Ferromagnetism in the Hubbard model with topological/non-topological flat bands

Hosho Katsura\textsuperscript{1,2}, Isao Maruyama\textsuperscript{3}, Akinori Tanaka\textsuperscript{4} and Hal Tasaki\textsuperscript{5}

\textsuperscript{1}Kavli Institute for Theoretical Physics, University of California - Santa Barbara, CA 93106, USA
\textsuperscript{2}Cross-Correlated Materials Research Group (CMRG), ASI, RIKEN - Wako, Saitama 351-0198, Japan
\textsuperscript{3}Graduate School of Engineering Science, Osaka University - Toyonaka, Osaka 560-8531, Japan
\textsuperscript{4}Department of General Education, Ariake National College of Technology - Omuta, Fukuoka 836-8585, Japan
\textsuperscript{5}Department of Physics, Gakushuin University - Mejiro, Toshima-ku, Tokyo 171-8588, Japan

received 29 May 2010; accepted in final form 30 August 2010
published online 24 September 2010

PACS 71.10.Fd – Lattice fermion models (Hubbard model, etc.)
PACS 71.10.-w – Theories and models of many-electron systems
PACS 05.30.Fk – Fermion systems and electron gas

Abstract – We introduce and study two classes of Hubbard models with magnetic flux or with spin-orbit coupling, which have a flat lowest band separated from other bands by a non-zero gap. We study the Chern number of the flat bands, and find that it is zero for the first class but can be non-trivial in the second. We also prove that the introduction of on-site Coulomb repulsion leads to ferromagnetism in both the classes.

Copyright © EPLA, 2010

Introduction. – Motivated by the recent discovery of the quantum spin Hall effect in band insulators with strong spin-orbit coupling (SOC) [1–3], the topological classification of non-interacting electron systems has attracted a renewed interest [4–6]. The states of the systems are characterized by the topological numbers linked to the presence or absence of gapless edge modes. In integer quantum Hall systems [7], the first Chern number is directly connected to the quantized Hall conductance [8], which is one of the most famous examples of time-reversal breaking insulators. On the other hand, in the recently found time-reversal invariant insulators, the states are classified by the $\mathbb{Z}_2$ topological number. Since a topological number remains invariant as long as the energy gap does not collapse, adiabatic transformation from the original model to a flat-band model, where all the bands are dispersionless, provides a useful tool for the classification [4–6].

The flat-band models also play an important role in a completely different context, i.e., rigorous examples of ferro- or ferrimagnetism in the Hubbard model [9–12]. In the models proposed by Mielke [10] and by Tasaki [11], the on-site Coulomb interaction leads to ferromagnetic ground states when the lowest flat band is half-filled (for a review, see [13]). A common feature of these models is that the electron hoppings are frustrated and the lowest band is spanned by moderately localized eigenstates.

Recently, similar localized states have been found in highly frustrated magnets in strong magnetic fields [14,15] and optical lattice models [16], and offer a playground for studying non-perturbative aspects of strongly correlated systems.

These two subjects have developed separately, and the effect of electron correlation in topological insulators has not been studied intensively. Here we study the Hubbard model with flat bands in the presence of magnetic flux or SOC, which bridges the two subjects. In order to define the topological number, the gap between the flat band and other bands is required. The possibility of a gapped flat band is already a non-trivial issue since in many cases the band touching occurs at some points in momentum space [17] and a uniform magnetic field destroys the flatness [18]. In this letter, we propose two classes of tight binding models (TBM) which have a flat band separated from other bands by a non-zero gap [19]. The first class is a TBM on a line graph (e.g. checkerboard lattice) with non-uniform flux. For models in this class, we find that the flat band is always non-topological, i.e., the Chern number is always zero. We can study the effect of interaction rigorously and show that the ground states are ferromagnetic when the lowest band is half-filled for both magnetic-flux and SOC cases. The second class is a TBM embedded on a thin torus with a
magnetic field perpendicular to the plane. Surprisingly, all the bands become flat if a special condition is satisfied. We calculate the topological numbers and show that the topological flat band indeed exists. We also study the effect of electron correlation and rigorously show that the ground states are ferromagnetic when the lowest Landau level (LLL) is half-filled, i.e., \( \nu = 1/\text{odd} \), without using the LLL projection [20].

**Non-topological flat band.** – We start from the first class. Let \( G = (V, E) \) be a graph (lattice), where \( V \) is the set of vertices (sites) and \( E \) the set of edges (bonds). We assume that \( G \) is twofold connected\(^1\). We define the incidence matrix \( B = (B_{ve})_{v \in V, e \in E} \) by assigning a non-zero complex number to \( B_{ve} \) when the vertex \( v \) is incident to the edge \( e \), and by setting \( B_{ve} = 0 \) otherwise (see fig. 1 for example). Define the line graph \( L(G) = (V_L, E_L) \) of \( G \) as usual, by first regarding (the mid-point of) each edge in \( E \) as a vertex in \( V_L \) (thus \( V_L = E \)), and then connecting any pair of vertices in \( V_L \) (by an edge in \( E_L \)) when the corresponding pair of edges in \( E \) share a common vertex. Figure 2(a) shows the square lattice and its line graph, the checkerboard lattice (ignore the differences in the bonds for the moment). Let us consider TBMs on \( G \) and \( L(G) \) with hopping matrices (single-particle Hamiltonians) \( T = BB^\dagger \) and \( T_L = B^\dagger B \), respectively. It is easily shown that (T1) all the eigenvalues of \( T \) and \( T_L \) are non-negative, (T2) non-zero eigenvalues of \( T \) and \( T_L \) are identical, and (T3) \( T_L \) has at least \( (|E| - |V|) \) zero-energy eigenstates\(^2\). (T1) follows if one notes that both \( T \) and \( T_L \) are positive semidefinite. (T3) is an immediate consequence of the fact that \( B \) is a \( |V| \times |E| \) matrix. To see (T2), let \( \varphi \) be an eigenvector of \( T \) with a non-zero eigenvalue \( a \), i.e., \( BB^\dagger \varphi = a \varphi \). Multiplying \( B^\dagger \) from the left, one finds \( T_L \varphi = a \varphi \) with a non-zero vector \( \tilde{\varphi} = B^\dagger \varphi \). To complete the proof we only need to repeat the same argument with \( T \) and \( T_L \) switched. For periodic systems, these zero-energy eigenstates of \( T_L \) form the lowest flat band.

Let us now show that the flat band in the TBM on \( L(G) \) may be gapped but is not topological. To be concrete we let \( G \) be the square lattice\(^3\), and consider models with a constant flux \( \phi \) per plaquette. For each edge \( \langle vv' \rangle \in E \), define \( \phi_{vv'} = -\phi_{v'v} \in \mathbb{R} \) so that \( \phi_{v_1v_2} + \phi_{v_2v_3} + \phi_{v_3v_4} + \phi_{v_4v_1} = \phi \) (mod 1) for any plaquette \( \langle v_1v_2v_3v_4 \rangle \) oriented in the counterclockwise direction. By setting \( B_{ve} = \exp[\pi i \phi_{vv'}] \) if \( e = \langle vv' \rangle \), we get a TBM on \( G \) with a uniform flux, and the corresponding TBM on the line graph \( L(G) \) with a uniform flux through the diamond plaquettes (see fig. 1(b), still ignoring the differences in the bonds)\(^4\).

It is useful for our proof to introduce interpolating TBMs with an extra parameter \( 0 \leq x \leq 1 \). Consider a disjoint decomposition \( E = E' \cup E'' \) as in fig. 1(b), and redefine \( B_{ve} \) as \( \sqrt{x} \exp[\pi i \phi_{vv'}] \) only if \( e = \langle vv' \rangle \in E'' \). We denote by \( T(x) \) and \( T_L(x) \) the corresponding hopping matrices. Note that \( T(1) \) and \( T_L(1) \) are the same as the original \( T \) and \( T_L \), respectively, and both \( T(0) \) and \( T_L(0) \) describe TBMs which decouple into local pieces. From the definition, one finds for any \( 0 \leq x \leq 1 \) that \( T(x) \geq T(0) \geq (\epsilon(\phi), \epsilon(\phi) \text{ is the lowest eigenvalue of } T(0) \) (see footnote \(^5\)). Suppose that \( \phi \) is non-integral. Since

---

\(^1\)A graph \( G \) is twofold connected if and only if one cannot divide \( G \) into disconnected graphs by removing a single vertex.

\(^2\)By \( |S| \) we denote the number of elements in a set \( S \).

\(^3\)The present proof automatically generalizes to models on other line graphs, such as the Kagomé lattice (see fig. 2(b)).

\(^4\)As for the TBM on \( G \), one easily finds \( T(x)_{vv'} = \exp[2\pi i \phi_{vv'}] \) when \( \langle vv' \rangle \in E \), which shows that there is a constant flux. The calculation of \( T_L \) is similar but slightly complicated.

\(^5\)Let \( \langle \varphi_v \rangle_{v \in V} \) be a vector on \( V \). Then, the expectation value of \( T(x) - T(0) \) for \( \langle \varphi_v \rangle \) is \( \sum_{\langle vv' \rangle \in E_j} |B_{vv'} \varphi_{v'} + B_{v'v} \varphi_v|^2 \), which implies \( T(x) - T(0) \geq 0 \), i.e., \( T(x) \geq T(0) \). Here we write \( A \geq B \) to denote that \( A - B \) is positive semidefinite. For more details, see Appendix C in [13].
an explicit calculation shows $\epsilon(\phi) > 0$ (see footnote $^6$),

\[ (T2) \text{ implies that } T_L(x) \text{ has a non-zero gap above the zero eigenvalue. Recalling that the lowest flat band of } T_L \text{ is gapless for } \phi = 0, \text{ we see that the above gap in } T_L \text{ originates from the flux.} \]

Now we investigate the Chern number defined by imposing twisted boundary conditions' as in [21, 22] of the flat band in the TBM on the line graph $L(G)$. Suppose that $\phi$ is non-integral. Since the Chern number is invariant as long as the energy gap does not close, one can evaluate it in the TBM with $T_L(x)$ for any $x$. But since a decoupled system is insensitive to a twist in the boundary conditions, the Chern number of the flat band in $T_L(0)$ is clearly zero. This proves that the Chern number of the flat band in the original model with $T_L$ is zero.

Although the flat band is non-topological in this class of models, we can rigorously study the effect of electron correlation. We present two specific examples in the following.

i) Kagomé ladder. The first example is the Hubbard model on the Kagomé ladder, the line graph of the square ladder, shown in fig. 3. The sites in the mid-chain are labeled by $e = n\delta$ with $n = 0, 1, \ldots, 2L - 1$ as shown in fig. 3. We denote vertices of the square ladder by $(e, l)$ with $l = \pm 1$. We let the incidence matrix element $B_{(e,l)}$ be 1 if $e' = e$ or $e = \nu_2$, $\epsilon^{\phi/2}$ if $e' = e + \nu_1$, and 0 otherwise. Here $\theta$ is the parameter determining the flux, and $\nu_1$ and $\nu_2$ are the vectors indicated in fig. 3. The tight-binding Hamiltonian is then given by

\[ H_{KL} = \sum_{\sigma = \uparrow, \downarrow} \sum_{e \in V_L} \sum_{l = \pm 1} B_{(e,l)} c_{e,\sigma}^\dagger c_{e',\sigma} \]  

where $e$ in the right-hand side is summed over the $2L$ sites in the mid-chain, and the $a$-operators are defined as

\[ a_{(e,l),\sigma} = \sum_{e'} B_{(e,l)} e'^\dagger c_{e',\sigma} = c_{e,\sigma} + \epsilon^{\phi/2} c_{e + \nu_1,\sigma} + c_{e - \nu_2,\sigma} \]  

We impose periodic boundary conditions in the mid-chain direction (see fig. 3). By a straightforward calculation, one finds that

\[ f_{n,\sigma}^\dagger = -\left( \epsilon^{\phi/2} + 1 \right) c_{n,\sigma}^\dagger \]

\[ + \sum_{l = \pm 1} \left( \epsilon^{\phi/2} c_{n + \nu_1,\sigma}^\dagger + c_{n + \nu_2,\sigma}^\dagger - \epsilon^{\phi/2} c_{n + i\delta,\sigma}^\dagger \right) \]

anticommuting with the $a$-operators. Therefore, the single-electron zero-energy states are given by $\Phi_0$, where $\Phi_0$ is the vacuum state. The collection of these states forms a complete basis for the flat band.

We shall consider the case where the flat band is half-filled. In the non-interacting case, the 2L-electron ground states are highly degenerate and exhibit paramagnetism. Let us add the standard on-site Coulomb repulsion

\[ H_U = U \sum_{e \in V_L} n_{e,\uparrow} n_{e,\downarrow} \]  

where $n_{e,\sigma} = c_{e,\sigma}^\dagger c_{e,\sigma}$ is the number operator at a site $e$ of the Kagomé ladder. Then the degeneracy is lifted and only the ferromagnetic states remain as the ground states, which are the zero-energy eigenstate of both $H_{KL}$ and $H_U$. To prove this claim, we introduce new fermion operators $d_{n,\sigma}^\dagger = \epsilon^{\phi/2} f_{n,\sigma}^\dagger - f_{n + (-1)^n,\sigma}^\dagger$, and follow the standard strategy [13]. The states $\Phi_0$ form another basis for the flat band. By representing a ground state $|\Psi\rangle$ in terms of the $d$-operators, we can firstly show that the zero-energy conditions for the on-site repulsion, $c_{e,\sigma}^\dagger c_{e',\sigma}^\dagger |\Psi\rangle = 0$, with sites $e' = n\delta + (-1)^n \nu_1$ forbid the double occupancy of $d$-states. Then the same conditions with sites in the mid-chain imply that the ground state must be $\left( \prod_{n=0}^{2L-1} d_{n,\sigma}^\dagger \right) \Phi_0$ and its $SU(2)$ rotations. We note that even when the flat band is less than half-filled, ferromagnetic states are ground states but are not unique.

It is also possible to consider the non-topological flat-band model associated with SOC. For the Kagomé ladder, the corresponding tight-binding Hamiltonian $H_{SOC}^{KL}$ is produced by replacing $\epsilon^{\phi/2}$ in the definition of $a$-operators with $\epsilon^{\phi/2}$ in eq. (1). This model has spin-dependent complex hoppings. Since for $\theta = \pi$, the model is reduced to the case of spin-independent hopping by a local gauge transformation, we restrict ourselves to $\theta \neq \pi$. The single-electron zero-energy states of $H_{SOC}^{KL}$ are given by $d_{n,\sigma}^\dagger \Phi_0$, where $d_{n,\sigma}^\dagger$ is defined as $d_{n,\sigma}^\dagger$ with

\[ \tilde{f}_{n,\sigma}^\dagger = -\left( \epsilon^{\phi/2} + 1 \right) c_{n,\sigma}^\dagger \]

\[ + \sum_{l = \pm 1} \left( \epsilon^{\phi/2} c_{n + \nu_1,\sigma}^\dagger + c_{n + \nu_2,\sigma}^\dagger - \epsilon^{\phi/2} c_{n + i\delta,\sigma}^\dagger \right) \]  

One has $\epsilon(\phi) = 2(1 - \cos(\pi/2))$ if $|\phi| < 1/2$. $^7$To obtain the Berry phase potential $A_{\mu}(\theta_1, \theta_2) = \left( \Psi(\theta_1, \theta_2) \partial_{\mu} \Psi(\theta_1, \theta_2) \right)$ \((\mu = 1, 2)\), where $\partial_{\mu} \equiv \frac{\partial}{\partial \theta_{\mu}}$ and $\Psi(\theta_1, \theta_2)$ is the many-body ground state of the non-interacting Hamiltonian $H(\theta_1, \theta_2)$ in which the boundary conditions in two orthogonal directions are twisted by $\theta_1$ and $\theta_2$. Then, the Chern number is defined by $C = \int d\theta_1 d\theta_2 [A_{12}(\theta_1, \theta_2) - A_{21}(\theta_1, \theta_2)]/(2\pi i)$.  

57007-p3
in place of \( f_{n,\sigma}^\dagger \). In the 2L-electron case, the ground states of the Hubbard Hamiltonian \( H = H_{\text{SOC}} + H_U \) are given by \( \left( \prod_{\alpha=1}^{2L-1} d_{n,\sigma}^\dagger \right) |\Phi_0\rangle \) with \( \sigma = \uparrow, \downarrow \). In contrast to the spin-independent case, the \( SU(2) \) spin degeneracy is lifted and there are only two ground states.

ii) Two-dimensional checkerboard lattice. The second example is the Hubbard model on the checkerboard lattice, the line graph of the square lattice whose vertex set is given by \( V = [0, L-1]^2 \cap \mathbb{Z}^2 \). For a technical reason, we take \( L \) to be an odd positive integer\(^8\). The periodic boundary conditions are imposed in both directions. We label an element of the edge set \( E \) of the square lattice by its mid-point position. Let \( \mu_1 = (1/2, 0) \) and \( \mu_2 = (0, 1/2) \). For later use we decompose \( E \) as \( E = E_1 \cup E_2 \), where \( E_i = \{ e = v + \mu_i | v \in V \} \). Let the incidence matrix element \( B_{\alpha e} \) be 1 if \( e = v \perp \mu_2 \) or \( e = v - \mu_1 \), \( e^{2(\mu_1 \mu_2)\theta} \) if \( e = v + \mu_1 \), and 0 otherwise. Here \( \theta \) is again the parameter determining the flux \( \phi = -\theta/(2\pi) \). The tight-binding Hamiltonian of this model is then given by

\[
H_{\text{CL}} = \sum_{\alpha = \uparrow, \downarrow} \sum_{\sigma = \uparrow, \downarrow} \sum_{e, e' \in V_\alpha} (T_L)_{e e'} c_{e, \sigma}^\dagger c_{e', \sigma}^\dagger,
\]

where the \( \alpha \)-operators are defined as

\[
a_{\nu, \sigma} = \sum_e B_{\nu e} c_{e, \sigma} + e^{(2e \mu_2 \theta)} c_{e - \mu_1, \sigma} + c_{e + \mu_2, \sigma}.
\]

For each \( e \in E_1 \) we define

\[
d_{e, \sigma}^\dagger = 2c_{e, \sigma}^\dagger - e^{(2e \mu_2 \theta)} \sum_{l=0}^{L-1} (-1)^l c_{e - \mu_2l, \sigma}^\dagger - \sum_{l=0}^{L-1} (-1)^l c_{e + \mu_2l, \sigma},
\]

where \( \mu = \mu_1 + \mu_2 \). These states are extended in one direction and localized in the perpendicular one as shown in fig. 4. It is easy to see that the \( d \)-operators anticommute with the \( \alpha \)-operators, and therefore the single-electron zero-energy states of \( H_{\text{CL}} \) are given by \( d_{e, \sigma}^\dagger |\Phi_0\rangle \). The collection of these states forms a complete basis for the flat band.

As in the case of the Kagomé ladder, the fully polarized states, \( \left( \prod_{e \in E_1} d_{e, \sigma}^\dagger \right) |\Phi_0\rangle \), and its \( SU(2) \) rotations are the unique ground states of the Hubbard Hamiltonian \( H = H_{\text{CL}} + H_U \) when the electron number is \( L^2 \), i.e., the flat band is half-filled. Note that these ground states are simultaneous eigenstates of both \( H_{\text{CL}} \) and \( H_U \) with zero energy. This claim can be proved by following the same strategy. Representing a ground state \( |\Psi\rangle \) in terms of the \( d \)-operators and noting that \( e \in E_1 \) supports only \( d_{e, \sigma} \), we can firstly show that the zero-energy conditions for the on-site repulsion, \( c_{e, \sigma}^\dagger c_{e, \sigma} |\Psi\rangle = 0 \), with sites \( e \in E_1 \) forbid the double occupancy of \( d \)-states. Then, examining the same conditions with sites \( e \in E_2 \) we arrive at the conclusion.

**Topological flat band.** We turn to the second class and show another construction of TBM with a flat band, which is different from the line graph construction of the first class. We consider the square-lattice TBM embedded on a torus with a magnetic field perpendicular to the plane. Such a problem is known as the Hofstadter problem [23] and has been extensively studied [24–26]. As we will show, all the bands become flat if the flux per plaquette and the number of sites along (1,1) direction satisfy certain conditions. We shall use a notation as close to those in [25] as possible. The tight-binding Hamiltonian is given by \( H_{\text{hop}} = T_x + T_y + T_z^\dagger + T_z \) with

\[
T_x = \sum_{\sigma = \uparrow, \downarrow} \sum_{m,n} \sum_{\alpha = \uparrow, \downarrow} e^{i\alpha \phi^x_{m,n}} c_{(m+1,n), \sigma}^\dagger c_{(m,n), \sigma},
\]

\[
T_y = \sum_{\alpha = \uparrow, \downarrow} \sum_{m,n} \sum_{\sigma = \uparrow, \downarrow} e^{i\phi^y_{m,n}} c_{(m+1,n), \sigma}^\dagger c_{(m,n), \sigma},
\]

where \( (m,n) \) denote the vertices of the square lattice, and \( \theta^x_{m,n} = (m+n)\pi \phi \) and \( \theta^y_{m,n} = -(m+n+1)\pi \phi \). The flux per plaquette is \( \phi = P/Q \) with mutually prime \( P \) and \( Q \). Periodic boundary conditions are imposed both in the (1,1) and (1,−1) directions. From the Bloch theorem, we can assume the single-electron state to be of the form

\[
|\Phi_{m,n}(p, p)\rangle = \sum_{m,n} \Psi_{m,n}(p, p) c_{(m+1,n), \sigma}^\dagger |\Phi_0\rangle,
\]

\[
\Psi_{m,n}(p, p) = e^{ip\pi(m+n)(m-n)} \psi_{m+n}(p, p),
\]

where \( \psi_{k+2Q}(p, p) = \psi_k(p, p), (k = 0, 1, \ldots, 2Q - 1) \). We now consider the thin-torus case where the system
Interestingly the lowest flat band in this construction can be topological. In fig. 5(c), we list the Chern numbers for several $\phi = P/Q$ computed numerically using the method of [21,22] and footnote 7.

We next show that the spatially localized state along $(1,-1)$ direction can be constructed from the solution of eq. (15). Due to the fact $q^2 - q^{-2} = 0$, we can take $\{u_l\}^{Q-1}_l=0$ to be of the form $u_l = v_l \in \mathbb{R}$ if $0 \leq l \leq Q - 1$ and zero otherwise. This solution is degenerate with the other one: $u_l = (-1)^l v_{l-Q}$ if $0 \leq l \leq 2Q - 1$ and zero otherwise. The vector $\{v_l\}^{Q-1}_l$ is normalized as $\sum_{l=0}^{Q-1} v_l^2 = 1$. From those solutions, a localized Wannier state extending from $m = n = j$ to $j + Q - 1$ can be constructed as

$$d^{\dagger}_{j,\sigma} |\Phi_0\rangle = \frac{1}{\sqrt{Q}} \sum_{l=0}^{Q-1} \chi_{j+l,k}(i)^{q+1}_l v_{l} c^{\dagger}_{j+l,k,\sigma} |\Phi_0\rangle,$$

where $c^{\dagger}_{l,\sigma} \equiv c^{\dagger}_{l+\pi/2,\sigma}$, and $\chi_{i,k} = 1$ if $i$ and $k$ have the same parity and 0 otherwise. One can easily show that $(d^{\dagger}_{j,\sigma} d_{j',\sigma'}) = \delta_{jj'} \delta_{\sigma\sigma'}$. From the Perron-Frobenious theorem, one also finds that the lowest eigenvalue for eq. (15) is twofold degenerate. This implies that the lowest energy of $H_{\text{hop}}$ is $2L$-fold degenerate. Therefore, $d$-states form a complete basis for the lowest flat bands.

Let us now study the effect of electron correlation within the Hubbard model. We define the Hubbard Hamiltonian $H$ by

$$H = H_{\text{hop}} + U \sum_{m,n} n_{m,n} \uparrow n_{m,n} \downarrow$$

with $U > 0$. If the total number of electrons $N_e = 2L$, the ferromagnetic state constructed from the lowest flat band of $H_{\text{hop}}$, $|\Psi\rangle = \prod_{j=0}^{2L-1} d^{\dagger}_{j,\sigma} |\Phi_0\rangle$, is a ground state of $H$. Since the Hamiltonian is $SU(2)$ invariant, we can construct other ground states by applying the spin-lowering operator to $|\Psi\rangle$. To show the uniqueness of these ground states, we make use of theorem due to Mielke [27]. The theorem asserts that if the single-particle density matrix constructed from $|\Psi\rangle$ as

$$\rho_{\sigma,\sigma'} = \frac{1}{N_e} \langle \Psi | c^{\dagger}_{m,n,\upsilon} c^{\dagger}_{m',n',\uparrow} |\Psi\rangle$$

is irreducible, $|\Psi\rangle$ is the unique ground state of $H$ with $N_e = 2L$ up to the trivial degeneracy from the $SU(2)$ symmetry. Here, $\sigma$ and $\sigma'$ denote $(m,n)$ and $(m',n')$, respectively. The diagonal matrix element, the electron density, is uniform and obtained as $\rho_{\sigma,\sigma} = 1/(2QL)$. To show that $\rho_{\sigma,\sigma'}$ is irreducible, it is sufficient to show that any matrix element corresponding to a pair of nearest neighbor sites is non-zero. Moreover, $\rho_{\sigma,\sigma'}$ is Hermitian, we have only to study the case of $m' - n' = m - n + 1,$

\[\text{Note that } P \text{ is odd because } P \text{ and } Q \text{ are mutually prime.}\]

\[\text{While the original theorem has been proved for the Hubbard model with real hoppings, it can be generalized to the case of complex hopping relevant to our problem.}\]
$m' + n' = m + n \pm 1$. In this case, an explicit calculation gives

$$
\rho_{\mathbf{r}, \mathbf{r}'} = \frac{q^{m+n}}{2QL} \sum_{i=0}^{Q-2} i^{\pm 1} q^{\pm (i+1)} v_i v_{i+1}.
$$

(20)

Recalling that $\{v_i\}_{i=0}^{Q-1}$ satisfy eq. (15), we find $\text{Re} \left( \sum_{i=0}^{Q-2} i^{\pm 1} q^{\pm (i+1)} v_i v_{i+1} \right) = \epsilon_1/4$, where $\epsilon_1$ is the single-particle energy of the lowest flat band. For even $Q$, $\epsilon_1 \neq 0$ can be shown by deriving the characteristic equation for $\epsilon(p)$ from eq. (15) and hence $\rho_{\mathbf{r}, \mathbf{r}'} \neq 0$. Therefore, via Mielke’s theorem, the ferromagnetic state $|\Psi\rangle$ is the unique ground state of $H$ up to the spin degeneracy. Physically speaking, the coupling between the electron orbital motion and the magnetic flux together with the Coulomb repulsion gives rise to the ferromagnetic ground states even without the Zeeman coupling between the electron spins and the magnetic field. This ground state corresponds to the quantum Hall ferromagnet for the filling factor $\nu = 1/P$ with odd $P$. So far, we have studied the square-lattice model on the thin torus. However, our argument can be generalized to other lattice geometry such as the honeycomb lattice which is relevant to the quantum Hall ferromagnetism in graphene [28].

Conclusion. – To conclude, we have shown two systematic ways to construct tight binding models which support the lowest flat band separated by a non-zero gap originating from the magnetic flux or spin-orbit coupling. In the first class, the flat band arises from the structure of the hopping matrix, which can be naturally understood in terms of the notion of line graph. We have proved that the Chern number of the flat band is zero in this class. In the second class, flat bands are hidden in a special case of the Hofstadter problem, in which all the bands are flat. Each flat-band manifold is spanned by the states localized in one direction while delocalized in the other. This is reminiscent of the LLL wave functions on a torus. The Chern number of the lowest band is numerically found to be non-trivial.

The lowest flat band allows us to study the effect of the Hubbard interaction, i.e., the on-site Coulomb repulsion, non-perturbatively. We have rigorously proved that the ground states are ferromagnetic in both classes when the lowest flat band is half-filled. In both cases, the only role of the Coulomb repulsion is to lift the massive ground-state degeneracy of the non-interacting Hamiltonian and select the ferromagnetic states as the ground states. Therefore, although two constructions of the lowest flat band are completely different from each other, the mechanism of the ferromagnetism in two classes is essentially the same and falls into the category of the flat-band ferromagnetism.

***

The authors are grateful to A. Mielke, N. Nagaosa, and K. Nomura for their valuable comments and discussions. This work was supported in part by Grant-in-Aids (No. 20740214) from the Ministry of Education, Culture, Sports, Science and Technology of Japan. HK was supported by the JSPS Postdoctoral Fellowships for Research Abroad and NSF Grant No. PHY05-51164.

REFERENCES

[1] KANE C. L. and MELE E. J., Phys. Rev. Lett., 95 (2005) 146802; 226801.
[2] BERNEVIG B. A., HUGHES T. L. and ZHANG S.-C., Science, 314 (2006) 1757.
[3] KÖNIG M. et al., Science, 318 (2007) 766.
[4] QI X.-L., HUGHES T. L. and ZHANG S.-C., Phys. Rev. B, 78 (2008) 195424.
[5] SCHNEDDER A. P. et al., Phys. Rev. B, 78 (2008) 195125.
[6] KITAEOV A., arXiv:0901.2686v2 [cond-mat.mes-hall].
[7] PRANGE R. E. and GIRVIN S. M. (Editors), The Quantum Hall Effect (Springer-Verlag) 1987.
[8] THOULESS J. D. et al., Phys. Rev. Lett., 49 (1982) 405.
[9] LIEB E. H., Phys. Rev. Lett., 62 (1989) 1201.
[10] MIELKE A., J. Phys. A, 24 (1991) L73; 3141; 25 (1992) 4355.
[11] TASAKI H., Phys. Rev. Lett., 69 (1992) 1608; MIELKE A. and TASAKI H., Commun. Math. Phys., 158 (1993) 341.
[12] GULACSI Z., KAMPF A. and VOLLHARDT D., Phys. Rev. Lett., 99 (2007) 026404; Prog. Theor. Phys. Suppl., 176 (2008) 1.
[13] See, e.g., TASAKI H., Prog. Theor. Phys., 99 (1998) 489.
[14] SCHULENBURG J. et al., Phys. Rev. Lett., 88 (2002) 167207.
[15] ZHITOMIRSKY M. E. and TSUNETSUGU H., Phys. Rev. B, 70 (2004) 100403(R); 75 (2007) 224416.
[16] WU C. et al., Phys. Rev. Lett., 99 (2007) 070401; WU C. and DAS SARMA S., Phys. Rev. B, 77 (2008) 235107.
[17] BERGMAN D. L., WU C. and BALENTS L., Phys. Rev. B, 78 (2008) 125104.
[18] AOKI H., ANDO M. and MATSUMURA H., Phys. Rev. B, 54 (1996) R17296.
[19] Another class of models has been proposed by GREEN D., SANTOS L. and CHAMON C., Phys. Rev. B, 82 (2010) 075104.
[20] MacDonald A. H., FERTIG H. A. and BREY L., Phys. Rev. Lett., 76 (1996) 2153.
[21] NUQ. THOULESS D. J. and WU Y-S., Phys. Rev. B, 31 (1985) 3372.
[22] FUKUI T. and HATSUGAI Y., Phys. Rev. B, 75 (2007) 121403(R).
[23] HOFSTADTER D. R., Phys. Rev. B, 14 (1976) 2239.
[24] WIEGMANN P. B. and ZABRODIN A. V., Phys. Rev. Lett., 72 (1994) 1890; Nucl. Phys. B, 422 (1994) 495.
[25] HATSUGAI Y., KOHMOTO M. and WU Y. S., Phys. Rev. Lett., 73 (1994) 1134; Phys. Rev. B, 53 (1996) 9697.
[26] ABAVNOV A. G., TALSTRA J. C. and WIEGMANN P. B., Phys. Rev. Lett., 81 (1998) 2112; Nucl. Phys. B, 525 (1998) 571.
[27] MIELKE A., Phys. Lett. A, 174 (1993) 443.
[28] NOMURA K. and MACDONALD A. H., Phys. Rev. Lett., 96 (2006) 256602.