On the Spectral Gap and the Diameter of Cayley Graphs

I. D. Shkredov\textsuperscript{a}

Received April 22, 2020; revised March 5, 2021; accepted April 23, 2021

Abstract—We obtain a new bound connecting the first nontrivial eigenvalue of the Laplace operator on a graph and the diameter of the graph. This bound is effective for graphs with small diameter as well as for graphs with the number of maximal paths comparable to the expected value.

DOI: 10.1134/S0081543821040167

1. INTRODUCTION

Expander graphs were first introduced by Bassalygo and Pinsker [1], and their existence was first proved by Pinsker [12] (see also [8]). The property of a graph to be an expander is significant in many mathematical and computational contexts (see, e.g., [5, 7, 13]). It is well known that the expansion property of a graph is controlled by the spectral gap of the Laplace operator $\Delta$, namely, by the first nontrivial eigenvalue $\lambda_1$ of $\Delta$ (see [7]; all required definitions can be found in Section 3 below). In this paper we study the connection between $\lambda_1$ and the diameter of a graph, and we focus on Cayley graphs (although some generalizations are possible as well; see Theorem 3). In [3] the following result was obtained (see also [13, Corollary 3.2.7]).

\textbf{Theorem 1.} Let $G$ be a finite group. Let $S \subseteq G$ be a set and $d$ the diameter of its Cayley graph $\text{Cay}(S)$. Then

$$\lambda_1(\text{Cay}(S)) \geq \frac{1}{2d^2 |S|}.$$ 

Now we formulate our first main result.

\textbf{Theorem 2.} Let $G$ be a finite group. Let $S \subseteq G$ be a set and $d$ the diameter of its Cayley graph $\text{Cay}(S)$. Then

$$\lambda_1(\text{Cay}(S)) \geq \frac{|G|}{d|S|^d}.$$ 

A set $S \subseteq G$ is called a \textit{basis} of order $d$ if $S^d = G$. It follows that Theorem 2 is better than Theorem 1 in the case when $S$ is a basis of order $d$ such that $|S|^{d-1} < 2d|G|$. In particular, our result is better for all possible $S$ in the case $d = 2$. On the other hand, if $d$ is the diameter of $\text{Cay}(S)$, then $|S|^d \geq |G|$. Thus, our result is better than Theorem 1 for “economical” bases $S$, i.e., in the case when almost every element of $G$ requires the expected number of multiplications of elements of $S$ to be represented. For example, assuming the condition $|S|^d \ll_d |G|$, we have $\lambda_1(\text{Cay}(S)) \gg_d 1$. The same bound holds if the number of representations of any $x \in G$ as $x = s_1 \ldots s_d$, $s_j \in S$, is $\Omega(|S|^d / |G|)$. Other examples of effective applications of Theorem 2 are given in Remark 8 and Section 6 below. Here we show that our new bound for the gap of the Laplace operator allows us

\textsuperscript{a}Steklov Mathematical Institute of Russian Academy of Sciences, ul. Gubkina 8, Moscow, 119991 Russia.

E-mail address: ilya.shkredov@gmail.com
to say something new on non-commutative sets having no solutions of linear equations, on Sidon sets, as well as on the famous Erdős–Turán conjecture.

Actually, the methods from [13, Ch. 3] are rather general and one can obtain an analog of Theorem 1 for almost arbitrary graphs. In this direction we prove

**Theorem 3.** Let $G = G(V, E)$ be a finite graph with valency $V$ and diameter $d$. Then

$$
\lambda_1(G) \geq \frac{|V|}{dV^d}.
$$

In Sections 4 and 5 we focus on the case of Cayley graphs and obtain a characterization of the spectral gap in terms of the intersection of our set $S$ with arithmetic progressions and (non-abelian) Bohr sets. Let us formulate a result from these sections (see Corollaries 14 and 21).

**Theorem 4.** Let $G$ be a finite group, $e \in (0, 1)$ a real number, $d \geq 2$ an integer, and $B, \Omega \subseteq G$, $|\Omega| = (1 - e)|G|$, sets such that any element of $G \setminus \Omega$ can be represented as a product of $d$ elements of $BB^{-1}$ or $B^{-1}B$ in at least $g$ ways. Suppose that $G$ has no normal proper subgroups of index at most $2/e$. Then

$$
\lambda_1(Cay(B)) \geq \frac{ge^{\log_{3/2} 3} |G|}{16d^2 |B|^{2d}}.
$$

In the abelian case the dependence on the parameters in (1.1) is better (see Corollary 14 below). Thus Theorem 4 shows that in the case of Cayley graphs one can have a relatively large exceptional Bohr sets. Let us formulate a result from these sections (see Corollaries 14 and 21).

**Theorem 4.** Let $G$ be a finite group, $e \in (0, 1)$ a real number, $d \geq 2$ an integer, and $B, \Omega \subseteq G$, $|\Omega| = (1 - e)|G|$, sets such that any element of $G \setminus \Omega$ can be represented as a product of $d$ elements of $BB^{-1}$ or $B^{-1}B$ in at least $g$ ways. Suppose that $G$ has no normal proper subgroups of index at most $2/e$. Then

$$
\lambda_1(Cay(B)) \geq \frac{ge^{\log_{3/2} 3} |G|}{16d^2 |B|^{2d}}.
$$

In the abelian case the dependence on the parameters in (1.1) is better (see Corollary 14 below). Thus Theorem 4 shows that in the case of Cayley graphs one can have a relatively large exceptional set $\Omega$ and nevertheless a rather good lower bound for $\lambda_1(Cay(B))$. Finally, in the Appendix we collect some simple properties of non-abelian Bohr sets.

## 2. DEFINITIONS

Here and throughout this paper, $G$ is a finite group with identity $e$. Given two sets $A, B \subseteq G$, define the **product set** of $A$ and $B$ as

$$
AB := \{ab : a \in A, b \in B\}.
$$

In a similar way we define the higher product sets; for example, $A^3$ is $AAA$. We will use the following simple fact:

$$
|A| + |B| > G \quad \Rightarrow \quad AB = G.
$$

Let $A^{-1} := \{a^{-1} : a \in A\}$. Having an element $g \in G$ and a positive integer $k$, we write $g^{1/k}$ for the set $\{x \in G : x^k = g\}$. Further, if $A \subseteq G$ is a set, then $A^{1/k}$ is $\{a^{1/k} : a \in A\}$. In this paper we use the same letter to denote a set $A \subseteq G$ and its characteristic function $A : G \rightarrow \{0, 1\}$. Given a function $f : G \rightarrow \mathbb{C}$, we write $\langle f \rangle$ for $\sum_{x \in G} f(x)$.

Now we recall some notions and simple facts from representation theory (see, e.g., [10] or [18]). For a finite group $G$ let $\hat{G}$ be the set of all irreducible unitary representations of $G$. It is well known that the size of $\hat{G}$ coincides with the number of all conjugacy classes of $G$. For $\rho \in \hat{G}$ denote by $d_\rho$ the dimension of this representation. By $d_{\min}(G)$ denote the quantity $\min_{\rho \neq 1} d_\rho$. We write $\langle \cdot, \cdot \rangle$ for the corresponding Hilbert–Schmidt scalar product $\langle A, B \rangle = \langle A, B \rangle_{HS} := \text{tr}(AB^*)$, where $A$ and $B$ are any two matrices of the same size. Put $\|A\|_{HS} = \sqrt{\langle A, A \rangle}$. Clearly, $\langle \rho(g)A, \rho(g)B \rangle = \langle A, B \rangle$ and $\langle AX, Y \rangle = \langle X, A^*Y \rangle$. We also have $\sum_{\rho \in \hat{G}} d_\rho^2 = |G|$.

For any function $f : G \rightarrow \mathbb{C}$ and $\rho \in \hat{G}$ define a matrix $\hat{f}(\rho)$, called the Fourier transform of $f$ at $\rho$, by the formula

$$
\hat{f}(\rho) = \sum_{g \in G} f(g)\rho(g).
$$

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS Vol. 314 2021
ON THE SPECTRAL GAP

Then we have the inversion formula

$$f(g) = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \langle \hat{f}(\rho), \rho(g^{-1}) \rangle$$

(2.3)

and the Parseval identity

$$\sum_{g \in G} |f(g)|^2 = \frac{1}{|G|} \sum_{\rho \in \hat{G}} d_\rho \| \hat{f}(\rho) \|_{HS}^2.$$  

(2.4)

The main property of the Fourier transform is the convolution formula

$$\hat{f} \ast \hat{g}(\rho) = \hat{f}(\rho) \hat{g}(\rho),$$

(2.5)

where the convolution of two functions $f, g: G \to \mathbb{C}$ is defined as

$$(f \ast g)(x) = \sum_{y \in G} f(y)g(y^{-1}x).$$

Given a function $f: G \to \mathbb{C}$ and a positive integer $k$, we write $f^{(k)} = f^{(k-1)} \ast f$ for the $k$th convolution power of $f$. Finally, it is easy to check that for any matrices $A$ and $B$ one has $\|AB\|_{HS} \leq \|A\| \cdot \|B\|_{HS}$ and $\|A\| \leq \|A\|_{HS}$, where $\|\cdot\|$ is the operator $l^2$-norm of $A$, that is, the maximal singular value of $A$. In particular, this shows that $\|\cdot\|_{HS}$ is indeed a matrix norm.

Also, given a set $S \subseteq G$, we denote $\min_{\rho \in \hat{G}, \rho \neq 1} \| \hat{S}(\rho) \|$ by $|S|$.

The signs $\ll$ and $\gg$ are the usual Vinogradov symbols. All logarithms are to base 2.

3. ON THE DIAMETER OF CAYLEY GRAPHS

Let $S \subseteq G$ be a set and let $\text{Cay}(S)$ be the corresponding Cayley graph of $S$ defined as $\text{Cay}(S) = (V, E)$ with the vertex set $V = G$ and the edge set

$$E = \{(g, gs): g \in G, s \in S\}.$$  

Clearly, $\text{Cay}(S)$ is a regular graph and its diameter equals the minimum $d$ such that $S^d = G$. As usual, we consider the (oriented) Laplace operator of $\text{Cay}(S)$ defined for an arbitrary function $f: G \to \mathbb{C}$ as

$$(\Delta f)(x) = f(x) - |S|^{-1} \sum_{s \in S} f(xs).$$  

(3.1)

In other words, the matrix of $\Delta$ is $I - |S|^{-1}M(x, y)$, where $I$ is the identity matrix and $M(x, y)$ is the adjacency matrix of the graph $\text{Cay}(S)$ (the Markov operator of $\text{Cay}(S)$), $M(x, y) = S(x^{-1}y)$. Actually, formula (3.1) can be used to define an operator with an arbitrary function $F(x)$ instead of $S(x)$ if one replaces $|S|$ by $\|F\|_1$. Further, the Laplace operator has the spectrum

$$0 = \lambda_0(\text{Cay}(S)) \leq \lambda_1(\text{Cay}(S)) \leq |\lambda_2(\text{Cay}(S))| \leq \ldots \leq |\lambda_{|G|-1}(\text{Cay}(S))|,$$

and there is a variational description of $\lambda_1(\text{Cay}(S))$, namely,

$$\lambda_1(\text{Cay}(S)) = \min_{(f) = 0, \|f\|_2 = 1} \langle \Delta f, f \rangle.$$

The quantity $\lambda_1$ is closely connected with the expansion properties of the graph under consideration (see, e.g., [7]). Below we write $\lambda_j$ for $\lambda_j(\text{Cay}(S))$. We will also consider the corresponding eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{|G|-1}$ of the operator $I - |S|^{-1}MM^*$ (one can think of these numbers as the squares of “singular” values of $\Delta$).
The same operator can be defined for an arbitrary graph $G = (V, E)$ (see [7]); namely, assuming for simplicity that the valency of $G$ is a constant, say, $\mathcal{V}$, we write

$$(\Delta f)(x) = f(x) - \mathcal{V}^{-1} \sum_{(x,y) \in E} f(y).$$

(3.2)

The spectrum of the Cayley graph $\text{Cay}(S)$ is closely connected with the Fourier transform of the characteristic function of $S$. For example, it is well known (see [17] or [5, Proposition 6.2.4]) that the multiplicity of any $\lambda_j$, $j \neq 0$, is at least $d_{\text{min}}(G)$, because each eigenspace of the Markov operator $M$ is a subrepresentation of the regular representation. We collect a series of further simple results on the spectrum of $\text{Cay}(S)$ in the following lemma.

**Lemma 5.** Let $G$ be a finite group and $S \subseteq G$ a set. Then $1 - \lambda_j$ and $1 - \lambda_j^*$ belong to the spectra of the matrices $|S|^{-1}\hat{S}(\rho)$ and $|S|^{-2}\hat{S}(\rho)\hat{S}(\rho)^*$, respectively, where $\rho$ runs through $\hat{G}$. Moreover, $\lambda_1 \geq 1 - |S|^{-1}\|S\|^2$ and

$$\lambda_1^* = 1 - |S|^{-2}\|S\|^2.$$  

(3.3)

**Proof.** Let $f^r(x) = f(x^{-1})$. The required inclusion follows from the formula $(\Delta f)(x) = f(x) - |S|^{-1}(S \ast f^r)(x^{-1})$ and a similar formula for $I - |S|^{-2}MM^*$. Let $f$ be an eigenfunction of $\Delta$, that is, $(\Delta f)(x) = \mu f(x)$, $\mu \in \mathbb{C}$. Taking the Fourier transform, we derive

$$\mu \hat{f}(\rho) = \hat{f}(\rho) - |S|^{-1}\hat{f}(\rho)\hat{S}(\rho)^*.$$  

In other words,

$$0 = \hat{f}(\rho)((1 - \mu)I - |S|^{-1}\hat{S}(\rho)^*).$$

In view of (2.3) we know that there is a $\rho \in \hat{G}$ such that $\hat{f}(\rho) \neq 0$, because otherwise $f \equiv 0$. Hence the matrix $(1 - \mu)I - |S|^{-1}\hat{S}(\rho)^*$ cannot be invertible for this $\rho$, and thus $1 - \mu$ belongs to the spectrum of $|S|^{-1}\hat{S}(\rho)^*$, which coincides with the spectrum of $|S|^{-1}\hat{S}(\rho)$.

Further, applying the Cauchy–Schwarz inequality, formula (2.4) twice, as well as identity (2.5), we find that for any function $f : G \to \mathbb{C}$, $\|f\|_2 = 1$, $(f) = 0$,

$$\langle \Delta f, f \rangle = 1 - |S|^{-1} \sum_x (S \ast f^r)(x^{-1})f(x) = 1 - (|S| \cdot |G|)^{-1} \sum_{\rho \in \hat{G}} \|\hat{S}(\rho)\hat{f}(\rho, \hat{f}(\rho)) \rangle \
\geq 1 - (|S| \cdot |G|)^{-1}\|S\| \sum_{\rho \in \hat{G}} \|\hat{f}(\rho)\|^2 = 1 - \frac{|S|^2}{|S|}.$$  

Finally, to get (3.3), we first notice that by the same calculations with $S$ replaced by $S \ast S^{-1}$, we have $\lambda_1^* \geq 1 - |S|^{-2}\|S\|^2$. Let us obtain the reverse inequality. We find a $\rho \in \hat{G}$, $\rho \neq 1$, and a vector $\varphi \in \mathbb{C}^d$, $\|\varphi\|_2 = 1$, such that $\|S\|^2 = \langle \hat{S}(\rho)\hat{S}(\rho)^*\varphi, \varphi \rangle$. Using the definition of the Fourier transform, we get

$$\|S\|^2 = \langle \hat{S}(\rho)\hat{S}(\rho)^*\varphi, \varphi \rangle = \sum_{g \in \hat{G}} (S \ast S^{-1})(g)\langle \rho(g)\varphi, \varphi \rangle := \sum_{g \in \hat{G}} (S \ast S^{-1})(g)F(g).$$

(3.4)

Let us calculate the Fourier transform of $F$. Applying the orthogonality relations for any $\pi \in \hat{G}$ (see, e.g., [10, Ch. II, Sect. 1.4, Theorem 1]), we obtain

$$\hat{F}(\pi) = \sum_{i,j} \varphi(i)\overline{\varphi(j)} \sum_{g \in \hat{G}} \rho(g)i_j\pi(g) = \left| \frac{|G|}{d_{\rho}} \sum_{k} \varphi(k) \right|^2 \geq 0.$$
Hence the Fourier transform of $F$ is nonnegative, and thus $F$ can be written as $f' * f$ for a certain function $f$. Since $\rho \neq 1$, it follows that $\sum_g F(g) = 0$ (here we have again used the orthogonality relations). This implies $\langle f \rangle = 0$. Similarly, $F(e) = \|f\|^2_2 = \|\varphi\|^2_2 = 1$. But by the definition of the Laplace operator, for any function $f$ we have

$$\langle MM^* f, f \rangle = |S|^{-2} \sum_{x \in G} (S * S^{-1})(x)(f' * f)(x). \quad (3.5)$$

Returning to (3.4) and using the fact that $\langle f \rangle = 0$ and the variational property of the singular values of $M$, we derive

$$\|S\|^2 = |S|^2 \langle MM^* f, f \rangle \leq |S|^2 (1 - \lambda_1^*)$$

or, in other words, $\lambda_1^* \leq 1 - |S|^{-2} \|S\|^2$, as required. □

The proof of the first main Theorem 2 is based on an idea from [9]. We formulate our result in a slightly more general form.

**Theorem 6.** Let $G$ be a finite group, $\Omega \subseteq G$ a set, $g$ a positive real, and $d \geq 2$ an integer, and let $B \subseteq G$ be a set such that any element of $G \setminus \Omega$ can be represented as a product of $d$ elements of $B$ in at least $g$ ways. Then

$$\lambda_1(\text{Cay}(B)) \geq \frac{g|G|}{d(|B|^2 + g|\Omega|)^d} - \frac{g|\Omega|}{|B|^2}. \quad (3.6)$$

Suppose that for sets $B_1, B_2 \subseteq G$ one has $(B_1 * B_2)^{(d)}(x) \geq g$ outside $\Omega$. Then

$$\lambda_1(\text{Cay}(B_1 * B_2)) \geq \frac{g|G|}{d(|B_1|^2 + |B_2| + g|\Omega|)^d} - \frac{g|\Omega|}{|B_1|^2 + |B_2|}. \quad (3.7)$$

In particular,

$$\lambda_1^*(\text{Cay}(B)) \geq \frac{g|G|}{d(|B|^2 + g|\Omega|)^d} - \frac{g|\Omega|}{|B|^2}. \quad (3.8)$$

**Proof.** We assume first that $\Omega = \emptyset$. Let $f(x) = f_B(x) = B(x) - |B|^2/|G|$ be the balanced function of the set $B$. Clearly, we have $\sum_{x \in G} f(x) = 0$. Further, for an arbitrary $j$ one has $f^{(j)}(x) = B^{(j)}(x) - |B|^2/|G|$ and hence $\sum_{x \in G} f^{(j)}(x) = 0$. For any $k \geq 1$ consider

$$T_k(f) = \sum_{x \in G} f^{(k)}(x) = \sum_{x \in G} B^{(k)}(x)^2 - \frac{|B|^{2k}}{|G|}. \quad (3.9)$$

Using the definition of the Laplace operator and counting the number of cycles of length $2k$ in Cay($B$) (see [5, 7] for details), we obtain

$$|G|T_k(f) = |B|^{2k} \sum_{j=0}^{G-1} |1 - \lambda_j|^{2k} - |B|^{2k} = |B|^{2k} \sum_{j=1}^{G-1} |1 - \lambda_j|^{2k}. \quad (3.9)$$

Notice that $T_1(f) < |B|$. We have

$$T_k(f) = \sum_y \sum_{z_1, z_2} f^{(k-d)}(yz^{-1}) f^{(k-d)}(yz^{-1})(B^{(d)}(z_1) - g)(B^{(d)}(z_2) - g), \quad (3.10)$$

and by the Cauchy–Schwarz inequality for any $z_1, z_2 \in G$ we obtain

$$\sum_y f^{(k-d)}(yz_1^{-1}) f^{(k-d)}(yz_2^{-1}) \leq T_{k-d}(f). \quad (3.11)$$
We know that $B^{(d)}(x) \geq g$ for any $x \in G$. Combining the last inequality with (3.10) and (3.11), we derive

$$T_k(f) \leq \sum_{z_1, z_2} T_{k-d}(f)(B^{(d)}(z_1) - g)(B^{(d)}(z_2) - g) = T_{k-d}(f)(|B|^d - g|G|)^2.$$ 

By induction we see that for any $l$ the following holds:

$$T_{dl+1}(f) \leq T_1(f)(|B|^d - g|G|)^{2l} < |B|(|B|^d - g|G|)^{2l}.$$ 

Substituting the last bound into (3.9), we obtain

$$(1 - \lambda_1)^{2ld+2}|B|^{2ld+2}|G|^{-1} \leq T_{dl+1}(f) < |B|(|B|^d - g|G|)^{2l} = |B|^{2ld+1}(1 - g|G|/|B|^d)^{2l}.$$ 

Taking $l$ sufficiently large, we get

$$1 - \lambda_1 \leq \left(1 - \frac{g|G|}{|B|^d}\right)^{1/d} \leq 1 - \frac{g|G|}{d|B|^d},$$

as required.

Now if $\Omega \neq \emptyset$, then we replace the characteristic function of $B$ by $\widetilde{B}(x) = B(x) + g\Omega(bx)$, where $b$ is an arbitrary element of $B^{d-1}$. Then for any $x \in G$ one has $\widetilde{B}^{(d)}(x) \geq g$ and we can apply the arguments above. This yields

$$\lambda_1(\text{Cay}(B)) + \frac{g|\Omega|}{|B|} \geq \lambda_1(\text{Cay}(\widetilde{B})) \geq \frac{g|G|}{d\|B\|_1^d} = \frac{g|G|}{d(|B| + g|\Omega|)^d},$$

and we have (3.6).

It remains to obtain (3.7), and again we first consider the case $\Omega = \emptyset$. Let us apply the same arguments with a new function $F(x) = (f_1 * f_2)(x)$ instead of $f$, where $f_1 = f_{B_1}$ and $f_2 = f_{B_2}$. One has

$$T_1(F) = \sum_{x \in G} F^2(x) = \sum_{x \in G} (f_1 * f_2)^2(x) = \sum_{x \in G} (B_1 * B_2)^2(x) - \frac{|B_1|^2|B_2|^2}{|G|}$$

$$< |B_1| \cdot |B_2| \min\{|B_1|, |B_2|\},$$

and we can repeat our reasoning. For nonempty $\Omega$ consider the function $(B_1 * B_2)(x) + g\Omega(bx)$, where $b$ is an arbitrary element of $(B_1B_2)^{d-1}$, and apply the same arguments as before. To obtain (3.8), we just use (3.7) with $B_1 = B$ and $B_2 = B^{-1}$ or vice versa. This completes the proof. 

**Remark 7.** Using the well-known Plünecke inequality [19] in the case of symmetric (for simplicity) basis $B \subseteq G$ of order $d$ and an abelian group $G$, one finds that for any $A \subseteq G$ the following inequality holds:

$$|A|\left(\frac{|G|}{|A|}\right)^{1/d} \leq |A|\left(\frac{|B|^d}{|A|}\right)^{1/d} \leq |AB|.$$ 

It shows that Cay($B$) has an expansion property and, in principle, one can obtain some lower estimates for $\lambda_1$ in terms of the expansion constant $h(\text{Cay}(B))$ (see, e.g., [5, Proposition 3.4.3]).

Similarly, notice that there is another well-known general bound for the spectrum of a strictly positive matrix $A = (a_{ij})_{i,j=1}^n$, namely, $|\mu_2(A)| \leq \mu_1(A)(M - m)/(M + m)$, where $M = \max_{i,j} a_{ij}, m = \min_{i,j} a_{ij}$, and $\mu_1(A) \geq |\mu_2(A)| \geq \ldots$ are the eigenvalues of the matrix $A$.

Nevertheless, Theorem 1 and our Theorem 6 give better bounds than both considered estimates.
Remark 8. If $B^{(d)}(x) \geq 1$ for any $x \in G$, then, clearly, $B^{(l)}(x) \geq |B|^{l-d}$ for an arbitrary integer $l \geq d$. Thus one can improve the bounds (3.6) and (3.8) of Theorem 6 taking a larger $l$ in the case when we know some better lower estimates for $B^{(l)}(x)$.

Now suppose that $B^{(d)}(x) \gg |B|^d/|G|$ for any $x \in G$, that is, the number of representations is comparable to its expectation. Then $\lambda_1(\text{Cay}(B)) \gg 1/d$ and hence the bound for $\lambda_1(\text{Cay}(B))$ does not depend on $|G|$ and $|B|$.

Combining Theorem 6 and Lemma 5, we obtain

**Corollary 9.** Let $G$ be a finite group, $g$ a positive real, and $d \geq 2$ an integer, and let $B \subseteq G$ be a set such that for any $x \in G$ one has $(B * B^{-1})^{(d)}(x) \geq g$ or $(B^{-1} * B)^{(d)}(x) \geq g$. Then for an arbitrary nontrivial representation $\rho$ one has

$$
\|\hat{B}(\rho)\| \leq |B|(1 - \frac{g|G|}{d|B|^{2d}})^{1/2}.
$$

The next corollary shows that the basis properties of a set $B$ imply the uniform distribution of the product $B^k$ for large $k$.

**Corollary 10.** Let $G$ be a finite group, $g$ a positive real, and $d \geq 2$ an integer, and let $B \subseteq G$ be a set such that $(B * B^{-1})^d$ or $(B^{-1} * B)^d$ is at least one on $G$. Suppose that $k$ grows to infinity faster than

$$
\frac{d|B|^{2d}}{|G|} \log \frac{|G|}{|B|}.
$$

Then for any $x \in G$ one has

$$
B^{(k)}(x) = \frac{|B|^k}{|G|}(1 + o(1)).
$$

**Proof.** Without loss of generality, we consider the case $B * B^{-1}$. Using formula (2.3), we get

$$
B^{(k+2)}(x) = \frac{1}{|G|} \sum_{\rho \in G} d_\rho \langle \hat{B}^{k+2}(\rho), \rho(x^{-1}) \rangle = \frac{|B|^{k+2}}{|G|} + \mathcal{E},
$$

and our task is to estimate the error term $\mathcal{E}$. We have $\|\hat{B}(\rho)\| \leq |B|(1 - |G|/|d|B|^{2d}))^{1/2}$ by Corollary 9, and thus in view of (2.4) we get

$$
|\mathcal{E}| \leq \left( |B| \left(1 - \frac{|G|}{d|B|^{2d}}\right)^{1/2} \right)^k \frac{1}{|G|} \sum_{\rho \in G} d_\rho \|\hat{B}(\rho)\|_\text{HS}^{2} \leq \left(1 - \frac{|G|}{d|B|^{2d}}\right)^{k/2} |B|^{k+1}.
$$

Comparing (3.13) and (3.14), we obtain the result. $\square$

The same arguments work in the general case in the proof of Theorem 3. We leave it to the reader to introduce an exceptional set $\Omega$ into Theorem 11 below.

**Theorem 11.** Let $G = G(V, E)$ be a graph with valency $V$. Suppose that there are at least $g$ paths of length $d$ between any two vertices of $G$. Then

$$
\lambda_1(G) \geq \frac{g|V|}{dV^d}.
$$

**Proof.** Let $F(x, y) = I - V^{-1}M(x, y)$ be the matrix of the operator from (3.2) and $M$ be the adjacency matrix of the graph $G$; denote by $F^{(j)}$ and $M^{(j)}$ the powers of these matrices. Clearly, we have $\sum_{x,y} F(x, y) = 0$; moreover, by the definition of the valency, one has $\sum_a F(a, y) = \sum_b F(x, b) = 0$ for any $x$ and $y$. Hence for an arbitrary $j$ and any $x$ and $y$ the following holds:

$$
\sum_a F^{(j)}(a, y) = \sum_b F^{(j)}(x, b) = 0.
$$

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS Vol. 314 2021
For any $k \geq 1$ consider
\[ T_k = \sum_{x,y} F^{(k)}(x,y)^2 = \text{tr}(F^{(k)}(F^{(k)})^*) = \sum_{j=0}^{V-1} |1 - \lambda_j|^{2k} = \sum_{j=1}^{V-1} |1 - \lambda_j|^{2k}. \tag{3.17} \]

Notice that
\[ T_1 = |V| - 2V^{-1} \text{tr}(M) + V^{-2}|E| \leq |V| + V^{-1}|V| \leq 2|V|. \tag{3.18} \]

Using formula (3.16), we obtain
\[ T_k = \sum_{x,y} \sum_{a,b} F^{(k-d)}(x,a)F^{(k-d)}(x,b)(M^{(d)}(a,y) - g)(M^{(d)}(b,y) - g), \tag{3.19} \]

and by the Cauchy–Schwarz inequality, for any $a, b \in V$ we have
\[ \sum_x F^{(k-d)}(x,a)F^{(k-d)}(x,b) \leq \left( \sum_x F^{(k-d)}(x,a)^2 \right)^{1/2} \left( \sum_x F^{(k-d)}(x,b)^2 \right)^{1/2} =: q^{1/2}(a)q^{1/2}(b). \tag{3.20} \]

Clearly, \( \|q^{1/2}\|_2^2 = \sum_a q(a) = T_{k-d} \). We know that \( M^{(d)}(x,y) \geq g \) for any $x$ and $y$. Combining the last inequality with (3.19) and (3.20), we derive
\[ T_k \leq \sum_{a,b} q^{1/2}(a)q^{1/2}(b)\left( (M^{(d)}(M^{(d)})^*)(a,b) - 2gV^d + g^2|V| \right) \]
\[ \leq \|M^{(d)}q^{1/2}\|_2^2 + (g^2|V| - 2gV^d)\left( \sum_a q^{1/2}(a) \right)^2 \leq (2V^d + (g^2|V| - 2gV^d)|V|)T_{k-d}. \]

By induction and estimate (3.18) we see that
\[ T_{dl+1} \leq T_1(V^d - g|V|)^{2l} \leq 2|V|(V^d - g|V|)^{2l}. \]

Substituting the last bound into (3.17), we obtain
\[ (1 - \lambda_1)^{2dl+2}V^{2dl+2} \leq T_{dl+1}(f) \leq 2|V|(V^d - g|V|)^{2l} = 2V^{2d+1}V\left( 1 - \frac{g|V|}{V^d} \right)^{2l}. \]

Taking $l$ sufficiently large, we get
\[ 1 - \lambda_1 \leq \left( 1 - \frac{g|V|}{V^d} \right)^{1/d} \leq 1 - \frac{g|V|}{dV^d}, \]

as required. \( \Box \)

4. ON THE $\mathbb{Z}/N\mathbb{Z}$-CASE

Now we consider the case of an abelian group $\mathbb{G}$; for simplicity we often take $\mathbb{G}$ equal to $\mathbb{Z}/N\mathbb{Z}$ with a prime $N$ (bounds for spectral gaps of the Cayley graphs in general abelian groups can be found, for example, in [16]). In this case we show that the results of the previous section can be obtained via another tool (namely, see Theorem 12 below) and, moreover, one can characterize the existence of a spectral gap in combinatorial terms.

It is easy to see (or consult Lemma 5) that in the abelian case for any set $S \subseteq \mathbb{G}$ we have the identity $\lambda_1(\text{Cay}(S)) = 1 - |S|^{-1}\|S\|$. In other words, for any nontrivial character $\chi$,
\[ \left| \sum_{s \in S} \chi(s) \right| \leq \left( 1 - \lambda_1(\text{Cay}(S)) \right)|S|, \tag{4.1} \]
and the estimate is attained for a certain $\chi$. Thus, estimating the exponential sums and finding nontrivial upper bounds for $\lambda_1$ are equivalent problems for abelian $G$.

In this section our basic tool is Theorem 1 from [6].

**Theorem 12.** Let $A \subseteq \mathbb{Z}/N\mathbb{Z}$ be a set, $\varepsilon \in (0,1)$ and $\delta \in (0,1/2)$ be real numbers, and $|\hat{A}(1)| \geq (1-2\varepsilon(1-\cos(\pi\delta)))|A|$. Then there exist $a \in \mathbb{Z}/N\mathbb{Z}$ and $l < \delta N$ such that

$$|A \setminus [a,a+l]| < \varepsilon|A|.$$

Given a positive integer $d$, a set $P \subseteq G$, and a nonnegative function $f$ on $G$, we define

$$\sigma_p^{(d)}(f) := \|f\|_1^{-d} \sum_{x \in P} f^{(d)}(x) \leq 1. \tag{4.2}$$

We characterize the spectral gap of $\text{Cay}(B)$ in terms of the purely combinatorial quantity (4.2).

**Theorem 13.** Let $N$ be a prime number, $d$ a positive integer, and $\varepsilon, \delta \in (0,1)$ real numbers. Suppose that for any arithmetic progression $P$, $|P| \leq \delta N$, $\delta < d/2$, one has $\sigma_p^{(d)}(B) \leq 1 - \alpha$. Then

$$\lambda_1(\text{Cay}(B)) \geq \frac{2\alpha}{d} \left(1 - \cos \frac{\pi \delta}{d}\right).$$

Conversely, for any arithmetic progression $P$, $|P| \leq \delta N$, one has $\sigma_p^{(d)}(B) \leq 1 - \alpha$, where $\alpha = (1 - (1 - \lambda_1(\text{Cay}(B)))^d - \pi \delta)/2$.

**Proof.** To obtain the first part of the required result, we apply Theorem 12 with the parameters $\delta/d$ and $\varepsilon = \lambda_1/(2(1-\cos(\pi\delta/d)))$. In view of (4.1), we have the decomposition $B = B_* \cup E$, where $B_* = B \cap [a,a+l]$, $a \in \mathbb{Z}/N\mathbb{Z}$, $l < \delta N/d$, and $|E| < \varepsilon|B|$. Let $P = [a,a+l]$. Then $dP$ is another arithmetic progression of length at most $\delta N$. Further,

$$|B|^d = \sum_x B^{(d)}(x) \leq \sum_x B_*(x) + d|E| \cdot |B|^{d-1} < \sum_{x \in dP} B_*^{(d)}(x) + \varepsilon d|B|^d$$

$$= |B|^d \sigma_p^{(d)}(B) + \varepsilon d|B|^d \leq |B|^d(1 - \alpha + \varepsilon d)$$

or, equivalently,

$$\lambda_1 \geq \frac{2\alpha}{d} \left(1 - \cos \frac{\pi \delta}{d}\right),$$

as required.

To establish the second part of our theorem, we take any arithmetic progression $P$ such that $\sigma_p^{(d)}(B) > 1 - \alpha$, where $\alpha$ will be chosen later. Then, for any nonzero $r \in \mathbb{Z}/N\mathbb{Z}$, we have

$$\hat{B}^{(d)}(r) = \sum_x B^{(d)}(x)e^{-2\pi irx/N} = \sum_{x \in P} B^{(d)}(x)e^{-2\pi irx/N} + \theta|B|^d, \tag{4.3}$$

where $|\theta| \leq 1$ is a certain number. By the assumption $N$ is a prime number. Shifting and choosing $r$ in an appropriate way, one can assume that $r = 1$ and $P$ is a symmetric progression with step one, i.e., $P = \{x \in \mathbb{Z}/N\mathbb{Z} : |x| \leq \delta N/2\}$. Returning to (4.3) and applying formula (4.1) to estimate the left-hand side of (4.3), we obtain

$$(1 - \alpha)|B|^d < \sum_{x \in P} B^{(d)}(x) \leq |B|^d((1 - \lambda_1)^d + \alpha) + \sum_{x \in P} B^{(d)}(x)e^{-2\pi irx/N} - 1$$

$$\leq |B|^d((1 - \lambda_1)^d + \alpha + \pi \delta)$$

or, in other words,

$$\alpha \geq 2^{-1}(1 - (1 - \lambda_1)^d - \pi \delta).$$

This completes the proof. $\square$
Theorem 13 has a consequence on the Laplace operator of an arbitrary basis of order \( d \).

**Corollary 14.** Let \( N \) be a prime number, \( d \geq 2 \) an integer, and \( B, \Omega \subseteq \mathbb{Z}/N\mathbb{Z} \) sets such that any element of \( \mathbb{Z}/N\mathbb{Z} \setminus \Omega \) can be represented as a sum of \( d \) elements of \( B \) in at least \( g \geq 1 \) ways. Then

\[
\lambda_1(\text{Cay}(B)) \geq \frac{g(N - 2|\Omega|)}{d|B|^d} \left( 1 - \cos \frac{\pi}{2d} \right). \tag{4.4}
\]

If \( |\Omega| = (1 - \varepsilon)N \), then

\[
\lambda_1(\text{Cay}(B)) \geq \frac{\varepsilon gN}{d|B|^d} \left( 1 - \cos \frac{\varepsilon \pi}{2d} \right). \tag{4.5}
\]

**Proof.** Let \( \delta \in (0, 1) \) be a number to be chosen later, let \( P \) be an arbitrary arithmetic progression with \( |P| \leq \delta N \), and let \( P^c := (\mathbb{Z}/N\mathbb{Z}) \setminus P \). Since \( B_{\alpha}(x) \geq g \) for any \( x \in G \setminus \Omega \), we see that

\[
\sigma_P^{(d)}(B) = 1 - |B|^{-d}\sigma_P^{(d)}(B) \leq 1 - \frac{g(|P^c| - |\Omega|)}{|B|^d} \leq 1 - \frac{gN(1 - \delta) - g|\Omega|}{|B|^d}. \tag{4.6}
\]

Applying Theorem 13 with \( \alpha = (gN(1 - \delta) - g|\Omega|)/|B|^d \) and \( \delta = 1/2 \), we derive

\[
\lambda_1(\text{Cay}(B)) \geq \frac{g(N - 2|\Omega|)}{d|B|^d} \left( 1 - \cos \frac{\pi}{2d} \right),
\]

as required. To obtain (4.5), we use Theorem 13 with the parameters \( \delta = \varepsilon/2 \) and \( \alpha = \varepsilon gN/(2|B|^d) \). This completes the proof. \( \square \)

Thus the bound of Corollary 14 is comparable with the estimate from Theorem 2. The main advantage of using Theorem 13 is that the problem of calculating \( \lambda_1 \) is reformulated in terms of the purely combinatorial quantity (4.2). In addition, the dependence on \( \Omega \) in (4.4) is better than in Theorem 6.

**Example 15.** Put \( S = \Lambda \cup P \subseteq \mathbb{Z}/N\mathbb{Z} \), where \( \Lambda \) is a randomly chosen set such that \( 2\Lambda = \mathbb{Z}/N\mathbb{Z} \) (or let \( 2\Lambda \) be close to \( \mathbb{Z}/N\mathbb{Z} \); this is not so important) and \( |\Lambda| = c_1 \sqrt{N} \) with an absolute constant \( c_1 > 0 \), and \( P \) is an arithmetic progression with step one such that \( |P| = C\sqrt{N} \) with a large parameter \( C > 0 \). One can easily show that the largest nonzero Fourier coefficient of \( S \) coincides with the largest nonzero Fourier coefficient of \( P \). The latter is \( |P|(1 + o(1)) \), and hence

\[
\lambda_1(\text{Cay}(S)) \geq \frac{1 - |P|(1 + o(1))}{|S|} \geq \frac{c_1}{c_1 + C} + o(1) \gg \frac{1}{C}.
\]

On the other hand, Corollary 14 yields \( \lambda_1(\text{Cay}(S)) \gg 1/(c_1 + C)^2 \gg 1/C^2 \). Thus, for a fixed large \( C \) these bounds are of comparable quality.

5. THE GENERAL CASE

In this section we generalize the results from Section 4 to the non-abelian case. Following [14, Sect. 17], we define Bohr sets in a (non-abelian) group \( G \).

**Definition 16.** Let \( \Gamma \) be a collection of unitary representations of \( G \) and \( \delta \in (0, 2] \) be a real number. Put

\[
\text{Bohr}(\Gamma, \delta) = \{ g \in G : \| \gamma(g) - I \| \leq \delta \ \forall \gamma \in \Gamma \}.
\]

Clearly, \( e \in \text{Bohr}(\Gamma, \delta) \), and \( \text{Bohr}(\Gamma, \delta) = \text{Bohr}^{-1}(\Gamma, \delta) = \text{Bohr}(\Gamma^*, \delta) \). Notice also (see, e.g., formula (A.1) in the Appendix) that

\[
\text{Bohr}(\Gamma, \delta_1) \text{Bohr}(\Gamma, \delta_2) \subseteq \text{Bohr}(\Gamma, \delta_1 + \delta_2). \tag{5.1}
\]

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS Vol. 314 2021
By the left/right invariance of $\| \cdot \|$ one can easily show (or consult [15, Lemma 4.1]) the normality of Bohr sets, i.e., the identity $x \text{Bohr}(\Gamma, \delta) x^{-1} = \text{Bohr}(\Gamma, \delta)$, which holds for any $x \in G$. If $\Gamma = \{ \rho \}$, then we simply write $\text{Bohr}(\rho, \delta)$ for $\text{Bohr}(\Gamma, \delta)$ (a lower bound for the size of Bohr$(\rho, \delta)$ can be found in [14, Lemma 17.3]). Further properties of Bohr sets are contained in the Appendix.

**Lemma 17.** Let $A \subseteq G$ be a set and $\varepsilon, \delta \in (0, 1)$ real numbers. Suppose that for a certain unitary representation $\rho$ one has $\| \hat{A}(\rho) \| \geq (1 - \varepsilon)|A|$. Then $\sum_{g \notin \text{Bohr}(\rho, \delta)} |A + A^{-1}(g)| \leq 2 \varepsilon |A|^2 / \delta$.

**Proof.** By the assumption, $\| \hat{A}(\rho) \| \geq (1 - \varepsilon)|A|$. This means that

$$\| A^2 I - \sum_{g \in G} (A + A^{-1}(g))(I - \rho(g)) \| = \| \sum_{g \in G} (A + A^{-1}(g)) \rho(g) \| \geq (1 - \varepsilon)^2 |A|^2.$$

For any $g \in G$ the operator $I - \rho(g)$ is normal and positive semidefinite. Moreover, the operator

$$\frac{1}{2}((A + A^{-1})(g)(I - \rho(g)) + (A + A^{-1})(g^{-1})(I - \rho(g^{-1})))$$

is Hermitian, because $(A + A^{-1})(g^{-1}) = (A + A^{-1})(g)$. Hence an arbitrary linear combination of such operators with nonnegative coefficients is Hermitian and positive semidefinite as well. This gives

$$\left\| \sum_{g \notin \text{Bohr}(\rho, \delta)} (A + A^{-1}(g))(I - \rho(g)) \right\| \leq \left\| \sum_{g \in G} (A + A^{-1}(g))(I - \rho(g)) \right\| \leq (2\varepsilon - \varepsilon^2) |A|^2,$$

(5.2)

because Bohr$(\rho, \delta)$ is a symmetric set. Again, for an arbitrary $g \notin \text{Bohr}(\rho, \delta)$ the operator $I - \rho^*(g)$ is normal and positive semidefinite; moreover, all singular values of any such operator are at least $\delta$ in view of the definition of Bohr sets. Also, $A(g) \geq 0$ for any $g \in G$. Thus, by the variational principle we find from (5.2) that

$$\delta \sum_{g \notin \text{Bohr}(\rho, \delta)} (A + A^{-1}(g)) \leq (2\varepsilon - \varepsilon^2)|A| \leq 2\varepsilon |A|^2,$$

as required. □

Now we are ready to obtain a non-abelian analog of Theorem 13.

**Theorem 18.** Let $d$ be a positive integer and $\varepsilon, \delta \in (0, 1)$ real numbers. Suppose that for any Bohr set $P = \text{Bohr}(\rho, \delta)$, $\rho \neq 1$, one has $\sigma_P^{(d)}(B * B^{-1}) \leq 1 - \alpha$. Then

$$\lambda_1(\text{Cay}(B)) \geq \frac{\alpha \delta}{2d^2}.$$

Conversely, for any Bohr set $P = \text{Bohr}(\rho, \delta)$, $\rho \neq 1$, one has $\sigma_P^{(d)}(B * B^{-1}) \leq 1 - \alpha$, where

$$\alpha = \frac{1 - (1 - \lambda_1^*(\text{Cay}(B)))^d - \delta}{2}.$$

**Proof.** By the first part of Lemma 5 one has $\| B \| \geq |B|(1 - \lambda_1)$. In other words, for a certain $\rho \neq 1$, we have $\| \hat{B}(\rho) \| \geq |B|(1 - \lambda_1)$. To obtain the first statement of the required result, we apply Lemma 17 with $\delta/d$ taken as $\delta$ and $\lambda_1$ taken as $\varepsilon$. For the function $f(x) = (B * B^{-1})(x)$ we have the decomposition $f(x) = f_1(x) + f_2(x)$, where the function $f_1$ is supported on the Bohr set $P_\varepsilon = \text{Bohr}(\rho, \delta)$, while the function $f_2$ is supported outside $P_\varepsilon$ and $\| f_2 \|_1 \leq 2\varepsilon d |B|^2 / \delta$. Further,

$$|B|^{2d} = \sum_x |f_1^{(d)}(x)| \leq \sum_x |f_1^{(d)}(x)| + \frac{2d^2 \varepsilon}{\delta} |B|^{2d} = |B|^{d} \sigma_P^{(d)}(B * B^{-1}) + \frac{2d^2 \varepsilon}{\delta} |B|^{2d}$$

$$\leq |B|^{2d} \left( 1 - \alpha + \frac{2d^2 \varepsilon}{\delta} \right).$$
or, equivalently,
\[ \lambda_1 \geq \frac{\alpha \delta}{2d^2}. \]

To get the second part of our theorem, we take any Bohr set \( P = \text{Bohr}(\rho, \delta) \), \( \rho \neq 1 \), such that \( \sigma_{(d)}(B * B^{-1}) > 1 - \alpha \), where \( \alpha \) will be chosen later. We have
\[ \tilde{f}^d(\rho) = \sum_{x}(B * B^{-1})^d(x)\rho(x) = \sum_{x \in P}(B * B^{-1})^d(x)\rho(x) + \theta \alpha |B|^{2d}, \tag{5.3} \]
where \( |\theta| \leq 1 \) is a certain number. Further, in view of the second part of Lemma 5, we can estimate \( \|\tilde{f}^d\| \) as \( (1 - \lambda_1^*)^d|B|^{2d} \). This gives
\[ (1 - \alpha)|B|^{2d} < \sum_{x \in P}(B * B^{-1})^d(x) \leq |B|^{2d}(1 - \lambda_1^*)^d + \alpha + \sum_{x \in P}(B * B^{-1})^d(x)\|\rho(x) - I\| \]
\[ \leq |B|^{2d}(1 - \lambda_1^*)^d + \alpha + \delta \]
or, in other words,
\[ \alpha \geq 2^{-1}(1 - (1 - \lambda_1^*)^d - \delta). \]

This completes the proof. \( \square \)

**Remark 19.** Clearly, if for any \( x \in G \) and any Bohr set \( P \) one can estimate from above the intersection \( |B \cap Px| \) or \( |B \cap xP| \) as \( (1 - \alpha)|B| \), then the inequality \( \sigma_{(d)}(B * B^{-1}) \leq 1 - \alpha \) holds for an arbitrary \( d \).

We need upper bounds for the sizes of Bohr sets.

**Lemma 20.** Let \( G \) be a finite group and \( \rho \) an irreducible representation, \( \rho \neq 1 \). Then for
\[ \delta \leq \frac{1}{\sqrt{2}} \left( 1 - \frac{1}{d_{\rho}} \right)^{1/2}, \quad d_{\rho} > 1, \quad \text{and} \quad \delta \leq \frac{\sqrt{3}}{2}, \quad d_{\rho} = 1, \tag{5.4} \]
the following holds:
\[ |\text{Bohr}(\rho, \delta)| \leq \frac{|G|}{2}. \tag{5.5} \]
Moreover, if \( G \) has no normal proper subgroups of index at most \( 1/\varepsilon, \varepsilon \leq 1/2 \), then
\[ |\text{Bohr}(\rho, \delta_\varepsilon)| \leq \varepsilon|G|, \tag{5.6} \]
where
\[ \delta_\varepsilon \leq \left( 2 - \frac{2}{d_{\rho}} \right)^{1/2} \varepsilon \log_{3/2}^2, \quad d_{\rho} > 1, \quad \text{and} \quad \delta_\varepsilon \leq \sqrt{3} \varepsilon \log_{3/2}^2, \quad d_{\rho} = 1. \]

**Proof.** Take \( \delta \) as in (5.4). If \( |\text{Bohr}(\rho, \delta)| > |G|/2 \), then by (2.1) one has \( \text{Bohr}(\rho, \delta)^2 = G \) and hence \( \text{Bohr}(\rho, 2\delta) = G \). In other words, \( \|\rho(g) - I\| \leq 2\delta \) for any \( g \in G \). However,
\[ 2d_{\rho} - 2 \text{tr}(\rho(g)) = \|\rho(g) - I\|_{HS}^2 \leq d_{\rho}\|\rho(g) - I\|^2 \tag{5.7} \]
and, on the other hand, by the orthogonality relations and the irreducibility of \( \rho \) one has
\[ \sum_{g \in G} |\text{tr}(\rho(g))|^2 = |G|. \]
Hence there is a \( g \) such that \( |\text{tr}(\rho(g))| \leq 1 \), and in view of (5.7) we obtain
\[ 2d_{\rho} - 2 \leq d_{\rho}(2\delta)^2, \]
as required. Finally, if \( d_\rho = 1 \), then \( \rho \) is a nontrivial character on \( G \) and

\[
\max_{g \in G} \| \rho(g) - I \| \geq \min_{1 < k \mid |G|} \max_n |e^{2\pi in/k} - 1| \geq \sqrt{3},
\]

where the minimum is taken over all divisors of \( |G| \) (the constant \( \sqrt{3} \) is attained for \( k = 3 \mid |G| \) and \( n = 1 \)).

It remains to obtain (5.6). Suppose that \( |\text{Bohr}(\rho, \delta_e)| > \varepsilon |G| \). We know that any Bohr set is a normal set. It is also well known (see, e.g., [19]) that for any set \( A \subseteq G \) either one has \( |AA| \geq 3|A|/2 \) or \( AA^{-1} \) is a subgroup of \( G \). By the assumption, \( G \) has no proper subgroups of index at most \( 1/\varepsilon \). Thus, for an integer \( k \geq 1/(2\varepsilon)\log_{3/2}^2 + 1 \), one has Bohr\(^k(\rho, \delta_e) = G \) and hence Bohr\((\rho, k\delta_e) = G \). It follows that \( k\delta_e > (2 - 2/d_\rho)^{1/2} \) for \( d_\rho > 1 \) and \( k\delta_e > \sqrt{3} \) for \( d_\rho = 1 \). This completes the proof. \( \square \)

Clearly, estimate (5.5) is tight, as the case \( G = \mathbb{F}_2^2 \) shows. Finally, notice a well-known fact that for any \( H < G \) one has \( |G|/|H| \geq d_{\min}(G) + 1 \). Thus \( d_{\min}(G) \geq 1/\varepsilon \) guaranties that \( G \) has no normal proper subgroups of index at most \( 1/\varepsilon \). Another sufficient property for avoiding normal subgroups of index \( 1/\varepsilon \) is the simplicity of \( G \), of course.

Finally, let us obtain an analog of Corollary 14.

**Corollary 21.** Let \( G \) be a finite group, \( d \geq 2 \) an integer, and \( B, \Omega \subseteq G \) sets such that any element of \( G \setminus \Omega \) can be represented as a product of \( d \) elements of \( BB^{-1} \) or \( B^{-1}B \) in at least \( g \geq 1 \) ways. Then

\[
\lambda_1(\text{Cay}(B)) \geq \frac{g(|G| - 2|\Omega|)}{8d^2|B|^{2d}}. \tag{5.8}
\]

If \( |\Omega| = (1 - \varepsilon)|G| \) and \( G \) has no normal proper subgroups of index at most \( 2/\varepsilon \), then

\[
\lambda_1(\text{Cay}(B)) \geq \frac{\varepsilon \log_{3/2}^3 g|G|}{16d^2|B|^{2d}}. \tag{5.9}
\]

**Proof.** Without loss of generality we consider the case of \( BB^{-1} \). Let \( \delta \) be as in formula (5.4) of Lemma 20. Then in any case one can take \( \delta = 1/2 \). Also, let \( P = \text{Bohr}(\rho, \delta) \) be a Bohr set with \( \rho \neq 1 \) and let \( P^c := G \setminus P \). By Lemma 20 we know that \( |P| \leq |G|/2 \) and hence \( |P^c| \geq |G|/2 \). Since \((B * B^{-1})^{(d)}(x) \geq g \) for any \( x \in G \setminus \Omega \), we see that

\[
\sigma^{(d)}_P(B * B^{-1}) = 1 - |B|^{-2d} \sigma^{(d)}_{P^c}(B * B^{-1}) \leq 1 - \frac{g(|P^c| - |\Omega|)}{|B|^{2d}} \leq 1 - \frac{g|G|/2 - g|\Omega|}{|B|^{2d}}. \tag{5.10}
\]

Applying the first part of Theorem 18 with \( \alpha = (g|G| - 2g|\Omega|)/(2|B|^{2d}) \) and \( \delta \) as before, we derive (5.8). To obtain (5.9), we use the first part of Theorem 18 with the parameters \( \delta = \delta_{e/2} \geq (\varepsilon/2)^{\log_{3/2}^2} \) and \( \alpha = \varepsilon g|G|/(2|B|^{2d}) \). This completes the proof. \( \square \)

6. EXAMPLES

Our first example of using the results from the previous sections concerns maximal sets in nonabelian groups that avoid non-affine equations. For simplicity we consider only an equation with three variables.

**Corollary 22.** Let \( G \) be a finite group with identity \( e \) and \( A \subseteq G \) be a maximal set such that \( e \notin A^3 \). Then

\[
\lambda_1(\text{Cay}(A)) \geq \frac{|G|}{2(|A| + |\sqrt{A^{-1}}| + |e^{1/3}|)^2} - \frac{1 + |\sqrt{A^{-1}}| + |e^{1/3}|}{|A|}. \tag{6.1}
\]
and
\[ \lambda_1(\text{Cay}(A \cup \sqrt{A^{-1}})) \geq \frac{|G|}{2(|A| + |e^{1/3}|)^2} - \frac{1 + |e^{1/3}|}{|A|}. \] (6.2)

**Proof.** Indeed, by the maximality of \( A \) we see that any \( x \notin A \) belongs to either \( A^{-1} A^{-1} \), or to \( e^{1/3} \). In other words, the set \( \{A \cup \{e\}\} \) covers the group \( G \) except for a set of size at most \( \sqrt{A^{-1} A^{-1} e^{1/3}} \), and the set \( \{A \cup \sqrt{A^{-1}} e^{1/3}\} \) covers the group \( G \) except for a set of size at most \( e^{1/3} \). Applying Theorem 6, we obtain (6.1) and (6.2). \( \square \)

In the next example we consider the family of so-called \( B_k \)-sets (see, e.g., [11]). Recall that \( A \subseteq \mathbb{N} \) is called a \( B_k \)-set, \( k \geq 2 \), if all sums \( a_1 + \ldots + a_k \), \( a_1, \ldots, a_k \in A \), are distinct.

**Corollary 23.** Let \( A \subseteq \{1, 2, \ldots, N\} \) be a \( B_k \)-set and \( N \) a prime. Suppose that \( |A| \gg_k N^{1/k} \). Then there is a constant \( c = c(k) > 0 \) such that for all \( r \neq 0 \) one has
\[ \left| \sum_{x \in A} e^{2\pi ir x/N} \right| \leq (1 - c)|A|. \] (6.3)

**Proof.** Since \( A \) is a \( B_k \)-set and \( |A| \gg_k N^{1/k} \), it follows that there are elements of \( kA \) that belong to \( \{1, \ldots, kN\} \). Consider the set \( A \) modulo \( N \). Then modulo \( N \) the set \( kA \subseteq \mathbb{Z}/N \mathbb{Z} \) has at least \( \varepsilon(k)N/k \) elements. Applying Corollary 14 with \( g = 1, d = k \), we obtain
\[ \lambda_1(\text{Cay}(A)) \gg_k \frac{N}{|A|^k} \gg_k 1. \]

This completes the proof. \( \square \)

Our third example concerns the well-known problem of Erdős and Turán (see [4]) on the quantity \( \limsup_n A^{(2)}(n) \) for an arbitrary basis \( A \subseteq \mathbb{N} \) of order 2. It was conjectured that the \( \limsup \) equals infinity for any such \( A \). We show that any basis of order 2 has a certain expansion property.

Given a set \( A \subseteq \mathbb{N} \), denote by \( A_N \) the intersection of \( A \) with \( \{1, \ldots, N\} \). Notice that if \( \limsup_n |A_N|/N^{1/2} = \infty \), then, obviously, \( \limsup_n A^{(2)}(n) = \infty \).

**Corollary 24.** Let \( A \subseteq \mathbb{N} \) be a set such that \( A + A \) equals \( \mathbb{N} \) up to a finite number of exceptions. Suppose that \( |A_N| \leq KN^{1/2} \) for all sufficiently large prime \( N \). Then there exists a constant \( c = c(K) > 0 \) such that for all sufficiently large \( N \) and any \( r \neq 0 \) one has
\[ \left| \sum_{x=1}^N A_N(x)e^{2\pi ir x/N} \right| \leq (1 - c)|A_N|. \] (6.4)

**Proof.** By the assumption there is a number \( M \) such that \( A^{(2)}(x) \geq 1 \) for all \( x \geq M \). Take a sufficiently large prime \( N \geq 4M \) and consider the set \( A_N \). Obviously, \( 2A_N \) contains at least three quarters of \( \mathbb{Z}/N \mathbb{Z} \). Applying Corollary 14 with \( g = 1, d = 2, \) and \( |\Omega| \leq N/4 \), we obtain
\[ \lambda_1(\text{Cay}(A_N)) \gg_k \frac{N}{|A_N|^2} \geq \frac{1}{K^2}. \]

This completes the proof. \( \square \)

Corollary 24 shows in particular that for a basis \( A \subseteq \mathbb{N} \) the function \( A^{(k)}(x) \) becomes more and more uniform as \( k \) tends to infinity (see Corollary 10).
ON THE SPECTRAL GAP

APPENDIX

Here we collect further natural properties of Bohr sets and related notions which have well-known abelian analogs. We do this for the convenience of the reader interested in this particular form of Bohr sets; most of these results are more or less contained in [2, 14, 15].

In Section 5 we have used the connection of Bohr sets with the set of unitary representations $\rho$ such that $\|\hat{A}(\rho)\| \geq (1-\varepsilon)|A|$ for a given set $A \subseteq G$. Thus it is natural to give a more general

Definition 25. Let $A \subseteq G$ be a set and $\varepsilon \in [0, 1]$ a real number. The spectrum $\text{Spec}_\varepsilon(A)$ of $A$ is the set of unitary representations

$$\text{Spec}_\varepsilon(A) = \{ \rho : \|\hat{A}(\rho)\| \geq \varepsilon |A| \}.$$ 

Using the arguments of the proof of Lemma 17, we obtain a non-abelian analog of the well-known result of Yudin [20].

Proposition 26. Let $A \subseteq G$ be a set and $\varepsilon_1, \varepsilon_2 \in [0, 1]$ real numbers. Then

$$\text{Spec}_{1-\varepsilon_1}(A) \cdot \text{Spec}_{1-\varepsilon_2}(A) \subseteq \text{Spec}_{1-\varepsilon_1-\varepsilon_2}(A).$$

Proof. As follows from the arguments of the proof of Lemma 17 (see estimate (5.2)), a unitary representation $\rho$ belongs to $\text{Spec}_{1-\varepsilon}(A)$ if and only if

$$\left\| \sum_{g \in G} (A \ast A^{-1})(g)(I - \rho(g)) \right\| \leq (2\varepsilon - \varepsilon^2)|A|^2 = (1 - (1 - \varepsilon)^2)|A|^2.$$ 

However,

$$I - \rho_1(g)\rho_2(g) = (I - \rho_1(g))\rho_2(g) + I - \rho_2(g)$$ (A.1)

and hence, by the triangle inequality for the operator norm, we get

$$\left\| \sum_{g \in G} (A \ast A^{-1})(g)(I - \rho_1(g)\rho_2(g)) \right\| \leq (2\varepsilon_1 - \varepsilon_1^2 + 2\varepsilon_2 - \varepsilon_2^2)|A|^2 = (1 - (1 - \varepsilon_1 - \varepsilon_2)^2)|A|^2,$$

as required. $\square$

Our next result shows that Bohr($\rho, \delta$) has small product set and hence it suffices to check the condition of smallness of the quantity $\varepsilon_P^{(d)}(B) \leq 1 - \alpha$ in Theorem 18 only for sets with small product.

Proposition 27. Let $\delta \in [0, 2/5]$ be a real number and $\rho$ a unitary representation. Then

$$|\text{Bohr}(\rho, \delta)\text{Bohr}(\rho, \delta)| \leq 2^{21d_P^2/2}|\text{Bohr}(\rho, \delta)|$$

and there are sets $X, Y \subseteq G$, $|X|, |Y| < 2^{25d_P^2}$, such that

$$\text{Bohr}(\rho, \delta) \subseteq \text{Bohr} \left( \rho, \frac{\delta}{2} \right) X \quad \text{and} \quad \text{Bohr}(\rho, \delta) \subseteq Y \text{Bohr} \left( \rho, \frac{\delta}{2} \right).$$

Proof. Let $k = d_P$. In view of (5.1) it is enough to compare $|\text{Bohr}(\rho, \delta)|$ and $|\text{Bohr}(\rho, 2\delta)|$. Further, one can check that $2(1 - \cos \theta) \leq \theta^2$ and $2(1 - \cos \theta) \geq \theta^2/2$ for $|\theta| \leq \sqrt{6}$. Put

$$\eta := \eta(\delta) = \frac{1}{2\pi} \arccos \left( 1 - \frac{\delta^2}{2} \right).$$

PROCEEDINGS OF THE STEKLOV INSTITUTE OF MATHEMATICS Vol. 314 2021
We have $\delta/(2\pi) \leq \eta(\delta) \leq \delta/\pi$. Let $U(\delta)$ be the set of unitary matrices $U$ such that $\|U - I\| \leq \delta$. In [14, Lemma 17.4] it was proved that the Haar measure $\mu$ of $U(\delta)$ equals

$$
\mu(U(\delta)) = \frac{1}{k!} \int_{-\eta}^{\eta} \cdots \int_{-\eta}^{\eta} \prod_{1 \leq n < m \leq k} |e^{2\pi i \theta_n} - e^{2\pi i \theta_m}|^2 \, d\theta_1 \cdots d\theta_k
$$

$$
= \frac{1}{k!} \int_{-\eta}^{\eta} \cdots \int_{-\eta}^{\eta} \prod_{1 \leq n < m \leq k} 2(1 - \cos(2\pi(\theta_n - \theta_m))) \, d\theta_1 \cdots d\theta_k. \quad (A.2)
$$

Put

$$
F(k) = \frac{(2\pi)^{k(k-1)}}{k!} \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1 \leq n < m \leq k} (\theta_n - \theta_m)^2 \, d\theta_1 \cdots d\theta_k.
$$

From (A.2) and our bounds for $2(1 - \cos \theta)$, it follows that

$$
2^{-k(k-1)/2} \eta^k F(k) \leq \mu(U(\delta)) \leq \eta^k F(k),
$$

because $4\pi \eta(1) = 2\pi/3 \leq \sqrt{6}$. Using the assumption $\delta \leq 2/5$ and the previous formula, we obtain

$$
\frac{\mu(U(5\delta/2))}{\mu(U(\delta/2))} \leq 2^{k(k-1)/2} \frac{\eta(5\delta/2)^k}{\eta(\delta/2)^k} \leq 2^{k(k-1)/2} \cdot 2^{10k^2} < 2^{21k^2/2}. \quad (A.3)
$$

Now let $V = \rho(G)$. It is easy to see that $|\text{Bohr}(\rho, \delta)| \geq |V \cap U(\delta/2)u|$ for any unitary matrix $u$, because $U(\delta/2)u(\delta/2)u^{-1} \subseteq U(\delta)$. Further, integrating with respect to the Haar measure, we get in view of (A.3)

$$
|\text{Bohr}(\rho, 2\delta)| = \left(\mu\left(U\left(\frac{\delta}{2}\right)\right)\right)^{-1} \int |\rho(\text{Bohr}(\rho, 2\delta)) \cap U\left(\frac{\delta}{2}\right) u| \, du
$$

$$
\leq |\text{Bohr}(\rho, \delta)| \frac{\mu(U(5\delta/2))}{\mu(U(\delta/2))} < 2^{21k^2/2} |\text{Bohr}(\rho, \delta)|.
$$

Now by the Ruzsa covering lemma (see, e.g., [19]) one finds $X$ (and similarly $Y$) such that

$$
\text{Bohr}(\rho, \delta) \subseteq \text{Bohr}\left(\rho, \frac{\delta}{4}\right) \text{Bohr}^{-1}\left(\rho, \frac{\delta}{4}\right) X \subseteq \text{Bohr}\left(\rho, \frac{\delta}{2}\right) X,
$$

where, as above,

$$
|X| \leq \frac{|\text{Bohr}(\rho, 5\delta/4)|}{|\text{Bohr}(\rho, \delta/4)|} \leq \frac{\mu(\text{Bohr}(\rho, 11\delta/8))}{\mu(\text{Bohr}(\rho, \delta/4))} \leq 2^{k(k-1)/2} \frac{\eta(11\delta/8)^k}{\eta(\delta/8)^k} < 2^{25k^2}.
$$

This completes the proof. \qed

Having a lower bound for the size of one-dimensional Bohr sets (see [14, Lemma 17.3] or the proposition above), one can obtain a lower bound for the size of Bohr sets with an arbitrary $\Gamma$.

**Proposition 28.** Let $\text{Bohr}(\rho_j, \delta_j)$, $j = 1, \ldots, k$, be Bohr sets such that $\delta_1 \leq \delta_2 \leq \ldots \leq \delta_k$. Then

$$
|\text{Bohr}(\{\rho_1, \ldots, \rho_k\}, \delta_k)| \geq |G|^{-1} \prod_{j=1}^{k} |\text{Bohr}(\rho_j, \frac{\delta_j}{2})|.
$$
Proof. Let $B = \text{Bohr}(\{\rho_1, \ldots, \rho_k\}, \delta)$ and $B_j = \text{Bohr}(\rho_j, \delta_j/2)$, $j = 1, 2, \ldots, k$. Clearly, for any $j$ one has $B_j B_j^{-1} \subseteq B$. Hence

$$\sigma := \sum_{x \in G} (B_1 * B_1^{-1})(x) \cdots (B_k * B_k^{-1})(x) = \sum_{x \in B} (B_1 * B_1^{-1})(x) \cdots (B_k * B_k^{-1})(x)$$

$$\leq |B| \cdot |B_1| \cdots |B_k|. \quad (A.4)$$

On the other hand, in view of formulas (2.4) and (2.5), we get

$$\sigma = \frac{1}{|G|} \sum_{\rho} d_{\rho} \langle \widehat{B}_1(\rho) \widehat{B}_1^{*}(\rho) \cdots \widehat{B}_{k-1}(\rho) \widehat{B}_{k-1}^{*}(\rho), \widehat{B}_k(\rho) \widehat{B}_k^{*}(\rho) \rangle \geq \frac{|B_1|^2 \cdots |B_k|^2}{|G|}, \quad (A.5)$$

because the operators $\widehat{B}_1(\rho) \widehat{B}_1^{*}(\rho)$ are Hermitian and positive semidefinite. Comparing (A.4) and (A.5), we obtain the result. □

A Bohr set $\text{Bohr}(\rho, \delta)$ is said to be regular if

$$||\text{Bohr}(\rho, (1 + \kappa)\delta) - \text{Bohr}(\rho, \delta)|| \leq 100d^2_\rho |\kappa| \cdot |\text{Bohr}(\rho, \delta)|$$

whenever $|\kappa| \leq 1/(100d^2_\rho)$. Even in the abelian case it is easy to see that not every Bohr set is regular (see, e.g., [19, Sect. 4.4]). Nevertheless, it was shown in [2] that for $G = \mathbb{Z}/N\mathbb{Z}$ one can find a regular Bohr set by slightly decreasing the parameter $\delta$. We show the same for general groups, repeating the arguments from [19, Lemma 4.25] (see also [14, Lemma 9.3]).

**Proposition 29.** Let $\delta \in [0, 1/2]$ be a real number and $\rho$ a unitary representation. Then there is a $\delta_1 \in [\delta, 2\delta]$ such that $\text{Bohr}(\rho, \delta_1)$ is regular.

**Proof.** Consider the nondecreasing function $f : [0, 1] \to \mathbb{R}$ defined as

$$f(a) := d^{-2}_\rho \log \mu(\text{Bohr}(\rho, 2^a \delta)).$$

By the first part of Proposition 27, we have $f(1) - f(0) \leq \log(21/2)$. Clearly, if we could find $a \in [0.1, 0.9]$ such that $|f(a) - f(a')| \leq 25|a - a'|$ for all $|a' - a| \leq 0.1$, then the set $\text{Bohr}(\rho, 2^a \delta)$ would be regular. If there is no such $a$, then for every $a$ from this interval there is an interval $I_a$, $a \in I_a$, $|I_a| \leq 0.1$, with $\int_{I_a} df > 25|I_a|$. Obviously, these intervals cover $[0.1, 0.9]$, and by the Vitali covering lemma (see, e.g., [19, Sect. 4.4]) one can find a finite subcollection of disjoint intervals of total measure at least $0.8/5$, say. Then

$$\log \frac{21}{2} \geq \int_0^1 df \geq 25 \cdot \frac{0.8}{5} = 4,$$

which is a contradiction. □

Finally, let us say something nontrivial about the spectrum of regular Bohr sets.

**Proposition 30.** Let $B = \text{Bohr}(\rho, \delta)$ be a regular Bohr set and $B' = \text{Bohr}(\rho, \delta')$, where $\delta' \leq \kappa \delta/(100d^2_\rho)$ and $\kappa \in (0, 1)$ is a real number. Then

$$\text{Spec}_\varepsilon(B) \subseteq \text{Spec}_{1-2\kappa/\varepsilon}(B').$$

**Proof.** Let $\pi \in \text{Spec}_\varepsilon(B)$. Let also $B^\pm = \text{Bohr}(\rho, \delta \pm \delta')$. We have

$$\varepsilon|B| \leq \| \widehat{B}(\pi) \| \leq \| B' \|^{-1} \| \widehat{B}(\pi) \| \cdot \| \widehat{B}'(\pi) \| + \left\| \sum_x (B(x) - |B'|^{-1}(B * B')(x)) \pi(x) \right\|$$

$$\leq |B'|^{-1} \| \widehat{B}(\pi) \| \cdot \| \widehat{B}'(\pi) \| + \sum_x |B(x) - |B'|^{-1}(B * B')(x)|. \quad (A.6)$$
It is easy to see that the summation in (A.6) is taken over $B^+ \setminus B^-$. By the regularity of $B$ one can estimate this sum as $2\kappa |B|$. Hence

$$\|\hat{B'}(\pi)\| \geq |B'| (1 - 2\kappa |B| \cdot \|\hat{B}(\pi)\|^{-1}) \geq |B'| (1 - 2\kappa \varepsilon^{-1})$$

or, in other words, $\pi \in \text{Spec}_{1 - 2\kappa \varepsilon^{-1}}(B')$. This completes the proof. □

**FUNDING**

This work is supported by the Russian Science Foundation under grant 19-11-00001.

**REFERENCES**

1. L. A. Bassalygo and M. S. Pinsker, “On the complexity of an optimal non-blocking commutation scheme without reorganization,” Probl. Inf. Transm. 9, 64–66 (1974) [transl. from Probl. Peredachi Inf. 9 (1), 84–87 (1973)].

2. J. Bourgain, “On triples in arithmetic progression,” Geom. Funct. Anal. 9 (5), 968–984 (1999).

3. P. Diaconis and L. Saloff-Coste, “Comparison techniques for random walk on finite groups,” Ann. Probab. 21 (4), 2131–2156 (1993).

4. P. Erdős and P. Turán, “On a problem of Sidon in additive number theory, and on some related problems,” J. London Math. Soc. 16, 212–215 (1941).

5. E. Kowalski, An Introduction to Expander Graphs (Soc. Math. France, Paris, 2019).

6. V. F. Lev, “Distribution of points on arcs,” Integers 5 (2), A11 (2005).

7. A. Lubotzky, Discrete Groups, Expanding Graphs and Invariant Measures (Birkhäuser, Basel, 1994), Prog. Math. 125.

8. G. A. Margulis, “Explicit group-theoretical constructions of combinatorial schemes and their application to the design of expanders and concentrators,” Probl. Inf. Transm. 24, 39–46 (1988) [transl. from Probl. Peredachi Inf. 24 (1), 51–60 (1988)].

9. N. G. Moshchevitin, “On numbers with missing digits: An elementary proof of a result due to S. V. Konyagin,” Chebyshev. Sb. 3 (2), 93–99 (2002).

10. M. A. Naimark, Theory of Group Representations (Fizmatlit, Moscow, 2010) [in Russian].

11. K. O’Bryant, “A complete annotated bibliography of work related to Sidon sequences,” Electron. J. Comb. DS11, 1–39 (2004).

12. M. S. Pinsker, “On the complexity of a concentrator,” in 7th International Teletraffic Congress (ITC, Stockholm, 1973), pp. 318/1–318/4.

13. L. Saloff-Coste, “Lectures on finite Markov chains,” in Lectures on Probability Theory and Statistics (Springer, Berlin, 1997), Lect. Notes Math. 1665, pp. 301–413.

14. T. Sanders, “A quantitative version of the non-abelian idempotent theorem,” Geom. Funct. Anal. 21 (1), 141–221 (2011); arXiv:0912.0308 [math.CA].

15. T. Sanders, “Approximate groups and doubling metrics,” Math. Proc. Cambridge Philos. Soc. 152 (3), 385–404 (2012).

16. T. Sanders, “Applications of commutative harmonic analysis,” Preprint (Math. Inst., Univ. Oxford, Oxford, 2015), http://people.maths.ox.ac.uk/~sanders/acha/notes.pdf.

17. P. Sarnak and X. Xue, “Bounds for multiplicities of automorphic representations,” Duke Math. J. 64 (1), 207–227 (1991).

18. J.-P. Serre, Représentations linéaires des groupes finis (Hermann, Paris, 1967), Collections Méthodes.

19. T. Tao and V. H. Vu, Additive Combinatorics (Cambridge Univ. Press, Cambridge, 2006), Cambridge Stud. Adv. Math. 105.

20. A. A. Yudin, “On the measure of the large values of the modulus of a trigonometric sum,” in Number-Theoretic Studies in the Markov Spectrum and in the Structural Theory of Set Addition (Kalininsk. Gos. Univ., Moscow, 1973), pp. 163–174 [in Russian].