A Note on Self-Dual Yang-Mills Theory

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Abstract
We translate the classical Atiyah-Ward correspondence into the $L_\infty$ language. We extend the correspondence to a quasi-isomorphism between the algebra of the self-dual four-manifold $X$ and the algebra of the holomorphic $BF$-theory in the twistor space $T(X)$.

1 Introduction

The problem of reformulation of general relativity and Yang-Mills theory became an important issue after the initial successes of twistor theory [13], [17], [2].

The self-dual Yang-Mills theory is a degenerate version of pure Yang-Mills theory. In this note we prove that a self-dual Yang-Mills theory is perturbatively equivalent to a certain BF theory on the twistor space. We hope that a similar technique could be applied to pure Yang-Mills theory, that potentially could lead to its interesting reformulation.

Let $X$ be a four-dimensional oriented Riemannian manifold. Eigen-decomposition of the Hodge $*$ operator defines the splitting of the bundle of two-forms $\Omega^2$ into the sum of self-dual and anti self-dual sub bundles $\Omega^2_+ + \Omega^2_-$. Fix a vector bundle $E$ over $X$ with a unitary connection $\nabla$. Let $G \in \Omega^2_- u$ be an ant-self-dual two-form with values in the adjoint bundle $u = u(E)$. The fields in the theory are the gauge equivalence classes of pairs $(\nabla, G)$. Define the Chalmers-Siegel Lagrangian density [5], [16] by the formula

$$\mathcal{L}_X(\nabla, G) = \text{tr}(F_\perp G)\text{dvol}$$

where $F_\perp \in \Omega^2_- u$ is the anti self-dual part of the curvature. We will refer to the theory, defined by the Lagrangian (1) as to self-dual Yang-Mills theory.

The Lagrangian (1) admits a perturbation mentioned in [18] and extensively studied in [10] and [7]

$$\text{tr}(GF_\perp) + c\text{tr}G^2$$

This Lagrangian is equivalent to the Lagrangian of pure Yang-Mills theory

$$\frac{1}{c} (\text{tr}F \wedge *F - F \wedge F)$$
This explains our interest in (1).

A holomorphic BF theory is defined on a $2k+1$ dimensional complex manifold $M$. The manifold is equipped with a holomorphic vector bundle $\mathcal{E}$. Let $\text{End}(\mathcal{E})$ be the vector bundle of local endomorphisms of $\mathcal{E}$, $\mathcal{O} = \mathcal{O}_M$ be the structure sheaf and $\omega = \omega_M$ be the canonical line bundle. The space of fields in this theory is a space of pairs. The first field is a differential operator of the first order $D : \mathcal{E} \rightarrow \Omega^1 \mathcal{E}$, that satisfies Leibniz rule $D(fe) = \bar{\partial}fe + fDe$, where $f$ is a function and $e$ is a section. The second field $H$ is a section of $\Omega^{0,n-2} \text{End}(\mathcal{E}) \otimes \omega$. Following the standard practice, $D$ extends to an operator $D : \Omega^i \mathcal{E} \rightarrow \Omega^{i+1} \mathcal{E}$. The operator $D^2$ is the operator of multiplication Newlander-Nirenberg tensor $F$, a section of $\Omega^{0,2} \text{End}(\mathcal{E})$. We define the Lagrangian density by the formula

$$L_M(D,H) = \text{tr}(FH)$$

We prove perturbative equivalence of theories (1) and (4) when $X$ is a self-dual Riemannian manifold and $M$ is the corresponding twistor space $T(X)$. We use the mathematical language of $L_\infty$ algebras, that is well adapted for this purpose.

The note is organised as follows. In Section 2 we fix our notations and remind the reader the basic definitions of four-dimensional differential geometry. In Section 3 we formulate the ingredients of our homological approach to twistors. As $L_\infty$ equivalence could be difficult to understand for non-experts we spend Section 4 on explanation of its geometric meaning. In Section 5 we introduce BV-actions of BF and self-dual theories. We prove the main result in Section 6. We moved the proofs of some technical statements that are used in the treatment of the main theorem to the Appendix. In particular Appendix A contains the explicit form of the kernel of the homotopy $\bar{\partial}^* \frac{1}{\Delta}$. 

3
2 Differential-Geometric Background

2.1 General Facts About Self-Dual Four-Manifolds

Let $X$ be a Riemannian four-dimensional oriented spin manifold. Let $T = T_X$ (resp. $\Omega = \Omega_X$) be its tangent (resp. cotangent) bundle. The vector bundle $\Omega = \Omega^1$ is a part of the De Rham complex $(\bigoplus_{i=0}^{4} \Omega^i, d)$. Let $P = P_X$ be the principal Spin(4) bundle. It is the two-sheeted cover of the bundle of orthonormal frames in $T$. The metric $g_{ij}$ defines Levi-Civita connection $\nabla$ in $T$.

**Remark 1** If we are given a representation $L$ of Spin(4) we automatically get a Spin(4) connections in the vector bundle $L_X$ associated with $P_X$.

An important Spin(4) (actually SO(4)) representation is in the exterior algebra $\Lambda = \bigoplus_{i=0}^{4} \Lambda^i$ of the euclidean, oriented four-dimensional vector space. The components $\Lambda^i$ have the euclidean structure. The inner product defines the Hodge $*$-operator $*: \Lambda^i \to \Lambda^{4-i}$ by the formula:

$$a \wedge \ast b = (a, b)\text{vol}.$$ 

As usual the canonical volume element $\text{vol}$ has the unit length and is compatible with the orientation. The operator satisfies $\ast^2 = \text{id}$. The linear space $\Lambda^2$ splits into the direct sum

$$\Lambda^2 = \Lambda^2_+ + \Lambda^2_-,$$

where $\Lambda^2_\pm$ are the $\pm 1$ eigen-spaces of $\ast$. We call them self-dual and anti-self-dual subspaces. The finite-dimensional algebra

$$\mathcal{A} = \mathbb{R} + \Lambda^1 + \Lambda^2_-$$  \hspace{1cm} \hspace{1cm} (5)

is another example of Spin(4) representation. It appeared for the first time in [6].

If we associate $\Lambda$ with $P_X$ we reproduce the vector bundle of differential forms $\Omega$. The same construction applied to $\Lambda^2_\pm$ leads to the bundles of self an anti self-dual forms $\Omega^2_\pm$. If yet we repeat it for $\mathcal{A}$ we get the bundle

$$\mathcal{A}_X = \Omega^0 + \Omega^1 + \Omega^2_-.$$  \hspace{1cm} \hspace{1cm} (6)
Following [3] the curvature of the metric defines self-adjoint transformation

\[ R : \Omega^2 \rightarrow \Omega^2 \]

given by

\[ R(e_i \wedge e_j) = \frac{1}{2} \sum R_{i,j,k,l} e_k \wedge e_l, \]

where \( < e_i > \) is a local orthonormal basis of 1-forms. The block matrix decomposition

\[ R = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \]  

relative to decomposition \( \Omega^2 = \Omega^2_+ + \Omega^2_- \) enables us to define the following components: traceless Ricci tensor \( B \) and the components of the Weyl tensor \( W = W_+ + W_- \) with \( W_+ = A - \frac{1}{3} \text{tr} A, W_- = C - \frac{1}{3} \text{tr} C. \)

A connection in a vector bundle \( E \) over \( X \) is defined by the operator of covariant differentiation \( \nabla : \Omega^0(E) \rightarrow \Omega^1(E). \) The covariant differentiation extends to the operator \( D : \Omega^i(E) \rightarrow \Omega^{i+1}(E) \) by the formula \( D(\nu \otimes s) = d(\nu) \otimes s + (-1)^i \omega \otimes \nabla s, \nu \in \Omega^i. \) The operator of multiplication on the curvature \( F = F_\nabla \) is defined as the operator \( D^2 \in \Omega^2(\text{End}(E)). \)

On the four-manifold \( X \) connection in the vector bundle \( E \) is said to be self-dual if its curvature \( F \) is in \( \Omega^2_+(\text{End}(E)). \) This definition has an obvious extension to principal bundles with the structure group \( G. \) Let \( g \) be the Lie algebra of \( G. \) A connection is self-dual if the curvature \( F \) belongs to self-dual two-forms \( \Omega^2_+(g) \) with values in the adjoint bundle.

By definition the elliptic complex of vector bundles

\[ A_X(E) \text{ is equal to } \Omega^0(E) \rightarrow \Omega^1(E) \rightarrow \Omega^2_+(E). \]  

(8)

The differential \( d_E \) is induced from \( D. \) The condition

\[ d_E^2 = 0 \]

is equivalent to self-duality of the connection. If \( E \) is the adjoint \( g_X \) of some principle \( G \)-bundle, then the complex of sections of \( A_X(g) \) is a differential graded
Lie algebra. The complex $A_X(g)$ is responsible for infinitesimal self-dual deformations of $\nabla$ (see [3]). Suppose $E$ is associated with a principal $G$-bundle. Then $A_X(E)$ is a bundle of $A_X(g)$-modules.

There is the companion elliptic complex of $A_X(g)$-modules

$$A_X^*(E) = \Omega^2_-(E) \to \Omega^3(E) \to \Omega^4(E).$$

The differential is also induced by $D$.

### 2.2 Twistor Spaces

In this section we collected some definitions and facts related to twistor spaces. They were borrowed from [3].

The Spin(4) group has two complex spinor representations $W_-, W_+$ of positive and negative chiralities. These are two-dimensional spaces. They are equipped with Spin(4)-invariant complex-linear symplectic forms and positive definite Hermitian forms.

We obtain spinor bundles $S_-$ and $S_+$ by the association procedure from $W_-, W_+$ and $P_X$. The bundles automatically inherit the covariantly constant bilinear forms. The total spin bundle $S_- + S_+$ is a module over the complexified Clifford algebra bundle $\text{Cl}$ of $\Omega^1$. If we ignore the algebra structure the bundle $\text{Cl}$ is isomorphic to $\Omega^1_{\text{C}} = \bigoplus_{i=0}^{4} \Omega^i_{\text{C}}$. The complexified one-forms act on spinors in the following way

$$\Omega^1_{\text{C}} \cong \text{Hom}(S_-, S_+) \cong \text{Hom}(S_+, S_-).$$

The subbundle $\Omega^2_{\text{C}}$ consists of traceless endomorphisms of $S_-$ and the real bundle $\Omega^2_{\text{C}}$ the traceless skew-hermitian endomorphisms of $S_-$. The similar statements hold for $S_+$.

The twistor space $T = T(X)$ is the projectivization $\mathbb{P}^1(S_-)$ of the spinor bundle $S_-$. It is a real six-dimensional manifold $\mathbb{P}^1$-fibered over $X$

$$p : T \to X. \quad (9)$$
The bundle $T$ has an interpretation of a bundle of the complex structures in the tangent bundle $T_X$ compatible with the metric.

The space $T$ has an almost complex structure $\mathfrak{B}$. Using Riemannian connection we can split the tangent bundle $\mathbb{P}^1(S_-)$ into vertical and horizontal parts. On the vertical part we have the complex structure of the fibers. On the horizontal part at a point $\phi \in \mathbb{P}^1(S_-)_x$ over $x \in X$ we put the complex structure on $\Omega^1_x$ as follows. Multiplication on $\alpha \in \Omega^1_x$ defines the real isomorphism

$$a : \alpha \rightarrow \alpha \phi$$

between $\Omega^1_x$ and a complex linear space $S_{+x}$. The fibers of the projection $\mathfrak{B}$ are holomorphic.

3 The Algebra $\mathcal{U}$

We have already seen, that algebras with Spin(4)-action (more precisely SO(4)-action), e.g., $\mathfrak{A}$ and $\mathcal{A}$, lead to interesting differential-geometric constructions. In this section we provide another example of an algebra that plays an important role in the analytic geometry of twistor spaces.

We need to refine the grading of $\mathcal{A}$ to a bi-grading. The individual summands of $\mathcal{A}$ (see $\mathfrak{A}$) have bi-grading $(0, 0), (3, 2), (6, 4)$. It is related to the standard differential-geometric grading by the formula

$$\mathcal{A}(j) = \bigoplus_{j=k-l} \mathcal{A}_{k,l}.$$  \hspace{1cm} (11)

Let $L^{**} = \bigoplus_{ij} L^{i,j}$ be a bi-graded linear space (or a sheaf). Sometimes it will be convenient to drop one of the indices. To distinguish the gradings we will be using the bold italics for the second index. Also

$$(L[s](t))^{i,j} \overset{\text{def}}{=} L^{i+s,j+t}.$$

The algebra $\mathcal{A}$ is a part of a larger graded commutative algebra $\mathcal{U}$.

**Definition 1** The algebra $\mathcal{U}$ is an extension of $\mathcal{A}$ by an ideal with a trivial multiplication $\mathcal{A}^*[-11][-8]$. We will often use the reduced grading $\mathfrak{U}$ in $\mathcal{U}$.
Let \( U_{\text{red}} \subset \mathcal{U} \) be the direct sum
\[
A(0) + (A^*[−11]|−8))(0) = \mathbb{R} + \Lambda^*_2.
\]

3.1 The Algebra-Geometric Interpretation of \( \mathcal{U} \)

Let \( W_+(1) = W_+ P(1) \) be the two-dimensional vector bundle over \( P^1 = P^1(W_-) \).

It is the twist of the trivial two-dimensional bundle with the fiber \( W_+ \) by the ample generator of the Picard group \( \mathcal{O}(1) = \mathcal{O}_P(1) \).

As usual \( \mathcal{O} = \mathcal{O}_{P^1} \) stands for the structure sheaf and \( \mathcal{O}(i) \) for the tensor powers of \( \mathcal{O}(1) \).

We shall be interested in the exterior algebra \( \Lambda(W_+(1)) \). This sheaf and a quasi-isomorphic complex of vector bundles \( D^\bullet \), which we shall construct shortly, are the key ingredients in our homological version of Atiyah-Ward construction.

**Lemma 2** Trivial vector bundles over \( P^1 \) with fibers \( W_+ \otimes W_- \) and \( \text{Sym}^2 W_- \) fit into short exact sequences
\[
\begin{align*}
0 \to (W_+(−1))^{-1} &\xmapsto{\iota_1} (W_+ \otimes W_-)^0 \xmapsto{\iota_2} (W_+(1))^1 \to 0 = \mathcal{C}^{*1} \\
0 \to (W_−(−1))^{-1} &\xmapsto{\iota_3} (\text{Sym}^2 W_-)^0 \xmapsto{\iota_4} (\mathcal{O}(2))^1 \to 0 = \mathcal{C}^{*2}
\end{align*}
\]
(12)

The superscript stands for the vector bundle’s index in the complex.

**Proof.** Immediately follows from the short exact sequence
\[
\mathcal{O}(−1) \to W_- \to \mathcal{O}(1).
\]

The truncated complexes \( D^{*1} = \tau^{≤0} \mathcal{C}^{*1}, D^{*2} = \tau^{≤0} \mathcal{C}^{*2} \) together with \( D^{*0} = \mathcal{O} \) form a graded algebra
\[
D = \mathcal{O} + D^{*1} + D^{*2}
\]
with \( \mathcal{O} \)-linear differential \( d_\cdot \).

Besides the obvious multiplication on \( \bigoplus_{i=0}^2 C^{0,i} \), the only nontrivial multiplication in \( D \) is
\[
\mathcal{C}^{-11} \otimes C^{0,1} = W_+(−1) \otimes W_+ \otimes W_- \to \Lambda^2 W_+ \otimes W_−(−1) \cong W_−(−1) = \mathcal{C}^{-12}.
\]
Let
\[ \omega \] denote the line bundle that is isomorphic to \( \mathcal{O}(-4) \). \( \tag{13} \)

To simplify notations we set
\[ \mathcal{G} = \Lambda(W_+(1)) \otimes (\mathcal{O} + \omega). \]
\( \tag{14} \)
\[ \mathcal{G}_{\text{ext}} = D \otimes (\mathcal{O} + \omega). \]
\( \tag{15} \)

The complexified bundle of differential forms on a holomorphic manifold \( M \) has a \((p,q)\) decomposition
\[ \Omega^i_C \cong \bigoplus_{i=p+q} \Omega^{p,q}. \]
We set \( \Omega^{0 \bullet} \) to be equal to \( \bigoplus_q \Omega^{0,q} \). If \( \mathcal{E} \) is a holomorphic vector bundle on \( M \) then \( \Omega^{0 \bullet} \mathcal{E} \) is equipped with the action of the \( \bar{\partial} \) operator.

In our applications we will be interested in the \( \bar{\partial} \) operator in algebras \( \mathcal{B}^\bullet \) and \( \mathcal{B}_{\text{ext}}^\bullet \). These algebras are equal to the space of \( C^\infty \) sections
\[ \mathcal{B}^\bullet = \Gamma(\mathbb{P}^1, \Omega^{0 \bullet}_{\mathbb{P}^1} \mathcal{G}) \]
\( \tag{16} \)
and
\[ \mathcal{B}_{\text{ext}}^\bullet = \Gamma(\mathbb{P}^1, \Omega^{0 \bullet}_{\mathbb{P}^1} \mathcal{G}_{\text{ext}}). \]
\( \tag{17} \)

Both of them contains a differential ideal with zero multiplication
\[ \mathcal{I} = \Gamma(\mathbb{P}^1, \Omega^{0 \bullet}_{\mathbb{P}^1} \Lambda(W_+(1)) \otimes \omega), \]
\( \tag{18} \)
\[ \mathcal{I}_{\text{ext}} \overset{\text{def}}{=} \Gamma(\mathbb{P}^1, \Omega^{0 \bullet}_{\mathbb{P}^1} D \otimes \omega). \]
\( \tag{19} \)

The group
\[ \text{Spin}(4) \cong \text{SU}_+(2) \times \text{SU}_-(2) \]
is a subgroup of the group of symmetries of \( \Lambda(W_+(1))(\mathcal{O} + \omega) \). The group \( \text{SU}_+(2) \) acts linearly on \( W_+ \) in \( \Lambda(W_+(1))(\mathcal{O} + \omega) \), whereas \( \text{SU}_-(2) \) acts by automorphisms of \( \mathcal{O}(1) \) and \( \omega \). Similarly, we define the action of \( \text{Spin}(4) \) on \( \mathcal{G}, \mathcal{G}_{\text{ext}}, \mathcal{B} \) and \( \mathcal{B}_{\text{ext}} \).
The algebra $\mathcal{B}$ has the multiplicative grading, described in the following table.

| degree | component |
|--------|-----------|
| 0      | $\Omega^0$ $\oplus \Omega^0 \otimes \omega$ |
| 1      | $\Omega^0 W(1)$ $\oplus \Omega^0 W(1) \otimes \omega$ $\oplus \Omega^1 W(1)$ $\oplus \Omega^1 W(1) \otimes \omega$ |
| 2      | $\Omega^2 W(1)$ $\oplus \Omega^2 W(1) \otimes \omega$ $\oplus \Omega^2 W(1) \otimes \omega$ |
| 3      | $\Omega^3 W(1)$ $\oplus \Omega^3 W(1) \otimes \omega$ |

It is the classical fact of the representation theory that any complex irreducible finite-dimensional representation of Spin(4) is isomorphic to the one in the series

$$\text{Sym}^i(W_+) \otimes \text{Sym}^j(W_-), i, j \geq 0.$$ 

This explains the isomorphisms

\begin{align*}
\psi &: \Lambda^1 \to W_+ \otimes W_- , \\
\psi_{\pm} &: \Lambda^2 \to \text{Sym}^2 W_\pm.
\end{align*} 

The isomorphism

$$\mathcal{U}_\mathbb{C} \cong H^\bullet (\mathbb{P}^1, \Lambda(W(1))(\mathcal{O} + \omega)) = H^\bullet (\mathcal{B})$$

follows from the identifications \[(21)\], the skew-symmetric isomorphism $W_+^* \cong W_-$ and Serre’s computation \[(15)\] of the cohomology of the projective space:

\begin{align*}
H^0 (\mathbb{P}^1(W), \mathcal{O}(j)) &\cong \text{Sym}^j W^*, j \geq 0; \\
H^1 (\mathbb{P}^1(W), \mathcal{O}(-j-2)) &\cong \text{Sym}^j W \otimes \Lambda^2 (W^*), j \geq 0.
\end{align*} 

(23)

Using Remark \[(8)\] we define the infinite-dimensional bundle $\mathcal{B}_{\text{ext}-X}$. As $\mathcal{B}_{\text{ext}}$ is a free $\mathcal{A}$-module we can define the operator $d_{\text{naive}}$ in $\mathcal{B}_{\text{ext}-X}$. The formula \[(8)\], however, does not produce the differential that satisfies $d^2 = 0$ equation. To fix this we notice that $(d_{\text{naive}})^2$ is determined by the $C$-component of the curvature tensor \[(7)\]. It is a symmetric section of $\text{End}(\text{Sym}^2 W_-)_X$. Let $\mathfrak{sl}_2$ be the complexification of the Lie algebra of SU$_-(2)$. We identify $C$ with a section of $(\mathfrak{sl}_2 \otimes \text{Sym}^2 W_-)_X$. If the Weyl tensor vanishes, the section $C$ has
a simple description in terms of the scalar curvature $R_X$. Let $e_1, e_2, e_3$ be a basis of $\mathfrak{sl}_2$, orthonormal with respect to the Killing’s form. Let $\psi$ be an SO(4) (in fact SO(3)) equivariant isomorphism of $\mathfrak{sl}_2$ with $\text{Sym}^2 W_-$. We define an $\text{SO}(4)$-invariant element of $\mathfrak{sl} \otimes \text{Sym}^2 W_-$ as a covariantly constant section $\mathfrak{s}\mathfrak{c} = \sum_{i=1}^3 e_i \otimes \psi(e_i)$. An elementary identification shows that under the isomorphism $\text{End}(\text{Sym}^2 W_-)_X \cong (\mathfrak{sl} \otimes \text{Sym}^2 W_-)_X$ induced by the symmetric square of the skew-symmetric bilinear form on $W_-$ and after a suitable normalisation of $\psi$ the curvature of the self-dual manifold $X$ maps to $R \mathfrak{s}\mathfrak{c}$.

The element $\mathfrak{s}\mathfrak{c}$ defines a differential operator of the first order

$$\mathfrak{s}\mathfrak{c} : \mathcal{O}_{\mathbb{P}^1} \to \text{Sym}^2 W_-.$$ 

It is $\sum_{i=1}^3 \psi(e_i)L_{e_i}$, where $L_{e_i}$ is the Lie derivative, corresponding to the element $e_i$. The image of this operator belongs to the image of $\iota_2$ [12] because the twisted tangent bundle $T_{\mathbb{P}^1}(2) \cong \mathcal{O}(4)$ has no SU(2)-invariant sections. Let

$$d\mathfrak{s}\mathfrak{c} : \mathcal{O}_{\mathbb{P}^1} \to W_-(-1)$$

be the operator such that $\iota_2 \circ d\mathfrak{s}\mathfrak{c} = \mathfrak{s}\mathfrak{c}$. This operator can be extended naturally to $\mathcal{B}_{\text{ext}}$. We define the operator $d\mathfrak{s}\mathfrak{c} X$ in $\mathcal{B}_{\text{ext}, X}$ as a product of the covariantly constant extension of $d\mathfrak{s}\mathfrak{c}$ on the scalar curvature $R$. We also extend $\epsilon [12]$ to a covariantly constant homomorphism

$$\epsilon : \mathcal{B}_{\text{ext}, X} \to \mathcal{B}_{X}.$$ 

Finally we can claim that the bundle $\mathcal{B}_{\text{ext}, X}$ is equipped with two anti-commuting differentials

$$d_I = \overline{\partial}_{\mathbb{P}^1},$$

and

$$d_{II} = d_{\mathfrak{s}\mathfrak{c}}^{\text{naive}} + d_{\mathfrak{s}\mathfrak{c}} + d_I.$$ 

The operators $\overline{\partial}_{\mathbb{P}^1}$, $d_I$ are covariantly constant extensions of operators in $\mathcal{B}_{\text{ext}}$ that have the same name.

This follows from the evident equations:

$$\overline{\partial}_{\mathbb{P}^1}^2 = d_I^2 = d_{\mathfrak{s}\mathfrak{c}}^2 = 0,$$ 

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\begin{align*}
\{\bar{\partial}_{\mathcal{P}_1}, d^{\text{naive}}_{\mathcal{B}_{\text{ext}X}}\} &= \{\bar{\partial}_{\mathcal{P}_1}, d_{\text{ext}}\} = \{\bar{\partial}_{\mathcal{P}_1}, d_I\} = 0, \\
d^{\text{naive}2}_{\mathcal{B}_{\text{ext}X}} + \frac{1}{2}(d_{\text{ext}}, d_I) = 0.
\end{align*}

**Lemma 3** The kernel of the homomorphism $\epsilon : \mathcal{B}_{\text{ext}X} \to \mathcal{B}_X$ is closed with respect to the action of the differential $d_I$ and $d_{II}$.

**Proof.** Follows immediately from the definition. ■

As a corollary we deduce the following.

**Proposition 4** The sheaf $\mathcal{B}_X$ is equipped with two anti-commuting differential $d_I$ and $d_{II}$. The algebra of global sections $\Gamma(X, \mathcal{B}_X)$ has a structure of a bicomplex.

We set

\[ d_T \text{ to be equal to } d_I + d_{II}. \tag{26} \]

It satisfies

\[ d^2_T = 0 \tag{27} \]

As we shall see shortly, the integrability of the complex structure in $T(X)$ is equivalent to the equation (27).

### 3.2 Formalism of CR Structures

We need to remind the reader some basic elements of the theory of Cauchy-Riemann (CR) structures.

A manifold $Y$ has a CR structure if the tangent bundle $T_Y$ contains a complex subbundle $\mathcal{F} \subset T_Y$. The CR structure is integrable if the space of sections $\bar{\mathcal{F}} \subset \mathcal{F} + \bar{\mathcal{F}} = \mathcal{F}^\mathbb{C}$ is closed under the bracket. A local function $f$ is said to be $\mathcal{F}$ holomorphic if $\xi f = 0$ for any section $\xi$ of $\bar{\mathcal{F}}$.

A vector bundle $\mathcal{E}$ over $Y$ is CR holomorphic or $\mathcal{F}$ holomorphic if the gluing cocycle $c_{ij}$ is $\mathcal{F}$ holomorphic.

The Dolbeault complex has its analogue in the CR setting. The De Rham complex $\Omega_Y$ contains an ideal $J$, generated the subspace $J^1 \subset \Omega^1$ orthogonal to
\( \mathcal{F} \). It follows from integrability condition that \( J \) is a differential ideal. Let \( \bar{\partial}_\mathcal{F} \) be the differential in the quotient \( \Omega^{\mathcal{F}}_\mathcal{F} = \Omega/J \). If \( \mathcal{E} \) is an \( \mathcal{F} \)-holomorphic vector bundle one can define \( \Omega^{\mathcal{F}}_\mathcal{F}\mathcal{E} \) along the same lines.

The manifold \( T \) can be interpreted as a CR manifold. The distribution \( \mathcal{F} \) is formed by vertical holomorphic vector fields with respect to projection \( \text{(9)} \).

The projection \( \text{Tot}(S_-)\backslash\{0\} \rightarrow \mathbf{P}^1(S_-) = T \) defines the principle \( \mathbb{C}^* \) bundle over \( T \) together with a series of associated line bundle \( \mathcal{O}_T(i) \). All of them, as well as \( p^*T^C_X \) and \( p^*S_{\pm}(i) \), are \( \mathcal{F} \)-holomorphic.

The map \( \text{(10)} \) is a part of \( \mathcal{F} \)-holomorphic short exact sequence of vector bundles

\[
0 \rightarrow p^*S_+(-1) \rightarrow p^*T^C_X \xrightarrow{\alpha} p^*S_+(1) \rightarrow 0.
\]  

\( \text{(28)} \)

The Riemann metric on \( X \) can be extended to the Hermitian metric in \( p^*T^C_X \). It splits the sequence \( \text{(28)} \). It also can be used to identify the complex normal bundle to the fibers of \( p \) with \((0,1)\)-part of the complexification of \( p^*\Omega^1_X \).

It was shown in \( [3] \) that the bundle \( \mathcal{O}(-4) \) over \( T \) has an intrinsic meaning. It is the bundle of holomorphic volume forms

\[
\omega_T = \Omega^{30}_T.
\]  

\( \text{(29)} \)

These observations enables us to make the identification of global \( C^\infty \) sections of

\[
\mathcal{B}_X \text{ with the space of } C^\infty \text{ sections of } \\
\Omega^0\bullet(\mathcal{O} + \omega_T) \overset{\text{def}}{=} \bigoplus_{i \geq 0} \Omega^{0i} + \Omega^{3i}
\]  

\( \text{(30)} \)

The differential \( d_T \) in \( \mathcal{B}_X \) under this identification transforms to the twistor \( \bar{\partial} \)-operator. The Newlander-Nirenberg theorem asserts that the condition \( \bar{\partial}^2 = 0 \) is equivalent integrability of the underlying almost complex structure. So \( T \) is an analytic manifold.

We outlined an algebraic proof of the following classical theorem.

**Theorem 5** \( [3] \) Suppose \( X \) is a self-dual Riemannian four-manifold. Then the canonical almost complex structure in the twistor space \( T(X) \) is integrable.
4 Equivalence in BV Formalism

In this section we review briefly (see [4]) the formalism of $Q$-manifolds. It is a geometric reformulation of the classical BV-formalism [14].

Let $M$ be a finite-dimensional supermanifold equipped with odd, possibly degenerate, closed two-form $\omega$, which should not be confused with $\omega$ (13) or $\omega_T$ (29). The last two will not appear in the present section. The classical BV structure is an odd vector field $Q$, that preserves $\omega$ and satisfies

$$\{Q, Q\} = 0. \quad (31)$$

In addition, we require that there is $S$ such that

$$dS = Q \omega \quad (32)$$

for some even function $S$. Equation (31) implies that $\frac{\partial S}{\partial Q} = 0$.

If $\omega$ is not degenerate, then the Hamiltonian vector field $Q$ is determined by the Hamiltonian $S$.

A map $\psi : M \to M'$ defines a morphism between system $(M, \omega, Q, S)$ and $(M', \omega', Q', S')$ if

$$\psi^* \omega' = \omega \quad (33)$$
$$\psi^* S' = S \quad (34)$$
$$\psi \text{ is a } Q - Q' \text{ equivariant map.} \quad (35)$$

The Lie derivative corresponding to the vector field $Q$ defines a nilpotent operator in the tangent space $T_x$ at a $Q$ fixed by $Q$ point $x$, fixed by $Q$. We denote the $\mathbb{Z}_2$-graded cohomology of this complex by $HT^\bullet_x(Q)$.

The map $\psi$ is a local quasi-isomorphism at a point $x$ if the induced map $\psi : HT^\bullet_x(Q) \to HT^\bullet_{\psi(x)}(Q')$ is an isomorphism.

Let $U_x$ be the formal neighborhood of a $Q$-fixed point $x$. Let us choose, in addition, a system of coordinates on $U_x$, i.e. a generating (in topological sense) subspace $V \subset \mathcal{O}(U_x)$. The object $(M, \omega, Q, S)$ can be restricted on $U_x$ and give an example of a formal classical $BV$ structure. The maps (33), that are
compatible with the chosen system of coordinates define morphisms of these $BV$ structures. The 5-tuples $(U_x, V, ω, Q, S)$ form a category. It contains a subcategory of objects equipped with a non-degenerate symplectic form. Such subcategory is equivalent to the category of finite-dimensional $L_∞$ algebras with a non-degenerate inner product (see [1]). The $L_∞$ formalism has an advantage that it allows to conveniently work with infinite-dimensional algebras. Such algebras correspond to infinite-dimensional mechanical systems which is the basic object of study in the field theory.

We now formulate the main reduction theorem.

**Proposition 6** Let $B$ be an $L_∞$ algebra. Suppose that $B^\bullet$ is the diagonal complex of the bi-complex $B^{\bullet\bullet}$. Let $π_1$ be a degree zero projector in $B$ such that $\text{id} − π = \{H, d_1\}$, where $d_1$ and $d_{II}$ are the two anti-commuting differentials. The homotopy $H$ has the bi-degree $(-1, 0)$ and, in addition, satisfies

$$H^2 = 0, \ (Ha, πb) = 0, \ \text{and} \ (Ha, b) = (-1)^a(a, Hb). \quad (36)$$

Additionally, we assume that the operator $H \circ d_{II}$ is nilpotent.

Then $A \overset{\text{def}}{=} \text{Im}π$ has a structure of $L_∞$-algebra with an inner product, induced by inclusion. This algebra is quasi-isomorphic to $B$. The quasi-isomorphism is compatible with the inner-product.

**Proof.** The proof goes along the same lines as for the ordinary complex (see [8], [9]). We only modify the structure of the trees. The trees are allowed to have vertices of valence two. We associate with them the operator $d_{II}$. Additional care should be taken about the compatibility of inner products. In the geometric language it corresponds to equation (33). The compatibility follows from equation (36). □

This proposition has a simple geometric interpretation. Let $U_0$ be the formal neighborhood of zero in the linear odd-symplectic super-space $B$, and $\tilde{U}_0$ is the similar neighborhood in $A$. The projection $π$ defines a fibration $π : U_0 → \tilde{U}_0$. The restriction of $S$ on the fiber $π^{-1}z, z ∈ \tilde{U}_0$ has a unique nondegenerate critical point $ψ(z)$. This defines a section $ψ : \tilde{U}_0 → U_0$, whose Taylor coefficients we
interpret as the quasi-isomorphism. The compatibility relations follow from the formal properties of the Legendre transform.

5 Holomorphic BF Theories

In this section we define the holomorphic BF theory data. This is the BV reformulation of the BF theory mentioned in the introduction.

We fix an odd dimensional complex manifold $M$ together with a holomorphic vector bundle $\mathcal{E}$. Let $\text{End}(\mathcal{E})$ be the vector bundle local endomorphisms of $\mathcal{E}$, $\mathcal{O} = \mathcal{O}_M$ be the structure sheaf and $\omega = \omega_M = \Omega^{\dim(M),0}$ be the canonical line bundle on $M$.

The Dolbeault complex

$$\Omega^{0,\bullet} \text{End}(\mathcal{E}) \otimes (\mathcal{O} + \omega)$$  \hspace{1cm} (37)

defines a sheaf of graded algebra. The subsheaf $\Omega^{0,\bullet} \text{End}(\mathcal{E}) \otimes \omega$ is an ideal with zero multiplication.

The space of $C^\infty$ sections of (37) is equipped with the linear functional

$$\Gamma(M, \Omega^{0,\bullet} \text{End}(\mathcal{E}) \otimes \omega) \xrightarrow{\text{tr}} \Gamma(M, \Omega^{0,n} \omega) \xrightarrow{\int} \mathbb{C}$$

that for any two sections satisfies the identity $\int \text{tr}(ab) = (-1)^{\overline{a} \overline{b}} \int \text{tr}(ba)$. If $M$ is compact or the sections decay fast at infinity, then $\int$ is a $\bar{\partial}$-closed linear functional. By definition the triple

$$\text{HBF}(M, \mathcal{E}) = (\Omega^{0,\bullet} \text{End}(\mathcal{E}) \otimes (\mathcal{O} + \omega), \int \text{tr}, \bar{\partial})$$  \hspace{1cm} (38)

comprise the holomorphic BF data. It is a special case of a Chern-Simons triple [12]. The Lagrangian density corresponding to (38) is equal to

$$\mathcal{L}_{\text{HBF}} = \text{tr}(\frac{1}{2} a \bar{\partial} a + \frac{1}{6} a^3), a \in \Gamma(M, \Omega^{0,\bullet} \text{End}(\mathcal{E}) \otimes (\mathcal{O} + \omega)).$$  \hspace{1cm} (39)

The reader should think about (39) as of a BV formulation of a suitable theory, which we also denote by HBF. More precisely the function $S$ is $\int \mathcal{L}_{\text{HBF}}$, the linear and quadratic Taylor coefficients of the vector field $Q$ are the differential $\bar{\partial}$.
and the graded commutator $[,]$. The inner product $\int \text{tr}(ab)$ defines a constant odd symplectic form on the $\mathbb{Z}_2$-graded linear space $\bigoplus_i \Omega^{0,i}\text{End}(\mathcal{E}) \otimes (\mathcal{O} + \omega)$. The classical master equation is a direct corollary of the axioms of differential graded algebra with a trace.

We can treat the sheaf of algebras $\mathcal{U}_X(\text{End}(E))$ along the same line and define the triple

$$\text{SD}(X, E) = (\mathcal{L}_{\text{SD}}, \mathcal{U}_X(\text{End}(E)), \int \text{tr})$$

with $\mathcal{L}_{\text{SD}}$ equal to

$$\mathcal{L}_{\text{SD}} = \text{tr}(\frac{1}{2}ad_E a + \frac{1}{6}a^3), a \in \Gamma(X, \mathcal{U}_X \otimes \text{End}(E))$$

(40)

The denote the corresponding formal BV system by $\text{SD}(X, E)$.

## 6 Equivalence of $\text{HBF}(M, \mathcal{E})$ with $\text{SD}(X, E)$

Our main example of a holomorphic BF theory is a theory on a twistor space $T(X)$. Let $E$ be a self-dual $k$-dimensional vector bundle on $X$. The Atiyah-Ward correspondence defines a holomorphic structure on $E$ equal to the pullback $p^*E$ with respect to the map (9).

In this section we construct the quasi-isomorphism

$$\Gamma(X, \mathcal{U}_X(\text{End}(E))) \rightarrow \Gamma(X, \mathcal{B}_X \otimes \text{End}(E)) \cong \Gamma(T(X), \Omega^0 \cdot (\mathcal{O} + \omega_T) \otimes \text{End}(\mathcal{E}))$$

(41)

It existence follows from the general Proposition 6. To make the connection with Proposition 6 more explicit we recall that $\Gamma(X, \mathcal{B}_X \otimes \text{End}(E))$ is a bicomplex with differentials $d_I$ and $d_{II}$, which are $E$-twisted versions of (24, 25). We also recall that the operator $H$, constructed in Appendix A commutes with the SU(2) action. We extend $H \otimes \text{id}_{\text{Mat}_k}$ to the covariantly constant endomorphism $H_X$ of $\mathcal{B}_X \otimes \text{End}(E)$. The conditions of Proposition 6 for this choice of homotopy are satisfied.

The linear part $\psi_1$ of the quasi-isomorphism $\psi$ is equal to $\sum_{k \geq 0}(H_X \circ d_{II})^k \circ i$. 

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Proposition 6 enables us to construct the desired quasi-isomorphism. The inclusion \( i \) coincides with the pullback \( p^* \). In particular the map \( \psi_1 \) is equal to \( p^* + H_X \circ d_{g_X} \circ \End(E) \circ p^* \).

We deduce the following.

**Theorem 7** There is a quasi-isomorphism \( \psi \) of the differential graded algebras. It also defines an \( L_\infty \) quasi-isomorphism of the graded Lie algebras \( \Gamma(X, \U_X(\End(E))) \) and \( \Gamma(T(X), \Omega^0(\Omega^0 + \omega_T) \otimes \End(E)) \) with the bracket equal to the graded commutator.

The category of self-dual vector bundles is closed with respect to the tensor multiplication. Let \( Z^* \) be a complex of self-dual vector bundles with covariantly constant differentials. The Atiyah-Ward correspondence provides us with a complex of holomorphic vector bundles \( Z^* \) on \( T(X) \). The following is an extension of the last theorem.

**Theorem 8** There is a quasi-isomorphism of the differential graded modules

\[
\Gamma(X, \U_X(\End(E) \otimes Z^*)) \to \Gamma(T(X), \Omega^0(\Omega^0 + \omega_T) \otimes \End(E) \otimes Z^*)
\]

compatible with the quasiisomorphism \( \psi \).

### A Explicit Formula for the Kernel of \( H \)

Recall that the complex linear spaces \( W_-, W_+ \) are equipped with positive Hermitian inner-products (see Section \ref{section:inner_products}). These inner-products allow to define the Fubini-Study metric on \( P^1(W_-) \) and the Hermitian structure in the vector bundle

In their presence we can define \( \bar{\partial}_{P^1} \) and the Laplace operator

\[
\Delta = \{ \bar{\partial}_{P^1}, \bar{\partial}_{P^1}^* \},
\]

that acts in the space of smooth sections \( \bigoplus_{i=0,1} \Gamma(P, \Omega^0 \mathcal{G}) \). Let

\[
\pi : \bigoplus_{i=0,1} \Gamma(P, \Omega^0 \mathcal{G}) \to \bigoplus_{i=0,1} \Gamma(P, \Omega^0 \mathcal{G})
\]
be the orthogonal projection on $\text{Ker} \, \Delta$.

We are interested in the homotopy

$$H : \Gamma(P, \Omega^{01} G) \to \Gamma(P, \Omega^{00} G) \quad (44)$$

that satisfies

$$\text{id} - \pi = \{ \bar{\partial}_P, H \}. \quad (45)$$

The goal of this section is to write an explicit formula for the kernel of the operator of homotopy $H$ (44).

The line bundle $O(-2)$ on $P^1$ is isomorphic to the line bundle of holomorphic one-forms $\Omega^{10}$. Thus we can identify $O(-n)$ with $\Omega^{n2}$. The homotopy $H_G$ is the direct sum of the homotopies $H_O(n)$ for suitable $n$. It suffices to compute the homotopies $H_{-n^2} : \Gamma(P^1, \Omega^{-n^2}) \to \Gamma(P^1, \Omega^{-n^2})$ for all $n$. Then the formula for $H_G$ will follow.

Let

$$\Delta = \Delta_{-\frac{n}{2}} \text{ be the Laplace operator in } \bigoplus_{i=0,1} \Omega^{-\frac{n}{2}}.$$

The self-adjoint Green’s operator $G = G_{-\frac{n}{2}}$ satisfies $G\Delta = \text{id} - \pi$. It can be used to construct the homotopy $H = H_{-\frac{n}{2}}$. Indeed if we set

$$H = \bar{\partial}^* G, \quad (47)$$

then the identity (45) would follow automatically. As the metric on $P$ and the Hermitian structure on $O(n)$ have SU(2)-symmetry, the operators $G$ and $H$ commute with the SU(2)-action. Additionally, equation (36) follows automatically from elementary properties of the Green’s operator.

The kernel $g = g_{-\frac{n}{2}}$ of the Green’s operator $G_{-\frac{n}{2}}$ is a SU(2)-invariant generalised section of

$$\Omega^{\frac{n}{2}+11} \boxtimes \Omega^{-\frac{n}{2}0} + \Omega^{\frac{n}{2}+10} \boxtimes \Omega^{-\frac{n}{2}1}. \quad$$

It is real analytic away of the diagonal and has log-singularity at the diagonal.

The kernel $h = h_{-\frac{n}{2}}$ of the operator $H_{-\frac{n}{2}}$ coincides with the generalised section

$$\text{id} \boxtimes \bar{\partial}^* g_{-\frac{n}{2}} \text{ of } \Omega^{\frac{n}{2}+10} \boxtimes \Omega^{-\frac{n}{2}0}. \quad (48)$$
It is SU(2)-invariant by construction.

Let $s_{-\frac{n}{2}} \alpha$ be a basis in the subspace of harmonic elements of $\Omega^{-\frac{n}{2}}$ and $t^\alpha_{\frac{n}{2}+1}$ be the dual basis in the harmonic subspace of $\Omega_{\frac{n}{2}+1}$. The section $\sum_\alpha t^\alpha_{\frac{n}{2}+1} \otimes s_{-\frac{n}{2}} \alpha + \sum_\beta s_{\frac{n}{2}+1} \otimes t^\beta_{-\frac{n}{2}}$ is a kernel of $\pi$. The section $h$ satisfies

$$\bar{\partial}h = \sum_\alpha t^\alpha_{\frac{n}{2}+1} \otimes s_{-\frac{n}{2}} \alpha + \sum_\beta s_{\frac{n}{2}+1} \otimes t^\beta_{-\frac{n}{2}}$$

off the diagonal. (49)

This is the equation (45) written in the language of kernels. As the line bundle $O(n)$ has the trivial first cohomology for $n \geq -1$, the sections $t^\alpha_{\frac{n}{2}+1}$ are zero. Similarly, for $n \leq -1$ the elements $s_{-\frac{n}{2}} \alpha$ vanish. In the following we will refer to $A$ as the first and to $B$ as the second multiples in the tensor product $A \otimes B$.

Then equation (49) implies the following.

**Lemma 9** If $n \geq -1$ the section $h_{-\frac{n}{2}}$ is holomorphic in the second argument. Similarly, for $n \leq -1$ $h_{-\frac{n}{2}}$ is holomorphic in the first argument.

Let us transfer the spherical metric from $P$ to $C$ using the stereographic projection. In the holomorphic coordinate the metric is equal to

$$\frac{|dz|^2}{(1+|z|^2)^2}.$$  

(50)

The group $SU(2) = \{ (\begin{smallmatrix} a & b \\ -\bar{b} & \bar{a} \end{smallmatrix}) \mid |a|^2 + |b|^2 = 1 \}$ acts birationally on $C$ by Möbius transformations $f(z) = \frac{az+b}{-\bar{b}z+\bar{a}}$. These transformations preserve the metric (50).

**Lemma 10** We let $h_{-\frac{n}{2}}$ to be equal to $h_{-\frac{n}{2}}(z_1, z_2) \sqrt{dz_1^{n+2} \sqrt{dz_2^{-n}}}. \quad (51)$

If $h_{-\frac{n}{2}}$ is SU(2) invariant, then $h_{-\frac{n}{2}}(z_1, z_2)$ satisfies

$$h_{-\frac{n}{2}}(z_1, z_2) = \frac{(-\bar{b}z_2 + \bar{a})^n}{(-\bar{b}z_1 + \bar{a})^2 + n} \cdot h_{-\frac{n}{2}} \left( \begin{array}{c} az_1 + b \\ -\bar{b}z_1 + \bar{a} \end{array} \right) \left( \begin{array}{c} az_2 + b \\ -\bar{b}z_2 + \bar{a} \end{array} \right). \quad (52)$$

**Proof.** Follows from the explicit computation. ■

If we set $az_1 + b = 0$, the condition $|a|^2 + |b|^2 = 1$ would imply that $|a|^2 = \frac{1}{|z_1|^2 + 1}$. It allows to show that

$$h_{-\frac{n}{2}}(z_1, z_2) = \frac{e^{2\pi i \theta} (z_1 z_2 + 1)^n}{(|z_1|^2 + 1)^{n+1}} h_{-\frac{n}{2}} \left( 0, e^{2\pi i \theta} \frac{z_2 - z_1}{z_1 z_2 + 1} \right),$$

(51)
where \( \theta \) is real and satisfies \( e^{2\pi i \theta} = \frac{a}{\bar{a}} \).

**Lemma 11** Let \( f(z) \) be a function that is holomorphic in a punctured neighborhood of zero. If \( f(z) \) satisfies \( e^{2\pi i \theta} f(e^{2\pi i \theta} z) = f(z) \), then \( f(z) \) is proportional to \( \frac{1}{z} \).

**Proof.** Follows from the uniqueness of the Laurent series expansion. ■

The combination of Lemmas 9, 10 and 11 results in the following.

**Proposition 12** Let \( h_{-\frac{\pi}{2}} \) be the kernel of the operator \( H = H_{-\frac{\pi}{2}} \), defined by the formula (47) using the Green’s function \( G \) of the Laplace operator \( \Delta \).

Additionally we assume that the metric in \( \Omega^{-\frac{\pi}{2}} \) has SU(2) symmetry.

If \( n \geq -1 \), then the kernel \( h_{-\frac{\pi}{2}} \) is equal to

\[
\frac{1}{2\pi\sqrt{-1}} \left( \frac{\bar{z}_2 z_1 + 1}{|z|^2 + 1} \right)^{n+1} \sqrt{dz_1}^{n+2} \sqrt{dz_2}^{-n} \frac{\sqrt{dz_1^{n+2} dz_2}}{z_2 - z_1}.
\]

The use of constant \( \pi \) should not produce a confusion with the projector. If \( n \leq -1 \), then \( h_{-\frac{\pi}{2}} \) is

\[
\frac{1}{2\pi\sqrt{-1}} \left( \frac{|z|^2 + 1}{z \bar{z} + 1} \right)^{n+1} \sqrt{dz_1}^{n+2} \sqrt{dz_2}^{-n} \frac{\sqrt{dz_1^{n+2} dz_2}}{z_2 - z_1}.
\]

Finally we can setup the formula for the kernel of \( H_{\bar{G}} \). Taking in to account the isomorphism \( O(n) \cong \Omega^{-\frac{\pi}{2}} \) we can claim that \( 2\pi\sqrt{-1} h_{\bar{G}} \) is equal to

\[
\frac{\bar{z}_1 z_2 + 1}{|z|^2 + 1} \sqrt{dz_1}^{2} \sqrt{dz_2}^{4} \frac{\sqrt{dz_1} \sqrt{dz_2}}{z_2 - z_1} + \frac{\bar{z}_2 z_1 + 1}{|z|^2 + 1} \sqrt{dz_1}^{3} \sqrt{dz_2}^{-1} \frac{\sqrt{dz_1} \sqrt{dz_2}}{z_2 - z_1} \text{id}_W + \frac{\bar{z}_1 z_2 + 1}{|z|^2 + 1} \sqrt{dz_1} \frac{\sqrt{dz_1} \sqrt{dz_2}}{z_2 - z_1} + \frac{\bar{z}_2 z_1 + 1}{|z|^2 + 1} \sqrt{dz_1}^{2} \sqrt{dz_2}^{-1} \frac{\sqrt{dz_1} \sqrt{dz_2}}{z_2 - z_1} \text{id}_W + \frac{\bar{z}_1 z_2 + 1}{|z|^2 + 1} \sqrt{dz_1}^{3} \sqrt{dz_2}^{-2} \frac{\sqrt{dz_1} \sqrt{dz_2}}{z_2 - z_1} \text{id}_W + \frac{\bar{z}_2 z_1 + 1}{|z|^2 + 1} \sqrt{dz_1}^{4} \sqrt{dz_2}^{-3} \frac{\sqrt{dz_1} \sqrt{dz_2}}{z_2 - z_1} \text{id}_W.
\]
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