ON THE INTEGRABILITY OF GEODESIC FLOWS OF SUBMERSION METRICS

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Abstract. Suppose we are given a compact Riemannian manifold \((Q, g)\) with a completely integrable geodesic flow. Let \(G\) be a compact connected Lie group acting freely on \(Q\) by isometries. The natural question arises: will the geodesic flow on \(Q/G\) equipped with the submersion metric be integrable? Under one natural assumption, we prove that the answer is affirmative. New examples of manifolds with completely integrable geodesic flows are obtained.

1. Introduction

Let \(T^*Q\) be the cotangent bundle of a compact connected \(n\)-dimensional Riemannian manifold \((Q, g)\) with the natural symplectic structure. The geodesic flow on \(Q\) is described by the Hamiltonian equations on \(T^*Q\)
\[ \dot{x} = s\text{grad} H(x), \]
where the Hamiltonian function is \(H(p, q) = \frac{1}{2}g_q^{-1}(p, p), p \in T^*_qQ\). The geodesic flow \((1)\) is completely integrable if there are \(n\) Poisson-commuting smooth integrals \(f_1, \ldots, f_n\) whose differentials are independent in an open dense set of \(T^*Q\). Then by Liouville’s theorem \(T^*Q\) is foliated by invariant Lagrangian tori in the open dense set. This situation is very exceptional. Furthermore, if additional conditions are placed on the integrals (such as real analyticity) then there are serious topological obstructions to the integrability [10, 15, 13]. Examples of manifolds admitting completely integrable geodesic flows are given in \([16, 7, 17, 14, 1, 3, 4, 5, 6]\). A more detailed list of references can be found in \([5, 6]\).

Suppose we are given a compact Riemannian manifold \((Q, g)\) with completely integrable geodesic flow. Let \(G\) be a compact connected Lie group acting freely on \(Q\) by isometries. The problem we are interested in is as follows: Will the geodesic flow on \(Q/G\), equipped with the submersion metric, be integrable?

Paternain and Spatzier proved that if the manifold \(Q\) has geodesic flow integrable by means of \(S^1\)-invariant integrals and if \(N\) is a surface of revolution, then the submersion geodesic flow on \(Q \times S^1, N = (Q \times N)/S^1\) will be completely integrable [14]. Combining submersions and Thimm’s method (see [16]), Paternain and Spatzier [14] and Bazaikin [1] proved integrability of geodesic flows on certain interesting bi-quotients of Lie groups (in fact, the author’s motivation in writing this letter was to explain these submersion examples in the framework of a general construction).

In this letter we use the Mishchenko–Fomenko–Nekhoroshev theorem on non-commutative integration of Hamiltonian systems (section 2). This allows us to...
take a new approach to the proposed problem and to prove that the answer, under one natural assumption, is affirmative. In section 3, the simple proof is given if original integrals are $G$–invariant functions (which forces $G$ to be abelian) and in section 4, as the main result, the proof is given for the arbitrary $G$–actions. As an illustration, we show the integrability of geodesic flows on a class of bi-quotients of Lie groups (section 5).

2. Local structure of integrable systems

We shall briefly recall the concept of non-commutative integrability introduced by Mishchenko and Fomenko in [11]. Let $M$ be a $2n$–dimensional symplectic manifold.

Let $\langle \mathcal{F}, \{\cdot, \cdot\} \rangle$ be a Poisson subalgebra of $(C^\infty(M), \{\cdot, \cdot\})$. Let $F_x$ be the subspace of $T^*_xM$ generated by $df(x)$, $f \in \mathcal{F}$. Suppose that on an open dense set of $M$ we have $\dim F_x = l$ and $\dim \ker \{\cdot, \cdot\}|_{F_x} = r$. Denote the numbers $l$ and $r$ by $\dim F_x$ and $\dim \ker \{\cdot, \cdot\}$, respectively. The algebra $\mathcal{F}$ is complete if the following condition is satisfied:

$$\dim F_x + \dim \ker \{\cdot, \cdot\} = \dim M.$$

**Remark 1.** Let $\mathcal{F}$ be a complete algebra. Set

$$W_x = \{\text{grad} f(x), f \in \mathcal{F}\},$$

$$D_x = \{\xi \in T_xM, df(x)(\xi) = 0, f \in \mathcal{F}\}.$$

The condition $\dim F_x + \dim \ker \{\cdot, \cdot\} = \dim M$ is equivalent to the coisotropy of $W_x$ and isotropy of $D_x$ in the symplectic linear space $T_xM$. In this case we say that $\mathcal{F}$ is complete at $x$.

The Hamiltonian system $\dot{x} = \text{grad} H(x)$ on a symplectic manifold $M$ is called completely integrable (in the non-commutative sense) if it possesses a complete algebra of first integrals $\mathcal{F}$. Then each connected compact component of a regular level set of the functions $f_1, \ldots, f_l \in \mathcal{F}$ is an $r$–dimensional invariant isotropic torus $T^r$ (see [11, 12, 17]). In a neighborhood of $T^r$ there are generalized action-angle variables $p, q, I, \varphi$ mod $2\pi$, defined in a toroidal domain $O = T^r\{\varphi\} \times B_\sigma\{I, p, q\}$,

$$B_\sigma = \{(I_1, \ldots, I_r, p_1, \ldots, p_k, q_1, \ldots, q_k) \in \mathbb{R}^l, \sum_{i=1}^r I_i^2 + \sum_{j=1}^k q_j^2 + p_j^2 \leq \sigma^2\}$$

such that the symplectic form becomes

$$\omega = \sum_{i=1}^r dI_i \wedge d\varphi_i + \sum_{i=1}^k dp_i \wedge dq_i,$$

and the Hamiltonian function depends only on $I_1, \ldots, I_r$. The Hamiltonian equations take the following form in action-angle coordinates:

$$\dot{\varphi}_1 = \omega_1 = \frac{\partial H}{\partial I_1}, \ldots, \dot{\varphi}_r = \omega_r = \frac{\partial H}{\partial I_r}, \quad \dot{I} = \dot{p} = \dot{q} = 0.$$

**Definition 2.** [2] The Hamiltonian system [2] defined in the toroidal domain $O = T^r \times B_\sigma$ is $T^r$–dense if the set of points $(I_0, p_0, q_0) \in B_\sigma$ for which the trajectories of [2] are dense on the torus $\{I = I_0, p = p_0, q = q_0\}$ is everywhere dense in the ball $B_\sigma$. 
The frequencies $\omega_1(I), \ldots, \omega_r(I)$ corresponding to the dense trajectories are rationally independent. For example, a non-degenerate system (det $\mathbf{J} \neq 0$ on an open dense set of $O$) is $T^r$–dense.

Any smooth first integral of a $T^r$–dense system is a function of the variables $I, q, p$ only.

It follows from theorem 8 in Bogoyavlenskij that for an arbitrary differentiable Hamiltonian function $H(I_1, \ldots, I_r)$ defined on a toroidal domain $O = T^r \{ \varphi \} \times B_\alpha \{ I, p, q \}$, there exists a family of balls $B_r \subset B_\alpha$ such that the union $\cup_r B_r$ is dense in $B_\alpha$ and the following properties hold. In the toroidal domain $O_r = T^r \times B_r$, there exists a canonical transformation: $\{ I, p, q, \varphi \} \rightarrow \{ I^r, p^r, q^r, \varphi^r \}$, that transforms the system to the form

$$
\dot{\varphi}_i^r = \omega_i^r = \frac{\partial H}{\partial I_i^r}, \ldots, \dot{\varphi}_{q(r)}^r = \omega_r^r = \frac{\partial H}{\partial I_r^r},
$$

$$
\dot{i}^r = \hat{p}^r = \hat{q}^r = 0,
$$

$H = H(I_1^r, \ldots, I_r^r)$, $\dot{r} = \dot{r}(r) \leq r$. The system is $T^r$–dense in $O_r$ (regarded as the product $T^r \times (T^{r-r} \times B_r)$). Moreover, if $H(I_1, \ldots, I_r)$ is analytic and the maximal dimension of the closures of trajectories is equal to $\mathring{r}$ then such canonical transformation exists globally and the system is $T^r$–dense in $O$.

We turn back to geodesic flows. Let $(Q, g)$ be a compact Riemannian manifold and let $H(p, q) = \frac{1}{2}g^{-1}(p, p)$. Suppose that the geodesic flow is completely integrable in the non-commutative sense by means of a complete algebra $F$ of first integrals. Since all iso-energy levels $(T^r Q)_h = H^{-1}(h)$ are compact, the phase space $T^*Q$ is foliated by invariant $F$–dimensional isotropic tori in an open dense set which we shall denote by $\text{reg} T^*Q$. Although the dynamics on $\text{reg} T^*Q$ are well understood, examples by Bolsinov and Taimanov show that an integrable geodesic flow may possess complicated dynamics on the singular set $T^*Q \setminus \text{reg} T^*Q$.

If the algebra of integrals $F = \{ f_1, \ldots, f_n \}$ is commutative $\text{dim} F = n$ then we have usual Liouville integrability. But if the geodesic flow is integrable in the non-commutative sense, then it is also integrable in the commutative sense, i.e., there exist $C^\infty$–smooth commuting integrals $g_1, \ldots, g_n$ that are independent on an open dense set of $T^*Q$. The $r$–dimensional invariant isotropic tori $\mathbb{T}^r$ can be organized into larger, $n$–dimensional Lagrangian tori $\mathbb{T}^n$ which are the level sets of the commutative algebra of integrals $\{ g_1, \ldots, g_n \}$. Furthermore, if $F = \{ f_1, \ldots, f_l \}$ is a finite dimensional Lie algebra $(\{ f_i, f_j \} = c_{ij}^k f_k)$, then the commuting integrals can be taken as polynomials in $f_1, \ldots, f_l$ (see Brailov).

### 3. Submersions

Let $G$ be a compact connected Lie group with a free Hamiltonian action on the symplectic manifold $(M, \omega)$. Let

$$
\Phi : M \rightarrow \mathfrak{g}^*
$$

be the corresponding equivariant moment map ($\mathfrak{g}^*$ is the dual space of the Lie algebra $\mathfrak{g} = T_e G$). Let $G_\eta$ be the coadjoint action isotropy group of $\eta \in \Phi(M) \subset \mathfrak{g}^*$. By $(M_\eta, \omega_\eta)$ we denote the reduced symplectic space

$$
M_\eta = \Phi^{-1}(\eta)/G_\eta,
$$

$$
\omega_\eta(d\pi(\xi_1), d\pi(\xi_2)) = \omega(\xi_1, \xi_2), \quad \xi_1, \xi_2 \in T_x \Phi^{-1}(\eta),
$$
where \( \pi : \Phi^{-1}(\eta) \to M_\eta \) is the natural projection. For a \( G \)-invariant function \( f \) on \( M \), let \( f_\eta \) be the induced function on \( M_\eta \).

We have the following simple general observation.

**Lemma 3.** Suppose a connected compact Lie group \( G \) acts effectively on a symplectic manifold \((M, \omega)\) with moment map \( \Phi \). Let \( H \) be a \( G \)-invariant Hamiltonian function. If the Hamiltonian system \( \dot{x} = \text{sgrad} H(x) \) is completely integrable by means of a \( G \)-invariant first integrals, then \( G \) is a torus. Moreover, if the action is free, the reduced Hamiltonian system on \( M_\eta \) is completely integrable for a generic value \( \eta \) of the moment map.

**Proof.** Let \( \mathcal{F} \) be a complete \( G \)-invariant algebra of integrals and let \( D_x \) and \( W_x \) be the subspaces of \( T_x M \) defined in remark 1. Since all functions from \( \mathcal{F} \) are \( G \)-invariant we have that \( T_x(G \cdot x) \subset D_x \) is isotropic, for generic \( x \in M \). The Hamiltonian vector fields of the functions \( \phi_a(x) = \langle \Phi(x), a \rangle \), \( a \in \mathfrak{g} \) generate one-parameter groups of symplectomorphisms. Therefore \( \omega_x(\text{sgrad} \phi_a(x), \text{sgrad} \phi_b(x)) \) vanishes in an open dense set of \( M \). Thus \( \phi_{[a,b]} = \{ \phi_a, \phi_b \} = 0 \) for every \( a, b \in \mathfrak{g} \). Since the action is effective, we get that \( G \) is a torus.

Let the torus \( G \) act freely on \( M \). Take \( x \in M \) such that \( \mathcal{F} \) is complete at \( x \). Since all functions from \( \mathcal{F} \) are \( G \)-invariant, the coisotropic space \( W_x = \{ \text{sgrad} f(x), f \in \mathcal{F} \} \) belongs to \( T_x\Phi^{-1}(\eta) \), \( \eta = \Phi(x) \). Let \( \pi : \Phi^{-1}(\eta) \to M_\eta \) be the natural projection. From the definition of the reduced symplectic structure, one can easily see that the space generated by \( \text{sgrad} f_\eta(\pi(x)) \), \( f \in \mathcal{F} \) is also coisotropic, i.e., the induced algebra \( \mathcal{F}_\eta = \{ f_\eta, f \in \mathcal{F} \} \) is complete at \( \pi(x) \). \( \Box \)

Now let a compact connected Lie group \( G \) act on a Riemannian manifold \((Q, g)\) by isometries. Suppose that \( G \) acts freely on \( Q \) and endow \( Q/G \) with the submersion metric. Let \( \mathcal{G}_x + \mathcal{H}_x = T_xQ \) be the orthogonal decomposition of \( T_xQ \), where \( \mathcal{G}_x \) is tangent space to the fiber \( G \cdot x \). By definition, the submersion metric \( g_{sub} \) is given by

\[
\langle \xi_1, \xi_2 \rangle_{p(x)} = \langle \xi_1, \xi_2 \rangle_{x}, \quad \xi_i \in T_{p(x)}(Q/G), \quad \xi_i \in \mathcal{H}_x, \quad \xi_i = d\rho(\xi_i),
\]

where \( \rho : Q \to Q/G \) is the canonical projection. The vectors in \( \mathcal{G}_x \) and \( \mathcal{H}_x \) are called vertical and horizontal respectively.

Let \( \Phi : T^*Q \to \mathfrak{g}^* \) be the moment map of the natural Hamiltonian \( G \)-action on \( T^*Q \). It is well known that the reduced symplectic space

\[
(T^*Q)_0 = \Phi^{-1}(0)/G
\]

is symplectomorphic to \( T^*(Q/G) \). Moreover, if \( H \) is the Hamiltonian function of the geodesic flow on \( Q \) then \( H_0 \) will be the Hamiltonian of the geodesic flow for the submersion metric. If we identify \( T^*Q \) and \( TQ \) by the metric \( g \), then \( \Phi^{-1}(0) \) will be the set of all horizontal vectors \( \mathcal{H} \) (see [13]).

Therefore, by lemma 3, if the geodesic flow of \((Q, g)\) is completely integrable by means of a \( G \)-invariant algebra of integrals \( \mathcal{F} \) then \( G \) is commutative. Moreover we have integrability of the reduced system on \((T^*Q)_\eta\), for generic \( \eta \in \Phi(T^*Q) \). Since \( G \) is commutative \((T^*Q)_\eta \) is diffeomorphic to \( T^*(Q/G) \), but for \( \eta \neq 0 \) the symplectic form possesses the additional "magnetic term". The reduced flow is no longer the inertial motion of a particle on \( Q/G \) (i.e., geodesic flow), but the motion of a particle under an additional magnetic force. Specifically, if \( \text{reg} T^*Q \) intersects the space of horizontal vectors \( \mathcal{H} \cong \Phi^{-1}(0) \) in a dense set then the geodesic flow of \((Q/G, g_{sub})\) will be completely integrable.
We can push the last observation further.

4. The main theorem

From lemma 3, the condition on a $G$–invariant algebra $F$ on $(M, \omega)$ to be complete forces $G$ to be abelian, which is too restrictive. However, it can happen that $F$ is not complete on $(M, \omega)$ but $F_\eta$ is a complete algebra on the reduced space $(M_\eta, \omega_\eta)$.

**Theorem 4.** Let a compact connected Lie group $G$ act freely by isometries on the compact Riemannian manifold $(Q, g)$. Suppose that the geodesic flow is completely integrable. If $\text{reg} T^*Q$ intersects the space of horizontal vectors $\mathcal{H} \cong \Phi^{-1}(0)$ in a dense set then the geodesic flow on $Q/G$ endowed with the submersion metric $g_{\text{sub}}$ is also completely integrable.

**Proof.** Take some toroidal domain $\mathcal{O} = \mathbb{T}\{\varphi\} \times B_x \{I, p, q\} \subset \text{reg} T^*Q$ $(\mathcal{O}_1 = \mathcal{O} \cap \Phi^{-1}(0) \neq 0)$ such that the geodesic flow is $\mathbb{T}^r$–dense in $\mathcal{O}$. Consider the $G$–invariant sets

$$U_1 = \mathcal{U} \cap \Phi^{-1}(0), \quad U = G \cdot \mathcal{O} = \{g \cdot x, \ x \in \mathcal{O}, \ g \in G\}.$$  

Note that the functions $\phi_a(x) = \langle \Phi(x), a \rangle, a \in \mathfrak{g}$ are first integrals of the geodesic flow and so do not depend on $\varphi$ in $\mathcal{O}$. In other words, the action of $G$ preserves the foliation by $r$–dimensional invariant isotropic tori of the sets $\mathcal{U}$ and $U_1$ as well.

The foliation $\mathcal{T}$ of $U_1$ by tori induces a foliation $\mathcal{L}$ of $U_1$ by $(\dim G + r - s)$–dimensional compact $G$–invariant submanifolds, with tangent spaces of the form

$$T_x \mathcal{L} = T_x(G \cdot x) + T_x(\mathcal{T}) \subset T_x \Phi^{-1}(0),$$

where $s = \dim \mathcal{S}_x$, 

\begin{equation}
S_x = T_x(G \cdot x) \cap T_x(\mathcal{T}), \quad x \in U_1
\end{equation}

(for $x \in \mathcal{O}_1$ we have that $T_x \mathcal{L} = \{\text{sgrad} \phi_a(x), \ \text{sgrad} I_i(x)\}, \ S_x = \{\text{sgrad} \phi_a(x)\} \cap \{\text{sgrad} I_i(x)\}$ and the foliation $\mathcal{L}_{\mid \mathcal{O}_1}$ does not depend on $\varphi$).

Let $\pi : \Phi^{-1}(0) \to \Phi^{-1}(0)/G = T^*(Q/G)$ be the natural projection and let $\hat{\mathcal{U}} = \pi(U_1) \subset T^*(Q/G)$. The foliation $\mathcal{L}$ induces a foliation $\hat{\mathcal{T}} = \pi(\mathcal{L})$ of $\hat{\mathcal{U}}$ by an $(r - s)$–dimensional invariant invariant manifolds of the geodesic flow of the submersion metric. We shall see below that $\hat{\mathcal{T}}$ is a foliation of $\hat{\mathcal{U}}$ by an invariant tori with respect to certain complete algebra of first integrals $\mathcal{F}_0$.

The foliation $\mathcal{L}$ can be seen as the level sets of of $G$–invariant integrals $f_1, \ldots, f_\rho$ on $U_1$, $\rho = \dim T^*Q - \dim G + s - r$. Indeed, this is always true locally: for $\sigma$ small enough there are functions $f_i = f_i(I, p, q)$ on $\mathcal{O}_1$ such that the tangent spaces $T_x \mathcal{L}$ (recall that $\mathcal{L}_{\mid \mathcal{O}_1}$ does not depend on $\varphi$) are given by the equations

$$T_x \mathcal{L} = \{\xi \in T_x \mathcal{O}_1, \ df_i(x)(\xi) = 0, \ i = 1, \ldots, \rho\}.$$

Then extend the $f_i$ to $G$–invariant functions on $\mathcal{U}_1$. Let $\mathcal{F}_0$ be the induced algebra of first integrals in $\mathcal{U}$. Since

$$D_{\pi(x)}(\xi) = \{\xi \in T_{\pi(x)}\hat{\mathcal{U}}, \ df_0(\pi(x))(\xi) = 0, \ f_0 \in \mathcal{F}_0\} = T_{\pi(x)}\hat{\mathcal{T}} = d\pi(x)(T_x \mathcal{L}),$$

$x \in U_1$ are isotropic spaces, by remark 1 we get that $\mathcal{F}_0$ is complete in $\hat{\mathcal{U}}$. Note that

\begin{equation}
d\dim \mathcal{F}_0 = \dim T^*(Q/G) + s - r, \quad \dim \mathcal{F}_0 = r - s.
\end{equation}
Now fill up $T^*(Q/G)$ with countably many disjoint toroidal domains $\hat{O}_\alpha = T^{r(\alpha)}\{\hat{\varphi}\} \times B_{r(\alpha)}\{I, \hat{p}, \hat{q}\}$ (with possible different dimensions of tori). In every domain $\hat{O}_\alpha$, one can construct complete involutive set of integrals that can be then "glued" in order to obtain a complete involutive set of integrals globally defined. The construction is suggested by Brailov for Darboux symplectic balls. We follow the construction in [4].

We have the following $m = \dim Q/G$ commuting integrals in $\hat{O}_\alpha$:

$$h_1 = \hat{p}_1^2, \ldots, h_r = \hat{p}_r^2, \quad h_{r+1} = \hat{p}_{r+1}^2 + \hat{q}_{r+1}^2, \ldots, h_m = \hat{p}_m^2 + \hat{q}_m^2.$$

Let $g_\alpha : \mathbb{R} \to \mathbb{R}$ be a smooth nonnegative function such that $g_\alpha(x)$ is equal to zero for $|x| > \sigma(\alpha)$, $g_\alpha$ monotonically increases on $[-\sigma(\alpha), 0]$ and monotonically decreases on $[0, \sigma(\alpha)]$. Let $h_\alpha(y) = g_\alpha(h_1(y) + \cdots + h_m(y))$. This function can be extended by zero to the whole manifold $T^*(Q/G)$. Then $f_1^\alpha = h_{\alpha} \cdot h_i, \ i = 1, \ldots, n$ will be commuting functions, independent on an open dense subset of $\hat{O}_\alpha$. With a "good" choice of $g_\alpha$’s, a complete commutative set of smooth integrals is given by $f_i(y) = f_i^\alpha(y)$ for $y \in \hat{O}_\alpha \subset \cup_{\beta} \hat{O}_\beta$ and zero otherwise, $i = 1, \ldots, m$.

**Remark 5.** It is clear that a similar statement holds for an arbitrary Hamiltonian $G$–space $(M, \omega)$ ($G$ acts freely on $M$) and an integrable Hamiltonian system $\dot{x} = \text{grad} H(x)$ with compact iso-energy levels $M_h = H^{-1}(h)$. The $M$ is foliated by invariant tori in the open dense set reg $M$. If reg $M$ intersects the submanifold $\Phi^{-1}(\eta)$ in a dense set then the reduced Hamiltonian system $(M_\eta, \omega_\eta)$ will be completely integrable.

Here is a simple construction that gives examples satisfying the hypotheses of theorem 4. Suppose we are given Hamiltonian $G$–actions on two symplectic manifolds $(M_1, \omega_1)$ and $(M_2, \omega_2)$ with moment maps $\Phi_{M_1}$ and $\Phi_{M_2}$. Then we have the natural diagonal action of $G$ on the product $(M_1 \times M_2, \omega_1 \oplus \omega_2)$, with moment map

$$\Phi_{M_1 \times M_2} = \Phi_{M_1} + \Phi_{M_2}. \quad (5)$$

If $(Q_1, g_1)$ and $(Q_2, g_2)$ have integrable geodesic flows, then $(Q_1 \times Q_2, g_1 \oplus g_2)$ also has integrable geodesic flow and reg $T^*(Q_1 \times Q_2) = \text{reg } T^*Q_1 \times \text{reg } T^*Q_2$. Using (5), one can easily see that if the $G$–actions on $Q_1$ and $Q_2$ are almost everywhere locally free, then a generic horizontal vector of the submersion $Q_1 \times Q_2 \to Q_1 \times_G Q_2 = (Q_1 \times Q_2)/G$ belongs to reg $T^*(Q_1 \times Q_2)$. Thus we get the following statement.

**Corollary 6.** Let the Lie group $G$ act by isometries on the compact Riemannian manifolds $(Q_1, g_1)$ and $(Q_2, g_2)$ almost everywhere locally freely, and freely on the product $Q_1 \times Q_2$. Suppose that the geodesic flows on $Q_1$ and $Q_2$ are completely integrable. Then the geodesic flow on $Q_1 \times_G Q_2$, endowed with the submersion metric, will be completely integrable.

The theorem and corollary allow us to construct examples of manifolds with completely integrable geodesic flows starting from some known integrable cases. This generalizes the construction of Paternain and Spatzier of manifolds of the form $Q \times S^1 N$, where $N$ is a surface of revolution and the geodesic flow on $Q$ possesses a complete involutive algebra of $S^1$–invariant integrals [14].

**Example 7.** Suppose the Lie group $G$ acts freely by isometries on $(Q, g)$. Let $G_1$ be an arbitrary compact Lie group, which contains $G$ as a subgroup. Let $ds_1^2$
be some left-invariant Riemannian metric on \( G_1 \) with integrable geodesic flow (see example 10 below). Then \( G \) acts in the natural way by isometries on \((G_1, ds^2_1)\). Therefore, by corollary 6, if the geodesic flow on \( Q \) is completely integrable, then the geodesic flows on \( Q \times_G Q \) and \( Q \times_G G_1 \) endowed with the submersion metrics will be also completely integrable.

As an illustration, we will show that we can get a very simple proof for the complete integrability of the geodesic flows on a class of bi-quotients of Lie groups.

5. Bi-quotients of compact Lie groups

Let \( G \) be a compact connected Lie group and \( \mathfrak{g} \) be its Lie algebra. Let \( \langle \cdot, \cdot \rangle \) be an \( Ad_G \)-invariant scalar product on \( \mathfrak{g} \) and \( ds^2_{\mathfrak{g}} \) the corresponding bi-invariant metric on \( G \). In what follows we shall identify \( T^*G \) and \( TG \) by the bi-invariant metric.

Consider a connected subgroup \( U \) of \( G \times G \) and define the action of \( U \) on \( G \) by:

\[(g_1, g_2) \cdot g = g_1 g g_2^{-1}, \quad (g_1, g_2) \in U, \quad g \in G.\]

If the action is free then the orbit space \( G/U \) is a smooth manifold called a bi-quotient of the Lie group \( G \). In particular, if \( U = K \times H \), where \( K \) and \( H \) are subgroups of \( G \), then the bi-quotient \( G/U \) is denoted by \( K \backslash G/H \). The bi-invariant metric \( ds^2_{\mathfrak{g}} \) on \( G \), induces the submersion metric \( ds^2_{\mathfrak{g}, sub} \) on \( G/U \). In this way, via the submersion \( Sp(2) \to \Sigma^7 \), Gromoll and Meyer \( \textnormal{[2]} \) obtained an exotic 7–sphere \( \Sigma^7 \).

It is well known that the geodesic flow of the metric \( ds^2_{\mathfrak{g}} \) on \( G \) is completely integrable in the non-commutative sense. For a complete algebra of integrals we can take \( \mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 \), where \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are the functions obtained from polynomials on \( \mathfrak{g} \) by left and right translations respectively. Then \( \dim \mathcal{F} = 2 \dim G - \text{rank} G \), \( \dim \mathcal{F} = \text{rank} G = r \). The system is \( \mathbb{T}^r \)-dense completely integrable. Let \( \mathbb{R}[\mathfrak{g}]^G \) be the algebra of the invariant polynomials on \( \mathfrak{g} \). The algebra \( \mathbb{R}[\mathfrak{g}]^G \) is generated by \( r \) homogeneous polynomials \( p_1, \ldots, p_r \). Let \( f_1, \ldots, f_r \) be the left (or right) translations of \( p_1, \ldots, p_r \) to \( TG \). Then \( f_1, \ldots, f_r \) generate \( \mathcal{F}_1 \cap \mathcal{F}_2 \), and can be seen as \( " \)action\( " \) variables: the vector fields \( X_1 = \text{sgrad} f_1, \ldots, X_r = \text{sgrad} f_r \) form a basis of commuting vector fields on the regular invariant tori.

Let \( u \subset \mathfrak{g} \cong T_e G \) be the vertical space at the neutral element of the group. Then the horizontal space \( \mathfrak{v} \) at the neutral element is the orthogonal complement of \( u \) with respect to \( \langle \cdot, \cdot \rangle \).

Let \( \mathfrak{g}_\xi = \{ \eta \in \mathfrak{g}, \ [\xi, \eta] = 0 \} \). The element \( \xi \in \mathfrak{g} \) is called regular if the adjoint orbit \( O_G(\xi) \) of the \( G \)-action has a maximal dimension (equal to \( \dim G - \text{rank} G \)), or equivalently if \( \mathfrak{g}_\xi \) is an \( r \)-dimensional commutative subalgebra of \( \mathfrak{g} \). Then \( \mathfrak{g}_\xi = \{ \nabla p(\xi), \ p \in \mathbb{R}[\mathfrak{g}]^G \} \). The element \( \xi \in \mathfrak{g} \) is called singular if it is not regular.

**Theorem 8.** Suppose that the horizontal space \( \mathfrak{v} \subset \mathfrak{g} \) contains a regular element of the Lie algebra \( \mathfrak{g} \). Then the geodesic flow of the metric \( ds^2_{\mathfrak{g}, sub} \) on the bi-quotient \( G/U \) is completely integrable. Moreover, the phase space \( T(G/U) \) is almost everywhere foliated by invariant isotropic tori of dimension

\[
\text{rank} G - \min_{\xi \in \mathfrak{v}} \dim u_\xi,
\]

where \( u_\xi = \{ \eta \in u, \ [\eta, \xi] = 0 \} \subset \mathfrak{g}_\xi \).
Proof. Since $F$ consists of analytic functions, if one horizontal vector belongs to $\text{reg} \, TG$, then this property will hold for a general horizontal vector as well. Furthermore, it is clear that the regular elements of $\mathfrak{g}$ belong to $\text{reg} \, TG$. Therefore, from theorem 4 we obtain the integrability of the geodesic flow.

To complete the proof, we have to find the dimension of the space $\mathfrak{g}$. Let $\xi \in \mathfrak{v} \subset \mathfrak{g} \cong T_e G$ be generic. Then the tangent space $T_\xi \mathbb{T}^r$ to the invariant torus containing $\xi$ can be naturally identified with $\mathfrak{g}_\xi$ and the space $S_\xi = T_\xi (U \cdot \xi) \cap T_\xi \mathbb{T}^r$ can be naturally identified with $u_\xi$. By (4), it follows that the $r$–dimensional invariant tori are reduced to $(r - \dim u_\xi)$–dimensional invariant tori. □

Without using submersions, the complete integrability of the geodesic flows of metrics $ds_{0,\text{sub}}^2$ on homogeneous spaces $G/H$ and bi-quotients $K \backslash G/H$ has been proved in [5] and [9], respectively.

Theorem 8 includes the examples given by Paternain and Spatzier [14] and Bazaikin [1]. Indeed, for the submersion examples studied in [1], Bazaikin already proved the regularity of generic $\xi \in \mathfrak{v}$. Paternain and Spatzier proved the integrability of the geodesic flows for the Eschenburg examples $M^7_{k,l,p,q}$ for the Gromoll and Meyer 7–sphere $\Sigma^7$. Since the set of singular elements has codimension at least 3 in $\mathfrak{g}$, the condition of theorem 7 is automatically satisfied for the Eschenburg examples $SU(3) \to M^7_{k,l,p,q}$. Here $U = U_{k,l,p,q} \cong T_1 \subset T^2 \times T^2$, where $T^2$ is a maximal torus (see [3]). On the other side, one can easily see that independence of the integrals $f_4$ and $f_5$ for the geodesic flow on $\Sigma^7$ (page 361, [14]) gives us independence of the invariant polynomials $tr(\xi^2)$ and $tr(\xi^4)$ at generic $\xi \in \mathfrak{v}$. Thus a generic $\xi \in \mathfrak{v}$ is a regular element of $\mathfrak{sp}(2)$.

The general construction presented here leads to smooth commuting integrals while the commuting integrals given in [14][1] are analytic functions. On the other hand, we proved that the Lagrangian tori are fibered into invariant isotropic tori, so in this sense the system is degenerate. For example, the 14–dimensional manifolds $TM^7_{k,l,p,q}$ are foliated by two-dimensional tori.

Remark 9. The exotic 7–sphere $(\Sigma^7, ds_{0,\text{sub}}^2)$ admits an effective action of $O(2) \times SO(3)$ by isometries [9]. The existence of the non-abelian group of isometries is related to the non-commutative integrability of the geodesic flow. Namely, suppose that a compact connected Lie group $G$ acts effectively by isometries on the $n$–dimensional Riemannian manifold $(Q, g)$ with moment map $\Phi : T^* Q \to \mathfrak{g}^*$. If the geodesic flow is $\mathbb{T}^n$–dense completely integrable, then in every toroidal domain the functions $\phi_a(x)$, $a \in \mathfrak{g}$ depend only on the action variables and so $\{\phi_a, \phi_b\} = 0$ and the Lie group $G$ is a torus.

In order to obtain manifolds with strictly positive sectional curvature, Eschenburg considered a one–parameter family of left-invariant metrics $ds^2_t$ on $SU(3)$ which are different from the bi-invariant metric $ds^2_0$ [3]. One can prove that the geodesic flows of the metrics $ds^2_t$ are completely integrable and that we can apply theorem 4 to get the integrability of the geodesic flows of the submersion metrics $ds^2_{t,\text{sub}}$ on $M^7_{k,l,p,q}$. A similar multi-dimensional example of bi-quotients with metrics different from $ds^2_{0,\text{sub}}$ and integrable geodesic flows is presented below.

Example 10. 4 There are several constructions of left(right)-invariant metrics on Lie groups with integrable geodesic flows (see [17]). We shall use the following

The example is slightly different than example 10 in the journal version.
construction due to Mishchenko and Fomenko. Let $G$ be a compact connected Lie group, $T \subset G$ a maximal torus, and $\mathfrak{g}$ and $\mathfrak{t}$ the corresponding Lie algebras. Take $a_1, a_2, b_1, b_2 \in \mathfrak{t}$ such that $a_1$ and $a_2$ are regular elements of $\mathfrak{g}$, i.e., $\mathfrak{g}_{a_1} = \mathfrak{g}_{a_2} = \mathfrak{t}$. Let $D_1, D_2 : \mathfrak{t} \to \mathfrak{t}$ be symmetric operators. Denote by $\varphi_1$ and $\varphi_2$ the symmetric operators (called sectional operators [17]) defined according to the orthogonal decomposition: $\mathfrak{g} = \mathfrak{t} + \mathfrak{t}^\perp$:

$$\varphi_i|_\mathfrak{t} = D_i, \quad \varphi_i|_{\mathfrak{t}^\perp} = ad_{a_i}^{-1} ad_{b_i}, \quad i = 1, 2.$$ 

In the case of compact Lie groups, among the sectional operators there are positive definite ones. Then the Hamiltonian functions $H_1$ and $H_2$, obtained from quadratic forms $B_i(\xi, \xi) = \langle \varphi_i \xi, \xi \rangle$, $\xi \in \mathfrak{g}$, $i = 1, 2$ by left and right translations, define left-invariant and right-invariant metrics on $G$ which we shall denote by $ds_1^2$ and $ds_2^2$, respectively. Mishchenko and Fomenko proved that the geodesic flows of $ds_1^2$ and $ds_2^2$ are completely integrable [17]. But now we can take the sum $H_1 + H_2$ which also gives the metric $ds_2^2$ on $G$ with completely integrable geodesic flow [5]. Let $U$ be any subgroup of $T \times T$ such that the action (6) is free. It can be proved that $U$ acts on $(G, ds_2^2)$ by isometries and that a generic horizontal vector of the submersion $G \to G/U$ belongs to $\operatorname{reg} T G$. Thus, by theorem 4, the geodesic flow of the metric $ds_{\text{sub}}^2$ on the bi-quotient $G/U$ is completely integrable. The motion of the system in $T(G/U)$ is more complicated than in the case of the metric $ds_{0, \text{sub}}^2$ and proceeds along tori that have the dimension of $G/U$ in the general case. In particular, when $U = \{e\} \times T$, $ds_{\text{sub}}^2$ is a Riemannian metric on the flag manifold $G/T$. The integrability of the geodesic flow of this metric, was proved in a different way in [6] (theorem 5).

Acknowledgments. I am grateful to Prof. A. V. Bolsinov for the very useful remarks and discussions which helped me look at the problem from a general point of view. This letter was written during my stay at the Mathematisches Institut LMU, München, as a postdoc of the Graduiertenkolleg "Mathematik im Bereich ihrer Wechselwirkung mit der Physik". I would like to thank Ludwig–Maximilians–Universität for the hospitality, Prof. K. Cieliebak for the kind support and the referee for various useful suggestions which improved the exposition of the paper. The research was partially supported by the Serbian Ministry of Science and Technology, Project 1643 – Geometry and Topology of Manifolds and Integrable Dynamical Systems.

Note Added. During the refereeing process of this letter, N. T. Zung’s preprint [18] appeared which contains a result somewhat more general than remark 5 (the manifold $M/G$ is allowed to have singularities), obtained independently.

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