ALGEBRAIC K-THEORY OF VARIETIES \( SL_2n / Sp_{2n} \), \( E_6 / F_4 \) AND THEIR TWISTED FORMS

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Abstract. Let \( SL_{2n} \), \( Sp_{2n} \), \( E_6 = G^{sc}(E_6) \), \( F_4 = G(F_4) \) be simply connected split algebraic groups over an arbitrary field \( F \). Algebraic K-theory of affine homogeneous varieties \( SL_{2n} / Sp_{2n} \) and \( E_6 / F_4 \) is computed. Moreover, explicit elements that generate \( K_*(SL_{2n} / Sp_{2n}) \) and \( K_*(E_6 / F_4) \) as \( K_*(F) \)-algebras are provided. For some twisted forms of these varieties K-theory is also computed.

Introduction

Algebraic K-theory is already known for some classes of algebraic varieties. At first it was computed for Severi-Brauer varieties by D. Quillen [9] and for smooth projective quadrics by R. Swan [11]. Then M. Levine [2] computed the K-theory of split semisimple simply connected algebraic groups. I. Panin [8] generalized this computation for all semisimple simply connected algebraic groups and computed the K-theory of flag varieties (see [7]). Later A. Ananyevskiy [1] computed the K-theory of homogeneous varieties \( G/H \), where \( H \subset G \) are connected reductive algebraic groups of the same rank. In all these cases K-theory turned out to be isomorphic to a sum of K-theories of some central semisimple algebras.

We provide a computation of the K-theory for affine homogeneous varieties \( SL_{2n} / Sp_{2n} \) and \( E_6 / F_4 \). The computation is based on using the Merkurjev spectral sequence for the equivariant K-theory (see [4]). The key point which allows us to accomplish the computation is the following fact: for the chosen varieties \( G/H \) there is an epimorphism \( i^*: R(G) \to R(H) \) on the rings of representations, and its kernel is generated by explicit elements. Here can be seen a big difference with the case of \( G/H \) where \( G \) and \( H \) have the same rank. In that case A. Ananyevskiy has shown [1, Theorem 2] that \( R(H) \) is a free \( R(G) \)-module.

The following theorem is proved.

Theorem. There are isomorphisms of graded \( K_*(F) \)-modules:

\[
K_*(SL_{2n} / Sp_{2n}) \cong K_*(F) \otimes \Lambda(Z^{n-1});
\]

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$K_*(E_6/F_4) \simeq K_*(F) \otimes \Lambda(Z^2)$,

where $\Lambda(Z^m)$ is an exterior algebra considered with the natural grading.

Moreover, we provide the elements $t_1, \ldots, t_{n-1} \in K_1(SL_{2n}/Sp_{2n})$ and $s_1, s_2 \in K_1(E_6/F_4)$ that are multiplicative generators of $K_*(F)$-algebras $K_*(SL_{2n}/Sp_{2n})$ and $K_*(E_6/F_4)$ respectively. These elements are constructed similarly to those in topological K-theory of these varieties (see [6]). The proof is based on M. Levine’s computation [2, Theorem 2.1] of multiplicative generators for $K_*(SL_{2n})$ and $K_*(E_6)$ as algebras over $K_*(F)$.

Explicitly constructed isomorphisms in the split case allow to compute K-theory of some twisted forms of these varieties using Panin’s splitting principle [8].

**Theorem.** Assume $\text{char}(F) \neq 2$. Let $\gamma: \text{Gal}(F_{\text{sep}}/F) \to (\text{Sp}_{2n}/\mu_2)(F_{\text{sep}})$ be a 1-cocycle, $A = \text{End}(V)$ where $V$ is a $2n$-dimensional vector space over $F$, and $\tau$ the standard symplectic involution on $A$. Denote $B_i$ the central simple algebra $A_\gamma$ for $i$ odd, and $F$ for $i$ even ($1 \leq i \leq n - 1$). Denote $B_I = B_{i_1} \otimes \cdots \otimes B_{i_q}$ for every $I = \{i_1 < \cdots < i_q\} \subseteq \{1, \ldots, n - 1\}$. Then the following graded $K_*(F)$-modules are isomorphic:

$$K_*(SL_{1,A_\gamma}/Sp(A_\gamma, \tau_\gamma)) \simeq \bigoplus_{I \subseteq \{1, \ldots, n-1\}} K_{* - |I|}(B_I).$$

Let $\delta: \text{Gal}(F_{\text{sep}}/F) \to F_4(F_{\text{sep}})$ be a 1-cocycle. Then the following graded $K_*(F)$-modules are isomorphic:

$$K_*(E_6/F_4) \simeq \bigoplus_{I \subseteq \{1, 2\}} K_{* - |I|}(F).$$

In section 1 we construct multiplicative generators of K-theory and introduce some notation. In section 2 we study the Merkurjev spectral sequence which is used in section 3 to compute the K-theories of the varieties in question as graded modules over the K-theory of a base field. In section 4 we compute the multiplicative generators of the K-theory and state the answer in the split case. In section 5 we state the problem for twisted forms of the varieties. Then in section 6 we describe how to twist the multiplicative generators with a cocycle. Finally, in section 7 we show how Panin’s splitting principle helps to reduce the problem to the split case, which is already solved.

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1. **Construction of the generators for $K_1(G/H)$**

1.1. **Representation rings of SL$_{2n}$ and Sp$_{2n}$, E$_6$ and F$_4$.**
Definition 1. Let $G$ be an algebraic group over a field $F$. The representation ring $R(G)$ of a group $G$ is the Grothendieck group of the category $\text{Rep}_F(G)$ with multiplication induced by tensor product of representations.

Suppose we have a subgroup $i: H \hookrightarrow G$. Then restriction of representations induces the homomorphism $i^*: R(G) \to R(H)$.

$\text{SL}_{2n}$ and $\text{Sp}_{2n}$. Denote the vector representation by $V$. Then for representation rings of the groups $\text{SL}_{2n}$ and $\text{Sp}_{2n}$ we have:

$$R(\text{SL}_{2n}) = \mathbb{Z}[V, \Lambda^2 V, \ldots, \Lambda^{2n} V], \quad R(\text{Sp}_{2n}) = \mathbb{Z}[V, \Lambda^2 V, \ldots, \Lambda^n V].$$

The representations $\Lambda^k V$ and $\Lambda^{2n-k} V$ become isomorphic after restriction to $\text{Sp}_{2n}$ for every $k = 1, \ldots, n-1$. The homomorphism $i^*: R(\text{SL}_{2n}) \to R(\text{Sp}_{2n})$ is surjective. The ideal $\text{Ker} i^*$ is generated by elements $\Lambda^k V - \Lambda^{2n-k} V$, where $k = 1, \ldots, n-1$.

$\text{E}_6$ and $\text{F}_4$. Let $\rho$ and $\rho' \vee$ be the 27-dimensional fundamental representations of $\text{E}_6$, and let $\rho'$ be the 26-dimensional fundamental representation of $\text{F}_4$.

Then for representation rings of the groups $\text{E}_6$ and $\text{F}_4$ we have:

$$R(\text{E}_6) = \mathbb{Z}[\rho, \rho' \vee, \Lambda^2 \rho, \Lambda^3 \rho, \text{Ad}_{\text{E}_6}], \quad R(\text{F}_4) = \mathbb{Z}[\rho', \Lambda^2 \rho', \Lambda^3 \rho', \text{Ad}_{\text{F}_4}],$$

and $\Lambda^3 \rho \simeq \Lambda^3 \rho' \vee$. The representations $\rho$ and $\rho' \vee$ become isomorphic after restriction to $\text{F}_4$. It is known that $i^*(\rho) = i^*(\rho' \vee) = \rho' + 1$; $i^*(\text{Ad}_{\text{E}_6}) = \rho' + \text{Ad}_{\text{F}_4}$ [3, p. 298]. Hence the homomorphism $i^*$ is surjective. The ideal $\text{Ker} i^*$ is generated by elements $\rho - \rho' \vee$ and $\Lambda^2 \rho - \Lambda^2 \rho' \vee$.

1.2. Construction. Suppose we have an affine homogeneous variety $G/H$. Assume there are two nonisomorphic representations of the group $G$ that are isomorphic when restricted to the subgroup $H$. In other words, there are homomorphisms $\phi, \psi: G \to \text{GL}(F^k)$ and a matrix $\alpha \in \text{GL}(F^k)$ such that $\phi(h) = \alpha^{-1} \psi(h) \alpha \quad \forall h \in H$.

Using these data we construct a well-defined map $\chi$ from $G/H$ to $\text{GL}(F^k)$:

$$[g] \mapsto \phi(g) \psi(g)^{-1} \alpha.$$ We identify $\text{Mor}_F(G/H, G/k)$ with $\text{GL}_k[F[G/H]]$ and consider the composition:

$$\text{GL}_k(F[G/H]) \xrightarrow{\chi} \text{GL}(F[G/H]) \xrightarrow{} K_1(F[G/H]) \xrightarrow{} K_1(G/H).$$

This way, the map $\chi$ gives us an element in $K_1(G/H)$. It is denoted by $\beta(\phi - \psi)$ and defined by the following formula:

$$\beta(\phi - \psi) = [ [g] \mapsto \phi(g) \psi(g)^{-1} \alpha ] \in K_1(G/H); \quad [g] \in G/H.$$

1.3. Application. Now we will provide some elements of $K_1(\text{SL}_{2n} / \text{Sp}_{2n})$ and $K_1(\text{E}_6 / \text{F}_4)$, and later we will show that they are multiplicative generators of $K_*(F)$-algebras $K_*(\text{SL}_{2n} / \text{Sp}_{2n})$ and $K_*(\text{E}_6 / \text{F}_4)$. These varieties are affine as quotients of reductive groups by reductive subgroups (see [10]), so we can apply here the construction described in 1.2.

For the group $\text{SL}_{2n}$ consider the vector representation $V$ and its exterior powers $\Lambda^k V$. For every $1 \leq k \leq n-1$ the representations $\Lambda^k V$ and $\Lambda^{2n-k} V$
are isomorphic when restricted to $\text{Sp}_{2n}$ (see 1.1). The corresponding elements of $K_1(\text{SL}_{2n} / \text{Sp}_{2n})$ are defined as follows:

$$t_k = \beta(\Lambda^k V - \Lambda^{2n-k} V), \ 1 \leq k \leq n - 1.$$  

For the group $E_6$ consider the fundamental representations $\rho$ and $\rho^\vee$, which are isomorphic when restricted to $F_4$ (see 1.1). Here are the desired elements of $K_1(E_6 / F_4)$:

$$s_1 = \beta(\rho - \rho^\vee); \quad s_2 = \beta(\Lambda^2 \rho - \Lambda^2 \rho^\vee).$$

1.4. Notation. Here we introduce some notation that will be used later.

- $G/H$ (or $X$) — both varieties $\text{SL}_{2n} / \text{Sp}_{2n}$ and $E_6 / F_4$;
- $\rho_1, \ldots, \rho_l$ — fundamental representations of the group $G$;
- $\{(\rho_{i_1}, \rho_{i_2})\}_{i=1}^m$ — pairs of fundamental representations of $G$, that are isomorphic when restricted to $H$ ($m = n - 1$ or $m = 2$);
- $\hat{\rho}_i = \rho_{i_1} - \rho_{i_2}$ — elements of $R(G)$ that generate $\text{Ker} \ i^*$ (see 1.1).

2. Merkurjev spectral sequence

The Merkurjev spectral sequence allows to express the K-theory of a variety $X$ equipped with an action of an algebraic group $G$ in terms of the $G$-equivariant K-theory of $X$ (see [4], [5]).

**Definition 2.** Let $X$ be a variety equipped with an action of an algebraic group $G$. The $G$-equivariant $K$-theory of $X$ is the K-theory of the category of $G$-equivariant vector bundles over $X$. It is denoted by $K_*(G; X)$.

For computing $K_*(G/H)$ as a $K_*(F)$-module we will need the following theorem of A. Merkurjev [4, Theorem 4.3].

**Theorem** (Merkurjev). Let $G$ be a split reductive group such that $\pi_1(G)$ is torsion-free, and let $X$ be a $G$-scheme. Then there is a spectral sequence:

$$E^2_{p,q} = \text{Tor}_p^{R(G)}(\mathbb{Z}, K_q(G; X)) \implies K_{p+q}(X).$$

Since both groups $G = \text{SL}_{2n}$ and $G = E_6$ are simple and simply connected, their fundamental groups are trivial [4, Cor. 1.3]. Applying this theorem to the case of the variety $G/H$, on which the group $G$ acts by left translation, we get the following spectral sequence:

$$E^2_{p,q} = \text{Tor}_p^{R(G)}(\mathbb{Z}, K_q(G; G/H)) \implies K_{p+q}(G/H).$$

Let us compute the terms of its second page.

2.1. Computation of $E^2_{p,q}$.

**Lemma 1.** $K_i(G; G/H) \simeq R(H) \otimes K_i(F)$ as $R(H)$-modules.

**Proof.** This statement is proved in [1, Lemma 9]. The proof is based on the fact that the categories $\text{Vect}_{G}(G/H)$ and $\text{Rep}(H)$ are equivalent [5, Example 2].
Therefore we need to compute the second page of the following spectral sequence:

\[ E^2_{p,q} = \text{Tor}^{R(G)}_p(\mathbb{Z}, R(H) \otimes K_q(F)) \implies K_{p+q}(G/H). \]

At first, we will treat the case \( q = 0 \).

**Computation of \( \text{Tor}^{R(G)}_p(\mathbb{Z}, R(H)) \).** First let us notice that \( \mathbb{Z} \) is considered as an \( R(G) \)-module by means of the dimension homomorphism \( R(G) \to \mathbb{Z} \), and \( R(H) \) becomes an \( R(G) \)-module by means of the homomorphism \( i^* : R(G) \to R(H) \). Recall that for both considered varieties \( G/H \) the homomorphism \( i^* \) is surjective (see 1.1).

We can see that the sequence \( (\hat{\rho}_1, \ldots, \hat{\rho}_m) \) is regular in \( R(G) \). Hence we can write the corresponding Koszul resolution \( K_\bullet \to R(H) \), consisting of free \( R(G) \)-modules:

\[
\Lambda^m(R(G)^m) \xrightarrow{d_m} \cdots \xrightarrow{d_2} \Lambda^2(R(G)^m) \xrightarrow{d_2} R(G)^m \xrightarrow{d_1} R(G) \xrightarrow{i^*} R(H)
\]

Let \( e_i \) generate \( R(G)^m \) as a free \( R(G) \)-module \( (i = 1 \ldots m) \), then the differentials are defined the following way: \( d_1 : e_i \mapsto \hat{\rho}_i; \) \( d_2 : e_i \wedge e_j \mapsto \hat{\rho}_i \cdot e_j - \hat{\rho}_j \cdot e_i \); etc.

Consider the isomorphism \( R(H) \otimes_{R(G)} \mathbb{Z} \simeq \mathbb{Z}; \) \( \rho \otimes n \mapsto \dim(\rho) \cdot n \). Let us multiply the resolution \( K_\bullet \) termwise by \( \mathbb{Z} \) and apply this isomorphism:

\[
- \otimes_{R(G)} \mathbb{Z}; \quad \Lambda^m(\mathbb{Z}^m) \rightarrow \cdots \rightarrow \Lambda^2(\mathbb{Z}^m) \rightarrow \mathbb{Z}^m \rightarrow \mathbb{Z} \rightarrow 0
\]

All the differentials will become zero because \( \dim(\rho_1) = \dim(\rho_2) \), and so \( \dim(\hat{\rho}_i) = 0 \) for every \( i \).

As a result, we get the formula:

\[
(1) \quad \text{Tor}^{R(G)}_p(\mathbb{Z}, R(H)) = H_p(K_\bullet \otimes_{R(G)} \mathbb{Z}) = \Lambda^p(\mathbb{Z}^m).
\]

**Final presentation of \( E^2_{p,q} \).** To finish the computation of \( E^2_{p,q} \) we will need the following lemma.

**Lemma 2.** \( \text{Tor}^{R(G)}_p(\mathbb{Z}, R(H) \otimes K_i(F)) = \text{Tor}^{R(G)}_p(\mathbb{Z}, R(H)) \otimes K_i(F) \) for every \( i \geq 0 \).

**Proof.** Because of associativity of tensor product we have:

\[
(\mathbb{Z} \otimes_{R(G)} R(H)) \otimes K_i(F) = \mathbb{Z} \otimes_{R(G)} (R(H) \otimes K_i(F)).
\]

This implies the existence of two spectral sequences that converge to triple \( \text{Tor} \):

\[
\tilde{E}^0_{p,q} = \text{Tor}^{R(G)}_p(\mathbb{Z}, R(H), K_i(F)) \implies \text{Tor}_{p+q}(\mathbb{Z}, R(H), K_i(F)),
\]

\[
\tilde{E}^0_{p,q} = \text{Tor}^{R(G)}_p(\mathbb{Z}, R(H), K_i(F)) \implies \text{Tor}_{p+q}(\mathbb{Z}, R(H), K_i(F)).
\]

Observe that: \( \tilde{E}^0_{p,q} = 0 \) for \( p \neq 0 \) because \( \text{Tor}^{R(G)}_p(\mathbb{Z}, R(H)) \) is a free \( \mathbb{Z} \)-module (see (1)); \( \tilde{E}^0_{p,q} = 0 \) for \( q \neq 0 \) because \( R(H) \) is a free \( \mathbb{Z} \)-module.
Therefore both spectral sequences degenerate on the second page and $\hat{E}_2^{p,0} = \hat{E}_2^{p,0}$, which is indeed the statement of the lemma.

Lemma 1, formula (1) and Lemma 2 imply that the Merkurjev spectral sequence for the varieties $G/H = SL_{2n} / Sp_{2n}$ and $G/H = E_6 / F_4$ looks this way:

$$E_2^{p,q} = \Lambda^p(Z^m) \otimes K_q(F) \Rightarrow K_{p+q}(G/H),$$

where $m = rk(G) - rk(H)$. The spectral sequence is first-quadrant, its differential $d_{p,q}^2$ acts from $E_{p,q}^2$ to $E_{p-2,q+1}^2$.

2.2. Degeneration of $E_{p,q}^r$. The Merkurjev spectral sequence is a special case of the Levine spectral sequence [4, 3.1]. There is a multiplicative structure on the zero row of the spectral sequence which is denoted by $\sim_2$ [2, Section 1]. To check that the spectral sequence $E_{p,q}^r$ degenerates we will need the following technical lemma.

**Lemma 3.** The multiplicative structure $\sim_2$ on $\bigoplus_p E_{p,0}^2$ coincides with the natural product on $\bigoplus_p \Lambda^p(Z^m)$.

**Proof.** The following statement is true [2, Example 1.1]: let $R$ be a local ring, $m$ its maximal ideal, $x_1, \ldots, x_n$ a regular sequence in $m$ and $B$ an ideal in $R$. Then the multiplicative structure $\sim_2$ on $\bigoplus_p \text{Tor}_p^R(R/(x_1, \ldots, x_n), R/B)$ coincides with the natural product.

Under the conditions of the lemma we need to show that the two products on $\bigoplus_p \Lambda^p(Z^m) = \bigoplus_p \text{Tor}_p^R(G, (Z, R(H)))$ coincide. Let us reduce this case to the proposition stated above.

Observe that the sequence $(\rho_1, \ldots, \rho_l)$ is regular in $R(G)$ and that $Z = R(G)/(\rho_1, \ldots, \rho_l)$. Recall that for our varieties $R(H) = R(G)/I$ where $I = \text{Ker } i^\ast$. Thus we get:

$$\bigoplus_p E_{p,0}^2 = \bigoplus_p \text{Tor}_p^R(G, (\rho_1, \ldots, \rho_l), R(G)/I).$$

A product on $\bigoplus_p \Lambda^p(Z^m)$ admits a natural extension by applying the localization homomorphism $\bigoplus_p \Lambda^p(Z^m) \hookrightarrow \bigoplus_p \Lambda^p(Q^m)$. Passing to the localization allows to consider the graded ring $\bigoplus_p \text{Tor}_p^R(R/a, R/J)$ in which the ideal $a$ is already maximal. By means of the identity $\text{Tor}_p^R(R/a, R/J) = \text{Tor}_p^R(R_a/(a \cdot R_a), R_a/J_a)$ the statement can be reduced to the case of a local ring $R$.

Let us consider the edge homomorphisms $g_i : K_i(G/H) \rightarrow E_{i,0}^2 = \Lambda^i(Z^m)$. Since the differentials $d_{i,0}^r$ are zero for every $r \geq 2$, we see that $E_{\infty,0}^2 = E_{1,0}^2$ hence $g_i$ is surjective. The edge homomorphism is multiplicative with respect to the product $\sim_2$ [2, Prop. 1.3], i.e., $g_i(a) \sim_2 g_i(b) = g_{i+j}(a \cup b)$. It follows from Lemma 3 that the edge homomorphism is multiplicative with respect to the natural product on $\Lambda(Z^m)$. The algebra $\Lambda(Z^m)$ is generated by the component $\Lambda^1(Z^m)$, thus surjectivity of $g_1$ implies that $g_i$ are surjective.
for every $i$. It follows from the surjectivity of the homomorphisms $g_i$ that $E^2_{i,0} = E^\infty_{i,0}$. Therefore all the differentials $d^r_{p,0}$ are zero for every $r \geq 2$.

The Levine spectral sequence is a module over $K_\ast(F)$ [2, Lemma 1.2]. Since $E^2_{p,q} = E^2_{p,0} \otimes K_q(F)$ and $d^2_{p,0} = 0$, all the differentials on the second page are zero. Using the facts that $d^r_{p,0} = 0$ and that $E^r_{p,q}$ is a $K_\ast(F)$-module for every $r \geq 2$, we get that the differentials are zero on the higher pages also. As a result we see that the spectral sequence degenerates at the second page.

**Corollary 1.** There is a filtration on $K_\ast(G/H)$ whose successive quotients are $K_\ast(F)$, $K_\ast(F)^m$, $\Lambda^2(K_\ast(F)^m)$, $\ldots$, $\Lambda^m(K_\ast(F)^m)$.

**Proof.** Since $E^\infty_{p,q} = E^2_{p,q} = K_q(F) \mathsurround 0pt \to K_{p+q}(G/H)$, there is a filtration on each $K_i(G/H)$ with the following successive quotients: $K_i(F)$, $K_{i-1}(F)^m$, $\Lambda^2(K_{i-2}(F)^m)$, $\ldots$, $\Lambda^m(Z^m)$. These filtrations give a general filtration on $K_\ast(G/H)$ with the desired successive quotients. \hfill \Box

**Corollary 2.** $K_\ast(G/H)$ is a free $K_\ast(F)$-module of rank $2^m$.

**Proof.** Let us consider the filtration on $K_\ast(G/H)$ defined in Corollary 1. All the successive quotients are free $K_\ast(F)$-modules of finite rank, therefore short exact sequences ending with those modules are split. It means that we have an isomorphism of $K_\ast(F)$-modules (which may not respect the graded structures):

$$K_\ast(G/H) \cong K_\ast(F) \oplus K_\ast(F)^m \oplus \Lambda^2(K_\ast(F)^m) \oplus \cdots \oplus \Lambda^m(K_\ast(F)^m).$$

\hfill \Box

2.3. Application of $E^*_{p,q}$. Let us get some information about $K_\ast(G/H)$ using the considered spectral sequence.

**Lemma 4.** $K_1(G/H) \cong K_1(F) \oplus \mathbb{Z}^m$. In particular, for reduced $K$-theory we have $\bar{K}_1(G/H) \cong \mathbb{Z}^m$.

**Proof.** The filtration on $K_1(G/H)$ implies the existence of a short exact sequence:

$$0 \longrightarrow K_1(F) \longrightarrow K_1(G/H) \longrightarrow \mathbb{Z}^m \longrightarrow 0$$

It splits by means of a homomorphism $j^\ast : K_1(G/H) \to K_1(F)$ induced by an inclusion $j : pt \hookrightarrow G/H$. \hfill \Box

Let us introduce some notation: $A$ is the graded ring $K_\ast(F)$; $A^+ = \bigoplus_{i \geq 0} A_i$ $(A/A^+ = \mathbb{Z})$; $B$ is the graded $A$-module $K_\ast(G/H)$. The quotient module $B/(A^+ \cdot B)$ has the structure of a $\mathbb{Z}$-module.

**Lemma 5.** There is an isomorphism of abelian groups $B/(A^+ \cdot B) \cong \Lambda(\mathbb{Z}^m)$.

**Proof.** For every $p > 0$ there is a filtration on $K_p(G/H)$ of length $p+1$ such that $K_p(G/H)(p) = K_p(F)$. Taking the quotient by $K_p(G/H)(p)$ we get a filtration of length $p$ on the quotient group, the first term of which is again
zero in $B/(A^+ \cdot B)$. Iterating this process we get that the homomorphism $B \to B/(A^+ \cdot B)$ sends the free summand $K_p(G/H)$ to $\Lambda^p(Z^m) = E^{p,0}_2$. □

3. Computation of $K_*(G/H)$ as a graded $K_*(F)$-module

Let us consider the exterior algebra $\Lambda(Z^m)$ as an abelian group with the natural grading: $\Lambda(Z^m)_i = \Lambda^i(Z^m)$.

Proposition 1. There is an isomorphism of graded $K_*(F)$-modules:

$$K_*(G/H) \simeq K_*(F) \otimes_{\mathbb{Z}} \Lambda(Z^m).$$

Proof. Let $S$ be a graded ring, $S^+ = \bigoplus_{i>0} S_i$. For a graded $S$-module $P$ we will denote the $S/S^+$-module $P/(S^+ \cdot P)$ as $\overline{P}$.

As earlier, we will write $A$ for $K_*(F)$. Let us introduce notation for graded $A$-modules:

$$B = K_*(G/H), \quad C = K_*(F) \otimes \Lambda(Z^m),$$

and let $j: A \into B$ be the canonical inclusion.

Consider the homomorphism of graded $A$-modules:

$$\phi = j \otimes \Lambda(id): C \to B,$$

where $\Lambda(id) = id: Z^m \to Z^m \subset B_1$ (see Lemma 4), and $\Lambda(id)(e_{i_1} \wedge \cdots \wedge e_{i_r}) = \Lambda(id)(e_{i_1}) \cup \cdots \cup \Lambda(id)(e_{i_r}) \in B_r$.

We will show that $\phi$ is an isomorphism. To do that we will use the graded version of the Nakayama lemma.

Lemma (Graded Nakayama Lemma). Let $R = \bigoplus_{i=0}^{\infty} R_i$ be a graded ring, $R^+ = \bigoplus_{i>0} R_i$. Let $M$ be a graded $R$-module such that $M_j = 0$ for $j << 0$. Then $R^+ \cdot M = M$ implies $M = 0$.

First we will check that $\phi$ is an epimorphism.

Lemma 6. The homomorphism $\phi$ is surjective.

Proof. Observe that $\overline{C} = \mathbb{Z} \otimes \Lambda(Z^m) \simeq \Lambda(Z^m)$. It follows from Lemma 5 that also $\overline{B} \simeq \Lambda(Z^m)$. The induced homomorphism of $\mathbb{Z}$-modules $\overline{\phi}: \overline{C} \to \overline{B}$ maps $Z^m$ to $Z^m$ isomorphically. Thus $\overline{\phi}$ is an isomorphism. It implies that $\text{Coker} \, \phi = 0$. Then by the graded Nakayama lemma $\text{Coker} \, \phi = 0$. □

It follows from Corollary 2 and Lemma 6 that the homomorphism $\phi$ is a graded epimorphism of free finitely-generated $K_*(F)$-modules of the same rank. It implies that $\text{Ker} \, \phi = 0$, and so $\text{Ker} \, \phi = 0$ by the graded Nakayama lemma. Thus $\phi$ is an isomorphism of graded $K_*(F)$-modules. □

4. Computation of generators of $K_*(G/H)$ as $K_*(F)$-module

4.1. Computation of generators of $\tilde{K}_1(G/H)$. It follows from Lemma 4 that for reduced K-theory we have $\tilde{K}_1(G/H) \simeq Z^m$. To get the final answer we only need to find $m$ generating elements for $\tilde{K}_1(G/H)$. First let us consider $\tilde{K}_1(G)$. It was proved by M. Levine [2, Theorem 2.1 and Cor. 2.2] that for
$G = \text{SL}_{2n}$ and $G = E_6$ there is an isomorphism $\tilde{K}_1(G) \simeq \mathbb{Z}^l$ where $l = rk(G)$; moreover, $\tilde{K}_1(G)$ is generated by the elements $[\rho_1], \ldots, [\rho_l] \in K_1(G)$.

Recall that for each pair of representations $(\rho_{i_1}, \rho_{i_2})$ we constructed an element $\beta(\rho_{i_1} - \rho_{i_2}) \in \tilde{K}_1(G/H)$ in 1.2.

**Proposition 2.** $\tilde{K}_1(G/H)$ is generated by the elements $u_i = \beta(\rho_{i_1} - \rho_{i_2})$, $1 \leq i \leq m$.

**Proof.** Let $\mathbb{Z}^m$ be generated by elements $e_1, \ldots, e_m$ as a free abelian group. Consider the diagram of abelian groups and their homomorphisms:

\[
\begin{array}{ccc}
\mathbb{Z}^m & \xrightarrow{\psi} & \tilde{K}_1(G/H) \simeq \mathbb{Z}^m \\
\chi & \searrow & \\
& \tilde{K}_1(G) \simeq \mathbb{Z}^l
\end{array}
\]

The homomorphisms are defined the following way:

1) $\psi$: $e_i \mapsto u_i$,

2) $p^*: \tilde{K}_1(G/H) \rightarrow \tilde{K}_1(G)$ is induced by projection $p: G \rightarrow G/H$,

3) $\chi$: $\tilde{K}_1(G) \rightarrow \mathbb{Z}^m$ is defined on generators: $[\rho_k] \mapsto \begin{cases} e_i, & \text{if } k = i_1, \\ 0, & \text{else.} \end{cases}$

We will show that $\psi$ is an isomorphism. Observe that:

$p^*(u_i) = p^*([\rho_{i_1} \cdot \alpha_i^{-1} \cdot \rho_{i_2}^{-1} \cdot \alpha_i]) = [\rho_{i_1}] + [\alpha_i^{-1}] + [\rho_{i_2}^{-1}] + [\alpha_i] = [\rho_{i_1}] - [\rho_{i_2}].$

Therefore,

$$(\chi \circ p^* \circ \psi)(e_i) = (\chi \circ p^*)(u_i) = \chi([\rho_{i_1}] - [\rho_{i_2}]) = e_i.$$

Hence $\chi \circ p^* \circ \psi = id$, i.e., $\psi$ is injective. Note that $\psi: \mathbb{Z}^m \rightarrow \mathbb{Z}^m$ has a left inverse $\chi \circ p^*$. It implies that $\text{Coker } \psi = 0$ and so $\psi$ is surjective.

\[\square\]

**4.2. Final result.** To study later K-theory of twisted forms of varieties, we will formulate now the obtained result in functorial terms.

Let $X$ be a variety and let $\xi = (x_1, \ldots, x_m)$ be a set of elements in $K_1(X)$. For every subset of indices $I = \{i_1 < \cdots < i_q\} \subseteq \{1, \ldots, m\}$ we denote $x_I = x_{i_1} \cup \cdots \cup x_{i_q} \in K_{|I|}(X)$ where $|I|$ is cardinality of $I$. For $I = \emptyset$ define $x_\emptyset = 1 \in K_0(F)$.

Let us consider the homomorphisms of graded $K_s(F)$-modules:

$$\Theta_{I, \xi}: K_{s-|I|}(F) \rightarrow K_s(X); \quad \alpha \mapsto x_I \cup \alpha.$$

We define the homomorphism $\Theta_\xi$ the following way:

$$\Theta_\xi = \sum_I \Theta_{I, \xi}: \bigoplus_I K_{s-|I|}(F) \rightarrow K_s(X),$$

where $I$ runs through all the subsets of a set $\{1, \ldots, m\}$.

The final result follows from Propositions 1 and 2.
Theorem 1. Let \( t = (t_1, \ldots, t_{n-1}) \) and \( s = (s_1, s_2) \) be the sets of elements in \( K_1(\text{SL}_{2n} / \text{Sp}_{2n}) \) and \( K_1(\text{E}_6 / F_4) \) respectively, defined in 1.3. Then the homomorphisms of graded \( K_\ast(F) \)-modules \( \Theta_t \) and \( \Theta_s \) are isomorphisms:

\[
\Theta_t: \bigoplus_{I \subseteq \{1, \ldots, n-1\}} K_{s-|I|}(F) \xrightarrow{\sim} K_\ast(\text{SL}_{2n} / \text{Sp}_{2n});
\]

\[
\Theta_s: \bigoplus_{I \subseteq \{1, 2\}} K_{s-|I|}(F) \xrightarrow{\sim} K_\ast(\text{E}_6 / F_4).
\]

5. Twisted forms and central simple algebras

From now on we will assume that \( \text{char}(F) \neq 2 \). As earlier, we denote both varieties \( \text{SL}_{2n} / \text{Sp}_{2n} \) and \( \text{E}_6 / F_4 \) as \( G/H \) or \( X \). We denote the center of an algebraic group \( G \) as \( Z(G) \).

Let us consider an action of the group \( H \) on the variety \( G/H \) by left translation. This action can be extended to \( \overline{H} = H/Z(H) \) (in the first case \( \overline{H} = \text{Sp}_{2n} / E_2 \), in the second case \( \overline{H} = F_4 \)). Let us fix a 1-cocycle \( \gamma: \text{Gal}(F^{\text{sep}}/F) \to \overline{H}(F^{\text{sep}}) \). Since there is an action of \( \overline{H} \) on \( G/H \), we can consider a twisted form of the variety \( X \) denoted \( X_\gamma \). The rest of the paper consists of the computation of \( K_\ast(X_\gamma) \).

5.1. Twisting of central simple algebras.

Definition 3. For an algebraic group \( G \) let us introduce a notation for the group of characters of the center: \( \text{Ch}(G) = \text{Hom}(Z(G), \mathbb{G}_m) \).

Definition 4. A representation \( \sigma: G \to \text{GL}(V) \) of an algebraic group \( G \) is called \textit{Ch-homogeneous} if there is a character \( \chi \in \text{Ch}(G) \) such that \( \sigma(z) v = \chi(z) \cdot v \) for every \( z \in Z(G) \), \( v \in V \). In particular, irreducible representations are Ch-homogeneous.

Let \( \sigma: H \to \text{GL}(V) \) be a Ch-homogeneous representation of the group \( H \) and \( A = \text{End}_F(V) \) be a central simple algebra (we will write \( \text{End}(V) \) for \( \text{End}_F(V) \)). Consider the action of \( H \) on \( A \) by conjugation: \( (h, \alpha) \mapsto \sigma(h) \alpha \sigma(h)^{-1} \). It induces an action of \( \overline{H} \) on the algebra \( A \). From the action of \( \overline{H} \) on the algebra \( A \) and a cocycle \( \gamma: \text{Gal}(F^{\text{sep}}/F) \to \overline{H}(F^{\text{sep}}) \) the Tits construction allows to build a twisted algebra \( A_\gamma^\sigma \) (see [12, §3] or [8, 8.6]).

Remark 1. Let \( V \) be a \( 2n \)-dimensional vector space over \( F \), \( A = \text{End}(V) \), and let \( \tau \) be an involution on \( A \) corresponding to the standard antisymmetric form. Consider the action of \( \text{Sp}_{2n} \) on \( \text{SL}_{2n} \) and \( \text{Sp}_{2n} \) by conjugation. Then for twisted forms of \( \text{SL}_{2n} / \text{Sp}_{2n} \) we have:

\[
(\text{SL}_{2n} / \text{Sp}_{2n})_\gamma \simeq (\text{SL}_{2n})_\gamma / (\text{Sp}_{2n})_\gamma = (\text{SL}_{1,A})_\gamma / \text{Sp}(A, \tau)_\gamma = \text{SL}_{1,A, \gamma} / \text{Sp}(A_\gamma, \tau_\gamma).
\]
5.2. Computation of central simple algebras. Let \( \sigma: H \to \text{GL}(V) \) be a Ch-homogeneous representation of a group \( H \) and let \( A = \text{End}(V) \). The class of the algebra \( A_\sigma^\gamma \) in the Brauer group \( \text{Br}(F) \) depends only on the character \( \chi \in \text{Ch}(H) \) representing the action of \( Z(H) \) on \( V \) under \( \sigma \) \cite{8, 8.7}. We will compute the equivalence classes of the algebras \( A_\sigma^\gamma \) in the Brauer group for fundamental representations \( \sigma \) of the groups \( H = \text{Sp}_{2n} \) and \( H = F_4 \).

The center of the group \( \text{Sp}_{2n} \) is equal to \( \mu_2 \), so \( \text{Ch}(\text{Sp}_{2n}) = \mathbb{Z}/2\mathbb{Z} \). Under vector representation \( V \) the center acts with the character \( \mathbb{I} \in \mathbb{Z}/2\mathbb{Z} \). Under representation \( \Lambda^i V \) the center acts with the character \( \tau \in \mathbb{Z}/2\mathbb{Z} \). Hence in \( \text{Br}(F) \) there are equivalences for \( A_{i,\gamma} = \text{End}(\Lambda^i V)^{\Lambda^i V} \):

\[
A_{i,\gamma} \sim A_\gamma \text{ if } i \text{ is odd; } A_{i,\gamma} \sim F \text{ if } i \text{ is even},
\]

where \( A = \text{End}(V) \), \( V \) is a \( 2n \)-dimensional vector space, \( i = 1 \ldots n \).

The center of the group \( F_4 \) is trivial, so the group of characters is trivial also. Therefore for all four algebras \( A_{i,\gamma} = \text{End}(V_i)^{\Sigma_i} \) corresponding to fundamental representations \( \sigma_i \) of the group \( F_4 \) we have \( A_{i,\gamma} \sim F \).

6. Construction of certain elements in \( K_1 \)

**Definition 5.** Let \( B \) be a central simple \( F \)-algebra. For an affine variety \( Y \) we put \( B[Y] = B \otimes_F F[Y] \). Then \( K_1 \) with coefficients in \( B \) is defined as follows:

\[
K_1(Y, B) = K_1(B[Y]) = \text{GL}(B[Y])/E(B[Y]).
\]

**General construction.** Suppose there is a morphism \( f \in \text{Mor}_F(Y, \text{GL}_{1, B}) \). We identify \( \text{Mor}_F(Y, \text{GL}_{1, B}) \) with \( \text{GL}_1(B[Y]) \) \cite{8, Section 9} and consider the composition:

\[
\text{GL}_1(B[Y]) \longrightarrow \text{GL}(B[Y]) \longrightarrow K_1(B[Y]) \longrightarrow K_1(B^{op}[Y]).
\]

This way we can assign an element \([f] \in K_1(Y, B^{op})\) to the morphism \( f \).

**Application.** Suppose that the representations \( (\rho_{i1}, \rho_{i2}) \) of the group \( G \) (where notation is as in 1.4) act on a vector space \( V_i \). Then each pair defines the map \( \bar{\rho}_i: G/H \to \text{GL}(V_i) \) described in 1.2:

\[
\bar{\rho}_i: gH \mapsto \rho_{i1}(g)\alpha_i^{-1}\rho_{i2}(g)^{-1}\alpha_i,
\]

where \( \alpha_i \) satisfy \( \rho_{i1}(h) = \alpha_i^{-1}\rho_{i2}(h)\alpha_i \) for every \( h \in H \).

Consider the action of \( \overline{\mathcal{P}} \) on \( \text{GL}(V_i) \): \( (h, \chi) \mapsto \rho_{i1}(h)\chi\rho_{i1}(h)^{-1} \). Then \( \bar{\rho}_i \) are \( \overline{\mathcal{P}} \)-equivariant morphisms (with respect to the action by left translation of \( \overline{\mathcal{P}} \) on \( G/H \)).

Denote \( A_i = \text{End}(V_i) \). Observe that representations \( \rho_{i1}: H \to \text{GL}(V_i) \) are Ch-homogeneous. Therefore we can twist \( \text{GL}(V_i) = \text{GL}_{1, A_i} \) with a 1-cocycle \( \gamma: \text{Gal}(F^{\text{sep}}/F) \to \overline{\mathcal{P}}(F^{\text{sep}}) \) (see 5.1). We will write \( A_{i,\gamma} \) for \( A_{i,\gamma}^{\rho_{i1}} \).

Furthermore, we can twist with this cocycle \( X = G/H \) and \( \bar{\rho}_i \) (because of \( \overline{\mathcal{P}} \)-equivariance of morphisms \( \bar{\rho}_i \)). We get the following objects:

\[
(G/H)_\gamma; \text{GL}(V_i)_\gamma = GL_{1, A_i, \gamma}; \bar{\rho}_i; (G/H)_\gamma \to GL_{1, A_i, \gamma},
\]
where \( \tilde{\rho}_{i,\gamma} \in \text{Mor}_F(X, GL_{A_i,X}) \). To the morphism \( \tilde{\rho}_{i,\gamma} \) we assign the element \( [\tilde{\rho}_{i,\gamma}] \in K_1(X, A_i^{op}) \) the way described in the general construction. After fixing a cocycle \( \gamma \) the corresponding elements of the K-theory will be denoted \( [\tilde{i}_1], \ldots, [\tilde{i}_{n-1}] \) in the case of the variety \( (\text{SL}_{2n} / \text{Sp}_{2n})_\gamma \) and \( [\tilde{s}_1], [\tilde{s}_2] \) in the case of the variety \( (E_6 / F_4)_\gamma \).

Recall that we know the equivalence classes of the algebras \( A_i,\gamma \) in the Brauer group from 5.2. Since \( K_1(Y, F^{op}) = K_1(Y) \) for every variety \( Y \), we have:

\[
[\tilde{i}_i] \in K_1((\text{SL}_{2n} / \text{Sp}_{2n})_\gamma, A_i^{op}) \text{ if } i \text{ is odd},
\]

\[
[\tilde{\bar{i}}_i] \in K_1((\text{SL}_{2n} / \text{Sp}_{2n})_\gamma) \text{ if } i \text{ even};
\]

\[
[\tilde{s}_1], [\tilde{s}_2] \in K_1((E_6 / F_4)_\gamma),
\]

where \( 0 \leq i \leq n-1 \), \( V \) is a \( 2n \)-dimensional \( F \)-vector space, \( A = \text{End}(V) \).

7. Computation of K-theory of twisted forms

Let \( Y \) be an affine variety, \( B_1, \ldots, B_m \) central simple \( F \)-algebras and \( \xi = (x_1, \ldots, x_m) \) a set of elements such that \( x_i \in K_1(Y, B_i^{op}) \). For every subset \( I = \{i_1 < \cdots < i_q\} \subseteq \{1, \ldots, m\} \) we denote \( x_I = x_{i_1} \cup \cdots \cup x_{i_q} \in K_{[I]}(Y, B_i^{op} \otimes \cdots \otimes B_{i_q}^{op}) \).

Define \( B_I = B_{i_1} \otimes \cdots \otimes B_{i_q} \) and consider the homomorphism of graded \( K_*(F) \)-modules:

\[
\Theta_{I,\xi} : K_{*-|I|}(B_I) \to K_*(Y); \quad \alpha \mapsto x_I \cup_B \alpha.
\]

We define the homomorphism \( \Theta_\xi \) the following way:

\[
\Theta_\xi = \sum_I \Theta_{I,\xi} : \bigoplus_I K_{*-|I|}(B_I) \to K_*(Y),
\]

where \( I \) runs through all subsets of the set \( \{1, \ldots, m\} \).

For the variety \( X,\gamma = (G/H)_\gamma \) we take central simple algebras \( B_I \) equal to \( A_i,\gamma = \text{End}(V_i^{\rho_i}), i = 1, \ldots, m \), where \( V_i \) is the vector space which the representations \( \rho_i \) act on. We consider the set of elements \( \tilde{\rho} = ([\tilde{\rho}_1], \ldots, [\tilde{\rho}_m]) \) where \( [\tilde{\rho}_i] \in K_1(X, A_i^{op}) \) (see 6). This way, we can define the homomorphism:

\[
\Theta_{\tilde{\rho}} : \bigoplus_{I \subseteq \{1, \ldots, m\}} K_{*-|I|}(B_I) \to K_*((G/H)_\gamma).
\]

Panin’s splitting principle tells us [8, Theorem 1.1] that in order to prove that the homomorphism \( \Theta_{\tilde{\rho}} \) is an isomorphism, it is enough to check the following property.

**Proposition 3.** Let \( E \subseteq E \) be any field extension such that cocycle \( \gamma_E \) is a coboundary. Then the homomorphism \( \Theta_{\tilde{\rho}} \) after scalar extension up to the field \( E \) becomes an isomorphism:

\[
\Theta_{\tilde{\rho}}(E) : \bigoplus_{I \subseteq \{1, \ldots, m\}} K_{*-|I|}(B_I \otimes E) \to K_*((G/H)_\gamma \times \text{Spec } E).
\]
Proof. Since \( \gamma_E \) is trivial, all the twistings trivialize over the field \( E \):
\[
(G/H)_{\gamma} \times \text{Spec} E \simeq (G/H)_E,
\]
\[
A_{i,\gamma} \otimes E \simeq A_i \otimes E \sim E \text{ (equivalence in Br}(E)),
\]
\[
[\tilde{\rho}_i] \otimes E = t_{i,E} \text{ if } G/H = \text{SL}_{2n} / \text{Sp}_{2n},
\]
\[
[\tilde{\rho}_i] \otimes E = s_{i,E} \text{ if } G/H = \text{E}_6 / \text{F}_4,
\]
where \( t_i \) and \( s_i \) are defined the same way as in Theorem 1. We see that the homomorphism \( \Theta_{\tilde{\rho}}(E) \) in case of every considered variety \( G/H \) coincides with the corresponding isomorphism from Theorem 1 after scalar extension up to the field \( E \). □

Therefore for varieties \( \text{SL}_{2n,\gamma} / \text{Sp}_{2n,\gamma} \) as well as for varieties \( (\text{E}_6 / \text{F}_4)_\gamma \) the homomorphism \( \Theta_{\tilde{\rho}} \) is an isomorphism. It implies the final result.

**Theorem 2.** Assume \( \text{char}(F) \not= 2 \). Let \( \gamma : \text{Gal}(\text{F}^{\text{sep}} / F) \rightarrow (\text{Sp}_{2n} / \mu_2)(\text{F}^{\text{sep}}) \) be a 1-cocycle. Let \( \tilde{t} = ([\tilde{t}_1], \ldots, [\tilde{t}_{n-1}]) \) be the set of elements defined in 6, \( A = \text{End}(V) \) where \( V \) is a 2n-dimensional vector space over \( F \), and let \( \tau \) be the standard symplectic involution on \( A \). Denote \( B_i \) the central simple algebra \( A_{\gamma} \) for \( i \) odd, and \( F \) for \( i \) even \((1 \leq i \leq n-1)\). Then the homomorphism \( \Theta_{\tilde{t}} \) is an isomorphism of graded \( K_*(F) \)-modules:
\[
\Theta_{\tilde{t}} : \bigoplus_{I \subseteq \{1, \ldots, n-1\}} K_{*-[I]}(B_I) \xrightarrow{\sim} K_*(\text{SL}_{2n} / \text{Sp}_{2n})_{\gamma} = K_*(\text{SL}_{1,A_{\gamma}} / \text{Sp}(A_{\gamma}, \tau_{\gamma})).
\]

Let \( \delta : \text{Gal}(\text{F}^{\text{sep}} / F) \rightarrow \text{F}_4(\text{F}^{\text{sep}}) \) be a 1-cocycle. Let \( \tilde{s} = ([\tilde{s}_1], [\tilde{s}_2]) \) be the set of elements defined in 6. Then the homomorphism \( \Theta_{\tilde{s}} \) is an isomorphism of graded \( K_*(F) \)-modules:
\[
\Theta_{\tilde{s}} : \bigoplus_{I \subseteq \{1, 2\}} K_{*-[I]}(F) \xrightarrow{\sim} K_*(\text{E}_6 / \text{F}_4)_{\delta}.
\]

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