Abstract. We compare two QCD-inspired quark models with four-fermion interaction, without and with the remnant coupling to low-energy gluons, in the regime of dynamical chiral symmetry breaking (DCSB). The first one, the Nambu-Jona-Lasinio (NJL) model ensures the factorization of scalar and pseudoscalar meson poles in correlators, the well-known Nambu relation between the scalar meson mass and the dynamical quark mass, $m_{\sigma} = 2m_{\text{dyn}}$, and the residual chiral symmetry in coupling constants characteristic for the linear $\sigma$-model. The second one, the Gauged NJL model, happens to be qualitatively different from the NJL model, namely, the Nambu relation is not valid and the factorization of light meson poles does not entail the residual chiral symmetry, i.e. it does not result in a linear $\sigma$-model. The more complicated DCSB pattern in the GNJL model is fully explained in terms of excited meson states with the same quantum numbers. The asymptotic restrictions on parameters of scalar and pseudoscalar meson states are derived from the requirement of chiral symmetry restoration at high energies.
1. Introduction

The QCD-inspired quark models with attractive four-fermion interaction are often applied for the truncation of the low-energy QCD in the hadronization regime [1-9]. In the common approach [1,2] the local four-fermion interaction only is involved to induce the DCSB due to strong attraction in the scalar channel. As a consequence, the dynamical quark mass \( m_{\text{dyn}} \) is created, massless pions (in the chiral limit, \( m_{\text{current}} = 0 \)) and the massive scalar meson with the mass, \( m_{\sigma} = 2m_{\text{dyn}} \) arise and the residual chiral symmetry characteristic for the linear \( \sigma \)-model [2] holds in coupling constants (see below). On the other hand the effective QCD action for quarks at low energies contains, in general, the remnant interaction to low-energy, background gluons which is responsible for the confinement and hadron properties at intermediate energies, in particular, for the formation of excited meson states [10]. Therefore, when gluons are treated as being low-energy, background ones the Gauged NJL model may be thought of as a better truncation of the low-energy QCD effective action [5]. As the DCSB holds also for such a model the appearance of large dynamical quark mass gives rise the possibility to calculate nonperturbative gluon corrections for hadron parameters in the condensate approach [3-6].

When comparing two models in the description of hadron properties one has to examine to what extent the Gauged NJL model leads to the same hadron lagrangian as the conventional NJL one, after proper renormalization of its basic characteristics [4]. It is one of the aims of our paper to show that two models are qualitatively different since the coupling to background gluons breaks the linear \( \sigma \)-model relations for masses and coupling constants which are valid in the non-gauged case.

In order to explain the roots of this difference we consider the qualitative features of QCD hadron spectrum in the planar limit (large \( N_c \)). In this approach the correlators of color-singlet quark currents are saturated by narrow meson resonances. In particular, the two-point correlators of scalar and pseudoscalar quark densities are represented by the sum of related meson poles,

\[
\Pi^S(Q) = -\int d^4x \exp(iqx) \left< T \left( \bar{\psi} \psi(x) \bar{\psi} \psi(0) \right) \right>_{\text{planar}} \\
= \sum_n \frac{Z_{n}^S}{Q^2 + M_{S,n}^2} + C_0^S + C_1^S Q^2;
\]

\[
\Pi^P(Q) = \int d^4x \exp(iqx) \left< T \left( \bar{\psi} \gamma_5 \psi(x) \bar{\psi} \gamma_5 \psi(0) \right) \right>_{\text{planar}} \\
= \sum_n \frac{Z_{n}^P}{Q^2 + M_{P,n}^2} + C_0^P + C_1^P Q^2, \tag{1}
\]

in the Euclidean momentum region, \( q^2 = -Q^2 < 0 \). The absence of physical thresholds for \( q^2 > 0 \) in the large \( N_c \) limit is ensured by the confinement of color intermediate states. The high-energy asymptotics is controlled [11] by the perturbation theory and the operator product expansion due to asymptotic freedom of QCD. Thereby the same
correlators have the power-like behavior at large \( Q^2 \),

\[
\Pi^S(Q)|_{Q^2 \to \infty} \simeq \Pi^P(Q)|_{Q^2 \to \infty} \simeq \frac{N_c}{8\pi^2} Q^2 \ln \frac{Q^2}{\mu^2}.
\] (2)

When comparing (1) and (2) one concludes that the infinite series of resonances with the same quantum numbers should exist in order to reproduce the perturbative asymptotics. Besides, one can prove [12] that

\[
\left( \Pi^S(Q) - \Pi^P(Q) \right)|_{Q^2 \to \infty} = O \left( \frac{1}{Q^6} \right),
\] (3)

and the chiral symmetry is restored at high energies.

Thus the QCD-inspired quark models are expected to reproduce the part of QCD meson spectrum in the planar limit and the asymptotic chiral symmetry restoration for higher energies. In particular, the conventional NJL model describes one scalar and one pseudoscalar (multiplet of) state whereas the Gauged NJL model should contain additional excited meson states with the same quantum numbers. Just the appearance of additional poles in correlators makes two models to be different in the DCSB pattern.

The rest of the paper contains the derivation of two-point correlators in the NJL and Gauged NJL models, the proof of the violation of linear \( \sigma \)-model relations and the soft-momentum determination of scalar meson parameters. At the end the two-resonance ansatz is performed to saturate the deviation of NJL and GNJL models. The asymptotic sum rules based on the chiral symmetry restoration are obtained for the resonance characteristics.

### 2. The definitions

We start from the NJL model without gluons and introduce the auxiliary scalar, \( \bar{\sigma} = \sigma(x) + m_{dyn}; \) \( (m_{dyn} \equiv m) \) and pseudoscalar, \( \pi(x) \) fields. Respectively the basic lagrangian density can be presented in two forms (the euclidean-space formulation is taken here),

\[
\mathcal{L}(x) = \bar{\psi} \left( i\hat{\partial} + iS(x) + \gamma_5 P(x) \right) \psi + \frac{g^2}{4N_c} \left[ (\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2 \right]
\]

\[
\Longleftrightarrow \bar{\psi}(\hat{D} + iS(x) + \gamma_5 P(x))\psi + \frac{N_c}{g^2} \left( (\sigma(x) + m)^2 + \pi^2(x) \right),
\] (4)

where \( \psi \equiv \psi_i \) stands for color fermion fields with \( N_c \) components, \( S(x), P(x) \) are external scalar and pseudoscalar sources and the Dirac operator is

\[
\hat{D} = i\hat{\partial} + im + i\sigma(x) + \pi(x)\gamma_5; \quad \hat{\partial} \equiv i\gamma^\mu \partial_\mu
\] (5)

The vector and axial-vector fields are not included since the main story is interplayed between scalar and pseudoscalar channels. We simplify our analysis setting up the number of flavours \( N_F = 1 \) and the current quark mass \( m_q = 0 \).
Let us make the change of fields, $\sigma \to \sigma - S; \quad \pi \to \pi - P$. Then the regularized fermion vacuum functional $Z_{\text{reg}}^F$ is described by the following relations,

$$\ln Z_{\text{reg}}^F(\sigma, \pi) = \ln \left\langle \exp\left(- \int d^4x \mathcal{L}_F(\sigma(x), \pi(x))\right) \right\rangle_{\bar{\psi}\psi} = N_c \text{Tr} \ln \hat{D}_{\text{reg}}$$  \hspace{1cm} (6)

The corresponding effective action for auxiliary fields reads

$$S_{\text{eff}} = -N_c \text{Tr} \ln \hat{D}_{\text{reg}}$$

$$+ \frac{N_c}{g^2} \int d^4x (\sigma^2 + 2m\sigma + \pi^2 - 2\sigma S - 2mS - 2\pi P + S^2 + P^2 + m^2)$$  \hspace{1cm} (7)

where a momentum-cutoff regularization is implied. The parameter $g$ is a four-fermion constant. When relating to the BRZ denotations \cite{8} it happens to be $g^2 = 8\pi^2 G_s/\Lambda^2$.

3. Parameters of scalar and pseudoscalar mesons in the conventional NJL model

As usual the dynamic mass is subject to the mass-gap equation in which $\Lambda$ is a $O(4)$-invariant cutoff,

$$m(\Lambda^2 - \frac{8\pi^2}{g^2} - m^2 \ln \frac{\Lambda^2}{m^2}) + O(\frac{m^2}{\Lambda^2}) = 0.$$  \hspace{1cm} (8)

Herein and further on we omit the terms of order $1/\Lambda^2$ which do not change the result drastically. This equation provides the cancellation of the linear in $\sigma$ terms in (7), with the tadpole in the fermion determinant,

$$\frac{2N_cm}{g^2} = i < \bar{\psi}\psi > = \frac{iN_c}{(Vol.)} \text{Tr} \left( \hat{D}^{-1} \right)_{\text{reg}} |_{\sigma,\pi=0} =$$

$$= 4N_c \int_{p<\Lambda} \frac{d^4p}{16\pi^2} \frac{p^2d(p^2)}{p^2 + m^2}.$$  \hspace{1cm} (9)

The quadratic in fields part of $S_{\text{eff}}$ determines the inverse propagators (“kinetic terms”) for meson fields,

$$S_{\text{eff}} = \frac{1}{2} \int \frac{d^4Q}{(2\pi)^4} \left[ \sigma(Q)\Gamma_{\sigma}(Q^2)\sigma(-Q) + \pi(Q)\Gamma_{\pi}(Q^2)\pi(-Q) \right],$$  \hspace{1cm} (10)

where in the simple NJL model without gluons one has,

$$\Gamma_{\sigma} = \frac{2N_c}{g^2} - N_c \int_{p<\Lambda} \frac{d^4p}{(2\pi)^4} \text{tr} \Delta(Q/2)\Delta(-Q/2),$$

$$\Gamma_{\pi} = \frac{2N_c}{g^2} - N_c \int_{p<\Lambda} \frac{d^4p}{(2\pi)^4} \text{tr} i\gamma_5 \Delta(Q/2)i\gamma_5\Delta(-Q/2).$$  \hspace{1cm} (11)
and the fermion propagator is given by $\Delta(Q/2) = (\hat{p} + \hat{Q}/2 + im)^{-1}$. The correspondence to the BRZ denotations [8] is following,

$$\bar{\Pi}_S = \frac{1}{g_s} - \Gamma_\sigma; \quad \bar{\Pi}_P = \frac{1}{g_s} - \Gamma_\pi;$$

\[
\frac{1}{g_s} = \frac{N_c \Lambda^2}{4 \pi^2 G_s} = \frac{2 N_c \Lambda^2}{g^2} = - \frac{<\bar{q}q>}{m} = i \frac{<\bar{\psi}\psi>E}{m}.
\] (12)

Both integrals in (11) are divergent. If we take into account the mass gap equation then they are logarithmically divergent only. Let us evaluate the integrand in (11),

\[
\text{tr} (1, i \gamma_5) \left( \left( \hat{p} + \frac{\hat{Q}}{2} \right) + i m \right)^{-1} \left( \left( \hat{p} - \frac{\hat{Q}}{2} \right) + i m \right)^{-1}
\]

\[
= 4 \left( (p_\mu + \frac{Q_\mu}{2})^2 + m^2 \right)^{-1} \left( p^2 - \frac{Q^2}{4} + m^2 \right) \left( (p_\mu - \frac{Q_\mu}{2})^2 + m^2 \right)^{-1}
\]

\[
= 2 \left( (p_\mu + \frac{Q_\mu}{2})^2 + m^2 \right)^{-1} + \left( (p_\mu - \frac{Q_\mu}{2})^2 + m^2 \right)^{-1}
\]

\[
- 2 \left( Q^2 + 2m^2 \right) \left( (p_\mu + \frac{Q_\mu}{2})^2 + m^2 \right)^{-1} \left( (p_\mu - \frac{Q_\mu}{2})^2 + m^2 \right)^{-1}
\] (13)

The sum of denominators is, $2p^2 + 2m^2 + (Q^2/2)$ and after its separation in the numerator of Eq.(13) one arrives to the required decomposition of loop integrals. Two first terms at zero-momentum represent the tadpole integrals in (9) and therefore precisely cancel the first terms in (11). This is a work of the mass-gap eq.(8). Thus, depending on the regularization we derive the inverse $\sigma$-model propagators,

$$\Gamma_\sigma = (4m^2 + Q^2)I(Q^2) + \xi Q^2 + O\left( \frac{1}{\Lambda^2} \right) \equiv (m_\sigma^2 + Q^2)I_\sigma(Q^2),$$

$$\Gamma_\pi = Q^2 I(Q^2) + \xi Q^2 + O\left( \frac{1}{\Lambda^2} \right) \equiv Q^2 I_\pi(Q^2),$$ (14)

where the parameter $\xi$ is a finite constant characteristic to the regularization, for instance, in the symmetric cutoff regularization (11) one has $\xi = N_c/32\pi^2$. Respectively the decay formfactor is provided by the last term in (13),

\[
I(Q^2) = 2N_c \int_{p<\Lambda} \frac{d^4p}{(2\pi)^4} \frac{1}{(p + \frac{1}{2}Q)^2 + m^2} \frac{1}{(p - \frac{1}{2}Q)^2 + m^2}.
\] (15)

It is regularization independent up to a redefinition of $\Lambda$ when neglecting the orders of $1/\Lambda^2$. For a convenience we display the analytic representation for $I(Q^2)$,

\[
I(Q^2) = \frac{N_c}{8\pi^2} \left( \ln \frac{\Lambda^2}{m^2} - 1 - \int_0^1 dx \ln \left( 1 + x(1 - x) \frac{Q^2}{m^2} \right) \right)
\]

\[
= \frac{N_c}{8\pi^2} \left( \ln \frac{\Lambda^2}{m^2} + 1 + \sqrt{1 + \frac{4m^2}{Q^2}} \ln \frac{\sqrt{1 + \frac{4m^2}{Q^2}} - 1}{\sqrt{1 + \frac{4m^2}{Q^2}} + 1} \right)
\] (16)
for euclidean momenta $Q^2 > 0$. If one defines the regularization where $\xi = 0$ (or an appropriate subtraction of $\xi$-term in the arbitrary regularization) then one arrives to the residual chiral symmetry and thereby to the generalized linear $\sigma$-model,

$$I_\sigma(Q^2) = I_\pi(Q^2) \equiv I(Q^2).$$

(17)

Then as a consequence of the NJL-mechanism of DSB we have precisely $m_\sigma = 2m$. The generating functional for scalar and pseudoscalar correlators can be derived after one integrates over $\sigma, \pi$ variables keeping in the effective action the quadratic terms only (the main large-$N_c$ contribution). The gaussian integral is calculated at the extremum,

$$\sigma_c(Q) = \frac{2N_c g^2}{I_\sigma} \Gamma^{-1}_\sigma S(Q) = \frac{2N_c}{g^2 I(Q) Q^2 + 4m^2} S(Q)$$

$$\pi_c(Q) = \frac{2N_c g^2}{I_\pi} \Gamma^{-1}_\pi P(Q) = \frac{2N_c}{g^2 I(Q) Q^2} P(Q)$$

(18)

and results in the generating functional, $W = -\ln Z(S, P)$,

$$W^{(2)} = \int \frac{d^4Q}{(2\pi)^4} \left[ \frac{1}{2} S(-Q) \left( \frac{<\bar{\psi}\psi>^2}{m^2} \Gamma^{-1}_\sigma - i \frac{<\bar{\psi}\psi>}{m} \right) S(Q) + \frac{1}{2} P(-Q) \left( \frac{<\bar{\psi}\psi>^2}{m^2} \Gamma^{-1}_\pi - i \frac{<\bar{\psi}\psi>}{m} \right) P(Q) - S(0)(2\pi^4)\delta^{(4)}(Q)i <\bar{\psi}\psi> \right].$$

(19)

The calculation of two-point correlators correspond to the second derivative of this action with subtraction of disconnected product of condensates (it corresponds to elimination of the last term $\sim S(0)$ in above Eq.).

$$\int d^4x \exp(iQx) \left[ <T(\bar{\psi}\psi(x)\bar{\psi}\psi(0))> - <\bar{\psi}\psi><\bar{\psi}\psi> \right]$$

$$= -\int d^4x \exp(iQx) \frac{\delta^2}{\delta S(x)\delta S(0)} \left( W^{(2)} + \int d^4y S(y)i <\bar{\psi}\psi> \right).$$

It is saturated by first terms in (19) quadratic in $S(x)$. One has the scalar-meson pole and another constant term which plays some role [8] when one reduces linear sigma model to the chiral one. Similarly we obtain the pseudoscalar correlator which contains also the pole and the constant term. One can compare them to the correlators (124)-(127) and (152) of the BRZ paper and find the one-to-one correspondence if to go back to the Minkowski space where $-i <\bar{\psi}\psi>_E \rightarrow <\bar{\psi}\psi>_M \equiv <\bar{q}q>$. The chiral symmetry restoration at high energies is ensured by (17).

4. Soft-momentum predictions

Let us now develop the soft-momentum expansion of above kinetic terms,

$$\Gamma_\sigma^0 = a_0^\sigma + Q^2 a_1^\sigma + \cdots, \quad \Gamma_\pi^0 = Q^2 a_1^\pi + Q^4 a_2^\pi + \cdots.$$ 

(20)
The coefficients $a^\sigma_0$, $a^\sigma_1$, $a^\pi_1$, and $a^\pi_2$ are related to the Taylor expansion of (14),

$$a^\sigma_0 = m_\sigma^2 I_\sigma(0), \quad a^\sigma_1 = I'_\sigma(0) + m_\sigma^2 I''_\sigma(0)$$
$$a^\pi_1 = I_\pi(0), \quad a^\pi_2 = I'_\pi(0)$$

(21)

We see that in the vicinity of $Q^2 \sim 0$ the would-be scalar-meson decay constant $a^\sigma_1$ and pion constant $a^\pi_1$ do not coincide. Moreover their difference represents the finite constant nearly independent on a regularization, $a^\sigma_1 - a^\pi_1 = 4m_\sigma^2 I''(0)$. We remark that the corresponding integral $I''(0)$ is convergent and in the limit $\Lambda \to \infty$ takes the value $I''(0) = -N_c/48\pi^2m^2$. On the other hand, when the residual chiral symmetry (17) is present (for $\xi = 0$) the latter one reveals itself in a series of $\sigma$-model identities between the coefficients of soft-momentum expansion,

$$\frac{a^\sigma_0}{a^\pi_1} = m^2_\sigma, \quad a^\pi_1 a^\sigma_0 = (a^\pi_1)^2 + a^\pi_2 a^\sigma_0, \ldots$$

(22)

We emphasize that the first relation is meaningful iff the next ones are valid. But only if $I_\sigma = I_\pi$ one can use the first relation in eq.(22) for the definition of the scalar meson mass. Otherwise it is misleading. Obviously, in the conventional NJL model the scalar meson mass is easily determined from the soft-momentum (derivative) expansion. In the next section we shall see that it is not the case for the Gauged NJL model.

5. Gauged NJL model and status of $\sigma$-model

The gauged NJL model is prepared conventionally with help of the long derivative in the Dirac operator, $i\partial_\mu \to D_\mu = i\partial_\mu + G_\mu$ and $G_\mu = g_sT^aG^a_\mu$ is a soft gluon field. The latter leads to the following modification of quark propagators in loop integrals, eq.(14),

$$\Delta(\pm Q^2) = \left(\pm \frac{Q^2}{2} + i\tilde{\partial} + \tilde{G} + im\right)^{-1}. $$

(23)

Respectively we examine the main large-$N_c$ contribution into the effective action given by averaging of functionals $\Gamma_{\sigma,\pi}(\Delta)$ over gluons,

$$\Gamma_{\sigma,\pi} \equiv \langle \Gamma_{\sigma,\pi}(\Delta) \rangle_G. $$

(24)

We assume that the main DSB mechanism is due to the strong four-fermion interaction and soft gluons bring the corrections accumulated into condensates. Thereby we imply that the derivative or large-mass expansion is reasonably good in calculating the gluon contributions.

Let us analyze the general structure of $\Gamma_{\sigma,\pi}$ which defines the inverse propagators for $\sigma$ and $\pi$-fields according to (10). For this purpose we first evaluate the integrand in (14),(24),

$$\text{Tr}(1, i\gamma_5) \left(\frac{Q^2}{2} + i\tilde{\partial} + \tilde{G} + im\right)^{-1} (1, i\gamma_5) \left(-\frac{Q^2}{2} + i\tilde{\partial} + \tilde{G} + im\right)^{-1} = \ldots$$
\[
\begin{align*}
&= \text{Tr} \left( \left( \frac{Q_\mu}{2} + i \partial_\mu + G_\mu \right)^2 + \tilde{F} + m^2 \right)^{-1} \\
&\quad \times \left( \left( i \partial_\mu + G_\mu \right)^2 + \tilde{F} - \frac{Q^2}{4} + m^2 + \frac{1}{2} [\tilde{q}, \tilde{D}] \right) \\
&\quad \times \left( \left( -\frac{Q_\mu}{2} + i \partial_\mu + G_\mu \right)^2 + \tilde{F} + m^2 \right)^{-1}
\end{align*}
\]

(25)

where \( \tilde{F} \equiv (1/4)[\gamma_\mu, \gamma_\nu] \cdot [D_\mu, D_\nu] \). The sum of denominators is,

\[
\Sigma \equiv 2(D_\mu)^2 + 2\tilde{F} + \frac{Q^2}{2} + 2m^2
\]

(26)

and after its separation in the second multiplier of product (25) one arrives to the following decomposition of loop integrals (compare with eq.(14)):

\[
\Gamma_\pi = (Q^2 + 4m^2) I_G(Q^2) + Q^2 \Xi(Q^2) + O\left(\frac{1}{\Lambda^2}\right),
\]

\[
\Gamma_\pi = Q^2 \left( I_G(Q^2) + \Xi(Q^2) \right) + O\left(\frac{1}{\Lambda^2}\right),
\]

(27)

where now the employed functions are defined as follows,

\[
I_G(Q^2) = \frac{1}{2} \langle tr < x \mid \left( \frac{Q_\mu}{2} + i \partial_\mu + G_\mu \right)^2 + \tilde{F} + m^2 \rangle^{-1} \\
\quad \times \left( -\frac{Q_\mu}{2} + i \partial_\mu + G_\mu \right)^2 + \tilde{F} + m^2 \rangle^{-1} \rangle_G; \\
\quad \times \left( (p_\mu - \frac{Q_\mu}{2} + i \partial_\mu + G_\mu)^2 + \tilde{F} + m^2 \right)^{-1} \\
\Xi(Q^2) = \xi - \frac{1}{2} \langle tr < x \mid \left( \frac{Q_\mu}{2} + i \partial_\mu + G_\mu \right)^2 + \tilde{F} + m^2 \rangle^{-1} \\
\quad \times \frac{1}{Q^2} [\tilde{q}, \tilde{D}] \left( -\frac{Q_\mu}{2} + i \partial_\mu + G_\mu \right)^2 + \tilde{F} + m^2 \rangle^{-1} \rangle_G; \\
\quad \times \frac{1}{Q^2} [\tilde{q}, \tilde{D}] \left( (p_\mu - \frac{Q_\mu}{2} + i \partial_\mu + G_\mu)^2 + \tilde{F} + m^2 \right)^{-1} \\
\quad \times \frac{1}{Q^2} [\tilde{p}, \tilde{\hat{D}}] \left( (p_\mu - \frac{Q_\mu}{2} + i \partial_\mu + G_\mu)^2 + \tilde{F} + m^2 \right)^{-1} \rangle_G.
\]

(28)

The integral for \( \Xi \) is not divergent. We have used the modified gap equation and the 1/\( \Lambda^2 \) expansion which results in the independence of parameter \( \xi \) on gluon condensates (\( \xi = N_c/32\pi^2 \) in the sharp cutoff regularization). The representation for \( I_G \) and \( \Xi \) in Eqs.(28) contains the trace over spin and color indices. In the second, integral representation of above functions the derivatives act on gluon fields when one
expands, for example, those functionals in the perturbation series. In other words, we have to calculate the mixed matrix element \( \langle x | \cdots | k = 0 \rangle \). Evidently the relations of the linear \( \sigma \)-model (17) would survive if it were true that,

\[
(Q^2 + 4m^2) I_G(Q^2) + Q^2 \Xi(Q^2) \overset{?}{=} (Q^2 + m^2) \left( I_G(Q^2) + \Xi(Q^2) \right)
\]

or \( \Xi(Q^2) \overset{?}{=} \frac{4m^2 - m^2_{\sigma}}{m^2_{\sigma}} I_G(Q^2) \).

By comparing of integrals in (28) one can convince oneself that for arbitrary momenta these functions do not coincide, the equation (29) is not valid and therefore the linear \( \sigma \)-model relations (17), (22) are not fulfilled. Thus we cannot effectively use the soft-momentum expansion for the calculation of scalar meson mass and its decay constant.

Now let us consider the soft-momentum and large-mass expansion which leads to the expansion in terms of gluon condensates. We stress that the coefficients of large-mass expansion containing gluon condensates are free of ultraviolet divergences and thereby they do not depend on regularization if one neglects terms \( O(1/\Lambda^2) \). The new mass-gap equation takes the form,

\[
\Lambda^2 - \frac{8\pi^2}{g^2} - m^2 \ln \frac{\Lambda^2}{m^2} + \frac{<G_{\mu\nu}^2>}{12N_c m^2} = 0.
\]

First terms of above expansion for \( \Gamma_\sigma, \Gamma_\pi \) read (see eq.(21)) in the momentum-cutoff regularization after applying the mass-gap equation:

\[
a^\sigma_0 = \frac{N_c}{2\pi^2} \left( m^2 \left( \ln \frac{\Lambda^2}{m^2} - 1 \right) + \frac{<G_{\mu\nu}^2>}{12N_c m^2} \right);
\]

\[
a^\sigma_1 = \frac{N_c}{8\pi^2} \left( \ln \frac{\Lambda^2}{m^2} - \frac{5}{3} + \tilde{\xi} - \frac{7}{120N_c m^4} <G_{\mu\nu}^2> \right);
\]

\[
a^\pi_1 = \frac{N_c}{8\pi^2} \left( \ln \frac{\Lambda^2}{m^2} - 1 + \tilde{\xi} + \frac{<G_{\mu\nu}^2>}{24N_c m^4} \right);
\]

\[
a^\pi_2 = \frac{N_c}{48\pi^2 m^2} \left( -1 - \frac{<G_{\mu\nu}^2>}{20N_c m^4} \right).
\]

where \( <G_{\mu\nu}^2> \equiv q_{\mu\nu}^2 < (G_{\mu\nu}^a)^2 >,\ \tilde{\xi} = 1/4 = \xi \cdot 8\pi^2/N_c \). One can verify explicitly that the linear \( \sigma \)-model relations (21), (22) are not fulfilled even after the renormalization to the chirally symmetric point, \( a^\sigma_1, a^\pi_1 \rightarrow a^\sigma_1 - \xi;\ \xi = N_c/32\pi^2 \). For the completeness, we display also the soft-momentum expansion for functions \( I_G(Q), \Xi(Q) \),

\[
I_G(Q) = \frac{N_c}{8\pi^2} \left( \ln \frac{\Lambda^2}{m^2} - 1 + \tilde{\xi} + \frac{<G_{\mu\nu}^2>}{24N_c m^4} - \frac{Q^2}{m^2} \left( \frac{1}{6} + \frac{<G_{\mu\nu}^2>}{40N_c m^4} \right) \right);
\]

\[
\Xi(Q) = -\frac{1}{8\pi^2} \frac{<G_{\mu\nu}^2>}{24m^4} + \frac{Q^2}{8\pi^2 m^2} \frac{<G_{\mu\nu}^2>}{60m^4}.
\]
Thus one can see that the relations (29) cannot be fulfilled for any reasonable choice of $\tilde{\xi}$ (which gives $I(0) \neq 0$).

Let us refer our analysis to the identities (199)-(201) in the paper [8] for one-fermion loop (OFL) polarization operators, scalar, $\Pi_S$, pseudoscalar, $\Pi_P$, and longitudinal axial-vector one, $\Pi_A^{(0)}$. They are valid in the non-gauged NJL model. The PCAC relations (199), (200) connect the OFL pseudoscalar correlator, $\Pi_P$, with the OFL longitudinal axial-vector one, $\Pi_A^{(0)}$. In the GNJL model only these two relations survive. Meantime, Eq.(201) is broken due to gluon corrections. Indeed, as a consequence of (27), (12) one has,

\[
\bar{\Pi}_S = -\frac{\langle \bar{q}q \rangle}{m} - (Q^2 + 4m^2)I_G(Q^2) - Q^2\Xi(Q^2) + O\left(\frac{1}{\Lambda^2}\right),
\]

\[
\bar{\Pi}_P = -\frac{\langle \bar{q}q \rangle}{m} - Q^2(I_G(Q^2) + \Xi(Q^2)) + O\left(\frac{1}{\Lambda^2}\right). \tag{33}
\]

The axial Ward identity (for $\xi = 0$) allows to find $\Pi_A^{(0)}(Q)$,

\[
4m^2\bar{\Pi}_P(Q) = -4m\langle \bar{q}q \rangle + Q^4\Pi_A^{(0)}(Q);
\]

\[
\Pi_A^{(0)}(Q) = -\frac{4m^2}{Q^2}(I_G(Q^2) + \Xi(Q^2)). \tag{34}
\]

Therefore the discrepancy in eq.(201) of [8] is solely due to non-zero $\Xi$,

\[
\bar{\Pi}_P(Q) + Q^2\Pi_A^{(0)}(Q) - \bar{\Pi}_S(Q) = -4m^2\Xi(Q^2) \neq 0. \tag{35}
\]

In the soft-momentum expansion, ($\xi = 0$), one obtains,

\[
-\Pi_A^{(0)}(Q) = \frac{N_c}{2\pi^2} \left[ \frac{1}{Q^2} \left( m^2 \left( \ln \frac{\Lambda^2}{m^2} - 1 \right) + \frac{\langle G_{\mu\nu}^2 \rangle}{24N_cm^2} \right) - \left( \frac{1}{6} + \frac{\langle G_{\mu\nu}^2 \rangle}{120N_cm^4} \right) \right];
\]

\[
4m^2\bar{\Pi}_P(Q) = -4m\langle \bar{q}q \rangle + Q^4\Pi_A^{(0)}(Q);
\]

\[
\Pi_P(Q) + Q^2\Pi_A^{(0)}(Q) - \Pi_S(Q) = \frac{\langle G_{\mu\nu}^2 \rangle}{48\pi^2m^2} + O\left(\frac{Q^2}{m^2}\right) \neq 0. \tag{36}
\]

Thus these identities prove that the results of [8] should be revised.

6. Determination of meson masses

The invalidity of the $\sigma$-model relations is of course a bad news since we cannot calculate reliably two-point correlators in the GNJL model for finite momenta $\sim 1 GeV$ but would like to obtain the information about spectra from the soft-momentum expansion. However, it is not a property of the Chiral Symmetry Breaking to provide the linear $\sigma$-model relations. The QCD-motivated pattern of the CSB [7] includes the radial excitations of pions and scalars with unequal masses and coupling constants that is an extra source of the CSB. Two interpolation schemes of mass spectrum determination can be developed.
1. Following the planar limit of the QCD, eq.(1) one can make the two-resonance ansatz for scalar and pseudoscalar correlators in the GNJL model, eq.(19),

\[
\Pi^S(Q) = \frac{<\bar{q}q>}{m^2} \Gamma_\sigma^{-1} + \frac{<\bar{q}q>}{m} = \frac{Z_0^\sigma}{Q^2 + M_\sigma^2} + \frac{Z_1^\sigma}{Q^2 + M_{\sigma'}^2} + C^\sigma;
\]

\[
\Pi^P(Q) = \frac{<\bar{q}q>}{m^2} \Gamma_\pi^{-1} + \frac{<\bar{q}q>}{m} = \frac{Z_0^\pi}{Q^2} + \frac{Z_1^\pi}{Q^2 + M_{\pi'}^2} + C^\pi.
\]

(37)

We remark that for the NJL-type models the constants \(C^S, P\) = 0. From the requirement of asymptotic CS restoration (3) it follows that,

\[
C^\sigma = C^\pi \equiv C = \frac{<\bar{q}q>}{m} < 0; \]

\[
Z_0^\sigma + Z_1^\sigma = Z_0^\pi + Z_1^\pi; \]

\[
Z_0^\sigma M_\sigma^2 + Z_1^\sigma M_{\sigma'}^2 = Z_1^\pi M_{\pi'}^2.
\]

(38) (39) (40)

The first two relations can be fulfilled in the conventional NJL model which corresponds to the one-resonance ansatz, \(Z_1^\pi = 0\), whereas the last one can be saturated only in a two-resonance model including at least \(\pi'\)-meson.

The soft-momentum limit of the correlators is connected to the structural constants of the chiral lagrangian [8, 13],

\[
\frac{Z_0^\sigma}{M_\sigma^2} + \frac{Z_1^\sigma}{M_{\sigma'}^2} + C = 8B_0^2(2L_8 + H_2);
\]

\[
\frac{Z_1^\pi}{M_{\pi'}^2} + C = 8B_0^2(-2L_8 + H_2).
\]

(41)

When eliminating the unobservable constant \(H_2\) one comes to the large-\(N_c\) sum rule for meson parameters based on the phenomenological constant \(L_8\) [13],

\[
\frac{Z_0^\sigma}{M_\sigma^2} + \frac{Z_1^\sigma}{M_{\sigma'}^2} - \frac{Z_1^\pi}{M_{\pi'}^2} = 32B_0^2L_8.
\]

(42)

Let us interpolate now the soft-momentum expansion (20) for \(\Gamma_{\sigma,\pi}\) of the GNJL model, eq.(31) by means of the two-resonance ansatz (37),

\[
a_0^\sigma = C^2 \frac{M_\sigma^2 M_{\sigma'}^2}{Z_0^\sigma M_\sigma^2 + Z_1^\sigma M_{\sigma'}^2}; \quad a_1^\sigma = C^2 \frac{Z_0^\sigma M_\sigma^4 + Z_1^\sigma M_{\sigma'}^4}{Z_0^\sigma M_\sigma^2 + Z_1^\sigma M_{\sigma'}^2}; \]

\[
a_1^\pi = C^2 \frac{1}{Z_0^\pi}; \quad a_2^\pi = -C^2 \frac{Z_1^\pi}{(Z_0^\pi)^2 M_{\pi'}^2}.
\]

(43)

As a consequence, seven meson parameters \(M_{\sigma,\sigma',\pi'}\); \(Z_{0,1,\sigma,\pi}^{\sigma,\pi}\) are determined by entries of the GNJL models, \(\Lambda, m, <G_{\mu\nu}^2>\). The preliminary estimates show that the gluon condensate of the GNJL model should be taken less than the similar nonperturbative parameter in the high-energy OPE expansion [11,12]. It is not surprising since partially the nonperturbative gluon effects are taken into account in the four-quark interaction.
2. In the recent literature there appeared few conjectures \[4,8,10\] of the validity of \(\sigma\)-model relations like \(I_\pi = I_\sigma\) (see eq. (14)) and \(m_\sigma = 2m_{dyn}\) in the case of a NJL model coupled with external vector fields. In particular, in the review [10] the influence of external electric or magnetic fields with constant strength and direction was considered for the case of two-point correlators. These configurations seem to form the same background condensate as gluons at the lowest order in fields and external momenta. However it can be checked that the restoring of the Lorentz symmetry of the vector-field condensate (by averaging over Lorentz rotations) mixes the longitudinal and transversal (to the external momentum) components. It results in the breaking of linear \(\sigma\)-model relations (14).

Nevertheless one can develop the soft-momentum scheme for the determination of scalar meson masses based on the approximation,

\[
\Gamma_\sigma \simeq q^2 (I_G(0) + \Xi(0)) + 4m^2 I_G(0); \quad \Gamma_\pi \simeq q^2 (I_G(0) + \Xi(0)) ;
\]

\[
4m^2_\sigma \simeq \frac{4m^2 I_G(0)}{I_G(0) + \Xi(0)} = \frac{a^\sigma_0}{a^\pi_1},
\]

that should be compared with adiabatic \(\sigma\)-model relations (22). The Nambu relation is not valid in this scheme, \(m^2_\sigma > 4m^2\).

If to append the two-resonance ansatz to the adiabatic approximation one arrives to the remarkable condition for scalar meson masses,

\[
m^2_\sigma = m^2_{\sigma_0} \frac{Z^\sigma_0 - Z^\sigma_1}{Z^\pi_1},
\]

which guaranties the validity of above approximation. As \(Z^\sigma_0 > Z^\sigma_1\) one expects that in the chiral limit the pion and \(\sigma\)-meson decay constants are different, \(F_\pi < F_\sigma\).

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