Optimization of Tree Modes for Parallel Hash Functions

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Abstract. This paper focuses on parallel hash functions based on tree modes of operation for a rate-1 hash (or compression) function. We discuss the various forms of optimality that can be obtained when designing such parallel hash functions. The first result is a scheme which optimizes the tree topology in order to decrease at best the running time. Then, without affecting the optimal running time we show that we can slightly change the corresponding tree topology so as to decrease at best the number of required processors as well. Consequently, the resulting scheme optimizes in the first place the running time and in the second place the number of required processors. The present work is of independent interest if we consider the problem of parallelizing the evaluation of an expression where the operator used is neither associative nor commutative.

Keywords. Hash functions, Hash tree, Merkle tree, Parallel algorithms

1 Introduction

A hash function is an algorithm (or a mode of operation in the cryptographic terminology) iterating (operating) a function having a fixed input size (e.g., a compression function or a block cipher) in order to process messages of arbitrary lengths. Such a function must satisfy the usual properties of pre-image resistance (given a digest value, it is hard to find any pre-image producing this digest value), second pre-image resistance (given a message \( m_1 \), it is hard to find a second message \( m_2 \) which produces the same digest value), and collision resistance (it is hard to find two distinct messages which produce the same digest value). A sequential (or serial) hash function can only use Instruction-Level Parallelism (ILP) and SIMD instructions \cite{11,12}. A cryptographic hash function has numerous applications, the main one is its use in a signature algorithm to compress a message before signing it.

The most well known sequential hashing mode is the Merkle-Damgård \cite{8,16} construction which can only take advantage of the fine-grained parallelism of the operated compression function. If such a low-level ”primitive” can benefit from the Instruction-Level Parallelism, by using also SIMD instructions, the outer algorithm iterating this building block could benefit from a coarse-grained parallelism. This parallelism can
be employed in multithreaded implementations. Let suppose that we have a collision-free hash (or compression) function taking as input a fixed-size data, \( f : \{0, 1\}^{2N} \rightarrow \{0, 1\}^N \). By using a balanced binary tree structure, Merkle and Damgård \([8,15]\) show that we can extend the domain of this function so that the new outer function, denoted \( H : \{0, 1\}^* \rightarrow \{0, 1\}^N \), has an arbitrary sized domain and is still collision-resistant.

A construction using a balanced binary tree allows simultaneous processing of multiple parts of data at a same level of the tree, reducing the running time to hash the message from \( \mathcal{O}(n) \) to \( \mathcal{O}(\log n) \) if we have \( \mathcal{O}(n) \) processors \([8,15]\). If we want to further reduce the amount of resources involved, we can use one of the following rescheduling techniques:

– Each processor is assigned the processing of a subtree (in the data structure sense) having \( \log n \) leaves. There are approximatively \( n/\log n \) such subtrees. The processing of the remaining ancestor nodes, at each remaining level of the tree, is distributed as fairly as possible between the processors. An example is depicted in Figure 1.

![Fig. 1. Example of the computation of the root node in \( \mathcal{O}(\log n) \) time using \( \mathcal{O}(n/\log n) \) processors.](image)

An alternative solution is, at each level of the tree, to distribute as fairly as possible the node computations among \( \mathcal{O}(n/\log n) \) processors.

The number of processors is then reduced by a factor \( \log n \) and the asymptotic running time is conserved (with, nevertheless, a multiplicative factor 2). In this paper we are not interested in tradeoffs between the amount of used resources and the running time but instead we study optimal algorithms in finite distance. More precisely, we determine the hash tree structures which give the best concrete (parallel) time complexity for finite message lengths.

A tree structure is notably used in parallel hashing modes of Skein \([9]\), Blake2 \([3]\) or MD6 \([18]\). To give some examples, Skein uses a tree whose topology is controlled by the user thanks to three parameters: the arity of base level nodes which is a power of two; the arity of other inner nodes, which is also a power of two, and a last parameter limiting the height of the tree. MD6 uses a full (but not necessarily perfect) quaternary
tree, in the sense that an inner node has always four children. Some fictive leaves or
nodes padded with 0 are added so that a rightmost node has the correct number of
children. Like Skein, MD6 offers a parameter which serves to limit the height of the
tree.

Some proposals \[19,20,17\] consider that a tree covering all the message blocks is
not a good thing, because the number of processors should not grow with the size of
the message. For instance, the domain extension parallel algorithm from Sarkar et al.
\[19,20,17\] uses a perfect binary tree of processors, of fixed size. This perfect binary tree
of compression/hash function calls can be seen as a big compression function, sequen-
tially iterated over large parts of the message. In other words only the nodes computa-
tions performed in the tree can be done in parallel. The number of usable processors
is a system parameter chosen by the issuer of the cryptographic form when hashing
the message. The value of this parameter has to be reused by the recipients, for in-
stance when verifying a signature. Thus, this one limits the scalability and the potential
speedup. In this paper we consider that the scalability and the potential speedup should
be independent of the characteristics (configuration) of the transmitting computer.

Bertoni \[5,7\] give sufficient conditions for a tree based hash function to ensure
its indifferentiability from a random oracle. They propose several tree hashing modes
for different usages. For example we can make use of a tree of height 2, defined in
the following way: we divide the message in as many parts (of roughly equal size) as
there are processors so that each processor hashes each part, and then the concatenation
of all the results is sequentially hashed by one processor. To divide the message in
parts of roughly equal size, the algorithm needs to know in advance the size of the
message. Bertoni \\[5\] use an idea from Gueron \[10\] to propose a variant which still
makes use of a tree having two levels and a fixed number of processors, but this one
interleaves the blocks of the message. This interleaving has several advantages. It allows
an efficient parallel hashing of a streamed message, a roughly equal distribution of the
data processed by each processor in the first level of tree (without prior knowledge of
the message size), and finally a correct alignment of the data in the processors’ registers.
This kind of solution is suitable for multithreaded and SIMD implementations. In this
paper we study theoretically optimal speedups, and, as a consequence, the message to
hash is supposed to be already available.

Our concern in this paper is with hash tree modes using a single-block-length and
rate-1 hash (or compression) function, \textit{i.e.} a hash (or compression) function that needs
\(l\) invocations of the underlying primitive to process a message of \(l\) blocks, where a
block and the hash output have the same size. The aim of this work is to show that
we can improve the performance of a hash tree mode of operation by reworking
the tree structured circuit topology. In particular, we are interested in minimizing the depth
(parallel time) of the circuit and the width (number of processors involved). This kind
of work has been done for parallel exponentiation \[21,22,13,23\]. To the best of our
knowledge, it is the first time that the problem of optimizing hash trees is addressed. The
main interest of this paper is the methodology provided. The results are the followings:

\begin{itemize}
  \item The first result is an algorithm which optimizes the tree topology in order to
decrease at best the depth. We first show that a node arity greater than 5 is not possi-
\end{itemize}
ble and then we prove that we can construct such an optimal tree using exclusively levels of arity 2 and 3.

- Without affecting this depth, we show that we can change the corresponding tree topology in order to decrease at best the width. This width is optimal for trees having all their leaves at the same level. In particular, we show that for some message lengths $l$, the width can be decreased to $\lceil l/5 \rceil$.

- Observations are made about trees having their leaves at different levels, indicating that if our previous algorithm does not produce optimal solutions for this kind of trees, it probably produces near-optimal solutions.

- For trees having all their leaves at the same level, we also provide an algorithm which optimizes the number of processors at each step of the hash computation.

- Finally, we show that these optimisations could be applied safely by using only 4 different compression functions having the same running time.

Suppose that the processing of one block of the message by the compression function costs one unit of time. A binary tree is not necessarily the structure which gives the best running time. Figure 2 shows two different tree topologies for hashing a 6-block message. The binary tree depicted in (2a) gives a (parallel) running time of 6 units while the rightmost one with a different arity at each level, depicted in (2b), gives a running time of 5 units. Furthermore, one may note that for messages of length less that 5 blocks, the use of the topology (2a) has no utility compared to a purely sequential mode (i.e. a completely degenerated binary tree).

(a) Non optimal tree  
(b) Optimal tree

**Fig. 2.** Tree hashing with a 6-block message. The hash tree on the left requires 2 units of time to process each level, while the one on the right requires 3 units of time to process the base level and 2 units of time to process the root node.

In what follows, we suppose the use of variable input length (VIL) compression functions having a domain space $\{0, 1\}^xN$ for $x \geq 2$ and a fixed length range space $\{0, 1\}^N$. We also assume that such a function has a computational cost of $x$ units when compressing $x$ blocks of size $N$ bits. In other words, if we consider a tree of calls of this compression function, the computation of a node having $k$ children (i.e. $k$ blocks) has a cost of $k$ units. For instance, the UBI compression function, used in the hash function family Skein [9], performs $x$ calls to the tweakable block cipher Threefish to compress a data of length $x$ blocks. Assuming a hash tree of height $h$ and $x_i$ the arity of level $i$ (for $i = 1 \ldots h$), we define the parallel running time to obtain the root node value as being $\sum_{i=1}^{h} x_i$. 

![Tree diagram](image-url)
The paper is organized in the following way. In Section 2 we give background information and definitions. In Section 3 we describe the approach to reduce at best the running time of a hash function. Then, in Section 4, we give an algorithm to construct a hash tree topology which achieves the same optimal running time while requiring a near-optimal number of processors. We also show how we can use this algorithm to optimize the number of processors at each step of the hash computation. Finally, in Section 5, we conclude the paper and discuss future works.

2 Preliminaries

2.1 Tree structures

Throughout this paper (except when referring to security aspects) we use the convention\(^3\) that a node is the result of a function called on a data composed of the node’s children. A node value then corresponds to an image by such a function and a child of this node can be either an other image or a message block. We call a base level node a node at level 1 pointing to the leaves representing message data blocks. The leaves (or leaf nodes) are then at level 0. We define the arity of a level in the tree as being the greatest node arity in this level.

A \(k\)-ary tree is a tree where the nodes are of arity at most \(k\). For instance a tree with only one node of arity \(k\) is said to be a \(k\)-ary tree. A full \(k\)-ary tree is a tree where all nodes have exactly \(k\) children. A perfect \(k\)-ary tree is a full \(k\)-ary tree where all leaves have the same depth.

We also define other “refined” types of tree. We say that a tree is of arities \(\{k_1, k_2, \ldots, k_n\}\) (we can call it a \(\{k_1, k_2, \ldots, k_n\}\)-aries tree) if it has \(n\) levels (not counting level 0) whose nodes at the first level are of arity at most \(k_1\), nodes at level 2 are of arity at most \(k_2\), and so on. We say that such a tree is full if all nodes at the first level have exactly \(k_1\) children, all nodes at level 2 have exactly \(k_2\) children, and so on. As before, we say that such a tree is perfect if it is full and if all the leaves are at the same depth.

2.2 Security of tree-based hash functions

In this section (and in Subsection 4.2), we use the convention of \[\text{5}\] that a node value is a \(f\)-input, assuming that a single inner function \(f\) is operated in the outer hash function, denoted \(H\). As a result, such a tree of nodes (\(f\)-inputs) has one less level compared to a tree using the usual convention.

A \(f\)-input is a finite sequence of bits among the followings: message bits, chaining value bits (i.e. bits coming from a \(f\)-image), and frame bits (bits which are fully determined by the hash algorithm and the message size). In a tree of \(f\)-inputs, there are pointers from children to their corresponding parent. When a chaining value is present in a formatted \(f\)-input, it is pointed by another \(f\)-input which is considered as its children. Each \(f\)-input has an associated index which locates it in the tree. In addition, we

\(^3\) This corresponds to the convention used to describe Merkle trees. The other (less frequent) convention is to define a node as being a \(f\)-input.
need to define, for a tree $T$ of $f$-inputs, its corresponding tree template $Z$ which has the same topology, where the corresponding $f$-inputs have the same lengths and the frame bits match the corresponding bits in $T$, but where message and chaining value bits are not valuated. Thus, a tree template is fully determined by the message size and the parameters of the tree mode algorithm. This tree template is used by the tree hash mode to instantiate the tree of $f$-inputs, by valuating progressively message bits and chaining value bits.

A tree $T$ of $f$-inputs is said to be compliant with a tree hash mode $\tau$ if the latter can produce a tree of $f$-inputs whose corresponding tree template is compatible with it (its topology, the frame bits and the sizes of its $f$-inputs match those of $T$). A tree $T$ of $f$-inputs is said to be final-subtree-compliant with $\tau$ if the latter can produce a tree of $f$-inputs whose proper subtree (i.e. containing at least the root $f$-input) has a corresponding tree template with which $T$ is compatible.

Bertoni et al. [5,6] give some guidelines to design correctly a tree hash mode $\tau$ operating an inner hash (or compression) function $f$. They define three sufficient conditions which ensure that the constructed hash function $\tau_f$, which makes use of an ideal hash (or compression) function $f$, is indifferentiable from an ideal hash function. Besides, they propose to use particular frame bits in order to meet these conditions. We refer to [5,6] for the detailed definitions, and we give here a short description for each of them:

- **message-completeness**: Suppose we have a tree of $f$-inputs produced by the tree hash mode. There is an algorithm $A_{\text{message}}$ which, among the bits in the tree, uniquely determines the message. This requires that each message bit is processed at least once by $f$. The message can be reconstructed correctly if, given the sequence of bits of a $f$-input, we can identify those which are message bits, and we are able to say what are their positions in the message. Generally, only the end of the message is problematic. To cope with this, dedicated frame bits can be used such as a reversible padding$^4$ for the message or a coding of the message length. The running time of $A_{\text{message}}$ should be linear in the total number of bits in the tree.

- **tree-decodability**: Intuitively, given a tree $T$ of $f$-inputs generated by $\tau$, it is impossible to extract a final proper subtree $T'$ of $T$ which could have been generated legitimately by $\tau$. In other words, given such a subset of $f$-inputs, we are able to say whether there is a missing input. More formally, this property is satisfied by $\tau$ if there are no trees of $f$-inputs which are both compliant and final-subtree-compliant with it, and there is a decoding algorithm $A_{\text{decode}}$ that can parse the tree progressively on subtrees, starting from the root $f$-input, to retrieve frame bits, chaining value bits and message bits unambiguously. Also, when terminating, $A_{\text{decode}}$ must

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$^4$ Since a hash function processes an entire number of blocks (whose the size $N$ depends on the underlying primitive), a reversible padding is a good way of revealing the end of the message. This consists in applying to the message $M$, whatever is its length, a function $\text{pad}$ which returns a bit-string of length a multiple of $N$. Such a padding has to be reversible in the sense that there must exist a function $\text{unpad}$ such that $\text{unpad}(\text{pad}(M)) = M$ for all messages $M$. A well-known technique consists to append the bit "1" to the end of the message, followed by the minimum number of bits "0", so that the total bit-length of the padded message is a multiple of $N$. 
decide if the tree is compliant, final-subtree-compliant, or incompliant with \( \tau \). The running time of \( A_{\text{decode}} \) should be linear in the total number of bits in the tree.

- **Final-node separability:** Whatever the tree \( T \) of \( f \)-inputs generated by \( \tau \), we can distinguish between a root \( f \)-input and any other \( f \)-input. Such a property is useful to prevent length extension attacks. One straightforward way to fulfil this property is by means of domain separation between this final (root) \( f \)-input and other \( f \)-inputs, for instance by augmenting them with a frame bit identifying them as such.

These conditions ensure that no weaknesses are introduced on top of the risk of collisions in the inner function. For instance, with tree-decodability, an inner collision in the tree is impossible without a collision for the inner function. Andreeva et al. have shown in [2] that a hash function indistinguishable from a random oracle satisfies the usual security notions, up to a certain degree, such as pre-image and second pre-image resistance, collision resistance and multicollisions resistance.

3 Optimization of hash trees for parallel computing

3.1 Minimizing the running time

In order to optimize the running time of a tree mode, we make a certain degree of flexibility on the choices of node arities. We can note straightforward that allowing different node arities in a same level of the tree provides no efficiency gains. Worse, the running time may be less interesting since a tree level processing running time is bounded by the running time to process the node having the highest arity. This observation suggests that, in order to hope a reduction of the tree processing running time, node arities at the same level need to be set to the same value while allowing arities to vary from one level to another. Therefore our strategy allows a different arity at each level of the tree.

Let us denote \( l \) the block-length of a message. The problem is to find a tree height \( h \) and integer arities \( x_1, x_2, \ldots, x_h \) such that \( \sum_{i=1}^{h} x_i \) is minimized. Any solution to the problem must necessarily satisfy the following constraints:

\[
\prod_{i=1}^{h} x_i \geq l \quad \text{and} \quad \left( \prod_{i=1}^{h} x_i \right) / x_j < l \quad \forall j \in \{1, h\}.
\] (1)

A solution to this problem is a multiset of arities. First, we show that, in a non-asymptotic setting, a perfect ternary tree comes closer to optimality than a perfect binary tree. Then we examine the case of trees having different arities at each level.

First of all, we can start by considering the \( h \) and \( x_i \) (for \( i = 1, \ldots, h \)) as real numbers. Thus, we have to minimize the summation of \( x_i \) subject to the constraint that their product is \( l \). We know that the minimum is reached when the \( x_i \) are equal to the same number, which we will denote \( x \). So we have \( x^h = l \), that is \( x = l^{1/h} \). We must now determine \( h \) so that \( hl^{h \times} \) is minimized. The calculation of a derivative shows that this minimum is reached for \( h = \ln(l) \), which implies \( x = e \). Consequently, we can wonder what is the best solution between a perfect binary tree and a perfect ternary tree. The

\[ ^5 \text{Except maybe the rightmost node which may be of smaller arity} \]
comparison of these two cases is done in Appendix A and shows that beyond a certain message size \( l = 2^{28} \), a perfect ternary tree gives a better running time than a perfect binary tree. In fact, as the present general study shows, a tree having different level arities can give better results.

Let us remind that node arities are not allowed to vary in a same level (same stage) of the tree. A level of the tree is said to be of arity \( a \) when all nodes at this level are of arity at most \( a \). Given an optimal tree (in the sense of the running time) for hashing, we can ask what the possible arities are for its levels. We have the following Theorem:

**Theorem 1.** For a hash tree whose running time is optimal, the followings hold:

- It can be comprised of levels of arity 2, 3, 4, or 5. Higher arities are not possible.
- It can be constructed using only levels of arity 2 and 3.

**Proof.** We first show that levels of arity \( a \) with \( a \geq 7 \) lead to trees having a suboptimal running time. Indeed, any node of arity \( a \geq 7 \) can be replaced by a tree of arity 2 having a better running time. We simply have to note that \( 2\lceil \log_2 a \rceil < a \) for all \( a > 6 \), meaning that a \( a \)-ary tree of height 1 can be advantageously replaced by a binary tree of height \( \lceil \log_2 a \rceil \). In contrast, for all nodes of arity \( a \) with \( a \in [3, 6] \) and for all \( i \in [2, 5] \) we have \( i \lceil \log_2 a \rceil \geq a \). Finally, a node of arity 6 can be replaced by a \( \{3, 2\} \)-aries tree, since \( 2 \cdot 3 = 6 \), thereby reducing the running time to 2 + 3 = 5 units. As regards the second assertion, a node of arity 5 can be replaced by a tree of arities \( \{3, 2\} \), since \( 2 \cdot 3 = 6 > 5 \). This transformation does not change the running time since 2 + 3 = 5. Finally, a node of arity 4 can be replaced by a binary tree of height 2 for a running time which is still unchanged.

An optimal tree has not necessarily a single topology. Firstly, a solution satisfying constraints [1] can be defined as a multiset of arities since we can permute them. For instance, suppose a tree has three levels with the first level of arity 3, the second one of arity 2 and the last one (that is, the root node) of arity 3. We can permute these arities so that the first level is of arity 2 and the latter two levels of arity 3. If this new tree has the same running time, its topology has however changed. Secondly, we can find examples where different multiset of arities lead to trees of optimal running time. For instance, if we consider a 7-block message, the multisets of arities \( \{2, 2, 2\} \), \( \{3, 3\} \) and \( \{4, 2\} \) allow the construction of trees having the optimal running time. We can, however, construct optimal trees by restricting the set of possible arities. We have the following theorem:

**Theorem 2.** Let a message of length \( l \) blocks and let \( i \) be the lowest integer such that \( 3^i \geq l \). Let us note \( x \in [0, 2] \) the value which minimizes the product \( 3^i x 2^x \) under the constraint \( 3^i x 2^x \geq 1 \). There exists an optimal tree (in the sense of optimal running time) which has \( i - x \) levels of arity 3 and \( x \) levels of arity 2. More precisely, we can state the followings:

- If \( l \leq 3^i < \frac{3^i}{2} \), then a ternary hash tree is optimal for a running time of \( 3^i \).
- If \( \frac{3^i}{4} \leq 3^i < \frac{3^i}{2} \), then an optimal hash tree has \( i - 1 \) levels of arity 3 and one level of arity 2, for a running time of \( 2 + 3(i - 1) \).
- Otherwise \( \frac{3^i}{4} \leq 3^i < 3l \), and then an optimal hash tree has \( i - 2 \) levels of arity 3 and 2 levels of arity 2, for a running time of \( 4 + 3(i - 2) \).
Such an optimal tree maximizes the number of levels of arity 3.

**Proof.** If we have at least 3 levels of arity 2 then we can replace these 3 levels by 2 levels of arity 3 ($3^2 = 9 > 2^3 = 8$). The running time to process 3 levels of arity 2 or 2 levels of arity 3 is 6. Therefore, it is always possible to construct optimal trees with at most 2 levels of arity 2. The three assertions follow immediately. \(\square\)

**Remark 1.** Let \(i\) be such that \(3^i \geq l\). Its is not difficult to see that the sought solution corresponds to the highest value \(x \in [0, 2]\) such that \(3^{i-x}2^x \geq l\).

**Algorithm 1.** To determine the levels arities of an optimal tree, we first compute \(i = \lceil \log l / \log 3 \rceil\) and then \(x = \lfloor \log(l/3^i) / \log(2/3) \rfloor\). The \(i-x\) first levels are of arity 3 and the last \(x\) levels of arity 2.

**Examples.** For messages of lengths \(l = 4, 5\) and 10 blocks respectively, Algorithm 1 returns the multisets of arities \(\{2, 2\}\), \(\{3, 2\}\) and \(\{3, 2, 2\}\) respectively. The number of processors is not optimized here. This aspect is addressed in the following section.

Why is it possible to minimize the running time with a tree whose leaves are at the same depth? Let us suppose that we have, for a given message length, an optimal tree whose leaves are not at the same depth. Then, for each leaf located at a level greater than zero, we can create descendants in order to complete the tree so that all leaves are at level 0. It is possible to perform this while keeping a tree of same height and respecting the level arities. The result is a tree whose the number of leaves is greater than the message length (the tree is said to be perfect since, on the one hand, nodes at a same level are all of same arity, and, on the other hand, all the leaves are at the same depth). It is possible to prune some right branches to remove this surplus of leaves. Consequently, there exists necessarily a tree having the same height, the same multiset of arities and a lower number of leaves corresponding to the message length. In the rest of the paper, we refer to a truncated \((x_1, x_2, \ldots, x_h)\)-aries tree to speak about a tree having a number of leaves equal to the message length and where the nodes of the base level are of arity at most \(x_1\), nodes at the second level are of arity at most \(x_2\) and so on.

As a last remark, since the hash function must be deterministic, the multiset of arities must also be chosen deterministically as a function of the message size. For instance, we can arrange in descending order the elements of the multiset of arities. The solution to the problem of minimizing the running time is then uniquely determined as an ordered multiset.

**Performance improvements.** We have seen that for a message of 6 blocks (see Figure (2)), the performance gain of an optimal tree compared to a binary tree is 20%. Figure (3a) shows the running times of an optimal tree and a binary tree as functions of the message size varying from 1 to \(10^5\) blocks. Figure (3b) shows the speed gain obtained with an optimal tree. The gain in time (or speedup gain) is computed as \(100(T_b / T_o - 1)\) where \(T_b\) is the running time of a binary tree and \(T_o\) the running time of an optimal tree. As we can see, the gain differs from one message size to another. The gain can be greater than 30% for very short messages but decreases quickly, to cap at 10%. As regards the message size, although the diagram does not cover a sufficiently long range, one can note a slight downward slope.
3.2 Minimizing the number of processors

In this section we look into how to reduce at best the number of required processors to obtain the optimal running time. We have two cases to study, the trees having all leaves at the same depth and the others. We fully treat the first case and we make a few observations regarding the second type of tree, which we intuitively sense to further reduce the number of required processors.

At the outset, one may be interested in the maximum possible number of levels of arity 5 or 4. We have the following Lemma:

**Lemma 1.** In a tree having an optimal running time there can at most be 1 level of arity 5 and 6 levels of arity 4.

**Proof.** Suppose that the tree has 2 levels of arity 5. We replace these 2 levels by 3 levels of arity 3 since $3^3 = 27 > 5^2 = 25$. The running time is improved since $3 \cdot 3 = 9 < 2 \cdot 5 = 10$. We can then state that 2 levels of arity 5 lead to a tree having a sub-optimal running time. Now, let us look for a pair of minimum integers $(i, j)$ satisfying $3^i > 4^j$ and $3 \cdot i < 4 \cdot j$. The first pair which satisfies these constraints is $i = 9$ and $j = 7$. We can then replace 7 levels of arity 4 by 9 levels of arity 3 in order to decrease the running time. \hfill \qed

**Trees having all leaves at the same level.** We have seen that it is possible to construct a tree optimizing the running time by using only levels of arity 2 and 3. In what follows, we show how to deduce an optimal tree minimizing the number of involved processors. Let us suppose that level arities $x_1, x_2, \ldots, x_h$ are noted in (no strictly) decreasing order
so that $x_1$ is the arity of the base level and $x_h$ the arity of the last level, i.e. the arity of the root node. The trees optimizing the running time, defined above, are not necessarily full in the sense that a rightmost node at a given level can be of arity strictly lower than the arity of this level. First, we note that for the trees constructed with Algorithm 1, the number of required processors is equal to $\lceil l/3 \rceil$ in the best case, and equal to $\lceil l/2 \rceil$ when there are only levels of arity 2. Moreover, according to Theorem 1, we know that a level arity cannot be greater than 5. This means that in the best case, after optimization, the number of required processors could be reduced to $\lceil l/5 \rceil$. Thus, we could in the best case decrease the number of processors by a factor of about $5/2$.

Given an optimal tree for the running time, the intent is to increase the arity of the first level (base level) while decreasing arities of the following levels so that the sum of the levels arities remains constant and their product remains greater than or equal to $l$. To solve this problem we propose in Appendix B two solutions (Algorithm 2a or 2b). However, as will be discussed below, we can further optimize hash trees.

According to Theorem 1, a level arity of a tree minimizing the running time cannot exceed 5. Thus, Algorithm 2a (or Algorithm 2b) of Appendix B allows us to substitute any sub-multiset $A$ for another one, denoted $A'$, whose the sum of arities remains the same, and by trying to increase the arity of the base level up to 5. Consider, for instance, a message of size $l = 95$ blocks. With such a message size, Algorithm 1 returns the multiset of arities $A_0 = \{3, 3, 3, 2, 2\}$ which defines a tree structure involving 32 processors. By applying Algorithm 2, we obtain the multiset $A_1 = \{4, 3, 3, 3\}$ which reduces the number of involved processors to 24 while leaving the running time unchanged.

**What if the arity of each level is increased? As much as possible?** We just saw that we can increase the arity of the first level. It would also be preferable to increase the arity of each level of the tree in order to free up the highest number of processors at each step of the computation. An example is depicted in Figure 4.

![Fig. 4. Two trees compressing a 20-block message, optimized both for the running time and the number of involved processors. Nevertheless, we note that the right tree is the best choice. Indeed, the one on the left needs 4 processors during 5 units of time, then 2 processors during 2 units of time, and finally one processor during 2 units of time. The one on the right needs 4 processors during 5 units of time and then one processor during 4 units of time.](image)

While we propose an iterative algorithm in Appendix B to construct an optimal tree maximizing the arity of each level, we also enumerate all possible cases in the following Theorem:
\textbf{Theorem 3.} For any integer \( l \geq 2 \) there is an unique ordered multiset \( A \) of \( h_5 \) arities 5, \( h_4 \) arities 4, \( h_3 \) arities 3 and \( h_2 \) arities 2 such that the corresponding tree covers a message size \( l \), has a minimal running time and has first \( h_5 \) as large as possible, and then \( h_4 \) as large as possible, and then \( h_3 \) as large as possible. More precisely, if \( i \) is the lowest integer such that \( l \leq 3^i < 3l \), this ordered multiset is such that:

\[
\begin{align*}
|A| &= i, h_5 = 0, h_4 = 0, h_3 = i, h_2 = 0 & \text{if } & l \leq 3^i < \frac{9l}{4}, \\
|A| &= i - 1, h_5 = 0, h_4 = 1, h_3 = i - 2, h_2 = 1 & \text{if } & \frac{9l}{4} \leq 3^i < \frac{27l}{4}, \\
|A| &= i - 1, h_5 = 1, h_4 = 1, h_3 = i - 3, h_2 = 0 & \text{if } & \frac{27l}{4} \leq 3^i < \frac{81l}{16}, \\
|A| &= i - 1, h_5 = 0, h_4 = 1, h_3 = i - 1, h_2 = 1 & \text{if } & \frac{81l}{16} \leq 3^i < \frac{243l}{64}, \\
|A| &= i - 1, h_5 = 0, h_4 = 2, h_3 = i - 3, h_2 = 0 & \text{if } & \frac{243l}{64} \leq 3^i < \frac{729l}{256}, \\
|A| &= i - 1, h_5 = 1, h_4 = 1, h_3 = i - 4, h_2 = 1 & \text{if } & \frac{729l}{256} \leq 3^i < \frac{2187l}{1024}, \\
|A| &= i - 1, h_5 = 0, h_4 = 1, h_3 = i - 2, h_2 = 0 & \text{if } & \frac{2187l}{1024} \leq 3^i < \frac{6561l}{4096}, \\
|A| &= i - 1, h_5 = 1, h_4 = 1, h_3 = i - 3, h_2 = 1 & \text{if } & \frac{6561l}{4096} \leq 3^i < \frac{19683l}{8192}, \\
|A| &= i - 1, h_5 = 0, h_4 = 2, h_3 = i - 4, h_2 = 1 & \text{if } & \frac{19683l}{8192} \leq 3^i < 3l,
\end{align*}
\]

where the number \( h_3 \) is at least 1 in the first case and can be 0 in the other cases.

\textbf{Proof.} Let us start from the 3 cases of Theorem 2 which maximize the number of levels of arity 3. For a given message length \( l \), we consider the corresponding optimal tree (in the sense of the running time). We denote by \( a \) the initial number of levels of arity 2 and by \( i - a \) the initial (maximized) number of levels of arity 3. We want to transform this tree in order to increase the arity of each level as much as possible, while leaving the running time unchanged. According to Lemma 1 there are at most one level of arity 5 and at most six levels of arity 4. Since we want to maximize the number of levels of arity 4 after having maximized the number of levels of arity 5, there cannot be more than one level of arity 2. Thus, \( h_5 \in [0, 1] \), \( h_4 \in [0, 6] \) and \( h_2 \in [0, 1] \), meaning there shall be at most 28 cases. Note that among these 28 cases, many may not be valid solutions. The aim is to transform the initial product \( 2^a 3^{i-a} \) into a product \( 2^u 3^{i-a-b} 4^v 5^w \) where \( b \) is the number of levels of arity 3 that we have transformed and \( u, v, w \) the number of levels of arity 5, 4, 2 respectively. For each triple \((h_5 = u, h_4 = v, h_2 = w)\) with \( u \in [0, 1], v \in [0, 6] \) and \( w \in [0, 1] \), we can verify that there is a solution \((a, b)\) with \( a \) an integer in \([0, 2]\) and \( b \) a positive integer such that \( 3b + 2a = 5u + 4v + 2w \). We remark that \( 3b + 2a \) can be rewritten \( 3b + 2a = 2(3/2)^a - (3/2)^a - a \), meaning that \( 3 \) divides \((3/2)^a - a \). This is impossible, unless \( a = 1 \). Such a solution must satisfy \( 3^{i-a-b} 4^v 5^w \geq l \), that is \( 3^i \geq 3^{a+b} l / (2^a 4^v 5^w) \). According to Theorem 2 we have \((3/2)^a l \leq 3^i < \min(3l, (3/2)^{a+1} l)\). Consequently, if we have

\[
\frac{3^{a+b} l}{2^a 4^v 5^w} \geq \min \left( 3l, \left( \frac{3}{2} \right)^{a+1} l \right),
\]

where \( \min \left( 3l, \left( \frac{3}{2} \right)^{a+1} l \right) = \begin{cases} 3l & \text{if } a = 2 \\ (3/2)^{a+1} l & \text{if } a = 0, 1 \end{cases} \).
this solution does not exist. Among the 28 cases, we observe that 13 of them are not valid solutions. Thus, we have 15 solutions, denoted \((u, v, w, a, b)\), for which we compute and sort the values \(3^{a+b}l/(2^w4^v5^u)\) so that we can establish their domains of validity. We then obtain the fifteen following cases:

\[
\begin{align*}
&\text{(I)} \quad \begin{cases} 
|A| = i, h_5 = 0, h_4 = 0, h_3 = i, h_2 = 0 & \text{if } \frac{3}{2} \leq 3^i < \frac{3}{2}, \\
|A| = i, h_5 = 0, h_4 = 1, h_3 = i - 2, h_2 = 1 & \text{if } \frac{3}{4} \leq 3^i < \frac{3}{2}, \\
|A| = i - 1, h_5 = 0, h_4 = 3, h_3 = i - 4, h_2 = 0 & \text{if } \frac{3}{8} \leq 3^i < \frac{3}{4}, \\
|A| = i - 1, h_5 = 1, h_4 = 1, h_3 = i - 3, h_2 = 0 & \text{if } \frac{3}{16} \leq 3^i < \frac{3}{8}, \\
|A| = i - 1, h_5 = 0, h_4 = 4, h_3 = i - 6, h_2 = 1 & \text{if } \frac{3}{128} \leq 3^i < \frac{3}{16}.
\end{cases}
\]

\[
\begin{align*}
&\text{(II)} \quad \begin{cases} 
|A| = i, h_5 = 0, h_4 = 0, h_3 = i - 1, h_2 = 1 & \text{if } \frac{3}{2} \leq 3^i < \frac{3}{2}, \\
|A| = i - 1, h_5 = 0, h_4 = 2, h_3 = i - 3, h_2 = 0 & \text{if } \frac{3}{4} \leq 3^i < \frac{3}{2}, \\
|A| = i - 1, h_5 = 1, h_4 = 0, h_3 = i - 2, h_2 = 0 & \text{if } \frac{3}{8} \leq 3^i < \frac{3}{4}, \\
|A| = i - 1, h_5 = 0, h_4 = 3, h_3 = i - 5, h_2 = 1 & \text{if } \frac{3}{16} \leq 3^i < \frac{3}{8}, \\
|A| = i - 1, h_5 = 1, h_4 = 1, h_3 = i - 4, h_2 = 1 & \text{if } \frac{3}{32} \leq 3^i < \frac{3}{16}, \\
|A| = i - 2, h_5 = 0, h_4 = 5, h_3 = i - 7, h_2 = 0 & \text{if } \frac{3}{256} \leq 3^i < \frac{3}{32}.
\end{cases}
\]

\[
\begin{align*}
&\text{(III)} \quad \begin{cases} 
|A| = i - 1, h_5 = 0, h_4 = 1, h_3 = i - 2, h_2 = 0 & \text{if } \frac{3}{2} \leq 3^i < \frac{3}{2}, \\
|A| = i - 1, h_5 = 0, h_4 = 2, h_3 = i - 4, h_2 = 1 & \text{if } \frac{3}{4} \leq 3^i < \frac{3}{2}, \\
|A| = i - 1, h_5 = 1, h_4 = 0, h_3 = i - 3, h_2 = 1 & \text{if } \frac{3}{8} \leq 3^i < \frac{3}{4}, \\
|A| = i - 2, h_5 = 0, h_4 = 4, h_3 = i - 6, h_2 = 0 & \text{if } \frac{3}{128} \leq 3^i < \frac{3}{16}.
\end{cases}
\]

In accordance with Theorem 2, we have this grouping:

- Group I consists of the five cases ensuring a running time of \(3^i\);
- Group II consists of the six cases ensuring a running time of \(3^i - 1\);
- Group III consists of the four cases ensuring a running time of \(3^i - 2\).

Now, we have to optimize the arities. In the first group, we delete the last case since it decreases \(h_5\) compared to the immediately preceding case. For the same reasons, we delete in the second group the fourth and sixth cases. Again, in the last group, we need to delete the last case. Overall, 11 cases are deduced by intersecting the intervals of validity. □

**Remark 2.** Let us consider a tree which is optimal in the sense of the theorem 3. If we extract a final subtree by deleting one or several lower levels (at the bottom), the resulting tree is still optimal. Indeed, let suppose that the original tree has height \(h\) and has \(l\) leaves (for \(l\) message blocks). If we delete the \(j\) lower levels, the resulting tree has \(l' = \left\lfloor l/(x_1x_2\cdots x_j) \right\rfloor\) leaves and is already optimal for a number \(l'\) of blocks. If this form of local optimality does not exist, we can further optimize the original tree.

For the purpose of minimizing the number of processors at each step of the computation, we apply Theorem 3. There are 11 possible cases and we would like to estimate their distribution. The following theorem helps us to calculate the proportions:
Theorem 4. Let a message size $l$ drawn uniformly at random from the set $[2, L]$ where $L$ is a fixed positive integer. Let $k = \lceil \log_3 L \rceil$ and $\alpha, \beta$ two real numbers such that $1 \leq \alpha < \beta \leq 3$. The probability that $3^{\lceil \log_3 l \rceil}$ is in the interval $[\alpha l, \beta l]$ is equal to

$$P(E) = \frac{1}{L-1} \left( \sum_{i=1}^{k-1} \left( \left\lfloor \frac{3^i}{\alpha} \right\rfloor - \left\lfloor \frac{3^i}{\beta} \right\rfloor \right) + \mu \right),$$

where

$$\mu = \begin{cases} 0 & \text{if } L \leq \frac{3^k}{\beta}, \\ \min \left( L, \left\lfloor \frac{3^k}{\alpha} \right\rfloor \right) - \left\lfloor \frac{3^k}{\beta} \right\rfloor & \text{if } L > \frac{3^k}{\beta}. \end{cases}$$

Proof. Let $E$ be the event: “$l$ is such that $3^{\lceil \log_3 l \rceil} \in [\alpha l, \beta l]$”. For any $i$ such that $1 \leq i \leq k - 1$ let $E_i$ be the event: “$l$ is such that $3^{i-1} < l \leq 3^i$” and $E_k$ the event: “$l$ is such that $3^{k-1} < l \leq L$”. The conditional probability $P(E \mid E_i)$ is given by the following:

1. Case $i \leq k - 1$. The event $E$ is realized if and only if $\alpha l < 3^i < \beta l$, namely if and only if

$$\frac{3^i}{\beta} < l \leq \frac{3^i}{\alpha}.$$ 

As $l$ must be an integer, this condition is equivalent to

$$l \in \left\lfloor \frac{3^i}{\beta} \right\rfloor, \left\lfloor \frac{3^i}{\alpha} \right\rfloor.$$

Hence

$$P(E \mid E_i) = \frac{1}{3^i - 3^{i-1}} \times \left( \left\lfloor \frac{3^i}{\alpha} \right\rfloor - \left\lfloor \frac{3^i}{\beta} \right\rfloor \right).$$

2. Case $i = k$.

(a) If $L \leq \frac{3^k}{\beta}$ then $P(E \mid E_i) = 0$.

(b) If $L > \frac{3^k}{\beta}$ then

$$P(E \mid E_i) = \frac{1}{L - 3^{k-1}} \times \left( \min \left( L, \left\lfloor \frac{3^k}{\alpha} \right\rfloor \right) - \left\lfloor \frac{3^k}{\beta} \right\rfloor \right).$$

As the $E_i$ are disjoint, we can compute $P(E)$ by the following formula:

$$P(E) = \sum_{i=1}^{k} P(E_i)P(E \mid E_i),$$

which gives the expected result. \qed
Remark 3. In the previous theorem we can write
\[ \left\lfloor \frac{3^i}{\alpha} \right\rfloor - \left\lfloor \frac{3^i}{\beta} \right\rfloor = 3^i \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) + u_i \]
where \(|u_i| \leq 1\). Then
\[ P(E) = \frac{1}{L-1} \left( \frac{3}{2} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \left( 3^{k-1} - 1 \right) + \sum_{i=1}^{k-1} u_i + \mu \right). \tag{2} \]

Let us now consider trees whose the number of leaves is equal to the message length. Having a multiset of arities arranged in descending order, that we denote \(A = \{x_1, x_2, \ldots, x_{|A|}\}\), the number of nodes of level \(i\) is \(\left\lceil \frac{l}{x_1 x_2 \ldots x_i} \right\rceil\). One important thing is the number of nodes of the base level. We have the following Theorem:

**Theorem 5.** Let the message size be \(l \geq 2\) and let \(i\) be the lowest integer such that \(3^i \geq l\). The number of processors required to process such a message is:
- \([1/3]\) if \(3^i \in [1, \frac{9}{16} \cup \frac{27}{64} \cup \frac{27}{32}],\)
- \([1/4]\) if \(3^i \in [\frac{9}{16} \cup \frac{27}{64} \cup \frac{27}{32} \cup \frac{9}{4} \cup \frac{27}{16}],\)
- \([1/5]\) if \(3^i \in [\frac{27}{32} \cup \frac{9}{4} \cup \frac{27}{16} \cup 3l].\)

**Proof.** These results follow immediately from Theorem 3. \(\square\)

The following theorem gives the proportions of message sizes for which \(\left\lceil \frac{l}{c} \right\rceil\) processors (with \(c = 3, 4, 5\)) are required:

**Theorem 6.** Let a message size \(l \geq 2\) bounded by \(L = 3^j\) be drawn randomly. When \(j\) tends to infinity, the number of required processors is \([1/3]\) with probability approaching \(5/18 \approx 28\%), \([1/4]\) with probability approaching \(21/54 \approx 39\%), and \([1/5]\) with probability approaching \(1/3 \approx 33\%).

**Proof.** Remark that if \(L = 3^j\), Formula (2) becomes:
\[ P(E) = \frac{1}{L-1} \left( \frac{3}{2} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \left( 3^j - 1 \right) + \sum_{i=1}^{j} u_i \right). \tag{3} \]

Now we apply Theorem 4 and Remark 3 with \(\alpha\) and \(\beta\) given by the intervals occurring in Theorem 5. These \(\alpha\) and \(\beta\) are of the form \(\frac{3^s}{u}\) where \(s \leq 6\) and \(u\) integer. Then for \(i \geq 6\) the numbers \(\frac{3^i}{\alpha}\) and \(\frac{3^i}{\beta}\) are integers. Thus, for \(i \geq 6\) we have \(u_i = 0\) and the following formula holds:
\[
P(E) = \frac{1}{L-1} \left( \frac{3}{2} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) \left( 3^j - 1 \right) + \sum_{i=1}^{5} u_i \right)
= \frac{3}{2} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right) + \frac{1}{L-1} \sum_{i=1}^{5} u_i.
\]
When \( j \) tends to infinity, \( P(E) \) has a limit which is
\[
\frac{3}{2} \left( \frac{1}{\alpha} - \frac{1}{\beta} \right).
\]
This formula applied to the intervals given in Theorem 5 gives the expected results.

Remark 4. When \( L \) approaches infinity, the approached proportions for the eleven cases of Theorem 3 can be estimated similarly. These proportions are depicted in Figure 5.

\[\text{Fig. 5. Proportions for the eleven cases of Theorem 3. The bars are drawn in decreasing order of frequency. The notation Tr(}a_1, a_2, ..., a_h\text{) stands for a tree having arities } a_1, a_2, ..., a_h \text{ from the base level to the root node.}\]

An other important thing is the minimization of the amount of work done by the hash tree algorithm for the processing of a single message. This amount of work corresponds to the total number of children of the nodes in the tree. Since we are interested in hash
trees having an optimal running time for a given message size \( l \), we apply Theorem 2 or 3 to retrieve a topology. For a perfect \((x_1, x_2, \ldots, x_h)\)-aries tree constructed thanks to this theorem, the amount of work is:

\[
W_l = x_h + x_h x_{h-1} + x_h x_{h-1} x_{h-2} + \cdots + x_h x_{h-1} \cdots x_2 x_1.
\]

We notice that \([\lceil \cdots \lceil l/x_1 \rceil/x_2 \rceil \cdots \rceil/x_i \rceil = \lceil l/(x_1 x_2 \cdots x_i) \rceil\) for (strictly) positive integers \((x_j)_{j=1}^i\). For a \((x_1, x_2, \ldots, x_h)\)-aries truncated tree constructed thanks to this theorem, the amount of work is:

\[
W_{tr,l} = l + \left[ l/x_1 \right] + \left[ l/(x_1 x_2) \right] + \cdots + \left[ l/(x_1 x_2 \cdots x_{h-1}) \right].
\]

This quantity is necessarily greater than or equal to \( l \). Regarding truncated trees having their leaves at the same depth and minimizing the running time, Theorem 3 indicates a topology which minimizes the amount of work, by first choosing \( x_1 \) as large as possible, then choosing \( x_2 \) as large as possible, and so on.

**Other balanced trees.** If we can move up (lift up) some leaves in the tree so that all leaves are not at the same depth, then we can reduce the number of processors. Let us suppose that we merely use full binary trees. Whatever the message length is, it is always possible to construct a full binary tree. This allows a reduction of the number of processors. Indeed, let us consider a compression function \( f \) taking as input two blocks and returning one block. If \( l \) is the number of blocks of the message, we compute the root node in \( l-1 \) evaluations of the function \( f \) as follows: the blocks of the message are paired consecutively and \( f \) is applied on each pair. The possibly remaining block (if \( l \) is odd) is not processed. We then consider the list of resulting blocks by \( f \) with the possibly remaining block and we repeat the process again and again until there is a single remaining block. The height of the resulting tree is \( \lceil \log_2 l \rceil \) and the number of saved processors is \( \lceil l_f/2 \rceil \) where \( l_f \) is the number of leaves located at a level greater than 0. Note that in the best case \( l_f = l_L/4 \) where \( l_L \) is the number of leaves of the largest perfect binary subtree.

Let us consider our variant of the definition of a full tree, when arities differ from one level to another. The question that arises is: is it always possible to construct a full tree minimizing the number of processors for an optimal running time? The answer is no. To show that it suffices to exhibit an example in which a tree which is not full further minimizes the number of processors compared to a full tree. For the sake of concreteness, let us take the example of a message of length 26 blocks. The optimal running time for such a message length is 9 and the only multiset of arities which allows deriving it is \( \{3, 3, 3\} \). It can be noted that a rightmost node at the base level is of arity 2, showing that the minimum cannot be obtained with a full tree.

If we consider trees without any structural constraint, there is still scope for reducing the number of processors, although marginally. Let us take a message of 56 blocks. An optimal set of arities for a tree with all leaves at the same level is \( \{5, 4, 3\} \). We can note that this tree has a rightmost node at the base level having only one child. This child can take the place of its parent node in order to save one more processor. We also notice that the resulting tree is full in the sense of our new definition.
4 Applying our optimizations safely

Let suppose that we have 4 different inner functions \( f_{BL}, f_I, f_F \) and \( f_{SN} \) with the following properties:

- \( f_t : \{0, 1\}^{yN} \rightarrow \{0, 1\}^N \) for \( y \geq 2 \) and \( t \in \{BL, I, F, SN\} \),
- they are rate-1 and have the same running time (of exactly \( y \) units of time when compressing \( y \) blocks of size \( N \) bits),
- they behave like independent random oracles.

In the hash tree construction we propose, we use \( f_{BL} \) to compute base level nodes, \( f_I \) to compute inner nodes, \( f_F \) to compute the root node. If the tree is of height one, there is only one node computed using \( f_{SN} \). In order to simulate four independent functions, we use the same inner function \( f \) but with domain separation. Indeed, since \( f \) behaves like a random oracle, by construction the functions \( f_{BL}(x) = f(10 \parallel x) \), \( f_I(x) = f(00 \parallel x) \), \( f_F(x) = f(01 \parallel x) \) and \( f_{SN}(x) = f(11 \parallel x) \) behave like independent random oracles.

4.1 An example of hash function

Given a message \( M \), a hash tree mode could be the following:

1. Whatever the message bit-length is, we append to \( M \) a bit “1” and the minimum number of bits “0” so that the total bit-length is a multiple of \( N \). The new message is denoted \( M_0 \) and its total number of blocks of size \( N \) bits is \( l = \lceil |M_0|/2 \rceil + 1 \).
2. We apply Theorem 3 on \( l \) to retrieve the height \( h \) of the tree and an ordered multiset \( A \) of arities \( a_1, a_2, ..., a_h \) (arranged in decreasing order).
3. If \( h = 1 \), we compute and return the hash value \( f_{SN}(M_0) \). Otherwise, we go to the following step.
4. We first split \( M_0 \) into blocks \( M_{0,1}, M_{0,2}, ..., M_{0,l_1} \) where: (i) \( l_1 = \lceil l/a_1 \rceil \); (ii) all blocks but the last one are \( a_1N \) bits long and the last block is between \( N \) and \( a_1N \) bits long. Then, we compute the message

\[
M_1 := \bigg\|_{j=1}^{l_1} f_{BL}(M_{0,j}) .
\]

5. If \( h = 2 \) we go to step 6. Otherwise, for \( k = 2 \) to \( h - 1 \), we perform the following operations:
   (a) We split \( M_{k-1} \) into blocks \( M_{k-1,1}, M_{k-1,2}, ..., M_{k-1,l_k} \) where: (i) \( l_k = \lceil l/\left(\prod_{j=1}^{k} a_j\right) \rceil \); (ii) all blocks but the last one are \( a_kN \) bits long and the last block is between \( N \) and \( a_kN \) bits long.
   (b) We compute the message

\[
M_k := \bigg\|_{j=1}^{l_k} f_{BL}(M_{k-1,j}) .
\]

6. We compute and return the hash value \( f_F(M_{h-1}) \).
4.2 Security

Since our mode can be rewritten as if it was using only $f$, it suffices to check whether the three conditions (seen in Section 2.2) are satisfied to prove soundness.

We first need to describe some rules regarding our mode:

**Rule 0.** The root $f$-input has a prepended code 01 or 11.

**Rule 1.** A $f$-input with a prepended code 01 has children having a prepended code 00 or 10.

**Rule 2.** A $f$-input with a prepended code 10 or 11 has no children.

**Rule 3.** A $f$-input with a prepended code 00 has children having a prepended code 10.

**Rule A.** A $f$-input must be $(2 + kN)$-bit long with an integer $k \geq 1$.

**Rule B.** At a same level of the tree, the number of chaining values is the same for all the $f$-inputs, except for the rightmost one whose the number may be smaller.

**Rule C.** At a same level of the tree, prepended codes are the same for all the $f$-inputs.

Note that the satisfaction of the rules 0, 1, 2, 3 and C imply that the leaves are at the same depth. So, we do not need to define a rule to express this.

By construction our mode is final-node separable. Our mode is trivially message-complete since it processes all message bits. Indeed, having the valued tree of $f$-inputs produced by $\tau$, the algorithm $A_{message}$ reaches directly the base level $f$-inputs and recovers message blocks by discarding the frame bits, whether they serve as padding purpose (in the rightmost $f$-input) or for identifying the type of node. This algorithm runs in linear time in the number of bits in the tree. Finally, our mode is also tree-decodable. Thanks to domain separation between base level nodes and other nodes, we cannot find a tree which is both compliant and final-subtree-compliant. Given only one $f$-input, the prepended coding allows its content to be recognized correctly. We can then construct a decoding algorithm $A_{decode}$ which runs in 2 phases: Phase 1 starts from the root $f$-input and fully determines the tree structure by recursively decoding each $f$-input. The size of a $f$-input determines the number of its children. This phase terminates with the “correct” state C0 if the visited $f$-inputs all respect the rules defined above. This phase terminates with the “incorrect” state C1 if one of the rule 0, 2, A, B or C is not respected. If Rule 1 is not respected it terminates with the “incorrect” state C2. Otherwise, it terminates with the “incorrect” state C3. Phase 2 examines the properties of the decoded tree by taking into account the termination state of the first phase. The details of Phase 2 are the followings:

1. If the state is C0, it runs the $A_{message}$ algorithm in order to check the message size. If for the corresponding number $l$ of blocks, Theorem 3 indicates a topology which differs from the one that Phase 1 has just decoded, then it returns “incompliant”. Otherwise, it returns “compliant”.
2. If the state is C1, it returns “incompliant”.
3. If the state is C2, the following examinations are made:
   1. If the tree seems incomplete with a single $f$-input, it checks, after having discarded the prepended code, the number of blocks of size $N$ bits. If there are
2, 3 or 4 blocks, then it returns “final-subtree-compliant”. Otherwise, it returns “incompliant”.

(b) Otherwise, the coding is incompatible with the mode, and it returns “incompliant”.

4. If the state is C3, this means that only the rule 3 is not respected. The following examinations are then made:

(a) If there is a \(f\)-input with a prepended code 00 which has a child with a prepended code not equal to 10, then it returns “incompliant”.

(b) Otherwise, there is at least one missing node. We note \(h\) the maximum number of \(f\)-inputs visited on a path in this proper final subtree. Thus, \(h\) corresponds to its height using the first convention for the definition of a node. The algorithm has to check if its topology is consistent with Theorem 3. First, it establishes a system of constraints regarding the arity of each level. If there is only one \(f\)-input at a level which is not the root, and if it is the rightmost one in the complete tree, then the number of (message or chaining) blocks it contains defines a lower bound for the arity of this level. Otherwise, the arity for this level has a single possible value and the constraint is an equality. Having these \(h\) constraints, it checks in Theorem 3 which cases (among the fifteen) can satisfy these constraints. We denote by \(L\) the list of compatible cases and for each case \(j\) in \(L\) we denote by \(a_i\) the arity of the level \(i\). For each case \(j\) in \(L\), the algorithm performs the following operations:

i. It completes this final subtree until its leaves are at the same depth \(h\). It performs this by (virtually) creating the missing nodes with a maximal arity (i.e., even if a missing node at level \(i\) is the rightmost one in the complete tree, it chooses its arity to be exactly \(a_i\)).

ii. It counts the number \(l\) of blocks covered by the completed subtree. If \(l\) is in the domain of validity of the case \(j\), then it returns “final-subtree-compliant”.

A this point, no cases in \(L\) are suitable. This phase 2 finally returns “incompliant”.

The total running time of \(A_{decode}\) is linear in the number of bits in the tree.

Remark 5. If prepending 2 bits is sufficient [6] for soundness, we could have used a reduction to the Sakura coding [7] where meta-information bitstrings are longer. According to the Sakura ABNF grammar [7], the number of chaining values (i.e. the number of children of a node) is also coded in the formatted input to \(f\). Thus, having 4 possible number of children for 4 kind of nodes, we would have used 16 inner functions. Using Sakura coding allows any tree based hash function to be automatically indifferentiable from a random oracle, without the need of further proofs. A good reason for not using it is that, in practice, finding 16 (instead of 4) different inner functions (having the same running time) is certainly a more difficult task.

\[\text{Meaning that it has to detect if this final subtree can be extracted from an optimal tree, in the sense of this theorem.}\]

\[\text{A } f\text{-input is the rightmost one in the complete tree if we see that it does not have a right sibling, when looking at the retrieved topology by Phase 1.}\]
Remark 6. Yet another solution is to use different IVs (Initial Values) instead of particular frame bits, as suggested in [6,14]. We could use a free-IV hash function, like the suffix-free-prefixed-free hash function from Bagheri et al. [4]. The distinction between nodes would be done by using 4 distinct IVs: BL_IV for base level nodes, I_IV for inner nodes, F_IV for the root node and SN_IV for a single node.

5 Conclusion

In this paper, we have shown, for a given message length, how to construct a hash tree minimizing the running time. For a hash tree having its leaves at the same depth, we have shown how to decrease at best the number of processors allowing such a minimized running time. We have also seen that it is possible to slightly decrease the number of processors by considering other types of trees. Analysis on few small message sizes have revealed that, in the best case, we can save one more processor by using a tree which does not have all its leaves at the same depth. Further work is necessary to adequately specify to what extent the amount of resources can actually be decreased.

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A Comparison between a perfect binary tree and a perfect ternary tree

Let \( l \geq 2 \) an integer. Let \( h_2 \) the lowest integer such that \( 2^{h_2} \geq l \) and \( h_3 \) the lowest integer such that \( 3^{h_3} \geq l \). We assume that we use a perfect binary (or ternary) tree as in the original Merkle (and Damgård) hash tree mode, i.e., the message is padded to obtain a message size which is a power of 2 (or 3). The problem is to compare \( 2^{h_2} \) and \( 3^{h_3} \).

Any \( l \) can be uniquely written

\[
l = 2^k + u,
\]

where \( u \) is an integer such that \( 0 \leq u < 2^k \). Then

\[
l = 2^k (1 + a) \text{ where } a = \frac{u}{2^k}.
\]

If \( a = 0 \) then \( h_2 = k \) else \( h_2 = k + 1 \).
A.1 The case \( a = 0 \)
In this case
\[
l = 2^k, h_2 = k, h_3 = \left\lceil \frac{k \log(2)}{\log(3)} \right\rceil.
\]
Then
\[
h_3 = \frac{k \log(2)}{\log(3)} + \alpha,
\]
where \( 0 < \alpha < 1 \). We must compare \( 3h_3 \) with \( 2h_2 \), namely
\[
3 \frac{k \log(2)}{\log(3)} + 3\alpha \text{ with } 2k,
\]
or
\[
3 \frac{\log(2)}{\log(3)} + 3 \frac{\alpha}{k} \text{ with } 2.
\]
As \( \alpha \) is bounded by 1 and \( 3 \frac{\log(2)}{\log(3)} < 2 \), for \( k \) sufficiently large we have \( 3h_3 < 2h_2 \).
More precisely if \( k \geq 28 \) then \( 3h_3 < 2h_2 \), meaning that a perfect ternary tree gives a better running time than a perfect binary tree. When \( 2 \leq k \leq 27 \), we compute the 27 values
\[
T = 3 \frac{k \log(2)}{\log(3)} - 2
\]
and we look at the sign of the result:
- For \( k = 3s \) (\( s = 1, \ldots, 9 \)), a perfect binary tree and a perfect ternary tree give the same result \( (T = 0) \).
- For \( k = 11, 14, 17, 19, 20, 22, 23, 25, 26 \), a perfect ternary tree is better \( (T < 0) \).
- For \( k = 2, 4, 5, 7, 8, 10, 13, 16 \), a perfect binary tree is better \( (T > 0) \).

A.2 The case \( a \neq 0 \)
In this case \( h_2 = k + 1 \) and
\[
h_3 = \left\lceil \frac{k \log(2)}{\log(3)} + \frac{\log(1 + a)}{\log(3)} \right\rceil.
\]
We must compare \( 3h_3 \) to \( 2h_2 \). But:
\[
\frac{3h_3}{k} \leq \frac{3 \log(2)}{\log(3)} + \frac{3 \log(2)}{k \log(3)} + \frac{3}{k}
\]
and
\[
\frac{2h_2}{k} = 2 + \frac{2}{k}.
\]
As \( \frac{3 \log(2)}{\log(3)} < 2 \), for \( k \) sufficiently large we have \( 3h_3 < 2h_2 \). More precisely for \( k \geq 27 \) then \( 3h_3 < 2h_2 \), meaning that a perfect ternary tree gives a better running time than
a perfect binary tree. For any $2 \leq k \leq 26$ and any $u$ such that $1 \leq u < 2^k$ we must compute the sign of

$$R = 3 \left\lceil \frac{k \log(2)}{\log(3)} + \frac{\log(1 + \frac{u}{2^k})}{\log(3)} \right\rceil - 2k - 2.$$  

As $R$ is an increasing function of $u$, it is sufficient to determine for any $k < 27$ the value of $u$ where the sign changes. This can be done by dichotomy. Results are in Table 1.

| $k$ = 2 | $Sign = 0$ for any $u$ |
| $k$ = 3 | $Sign < 0$ for $u \leq 1$ and $Sign > 0$ for $u > 1$ |
| $k$ = 4 | $Sign < 0$ for $u \leq 11$ and $Sign > 0$ for $u > 11$ |
| $k$ = 5 | $Sign = 0$ for any $u$ |
| $k$ = 6 | $Sign < 0$ for $u \leq 17$ and $Sign > 0$ for $u > 17$ |
| $k$ = 7 | $Sign < 0$ for $u \leq 115$ and $Sign > 0$ for $u > 115$ |
| $k$ = 8 | $Sign = 0$ for any $u$ |
| $k$ = 9 | $Sign < 0$ for $u \leq 217$ and $Sign > 0$ for $u > 217$ |
| $k$ = 10 | $Sign < 0$ for any $u$ |
| $k$ = 11 | $Sign < 0$ for $u \leq 139$ and $Sign > 0$ for $u > 139$ |
| $k$ = 12 | $Sign < 0$ for $u \leq 2465$ and $Sign > 0$ for $u > 2465$ |
| $k$ = 13 | $Sign < 0$ for any $u$ |
| $k$ = 14 | $Sign < 0$ for $u \leq 3299$ and $Sign > 0$ for $u > 3299$ |
| $k$ = 15 | $Sign < 0$ for $u \leq 26281$ and $Sign > 0$ for $u > 26281$ |
| $k$ = 16 | $Sign < 0$ for any $u$ |
| $k$ = 17 | $Sign < 0$ for $u \leq 46075$ and $Sign > 0$ for $u > 46075$ |
| $k$ = 18 | $Sign < 0$ for any $u$ |
| $k$ = 19 | $Sign < 0$ for any $u$ |
| $k$ = 20 | $Sign < 0$ for $u \leq 545747$ and $Sign > 0$ for $u > 545747$ |
| $k$ = 21 | $Sign < 0$ for any $u$ |
| $k$ = 22 | $Sign < 0$ for any $u$ |
| $k$ = 23 | $Sign < 0$ for $u \leq 5960299$ and $Sign > 0$ for $u > 5960299$ |
| $k$ = 24 | $Sign < 0$ for any $u$ |
| $k$ = 25 | $Sign < 0$ for any $u$ |
| $k$ = 26 | $Sign < 0$ for $u \leq 62031299$ and $Sign > 0$ for $u > 62031299$ |

Table 1. Comparison between a perfect binary tree and a perfect ternary tree. If $Sign < 0$ a perfect ternary tree has a better running time. If $Sign = 0$ the two trees give the same running time. Otherwise a perfect binary tree is better.

B Algorithms for reducing the number of processors

B.1 Reducing the number of processors at the base level

We propose two (different) algorithms to construct an optimal tree (in the sense of the running time) which covers exactly $\ell$ blocks (the tree is not necessarily perfect) and
increases as much as possible the arity of the base level. The first solution consists to check if there exists an optimal tree having a level of arity 5 or 4.

**Algorithm 2a.** This algorithm takes as input a message length $l$, a multiset of arities (arranged in descending order) minimizing the running time, denoted $A = \{x_1, x_2, \ldots, x_{|A|}\}$, and returns a multiset of arities (still sorted in descending order) minimizing the number of processors while leaving unchanged the running time. Let $t_l$ the optimal running time for a message of size $l$, i.e. the sum of arities of $A$. The algorithm proceeds as follows:

1. Use Algorithm 1 to construct a tree for a message length $l' = \lceil l/5 \rceil$ and denote by $A'$ the corresponding ordered multiset of arities. If $t_{l'} = t_l + 5$ then return the multiset $A'' = \{5, A'\}$, otherwise go to the following step.
2. Use Algorithm 1 to construct a tree for a message length $l' = \lceil l/4 \rceil$ and denote by $A'$ the corresponding ordered multiset of arities. If $t_{l'} = t_l + 4$ then return the multiset $A'' = \{4, A'\}$, otherwise go to the following step.
3. Return $A$ (which cannot be further optimized).

The second approach uses the following hints:

**Hints.** Let us note that if $k > 0$, then $a > b \iff (a - k)b > a(b - k)$. Moreover, if $b \leq a$ then $(b - 1)(a + 1) \leq ab$. This suggests that a product of several numbers, whose the sum is constant, is maximized when these numbers are as close together as possible. In order to decrease the product of arities as slowly as possible we use the fact that if $c \geq b \geq a$ we have $(c + 1)(b - 1)a \geq (c + 1)b(a - 1)$.

**Algorithm 2b.** This algorithm takes as inputs a message length $l$, a multiset of arities (arranged in descending order) minimizing the running time, denoted $A = \{x_1, x_2, \ldots, x_{|A|}\}$, and returns a multiset of arities (still sorted in descending order) minimizing the number of processors while leaving unchanged the running time. The algorithm proceeds as follows:

1. We start by replacing in $A$ each pair of arities 2 by an arity 4 (leaving possibly only one arity 2 in $A$). We sort $A$ in descending order.
2. We repeat at most two times the following routine to determine the solution:
   - Case $|A| = 1$: we return $A$.
   - Case $|A| = 2$:
     - Case $x_1 = 5$: we return $A$.
     - Case $x_1 \geq 3, x_2 \geq 3$: if $(x_1 + 1)(x_2 - 1) \geq l$ then $A = \{x_1 + 1, x_2 - 1\}$, otherwise we return $A$.
     - Case $x_1 = 4, x_2 = 2$: we return $A$.
     - Case $x_1 = 3, x_2 = 2$: if $5 \geq l$ then $A = \{5\}$. We return $A$.
   - Case $|A| \geq 3$:
     - Case $x_1 = 5$: we return $A$.
     - Case $x_1 \geq 3, x_2 \geq 3, x_3 \geq 2$: if $(x_1 + 1)(x_2 - 1) \prod_{i=3}^{|A|} x_i \geq l$ then we perform the following operations: (i) we add 1 to $x_1$ and we subtract 1 to $x_2$; (ii) we replace a possible pair of arities 2 by an arity 4; (iii) we reorder $A$. If either the check fails or $x_1 = 5$ then we return $A$. 

B.2 Reducing the number of processors at all the levels

The following algorithm uses Algorithm 1 and 2 in order to compute a multiset of arities (sorted in descending order) minimizing the running time and the required number of processors at each step of the computation.

**Algorithm 3.** Let $A_0 = \{x_1, x_2, \ldots, x_{|A_0|}\}$ be the multiset of arities returned by Algorithm 1. We then use Algorithm 2 with a message of length $l$ and the multiset $A_0$ to compute the multiset of arities $A_1 = \{x'_1, x'_2, \ldots, x'_{|A_1|}\}$. The rest of the algorithm proceeds iteratively as follows:

1. We apply Algorithm 2 on inputs $l' = \lceil l/x'_1 \rceil$ and $A'_1 = \{x'_2, \ldots, x'_{|A_1|}\}$ to compute the multiset $A'_2 = \{x''_2, \ldots, x''_{|A'_1|}\}$. We set $n = 1$.

2. As long as one of the following termination conditions is not met, namely
   (i) $A_{n+1}^{(n)} = A_n^{(n)}$; (ii) the highest number of levels of arity 4 has been reached (see Lemma 1); or (iii) $A_{n+1}^{(n)} = \emptyset$, we set $n = n + 1$ and apply Algorithm 2 with the inputs $l^{(n)} = \lceil l^{(n-1)}/x_n^{(n)} \rceil$ and $A_n^{(n)} = \{x_n^{(n)}, \ldots, x_{|A_n^{(n-1)}|}\}$ to compute the multiset $A_{n+1}^{(n)} = \{x_{n+1}^{(n+1)}, \ldots, x_{|A_n^{(n)}|}\}$.

The resulting multiset of arities $A_r = \{x'_1, x''_2, \ldots, x_n^{(n)}, x_{n+1}^{(n+1)}, \ldots, x_{|A_n^{(n)}|}^{(n+1)}\}$ minimizes the number of required processors at each step of the computation.