Interlacing inequalities for eigenvalues of discrete Laplace operators

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Abstract

The term interlacing refers to systematic inequalities between the sequences of eigenvalues of two operators defined on objects related by a specific operation. In particular, knowledge of the spectrum of one of the objects then implies eigenvalue bounds for the other one.

In this paper, we therefore develop topological arguments in order to derive such analytical inequalities. We investigate, in a general and systematic manner, interlacing of spectra for weighted simplicial complexes with arbitrary weights. This enables us to control the spectral effects of operations like deletion of a subcomplex, collapsing and contraction of a simplex, coverings and simplicial maps, for absolute and relative Laplacians. It turns out that many well-known results from graph theory become special cases of our general results and consequently admit improvements and generalizations. In particular, we derive a number of effective eigenvalue bounds.

1. Introduction

Spectra of Laplace operators typically encode important geometric information about the space on which they are defined. This raises the question of how to control the effect on the spectrum when we perform some standard
operation on that underlying space, like cutting out, contracting or adding a part or taking a local covering. A good quantitative answer to such a question will be very useful when we can start from some space whose spectrum is explicitly known or at least tightly controlled, and then pass by such operations to other spaces whose spectrum we would like to know. In fact, as we shall explore in this paper, there exist some general such relations between the spectra of spaces related by specific operations. More precisely, the eigenvalues of such spaces control each other, with inequalities like

\[ \lambda_{k - \kappa_1} \leq \theta_k \leq \lambda_{k + \kappa_2} \]

for the corresponding eigenvalues \( \lambda_k \) and \( \theta_k \), resp., with integers \( \kappa_1, \kappa_2 \) that only depend on certain topological characteristics of the spaces and operations involved, but are independent of the index \( k \). See Theorem 1.1. The main point of this paper then is a systematic scheme how to derive such analytical inequalities from topological considerations. In particular, this provides a unifying perspective on various special results scattered throughout the literature.

We shall work here within the framework of generalized graphs, i.e., simplicial complexes; let us start with a brief overview of necessary definitions and theorems. An abstract simplicial complex \( K \) on a finite set \( V \) is a collection of subsets of \( V \) which is closed under inclusion. An \( n \)-face of \( K \) is an element of cardinality \( n + 1 \), and the set of all \( n \)-faces of complex \( K \) is denoted by \( S_n(K) \). Two \( n + 1 \)-simplices sharing an \( n \)-face are called \( n \)-down neighbours, and two \( n \)-simplices belonging to a boundary of the same \( (n + 1) \)-face are called \( (n + 1) \)-up neighbours. A simplicial complex \( K \) is \( (n + 1) \)-path connected, if for every pair of \( (n + 1) \)-simplices \( F, F' \) there exists a sequence of \( (n + 1) \)-simplices \( \bar{F} = F_1, F_2, \ldots, F_k = F' \), such that any two neighbouring ones are \( n \)-down neighbours.

An oriented simplex \([F]\) is a simplex \( F \) together with an ordering of its vertices. Two orderings of the vertices are said to determine the same/opposite orientation when related by an even/odd permutation. The \( n \)-th chain group with coefficients in \( \mathbb{R} \), denoted by \( C_n(K, \mathbb{R}) \), is a vector space over \( \mathbb{R} \) generated by oriented \( n \)-faces of \( K \) modulo the relation \([F_1] + [F_2] = 0\), when \([F_1]\) and \([F_2]\) are two different orientations of the same \( n \)-simplex. Elements of \( C_n(K, \mathbb{R}) \) are formal \( \mathbb{R} \)-linear sums of oriented \( n \)-dimensional faces \([F]\) of \( K \). Let \( C^n(K, \mathbb{R}) \) denote \( n \)-th cochain group, i.e.,
the dual of the vector space $C_n(K, \mathbb{R})$. A basis of $C^n(K, \mathbb{R})$ is given by the set of elementary cochains $\{e_{[F]} \mid F \in S_n(K, \mathbb{R})\}$ with

$$e_{[F]}([F']) = \begin{cases} 1 & \text{if } [F'] = [F], \\ 0 & \text{otherwise.} \end{cases}$$

The coboundary operator $\delta_n : C^n(K, \mathbb{R}) \to C^{n+1}(K, \mathbb{R})$ is the linear map

$$(\delta_n f)([v_0, \ldots, v_{n+1}]) = \sum_{i=0}^{n+1} (-1)^i f([v_0, \ldots, \hat{v}_i \ldots v_{n+1}]),$$

where $\hat{v}_i$ denotes that the vertex $v_i$ has been omitted.

After introducing scalar products on cochain vector spaces we define adjoints $\delta_n^* : C^{n+1}(K, \mathbb{R}) \to C^n(K, \mathbb{R})$ of the coboundary maps $\delta_n$ and the combinatorial Laplace operator

$$L_n(K) = \delta_n^* \delta_n + \delta_n \delta_n^* - 1.$$ 

The operators $\delta_n^* \delta_n$ and $\delta_n \delta_n^*$ are called the $n$-up and $n$-down Laplace operators and are denoted by $L_n^{up}(K)$ and $L_n^{down}(K)$, respectively.

Scalar products are commonly chosen such that elementary cochains form an orthonormal basis. In other words,

$$(f, g)_{C^n} = \sum_{F \in S_n(K)} f([F])g([F]), \tag{1.1}$$

where $( , )_{C^n}$ denotes a scalar product defined on the vector space $C^n(K, \mathbb{R})$. However, it is possible and, as we shall see, advantageous, to consider inner products on cochain spaces, different from (1.1). In particular, we will consider a scalar product that turns elementary cochains into an orthogonal, but not necessarily orthonormal, basis. In fact, a positive real valued weight function $w : \bigcup_i S_i(K) \to \mathbb{R}^+$ uniquely determines scalar products on cochain groups, i.e.,

$$(f, g)_{C^n} = \sum_{F \in S_n(K)} w(F)f([F])g([F]).$$

A simplicial complex $K$ together with a weight function $w$ forms an ordered pair $(K, w)$, called a weighted simplicial complex. Depending on a weight function $w$ (scalar products on cochain groups), one can consider different versions of Laplace operators, i.e., $L_n(K, w)$. For example, if the weight
function is constant and equal to 1 on every simplex, then the underlying Laplacian is the combinatorial Laplace operator denoted by $L_n(K)$, as analysed in [4, 6].

If $w$ satisfies the normalizing condition

$$w(F) = \sum_{\substack{\bar{F} \in S_{n+1}(K): \bar{F} \in \partial \bar{F} \\ F \in \partial \bar{F} \\ F \in S_n(K) \\ F \neq \bar{F}, \ F, F' \in \partial F}} w(\bar{F})$$

for every $F \in S_n(K)$, which is not a facet of $K$, then $w$ determines the normalized Laplace operator denoted by $\Delta_n$. For more details on this topic the reader is invited to consult [10].

Let $[\bar{F}] = [v_0, \ldots, v_{n+1}]$ be an oriented $(n + 1)$-face of a complex $K$, such that $v_0 < \ldots < v_{n+1}$. Then the boundary of $[\bar{F}]$ is

$$\partial[\bar{F}] = \sum_i (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}].$$

The faces $F_i = \{v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}\}$ are part of the boundary of $\bar{F}$, and $\text{sgn}([F_i], \partial[\bar{F}]) = (-1)^i$, where $[F_i] = [v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}]$. By abuse of notation, we denote the set of all $n$-dimensional faces in the boundary of $\bar{F}$ by $\partial \bar{F}$.

The up- and down-Laplace operators for an arbitrary weight function $w$ are

$$(L_n^{\text{up}} f)([F]) = \sum_{\bar{F} \in S_{n+1}(K): \bar{F} \in \partial \bar{F} \\ F \in \partial \bar{F}} \frac{w(\bar{F})}{w(F)} f([\bar{F}]) + \sum_{\substack{F' \in S_n(K): F' \neq F, \ F, F' \in \partial \bar{F} \\ F' \in \partial \bar{F}} \text{sgn}([F], \partial[\bar{F}]) \text{sgn}([F'], \partial[\bar{F}]) f([F'])$$

and

$$(L_n^{\text{down}} f)([F]) = \sum_{E \in \partial F} \frac{w(F)}{w(E)} f([E]) + \sum_{\substack{F' \in S_n(K): \ F' \in \partial \bar{F} \\ F \cap F' = E}} \frac{w(F')}{w(E)} \text{sgn}([E], \partial[F]) \text{sgn}([E], \partial[F']) f([F']).$$

The combinatorial Laplace operator for pairs (relative Laplacian) can be defined in a similar manner, see [3] for its definition on chain complexes. The relative Laplacian on cochain complexes is dual to this definition. In particular, let $K$ be a simplicial complex and $K_0$ its subcomplex. Then the $n$-th cochain group of a pair $(K, K_0)$ is

$$C^n(K, K_0; \mathbb{R}) := \{ f \in C^n(K, \mathbb{R}) \mid f([F]) = 0, \text{ for every } F \in S_n(K_0) \},$$
and the coboundary operator $\delta_n : C^n(K, \mathbb{R}) \to C^{n+1}(K, \mathbb{R})$ induces a homomorphism of relative groups $\delta'_n : C^n(K, K_0; \mathbb{R}) \to C^{n+1}(K, K_0; \mathbb{R})$, i.e. $\delta'_n([F]) = f(\partial_n[F])$, for $f \in C^n(K, K_0; \mathbb{R})$. Moreover, $H^n(K, K_0; \mathbb{R}) \cong \tilde{H}^n(K/K_0; \mathbb{R})$. For details on relative cohomology the reader can consult [9] or [12]. The Laplace operators of a pair are $L^\text{up}_n = \delta'_n \delta'_n$, $L^\text{down}_n = \delta'_n - 1 \delta'_n - 1$.  

**Example 1.1.** Let the weight function $w$ be constant and 1 on every face of $K$, as illustrated in Figure 1. The up-Laplacians of the complex $K$ are given by the following matrices (with respect to lexicographical bases)

\[
L^\text{up}_0(K) = \begin{pmatrix}
e[1] & e[2] & e[3] & e[4] & e[5] \\
e[2] & -1 & -1 & 0 & 0 \\
e[3] & -1 & 2 & -1 & 0 \\
e[4] & 0 & 0 & -1 & 1 \\
e[5] & 0 & 0 & 1 & 0
\end{pmatrix}, \quad L^\text{up}_1(K) = \begin{pmatrix}
e[13] & e[23] & e[34] & e[35] \\
e[12] & 1 & -1 & 0 & 0 \\
e[13] & -1 & 1 & -1 & 0 \\
e[23] & 1 & -1 & 1 & 0 \\
e[34] & 0 & 0 & 0 & 0 \\
e[35] & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The simplicial complex $K_0 = \{1, 2, 3, 4, 1, 2, 3, 4\}$ is a subcomplex of $K$, and the up-Laplace matrices of the pair $(K, K_0)$ are

\[
L^\text{up}_0(K, K_0) = e[5] \begin{pmatrix} e[5] \\ 1 \end{pmatrix}, \quad L^\text{up}_1(K, K_0) = e[13] \begin{pmatrix} e[13] & e[23] & e[35] \\ 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Note that the matrix $L^\text{up}_1(K, K_0)$ is equal to the matrix $L^\text{up}_1(K)$ after deletion of the rows and columns indexed by $e[12]$ and $e[34]$. Analogously for $L^\text{up}_0(K, K_0)$.
We shall not only consider positive valued weight functions, but also permit values equal to 0, i.e., $w : \bigcup S_i(K) \to \mathbb{R}^+ \cup \{0\}$. In this case we say that $(K, w)$ is a degenerate weighted simplicial complex. The above analysis carries over to degenerate simplicial complexes, with the difference that $\delta^n_*$ is called a formal adjoint in this case, and is defined as the standard adjoint on its non-degenerate counterpart and extended by zero on simplices of weight zero, i.e.,

$$\delta^n_* e_{[F]} = \sum_{F \in \partial \bar{F}, \ w(F) \neq 0} \frac{w(\bar{F})}{w(F)} \text{sgn}([F], \partial[\bar{F}]) e_{[F]},$$

or equivalently, for $w(F) \neq 0$

$$\delta^n_* \tilde{f}([F]) = \sum_{F \in S_{n+1}, \ F \in \partial \bar{F}, w(F) \neq 0} \frac{w(\bar{F})}{w(F)} \text{sgn}([F], \partial[\bar{F}]) f([F]),$$

and for $w(F) = 0$, $\delta^n_* \tilde{f}([F]) = 0$.

Remark 1.1. Note that, in the case of degenerate simplicial complexes, a revised version of the discrete Hodge theorem will hold. In particular, the kernel of the $n$-Laplace operator will be isomorphic to the direct sum of the $n$-th homology group of a complex and the vector space over $\mathbb{R}$ generated by $n$-simplices of weight zero.

Remark 1.2. Graphs with loops are a special case of weighted graphs. For the graph Laplace operator, a loop effects only the degree of the corresponding vertex. The same is true for simplicial complexes. We can define simplicial complexes with loops in analogy to graphs with loops:

Definition 1.1. Let $[K]$ be an oriented simplicial complex. If we allow oriented faces in which a vertex appears more than once, to be elements of $[K]$, then we say that $[v_0, \ldots, v_i, v_{i+1}, \ldots, v_i, v_{j+1}, v_{n+1}]$ is a loop on the face $[v_0, \ldots, v_i, \hat{v}_i, v_{i+1}, \ldots, v_{j+1}, v_{n+1}]$

It is not difficult to see that the boundary of a loop $[v_0, \ldots, v_i, v_{i+1}, \ldots, v_i, v_{j+1}, v_{n+1}]$ is zero. Thus, the Laplace operator defined on a simplicial complex $[K]$ with loops will coincide with the one defined on $K$. The only place where we could account for loops is when defining the degree of a face, by adding the weight of the loop to its total degree. Therefore, simplicial complexes with loops are already contained in the definition of weighted complexes, and we will
not treat them separately. We include simplicial complexes with loops here because they will arise naturally when we consider simplicial maps, which do not preserve dimensionality, see Section 3.

In this paper we develop a general framework for eigenvalue interlacing of generalized Laplace operators. This will, of course, include the results previously obtained on interlacing for combinatorial and normalized graph Laplacians, and adjacency matrices of graphs. Interlacing of the combinatorial Laplacian spectrum under deletion of an edge with non-pending vertices is a well-known result from algebraic graph theory, see [7], Thm. 13.6.2.

Let \( \lambda_1 \leq \ldots \leq \lambda_n \) be the eigenvalues of \( L^\uparrow_0(G) \) and \( \theta_1 \leq \ldots \leq \theta_n \) the eigenvalues of \( L^\uparrow_0(G - e) \), then

\[
\lambda_{k-1} \leq \theta_k \leq \lambda_k.
\]  

(1.5)

A removal of a vertex and its incident edges was studied in [11] by Lotker, who proved the following

\[
\lambda_{k-1} \leq \theta_k \leq \lambda_{k+1}.
\]  

(1.6)

Similar results were obtained by Chen et al. in [3], for the case of the normalized graph Laplacians. In particular,

\[
\lambda_{k-t} \leq \theta_k \leq \lambda_{k+t},
\]  

(1.7)

where \( \lambda_i \)’s and \( \theta_i \)’s are eigenvalues ordered non-decreasingly, of the normalized Laplacian of graph \( G \) and of \( G - e \), respectively, and more generally,

\[
\lambda_{k-t} \leq \theta_k \leq \lambda_{k+t},
\]  

(1.8)

where the \( \lambda_i \)’s and \( \theta_i \)’s are the eigenvalues ordered non-decreasingly, of the normalized Laplacian of the graph \( G \) and \( G - H \), respectively, with \( H \) being a spanning subgraph of \( G \) on \( t \) edges, and \( \lambda_{-t+1} = \ldots = \lambda_{-1} = \lambda_0 = 0 \) and \( \lambda_{n+1} = \ldots = \lambda_{n+t} = 2 \). Butler in [2] treats a general case of interlacing on weighted graphs. He studies interlacing of \( \Delta^\uparrow_0(G, w_G) \) and \( \Delta^\uparrow_0(L, w_L) \), where \( (L, w_L) \) is a weighted graph, obtained from \( (G, w_G) \) by deletion of its weighted subgraph \( (H, w_H) \), which has no isolated vertices. The following inequalities are established

\[
\lambda_{k-t+1} \leq \theta_k \leq \begin{cases} 
\lambda_{k+t-1} & \text{if } H \text{ is bipartite,} \\
\lambda_{k+t} & \text{otherwise,}
\end{cases}
\]  

(1.9)
where $\lambda_{-t+1} = \ldots = \lambda_{-1} = 0$ and $\lambda_n = \ldots = \lambda_{n+t-1} = 2$, and $t$ is the number of vertices of $H$. In Section 2 we analyze interlacing of eigenvalues of the generalised Laplacian $L^\text{up}_n(K, w_K)$ of a weighted simplicial complex $(K, w_K)$ and its weighted subcomplex $(L, w_L)$, and we obtain the following result.

**Theorem 1.1.** Let $(K, w_K)$ and $(H, w_H)$ be $(n + 1)$-dimensional, weighted simplicial complexes such that their difference $(L, w_L) := (K - H, w_K - w_L)$ is also a simplicial complex with non-negative weights. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ and $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_N$ be the eigenvalues of $L^\text{up}_n(K)$ and $L^\text{up}_n(L)$, respectively. Then we have

$$\theta_k \leq \lambda_{k + D_H},$$

for all $k = 1, 2, \ldots, N - D_H$, and for all $k = 1, 2, \ldots, N$

$$\lambda_{k - D_W} \leq \theta_k,$$

where $D_W = \dim C^{n+1}(H, \mathbb{R}) - \dim H^{n+1}(H, \mathbb{R})$, $D_H = \dim C^n(H, \mathbb{R})$, and $0 = \lambda_{1 - D_W} = \ldots = \lambda_0$.

Compared to (1.9) and to (1.8), a special case of the theorem above will yield a sharper lower interlacing inequality for the graph Laplacian with normalized weights, see Corollary 2.12, and inequalities on the combinatorial graph Laplacian are a special case of Theorem 2.15. In the same section we obtain many well-known eigenvalue bounds as a special case of more general inequalities, which are consequences of Theorem 1.1 and Theorem 2.15.

In Section 3 we analyze coverings of simplicial complexes, point out the failure of the definition of combinatorial covering used so far [14], [8], give the correct discretization of covering spaces and maps from the continuous setting, and state a correct proof (Corollary 3.6), of Theorem 4.4, from [8]. Furthermore, we analyze the general case when $\varphi : (K, w_K) \to (K', w_{K'})$ is a simplicial map, and prove interlacing theorems for two natural choices of the weight function $w_{K'}$ i.e., Theorem 3.7 and Theorem 3.8. In Section 4 we study special cases of simplicial maps, contractions and collapses, and prove interlacing inequalities for the eigenvalues of the combinatorial and normalized Laplacians of such complexes. As a special case, we prove the interlacing inequalities on graphs under the operation of contraction of an edge. Section 5 treats the case of eigenvalues of the relative Laplacian.
2. Interlacing theorems: deletion of a subcomplex

As shown in [10], the generalized combinatorial Laplace operator \( L_n \) is self-adjoint positive definite and the set of its non-zero eigenvalues is a disjoint union of the non-zero eigenvalues of \( L_{n}^{up} \) and \( L_{n}^{down} \). Thus, it suffices to investigate only one of the following families of spectra

\[
\{ s(L_n(K)) \mid -1 \leq n \leq d \}, \{ s(L_{n}^{up}(K)) \mid -1 \leq n \leq d \} \text{ or } \{ s(L_{n}^{down}(K)) \mid 0 \leq n \leq d \},
\]

where \( s \) denotes the multiset of eigenvalues. The operators \( L_{n}^{up} \) and \( L_{n}^{down} \) are self-adjoint and compact, whence we can apply the Spectral theorem, the Variational characterization theorem, and the Min-max theorem (also known as Courant-Fischer-Weyl min-max principle) to characterize their eigenvalues. We state these theorems below. For proofs, see [1].

**Theorem 2.1** (Spectral theorem). Let \( A \) be a compact self-adjoint operator on a Hilbert space \( V \). Then, there is an orthonormal basis of \( V \) consisting of eigenvectors of \( A \).

**Theorem 2.2** (Variational characterization theorem). Let \( f_1, \ldots, f_m \) denote orthogonal eigenfunctions corresponding to eigenvalues \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m \) of a compact, self-adjoint operator \( A \) on a Hilbert space \( V \). Let \( F_i = \{ f_1, \ldots, f_i \} \) be the set of the first \( i \) eigenfunctions of \( A \), and \( F_i^\perp \) its orthogonal complement. Then,

\[
\lambda_i = \min_{g \in F_i^\perp} \frac{(g, Ag)}{(g, g)} = \max_{g \in F_i^\perp} \frac{(g, Ag)}{(g, g)}.
\]

The quantity \( \frac{(g, Ag)}{(g, g)} \) is commonly called the Rayleigh quotient \( R_A(g) \).

**Theorem 2.3** (Min-max theorem). Let \( V_k \) denote a \( k \)-dimensional subspace of \( V \). Then,

\[
\lambda_k = \min_{V_k} \max_{g \in V_k} R_A(g) = \max_{V_{m-k+1}} \min_{g \in V_{m-k+1}} R_A(g) \quad (2.1)
\]

and

\[
\lambda_k = \min_{V_{n-k}} \max_{g \in V_{n-k}} R_A(g) = \max_{V_{k-1} \perp g \in V_{k-1}} \min_{g \in V_{k-1}} R_A(g) \quad (2.2)
\]

We shall use the following slight modification of the min-max theorem.
Lemma 2.4. Let $V_k$ be, as before, a $k$-dimensional subspace of a Hilbert space $V$ and let $A : V \to V$ be a compact, self-adjoint operator. Then

$$\lambda_{k-s} = \min_{V_k} \max_{x \in V_{k-s}} A(x) = \min_{V_{k-s}} \max_{x \in V_{k-s}} A(x) = \lambda_{k-s}.$$ 

Furthermore,

$$\lambda_k = \min_{V_k} \max_{x \in V_k} A(x) \geq \min_{V_{k-s}} \max_{x \in V_{k-s}} A(x) = \min_{V_{k-s}} \max_{x \in V_{k-s}} A(x) = \lambda_{k-s}.$$

Remark 2.1. Theorem 2.1, Theorem 2.2, Theorem 2.3, and Lemma 2.4 also hold when the vector space $V$ is equipped with a degenerate inner product, with the convention that $R_A(g) = 0$, if $(g, g) = 0$.

From the previous considerations we deduce the following. Given a weighted simplicial complex $(K, w_K)$, there exists an orthonormal basis of $C^n(K, \mathbb{R})$ (with the scalar product induced by the weight function as discussed in Section 1) that consists of eigenfunctions $f_1, \ldots, f_m$ corresponding to eigenvalues $\lambda_1 \leq \ldots \leq \lambda_m$ of $\mathcal{L}_{n}^{up}(K)$. Moreover, for every $g \in C^n(K, \mathbb{R})$,

$$\mathcal{L}_{n}^{up}(g) = \sum_{i=1}^{m} \lambda_i(g, f_i) f_i$$

and

$$(g, \mathcal{L}_{n}^{up}(g)) = \sum_{i=1}^{m} \lambda_i(g, f_i)^2.$$

Definition 2.1. A weighted simplicial complex $(H, w_H)$ is a subcomplex of $(K, w_K)$ iff $H$ is a subcomplex of $K$ and $w_H(F) \leq w_K(F)$ for every face $F$ of $H$.

Definition 2.2. A proper difference of the weighted simplicial complexes $(K, w_K)$ and $(H, w_H)$ is a degenerate weighted simplicial complex $(L, w_L)$, such that $L \equiv K$, $w_L := w_K - w_H$, and $L' = \{ F \in K \mid w_L(F) > 0 \}$ is a simplicial complex.

Remark 2.2. When analyzing $\mathcal{L}_{n}^{up}(K)$, it suffices to consider the $(n+1)$-skeleton of a simplicial complex $K$. For details see [10], Section 6.1. Therefore, without loss of generality, we can assume that $K$ and $H$ are $(n+1)$-dimensional.
Proposition 2.5. Let \((K, w_K)\) be a weighted simplicial complex, \((H, w_H)\) its subcomplex, and \((L, w_L)\) their proper difference. Let \(i_n : C^n(L, \mathbb{R}) \to C^n(K, \mathbb{R})\) be an inclusion map, such that \((i_n g)[F] = g([F])\), for every \(F \in S_n(K)\), \(g \in C^n(L, \mathbb{R})\) and let \(\pi_{n+1} : C^{n+1}(K, \mathbb{R}) \to C^{n+1}(L, \mathbb{R})\) be a projection map such that \((\pi_{n+1} f)[\bar{G}] = \bar{f}([\bar{G}])\), for all \(\bar{G} \in S_{n+1}(L)\) and \(\bar{f} \in C^{n+1}(K, \mathbb{R})\).

We consider the following formal adjoints \(i_n^* : C^n(K, \mathbb{R}) \to C^n(L, \mathbb{R})\) and \(\pi_{n+1}^* : C^{n+1}(L, \mathbb{R}) \to C^{n+1}(K, \mathbb{R})\) of the inclusion and projection maps, respectively.

\[
i_n^* f([G]) = \begin{cases} 
\frac{w_K(G)}{w_L(G)} f(i_n[G]) = \frac{w_K(G)}{w_L(G)} f(i_n[G]) & \text{if } w_L(G) > 0, \\
0 & \text{otherwise,}
\end{cases}
\] (2.3)

and

\[
\pi_{n+1}^* g([\bar{F}]) = \frac{w_L(\bar{F})}{w_K(\bar{F})} g(\pi_{n+1}[\bar{F}]) = \frac{w_L(\bar{F})}{w_K(\bar{F})} g([\bar{F}]).
\] (2.4)

Then for every \(0 \leq n < \dim K\) the following diagrams commute.

\[
\begin{array}{ccc}
C^{n+1}(K, \mathbb{R}) & \xleftarrow{\delta_K} & C^n(K, \mathbb{R}) \\
\downarrow{\pi_{n+1}} & & \uparrow{i_n} \\
C^{n+1}(L, \mathbb{R}) & \xleftarrow{\delta_L} & C^n(L, \mathbb{R})
\end{array}
\] (2.5)

Proof. The proof is elementary. \(\square\)

For brevity, in what follows, we omit the index \(n\) in \(i_n, \pi_n, i_n^*, \text{ and } \pi_n^*\).

From the definitions of inclusion and projection maps, it is straightforward to calculate

\[
\pi^* \pi f([\bar{F}]) = \frac{w_L(\bar{F})}{w_K(\bar{F})} \bar{f}([\bar{F}]),
\] (2.6)

and

\[
i^* i g([G]) = \begin{cases} 
\frac{w_K(G)}{w_L(G)} g([G]) & \text{for } w_L(G) > 0, \\
0 & \text{otherwise.}
\end{cases}
\] (2.7)

\(^1\)The epithete "formal" emphasizes that the scalar product may be degenerate.
Furthermore, from the commutativity of the diagrams \(2.5\), we derive

\[
\mathcal{R}_{\mathcal{L}^p(L)}(g) = \frac{(\delta^*_L \delta_L g, g)}{(g, g)} = \frac{(i^* \delta^*_K \pi^* \pi \delta_K ig, g)}{(g, g)} = \frac{(\delta^*_K \pi^* \pi \delta_K ig, ig)}{(g, g)} = \frac{(\pi^* \delta_K ig, \delta_K ig)}{(ig, ig)} = \frac{(\pi^* \delta_K ig, \delta_K ig)}{(ig, ig)} = \mathcal{R}_{\pi^* \pi}(\delta_K ig) R_{\mathcal{L}^p(K)}(ig) R_{i^*}(g). \tag{2.8}
\]

Here we adopt the convention of Remark \(2.1\) that \(\mathcal{R}_A(f) = 0\) whenever the denominator of the Rayleigh quotient is 0. One can easily check that this convention does not affect the equality above.

**Remark 2.3.** We point out that with the simplicial complex \(L'\) that avoids the degeneracy of the scalar product, instead of \(L\), we cannot obtain a good lower bound on \(\theta_k\) because we cannot precisely calculate the dimension of \(C^n(L \cap H, \mathbb{R})/(C^n(L \cap H, \mathbb{R}) \cap \ker \delta_H)\), which is necessary to establish the equivalence \(2.9\).

The following lemmas characterize the operator \(\pi^* \pi\).

**Lemma 2.6.** The eigenvalues of the operator \(\pi^* \pi : C^{n+1}(K, \mathbb{R}) \to C^{n+1}(K, \mathbb{R})\) are \(\{w_{\omega_L(\bar{F})} | \bar{F} \in S_{n+1}(K)\}\), hence \(\mathcal{R}_{\pi^* \pi}(f) \leq 1\).

**Proof.** The proof is a direct consequence of \(2.6\). \qed

**Lemma 2.7.** The following equivalence holds

\[
\mathcal{R}_{\pi^* \pi}(\delta_K ig) = 1 \text{ iff } g|_{C^n(H, \mathbb{R})} \in \ker \delta_H. \tag{2.9}
\]

The set of functions \(g \in C^n(L, \mathbb{R})\) satisfying \(2.9\) is a vector space orthogonal to \(\mathcal{W} := \text{coker} \delta_H\), and the dimension of \(\mathcal{W}\) is \(D_{\mathcal{W}} = \dim C^{n+1}(H, \mathbb{R}) - \dim H^{n+1}(H, \mathbb{R})\).
Proof. Let $\tilde{f} := \delta_K ig$, then according to (2.6) we get

$$ R_{\pi^*\pi}(\tilde{f}) = \frac{\sum_{\tilde{F} \in S_{n+1}(L)} w_L(\tilde{F}) \tilde{f}(\tilde{F})^2}{\sum_{\tilde{F} \in S_{n+1}(K)} w_K(\tilde{F}) \tilde{f}(\tilde{F})^2}. $$

(2.10)

Thus, (2.10) is equal to 1 iff

$$ w_K(\tilde{F}) \tilde{f}(\tilde{F})^2 = (w_K(\tilde{F}) - w_H(\tilde{F})) \tilde{f}(\tilde{F})^2, $$

(2.11)

for every $\tilde{F} \in S_{n+1}(K)$. The relation (2.11) holds if $\tilde{f}$ is identically equal to zero on the subcomplex $H$, i.e., if the restriction of $g \in C^n(L, \mathbb{R})$ on $C^n(H, \mathbb{R})$ is in the kernel of $\delta_H$. Therefore, the function $g$ is orthogonal to $W := \text{coker} \delta_H$. For the dimension of $W$, we have the following short exact sequences that split,

$$ 0 \to \ker \delta_n \to C^n \to \text{im} \delta_n \to 0, $$

$$ 0 \to \text{im} \delta_n - 1 \to \ker \delta_n \to H^n \to 0, $$

and deduce

$$ \dim \text{coker} \delta_n = \dim \ker \delta_n + 1 - \dim H^{n+1}. $$

(2.12)

According to Remark 2.2, $H$ and $K$ are $(n+1)$-dimensional, therefore $\ker \delta_{n+1} = C^{n+1}(H, \mathbb{R})$ and

$$ \dim \ker \delta_n = \dim C^{n+1}(H, \mathbb{R}) - \dim H^{n+1}(H, \mathbb{R}). $$

(2.13)

The other direction in the equivalence (2.9) is trivial.

The eigenvalues of the operator $i^*i$ are characterized by the following lemmas.

Lemma 2.8. Let $g \in C^n(L, \mathbb{R})$. Then, the following implications are true

(i) $(g, g) > 0 \Rightarrow R_{i^*i}(g) \geq 1.$

(ii) $(g, g) = 0 \Rightarrow R_{n^*n}(\delta_K ig) = 0.$

Proof. From (2.7), it follows that

$$ \sum_{G \in S_n(L)} w_K(G) g([G])^2 \sum_{G \in S_n(L)} w_L(G) g([G])^2. $$

(2.14)

Since $w_K(G) \geq w_L(G)$, then (i) holds. Implication (ii) is a direct consequence of (2.10).
Lemma 2.9. Let $g \in C^n(L, \mathbb{R})$, then the following equivalence holds.

$$\mathcal{R}_{i^*i}(g) = 1 \iff g \perp C^n(H, \mathbb{R})$$ (2.15)

Proof. Due to (2.14), we have $\mathcal{R}_{i^*i}(g) = 1 \iff g([G]) = 0$ for every $G \in S_n(H)$. \hfill \Box

Proof of Theorem 2.4. Denote the dimension of the $n$-th cochain group $C^n(K, \mathbb{R})$ by $N$ and let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ and $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_N$ be the eigenvalues of the operators $\mathcal{L}_{n}^{up}(K)$ and $\mathcal{L}_{n}^{up}(L)$, respectively. Then,

$$\theta_k = \min_{\mathcal{V}_k} \max_{g \in \mathcal{V}_k} \mathcal{R}_{\mathcal{L}_{n}^{up}(L)}(g)$$

$$= \min_{\mathcal{V}_k} \max_{g \in \mathcal{V}_k} \mathcal{R}_{\pi^*\pi}(\delta_K ig)\mathcal{R}_{\mathcal{L}_{n}^{up}(K)}(ig)\mathcal{R}_{i^*i}(g)$$ (2.16)

$$\geq \min_{\mathcal{V}_k} \max_{g \in \mathcal{V}_k} \mathcal{R}_{\pi^*\pi}(\delta_K ig)\mathcal{R}_{\mathcal{L}_{n}^{up}(K)}(ig)$$ (2.17)

$$\geq \min_{\mathcal{V}_k} \max_{g \in \mathcal{V}_k, g \perp W} \mathcal{R}_{\mathcal{L}_{n}^{up}(K)}(ig)$$ (2.18)

$$\geq \min_{\mathcal{V}_k} \max_{g \in \mathcal{V}_k, g \perp V_W} \mathcal{R}_{\mathcal{L}_{n}^{up}(K)}(ig).$$ (2.19)

(2.16) comes from (2.8). (2.17) is a consequence of Lemma 2.8. (2.18) follows from Lemma 2.7 and the fact that we are taking the maximum over a smaller set, whereas (2.19) follows since we are performing a wider minimization than in (2.18). Due to Lemma 2.4 from (2.19) we obtain

$$\theta_k \geq \min_{\mathcal{V}_k} \max_{g \in \mathcal{V}_k-D_W} \mathcal{R}_{\mathcal{L}_{n}^{up}(K)}(ig)$$

$$= \min_{\mathcal{V}_k-D_W} \max_{g \in \mathcal{V}_k-D_W} \mathcal{R}_{\mathcal{L}_{n}^{up}(K)}(ig) = \lambda_{k-D_W}.$$  

Therefore,

$$\theta_k \geq \lambda_{k-D_W}.$$ (2.20)

We derive an upper interlacing inequality for $\theta_k$ from (2.8) and Theorem 2.3 as well, i.e.,

$$\theta_k = \max_{\mathcal{V}_{N-k+1}} \min_{f \in \mathcal{V}_{N-k+1}} \mathcal{R}_{\mathcal{L}_{n}^{up}(L)}(f)$$

$$= \max_{\mathcal{V}_{N-k+1}} \min_{f \in \mathcal{V}_{N-k+1}} \mathcal{R}_{\pi^*\pi}(\delta_K if)\mathcal{R}_{\mathcal{L}_{n}^{up}(K)}(if)\mathcal{R}_{i^*i}(f).$$
Since \( \mathcal{R}_{n^*}(\delta_K ig) \leq 1 \), we have

\[
\theta_k \leq \max_{V_{N-k+1} \subseteq V_{N-k+1}} \min_{g \in V_{N-k+1}} \mathcal{R}_{\Delta_K}(ig) \mathcal{R}_{\Delta^*_1}(g) \tag{2.21}
\]

\[
\leq \max_{V_{N-k+1} \subseteq V_{N-k+1}} \min_{g \in V_{N-k+1} \cap C^n(H, \mathbb{R})} \mathcal{R}_{\Delta_K}(ig) \tag{2.22}
\]

\[
= \max_{V_{N-k+1} \subseteq V_{N-k+1} \cap \Delta_K} \min_{g \in V_{N-k+1} \cap \Delta_K} \mathcal{R}_{\Delta_K}(ig) \tag{2.23}
\]

\[
= \max_{V_{N-k+1} \subseteq V_{N-k+1} \cap \Delta_K} \min_{g \in V_{N-k+1} \cap \Delta_K} \mathcal{R}_{\Delta_K}(ig) \tag{2.24}
\]

\[
= \lambda_k + D_H. \tag{2.25}
\]

(2.22) follows from Lemmas 2.8 (i) and 2.9 and the fact that we are minimizing over a subset of the set \( V_{N-k+1} \) occurring in (2.21), whereas (2.24) is a consequence of Lemma 2.4.

Together with (2.20) this proves Theorem 1.1. \( \square \)

Remark 2.4. Other than the requirement that the proper difference of the complexes \( K \) and \( H \) must exist, no restrictions on weight functions nor on simplicial complexes are imposed. Thus, the interlacing theorem Theorem 1.1 holds for the generalized Laplace operator. Note that, unlike in [2], we allow \( H \) to contain isolated \( n \)-simplices (isolated vertices in the case of graphs).

The spectrum of the combinatorial Laplace operator \( L_{i^*}^n(K) \) is bounded from above by the number of vertices of the complex \( K \) (for details, see Proposition 4.2. [4]); hence, we have the following version of Theorem 1.1.

Theorem 2.10. Let \( (K, w_K \equiv 1) \) be an \( (n+1) \)-dimensional simplicial complex on a vertex set of cardinality \( |V| \), and let \( (H, w_H \equiv 1) \) be its \( (n+1) \)-subcomplex, such that their proper difference \( (L, w_L) := (K - H, w_K - w_L) \) exists. Let \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N \) and \( \theta_1 \leq \theta_2 \leq \ldots \leq \theta_N \) be the eigenvalues of \( L_{i^*}^n(K) \) and \( L_{i^*}^n(L) \) respectively. Then, for all \( k = 1, 2, \ldots, N \), we have

\[
\theta_k \leq \lambda_k - D_{w} \leq \lambda_k + D_{H},
\]

where \( D_{w} = \dim C^{n+1}(H, \mathbb{R}) - \dim H^{n+1}(H, \mathbb{R}), \) \( 0 = \lambda_1 - D_{w} = \ldots = \lambda_0, \) and \( |V| = \lambda_N + 1 = \ldots = \lambda_N + D_{H}. \)

The eigenvalues of the normalized combinatorial Laplacian \( \Delta_{i^*}^n \) are bounded by \( n + 2 \) (see [10]). Hence, we have the following theorem.
Theorem 2.11. Let \((K, w_K)\) and \((H, w_H)\) be \((n+1)\)-dimensional, weighted simplicial complexes such that their proper difference \((L, w_L) := (K-H, w_K-w_L)\) exists. Let \(\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N\) and \(\theta_1 \leq \theta_2 \leq \ldots \leq \theta_N\) be the eigenvalues of \(\Delta_n^{up}(K)\) and \(\Delta_n^{up}(L)\), respectively. Then, for all \(k = 1, 2, \ldots, N\), we have
\[
\lambda_{k-D_W} \leq \theta_k \leq \lambda_{k+D_H},
\]
where \(D_W = \dim C^{n+1}(H, \mathbb{R}) - \dim H^{n+1}(H, \mathbb{R})\), \(0 = \lambda_{1-D_W} = \ldots = \lambda_0\), and \(n+2 = \lambda_N + 1 = \ldots = \lambda_{N+D_H}\).

As a special case of Theorem 2.11 we derive the following corollary on interlacing for the normalized graph Laplacian.

Corollary 2.12. Let \((K, w_K)\) and \((H, w_H)\) be weighted graphs, and let \((L, w_L) := (K-H, w_K-w_L)\) be their proper difference. Let \(\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N\) and \(\theta_1 \leq \theta_2 \leq \ldots \leq \theta_N\) be the eigenvalues of \(\Delta_0^{up}(K)\) and \(\Delta_0^{up}(L)\) respectively, then for all \(k = 1, 2, \ldots, N\) we have
\[
\lambda_{k-\dim C^0(H, \mathbb{R})+\dim H^0(H, \mathbb{R})} \leq \theta_k \leq \lambda_{k+\dim C^0(H, \mathbb{R})},
\] (2.26)
where \(0 = \lambda_{1-\dim C^0(H, \mathbb{R})+\dim H^0(H, \mathbb{R})} = \ldots = \lambda_0\), and \(2 = \lambda_{N+1} = \ldots = \lambda_{N+\dim C^0(H, \mathbb{R})}\).

Proof. From the formula for the Euler characteristic in terms of Betti numbers, \(\chi = \sum_i (-1)^i \dim C^i(K, \mathbb{R}) = \sum_i (-1)^i \dim H^i(K, \mathbb{R})\), we get
\[
\dim C^1(G, \mathbb{R}) - \dim H^1(G, \mathbb{R}) = \dim C^0(G, \mathbb{R}) - \dim H^0(G, \mathbb{R}),
\]
for every graph \(G\). Inserting this in the expression for \(D_W\), where \(D_W = \dim C^1(H, \mathbb{R}) - \dim H^1(H, \mathbb{R})\), we obtain the desired formula. \(\square\)

Remark 2.5. A similar claim holds for the combinatorial graph Laplacian with a slight modification. Instead of \(2 = \lambda_{N+1} = \ldots = \lambda_{N+\dim C^0(H, \mathbb{R})}\), we take \(\dim C^0(K, \mathbb{R}) = \lambda_{N+1} = \ldots = \lambda_{N+\dim C^0(H, \mathbb{R})}\).

Remark 2.6. The interlacing inequalities of Theorem 1.1 for non-bipartite graphs \(H\) are at least as good as the interlacing inequalities in (1.9). Moreover, for a general disconnected graph \(H\), we obtain a finer lower bound than Butler [2]. In addition, (2.26) includes even the cases when \(H\) contains isolated vertices. When the graph \(H\) is bipartite, the upper interlacing inequality in (1.9) is better than in Theorem 1.1; otherwise, the two inequalities are the same. However, the estimate of Butler for bipartite graphs is tightly connected to the normalization properties of the normalized graph Laplacian, thus it cannot be generalized to other graph Laplacians.
Example 2.1. Let \((K, w_K)\) and \((L, w_L)\) be the simplicial complexes shown in Figure 2(a) and Figure 2(b), respectively, such that the weight functions \(w_K\) and \(w_L\) take value 1 on all edges. The eigenvalues of the normalized graph Laplacian \(\Delta_0^u(K)\) are 0, 0.73, 1, 1.42, and 1.85, and the eigenvalues of \(\Delta_0^u(L)\) are 0, 0.19, 0.89, 1.5, 1.5, and 1.92. The lower interlacing inequality from Corollary 2.12 yields \(\lambda_k - 3 \leq \theta_k\), while the inequality (1.9) gives \(\lambda_k - 5 \leq \theta_k\).

![Figure 2:](image)

**Figure 2:** The graph \(L\) on the right is obtained as the difference of \(K\) and \(H\), where \(H = \{\{2, 4\}, \{3, 5\}, \{1, 6\}\}\)

An interesting consequence of the previous corollary is the relation between the smallest non-zero eigenvalues of the graphs \(L\) and \(K\) for certain types of graphs \(H\).

**Corollary 2.13.** Let \((K, w_K)\) be a weighted graph, \((H, w_H)\) its subgraph, and \((L, w_L)\) their proper difference. Assume the graph \(H = (v_i, v_j)\) is an edge with one pending vertex i.e., the only neighbour of the vertex \(v_j\) is \(v_i\), and let \(w_H(v_i, v_j) = w_K(v_i, v_j)\). Then \(\lambda_{k-1} \leq \theta_k\) for every \(k \leq N\). In particular, \(\lambda_2 \leq \theta_3\), where \(\lambda_2\) and \(\theta_3\) are the smallest non-zero eigenvalues of \(L_0^u(K)\) and \(L_0^u(L)\), respectively.

The above result implies that the algebraic connectivity of a graph increases with the removal of pending vertices. A similar claim holds for the general Laplace operator \(L_n^u\).

**Corollary 2.14.** Let \((K, w_K)\) be a weighted simplicial complex, and \((H, w_H)\) its subcomplex such that \((L, w_L)\) is their proper difference. Assume the graph
$H$ is an $(n+1)$-simplex $\bar{F}$ with one pending vertex $v$, i.e., the only $(n+1)$-face containing $v$ is $H$ itself, and $w_H(\bar{F}) = w_K(\bar{F})$. Then, $\lambda_k - 1 \leq \theta_k$ for every $k \leq N$. In particular, $\lambda_{K-1} \leq \theta_K$, where $\lambda_{K-1}$ and $\theta_K$ are the smallest non-zero eigenvalues of $L_n^{up}(K)$ and $L_n^{up}(L)$, respectively.

Proof. From [10] (Theorem 3.1.), it follows that the number of zero eigenvalues in the spectrum of $L_n^{up}(K)$ is $\dim C^n(K, \mathbb{R}) + \dim H^{n+1}(K, \mathbb{R}) - \dim C^{n+1}(K, \mathbb{R})$.

Theorem 2.15. The number of zeros in the spectrum of the generalized upper Laplacian of $L$ by deletion of simplices of weight 0 is

$$\dim C^n(K, \mathbb{R}) - (n+1) + \dim H^{n+1}(K, \mathbb{R}) - \dim C^{n+1}(K, \mathbb{R}) + 1 = \dim C^n(K, \mathbb{R}) + \dim H^{n+1}(K, \mathbb{R}) - \dim C^{n+1}(K, \mathbb{R}).$$

Hence, the number of zeros in $L_n^{up}(L)$ is

$$\dim C^n(K, \mathbb{R}) + \dim H^{n+1}(K, \mathbb{R}) - \dim C^{n+1}(K, \mathbb{R}) - n + (n+1),$$

since $L$ contains exactly $n+1$ $n$-faces whose weight is 0. Therefore, there is an additional zero in the spectrum of $L_n^{up}(L)$ compared to $L_n^{up}(K)$. Together with Theorem [11], this proves the claim.

In the sequel, we shall further exploit the properties of $R_{\pi^*\pi}$ and $R_{i^*i}$, and obtain the following inequalities.

**Theorem 2.15.**

$$\min_{F \in S_{n+1}(L)} \frac{w_L(\bar{F})}{w_L(\bar{F})} \lambda_{k-D_Z} \leq \theta_k \leq \max_{F \in S_n(L)} \frac{w_K(F)}{w_L(F)}, \quad (2.27)$$

where $D_Z = \dim C^n(L, \mathbb{R}) - \dim C^{n+1}(L', \mathbb{R}) + \dim H^{n+1}(L', \mathbb{R})$, and $L'$ is a maximal subcomplex of $L$ whose simplices are of non-zero weight, and $\lambda_0 = \ldots = \lambda_{-D_Z+1} = 0$.

Proof. From (2.10), it follows that $R_{\pi^*\pi}(\delta_K ig) = 0$ iff $\delta_K ig(\bar{F}) = 0$ for every $\bar{F} \in S_{n+1}(L')$, i.e., the restriction of $\delta_K ig(\bar{F})$ on $C^{n+1}(L', \mathbb{R})$ must be zero. Let $\bar{L}$ be a subcomplex of $K$, such that $S_n(K) = S_n(\bar{L})$ and $S_{n+1}(L') = S_{n+1}(\bar{L})$, and let $Z = \ker \delta_{\bar{L}}$, where $\delta_{\bar{L}}$ is the $n$-th coboundary map of the simplicial complex $\bar{L}$. Then,

$$\text{if } g \in \ker \delta_{\bar{L}}, \text{ then } R_{\pi^*\pi}(\delta_K ig) = 0. \quad (2.28)$$

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The dimension of $Z$ is $\dim C^m(L, \mathbb{R}) - \dim C^{m+1}(L', \mathbb{R}) + \dim H^{n+1}(L', \mathbb{R})$. From (2.17) and (2.28), we obtain

$$\theta_k \geq \min_{V_k} \max_{g \in V_k} R_{\pi^* \pi}(\delta_K ig) R_{L^w_n(K)}(ig)$$

From (2.27) is due to (2.21) and the fact that $R_{\iota^* \iota}(f) \leq \max_{F \in S_n(L)} w_L(F) \min_{F \in S_n(L): w_L(F) \neq 0} \lambda_{k-D_Z}$. Therefore, inequality (1.5) is a direct consequence of Corollary 2.16 and Theorem 1.1. In the sequel, we shall derive an upper bound for the maximal eigenvalue of the generalized Laplacian, using the above interlacing theorems.

**Corollary 2.16.** Let $(K, w_K)$ be a weighted simplicial complex and $(L, w_L)$ its subcomplex, such that $C^n(K, \mathbb{R}) = C^n(L, \mathbb{R})$, $w_K = 1$, and $w_L(F) \in \{0, 1\}$. Then $\theta_k \leq \lambda_k$, where $\theta_k$ and $\lambda_k$ are the eigenvalues of $L^w_n(L)$ and $L^w_n(K)$, respectively, ordered increasingly.

Therefore, inequality (1.5) is a direct consequence of Corollary 2.16 and Theorem 1.1. In the sequel, we shall derive an upper bound for the maximal eigenvalue of the generalized Laplacian, using the above interlacing theorems.

**Corollary 2.17.** Let $(L, w_L)$ be a weighted simplicial complex on $N$ vertices, and let $\theta_{N_L}$ be the maximum eigenvalue of $L^w_n(L)$. Then

$$\theta_{N_L} \leq \frac{N}{\min_{F \in S_n(L)} w_L(F)}.$$  

In particular, if $w_L \equiv 1$ (the combinatorial Laplacian), then

$$\theta_{N_L} \leq N.$$
Proof. Without loss of generality assume that the weight function $w_L$ on every simplex has value $\leq 1$. Let $K_N$ be an $(N-1)$-simplex on $N$ vertices with the weight function $w_K \equiv 1$. Then it is possible to obtain any $(L, w_L)$ as a difference $(K_N, w_K) - (H, w_H)$ for some subcomplex $(H, w_H)$ of $(K_N, w_K)$. The maximum eigenvalue of $L^\uparrow(K_N)$ is $\lambda_N = N$. By combining this result with (2.27), we obtain the inequalities (2.32) and (2.33).

Remark 2.7. In [4] Duval and Reiner obtained (2.33) by using a different method. For the normalized Laplacian, the operator estimate (2.32) is better than $\theta_{N_L} \leq (n+2)$ if $N \leq \min_{F \in S_n(L)} \deg(F)$.

3. Interlacing effects of covering and simplicial maps

In this section we analyze the effect of covering maps and more general simplicial maps on the spectrum of the Laplace operator. Furthermore, we propose a definition for discrete coverings, different from that in [14], [8], which agrees with the continuous counterpart of covering maps. In particular, we give counterexamples to the Universal Lifting Theorem for discrete covering maps from [14] (Theorem 2.1.), and Theorem 4.4. from [8], and propose a modified definition of coverings, called strong covering, which fixes the problem.

Definition 3.1. Let $K$ and $L$ be simplicial complexes. A simplicial map $\varphi : K \to L$ is a function from the vertices of $K$ to the vertices of $L$, such that $\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_n)$ span a simplex in $L$ whenever $v_0, v_1, \ldots, v_n$ span a simplex in $K$.

The homomorphism of chain groups $\varphi_n : C_n(K, \mathbb{R}) \to C_n(L, \mathbb{R})$, induced by a simplicial map $\varphi : K \to L$, is defined $\mathbb{R}$-linearly by extending the following map on basis elements $\varphi_n[v_0, v_1, \ldots, v_n] = [\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_n)]$. If $\varphi(v_i) = \varphi(v_j)$, for some $i \neq j$, then $\varphi_n[v_0, v_1, \ldots, v_n] = 0$. By the duality principle the induced homomorphism $\varphi^n : C^n(L, \mathbb{R}) \to C^n(K, \mathbb{R})$ of the cochain groups is $\varphi g([v_0, v_1, \ldots, v_n]) = g([\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_n)])$. For brevity, in what follows we shall omit the index $n$ in $\varphi_n$ and $\varphi^n$, and assume that all faces, which are not written as a set of their vertices, are oriented positively, unless stated otherwise.

Let $[G]$ and $[F]$ be positively oriented $n$-simplices on the vertex sets $v_0, v_1, \ldots, v_n$ and $\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_n)$, respectively. We can also represent...
the induced chain map \( \varphi \) as 
\[
\varphi[F] = \text{sgn}([F], [G])e[G],
\]
where
\[
\text{sgn}([F], [G]) = \begin{cases} 
1 & \text{if } [\varphi(v_0), \varphi(v_1), \ldots, \varphi(v_n)] \text{ is a positively oriented simplex,} \\
-1 & \text{otherwise.}
\end{cases}
\]

Henceforth, \( \varphi e[G] \) can be written as
\[
\varphi e[G] = \sum_{F \in S_n(K): \varphi(F) = G} w_K(F)e[F].
\]
Assume \( w_K \) and \( w_L \) are weight functions assigned to \( K \) and \( L \). Then, the adjoint \( \varphi^* \) of a simplicial map \( \varphi \) is given by
\[
\varphi^* e[F] = \text{sgn}([F], [G]) \frac{w_K(F)}{w_L(G)} e[G].
\]
However, we will not be concerned with the Laplace operators \( L_{up}^n(K, w_L) \) for an arbitrary weight function \( w_L \), instead we will consider the weight function \( w_L \), induced by the simplicial map \( \varphi \) and the weight function \( w_K \). There are two reasonable ways to define \( w_L \), given \( w_K \) and the simplicial map \( \varphi \). Namely,
\[
w_L(G) = \sum_{F \in S_n(K): \varphi(F) = G} w_K(F), \text{ for every } G \in S_n(L), \quad (3.1)
\]
and
\[
w_L(G) = \sum_{F \in S_n(K): \varphi(F) = G} w_K(F) - \sum_{F \in S_n+1(K): \varphi(F) = G} w_K(\bar{F}), \text{ for } G \in S_n(L). \quad (3.2)
\]

In the remainder we prove that the following diagram
\[
\begin{array}{c}
C^n(K, \mathbb{R}) \xleftarrow{L_{up}(K)} C^n(K, \mathbb{R}) \\
\downarrow{\varphi^*} & \quad & \uparrow{\varphi} \\
C^n(L, \mathbb{R}) \xleftarrow{L_{up}(L)} C^n(L, \mathbb{R})
\end{array}
\]
is commutative for a simplicial map \( \varphi \), where \( w_L \) is determined in accordance with the rule \( (3.1) \). In particular, we will prove that
\[
\varphi^* L_{up}(K) e[G] = \sum_{F \in S_n(K): \varphi(F) = G} \sum_{F' \in S_n(K): \varphi(F') = G, F, F' \in \partial F} \text{sgn}([F], [G]) \text{sgn}(F, \partial[F]) \text{sgn}([F'], [G']) \text{sgn}(F', \partial[F']) \frac{w_K(F)}{w_L(G')} e[G'],
\]
\[
(3.4)
\]
\[\text{Note that it suffices to consider the weights of the } n \text{ and } n+1-\text{faces of the simplicial complexes } K, L \text{ in order to completely determine the eigenvalues of } L_{up}^n.\]
where \( G' \) denotes \( \varphi(F') \), and is equal to \( L^\varphi_w(L) \). For simplicity, we first look at the case when \( \varphi \) is a covering map. According to Rotman (see [14]), a covering complex is defined as follows.

**Definition 3.2.** Let \( K \) be a simplicial complex. A pair \((K, \varphi)\) is a covering complex of \( L \) if:

(i) \( K \) is a connected simplicial complex,

(ii) \( \varphi : K \to L \) is a simplicial map, and

(iii) for every simplex \( G \in L \), \( \varphi^{-1}(G) \) is a union of pairwise disjoint simplices \( \varphi^{-1}(G) = \bigcup_i F_i \), with a bijection \( \varphi|_{F_i} : F_i \to G \) for each \( i \). A simplicial map \( \varphi \) is called a \textit{covering map}.

The covering complexes are meant to discretize the notion of covering topological spaces.

**Definition 3.3.** Let \( Y \) be a topological space. A covering space of \( Y \) is a topological space \( X \) together with a continuous surjective map \( p : X \to Y \), such that

(i) for every \( y \in Y \), there exists an open neighbourhood \( U \), such that \( p^{-1}(U) \) is a union of pairwise disjoint open sets \( p^{-1}(U) = \bigcup_i V_i \), such that \( p|_{V_i} : V_i \to U \) is a homeomorphism for every \( i \).

**Example 3.1.** All horizontal pairs in Figure 3(a) represent coverings. In particular, the union of the simplicial complexes \( \tilde{K} = \{\{1,6\}, \{2,6\}, \{3,5\}, \{3,4\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\} \) in Figure 3(a) is a covering space of their join \( K = \{\{1,6\}, \{2,6\}, \{3,5\}, \{3,4\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\} \) shown in Figure 3(b). The hexagon \{1, 2, 3, 4, 5, 6\} is a covering space of the hollow triangle \{1', 2', 3'\}, and the covering map \( \varphi \), is given by \( \varphi(\{1\}) = \varphi(\{4\}) = \{1'\}, \varphi(\{2\}) = \varphi(\{5\}) = \{2'\}, \) and \( \varphi(\{3\}) = \varphi(\{6\}) = \{3'\} \). The simplicial complex \( L = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{1\}, \{2\}, \{3\}, \{4\}, \{5\} \) in Figure 3(c) is a covering of \( L = \{\{1',2'\}, \{2',3'\}, \{1',3'\}, \{1',5'\}, \{1'\}, \{2'\}, \{3'\}, \{5'\} \} \). The covering map is given by \( \varphi(\{1\}) = \varphi(\{4\}) = \{1'\}, \varphi(\{2\}) = \{2'\}, \) \( \varphi(\{3\}) = \{3'\} \) and \( \varphi(\{5\}) = \{5'\} \).

Note that in the case of covering maps the weight functions in (3.1) and (3.2) are identical.

**Lemma 3.1.** Let \( \varphi : K \to L \) be a covering map, and let \( w_K \) and \( w_L \) be weight functions satisfying (3.7). Then, diagram (3.3) commutes.
Figure 3: Coverings
Proof. Let \([\bar{F}] = [v_0, \ldots, v_{n+1}]\) be a positively oriented \((n+1)\)-simplex in \(K\), and \([F] = [v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}]\), \([\bar{F}'] = [v_0, \ldots, \hat{v}_j, \ldots, v_{n+1}]\) its oriented facets. Let \(G = \{\varphi(v_0), \ldots, \varphi(v_{n+1})\}\), \(G = \{\varphi(v_0), \ldots, \hat{\varphi}(v_i), \ldots, \varphi(v_{n+1})\}\), and \(G' = \{\varphi(v_0), \ldots, \varphi(v_j), \ldots, \varphi(v_{n+1})\}\) be the images of \(F\), \(\bar{F}\), and \(\bar{F}'\) under the covering map \(\varphi\), respectively. Assume \([\bar{G}], [G], \text{ and } [G']\) are positively oriented. Then,

\[
[\bar{G}] = [\varphi(v_0), \ldots, \varphi(v_{n+1})] \text{ sgn}([F], [\bar{G}]),
\]

\[
[G] = [\varphi(v_0), \ldots, \hat{\varphi}(v_i), \ldots, \varphi(v_{n+1})] \text{ sgn}([F], [G])
\]

and

\[
[G'] = [\varphi(v_0), \ldots, \varphi(v_j), \ldots, \varphi(v_{n+1})] \text{ sgn}([F], [G]).
\]

Therefore,

\[
\text{sgn}([G], \partial[G]) = \text{sgn} \left( [\varphi(v_0), \ldots, \varphi(v_i), \ldots, \varphi(v_{n+1})] \text{ sgn}([F], [G]), [\varphi(v_0), \ldots, \varphi(v_{n+1})] \text{ sgn}([F], [\bar{G}]) \right)
\]

\[
= (-1)^i \text{sgn}([F], [G]) \text{ sgn}([F], [G])
\]

\[
= \text{sgn([F], [\bar{F}]) \text{ sgn([F], [G]) sgn([\bar{F}], [G])}}.
\]

Hence, \((3.3)\) equals to

\[
\varphi^* \mathcal{L}_{\text{up}}^+(K) \varphi e_{[G]} = \sum_{F \in S_n(K): \varphi(F) = G} \sum_{F' \in S_n(K): F', F' \in \partial F} \text{sgn}([G], \partial[G]) \text{ sgn}([G'], \partial[G']) \frac{w_K(F)}{w_L(G')} e_{[G']}
\]

\[
(3.8)
\]

\[
= \sum_{F \in S_n(K): \varphi(F) = G} \sum_{F' \in S_{n+1}(K): F' \in \partial F} \text{sgn}([G], \partial[G]) \text{ sgn}([G'], \partial[G']) \frac{1}{w_L(G')} e_{[G']},
\]

\[
(3.9)
\]

where \(\varphi(F') = G'\). Since \(w_L(G) = \sum_{F \in S_n(K): \varphi(F) = G} w_K(F)\), from \((3.8)\) we get

\[
\varphi^* \mathcal{L}_{\text{up}}^+(K) \varphi e_{[G]} = \sum_{G' \in S_n(L): G, G' \in \partial G} \text{sgn}([G], \partial[G]) \text{ sgn}([G'], \partial[G']) \frac{w_L(G)}{w_L(G')} e_{[G']}
\]

\[
(3.10)
\]

\[
= \mathcal{L}_{\text{up}}^+ e_{[G]}.
\]

Thus, diagram \((3.3)\) commutes. \(\square\)
Let $V_k$ be a $k$-dimensional subspace of $C^n(L, \mathbb{R}) \cong \mathbb{R}^{N_L}$, then according to Theorem 2.3 we have

$$
\theta_k = \max_{V_{k-1}} \min_{g \perp V_{k-1}} \frac{(\mathcal{L}_{up}^u(L)g, g)}{(g, g)} = \max_{V_{k-1}} \min_{g \perp V_{k-1}} \frac{(\varphi^* \mathcal{L}_{up}^u(K) \varphi g, g)}{(g, g)} = \max_{V_{k-1}} \min_{g \perp V_{k-1}} \frac{(\mathcal{L}_{up}^u(K) \varphi g, \varphi g)}{(\varphi g, \varphi g)}.
$$

Equality (3.14) is due to

$$(\varphi g, \varphi g) = \sum_{G \in S_n(L)} \sum_{F \in S_n(K): \varphi(F) = G} (\text{sgn}([F], [G]) g(\varphi F))^2 w_K(F) = \sum_{G \in S_n(L)} g(G)^2 w_L(G) = (g, g).$$

Let $W$ be a vector space generated by $\{\text{sgn}([F_{ji}], [G_i]) e_{[F_{ji}]} - \text{sgn}([F_{(i+1)j}], [G_i]) e_{[F_{(i+1)j}]}, \bigcup F_{ji} = \varphi^{-1}(G_i), \text{ and } \bigcup_i G_i = S_n(L)\}$. The dimension of $W$ is $N_K - N_L$, where $N_K, N_L$ is the number of $n$-dimensional simplices in the simplicial complexes $K$ and $L$, respectively. Let $\varphi g = f$ and let $V_{k-1}$ be a subspace of $\mathbb{R}^{N_K}$. From (3.14) we get

$$
\theta_k = \max_{V_{k-1}} \min_{f \perp V_{k-1}, f \perp W} \frac{(\mathcal{L}_{up}^u(K)f, f)}{(f, f)} \geq \max_{V_{k-1}} \min_{f \perp V_{k-1}} \frac{(\mathcal{L}_{up}^u(K)f, f)}{(f, f)} \geq \lambda_k.
$$

Inequality (3.16) holds since we are minimizing over a larger set. As a broader optimization, from (3.15) we obtain the upper interlacing inequality, i.e.,

$$
\theta_k \leq \max_{V_{k-1+N_L-N_{K}}} \min_{f \perp V_{k-1+N_L-N_{K}}} \frac{(\mathcal{L}_{up}^u(K)f, f)}{(f, f)} = \lambda_{k+N_L-N_{K}},
$$

and we obtain the following theorem.
Theorem 3.2. Let $\varphi : K \rightarrow L$ be a covering map, and let $w_K$ and $w_L$ be weight functions of $K$ and $L$, respectively, such that (3.1) holds. Let $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{N_K}$ and $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_{N_L}$ be the eigenvalues of $\mathcal{L}_n^{up}(K)$ and $\mathcal{L}_n^{up}(L)$, respectively. Then,

$$\lambda_k \leq \theta_k \leq \lambda_{k+N_K-N_L},$$

where $\lambda_{N_K+1} = \ldots = \lambda_{2N_K-N_L} = N$, where $N$ is the number of vertices of $K$.

Note that, however, covering complexes are an inaccurate discretization of covering spaces (continuous setting), since Definition 3.2 does not contain a discrete analogue of the homeomorphic neighbourhood requirement (i) from Definition 3. As an example see Figure 3(e) and Figure 3(f). In what follows, we provide the definition of a strong covering, which accurately discretizes the notion of covering from the continuous setting and overcomes this problem.

Definition 3.4. A covering map $\varphi : K \rightarrow L$ is called a strong covering if for every $G$, which is a facet of $\bar{G}$ and every $F \in \varphi^{-1}(G)$, there exists $\bar{F} \in S_{n+1}(K)$ such that $F \in \partial \bar{F}$ and $\varphi(\bar{F}) = G$.

Remark 3.1. According to the previous definition for any two $n$-faces of $L$, $G$ and $G'$, which are $(n+1)$-up neighbours, and for every $F \in \varphi^{-1}(G)$, there must exist $F' \in S_n(K)$ such that $F$ and $F'$ are $(n+1)$-up neighbours and $\varphi(F') = G'$.

Lemma 3.3. Let $\varphi : K \rightarrow L$ be a strong covering. Then, for every $F \in S_n(K)$ and every $G \in S_n(L)$, such that $\varphi(F) = G$, the following holds

$$|\{\bar{F} \in S_{n+1}(K) \mid F \in \partial \bar{F}\}| = |\{\bar{G} \in S_{n+1}(L) \mid G \in \partial \bar{G}\}|.$$

Proof. Assume $F_1, \ldots, F_k \in S_{n+1}(K)$, such that $F \in \partial F_i$, $i \in \{1, \ldots, k\}$. Then, due to the definition of a covering (Definition 3.2), $\varphi(\bar{F}_i) \in S_{n+1}(L)$ and $\varphi(F) \in \partial \varphi(\bar{F}_i)$, and $\varphi(\bar{F}_i) \neq \varphi(\bar{F}_j)$, for every $i \neq j$. Assume $\bar{G} \in S_{n+1}(L)$, such that $G \in \partial \bar{G}$. Since $\varphi$ is a strong covering and satisfies Definition 3.4, then there exists $\bar{F} \in S_{n+1}(K)$, such that $F \in \partial \bar{F}$, such that $\varphi(\bar{F}) = G$. In addition, if $G \in \partial \bar{G}, \partial \bar{G}'$ and $\bar{G} \neq \bar{G}'$, then $\bar{F} \neq \bar{F}'$ (simply because $\varphi$ is a map from $K$ to $L$).
Lemma 3.4. Let $\varphi : K \to L$ be a strong covering, then there exists a constant $c \in \mathbb{N}$, such that

$$|\{F \in S_n(K) \mid F \in \varphi^{-1}(G), G \in S_n(L)\}| = c$$

for every $F \in K$. The quantity $c$ is also called the degree of the covering.

Proof. Let $G$ and $G'$ be $(n+1)$-up neighbours, i.e., there exists $\tilde{G} \in S_{n+1}(L)$, s.t. $G, G' \in \partial \tilde{G}$. Due to Definition 3.4, for every $F \in \varphi^{-1}(G)$, there exists $\tilde{F}$, s.t. $\varphi(\tilde{F}) = G$. Thus, there exists $F' \in \partial \tilde{F}$ which is in the preimage of $G'$ under the covering map $\varphi$. Therefore, if $G$ and $G'$ are $(n+1)$-up neighbours, then $|\{F \in S_n(K) \mid F \in \varphi^{-1}(G)\}| = |\{F \in S_n(K) \mid F' \in \varphi^{-1}(G')\}|$. Since $L$ is an $(n+1)$-path connected simplicial complex, then for arbitrary $n$-faces $G$ and $G'$ of $L$, there exists a sequence $G = G_0, G_1, \ldots, G_m = G'$ of $n$-faces of the simplicial complex $L$, such that any two neighbouring ones are $(n+1)$-up neighbours as well. This gives us the same cardinal number of sets $\varphi^{-1}(G)$, $\varphi^{-1}(G')$, and $\varphi^{-1}(G)$.

Theorem 3.5. Let $\varphi : K \to L$ be a strong covering, and let $w_{K}(\tilde{F}) = w_{L}(\varphi(\tilde{F}))$, for every pair $\tilde{F}, F$, such that $F \in \partial \tilde{F}$. Then, $\varphi \mathcal{L}^{\mathrm{up}}(L) = \mathcal{L}^{\mathrm{up}}(K) \varphi$, i.e., $s(\mathcal{L}^{\mathrm{up}}(L)) \subset s(\mathcal{L}^{\mathrm{up}}(K) \varphi)$.

Proof. The proof follows directly from the equations below and the definition of a strong covering.

$$\varphi \mathcal{L}^{\mathrm{up}}(L) e_{[G]} = \sum_{\tilde{G} \in S_{n+1}(L)} \sum_{\substack{F' \in S_n(K) : F' \in S_n(K) : \varphi(F') = G'}} \text{sgn}([F'], [G']) \text{sgn}([G], \partial[G]) \text{sgn}([G'], \partial[G]) \frac{w_{L}(\tilde{G})}{w_{L}(G')} e_{[F']}.$$  \hspace{1cm} (3.21)

$$\mathcal{L}^{\mathrm{up}}(K) \varphi e_{[G]} = \sum_{F \in S_n(K) : F \in S_n(K) : \varphi(F) = G} \sum_{\tilde{F} : F, F' \in \partial \tilde{F}} \text{sgn}([F], [G]) \text{sgn}([F], \partial[F]) \text{sgn}([F'], [F]) \frac{w_{K}(\tilde{F})}{w_{K}(F')} e_{[F']}.$$  \hspace{1cm} (3.22)

First, we will prove

$$\{F' \in S_n(K) \mid \exists G \in S_{n+1}(L), \text{ s.t. } G, \varphi(F') \in \partial G\} = \{F' \in S_n(K) \mid \exists \tilde{F} \in S_{n+1}(K), \text{ s.t. } F, F' \in \partial \tilde{F}\}.$$  \hspace{1cm} (3.23)

If a multiple of $e_{[F']}$ occurs in the sum in (3.22), then there exists $F \in S_n(K)$, such that $\varphi(F) = G$, and $F$ and $F'$ are $(n+1)$-up neighbours. Therefore, $\varphi(F')$ and $G$ are $(n+1)$-up neighbours as well, and $e_{[F]}$ is a summand in (3.21).
On the other hand, if a multiple of $e_{[F']}\in (3.21)$, then $F'$ is an $n$-face of the simplicial complex $K$, such that $\varphi(F') = G$, and $F, F'$ are $(n+1)$-up neighbours; hence, $e_{[F']}$ is a summand in $\Delta^0_n$. Note that this claim will not hold if $\varphi$ is only a covering map (see Example 3.1 and Figure 3(e), Figure 3(f)). This proves (3.23).

Due to (3.5) we have

$$\sgn([F], [G]) \sgn([F], \partial[F]) \sgn([F'], \partial[F']) = \sgn([G], \partial[G]) \sgn([F'], [G]) \sgn([F'], \partial[F'])$$

$$= \sgn([G], \partial[G]) \sgn([G'], \partial[G]) \sgn([F'], [G'])$$

which makes (3.21) and (3.22) equal. \hfill \Box

As a consequence of Theorem 3.5 and Lemma 3.4 we obtain the following corollary.

**Corollary 3.6.** If $\varphi : K \to L$ is a strong covering, then

(i) $s(L^0_n(L)) \subset s(L^0_n(K))$ and

(ii) $s(\Delta^0_n(L)) \subset s(\Delta^0_n(K))$.

**Proof.** First we consider the case (i), when $w_K \equiv 1$. Since $\varphi$ is a strong covering, then according to Theorem 3.5 we have $s(L^0_n(L)) \subset s(L^0_n(K))$, where $w_L(G) = \sum_{F, \varphi(F) = G} w_K(F)$. Due to Lemma 3.4 there exists a constant $m = |\{F \in S_n(K) \mid F \in \varphi^{-1}(G)\}|$, such that $w_L(G) = m$ for every $G \in L$. From the definition of the generalized Laplace operator, it follows that $L^0_n(L, w_L) = L^0_n(L)$. Case (ii) is a direct consequence of Theorem 3.5, Lemma 3.3 and previous considerations. \hfill \Box

**Remark 3.2.** A similar theorem for coverings and the spectrum of the combinatorial Laplacian was proposed in [8]. In particular Gustavson claims that for a given covering map $\varphi : K \to L$ among simplicial complexes $K$ and $L$, the spectrum of the combinatorial Laplacian $L^0_n(L)$ is a subset of $L^0_n(L)$. However, there are counterexamples to this claim. For instance, the simplicial complex $\tilde{L}$ given in Figure 3(e) is a covering (according to Definition 3.2) of the simplicial complex $L$ in Figure 3(f). The eigenvalues of $L^0_n(\tilde{L})$ are 0, 0.38, 1.38, 2.61, 3.61, whereas the eigenvalues of $L^0_n(L)$ are 0, 1, 3, 4. The same definition 3.2 of a covering map was used by Rotman in [13], who
claimed the lifting lemma (Theorem 2.1.), which says that every path in \( L \) with a base point \( v \) can be uniquely lifted to a path in \( K \) with a base point \( \tilde{v} \), for every \( \tilde{v} \in \varphi^{-1}(v) \). However, the same pair of covering complexes can be used as a counterexample to this claim as well. These counterexamples are eliminated if we consider strong coverings as Definition 3.4 instead of coverings.

**Remark 3.3.** A combinatorial \( k \)-wedge \( K_1 \sqcup K_2 \) has as a cover (not strong) \( K_1 \sqcup K_2 \), thus Theorem 3.2 holds, whereas Theorem 3.5 does not.

If a simplicial map \( \varphi \) fails to be a covering, then \( \varphi \) need not preserve the dimensionality, i.e., there may exist \( n \)-simplices in \( K \), whose image under map \( \varphi \) will be \( m \)-dimensional, \( m < n \). Without loss of generality assume \( [\tilde{F}] = [v_0, \ldots, v_{n+1}], [F_i] = [v_0, \ldots, \hat{v}_i, \ldots, v_{n+1}], [F_j] = [v_0, \ldots, \hat{v}_j, \ldots, v_{n+1}] \), and \( \varphi(\tilde{F}) = \varphi(F_i) = \varphi(F_j) = G \), i.e. \( \varphi(v_i) = \varphi(v_j) \). Then,

\[
\sum_{F \in \{F, F_i\}} \sum_{F \in S_{n+1}(K): F, F' \in \partial \hat{F}} \text{sgn}([F], [G]) \text{sgn}([\partial F]) \text{sgn}([F'], [G']) \text{sgn}([\partial F']) \frac{w_K(\tilde{F})}{w_L(G')} e_{[G']}
\]

\( (3.24) \)

\[
= \frac{w_K(\tilde{F})}{w_L(G)} \text{sgn}([F_i], [G])(-1)^i \text{sgn}([F_j], [G])(-1)^j e_{[G]} + \frac{w_K(\tilde{F})}{w_L(G)} \text{sgn}([F_i], [G])(-1)^i \text{sgn}([F_j], [G])(-1)^j e_{[G]}
\]

\( (3.25) \)

\[
+ \frac{w_K(\tilde{F})}{w_L(G)} \text{sgn}([F_i], [G])(-1)^i \text{sgn}([F_j], [G])(-1)^j e_{[G]} + \frac{w_K(\tilde{F})}{w_L(G)} \text{sgn}([F_j], [G])(-1)^j e_{[G]}
\]

\( (3.26) \)

\[
= \frac{w_K(\tilde{F})}{w_L(G)} (\text{sgn}([F_i], [G])(-1)^i(-1)^j \text{sgn}([F_i], [G])(-1)^j (-i+1) e_{[G]} + e_{[G]})
\]

\( (3.27) \)

\[
+ \frac{w_K(\tilde{F})}{w_L(G)} (\text{sgn}([F_j], [G])(-1)^i(-1)^j \text{sgn}([F_j], [G])(-1)^j (-i+1) e_{[G]} + e_{[G]})
\]

\( (3.28) \)

\[
= 0
\]

\( (3.29) \)

where \( G' = \varphi(F') \). Therefore, \( \mathcal{L}^{up} L e_{[G]} = \varphi^* \mathcal{L}^{up} K \varphi e_{[G]} \) and the diagram (3.3) commutes for both choices of the weight function \( w_L \), i.e., (3.2) and (3.1). Furthermore, the following holds.

**Theorem 3.7.** Let \( \varphi : K \to L \) be a simplicial map, and \( w_K \) and \( w_L \) be weight functions of the simplicial complexes \( K \) and \( L \) respectively, such that (3.1) is satisfied. Let \( \lambda_1, \ldots, \lambda_{N_K} \) and \( \theta_1, \ldots, \theta_{N_L} \) be the eigenvalues of \( \mathcal{L}^{up}_n(K, w_K) \)
and \( L^\text{up}_n(L, w_L) \), respectively, ordered increasingly. Then,

\[
\lambda_k \leq \theta_k \leq \lambda_{k+K-N_L}, \tag{3.30}
\]

with \( \lambda_{N_K+1} = \ldots = \lambda_{2N_K-N_L} = N \), where \( N \) is the number of vertices of \( K \).

**Proof.** Follows directly from (3.29), the fact that \((g, g) = (\varphi g, \varphi g)\), and Theorem 3.2. \( \square \)

On the other hand, for the weight function \( w_L \) as defined in (3.2), we need certain modifications for the interlacing theorem to hold.

First, we motivate the choice of weight function \( w_L \) in (3.2). Let \((K, w_K)\) be a simplicial complex whose weights are normalized, i.e. \( w_K(F) = \sum_{\bar{F} : \bar{F} \in \bar{F}} w_K(\bar{F}) \), and let \( \varphi : K \to L \) be a simplicial map. Assume \( \bar{F}_1, \ldots, \bar{F}_k \) are \((n+1)\)-faces of \( K \) whose \( \varphi \)-images are \( n \)-dimensional, i.e. \( \varphi(\bar{F}_i) = G_i \in S_n(L) \), and assume (3.1) holds. Note that we are now dealing with simplicial complexes with loops, i.e., \( L \) is such a complex and \( \varphi(\bar{F}_i) \) are loops; thus,

\[
\deg G = \sum_{G \in S_{n+1}(L)} w_L(G) + \sum_{F_i \in S_{n+1}(K)} w_K(F_i),
\]

Therefore the weight function \( w_L \) is not normalized. However, if we adopt definition (3.2) for \( w_L \), we get the desired equality \( w_L(G) = \deg G \); hence, if \( L^\text{up}_n(K, w_K) \) is the normalized Laplacian, then \( L^\text{up}_n(L, w_L) \) is the normalized Laplacian as well.

Let \( W \) be, as before, a vector space spanned by \{sgn([F_ji], [G_i])e[F_ji] - sgn([F_{(i+1)j}], [G_i])e[F_{(i+1)j}] | \bigcup_j F_ji = \varphi^{-1}(G_i), \text{ and } \bigcup_i G_i = S_n(L)\}. \) Let \( \phi : C^m(L) \to C^m(L) \) be an operator such that \( \phi e[G] = \sum_{F \in S_{n+1}(\varphi(F) = G) \wedge (G)} \frac{w_K(F)}{w_L(G)} e[F] \), and let \( Z \) be a vector space of dimension \( z \) spanned by vectors \{e[G] | G \in
$S_n(L), \exists \bar{F} \in S_{n+1}(K)$ s.t. $\varphi(\bar{F}) = G$. Assume that $\varphi : K \to L$ is a simplicial map, and $w_K$ and $w_L$ are weight functions satisfying $3.2$. Then, we have

$$\theta_k = \max \min_{V_{k-1} \downarrow V_{k-1}} (L_{up}(L)g, g) \quad (g, g) \quad (3.31)$$

$$= \max \min_{V_{k-1} \downarrow V_{k-1}} (\varphi^* L_{up}(K) \varphi g, g) \quad (g, g) \quad (3.32)$$

$$= \max \min_{V_{k-1} \downarrow V_{k-1}} (L_{up}(K) \varphi g, \varphi g) \quad (g, g) \quad (3.33)$$

$$\geq \max \min_{V_{k-1} \downarrow V_{k-1}} (L_{up}(K) \varphi g, \varphi g) \quad (g, g) \quad (3.34)$$

$$= \max \min_{V_{k-1} \downarrow V_{k-1}} (L_{up}(K) f, f) \quad (f, f) \quad (3.35)$$

$$\geq \max \min_{V_{k-1} \downarrow V_{k-1}} (L_{up}(K) f, f) \quad (f, f) \quad (3.36)$$

$$\geq \lambda_k. \quad (3.37)$$

On the other hand, the upper interlacing inequality follows from $3.33$.

$$\theta_k \leq \max \min_{V_{k-1} \downarrow V_{k-1}} (L_{up}(K) \varphi g, \varphi g) \quad (\varphi g, \varphi g) \quad (3.38)$$

$$\leq \max \min_{V_{k-1} \downarrow V_{k-1} \downarrow Z} (L_{up}(K) \varphi g, \varphi g) \quad (\varphi g, \varphi g) \quad (3.39)$$

$$\leq \max \min_{V_{k+z-1} \downarrow V_{k+z-1}} (L_{up}(K) \varphi g, \varphi g) \quad (\varphi g, \varphi g) \quad (3.40)$$

$$= \max \min_{V_{k+z-1} \downarrow V_{k+z-1} \downarrow W} (L_{up}(K) f, f) \quad (f, f) \quad (3.41)$$

$$\leq \min_{V_{k+z-NK-NL-1} \downarrow V_{k+z-NK-NL-1}} (L_{up}(K) f, f) \quad (f, f) \quad (3.42)$$

$$\leq \lambda_k + NK - NL + z. \quad (3.43)$$

The inequalities here are derived analogously to the ones in Theorem $3.2$. We assemble our results into the following theorem.

**Theorem 3.8.** Let $(K, w_K)$ and $(L, w_L)$ be weighted simplicial complexes and $\varphi : K \to L$ a simplicial map, such that $3.2$ holds. Let $Z$ be the vector space
of dimension $z$ with basis $\{e_G | \exists \bar{F} \in S_{n+1}(K) \text{ s.t. } \varphi(\bar{F}) = G\}$, and let $\mathcal{W}$ be the vector space spanned by $\{\text{sgn}([F_{ji}], [G_i])e_{[F_{ji}]} - \text{sgn}([F_{(i+1)i}], [G_i])e_{[F_{(i+1)i}]} \mid \bigcup_j F_{ji} = \varphi^{-1}(G_i), \text{ and } \bigcup_i G_i = S_n(L)\}$. Assume $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{NK}$ and $\theta_1 \leq \theta_2 \leq \ldots \leq \theta_{NK}$ are the eigenvalues of $\mathcal{L}_{n}^{up}(K)$ and $\mathcal{L}_{n}^{up}(L)$, respectively. Then,

$$\lambda_k \leq \theta_k \leq \lambda_{k+z-NK-NL} \quad (3.44)$$

Remark 3.4. In [2] Section 3, Butler obtains interlacing effects of weak coverings. Given weighted graphs $(G, w_G)$ with normalized weights and $(H, w_H)$, then $\pi : G \to H$ is a weak covering map, if it maps vertices to vertices.

The weight functions of $H$ and $G$ satisfy the following. Let $(x, y)$ be an edge in $H$, then $w_H(x, y) = \sum_{u \in \pi^{-1}(x)} w_G(u, v)$, and let $x$ be a vertex in $H$, then $\deg_H x = \sum_{v \in \pi^{-1}(x)} \deg_G v$. In our terminology, a ”weak cover” is nothing but a simplicial map defined on graphs such that the weights obey (3.1). However, if $L$ happens to have loops, stemming from an edge collapsing into a vertex, the Laplace operator $\mathcal{L}_{0}^{up}(H, w_H)$ will not be normalized. For instance, let $G$ be a hollow triangle, with edges $(1, 2)$, $(2, 3)$, and $(1, 3)$ with weights 1 on the edges, and 2 on vertices; and let $H$ be a graph on one edge $(1', 2')$ and a loop $(2', 2')$. Then the map $\pi : G \to H$ with $\pi(1) = 1'$ and $\pi(2) = \pi(3) = 2'$ is a weak covering map. Thus, $w_H(1', 2') = 2$, $\deg_H(1') = 2$, and $\deg_H(2') = 4 \neq \sum_{(u,v):\pi(u)=2} w_G(u,v) = 3$, which is clearly in contradiction with the weights required for the normalized graph Laplacian. However, the interlacing theorem Butler obtains is accurate, for weights defined as $w_H(x) = \sum_{u: \pi(u) = x} w_K(v)$, but the resulting Laplacian $\mathcal{L}_{0}^{up}(H, w_H)$ will not be normalized!

4. Collapsing, Contracting and Interlacing

In this section we analyze the effect of collapsing and contraction on the eigenvalues of the Laplace operator.

**Definition 4.1.** Let $K$ be a simplicial complex and $(\bar{F}, F)$ pair of faces, such that $F \in \partial \bar{F}$, and $F$ is not a facet of any other face in $K$. The face $\bar{F}$ is called a **free face**, and a simplicial complex $K'$ obtained from $K$ by deleting $F$ and $F$ is called an **elementary collapse** of $K$. A sequence of elementary collapses is called a **collapse**. A simplicial map $\varphi : K \to K'$, corresponding to this operation, is also called an elementary collapse.
**Definition 4.2.** Let $K$ be a simplicial complex of dimension $n + 1$, and let $\bar{F}$ be an $(n + 1)$-face, such that only two of its facets $F, F'$, are incident to $(n + 1)$-simplices other than $\bar{F}$. The simplicial complex $K'$ obtained from $K$ by deleting the face $\bar{F}$ and identifying $F$ and $F'$ is called an *elementary contraction* of $K$. A sequence of elementary contractions is called a *contraction*. A simplicial map $\varphi : K \to K'$, corresponding to this operation, is also called an elementary contraction.

**Remark 4.1.** Assume $F$ and $F'$ are $n$-faces of the $(n + 1)$-face $\bar{F}$, which are identified under the elementary contraction $\varphi : K \to K'$. If $F'$ has no $(n + 1)$-up neighbours, other than facets of $\bar{F}$, then this elementary contraction can be treated as a composition of two elementary collapses. Thus, we will not analyse this case separately.

Let $\varphi : K \to K'$ be an elementary contraction, which identifies two $n$-faces $F_1$ and $F'_1$, which are $(n + 1)$-up neighbours. In other words, there exist vertices $v \in F_1$ and $v' \in F'_1$, such that $\varphi(v) = \varphi(v')$, and $\varphi$ is injective on all other vertices of $K$. In what follows we will distinguish among two types of elementary contractions:

1. There exist $2m$ $(n + 1)$-faces $\bar{F}_2$, $\bar{F}_2'$, ..., $\bar{F}_{m+1}$, $\bar{F}'_{m+1}$ of $K$, such that $\varphi(\bar{F}_i) = \varphi(\bar{F}'_i)$, for all $2 \leq i \leq m+1$, and $\varphi$ is injective on the remaining $(n + 1)$-faces of $K$.

2. $\varphi$ is injective on $S_{n+1}(K)$.

**Example 4.1.** In Figures 5(a) and 5(b), elementary collapses of the simplicial complex $K$ (Figure 4) are presented, in particular the collapse of an edge and a triangle, respectively. Figures 6(a) and 6(b) represent two main types of elementary contractions, type (i) and (ii), respectively.

Note that both these operations, collapse and contraction, are simplicial maps. However, the interlacing theorems proven Section 3 are valid for specific choices of weights on $K'$, see (3.2) and (3.1). For the combinatorial Laplacian $L_n^w(K) = \mathcal{L}_n^w(K, w_K)$, where $w_K \equiv 1$, one might want to consider the case of interlacing eigenvalues of $L_n^w(K)$ ($\Delta_n^w(K)$) and $L_n^w(K')$ ($\Delta_n^w(K')$), where $K'$ is either a collapse or a contraction of $K$. These cases are resolved in the following theorems.
Figure 4: Simplicial complex $K$

Figure 5: Elementary collapses of simplicial complex $K$

Figure 6: Elementary contractions of simplicial complex $K$
Theorem 4.1. Let $\varphi : K \rightarrow K'$ be an elementary contraction, and let $\lambda_1 \leq \ldots \lambda_{N_K}, \theta_1 \leq \ldots \theta_{N_{K'}}$ be the eigenvalues of $L_{n}^{up}(K)$ and $L_{n}^{up}(K')(\Delta_{n}^{up}(K)$ and $\Delta_{n}^{up}(K'))$, respectively, then

(i)
\[
\lambda_k - m(n+2) \leq \theta_k \leq \lambda_k + (N_N - N_{K'} + m(n+2));
\]
if the contraction $\varphi$ is of type (i), or

(ii)
\[
\lambda_k \leq \theta_k \leq \lambda_{k+n+2};
\]
if $\varphi$ is of type (ii),

where $\lambda_{N_{K+1}} = \ldots = \lambda_{2N_{N} - N_{K'} + m(n+1)} = N$, $\lambda_0 = \lambda_{-1} = \ldots = \lambda_{-n-1} = 0$, and $N$ is the number of vertices of $K$.

Proof. Assume that the weight functions $w_K$ and $w_{K'}$ are identically equal to 1, i.e., we are dealing with the combinatorial Laplacian $L_{n}^{up}$. Let $\tilde{F}_1$, $\tilde{F}_1'$, and $F_{i}'$ be the faces of $K$, such that the elementary contraction $\varphi$ is given by identification of $F_1$ and $F_1'$. In the proof we will distinguish among two, already mentioned, cases of elementary collapses $\varphi$. Denote by $\bar{F}_2, \bar{F}_{+1}, \bar{F}_{m+1}'$ the $(n+1)$-faces of $K$ that are identified under $\varphi$, if $\varphi$ is of type (i). Let $\psi : C^n(K', \mathbb{R}) \rightarrow C^n(K', \mathbb{R})$ be a map, such that

\[
\psi e_{[G]} = \sum_{G' \in \partial G_i} w_{K}(\tilde{F}_i) \frac{1}{w_{K'}(G') e_{[G']}} \frac{\text{sgn}([G], \partial [G_i]) \text{sgn}([G'], \partial [G_i])}{\text{sgn}([G'], \partial [G_i])} e_{[G']}^{G'}, \quad (4.3)
\]

if $\varphi$ is of type (i), $G \in \partial \varphi(\tilde{F}_i)$, where $\bar{G}_i = \varphi(\tilde{F}_i)$, for all $2 \leq i \leq m + 1$, and

\[
\psi e_{[G]} = 0, \quad (4.4)
\]

for other choices of $G \in K'$ and for $\varphi$ of type (ii).

According to (3.9) and the considerations for simplicial maps, we have

\[
\varphi^* L_{n}^{up}(K) \varphi e_{G} = \sum_{\tilde{F} \in S_{n+1}(K)} \sum_{G' \in \partial G} w_{K}(\tilde{F}) \frac{\text{sgn}([G], \partial [G_i]) \text{sgn}([G'], \partial [G_i])}{w_{K'}(G') e_{[G']}} e_{[G']}^{G'}.
\]

Therefore,

\[
\varphi^* L_{n}^{up}(K) \varphi e_{[G]} = L_{n}^{up}(K') e_{[G]} + w_{K}(\tilde{F}_i) \sum_{G' \in \partial G_i} \frac{\text{sgn}([G], \partial [G_i]) \text{sgn}([G'], \partial [G_i])}{w_{K'}(G')} e_{[G']}^{G'}, \quad (4.5)
\]
if $\varphi$ is of type $(i)$ and $G \in \partial \varphi(\bar{F}_i), 2 \leq i \leq m + 1$, and

$$\varphi^* L^\text{up}_n(K) \varphi e[G] = L^\text{up}_n(K') e[G], \quad (4.7)$$

otherwise. Thus, $L^\text{up}_n(K') = \varphi^* L^\text{up}_n(K) \varphi - \psi$. As for $(\varphi g, \varphi g)$ we have the following equalities

$$(\varphi g, \varphi g) = \sum_{G \in S_n(K')} \sum_{F \in S_n(K) : \varphi(F) = G} (\text{sgn}([F], [G]) g(\varphi F))^2 w_K(F)$$

$$= \sum_{G \in S_n(K')} g([G])^2 \sum_{F \in S_n(K) : \varphi(F) = G} w_K(F)$$

$$= (g,g) + (\phi_1 g, g) + (\phi_2 g, g), \quad (4.10)$$

where

$$\phi_1 e[G] = \begin{cases} w_K(F) e[G] & \text{if } \varphi \text{ is of type } (i), \text{ and } \varphi(F) = G \in \partial \varphi(\bar{F}_i), 2 \leq i \leq m + 1, \\ 0 & \text{otherwise}, \end{cases} \quad (4.11)$$

and

$$\phi_2 e[G] = \begin{cases} w_K(F_1) e[G] & \text{if } F_1 \in \varphi^{-1}(G), \text{ and } \varphi \text{ is of type } (ii), \\ 0 & \text{otherwise}. \end{cases} \quad (4.12)$$
It is now straightforward to deduce the interlacing inequalities. Namely,

\[ \theta_k = \min_{\mathcal{V}_{N_K-k}} \max_{g \perp \mathcal{V}_{N_K-k}} \left( \frac{\mathcal{L}^{up}(K')g,g}{(g,g)} \right) \]

\[ = \min_{\mathcal{V}_{N_K-k}} \max_{g \perp \mathcal{V}_{N_K-k}} \left( \frac{\varphi^* \mathcal{L}^{up}(K)\varphi g,g}{(g,g)} \right) - (\psi g,g) \]

\[ \geq \min_{\mathcal{V}_{N_K-k}} \max_{g \perp \mathcal{V}_{N_K-k}} \left( \frac{\varphi^* \mathcal{L}^{up}(K)\varphi g,g}{(g,g)} \right) - (\psi g,g) \]

\[ = \min_{\mathcal{V}_{N_K-k}} \max_{g \perp \mathcal{V}_{N_K-k}} \left( \frac{\mathcal{L}^{up}(K)\varphi g,\varphi g}{(\varphi g,\varphi g)} \right) \]

\[ \geq \min_{\mathcal{V}_{N_K-k}} \max_{g \perp \mathcal{V}_{N_K-k}} \left( \frac{\mathcal{L}^{up}(K)\varphi g,\varphi g}{(\varphi g,\varphi g)} \right) \]

\[ \geq \min_{\mathcal{V}_{N_K-k}} \max_{f \perp \mathcal{V}_{N_K-k}} \left( \frac{\mathcal{L}^{up}(K)f,f}{(f,f)} \right) \]

\[ \geq \min_{\mathcal{V}_{N_K-k}} \max_{f \perp \mathcal{V}_{N_K-k}} \left( \frac{\mathcal{L}^{up}(K)f,f}{(f,f)} \right) \]

\[ \geq \min_{\mathcal{V}_{N_K-k}} \max_{f \perp \mathcal{V}_{N_K-k}} \left( \frac{\mathcal{L}^{up}(K)f,f}{(f,f)} \right) \]

\[ \geq \lambda_{k-y}. \]

In inequality (4.15), \( \mathcal{V} \) denotes the subspace of \( C^n(K',\mathbb{R}) \) on which \( (\psi g,g) = 0 \) and the dimension of this space \( (\dim \mathcal{V} = y) \) is at most \((n+2)m\), if \( \varphi \) is of type \((i)\), and \( y = 0 \), if \( \varphi \) is an elementary contraction of type \((ii)\). The vector space \( \mathcal{W} \) in (4.20) is generated by \{\sgn([F_{ji}], [G_i])e_{[F_{ji}]} - \sgn([F_{(i+1)j}], [G_i])e_{[F_{(i+1)j}]} \mid \bigcup_j F_{ji} = \varphi^{-1}(G_i), \text{ and } \bigcup_i G_i = S_n(K')\}, and of dimension \( \dim \mathcal{W} = N_K - \hat{N}_{K'} \). Note that the quantity \( N_K - \hat{N}_{K'} \) equals \( n + 1 \), if \( \varphi \) is an elementary collapse of type \((ii)\).
Similarly, the upper interlacing inequality follows from (3.33), i.e.,

\[
\theta_k \leq \max_{V_{k-1} \sqsubset V_k} \min_{g \in V_{k-1}} \frac{(L^{up}(K)\varphi g, \varphi g) - (\psi g, g)}{(\varphi g, \varphi g)} \quad (4.23)
\]

\[
\leq \max_{V_{k-1} \sqsubset V_k} \min_{g \in V_{k-1}} \frac{(L^{up}(K)\varphi g, \varphi g)}{(\varphi g, \varphi g)} \quad (4.24)
\]

\[
\leq \max_{V_{k-1} \sqsubset V_k} \min_{g \in V_{k-1}} \frac{(L^{up}(K)\varphi_1 g, \varphi_1 g) - (\varphi_2 g, g)}{(\varphi g, \varphi g)} \quad (4.25)
\]

\[
= \max_{V_{k-1} \sqsubset V_k} \min_{f \in V_{k-1}, f \perp Z} \frac{(L^{up}(K)f, f)}{(f, f)} \quad (4.26)
\]

\[
\leq \max_{V_{k-1} \sqsubset V_k} \min_{f \in V_{k-1}, f \perp W} \frac{(L^{up}(K)f, f)}{(f, f)} \quad (4.27)
\]

\[
\leq \max_{V_{k-1} \sqsubset V_k} \min_{f \in V_{k-1}, f \perp W} \frac{(L^{up}(K)f, f)}{(f, f)} \quad (4.28)
\]

\[
\leq \lambda_{k+N_K-N_{K'}}+z. \quad (4.29)
\]

The vector space \( Z \) appearing in inequality (4.25) is

\[
Z = \{ g \in C^n(K', \mathbb{R}) | (\varphi_1 g, g) + (\varphi_2 g, g) = 0 \}, \quad \text{and the dimension } z = \dim Z, \text{ is equal to 1, if } \varphi \text{ is of type (ii), and } m(n+2) \text{ otherwise. Thus, in the case (i) we have the following interlacing inequalities:}
\]

\[
\lambda_{k-m(n+2)} \leq \theta_k \leq \lambda_{k+N_K-N_{K'}+m(n+2)}. \quad (4.30)
\]

And the case (ii) results in

\[
\lambda_k \leq \theta_k \leq \lambda_{k+N_K-N_{K'}+1}. \quad (4.31)
\]

A very similar method can be used to prove inequalities for the normalized Laplacian, with the difference in the definition of weight functions and maps \( \phi_1, \phi_2 \). In this case, the weight functions \( w_K \) and \( w_{K'} \) have value 1 on all faces of dimension \( n + 1 \), whereas \( w_K(F) = \deg F \), and \( w_{K'}(G) = \deg G \), for all \( n \)-faces \( F, G \), of \( K \) and \( K' \), respectively. The maps \( \phi_1, \phi_2 : C^n(K', \mathbb{R}) \to C^n(K', \mathbb{R}) \) are given by

\[
\phi_1 e_{[G]} = \begin{cases} 
 w_K(\bar{F}_i)e_{[G]} & \text{if } \varphi \text{ is of type (i), and } G \in \partial \bar{G}_i, 2 \leq i \leq m+1, \\
 0 & \text{otherwise},
\end{cases}
\]

and

\[
\phi_2 e_{[G]} = \begin{cases} 
 2w_K(\bar{F}_1)e_{[G]} & \text{if } G = \varphi(F_1), \\
 0 & \text{otherwise}.
\end{cases}
\]

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However, the remainder of the proof is exactly the same as in the case of the combinatorial Laplacian $L^{up}_{n}$, and so are the dimensions of vector spaces $Z$, $Y$, and $W$; thus the same interlacing inequalities hold.

We now briefly discuss interlacing inequalities for elementary collapses.

**Theorem 4.2.** Let $\varphi : K \to K'$ be an elementary collapse, and let $\lambda_{1} \leq \ldots \lambda_{N_{K}}$ and $\theta_{1} \leq \ldots \theta_{N_{K'}}$ be the eigenvalues of $L^{up}_{n}(K)$ and $L^{up}_{n}(K')(\Delta^{up}_{n}(K)$ and $\Delta^{up}_{n}(K'))$, respectively, then

$$\lambda_{k} \leq \theta_{k} \leq \lambda_{k+n+3},$$

(4.34)

where $N = \lambda_{N_{K}+1} = \ldots = \lambda_{N_{K}+n+3}$, and $N$ is the number of vertices of $K$.

**Proof.** A direct consequence of Theorem 1.1 and Corollary 2.14. $\square$

5. Eigenvalue interlacing of relative Laplacians

For a simplicial complex $(K, w_{K})$ and a subcomplex $(L, w_{L})$ which is pure of dimension $n$, such that $w_{L}(F) = w_{K}(F)$ for every $F \in L$, we define the relative Laplacian $L_{n}(K, L; \mathbb{R})$ as in Section 1.

Let $\pi_{n+1} : C^{n+1}(K, \mathbb{R}) \to C^{n+1}(K, L; \mathbb{R})$ be a projection map, i.e.,

$$\pi(e_{[F]}) = \begin{cases} e_{[F]} & \text{if } [F] \notin L, \\ 0 & \text{otherwise,} \end{cases}$$

and let $\pi^{*} : C^{n}(K, L; \mathbb{R}) \to C^{n}(K, \mathbb{R})$ be the adjoint of the projection map defined on $n$-cochains, i.e. $\pi^{*}(e_{[F]}) = e_{[F]}$. Hence $\pi^{*}$ is an inclusion map. Since $C^{n+1}(K, L; \mathbb{R}) = C^{n+1}(K, \mathbb{R})$, then $\pi_{n+1} = id$, and the following diagram commutes

$$
\begin{array}{ccc}
C^{n+1}(K, \mathbb{R}) & \xrightarrow{\delta_{K}} & C^{n}(K, \mathbb{R}) \\
\left\downarrow id \right. & & \left\uparrow \pi^{*} \right.
\end{array}
$$

$$
\begin{array}{ccc}
C^{n+1}(K, L; \mathbb{R}) & \xrightarrow{\delta_{K,L}} & C^{n}(K, L; \mathbb{R}) \\
\left\downarrow id \right. & & \left\uparrow \pi^{*} \right.
\end{array}
$$

From the commutativity of this diagram and (2.8), we have

$$
\mathcal{R}_{\mathcal{L}_{n}(K, L)}(g) = \mathcal{R}_{id^{*}id}(\delta_{K} \pi^{*} g) \mathcal{R}_{\mathcal{L}_{n}(K)}(\pi^{*} g) \mathcal{R}_{\pi^{*}}(g) = \mathcal{R}_{\mathcal{L}_{n}(K)}(\pi^{*} g) \mathcal{R}_{\pi^{*}}(g).
$$

(5.1)

(5.2)

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Let \( \lambda_1 \leq \ldots \lambda_{N_K} \) be the eigenvalues of \( \mathcal{L}_{n}^{up}(K) \) and \( \theta_1 \leq \ldots \theta_{N_L} \) the eigenvalues of \( \mathcal{L}_{n}^{up}(K, L) \). Since \( \mathcal{R}_{\pi \pi^*}(g) = 1 \), we have

\[
\theta_k = \min_{V_k} \max_{g \in V_k} \mathcal{R}_{\mathcal{L}_n^{up}(K, L)}(g) = \min_{V_k} \max_{g \in V_k} \mathcal{R}_{\mathcal{L}_n^{up}(K)}(\pi^* g) \mathcal{R}_{\pi \pi^*}(g)
\]

(5.3)

\[
\geq \min_{V_k} \max_{g \in V_k} \mathcal{R}_{\mathcal{L}_n^{up}(K)}(\pi^* g)
\]

(5.4)

\[
\geq \lambda_k,
\]

(5.5)

and this is the lower interlacing inequality among the eigenvalues of \( \mathcal{L}_{n}^{up}(K) \) and \( \mathcal{L}_{n}^{up}(K, L) \). The upper interlacing inequality is

\[
\theta_k = \min_{V_{N_L-k+1}} \max_{g \in V_{N_L-k+1}} \mathcal{R}_{\mathcal{L}_n^{up}(K, L)}(g) = \min_{V_{N_L-k+1}} \max_{g \in V_{N_L-k+1}} \mathcal{R}_{\mathcal{L}_n^{up}(K)}(\pi^* g)
\]

(5.7)

\[
= \min_{V_{N_L-k+1}} \max_{g \in V_{N_L-k+1}} \mathcal{R}_{\mathcal{L}_n^{up}(K)}(\pi^* g)
\]

(5.8)

\[
\leq \min_{V_{N_L-k+1}} \max_{g \in V_{N_L-k+1}} \mathcal{R}_{\mathcal{L}_n^{up}(K)}(g)
\]

(5.9)

\[
\leq \min_{V_{N_K-N_K+N_L-k+1}} \max_{g \in V_{N_K-N_K+N_L-k+1}} \mathcal{R}_{\mathcal{L}_n^{up}(K)}(g)
\]

(5.10)

\[
\leq \lambda_{k+N_K-N_L}.
\]

(5.11)

We collect our results in the following theorem.

**Theorem 5.1.** Let \((K, w_K)\) be a simplicial complex and \((L, w_L)\) a subcomplex, which is pure of dimension \(n\), such that \(w_L(F) = w_K(F)\) for every \(F \in L\). Let \(\lambda_1 \leq \lambda_2 \leq \ldots \lambda_{N_K}\) and \(\theta_1 \leq \theta_2 \leq \ldots \leq \theta_{N_L}\) be the eigenvalues of \(\mathcal{L}_{n}^{up}(K)\) and \(\mathcal{L}_{n}^{up}(K, L)\), respectively. Then,

\[
\lambda_k \leq \theta_k \leq \lambda_{k+N_K-N_L},
\]

(5.12)

where \(N = \lambda_{N_K+1} = \ldots = \lambda_{2N_K-N_L}\), and \(N\) is the number of vertices of \(K\).

**Remark 5.1.** It is not difficult to see that the matrix of the relative Laplacian \(\mathcal{L}_{n}^{up}(K, L)\) is obtained by deleting rows and columns from the matrix of \(\mathcal{L}_{n}^{up}(K)\), thus one can apply the Cauchy interlacing theorem and obtain the same results. However, the method employed here is general and can be used to treat a variety of different interlacing problems, and the Cauchy interlacing theorem is just one of its special cases.
References

[1] T. Biyikoglu, J. Leydold, and P. F. Stadler. *Laplacian eigenvectors of graphs: Perron-Frobenius and Faber ..., Issue 1915*. Springer, 2007.

[2] S. Butler. Interlacing for weighted graphs using the normalized Laplacian. *Electron. J. Linear Algebra*, 16:90–98.

[3] G Chen, D. Davis, F. Hall, Z. Li, K. Patel, and M. Stewart. An interlacing result on normalized Laplacians. *SIAM J. Discrete Math.*, 18:353–361, 2004.

[4] A.M. Duval and V. Reiner. Shifted simplicial complexes are Laplacian integral. *Trans. Amer. Math. Soc.*, 354(11):4313–4344, 2002.

[5] Art M. Duval. A common recursion for laplacians of matroids and shifted simplicial complexes. *Doc. Math.*, 10:583–618, 2005.

[6] Joel Friedman. Computing betti numbers via combinatorial laplacians. In *In Proc. 28th Ann. ACM Sympos. Theory Comput.*, pages 386–391, 1996.

[7] C.D. Godsil and G. Royle. *Algebraic graph theory*. Graduate texts in mathematics. Springer, 2001.

[8] R. Gustavson. Laplacians of covering complexes. *Rose Hulman Undergrad. Math. J.*, 12, 2011.

[9] A. Hatcher. *Algebraic Topology*. Cambridge University Press, 2002.

[10] D. Horak. Combinatorial Laplace operators: a unifying approach normalization and spectra. *preprint, arXiv:1105.2712*, 2011.

[11] Z. Lotker. Note on deleting a vertex and weak interlacing of the Laplacian spectrum. *Electron. J. Linear Algebra*, 16:62–72, 2007.

[12] W.S. Massey. *A basic course in algebraic topology*. Graduate texts in mathematics. Springer-Verlag, 1991.

[13] J. J. Rotman. Covering complexes with application to algebra. *Rocky Mountain Journal of Mathematics*, 3:641–674, 1973.

[14] J. J. Rotman. *An Introduction to Algebraic Topology (Graduate Texts in Mathematics)*. Springer, 1998.