Quasi likelihood analysis of point processes for ultra high frequency data

Teppei Ogihara\textsuperscript{1,2,4} and Nakahiro Yoshida\textsuperscript{3,4,1}

\textsuperscript{1}Institute of Statistical Mathematics\textsuperscript{†}
\textsuperscript{2}SOKENDAI (The Graduate University for Advanced Studies)\textsuperscript‡
\textsuperscript{3}Graduate School of Mathematical Sciences, University of Tokyo\textsuperscript§
\textsuperscript{4}CREST, Japan Science and Technology Agency

December 8, 2015

Abstract
We introduce a point process regression model that is applicable to price models and limit order book models. Hawkes type autoregression in the intensity process is generalized to a stochastic regression to covariate processes. We establish the so-called quasi likelihood analysis, which gives a polynomial type large deviation estimate for the statistical random field. We derive large sample properties of the maximum likelihood type estimator and the Bayesian type estimator when the intensity processes become large under a finite time horizon. There appears non-ergodic statistics. A classical approach is also mentioned.

Key words and phrases: point process, regression, quasi likelihood analysis, maximum likelihood estimator, Bayesian estimator, Hawkes process, price model, limit order book, polynomial type large deviation, statistical random field, convergence of moments.

1 Introduction
High frequency financial data is one of the latest objects to be challenged by the most advanced statistics. Ad hoc descriptive methods often mislead and fail data analysis. For two stochastic processes observed high-frequently and asynchronously, the estimated covariance between them almost vanishes if one applies the realized covariance with a “natural” interpolation method, even if the real covariance

\textsuperscript{†}Institute of Statistical Mathematics: 10-3 Midori-cho, Tachikawa, Tokyo 190-8562, Japan
\textsuperscript‡}Shonan Village, Hayama, Kanagawa 240-0193, Japan
\textsuperscript§}Graduate School of Mathematical Sciences, University of Tokyo: 3-8-1 Komaba, Meguro-ku, Tokyo 153-8914, Japan

\textsuperscript{‡}This work was in part supported by Japan Society for the Promotion of Science Grants-in-Aid for Scientific Research Nos. 24340015 (Scientific Research), Nos. 24650148 and 26540011 (Challenging Exploratory Research); CREST Japan Science and Technology Agency; NS Solutions Corporation; and by a Cooperative Research Program of the Institute of Statistical Mathematics.
is not null. Recently it was recognized that the non-synchronicity of sampling schemes, though it is inevitable for real financial data, causes such phenomena generically called the Epps effect. Theory of non-synchronous estimation has been dramatically developing for the last decade and successfully applied to actual data analyses. Market microstructure is another main factor that causes the Epps effect. Remarkable progresses were recently made in volatility estimation problems by proposing effective filters that remove microstructure noises and at the same time treat non-synchronous sampling schemes. Non-synchronicity and microstructure are now the point where theoretical statistics, probability theory and real data analysis are confluent (Epps [11], Malliavin and Mancino [24, 25], Hayashi and Yoshida [13, 14, 15], Voev and Lunde [35], Griffin and Oomen [12], Mykland [27], Zhou [39], Zhang et al. [37], Zhang [38], Podolskij and Better [31], Jacod et al. [18], Christensen et al. [9], Bibinger [5, 6], Ogihara and Yoshida [30], Koike [20, 22, 21], Ogihara [29, 28] among many others).

The very latest issue is modeling of ultra high frequency phenomena by point processes. It enables us to model microstructure itself rather than eliminating it as noise. In ultra high frequency sampling, the central limit theorem does not work and there is no longer Brownian motion as the driving factor of asset prices, differently from the standard mathematical finance. The world of real data is already beyond a standard theory but this is the reality statisticians are confronted with.

Statistical theory of non-synchronous data suggests relativity of time. Theory of lead-lag estimation emerged against this background. When observing two time series, we often find lead-lag between them, namely, one is the leader and the other is the follower. If these series are stock prices, this means the existence of statistical arbitrage. Some developments for high frequency data are in de Jong and Nijman [19, 17] and Abergel and Huss [1].

In this article, we consider a point process regression model that enables us to express non-synchronicity of observations, lead-lag relation and microstructure. Our model can describe self-exciting/self-correcting effects of the point processes as well as exogenous effects. Non-ergodic statistics is constructed in the QLA (quasi likelihood analysis) framework. The point process regression model has applications to price models and limit order book models.

2 Point process regression model

The $d$-dimensional point process $N^n = (N^{n,\alpha})_{\alpha \in \mathcal{I}}$ on the interval $I = [T_0, T_1]$, $\mathcal{I} = \{1, \ldots, d\}$, is assumed to have a $d$-dimensional intensity process $n\lambda^n(t, \theta)$ defined by

$$\lambda^n(t, \theta) = g^n(t, \theta) + \int_{T_0}^{t} K^n(t, s, \theta) dX^n_s,$$

where $\theta$ is a parameter. The point process $N^n$ forms a model whose intensity processes refer to the covariates $g^n$ and $K^n$ as well as the explanatory process $X^n$. More precisely, we will work on a stochastic basis $\mathcal{B} = (\Omega, \mathcal{F}, \mathbf{F}, P)$, $\mathbf{F} = (\mathcal{F}_t)_{t \in \hat{I}}$ being a filtration on $(\Omega, \mathcal{F})$, where $\hat{I} = [T_0, T_1] \supset I$. For each $n \in \mathbb{N}$ and $\theta \in \Theta$, $(g^n(t, \theta))_{t \in I}$ is a $d$-dimensional predictable process, $(K^n(t, s, \theta))_{s \in [T_0, t]}$ is a $d \times d_0$ matrix-valued optional process for $t \in I$, $Z_0 = \{1, \ldots, d_0\}$, and $(X^n_t)_{t \in \hat{I}}$ is a $d_0$-dimensional $\mathbf{F}$-adapted right-continuous increasing (i.e., non-decreasing) process on $\mathcal{B}$. We will impose conditions that ensure the existence of those stochastic integrals later. The multivariate point process $N^n$ is

\textsuperscript{1}It is possible to make $g^n(t, \theta)$ include the part $\int_{T_0}^{T_1} K^n(t, s, \theta) dX^n_s$. However this definition fits Hawkes type processes we will discuss later.
compensated by the process \( \int_0^t \eta \lambda^n(s, \theta) ds \) when \( \theta \) is the true value of the unknown parameter.

We will assume that any two elements of \( N^n \) do not share common jumps. \(^2\)

Applications of point processes to financial data were in Hewlett [16] on the clustered arrivals of buy and sell trades using Hawkes processes, Large [23] on extension by using a fine description of orders, Bowsher [7] on a generalized Hawkes model, and in Bacry et al. [4] on a price model. Chen and Hall [8] investigated the maximum likelihood estimator of a non-stationary self-exciting point process when the intensity goes up.

Obviously, the transaction times of stocks whose occurrence intensities possibly depend on their own or exogenous randomly changing factors can be described by a point process regression model. The point process regression model also applies to micro-scale modeling of the movements of the stock prices, incorporating information of covariate processes.

Recently, point processes are applied to order-book modeling; see Cont et al. [10] and Abergel and Jedidi [2, 3], and also Smith et al. [32]. Abergel and Jedidi [2, 3] presented an order book model by a multivariate point process and proved ergodicity of the system in infinite time horizon by using the drift condition for the Markov chain. Muni Toke and Pomponio [26] gave a model of trades-through in a limited order book using Hawkes processes.

Our point process regression model has finite time horizon and the resulting statistics becomes non-ergodic. We shall give short descriptions of the last two examples, before going into the main part of this article.

2.1 Modeling digital movements of stock prices

The point process \( \Pi^n \) is fairly generic. For ultra high-frequency financial data, we model the movements of the prices \( Y_t \) by the combination of the components of the multi-variate point process \( N^n = (N^n_\alpha)_{\alpha \in I} \), e.g., by

\[
Y_t = AN^n_t, \tag{2.1}
\]

where \( A \) is a constant matrix. In the following examples, \( N \) denotes \( N^n \) for notational simplicity.

**Example 2.1.** (± one-unit jumps) Let \( a \) denote a monetary unit. Let

\[
A = a \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}
\]

Then

\[
Y = \begin{bmatrix} a(N^0 - N^1) \\ a(N^2 - N^3) \end{bmatrix}
\]

for \( N = [N^0, N^1, N^2, N^3]' \).

**Example 2.2.** (± one/two-unit jumps) Let

\[
A = a \begin{bmatrix} 1 & 2 & -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & -1 & -2 \end{bmatrix}
\]

\(^2\)This assumption is necessary to specify the asymptotic variance of the estimators, as will be done later.
Then
\[ Y = \begin{bmatrix} aN^0 + 2aN^1 - aN^2 - 2aN^3 \\ aN^4 + 2aN^5 - aN^6 - 2aN^7 \end{bmatrix} \]
for \( N = [N^0, ..., N^7]' \).

**Example 2.3.** (Simultaneous jumps) Let
\[ A = a \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & \mp 1 & \mp 1 \end{bmatrix} \]
Then
\[ Y = \begin{bmatrix} a(N^0 - N^1) + a(N^4 - N^5) \\ a(N^2 - N^3) \pm a(N^4 - N^5) \end{bmatrix} \]
for \( N^n = [N^0, ..., N^5]' \).

In this way, ultimately, we can assume that our data is described by a multi-variate point process.

**Example 2.4.** The Hawkes type process is an example if one takes \( X^n_t = n^{-1}N^n_t \). Our setting also includes the models with \( X^n_t = n^{-1}V^n_t \) and \( X^n_t = (n^{-1}N^n_t, n^{-1}V^n_t) \) for covariates subordinator \( V^n_t \) like the cumulative volume of the trades.

**Example 2.5.** An example of the kernel function is \( K^n(t, s, \theta) = c(\theta_1, \theta_2)(t - s)^{\theta_2}e^{-\theta_1(t-s)} \). This extends the original Hawkes process and it will be important when we discuss the lead-lag estimation in our framework.

### 2.2 Limit order book

Multivariate point processes give an approach to modeling of the limit order book. For simplicity, we will assume the volumes of limit orders, market orders and cancellation are a common value \( q \) for all prices. The state of the limit order book is described by the multi-dimensional process
\[ X = ((A^\alpha)_{\alpha=1,\ldots,k_A}, (B^\beta)_{\beta=1,\ldots,k_B}), \]
where the process \( A^\alpha \) counts the number of shares available at price \( p^\alpha_A \) at time \( t \) on the ask side and the process \( B^\beta \) counts the number of shares available at price \( p^\beta_B \) at time \( t \) on the bid side. In this modeling, the state space of \( X \) is absolute or fixed. The price \( p^\alpha_A \) may denote a relative quoted price if one defines \( p^\alpha_A \) as the price \( \alpha \) ticks away from the best opposite quote, while it is also possible to consider a fixed state space. The random evolution of \( X \) is determined by the processes \( M^A \) counting number of arrivals of market orders on the ask side, \( M^B \) of market orders on the bid side, \( L^\alpha \) of limit orders at level \( \alpha \) on the ask side, \( L^\beta \) of limit orders at level \( \beta \) on the bid side, \( C^\alpha \) of cancellation at level \( \alpha \) on the ask side, and \( C^\beta \) of cancellation at level \( \beta \) on the bid side. The multivariate counting process \( N^n \) consists of these counting processes. Here prices can be recognized as a function of \( X \).
3 Quasi likelihood

We shall consider estimation for the unknown parameter $\theta$. Suppose that we have observations

$$(N_t^{n,\alpha})_{t \in I, \alpha \in \mathcal{I}}, \quad (X_t^{n,\beta})_{t \in I, \beta \in \mathcal{I}_0}, \quad (g^n t, \alpha, t, \theta)_{t \in I, \alpha \in \mathcal{I}, \theta \in \Theta}, \quad (K^n_{\beta}(t, s, \theta))_{t \in I, s \in [T_0, t), \alpha \in \mathcal{I}, \beta \in \mathcal{I}_0, \theta \in \Theta}$$

up to $\theta$. This is the case, for example, when $g^n t, \alpha, t, \theta$ is a function of $\theta$ and some observable covariate process, that is, $g^n t, \alpha, t, \theta = g(Z_t, \theta)$ for some observable covariate process $Z_t$ and a given function $g$.

A statistician models the phenomena by the point processes $N_{\alpha}$ with intensity processes $n\lambda^n t, \alpha$, choosing a large number $n$. Only the estimated function $n\lambda^n t, \alpha$, not $\lambda^n t, \alpha$, makes sense as a result of statistical analysis.

We adopt the quasi log likelihood

$$l_n(\theta) = \sum_{\alpha \in \mathcal{I}} \left( \int_{T_0}^{T_1} \log[n\lambda^n t, \alpha, t, \theta] dN_t^{n,\alpha} - \int_{T_0}^{T_1} [n\lambda^n t, \alpha, t, \theta] - 1] dt \right)$$

for observed point process $N^n$. Obviously, “$-1$” in the second integral can be eliminated for maximization. The factor “$n$” in the first integral is also unnecessary. Thus we can use

$$\ell_n(\theta) = \sum_{\alpha \in \mathcal{I}} \left( \int_{T_0}^{T_1} \log[n\lambda^n t, \alpha, t, \theta] dN_t^{n,\alpha} - \int_{T_0}^{T_1} n\lambda^n t, \alpha, t, \theta dt \right)$$

instead of $l_n(\theta)$. To estimate $\theta$, we will consider the quasi maximum likelihood estimator (QMLE), that is, a sequence of estimators $\hat{\theta}_n$ that maximizes or asymptotically maximizes $\ell_n(\theta)$. The quasi Bayesian estimator (QBE) is another option, as discussed later.

Hereafter, we suppose that $\Theta$ is a bounded open set in $\mathbb{R}^p$ and satisfies

$$\inf_{\theta' \in \Theta} \text{Leb}\left( \left\{ \theta' \in \Theta; |\theta' - \theta| < \epsilon \right\} \right) \geq a_0(\epsilon^p \wedge 1) \quad (\epsilon > 0)$$

for some positive constant $a_0$, where $\text{Leb}$ is the Lebesgue measure. The true value of $\theta$ will be denoted by $\theta^*$.

4 Quasi maximum likelihood estimator by a classical approach

4.1 Consistency of the quasi maximum likelihood estimator

In Section 4.1, we suppose that the function $\Theta \ni \theta \mapsto \lambda^n t, \theta$ has continuous extension to $\overline{\Theta}$. Let $J = \{(t, s); s \in [T_0, t), t \in I\}$. Denote by $\mathcal{J}$ the set of $\mathbb{R}_+\text{-valued left-continuous non-decreasing adapted processes on } \Omega \times I$.

$[A1]$: For each $n \in \mathbb{N}$, $K^n(t, s, \theta)$ is an $\mathbb{R}_+^d \otimes \mathbb{R}_+^{d_0}$-valued $\mathcal{F} \times \mathcal{B}(J) \times \mathcal{B}(\Theta)$-measurable function satisfying the following conditions.

$^3$ B(A) is the Borel $\sigma$-field for a topological space A.
(i) For each \((n, t, \theta) \in \mathbb{N} \times I \times \Theta\), the process \([\hat{T}_0, t) \ni s \mapsto K^n(t, s, \theta)\) is optional on \(\Omega \times [\hat{T}_0, t)\), and each path is continuous or has jump discontinuity at every point \(s\).

(ii) For each \((n, t, s) \in \mathbb{N} \times J\), the mapping \(\Theta \ni \theta \mapsto K^n(t, s, \theta)\) is \(\check{j}\) times differentiable a.s., those derivatives are right or left-continuous in \(s\) at every point \(s \in [\hat{T}_0, t)\), and there exist \(\check{K}^n \in \mathcal{I}\) for \(n \in \mathbb{N}\) such that

\[
\sum_{j=0}^{\check{j}} \sup_{t' \in [\hat{T}_0, t]} \sup_{s \in [T_0, t')} \sup_{\theta \in \Theta} |(\partial_\theta)^j K^n(t', s, \theta)| \leq \check{K}^n(t) \quad (t \in I, n \in \mathbb{N})
\]

and that the family \(\{\check{K}^n(T_1)\}_{n \in \mathbb{N}}\) is tight.

(iii) There exist an \(\mathbb{R}^d \otimes \mathbb{R}^d\)-valued random field \(K^\infty(t, s, \theta)\) that is differentiable in \(\theta \in \Theta\), and there exist \(\check{K}^n \in \mathcal{I}\) for \(n \in \mathbb{N}\) such that

\[
\sum_{j=0}^{\check{j}} \sup_{t' \in [\hat{T}_0, t]} \sup_{s \in [T_0, t')} \sup_{\theta \in \Theta} \left| \partial_\theta^j K^n(t', s, \theta) - \partial_\theta^j K^\infty(t', s, \theta) \right| \leq \check{K}^n(t) \quad (t \in I, n \in \mathbb{N})
\]

and that \(\check{K}^n(T_1) \to^p 0\) as \(n \to \infty\).

[A2] For each \((\alpha, n) \in \mathcal{I} \times \mathbb{N}\), \(g^{n, \alpha}(t, \theta)\) is an nonnegative \(\mathcal{F} \times \mathcal{B}(I) \times \mathcal{B}(\Theta)\)-measurable function for which the following conditions are fulfilled.

(i) For each \((n, \alpha, \theta) \in \mathbb{N} \times \mathcal{I} \times \Theta\), the process \((g^{n, \alpha}(t, \theta))_{t \in I}\) is predictable.

(ii) For each \((n, t) \in \mathbb{N} \times I\), the mapping \(\Theta \ni \theta \mapsto g^n(t, \theta)\) is \(\check{j}\) times differentiable a.s., and there exist \(\check{g}^n \in \mathcal{I}\) for \(n \in \mathbb{N}\) such that

\[
\sum_{j=0}^{\check{j}} \sup_{t' \in [\hat{T}_0, t]} \sup_{\theta \in \Theta} \left| (\partial_\theta)^j g^n(t', \theta) \right| \leq \check{g}^n(t) \quad (t \in I, n \in \mathbb{N})
\]

and that the family \(\{\check{g}^n(T_1)\}_{n \in \mathbb{N}}\) is tight.

(iii) There exist an \(\mathbb{R}^d\)-valued random field \(g^\infty(t, \theta)\) that is differentiable in \(\theta \in \Theta\), and there exist \(\check{g}^n \in \mathcal{I}\) for \(n \in \mathbb{N}\) such that

\[
\sum_{j=0}^{\check{j}} \sup_{t' \in [\hat{T}_0, t]} \sup_{\theta \in \Theta} \left| \partial_\theta^j g^n(t', \theta) - \partial_\theta^j g^\infty(t', \theta) \right| \leq \check{g}^n(t) \quad (t \in I, n \in \mathbb{N})
\]

and that \(\check{g}^n(T_1) \to^p 0\) as \(n \to \infty\).

**Remark 4.1.** If one assumes separability for random fields on \((t, s, \theta)\) or \((t, \theta)\), it is possible to avoid introducing the envelope processes \(\check{K}^n\) etc. However, finding envelope processes is easier than verifying separability involving many intervals of variables.

\(^4\)A mapping \(f\) on \([a, b]\) is continuous or has jump discontinuity at every point \(s\) if \(f(s+)\) exist for all \(s \in [a, b]\) and if \(f(s-)\) exist for all \(s \in (a, b)\).
Let \( \rho : \mathcal{M}_b(\hat{I}) \times \mathcal{M}_b(\hat{I}) \to \mathbb{R}_+ \) be a metric that is compatible with the weak\(^*\)-topology on the set of \( \mathbb{R} \)-valued measures with finite total variation. In other words, for \( \mu_n, \mu \in \mathcal{M}_b(\hat{I}) \), the convergence \( \rho(\mu_n, \mu) \to 0 \) is equivalent to that \( \mu_n(f) \to \mu(f) \) for all \( f \in C(\hat{I}) \). Nondecreasing functions will be identified with measures.

\[ A3 \] For each \( n \in \mathbb{N} \) and \( \beta \in \mathcal{I}_0 \), \( (X_{t, \theta}^n)_{t \in \hat{I}} \) is a non-decreasing right-continuous \((\mathcal{F}_t)_{t \in \hat{I}}\)-adapted process, and for each \( \beta \in \mathcal{I}_0 \), there exists a non-decreasing process \( (X_{t, \theta}^{\infty})_{t \in \hat{I}} \) such that
\[
\rho(X_{t, \theta}^n, X_{t, \theta}^{\infty}) \to \rho 0
\]
as \( n \to \infty \) and that
\[
\int_{[\hat{T}_0, t]} 1_{\mathcal{D}_\beta(t, \theta)}(s)dX_{s, \theta}^{\infty} = 0
\]
for all \((t, \theta, \alpha, \beta) \in I \times \Theta \times \mathcal{I} \times \mathcal{I}_0 \) a.s. where \( \mathcal{D}_\alpha(t, \theta) = \{ s \in [\hat{T}_0, t); \partial_\theta^j K(t, s, \theta)_{\beta}^{\alpha} \} \) is discontinuous at \( s \) for some \( j \leq \frac{\beta}{2} \).

We do not assume continuity of \( K^{\infty} \) in \( s \), which is necessary to treat a kernel looking back a finite-length of history.

**Lemma 4.2.** Let \( j \in \{0, 1\} \). Then under \([A1]\)_j (ii) and \([A3]\)_j,
\[
\int_{I \times \Theta} \mu_1(dt)\mu_2(d\theta) \left\| \int_{[\hat{T}_0, t]} \partial_\theta^j K^{\infty}(t, s, \theta) dX_s^n - \int_{[\hat{T}_0, t]} \partial_\theta^j K^{\infty}(t, s, \theta) dX_s^{\infty} \right\| \to \rho 0 \tag{4.1}
\]
as \( n \to \infty \) for any a.s.-bounded continuous random measure \( \mu_1 \) on \( I \) and any a.s.-bounded random measure \( \mu_2 \) on \( \Theta \). Additionally under \([A1]\)_j (iii),
\[
\int_{I \times \Theta} \mu_1(dt)\mu_2(d\theta) \left\| \int_{[\hat{T}_0, t]} \partial_\theta^j K^n(t, s, \theta) dX_s^n - \int_{[\hat{T}_0, t]} \partial_\theta^j K^{\infty}(t, s, \theta) dX_s^{\infty} \right\| \to \rho 0 \tag{4.2}
\]
as \( n \to \infty \).

**Proof.** Let \( j = 0 \). To prove \([A1]\)_0 with the subsequence argument, we may assume the convergence in \([A3]\)_0 holds a.s. and we will consider \( \omega \) for which this convergence occurs as well as \( \mu_1(\omega, \cdot) \) and \( \mu_2(\omega, \cdot) \) are bounded. Moreover, we may assume the boundedness of \( K^{\infty} \) thanks to \([A1]\)_0 (ii). Then there is a subset \( D_\omega \) of \( \hat{I} \) such that \( \hat{I} \setminus D_\omega \) is at most countable and that
\[
\rho \left( X_{t, \theta}^{n, \beta}_{[\hat{T}_0, t]}, X_{t, \theta}^{\infty, \beta}_{[\hat{T}_0, t]} \right) \to 0
\]
for every \( t \in D_\omega \). Due to the second condition of \([A3]\)_0,
\[
\int_{[\hat{T}_0, t]} K_s^{\infty}(t, s, \theta) dX_s^n \to \int_{[\hat{T}_0, t]} K_s^{\infty}(t, s, \theta) dX_s^{\infty}
\]
That is, this equality holds for all \((t, \theta) \in I \times \Theta\) on some event \( \Omega_0 \in \mathcal{F} \) with \( P[\Omega_0] = 1 \).
as \( n \to \infty \) for every \((t, \theta) \in D_\omega \times \Theta\). Convergence of \([A3]_0\) also implies the boundedness of \(\{X^{n,\beta}(\hat{I})\}_{n \in \mathbb{N}}\). Then the dominated convergence theorem gives
\[
\int_{I \times \Theta} \mu_1(dt)\mu_2(d\theta) \left| \int_{[T_0,t]} K^\infty(t, s, \theta) dX^n_s - \int_{[T_0,t]} K^\infty(t, s, \theta) dX^\infty_s \right| \to 0
\]
for the \(\omega\). This proves the convergence \((4.1)\), and as a result \((4.2)\). The convergences in the case \(j = 1\) are verified in the same way.

[\textbf{A4}] For each \((\omega, n, \alpha, t, \theta) \in \Omega \times \mathbb{N} \times \mathcal{I} \times I \times \Theta\), \(\lambda^{n,\alpha}(t, \theta) = 0\) if and only if \(\lambda^{n,\alpha}(t, \theta^*) = 0\), and
\[
\sum_{\alpha \in \mathcal{I}} \sup_{t' \in [T_0, t]} \left\{ \lambda^{n,\alpha}(t', \theta)^{-1} 1_{\{\lambda^{n,\alpha}(t', \theta) \neq 0\}} \right\} \leq \dot{\lambda}^n(t) \quad (t \in I, \ n \in \mathbb{N})
\]
for some \(\dot{\lambda}^n \in \mathcal{J}\) for \(n \in \mathbb{N}\) such that the family \(\{\dot{\lambda}^n(T_1)\}_{n \in \mathbb{N}}\) is tight.

\textbf{Remark 1.} For modeling of \(C^\alpha\) and \(C^\beta\) of the limit order book in Section \textbf{2.2}, we may consider \(g^{n,\alpha}(t, \theta)\) proportional to \(A^\alpha\) or \(B^\beta\), or more complicated mechanism. Non-degeneracy of the intensity processes for validating likelihood analysis seems to be problematic due to the shape of the quasi-likelihood, but it causes no difficulty thanks to a positive minimum unit of orders in the limit order book.

Let
\[
\tilde{N}^{n,\alpha}_t = N^{n,\alpha}_t - N^{n,\alpha}_{T_0} - \int_{T_0}^t n \lambda^{n,\alpha}(s, \theta^*) ds \quad (t \in I)
\]
(4.3)

\textbf{Lemma 4.3.} Suppose that Conditions \([A1]_0\) \(i), (ii), [A2]_0\) \((i), (ii), and [A3]_0\) are fulfilled.

(a) The family \(\left\{ |\lambda^n(t, \theta^*)| \right\}_{(n, t) \in \mathbb{N} \times I}\) is tight.

(b) The family \(\left\{ n^{-1}N^{n,\alpha}(T_0, T_1) \right\}_{\alpha \in \mathcal{I}, n \in \mathbb{N}}\) is tight.

(c) The process \((\tilde{N}^{n,\alpha}_t)_{t \in I}\) is a locally square-integrable martingale with
\[
\langle \tilde{N}^{n,\alpha} \rangle_t = n \int_{T_0}^t \lambda^{n,\alpha}(s, \theta^*) ds.
\]

(d) The family \(\left\{ \sup_{t \in I} \left| n^{-1/2} \tilde{N}^{n,\alpha}_t \right| \right\}_{n \in \mathbb{N}}\) is tight.

\textbf{Proof.} By positivity of processes,
\[
\lambda^{n,\alpha}(t, \theta) \leq g^{n,\alpha}(t, \theta) + \sum_{\beta \in \mathcal{I}_0} \sup_{(t, s) \in J} K^{n,\alpha}_{\beta}(t, s, \theta) (X^{n,\beta}_{T_1} - X^{n,\beta}_{T_0})
\]
\[
\leq g^n(T_1) + K^n(T_1) (X^{n,\beta}_{T_1} - X^{n,\beta}_{T_0})
\]

8
for all \( t \in I \) and \( \theta \in \Theta \). Therefore (a) follows. In particular, 
\[
\int_{T_0}^{T_1} \lambda_{n,\alpha}^n(t, \theta^*) dt < \infty \text{ a.s., therefore (4.3) is well defined. Now it is easy to see (a) } \Rightarrow (b) \Rightarrow (c) \Rightarrow (d).
\]

For \( \ell_n \) in (3.1), let 
\[
Y_n(\theta) = 1_n \left[ \ell_n(\theta) - \ell_n(\theta^*) \right] \equiv d \sum_{\alpha=1}^d \left( \int_{T_0}^{T_1} \frac{\log \lambda_{n,\alpha}(t, \theta)}{\lambda_{n,\alpha}(t, \theta^*)} n^{-1} dN_t^{n,\alpha} - \int_{T_0}^{T_1} \left[ \lambda_{n,\alpha}(t, \theta) - \lambda_{n,\alpha}(t, \theta^*) \right] dt \right).
\]

Here it should be noted that the random fields \( \ell_n \) and \( Y_n \) are well defined thanks to \([A4]\). Let 
\[
\lambda_{\infty,\alpha}^n(t, \theta) = g_{\infty,\alpha}^n(t, \theta) + \sum_{\beta \in I_0} \int_{T_0}^{t^-} K_{\beta}^{\infty,\alpha}(t, s, \theta) dX_{s}^{\infty,\beta}
\]
for \( t \in I \) and \( \theta \in \Theta \).

**Lemma 4.4.** Suppose that \([A1]_1, [A2]_1, [A3]_0 \) and \([A4] \) are fulfilled. Then the family 
\[
\left\{ \sup_{\theta \in \Theta} \left| \int_{T_0}^{T_1} \frac{\lambda_{n,\alpha}(t, \theta)}{\lambda_{n,\alpha}(t, \theta^*)} n^{-1/2} d\tilde{N}_t^{n,\alpha} \right| \right\}_{n \in \mathbb{N}}
\]
is tight.

**Proof.** Before starting the proof, we note that the stochastic integrals of the statement are continuous in \( \theta \) under the assumptions, and hence the supremum is measurable.

Let 
\[
M_t^n(\theta) = \int_{T_0}^{t} \xi_{n,\alpha}^n(s, \theta) n^{-1/2} d\tilde{N}_s^{n,\alpha} \quad (t \in I)
\]
with 
\[
\xi_{n,\alpha}^n(s, \theta) = 1_{\{\lambda_{n,\alpha}(s, \theta^*) \neq 0\}} \log \frac{\lambda_{n,\alpha}(s, \theta)}{\lambda_{n,\alpha}(s, \theta^*)}
\]

Let \( A > 0 \). Define stopping times \( \tau_A^n \) depending on \( A > 0 \) by 
\[
\tau_A^n = \inf \{ t \geq T_0; \hat{K}^n(t) + \hat{\lambda}^n(t) + \hat{\theta}^n(t) + 1_{\{r^n > 0\}}|X^n_{t^-} - X^n_{T_0}| > A \} \wedge T_1.
\]

Then for an event \( \Omega_0 \in \mathcal{F} \) with \( P(\Omega_0) = 1 \) and a non-decreasing function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \), it holds that 
\[
\sup_{\omega \in \Omega_0} \sup_{n \in \mathbb{N}} \sup_{t \in [T_0, \tau_A^n]} \sup_{\theta \in \Theta} \left( 1_{\{r_A^n > 0\}} \lambda_{n,\alpha}^n(t, \theta) + 1_{\{r_A^n > 0\}}|\xi_{n,\alpha}^n(t, \theta)| \right) \leq f(A).
\]
Here the left-continuity of the dominating functions worked.
By the Burkholder-Davis-Gundy inequality, for $k > 1/2$,
\[
E \left[ \left| \int_{T_0}^{t \wedge \tau^n_A} (\partial_\theta)^j \xi^{n,\alpha}(s, \theta) n^{-1/2} d\tilde{N}_s^{n,\alpha} \right|^{2k} \right] 
\leq C_k E \left[ n^{-1} \int_{T_0}^{t \wedge \tau^n_A} (\partial_\theta)^j \xi^{n,\alpha}(s, \theta)^2 dN_s^{n,\alpha} \right]^k
\]
\[
\leq 2^{k-1} C_k n^{-k/2} E \left[ \int_{T_0}^{t \wedge \tau^n_A} (\partial_\theta)^j \xi^{n,\alpha}(s, \theta)^2 n^{-1/2} d\tilde{N}_s^{n,\alpha} \right]^k
+ 2^{k-1} C_k E \left[ \int_{T_0}^{t \wedge \tau^n_A} (\partial_\theta)^j \xi^{n,\alpha}(s, \theta) \lambda^{n,\alpha}(s, \theta^*) ds \right]^k. \tag{4.6}
\]
Repeatedly using (4.6) starting with $k = 2^m$ for $m \in \mathbb{N}$ and $E[(\int \cdots n^{-1/2} d\tilde{N}_s^{n,\alpha})^2] = E[\int \cdots 2 \lambda^{n,\alpha}(s, \theta^*) ds]$ at last, we obtain
\[
\sup_{(n, t, \theta) \in \mathbb{N} \times I \times \Theta} E \left[ \int_{T_0}^{t \wedge \tau^n_A} (\partial_\theta)^j \xi^{n,\alpha}(s, \theta)n^{-1/2} d\tilde{N}_s^{n,\alpha} \right]^p < \infty \tag{4.7}
\]
for every $p > 1$ and $j = 0, 1$.

Applying Sobolev’s inequality to $\Theta$ and using (4.7), we have
\[
\sup_{(n, t) \in \mathbb{N} \times I} E \left[ \sup_{\theta \in \Theta} |M^{n}_{t \wedge \tau^n_A}(\theta)|^p \right] \leq \sup_{(n, t) \in \mathbb{N} \times I} C_p(\Theta) E \left[ \sum_{j=0}^1 \int_{\Theta} (\partial_\theta)^j M^{n}_{t \wedge \tau^n_A}(\theta))^p d\theta \right]
\leq \sup_{(n, t) \in \mathbb{N} \times I} C_p(\Theta) \sum_{j=0}^1 \int_{\Theta} d\theta E \left[ (\partial_\theta)^j M^{n}_{t \wedge \tau^n_A}(\theta))^p \right]
< \infty \tag{4.8}
\]
for $p > p$.

It follows from the tightness of $\{\tilde{K}^{n}(T_1), \ldots, \tilde{g}^{n}(T_1), |X^n_{T_1} - X^n_{T_0}| \}_{n \in \mathbb{N}}$ that for any $\epsilon > 0$, there exists $A > 0$ such that $\sup_{n \in \mathbb{N}} P[\tau^n_A < T_1] < \epsilon$. This with Inequality (4.8) proves the result. \hfill \square

Now
\[
\mathbb{V}_n(\theta) = \sum_{\alpha=1}^d \left( \int_{T_0}^{T_1} \log \frac{\lambda^{n,\alpha}(t, \theta)}{\lambda^{n,\alpha}(t, \theta^*)} n^{-1} d\tilde{N}_t^{n,\alpha} 
- \int_{T_0}^{T_1} \left[ \lambda^{n,\alpha}(t, \theta) - \lambda^{n,\alpha}(t, \theta^*) - \log \frac{\lambda^{n,\alpha}(t, \theta)}{\lambda^{n,\alpha}(t, \theta^*)} \lambda^{n,\alpha}(t, \theta^*) \right] dt \right).
\]
Let
\[
\mathbb{V}(\theta) = - \sum_{\alpha=1}^d \int_{T_0}^{T_1} \left[ \lambda^{\infty,\alpha}(t, \theta) - \lambda^{\infty,\alpha}(t, \theta^*) - \log \frac{\lambda^{\infty,\alpha}(t, \theta)}{\lambda^{\infty,\alpha}(t, \theta^*)} \lambda^{\infty,\alpha}(t, \theta^*) \right] dt \tag{4.9}
\]
Lemma 4.5. Under \([A1]_1, [A2]_1, [A3]_0\) and \([A4]\), Then \(Y\) has a continuous extension to \(\bar{\Theta}\) and
\[
\sup_{\theta \in \bar{\Theta}} |Y_n(\theta) - Y(\theta)| \to^p 0.
\]

Proof. We shall use the stopping times \(\tau^n_A\) given in (4.5) for \(A > 0\). Let
\[
\bar{Y}_n(\theta) = -\sum_{\alpha=1}^{d} \int_{T_0}^{T_1} \left[ \lambda^{n,\alpha}(t, \theta) - \lambda^{n,\alpha}(t, \theta^*) - \log \frac{\lambda^{n,\alpha}(t, \theta)}{\lambda^{n,\alpha}(t, \theta^*)} \right] dt.
\]
Then, on the event \(\{\tau^n_A = T_1\}\),
\[
|\bar{Y}_n(\theta) - Y(\theta)| \leq C_A \left( 1 + |X^n_{T_1} - X^n_{T_0}| + |X^\infty_{T_1} - X^\infty_{T_0}| \right)
\times \sum_{\alpha=1}^{d} \left\{ \int_{T_0}^{T_1} |\lambda^{n,\alpha}(t, \theta) - \lambda^{\infty,\alpha}(t, \theta)| dt + \int_{T_0}^{T_1} |\lambda^{n,\alpha}(t, \theta^*) - \lambda^{\infty,\alpha}(t, \theta^*)| dt \right\},
\]
where \(C_A\) is a constant depending on \(A\). Therefore, due to Lemma 4.2, we have
\[
\bar{Y}_n(\theta) \to^p Y(\theta) \quad (4.10)
\]
as \(n \to \infty\) for each \(\theta \in \Theta\). Since
\[
\sup_{\theta \in \bar{\Theta}} |\partial_\theta \bar{Y}_n(\theta)| \leq d |I| \left( 1 + \tilde{\lambda}^{n}(T_1) \right) \left\{ 1 + \tilde{g}^{n}(T_1) + \tilde{K}^{n}(T_1)(X^n_{T_1} - X^n_{T_0}) \right\}^2
\]
and the family of random variables on the right-hand is tight, the family \(\{P_{\bar{Y}_n}\}_{n \in \mathbb{N}}\) is tight as the family of distributions on \(C(\bar{\Theta})\) due to the existence of continuous extension of \(\bar{Y}_n\) to \(\bar{\Theta}\). In particular, \(Y\) is well defined as a continuous function on \(\bar{\Theta}\) and (4.10) holds for all \(\theta \in \bar{\Theta}\). Thus we can conclude according to the standard argument,
\[
\sup_{\theta \in \bar{\Theta}} |\bar{Y}_n(\theta) - Y(\theta)| \to^p 0
\]
as \(n \to \infty\), which gives the result if combined with Lemma 4.4. \(\square\)

We assume
\([A5]\) For every \(\epsilon > 0\), \(\inf_{\theta \in \Theta} Y(\theta) < 0 \text{ a.s.} \quad (\theta^*(\theta) > \epsilon)\)

The following theorem gives consistency of the approximate maximum likelihood estimator.

Theorem 4.6. Suppose that \([A1]_1, [A2]_1, [A3]_0, [A4]\) and \([A5]\) are satisfied. Then any estimator \(\hat{\theta}_n\) for \(\theta\) satisfying \(n^{-1}\ell_n(\hat{\theta}_n) \geq n^{-1}\ell_n(\theta^*) - o_p(1)\) as \(n \to \infty\) is consistent, that is, \(\hat{\theta}_n \to^p \theta^*\) as \(n \to \infty\).
Remark 4.7. Theorems 4.6 and 4.12 are regarded as generalizations of Theorems 1 and 2 of Chen and Hall [8], respectively. The consistency result given here is asserted for any sequence of quasi maximum likelihood estimator. The limit theorem gives asymptotic mixed normality in a general regression scheme in non-ergodic statistics. Though the treatments are simpler in the classical methods, they are not sufficient to develop advanced themes such as prediction, information criteria and higher-order asymptotic theory. In particular, convergence of the moments of the quasi likelihood estimators or equivalently sharp estimates of tail probability of them is indispensable. The quasi likelihood analysis with the polynomial type large deviation inequalities for statistical random fields will be established in Section 5, the main part of this article. This construction of inferential theory enables us to approach the above mentioned problems. Indeed, the reader can find in [34] and [33] such a flow from non-ergodic statistical inference for volatility to an information criterion for volatility model selection. When the input intensities of the Hawkes type processes are time-varying, the question of non-degeneracy of the statistical model becomes complicated than expected. We will give a sufficient condition for non-degeneracy. Asymptotic properties of the quasi Bayesian estimator will be elucidated as well.

4.2 Asymptotic mixed normality of the QMLE: a classical approach

We shall investigate the asymptotic distribution of the QMLE. In Section 4.2, the parameter space $\Theta$ is assumed only to be open without Condition (3.2) because only local properties are discussed.

Under regularity conditions stated later, we have

$$\frac{1}{n} \partial_\theta \ell_n(\theta^*) = \sum_\alpha \int_{T_0}^{T_1} \lambda_{n,\alpha}(t, \theta^*)^{-1} \partial_\theta \lambda_{n,\alpha}(t, \theta^*) d\tilde{N}^{n,\alpha}_t$$

and

$$n^{-1} \partial^2_\theta \ell_n(\theta) = \sum_\alpha \int_{T_0}^{T_1} \partial^2_\theta \left( \partial_\theta \lambda_{n,\alpha}/\lambda_{n,\alpha} \right)(t, \theta) n^{-1} d\tilde{N}^{n,\alpha}_t$$

$$- \sum_\alpha \int_{T_0}^{T_1} \left( \partial_\theta \lambda_{n,\alpha} \right)^2(t, \theta) \left( \lambda_{n,\alpha}(t, \theta) \right)^{-2} \lambda_{n,\alpha}(t, \theta^*) dt$$

$$- \sum_\alpha \int_{T_0}^{T_1} \partial^2_\theta \lambda_{n,\alpha}(t, \theta) \left( \lambda_{n,\alpha}(t, \theta) \right)^{-1} \left( \lambda_{n,\alpha}(t, \theta) - \lambda_{n,\alpha}(t, \theta^*) \right) dt.$$  

(4.12)

We obtain the following lemma in the same way as the proof of Lemma 4.4, replacing $\xi^{n,\alpha}(s, \theta)$ by $\partial_\theta \left( \partial_\theta \lambda_{n,\alpha}/\lambda_{n,\alpha} \right)(s, \theta)$ in this case.

Lemma 4.8. Suppose that $[A1]_3$, $[A2]_3$, $[A3]_1$ and $[A4]$ are fulfilled. Then for any ball $V$ centered at $\theta^*$ in $\Theta$, the family

$$\left\{ \sup_{\theta \in V} \left| \int_{T_0}^{T_1} \partial_\theta \left( \partial_\theta \lambda_{n,\alpha}/\lambda_{n,\alpha} \right)(t, \theta) n^{-1/2} d\tilde{N}^{n,\alpha}_t \right| \right\}_{n \in \mathbb{N}}$$

is tight.
Let
\[ \Gamma = \sum_{\alpha \in I} \int_{T_0}^{T_1} (\partial_{\theta} \lambda^{n,\alpha})^2 \lambda^{\infty,\alpha}^{-1}(t, \theta^*) dt. \]  

(4.13)

**Lemma 4.9.** Suppose that \([A1]_2, [A2]_2, [A3]_1\) and \([A4]\) are fulfilled. For any sequence \((V_n)_{n \in \mathbb{N}}\) of neighborhoods of \(\theta^*\) that is shrinking to \(\{\theta^*\}\),
\[ \sup_{\theta \in V_n} \left| n^{-1} \partial_{\theta}^2 \ell_n(\theta) + \Gamma \right| \to^p 0 \]
as \(n \to \infty\).

**Proof.** Since \(g^{\infty}(t, \theta)\) and \(K^{\infty}(t, s, \theta)\) are continuous in \(\theta\), and bounded a.s., so is \(\lambda^{\infty}(t, \theta)\), that is defined by \([4.4]\). We use the stopping times \(\tau^n_A\) given in \([4.3]\) for \(A > 0\). On the event \(\{\tau^n_A = T_1\}\),
\[ \left| n^{-1} \partial_{\theta}^2 \ell_n(\theta) - \Gamma \right| \]
\[ \leq \sum_{\alpha} \left| \int_{T_0}^{T_1} \partial_{\theta} \left( \partial_{\theta} \lambda^{n,\alpha}/\lambda^{n,\alpha} \right)(t, \theta) n^{-1} d\hat{N}^{n,\alpha}_t \right| \]
\[ + \sum_{\alpha} \left| \int_{T_0}^{T_1} \left( \partial_{\theta} \lambda^{\infty,\alpha} \right)^2 (t, \theta) \left( \lambda^{\infty,\alpha}(t, \theta) \right)^{-2} \lambda^{n,\alpha}(t, \theta^*) dt \right| \]
\[ - \int_{T_0}^{T_1} \left( \partial_{\theta} \lambda^{\infty,\alpha} \right)^2 (t, \theta^*) \left( \lambda^{\infty,\alpha}(t, \theta^*) \right)^{-2} \lambda^{\infty,\alpha}(t, \theta^*) dt \right| \]
\[ + \sum_{\alpha} \left| \int_{T_0}^{T_1} \partial_{\theta}^2 \lambda^{n,\alpha}(t, \theta) \left( \lambda^{n,\alpha}(t, \theta) \right)^{-1} \lambda^{n,\alpha}(t, \theta^*) dt \right| \]
\[ \leq \sum_{\alpha} \left| \int_{T_0}^{T_1} \partial_{\theta} \left( \partial_{\theta} \lambda^{n,\alpha}/\lambda^{n,\alpha} \right)(t, \theta) n^{-1} d\hat{N}^{n,\alpha}_t \right| \]
\[ + C \left( 1 + \hat{\lambda}^{n}(T_1) \right)^3 \left( 1 + \hat{g}^n(T_1) + \hat{K}^n(T_1) \right) \left| X^n(T_1) - X^n(\hat{T}_0) \right|^3 \]
\[ \times \left\{ \sum_{j=0}^{1} \int_{T_0}^{T_1} \left| \partial_{\theta} \lambda^{n,\alpha}(t, \theta) - \partial_{\theta} \lambda^{\infty,\alpha}(t, \theta) \right| dt + \int_{T_0}^{T_1} \left| \lambda^{n,\alpha}(t, \theta^*) - \lambda^{\infty,\alpha}(t, \theta^*) \right| dt \right\} \]
\[ + \sum_{j=0}^{1} \int_{T_0}^{T_1} \left| \partial_{\theta} \lambda^{\infty,\alpha}(t, \theta) - \partial_{\theta} \lambda^{\infty,\alpha}(t, \theta^*) \right| dt \]

(4.14)

Under \([A1]_1, [A2]_1\) and \([A3]_1\), Lemma \([4.2]\) gives
\[ \int_{T_0}^{T_1} \left| \partial_{\theta} \lambda^{n,\alpha}(t, \theta) - \partial_{\theta} \lambda^{\infty,\alpha}(t, \theta) \right| dt \to^p 0 \]
as \( n \to \infty \) for each \( \theta \in \Theta \) for \( j = 0, 1 \). Then with the the tightness of \( \{ \sup_{t,\theta} |\partial^j_\theta \lambda^n(t, \theta)| \}_{n \in \mathbb{N}} \) for \( j = 1, 2 \), deduced from \([A1]_2\) and \([A2]_2\), we obtain the uniform convergence

\[
\sup_{\theta \in V_n} \frac{1}{n} \sum_{j=0}^{\infty} \int_{T_0}^{T_1} |\partial^j_\theta \lambda^n(t, \theta) - \partial^j_\theta \lambda^\infty(t, \theta)| \, dt \to^p 0 \tag{4.15}
\]

Moreover,

\[
\lim_{\theta \to \theta^*} \int_{T_0}^{T_1} |\partial^j_\theta \lambda^\infty(t, \theta) - \partial^j_\theta \lambda^\infty(t, \theta^*)| \, dt = 0 \quad \text{a.s.} \quad (j = 0, 1) \tag{4.16}
\]

by the regularity of \( \lambda^\infty \). Indeed, this property follows from (4.15) and tightness of \( \{ \sup_{t,\theta} |\partial^j_\theta \lambda^n(t, \theta)| \}_{n \in \mathbb{N}} \) for \( j = 1, 2 \). Then Lemma 4.8, (4.14), (4.15) and (4.16) yield the lemma with an argument with localization.

We shall recall a mixed normal limit theorem. Given a stochastic basis \((\Omega, \mathcal{F}, \mathbf{F}, P)\) with \( \mathbf{F} = (\mathcal{F}_t)_{t \in [T_0, T_1]} \), we suppose that \( \mu^{n,\alpha} (\alpha = 1, \ldots, d) \) are integer-valued random measures on \( E_\alpha = \mathbb{R}^d_E \{ 0 \} \) with compensators \( \nu^{n,\alpha} \) respectively. Let \( c^n_\alpha : \Omega \times \mathbb{R}^d \to \mathbb{R}^d \) be predictable processes. Let

\[
L^n_t = \sum_\alpha \int_{T_0}^{t} \int_{E_\alpha} c^n_\alpha(t, x) \tilde{\mu}^{n,\alpha}(dt, dx), \tag{4.17}
\]

where \( \tilde{\mu}^{n,\alpha} = \mu^{n,\alpha} - \nu^{n,\alpha} \).

**Lemma 4.10.** Suppose that the following conditions are fulfilled.

(i) For each \( n \in \mathbb{N} \), any two of \( \mu^{n,\alpha} \) do not have common jump times and \( \nu^{n,\alpha}(\{s\}, E_\alpha) = 0 \) for all \( s \in [T_0, T_1] \).

(ii) There exists an \( \mathbb{R}^d \otimes \mathbb{R}^r \)-valued \( \mathbf{F} \)-predictable process \( g = (g_t) \) such that \( \int_{T_0}^{T_1} |g_s|^2 \, ds < \infty \) a.s. and that

\[
\sum_\alpha \int_{T_0}^{t} \int_{E_\alpha} c^n_\alpha(s, x)^{\otimes 2} \nu^{n,\alpha}(ds, dx) \to^p \int_{T_0}^{t} g_s^{\otimes 2} ds
\]

as \( n \to \infty \) for every \( t \in [T_0, T_1] \).

(iii) For every \( \epsilon > 0 \),

\[
\int_{T_0}^{T_1} \int_{\{|c^n_\alpha(s, x)| > \epsilon\}} |c^n_\alpha(s, x)|^2 \nu^{n,\alpha}(ds, dx) \to^p 0
\]

as \( n \to \infty \).

Then \( L^n \to^d \int_{T_0}^{T_1} g_s dW_s \) in \( \mathcal{D}([T_0, T_1]; \mathbb{R}^d) \), where \( W \) is an \( r \)-dimensional standard Wiener process (defined on an extension of \((\Omega, \mathcal{F}, P)) \) independent of \( g \).

We leave a sketch of proof for reader’s convenience in Appendix.
Lemma 4.11. $n^{-1/2} \partial_\theta \ell_n(\theta^*) \to^{d_s} \Gamma^{1/2} \zeta$ as $n \to \infty$, where $\zeta$ is a $p$-dimensional standard Gaussian random vector defined on an extended probability space of $(\Omega, \mathcal{F}, P)$ and independent of $\mathcal{F}$, and $d_s$ denotes the $\mathcal{F}$-stable convergence.

Proof. Let $\mu^{n,\alpha}(dt, dx) = N^{n,\alpha}(dt) \times \delta_1(dx)$, $\nu^{n,\alpha}(dt, dx) = n\lambda^{n,\alpha}(t, \theta^*) dt \delta_1(dx)$ and

$$c_\alpha^n(t, x) = n^{-1/2} \lambda^{n,\alpha}(t, \theta^*)^{-1} \partial_\theta \lambda^{n,\alpha}(t, \theta^*).$$

Then $L_t^n$ in (4.17) has an expression

$$L_t^n = n^{-1/2} \sum_{\alpha} \int_{t \in \mathcal{I}_0} \lambda^{n,\alpha}(t, \theta^*)^{-1} \partial_\theta \lambda^{n,\alpha}(t, \theta^*) d\tilde{N}_t^{n,\alpha}$$

for $t \in \tilde{I}$. Following the argument in the proof of Lemma 4.9 we see the convergence in (ii) of Lemma 4.10 holds for

$$\int_{I_0}^t g_s \otimes^2 ds = \int_{I_0}^t (\partial_\theta \lambda^{\infty,\alpha}) \otimes^2(s, \theta^*) (\lambda^{\infty,\alpha}(s, \theta^*))^{-1} ds.$$

Tightness of the family

$$\left\{ \frac{1}{n} \sup_{j=0}^{n} \sup_{t \in [I_0, I_1]} |\partial_\theta \lambda^{n,\alpha}(t, \theta^*)|, \sup_{t \in [I_0, I_1]} \lambda(t, \theta^*)^{-1} 1_{\lambda^{n,\alpha}(t, \theta^*) \neq 0}; \alpha \in \mathcal{I}, n \in \mathbb{N} \right\}$$

verifies (iii) of Lemma 4.10 via the Lyapunov condition. Thus, by Lemma 4.10 we obtained the stable convergence of $n^{-1/2} \partial_\theta \ell_n(\theta^*)$. \hfill \Box

Theorem 4.12. Suppose that $[A1]_3$, $[A2]_3$, $[A3]_1$ and $[A4]$ are fulfilled. Suppose that the estimator $\hat{\theta}_n$ of $\theta$ satisfies $\hat{\theta}_n \to^p \theta^*$ and $\partial_\theta \ell_n(\hat{\theta}_n) = o_p(n^{1/2})$ as $n \to \infty$. Then

$$\sqrt{n}(\hat{\theta}_n - \theta^*) \to^{d_s} \Gamma^{-\frac{1}{2}} \zeta$$

as $n \to \infty$, where $\zeta$ is a $p$-dimensional standard Gaussian random vector given in Lemma 4.11.

Proof. It is easy to obtain the result from Lemmas 4.9 and 4.11. Indeed, there is a sequence $\{V_n\}_{n \in \mathbb{N}}$ of open balls centered at $\theta^*$ such that the diameter of $V_n$ tends to 0 as $n \to \infty$ and $P[\hat{\theta}_n \in V_n] \to 1$. On the event $\{\hat{\theta}_n \in V_n\}$, one has

$$n^{-1/2} \partial_\theta \ell_n(\hat{\theta}_n)[u] - n^{-1/2} \partial_\theta \ell_n(\theta^*)[u] = n^{-1} \int_0^1 \partial_\theta^2 \ell_n(\theta_n(s)) ds \left[ n^{1/2}(\hat{\theta}_n - \theta^*), u \right]$$

for $u \in \mathbb{R}^p$, where $\theta_n(s) = \theta^* + s(\hat{\theta}_n - \theta^*)$. Then the result is easily obtained. \hfill \Box
5 The quasi likelihood analysis: QMLE and QBE

In Theorem 4.12, we obtained a limit theorem for the quasi maximum likelihood estimator $\hat{\theta}_n$. It is the first step of analysis of the estimator, however, more precise estimates for the tail of the distribution of the estimator will be indispensable to develop basic theory of statistical inference such as asymptotic decision theory, prediction, higher-order efficiency, information criteria, etc. This section presents the so-called quasi likelihood analysis, that gives certain tail probability estimates of the quasi maximum likelihood estimator (QMLE) and the quasi Bayesian estimator (QBE).

5.1 Polynomial type large deviation inequality for the quasi likelihood random field

We shall work with the statistical random field

$$H_n(\theta) = \ell_n(\theta)$$

on $\Theta$ and apply the frame of the quasi likelihood analysis in [36]. The random fields $Z_n$ is defined on $U_n = \{u \in \mathbb{R}^p; \theta_u \in \Theta\}$, $\theta_u = \theta^* + n^{-1/2}u$, by

$$Z_n(u) = \exp \left( \sum_{\alpha=1}^{d} \int_{T_0}^{T_1} \log \frac{\lambda^{n,\alpha}(t, \theta_u)}{\lambda^{n,\alpha}(t, \theta^*)} dN^{n,\alpha}_t \right) \cdot \left( -\sum_{\alpha=1}^{d} \int_{T_0}^{T_1} n[\lambda^{n,\alpha}(t, \theta_u) - \lambda^{n,\alpha}(t, \theta^*)] dt \right).$$

Under necessary regularity conditions specified later, we define a random vector $\Delta_n$ by

$$\Delta_n[u] = \frac{1}{\sqrt{n}} \sum_{\alpha=1}^{d} \int_{T_0}^{T_1} \lambda^{n,\alpha}(t, \theta_u)^{-1} \partial_{\theta} \lambda^{n,\alpha}(t, \theta_u)[u] d\tilde{N}^{n,\alpha}_t$$

and a random matrix $\Gamma_n(\theta)$ by

$$\Gamma_n(\theta)[u \otimes 2] = -\partial_{\theta}^{2} H_n(\theta) [(n^{-1/2}u) \otimes 2]$$

for $u \in \mathbb{R}^p$.

Let

$$r_n^{(1)}(u) = \sum_{\alpha=1}^{d} \int_{T_0}^{T_1} \left\{ \log \frac{\lambda^{n,\alpha}(t, \theta_u)}{\lambda^{n,\alpha}(t, \theta^*)} - \lambda^{n,\alpha}(t, \theta_u)^{-1} \partial_{\theta} \lambda^{n,\alpha}(t, \theta_u)[n^{-1/2}u] \right\} d\tilde{N}^{n,\alpha}_t,$$

and

$$r_n^{(2)}(u) = -\sum_{i=1}^{3} r_n^{(2i)}(u),$$

16
where

\begin{align*}
\lambda_n^{(21)}(u) &= \sum_{\alpha=1}^{d} \int_{T_0}^{T_1} \left\{ \frac{\lambda_n^{\alpha}(t, \theta_u)}{\lambda_n^{\alpha}(t, \theta^*)} - 1 - \log \frac{\lambda_n^{\alpha}(t, \theta_u)}{\lambda_n^{\alpha}(t, \theta^*)} \right\} n \lambda_n^{\alpha}(t, \theta^*) dt \\
\lambda_n^{(22)}(u) &= \sum_{\alpha=1}^{d} \int_{T_0}^{T_1} \left\{ \left( \lambda_n^{\alpha}(t, \theta_u) - \lambda_n^{\alpha}(t, \theta^*) \right)^2 \right. \\
&\quad \left. - (\partial_{\theta} \lambda_n^{\alpha}(t, \theta^*) [u^{-1/2}])^2 \right\} n \lambda_n^{\alpha}(t, \theta^*)^{-1} dt, \\
\lambda_n^{(23)}(u) &= \sum_{\alpha=1}^{d} \int_{T_0}^{T_1} \left\{ \left( \lambda_n^{\alpha}(t, \theta_u) - \lambda_n^{\alpha}(t, \theta^*) \right) - (\partial_{\theta} \lambda_n^{\alpha}(t, \theta^*) [u])^2 \\
&\quad - \lambda_{1,0}^{\alpha}(t) - (\partial_{\theta} \lambda_n^{\alpha}(t, \theta^*) [u])^2 \right\} dt.
\end{align*}

Let \( \lambda_n(u) = \lambda_n^{(1)}(u) + \lambda_n^{(2)}(u) \). Then we have an expression of \( Z_n \) as

\[
Z_n(u) = \exp \left( \Delta_n[u] - \frac{1}{2} \Gamma([u])^{\otimes 2} + \lambda_n(u) \right) \quad (5.1)
\]

for \( u \in \mathbb{U}_n \), which suggests the LAMN property of the random field \( \mathbb{H}_n \).

Assume the condition (3.2) for \( \Theta \). Moreover we suppose that the function \( \Theta \ni \theta \mapsto \lambda_n(t, \theta) \) has continuous extension to \( \overline{\Theta} \) when the QMLE is discussed.

Let \( \varepsilon \) be a positive number less than 1/2. Let \( \mathbb{N} = \mathbb{N} \cup \{ \infty \} \). For mathematical validation of the asymptotic properties deduced below, we shall assume the following conditions.

\[ [B1] \] For each \( n \in \mathbb{N} \), \( K_n(t, s, \theta) \) is an \( \mathbb{R}^d_+ \otimes \mathbb{R}^d_+ \)-valued \( \mathcal{F} \times \mathcal{B}(J) \times \mathcal{B}(\Theta) \)-measurable function satisfying the following conditions.

(i) For each \( (n, t, \theta) \in \mathbb{N} \times I \times \Theta \), the process \( [T_0, t] \ni s \mapsto K_n(t, s, \theta) \) is \( (\mathcal{F}_s)_{s \in [T_0, t]} \)-optional.

(ii) For each \( (n, t, s) \in \mathbb{N} \times J \), the mapping \( \Theta \ni \theta \mapsto K_n(t, s, \theta) \) is \( j \) times differentiable a.s., \( \sup_{(s, \theta) \in [T_0, t] \times \Theta} |\partial_{\theta}^j K_n(t, s, \theta)| < \infty \) a.s. for \( t \in I \), and

\[
\sum_{j=0}^{j} \sup_{(n, s, t) \in \mathbb{N} \times J} \sup_{\theta \in \Theta} ||\partial_{\theta}^j K_n(t, s, \theta)||_p < \infty
\]

for every \( p > 1 \).
(iii) For each \((n, t) \in \mathbb{N} \times I\), the mappings \([\hat{T}_0, t) \ni s \mapsto \partial_s^n K^n(t, s, \theta^*)\) \((i = 0, 1)\) are differentialble a.s., 
\[
\sup_{(n, t) \in \mathbb{N} \times I} \sum_{i=0}^{1} \int_{\hat{T}_0}^t \|\partial_s^n \partial_s^i K^n(t, s, \theta^*)\|_p ds < \infty
\]
for every \(p > 1\).

(iv) For every \(p > 1\),
\[
n^\varepsilon \sum_{j=0}^1 \sup_{(t, \theta) \in I \times \Theta} \left\| \sup_{s \in [\hat{T}_0, t)} \|\partial_s^j K^n(t, s, \theta) - \partial_s^j K^n(t, s, \theta^*)\|_p \right\| \to 0
\]
as \(n \to \infty\).

[B2] For each \((\alpha, n) \in \mathcal{I} \times \tilde{\mathbb{N}}, g^{n, \alpha}(t, \theta)\) is a nonnegative \(\mathcal{F} \times \mathcal{B}(I) \times \mathcal{B}(\Theta)\)-measurable function for which the following conditions are fulfilled.

(i) For each \((n, \alpha, \theta) \in \mathbb{N} \times \mathcal{I} \times \Theta\), the process \((g^{n, \alpha}(t, \theta))_{t \in I}\) is predictable.

(ii) For each \((n, t) \in \mathbb{N} \times I\), the mapping \(\Theta \ni \theta \mapsto g^n(t, \theta)\) is \(j\) times differentiable a.s. and
\[
\sum_{j=0}^j \sup_{(n, t) \in \mathbb{N} \times I} \sup_{\theta \in \Theta} \left\| (\partial_\theta)^j g^n(t, \theta) \right\|_p < \infty
\]
for every \(p > 1\).

(iii) For each \(p > 1\),
\[
n^\varepsilon \sum_{j=0}^1 \sup_{t \in I} \sup_{\theta \in \Theta} \left\| (\partial_\theta)^j g^n(t, \theta) - (\partial_\theta)^j g^n(t, \theta^*) \right\|_p \to 0
\]
as \(n \to \infty\).

[B3] For each \((\beta, n) \in \mathcal{I}_0 \times \tilde{\mathbb{N}}, (X_{t}^{n, \beta})_{t \in \hat{I}}\) is a non-decreasing \((\mathcal{F}_t)_{t \in \hat{I}}\)-adapted process such that
\[
\sup_{(n, t) \in \mathbb{N} \times I} \left\| X_{t}^{n, \beta} \right\|_p < \infty \quad \text{and} \quad n^\varepsilon \sup_{t \in \hat{I}} \left\| X_{t}^{n, \beta} - X_{t}^{\infty, \beta} \right\|_p \to 0
\]
as \(n \to \infty\), for every \(p > 1\). \((\hat{I} = [\hat{T}_0, T_1])\).

[B4] For each \((\omega, n, \alpha, t, \theta) \in \Omega \times \mathbb{N} \times \mathcal{I} \times \Theta\), \(\lambda^{n, \alpha}(t, \theta) = 0\) if and only if \(\lambda^{n, \alpha}(t, \theta^*) = 0\), and
\[
\sup_{(n, t, \theta) \in I \times \Theta} \left\| \lambda^{n, \alpha}(t, \theta) \right\|_p < \infty
\]
for every \(p > 1\) and \(\alpha \in \mathcal{I}\).
Define the index \( \chi_0 \) by
\[
\chi_0 = \inf_{\theta \in \Theta \setminus \{\theta^*\}} -\frac{Y(\theta)}{|\theta - \theta^*|^2},
\]
where \( Y \) is given in (4.9). The nondegeneracy of the key index \( \chi_0 \) will play an essential role in our argument.

[B5] For every \( L > 0 \), there exists a constant \( C_L \) such that
\[
P[\chi_0 < r^{-1}] \leq \frac{C_L}{r^L}
\]
for all \( r > 0 \).

Lemma 5.1. Suppose that Conditions [B1], [B2], [B3] and [B4] are satisfied. Then
\[
\sup_{n \in \mathbb{N}} \left\| n^{-1} \sup_{\theta \in \Theta} \left| \frac{\partial^3 \mathbb{H}_{n}(\theta)}{\partial \theta^3} \right| \right\|_p < \infty
\]
for every \( p > 1 \).

Proof. We have a representation
\[
\partial^j \lambda^\text{n}(t, \theta) = \partial^j \gamma^\text{n}(t, \theta) + \int_{T_0}^t \partial^j K^\text{n}(t, s, \theta)dX_s^\text{n}.
\]
Like (4.8), use Sobolev’s inequality and the Burkholder-Davis-Gundy inequality to obtain the desired estimate.

Lemma 5.2. For every \( p > 1 \),
\[
\sup_{n \in \mathbb{N}} \left\| n^p |\Gamma_n(\theta^*) - \Gamma| \right\|_p < \infty.
\]

Proof. We have the representation
\[
\partial^j \lambda^\text{n}(t, \theta) - \partial^j \lambda^\infty(t, \theta) = \partial^j \gamma^\text{n}(t, \theta) - \partial^j \gamma^\infty(t, \theta)
\]
\[
+ \int_{T_0}^t (\partial^j K^\text{n}(t, s, \theta) - \partial^j K^\infty(t, s, \theta))dX_s^\infty
\]
\[
+ \partial^j K^\text{n}(t, \hat{T}_0, \theta) (X_{\hat{T}_0}^n - X_{\hat{T}_0}^\infty) - \partial^j K^\text{n}(t, \hat{T}_0, \theta) (X_{\hat{T}_0}^n - X_{\hat{T}_0}^\infty)
\]
\[
- \int_{T_0}^t (\partial_s \partial^j K^\text{n})(t, s, \theta^*) (X_s^n - X_s^\infty) ds \quad (5.2)
\]
for \( j = 0, 1 \) and \( \theta \in \Theta \). Then it is possible to obtain the desired estimate by using (4.14), (5.2) evaluated at \( \theta = \theta^* \) and an estimate similar to (4.7).

Obviously we have
Lemma 5.3. For every $p > 1$, $\sup_{n \in \mathbb{N}} \| \Delta_n \|_p < \infty$.

Lemma 5.4. For every $p > 1$,

$$\sup_{n \in \mathbb{N}} \left\| \frac{n^\varepsilon}{\sup_{\theta \in \Theta} |Y_n(\theta) - Y(\theta)|} \right\|_p < \infty.$$ 

Proof. Use (5.2) and Sobolev’s inequality, as well as the Burkholder-Davis-Gundy inequality for uniform estimate of the martingale part.

Proposition 5.5. (Polynomial type large deviation inequality) Suppose that Conditions $[B_1], [B_2], [B_3], [B_4]$ and $[B_5]$ are fulfilled. Then, for every $L > 0$, there exists a constant $C_L$ such that

$$P \left[ \sup_{u \in V_n(r)} Z_n(u) \geq e^{-r} \right] \leq \frac{C_L}{r^L}$$

for all $r > 0$ and all $n \in \mathbb{N}$, where $V_n(r) = \{ u \in U_n; |u| \geq r \}$.

Proof. We will follow the procedure in [36]. The parameters there, here in quotes, will be set as follows: let “$\rho_1 = 2$, “$\beta_1 = \varepsilon$ and “$\beta_2 = \frac{1}{2} - \varepsilon$, next choose “$\rho_2 \in (0, 2\varepsilon)$, take “$\alpha \in (0, \rho_2/2)$, finally take “$\rho_1 \in (0, \min\{1, \alpha/(1 - \alpha), 2\varepsilon/(1 - \alpha)\}$}. Then Condition [A4'] of [36] is satisfied. Lemmas 5.1 and 5.2 ensure [A1’] of [36]. Condition [B5] implies Conditions [A2] and [A5] of [36]. Condition [A3] of [36] obviously holds. Condition [A6] of [36] is checked by Lemmas 5.3 and 5.4. Apply Theorem 2 of [36] to obtain the result.

5.2 Limit distribution and moment convergence of QMLE and QB E

A quasi maximum likelihood estimator is any estimator that satisfies

$$H_n(\hat{\theta}_n) = \max_{\theta \in \Theta} H_n(\theta).$$

The quasi Bayesian estimator is defined by

$$\tilde{\theta}_n = \left[ \int_{\Theta} \exp(H_n(\theta)) \varpi(\theta) d\theta \right]^{-1} \int_{\Theta} \theta \exp(H_n(\theta)) \varpi(\theta) d\theta$$

for a prior density $\varpi$ on $\Theta$. We will assume that $\varpi$ is continuous and $0 < \inf_{\theta \in \Theta} \varpi(\theta) \leq \sup_{\theta \in \Theta} \varpi(\theta) < \infty$.

Let

$$Z(u) = \exp \left( \Delta[u] - \frac{1}{2} \Gamma [u^{\otimes 2}] \right)$$

for $u \in \mathbb{R}^p$, where $\Delta = \Gamma^{1/2} \zeta$.

Denote by $C_1(\mathbb{R}^p)$ the set of continuous functions $f : \mathbb{R}^p \to \mathbb{R}$ of at most polynomial growth.

Theorem 5.6. Suppose that Conditions $[B_1], [B_2], [B_3], [B_4]$ and $[B_5]$ are fulfilled. Then
\( \sqrt{n}(\hat{\theta}_n - \theta^*) \to^d \Gamma^{-1/2}\zeta \) as \( n \to \infty \).

\( E[f(\sqrt{n}(\hat{\theta}_n - \theta^*))] \to E[f(\Gamma^{-1/2}\zeta)] \) as \( n \to \infty \) for all \( f \in C_\uparrow(R^p) \).

**Theorem 5.7.** Suppose that Conditions \([B1]_4, [B2]_4, [B3], [B4]_4\) and \([B5]_4\) are fulfilled. Then

(a) \( \sqrt{n}(\hat{\theta}_n - \theta^*) \to^d \Gamma^{-1/2}\zeta \) as \( n \to \infty \).

(b) \( E[f(\sqrt{n}(\hat{\theta}_n - \theta^*))] \to E[f(\Gamma^{-1/2}\zeta)] \) as \( n \to \infty \) for all \( f \in C_\uparrow(R^p) \).

**Proof of Theorems 5.6 and 5.7.** Lemma 4.11 implies the finite-dimensional stable convergence of \( Z_n \to^d f_s Z \) as \( n \to \infty \) [the proof is still valid under the present assumptions]. Since

\[
\sum_{j=0}^3 \sup_{(n,t) \in \mathbb{N} \times I} \| \partial_\theta^j \lambda^n(t, \theta) \|_p < \infty
\]

for any \( p > 1 \), we obtain the tightness of the residual random fields \( \{r_n|K\}_{n \in \mathbb{N}} \) restricted to \( K \), and hence that of \( \{Z_n|K\}_{n \in \mathbb{N}} \) in \( C(K) \) for every compact set \( K \) in \( \mathbb{R}^p \). Consequently, \( Z_n|K \to^d Z|K \) in \( C(K) \) for compacts \( K \). Now Theorem 4 of [36] gives the properties of the QMLE. Moreover, by Lemma 2 of [36],

\[
\sup_{n \in \mathbb{N}} E\left[ \left( \int_{|u| \leq \delta} Z_n(u) \right)^{-1} \right] < \infty.
\]

Then Theorem 8 of [36] provides the asymptotic properties of the QBE. \( \square \)

### 6 Hawkes type processes

In this section, the point process regression model will be applied to a feedback system of point processes. We will consider the explanatory variables \( X^n = n^{-1}N^n \), that is, the the process \( \lambda^n(t, \theta) \) will be

\[
\lambda^n(t, \theta) = g^n(t, \theta) + \int_{t_0}^t K^n(t, s, \theta)n^{-1}dN^n_s
\]

for \( t \in I \). The Hawkes process is a special case of this model. It should be remarked that Hawkes processes are often used to describe ergodic systems in long-run, whereas we will work with non-ergodic processes with finite time horizon and the intensities diverge.

Here a QLA will be formulated according to Section 5. Other formulations are obviously possible under milder assumptions if one applies previous sections. Hereafter, we will consider the case where \( g^n(t, \theta) = g(t, \theta) \) and \( K^n(t, s, \theta) = K(t, s, \theta) \) for simplicity of presentation.

Consider the following conditions.

\([H1]_j K(t, s, \theta) \) is an \( \mathbb{R}^d_+ \otimes \mathbb{R}^d_0 \)-valued \( \mathcal{F} \times \mathcal{B}(J) \times \mathcal{B}(\Theta) \)-measurable function satisfying the following conditions.
(i) For each \((t, \theta) \in I \times \Theta\), the process \([\hat{T}_0, t] \ni s \mapsto K(t, s, \theta)\) is \((\mathcal{F}_s)_{s \in [\hat{T}_0, t]}\)-optional.

(ii) For each \((t, s) \in \mathbb{N} \times J\), the mapping \(\Theta \ni \theta \mapsto K(t, s, \theta)\) is \(j\) times differentiable a.s., and
\[
\sum_{j=0}^{\hat{j}} \sup_{(\omega, s, t, \theta) \in \Omega \times \mathbb{N} \times J \times \Theta} |\partial_\theta^j K(t, s, \theta)| < \infty.
\]

(iii) For each \((t, \theta) \in I \times \Theta\), the mappings \([\hat{T}_0, t] \ni s \mapsto \partial_\theta^i K(t, s, \theta)\) (\(i = 0, 1\)) are differentialble a.s., and
\[
\text{ess.sup}_{\omega \in \Omega} \sup_{t, s, \theta} |\partial_\theta^i K(t, s, \theta)| < \infty \text{ for } i = 0, 1.
\]

\([H2]\) For each \((\alpha) \in I\), \(g^\alpha(t, \theta)\) is an nonnegative \(\mathcal{F} \times \mathbb{B}(I) \times \mathbb{B}(\Theta)\)-measurable function for which the following conditions are fulfilled.

(i) For each \((\alpha, \theta) \in I \times \Theta\), the process \((g^\alpha(t, \theta))_{t \in I}\) is predictable.

(ii) For each \(t \in I\), the mapping \(\Theta \ni \theta \mapsto g(t, \theta)\) is \(\bar{j}\) times differentiable a.s. and
\[
\sum_{j=0}^{\bar{j}} \sup_{t \in I} \sup_{\theta \in \Theta} \|\partial_\theta^j g(t, \theta)\|_p < \infty
\]
for every \(p > 1\).

\([H3]\) For each \((\omega, n, \alpha, t, \theta) \in \Omega \times \mathbb{N} \times I \times I \times \Theta\), \(\lambda^{n, \alpha}(t, \theta) = 0\) if and only if \(\lambda^{n, \alpha}(t, \theta^*) = 0\), and
\[
\sup_{(n, t, \theta) \in I \times I \times \Theta} \|\lambda^{n, \alpha}(t, \theta)^{-1}1_{\{\lambda^{n, \alpha}(t, \theta) \neq 0\}}\|_p < \infty
\]
for every \(p > 1\) and \(\alpha \in I\).

Let
\[
\lambda^\infty(t, \theta^*) = G g(\cdot, \theta^*)(t)
\]
(6.2)

where \(G = \sum_{m=0}^{\infty} K^m(\cdot, \cdot, \theta^*)\) with
\[
K^m(t, s, \theta^*) = \int_{\hat{T}_0}^{t} K(t, t_1, \theta^*) dt_1 \int_{\hat{T}_0}^{t_1} K(t_1, t_2, \theta^*) dt_2 \cdots \int_{\hat{T}_0}^{t_{m-2}} K(t_{m-2}, t_{m-1}, \theta^*) dt_{m-1} K(t_{m-1}, s, \theta^*).
\]

Let
\[
\lambda^\infty(t, \theta) = g(t, \theta) + \int_{\hat{T}_0}^{t} K(t, s, \theta) \lambda^\infty(s, \theta^*) ds.
\]
(6.3)

The representation \([H3]\) at \(\theta = \theta^*\) is compatible with \([H2]\). Define \(Y\) and the index \(\chi_0\) as before for the present \(\lambda^\infty(t, \theta)\).
For every $L > 0$, there exists a constant $C_L$ such that

$$P[\chi_0 < r^{-1}] \leq \frac{C_L}{r^L}$$

for all $r > 0$.

**Theorem 6.1.** Under Conditions $[H1]$, $[H2]$, $[H3]$ and $[H4]$, the same results as Theorems 5.6 and 5.7 hold true.

**Proof.** We need to verify $[B3]$ in the present situation where $X^{n,\beta} = n^{-1}N^{n,\beta}$. We have

$$\lambda_n(t, \theta) = g(t, \theta) + \int_{t_0}^{t-} K(t, s, \theta) n^{-1} dN^n_s$$

$$= g(t, \theta) + \int_{t_0}^{t-} K(t, s, \theta) \lambda_n(s, \theta^*) ds + \int_{t_0}^{t-} K(t, s, \theta) n^{-1} d\tilde{N}^n_s$$

(6.4)

By $C_r$-inequality,

$$E[\lambda_n(t, \theta)^{2k}] \lesssim E[g^n(t, \theta^*)^{2k}] + \| K \|_\infty^{2k} \int_{t_0}^{t} E[\lambda_n(s, \theta^*)^{2k}] ds$$

$$+ E \left[ \left( \int_{t_0}^{t} K(t, s, \theta^*) n^{-1} d\tilde{N}^n_s \right)^{2k} \right]$$

for $k \in \mathbb{N}$. Then by an essentially the same inequality as (4.6) and by induction, we obtain

$$E[\lambda_n(t, \theta^*)^{2k}] \lesssim 1 + E[g^n(t, \theta^*)^{2k}] + \sum_{j \leq k} \int_{t_0}^{t} E[\lambda_n(s, \theta^*)^{2j}] ds$$

$$\lesssim 1 + E[g^n(t, \theta^*)^{2k}] + \int_{t_0}^{t} E[\lambda_n(s, \theta^*)^{2k}] ds,$$

and hence by Gronwall’s lemma,

$$\sup_{t \in I} E[\lambda_n(t, \theta^*)^p] < \infty$$

(6.5)

for every $p > 1$. Once again by the uniform version of the scheme of (4.6), with the aid of (6.5), we obtain

$$\sup_{t \in I} E \left[ \left\| \int_{t_0}^{t} n^{-1/2} d\tilde{N}^n_s \right\|^p \right] < \infty$$

(6.6)

for every $p > 1$.

Now Equation (6.4) for $\theta = \theta^*$ gives

$$\lambda_n(t, \theta^*) = G\left(g(\cdot, \theta^*) + \int_{t_0}^{t-} K(\cdot, s, \theta^*) n^{-1} d\tilde{N}^n_s\right)(t).$$

(6.7)
Indeed, the convergence (6.7) holds in $\| \cdot \|_p$-norm uniformly in $t \in I$ due to [H2] and (6.6). The limit should be

$$dX_t^\infty/dt = \lambda^\infty(t, \theta^*)$$

having a representation

$$\lambda^\infty(t, \theta^*) = Gg(\cdot, \theta^*)(t)$$

(6.8)
deduced from (6.7). Comparing (6.7) and (6.8), we obtain

$$n^{1/4} \sup_{t \in I} \| \lambda^n(t, \theta^*) - \lambda^\infty(t, \theta^*) \|_p \to 0 \quad (6.9)$$

from (6.6). Since $X^n = n^{-1}N$ and

$$n^{1/4} \| X_t^n - X_t^\infty \|_p \leq n^{1/4} \| n^{-1}N_t^n \|_p + n^{1/4} \left\| \int_{T_0}^{t^-} (\lambda^n(s, \theta^*) - \lambda^\infty(s, \theta^*)) ds \right\|_p,$$

we have $n^{1/4} \| X_t^n - X_t^\infty \|_p \to 0$, which gives [B3].

In what follows, we shall discuss a two-dimensional Hawkes type process $N = (N_t)_{t \in [0, T]}$ as an illustrative example. Consider the parametric model of two-dimensional Hawkes process with intensity processes

$$\lambda^n(t, \theta) = g(t, \gamma) + \int_{T_0}^{t^-} e^{-b(t-s)} A_n^{-1} dN^n_s$$

(6.10)

with $\theta = (\gamma, b, A)$. It is remarked that in practice, we need $n\lambda^n(t, \theta)$ to make the function $\ell_n(\theta)$, and what is estimated is $ng$, not $g$, for the underlying intensity parameter. The asymptotics about the estimator of the parameter in $g$ is relative in the sense that its value can depends on the value of $n$ the user chooses. However it is rather natural because the baseline intensity is very changeable even interday and possibly randomly changing. There consistent estimation of the baseline intensity has no important meaning. We are rather interested in finding relations between $N^n$ and $X^n$, and then $g$ serves as a nuisance parameter. In statistical theory, similar treatments of scaling are found in change point problems and in volatility parameter estimation for small diffusions.

At the true values $\theta^* = (b^*, \gamma^*, A^*)$ of the parameters, it has two-dimensional intensity process expressed by

$$\lambda^n(t, \theta^*) = g_t^* + \int_{T_0}^{t^-} A^* e^{-b^*(t-s)} n^{-1} dN^n_s,$$

where $g_t^* = g(t, \gamma^*)$ is an $\mathbb{R}^2_+$-valued $C^1$-function, $A^* \in M_2(\mathbb{R}) = \mathbb{R}^2 \otimes \mathbb{R}^2$ and $b^* \in (0, \infty)$. Let $C^* = A^* - b^* I$ for two-by-two identity matrix $I$. For a matrix $M$, let

$$G(M)_t = e^{tM} \int_0^t e^{-sM} g_s^* ds.$$
Lemma 6.2.

\[
\lambda^\infty(t, \theta^*) = e^{(t - T_0)C^*} g_t^* + e^{tC^*} \int_{T_0}^{t} e^{-sC^*} (\partial_s g_s^* + b^* g_s^*) ds = g_t^* + A^* G(C^*)_t. \tag{6.11}
\]

In particular, if \( g^* \) is a constant vector \( g^* \), and if \( C^* \) is invertible, then

\[
\lambda^\infty(t, \theta^*) = e^{(t - T_0)C^*} (I + b^* C^* - 1) g^* - b^* C^* - 1 g^*. \tag{6.12}
\]

Proof. Equation (6.3) becomes

\[
\lambda^\infty(t, \theta^*) = g_t^* + \int_{T_0}^{t} e^{-b^*(t-s)} A^* \lambda^\infty(s, \theta^*) ds.
\]

Therefore

\[
\partial_t \left( e^{-tC^*} \lambda^\infty(t, \theta^*) \right) = e^{-tC^*} \partial_t g_t^* - C^* e^{-tC^*} g_t^* + b^* e^{-tC^*} G(C^*)_t.
\]

which gives the first equality of (6.11). The second equality is due to integration-by-parts. Simple calculus gives (6.12) when \( C^* \) is invertible and \( g_t^* \) is constant.

\[\square\]

Lemma 6.3. Suppose that \( bI + C^* \) is invertible. Then

\[
\lambda^\infty(t, \theta) = g(t, \gamma) + A(b - b^*)(bI + C^*)^{-1}G(-bI)_t + AA^*(bI + C^*)^{-1}G(C^*)_t.
\]

Proof. From (6.2), we have

\[
\lambda^\infty(t, \theta) = g(t, \gamma) + \int_{T_0}^{t} e^{-b(t-s)} A \lambda^\infty(s, \theta^*) ds
\]

\[
= g(t, \gamma) + \int_{T_0}^{t} e^{-b(t-s)} A \left\{ g_s^* + A^* G(C^*)_s \right\} ds
\]

\[
= g(t, \gamma) + AG(-bI)_t + e^{-bt} \int_{T_0}^{t} \int_u^{t} AA^* e^{s(bI+C^*)} e^{-uC^*} g_u^* ds du
\]

\[
= g(t, \gamma) + AG(-bI)_t + e^{-bt} AA^* \int_{T_0}^{t} (bI + C^*)^{-1} (e^{t(bI+C^*)} - e^{u(bI+C^*)}) e^{-uC^*} g_u^* du
\]

\[
= g(t, \gamma) + AG(-bI)_t + AA^*(bI + C^*)^{-1} (G(C^*)_t - G(-bI)_t)
\]

\[
= g(t, \gamma) + A(b - b^*)(bI + C^*)^{-1}G(-bI)_t + AA^*(bI + C^*)^{-1}G(C^*)_t.
\]

\[\square\]
Lemma 6.4. Let $\beta \in \mathbb{R}$, $v_1, v_2 \in \mathbb{R}^2$ and let $B_1, B_2, C \in M_2(\mathbb{R})$. Suppose that $\text{span}\{v_2, Cv_2\} = \mathbb{R}^2$, $\beta I + C$ is invertible, and that
\[ B_1 e^{-\beta t} v_1 + B_2 e^{tC} v_2 = 0 \quad (6.13) \]
for any $t$ in an interval of $\mathbb{R}$. Then $B_2 = O$ and $B_1 v_1 = 0$.

Proof. The equation (6.13) holds for all $t \in \mathbb{R}$. Differentiating $B_1 v_1 + B_2 e^{t(\beta I + C)} v_2 = 0$ in $t$, we see $B_2(\beta I + C)e^{-t(\beta I + C)} v_2 = 0$ for all $t$. Since $\text{span}\{v_2, e^{-t(\beta I + C)} v_2\} = \mathbb{R}^2$ for small $t > 0$, $B_2(\beta I + C) = O$. Therefore $B_2 = O$ by the invertibility of $\beta I + C$. This entails $B_1 v_1 = 0$ from (6.13).

Remark 6.5. The claim of Lemma 6.4 is not valid without the invertibility of $\beta I + C$. For example, the assumptions are satisfied for
\[ C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \]
and $\beta = 1$. However, $B_2 \neq O$ and $B_1 v_1 \neq 0$.

Suppose that $g_\ell^* = g(t, \gamma^*)$ is an $\mathbb{R}^2$-valued polynomial in $t$ of degree $p$. Then, for any $M \in \text{GL}(\mathbb{R}^2)$, there exist $\mathbb{R}^2$-valued smooth (in $M$) mappings $c_\ell(M)$ such that
\[ \partial_\ell \left( \sum_{\ell=0}^{p} (s - \hat{T}_0)^{\ell} e^{-(s - \hat{T}_0)M} c_\ell(M) \right) = e^{-(s - \hat{T}_0)M} g_\ell^*. \]

Then
\[ G(M)_t = \sum_{\ell=0}^{p} (t - \hat{T}_0)^{\ell} c_\ell(M) - e^{(t - \hat{T}_0)M} c_0(M). \quad (6.14) \]

In particular,
\[ g_\ell^* = \ell 1(M) - M c_0(M) \]
and
\[ \partial_t^{p+1} G(M)_t = -M^{p+1} e^{(t-\hat{T}_0)M} c_0(M). \]

Suppose that $g(t, \gamma)$ is an $\mathbb{R}^2$-valued polynomial in $t$ of degree $p$ and that
\[ \lambda(\infty, t, \theta) = \lambda(\infty, t, \theta^*) \quad (\forall t \in I) \quad (6.15) \]
Differentiating $\lambda(\infty, t, \theta) - \lambda(\infty, t, \theta^*) \equiv 0$ $(p + 1)$-times in $t$ with the expression of $\lambda(\infty, t, \theta)$ given in Lemma 6.3 we obtain
\[ A(b - b^*)(bI + C^*)^{-1}(-1)^{p+2} b^{p+1} e^{-(t - \hat{T}_0)bI} (-bI) \]
\[ - (C - C^*) A^*(bI + C^*)^{-1} (C^*)^{p+1} e^{(t - \hat{T}_0)C^*} c_0(C^*) = 0 \]
for all $t$.

Let us consider the following condition.
The parametric model admits a continuous extension to $\Theta$ and that the following conditions are satisfied on $\Theta$:

(i) $b \neq 0$, 
(ii) the matrices $C^*$ and $bI + C^*$ are invertible, 
(iii) $c_0(-bI) \neq 0$, 
(iv) $c_0(C^*)$ is not an eigenvector of $C^*$. 
(v) $g(t, \gamma) = \sum_{t=0}^{p} a_t(\gamma)t^\ell$, and $g$ is identifiable, i.e., $\gamma = \gamma^*$ if $g(t, \gamma) = g(t, \gamma^*)$ for all $t \in I$. 
(vi) $\inf_{t \in I, \theta \in \Theta} g(t, \gamma) > 0$. 
(vii) The mapping $\gamma \mapsto g(t, \gamma)$ is of class $C^4$ with each derivative admitting a continuous extension to $\Theta$. Moreover $\sum_{\alpha=1}^{2}(\partial_\gamma g^\alpha)^\otimes 2(t, \gamma^*)$ is positive definite for some $t \in I$.

**Lemma 6.6.** Under $[M]$, the matrix $\Gamma$ is positive definite.

**Proof.** Let $x$ satisfy $x^\top \Gamma x = 0$. Then it is sufficient to show $x = 0$. Condition $[M]$ (vi) in particular ensures $\inf_{t \in I} g(t, \gamma^*) > 0$, therefore $\inf_{t \in I} \lambda^{\infty, \alpha}(t, \theta^*) > 0$ for $\alpha = 1, 2$. Thus $\Gamma$ of (6.13) is well defined and we obtain

$$\partial_\gamma \lambda^{\infty, \alpha}(t, \theta^*) \cdot x = 0 \quad (6.16)$$

for any $t, \alpha$. Simple calculations with the formula in Lemma 6.3 show

$$\partial_\gamma \lambda^{\infty}(t, \theta^*) = \partial_\gamma g(t, \gamma^*), \quad \partial_\beta \lambda^{\infty}(t, \theta^*) = G(-b_\beta I)_t - G(C^*)_t, \quad (\partial_A^{\alpha, \alpha_1} \lambda^{\infty, \alpha}, \partial_A^{\alpha, \alpha_2} \lambda^{\infty, \alpha})^\top (t, \theta^*) = \delta_{\alpha, \alpha'} G(C^*)_t$$

for $1 \leq \alpha, \alpha' \leq 2$. Rewriting (6.16) with these expressions, we have

$$\partial_\gamma g(t, \gamma^*) \cdot x + \begin{pmatrix} x_{A_{11}} & x_{A_{12}} \\ x_{A_{21}} & x_{A_{22}} \end{pmatrix} G(C^*)_t + x_b((G(-b_\beta I)_t - G(C^*)_t) = 0$$

for any $t$, where $x = (x_b, (x_{A_{\alpha, \alpha'}})_{\alpha, \alpha'})$. Moreover, by differentiating the both sides of the above equation $p + 1$ times with respect to $t$, we obtain

$$\left( x_{bI} - \begin{pmatrix} x_{A_{11}} & x_{A_{12}} \\ x_{A_{21}} & x_{A_{22}} \end{pmatrix} \right)(C^*)^{p+1}e^{(t-\hat{t}_0)C^*}c_0(C^*) + x_b(-1)^{p+1}e^{-(t-\hat{t}_0)}c_0(-b^*I) = 0$$

for any $t$. Here the expression (6.13) was used.

Now Lemma 6.4 and $[M]$ (i)-(iv) yield $x_b = 0$ and $x_{A_{\alpha, \alpha'}} = 0$ for $1 \leq \alpha, \alpha' \leq 2$. Therefore $\partial_\gamma g(t, \gamma^*) \cdot x = 0$ for any $t$ by (6.16). Then $[M]$ (vii) implies $x_\gamma = 0$. \qed

**Theorem 6.7.** Under Condition $[M]$, the same results as Theorems 5.6 and 5.7 hold for the Hawkes type model (6.17).
Proof. Under \([M]\) (i)-(v), Lemma \([6.4]\) implies that \(C - C^* = 0\) and that \(b - b_* = 0\) since \(A = bI + C = bI + C^*\) is invertible and \(c_0(bI) \neq 0\) as well as \(b \neq 0\). Therefore \(g(t, \gamma) = g(t, \gamma^*)\) from (6.15), which entails \(\gamma = \gamma^*\).

Since \(\Theta\) is compact and \(\Upsilon\) has a continuous extension to it, we see from Lemma \([6.6]\) that there exists a positive constant \(c\) such that \(\Upsilon(\theta) \leq -c|\theta - \theta^*|^2\) for all \(\theta \in \Theta\). Therefore Condition \([H4]\) is satisfied, obviously.

Example 6.8. Let us consider a mapping
\[
g(t, \gamma) = \gamma_1(t - T^*)^2 + \gamma_2
\]
taking values in \((0, \infty)^2\) for \(t \in [T_0, T_1]\), where \(T^* = (T_1 + T_0)/2\) and \(\gamma_i (i = 1, 2)\) are parameters in \((0, \infty)^2\). By definition,
\[
c_2(M) = -M^{-1} \gamma_1^*

c_1(M) = 2(T^* - \hat{T}_0) M^{-1} \gamma_1^* - 2M^{-2} \gamma_1^*

c_0(M) = -(T^* - \hat{T}_0)^2 M^{-1} \gamma_1^* - M^{-1} \gamma_2^* + 2(T^* - \hat{T}_0) M^{-2} \gamma_1^* - 2M^{-3} \gamma_1^*.
\]
Then, for \([M]\) (iii), \(c_0(-bI) \in (0, \infty)^2\) whenever \(\beta > 0\). Condition \([M]\) (iv) requires that
\[
c_0(C^*) = \{- (T^* - \hat{T}_0)^2 (C^*)^{-1} + 2(T^* - \hat{T}_0) (C^*)^{-2} - 2(C^*)^{-3}\} \gamma_1^* - M^{-1} \gamma_2^*
\]
is not an eigenvector of \(C^*\).

7 Appendix

Proof of Lemma \([4.10]\) In the present situation,
\[
\langle L^n, L^n \rangle = \sum_{\alpha} \int_{T_0}^{t} \int_{E_{\alpha}} c_{\alpha,i}^{n,i}(s, x) \otimes \nu_{\alpha}(ds, dx).
\]
Therefore, \(\langle L^n, L^n \rangle_t \rightarrow p \int_{T_0}^{t} g_s^{\otimes 2} ds\) for each \(t\).

Let \(M\) be an \(F\)-locally square-integrable purely discontinuous martingale. For \(\epsilon > 0\), there is a canonical representation
\[
M_t = B_t^\epsilon + M_t^\epsilon + M_t^\xi
\]
with \(M_t^\epsilon = \int_{T_0}^{t} \int 1_{\{|z| > \epsilon\}} \mu(ds, dz)\) and \(M_t^\epsilon = \int_{T_0}^{t} \int 1_{\{|z| \leq \epsilon\}} (\mu - \nu)(ds, dz)\), where \(B_t^\epsilon\) is a predictable bounded variational process and \(\mu = \mu^M\) is the integer-valued random measure of jumps of \(M\) and compensated by \(\nu\).
Since a sequence of predictable times exhausts the jumps of $B^c$ and $\Delta L^n_T = 0$ for any predictable time $T$ for the quasi-left continuous $L^n = (L^n,i)_i$, we have $[L^n,i, B^c] = 0$. Then the variation

$$\text{Var} [L^n,i, M] = \text{Var} [L^n,i, M^\ell] + \text{Var} [L^n,i, M^r]$$

$$\leq \sum_{s \leq T_1} |\Delta L^n_s| |\Delta M^\ell_s| 1_{|\Delta M^\ell_s| > \epsilon} + [L^n,i, L^n,i]^{1/2} [M^\ell, M^r]^{1/2}$$

$$\leq \sup_{s \leq T_1} |\Delta L^n_s| \left( 2 \sup_{s \leq T_1} |M_s| \right) \# \{ s \leq T_1; |\Delta M^\ell_s| > \epsilon \} + [L^n,i, L^n,i]^{1/2} [M^\ell, M^r]^{1/2}. \quad (7.1)$$

For $\eta > 0$,

$$\sup_s |\Delta L^\alpha_s|^2 \leq \eta^2 + \sup_s \left( |\Delta L^\alpha_s|^2 1_{|\Delta L^\alpha_s| \geq \eta} \right)$$

$$\leq \eta^2 + \sum_{\alpha} \int_{T_0}^{T_1} |e^n_{\alpha}(s, x)|^2 1_{|e^n_{\alpha}(s, x)| \geq \eta} (\mu^{n,\alpha} + \nu^{n,\alpha})(ds, dx)$$

Thus, the Lenglart inequality gives $\sup_s |\Delta L^\alpha_s| \to^p 0$ as $n \to \infty$ under (iii).

Moreover, the family

$$[L^n,i, L^n,j]_{T_1} = \sum_{\alpha} \int_{T_0}^{T_1} \int |e^n_{\alpha}(s, x)|^2 \mu^{n,\alpha}(ds, dx)$$

is stochastically bounded due to (ii) and the Lenglart inequality. By definition, $\lim_{\epsilon \downarrow 0} [M^\ell, M^r]_{T_1} = 0$ a.s. These properties applied to (7.1) entail

$$\text{Var}[L^n,i, M]_{T_1} \to^p 0 \quad (7.2)$$

as $n \to \infty$.

Since the signed measure $\langle L^n,i, M \rangle \ll (M, M)$, there is a predictable process $d\langle L^n,i, M \rangle / d\langle M, M \rangle(t)$ that is a version of the Radon-Nikodym derivative. Define an increasing process $\langle L^n,i, M \rangle^+$ by

$$\langle L^n,i, M \rangle^+_t = \int_0^t 1_{\{ d\langle L^n,i, M \rangle(s) \geq 0 \}} d\langle L^n,i, M \rangle_s \equiv \int_0^t 1_{\{ d\langle L^n,i, M \rangle(s) \geq 0 \}} d\langle L^n,i, M \rangle(s) d\langle M, M \rangle_s.$$  

The process $\langle L^n,i, M \rangle^+$ is $L$-dominated by the optional process $\text{Var}[L^n,i, M]$. In fact, there exists a reducing sequence $\tau_m$ of stopping times for which both $\text{Var}[L^n,i, M]$ and $\text{Var}[L^n,i, M]$ are integrable. Then for any stopping time $\tau$,

$$E[\langle L^n,i, M \rangle^+_{\tau \wedge \tau_m}] = E \left[ \int_{T_0}^{\tau \wedge \tau_m} 1_{\{ d\langle L^n,i, M \rangle(s) \geq 0 \}} d[L^n,i, M]_s \right]$$

$$\leq E \left[ \int_{T_0}^{\tau \wedge \tau_m} 1_{\{ d\langle L^n,i, M \rangle(s) \geq 0 \}} d\text{Var}[L^n,i, M]_s \right]$$

$$\leq E[\text{Var}[L^n,i, M]_{\tau \wedge \tau_m}]$$
and let \( m \not\to \infty \) to obtain the L-domination. Since we know that \( \{ \sup_s \Delta \text{Var}[L^n,i,M]_s \} \) is stochastically bounded, we obtain \( \langle L^n,i,M \rangle^+_t \to^p 0 \) as \( n \to \infty \) from (7.2) by the Lenglart inequality. In the same fashion, we can prove

\[
\langle L^n,i,M \rangle^-_t := -\int_{T_0}^t 1_{\{d(L^n,i,M)_s < 0\}} d\langle L^n,i,M \rangle_s \to^p 0
\]

Thus we obtained \( \langle L^n,i,M \rangle_t \to^p 0 \).

Obviously, \( \langle L^n,M \rangle = 0 \) for \( F \)-continuous martingales \( M \). Under the properties shown above, the result follows from Jacod’s theorem.

References

[1] Abergel, F., Huth, N.: High frequency lead/lag relationships. Empirical Facts (2012)

[2] Abergel, F., Jedidi, A.: A mathematical approach to order book modeling. International Journal of Theoretical and Applied Finance 16(05) (2013)

[3] Abergel, F., Jedidi, A.: Long time behavior of a hawkes process-based limit order book. hal-01121711v4 (2015). URL https://hal.archives-ouvertes.fr/hal-01121711v4

[4] Bacry, E., Delattre, S., Hoffmann, M., Muzy, J.F.: Modelling microstructure noise with mutually exciting point processes. Quantitative Finance 13(1), 65–77 (2013)

[5] Bibinger, M.: Efficient covariance estimation for asynchronous noisy high-frequency data. Scandinavian Journal of Statistics 38(1), 23–45 (2011)

[6] Bibinger, M.: An estimator for the quadratic covariation of asynchronously observed itô processes with noise: Asymptotic distribution theory. Stochastic Processes and their Applications 122(6), 2411–2453 (2012)

[7] Bowsher, C.G.: Modelling security market events in continuous time: Intensity based, multivariate point process models. Journal of Econometrics 141(2), 876–912 (2007)

[8] Chen, F., Hall, P., et al.: Inference for a nonstationary self-exciting point process with an application in ultra-high frequency financial data modeling. Journal of Applied Probability 50(4), 1006–1024 (2013)

[9] Christensen, K., Kinnebrock, S., Podolskij, M.: Pre-averaging estimators of the ex-post covariance matrix in noisy diffusion models with non-synchronous data. Journal of Econometrics 159(1), 116–133 (2010)

[10] Cont, R., Stoikov, S., Talreja, R.: A stochastic model for order book dynamics. Operations research 58(3), 549–563 (2010)

[11] Epps, T.: Comovements in stock prices in the very short run. JASA 74, 291–298 (1979)
[12] Griffin, J.E., Oomen, R.C.: Covariance measurement in the presence of non-synchronous trading and market microstructure noise. Journal of Econometrics 160(1), 58–68 (2011)

[13] Hayashi, T., Yoshida, N.: On covariance estimation of non-synchronously observed diffusion processes. Bernoulli 11(2), 359–379 (2005)

[14] Hayashi, T., Yoshida, N.: Asymptotic normality of a covariance estimator for nonsynchronously observed diffusion processes. Ann. Inst. Statist. Math. 60(2), 367–406 (2008). DOI 10.1007/s10463-007-0138-0. URL http://dx.doi.org/10.1007/s10463-007-0138-0

[15] Hayashi, T., Yoshida, N.: Nonsynchronous covariation process and limit theorems. Stochastic Process. Appl. 121(10), 2416–2454 (2011). DOI 10.1016/j.spa.2010.12.005. URL http://dx.doi.org/10.1016/j.spa.2010.12.005

[16] Hewlett, P.: Clustering of order arrivals, price impact and trade path optimisation. In: Workshop on Financial Modeling with Jump processes, Ecole Polytechnique, pp. 6–8 (2006)

[17] Hoffmann, M., Rosenbaum, M., Yoshida, N.: Estimation of the lead-lag parameter from non-synchronous data. Bernoulli 19(2), 426–461 (2013). DOI 10.3150/11-BEJ407. URL http://dx.doi.org/10.3150/11-BEJ407

[18] Jacod, J., Li, Y., Mykland, P.A., Podolskij, M., Vetter, M.: Microstructure noise in the continuous case: the pre-averaging approach. Stochastic processes and their applications 119(7), 2249–2276 (2009)

[19] Jong F. de Nijman, T.: High frequency analysis of lead-lag relationships between financial markets. JEF 4, 259–277 (1997)

[20] Koike, Y.: Estimation of integrated covariances in the simultaneous presence of nonsynchronicity, microstructure noise and jumps. Econometric Theory pp. 1–79 (2013)

[21] Koike, Y.: An estimator for the cumulative co-volatility of asynchronously observed semimartingales with jumps. Scandinavian Journal of Statistics 41(2), 460–481 (2014)

[22] Koike, Y.: Limit theorems for the pre-averaged hayashi–yoshida estimator with random sampling. Stochastic Processes and their Applications 124(8), 2699–2753 (2014)

[23] Large, J.: Measuring the resiliency of an electronic limit order book. Journal of Financial Markets 10(1), 1–25 (2007)

[24] Malliavin, P., Mancino, M.E.: Fourier series method for measurement of multivariate volatilities. Finance Stoch. 6(1), 49–61 (2002)

[25] Malliavin, P., Mancino, M.E., et al.: A fourier transform method for nonparametric estimation of multivariate volatility. The Annals of Statistics 37(4), 1983–2010 (2009)

[26] Muni Toke, I., Pomponio, F.: Modelling trades-through in a limited order book using hawkes processes. Economics discussion paper (2011-32) (2011)
[27] Mykland, P.A.: A gaussian calculus for inference from high frequency data. Annals of finance 8(2-3), 235–258 (2012)

[28] Ogihara, T.: Parametric inference for nonsynchronously observed diffusion processes in the presence of market microstructure noise. arXiv preprint arXiv:1412.8173 (2014)

[29] Ogihara, T.: Local asymptotic mixed normality property for nonsynchronously observed diffusion processes. Bernoulli 21(4), 2024–2072 (2015)

[30] Ogihara, T., Yoshida, N.: Quasi-likelihood analysis for nonsynchronously observed diffusion processes. Stochastic Processes and their Applications 124(9), 2954–3008 (2014)

[31] Podolskij, M., Vetter, M.: Estimation of volatility functionals in the simultaneous presence of microstructure noise and jumps. bernoulli 15 634–658. Mathematical Reviews (MathSciNet): MR2555193 Digital Object Identifier: doi 10 (2009)

[32] Smith, E., Farmer, J.D., Gillemot, L.s., Krishnamurthy, S.: Statistical theory of the continuous double auction. Quantitative finance 3(6), 481–514 (2003)

[33] Uchida, M., Yoshida, N.: Model selection for volatility prediction. to appear

[34] Uchida, M., Yoshida, N.: Quasi likelihood analysis of volatility and nondegeneracy of statistical random field. Stochastic Process. Appl. 123(7), 2851–2876 (2013). DOI 10.1016/j.spa.2013.04.008. URL http://dx.doi.org/10.1016/j.spa.2013.04.008

[35] Voev, V., Lunde, A.: Integrated covariance estimation using high-frequency data in the presence of noise. Journal of Financial Econometrics 5(1), 68–104 (2007)

[36] Yoshida, N.: Polynomial type large deviation inequalities and quasi-likelihood analysis for stochastic differential equations. Ann. Inst. Statist. Math. 63(3), 431–479 (2011). DOI 10.1007/s10463-009-0263-z. URL http://dx.doi.org/10.1007/s10463-009-0263-z

[37] Zhang, L., Mykland, P.A., Aït-Sahalia, Y.: A tale of two time scales. Journal of the American Statistical Association 100(472) (2005)

[38] Zhang, L., et al.: Efficient estimation of stochastic volatility using noisy observations: A multiscale approach. Bernoulli 12(6), 1019–1043 (2006)

[39] Zhou, B.: High-frequency data and volatility in foreign-exchange rates. Journal of Business & Economic Statistics 14(1), 45–52 (1996)