Spherical bodies of constant width

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Abstract. The intersection $L$ of two different non-opposite hemispheres $G$ and $H$ of the $d$-dimensional unit sphere $S^d$ is called a lune. By the thickness of $L$ we mean the distance of the centers of the $(d-1)$-dimensional hemispheres bounding $L$. For a hemisphere $G$ supporting a convex body $C \subset S^d$ we define $\text{width}_G(C)$ as the thickness of the narrowest lune or lunes of the form $G \cap H$ containing $C$. If $\text{width}_G(C) = w$ for every hemisphere $G$ supporting $C$, we say that $C$ is a body of constant width $w$. We present properties of these bodies. In particular, we prove that the diameter of any spherical body $C$ of constant width $w$ on $S^d$ is $w$, and if $w < \frac{\pi}{2}$, then $C$ is strictly convex. Moreover, we check when spherical bodies of constant width and constant diameter coincide.

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1. Introduction

Consider the unit sphere $S^d$ in the $(d + 1)$-dimensional Euclidean space $E^{d+1}$ for $d \geq 2$. The intersection of $S^d$ with any two-dimensional subspace of $E^{d+1}$ is called a great circle of $S^d$. By a $(d-1)$-dimensional great sphere of $S^d$ we mean the common part of $S^d$ with any hyper-subspace of $E^{d+1}$. The 1-dimensional great spheres of $S^2$ are called great circles. By a pair of antipodes of $S^d$ we understand a pair of points of intersection of $S^d$ with a straight line through the origin of $E^{d+1}$.

Clearly, if two different points $a, b \in S^d$ are not antipodes, there is exactly one great circle containing them. As the arc $ab$ connecting $a$ and $b$ we define the shorter part of the great circle containing these points. The length of this arc is called the spherical distance $|ab|$ of $a$ and $b$, or shortly distance. Moreover, we agree that the distance of coinciding points is 0, and that of any pair of antipodes is $\pi$. 
A spherical ball \( B_\rho(x) \) of radius \( \rho \in (0, \frac{\pi}{2}] \), or a ball for short, is the set of points of \( S^d \) at distances at most \( \rho \) from a fixed point \( x \), which is called the center of this ball. An open ball (a sphere) is the set of points of \( S^d \) having distance smaller than (respectively, exactly) \( \rho \) from a fixed point. A spherical ball of radius \( \frac{\pi}{2} \) is called a hemisphere. So it is the common part of \( S^d \) and a closed half-space of \( E^{d+1} \). We denote by \( H(m) \) the hemisphere with center \( m \). Two hemispheres with centers at a pair of antipodes are called opposite.

A spherical \((d-1)\)-dimensional ball of radius \( \rho \in (0, \frac{\pi}{2}] \) is the set of points of a \((d-1)\)-dimensional great sphere of \( S^d \) which are at distances at most \( \rho \) from a fixed point. We call it the center of this ball. The \((d-1)\)-dimensional balls of radius \( \frac{\pi}{2} \) are called \((d-1)\)-dimensional hemispheres, and semicircles for \( d = 2 \).

A set \( C \subset S^d \) is said to be convex if no pair of antipodes belongs to \( C \) and if for every \( a, b \in C \) we have \( ab \subset C \). A closed convex set on \( S^d \) with non-empty interior is called a convex body. Some basic references on convex bodies and their properties are [4], [9] and [10]. A short survey of other definitions of convexity on \( S^d \) is given in Section 9.1 of [2].

Since the intersection of every family of convex sets is also convex, for every set \( A \subset S^d \) contained in an open hemisphere of \( S^d \) there is the smallest convex set \( \text{conv}(A) \) containing \( Q \). We call it the convex hull of \( A \).

Let \( C \subset S^d \) be a convex body. Let \( Q \subset S^d \) be a convex body or a hemisphere. We say that \( C \) touches \( Q \) from inside if \( C \subset Q \) and \( \text{bd}(C) \cap \text{bd}(Q) \neq \emptyset \). We say that \( C \) touches \( Q \) from outside if \( C \cap Q \neq \emptyset \) and \( \text{int}(C) \cap \text{int}(Q) = \emptyset \). In both cases, points of \( \text{bd}(C) \cap \text{bd}(Q) \) are called points of touching. In the first case, if \( Q \) is a hemisphere, we also say that \( Q \) supports \( C \), or supports \( C \) at \( t \), provided \( t \) is a point of touching. If at every boundary point of \( C \) exactly one hemisphere supports \( C \), we say that \( C \) is smooth. We call \( e \in C \) an extreme point of \( C \) if \( C \setminus \{e\} \) is convex.

If hemispheres \( G \) and \( H \) of \( S^d \) are different and not opposite, then \( L = G \cap H \) is called a lune of \( S^d \). This notion is considered in many books and papers (for instance, see [12]). The \((d-1)\)-dimensional hemispheres bounding \( L \) and contained in \( G \) and \( H \), respectively, are denoted by \( G/H \) and \( H/G \).

Observe that \((G/H) \cup (H/G)\) is the boundary of the lune \( G \cap H \). Denote by \( c_{G/H} \) and \( c_{H/G} \) the centers of \( G/H \) and \( H/G \), respectively. By corners of the lune \( G \cap H \) we mean points of the set \((G/H) \cap (H/G)\). In particular, every lune on \( S^2 \) has two corners. They are antipodes.

We define the thickness \( \Delta(L) \) of a lune \( L = G \cap H \) on \( S^d \) as the spherical distance of the centers of the \((d-1)\)-dimensional hemispheres \( G/H \) and \( H/G \) bounding \( L \). Clearly, it is equal to each of the non-oriented angles \( \angle c_{G/H} r c_{H/G} \), where \( r \) is any corner of \( L \).

Compactness arguments show that for any hemisphere \( K \) supporting a convex body \( C \subset S^d \) there is at least one hemisphere \( K^* \) supporting \( C \) such
that the lune $K \cap K^*$ is of the minimum thickness. In other words, there is a “narrowest” lune of the form $K \cap K'$ over all hemispheres $K'$ supporting $C$. The thickness of the lune $K \cap K^*$ is called the width of $C$ determined by $K$. We denote it by $\text{width}_K(C)$.

We define the thickness $\Delta(C)$ of a spherical convex body $C$ as the smallest width of $C$. This definition is analogous to the classical definition of thickness (also called minimal width) of a convex body in Euclidean space.

By the relative interior of a convex set $C \subset S^d$ we mean the interior of $C$ with respect to the smallest sphere $S^k \subset S^d$ that contains $C$.

2. A few lemmas on spherical convex bodies

Lemma 1. Let $A$ be a closed set contained in an open hemisphere of $S^d$. Then $\text{conv}(A)$ coincides with the intersection of all hemispheres containing $A$.

Proof. First, let us show that $\text{conv}(A)$ is contained in the intersection of all hemispheres containing $A$. Take any hemisphere $H$ containing $A$ and denote by $J$ the open hemisphere from the formulation of our lemma. Recall that $A \subset J$ and $A \subset H$. Thus since $J \cap H$ is a convex set, we obtain $\text{conv}(A) \subset \text{conv}(J \cap H) = J \cap H \subset H$. Thus, since $\text{conv}(A)$ is contained in any hemisphere that contains $A$, also $\text{conv}(A)$ is a subset of the intersection of all those hemispheres.

Now we intend to show that the intersection of all hemispheres containing $A$ is contained in $\text{conv}(A)$. Assume the opposite, i.e., that there is a point $x \notin \text{conv}(A)$ which belongs to every hemisphere containing $A$. Since $A$ is closed, by Lemma 1 of [6] the set $\text{conv}(A)$ is also closed. Hence there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \cap \text{conv}(A) = \emptyset$. Since these two sets are convex, we may apply the following more general version of Lemma 2 of [6]: any two convex disjoint sets on $S^d$ are subsets of two opposite hemispheres (which is true again by the separation theorem for convex cones in $E^{d+1}$). So $B_\varepsilon(x)$ and $\text{conv}(A)$ are in some two opposite hemispheres. Hence $x$ does not belong to the one which contains $\text{conv}(A)$. Clearly, that one also contains $A$. This contradicts our assumption on the choice of $x$, and thus the proof is finished. \qed

We omit the simple proof of the next lemma, which is analogous to the situation in $E^d$ and needed a few times later. Here our hemisphere plays the role of a closed half-space there.

Lemma 2. Let $C$ be a spherical convex body. Assume that a hemisphere $H$ supports $C$ at a point $p$ of the relative interior of a convex set $T \subset C$. Then $T \subset \text{bd}(H)$.

Lemma 3. Let $K, M$ be hemispheres such that the lune $K \cap M$ is of thickness smaller than $\pi/2$. Denote by $b$ the center of $M/K$. Every point of $K \cap M$ at distance $\pi/2$ from $b$ is a corner of $K \cap M$. 
Proof. Denote the center of $K/M$ by $a$. Take any point $p \in K \cap M$. Let us show that there are points $x \in (K/M) \cap (M/K)$ and $y \in ab$ such that $p \in xy$.

If $p = b$ then it is obvious. Otherwise there is a unique point $q \in K/M$ such that $p \in bq$. Moreover, there exists $x \in (K/M) \cap (M/K)$ such that $q \in ax$. The reader can easily show that points $p, q$ belong to the triangle $abx$ and thus observe that there exists $y \in ab$ such that $p \in xy$, which confirms the statement from the first paragraph of the proof.

We have $|by| \leq |ba| < \pi/2$. The inequality $|by| < \pi/2$ means that $y$ is in the interior of $H(b)$. Of course, $|bx| = \pi/2$, which means that $x \in \partial H(b))$. From the two preceding sentences we conclude that $xy$ is a subset of $H(b)$ with $x$ being its only point on $\partial H(b))$. Thus, if $|pb| = \pi/2$, we conclude that $p \in \partial H(b))$, and consequently $p = x$, which implies that $p$ is a corner of $K \cap M$. The last sentence means that the statement of our lemma holds true. \hfill \qed

Lemma 4. Let $o \in S^d$ and $0 < \mu < \pi/2$. For every $x \in S^d$ at distance $\pi/2$ from $o$ denote by $x'$ the point of the arc $ox$ at distance $\mu$ from $x$. Consider two points $x_1, x_2$ at distance $\pi/2$ from $o$ such that $|x_1x_2| < \pi - \mu$. Then for every $x \in x_1x_2$ we have

$$B_\mu(x') \subset \text{conv}(B_\mu(x_1') \cup B_\mu(x_2')).$$

Proof. Let $o, m$ be points of $S^d$ and $\rho$ be a positive number less than $\pi/2$. Let us show that

$$B_\rho(o) \subset H(m) \text{ if and only if } |om| \leq \frac{\pi}{2} - \rho. \tag{1}$$

First assume that $B_\rho(o) \subset H(m)$. Let $b$ be a boundary point of $B_\rho(o)$ such that $o \in mb$. We have: $|om| = |mb| - |ob| = |bm| - \rho \leq \frac{\pi}{2} - \rho$, which confirms the “only if” part of (1). Assume now that $|om| \leq \frac{\pi}{2} - \rho$. Let $b$ be any point of $B_\rho(o)$. We have: $|bm| \leq |bo| + |om| \leq \rho + \left(\frac{\pi}{2} - \rho\right) = \frac{\pi}{2}$. Therefore every point of $B_\rho(o)$ is at a distance at most $\pi/2$ from $m$. Hence $B \subset H(m)$, which confirms the “if” part of (1). So (1) is shown.

Lemma 1 of \cite{6} guarantees that $Y = \text{conv}(B_\mu(x_1') \cup B_\mu(x_2'))$ is a closed set as a convex hull of a closed set. Consequently, from Lemma 1 we see that $Y$ is the intersection of all hemispheres containing $Y$. Moreover, observe that an arbitrary hemisphere contains a set if and only if it contains the convex hull of it. Hence $Y$ is the intersection of all hemispheres containing $B_\mu(x_1') \cup B_\mu(x_2')$.

As a result of the preceding paragraph, in order to prove the statement of our lemma it is sufficient to show that every hemisphere $H(m)$ containing $B_\mu(x_1') \cup B_\mu(x_2')$ also contains $B_\mu(x')$. Thus, having (1) in mind we see that in order to verify this it is sufficient to show that for any $m \in S^d$

$$|x_1'm| \leq \frac{\pi}{2} - \mu \text{ and } |x_2'm| \leq \frac{\pi}{2} - \mu \text{ imply } |x'm| \leq \frac{\pi}{2} - \mu. \tag{2}$$

Let us assume the first two of these inequalities and show the third one.
Observe that $x, x_1'$ and $x_2'$ belong to the spherical triangle $x_1 x_2 o$. Therefore the arcs $x o$ and $x_1' x_2'$ intersect. Denote the point of intersection by $g$.

In this paragraph we consider the intersection of $S^d$ with the three-dimensional subspace of $E^{d+1}$ containing $x_1', x_2', m$. Observe that this intersection is a two-dimensional sphere concentric with $S^d$. Denote this sphere by $S^2$. Denote by $\bar{o}$ the other unique point on $S^2$ such that the triangles $x_1' x_2' o$ and $x_1' x_2' \bar{o}$ are congruent. By the first two inequalities of (2) we obtain $m \in B_{\frac{\pi}{2} - \mu}(x_1') \cap B_{\frac{\pi}{2} - \mu}(x_2')$. Observe that $g\bar{o} \cup go$ dissects $B_{\frac{\pi}{2} - \mu}(x_1') \cap B_{\frac{\pi}{2} - \mu}(x_2')$ into two parts so that $x_1'$ belongs to one of them and $x_2'$ belongs to the other. Therefore at least one of the arcs $x_1'm$ and $x_2'm$, say $x_1'm$ intersects $g\bar{o}$ or $go$, say $go$. Denote this point of intersection by $s$. Taking the first assumption of (2) into account and using two times the triangle inequality we obtain $|og| = (|os| + |x_1's|) - |x_1's| + |sg| \geq |ox_1'| - |x_1's| + |sg| \geq |x_1'm| - |x_1's| + |sg| = |sm| + |sg| \geq |gm|$.

Applying the just obtained inequality and looking now again on the whole $S^d$ we get $|x'm| \leq |x'g| + |gm| \leq |x'g| + |og| = |x'o| = \frac{\pi}{2} - \mu$ which is the required inequality in (2). Thus by (2) also our lemma holds true. □

**Lemma 5.** Let $C \subset S^d$ be a convex body. Every point of $C$ belongs to the convex hull of at most $d + 1$ extreme points of $C$.

**Proof.** We apply induction with respect to $d$. For $d = 1$ the statement is trivial since every convex body on $S^1$ is a spherical arc. Let $d \geq 2$ be a fixed integer. Assume that for each $k = 1, 2, \ldots, d - 1$ every boundary point of a spherical convex body $\hat{C} \subset S^k$ belongs to the convex hull of at most $k + 1$ extreme points of $\hat{C}$.

Let $x$ be a point of $C$. Take an extreme point $e$ of $C$. If $x$ is not a boundary point of $C$, take the boundary point $f$ of $C$ such that $x \in ef$. In the opposite case put $f = x$.

If $f$ is an extreme point of $C$, the statement follows immediately. In the opposite case take a hemisphere $K$ supporting $C$ at $f$. Put $C' = bd(K) \cap C$. Observe that every extreme point of $C'$ is also an extreme point of $C$. Let $Q$ be the intersection of the smallest linear subspace of $E^{d+1}$ containing $C'$ with $S^d$. Clearly, $Q$ is isomorphic to $S^k$ for $k < d$. Moreover, $C'$ has non-empty relative interior with respect to $Q$, because otherwise there would exist a smaller linear subspace of $E^{d+1}$ containing $C'$. Thus, by the inductive hypothesis we obtain that $f$ is in the convex hull of a set $F$ of at most $d$ extreme points of $C$. Therefore $x \in \text{conv}\{e \cup F\}$ which means that $x$ belongs to the convex hull of $d + 1$ extreme points of $C$. This finishes the inductive proof. □

The proof of the following $d$-dimensional lemma is analogous to that of the two-dimensional Lemma 4.1 from [8] shown there for a wider class of reduced spherical convex bodies.
Lemma 6. Let $C \subset S^d$ be a spherical convex body with $\Delta(C) > \frac{\pi}{2}$ and let $L \supset C$ be a lune such that $\Delta(L) = \Delta(C)$. Each of the centers of the $(d - 1)$-dimensional hemispheres bounding $L$ belongs to the boundary of $C$ and both are smooth points of the boundary of $C$.

Having the next lemma in mind, we note the obvious fact that the diameter of a convex body $C \subset S^d$ is realized only for some pairs of points of $\text{bd}(C)$.

Lemma 7. Assume that the diameter of a convex body $C \subset S^d$ is realized for points $p$ and $q$. The hemisphere $K$ orthogonal to $pq$ at $p$ and containing $q \in K$ supports $C$.

Proof. Denote the diameter of $C$ by $\delta$.

Assume first that $\delta > \frac{\pi}{2}$. The set of points at distance at least $\delta$ from $q$ is the ball $B_{\pi - \delta}(q')$, where $q'$ is the antipode of $q$. Clearly, $K$ has only $p$ in common with $B_{\pi - \delta}(q')$.

Since the diameter $\delta$ of $C$ is realized for $pq$, every point of $C$ is at distance at most $\delta$ from $q$. Thus $C$ has empty intersection with the interior of $B_{\pi - \delta}(q')$.

Assume that $K$ does not contain $C$. Then $C$ contains a point $b \notin K$. Observe that the arc $bp$ has nonempty intersection with the interior of $B_{\pi - \delta}(q')$ [the reason: $K$ is the only hemisphere touching $B_{\pi - \delta}(q')$ from outside at $p$]. On the other hand, by the convexity of $C$ we have $bp \subset C$. This contradicts the fact from the preceding paragraph that $C$ has empty intersection with the interior of $B_{\pi - \delta}(q')$. Consequently, $K$ contains $C$.

Now consider the case when $\delta \leq \frac{\pi}{2}$. For every $y \notin K$ we have $|pq| < |yq|$ which by $|pq| = \delta$ implies $y \notin C$. Thus if $y \in C$, then $y \in K$. \qed

Let us apply our Lemma 7 for a convex body $C$ of diameter larger than $\frac{\pi}{2}$. Having in mind that the center $k$ of $K$ is in $pq$ and thus in $C$, by Part III of Theorem 1 in [6] we obtain that $\Delta(K \cap K^*) > \frac{\pi}{2}$. This gives the following corollary which implies the other one. The symbol $\text{diam}(C)$ denotes the diameter of $C$.

Corollary 1. Let $C \subset S^d$ be a convex body of diameter larger than $\frac{\pi}{2}$ and let $\text{diam}(C)$ be realized for points $p, q \in C$. Take the hemisphere $K$ orthogonal to $pq$ at $p$ which supports $C$. Then $\text{width}_K(C) > \frac{\pi}{2}$.

Corollary 2. Let $C \subset S^d$ be a convex body of diameter larger than $\frac{\pi}{2}$ and let $K$ denote the family of all hemispheres supporting $C$. Then $\max_{K \in K} \text{width}_K(C) > \frac{\pi}{2}$.

3. Spherical bodies of constant width

If for every hemisphere supporting a convex body $W \subset S^d$ the width of $W$ determined by $K$ is the same, we say that $W$ is a body of constant width.
(see [6] and for an application also [5]). In particular, spherical balls of radius smaller than \( \frac{\pi}{2} \) are bodies of constant width. Also every spherical Reuleaux odd-gon (for the definition see [6], p. 557) is a convex body of constant width. Each of the \( 2^{d+1} \) parts of \( S^d \) dissected by \( d + 1 \) pairwise orthogonal \((d - 1)\)-dimensional spheres of \( S^d \) is a spherical body of constant width \( \frac{\pi}{2} \), which easily follows from the definition of a body of constant width. The class of spherical bodies of constant width is a subclass of the class of spherical reduced bodies considered in [6] and [8], and mentioned by [3] in a larger context, (recall that a convex body \( R \subset S^d \) is called reduced if \( \Delta(Z) < \Delta(R) \) for every body \( Z \subset R \) different from \( R \), see also [7] for this notion in \( E^d \)).

By the definition of width and by Claim 2 of [6], if \( W \subset S^d \) is a body of constant width, then every supporting hemisphere \( G \) of \( W \) determines a supporting hemisphere \( H \) of \( W \) for which \( G \cap H \) is a lune of thickness \( \Delta(W) \) such that the centers of \( G/H \) and \( H/G \) belong to the boundary of \( W \). Hence every spherical body \( W \) of constant width is an intersection of lunes of thickness \( \Delta(W) \) such that the centers of the \((d - 1)\)-dimensional hemispheres bounding these lunes belong to \( \text{bd}(W) \). Recall the related question from p. 563 of [6] if a convex body \( W \subset S^d \) is of constant width provided every supporting hemisphere \( G \) of \( W \) determines at least one hemisphere \( H \) supporting \( W \) such that \( G \cap H \) is a lune with the centers of \( G/H \) and \( H/G \) in \( \text{bd}(W) \).

Here is an example of a spherical body of constant width on \( S^3 \).

**Example.** Take a circle \( X \subset S^3 \) (i.e., a set congruent to a circle in \( E^2 \)) of a positive diameter \( \kappa < \frac{\pi}{2} \), and a point \( y \in S^3 \) at distance \( \kappa \) from every point \( x \in X \). Prolong every spherical arc \( xy \) by a distance \( \sigma \leq \frac{\pi}{2} - \kappa \) up to points \( a \) and \( b \) so that \( a, y, x, b \) are on one great circle in this order. All these points \( a \) form a circle \( A \), and all points \( b \) form a circle \( B \). On the sphere on \( S^3 \) of radius \( \sigma \) whose center is \( y \) take the “smaller” part \( A^+ \) bounded by the circle \( A \). On the sphere on \( S^3 \) of radius \( \kappa + \sigma \) with center \( y \) take the “smaller” part \( B^+ \) bounded by \( B \). For every \( x \in X \) denote by \( x' \) the point on \( X \) such that \( |xx'| = \kappa \). Prolong every \( xx' \) up to points \( d, d' \) so that \( d, x, x', d' \) are in this order and \( |dx| = \sigma = |xx'| \). For every \( x \) provide the shorter piece \( C_x \) of the circle with center \( x \) and radius \( \sigma \) connecting \( b \) and \( d \) determined by \( x \) and also the shorter piece \( D_x \) of the circle with center \( x \) of radius \( \kappa + \sigma \) connecting \( a \) and \( d' \) determined by \( x \). Denote by \( W \) the convex hull of the union of \( A^+, B^+ \) and all the pieces \( C_x \) and \( D_x \). It is a body of constant width \( \kappa + 2\sigma \) (hint: for every hemisphere \( H \) supporting \( W \) and every \( H^* \) the centers of \( H/H^* \) and \( H^*/H \) belong to \( \text{bd}(W) \) and the arc connecting them passes through one of our points \( x \), or through the point \( y \)).

**Theorem 1.** At every boundary point \( p \) of a body \( W \subset S^d \) of constant width \( w > \pi/2 \) we can inscribe a unique ball \( B_{w-\pi/2}(p') \) touching \( W \) from inside at \( p \). What is more, \( p' \) belongs to the arc connecting \( p \) with the center of the unique hemisphere supporting \( W \) at \( p \), and \( |pp'| = w - \frac{\pi}{2} \).
Proof. Observe that if a ball touches $W$ at $p$ from inside, then there exists a unique hemisphere supporting $W$ at $p$ such that our ball touches this hemisphere at $p$. So for any $\rho \in (0, \frac{\pi}{2})$ there is at most one ball of radius $\rho$ touching $W$ from inside at $p$. Our aim is to show that we can always find one.

In the first part of the proof consider the case when $p$ is an extreme point of $W$. By Theorem 4 of [6] there is a lune $L = K \cap M$ of thickness $w$ containing $W$ such that $p$ is the center of $K/M$. Denote by $m$ the center of $M$ and by $k$ the center of $K$. Clearly, $m \in pk$ and $|pm| = w - \frac{\pi}{2}$. Since width$_M(W) = w$, by the third part of Theorem 1 of [6] the ball $B_{w-\pi/2}(m)$ touches $W$ from inside. Moreover, it touches $W$ from inside at the center of $M^*$. Since $K$ is one of these hemispheres $M^*$, our ball touches $W$ at $p$.

In the second part consider the case when $p$ is not an extreme point of $W$. From Lemma 5 we see that $p$ belongs to the convex hull of a finite set $E$ of extreme points of $W$. We do not lose the generality assuming that $E$ is a minimum set of extreme points of $W$ with this property. Hence $p$ belongs to the relative interior of conv($E$).

Take a hemisphere $K$ supporting $W$ at $p$ and denote by $o$ the center of $K$. Since $p$ belongs to the relative interior of conv($E$), by Lemma 2 we obtain conv($E$) $\subset$ bd($K$). Moreover, conv($E$) is a subset of the boundary of $W$.

We intend to show that for every $x \in$ conv($E$) the inclusion

$$B_{w-\frac{\pi}{2}}(x') \subset W$$

holds true, where $x'$ denotes the point on $ox$ at distance $w - \frac{\pi}{2}$ from $x$.

By Lemma 4 for $w = \mu$, if (3) holds true for $x_1, x_2 \in$ conv($E$), then (3) is also true for every point of the arc $x_1x_2$. Applying this lemma a finite number of times and considering the first part of this proof, we conclude that (3) is true for every point of conv($E$), so in particular for $p$. Clearly, the ball $B_{w-\frac{\pi}{2}}(p')$ supports $W$ at $p$ from inside.

Both parts of the proof confirm the statement of our theorem. \qed

By Lemma 6 we obtain the following proposition generalizing Proposition 4.2 from [8] for arbitrary dimension $d$. We omit an analogous proof.

**Proposition 1.** Every spherical body of constant width larger than $\frac{\pi}{2}$ (and more generally, every reduced body of thickness larger than $\frac{\pi}{2}$) of $S^d$ is smooth.

From Corollary 2 we obtain the following corollary which implies the two other ones.

**Corollary 3.** If diam($W$) $> \frac{\pi}{2}$ for a body $W \subset S^d$ of constant width $w$, then $w > \frac{\pi}{2}$.

**Corollary 4.** For every body $W \subset S^d$ of constant width $w \leq \frac{\pi}{2}$ we have diam($W$) $\leq \frac{\pi}{2}$.
Corollary 5. Let $p$ be a point of a body $W \subset S^d$ of constant width at most $\frac{\pi}{2}$. Then $W \subset H(p)$.

The following theorem generalizes Theorem 5.2 of [8] proved there for $d = 2$ only.

**Theorem 2.** Every spherical convex body of constant width smaller than $\frac{\pi}{2}$ on $S^d$ is strictly convex.

**Proof.** Take a body $W$ of constant width $w < \frac{\pi}{2}$ and assume it is not strictly convex. Then there is a supporting hemisphere $K$ of $W$ that supports $W$ at more than one point. By Claim 2 of [6] the centers $a$ of $K/K^*$ and $b$ of $K^*/K$ belong to $\text{bd}(W)$. Since $K$ supports $W$ at more than one point, $K/K^*$ contains also a boundary point $x \neq a$ of $W$. By the first statement of Lemma 3 of [6] we have $|xb| > |ab| = w$. Hence $\text{diam}(W) > w$.

By Corollary 4 we have $\text{diam}(W) \leq \frac{\pi}{2}$. By Theorem 3 of [6] we see that $w = \text{diam}(W)$. This contradicts the inequality $\text{diam}(W) > w$ from the preceding paragraph. The contradiction means that our assumption that $W$ is not strictly convex must be false. Consequently, $W$ is strictly convex. \[\square\]

On p. 566 of [6] the question is put if for every reduced spherical body $R \subset S^d$ and for every $p \in \text{bd}(R)$ there exists a lune $L \supset R$ fulfilling $\Delta(L) = \Delta(R)$ with $p$ as the center of one of the two $(d-1)$-dimensional hemispheres bounding this lune. The following theorem gives a positive answer in the case of spherical bodies of constant width. It is a generalization of the version for $S^2$ given as Theorem 5.3 in [8]. The idea of the proof of our theorem below for $S^d$ substantially differs from the one given for $S^2$.

**Theorem 3.** For every body $W \subset S^d$ of constant width $w$ and every $p \in \text{bd}(W)$ there exists a lune $L \supset W$ fulfilling $\Delta(L) = w$ with $p$ as the center of one of the two $(d-1)$-dimensional hemispheres bounding this lune.

**Proof.** Part I for $w < \frac{\pi}{2}$.

By Theorem 2 the body $W$ is strictly convex, which means that all its boundary points are extreme. Thus the statement follows from Theorem 4 of [6].

Part II for $w = \frac{\pi}{2}$.

If $p$ is an extreme point of $W$ we again apply Theorem 4 of [6].

Consider the case when $p$ is not an extreme point. Take a hemisphere $G$ supporting $W$ at $p$. Applying Corollary 5 we see that $W \subset H(p)$. Clearly, the lune $H(p) \cap G$ contains $W$. The point $p$ is at distance $\frac{\pi}{2}$ from every corner of this lune and also from every point of the opposite $(d-1)$-dimensional hemisphere bounding the lune. Hence this is a lune that we are looking for.

Part III, for $w > \frac{\pi}{2}$.

By Lemma 5 the point $p$ belongs to the convex hull $\text{conv}(E)$ of a finite set $E$ of extreme points of $W$. We do not lose the generality by assuming that $E$ is a
minimum set of extreme points of $W$ with this property. Hence $p$ belongs to the relative interior of $\text{conv}(E)$. By Proposition 1 we know that there is a unique hemisphere $K$ supporting $W$ at $p$. Since $p$ belongs to the relative interior of $\text{conv}(E)$, by Lemma 2 we have $\text{conv}(E) \subset \text{bd}(K)$. Moreover, $\text{conv}(E)$ is a subset of the boundary of $W$.

By Theorem 4 of [6] for every $e \in E$ there exists a hemisphere $K_e^*$ (it plays the part of $K^*$ in Theorem 1 of [6]) supporting $W$ such that the lune $K \cap K_e^*$ is of thickness $\Delta(W)$ with $e$ as the center of $K/K_e^*$. By Proposition 1, for every $e$ the hemisphere $K_e^*$ is unique. For every $e \in E$ denote by $t_e$ the center of $K_e^*/K$ and by $k_e$ the boundary point of $K$ such that $t_e \in ok_e$, where $o$ is the center of $K$. So $e, k_e$ are antipodes. Denote the set of all these points $k_e$ by $Q$.

Clearly, the ball $B = B_{\Delta(W)-\frac{\pi}{2}}(o)$ (as in Part III of Theorem 1 in [6]) touches $W$ from inside at every point $t_e$. Moreover, from the proof of Theorem 1 of [6] and from the earlier established fact that every $e \in E$ is the center of $K/K_e^*$ and every $t_e$ is the center of $K_e^*/K$ we obtain that $o$ belongs to all the arcs of the form $et_e$.

Put $U = \text{conv}(Q \cup \{o\})$. Denote by $U_B$ the intersection of $U$ with the boundary of $B$, and by $U_W$ the intersection of $U$ with the boundary of $W$. Having this construction in mind we see the following one-to-one correspondence between some pairs of points in $U_B$ and $U_W$. Namely, between the pairs of points of $U_B$ and $U_W$ such that each pair is on the arc connecting $o$ with a point of $\text{conv}(Q)$.

Now, we will show that $U_W = U_B$. Assume the opposite. By the preceding paragraph, our opposite assumption means that there is a point $x$ which belongs to $U_W$ but not to $U_B$. Hence $|xo| > \Delta(W) - \frac{\pi}{2}$. Moreover, there is a boundary point $y$ of the $(d-1)$-dimensional great sphere bounding $K$ such that $o \in xy$ and a point $y' \in oy$ at distance $\Delta(W) - \frac{\pi}{2}$ from $y$.

We have $|xy'| = |xo| + |oy| - |yy'| > (\Delta(W) - \frac{\pi}{2}) + \frac{\pi}{2} - (\Delta(W) - \frac{\pi}{2}) = \frac{\pi}{2}$.

By Lemma 5 the point $x$ belongs the convex hull of a finite set of extreme points of $W$. Assume for a while that all these extreme points are at distance at most $\frac{\pi}{2}$ from $y'$. Therefore all of them are contained in $H(y')$. Thus their convex hull is contained in $H(y')$ and so $x \in H(y')$. This contradicts the fact established in the preceding paragraph that $|xy'| > \frac{\pi}{2}$. The contradiction shows that at least one of these extreme points is at distance larger than $\frac{\pi}{2}$ from $y'$. Take such a point $z$ for which $|zy'| > \frac{\pi}{2}$.

Since $z$ is an extreme point of $W$, by Theorem 4 of [6] it is the center of one of the $(d-1)$-dimensional hemispheres bounding a lune $L$ of thickness $\Delta(W)$ which contains $W$. Hence by the third part of Lemma 3 of [6] every point of $L$ different from the center of the other $(d-1)$-dimensional hemisphere bounding $L$ is at distance smaller than $\Delta(W)$ from $z$. Taking into account that the distance of these centers is $\Delta(W)$ we see that the distance of every point of $L$, and in particular of $W$, from $z$ is at most $\Delta(W)$.

By Theorem 1 the ball $B_{\Delta(W)-\frac{\pi}{2}}(y')$ touches $W$ from inside at $y$.
For the boundary point $v$ of this ball such that $y' \in zv$ we have $|zv| = |zy'| + |y'v| > \frac{\pi}{2} + (\Delta(W) - \frac{\pi}{2}) = \Delta(W)$, which by $v \in W$ contradicts the result of the preceding paragraph. Consequently, $U_W = U_B$.

Since $U_W = U_B$, the ball $B$ touches $W$ from inside at every point of $U_B$, in particular at the point $t_p$ such that $o \in pt_p$. Therefore by Part III of Theorem 1 in [6] there exists a hemisphere $K_p^*$ supporting $W$ at $t_p$, such that $t_p$ is the center of $K_p^*/K$, $p$ is the center of $K/K_p^*$ and the lune $L = K \cap K_p^*$ is of thickness $\Delta(W)$. Consequently, $L$ is a lune announced in our theorem. □

If the body $W$ from Theorem 3 is of constant width greater than $\frac{\pi}{2}$, then by Proposition 1 it is smooth. Thus at every $p \in \text{bd}(W)$ there is a unique supporting hemisphere of $W$, and so the lune $L$ from the formulation of this theorem is unique. If the constant width of $W$ is at most $\frac{\pi}{2}$, there are non-smooth bodies of constant width (e.g., a Reuleaux triangle on $S^2$) and then for non-smooth $p \in \text{bd}(W)$ we may have more such lunes.

Our Theorem 3 and Claim 2 in [6] imply the first sentence of the following corollary. The second sentence follows from Proposition 1 and the last part of Lemma 3 in [6].

**Corollary 6.** For every convex body $W \subset S^d$ of constant width $w$ and for every $p \in \text{bd}(W)$ there exists $q \in \text{bd}(W)$ such that $|pq| = w$. If $w > \frac{\pi}{2}$, this $q$ is unique.

**Theorem 4.** If $W \subset S^d$ is a body of constant width $w$, then $\text{diam}(W) = w$.

**Proof.** If $\text{diam}(W) \leq \frac{\pi}{2}$, then the statement is an immediate consequence of Theorem 3 in [6].

Assume that $\text{diam}(W) > \frac{\pi}{2}$. Take an arc $pq$ in $W$ such that $|pq| = \text{diam}(W)$. By Corollary 1 this hemisphere $K$ orthogonal to $pq$ at $p$ which contains $q$, contains also $W$. The center of $K$ lies strictly inside $pq$ and thus by Part III of Theorem 1 in [6] we have $w > \frac{\pi}{2}$.

Having Theorem 3 in mind, consider a lune $L \supset W$ with $\Delta(L) = \Delta(W)$ such that $p$ is the center of a $(d-1)$-dimensional hemisphere bounding $L$. Clearly, $q \in W \subset L$. Since $W$ is of constant width $w > \frac{\pi}{2}$, we have $\Delta(L) > \frac{\pi}{2}$.

Thus from the last part of Lemma 3 of [6] it easily follows that the center of the other $(d-1)$-dimensional hemisphere bounding $L$ is a farthest point of $L$ from $p$. Since it is at distance $w$ from $p$, we obtain $w \geq |pq| = \text{diam}(W)$.

On the other hand, by Proposition 1 of [6] we have $w \leq \text{diam}(W)$.

As a consequence, $\text{diam}(W) = w$.

4. Constant width and constant diameter

We say that a convex body $W \subset S^d$ is of constant diameter $w$ if the following two conditions hold true
(i) diam\((W) = w,\)

(ii) for every boundary point \(p\) of \(W\) there exists a boundary point \(p'\) of \(W\) with \(|pp'| = w|.

This definition is analogous to the Euclidean notion (compare the beginning of Part 7.6 of [11] for the Euclidean plane, and the bottom of p. 53 of [1] also for higher dimensions). Here is a theorem similar to the planar Euclidean version from [11] (see the beginning of Part 7.6).

**Theorem 5.** Every spherical convex body \(W \subset S^d\) of constant width \(w\) is of constant diameter \(w\). Every spherical convex body \(W \subset S^d\) of constant diameter \(w \geq \frac{\pi}{2}\) is of constant width \(w\).

**Proof.** For the proof of the first statement of our theorem assume that \(W\) is of constant width \(w\). Theorem 4 implies (i) and Corollary 6 implies (ii), which means that \(W\) is of constant diameter \(w\).

Let us prove the second statement. Let \(W \subset S^d\) be a spherical body of constant diameter \(w \geq \frac{\pi}{2}\). We have to show that \(W\) is a body of constant width \(w\).

Consider an arbitrary hemisphere \(K\) supporting \(W\). As an immediate consequence of two facts from [6], namely Theorem 3 and Proposition 1, we obtain that

\[
\text{width}_K(W) \leq w. \tag{4}
\]

Let us show that \(\text{width}_K(W) = w\).

Make the opposite assumption (that is \(\text{width}_K(W) \neq w\)) in order to provide an indirect proof of this equality. By (4) this opposite assumption implies that \(\text{width}_K(W) < w\).

Consider two cases.

At first consider the case when \(w > \pi/2\).

Put \(w' = \text{width}_K(W)\). There exists a hemisphere \(M\) supporting \(W\) such that \(\Delta(K \cap M) = w'\). Denote the center of \(K/M\) by \(a\) and the center of \(M/K\) by \(b\). From Corollary 2 of [6] we see that \(b \in \text{bd}(W)\).

We have \(\frac{\pi}{2} < w'\) since the opposite means \(w' \leq \frac{\pi}{2}\) and then every point of the lune \(K \cap M\) is at distance at most \(\frac{\pi}{2}\) from the center \(b\) of \(M/K\) (for \(w' = \frac{\pi}{2}\) this is clear by \(K \cap M \subset H(b)\), and consequently this is also true if \(w' < \frac{\pi}{2}\)). Since \(b\) is a boundary point of our body \(W\) of constant diameter \(w > \pi/2\), we get a contradiction to (ii).

Since \(b\) is a boundary point of the body \(W\) of constant diameter, by the assumption (ii) there exists \(b' \in \text{bd}(W)\) such that \(|bb'| = w|\). By the definition of the thickness of a lune, we have \(|ab| = w|\). Observe that the last part of Lemma 3 of [6] implies that \(|uc_{H/G}| \leq |c_{G/H}c_{H/G}|\) for every point \(u\) of the lune \(H \cap G\). This observation applies to our lune \(K \cap M\) since \(\Delta(K \cap M) > \frac{\pi}{2}\) (i.e., \(w' > \frac{\pi}{2}\)). Hence we obtain \(|b'b| \leq |ab|\), which by the first two sentences of
this paragraph gives $w \leq w'$. This contradicts the inequality $w' < w$ resulting from our opposite assumption that $\text{width}_K(W) \neq w$.

Consequently, $\text{width}_K(W) = w$.

Now consider the case when $w = \frac{\pi}{2}$.

From $\text{width}_K(W) < w$ (resulting from our opposite assumption) we obtain $\text{width}_K(W) < \pi/2$. Thus $\Delta(K \cap K^*) < \frac{\pi}{2}$. Denote by $b$ the center of $K^*/K$. From Corollary 2 of [6] we see that $b \in \text{bd}(W)$.

The set $D = (K/K^*) \cap (K^*/K)$ of corner points of $K \cap K^*$ is isomorphic to $S^{d-2}$. Moreover, $S^k$ contains at most $k + 1$ points pairwise distant by $\frac{\pi}{2}$, which follows from the fact (which is easy to show) that every set of at least $k + 2$ points pairwise equidistant on $S^k$ must be the set of vertices of a regular simplex inscribed in $S^k$ (still the distances of these vertices are not $\frac{\pi}{2}$). Putting $k = d - 2$, we see that $D$ contains at most $d - 1$ points pairwise distant by $\frac{\pi}{2}$. Therefore there exists a set $P_{\text{max}}$ of the maximum number (being at most $d - 1$) of points of $W \cap D$ pairwise distant by $\frac{\pi}{2}$.

Put $T = \text{conv}(P_{\text{max}} \cup \{b\})$. Clearly, $T \subset W$, and since moreover $T \subset \text{bd}(K^*)$ and $W \subset K^*$, we obtain $T \subset \text{bd}(W)$. Take a point $x$ from the relative interior of $T$. The inclusion $T \subset \text{bd}(W)$ implies that $x \in \text{bd}(W)$. Hence by (ii) there exists $y \in \text{bd}(W)$ such that $|xy| = \frac{\pi}{2}$. By Lemma 2 we have $T \subset \text{bd}(H(y))$. By this inclusion and $b \in T$ we obtain $|by| = \frac{\pi}{2}$. Thus by Lemma 3 we have $y \in D$. As a consequence, the set $P_{\text{max}} \cup \{y\}$ is a set of points of $W \cap D$ pairwise distant by $\frac{\pi}{2}$. This set has a greater number of points than the set $P_{\text{max}}$. This contradiction shows that our assumption $\text{width}_K(W) \neq w$ is wrong. So $\text{width}_K(W) = w$.

In both cases, from the arbitrariness of the hemisphere $K$ supporting our convex body $W$ we get that $W$ is a body of constant width $w$. □

**Problem.** Is every spherical body of constant diameter $w < \frac{\pi}{2}$ a body of constant width $w$?

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