GLOBAL DYNAMICS OF A GENERAL LOTKA-VOLterra
COMPETITION-DIFFUSION SYSTEM IN
HETEROGENEOUS ENVIROMENTS

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Abstract. Previously in [14], we considered a diffusive logistic equation with
two parameters, \( r(x) \) – intrinsic growth rate and \( K(x) \) – carrying capacity. We
investigated and compared two special cases of the way in which \( r(x) \) and \( K(x) \)
are related for both the logistic equations and the corresponding Lotka-Volterra
competition-diffusion systems. In this paper, we continue to study the Lotka-
Volterra competition-diffusion system with general intrinsic growth rates and
carrying capacities for two competing species in heterogeneous environments.
We establish the main result that determines the global dynamics of the system
under a general criterion. Furthermore, when the ratios of the intrinsic growth
rate to the carrying capacity for each species are proportional — such ratios
can also be interpreted as the competition coefficients — this criterion reduces
to what we obtained in [18]. We also study the detailed dynamics in terms of
dispersal rates for such general case. On the other hand, when the two ratios
are not proportional, our results in [14] show that the criterion in [18] cannot
be fully recovered as counterexamples exist. This indicates the importance and
subtleties of the roles of heterogeneous competition coefficients in the dynam-
ics of the Lotka-Volterra competition-diffusion systems. Our results apply to
competition-diffusion-advection systems as well. (See Corollary 5.1 in the last
section.)

1. Introduction. One of the most fundamental equations in population dynamics
is the logistic equation. Recently, the following diffusive logistic equation with

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intrinsic growth rate \( r(x) \) and carrying capacity \( K(x) \) has been considered in [11, 14]:

\[
\begin{cases}
  u_t = d \Delta u + r(x) u \left( 1 - \frac{u}{K(x)} \right) & \text{in } \Omega \times \mathbb{R}^+, \\
  \partial_\nu u = 0 & \text{on } \partial \Omega \times \mathbb{R}^+.
\end{cases}
\]  

(1)

Here, \( u(x,t) \) represents the population density of a species at location \( x \in \Omega \) and at time \( t > 0 \), which is therefore assumed to be non-negative; \( \Omega \), the habitat, is a bounded domain in \( \mathbb{R}^N \) with smooth boundary \( \partial \Omega \); \( d > 0 \) is the dispersal rate of \( u \); \( \Delta = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \) is the usual Laplace operator, and \( \partial_\nu = \nu \cdot \nabla \), where \( \nu \) denotes the outward unit normal vector on \( \partial \Omega \), is the normal derivative on the boundary. The zero Neumann (no-flux) boundary condition is to ensure that no individual crosses the boundary of the habitat. Denoting

\[
\xi(x) = \frac{r(x)}{K(x)},
\]

we can rewrite (1) as

\[
\begin{cases}
  u_t = d \Delta u + u(r(x) - \xi(x)) u & \text{in } \Omega \times \mathbb{R}^+, \\
  \partial_\nu u = 0 & \text{on } \partial \Omega \times \mathbb{R}^+.
\end{cases}
\]  

(2)

In this form, \( \xi(x) \) can be interpreted as density-dependent crowding effect, or intraspecific competition coefficient.

The above two different forms of reaction or growth term are referred to as the “Pearl-Verhulst growth” and the “original Verhulst growth” respectively, which were originally used in a non-spatial context. Although both these formulations represent the same underlying model, they may lead to different understanding or interpretation of evolution and speciation from the ecological point of view. (See [34] for more discussions.) The main issue is that in a non-spatial context, the quantity \( K \), seen as the carrying capacity, seems to cause some confusion between the population limited by resources and equilibrium density. Recent studies in [41, 15] suggest that, the concept “carrying capacity” perhaps is not well-defined as we had realized and it ought to be “dynamic” in nature — that is, it depends not just on the total amount of available resources, but also on how the resources are distributed as well as how the species disperses in the habitat and consumes the resources.

Generally speaking, the two quantities \( r(x) \) and \( K(x) \) are not completely independent, some correlations may exist between them. It is shown in [11] that the way in which growth rate \( r(x) \) and the quantity \( K(x) \) in (1) or \( \xi(x) \) in (2) are related is important for total equilibrium population to exceed total carrying capacity. Furthermore, it is established in [14] that the two cases \( r \equiv cK \) and \( r \equiv \text{constant} \) in the logistic equation (1) would produce very different total populations at equilibrium compared to the total carrying capacity. For more details and further discussion in this direction, see [11, 14, 29] and references therein.

Since the logistic equations are the basis for the corresponding Lotka-Volterra competition-diffusion systems, the way in which growth rate \( r(x) \) and the quantity \( K(x) \) are related leads to different forms of two-species Lotka-Volterra competition systems. Moreover, the aforementioned opposite phenomena of total populations at equilibrium compared to the total carrying capacity have significant consequences on the outcomes of the two competing species— clearly illustrated in [14]. Given the results in [14], it seems necessary to consider the following Lotka-Volterra
competition-diffusion system where more general carrying capacities and intrinsic growth rates are included:

\[
\begin{align*}
U_t &= d_1 \Delta U + r_1(x)U \left(1 - \frac{U + cV}{K_1(x)}\right) \quad \text{in } \Omega \times \mathbb{R}^+, \\
V_t &= d_2 \Delta V + r_2(x)V \left(1 - \frac{bU + V}{K_2(x)}\right) \quad \text{in } \Omega \times \mathbb{R}^+, \\
\partial_{\nu}U &= \partial_{\nu}V = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\
U(x,0) &= U_0(x), V(x,0) = V_0(x) \quad \text{in } \Omega.
\end{align*}
\]

Here, \(U(x,t)\) and \(V(x,t)\) represent the population densities of two competing species at location \(x \in \Omega\) and at time \(t > 0\); \(d_1\) and \(d_2\) are the dispersal rates of \(U\) and \(V\) respectively; \(r_1(x)\) and \(r_2(x)\) are the intrinsic growth rates of \(U\) and \(V\) respectively; \(K_1(x)\) and \(K_2(x)\) are the carrying capacities of \(U\) and \(V\) respectively; the constants \(b,c > 0\) are interspecific competition strengths, where we have normalized the intraspecific competition strengths to be 1. Denoting

\[\xi_i(x) = \frac{r_i(x)}{K_i(x)},\]

we again rewrite (3) as

\[
\begin{align*}
U_t &= d_1 \Delta U + U[r_1(x) - \xi_1(x)(U + cV)] \quad \text{in } \Omega \times \mathbb{R}^+, \\
V_t &= d_2 \Delta V + V[r_2(x) - \xi_2(x)(bU + V)] \quad \text{in } \Omega \times \mathbb{R}^+, \\
\partial_{\nu}U &= \partial_{\nu}V = 0 \quad \text{on } \partial\Omega \times \mathbb{R}^+, \\
U(x,0) &= U_0(x), V(x,0) = V_0(x) \quad \text{in } \Omega.
\end{align*}
\]

Similarly as in the single species case, \(\xi_i(x)\) can be interpreted as intraspecific competition coefficients and \(\xi_i(x)\) together with the constants \(b, c\) can be interpreted as interspecific competition coefficients.

When \(r_i\) and \(K_i\) are proportional for \(i = 1, 2\) on \(\Omega\), i.e., both \(\xi_1\) and \(\xi_2\) are constant, the global dynamics of (3) or (4) has been studied extensively in the past few decades. See [5, 6, 7, 8], [13], [16, 17, 18, 19, 20], [24, 27, 29, 30] and references therein. For instance, the celebrated fact — “slower diffuser always prevails” [13] — for the case \(r_1 \equiv r_2 \equiv K_1 \equiv K_2\) and \(b = c = 1\) with nonnegative and nonzero initial data \((U_0(x), V_0(x))\) says that, regardless of the initial values, \(U\) will always wipe out \(V\), as long as \(d_1 < d_2\). It is easy to see that this phenomenon also holds true for the more general case \(r_1 \equiv r_2, K_1 \equiv K_2\) and \(b = c = 1\). Another interesting fact, due to Lou [29], says that the combined effects of diffusion and spatial heterogeneity change the nature of weak competition in Lotka-Volterra competition systems with constant coefficients. Finally, He and Ni [18] obtain a complete classification of the global dynamics of (4) when \(\xi_1 \equiv \xi_2 \equiv 1\) for a large range of the parameters \(b\) and \(c\) including the region \(bc \leq 1\).

Our goal in this paper is to study a general Lotka-Volterra competition-diffusion model where intrinsic growth rates and inter-/intra-specific competition coefficients are spatially heterogeneous, which certainly includes model (3) or (4) as a special case. To describe our results mathematically, we focus mainly on system (4) in the introduction for simplicity.

To completely classify the global dynamics of (4) in the first quadrant \(Q = \mathbb{R}^+ \times \mathbb{R}^+\) of the \(d_1d_2\)-plane, throughout the paper, we assume that

\[(A1) \quad r_i, \xi_i \in C^\alpha(\bar{\Omega}) (\alpha \in (0,1)) \text{ are positive on } \Omega \text{ for } i = 1, 2.\]
Denote the unique positive steady state of (2) by $\theta_{d, r, \xi}$, i.e., $\theta_{d, r, \xi}$ satisfies the following equation:

$$d\Delta \theta + \theta(r(x) - \xi(x)\theta) = 0 \text{ in } \Omega, \quad \partial_{\nu} \theta = 0 \text{ on } \partial\Omega. \quad (5)$$

(See, e.g., [11] for a proof of existence and uniqueness results of (5).) For simplicity of notation, throughout this paper, we denote $u_{d_1} = \theta_{d_1, r_1, \xi_1}$ and $v_{d_2} = \theta_{d_2, r_2, \xi_2}$.

It is easy to see that (4) has two semi-trivial steady states $(u_{d_1}, 0)$ and $(0, v_{d_2})$. To characterize their linear stability properties in terms of $(d_1, d_2) \in Q$ precisely, we define:

$$\Sigma_U := \{(d_1, d_2) \in Q \mid (u_{d_1}, 0) \text{ is linearly stable}\}, \quad \Sigma_V := \{(d_1, d_2) \in Q \mid (0, v_{d_2}) \text{ is linearly stable}\}, \quad \Sigma_- := \{(d_1, d_2) \in Q \mid \text{both } (u_{d_1}, 0) \text{ and } (0, v_{d_2}) \text{ are linearly unstable}\}. \quad (6)$$

We also need to consider the situation when one (or both) of the two semitrivial steady states is (or are) neutrally stable, i.e., neither linearly stable nor linearly unstable. For this purpose, we introduce the following elliptic eigenvalue problem.

**Definition 1.1.** Given a constant $d > 0$ and a function $h \in L^\infty(\Omega)$, we define $\mu_1(d, h)$ to be the first eigenvalue of

$$\begin{cases}
    d\Delta \psi + h(x)\psi + \mu \psi &= 0 \text{ in } \Omega, \\
    \partial_{\nu} \psi &= 0 \text{ on } \partial\Omega.
\end{cases}$$

Now we are ready to define the following subsets of $Q$ where at least one of the two semitrivial steady states is *neutrally stable*:

$$\Sigma_{U, 0} := \{(d_1, d_2) \in Q \mid \mu_1(d_2, r_2 - b \xi_2 u_{d_1}) = 0\}, \quad \Sigma_{V, 0} := \{(d_1, d_2) \in Q \mid \mu_1(d_1, r_1 - c \xi_1 v_{d_2}) = 0\}, \quad \Pi := \Sigma_{U, 0} \cap \Sigma_{V, 0}. \quad (7)$$

We now characterize the global dynamics of (4) in $Q$ assuming further that:

$$0 < bc \leq \left(\frac{\min_{\Omega} \xi_1}{\max_{\Omega} \xi_2}\right) \frac{1}{4}. \quad (A2)$$

**Theorem 1.2.** Assume that (A1) and (A2) hold. Then we have the following mutually disjoint decomposition of $Q$:

$$Q = (\Sigma_U \cup \Sigma_{U, 0} \setminus \Pi) \cup (\Sigma_V \cup \Sigma_{V, 0} \setminus \Pi) \cup \Sigma_- \cup \Pi. \quad (8)$$

In particular,

$$(d_1, d_2) \in \Pi \iff \xi_1 \propto \xi_2, \quad bc = 1 \quad \text{and} \quad u_{d_1} \equiv c v_{d_2}.$$ 

Moreover, the following statements hold for (4):

(i) For all $(d_1, d_2) \in (\Sigma_U \cup \Sigma_{U, 0} \setminus \Pi)$, $(u_{d_1}, 0)$ is globally asymptotically stable.

(ii) For all $(d_1, d_2) \in (\Sigma_V \cup \Sigma_{V, 0} \setminus \Pi)$, $(0, v_{d_2})$ is globally asymptotically stable.

(iii) For all $(d_1, d_2) \in \Sigma_-$, (4) has a unique coexistence steady state that is globally asymptotically stable.
(iv) For all \((d_1, d_2) \in \Pi\), (4) has a global attractor consisting of a continuum of steady states
\[
\{ (\zeta u_{d_1}, (1 - \zeta) u_{d_1} / c) | \zeta \in [0, 1] \}
\]
connecting the two semitrivial steady states.

We remark that the proof of Theorem 1.2 uses the same idea as in [18, Theorem 1.3]. (See also the remark after Theorem 1.3 in [18].) When \(\xi_1 \propto \xi_2\), i.e., \(\xi_1 / \xi_2 \equiv \text{const on } \Omega\), we obtain a full extension of [18, Theorem 1.3] to system (4). However, when \(\xi_1 \not\propto \xi_2\), we have
\[
\left( \min_{\Omega} \frac{\xi_1}{\xi_2} / \max_{\Omega} \frac{\xi_1}{\xi_2} \right)^{\frac{1}{2}} < 1.
\]
Hence, [18, Theorem 1.3] cannot be fully extended so that the region \(bc \leq 1\) is completely included in Theorem 1.2. In fact, examples of \(\xi_1, \xi_2, d_1\) and \(d_2\) have been constructed in Theorem 17(ii)(b) in [14] such that when \(b = c = 1\), there exists an unstable coexistence steady state to system (4). Biologically, this means that the complexity of spatially heterogeneous competition coefficients can change the nature of the competition so that a simple classification of global dynamics for all \((d_1, d_2) \in Q\) can no longer be obtained for all \(b\) and \(c\) satisfying \(bc \leq 1\).

Nevertheless, as we can see from Theorem 3.1 below, when all the intra-/inter-specific competition coefficients are identical up to scalar multipliers, [18, Theorem 1.3] can still be fully recovered. This indicates that, besides the self-adjointness of the dispersal operator, certain “symmetric” structure of the quadratic terms in the reaction term seems necessary, from a mathematical point of view. (See also the concluding remarks in Section 5.)

Although Theorem 1.2 already characterizes all possible long-term dynamical behaviors of (4) for all \((d_1, d_2) \in Q\), next we would like to further classify the global dynamics of (4) in terms of \(b, c\) and \((d_1, d_2)\), under the additional condition that \(\xi_1 \propto \xi_2\). With this condition, (A2) reduces to \(bc \leq 1\). Actually, in this case, globally dynamics of (4) can be completely characterized in a larger region including \(bc \leq 1\) as follows. Motivated by [18], we define
\[
L_U := \inf_{d_1 > 0} \frac{r_2}{\xi_2 u_{d_1}} \in (0, \infty), \quad S_U := \sup_{d_1 > 0} \sup_{\Omega} \frac{r_2}{\xi_2 u_{d_1}} \in (0, \infty),
\]
\[
L_V := \inf_{d_2 > 0} \frac{r_1}{\xi_1 v_{d_2}} \in (0, \infty), \quad S_V := \sup_{d_2 > 0} \sup_{\Omega} \frac{r_1}{\xi_1 v_{d_2}} \in (0, \infty),
\]
and
\[
\Xi := \Xi_1 \cup \{(b, c) | 0 < c \leq \frac{1}{S_U} \} \cup \{(b, c) | 0 < b \leq \frac{1}{S_V} \},
\]
where
\[
\Xi_1 := \{(b, c) | b, c > 0 \text{ and } bc \leq 1\}. \quad (11)
\]

**Theorem 1.3.** Assume that (A1) holds and \(\xi_1 \propto \xi_2\). If one of \(r_1 / \xi_1\) and \(r_2 / \xi_2\) is nonconstant, then
\[
0 < L_U L_V < 1, \quad L_U S_V > 1, \quad L_V S_U > 1. \quad (12)
\]
Assume further that \((b, c) \in \Xi\), then the following statements hold for system (4):

(i) If \(b \geq S_U\) and \(c \leq 1 / S_U\), then \(Q = \Sigma_U\); i.e., for all \(d_1, d_2 > 0\), \((u_{d_1}, 0)\) is globally asymptotically stable.
(ii) If \( c \geq S_V \) and \( b \leq 1/S_V \), then \( Q = \Sigma_V \); i.e., for all \( d_1, d_2 > 0 \), \( (0, v_{d_2}) \) is globally asymptotically stable.

(iii) If \( b < L_U \) and \( c < L_V \), then \( Q = \Sigma_\ldots \); i.e., for all \( d_1, d_2 > 0 \), (4) has a unique coexistence steady state that is globally asymptotically stable; if \( b = L_U \) and \( c < L_V \), then \( Q = \Sigma_\ldots \cup \Sigma_{U,0} \), where \( \Sigma_{U,0} \) may be empty; if \( b < L_U \) and \( c = L_V \), then \( Q = \Sigma_\ldots \cup \Sigma_{V,0} \), where \( \Sigma_{V,0} \) may be empty; if \( b = L_U \) and \( c = L_V \), then \( Q = \Sigma_\ldots \cup \Sigma_{U,0} \cup \Sigma_{V,0} \), where \( \Sigma_{U,0} \cup \Sigma_{V,0} \) may be empty.

(iv) If \( b > L_U \) and \( c < L_V \), then \( \Sigma_{U,0} \), \( \Sigma_V \), \( \Sigma_{V,0} \) \( \neq \emptyset \).  Moreover, \( \Sigma_\ldots = \emptyset \) if and only if \( r_1 \equiv cr_2 \xi_1 / \xi_2 \) and \( bc = 1 \), in which case, \( \Sigma_{U,0} = \Sigma_{V,0} = \Pi = \{(d_1, d_2) | d_2 = d_1/c \} \), \( \Sigma_U = \{(d_1, d_2) | d_2 > d_1/c \} \), and \( \Sigma_V = \{(d_1, d_2) | d_2 < d_1/c \} \).

(v) If \( L_U < b < S_U \) and \( c < L_V \), then \( \Sigma_{U,0} \), \( \Sigma_V \), \( \Sigma_{V,0} \) \( \neq \emptyset \) and \( \Sigma_V = \Sigma_{V,0} = \Pi = \emptyset \); if \( L_U < b < S_U \) and \( c = L_V \), then the same statement holds except for \( \Sigma_{V,0} \) which may be nonempty.

(vi) If \( L_U < c < S_V \) and \( b < L_U \), then \( \Sigma_V \), \( \Sigma_{V,0} \), \( \Sigma_\ldots \neq \emptyset \) and \( \Sigma_U = \Sigma_{U,0} = \Pi = \emptyset \); if \( L_U < c < S_V \) and \( b = L_U \), then the same statement holds except for \( \Sigma_{U,0} \) which may be nonempty.

Figure 1 illustrates the decomposition of \( Q \) for each \((b, c) \in \Xi\) in bc-plane considered in Theorem 1.3, where the corresponding global dynamics of system (4) are described in details.

**Figure 1.** Global dynamics of system (4) with \( r_i/\xi_i \neq \text{const} \), \( i = 1,2 \). See Theorem 1.3.

In the second part of this paper, we assume that the two species have identical competition abilities, i.e., \( b = c = 1 \) and \( \xi_1 \equiv \xi_2 \equiv \xi \), and consider the global dynamics of the following system:

\[
\begin{align*}
U_t &= d_1 \Delta U + U(r_1(x) - \xi(x)(U + V)) & \text{in} \Omega \times \mathbb{R}^+, \\
V_t &= d_2 \Delta V + V(r_2(x) - \xi(x)(U + V)) & \text{in} \Omega \times \mathbb{R}^+, \\
\partial_\nu U = \partial_\nu V &= 0 & \text{on} \partial\Omega \times \mathbb{R}^+, \\
U(x, 0) = U_0(x), V(x, 0) &= V_0(x) & \text{in} \Omega.
\end{align*}
\]
Moreover, for comparison purpose we assume further that $\bar{r}_1 = \bar{r}_2$, as we have done in previous papers [16, 19, 20].

Our goal is to illustrate the global dynamics of (13) in terms of $(d_1, d_2)$ and to determine the asymptotic behaviors of the four sets in the decomposition (8) as $d_1$ and/or $d_2$ tend to 0 or $\infty$. Similarly as in [19] and [20], we will distinguish between the following two cases: (i) $r_1/\xi \not\equiv \text{const}$ and $r_2/\xi \equiv \text{const}$; (ii) both $r_1/\xi$ and $r_2/\xi$ are nonconstant.

We first consider Case (i):

**Theorem 1.4.** Assume that (A1) holds with $\xi_1 \equiv \xi_2 \equiv \xi$. We assume in addition that

$$\bar{r}_1 = \bar{r}_2, \quad r_1/\xi \not\equiv \text{const} \quad \text{and} \quad r_2/\xi \equiv \text{const}. \quad (14)$$

Then for system (13), it holds that $\Sigma_U, \Sigma_- \neq \emptyset, \Sigma_V, \Sigma_{V,0} = \emptyset$ and $\mathcal{Q} = \overline{\Sigma_U} \cup \Sigma_-$, where $\overline{\Sigma_U}$ is the closure of $\Sigma_U$ in $\mathcal{Q}$. Moreover,

$$\Sigma_U = \{(d_1, d_2) \in \mathcal{Q} \mid d_2 > \check{d}_2(d_1)\},$$

where $\check{d}_2(d_1)$ is continuous function of $d_1$ defined in $\mathbb{R}^+$. Moreover,

$$\lim_{d_1 \to 0^+} \check{d}_2(d_1) = \infty, \quad \lim_{d_1 \to \infty} \check{d}_2(d_1) = 0. \quad (15)$$

In other words, the species $U$ never loses, while $V$ never wins! More detailed asymptotic behavior of $\check{d}_2(d_1)$ as $d_1 \to \infty$ will be given in Theorem 4.2 in Section 4 below. For an illustration of shapes of the sets $\Sigma_U$ and $\Sigma_-$ in Theorem 1.4, see Figures 2 and 3.

![Figure 2](image1.png)  
**Figure 2.** Shapes of $\Sigma_U$ and $\Sigma_-$ for Theorem 1.4 as illustration.

![Figure 3](image2.png)  
**Figure 3.** Another scenario for Theorem 1.4. See also Theorem 4.2(i) for details.

For Case (ii), we have the following result:

**Theorem 1.5.** Assume that (A1) holds with $\xi_1 \equiv \xi_2 \equiv \xi$. We assume in addition that

$$\bar{r}_1 = \bar{r}_2, \quad r_1 \not\equiv r_2, \quad r_1/\xi \not\equiv \text{const} \quad \text{and} \quad r_2/\xi \not\equiv \text{const}. \quad (16)$$
Then for system (13), it holds that \( \Sigma_U, \Sigma_V, \Sigma_- \neq \emptyset \), \( \Pi = \emptyset \) and \( Q = \Sigma_U \cup \Sigma_V \cup \Sigma_- \). Moreover,
\[
\Sigma_U = \{(d_1, d_2) \in Q \mid d_2 > \tilde{d}_2(d_1)\} \quad \text{and} \quad \Sigma_V = \{(d_1, d_2) \in Q \mid d_1 > \tilde{d}_1(d_2)\},
\]
where \( \tilde{d}_2(d_1) \) (resp. \( \tilde{d}_1(d_2) \)) is a continuous function of \( d_1 \) (resp. \( d_2 \)) defined in \( \mathbb{R}^+ \) satisfying that
\[
\lim_{d_1 \to 0^+} \tilde{d}_2(d_1) = \lim_{d_1 \to \infty} \tilde{d}_2(d_1) = \infty \quad \text{and} \quad \lim_{d_2 \to 0^+} \tilde{d}_1(d_2) = \lim_{d_2 \to \infty} \tilde{d}_1(d_2) = \infty.
\]
Moreover, there exist two constants \( B_1 := B_1(r_1, r_2, \xi) \) and \( B_2 := B_2(r_1, r_2, \xi) \) such that
\[
\tilde{d}_2(d_1) = C_1(r_2, \xi) d_1 + B_1 + O\left(\frac{1}{d_1}\right) \quad \text{for all} \quad d_1 > \tilde{D},
\]
\[
\tilde{d}_1(d_2) = C_1(r_1, \xi) d_2 + B_2 + O\left(\frac{1}{d_2}\right) \quad \text{for all} \quad d_2 > \tilde{D},
\]
where \( \tilde{D} = \tilde{D}(r_1, r_2, \xi) \) is a positive constant. Furthermore, as \( d_1, d_2 \to \infty \), \( \Sigma_- \) approaches asymptotically a band in \( \Omega \) with slope \( C_1(r_2, \xi)/C_1(r_1, \xi) \) and width
\[
\frac{C_1(r_1, \xi)}{\sqrt{C_1^2(r_1, \xi) + C_1^2(r_2, \xi)}} \left( B_1 + \frac{C_1(r_2, \xi)}{C_1(r_1, \xi) B_2} \right) > 0.
\]

Therefore, in Case (ii), the competition seems more “balanced”. For an illustration of shapes of the sets \( \Sigma_U, \Sigma_V \) and \( \Sigma_- \) in Theorem 1.5, see Figure 4.

![Figure 4. Shapes of \( \Sigma_U, \Sigma_V \) and \( \Sigma_- \) for Theorem 1.5 as illustration.](image)

The rest of this paper is organized as follows: In Section 2 we establish some preliminary results which will be used in later sections. In Section 3, we classify global dynamics of a general Lotka-Volterra competition system which includes model (4) as a special case. Then we establish Theorems 1.2 and 1.3. Section 4 is devoted to proving Theorems 1.4 and 1.5. Some concluding remarks including an important application of our main result to a general Lotka-Volterra competition-diffusion-advection systems given by Corollary 5.1 are provided in Section 5.
2. Preliminaries. In this section, we establish some basic facts and preliminary results which will be need latter.

**Lemma 2.1.** Assume that \( r, \xi \in C^a(\bar{\Omega}) \) (\( a \in (0, 1) \)) are positive on \( \bar{\Omega} \). Then the following statements hold for the positive steady state \( \theta_{d,r,\xi} \) of (2):

(i) \( \theta_{d,r,\xi} \to r/\xi \) in \( L^\infty(\Omega) \) as \( d \to 0 \).

(ii) \( \theta_{d,r,\xi} \to \int_\Omega r dx \quad \text{in} \quad L^\infty(\Omega) \quad \text{as} \quad d \to \infty \).

(iii) If \( r/\xi \not\equiv \text{const} \), then \( \|\theta_{d,r,\xi}\|_{L^\infty(\Omega)} < \|r/\xi\|_{L^\infty(\Omega)} \); in particular, we have \( \sup_{\bar{\Omega}} \theta_{d,r,\xi} < \sup_{\bar{\Omega}} \frac{r}{\xi} \) and \( \inf_{\bar{\Omega}} \theta_{d,r,\xi} > \inf_{\bar{\Omega}} \frac{r}{\xi} \).

For proofs of Lemma 2.1(i) and (ii), see [11]. The proof for Lemma 2.1(iii) follows from similar arguments as that for [27, Proposition 2.4].

For linear stability of the trivial steady state \((0,0)\) and the two semi-trivial steady state \((u_d,0)\) and \((0,v_d)\) of system (4), we have the following relatively simple criterion. The proof follows essentially from the same arguments as in that of [27, Corollary 2.10] and therefore is omitted here.

**Lemma 2.2.** The linear stability of \((u_d,0)\), \((0,v_d)\) and \((0,0)\) in system (4) are determined by the sign of \( \mu_1(d_2,r_2-b\xi_2u_d) \), \( \mu_1(d_1,r_1-c\xi_1v_d) \) and \( \min\{\mu_1(d_1,r_1), \mu_1(d_2,r_2)\} \) respectively. In particular, \((0,0)\) is always linearly unstable under condition (A1) for any \( d_1, d_2 > 0 \).

To further characterize the principal eigenvalue \( \mu_1(d,h) \), we need to introduce the following eigenvalue problem with indefinite weight:

\[
\begin{aligned}
  d\Delta \varphi + \lambda h(x)\varphi &= 0 \quad \text{in} \quad \Omega, \\
  \partial_\nu \varphi &= 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]  

(21)

where \( h \in L^\infty(\Omega) \) is nonconstant and could change sign in \( \Omega \). We say that \( \lambda \) is a principal eigenvalue if (21) has a positive solution. (Notice that 0 is always a principal eigenvalue.) The following result is standard. (See, e.g., [4, 37].)

**Proposition 2.3.** The problem (21) has a nonzero principal eigenvalue \( \lambda_1 = \lambda_1(h) \) if and only if \( h \) changes sign and \( \int_\Omega h \neq 0 \). More precisely, if \( h \) changes sign, then

(i) \( \int_\Omega h = 0 \iff 0 \) is the only principal eigenvalue.

(ii) \( \int_\Omega h > 0 \iff \lambda_1(h) < 0 \).

(iii) \( \int_\Omega h < 0 \iff \lambda_1(h) > 0 \).

(iv) \( \lambda_1(h_1) > \lambda_1(h_2) \) if \( h_1 \leq h_2, h_1 \neq h_2, \) and \( h_1, h_2 \) both change sign.

We will also use the following lemma derived from the theory of monotone dynamical systems. See, e.g., Proposition 9.1 and Theorem 9.2 in [21].

**Lemma 2.4.** For any \( d_1, d_2 > 0 \), assume that every co-existence steady state of (4), if exists, is asymptotically stable, then one of the following alternatives holds:

(i) There exists a unique co-existence steady state of (4) which is globally asymptotically stable.

(ii) System (4) has no co-existence steady state and either one of \((u_d,0)\) or \((0,v_d)\) is globally asymptotically stable, while the other one is unstable.

3. Global dynamics of general Lotka-Volterra competition systems. In this section, we state and prove our main results concerning the global dynamics of
the following general system:
\[
\begin{aligned}
U_t &= \nabla \cdot (a_1(x) \nabla U) + U(r_1(x) - b_1(x)U - c_1(x)V) \quad \text{in} \; \Omega \times \mathbb{R}^+, \\
V_t &= \nabla \cdot (a_2(x) \nabla V) + V(r_2(x) - b_2(x)U - c_2(x)V) \quad \text{in} \; \Omega \times \mathbb{R}^+,
\end{aligned}
\]
\[
\partial u = \partial V = 0 \quad \text{on} \; \partial \Omega \times \mathbb{R}^+,
\]
\[
U(x, 0) = U_0(x), \quad V(x, 0) = V_0(x) \quad \text{in} \; \Omega,
\]
which includes system (4) as a special case. Throughout the paper, we assume that $a_i \in C^{1,\alpha}(\bar{\Omega})$ ($\alpha \in (0, 1)$) is positive on $\bar{\Omega}$ for $i = 1, 2$; $r_i$, $b_i$ and $c_i \in C^\alpha(\bar{\Omega})$ are positive on $\bar{\Omega}$ for $i = 1, 2$.

By similar arguments as in the proofs of existence and uniqueness results in [8] (see also [21, Theorem 28.1]), we see that the following equations
\[
\begin{aligned}
\partial_t u &= \nabla \cdot (a_1(x) \nabla u) + u(r_1(x) - b_1(x)u) \quad \text{in} \; \Omega \times \mathbb{R}^+,
\partial_v u &= 0 \quad \text{on} \; \partial \Omega \times \mathbb{R}^+, \\
\partial_t v &= \nabla \cdot (a_2(x) \nabla v) + v(r_2(x) - b_2(x)v) \quad \text{in} \; \Omega \times \mathbb{R}^+,
\partial_v v &= 0 \quad \text{on} \; \partial \Omega \times \mathbb{R}^+,
\end{aligned}
\]
and
\[
\begin{aligned}
\partial_t u &= 0 \quad \text{on} \; \partial \Omega \times \mathbb{R}^+,
\partial_v v &= 0 \quad \text{on} \; \partial \Omega \times \mathbb{R}^+,
\end{aligned}
\]
have unique positive steady states, which we denote by $\bar{u}$ and $\bar{v}$ respectively. Moreover, $\bar{u}$ and $\bar{v}$ are globally asymptotically stable. Hence system (22) has two semi-trivial steady states $(\bar{u}, 0)$ and $(0, \bar{v})$.

Let $\bar{\mu}_1(a(x), q(x))$ be the principal eigenvalue of the following eigenvalue problem:
\[
\nabla \cdot (a(x) \nabla \phi) + q(x) \phi + \bar{\mu} \phi = 0 \quad \text{in} \; \Omega, \quad \partial_\nu \phi = 0 \quad \text{on} \; \partial \Omega,
\]
where $a \in C^{1,\alpha}(\Omega)$. Then $\bar{\mu}_1(a(x), q(x))$ admits the following variational characterization:
\[
\bar{\mu}_1(a, q) = \inf_{\psi \in H_1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (a(x) |\nabla \psi|^2 - q(x) \psi^2)}{\int_{\Omega} \psi^2}.
\]

By similar arguments to the proof of [27, Corollary 2.10], we can show that the linear stability of $(\bar{u}, 0)$ is determined by the sign of $\bar{\mu}_1(a_2, r_2 - b_2 \bar{u})$. In other words, $(\bar{u}, 0)$ is linearly stable (resp. unstable) if the principal eigenvalue $\bar{\mu}_1(a_2, r_2 - b_2 \bar{u}) > 0$ (resp. $\bar{\mu}_1(a_2, r_2 - b_2 \bar{u}) < 0$). The linear stability of $(0, \bar{v})$ can be determined accordingly by the sign of $\bar{\mu}_1(a_1, r_1 - c_1 \bar{v})$. The global dynamics of system (22) can be characterized as follows:

**Theorem 3.1.** Assume that either
\[
I(b_1, c_1, b_2, c_2) := \max_{\Omega} \frac{c_1}{b_1} \cdot \max_{\Omega} \frac{b_2}{c_2} \cdot \left( \frac{\max_{\Omega} \frac{b_1}{c_1}}{\min_{\Omega} \frac{b_1}{c_1}} \right)^{1/3} \leq 1
\]
or
\[
I'(b_1, c_1, b_2, c_2) := \max_{\Omega} \frac{c_1}{b_1} \cdot \max_{\Omega} \frac{b_2}{c_2} \cdot \left( \frac{\max_{\Omega} \frac{b_2}{c_2}}{\min_{\Omega} \frac{b_2}{c_2}} \right)^{1/3} \leq 1
\]
holds, then $\bar{\mu}_1(a_2, r_2 - b_2 \bar{u})$ and $\bar{\mu}_1(a_1, r_1 - c_1 \bar{v})$ cannot be both positive, i.e., exactly one of the following four alternatives holds:

(A) $\bar{\mu}_1(a_2, r_2 - b_2 \bar{u}) \geq 0$ and $\bar{\mu}_1(a_1, r_1 - c_1 \bar{v}) < 0$;
(B) $\bar{\mu}_1(a_2, r_2 - b_2 \bar{u}) < 0$ and $\bar{\mu}_1(a_1, r_1 - c_1 \bar{v}) \geq 0$;
(C) $\bar{\mu}_1(a_2, r_2 - b_2 \bar{u}) < 0$ and $\bar{\mu}_1(a_1, r_1 - c_1 \bar{v}) < 0$;
(D) $\bar{\mu}_1(a_2, r_2 - b_2 \bar{u}) = 0$ and $\bar{\mu}_1(a_1, r_1 - c_1 \bar{v}) = 0$. 

Moreover,

\[ \text{Case (D) holds} \iff b_1 \propto c_1 \propto b_2 \propto c_2, \quad \frac{c_1}{b_1} \cdot \frac{b_2}{c_2} = 1, \text{ and } \frac{\tilde{u}}{\tilde{v}} \equiv \frac{c_1}{b_1} \equiv \eta. \quad (29) \]

Furthermore, the following hold for system (22):

(i) If Case (A) holds, then \((\tilde{u}, 0)\) is globally asymptotically stable.

(ii) If Case (B) holds, then \((0, \tilde{v})\) is globally asymptotically stable.

(iii) If Case (C) holds, then there exists a unique coexistence steady state that is globally asymptotically stable.

(iv) If Case (D) holds, then there is a compact global attractor consisting of a continuum of steady states \(\{(\zeta \tilde{u}, (1 - \zeta)\tilde{u}/\eta) | \zeta \in [0, 1]\}\) connecting the two semi-trivial steady states.

**Proof.** In the following, we first prove the theorem assuming that (27) holds. At the end of the proof, we point out the slight differences during the proof when \(I(b_1, c_1, b_2, c_2)\) is replaced by \(I'(b_1, c_1, b_2, c_2)\), i.e., when condition (27) is replaced by (28).

Multiplying the equation for \(\tilde{u}\) by \(\tilde{u}\) and integrating over \(\Omega\), we obtain that

\[ \int_\Omega a_1 |\nabla \tilde{u}|^2 = \int_\Omega (r_1 - b_1 \tilde{u}) \tilde{u}^2. \quad (30) \]

Choosing \(\tilde{u}\) as a test function in the variational characterization for \(\bar{\mu}_1(a_1, r_1 - c_1 \tilde{v})\), by (26) and (30), we obtain that

\[ \bar{\mu}_1(a_1, r_1 - c_1 \tilde{v}) \leq \frac{\int_\Omega a_1 |\nabla \tilde{u}|^2 + \int_\Omega (c_1 \tilde{v} - r_1) \tilde{u}^2}{\int_\Omega \tilde{u}^2} \]

\[ = \frac{\int_\Omega (c_1 \tilde{v} - b_1 \tilde{u}) \tilde{u}^2}{\int_\Omega \tilde{u}^2}. \]

Therefore, by Hölder’s inequality, we see that

\[ \bar{\mu}_1(a_1, r_1 - c_1 \tilde{v}) \int_\Omega \tilde{u}^2 \leq \int_\Omega c_1 \tilde{v}^2 \tilde{u} - \int_\Omega b_1 \tilde{u}^3 \leq \max_\Omega \frac{c_1}{b_1} \cdot \int_\Omega b_1 \tilde{u}^2 \tilde{u} - \int_\Omega b_1 \tilde{u}^3 \]

\[ \leq \left( \int_\Omega b_1 \tilde{u}^3 \right)^{2/3} \cdot \left[ \max_\Omega \frac{c_1}{b_1} \cdot \left( \int_\Omega b_1 \tilde{v}^3 \right)^{1/3} - \left( \int_\Omega b_1 \tilde{u}^3 \right)^{1/3} \right] \]

\[ \leq \left( \int_\Omega b_1 \tilde{u}^3 \right)^{2/3} \cdot \left[ \max_\Omega \frac{c_1}{b_1} \cdot \left( \max_\Omega \frac{b_1}{c_2} \right)^{1/3} \cdot \left( \int_\Omega c_2 \tilde{v}^3 \right)^{1/3} - \left( \int_\Omega b_1 \tilde{u}^3 \right)^{1/3} \right]. \quad (31) \]

Similarly, choosing \(\tilde{v}\) as a test function in the variational characterization for \(\bar{\mu}_1(a_2, r_2 - b_2 \tilde{u})\), we obtain that

\[ \bar{\mu}_1(a_2, r_2 - b_2 \tilde{u}) \leq \frac{\int_\Omega a_2 |\nabla \tilde{v}|^2 + \int_\Omega (b_2 \tilde{u} - r_2) \tilde{v}^2}{\int_\Omega \tilde{v}^2} \]

\[ = \frac{\int_\Omega (b_2 \tilde{u} - c_2 \tilde{v}) \tilde{v}^2}{\int_\Omega \tilde{v}^2}. \]
Therefore, 
\[
\tilde{\mu}_1(a_2, r_2 - b_2 \bar{v}) \int_{\Omega} \bar{v}^2 \leq \int_{\Omega} b_2 \bar{v}^2 - \int_{\Omega} c_2 \bar{v}^3 \leq \max_{\Omega} \frac{b_2}{c_2} \int_{\Omega} c_2 \bar{v}^2 - \int_{\Omega} \bar{v}^3 \\
\leq \left( \int_{\Omega} c_2 \bar{v}^3 \right)^{2/3} \left[ \max_{\Omega} \frac{b_2}{c_2} \left( \max_{\Omega} \frac{c_2}{b_2} \right)^{1/3} \left( \int_{\Omega} b_1 \bar{v}^3 \right)^{1/3} - \left( \int_{\Omega} c_2 \bar{v}^3 \right)^{1/3} \right]. \tag{32}
\]

Denote
\[
\eta_1 = \max_{\Omega} \frac{c_1}{b_1} \cdot \left( \max_{\Omega} \frac{b_1}{c_2} \right)^{1/3} \quad \text{and} \quad \eta_2 = \max_{\Omega} \frac{b_2}{c_2} \cdot \left( \max_{\Omega} \frac{c_2}{b_1} \right)^{1/3}.
\]

Then \(\eta_1 \eta_2 = I(b_1, c_1, b_2, c_2) \leq 1\) by (27) and it follows from (31) and (32) that
\[
\frac{\eta_2 \tilde{\mu}_1(a_1, r_1 - c_1 \bar{v}) \int_{\Omega} \bar{u}^2}{\left( \int_{\Omega} b_1 \bar{u}^3 \right)^{2/3}} + \frac{\tilde{\mu}_1(a_2, r_2 - b_2 \bar{u}) \int_{\Omega} \bar{v}^2}{\left( \int_{\Omega} c_2 \bar{v}^3 \right)^{2/3}} \leq (I(b_1, c_1, b_2, c_2) - 1) \left( \int_{\Omega} c_2 \bar{v}^3 \right)^{1/3} \leq 0. \tag{33}
\]

Therefore, one of the four alternative cases (A)–(D) must hold, as \(\tilde{\mu}_1(a_2, r_2 - b_2 \bar{u}) > 0\) and \(\tilde{\mu}_1(a_1, r_1 - c_1 \bar{v}) > 0\) simultaneously would lead to a contradiction to (33).

Now assume that Case (D) holds, then it follows from (33) that \(I(b_1, c_1, b_2, c_2) = 1\) and all the above inequalities involved in the proof of (33) must be equalities. In other words, we obtain that
\[
b_1 \propto c_1 \propto b_2 \propto c_2, \quad \frac{c_1}{b_1} \cdot \frac{b_2}{c_2} = 1 \quad \text{and} \quad \frac{\bar{u}}{\bar{v}} \equiv \frac{c_1}{b_1} \equiv \eta.	ag{34}
\]

The above relations and the equations for \(\bar{u}\) and \(\bar{v}\) actually further imply that in Case (D), \(a_1, a_2, r_1\) and \(r_2\) must satisfy
\[
\nabla \cdot (a_1 \nabla \bar{u}) + r_1 \bar{u} \equiv \frac{c_1^2}{b_1 c_2}.
\]

On the other hand, assuming that (34) holds, we see from the equations for \(\bar{u}\) and \(\bar{v}\) that
\[
\nabla \cdot (a_1 \nabla \bar{v}) + r_1 \bar{v} - c_1 \bar{v} = \frac{1}{\eta} \left[ \nabla \cdot (a_1 \nabla \bar{u}) + \bar{u}(r_1 - b_1 \bar{u}) \right] + \frac{b_1}{\eta} \bar{u}^2 - \frac{c_1}{\eta^2} \bar{u}^2 = 0.
\]

Therefore, \(\tilde{\mu}_1(a_1, r_1 - c_1 \bar{v}) = 0\). Similarly, we can show that \(\tilde{\mu}_1(a_2, r_2 - b_2 \bar{u}) = 0\). Hence, Case (D) holds. This finishes the proof for (29).

We now claim that:

(S1) if one of Cases (A)–(C) holds, then any coexistence steady state of system (22), if it exists, is linearly stable;

(S2) if \(\tilde{\mu}_1(a_2, r_2 - b_2 \bar{u}) > 0 > \tilde{\mu}_1(a_1, r_1 - c_1 \bar{v})\) in Case (A) or \(\tilde{\mu}_1(a_2, r_2 - b_2 \bar{u}) < 0 = \tilde{\mu}_1(a_1, r_1 - c_1 \bar{v})\) in Case (B), system (22) has no coexistence steady state at all.

Then Theorem 3.1(i)–(iii) follow directly from the above two claims and Lemma 2.4.

We now prove Claim (S1). Let \((U, V)\) be any coexistence steady state of system (22), then \(U, V > 0\) on \(\Omega\). Linearizing the steady state problem of (22) at \((U, V)\),
we have
\[
\begin{align*}
\nabla \cdot (a_1(x)\nabla \Phi) + \Phi(r_1 - b_1 U - c_1 V) - U(b_1 \Phi + c_1 \Psi) + \lambda \Phi &= 0 \quad \text{in } \Omega, \\
\nabla \cdot (a_2(x)\nabla \Psi) + \Psi(r_2 - b_2 U - c_2 V) - V(b_2 \Phi + c_2 \Psi) + \lambda \Psi &= 0 \quad \text{in } \Omega, \\
\partial_r \Phi = \partial_r \Psi &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

Then by the Krein-Rutman theorem \([25, 38]\), (35) has a principal eigenvalue \(\lambda_1 \in \mathbb{R}\); i.e., \(\lambda_1\) is simple, real and has the least real part among all eigenvalues of (35). Moreover, the corresponding eigenfunction \((\Phi_1, \Psi_1)\) of \(\lambda_1\) can be chosen to satisfy \(\Phi_1 > 0 > \Psi_1\) on \(\Omega\). We normalize \((\Phi_1, \Psi_1)\) such that \(\|\Phi_1\|_{L^2(\Omega)}^2 + \|\Psi_1\|_{L^2(\Omega)}^2 = 1\).

By direct calculation, using the equations satisfied by \(U\) and \(\Phi_1\), we obtain that
\[
\nabla \cdot (a_1 U^2 \nabla \Phi_1) = U^2(b_1 \Phi_1 + c_1 \Psi_1) - \lambda_1 U \Phi_1.
\]

Multiplying both sides by \(\Phi_1^2/U^2\) and integrating over \(\Omega\), we have
\[
-2 \int_\Omega a_1 U \Phi_1 |\nabla \Phi_1/U|^2 = \int_\Omega \Phi_1^2(b_1 \Phi_1 + c_1 \Psi_1) - \lambda_1 \int_\Omega \Phi_1^3/U.
\]

Similarly, we can derive the following identity:
\[
-2 \int_\Omega a_2 V \Psi_1 |\nabla \Psi_1/V|^2 = \int_\Omega \Psi_1^2(b_2 \Phi_1 + c_2 \Psi_1) - \lambda_1 \int_\Omega \Psi_1^3/V.
\]

Denote \((\Phi_1, \Psi_1) := (\Phi_1, -\Psi_1)\), then we see that \(\Phi_1, \Psi_1 > 0\) on \(\Omega\). By similar arguments to the proof of (31) and (32), we can show that
\[
-\lambda_1 \int_\Omega \frac{\Phi_1^3}{U} \leq \int_\Omega c_1 \Phi_1^3 \Phi_1 - \int_\Omega b_1 \Phi_1^3 \leq \left(\int_\Omega b_1 \Phi_1^3\right)^{2/3} \cdot \left[\max_\Omega \frac{c_1}{b_1} \cdot \left(\max_\Omega \frac{b_1}{c_2}\right)^{1/3} \left(\int_\Omega c_2 \Phi_1^3\right)^{1/3}ight] - \left(\int_\Omega b_1 \Phi_1^3\right)^{1/3} \tag{36}
\]

and
\[
-\lambda_1 \int_\Omega \frac{\Psi_1^3}{V} \leq \int_\Omega b_2 \Phi_1 \Psi_1^2 - \int_\Omega c_2 \Psi_1^3 \leq \left(\int_\Omega c_2 \Phi_1^3\right)^{2/3} \cdot \left[\max_\Omega \frac{b_2}{c_2} \cdot \left(\max_\Omega \frac{c_2}{b_1}\right)^{1/3} \left(\int_\Omega b_1 \Phi_1^3\right)^{1/3}\right] - \left(\int_\Omega c_2 \Phi_1^3\right)^{1/3} \tag{37}
\]

Hence,
\[
-\lambda_1 \left[\frac{\int_\Omega \frac{\Phi_1^3}{U}}{\left(\int_\Omega b_1 \Phi_1^3\right)^{2/3}} + \frac{\int_\Omega \frac{\Psi_1^3}{V}}{\left(\int_\Omega c_2 \Phi_1^3\right)^{2/3}}\right] \leq (I(b_1, c_1, b_2, c_2) - 1) \left(\int_\Omega c_2 \Phi_1^3\right)^{1/3} \leq 0, \tag{38}
\]

which implies that \(\lambda_1 \geq 0\) and \(\lambda_1 = 0\) if and only if \(I(b_1, c_1, b_2, c_2) = 1\) and all inequalities involved in the proof of (38) become equalities. In other words, \(\lambda_1 = 0\) if and only if
\[
b_1 \propto c_1 \propto b_2 \propto c_2, \quad \Phi \propto U \propto \Psi \propto V, \quad \frac{c_1}{b_1} \cdot \frac{b_2}{c_2} = 1 \quad \text{and} \quad \frac{\Phi}{\Psi} \equiv \frac{c_1}{b_1} \equiv \eta. \tag{39}
\]
Moreover, when \( \lambda_1 = 0 \), denote \( \eta' = b_2/c_2 \) and \( \tilde{\eta} = U/V \), which are positive constants. Then \( U \) and \( V \) satisfy the following equations:

\[
\begin{align*}
\nabla \cdot (a_1 \nabla U) + U(r_1 - b_1 (1 + \eta/\tilde{\eta})U) &= 0 \quad \text{in } \Omega, \\
\nabla \cdot (a_2 \nabla V) + V(r_2 - c_2 (\eta' \tilde{\eta} + 1)V) &= 0 \quad \text{in } \Omega, \\
\partial_r U = \partial_r V &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

By uniqueness of positive steady state to (23) and (24), we obtain that \( (1 + \eta/\tilde{\eta})U = \tilde{u} \) and \( (\eta' \tilde{\eta} + 1)V = \tilde{v} \). Therefore,

\[
\tilde{u}/\tilde{v} \equiv c_1/b_1 = \eta \quad \text{and} \quad U + \eta V = \tilde{u},
\]

which implies by (29) that Case (D) holds. Therefore \( \lambda_1 > 0 \) and this finishes the proof for Claim (S1).

Next we prove Claim (S2). It actually follows from a similar perturbation arguments as in the proof of Theorem 3.4 in [18]. We briefly outline the main ideas here. Assume that \( \tilde{\mu}_1(a_2, r_2 - b_2 \tilde{u}) = 0 > \tilde{\mu}_1(a_1, r_1 - c_1 \tilde{v}) \) in Case (A) and assume for contradiction there exists a coexistence steady state \((U^*, V^*) \). We then replace \( b_2 \) and \( c_1 \) by \( b_2' \) and \( c_1' \) respectively such that \( b_2' \) and \( c_1' \) are in a sufficiently small neighborhood of \( b_2 \) and \( c_1 \) in \( C^0(\bar{\Omega}) \) satisfying that

\[
b_2' > b_2, \quad c_1' < c_1 \quad \text{and} \quad I(b_1, c_1', b_2', c_2) < 1.
\]

Then it is easy to show by the implicit function theorem that, the perturbed system of (22) with \( b_2 \) and \( c_1 \) replaced by \( b_2' \) and \( c_1' \) respectively, still has a coexistence steady state lying in a small neighborhood of \((U^*, V^*) \). On the other hand, we can show that \( \tilde{\mu}_1(a_2, r_2 - b_2' \tilde{u}) > 0 > \tilde{\mu}_1(a_1, r_1 - c_1' \tilde{v}) \). Then it follows from Claim (S1) and Lemma 2.4 that, \((\tilde{u}, 0)\) is globally asymptotically stable for the perturbed system of (22). However, this is a contradiction to the existence of the coexistence steady state that is close to \((U^*, V^*) \). This finishes the proof for Claim (S2).

It only remains to prove Theorem 3.1(iv). It is easy to see by (29) that \( \{(\zeta \tilde{u}, (1 - \zeta)\tilde{u}/\eta) \mid \zeta \in [0, 1] \} \) is a continuum of steady states of (22) connecting the two semi-trivial steady states, where \( \eta \equiv \tilde{u}/\tilde{v} \equiv c_1/b_1 \). On the other hand, let \((U, V)\) be a coexistence steady state of system (22). We first claim that \( U \propto V \). Indeed, assume that it is not true for contradiction, then (38) and (39) imply that \( \lambda_1 > 0 \). By a similar perturbation argument to the proof of Claim (S2), we arrive at a contradiction. Therefore, it must hold that \( U \propto V \). This immediately implies that \( (U, V) = (\zeta \tilde{u}, (1 - \zeta)\tilde{u}/\eta) \) for some \( \zeta \in (0, 1) \). Consequently, we conclude that the set of equilibria of (22) is

\[
\{(0, 0)\} \cup \{(\zeta \tilde{u}, (1 - \zeta)\tilde{u}/\eta) \mid \zeta \in [0, 1] \},
\]

with \((0, 0)\) being a repeller. Now by exactly the same arguments in the second part of the proof of Theorem 3 in [23], it follows that every solution of (22) converges to a single equilibrium \((\zeta \tilde{u}, (1 - \zeta)\tilde{u}/\eta) \) for some \( \zeta \in (0, 1) \). This finishes the proof of Theorem 3.1(iv).

Finally, we assume that \( I(b_1, c_1, b_2, c_2) \) is replaced by \( I'(b_1, c_1, b_2, c_2) \), i.e., condition (27) is replaced by (28), and point out the places which need to be modified.
during the above proof. The estimate in (31) can be replace by the following:
\[
\bar{\mu}_1(a_1, r_1 - c_1\tilde{v}) \int_\Omega \tilde{u}^2 \leq \int_\Omega c_1 \tilde{u}^2 \tilde{v} - \int_\Omega b_1 u^3 \\
\leq \left( \int_\Omega c_1 \tilde{u}^3 \right)^{2/3} \left( \int_\Omega c_1 \tilde{v}^3 \right)^{1/3} - \min_\Omega \frac{b_1}{c_1} \int_\Omega c_1 \tilde{u}^3 \\
\leq \left( \int_\Omega c_1 \tilde{u}^3 \right)^{2/3} \left[ \left( \int_\Omega c_1 \tilde{v}^3 \right)^{2/3} - \min_\Omega \frac{b_1}{c_1} \cdot \left( \int_\Omega c_1 \tilde{u}^3 \right)^{1/3} \right] \\
\leq \left( \int_\Omega c_1 \tilde{u}^3 \right)^{2/3} \cdot \min_\Omega \frac{b_1}{c_1} \left[ \max_\Omega \frac{c_1}{b_1} \cdot \left( \max_\Omega \frac{c_1}{b_2} \right)^4 \left( \int_\Omega b_2 \tilde{v}^3 \right)^{1/3} - \left( \int_\Omega c_1 \tilde{u}^3 \right)^{1/3} \right]. \quad (31')
\]
Similarly, (32) should be replaced by the following:
\[
\bar{\mu}_1(a_2, r_2 - b_2\tilde{u}) \int_\Omega \tilde{v}^2 \leq \int_\Omega b_2 \tilde{u}^2 \tilde{v} - \int_\Omega c_2 \tilde{v}^3 \\
\leq \left( \int_\Omega b_2 \tilde{v}^3 \right)^{2/3} \left( \int_\Omega b_2 \tilde{u}^3 \right)^{1/3} - \frac{c_2}{b_2} \int_\Omega b_2 \tilde{v}^3 \\
\leq \left( \int_\Omega b_2 \tilde{v}^3 \right)^{2/3} \cdot \min_\Omega \frac{c_2}{b_2} \left[ \max_\Omega \frac{b_2}{c_2} \cdot \left( \max_\Omega \frac{b_2}{c_1} \right)^4 \left( \int_\Omega c_2 \tilde{v}^3 \right)^{1/3} - \left( \int_\Omega b_2 \tilde{v}^3 \right)^{1/3} \right]. \quad (32')
\]
Denote
\[
\eta_1' = \max_\Omega \frac{c_1}{b_1} \cdot \left( \max_\Omega \frac{c_1}{b_2} \right)^{1/3} \quad \text{and} \quad \eta_2' = \max_\Omega \frac{b_2}{c_2} \cdot \left( \max_\Omega \frac{b_2}{c_1} \right)^{1/3}.
\]
Then \(\eta_1'\eta_2' = I'(b_1, c_1, b_2, c_2) \leq 1\) by (28) and it follows from (31') and (32') that
\[
\frac{\eta_2' \bar{\mu}_1(a_1, r_1 - c_1\tilde{v}) \int_\Omega \tilde{u}^2}{\left( \int_\Omega c_1 \tilde{u}^3 \right)^{2/3} \cdot \min_\Omega \frac{b_1}{c_1}} + \frac{\bar{\mu}_1(a_2, r_2 - b_2\tilde{u}) \int_\Omega \tilde{v}^2}{\left( \int_\Omega b_2 \tilde{v}^3 \right)^{2/3} \cdot \min_\Omega \frac{c_2}{b_2}} \\
\leq \left( I'(b_1, c_1, b_2, c_2) - 1 \right) \left( \int_\Omega b_2 \tilde{v}^3 \right)^{1/3} \leq 0.
\]
Then we can continue the above proof from (33).

A similar estimate to (38) with \(I(b_1, c_1, b_2, c_2)\) replaced by \(I'(b_1, c_1, b_2, c_2)\) can be obtained, by modifying inequalities (36) and (37) in a similar way as above. Hence, we omitted the details here. This completely finishes the proof of the theorem.

We now prove Theorem 1.2.

Proof of Theorem 1.2. Assume that
\[
a_1(x) \equiv d_1, \quad a_2(x) \equiv d_2, \\
b_1(x) = \xi_1(x), \quad c_1(x) = c \xi_1(x), \quad b_2(x) = b \xi_2(x) \quad \text{and} \quad c_2(x) = \xi_2(x),
\]
where \(d_1, d_2, b, c > 0\) are constants, \(\xi_1\) and \(\xi_2\) satisfy condition (A1). Then system (22) reduces to system (4) and
\[
I(b_1, c_1, b_2, c_2) = I'(b_1, c_1, b_2, c_2) = bc \left( \max_\Omega \frac{\xi_1}{\xi_2} / \min_\Omega \frac{\xi_1}{\xi_2} \right)^{1/3} \leq 1
\]
by condition (A2). Therefore Theorem 1.2 follows directly from Theorem 3.1.
In the end of this section, we characterize in details how the four disjoint components $\Sigma_U \cup \Sigma_{U,0} \setminus \Pi$, $\Sigma_V \cup \Sigma_{V,0} \setminus \Pi$, $\Sigma_-$ and $\Pi$ of $Q$ in (8) change in terms of $b$ and $c$, assuming that $\xi_1 \propto \xi_2$. For this purpose, we first establish the following lemma which determines the relative positions of the points $(L_U, L_V)$, $(L_U, S_V)$ and $(L_V, S_U)$ with respect to the line $bc = 1$ in the $bc$-plane, where $L_U, S_V, L_V$ and $S_U$ are defined in (9).

**Lemma 3.2.** Assume that (A1) holds, $\xi_1 \propto \xi_2$ and that at least one of $r_1/\xi_1$ and $r_2/\xi_2$ is nonconstant. Then the following hold:

(i) $L_U, L_V < 1$;
(ii) $L_U, S_V > 1$;
(iii) $L_V, S_U > 1$.

**Proof.** Multiplying the equation for $\theta_{d,\xi}$ by $\frac{1}{2d,\xi}$ and integrating over $\Omega$, we have obtain that

$$
\int_{\Omega} (r - \xi \theta_{d,\xi}) \, dx = -d \int_{\Omega} \frac{\|
abla \theta_{d,\xi}\|^2}{\theta_{d,\xi}^2} \, dx \leq 0,
$$

where the last inequality becomes equality if and only if $r/\xi \equiv \text{const}$. This implies that

$$
\int_{\Omega} \xi \theta_{d,\xi} \, dx > \int_{\Omega} r \, dx, \ \forall d > 0 \text{ and } r/\xi \not\equiv \text{const}.
$$

(41)

Hence, (i) follows directly from (41) and the fact $\xi_1 \propto \xi_2$. We now prove (ii). By Lemma 2.1 and $\xi_1 \propto \xi_2$, we see that

$$
L_U S_V = \inf_{d_2 > 0} \frac{r_2}{\xi_2 u_{d_2}} \sup_{d_1 > 0} \frac{r_1}{\xi_1 u_{d_1}} \geq \frac{\bar{r}_2}{\sup_{d_1 > 0} \xi_2 u_{d_1}} \cdot \lim_{d_2 \to \infty} \sup_{\Omega} \frac{r_1}{\xi_1 u_{d_2}} = \frac{\bar{r}_2}{\sup_{d_1 > 0} \xi_2 u_{d_1}} \cdot \frac{\xi_2}{\sup_{d_1 > 0} \xi_1 u_{d_1}} \cdot \sup_{\Omega} \frac{r_1}{\xi_1} = \frac{\xi_1}{\sup_{d_1 > 0} \xi_1 u_{d_1}} \cdot \sup_{\Omega} \frac{r_1}{\xi_1} \geq 1.
$$

Here, if $r_1/\xi_1$ is nonconstant, the last inequality is strict by Lemma 2.1 (iii); if $r_2/\xi_2$ is nonconstant, then the first inequality is strict by (41). This finishes the proof of (ii). The proof of (iii) is similar to (ii) and is thus omitted.

We now describe how the sets $\Sigma_U$ and $\Sigma_{U,0}$ (resp., $\Sigma_V$ and $\Sigma_{V,0}$) change in the $d_1,d_2$-plane when we vary $b$ (resp., $c$). To characterize the set $\Sigma_U$ in terms of $b > 0$, we define for each $b > 0$,

$$
I_b := \{d_1 > 0 \mid \int_{\Omega} (r_2 - b \xi_2 u_{d_1}) < 0\} = I_b^0 \cup I_b^1,
$$

where

$$
I_b^0 := \{d_1 > 0 \mid (r_2 - b \xi_2 u_{d_1}) \leq (\neq) 0 \text{ on } \bar{\Omega}\},
$$

$$
I_b^1 := \{d_1 \in I_b \mid \sup_{\Omega} (r_2 - b \xi_2 u_{d_1}) > 0\}.
$$

(43)

Note that $I_b$ is the union of finitely many open intervals, $I_b^0$ is closed, and $I_b^1$ is open in $\mathbb{R}^+$. Similarly, to characterize the set $\Sigma_V$ in terms of $c > 0$, we define for each $c > 0$,

$$
I_c := \{d_2 > 0 \mid \int_{\Omega} (r_1 - c \xi_1 u_{d_2}) < 0\} = I_c^0 \cup I_c^1,
$$

(44)
where
\[ I_c^0 := \{ d_2 > 0 | (r_1 - c \xi_1 v_{d_1}) \leq (\neq)0 \text{ on } \Omega \}, \]
\[ I_c^1 := \{ d_2 \in I_c | \text{sup}(r_1 - c \xi_1 v_{d_2}) > 0 \}. \] (45)

Note that \( I_c \) is the union of finitely many open intervals, \( I_c^0 \) is closed, and \( I_c^1 \) is open in \( \mathbb{R}^+ \).

We now characterize the sets \( \Sigma_U, \Sigma_{U,0}, \Sigma_V, \Sigma_{V,0}, \Sigma_- \) and \( \Pi \) in details.

**Theorem 3.3.** Assume that (A1) holds and \( \xi_1 \propto \xi_2 \). Then the following hold for (4):

(i) For \( \Sigma_U \), we have the following characterization:

\[ \Sigma_U = \begin{cases} \emptyset & \text{if } 0 < b \leq L_U, \\ \{(d_1, d_2) | d_1 \in I_b, d_2 > d_2^*(d_1)\} \subsetneq \mathcal{Q} & \text{if } L_U < b < S_U, \\ \mathcal{Q} & \text{if } b \geq S_U, \end{cases} \] (46)

where \( d_2^*(d_1) \) is defined in \( I_b \) as follows:

\[ d_2^*(d_1) = \begin{cases} 0 & \text{if } d_1 \in I_b^0, \\ \frac{1}{\lambda_1(r_2 - b \xi_2 u_{d_1})} & \text{if } d_1 \in I_b^1. \end{cases} \] (47)

Hence \( \Sigma_U \neq \emptyset \) if and only if \( b > L_U \) and \( \Sigma_U \) is strictly monotonically increasing in \( b \in (L_U, S_U) \).

(ii) For \( \Sigma_V \), we have the following characterization:

\[ \Sigma_V = \begin{cases} \emptyset & \text{if } 0 < c \leq L_V, \\ \{(d_1, d_2) | d_2 \in I_c, d_1 > d_1^*(d_2)\} \subsetneq \mathcal{Q} & \text{if } L_V < c < S_V, \\ \mathcal{Q} & \text{if } c \geq S_V, \end{cases} \] (48)

where \( d_1^*(d_2) \) is defined in \( I_c \) as follows:

\[ d_1^*(d_2) = \begin{cases} 0 & \text{if } d_2 \in I_c^0, \\ \frac{1}{\lambda_1(r_1 - c \xi_1 v_{d_2})} & \text{if } d_2 \in I_c^1. \end{cases} \] (49)

Hence \( \Sigma_V \neq \emptyset \) if and only if \( c > L_V \) and \( \Sigma_V \) is strictly monotonically increasing in \( c \in (L_V, S_V) \).

(iii) For \( \Sigma_{U,0} \), we have the following characterization:

\[ \Sigma_{U,0} = \begin{cases} \emptyset & \text{if } 0 < b < L_U \text{ or } b \geq S_U, \\ \{(d_1, d_2) | r_2 \equiv L_V \xi_2 u_{d_1}\} & \text{if } b = L_U, \\ \partial \Sigma_U \cup \{(d_1, d_2) | r_2 \equiv b \xi_2 u_{d_1}\} & \text{if } L_U < b < S_U, \end{cases} \] (50)

where for any \( s > 0 \), the set \( \{(d_1, d_2) | r_2 \equiv s \xi_2 u_{d_1}\} \) is either empty or equal to a single straight vertical line segment \( \{(d_1^*, d_2) | d_2 > 0\} \) where \( d_1^* \) is the unique \( d_1 \) such that \( r_2 \equiv s \xi_2 u_{d_1} \).

(iv) For \( \Sigma_{V,0} \), we have the following characterization:

\[ \Sigma_{V,0} = \begin{cases} \emptyset & \text{if } 0 < c < L_V \text{ or } c \geq S_V, \\ \{(d_1, d_2) | r_1 \equiv L_V \xi_1 v_{d_2}\} & \text{if } c = L_V, \\ \partial \Sigma_V \cup \{(d_1, d_2) | r_1 \equiv c \xi_1 v_{d_2}\} & \text{if } L_V < c < S_V, \end{cases} \] (51)
where for any \( s > 0 \), the set \( \{(d_1, d_2) \mid r_1 \equiv s \xi v_{d_2}\} \) is either empty or equal to a single straight vertical line segment \( \{(d_1, d_2') \mid d_1 > 0\} \) where \( d_2' \) is the unique \( d_2 \) such that \( r_1 \equiv s \xi v_{d_2} \).

Assume further that \((b, c) \in \Xi\), where \( \Xi \) is defined in (10), then the following hold:

(v) For \( \Sigma_- \), we have the following characterization:

\[
\Sigma_- = \begin{cases} 
\emptyset & \text{if } b \geq S_U \text{ or } c \geq S_V, \\
\{u \mid u \in \Omega \} & \text{if } b < L_U \text{ and } c < L_V, \\
\{u \mid u \in \Omega \} \setminus \Sigma_{U,0} & \text{if } b = L_U \text{ and } c < L_V, \\
\{u \mid u \in \Omega \} \setminus \Sigma_{V,0} & \text{if } b < L_U \text{ and } c = L_V, \\
\{u \mid u \in \Omega \} \setminus (\Sigma_{U,0} \cup \Sigma_{V,0}) & \text{if } b = L_U \text{ and } c = L_V, \\
\{u \mid u \in \Omega \} \setminus (\Sigma_{U} \cup \Sigma_{U,0} \cup \Sigma_{V,0}) & \text{if } L_U < b < S_U \text{ and } c \leq L_V, \\
\{u \mid u \in \Omega \} \setminus (\Sigma_{V} \cup \Sigma_{U,0} \cup \Sigma_{V,0}) & \text{if } b \leq L_U \text{ and } L_U < c < S_V, \\
\{u \mid u \in \Omega \} \setminus (\Sigma_{U} \cup \Sigma_{U,0} \cup \Sigma_{V} \cup \Sigma_{V,0}) & \text{if } L_U < b < S_U \text{ and } L_U < c < S_V, \\
\end{cases}
\]

and \( \Sigma_- = \emptyset \) if and only if either the first case holds or \( r_1/\xi_1 = cr_2/\xi_2 \) and \( bc = 1 \) in the last case.

(vi) For \( \Pi \), we have the following characterization:

\[
\Pi = \{(d_1, d_2) \mid u_{d_1} \equiv c v_{d_2} \text{ and } bc = 1\}.
\]

Hence, \( \Pi \neq \emptyset \) if and only if \( bc = 1 \) and exists \( (d_1, d_2) \in \Omega \) such that \( u_{d_1} \equiv c v_{d_2} \).

The proof for the above theorem follows from similar arguments as in Theorems 3.3 and 3.5 in [18] and thus is omitted.

Theorem 1.3 now follows from Lemma 3.2 and Theorem 3.3.

4. Proof of Theorems 1.4 and 1.5. We now prove Theorem 1.5. The following asymptotic expansion of \( \theta_{d,r,\xi} \) as \( d \to \infty \) which will be used in this section.

**Proposition 4.1.** Assume that \( r, \xi \in C^\alpha(\bar{\Omega}) \) \((\alpha \in (0, 1))\) are positive on \( \bar{\Omega} \). Then there exists a constant \( D_{r,\xi} > 0 \) depending only on \( r \) and \( \xi \) such that

\[
\theta_{d,r,\xi} = L + \frac{\rho_{r,\xi} + C_1(r, \xi)}{d} + \frac{\gamma_{r,\xi} + C_2(r, \xi)}{d^2} + O\left(\frac{1}{d^3}\right) \text{ for all } d > D_{r,\xi},
\]

where \( \rho_{r,\xi} \) and \( \gamma_{r,\xi} \) are the unique solution of

\[
\begin{align*}
\Delta \rho_{r,\xi} + L(r - \xi L) &= 0 \text{ in } \Omega, \\
\partial_\nu \rho_{r,\xi} &= 0 \text{ on } \partial \Omega, \\
\int_\Omega \xi \rho_{r,\xi} &= 0,
\end{align*}
\]

and

\[
\begin{align*}
\Delta \gamma_{r,\xi} + (r - 2\xi L)(\rho_{r,\xi} + C_1(r, \xi)) &= 0 \text{ in } \Omega, \\
\partial_\nu \gamma_{r,\xi} &= 0 \text{ on } \partial \Omega, \\
\int_\Omega \xi \gamma_{r,\xi} &= 0,
\end{align*}
\]

respectively,

\[
L := \frac{\int_\Omega r \, dx}{\int_\Omega \xi \, dx}, \quad C_1(r, \xi) := \frac{1}{L} \int_\Omega |\nabla \rho_{r,\xi}|^2 \, dx \quad \text{and} \quad C_2(r, \xi) := \frac{1}{L} \int_\Omega (r - 3\xi L)\rho_{r,\xi}^2 \, dx.
\]

For a proof of Proposition 4.1, see [14, Proposition 22]. Note that here, the splitting between the functions \( \rho_{r,\xi} \) and \( \gamma_{r,\xi} \) and the remaining constant terms are different with that in [14, Proposition 22], but the proof is the same.

We first establish Theorem 1.5.
Proof of Theorem 1.5. The proof for (17) and (18) follows from similar arguments as in the proof for Theorem 1.3 in [16] and Theorem 1.4 in [20] and hence is omitted.

We now prove (19). By Theorem 3.3(i), we obtain that

\[
\tilde{d}_2(d_1) = \begin{cases} 
0 & \text{if } r_2 - \xi u_{d_1} \leq 0, \\
\frac{1}{\lambda_1(r_2 - \xi u_{d_1})} & \text{otherwise},
\end{cases}
\]

where \(\lambda_1(h)\) is defined as the unique nonzero principal eigenvalue of (21). By Lemma 2.1 and (16), \(r_2 - \xi u_{d_1}\) changes sign in \(\Omega\) for all \(d_1\) large. Then by the definition of \(d_2(d_1)\), it suffices to show that there exist two constants \(\tilde{D} := \tilde{D}(r_1, r_2, \xi) > 0\) and \(B_1 := B_1(r_1, r_2, \xi)\) such that

\[
\frac{1}{\lambda_1(r_2 - \xi u_{d_1})} = \frac{C_1(r_2, \xi)}{C_1(r_1, \xi)} d_1 + B_1 + O\left(\frac{1}{d_1}\right) \text{ for all } d_1 > \tilde{D}.
\]

(56)

For convenience, we denote

\[
\lambda_1 := \lambda_1(r_2 - \xi u_{d_1})
\]

in the rest of this proof. First we claim that

\[
d_1 \lambda_1 - \frac{C_1(r_1, \xi)}{C_1(r_2, \xi)} = o(1), \text{ as } d_1 \to \infty.
\]

(57)

Let \(\varphi_1 > 0\) be the eigenfunction corresponding to \(\lambda_1\) normalized such that

\[
\|\varphi_1\|_{L^\infty(\Omega)} = \|\varphi_1\|_{L^\infty(\Omega)},
\]

(58)

where \(L = \tilde{r}_1/\tilde{\xi} = \tilde{r}_2/\tilde{\xi}\). By Lemma 2.1 and (16), \(\int_\Omega (r_2 - \xi u_{d_1}) \to 0\) as \(d_1 \to \infty\). Therefore by Proposition 2.3(i) and similar arguments as in the proof of Lemma 2.4 in [31], we can show that, \(\lambda_1 \to 0\) as \(d_1 \to \infty\). By standard elliptic regularity estimates, we deduce that, passing to a subsequence of \(d_1\) if necessary, \(\varphi_1\) converges to \(\tilde{\varphi}_1\) in \(W^{2,p}(\Omega) \cap C^1(\bar{\Omega})\) for some constant \(\tilde{\varphi}_1 \geq 0\). Therefore, it follows from (58) that \(\tilde{\varphi}_1 = L\). This implies that \(\varphi_1 = L + o(1)\) as \(d_1 \to \infty\). We now rewrite

\[
\varphi_1 = L + \lambda_1 \rho_{r_2, \xi} + \frac{\omega}{\tilde{d}_1^2}.
\]

By direct calculation, \(\omega\) satisfies the following equation:

\[
\begin{cases}
\Delta \omega + \lambda_1 \cdot (r_2 - \xi u_{d_1}) \omega + R = 0 & \text{in } \Omega, \\
\partial \nu \omega = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where

\[
R := (\lambda_1 d_1)^2 \rho_r \xi (r_2 - \xi u_{d_1}) + \lambda_1 d_1^2 \xi L (L - u_{d_1}).
\]

(59)

Multiplying the equation for \(\varphi_1\) by \(\omega\) and the equation for \(\omega\) by \(\varphi_1\), integrating over \(\Omega\) and subtracting, we obtain that

\[
\int_{\Omega} R \varphi_1 = 0.
\]

Dividing both sides of the above identity by \(\lambda_1 d_1\), by (53), (59) and the fact that \(\varphi_1 = L + o(1)\) on \(\Omega\), we derive that:

\[
0 = \lambda_1 d_1 \int_{\Omega} \rho_{r_2, \xi} (r_2 - \xi L + o(1))(L + o(1)) - \int_{\Omega} \xi L (r_2, \xi) + C_1(r_1, \xi) + o(1))(L + o(1))
\]

\[
= \lambda_1 d_1 (L \tilde{r}_2 |\Omega| C_1(r_2, \xi) + o(1)) - L \tilde{r}_2 |\Omega| C_1(r_1, \xi) + o(1),
\]

(60)
as $d_1 \to \infty$, where we used the identity
\[
\int_\Omega L(r_2 - \xi L)\rho_{r_2, \xi} = \int_\Omega |\nabla \rho_{r_2, \xi}|^2 = L|\Omega|r_2 C_1(r_2, \xi)
\]
(61)
obtained by multiplying the equation for $\rho_{r_2, \xi}$ by $\rho_{r_2, \xi}$ and integrating over $\Omega$. Therefore, (57) follows by letting $d_1 \to \infty$ in (60).
Define
\[
\psi_1 := \frac{L[\xi]}{\|\varphi_1\|_{L^1(\Omega)}} \varphi_1,
\]
i.e., $\psi_1 > 0$ is a renormalization of $\varphi_1$ such that $\|\psi_1\|_{L^1(\Omega)} = L[\xi]_{L^1(\Omega)}$. Rewrite
\[
\psi_1 = L + L\rho_{r_2, \xi} + \frac{\chi}{d_1^2}.
\]
Since $\int_\Omega \xi \rho_{r_2, \xi} = 0$, we see that $\int_\Omega \xi \chi = 0$. Hence, $\chi$ satisfies the following equation:
\[
\begin{cases}
\Delta \chi + \xi \cdot (r_2 - \xi u_{d_1}) \chi + R = 0 & \text{in } \Omega, \\
\partial_\nu \chi = 0 & \text{on } \partial \Omega, \\
\int_\Omega \xi \chi = 0.
\end{cases}
\]
Note that $\chi$ satisfies the same equation as $\omega$.
Let us denote by $[\psi_1]^\perp$ the orthogonal complement of $\psi_1$ in $L^2(\Omega)$. Since the principal eigenvalue of the operator $\Delta + \rho_{r_2, \xi}(r_2 - \xi u_{d_1})$ with zero-Nuemann boundary condition is $0$ with principal eigenfunction $\psi_1$, it follows from Fredholm Alternative that the inverse of $\Delta + \rho_{r_2, \xi}(r_2 - \xi u_{d_1})$ restricted to $[\psi_1]^\perp$ exists. Let us denote its inverse by $\mathcal{L}^{-1}_{d_1^2}$. Then it is obvious that $\|\mathcal{L}^{-1}_{d_1^2}||_\psi_1^\perp$ is bounded uniformly for all $d_1$ large. Decompose $\chi$ as
\[
\chi = c\psi_1 + \mathcal{L}^{-1}_{d_1^2}R,
\]
where $c$ is a constant such that $\int_\Omega \xi \chi = 0$. Since $R$ converges and $\lim_{d_1 \to \infty} \mathcal{L}^{-1}_{d_1^2} = \Delta^{-1}$ restricted to $\{f \in L^2(\Omega) \mid \int f = 0 \}$ as $\lambda_1 \to 0$, we deduce that $c$ converges $0$ as $d_1 \to \infty$. Therefore $\|\chi\|_{L^\infty(\Omega)}$ is bounded uniformly for all $d_1$ large. Hence,
\[
\psi_1 = L + \lambda_1 \rho_{r_2, \xi} + O\left(\frac{1}{d_1^2}\right) \text{ for all } d_1 > \bar{D},
\]
(62)
by choosing $\bar{D} = \bar{D}(r_1, r_2, \xi)$ sufficiently large. Since $\int_\Omega R \psi_1 = 0$, dividing both sides of this identity by $\lambda_1$ and plugging (59), we obtain that
\[
0 = \lambda_1 d_1^2 \int_\Omega \rho_{r_2, \xi}(r_2 - \xi u_{d_1}) \psi_1 + d_1^2 \int_\Omega \xi L(\xi u_{d_1}) \psi_1 =: I_1 + I_2.
\]
(63)
By (16), (53), (57), (61) and (62), we can estimate $I_1$ and $I_2$ as follows:
\[
I_1 = \lambda_1 d_1^2 \int_\Omega \rho_{r_2, \xi}(r_2 - \xi L - \frac{1}{d_1} \xi \rho_{r_1, \xi} + \xi C_1(r_1, \xi)) + O\left(\frac{1}{d_1^2}\right) \text{ for all } d_1 > \bar{D},
\]
and
\[
I_2 = -d_1 \int_\Omega L \xi \left[(\rho_{r_1, \xi} + C_1(r_1, \xi)) + \frac{1}{d_1} (\gamma_{r_1, \xi} + C_2(r_1, \xi)) + O\left(\frac{1}{d_1^2}\right)\right].
\]
for all \( d_1 > \hat{D} \). Therefore by the definition of \( C_2(r_1, \xi) \) in Proposition 4.1 and (63), we obtain that

\[
d_1 (L \bar{r}_2 C_1 (r_1, \xi) |\Omega) - \lambda_1 d_1 L \bar{r}_2 C_1 (r_2, \xi) |\Omega) = (\lambda_1 d_1)^2 L \int_\Omega \xi \rho_{r_2, \xi}^2 - 2 \lambda_1 d_1 L \int_\Omega \xi \rho_{r_1, \xi} \rho_{r_2, \xi} + L \int_\Omega \xi \rho_{r_1, \xi}^2 + (\lambda_1 d_1)^2 \int_\Omega \rho_{r_2, \xi}^2 (r_2 - 2 \xi L) - \int_\Omega \rho_{r_1, \xi}^2 (r_1 - 2 \xi L) + O\left(\frac{1}{d_1}\right),
\]

for all \( d_1 > \hat{D} \). Dividing both sides of the above identity by \( \lambda_1 d_1 L \bar{r}_2 |\Omega| \) and using (57), we see that

\[
d_1 C_1 (r_1, \xi) \left( \frac{1}{\lambda_1 d_1} - \frac{C_1 (r_2, \xi)}{C_1 (r_1, \xi)} \right) = \frac{1}{\bar{r}_2 |\Omega|} \int_\Omega \xi \left( \sqrt{\lambda_1 d_1} \rho_{r_2, \xi} - \frac{\rho_{r_1, \xi}}{\sqrt{\lambda_1 d_1}} \right)^2 + \frac{\lambda_1 d_1}{L \bar{r}_2 |\Omega|} \int_\Omega \rho_{r_2, \xi}^2 (r_2 - 2 \xi L) - \frac{1}{\lambda_1 d_1 L \bar{r}_2 |\Omega|} \int_\Omega \rho_{r_1, \xi}^2 (r_1 - 2 \xi L) + O\left(\frac{1}{d_1}\right),
\]

for all \( d_1 > \hat{D} \). Therefore by (57),

\[
\frac{1}{\lambda_1 d_1} - \frac{C_1 (r_2, \xi)}{C_1 (r_1, \xi)} = O\left(\frac{1}{d_1}\right),
\]

i.e.,

\[
\lambda_1 d_1 = \frac{C_1 (r_1, \xi)}{C_1 (r_2, \xi)} + O\left(\frac{1}{d_1}\right), \quad \forall d_1 > \hat{D}.
\]

Plugging them back to the right hand side of (64), we obtain that

\[
d_1 \left( \frac{1}{\lambda_1 d_1} - \frac{C_1 (r_2, \xi)}{C_1 (r_1, \xi)} \right) = B_1 (r_1, r_2, \xi) + O\left(\frac{1}{d_1}\right),
\]

for all \( d_1 > \hat{D} \), where

\[
B_1 (r_1, r_2, \xi) := \frac{1}{\bar{r}_2 |\Omega|} \int_\Omega \xi \left( \frac{\rho_{r_2, \xi}}{\sqrt{C_1 (r_2, \xi)}} - \frac{\sqrt{C_1 (r_2, \xi)}}{C_1 (r_1, \xi)} \rho_{r_1, \xi} \right)^2 + \frac{1}{L \bar{r}_2 |\Omega|} \int_\Omega \rho_{r_2, \xi}^2 (r_2 - 2 \xi L) - \frac{C_1 (r_2, \xi)}{C_1^2 (r_1, \xi)} \int_\Omega \rho_{r_1, \xi}^2 (r_1 - 2 \xi L). \quad (65)
\]

This finishes the proof of (56) and hence (19). The proof for (20) follows from similar arguments as in the proof for (19) with \( B_2 := B_2 (r_1, r_2, \xi) \) defined by

\[
B_2 (r_1, r_2, \xi) := \frac{1}{\bar{r}_2 |\Omega|} \int_\Omega \xi \left( \frac{\rho_{r_1, \xi}}{\sqrt{C_1 (r_1, \xi)}} - \frac{\sqrt{C_1 (r_1, \xi)}}{C_1 (r_2, \xi)} \rho_{r_2, \xi} \right)^2 + \frac{1}{L \bar{r}_2 |\Omega|} \int_\Omega \rho_{r_1, \xi}^2 (r_1 - 2 \xi L) - \frac{C_1 (r_1, \xi)}{C_1^2 (r_2, \xi)} \int_\Omega \rho_{r_2, \xi}^2 (r_2 - 2 \xi L). \quad (66)
\]

Hence

\[
B_1 + \frac{C_1 (r_2, \xi)}{C_1 (r_1, \xi)} B_2 = \frac{2}{\bar{r}_2 |\Omega|} \int_\Omega \xi \left( \frac{\rho_{r_2, \xi}}{\sqrt{C_1 (r_2, \xi)}} - \frac{\sqrt{C_1 (r_2, \xi)}}{C_1 (r_1, \xi)} \rho_{r_1, \xi} \right)^2 \geq 0.
\]
It is obvious that $B_1 + \frac{C_1(r_2, \xi)}{C_1(r_1, \xi)}B_2 = 0$ if and only if

$$\rho_{r_1, \xi} = \frac{C_1(r_1, \xi)}{C_1(r_2, \xi)}\rho_{r_2, \xi},$$

By Proposition 4.1 and (16), the above identity implies that

$$r_1 - \xi L = \frac{C_1(r_1, \xi)}{C_1(r_2, \xi)}(r_2 - \xi L) \quad \text{and} \quad \frac{C_1(r_1, \xi)}{C_1(r_2, \xi)} = \frac{C_1(r_1, \xi)}{C_1(r_2, \xi)}.$$ 

Hence, $C_1(r_1, \xi) = C_1(r_2, \xi)$ and $r_1 \equiv r_2$, which is a contradiction to (16). Therefore the distance between the two parallel lines $d_2 = \frac{C_1(r_2, \xi)}{C_1(r_1, \xi)}d_1 + B_1$ and $d_1 = \frac{C_1(r_1, \xi)}{C_1(r_2, \xi)}d_2 + B_2$ is

$$\frac{C_1(r_1, \xi)}{\sqrt{C_1^2(r_1, \xi) + C_1^2(r_2, \xi)}}(B_1 + \frac{C_1(r_2, \xi)}{C_1(r_1, \xi)}B_2),$$

which is strictly positive. Moreover, the curve $\tilde{d}_2(d_1)$ lies above the curve $\tilde{d}_1(d_2)$ when $d_1$ and $d_2$ are sufficiently large. This finishes the proof for (19) and (20).

At the end of this section, we give a more detailed characterization of the asymptotic behavior of $\hat{d}_2(d_1)$ in Theorem 1.4.

**Theorem 4.2.** Assume that the conditions in Theorem 1.4 hold. Then there exists a constant $\hat{D} > 0$ depending only on $r_1$ and $\xi$ such that the following hold for system (13):

(i) If $\inf_\Omega \rho_{r_1, \xi} + C_1(r_1, \xi) > 0$, then for all $d_1 > \hat{D}$,

$$r_2 - \xi u_{d_1} < 0$$

on $\Omega$ and $\tilde{d}_2(d_1) = 0$,

which implies that $\Sigma_U \cap \{(d_1, d_2) \mid d_1 > \hat{D}\} = \{(d_1, d_2) \mid d_1 > \hat{D}\}$.

(ii) If $\inf_\Omega \rho_{r_1, \xi} + C_1(r_1, \xi) < 0$, then for all $d_1 > \hat{D}$, $r_2 - \xi u_{d_1}$ changes sign in $\Omega$ and there exist two numbers

$$\hat{\lambda}_1 := \lambda_1(\rho_{r_1, \xi} - C_1(r_1, \xi)) > 0$$

and $\Gamma_{r_1, \xi} \in \mathbb{R}$ depending on $r_1$ and $\xi$ such that

$$\tilde{d}_2(d_1) = \frac{1}{d_1 \hat{\lambda}_1} + \frac{\Gamma_{r_1, \xi}}{d_1^2} + O\left(\frac{1}{d_1^3}\right),$$

which implies that $\Sigma_- \cap \{(d_1, d_2) \mid d_1 > \hat{D}\} = \{(d_1, d_2) \mid d_1 > \hat{D}\} \setminus \Sigma_U$ is nonempty.

(iii) If $\inf_\Omega \rho_{r_1, \xi} + C_1(r_1, \xi) = 0$, then

$$\tilde{d}_2(d_1) = O\left(\frac{1}{d_1}\right)$$

for all $d_1 > \hat{D}$.

The proofs of the Theorems 1.4 and 4.2 are similar to that of [16, Theorem 1.1] and [19, Theorem 1.3] and are thus omitted.
5. **Concluding remarks.** Finally, a few remarks are in order.

First, competition systems with variable competition coefficients, spatial and/or temporal, have been considered in many works; see [21, Chapter IV] and references therein. In the recent work by Bai and Li [3], global dynamics of system (22) can be clarified provided that

\[
\max_{\Omega} b_2 \cdot \max_{\Omega} c_1 \leq \min_{\Omega} b_1(x) \cdot \min_{\Omega} c_2(x). \tag{67}
\]

For the case of system (4), i.e., when

\[
b_1(x) = \xi_1(x), \quad c_1(x) = c \xi_1(x), \quad b_2(x) = b \xi_2(x) \quad \text{and} \quad c_2(x) = \xi_2(x), \tag{68}
\]

condition (67) reduces to

\[
bc \leq \frac{\min_{\Omega} \xi_1}{\max_{\Omega} \xi_1} \frac{\min_{\Omega} \xi_2}{\max_{\Omega} \xi_2}.
\]

Since

\[
\frac{\min_{\Omega} \xi_1}{\max_{\Omega} \xi_1} \frac{\min_{\Omega} \xi_2}{\max_{\Omega} \xi_2} \leq \left( \frac{\min_{\Omega} \xi_1 / \max_{\Omega} \xi_1}{\max_{\Omega} \xi_2} \right)^{\frac{1}{2}}
\]

when \(\xi_1\) or \(\xi_2\) is nonconstant, we see that condition (A2) is much better. For the general system (22), none of the conditions (67), (27) and (28) implies the other two, as examples can be easily constructed. This seems to indicate that those criteria are not sharp, and a more complete understanding of the global dynamics of the system is still open. If we restrict ourselves to conditions (67), (27) and (28) and assume that the four competition coefficients \(b_1, c_1, b_2\) and \(c_2\) are nonconstant, then roughly speaking, we have the following observation:

(i) if the pair of intraspecific competition coefficients \(b_1(x)\) and \(c_2(x)\) are very close/comparable on \(\bar{\Omega}\) up to constant scaling, then condition (27) may be better than the other two;

(ii) if the pair of interspecific competition coefficients \(b_2(x)\) and \(c_1(x)\) are very close/comparable on \(\bar{\Omega}\) up to constant scaling, then condition (28) may be better than the other two;

(iii) if \(\min_{\Omega} \{b_1, c_2\} \geq \max_{\Omega} \{b_2, c_1\}\), then condition (67) may be better than the other two.

Along with the study on global dynamics of Lotka-Volterra competition-diffusion systems, there have been a lot of work devoted to understanding the dynamics of Lotka-Volterra competition-diffusion-advection systems in heterogeneous environments, such as the following:

\[
\begin{align*}
U_i &= \nabla \cdot (d_i \nabla U - \alpha_i U \nabla P_i(x)) + U(r_1(x) - U - c V) \quad \text{in} \ \Omega \times \mathbb{R}^+, \\
V_i &= \nabla \cdot (d_2 \nabla V - \alpha_2 V \nabla P_2(x)) + V(r_2(x) - b U - V) \quad \text{in} \ \Omega \times \mathbb{R}^+, \\
(d_1 \nabla U - \alpha_1 U \nabla P_1) \cdot \nu &= (d_2 \nabla V - \alpha_2 V \nabla P_2) \cdot \nu = 0 \quad \text{on} \ \partial \Omega \times \mathbb{R}^+, \\
U(x, 0) &= U_0(x), \quad V(x, 0) = V_0(x) \quad \text{in} \ \Omega.
\end{align*}
\]

Here, \(\nabla\) is the gradient operator and \(\nabla \cdot\) is the divergence operator; \(r_i \in C^0(\bar{\Omega})\) are positive on \(\bar{\Omega}\) for \(i = 1, 2\); \(P_i \in C^2(\bar{\Omega})\); \(d_1, d_2 > 0\) are the random diffusion coefficients; \(\alpha_1, \alpha_2 > 0\) measure the tendencies of the biased movements of the species \(U\) and \(V\) along the environmental gradients \(P_1\) and \(P_2\) respectively. See [1], [2], [9], [10], [26], [28], [32], [33], [39], [43] and references therein.

As was pointed out in [33], the key ideas developed in [18] for Lotka-Volterra competition-diffusion systems do not work directly when advection terms are involved, since the diffusion-advection operator is usually not self-adjoint. However,
Moreover, holds:

exists. Denote diffusion systems in the form of (22), in which diffusion-advection systems can be rewritten into general Lotka-Volterra competition-through a standard transformation, the above form of Lotka-Volterra competition-

Then (W, Z) satisfies the following system

\[
\begin{align*}
\nabla \cdot (d_1 e^{\alpha_1 P_1} \nabla W) + W(e^{\alpha_1 P_1} r_1 - e^{\alpha_1 P_1} W - c e^{\alpha_1 P_1 + \alpha_2 P_2} Z) &= 0 & \text{in } \Omega \times \mathbb{R}^+, \\
\nabla \cdot (d_2 e^{\alpha_2 P_2} \nabla Z) + Z(e^{\alpha_2 P_2} r_2 - b e^{\alpha_1 P_1 + \alpha_2 P_2} W - e^{\alpha_2 P_2} P_2 Z) &= 0 & \text{in } \Omega \times \mathbb{R}^+, \\
\n\nabla W \cdot \nu = \nabla Z \cdot \nu &= 0 & \text{on } \partial \Omega \times \mathbb{R}^+.
\end{align*}
\]

Therefore, Theorem 3.1 can be applied to system (69) to clarify its global dynamics in terms of the local stability of the two semitrivial steady states of (69). Denote \( u^* \) and \( v^* \) the unique positive steady states to the following equations

\[
\begin{align*}
\begin{dcases}
\nabla \cdot (d_1 \nabla u - \alpha_1 u \nabla P_1(x)) + u(r_1(x) - u) & \text{in } \Omega \times \mathbb{R}^+, \\
(d_1 \nabla u - \alpha_1 u \nabla P_1) \cdot \nu &= 0 & \text{on } \partial \Omega \times \mathbb{R}^+,
\end{dcases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{dcases}
\nabla \cdot (d_2 \nabla v - \alpha_2 v \nabla P_2(x)) + v(r_2(x) - v) & \text{in } \Omega \times \mathbb{R}^+, \\
(d_2 \nabla v - \alpha_2 v \nabla P_2) \cdot \nu &= 0 & \text{on } \partial \Omega \times \mathbb{R}^+,
\end{dcases}
\end{align*}
\]

respectively. Then we have the following result:

**Corollary 5.1.** Assume that

\[
bc \leq \frac{\min_{\bar{\Omega}} e^{\alpha_1 P_1 - \frac{\alpha_2}{\alpha_2} P_2}}{\max_{\bar{\Omega}} e^{\alpha_1 P_1 - \frac{\alpha_2}{\alpha_2} P_2}},
\]

where the right hand side is computed from (28). Then \((u^*, 0)\) and \((0, v^*)\) cannot be linearly stable simultaneously, i.e., exactly one of the following four alternatives holds:

(A) \((u^*, 0)\) is linearly stable or neutrally stable and \((0, v^*)\) is linearly unstable;
(B) \((u^*, 0)\) is linearly unstable and \((0, v^*)\) is linearly stable or neutrally stable;
(C) both \((u^*, 0)\) and \((0, v^*)\) are linearly unstable;
(D) both \((u^*, 0)\) and \((0, v^*)\) are neutrally stable;

Moreover,

\[
\text{Case (D) holds } \iff \frac{\alpha_1}{d_1} P_1 - \frac{\alpha_2}{d_2} P_2 = \text{const}, \ bc = 1, \ \text{and} \ u^* v^* = c.
\]

Furthermore, the following hold for system (69):

(i) If Case (A) holds, then \((u^*, 0)\) is globally asymptotically stable.
(ii) If Case (B) holds, then \((0, v^*)\) is globally asymptotically stable.
(iii) If Case (C) holds, then there exists a unique co-existence steady state that is globally asymptotically stable.
(iv) If Case (D) holds, then there is a compact global attractor consisting of a continuum of steady states \(\{(\zeta u^*, (1 - \zeta)u^*/c) | \zeta \in [0, 1]\}\) connecting the two semi-trivial steady states.
It seems that our current understanding of this general heterogeneous Lotka-Volterra competition is fairly good already — at least for system (3) or (4). Given the recent empirical evidence as well as the theoretical models presented in [41] and [40], it seems that a much more realistic model would involve resource dynamics as well, and perhaps that could become our next direction. In that case, our major tasks should consist of: First, to continue the work started in [15] and obtain a complete understanding of systems for single species with resource dynamics, and then move onto consumer-resource competition systems.

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