Unified Brane Gravity:
Cosmological Dark Matter from Scale Dependent Newton Constant

Ilya Gurwich\textsuperscript{[1]} and Aharon Davidson\textsuperscript{[1]}
Physics Department, Ben-Gurion University, Beer-Sheva 84105, Israel

We analyze, within the framework of unified brane gravity, the weak-field perturbations caused by the presence of matter on a 3-brane. Although deviating from the Randall-Sundrum approach, the masslessness of the graviton is still preserved. In particular, the four-dimensional Newton force law is recovered, but serendipitously, the corresponding Newton constant is shown to be necessarily lower than the one which governs FRW cosmology. This has the potential to puzzle out cosmological dark matter. A subsequent conjecture concerning galactic dark matter follows.

PACS numbers: 04.50.Kd, 11.25.Db, 95.35.+d

1. Introduction

The discovery of dark matter\textsuperscript{[1]} continues to be a challenging problem for astrophysics and cosmology. Although many ideas from particle physics\textsuperscript{[2]} have been put forth, none of them so far have been able to provide a convincing explanation for its mysterious nature and its preponderance in the constitution of the Universe. Besides, the search for a suitable particle candidate so far has proved elusive. A different approach proposes that dark matter is not matter at all in the conventional sense but rather an artifact originating from a deviation from general relativity. Various forms of modified theories of gravity have been proposed to explain the phenomenon\textsuperscript{[3]}. However, all such theories are beset with their own problems. In this context, it is important to note the remarkable coincidence in the amounts of galactic and cosmological dark matter. This is natural for particle dark matter but has to be explained by any full theory of artifact dark matter in modified theories of gravity. Recently, the idea that brane world theories could provide an explanation for the dark matter has been suggested\textsuperscript{[4]}. Brane world theories have recently made great breakthroughs in the area of reproducing some results of general relativity on the cosmic scale, as well as in deriving the low energy Newtonian limit\textsuperscript{[5, 6, 7, 8]}. The possibility that branes can naturally produce a solution to an unsolved problem in gravity, such as dark matter, will generate a great boost in the theory aside from being a significant achievement and a good verification of the branes and extra-dimensions ideas.

In this paper we analyze the weak-field perturbations around a flat background generated by matter on the brane. We do so in the framework of unified brane gravity\textsuperscript{[9]}, following Dirac’s prescription of careful variation in the region of the brane\textsuperscript{[10]} (we present the basic principles in the next section). After some remarks on the general scenario, we focus on the radial case, thus studying the field created far from a source. Our main results are as follows:

1. We recover a Newtonian $1/r$ potential.

2. The conventional (cosmological) Newton constant is suppressed by a constant factor greater than 1. This difference between cosmological and radial Newton constants gives rise to natural cosmological dark matter. The amount of dark matter, characterized by the ratio between the two constants is an arbitrary parameter at this point.

We later discuss the possibility of a transition scale between the two Newton constants. Such a transition will result in an effective deviation from the Newtonian potential. An observer who is unaware of this transition could interpret it as a continuous distribution of dark matter. To calculate the exact transition, one would need to analyze weak-field perturbations around a cosmological brane, which is very complicated. We do calculate roughly the scale of the above transition without the exact solution. Remarkably the scale $\sim 10^5$ ly (light years) is only 1 order of magnitude above the experimental scale. It remains to be seen whether this prediction is correct and whether this transition will lead to flat rotation curves.

2. The Basics of Unified Brane Gravity

Dirac has shown\textsuperscript{[10]} that when performing variation of action on a surface, around which one or more of the fields are discontinuous, it is crucial to perform the variation in a coordinate system where this surface remains static, to preserve the linearity of the variation. Violating this principle results in nonlinear variation, and if this problem is untreated, it would lead to incorrect equations of motion. Dirac demonstrated this in his paper, where he performed both the naive variation and the correct variation on a bubble model of an electron, and showed that the naive variation results in a missing term in the equations of motion. In\textsuperscript{[11]}, Karasik and Davidson demonstrated how the naive variation would lead to a wrong Snell law, whereas the Dirac prescription leads to the correct equation.

Unified brane gravity\textsuperscript{[11]} is based on the same action...
principle as the standard brane models (Randall-Sundrum, Dvali-Gabadadze-Porrati, and Collins-Holdom) but following carefully Dirac’s prescription for correct variation of the brane. We work in a coordinate system, where the variation of the bulk metric on the surface of the brane has only 5 degrees of freedom,

$$
\delta g_{ab} = g_{AB,C} \delta y^C y^A_b y^B_a + 2 g_{AB} y^A_a \delta y^B_b ,
$$

(1)

where $g_{AB}$ is the bulk metric and $y^A$ are the bulk coordinates. This way, the brane remains undeformed during the variation. The other degrees of freedom are not lost but are simply expressed by the constraints that define the brane. The equations of motion remain covariant and independent of the reference frame. Using this principle of variation, the unified brane gravity field equations for a $Z_2$ symmetric AdS bulk with an AdS scale $b^{-1}$ take the form

$$
\frac{1}{4\pi G_5} (K_{\mu\nu} - g_{\mu\nu} K) = \frac{3b}{4\pi G_5} g_{\mu\nu} + \frac{1}{8\pi G_4} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + T_{\mu\nu} + \lambda_{\mu\nu} .
$$

(2)

In addition to the familiar terms (namely, the Israel junction term, the brane surface tension, the Einstein tensor associated with the scalar curvature $R_4$, and the physical energy-momentum tensor $T_{\mu\nu} = \delta \mathcal{L}_{\text{matter}} / \delta \gamma_{\mu\nu}$ of the brane), unified brane gravity introduces $\lambda_{\mu\nu}$. The latter consists of Lagrange multipliers associated with the fundamental induced metric constraint $g_{\mu\nu}(x) = g_{AB}(y(x)) y^A_\mu y^B_\nu$. In the above field equations, $\lambda_{\mu\nu}$ serves as a geometric (embedding originated) contribution to the total energy-momentum tensor of the brane. If the variation would be performed naively, it would yield

$$
\lambda^{\mu\nu} = 0 .
$$

(3)

This is reminiscent of Dirac’s missing term. A vanishing $\lambda_{\mu\nu}$ in Eq.(2) results in the familiar Collins-Holdom equations (Dvali-Gabadadze-Porrati with AdS bulk), where these in turn contain the Randall-Sundrum and Dvali-Gabadadze-Porrati equations as special cases. Following Dirac’s variation principle, we find that $\lambda_{\mu\nu}$ is nonzero but is conserved, and its contraction with the extrinsic curvature vanishes

$$
\lambda^{\mu\nu};_\nu = 0 , \quad \lambda_{\mu\nu} K^{\mu\nu} = 0 .
$$

(4)

This is reduced to the Regge-Tietelboim theory in the static bulk limit. Since the Regge-Tietelboim theory was derived in a static bulk and is the simplest of all brane models, it is very reassuring to have it as a limit (the standard brane models are unsuccessful in that).

### 3. General Perturbations and the Graviton

We begin with the simplest scenario of a four-dimensional flat brane of positive tension embedded in five-dimensional AdS bulk

$$
ds^2 = dy^2 + e^{-2b|y|} \eta_{\mu\nu} dx^\mu dx^\nu .
$$

(5)

$b^{-1} = \sqrt{-6/\Lambda_5}$ denotes the AdS scale, $\eta_{\mu\nu}$ is the four-dimensional Minkowski metric, and the brane is conveniently located at $y = 0$. Before turning to the main discussion concerning perturbations of this brane, it is imperative to understand the full potential of the unperturbed brane. In the conventional Randall-Sundrum and Collins-Holdom scenarios, in order to ensure its flatness, the brane has to be of positive (or negative) tension:

$$
\sigma = \frac{3b}{4\pi G_5} .
$$

(6)

Unified brane gravity, although it requires the same, allows for one more degree of freedom.

To see the point, first recall the unified brane gravity field equations [24]. For a flat brane embedded in a five-dimensional AdS background, which is the special case of interest, $K_{\mu\nu} = -b \eta_{\mu\nu}$. In turn, Eq.(1) simply implies that the corresponding $\lambda_{\mu\nu}$ is traceless. A traceless and conserved source serves as an effective (positive or negative) radiation term.

The flatness of the unperturbed brane can be achieved the conventional way, if the energy-momentum and the embedding terms both vanish, that is, $T_{\mu\nu} = \lambda_{\mu\nu} = 0$. But now there exists the milder option $T_{\mu\nu} + \lambda_{\mu\nu} = 0$. Following the above, if (and only if) the real matter on the brane exclusively consists of radiation, one can choose an appropriate $\lambda_{\mu\nu}$ to cancel it out. To be more specific, let our unperturbed flat brane host a constant radiation density $\rho$, and choose the embedding counterterm to be $\lambda^0_{\mu\nu} = -T^0_{\mu\nu} = -\text{diag}(\rho, \frac{1}{\sqrt{\rho}}, \frac{1}{\sqrt{\rho}}, \frac{1}{\sqrt{\rho}})$. This reflects the peculiarity that a flat brane can in fact be hot, which is unique to unified brane gravity. The perturbations around such a brane are expected to be quite different from those around a Collins-Holdom brane, thus giving rise to new physics. To study the perturbations induced by an arbitrary source $\delta T_{\mu\nu} \equiv \tau_{\mu\nu}$, we find it useful to invoke Gaussian normal coordinates, such that $\delta g_{AB} = h_{\mu\nu} \delta A^\mu B^\nu$ are the only allowed nonzero components (and reserve the option of supplementing this gauge later by the traceless nontransverse gauge). It is important to keep in mind that, although $\tau_{\mu\nu}$ is arbitrary, the perturbations of the metric are accompanied by built-in perturbations of all of the brane components that are perturbed not by the source itself but rather by the shift in the brane space-time structure. For example, the radiation energy-momentum term must still satisfy the conservation and traceless conditions, but it must be satisfied
in the new metric. To this extent, the radiation term is corrected via a perturbation \( T^{\text{rad}}_{\mu\nu} = T^0_{\mu\nu} + \delta T_{\mu\nu} \) that satisfies

\[
\partial^\nu \delta T^{\text{rad}}_{\mu\nu} = 
\eta^{\nu\lambda} \left( \Gamma_{\lambda\mu\nu} T^0_{\sigma\tau} + \Gamma_{\lambda\nu\sigma} T^0_{\mu\tau} \right) + h^{\nu\lambda} \partial_\lambda T^0_{\mu\tau},
\]

or the localized part of the equation

\[
\eta^{\mu\nu} \delta T^{\text{rad}}_{\mu\nu} = h^{\mu\nu} T^0_{\mu\tau},
\]

to preserve conservation and tracelessness, respectively. Here \( \Gamma_{\mu\nu} = \frac{1}{2} \left( -\partial^\lambda h_{\mu\nu} + \partial_\mu h^\lambda_{\nu} + \partial_\nu h^\lambda_{\mu} \right) \) is the affine connection. The above perturbation does not represent an addition of radiation (which can be present independently via \( \tau_{\mu\nu} \)) but rather a geometric effect. By the same token, the embedding term is also perturbed via \( \lambda_{\mu\nu} = \lambda^0_{\mu\nu} + \delta \lambda_{\mu\nu} \) and satisfies

\[
\partial^\nu \delta \lambda_{\mu\nu} = 
\eta^{\nu\lambda} \left( \Gamma_{\lambda\mu\nu} \lambda^0_{\sigma\tau} + \Gamma_{\lambda\nu\sigma} \lambda^0_{\mu\tau} \right) + h^{\nu\lambda} \partial_\lambda \lambda^0_{\mu\tau},
\]

\[
\eta^{\mu\nu} \delta \lambda_{\mu\nu} = b^{-1} \delta K^{\mu\nu} \lambda^0_{\mu\nu},
\]

However, since for a general perturbation \( \delta K^{\mu\nu} \) is not proportional to \( h^{\mu\nu} \), the term

\[
s_{\mu\nu} \equiv \lambda_{\mu\nu} + T^{\text{rad}}_{\mu\nu} = \delta \lambda_{\mu\nu} + \delta T^{\text{rad}}_{\mu\nu}
\]

is not necessarily zero. One can furthermore verify that \( s_{\mu\nu} \) is conserved and not necessarily traceless:

\[
s = \eta^{\mu\nu} s_{\mu\nu} = \frac{1}{2b} \lambda^0_{\mu\nu} \left( \frac{\partial}{\partial |y|} + 2b \right) h^{\mu\nu}. \]

The nonlocalized part of the perturbation equations is the same as the familiar Randall-Sundrum case, since the bulk still follows the normal five-dimensional Einstein equations

\[
\left( \frac{\partial^2}{\partial |y|^2} - 4b^2 + c^{2b|y|} \right) h_{\mu\nu} = 0,
\]

where \( \{ \} \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu \) is the four-dimensional (unperturbed) \( d \)’Alembertian. The localized part of the equation is

\[
\delta(y) \left[ \frac{1}{8\pi G_5} \left( \frac{\partial}{\partial |y|} + 2b \right) + \frac{1}{8\pi G_4} \{ \} \right] h_{\mu\nu} = \delta(y) \left( \tau_{\mu\nu} + s_{\mu\nu} \right).
\]

The propagation of modes into the bulk remains the same as in all of the familiar cases. Thus, we will be focusing on only the perturbations on the brane. Performing separation of variables, \( h_{\mu\nu} = A(y) \tilde{h}_{\mu\nu} (x^\mu) \) \( \{ \} \), where we have normalized without loss of generality \( A(0) = 1 \) and define \( \alpha = 1 + \frac{A'(0)}{2b} \). Next let us separate the perturbation \( \tilde{h}_{\mu\nu} = h_{\mu\nu}^{(m)} + h_{\mu\nu}^{(u)} \) to the standard term \( h_{\mu\nu}^{(m)} \), which follows the usual brane equation and thus admits the familiar solutions and the new term \( h_{\mu\nu}^{(u)} \), which is a direct result of the additional effective source \( s_{\mu\nu} \). For \( h_{\mu\nu}^{(m)} \), we can write

\[
\left( \frac{\alpha b^2}{4\pi G_{RS}} + \frac{1}{8\pi G_4} \{ \} \right) h_{\mu\nu}^{(m)} = \tau_{\mu\nu},
\]

where \( G_{RS} = b G_5 \) is the Randall-Sundrum gravitational constant on the brane, whereas for the new term

\[
\left( \frac{\alpha b^2}{4\pi G_{RS}} + \frac{1}{8\pi G_4} \{ \} \right) h_{\mu\nu}^{(u)} = s_{\mu\nu}.
\]

Unfortunately, we cannot find a general Green function to Eq.\( \{ \} \), because there is no closed-form expression of \( s_{\mu\nu} \) in terms of \( h_{\mu\nu}^{(u)} \). To that end, the only general prescription to solve Eq.\( \{ \} \) is perturbatively in \( \rho \) (see Appendix A). Despite not being able to find a general solution, we can get a clue on its properties by taking the trace of Eq.\( \{ \} \) and reorganizing the various terms

\[
\delta(y) \left[ \frac{1}{8\pi G_5} \eta^{\mu\nu} - \frac{1}{2b} \lambda^0_{\mu\nu} \right] \left( \frac{\partial}{\partial |y|} + 2b \right) h_{\mu\nu} + \frac{1}{8\pi G_4} \eta^{\mu\nu} \{ \} h_{\mu\nu} = \delta(y) \eta^{\mu\nu} \tau_{\mu\nu}.
\]

Keeping in mind that \( \lambda^0_{\mu\nu} \sim -\rho \) and \( G_N \sim b G_5 \), by looking at the first term in the equation, one may expect that the effective Newton constant may take the following form:

\[
\frac{1}{G_N} = \frac{1}{G_{CH}} + \beta \frac{\rho}{b^2},
\]

where

\[
\frac{1}{G_{CH}} = \frac{1}{G_{RS}} + \frac{1}{G_4}
\]

is the effective Newton constant in the Collins-Holdom scenario and \( \beta \) is a dimensionless constant. In the next section, we show that this prediction is indeed true and, interestingly, \( \beta \) is geometry dependent.

Although we did not obtain a propagator for the graviton, the form of the equation looks all too similar to the usual brane equations and along with Eq.\( \{ \} \) suggests that, despite deviating from the standard Randall-Sundrum scenario, the graviton propagator remains the same. In the following section, we show that the Newtonian potential is recovered for large \( r \) and thus prove that the graviton’s zero-mode is massless.

### 4. Static Radial Source

In all studies of gravitational perturbations, the point-like radial source is of special interest. Since an exact radial solution is missing in all brane theories, the best idea we have for a radial potential comes from perturbative treatment. We solve the equations far from the source, in the region where \( \tau_{\mu\nu} = 0 \).
For the radial case, we show that an exact (non-perturbative in $\rho$) weak-field solution can be obtained. This is mainly due to the fact that we are able to express $s_{\mu\nu}$ explicitly. We choose to work in a traceless Gaussian frame. For a radially symmetric perturbation, we can choose, in addition to the Gaussian traceless gauge, the radial gauge \[ e^{\chi} = 1 \] and $\kappa = 0$. It follows that $s_{\theta\theta} = s_{\phi\phi} = 0$. Solving the conservation equation for $s_{\mu\nu}$ along with Eq. (12) and gauging following the above, we have

\[
s_{tt}(r) = s_{rr}(r) = -\frac{1}{4}s(r) + \frac{1}{2r^2}\int d^4 s(r). \tag{20}\]

From Eq. (10), we see that since $s_{tt} = s_{rr}$ it follows that $h_{tt}^{(u)} = h_{rr}^{(u)} \equiv h^{(u)}$. This is the familiar form of radial fluctuations. Finally, $h_{\mu\nu}^{(m)}$ constitutes the familiar Collins-Holdom solution. The exact solution is quite complicated, but to first order in $1/r$, the solution simply yields

\[
h_{tt}^{(m)} = h_{rr}^{(m)} \equiv \frac{2G_{CH}M}{r}, \tag{21}\]

where $M = \int d^3 x \pi_t$ is the mass of the source. Substituting Eq. (12) along with

\[
\Box = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) \tag{22}\]

into Eq. (15), we can write the equation for $h^{(u)}$

\[
\kappa_4^2 h^{(u)''} + \frac{2\kappa_4^2}{r} + \left( k - \frac{2}{3} \alpha \rho \right) r h^{(u)'} + 2kh^{(u)} = -\frac{4G_{CH}M\alpha \rho}{3r}, \tag{23}\]

where $\kappa_4^2 \equiv \frac{3}{16\pi G_4}$ and $k \equiv \frac{\alpha b^2}{2\pi G_{RS}}$. The solution of physical relevance is the nonhomogeneous one, namely, 

\[
h^{(u)} = -\frac{2G_{CH}M}{3b^2} r^{-1} + \frac{1}{1 + \frac{4\pi G_{RS} \rho}{3b^2}}. \tag{24}\]

The full perturbation $\tilde{h}_{\mu\nu} = h_{\mu\nu}^{(m)} + h_{\mu\nu}^{(u)}$ is therefore 

\[
\tilde{h}_{tt} = \tilde{h}_{rr} = \frac{1}{1 + \frac{4\pi G_{RS} \rho}{3b^2}} \frac{2G_{CH}M}{r}. \tag{25}\]

It is important to note that it is only due to the solution being independent of $\alpha$ that we can proceed without integrating over all the mass modes. The Newtonian potential is thus recovered, giving us further reassurance that the graviton is indeed massless, since a mass term in the propagator would have generated an exponential decay. The associated Newton constant is

\[
G_N^r = \frac{G_{CH}}{1 + \frac{4\pi G_{RS} \rho}{3b^2}}, \tag{26}\]

where the $r$ index stands for radial.

Now that the mathematics has been understood, we return to physics. Alone, Eq. (26) has nothing new to offer. However, gravitational measurements in our Universe, although they began with the Solar System, which is physically a radially symmetric system, are now quite based in the field of cosmology as well. We recall (see Appendix B) the cosmological result for expansion around a flat background gives an FRW solution with an associated Newton constant

\[
\frac{1}{G_N^c} = \frac{1}{G_{RS}} + \frac{1}{G_4} + \frac{4\pi \rho}{3b^2}, \tag{27}\]

where the $c$ index stands for cosmological and $\rho$ here has the exact same role of background radiation. Equation (26) can also be written as

\[
\frac{1}{G_N^c} = \frac{1}{G_{RS}} + \frac{1}{G_4} + \frac{4\pi \rho}{3b^2} \left( 1 + \frac{G_{RS}}{G_4} \right). \tag{28}\]

Now, if we further assume that the role of radiation in our case is also played by the background radiation from cosmology, we can compare the two results. First of all, since we do have bounds on $b$ from both particle and gravitational localization, we can clearly state that the term $\frac{\rho}{b^2}$ is negligible in both equations. This means that

\[
G_N^c = G_{CH}, \tag{29}\]

The last term in the radial gravitational constant would have been negligible if not for the factor $\frac{G_{RS}}{G_4}$. We have no experimental or theoretical bounds on the latter ratio. In fact, the proposed self-accelerated Dvali-Gabadadze-Porrati solution for the cosmological constant requires this quantity to be very large. If it is large enough, then the above term can be significant in the calculation of the Newton constant. Thus, in principle, we have a real difference between the cosmological and the radial gravitational constants, the radial constant being necessarily lower. However, historically, the Newton constant was measured in radial systems (the Solar System). Thus an observer that is unfamiliar with this physics would interpret this effective growth of the gravitational constant as missing cosmological mass (since, in general relativity, mass is inseparable from the gravitational constant), thus bringing him to the phenomenon of cosmological dark matter, without facing dark matter in the Solar System.

Although we have not shown it here (this is a conjecture subject to future research), when solving the perturbation equations around a cosmological background, one expects the two branches of the solution, one being the $G_N^c$ and the other $G_N^r$, to be connected, creating some sort of transition between them. Such a transition, to an observer that is unaware of this effect, will seem as
a gradual increase of mass, that may result in flat rotation curves. Although the exact solution to fluctuations around a cosmological brane is highly complex, we can give a rough estimate to the typical scale of such a transition. We assume the scale to be roughly in the region where the cosmological and radial curvatures are of the same order of magnitude, so that the cosmological and radial solutions ‘mix’. The radial curvature is of the order \( \frac{r_s^2}{r^2} \), \( r_s \) being the Schwarzschild radius and the cosmological curvature is of the order of \( H^2 \), \( H \) being the Hubble constant. The scale of the predicted transition is therefore

\[
 r_{dm} \sim \left( r_s^2 \frac{t_{Hubble}}{H} \right)^{1/3} \propto M^{1/3}, \tag{30}
\]

where \( t_{Hubble} \) is the age of the Universe. When this scale is calculated for the Sun, the result is 100 ly, which is way beyond the scale of the Solar System. At these distances, other stars contribute, and thus the effect is unmeasurable today. For a galactic mass, on the other hand, the result is of the order of 10^5 ly, which is only 1 order of magnitude higher than the real galactic scale. One needs to remember that it is only a rough estimate and also that galaxies are not radial systems and are composed of many stars, each giving an effect on the scale of about 100 ly, so that the combined effect may be closer than the above result, to give the exact scale of flat rotation curves.

5. Summary and Conclusions

We have studied the behavior of weak-field perturbations around a flat brane, in the framework of unified brane gravity. It was shown that, even for the most general perturbation, the novel embedding term is ‘harmless’ and the graviton propagator is intact, leaving the graviton massless. We verify this result, in particular, for a spherically symmetric source, where the conventional Newtonian potential \( 1/r \) is recovered. However, upon a closer examination, we see that, although the functional form of the potential is standard, the gravitational constant differs from the one found in cosmology. Furthermore, the radial gravitational constant is necessarily lower than the cosmological one. For an observer, familiar only with Einstein’s general relativity, this would be immediately interpreted as cosmological dark matter. This can also be the source of galactic dark matter. The flat rotation curves may simply represent the transition between the radial and cosmological gravitational constants. The scale of the suggested flat rotation curves is predicted in this case to be of the order of \( \left( r_s^2 \frac{t_{Hubble}}{H} \right)^{1/3} \). When evaluated for a galactic mass, this is indeed close to the galactic scale. Despite this transition being the natural outcome of the two different gravitational constants, there is no reason why such a transition would generate flat (rather than some general form) rotation curves, and the flatness of the rotation curves is wishful thinking at this point.

Although the radial dark matter solution is completely speculative in this paper, the cosmological dark matter is fully postulated. The only thing that is arbitrary is the amount of dark matter. This is due to the arbitrariness of \( G_5/4 \). In fact, in order to account for the right amount of dark matter, we would need an extremely large \( G_5 \), implying a very low five-dimensional plank mass \( M_5 \approx (\rho b)^{1/3} \).

### Appendix A: Perturbative Method

We can expand the solution to Eq.(16) via

\[
 h^{(u)}_{\mu\nu} = \sum_{i=1}^{\infty} h^{(i)}_{\mu\nu}, \tag{31}
\]

and

\[
 \left( \frac{\alpha h^2}{4\pi G_{RS}} + \frac{1}{8\pi G_4} \right) h^{(i)}_{\mu\nu} = s^{(i)}_{\mu\nu}, \tag{32}
\]

where, for \( i > 1 \),

\[
 s^{(i)} = s^{(i)}_{\mu\nu} h_{\mu\nu} = \alpha \Lambda_0^{\mu\nu} h^{(i-1)}_{\mu\nu}, \tag{33}
\]

and

\[
 s^{(1)} = s^{(1)}_{\mu\nu} h_{\mu\nu} = \alpha \Lambda_0^{\mu\nu} h^{(m)}_{\mu\nu}. \tag{34}
\]

### Appendix B: Cosmological Gravitational Constant

In [9] we have proven that, when expanding the cosmological equations around a flat background with positive tension and radiation density of Eq.(76),

\[
 \rho(a) = \sqrt{\frac{2}{-\Lambda} \frac{\omega}{a^4}}, \tag{35}
\]

where \( \omega \) is a constant. The resulting FRW equation was given by Eq.(81):

\[
 \tilde{\rho} = \left( \frac{1}{8\pi G_4} + \frac{6}{-\Lambda_5} \left( \frac{1}{8\pi G_5} + \frac{\rho}{6b} \right) \right) \epsilon, \tag{36}
\]

where \( \epsilon = 3 \frac{\dot{a}^2 + k}{a^2} \) and, therefore,

\[
 \frac{1}{G_N} = \frac{1}{G_4} + \frac{1}{G_{RS}} + \sqrt{\frac{2}{-\Lambda_5} \frac{8\pi \omega}{a^4}}. \tag{37}
\]
We would like to express the last term in Eq. (37) in terms of $\rho$ and $b$ and, therefore,

$$\frac{1}{G_N} = \frac{1}{G_{CH}} + \frac{4\pi \rho}{3b^2}. \quad (38)$$

The authors thank Professors Philip Mannheim, Eduardo Guendelman and especially our colleague Shimon Rubin for enlightening discussions and constructive comments. A special thanks to Wali Kameshwar for constructive comments that have significantly improved this paper.

* Email: gurwichphys@gmail.com
† Email: davidson@bgu.ac.il

[1] E. Komatsu et al. (WMAP Collaboration), Astrophys. J. Suppl. 180, 330 (2009). R.B. Tully and J.R. Fisher, Astron. Astrophys. 54, 661 (1977). M. Davis et al. Astrophys. J. 292, 371 (1985). F. Zwicky, Helv. Phys. Acta. 6, 110 (1933). S.M. Carroll, Nature Phys. 2, 653 (2006).
[2] G. Jungman, M. Kamionkowski and K. Graesser, Phys. Rept. 267, 195 (1996). R. Holman, G. Lazarides and Q. Shafi, Phys. Rev. D27, 995 (1983). C.D. Froggatt and H.B. Nielsen Phys. Rev. Lett. 95, 231301 (2005).
[3] M. Milgrom, Astrophys. J. 270, 365, (1983). P.D. Mannheim and D. Kazanas, Astrophys. J. 342, 635 (1989). J.D. Bekenstein, Phys. Rev. D70, 083509 (2004).
[4] J.A.R. Cembranos, A. Dobado and A.L. Maroto, Phys. Rev. Lett. 90, 241301 (2003). M.K. Mak and T. Harko, Phys. Rev. D70, 024010 (2004).
[5] W. Israel, Nuovo Cimento B44, 1 (1966). T. Regge and C. Teitelboim, in Proc. Marcel Grossman (Trieste), 77 (1975). R. Cordero and A. Vilenkin, Phys.Rev. D65, 083519 (2002). R. Sundrum, Phys. Rev. D59, 085010 (1999). L. Randall and R. Sundrum, Phys. Rev. Lett. 83, 3370 (1999). Phys. Rev. Lett. 83, 4690 (1999). H. Collins and B. Holdom, Phys. Rev. D62, 105009 (2000).
[6] P. Binetruy, C. Deffayet, and D. Langlois, Nucl. Phys. B565, 269 (2000). T. Shiromizu, K. Maeda and M. Sasaki, Phys. Rev. D62, 024012 (2000). R. Maartens, Phys. Rev. D62, 084023 (2000). P. Bowcock, C. Charmousis and R. Gregory, Class. Quant. Grav. 17, 4745 (2000). E. Flanagan, H. Tye and I. Wasserman, Phys. Rev. D62, 024011 (2000). G. Kofinas, R. Maartens and E. Papantonopoulos, J. High Energy Phys. 10, 066 (2003). C. Deffayet, Phys. Rev. D66, 103504 (2002).
[7] C. Deffayet, Phys. Lett. B502, 199 (2001). A. Lue, Phys. Rep. 423, 1 (2006). E. Papantonopoulos, Lect. Notes Phys. 592, 458 (2002). F. Quevedo, Class. Quant. Grav. 19, 5721 (2002). D. Langlois, Prog. Theor. Phys. Supp. 148, 181 (2002). P. Brax and C. Van de Bruck, Class. Quant. Grav. 20, R201 (2003).
[8] J. Garriga and T. Tanaka, Phys. Rev. Lett. 84, 2778 (2000). S.B. Giddings, E. Katz and L. Randall, J. High Energy Phys. 03, 023 (2000). G. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B485, 208 (2000). H. Collins and B. Holdom, Phys. Rev. D62, 124008 (2000). Y. Shtanov and A. Viznyuk, Class. Quant. Grav. 22, 987 (2005).
[9] A. Davidson and I. Gurwich, Phys. Rev. D74, 044023 (2006). A. Davidson and I. Gurwich JCAP 06, 001 (2008).
[10] P.A.M. Dirac, Proc. Roy. Soc. A268, 57 (1962).
[11] D. Karasik and A. Davidson, Class. Quant. Grav. 21 1295 (2004).
[12] This is just a different parametrization to the mass modes expansion.
[13] Since any static radial perturbation can be brought to the form of $ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2d\Omega^2$, we can impose $h_{\theta\theta} = h_{\phi\phi} = 0$. 
[14] For any static radial perturbation can be brought to the form of $ds^2 = -f(r)dt^2 + g(r)dr^2 + r^2d\Omega^2$, we can impose $h_{\theta\theta} = h_{\phi\phi} = 0$. 
[15] This is just a different parametrization to the mass modes expansion.