FINITENESS THEOREMS FOR COMPLEMENTS OF LARGE DIVISORS

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Abstract. We prove finiteness results on integral points on complements of large divisors in projective varieties over finitely generated fields of characteristic zero. To do so, we prove a function field analogue of arithmetic finiteness results of Corvaja-Zannier and Levin using Wang’s function field Subspace Theorem. We then use a method of Evertse-Győry for concluding finiteness of integral points over finitely generated fields from known finiteness results over number fields.

1. Introduction

In [25, 26], Lang suggests that Diophantine statements involving rational points over number fields should continue to hold over arbitrary finitely generated fields over \( \mathbb{Q} \); see his question on [25, p. 202] for a precise question of his. For instance, the work of Siegel-Mahler-Lang (see [29, 31, 32]) in the classical setting of rings of \((S-)\)integers was extended by Lang to show that the unit equation

\[ u + v = 1, \quad u, v \in A^* \]

has only finitely many solutions when \( A \) is any \( \mathbb{Z} \)-finitely generated integral domain of characteristic zero.

Our main theorem extends finiteness results of Autissier, Corvaja-Zannier and Levin from number fields to finitely generated fields in a similar vein as Lang did for the unit equation.

Before stating our precise results, we introduce some terminology. By a variety over a field \( K \) we mean a finite type integral separated scheme over \( K \). Let \( k \) be an algebraically closed field of characteristic zero. Following [16, Definition 7.1], a variety \( X \) over \( k \) endowed with a closed subset \( \Delta \) is said to be arithmetically hyperbolic modulo \( \Delta \) over \( k \) if there is a \( \mathbb{Z} \)-finitely generated subring \( A \subset k \) and a finite type separated \( A \)-scheme \( X \) with \( X_k \cong X \) over \( k \) (i.e., a model for \( X \) over \( A \) [16, Definition 3.1]) such that, for all \( \mathbb{Z} \)-finitely generated subrings \( A' \subset k \) containing \( A \), the set \( X(A') \setminus \Delta \) of \( A' \)-points on \( X \) is finite. Thus, roughly speaking, a variety \( X \) is arithmetically hyperbolic modulo \( \Delta \) if it has only finitely many integral points outside \( \Delta \). We say that \( X \) is pseudo-arithmetically hyperbolic over \( k \) if there exists a proper closed subset \( \Delta \subset X \) such that \( X \) is arithmetically hyperbolic modulo \( \Delta \), and we say that \( X \) is arithmetically hyperbolic over \( k \) if it is arithmetically hyperbolic modulo the empty subset. The notion of arithmetic hyperbolicity is independent of the chosen model in the sense that, if \( X \) is arithmetically hyperbolic over \( k \), then, for every \( \mathbb{Z} \)-finitely generated subring \( A \subset k \) and every model \( \mathcal{X} \) for \( X \) over \( A \), the set \( \mathcal{X}(A) \) is finite; see [23, Lemma 4.8]. These notions extend Lang’s notions [27] of the Mordell/Siegel property for projective/affine

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varieties; see [2, 3, 4, 7, 10, 11, 12, 14, 15, 18, 19, 20, 24, 28, 30, 34, 35, 36, 37] for examples of arithmetically hyperbolic varieties.

Following [17], if \( X \) is a variety over an algebraically closed field \( k \) of characteristic zero such that \( X_L \) is arithmetically hyperbolic over \( L \) for all algebraically closed field extensions \( L \supseteq k \), then we say that \( X \) is absolutely arithmetically hyperbolic. More generally, if \( \Delta \subseteq X \) is a closed subset, then \( X \) is absolutely arithmetically hyperbolic modulo \( \Delta \) if, for every algebraically closed field extension \( L \supseteq k \), the variety \( X_L \) is arithmetically hyperbolic modulo \( \Delta_L \).

Motivated by Lang’s aforementioned philosophy on rational points over finitely generated fields, we are interested in the persistence of arithmetic hyperbolicity along field extensions. For example, Siegel-Mahler-Lang showed that \( \mathbb{A}^1 \setminus \{0, 1\} \) is arithmetically hyperbolic over \( \mathbb{Q} \), and Lang showed that this persists over all algebraically closed fields of characteristic zero, thereby proving the so-called “Persistence Conjecture” in this case:

**Conjecture 1.1** (Persistence Conjecture). Let \( k \) be an algebraically closed field of characteristic zero, and let \( X \) be a variety over \( k \). If \( X \) is arithmetically hyperbolic over \( k \), then \( X \) is absolutely arithmetically hyperbolic.

For projective varieties, the Persistence Conjecture is a consequence of Lang’s conjectures as formulated in [16, Section 12]. Indeed, if \( X \) is a projective arithmetically hyperbolic variety over \( k \), then every subvariety of \( X \) is of general type by this conjecture, which one can show (using, for example, the results in [22]) implies that every subvariety of \( X_L \) is of general type, so that (again by Lang-Vojta’s conjecture), the variety \( X_L \) is arithmetically hyperbolic over \( L \).

This conjecture was first systematically studied in [14, Conjecture 1.5] (see also [4, Conjecture 1.20] and [16, Conjecture 17.5]). It was shown to hold for algebraically hyperbolic projective varieties in [14, Section 4], and then also for varieties which admit a quasi-finite morphism to a semi-abelian variety [4, Theorem 7.4]. It was also shown to hold for varieties which admit a quasi-finite period map [18], hyperbolically embeddable smooth affine varieties [17] and certain moduli spaces of polarized varieties [21].

In this paper, we extract new results on the Persistence Conjecture from the work of Evertse-Györy (discussed further below).

**Main Results.** To state our first finiteness result, we consider a result of Levin for varieties over number fields [28, Theorem 6.1A(b), Theorem 6.2A(d)], [13, Theorem 1.4] (after work of Corvaja-Zannier [5, 6, 8]; see also the work of Autissier [2, 3]) and extend the result to finitely generated fields.

**Theorem 1.2** (Main Result I). Let \( m \) be a positive integer, let \( X \) be a smooth projective variety over \( \overline{\mathbb{Q}} \), and let \( D = \sum_{i=1}^{r} D_i \) be a sum of \( r \) ample effective divisors on \( X \) such that at most of \( m \) of the \( D_i \) meet in any point, where \( r > 2m \dim(X) \). Then the affine variety \( X \setminus D \) is absolutely arithmetically hyperbolic.

In [17], Javanpeykar-Levin prove Theorem 1.2 by 1) verifying that \( X \setminus D \) is hyperbolically embeddable (over \( \mathbb{C} \)) and 2) verifying that the Persistence Conjecture holds for hyperbolically embeddable varieties. Combining 1) and 2) with Levin’s theorem then completes the proof.

Our proof of Theorem 1.2 proceeds in a different fashion. The main step of Levin’s argument is to prove the pseudo-arithmetic hyperbolicity of complements of large divisors [28, Theorem 8.3A]). In the first step, we prove a function field version of that result.
Definition 1.3 (Large divisor). Let $X$ be a smooth projective variety over a field $k$. An effective divisor $D$ on $X$ is very large if for all $P \in D(k)$, there is a basis $B$ for $V(D) = H^0(X, \mathcal{O}_D)$ such that $\text{ord}_E \left( \prod_{f \in B} f \right) > 0$ for all irreducible components $E$ of $D$ with $P \in E(k)$. An effective divisor $D$ is large if some positive integral linear combination of its irreducible components is very large.

Theorem 1.4 (Main Result II, Corvaja-Zannier-Levin for countable function fields). Let $X$ be a projective variety over $\mathbb{Q}$ and let $D$ be a very large divisor on $X$. Then there is a proper closed subscheme $Z \subseteq X$ with the following property.

For every countable algebraically closed field $k$ of characteristic 0, for every algebraic function field $F$ over $k$ (as defined in Subsection 2.2), for every finite set of places $T \subseteq F$ and for every set of $(D, \mathcal{O}_{F,T})$-integral points $R \subseteq (X \setminus D)(F)$ (as defined in Subsection 3.2), there is subset $R' \subseteq R$ with the same Zariski closure $\overline{R} = \overline{R'}$ such that $h_{\text{aff}}^F \circ \phi_D$ is bounded on $R' \setminus Z$.

For the proof, we appeal to the function field version of the Subspace Theorem [1, 38] and “re-do” some of the arguments in [28].

In the second step, we apply a method by Evertse and Győry [9, Chapter 8]. They studied the unit equation over finitely generated domains. Given

1. an effective bound for the number of solutions over $\mathcal{O}_{K,S}$ for any $K$ and any finite set of places $S$, and
2. an effective height bound for the solutions over function fields,

they showed that there is an effective bound for the number of solutions over finitely generated domains.

To explain their method, we consider the following example. Let $A = \mathbb{Z}[t, 1/(1 + t^2)]$, and let $R \subseteq A$ be a subset. Then $A$ is contained in the function field $F = \mathbb{Q}(t)$. For $p, q \in \mathbb{Z}[t]$, $q \neq 0$, with $\gcd(p, q) = 1$ and $p/q \in R$ the height is given as

$$H_{\text{aff}}^F(p/q) = \max\{\deg(p), \deg(q)\}.$$ 

So bounding the height bounds the degree of possible denominators and numerators.

Furthermore, evaluating at $u \in \mathbb{Z}$ defines a specialization map $\psi_u: A \to \mathbb{Z}[1/(1 + u^2)]$. If $\psi_u(R)$ is finite for sufficiently many $u \in \mathbb{Z}$, then the set $R$ is finite since each element of $R$ has to interpolate these points. By extracting their key arguments, we can show that their method works in a more general setting. This culminates in the following result.

Theorem 1.5 (Main Result III, Evertse-Győry method on affine varieties). Let $X \subseteq \mathbb{A}^n_{\mathbb{Q}}$ be an affine variety over $\mathbb{Q}$, let $K$ be a number field, let $S$ be a finite set of places of $K$, let $X' \subseteq \mathbb{A}^n_{\mathcal{O}_K,S}$ be a model for $X$ over $\mathcal{O}_{K,S}$. Let $\mathcal{A} \supseteq \mathcal{O}_{K,S}$ be a $\mathbb{Z}$-finitely generated integral domain with quotient field $L$, and let $R \subseteq X(\mathcal{A})$ be a subset. Assume that

1. $X$ is arithmetically hyperbolic over $\mathbb{Q}$, and
2. for each finite set of function fields $\{F_1, \ldots, F_t\}$ with $F_i \supseteq L$, there is a subset $R' \subseteq R$ with the same Zariski closure $\overline{R} = \overline{R'}$ such that $h_{\text{aff}}^{F_i}$ is bounded on $R' \subseteq X_{F_j}(F_j)$.

Then $R$ is finite.

Consequently, we prove the following result on the persistence conjecture.
Corollary 1.6 (Main Result IV, Evidence for the persistence conjecture). Let $X \subseteq \mathbb{A}_\mathbb{Q}^n$ be an arithmetically hyperbolic affine variety over $\overline{\mathbb{Q}}$, let $K$ be a number field, let $S$ be a finite set of places of $K$, let $X \subseteq \mathbb{A}_{{O_K,S}}^n$ be a model for $X$ over $O_{K,S}$. Assume that for every $\mathbb{Z}$-finitely generated integral domain $A \supseteq O_{K,S}$ with quotient field $L$ and every finite set of function fields $\{F_1, \ldots, F_i\}$ with $F_i \supseteq L$, there is a subset $R' \subseteq X(A)$ with $R' = \overline{X(A)}$ such that $h_{F_i}^{aff}$ is bounded on $R' \subseteq X_{F_i}(F_j)$. Then $X$ is absolutely arithmetically hyperbolic.

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2. Height

Following [33, Part B], in this section, we introduce the notion of height on projective varieties over number fields and function fields.

2.1. Height on projective space over a number field. Let $K$ be a number field. The set of places $M_K$ is the union of the set of finite places $M_K^0$ and the set of infinite places $M_K^\infty$. The set $M_K^0$ is the set of all non-zero prime ideals $\mathfrak{p} \subseteq O_K$, and the set $M_K^\infty$ is the set of all real places, i.e., embeddings $\sigma: K \to \mathbb{R}$ and complex places, i.e., pairs of complex conjugated embeddings $\tau, \bar{\tau}: K \to \mathbb{C}$. Each place $p \in M_K$ comes with an associated absolute value $|| \cdot ||_p: K \to \mathbb{R}$ given by

$$||x||_p = (N_{K/\mathbb{Q}}(p))^{-\text{ord}_p(x)} \quad \text{if} \; p = p \in M_K^0,$$

$$||x||_p = |\sigma(x)| \quad \text{if} \; p = \sigma \text{ is real, and}$$

$$||x||_p = |\tau(x)|^2 \quad \text{if} \; p = (\tau, \bar{\tau}) \text{ is complex}$$

for $x \in K$. For a point $Q = (x_0: \ldots: x_n) \in \mathbb{P}^n(K)$, we define the height

$$H_K(Q) = \prod_{p \in M_K} \max_i \{||x_i||_p\}.$$ 

We note that the height function over number fields has the following useful property; for proof see [33, Theorem B.2.3].

Theorem 2.1 (Northcott Property). Let $K$ be a number field. If $H_K$ is bounded on a subset $R \subseteq \mathbb{P}^n(K)$, then $R$ is finite.

2.2. Height on projective space over a function field. Let $k$ be an algebraically closed field of characteristic zero. An algebraic function field over $k$ is a field extension $F \supseteq k$ of transcendence degree 1. Equivalently, $F = K(C)$ is the function field of a smooth projective curve $C$ over $k$. The set of places on $F$ is the set of all closed points $M_F = C(k)$. Note that every algebraic function field over $k$ is a finite field extension of $k(t)$ and vice versa. For $p \in M_F$, the absolute value associated to $p$ is the function $|| \cdot ||_p: F \to \mathbb{R}$ given by

$$||x||_p = e^{-\text{ord}_p(x)}$$

for $x \in F$. For $Q = (f_0: \ldots: f_n) \in \mathbb{P}^n(F)$, we define the height

$$H_F(Q) = \prod_{p \in M_F} \max_i \{||f_i||_p\}.$$
2.3. Weil’s height machine. Let \( L = \mathbb{Q} \) or \( L = k(t) \). For any finite field extension \( K \supseteq L \), we defined the height function \( H_K \) above. The absolute height is the function \( H : \mathbb{P}^n(\mathbb{T}) \rightarrow \mathbb{R} \), that maps a point \( Q = (x_0 : \ldots : x_n) \in \mathbb{P}^n(\mathbb{T}) \) to \( H_K(Q)^{1/[K:L]} \), where \( K \supseteq L \) is a finite field extension such that \( x_i \in K \) for all \( i \).

Consider the embedding \( \iota : \mathbb{A}^n \rightarrow \mathbb{P}^n : (x_1, \ldots, x_n) \rightarrow (1 : x_1 : \ldots : x_n) \). For each height function, there is an associated affine height function \( h^\text{aff}_K := H_K \circ \iota, H^\text{aff} := H \circ \iota \) defined on points in affine space. Furthermore, for each height function, there is an associated logarithmic height function \( h_K := \log \circ H_K, h := \log \circ H, h^\text{aff}_K := \log \circ H^\text{aff}_K, h^\text{aff} := \log \circ H^\text{aff} \).

As explained in \[33, \text{Theorem B.3.2, Remark B.3.2.1, Theorem B.10.4}\], there is the following way of defining height functions on projective varieties. 

**Theorem 2.2** (Weil’s height machine). For each projective variety \( X \) over \( \mathbb{T} \) and each Cartier divisor \( D \) on \( V \), there is an associated function 
\[
h_{X,D} : X(\mathbb{T}) \rightarrow \mathbb{R}
\]
such that the following properties are satisfied (\( O(1) \) denotes some bounded function on \( X(\mathbb{T}) \)).

1. Let \( H \subseteq \mathbb{P}^n \) be a hyperplane, and let \( h \) be the absolute logarithmic height on projective space. Then \( h^\text{aff}_{H,M} = h + O(1) \).
2. Let \( \phi : X \rightarrow Y \) be a morphism of projective varieties over \( \mathbb{T} \), and let \( D \in \text{CaDiv}(Y) \). Then \( h_{X,\phi^*D} = h_{Y,D} \circ \phi + O(1) \).
3. Let \( D, E \in \text{CaDiv}(X) \). Then \( h_{X,D+E} = h_{X,D} + h_{X,E} + O(1) \).
4. Let \( D, E \in \text{CaDiv}(X) \) with \( D \sim E \) linearly equivalent. Then \( h_{X,D} = h_{X,E} + O(1) \).
5. If \( D \in \text{CaDiv}(X) \) is base point free, then \( h_{X,D} = h \circ \phi_D + O(1) \).

3. CORVAJA-ZANNIER FOR COUNTABLE FUNCTION FIELDS

In this section, we will prove Theorem \[1.4\] Let \( k \) be a countable algebraically closed field of characteristic 0. Note that \( \overline{\mathbb{Q}} \subseteq k \). Let \( F \) be an algebraic function field over \( k \) as defined in Subsection \[2.2\]. Let \( X \) be a projective variety over \( \overline{\mathbb{Q}} \) and let \( D \) be a very large divisor on \( X \). We choose a basis \( \phi_0, \ldots, \phi_n \) of \( V(D) := H^0(X, \mathcal{O}_X(D)) \), where \( n = \dim_{\overline{\mathbb{Q}}} H^0(X, \mathcal{O}_D) - 1 \). Let \( \phi_D : X \rightarrow \mathbb{P}^n_{\overline{\mathbb{Q}}} \) denote the associated rational map.

3.1. Construction of the exceptional locus. We will start by constructing a proper closed subscheme \( Z \subseteq X \) depending only on \( D \) that "contains almost all \( D \)-integral points".

Since \( D \) is very large, there is a finite index set \( J \) and, for each \( j \in J \), there is a basis \( B_j = \{\psi_{j,0}, \ldots, \psi_{j,n}\} \) of \( V(D) \) satisfying the following. Given any point \( P \in D(\overline{F}) \), there is an index \( j \in J \) such that, for all irreducible components \( E \) of \( D \) that contain \( P \), we have

\[
\text{ord}_E \left( \prod_{i \in B_j} \psi_i \right) > 0.
\]

We can take for example \( J = D(\overline{F}) / \sim \), where \( P \sim Q \) if \( P \) and \( Q \) are contained in all the same irreducible components of \( D \).

For each \( j \in J \) and \( i \in \{0, \ldots, n\} \), there is a linear form \( \Lambda_{j,i} \in \overline{\mathbb{Q}}[x_0, \ldots, x_n]_1 \) such that \( \psi_{j,i} = \Lambda_{j,i}(\phi_0, \ldots, \phi_n) \). Let \( H_{j,i} \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}} \) denote the hyperplane determined by \( \Lambda_{j,i} = 0 \) and let \( H_i \subseteq \mathbb{P}^n_{\overline{\mathbb{Q}}} \) denote the hyperplane determined by \( x_i = 0 \). Let \( \mathcal{H} = \{H_{j,i}\}_{j,i} \cup \{H_i\}_i \).
We are going to apply Wang’s version of Schmidt’s subspace theorem (see [38, p. 821] or [1]) on the collection \( \mathcal{H} \). For its statement, we need the following terminology. Let \( H \subseteq \mathbb{P}^n_F \) be a hyperplane given by a linear form \( \Lambda \) and let \( v_p = \text{ord}_p \) denote valuation associated to some place \( p \in M_F \). Then the Weil function associated to \( (H, p) \) is the map
\[
\lambda_{H,p} : \mathbb{P}^n(F) \to \mathbb{R}
\]
where for a point \( P \in \mathbb{P}^n(F) \) with coordinates \((x_0, \ldots, x_n) \in F^{n+1} \setminus \{0\} \), we set
\[
\lambda_{H,p}(P) = v_p(\Lambda(x_0, \ldots, x_n)) - \min_{0 \leq i \leq n}\{v_p(x_i)\}.
\]

**Theorem 3.1** (Wang’s subspace theorem for function fields). Let \( F \) be a function field and let \( \mathcal{H} \) be a finite set of hyperplanes in \( \mathbb{P}^n_F \). Then there is a finite union of proper linear subspaces \( Y \subseteq \mathbb{P}^n_F \) that may be constructed from elements of \( \mathcal{H} \) using only operations \(<, \cdot, \cdot >\) and \( \cap \), satisfying the following.

For all finite sets of places \( T \subseteq M_F \) and all \( \epsilon > 0 \), there are constants \( C, C' \in \mathbb{R} \) such that for any \( P \in \mathbb{P}^n(F) \), at least one of the following statements holds.

1. The point \( P \) lies in \( Y \).
2. The height of \( P \) satisfies the inequality
   \[ h_F(P) \leq C. \]
3. The height of \( P \) satisfies the inequality
   \[ \sum_{p \in T} \max_{H \in I} \sum_{p \in T} \lambda_{H,p}(P) \leq (n + 1 + \epsilon)h_F(P) + C', \]
   where the maximum is taken over all subsets \( I = \{H_1, \ldots, H_m\} \subseteq \mathcal{H} \) such that the linear forms defining the \( H_i \) are linearly independent.

Let \( Y \subseteq \mathbb{P}^n \) be the union of linear subspaces yielded by Theorem 3.1 applied to the collection \( \mathcal{H} \) above. Let \( Z \subseteq X \) be the union of the Zariski closure of \( \phi_D^{-1}(Y) \) and the locus of indeterminacy of \( \phi_D \). Note that considering the way it was constructed, we see \( Z \) is again a scheme over \( \mathbb{Q} \).

### 3.2. Binding or bounding \( D \)-integral points.

For a finite subset of places \( T \subseteq M_F \), we define
\[
O_{F,T} = \{x \in F \mid v_p(x) \geq 0 \text{ for all } p \in M_F \setminus T\}.
\]
Note that the subring \( O_{F,T} \subseteq F \) is integrally closed. A subset \( R \subseteq (X \setminus D)(F) \) is called a set of \( (D, O_{F,T}) \)-integral points if there is a model \( W \) for \( W = X \setminus D \) over \( O_{F,T} \) such that \( R \subseteq W(O_{F,T}) \subseteq (X \setminus D)(F) \). We claim the following.

**Lemma 3.2.** Let \( R \subseteq (X \setminus D)(F) \) be a set of \( (D, O_{F,T}) \)-integral points. Then \( R \) contains a subset \( R' \subseteq R \) with the same Zariski closure \( \overline{R} = \overline{R'} \) such that \( h \circ \phi_D \) is bounded on \( R' \setminus Z \), where \( Z \) is the exceptional locus constructed in Subsection 3.7.

In the remainder of this section, we will prove Lemma 3.2. The set \( R \) is a finite union of subsets that have an irreducible Zariski closure. We can consider each one of these subsets individually and therefore assume that the Zariski closure \( \overline{R} \) is irreducible. For each place \( p \in M_F \), there is an absolute value \( ||.||_p = \exp \circ (-v_p(\cdot)) \) on \( F \). We fix some embedding \( X_F \subseteq \mathbb{P}^m_F \). The absolute value \( ||.||_p \) defines a topology on projective space \( \mathbb{P}^m_F(F) \) and so on \( X(F) \). By \( F_p \) we denote the completion of \( F \) with respect to \( ||.||_p \).
Lemma 3.3. If the Zariski closure $\overline{R}$ is irreducible, then there is a sequence $(Q_i)_{i \in \mathbb{N}}$ of points in $R$ and, for each $p \in T$, there is a point $Q_p \in X(F_p)$ such that

1. $(Q_i)_{i \in \mathbb{N}}$ is Zariski dense in $\overline{R}$, and
2. for all $p \in T$, the sequence $Q_i$ converges towards $Q_p$ in the $||.||_p$-topology.

Proof. We fix some $C$. That means by changing $V$ and, for each $p \in T$, there is a point $Q_p \in X(F_p)$ such that for each $||.||_p$-open neighbourhood $U$ of $Q_p$, the set $U \cap R$ is Zariski dense in $V(F)$. This is because otherwise $R$ would be the union of finitely many sets that are not Zariski dense in $V(F)$, which is not possible because $V$ is irreducible.

Since $F$ is countable, there are countably many hypersurfaces of $\mathbb{P}^n_F$ not containing $V$. Let $(H_i)_{i \in \mathbb{N}}$ be an enumeration of these. Let $U_1 \supseteq U_2 \supseteq \ldots$ be a descending chain of $||.||_p$-open neighbourhoods of $Q_p$ with $\bigcap U_i = \{Q_p\}$. Since $R$ is Zariski dense in $V(F)$, the set $R \cap U_i$ is discrete. Then the sequence $Q_i$ converges towards $Q_p$ in $||.||_p$-topology and is Zariski dense in $V$.

Iterating this process for all other $p \in T$ finishes the proof. $\square$

We set $R'$ to be the sequence constructed in Lemma 3.3. Let $Z$ be as constructed in Subsection 3.1. Let $R'' = R' \setminus Z$. Then the rational map $\phi_D$ is determined on $R''$. We have to show that the height $h$ is bounded on $\phi_D(R'')$. Note if $R''$ is finite, we are done. Otherwise, $R''$ is a subsequence of $R'$ and has the same convergency properties as $R'$ described in Lemma 3.3.

By Theorem 3.1, given $\epsilon > 0$, we can find constants $C, C'$ such that for all $Q \in \phi_D(R'')$ with $h_F(Q) > C$, the inequality

$$\sum_{p \in T} \max_{I} \sum_{H \in I} \lambda_{H,p}(Q) \leq (n + 1 + \epsilon/2)h_F(Q) + C',$$

is satisfied. Let $C'' \in \mathbb{R}$ be arbitrary. Then for $P \in R''$ with $h_F(\phi_D(Q)) > \frac{2(C'-C'')}{\epsilon}$, we have

$$(n + 1 + \epsilon/2)h_F(Q) + C' < (n + 1 + \epsilon)h_F(Q) + C''.$$ 

That means by changing $C$ if necessary we can choose $C'$ arbitrarily. Now since $R''$ is a set of $(D,O_{F,T})$-integral points, there is an $a \in O_{F,T}$ so that for all $Q \in R''$ and $i \in \{0, \ldots, n\}$, we have $a\phi_i(Q) \in O_{F,T}$. This implies

$$h_F(\phi_D(Q)) = h_f(a\phi_D(Q)) = -\sum_{p \in M_F} \min\{v_p(a\phi_0(Q)), \ldots, v_p(a\phi_n(Q))\}$$

$$\leq -\sum_{p \in T} \min\{v_p(a\phi_0(Q)), \ldots, v_p(a\phi_n(Q))\}$$

$$\leq -\sum_{p \in T} \min\{v_p(\phi_0(Q)), \ldots, v_p(\phi_n(Q))\} - \sum_{p \in T} v_p(a)$$

The term $\sum_{p \in T} v_p(a)$ is some constant. All together, we have seen that given $\epsilon > 0$ and $C' \in \mathbb{R}$, we can find a constant $C \in \mathbb{R}$ such that for all $Q \in R''$, we either have

1. $\sum_{p \in T} f_p(Q) \leq C'$,
where

\[
f_p(Q) = \max_I \left( \sum_{H \in I} \lambda_{H,p}(\phi_D(Q)) \right) + (n + 1 + \epsilon) \cdot \min\{v_p(\phi_0(Q)), \ldots, v_p(\phi_n(Q))\},
\]

or we have \(h_F(\phi_D(Q)) \leq C\). In other words, this means if we find a lower bound for the left-hand-side of Inequality \[ \[ \] \] i.e., a constant \(C'\) such that Inequality \[ \[ \] \] is never satisfied, then we have bounded the height.

We put

\[
M = \max \{-\text{ord}_E(\phi_j) | E \text{ is an irreducible component of } D, j \}
\]

and choose \(\epsilon = 1/M\). We can bound each function \(f_p\) separately. So fix \(p \in T\), let \(Q_p\) be as in Lemma \[ \[ \] \].

First assume \(Q_p \not\in D(F_p)\). Considering \(I = \{H_0, \ldots, H_n\}\), we see

\[
f_p(q) \geq \left( \sum_{i=0}^{n} \lambda_{H_i,p}(\phi_D(Q)) \right) + (n + 1 + \epsilon) \cdot \min\{v_p(\phi_0(Q)), \ldots, v_p(\phi_n(Q))\}
\]

\[
= \left( \sum_{i=0}^{n} v_p(\phi_i(Q)) \right) + \epsilon \min_{0 \leq i \leq n} \{v_p(\phi_i(Q))\}.
\]

As all \(\phi_i\) have no pole at \(Q_p\), the right-hand-side converges to the value at \(Q_p\). In particular, it is bounded.

Now assume \(Q_p \in D(F_p)\). We choose \(j \in J\) such that for each irreducible component \(E\) of \(D\) with \(Q_p \in E(F_p)\), we have

\[
\text{ord}_E \left( \prod_{\psi \in B_j} \psi \right) > 0.
\]

Considering \(I = \{H_{j,0}, \ldots, H_{j,n}\}\), we see

\[
f_p(Q) \geq \left( \sum_{i=0}^{n} \lambda_{H_{j,i},p}(\phi_D(Q)) \right) + (n + 1 + \epsilon) \cdot \min\{v_p(\phi_0(Q)), \ldots, v_p(\phi_n(P))\}
\]

\[
= \left( \sum_{i=0}^{n} v_p(\psi_{j,i}(Q)) \right) + \epsilon \min_{0 \leq i' \leq n} \{v_p(\phi_{i'}(Q))\}.
\]

For each irreducible component \(E\) of \(D\) with \(Q_p \in E(F_p)\), we have

\[
\text{ord}_E \left( \phi_{i'} \cdot \left( \prod_{i=0}^{M} \psi_{j,i} \right)^{l(D)} \right) = \text{ord}_E(\phi_{i'}) + M \text{ord}_E \left( \prod_{i=1}^{l(D)} \psi_{j,i} \right) \geq -M + M = 0.
\]

This shows that \(f_p\) is bounded by the minimum of a finite set of functions that have no pole along \(D\). As before by convergency reasons, this shows \(f_p\) is bounded from below on \(R''\). This finishes the proof of Lemma \[ \[ \] \] and Theorem \[ \[ \] \].

4. EVERTSE-GYÖRY’S METHOD

In this section, we review Evertse-Györy’s method \[ \[ \] \] and prove Theorem \[ \[ \] \].
4.1. Degree and height functions on finitely generated domains. Let \( A \) be a \( \mathbb{Z} \)-finitely generated domain. We choose a maximal number of algebraically independent elements \( z_1, \ldots, z_q \in A \). Then \( A \) is an extension of \( A_0 = \mathbb{Z}[z_1, \ldots, z_q] \), and we can find elements \( y_1, \ldots, y_t \in A \) that are algebraic over \( A_0 \) such that \( A = A_0[y_1, \ldots, y_t] \). We denote the quotient field of \( A \) by \( L = Q(A) \) and the quotient field of \( A_0 \) by \( L_0 = Q(A_0) = \mathbb{Q}(z_1, \ldots, z_q) \). Note that the field extension \( L/L_0 \) is finite. By the primitive element theorem, there is a \( y \in L \) such that \( L = L_0(y) \). The primitive element \( y \) has a minimal polynomial

\[
G(z_1, \ldots, z_q)(x) = x^d + G_1(z_1, \ldots, z_q)x^{d-1} + \cdots + G_d(z_1, \ldots, z_q) \in L_0[x].
\]

Let \( g \in A_0 \) be the common denominator of the rational functions \( G_i \in L_0 \). By replacing \( y \) with \( y \cdot g \), we may assume that \( G \in A_0[x] \). The elements \( 1, y, \ldots, y^{d-1} \) form a basis of \( L \) as \( L_0 \) vector space. Therefore, for each \( \alpha \in L \), we can find \( Q_\alpha, P_{0, \alpha}, \ldots, P_{d-1, \alpha} \in A_0 \) (unique up to sign) with no common factor such that

\[
\alpha = Q_\alpha^{-1} \sum_{j=0}^{d-1} P_{j, \alpha} y^j.
\]

In particular, we can find such for \( \alpha \in \{y_1, \ldots, y_t\} \). We set \( f = \prod_{i=1}^t Q_{y_i} \in A_0 \). Then there is an inclusion of rings

\[
A \subseteq A_0[f^{-1}, y] =: B.
\]

The ring \( B \) has the same quotient field as \( A \). For \( \alpha \in L \), we define

\[
\overline{\deg}(\alpha) := \max\{\deg(P_{\alpha,0}), \ldots, \deg(P_{\alpha,d-1}), \deg(Q_\alpha)\}
\]

\[
\overline{h}(\alpha) := \max\{h^{aff}(P_{\alpha,0}), \ldots, h^{aff}(P_{\alpha,d-1}), h^{aff}(Q_\alpha)\}
\]

where, for \( g = \sum_{\mu} c_\mu z_1^{\mu_1} \cdots z_q^{\mu_q} \in A_0 \), we set \( \deg(g) = \max\{\mu_1 + \cdots + \mu_q | c_\mu \neq 0 \} \) and \( h^{aff}(g) = \sum_{p \in M_0} \ln \max_{\mu, c_\mu \neq 0} \{1, ||c_\mu||_p \} \). Furthermore, for \( (\alpha_1, \ldots, \alpha_n) \in L^n \), we define:

\[
\overline{\deg}(\alpha_1, \ldots, \alpha_n) := \max\{\overline{\deg}(\alpha_1), \ldots, \overline{\deg}(\alpha_n)\}
\]

\[
\overline{h}(\alpha_1, \ldots, \alpha_n) := \max\{\overline{h}(\alpha_1), \ldots, \overline{h}(\alpha_n)\}
\]

These functions combined have the Northcott property.

**Lemma 4.1.** Let \( R \subseteq L^n \) be a subset such that \( \overline{\deg} \) and \( \overline{h} \) are bounded on \( R \). Then \( R \) is finite.

**Proof.** Let \( \alpha = Q_\alpha^{-1} \sum_{j=0}^{d-1} P_{j, \alpha} y^j \in L \) be a coordinate of some point in \( R \). By Theorem 2.1, the boundedness of \( \overline{h} \) on \( R \) implies that there are only finitely many possibilities for the coefficients of \( Q_\alpha, P_{0, \alpha}, \ldots, P_{d-1, \alpha} \). Since \( \overline{\deg} \) is bounded on \( R \), also the degree of these polynomials is bounded. \( \square \)

4.2. Using embeddings into function fields to bound the degree. We may embed \( L \) into function fields. We denote the \( d \) many \( L_0 \)-invariant embeddings of \( L \) into an algebraic closure of \( L_0 \) by \( x \mapsto x^{(j)} \), where \( j \in \{1, \ldots, d\} \). For \( i \in \{1, \ldots, q\} \), let \( k_i \) be an algebraic closure of \( \mathbb{Q}(z_1, \ldots, \hat{z_i}, \ldots, z_q) \). The notation \( \hat{z_i} \) means that we leave out \( z_i \). Then \( F_i = \)
\(k_i(z_i, y^{(1)}, \ldots, y^{(d)})\) is an algebraic function field of transcendence degree 1 over \(k_i\). Consider the \(d\) many different embeddings of \(L\) into \(F_i\)

\[(4) \quad \varphi_{i,j} : L \rightarrow F_i; z_i \mapsto z_i, y \mapsto y^{(j)}, \quad j \in \{1, \ldots, d\}.\]

Evertse and Györy showed the following; see [9, Lemma 8.4.1].

**Lemma 4.2.** There is a constant \(C > 0\) such that for all \(\alpha \in L\)

\[
\overline{\deg}(\alpha) \leq C + \sum_{i=1}^{q} [F_i : k_i(z_i)]^{-1} \sum_{j=1}^{d} h_{F_i}(\alpha^{(j)}).
\]

**Corollary 4.3.** Let \(R \subseteq \mathbb{Z}^n\) be a subset such that \(h_{\text{aff}}^{\alpha} F_i\) is bounded from above on

\[\varphi_{i,j}(R) := \{ (\varphi_{i,j}(\alpha_1), \ldots, \varphi_{i,j}(\alpha_n)) | (\alpha_1, \ldots, \alpha_n) \in R \} \subseteq F_i^n\]

for all \(i \in \{1, \ldots, q\}\) and \(j \in \{1, \ldots, d\}\). Then \(\overline{\deg}\) is bounded on \(R\).

**Proof.** Let \((x_1, \ldots, x_n) \in \varphi_{i,j}(R)\). For all \(j \in \{1, \ldots, n\}\), we have

\[h_{\text{aff}}^{\alpha} F_i(x_1, \ldots, x_n) = \sum_{p \in M_{F_i}} \max\{0, -v_p(x_1), \ldots, -v_p(x_n)\} \geq \sum_{p \in M_{F_i}} \max\{0, -v_p(x_j)\} = h_{\text{aff}}^{\alpha} F_i(x_j).
\]

So the height is bounded on each coordinate. By Lemma 4.2 this implies that \(\overline{\deg}\) is bounded on each coordinate. \(\square\)

### 4.3. Specializations.

As we have seen in the Subsection 4.1, the finitely generated domain \(A\) is contained in \(B = A_0[f^{-1}, y]\), where \(f \in A_0 = \mathbb{Z}[z_1, \ldots, z_q]\) and \(y\) is integral over \(A_0\) with minimal polynomial

\[G(z_1, \ldots, z_q)(x) = x^d + G_1(z_1, \ldots, z_q)x^{d-1} + \cdots + G_d(z_1, \ldots, z_q) \in A_0[x].\]

We want to define specialization maps \(B \rightarrow \overline{\mathbb{Q}}\). The discriminant \(\text{disc}(G)\) is a polynomial in the variables \(z_1, \ldots, z_q\). Since \(G\) is irreducible and separable, this polynomial does not vanish entirely. We define

\[(5) \quad H = G_d \cdot \text{disc}(G) \cdot f \in A_0.
\]

Now for \(u \in \mathbb{Z}^d\) with \(H(u) \neq 0\), the polynomial \(G(u)(x) \in \mathbb{Z}[x]\) has \(d\) many distinct roots \(y_{u,1}, \ldots, y_{u,d} \in \overline{\mathbb{Q}} \setminus \{0\}\) that are integral over \(\mathbb{Z}\), and we have \(f(u) \in \mathbb{Z} \setminus \{0\}\). Now mapping \(z_j \mapsto u_j\) and \(y \mapsto y_{u,i}\) defines a map

\[(6) \quad \psi_{u,i} : B \rightarrow \mathbb{Z}[f(u)^{-1}, y_{u}] \subseteq O_{K_{u,i}} S_{u,i} \subseteq K_{u,i},\]

where \(K_{i,u} = \mathbb{Q}(y_{u,i})\) is a number field and \(S_{u,i}\) is the set that contains all infinite places and the \(p \in M_{K_{u,i}}^0\) with \(\text{ord}_p(f(u)) > 0\) or \(\text{ord}_p(y_{i}(u)) < 0\).

**Remark 4.4.** Let \(K\) be a number field and let \(S \subseteq M_K\) be a finite such that there is an embedding \(O_{K,S} \subseteq B\). Then

\[p = \ker \left( O_{K,S} \xrightarrow{\psi_{u,i}} K_{u,i} \right) \subseteq O_{K,S}\]

is a prime ideal. Hence either \(p = 0\) or \(p\) contains some prime number \(p \in \mathbb{Z}\). The latter is impossible, as \(K_{u,i}\) is no \((\mathbb{Z}/p\mathbb{Z})\)-algebra. Hence the specialization map \(\psi_{u,i}\) induces an isomorphism of \(K\) with some subfield of \(K_{u,i}\).
We will use the following version of a result by Evertse and Győry.

**Proposition 4.5.** Let \( R \subseteq B \) be a subset of bounded \( \deg \), let \( N \in \mathbb{N} \) be so big that \( \deg(r) \leq N \) for all \( r \in R \) and \( \deg(H) \leq N \), and let
\[
U = \{ u \in \mathbb{Z}^q \mid \max_i \{|u_i|\} \leq N, H(u) \neq 0 \}.
\]
Furthermore, suppose the height function \( H_{K_{u,i}}^{\text{aff}} \) is bounded on \( \psi_{u,i}(R) \) for all \( i \in \{1, \ldots, d\} \) and \( u \in U \). Then \( R \) is finite.

**Proof.** By [9, Lemma 8.5.6], \( h \) is bounded on \( R \). The assertion now follows from Lemma [4.1].

**Corollary 4.6.** Let \( R \subseteq B^n \) be a subset of bounded \( \deg \), let \( N \in \mathbb{N} \) be so big that \( \deg(r) \leq N \) for all \( r \in R \) and \( \deg(H) \leq N \), and let
\[
U = \{ u \in \mathbb{Z}^q \mid \max_i \{|u_i|\} \leq N, H(u) \neq 0 \}.
\]
Furthermore, suppose for all \( i \in \{1, \ldots, d\} \) and \( u \in U \), the height function \( H_{K_{u,i}}^{\text{aff}} \) is bounded on
\[
\{ (\psi_{u,i}(x_1), \ldots, \psi_{u,i}(x_n)) \mid (x_1, \ldots, x_n) \in R \}.
\]
Then \( R \) is finite.

**Proof.** When \( \deg \) is bounded on \( R \) by \( N \), then \( \deg \) is also bounded by \( N \) on each coordinate of every point in \( R \). Furthermore, for \( (q_1, \ldots, q_n) \in K_{u,i}^n \), we have
\[
H_{K_{u,i}}^{\text{aff}}(q_1, \ldots, q_n) = \prod_{p \in M_{K_i}} \max \{1, ||q_1||_p, \ldots, ||q_n||_p \} \geq \prod_{p \in M_{K_i}} \max \{1, ||q_j||_p \} = H_{K_{u,i}}^{\text{aff}}(q_j).
\]
Hence, the set of coordinates of points in \( R \) satisfies the assumptions of Proposition [4.5]. So we are done.

### 4.4. Formulation of the method.

Let \( R \subseteq A^n \) be a subset, where \( A \) is a \( \mathbb{Z} \)-finitely domain such that

1. for all embeddings into function fields \( \varphi_{i,j} : A \to F_i \) (see Subsection 4.2), the height \( H_{F_{i}}^{\text{aff}} \) is bounded on \( \varphi_{i,j}(R) \), and
2. for all specialization maps \( \psi_{u,i} : A \to O_{K_{u,i},S_{u,i}} \) (see Subsection 4.3), the height \( H_{K_{u,i}}^{\text{aff}} \) is bounded on \( \psi_{u,i}(R) \).

Then Evertse and Győry’s results Corollary [4.3] and Corollary [4.6] combined prove the finiteness of \( R \).

**Proof of Theorem [4.7].** By Remark [4.4], the set \( \psi_{u,i}(R) \) is contained in some set of integral points of \( X' = X \otimes_{K,\psi_{u,i}} K_{u,i} \). The variety \( X' \) is arithmetically hyperbolic as well; see [23]. Hence the sets \( \psi_{u,i}(R) \) are all finite and therefore the height is bounded. Now we apply Evertse-Győry’s method. 

\[\square\]
5. Absolute arithmetic hyperbolicity

In this section, we will prove Theorem 1.2

Lemma 5.1. Let $m$ be an integer, let $k$ be a field of characteristic zero, let $X$ be a projective variety over $k$, and let $D = \sum_{i=1}^{r} D_i$ be a sum of effective, big and nef divisors on $X$ such that at most of $m$ of the $D_i$ meet in any point, where $r > 2m \dim(X)$. Then there is a non-singular projective variety $X'$ over $k$ and a birational morphism $\pi: X' \to X$ and positive integers $b_i$ such that

1. the divisor $E = \sum_{i=1}^{r} b_i D_i$ has the same support as $D$,
2. the pullback $E' = \pi^* E$ is very large,
3. the associated rational map $\phi_{E'}$ is birational onto its image and
4. the variety $X'$ and every irreducible component of $E'$ is non-singular. [Lemma 4.14]

Proof. Let $q = \dim(X)$. By Hironaka, we can find a smooth projective $X'$ and birational morphism $\pi: X' \to X$ such that $D_i' := \pi^* D_i$ is non-singular for every $i$. The $D_i'$ are big and nef again, since they are pullbacks of big and nef divisors along a birational morphism. Therefore, all $q$-fold intersections of the $D_i'$ are nonnegative and $D_i'^q > 0$. By [28, Lemma 9.7], the divisor $\sum_{i=1}^{r} D_i'$ is equidegreeelizable. Hence by [28, Theorem 9.9], we can find suitable $b_i$. □

Proposition 5.2. Let $m$ be an integer, let $X$ be a projective variety over $\overline{\mathbb{Q}}$, and let $D = \sum_{i=1}^{r} D_i$ be a Cartier divisor on $X$, where all the $D_i$ are effective and ample and at most $m$ of the $D_i$ meet at any point, with $r > 2m \dim(X)$. Then there is a proper closed subvariety $Z \subseteq X$ and an ample Cartier divisor $E$ with the same support as $D$ such that the following is true.

For any countable algebraic function field $F$, for any finite set of places $T \subseteq M_F$, and for all sets $R$ of $(E, O_{F,T})$-integral points on $X_F$, there is a subset $R' \subseteq R \setminus Z$ such that $R'$ is Zariski dense in $R \setminus Z$ and the height function $h_{X,E}$ is bounded on $R'$.

Proof. By Lemma 5.1 we can find a Cartier divisor $E = \sum_{i=1}^{r} b_i D_i$ on $X$ with the same support as $D$ and a birational proper morphism of projective varieties $\pi: X' \to X$ such that $E' = \pi^* E$ is very large and $\phi_{E'}$ is birational onto its image. Note that $E'$ is semi-ample since it is the pullback of an ample divisor. Therefore, by replacing the $b_i$ with $nb_i$ for $n \gg 0$, we may assume that $E'$ is base point free.

There is a proper closed $V \subseteq X$ (namely the singular locus of $X$ and $D$) that can be defined over $K$ such that $\pi^{-1}$ is defined outside of $V$. Let $Z' \subseteq X'$ be a proper closed subscheme like in Theorem 1.4 and let $Z = \pi(Z') \cup V \subseteq X$.

Let $R \subseteq X(F)$ be a set of $(E, O_{F,T})$ integral points. Then $\pi^{-1}(R \setminus V)$ is a set of $(E', O_{F,T})$-integral points on $X'$. Hence, there is a subset $R' \subseteq R \setminus V$ such that

1. the Zariski closures of $\pi^{-1}(R') \subseteq X'$ and $\pi^{-1}(R \setminus V) \subseteq X'$ equal, and
2. the height $h_{X',E'}$ is bounded on $\pi^{-1}(R') \setminus Z'$.

Since $\pi$ is birational and proper, the set $Z \subseteq X$ is a proper closed subvariety. Since $h_{X',E'} = h_{X',\pi^*E'} = h_{X,E'} \circ \pi + O(1)$ (see Theorem 2.2), we conclude that $h_{X,E}$ is bounded on $R' \setminus Z$. Since $\pi$ is birational, $R' \setminus Z$ is Zariski dense in $R \setminus Z$. □

Lemma 5.3. Let $X$ be a projective variety over $\overline{\mathbb{Q}}$, and let $m$ and $r$ be positive integers with $r > 2m \dim(X)$. Let $D = \sum_{i=1}^{r} D_i$ be a Cartier divisor on $X$, where all the $D_i$ are effective
and ample and the intersection of $m + 1$ distinct $D_i$ is empty. Then there is a proper closed subscheme $Z \subseteq X$ such that $X$ is absolutely arithmetically hyperbolic modulo $Z$.

Proof. The finiteness of integral points is independent of the chosen model; see for example [23, Lemma 4.8]. Thus, let $K$ be a number field, let $S$ be a finite set of places and let $W$ be a model for $W = X \setminus D$ over $O_{K,S}$. Let $A$ be a $Z$-finitely generated integral domain and let $R = W(A)$.

Let $\{F_1, \ldots, F_l\}$ be a finite set of function fields. By repeatedly applying Proposition 5.2 we can find a proper closed subscheme $Z \subseteq X$, an ample Cartier divisor $E$ on $X$ with the same support as $D$ and a subset $R' \subseteq R \setminus Z$ that is Zariski dense in $R \setminus Z$ such that $h_{X,E}$ is bounded on $R' \subseteq (X,F, \setminus E)(F_j)$ for all $j$.

We claim that $R \setminus Z$ is finite. To show this, we apply Evertse-Győry’s method. Let $t > 0$ be an integer such that $tE$ is very ample. By Theorem 2.2.3, the height $h_{X,tE}$ is bounded on $R'$. We may choose an embedding $\phi : X \to \mathbb{P}^n$ such that $tE = \phi^*H$ for a hyperplane $H \subseteq \mathbb{P}^n$. The restriction of $\phi$ to $X \setminus D$ induces an embedding $\alpha : X \setminus D \to \mathbb{A}^n \cong \mathbb{P}^n \setminus H$.

By Theorem 2.2.1 and Theorem 2.2.2, restricting to $X \setminus D$, we have

$$h_{X,tE} = h_{X,\phi^*H} = h_{\mathbb{P}^n,H} \circ \phi + O(1) = h \circ \phi + O(1).$$

Again restricting to $X \setminus D$, gives us $h_{X,tE}|_{X \setminus D} = h^{aff} \circ \alpha + O(1)$. Therefore, $h^{aff}_{F_i}$ is bounded on $R' \subseteq (X \setminus D)(F_i)$. Combined with the fact that $X \setminus D$ is arithmetically hyperbolic over $\overline{\mathbb{Q}}$ ([28, Theorem 9.11A]), it follows from Theorem 1.5 that $R \setminus Z$ is finite. \qed

Proof of Theorem 1.2. We proceed by induction on the dimension of $X$. Note any variety of dimension 0 is absolutely arithmetically hyperbolic. Let $m$ be a positive integer, let $X$ be a smooth projective connected variety over $\overline{\mathbb{Q}}$ of dimension $\dim(X) > 0$, and let $D = \sum_{i=1}^r D_i$ be a sum of $r$ ample effective divisors on $X$ such that at most $m$ of the divisors $D_i$ meet in a point, with $r > 2m \dim(X)$. By Lemma 5.3 there is a proper closed subscheme $Z \subseteq X$ that contains all but finitely many integral points. We finish the proof by showing that all irreducible components of $Y \subseteq Z$ are absolutely arithmetically hyperbolic. If $\dim(Y) = 0$, there is nothing to do. Otherwise, we have $0 < \dim(Y) < \dim(X)$. Note that the pullback of the divisor $D$ along the closed immersion $Y \to X$ again satisfies the hypotheses of the theorem. Hence by induction $Y$, is absolutely arithmetically hyperbolic. \qed

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