ON THE STRUCTURE OF SOME LOCALLY NILPOTENT GROUPS WITHOUT CONTRANORMAL SUBGROUPS

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Abstract. Following J.S. Rose, a subgroup $H$ of a group $G$ is said contranormal in $G$ if $G = H^G$. In a certain sense, contranormal subgroups are antipodes to subnormal subgroups. It is well known that a finite group is nilpotent if and only if it has no proper contranormal subgroups. We prove that a nilpotent-by-finite group with no proper contranormal subgroup is nilpotent. There are locally nilpotent groups with a proper contranormal subgroup. We study the structure of hypercentral groups with a finite proper contranormal subgroup.

Dedicated to Professor Pavel Shumyatsky on his 60th birthday

1. Introduction.

A subgroup $H$ of a group $G$ is called contranormal in $G$ if $H^G = G$, where $H^G = \langle x^{-1}hx \mid h \in H, x \in G \rangle$ is the normal closure of $H$ in $G$, the smallest normal subgroup of $G$ containing $H$. For example $G$ is contranormal in $G$, for any group $G$. The term ”contranormal subgroup” has been introduced by J.S. Rose in the paper [14]. Contranormal subgroups have been studied for example in the paper [9]. If $G$ is a group and $H$ is a contranormal subgroup of $G$, then every subgroup $K$ containing $H$ is contranormal in $G$. In particular, if $H$ and $L$ are contranormal subgroups of $G$, then the subgroup $\langle H, L \rangle$ is also contranormal in $G$. However, the intersection of two contranormal subgroups is not always contranormal. For example, in the group $A_4$ every Sylow 3-subgroup is contranormal, but the intersection of every two Sylow 3-subgroups of $A_4$ is trivial, so that it is not contranormal. Notice also that if $M$ is a maximal subgroup of $G$ which is not normal, then clearly $M$ is a contranormal subgroup of $G$. Moreover, every subgroup of a finite group $G$ is a contranormal subgroup of a subnormal subgroup.

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of $G$. As we can see by the definition, contranormal subgroups are in a certain sense, antipode of normal and subnormal subgroups: a contranormal subgroup $H$ of a group $G$ is normal (respectively subnormal) if and only if $H = G$. It follows that groups, whose subgroups are subnormal (in particular, nilpotent group), do not contain proper contranormal subgroups. For finite groups the converse is true.

A finite group $G$ is nilpotent if and only if $G$ does not have proper contranormal subgroups.

Indeed, suppose that there is a prime $p$ such that $G$ has a Sylow $p$-subgroup $P$ which is not normal in $G$. Then $N_G(P) \neq G$. Since $P$ is pronormal in $G$, $N_G(P)$ is abnormal in $G$ ([13], 1.6). But every abnormal subgroup is contranormal, and we obtain a contradiction, which shows that Sylow $q$-subgroups of $G$ are normal for each prime $q$.

It follows that $G$ is nilpotent.

There exist infinite non-nilpotent groups, whose subgroups are subnormal (it is possible to find examples of such groups in the survey [3]). Therefore the following question naturally appears:

When a locally nilpotent group without proper contranormal subgroups is nilpotent?

We notice that there exist Chernikov locally nilpotent groups having proper contranormal subgroups, as the following example shows. Let $D$ be a divisible abelian 2-group. Then $D$ has an automorphism $\varphi$ such that $\varphi(d) = d^{-1}$ for each element $d \in D$. Define the semidirect product $G = D \rtimes \langle b \rangle$ such that $d^b = \varphi(d) = d^{-1}$ for each element $d \in D$. Let $a$ be an arbitrary element of $D$. Since $D$ is divisible, there exists an element $d \in D$ such that $d^2 = a$. We have $[b, d] = b^{-1}d^{-1}bd = d^2 = a$.

It follows that $[b, D] = D$. From $[b, D] \leq \langle b \rangle^G$ and $\langle b \rangle \leq \langle b \rangle^G$ we obtain that $\langle b \rangle^G = \langle b \rangle [b, D] = \langle b \rangle D = G$, so that the subgroup $\langle b \rangle$ is contranormal in $G$. We note that the group $G$ is not nilpotent, however the series

$$\langle 1 \rangle \leq \Omega_1(D) \leq \cdots \leq \Omega_n(D) \leq \Omega_{n+1}(D) \leq \cdots \leq D \leq G$$

is central, so that $G$ is a hypercentral abelian-by-finite group. Besides, the contranormal subgroup $\langle b \rangle$ is ascendant. This group is abelian-by-finite, thus there exist hypercentral abelian-by-finite groups having proper contranormal subgroups, and also finite contranormal subgroups. This example raises the following question:

What can we say about locally nilpotent abelian-by-finite groups having no proper contranormal subgroups?

Our first result gives an answer to this question. In fact we have the following Theorem.
Theorem A. Let $G$ be a nilpotent-by-finite group. If $G$ has no proper contranormal subgroups, then $G$ is nilpotent.

Now the question appears about the structure of locally nilpotent abelian-by-finite groups having proper contranormal subgroups. We show here the following result.

Proposition B. Let $G$ be a locally nilpotent group and $A$ be a normal abelian subgroup of $G$ with $G/A$ finite. Suppose that $G$ has a proper contranormal subgroup $C$, then $C = BK$ where $B \leq A$ is normal in $G$, $K$ is a finitely generated subgroup such that $G = AK$, and $A = B[K, A]$. In particular the factor group $G/B$ has the finite contranormal subgroup $KB/B$.

Therefore we naturally come to locally nilpotent abelian-by-finite groups having a finite contranormal subgroup. Our last result gives a description of hypercentral groups which include a finite contranormal subgroup.

Theorem C. Let $G$ be a hypercentral group. If $G$ contains a finite contranormal subgroup, then $G$ satisfies the following conditions:

(i) $G = VC$, where $V$ is a normal divisible abelian subgroup and $C$ is a finite contranormal subgroup of $G$;

(ii) $\Pi(G) = \Pi(C)$, in particular the set $\Pi(G)$ is finite;

(iii) $V$ has a family of $G$-invariant $G$-quasifinite subgroups $\{D_\mu \mid \mu \in M\}$ such that $V = \langle D_\mu \mid \mu \in M\rangle$;

(iv) $[D_\mu, C] = D_\mu$ for all $\mu \in M$, in particular, $[V, C] = V$.

Here an infinite normal abelian subgroup $A$ of a group $G$ is called $G$-quasifinite if every proper $G$-invariant subgroup of $A$ is finite.

2. Nilpotent-by-finite groups without proper contranormal subgroups

We start our investigation with this easy and very useful Lemma.

Lemma 2.1. Let $G$ be a group. Then:

(i) If $C$ is a contranormal subgroup of $G$ and $H$ is a normal subgroup of $G$, then $CH/H$ is a contranormal subgroup of $G/H$.

(ii) If $H$ is a normal subgroup of $G$ and $C$ is a subgroup of $G$ such that $H \leq C$ and $C/H$ is a contranormal subgroup of $G/H$, then $C$ is a contranormal subgroup of $G$.

(iii) If $C$ is a contranormal subgroup of $G$ and $D$ is a contranormal subgroup of $C$, then $D$ is a contranormal subgroup of $G$.

Proof. These assertions are obvious.  \[\square\]
Let $G$ be a nilpotent-by-finite group and assume that $G$ has no con-tranormal subgroups. In order to prove Theorem A, we first assume that $G$ is $p$-group, $p$ a prime. Furthermore, we first suppose that $G$ is abelian-by-finite, thus there exists a normal abelian subgroup $A$ of $G$ of finite index in $G$. We start stating three easy Lemmas, well known in the literature. We add the proofs for the sake of completeness.

**Lemma 2.2.** Let $G$ be a $p$-group, $p$ a prime, and suppose that $G$ contains a normal bounded abelian subgroup $A$ such that $G/C_G(A)$ is finite. Then for some positive integer $m$, $A$ is contained in $\zeta_m(G)$, the $m$–th term of the upper central series of $G$.

Proof. Write $s$ the exponent of $A$ and $k = |G/C_G(A)|$. For each $a \in A$ we have $A \leq C_G(a)$ and $|G : C_G(a)| \leq k$. Thus $a$ has at most $k$ conjugates in $G$. Therefore $\langle a \rangle^G$ is an abelian group, of exponent $\leq s$, generated by at most $k$ elements. Thus $\langle a \rangle^G$ is a finite normal subgroup of order at most $s^k$. Write $m = s^k$. Since $G$ is a soluble $p$-group, then $G$ is locally nilpotent, hence $\langle a \rangle^G$ is contained in the $m$–th term of the upper central series of $G$. That holds for each $a \in A$, therefore $A \leq \zeta_m(G)$. □

**Lemma 2.3.** Let $A$ be an abelian $p$-group, $p$ a prime. If $A$ is not bounded, then $A$ contains a subgroup $B$ such that $A/B$ is a divisible Chernikov group.

Proof. Suppose first that $A$ is a direct product of cyclic groups. Then since $A$ is not bounded, there exists a subgroup $C$ of $A$ such that $A/C = \bigoplus_{n \in \mathbb{N}} \langle d_n \rangle$, where the element $d_n$ has order $p^n$. Consider the subgroup $B/C = \langle d_n^{d_n} \rangle$. Then by this choice the factor group $A/B$ is a Prüfer $p$-group.

Suppose now that $A$ cannot be decomposed in a direct product of cyclic subgroups. Let $D$ be a basic subgroup of $A$ (see Theorem 32.3 of the book [6]). Then $D$ is the direct product of cyclic subgroups, therefore $D \neq A$. Moreover $A/D$ is a divisible group. Thus $A/D$ is direct product of Prüfer $p$-groups and there exists a subgroup $B/D$ of $A/B$ such that $A/B$ is a Prüfer $p$-group. □

**Lemma 2.4.** Let $\mathcal{H}$ be a class of groups closed under subgroups and under finite direct products. Let $G$ be a group containing a normal abelian subgroup $A$ such that $|G/C_G(A)|$ is finite. Suppose that $A$ contains a subgroup $B$ such that $A/B \in \mathcal{H}$, then $A$ contains a $G$-invariant subgroup $C$ such that $C \leq B$ and $A/C \in \mathcal{H}$.

Proof. For each element $g \in G$ the isomorphism $A/B^g \cong A^g/B^g \cong A/B$ shows that $A/B^g \in \mathcal{H}$. Since the subgroup $C_G(A)$ has finite
index in $G$, the set $\{B^g \mid g \in G\}$ is finite. Write $\{B^g \mid g \in G\} = \{B_1, B_2, \ldots, B_n\}$, and $C = B_1 \cap B_2 \cdots \cap B_n$. Using Remak’s theorem we obtain the embedding $A/C \preceq A/B_1 \times A/B_2 \times \cdots \times A/B_n$. Since $A/B_i \in \mathcal{H}$, for every $i \in \{1, \cdots, n\}$, and $\mathcal{H}$ is closed under subgroups and finite direct products, it follows that $A/C \in \mathcal{H}$. □

Another general lemma we will use is the following:

**Lemma 2.5.** Let $G$ be a $p$-group, $p$ a prime, and suppose that $G$ contains a normal abelian subgroup $A$ such that $C_G(A)$ has finite index. Assume that $A$ contains a $G$-invariant divisible Chernikov subgroup $D$. Then $A$ contains a $G$-invariant subgroup $S$ such that $A = SD$ and the intersection $S \cap D$ is finite.

**Proof.** Since $D$ is divisible, it has a complement in $A$, that is $A$ contains a subgroup $B$ such that $A = D \times B$. Then $A$ contains a $G$-invariant subgroup $C$ such that $(D \cap C)^n = \langle 1 \rangle$ and $A^n \leq DC$ where $n = |G/C_G(A)|$ (see, for example [9], Theorem 5.9). In particular, the intersection $D \cap C$ is finite. Then $DC/C \simeq D/(D \cap C) \simeq D$. In particular, $DC/C$ is a divisible subgroup of $A/C$, therefore $A/C$ contains a subgroup $E/C$ such that $A/C = (DC/C) \times E/C$. Since the factor $A/DC$ is bounded, $E/C$ is bounded, moreover $(E/C)^n = \langle 1 \rangle$. Let $n = p^k$, then $E/C \leq \Omega_k(A/C)$. Put $S/C = \Omega_k(A/C)$, then the intersection $(S/C) \cap (DC/C)$ is finite and $A/C = (DC/C)(S/C)$. It follows that $A = DS$. Since $D \cap C$ and $(S/C) \cap (DC/C)$ are finite, then $S \cap D$ is finite. The Lemma is proved. □

Now assume that $G$ is a $p$-group, $p$ a prime, and that $G$ has no proper contranormal subgroups. Suppose that $G$ has a normal abelian subgroup $A$ of finite index in $G$. If $A$ is bounded, then there exists a positive integer $m$ such that $A \leq \zeta_m(G)$, the $m-th$ term of the upper central series of $G$, by Lemma 2.2. Since $G/A$ is a finite $p$-group, $G/A$ is nilpotent. Therefore $G$ is nilpotent and we have the result of Theorem A in this case. Then we can suppose that $A$ is not bounded. Thus, by Lemma 2.3, there exists a subgroup $B$ of $A$ such that $A/B$ is a divisible Chernikov group. By Lemma 2.4 we can also suppose that $B$ is $G$-invariant. In this case we have.

**Lemma 2.6.** Let $G$ be a $p$-group, $p$ a prime, and suppose that $G$ contains a normal abelian subgroup $A$ of finite index. Assume that $A$ contains a $G$-invariant subgroup $C$ such that $A/C$ is a divisible Chernikov group. If $G$ has no proper contranormal subgroups, then $[G, A] \leq C$. 

\[5\]
Proof. $A/C$ is a Chernikov group, thus $G/C$ satisfies the minimal condition on subgroups. Then there exists a series

$$C = C_1 \leq C_2 \leq \cdots \leq C_n = A$$

of $G$-invariant subgroups such that the factors $C_{j+1}/C_j$ are $G$-quasifinite, $j \in \{1, \ldots, n\}$. Consider the factor $A/C_{n-1}$. The subgroup $[G/C_{n-1}, A/C_{n-1}]$ is $G$-invariant, then either $[G/C_{n-1}, A/C_{n-1}] = A/C_{n-1}$, or $[G/C_{n-1}, A/C_{n-1}]$ is finite. Assume that $[G/C_{n-1}, A/C_{n-1}] = A/C_{n-1}$. Choose a finite subgroup $K/C_{n-1}$ such that $G/C_{n-1} = (A/C_{n-1})(K/C_{n-1})$. Then $[G/C_{n-1}, A/C_{n-1}] = [K/C_{n-1}, A/C_{n-1}]$. Then the inclusion $A/C_{n-1} = [K/C_{n-1}, A/C_{n-1}] \leq (K/C_{n-1})^{G/C_{n-1}}$ implies that $(K/C_{n-1})^{G/C_{n-1}} = (A/C_{n-1})(K/C_{n-1}) = G/C_{n-1}$. This means that the subgroup $K/C_{n-1}$ is contranormal in $G/C_{n-1}$. By Lemma 2.1, the subgroup $K$ is contranormal in $G$, and we obtain a contradiction. This contradiction shows that $[G/C_{n-1}, A/C_{n-1}]$ is finite. In this case the factor group $G/C_{n-1}$ is nilpotent. It follows that the center of $G/C_{n-1}$ contains $A/C_{n-1}$ (see, for example, [5], Proposition 3.2.11). Hence $[G, A] \leq C_{n-1}$.

Suppose that we have already proved that $[G, A] \leq C_2$. Since the subgroup $A/C$ is divisible and Chernikov, $A/C$ contains a $G$-invariant divisible subgroup $D/C$ such that $A/C = (C_2/C)(D/C)$ and the intersection $(C_2/C) \cap (D/C)$ is finite (see, for example, [9], Corollary 5.11). Then the factor $A/D$ is divisible Chernikov and $G$-quasifinite. Using the result of the previous paragraph, we obtain that $[G, A] \leq D$.

Thus we have $[G/C, A/C] \leq C_2/C$ and $[G/C, A/C] \leq D/C$, therefore $[G/C, A/C] \leq (C_2/C) \cap (D/C)$. Since the last intersection is finite, the factor group $G/C$ is nilpotent. It follows that the center of $G/C$ contains $A/C$ (see, for example, [5], Proposition 3.2.11). Hence $[G, A] \leq C$, and the Lemma is proved.

From Lemma 2.6 we have the following lemma:

**Lemma 2.7.** Let $G$ be a $p$-group, $p$ a prime, and suppose that $G$ contains a normal abelian subgroup $A$ of finite index. If $G$ has no contranormal subgroups, then $[G, A]$ is bounded.

Proof. If $A$ is bounded, we have the result. Therefore we suppose that $A$ is not bounded. Then Lemma 2.3 shows that $A$ contains a subgroup $B$ such that $A/B$ is a divisible Chernikov group. Then Lemma 2.6 implies that $[G, A] \neq A$, moreover $A/[G, A]$ is not bounded. Suppose that the subgroup $D = [G, A]$ is not bounded. Using again Lemma 2.3 we obtain that $D$ contains a subgroup $C$ such that $D/C$ is a divisible Chernikov group. Then, by Lemma 2.4, there exists a $G$-invariant subgroup $E$ such that $D/E$ is a Chernikov group. Then $D$ contains...
a $G$-invariant subgroup $H$ such that $E \leq H$, $H/E$ is finite and $D/H$ is a divisible Chernikov group. Therefore without loss of generality we may suppose that $D/E$ is a divisible Chernikov group. We have $[G/E, A/E] = [G, A]E/E = DE/E = D/E$. Therefore $[G/E, A/E]$ is a divisible Chernikov group. By Lemma 2.5, $A/E$ contains a $G$-invariant subgroup $S/E$ such that $A/E = (D/E)(S/E)$ and the intersection $(D/E) \cap (S/E)$ is finite. It follows that $A/S \cong (A/E)/(S/E) = (D/E)(S/E)/(S/E) \cong (D/E)/(D/E) \cong D/E$ is a divisible Chernikov group. Furthermore, $A/S = (DS)/S = [G, A]S/S = [G/S, A/S]$. Now, by Lemma 2.6, $[G/E, A/E] \leq S/E$, since $A/S$ is a divisible Chernikov. Then $[G, A] \leq S$ and we obtain the contradiction $A = S$. This contradiction proves that the subgroup $[G, A]$ is bounded. 

Now we can prove the result of Theorem A, if $G$ is an abelian-by-finite $p$-group, $p$ a prime.

**Corollary 2.8.** Let $G$ be a $p$-group, $p$ a prime, and suppose that $G$ contains a normal abelian subgroup $A$ of finite index. If $G$ has no proper contranormal subgroups, then $G$ is nilpotent.

**Proof.** By Lemma 2.7, $[G, A]$ is bounded. Then, by Lemma 2.2, there exists a positive integer $t$ such that $[G, A] \leq \zeta_t(G)$. Then $A \leq \zeta_{t+1}(G)$, and $G$ is nilpotent since $G/A$ is a finite $p$-group. 

Next step is to prove the result of Theorem A for every locally nilpotent abelian-by-finite group.

**Corollary 2.9.** Let $G$ be a locally nilpotent group, and suppose that $G$ contains a normal abelian subgroup $A$ of finite index. If $G$ has no proper contranormal subgroups, then $G$ is nilpotent.

**Proof.** First, suppose that $G$ is periodic. Let $\pi = \Pi(G/A)$ and $\sigma = \Pi(G) \setminus \pi$, then the set $\pi$ is finite and we have $G = Dr_{p \in \pi} G_p \times Dr_{p \in \sigma} G_p$, where $G_p$ is a Sylow $p$-subgroup of $G$ for all $p \in \Pi(G)$. The isomorphism $G_p \simeq G/Dr_{q \in \Pi(G), q \neq p} G_q$ and Lemma 2.1 show that $G_p$ has no proper contranormal subgroups for every $p \in \pi$. Using Corollary 2.8 we obtain that $G_p$ is nilpotent for each $p \in \pi$. The finiteness of the set $\pi$ implies that $Dr_{p \in \pi} G_p$ is nilpotent. Obviously the subgroup $G_p$ is abelian for every $p \in \sigma$, hence $Dr_{p \in \sigma} G_p$ is abelian. Therefore $G$ is nilpotent. Now suppose that $G$ is non-periodic. Then the set $\text{Tor}(G)$ of all elements of $G$ having finite order, is a characteristic subgroup of $G$ and the factor group $S = G/\text{Tor}(G)$ is torsion-free. On the other hand, $S$ is abelian-by-finite. then $S$ is a locally nilpotent torsion-free abelian-by-finite group, and then it is abelian (see, for example, [5], Corollary...
Choose in the abelian subgroup $A$ a maximal $\mathbb{Z}$-independent subset $M$ and let $C$ be the subgroup of $A$ generated by $M$. Then $A/C$ is a periodic group. By Lemma 2.4 there exists a $G$-invariant subgroup $E \leq C$ such that $A/E$ is periodic. Obviously $E$ is torsion-free. Then $E \cap \text{Tor}(G) = \langle 1 \rangle$. Using Remak’s theorem, we obtain an embedding $G \leq G/E \times G/\text{Tor}(G)$. By Lemma 2.1 $G/E$ does not include proper contranormal subgroups. Then $G/E$ is nilpotent by Corollary 2.8, moreover $G/\text{Tor}(G)$ is abelian, therefore $G$ is nilpotent and we have the result.

Now we extend Corollary 2.9 to any abelian-by-finite group. We start with the following two results.

**Lemma 2.10.** Let $G$ be a group and suppose that $G$ contains a normal abelian $p$-subgroup $A$ of finite index, where $p$ is a prime. If $G$ has no proper contranormal subgroups, then $G$ is nilpotent.

**Proof.** By Lemma 2.1 the factor group $G/A$ does not contain proper contranormal subgroups. Being finite, $G/A$ is nilpotent. Then $G/A = P/A \times S/A$, where $P/A$ is a $p$-group and $S/A$ is a $p'$-group. We have $A = C_A(S) \times [S, A]$ (see, for example, [1], Proposition 2.12). Suppose that the subgroup $[S, A]$ is not trivial. Since the subgroup $S$ is normal in $G$, then both subgroups $C = C_A(S)$ and $[S, A]$ are $G$-invariant. Moreover, we have $A/C = C[S, A]/C = [S/C, A/C]$. If the abelian $p$-group $A/C$ is bounded, then it is the direct product of cyclic subgroups. In particular, $A/C$ contains a proper subgroup having finite index. Then, by Lemma 2.4, $A/C$ contains a proper $G$-invariant subgroup $B/C$, having finite index. By Lemma 2.1 the factor group $G/C$ does not contain proper contranormal subgroups. Being finite, this factor group must be nilpotent. But in this case $[A/C, S/C] = \langle 1 \rangle$, and we obtain a contradiction. If the abelian $p$-group $A/C$ is not bounded, then by Lemma 2.3, $A/C$ contains a subgroup $D/C$ such that $A/D$ is a divisible Chernikov group. By Lemma 2.4, $A/C$ contains a proper $G$-invariant subgroup $E/C$ such that $A/E$ is Chernikov. By Lemma 2.1 the factor group $G/E$ does not contain proper contranormal subgroups. Being Chernikov, this factor group must be nilpotent ([11], Lemma 4.9). But in this case $[A/C, S/C] = \langle 1 \rangle$, and we again obtain a contradiction. This contradiction proves that $A = C_A(S)$. It follows that $S = A \times V$ where $V$ is a finite $p'$-subgroup. Moreover, $V$ is a Sylow $p'$-subgroup of $S$, so that $V$ is normal in $G$. By Lemma 2.1 the factor group $G/V$ does not contain proper contranormal subgroups. This factor group is an abelian-by-finite $p$-group, then it is nilpotent, by Corollary 2.8.
The equality \( A \cap V = \langle 1 \rangle \) and Remak’s theorem imply the embedding \( G \preceq G/A \times G/V \), which implies that \( G \) is nilpotent. \( \square \)

Let \( G \) be a group and \( A \) be a normal subgroup of \( G \). We put \( \gamma_1(G, A) = A, \gamma_2(G, A) = [G, A] \), and, recursively, \( \gamma_{\alpha+1}(G, A) = [G, \gamma_\alpha(G, A)] \), for all ordinals \( \alpha \), moreover, if \( \lambda \) is a limit ordinal, we write \( \gamma_\lambda(G, A) = \bigcap_{\mu<\lambda} \gamma_\mu(G, A) \)

**Lemma 2.11.** Let \( G \) be a group and suppose that \( G \) contains a normal abelian torsion-free subgroup \( A \) of finite index. If \( G \) has no proper contranormal subgroups, then \( G \) is nilpotent.

**Proof.** Let \( M \) be a finite subset of \( A \) and write \( B = \langle M \rangle^G \). Since \( G/A \) is finite, the subgroup \( B \) is finitely generated. Being torsion-free, it is free abelian. Moreover, \( B \) is \( G \)-invariant. Put \( T/B = \text{Tor}(A/B) \), then the subgroup \( T \) has finite 0-rank and it is \( G \)-invariant. Let \( r_0(T) = n \), then \( T/B \) has special rank at most \( n \). Let \( p \) be an arbitrary prime and consider the factor \( A/Bp \). Let \( S_p/Bp \) be the Sylow \( p \)-subgroup of \( A/Bp \), then \( S_p/Bp \) is a Chernikov group of special rank at most \( n \). We have the direct decomposition \( A/Bp = S_p/Bp \times Cp/Bp \) (see, for example [6], Theorems 21.2 and 27.5). Thus \( A/Cp \) is a Chernikov \( p \)-group of special rank at most \( n \). By Lemma 2.4 there exists a \( G \)-invariant subgroup \( D_p \), \( D_p \leq C_p \) such that \( A/D_p \) is a Chernikov \( p \)-group, it is \( D_p = \bigcap_{g \in G} C_p^g \), thus \( A/D_p \) has special rank at most \( kn \) where \( k = |G/A| \). The inclusion \( D_p \leq C_p \) implies that \( B \cap D_p = Bp \). It follows that \( (BD_p)/D_p \simeq B/(B \cap D_p) = B/Bp \), in particular \( (BD_p)/D_p \) is an elementary abelian \( p \)-group, having finite order less or equal to \( p^n \). The factor-group \( G/D_p \) is periodic, therefore, by Corollary 2.9, \( G/D_p \) is nilpotent. Then \( (BD_p)/D_p \leq \gamma_n(G) \), the \( n-th \) term of the lower central series of \( G \). It follows that \( \gamma_{n+1}(G, B) \leq D_p \). On the other hand, since \( B \) is normal in \( G \), \( \gamma_{n+1}(G, B) \leq B \), so that \( \gamma_{n+1}(G, B) \leq D_p \cap B = Bp \). The last inclusion is true for each prime \( p \), therefore \( \gamma_{n+1}(G, B) \leq \bigcap_{p \in P} Bp \), where \( P \) is the set of all primes. Since \( B \) is a free abelian subgroup, \( \bigcap_{p \in P} Bp = \langle 1 \rangle \), thus \( \gamma_{n+1}(G, B) = \langle 1 \rangle \). It follows that \( B \leq \gamma_n(G) \). That holds for every finitely generated subgroup \( B \) of \( A \), therefore \( A \) is contained in the hypercenter of \( G \). By Lemma 2.1 the factor group \( G/A \) does not contain proper contranormal subgroups. Being finite, \( G/A \) is nilpotent. Then \( G \) is hypercentral. In particular, \( G \) is locally nilpotent, and, by Lemma 2.9, \( G \) is nilpotent. \( \square \)

**Corollary 2.12.** Let \( G \) be an abelian-by-finite group. If \( G \) has no proper contranormal subgroups, then \( G \) is nilpotent.
Proof. Let $A$ be a normal abelian subgroup of $G$ such that the factor group $G/A$ is finite. First suppose that $G$ is periodic. Let $\pi = \Pi(G/A)$ and $\sigma = \Pi(A) \setminus \pi$, then the set $\pi$ is finite and we have $A = Dr_{p \in \pi} A_p \times Dr_{p \in \sigma} A_p$, where $A_p$ is the Sylow $p$-subgroup of $A$ for all $p \in \Pi(A)$. Put $B_p = Dr_{q \in \Pi(A), q \neq p} A_q$, then the subgroup $B_p$ is $G$-invariant, $A/B_p \simeq A_p$ and by Lemma 2.1 $G/B_p$ does not contain proper contranormal subgroups for every $p \in \Pi(A)$. By Lemma 2.10 $G/B_p$ is nilpotent for each $p \in \Pi(A)$. In particular, if $p \in \pi$, then $G/B_p$ is abelian. Since $\bigcap_{p \in \Pi(A)} B_p = \langle 1 \rangle$, by Recam’s theorem, we obtain an embedding $G \leq Dr_{p \in \pi} G/B_p \times Cr_{p \in \sigma} G/B_p$. Since the set $\pi$ is finite $Dr_{p \in \pi} G/B_p$ is nilpotent. Since $G/B_p$ is abelian for all $p \in \pi$, then $Cr_{p \in \sigma} G/B_p$ is abelian. Therefore $G$ is nilpotent. Now suppose that $G$ is not periodic. Since $G$ is not periodic, $A$ is also not periodic. Write $T = Tor(A)$. Then $A \neq T$. Obviously the subgroup $T$ is $G$-invariant and $A/T$ is torsion-free. Lemma 2.1 shows that $G/T$ does not contain proper contranormal subgroups. Hence the factor group $G/T$ is nilpotent, by Lemma 2.11. Choose in the abelian subgroup $A$ a maximal $\mathbb{Z}$-independent subset $M$ and let $C = \langle M \rangle$. Then $A/C$ is a periodic group. By Lemma 2.4 there exists a $G$-invariant subgroup $E$ such that $E \leq C$ and $A/E$ is a periodic group. The inclusion $E \leq C$ implies that $E$ is torsion-free. Thus $E \cap T = \langle 1 \rangle$. By Remark theorem, we obtain an embedding $G \leq G/E \times G/T$. Lemma 2.1 shows that $G/E$ does not contain proper contranormal subgroups. Being periodic, $G/E$ is nilpotent, we know that $G/T$ is nilpotent, hence $G$ is nilpotent, as required.

$\square$

Now we can prove Theorem A.

Proof of Theorem A. Let $K$ be a nilpotent normal subgroup of $G$ such that $G/K$ is finite. Write $D = [K, K]$. Lemma 2.1 implies that the factor group $G/D$ does not contain proper contranormal subgroups. Moreover, $G/D$ is abelian-by-finite. Then Corollary 2.12 implies that $G/D$ is nilpotent. Using now Theorem 7 of paper [7], we obtain that $G$ is nilpotent, as required. $\square$

3. Locally nilpotent abelian-by-finite groups with a finite contranormal subgroup

We start this section by proving Proposition B.

Proof of Proposition B. Suppose that $AC \neq G$. Then Lemma 2.1 implies that $CA/A$ is a proper contranormal subgroup of the finite nilpotent group $G/A$. But a nilpotent group does not contain a proper contranormal subgroups. Hence $AC = G$. Choose in $C$ a finitely
generated subgroup $K$ such that $AK = G$, then $C = BK$ where $B = C \cap A$. Since $A$ is normal in $G$, $[K, B] \leq A$. On the other hand, $[K, B] \leq C$, so that $[K, B] \leq C \cap A = B$. Therefore, the subgroup $B$ is $K$-invariant. $B$ is also $A$-invariant, since $A$ is abelian, thus from $G = AK$ we get that $B$ is $G$-invariant. The intersection $K \cap A$ is normal in $G$. Considering the factor group $G/(K \cap A)$, without loss of generality we may assume that $K \cap A$ is trivial. Then the subgroup $K$ is finite. From $G = AK$, with $A$ normal in $G$, it follows $[K, A]$ normal in $G$ and $[G, G] = [K, A][K, K] \leq K[A, K]$ Thus $G/(K[K, A])$ is abelian. By Lemma 2.1 $C[K, A]/(K[K, A])$ is contranormal in $G/(K[K, A])$. It follows that $C[K, A]/(K[K, A]) = G/(K[K, A])$. Therefore we have $G = C[K, A] = BK[K, A] = B[K, A] \times K$. In particular, we obtain that $A = B[K, A]$. The subgroup $B$ is normal in $G$. Then we obtain $G/B = A/B \times KB/B = [K, A]B/B \times KB/B = [KB/B, A/B] \times KB/B$. It follows $G/B = (KB/B)^{G/B}$, hence $KB/B$ is contranormal in $G/B$. □

We start our investigation assuming that $G$ is a $p$-group, $p$ a prime.

**Proposition 3.1.** Let $G$ be an abelian-by-finite $p$-group, $p$ a prime. If $G$ contains a finite contranormal subgroup, then $G$ satisfies the following conditions:

(i) $G = VC$ where $V$ is a normal divisible abelian subgroup and $C$ is a finite contranormal subgroup of $G$;

(ii) $V$ has a family of $G$-invariant $G$-quasifinite subgroups $\{D_\mu \mid \mu \in M\}$ such that $V = \langle D_\mu \mid \mu \in M \rangle$;

(iii) $[D_\mu, C] = D_\mu$ for all $\mu \in M$, in particular, $[V, C] = V$.

**Proof.** Let $A$ be a normal abelian subgroup of $G$ having finite index and let $C$ be a finite contranormal subgroup of $G$. By Lemma 2.1 $CA/A$ is contranormal in $G/A$. Since $G/A$ is a finite $p$-group, it is nilpotent. The fact that a nilpotent group does not include proper contranormal subgroups implies that $CA/A = G/A$ or $G = CA$. If $A = A^p$, then $A$ is divisible and (i) holds. Suppose that $B = A^p \neq A$. Then $B$ is normal in $G$ and $G/B$ is an extension of an elementary abelian $p$-subgroup by a finite $p$-group. Such groups are nilpotent (2). On the other hand, Lemma 2.1 shows that $CB/B$ is a contranormal subgroup of $G/B$. The fact that a nilpotent group does not include a proper contranormal subgroup implies that $CB/B = G/B$. It follows that $A/B$ is finite. The finiteness of $A/A^p$ implies that $A = F \times V$ where $V$ is a divisible subgroup and $F$ is a finite subgroup (see, for example [8], Lemma 3). Clearly the subgroup $V$ is $G$-invariant. Being a finite $p$-group, the factor group $G/V$ is nilpotent. As above it follows that $CV/V = G/V$ or
It is not hard to prove that the subgroup $D$, for example (9), Corollary 5.11. Put suppose that $D = G$ and $Y \subseteq G$ is finite, $Q_1$ has only finitely many conjugates, so that $Y = Q_1^G$ is a divisible Chernikov subgroup. Since $Y$ satisfies the minimal condition, $Y$ includes an infinite $G$-invariant subgroup $D_1$ which is $G$-quasifinite. If $D_1^p \neq D_1$, then $D_1^p$ is finite since $D_1$ is quasi finite, and $D_1/D_1^p$ is finite since it is an elementary abelian $p$-group with the minimal condition, hence $D_1$ is finite, a contradiction. Therefore $D_1^p = D_1$ and $D_1$ is divisible. Thus $V = D_1R$ for some subgroup $R$ such that $R$ is $G$-invariant, the intersection $D_1 \cap R$ is finite and $(D_1 \cap R)^{|C|} = \langle 1 \rangle$ (see, for example [9], Corollary 5.11). Put $|C| = p^n$, then $D_1 \cap R \leq \Omega_n(V)$. It is not hard to prove that the subgroup $[D_1, C]$ is $G$-invariant. If we suppose that $[D_1, C]$ is a proper subgroup of $D_1$, then the fact that $D_1$ is $G$-quasifinite implies that $[D_1, C]$ must be finite. Then $D_1C$ is a finite-by-abelian $p$-group, so that $D_1C$ is nilpotent. Being Chernikov, $D_1C$ is central-by-finite (see, for example [5], Corollary 3.2.10). It follows that $D_1 \leq \zeta(D_1C)$. Consider the factor group $G/R$. We have $V/R = D_1R/R \simeq D_1/(D_1 \cap R)$. The equality $[D_1, C] = \langle 1 \rangle$ implies that $[V/R, C] = [D_1, C] = \langle 1 \rangle$. It follows that $V/R \leq \zeta(G/R)$. But in this case $(C R/R)^{G/R} = CR/R$, and we obtain a contradiction with Lemma 2.1. This contradiction shows that $[D_1, C] = D_1$. Choose in the subgroup $R$ a Prüfer $p$-subgroup $Q_2$. Again $Q_2$ has only finitely many conjugates, so that $Q_2^G$ is a divisible Chernikov subgroup. As above $Q_2^G$ includes an infinite $G$-invariant subgroup $D_2$, which is $G$-quasifinite. Arguing as before it is possible to prove that $D_2$ is divisible. Then, by Corollary 5.11 of [9], $R = D_2R_1$ for some subgroup $R_1$ such that $R_1$ is $G$-invariant and the intersection $D_2 \cap R_1$ is finite, moreover $D_2 \cap R_1 \leq \Omega_n(V)$. Using the above arguments, we obtain that $[D_2, C] = D_2$. Put $L_1 = \Omega_n(D_1)$, then $D_1/L_1 \cap RL_1/L_1 = \langle 1 \rangle$ and $L_1 \leq \Omega_n(V)$. Similarly, put $L_2 = \Omega_n(D_2)$, then $D_2/L_2 \cap R_1L_2/L_2 = \langle 1 \rangle$ and $L_2 \leq \Omega_n(V)$. Repeating these arguments and using transfinite induction, we obtain that the subgroup $V$ has a family of $G$-invariant $G$-quasifinite subgroups $\{D_\mu \mid \mu \in M\}$ such that $V = \langle D_\mu \mid \mu \in M \rangle$, $[D_\mu, C] = D_\mu$ for all $\mu \in M$, as required. Moreover we have $V/\Omega_n(V) = \times_{\mu \in M} D_\mu \Omega_n(V)/\Omega_n(V)$. \[\square\]

Now we can prove
Corollary 3.2. Let $G$ be a periodic locally nilpotent abelian-by-finite group. If $G$ contains a finite contranormal subgroup, then $G$ satisfies the following conditions:

(i) $G = VC$ where $V$ is a normal divisible abelian subgroup and $C$ is a finite contranormal subgroup of $G$;
(ii) $\Pi(G) = \Pi(C)$, in particular the set $\Pi(G)$ is finite;
(iii) $V$ has a family of $G$-invariant $G$-quasifinite subgroups $\{D_\mu : \mu \in M\}$ such that $V = \langle D_\mu : \mu \in M \rangle$;
(iv) $[D_\mu, C] = D_\mu$ for all $\mu \in M$, in particular, $[V, C] = V$.

Proof. Let $A$ be a normal abelian subgroup of $G$ having finite index, and let $C$ be a finite contranormal subgroup of $G$. Then, arguing as above we have $G = CA$. Suppose that $\Pi(G) \neq \Pi(C)$ and choose a prime $q \in \Pi(G) \setminus \Pi(C)$. The equality $G = AC$ implies that $A$ contains a Sylow $q$-subgroup $Q$ of $G$. We have $A = Q \times R$ where $R$ is a Sylow $q'$-subgroup of $A$. Then $G/R = QR/R \times CR/R$, which shows that $CR/R$ cannot be a contranormal subgroup of $G/R$. Thus we obtain a contradiction with Lemma 2.1. This contradiction proves that $\Pi(G) = \Pi(C)$. We have $G = \times_{p \in \Pi(G)} S_p$ where $S_p$ is a Sylow $p$-subgroup of $G$. The isomorphism $S_p \simeq G/(\times_{q \in \Pi(G), q \neq p} S_p)$ and an application of Proposition 3.1 prove the result. 

Recall that a group $G$ is called $F$-perfect, if $G$ does not contain a proper subgroup of finite index. In every group the subgroup $F(G)$, generated by all $F$-perfect subgroups, is $F$-perfect. It is the greatest $F$-perfect subgroup of $G$. Clearly $F(G)$ is a characteristic subgroup of $G$, and the factor group $G/F(G)$ does not contain $F$-perfect subgroups. The subgroup $F(G)$ is called the $F$-perfect part of $G$. Let $\mathcal{X}$ be a class of groups. If $G$ is a group, then we denote by $G_\mathcal{X}$ the intersection of all normal subgroups $H$ of $G$ such that $G/H \in \mathcal{X}$. The subgroup $G_\mathcal{X}$ is called the $\mathcal{X}$-residual of the group $G$. If $\mathcal{X} = F$ is the class of all finite groups, then $G_F$ is called the finite residual of $G$.

Lemma 3.3. Let $G$ be a locally nilpotent periodic group. If $G$ contains a finite contranormal subgroup, then the $F$-perfect part of $G$ has finite index.

Proof. If $G$ does not contain proper subgroups of finite index, then $G$ is $F$-perfect and the result is proved. Therefore we suppose that $G$ contains proper subgroups of finite index. Let $S$ be a finite contranormal subgroup of $G$. Then $S$ is nilpotent. Let $k$ be the nilpotency class of $S$. If $H$ is a normal subgroup of $G$ such that $G/H$ is finite, then Lemma 2.1 shows that $SH/H$ is a contranormal subgroup of $G/H$. On the other hand, $G/H$ is nilpotent, and a nilpotent group does not contain proper
contranormal subgroups. It follows that $SH/H = G/H$. In particular, $G/H$ has nilpotency class at most $k$. Let $\mathcal{S}$ be the family of all normal subgroups of $G$ having finite index, and let $L = \bigcap_{H \in \mathcal{S}} H$. By Remak’s theorem there is an embedding $G/L \leq Cr_{H \in \mathcal{S}} G/H$. Since $G/H$ has nilpotency class at most $k$ for every $H \in \mathcal{S}$, this implies that $G/L$ is a nilpotent group. It follows that $G/L$ does not contain proper contranormal subgroups and we obtain the equality $G/L = SL/L$. This means that $G/L$ is finite. If we suppose that $L$ contains a proper subgroup $K$ having finite index in $L$, then $K$ has finite index in $G$. Then $D = Core_G(K)$ is normal in $G$ and has finite index in $G$. Then $D \in \mathcal{S}$, and therefore $L \leq D$, a contradiction. This contradiction proves that $L$ is $\mathcal{F}$-perfect and $L$ coincides with the $\mathcal{F}$-perfect part of $G$. \[\square\]

**Corollary 3.4.** Let $G$ be a hypercentral periodic group. If $G$ contains a finite contranormal subgroup, then $G$ is abelian-by-finite.

**Proof.** Let $L$ be the $\mathcal{F}$-perfect part of $G$. Lemma 3.3 implies that $L$ has finite index in $G$. The result follows since a periodic hypercentral $\mathcal{F}$-perfect group is abelian (see [4], Chapter 2, n. 2, Theorem 2.2). \[\square\]

**Lemma 3.5.** Let $G$ be a locally nilpotent group. If $G$ is not periodic, then $G$ does not contain finite contranormal subgroups.

**Proof.** Suppose the contrary, and let $S$ be a finite contranormal subgroup of $G$. Since $G$ is locally nilpotent, the set $Tor(G)$ of all elements of $G$ having finite order is a characteristic subgroup of $G$. Since $G$ is not periodic, $G \neq Tor(G)$. Then the inclusion $S \leq Tor(G)$ implies that $S^G \neq G$ and we obtain a contradiction which proves the result. \[\square\]

Now we can prove Theorem C.

**Proof of Theorem C.** Lemma 3.5 implies that a group $G$ must be periodic. By Corollary 3.4 $G$ is abelian-by-finite, and the result follows from Corollary 3.2. \[\square\]

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