0 \quad (2.6)$$
and consequently
\[
\frac{d}{dt} \tilde{E}(\tilde{u}(t)) \leq c_2(R) \left( \|w_0 - \tilde{w}_0\|^2_{H^1(\Omega)} + \|w_1 - \tilde{w}_1\|^2_{H^{-1}(\Omega)} \right) + 2c_2(R)t \tilde{E}(\tilde{u}(t)),
\]
where \( \tilde{E}(\tilde{u}(t)) = E(\tilde{u}(t)) + \frac{1}{2} \int_0^t \|\nabla u(\tau)\|^2_{L^2(\Omega)} \, d\tau. \) Applying Gronwall's lemma to the last inequality, we get
\[
\tilde{E}(\tilde{u}(t)) \leq c_3(R)e^{c_3(R)t^2} \left( \|w_0 - \tilde{w}_0\|^2_{H^1(\Omega)} + \|w_1 - \tilde{w}_1\|^2_{H^{-1}(\Omega)} \right), \quad \forall t \geq 0.
\]
By (2.1), (2.3), (2.4) and (2.7), it follows that
\[
|E(u(t), u(t))| \leq c_4(R) \left( \|u(t)\|_{L^2(\Omega)} + \|\nabla \tilde{u}(t)\|_{L^2(\Omega)} \right) \leq c_5(R)e^{c_5(R)t^2} \left( \|w_0 - \tilde{w}_0\|_{H^1(\Omega)} + \|w_1 - \tilde{w}_1\|_{H^{-1}(\Omega)} \right), \quad \forall t \geq 0.
\]
Taking into account (2.7) and the last inequality in (2.6), we obtain
\[
\|\nabla u(t)\|^2_{L^2(\Omega)} \leq c_6(R)(1 + t)e^{c_6(R)t^2} \left( \|w_0 - \tilde{w}_0\|_{H^1(\Omega)} + \|w_1 - \tilde{w}_1\|_{H^{-1}(\Omega)} \right), \quad \forall t \geq 0.
\]
Now, from (2.5), we have
\[
\|u(t)\|_{H^{-1}(\Omega)} \leq \|\nabla u(t)\|_{L^2(\Omega)} + \|\nabla \tilde{u}(t)\|_{L^2(\Omega)} + \|\Sigma(w(t)) - \Sigma(\tilde{w}(t))\|_{H^{-1}(\Omega)} +
\]
\[
+ \int_0^t \|f(w(\tau)) - f(\tilde{w}(\tau))\|_{H^{-1}(\Omega)} \, d\tau + \|\Sigma(w_0) - \Sigma(\tilde{w}_0)\|_{H^{-1}(\Omega)} +
\]
\[
+ \|\nabla (w_0 - \tilde{w}_0)\|_{L^2(\Omega)} + \|w_1 - \tilde{w}_1\|_{H^{-1}(\Omega)},
\]
which due to the above inequalities gives
\[
\|u(t)\|^2_{H^{-1}(\Omega)} \leq c_7(R)(1 + t)e^{c_7(R)t^2} \left( \|w_0 - \tilde{w}_0\|_{H^1(\Omega)} + \|w_1 - \tilde{w}_1\|_{H^{-1}(\Omega)} \right), \quad \forall t \geq 0.
\]

Thus by Theorem 2.1 and Theorem 2.2, it follows that by the formula \( S(t)(w_0, w_1) = (w(t), w_1(t)) \), problem (1.1)-(1.3) generates a weakly continuous (in the sense, if \( \varphi_n \to \varphi \) strongly then \( S(t)\varphi_n \to S(t)\varphi \) weakly) semigroup \( \{S(t)\}_{t \geq 0} \) in \( \mathcal{H} \), where \( w(t, \cdot) \) is a weak solution of (1.1)-(1.3), determined by Theorem 2.1, with initial data \( (w_0, w_1) \). To show the strong continuity of \( \{S(t)\}_{t \geq 0} \) we firstly prove the following lemma:

**Lemma 2.1.** Let \( \varphi \in C(R) \) and \( |\varphi(x)| \leq c(1 + |x|^r) \) for every \( x \in R \) and some \( r \geq 1 \). If \( v_n \to v \) strongly in \( L_q(\Omega) \) for \( q \geq r \), then \( \varphi(v_n) \to \varphi(v) \) strongly in \( L_\varphi(\Omega) \).
Proof. By the assumption of the lemma, there exists a subsequence \( \{ v_{n_k} \} \) such that \( v_{n_k} \to v \) a.e. in \( \Omega \). Then by Egorov’s theorem, for any \( \varepsilon > 0 \) there exists a subset \( A_\varepsilon \subset \Omega \) such that \( \text{mes}(A_\varepsilon) < \varepsilon \) and \( v_{n_k} \to v \) uniformly in \( \Omega \setminus A_\varepsilon \). Hence for large enough \( k \)
\[
|v_{n_k}(x)| \leq 1 + |v(x)| \quad \text{in} \ \Omega \setminus A_\varepsilon
\]
and consequently
\[
|\varphi(v_{n_k}(x))| \leq c_1(1 + |v(x)|^r) \quad \text{in} \ \Omega \setminus A_\varepsilon.
\]
Applying Lebesgue’s theorem we get
\[
\lim_{k \to \infty} \|\varphi(v_{n_k}) - \varphi(v)\|_{L^q(\Omega \setminus A_\varepsilon)} = 0. \tag{2.8}
\]
On the other hand since we have
\[
\lim_{k \to \infty} \|v_{n_k}\|_{L^q(A_\varepsilon)} = \|v\|_{L^q(A_\varepsilon)},
\]
the inequality
\[
\limsup_{k \to \infty} \|\varphi(v_{n_k})\|_{L^q(A_\varepsilon)}^q < c_3(\varepsilon + \|v\|_{L^q(A_\varepsilon)})
\]
is satisfied. The last inequality together with (2.8) implies that
\[
\limsup_{k \to \infty} \|\varphi(v_{n_k}) - \varphi(v)\|_{L^q(\Omega)}^q \leq c_4 \lim_{\varepsilon \to 0} (\varepsilon + \|v\|_{L^q(A_\varepsilon)}) = 0.
\]

\( \square \)

**Theorem 2.3.** Under conditions (2.1)-(2.3) the semigroup \( \{ S(t) \}_{t \geq 0} \) is strongly continuous in \( H \).

Proof. Let \( (w_{0n}, w_{1n}) \to (w_0, w_1) \) strongly in \( H \). Denoting \( (w_n(t), w_{tn}(t)) = S(t)(w_{0n}, w_{1n}), (w(t), w_t(t)) = S(t)(w_0, w_1) \) and \( u_n(t) = w_n(t) - w(t) \), by (1.1) we have

\[
u_{tnt} - \Delta u_{nt} + \sigma(w_n)w_{nt} - \sigma(w)w_t - \Delta u_n + f(w_n(\tau)) - f(w(t)) = 0.
\]
Since, by Theorem 2.1, every term of the above equation belongs to \( L_2(0, T; H^{-1}(\Omega)) \), testing it by \( u_{nt} \), we obtain

\[
E(u_n(t)) \leq E(u_n(0)) + c \|\sigma(w_n) - \sigma(w)\|_{C([0, T]; L_2^2(\Omega))}^2 + c \int_0^t E(u_n(s))ds, \ \forall t \in [0, T].
\]

Applying Gronwall’s lemma we have

\[
E(u_n(T)) \leq \left( E(u_n(0)) + c \|\sigma(w_n) - \sigma(w)\|_{C([0, T]; L_2^2(\Omega))}^2 \right) e^{cT}, \ \forall T \geq 0. \tag{2.9}
\]
By Theorem 2.2, it follows that

\[
\lim_{n \to \infty} \|w_n - w\|_{C([0, T]; L_2(\Omega))} = 0.
\]
Now applying Lemma 2.1 it is easy to see that

\[
\lim_{n \to \infty} \|\sigma(w_n) - \sigma(w)\|_{C([0, T]; L_2^2(\Omega))} = 0,
\]
which together with (2.9) yields that \( S(T)(w_{0n}, w_{1n}) \to S(T)(w_0, w_1) \) strongly in \( H \), for every \( T \geq 0 \). \( \square \)
Now let us recall the definition of a global attractor.

**Definition (17).** Let \( \{V(t)\}_{t \geq 0} \) be a semigroup on a metric space \((X, d)\). A compact set \( A \subset X \) is called a global attractor for the semigroup \( \{V(t)\}_{t \geq 0} \) iff

- \( A \) is invariant, i.e. \( V(t)A = A \), \( \forall t \geq 0 \);
- \( \lim_{t \to \infty} \sup_{v \in B} \inf_{w \in A} d(V(t)v, u) = 0 \) for each bounded set \( B \subset X \).

Our main result is as follows:

**Theorem 2.4.** Under the conditions (2.1)-(2.3), the semigroup \( \{S(t)\}_{t \geq 0} \) generated by the problem (1.1)-(1.3) possesses a global attractor \( A \) in \( \mathcal{H} \), which is also a global attractor in \( H_1 := (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega) \).

**Remark 2.1.** We note that if the condition (2.3) is replaced by

\[ \sigma \in C^1(R), \quad \sigma(s) \geq 0, \quad |\sigma'(s)| \leq c(1 + |s|^p), \quad 0 \leq p < 4, \quad \forall s \in R, \]

then using the methods of [15], [19] and [21] one can prove Theorem 2.4. If we assume

\[ \sigma \in C^1(R), \quad \sigma(s) \geq 0, \quad |\sigma'(s)| \leq c(1 + |s|), \quad \forall s \in R, \]

instead of (2.3), then the method of [15] can be applied to (1.1)-(1.3). In this case, as in [20], one can show that a global attractor \( A \) attracts every bounded subset of \( \mathcal{H} \) in the topology of \( H_1(\Omega) \times H^1_0(\Omega) \).

**Remark 2.2.** We also note that problem (1.1)-(1.3), in 3-D case, without the strong damping \(-\Delta w_t\) was considered in [11] and [10]. In this case, when \( \sigma(\cdot) \) is not globally bounded, the existence of a global attractor in the strong topology of \( \mathcal{H} \) and the regularity of the weak attractor remain open (see [11] and [10] for details).

3. **Existence of the Global Attractor in \( \mathcal{H} \)**

We start with the following asymptotic compactness lemma:

**Lemma 3.1.** Let conditions (2.1)-(2.3) hold and \( B \) be a bounded subset of \( \mathcal{H} \). Then every sequence of the form \( \{S(t_n)\phi_n\}_{n=1}^{\infty}, \{\phi_n\}_{n=1}^{\infty} \subset B, \ t_n \to \infty \), has a convergent subsequence in \( \mathcal{H} \).

**Proof.** By (2.4), we have

\[
\begin{cases}
\sup_{t \geq 0} \sup_{\phi \in B} \|S(t)\phi\|_{\mathcal{H}} < \infty, \\
\sup_{\phi \in B_0} \int_{t_0}^{\infty} \|PS(t)\phi\|^2_{H^1_0(\Omega)} \, dt < \infty,
\end{cases}
\]  

(3.1)

where \( P : \mathcal{H} \to L_2(\Omega) \) is a projection map, i.e. \( P\phi = \phi_2 \), for every \( \phi = (\phi_1, \phi_2) \in \mathcal{H} \). So for any \( T_0 \geq 1 \) there exists a subsequence \( \{n_k\}_{k=1}^{\infty} \) such that \( t_{n_k} \geq T_0 \) and

\[
\begin{cases}
w_k \to w \quad \text{weakly star in } L_\infty(0, \infty; H^1_0(\Omega)), \\
w_{kt} \to w_t \quad \text{weakly in } L_2(0, \infty; H^1_0(\Omega)),
\end{cases}
\]

(3.2)

for some \( w \in L_\infty(0, \infty; H^1_0(\Omega)) \cap W^{1,\infty}(0, \infty; L_2(\Omega)) \cap W^{1,2}_{loc}(0, \infty; H^1_0(\Omega)) \), where \( (w_k(t), w_{kt}(t)) = S(t + t_{n_k} - T_0)\phi_{n_k} \). Now multiplying the equality

\[
(w_k - w_m)_{tt} - \Delta(w_{kt} - w_{mt}) + \sigma(w_k)w_{kt} - \sigma(w_m)w_{mt} - \Delta(w_k - w_m) + f(w_k) - f(w_m) = 0
\]
by \((w_{kt} - w_{mt} + \frac{\lambda_1}{2}(w_k - w_m))\) and integrating over \((s, T) \times \Omega\), we obtain

\[
\frac{1}{2}E(w_k(T) - w_m(T)) + \lambda_1 \int_s^T E(w_k(t) - w_m(t))dt + \\
+ \int_s^T \langle \sigma(w_k(t))w_{kt}(t) - \sigma(w_m(t))w_{mt}(t), w_{kt}(t) - w_{mt}(t) \rangle dt + \\
+ \frac{\lambda_1}{2} \langle \hat{\Sigma}(w_k(T)) + \hat{\Sigma}(w_m(T)), 1 \rangle - \frac{\lambda_1}{2} \int_s^T \langle \sigma(w_k(t))w_{kt}(t), w_m(t) \rangle dt \\
- \frac{\lambda_1}{2} \int_s^T \langle \sigma(w_m(t))w_{mt}(t), w_k(t) \rangle dt + \langle F(w_k(T)) + F(w_m(T)), 1 \rangle - \\
- \int_s^T \langle f(w_k(t)), w_{mt}(t) \rangle dt - \int_s^T \langle f(w_m(t)), w_{kt}(t) \rangle dt + \\
+ \frac{\lambda_1}{2} \int_s^T \langle f(w_k(t)) - f(w_m(t)), w_k(t) - w_m(t) \rangle dt \leq \\
\leq \frac{3}{2} + \frac{\lambda_1}{2}E(w_k(s) - w_m(s)) + \frac{\lambda_1}{2} \langle \hat{\Sigma}(w_k(s)) + \hat{\Sigma}(w_m(s)), 1 \rangle + \\
+ \langle F(w_k(s)) + F(w_m(s)), 1 \rangle, \quad 0 \leq s \leq T,
\]

where \(\hat{\Sigma}(w) = \int_0^w \sigma(s)ds\). Integrating the last inequality with respect to \(s\) from 0 to \(T\) we find

\[
\frac{T}{2}E(w_k(T) - w_m(T)) + \lambda_1 \int_0^T sE(w_k(s) - w_m(s))ds + \\
+ \int_0^T s \langle \sigma(w_k(s))w_{kt}(s) - \sigma(w_m(s))w_{mt}(s), w_{kt}(s) - w_{mt}(s) \rangle ds + \\
+ \frac{\lambda_1}{2}T \langle \hat{\Sigma}(w_k(T)) + \hat{\Sigma}(w_m(T)), 1 \rangle - \frac{\lambda_1}{2} \int_0^T s \langle \sigma(w_k(s))w_{kt}(s), w_m(s) \rangle ds \\
- \frac{\lambda_1}{2} \int_0^T s \langle \sigma(w_m(s))w_{mt}(s), w_k(s) \rangle ds + T \langle F(w_k(T)) + F(w_m(T)), 1 \rangle - \\
- \int_0^T s \langle f(w_k(s)), w_{mt}(s) \rangle ds - \int_0^T s \langle f(w_m(s)), w_{kt}(s) \rangle ds + \\
+ \frac{\lambda_1}{2} \int_0^T s \langle f(w_k(s)) - f(w_m(s)), w_k(s) - w_m(s) \rangle dt \leq
\]
implies that applying Lemma 2.1 it yields that by which we obtain

\[
\begin{align*}
&\leq \left(\frac{3}{2} + \lambda_1\right) \int_0^T E(w_k(s) - w_m(s)) ds + \int_0^T \left( F(w_k(s)) + \frac{\lambda_1}{2} \Sigma(w_k(s)), 1 \right) ds + \\
&\quad + \int_0^T \left( F(w_m(s)) + \frac{\lambda_1}{2} \Sigma(w_m(s)), 1 \right) ds, \quad \forall T \geq 0. \quad (3.3)
\end{align*}
\]

By (3.1), it follows that

\[
\left(\frac{3}{2} + \lambda_1\right) \int_0^T E(w_k(s) - w_m(s)) ds \leq c_1 + \\
\quad + \frac{\lambda_1}{2} \int_0^T s E(w_k(s) - w_m(s)) ds, \quad \forall T \geq \frac{3 + 2\lambda_1}{\lambda_1}. \quad (3.4)
\]

Since for every \(\varepsilon > 0\) the embedding \(H^1(\Omega) \subset H^{1-\varepsilon}(\Omega)\) is compact (see for example [12, Theorem 16.1]), applying [18, Corollary 1] to (3.2), we have

\[
w_k \rightarrow w \text{ strongly in } C([0, T]; H^{1-\varepsilon}(\Omega)).
\]

Applying Lemma 2.1 it yields that

\[
\begin{align*}
\{ \begin{array}{l}
\sigma(w_k) \rightarrow \sigma(w) \text{ strongly in } C([0, T]; L_{2-\varepsilon}^2(\Omega)), \\
\sigma^{\frac{1}{2}}(w_k) \rightarrow \sigma^{\frac{1}{2}}(w) \text{ strongly in } C([0, T]; L_{3-\varepsilon}^2(\Omega)),
\end{array} \}
\end{align*}
\]

for small enough \(\varepsilon > 0\). The last approximation together with (2.3) and (3.2) implies that

\[
\begin{align*}
\{ \begin{array}{l}
\sigma(w_k)w_{kt} \rightarrow \sigma(w)w_t \text{ weakly in } L_2([0, T]; L_2^2(\Omega)), \\
\sigma^{\frac{1}{2}}(w_k)w_{kt} \rightarrow \sigma^{\frac{1}{2}}(w)w_t \text{ weakly in } L_2([0, T]; L_2^2(\Omega)),
\end{array} \}
\end{align*}
\]

by which we obtain

\[
\begin{align*}
\lim \inf_{m \rightarrow \infty} \lim \inf_{k \rightarrow \infty} \int_0^T s \left( \sigma(w_k(s))w_{kt}(s) - \sigma(w_m(s))w_{mt}(s), w_{kt}(s) - w_{mt}(s) \right) ds = \\
= \lim \inf_{k \rightarrow \infty} \int_0^T s \left\| \sigma^{\frac{1}{2}}(w_k(s))w_{kt}(s) \right\|_{L_2^2(\Omega)}^2 ds + \lim \inf_{m \rightarrow \infty} \int_0^T s \left\| \sigma^{\frac{1}{2}}(w_m(s))w_{mt}(s) \right\|_{L_2^2(\Omega)}^2 ds - \\
\quad - 2 \int_0^T s \left\| \sigma^{\frac{1}{2}}(w(s))w_t(s) \right\|_{L_2^2(\Omega)}^2 ds \geq 0, \quad (3.5)
\end{align*}
\]

\[
\begin{align*}
\lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \int_0^T s \left( \sigma(w_k(s))w_{kt}(s), w_m(s) \right) ds = \int_0^T s \left( \sigma(w(s))w_t(s), w(s) \right) ds = \\
= T \int_0^T \left\langle \tilde{\Sigma}(w(s)), 1 \right\rangle ds - \int_0^T \left\langle \tilde{\Sigma}(w(s)), 1 \right\rangle ds \quad (3.6)
\end{align*}
\]
By the same way, we have
\[
\lim_{m \to \infty} \lim_{k \to \infty} \int_0^T s \langle \sigma(w_m(s))w_{mt}(s), w_k(s) \rangle ds = \int_0^T s \langle \sigma(w(s))w_t(s), w(s) \rangle ds =
\]
\[
= T \int_0^T \langle \bar{\Sigma}(w(s)), 1 \rangle ds - \int_0^T \langle \bar{\Sigma}(w(s)), 1 \rangle ds
\tag{3.7}
\]
Also applying Fatou's lemma and using (2.1), (2.2), (2.3), (3.2), we have
\[
\begin{align*}
\liminf_{k \to \infty} \langle \bar{\Sigma}(w_k(T)), 1 \rangle & \geq \langle \bar{\Sigma}(w(T)), 1 \rangle, \\
\liminf_{k \to \infty} \langle F(w_k(T)), 1 \rangle & \geq \langle F(w(T)), 1 \rangle, \\
\liminf_{k \to \infty} \int_0^T s \langle f(w_k(s)), w_k(s) \rangle ds & \geq \int_0^T \langle f(w(s)), w(s) \rangle ds.
\end{align*}
\tag{3.8}
\]
Taking into account (3.4)-(3.8) in (3.3), we obtain
\[
\frac{T}{2} \liminf_{m \to \infty} \liminf_{k \to \infty} E(w_k(T) - w_m(T)) + \frac{\lambda_1}{2} \liminf_{m \to \infty} \liminf_{k \to \infty} \int_0^T s E(w_k(s) - w_m(s)) ds \leq c_1 +
\]
\[
+ 2 \liminf_{k \to \infty} \int_0^T \left( F(w_k(s)) + \frac{\lambda_1}{2} \bar{\Sigma}(w_k(s)) - F(w(s)) - \frac{\lambda_1}{2} \bar{\Sigma}(w(s)) \right) ds,
\tag{3.9}
\]
for \( T \geq \frac{3 + 2\lambda_1}{\lambda_1} \). Now let us estimate the right hand side of (3.9). By (2.1), (3.1), (3.2), we find that
\[
\int_0^T |(F(w_m(s)) - F(w(s)), 1)| ds \leq c_2 \int_0^T \|w_m(s) - w(s)\|_{\mathcal{H}_1^1(\Omega)} ds \leq c_3 + c_4(\varepsilon) \log(T) +
\]
\[
+ \varepsilon \int_0^T s \|w_m(s) - w(s)\|_{\mathcal{H}_1^1(\Omega)}^2 ds \leq c_3 + c_4(\varepsilon) \log(T) +
\]
\[
+ \varepsilon \liminf_{k \to \infty} \int_0^T s \|w_m(s) - w_k(s)\|_{\mathcal{H}_1^1(\Omega)}^2 ds, \quad \forall T \geq 1, \ \forall \varepsilon > 0.
\tag{3.10}
\]
By the same way, we have
\[
\int_0^T \left| \langle \bar{\Sigma}(w_m(s)) - \bar{\Sigma}(w(s)), 1 \rangle \right| ds \leq c_5 + c_6(\varepsilon) \log(T) +
\]
\[
+ \varepsilon \liminf_{k \to \infty} \int_0^T s \|w_m(s) - w_k(s)\|_{\mathcal{H}_1^1(\Omega)}^2 ds, \quad \forall T \geq 1, \ \forall \varepsilon > 0.
\tag{3.11}
\]
Now, choosing \( \varepsilon \) small enough, by (3.9)-(3.11), we obtain
\[
\liminf_{m \to \infty} \liminf_{k \to \infty} E(w_k(T) - w_m(T)) \leq \frac{c_7(1 + \log(T))}{T}, \quad \forall T \geq \max \left\{ 1, \frac{3 + 2\lambda_1}{\lambda_1} \right\}.
\]
Choosing $T = T_0$ in the last inequality we find
\[
\lim_{n \to \infty} \lim_{m \to \infty} \| S(t_n) \varphi_n - S(t_m) \varphi_m \|_{\mathcal{H}} \leq c_8 \sqrt{\frac{(1 + \log(T_0))}{T_0}},
\]
and passing to the limit as $T_0 \to \infty$ we have
\[
\lim_{n \to \infty} \lim_{m \to \infty} \| S(t_n) \varphi_n - S(t_m) \varphi_m \|_{\mathcal{H}} = 0.
\]
Similarly one can show that
\[
\lim_{k \to \infty} \liminf_{m \to \infty} \| S(t_{n_k}) \varphi_{n_k} - S(t_{n_m}) \varphi_{n_m} \|_{\mathcal{H}} = 0,
\]
for every subsequence $\{n_k\}_{k=1}^{\infty}$. Now if the sequence $\{S(t_n) \varphi_n\}_{n=1}^{\infty}$ has no convergent subsequence in $\mathcal{H}$, then there exist $\varepsilon_0 > 0$ and a subsequence $\{n_k\}_{k=1}^{\infty}$, such that
\[
\| S(t_{n_k}) \varphi_{n_k} - S(t_{n_m}) \varphi_{n_m} \|_{\mathcal{H}} \geq \varepsilon_0, \quad k \neq m.
\]
The last inequality contradicts (3.12).

Now since, by (2.4), the problem (1.1)-(1.3) has a strict Lyapunov function $L(w(t)) := E(w(t)) + \langle F(w(t)), 1 \rangle - \langle g, w(t) \rangle$, according to [4, Corollary 2.29] we have the following theorem:

**Theorem 3.1.** Under conditions (2.1)-(2.3), the semigroup $\{S(t)\}_{t \geq 0}$ possesses a global attractor $A_{\mathcal{H}}$ in $\mathcal{H}$.

### 4. Existence of the global attractor in $\mathcal{H}_1$

To prove the existence of a global attractor in $\mathcal{H}_1$ we need the following lemmas:

**Lemma 4.1.** Let conditions (2.1)-(2.3) hold and $B$ be a bounded subset of $\mathcal{H}_1$. Then
\[
\sup_{t \geq 0} \sup_{\varphi \in B} \| S(t) \varphi \|_{\mathcal{H}_1} < \infty. \tag{4.1}
\]

**Proof.** We use the formal estimates which can be justified by Galerkin’s approximations. Multiplying both sides of (1.1) by $-\Delta w_t$ and integrating over $\Omega$, we obtain
\[
\frac{d}{dt} \left( \frac{1}{2} \| \nabla w_t(t) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \Delta w(t) \|_{L^2(\Omega)}^2 + \langle g, \Delta w(t) \rangle \right) + \| \Delta w_t(t) \|_{L^2(\Omega)}^2 \leq \| f(w(t)) \|_{L^2(\Omega)}^2 + \| \sigma(w(t)) \|_{L^2(\Omega)}^2 \| w_t(t) \|_{L^2(\Omega)}^2, \quad \forall t \geq 0. \tag{4.2}
\]
By (2.1) and (2.3), we have
\[
\| f(w(t)) \|_{L^2(\Omega)}^2 + \| \sigma(w(t)) \|_{L^2(\Omega)}^2 \| w_t(t) \|_{L^2(\Omega)}^2 \leq c_1 \left( 1 + \| w(t) \|_{L^{10}(\Omega)}^{10} + \| w(t) \|_{L^{4}(\Omega)}^2 \right) + c_2 \| w(t) \|_{L^{10}(\Omega)}^5 \| w(t) \|_{L^2(\Omega)}^2, \quad \forall t \geq 0. \tag{4.3}
\]
On the other hand, by the embedding and interpolation theorems, we find
\[
\| \varphi \|_{L^{10}(\Omega)} \leq c_2 \| \varphi \|_{H^{\frac{1}{2}}(\Omega)} \leq c_3 \| \varphi \|_{H^1(\Omega)}^{\frac{1}{2}} \| \varphi \|_{H^{1}(\Omega)}^{\frac{1}{2}}, \quad \forall \varphi \in H^2(\Omega). \tag{4.4}
\]
Taking into account (2.4), (4.3) and (4.4) in (4.2) and applying Gronwall’s lemma, we obtain
\[
\| (w(t), w_t(t)) \|_{\mathcal{H}_1} \leq C(t, r)(1 + \| (w_0, w_1) \|_{\mathcal{H}_1}), \quad \forall t \geq 0, \tag{4.5}
\]
where $C : R_+ \times R_+ \to R_+$ is a nondecreasing function with respect to each variable and $r = \sup_{\varphi \in B} \| \varphi \|_H$. Since the embedding $\mathcal{H}_1 \subset \mathcal{H}$ is compact, by (4.5), it follows that the set $\bigcup_{0 \leq t \leq T} S(t)B$ is a relatively compact subset of $\mathcal{H}$, for every $T > 0$. This together with Lemma 3.1 implies the relative compactness of $\bigcup_{t \geq 0} S(t)B$ in $\mathcal{H}$. Now using this fact let us estimate $\| w(t) \|_{L^{10}(\Omega)}$:

$$
\| w(t) \|_{L^{10}(\Omega)}^{10} \leq m^{10} \text{mes}(\Omega) + \int_{\{x : x \in \Omega, \ |w(t,x)| > m\}} |w(t,x)|^{10} \, dx \leq
$$

$$
\leq m^{10} \text{mes}(\Omega) + \left( \int_{\{x : x \in \Omega, \ |w(t,x)| > m\}} |w(t,x)|^6 \, dx \right)^{\frac{5}{3}} \|w(t)\|_{L^2(\Omega)}^8 \leq
$$

$$
\leq m^{10} \text{mes}(\Omega) + c_4 \left( \int_{\{x : x \in \Omega, \ |w(t,x)| > m\}} |w(t,x)|^6 \, dx \right)^{\frac{1}{3}} \|w(t)\|_{H^2(\Omega)}^3 \|w(t)\|_{H^1(\Omega)}^6 .
$$

So for any $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$
\| w(t) \|_{L^{10}(\Omega)} \leq \varepsilon \| \Delta w(t) \|_{L^2(\Omega)} + c_\varepsilon, \forall t \geq 0,
$$

which together with (4.2)-(4.4) yields

$$
\frac{d}{dt} \left( \frac{1}{2} \| \nabla w(t) \|_{L^2(\Omega)}^2 + \frac{1}{2} \| \Delta w(t) \|_{L^2(\Omega)}^2 + (g, \Delta w(t)) \right) + \frac{1}{4} \| \Delta w(t) \|_{L^2(\Omega)}^2 \leq
$$

$$
\leq c_5 \| \nabla w(t) \|_{L^2(\Omega)}^2 \| \Delta w(t) \|_{L^2(\Omega)}^2 + \varepsilon \| \Delta w(t) \|_{L^2(\Omega)}^2 + \tilde{c}_\varepsilon + c_5, \forall t \geq 0.
$$

Now multiplying both sides of (1.1) by $-\mu \Delta w$ ($\mu \in (0,1)$) and integrating over $\Omega$, we obtain

$$
\frac{d}{dt} \left( \frac{1}{2} \mu \| \Delta w(t) \|_{L^2(\Omega)}^2 + \mu \langle \nabla w(t), \nabla w(t) \rangle \right) + \mu \| \Delta w(t) \|_{L^2(\Omega)}^2 \leq
$$

$$
\leq \mu \| g \|_{L^2(\Omega)} \| \Delta w(t) \|_{L^2(\Omega)} + \mu \| \nabla w(t) \|_{L^2(\Omega)}^2 + \mu \| \sigma(w(t)) w(t) \|_{L^2(\Omega)} \| \Delta w(t) \|_{L^2(\Omega)}
$$

$$
+ \mu \| f(w(t)) \|_{L^2(\Omega)} \| \Delta w(t) \|_{L^2(\Omega)}, \forall t \geq 0.
$$

Taking into account the relative compactness of $\bigcup_{t \geq 0} S(t)B$, similar to the argument done above, we can say that for any $\varepsilon > 0$ there exists $\tilde{c}_\varepsilon > 0$ such that

$$
\| f(w(t)) \|_{L^2(\Omega)}^2 + \| \sigma(w(t)) w(t) \|_{L^2(\Omega)}^2 \leq \varepsilon \left( \| \Delta w(t) \|_{L^2(\Omega)}^2 + \| \Delta w(t) \|_{L^2(\Omega)}^2 \right)
$$

$$
+ \tilde{c}_\varepsilon \| \Delta w(t) \|_{L^2(\Omega)}^2 \| \nabla w(t) \|_{L^2(\Omega)}^2 + \tilde{c}_\varepsilon, \forall t \geq 0.
$$

By the last three inequalities we have

$$
\frac{d}{dt} \left( \frac{1}{2} \| \nabla w(t) \|_{L^2(\Omega)}^2 + \frac{1}{2} (1 + \mu) \| \Delta w(t) \|_{L^2(\Omega)}^2 + \mu \langle \nabla w(t), \nabla w(t) \rangle + (g, \Delta w(t)) \right)
$$

$$
+ \left( \frac{1}{4} - \mu \varepsilon - \varepsilon \right) \| \Delta w(t) \|_{L^2(\Omega)}^2 + \left( \frac{1}{4} \mu - 2 \varepsilon \right) \| \Delta w(t) \|_{L^2(\Omega)}^2 \leq
$$

$$
\leq (c_5 + \tilde{c}_\varepsilon) \| \Delta w(t) \|_{L^2(\Omega)}^2 \| \nabla w(t) \|_{L^2(\Omega)}^2 + c_6 + \tilde{c}_\varepsilon + \tilde{c}_\varepsilon, \forall t \geq 0.
Choosing $\mu$ small enough and $\varepsilon \in (0, \frac{1}{8}\mu)$, we obtain
\[
\frac{d}{dt} \Phi(t) + c_7 \Phi(t) \leq c_8 \|\nabla u(t)\|_{L_2(\Omega)}^2 \Phi(t) + c_8 (1 + \|\nabla u(t)\|_{L_2(\Omega)}^2), \quad \forall t \geq 0,
\]
where $\Phi(t) = \frac{1}{2} \|\nabla u(t)\|_{L_2(\Omega)}^2 + \frac{1}{2} (1 + \mu) \|\Delta u(t)\|_{L_2(\Omega)}^2 + \mu \langle \nabla u(t), \nabla w(t) \rangle + (g, \Delta w(t))$. Multiplying both sides of the last inequality by $\int_{0}^{\tau} (\varepsilon - \varepsilon_0 \|w(t)\|_{L_2(\Omega)}) dt$, integrating over $[0, T]$ and multiplying both sides of obtained inequality by $e^{-\int_{0}^{T} (\varepsilon - \varepsilon_0 \|w(t)\|_{L_2(\Omega)}) dt}$, we find
\[
\Phi(T) \leq \Phi(0) e^{-\int_{0}^{T} (\varepsilon - \varepsilon_0 \|w(t)\|_{L_2(\Omega)}) dt} + c_8 \int_{0}^{T} (1 + \|\nabla u(t)\|_{L_2(\Omega)}^2) e^{-\int_{0}^{\tau} (\varepsilon - \varepsilon_0 \|w(t)\|_{L_2(\Omega)}) dt} dt, \quad \forall \tau \geq 0,
\]
which together with (4.4) yields (4.1). \hfill \Box

**Lemma 4.2.** Let conditions (2.1)-(2.3) hold and $B$ be a bounded subset of $\mathcal{H}_1$. Then every sequence of the form $\{S(t_n)\varphi_n\}_{n=1}^{\infty}$, $\{\varphi_n\}_{n=1}^{\infty} \subset B$, $t_n \to \infty$, has a convergent subsequence in $\mathcal{H}_1$.

**Proof.** Let us decompose $\{S(t)\}_{t \geq 0}$ as $S(t) = U(t) + C(t)$, where $U(t)$ is a linear semigroup generated by the problem
\[
\begin{cases}
    u_{tt} - \Delta u_t - \Delta u = 0, & \text{in } (0, \infty) \times \Omega, \\
    u = 0, & \text{on } (0, \infty) \times \partial \Omega, \\
    u(0, \cdot) = w_0, & \text{in } \Omega,
\end{cases} \tag{4.6}
\]

and $C(t)$ is a solution operator of
\[
\begin{cases}
    v_{tt} - \Delta v_t - \Delta v = g(x) - f(w) - \sigma(w) w_t, & \text{in } (0, \infty) \times \Omega, \\
    v_t = 0, & \text{on } (0, \infty) \times \partial \Omega, \\
    v(0, \cdot) = 0, & \text{in } \Omega,
\end{cases} \tag{4.7}
\]

(i.e. $(u(t), u_t(t)) = (U(t)(w_0, w_1)$ and $(v(t), v_t(t)) = C(t)(w_0, w_1)$ and $(w(t), w_t(t)) = S(t)(w_0, w_1$). Multiplying (4.6) by $\langle u_t - \frac{1}{2} \Delta u - \mu \Delta u_t - \nu \Delta u_t, \rangle$ and integrating over $\Omega$, we obtain
\[
\begin{align*}
\frac{d}{dt} \left( E(u(t)) + \frac{1}{4} \|\Delta u(t)\|_{L_2(\Omega)}^2 - \frac{1}{2} \langle u_t, \Delta u \rangle + \frac{1}{2} (\mu + \nu t) \|\nabla u(t)\|_{L_2(\Omega)}^2 \right) & + \frac{1}{2} (\mu + \nu t) \|\Delta u(t)\|_{L_2(\Omega)}^2 \langle u_t, \Delta u \rangle + \frac{1}{2} (1 - \nu) \|\Delta u(t)\|_{L_2(\Omega)}^2 \\
& + \mu \|\nabla u(t)\|_{L_2(\Omega)}^2 = 0, \quad \forall t \geq 0.
\end{align*}
\]

Choosing $(\mu, \nu) = (1, 0)$ and $(\mu, \nu) = (0, 1)$ in the last equality, we find
\[
\|U(t)\|_{\mathcal{L}(\mathcal{H}_1, \mathcal{H}_1)} \leq Me^{-\omega t}, \quad \forall t \geq 0, \tag{4.8}
\]

and
\[
\|U(t)\|_{\mathcal{L}(H^2(\Omega), H^1(\Omega)) \times L_2(\Omega), \mathcal{H}_1)} \leq \frac{M}{\sqrt{t}}, \quad \forall t > 0. \tag{4.9}
\]
Lemma 5.1. Now let us prove the following lemmas:  

\[ C(t)(w_0, w_1) = \int_0^t U(t - s)(0, \Phi_{(w_0, w_1)}(s))ds, \]  

where \( \Phi_{(w_0, w_1)}(s) = g - f(w(s)) - \sigma(w(s))w_t(s) \). By Lemma 4.1 and equation (1.1), it follows that the set of functions \( \{ \Phi_{(w_0, w_1)}(s) : (w_0, w_1) \in B \} \) is precompact in \( C([0, t]; L_2(\Omega)) \). So, from (4.9) and (4.10) we obtain that the operator \( C(t) : \mathcal{H}_1 \to \mathcal{H}_1, t \geq 0 \), is compact. Since

\[ S(t_n)\varphi_n = U(T)S(t_n - T)\varphi_n + C(T)S(t_n - T)\varphi_n \]

for \( t_n \geq T \), by (4.1), (4.8) and the compactness of \( C(t) \), we obtain that the sequence \( \{ S(t_n)\varphi_n \}_{n=1}^{\infty} \) has a finite \( \varepsilon \)-net in \( \mathcal{H}_1 \), for every \( \varepsilon > 0 \). This completes the proof. \( \square \)

Now by Lemma 4.2, similar to Theorem 3.1, we obtain the following theorem:

**Theorem 4.1.** Under conditions (2.1)-(2.3), the semigroup \( \{S(t)\}_{t \geq 0} \) possesses a global attractor \( A_{\mathcal{H}_1} \) in \( \mathcal{H}_1 \).

### 5. Regularity of the \( A_{\mathcal{H}} \)

To prove the regularity of \( A_{\mathcal{H}} \) we will use the method used in [9] and [10]. Since \( \mathcal{A}_{\mathcal{H}} \) is invariant, by [11] p. 159, for every \( (w_0, w_1) \in \mathcal{A}_{\mathcal{H}} \) there exists an invariant trajectory \( \gamma = \{ W(t) = (w(t), w_t(t)), t \in R \} \subset \mathcal{A}_{\mathcal{H}} \) such that \( W(0) = (w_0, w_1) \). By an invariant trajectory we mean a curve \( \gamma = \{ W(t), t \in R \} \) such that \( S(t)W(t) = W(t + \tau) \) for \( t \geq 0 \) and \( \tau \in R \) (see [1] p. 157)). Let us decompose \( w(t) \) as \( w(t) = w_k(t, s) + w_{\pi}(t, s) \), where

\[
\begin{align*}
\left\{ \begin{array}{ll}
v_{ktt} - \Delta v_{kt} + \sigma_k(w)v_{kt} - \Delta v_k + f_k(w) &= g(x), & \text{in } (s, \infty) \times \Omega, \\
v_k &= 0, & \text{on } (s, \infty) \times \partial \Omega, \\
v_k(s, s, \cdot) &= 0, & \text{in } \Omega 
\end{array} \right. , \tag{5.1}
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{ll}
u_{ktt} - \Delta u_{kt} + \sigma(w)w_t - \sigma_k(w)v_{kt} - \Delta u_k &= f_k(w) - f(w), & \text{in } (s, \infty) \times \Omega, \\
u_k &= 0, & \text{on } (s, \infty) \times \partial \Omega, \\
u_k(s, s, \cdot) &= w_t(s, \cdot), & \text{in } \Omega 
\end{array} \right. , \tag{5.2}
\end{align*}
\]

\[ f_k(s) = \begin{cases} 
\sigma(k), & s > k, \\
f(s), & |s| \leq k, \\
f(-k), & s < -k,
\end{cases} \quad \sigma_k(s) = \begin{cases} 
\sigma(k), & s > k, \\
\sigma(s), & |s| \leq k, \\
\sigma(-k), & s < -k 
\end{cases} \]

Now let us prove the following lemmas:

**Lemma 5.1.** Assume that conditions (2.1)-(2.3) are satisfied. Then \((v_k(t, s), v_{\pi}(t, s)) \in \mathcal{H}_1 \) and for any \( k \in \mathbb{N} \) there exists \( T_k < 0 \) such that

\[ \|v_{ktt}(t, s)\|_{H^1(\Omega)} + \|v_k(t, s)\|_{H^2(\Omega)} \leq r_0 k^{\frac{1}{2}}, \quad \forall s \leq t \leq T_k, \]  

where the positive constant \( r_0 \) is independent of \( k \) and \((w_0, w_1)\).

**Proof.** Multiplying both sides of (5.1) by \( v_{ktt} + \mu v_k \) (\( \mu \in (0, 1) \)) and integrating over \( \Omega \), we obtain

\[
\frac{d}{dt} \left( E(v_k(t, s)) + \frac{\mu}{2} \| \nabla v_k(t, s) \|_{L_2(\Omega)}^2 + \mu \langle v_{ktt}(t, s), v_k(t, s) \rangle \right) + \]
\[ \frac{1}{2} \| \nabla v_k(t, s) \|^2_{L^2(\Omega)} - \mu \| v_k(t, s) \|^2_{L^2(\Omega)} + (\mu - c_1 \mu^2) \| \nabla v_k(t, s) \|^2_{L^2(\Omega)} \leq c_2, \quad \forall t \geq s. \]

Choosing \( \mu \) small enough in the last inequality, we find
\[ \| v_k(t, s) \|^2_{L^2(\Omega)} + \| v_k(t, s) \|_{H^1(\Omega)} \leq c_3, \quad \forall t \geq s. \]  \hfill (5.4)

Multiplying both sides of (5.1) by \( v_k \), integrating over \((\tau_1, \tau_2) \times \Omega\) and taking into account (5.4), we have
\[
\int_{\tau_1}^{\tau_2} \| \nabla v_k(t, s) \|^2_{L^2(\Omega)} dt \leq c_4 + \int_{\tau_1}^{\tau_2} | \langle f_k(w(t))w(t), v_k(t, s) \rangle | dt \leq c_4 +
\]
\[+ c_5 \int_{\tau_1}^{\tau_2} \| \nabla w(t) \|^2_{L^2(\Omega)} dt, \quad \forall \tau_2 \geq \tau_1 \geq s. \]  \hfill (5.5)

On the other hand, by (2.4), we have
\[
\int_{-\infty}^{\tau_2} \| \nabla w(t) \|^2_{L^2(\Omega)} dt < \infty, \]  \hfill (5.6)

which together with (5.5) yields
\[
\int_{\tau_1}^{\tau_2} \| \nabla v_k(t, s) \|^2_{L^2(\Omega)} dt \leq c_6 (1 + (\tau_2 - \tau_1)^{\frac{1}{2}}), \quad \forall \tau_2 \geq \tau_1 \geq s. \]  \hfill (5.7)

Multiplying both sides of (5.1) by \(- \Delta v_k - \mu \Delta v_k \) (\( \mu \in (0, 1) \)), integrating over \( \Omega \) and taking into account (5.4), we have
\[
\frac{d}{dt} \left( \frac{1}{2} \| \nabla v_k(t, s) \|^2_{L^2(\Omega)} + \frac{1}{2} \| \Delta v_k(t, s) \|^2_{L^2(\Omega)} + \mu \langle \nabla v_k(t, s), \nabla v_k(t, s) \rangle \right) +
\]
\[+ \left( \frac{1}{2} - c_7 \mu \right) \| \Delta v_k(t, s) \|^2_{L^2(\Omega)} + (\mu - \mu^2) \| \Delta v_k(t, s) \|^2_{L^2(\Omega)} \leq c_7 +
\]
\[+ c_7 \| \sigma_k(w(t))v_k(t, s) \|^2_{L^2(\Omega)} + c_7 \| f_k(w(t)) \|^2_{L^2(\Omega)}, \quad \forall t \geq s. \]  \hfill (5.8)

Now let us estimate the last two terms on the right side of (5.8). By (4.4) and (5.4), we find
\[
\| \sigma_k(w(t))v_k(t, s) \|^2_{L^2(\Omega)} \leq \| \sigma_k(w(t)) \|^2_{L^2(\Omega)} \| v_k(t, s) \|^2_{L^2(\Omega)} \leq \]
\[\leq c_8 \| \sigma_k(w(t)) \|^2_{L^2(\Omega)} \| v_k(t, s) \|_{H^2(\Omega)} \| v_k(t, s) \|^2_{H^1(\Omega)} \leq \]
\[\leq c_9 \| \sigma_k(w(t)) \|^2_{L^2(\Omega)} + c_9 \| \Delta v_k(t, s) \|^2_{L^2(\Omega)} \| \nabla v_k(t, s) \|^2_{L^2(\Omega)} +
\]
\[+ \frac{1}{3c_7} \| \Delta v_k(t, s) \|^2_{L^2(\Omega)} , \quad \forall t \geq s. \]  \hfill (5.9)

Also by the definitions of \( \sigma_k(\cdot) \) and \( f_k(\cdot) \), we have
\[
\| \sigma_k(w(t)) \|^2_{L^2(\Omega)} = \int_\Omega |\sigma_k(w(t, x))|^2 dx \leq 
\]
\[\leq \int_{\{ x : x \in \Omega, |w(t, x)| \leq 2m \}} |\sigma_k(w(t, x))|^2 dx + \int_{\{ x : x \in \Omega, |w(t, x)| > 2m \}} |\sigma_k(w(t, x))|^2 dx \leq \]
Now denote $w^{(m)}(t, x) = \begin{cases} 
 w(t, x) - m, & w(t, x) > m \\
 0, & |w(t, x)| \leq m \\
 w(t, x) + m, & w(t, x) < -m 
\end{cases}$.

Since,

\[
|w(t, x)| < 2 \left| w^{(m)}(t, x) \right|, \quad \forall (t, x) \in \{(t, x) \in R \times \Omega, \ |w(t, x)| > 2m \},
\]

we have

\[
\int_{\{x:x \in \Omega, \ |w(t, x)| > 2m \}} |w(t, x)|^6 \, dx \leq \int_{\{x:x \in \Omega, \ |w(t, x)| > 2m \}} |w^{m}(t, x)|^6 \, dx \leq
\]

\[
\leq 2^6 \int_{\Omega} |w^{m}(t, x)|^6 \, dx \leq c_{14} \left\| \nabla w^{(m)}(t) \right\|^2_{L_2(\Omega)}, \quad \forall t \in R.
\]

So, by (5.8)-(5.12), it follows that

\[
\frac{d}{dt} \left( \frac{1}{2} \left\| \nabla v_k(t, s) \right\|^2_{L_2(\Omega)} + \frac{1}{2} \left\| \Delta v_k(t, s) \right\|^2_{L_2(\Omega)} + \mu \left\langle \nabla v_k(t, s), \nabla v_k(t, s) \right\rangle + \mu \left( \nabla v_k(t, s), \nabla v_k(t, s) \right) \right) +
\]

\[
+ \left( \frac{1}{6} - c_7 \mu \right) \left\| \Delta v_k(t, s) \right\|^2_{L_2(\Omega)} + \left( \mu - \mu^2 \right) \left\| \Delta v_k(t, s) \right\|^2_{L_2(\Omega)} \leq c_{15} m^{\frac{32}{3}} +
\]

\[
+ c_{15} \left\| \nabla v_k(t, s) \right\|^2_{L_2(\Omega)} + \left\| \nabla v_k(t, s) \right\|^2_{L_2(\Omega)}
\]

On the other hand, testing (1.1) by $w^{(m)}$, we obtain

\[
\frac{d}{dt} \langle w(t), w^{(m)}(t) \rangle + \left\| \nabla w^{(m)}(t) \right\|^2_{L_2(\Omega)} = \left\| \nabla w^{(m)}(t) \right\|^2_{L_2(\Omega)} + \left\langle \nabla w(t), \nabla w^{(m)}(t) \right\rangle =
\]

\[
= \left\langle g, w^{(m)}(t) \right\rangle - \left\langle \sigma(w(t))w(t), w^{(m)}(t) \right\rangle - \left\langle f(w(t)), w^{(m)}(t) \right\rangle, \quad \forall t \in R.
\]
Let us estimate each term on the right hand side of (5.14). By the definition of \( w^{(m)} \), we have

\[
\left\langle g, w^{(m)}(t) \right\rangle \leq \frac{c_{16}}{m^2} \left\| \nabla w^{(m)}(t) \right\|_{L_2(\Omega)}, \quad \forall t \in R.
\]

By (2.3), it follows that

\[
\left| \left\langle \sigma(w(t))w_t(t), w^{(m)}(t) \right\rangle \right| \leq c_{17} \left\| \nabla w_t(t) \right\|_{L_2(\Omega)} \left\| \nabla w^{(m)}(t) \right\|_{L_2(\Omega)}, \quad \forall t \in R.
\]

Also by (2.3), we obtain

\[
\left\langle f(w(t)), w^{(m)}(t) \right\rangle = -\lambda_1 \left\langle w(t), w^{(m)}(t) \right\rangle \geq -\lambda_1 \left( \int_{\{x : x \in \Omega, |w(t,x)| > m\}} |w(t,x)|^\frac{\mu}{\mu + 1} \, dx \right)^{\frac{\mu + 1}{\mu}} \left\| w^{(m)}(t) \right\|_{L_2(\Omega)} \geq -\frac{c_{18}}{m^2} \left\| \nabla w^{(m)}(t) \right\|_{L_2(\Omega)}, \quad \forall t \in R,
\]

for large enough \( m \). Taking into account the last three inequalities in (5.14), we have

\[
\frac{d}{dt} \left\langle w_t(t), w^{(m)}(t) \right\rangle + c_{19} \left\| \nabla w^{(m)}(t) \right\|_{L_2(\Omega)}^2 \leq c_{20} \left\| \nabla w_t(t) \right\|_{L_2(\Omega)}^2 + \frac{c_{20}}{m^2}, \quad \forall t \in R.
\]

(5.15) for large enough \( m \). Now multiplying (5.15) by \( \frac{c_{18}}{c_{19}} k^{\frac{\mu}{2}} \), adding to (5.13) and then choosing \( m = k^{\frac{\mu}{2}} \), we get

\[
\frac{d}{dt} \Lambda_{k,s}(t) + \tilde{c}_1 \Lambda_{k,s}(t) \leq \tilde{c}_2 \Lambda_{k,s}(t) \left\| \nabla v_{kt}(t,s) \right\|_{L_2(\Omega)}^2 + \tilde{c}_2 k^{\frac{\mu}{2}} \left\| \nabla w_t(t) \right\|_{L_2(\Omega)}^2 + \tilde{c}_2 k^{\frac{\mu}{2}} \left\| \nabla w^{(m)}(t) \right\|_{L_2(\Omega)}^2, \quad \forall t \geq s.
\]

for large enough \( k \) and small enough \( \mu \), where \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are positive constants and \( \Lambda_{k,s}(t) := \frac{1}{\mu} \left\| \nabla v_{kt}(t,s) \right\|_{L_2(\Omega)}^2 + \frac{1}{\mu} \left\| \Delta v_k(t,s) \right\|_{L_2(\Omega)}^2 + \mu \left\| \nabla v_{kt}(t,s), \nabla v_k(t,s) \right\|_{L_2(\Omega)}^2 + c_{16} k^{\frac{\mu}{2}} \left\langle w_t(t), w^{(k^{\frac{\mu}{2}})}(t) \right\rangle \). Since

\[
\left| \left\langle w_t(t), w^{(k^{\frac{\mu}{2}})}(t) \right\rangle \right| \leq \left\| w_t(t) \right\|_{L_2(\Omega)} \left( \int_{\{x : x \in \Omega, |w(t,x)| > k^{\frac{\mu}{2}}\}} |w(t,x)|^\frac{\mu}{\mu + 1} \, dx \right)^{\frac{\mu + 1}{\mu}} \leq \frac{c_3}{k^{\frac{\mu}{2}}} \left\| \nabla w_t(t) \right\|_{L_2(\Omega)}, \quad \forall t \in R,
\]

by the last differential inequality, we obtain

\[
\frac{d}{dt} \Lambda_{k,s}(t) + \tilde{c}_1 \Lambda_{k,s}(t) \leq \tilde{c}_2 \Lambda_{k,s}(t) \left\| \nabla v_{kt}(t,s) \right\|_{L_2(\Omega)}^2 +
\]

\[ + \tilde{c}_4 k^{\frac{26}{55}} + \tilde{c}_4 k^8 \| \nabla w_t(t) \|_{L^2(\Omega)}^2, \quad \forall t \geq s. \]

Multiplying both sides of the above inequality by \( e^{\int [\tilde{c}_1 - \tilde{c}_2 \| \nabla u_{kt}(\tau, s) \|_{L^2(\Omega)}^2] d\tau} \), integrating over \([s, T]\), multiplying both sides of the obtained inequality by \( e^{-\int [\tilde{c}_1 - \tilde{c}_2 \| \nabla u_{kt}(t, s) \|_{L^2(\Omega)}^2] dt} \) and taking into account (5.7), we find
\[
\Lambda_{k,s}(T) \leq \tilde{c}_5 k^{\frac{26}{55}} \left| \left\langle w_t(s), w^{(m)}(s) \right\rangle \right| + \tilde{c}_5 k^8 \int_s^T \| \nabla w_t(t) \|_{L^2(\Omega)}^2 dt, \quad \forall T \geq s,
\]
(5.16)
for large enough \( k \) and small enough \( \mu \). On the other hand, since \( A_\mathcal{H} \) is compact subset of \( \mathcal{H} \) and problem (1.1)-(1.3) admits a strict Lyapunov function, we have
\[
w_t(t) \to 0 \text{ strongly in } L^2(\Omega) \text{ as } t \to -\infty
\]
(5.17)
Thus, by (5.6) and (5.17), for any \( k \in \mathbb{N} \) there exists \( T_k = T_k(\gamma) < 0 \) such that
\[
\tilde{c}_5 k^{\frac{26}{55}} \left| \left\langle w_t(T), w^{(m)}(T) \right\rangle \right| + \tilde{c}_5 k^8 \int_{-\infty}^T \| \nabla w_t(t) \|_{L^2(\Omega)}^2 dt \leq 1, \quad \forall T \leq T_k,
\]
which together with (5.16) yields (5.3).

**Lemma 5.2.** Assume that conditions (2.1)-(2.3) are satisfied. Then there exists \( k_0 \in \mathbb{N} \) such that

\[
\lim_{s \to -\infty} \left( \| u_{k_0,t}(t,s) \|_{L^2(\Omega)} + \| u_{k_0}(t,s) \|_{H^1(\Omega)} \right) = 0, \quad \forall t \leq T_{k_0}
\]
(5.18)

**Proof.** Multiplying both sides of (5.2) by \( u_{kt} + \mu u_k \) \((\mu \in (0,1))\) and integrating over \( \Omega \), we obtain
\[
\frac{d}{dt} \left( E(u_k(t,s)) + \frac{\mu}{2} \| \nabla u_k(t,s) \|_{L^2(\Omega)}^2 + \mu \langle u_{kt}(t,s), u_k(t,s) \rangle \right) + \\
+ \| \nabla u_{kt}(t,s) \|_{L^2(\Omega)}^2 + \mu \| \nabla u_k(t,s) \|_{L^2(\Omega)}^2 - \mu \| u_{kt}(t,s) \|_{L^2(\Omega)}^2 \leq \\
\leq \| \sigma(w(t)) - \sigma_k(w(t)) \|_{L^2(\Omega)}^2 \| v_{kt}(t,s) \|_{L^\infty(\Omega)}^2 \| u_{kt}(t,s) \|_{L^6(\Omega)} + \\
+ \mu \| \sigma(w(t)) \|_{L^2(\Omega)}^2 \| u_{kt}(t,s) \|_{L^6(\Omega)} \| u_k(t,s) \|_{L^6(\Omega)} + \\
+ \mu \| \sigma(w(t)) - \sigma_k(w(t)) \|_{L^2(\Omega)} \| v_{kt}(t,s) \|_{L^6(\Omega)} \| u_k(t,s) \|_{L^6(\Omega)} + \\
+ \| f(w(t)) - f_k(w(t)) \|_{L^6(\Omega)} \| u_{kt}(t,s) \|_{L^6(\Omega)} + \\
+ \mu \| f(w(t)) - f_k(w(t)) \|_{L^6(\Omega)} \| u_k(t,s) \|_{L^6(\Omega)}, \quad \forall t \geq s.
\]
(5.19)
Taking into account (2.4) in (5.19) and choosing \( \mu \) small enough, we find
\[
\frac{d}{dt} \left( E(u_k(t,s)) + \frac{\mu}{2} \| \nabla u_k(t,s) \|_{L^2(\Omega)}^2 + \mu \langle u_{kt}(t,s), u_k(t,s) \rangle \right) + \\
+ c_1 \left( E(u_k(t,s)) + \frac{\mu}{2} \| \nabla u_k(t,s) \|_{L^2(\Omega)}^2 + \mu \langle u_{kt}(t,s), u_k(t,s) \rangle \right) \leq \\
\leq c_2 \| \sigma(w(t)) - \sigma_k(w(t)) \|_{L^2(\Omega)}^2 \| v_{kt}(t,s) \|_{L^6(\Omega)}^2 + \\
+ c_2 \| f(w(t)) - f_k(w(t)) \|_{L^6(\Omega)}^2, \quad s \leq t \leq T_k,
\]
(5.20)
where \( c_1 \) and \( c_2 \) are positive constants. Now let us estimate the terms on the right side of (5.20). Since \( H^{3+\varepsilon}(\Omega) \subset C(\Omega) \) and

\[
\|\varphi\|_{H^{3+\varepsilon}(\Omega)} \leq c_3(\varepsilon) \|\varphi\|_{H^{3}(\Omega)} \|\varphi\|_{H^{2}(\Omega)}, \quad \forall \varphi \in H^2(\Omega), \forall \varepsilon \in (0, \frac{1}{2}),
\]

from (5.3) and (5.4) it follows that

\[
\|v_k(t, s)\|_{C(\Omega)} \leq \frac{1}{2} k, \quad s \leq t \leq T_k,
\]

for large enough \( k \). The last inequality together with (2.1)-(2.4) yields that

\[
\|\sigma(w(t)) - \sigma_k(w(t))\|_{L^2(\Omega)} \leq c_4 \int_{\{x \in \Omega, |w(t, x)| > k\}} |w(t, x)|^6 \, dx \leq
\]

\[
\leq c_5 \left( \int_{\{x \in \Omega, |w(t, x)| > k\}} |w(t, x)|^6 \, dx \right)^{\frac{1}{6}} \leq
\]

\[
\leq c_5 \left( \int_{\{x \in \Omega, |u_k(t, s, x)| > |v_k(t, s, x)|\}} |w(t, x)|^6 \, dx \right)^{\frac{1}{6}} \leq
\]

\[
\leq c_6 \left( \int_{\{x \in \Omega, |u_k(t, s, x)| > |v_k(t, s, x)|\}} |u_k(t, s, x)|^6 \, dx \right)^{\frac{1}{6}} \leq c_6 \|\nabla u_k(t, s)\|_{L^2(\Omega)}^{\frac{1}{2}}, \quad s \leq t \leq T_k,
\]

and

\[
\|f(w(t)) - f_k(w(t))\|_{L^2(\Omega)} \leq c_7 \int_{\{x \in \Omega, |w(t, x)| > k\}} |w(t, x)|^6 \, dx \leq
\]

\[
\leq c_8 \left( \int_{\{x \in \Omega, |w(t, x)| > k\}} |w(t, x)|^6 \, dx \right)^{\frac{1}{6}} \times
\]

\[
\left( \int_{\{x \in \Omega, |u_k(t, s, x)| > |v_k(t, s, x)|\}} |w(t, x)|^6 \, dx \right)^{\frac{1}{6}} \leq
\]

\[
\leq c_9 \left( \int_{\{x \in \Omega, |u_k(t, s, x)| > |v_k(t, s, x)|\}} |u_k(t, s, x)|^6 \, dx \right)^{\frac{1}{6}} \times
\]

\[
\left( \int_{\{x \in \Omega, |u_k(t, s, x)| > |v_k(t, s, x)|\}} |u_k(t, s, x)|^6 \, dx \right)^{\frac{1}{6}} \leq
\]
for large enough $k$. On the other hand, since $\mathcal{A}_H$ is compact subset of $\mathcal{H}$ and $(w(t), w_1(t)) \in \mathcal{A}_H$, we have

$$\sup_{t \in \mathcal{R}} \int_{\{x : x \in \Omega, |w(t, x)| > k\}} |w(t, x)|^6 \, dx \to 0 \text{ as } k \to \infty$$

(5.23)

Thus choosing $\mu$ small enough, $k$ large enough and taking into account (5.21)-(5.23) in (5.20), we obtain

$$\frac{d}{dt} \tilde{\Lambda}_{k,s}(t) + \hat{c}_1 \tilde{\Lambda}_{k,s}(t) \leq \hat{c}_2 \|
abla u_k(t, s)\|^2_{L^2(\Omega)} \tilde{\Lambda}_{k,s}(t), \quad s \leq t \leq T_k,$$

where $\hat{c}_1$ and $\hat{c}_2$ are positive constants and $\tilde{\Lambda}_{k,s}(t) = E(u_k(t, s)) + \frac{\mu}{2} \|
abla u_k(t, s)\|^2_{L^2(\Omega)} + \mu (u_k(t, s), u_k(t, s)).$ Now multiplying both sides of the last inequality by $e^{\int_s^t \hat{c}_1 - \hat{c}_2 \|
abla u_k(t, s)\|^2_{L^2(\Omega)} \, dt}$, integrating over $[s, T_k]$ and multiplying both sides of the obtained inequality by $e^{-\int_s^T \hat{c}_1 - \hat{c}_2 \|
abla u_k(t, s)\|^2_{L^2(\Omega)} \, dt}$, we find

$$\tilde{\Lambda}_{k,s}(T) \leq \tilde{\Lambda}_{k,s}(s) e^{-\int_s^T \hat{c}_1 - \hat{c}_2 \|
abla u_k(t, s)\|^2_{L^2(\Omega)} \, dt}, \quad s \leq t \leq T_k,$$

which together with (5.7) yields $\Box$

By Lemma 5.1 and Lemma 5.2, we have $(w(T_{k_0}), w_1(T_{k_0})) \in \mathcal{H}_1$ and

$$\|w_1(T_{k_0})\|_{H^1(\Omega)} + \|w(T_{k_0})\|_{H^2(\Omega)} \leq \tilde{r}_0,$$

where $\tilde{r}_0$ is independent of $(w_0, w_1)$. Now since $(w(t, x)$ satisfies (1.1)-(1.3) on $T_{k_0}, \infty \times \Omega$, with initial data $(w(T_{k_0}), w_1(T_{k_0}))$, applying Lemma 4.1 and taking into account the last inequality, we find $(w_0, w_1) \in (H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$ and

$$\|(w_0, w_1)\|_{H^2(\Omega) \times H^1(\Omega)} \leq R_0,$$

where the positive constant $R_0$ is independent of $(w_0, w_1)$. So $\mathcal{A}_H$ is a bounded subset of $(H^2(\Omega) \cap H^1_0(\Omega)) \times H^1_0(\Omega)$ and that is why it coincides with $\mathcal{A}_{H_1}$.

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