Instabilities in Chains Coupled by Two-Body Interactions

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Abstract

We derive a general set of Poor Man’s scaling equations and analyze the stability of the Luttinger state in a system composed of a finite number N of one dimensional spinless fermionic chains, coupled through a general two body interaction. The effect of processes with momentum transfer parallel to the Fermi surface in destroying massless states is investigated. It will be shown that there are two processes competing: one in which two electrons exchange chains and the other in which they jump into a same chain. When periodic boundary conditions in the transverse direction are taken into account this competition leads always to massive states (except in hyperplanes of the phase diagram), a well known example being the generalized sine-Gordon model. If instead open boundary conditions are taken, massless states are possible but due to this competition the system is placed near instabilities. We argue that this kind of analysis has relevance for understanding the instabilities of 2D fermionic systems.

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The many unexplained features of high-Tc superconductors have motivated an intense study of low dimensional interacting fermions[1]. A renormalization group (RG) approach then developed allowed a reformulation of the Fermi liquid concept[2]. However this analysis seems to rely heavily on the isotropic nature of the Fermi surface (FS). Indeed, for a two-dimensional (2D) circular FS one can neglect, to some extent, the effect of scattering processes with momentum transfer tangential to the Fermi surface. This is because the FS topology highly restricts contributions to the zero sound channel by a factor of $(\Lambda/k_F)^2$ (where $\Lambda$ is the momentum cutoff used in the RG procedure), whereas the Cooper channel receives contributions proportional to $\Lambda/k_F$. Thus we can neglect the zero sound contributions to the one loop renormalization of the four fermion vertex and then we are left with a simple differential equation that diverges only if attractive interactions are present, indicating the appearance of a superconducting instability. However for a FS with large flat portions (nested), like in the case of a square FS, this picture may not apply. Then the scattering processes with momentum transfers parallel to the FS and involving electrons on parallel sides of the FS, give supplementary contributions and may play an important role in destabilizing massless regimes. To understand better the effect of these contributions, we suggest, in a first reasonable approximation, the study of a model of 1D chains coupled through a general two body interaction.
It is worth noting that this kind of model has quite a large range of applications, for it can be related (even if approximately) to 1D spin chains through the Jordan-Wigner transformation and to systems of coupled Hubbard chains[3]. Even though, these are particular cases of our model since for these models the chains are coupled through density-density interactions, and the densities are operators defined on each chain. Consequently the number of electrons on each chain is conserved. By contrast, this is not a requirement for our model, which has this new degree of complexity. In particular, some commonly used techniques, like 1D bosonization, find inconsistencies if one takes account of the fermionic nature of the field operators correctly, through the Haldane ladder operators[4, 5, 6]. For this reason we derive a general set of RG differential equations based on the Poor Man’s Scaling (PMS) method, thus avoiding the use of the bosonization tools whenever they lead to incongruities.

Consider an array of N spinless fermionic chains, aligned along x. As in the 1D case we separate the field operator in slow and fast components, \( \psi(x) = 1/\sqrt{N} \sum_{p=\pm} \sum_{n=1}^N \exp(ipk_n \cdot x)\psi_{p,n}(x) \), where \( pk_n \) is the momentum Fermi surface point for the chain \( n \) and branch \( p \). We take all \((k_n)_x \) equal to \( k_F \), a linearized dispersion relation near the FS, and neglect any Fermi velocity modulation. A general density-density interaction can be written as:

\[
    \mathcal{H}_G = \frac{1}{NL} \sum_{i,j,\delta} \sum_q \sum_{k,k'} G_\delta(i - \frac{\delta}{2}, j + \frac{\delta}{2}) R^+_{i+\delta/2,k+q} L^+_{j-\delta/2,k'-q} L^+_{j+\delta/2,k'} R^+_{i-\delta/2,k} \tag{1}
\]

where here \( L^+_{j,k} \) is the creation operator for a left mover on chain \( j \) and with momentum \( k \) along the chain. If we take periodic boundary conditions (PBC) along \( y \), then the couplings depend only on \( j - i \equiv \Delta \): \( G_\delta(I, J) \rightarrow G_\delta(\Delta) \), with \( J = j + \delta/2, I = i - \delta/2 \). From [1] we can see that due to the inversion symmetry (i\( \leftrightarrow \)j), hermiticity and boundary conditions, these couplings must verify the conditions:

\[
    G_\delta(I, J) = G_{-\delta}(J, I) = G_{-\delta}(I + \delta, J - \delta) = G_{-\delta}(N + 1 - I, N + 1 - J) \quad \text{(no PBC)}
\]

\[
    G_\delta(\Delta) = G_{-\delta}(\Delta) = G_\delta(-\Delta) = G_{-\delta}(-\Delta) = G_{\Delta \pm N}(\Delta) = G_\delta(\Delta \pm N) \quad \text{(PBC)}
\]

Note that the chain index can also be regarded as a spin index. In this way we can establish a dictionary between the usual g-ology couplings[3] and the G’s, for a model with 2 coupled chains:

\[
    G_0(1)/2 \equiv g_{2\perp} \quad G_1(0)/2 \equiv -g_{1\perp} \quad G_0(0)/2 \equiv g_{2\parallel} - g_{1\parallel} \quad G_1(1)/2 \equiv 0 \tag{3}
\]

Notice that according to the formulation [3], the backscattering processes are now seen as forward scatterings with spin-flips, and thus within each chain there are no backscattering (see figure 1a ). This formulation will allow us to obtain a compact set of RG equations, which is particularly useful in treating the N-chain problem.

The PMS equations are derived by requiring invariance of the vertex calculated in a one loop expansion. There are only two diagrams giving logarithmic contributions, both with the same magnitude \( \rho_0/2 \left[ \ln(\omega/D) - i\pi/2 \right] \) but having opposite signs. Here \( \rho_0 \) is the density of states for one spin direction. For convenience we will redefine now from the couplings as the adimensional quantities \( G_\delta(\Delta) \rightarrow G_\delta(\Delta)/\pi v_F \). The RG flow equations for models with and without PBC are then:

\[
    \frac{\partial G_\delta(\Delta)}{\partial \ln D} = \frac{1}{2N} \sum_{\alpha=0}^{N-1} \left[ G_\alpha(\Delta + \delta - \alpha)G_{\delta-\alpha}(\Delta - \alpha) - G_{\delta-\alpha}(\Delta)G_\alpha(\Delta) \right] \quad \text{(PBC)} \tag{4}
\]
\[
\frac{\partial G_\delta(I,J)}{\partial \ln D} = \frac{1}{2N} \sum_{\alpha = -N+1}^{N-1} [G_\alpha(I,J)G_{\delta-\alpha}(I+\alpha,J-\alpha) - G_{\delta-\alpha}(I+\alpha,J)G_\alpha(I,J-\delta+\alpha)](5)
\]

where in (5) the boundary conditions impose that only couplings with \(I, I + \delta, J\) and \(J - \delta\) between 1 and \(N\) are non-zero. The reader may verify that all the RG equations found in (4) are reproduced by (5), via the dictionary (3). The generalized flow invariant is \(\sum_\Delta G_0(\Delta)\). It is important to call attention to the compactness of the formula (4). Note that if one takes \(\delta\) and \(\Delta\) as vectors instead of scalars, then the couplings can be properly chosen to obtain the RG equations for models of coupled chains of electrons with spin.

Let us analyze first systems with PBC. For \(N=2\), bosonisation of the model gives [8]:

\[
H = H_{TL} + H_{GSG},
\]

where \(H_{TL}\) is the Tomonaga-Luttinger hamiltonian, and \(H_{GSG} = 1/(2\pi\alpha)^2 \int dx \left[ -G_1(0) \cos \left( \sqrt{8} \phi_+ \right) + G_1(1) \cos \left( \sqrt{8} \theta_- \right) \right]\). The model consists of the sine-Gordon model, plus a cosine term in the dual field. For this reason it is called the generalized sine-Gordon model (GSG). The GSG model is always massive because it involves dual operators, which thus have inverse scaling dimensions. There can only be gapless states in hyperplanes of the phase diagram. A generalization to the \(N\)-chain problem leads to the same conclusion. A simple way of seeing this uses a linearization of the equation (4), around the possible fixed points. Writing \(G_\delta(\Delta) = G_\delta^*(\Delta) + g_\delta(\Delta)\) and linearizing around the Luttinger Liquid fixed point, we obtain simply:

\[
\partial g_\delta(\Delta) = 1/2N \ (G_0^*(\Delta + \delta) + G_0^*(\Delta - \delta) - 2G_0^*(\Delta)) \ g_\delta(\Delta)\] (6)

We realize that we can always have two symmetrical eigenvalues if \(N\) is even: for \(\Delta = \delta = N/2\), \(\lambda = 1/2N \ [G_0(0) - G_0(N/2)]\) whereas for \(\Delta = 0, \delta = N/2, \lambda = 1/2N \ [G_0(N/2) - G_0(0)]\). This corresponds to a competition between two quite different kinds of processes: in the first case two electrons jump within the same chain, whereas in the second they exchange chains. Thus we conclude that this competition has a dramatic effect in destabilizing gapless states in a system where PBC are taken. An example of systems behaving in this way are the systems of chains coupled by transverse hopping terms [6].

Next we investigate the case where no PBC are considered which is the model with more practical relevance. Here the boundary conditions do not allow processes like \(G_{N/2}(N/2)\) and so we may think that this kind of competition is not present. We show next that this competition is indeed softened, but that it is still present although in a slightly different fashion. Consider a system of four coupled chains (no PBC). A simple argument states that if the only non-zero couplings are \(G_1(2,2), G_1(2,3), G_0(2,2)\) and \(G_0(2,3)\) (and all the other couplings that by symmetry are equal to these), then apart from multiplicative factors, the RG equations have the same structure as those of the GSG model. Intuitively this arises because we can look at the first chain as the image of the third and the fourth as the image of the second, like if we had a model with PBC. Thus we showed that in this region of the phase diagram a four chain model with open boundary conditions has the same properties as the GSG model. This subspace is stable and doesn’t really tell us much about the effect of the omitted couplings. However this provides us with a picture for a kind of competition present between couplings of the form \(G_\delta(I,I + \delta)\) and \(G_\delta(I,I)\).

Consider a model with three spinless chains. We rename the couplings as shown in figure...
1b). Then the RG equations look like: (the factor 1/N is omitted)

\[
\begin{align*}
\partial(A + C + D) &= \partial(B + 2C) = 0 \\
\partial(D + A) &= (E^2 - F^2) \\
\partial G &= E^2 - F^2 + 2G(D - A) \\
\partial E &= E(B - 2C + G + D) \\
\partial F &= -F(B - 2C + G + A)
\end{align*}
\]

Linearizing around the various fixed points we can verify that the only stable fixed points are of Luttinger type (E=F=G=0) and lie in the region A < 2C − B < D. This seems to hold true for an arbitrary number of chains as the number of conditions for the eigenvalues of \(T\) is proportional to \(N^3\) (the order of the number of different \(G_{\delta \neq 0}(I, J)\)), whereas the number of relations we can have among the \(G_{\delta = 0}(I, J)\) couplings goes as \(N^4\). Thus finding it possible to have a weak coupling theory for \(N=2\) and \(3\), it looks likely that a weak coupling theory exists for any \(N\).

Another point worth noticing is how the E and F couplings do in fact compete. They flow independently of their sign and almost in opposite directions. On the other hand, the \(\delta = 0\) processes depend only on the relative magnitude between the E and F processes, which is a clear manifestation of their competition. For positive bare couplings the system can reach two kinds of strong coupling regimes. If \(F \to +\infty\) and \(E \to 0\), then A,B,G \(\to +\infty\), D,C \(\to -\infty\) (Regime I). If \(E \to +\infty\) and \(F \to 0\), then G,D,B \(\to -\infty\), A,C \(\to +\infty\) (Regime II).

We are interested in knowing whether what we found for a system of 3 chains provides us with a picture for systems with a higher number of chains. Here we argue that this can indeed be the case for a large class of systems. We look at systems close to a generalized Luttinger liquid state and for strictly repulsive interactions. We want to study the influence of small momenta transfer processes (tangential to the FS) in destabilizing the massless Luttinger state. For this reason we analyze bare coupling functions of the general form \(G_{\delta}(I, J) = g(|\delta|) f(J - I - \delta)\).

The function \(g\) controls the dependence of the couplings on the momentum transfer. The forward scattering models used in studies of 2D bosonization correspond to the case where \(g(|\delta|)\) is a delta function at \(\delta = 0\). It’s easy to show that \(G\) respects all the necessary symmetries, provided \(f\) is an even function. Here we will present some numerical results. An analytical study of the equation \(\text{(7)}\) for this class of bare coupling functions can also be achieved but this is left for future publication.

In the following numerical results we considered systems of five chains. We take \(g(|\delta|) = \exp\left(-\delta^2/2\sigma\right)\). The form of \(g\) doesn’t seem to play an important role so long as it is even and peaked only at \(\delta = 0\). The control parameter \(\sigma\) may be seen as anisotropy dependent: we can expect a curved FS to favor small values of \(\sigma\). For the function \(f\) we take \(f(x) = a_0 \left(1 + b_0 \left(x/N\right)^2\right)\), like in a Taylor expansion. The coefficients \(a_0\) and \(b_0\) are chosen so that \(f\) remains positive, as we are interested in studying the effect of repulsive interactions. In principle it may look physically more reasonable to take \(f\) as a decreasing function. However one should also remember that \(f\) is already a renormalized function where other effects like those coming from phonons and anisotropy are included.

The results can be summarized as follows. When \(f\) is monotonous and decreasing then no matter how small \(\sigma\) is, we fall in a regime where all chain exchange processes diverge to \(+\infty\) and the jump within the same chain processes decrease (figure 2a ). This corresponds to a generalization of the regime I (\(F \to +\infty\)), in the three chains model. The competition between the two classes of couplings is obvious. Then we also clearly observe that all \(G_0(n,m)\), with \(n \neq m\), diverge to \(-\infty\) whereas all the diagonal elements \(G_0(n,n)\), diverge to \(+\infty\). When \(f\) is monotonous and increasing we can have several regimes. Like in the three chain model a
The Luttinger state is possible for small enough $\sigma$. By increasing $\sigma$ a massive regime appears similar to regime II ($E \to +\infty$) of the three chain model (figure 2b). Now some processes with jumps within the same chain may also diverge to $-\infty$. The remaining couplings behave in the same way as in the three chain system. The competition between each pair of couplings $G_\delta (I, I + \delta)$ and $G_\delta (I, I)$, is still present but works in different ways and magnitudes for each $I$ and $\delta$. If $\sigma$ is further increased then we recover regime I, where the jumps within the same chain processes strongly decrease (figure 2c). In the three chain model it is easy to see why this happens: by increasing $\sigma$ we increase the coupling $G$, which is a chain exchange process. Thus this class of processes is favoured and manages to overcome the jump within the same chain processes. This analysis seems to apply for any finite $N$. We would like to stress that the three chain model seems to reproduce much of the physics of a more general $N$ chain model.

Finally, we consider a 2D electron gas, where flat portions are introduced in a circular FS (figure 3). We take $N$ patches in the nested regions, and $M$ in the curved portions, with $N/M \ll 1$. Due to the geometry of the FS, it is more appropriate to parametrize the various patches in terms of their angular positions. This is achieved through the substitution $G_\delta (I, J) \to \tilde{G}_\delta (I, N + M + 1 - J)$. The symmetries on the square restrict the number of independent couplings: $0 \leq \delta \leq N + M, 1 \leq I, J \leq N + M$. Using Shankar’s arguments$[2]$, we neglect any zero sound contributions to a vertex involving electrons on the curved portions. This corresponds to introducing a factor $\delta_{I\in\phi}\delta_{I+\delta\in\phi}$ on the second term of (5) ($\phi$ stands for the nested region). Also, the only couplings with electrons on these regions have the BCS form $\tilde{G}_\delta (I, I)$. With these modifications, the equation (5) is still valid, if we regard the couplings as already renormalized by the finite 2D density of states. We linearized the RG equations around the Luttinger fixed point for a system with $N=3, M=0$ and concluded that the Luttinger regime remains stable if a further condition on the couplings $B, D > 0$, is verified. The other strong coupling regimes do also exist. If the couplings involving scatterings not restricted to a nested region are sufficiently strong, then the regime II leads to a new regime, where the $E$ coupling goes to $-\infty$ (figure 3). This state seems to be a good candidate for a superconductive state. In fact, in this regime $F \to 0$, so that the main corrections to the vertices come from the Cooper channel, and are due to effective attractive interactions. This will be studied in detail elsewhere$[4]$.

We should point that the possibility of generating ordered phases due to nesting is not a new concept: it remains the main explanation for itinerant antiferromagnetism, it has been suggested to explain the HTCSC, and it may explain the stability of some CDW phases recently observed$[9]$.

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Figure 1. a) One-dimensional interacting electrons with spin can be seen as two spinless chains in interaction. The backscattering process (dashed arrows), where two electrons change from one side of the FS to the other, is now seen as a spin-flip process (solid arrows). In b) all scattering processes involved in the three chain model with no PBC are shown.
Figure 2. Strong coupling regimes: a) \( f(x) = 0.4 \left(1 - \frac{1}{2} \left(\frac{x}{N}\right)^2\right) \) and \( \sigma = 0.5 \); b) \( f(x) = 0.4 \left(1 + \left(\frac{x}{N}\right)^2\right) \) and \( \sigma = 0.5 \); c) same \( f(x) \) but \( \sigma = 2.5 \)
Figure 3. For a system with $N=3$, $M=14$, bare couplings $A=B=C=D=0.1$, $E=F=0.08$, $G=0.02$ and the remaining couplings equal to 0.12, the Fourier modes $V_l$ of $\sum_I \tilde{G}_0 (I, I)$ diverge to $-\infty$, whereas $F \to 0$. 