Classification of small \((0,1)\) matrices

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Abstract

Denote by \(A_n\) the set of square \((0,1)\) matrices of order \(n\). The set \(A_n, n \leq 8\), is partitioned into row/column permutation equivalence classes enabling derivation of various facts by simple counting. For example, the number of regular \((0,1)\) matrices of order 8 is \(10160459763342013440\). Let \(D_n, S_n\) denote the set of absolute determinant values and Smith normal forms of matrices from \(A_n\). Denote by \(a_n\) the smallest integer not in \(D_n\). The sets \(D_9\) and \(S_9\) are obtained; especially, \(a_9 = 103\). The lower bounds for \(a_n\), \(10 \leq n \leq 19\), (exceeding the known lower bound \(a_n \geq 2f_n-1\), where \(f_n\) is \(n\)th Fibonacci number) are obtained. Row/permutation equivalence classes of \(A_n\) correspond to bipartite graphs with \(n\) black and \(n\) white vertices, and so the other applications of the classification are possible.

Key words: \((0,1)\) matrices, Smith normal form, permutation equivalence, determinant range, classification

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1 Introduction

Let \(A_n\) denote the set of square \((0,1)\) matrices of order \(n\). Hadamard maximum determinant problem is: find the maximum determinant among the matrices in \(A_n\). In this paper we consider a slightly more general problem: determine the set \(D_n = \{|\det A| \mid A \in A_n\}\).

It is known [1] that determinants of \((0,1)\) matrices of order \(n\) are related to determinants of \(\pm1\) matrices of order \(n+1\). If \(A\) is a \((0,1)\)-matrix of order \(n\), let \(B = \Psi(A)\) be a \(\pm1\)-matrix of order \(n+1\) obtained from \(A\) by replacing

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its 0 by –1, bordering with a row –1’s on the top, and a column of 1’s on the right. Clearly, Ψ is a one-to-one correspondence. By adding row 1 of B to each of the other rows of B, we see that det B = 2^n det A.

By the Hadamard inequality |det B| ≤ \sqrt{(n + 1)^{n+1}}, and therefore for all A ∈ \mathcal{A}_n |det A| ≤ 2^{-n} \sqrt{(n + 1)^{n+1}}. The equality is attained if B is an Hadamard matrix, i.e. if BB^T = (n+1)I_{n+1}, where T denotes transposition, and I_n is the unit matrix of order n; for n > 2 this implies n = 4k – 1. For upper bounds for determinants of A ∈ \mathcal{A}_n see for example [2].

Let d_n denote the largest element in \mathcal{D}_n, and let a_n be the smallest integer not in \mathcal{D}_n. Craigen [3] shows that the set \mathcal{D}_n is the interval \{1, 2, ..., d_n\} for n ≤ 6, but not for n = 7, because a_8 = 41 < d_8 = 56; he suggests that a_9 = 103.

Some interesting sequences, related to (0, 1) matrices are found in [4]: A003432 (the sequence d_n), A013588 (the sequence a_n), A051752 (c_n, the number of matrices in \mathcal{A}_n with the determinant d_n) and A055165 (m_n, the number of regular matrices in \mathcal{A}_n). A few first members of these sequences are given in the following table. The values of a_9, c_8, c_9 and m_8 seem to be new.

| n | A003432 | A013588 | A051752 | A055165 |
|---|--------|--------|--------|--------|
| 1 | 1      | 2      | 1      | 1      |
| 2 | 1      | 2      | 3      | 6      |
| 3 | 2      | 3      | 3      | 174    |
| 4 | 3      | 4      | 60     | 22560  |
| 5 | 5      | 6      | 3600   | 12514320|
| 6 | 9      | 10     | 529200 | 28836612000|
| 7 | 32     | 19     | 75600  | 27034566985440|
| 8 | 56     | 41     | 195955200 | 10160459763342013440 |
| 9 | 144    | *103   | 13716864000 | *13716864000 |
| 10| 320    |        |        |        |
| 11| 1458   |        |        |        |
| 12| 3645   |        |        |        |
| 13| 9477   |        |        |        |

In this paper, which is a continuation of [5], the matrices in \mathcal{A}_n, n ≤ 8, are
partitioned into row/column permutation equivalence classes, enabling the classification by ADV, and more precisely — by SNF (see section 2). Let \( S_n \) denote the set of SNF’s of matrices in \( A_n \). In section 3 the sets \( D_9 \) and \( S_9 \) are determined. In section 4 the lower bounds for \( a_n, 10 \leq n \leq 19 \) are obtained; \( c_n, n \leq 9 \), are obtained in section 5.

We introduce now some notation. If \( A = [a_{ij}] \) and \( B = [b_{ij}] \) are matrices of the same dimension \( m \times n \), we say that \( A < B \) if \( A \) is lexicographically less than \( B \), i.e. if for some pair of indices \((i, j)\) the first \( i - 1 \) rows of \( A \) and \( B \) are equal, the first \( j - 1 \) elements in the \( i \)th row of \( A \) and \( B \) are equal, and \( a_{ij} < b_{ij} \). For example, 

\[
\begin{bmatrix}
1 & 0 \\
1 & 0
\end{bmatrix} < \begin{bmatrix}
1 & 0 \\
1 & 1
\end{bmatrix}.
\]

The smallest matrix in a set \( A \) is the representative of \( A \).

Denote by \( P_{i,j} \) the permutation matrix obtained from \( I_n \) by exchanging the \( i \)th and \( j \)th row.

The matrices \( A, B \in A_n \) are equivalent \([7]\), \( A \sim B \), if \( B \) is be obtained from \( A \) by a sequence of elementary row/column operations of the following types: exchange of two rows/columns, multiplication of a row/column by \(-1\), and addition/subtraction of a row/column to/from another row/column. Let \( \text{SNF}(A) \) denote the SNF of \( A \). It is known that \( A \sim B \) is equivalent to \( \text{SNF}(A) = \text{SNF}(B) \) (in [7] this statement is proved for polynomial matrices).

The SNF \( \text{diag}(d_1, d_2, \ldots, d_n) \) is written simply as a vector \((d_1, d_2, \ldots, d_n)\). If diagonal elements of SNF are repeated, we use the shortened exponential notation. For example, \((1^3, 2, 0)\) is short \((1, 1, 1, 2, 0)\). If \( s \in S_n \), then we also say that the SNF-class \( s \) is the set \( \{A \in A \mid \text{SNF}(A) = s\} \).

Let \( J_n \) denote the square matrix of order \( n \) with all elements equal to one.

2 Classification of \((0, 1)\) matrices of order 8 or less

The set \( D_n \) could be obtained by computing determinants of all \( A \in A_n \). A better approach is to group matrices with the same determinant, and then to compute the determinant of only one matrix in each group. It is useful to classify \( A_n \) into subsets with constant absolute determinant value(ADV), or into even smaller subsets with constant SNF. We now review some such partitions of \( A_n \).
Let $\Pi_r$ denote the group of row permutations of matrices from $\mathcal{A}_n$. Permutations from $\Pi_r$ preserve $\text{ADV}$.

The representative of the matrix $A$ orbit is obtained from $A$ by sorting its rows into a nondecreasing sequence. Rows of $A$ correspond to binary numbers less than $N = 2^n$. Therefore, the number of orbits of $\Pi_r$ in $\mathcal{A}_n$ is equal to $\binom{N+n-1}{n-1}$, i.e. the number of nondecreasing sequences of length $n$ from $\{0, 1, \ldots, N-1\}$.

Let $\Pi$ denote the group of row and column permutations; $\Pi$ also preserves $\text{ADV}$. The group $\Pi$ induces an equivalence relation $\pi$ over $\mathcal{A}_n$. We say that matrices $A$ and $B$ are permutationally equivalent, $A \sim_\pi B$, if they are in the same orbit of $\Pi$. Let $A_\pi$ denote the representative of the matrix $A$ equivalence class ($\pi$-class; we say shorter that $A_\pi$ is a $\pi$-representative of $A$).

**Example 1** The $\pi$-representative of

$$
\begin{bmatrix}
1 & 0 & 1 \\
1 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
$$

is the matrix

$$
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{bmatrix}
$$

the smallest of all 36 permutationally equivalent matrices.

Let $\mathcal{A}_n^\pi$ denote the set of $\pi$-representatives in $\mathcal{A}_n$. In [8] it is shown that the number of $\pi$-classes in $\mathcal{A}_n$ is given by:

$$
|\mathcal{A}_n^\pi| = \sum_{i_1+2i_2+\ldots+ni_n=n} \sum_{j_1+2j_2+\ldots+nj_n=n} C(i)C(j) \exp_2 \sum_{r,s=1}^{n} i_rj_s2^{(r,s)},
$$

(1)

where the summation is over all vectors $i = (i_1, i_2, \ldots, i_n)$, $j = (j_1, j_2, \ldots, j_n)$, and

$$
C(i) = n!/(1^{i_1}i_1! \ldots n^{i_n}i_n!)
$$

is the number $n$-permutations with $i_r$ cycles of length $r$, $r = 1, 2, \ldots, n$; $(r, s)$ denotes GCD of integers $r$, $s$. The values $|\mathcal{A}_n^\pi|$ are listed in Table 1; they are easily computed for quite a large $n$ using, for example, UBASIC [9]. It is seen that $p_n$ is close to $2^{n^2}/(n!)^2$ for $n \leq 15$. An effective algorithm to generate the representative $A_\pi$ of a given matrix $A$ (section 2.3) simplifies the classification of matrices, because it enables to deal with the small subset $\mathcal{A}_n^\pi$ of $\mathcal{A}_n$.  

4
Table 1
The number of permutationally nonequivalent matrices in $A_n$, $n \leq 15$.

| $n$ | $(2^{n^2}/n^2)/|A_n^\pi|$ | $|A_n^\pi|$ |
|-----|---------------------------|-----------|
| 1   | 1.00000                   | 2         |
| 2   | 0.57143                   | 7         |
| 3   | 0.39506                   | 36        |
| 4   | 0.35892                   | 317       |
| 5   | 0.41433                   | 5624      |
| 6   | 0.52685                   | 251610    |
| 7   | 0.65875                   | 33642660  |
| 8   | 0.77266                   | 14685630688 |
| 9   | 0.85533                   | 21467043671008 |
| 10  | 0.91045                   | 105735224248507784 |
| 11  | 0.94565                   | 1764356230257807614296 |
| 12  | 0.96754                   | 100455994644460412263071692 |
| 13  | 0.98088                   | 19674097197480928600253198363072 |
| 14  | 0.98886                   | 13363679231028322645152300040033513414 |
| 15  | 0.99358                   | 31735555932041230032311939400670284689732948 |

2.1 Matrix extension

In order to classify matrices in $A_n$ by ADV values, one has to select carefully the order by which determinants are computed. It is natural to start from matrices of order $n - 1$, and then to extend them by one row and one column of ones and zeros in each possible way. For an arbitrary $B \in A_{n-1}$, let bord($B$) denote the subset of $A_n$, containing matrices with the upper left minor equal to $B$. We say that the matrices in bord($B$) are obtained by extending $B$; if $A \in$ bord($B$), then $A$ is an extension of $B$.

The calculation of determinants of all matrices in bord($B$) is an easy task. If $A \in$ bord($B$), then $A$ is of the form

$$A = \begin{bmatrix} B & y \\ x & b \end{bmatrix},$$  \hspace{1cm} (2)
where \( x = [x_1 \ x_2 \ \ldots \ x_{n-1}] \) and \( y = [y_1 \ y_2 \ \ldots \ y_{n-1}]^T \). Then \([1]\)

\[
\det A = b \det B - \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} x_i y_j \det B_{ij},
\]

(3)

where \( B_{ij} \) is the cofactor of \( B \), corresponding to \( a_{ij} \).

Obviously, \( \mathcal{A}_n = \{ A \mid (B, x, y, b) \in \mathcal{A}_{n-1} \times \{0, 1\}^{n-1} \times \{0, 1\}^{n-1} \times \{0, 1\} \} \).

If we precompute cofactors \( B_{ij} \), then determinant of each matrix from bord(\( B \)) is computed by only one addition: for the fixed \( x \), the column \( y \) might traverse the set of possible values via a Gray code (so that in the sequence of \( y \)'s each two subsequent vectors differ in exactly one position).

Williamson [1] noted that it is enough to let \( B \) cross the set of \( \pi \)-representatives in \( \mathcal{A}_{n-1} \). Let bord\(_\pi(\( B \))\) denote the set of \( \pi \)-representatives of matrices in bord(\( B \)).

**Lemma 2** If \( B \sim_\pi B' \) then bord\(_\pi(\( B \)) = \text{bord}_\pi(\( B' \))\).

**Proof.** Let \( A \in \text{bord}_\pi(\( B \)) \). If the row/column permutations, transforming \( B \) into \( B' \), are applied to the first \( n-1 \) rows/columns of \( A \), then the matrix with the upper left minor equal to \( B' \) is obtained. Therefore, the matrix permutationally equivalent to \( A \) is obtained by extending \( B' \), meaning that \( A \) is permutationally equivalent to a matrix from bord(\( B' \)), i.e. \( A \in \text{bord}_\pi(\( B' \))\). Analogously, bord\(_\pi(\( B' \)) \subseteq \text{bord}_\pi(\( B \))\), and so \( \text{bord}_\pi(\( B' \)) = \text{bord}_\pi(\( B \)) \). \( \square \)

Not only determinants, but also SNF’s of matrices in bord(\( B \)) can be efficiently computed. The preprocessing step is to compute \( D = \text{SNF}(\( B \)) = \text{diag}(d_1, d_2, \ldots, d_n) \), and the matrices \( P, Q \), such that \( PBQ = D \), \( |\det P| = |\det Q| = 1 \). In order to determine \( \text{SNF}(A) \) for an arbitrary \( A \in \text{bord}(\( B \)) \) of the form (2), we use the identity

\[
\begin{bmatrix}
P & 0 \\
0 & 1
\end{bmatrix}
\begin{bmatrix}
B & y \\
x & b
\end{bmatrix}
\begin{bmatrix}
Q & 0 \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
D & Py \\
xQ & b
\end{bmatrix}.
\]

(4)

Denote \( xQ = [a_1 \ a_2 \ \ldots \ a_n] \), \( Py = [c_1 \ c_2 \ \ldots \ c_n]^T \). Suppose \( d_1 = d_2 = \cdots = d_k = 1 \), for some \( k, 1 \leq k \leq n \). Transforming the matrix from the righthand side of (4) by subtracting the row \( i \) multiplied by \( c_i \) from the row \( n \), \( 1 \leq i \leq k \), and then subtracting the column \( i \) multiplied by \( c_i \) from the
column $n$, $1 \leq i \leq k$, we derive that $A$ is equivalent to

$$
\begin{bmatrix}
1 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 1 & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & d_{k+1} & \ldots & 0 & c_{k+1} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & 0 & \ldots & d_n & c_n \\
0 & \ldots & 0 & a_{k+1} & \ldots & a_n & b - \sum_{i=1}^{k} a_i c_i \\
\end{bmatrix}.
$$

(5)

Hence, SNF($A$) determination is reduced to determination of SNF of a matrix of order $n - k$. The special cases when $k \geq n - 1$ are extremely simple, and they are not rare at all, because the corresponding SNF-classes are among the largest ones (at least for $n \leq 9$). More generally, one can reduce $a_i, c_i$ modulo $d_i$, $1 \leq i \leq \text{rank } B$.

2.2 $\Phi$-extension

Following Williamson [1], the approach based on extending $\pi$-representatives only, can be further improved.

For an arbitrary $A \in \mathcal{A}_n$ let $A' = X_iA$ denote the matrix with the $i$th row equal to the $i$th row of $A$, and with the row $j \neq i$ equal to the coordinatewise modulo two sum of $j$th and $i$th row of $A$. Equivalently, $A' = RAS$, where $R$ is the matrix obtained from $I_n$ by subtracting $i$th row from the others, and then by multiplying $i$th row by $-1$; $S$ is the matrix obtained from $I_n$ by changing sign of columns corresponding to ones in the $i$th row of $A$. A third equivalent definition of $X_i$ [1] can be stated as follows: in the $\pm 1$ matrix $B = \Psi(A)$ of order $n + 1$, the rows 1 and $(i + 1)$ are exchanged, then the first row is ”normalized” to all ones by changing signs of appropriate columns. By applying $\Psi^{-1}$, the matrix $A'$ is obtained. Therefore, application of $X_i$ to $A$ corresponds to a special row permutation in $\Psi(A)$ (followed by scaling). It is natural to denote the identity transformation by $X_0$, $X_0A = A$.

The transformation $X_i$ also preserves ADV. The composition of arbitrary two transformations $X_i, X_j$ is equivalent to only one:

$$
X_i(X_jA) = \begin{cases} 
P_{ij}(X_iA), & \text{if } 1 \leq i, j \leq n \text{ if } i \neq j \\
A, & \text{if } 1 \leq i = j \leq n 
\end{cases}
$$
Let $\Phi_r$ denote the set of $(n+1)!$ transforms of the form $PX_i$, $0 \leq i \leq n$, where $P$ is an arbitrary permutation matrix.

**Theorem 3** The set $\Phi_r$ is a transformation group of $A_n$.

**Proof.** We have

$$X_i PA = PX_{p_i}A,$$

where $p_i$ is the index of the row of $A$, which is moved to the position $i$ after the left multiplication by $P$. Let $P_1$ and $P_2$ be the two permutation matrices and let $p_j$ be the position to which $P_1$ moves the row $j$ after the left multiplication. Then

$$P_2 X_j P_1 X_i = P_2 P_1 X_{p_j} X_i = \begin{cases} P_2 P_1, & p_j = i \\ P_2 P_1 X_{p_j} X_i, & p_j \neq i \end{cases}.$$

If $P_1 = P$ is an arbitrary permutation matrix, $1 \leq i \leq n$, $P_2 = P^{-1}$, and $p_j = i$, then

$$(PX_i)^{-1} = P^{-1}X_j.$$  

Clearly, each orbit of $\Phi_r$ contains at most $n + 1$ orbits of $\Pi_r$.

The corresponding transformation $AX_j$ over the columns of $A$ (coordinatewise addition modulo two of the column $i$ to all other columns) is defined by $AX_j = (X_j A^T)^T$. Let $\Phi_c$ denote the group generated by column permutations and column transformations $(\cdot) X_j$.

Let $\Phi$ be the group generated by the elements of groups $\Phi_r$ and $\Phi_c$; it also preserves $ADV$ and its size is $(n + 1)!^2$. Matrices $A$ and $A'$ are said to be $\phi$-equivalent, $A \sim_{\phi} A'$, if they belong to the same orbit of $\Phi$. Equivalently, $A \sim_{\phi} A'$ if and only if there exist row and column permutations $P$, $Q$, and row and column transformations $X_i$, $X_j$, such that $A = PX_i A' X_j Q$. For an arbitrary $A \in A_n$ let $A_{\phi}$ denote the $\phi$-representative of $A$; $\phi$-class of $A$ is the orbit of $\Phi$ containing $A$.

Let $\text{bord}_{\phi}(B)$ denote the set of $\phi$-representatives of matrices in $\text{bord}(B)$. Williams [1] noted that $\Phi$ and $\Pi$ have similar properties: in order to obtain the set $A_{\phi}^n$ of all $\phi$-representatives in $A_n$, it is enough to extend $\phi$-representatives in $A_{n-1}$.

**Lemma 4** If $B \sim_{\phi} B'$, then $\text{bord}_{\phi}(B) = \text{bord}_{\phi}(B')$.

**Proof.** If $B$ and $B'$ are $\phi$-equivalent, then there exist $g \in \Phi$, transforming $B$ into $B'$. Suppose $A \in \text{bord}_{\phi}(B)$. Then there exists a matrix $A' \in \text{bord}(B)$, $A' \sim_{\phi} A$. By applying $g$ to upper left minor of $A$, the matrix $A'' \sim_{\phi} A'$, $A'' \in$
bord($B'$) is obtained. Therefore, $A \sim_\phi A''$, and $A \in \text{bord}(B')$. Because $A$ is a $\phi$-representative, we obtain $A \in \text{bord}_\phi(B')$, implying $\text{bord}_\phi(B) \subseteq \text{bord}_\phi(B')$. Analogously, $\text{bord}_\phi(B') \subseteq \text{bord}_\phi(B)$, and hence $\text{bord}_\phi(B) = \text{bord}_\phi(B')$. \hfill \Box

2.3 Effective determination of $\pi$-representatives

The classification of matrices in $\mathcal{A}_n$ by extending matrices from $\mathcal{A}^\phi_{n-1}$ must be accompanied by an effective procedure to determine $A_\pi$ and $A_\phi$ for an arbitrary $A \in \mathcal{A}_n$.

The matrix $A_\pi$ is the smallest among the family of at most $n!$ matrices obtained by sorting rows of all the column permutations of $A$. Search is performed more efficiently by a branch-and-bound algorithm. If we know the first $i$ rows of $A_\pi$ (i.e. the row and column permutations $P$, $Q$ such that the first $i$ rows of $PAQ$ are minimal), then the next row of $A_\pi$ is a smallest column permutation (only permutations preserving the first $i$ rows of $P AQ$ are considered) of some of the remaining rows of $PAQ$.

Algorithm 1 Branch-and-bound algorithm to determine $A_\pi$ given $A \in \mathcal{A}_n$.

Input : $A \in \mathcal{A}_n$
Output : $A_\pi$; the permutation matrices $P$, $Q$, such that $PAQ = A_\pi$;
          count – the number of pairs $(P, Q)$, such that $PAQ = A_\pi$;
$P(0) \leftarrow I_n$; $Q(0) \leftarrow I_n$; $A_\pi \leftarrow J_n$;
i $\leftarrow$ 0;
count $\leftarrow$ 0;
Optimize($i$);

{Continuation of the search for $A_\pi$ starting from the row $i$ of $P^{(i-1)}AQ^{(i-1)}$,
{i.e. when the first $i-1$ rows are already chosen and permuted}

Optimize($i$)
Generate the minimal set of boundaries $\Sigma^{(i)} = (s_0^{(i)}, s_1^{(i)}, \ldots, s_k_i^{(i)} = n)$
between adjacent columns of $P^{(i-1)}AQ^{(i-1)}$, such that the $(i-1)$-prefixes
of columns from $s_{j-1}^{(i)} + 1$ to $s_j^{(i)}$ are mutually equal, $1 \leq j \leq k_i$;
for $j = i$ to $n$ do
  \(v_{jl} \leftarrow \sum_{r=s_{l-1}+1}^{s_l} (P^{(i-1)}AQ^{(i-1)})_{j,r}^l, 1 \leq l \leq k_i \); \{the number of 1's\}
  \{in positions from $s_{j-1}^{(i)} + 1$ to $s_j^{(i)}$ in the $j$th row of $P^{(i-1)}AQ^{(i-1)}$\}
$L^{(i)} \leftarrow$ the list of indices of largest vectors $v_j = (v_{j1}, v_{j2}, \ldots, v_{jk_i}), i \leq j \leq n$;
for all $j \in L^{(i)}$ do \{the candidates for the $i$th row of $A_\pi$\}
  $P^{(i)} \leftarrow P_{ij}P^{(i-1)}, \{exchange the rows $i$ and $j$\}

9
compute $Q^{(i)}$ from $Q^{(i-1)}$, so that all 1’s in the part of the row $i$ from $s_{i-1}^{(i)} + 1$ to $s_i^{(i)}$ are moved to the right end of the part, $1 \leq l \leq k_i$; 
\{hence preserving the first $i-1$ rows of $P^{(i)}AQ^{(i)}$\}

compare the $i$th row of $P^{(i)}AQ^{(i)}$ to the $i$th row of $A_\pi$:

if the $i$th row of $P^{(i)}AQ^{(i)}$ is less then

\begin{itemize}
  \item copy the first $i$ rows from $P^{(i)}AQ^{(i)}$ into $A_\pi$;
  \item fill with ones the rest of $A_\pi$;
\end{itemize}

if $i = n$ then $P \leftarrow P^{(i)}; Q \leftarrow Q^{(i)}$; count $\leftarrow 1$; else Optimize($i+1$);

else if the $i$th row of $P^{(i)}AQ^{(i)}$ greater then

\begin{itemize}
  \item continue; \{bound step: try the next row index from $L^{(i)}$\}
\end{itemize}

else

\begin{itemize}
  \item if $i = n$ then count $\leftarrow$ count $+ 1$; else Optimize($i+1$);
\end{itemize}

Example 5 Algorithm 1, applied to the matrix from Example 1, gives the same $\pi$-representative as obtained by trivial algorithm, see Figure 1.

The Algorithm 1 is not efficient for extremely symmetric matrices, such as $I_n$; in that case bound step does not ever occur, because all the remaining rows are always equally good. Hence, Algorithm 1 must be improved, in order to detect some symmetries, and to avoid some unnecessary repetitions. Suppose that there remain $l$ rows not included in $A_\pi$, and that the column classes defined by $\Sigma^{(n-l-1)}$ are such, that all column classes in the remaining rows are uniform (they contain either all ones or all zeros), except for at most one column class, which in that case has $l$ columns, with the row and column sums both equal to $l-1$ or 1. Then, because of the symmetry, it is enough to put in $L^{(n-l-1)}$ only one of the $l$ remaining rows. After the incorporation of this simple heuristic, the algorithm much more efficiently deals with the matrices such as $I_n$, the complement of $I_n$, and the other highly symmetric matrices.

Using Algorithm 1, it is possible to determine $A_\phi$ for an arbitrary $A \in A_n$: it is enough to find $\pi$-representatives of all $(n+1)^2$ matrices $X_iAX_j$, $0 \leq i, j \leq n$, and then to choose the smallest among them.

One of the outputs from Algorithm 1 is the number of the pairs of row/column permutations, transforming $A$ into $A_\pi$. That number is used to determine the size of the $\pi$-class of $A^T$, as it will be demonstrated below.

Consider the problem of counting the matrices in the $\pi$-class of an arbitrary $A \in A_n$. For an arbitrary $B \in A_n$ let $B_0$ denote the matrix obtained from $B$ by sorting its rows. If $A$ has $i_k$ groups of $k$ equal rows, $1 \leq k \leq n$, then the number of matrices that could be obtained from $A$ by row permutations is

$$ a = n! / \prod_{k=1}^{n} i_k! $$
\[ i = 0, \ A_\pi = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \ P^{(0)}AQ^{(0)} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \]

\[ i = 1, \ \Sigma^{(1)} = \{0, 3\}, \ L^{(1)} = (3) \]

\[ j = 3 \]

\[ \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow P^{(1)}AQ^{(1)}, \ A_\pi \leftarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

\[ i = 2, \ \Sigma^{(2)} = \{0, 2, 3\}, \ L^{(2)} = (2, 3) \]

\[ j = 2 \]

\[ j = 3 \]

\[ \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow P^{(2)}AQ^{(2)}, \ A_\pi \leftarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

\[ \begin{bmatrix} 3 & 2 & 1 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix} \rightarrow P^{(2)}AQ^{(2)} \]

\[ i = 3, \ \Sigma^{(3)} = \{0, 1, 2, 3\}, \ L^{(3)} = (3) \]

\[ j = 3 \]

\[ \begin{bmatrix} 3 & 2 & 1 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow P^{(3)}AQ^{(3)}, \ A_\pi \leftarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \]

\[ P \leftarrow P^{(3)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \ Q \leftarrow Q^{(3)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \]

\[ \begin{bmatrix} 2 & 3 & 1 \\ 3 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \rightarrow P^{(3)}AQ^{(3)} \]

Fig. 1. An example of \( \pi \)-representative determination by Algorithm 1.

The representative of these \( a \) matrices is \( A_0 \). An arbitrary matrix \( A' \), obtained from \( A \) by a column permutation, generates in the same manner a new set of \( a \) matrices if and only if \( A'_0 \neq A_0 \). If the number of different matrices \( A'_0 \) is \( b \), then the size of the \( \pi \)-class of \( A \) is \( ab \). It is simpler to obtain \( b \) by counting the number \( p \) of column permutations \( A' \) of \( A \) satisfying \( A'_0 = A_0 \), because \( b = n!/p \). Note that \( p \) is preserved by row and column permutations of \( A \).

Applying Algorithm 1 to \( (A^T)_\pi \), \( p \) is obtained even more easily. Indeed, suppose that \( A \) is already a \( \pi \)-representative, i.e. \( A = A_\pi \). Then Algorithm 1 counts the row permutations \( A'' \) of \( A \), such that there exists a column permutation \( A''' \) of \( A'' \), equal to \( A_\pi \). Now we find \( A' = ((A^T)_\pi)^T \) and apply Algorithm 1 (again)

11
Algorithm 1 gives the number of row permutations \((A')^T\) such that there exists a column permutation \((A'')^T\) of \((A')^T\), equal to \((A')^T\). In other words, we obtain the number of column permutations \(A''\) of \(A\), such that there exists a row permutation \(A''\) of \(A''\), equal to \(A'\) — which is exactly \(p\) (count in Algorithm 1).

**Example 6** Looking again at Example 5, we see that there are two pairs \((P, Q)\) that minimize \(P AQ\). Therefore, there are \(3!^2/2 = 18\) matrices in the \(\pi\)-class of \(A^T\).

The problem of counting the matrices in the SNF-class of an arbitrary \(A \in \mathcal{A}_n\) is much harder. It is even harder is to enumerate the sets \(\mathcal{A}_{n,k} = \{A \in \mathcal{A} \mid \text{rank } A = k\}, 0 \leq k \leq n\): (especially \(m_n = \mathcal{A}_{n,n}\)) We now explicitly enumerate the sets \(\mathcal{A}_{n,1}, \mathcal{A}_{n,2}\), using the following characterization of matrices in \(\mathcal{A}_{n,2}\).

**Lemma 7** If the matrix \(A \in \mathcal{A}_{n,2}\) contains three different nonzero columns \(a, b, c\), then one of them is equal to the sum of the other two, for example \(c = a + b\). Furthermore, the set of nonzero rows of the matrix \([a b]\) equals to \(\{[0 1], [1 0]\}\). There can not be four different nonzero columns in \(A\).

**PROOF.** Suppose \(A \in \mathcal{A}_{n,2}\). If two nonzero columns of \(A\) are linearly dependent, then they are obviously equal. Suppose \(a, b, c\) are the three different nonzero linearly dependent columns, i.e. \(aa + \beta b + \gamma c = 0\) for some integers \(\alpha, \beta, \gamma\). The coefficients \(\alpha, \beta, \gamma\) must be nonzero; otherwise, if for example \(\alpha = 0\), then \(\beta b + \gamma c = 0\) implies \(b = c\). Denote by \(U\) the set of nonzero rows of the \(n \times 3\) matrix \([a b c]\). Then

- \(|U| > 1\); otherwise it would be \(a = b = c\).
- \(U \cap \{[1 0 0], [0 1 0], [0 0 1]\} = \emptyset\); if, for example \([1 0 0] \in U\), then \(\alpha = 0\).
- therefore, \(U \subseteq \{[1 1 1], [0 1 1], [1 0 1], [1 1 0]\}\) and \(U \neq \{[1 1 1]\}\).
- \([1 1 1] \notin U\); if \([1 1 1] \in U\), and for example \([0 1 1] \in U\), then from \(\alpha + \beta + \gamma = 0\) and \(\beta + \gamma = 0\), it follows \(\alpha = 0\).
- \(U \neq \{[0 1 1], [1 0 1], [1 1 0]\}\); otherwise \(\beta + \gamma = 0, \alpha + \gamma = 0, \alpha + \beta = 0\) implies \(\alpha = \beta = \gamma = 0\).

Hence, there are three possibilities for \(U\): \(\{[0 1 1], [1 0 1]\}\), or \(\{[0 1 1], [1 1 0]\}\), or \(\{[1 0 1], [1 1 0]\}\). If \(U = \{[0 1 1], [1 0 1]\}\), then \(\beta + \gamma = 0, \alpha + \gamma = 0\) implies \((\alpha, \beta, \gamma) = (\gamma(-1, -1, 1))\), i.e. \(c = a + b\); the set of nonzero rows of \([a b]\) is \(\{[0 1], [1 0]\}\). The two other cases are symmetrical.

Suppose that \(A\) contains four different columns \(a, b, c, d\). Then we must have for example \(c = a + b\) and the set of nonzero rows of \([a b]\) is \(\{[0 1], [1 0]\}\). Applying the first part of Lemma to \(a, b, d\), we conclude that \(d = a + b\) or \(a = b + d\) or \(b = a + d\); but \(d = a - b\) and \(d = b + a\) are impossible, and
Theorem 8  a) For an arbitrary \( A \in \mathcal{A}_n \) the following three statements are equivalent:

(1) \( \text{rank } A = 1; \)
(2) \( \text{SNF}(A) = (1, 0^{n-1}); \)
(3) \( A \) contains a column \( a \neq 0 \), such that all nonzero columns of \( A \) are equal to \( a \).

The number of matrices in \( \mathcal{A}_{n,1} \) equals
\[
|\mathcal{A}_{n,1}| = (2^n - 1)^2.
\]

b) For an arbitrary \( A \in \mathcal{A}_n \) the following three statements are equivalent:

(1) \( \text{rank } A = 2; \)
(2) \( \text{SNF}(A) = (1, 1, 0^{n-2}); \)
(3) • \( A \) contains the two nonzero columns \( a \neq b \), such that all columns of \( A \) are in \( \{0, a, b\} \), or
• \( A \) contains the two nonzero columns \( a \neq b \), such that the set of nonzero rows of \([a \ b]\) equals \( \{[0 \ 1], [1 \ 0]\} \), and that the set of nonzero columns of \( A \) is \( \{a, b, a+b\} \).

The number of matrices in \( \mathcal{A}_{n,2} \) equals
\[
|\mathcal{A}_{n,2}| = (3^n - 2 \cdot 2^n + 1)(2 \cdot 4^n - 3 \cdot 3^n + 1)/2.
\]

PROOF. a) If \( \text{rank } A = 1 \) then \( A \) contains nonzero column \( a \), such that all nonzero columns of \( A \) are equal to \( a \). By subtracting one of nonzero columns from the others, we obtain an equivalent matrix with exactly one nonzero column \( a \). By the column permutation column \( a \) is moved to the first position, and by the row permutation some 1 is moved to the upper left corner. By subtracting the first row from the other nonzero rows, we obtain that SNF of \( A \) is \( (1, 0^{n-1}) \). How many matrices of rank 1 there are? The number of choices for nonzero column \( a \) is \( 2^n - 1 \), and the number of matrices corresponding to the fixed \( a \) is \( 2^n - 1 \): each its column is 0 or \( a \), but at least one of them has to be equal to \( a \). Hence, \( |\mathcal{A}_{n,1}| = (2^n - 1)^2 \).

b) If \( \text{rank } A = 2 \) then \( A \) contains two linearly independent columns, such that the other columns are their linear combinations. The number of different nonzero columns in \( A \) is either two or it is greater than two.

Case 1. Suppose there are exactly two different nonzero columns \( a, b \) in \( A \).
The number of such matrices $A$ is
\[
\binom{2^n - 1}{2}(3^n - 2 \cdot 2^n + 1).
\]

Indeed, the number of choices for $a$, $b$ equals to the above binomial coefficient. Without loss of generality we suppose that $a < b$. For fixed $a$, $b$, by the inclusion-exclusion principle the number of matrices $A$ is $3^n - 2 \cdot 2^n + 1$, because
- $3^n$ is the number of matrices with the columns from the set $\{0, a, b\}$,
- $2^n$ is the number of matrices without $a$, and also the number of matrices without $b$,
- $1$ is the number of matrices without $a$ and $b$.

**Case 2.** If there are more than two different nonzero columns in $A$, then by Lemma 7 there are two different nonzero columns $a$, $b$ ($a < b$) in $A$, such that the set of nonzero columns in $A$ is $\{a, b, c = a + b\}$, and such that the row set of the matrix $[a b]$ is $\{[0 1], [1 0]\}$. There are $(3^n - 2 \cdot 2^n + 1)/2$ choices for columns $a$, $b$ satisfying these conditions. Indeed, consider all matrices $[a b], [b a]$:
- $3^n$ is the number of matrices with the row set $\{[0 0], [0 1], [1 0]\}$,
- $2^n$ is the number of matrices without the row $[0 1]$, and also the number of matrices without the row $[1 0]$,
- $1$ is the number of matrices without the rows $[0 1], [1 0]$.

The number of matrices $[a b]$ is therefore $(3^n - 2 \cdot 2^n + 1)/2$. The number $4^n - 3 \cdot 3^n + 3 \cdot 2^n - 1$ of matrices with the set of nonzero columns $\{a, b, c\}$ (where $c = a + b$) is also obtained by the inclusion-exclusion principle:
- $4^n$ is the number of matrices with all the columns $0, a, b, c$;
- $3^n$ is the number of matrices without the column $a$ (and analogously without $b, c$);
- $2^n$ is the number of matrices without columns $a, b$ (and analogously without $a, c$; and without $b, c$);
- $1$ is the number of matrices without columns $a, b, c$.

Therefore, the number of matrices of the rank 2, with more than two different nonzero columns equals
\[
(3^n - 2 \cdot 2^n + 1)(4^n - 3 \cdot 3^n + 3 \cdot 2^n - 1)/2.
\]

The total number of matrices in $\mathcal{A}_{n,2}$ equals
\[
\left(2 \binom{2^n - 1}{2} + (4^n - 3 \cdot 3^n + 3 \cdot 2^n - 1)\right)(3^n - 2 \cdot 2^n + 1)/2 =

= (3^n - 2 \cdot 2^n + 1)(2 \cdot 4^n - 3 \cdot 3^n + 1)/2.
\]

In either case, in order to obtain $\text{SNF}(A)$, the other nonzero columns are first transformed to 0 by subtracting $a$, $b$ or $a + b$ from them. Next, in $[a b]$ there is a row $[0 1]$, because $a < b$; using that 1, the other elements of $b$ are changed to
0. Finally, choosing some 1 in \(a\), and subtracting if necessary that row from the others, after permuting rows/columns, we obtain the SNF. Hence, rank \(A = 2\) implies \(\text{SNF}(A) = (1, 1, 0^{n-2})\). \(\square\)

2.4 Iterative classification of \((0, 1)\) matrices

According to Lemma 2 we have

\[ A_{n+1}^\pi = \bigcup_{A \in A_n^\phi} \text{bord}_\pi(A). \]

By changing the order of calculations, it is possible to simplify repeated determination of \(\pi\)-representatives of matrices from \(\text{bord}(A)\) by Algorithm 1. Matrices \(B\) in \(\text{bord}(A)\) are of the form (2). For each \(y\) the \(\pi\)-representatives of \(B\)'s corresponding to various inserted rows \([x b]\) are found spending smaller number of steps. The point is that the rows of the \(\pi\)-representative preceding the row \([x b]\) are already determined for some previous variants for that row.

Somewhat more detailed description follows. Determine first the \(\pi\)-representative of the matrix, corresponding to \(x = 0, b = 0\); the inserted zero row \([x b]\) is certainly the first row in the \(\pi\)-representative. The corresponding row and column permutations \(P, Q\) are recorded. The remaining pairs \((x, b)\) are then considered in turn, lexicographically ordered. The question arises, to which position \(l\) might \([x b]\) be moved during the \(\pi\)-representative determination, skipping the determination of first \(l - 1\) rows of the representative. The obvious lower bound for \(l\) is the smallest among all positions where the previous rows \(w\), obtained from \([x b]\) by changing exactly one 1 into 0, have been moved (except if there was an alternative to \(w\) during that step, i.e. if \(L^{(i)}\) had more than one member at the moment when \(w\) arrived to its destination).

Instead of extending all \(A \in A_n^\pi\), it is enough to extend the matrices from the set \(A_n^\phi\) of all \(\phi\)-representatives in \(A_n\). By extending all \(A \in A_n^\phi\) a subset of \(A_{n+1}^\pi\) is obtained; the set of \(\phi\)-representatives of matrices from that subset is exactly \(A_{n+1}^\phi\).

It is convenient to use a balanced tree to collect \(\pi\)-representatives in an ordered fashion. We chose AVL tree [6] — the binary search tree satisfying the condition that, for every node, the difference between the heights of its left and right subtrees is at most 1. For \(n = 8\), in order to save memory, a combination of AVL tree and the sorted array of matrices is used: from time to time the content of the tree is merged into the array. After collecting all \(\pi\)-representatives, the \(\pi\)-representatives set is reduced to the corresponding \(\phi\)-representatives set. To determine the set of \(\phi\)-representatives, corresponding to a given set of \(\pi\)-representatives, the following simple algorithm is used.
Algorithm 2 Reduction of a given set $L_\pi$ of $\pi$-representatives to the set $L_\phi$ of corresponding $\phi$-representatives.

\[
\{ T \text{ — auxiliary AVL tree used to collect $\pi$-representatives. } \}
\]

while $L_\pi \neq \emptyset$

while there is a space in $T$ for at least $(n + 1)^2$ matrices

- remove the first matrix $A$ from $L_\pi$;
- generate the set $T_A$ of $\pi$-representatives contained in the $\phi$-class of $A$;
- insert $T_A$ into $T$;
- insert $A_\phi$ into $L_\phi$;
- remove from $L_\pi$ all the matrices contained in $T$;

$T \leftarrow \emptyset$.

The classification of $A_8$ lasted about a month in parallel on five PC's. A huge number of collected $\pi$-representatives of order $n = 8$ caused serious difficulties. The space requirement is reduced by dividing $\pi$-representatives into subsets, according to their SNF. For each extended matrix, its SNF is determined, and the $\pi$-representatives are classified into subsets with the same SNF. These subsets are then independently processed. The hardest was the SNF-class $(17, 0)$, with 5204144555 $\pi$-representatives contained in a number of non disjoint subsets. These subsets were independently processed by Algorithm 2, producing the non disjoint sets of $\phi$-representatives; their union consists of 71348129 $\phi$-representatives, approximately 1/3 of matrices in $A_8^\phi$.

In order to save the space, $L_\pi$ and $L_\phi$ are stored in a sorted, compressed form: one byte for each matrix row; the group of consecutive matrices with the same first $n - 2$ rows is stored so that the common $n - 2$ rows are stored only once. As a result, the average space for a matrix of order 8 was little more than two bytes.

If somebody tries to extend $\phi$-representatives of order 8, he could expect to process about 300 times more $\phi$-representatives, each giving approximately 4 times more $\pi$-representatives. Therefore, the classification of matrices of order 9 is expected to last 1000 times longer, requiring huge memory.

2.5 Results of classification

We start with the simplest nontrivial case.

Example 9 The 16 matrices of order 2 are divided into 3 $\phi$-classes, which are further subdivided into 7 $\pi$-classes:
In Table A.1 all the 36 \( \pi \)-representatives of order 3 are shown. The 5 SNF-classes are in separate blocks, divided into compartments with \( \phi \)-classes. The first matrix in each \( \phi \)-class is the smallest \( \pi \)-representative, i.e. the \( \phi \)-representative. For each \( \pi \)-and SNF-class, their size is given. The matrices are represented by hexadecimal vectors, each component representing a row of a matrix. For example, the last vector \((3, 5, 6)\) in Table A.1 represents the matrix

\[
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{bmatrix}
\]

The matrix \((1, 2, 5)\) is a \( \pi \)-representative of the matrix from Example 5.

In Table A.2 all the 39 \( \phi \)-representatives of order 4 are shown, together with the sizes of their \( \phi \)-classes.

In Table 2 \( \rho_n, |A_n^\phi|, s_n, a_n, |D_n| \), and the set \( D_n \) are given for \( 1 \leq n \leq 8 \), where \( s_n = |S_n| \) and \( \rho_n = (2^{n^2}/(n+1)!^2)/|A_n^\phi| \). In the last row of Table 2 \( s_9, |D_9|, a_9, D_9 \) are given; the explanation how they are obtained will be given in section 3.

Denote by \( F(n) \) the following statement:

\[ A \in A_n, \text{ satisfying } \text{SNF}(A) = d = (d_1, d_2, \ldots, d_n) \text{ exists if and only if (6)} \]

there exists \( A' \in A_{n+1}, \text{ satisfying } \text{SNF}(A') = d' = (d_1, d_2, \ldots, d_n, 0) \).

Obviously, the first condition implies the second one. The implication in the opposite direction is not obvious at all; it would follow from the following stronger statement:

\[ H(n) : \text{ Let } A' \in A_{n+1}, \text{ rank } A' = n, \text{ and } \text{SNF}(A') = d' = (d_1, d_2, \ldots, d_n, 0). \]

Then \( A' \) has at least one minor \( A \in A_n \) with \( \text{SNF}(A) = d = (d_1, d_2, \ldots, d_n) \).
Table 2
The numbers of equivalence classes in \( A_n \).

| \( n \) | \( \rho_n \) | \( |A_n^\varphi| \) | \( s_n \) | \( a_n \) | \( |D_n| \) | \( D_n \) |
|---|---|---|---|---|---|---|
| 1 | 0.250 | 2 | 2 | 2 | 2 \{0, 1\} |
| 2 | 0.148 | 3 | 3 | 2 | 2 \{0, 1\} |
| 3 | 0.074 | 12 | 5 | 3 | 3 \{0 − 2\} |
| 4 | 0.117 | 39 | 8 | 4 | 4 \{0 − 3\} |
| 5 | 0.167 | 388 | 14 | 6 | 6 \{0 − 5\} |
| 6 | 0.334 | 8102 | 26 | 10 | 10 \{0 − 9\} |
| 7 | 0.528 | 656103 | 56 | 19 | 22 \{0 − 18, 20, 24, 32\} |
| 8 | 0.701 | 199727714 | 129 | 41 | 46 \{0 − 40, 42, 44, 45, 48, 56\} |
| 9 | 333 | 103 | 114 \{0 − 102, 104, 105, 108, 110, 112, 116, 117, 120, 125, 128, 144\} |

But the following matrix \( F \in A_{10} \) is a counterexample to \( H(10) \):

\[
F = \begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\]

The matrix \( F \) consists of blocks \( A, B, C, D \), having 2, 3, 2, 3 ones in each row respectively, and also having 2, 2, 3, 3 ones in each column, respectively; \( F \) is singular, because the sums of rows of \([A \ B]\) and \([C \ D]\) are equal. It can be verified that \( \text{rank}\ F = 9, \text{SNF}(F) = (1^9, 0) \), but all minors of \( F \) have SNF different from \( (1^9) \).

In Table A.3 the SNF-representatives of matrices in \( A_n, n \leq 8 \), are listed, accompanied with the size measures of corresponding SNF-classes (the number of
Table 3
The number of matrices of the rank \( k \) in \( \mathcal{A}_n \), \( n \leq 8 \).

| \( n \) | \( 1 \) | \( 2 \) | \( 3 \) | \( 4 \) | \( 5 \) | \( 6 \) | \( 7 \) | \( 8 \) |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 9 | 49 | 225 | 961 | 3969 | 16129 | 65025 |
| 2 | 6 | 288 | 6750 | 118800 | 19007100 | 9271660734 | 397046059200 |
| 3 | 174 | 36000 | 3159750 | 190071000 | 9271660734 | 397046059200 |
| 4 | 22560 | 17760600 | 5295204600 | 1001080231200 | 144998212423680 |
| 5 | 12514320 | 34395777360 | 32307576315840 | 17952208799918400 |
| 6 | 28836612000 | 259286329895040 | 720988662376725120 |
| 7 | 270345669985440 | 7547198043595392000 |
| 8 | 10160459763342013440 |

matrices, the number of \( \pi \)-representatives and the number of \( \phi \)-representatives in each SNF-class). The sizes of \( \pi \)-classes are determined using Algorithm 1. The classes are ordered lexicographically by the SNF (with zeros moved to the end of SNF).

One can verify this classification starting from the sorted list of all \( \phi \)-representatives. For each of them one has to check if it is indeed a \( \phi \)-representative. The next step is to sum the numbers of \( \pi \)-representatives in all \( \phi \)-classes, and to compare the sum with the corresponding entry in Table 1. One could also check that the sum of sizes of SNF-classes in \( \mathcal{A}_n \) equals \( 2^{n^2} \) for each \( n \leq 8 \), see Table A.3. The sorted lists of \( \phi \)-representatives for \( n \leq 8 \) can be downloaded from [http://www.matf.bg.ac.yu/~ezivkovm/01matrices.htm](http://www.matf.bg.ac.yu/~ezivkovm/01matrices.htm).

We now review some interesting facts, which are seen from Table A.3.

Let \( T(n,k) = |\mathcal{A}_{n,k}| \). In Table 3 the numbers \( T(n,k) \), \( 0 \leq k \leq n \leq 8 \), are shown (of course, they are easily obtained from Table A.3). The part of Table 3 corresponding to \( n \leq 7 \) is the same as in [10]; it is also an entry in [4, Sequence A064230]. Another interesting entry in [4, Sequence A055165] is the sequence \( m_n \), where \( m_n \) is the number of regular \((0,1)\) matrices of order \( n \) — the diagonal of Table 3. The seemingly new member of that sequence is \( m_8 = 10160459763342013440 \). If we suppose that all matrices in \( \mathcal{A}_n \) are equiprobable, then the rank probability distribution is shown in Table 4 for \( n \leq 8 \). Looking at Table 4, one could erroneously conclude that large fraction of matrices in \( \mathcal{A}_n \) is singular. In fact, the fraction of singular matrices in \( \mathcal{A}_n \) tends to 0 for \( n \) large [11].
Table 4
The probability that a random matrix in $A_n$ has the rank $k$, $0 \leq k \leq n \leq 8$.

| n | k | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|---|---|-----|-----|-----|-----|-----|-----|-----|-----|
| 0 | 0 | 0.5 | 0.0625 | 0.00195 | 0.00002 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 1 | 0.5 | 0.5625 | 0.09570 | 0.00343 | 0.00003 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 2 | 0.3750 | 0.56250 | 0.10300 | 0.00354 | 0.00003 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 3 | 0.33984 | 0.54932 | 0.09417 | 0.00277 | 0.00002 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 4 | 0.34424 | 0.52931 | 0.07706 | 0.00178 | 0.00001 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 5 | 0.37296 | 0.50052 | 0.05739 | 0.00097 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 6 | 0.41963 | 0.46059 | 0.03908 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 7 | 0.48023 | 0.40913 | 0.01975 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |
| 8 | 0.55080 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 | 0.00000 |

Table 5
The possible numbers of $\pi$-orbits inside $\phi$-orbits of $A_n$.

| n | The set of $\phi$-orbit sizes |
|---|----------------------------|
| 1 | \{1\} |
| 2 | \{1, 2, 4\} |
| 3 | \{1, 2, 4, 5, 9\} |
| 4 | \{1, 5, 7, 9 − 11, 13, 16, 17\} |
| 5 | \{1, 18, 20, 21, 25, 26, 30, 36\} |
| 6 | \{1, 2, 4, 27, 29 − 32, 35 − 37, 42, 49\} |
| 7 | \{1, 38, 40, 42 − 44, 48 − 50, 56, 64\} |
| 8 | \{1, 46, 48 − 51, 53, 54, 56 − 58, 63 − 65, 72, 81\} |

It turns out that $F(n)$ (6) is true for $n \leq 7$, i.e. the set of SNF’s of rank $k$ is the same for all $n$, $k \leq n \leq 8$. For example, the SNF-representative of the SNF-class $(1, 1, 2, 0^n-3)$ is the matrix $(0^n-3, 3, 5, 6)$ for $3 \leq n \leq 8$.

The smallest $n$ for which there are two matrices in $A_n$ with the same determinant, but with different SNF’s is 5: $\text{SNF}(3,C,15,16,19) = (1, 1, 1, 4)$ and $\text{SNF}(3,5,9,11,1E) = (1, 1, 2, 2)$.

In Table 5 the possible numbers of $\pi$-orbits inside $\phi$-orbits are shown for $1 \leq n \leq 8$. These numbers are between 1 and $(n + 1)^2$; as it is seen, the value $(n + 1)^2$ is attained only if $n \geq 5$. 

20
Table 6
The maximal ADV’s of matrices from $A_{n+1}$, obtained by extending matrices equivalent to $I_n$.

| $n$ | $\det A$ | $A$ |
|-----|----------|-----|
| 3   | 3 5 9 E  |     |
| 4   | 5 3 5 E  | 16 19 |
| 5   | 9 D 15 1 | $A$ 26 39 |
| 6   | 18 7 19 2A | 34 4C 53 65 |
| 7   | 40 7 19 2A | 56 65 9C B3 CB |
| 8   | 105 7 39 5A | AC D5 E3 136 14D 19B |

If $A \in A_n$, $A \sim I_n$ and $B \in \text{bord}(A)$, then SNF($B$) contains at least $n$ ones, see (5). The question arises, what are the possible values of the last element of SNF($B$), i.e. which values can take $|\det B|$? The largest possible values of $|\det B|$ under these assumptions, along with the examples of matrices $B$ for which these values are attained, are given in Table 6. In fact, the matrices from Table 6 maximize $|\det B/\det A|$ for all regular $A \in A$, $n \leq 8$.

More generally, it is interesting to describe the relationship of SNF($A$) to SNF($A'$) if $A' \in \text{bord}(A)$. During iterative classification, the sets

$$\{\text{SNF}(B) \mid B \in \text{bord}(A), A \in A_n, \text{SNF}(A) = s\}$$

are recorded for all SNF-classes $s \in S_n$. The results are represented by the incidence matrix $M_n$ of dimensions $|S_n| \times |S_{n+1}|$, with entries

$$m_{s,s'} = \begin{cases} 1, & \text{if there exist } A \in A_n \text{ and } B \in \text{bord}(A), \text{ with SNF’s } s \text{ and } s' \\ 0, & \text{otherwise} \end{cases}$$

(7)

Let $G(n)$, denote the following statement:

$G(n)$: There exist matrices $A \in A_n$, $A' \in \text{bord}(A)$, such that

(8)

$$\text{SNF}(A) = (d_1, d_2, \ldots, d_n), \text{ SNF}(A') = (d'_1, d'_2, \ldots, d'_n, d'_{n+1})$$

if and only if there exist matrices $B \in A_{n+1}$, $B' \in \text{bord}(B)$, such that

$$\text{SNF}(B) = (d_1, d_2, \ldots, d_n, 0), \text{ SNF}(B') = (d'_1, d'_2, \ldots, d'_n, d'_{n+1}, 0).$$

By exhaustive search it is verified that $G(n)$ is true for $n \leq 6$, enabling to put all the transposed incidence matrices $M_n$, $n \leq 7$ together into single Table A.4. The 1’s are represented by •; the 0’s are represented by • if they are the
consequence of the following Lemma (describing constraints for SNF($A'$) if $A' \in \text{bord}(A)$); otherwise, they are represented by $\circ$.

**Lemma 10** For an arbitrary $A \in \mathcal{A}_n$, let $A' \in \text{bord}(A)$, and let SNF($A$) = $(d_1, d_2, \ldots, d_n)$, SNF($A'$) = $(d'_1, d'_2, \ldots, d'_n, d'_{n+1})$. Then

1. $\text{rank } A \leq \text{rank } A' \leq \text{rank } A + 2$;
2. $d'_1 d'_2 \ldots d'_i$ divides $d_1 d_2 \ldots d_i$ for all $i$, $1 \leq i \leq \text{rank } A$;
3. $\prod_{i=1}^{n-1} d_i$ divides $\det A'$.

**Proof.**

1. The first inequality follows from the fact that the rank of a submatrix is a lower bound on the rank of a matrix. The second inequality follows from the observation that $A'$ is an at most rank 2 perturbation of $A$.
2. This is a direct consequence of the fact that $d'_1 d'_2 \ldots d'_i$ is the largest common divisor of all minors of $A'$ of order $i$, see for example [7].
3. Let $P, Q$ be the matrices such that $\text{SNF}(A) = PAQ = D = (d_1, d_2, \ldots, d_n)$, $|\det P| = |\det Q| = 1$. Let

$$A' = \begin{bmatrix} A & y \\ x & b \end{bmatrix}.$$ 

The case $\det A' = 0$ is trivial; suppose $\det A' \neq 0$. If $xQ = [a_1 \ a_2 \ \ldots \ a_n]$, $Py = [c_1 \ c_2 \ \ldots \ c_n]^T$, then from the identity

$$\begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} A & y \\ x & b \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} D & Py \\ xQ & b \end{bmatrix}$$

it follows (another way to express determinants of matrices obtained by extension, see (3))

$$\det A' = b \prod_{i=1}^n d_i - \sum_{i=1}^n a_i c_i \prod_{1 \leq j \leq n, j \neq i} d_j.$$  \hspace{1cm} (9)

Since $\text{rank } A' = n + 1$, then we have $\text{rank } A \geq n - 1$. If $\text{rank } A = n - 1$, then $d_n = 0$, implying

$$\det A' = -a_n c_n \prod_{i=1}^{n-1} d_i;$$

otherwise

$$\det A' = \left( bd_n - \sum_{i=1}^n a_i c_i d_n / d_i \right) \prod_{i=1}^{n-1} d_i.$$
In both cases $\prod_{i=1}^{n-1} d_i$ divides $\det A$. $\square$

Suppose $A \in \mathcal{A}_n$, $A' \in \text{bord}(A)$. From Table A.4, we see the following interesting facts:

- The first $\circ$ in some $M_n$ corresponds to $s = (1, 0)$, $s' = (1, 1, 2)$. It is equivalent to following statement: if $A \in \mathcal{A}_{2, 1}$ then $|\det A'| < 2$.
- if $A \in \mathcal{A}_{3, 2}$, then $|\det A'| < 3$.
- if $A \in \mathcal{A}_4$, SNF($A$) = $(1, 1, 1, 0)$, then $|\det A'| < 5$.
- if $A \in \mathcal{A}_4$, SNF($A$) = $(1, 1, 2, 0)$, then $|\det A'| < 4$.
- if $A \in \mathcal{A}_5$, SNF($A$) = $(1, 1, 1, 1, 0)$, then $|\det A'| \neq 7$.
- if $A \in \mathcal{A}_5$, SNF($A$) = $(1, 1, 1, 2, 0)$, then $|\det A'| \neq 6$.
- if $A \in \mathcal{A}_5$, SNF($A$) = $(1, 1, 1, 3, 0)$, then $|\det A'| \notin \{6, 9\}$.
- if $A \in \mathcal{A}_5$, SNF($A$) = $(1, 1, 1, 2, 2)$, then SNF($A'$) $\neq (1, 1, 1, 4, 0)$.
- if SNF($A$) = $(1^{n-1}, d_n)$ and SNF($A'$) = $(1^{n-1}, d'_n, 0)$ then $d'_n$ divides $d_n$ for all $n \leq 7$.
- if SNF($A$) = $s = (1^{n-1}, d_n) \in \mathcal{S}_n$ and SNF($A'$) = $s' = (1^n, d'_{n+1}) \in \mathcal{S}_{n+1}$ then
  - if $n \leq 6$, then $m_{s, s'} = 1$.
  - if $n = 7$, then $m_{s, s'} = 1$ if and only if
    
    \[
    (d_n, d'_{n+1}) \notin \{(17, 34), (7, 39), (13, 39), (1, 42), (4, 42), (6, 42), (7, 42), (13, 42), (14, 42)\}.
    \]

- if $n = 8$, then there are more exceptions to $m_{s, s'} = 1$, but there is one exotic group of them: if $d_n = 19$ then $d'_{n+1}$ must be divisible by 19; 19 is the only integer satisfying such a condition.

3 **Determinant and SNF sets of $(0, 1)$ matrices of order $9$**

Determination of $\{|\det(A')| \mid A' \in \text{bord}(A)\}$ is a simple operation, see the explanation following (3). It was effectively performed for all 199727714 matrices in $\mathcal{A}_9$; merging these sets $\mathcal{D}_9$ is obtained, see Table 2.

The similar idea — determine ADV’s, and only if necessary, determine SNF’s of the results of extension — is used to obtain $\mathcal{S}_9$. Suppose we know in advance the number $f_d$ of different SNF’s in $\mathcal{D}_9$ corresponding to a given ADV $d > 0$. During the extension of matrices from $\mathcal{A}_n$, the SNF’s of extended matrices with the ADV $d$ are determined only if the number of SNF’s with ADV $d$ is still less than $f_d$. If we know only upper bound on $f_d$, then the heuristic does not work — we have to determine SNF’s of all matrices with the ADV $d$. Therefore, it is useful to determine $f_d$ for at least some $d > 0$. 23
Table 7
The number of partitions of \( r \) into at most \( n \) positive integers.

| \( r \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| \( n \) | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 1 | 2 | 3 | 3 | 4 | 4 | 4 |
| 3 | 1 | 1 | 2 | 3 | 4 | 5 | 7 | 8 |
| 4 | 1 | 1 | 2 | 3 | 5 | 6 | 9 | 11 |
| 5 | 1 | 1 | 2 | 3 | 5 | 7 | 10 | 13 |
| 6 | 1 | 1 | 2 | 3 | 5 | 7 | 11 | 14 |
| 7 | 1 | 1 | 2 | 3 | 5 | 7 | 11 | 15 |

Denote by \( p_n(r) \) the number of partitions of \( r \) into at most \( n \) positive integers.

In order to determine the upper bound for \( f_d \), suppose first that \( d \) is a prime power, \( d = p^r \). If \( A \in \mathcal{A}_n \) and \( |\det A| = d \), then \( \text{SNF}(A) \) is of the form

\[
(p^{x_1}, p^{x_2}, \ldots, p^{x_n}), \quad 0 \leq x_1 \leq x_2 \leq \cdots \leq x_n, \quad \sum_{i=1}^n x_i = r.
\]

The number of different exponent vectors \((x_1, x_2, \ldots, x_n)\) is equal to \( p_n(r) \). The values \( p_n(m) \) are computed using the recurrence (see for example [12])

\[
p_n(0) = 1, \quad n \geq 0, \quad p_0(r) = 0 \text{ for } r \geq 1, \quad p_n(r) = p_{n-1}(r) + p_n(r-n),
\]

see Table 7.

**Example 11** If \( d = 8 = 2^3 \) and \( n = 6 \) we have \( p_6(3) = 3 \); \( \text{SNF}(A) \) is one of \((1^5, 8), (1^4, 2, 4)\) and \((1^3, 2, 2, 2)\). We see from Table A.3 that all these \( \text{SNF} \) do exist, i.e. for each of them there exists some \((0,1)\) matrix. Another example \( d = 3^2, n = 6 \), shows that \( p_6(2) = 2 \) is only an upper bound: the \( \text{SNF} \)-class \((1^4, 3, 3)\) is empty.

More generally, if \( d = \prod_{i=1}^m p_i^{\alpha_i} \), where \( p_i \) are different primes, then the upper bound on the number of different \( \text{SNF} \)’s with the ADV \( d \) is \( \prod_{i=1}^m p_n(\alpha_i) \).

**Example 12** If \( n = 8 \) and \( d = 36 \), then there are \( p_8(2)p_6(2) = 4 \) such \( \text{SNF} \)’s: \((1^6, 2, 18), (1^7, 36), (1^6, 3, 12), (1^6, 6, 6)\); all these \( \text{SNF} \)’s are found in Table A.3.

In order to obtain a tighter upper bound for the number of different \( \text{SNF} \)’s, we have to include somewhat more information. If we further suppose that \( A' \in \text{bord}(A) \) and \( \text{SNF}(A) = s = (d_1, d_2, \ldots, d_n) \), then by Lemma 10 for some \( s' = (d'_1, d'_2, \ldots, d'_{n+1}) \) the equality \( \text{SNF}(A') = s \) is impossible. For example, if
s contains k ones, then s′ contains at least k ones and if rank A ≥ k + 1 then

\[ d'_{k+1} \text{ divides } d_{k+1}. \]

Using these facts, the regular part of \( S_9 \) was determined, see Table A.5. The \( \phi \)-representatives from the chosen SNF-class of \( A_8 \) were extended computing determinants, and, if necessary, determining SNF’s. The upper bounds for the number of different SNF’s, obtained by Lemma 10, are rough for larger ADV values, but the consequences are not dangerous, because of the small number of extended matrices with the large ADV: it is not hard to compute the SNF’s of all of them.

To complete \( S_9 \), it is necessary to determine the singular part of \( S_9 \). If we would know that \( F(8) \) is true, then the set of singular SNF’s of order 8 would be equal to \( S_8 \) (with each SNF extended by one zero, of course). Not knowing a simple proof of \( F(8) \), we proceed with a shortened exhaustive proof.

The idea is to narrow the set of SNF-classes in \( S_8 \), the extension of which can lead to a new singular SNF of order 9. If SNF(\( A \)) = \( (d_1, d_2, \ldots, d_8, 0) \) for some \( A \in A_9 \), then (because we know the set of SNF’s of lower orders) by Lemma 10 we can narrow the set SNF-classes, containing \( A \). We obtain that the only new possible SNF’s are the following SNF’s of the rank 8:

\[ (1^7, m, 0), m = 44, 45, 48, 56 \text{ and } (1^6, 2, 28, 0) \]; and the following SNF’s of the rank 7: \( (1^6, 20, 0, 0), (1^6, 24, 0, 0), (1^6, 32, 0, 0), (1^5, 2, 12, 0, 0), (1^5, 2, 16, 0, 0), (1^5, 4, 8, 0, 0), (1^4, 2, 2, 8, 0, 0), (1^4, 2, 4, 4, 0, 0). \]

The extension of which matrices gives the matrices with such SNF’s? For example, we know that the SNF \( (1^7, 44, 0) \) can be obtained only by the extension of a matrix in which 44 divides all minors of order 8; therefore 44 also divides a nonsingular minor of order 8; hence the SNF of that minor could be only \( (1^6, 2, 22) \). Considering analogously the rest of listed SNF’s of order 8, we obtain that matrices from \( A_{9,8} \), with the SNF equal to some from the list above, can be obtained only by the extension of matrices from \( A_8 \) with the SNF \( (1^6, 2, 22), (1^6, 2, 24), (1^6, 3, 15), (1^5, 2, 2, 12), \) or \( (1^5, 2, 2, 14) \).

Analogously, we obtain that matrices from \( A_{9,7} \) with one of the listed SNF’s, can be obtained only by double extension of matrices from \( A_7 \) with the SNF \( (1^5, 2, 10), (1^4, 2, 2, 6), \) or \( (1^3, 2, 2, 2, 4) \). After the complete search through all matrices that can be obtained by the extensions listed, it is found that there are no new singular SNF’s of order 9 i.e. that \( F(8) \) is also true. That completes the determination of \( S_9 \).

In Table A.6 the part of the incidence matrix \( M_8 \) is shown, corresponding to regular matrices in \( S_9 \). The table was obtained by extending \( \phi \)-representatives from \( A_{8,7} \) and \( A_{8,8} \); the singular extended matrices were ignored.
Denote by $f_n$ the $n$th Fibonacci number ($f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$). Paseman [13] shows that $a_n \geq 2f_{n-1}$. We give the sketch of his proof, and then we give the sharper lower bounds for $a_n$, $n \leq 19$.

Consider the so called Fibonacci matrices $F_n \in \mathcal{A}_n$ with the $(i, j)$ element equal to 1 if and only if $j - i = -1, 0, 2, 4, \ldots$; $\det F_n = f_n$. The cofactors corresponding to the first row of $F_n$ are $f_{n-1}, f_{n-2}, -f_{n-3}, -f_{n-4}, \ldots, -f_1$.

Consider the matrix $U \in \text{bord}(F_n)$,

$$U = \begin{bmatrix} F_n & y \\ x & b \end{bmatrix},$$

where $x = [x_1, x_2, \ldots, x_{n-1}], y = [y_1, y_2, \ldots, y_{n-1}]^T$. Let $y_1 = 1, y_2 = y_3 = \cdots = y_n = 0$ and $x_1 = x_2 = 0$. Then from (3) we have

$$\det U = \sum_{i=1}^{n-2} x_{n+1-i} f_i + bf_n.$$

Therefore, each integer from $[0, 2f_n - 1]$ is determinant of some $U \in \text{bord}(F_n)$, and $a_n \geq 2f_{n-1}$.

In order to prove that $a_n \geq m$, one can give a list of matrices from $\mathcal{A}_{n-1}$, such that determinants of their extensions cover $[1, m - 1]$. The proof verification then includes the procedure of finding determinants of all extensions of a given matrix. Still, such a list is essentially more compact than the list of matrices from $\mathcal{A}_n$, with determinants covering $[1, m - 1]$.

Denote by $a_A$ the minimal integer not in $\cup\{|\det B| \mid B \in \text{bord}(A)\}$, the "extension spectrum" of $A \in \mathcal{A}_n$. In this context, the matrices $A$ with high $a_A$ are of special interest. If $a_A > 1$ and $\text{SNF}(A) = (d_1, d_2, \ldots, d_n)$, then $d_1 = d_2 = \cdots = d_{n-1} = 1$, because determinants of all extensions of $A$ are divisible by $d_{n-1}$, see (9).

In order to find lower bounds for some $a_n$, one can start from a well chosen set $B_{n-1} \subset \mathcal{A}_{n-1}$, and then to find ADV’s of all extended matrices. If $m$ is the smallest number not equal to some of these ADV’s, then $a_n \geq m$. Afterwards, some subset of extended matrices with different SNF’s is taken to be the set $B_n$, and the next iteration can be started.

The starting set $B_9$ was constructed in the following way. From each SNF-class in $\mathcal{A}_8$ a number of matrices is taken, with different numbers of $\pi$ representatives in their $\phi$-classes. Extending these matrices, a set of matrices with
different SNF’s is obtained, but without any matrix with the SNF (1^8, 97).
By adding one such matrix, the set $B_0$ is completed. The sets $B_{10}$, $B_{11}$ and $B_{12}$ are generated iteratively, as explained above. At the end, the ADV’s of all matrices obtained by extending the matrices in $B_{12}$ are determined. The resulting lower bounds are $a_{10} \geq 259$, $a_{11} \geq 739$, $a_{12} \geq 2107$, $a_{13} \geq 6157$.

For $n > 13$ we used an alternative heuristic, described by Algorithm 3.

**Algorithm 3** Heuristic to find lower bound for $a_{n+1}$.

**Input:** $\mathcal{L}_n \subset \mathcal{A}_n$, list of matrices to be extended.

**Output:** lower bound for $a_{n+1}$, and list $\mathcal{L}_{n+1} \subset \mathcal{A}_{n+1}$ of ”promising” matrices for the following iteration.

{ Initialization: }
first0 $\leftarrow 1$; { the first integer not ”covered” by ADV’s }
dmax $\leftarrow 1$; { the largest ADV found until now }
$\mathcal{L}_{n+1} \leftarrow \emptyset$; { output list }

for all $A \in \mathcal{L}_n$ do

{ Consider the extensions $A' = \begin{bmatrix} A & y \\ x & b \end{bmatrix}$ }

Compute $\det A$ and $B = \begin{bmatrix} B_{ij} \end{bmatrix} = \text{adj} A$; { transposed cofactor matrix of $A$}
for all $y \in \{0, 1\}^n$ do

{ the next linear combination of rows of $B$ }

determine the coefficients of the linear combination

$-b \det A + \sum_{i=1}^n x_i \left( \sum_{j=1}^n y_j B_{ij} \right)$

and the sums $s^+$, $s^-$ of its positive and negative members;

if $\max\{-s^-, s^+\} \geq \text{first0}$ then { ”poor” linear combinations are skipped }

for all $(x, b) \in \{0, 1\}^{n+1}$ do

compute $\det A'$; { by one addition only, using Gray code }

if $|\det A'| = \text{first0}$ then

update first0;

if $|\det A'| > \text{dmax}$ then

$dmax \leftarrow |\det A'|$;

if $|\det A'| > 0.9 \times \text{dmax}$ then

append $A'$ to $\mathcal{L}_{n+1}$;

Elimination of ”poor” linear combinations is a powerful heuristics if the matrices with the high extension spectra are placed in the beginning of $\mathcal{L}_n$. The major part of linear combinations is skipped after only a few first matrices in $\mathcal{L}_n$, reducing the extension complexity roughly to $O(n2^n)$ (instead of $O(4^n)$). In Table A.7 for $10 \leq n \leq 19$ we give

- lower bound for $a_n$,
• \(|\mathcal{L}_{n-1}|\), the number of extended matrices,
• a matrix \(A_{n-1}\) with the highest extension spectrum found in \(|\mathcal{A}_{n-1}|\),
• extension spectrum and determinant of \(A_{n-1}\).

Complete lists of matrices, whose extension determinants prove these lower bounds, can be found at http://www.matf.bg.ac.yu/~ezivkovi/01matrices.htm.

5 Counting \((0,1)\) matrices with the maximum determinant

Using the classification of \(\mathcal{A}_n\), it is not hard to compute the number \(c_n\) [4, Sequences A051752] of matrices in \(\mathcal{A}_n\) with the maximal determinant \(d_n\) (i.e. \(1/2\) of the number of matrices with the ADV \(d_n\)) for \(n \leq 9\).

The first 8 members of the sequence \(c_n\) are found in Table A.3; the number \(c_8 = 195955200\) is new.

In order to determine \(c_9\), from Table A.4 we see that the matrix from \(\mathcal{A}_9\) with the ADV 144 could be obtained only by extending matrices from \(\mathcal{A}_8\) with the SNF \(\left(1^5, 2, 2, 6\right)\) or \(\left(1^5, 2, 2, 12\right)\). After the extension of these two SNF-classes, it turned out that there is a unique \(\phi\)-class with the ADV 144 — the class with the representative \((F,33,C3,FC,155,15A,166,196,1A9)\). Half of the number of matrices in that \(\phi\)-class is \(c_9 = 13716864000\). It is interesting that for all \(n \leq 9\) there is a unique \(\phi\)-class with the maximal ADV.

6 Acknowledgement

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A Large tables
Table A.1
π-representatives of (0,1) matrices of order 3.

| SNF | size |
|-----|------|
| 0 0 0 | 1 |
| 0 0 0 | 1 |

| SNF | size |
|-----|------|
| 1 0 0 | 49 |
| 1 0 0 | 49 |

| SNF | size |
|-----|------|
| 0 1 2 | 18 |
| 0 1 7 | 18 |
| 1 1 3 | 18 |
| 1 1 6 | 9 |
| 3 7 7 | 9 |
| 0 1 3 | 36 |
| 0 1 6 | 18 |
| 0 3 7 | 18 |
| 1 1 2 | 18 |
| 1 3 3 | 18 |
| 1 1 7 | 9 |
| 3 3 7 | 9 |
| 1 6 6 | 9 |
| 1 7 7 | 9 |
| 0 3 5 | 18 |
| 3 3 5 | 18 |

| SNF | size |
|-----|------|
| 1 1 1 | 168 |
| 1 1 1 | 168 |

| SNF | size |
|-----|------|
| 1 2 4 | 6 |
| 1 2 7 | 18 |
| 1 3 5 | 18 |
| 1 3 6 | 36 |
| 3 5 7 | 18 |
| 1 2 5 | 36 |
| 1 3 7 | 36 |

| SNF | size |
|-----|------|
| 1 1 2 | 6 |
| 3 5 6 | 6 |
Table A.2
φ-representatives of (0,1) matrices of order 4.

| SNF | size |
|-----|------|
| 0 0 0 0 | 1 |
| 0 0 0 0 | 1 |

| SNF | size |
|-----|------|
| 0 0 0 1 | 25 |
| 0 0 0 3 | 50 |
| 0 0 1 1 | 50 |
| 0 0 1 3 | 50 |
| 0 0 3 3 | 100 |

| SNF | size |
|-----|------|
| 1 0 0 0 | 225 |
| 1 0 0 0 | 225 |
| 1 1 0 0 | 6750 |

| SNF | size |
|-----|------|
| 1 0 0 0 | 225 |
| 1 0 0 1 | 225 |
| 1 0 0 3 | 225 |
| 1 0 1 1 | 225 |
| 1 0 1 3 | 225 |
| 1 0 3 3 | 225 |
| 1 1 0 0 | 6750 |
| 1 1 0 1 | 6750 |
| 1 1 0 3 | 6750 |
| 1 1 1 3 | 6750 |
| 1 1 3 3 | 6750 |
| 1 1 3 6 | 6750 |
| 1 1 6 6 | 6750 |
| 1 1 6 7 | 6750 |
| 1 3 3 5 | 6750 |

| SNF | size |
|-----|------|
| 1 1 1 0 | 35400 |

| SNF | size |
|-----|------|
| 0 1 2 4 | 600 |
| 0 1 2 5 | 600 |
| 0 1 2 7 | 1800 |
| 0 1 3 5 | 1800 |
| 0 1 3 6 | 3600 |
| 0 1 3 7 | 3600 |
| 0 1 6 A | 3600 |
| 0 3 5 7 | 1800 |
| 0 3 5 9 | 1200 |
| 0 3 5 A | 3600 |
| 1 2 3 4 | 3600 |
| 1 2 4 7 | 1200 |
| 1 2 5 6 | 3600 |
| 1 6 7 A | 1800 |

| SNF | size |
|-----|------|
| 1 1 2 0 | 600 |
| 3 5 6 0 | 600 |

| SNF | size |
|-----|------|
| 1 1 2 1 | 20040 |

| SNF | size |
|-----|------|
| 1 2 4 8 | 600 |
| 1 2 4 9 | 7200 |
| 1 2 5 A | 1440 |
| 1 2 5 B | 7200 |
| 1 2 7 B | 3600 |

| SNF | size |
|-----|------|
| 1 1 2 1 | 20040 |

| SNF | size |
|-----|------|
| 1 1 1 2 | 2400 |
| 1 6 A C | 2400 |

| SNF | size |
|-----|------|
| 1 1 1 3 | 120 |
| 3 5 9 E | 120 |

31
Table A.3
The representatives and the sizes of SNF-classes in $\mathcal{A}_n$, $n \leq 8$.

| $\mathcal{A}_1$ | The number of | The SNF-class |
|-----------------|---------------|---------------|
|                 | $\det$ SNF | matrices | $\pi$-classes | $\phi$-classes | representative |
| 0 0 0 0         | 1 1 1      | 1           | 0             |
| 1 0 0 1         | 9 4 2      | 1           | 0             |
| 2 1 1 1         | 6 2 1      | 1           | 0             |
| Total:          | 2 2 2      | 2           | 2             |

| $\mathcal{A}_2$ | The number of | The SNF-class |
|-----------------|---------------|---------------|
|                 | $\det$ SNF | matrices | $\pi$-classes | $\phi$-classes | representative |
| 0 0 0 0         | 1 1 1      | 2           | 0             |
| 1 0 0 1         | 9 4 1      | 1           | 1             |
| 2 1 1 1         | 6 2 1      | 1           | 1             |
| Total:          | 16 7 3    | 3           | 1             |

| $\mathcal{A}_3$ | The number of | The SNF-class |
|-----------------|---------------|---------------|
|                 | $\det$ SNF | matrices | $\pi$-classes | $\phi$-classes | representative |
| 0 0 0 0         | 1 1 1      | 2           | 0             |
| 1 0 0 1         | 49 9 2     | 3           | 0             |
| 2 0 1 1         | 288 18 2   | 4           | 0             |
| 3 1 1 1         | 168 7 2    | 2           | 1             |
| 4 2 1 1         | 6 1 2      | 1           | 3             |
| Total:          | 512 36 12  | 12          | 1             |

| $\mathcal{A}_4$ | The number of | The SNF-class |
|-----------------|---------------|---------------|
|                 | $\det$ SNF | matrices | $\pi$-classes | $\phi$-classes | representative |
| 0 0 0 0 0       | 1 1 1 1 1   | 1           | 0             |
| 1 0 0 1 1       | 225 16 4    | 0           | 0             |
| 2 0 1 1 1       | 6750 84 12  | 0           | 1             |
| 3 0 1 1 1       | 35400 150 14| 0           | 2             |
| 4 0 1 1 1       | 600 5 1     | 0           | 3             |
| 5 1 1 1 1       | 20040 49 5  | 1           | 2             |
| 6 2 1 1 1       | 2400 10 1   | 1           | 6             |
| 7 3 1 1 1       | 120 2 1     | 1           | 5             |
| Total:          | 65536 317 39| 39          | 1             |
Table A.3
Continued

| \(A_5\) | The number of matrices | \(\pi\)-classes | \(\phi\)-classes | The SNF-class representative |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 961 | 25 | 9 | 0 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 0 | 0 | 1 | 118800 | 260 | 37 | 0 | 0 | 1 | 0 | 2 | 2 |
| 3 | 0 | 0 | 0 | 0 | 1 | 3134400 | 1346 | 113 | 0 | 1 | 1 | 0 | 2 | 2 |
| 4 | 0 | 0 | 0 | 0 | 1 | 25350 | 25 | 5 | 0 | 0 | 3 | 5 | 6 |
| 5 | 0 | 0 | 1 | 1 | 1 | 16853400 | 2589 | 141 | 0 | 1 | 2 | 4 | 8 |
| 6 | 0 | 0 | 1 | 1 | 2 | 880200 | 210 | 17 | 0 | 1 | 6 | A | C |
| 7 | 0 | 0 | 1 | 1 | 3 | 27000 | 15 | 2 | 0 | 3 | 5 | 9 | E |
| 8 | 1 | 1 | 1 | 1 | 9702720 | 831 | 39 | 1 | 2 | 4 | 8 | 10 |
| 9 | 2 | 1 | 1 | 2 | 2427840 | 254 | 15 | 1 | 2 | C | 14 | 18 |
| 10 | 3 | 1 | 1 | 3 | 289440 | 51 | 5 | 1 | 6 | A | 12 | 1C |
| 11 | 4 | 1 | 1 | 4 | 65520 | 12 | 2 | 3 | 5 | 9 | 11 | 1E |
| 12 | 5 | 1 | 1 | 5 | 7200 | 3 | 1 | 3 | 5 | E | 16 | 19 |
| 13 | 4 | 1 | 1 | 6 | 21600 | 2 | 1 | 3 | 15 | 16 | 19 |
| Total: | 33554432 | 5624 | 388 |

| \(A_6\) | The number of matrices | \(\pi\)-classes | \(\phi\)-classes | The SNF-class representative |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 0 | 0 | 1 | 3969 | 36 | 9 | 0 | 0 | 0 | 0 | 0 | 1 |
| 2 | 0 | 0 | 0 | 0 | 1 | 1807806 | 660 | 3 | 6 | 0 | 0 | 0 | 0 | 1 |
| 3 | 0 | 0 | 0 | 0 | 1 | 189336000 | 7586 | 472 | 0 | 0 | 0 | 1 | 2 | 4 |
| 4 | 0 | 0 | 0 | 0 | 1 | 735000 | 86 | 10 | 0 | 0 | 0 | 3 | 5 | 6 |
| 5 | 0 | 0 | 1 | 1 | 1 | 5168108400 | 47605 | 1913 | 0 | 0 | 1 | 2 | 4 | 8 |
| 6 | 0 | 0 | 1 | 1 | 2 | 124744200 | 2120 | 115 | 0 | 0 | 1 | 6 | A | C |
| 7 | 0 | 0 | 1 | 1 | 3 | 2352000 | 91 | 9 | 0 | 0 | 3 | 5 | 9 | E |
| 8 | 0 | 1 | 1 | 1 | 1 | 30991962960 | 112080 | 3262 | 0 | 1 | 2 | 4 | 8 | 10 |
| 9 | 0 | 1 | 1 | 2 | 1 | 3122915040 | 14986 | 511 | 0 | 1 | 2 | C | 14 | 18 |
| 10 | 0 | 1 | 1 | 3 | 1 | 226603440 | 1618 | 75 | 0 | 1 | 6 | A | 12 | 1C |
| 11 | 0 | 1 | 1 | 4 | 3 | 38419920 | 307 | 16 | 0 | 3 | 5 | 9 | 11 | 1E |
| 12 | 0 | 1 | 1 | 5 | 3 | 3175200 | 46 | 3 | 0 | 3 | 5 | E | 16 | 19 |
| 13 | 0 | 1 | 1 | 6 | 1 | 127008000 | 78 | 4 | 0 | 3 | C | 15 | 16 | 19 |
| Total: | 68719476736 | 251610 | 8102 |

Total: 33554432 5624 388
### Table A.3
Continued

| $\mathcal{A}_7$ | The number of matrices | $\pi$-classes | $\phi$-classes | The SNF-class representative |
|-----------------|------------------------|---------------|---------------|-----------------------------|
| det | SNF | | | |
| 0 | 00000000 | 1 | 1 | 1 | 0 0 0 0 0 0 0 |
| 1 | 00000001 | 16129 | 49 | 16 | 0 0 0 0 0 0 1 |
| 2 | 00000011 | 25316928 | 1428 | 170 | 0 0 0 0 0 1 2 |
| 3 | 00000111 | 9254300328 | 31994 | 1908 | 0 0 0 1 2 4 |
| 4 | 00000112 | 17360406 | 246 | 34 | 0 0 0 3 5 6 |
| 5 | 00001111 | 989588124000 | 501563 | 17596 | 0 0 1 2 4 8 |
| 6 | 00001112 | 11359807200 | 13645 | 694 | 0 0 1 6 A C |
| 7 | 00011113 | 132300000 | 400 | 30 | 0 0 3 5 9 E |
| 8 | 00111111 | 30826279895040 | 4358421 | 105808 | 0 1 2 4 8 10 |
| 9 | 00111112 | 1405763634240 | 316904 | 9295 | 0 1 2 C 14 18 |
| 10 | 00111113 | 64153434240 | 22902 | 853 | 0 1 6 A 12 1C |
| 11 | 00111114 | 8175222720 | 3714 | 168 | 0 3 5 9 11 1E |
| 12 | 00111115 | 509443200 | 413 | 23 | 0 3 5 E 16 19 |
| 13 | 00111116 | 2694686400 | 1032 | 58 | 0 3 C 15 16 19 |
| 14 | 00111117 | 219225571810560 | 13834240 | 261882 | 0 1 2 4 8 10 20 |
| 15 | 00111118 | 34159997168640 | 2624469 | 53874 | 0 1 2 4 18 28 30 |
| 16 | 00111119 | 4018162256640 | 3766999 | 8633 | 0 1 2 C 14 24 38 |
| 17 | 00111120 | 1176364465920 | 123510 | 3024 | 0 1 6 A 12 22 3C |
| 18 | 00111121 | 182858215680 | 23489 | 633 | 0 1 6 A 1C 2C 32 |
| 19 | 00111122 | 110954188800 | 13823 | 361 | 0 3 5 9 16 2E 31 |
| 20 | 00111123 | 12940704000 | 2133 | 64 | 0 3 5 9 1E 2E 31 |
| 21 | 00111124 | 5966553600 | 1006 | 33 | 0 3 5 E 19 29 36 |
| 22 | 00111125 | 8297856000 | 189 | 7 | 0 3 7 19 1E 2A 2D 33 |
Table A.3
Continued

| \(A_7\) | \(\text{det} \) | SNF matrices | \(\pi\)-classes | \(\phi\)-classes | Representative |
|----------|----------------|--------------|----------------|----------------|---------------|
| 26 | 11111111 | 135491563468800 | 5593528 | 91764 | 1 2 4 8 10 20 40 |
| 27 | 211111111 | 83220427382400 | 3493129 | 58179 | 1 2 4 8 30 50 60 |
| 28 | 3111111111 | 23436399744000 | 1020752 | 17707 | 1 2 4 18 28 48 70 |
| 29 | 4111111111 | 132856722432000 | 581948 | 10189 | 1 2 14 24 44 78 |
| 30 | 511111111 | 375452001792000 | 172714 | 3169 | 1 2 14 38 58 64 |
| 31 | 61111111111 | 4201407745920 | 185688 | 3320 | 1 6 A 12 2C 5C 62 |
| 32 | 7111111111 | 813250851840000 | 39068 | 749 | 1 6 A 12 C 5C 62 |
| 33 | 8111111111 | 693389168640000 | 32490 | 645 | 1 6 A 1C 32 52 6C |
| 34 | 91111111111 | 257766405120000 | 12609 | 253 | 1 6 A 12 C 5C 62 |
| 35 | 1011111111110 | 215881142400000 | 10094 | 199 | 3 5 9 1E 2E 4E 71 |
| 36 | 111111111111 | 497984256000000 | 2598 | 55 | 3 5 9 1E 31 51 6E |
| 37 | 121111111112 | 675118080000000 | 3263 | 71 | 3 5 E 16 39 59 66 |
| 38 | 131111111113 | 122830848000000 | 686 | 17 | 3 5 E 19 36 56 69 |
| 39 | 141111111114 | 122605056000000 | 615 | 12 | 3 5 19 29 36 4E 71 |
| 40 | 151111111115 | 406425600000000 | 215 | 6 | 3 15 26 38 5E 61 |
| 41 | 161111111116 | 223534080000000 | 143 | 6 | 3 C 15 36 39 5A 65 |
| 42 | 171111111117 | 406425600000000 | 27 | 2 | 3 16 2E 39 5A 65 |
| 43 | 181111111118 | 541900800000000 | 24 | 1 | 7 19 2A 34 4C 53 65 |
| 44 | 11111111121 | 441333043200000 | 184475 | 3220 | 1 2 C 30 54 58 64 |
| 45 | 11111111124 | 343226419200000 | 15119 | 317 | 1 6 18 2A 34 4C 52 |
| 46 | 11111111126 | 219469824000000 | 997 | 28 | 3 5 19 29 36 4E 51 |
| 47 | 11111111128 | 121927680000000 | 102 | 6 | 3 31 55 5A 66 69 |
| 48 | 111111111210 | 135475200000000 | 10 | 1 | 7 19 2A 34 4C 52 63 |
| 49 | 1111111133 | 299400192000000 | 1358 | 37 | 3 5 18 28 49 4E 71 |
| 50 | 1111111136 | 139708800000000 | 10 | 2 | 3 1D 2D 36 3A 4E 71 |
| 51 | 1111111144 | 254016000000000 | 19 | 3 | 3 C 35 3A 55 66 69 |
| 52 | 1111112222 | 159691392000000 | 750 | 29 | 1 E 32 3C 54 5A 66 |
| 53 | 1111112224 | 118540800000000 | 20 | 4 | 3 C 30 55 5A 66 69 |
| 54 | 1111112226 | 967680000000000 | 5 | 1 | 7 19 2A 34 4C 52 61 |
| 55 | 1111112224 | 151200000000000 | 1 | 1 | 3 F 33 3C 55 5A 66 69 |

Total: 562949953421312 33642660 6561033
Table A.3
Continued

| $\mathcal{A}_8$ | det | SNF | The number of matrices | $\pi$-classes | $\phi$-classes | The SNF-class representative |
|-----------------|-----|-----|------------------------|--------------|---------------|-----------------------------|
| 0               | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 1               | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 2               | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 3               | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 4               | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 5               | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 6               | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 7               | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 8               | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 9               | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 10              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 11              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 12              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 13              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 14              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 15              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 16              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 17              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 18              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 19              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 20              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 21              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 22              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 23              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 24              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 25              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 26              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 27              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 28              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 29              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 30              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 31              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 32              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 33              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 34              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 35              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 36              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 37              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 38              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 39              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 40              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 41              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 42              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| 43              | 0   | 0   | 0                      | 0            | 0             | 0                          |
| det | SNF | $\mathcal{A}_8$ | The number of matrices | $\pi$-classes | $\phi$-classes | The SNF-class representative |
|-----|-----|----------------|------------------------|-------------|---------------|----------------------------|
| 44  | 01  | 25998918420787200 | 31764328 | 476566 | 0 1 2 C 30 54 58 64 |
| 45  | 01  | 9049454120448444 | 1434616 | 23722 | 0 1 6 182A 34 4C 52 |
| 46  | 01  | 388681058304000 | 70882 | 1251 | 0 3 5 19 29 36 4E 51 |
| 47  | 01  | 177770557744000 | 4348 | 114 | 0 3 C 31 555A 66 69 |
| 48  | 01  | 175575859200000 | 408 | 10 | 0 7 192A 34C 52 63 |
| 49  | 01  | 706199026176000 | 117219 | 1972 | 0 3 5 18 28 49 4E 71 |
| 50  | 01  | 181062604800000 | 385 | 10 | 0 31D 2D 363A 4E 71 |
| 51  | 01  | 362125290690000 | 902 | 23 | 0 3 C 353A 55 66 69 |
| 52  | 01  | 409319059968000 | 65455 | 1203 | 0 1 E 32C 54 5A 66 |
| 53  | 01  | 192036096000000 | 606 | 26 | 0 3 C 30 555A 66 69 |
| 54  | 01  | 125411328000000 | 66 | 5 | 0 7 192A 34C 52 61 |
| 55  | 01  | 195955200000000 | 5 | 1 | 0 F 33C 355A 66 69 |
| 56  | 11  | 376692568171528000 | 2363927011 | 29610494 | 1 2 4 8 10 20 40 80 |
| 57  | 11  | 310722185632158700 | 1958051993 | 24598561 | 1 2 4 8 10 60 A0 C0 |
| 58  | 11  | 1128344550375409920 | 717105693 | 9069162 | 1 2 4 8 30 50 90 E0 |
| 59  | 11  | 708113338051276800 | 508274660 | 6438161 | 1 2 4 18 28 48 88 F0 |
| 60  | 11  | 2805583980458644600 | 180518667 | 23036828 | 1 2 4 18 28 70 B0 C8 |
| 61  | 11  | 3918399813303091200 | 249971410 | 3172566 | 1 2 C 14 24 58 B8 C4 |
| 62  | 11  | 9271761872925888800 | 601524371 | 772896 | 1 2 C 14 24 78 B8 C4 |
| 63  | 11  | 9080140580406784080 | 632800711 | 811462 | 1 2 C 14 38 64 A4 D8 |
| 64  | 9  | 463926284332441600 | 30131275 | 388116 | 1 2 30 C 34 54 68 98 E4 |
| 65  | 10 | 493708393788825600 | 31886563 | 408631 | 1 6 A 123C 5C 9C E2 |
| 66  | 11 | 142143843810214800 | 9334363 | 121275 | 1 6 A 123C 62 A2 DC |
| 67  | 12 | 252874744316006400 | 16375045 | 211035 | 1 6 A 1C 2C 72 B2 CC |
| 68  | 13 | 607690793169792000 | 4012940 | 54219 | 1 6 A 1C 326C AC D2 |
| 69  | 14 | 866161820385792000 | 5648182 | 72918 | 1 6 A 32 526C 9C E2 |
| 70  | 15 | 466087642292160000 | 3060154 | 39595 | 1 6 A 2A 4C 70 BC C2 |
| 71  | 16 | 336397798517760000 | 2214742 | 29108 | 1 6 182A 6C 72 B4 CA |
| 72  | 17 | 126047774661120000 | 804732 | 927176 | 1 6 A 2C 5C 72 B4 CA |
| 73  | 18 | 230871372805260000 | 1503637 | 19409 | 1 6 32 54 68 98 A6 CA |
| 74  | 19 | 548565282316800000 | 367458 | 4863 | 3 5 9 31 51 6E 9E E1 |
| 75  | 20 | 883790793169792000 | 583710 | 7689 | 3 5 E 19 36 68 9A D6 |
| 76  | 21 | 420654153830400000 | 280261 | 3719 | 3 5 E 19 36 69 A9 D6 |
| 77  | 22 | 358862424883200000 | 237040 | 3085 | 3 5 E 32 56 69 99 E6 |
| 78  | 23 | 113325986952000000 | 76671 | 1023 | 3 5 E 36 56 99 E6 |
| 79  | 24 | 228366581990400000 | 152796 | 2117 | 3 5 18 68 A9 B6 CE D1 |
| 80  | 25 | 597479162112000000 | 40146 | 532 | 3 5 18 2E 69 76 B1 CE |
| 81  | 26 | 732041597952000000 | 48393 | 630 | 3 5 19 2E 69 76 B1 CE |
| 82  | 27 | 331554892032000000 | 22046 | 293 | 3 5 39 59 6E 76 9E E1 |
| 83  | 28 | 347091526650000000 | 23559 | 324 | 3 5 C 15 365A 65 B9 C6 |
| 84  | 29 | 911897118720000000 | 6180 | 82 | 3 5 C 31 55 7A 96 D9 E5 |
| 85  | 30 | 280153230336000000 | 18147 | 238 | 3 5 31 54 6A 9A C9 |
| 86  | 31 | 362125209600000000 | 2436 | 32 | 3 5 D 16 2E 5A 75 B9 C6 |
### Table A.3
Continued

| \(A_8\) | The number of matrices | \(\pi\)-classes | \(\phi\)-classes | The SNF-class representative |
|---|---|---|---|---|
| det | SNF | | | |
| 87 32 | 1111111132 | 54236480256000 | 3953 | 68 | 3 C 31 56 6A 9A A6 C5 |
| 88 33 | 1111111133 | 28064703744000 | 1866 | 27 | 3 D 1E 35 66 79 AA D5 |
| 89 34 | 1111111134 | 28311607296000 | 1815 | 23 | 3 D 31 54 6A 9A A6 C9 |
| 90 35 | 1111111135 | 75717098928000 | 491 | 7 | 31D 2E 56 69 9A B1 CD |
| 91 36 | 1111111136 | 13277924352000 | 873 | 13 | 3 D 35 56 69 99 AE C5 |
| 92 37 | 1111111137 | 13168189440000 | 81 | 1 | 7 192A 4C 71 A5 CB D6 |
| 93 38 | 1111111138 | 59256852480000 | 389 | 5 | 31D 2E 56 69 9A B5 CD |
| 94 39 | 1111111139 | 65840947200000 | 45 | 1 | 7 192A 56 6D 9C B3 CB |
| 95 40 | 1111111140 | 26336378880000 | 200 | 4 | 7 192A 4C 71 96 AD CB |
| 96 41 | 1111111141 | 65840947200000 | 45 | 1 | 7 19 3E 63 AA B5 CC D2 |
| 97 42 | 1111111142 | 26448993127895040 | 166494351 | 2095861 | 1 2 4 18 60 A8 B0 C8 |
| 98 43 | 1111111143 | 486286944875827200 | 30927272 | 394107 | 1 2 C 30 54 68 98 A4 |
| 99 44 | 1111111144 | 8317470133248000 | 5286249 | 67857 | 1 6 A 32 52 6C 9C A2 |
| 100 45 | 1111111145 | 1671575454259200 | 1087782 | 14500 | 1 6 18 62 AA B4 CC D2 |
| 101 46 | 1111111146 | 3001287437568000 | 194974 | 2641 | 1 E 32 54 68 98 A4 C6 |
| 102 47 | 1111111147 | 1197180456192000 | 80651 | 1172 | 3 5 18 60 A9 B6 CE D1 |
| 103 48 | 1111111148 | 143889903360000 | 9935 | 157 | 3 C 15A 65 A5 D6 D9 |
| 104 49 | 1111111149 | 418090014720000 | 3259 | 65 | 3 C 31 56 6A 9A A6 C1 |
| 105 50 | 1111111150 | 136071290880000 | 1103 | 23 | 3 D 31 54 6A 9A A6 C1 |
| 106 51 | 1111111151 | 271593907200000 | 232 | 5 | 3 D 35 59 6E 9E A9 C5 |
| 107 52 | 1111111152 | 768144384000000 | 74 | 2 | 31D 2E 56 79 9A B5 CD |
| 108 53 | 1111111153 | 164602368000000 | 28 | 1 | 7 19 3E 61 AB B5 CC D2 |
| 109 54 | 1111111154 | 572897459056000 | 3662516 | 47056 | 1 6 A 30 50 92 9C E2 |
| 110 55 | 1111111155 | 292630089304000 | 187420 | 2455 | 1 63A 5A 6C 74 9C E2 |
| 111 56 | 1111111156 | 64661296860000 | 4234 | 71 | 3 C 31 54 7A 9A A6 C9 |
| 112 57 | 1111111157 | 28531077120000 | 219 | 7 | 31C 65A 6A B1 C9 D2 |
| 113 58 | 1111111158 | 27433728000000 | 6 | 1 | 7 39A 5A 6C 9C AB B6 D1 |
| 114 59 | 1111111159 | 2826990802944000 | 186028 | 2556 | 1 6 18 6A 74 AA CC D2 |
| 115 60 | 1111111160 | 72425041920000 | 713 | 21 | 3 C 35A 56 69 99 A6 |
| 116 61 | 1111111161 | 23394111520000 | 1497 | 22 | 3 D 15 26 5E 61 B8 C5 |
| 117 62 | 1111111162 | 14265538560000 | 136 | 4 | 7 192A 4B 74 8C D2 E1 |
| 118 63 | 1111111163 | 22603498944768000 | 1421783 | 18397 | 1 21C 64 78 A8 B4 CC |
| 119 64 | 1111111164 | 1362450378240000 | 90153 | 1346 | 1 6 18 60 AA B4 CC D2 |
| 120 65 | 1111111165 | 653001108480000 | 5175 | 114 | 1 E 32 54 68 98 A4 C2 |
| 121 66 | 1111111166 | 625488989400000 | 578 | 15 | 3 D 31 55 6A 9A A6 C1 |
| 122 67 | 1111111167 | 54867456000000 | 96 | 5 | 3 D 31 56 6A 9A A6 C1 |
| 123 68 | 1111111168 | 16402363800000 | 49 | 3 | 31C 65A 6A B9 CE D1 |
| 124 69 | 1111111169 | 39191040000000 | 6 | 1 | 31D 65A 6B CE D1 |
| 125 70 | 1111111170 | 98761420800000 | 134 | 4 | 31C 64 78 A9 B2 CA D1 |
| 126 71 | 1111111171 | 10150479360000 | 96 | 6 | 7 192A 4B 74 8D B1 D6 |
| 127 72 | 1111111172 | 108271779840000 | 754 | 17 | 31C 64 78 A9 AA B5 CD |
| 128 73 | 1111111173 | 282175488000000 | 74 | 3 | 7 19 3E 63 AA B5 CC D2 |

Total: 18446744073709551616 14685630688 199727714
Table A.4
Transposed incidence matrix $M_T$ containing all $M_n$, $1 \leq n \leq 7$. Symbol at position $(s', s)$ carries information about $m_{s, s'}$ (7): $\bullet$, $\ast$, $\circ$ denotes respectively 1, 0 explained by Lemma 10, and 0 not explained by Lemma 10.
Table A.5
The part of $S_9$ corresponding to nonsingular part of $A_9$.

| det | SNF |
|-----|-----|
| 1-106 | $m$ | 1 1 1 1 1 1 1 1 $m$ |
| 107-135 | $4m$ | 1 1 1 1 1 1 1 2 $2m$ |
| 136-148 | $9m$ | 1 1 1 1 1 1 1 3 $3m$ |
| 149-155 | $16m$ | 1 1 1 1 1 1 1 4 $4m$ |
| 156-159 | $25$ | 1 1 1 1 1 1 1 5 $5m$ |
| 160 | 36 | 1 1 1 1 1 1 1 6 |
| 161 | 72 | 1 1 1 1 1 1 1 6 12 |
| 162 | 108 | 1 1 1 1 1 1 1 6 18 |
| 163 | 49 | 1 1 1 1 1 1 1 7 7 |
| 164 | 98 | 1 1 1 1 1 1 1 7 14 |
| 165 | 64 | 1 1 1 1 1 1 1 8 8 |
| 166 | 128 | 1 1 1 1 1 1 1 8 16 |
| 167 | 81 | 1 1 1 1 1 1 1 9 9 |
| 168 | 100 | 1 1 1 1 1 1 1 10 10 |
| 169-182 | $8m$ | 1 1 1 1 1 1 1 2 2 $2m$ |
| 183 | 32 | 1 1 1 1 1 1 1 2 4 4 |
| 184 | 64 | 1 1 1 1 1 1 1 2 4 8 |
| 185 | 96 | 1 1 1 1 1 1 1 2 4 12 |
| 186 | 72 | 1 1 1 1 1 1 1 2 6 6 |
| 187 | 128 | 1 1 1 1 1 1 1 2 8 8 |
| 188 | 27 | 1 1 1 1 1 1 1 3 3 3 |
| 189 | 54 | 1 1 1 1 1 1 1 3 3 6 |
| 190 | 81 | 1 1 1 1 1 1 1 3 3 9 |
| 191 | 108 | 1 1 1 1 1 1 1 3 3 12 |
| 192 | 64 | 1 1 1 1 1 1 1 4 4 4 |
| 193 | 128 | 1 1 1 1 1 1 1 4 4 8 |
| 194 | 16 | 1 1 1 1 1 1 1 4 4 8 |
| 195 | 32 | 1 1 1 1 1 1 1 2 2 2 2 |
| 196 | 48 | 1 1 1 1 1 1 1 2 2 2 6 |
| 197 | 64 | 1 1 1 1 1 1 1 2 2 2 8 |
| 198 | 80 | 1 1 1 1 1 1 1 2 2 2 10 |
| 199 | 96 | 1 1 1 1 1 1 1 2 2 2 12 |
| 200 | 64 | 1 1 1 1 1 1 1 2 2 4 4 |
| 201 | 144 | 1 1 1 1 1 1 1 2 2 6 6 |
| 202 | 81 | 1 1 1 1 1 1 1 3 3 3 3 |
| 203 | 32 | 1 1 1 1 1 1 1 2 2 2 2 |
| 204 | 64 | 1 1 1 1 1 1 2 2 2 2 4 |
Table A.6

The part of the transposed incidence matrix $M_{S}$, corresponding to the regular part of $S_{9}$. Symbol at position $(s', s)$ carries information about $m_{s, s'}$ (7): •, *, ⋄ denotes respectively 1, 0 explained by Lemma 10, and 0 not explained by Lemma 10.
Table A.6
Continued.
Table A.6
Continued.

| \( a \) | \( a' \) |
|-------|-------|
| \( 1 \) | 3     |
| \( 3 \) | 1     |
|     |       |
| \( 4 \) | 2     |
| \( 2 \) | 4     |
|     |       |
| \( 5 \) | 7     |
| \( 7 \) | 5     |
|     |       |
| \( 8 \) | 9     |
| \( 9 \) | 8     |
|     |       |
| \( 10 \) | 11    |
| \( 11 \) | 10    |
|     |       |
| \( 12 \) | 13    |
|     |       |

\( s = s \)
Table A.7
Lower bounds for $a_n$ and matrices with high extension spectra, $10 \leq n \leq 19.$

| $n$ | $a_n \geq$ | $|L_{n-1}|$ | $\det A_{n-1}$ | $a_{A_{n-1}}$ | $A_{n-1}$ |
|-----|-----------|-------------|----------------|----------------|-----------|
| 10  | 259       | 2           | 110            | 257            | $[7, 39, 5A, 9C, E1, 149, 174, 193, 1AA]$ |
| 11  | 739       | 6           | 291            | 679            | $[F, 71, B6, 13A, 1C3, 1DC, 256, 299, 2EC, 325]$ |
| 12  | 2107      | 19          | 779            | 1894           | $[F, 73, 195, 1EA, 2A6, 35C, 4D6, 53E, 565, 6B9, 703]$ |
| 13  | 6157      | 18          | 2201           | 5618           | $[1F, E3, 17C, 3A5, 649, 6D6, 732, A6E, AB8, B53, C35, D8E]$ |
| 14  | 19073     | 40          | 6731           | 16821          | $[3F, 1C7, 2D9, 76A, C4D, CF2, F94, 1575, 168E, 195A, 19A9, 1A64, 1E13]$ |
| 15  | 58741     | 46          | 23288          | 53117          | $[7D, 38F, 5B2, ED5, 189B, 19E4, 1F29, 2AE4, 2D1C, 32B4, 3353, 34C9, 3C27, 164E]$ |
| 16  | 185693    | 190         | 67832          | 161599         | $[FD, 71F, BE3, 1D29, 324F, 36B2, 3995, 5370, 55C6, 5A9A, 61AB, 6C53, 6E24, 27C8, 297E]$ |
| 17  | 610187    | 480         | 213175         | 517794         | $[1FB, E3E, 17C6, 3A53, 649F, 6D64, 732B, A6E2, AB8D, B535, C356, D8A7, DC49, 4F91, 72FC, F99A]$ |
| 18  | 2039033   | 697         | 709503         | 1719277        | $[3F9, 1C7E, 2D95, 76AC, C4D2, CF27, F949, 15755, 168E5, 195A3, 19A9C, 1A64B, 1E13E, 14D8A, 17A33, 33C6, 1AF70]$ |
| 19  | 6478579   | 54          | 2331887        | 4663774        | $[7E9, 38F7, 5F13, E95D, 19277, 1B599, 1CCAF, 29B8E, 2AE31, 2D4C5, 30D56, 3629F, 37125, 13E4C, 1EBC2, 358F8, 2F46A, E7B4]$ |