Nature of Quasielectrons and the Continuum of Neutral Bulk Excitations in Laughlin Quantum Hall Fluids

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We construct model wavefunctions for a family of single-quasielectron states supported by the \( \nu = 1/3 \) fractional quantum Hall (FQH) fluid. The charge \( e^* = e/3 \) quasielectron state is identified as a composite of a charge-2e\(^{\ast} \) quasiparticle and a \(-e^{\ast} \) quasihole, orbiting around their common center of charge with relative angular momentum \( nh > 0 \), and corresponds precisely to the “composite fermion” construction based on a filled \( n = 0 \) Landau level plus an extra particle in level \( n > 0 \). An effective three-body model (one 2e\(^{\ast} \) quasiparticle and two \(-e^{\ast} \) quasiholes) is introduced to capture the essential physics of the neutral bulk excitations.

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Elementary excitations of the fractional quantum Hall (FQH) fluids not only are the building blocks of the multi-component FQH states, they also form neutral bound states with an energy gap that characterizes FQH incompressibility. In the Abelian \( \nu = 1/n \) Laughlin FQH states\(^1\), the quasihole states correspond to insertion of an extra flux quanta through the two-dimensional “Hall surface”, and are well-understood. Quasielectron excitations require addition of both electrons and flux quanta, and are in general more complex (i.e. one cannot take quasielectrons as “anti-particles” of quasiholes).

A model wavefunction for a single quasielectron was first proposed by Laughlin\(^1\), and was later improved by Jain.\(^2\) In Jain’s “composite fermion” (CF) picture, the FQH state is described as an integer quantum Hall (IQHE) state of CFs (electron-vortex composites) in an effective magnetic field, where they fill in what Jain has called “A-levels” (ALs)\(^3\). A model CF wavefunction for a lowest-Landau-level (LLL) FQH state is obtained by multiplying the corresponding Slater determinant IQHE state by an even power of the Vandermonde determinant, followed by projection into the LLL. The IQHE state corresponding to the \( n = 1 \) AL quasielectron state of Ref.\(^2\) corresponds to a filled LLL plus a single electron in the second (\( n = 1 \)) Landau level. This description of quasielectron states has been reformulated in the formalisms of conformal field theory\(^4, 5\) and that of Jack polynomials\(^6\); though both these descriptions seem very different from that of Ref.\(^2\), all three constructions turn out to produce identical quasielectron states.

Model quasielectron states where the extra CF is placed in a higher (\( n > 1 \)) AL were recently used in the construction\(^8\) of “CF excitons” consisting of a quasielectron-quasihole bound state\(^8\), in an attempt to explain experimental inelastic light-scattering observations\(^9\) that were interpreted as the splitting of the neutral bulk excitation spectrum at long wavelengths. Placing the extra CF in different ALs leads to different species of exciton states. Most constructions of the quasielectron wavefunctions are implemented in the spherical geometry\(^1\), where many-particle states are characterized by a total angular-momentum quantum number \( L \), and FQH ground states have \( L = 0 \). Charged excitations have \( L \sim O(N_e) \), where \( N_e \) is the number of electrons, while neutral excitations such as excitons have \( L \sim O(1) \), with \( L \geq 2 \). Ref.\(^8\) studied the energies of the “\( n > 1 \) exciton” bands as a function of \( L \) (which gives their energy as a function of momentum in the large-\( N_e \) limit), and found that the exciton energy became independent of \( n \) in the small-\( L \) limit. However, as pointed out here, after the CF projection into the LLL, these exciton states are not linearly independent, and in the \( L = 2 \) and \( L = 3 \) sector their microscopic wavefunctions become identical. This suggests that a better understanding of the nature of these “excitons”, or quasielectron-quasihole pairs, is needed.

In this Letter, we show there is a family of \( e/3 \) quasielectron states that can themselves be interpreted as a composite of a 2e\(^{\ast} \) quasiparticle with a \(-e^{\ast} \) quasihole orbiting around it. The closest allowed orbit corresponds to placement of the extra CF in the \( n = 1 \) AL, and has a much lower energy than \( n > 1 \) states, which are quasi-degenerate in the case of the model “short-range pseudopotential” interactions\(^7\) for which the Laughlin state is the exact ground state. The model wavefunctions of the family of single quasielectron states and the resulting neutral excitations (a charge 2e\(^{\ast} \) quasielectron and two \(-e^{\ast} \) quasiholes) can also be obtained by a universal scheme, described here, based on the Jack polynomial formalism, which naturally explains the counting and linear dependence of various AL quasielectron and exciton states in different angular momentum sectors, seen in numerical studies\(^8\).

We first recall the single quasielectron state for the Laughlin state constructed in \( \ref{8} \), which is identical to the one constructed in the CF picture\(^2\). For convenience we look at the fermionic Laughlin state at filling factor \( \nu = 1/3 \). In the language of the fermionic generalization\(^10\) of the Jack polynomials, where the clustering properties of the many-body wavefunction can
generalize \(1\) into a family of root configurations orbiting around it. The key proposal of this Letter is to be explicitly defined\([11]\), the “root configuration” is

\[
110001001001001\cdots = L = N_e/2.
\] (1)

In the “hierarchy picture”\([7]\), this configuration can be identified as a single “elementary droplet” (11000) of the \(\nu = 2/5\) state “daughter state” in the background of its \(\nu = 1/3\) parent state (with elementary droplet 100). On the sphere, this single quasielectron state is in the total angular momentum sector \(L = N_e/2\), where \(N_e\) is the number of electrons. Each binary number represents an orbital on the sphere, arranged sequentially from the leftmost orbital at the north pole and the rightmost orbital at the south pole. Each “1” represents an occupied orbital, “0” an empty orbital. The root configuration of the ground state at \(\nu = 1/3\) has the clustering property that there is one electron in every three consecutive orbitals. Any deviation from that clustering property in the root configuration indicates the presence of a charge \(e/3\) quasiparticle or a charge \(-e/3\) quasihole, respectively indicated by a bar \(\bar{1}\) or an open circle \(\bar{0}\) below the occupation number in \(1\).

The many-body wavefunction \(|\psi^{\text{qe}}_{N_e/2}\rangle\) is given by\([12]\)

\[
|\psi^{\text{qe}}_{N_e/2}\rangle = \sum \alpha_i|\psi^e_i\rangle + \beta\delta_{\psi^1_1|\psi^j}\rangle
\] (2)

where \(|\psi^e_i\rangle\) is the set of all basis squeezed from the root configuration\([11]\) in \(1\) with at most one of the first two orbitals occupied; \(|\psi^j\rangle\) is the state corresponding to fermionic Jack polynomial \(J^{2}_{\alpha_1,001,0101,001}\cdots\) with an “admissible” root configuration obtained by annihilating the first two electrons (in orbitals labeled 0 and 1) of the root configuration of \(1\): the coefficients \(\alpha_i\) and \(\beta\) can be uniquely fixed\([12]\) by imposing the highest weight condition \(L^+|\psi^{\text{qe}}_{N_e/2}\rangle = 0\), where \(L^+\) is the raising operator of the total \(L_z\) on the sphere. It is noteworthy that for a basis derived in this way from such a root configuration, application of the highest-weight condition leads to a highly-overdetermined set of linear equations, which nevertheless have a (unique) solution.

Physically, the excess charge of the quasielectron state is concentrated at the north pole, and the state relaxes to the ground state far away from the north pole, as manifested by the Jack polynomial in \(2\), which is a zero energy state of the short-range pseudopotential model Hamiltonian. From the positions of the underscores and circles in \(1\) one can infer that the \(e/3\) quasielectron state is a composite of a \(2e/3\) quasiparticle (we here reserve the term “quasielectron” for excitations with opposite charge to the quasihole) at the north pole, with a \(-e/3\) quasihole orbiting around it. The key proposal of this Letter is to generalize \(1\) into a family of root configurations

\[
\begin{align*}
110001001001001 \cdots & = L = N_e/2 \\
11001001001001001 \cdots & = L = N_e/2 + 1 \\
11001001001001001 \cdots & = L = N_e/2 + 2 \\
11001001001001001 \cdots & = L = N_e/2 + 3 \quad \text{etc.} 
\end{align*}
\] (3)

Above, we list the first four states of the family. The same scheme used to define states with root configuration \(2\) uniquely defines model wavefunctions for each of the root configurations in \(3\). We verified that the quasielectron state in the \(L = N_e/2 + (n-1)\) sector with \(n \geq 1\) is identical to the quasielectron state constructed in the CF picture (with the lowest-Landau-level projection implemented exactly, using symbolic computer algebra for \(N_e \leq 6\)), when the extra CF is in the \(n\)th \(\Lambda L\) (the zero-energy ground state is given by the filled \(n = 0\) \(\Lambda L\)). The root configurations give insight into the nature of the CF construction using higher \(\Lambda Ls\): the orbit of the quasihole around the \(2e/3\) quasiparticle decreases with increasing \(\Lambda L\), defining the effective size of the quasielectron. For a finite system of \(N_e\) electrons, the total number of single quasielectron states is \(N_e - 1\). In the thermodynamic limit where \(N_e \to \infty\), for any finite \(n > 0\), a net excess charge \(e/3\) remains near the origin.

We now evaluate the variational energies of this family of single quasielectron states. From Fig. \(1\), one can see that for both the short-range \(V_1\) pseudopotential interaction and the more realistic Coulomb interaction, there is a large energy gap between the \(L = N_e/2\) quasielectron and other quasielectron states in the family; the \(L = N_e/2 + (n-1)\) states with \(n > 1\) are quasi-degenerate with similar energies, implying a small binding energy between the \(2e/3\) quasiparticle and the \(-e/3\) quasihole, once they are a few units of magnetic length away from each other. Thus from a dynamical point of view, for the Laughlin state there are effectively two basic excitations with the same sign of charge as the electron: (a) a \(e/3\) quasielectron composite state made of a \(2e/3\) quasiparticle and a tightly bound \((n = 1)\) quasihole; (b) a bare \(2e/3\) quasiparticle.

With the newly-defined family of quasielectrons, we now turn our attention to the neutral bulk excitations of

![FIG. 1. Variational energies of the quasielectron states, evaluated with the \(V_1\) pseudopotential Hamiltonian (open circles, left axis, units \(V_1\)) and the Coulomb interaction (solid circles, right axis, units \(e^2/4\pi\varepsilon_0\varepsilon_B\)). The system size is 12 electrons in 33 orbitals. The index \(n\) (the “\(\Lambda\)-level” index in the CF picture) is related to the total angular momentum \(L\) by \(n - 1 = L - N_e/2\), where \(N_e\) is the number of particles.](image-url)
the FQHE at \( \nu = 1/3 \). Since the neutral excitations are made of quasielectron-quasihole pairs, different types of single quasielectron states can lead to different types of single-pair neutral excitations. The root configurations of the neutral bulk excitations (see table) are obtained from those in (3) by inserting an additional empty orbital while keeping intact the \( 2e/3 \) quasiparticle (the \( m = 1 \) pair \( \ldots \) at the north pole). The number of states in the \( L = N \) sector is \([N/2]\), the greatest integer less or equal to \( N/2 \). The first root configuration in each \( L \) sector corresponds to the magneto-roton model \(^{12} \). The many-body wavefunctions of the additional neutral modes can be constructed just like the quasielectron states: we found the resulting wavefunctions are equivalent to the “CF excitons” of Ref.\(^{3} \). Note for \( L = 2 \) and \( L = 3 \), each sector contains only one state corresponding to the magneto-roton mode. Thus, in these sectors, apparently-different “CF exciton” states become identical after projection (perhaps explaining why the Monte Carlo CF calculations of Ref.\(^{3} \) found their variational energies appeared to coincide in this limit).

We also diagonalized the two-body-interaction Hamiltonian within the subspace spanned by these neutral excitation modes, for both short-range \( (V_1) \) pseudopotential\(^{12} \) and Coulomb interactions. A clear separation of the magneto-roton mode and the continuum of other neutral excitations is seen in Fig.\(^{2} \), though all the modes seem to merge together into the continuum in the long wavelength limit. We want to emphasize that states in this continuum consists of one quasielectron-quasihole pair \(^{12} \), distinguishing them from the multi-roton continuum starting at an energy double the roton-minimum gap \(^{12} \), which involves two quasielectrons. These additional neutral excitations are buried in the multi-roton continuum, and in the thermodynamic limit the number of single-pair neutral excitations is also macroscopic.

To further understand the property of the single-pair neutral excitations for the Laughlin state, especially in the long wavelength limit, we first look at quasielectrons in \(^{3} \). Denoting the guiding center coordinate of the quasiparticle with charge \( 2e^* \) by \( R_0^q \) and that of the quasihole with charge \( -e^* \) by \( R_1^r \), where \( a = x, y \) is the spatial index, we have the following commutation relations

\[
[R_0^q, R_1^r] = -i\epsilon^{ab}t_B^2, \quad [R_1^q, R_1^r] = i\epsilon^{ab}t_B^2. \quad (4)
\]

Here \( \epsilon^{ab} \) is the 2D antisymmetrization symbol and \( t_B^2 = \hbar/[e^*B] \) is the effective magnetic length. Assuming rotational invariance, the two-body interaction Hamiltonian is given by

\[
\mathcal{H}_{ph} = \int \frac{d^2 q}{2\pi} U(|q|) e^{i\nu (R_0^q - R_1^r)}, \quad (5)
\]

where the two-body interaction has a “pseudopotential expansion” in terms of Laguerre polynomials:

\[
U(|q|) = \frac{1}{2} \sum_n E_n L_n \left( \frac{1}{4} |q_B^r|^2 \right) e^{-\frac{1}{2} |q_B^r|^2}. \quad (6)
\]

From Fig.\(^{1} \) all pseudopotentials are close to zero except for \( E_1 \). Without loss of generality we now set \( E_1 = -1 \), \( E_{n>1} = 0 \). To project out the \( n = 0 \) quasielectron state (such a state would correspond to the invalid root configuration 1010010010001 \ldots , where the highest-weight charge \( 2e^* \) quasiparticle 11 has been destroyed), we also require \( E_0 = \infty \). This set of pseudopotentials \( E_n \) in \(^{5} \) describes qualitatively the energetics of the quasielectrons in \(^{3} \).

Next, denoting the guiding center coordinates of the two quasiholes by \( R_1^q, R_2^q \), the two-body interaction Hamiltonian for the quasiholes is given by

\[
\mathcal{H}_{hh} = \int \frac{d^2 q}{2\pi} V(|q|) e^{i\nu (R_1^q - R_2^q)}. \quad (7)
\]

With rotational invariance one can also expand the two-body interaction \( V(|q|) \) in terms of pseudopotentials \( V_n \)

\[
V(|q|) = 2 \sum_n V_n L_n (|q_B^r|^2) e^{-\frac{1}{2} |q_B^r|^2}. \quad (8)
\]
Combining (5) and (7) we can write down an effective model of the single-pair neutral excitations with one quasiparticle with guiding center coordinates \( R_0 \) and two quasiholes with \( R_1^0, R_2^0 \) respectively. The Hamiltonian is given by

\[
\mathcal{H}_{\text{phh}} = \int \frac{d^2q}{2\pi} U(|q|) \left( e^{iqa_\mathcal{R}(R_0^0 - R_1^0)} + e^{iqa_\mathcal{R}(R_0^0 - R_2^0)} \right) + \int \frac{d^2q}{2\pi} V(|q|) e^{iqa_\mathcal{R}(R_1^0 - R_2^0)}. \tag{9}
\]

The electric dipole moment carried by the neutral excitation is \( e^* R_0 \), where \( R_0 = 2R_0^0 - (R_1^0 + R_2^0) \), and \([R_0^0, R_2^0] = 0, [R_0^0, \mathcal{H}_{\text{phh}}] = 0\). The residual dynamical degree of freedom is \( R_{12}^0 = R_1^0 - R_2^0 \), where \( R_{12}^0 + iR_{12}^0 = 2l_B a \), with \([a, a^\dagger] = 1\). After some algebra, (9) can be written as an effective one-body Hamiltonian

\[
\mathcal{H}_{\text{phh}}(\mathbf{R}) = \sum_n V_n |\psi_n^0\rangle \langle \psi_n^0| + \sum_n \pm E_n |\psi_n^+\rangle \langle \psi_n^+|, \tag{10}
\]

where \( a|\psi_n^0\rangle = 0, a^\dagger|\psi_n^0\rangle = \sqrt{(n + 1)}|\psi_{n+1}^0\rangle \), and \( |\psi_n^\pm\rangle = \exp(\pm i(\pm \sqrt{n})a)|\psi_n^0\rangle \), with complex \( k = (R^0 + iR^0)/l_B^\epsilon \).

Note that the momentum \( \hbar \mathbf{K} \) carried by the neutral mode is proportional to its dipole moment, with a magnitude \( K \) given by \( Kl_B = |k| \). As in Ref. [7], our formalism requires the state to be symmetric in the quasihole coordinates (even under \( a \to -a \)), so eigenstates of (10) have the form

\[
|\psi(k)\rangle = \sum_{n=0}^{\infty} A_n |\psi_{2n}^0\rangle + B_n \left( |\psi_{2n}^+\rangle + |\psi_{2n}^-\rangle \right) + C_n \left( |\psi_{2n+1}^+\rangle - |\psi_{2n+1}^-\rangle \right). \tag{11}
\]

In general, the model (10) is hard to solve analytically. It is instructive to look at the simplest case where \( E_0 = \infty, E_1 = -1, E_{n>1} = V_n = 0 \). For the pseudopotential Hamiltonian where the Laughlin state is the exact ground state, all quasihole states have zero energy; thus quasiholes do not interact with each quasiparticle, at least when they are far away from quasiparticles. This, together with the quasi-degenerate energies of the \( n > 1 \) quasielectrons in Fig. 1, suggests that the simplest case should capture most of the physics of the realistic systems. A straightforward computation shows there is only one non-zero-energy band, with an energy dispersion (relative to the sum of the energies of an isolated \( 2e/3 \) quasiparticle and two isolated quasiholes)

\[
E(k) = S_k \left( 1 - 4|k|^2/(1 + S_k) \right) - 1, \tag{12}
\]

where \( S_k \equiv |\langle \psi_0^+ | \psi_0^0 \rangle| = \exp(-2|k|^2) \). This band, shown in Fig. 3a, qualitatively resembles the familiar magnetoroton mode [13], although its minimum at \(|k| \approx 0.96\) predicts a too-small value \( Kl_B = 0.96(\epsilon^*/\epsilon) = 0.55 \) (instead of \( Kl_B \approx 2 \)) for the \( \nu = 1/3 \) roton minimum, and has a shape that seems too pronounced as compared to Fig. 3(a).

This discrepancy could be mainly because at the roton minimum, the separation between the quasiparticle and the quasihole is only on the order of the magnetic length, and these localized charged objects with finite sizes are less well defined. There is a non-dispersing continuum, qualitatively agreeing with the continuum of neutral excitations in Fig 2a). When further pseudopotentials \( E_{n>1} \) and \( V_n \) are included, additional bands split off from the continuum: see insets in Fig. 3. At \( k = 0 \), for each integer \( m > 0 \), there is a band with energy \( V_{2m} + 2E_{2m} \); in general the energies of different bands split in the long wavelength limit. While the simplest form of the “toy model” (10) does not quantitatively fit the collective-mode spectrum, it appears to describe some of the essential physics of the neutral excitations deriving from the composite structure of the quasielectron.

In conclusion, we have exposed an internal structure of the charge-\( e^* \) “quasielectron” of the Laughlin state, revealing it as a composite of a charge 2\( e^* \) quasiparticle and a charge \( -e^* \) quasihole. We identified the internal orbital angular-momentum of the composite with the “A-level” index in Jain’s composite-fermion construction [3], and constructed a simple model (10) capturing some of its features. This structure may be relevant to the more recent experimental results reported in Ref. [14].

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