ON THE FINITENESS OF ATTRACTORS FOR ONE-DIMENSIONAL MAPS WITH DISCONTINUITIES

P. BRANDÃO, J. PALIS, AND V. PINHEIRO

ABSTRACT. Since the proof, at the end of the 80’s, of the finiteness of the number of attractors for $C^3$ maps of the interval having negative Schwarzian derivative, it has been generally considered that the same result could be true for maps with discontinuities. In the present paper we show that this is indeed the case.

CONTENTS

1. Introduction and statement of results 1
1.1. Statement of the main theorems 3
1.2. An outline of the paper 5
2. Setting and preliminary facts 6
3. Markov maps and the interval dichotomy 11
4. Unimodal maps 16
5. Proof of the finiteness of the number of attractors 18
6. Contracting Lorenz maps 27
References 37

1. INTRODUCTION AND STATEMENT OF RESULTS

Attractors play a fundamental role in the study of dynamical systems for understanding the evolution of initial states. Along this line, stressing the importance of attractors, it was conjectured by Palis in 1995, in a conference in honour of Adrien Douady (see [36, 37]) that, in compact smooth manifolds, there is a dense set $D$ of differentiable dynamics such that any element of $D$ has finitely many attractors whose union of basins of attraction has total probability.

In the one-dimensional case we can expect the finiteness of attractors to be valid for a much larger range of maps and similarly concerning other important properties. As a consequence, we can obtain a more comprehensive description of generic parametrized dynamics in the case of quadratic maps, like the density of hyperbolic dynamics (Graczyk-Swiatek [12] and Lyubich [25]), together with stochastic dynamics in the complement (Lyubich [26]). See also [3, 4, 18, 19, 27, 43, 41].

Going beyond the quadratic case, the finiteness of the number of attractors for smooth non-flat maps of the interval began to be established at the end of the 80’s and early 90’s. Fundamental results were obtained by Blokh and Lyubich [7, 8] and Lyubich [22].

Date: January 3, 2014.
They have shown that if \( f : [0, 1] \to [0, 1] \) is a \( C^3 \) non-flat map (near any critical point \( f \) looks like a polynomial map, see [33]) with a single critical point and negative Schwarzian derivative, then there is a (minimal) attractor whose basin of attraction contains Lebesgue almost every point of the interval \([0, 1]\). Main contributions were also due to de Melo and van Strien [29], Guckenheimer and Johnson [16], Keller [20] and others. The Schwarzian derivative \( Sf(x) \) of a \( C^3 \) map \( f \) at a point \( x \) with \( f'(x) \neq 0 \) is defined by

\[
Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left( \frac{f''(x)}{f'(x)} \right)^2.
\]

For \( C^3 \) non-flat maps with more than a critical point and without the condition of negative Schwarzian derivative, Lyubich [23] and van Strien and Vargas [40] showed that the number of non-periodic attractors is finite.

In contrast with the \( C^3 \) case, the problem of the finiteness of the number of attractors remained open until now for maps with discontinuities or another kind of “lack of regularity” (as displayed in Figure 1: \( c_3, c_4 \) and \( c_6 \) are non-regular points), the exception being the simplest case: the contracting Lorenz map (see Figure 2). Indeed, St. Pierre has shown [42] (see also [21]) that a \( C^3 \) non-flat contracting Lorenz map \( f \) with negative Schwarzian derivative has at most two attractors and, if it does not have a periodic attractor, then \( f \) has only a single attractor with a full measure basin of attraction.

We note that maps of the interval with discontinuities come naturally from smooth vector fields. For instance, it can appear as the quotient by stable manifolds of Poincaré maps of \( C^r \) dissipative flows with singularities induced by contracting Lorenz flows (see for instance [11, 38]).
1.1. **Statement of the main theorems.** Following Milnor [34], given a smooth compact manifold $M$, possibly with boundary, we say that a compact subset $A \subset M$ is a *metrical attractor* for a map $f : M \to M$ if its basin of attraction $\beta_f(A) := \{ x ; \omega_f(x) \subset A \}$ has positive Lebesgue measure and there is no strictly smaller closed set $A' \subset A$ so that $\beta_f(A')$ is the same as $\beta_f(A)$ up to a measure zero set. Here $\omega_f(x)$ is the positive limit set of $x$, that is, the set of accumulating points of the forward orbit of $x$.

An attractor is called *minimal* if $\text{Leb}(\beta_f(A')) = 0$ for every compact set $A' \subsetneq A$. Notice that if $A$ is a minimal attractor then

$$\omega_f(x) = A$$

for almost all $x \in \beta_f(A)$.

In this work we are dealing with piecewise $C^3$ maps of the interval into itself, that is, maps $f : [0, 1] \to [0, 1]$ that are $C^3$ local diffeomorphisms with negative Schwarzian derivative in the whole interval, except for a finite set of points $C_f \subset (0, 1)$. This exceptional set contains all the critical, as well as discontinuities and other non-regular points of $f$ (also, we are not assuming the non-flatness condition for the critical points). In Figure 1 we indicate several kind of points belonging to the exceptional set.

The main purpose of this article is to prove Theorems A, B and C below.

**Theorem A.** Let $f : [0, 1] \to [0, 1]$, be a $C^3$ local diffeomorphism with negative Schwarzian derivative in the whole interval, except for a finite set $C_f \subset (0, 1)$. As usual, we assume the invariance of the boundary of the interval, i.e., $f(\{0, 1\}) \subset \{0, 1\}$. There is a finite collection of attractors $A_1, \cdots, A_n$, such that

$$\text{Leb}(\beta_f(A_1) \cup \cdots \cup \beta_f(A_n)) = 1.$$

Furthermore, for almost all points $x$, we have $\omega_f(x) = A_j$ for some $j = 1, \cdots, n$.

It is to be noted that the theorem above completes the program set up by the Palis Conjecture for maps of the interval with a finite number of discontinuities under the assumption of negative Schwarzian derivative.

Unimodal maps with negative Schwarzian derivative have been playing a major role in the recent development of the theory of dynamical systems. As mentioned before, Blokh and Lyubich [8] proved that any such map displays a unique metrical attractor as defined...
Figure 3. Let $0 < t \leq 1$ and $f_t : [0, 1] \to [0, 1]$ be given by $f_t(1/2) = t$ and $f_t(x) = t(1 - e^{2^{-1/|x - 1/2|}})$ if $x \neq 1/2$. The only critical point of $f_t$ is $c = 1/2$. In the picture, we draw the graph of $f_t$ with $t = 0.9$. Notice that $f_t$ is $C^\infty$, $f_t^{(n)}(1/2) = 0$ for all $n \geq 1$ and $Sf(x) = -(8/(1 - 2x)^4)$. Thus, $f_t$ is a family of flat $S$-unimodal maps.

below. With techniques developed in the present paper, we also provide in Theorem B a new proof of this fact in a slightly broader context, encompassing the case of a flat critical point (see Figure 3). To state this theorem we have to introduce the notion of cycle of intervals. A cycle of intervals is a transitive finite union of non-trivial closed intervals and it is a common type of attractor for maps of the interval that is associated to the existence of an absolutely continuous invariant measure.

**Theorem B** (Uniqueness of attractors for $S$-unimodal maps). If $f : [0, 1] \to [0, 1]$ is a $S$-unimodal map with critical point $c \in (0, 1)$ (the critical point may be flat), then there is an attractor $A \subset [0, 1]$ such that $\omega_f(x) = A$ for almost every $x \in [0, 1]$. In particular, $\text{Leb}(\beta_f(A)) = 1$. The attractor $A$ is either a periodic orbit or a cycle of intervals or a Cantor set. Furthermore, if $A$ is a Cantor set then $A = \overline{O_f(c)}$.

A point $p \in [0, 1]$ is called right-periodic with period $n$ for $f$ if $n$ is the smallest integer $\ell \geq 1$ such that $p = \lim_{0 < \varepsilon \to 0} \sup \{f^\ell((p - \varepsilon, p)) \cap [0, p]\}$. Similarly, $p$ is called left-periodic with period $n$ if $n$ is the smallest integer $\ell \geq 1$ such that $p = \lim_{0 < \varepsilon \to 0} \inf \{f^\ell((p, p + \varepsilon)) \cap (p, 1]\}$. We say that a point $p$ is periodic-like if it is a left or a right-periodic point. A fixed-like point is a periodic-like point with period equal to one.

We shall deal with two types of finite minimal attractors: one of them corresponds to the ordinary attracting periodic orbit, and the other is when it contains at least one point of the exceptional set $C_f$. In this last case we have an attracting periodic-like orbit that is not a periodic orbit of $f$.

Thus, in Figure 4 we have that $A = \{c\}$ is a periodic-like attractor with $\beta_f(A) = (0, 1)$. Furthermore, as $c = \lim_{0 < \varepsilon \to 0} \sup \{f((c - \varepsilon, c)) \cap [0, c]\}$, it follows that $c$ is a fixed-like attracting periodic point.

We say that $f : [0, 1] \setminus \{c\} \to [0, 1], 0 < c < 1$, is a contracting Lorenz map if $f(0) = 0$, $f(1) = 1$ and $f$ is an orientation preserving $C^2$ local diffeomorphism. If such a map is
$C^3$ with negative Schwarzian derivative and it does not have periodic attractors, we prove that there is only a single (minimal) attractor whose basin of attraction has full Lebesgue measure. Notice that our result extends the one in [42], since we do not require the non-flatness condition on the critical point.

**Theorem C.** Let $f$ be a $C^3$ contracting Lorenz map $f : [0, 1] \setminus \{c\} \rightarrow [0, 1]$, $c \in (0, 1)$, with negative Schwarzian derivative. If $f$ does not have periodic-like attractors, then $f$ has an attractor $A$ such that $\omega_f(x) = A$ for almost every $x \in [0, 1]$. In particular, Leb($\beta_f(A)$) = 1.

Furthermore, $f$ can have at most two periodic-like attractors. If $f$ has a single periodic attractor, its basin of attraction has full Lebesgue measure. In the case of $f$ having two periodic-like attractors, the union of their basins of attraction has full Lebesgue measure. If $f$ does not have periodic-like attractors then the attractor $A$ above is either a cycle of intervals or a Cantor set. If $A$ is a Cantor set then $A = \mathcal{O}_f^+(c_-) \cup \mathcal{O}_f^+(c_+)$, where \( \mathcal{O}_f^+(c_\pm) = \{ \lim_{\varepsilon \downarrow 0} f^n(c_\pm \varepsilon) ; n \geq 0 \} \).

1.2. **An outline of the paper**. Our main results are stated in the introduction above.

In Section 2 we state some basic results in one-dimensional dynamics, like Koebe’s Lemma, Singer and Mañé’s Theorems, as well as the Hominterval Lemma. We also introduce some notation to deal with lateral limits and lateral periodic orbits. In particular, we make use of some dynamical operations involving complex numbers to simplify our computations. Finally, we also present in this section some facts on cycles of intervals.

In Section 3 we prove the Interval Dichotomy (Proposition 13), which is one of the main ingredients of the paper. To do that, firstly we make a brief study of the ergodicity with respect to the Lebesgue measure of complete Markov maps adapted to our context (Lemmas 11 and 12) and use it to prove the dichotomy. As a consequence of the interval dichotomy, the omega-limit set of almost every point $x \in [0, 1]$, outside the basins of attraction of the periodic-like attracting orbits and the cycles of intervals, is contained in the closure of the union of the exceptional set (Corollary 14). This gives a sketch of the locus of the omega-limit of almost every point and this information is fundamental in the subsequent sections.
In Section 4, we apply the interval dichotomy to give a straightforward proof of the uniqueness of the attractor of a $S$-unimodal map (Theorem A), even if this map has a flat critical point.

In Section 5, we prove our main theorem (Theorem A). The strategy of most proofs in this section is a sort of a “parallax argument”. Parallax is the technique to make precise the position of an object by comparing the projections of this object on the background from two different viewpoints. In our context, the object to be considered is the omega-limit set of typical orbits, indeed some conveniently chosen subset $V$ of it, the background is the interval $[0, 1]$ and the projection is given by Corollary 14 commented above. The first viewpoint is given by the original map $f$. To get a second one, we construct a suitable auxiliary map $g$ that has a distinct exceptional set $C_g \neq C_f$ but keeps unchanged the omega-limit set of the points in $V$. Applying Corollary 14 to both $f$ and $g$, and comparing the omega-limit set with respect of the closure of the exceptional sets of $f$ and $g$, we are able to make precise the omega-limit set of the points that are not attracted by periodic-like attracting orbits or cycles of interval (Theorem B). With that, we prove Theorem A.

The last section of the paper, Section 6, is dedicated to the contracting Lorenz maps and the proof of Theorem C. This Section has three parts. In the first part we have the main ingredient of this Section that is the induced map given by Lemma 25. This induced map is crucial to prove the principal results of this Section. In the second part we study the contracting Lorenz maps without periodic-like attractor, proving the uniqueness of the attractor for those maps (Corollary 25). In the third part we study the basin of attraction of the periodic-like attractors, concluding the proof of Theorem C.

2. Setting and preliminary facts

Given a set $U \subset [0, 1]$, the pre-orbit of $U$ is $O_f^-(U) := \bigcup_{j \geq 0} f^{-j}(U)$. If $x \notin O_f^-(C_f)$, the forward orbit of $x$ is $O_f^+(x) = \{f^j(x) \mid j \geq 0\}$ and the omega limit set of $x$, denoted by $\omega_f(x)$, is the set of accumulating points of the sequence $\{f^n(x)\}_{n}$. That is, $\omega_f(x) = \{y \in [0, 1] \mid y = \lim_{j \to \infty} f^{n_j}(x) \text{ for some } n_j \to \infty\}$.

If $x \in O_f^-(C_f)$ then $O_f^+(x) = \{x, \ldots, f^n(x)\}$ where $n = \min\{j \geq 0 \mid x \in f^{-j}(C_f)\}$. A point $p \in [0, 1]$ is called wandering if there is a neighborhood $V$ of it such that $O_f^+(V) \cap V = \emptyset$, where $O_f^+(V) = \bigcup_{x \in V} O_f^+(x)$. If this is not the case, the point $p$ is called non-wandering. The non-wandering set of $f$, $\Omega(f)$, is the set of all non-wandering points $x \in [0, 1]$. One can easily prove that $\Omega(f)$ is compact and that $\omega_f(x) \subset \Omega(f)$ for all $x$.

We denote the set of periodic points of $f$ by $\text{Per}(f)$, that is, $\text{Per}(f) = \{p \in [0, 1] \mid \forall i \in \mathbb{N} \exists n \geq 1 \exists x \in [0, 1] \mid O_f^-(C_f) : f^n(p) = p \text{ for some } n \geq 1\}$.

We now quote some classic results that we shall use in the sequel.

**Theorem** (Koebe’s Lemma [83]). For every $\varepsilon > 0 \exists K > 0$ such that the following holds: let $M, T$ be intervals in $[0, 1]$ with $M \subset T$ and denote respectively by $L$ and $R$ the left and right components of $T \setminus M$ and let $U$ be an open subset of the interval $[a, b]$ and $f : U \to [a, b]$ be a map with negative Schwarzian derivative. If $f^n|_T$ is monotonous for a given $n \geq 1$ and

$$|f^n(L)| \geq \varepsilon |f^n(M)| \quad \text{and} \quad |f^n(R)| \geq \varepsilon |f^n(M)|$$


there is some \( n \) (Singer’s Theorem \([5]\))

Theorem (Singer’s Theorem \([5]\)). Consider an interval \( I = [a, b] \) and a finite set \( C_f \subset (a, b) \). If \( f: I \setminus C_f \to I \) is a \( C^3 \) local diffeomorphism with negative Schwarzian derivative and \( f(\partial I) \subset I \), then

1. the immediate basin of any attracting periodic orbit contains a point of \( C_f \) or a boundary point of \( I \);
2. each neutral periodic point \( p \notin \partial I \) is attracting (indeed, it is a saddle-node periodic point);
3. there exists no interval of periodic points (indeed, \( \# \Fix(f^n) < \infty \ \forall \ n \geq 1 \)).

In particular, the number of non-repelling periodic orbits is bounded if the number of critical points of \( f \) is finite.

Theorem (Mañe’s theorem \([32]\)). Let \( f : [a, b] \to \mathbb{R} \) be a \( C^2 \) map. If \( U \) is an open neighborhood of the critical points of \( f \) and every periodic point of \( [a, b] \setminus U \) is hyperbolic and expanding then \( \exists C > 0 \) and \( \lambda > 1 \) such that, for any \( n > 0 \),

\[ |Df^n(x)| > C\lambda^n \]

for every \( x \) such that \( \{x, \cdots, f^{n-1}(x)\} \cap U = \emptyset \).

Considering any \( C^2 \) map \( \tilde{f} : [0, 1] \to [0, 1] \) such that \( \tilde{f}|_{[0,1] \setminus U} = f|_{[0,1] \setminus U} \) and applying Mañe’s theorem to it, we conclude that \( |Df^n(x)| = |D\tilde{f}^n(x)| \geq C\lambda^n \) if \( \{x, \cdots, f^{n-1}(x)\} \cap U = \emptyset \). Thus, we get the following version of Mañe’s theorem:

Theorem (The piecewise \( C^2 \) Mañe’s theorem). Let \( f : [0, 1] \setminus C_f \to [0, 1] \) be a \( C^2 \) local diffeomorphism with \( C_f \subset (0, 1) \) being a finite set. If \( U \) is an open neighborhood of \( C_f \) and every periodic point of \( [a, b] \setminus U \) is hyperbolic and expanding then \( \exists C > 0 \) and \( \lambda > 1 \) such that

\[ |Df^n(x)| > C\lambda^n \]

for every \( x \) such that \( \{x, \cdots, f^{n-1}(x)\} \cap U = \emptyset \).

Corollary 1. Let \( f : [0, 1] \setminus C_f \to [0, 1] \) be a \( C^3 \) local diffeomorphism with \( C_f \subset (0, 1) \) being a finite set. If \( f \) has negative Schwarzian derivative and there are no saddle-nodes then \( \omega_f(x) \cap C_f \neq \emptyset \) for Lebesgue almost every \( x \in [0, 1] \setminus B_0(f) \), where \( B_0(f) \) is the basin of attraction of all attracting periodic orbits.

Proof. As \( Sf < 0 \), there are no weak repellers nor, by hypothesis, saddle-nodes and so, either a periodic orbit is a hyperbolic repeller or it is an attracting periodic orbit. Furthermore, the number of attracting periodic orbits is finite by the adapted Singer’s theorem. Let \( U_n = \{x \in O_f^+(C_f) : O_f^+(x) \cap B_{1/n}(A_0 \cup C_f) = \emptyset \} \), where \( A_0 \) is the union of the attracting periodic orbits of \( f \) and \( B_{1/n}(A_0 \cup C_f) = \bigcup_{p \in A_0 \cup C_f} B_{1/n}(p) \). Thus, if \( x \) does not belong to the basin of attraction of an attracting periodic orbit and \( O_f^+(x) \cap C_f = \emptyset \) then there is some \( n \in \mathbb{N} \) such that \( x \in U_n \). As each \( U_n \) is an uniformly expanding set and all uniformly expanding set of a \( C^1+ \) map of the interval has zero Lebesgue measure, we have that \( \text{Leb}(U_n) = 0 \ \forall \ n \geq 1 \). So, \( \text{Leb}(\bigcup_{n \geq 1} U_n) = 0 \), that is, \( \omega_f(x) \cap (A_0 \cup C_f) \neq \emptyset \).
Figure 5. $C^3$ local diffeomorphisms $f : [0, 1] \setminus C_f \to [0, 1]$ with negative Schwarzian derivative

As $\text{Leb}(\mathcal{O}_f(C_f)) = 0$, if $c \in \omega_f(x)$ for a Lebesgue typical point $x$, with $c \in C_f$, then $\mathcal{O}_f^+(x) \cap (c-\varepsilon, c+\varepsilon) = \mathcal{O}_f^+(x) \cap \left((c-\varepsilon, c+\varepsilon) \setminus \{c\}\right)$ for all $\varepsilon > 0$ and so, $\lim_{0<\varepsilon\to 0} f(c+\varepsilon)$ or $\lim_{0<\varepsilon\to 0} f(c-\varepsilon)$ belongs to $\omega_f(x)$ but not necessarily $f(c)$. That is, if $c \in \omega_f(x)$ the omega-limit set of such a typical point $x$ does not involve the image of the exceptional set (the exceptional values $f(C_f)$), but their lateral exceptional values $\mathcal{V}_f = \{\lim_{0<\varepsilon\to 0} f(c \pm \varepsilon) ; c \in C_f\}$.

Because of that, we can consider $f$ as a map from $[0, 1] \setminus C_f$ to $[0, 1]$, instead of a map of the interval to itself. As a consequence, we have:

**Remark.** To prove Theorem 4 we can consider $f$ as a $C^3$ local diffeomorphism $f : [0, 1] \setminus C_f \to [0, 1]$ with negative Schwarzian derivative, $f(\{0, 1\}) \subset \{0, 1\}$ and $C_f$ being a finite subset of $(0, 1)$, as in Figure 5. The set $C_f$ is called the exceptional set of $f$.

**Extended orbits.** Now we will introduce some notation to deal with lateral limits, lateral periodic orbits and other relevant concepts. We shall make use of some dynamical operations involving complex numbers, which we believe simplify our presentation.

In the remaining part of this section let $f : [0, 1] \setminus C_f \to [0, 1]$ be a $C^3$ local diffeomorphism. Let $\mathbb{i} \in \mathbb{C}$ be the imaginary unit ($\mathbb{i}^2 = -1$). Given $p \in [0, 1]$ set $f^0(p \pm \mathbb{i}) = p \pm \mathbb{i}$. For $n \geq 1$, we set

$$f(p - \mathbb{i}) = \lim_{x \uparrow p} f(x) - \mathbb{i} \lim_{x \uparrow p} \frac{f'(x)}{|f'(x)|}$$

and

$$f(p + \mathbb{i}) = \lim_{x \downarrow p} f(x) + \mathbb{i} \lim_{x \downarrow p} \frac{f'(x)}{|f'(x)|}$$
Definition 3. Given $\omega$ of accumulating points of the sequence $\{p_n\}$, let $\alpha$ be an element of the (Lateral exceptional set) $f$-fixed exceptional set, that is, $C = \{c - \delta; c \in C_f \} \cup \{c + \delta; c \in C_f\}$.

As usual $\mathcal{O}^+_f(p \pm \delta) = \{f^n(p \pm \delta); n \geq 0\}$ and $\omega_f(p \pm \delta) \supset [0, 1] \times \{-\delta, +\delta\} \subset \mathbb{C}$ is the set of accumulating points of the sequence $\{f^n(p \pm \delta)\}_{n \in \mathbb{N}}$.

As defined, if $p \in \mathcal{O}^+_f(C_f)$ then $\omega_f(p) = \emptyset$. Nevertheless, $f^n(p \pm \delta)$ is well defined for every $n \in \mathbb{N}$ and every $p \in [0, 1]$. If $p \notin \mathcal{O}^-_f(C_f)$ then $\omega_f(p) = \Re(\omega_f(p \pm \delta))$.

Definition 3. Given $x \in [0, 1] \setminus \mathcal{O}^+_f(C_f)$ and $p \in [0, 1]$, we say that $p - \delta \in \omega_f(x)$ if $p \in \overline{\omega_f(x)} \cap (0, p)$. Similarly, $p + \delta \in \omega_f(x)$ if $p \in \overline{\omega_f(x)} \cap (p, 1)$.

Definition 4 (Homterval). A homterval is an open interval $J = (a, b)$ such that $f^n|_J$ is a homeomorphism for $n \geq 1$. This is equivalent to assume that $J \cap \mathcal{O}^+_f(C_f) = \emptyset$. A homterval $J$ is called a wandering interval if $f^j(J) \cap f^k(J) = \emptyset$ for all $1 \leq j < k$.

Lemma 5 (Homterval Lemma). Suppose that $\text{Fix}(f^n)$ has empty interior $\forall n \geq 1$. Let $J = (a, b)$ be a homterval of $f$. If $I$ is not a wandering interval then $I \setminus \mathbb{B}_0(f)$ has empty interior, where $\mathbb{B}_0(f)$ is the union of the basins of attraction of all attracting periodic-like orbits of $f$. Furthermore, if $f$ is $C^3$ with $Sf < 0$, and $I$ is not a wandering interval, then the set $I \setminus \mathbb{B}_0$ has at most one point.

Proof. A homterval (as named by Misiurewicz) is an interval on which $f^j$ is monotone $\forall j \geq 0$, see, for example, [15]. Suppose $I$ is not a wandering interval. Then, there will be $k < \ell$ for which $f^k(I) \cap f^\ell(I) \neq \emptyset$. Define $f^k(I) = (a, b)$. If there is no integer $n$ such that $c \in f^n(I)$, we can then consider the union $(a, b) \cup (f^{\ell-k}(a), f^{\ell-k}(b))$, and this is an interval, as their intersection is non-empty.

Then $T := \bigcup_{j=0}^{\infty} f^{(\ell-k)j}(I)$ is a positively invariant homterval. Let $F$ be the continuous extension of $f^{\ell-k}|_T$ to $T$. Let $R$ be the set all repeller fixed point of $F^2$, i.e., $R = \{p \in T; F^2(p) = p \text{ and } \text{Leb}(\beta_{F^2}(p)) = 0\}$. As $F$ is a homeomorphism, $F(T) = T$ and $\text{Interior}(\text{Fix}(F^2)) = \emptyset$ (because $\text{Interior}(\text{Fix}(f^n)) = \emptyset$), we can see that $\overline{T \setminus R}$ is open and dense in $T$ and so, $I \setminus R$ is open and dense in $J$. Moreover, $I \setminus R \subset T \setminus R \subset \mathbb{B}(F^2) \subset T \setminus \mathbb{B}_0(f)$, where $\mathbb{B}(F^2)$ is the set of all attracting fixed point of $F^2$.

If $Sf < 0$ then the condition $\text{Fix}(f^n)$ has empty interior $\forall n \geq 1$ is automatic satisfied. Furthermore, as the extension $F$, given in the former paragraph, also has negative Schwarzian derivative, it follows that $\# R \leq 1$, proving the lemma.

As we are dealing with subsets of the interval, if the $\omega$-limit of a point is not totally disconnected then its interior is not empty. This implies, by Lemma 7 below, that either $\omega_f(x)$ is a totally disconnected set or it is a cycle of intervals.
Lemma 8. Let \( p, q \in [0, 1] \setminus O^+_f(C_f) \). If \( \text{Interior}(\omega_f(p)) \cap \text{Interior}(\omega_f(q)) \neq \emptyset \) then \( \omega_f(p) = \omega_f(q) \).

Proof. Let \( (\alpha, \beta) \subset \text{Interior}(\omega_f(p)) \cap \text{Interior}(\omega_f(x)) \). As \( \omega_f(p) \cap (\alpha, \beta) \neq \emptyset \), \( \exists n_p \geq 0 \) such that \( f^{n_p}(p) \in \omega_f(q) \). As \( f \) is continuous in \( O^+_f(p) \), we have that \( f^j(p) \in \omega_f(q) \forall j \geq n_p \) and then \( \omega_f(p) \subset \omega_f(q) \). Similarly, we have that \( \omega_f(p) \supset \omega_f(q) \).

\[ \triangle \]

A cycle of intervals may not be a minimal attractor (or even an attractor in Milnor’s sense). Indeed, this is the case of the so called wild attractors [9]. Nevertheless, as every
A cycle of intervals contains a critical point in its interior, it follows from Lemma 8 above that the number of cycles of intervals for $f$ is always finite.

**Corollary 9.** $f$ has at most $\#C_f$ distinct cycles of intervals.

If $A$ is an attracting periodic-like orbit, then Singer’s result assures that $A = \text{Re}(\omega_f(c - \delta))$ or $\text{Re}(\omega_f(c + \delta))$ for some $c \in C_f$. On the other hand, if $A$ is not a periodic-like attractor neither a cycle of intervals then we can use Theorem 1 to generated $A$ in terms of a subset of $C_f$. A cycle of intervals is the unique attractor that may not be traced by the critical orbits. Indeed, if $f : [0, 1] \setminus \{1/2\} \to [0, 1]$ is the restriction of the complete logistic map, $f(x) = 4x(1 - x)$, to $[0, 1] \setminus \{1/2\}$, then the whole interval $[0, 1]$ is transitive and $\omega_f(x) = [0, 1]$ for Lebesgue almost every $x$, that is, the cycle of intervals $A = [0, 1]$ is a minimal attractor for $f$. Nevertheless, the closure of lateral (extended) orbits of the critical point $c = 1/2$ does not contain the attractor $\overline{O^+_f((1/2)\pm)} = O^+_f((1/2)\pm) = \{1/2, 1, 0\} \not\supset A$.

3. **Markov maps and the interval dichotomy**

Let $\mathcal{X}$ be a compact metric space and $\mu$ a finite measure defined on the Borel sets of $\mathcal{X}$. Let $F : V \to \mathcal{X}$ be a measurable map defined on a Borel set $V \subset \mathcal{X}$ with full measure (i.e., $\mu(V) = \mu(\mathcal{X})$), to which we shall refer. Note that we are not requiring $\mu$ to be $F$-invariant. The map $F$ is called *ergodic* with respect to $\mu$ (or $\mu$ is called ergodic with respect to $F$) if $\frac{\mu(U)}{\mu(\mathcal{X})} = 0$ or 1 for every $F$-invariant Borel set $U$, noting that here $F$-invariant means $U = F^{-1}(U)$.

**Proposition 10.** If a measure $\mu$ is ergodic with respect to $F$ (not necessarily invariant) then there is a compact set $A \subset \mathcal{X}$ such that $\omega_F(x) = A$ for $\mu$ almost every $x \in \mathcal{X}$.
Proof. If $U \subset \mathcal{X}$ is an open set, then either $\omega_F(x) \cap U \neq \emptyset$ for $\mu$ almost every $x \in \mathcal{X}$ or $\omega_F(x) \cap U = \emptyset$ for $\mu$ almost every $x \in \mathcal{X}$. Indeed, taking $\tilde{U} = \{x \mid \omega_F(x) \cap U \neq \emptyset\}$ we have that $\tilde{U}$ is invariant and then, by ergodicity, either $\mu(\tilde{U}) = 0$ or $\mu(\tilde{U}) = \mu(\mathcal{X})$.

Note that if $\omega_F(x) \cap U \neq \emptyset$ for every open set $U$ and $\mu$ almost every $x$, then $\omega_F(x) = \mathcal{X}$ almost surely, proving the proposition. Thus, we may suppose the existence of a non-empty open $U \subset \mathcal{X}$ such that $\omega_F(x) \cap U = \emptyset$ for $\mu$ almost every $x$. Let $W$ be the maximal open set such that $\omega_F(x) \cap W = \emptyset$ for $\mu$ almost every $x \in \mathcal{X}$.

Now, we shall show that $\omega_F(x) = A$ for $\mu$ almost every $x \in \mathcal{X}$, where $A = \mathcal{X} \setminus W$. Indeed, given $p \in A$ we have that necessarily $\omega_F(x) \cap B_\varepsilon(p) \neq \emptyset$ for $\mu$ almost every $x \in \mathcal{X}$ and any $\varepsilon > 0$, for otherwise we would have $\varepsilon > 0$ such that $\omega_F(x) \cap B_\varepsilon(p) = \emptyset$ for Lebesgue almost every $x \in \mathcal{X}$, but then $B_\varepsilon(p) \cup W$ would contradict the maximality of $W$. Let $W_n = \{x; \omega_F(x) \cap B_\varepsilon(p) \neq \emptyset\}$. We have that $\mu(W_n) = \mu(\mathcal{X})$, $\forall n \in \mathbb{N}$, and thus, $\mu(\bigcap_n W_n) = \mu(\mathcal{X})$. As every $x \in (\bigcap_n W_n)$, we have that $\text{dist}(p, \omega_F(x)) = 0$ for any $x \in \bigcap_n W_n$ and then $p \in \omega_F(x) = \omega_F(x)$ proving Proposition 10. \hfill \Box

**Lemma 11.** Let $a < b \in \mathbb{R}$ and $V \subset (a, b)$ be an open set. Let $\mathcal{P}$ be the set of connected components of a Borel set $V$. Let $G : V \to (a, b)$ be a map satisfying:

1. $G(P) = (a, b)$ diffeomorphically, for any $P \in \mathcal{P}$;
2. $\exists V' \subset \bigcap_{j \geq 0} G^{-j}(V)$, with $\text{Leb}(V') > 0$, such that
   a. $\lim_{n \to \infty} |\mathcal{P}_n(x)| = 0$, $\forall x \in V'$, where $\mathcal{P}_n(x)$ is the connected component of $\bigcap_{j=0}^n G^{-j}(V)$ that contains $x$;
   b. $\exists K > 0$ such that $\left| \frac{DG^n(p)}{DG^n(q)} \right| \leq K$, for all $n$, and $p, q \in \mathcal{P}_n(x)$, and $x \in V'$.

Then, $\text{Leb}([a, b] \setminus V) = 0, \omega_G(x) = [a, b]$ for Lebesgue almost all $x \in [a, b]$.

**Proof.** Firstly, we show that

**Claim 1.** Every positively invariant set $U \subset V'$ with positive measure has measure $|b - a|$.

**Proof.** Suppose that $U \subset V'$ is positively invariant with positive measure. Note that necessarily $U \subset \bigcap_{j \geq 0} G^{-j}(V)$.

By the Lebesgue Density Theorem, there is $p \in U$ (indeed for Lebesgue almost every $p \in U$) such that

$$
\lim_{n \to \infty} \frac{\text{Leb}(\mathcal{P}_n(p) \setminus U)}{\text{Leb}(\mathcal{P}_n(p))} = 0 \quad \text{(1)}
$$

By the bounded distortion hypothesis, item \[2\], since $U$ is positively invariant, it follows from \[1\] that

$$
\text{Leb}((a, b) \setminus U) = 0
$$

Indeed,

$$
\text{Leb}((a, b) \setminus U) \leq \text{Leb}(G^n(\mathcal{P}_n(p) \setminus U))
$$

$$
= \text{Leb}((a, b) \frac{\text{Leb}(G^n(\mathcal{P}_n(p) \setminus U))}{\text{Leb}(G^n(\mathcal{P}_n(p))))} \leq \text{Leb}((a, b))K \frac{\text{Leb}(\mathcal{P}_n(p) \setminus U)}{\text{Leb}(\mathcal{P}_n(p))} \to 0 \quad \text{(2)}
$$

Here the inequality \[2\] follows from the fact that $U \supset G^n(\mathcal{P}_n(p) \cap U)$ and then $(a, b) \setminus U \subset (a, b) \setminus G^n(\mathcal{P}_n(p) \cap U)) = G^n(\mathcal{P}_n(p) \setminus U)$. And the last equality we get by writing
\((a, b) = G^n(\mathcal{P}_n(p))\), and as \(G^n(\mathcal{P}_n(p))\) is a bijection, \((a, b)\) can be written as a disjoint union of \(G^n(\mathcal{P}_n(p) \setminus U)\) and \(G^n(\mathcal{P}_n(p) \cap U)\).

It follows from the claim above that \(\text{Leb}((a, b) \setminus V') = 0\) because \(V'\) itself is a positively invariant set with positive measure. Thus, \(\text{Leb}(V) = \text{Leb}(V') = \text{Leb}((a, b))\). As an invariant set is also a positively invariant set, it also follows from the claim that every invariant subset of \([a, b]\) with positive measure has full measure. That is, \(G\) is ergodic with respect to Lebesgue measure (more precisely, with respect to \(\text{Leb}|_{[a,b]}\)).

From Proposition 10 there is a compact set \(A \subset [a, b]\) such that \(\omega_G(x) = A\) for Lebesgue almost every \(x \in [a, b]\).

**Claim 2.** \(\text{Leb}(A) > 0\)

**Proof.** Suppose that \(\text{Leb}(A) = 0\). Then, given \(\varepsilon > 0\), there exists an open neighborhood \(M_\varepsilon\) of \(A\) such that \(\text{Leb}(M_\varepsilon) < \varepsilon\). Let \(\Omega_\varepsilon(n) = \{x \in (a, b); \mathcal{O}_x^G(G^n(x)) \subset M_\varepsilon\}\). Observe that \(\text{Leb}(\bigcup_n \Omega_\varepsilon(n)) = |b - a|\), as \(\omega(x) = A\) for Lebesgue almost all \(x\). Then, \(\exists n_0\) such that \(\text{Leb}(\Omega_\varepsilon(n_0)) > 0\). As \(\Omega_\varepsilon(n_0)\) is positively invariant, it also follows that \(G^{n_0}(\Omega_\varepsilon(n_0))\) is positively invariant. Then, by Claim 1 \(\text{Leb}(G^{n_0}(\Omega_\varepsilon(n_0))) = |b - a|\). This is a contradiction since \(G^{n_0}(\Omega_\varepsilon(n_0)) \subset M_\varepsilon\) and \(\text{Leb}(M_\varepsilon) < \varepsilon\). \(\square\)

As \(A\) is positively invariant and \(\text{Leb}(A) > 0\), by Claim 2 it follows from Claim 1 that \(\text{Leb}(A) = |b - a|\). As \(A\) is a compact set, it follows that \(A = [a, b]\). Thus \(\omega_G(x) = A = [a, b]\) for Lebesgue almost every \(x\), proving the Lemma. \(\square\)

**Lemma 12.** Let \(U \subset (a, b) \subset \mathbb{R}\) be an open set and \(F : U \to (a, b)\) be a \(C^3\) local diffeomorphism with \(SF < 0\). Let \(\mathcal{P}\) be the collection of connected components of \(U\). If there is a positively invariant set \(V \subset U\) with positive measure such that

1. \(F(\mathcal{P}(x)) = (a, b) \ \forall x \in V\)
2. \(V\) does not intersect the basin of attraction of any periodic-like attractor of \(F\)

then \(\text{Leb}([a, b] \setminus V) = 0\), \(F\) is ergodic with respect to Lebesgue measure and \(\omega_F(x) = [a, b]\) for Lebesgue almost every \(x \in [a, b]\).

**Proof.** Let \(I = (a, b)\) and choose \(a < a' < b' < b\) such that \(\text{Leb}(V') > 0\), where \(V' = \{x \in V; \omega_F(x) \cap (a', b') \neq \emptyset\}\). Write \(J = (a', b')\) (we will consider \(J \subset I\) instead of \(I\), so that we can apply Koebe’s Lemma). As \(F_*\text{Leb}\) is an absolutely continuous measure, \(\text{Leb}(V' \cap J) > 0\) (indeed, \(V' = \bigcup_{n \geq 0} f^{-n}(V' \cap J)\) implies that \(\text{Leb}(f^{-j}(V' \cap J)) > 0\) for some \(j \geq 0\) and, as a consequence of \(F_*\text{Leb} \ll \text{Leb}\), we get \(\text{Leb}(V' \cap J) > 0\).

Let \(F_J : J^* \to J\) be the first return map to \(J\) by \(F\), where \(J^* = \{x \in J; \mathcal{O}_x^+(F(x)) \cap J \neq \emptyset\}\).

**Claim 3.** If \(L\) is a connected component of \(J^*\), then \(F_J(L) = J\), there are \(T \supset L\) and \(t > 0\) such that \(F^t|_T\) is a diffeomorphism, \(F_J|_L = F^t|_L\) and \(F^t(T) = I \supset J = F_J(J)\).

**Proof of the claim.** As \(F(T) = I\) for every \(T \in \mathcal{P}\), we also have \(F^n(T) = I\) for any connected component \(T\) of \(F^{-n}(I)\). On the other hand, as \(F_J\) is also the first return map to \(J\) by \(F\), we have \(F_J|_L = F^t|_L\) for some \(t > 0\). Taking \(T\) as the connected component of \(F^{-t}(I)\) that contains \(L\), we have \(F^t(T) = I\). \(\square\)
we may conclude that

\[ J \]

As \( F \) the set of "lateral exceptional values" of \( f \) for almost every \( x \in F_J^{-n}(J) \),
\[
\text{where } P_n(x) \text{ is the connected component of } F_J^{-n}(J) \text{ that contains } x.
\]

Proof. Given \( x \in F_J^{-n}(J) \), let \( t > 0 \) and \( T \supseteq P_n(x) \) be as in Claim 3. Then \( F^n_j|_{P_n(x)} = F^t|_{P_n(x)} \), \( F^t|_{T} \) is diffeomorphism and \( F^t(T) = I \supseteq J = F^n_j(P_n(x)) \). As \( Sf < 0 \), it follows from Koebe’s Lemma that there is \( K \), depending only on \( \frac{a - a'}{b - a} \) and \( \frac{b'}{b - a} \), such that
\[
\frac{\left| (F^n_j)'(p) \right|}{\left| (F^n_j)'(q) \right|} \leq K \text{ for all } p, q \in P_n(x).
\]

\]

Claim 4. We have \( |P_n(x)| \rightarrow 0 \), for Lebesgue almost every \( x \in V' \).

Proof of the claim. Suppose that there is a Lebesgue density point \( x \) of \( V' \) such that
\[
\lim_{n \to \infty} |P_n(x)| > 0.
\]

Then, \( M := \text{Interior}(\cap_{n>0} P_n(x)) \) is an open interval with

\[
P_1(x) \supset P_2(x) \supset \cdots \supset P_n(x) \to M.
\]

As \( F^n_j(P_n(x)) = J \forall n \), it follows from the bounded distortion in Claim 4 that \( F^n_j(M) \to \infty \) \( J \). Thus, there exists \( \ell \) be big enough so that \( F^n_j(M) \cap M \neq \emptyset \). As \( F^n_j|M \) is diffeomorphism \( \forall n \), defining \( \tilde{M} = \bigcup_{n>0} F^n_{\ell}(M) \) we have that \( F^n_{\ell}(\tilde{M}) = \tilde{M} \). This implies, since \( S\tilde{F} < 0 \), that \( F^n_{\ell}|_{\tilde{M}} \) has one, \( p_1 \), or at most two attracting fixed-like points \( p_1, p_2 \) such that \( \text{Leb}(\tilde{M} \setminus \beta(A)) = 0 \), where \( A = \{p_1\} \) or \( \{p_1, p_2\} \). Therefore, Lebesgue almost every point of the neighborhood of \( x \) is containing in the basin of attraction of some attracting periodic-like orbit, contradicting the fact that \( V' \) does not intersec the basin of the any periodic-like attractor (recall that \( x \) is a density point of \( V' \)).

It follows from Claim 4 and 5 that \( F_j \) satisfies the hypothesis of Lemma 11 and thus \( \omega_{F_j}(\alpha) \supset [a', b'] \) for Lebesgue almost every \( x \in [a', b'] \). Therefore \( \omega_{F}(\alpha) \supset \omega_{F_j}(\alpha) = [a', b'] \) for almost every \( x \in [a', b'] \). As we can take \( a' \) as close to \( a \) and \( b' \) as close to \( b \) as we want, we may conclude that \( \omega_{F}(\alpha) = [a, b] \) for Lebesgue almost every \( x \in [a, b] \).

Let \( \mathcal{B}_0(f) \) be the union of the basins of the periodic-like attractors for \( f \). Denote by \( V_f \) the set of “lateral exceptional values” of \( f \), i.e., \( V_f = \{f(\alpha) : \alpha \in \mathcal{C}_f\} \), and let \( O^+_f(V_f) = \{ f^n(\alpha) : \alpha \in \mathcal{C}_f \text{ and } n \geq 1 \} \).

Proposition 13 (Interval Dichotomy). Let \( I = (a, b) \subset [0, 1] \) be an interval such that \( \text{Re}(\mathcal{O}^+_f(V_f)) = \emptyset \). If \( \text{Leb}(I \setminus \mathcal{B}_0(f)) > 0 \) then either \( \#(\mathcal{O}^+_f(x) \cap I) < \infty \) for almost all \( x \in I \) (in particular, \( \omega_f(x) \cap I = \emptyset \)) or \( \omega_f(x) \supset I \) for almost every \( x \in I \).

Proof. Let \( F : I^* \to I \) be the first return map to \( I \), with \( I^* = \{x \in I : \mathcal{O}^+_f(f(x)) \cap I \neq \emptyset \} \). Let \( \mathcal{P} \) be the set of connected components of \( I^* \).
Claim 6. \( F \) is a local diffeomorphism having negative Schwarzian derivative and \( F(P) = I \) for \( \forall P \in \mathcal{P} \).

Proof of the claim. As \( f \) is a local diffeomorphism with \( Sf < 0 \), it follows that \( F \) is also a local diffeomorphism with \( SF < 0 \).

Given \( P \in \mathcal{P} \), there is some \( m > 0 \) such that \( F|_P = f^m|_P \). Write \( P = (p_0, p_1) \). If \( F(P) \neq I \) then \( \exists i = 0 \) or 1 and \( 0 \leq n < m \) such that \( f^n(p_i) \in \mathcal{C}_f \). That is, \( \alpha := f^n(p_i + (-1)^i) \in \mathcal{C}_f \) and \( \Re(f^m(p_i + (-1)^i)) = \Re(f^{m-(n+1)}(\alpha)) \in \Re(O_f^+(V_f)) \cap I \) contradicting our hypothesis. \( \square \)

Now let \( \mathcal{V} = \{ x \in I \setminus \mathcal{B}_0; \#(O_f^+(x) \cap I) = \infty \} \), i.e., \( \mathcal{V} = (\bigcap_{n \geq 0} F^{-n}(I)) \setminus \mathcal{B}_0 \), and assume that \( \text{Leb}(\mathcal{V}) > 0 \). Note that \( \mathcal{V} \) is a \( F \)-positively invariant set with positive measure and it does not intersect the basin of attraction of any periodic-like attractor of \( F \). Thus, the first return map \( F \) satisfy all the hypotheses of Lemma 11. As a consequence, \( \text{Leb}(I \setminus \mathcal{V}) = 0 \) and \( \omega_F(x) \supset \omega_F(x) = I \cup I \) for almost every \( x \in I \).

As before, let \( \mathcal{B}_0(f) \) be the union of the basins of attraction of all periodic-like attractors. Let \( \mathcal{B}_1(f) = \{ x \in [0, 1]; \text{Interior}(\omega_f(x)) \neq \emptyset \} \). By Lemma 11, \( \mathcal{B}_1(f) \) is the set of points \( x \in [0, 1] \) such that \( \omega_f(x) \) is a cycle of intervals. In particular, \( \mathcal{B}_1(f) \) is contained in the union of the basins of attraction of all cycles of intervals.

Corollary 14 (The \( \omega \)-locus sketch). For almost all \( x \in [0, 1] \setminus (\mathcal{B}_0(f) \cup \mathcal{B}_1(f)) \),

\[
\omega_f(x) \subset \bigcup_{\alpha \in \mathcal{C}_f} \Re(O_f^+(\alpha)).
\]

Proof. As the collection \( \mathcal{P} \) of all connected components of \( [0, 1] \setminus \bigcup_{\alpha \in \mathcal{C}_f} \Re(O_f^+(\alpha)) \) is a countable set of intervals, it follows from the interval dichotomy that \( \text{Leb}(\{ x \in I \setminus (\mathcal{B}_0(f) \cup \mathcal{B}_1(f)); \omega_f(x) \cap I \neq \emptyset \}) = 0 \forall I \in \mathcal{P} \). Thus,

\[
0 \leq \text{Leb} \left( \left\{ x \in [0, 1] \setminus (\mathcal{B}_0(f) \cup \mathcal{B}_1(f)); \omega_f(x) \not\subset \bigcup_{\alpha \in \mathcal{C}_f} \Re(O_f^+(\alpha)) \right\} \right) \leq \sum_{I \in \mathcal{P}} \text{Leb} \left( \{ x \in I \setminus (\mathcal{B}_0(f) \cup \mathcal{B}_1(f)); \omega_f(x) \cap I \neq \emptyset \} \right) = 0.
\]

\( \square \)

Corollary 15 (Avoiding isolated points). If \( q \) is an isolated point of \( \bigcup_{\alpha \in \mathcal{C}_f} \Re(O_f^+(\alpha)) \) then

\[
\text{Leb} \left( \{ x \in [0, 1] \setminus (\mathcal{B}_0(f) \cup \mathcal{B}_1(f)); q \in \omega_f(x) \} \right) = 0.
\]

Proof. Let \( \varepsilon > 0 \) be so that \( \{ q \} = B_\varepsilon(q) \cap \bigcup_{\alpha \in \mathcal{C}_f} \Re(O_f^+(\alpha)) \). Suppose that \( \text{Leb}(\mathcal{U}) > 0 \), where \( \mathcal{U} = \{ x \in [0, 1] \setminus (\mathcal{B}_0(f) \cup \mathcal{B}_1(f)); \omega_f(x) \} \). Let \( \mathcal{U}_0 = \{ x \in (q - \varepsilon, q) ; \#(O_f^+(x) \cap (q - \varepsilon, q)) = \infty \} \) and \( \mathcal{U}_1 = \{ x \in (q, q + \varepsilon) ; \#(O_f^+(x) \cap (q, q + \varepsilon)) = \infty \} \). As \( \text{Leb} \circ f^{-1} \ll \text{Leb} \), we get \( \text{Leb}(\mathcal{U} \cap B_\varepsilon(q)) > 0 \). As \( \mathcal{U}_0 \cup \mathcal{U}_1 \supset (\mathcal{U} \cap B_\varepsilon(q) \setminus O_f^+(q)) \), it follows that \( \text{Leb}(\mathcal{U}_0 \cap \mathcal{U}) \) or \( \text{Leb}(\mathcal{U}_1 \cap \mathcal{U}) > 0 \).
Suppose, for instance, that $\text{Leb}(\mathcal{U}_0 \cap \mathcal{U}) > 0$ (the proof for $\text{Leb}(\mathcal{U}_1 \cap \mathcal{U}) > 0$ is analogous). Writing $I = (q - \varepsilon, q)$, we can observe that $I \cap \text{Re}(\mathcal{O}_f^+(V)) = \emptyset$. As $\text{Leb}(I \setminus \mathbb{B}_0(f))$ and $\text{Leb}(\{x \in I : \#(\mathcal{O}_f^+(x) \cap I) = \infty\}) \geq \text{Leb}(\mathcal{U}_0 \cap \mathcal{U}) > 0$, it follows from Proposition 13 that $\omega_f(x) \supset I$ for almost all $x \in I$. As this implies that $I$ is contained in a cycle of intervals, we have $I \subset \mathbb{B}_1(F)$, contradicting that $\text{Leb}(I \cap \mathcal{U}) \geq \text{Leb}(\mathcal{U}_0 \cap \mathcal{U}) > 0$. □

4. UNIMODAL MAPS

A $C^3$ map $f : [0, 1] \to [0, 1]$, with $f(0) = f(1) = 0$, is called a $S$-unimodal map if it has negative Schwarzian derivative and a single critical point $c \in (0, 1)$. By Singer’s theorem, such a map $f$ has at most one attracting periodic orbit. In Lemma 16 below, we shall make use the interval dichotomy to prove that if $f$ has an attracting periodic orbit, then its basin of attraction has full Lebesgue measure.

We observe that this lemma is not a straightforward application of Mañé’s theorem, at least in the presence of a saddle-node. Because of the negative Schwarzian derivative, $f$ cannot have weak repellers and so, the only type of non-hyperbolic periodic orbit allowed is the saddle-node one. Nevertheless, if $f$ has a saddle-node, the closure of the complement of the basin of attraction contains the saddle-node. This implies that the complement of the basin is not an uniformly expanding set and so, it is not straightforward that it has zero Lebesgue measure.

Lemma 16 (Uniqueness of periodic attractors for $S$-unimodal maps). If $f : [0, 1] \to [0, 1]$ is a $S$-unimodal map having an attracting periodic point $p$ then $\text{Leb}(\beta_f(\mathcal{O}_f^+(p))) = 1$.

Proof. Let $0 < c < 1$ be the critical point of $f$ and let $\hat{p} = \max \mathcal{O}_f^+(p)$. It is easy to see that $J = f^{-1}((\hat{p}, 1])$ is a nice interval containing $c$. Write $(a, b) = J$. Notice that $a$ or $b \in \mathcal{O}_f^+(p)$. Furthermore, by Singer’s Theorem we have that $\omega_f(x) = \mathcal{O}_f^+(p)$, for every $x \in J$.

We claim that $\mathcal{O}_f^+(x) \cap J = \emptyset$ for almost every $x \in [0, 1]$. To show this, let’s consider $g : [0, 1] \setminus C_g \to [0, 1]$, defined by

$$g(x) = \begin{cases} 
  f(x) & \text{if } x \notin J \\
  \lambda(f(x) - f(p)) + f(p) & \text{if } x \in J
\end{cases}$$

where $\lambda = (1 - f(p))/(f(c) - f(p))$ and $C_g = \{c\} \cup \partial J$, see Figure 7.

Let $\mathcal{U} = [0, 1] \setminus \mathbb{B}_0(f)$ and assume, by contradiction, that $\text{Leb}(\mathcal{U}) > 0$. Notice that $\mathcal{O}_f^+(x) = \mathcal{O}_g^+(x)$, $\forall x \in \mathcal{U}$. In particular, $\mathbb{B}_0(g) \cap \mathcal{U} = \emptyset$.

Let $I_1, \ldots, I_t$ be the connected components of $([0, 1] \setminus J) \setminus \mathcal{O}_f^+(p)$ and observe that $\mathcal{U} \subset \bigcup_{i=1}^t I_i$. In particular, there is $t_0$ so that

$$\text{Leb}\big\{x \in \mathcal{U} : \#(\mathcal{O}_g^+(x) \cap I_{t_0}) = \infty\big\} > 0.$$ 

As $\text{Re}(\mathcal{O}_g^+(g(a \pm \delta))) = \text{Re}(\mathcal{O}_g^+(g(b \pm \delta))) = \mathcal{O}_g^+(p)$ and $\text{Re}(\mathcal{O}_g^+(g(c \pm \delta))) = \{0, 1\}$, we get that $I_{t_0} \cap \text{Re}(\mathcal{O}_g^+(V)) = \emptyset$. Thus, by the interval dichotomy, $\omega_g(x) \supset I_{t_0}$ for almost every $x \in I_{t_0}$. This implies that $I_0$ does not intersect the basin of attraction of a periodic attractor nor it is a wandering interval. Thus, by the homterval lemma, there is some $n \geq 0$ such that $g^n|_{I_{t_0}}$ is a diffeomorphism and $\{a, b, c\} \cap g^n(I_{t_0}) \neq \emptyset$, but this implies that $g^n(I_{t_0}) \cap J \neq \emptyset$ and as a consequence the orbit (with respect to $g$ and also $f$) of almost
Figure 7. This is the figure referred to by Lemma 16. The map $f$ on the left side in a $S$-unimodal map with a nice interval $J$ having a fixed point on its boundary. On the right side of the figure, we have the map $g$ associated to $f$ as in the proof of Lemma 16.

Every point of $U \cap I_0$ intersects $J$, contradicting the definition of $U$. Therefore, $U$ must be a zero measure set, proving the lemma.

Now we are able to prove the uniqueness of attractor for $S$-unimodal maps (Theorem B).

Proof of Theorem B. By Lemma 16, we may assume that $f$ does not have a periodic attractor. As $\text{Re}(O_f^+(c-i)) = \text{Re}(O_f^+(c+i)) = O_f^+(c)$, it follows from Corollary 14 that $\omega_f(x) \subset O_f^+(c)$ for almost every $x \in [0,1] \setminus \mathbb{B}(f)$. As we are assuming that $f$ does not have a periodic attractor, in particular it does not have a saddle-node. So, all periodic orbits are hyperbolic repellers. Applying Mañé’s theorem, it follows that $c \in \omega_f(x)$ for almost every $x \in [0,1]$. As a consequence, $\overline{O_f^+(c)} \subset \omega_f(x)$ almost surely. On the other hand, Corollary 14 implies that $\omega_f(x) \subset \overline{O_f^+(c)}$ for almost every $x \in [0,1] \setminus \mathbb{B}(f)$. Thus,

$$\omega_f(x) = \overline{O_f^+(c)} \text{ for almost every } x \in [0,1] \setminus \mathbb{B}(f).$$

(3)

As $f$ does not have periodic attractors, $\mathbb{B}(f) = \mathbb{B}_1(f)$, that is, $\mathbb{B}(f)$ is the union of the basins of attraction of all cycles of intervals of $f$. If $\text{Leb}(\mathbb{B}_1(f)) = 0$ (in particular, if $f$ does not have a cycle of intervals), then the theorem is proved by taking $A = \overline{O_f^+(c)}$.

Indeed, because of (3), we need only to verify that $\overline{O_f^+(c)}$ is a perfect set and this follows straightforward from (3) and Corollary 15.

Therefore, we may suppose that $f$ has a cycle of intervals $U = I_1 \cup \cdots \cup I_n$, $I_j = [a_j, b_j]$, and also that $\text{Leb}(\mathbb{B}_1(f)) > 0$. It follows from Corollary 9 that this cycle of intervals is unique. Thus, $\mathbb{B}_1(f) = \{x \in [0,1]; \omega_f(x) = U\}$. Furthermore, $c \in (a_k, b_k)$ for some $1 \leq k \leq n$ (Lemma 7).
Claim. \( \omega_f(x) = U \) for almost every \( x \in [0,1] \).

Proof of the claim. Suppose by contradiction that \( \text{Leb}([0,1] \setminus \mathcal{B}_1(f)) > 0 \). In this case, we have that \( \text{Interior}(\omega_f(c)) = \emptyset \). Otherwise, by the Lemma 17, \( \omega_f(c) \) is a cycle of intervals, and so \( \omega_f(c) = U \). As a consequence, \( c \in \omega_f(c) \) and then \( \mathcal{O}_f^+(c) = \omega_f(c) = U \). By (3) and the definition of \( \mathcal{B}_1(f) \), we get \( \omega_f(x) = U \) for almost every \( x \in [0,1] \), i.e., \( \text{Leb}([0,1] \setminus \mathcal{B}_1(f)) = 0 \), contradicting the assumption.

As \( \text{Interior}(\omega_f(c)) = \emptyset \), we can pick an open interval \( I = (a, b) \) contained in \( U \setminus \omega_f(c) \). As \( \mathcal{O}_f^+(c) \cap I = \emptyset \) and we know that \( \text{Leb}(\{x; \omega_f(x) \supset U \supset I\}) > 0 \), it follows from the interval dichotomy (Proposition 13) that \( \omega_f(x) \supset I \) for almost every \( x \in I \). By the homotopy lemma, either \( c \in I \) or there is some \( \ell \geq 1 \) such that \( f^\ell|_I \) is a diffeomorphism and \( c \in f^\ell(I) \). Thus, \( \omega_f(x) \supset f^\ell(I) \) for almost every \( x \in I \) and also for almost every \( x \in T := f^\ell(I) \). As a consequence, \( \text{Interior}(\omega_f(x)) \cap \text{Interior}(U) \neq \emptyset \) for almost every \( x \in I \). Therefore, it follows from Lemma 5 that \( \omega_f(x) = U \) for almost every \( x \in T \).

Let \( T' = \{x \in T; \omega_f(x) = U\} \). As \( T \) is an open interval containing \( c \), it follows from Mañe’s theorem that \( \text{Leb}(\bigcup_{n \geq 0} f^{-n}(T')) = 1 \) and so, \( \text{Leb}(\bigcup_{n \geq 0} f^{-n}(T')) = 1 \) (because \( f, \text{Leb} \ll \text{Leb} \)). That is, \( \text{Leb}([0,1] \setminus \mathcal{B}_1(f)) = 0 \), contradicting our assumption. \( \square \)

It follows from the claim that \( A := U \) is a minimal attractor and \( \beta_f(A) \) contains almost every point of \([0,1] \), concluding the proof of the theorem. \( \square \)

5. Proof of the finiteness of the number of attractors

Lemma 17. Let \( g : [0,1] \setminus C_g \to [0,1] \) be a \( C^3 \) local diffeomorphism with \( Sg < 0 \) and \( C_g \subset (0,1) \) being a finite set. If \( \exists I = (c_0, c_1) \) and \( p \in I \) such that \( c_0, c_1 \in C_g, I \cap C_g = \emptyset, g|_I \) is a contraction with attracting fixed point \( p \), then for almost all \( x \in [0,1] \setminus (\mathcal{B}_0(g) \cup \mathcal{B}_1(g)) \) we have

\[
\omega_g(x) \subset \bigcup_{\alpha \in C_g \setminus C_g(I)} \text{Re}(\mathcal{O}_g^+(\alpha)),
\]

where \( C_g(I) = \{\eta \in C_g; \text{Re}(\mathcal{O}_g^+(\eta)) \cap I \neq \emptyset\} \supset \{c_0 + \mathbb{i}, c_1 - \mathbb{i}\} \) and \( C_g = C_g \pm \mathbb{i} \).

Figure 8. This is the graphic of a map \( g \) as in the hypothesis of Lemma 17.
ON THE FINITENESS OF ATTRACTORS FOR ONE-DIMENSIONAL MAPS WITH DISCONTINUITIES

Proof. Note that

\[
\bigcup_{\alpha \in \mathcal{C}_g} \text{Re}(\mathcal{O}_g^+(\alpha)) = \left( \bigcup_{\alpha \in \mathcal{C}_g} \text{Re}(\mathcal{O}_g^+(\alpha)) \right) \cup \left( \bigcup_{\alpha \in \mathcal{C}_g(I)} \text{Re}(\mathcal{O}_g^+(\alpha)) \right) = \bigcup_{\alpha \in \mathcal{C}_g(I)} \text{Re}(\mathcal{O}_g^+(\alpha)) \cup \{p\}
\]

Let \( n_0 \geq 0 \) be such that \( \text{Re}(g^n(\eta)) \in I \) for all \( n \geq n_0 \) and all \( \eta \in \mathcal{C}_g(I) \). Let \( D = \{ x \notin \mathbb{B}_0(g) \cup \mathbb{B}_1(g) : \omega_f(x) \subset \bigcup_{\alpha \in \mathcal{C}_g} \text{Re}(\mathcal{O}_g^+(\alpha)) \} \). For every \( x \in D \) and \( q \in \omega_f(x) \setminus K \) we have \( q \in \omega_f(x) \cap \left( \{p\} \cup \bigcup_{\alpha \in \mathcal{C}_g(I)} \text{Re}(\mathcal{O}_g^+(\alpha)) \right) \). As \( \omega_g(x) \cap I = \emptyset \), because \( x \notin \mathbb{B}_0(g) \cup \mathbb{B}_1(g) \), we get \( q \in \omega_f(x) \cap \left( \bigcup_{\alpha \in \mathcal{C}_g(I)} \bigcup_{j=0}^{n-1} \text{Re}(g^j(\alpha)) \right) \). That is

\[
\omega_g(x) \setminus K \subset \left( \bigcup_{\alpha \in \mathcal{C}_g(I)} \bigcup_{j=0}^{n-1} \text{Re}(g^j(\alpha)) \right) \setminus K,
\]

for all \( x \in [0, 1] \setminus (\mathbb{B}_0(g) \cup \mathbb{B}_1(g)) \).

Thus, we can conclude that

\[
\bigcup_{x \in D} \omega_g(x) \subset K \cup W \subset \bigcup_{\alpha \in \mathcal{C}_g} \text{Re}(\mathcal{O}_g^+(\alpha))
\]

As \( K \) is a compact set, \( W \) is a finite set and \( K \cap W = \emptyset \), it follows that every point of \( W \) is an isolated point of \( \bigcup_{\alpha \in \mathcal{C}_g} \text{Re}(\mathcal{O}_g^+(\alpha)) \). Thus, follows from Corollary 15 that

\[
\text{Leb}\{x \in D : \omega_g(x) \not\subset K\} = 0.
\]

Finally, as \( [0, 1] \setminus (\mathbb{B}_0(g) \cup \mathbb{B}_1(g)) = D \mod 0 \) (Corollary 14), we finish the proof. \( \square \)

By definition, if \( J \) is a wandering interval then \( J \cap \mathcal{C}_f = \emptyset \). Nevertheless, the border of \( J, \partial J \), may contain some \( c \in \mathcal{C}_f \). In this case, we have either \( c - i \in J \) or \( c + i \in J \). Let \( W_f \) be the set of all \( \alpha \in \mathcal{C}_f \) contained in a wandering interval. If \( \alpha = c - i \in W_f \), define \( p_\alpha \) as the infimum of all \( 0 < t < c \) such that \( (t, c) \) is a wandering interval and define \( J_\gamma = (p_\alpha, c) \). Analogously, if \( \alpha = c + i \in W_f \), define \( p_\alpha \) as the supremum of all \( c < t < 1 \) such that \( (c, t) \) is a wandering interval and define \( J_\gamma = (c, p_\alpha) \). The maximal wandering interval \( J_\alpha \) is called the exceptional wandering interval associated to \( \alpha \). Given \( \alpha \in \mathcal{C}_f \) define the shadow of \( \alpha \) as

\[
\alpha^* = \begin{cases} 
\alpha & \text{if } \alpha \notin W_f \\
+\infty & \text{if } \alpha \in W_f \text{ and } p_\alpha \in \mathcal{O}_f^-(\mathcal{C}_f) \\
p_\alpha & \text{if } \alpha \in W_f \text{ and } p_\alpha \notin \mathcal{O}_f^-(\mathcal{C}_f) 
\end{cases}
\]

**Definition 18 (Ewi attractors).** We say that \( A \) is an Ewi (exceptional wandering interval) attractor if \( A = \text{Re}(\omega_f(\alpha)) \) for some \( \alpha \in W_f \). Notice that the basin of attraction of \( A \),
In the picture of the left side, we have the graphic of a contracting Lorenz map $f$ whose restriction to the interval $J$, the interval with boundary on the critical values, is a gap map. In the right side of the picture we have the map $F$ as in the Example 19. This map $F$ has an Ewi attractor, as the wandering interval $I$ has an exceptional point $a$ in its boundary.

$\beta f(A)$, contains the exceptional wandering interval $J_\alpha$. Indeed, an Ewi attractor is the smallest attractor containing an exceptional wandering interval.

**Example 19** (Ewi attractor). Let $f : [0, 1] \setminus \{c\} \to [0, 1]$, $0 < c < 1$, be a $C^3$ contracting Lorenz map with $Sf < 0$ such that $\text{Leb}(f([v_0, v_1] \setminus \{c\})) < |v_0 - v_1|$, where $v_j = \text{Re}(f(c + (-1)^{\frac{j}{2}}))$. In this case, the map $G := f|_{J \setminus \{c\}}$ is injective but not surjective, where $J = [v_0, v_1]$. Such a map $G$ is called a gap map and it is known that it has a well defined rotation number. If its rotation number is irrational then $I_0 := (G(v_1), G(v_0)) = (f(v_1), f(v_0))$ is a wandering interval. In this case, $I = (a, v_0)$ is a wandering interval for $f$, where $a < v_0$ and $f(a) = f(v_1)$. Thus, choosing $f$ so that $G$ has an irrational rotation number and consider $F : [0, 1] \setminus \{a, c\} \to [0, 1]$ given by

$$F(x) = \begin{cases} f(x) & \text{if } x > a \\ f(x)/f(a) & \text{if } x < a \end{cases}$$

see Figure 9. The interval $I$ is a wandering interval for $F$ with the exceptional point $a$ belonging to the boundary of $I$. Thus, $F$ has an Ewi attractor.

Let $\mathbb{B}_2(f) = \bigcup_{\alpha \in \mathcal{W}_f} \{x ; \omega_f(x) = \text{Re}(\omega_f(\alpha))\}$ and define

$$\mathbb{B}(f) = \mathbb{B}_0(f) \cap \mathbb{B}_1(f) \cap \mathbb{B}_2(f).$$

Define

$$\mathcal{C}_f^\infty = \{\alpha \in \mathcal{C}_f ; \alpha^* = \infty\}.$$
Lemma 20 (Parallax I). For almost every point $x \in [0,1] \setminus \mathcal{B}(f)$ we have that
\[
\omega_f(x) \subset \bigcup_{\alpha \in \mathcal{C}_f \setminus \mathcal{C}_f^\infty} \text{Re}(\mathcal{O}_f^+(\alpha))
\] (5)

Proof. Write $\{\gamma_1, \cdots, \gamma_s\} = \mathcal{C}_f^\infty$. Let $\mathcal{C}_0 = \mathcal{C}_f$ and $\mathcal{C}_n = \mathcal{C}_f \setminus \{\gamma_1, \cdots, \gamma_n\}$, for $1 \leq n \leq s$. By Corollary 14 \[\omega_f(x) \subset \bigcup_{\alpha \in \mathcal{C}_0} \text{Re}(\mathcal{O}_f^+(\alpha))\] for almost all $x \in [0,1] \setminus \mathcal{B}(f)$. Suppose by induction that $\omega_f(x) \subset \bigcup_{\alpha \in \mathcal{C}_{n-1}} \text{Re}(\mathcal{O}_f^+(\alpha))$ for almost all $x \in [0,1] \setminus \mathcal{B}(f)$, with $1 \leq n \leq s$.

We may assume that $\gamma_n = c + i$ for some $c \in \mathcal{C}_f$, the case $c - i$ is analogous. Let $\bar{c} \in \mathcal{C}_f$ and $\ell, \ell \geq 0$ be such that $f^\ell(p_{\gamma_n}) = \bar{c}$. Write $I = (p_{\gamma_n}, c)$ and $q = \frac{1}{2}(p_{\gamma_n} + c) \in I$ and consider $g : [0,1] \setminus \mathcal{C}_f \to [0,1]$ (see Figure 10) given by
\[
g(x) = \begin{cases} f(x) & \text{if } x \notin I \\ q + \sigma(f(x) - q) & \text{if } x \in I \end{cases}
\]
where $\sigma = (2 \sup |f'|)^{-1}$. Note that $\mathcal{C}_g = \mathcal{C}_f \cup \{p_{\gamma_n} - i, p_{\gamma_n} + i\}$, $q$ is an attracting fixed point for $g$ and $I$ belongs to the basin of attraction of $q$. In particular, $\bigcup_{\alpha \in \{p_{\gamma_n} + i, p_{\gamma_n} - i\}} \text{Re}(\mathcal{O}_g^+(\alpha)) \subset [p_{\gamma_n}, c] = \mathcal{B}$. Furthermore, as also $g|_{([0,1] \setminus \mathcal{C}_f) \setminus \mathcal{B}(f)} \equiv f|_{([0,1] \setminus \mathcal{C}_f) \setminus \mathcal{B}(f)}$, we can conclude that $\mathcal{B}(g) = \mathcal{B}(f)$. Beside that, if $x \notin \mathcal{B}(f)$ then $\mathcal{O}_f^+(x) \cap I = \emptyset$. As a consequence, $\mathcal{O}_f^+(x) = \mathcal{O}_g^+(x)$ and also $\omega_f(x) = \omega_g(x)$. In particular, $\omega_g(x) \cap I = \emptyset \forall x \in [0,1] \setminus \mathcal{B}(f)$ and
\[
\text{Re}(\mathcal{O}_g^+(\eta)) = \text{Re}(\mathcal{O}_f^+(\eta)) \forall \eta \in \mathcal{C}_g \setminus \mathcal{C}_g(I),
\] (6)
where $\mathcal{C}_g(I) = \{\eta \in \mathcal{C}_g; \text{Re}(\mathcal{O}_g^+(\eta)) \cap I \neq \emptyset\}$.

As $\mathcal{C}_f \cap J_{\gamma_n} = \emptyset$ (because $J_{\gamma_n}$ is a wandering interval) and as $f^\ell(p_{\gamma_n} - i) \in \mathcal{C}_f$, we obtain that
\[
\mathcal{C}_g(I) = \{\eta \in \mathcal{C}_f; \text{Re}(\mathcal{O}_f^+(\eta)) \cap I \neq \emptyset\} \cup \{p_{\gamma_n} + i, \gamma_n\}.
\]
Thus, it follows from Lemma 17 that
\[
\omega_f(x) = \omega_g(x) \subset \left( \bigcup_{\alpha \in C_{n-1}} \Re(\mathcal{O}_f^+(\alpha)) \right) \cap \left( \bigcup_{\alpha \in C_{f,I}} \overline{\Re(\mathcal{O}_g^+(\alpha))} \right),
\]
for almost all \( x \in [0, 1] \setminus B(f) = [0, 1] \setminus B(g) \).

If \( \gamma_n - \delta \in C_g(I) \) then
\[
\bigcup_{\alpha \in C_g \setminus C_g(I)} \Re(\mathcal{O}_g^+(\alpha)) = \bigcup_{\alpha \in C_f \setminus C_g(I)} \Re(\mathcal{O}_f^+(\alpha)) \subset \bigcup_{\alpha \in C_f \setminus \{ \gamma_n \}} \Re(\mathcal{O}_f^+(\alpha))
\]
and it follows from (7) that
\[
\omega_f(x) = \omega_g(x) \subset \bigcup_{\alpha \in C_{n-1} \setminus \{ \gamma_n \}} \overline{\Re(\mathcal{O}_f^+(\alpha))} = \bigcup_{\alpha \in C_n} \overline{\Re(\mathcal{O}_f^+(\alpha))},
\]
for almost all \( x \in [0, 1] \setminus B(f) = [0, 1] \setminus B(g) \).

Thus, assume that \( \gamma_n - \delta \notin C_g(I) \) and let
\[
K(g) = \{ p_{\gamma_n}, \ldots, f^{f-1}(p_{\gamma_n}) \} = \{ \Re(p_{\gamma_n} - \delta), \ldots, \Re(g^{f-1}(p_{\gamma_n} - \delta)) \}.
\]
Then
\[
\bigcup_{\alpha \in C_g \setminus C_g(I)} \Re(\mathcal{O}_g^+(\alpha)) = K(g) \cup \bigcup_{\alpha \in C_f \setminus C_g(I)} \Re(\mathcal{O}_g^+(\alpha)). \tag{8}
\]
Furthermore, using (6) we also get
\[
\bigcup_{\alpha \in C_f \setminus C_g(I)} \Re(\mathcal{O}_g^+(\alpha)) = \bigcup_{\alpha \in C_I \setminus C_g(I)} \Re(\mathcal{O}_f^+(\alpha)). \tag{9}
\]
From (7) and (8) follows that
\[
\omega_f(x) = \omega_g(x) \subset \left( \bigcup_{\alpha \in C_{n-1}} \Re(\mathcal{O}_f^+(\alpha)) \right) \cap \left( K(g) \cup \bigcup_{\alpha \in C_f \setminus C_g(I)} \overline{\Re(\mathcal{O}_g^+(\alpha))} \right),
\]
for almost all \( x \in [0, 1] \setminus B(f) = [0, 1] \setminus B(g) \).

As \( K(g) \cap (q, c) = \emptyset \), \( K(g) \) is a finite set, \( \bigcup_{\alpha \in C_f \setminus C_g(I)} \overline{\Re(\mathcal{O}_g^+(\alpha))} \) is compact and
\[
\bigcup_{\alpha \in C_g} \overline{\Re(\mathcal{O}_g^+(\alpha))} = \left( \bigcup_{\alpha \in C_g(I)} \overline{\Re(\mathcal{O}_g^+(\alpha))} \right) \cup \left( \bigcup_{\alpha \in C_f \setminus C_g(I)} \overline{\Re(\mathcal{O}_g^+(\alpha))} \right) =
\]
\[
= \left( \bigcup_{\alpha \in C_g(I)} \overline{\Re(\mathcal{O}_g^+(\alpha))} \right) \cup \left( K(g) \cup \bigcup_{\alpha \in C_f \setminus C_g(I)} \overline{\Re(\mathcal{O}_g^+(\alpha))} \right) \subset
\]
\[
\subset (q, c) \cup \left( K(g) \cup \bigcup_{\alpha \in C_f \setminus C_g(I)} \overline{\Re(\mathcal{O}_g^+(\alpha))} \right),
\]
we get that any \( y \in K(g) \setminus (\bigcup_{\alpha \in C_f \setminus C_g(I)} \overline{\Re(\mathcal{O}_g^+(\alpha))}) \) is an isolated point of \( \bigcup_{\alpha \in C_g} \overline{\Re(\mathcal{O}_g^+(\alpha))} \).

Thus, it follows from Corollary 15 that
\[
\text{Leb}(\{ x \in [0, 1] \setminus B(g) ; K(g) \cap \omega_g(x) \neq \emptyset \}) = 0.
\]
As a consequence, for almost all \( x \in [0, 1] \setminus B(f), \)
\[
\omega_f(x) = \omega_g(x) \subset \left( \bigcup_{\alpha \in \mathcal{C}_{n-1}} \text{Re}(\mathcal{O}^+(\alpha)) \right) \cap \left( \bigcup_{\alpha \in \mathcal{C}_f \setminus \mathcal{C}_g(I)} \text{Re}(\mathcal{O}^+_g(\alpha)) \right) = \left( \bigcup_{\alpha \in \mathcal{C}_{n-1}} \text{Re}(\mathcal{O}^+(\alpha)) \right) \cap \left( \bigcup_{\alpha \in \mathcal{C}_f \setminus \mathcal{C}_g(I)} \text{Re}(\mathcal{O}^+_f(\alpha)) \right) \subset \left( \bigcup_{\alpha \in \mathcal{C}_{n-1}} \text{Re}(\mathcal{O}^+(\alpha)) \right) \cap \left( \bigcup_{\alpha \in \mathcal{C}_f \setminus \{\gamma_n\}} \text{Re}(\mathcal{O}^+_f(\alpha)) \right) = \bigcup_{\alpha \in \mathcal{C}_n} \text{Re}(\mathcal{O}^+_f(\alpha)).
\]

it and set \( U(y) = \{ x \in [0, 1] \setminus B(f); y \in \omega_f(x) \} \). If \( \text{Leb}(U(y)) > 0 \) then it follows from Corollary \cite{15} that
\[
\omega_f(x) \subset A(\mathcal{C}_f^0), \text{ for almost all } x \in [0, 1] \setminus (B_0(f) \cup B_1(0)). \tag{10}
\]

Given \( x \in [0, 1] \), let \( [\mathcal{C}_f]_x = \{ \alpha \in \mathcal{C}_f; \alpha^* \in \omega_f(x) \} \). Notice that \( [\mathcal{C}_f]_x = \mathcal{C}_f^0 \cap [\mathcal{C}_f]_x \).

**Lemma 21 (Parallax II).** For almost all \( x \in B(f) \), we have \( \omega_f(x) \subset A(\mathcal{C}_f \setminus \{ \gamma \}) \) for all \( \gamma \in \mathcal{C}_f \setminus [\mathcal{C}_f]_x \).

**Proof.** Let \( \gamma \in \mathcal{C}_f \) and set \( U(\gamma) = \{ x \in [0, 1] \setminus B(f); \gamma \notin \omega_f(x) \} \), that is, \( U(\gamma) = \{ x \in [0, 1] \setminus B(f); \gamma \in \mathcal{C}_f \setminus [\mathcal{C}_f]_x \} \). To prove the claim above, we may assume \( \text{Leb}(U(\gamma)) > 0 \).

By Lemma \cite{20}, that \( \gamma \in \mathcal{C}_f^0 \). We can also assume that \( \gamma = c - \delta \) for some \( c \in \mathcal{C}_f \), the case \( c + \delta \) is analogous. As \( \gamma \not\in \mathcal{C}_f^0 \), either \( \mathcal{O}_f^{-}(\mathcal{C}_f) \not\ni p_\gamma < c \) or \( p_\gamma = \gamma \).

Let \( U(\gamma, n) = \{ x \in U(\gamma); \mathcal{O}_f^+(x) \cap (p_\gamma - 1/n, c) = \emptyset \} \) and let \( n_0 \geq 1 \) be such that \( \mathcal{C}_f \cap (p_\gamma - 1/n_0, c) = \emptyset \). Thus, \( U(\gamma) = \bigcup_{n \geq n_0} U(\gamma, n) \).

Take any \( n \geq n_0 \) so that \( \text{Leb}(U(\gamma, n)) > 0 \) and choose a point \( q \in (c - 1/n, p_\gamma) \cap \mathcal{O}_f^{-}(\mathcal{C}_f) \).

By the maximality of \( J_\gamma \), the interval \( (c - 1/n, p_\gamma) \) is not a wandering one. Thus, either \( (c - 1/n, p_\gamma) \) is contained in the basin of a periodic-like attractor, or \( (c - 1/n, p_\gamma) \cap \mathcal{O}_f^{-}(\mathcal{C}_f) \neq \emptyset \). The first situation is impossible, as it would imply that \( p_\gamma \) would be attracted by the periodic-like attractor and then \( J_\gamma \) wouldn’t be a wandering interval. So, we have \( (c - 1/n, p_\gamma) \cap \mathcal{O}_f^{-}(\mathcal{C}_f) \neq \emptyset \) and we are free to choose \( q \) as above.

Taking \( I = (q, c) \), we shall now consider two functions, \( g : [0, 1] \setminus (\mathcal{C}_f \cup \{ q \}) \rightarrow [0, 1], j = 1, 2, \) given by
\[
g(x) = \begin{cases} f(x) & \text{if } x \notin I \\ p_\gamma + \sigma(f(x) - p_\gamma) & \text{if } x \in I \end{cases}.
\]
Figure 11. In this picture we have in the left side a map $f$ and in the right side the associated map $g$ which is equal to $f$ outside the interval $[q, c]$, has $p_\gamma \in (q, c)$ as a fixed point and $C_f \cup \{q\}$ as its exceptional set (see Lemma 21).

where $\sigma = (2\sup |f'|)^{-1}$.

In Figure 11 we can compare the maps $f$ and $g$. Notice that $g|_I$ is a contraction and $\omega_g(x) = p_\gamma \forall x \in I$. Applying Lemma 17 we get for almost all $x \in [0, 1] \setminus (B_0(g) \cup B_1(g))$ that

$$\omega_g(x) \subset \bigcup_{\alpha \in C_g \setminus C_g(I)} \Re(O_g^+(\alpha)),$$

where

$$C_g(I) = \{ \eta \in C_g : \Re(O_g^+(\eta)) \cap (q, c) \neq \emptyset \} \supset \{ \eta \in C_f : \Re(O_f^+(\eta)) \cap (q, c) \neq \emptyset \} \cup \{q + i\} \ni \gamma.$$

As $C_g \setminus C_g(I) = (\{q - i\} \cup C_f) \setminus C_g(I)$ and as $\Re(O_g^+(\eta)) = \Re(O_f^+(\eta))$ for all $\eta \in C_f \setminus C_g(I)$, if $q - i \in C_g(I)$ then

$$\bigcup_{\alpha \in C_g \setminus C_g(I)} \Re(O_g^+(\alpha)) = \bigcup_{\alpha \in C_f \setminus C_g(I)} \Re(O_f^+(\alpha)) = A(C_f \setminus C_g(I)) \subset A(C_f \setminus \{\gamma\})$$

and in this case, for almost all $x \in U(\gamma, n)$ we have

$$\omega_f(x) = \omega_g(x) \subset A(C_f) \bigcap \left( \bigcup_{\alpha \in C_g \setminus C_g(I)} \Re(O_g^+(\alpha)) \right) \subset$$

$$\subset A(C_f) \cap A(C_f \setminus \{\gamma\}) = A(C_f \setminus \{\gamma\}).$$

So, we may suppose that $q - i \notin C_g(I)$. In this case,

$$C_g \setminus C_g(I) = \{q - i\} \cup (C_f \setminus C_g(I)).$$

and so,

$$\bigcup_{\alpha \in C_f \setminus C_g(I)} \Re(O_g^+(\alpha)) = \bigcup_{\alpha \in C_f \setminus C_g(I)} \Re(O_f^+(\alpha)) = A(C_f \setminus C_g(I)).$$
Let \( \ell \geq 1 \) be so that \( f^\ell(q) \in C_f \) and write \( K(g) = \{ q, \ldots, f^{\ell-1}(q) \} \). Thus,
\[
\text{Re}(O^+_g(q - \bar{i})) = \text{Re}(O^+_f(q - \bar{i})) = K(g) \cup \text{Re}(O^+_f(f^\ell(q - \bar{i}))).
\]
This implies that \( \text{Re}(O^+_g(q - \bar{i})) \subset \left( K(g) \cup \bigcup_{\alpha \in C_f \setminus C_g(I)} \text{Re}(O^+_f(\alpha)) \right) \) and so, it follows from [12] and [13] that
\[
\bigcup_{\alpha \in C_g \setminus C_g(I)} \text{Re}(O^+_g(\alpha)) = K(g) \cup A(C_f \setminus C_g(I)) = K_0 \cup A(C_f \setminus C_g(I)),
\]
where \( K_0 = K(g) \setminus A(C_f \setminus C_g(I)) \).

As \( K_0 \) is a finite set and \( A(C_f \setminus C_g(I)) \) is compact, we get that every \( y \in K_0 = K(g) \setminus \left( \bigcup_{\alpha \in C_f \setminus C_g(I)} \text{Re}(O^+_g(\alpha)) \right) \) is an isolated point of \( \bigcup_{\alpha \in C_g} \text{Re}(O^+_g(\alpha)) \). Thus, it follows from Corollary [15] that
\[
\text{Leb}(\{ x \in [0, 1] \setminus (B_0(g) \cup B_1(g)) ; K_0 \cap \omega_g(x) \neq \emptyset \}) = 0. \tag{15}
\]
Therefore, it follows from [11], [14], [15] and \( \gamma \in C_g(I) \) that
\[
\omega_f(x) = \omega_g(x) \subset A(C_f \setminus C_g(I)) \subset A(C_f \setminus \{ \gamma \})
\]
for almost all \( x \in U(\gamma, n) \). So, \( \omega_f(x) \subset A(C_f \setminus \{ \gamma \}) \) for almost all \( x \in U(\gamma) = \bigcup_{n \geq n_0} U(\gamma, n) \), proving the claim as well as the theorem.

**Theorem 1.** For almost every point \( x \in [0, 1] \setminus B(f) \) we have that
\[
\omega_f(x) = \bigcup_{\alpha \in \omega_f(x)} \text{Re}(O^+_f(\alpha^*)).
\]

**Proof.** Given \( K \subset C_f \), let \( A(K) = \bigcup_{\alpha \in K} \text{Re}(O^+_f(\alpha^*)) \). Thus, Lemma[20] can be rewritten as:
\[
\omega_f(x) \subset A(C_f^0),
\]
for almost all \( x \in [0, 1] \setminus (B_0(f) \cup B_1(0)) \supset [0, 1] \setminus B(f) \). Given \( x \in [0, 1] \), let \( [C_f]_x = \{ \alpha \in C_f : \alpha^* \in \omega_f(x) \} \). Notice that \( [C_f]_x = C_f^0 \cap [C_f]_x \).

Using Lemma[21] we get
\[
\omega_f(x) \subset A(C_f^0) \cap \left( \bigcap_{\gamma \in C_f \setminus [C_f]_x} A(C_f \setminus \{ \gamma \}) \right) = A(C_f^0) \cap A\left( \bigcap_{\gamma \in C_f \setminus [C_f]_x} C_f \setminus \{ \gamma \} \right) = A(C_f^0) \cap A\left( \bigcap_{\gamma \in C_f \setminus [C_f]_x} C_f \setminus [C_f]_x \right) = A(C_f^0) \cap A([C_f]_x) = A([C_f]_x).
\]
As \( \alpha^* \in \omega_f(x) \) implies that \( \text{Re}(O^+_f(\alpha^*)) \subset \omega_f(x) \), we always have \( \omega_f(x) \supset A([C_f]_x) \). Therefore, for almost all \( x \in B(f) \), \( A([C_f]_x) \subset \omega_f(x) \subset A([C_f]_x) \). That is, \( \omega_f(x) = A([C_f]_x) \) for almost all \( x \in [0, 1] \setminus B(f) \), proving the theorem.

Notice that we did not use Mañé theorem to prove Theorem[3] above. Indeed, we only use Mañé's result in the proof of the uniqueness of the attractor for \( S \)-unimodal maps in Theorem[4]. As a consequence, the corollary below does depend on Mañé's theorem.
Corollary 22. Let $\mathcal{C}_f \subset (0,1)$ be a finite set and $f : [0,1] \setminus \mathcal{C}_f \to [0,1]$ be a $C^3$ local diffeomorphism with negative Schwarzian derivative and $f(\{0,1\}) \subset \{0,1\}$. If $f$ does not have Ewi attractors, then $\omega_f(x) \cap \mathcal{C}_f \neq \emptyset$ for Lebesgue almost every $x \in [0,1] \setminus \mathcal{B}_0(f)$.

Proof. As, by hypothesis, $\mathcal{B}_0(f) = \emptyset$, then $\mathcal{C}_f^\infty = \emptyset$, $\mathcal{B}(f) = \mathcal{B}_0(f) \cup \mathcal{B}_1(f)$ and $\alpha^* = \alpha$ for all $\alpha \in \mathcal{C}_f$. Thus, Theorem $\star$ becomes

$$\omega_f(x) = \bigcup_{\alpha \in \omega_f(x) \setminus \mathcal{C}_f^\infty} \text{Re}(\mathcal{O}_f^+(\alpha^*)),$$

for almost every $x \in [0,1] \setminus (\mathcal{B}_0(f) \cup \mathcal{B}_1(f))$. As $\alpha \in \omega_f(x)$, with $\alpha \in \mathcal{C}_f$, implies that $\text{Re}(c) \in \omega_f(x)$ and $\text{Re}(\alpha) \in \mathcal{C}_f$, then $\omega_f(x) \cap \mathcal{C}_f \neq \emptyset$ for almost all $x \in [0,1] \setminus (\mathcal{B}_0(f) \cup \mathcal{B}_1(f))$.

On the other hand, by Lemma $\star$, $\text{Interior}(\omega_f(x)) \cap \mathcal{C}_f \neq \emptyset$ for every $x \in \mathcal{B}_1(f)$ and so, we conclude the proof. $\square$

Now we can prove our main theorem.

Proof of Theorem $\star$. Using the adapted Singer’s theorem, we can conclude that the basin of attraction of each periodic-like attractor contains at least one element of $\mathcal{C}_f \setminus \mathcal{W}_f$. This means that, writing $\alpha = c \pm \delta \in \mathcal{C}_f \setminus \mathcal{W}_f$, there is $\varepsilon > 0$ such that $(c, c + \varepsilon) \subset \beta_f(A_j)$ or $(c - \varepsilon, c) \subset \beta_f(A_j)$. Moreover, each cycle of intervals contains at least one point $c \in \mathcal{C}_f$ (Lemma $\star$), and so it contains at least two elements of $\mathcal{C}_f \setminus \mathcal{W}_f$, i.e., $c, c \pm \delta$. By the definition of Ewi attractors, each of them contains at least one element of $\mathcal{C}_f$. Furthermore, if $\mathcal{C}_0$ is the set of all $\alpha \in \mathcal{C}_f$ contained in the basin of a periodic-like attractor and $\mathcal{C}_1$ is the set of all $c \in \mathcal{C}_f$ contained in the interior of a cycle of intervals then $\mathcal{C}_0 \setminus (\mathcal{C}_1 \pm \delta) = \mathcal{C}_0 \cap \mathcal{W}_f = (\mathcal{C}_1 \pm \delta) \cap \mathcal{W}_f = \emptyset$. Thus, $f$ has at most $2 \#\mathcal{C}_f$ attractors that are periodic-like attractors or cycle of intervals or Ewi attractors. Let $\{A_1, \ldots, A_s\}$ be the collection of all periodic-like attractors, cycle of intervals and Ewi attractors, where $0 \leq s \leq 2 \#\mathcal{C}_f$. Thus, $\beta_f(A_1) \cup \cdots \cup \beta_f(A_s) \supset \mathcal{B}(f)$ and for every $j = 1, \ldots, s$ we have $\omega_f(x) = A_j \forall x \in \mathcal{B}(f) \cap \beta_f(A_j)$.

On the other hand, if $x \in [0,1] \setminus \mathcal{B}(f)$, it follows from the Theorem $\star$ that almost surely we have

$$\omega_f(x) = \bigcup_{\alpha^* \in \omega_f(x) \setminus \mathcal{C}_f^\infty} \text{Re}(\mathcal{O}_f^+(\alpha^*)),$$

Moreover, $\alpha^* \notin \omega_f(x) \forall x \in [0,1] \setminus \mathcal{B}(f)$ and all $\alpha \in \mathcal{C}_f \setminus (\mathcal{C}_0 \cup (\mathcal{C}_1 \pm \delta) \cup \mathcal{C}_f^\infty)$. Thus, is if $\mathcal{U}$ is the collection of all non-empty subset of $\mathcal{C}_f \setminus (\mathcal{C}_0 \cup (\mathcal{C}_1 \pm \delta) \cup \mathcal{C}_f^\infty)$ then for almost all $x \in [0,1] \setminus \mathcal{B}(f)$ there is some $U_x \in \mathcal{U}$ such that

$$\omega_f(x) = \bigcup_{\alpha \in U_x} \text{Re}(\mathcal{O}_f^+(\alpha^*)) = A(U_x).$$

Thus, for almost all $x \in [0,1] \setminus \mathcal{B}(f)$ there is some $A \in \{A(U); U \in \mathcal{U}\}$ such that $\omega_f(x) = A$

As $\#\{A(U); U \in \mathcal{U}\} \leq \#\mathcal{U} \leq 2^{2 \#\mathcal{C}_f - \#\mathcal{C}_0 + 2 \#\mathcal{C}_1 + \#\mathcal{C}_f^\infty} - 1 \leq 2^{2 \#\mathcal{C}_f - (s - \#\mathcal{W}_f) - 1}$, if we write $\{A_{s+1}, \ldots, A_{s+t}\} = \{A(U); U \in \mathcal{U}\}$ and $n = s + t$ then, for almost all $x \in [0,1]$ we have

$$\omega_f(x) = A_j, \text{ for some } 1 \leq j \leq n \leq s + 2^{2 \#\mathcal{C}_f - (s - \#\mathcal{W}_f) - 1} \leq 2^{1+2 \#\mathcal{C}_f}. \tag{16}$$
In particular,
\[
\beta_f((A_1) \cup \cdots \cup \beta_f(A_n)) = 1. \quad \square
\]

6. Contracting Lorenz maps

In this section we shall deal with orientation preserving maps of the interval. We say that \( f : [0, 1] \setminus \{c\} \to [0, 1], c \in (0, 1), \) is an orientation preserving map if for every \( x \in [0, 1] \setminus \{c\} \) there is \( \varepsilon > 0 \) such that \( f(a) < f(b) \forall a, b \in (x - \varepsilon, x + \varepsilon) \cap [0, 1] \setminus \{c\} \) with \( a < b \). If \( f \) is a local diffeomorphism, this condition is equivalent to \( f' < 0 \) for all \( x \in [0, 1] \setminus \{c\} \).

**Definition** (Nice intervals). We say that \( J = (a, b) \subset [0, 1] \) is a nice interval for \( f \) if
\[
\mathcal{O}_f^+(\partial J) \cap J = \emptyset.
\]

The notion of nice intervals was introduced by Martens [31] to study induced maps of the interval. In most of the results of this section, like Lemma 23, 24 and 25, we are not asking the intervals to be nice, but only “partially” nice, that is, the orbit of one of the points in the boundary of the interval should not intersect the interval. This is important to prove Proposition 27 and to conclude that contracting Lorenz maps do not admit Ewi attractors.

**Lemma 23.** Let \( f : [0, 1] \setminus \{c\} \to [0, 1] \) be an orientation preserving local homeomorphism and \( J = (a, b) \) be an open interval with \( c \in J \). Let \( F : J^* \to J \) be the first return map to \( J \), where \( J^* = J \cap \bigcup_{j \geq 1} f^{-j}(J) \). If \( \mathcal{O}_f^+(a) \cap (a, b) = \emptyset \) and \( I \) is a connected component of \( J^* \cap (a, c) \) such that \( F(I) \neq J \) then \( c \in \partial I \) and \( F(I) = (a, F(c_-)) \). Analogously, if \( \mathcal{O}_f^+(b) \cap (a, b) = \emptyset \) and \( I \) is a connected component of \( J^* \cap (c, b) \) such that \( F(I) \neq J \) then \( c \in \partial I \) and \( F(I) = (F(c_+), b) \).

**Proof.** Suppose that \( \mathcal{O}_f^+(a) \cap (a, c) = \emptyset \), the proof of the case when \( \mathcal{O}_f^+(b) \cap (c, b) = \emptyset \) is analogue. Let \( I = (t, s) \) be a connected component of \( J^* \cap (a, c) \). Let \( n \geq 1 \) be such that \( F|_I = f^n|_I \) and let \( T = (t, s') \) with \( s \leq s' \) be the maximal interval such that \( f^n|_T \) is a homeomorphism and that \( f^n(T) \subset J \). If \( s < s' \) then there is some \( 1 \leq \ell < n \) such that \( f^\ell(T) \cap J \neq \emptyset \). As \( f^\ell(I) \cap J = \emptyset \) (because \( n \) is the first return to \( J \) for the points of \( I \)), we get \( J \not\supset f^\ell(T) \). As \( c \notin f^\ell(T) \), because \( f^n|_T \) is a diffeomorphism, and as \( f \) is orientation preserving, we get \( f^\ell(t) < f^\ell(s) < a < f^\ell(s') \). But this implies that \( f^{n-\ell}(a) \in (a, b) \), contradicting our assumption. So, \( s = s' \) and \( T = I \).

Now, suppose that \( F(I) \neq J \). By the maximality of \( T \) there is some \( 0 \leq \ell < n \) such that \( c \in f^\ell(\partial T) \). As \( f^\ell(I) \cap J = \emptyset \forall 1 \leq j < n \), we get \( c \in \partial I \), i.e., \( I = T = (t, c) \) and so, \( F(I) = (F(t_+), F(c_-)) \). Furthermore, as \( t \neq c \) and \( f^j(I) \cap (a, b) = \emptyset \forall 0 \leq j < n \), it follows that \( f^j(t) \neq c \forall 0 \leq j < n \). So,
\[
F(I) = (f^n(t), F(c_-)) \tag{17}
\]

**Claim.** \( f^n(t) = a \).

**Proof of the claim.** As \( f^n(t) \) is well defined, \( f \) is orientation preserving and \( f^n((t, c)) \subset (a, b) \), we get \( a \leq f^n(t) < b \). Thus, if \( f^n(t) \neq a \) then \( a < f^n(t) < b \). By continuity, there is \( \varepsilon > 0 \) such that \( f^n([t-\varepsilon, c)) \) is a diffeomorphism, \( f^n((t-\varepsilon, c)) \cap (a, b) = \emptyset \forall 0 < j < n \) and \( f^n((t-\varepsilon, c)) \subset (a, b) \) which contradicts the fact that \( I \) is a connected component of \( J^* \). \( \square \)
Thus, the proof of the lemma follows from (17) and the claim above.

\[\square\]

**Lemma 24.** Let \( f : [0, 1] \setminus \{c\} \to [0, 1] \) be an orientation preserving local homeomorphism and let \( J = (a, b) \in \mathcal{N}_f \) be a interval with \( c \in J \). If \( \mathcal{O}_f^1(a) \cap (a, b) = \emptyset \) then either \( \#\{j \geq 0 ; f^j(x) \in (a, c)\} < \infty \forall x \in (a, c) \) or \( \exists J' = (a', b') \) such that \( c \in J' \cap J \), \( \mathcal{O}_f^1(a') \cap (a, b) = \emptyset \) and \( a' \in \text{Per}(f) \). Similarly, If \( \mathcal{O}_f^1(b) \cap (a, b) = \emptyset \) then either \( \#\{j \geq 0 ; f^j(x) \in (c, b)\} < \infty \forall x \in (c, b) \) or \( \exists J' = (a', b') \) such that \( c \in J' \cap J \), \( \mathcal{O}_f^1(b') \cap (a, b) = \emptyset \) and \( b' \in \text{Per}(f) \).

**Proof.** If \( a \in \text{Per}(f) \) there is nothing to prove. Thus, we may assume that \( a \notin \text{Per}(f) \).

Suppose \( \exists x_0 \in (a, c) \) such that \( \#\{j \geq 0 ; f^j(x_0) \in (a, c)\} = \infty \). Let \( F : J^* \to J \) be the first return map to \( J \), where \( J^* = J \cap \bigcup_{j \geq 1} f^{-j}(J) \).

**Claim.** \( \text{Fix}(F) \cap (a, c) \neq \emptyset \).

**Proof of the claim.** First note that \( a \in \partial J \) does not belong to the boundary of any connected component of \( J^* \). Indeed, if \( I = (a, p) \) is a connected component of \( J^* \), then by Lemma 23, \( F(a + \mathbb{I}) = a + \mathbb{I} \). Let \( n \) be such that \( F|_I = f^n|_I \). As \( \mathcal{O}_f^1(a) \cap (a, b) = \emptyset \), we have that \( a \notin \mathcal{O}_f^1(c) \). So, \( F(a + \mathbb{I}) = a + \mathbb{I} \) implies that \( f^n(a) = a \), contradicting our assumption.

Assume that \( J^* \cap (a, c) \) has more than one connected component. In this case, let \( I = (p, q) \) be a connected component of \( J^* \) such that \( q < c \). Then, by Lemma 23, \( F(I) = J \supset I \).

This implies that either \( F \) has a fixed point \( y \in I \) (proving the claim) or that \( p = a \) and \( F(a + \mathbb{I}) = a + \mathbb{I} \). As we have seen before, \( F(a + \mathbb{I}) = a + \mathbb{I} \) implies that \( f^n(a) = a \) (where \( F|_I = f^n|_I \)), contradicting our assumption.

Now, suppose that \( J^* \cap (a, c) \) has only one connected component. Let \( I = (p, q) \) be this single connected component and let \( n \geq 1 \) be such that \( F|_I = f^n|_I \). We may suppose that \( F|_I \) does not have a fixed point. Let \( I = (p, c) \) for some \( p \in [a, c] \). By Lemma 23, \( c \) must belongs to the boundary of \( I \) or we have that \( p = a \) and also \( F(a + \mathbb{I}) = a + \mathbb{I} \). Again \( F(a + \mathbb{I}) = a + \mathbb{I} \) implies that \( a \in \text{Per}(f) \), contradicting our assumption. Thus, \( a < p < c \).

As \( F(I) = (a, F(c-)) \) (Lemma 23) and as \( F(x) \neq x \ \forall x \in I \), we get \( F(x) < x \ \forall x \in I \). Because \( I = J^* \cap (a, c) \) and \( \#\{j \geq 0 ; f^j(x_0) \in (a, c)\} = \infty \), we get

\[a < p < \cdots < F^n(x_0) < F^{n-1}(x_0) < \cdots < F(x_0) < x_0 < c.\]

So, \( q := \lim_{n}(F^n(x_0)) \) belongs to \((a, c)\) and satisfies \( F(q_+) = q \geq p > a \). Furthermore, as \( F(p_+) = a \) (Lemma 23), we get \( q \in (p, c) = I \). Therefore, \( q \) is indeed a fixed point of \( F \), proving the claim.

To finish the proof of the lemma, let \( q_0 \in \text{Fix}(F) \cap [a, c] \) and let \( n \) be the period of \( q_0 \) with respect to \( f \). As \( F(q_0) = F^n(q_0) = q_0 \) and \( F \) is the first return map to \((a, b)\), we get \( \{f(q_0), \cdots, f^{n-1}(q_0)\} \cap (a, b) = \emptyset \). Thus, \( \mathcal{O}_f^1(q_0) \cap (q_0, b) = \emptyset \), proving the lemma.

\[\square\]

**Lemma 25.** Let \( f : [0, 1] \setminus \{c\} \to [0, 1] \) be an orientation preserving local homeomorphism and let \((a, b)\) be an interval with \( c \in (a, b) \). If \( \mathcal{O}_f^1(a) \cap (a, b) = \emptyset \), \( a \in \text{Per}(f) \), \( c_- \) does not belong to the basin of a periodic-like attractor, \( c_- \in \omega_f(c_-) \) and \( \mathcal{O}_f^1(c_-) \cap (c, b) = \emptyset \), then there exist an open set \( U \subset (a, c) \) and a continuous map \( R : U \to \mathbb{N} \) such that

1. \( U \supset \{x \in (a, c) ; \mathcal{O}_f^j(f^j(x)) \cap (a, b) \neq \emptyset \forall j > 0\} \);
The aim of Lemma 25 is to construct an induced map $F : U \to (a, b)$, with $U \subset (a, c)$ or $(c, b)$ as sketched in this picture, that is, each connected component of the domain has to be sent by the induced map onto the interval $(a, b)$ and the domain $U$ has to contain all the points of $(a, c)$ or $(c, b)$ that will return to $(a, b)$.

1. $F : U \to (a, b)$ given by $F(x) = f^{R(x)}(x)$ is a local homeomorphism;
2. $F(I) = (a, b)$ for every connected component $I$ of $U$.

Similarly, if $O_f^+(b) \cap (a, b) = \emptyset$, $b \in \text{Per}(f)$, $c_+$ does not belong to the basin of a periodic-like attractor, $c_+ \in \omega_f(c_+)$ and $O_f^+(c_+) \cap (a, c) = \emptyset$, then there exist an open set $U \subset (c, b)$ and a continuous map $R : U \to \mathbb{N}$ such that

1. $U \supset \{x \in (c, b) ; O_f^+(f^j(x)) \cap (a, b) \neq \emptyset \forall j > 0\}$;
2. $F : U \to (a, b)$ given by $F(x) = f^{R(x)}(x)$ is a local homeomorphism;
3. $F(I) = (a, b)$ for every connected component $I$ of $U$.

Proof. Our purpose in this lemma is to construct an induced map $F : U \to (a, b)$, with $U \subset (a, c)$ or $(c, b)$ as sketched in Figure 12. The induced map $F$ will not be the restriction of the first return map to $(a, b)$ and it will be constructed inductively. For this, suppose that $a \in \text{Per}(f)$, $c_-$ does not belong to the basin of a periodic-like attractor, $c_- \in \omega_f(c_-)$ and $O_f^+(c_-) \cap (c, b) = \emptyset$ (the other case is analogous).

Let $r : U \to \mathbb{N}$ be the first return time to $(a, b)$ ($r(x) = \min\{j \geq 1 ; f^j(x) \in (a, b)\}$), where $U$ is the set of $x \in (a, b) \setminus \{c\}$ such that $f^n(x) \in (a, b)$ for some $n \in \mathbb{N}$. Let $F : U \to (a, b)$ be the first return map to $(a, b)$, that is, $F(x) = f^r(x)$ (see Figure 13 for a sketch of some possible graphics of $F$).

Let $U_0 = U \cap (a, c)$ and $P_0$ be the collection of connected components of $U_0$. As $a \in \text{Per}(f)$ and $O_f^+(c_-) \cap (a, c) \neq \emptyset$, there are $I_a, I_0 \in P_0$ such that $a \in \partial I_a$ and $c \in \partial I_0$. Write $I_a = (a, \alpha)$ and $I_0 = (t_0, c)$. Note that $F(I) = (a, b) \forall I \in P_0 \setminus \{I_0\}$ and, as $f$ preserves orientation, $F(I_0) = (a, f^{r(I_0)}(c_-)) \subset (a, c)$. Furthermore, $I_0 \neq I_a$. Otherwise $I_0 = (a, c)$ and, as $F(I_0) \subset (a, c)$, this will imply the existence of a periodic-like attractor, contradicting our hypothesis. Set $R_0 = r|_{(a, c)}$ and $F_0 = F|_{U_0}$.

We now construct a sequence $F_n : U_n \to (a, b)$ of $f$-induced maps defined on open sets $U_n \subset (a, c)$, with induced time $R_n$. The collection of connected components of $U_n$ will be
denoted by $\mathcal{P}_n$, $n \in \mathbb{N}$. For each $n \geq 0$ there will be an element of $\mathcal{P}_n$, denoted by $I_n$, such that $c \in \partial I_n$. This sequence will satisfy the following properties:

1. $I_0 \in \mathcal{P}_n \forall n$;
2. $F_n(I) = (a, b) \forall I \in \mathcal{P}_n \setminus \{I_n\} \forall n$;
3. $F_n(I_n) = (a, f_{R_n}^{-1}(c_-)) \subset (a, c) \forall n$;
4. $R_0(I_0) < R_1(I_1) < R_2(I_2) < \cdots$;
5. $a < t_0 < t_1 < t_2 < t_3 < \cdots$, where $(t_n, c) = I_n \forall n$;
6. $U_{n+1} \cap (a, t_n) = U_n \cap (a, t_n) \forall n$;
7. $F_{n+1}(a, t_n) \cap U_{n+1} = F_n(a, t_n) \cap U_n \forall n$;
8. $U_n \supset \{x \in (a, c) : \mathcal{O}_f^+(f^j(x)) \cap (a, b) \neq \emptyset \forall j > 0\} \forall n$.

Let $\ell_0 = 1 + \max\{j \geq 1 ; F_0^j(I_0) \subset I_0\}$. As $c \in \overline{\mathcal{O}_f^+(c_-) \cap (a, c)}$, $\ell_0$ is precisely the first return time with respect to $F_0$ of $c_-$ to $(a, c)$. Let $R_1(x) = R_0(x)$ if $x \in (a, t_0) \cap U_0$ and $R_1(x) = \sum_{j=0}^{\ell_0} R_0(F_0^j(x))$ for $x \in (t_0, c) \cap F_0^{-\ell_0}(U_0)$. Set $U_1 = (U_0 \cap (a, t_0)) \cup ((t_0, c) \cap F_0^{-\ell_0}(U_0))$ and let $F_1 : U_1 \to (a, b)$ be given by $F_1(x) = f_{R_1}(x)$.

Let $\mathcal{P}_1$ be the collection of connected components of $U_1$. Let $I_1 = (F_0^{\ell_0}|_{I_0})^{-1}(I)$, where $I$ is the element of $\mathcal{P}_0$ containing $F_0^{\ell_0}(c_-)$. By construction, $I_0 \not\supsetneq I_1 \in \mathcal{P}_1$, $c \in \partial I_1$ and $R_1(I_1) = R_0(I_0) + (\ell_0 - 1)r(I_a) + R_0(F_0^{\ell_0}(c_-)) \geq R_0(I_0) + 1$. As $U_1 \supset F_0^{-\ell_0}(U_0)$, we get $U_1 \supset \{x \in (a, c) : \mathcal{O}_f^+(f^j(x)) \cap (a, b) \neq \emptyset \forall j > 0\}$. Define $F_1 : U_1 \to (a, b)$ by $F_1(x) = f_{R_1}(x)$ (see Figure 14).

Inductively, suppose that $F_{n-1} : U_{n-1} \to (a, b)$ is already defined. Let $\ell_{n-1} = 1 + \max\{j \geq 1 ; F_{n-1}^j(I_n) \subset I_0\}$, i.e., $\ell_{n-1}$ is the first return time with respect to $F_{n-1}$ of $c_-$ to $(a, c)$. Let $R_n(x) = R_{n-1}(x)$ if $x \in (a, t_{n-1}) \cap U_{n-1}$ and $R_n(x) = \sum_{j=0}^{\ell_{n-1}} R_{n-1}(F_{n-1}^{-j}(x))$ for $x \in (t_{n-1}, c) \cap (F_{n-1})^{-\ell_{n-1}}(U_{n-1})$. Set $U_n = (U_{n-1} \cap (a, t_{n-1})) \cup ((t_{n-1}, c) \cap (F_{n-1})^{-\ell_{n-1}}(U_{n-1}))$. Let $F_n : U_n \to (a, b)$ be given by $F_n(x) = f_{R_n}(x)$, $\mathcal{P}_n$ be the collection of connected components of $U_n$ and $I_n = ((F_{n-1})^{\ell_{n-1}}|_{I_{n-1}})^{-1}(I)$, where $I$ is the element of $\mathcal{P}_{n-1}$ containing $(F_{n-1})^{\ell_{n-1}}(c_-)$. By construction, $I_{n-1} \not\supsetneq I_n \in \mathcal{P}_n$, $c \in \partial I_n$ and $R_n(I_n) = R_{n-1}(I_{n-1}) + (\ell_{n-1} - 1)r(I_a) + R_{n-1}(F_{n-1}^{-\ell_{n-1}}(c_-)) \geq R_{n-1}(I_{n-1}) + 1$. As $U_n \supset (F_{n-1})^{-\ell_{n-1}}(U_{n-1})$, we get $U_n \supset \{x \in (a, c) : \mathcal{O}_f^+(f^j(x)) \cap (a, b) \neq \emptyset \forall j > 0\}$.

Claim 7. $t_n \to c$. 

**Figure 13.** In this picture we have sketches of some possible graphics of the first return map $\mathcal{F}$ to the interval $(a, b)$ (see Lemma 25).
Figure 14. In this picture, $F_0$ is the restriction (to the interval $(a,c)$) of first return map to $(a,b)$ and $F_1$ is the first step in the inductive construction in the proof of Lemma 25.

Proof. Otherwise $f_j|_{(t_\infty,c)}$ will be a homeomorphism for all $j \in \mathbb{N}$, where $t_\infty = \lim_{n \to \infty} t_n$ (because $F_k$ is monotone on $(t_\infty,c)$ $\forall k \geq 1$ and $R_k((t_\infty,c)) = R_k(I_k) \to \infty$). That is, $(t_\infty,c)$ is a homterval. It follows from the Homterval Lemma that $(t_\infty,c)$ is either a wandering interval or $(t_\infty,c)$ belongs to the basin of a periodic-like attractor. As $(t_\infty,c)$ cannot be a wandering interval, because $c- \in \omega_f(c-)$, we get that $(t_\infty,c)$ belongs to the basin of a periodic-like attractor. But this implies that $c- \in \omega_f(c-)$, which contradicts the hypothesis of the Lemma. □

To finish the proof, set $t_{-1} = a$, $U = \bigcup_{n \geq 0} U_n \cap (t_{n-1},t_n)$ and $F : U \to (a,c)$ by $F|_{U_n \cap (t_{n-1},t_n)} = F_n|_{U_n \cap (t_{n-1},t_n)}$ for $n \geq 0$. □

Notation. To make it short, instead of writing $p \pm \in \omega_f(x)$ as in Definition 3, we will write $p_\pm \in \omega_f(x)$. That is, we write $p_- \in \omega_f(x)$, $p \in [0,1]$, if there is a sequence $n_j \to \infty$ such that $f^{n_j}(x) \not\to p$, i.e., $p \in \overline{O_f^+(x)} \cap [0,p]$. Similarly, $p_+ \in \omega_f(x)$ if $p \in \overline{O_f^-(x)} \cap (p,1]$.

Lemma 26. Let $f : [0,1] \setminus \{c\} \to [0,1]$ be a $C^3$ contracting Lorenz map with negative Schwarzian derivative and without attracting periodic-like orbits. If $(a,c) \subset [0,1]$ is not a wandering interval and $c \in \omega_f(x)$ for all $x \in (a,c) \setminus \mathcal{O}_f^-(c)$ then $\exists b \in (c,1)$ such that $c_- \in \omega_f(x)$ and $c_+ \in \omega_f(x)$ for all $x \in (a,b) \setminus \mathcal{O}_f^-(c)$.

Similarly, if $(c,b) \subset [0,1]$ is not a wandering interval and $c \in \omega_f(x)$ for every $x \in (c,b) \setminus \mathcal{O}_f^-(c)$ then $\exists a \in (0,c)$ such that $c_- \in \omega_f(x)$ and $c_+ \in \omega_f(x) \forall x \in (a,b) \setminus \mathcal{O}_f^-(c)$. 
Proof. Suppose that \( c \in \omega_f(x) \) for all \( x \in (a,c) \setminus \mathcal{O}_f^-(c) \), \((a,c)\) is not wandering and it is not contained in the basin of attraction of a periodic-like orbit. It follows from the hominterval lemma that there is some \( n \geq 1 \) such that \( c \in f^n((a,c)) \) and that \( f^n|_{(a,c)} \) is a diffeomorphism. Therefore, \( c \in \omega_f(x) \) for every \( x \in f^n((a,c)) \setminus \mathcal{O}_f^-(c) \). Let \( I = (a_{\max}, b_{\max}) \) be the maximal open interval such that

\[
c \in \omega_f(x) \text{ for every } x \in I \setminus \mathcal{O}_f^-(c).
\]

Let \( t_0, t_1 \geq 0 \) be the smaller integers such that \( f^{t_0}((a_{\max},c)) \cap I \) and \( f^{t_1}((c,b_{\max})) \cap I \neq \emptyset \) (because of (18), these numbers are well defined). Furthermore, it follows from the maximality that \( I \) is a nice interval and \( f^{t_0}((a_{\max},c)) \subset I \supset f^{t_1}((c,b_{\max})) \). As \( f \) is an orientation preserving map, \( I = (a_{\max}, b_{\max}) \) is a nice interval and \( f \) does not admit attracting periodic-like orbits, we get \( f^{t_0}(a_{\max}) = a_{\max} < c < f^{t_0}(c_{-}) < b_{\max}, a_{\max} < f^{t_1}(c_{+}) < c < b_{\max} = f^{t_1}(b_{\max}) \) and that both \( F_0 := f^{t_0}|_{(a_{\max},c)} \) and \( F_1 := f^{t_1}|_{(c,b_{\max})} \) are diffeomorphisms. Thus, the first return map to \([a,b]\), \( F : [a,b] \setminus \{c\} \to [a,b] \), is given by

\[
F(x) = \begin{cases} 
F_0(x) & \text{if } x < c \\
F_1(x) & \text{if } x > c 
\end{cases}
\]

and it is conjugated to a contracting Lorenz map. Furthermore, \( \mathcal{O}_F^+(x) \cap (a,c) \neq \emptyset \neq \mathcal{O}_F^-(x) \cap (a,c) \), for every \( x \in (a,c) \setminus \mathcal{O}_F^-(c) \), because \( F |_{(a,c)} \) is strict increasing, \( F |_{(c,b)} \) is strict decreasing and \( F(c_{+}) < c < F(c_{-}) \). This implies that \( \mathcal{O}_F^+(x) \cap (a,c) \neq \emptyset \neq \mathcal{O}_F^-(x) \cap (a,c) \) for every \( x \in (a,c) \setminus \mathcal{O}_F^-(c) \). Thus, for every \( x \in (a,c) \setminus \mathcal{O}_F^-(c) \), \( \mathcal{O}_F^+(x) \) accumulates on \( c \) from both sides.

\[\square\]

**Contracting Lorenz maps without periodic attractors.** To prove Theorem C we shall consider two main cases: maps with or without periodic-like attractors. Firstly, we will prove the uniqueness of attractors for maps without periodic-like attractor (Corollary 28). Thereafter, we study maps with periodic-like attractors, showing that we get one or, at most, two attractors.

**Proposition 27.** Let \( f : [0,1] \setminus \{c\} \to [0,1] \) be a \( C^3 \) contracting Lorenz map with negative Schwarzian derivative. If \( f \) does not have an attracting periodic-like orbit, then \( c_{-} \in \omega_f(x) \) and \( c_{+} \in \omega_f(x) \) for Lebesgue almost all \( x \in [0,1] \). In particular, \( f \) does not admit Ewi attractors.

**Proof.** Suppose that there exists \( W \subset [0,1] \) with positive measure and such that \( c_{+} \notin \omega_f(x) \) \( \forall x \in W \) (the case where \( c_{-} \notin \omega_f(x) \) for a positive set of points \( x \in [0,1] \) is analogous).

As \( \text{Leb}\{x : c \notin \omega_f(x)\} = 0 \) (Corollary 1), we get that \( c_{-} \in \omega_f(x) \) for Lebesgue almost every \( x \in W \). Thus, there is some \( \varepsilon > 0 \) and \( V \subset W \), with \( \text{Leb}(V) > 0 \), such that

\[
\mathcal{O}_f^+(x) \cap (c,c + \varepsilon) = \emptyset
\]

and \( c_{-} \in \omega_f(x) \) \( \forall x \in V \).

Note that \( \mathcal{O}_f^-(c_{-}) \cap (c,c + \varepsilon) = \emptyset \). Otherwise \( \mathcal{O}_f^+(x) \cap (c,c + \varepsilon) \neq \emptyset \) \( \forall x \in V \), because \( f \) is continuous and \( c \in \mathcal{O}_f^+(x) \cap (0,c) \) \( \forall x \in V \).
If \( c \in \omega_f(x) \) for all \( x \in (c - \varepsilon, c) \setminus \mathcal{C}_f(c) \), then the proof follows from Lemma \( \text{26} \). Thus, we may assume that this is not the case. That is, for every \( \delta > 0 \) there are \( a_\delta \in (c - \delta, c) \) so that \( c \notin \overline{O_f^+(a_\delta)} \).

First, suppose that \( c_- \) is not recurrent, i.e., \( c_- \notin \omega_f(c_-) \). In this case, let \( \delta \in (0, \varepsilon) \) be such that \( \mathcal{O}_f^+(f(c_-)) \cap (c - \delta, c) = \emptyset \) and define \( a = \max(\mathcal{O}_f^+(a_\delta) \cap (0, c)) \) and \( b = \min \left( \{c + \delta\} \cup (\mathcal{O}_f^+(a_\delta) \cap (1, 1)) \right) \).

It is easy to see that \( \mathcal{O}_f^+(a) \cap (a, b) = \emptyset \). Let \( J = (a, b) \), \( F : J^* \to J \) be the first return map to \( J \), where \( J^* = \{x \in J : \mathcal{O}_f^+(f(x)) \cap J \neq \emptyset \} \), and set \( U = J^* \cap (a, c) \). As \( \mathcal{O}_f^+(c_-) \cap J = \emptyset \), it follows from Lemma \( \text{23} \) that \( F(I) = J \) for every connected component \( I \) of \( U \).

Now suppose that \( c_- \) is recurrent, \( c_- \in \omega_f(c_-) \). In this case, let \( a_1 = \max(\mathcal{O}_f^+(a_\varepsilon) \cap (0, c)) \) and \( b = \min \left( \{c+\varepsilon\} \cup (\mathcal{O}_f^+(a_\varepsilon) \cap (c, 1)) \right) \).

Thus, \( \mathcal{O}_f^+(a_1) \cap (a_1, b) = \emptyset \). As \#\{\( j \geq 0 : f^j(x_0) \in (a_1, c) \} = \infty \) if \( x_0 \in V \), it follows from Lemma \( \text{24} \) that there is \( a \in (a_1, c) \), such that \( a \in \text{Per}(f) \) and \( \mathcal{O}_f^+(a) \cap (a, b) = \emptyset \). Set \( J = (a, b) \). Let \( U \subset (a, c) \), \( F : U \to (a, b) \) and \( R : U \to \mathbb{N} \) be given by Lemma \( \text{25} \). In this case, we also have \( F(I) = J \) for every connected component of \( U \).

Note that, independently of \( c_- \) being recurrent or not, \( V \subset U \). Let \( C_0 \subset (c, b) \) be a finite set and \( g : (c, b) \setminus C_0 \to (c, b) \) be any \( C^3 \) orientation preserving local diffeomorphism with \( Sg < 0 \). Set \( U = U \cup (c, b) \setminus C_0 \) and \( G : U \to (a, b) \) by

\[
G(x) = \begin{cases} 
F(x) & \text{if } x \in U \\
g(x) & \text{if } x \in (c, b) \setminus C_0
\end{cases}
\]

Because \( G^n(x) = F^n(x) \) \( \forall x \in V \) and \( \forall n \in \mathbb{N} \), we get \( G(V) \subset V \). Let \( \mathcal{P} \) be the collection of connected components of \( U \). Notice that \( G(P) = (a, b) \) \( \forall P \in \mathcal{P} \). Thus, as \( SG < 0 \) and \( \text{Leb} \left( \bigcap_{n \geq 0} G^{-n}(U) \right) \geq \text{Leb}(V) > 0 \), it follows from Lemma \( \text{12} \) that \( \omega_F(x) = \omega_G(x) = [a, b] \) for almost every \( x \in V \). This implies that \( \mathcal{O}_f^+(x) \cap (c, c+\varepsilon) \neq \emptyset \) for almost every \( x \in V \), which contradicts \( \text{[19]} \).

\[ \square \]

**Corollary 28.** Let \( f : [0, 1] \setminus \{c\} \to [0, 1] \) be a \( C^3 \) contracting Lorenz map with negative Schwarzian derivative. If \( f \) does not attracting periodic-like orbits then \( \omega_f(x) = \overline{O_f^+(c_-)} \cup \overline{O_f^+(c_+)} \) for Lebesgue almost every \( x \in [0, 1] \).

**Proof.** This Corollary follows straightforwardly from Theorem \( \text{[1]} \) and Proposition \( \text{27} \) above. Indeed, it follows from Proposition \( \text{27} \) that \( f \) does not have an Ewi attractor. Thus, \( \mathbb{B}_0(f) = \mathbb{B}_2(f) = \emptyset \), \( W_f = \mathcal{C}_f^\infty = \emptyset \) and also \( \alpha^* = \alpha \), \( \forall \alpha \in \mathcal{C}_f \), where \( \mathcal{C}_f = \{c - \frac{1}{2}, c + \frac{1}{2}\} \). As a consequence, it follows from Theorem \( \text{1} \) that \( \omega_f(x) = \overline{O_f^+(c_-)} \cup \overline{O_f^+(c_+)} \) for almost every \( x \in [0, 1] \), since by Proposition \( \text{27} \) we have that \( \alpha \in \omega_f(x) \) for almost every \( x \in [0, 1] \).

\[ \square \]

**Contracting Lorenz maps with periodic attractors.** By Singer’s theorem, a contracting Lorenz map can have at most two attracting periodic-like orbits. In Figure \( \text{[15]} \) we give some simple examples of contracting Lorenz maps with one or two periodic-like fixed points. In these examples it is not difficult to guess that the union of the basins of attraction of the
Figure 15. In the left hand side picture, we have a contracting Lorenz map with an attracting fixed point \( p \) whose basin of attraction is \((0,1)\setminus \mathcal{O}_f^{-}(c)\). In the center one, we have a contracting Lorenz map with two attracting fixed points: \( q = 0 \) and \( p \in (c,1) \). In this case, \( \beta_f(q) = (0,c) \) and \( \beta_f(p) = (c,1) \). Finally, in the right hand side picture, the contracting Lorenz map has a single attracting fixed-like point \( p = c \) and its basin of attraction is \((0,1)\).

Attracting periodic-like orbits contains almost every point. Nevertheless, it is not obvious that this is the case when a contracting Lorenz map has attracting periodic orbits with large periods. Assuming the non-flatness condition, this was proved in [42]. Here we present a proof of this result without the additional hypothesis of non-flatness of the critical point.

**Lemma 29.** Let \( f : [0,1] \setminus \{c\} \to [0,1] \) be a \( C^3 \) contracting Lorenz map with negative Schwarzian derivative. If there are periodic-like attractors \( A_1 \) and \( A_2 \) (\( A_2 \) may be equal to \( A_1 \)) and \( \delta > 0 \) such that \( \text{Leb}((c-\delta,c) \cup (c,c+\delta)) \setminus (\beta_f(A_1) \cup \beta_f(A_2)) = 0 \), then \( \text{Leb}(\beta_f(A_1) \cup \beta_f(A_2)) = 1 \).

**Proof.** Let \( J = (a,b) \) be the maximal open interval containing \( c \) such that \( \text{Leb}((a,b) \setminus (\beta_f(A_1) \cup \beta_f(A_2))) = 0 \). It is easy to check that the maximality of \( (a,b) \) implies that \( a,b \in \text{Per}(f) \) (indeed, the first return map to \([a,b]\) is conjugated to a contracting Lorenz map).

Similarly to the proof of Lemma [16], we claim that \( \mathcal{O}_f^+(x) \cap J = \emptyset \) for almost every \( x \in [0,1] \). To show this, consider \( g : [0,1] \setminus \mathcal{C}_g \to [0,1] \), defined by

\[
g(x) = \begin{cases} 
  f(x) & \text{if } x \notin (a,b) \\
  \lambda_a(f(x) - f(a)) + f(a) & \text{if } x \in (a,c) \\
  \lambda_b(f(x) - f(b)) & \text{if } x \in (c,b) 
\end{cases}
\] (20)

where \( \lambda_a = (1 - f(a))/(f(c_\cdot) - f(a)) \), \( \lambda_b = (f(b))/(f(b) - f(c_\cdot)) \) and \( \mathcal{C}_g = \{c\} \cup \partial J \), see Figure [16].

Letting \( \mathcal{U} = [0,1] \setminus \mathbb{B}_0(f) \) and assuming by contradiction that \( \text{Leb}(\mathcal{U}) > 0 \), one can show (following the proof of Lemma [16]) the existence of a connected component \( I_{t_0} \) of \( ([0,1] \setminus J) \setminus (\mathcal{O}_f^+(a) \cup \mathcal{O}_f^+(b)) \) such that \( \text{Leb}\{x \in \mathcal{U} : \#(\mathcal{O}_f^+(x) \cap I_{t_0}) = \infty\} > 0 \) and that \( I_{t_0} \cap \text{Re}(\mathcal{O}_g^+(\mathcal{V}_g)) = \emptyset \). Thus, by the interval dichotomy (Proposition [13]), \( \omega_g(x) \supset I_{t_0} \) for almost every \( x \in I_{t_0} \). By homterval lemma, we get \( n \geq 0 \) such that \( g^n|_{I_{t_0}} \) is a
diffeomorphism and \( C_g \cap g^n(I_{t_0}) = \{a, b, c\} \cap g^n(I_{t_0}) \neq \emptyset \) and so, \( g^n(I_{t_0}) \cap J \neq \emptyset \). As a consequence the orbit (with respect to \( g \) and also \( f \)) of almost every point of \( U \cap I_{t_0} \) intersects \( J \), contradicting the definition of \( U \). □

**Proposition 30** (Periodic-like attractors for contracting Lorenz maps). Let \( f : [0, 1] \setminus \{c\} \to [0, 1] \) be a \( C^3 \) contracting Lorenz map with negative Schwarzian derivative (\( f \) may be flat). If \( f \) has a periodic-like attractor \( A_1 \), then either \( \text{Leb}(\beta_f(A_1)) = 1 \) or else there is a second periodic-like attractor \( A_2 \) such that \( \text{Leb}(\beta_f(A_1) \cup \beta_f(A_2)) = 1 \).

**Proof.** Let \( A_1 \) be a periodic-like attractor for \( f \). By Singer’s theorem, there is \( \epsilon > 0 \) such that \( (c-\epsilon, c) \) or \( (c, c+\epsilon) \subseteq \beta_f(A_1) \). Suppose for instance that \( (c, c+\epsilon) \subseteq \beta_f(A_1) \), the other case being similar. Let \( (c, b) \) be the maximal open interval containing \( (c, c+\epsilon) \) and contained in \( \beta_f(A_1) \).

First suppose that \( \exists \delta > 0 \) such that \( (c-\delta, c) \) belongs to the basin of attraction of a periodic-like attractor \( A_2 \) (\( A_2 \) may be equal to \( A_1 \)). We then apply Lemma 29 and conclude the proof.

Thus, we may assume that, for every \( \delta > 0 \), \( (c-\delta, c) \) is not contained in the basin of attraction of a periodic attractor. In particular this implies, by Singer’s theorem, that \( A_1 \) is the unique periodic-like attractor of \( f \).

**Claim.** If \( (a, c) \) is a homterval for some \( a < c \), then \( \text{Leb}(\beta_f(A_1)) = 1 \).

**Proof of the claim.** Let \( I = (a', c) \) be the maximal homterval containing \( (a, c) \). As \( I \) is a homterval, \( \omega_f(x) = \omega_f(y) \), \( \forall x, y \in I \). Thus, if \( I \cap \beta_f(A_1) \neq \emptyset \) then \( I \subseteq \beta_f(A_1) \) and by Lemma 29 we get that \( \text{Leb}(\beta_f(A_1)) = 1 \).

Now, let us show that the assumption \( I \cap \beta_f(A_1) = \emptyset \) leads to a contradiction. Indeed, if \( I \cap \beta_f(A_1) = \emptyset \) then \( f^n(J) \cap (c, b) = \emptyset \) \( \forall n \geq 0 \). By the maximality of \( I \) either \( a' = 0 \) or \( f^\ell(a') = c \) for some \( \ell \geq 1 \). As \( f^n(J) \cap (c, b) = \emptyset \) \( \forall n \geq 0 \), we necessarily have \( a' = 0 \). As \( f^2|_I \) is a diffeomorphism, \( c \notin f(I) = (0, f(c_-)) \). This implies \( f(I) \subset I \) and therefore, \( (0, c) \) in...
containing in the basin of attraction of some fixed-like point \( p \in [0, c] \), as we are assuming that \( A_1 \) is the unique periodic-like attractor, \( A_1 = p \) and \( I = (0, c) \subseteq \beta_f(p) \), contradicting the assumption.

Let us suppose, by contradiction, that \( \text{Leb}([0, 1] \setminus \beta_f(A_1)) > 0 \). Thus, it follows from the claim above that \( f \) can not have an Ewi attractor. Also, by Lemma \ref{lemma:f_i}, \( f \) does not admit a cycle of intervals. Thus, it follows from Corollary \ref{corollary:cycle} that \( c - \delta \in \omega_f(x) \) for almost all \( x \in [0, 1] \setminus \beta_f(A_1) \). In particular, writing \( V(a) = \{ x \in (a, c) : \#(O_f^n(x) \cap (a, c)) = \infty \} \), we get

\[
\text{Leb}(V(a)) > 0, \quad \forall a \in [0, c).
\]

As \( \Re(\omega_f(c + \delta)) = A_1 \), if \( c - \delta \notin \Re(\omega_f(c - \delta)) \) then \( \exists 0 < a < c \) such that \( \Re(O_f^n(V_f)) \cap (a, c) = \emptyset \), where \( V_f = \{ f(c - \delta), f(c + \delta) \} \). Thus, by the interval dichotomy, \( \omega_f(x) \supseteq (a, c) \) for almost every \( x \in (a, c) \). This implies the existence of a cycle of intervals, which is a contradiction. As a conclusion we have \( c - \delta \in \Re(\omega_f(c - \delta)) \).

**Claim.** If \( \#A_1 \geq 2 \) then \( c \notin A_1 \).

**Proof of the claim.** Suppose that \( c \in A_1 \) and let \( n \geq 2 \) be the period of \( c + \delta \). Let \( T := (c, t) \) be the maximal interval such that \( f^n|_T \) is a diffeomorphism. We claim that \( f^n(t_-) > t \). Indeed, by the maximality of \( T \) either \( t = 1 \) or \( \exists 1 \leq \ell < n \) such that \( f^n(t) = c \). As \( t = 1 \) implies that \( n = 1 \), we conclude that \( f^\ell(t) = c \), for some \( 1 \leq \ell < n \). If \( \ell = 1 \) and also that \( f^n(T) = (f^n(c_+), f^n(t_-)) = (f^n(c_+), f^n(c_-)) \). Thus, if \( f^n(t_-) \leq t_- \) then either \( t - \delta \) is a periodic-like point with \( c - \delta \notin O_f^n(t - \delta) \) or \( c \notin f^n(t_-) = f^n(t_-) < t \). The first case is impossible because it implies the existence of a second attracting periodic-like orbit, and we are assuming that \( A_1 \) is the unique one. On the other hand, \( c < f^{n - \ell}(c_-) = f^n(t_-) \) implies that \( c < f^{k_n}(x) = f^{(k_{n-1})n}(x) < \cdots < f^n(x) < x \) for every \( x \in (c, t) \) and this means that \( T = (c, t) \subseteq \beta_f(A_1) \). As a consequence, \( T' = f^\ell(T) \subseteq \beta_f(A_1) \) which contradicts the assumption that \( (c - \delta, c) \) is not contained in the basin of attraction of a periodic-like attractor \( \forall \delta > 0 \).

Notice that, \( A_1 \cap (0, c) = \emptyset \) if and only if \( A_1 \) is an attracting fixed-like point \( q \in [c, 1] \). As we are assuming that \( p \) is the unique attracting periodic-like point it is easy to see that if \( A_1 \) is a fixed-like point then \( f(c_-) > c \) and that \( (c, 1) \supset \beta_f(q) \). In this case one can conclude easily that \( \beta_f(q) = (0, 1) \).

So, we may assume that \( A_1 \cap (0, c) \neq \emptyset \) and, by the claim just above, we get that \( c \notin A_1 \).

Let \( J := (p, q) \) be the connected component of \( (0, 1) \setminus A_1 \) containing \( c \). Thus, \( J \) is a nice interval containing \( c \) and \( p \in \text{Per}(f) \).

By Singer’s theorem, \( (c, q) \subset \beta_f(A_1) \) and so, \( O_f^n(c_-) \cap (c, q) = \emptyset \). As we also have that \( q \in \text{Per}(f) \), \( c_- \) does not belong to the basin of attraction of \( A_1 \) (the unique periodic-like attractor of \( f \)) and \( c_- \in \omega_f(c_-) \), we can consider \( U \subset (p, c), F : U \to (p, q) \) and \( R : U \to \mathbb{N} \) as in Lemma \ref{lemma:Singer}. Note that \( V(p) \subset U \). Let \( C_0 \subset (c, q) \) be a finite set and \( g : (c, q) \setminus C_0 \to (c, q) \) be any \( C^3 \) orientation preserving local diffeomorphism with \( Sg < 0 \).
ON THE FINITENESS OF ATTRACTORS FOR ONE-DIMENSIONAL MAPS WITH DISCONTINUITIES

Set $\mathcal{U} = U \cup (c, q) \setminus C_0$ and $G: \mathcal{U} \to (p, q)$ by

$$G(x) = \begin{cases} F(x) & \text{if } x \in U \\ g(x) & \text{if } x \in (c, q) \setminus C_0 \end{cases}$$

Because $G^n(x) = F^n(x)$ $\forall x \in V(p)$ and $\forall n \in \mathbb{N}$, we get $G(V(p)) \subset V(p)$. Let $\mathcal{P}$ be the collection of connected components of $\mathcal{U}$. As, $G(P) = (p, q) \forall P \in \mathcal{P}$, $SG < 0$ and $\text{Leb}(\bigcap_{n \geq 0} G^{-n}(U)) \geq \text{Leb}(V(p)) > 0$, it follows from Lemma 12 that $\omega_F(x) = \omega_G(x) = [p, q]$ for almost every $x \in V(p)$. In particular, the $\omega$-limit set of almost every $x \in V(p)$ is a cycle of intervals. This is a contradiction, as $f$ cannot admit a cycle of intervals. Thus, we necessarily have $\text{Leb}([0, 1] \setminus \beta_f(A_1)) = 0$, which concludes the proof.

\[\square\]

**Proof of Theorem C.** If $f$ has attracting periodic-like orbits, then the proof follows from Proposition 30 above. If $f$ does not admit attracting periodic-like orbits, then it follows from Corollary 28 that $\omega_f(x) = A$ for almost every $x \in [0, 1]$, where $A = \mathcal{O}_f^+(c_-) \cup \mathcal{O}_f^+(c_+)$. Thus, we need only to verify that $\mathcal{O}_f^+(c_-) \cup \mathcal{O}_f^+(c_+)$ is a perfect set and this follows straightforward from Corollary 28 and Corollary 15.

**Further comments.** Here we have dealt with metrical attractors. On the other hand, notice that for non-flat $C^3$ maps of the interval $[0, 1]$ with negative Schwarzian derivative it is known that the number of topological attractors is bounded by the number of critical points [15, 22]. Also notice that the topological attractors and the metrical ones may not be the same, as is the case of the wild attractors [9]. For maps with discontinuities, Brandão showed in [5] that contracting Lorenz maps have either one single topological attractor (with its basin of attraction being a residual subset of the interval) or two attracting periodic orbits whose union of basins of attraction is a residual subset of the interval. In the context of topological attractors, the question of finiteness of the number of attractors for discontinuous maps with more than one critical point remains largely open.

**REFERENCES**

[1] Alsedà, L, Llibre, J and Misiurewicz, M. Periods and entropy for Lorenz-like maps. Ann. Inst. Fourier (1989).

[2] A. Arneodo, P. Coullet and C. Tresser. *A possible new mechanism for the onset of turbulence.* Phys. Lett. A 81(4) (1981),197-201.

[3] A. Avila, M. Lyubich, W. de Melo, Regular or stochastic dynamics in real analytic families of unimodal maps, Invent. Math. 154 (2003) 451?550.

[4] A. Avila, C.G. Moreira, Statistical properties of unimodal maps: the quadratic family, Ann. of Math., in press.

[5] P. Brandão. *On the Structure of Contracting Lorenz maps, preprint, IMPA Thesis, 2013.*

[6] A. Blokh and M. Lyubich, *Ergodicity of transitive maps of the interval,* Ukrainian Math. J. 41 (1989), 985–988.

[7] Blokh, A. M.; Lyubich, M. Yu. Nonexistence of wandering intervals and structure of topological attractors of one-dimensional dynamical systems. II. The smooth case. Ergodic Theory Dynam. Systems 9 (1989), no. 4, 751-758.

[8] A. Blokh and M. Lyubich, *Measurable dynamics of $S$-unimodal maps of the interval,* Ann. Sci. École Norm. Sup. 24 (1991), 545–573.
[9] H. Bruin, G. Keller, T. Nowicki and S. van Strien. Wild Cantor Attractors Exist. The Annals of Mathematics, Second Series, Vol. 143, No. 1 (Jan., 1996), pp. 97-130.

[10] A. Chenciner; J-M. Gambaudo; C. Tresser. Une remarque sur la structure des endormorphismes de degré 1 du cercle. C.R. Acad. Sc. Paris, t 299, Serie I, n° 5, 1984

[11] Gambaudo, J M, Procaccia, I, Thomas, S and Tresser, C (1986). New universal scenarios for the onset of chaos in Lorenz-type flows. Physical review letters. 57 925-8

[12] J. Graczyk and G. Swiatek Generic Hyperbolicity in the Logistic Family. Annals of Mathematics, Second Series, Vol. 146, No. 1 (Jul., 1997), pp. 1-52.

[13] J. Guckenheimer. On the bifurcation of maps of the interval. Inventiones Mathematicae 39, 165 (1977).

[14] J. Guckenheimer. A strange, strange attractor. In The Hopf bifurcation theorem and its applications, pages 368-381. Springer Verlag, 1976.

[15] J. Guckenheimer. Sensitive Dependence to Initial Conditions for One Dimensional Maps. Commun. Math. Phys. 70, 133 (1979).

[16] Guckenheimer, J and Johnson, S. Distortion of S-Unimodal Maps. The Annals of Mathematics. 1990. 1v32, 71-130.

[17] J. Guckenheimer and R. F. Williams. Structural stability of Lorenz attractors. Publ. Math. IHES 50 (1979), 59-72.

[18] Hubbard, J H. Local connectivity of Julia sets and bifurcation loci: three theorems of J.-C. Yoccoz. Topological methods in modern mathematics (Stony Brook, NY, 1991). Publish or Perish, Houston, TX, 1993. 467-511

[19] M. Jakobson, Absolutely continuous invariant measures for one-parameter families of one-dimensional maps, Comm. Math. Phys. 81 (1981), 39-88.

[20] G. Keller, Exponents, attractors and Hopf decompositions for interval maps. Ergod. Th. & Dynam. Sys. 10 (1990), 717-744.

[21] Keller, G and St Pierre, M. Topological and Measurable Dynamics of Lorenz Maps. Ergodic Theory, Analysis, and Efficient Simulation of Dynamical Systems (ISBN: 978-3-642-62524-4). 2001. 333-361.

[22] Lyubich, M. Yu. Nonexistence of wandering intervals and structure of topological attractors of one-dimensional dynamical systems. I. The case of negative Schwarzian derivative. Ergodic Theory Dynam. Systems 9 (1989), no. 4, 737-749.

[23] Lyubich, M. Yu. Ergodic theory for smooth one dimensional dynamical systems (1991), arXiv:math/9201286.

[24] M. Lyubich, Combinatorics, Geometry and Attractors of Quasi-Quadratic Maps. The Annals of Mathematics Second Series 140, 347 (1994).

[25] M. Lyubich. Dynamics of quadratic polynomials. I, II. Acta Math. 178 (1997), 185-247, 247-97.

[26] M. Lyubich. Almost every real quadratic map is either regular or stochastic. Ann. of Math. (2) 156 (2002), no. 1, 1-78.

[27] C. McMullen, Complex Dynamics and Renormalization, Ann. of Math. Stud., vol. 142, Princeton University Press, 1994.

[28] de Melo, W. ; Martens, M. . Universal Models for Lorenz Maps.. Ergod. Th. & Dynam. Sys., v. 21, n. 2, p. 343-370, 2000.

[29] W. de Melo; S. van Strien. A Structure Theorem in One Dimensional Dynamics. Annals of Mathematics, vol. 129, 1989, pp. 519-546

[30] M. Martens, W. de Melo, P. Mendes and S. van Strien. On Cherry flows. Ergod. Th. & Dynam. Sys. 10 (1990), 531-554.

[31] M. Martens. Distortion results and invariant Cantor Sets of unimodal maps. Ergod. Th. & Dynam. Sys. 14(2) (1994), 331-349.

[32] R. Mañé. Hyperbolicity, sinks and measure in one-dimensional dynamics. Comm. Math. Phys.100:495-524, 1985. Commun. Math. Phys. 112 (1987), 721-724 (Erratum).

[33] de Melo, W. C., Strien, S. V.. One Dimensional Dynamics, Springer-Verlag, 1993.

[34] J. Milnor. On the Concept of Attractor, , Commun. Math. Phys. 99, 177 (1985): 102, 517 (1985).

[35] John Milnor and William Thurston. On iterated maps of the interval. In Dynamical systems (College Park, MD, 1986-87), pages 465-563. Springer, Berlin-New York, 1988.
[36] J. Palis. A Global View of Dynamics and a Conjecture on the Denseness of Finitude of Attractors. Astérisque, France, v. 261, p. 339-351, 2000.

[37] J. Palis. A global perspective for non-conservative dynamics. Annales de l’Institut Henri Poincaré. Analyse Non Linéaire, 22, 2005, p. 487-507.

[38] A. Rovella. The dynamics of perturbations of contracting Lorenz Maps. Bul. Soc. Brazil Mat. (N.S.) 24(2) (1993), 233-259.

[39] Singer, D.. Stable orbits and bifurcations of maps of the interval. SIAM J. Appl. Math. 35, 260-267 (1978)

[40] S. van Strien and E. Vargas. Real bounds, ergodicity and negative Schwarzian for multimodal maps, J. Am. Math. Soc. 17 (2004), 749-782.

[41] S. van Strien. One-parameter families of smooth interval maps: density of hyperbolicity and robust chaos. Proceedings of the American Mathematical Society. 138 (2010), 4443-4446.

[42] M. St. Pierre, Topological and measurable dynamics of Lorenz maps, Dissertationes Mathematicae,17, 1999.

[43] D. Sullivan, Bounds, quadratic differentials and renormalization conjectures, Ann. Math. Soc. Centennial Publ. 2 (1992) 417?466.

[44] R. F. Williams. The structure of Lorenz attractors. Publ. Math. IHES 50 (1979), 73-99.

P. Brandão, IMPA, Estrada Dona Castorina, 110, Rio de Janeiro, Brazil.
E-mail address: paulo@impa.br

J. Palis, IMPA, Estrada Dona Castorina, 110, Rio de Janeiro, Brazil.
E-mail address: jpalis@impa.br

V. Pinheiro, Departamento de Matemática, Universidade Federal da Bahia, Av. Ademar de Barros s/n, 40170-110 Salvador, Brazil.
E-mail address: viltonj@ufba.br