The quantum bit commitment: a finite open system approach for a complete classification of protocols

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Mayers [1], Lo and Chau[2, 3] argued that all quantum bit commitment protocols are insecure, because there is no way to prevent an Einstein-Podolsky-Rosen (EPR) cheating attack. However, Yuen [4, 5, 6] presented some protocols which challenged the previous impossibility argument. Up to now, it is still debated whether there exist or not unconditionally secure protocols [7]. In this paper the above controversy is addressed. For such purpose, a complete classification of all possible bit commitment protocols is given, including all possible cheating attacks. Focusing on the simplest class of protocols (non-aborting and with complete and perfect verification), it is shown how naturally a game-theoretical situation arises. For these protocols, bounds for the cheating probabilities are derived, involving the two quantum operations encoding the bit values and their respective alternate Kraus decompositions. Such bounds are different from those given in the impossibility proof [1, 2, 3].

The whole classification and analysis has been carried out using a finite open system approach. The discrepancy with the impossibility proof is explained on the basis of the implicit adoption of a closed system approach—equivalent to modeling the commitment as performed by two fixed machines interacting unitarily in an overall closed system—according to which it is possible to assume as openly known both the initial state and the probability distributions for all secret parameters, which can be then purified. This approach is also motivated by existence of unitary extensions for any open system. However, it is shown that the closed system approach for the classification of commitment protocols unavoidably leads to infinite dimensions, which then invalidate the continuity argument at the basis of the impossibility proof.

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I. INTRODUCTION

Among all kinds of quantum cryptography protocols, the quantum bit commitment is a crucial element to build up more sophisticated protocols, such as remote quantum gambling [8], coin tossing [9], and unconditionally secure two-party quantum computation [10]. Therefore, it is of practical relevance to establish if there exist secure quantum bit commitment protocols.

In the bit commitment Alice provides Bob with a piece of evidence that she has chosen a bit \( b = 0, 1 \) which she commits to him. Later, Alice will open the commitment, revealing the bit \( b \) to Bob, and proving that it is indeed the committed bit with the evidence in Bob’s possession. Therefore, Alice and Bob should agree on a protocol which satisfies simultaneously the three requirements: (1) it must be concealing, namely Bob should not be able to retrieve \( b \) before the opening; (2) it must be binding, namely Alice should not be able to change \( b \) after the commitment; (3) it must be verifiable, namely Bob must be able to check \( b \) against the evidence in his possession, according to the rules of the protocol. In a in-principle proof of security of the commitment it is supposed that both parties possess unlimited technology, e. g. computational power, space, time, etc., and the protocol is said unconditionally secure if neither Alice nor Bob can cheat with significant probability of success as a consequence of physical laws.

In 1993, a quantum mechanical protocol was proposed [11], and the unconditional security of this protocol has been generally accepted for long time. The insecurity of this protocol was shown by Mayers, Lo and Chau [1, 2, 3] in 1997, where it was recognized the possibility for Alice to cheat by entangling the committed evidence with a quantum system in her possession, and it was argued that no unconditionally secure protocol is possible. Finally after 2000 Yuen [4, 5, 6] presented some protocols which challenged the previous impossibility proof, mostly on the basis of the possibility of encoding the bit on an anonymous state given to Alice by Bob and known only to him, and suggesting the use of decoy systems that make the protocol concealing in the limit of infinitely many systems, with the possibility for Bob of performing his quantum measurement before Alice opening, whence disputing the general availability of EPR cheating for Alice. Besides the above schemes, protocols have also been suggested based on the theory of special relativity [12] (for historical reviews on the quantum bit commitment see Refs. [6, 7]). Here, however, we will consider only non relativistic protocols.

In this paper, in order to provide clarifications on the issue of existence of unconditionally secure protocols, we will give a complete classification of all possible bit com-
mitment protocols based on a single commitment step, and show how a multi-step commitment can be reduced to a single one. We will analyze all possibilities of cheating for both parties. Then, we will focus on the simplest class of protocols, namely the non-aborting protocols with complete and perfect verification, showing that naturally a game-theoretical situation arises. As we will see, even though perfectly concealing protocols are certainly not binding (i.e., Alice has a unit cheating probability), the protocol could still be binding if it is $\varepsilon$-concealing. Bounds for the cheating probabilities of these protocols are derived, involving the two quantum operations encoding the bit values and their respective alternate Kraus decompositions. Such bounds turn out to be different from those given in the impossibility proof. In the final discussion we will see how the discrepancy between the two opposite analysis arises, as a result of the restrictive assumption—which lies beneath the impossibility proof—of openly known, whence purifiable, probability distributions for all secret parameters. Such an assumption is equivalent to modeling the commitment as a closed system made of two fixed machines interacting unitarily. It is shown that such modeling, along with the requirement of unlimited technology, necessarily lead to infinite dimensions, which invalidate the continuity argument at the basis of the impossibility proof. Instead, one either needs to prove the continuity argument for infinite dimensions, or else must adopt a finite open system approach.

II. THE CLASSIFICATION OF PROTOCOLS

The most general bit commitment scheme with a single step is of the form: (1) Bob prepares the Hilbert space $\mathcal{H}$ with the anonymous state $|\varphi\rangle \in \mathcal{H}$, and sends $\mathcal{H}$ to Alice; (2) Alice modulates the value $b$ of the committed bit on the anonymous state $|\varphi\rangle$ and sends the output back to Bob. The bit modulation is a quantum operation (QO) parametrized by $b = 0, 1$. It is clear that this general scheme contains all possibilities, including the anonymous-state based protocols of Yuen, Lo and Chau, and as a special case, the original protocols by Mayers, and Lo and Chau, which correspond to openly known state $|\varphi\rangle$.

A. Bit modulation

To make the protocol concealing and at the same time verifiable, the modulation must be a choice between two ensembles of QO’s $\{M_j^{(b)}\}$ for $b = 0, 1$, from $T(\mathcal{H})$ to $T(\mathcal{K})$, where $T(\mathcal{H})$ denotes the set of traceclass operators on $\mathcal{H}$, and generally the two Hilbert spaces $\mathcal{K}$ and $\mathcal{H}$ are not isomorphic. We will name the cases $\mathcal{K} \supseteq \mathcal{H}$ and $\mathcal{K} \subseteq \mathcal{H}$ extending modulation and restricting modulation, respectively, the extending case including the possibility of using decoy systems. The variable $j$ is a secret parameter known only to Alice, parametrizing the choice of different forms for the modulation, and which will be declared to Bob at the opening.

B. The secret-parameter space

Alice has always the option of choosing $j$ by preparing a secret-parameter space $\mathcal{P}$ in a state chosen from an orthonormal set $\{|j\rangle\}$, and performing the QO on $\mathcal{H} \otimes \mathcal{P}$

$$M_j^{(b)} = \frac{1}{\rho_j} \sum_j M_j^{(b)} \otimes P_j,$$  \hspace{1cm} (1)

with $P_j$ representing the orthonormal projection map $P_j(\rho_j) = |j\rangle\langle j| \rho_j |j\rangle\langle j|$. The actually performed map depends on the state preparation $\rho_j$ that Alice chooses for $\mathcal{P}$, and any (pure or mixed) state will be equivalent to a set of probabilities $p_j^{(b)} = \langle j| \rho_j^{(b)} |j\rangle$ for the secret parameter $j$ as follows

$$\text{Tr}[M_j^{(b)}(\rho |\varphi\rangle \otimes \rho_j^{(b)})] = \sum_j M_j^{(b)}(\langle \varphi| \langle j| \rho_j^{(b)} |j\rangle) = \sum_j M_j^{(b)}(\langle \varphi| \langle j| \rho_j^{(b)} |j\rangle).$$  \hspace{1cm} (2)

C. Reduction to trace-preserving maps

The maps $M_j^{(b)}$ are generally trace-decreasing, i.e., they may be achieved with nonunit probability. In terms of the Kraus decomposition for any input state $\rho$

$$M_j^{(b)}(\rho) = \sum_i E_{ji}^{(b)} \rho E_{ji}^{(b)\dagger},$$  \hspace{1cm} (3)

this means that generally

$$\sum_i E_{ji}^{(b)} E_{ji}^{(b)\dagger} \leq I.$$  \hspace{1cm} (4)

Strictly trace-decreasing maps correspond to aborting protocols, namely when Alice doesn’t succeed in achieving the map the protocol is aborted. By completing the sum in Eq. (4) with additional terms in order to get the identity, we see that a trace decreasing map is equivalent to a trace preserving one, with additional outcomes $i$ corresponding to the protocol aborted.

D. Reduction to unitary

Alice has unlimited technology, whence she can always achieve $E_{ji}^{(b)}$ knowingly, i.e., she has the option of achieving each trace-preserving map $M_j^{(b)}$ as a perfect pure measurement. This can be done as follows (in the following
we will temporarily drop the indices $b$ and $j$. The trace-preserving QO can be written in the form

$$M(\rho) = \text{Tr}_F[E\rho E^\dagger],$$

$$E = \sum_i E_i \otimes |i\rangle_F \in \mathcal{B}(H, K \otimes F) \text{ isometry},$$

for a suitable ancillary space $F$ (notice the tensor notation $E \otimes |\psi\rangle_F$ that for $E$ operator in $\mathcal{B}(H, K)$ represents an “extension” operator from $H$ to $K \otimes F$. By unitary embedding $H$ into $K \otimes F \simeq H \otimes A$ for another suitable ancillary space $A$ as $E = U(I_H \otimes |\omega\rangle_A)$, with $U$ unitary on $H \otimes A \simeq K \otimes F$, we have

$$M(\rho) = \text{Tr}_F[U(\rho \otimes |\omega\rangle \langle \omega|_A)U^\dagger],$$

namely Alice prepares the ancilla (and decoy systems) in the state $|\omega\rangle$, and then performs a complete von Neumann measurement on $F$, with outcome $i$, which she keeps secret [the possibility of using a more general type of measurement is already considered in an extended space $F$]. The strictly trace-decreasing case would correspond to write

$$M(\rho) = \text{Tr}_F[(I_K \otimes \Sigma_F)U(\rho \otimes |\omega\rangle \langle \omega|_A)U^\dagger],$$

with $\Sigma_F$ orthogonal projector on a subspace of $F$. In conclusion, Alice can achieve the QO $M(\rho) = \sum_i E_i \rho E_i^\dagger$ knowingly: (1) preparing an ancilla/decoy state $|\omega\rangle_A \in A$; (2) performing a unitary transformation $U$ on $H \otimes A$; (3) performing a complete von Neumann measurement on $F$, with $K \otimes F \simeq H \otimes A$ and outcome $i$; (4) sending $K$ to Bob. Notice that we can have both extending and restricting protocols, depending on the choice of $A$ and $F$. At this point, the encoding maps are given by

$$M^{(b)}(|\varphi\rangle\langle \varphi|) = \sum_j p_j \text{Tr}_F[(I_K \otimes \Sigma^{(b)}_F)U_j^{(b)}(|\varphi\rangle \langle \varphi| \otimes |\omega\rangle_A U_j^{(b)\dagger}] ,$$

with $|\omega\rangle$ independent on $j$ and $b$, since any dependence can be included in $U_j^{(b)}$. Notice that if all orthogonal projectors $\Sigma^{(b)}_j$ on subspaces of $F$ have the same rank, their dependence on $j$ and $b$ can also be included in $U_j^{(b)}$, but generally rank($\Sigma^{(b)}_j$) depends on both $j$ and $b$. For the moment, we will focus attention on the case in which rank($\Sigma^{(b)}_j$) is independent on $j$. Now, by considering the unitary operator $U^{(b)} = \sum_i U_j^{(b)} \otimes |j\rangle i\rangle$ over $H \otimes A \otimes P \simeq K \otimes F \otimes P$, we see that Alice can achieve any possible commitment step as follows

$$M^{(b)}(|\varphi\rangle\langle \varphi|) = \text{Tr}_F \otimes P[(I_K \otimes \Sigma^{(b)}_F)U^{(b)}(|\varphi\rangle \langle \varphi| \otimes |\omega\rangle_A \otimes \rho_P)U^{(b)\dagger}]] ,$$

where also $\rho_P$ is independent on $b$, whence, since also the probabilities $p_j^{(b)}$ will be independent on $b$, we will denote them simply as $p_j$.

### E. Opening step

In a protocol which is completely and perfectly verifiable Alice tells $b$ along with the secret parameter $j$ and the secret outcome $i$ to Bob, who verifies the state $E_j^{(b)}|\varphi\rangle$. (In a protocol that is not perfectly verifiable, the disclosed state is generally mixed, e. g. Alice keeps the outcome $i$ secret, or join outcomes in composite events as in a degenerate Lüders measurement). However, we emphasize that, whatever is the rule for the opening, Alice has always the option of achieving the encoding QO by performing a complete von Neumann measurement. Since the local QO’s on $K$ and $F \otimes P$ commute, Alice has the possibility of: (1) first sending $K$ to Bob; (2) then performing the measurement on $F \otimes P$ at the very last moment of the opening. As we will see, this is the basis for Alice EPR cheating attacks. Notice that strictly trace-decreasing QO’s—i. e. aborting protocols—pose limitations to Alice’s EPR cheating. In fact, Alice cannot delay the abortion of the protocol at the opening, and must declare it at the commitment. Since both secret parameters $j$ and $i$ can be conveniently measured by Alice, they can be treated on equal footings as a single parameter $J \equiv (j, i)$. The two maps are then

$$M^{(b)}(|\varphi\rangle\langle \varphi|) = \sum_j p_j M^{(b)}_j(|\varphi\rangle\langle \varphi|) = \sum_j E_j^{(b)}|\varphi\rangle\langle \varphi|E_j^{(b)\dagger},$$

with $E_j^{(b)} = \sqrt{p_j} E_j^{(b)} \otimes \rho_j^{(b)} \in \mathcal{B}(H, K)$.

### F. Reduction to a single commitment step

A protocol with more than a commitment step generally consists of a sequence of conditioned QO’s, namely in which one party is requested to make a different QO in which one party is requested to make a different QO, say $\{N^{(x)}\}$, depending on the outcome $x$ of a previous QO from the other party. However, the same result is achieved by automatizing the conditioned QO, and using instead the unconditioned one $N = \sum_x N^{(x)} \otimes P_x$ on an extended Hilbert space $H \otimes N$, without even knowing $x$. If the knowledge of $x$ is requested only at the opening—as for nonaborting protocols—then the orthogonal measurement $P_x$ can be delayed up to the opening moment, since the notepad space $N$ is kept by the party. Then, analogously as for a single commitment step, each QO can be achieved knowingly, by means of a pure measurement, with a suitable unitary embedding. Again, since the measurement-ancillary space ($F$ in the above analyzed single commitment step) is kept by the considered party, its measurement can be delayed up to the opening moment (for strictly trace decreasing maps the two parties can agree to declare the abortion at the end of the whole commitment). At this point, we have a sequence of interlaced unitary operators, one from each party alternatively, e. g. for three steps $U_A^{(b)} U_B U_A^{(b)}$, where clearly
the unitary transformation by Bob $U_B$ cannot depend on $b$. Now, for a numerable set of possible unitary transformations $U_B \in \{U_i\}$, Bob can use instead the unitary $U_B = \sum_j U_i \otimes |j\rangle\langle j|$ by preparing a state from an orthogonal set $\{|j\rangle\}$ on an additional Hilbert space. Therefore, the choice of the unitary is equivalent to the state preparation of another anonymous-state Hilbert space. In conclusion, from the arguments above we see that the whole multi-step (non aborting) protocol is equivalent to a single-step one, with larger spaces $H$, $K$, $A$, $F$, and $P$. We don’t know what is the minimal Hilbert space for anonymous-state preparation of a generally continuous set of unitary operators, for which one may need a non separable space. Notice that with a teleportation protocol it is possible to achieve any contraction on a space $H$ by performing a state preparation on the space $H \otimes H$ of the entangled resource, however, only with probability equal to $\dim(H)^{-2}$.

G. Classical protocols

It is obvious that a classification of quantum protocols must include also the classical ones as a particular case. In fact, the classical protocols correspond to consider just orthogonal states, and QO’s on abelian operator algebras. Consider, for example, a one-way trapdoor function $f_A(j)$, where the integer $j$ plays the role of the secret parameter. Let the value $b$ if the committed bit be the parity of $j$. Then, Alice sends the state $|n\rangle \otimes |f_A\rangle$ to Bob, with $n = f_A(j)$. Bob can verify that $f_A$ is indeed one-way. However, since he cannot compute $j$ from the knowledge of $n$, he can just guess whether $j$ is even or odd. At the opening Alice tells $j$ to Bob, and Bob verifies that indeed $n = f_A(j)$ and evaluates the parity of $j$.

III. CHEATING

For the moment we will focus attention on non- aborting protocols, postponing the discussion of the aborting (strictly decreasing) case. Alice can cheat at two different moments: before and after the commitment. We will name the two cases: pre-cheating and post-cheating, respectively. Bob, as we will see, can perform a combined attack before and after the commitment. The possibility also for Alice of performing a combined attack will be also discussed.

A. Alice pre-cheating attacks

These correspond to prepare the ancillary spaces $A \otimes P$ in a state not of the prescribed form $|\omega\rangle\langle\omega|_A \otimes \rho_P$. This will generally lead to QO’s different from the ones prescribed from the protocol. In the following, we will not further analyze pre-cheating, for the two following reasons: (1) it seems that there is no practical use for Alice to cheat before knowing if the committed bit needed to be changed; (2) either there is a chance that the pre-cheating will be detected at the opening, or it would lead to QO’s indistinguishable from the prescribed ones, in which case it will give the same result of a post-cheating attack considered in the following. Finally, notice that a cheating attack based on changing the prescribed unitaries $U^{(b)}$ belongs to the same class of pre-cheating attacks, and the same considerations hold.

B. Alice post-cheating attacks

After the commitment and before the opening Alice can try to cheat by performing a unitary transformation $V$ on $F \otimes P$: this is the so-called EPR attack. The maneuver will not change the QO’s $M^{(b)}$, however, it will change the Kraus decompositions—which are relevant at the opening—giving a new set of contractions $\{E_j^{(b)}\} \rightarrow \{E_j^{(b)}(V)\}$ with the same cardinality, in the following way

$$E_j^{(b)}(V) = \sum_L E^{(b)}_L V_{jL}, \quad V_{jL} = \langle J|V|L\rangle.$$  \hspace{1cm} (11)

Another attack available to Alice is also that of declaring a $J$ different from the actual outcome: however, since Alice doesn’t know the anonymous state $|\varphi\rangle$, she must adopt a fixed rule to scramble the $J$’s, and being just a permutation, this cheating is again equivalent to a unitary cheating transformation $V$.

The probability that Alice can cheat successfully in pretending having committed, say, $b = 1$, whereas she committed $b = 0$ instead, is given by

$$P_c^A(V, \varphi) = \sum_j \frac{|\langle J|E_j^{(0)}(V)E_j^{(1)}|\varphi\rangle|^2}{|E_j^{(1)}| \varphi\rangle|^2}, \hspace{1cm} (12)$$

and it clearly depends on the anonymous state $|\varphi\rangle$ and on the cheating transformation $V$. However, Alice doesn’t know $|\varphi\rangle$, and the optimal choice of $V$ obviously depends on $|\varphi\rangle$. So, which is the transformation $V$ to be used? Without any knowledge of $|\varphi\rangle$, the best Alice can do is to adopt a conservative strategy, by choosing the $V$ such that the minimum $P(V, \varphi)$ for $|\varphi\rangle$ chosen by Bob is maximum, namely she maximizes her probability of cheating in the worst case, corresponding to the minimax choice of $V$

$$(P_{c}^A)_\mu = \max_V \min_\varphi P_c^A(V, \varphi). \hspace{1cm} (13)$$

It is evident that in this way a game theoretical situation arises, in which Bob choses $|\varphi\rangle$ and Alice choses $V$, with the probability $P(V, \varphi)$ playing the role of a payoff matrix. Obviously Alice and Bob can generally adopt randomized strategies, which can then be purified via entanglement with an ancillary system. However, in the general situation the game is further complicated by the
fact that Bob’s choice for $|\phi\rangle$ is also dictated by maximization of his own probability of cheating (see later), and all other parameters—such as Alice secret parameter $j$—must enter the game. Since we are only interested to set the debate on the impossibility proof via a complete classification of all protocols and cheating attacks, this game situation, which arises as a consequence of using anonymous states, will be analyzed elsewhere.

Another possibility for Alice’s choice of $V$ would be that of maximizing the probability $P(V, \phi)$ averaged over all anonymous states, with the unitarily invariant probability measure $d\mu(\phi)$ on the (compact) manifold of pure states, namely

$$(P^A_c)_{av} = \max_V \int d\mu(\phi) P^A_c(V, \phi).$$

However, such a (pure) strategy will not be optimal if Bob chooses a non-uniform probability distribution, e.g. a delta-function, and the actual probability of cheating could be much lower than the one in Eq. (14). It is obvious that for compact manifold of states—i.e. for finite dimensions—then the two probabilities in Eqs. (13) and (14) will be related by a constant depending on the dimension of $H$. The evaluation of the average in Eq. (13) is made difficult by the presence of the norm in the denominator. When the encoding for $b = 1$ is random unitary, i.e. $E_j^{(1)} = \sqrt{p_j} U_j^{(1)}$, with unitary $U_j^{(1)}$, the evaluation of the average in Eq. (14) is simplified by the following identity which holds for any couple of operators $A, B \in B(H)$ for $d = \dim(H) < \infty$

$$\int d\mu(\phi) \langle A | \phi \rangle \langle \phi | B | \phi \rangle = \frac{1}{d(d+1)} \left[ \text{Tr}(A) \text{Tr}(B) + \text{Tr}(AB) \right].$$

Using Eq. (15) the averaged probability in Eq. (14) rewrites

$$(P^A_c)_{av} = \frac{1}{d+1} \left[ \int d\mu(\phi) \left( \sum_j \left| \sum_L \text{Tr} \left( U_j^{(1)\dagger} E^{(0)}_L \right) V_{Lj} \right| ^2 \right]$$

which can be bounded as follows

$$\frac{1}{d+1} \leq (P^A_c)_{av} \leq \frac{1}{d+1} + \frac{1}{d(d+1)} \|Z\|_1,$$

where $\|\cdot\|_1$ denotes the trace-norm, and the matrix $Z_{(JL)K}$ has to be considered as rectangular, with $(JL)$ as a single index. From Eq. (16), we see that in order to reduce Alice’s cheating probability we better increase the dimension of the anonymous-state Hilbert space. The upper bound in Eq. (13) could be useful for proving unconditional security of the protocol: however, we don’t know if the trace norm in Eq. (17) is bounded as $\|Z\|_1 \leq d^2$, otherwise the bound (17) would be useless (one can check that $\|Z\|_1 = d^2$ in the perfectly concealing case).

C. Bob cheating

Bob can try to cheat by making the best discrimination between the two maps $M_{opt}(b) = \sum_j p_j M_{opt}(b)^j$. However, since he doesn’t know the probabilities $p_j$ actually used by Alice, his strategy will be suboptimal, and his actual cheating probability $P^B_c$ will be lower than the probability $P^B_{opt}$. Since map-discrimination is generally more reliable with the map acting locally on an entangled state, instead of preparing $|\phi\rangle \in H$ Bob prepares an entangled state on $H \otimes R$ and sends only $H$ to Alice. (Here, we can see clearly that the use of anonymous states in the protocol, while limiting Alice EPR cheating attacks, at the same time allows Bob to perform EPR attacks himself). Therefore, Bob’s optimal probability of cheating is bounded as follows (for equally probable bit values $b = 0, 1$)

$$P^B_c \leq (P^B_{opt})_{av} = \frac{1}{2} + \max_{|\phi\rangle \in H \otimes R} \frac{1}{4} \left\| M^{(1)} \otimes I_R (|\phi\rangle \langle \phi |) - M^{(0)} \otimes I_R (|\phi\rangle \langle \phi |) \right\|_1$$

$$+ \frac{1}{4} \| M^{(1)} - M^{(0)} \|_{cb},$$

where $\|\cdot\|_{cb}$ denotes the completely bounded (CB) norm, and we used the fact that the difference of two CP maps is Hermitian-preserving, whence its CB-norm is achieved on a normalized vector in $H \otimes R$. Notice that for trace-preserving QO’s the difference $M^{(1)} - M^{(0)}$ is never completely positive, and generally an entangled anonymous state improves the discrimination, whereas for aborting protocols the QO’s are strictly trace-decreasing, and the difference map can be completely positive itself, in which case the EPR attack is of no use (for such analysis on discriminations between QO’s, see Ref. [13]).

| Symbol | Hilbert space |
|--------|--------------|
| H      | Anonymous state |
| K      | Output |
| A      | Preparation ancilla/decoy |
| P      | Secret parameter |
| F      | Measurement ancilla |
| R      | Bob cheating space |
| Rng(Σ) | Range of Σ (abortion) |

**TABLE I:** List of Hilbert spaces needed for protocol and cheating attacks classification.
| start | commitment | after commitment |
|-------|-------------|------------------|
| Alice | A, P        | F, P             |
| Bob   | H, R        | K, R             |

TABLE II: Who owns which space and when.

D. Discussion on the aborting (strictly trace-decreasing) protocols

In the simplest case in which the projector $\Sigma$ is independent on both $b$ and $j$, Alice can launch an EPR attack easily, performing it on the range space of $\Sigma$. However, when the rank of $\Sigma$ depends on $b$, an EPR attack has a probability of being detected by Bob at the opening when the attack leads to a larger Kraus cardinality than the prescribed one. Notice also that, in general, a dependence of rank($\Sigma$) on $b$ and/or $j$ will enhance Bob’s probability of cheating.

Up to now we have seen that in order to classify all possible protocols and cheating attacks we need to consider seven Hilbert spaces with different physical meanings: these are summarized in Tables I and II.

E. Perfectly concealing protocols

A perfectly concealing protocol means that the CB-norm in Eq. (13) is zero, namely the two maps are the same. Therefore, the two Kraus are connected via a unitary transformation $V$ on $F \otimes P$, and Alice can cheat with probability one, namely the protocol is not binding.

F. Approximate concealing protocols

We consider now the case in which Bob’s probability of cheating for the optimal strategy is infinitesimally close to the pure guessing probability $\frac{1}{2}$, which means that also the CB-norm distance between the maps is infinitesimal, i.e. $\|M(1) - M(0)\|_{cb} = \varepsilon$. We emphasize that generally $\varepsilon$ is vanishing for increasing dimension of $K$ (see, for example, some protocols given by Yuen[6], where the approximately concealing condition is achieved for increasingly large number of decoy systems), and no obvious continuity argument can be invoked to assert that Alice cheating probability will approach unit for vanishing $\varepsilon$. More precisely, in the present context based on anonymous states, such an argument (which is at the basis of the possibility proof of Refs. [1, 3, 4]) would imply that for both the minimax and the averaged strategies in Eqs. (13) and (14) Alice probability of cheating would be infinitesimally close to unit for $\varepsilon \to 0$, namely

$$1 - (P^A_c)_{\mu, av} \leq \min_V \left\| \sum_j \left| E_j^0(V) - E_j^1 \right|^2 \right\|$$

for some function $\omega(\varepsilon)$ independent on the dimension of $K$ and vanishing with $\varepsilon$. However, using anonymous states such assertion may turn out to be false. In fact, it is obvious that if there is an alternate Kraus decomposition $\{E_j^0(V)\}$ for the map $M(0)$ such that the two Kraus $\{E_j^0(V)\}$ and $\{E_j^1\}$ are close, then the protocol is approximately concealing and not binding, since (see Appendix)

$$\frac{(P^A_c)_{\text{opt}} - 1}{2} = \frac{1}{4} \|M(1) - M(0)\|_{cb}$$

$$\leq \frac{1}{2} \left\| \sum_j \left| E_j^0(V) - E_j^1 \right|^2 \right\|,$$  \hspace{1cm} (20)

$$P^A_c(V, \varphi) \geq \left[ 1 - \frac{1}{2} \left\| \sum_j \left| E_j^0(V) - E_j^1 \right|^2 \right\| \right]^{\frac{1}{2}} \hspace{1cm} (21)$$

where $\| \cdot \|$ denotes the usual spectral norm, and for any operator $A$ we use the customary abbreviation $|A|^2 \equiv A^*A$. However, the impossibility proof would be true if a bound of the form (20) would be satisfied in the reverse direction, in which case one would have

$$1 - (P^A_c)_{\mu, av} \leq (P^B_c)_{\text{opt}} - 1 \geq \frac{1}{2} \left\| \sum_j \left| E_j^0(V) - E_j^1 \right|^2 \right\|$$

which would correspond to the following continuity argument: if two CP maps are close in CB-norm, then for a given fixed Kraus decomposition for one of the two maps, there is always an alternate Kraus decomposition for the other map such that the two are close. Since one also has that $|A| \leq |A|_2$, where $| \cdot |_2$ now denotes the Frobenius (Hilbert-Schmidt) norm, the bounding (22) could also be written with the Frobenius norm in the middle term (see Eqs. (20) and (21), in which case the minimum over $V$ would be in the form of a Procrustes problem [4].

Since as regards the cheating probabilities we have considered only the case of non-aborting protocols with perfect-verification, proving the continuity argument (22) or directly the bound (19) would mean that a secure protocol can still be searched outside such class of protocols.

On the other hand, finding a counterexample to Eq. (19) would provide a perfectly verifiable and unconditionally secure protocol.

Finally, a few words on the possibility of a combined pre/post-cheating Alice attack. It is clear that if it leads to a set of QO’s different from the prescribed ones, then it can be detected by Bob at the opening (if it gives the prescribed set of maps, then the same effect can be better achieved by post-cheating). However, in principle, it may help in increasing the overall Alice’s cheating probability, particularly when there is possibility of abortion, i.e. for strictly trace-decreasing protocols.
IV. DISCUSSION

The discrepancy between the previous analysis and the analysis beneath the impossibility proof \[1, 2, 3\] is essentially due to the fact that the latter is based on the assumption that the starting state of the commitment protocol is openly known, in the sense that the probability distribution of the state is given, and then the corresponding mixed state can be purified. The general underlying idea is that the protocol should be processed by machines, and therefore all probability distributions are defined, and purified inside the machines. However, such an assumption is certainly not realistic for a cryptographic protocol, where each party has actually the freedom of changing or tuning the machine, namely choosing any desired probability distribution. Or else, one would need to purify the human being himself. Now, if a parameter is secret for a party, nobody forbids him/her to believe that the other party is still using the same prepared machine, and accordingly to adopt a Bayesian approach with the known a priori probability distribution. However, in practice, the other party could have used another machine and/or with a different preparation. One can continue to argue on this line, asserting that changing the machine is equivalent to use a larger machine, or, in other words, that an unknown probability distribution can be regarded as an a priori uniform distribution on the space of probability distributions. This line of reasoning, however, constitutes a very dangerous argument in a proof, since it is equivalent to consider infinite machines or, equivalently, uniform probabilities on infinite sets, which then must vanish everywhere. In addition, for infinite probability spaces one needs infinite dimensional purifications, (even worst, for continuous spaces one may need non separable Hilbert spaces). This would invalidate an impossibility proof based on a non proved continuity argument which a fortiori must apply to infinite dimensions. Finally, one could now argue that in the real world the machines must be bounded: however, this assertion would contradict not only the previous assumption of uniform probabilities (otherwise, which non uniform probabilities are to be adopted?), but also the fact that the proof is purported for unconditionally secure, with both Alice and Bob supposed to possess unlimited technology.

The above hill-posed mathematical framework arises from the Bayesian approach to secret parameters, dictated from the closed system modeling with fixed machines and purification of probabilities. This model, along with the assumed unbounded technology for both parties, necessarily lead to infinities which don’t allow unproven continuity arguments, thus falsifying the proof. Alternative to the previous approach, we have the realistic finite open system approach, in which unknown parameters are treated as such, without the need of any a priori probability distribution, in which we can address the problem for finite dimension with the parameter \(\varepsilon\) depending on it. As well known, the need of treating unknown parameters without a priori probability distribution is the reason why in detection and estimation theory \[4, 5\] we have both the minimax and the Bayesian approaches. Then, if one proceeds by treating unknown parameters as such, no openly known state can be assumed, and the anonymous state encoding of Yuen\[5, 6\] leads to the present classification of protocols. Notice that if the initial state \(|\varphi\rangle\) is openly known, then for that given fixed states all QO’s can be regarded as random unitary transformations (since all states are connected by unitary transformations), and this lead to the simple form of Alice cheating probability in terms of fidelities\[4, 5, 6\], whereas in the present context the probability of cheating has the more involved form \(\langle J \rangle\), due to the fact that the state \(|\varphi\rangle\) is unknown, and that there are QO’s that don’t admit random unitary Kraus decompositions.

Finally, a few words on the possibility of aborting protocols. This possibility was not considered in Refs. \[1, 2, 3\], since also this in practice arises as a consequence of not assuming openly known probabilities. In fact, in a closed model of interacting machines with purified parameters, every transformation would be unitary. However, one could reasonably argue that if the protocol aborts, then another protocol must be started, and the procedure will be repeated as long as the bit is not successfully committed, and that such a succession of protocols is itself a non aborting protocol. Such kind of protocols that could be chained ad infinitum can be regarded as infinite convex combinations of protocols on infinite dimensional anonymous spaces \(H\), (the QO will be trace-preserving only for infinite dimensions). Again one can see that a closed system approach necessarily lead to infinite dimensions.

APPENDIX A: DERIVATION OF THE BOUNDS \(20\) AND \(21\) FOR CHEATING PROBABILITIES

Using Jensen inequality as suggested in Ref. \[5\], the Alice’s cheating probability can be bounded from below as follows [here we use the abbreviate notation \(F_j(0) = E_j^{(0)}(V)\)]

\[
P_c^A(V, \varphi) = \sum_j |E_j^{(1)}| \varphi|^2 \left(\frac{\langle \varphi | F_j^{(0)} E_j^{(1)} | \varphi \rangle}{\| E_j^{(1)} \varphi \|^2} \right)^2 \geq \left| \sum_j \langle \varphi | F_j^{(0)} | E_j^{(1)} | \varphi \rangle \right|^2 \geq \left( \Re \sum_j \langle \varphi | F_j^{(0)} E_j^{(1)} | \varphi \rangle \right)^2 = \left[ 1 - \frac{1}{2} \langle \varphi | \sum_j F_j^{(0)} - E_j^{(1)} \rangle \varphi \rangle \right]^2 \geq \left[ 1 - \frac{1}{2} \left\| \sum_j F_j^{(0)} - E_j^{(1)} \right\|^2 \right]^2 ,
\]

(A1)
where we used $|A|^2 \doteq A^\dagger A$, and the fact that for $P \geq 0$ one has $|P| = \sup_{|\varphi|=1} |\langle \varphi | P | \varphi \rangle|$. On the other hand, we have that

$$\begin{align*}
\left\| M^{(0)} - M^{(1)} \right\|^2_{cb} &= \left\| \text{Tr}_F \left[ F^{(0)} \cdot F^{(0)\dagger} - E^{(1)} \cdot E^{(1)\dagger} \right] \right\|^2_{cb} \\
&\leq \left\| \text{Tr}_F \left[ F^{(0)} \cdot F^{(0)\dagger} - E^{(1)} \cdot E^{(1)\dagger} \right] \right\|^2_{cb} \\
&\leq \left\| F^{(0)} \cdot F^{(0)\dagger} - E^{(1)} \cdot E^{(1)\dagger} \right\|^2_{cb} \\
&\leq \left\| I \cdot F^{(0)\dagger} \right\|_{cb} \left\| F^{(0)} \cdot I - E^{(1)} \cdot I \right\|_{cb} \\
&+ \left\| F^{(0)} \cdot I \right\|_{cb} \left\| I \cdot F^{(0)\dagger} - E^{(1)} \cdot F^{(1)\dagger} \right\|_{cb} \\
&\leq 4 \left\| F^{(0)} - E^{(1)} \right\|^2 \leq 4 \sup_{|\varphi|=1} \left\| F^{(0)} - E^{(1)} \right\|^2_{cb} \\
&= 4 \sum_j \left\| F^{(0)}_j - E^{(1)}_j \right\|^2. \\
\end{align*}$$

(A2)

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