Parameterized excitation operators for coupled cluster method

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We present a coupled cluster method (CCM) with optimized excitation operators. The efficiency comes from a parameterized form of excitation operators. The parameters are found by variational optimization procedure. The resultant number of excitation operators is much smaller than that of the conventional CCM theory. This property makes it possible to apply the method in systems of solid state physics. Starting from Hartree-Fock state as the reference state, i.e., the Fermi sea, we search for particle-hole excitation operators such that the wave function of configuration interaction in terms of these excitation operators spans a good approximation to the ground state. The Matching-pursuit algorithm is capable of doing the search of the excitation operators. The resultant operators are our excitation operators for the CCM wave function. We test the method by two dimensional fermionic Hubbard model on a square lattice.

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The coupled cluster method (CCM) [1–3] is an valuable first principle numeric tool to treat fermionic many body systems. It is now popular in Chemistry community, although it was originally proposed by nuclear physicist in the fifties of last century [1–3]. For moderate sized atoms and molecules, CCM is cost effective with remarkable accuracy. Some comments refer CCM as a best wave function based numeric method. As a post Hartree-Fock theory, the CCM is an improvement to the mean field result. It uses the determinant state of the mean field result as reference state (other choices of the reference state are possible, see, e.g. the multi-references CCM [6–12]). Correlation is taken into account from particle-hole excitations from the Fermi sea, i.e., the reference state. A closely related theory with the same idea of particle-hole excitations from a Fermi sea is the configuration interaction (CI) theory [1]. The reference state plus all possible particle-hole excitations form a complete basis set of the Hilbert space. Taking into account of all possible excitations, i.e., full configuration interaction (FCI), leads to exact solution. This is only possible for small systems. Practical applications of CI usually implement a truncated form of FCI that consider only certain order of excitations. However, this truncated CI is not size extensive. The accuracy becomes poor for large systems. CCM, on the other hand, takes into account of all orders of particle-hole excitations in a concise way by an exponential function of some low order excitation operators. These excitation operators acting to the reference state are orthogonal with each others.

The CCM has a drawback that prevents it from widespread applications in condensed matter physics. The scaling of CCM with system size is a very steep polynomial. For example, if only single and double particle-hole excitations are considered, the scaling is about $N^6$ with $N$ the dimension of single particle Hilbert space. Thus it is impractical to do calculation with CCM for large systems, including many important systems of condensed matter physics. However, by exploiting the translational symmetry, some special cases, such as Hubbard model at half filling [13–15] and various spin systems modeled by Heisenberg model [16–20] can be treated by CCM. There are many active investigations to extend the applicability of CCM [21–30]. The basic idea is to restrict the selection of excitation operators. Up to now, almost all methods employ excitation operators that are orthogonal with each other, i.e., the states resultant from excitation operators acting to the reference state are orthogonal with each others.

Here we explore a new approach to the CCM by employing an efficient form of particle-hole excitation operators. These excitation operators for the exponent of CCM wave function are non-orthogonal with each other. All possible states resultant from such excitation operators acting on the reference state form a over complete basis set. The Matching-pursuit algorithm [31–34] is capable of searching this over complete basis set to find most relevant excitation operators for the coupled cluster method. The Matching-pursuit algorithm is originally proposed as signal process method. This algorithm searches a over complete basis set to find the most important ones to span a given state. This is basically in the same spirit with renormalization algorithm in the sense that it keeps the most important parts and ignore small components. Redundancy of the basis set affects the performance of the algorithm. For sufficiently redundant basis set, the convergence of the matching-pursuit algorithm can be exponential [35]. As a result, a few hundreds of excitation operators are enough to arrive meaningful result for the Hubbard model in our test calculations.

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This is the advantage of non-orthogonal form of excitation operators.

The wave function of coupled cluster method (CCM) has the form
\[ \Psi = \exp(T) |\Phi_0\rangle, \] (1)
where \(|\Phi_0\rangle = \prod_{i=1}^{M} a_i^\dagger |0\rangle\) is the reference state, \(M\) the particle number, and \(a_i^\dagger (i = 1, \ldots, N)\) the single particle creation operator. The states \(\{a_i^\dagger |0\rangle, \ldots, a_N^\dagger |0\rangle\}\) form single particle basis states. The operator \(T\) is a superposition of some particle-hole excitation operators \(\{T_1, \ldots, T_n\}\)
\[ T = \sum_{i=1}^{n} \alpha_i T_i, \] (2)
The conventional CCM usually chooses \(\{T_i\}\) as some low order excitation operators, such as single and double excitation operators in the form \(a_\alpha^\dagger a_\beta, a_\alpha^\dagger a_\beta^\dagger a_\gamma a_\delta, \ldots\). Here \(i, j \leq M\) denote the occupied orbits, and \(\alpha, \beta > M\) denote the unoccupied ones. A high order excitation operator is indeed a product of single particle-hole excitation operators \(a_\alpha^\dagger a_\beta\). Despite success for moderate sized atoms and molecules, the number of these excitation operators increases rapidly with particle number, and it is hard to implement in typical systems of condensed matter physics, such as, e.g., Hubbard model away from half filling.

We employ an efficient parameterization for the single particle-hole excitation operators that form the excitation operators \(T_i\),
\[ T_i = \prod_{j} f_j^{(i)}, \] (3)
where \(m\) is the order of excitation. Unlike the conventional CCM, the single particle-hole excitation operator has the form
\[ f_j^{(i)} = \lambda_j^{(i)} + f_j^{(i)\dagger} g_j^{(i)}. \] (4)
Here \(\lambda_j^{(i)}\) is an adjusting parameter usually not vanished. The single particle creation operator
\[ f_j^{(i)\dagger} = \sum_{\alpha=M+1}^{N} f_{j\alpha} a_\alpha^\dagger, \] (5)
and the single hole creation operator
\[ g_j^{(i)} = \sum_{k=1}^{M} g_{jk} a_k \] (6)
are linear combination of basis single particle and hole creation operators, respectively. The parameters \(\lambda_j^{(i)}\), \(f_{j\alpha}\), and \(g_{jk}\) are yet to be determined in optimization procedures. Our results show that the parameter \(\lambda_j^{(i)}\) is usually nonzero. Thus the single particle-hole excitation operator \(t_j^{(i)}\) has only finite probability to excite a particle-hole pair. This means that the excitation operator \(T_i\) contains all orders of excitations up to order \(m\). In fact, one can force \(\lambda_j^{(i)}\) to be zero. But this costs the efficiency. In this case, one has to consider all orders of excitation explicitly. Such setting needs much more excitation operators to reach a similar accuracy. It is easy to see from the form of (4) that a low order excitation operator is a special form of a high order one. For example, by setting \(\lambda_m = 1\) and \(f_{m\alpha} = g_{m\alpha} = 0\), a \(m\)-th order excitation operator becomes in fact a \((m - 1)\)-th order excitation.
In our calculations, we fix the order of excitation \(m\) for all operators \(T_i\). Note that \(f_j^{(i)\dagger}\) and \(g_j^{(i)}\) anti-commute with each other \([f_j^{(i)\dagger}, g_j^{(i)}]_+ = 0\). Thus all the particle-hole excitation operators \(t_j^{(i)}\) and \(T_i\) commute with each other \([t_j^{(i)}, T_i]_+ = 0, [T_i, T_j]_+ = 0\).

We search the excitation operators by optimization of an approximate ground state wave function of configuration interaction in terms of these excitation operators. The wave function is formed by acting the excitation operators to the reference state,
\[ \Psi_{CI} = (I + \sum_i c_i T_i) |\Phi_0\rangle. \] (7)
In this sense, our method is indeed a procedure of improvement to the CI wave function via CCM formulation, i.e., a CCM wave function using pre-determined excitation operator of a CI wave function. Note that the conventional CCM wave function also uses the same excitation operators of the CI wave function. The CCM wave function can be viewed as a normalized form of CI wave function with amplitudes of high order excitation operators determined by the low order ones. There is a basic observation from the CCM theory: If two excitation operators \(T_i\) and \(T_j\) have significant contribution to the CI wave function, then their product \(T_i T_j\) also has reasonable contribution to the CI wave function. The CI wave function (7) is a first order approximation to the CCM wave function (4), this linearized form is a major part of the CCM wave function. Thus find the approximate wave function (7) should determine the correct form of the excitation operators \(\{T_i\}\).

We use matching-pursuit algorithm in a similar way to that of Ref. [30] to find the operators \(T_i\). The match-pursuit algorithm searches a over complete basis set to find a sparse representation of a given state. The convergence depends on the redundancy of the basis set: The more redundant a basis set is, the faster the convergence rate is [32]. The states \(\{T_i|\Phi_0\rangle\}\) are not orthogonal with each other, and all possible such states form a over complete basis set. The non-vanish parameters \(\lambda_j^{(i)}\) in the single particle excitation operator \(t_j^{(i)}\) increase the redundancy of the excitation operators. If we set all \(\lambda_j^{(i)} = 0\),...
the redundancy decrease drastically. In this case, the states \( \{ T_i \}_0 \) are almost orthogonal with each other, this is especially the case for high order excitations. This is a key figure for the fast convergence with the number of the excitation operators. The matching-pursuit algorithm finds the basis states one by one. Our basis set are multi-linear function of the unknown variables. Thus if two basis states differ only by one single particle creation (destruction) operator, they can be merged into one operators. This property can be used to further optimize a given excitation operator.

An alternative form of the excitation operators \( \{ T_i \} \) in \( \text{T} \) is

\[
T_i = \prod_{k=1}^{m} f^{(i)}_k \prod_{j=1}^{m} g^{(i)}_j.
\]

Here the single particle creation operator \( f^{(i)}_k \) is a linear combination of all basis creation operators

\[
f^{(i)}_k = \sum_{s=1}^{N} f^{(i)}_{ks} g^{(i)}_j.
\]

In fact, for every operator \( T_i \), there exist an operator \( T'_i \), such that \( T_i | \Phi_0 \rangle = T'_i | \Phi_0 \rangle \), and vice versa. Our calculation is indeed to search the operators \( T'_i \). The advantage of using \( T'_i \) is that the CI wave function is multi-linear function of the coefficients \( f^{(i)}_k \) and \( g^{(i)}_j \). A rotation in the subspace spanned by single particle creation (destruction) operators \( f^{(i)*}_k \) (\( g^{(i)}_j \)) doesn’t change the excitation operator. The operators \( T'_i \) are not commute with each other, and thus not suitable for the CCM calculation. We transforming \( T'_i \) into the form of \( T_i \) for CCM calculation.

These operators are randomly initialized. We using linear optimization method to find these operator one by one. The detail procedures is effectively the same as that of Ref. \( \text{T} \). It is a variational procedure that minimizes the Reyleigh quotient \( E = \langle \Psi_{C1} | H | \Psi_{C1} \rangle / \langle \Psi_{C1} | \Psi_{C1} \rangle \). The wave function \( \text{T} \) is a multi-linear function of the parameters that define these excitation operators. Each step of search is to optimize the linearly dependent parameters in one particle-hole excitation operator. We optimize each operators consecutively one by one. Backward optimization are needed to further improve the result.

Calculation of matrix element is a basic task. Note that an excitation operator acting on the reference state results another determinant state. And the overlap of two determinant states is a determinant. A further detail to optimize the computation is to employ Wick’s theorem to calculate those determinants \( \text{T} \).

Using the excitation operators obtained in the above CI wave function, we construct the CCM wave function \( \text{T} \). Finding the amplitudes of the excitation operators in the CCM wave function is similar to that of conventional CCM method by a projective method. This method chooses the amplitudes of the excitation operators such that the reference state \( | \Phi_0 \rangle \) to be the ground state of the similarity transformed Hamiltonian \( \tilde{H} = e^{-T} H e^{T} \). Of course, this is equivalent to say that the state \( e^T | \Phi_0 \rangle \) is the ground state of the original Hamiltonian \( H \). Thus the state \( \tilde{H} | \Phi_0 \rangle \) is orthogonal to state \( (T_i - \langle \Phi_0 | T_i | \Phi_0 \rangle) | \Phi_0 \rangle \),

\[
\langle \Phi_0 | (T_i - \langle \Phi_0 | T_i | \Phi_0 \rangle) e^{-T} H e^{T} | \Phi_0 \rangle = 0.
\]

Note that the state \( T_i | \Phi_0 \rangle \) is not orthogonal to the reference state \( | \Phi_0 \rangle \). These equations determine the amplitudes of the excitation operators. This is in fact a diagonalization of the similarity transformed Hamiltonian \( \tilde{H} \) in the subspace spanned by \( | \Phi_0 \rangle \) and \( \{ T_i | \Phi_0 \rangle \} \). Similar to the conventional CCM, these equations are nonlinear. The left side of each equation is an algebraic forth order polynomial of the amplitudes. The polynomial is determined by

\[
e^{-T} H e^{T} = H + [H, T] + \frac{1}{3!} [H, T], T] + \cdots.
\]

For Hamiltonian containing one and two body operators, the above expansion terminates at forth order.

In usual cases, several hundreds of excitation operators are enough. Thus the number of variables in the above nonlinear equations is much smaller than the implementation of conventional CCM. The solution of the above equation is easier to hand. We simply use Newton’s iteration method to find solution. The convergence is usually quite fast from a proper initial guess. From our tests, we find that result from CI calculation is a good choice for initial guess. Another useful rule is that the expansion to second order in Eq. \( \text{T} \) is enough to obtain accurate result. There are similar treatments in unitary CCM, see, e.g., Refs. \( \text{T} \).

The main numeric cost for the amplitudes of the excitation operators is the matrix elements of the commutators in Eq. \( \text{T} \). Each commutator of the Hamiltonian and the excitation operators needs to be calculated independently. Our practical implementation is to compute the matrix elements of the products of the excitation operators and the Hamiltonian, \( \langle \Phi_0 | T_i | \Pi_j T_l | \Phi_0 \rangle \). The result of the matrix element is a determinant. Application of Wick’s theorem can optimize the calculation of the matrix elements. This is in sharp difference with the conventional CCM. Truncation to second order in \( \text{T} \) saves much of the numeric costs.

Calculations of the expectation value of an observable need the left side ground state \( | \Phi_0 \rangle \) of the similarity transformed Hamiltonian \( \tilde{H} \). \( \langle \Phi_0 | \tilde{H} | \Phi_0 \rangle = \langle \Phi_0 | H | \Phi_0 \rangle \). Here \( \langle \Phi_0 | = \sum_{i} \langle \Phi_0 | (I + T_i^{(i)}) \phi_i \rangle \). We use standard method to find the left side ground state in the subspace spanned by \( \langle \Phi_0 | \) and \( \{ \langle \Phi_0 | T_i \} \). The expectation value of an observable \( F \) with respect to the original Hamiltonian’s ground state is \( \langle \Phi_0 | \exp(-T) F \exp(T) | \Phi_0 \rangle \).

An advantage of the above formulation is easy to take into account of high order excitation operators. To estimate the accuracy of the above method, consider configuration interaction by excitation operators \( T_i \), and all
of their possible products \(T_i T_j, T_i T_j T_k, \ldots\), i.e., using basis states \(\{T_i \Psi_0\}, \{T_i T_j \Psi_0\}, \{T_i T_j T_k \Psi_0\}, \ldots\), to span the ground state of the Hamiltonian. Such configuration interaction approaches the full configuration interaction (FCI) when basis states \(\{T_i \Psi_0\}\) span a low order configuration interaction. Similar to the conventional CCM, if basis states \(\{T_i \Psi_0\}\) plus the reference state \(\Psi_0\) span major part of the ground state, our CCM wave function approaches the configuration interaction by the excitation operators \(T_i\) and all their possible products. In other word, if CI wave function in terms of the excitation operators \(\{T_i\}\) approaches a \(k\)-th order conventional CI, our CCM wave function approaches \(k\)-th order conventional CCM. Such order \(k\) can be significant higher than two. This is in sharp distinction with the conventional CCM that usually restricts to the single and double excitations, since numeric cost in conventional CCM increases drastically with the order of excitations.

We test the above method by finding ground state of Hubbard model on a square lattice with periodic boundary condition. The Hamiltonian reads

\[
H = -t \sum_{\langle i,j \rangle, \sigma} (a^\dagger_{i\sigma} a_{j\sigma} + H.c.) + U \sum_i n_{i\uparrow} n_{i\downarrow}, \tag{12}
\]

where \(a^\dagger_{i\sigma}\) (\(a_{i\sigma}\)) is creation (annihilation) operator of spin \(\sigma\) electron on site \(i\), \(n_{i\sigma} = a^\dagger_{i\sigma} a_{i\sigma}\). This model relates to many important phenomena of correlated electrons. It is extensively explored by many algorithms.

Table \(\text{II}\) shows the ground state energies at half filling of an 8 \(\times\) 8 lattice. We use 200 8-th order excitation operators to obtain the above result labeled as PCCM. The number of the excitation operators is determined by the convergence of the CI wave function. To compare with other results, the conventional CCM result \([\text{II}]\) and variational Quantum Monte-Carlo (VMC) result \([\text{II}]\), as well as two QMC results labeled as QMC \([\text{II}]\) and CPMC \([\text{II}]\) are also listed. Note that the CPMC result is for infinite sized system, and it has a little bit discrepancies from the first one. We see that our result is in the same accuracy as the conventional CCM result with some improvement for large \(U\). It is also compatible with the QMC results. The conventional CCM method for Hubbard model works at half filling case with a reference state that each electron occupies one site. This kind of reference state has translational symmetry that is a precondition of applying the conventional CCM to the Hubbard model. Here we use the usual unrestricted Hartree-Fock state as reference state which has no need for the precondition of the translational symmetry. Unlike the excitation operators in the conventional CCM, which are set by hand from consideration of the Hamiltonian’s structure, the excitation operators in our method come from search procedure with a randomly initial guess. This implementation can apply to general systems. The search for the excitation operators is for configuration interaction wave function by minimizing the expectation value of the Hamiltonian. Then we use these operators to perform coupled cluster calculation. The result shows that this approach works well for a reasonable reference state. In fact, in the half filled case, the ground state energy from the mean field theory has about 90% accuracy. Our test calculations show that it is enough to expand the polynomial \(e^{-T} He^T\) of Eq. (11) to the second order for the calculation of the amplitudes of the excitation operators in the CCM wave function.

Table \(\text{II}\) shows the ground state energies at a filling 62/64 of a 8 \(\times\) 8 square lattice. The QMC and VMC results are from the same references of that in table \(\text{II}\). We also include the Hartree-Fock (HF) and configuration interaction (CI) results of our calculation for comparison. The conventional CCM method is hard to treat such cases of alway from half filling due to the difficulty to find a seasonable reference state with translational symmetry. In our implementation, the procedures are the same as above the half filling case. The resultant accuracy and numeric cost are similar to the above half filling case. In
TABLE III: Ground state energies of some the $8 \times 8$ systems at various filling for $U = 8t$. The comparing QMC and VMC results are from the same sources of that in Table II.

| N  | CPMC | VMHC | HF | CI | PCCM(1) | PCCM(2) |
|----|------|------|----|----|---------|---------|
| 50 | -0.9168 | -0.7031 | -0.7966 | -0.8453 | -0.9167 |
| 52 | -0.8709 | -0.6638 | -0.7343 | -0.7943 | -0.8672 |
| 54 | -0.8200 | -0.6320 | -0.6973 | -0.7536 | -0.8202 |
| 56 | -0.7623 | -0.6462 | -0.6549 | -0.7005 | -0.7595 |
| 58 | -0.6995 | -0.5640 | -0.6184 | -0.6561 | -0.6991 |
| 60 | -0.6316 | -0.5331 | -0.5783 | -0.6094 | -0.6222 |
| 62 | -0.5705 | -0.5641 | -0.4996 | -0.5373 | -0.5630 |
| 64 | -0.5110 | -0.4926 | -0.4659 | -0.4956 | -0.5143 |

fact, the performance of our method near half filling is all the same. The reference state is still the unrestricted Hartree-Fock state. This mean field state near half filling has a reasonable accuracy for ground state energy, and almost correct symmetries. It is enough to use 200 8-th order excitation operators to obtain table III. We search the excitation operators by optimization of the CI wave function in the same way as the half filled case. Similar to the conventional CCM, the result is insensitive to the change of $U$. When the on site repulsion $U$ increase, the accuracy of our result is almost unchanged. This is indeed a common property of wave function based methods that immune from the sign problem.

Table III shows ground state energies of a $8 \times 8$ lattice at various fillings with periodic boundary condition and the on site repulsion strength $U = 8t$. The comparing QMC and VMC results are from the same sources of that in Table II. The Hartree-Fock (HF) and configuration interaction (CI) results are from our calculations. Along with comparing results, we list two results, labeled by superscripts (1), and (2), correspondent 8-th order and higher order excitations, respectively. For fittings $50/64$, $52/64$, and $54/54$, the excitation order is 20. The excitation order is set to 58 for the fittings $58/64$ and $60/64$. For other cases near half filling, 8-th order is enough. The results for cases of near half filling are essentially similar as above. However, in the cases of far away from half filling there are instabilities in the projective numeric procedures. This is a well known problem of the CCM’s projective implementation [43,47]. Similar to the conventional CCM, the reference state is crucially important to the performance. The mean field results exhibit shell effect. The half filling case corresponds to the closed shell case in the quantum chemistry, and away from half filling cases correspond open shell cases. Conventional CCM for open shell cases usually need multiple reference states that demands much higher numeric cost [6–12]. For Hubbard model, the mean field results of open shell cases are highly degenerated. Different initial guess may results in qualitatively different wave function. If one chooses such qualitatively wrong reference state, the projective method may exhibit instability or even divergent result. In our calculations, we need high order excitation opera-

FIG. 1: Spin structure factor versus wave vector along three different lines for various filling numbers at $U = 8t$ with a $8 \times 8$ system size.
Figure 1 shows the magnetic structure factors

\[ S(q) = \frac{1}{N} \sum_{i,j} e^{i(q \cdot (R_i - R_j))} (n_{i\uparrow} - n_{i\downarrow})(n_{j\uparrow} - n_{j\downarrow}), \]

along three different lines for various filling with interaction strength \( U = 8 \) on a \( 8 \times 8 \) lattice. Near half filling, the structure factors peak at \((\pi, \pi)\) indicating antiferromagnetic correlation. The correspondent mean field result also shows same correlation with a magnitude of several times smaller. In the region far from half filling, the CCM results of magnetic structure factors are qualitatively different from mean field result. Such difference indicates that the mean field result is qualitatively wrong. In fact, the CI wave functions of the open shell cases also have qualitatively different magnetic structure factors with that of mean field results. These cases cause instability and need further investigations. On the other hand, comparison of difference between the structure factor of CCM wave function and that of the reference state provides a way to check the reference state.

Figure 2 shows convergence rate versus number of excitation operators and order of excitation operators in the top and bottom panels, respectively. We show two filling numbers, 58/64 and 60/64, in a periodic \( 8 \times 8 \) lattice with \( U = 8 \). Other cases are quite similar. Here the convergence rate is relative error with respect to best result, \( E_r = |(E - E_0)/E| \) with \( E \) being our numeric result and \( E_0 \) the best available result from others. We see that the wave function converges quickly with the number of the excitation operators. One usually needs about 200 to 300 excitation operators to arrive a convergent wave function. This is much faster than the CI wave functions. Note that using high order excitation operators automatically includes lower order operators. For a proper reference state near half filling, eighth order excitation operators are enough to arrive a convergent wave function. The CI wave function is rather insensitive to the order of excitation. The CI wave functions with second to eighth order excitation operators give similar ground state energy. However, the CCM wave function with higher order excitation operators produces much lower ground state energy. For open shell cases, one needs much higher order of excitation operators to obtain ground state energy comparable with other best result. A convergent CCM wave function of higher order excitation operators can be qualitatively different from the reference state. In the 50/64 filling case, we use 20-th order operators to obtain an convergent wave function. Our results show that high order excitation operators are more efficient to improve the CCM wave function than increasing number of lower order excitation operators. Note that, one can set \( \alpha_i = 0 \) in Eq. (2), and consider every order of excitation operators explicitly. Such choice in our calculations performs poorly. In our test calculations, one needs much more excitation operators. Even in the case of half filling, one needs more than 1000 operators up to 4-th order to arrive convergence. This indicates that let \( \alpha_i \) as an adjustable parameter is an optimal form for the excitation operators.

In summary, we show a coupled cluster method in terms of non-orthogonal excitation operators. Search of these optimal excitation operators is indeed in the same spirit of renormalization approach. In comparison with conventional CCM, these parameterized excitation operators represent the most relevant portion of the whole excitation operators. Several hundreds of excitation operators are enough to arrive meaningful results. This number is several orders smaller than that of conventional CCM. This performance makes it possible to apply the CCM to condensed matter physics. The current implementation is indeed a single reference CCM that is efficient in cases of near closed shell. Other cases in deep regions of open shell need further investigations to improve the projective procedure or developing multi-reference states CCM.

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