ANOTHER LOOK AT SECOND ORDER CONDITION IN EXTREME VALUE THEORY

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Abstract. This note compares two approaches both alternatively used when establishing normality theorems in univariate Extreme Value Theory. When the underlying distribution function (df) is the extremal domain of attraction, it is possible to use representations for the quantile function and regularity conditions (RC), based on these representations, under which strong and weak convergence are valid. It is also possible to use the now fashion second order condition (SOC), whenever it holds, to do the same. Some authors usually favor the first approach (the SOC one) while others are fond of the second approach that we denote as the representational one. This note aims at comparing the two approaches and show how to get from one to the other. The auxiliary functions used in each approach are computed and compared. Statistical applications using simultaneously both approaches are provided. A final comparison is provided.

1. Introduction

Statistical modelling based on the univariate Extreme Values Theory usually requires regularity conditions of the underlying distributions. Since the work of de Haan and Stadtmüller ([7]) on the so-called Second Order Condition (SOC), using this SOC has become fashion in research papers so that this way of doing is the mainstream one, led by de Haan (see for instance [4], [12], [6]). However the second order condition does not always hold as we will show it (see (2.1)), although a large class of distribution functions fulfills it. Yet, there exists an other approach, that is the representational one, based on the Karamata representation for a slowly varying function. In this view, each distribution function $F$ in the extremal domain may be represented a couple of functions $p(s)$ and $b(s)$, $s \in (0, 1)$, to be precised in Theorem 1. This approach is the one preferred by many other authors, for instance Csörgő, Deheuvels, and Mason, ([2]), Lo (9), Hall (see [10], [11]) etc. This latter in particularly adapted for the use on the Gaussian approximations like that of Csörgő-Csörgő-Horváth-Mason ([1]).

This motivates us to undertake here a comparative study of the second order condition in the two approaches and provide relations and methods for moving from one to the other. We give specific statistical applications using simultaneously the two ways. The paper is to serve as a tool for comparative reading of papers based on the two approaches.

The paper is organized as follows. In Section 1, we introduce the second order condition in the frame of de Haan and Stadtmüller ([7]) using quantile functions.

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Key words and phrases. Extreme value Theory; quantile functions; quantile representation; Theorem of Karamata; Slowly and regularly variation; second order condition; statistical estimation; asymptotic normality.
In Section 3, we recall the representational scheme and link them to the second order condition. Precisely, we express the second order condition, when it holds, through the couple of functions \((p, b)\) associated with a df attracted to the extremal domain. The results are then given through the df \(G(x) = F(e^x), x \in \mathbb{R}\), that is the most used in statistical context. In Section 4, we settle a new writing the SOC for the quantiles while the auxiliary functions of that condition, denoted as \(s\) and \(S\), are computed for a large number of df’s. In Section 6, we deal with applications in statistical contexts. The first concerns the asymptotic normality of the large quantiles process and the second treats the functional Hill process. In both cases, we use the two approaches. We finish by comparing the two methods at the light of these applications.

2. The second order condition

2.1. Definition and expressions. Consider a df \(F\) lying in the extremal domain of attraction of the Generalized Extreme Value (GEV) distribution, that is

\[
G_\gamma(x) = \exp(-(1 + \gamma x)^{-1/\gamma}), \text{ for } 1 + \gamma x > 0,
\]
denoted \(F \in D(G_\gamma)\), and let \(U(x) = (1/(1 - F))^{-1}\), where for any nondecreasing and right-continuous function \(L : \mathbb{R} \rightarrow [a, b]\), with \(a < b\),

\[
L^{-1}(t) = \inf\{x \geq t, L(x) \geq t\}, a \leq t \leq b,
\]
is the generalized inverse of \(L\). One proves (see [6], p. 43) that there exists a positive function \(a(t)\) of \(t \in \mathbb{R}\), such that

\[
\forall (x > 0), \lim_{t \to \infty} \frac{U(xt) - U(t)}{a(t)} = \frac{x^\gamma - 1}{\gamma} = D_\gamma(x),
\]
where \((x^\gamma - 1)/\gamma\) is interpreted as \(\log(x)\) for \(\gamma = 0\). Now, by definition, \(F\) is said to satisfy a second order condition ([7]) if and only if there exists a function \(A(t)\) of \(t \in \mathbb{R}\) with a constant sign such that

\[
\text{(SOCU)} \quad \lim_{t \to \infty} \frac{U(xt) - U(t)}{A(t)} = H(x),
\]
holds. According to Theorem 2.3.3 in (de haan and Ferreira), the function \(H\), when it is not a multiple of \(D_\gamma(x)\), can be written as

\[
H_{\gamma, \rho}(x) = c_1 \int_1^x s^{\gamma - 1} \int_1^s u^{\gamma - 1} du \, ds + c_2 \int_1^x s^{\gamma + \rho - 1} ds,
\]
where \(\rho\) is a negative number and the functions \(a(t)\) and \(A(t)\) satisfies for any \(x > 0\),

\[
\lim_{t \to \infty} \frac{a(tx)/a(t) - x^\gamma}{A(t)} = c_1 x^\gamma x^\rho - 1 \rho
\]
and

\[
\lim_{t \to \infty} A(t x)/A(t) = x^\rho.
\]
According to Corollary 2.3.5 in de Haan and Ferreira, one can choose a positive function \(a^*(t)\) and a function \(A^*(t)\) of constant sign such that

\[
\frac{U(xt) - U(t)}{a^*(t)} = H_{\gamma, \rho}(x),
\]
with

\[
H_{\gamma,\rho}(x) = \begin{cases} 
\frac{x^{\gamma+\rho-1}}{\gamma+\rho}, & \gamma + \rho \neq 0, \rho < 0, \\
\log x, & \gamma + \rho = 0, \rho < 0, \\
\frac{1}{\gamma}x^\gamma \log x, & \gamma \neq 0, \rho = 0, \\
(\log(x))^2/2, & \gamma = \rho = 0,
\end{cases}
\]

\[
a^*(t) = \begin{cases} 
0, & \rho > 0, \\
a(t)(1 - A(t)/\rho), & \rho = 0, \gamma \neq 0, \\
a(t), & \gamma = 0.
\end{cases}
\]

and

\[
A^*(t) = A(t)/\rho I(\rho > 0) + A(t)I(\rho = 0).
\]

To see that the SOC does not necessary hold, consider the standard exponential distribution function. We have

\[
U(t) = \log t
\]

and

\[
(2.1) \quad U(tx) - U(t) = \log x = [(x^\gamma - 1)/\gamma]_{\gamma=0}.
\]

It is clear that the function \(a\) is necessarily constant and equal to the unity and the second order condition is here meaningless. As a consequence, the results obtained under a second order condition are partial.

2.2. Expression in terms of generalized inverse functions. We are going to express the SOC through the generalized function \(F^{-1}(1-u)\). Let

\[
F \in D(G_{\gamma}).
\]

With this parameterization, the case \(\gamma = 0\) corresponds to \(D(\Lambda)\), the case \(-\infty < \gamma < 0\) to \(D(\psi_{1/\gamma})\) and finally, the case \(0 < \gamma < +\infty\) to \(D(\phi_{1/\gamma})\). The second order condition will become: there exist a positive function \(s(u)\) and a function \(S(u)\) with constant sign such that for any \(x > 0\),

\[
(\text{SOCF}) \quad \frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \gamma^{-1}(x^\gamma - 1) = H_{\gamma,\rho}(1/x) = h_{\gamma,\rho}(x).
\]

3. Representation for \(F \in D(G_{\gamma})\)

3.1. Representations. Now we recall the classical representations of \(df\) attracted to some nondegenerated extremal \(df\).

**Theorem 1.** We have:

1. **Karamata’s representation (KARARE)**
   
   (a) If \(F \in D(G_{\gamma})\), \(\gamma > 0\), then there exist two measurable functions \(p(u)\) and \(b(u)\) of \(u \in (0,1)\) such that \(\sup(|p(u)|, |b(u)|) \to 0\) as \(u \to 0\) and a positive constant \(c\) so that

   \[
   (3.1) \quad G^{-1}(1-u) = \log c + \log(1 + p(u)) - \gamma \log u + \left(\int_u^1 b(t)t^{-1}dt\right), \quad 0 < u < 1,
   \]

   where \(G^{-1}(u) = \inf\{x, G(x) \geq u\}, 0 < u \leq 1\) is the generalized inverse of \(G\) with \(G^{-1}(0) = G^{-1}(0+)\).
(b) If \( F \in D(G_\gamma) \), \( \gamma < 0 \), then \( y_0(G) = \sup\{x, G(x) < 1\} < +\infty \) and there exist two measurable functions \( p(u) \) and \( b(u) \) for \( u \in (0, 1) \) and a positive constant \( c \) as defined in (3.1) such that

\[
y_0 - G^{-1}(1 - u) = c(1 + p(u))u^{-\gamma} \exp\left(\int_u^1 b(t)t^{-1}dt\right), 0 < u < 1.
\]

(2) Representation of de Haan (Theorem 2.4.1 in [5]),
If \( G \in D(G_0) \), then there exist two measurable functions \( p(u) \) and \( b(u) \) of \( u \in (0, 1) \) and a positive constant \( c \) as defined in (3.1) such that for

\[
y(s) = c(1 + p(u)) \exp\left(\int_u^1 b(t)t^{-1}dt\right), \quad 0 < u < 1,
\]

we have for some constant \( d \in R \),

\[
G^{-1}(1 - u) = d - s(u) + \int_u^1 s(t)t^{-1}dt, 0 < u < 1.
\]

It is important to remark at once that any \( df \) in the extremal domain of attraction is associated with a couple of functions \( (p, b) \) used in each appropriate representation.

3.2. Preparation of second order condition. We are now proving, under \( F \in D(G_\gamma) \) that for \( x > 0 \),

\[
\lim_{u \to 0} \frac{F^{-1}(1 - ux) - F^{-1}(1 - u)}{s(u)} = d_\gamma(x) = \gamma^{-1}(x^{-\gamma} - 1).
\]

3.2.1. \( F \in D(G_0) \). The representation (3.4) is valid for \( F^{-1}(1 - u), u \in (0, 1) \). We get for \( u \in (0, 1) \) and \( x > 0 \):

\[
F^{-1}(1 - xu) - F^{-1}(1 - u) = s(u) - s(ux) + \int_{ux}^u s(t)/t dt.
\]

For \( v \in [\min(ux, u), \max(ux, u)] = A(u, x) \),

\[
s(v)/s(u) = \frac{1 + p(t)}{1 + p(u)} \exp\left(\int_v^t b(t)/t dt\right).
\]

By letting \( pr(u, x) = \sup\{|p(t)|, 0 \leq t \leq \max(ux, u)\} \) and \( br(u, x) = \sup\{|b(t)|, 0 \leq t \leq \max(ux, u)\} \), one quickly shows that, for \( u \) sufficiently small,

\[
\sup_{t \in A(u, x)} |1 - (1 + p(v))/(1 + p(u))| \leq 2pr(u, x).
\]

and

\[
x^{-br(u, x)} \leq \exp\left(\int_{ux}^u b(t)/t dt\right) \leq x^{br(u, x)}
\]

and then

\[
\sup_{v \in A(u, x)} \left| 1 - \exp\left(\int_v^u b(t)/t dt\right) \right| = O(-br(u, x)).
\]

It follows that

\[
\sup_{v \in A(u, x)} |1 - s(v)/s(u)| = O(max(pr(u, x), br(u, x)) \to 0,
\]
as $u \to 0$. We get, for $pbr(u, x) = pr(u, x) \times br(u, x)$,

$$\frac{F^{-1}(1 - xu) - F^{-1}(1 - u)}{s(u)} + \log x$$

$$= O(pbr(u, x)) + O(pbr(u, x) \log x) \to 0.$$ 

Then

$$\lim_{u \to 0} \frac{F^{-1}(1 - xu) - F^{-1}(1 - u)}{s(u)} = - \log x = \left[ \gamma(x^{-1/\gamma} - 1) \right]_{\gamma=\infty}.$$

We notice that

$$(3.5) \quad \frac{F^{-1}(1 - xu) - F^{-1}(1 - u)}{s(u)} + \log x$$

$$= O(p(u, x) + b(u, x)),$$

where

$$(3.6) \quad p(u, x) = 1 - (1 - p(ux))/(1 - p(u)) \to 0 \text{ as } u \to 0$$

and

$$(3.7) \quad b(u, x) = \int_{ux}^{u} t \left[ \frac{1 - p(t)}{(1 + p(u))} \exp(\int_{u}^{t} v^{-1} b(v) \, dv) - 1 \right] dt \to 0 \text{ as } u \to 0.$$

3.2.2. $F \in D(G_{1/\gamma})$ ($\gamma > 0$). We have the KARARE representation

$$F^{-1}(1 - u) = c(1 + p(u)) u^{-\gamma} \exp(\int_{u}^{1} t^{-1} b(t) \, dt), u \in (0, 1).$$

Then, for $s(u) = \gamma cu^{-\gamma} \exp(\int_{u}^{1} t^{-1} b(t) \, dt)$, for $x > 0$

$$\frac{F^{-1}(1 - ux) - F^{-1}(1 - u)}{s(u)} = \gamma^{-1} \left\{ (1 + p(ux)) x^{-1/\gamma} \exp(\int_{ux}^{x} t^{-1} b(t) \, dt) - 1 - p(u) \right\}.$$ 

As previously, we readily see that

$$\exp(\int_{ux}^{x} t^{-1} b(t) \, dt) = \exp(O(br(u, x) \log x)) = 1 + O(br(u, x) \log x) \to 1,$$

as $u \to 0$. It follows that

$$(3.8) \quad \frac{F^{-1}(1 - ux) - F^{-1}(1 - u)}{s(u)} \to \gamma^{-1}(x^{-\gamma} - 1).$$

Moreover we have

$$\frac{F^{-1}(1 - ux) - F^{-1}(1 - u)}{s(u)} - \gamma^{-1}(x^{-\gamma} - 1)$$

$$= \gamma^{-1} \left\{ x^{\gamma} \left( (1 + p(ux)) \exp(\int_{ux}^{x} t^{-1} b(t) \, dt) - 1 \right) - p(u) \right\} = pb(u, x).$$

Notice that we may also take $s(u) = F^{-1}(1 - u)$. 

3.2.3. \( F \in D(G_\gamma), \gamma < 0 \). We have \( x_0 = \sup \{ x, F(x) < 1 \} < +\infty \) and the following representation holds:

\[
x_0 - F^{-1}(1 - u) = c(1 + p(u))u^{-\gamma} \exp\left( \int_u^1 t^{-1}b(t)dt \right), u \in (0, 1).
\]

Then, for \( s(u) = -\gamma cu^{-\gamma} \exp(\int_u^1 t^{-1}b(t)dt), u \in (0, 1) \) and \( x > 0 \),

\[
\frac{F^{-1}(1 - ux) - F^{-1}(1 - u)}{s(u)} = \frac{(x_0 - F^{-1}(1 - u)) - (x_0 - F^{-1}(1 - ux))}{s(u)}
\]

\[
= -\gamma^{-1}\left\{1 + p(u) - x^{-\gamma}(1 + p(ux)) \exp(\int_{ux}^x t^{-1}b(t)dt)\right\}
\]

\[
= \gamma^{-1}\left\{x^{-\gamma}(1 + p(ux)) \exp(\int_{ux}^x t^{-1}b(t)dt) - 1 - p(u)\right\}
\]

\[
\rightarrow \gamma^{-1}(x^{-\gamma} - 1),
\]

as \( u \to 0 \). Likely to the case \( 0 < \gamma < \infty \),

\[
\frac{F^{-1}(1 - ux) - F^{-1}(1 - u)}{s(u)} - \gamma(x^{-1/\gamma} - 1)
\]

(3.10)

\[
= \gamma^{-1}\left\{x^{-\gamma}\left(1 + p(ux)\right) \exp(\int_{ux}^x t^{-1}b(t) dt) - 1 - p(u)\right\} = pb(u, x).
\]

Notice that we may also take \( s(u) = x_0 - F^{-1}(1 - u) \).

4. Second order condition via representations

4.1. Case by case.

4.1.1. \( F \in D(G_\gamma), 0 < \gamma < \infty \). The second order condition is equivalent to finding a function \( S(u) \) of constant sign such that

\[
S(u)^{-1}\left\{\frac{F^{-1}(1 - ux) - F^{-1}(1 - u)}{s(u)} - \gamma^{-1}(x^{-\gamma} - 1)\right\} = S(u)^{-1}pb(u, x)
\]

converges to a function \( h_{\gamma, \rho} \), where

\[
pb(u, x) = \gamma \left\{x^{-\gamma}\left(1 + p(ux)\right) \exp(\int_{ux}^x t^{-1}b(t) dt) - 1 - p(u)\right\}
\]

and \( s(u) \) may be taken as \( F^{-1}(1 - u) \).

4.1.2. \( F \in D(G_\gamma), -\infty < \gamma < 0 \). The second order condition is equivalent to finding a function \( S(u) \) of constant sign such that

\[
S(u)^{-1}\left\{\frac{F^{-1}(1 - ux) - F^{-1}(1 - u)}{s(u)} - \gamma^{-1}(x^{-\gamma} - 1)\right\} = S(u)^{-1}pb(u, x)
\]

converges to a function \( h_{\gamma, \rho} \) where

\[
pb(u, x) = \gamma^{-1}\left\{x^{-\gamma}\left(1 + p(ux)\right) \exp(\int_{ux}^x t^{-1}b(t) dt) - 1 - p(u)\right\}
\]

and \( s(u) \) may be taken as \( x_0 - F^{-1}(1 - u) \) and \( x_0 \) is the upper endpoint of \( F \).
4.1.3. \( F \in D(G_0) \). The second order condition is equivalent to finding a function \( S(u) \) of constant sign such that
\[
S(u)^{-1} \left\{ \frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} \log x \right\} = S(u)^{-1}(p(u,x) + b(u,x))
\]
converges to a function \( h_{1/\gamma,p} \), where
\[
p(u,x) = 1 - (1 - p(ux))/(1 - p(u)) \to 0 \text{ as } u \to 0
\]
and
\[
b(u,x) = \int_u^x \frac{1}{t} \frac{1 - p(t)}{1 + p(u)} \exp(\int_u^t v^{-1} b(v) \, dv) - 1) \, dt \to 0 \text{ as } u \to 0.
\]
and \( s(u) \) may be taken \( u^{-1} \int_u^1 (1-s) dF^{-1}(s) \).

5. Special cases

5.1. Statistical context. In the statistical context, especially in the extreme value index estimation, the bulk of the work is done with
\[
G^{-1}(1-u) = \log F^{-1}(1-u).
\]
Let \( F \in D(G_\gamma) \). The three cases \( -\infty < \gamma < 0, 0 < \gamma < +\infty, \gamma = 0 \) respectively imply
\[
G \in D(G_{1/\gamma}),
\]
\[
G \in D(G_0) \text{ and } s(u,G) \to \gamma
\]
and
\[
G \in D(G_0) \text{ and } s(u,G) \to 0.
\]
For \( F \in D(G_\gamma) \), \( 0 < \gamma < \infty \), we have a representation like
\[
G^{-1}(1-u) = c + \log(1 + p(u)) - \gamma \log u + \int_u^1 t^{-1} b(t) \, dt.
\]
We take here \( s(u) = \gamma \). The second order conditions becomes
\[
S(u)^{-1} \left\{ \frac{G^{-1}(1-ux) - G^{-1}(1-u)}{\gamma} \log x \right\} = S(u)^{-1} \left\{ \gamma^{-1} \log \frac{1 + p(ux)}{1 + p(u)} + \gamma^{-1} \int_u^x t^{-1} b(t) \, dt \right\}
\]
\[
= A(u)^{-1} \int u x p(b(u,x) \to h_{0,\rho}(x)).
\]
Denote \( dG^{-1}(1-u)/du = G^{-1}(1-u)' \) whenever if exists. Now if \( G^{-1}(1-u)' \) exists for \( u \) near zero, we may take
\[
b(u) = G^{-1}(1-u)' + \gamma \to 0,
\]
For \( F \in D(G_\gamma) \), \( -\infty < \gamma < 0 \), we may transfer the SOC to \( G \) in a way similar as to \( F \), with
\[
\log x_0 - G^{-1}(1-u) = c(1 + p(u))u^\gamma \exp(\int_u^1 t^{-1} b(t) \, dt).
\]
For \( F \in D(G_0) \). If \( s(u) = u(G^{-1}(1-u)' \to 0 \), we will have
\[
G^{-1}(1-s) = d - \int_u^1 t^{-1} s(t) \, dt.
\]
We may take
\[
b(u) = us'(u). \]
The second order condition becomes simpler as
\begin{align*}
S(u)^{-1}b(u, x) &= S(u)^{-1} \int_{u_x}^{u} \frac{1}{t} \left( \frac{1 - p(t)}{1 + p(u)} \exp\left( \int_{u}^{t} v^{-1}b(v) \, dv \right) - 1 \right) dt \\
&\rightarrow h_{0, \rho}(x).
\end{align*}

Moreover, for \( g(x) = dG(x)/dx \), if
\begin{align*}
b(u) &= us'(u)/s(u) = 1 - u(G^{-1}(1 - u))^{-1} \rightarrow 0,
\end{align*}
we have
\begin{align*}
s(u) &= c \exp\left( \int_{u}^{1} t^{-1}b(t) \, dt \right),
\end{align*}
and the SOC becomes
\begin{align*}
S(u)^{-1}b(u, x) &= S(u)^{-1} \int_{u_x}^{u} \frac{1}{t} \left[ \exp\left( \int_{u}^{t} \frac{1}{v} - \frac{1}{G^{-1}(1 - \nu)g(G^{-1}(1 - \nu))} - 1 \right) dv \right] dt \\
&\rightarrow h_{0, \rho}(x).
\end{align*}

6. Finding the functions \( b \) and \( S \).

6.1. Determination of the function \( b \). In the usual cases, the function is ultimately differentiable, that is in a right neighbourhood of \( x_0(F) \). It is then easy to find the function \( b \) by derivating \( G^{-1}(1 - u) \). In summary, for \( D(G_{\gamma}) \), \( \gamma > 0 \), the function \( b \) in the representation of \( G^{-1}(1 - u) \) is
\begin{align*}
b(u) &= - \left\{ \gamma + u(G^{-1}(1 - u))' \right\}.
\end{align*}
The function \( b \) in the representation of \( F^{-1}(1 - u) \) is defined by
\begin{align*}
-b(u) &= \gamma + u(G^{-1}(1 - u))'/G^{-1}(1 - u)
\end{align*}
For \( \gamma = +\infty \), the function \( b \) is the representation of \( G^{-1}(1 - u) \), that is,
\begin{align*}
b(u) &= -us'(u)/s(u),
\end{align*}
where
\begin{align*}
s(u) &= u(G^{-1}(1 - u))'.
\end{align*}
For \( \gamma < 0 \) and \( y_0(G) = y_0 \),
\begin{align*}
b(u) &= -\gamma + u(G^{-1}(1 - u))'/(x_0 - G^{-1}(1 - u))
\end{align*}
Then we apply these formulas and determine the function \( b \) for usual df’s. Regularity conditions in the representational approach mainly rely on the function \( b \), while they rely of the function \( S \) for the SOC approach. It is then interesting to have both functions for usual df’s in tables in Subsection 6.3 following [12].

6.2. The function \( S \) for the second order condition. Functions \( a \) and \( A \) in the \((SOCU)\), as well as the functions \( H_{\gamma, \rho} \) are available in the usual cases (see [12] for example). It is not the case for the \((SOCF)\) expressed in terms of the quantile functions. We then seize this opportunity to compute their analogs \( s \) and \( S \) in the this case for the usual df’s. The results are summarized in our tables in Subsection 6.3.
6.2.1. The Singh-Maddala Law. Let for constants \( a, b \) and \( c \), for \( x \geq 0 \),
\[
1 - F(x) = (1 + ax^b)^{-c},
\]
the so-called Singh-Maddala df. This function plays a special role in income fitting
distribution. It is clear that
\[
F \in D(G_{bc}).
\]
Put
\[
\lambda = 1/\beta c.
\]
Straightforward calculations give
\[
G^{-1}(1-u) = -b^{-1}\log a - \lambda \log u + b^{-1}\log(1 - u^{1/c})
\]
\[
= d - \gamma \log u - b^{-1}\log(1 - u_0) + \int_{u_0}^{u} t^{-1}B(t)dt,
\]
where
\[
a = -b^{-1}\log a,
\]
\[
B(u) = \gamma u^{1/c}(1 - u^{1/c})^{-1}
\]
and \( u_0 \in [0, 1] \). Put \( K_0 = -\frac{1}{b} \log(1 - u_0)/u_0 \) and
\[
b(u) = B(u)\mathbb{I}_{(0 \leq u \leq u_0)} + K_0\mathbb{I}_{(u_0 \leq u \leq 1)},
\]
we get
\[
G^{-1}(1-u) = d - \gamma \log u + \int_{u}^{1} t^{-1}b(t)dt,
\]
with
\[
b(u) \to 0.
\]
We have
\[
\frac{G^{-1}(1-ux) - G^{-1}(1-u)}{\gamma} + \log x = \frac{1}{\gamma b} (\log(1-(ux)^{1/c}) - \log(1 - u^{1/c}))
\]
\[
= \frac{1}{\gamma b} (-x^{1/c}u^{1/c} + u^{1/c} + O(u^{2/c})).
\]
Thus, for \( S(u) = cu^{1/c}/(\gamma b) \), we get
\[
\frac{G^{-1}(1-ux) - G^{-1}(1-u)}{S(u)} - \log x = \frac{x^{1/c} - 1}{-1/c} = h_{0,\rho}(x) = H_{0,\rho}(1/x).
\]
This corresponds to a second order condition. As for \( F^{-1}(1-u) \) itself, we have
\[
F^{-1}(1-u) = (1/a)^{1/b}(1 - u^{-1/c})^{1/b}
\]
and
\[
b(u) = F^{-1}(1-u)'/F^{-1}(1-u) + \frac{1}{bc} = (1 - \frac{u^{-1/c}}{1 + u^{-1/c}})/bc \to 0.
\]
We have for \( s(u) = F^{-1}(1-u)/(bc) \),
\[
\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} = bc\left(\frac{1 - x^{-1/c}u^{-1/c}}{1 - u^{-1/c}}\right)^{1/b} - 1 \to \frac{x^{-1/bc} - 1}{1/bc}.
\]
Next
\[
\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \frac{x^{-1/bc} - 1}{1/bc} = (bc)\left\{ \left( \frac{1 - x^{-1/c}u^{-1/c}}{1 - u^{-1/c}} \right)^{1/b} - \left( x^{-1/c} \right)^{1/b} \right\} = (bc)\left\{ \frac{1 - x^{-1/c}u^{-1/c}}{1 - u^{-1/c}} - \frac{x^{-1/c}(1 - u^{-1/c})}{1 - u^{-1/c}} \right\} \zeta(u,x)^{(1/b)-1},
\]
where \( \zeta(u,x) \in I(x^{-1/c}, b) \), with \( b = (1 - x^{-1/c}u^{-1/c})/(1 - u^{-1/c}) \sim x^{-1/c} \). Hence
\[
\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \frac{x^{-1/bc} - 1}{1/bc} = c\left\{ \frac{1 - x^{-1/c}u^{-1/c}}{1 - u^{-1/c}} - \frac{x^{-1/c}(1 - u^{-1/c})}{1 - u^{-1/c}} \right\} \zeta(u,x)^{(1/b)-1} = c\left\{ \frac{1 - x^{-1/c}}{1 - u^{-1/c}} \right\} \zeta(u,x)^{(1/b)-1}.
\]
Finally for \( S(u) = -(1 - u^{-1/c})^{-1} \),
\[
\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{S(u)} - \frac{x^{-1/bc} - 1}{1/bc} \rightarrow h(x) = \frac{(x^{-1/c} - 1)x^{-1/(bc)+1/c}}{1/c}
\]
6.2.2. Burr’s df. \( 1 - F(x) = (x^{-\rho/\gamma} + 1)^{1/\rho}, \rho < 0 \), and
\[
F^{-1}(1-u) = (w^\rho - 1)^{-\gamma/\rho}.
\]
For \( s(u) = \gamma F^{-1}(1-u) \),
\[
\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} = \frac{1}{\gamma} \left\{ \left( \frac{x^\rho w^\rho - 1}{w^\rho - 1} \right)^{-\gamma/\rho} - 1 \right\} = \frac{1}{\gamma} \left\{ \left( \frac{x^\rho w^\rho - 1}{w^\rho - 1} \right)^{-\gamma/\rho} - 1 \right\} \rightarrow \frac{x^{-\gamma} - 1}{\gamma}.
\]
Next
\[
\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \frac{x^{-\gamma} - 1}{\gamma} = \frac{1}{\gamma} \left\{ \left( \frac{x^\rho w^\rho - 1}{w^\rho - 1} \right)^{-\gamma/\rho} - x^{-\gamma} \right\} = \frac{1}{\gamma} \left\{ \left( \frac{x^\rho w^\rho - 1}{w^\rho - 1} \right) - (x^\rho)^{-\gamma/\rho} \right\} = \frac{1}{\rho} \left\{ \frac{x^\rho w^\rho - 1 - x^\rho (w^\rho - 1)}{w^\rho - 1} \right\} \zeta(u,x)^{-\gamma/\rho-1} \rightarrow \frac{x^{-\gamma} - 1}{\gamma}.
\]
where \( \zeta(u,x) = I(a,b) = |a \land b, a \lor b| \), with \( a = x^\rho \) and \( b = (x^\rho w^\rho - 1)/(w^\rho - 1) \sim x^\rho \). Hence
\[
\frac{F^{-1}(1-ux) - F^{-1}(1-u)}{s(u)} - \frac{x^{-\gamma} - 1}{\gamma} = -\frac{1}{\rho} \left\{ \frac{x^\rho w^\rho - 1 - x^\rho (w^\rho - 1)}{w^\rho - 1} \right\} \zeta(u,x)^{-\gamma/\rho-1}.
\]
Hence for $S(u) = (u^\rho - 1)^{-1}$,

$$\frac{F^{-1}(1 - ux) - F^{-1}(1 - u)}{s(u)} - \frac{x^{-\gamma} - 1}{\gamma} \rightarrow - \frac{(x^\rho - 1)x^{-\gamma - \rho}}{\rho}. $$

6.2.3. Log Exponential law. $F(x) = 1 - \exp(-e^{-x})$, that is

$$F^{-1}(1 - u) = \log \log(1/u).$$

We have

$$F^{-1}(1 - u)' = -1/(u \log u)$$

and let

$$s(u) = uF^{-1}(1 - u)' = -1/\log u$$

so that

$$s'(u) = -1/(u(\log u)^2).$$

Finally

$$b(u) = -(\log u)^{-2} \rightarrow 0.$$  

With the representation

$$F^{-1}(1 - u) = c + \int_u^1 \frac{s(t)}{t} dt,$$

where $s$ is slowly varying at zero, we get that

$$\frac{F^{-1}(1 - ux) - F^{-1}(1 - u)}{s(u)} \rightarrow - \log x.$$  

We can use direct methods and get

$$F^{-1}(1 - ux) - F^{-1}(1 - u) = \log(\log(ux)/\log u).$$

We remark $v(u, x) = \log(ux)/\log u \rightarrow 1$, and that $v(u, x) - 1 = (\log x)/\log u$. We may use the expansion of the logarithm function and get

$$F^{-1}(1 - ux) - F^{-1}(1 - u) = (v - 1) - (v - 1)^2 + O((v - 1)^3)$$

$$= (\log x)/\log u + ((\log x)/\log u)^2/2 + O((\log x)/\log u^3).$$

By putting

$$S(u) = s(u) = 1/(\log u),$$

It comes that

$$\frac{F^{-1}(1 - ux) - F^{-1}(1 - u)}{s(u)} + \log x$$

$$\rightarrow (\log x)^2/2.$$
6.2.4. Normal standard. Let $F$ be the d.f. of a standard normal law. We have the simple approximation, for $M = \sqrt{2\pi}$, for $x > 1$,

$$M^{-1}(x^{-1} - x^{-3})e^{-x^2/2} \leq 1 - F(x) \leq M^{-1}e^{-x^2/2}.$$  

For $s = 1 - F(x)$,

$$-\log M - \log x - \log(1 - x^{-2}) - x^2/2 \leq s \leq -\log M - \log x - x^2/2$$

$$-\log M - \log x + \frac{1}{x} + O(x^{-2}) - x^2/2 \leq \log s \leq -\log M - \log x - x^2/2$$

$$\log M + \log x + x^2/2 \leq \log(1/s) \leq \log M + \log x - \frac{1}{x} + O(x^{-2}) + x^2/2.$$  

And as $x \to 0 \iff s \to 0$,

$$x = F^{-1}(1 - s) = (2 \log 1/s)^{1/2}(1 + o(1)).$$

We easily see that the $o(1)$ term is at least of order $(\log 1/s)^{-1}$. This gives

$$x^2/2 + \log M + \log x \leq \log(1/s) \leq x^2/2 + \log M + \log x - \frac{1}{x} + O(x^{-2})$$

$$\log(1/s) + \log(1/s) \times o(1) + \log M + \log((2 \log 1/s)^{1/2}(1 + o(1)).$$

The left term is

$$= \log(1/s)(1 + o(1)) + \log M(1/2) \log 2 + (1/2) \log \log(1/s) + o(1).$$

The right term is

$$\log(1/s)(1 + o(1)) + \log M + (1/2) \log 2 + (1/2) \log \log(1/s)$$

$$- (2 \log 1/s)^{-1/2}(1 + o(1)) + O((\log 1/s)^{-1}) + o(1).$$

The middle term is

$$(1 + o(1))x^2/2.$$  

By dividing by $\log(1/s)$, we get

$$1 + \frac{(1/2) \log 4\pi + (1/2) \log \log(1/s) + (2 \log 1/s)^{-1/2}(1 + o(1)) + O((\log 1/s)^{-1})}{\log(1/s)}$$

$$\leq (x/(2 \log(1/s))^{1/2})^2 + o(1) \times (x/(2 \log(1/s))^{1/2})^2$$

$$\leq 1 + \frac{(1/2) \log 4\pi + (1/2) \log \log(1/s)}{\log(1/s)}.$$  

Then

$$1 + \frac{(2 \log 1/s)^{-1/2}(1 + o(1)) + O((\log 1/s)^{-1}) + \log(1/s) \times o(1)}{\log(1/s)}$$

$$\leq (x/(2 \log(1/s))^{1/2})^2 - \frac{(1/2) \log 4\pi + (1/2) \log \log(1/s)}{\log(1/s)}$$

$$= (x/(2 \log(1/s))^{1/2})^2 - \frac{(1/2) \log 4\pi + (1/2) \log \log(1/s)}{\log(1/s)}$$

$$= 1 + \frac{o(1)}{\log(1/s)}.$$  

Then

$$x^2 = (2 \log(1/s)) \left\{ 1 - \frac{(1/2) \log 4\pi + (1/2) \log \log(1/s) + o(1)}{\log(1/s)} \right\},$$
and
\[ x = F^{-1}(1 - u) = (2 \log(1/s))^{1/2} \left\{ 1 - \frac{(1/2) \log 4\pi + (1/2) \log \log(1/s) + o(1)}{2 \log(1/s)} \right\}. \]

We have
\[ F^{-1}(1 - s) - F^{-1}(1 - x s) = A(x, s) + B(x, s) + C(x, s) \]
with
\[ A(x, s) = (2 \log(1/s))^{1/2} - (2 \log(x/s))^{1/2} = -(2 \log(1/s))^{1/2}(\frac{\log(x/s)}{\log(1/s)})^{1/2} - 1). \]
But
\[ \left(\frac{\log(x/s)}{\log(1/s)}\right)^{1/2} = \left(1 + \frac{\log(x/s)}{\log(1/s)} - 1\right)^{1/2} \]
\[ = 1 + \frac{1}{2} \left(\frac{\log(x/s)}{\log(1/s)} - 1\right) - \frac{1}{8} \left(\frac{\log(x/s)}{\log(1/s)} - 1\right)^2 + O\left(\frac{\log(x/s)}{\log(1/s)} - 1\right)^3 \]
\[ = 1 + \frac{1}{2} \log x - \frac{1}{2} (\frac{\log x}{2 \log 1/s}) + O\left(\frac{\log 1/s}{2} \right)^{-\frac{1}{2}}. \]
We get
\[ A(x, s) = -\frac{\log x}{(2 \log 1/s)^{1/2}} - \frac{1}{2} (\frac{\log x}{2 \log 1/s})^{3/2} + \frac{O\left(\log 1/s \right)^{-5/2}}{2} \]
and
\[ B(x, s) = \frac{(1/2) \log 4\pi + (1/2) \log \log(x/s) + o(\log 1/s)^{-1})}{(2 \log(x/s))^{1/2}} \]
\[ = \frac{(1/2) \log 4\pi + (1/2) \log \log(1/s) + o(\log 1/s)^{-1})}{(2 \log(x/s))^{1/2}} \]
\[ C(x, s) = \frac{(1/2) \log 4\pi + (1/2) \log \log(1/s) + o(\log 1/s)^{-1})}{(2 \log(1/s))^{1/2}} \]
and
\[ B(x, s) = \frac{1}{(2 \log(x/s))^{1/2}} \left(\frac{1}{2} \log(x/s) + o(\log 1/s)^{-1}\right). \]
But
\[ \log(x/s)/\log(1/s) = \log(1 + (\log(x/s)/\log(1/s) - 1)) \]
\[ = (\log(x/s)/\log(1/s) - 1) - \frac{1}{2} (\log(x/s)/\log(1/s) - 1)^2 \]
\[ + O\left((\log(x/s)/\log(1/s) - 1)^3\right) \]
\[ = \frac{\log x}{\log 1/s} - \frac{1}{2} (\frac{\log x}{2 \log 1/s})^2 + O((\log 1/s)^{-3}). \]
Then
\[
B(x, s) = \frac{1}{(2 \log(1/s))^{1/2}} \left\{ \frac{\log x}{\log 1/s} - \frac{1}{2} \frac{(\log x)^2}{(\log 1/s)^2} + \mathcal{O}((\log 1/s)^{-3}) \right\}
+ \mathcal{O}(\log 1/s)^{-3/2},
\]
and
\[
C(x, s) = \frac{(1/2) \log 4\pi + (1/2) \log \log(1/s) + \mathcal{O}((\log 1/s)^{-1})}{(2 \log(1/s))^{1/2}} \times \left\{ (2 \log(1/s))^{1/2} - (2 \log(x/s))^{1/2} \right\}
= \frac{(1/2) \log 4\pi + (1/2) \log \log(1/s) + \mathcal{O}((\log 1/s)^{-1})}{(2 \log(1/s))^{1/2}} \times \left\{ -\frac{\log x}{(2 \log 1/s)^{1/2}} - \frac{1}{2} \frac{(\log x)^2}{(2 \log 1/s)^{3/2}} + \mathcal{O}((\log 1/s)^{-5/2}) \right\}.
\]
Recall
\[
A(x, s) = \left\{ -\frac{\log x}{(2 \log 1/s)^{1/2}} - \frac{1}{2} \frac{(\log x)^2}{(2 \log 1/s)^{3/2}} + \mathcal{O}((\log 1/s)^{-5/2}) \right\}.
\]
We conclude that
\[
(\frac{2 \log(1/s))^{1/2}}{(1/2) \log 4\pi + (1/2) \log \log(1/s)} \left\{ \frac{F^{-1}(1-s) - F^{-1}(1-xs)}{(2 \log(1/s))^{-1/2}} + \log x \right\} \to -\log x.
\]
6.2.5. Lognormal. We have
\[
G^{-1}(1-s) = \exp(F^{-1}(1-s)), \ s \in (0, 1),
\]
where \( F \) is standard normal. This gives
\[
G^{-1}(1-u) - G^{-1}(1-ux) = \exp(F^{-1}(1-s)) \times (1 - \exp(F^{-1}(1-xs) - F^{-1}(1-s))).
\]
But
\[
\exp(F^{-1}(1-xs) - F^{-1}(1-s)) = \exp \left( -\frac{D(x, s)}{(2 \log(1/s))^{1/2}} \right)
= \exp \left( -\frac{D(x, s)}{(2 \log(1/s))^{1/2}} \right)
= 1 - \frac{D(x, s)}{(2 \log(1/s))^{1/2}} + \frac{1}{2} \frac{D(x, s)^2}{(2 \log(1/s))^{1/2}} + \mathcal{O} \left( \frac{1}{(2 \log(1/s))^{1/2}} \right),
\]
where \( D(x, s) \) is defined in (6.1). This yields
\[
\frac{G^{-1}(1-s) - G^{-1}(1-sx)}{\exp(F^{-1}(1-s))(2 \log(1/s))^{-1/2}} = D(x, s) + \frac{1}{2} \frac{D(x, s)^2}{(2 \log(1/s))^{1/2}} + \mathcal{O} \left( \frac{1}{(2 \log(1/s))^{1/2}} \right);
\]
Thus
\[
\frac{(2 \log(1/s))^{1/2}}{(1/2) \log 4\pi + (1/2) \log \log(1/s)} \left\{ \frac{G^{-1}(1-s) - G^{-1}(1-sx)}{\exp(F^{-1}(1-s))(2 \log(1/s))^{-1/2}} + \log x \right\} \to -\frac{1}{2} \frac{(\log x)^2}{(2 \log(1/s))^{1/2}}.
\]
6.2.6. **Logistic law.**

\[ F(x) = 1 - 2/(1 + e^x), \]

that is

\[ F^{-1}(1 - s) = \log s^{-1}(2 - s). \]

Routine computations yield

\[
F^{-1}(1 - s) - F^{-1}(1 - xs) = \log x + \log(2 - s)/(2 - xs) = \log x + \log(1 + (s(x - 1)/(2 - xs))) = \log x + (s(x - 1)/(2 - xs)) + O(s^2).
\]

Thus

\[
(s/2)^{-1} \left\{ F^{-1}(1 - s) - F^{-1}(1 - xs) - \log x \right\} \to \frac{x - 1}{2}.
\]

6.2.7. **Log-Expo.**

We have

\[ F^{-1}(1 - s) = \log(\log(1/s)), \quad s \in (0, 1). \]

\[
\begin{align*}
F^{-1}(1 - s) - F^{-1}(1 - xs) &= \log((\log(x/s)/(\log(1/s)))) \\
&= \log(1 + (\log(x/s)/\log(1/s) - 1)) \\
&= (\log(x/s)/\log(1/s) - 1) - \frac{1}{2} (\log(x/s)/\log(1/s) - 1)^2 \\
&\quad + O((\log(x/s)/\log(1/s) - 1)^3) \\
&= \frac{\log x}{\log(1/s)} - \frac{1}{2} \frac{(\log x)^2}{(\log(1/s))^2} + O((\log(1/s))^{-3}).
\end{align*}
\]

This gives

\[
(\log(1/s))^2 \left\{ \frac{F^{-1}(1 - s) - F^{-1}(1 - xs)}{(\log(1/s))^{-1}} - \log x \right\} \to -\frac{1}{2}(\log x)^2
\]

6.2.8. **Reversed Burr’s df.**

We have

\[ F(x) = 1 - ((-x)^{-\rho/\gamma} + 1)^{1/\rho}, \quad x \leq 0, \quad \rho < 0 \text{ and } \gamma > 0. \]

Then

\[ F^{-1}(1 - u) = -(u^{\rho} - 1)^{-\gamma/\rho}. \]

and

\[
\begin{align*}
F^{-1}(1 - u) - F^{-1}(1 - ux) &= ((xu)^{\rho} - 1)^{-\gamma/\rho} - (u^{\rho} - 1)^{-\gamma/\rho} \\
&= (u^{\rho} - 1)^{-\gamma/\rho} \left\{ \left( \frac{x^{\rho} u^{\rho} - 1}{u^{\rho} - 1} \right)^{-\gamma/\rho} - 1 \right\} \\
&= (u^{\rho} - 1)^{-\gamma/\rho} \left\{ x^{-\gamma} \left( \frac{1 - x^{-\rho} u^{-\rho}}{1 - u^{-\rho}} \right)^{-\gamma/\rho} - 1 \right\}.
\end{align*}
\]

But

\[
\begin{align*}
\left( \frac{1 - x^{-\rho} u^{-\rho}}{1 - u^{-\rho}} \right)^{-\gamma/\rho} &= \left( 1 + \left\{ \frac{1 - x^{-\rho} u^{-\rho}}{1 - u^{-\rho}} - 1 \right\} \right)^{-\gamma/\rho} \\
&= \left( 1 + \left\{ \frac{(1 - x^{-\rho}) u^{-\rho}}{1 - u^{-\rho}} \right\}^{-\gamma/\rho} \\
&= 1 - \frac{\gamma (1 - x^{-\rho}) u^{-\rho}}{\rho (1 - u^{-\rho})} + O(u^{-2\rho}).
\end{align*}
\]
Thus
\[
F^{-1}(1-u)-F^{-1}(1-uX) = \frac{x^{-\gamma}-1}{\gamma} - \frac{x^{-\gamma}(1-x^{-\rho})u^{-\rho}}{\rho} + O(x^{-\gamma}\gamma^{-1}u^{-2\rho}).
\]
So
\[
(u^{-\rho})^{-1}\left\{ \frac{F^{-1}(1-u)-F^{-1}(1-uX)}{\gamma(u^\rho-1)^{-\gamma/\rho}} - \frac{x^{-\gamma}-1}{\gamma} \right\} = -\frac{x^{-\gamma}(1-x^{-\rho})}{\rho}
\]
We now summarize the results of these computations in the next subsection.

6.3. Tables of functions s and b.

| Name               | F            | b                          |
|--------------------|--------------|-----------------------------|
| Burr               | 1 - (x^{-\rho}\gamma + 1)^{1/\rho}, | x \geq 0, \rho < 0, \gamma > 0 | u^\rho |
| Reversed Burr      | 1 - ((-x)^{-\rho}\gamma + 1)^{1/\rho}, | x \leq 0, \rho < 0, \gamma > 0 | u^{-\rho} |
| Singh-Maddala      | 1 - F(x) = (1 + ax^b)^{-c}, | x \geq 0 | (bc(1 - u^{-1/c})^{-1} |
| Log-Sm             | 1 - G(x) = (1 + ae^{xb})^{-c} | | (bc(1 - u^{-1/c})^{-1} |
| Exponentiel        | 1 - e^{-x}, | x \geq 0 | (log u)^{-1} |
| Log-Expo           | 1 - F(x) = exp(-e^{-x}) | | 1/log u |
| Normal             | \phi(x) | | (log 1/u)^{-3/2} |
| Lognormal          | \phi(e^x) | | (log 1/u)^{-1} |
| Logistic           | 1 - 2/(1 + e^x), | x \geq 0 | (log 1/u)^{-1} |

| Name        | gamma | h_{\gamma,\rho}(x) | S(u)          |
|-------------|-------|---------------------|---------------|
| Burr        | 1/\gamma | -\frac{(x^{-1})^{1/\gamma}}{x^{\gamma-1}} | (u^\rho - 1)^{-1} |
| Reversed Burr | -1/\gamma | -\frac{x^{-\gamma}(1-x^{-\rho})}{x^{\gamma-1}} | u^{-\rho} |
| Singh-Maddala | 1/(bc) | \frac{(x^{-1/c})^{1/\gamma}}{x^{\gamma-1/(bc)+1/c}} | (1 - u^{-1/c})^{-1} |
| Log-Sm      | 0 | -c(x^{-1/c} - 1) | cu^{1/c} | (\gamma b) |
| Exponentiel | 0 | Not applicable | Not applicable |
| Log-Expo    | 0 | -(log x)^2/2 | 1/log u |
| Normal      | 0 | -log x | D(u) = \left\{ \frac{(1/2) \log 4\pi + (1/2) \log \log(1/u)}{(2 \log(1/u))^{1/2}} \right\}^{-1} |
| Lognormal   | 0 | (log x)^2/2 | the same D(u) |
| Logistic    | 0 | x - 1 | u/2 |

Statistical applications
Practically, the normality results on statistics based on the extremes are applied for ultimately differentiable distribution functions (at +\infty). They usually depend of the functions b(.) for in the representation scheme and, on S in the second order condition one. This means that we may move from one approach to the other. Let us illustrate this with two examples.

6.4. Large quantiles process. Let X_1, X_2, \ldots be a sequence of real and independent random variables indentically distributed and associated to the distribution function F(x) = P(X \leq x), x \in R. We suppose that these random variables are represented as X_i = F^{-1}(U_i), i = 1, 2, \ldots, where U_1, U_2, \ldots are standard uniform
independant random variables. For each \( n \), \( U_{1,n} < ... < U_{n,n} \) denote the order statistics based on \( U_1, ..., U_n \). Finally let \( \alpha > 0 \) and \( a > 0 \) and

\[
k \to \infty; \ k/n \to 0 \text{ and } \log \log n/k \to 0 \text{ as } n \to \infty.
\]

Consider this large quantile process (see Drees [3])

\[
A_n(s, \alpha) = X_{n - [k/s^\alpha] + 1,n} - F^{-1}(1 - [k/s^\alpha]/n).
\]

We suppose that \( F \) is in the extremal domain. We use first the representation scheme.

6.4.1. Representation approach. Consider the function \( p \) and \( b \) defined in Theorem [1]. For any \( \lambda > 1 \), put the convention

\[
d_n(h, a, \alpha) = \sup\{|h(t)|, |t| \leq [a^{-\alpha}]\lambda k/n.
\]

We may then define the regularity condition,

\[
(RCREP) \quad \sqrt{k}(d_n(p, a, \alpha) \vee d_n(b, a, \alpha)) \to 0
\]

under which we may find a uniform Gaussian approximation of \( A_n(s, \alpha) \). Put for convenience

\[
k(s, \alpha) = [k/s^\alpha], \ l(n, \alpha) = k(s, \alpha)/n, \ k'(s, \alpha) = k(s, \alpha)/k, \text{ and } U_{[k/s^\alpha],n} = U_{k(s, \alpha),n}.
\]

For

\[
a(k/n) = c(1 + p(k/n))(k/n)^{-\gamma} \exp(\int_{k/n}^1 b(t)dt),
\]

we have

\[
A_n(1) = X_{n - [k/s^\alpha] + 1,n}/a(k/n)
\]

\[= (1 + O(d_n(p, a, \lambda))(1 + O(d_n(b, a, \lambda)))(U_{k(s, \alpha),n}/l(n, s, \alpha)^{-\gamma}k'(s, \alpha)^{-\gamma}.
\]

We also have

\[
A_n(2) = F^{-1}(1 - l(n, s, \alpha)/a(k/n)
\]

\[= (1 + O(d_n(p, a, \lambda))(1 + O(d_n(b, a, \lambda))k'(s, \alpha)^{-\gamma}.
\]

It follows, since \( k'(s, \alpha)^{-\gamma} = s^{\alpha \gamma}(1 + O(k^{-1})) \) uniformly in \( s \in (a, 1) \), that

\[
A_n(1) - A_n(2) = s^{\alpha \gamma} \{ (l(n, s, \alpha)^{-1}U_{k(s, \alpha),n})^{-\gamma} - 1 \}
\]

\[+ O(d_n(p, a, \lambda) \vee d_n(b, a, \lambda)) + O(k^{-1});
\]

and by (3)

\[
\sqrt{k} \{ A_n(1) - A_n(2) \} = -\gamma s^{\alpha \gamma} \sqrt{k} \{ (l(n, s, \alpha)^{-1}U_{k(s, \alpha),n}) - 1 \}
\]

\[+ O(\sqrt{k}d_n(p, a, \lambda)) + O(\sqrt{k}d_n(b, a, \lambda)))
\]

\[= -\gamma s^{\alpha \gamma} W_n(1, s^\alpha) + o_p(a, \alpha, s)
\]

whenever (RCREP) is valid. We then obtain the limiting law of the process of large quantiles under this condition. When \( F \) is differentiable in the neighborhood of \(+\infty\), we may take \( p = 0 \) and (RCREP) becomes

\[
\sqrt{k}d_n(b, a, \alpha) \to 0.
\]

Under this (RCREP), the large quantile process behaves as the Gaussian stochastic process \(-\gamma s^{\alpha \gamma}(s, \alpha)W_n(1, s^\alpha)\).
6.4.2. Second order condition approach. There exist functions \(a(\cdot)\) and \(S(\cdot)\) \(a(\cdots)\) is not necessarily the same as the previous function \(s(\cdots)\), such that the SOC holds. But for statistical purposes, it is more convenient to use the continuous second order condition, that is for \(u_n \to 0\), for \(x_n \to x > 0\),

\[
\lim_{n \to \infty} S(u_n) \left\{ \frac{F^{-1}(1-u_n) - F^{-1}(1-u_n x_n)}{a(u_n)} - \frac{x_n^{-\gamma} - 1}{\gamma} \right\} = h_\gamma(x).
\]

A simple argument based on compactness yields for \(u_n \to 0\) and for \(0 < a < b\),

\[
\lim_{n \to \infty} \sup_{a \leq x \leq b} \left| S(u_n) \left\{ \frac{F^{-1}(1-u_n) - F^{-1}(1-u_n x)}{a(u_n)} - \frac{x^{-\gamma} - 1}{\gamma} \right\} - h_\gamma(x) \right| = 0.
\]

Put \(x(n, s, \alpha) = l(n, s, \alpha)^{-1} U_{k(s,\alpha),n} \to 1\). For \(s \geq a\), we may see that \(l(n, s, \alpha)^{-1} U_{k(s,\alpha),n} = 1 + k^{-1/2}(W_n(s^\alpha) + o_P(1))\), uniformly in \(s \in (a,1)\), where \(W_n\) is a standard Wiener process (see Lemma 1 in [8]). Then we may apply the CSOC as follows:

\[
\sup_{a \leq s \leq b} S(l(n,s,\alpha))^{-1} \left\{ \frac{F^{-1}(1-l(n,s,\alpha)) - F^{-1}(1-U_{k(s,\alpha),n})}{a(l(n,s,\alpha))} - \frac{(l(n,s,\alpha)^{-1} U_{k(s,\alpha),n})^{-\gamma} - 1}{\gamma} \right\} - h_\gamma(x(n, s, \alpha)) = 0
\]

This gives, uniformly in \(s \in (a,1)\),

\[
\frac{F^{-1}(1-l(n,s,\alpha)) - F^{-1}(1-U_{k(s,\alpha),n})}{a(l(n,s,\alpha))} = \frac{(l(n,s,\alpha)^{-1} U_{k(s,\alpha),n})^{-\gamma} - 1}{\gamma} - (h_\gamma(x(n, s, \alpha)) + o_P(1))S(l(n,s,\alpha)).
\]

Then

\[
\frac{\sqrt{\epsilon} \left\{ F^{-1}(1-l(n,s)) - F^{-1}(1-U_{k(s,\alpha),n}) \right\}}{a(l(n,s,\alpha))} = \frac{\sqrt{\epsilon} (l(n,s)^{-1} U_{k(s,\alpha),n})^{-\gamma} - 1}{\gamma} - (h_\gamma(1) + o_P(1))S(l(n,s))\sqrt{\epsilon}.
\]

We will apply Lemma 1 in [8]. Since \(S(l(n,s,\alpha)) = O(S(l(n,s)))\) and \(a(l(n,s,\alpha)) \sim s^\alpha a(l(n,1,1))\), we also get

\[
\frac{\sqrt{\epsilon} \left\{ F^{-1}(1-l(n,s)) - F^{-1}(1-U_{k(s,\alpha),n}) \right\}}{a(l(n,s,\alpha))} = -W_n(1,s^\alpha) - (h_\gamma(1) + o_P(1))S(l(n,s,\alpha))\sqrt{\epsilon}.
\]

We get the regularity condition

(RCSOC) \(\sqrt{k} S(k/n) \to 0\).

**Conclusion 1.** In both cases, we conclude that the large quantile process behaves as the Gaussian process \(-s^\alpha W_n(1,s^\alpha)\) when appropriately normalized under conditions based on \(b\) or on \(S\).

By comparing (RCREP) and (RCSOC), we see that the present normality result in the representation scheme uses the function \(b\) while the Second order one relies on \(S\). In fact, almost all the normality results in both cases rely either on \(b\) in
the Representation scheme or on \( S \) in the Second order model. We also see that
the second order scheme seems to use a shorter way. But, as a compensation,
the function \( S \), as we may see it here, is more complicated to get. Indeed for
differentiable distribution functions, the function \( b \), is easily obtained.

6.5. Functional Hill process.

6.5.1. Representation approach. Now consider the functional Hill process

\[ T_n(f) = \sum_{j=1}^{k} f(j) \left( \log X_{n-j+1,n} - \log X_{n-j,n} \right), \]

where \( f \) is some positive and bounded function and \( k = k(n) \) is a sequence if positive
integer such that \( 1 \leq 1 \leq k \leq n \) and \( k/n \to 0 \) as \( n \to +\infty \). We are going to study
the process under the hypothesis \( F \in D(\psi_{1/\gamma}) = D(G_{-\gamma}) \), \( \gamma > 0 \). Now using the
same representation \( X_i = F^{-1}(U_i) \), \( i = 1, 2, ... \) We get

\[
T_n(f) = \sum_{j=1}^{k} f(j)(\log X_{n-j+1,n} - \log X_{n-j,n}) = \sum_{j=1}^{k} f(j)(-(y_0 - \log X_{n-j+1,n}) + (y_0 - \log X_{n-j,n}))
\]

\[
= \sum_{j=1}^{k} f(j) \left\{ c(1 + p(U_{j+1,n}))(U_{j+1,n})^{\gamma} \exp \left( \int_{U_{j+1,n}}^{1} \frac{b(t)}{t} \, dt \right) \right. \\
- c(1 + p(U_{j,n}))(U_{j,n})^{\gamma} \exp \left( \int_{U_{j,n}}^{1} \frac{b(t)}{t} \, dt \right) \}
\]

\[
= \sum_{j=1}^{k} f(j) \left\{ c(1 + p(U_{j,n}))(U_{j,n})^{1/\gamma} \exp \left( \int_{U_{j,n}}^{1} \frac{b(t)}{t} \, dt \right) \right. \\
\times \left\{ \frac{1 + p(U_{j+1,n})}{1 + p(U_{j,n})} \left( \frac{U_{j+1,n}}{U_{j,n}} \right)^{\gamma} \exp \left( \int_{U_{j+1,n}}^{1} \frac{b(t)}{t} \, dt \right) - 1 \right\}.
\]

But

\[
\left( \frac{U_{j+1,n}}{U_{j,n}} \right)^{\gamma} = \exp \left( \frac{\gamma}{j} \log \left( \frac{U_{j+1,n}}{U_{j,n}} \right) \right) = \exp \left( \frac{\gamma}{j} E_j \right) = \exp(F_j)
\]

where, by the Malmquist representation (see [13], p. 336), the \( E_j \)s are independent
standard exponential random variables. Let also

\[
p_n = \sup\{|p(u)|, 0 \leq u \leq U_{k+1,n}\} \to 0,
\]

\[
b_n = \sup\{|b(u)|, 0 \leq u \leq U_{k+1,n}\} \to 0,
\]

as \( n \to +\infty \), and \( c_n = a_n \vee (b_n \log k) \). Then

\[
\left\{ \frac{1 + p(U_{j+1,n})}{1 + p(U_{j,n})} \left( \frac{U_{j+1,n}}{U_{j,n}} \right)^{\gamma} \exp \left( \int_{U_{j+1,n}}^{1} \frac{b(t)}{t} \, dt \right) - 1 \right\} = \exp(F_j)(1 + O(p_n)) \exp(O(b_n E_j/j)) - 1
\]

\[
= F_j(1 + O(p_n))(1 + O(b_n \log k))
\]

\[
= F_j(1 + O(c_n)) - 1
\]

Let also

\[
s_n = y_0 - G^{-1}(1-U_{k,n}) = y_0 - \log X_{n-k+1,n} = c(U_{k+1,n})^{1/\gamma} \left( 1 + \exp \left( \int_{U_{j,n}}^{1} \frac{b(t)}{t} \, dt \right) \right).
\]
This gives

\[ T_n(f)/s_n \]

\[ = T_n(f) / s_n \]

\[ = \sum_{j=1}^{k} f(j) \left\{ \frac{1 + p(U_{j,n})}{1 + p(U_{k,n})} \right\} \gamma \exp \left( \int_{U_{j,n}}^{U_{k,n}} \frac{b(t)}{t} dt \right) \times \{ \exp(F_j)(1 + O(c_n)) - 1 \} . \]

Let us remark that

\[ \log \left( \frac{U_{k,n}}{U_{j,n}} \right) = \log \left( \prod_{h=j}^{k-1} \frac{U_{h+1,n}}{U_{h,n}} \right) = \sum_{h=j}^{k-1} \log \left( \frac{U_{h+1,n}}{U_{h,n}} \right) = \sum_{h=j}^{k-1} \frac{1}{h} E_h = O(\log k) \]

and

\[ \left( \frac{U_{i,n}}{U_{k,n}} \right)^\gamma = \exp \left( -\gamma \sum_{h=j}^{k-1} \frac{1}{h} E_h \right) = \exp \left( -\sum_{h=j}^{k-1} F_h \right) = F_j^* \]

where \( F_j^* = 0 \). Then

\[ T_n(f)/s_n = \sum_{j=1}^{k} f(j) \left\{ (1 + O(p_n))(1 + O(b_n \log k)) \times \exp \left( -\sum_{h=j}^{k-1} F_h \right) \right\} \times \{ \exp(F_j) - 1 + O(c_n \exp(F_j)) \} \]

\[ = \sum_{j=1}^{k} f(j) F_j^* (\exp(F_j) - 1) + O(c_n F_j^*(\exp(F_j) - 1)) \]

\[ + O(c_n^2 F_j^*(\exp(F_j) - 1)) + O(c_n \exp(F_j)) \]

We conclude that \( T_n(f)/s_n \) behaves as that of \( \sum_{j=1}^{k} f(j) F_j^*(\exp(F_j) - 1) \) under regularity conditions based on the functions \( p \) and \( b \).

6.5.2. Second order condition approach. Let use the continuous second order condition:

\[ S^{-1}(u_n) \left\{ \frac{G^{-1}(1 - x_n u_n) - G^{-1}(1 - u_n)}{s(u_n)} - \frac{x_n^{-\gamma} - 1}{\gamma} \right\} = h_\gamma(x) + o(1), \]

where \( x_n \to 0 \) et \( u_n \to 0 \) as \( n \to \infty \) and \( u_n = \gamma \{ g_0(G) - G^{-1}(1 - u_n) \} \). We get, for \( G(x) = F(e^x), x \in R, \)

\[ T_n(f) = \sum_{j=1}^{k} f(j) (\log X_{n-j+1,n} - \log X_{n-j,n}) \]

\[ = \sum_{j=1}^{k} f(j) \left\{ G^{-1}(1 - U_{j,n}) - G^{-1}(1 - U_{j+1,n}) \right\} . \]
As a general conclusion, we say:

(1) The representation approach is more general.
(2) The second order condition seems to be shorter and more unified.
(3) The computation of $b$ is less complicated than that of $S$.
(4) The representation approach is still used within the second order approach.
(5) The two approaches may be simultaneously used.

7. Conclusion

As a general conclusion, we say:

Let $u_n(j) = U_{j,n}$ and $x_n(j) = U_{j+1,n}/U_{j,n} = \exp(F_j)$. Then

$$\frac{G^{-1}(1 - U_{j,n}) - G^{-1}(1 - U_{j+1,n})}{s(u_n(j))} = \frac{-G^{-1}(1 - x_n(j)U_{j,n}) - G^{-1}(1 - U_{j,n})}{s(u_n(j))}$$

$$- \left\{ \frac{G^{-1}(1 - x_n(j)U_{j,n}) - G^{-1}(1 - U_{j,n})}{s(u_n(j))} \right\} - \frac{\exp\left(-\frac{1}{2}E_j\right) - 1}{\gamma} - \frac{\exp\left(-\frac{1}{2}E_j\right) - 1}{\gamma}$$

$$- S(u_n(j))h_\gamma(\exp(F_j)) - \frac{E_j^{-\gamma/j} - 1}{\gamma} + o_p(A(u_n(j)))$$

Let us use

$$\frac{T_n(f)}{s(u_n(k))} = \sum_{j=1}^{k} f(j) \frac{s(u_n(j))}{s(u_n(k))} G^{-1}(1 - U_{j,n}) - G^{-1}(1 - U_{j+1,n})$$

$$= \sum_{j=1}^{k} f(j) \frac{s(u_n(j))}{s(u_n(k))} \left\{ -S(u_n(j))h_\gamma(\exp(F_j)) - \frac{\exp\left(-\frac{1}{2}E_j\right) - 1}{\gamma} + o_p(S(u_n(j))) \right\}.$$

Let us apply

$$\frac{s(u_n(j))}{s(u_n(k))} = \left\{ (1 + O(p_n))(1 + O(b_n \log k)) \times \exp\left(-\sum_{h=j}^{k-1} F_h\right) \right\}.$$
We conclude that the two approaches are equivalent and we have proposed for both cases the computation of \( b \) and \( S \) for usual distribution functions.

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