KOSZUL HOMOLOGY OF CODIMENSION 3 GORENSTEIN IDEALS

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Abstract. In this note, we calculate the Koszul homology of the codimension 3 Gorenstein ideals. We find filtrations for the Koszul homology in terms of modules with pure free resolutions and completely describe these resolutions. We also consider the Huneke–Ulrich deviation 2 ideals.

Introduction

For the codimension 3 Pfaffian ideal of $2n \times 2n$ Pfaffians of a $(2n + 1) \times (2n + 1)$ generic skew-symmetric matrix, we give an explicit description of the Koszul homology modules. By a result of Buchsbaum–Eisenbud [BE], the general case of codimension 3 Gorenstein ideals reduces to this case. They are filtered by equivariant modules $M_i$ with self-dual pure free resolutions of length 3 supported in the ideal of Pfaffians. The free resolutions of the modules $M_i$ give natural generalizations of the Buchsbaum–Eisenbud complexes for codimension 3 Gorenstein ideals and are interesting in their own right. It was known that the Koszul homology modules of codimension 3 Pfaffian ideals are Cohen–Macaulay [Hun1, Example 2.2], but no explicit description was given. The only other example we could find in the literature of explicit calculations of Koszul homology is the paper of Avramov–Herzog [AH], which handles the case of codimension 2 perfect ideals. The Koszul homology modules of codimension 3 Pfaffian ideals also give examples of modules with pure filtrations that do not follow from the results in [EES]. Finally, we calculate the Koszul homology modules for the Huneke–Ulrich deviation 2 ideals which were studied by Kustin [Kus].

1. Koszul homology

Throughout $R$ is a Cohen–Macaulay (graded) local ring. After this section, we will be working over polynomial rings with $\mathbb{Z}$-coefficients, which we pretend is a graded local ring by saying that its maximal ideal is the one generated by the variables. Let $I \subset R$ be a (graded) ideal of grade $g$, and let $\mu(I)$ denote the smallest size of a generating set of $I$. The Koszul homology of $I$ depends on a set of generators, but any two choices of minimal generating sets yield isomorphic Koszul homology. In the case of a minimal generating set, we denote the Koszul homology by $H_\bullet(I; R)$. We will only be interested in Koszul homology for minimal generating sets of $I$. We say that $I$ is strongly Cohen–Macaulay if the Koszul homology of $I$ is Cohen–Macaulay.
If \( R \) is Gorenstein and \( R/I \) is Cohen–Macaulay, then the top nonvanishing Koszul homology \( H_{\mu(I)-g}(I; R) \) is the canonical module \( \omega_{R/I} \) of \( R/I \) (see \[Hum2\] Remark 1.2)]. Furthermore, the exterior multiplication on the Koszul complex induces maps

\[
H_i(I; R) \to \text{Hom}_R(H_{\mu(I)-g-i}(I; R), H_{\mu(I)-g}(I; R))
\]

and these maps are isomorphisms in the case that \( I \) is strongly Cohen–Macaulay. This is also true if we only assume that the Koszul homology modules are reflexive \( [Hum2] \) Proposition 2.7).

2. Codimension 3 Pfaffian ideals

In this section we work over the integers \( \mathbb{Z} \) and set \( A = \text{Sym}(\wedge^2 E) \), where \( E \) is a free \( \mathbb{Z} \)-module of rank \( 2n+1 \). We consider the ideal \( I = \text{Pf}_{2n}(\varphi) \) of \( 2n \times 2n \) Pfaffians of the generic skew-symmetric matrix

\[
\varphi = (\varphi_{i,j})_{1 \leq i, j \leq n},
\]

where \( \varphi_{i,j} \) are the variables satisfying \( \varphi_{i,j} = -\varphi_{j,i} \). The free resolution for this ideal and its main properties can be found in \[BE\] (the quotient \( A/I \) is also the module \( M_0 \) defined in the next section). Thus if \( \{e_1, \ldots, e_{2n+1}\} \) is a basis in \( E \), we can think of \( \varphi_{i,j} = e_i \wedge e_j \in \wedge^2 E \). Denote the \( 2n \times 2n \) Pfaffians of \( \varphi \) by

\[
Y_i = (-1)^{i+1} \text{Pf} \varphi(i),
\]

where \( \varphi(i) \) is the skew-symmetric matrix we get from \( \varphi \) by omitting the \( i \)-th row and \( i \)-th column.

Consider the Koszul complex \( K_\bullet = K(Y_1, \ldots, Y_{2n+1}; A) \). In this case,

\[
K_i = \bigwedge^i (\wedge^2 E) \otimes A(-in) = \bigwedge^{2n+1-i} E \otimes (\det E)^i \otimes A(-in)
\]

2.1. Modules \( M_i \). Before we start we describe a family of \( A \)-modules supported in the ideal \( I \). For \( i = 0, \ldots, n-1 \), we get equivariant inclusions

\[
d_1: \bigwedge^{2n-i} E \subset \bigwedge^i E \otimes \bigwedge^{2n-2i} E \subset \bigwedge^2 E \otimes \text{Sym}^{n-i}(\wedge E),
\]

\[
d_2: \det E \otimes \bigwedge^{2n-i} E \subset \bigwedge^i E \otimes \bigwedge^{2n-2i} E \subset \bigwedge^2 E \otimes \text{Sym}^{i+1}(\wedge E),
\]

\[
d_3: \det E \otimes \bigwedge^{2n+1-i} E \subset \det E \otimes \bigwedge^i E \otimes \bigwedge^{2n-2i} E \subset \det E \otimes \bigwedge^i E \otimes \text{Sym}^{n-i}(\wedge E),
\]

where in each case the first inclusion can be defined in terms of comultiplication, and the second is given by Pfaffians. We make these maps more explicit. Let \( e_1, \ldots, e_{2n+1} \) be an ordered basis for \( E \) compatible with \( \varphi \). For an ordered sequence \( I = (i_1, \ldots, i_n) \) consisting of elements from \( [1, 2n] \) we denote by \( e_I \) the decomposable tensor \( e_{i_1} \wedge \cdots \wedge e_{i_n} \). The embedding \( \bigwedge^{2d} E \subset \text{Sym}^d(\wedge^2 E) \) sends the tensor \( e_I \) \((\#I = 2d)\) to the Pfaffian of the \( 2d \times 2d \) skew-symmetric submatrix of \( \varphi \) corresponding to the rows and columns indexed by \( I \). We will denote this...
Pfaffian by Pf(I). With these conventions, the maps \(d_1, d_2, d_3\) are given by the formulas
\[
\begin{align*}
    d_1(e_I) &= \sum_{I' \subset I} \sgn(I', I'')e_{I'} \otimes \Pf(I''), \\
    d_2(e_{[1,2n+1]} \otimes e_J) &= \sum_{I' \subset [1,2n+1]} \sgn(I', I'')\sgn(I'', J)e_{I'} \otimes \Pf(I'' \cup J), \\
    d_3(e_I) &= \sum_{I' \subset I} \sgn(I', I'')e_{I'} \otimes \Pf(I''),
\end{align*}
\]
where \(I''\) is the complement of \(I'\) in \(I\), all subsets are listed in increasing order, and \(\sgn(I', I'')\) is the sign of the permutation that reorders \((I', I'')\) in its natural order. The symbol \(\Pf(I'' \cup J)\) is by convention 0 if \(I'' \cap J \neq \emptyset\).

**Proposition 2.1.** For \(i = 0, \ldots, n - 1\), we define the complex \(C^i\),
\[
\begin{array}{ccccccc}
    0 \to (\det E) \otimes \bigwedge^{2n+1-i} E & \xrightarrow{d_3} & \bigwedge^i E \\
    \otimes A(-2n+i-1) & \xrightarrow{d_2} & \bigwedge^{2n-i} E & \xrightarrow{d_1} & \bigwedge^i E \\
    \otimes A(-n-1) & & \otimes A(-n+i) & & \otimes A,
\end{array}
\]
using the inclusions defined above. This complex is acyclic, and the cokernel \(M_i\) is supported in the variety defined by the Pfaffians of size \(2n\).

**Proof.** To check that the above is a complex, it is enough to extend scalars to \(Q\). In this case, we can use representation theory (namely, Pieri’s formula \[\text{Wey}, \text{Corollary 2.3.5}\] and the decomposition of \(\text{Sym}(\wedge^2)\) into Schur functors \[\text{Wey}, \text{Proposition 2.3.8}\]) to see that these maps define a complex.

To prove acyclicity, we use the Buchsbaum–Eisenbud exactness criterion. The formulation of the result that we use, which is a consequence of \[\text{Eis}, \text{Theorem 20.9}\], is: Given a finite free resolution \(F_\bullet\) of length \(n\), \(F_\bullet\) is acyclic if and only if the localization \((F_\bullet)_P\) is acyclic for all primes \(P\) with depth \(A_P < n\). Localizing at a prime \(P\) with depth at most 2, some variable becomes a unit, so using row and column operations, we can reduce \(\varphi\) to the matrix
\[
\varphi = \begin{bmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & \varphi'
\end{bmatrix},
\]
where \(\varphi'\) is a generic \((2n-1) \times (2n-1)\) skew-symmetric matrix. Let \(C^0, \ldots, C^{n-2}\) be the complexes in Proposition 2.1 defined for the matrix \(\varphi'\). Then
\[
(C^i)_P \cong C^i \oplus 2C^{i-1} \oplus C^{i-2},
\]
with the convention that \(C^{n-1} = 0\) and \(C^j = 0\) for \(j < 0\). By induction on the size of \(\varphi\), we see that each \(C^i\) is acyclic. \(\square\)

### 2.2. Results and proofs.

**Theorem 2.2.** Set \(K_\bullet = K(Y_1, \ldots, Y_{2n+1}; A)_\bullet\).

(a) For \(0 \leq j \leq n - 1\) we have a filtration \(\cdots \subset F_1H_j \subset F_0H_j = H_j(K_\bullet)\) such that
\[
F_iH_j / F_{i+1}H_j \cong M_{j-2i} \otimes (\det E)^j.
\]
(b) For \(0 \leq j \leq n - 2\) we have a filtration \(0 = F_0H_{2n-2-j} \subset F_1H_{2n-2-j} \subset \cdots\) such that
\[
F_{i+1}H_{2n-2-j} / F_iH_{2n-2-j} \cong M_{j-2i} \otimes (\det E)^{2n-2-j}.
\]
Proof. We assume that \( n > 0 \) since the case \( n = 0 \) is trivial. We will construct a sequence of complexes \( F(r)_\bullet \) for \( r = 0, \ldots, n - 1 \) such that

(1) \( F(0)_\bullet = K_\bullet \).

(2) \( F(r)_\bullet \) is concentrated in degrees \([r, 2n + 1]\),

(3) the cokernel of \( F(r)_\bullet \) has a filtration as specified by the theorem. Letting \( G(r)_\bullet \) be its minimal free resolution, we have that \( F(r + 1)_\bullet \) is the minimal subcomplex of the mapping cone \( F(r)_\bullet \to G(r)_\bullet \).

The existence of this sequence implies the first part of the theorem. For the second part, we appeal to (1.1), which says that \( H_{2n-2-i}(K_\bullet) \) is the \( A \)-dual of \( H_i(K_\bullet) \) (note that the \( M_i \) are self-dual by the form of their free resolutions). We construct this sequence by induction on \( r \).

For \( r = 0 \), there is nothing to check, so assume that \( r > 0 \) and that \( F(r-1)_\bullet \) has the listed properties. Then \( F(r-1) \) is the minimal subcomplex of some extension of

\[
K_{r-2} = \bigoplus_{i=0}^{r-2} \left( \left( \det E \right)^i \otimes \bigoplus_k C_{i-2k} \right)
\]

which is concentrated in degrees \([r-1, 2n+1]\). Since each complex \( C \) has length 3, we see that \( F(r-1)_i = K_i \) for all \( i \geq r+1 \). Recall that \( r \leq n-1 \). Then we see that from the structure of the representations in the resolutions of the \( M_i \) that after cancellations, we get (there are no cancellations in homological degree \( r+1 \))

\[
F(r-1)_{r-1} = \bigoplus_k \bigwedge^{r-1-2k} E \otimes (\det E)^{r-1} \otimes A,
\]

\[
F(r-1)_r = \bigoplus_k \bigwedge^{2n-r+1+2k} E \otimes (\det E)^{r-1} \otimes A \oplus \begin{cases} 0 & \text{if } r-1 \text{ is even} \\ (\det E)^r \otimes A & \text{if } r-1 \text{ is odd} \end{cases},
\]

\[
F(r-1)_{r+1} = K_{r+1} = \bigwedge E \otimes (\det E)^r \otimes A
\]

(we ignore the grading since it is determined by the degree of the functor on \( E \)). By our induction hypothesis, up to a change of basis, we can write the presentation matrix for \( F(r-1) \) in "upper-triangular form"; i.e., the map from \( \bigwedge^{2n-r+1+2k'} E \) to \( \bigwedge^{r-1-2k} E \) is nonzero if and only if \( k' \geq k \). Also, when \( r-1 \) is odd, the extra term \( (\det E)^{r} \otimes A \) is a redundant relation. Now consider the mapping cone

\[
\begin{array}{cccccc}
F(r-1)_{r-1} & \leftarrow & F(r-1)_r & \leftarrow & F(r-1)_{r+1} & \cdots \\
\downarrow & & \downarrow & & \downarrow & \\
G(r-1)_0 & \leftarrow & G(r-1)_1 & \leftarrow & G(r-1)_2 & \leftarrow \ldots & G(r-1)_{n-1} \leftarrow 0.
\end{array}
\]

The maps \( F(r-1)_{r-1} \to G(r-1)_0 \) and \( F(r-1)_r \to G(r-1)_1 \) are isomorphisms, except when \( r-1 \) is odd, in which case the term \( (\det E)^r \otimes A \) is in the kernel of the second map. When \( r = n-1 \), there is an additional cancellation involving the terms \( \bigwedge^n E \otimes (\det E)^n \otimes A \) in \( F(n-2)_n \) and \( G(n-2)_2 \).

Finally, we can rearrange the resulting presentation matrix into upper-triangular form as follows. Note that all of the maps in the presentation matrix are saturated maps; i.e., their cokernels are free \( \mathbb{Z} \)-modules. This can be shown by induction.
on \( r \). Let \( N_r \) be the cokernel of the presentation matrix. Consider the submodule of \( N_r \) generated by \( \bigoplus_{k>0} \bigwedge^{r-2k} E \otimes (\det E)^r \otimes A \). The quotient is generated by \( \bigwedge^r E \otimes (\det E)^r \otimes A \). By induction, the cokernel of \( G(r-1) \) has \( M_{r-1} \) as a factor, so this implies that in the diagonal maps, the map \( \bigwedge^{2n-r} E \otimes (\det E)^r \otimes A \to \bigwedge^r E \otimes (\det E)^r \otimes A \) is nonzero. Since all of the maps from the relation module to this term are saturated, they all factor through the relations given by \( \bigwedge^{2n-r} E \otimes (\det E)^r \otimes A \) (this follows from the uniqueness of such maps up to sign by Pieri’s rule \cite[Corollary 2.3.5]{We}. Hence \( M_r \otimes (\det E)^r \) is a quotient, and continuing in this way, one can show that \( N_r \) has the desired filtration. This finishes the induction and the proof.

\[ \square \]

**Remark 2.3.** Since the \( M_i \) have pure resolutions, the above result shows that the Koszul homology of the codimension 3 Pfaffians have a pure filtration in the sense of \cite{EES}. \[ \square \]

### 3. Huneke–Ulrich ideals

We continue to work over the integers \( \mathbb{Z} \).

In this section, we study the Huneke–Ulrich ideals, which are defined as follows. Let \( \Phi \) be a generic skew-symmetric matrix of size 2\( n \) and let \( \mathbf{v} \) be a generic column vector of size 2\( n \). The Huneke–Ulrich ideal \( J \) is generated by the Pfaffian of \( \Phi \) along with the entries of \( \Phi \mathbf{v} \). It is well known that the ideal \( J \) is Gorenstein of codimension \( 2n-1 \) with \( 2n+1 \) minimal generators; i.e., it has deviation 2. Since \( H_2 \) is the canonical module, the only interesting Koszul homology group to calculate is \( H_1 \).

The notation is as follows. Let \( F \) be a free \( \mathbb{Z} \)-module of rank 2\( n \). We work over the polynomial ring

\[
\mathbb{A} = \text{Sym}(\bigwedge^2 F) \otimes \text{Sym}(F^*) = \mathbb{Z}[x_{i,j}, y_i]_{1 \leq i < j \leq 2n},
\]

where the variables \( x_{i,j} \) are the entries of the generic skew-symmetric matrix \( \Phi \) and \( y_i \) are coordinates of the generic vector \( \mathbf{v} \). Both \( \mathbb{A} \) and \( J \) are naturally bigraded.

The minimal free resolution \( F_\bullet \) of Huneke–Ulrich ideals was calculated by Kustin \cite{Kus}. When \( n = 2 \), the ideal \( J \) is a codimension 3 Gorenstein ideal, and so is covered by the previous section via specialization. We will use \( V(-d, -e) \) to denote \( V \otimes A(-d, -e) \). For \( n \geq 4 \), the first three terms of the minimal free resolution are given by

\[
F_1 = F(-1, -1) \oplus (\det F)(-n, 0),
\]

\[
F_2 = \bigwedge^2 F(-2, -2) \oplus \bigwedge^{2n-1} F(-n, -1) \oplus A(-1, -2),
\]

\[
F_3 = \bigwedge^3 F(-3, -3) \oplus \bigwedge^{2n-2} F(-n, -2) \oplus F(-2, -3) \oplus (\det F)(-n-1, -2).
\]

When \( n = 3 \), the same is true except that we omit the term \( \bigwedge^3 F(-3, -3) \) from \( F_3 \).

Now we consider the Koszul complex \( K_\bullet \) on the minimal generating set of \( J \). Since \( J \) has deviation 2, there are only 2 nonzero Koszul homology modules. We already know that \( H_2 \) is the canonical module of \( A/J \). More precisely, we have \( H_2 = (\det F) \otimes A/J(-n-1, -2) \). Let us describe the cycle giving \( H_2 \) precisely. Denote
the basis of the module $K_1 = F \otimes A(-1, -1) \oplus (\det F) \otimes (-n, 0)$ by \{e_1, \ldots, e_{2n}, f\}. For $1 \leq i < j \leq 2n$ we denote by $X(i, j)$ the $(2n - 2) \times (2n - 2)$ skew-symmetric matrix obtained from $X$ by removing the $i$-th and $j$-th row and column. Then the cycle in $K_2$ generating $H_2(K_*)$ is given by

$$\sum_{i=1}^{2n} y_i e_i \land f - \sum_{1 \leq i < j \leq 2n} (-1)^{i+j} \text{Pf}(X(i, j))e_i \land e_j.$$ 

Equivariantly, we just have the map

$$(\det F) \otimes A(-n - 1, -2) \rightarrow (\det F) \otimes F \otimes A(-n - 1, -1) \oplus \bigwedge^2 F \otimes A(-2, 2).$$

It is easy to check that there exists only one (up to a choice of sign) equivariant $\mathbf{Z}$-flat (saturated) map to each summand and that there is no such equivariant map in lower degrees. It is clear that our map defines a cycle and that the coset of this cycle in homology is annihilated by $J$, since the Koszul homology modules of a complex $K(u_1, \ldots, u_r)$ are always annihilated by the ideal $(u_1, \ldots, u_r)$. So we get an equivariant map

$$(\det F) \otimes A/J \rightarrow H_2(K_*).$$

A standard application of the acyclicity lemma shows that this map is an isomorphism.

**Proposition 3.1.** The first Koszul homology module has the presentation

$$\begin{align*}
\bigoplus & \bigwedge^{2n-2} F(-n, -2) & \oplus (\det F) \otimes F(-n - 1, -1) & \rightarrow & \bigoplus \bigwedge^{2n-1} F(-n, -1) & \rightarrow & H_1 & \rightarrow & 0.
\end{align*}$$

**Proof.** First note that

$$K_i = \bigwedge^i (F(-1, -1) \oplus (\det F)(-n, 0)) = \bigwedge^i F(-i, -i) \oplus (\det F) \otimes \bigwedge^i F(1-i-n, 1-i).$$

Since the cokernel of both $K_*$ and $F_*$ agree and $F_*$ is acyclic, we get a lifting $K_* \rightarrow F_*:

$$\begin{align*}
A & \leftarrow & F(-1, -1) & \oplus & (\det F)(-n, 0) & & & & & & \bigwedge^2 F(-2, -2) & \oplus & (\det F)(-1-n, -1) & \leftarrow & \bigwedge^3 F(-3, -3) & \oplus & \bigwedge^2 F(-2-n, -2).
\end{align*}$$

Hence a presentation matrix for $H_1$ is given by

$$\begin{align*}
\bigoplus & \bigwedge^{2n-2} F(-n, -2) & \oplus (\det F) \otimes F(-n - 1, -1) & \rightarrow & \bigoplus \bigwedge^{2n-1} F(-n, -1) & \rightarrow & H_1 & \rightarrow & 0.
\end{align*}$$

From [Kus] Definition 2.3, we conclude that the relations given by $(\det F)(-n - 1, -2)$ are redundant, which finishes the proof.

\[ \square \]
Inside the affine space $X = \text{Spec } A = \Lambda^2 F^* \oplus F$ the subvariety defined by $J$ is

$$Y = \{(\varphi, v) \in X \mid \text{rank } \varphi \leq 2n - 2, \ \varphi(v) = 0\}.$$ 

Let us consider the Grassmannian $\text{Gr}(2, F)$ with the tautological sequence

$$0 \to R \to F \times \text{Gr}(2, F) \to Q \to 0,$$

where $R = \{(f, W) \mid f \in W\}$. Consider the incidence variety

$$Z = \{(\varphi, v, W) \in X \times \text{Gr}(2, F) \mid v \in W \subset \ker(\varphi)\}.$$ 

Then $\mathcal{O}_Z = \text{Sym}(\eta)$, where $\eta = \Lambda^2 Q \oplus R^*$. The first projection $q: Z \to X$ satisfies $q(Z) = Y$.

**Theorem 3.2.** The nonzero homology of $K_\bullet$ is

$$H_0(K_\bullet) = H^0(\text{Gr}(2, F); \text{Sym}(\eta)) = A/J,$$

$$H_1(K_\bullet) = H^0(\text{Gr}(2, F); R \otimes \text{Sym}(\eta))(-1, -1),$$

$$H_2(K_\bullet) = H^0(\text{Gr}(2, F); \Lambda^2 R \otimes \text{Sym}(\eta))(-2, -2) = \det F \otimes A/J(-n - 1, -2).$$

**Proof.** First we work over $\mathbb{Q}$. Using the results in [Wey, Chapter 5], one can check that the presentation matrix for $H^0(\text{Gr}(2, F); R \otimes \text{Sym}(\eta))$ contains the same representations as the presentation matrix for $H_1(K_\bullet)$. By equivariance, such maps are unique up to sign, so we conclude that they agree. From [Kus], we know that the coordinate ring of $Y$ and hence its canonical module are torsion-free over $\mathbb{Z}$. In particular, the descriptions of $H_0$ and $H_2$ are independent of characteristic. By a Hilbert function argument, one sees that $H_1$ is also a torsion-free $\mathbb{Z}$-module, so our description extends to $\mathbb{Z}$-coefficients. \hfill $\square$

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