SPHERICAL SUBCATEGORIES
IN REPRESENTATION THEORY

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Abstract. We introduce a new invariant for triangulated categories: the poset of spherical subcategories ordered by inclusion. This yields several numerical invariants, like the cardinality and the height of the poset. We explicitly describe spherical subcategories and their poset structure for derived categories of certain finite-dimensional algebras.

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Introduction

Our previous work [13] associates a natural triangulated subcategory to an object of a \( k \)-linear triangulated category with two dimensional graded endomorphism algebra. Such an object is called spherelike and the associated subcategory is called spherical subcategory. It is the unique maximal subcategory in which the object becomes spherical in the sense of Seidel & Thomas [28], who introduced them to construct symmetries predicted by Kontsevich’s Homological Mirror Symmetry Conjecture.

In this paper, we extend this study in several ways. Namely, we observe that a spherical subcategory can be properly contained in another spherical subcategory. This yields a partial order on the set of spherical subcategories and indeed an invariant of triangulated categories. Several coarser invariants like the cardinality and the height of the poset are derived from this. We illustrate how these invariants capture the complexity of the category in several algebraic examples. Moreover, examples in geometry also suggest that these invariants carry geometric information.

In another direction, we develop a general machinery to construct spherical and spherelike objects in representation theory. This builds on work of Keller & Reiten [18] on cluster tilting theory, which in turn was developed...
to study cluster algebras. More precisely, \(d\)-cluster tilting subcategories in stably \(d\)-Calabi–Yau categories give rise to \((d + 1)\)-Calabi–Yau objects in a certain functor category. For \(d = 1\) or \(2\) these objects turn out to be spherical, see Proposition 3.23. Examples include the classical cluster categories of Buan, Marsh, Reineke, Reiten & Todorov [5] and the constructions of Geiß, Leclerc & Schröer, see e.g. [10].

We study two techniques — insertion and tacking — to modify an algebra in such a way that spherical objects become spherelike, i.e. lose the Calabi–Yau property. In both constructions, we extend the quiver of an algebra \(\Lambda\) by a quiver \(\Gamma\), yielding an algebra \(\Lambda'\) and an embedding \(\mathbb{D}^{b}(\Lambda) \hookrightarrow \mathbb{D}^{b}(\Lambda')\). Moreover, the emerging recollements are similar to blowing-up of varieties in the geometric situation [13]; see Remark 3.18.

If \(F \in \mathbb{D}^{b}(\Lambda)\) is a spherical object, under explicitly given conditions, \(F\) will become properly spherelike in \(\mathbb{D}^{b}(\Lambda')\) and then we can compute \(\mathbb{D}^{b}(\Lambda')_F\):

**Theorem** (Props. 3.9 & 3.15). If \(F \in \mathbb{D}^{b}(\Lambda')\) is properly spherelike, then there exists a subquiver \(\Gamma'\) of \(\Gamma\) such that the spherical subcategory of \(F\) in \(\mathbb{D}^{b}(\Lambda')\) is \(\mathbb{D}^{b}(\Lambda')_F = \mathbb{D}^{b}(\Lambda) \oplus \mathbb{D}^{b}(k\Gamma')\).

Note that the resulting spherical subcategories are always derived categories of finite-dimensional algebras. While the algebras and categories are rather straightforward, this approach helps computing the posets in some examples. As one instance of this, we prove that spherelike posets can get arbitrarily complicated: in Lemma 3.19, we show that any finite poset occurs as a subposet of the spherelike poset of some hereditary algebra.

However, we also consider examples of a different form. We show that for Vossieck’s discrete derived algebras [30], there are spherelike objects such that the spherical subcategory is not the derived category of a finite-dimensional algebra or a projective variety; see Proposition 6.1. In Example 7.1, we present a spherelike object with the same properties for an algebra coming from a non-commutative nodal curve.

We present some examples of posets that can occur in spherelike posets, drawing their Hasse diagrams. The left-hand example is the spherelike poset of both a tame hereditary algebra and a discrete-derived algebra; the middle is an instance of Lemma 3.19; the right-hand example is part of a family of hereditary algebras whose spherelike posets contain the given chains.

\[
P(k\tilde{D}_4) = P(\Lambda(2,3,3))
\]

Examples 2.4, 6.3

Example 3.22

\[
P(\Lambda_3) P(\Lambda_4) P(\Lambda_5)
\]

Corollary 3.20

**Question.** Is the height of the spherical poset \(P(D)\) bounded?

We note that the rank of the Grothendieck group \(K(D)\) is a bound for the height of \(P(D)\) in all examples where we can compute this poset completely. One may wonder whether this gives an upper bound in general. We note that even if true, this bound can be arbitrarily bad, as e.g. Dynkin quivers always have empty spherelike posets.
**Conventions on categories.** Throughout, we fix an algebraically closed field \( k \) and all algebras and varieties are defined over \( k \). Additive categories are assumed to be \( k \)-linear, and subcategories to be full.

The shift (or translation, or suspension) functor of triangulated categories is denoted by \([1]\). All triangles in triangulated categories are meant to be distinguished. Also, we will generally denote triangles abusively by \( A \to B \to C \), hiding the degree increasing morphism \( C \to A[1] \). We write \( \text{Hom}^\bullet(A, B) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}(A, B[i])[-i] \) for the homomorphism complexes in triangulated categories.

All functors between triangulated categories are meant to be exact. We denote derived functors with the same symbol as the functors between abelian categories. In particular, we write \( \otimes \Lambda \) and \( \text{Hom}_\Lambda \) instead of \( \otimes L \Lambda \) and \( R\text{Hom}_\Lambda \), respectively. Often, we suppress the index \( \Lambda \) from \( \text{Hom}_\Lambda \).

**Conventions on algebras.** Let \( \Lambda \) be a finite-dimensional basic algebra. We will denote by \( \text{D}^b(\Lambda) \) the bounded derived category of finitely generated left \( \Lambda \)-modules. We denote by \( Q(\Lambda) \) the quiver of \( \Lambda \), whose vertex set is \( Q_0(\Lambda) := \{ S \in \Lambda \text{-mod simple} \} / \cong \) and whose arrows are given by extensions between simples. In other words, \( \Lambda \) is given as the path algebra of \( Q(\Lambda) \) bound by a (non-unique) ideal. For any \( x \in Q_0(\Lambda) \), the associated simple, projective, injective modules are denoted by \( S(x) \), \( P(x) \), \( I(x) \), respectively.

For an idempotent \( e \in \Lambda \), we tersely write \( \Lambda / e \) for the quotient of \( \Lambda \) by the two-sided ideal \( \Lambda e \Lambda \) generated by \( e \).

The support of a \( \Lambda \)-module \( M \) is \( \{ x \in Q_0(\Lambda) \mid \text{Hom}(P(x), M) \neq 0 \} \). Interpreting \( M \) as a representation of \( Q(\Lambda) \) bound by some ideal, \( \text{supp}(M) \) is the set of vertices \( x \in Q_0(\Lambda) \) with \( \dim(M_x) > 0 \). The support of an object in \( \text{D}^b(\Lambda) \) is defined as the union of the supports of all cohomology modules.

We denote the Serre functor of abstract triangulated categories by \( S \). For a finite-dimensional algebra \( \Lambda \) of finite global dimension, the Serre functor of \( \text{D}^b(\Lambda) \) is given by the Nakayama functor \( \nu = \text{D} \Lambda \otimes \Lambda - \), and we write \( \nu \) instead of \( S \). We also use the Auslander–Reiten (AR) translation \( \tau = \nu[-1] \) of \( \text{D}^b(\Lambda) \). When using duality, the dual of a homomorphism space will be denoted \( \text{Hom}(\cdot, \cdot)^* \).

\( A_n \) denotes the linearly oriented quiver with \( n \) vertices of Dynkin type \( A \).

**Conventions on recollements.** Repeatedly in this article, we will discuss decompositions of triangulated categories. For this, we use the language of recollements, see e.g. [1] and [8] as general references. Given a triangulated category \( \mathcal{D} \) and a full triangulated subcategory \( \mathcal{C} \) of \( \mathcal{D} \) such that the inclusion of \( \mathcal{C}^\perp := \{ D \in \mathcal{D} \mid \text{Hom}^\bullet(\mathcal{C}, D) = 0 \} \) has both adjoints, then there is a canonical equivalence \( \mathcal{C} \cong \mathcal{D}/\mathcal{C}^\perp \), which gives rise to a recollement

\[
\mathcal{C}^\perp \overset{\cong}{\longrightarrow} \mathcal{D} \overset{\cong}{\longrightarrow} \mathcal{C}.
\]

We also need a slightly more general notion: if \( \mathcal{C} \hookrightarrow \mathcal{D} \) has only a right adjoint then \( \mathcal{D} \to \mathcal{D}/\mathcal{C} \) also has only a right adjoint. This right adjoint is explicitly given by the inverse of the canonical equivalence \( \mathcal{C}^\perp \cong \mathcal{D}/\mathcal{C} \) and the inclusion \( \mathcal{C}^\perp \hookrightarrow \mathcal{D} \). We call this a weak recollement and we will write \( \mathcal{C}^\perp \overset{\cong}{\hookrightarrow} \mathcal{D} \overset{\cong}{\twoheadrightarrow} \mathcal{C} \).
For readers familiar with the language of semi-orthogonal decompositions, we point out that a weak recollement \( C^\perp \rightarrow D \rightarrow C \) is equivalent to a weak semi-orthogonal decomposition \( D = \langle C^\perp, C \rangle \). If \( C, C^\perp \) and \( D \) have Serre functors, these are recollements and semi-orthogonal decompositions, respectively. See [25, §1.1] for details.

1. Spherelike objects and their spherical subcategories

We recall some notions and results from our previous article [13]. Let \( D \) be a Hom-finite triangulated category.

- An object \( F \in D \) has a Serre dual \( SF \), if the homological functor \( \text{Hom}(F, -) : \mathcal{D}^{\text{op}} \to \text{k-mod} \) is representable by \( SF \). More precisely, \( \text{Hom}(F, -) \cong \text{Hom}(-, SF) \) where \((-)^*\) is duality of \( \text{k}\)-vector spaces.
- An object \( F \) is called \textit{d-spherelike} if \( \text{Hom}^*(F, F) \cong \text{k} \oplus \text{k}[-d] \).
- An object \( F \) is called \textit{d-spherical} if it is \( d\)-spherelike and possesses a Serre dual such that \( SF \cong F[d] \).

**Remark 1.1.** There is a dichotomy regarding the algebra structure of \( \text{Hom}^*(F, F) \), since a two-dimensional \( \text{k} \)-algebra over an algebraically closed field \( \text{k} \) is either isomorphic to \( \text{k}[x]/x^2 \) or to \( \text{k} \times \text{k} \).

The case \( \text{Hom}^*(F, F) \cong \text{k} \times \text{k} \) can only occur for \( d = 0 \). If \( E_1, E_2 \in D \) are exceptional with \( \text{Hom}^*(E_1, E_2) = \text{Hom}^*(E_2, E_1) = 0 \), then \( F := E_1 \oplus E_2 \) has endomorphism algebra \( \text{k} \times \text{k} \). If \( D \) is idempotent complete, e.g. if \( D \) is a derived category of an abelian category, then all spherelike objects with endomorphism algebra \( \text{k} \times \text{k} \) are of this kind. We call such spherelike objects \textit{decomposable} (even if \( D \) is not idempotent complete). Correspondingly, spherelike objects \( F \) with \( \text{Hom}^*(F, F) \cong \text{k}[x]/x^2 \) are called \textit{indecomposable}. We will only be interested in indecomposable spherelike objects.

The central idea of [13] was to associate to any \( d\)-spherelike object \( F \in D \) with Serre dual a triangulated subcategory in the following fashion: for \( d \neq 0 \), there is a unique morphism (up to scalars) \( F \to SF[-d] \) and we denote its cone by \( Q_F \); this object is called the \textit{asphericality} of \( F \). (See the Appendix of [13] for the case \( d = 0 \).) Evidently, \( Q_F = 0 \) if and only if \( F \) is \( d\)-spherical. Next, the \textit{spherical subcategory} \( \mathcal{D}_F \) of \( F \) is defined to be the left orthogonal complement of the asphericality:

\[
\mathcal{D}_F := \mathcal{D}^\perp_F = \{ A \in D \mid \text{Hom}^*(A, Q_F) = 0 \} \quad \text{with} \quad F \to SF[-d] \to Q_F.
\]

The terminology is justified by the results of [13, Thms. 3.4 & 3.6]:

- \( F \in \mathcal{D}_F \), and \( F \) is a \( d\)-spherical object of \( \mathcal{D}_F \).
- If \( U \subseteq D \) is a full triangulated subcategory such that \( F \in U \), and \( F \) is a \( d\)-spherical object of \( U \), then \( U \subseteq \mathcal{D}_F \).

In other words, there is a \textit{unique maximal} triangulated subcategory in which \( F \) becomes spherical, and this subcategory is \( \mathcal{D}_F \). By contrast, Keller, Yang & Zhou in [19] study the minimal subcategory of \( D \) in which \( F \) becomes spherical — this is the triangulated category \( \langle F \rangle \) generated by \( F \).

**Remark 1.2.** We mention that the functor \( S[-d] \) used above to define the asphericality also plays an important role in representation theory [16]: it is the AR-translation for \( d = 1 \) and Iyama’s higher AR-translation for \( d \neq 1 \).
We introduce two conditions, \((\dagger)\) and \((\ddagger)\), which ensure either better tractability of spherical subcategories or better behaviour of twist functors associated to spherelike objects:

\((\dagger)\)  
F is \(d\)-spherical in a subcategory \(\mathcal{C} \subseteq \mathcal{D}\) such that the inclusion \(\mathcal{C} \hookrightarrow \mathcal{D}\) has a right adjoint and \(\mathcal{D}\) a Serre functor.

By [13, Thm. 3.7], condition \((\dagger)\) allows to compute the spherical subcategory without recourse to the asphericality:

**Theorem 1.3.** If \((\dagger)\) holds then \(F\) is \(d\)-spherelike as an object of \(\mathcal{D}\), and there is a weak recollement \(\mathcal{C}^+ \cap \mathcal{F} \xrightarrow{\sim} \mathcal{D}_F \xleftarrow{\sim} \mathcal{C}\).

In the examples computed via the theorem in this article, we will actually have recollements \(\mathcal{C}^+ \cap \mathcal{F} \xrightarrow{\sim} \mathcal{D}_F \xleftarrow{\sim} \mathcal{C}\). In fact, even more will hold, namely \(\mathcal{D}_F = (\mathcal{C}^+ \cap \mathcal{F}) \oplus \mathcal{C}\).

\((\ddagger)\)  
\(\mathcal{D}\) has a Serre functor and \(\mathcal{D}_F \hookrightarrow \mathcal{D}\) has a right adjoint.

We recall that under mild assumptions on \(\mathcal{D}\), for any object \(F \in \mathcal{D}\) there is an associated twist functor \(T_F: \mathcal{D} \to \mathcal{D}\). This functor is an autoequivalence if and only if \(F\) is spherical. If the condition \((\ddagger)\) is satisfied, \(T_F\) is still conservative, by [13, Prop. 3.9], i.e. if the twist of a map is an isomorphism, then the map is an isomorphism. Even though we make no use of the twists in this article whatsoever, we check that condition \((\ddagger)\) is satisfied in all our examples. For an explanation of the existence of twist functors in enhanced (or, equivalently, algebraic) triangulated categories, see [13, §2.1].

We state a simple observation which explains why there are no negatively-spherical objects in well-behaved algebraic or geometric situations. Recall that a finite-dimensional algebra \(\Lambda\) is called Iwanaga–Gorenstein if \(\Lambda\) has finite injective dimension, as a left and a right \(\Lambda\)-module.

**Lemma 1.4.** Let either \(\mathcal{D} = \mathcal{D}^b(\Lambda)\) for a finite-dimensional algebra \(\Lambda\) of finite global dimension, or \(\mathcal{D} = \mathcal{D}^b(X)\) for a smooth, projective variety \(X\). If \(F \in \mathcal{D}\) is a \(d\)-Calabi–Yau object, then \(d \geq 0\).

Moreover, the same statements hold for \(\mathcal{D} = \mathcal{K}^b(\text{proj-}\Lambda)\) if \(\Lambda\) is an Iwanaga–Gorenstein algebra, and for \(\mathcal{D} = \text{Perf}(X)\) if \(X\) is a Gorenstein variety.

**Remark 1.5.** More generally, let \(L\) be a left-derived functor of a right exact functor. The proof shows that \(LK \cong X[d]\) implies \(d \geq 0\).

**Proof.** Assume \(d < 0\) and let \(F \in \mathcal{D} = \mathcal{D}^b(\Lambda)\) be a non-zero object with \(\nu F \cong F[d]\). Set \(m \in \mathbb{Z}\) to be the maximal non-zero cohomology of \(F\), i.e. \(F \in \mathcal{D}^{\leq m}(\Lambda)\) but \(F \notin \mathcal{D}^{< m}(\Lambda)\). Computing \(\nu F\) using a projective resolution, we see that again \(\nu F \in \mathcal{D}^{\leq m}(\Lambda) \subset \mathcal{D}^{< m-d}(\Lambda)\), although not necessarily \(\nu F \notin \mathcal{D}^{< m}(\Lambda)\) anymore. On the other hand, \(F[d] \in \mathcal{D}^{\leq m-d}(\Lambda)\) and \(F[d] \notin \mathcal{D}^{< m-d}(\Lambda)\), a contradiction.

For \(\mathcal{D} = \mathcal{D}^b(X)\), the Serre functor is given by \(S(F) = F \otimes \omega_X[\dim X]\). Since tensoring with a line bundle is exact, we see that \(S(F) \cong F[d]\) if and only if \(d = \dim X\) and \(F \cong F \otimes \omega_X\).

The generalisation to Gorenstein algebras and varieties follows with the same proof, observing that \(\mathcal{K}^b(\text{proj-}\Lambda)\) and \(\text{Perf}(X)\) have Serre functors, again given by the Nakayama functor and \(- \otimes \omega_X[\dim X]\), respectively. \(\square\)
These similarities notwithstanding, there is an important difference between the algebraic and the geometric cases: an algebra can have $d$-spherical objects for different $d$. One example is the discrete derived algebra $\Lambda(1,2,0)$; see Section 6. This cannot happen in $D^b(X)$ if $X$ is connected.

**Lemma 1.6.** Let either $\mathcal{D} = \mathcal{D}^b(\Lambda)$ for a finite-dimensional algebra $\Lambda$ of finite global dimension, or $\mathcal{D} = \mathcal{D}^b(X)$ for a smooth, projective variety $X$. If $F \in \mathcal{D}$ is a spherelike object such that $D_F$ is equivalent to $D^b(\Lambda')$ for a finite-dimensional algebra $\Lambda'$ to $D^b(X')$ for a projective variety $X'$, then $\Lambda'$ has finite global dimension and $X'$ is smooth, respectively.

**Proof.** We use Orlov’s characterisation of the perfect subcategory of a triangulated category $\mathcal{D}$ as the full subcategory of homologically finite objects $D_{hf} := \{ D \in \mathcal{D} | s(\text{Hom}^*(D, D')) \text{ finite for all } D' \in \mathcal{D} \}$ where, for a complex $V^* \in D^b(k)$ of vector spaces, $s(V^*) := \{ i \in \mathbb{Z} | H^i(V^*) \neq 0 \}.$

By [26, Prop. 1.11], $\text{Perf}(X) \cong D^b(X)_{hf}$ for varieties. Likewise, for algebras we have $\text{Perf}(\Lambda) = D^b(\Lambda)_{hf}$. The inclusion $\text{Perf}(\Lambda) \subseteq D^b(\Lambda)_{hf}$ is easy; and if $M \in D^b(\Lambda)$ but $M \notin \text{Perf}(\Lambda)$, then we can replace $M$ by a bounded-above, minimal projective resolution $P^* \to M$. Let $S := \bigoplus_{i \in \mathbb{Q}(\Lambda)} S(i)$ be the sum of all simple $\Lambda$-modules. Then for any $i \leq 0$, we can find a map $P^* \to S[i]$ which is not null-homotopic, hence $M \notin D^b(\Lambda)_{hf}$.

With $\mathcal{D} = D_{hf}$ by assumption, the subcategory $D_F$ contains only homologically finite objects as well, hence the claim. Note this proof only uses that $D_F$ is a triangulated subcategory of $\mathcal{D}$. $\square$

2. A new triangulated invariant: the spherelike poset

Let $\mathcal{D}$ be a $k$-linear Hom-finite triangulated category. We define the sets

$\mathcal{P}(\mathcal{D}) := \{ D_F | F \in \mathcal{D} \text{ spherelike, indecomposable and has a Serre dual} \}$,

$\mathcal{P}_d(\mathcal{D}) := \{ D_F | F \in \mathcal{P}(\mathcal{D}) \text{ $d$-spherelike} \}$.

The set $\mathcal{P}(\mathcal{D})$ is partially ordered by inclusion and called the spherelike poset of $\mathcal{D}$. The subposet $\mathcal{P}_d(\mathcal{D})$ is called the $d$-spherelike poset. We also write $\mathcal{P}(\Lambda) = \mathcal{P}(D^b(\Lambda))$ and $\mathcal{P}(X) = \mathcal{P}(D^b(X))$, if $\mathcal{D}$ is the bounded derived category of an algebra $\Lambda$ or a variety $X$, respectively. If there is a spherical object $E \in \mathcal{D}$, then $\mathcal{D} = \mathcal{D}_E \in \mathcal{P}(\mathcal{D})$ is the maximal element.

**Remark 2.1.** A fully faithful functor $\iota: \mathcal{D} \to \mathcal{D}'$ maps spherelike objects to spherelike objects. However, the assignment $\mathcal{D}_F \mapsto \mathcal{D}'_{\iota(F)}$ does not induce a well-defined map of sets $\mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{D}')$ in general.

An example is given by spherical objects $F_1, F_2 \in \mathcal{D}$ such that $\iota(F_1) \in \mathcal{D}'$ remains spherical but $\iota(F_2) \not\in \mathcal{D}'$ becomes properly spherelike. A concrete instance of this is provided by derived categories of discrete derived algebras $\mathcal{D} = D^b(\Lambda(1,2,0))$ and $\mathcal{D}' = D^b(\Lambda(1,4,0))$; see Section 6.

Note that $\mathcal{D}_F \mapsto \mathcal{D}_F$ will in general not work either: while any functor preserves inclusions of subcategories, it can happen that $\mathcal{D}_F \notin \mathcal{P}(\mathcal{D}')$. Indeed, the very same example $\iota: \mathcal{D} = D^b(\Lambda(1,2,0)) \to \mathcal{D}' = D^b(\Lambda(1,4,0))$ shows this: $\mathcal{D}_{\iota(F_1)} = \mathcal{D}_{\iota(F_2)} = \mathcal{D}_F \not\subset \mathcal{D}'_{\iota(F_1)}, \mathcal{D}'_{\iota(F_2)}$ by Proposition 6.1.

Nonetheless, the next lemma shows that spherical subcategories are well-behaved with respect to autoequivalences.
Lemma 2.2. Let \( \varphi : \mathcal{D} \to \mathcal{D}' \) be an equivalence of triangulated categories and let \( F \in \mathcal{D} \) be a spherelike object. Then \( \varphi(F) \in \mathcal{D}' \) is spherelike with \( \mathcal{D}'_{\varphi(F)} = \varphi(\mathcal{D}_F) \).

In particular, the spherelike poset is an invariant of \( k \)-linear Hom-finite triangulated categories.

Proof. We start by showing that the asphericality objects behave well under equivalences: \( \varphi(Q_{\varphi(F)}) \cong Q_{\varphi(F)} \), so \( \varphi(\mathcal{D}_F) \) is indeed in \( \mathcal{P}(\mathcal{D}') \). This follows at once from applying \( \varphi \) to the defining triangle \( F \to SF[-d] \to Q_F \):

\[
\varphi(F) \to \varphi SF[-d] \to \varphi(Q_F) \to \varphi(F)[1]
\]

and

\[
\varphi(F) \to S\varphi F[-d] \to Q_{\varphi(F)} \to \varphi(F)[1]
\]

where the vertical isomorphism and commutativity of the left-hand square follow from equivalences commuting with Serre functors. The dashed arrow exists by the axioms of triangulated categories, and is an isomorphism by the five lemma. Hence \( \mathcal{D}'_{\varphi(F)} = \perp Q_{\varphi(F)} = \perp (\varphi(Q_F)) = \varphi(\perp Q_F) = \varphi(\mathcal{D}_F) \). □

By the lemma, an equivalence \( \varphi : \mathcal{D} \to \mathcal{D}' \) induces a well-defined map of posets \( \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{D}') \), \( \mathcal{D}_F \mapsto \mathcal{D}_F' \varphi(F) = \varphi(\mathcal{D}_F) \). Hence it makes sense to look at the spherelike poset up to autoequivalences; we define the stable spherelike poset of \( \mathcal{D} \) to be

\[
\mathcal{GP}(\mathcal{D}) := \mathcal{P}(\mathcal{D})/\text{Aut}(\mathcal{D}).
\]

Analogously, the stable \( d \)-spherelike poset is \( \mathcal{GP}_d(\mathcal{D}) := \mathcal{P}_d(\mathcal{D})/\text{Aut}(\mathcal{D}) \).

Having introduced the poset \( \mathcal{P}(\mathcal{D}) \) as a triangulated invariant of \( \mathcal{D} \), we obtain further numerical invariants: the cardinality, the height and the width of \( \mathcal{P}(\mathcal{D}) \) and its variants \( \mathcal{GP}_d(\mathcal{D}) \) etc. All of these take values in \( \mathbb{N} \cup \{\infty\} \). We recall that the height of a poset is the maximum among lengths of chains, i.e. subsets consisting of pairwise comparable elements. Dually, the width of a poset is the maximal number of elements of antichains, i.e. subsets of pairwise incomparable elements.

Lemma 3.19 shows that spherelike posets can become very complicated, already for hereditary algebras. Corollary 3.20 exemplifies this with a series of algebras of increasing heights, and Example 3.22 uses the method of the lemma to create a spherelike poset containing a cycle. Moreover, Example 7.2 has an algebra of infinite width, even up to autoequivalences.

Example 2.3 (Spherelike posets are non-additive invariants). Let \( \mathcal{D} \) and \( \mathcal{D}' \) be two triangulated categories. For the spherelike poset of the direct sum \( \mathcal{D} \oplus \mathcal{D}' \), two different cases occur: if neither \( \mathcal{D} \) nor \( \mathcal{D}' \) contain spherical objects, then \( \mathcal{P}(\mathcal{D} \oplus \mathcal{D}') \cong \mathcal{P}(\mathcal{D}) \amalg \mathcal{P}(\mathcal{D}') \). On the other hand, if \( \mathcal{D} \) or \( \mathcal{D}' \) contain spherical objects, then the Hasse diagram of \( \mathcal{P}(\mathcal{D} \oplus \mathcal{D}') \) looks like

\[
\begin{array}{c}
\mathcal{D} \\
\mathcal{P}(\mathcal{D}) \setminus \{\mathcal{D}\} \\
\mathcal{D} \oplus \mathcal{D}' \\
\mathcal{P}(\mathcal{D}') \setminus \{\mathcal{D}'\}
\end{array}
\]

where (at most) one of the two set differences may be trivial. This kind of non-additivity is atypical for triangulated invariants such as K-theory.
As a first concrete example, we mention that if $\mathcal{D}$ is a Calabi–Yau category, i.e. the Serre functor is isomorphic to a shift, then either $P(\mathcal{D}) = \{\mathcal{D}\}$ (if $\mathcal{D}$ contains some spherical object) or else $P(\mathcal{D}) = \emptyset$. This applies to cluster categories, or $\text{Perf}(\Lambda)$ for a symmetric algebra $\Lambda$, or to $\mathcal{D}^b(X)$ for smooth, projective varieties $X$ with trivial canonical bundle.

If $C$ is a smooth projective curve, then $P(C) = P_1(C) = \{\mathcal{D}^b(C)\}$. The spherical poset becomes more interesting for the generalisation to weighted projective lines, see Section 5, and for surfaces, see Example 7.2.

Example 2.4 (Hereditary algebras). Let $\Lambda := kQ$ be the path algebra of an acyclic quiver. Then $\Lambda$ is hereditary, i.e. $\text{gl.dim}(\Lambda) = 1$ and therefore every object in $\mathcal{D}^b(\Lambda)$ is isomorphic to its cohomology. If $Q$ is Dynkin, then every indecomposable object in $\Lambda\text{-mod}$ is exceptional and therefore there are no indecomposable spherelike objects in $\mathcal{D}^b(\Lambda)$; in particular, $P(\Lambda) = \emptyset$.

If $Q$ is Euclidean, i.e. of type $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_n$, then $\Lambda$ is tame hereditary. For types $\tilde{D}_n$ and $\tilde{E}_n$, $\Lambda\text{-mod}$ has three non-homogeneous tubes of ranks $p, q, r$ and $p + q + r - 2 = n$ and assuming $p \leq q \leq r$, we have

$$P(\Lambda) = \{\mathcal{D} > \mathcal{D}X_1, \ldots, \mathcal{D}X_p, \mathcal{D}Y_1, \ldots, \mathcal{D}Y_q, \mathcal{D}Z_1, \ldots, \mathcal{D}Z_r\},$$

where $X_i, Y_j, Z_k$ are the indecomposable modules of quasi-lengths $p, q, r$ in the exceptional tubes, respectively. These modules are properly 1-spherelike. In type $\tilde{A}_{p,q}$, the category $\Lambda\text{-mod}$ has up to two non-homogeneous tubes of ranks $p$ and $q$ and $P(kA_{p,q})$ looks similar, with the spherical subcategories $\mathcal{D}Z_k$ omitted and special cases for $p \leq 1$ or $p = q = 1$.

The modules $X_i$ are in the same $\tau$-orbit, and hence identified in the poset $GP(\Lambda)$. The same is true for the modules $Y_j$ and $Z_k$, respectively. Moreover, all quasi-simple modules in homogenous tubes are 1-spherical. In particular, $P(\Lambda)$ and $GP(\Lambda)$ have height 2, except for $Q = \tilde{A}_1$. In Section 5, we treat the more general Geigle–Lenzing weighted projective lines — the weight sequences for the categories considered above $(p, q, r)$.

If $Q$ is an $n$-Kronecker quiver with $n \geq 3$, then using the Euler form one can check that $\mathcal{D}^b(\Lambda)$ has no spherelike objects at all. We don’t have a description of the poset for general wild hereditary algebras. However, if $Q$ contains a full Euclidean subquiver, then $P(\Lambda) \neq \emptyset$. Moreover, there are acyclic quivers $Q$ such that the corresponding poset has height $\geq n$ for an arbitrary integer $n$, see Corollary 3.20.

3. TWO QUIVER CONSTRUCTIONS

We present two general constructions that can be applied to quiver algebras. In one of them, we insert a linearly oriented quiver of Dynkin type $A_n$ at a specified vertex. In the other, we tack an arbitrary quiver without relations to an algebra, at a sink. In both cases, we get recollements for the derived categories of the resulting algebras. In particular, a spherical object for the original algebra gives rise to a spherelike object for the amalgamated algebra. These constructions will turn up in our examples.

Idempotent calculus. It is well-known that idempotents are sources of recollements.
**Proposition 3.1** ([24, Thm. 1, Prop. 2]). Let $\Lambda$ be a finite-dimensional algebra, and let $e \in \Lambda$ be an idempotent such that
- $\operatorname{Ext}^k_\Lambda(\Lambda/e, \Lambda/e) = 0$ for $k > 0$;
- $\operatorname{projdim}_\Lambda(\Lambda/e) < \infty$;
- $\operatorname{projdim}(\Lambda/e)_\Lambda < \infty$ or $\operatorname{projdim}(\Lambda e)_\Lambda < \infty$.

Then there is a recollement

\[
\begin{align*}
\mathcal{D}^b(\Lambda/e) & \xrightarrow{i_*=i} \mathcal{D}^b(\Lambda) \xrightarrow{j^*} \mathcal{D}^b(e\Lambda e) \\
i_! & \xrightarrow{i} \xrightarrow{j} \xrightarrow{j_*} \end{align*}
\]

where the involved functors are

- $i_* = \Lambda/e \otimes_{\Lambda} -$;
- $\iota := i_! = \operatorname{Hom}(\Lambda/e, -) = \Lambda/e \otimes_{\Lambda} - = i_!$;
- $j^* = \Lambda e \otimes_{e\Lambda} -$;
- $\iota := j_! = \Lambda e \otimes_{e\Lambda} -$
- $\pi := j_* = \operatorname{Hom}(\Lambda e, -) = e\Lambda \otimes_{\Lambda} - = j_*$
- $j_* = \operatorname{Hom}(\Lambda e, -)$

We will use the following notation throughout the article:

- $j := \Lambda e \otimes_{e\Lambda} - : \mathcal{D}^b(e\Lambda e) \hookrightarrow \mathcal{D}^b(\Lambda)$,
- $\iota := \operatorname{Hom}(\Lambda/e, -) : \mathcal{D}^b(\Lambda/e) \hookrightarrow \mathcal{D}^b(\Lambda)$.

**Remark 3.2.** The recollement exists in greater generality, if we replace the left-hand category $\mathcal{D}^b(\Lambda/e)$ by $\mathcal{D}^b_{\Lambda/e}(\Lambda)$, i.e. the full subcategory of $\mathcal{D}^b(\Lambda)$ whose objects have cohomology in $\Lambda/e\text{-mod}$; see [8, Sec. 2]. More precisely, the objects in $\mathcal{D}^b_{\Lambda/e}(\Lambda)$ are supported off $e$ which means that

\[M \in \mathcal{D}^b_{\Lambda/e}(\Lambda) \iff \text{supp}(e) \cap \text{supp}(M) = \emptyset,\]

where $\text{supp}(e) := \{x \in Q_0(\Lambda) \mid e \cdot S(x) = S(x)\}$.

Finally, we want to note that $\mathcal{D}^b_{\Lambda/e}(\Lambda) \cong \text{thick}_{\Lambda}(\Lambda/e\text{-mod})$, so in the above situation the image of $\iota$ is the subcategory $\text{thick}_{\Lambda}(\Lambda/e\text{-mod})$. On the other hand, the image of $j$ is $\text{thick}_{\Lambda}(\Lambda e)$.

**Proposition 3.3** ([31, Lem. 2.1]). Let $\Lambda$ be a finite-dimensional algebra, and let $e \in \Lambda$ be an idempotent such that the assumptions of Proposition 3.1 hold. Then $\Lambda$ has finite global dimension if and only if $\Lambda/e$ and $e\Lambda e$ have.

The following statement is well-known.

**Lemma 3.4.** Let $\Lambda$ be a finite-dimensional algebra such that the quiver $Q(\Lambda)$ has no oriented cycles. Then there is an ordering of $Q_0(\Lambda) = \{x_1, \ldots, x_t\}$ such that there is a full, exceptional collection $\mathcal{D}^b(\Lambda) = \langle S(x_1), \ldots, S(x_t) \rangle$.

**Proof.** It is well-known that $\operatorname{Hom}(S(x), S(y)) \cong \delta_{xy} \cdot k$. Moreover for $k > 0$, $\operatorname{Ext}^k(S(x), S(y))$ vanishes if there is no path $x \rightarrow \cdots \rightarrow y$. Since there are no oriented cycles in $Q(\Lambda)$, we can order the simples to form an exceptional sequence. It is clear that they generate the whole category, so it is a full exceptional sequence. \(\square\)

**Remark 3.5.** Let $\Lambda$ be a finite-dimensional algebra and $e$ an idempotent such that the conditions of Proposition 3.1 and the previous lemma are satisfied. In particular, $\text{gl.dim}(\Lambda) < \infty$ since $Q(\Lambda)$ has no oriented cycles. Let $\mathcal{D} = \mathcal{D}^b(\Lambda)$ and suppose there is a $d$-spherical object $F$ in $\mathcal{D}^b(e\Lambda e)$. Restricting the recollement $\mathcal{D}^b(\Lambda/e) \xrightarrow{i_*} \mathcal{D}^b(\Lambda) \xrightarrow{j_*} \mathcal{D}^b(e\Lambda e)$ in Proposition 3.1, we
obtain a weak recollement (Proposition 1.3)
\[ \langle S(x_1), \ldots, S(x_t) \rangle \cap \perp jF \hookrightarrow D_{jF} \hookrightarrow D^b(eAe). \]

The functor \( \pi = \text{Hom}(Ae, -) \) is right adjoint to \( j = Ae \otimes eAe - \), so
\[ \text{Hom}^* (S(x_i), jF) = \text{Hom}^* (jF, \nu S(x_i))^* = \text{Hom}^* (F, \pi \nu S(x_i))^*. \]

We claim that if \( \text{Hom}^* (S(x_i), jF) = 0 \iff i < t \), then the left-hand term of the recollement simplifies to \( \langle S(x_1), \ldots, S(x_i) \rangle \cap \perp jF = \langle S(x_1), \ldots, S(x_{t-1}) \rangle \).

To see this, consider \( \text{Hom}^* (S(x_i), jF) = \text{Hom}^* (jF, \nu S(x_i))^* = \text{Hom}^* (F, \pi \nu S(x_i))^* \).

Many of our examples use this duality-adjunction argument, and therefore deal with computing \( \pi \nu \) on the simples killed by \( e \).

3.1. \( A_n \)-insertion. Let \( \Lambda = kQ/I \) be a finite-dimensional algebra, given by a quiver \( Q \) bound by an ideal \( I \). We fix a vertex \( x \in Q_0 \) and a number \( n \in \mathbb{N} \). The \( A_n \)-insertion in \( \Lambda = kQ/I \) at the vertex \( x \) is the path algebra \( \Lambda(nx) := kQ(nx)/I(nx) \) where

- \( Q(nx) := (Q_0 \setminus \{x\}) \cup \{x_0, \ldots, x_n\} \) on vertex sets;
- the vertex \( x \) is replaced by the quiver \( x_0 \to \cdots \to x_n \) of type \( A_{n+1} \);
- arrows going into \( x \) become arrows going into \( x_0 \), and arrows going out of \( x \) become arrows going out of \( x_n \).

The relations \( \rho(nx) \) generating \( I(nx) \) are obtained from the relations \( \rho \) generating \( I \) by the following modification procedure

- if a relation passes \( x \), extend it by \( \xi := x_0 \to \cdots \to x_n \);
- if a relation starts in \( x \), let it start in \( x_n \);
- if a relation ends in \( x \), let it end in \( x_0 \).

More precisely, write \( \rho = \sum \lambda_ip_{i,1}\cdots p_{m,i} \in I \) for non-trivial paths \( p_{j,i} \) in \( Q \) which end in \( x \) and do not contain subpaths of the form \( x \to x \). Then
\[ \rho(nx) := \sum \lambda_i p_{i,1} \xi p_{2,i} \cdots \xi p_{m-1,i} \xi p_{m,i}. \]

We remark that this construction only depends on the algebra \( \Lambda \), i.e. an isomorphism of quiver algebras \( \varphi : \Lambda \to \Lambda' \) extends to an algebra isomorphism \( \Lambda(nx) \to \Lambda'(nx(x)) \). This justifies the notation \( \Lambda(nx) \).

**Example 3.6.** We consider the Auslander algebra \( \Lambda \) of \( k[x]/x^2 \) given by \( 0 \rightarrow 1 \) with a zero relation at 1. Inserting \( A_2 \) at 0 yields the quiver shown left, whereas inserting \( A_2 \) at 1 yields the quiver on the right:

- \( \bullet \rightarrow \bullet \) with \( ba = 0 \)
- \( \bullet \rightleftharpoons \bullet \) with \( dcba = 0 \)

**Example 3.7.** The discrete derived algebra \( \Lambda(r, n, m) \) of Section 6 is an \( A_{n-r-1} \)-insertion of \( \Lambda(r, r + 1, n) \) at the successor of the trivalent vertex.
We will consider the idempotent \( e := 1 - e_{x_0} - e_{x_1} - \ldots - e_{x_{n-1}} \in \Lambda(nx) \), and the induced functor

\[
j := \Lambda(nx)e \otimes \Lambda - : \mathcal{D}^b(\Lambda) \to \mathcal{D}^b(\Lambda(nx)).
\]

**Lemma 3.8.** Let \( \Lambda(nx) \) be the \( A_n \)-insertion in \( \Lambda \) at \( x \in Q_0(\Lambda) \). Then

1. \( \Lambda \cong e\Lambda(nx)e \) and \( \Lambda(nx)/e \cong kA_n \);
2. if \( \text{projdim} (\Lambda(nx)e)_\Lambda < \infty \), then there is a recollement
   \[
   \mathcal{D}^b(\Lambda(nx)/e) \xrightarrow{\sim} \mathcal{D}^b(\Lambda(nx)) \xrightarrow{\sim} \mathcal{D}^b(\Lambda);
   \]
3. if \( \Lambda \) has finite global dimension, then so does \( \Lambda(nx) \);
4. the functor \( i = \text{Hom}_{\Lambda(nx)/e}(\Lambda(nx)/e, -) \colon \mathcal{D}^b(\Lambda(nx)/e) \to \mathcal{D}^b(\Lambda(nx)) \)
   in (2) is fully faithful with image \( \langle S(x_0), \ldots, S(x_{n-1}) \rangle \). Each \( S(x_i) \)
   is exceptional as an object of \( \mathcal{D}^b(\Lambda(nx)) \).

**Proof.** (1) The isomorphism \( \Lambda(nx)/e \cong kA_n \) is clear. To show \( \Lambda \cong e\Lambda(nx)e \),
we note that

\[
e(kQ(nx)/I(nx))e \cong (ekQ(nx)e)/(eI(nx)e)
\]

By construction, we get a bijection between the quiver of \( e(kQ(nx)e) \) and the original quiver \( Q \) by sending \( x_n \) to \( x \), which extends to an isomorphism \( \varphi : e(kQ(nx)e) \cong kQ \) of algebras.

To establish the claim, we check that \( eI(nx)e \) becomes \( I \) under this isomorphism. Let \( I = \langle \rho_1, \ldots, \rho_m \rangle \) where we may assume that each relation \( \rho_i \)
consists of linear combinations of paths starting in the same vertex and ending in a single other one. Then by construction \( I(nx) = \langle \rho_1(nx), \ldots, \rho_m(nx) \rangle \).
We assume that these relations are ordered in such a way that \( \rho_1(nx), \ldots, \rho_m(nx) \) end in \( x_0 \) and all other relations \( \rho_i(nx) \) end in vertices different from \( x_0 \). One can check that

\[
eI(nx)e = e\langle \rho_1(nx), \ldots, \rho_m(nx), \xi_0, \rho_1(nx), \ldots, \xi_0, \rho_m(nx) \rangle e
= e\langle \rho_1(nx), \ldots, \rho_m(nx), \xi_0, \rho_1(nx), \ldots, \xi_0, \rho_m(nx) \rangle e.
\]

Applying the isomorphism \( \varphi \) from above gives \( \varphi(e\rho_i(nx)e) = \rho_i \) and \( \varphi(e\xi_0\rho_j(nx)e) = \rho_j \), where \( 1 \leq i \leq l \) and \( l + 1 \leq j \leq n \). So \( \varphi(eI(nx)e) = I \),
completing the proof.

(2) We will apply Proposition 3.1 here, so we first check the condition
\( \text{Ext}^k_{\Lambda(nx)}(\Lambda(nx)/e, \Lambda(nx)/e) = 0 \) for \( k \neq 0 \). We decompose \( \Lambda(nx)/e \) into a direct sum of indecomposable \( \Lambda(nx)/e \)-modules. As left \( \Lambda(nx) \)-modules
these summands have projective resolutions \( 0 \to P(x_n) \to P(x_j) \to 0 \), where \( x_j \in Q_0(\Lambda(nx)/e) \). In particular, the left \( \Lambda(nx) \)-module \( \Lambda(nx)/e \) has projective dimension one. So there can be at most a non-trivial \( \text{Ext}^1 \), so we
check for \( x_j \) and another vertex \( x' \):

\[
\begin{array}{ccc}
0 & \longrightarrow & P(x_n) \\
\lambda & \longrightarrow & P(x_j) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & P(x_n) \\
\end{array}
\]

There is only one map \( P(x_n) \to P(x_j) \) up to scalars; it is given by the path from \( x_j \) to \( x_n \). Therefore, there is a \( \lambda \in k \) such that the triangle on the left commutes. This extends to a null-homotopy showing that \( \text{Ext}^1 \)
We deal with the two cases of the propositions:

Remark 3.5, using that \( \pi \)

We check for \( i \)

It is immediate that \( d \)

Proof. We are going to apply Theorem 1.3 for the recollement from Lemma 3.8.

By support reasons, we get \( \nu \)

Where for \( (3) \), we use that \( \Lambda(\Lambda(x)e) \)

Proposition 3.9. Let \( \Lambda \) be an algebra of finite global dimension and let \( \Lambda(\Lambda(x)) \) be the \( A_n \)-insertion in \( \Lambda \) at \( x \in Q_0(\Lambda) \).

Suppose \( F \in D^b(\Lambda) \) is a \( d \)-spherical object.

(1) If \( \text{Hom}^\bullet_{\Lambda}(S(x), F) = 0 \), then \( jF \in D^b(\Lambda(\Lambda(x))) \) is again \( d \)-spherical.

(2) If \( \text{Hom}^\bullet_{\Lambda}(S(x), F) \neq 0 \), then \( jF \in D^b(\Lambda(\Lambda(x))) \) is properly \( d \)-spherical with spherical subcategory

\[
D^b(\Lambda(\Lambda(x)))_{jF} \cong D^b(\Lambda(\Lambda(x))) \oplus D^b(kA_{n-1}).
\]

Proof. We are going to apply Theorem 1.3 for the recollement from Lemma 3.8.

It is immediate that \( jF \in D^b(\Lambda(\Lambda(x))) \) is \( d \)-spherical.

From Remark 3.5, we get a weak recollement

\[
\langle S(x_0), \ldots, S(x_{n-1}) \rangle \cap \downarrow jF \hookrightarrow D^b(\Lambda(\Lambda(x)))_{jF} \twoheadrightarrow D^b(\Lambda).
\]

We check for \( i = 0, \ldots, n-1 \) which \( S(x_i) \) lie in \( \downarrow jF \). As noted in the Remark 3.5, using that \( \pi = \text{Hom}(\Lambda(\Lambda(x)e), -) \) is the right adjoint of \( j \),

\[
\text{Hom}^\bullet_{\Lambda(\Lambda(x))}(S(x_i), jF) = \text{Hom}^\bullet_{\Lambda}(F, \pi S(x_{i+1})[1])^*,
\]

where \( \nu(S(x_i)) = S(x_{i+1})[1] \), applying the Nakayama functor \( \nu \) for \( \Lambda(\Lambda(x)) \).

By support reasons, we get \( \pi S(x_{i+1}) = 0 \) for \( i \leq n-2 \). Hence \( S(x_i) \in \downarrow jF \) if \( i \leq n-2 \).

Whereas for \( i = n-1 \), we have \( \nu(S(x_{n-1})) = S(x_n)[1] \), so

\[
\text{Hom}^\bullet_{\Lambda(\Lambda(x))}(S(x_{n-1}), jF) = \text{Hom}^\bullet_{\Lambda}(F, \pi S(x_{n})[1])^* = \text{Hom}^\bullet_{\Lambda}(F, S(x)[1])^*
\]

\[
= \text{Hom}^\bullet_{\Lambda}(\nu^{-1}S(x), F[-1]) = \text{Hom}^\bullet_{\Lambda}(S(x), F[d-1])
\]

using the Nakayama functor for \( \Lambda \) and \( d \)-sphericity of \( F \) in the last step.

We deal with the two cases of the propositions:

If \( \text{Hom}^\bullet_{\Lambda}(S(x), F) = 0 \), then \( D^b(\Lambda(\Lambda(x)))_{jF} = D^b(\Lambda(\Lambda(x))) \), since the weak recollement (1) is a restriction of the recollement for \( D^b(\Lambda(\Lambda(x))) \). Hence \( jF \) is again \( d \)-spherical in this case, as claimed.

So from now on, assume \( \text{Hom}^\bullet_{\Lambda}(S(x), F) = \text{Hom}^\bullet_{\Lambda(\Lambda(x))}(S(x_{n-1}), jF) \neq 0 \).

By the duality-adjunction argument from Remark 3.5, the weak recollement (1) simplifies to \( \langle S(x_0), \ldots, S(x_{n-2}) \rangle \twoheadrightarrow D^b(\Lambda(\Lambda(x)))_{jF} \twoheadrightarrow D^b(\Lambda) \). In particular, \( F \) is properly \( d \)-spherelike.

Equation (2) implies \( \text{Hom}^\bullet_{\Lambda(\Lambda(x))}(S(x_i), jM) = 0 \) for all \( M \in D^b(\Lambda) \) and \( i \leq n-2 \).

Therefore the outer parts of the recollement are fully orthogonal, i.e. \( D^b(\Lambda(\Lambda(x)))_{jF} = \langle S(x_0), \ldots, S(x_{n-2}) \rangle \oplus j(D^b(\Lambda)) \).

It remains to show that \( S : \langle S(x_0), \ldots, S(x_{n-2}) \rangle \cong D^b(kA_{n-1}) \). Let \( M_j := M(j, n-2) \) be the unique indecomposable \( \Lambda(x) \)-module with top \( S(x_j) \) and socle \( S(x_{n-2}) \). Then \( T = \bigoplus_{j=0}^{n-2} M_j \) is a tilting object in \( S \).

To see this, first note that any such \( M_j \) has a projective resolution \( P(x_{n-1}) \rightarrow P(x_j) \).

Therefore, the projective dimension of \( T \) is at most one. One can check that all Ext\(^1\)(\( M_k \), \( M_l \)) = 0, so all Ext-groups of \( T \) vanish. Since \( 0 \rightarrow M_{j+1} \rightarrow M_j \rightarrow S(x_j) \rightarrow 0 \) is exact, \( T \) is a tilting object. One can calculate that the endomorphism ring of \( T \) is indeed \( kA_{n-1} \). This completes the proof.
Remark 3.10. The proof shows that for properly spherelike $jF$, $\langle S(x_0), \ldots, S(x_{n-2}) \rangle \hookrightarrow D^b(A(nx))_{jF} \to D^b(A)$, is actually a recollement. Moreover, only $S(x_{n-1})$ is missing to get the whole category $D^b(A(nx))$. If we mutate $S(x_{n-1})$ to the left all the way through $S(x_{n-2}), \ldots, S(x_0)$, we obtain the recollement $\langle L_S(x_{n-2}), \ldots, S(x_0) S(x_{n-1}) \rangle \hookrightarrow D^b(A(nx))_{jF} \to D^b(A)$ so condition (4) is met. Actually, we can compute the left-hand part explicitly. By Lemma 3.4, we have $\langle S(x_0), \ldots, S(x_{n-1}) \rangle = D^b(kA_n)$, and helix theory (see e.g. [3, Cor. 2.10]) yields $L_S(x_{n-2}, \ldots, S(x_0) S(x_{n-1}) = \nu_{kA_n} S(x_{n-1}) = I(x_{n-1})$.

Here we have used that the simple on the sink $x_{n-1}$ of $kA_n$ is also the projective module of that vertex; thus its Serre dual is given by the injective of the sink.

Remark 3.11. The spherical subcategories behave as expected for joint insertions at a subset $\Gamma \subset Q_0(\Lambda)$ with $\text{Hom}_A^*(S(x), F) \neq 0$ for all $x \in \Gamma$.

We treat $\Gamma = \{x, y\}$ and leave the general case to the reader. We denote $\Lambda' = \Lambda(nx)$ and $\Lambda'' = \Lambda'(my)$ with $j': D^b(\Lambda') \hookrightarrow D^b(\Lambda'')$. By the considerations of the previous remark and Lemma 3.8, we obtain the recollement $\langle S(y_0), \ldots, S(y_{m-1}), j'I(x_{n-1}) \rangle \hookrightarrow D^b(\Lambda'') \to D^b(\Lambda)$.

As in the proof of Proposition 3.9, we use the duality-adjunction argument from Remark 3.5 to deduce $\text{Hom}_A^*(S(y_j), j'F) \iff j < m - 1$. Moreover, $\text{Hom}_A^*(j'I(x_{n-1}), j'F) = \text{Hom}_A^*(I(x_{n-1}), jF) \neq 0$, as follows from the short exact sequence $0 \to S(x_{n-1}) \to I(x_{n-1}) \to C \to 0$ with $C \in \langle S(x_0), \ldots, S(x_{n-2}) \rangle$ and therefore $C \in \leftarrow jF$.

Therefore we arrive at the recollement $\langle S(y_0), \ldots, S(y_{m-2}) \rangle \hookrightarrow D^b(\Lambda')_{jF} \to D^b(\Lambda')$ which turns out to be $D^b(\Lambda')_{jF} \cong D^b(\Lambda) \oplus D^b(kA_{n-1}) \oplus D^b(kA_{m-1})$.

3.2. Tacking on quivers. We start with the following data:

- $\Lambda = kQ/I$, a finite-dimensional algebra, given as a quiver $Q$ bound by an ideal $I$;
- $(T, t)$, a finite quiver $T$ without oriented cycles and $t$ a sink in $T$;
- $n: Q_0 \to \mathbb{N}$, assigning a multiplicity to every vertex of $Q$.

From this, we construct the following upper triangular matrix algebra

$$(T, t) \triangleright_n \Lambda := \left( \begin{array}{cc} \Lambda & M \\ 0 & kT \end{array} \right)$$

with $M := \bigoplus_{x \in Q_0(\Lambda)} (\Lambda e_x)^n(x) \otimes_k e_t kT$.

In particular, as the $A_n$-insertion this construction only depends on the algebra $\Lambda$. As quiver with relations $(T, t) \triangleright_n \Lambda$ is given as follows:

- the vertex set is $Q_0 \cup T_0$ (disjoint union);
- the arrow set is $Q_1 \cup T_1$ together with $n(x)$ arrows $t \to x$ for all $x \in Q_0$;
- the relations are $I$.

When $t$ and $n$ are understood, we simply write $T \triangleright \Lambda$ instead of $(T, t) \triangleright_n \Lambda$. 
Example 3.12. When $T = A_1$, the construction coincides with the one-point extension on a projective module. This is given by a matrix algebra
\[
\begin{pmatrix}
\Lambda & M \\
0 & k
\end{pmatrix}
\] with $M = \bigoplus_{x \in Q_0(\Lambda)} P(x)^{\oplus n(x)}$.

Example 3.13. The discrete derived algebra $\Lambda(r, n, m)$ of Section 6 is obtained from tacking $A_m$ to the quiver of $\Lambda(r, n, 0)$.

In the following lemma, we consider the idempotent $e := e_{Q_0} \in T \triangleright \Lambda$, and the induced functor
\[
j := (T \triangleright \Lambda) e \otimes_{\Lambda} - : D^b(\Lambda) \to D^b(T \triangleright \Lambda).
\]

It is well-known that (1) – (3) of the lemma below actually hold for upper triangular matrix algebras
\[
\begin{pmatrix}
\Lambda & M \\
0 & B
\end{pmatrix}
\]
where $B$ has finite global dimension; see for example [22, Prop. 3.6].

**Lemma 3.14.** Let $\Lambda$, $(T, t)$ and $n: Q_0(\Lambda) \to \mathbb{N}$ as above. Then

1. $\Lambda \cong e(T \triangleright \Lambda)e$ and $(T \triangleright \Lambda)/e \cong kT$;
2. there is a recollement $D^b((T \triangleright \Lambda)/e) \xrightarrow{\sim} D^b(T \triangleright \Lambda) \xrightarrow{\sim} D^b(\Lambda)$;
3. if $\Lambda$ has finite global dimension, then so does $T \triangleright \Lambda$;
4. the functor $\iota = \text{Hom}_{T \triangleright \Lambda}((T \triangleright \Lambda)/e, -) : D^b((T \triangleright \Lambda)/e) \to D^b(T \triangleright \Lambda)$ is fully faithful with image $\langle S(y) | y \in T_0 \rangle$. Here these $S(y)$ form an exceptional sequence of $D^b(kT)$, and we consider these as objects of $D^b(T \triangleright \Lambda)$ via $T \triangleright \Lambda \to (T \triangleright \Lambda)/e = kT$. Each $S(y)$ is an exceptional object of $D^b(T \triangleright \Lambda)$.

**Proof.** (1) The isomorphisms $(T \triangleright \Lambda)/e \cong kT$ and $\Lambda \cong e(T \triangleright \Lambda)e$ are obvious from the construction of $T \triangleright \Lambda$ and the choice of $e = e_{Q_0}$, as there are no paths whatsoever from $Q$ to $T$.

(2) First, we show $\text{Ext}^1_{T \triangleright \Lambda}((T \triangleright \Lambda)/e, (T \triangleright \Lambda)/e) = 0$ for $k \neq 0$. We can resolve $(T \triangleright \Lambda)/e$ as a $(T \triangleright \Lambda)/e$-module using direct sums of the indecomposable injective modules $I(j)$ for $j \in Q_0(T)$. Note that these modules stay injective when considered as $T \triangleright \Lambda$-modules, so the resolution stays the same. Hence there is no $\text{Ext}^1$ between those $T \triangleright \Lambda$-modules, because there is no $\text{Ext}^1$ between the corresponding $(T \triangleright \Lambda)/e$-modules.

Next, we check that $\text{Ext}^i(T \triangleright \Lambda)/e$ has finite projective dimension as a left $T \triangleright \Lambda$-module. For this, we decompose $(T \triangleright \Lambda)/e$ into a direct sum of indecomposable projective $(T \triangleright \Lambda)/e$-modules. A projective resolution for one single summand as a $T \triangleright \Lambda$-module then looks like
\[
0 \to \bigoplus_{t \to x} P(x)^{n(x)-m(y)} \to P(y)
\]
where the sum is over the successors of $t$, so there are $n(x)$ arrows $t \to x$, and $m(y)$ is the number of paths from $y \in T_0$ to $t$. This shows that the projective dimension of $(T \triangleright \Lambda)/e$ as a left $T \triangleright \Lambda$-module is at most 1.

Finally, for any $y \in T = Q_0((T \triangleright \Lambda)/e)$, the injective resolution of the simple $S(y)$ as a $T \triangleright \Lambda$-module coincides with its $kT$-resolution, hence $\text{injdim}_{T \triangleright \Lambda}(S(y)) \leq 1$. Therefore, all $(T \triangleright \Lambda)/e$-modules have finite injective
dimension, and in particular the dual $D((T \triangleright \Lambda)/e)$ has. It follows that $(T \triangleright \Lambda)/e$ has finite projective dimension as a right $T \triangleright \Lambda$-module.

(3) and (4) follow from Proposition 3.3 and Lemma 3.4, respectively. □

**Proposition 3.15.** Let $\Lambda$, $(T,t)$ and $n: Q_0 \to \mathbb{N}$ as above and let $M \in D^b(\Lambda)$ be spherical. Then $jM \in D^b(T \triangleright \Lambda)$ is properly spherelike if and only if $n(x) > 0$ for some $x \in \text{supp}(M)$, and then the spherical subcategory is

$$D^b(T \triangleright \Lambda)_{jM} \cong D^b(\Lambda) \oplus D^b(kT/e_t).$$

**Proof.** By the lemma above, the functor $j: D^b(\Lambda) \to D^b(T \triangleright \Lambda)$ is fully faithful. Therefore, $jM$ is spherelike. From Remark 3.5 we get the weak recollement

$$\langle S(y) \mid y \in T_0 \rangle \cap jM \xhookrightarrow{i} D^b(T \triangleright \Lambda)_{jM} \xhookrightarrow{\pi} D^b(\Lambda),$$

and to prove the claim, we look at

$$\text{Hom}^*(S(y), jM) = \text{Hom}^*(jM, \nu S(y))^* = \text{Hom}^*(M, \pi \nu S(y))^*$$

where $\pi = \text{Hom}(T \triangleright \Lambda, e, -)$ and $y \in T_0$. For $y \neq t$, the simple $S(y)$ is resolved by $0 \to P \to P(y) \to S(y) \to 0$, where $P$ is the direct sum of all $P(y')$ for all arrows $y \to y'$ in $T$. Hence $\nu S(y)$ is isomorphic to $I \to I(y)$. Kernel and cokernel of this map are supported on $T_0$, invoking the construction of $T \triangleright \Lambda$, and $y \neq t$. Therefore $\pi \nu S(y) = 0$ and thus $S(y) \in jM$ for all $t \neq y \in T_0$ by (3).

Next we want to see that $\text{Hom}^*(S(t), jM) \neq 0$, assuming that $n(x) > 0$ for some $x \in \text{supp}(M)$. The support condition translates to $e_x(H^j(M)) \neq 0$ for some $j \in \mathbb{Z}$, hence $e_x(H^j(jM)) \neq 0$. (Note that the object $jM$ is obtained from $M$ by extending it on $T$ with 0.) Let $0 \to P' \oplus P(x) \xrightarrow{\alpha} P(t) \to S(t) \to 0$ be a projective resolution of $S(t)$ as an $T \triangleright \Lambda$-module; note that $P(x)$ occurs as a non-trivial summand due to $n(x) > 0$. Consider the complex morphism

$$0 \to P' \oplus P(x) \xrightarrow{\alpha} P(t) \to 0 \xrightarrow{0} 0$$

$$\cdots \to M^{j-1} \xrightarrow{d^{j-1}} M^j \xrightarrow{(0, \pi)} M^j \xrightarrow{d^j} M^{j+1} \xrightarrow{d^{j+1}} M^{j+2} \xrightarrow{d^{j+2}} \cdots$$

where $\pi: P(x) \to \ker(d^j) \subset M^j$ is such that the induced map $P(x) \to H^j(M)$ is non-zero — this is possible due to $e_x(H^j(M)) \neq 0$. In particular, $\pi$ does not factor through $d^{j-1}$. Furthermore we have Hom($P(t), M^j) = 0$, as $M^j$ is supported off $t$. Hence the complex morphism $(0, \pi)$ is not null-homotopic and thus Hom$^*(S(t), jM) \neq 0$.

By now we know $D^b(T \triangleright \Lambda)_{jM} \cong D^b(\Lambda) \oplus \langle S(y)_{y \in T_0 \setminus \{t\}} \rangle$. The right-hand summand is equivalent to $D^b(kT/e_t)$ because of $D^b(kT) = \langle S(y)_{y \in T_0} \rangle$, and $t$ being a sink (so that the simple $S(t)$ comes last in the exceptional sequence).

□

**Remark 3.16.** Following the argumentation of Remark 3.10, condition $(\ddagger)$ is again met.

**Remark 3.17.** It would be interesting to extend Proposition 3.15 to more general upper triangular matrix algebras.
Remark 3.18. As Lemmas 3.8 and 3.14 show, both constructions presented in this section share the following formal properties:

- They produce a new algebra from an algebra and combinatorial data.
- The derived category of the new algebra is a recollement of the derived category of the old algebra and an exceptional collection.

These properties are reminiscent of blowing up smooth, projective varieties in points; see [14, §11.2]. The analogy is a bit closer: let $\Lambda$ be the algebra given by the Beilinson quiver $0 \rightarrow 1 \rightarrow 2$ (with commutativity relations), then $D^b(\Lambda) \cong D^b(\mathbb{P}^2)$, by Beilinson’s famous equivalence. This corresponds to the exceptional collections $\langle P(0), P(1), P(2) \rangle$ and $\langle O, O(1), O(2) \rangle$ for $\Lambda$ and $\mathbb{P}^2$, respectively. Blowing up $\mathbb{P}^2$ in a point produces a strong exceptional collection $\langle O_E(-1)[-1], O, O(1), O(2) \rangle$, which corresponds exactly to $A_1 \triangleright_0 \Lambda$ with $n(0) = 1$ and $n(1) = n(2) = 0$.

Lemma 3.19. Let $P$ be a finite poset. Then there exists a finite-dimensional hereditary algebra $\Lambda$ such that $P \subseteq P_1(\Lambda)$ as an induced subposet.

Proof. Write $P = \{ \{1, \ldots, m\}, \leq \}$ for the finite poset $P$. Let $S = 2^P$ the set of all subsets of $P$. Then $S = (S, \subseteq)$ is a poset and the map

$$\iota: P \rightarrow S, \quad p \mapsto \iota(p) := \{ q \in P \mid q \leq p \}$$

is an inclusion of posets. We identify $P$ with its image in $S$. We consider the decomposable algebra $\Lambda := kQ^1 \times \cdots \times kQ^m$ with $m$ factors, where each $Q^i$ is a copy of the Kronecker quiver. We denote the sink and the source of $Q^i$ by $i'$ and $i''$, respectively. Any quasi-simple regular $Q^i$-representation, considered as a $\Lambda$-module, is a 1-spherical object of $D^b(\Lambda)$. Choose such spherical objects $F^i \in D^b(\Lambda)$ for each $i = 1, \ldots, m$.

We will repeatedly tack quivers onto $\Lambda$, and thereby obtain an algebra $\Lambda_m$ with a spherelike poset containing $P$.

The construction is iterative: put $\Lambda_0 := \Lambda$. For $\Lambda_{i-1}$ already defined, set

- $\Lambda_i := A_1 \triangleright_{n_i} \Lambda_{i-1}$ with $Q_0(A_1) := \{ i \}$ and
- $n_i: Q_0(\Lambda_{i-1}) \rightarrow \mathbb{N}$ with $n_i(j') = 1$ if $i \notin \iota(j)$ and $n_i \equiv 0$ else.

Proposition 3.15 shows that the spherical object $F^i \in D^b(\Lambda)$ gives rise to a spherelike object $\bar{F}^i \in D^b(\Lambda_m)$ with spherical subcategory

$$D^b(\Lambda_m)_{\bar{F}^i} = \text{thick}(S(j'), S(j''), \{ S(j) \mid j \in \iota(i) \})$$

In particular,

$$D^b(\Lambda_m)_{\bar{F}_i} \subseteq D^b(\Lambda_m)_{\bar{F}_j} \iff \iota(i) \subseteq \iota(j) \iff i \leq j$$

This result has the following immediate consequence:

Corollary 3.20. There is a sequence of hereditary algebras such that the heights (respectively widths) of their spherelike posets are strictly increasing.

Example 3.21 (Posets of increasing heights). As an example to this sequence of algebras $\Lambda_n$ with increasing posets as drawn in the introduction, we can follow the proof: take a series of $n$ Kronecker quivers tacked together in an inductive way. Below we show the quivers for $\Lambda_2$ and $\Lambda_3$: 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example_quivers.png}
\caption{Quivers for $\Lambda_2$ and $\Lambda_3$.}
\end{figure}
Example 3.22 (Poset containing a cycle). Lemma 3.19 gives a recipe to construct an algebra $\Lambda$ whose spherical poset contains a cycle, i.e. the poset $P = \{1, 2, 3, 4\}$ with predecessor sets $\iota(1) = \{1\}, \iota(2) = \{1, 2\}, \iota(3) = \{1, 3\}, \iota(4) = \{1, 2, 3, 4\}$. Following the proof of the lemma, we need four copies of the Kronecker quiver, to which four $A_1$-quivers are tacked:

As is clear from inspecting the resulting quiver, the algorithm is not optimal. For example, the disconnected vertex 1 is superfluous.

3.3. Sources of spherical objects: cluster-tilting theory. We have seen two constructions that produce spherelike objects out of spherical ones. In this section, we present a general recipe of Keller & Reiten [18] to generate spherical objects. They construct finite-dimensional algebras $\Lambda$, which have perfect derived categories $K^b(\text{proj-}\Lambda)$ containing $(d+1)$-Calabi–Yau objects. For $d = 1$ and $d = 2$, there are always $(d+1)$-spherical objects among these $(d+1)$-Calabi–Yau objects.

Let $E$ be a $k$-linear, Hom-finite, idempotent complete Frobenius category, such that $\text{proj-}E = \text{add}(P)$ for some $P \in E$. Let $C = E/\text{proj-}E$ be the associated stable category, which is triangulated by work of Happel [11]. We assume that $C$ is $d$-Calabi–Yau, i.e. $[d]$ is a Serre functor for $C$. An object $T \in C$ satisfying

$$\text{add}(T) = \{N \in C \mid \text{Ext}^i_C(T, N) = 0 \text{ for } 1 \leq i \leq d - 1\}$$

is called $d$-cluster-tilting object. Let $\Lambda = \text{End}_E(P \oplus T)$ be the higher Auslander algebra of $T$ and $\Delta = \text{End}_C(T)$ be the cluster tilted algebra, which is the factor algebra of $\Lambda$ by the ideal of morphisms factoring through $\text{add}(P)$. By [18, Thm. 5.4(c)], every $\Delta$-module has finite projective dimension as a $\Lambda$-module. Therefore, the subcategory $D^b_{\Delta}(\Lambda) \subseteq D^b(\Lambda)$ of complexes with cohomologies in $\Delta$ is contained in $K^b(\text{proj-}\Lambda)$ — note that this subcategory is equal to the thick subcategory $\text{thick}(\text{mod-}\Delta) \subseteq K^b(\text{proj-}\Lambda)$. Now, by [loc.cit.], every object of $D^b_{\Delta}(\Lambda)$ is a $(d+1)$-Calabi–Yau object, when considered as an object in $K^b(\text{proj-}\Lambda)$. In particular, the simple $\Delta$-modules considered as $\Lambda$-modules are natural candidates for $(d+1)$-spherical objects. The following statement shows that for $d = 1$ and $d = 2$ these simple modules are always spherical.

**Proposition 3.23.** Let $d = 1$ or $d = 2$. Then every simple $\Delta$-module is $(d+1)$-spherical in $K^b(\text{proj-}\Lambda)$. 

$\Lambda_2 : \qquad \bullet \rightarrow \bullet \rightarrow \longrightarrow \circ \quad \Lambda_3 : \qquad \bullet \rightarrow \bullet \rightarrow \longrightarrow \circ \quad \bullet$$\downarrow \downarrow \downarrow \downarrow \downarrow \downarrow$
Proof. Let $S$ be a simple $\Lambda$-module. As $S$ is $(d+1)$-Calabi–Yau, we have $k \cong \text{End}(S) \cong \text{Hom}(S, S[d+1])^*$. The CY property also implies that $S$ has projective dimension at most $d+1$ as a $\Lambda$-module. Hence $\text{Hom}(S, S[n]) = 0$ for all $n < 0$ or $n > d+1$. As $\mathcal{E}$ is Hom-finite, $\Lambda$ is a finite-dimensional $k$-algebra. Using that $S$ has finite projective dimension over $\Lambda$, the validity of the strong no-loop conjecture in this context [15] shows $\text{Ext}^1(S, S) = 0$. This completes the proof for $d = 1$. For $d = 2$, the proof follows from Serre duality and the strong no-loop conjecture: $0 = \text{Hom}(S, S[1])^* = \text{Hom}(S, S[2])^*$.

Example 3.24 (Auslander algebra). Let $R := k[x]/(x^3)$. By definition, the Auslander algebra of $\mathcal{E} = R$-$\text{mod}$ is $\Lambda := \text{End}_R(R \oplus k[x]/(x^2) \oplus k[x]/(x))$. It is a higher Auslander algebra with quiver

$$
\begin{array}{c}
1 & \overset{a}{\rightleftharpoons} & 2 & \overset{b}{\rightleftharpoons} & 3 \\
\downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
1 & \overset{a}{\rightleftharpoons} & 2 & \overset{b}{\rightleftharpoons} & 3 \\
\end{array}
$$

and relations $ca = 0$ and $ac \cdot db = 0$. Here, the vertices 1, 2 and 3 correspond to the $R$-modules $k[x]/(x)$, $k[x]/(x^2)$ and $R$. Moreover, $a$ and $b$ are inclusions whereas $c$ and $d$ are projections. One can check that the simple $\Lambda$-modules $S(1)$ and $S(2)$ are 2-spherical objects in $D^b(\Lambda)$, in accordance with Proposition 3.23 for $d = 1$. This example generalises to $R = k[x]/(x^n)$.

Example 3.25 (Preprojective algebra). The preprojective algebra $\Pi := \Pi(A_3)$ of type $A_3$ is given by the quiver

$$
\begin{array}{c}
1 & \overset{a}{\rightleftharpoons} & 2 & \overset{b}{\rightleftharpoons} & 3 \\
\downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
1 & \overset{a}{\rightleftharpoons} & 2 & \overset{b}{\rightleftharpoons} & 3 \\
\end{array}
$$

with relations $ca = 0 = bd$ and $ac \cdot db = 0$. Then $\mathcal{E} = \Pi$-$\text{mod}$ is a Frobenius category. The stable category $\Pi$-$\text{mod}$ is a 2-Calabi–Yau triangulated category, known as the 2-cluster category of type $A_3$, see [5, 10]. More generally, one could take the triangulated categories arising from preprojective algebras of acyclic quivers $Q$ and Weyl group elements $w \in W_Q$, see [6, 10].

$$
T = T_1 \oplus T_2 \oplus T_3 := 1 \oplus 2 \oplus 2 \\
\quad 1 \quad 3
$$

is a 2-cluster-tilting object in $\Pi$-$\text{mod}$. We consider the corresponding higher Auslander algebra $\Lambda := \text{End}_\Pi(T \oplus \Pi)$ with quiver

$$
\begin{array}{c}
6 & \overset{h}{\rightleftharpoons} & 5 & \overset{d}{\rightleftharpoons} & 4 \\
\downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
\downarrow & & \downarrow & & \downarrow \\
6 & \overset{h}{\rightleftharpoons} & 5 & \overset{d}{\rightleftharpoons} & 4 \\
\end{array}
$$

and relations $da - hgf, ed - cb, ac, ba, feh, ehg$. Here, the vertices 1 to 3 correspond to the modules $T_1$ to $T_3$, the vertices 4 to 6 correspond to the projective $\Pi$-modules $P(1)$ to $P(3)$. Then the simple $\Lambda$-modules $S(1)$ to $S(3)$ are 3-spherical objects, in accordance with Proposition 3.23 for $d = 2$. 

1. Circular quivers

Let $Q_n$ be the clockwise oriented circle with $n$ vertices $1, 2, \ldots, n$ and arrows $i \to a_i i+1$, with $n+1 = 1$. For any tuple of integers $n > 1$, $t \geq 1$ and $r_1, \ldots, r_t$ with $1 \leq r_1 < \cdots < r_t < n$, we define the following ‘circular’ $k$-algebra

$$C_n(r_1, \ldots, r_t) := kQ_n / (a_{r_2} \cdots a_{r_1}, \ldots, a_{r_t} \cdots a_{r_{t-1}}, a_n \cdots a_{r_1})$$

These are finite-dimensional Nakayama algebras of global dimension $t + 1$. As a special case, we introduce the ‘circular basic algebra’ $CB_n$. In other words, $CB_n$ has all but one possible zero relations of length two.

**Lemma 4.1.** The simple module $S(1)$ is a $t$-spherical object in $D^b(CB_t)$.

**Proof.** The minimal projective resolution of $S(1)$ is given as follows

$$0 \to P(1) \to P(t) \to \cdots \to P(3) \to P(2) \to P(1) \to S(1) \to 0.$$ 

Since $\text{Hom}^*(P(i), S(j)) = k \cdot \delta_{ij}$, the object $S(1)$ is $t$-spherelike. Applying the Nakayama functor to $S(1)$, we get a complex of injectives $I(1) \to \cdots \to I(3) \to I(2) \to I(1)$, which has only one non-zero cohomology: namely, $S(1)$ in degree $t$. Hence $S(1)$ is a $t$-spherical object. \hfill \square

**Remark 4.2.** Let $CI_d := kQ_d / (a_2a_1, a_3a_2, \ldots, a_1a_d)$, so that all possible zero relations of length two occur. This is a self-injective algebra, yielding a Frobenius category $CI_d \text{-mod}$, which satisfies the conditions from Section 3.3.

The stable category $CI_d \text{-mod}$ is a generalized $d$-cluster category of type $A_1$, i.e. it is given as a triangulated orbit category $D^b(k)/[d]$, see [17]. In particular, it is $d$-CY. Each of the $d$ indecomposable objects in this category is $d$-cluster tilting and their relative cluster tilted algebras are isomorphic to the algebra $CB_d$ from above. This explains why the simple module $S(1)$ is a $(d+1)$-CY object in $D^b(CB_{d+1})$ (see Lemma 4.1).

**Proposition 4.3.** Let $C = C_n(r_1, \ldots, r_t)$ be a circular algebra. Then $j(S(1))$ is a $(t + 1)$-spherelike object of $D^b(C)$ and its spherical subcategory is the following derived category (with $r_0 := 0$ and $r_{t+1} := n$)

$$D^b(C)_{j(S(1))} \cong D^b(CB_{t+1}) \oplus \bigoplus_{i=0}^t D^b(kA_{r_{i+1}-r_i-2})$$
Proof. Note that the algebra \( C_n(r_1, \ldots, r_t) \) is built from \( \text{CB}_{t+1} \) by simultaneous \( A_{n+1-r_i} \)-insertions at all vertices. So the idempotent \( e = \sum_{i=1}^{t+1} e_{r_i} \) yields an isomorphism \( \text{CB}_{t+1} \cong eCe \). By Lemma 3.8, this induces a fully faithful embedding \( j : \mathcal{D}^b(\text{CB}_{t+1}) \to \mathcal{D}^b(C_n(r_1, \ldots, r_t)) \). Therefore the statement follows from Proposition 3.9 and Remark 3.11, it just remains to check that \( \text{Hom}_{\text{CB}_{t+1}}(S(i), S(1)) \neq 0 \) for all \( i = 1, \ldots, t+1 \). Indeed the minimal projective resolution of the simple \( \text{CB}_{t+1} \)-module \( S(i) \) has the form
\[
0 \to P(1) \to P(t+1) \to \cdots \to P(i+2) \to P(i+1) \to P(i) \to S(i) \to 0.
\]
\[ Q \]

Example 4.4. Consider the circular algebra \( C := C_7(5) = kQ_7/(a_7a_3a_6a_5) \) from Figure 1. It is isomorphic to an \( A_1 \)-insertion at vertex 1 followed by an \( A_1 \)-insertion at vertex 2 for the algebra \( \text{CB}_2 \). Set \( e = e_5 + e_7 \). According to Proposition 4.3, the complex of \( C \)-modules \( \mathcal{J}(S(1)) = P_5 \xrightarrow{\delta_5a_3a_2a_1a_3} P_6 \), \( P_6 \xrightarrow{a_6a_5} P_5 \) is 2-spherelike and the spherical subcategory is given by
\[
\mathcal{D}^b(C)_{\mathcal{J}(S(1))} \supseteq \mathcal{D}^b(eCe) \oplus \langle S(1), S(2), S(3) \rangle \cong \mathcal{D}^b(\text{CB}_2) \oplus \mathcal{D}^b(kA_3).
\]

5. Canonical algebras and weighted projective lines

The derived categories of canonical algebras or, equivalently, weighted projective lines possess properly spherelike objects in the exceptional tubes. See [27] for a general reference to this class of algebras.

Fix a weight sequence \( p = (p_1, \ldots, p_t) \) with all \( p_i \geq 2 \) and a sequence of distinct points \( \lambda = (\lambda_3, \ldots, \lambda_t) \) with \( \lambda_i = (a_i : b_i) \in \mathbb{P}^1_k \). We denote by \( C(p_1, \ldots, p_t; \lambda_3, \ldots, \lambda_t) = C(p; \lambda) = kQ(p)/I \) the associated canonical algebra, where \( Q(p) \) is the quiver consisting of a source 0, a sink 1 and \( t \) paths \( \vec{p} \) of length \( p_i \) from 0 to 1. There are \( t-2 \) relations for \( C(p; \lambda) \), given by \( x_i^p = a_ix_2^{p_2} - b_ix_1^{p_1} \) for \( 3 \leq i \leq t \), where all arrows along the path \( \vec{p} \) are denoted \( x_i \); see [9, Sec. 4]. One can assume all \( a_i = 1 \) and \( b_i \in k \) and, moreover, \( b_3 = 1 \).

The quiver for \( C(5, 4, 6) \):

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet \\
\bullet
\end{array}
\]

There is a family of tubes in \( \mathcal{D}^b(C(p; \lambda)) \) and among these, there are \( t \) tubes of ranks \( p_1, \ldots, p_t \) at the points \( \lambda_1, \ldots, \lambda_t \in \mathbb{P}^1 \). See Figure 2.

Remark 5.1. Let \( M \in \mathcal{D}^b(C(p; \lambda)) \) be an object of a tube of rank \( r \). Then \( \tau^r M \cong M \) and hence the tube is \( \frac{r}{s} \)-fractionally Calabi–Yau, i.e. \( \nu^r M \cong M[r] \). Since tubes are standard, see [29, Thm. 1.6, Cor. 1.7], we can compute \( \text{Hom}^*(M, N) \) using the diagrams in Figure 2 and get for \( M \) indecomposable of quasi-length \( s \):
\[
\dim \text{Hom}(M, M) = \left[ \frac{s}{r} \right], \quad \dim \text{Ext}^1(M, M) = \left[ \frac{s}{r} \right].
\]

So for a tube of rank \( r \), all indecomposable modules on quasi-length \( r \) are 1-spherelike objects, and they are mapped into each other by AR-translations. Moreover, these are spherical if and only if \( r = 1 \).
we obtain the statement. By arguments analogous to the ones given in the proof of Proposition 3.9, with Theorem 1.3 yields the weak recollement
\[ \langle S \rangle \cap P \overset{\sim}{\rightarrow} \mathcal{E} \overset{\sim}{\rightarrow} \mathcal{C}. \]

For the \( i \)-th tube of rank \( p_i \), define the spherelike object \( F_i \) as the cokernel of its projective resolution \( P(1) \xrightarrow{\sim} P(0) \) where \( w = \tilde{p}_i \) is the path from 0 to 1 along the \( i \)-th arm. With this resolution one can check that we obtain indeed spherelike objects for each tube. Finally, \( F_i \) has the following representation: one-dimensional vector spaces on all vertices of \( Q_0(p) \); the identity for all arrows of \( Q_1(p) \), except for zero on the single arrow on the \( i \)-th arm going into the sink.

**Proposition 5.2.** For \( p = (p_1, \ldots, p_t) \) and \( \lambda = (\lambda_1, \ldots, \lambda_t) \), the \( C(p, \lambda) \)-module \( F_i \) is a 1-spherelike object of \( D^b(C(p; \lambda)) \), with spherical subcategory
\[ D^b(C(p; \lambda)) \overset{\sim}{\rightarrow} D^b(C(p_1, \ldots, p_t; \lambda_1, \ldots, \lambda_t)) \oplus D^b(kA_{p_1-2}). \]

**Corollary 5.3.** Given \( p = (p_1, \ldots, p_t) \) and \( \lambda = (\lambda_1, \ldots, \lambda_t) \) as above, let \( N \) the number of pairwise different weights \( p_i \). Then the stable spherelike poset of \( D^b(C(p; \lambda)) \) contains the discrete poset of \( N \) elements:
\[ \{\bullet_1, \ldots, \bullet_N\} \subseteq \mathcal{P}(C(p, \lambda)). \]

**Proof of the proposition.** Without loss of generality we can assume \( i = t \). We first look at the special case that \( p_t = 2 \). Denote \( C = C(p_1, \ldots, p_{t-1}, 2; \lambda) \) and let \( e \) the idempotent yielding \( eCe = C(p_1, \ldots, p_{t-1}; \lambda_1, \ldots, \lambda_{t-1}) \). Using the relations in \( C \), \( w = \tilde{p}_t \) is also a combination \( L = a_t\tilde{p}_t + b_t\tilde{p}_2 \) of paths in \( eCe \), so we can write \( F = F_t \) as \( jE \) for the \( eCe \)-module \( E := \text{coker}(P_{eCe}(1) \xrightarrow{\sim} P_{eCe}(0)) \), where \( j: C := D^b(eCe) \overset{i}{\rightarrow} D := D^b(C) \). Since \( E \) is a homogenous quasi-simple for \( eCe \), it is 1-spherical in \( C \).

We have to show \( D_F \cong C \). From Theorem 1.3 we get the weak recollement \( C \cap F \overset{\sim}{\rightarrow} D_F \overset{\sim}{\rightarrow} C \). Thus we are left to show \( C \cap F = 0 \). As in Lemma 3.8, \( C = \langle S \rangle \) with \( S = C/e \). Using the representation of \( F \), we see that \( S \) is a submodule of \( F \), so \( \text{Hom}(S, F) \cong k \neq 0 \).

Now denote \( \mathcal{E} := D^b(C(p; \lambda)) \). Since \( C(p; \lambda) = C(p_1, \ldots, p_t; \lambda_1, \ldots, \lambda_t) \) is obtained from \( C \) by \( A_{p_t-1} \)-insertion, Lemma 3.8 gives the recollement \( \langle S(1), \ldots, S(p_t-1), S \rangle \overset{\sim}{\rightarrow} E \overset{\sim}{\rightarrow} C \). Combining the previous calculation with Theorem 1.3 yields the weak recollement
\[ \langle S(1), \ldots, S(p_t-1) \rangle \cap F \overset{\sim}{\rightarrow} \mathcal{E}_F \overset{\sim}{\rightarrow} C. \]

By arguments analogous to the ones given in the proof of Proposition 3.9, we obtain the statement. \( \square \).
Remark 5.4. This statement can be interpreted geometrically, i.e. from the point of view of weighted projective lines or of Deligne–Mumford curves: while skyscraper sheaves in ordinary points are spherical, skyscraper sheaves of length 1 supported on points with non-trivial isotropy are exceptional. Moreover, if the local isotropy at a point $x$ has order $p$, then a $p$-fold extension of these exceptional sheaves is a 1-spherelike sheaf. The spherical subcategory of that sheaf contains the weighted/stacky curve with trivial isotropy at $x$.

We mention that the objects at the mouth of exceptional tubes form what is called an exceptional cycle in [4]. There is an associated autoequivalence which in this case goes back to Meltzer’s tubular mutations [23]. From the point of view of geometry, these are line bundle twists on the stacks.

6. Discrete derived algebras

We turn to algebras with discrete derived categories, also called ‘discrete derived algebras’. These have been introduced by Vossieck [30] and also classified up to Morita equivalence. Bobiński, Geiß & Skowroński [2] provide normal forms for the derived equivalence classes of these algebras, and they also compute the Auslander–Reiten quivers of the derived categories.

As we are interested in spherelike objects and their spherical subcategories, we employ the derived normal form of [2] and consider the algebras $\Lambda(r,n,m)$ whose bound quiver consists of an oriented cycle of length $n \geq 2$ with $1 \leq r < n$ consecutive zero relations and, for $m \geq 0$, an oriented $A_m$-quiver inserted into the last vertex of the cycle part of the $r$ relations.

![Diagram](image)

By [2], the Auslander–Reiten quiver of $D^b(\Lambda(r,n,m))$ consists of $3r$ components $X^0, \ldots, X^{r-1}, Y^0, \ldots, Y^{r-1}, Z^0, \ldots, Z^{r-1}$ and by abuse of notation we denote the full subcategories of $D^b(\Lambda(r,n,m))$ of objects in an AR-component by the same symbol.

The $X$ and $Y$ components are of type $ZA_{\infty}$, whereas the $Z$ components are of type $ZA_{\infty}$. Hence there is a notion of height $h(A)$ for indecomposable objects $A$ in the $X$ and $Y$ components; we take height 0 to mean objects at the mouths. We write $X$ for the additive hulls of the union of $X^0, \ldots, X^{r-1}$, and analogously for $Y$ and $Z$. The subcategories $X$ and $Y$ are triangulated, and the shift in these categories satisfies $[r]_X = \tau^{-m-r}$ and $[r]_Y = \tau^{n-r}$. This can be rephrased by saying that $X$ and $Y$ are fractionally Calabi–Yau of CY-dimensions $m+r$ and $n-r$, respectively.

The existence of spherelike objects is fully understood: [4, Prop. 5.4] states that all indecomposable objects in $Z$ are exceptional, and moreover

\begin{align*}
X \in \text{ind}X \text{ spherelike} &\iff h(X) = m + r - 1; \\
Y \in \text{ind}Y \text{ spherelike} &\iff h(Y) = n - r - 1.
\end{align*}
and then $X$ is $(1 - r)$-spherelike, and $Y$ is $(1 + r)$-spherelike, respectively. In certain cases the spherelike objects are Calabi–Yau:

$$X \in \text{ind } \mathcal{Y} \text{ spherical } \iff m = 0, r = 1, h(X) = 0;$$

$$Y \in \text{ind } \mathcal{Y} \text{ spherical } \iff n = r + 1, h(Y) = 0.$$

**Proposition 6.1.** Let $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ be indecomposable spherelike objects in $D = D^b(\Lambda(r, n, m))$. Then the spherical subcategories $\mathcal{D}_X, \mathcal{D}_Y$ are generated by exceptional collections and satisfy conditions $(\dagger), (\ddagger)$.

If $X$ is properly spherelike with $r > 1$, its spherical subcategory $\mathcal{D}_X$ is not the bounded derived category of a finite-dimensional $k$-algebra.

If $X$ is properly spherelike with $r = 1$, its spherical subcategory $\mathcal{D}_X$ is $D^b(\Lambda(1, n, m))_X \cong D^b(kA_{m-1}) \oplus D^b(\Lambda(1, n, 0))$.

If $Y$ is properly spherelike, i.e. $n > r + 1$, its spherical subcategory $\mathcal{D}_Y$ is

$$D^b(\Lambda(r, n, m))_Y \cong D^b(kA_{n-r-2}) \oplus D^b(\Lambda(r, r, m)).$$

**Proof.** Fix an indecomposable object $X \in \mathcal{X}$ of height $m + r - 1$. Then $X$ is $(1 - r)$-spherelike and $\omega(X) := SX[r - 1] = \tau X[r] = \tau^{1-m-r}X$, using $[r]|_X = \tau^{-m-r}$. Up to scalars, there is a unique non-zero morphism $X \to \omega(X)$, and its cone is the asphericality $Q_X$. Denote by $E \in \mathcal{X}$ the unique indecomposable object of height $m + r - 2$ admitting an irreducible morphism $E \to X$. Note that $E$ is an exceptional object.

We claim that $Q_X \cong E[1] \oplus E[r]$. For this, we consider the triangle $E \to X \to X_0$, where $X_0 \in \text{ind } \mathcal{X}$ is the unique object with $h(X_0) = 0$ and $\text{Hom}(X, X_0) \neq 0$. By duality, there is a non-zero map $\omega(X) \to E[r]$, giving rise to a triangle $\omega(X) \to \omega(X) \to E[r]$, where now $\omega(X) \in \text{ind } \mathcal{X}$ is the unique object with $h(\omega(X)) = 0$ and $\text{Hom}(\omega(X), \omega(X)) \neq 0$. At this point, $h(X) = m + r - 1$ implies $X_0 = \omega(X)$, see e.g. the Hom hammock computations of [4, §2]. The octahedral axiom applied to the composition of $X \to X_0$ and $X_0 \to \omega(X)$ yields a triangle $E[1] \to Q_X \to E[r]$. Since $E$ is exceptional, the connecting morphism $E[r] \to E[2]$ has to vanish, triggering $Q_X$ and a contradiction to $X$ non-spherical. Thus the triangle splits, showing the claim.

Hence we get $\mathcal{D}_X = \perp Q_X = \perp E$. In particular, $E$ exceptional shows that both conditions $(\dagger)$ and $(\ddagger)$ are met; note $D^b(\Lambda(r, n, m))$ has a Serre functor because $\Lambda(r, n, m)$ is of finite global dimension if and only if $n > r$ — an assumption we have made at the outset. Exactly the same reasoning works for $Y \in \text{ind } \mathcal{Y}$ of height $n - r - 1$.

If $r = 1$, we use that $\Lambda(1, n, m) = A_m \triangleright \Lambda(1, n, 0)$ is obtained by taking $A_m$ onto the circular quiver algebra $\Lambda(1, n, 0)$. Now $D^b(\Lambda(1, n, 0))$ has up to shift a unique 0-spherical object $X'$ and the subgroup $\langle [1], \tau \rangle \subset \text{Aut}(D^b(\Lambda(1, n, m)))$ acts transitively on 0-spherelike objects. Hence the embedding $D^b(\Lambda(1, n, 0)) \hookrightarrow D^b(\Lambda(1, n, m))$ maps $X' \to \tau^sX[t]$ for some $s, t \in \mathbb{Z}$. Applying Lemma 2.2 for the autoequivalence $\tau^s[t]$ and Proposition 3.15 show

$$D^b(\Lambda(1, n, m))_X \cong D^b(\Lambda(1, n, m))_{\tau^sX[t]} \cong D^b(kA_{m-1}) \oplus D^b(\Lambda(1, n, 0)).$$

If $r > 1$, then the spherical subcategory $\mathcal{D}_X$ is not the derived category of a finite-dimensional $k$-algebra: $\mathcal{D}_X \cong D^b(A)$ for a finite-dimensional $k$-algebra
Figure 3. The $X^0$ part of the spherical subcategory of $X \in D^b(\Lambda(2,n,2))$.

Figure 4. The $Z^0$ part of that spherical subcategory.

$A$ implies, by Lemma 1.6, that $A$ has finite global dimension. However, that contradicts Lemma 1.4 and the fact that $X \in D$ is $(1-r)$-spherical.

In order to compute $D_Y$, note that $\Lambda(r,n,m)$ is an $A_{n-r-1}$-insertion of $\Lambda(r,r+1,m)$ at the successor of the trivalent vertex. Up to shift, there is a unique $(1+r)$-spherical object $Y'$ in $D_Y$. Proposition 3.9 shows that $D_Y \cong D^b(kA_{n+r-2}) \oplus D_Y(\Lambda(r,r+1,m))$. In particular, since $D_Y(\Lambda(r,r+1,m)) \cong D_Y(\Lambda(r,r+1,m))$, the claim about $D_Y$ follows.

Remark 6.2. It is easy to compute the left orthogonal of the (shifted) asphericality $Q_X = E$, using the Hom hammock computations of [4, §2]. From $\text{Hom}^*(A,E) = \text{Hom}^*(E,\nu A)^* = \text{Hom}^*(\tau^{-1}E,A)^*$, we get $\perp E = \tau^{-1}E \perp$. By [4, §2], the right orthogonal of $\tau^{-1}E$ consists of the full $\mathcal{Y}$ component and strips of triangles of width $m + r - 2$ at the bases of all $\mathcal{X}$ components together with the lattice consisting of $(X)$, i.e. extensions and shifts of $X$.

Regarding the spherelike object $Y$, the indecomposables of $D_Y$ look like those of $D_X$, with the roles of $\mathcal{X}$ and $\mathcal{Y}$ reversed: $\mathcal{X} \subset D_Y$ and the indecomposables of $D_Y$ at the bottom of $\mathcal{Y}^0$ make up triangles of width $n - r - 2$.

Example 6.3. Let $r = m = 2$ and $n > 2$ arbitrary, and $X$ as before. Write $\mathcal{D} = D^b(\Lambda(2,n,2))$ and note $[2]|_X = \tau^{-4}$. Figure 3 depicts $D_X \cap X^0$. The
shaded regions are the various Hom hammocks; the darker region on the left hand is \( \{ A \in \mathcal{X}^0 \mid \text{Hom}(A,E) = \text{Hom}(\tau^{-1}E[-1],A)^* \neq 0 \} \), and the one on the right is \( \{ A \in \mathcal{X}^0 \mid \text{Hom}(\tau^{-1}E,A) \neq 0 \} \). The intersection \( \mathcal{D}_X \cap \mathcal{X}^1 \) is not shown; it looks similar to Figure 3 and contains the odd shifts of \( X \).

The additive subcategory generated by unshaded objects of heights \( 3+4k \) with \( k \in \mathbb{N} \) in \( \mathcal{X}^0 \) and \( \mathcal{X}^1 \) is \( (X) \), the triangulated category generated by \( X \).

Figure 4 shows the intersection \( \mathcal{D}_X \cap \mathcal{Z}^0 \). Again, the picture for \( \mathcal{D}_X \cap \mathcal{Z}^1 \) looks similar. Finally, we have \( \mathcal{D}_X \cap \mathcal{Y} = \mathcal{Y} \). Altogether, we get a complete and explicit description of \( \mathcal{D}_X \) in this example.

**Theorem 6.4.** The spherelike poset of \( \mathbb{D} = \mathbb{D}^b(\Lambda(r,n,m)) \) is

\[
\mathcal{P}(\mathbb{D}) = \begin{cases} 
\{ \mathbb{D} \} & \text{if } (r,n,m) = (1,2,0); \\
\{ \mathbb{D} > \mathbb{D}_{X_1}, \ldots, \mathbb{D}_{X_{m+r}} \} & \text{if } r = n - 1 \text{ and } m + r > 1; \\
\{ \mathbb{D} > \mathbb{D}_{Y_1}, \ldots, \mathbb{D}_{Y_{n-r}} \} & \text{if } r = 1, m = 0, n \geq 2; \\
\{ \mathbb{D}_{X_1}, \ldots, \mathbb{D}_{X_{m+r}}, \mathbb{D}_{Y_1}, \ldots, \mathbb{D}_{Y_{n-r}} \} & \text{else},
\end{cases}
\]

where \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \) are fixed indecomposable spherelike objects and \( X_i := \tau^{-i-1}X \) and \( Y_i := \tau^{-i-1}Y \).

**Proof.** Fix indecomposable spherelike objects \( X \in \mathcal{X} \) and \( Y \in \mathcal{Y} \). They are both spherical if and only if \( m = 0, r = 1 \) and \( n = 2 \). In this case, the poset \( \mathcal{P}(\mathbb{D}) \) is reduced to one element.

Now \( Y \) is spherical if and only if \( n = r + 1 \). Given this, \( X \) is properly spherical if and only if \( m > 0 \) or \( r > 1 \); this is equivalent to \( m + r > 1 \). The spherical subcategory of \( Y \) is \( \mathcal{D} = \mathcal{D}_Y \) and \( X, \tau X, \ldots, \tau^{m+r-1}X \) are \( m + r \) pairwise incomparable properly spherelike objects, hence \( \mathcal{D}_Y > \mathcal{D}_{\tau^i X} \).

Next, \( X \) spherical and \( Y \) properly spherelike is equivalent to \( r = 1, m = 0 \) and \( n > 1 \). Here, the top element is \( \mathcal{D} = \mathcal{D}_X \) and the bottom elements are the spherical subcategories of \( Y_1, \ldots, Y_{n-r} \) which differ by consecutive \( \tau \)-translations, hence are incomparable.

Finally, we have the ‘generic’ case, when \( \mathbb{D} \) has no spherical objects. Here, we have the mutually incomparable spherelike objects \( X_1, \ldots, X_{m+r} \) and \( Y_1, \ldots, Y_{n-r} \), i.e. \( \mathcal{P}(\mathbb{D}) \) is discrete of cardinality \( m + n \).

\( \square \)

The following corollary also holds for \( r = n \), because the \( \mathcal{X} \) and \( \mathcal{Y} \) components (which contain the spherelike objects) remain the same.

**Corollary 6.5.** The d-spherelike poset of \( \mathbb{D} = \mathbb{D}^b(\Lambda(r,n,m)) \) has cardinality

\[
\# \mathcal{P}_d(\mathbb{D}) = \begin{cases} 
m + r, & \text{if } d = 1 - r; \\
n - r, & \text{if } d = 1 + r; \\
0, & \text{otherwise}.
\end{cases}
\]

In particular, the parameters \( r,n,m \) are determined by the integer-indexed sequence \( \# \mathcal{P}_d(\Lambda(r,n,m))_{d \in \mathbb{Z}} \), making this sequence a complete derived invariant for discrete derived algebras.

**Corollary 6.6.** There are three possibilities for the stable spherelike poset \( \mathcal{G}(\mathbb{D}) \) of \( \mathbb{D} = \mathbb{D}^b(\Lambda(r,n,m)) \):

\( \mathcal{G} = \{ \bullet \} \) if all spherelike objects are spherical, \( (r,n,m) = (1,2,0) \);

\( \mathcal{G} = \{ \bullet \bullet \} \) if there are no spherical objects;

\( \mathcal{G} = \{ \bullet \bullet < \bullet \} \) if spherical and properly spherelike objects exist.
Proof. As we look at the spherelike poset up to autoequivalences, all spherelike objects in $\mathcal{X}$ become identified under AR-translation; same for $\mathcal{Y}$. If both $\mathcal{X}$ and $\mathcal{Y}$ possess properly spherelike objects, then the associated spherical subcategories are incomparable by Proposition 6.1, leading to $GP = \{\bullet \bullet \}$. The other cases are obvious. □

Corollary 6.7. Let $\Lambda$ be an arbitrary finite-dimensional algebra and let $\mathcal{D} := D^b(\Lambda)$. If $GP(\mathcal{D})$ has cardinality greater than two, or if the height of $P(\mathcal{D})$ is greater than two, then $\Lambda$ is not a discrete derived algebra.

Remark 6.8. Note that $D_X$ and $D_Y$ are the left orthogonal complements of certain exceptional objects; in particular, they form parts of a recollement of $D^b(\Lambda)$ with their respective complements. There is an interesting contrast if we take some indecomposable object $Z \in Z$ (which is always exceptional) instead: by [4, Prop. 6.5], the orthogonal complement of $Z$ is the derived category of an iterated algebra of type $A_n^{m-1}$; more precisely, there is a recollement $D^b(kA_n^{m-1}) \leftarrow \cdots \leftarrow D^b(\Lambda(r,n,m)) \leftarrow \cdots \langle Z \rangle$. Note that $D^b(kA_n)$ does not contain any indecomposable spherelike objects.

7. A non-commutative curve and a tilted surface

Example 7.1. We give an example of a 3-spherelike object, such that the spherical subcategory is neither the derived category nor the perfect derived category of a finite-dimensional algebra. Consider the algebra $\Lambda$ given by the following quiver

$$
1 \xleftarrow{a} b \xrightarrow{3} 2 \xleftarrow{a} b \xrightarrow{3} 3
$$

with relations $I = (a^2, b^2)$ and the following $\Lambda$-module $E$

$$
\begin{array}{c}
\begin{array}{c}
k
\end{array}
\end{array} \xleftarrow{1} \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array} \xrightarrow{3} \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} \xleftarrow{1} \begin{array}{c}
\begin{array}{c}
0
\end{array}
\end{array} \xrightarrow{3} k.
$$

Then $E$ is an exceptional object of $D^b(\Lambda)$ and, moreover, $(\tau^{-1}E, E)$ forms an exceptional $(2, 2)$-cycle in the sense of [4], i.e. $SE \cong (\tau^{-1}E)[2]$ and $S(\tau^{-1}E) \cong E[2]$. There is an Auslander–Reiten triangle $E \to F \to \tau^{-1}E$ and by [4, Prop. 3.6], $F$ is 3-spherelike with asphericality $Q_F = E[1] \oplus E[-2]$. In particular, $D^b(\Lambda)_F = \frac{1}{2}E$.

We use a $K$-theoretic argument first observed by Bondal, see e.g. [21]. Computing the Euler form $\chi(-, -)$ on $K(\Lambda)$ in the basis $S(1), S(2), S(3)$, and its restriction to the subgroup $\frac{1}{2}[E]$ gives

$$
\chi = \begin{pmatrix}
1 & -2 & 2 \\
0 & 1 & -2 \\
0 & 0 & 1
\end{pmatrix}
$$

and

$$
\chi|_{\frac{1}{2}[E]} = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
$$

where $\frac{1}{2}[E] = \langle [S(1)] + [S(2)], [S(2)] + [S(3)] \rangle$. Therefore $\frac{1}{2}E$ has an anti-symmetric Euler form and consequently cannot be of the form $K^b(proj - A)$ or $D^b(A)$ for a finite-dimensional algebra $A$, since $\chi([A], [A]) = \dim_k A \neq 0$.

As $E$ is exceptional and $\Lambda$ has finite global dimension, conditions (†) and ($\ddagger$) are satisfied. Burban & Drozd [7] have shown that $\Lambda$ is derived equivalent to a certain non-commutative nodal cubic curve. Under this equivalence, $E$ corresponds to a simple skyscraper sheaf supported on the singular point.
Example 7.2. Let $\mathcal{D} = \mathcal{D}^b(\Lambda)$ with the tensor algebra $\Lambda = k\tilde{A}_1 \otimes_k k\tilde{A}_1$ of the Kronecker quiver $\tilde{A}_1 = \begin{array}{c} 1 \\ \longrightarrow \\ 2 \end{array}$. We show that $\mathcal{P}(\mathcal{D})$ has infinite width and, particularly, infinite cardinality.

Consider the idempotent $e = e_1 \otimes e_1 + e_1 \otimes e_2 \in \Lambda$. One can check that $e\Lambda e \cong k\tilde{A}_1 = k(\bullet \longrightarrow \bullet \longrightarrow \bullet)$ and that $\Lambda/e \cong \Lambda(1-e)$ is a projective $\Lambda$-module. This shows that the conditions of Proposition 3.1 are satisfied.

Thus $j = \Lambda e \otimes_{e\Lambda e} (-) : \mathcal{D}^b(e\Lambda e) \to \mathcal{D}^b(\Lambda)$ is an inclusion which has a right adjoint. Therefore we may apply Theorem 1.3 to obtain a weak recollement

$$\text{thick}(\Lambda/e\text{-mod}) \cap \mathcal{D}(F_x) \leftrightarrow \mathcal{D}(F_x) \leftrightarrow \text{thick}(\Lambda e),$$

where for each $x \in \mathbb{P}^1$, $F_x$ denotes the corresponding quasi-simple $e\Lambda e$-module — in other words, the modules sitting at the bottom of homogeneous tubes. Hence these are 1-spherical objects.

For $y \in \mathbb{P}^1$, let $G_y$ be the corresponding quasi-simple module over $\Lambda/e \cong k\tilde{A}_1 \cong k(\bullet \longrightarrow \bullet \longrightarrow \bullet)$. A computation in the homotopy category $\mathcal{K}^b(\text{proj-}\Lambda)$ shows that $G_y \in \mathcal{D}(F_x)$ if and only if $x \neq y$. The weak recollement translates this to $G_y \in \mathcal{D}(\Lambda)_{j(F_x)}$ if and only if $x \neq y$. Consequently, the spherical subcategories $\mathcal{D}(\Lambda)_{j(F_x)}$ are pairwise incomparable in $\mathcal{D}$.

We remark that all these spherical subcategories $\mathcal{D}^b(\Lambda)_{j(F_x)}$ become one element in $\mathcal{GP}(\Lambda)$, since $\text{Aut}(\mathbb{P}^1) \subseteq \text{Aut}(\mathcal{D}^b(\Lambda/e))$ acts transitively on tubes.

The above example corresponds to $\mathbb{P}^1 \times \mathbb{P}^1$ since $\mathcal{D}^b(\tilde{A}_1) \cong \mathcal{D}^b(\mathbb{P}^1)$, where $F_x$ corresponds to $\pi^*O_p$ with $\pi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ the projection to the first factor and $p \in \mathbb{P}^1$. Here, by [13, Ex. 4.7]

$$\mathcal{D}^b(\mathbb{P}^1 \times \mathbb{P}^1)_{\pi^*O_p} = (\pi^*\mathcal{D}^b_{\mathbb{P}^1\setminus\{p\}}(\mathbb{P}^1) \otimes \mathcal{O}(-1,0), \pi^*\mathcal{D}^b(\mathbb{P}^1)),$$

Now, let $X$ be $\mathbb{P}^1 \times \mathbb{P}^1$ blown-up in 5 points in general position, so $X$ is a del Pezzo surface of degree 3. Since its anti-canonical bundle is ample, $\text{Aut}(\mathcal{D}^b(X))$ is generated by the shift, tensoring with line bundles and pullbacks via automorphisms; see [14, Prop. 4.17]. But by [20], $\text{Aut}(X)$ is trivial, so only tensoring with line bundles and the shift remain.

If we consider the composition $\tilde{\pi} : X \to \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\pi} \mathbb{P}^1$ and a point $p$ such that $\tilde{\pi}^{-1}(p)$ does not contain an exceptional divisor, then the spherical subcategory of $\tilde{\pi}^*\mathcal{O}_p$ is essentially the same as above (where we add terms of the form $\mathcal{O}_E(-1)$ for each exceptional divisor $E$; see [13, Sect. 4.2]). Since shift and twisting with a line bundle do not connect two different spherical subcategories of this kind (for they do not change the support), there is a subposet of infinite width in $\mathcal{GP}(X)$. The surface $X$ admits a tilting bundle by [12], and so this geometric example also gives rise to a representation theoretic example.

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References

[1] A. Beilinson, J. Bernstein, P. Deligne, Faisceaux pervers, Astérisque 100 (1980).
[2] G. Bobiński, C. Geiß, A. Skowroński, Classification of discrete derived categories, Central Eur. J. Math. 2 (2004), 19–49.
[3] T. Bridgeland, D. Stern, *Helices on del Pezzo surfaces and tilting Calabi–Yau algebras*, Adv. Math. 224 (2010), 1672–1716, also arXiv:0909.1732.
[4] N. Broomhead, D. Pauksztello, D. Ploog, *Discrete derived categories I — Homomorphisms, autoequivalences and t-structures*, arXiv:1312.5203.
[5] A.B. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 210 (2007), 51–82, also arXiv:math/0402054.
[6] A.B. Buan, O. Iyama, I. Reiten, J. Scott, *Cluster structures for 2-Calabi–Yau categories and unipotent groups*, Compos. Math. 145 (2009), 1035–1079, also arXiv:math/0701557.
[7] I. Burban, Y. Drozd, *Tilting on non-commutative rational projective curves*, Math. Ann. 351 (2011), 665–709.
[8] E. Cline, B. Pashall, L. Scott, *Finite dimensional algebras and highest weight categories*, J. reine angew. Math. 391 (1988), 85–99.
[9] W. Geigle, H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, in Singularities, Representation of Algebras and Vector Bundles (Eds. G.-M. Greuel, G. Trautmann), Springer Verlag (1987), 265–297.
[10] C. Geiss, B. Leclerc, J. Schröer, *Cluster algebra structures and semicanonical bases for unipotent groups*, arXiv:math/0703039v4.
[11] D. Happel, *Tilting objects in categories of finite dimensional representations of algebras*, Cambridge University Press (1988).
[12] L. Hille, M. Perling, *Tilting bundles on rational surfaces and quasi-hereditary algebras*, to appear in Ann. Inst. Fourier, also arXiv:1110.5843.
[13] A. Hochenegger, M. Kalck, D. Ploog, *Spherical subcategories in algebraic geometry*, arXiv:1208.4046.
[14] D. Huybrechts, *Fourier-Mukai Transforms in Algebraic Geometry*, Oxford University Press (2006).
[15] K. Igusa, Sh. Liu, Ch. Paquette, *A proof of the strong no loop conjecture*, Adv. Math. 228 (2011), 2731–2742, also arXiv:1103.5361.
[16] O. Iyama, *Cluster-tilting for higher Auslander algebras*, Adv. Math. 226 (2011), 1–61, also arXiv:0809.4897.
[17] B. Keller, *On triangulated orbit categories*, Doc. Math. 10 (2005), 551–581, also arXiv:math/0503240.
[18] B. Keller, I. Reiten, *Cluster-tilted algebras are Gorenstein and stably Calabi–Yau*, Adv. Math. 211 (2007), 123–151, also arXiv:math/0512471.
[19] B. Keller, D. Yang, G. Zhou, *The Hall algebra of a spherical object*, J. London Math. Soc. 80 (2009), 771–784, also arXiv:0810.5546.
[20] M. Koitabashi, *Automorphism groups of generic rational surfaces*, J. Algebra 116 (1988), 130–142.
[21] A. Kuznetsov, *A simple counterexample to the Jordan-Hölder property for derived categories*, arXiv:1304.0903.
[22] L. Li, *Triangular matrix algebras: Recollements, torsion theories, and derived equivalences*, arXiv:1311.1258.
[23] H. Meltzer, *Tubular mutations*, Colloq. Math. 74 (1997), 267–274.
[24] J.-I. Miyachi, *Récoclements et Idempotent Ideals*, Tsukuba J. Math. 16 (1992), 545–550.
[25] D.O. Orlov, *Derived categories of coherent sheaves and triangulated categories of singularities*, in: Y. Tschinkel (ed.), Algebra, arithmetic, and geometry. Vol. II. Birkhäuser. Progress in Math. 270 (2009), 503–531, also arXiv:math/0503632.
[26] D.O. Orlov, *Triangulated categories of singularities and equivalences between Landau-Ginzburg models*, Sb. Math. 197 (2006), 1827–1840, also arXiv:math/0503630.
[27] C.M. Ringel, *The canonical algebras*, (with an appendix by W. Crawley-Boevey) Banach Center Publ. 26, Part 1, Topics in algebra, Warsaw (1988), 407–432.
[28] P. Seidel, R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. 108 (2001), 37–108, also arXiv:math/0001043.
[29] A. Skowroński, *Generalized canonical algebras and standard stable tubes*, Colloq. Math. 90 (2001), 77–93.
[30] D. Vossieck, *The algebras with discrete derived category*, J. Algebra 243 (2001), 168–176.
[31] A. Wiedemann, *On stratifications of derived module categories*, Canad. Bull. Math. 34 (1991), 275–280.

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