Functional integration on
two dimensional Regge geometries

Pietro Menotti and Pier Paolo Peirano
Dipartimento di Fisica dell’Università, Pisa 56100, Italy and
INFN, Sezione di Pisa

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Abstract

By adopting the standard definition of diffeomorphisms for a Regge surface we give an exact expression of the Liouville action both for the sphere and the torus topology in the discretized case. The results are obtained in a general way by choosing the unique self–adjoint extension of the Lichnerowicz operator satisfying the Riemann–Roch relation. We also give the explicit form of the integration measure for the conformal factor. For the sphere topology the theory is exactly invariant under the $SL(2,C)$ transformations, while for the torus topology we have exact translational and modular invariance. In the continuum limit the results flow into the well known expressions.

1 Introduction

Regge discretized approach to gravity consists in replacing regular geometries with piece–wise flat ones with the curvature confined to $D − 2$ dimensional simplices. Apart from applications to classical gravity such an approach has been considered as a way to regulate quantum gravity. It has also been used as the scheme suitable to perform numerical simulations of quantum gravity. In two dimensions there is the possibility of comparing the results of Regge gravity to those of the continuum theory.

Most of the discussion on Regge gravity at the quantum level has been centered on the integration measure, where the most popular choices have been of the type $\prod_i dl_i f(l_i)$, being $l_i$ the bone lengths. On the other hand the continuum approach, which was developed from the analogy with gauge theories, starts from the unique ultra-local, diff–invariant measure, i.e. the De Witt measure. Given the infinite volume of the diffeomorphism group a gauge fixing and the evaluation of the related Faddeev–Popov (F.P.) determinant are required. In particular in $D = 2$ this is the only source of a non trivial dynamics, being the Einstein action a topological invariant.

In the present paper we shall maintain for the diffeomorphisms the same meaning as on the continuum. Thus the only difference between the continuum and the Regge approach...
will be that in the last case one restricts the functional integration to the piece-wise flat surfaces.

There is a difference between such an approach to gravity and the usual lattice discretization of gauge theories. In fact in the last case after discretizing the space-time the action becomes invariant under a compact group. Thus the imposition of a gauge fixing is not necessary as one can factorize a finite gauge volume. On the other hand in order to keep the usual diff-invariance we have to maintain the description of space-time by the manifold structure. Being the symmetry group non compact the gauge fixing turns out to be necessary.

Given a Regge surface there are many metrics $g_{\mu\nu}$ that describe such a geometry; the metric has to be given after having equipped our space-time with a manifold structure, i.e. charts with transition function which are independent of the metric. For example the metric
\begin{equation}
   g^{(1)}_{\mu\nu} = \begin{pmatrix}
   l_1^2 & \frac{1}{2}(l_1^2 + l_2^2 - l_3^2) \\
   \frac{1}{2}(l_1^2 + l_2^2 - l_3^2) & l_2^2
   \end{pmatrix}
\end{equation}
defined on an open set which includes a triangular simplex with link lengths $l_1, l_2, l_3$ and the analogous metric $g^{(2)}_{\mu\nu}$ defined on an open set which covers the adjacent triangular simplex with link lengths $l_2, l_4, l_5$ are not compatible on the intersection region with $l$–independent transition functions.

As envisaged by Jevicki and Ninomija we shall maintain the De Witt metric as the starting point, impose a gauge fixing and compute the associated integration measure. In $D = 2$ by far the simplest gauge fixing is the conformal one, as any metric can be described modulo diffeomorphisms by $g_{\mu\nu} = \hat{g}_{\mu\nu} e^{2\sigma}$ where $\hat{g}_{\mu\nu}$ is a background metric depending on the Teichmüller parameters. After imposing the conformal gauge fixing the functional integral becomes an integral over the conformal factor and on the Teichmüller parameters. Within our framework one integrates only over those conformal factors which describe a Regge surface. Thus the problem reduces to compute the quantities which appear in the continuum partition function in the case of a Regge geometry, while the functional integration becomes an integral on a finite number of parameters which describe such
surfaces. As explained in sect. 3 these parameters will be the positions of the conical singularities on a coordinate plane, the associated conical deficits and an overall scale factor.

The first term to be computed is the functional determinant of the conformal Lichnerowicz operator, i.e. the F.P. determinant. What is interesting is that with the above choice of parameters, describing all the Regge geometries, such a quantity can be obtained exactly in closed form. This will be performed by extending the technique developed by Aurell and Salomonson [9] for the computation of the functional determinant of the scalar Laplace–Beltrami operator to the Lichnerowicz operator that acts on vector fields. Such an extension in not straightforward [10, 11] because a simple minded translation of the formulas for the scalar case gives rise to a wrong result. The reason is that one has to find out which are the boundary conditions on the vector field (and on the related traceless symmetric tensor field) at the singularity suitable for a compact surface. In reference [10] the problem of finding the correct boundary conditions has been solved by regularizing the conical singularities by means of a smooth geometry and then taking the limit of vanishing regulator.

Here the problem will be addressed in a completely general way by looking to all possible self–adjoint extensions of the Lichnerowicz operator and of the related operator which acts on the traceless tensor field. The result is that for $\frac{1}{2} < \alpha < 2$ ($\alpha$ is the opening of the cone with $\alpha = 1$ for the plane) the imposition of the Riemann–Roch relation for a compact manifold without boundary, selects a well defined self–adjoint extension which coincides with the one previously found with the regularization method. Outside of the interval $(\frac{1}{2}, 2)$ is not possible to satisfy the Riemann–Roch relation within the realm of $L^2$ functions. From the technical viewpoint the calculation of the determinant is performed similarly as in the continuum, i.e. by taking first a variation of the conformal factor and then integrating back the result. To this purpose it is necessary to compute the small time behavior of the heat kernel of the Lichnerowicz operator and of the associated operator that acts on the traceless symmetric tensors.

We have examined separately the topologies of the sphere and of the torus. In both
cases in the continuum limit the results go over to the well known expressions.

In the case of the sphere we have explicit invariance under the group $SL(2, C)$ which corresponds to the six conformal Killing vectors of the sphere. For the torus we have invariance under translations.

The expression of the integration measure for the conformal factor flows directly from the De Witt continuum definition. It is given by the determinant of a finite dimensional matrix whose elements are given by integrals which appeared in the old conformal theory $[^{13}]$. One can easily derive the invariance properties of such a measure. For the sphere topology the measure turns out to be invariant under $SL(2, C)$ which combined with the invariance of the action under the same group renders the whole theory $SL(2, C)$ invariant.

The same thing happens for the torus with regard to translations. In addition here the transformation properties of the action under modular transformations combined with those of the measure, give rise to a modular invariant integral of the Liouville action over the conformal factor, thus assuring the modular invariance of the partition function. This procedure provides a non formal proof of the modular invariance of the theory.

It appears a notable advantage of the geometric nature of the Regge regulator the fact that such symmetries, like $SL(2, C)$ for the sphere topology and translation and modular invariance for the torus topology, are exactly preserved at the discretized level. Obviously the Regge surface can be equivalently described by the conventional method of the bone lengths (in fact it is easy to check that one has the same number of physical degrees of freedom); but the choice we adopted $[^{12}]$ appears more suitable for the evaluation of the functional integral and for the study of its symmetries.

While with such an approach one obtains an action that in the continuum limit flows in the usual continuum result, it is very hard to understand how something similar could be obtained using the measure $\prod_i dl_i f(l_i)$. In fact in $D = 2$ the Einstein action is a topological invariant and thus all the dynamics resides in the triangular inequalities among the bone lengths. On the other hand for small variations of the geometry the Liouville action in the continuum approach can be approximately computed with one loop calculation. But
at the perturbative level triangular inequalities do not play any role and thus one does not see how a Liouville action could emerge.

The paper is structured as follows. In sect. 2 we discuss the self–adjoint extensions of the conformal Lichnerowicz operator and the related heat kernels; then we impose on them the restrictions given by the Riemann–Roch relation. In sect. 3 we apply the above general results to the sphere topology and give the explicit form of the Liouville action for a Regge surface. Then we derive the integration measure of the conformal factor and prove the \( SL(2,C) \) invariance of the theory. In sect. 4 we give the Liouville action for the torus topology and prove the invariance of the functional integral under modular transformations. In sect. 5 we examine briefly the relation of the smooth Liouville action to our discretized one. In appendix A we give a concise summary of the continuum gauge fixing procedure, to which we often refer in the text; in appendix B we give the asymptotic expansion of the trace of the heat kernels; in appendix C we report the regulator procedure for extracting the boundary conditions at the singularities and in appendix D we write the integral representation of the heat kernels previously discussed.

2 Self–adjoint extension of the Lichnerowicz operator

We need to compute for a Regge manifold

\[
\frac{\det'(P^\dagger P)}{\det(\phi_a, \phi_b) \det(\psi_k, \psi_l)}
\]

(see eq.(125)) where the operator \( P \) takes from the 2 dimensional vector field \( \xi_\mu \) to the traceless symmetric tensor field \( h_{\mu\nu} \)

\[
h_{\mu\nu} = \frac{1}{2} (\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \delta_{\mu\nu} \nabla \xi) = (P \xi)_{\mu\nu}.
\]

Following Alvarez [6] we go over to the complex formalism where \( \omega = \omega^1 + i\omega^2 \), \( \xi(\omega, \bar{\omega}) = \xi_\omega = \xi_1 + i\xi_2 \) and \( h(\omega, \bar{\omega}) = h_{\omega\omega} = h_{11} + ih_{12} \). The two spaces \( \xi \) and \( h \) are equipped with
the corresponding invariant metrics, which in the conformal gauge \( g = e^{2\sigma} d\omega \otimes d\bar{\omega} \) take the form

\[
(\xi^{(1)}, \xi^{(2)}) = \int d^{2}\omega \bar{\xi}^{(1)} \xi^{(2)}
\]

and

\[
(h^{(1)}, h^{(2)}) = \int d^{2}\omega e^{-2\sigma} \bar{h}^{(1)} h^{(2)}.
\]

It is well known that \( P \) acts diagonally on the column vector \((\xi, \bar{\omega}, \xi, \omega)\) by transforming it into \((h, \bar{\omega}, h, \omega)\)

\[
P \begin{pmatrix} \xi \\ \bar{\omega} \\ \xi \\ \omega \end{pmatrix} = \begin{pmatrix} L & 0 \\ 0 & \bar{L} \end{pmatrix} \begin{pmatrix} \xi \\ \bar{\omega} \\ \xi \\ \omega \end{pmatrix} = \begin{pmatrix} h \\ \bar{\omega} \\ h \\ \omega \end{pmatrix}
\]

and

\[
P^{\dagger} \begin{pmatrix} h \\ \bar{\omega} \\ h \\ \omega \end{pmatrix} = \begin{pmatrix} L^{\dagger} & 0 \\ 0 & \bar{L}^{\dagger} \end{pmatrix} \begin{pmatrix} h \\ \bar{\omega} \\ h \\ \omega \end{pmatrix} = \begin{pmatrix} \xi \\ \bar{\omega} \\ \xi \\ \omega \end{pmatrix}.
\]

In the conformal gauge \( L \) and \( L^{\dagger} \) assume the form

\[
L = e^{2\sigma} \frac{\partial}{\partial \omega} e^{-2\sigma} \quad \text{and} \quad L^{\dagger} = -e^{-2\sigma} \frac{\partial}{\partial \omega}.
\]

From eqs. (3), (4) is clear that \( \det'(P^{\dagger}P) = [\det'(L^{\dagger}L)]^{2} \) and the determinant of \( L^{\dagger}L \)

is defined through the \( Z \)-function technique, i.e. \(- \log \det'(L^{\dagger}L) = \dot{Z}_{K}(0) \equiv \left. \frac{dZ_{K}(s)}{ds} \right|_{s=0} \).

\( Z_{K}(s) \) is given by the heat kernel of \( L^{\dagger}L \) as follows

\[
Z_{K}(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dt \, t^{s-1} \text{Tr} \left( e^{-tL^{\dagger}L} \right)
\]

where the prime means exclusion of the zero modes. The value of \( \det'(L^{\dagger}L) \) can be written as

\[
- \log(\det'(L^{\dagger}L)) = \dot{Z}_{K}(0) = \gamma_{E} Z_{K}(0) + \text{Finite}_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{dt}{t} \text{Tr} \left( e^{-tL^{\dagger}L} \right).
\]

The standard procedure is to compute the change of \( \dot{Z}_{K}(0) \) under a variation of the conformal factor

\[
- \delta \log \left[ \frac{\det'(L^{\dagger}L)}{\det(\Phi_{a}, \Phi_{b}) \det(\Psi_{l}, \Psi_{m})} \right] = \gamma_{E} \delta c_{0}^{K} + \text{Finite}_{\epsilon \to 0} \text{Tr} \left[ 4\delta \sigma \mathcal{K}(\epsilon) - 2\delta \sigma \mathcal{H}(\epsilon) \right],
\]
and then integrating back the result. In the previous equation \( K = L^* L \), \( H = LL^* \), \( K \) is the heat kernel of \( K \) and \( H \) is the heat kernel of \( H \); \( c^K_0 \) is the constant term in the asymptotic expansion of the trace of the heat kernel \( \mathcal{K}(t) \) and is related to \( Z_K(0) \) by

\[
c^K_0 = Z_K(0) + \dim(\text{Ker } K).
\]

(12)

\( \Phi_a \) and \( \Psi_i \) are the zero modes of \( K \) and \( H \) respectively. The central point in the evaluation of the r.h.s. of eq.(11) will be the knowledge of \( c^K_0 \) and of \( K(t) \) and \( H(t) \) on the Regge manifold for small \( t \). As is well known such quantities are local in nature and thus we shall start by computing them on a single cone.

### 2.1 Heat kernels \( K(\epsilon) \) and \( H(\epsilon) \) on a cone

In the complex \( \omega \) plane a cone is described by the conformal metric \( e^{2\sigma} = c^2 (\omega \bar{\omega})^{\alpha - 1} \), with \( 2\pi \alpha \) the angular aperture and \( c \) a normalization constant. In the polar representation \( \omega = re^{i\phi} \), \( L \) and \( L^* \) are given by

\[
L = \frac{1}{2} e^{2\sigma} e^{i\phi} \left( \frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \phi} \right) e^{-2\sigma}
\]

and

\[
L^* = -\frac{1}{2} e^{-2\sigma} e^{-i\phi} \left( \frac{\partial}{\partial r} - \frac{i}{r} \frac{\partial}{\partial \phi} \right).
\]

(13)

(14)

By decomposing \( \xi \) in angular harmonics \( \xi = \sum_{m=-\infty}^{\infty} e^{im\phi} f_m(r) \), the eigenvalue equation \( L^* L(e^{im\phi} f_m(r)) = \lambda^2 e^{im\phi} f_m(r) \) becomes

\[
-\frac{c^{-2}}{4} r^{2(1-\alpha)} \left[ \frac{d^2}{dr^2} + \left( \frac{1}{r} \right) \frac{d}{dr} - \frac{1}{r^2} (m^2 + 2(\alpha - 1)m) \right] f_m = \lambda^2 f_m
\]

(15)

which is solved by

\[
f_m = r^{\alpha - 1} J_{\pm \nu} \left( \frac{2c\lambda}{\alpha} r^\alpha \right)
\]

with \( \nu = \frac{m+\alpha-1}{\alpha} \), or linear combination thereof.

The condition of \( L^2(2dr) \) integrability at the origin dictates the choice

\[
f_m = \begin{cases} 
  r^{\alpha - 1} J_{\nu} \left( \frac{2c\lambda}{\alpha} r^\alpha \right) & \text{for } \nu > -1 \\
  r^{\alpha - 1} J_{-\nu} \left( \frac{2c\lambda}{\alpha} r^\alpha \right) & \text{for } \nu < 1
\end{cases}
\]

(16)
If only one of the two inequalities is satisfied there is no ambiguity, else we have that any linear combination of $J_\nu$ and $J_{-\nu}$ is $L^2$-integrable. This gives rise to the problem of the choice of the domain of self–adjointness of the operator $L^\dagger L$. Moreover as $L^\dagger L$ originates from $(L\xi, L\xi)$ we shall require $L^\dagger L$ to be really the product of an operator $L$ and of its adjoint $L^\dagger$, with the ensuing restrictions on the domains $D(L)$ and $D(L^\dagger)$.

We shall start by looking at the domain of self–adjointness of $(L^\dagger L)_m$ ($L^\dagger L$ restricted to the partial wave $m$) with $-1 < \frac{m+n-1}{\alpha} < 1$. First we define $K = (L^\dagger L)_m$ as a closed symmetric operator. The domain of $K$, $D(K)$ will be defined by the functions $\xi \in L^2(r\,dr)$ with two derivatives and such that

$$
\lim_{r \to 0} r^{-2\alpha + 1 + \epsilon} \xi = 0, \quad \lim_{r \to 0} r^{-2\alpha + 2 + \epsilon} \xi' = 0 \quad (17)
$$

for any $\epsilon > 0$.

We want to find $D(K^\dagger)$. We have

$$
\int r\,dr \, \bar{\eta}(L^\dagger L)_m \xi
= -\frac{1}{4} \int r\,dr \left[e^{-2\sigma} \left(\frac{\partial}{\partial r} + \frac{m+1}{r}\right)e^{2\sigma} - \frac{m}{r} \right] e^{-2\sigma} \eta \xi + \lim_{r \to 0} [r(\bar{\eta} \frac{\partial}{\partial r} (e^{-2\sigma} \eta) - \frac{\partial}{\partial r} (e^{-2\sigma} \bar{\eta}) \xi)].
$$

$D(K^\dagger)$ is given by the set of the $L^2(r\,dr)$ functions $\eta$ with two derivatives, and for which

$$
\lim_{r \to 0} r e^{-2\sigma} (\bar{\eta} \xi' - \bar{\eta}' \xi) = 0, \quad \forall \xi \in D(K). \quad (19)
$$

From the definition of $D(K)$ it follows that $D(K^\dagger) \supset D(K)$. In particular $D(K^\dagger)$ includes the $L^2(r\,dr)$ functions with two derivatives which in a neighborhood of the origin are equal to $r^{-1+\epsilon}$ with $\epsilon > 0$ and this implies according to eq.(17) that $D((K^\dagger)^\dagger) = D(K)$, in other words $K$ defined on $D(K)$ is a closed operator.

$D(K^\dagger)$ properly contains $D(K)$ and thus $K$ is not self–adjoint and now we look for all possible self–adjoint extensions of it. As $K$ defined on $D(K)$ is a closed operator we can apply the standard theory of self–adjoint extensions. We must look for the $L^2(r\,dr)$ solutions of $(K^\dagger \pm i)\xi = 0$. As $K$ is a real operator the dimension of Ker($K^\dagger - i$) equals the dimension of Ker($K^\dagger + i$), which assures that there exist self–adjoint extensions of $K$. 

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We have only one $L^2(r \, dr)$ solution of $(K^\dagger - i)\xi_0 = 0$ and it is given by
\[
\xi_0 = r^{\alpha-1} \left( J_{-\nu}(\frac{2c}{\alpha}e^{i\frac{\pi}{4}}r^\alpha) - e^{-i\nu\pi} J_{\nu}(\frac{2c}{\alpha}e^{i\frac{\pi}{4}}r^\alpha) \right) = i \sin(\nu \pi) r^{\alpha-1} H^{(1)}(\frac{2c}{\alpha}e^{i\frac{\pi}{4}}r^\alpha)
\] (20)
while the solution of $(K^\dagger + i)\xi = 0$ is obviously given by its complex conjugate.

The general theory of the self-adjoint extensions of a symmetric operator \cite{14} tell us that we have as many extensions as the unitary maps between $\text{Ker}(K^\dagger - i)$ and $\text{Ker}(K^\dagger + i)$. Therefore in principle, if for a given $\alpha$ there are $l$ values of $m$ for which $-1 < \nu < 1$, then we have an $l^2$-dimensional family of self-adjoint extensions.

On the other hand if we want to preserve invariance under rotations around the tip of the cone, we can mix only solution of $\text{Ker}(K^\dagger - i)$ and $\text{Ker}(K^\dagger + i)$ with the same angular momentum. Thus we are left only with an $l$-dimensional family of self-adjoint extensions.

The domain of such self-adjoint extension is given for each partial wave by $D(K) \oplus (e^{i\theta} \xi_0 + e^{-i\theta} \bar{\xi}_0)$. This is completely equivalent to add to the initial domain $D(K)$ the $L^2(r \, dr)$ functions which at the origin behave like $r^{\alpha-1}(ar^{-\nu} + br^{\nu})$ with $\frac{\alpha}{2}$ real and fixed. One can easily check that despite having introduced a singular behavior at the origin, in passing form $(\eta, K\xi)$ to $(K\eta, \xi)$ the integrations by parts are carried through without leaving boundary terms.

We now impose that $K = L^\dagger \cdot L$, i.e.
\[
\text{Im}(L) \subseteq D(L^\dagger).
\] (21)
We shall show that eigenfunctions of the type $aJ_{\nu} + bJ_{-\nu}$ violate this requirement. In fact given
\[
\xi_\lambda = a \left( \frac{\alpha}{2\lambda c} \right)^\nu r^{\alpha-1} J_{\nu} \left( \frac{2\lambda c}{\alpha} r^\alpha \right) + b \left( \frac{\alpha}{2\lambda c} \right)^{-\nu} r^{\alpha-1} J_{-\nu} \left( \frac{2\lambda c}{\alpha} r^\alpha \right)
\] (22)
we have
\[
(\xi_\lambda, K\xi_\lambda) = \int_0^\infty dr \, r \frac{1}{(L\xi_\lambda)} e^{-2\sigma}(L\xi_\lambda) + \frac{1}{2} ab(1 - \alpha - m).
\] (23)
The requirement $L\xi \in L^2(e^{-2\sigma} r \, dr)$, i.e. that the integral on the r.h.s. of eq.(23) converges, imposes that for $0 < \nu < 1$, $b$ must be taken equal to zero. Thus the values of $\nu$ for
which the choice is ambiguous is $(-1, 0)$. Furthermore from eq.(23) the imposition that
$(\xi_\nu, K\xi_\lambda) = (L\xi_\nu, L\xi_\lambda)$ imposes that either $a$ or $b$ equals zero.

In conclusion the requirement of self–adjointness of $K$, imposes on each partial wave
for which two independent $L^2$ eigenfunction exist, a universal linear combination of them,
while the requirement that $K = L^\dagger \cdot L$ reduces the choice either to the regular or to the
irregular solution. Such a choice exists only for $-1 < \nu < 0$.

We now examine how the imposition of the Riemann–Roch theorem is able to further
restrict the choice of the self–adjoint extension of $K$. The Riemann–Roch theorem for a
2–dimensional compact surface states that

$$\dim (\ker P) - \dim (\ker P^\dagger) = 3\chi$$

being $\chi = 2 - 2h$ the Euler characteristic of the surface of genus $h$. With $L$ and $L^\dagger$ referred
to the whole manifold we have

$$c^K_0 = Z_K(0) + \dim (\ker L) \quad c^H_0 = Z_H(0) + \dim (\ker L^\dagger).$$

We recall that $K = L^\dagger L$, $H = LL^\dagger$ and $c^K_0$, $c^H_0$ are the constant coefficients in the
asymptotic expansions of the trace of the heat kernels $K(t) = e^{-tL^\dagger L}$ and $H(t) = e^{-tLL^\dagger}$
for small $t$.

The spectra of $K$ and $H$ coincide except for the zero modes, hence $Z_K(0) = Z_H(0)$. Since the same treatment can be applied to the operators $\bar{L}$ and $\bar{L}^\dagger$ acting on $\xi_\omega$ and on $h_{\omega\omega}$ with the same results, we have

$$2(c^K_0 - c^H_0) = \dim (\ker (P^\dagger P)) - \dim (\ker (PP^\dagger)) = 3\chi.$$ 

So we must check whether on our singular manifold the relation $2(c^K_0 - c^H_0) = 3\chi$ is
satisfied. Due to the local nature of the coefficients in the asymptotic expansion of the
heat kernel, in order to respect the Riemann-Roch result, we need for a single conical
singularity

$$2(c^K_{0i} - c^H_{0i}) = 3(1 - \alpha_i).$$ 

(27)
We shall see that this requirement selects unambiguously the domain of self-adjointness of $K$ for $\frac{1}{2} < \alpha < 2$.

We note that the choice $\nu_m = \frac{m + \alpha - 1}{\alpha}$ for $m \geq 0$, $\nu_m = -\frac{m + \alpha - 1}{\alpha}$ for $m < 0$, in the interval $\frac{1}{2} < \alpha < 2$ satisfies the discussed requirement of self-adjointness of $K = L^\dagger \cdot L$, as it chooses functions which are $L^2$ and are never mixtures of a regular and a singular solution.

In appendix B we report the calculation of $c^K_0$ for a generic phase shift $\delta$ (i.e. $\nu_m = \frac{m + \delta}{\alpha}$, $m \geq 0$; $\nu_m = -\frac{m - \delta}{\alpha}$, $m < 0$). With the above choice $\delta = \alpha - 1$, we obtain

$$c^K_0 = \frac{1 - \alpha^2}{12\alpha} + \frac{(\alpha - 1)(\alpha - 2)}{2\alpha}.$$  \hspace{1cm} (28)

Using

$$L \left( e^{im\phi} \frac{\alpha - 1}{\alpha} J_{\nu_m} \left( \frac{2\lambda c}{\alpha} r^\alpha \right) \right) = \text{const} e^{i(m+1)\phi} r^{2(\alpha - 1)} J_{\gamma_m} \left( \frac{2\lambda c}{\alpha} r^\alpha \right)$$  \hspace{1cm} (29)

with $\gamma_m = \nu_m + 1$ for $\nu_m = \frac{m + \alpha - 1}{\alpha}$ and $\gamma_m = \nu_m - 1$ for $\nu_m = -\frac{m + \alpha - 1}{\alpha}$. $c^H_0$ is given by eq.(135) i.e.

$$c^H_0 = \frac{1 - \alpha^2}{12\alpha} + \frac{(2\alpha - 1)(2\alpha - 2)}{2\alpha}.$$  \hspace{1cm} (30)

Taking the difference we have

$$2(c^K_0 - c^H_0) = 3(1 - \alpha)$$  \hspace{1cm} (31)

in agreement with the Riemann–Roch theorem. The other possible self–adjoint extensions of $K = L^\dagger \cdot L$ differ from the previously described one by replacing for $-1 < \nu < 0$ a singular (regular) solution with a regular (singular) one.

For $\frac{1}{2} < \alpha < 1$ we saw that the choice of the singular eigenfunction satisfies the Riemann-Roch relation. Replacing it with the regular one amounts to change a term in the sum eq.(132), i.e. the term corresponding to $m = 0$ given by

$$- \frac{1}{2} \left[ \left( \frac{\alpha - 1}{\alpha} \right)^{1-2s} + s \left( \frac{\alpha - 1}{\alpha} \right)^{1-2s} B_2 \right]$$  \hspace{1cm} (32)

with

$$- \frac{1}{2} \left[ \left( \frac{1 - \alpha}{\alpha} \right)^{1-2s} + s \left( \frac{1 - \alpha}{\alpha} \right)^{1-2s} B_2 \right]$$  \hspace{1cm} (33)
which in the limit $s \to 0$ gives
\[ \Delta c_0^K = c_0'^K - c_0^K = \frac{\alpha - 1}{\alpha}. \quad (34) \]

The change in the eigenfunctions of $K$ determines a well defined change in the eigenfunctions of $H$ according to the eq.(29), giving
\[ \Delta c_0^H = c_0'^H - c_0^H = \frac{\alpha - 1}{\alpha} + 1 . \quad (35) \]

Taking the difference we have
\[ \Delta (c_0^K - c_0^H) = -1 \quad (36) \]
thus giving rise to a violation of the Riemann–Roch relation. Similarly for $1 < \alpha < \frac{3}{2}$ we find that the alternative choice gives
\[ \Delta (c_0^K - c_0^H) = 1 . \quad (37) \]

For $\frac{3}{2} < \alpha < 2$, we have 3 alternative possibilities due to the fact that we have two angular momenta $m = -1$ and $m = -2$ with two acceptable eigenfunctions. Substituting in our expression one of the two eigenfunctions with the alternative one, we obtain the same violation of eq.(37), while substituting both, the result is
\[ \Delta (c_0^K - c_0^H) = 2 . \quad (38) \]

In conclusion in the interval $\frac{1}{2} < \alpha < 2$ the imposition of the Riemann–Roch relation singles out a unique self–adjoint extension of $K$ and $H$. We come now to discuss $\alpha < \frac{1}{2}$ and $\alpha > 2$. For $\alpha < \frac{1}{2}$ and $m = 0$ we have a unique $L^2$ eigenfunction which corresponds to choosing $J_{-\nu_0}$, which as we have just seen, see eq.(37), violates the Riemann–Roch relation.

For $\alpha > 2$ the requirement of $L^2$ summability on the eigenfunctions requires some terms with index $\frac{n-\delta}{\alpha}$ in the second sum appearing in eq.(132), to be substituted by the corresponding ones with index $\frac{\delta-n}{\alpha}$. Each of these shifts gives rise to a violation of 1 in the Riemann–Roch relation. In addition for the angular momenta for which
$1 - 2\alpha < m < 1 - \alpha$, where two independent $L^2$ function exist, each further allowed shift from the second to the first sum gives rise to an additional violation of 1. Thus outside the interval $\frac{1}{2} < \alpha < 2$ is not possible to satisfy the Riemann–Roch relation within the realm of $L^2$-functions. In appendix C we shortly report the calculation with the regulator technique, which gives the same result.

3 Sphere topology

We recall that in two dimensions, modulo diffeomorphisms, every metric can be given in terms of a background metric of constant curvature multiplied by a conformal factor $g_{\mu \nu} = e^{2\sigma} \hat{g}_{\mu \nu}$. We start with the topology of the sphere. The usual choice is to describe it through a stereographic projection on the plane, with $\hat{g}_{\mu \nu} = \delta_{\mu \nu}$. Then for a Regge geometry we have

$$e^{2\sigma} = e^{2\lambda_0} \prod_{i=1}^N |\omega - \omega_i|^{2(\alpha_i - 1)}, \quad \sigma \equiv \sigma(\omega; \lambda_0, \omega_i, \alpha_i) = \lambda_0 + \sum_i (\alpha_i - 1) \log |\omega - \omega_i| \quad (39)$$

with the restriction $\sum_{i=1}^N (1 - \alpha_i) = 2$, i.e. the sum of the deficits must be equal to the Euler characteristic. Due to the presence of 6 conformal Killing vectors for a manifold with the topology of a sphere, the gauge fixing is not complete, i.e. two different $\sigma$ related by an $SL(2, C)$ transformation, describe the same geometry. The transformation is

$$\omega' = \frac{\omega a + b}{\omega c + d}, \quad \omega = \frac{\omega' d - b}{-\omega' c + a}, \quad ad - bc = 1 \quad (40)$$

and $\sigma$ goes over to

$$\sigma'(\omega'; \lambda_0, \omega_i, \alpha_i) \equiv \sigma(\omega(\omega'); \lambda_0, \omega_i, \alpha_i) + \log |\frac{\partial \omega}{\partial \omega'}| \quad (41)$$

$$= \lambda_0 + \sum_i (\alpha_i - 1) \log |\frac{\omega' d - b}{-\omega' c + a} - \omega_i| - 2 \log |\omega' - a|$$

$$= \lambda_0 + \sum_{i=1}^N (\alpha_i - 1) \log \frac{\omega' d - b - \omega_i (a - \omega' c)}{|\omega_i c + d|} + \sum_{i=1}^N (\alpha_i - 1) \log |\omega_i c + d|$$

$$= \lambda_0 + \sum_{i=1}^N (\alpha_i - 1) \log |\omega' - \frac{\omega_i a + b}{\omega_i c + d}| + \sum_{i=1}^N (\alpha_i - 1) \log |\omega_i c + d|$$
having used $\sum_{i=1}^{N}(1 - \alpha_i) = 2$. The new conformal factor is given by

$$\sigma'(\omega'; \lambda_0, \omega_i, \alpha_i) = \sigma(\omega'; \lambda'_0, \omega'_i, \alpha'_i)$$

(42)

with

$$\lambda'_0 = \lambda_0 + \sum_{i=1}^{N}(\alpha_i - 1) \log |\omega_i c + d|, \quad \omega'_i = \frac{a\omega_i + b}{\omega_i + d}, \quad \alpha'_i = \alpha_i.$$  

(43)

Under such transformation the area $A$

$$A = e^{2\lambda_0} \int d^2\omega |\omega - \omega_i|^{2(\alpha_i - 1)}.$$  

(44)

being a geometric invariant is left unchanged. We notice that the number of physical degrees of freedom is the same as the number of links in the usual parameterization of a Regge surface. In fact from the Euler relation $F + V = H + 2$ with $H = \frac{3}{2}F$ we get $H = 3V - 6$, where $-6$ corresponds to the 6 conformal Killing vectors of the sphere.

In the neighborhood of $\omega_i$ the conformal factor can be rewritten as

$$|\omega - \omega_i|^{2(\alpha_i - 1)} e^{2\lambda_i} \quad \text{with} \quad \lambda_i = \lambda_0 + \sum_{j \neq i}(\alpha_j - 1) \log |\omega_j - \omega_i|.$$  

(45)

In working out the r.h.s. of eq.(11) it is simpler to use cartesian coordinates [9] given by

$$z = e^{\lambda_i}(\omega - \omega_i)^{\alpha_i} \quad \text{with} \quad \frac{dz}{d\omega} = e^{\lambda_i}(\omega - \omega_i)^{\alpha_i - 1}.$$  

To a variation $\delta \lambda_i$ and $\delta \alpha_i$ there corresponds a variation in $\sigma$

$$\delta \sigma(z, \bar{z}) = \log \left| \frac{dz'}{dz} \right| = (\delta \lambda_i - \lambda_i \frac{\delta \alpha_i}{\alpha_i}) + \frac{\delta \alpha_i}{\alpha_i} \log(\alpha_i |z|)$$

(46)

and substituting in eq.(11) we have, taking into account that on the sphere there are no Teichmüller parameters

$$-\delta \log \frac{\det(L^\dagger L)}{\det(\Phi, \Phi)} = \gamma_E \delta c^K_0 + \sum_i \left\{ (\delta \lambda_i - \lambda_i \frac{\delta \alpha_i}{\alpha_i})[4c^K_0 - 2c^H_0] \right\}$$

(47)

$$+ \text{Finite}_{\epsilon \to 0} \left[ 4 \frac{\delta \alpha_i}{\alpha_i} \int d^2x \log(\alpha_i|\mathbf{x}|) K_{\alpha_i}(\mathbf{x}, bfx, \epsilon) - 2 \frac{\delta \alpha_i}{\alpha_i} \int d^2x \log(\alpha_i|\mathbf{x}|) \mathcal{H}_{\alpha_i}(\mathbf{x}, \mathbf{x}, \epsilon) \right],$$

where from eqs.(28), (31)

$$4c^K_0 - 2c^H_0 = \frac{13}{6} \left( \frac{1}{\alpha_i} - \alpha_i \right).$$

(48)
A differential of this structure can be integrated. In fact the expression in the square brackets depends only on the $\alpha_i$ separately while one can write

$$
(\delta\lambda_i - \lambda_i \frac{\delta\alpha_i}{\alpha_i})(\frac{1}{\alpha_i} - \alpha_i) = \delta \left[ \left( \frac{1}{\alpha_i} - \alpha_i \right) \lambda_i \right] + 2\delta\alpha_i\lambda_i
$$

and as $\sum_{i=1}^{N} \delta\alpha_i = 0$

$$
\sum_i \delta\alpha_i \lambda_i = \sum_i \delta\alpha_i \sum_{j \neq i} (\alpha_j - 1) \log |\omega_i - \omega_j|
$$

$$
+ \sum_j (\alpha_j - 1) \sum_{i \neq j} (\alpha_i - 1) \delta(\log |\omega_i - \omega_j|)
$$

due to the antisymmetry of $\delta(\log |\omega_i - \omega_j|)$ in $i, j$. Thus

$$
\sum_{i=1}^{N} \delta\alpha_i \lambda_i = \sum_{i=1}^{N} (\alpha_i - 1)(\delta\lambda_i - \delta\lambda_0)
$$

and then

$$
2 \sum_{i=1}^{N} \delta\alpha_i \lambda_i = \sum_{i=1}^{N} \delta\alpha_i \lambda_i + \sum_{i=1}^{N} (\alpha_i - 1)(\delta\lambda_i - \delta\lambda_0) = \delta \left[ \sum_{i=1}^{N} (\alpha_i - 1)(\lambda_i - \lambda_0) \right].
$$

All the above reasonings refer to the operator $L^\dagger L$ acting on the field $\xi$. The same treatment is to be applied to the field $\xi_\omega$ and so one has to multiply the result by a factor 2. The final result for the determinant is

$$
\log \sqrt{\frac{\det(\mathcal{P}\mathcal{P})}{\det(\phi_a, \phi_b)}}
$$

$$
= \frac{26}{12} \left\{ \sum_{i,j \neq i} \frac{(1 - \alpha_i)(1 - \alpha_j)}{\alpha_i} \log |w_i - w_j| + \lambda_0 \sum_i (\alpha_i - \frac{1}{\alpha_i}) - \sum_i F(\alpha_i) \right\}
$$

where $F(\alpha)$ is given by $\gamma E c_0^K(\alpha)$ added to the primitive of

$$
\text{Finite}_{\epsilon \to 0} \left[ \frac{4}{\alpha} \int d^2x \log(\alpha|x|)K_\alpha(x, x; \epsilon) - \frac{2}{\alpha} \int d^2x \log(\alpha|x|)H_\alpha(x, x; \epsilon) \right]
$$

(see appendix D for an integral representation of these terms). By direct substitution one verifies that eq.(53) is invariant under the $SL(2, C)$ transformations (13).
We notice that apart from the term\( \sum_{i=1}^{N} F(\alpha_i) \) eq.(53) is exactly \(-26\) times the conformal anomaly for the scalar field as computed by Aurell and Salomonson \[9\]. In the continuum limit \( N \to \infty \), the \( \omega_i \) become dense and the \( \alpha_i \to 1 \), always with \( \sum_{i=1}^{N} (1-\alpha_i) = 2 \). In such a limit \( \sum_{i=1}^{N} F(\alpha_i) \) goes over to the topological invariant \( N F(1) - \chi F'(1) \), while the remainder goes over to the well known continuum expression. In fact we have

\[
\frac{1}{2\pi} \log |\omega - \omega'| = \frac{1}{\Box}(\omega, \omega')
\]  

and for any region \( V \) of the plane \( \omega \)

\[
\int_{V} \! d^2\omega e^{2\sigma} R = -2 \int_{V} \! d^2\omega \Box \sigma = 4\pi \sum_{i: \omega_i \in V} (1 - \alpha_i).
\]  

Thus the r.h.s. of eq.(53) goes over to

\[
\frac{26}{96\pi} \left\{ \int \! d^2\omega d^2\omega' \left( \sqrt{g} R \right) \frac{1}{\Box} (\omega, \omega') \left( \sqrt{g} R \right)_{\omega'} - 2 \left( \log \frac{A}{A_0} \right) \int \! d^2\omega \sqrt{g} R \right\}
\]  

where \( A_0 \) is the value of the area for \( \lambda_0 = 0 \).

### 3.1 Integration measure for the conformal factor

We work out the functional integration measure \( D[\sigma] \) appearing in appendix A in the Regge framework. The distance between two nearby configurations \( \sigma \) and \( \sigma + \delta \sigma \) is given by

\[
(\delta \sigma, \delta \sigma) = \int d^2\omega e^{2\sigma} \delta \sigma \delta \sigma.
\]  

Such an expression is a direct outcome of the original De-Witt measure (113).

From eq.(58) it follows that having parameterized the Regge surface by means of the \( 3N \) variables \( p_i \)

\[
\{p_1, \ldots, p_{3N}\} \equiv \{\omega_{1,x}, \omega_{1,y}, \omega_{2,x}, \omega_{2,y}, \ldots, \omega_{N,x}, \omega_{N,y}, \lambda_0, \alpha_1, \alpha_2, \ldots, \alpha_{N-1}\}
\]  

\( D[\sigma] \) is given by

\[
D[\sigma] = \prod_{k=1}^{N} d^2\omega_k \prod_{i=1}^{N-1} d\alpha_i d\lambda_0 \sqrt{\det J}
\]  

16
being $J$ the $3N \times 3N$ matrix

$$J_{ij} = \int d^2 \omega \, e^{2\sigma} \frac{\partial \sigma}{\partial p_i} \frac{\partial \sigma}{\partial p_j}, \quad (61)$$

with $\alpha_N = \sum_{i=1}^{N-1} (1 - \alpha_i) - 1$. We recall the expression for $\sigma$

$$e^{2\sigma} = e^{2\lambda_0} \prod_{i=1}^{N} |\omega - \omega_i|^{2(\alpha_i - 1)}. \quad (62)$$

Eq.(61) can be given a more transparent form by doubling the number of variables, i.e. using $\omega_k, \tilde{\omega}_k, \lambda_0, \tilde{\lambda}_0, \alpha_i, \tilde{\alpha}_i$ and introducing a new conformal factor

$$e^{\sigma(p)+\sigma(\tilde{p})} = e^{\lambda_0 + \tilde{\lambda}_0} \prod_{i=1}^{N} |\omega - \omega_i|^{(\alpha_i - 1)}|\omega - \tilde{\omega}_i|^{(\tilde{\alpha}_i - 1)} \quad (63)$$

and computing

$$\tilde{A} = \int d^2 \omega \, e^{\sigma(p)+\sigma(\tilde{p})} \quad (64)$$

which is the area of the Regge manifold described by the $6N$ parameters $p$ and $\tilde{p}$.

It is easily verified that

$$J_{ij} = \left[ \frac{\partial^2 \tilde{A}}{\partial p_i \partial p_j} \right]_{p=\tilde{p}}. \quad (65)$$

Being $\sigma = \lambda_0 + \frac{1}{2} \sum_{i=1}^{N} (\alpha_i - 1) \log |\omega - \omega_i|^2$ we obtain

$$\frac{\partial \sigma}{\partial \omega_{i,x}} = (\alpha_i - 1) \frac{\omega_{i,x} - \omega_x}{|\omega_{i,x} - \omega_x|^2}$$

$$\frac{\partial \sigma}{\partial \alpha_i} = \frac{1}{2} \log |\omega_i - \omega|^2 - \frac{1}{2} \log |\omega_N - \omega|^2, \quad i \leq N - 1 \quad (66)$$

that substituted in eq.(61) with $e^{2\sigma}$ given by eq.(62) give all elements $J_{ij}$.

Each row $J_{\omega_{i,x},p_j}$ contains a factor $\alpha_i - 1$. Due to the multi-linear property of the determinant in the rows, $\det J$ will factorize a factor $\prod_{i=1}^{N} (\alpha_i - 1)^2$ i.e.

$$\det J = \prod_{i=1}^{N} (\alpha_i - 1)^2 F(p). \quad (67)$$
The vanishing of \( \det J \) whenever an \( \alpha_i \) equals 1 is expected from the fact that in such situation the position of the vertex \( i \) is irrelevant in determining the metric.

A measure of the structure

\[
\prod_{i=1}^{N} d\omega_{i,x} d\omega_{i,y} \prod_{j=1}^{N-1} d\alpha_i d\lambda_0 \sqrt{\det(J)} \tag{68}
\]

with \( J \) given by eq.(65) is invariant under the \( SL(2,\mathbb{C}) \) transformations

\[
\omega_i' = \frac{\omega_i a + b}{\omega_i c + d},
\]

\[
\lambda_0' = \lambda_0 + \sum_{i=1}^{N} (\alpha_i - 1) \log |\omega_i c + d| \tag{69}
\]

\[
\alpha_i' = \alpha_i, \quad ad - bc = 1.
\]

In fact \( A \) is an invariant

\[
A(p) = A(p'), \quad \tilde{A}(p, \tilde{p}) = \tilde{A}(p', \tilde{p}'). \tag{70}
\]

Then

\[
\frac{\partial^2 \tilde{A}}{\partial p_i \partial \tilde{p}_j} dp_i d\tilde{p}_j = \frac{\partial^2 \tilde{A}}{\partial p'_k \partial \tilde{p}'_l} \left( \frac{\partial p'_k}{\partial p_i} \right) \left( \frac{\partial \tilde{p}'_l}{\partial \tilde{p}_j} \right) dp_i d\tilde{p}_j \tag{71}
\]

and for \( p = \tilde{p} \) we obtain

\[
\sqrt{\det J} = \sqrt{\det J'} \det \left( \frac{\partial p'_k}{\partial p_i} \right) \tag{72}
\]

which proves that

\[
\sqrt{\det J} \prod_{i=1}^{3N} dp_i = \sqrt{\det J'} \prod_{i=1}^{3N} dp'_i.
\]

We notice that all \( J_{ij} \) are given by convergent integrals except those involving two \( \omega_i \) with the same indexes, which converge only for \( \alpha_i > 1 \). For example we have

\[
J_{\omega_i,\omega_i,x} = (\alpha_i - 1)^2 \int d^2\omega e^{2\sigma} \frac{(\omega_{i,x} - \omega_x)^2}{|\omega - \omega_i|^4} \tag{73}
\]

and the behavior of \( e^{2\sigma(\omega)} \) in the neighborhood of \( \omega_i \) is \( e^{2\lambda_i} |\omega - \omega_i|^{2(\alpha_i - 1)} \). However for \( \alpha_i \to 1^+ \) the term \( (73) \) does not diverge, actually goes to 0 because of the presence of the factor \( (\alpha_i - 1)^2 \). Thus for these diagonal matrix elements we must consider the analytic continuation for \( \alpha_i < 1 \) and prove that such continuation is invariant under \( SL(2,\mathbb{C}) \).
Setting $\delta_i = 1 - \alpha_i$, the analytic continuation of

$$J_{\omega_i, x \omega_i, x} = \frac{\delta_i^2}{4} \int d^2 \omega e^{2\lambda_0} \frac{(\omega_{i,x} - \omega_x)^2}{|\omega - \omega_i|^4}$$

for $\delta_i > 0$ is given by

$$\delta_i^2 \int d^2 \omega \left[ e^{2\lambda_0} \prod_{j \neq i} |\omega - \omega_j|^{-2\delta_i} e^{2\lambda_i} e^{-|\omega - \omega_i|^2} \right] |\omega - \omega_i|^{-2\delta_i} \frac{(|\omega_{i,x} - \omega_x)^2}{|\omega - \omega_i|^4} + \frac{\pi}{2} \Gamma(-\delta_i) \delta_i^2 e^{2\lambda_i}.$$  

(75)

We saw above that the transformation law

$$J_{\omega_i, x \omega_j, b} = \sum_{c d} J'_{\omega_i, c \omega_j, d} \frac{\partial \omega'_c}{\partial \omega_{ia}} \frac{\partial \omega'_d}{\partial \omega_{jb}} + \sum_c J'_{\omega_i, c \lambda} \frac{\partial \omega'_c}{\partial \omega_{ia}} \frac{\partial \lambda'}{\partial \omega_{jb}}$$

$$+ \sum_d J'_{\lambda \omega_i, d} \frac{\partial \lambda'}{\partial \omega_{ia}} \frac{\partial \omega'_d}{\partial \omega_{jb}} + J'_{\lambda \lambda} \frac{\partial \lambda'}{\partial \omega_{ia}} \frac{\partial \lambda'}{\partial \omega_{jb}}$$

(76)

due to the invariance of the area $A$, holds in the convergence region, i.e. for $i \neq j$ and if $i = j$ for $\delta_i < 0$. As $\frac{\partial \omega'_c}{\partial \omega_{ia}}$ does not depend on the $\alpha_j$ and $\frac{\partial \lambda'}{\partial \omega_{ia}}$ is a linear function in the $\alpha_j$ (see eq.(69)), the relation holds also for the matrix elements continued for $\delta_i > 0$. On the other hand the validity of eq.(76) implies eq.(72).

In $\det J$ one can separate the dependence on $\lambda_0$ and on the harmonic ratios of the $\omega_i$ by writing

$$\sqrt{\det J} = e^{3N\lambda_0} W \prod_{i,j>i} |\omega_i - \omega_j|^{2\beta_{ij}}.$$  

(77)

From eq.(72), under $SL(2,C)$ transformations (69), we have

$$\sqrt{\det J'} = \sqrt{\det J} \prod_{i=1}^{N} |\omega_i c + d|^4$$

(78)

which using eq.(77) becomes

$$e^{3N\lambda_0} W' \prod_{i,j>i} |\omega'_i - \omega'_j|^{2\beta_{ij}} = e^{3N\lambda_0} W \prod_{i,j>i} |\omega_i - \omega_j|^{2\beta_{ij}} \prod_{i=1}^{N} |\omega_i c + d|^4.$$  

(79)

In order to have $W' = W$ the $\beta_{ij}$ must be chosen to satisfy

$$\prod_{i,j>i} (|\omega_i c + d|(|\omega_j c + d|)^{-2\beta_{ij}} = \prod_{i} |\omega_i c + d|^{4+3\delta_i N}.$$  

(80)
A particular solution of eq. (81) is

$$\beta_{ij} = \frac{3}{2} \frac{N}{N-2} \left( \frac{2}{N-1} - \delta_i - \delta_j \right) - \frac{2}{N-1}.$$  (82)

The conclusion is that $J$ can be written as

$$J = e^{6N\lambda_0} W^2 \prod_{i,j>i} |\omega_i - \omega_j|^{4\beta_{ij}}$$  (83)

with $W$ a function of only $\omega_i$ and $\alpha_i$ which is invariant under the full $SL(2, C)$ and thus function only of the harmonic ratios of the $\omega_i$.

## 4 Torus topology

The most general metric, modulo diffeomorphisms, is given by a flat metric $\hat{g}_{\mu\nu}(\tau_1, \tau_2)$ times a conformal factor $e^{2\sigma}$. $\tau_1$ and $\tau_2$ are the two Teichmüller parameters in terms of which, with $\tau = \tau_1 + i\tau_2$,

$$ds^2 = dx^2 + 2\tau_1 dxdy + |\tau|^2 dy^2$$  (84)

and the fundamental region has been taken the square $0 \leq x < 1, 0 \leq y < 1$. We recall the expression for the Green function of $\Box$ on the torus \cite{13} using $\omega = x + \tau y$

$$\Box G(\omega - \omega'|\tau) = \delta^2(\omega - \omega') - \frac{1}{\tau_2}$$  (85)

$$G(\omega - \omega'|\tau) = \frac{1}{2\pi} \log \left| \frac{\psi_1(\omega - \omega'|\tau)}{\eta(\tau)} \right| - \frac{(\omega_y - \omega'_y)^2}{2\tau_2}$$  (86)

being $\psi_1(\omega|\tau)$ the Jacobi $\psi$–function and

$$\eta(\tau) = e^{\frac{2\pi i}{\tau_1}} \prod_{n=1}^{\infty} [1 - e^{2in\pi\tau}].$$  (87)

From the written Green function and using

$$R(e^{2\sigma}\hat{g}) = e^{-2\sigma} (R(\hat{g}) - 2\Box \sigma)$$  (88)
it follows that the conformal factor \( \sigma \), in presence of angular deficits \( 2\pi(1 - \alpha_i) \), concentrated at the points \( \omega_i \) is

\[
\sigma(\omega) = \lambda_0 + \sum_{i=1}^{N} (\alpha_i - 1) \left\{ \log \left| \frac{\vartheta_1(\omega - \omega_i|\tau)}{\eta(\tau)} \right| - \frac{\pi}{\tau_2} (\omega_y - \omega_{i,y})^2 \right\}.
\] (89)

Thus the physical degrees of freedom are \( 3N \): in fact in addition to the \( 2N \) \( x_i, y_i \) we have \( N - 1 \) independent angular deficits \( \sum_{i=1}^{N} (\alpha_i - 1) = 0 \), two Teichmüller parameters and \( \lambda_0 \), to which we must subtract the two conformal Killing vectors of the torus. We have the same number of physical degrees of freedom as the number of bones in a Regge triangulation of the torus with \( N \) vertices as it can be easily checked through the Euler relation for a torus \( (F + V = H = 3F/2, \text{from which } H = 3V) \).

The derivation of the Liouville action proceeds similarly as for the sphere. Eqs.(47), (48), (49) are unchanged. The main difference is given by the form of \( \lambda_i \), defined as before as \( e^{2\lambda_i} = \lim_{\omega \to \omega_i} (e^{2\pi|\omega - \omega_i|^{2(1 - \alpha_i)}) \). From eq.(89) we have for the torus

\[
\lambda_i = \lambda_0 + \sum_{j \neq i} 2\pi(\alpha_j - 1)G(\omega_i - \omega_j|\tau) + (\alpha_i - 1) \log |2\pi \eta^2(\tau)|.
\] (90)

Now proceeding as after eq.(49)

\[
\sum_{i=1}^{N} \delta \alpha_i \lambda_i = \sum_{i=1}^{N} (\alpha_i - 1) \delta \left[ \sum_{j \neq i} 2\pi(\alpha_j - 1)G(\omega_i - \omega_j|\tau) \right] + \frac{1}{2} \delta \left[ \sum_{i=1}^{N} (\alpha_i - 1)^2 \log |2\pi \eta^2| \right] \] (91)

and then

\[
2 \sum_{i=1}^{N} \delta \alpha_i \lambda_i = \delta \left[ \sum_{i=1}^{N} (\alpha_i - 1) \sum_{j \neq i} 2\pi(\alpha_j - 1)G(\omega_i - \omega_j|\tau) + \sum_{i=1}^{N} (\alpha_i - 1)^2 \log |2\pi \eta^2| \right], \] (92)

having used \( \sum_{i=1}^{N} (\alpha_i - 1) = 0 \) for the torus.

The final result is

\[
\log \left| \frac{\det'(P')}{\det(\phi_a, \phi_b) \det(\psi_l, \psi_k)} \right| = \log \left| \frac{\det'(P')}{\det(\phi_a, \phi_b) \hat{\det}(\psi_l, \psi_k)} \right| + \frac{26}{12} S_l
\] (93)

with

\[
S_l = \sum_{i,j \neq i} \frac{(1 - \alpha_i)(1 - \alpha_j)}{\alpha_i} \left[ \log \left| \frac{\vartheta_1(\omega_j - \omega_i|\tau)}{\eta(\tau)} \right| - \frac{\pi}{\tau_2} (\omega_{i,y} - \omega_{j,y})^2 \right]
\] + \( (\lambda_0 - \log |2\pi \eta^2|) \sum_i (\alpha_i - \frac{1}{\alpha_i}) - \sum_i F(\alpha_i) \). (94)
This action is obviously invariant under the translation \( \omega_i \to \omega_i + a \) (with complex \( a \)) corresponding to the two conformal Killing vectors of the tours, and compared to the sphere topology is no longer invariant under dilatations, rotations and special conformal transformations.

In the continuous limit eq.(94) goes over to the well known expression

\[
\frac{1}{8\pi} \int d^2 \omega d^2 \omega' (\sqrt{gR})_\omega \frac{1}{4}(\omega, \omega')(\sqrt{gR})_{\omega'}
\]  

(95)

### 4.1 Modular invariance

The partition function is given by eq.(123)

\[
\int D[\sigma] \frac{d^2 \tau}{v(\tau)} \sqrt{\text{det}'(P^\dagger P)_{\tilde{g}}} \text{det}(\psi_m, \frac{\partial g}{\partial \tau_n}) e^{\frac{2\pi}{\tau_2} S_i}
\]

(96)

where for a torus (for a detailed discussion see [16]) \( v(\tau) = \tau_2 \)

\[
\text{det}(\phi_a, \phi_b)_{\tilde{g}} = \text{const} \tau_2^2 \quad \text{det}(\psi_k, \psi_l) = \text{const} \tau_2^2
\]  

(97)

\[
\text{det}(\psi_m, \frac{\partial g}{\partial \tau_n}) = \text{const} \quad [\text{det}'(P^\dagger P)]_{\tilde{g}} = \tau_2^4 |\eta(\tau)|^8.
\]

It is well known that the expression

\[
\frac{d^2 \tau}{v(\tau)} \sqrt{\text{det}'(P^\dagger P)_{\tilde{g}}} \text{det}(\psi_m, \frac{\partial g}{\partial \tau_n}) = \text{const} \quad \frac{d^2 \tau}{\tau_2} |\eta(\tau)|^4
\]  

(98)

is invariant under the modular transformation

\[
\tau \longrightarrow \tau' = \frac{\tau a + b}{\tau c + d}
\]  

(99)

with \((a, b, c, d) \in \mathbb{Z} \) and \( ad - bc = 1 \). Thus we are left to prove the modular invariance of

\[
\int D[\sigma] e^{\frac{2\pi}{\tau_2} S_i}.
\]

This is achieved by accompanying the change in \( \tau \) by a proper change in the integration variable \( \omega_i, \lambda_0 \) given by

\[
\omega' = \frac{\omega}{\tau c + d} \quad \lambda_0' = \lambda_0 + \log |\tau c + d|.
\]  

(100)
The last equation follows from the transformation of $\sigma(\omega; \lambda_0, \omega_i, \alpha_i, \tau)$ under a change of coordinates

$$
\sigma'(\omega'; \lambda_0, \omega_i, \alpha_i, \tau) \equiv \sigma(\omega(\omega'); \lambda_0, \omega_i, \alpha_i, \tau) + \log \left| \frac{d\omega}{d\omega'} \right|
$$

(101)

$$
= \sigma((\tau c + d)\omega'; \lambda_0, \omega_i, \alpha_i, \tau) + \log |\tau c + d| = \sigma(\omega'; \lambda'_0, \omega'_i, \alpha_i, \tau')
$$

keeping in mind eq.(89) and the modular invariant $G(\omega - \omega_i|\tau) = G(\omega' - \omega'_i|\tau')$.

$S_t$, as given by eq.(94), is invariant under transformations (100), (99) because of the just cited modular invariance of the Green function and because

$$
\eta \left( \frac{a\tau + b}{c\tau + d} \right) = e^{i\phi}(c\tau + d)^{1/2}\eta(\tau)
$$

(102)

compensates the change in $\lambda_0$.

Also we have

$$
\sqrt{J} \prod_{i=1}^{N} \omega'_id\lambda'_0 \prod_{j=1}^{N-1} d\alpha_j = \sqrt{J} \prod_{i=1}^{N} \omega_id\lambda_0 \prod_{j=1}^{N-1} d\alpha_j.
$$

(103)

In fact from the invariance of the area

$$
\tilde{A} = e^{\lambda_0+\tilde{\lambda}_0} \int_{\mathcal{M}(\tau)} d^2\omega \prod_{j,i=1}^{N} e^{2\pi(\alpha_i-1)G(\omega-\omega_i|\tau)} e^{2\pi(\tilde{\alpha}_i-1)G(\omega-\tilde{\omega}_i|\tau)}
$$

(104)

under the transformations (99), (100) it follows

$$
\prod_{i,j} \frac{\partial^2 \tilde{A}}{\partial p_i \partial \tilde{p}_j} dp_i d\tilde{p}_j = \prod_{i,j} \frac{\partial^2 \tilde{A}}{\partial p'_i \partial \tilde{p}'_j} dp'_i d\tilde{p}'_j
$$

(105)

This concludes our proof of the modular invariance of eq.(96).

5 Comparison with the smooth limit

One might ask how far one can reach the results (53), (93), starting from the Liouville action for a smooth $\sigma$

$$
\int d^2\omega(-\sigma \Box \sigma + \mu^2 e^{2\sigma}) = \int d^2\omega(-\sigma \Box \sigma) + \mu^2 A
$$

(106)
and then taking a proper singular limit. One should construct a smooth surface depending on an invariant parameter $\rho$ such that for $\rho \to 0$ it tends to our Regge manifold with vertices at $\omega_i$ and angular openings $\alpha_i$. This is a not trivial task; nevertheless we shall show that a rough cut off procedure reproduces the main features of formulas (53), (93).

For some aspect the problem is similar to that of electrostatics when one takes the limit of a continuous distribution of charge to a point-like distribution and the infinities arising from the self energies are removed. The difference is that in our case it is difficult to implement an exact cut off that is a geometric invariant in presence of more than one singularity.

We shall regularize at an approximate level the tip of the cones with segments of sphere (or pseudosphere) all with the same radius of curvature $\rho/2$. The conformal factor describing a sphere of radius $\rho/2$ around $\omega_i$ is

$$e^{2\sigma} = \frac{k^2}{(1 + (k\rho/2)^2|\omega - \omega_i|^2)^2}$$

(107)

for which $R = -2e^{-2\sigma} \Delta \sigma = \frac{8}{\rho^2}$.

$R$ is constant within a region $|\omega - \omega_i| \leq r_0$, where $r_0$ is related to the deficit angle by

$$r_0^2 = \frac{\rho^2}{k^2} \frac{1 - \alpha}{1 + \alpha}.$$  

(108)

In presence of more than one singularity we shall impose that the conformal factor $\sigma$ for $|\omega - \omega_i| > r_0$ goes over to

$$\sigma = \lambda_0 + (\alpha_i - 1) \log |\omega - \omega_i| + \sum_{j \neq i} (\alpha_j - 1) \log |\omega_j - \omega_i|.$$  

(109)

Thus for $|\omega - \omega_i| = r_0$ we have

$$\log k = \frac{1}{2\alpha_i} [\log 4 + 2\lambda_0 + (\alpha_i - 1) \log (1 - \alpha_i)]$$

$$- \log (\alpha_i + 1) + 2 \sum_{j \neq i} (\alpha_j - 1) \log |\omega - \omega_i|.$$  

(110)

Integrating $\sigma \Delta \sigma$ over the region $V$ around $\omega_i$ of non vanishing curvature, we obtain

$$-\frac{1}{2\pi} \int_V \sigma \Delta \sigma d^2\omega$$  

(111)
\[
= - \sum_{i,j \neq i} \frac{(1 - \alpha_i)(1 - \alpha_j)}{\alpha_i} \log |\omega - \omega_i| + \sum_i f(\alpha_i) + \lambda_0(\frac{1}{\alpha_i} - 1) - \frac{1}{2\alpha_i}(1 - \alpha_i)^2 \log \rho^2.
\]

Similarly one works with \(\alpha > 1\), i.e. with negative curvatures, obtaining the same result. The considered approximation, after removing the divergent terms \(\sim \log \rho^2\), misses with respect to the exact expression (53) the term \(-\lambda_0 \alpha_i\) which is here replaced by \(-\lambda_0\). This is obviously due the approximate matching of the internal to the external metric. We stress that, contrary to the exact expression (53), the approximate equation (111) is not invariant under \(SL(2, C)\).

6 Conclusions

Applying to the Regge surfaces the conventional definition of diffeomorphisms we have derived the analogous of the Liouville action for the discretized case. Such results are exactly invariant at the discretized level under the \(SL(2, C)\) group for the sphere topology and under translations and modular transformations for the torus topology. In the continuum limit they go over to the usual continuum results. For the sphere the action is given by eq.(53) and for the torus by eqs.(93), (94). The integration measure for the conformal factor is provided for the sphere topology by the determinant of the finite dimensional matrix \(J_{ij}\) eq.(63) and for the torus topology by the analogous expression obtained from the area \(\tilde{A}\) given by eq.(104). One could subject the partition function to numerical computations. The action, even though non local (as one expects from a functional determinant) is very simple especially for the sphere topology. Probably the difficult part for a numerical simulation is the evaluation of the finite dimensional determinant \(\text{det } J_{ij}\).

The developed approach can be extended to any dimension [17]. However it appears very hard to provide an explicit expression for the analogous of the functional determinant (2); in fact for \(D \geq 3\) it is unlikely that the computation of such a determinant can be reduced to the evaluation of local quantities as it happens for \(D = 2\).

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A Two dimensional continuum gravity

In the text we refer repeatedly to the continuum formulation; thus we report here a concise derivation of the main formula following the classical papers of [6]. The starting point is the formal functional integral

\[ Z = \int \frac{\mathcal{D}[g_{\mu\nu}]}{V_{GC}} e^{-\int dx^2 \sqrt{g} \left( \frac{\lambda}{2} g^{ab} \partial_a X \partial_b X + \lambda R + \mu^2 \right)} \]  

(112)

where \( V_{GC} \) is the volume of the general coordinate transformation. For the distance in the space of the metrics one adopts the De Witt metric

\[ (\delta g_{\mu\nu}, \delta g_{\mu\nu}) = \int \sqrt{g} d^2 x \delta g_{\mu\nu} G_{\mu\nu,\mu'} \delta g_{\mu'\nu'} \]  

(113)

\[ G_{\mu\nu,\mu'} = g^{\mu\mu'} g^{\nu\nu'} + g^{\mu\nu'} g^{\nu\mu'} - C g^{\mu\nu} g^{\mu'\nu'} . \]  

The most general metric can be written uniquely as

\[ g_{\mu\nu} = \hat{f}^* (e^{2\sigma} \hat{g}(\tau))_{\mu\nu} = g_{\mu\nu}(\sigma, \tau, \hat{f}) \]  

(114)

where \( \hat{f} \) is the general diffeomorphism orthogonal to the action of a conformal transformation. A variation \( \delta g_{\mu\nu} \) of the metric can be decomposed into a Weyl transformation, plus a coordinate transformation plus a change in the Teichmüller parameters

\[ \delta g_{\mu\nu} = \nabla_{\mu} \xi_{\nu} + \nabla_{\nu} \xi_{\mu} + 2 \hat{f}^* \delta \sigma \ g_{\mu\nu} + \frac{\partial g_{\mu\nu}}{\partial \tau_i} \delta \tau_i \]  

= \left[ \frac{\partial g_{\mu\nu}}{\partial \tau_i} + \frac{1}{P^i} P^j k_{\mu\nu}^{\ j} \delta \tau_j \right] + g_{\mu\nu} (2 \hat{f}^* \delta \sigma + \nabla_{\mu} \xi_{\nu} + \frac{g^{\alpha\beta}}{2} \frac{\partial g_{\alpha\beta}}{\partial \tau_i} \delta \tau_i) + (1 - \frac{1}{P^i} P^j) k_{\mu\nu}^{\ i} \delta \tau_i \]  

(115)

where

\[ k_{\mu\nu}^{\ i} = \frac{\partial g_{\mu\nu}}{\partial \tau_i} - \frac{g_{\mu\nu}}{2} g^{\alpha\beta} \frac{\partial g_{\alpha\beta}}{\partial \tau_i} . \]  

(116)

We shall find

\[ \mathcal{D}[g] = \mathcal{D}[\hat{f}] \mathcal{D}[\sigma] d\tau_i J(\sigma, \tau) \]  

(117)

where \( J \) does not depend on \( \hat{f} \). If now one integrates a diff-invariant quantity \( \mathcal{F} \), we have

\[ \int \mathcal{D}[g] \mathcal{F} = \int \mathcal{D}[\hat{f}] \int \frac{\mathcal{D}[\sigma]}{\mathcal{D}[\hat{f}]} \mathcal{D}[\sigma] d\tau_i J(\sigma, \tau) \mathcal{F} . \]  

(118)
In order to find $J$ let us write, using the standard normalization
\[
1 = \int \mathcal{D}[\delta g_{\mu\nu}] e^{-\frac{1}{2}(\delta g_{\mu\nu},\delta g_{\mu\nu})} = \int \mathcal{D}[\xi] \mathcal{D}[\delta \sigma] d\delta \tau_i J(\sigma,\tau)
\]
\[
\times \exp \left[ -\frac{1}{2}(\delta \sigma, \delta \sigma) - \frac{1}{2}(P\xi, P\xi) - \frac{1}{2}((1 - P\frac{1}{P\dot{P}} P)k_{\mu\nu}^i, ((1 - P\frac{1}{P\dot{P}} P)k_{\mu\nu}^i)\delta \tau_i\delta \tau_j \right]
\]
where in the first two terms, invariance of the measure on the tangent space under translations has been used and $J$ does not depend on $f$, due to diff-invariance; $1 - P\frac{1}{P\dot{P}} P$ is the projector on the zero modes $\psi_l$ of $P^i$. Taking into account that $\psi_l$ are traceless, we have
\[
1 = J(\sigma,\tau)[\det'(P^i P)]^{-\frac{1}{2}}[\det(\psi_m, \partial g/\partial \tau_n)]^{-1}[\det(\psi_k, \psi_l)]^{\frac{1}{2}}.
\]
(120)

Finally one must compute
\[
\int \mathcal{D}[\tilde{f}]/\mathcal{D}[f].
\]
(121)
This can be achieved by the following change of variable
\[
\mathcal{D}[f] = \mathcal{D}[\tilde{f}] \Pi dw_c K
\]
(122)
where $w_c$ are the normal coordinates associated to the $N$ conformal Killing vectors $\phi_a$ (which satisfy $P(e^{2\alpha} \phi_a) = 0$ and as such do not depend on $\sigma$, see eq. (8)). In order to find $K$ one computes on the space tangent to the diffeomorphisms
\[
1 = \int \mathcal{D}[\xi] e^{-\frac{1}{2}(\xi,\xi)}
\]
\[
= \int \mathcal{D}[\xi] \prod_c d\delta w_c K \exp \left[ -\frac{1}{2}(\xi,\xi) - \frac{1}{2}(\phi_a, \phi_b)\delta w_a \delta w_b \right] = \text{const } K[\det(\phi_a, \phi_b)]^{-\frac{1}{2}}.
\]
(123)
Thus
\[
\int \mathcal{D}[f] = \int dw_c \int \mathcal{D}[\tilde{f}] \det(\phi_a, \phi_b)^{\frac{1}{2}} = v(\tau) \int \mathcal{D}[\tilde{f}][\det(\phi_a, \phi_b)]^{\frac{1}{2}},
\]
(124)
being $v(\tau)$ the volume of the group generated by the conformal Killing vectors. From eqs. (118), (120), (123) one can write [4]
\[
\int \mathcal{D}[g] F = \int \mathcal{D}[f] \int \mathcal{D}[\sigma] \frac{d\tau_i}{v(\tau)} \left[ \frac{\det'(P^i P)}{\det(\phi_a, \phi_b) \det(\psi_k, \psi_l)} \right]^{\frac{1}{2}} \det(\psi_m, \partial g/\partial \tau_n).
\]
(125)
We recall that $v(\tau)$ and $\det(\psi_m, \partial g/\partial \tau_n)$ do not depend on $\sigma$ but only on the Teichmüller parameters $\tau_i$, while the remaining square root is the exponential of the Liouville action multiplied by the same quantity at $\sigma = 0$. 

27
B Asymptotic expansion of the heat kernel

In this appendix we summarize the computation of $\Xi_K(0)$, where $\Xi_K(s)$ is given by

$$\Xi_K(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \text{Tr}(e^{-tK}) dt,$$  \(126\)

employing the direct method of Cheeger [18] that does not involve contour integrals.

The trace $\text{Tr}$ involves also the summation on the angular momentum $m$. In [18] such a sum is split into two parts, which give rise to contributions to $\Xi_K(s)$ that are analytic in two non overlapping vertical strips in the complex $s$-plane. $\Xi_K(s)$ is defined as the sum of the two analytic continuations. $\Xi_K(0)$ is given by the constant term in the asymptotic expansion of the trace of the heat kernel. Furthermore in [18] it is proven that

$$\Xi_K(s) = \text{Tr}(K^{-s})$$  \(127\)

as can be expected from eq.(126).

In our case with $K = L^\dagger L$, it is easy to prove from the choice of the eigenfunctions given in the text, that

$$\text{Tr}(K^{-s}) = \int_0^{R_0} R dR \sum_m \int_0^\infty J_{2\nu_m}^2(\lambda R) \lambda^{1-2s} d\lambda$$  \(128\)

where for clearness sake we have explicitly indicated a space cut-off $R_0$, that will disappear in the value of $\Xi_K(0)$.

Using the relation

$$\int_0^\infty J_{2\nu}^2(\lambda R) \lambda^{1-2s} d\lambda = R^{2s-1} \frac{\Gamma(\nu - s + 1)\Gamma(s - 1/2)}{2\sqrt{\pi}\Gamma(\nu + s)\Gamma(s)}$$  \(129\)

we obtain

$$\Xi_K(s) = \frac{R_0^{2s}}{2s\Gamma(s)} \frac{\Gamma(s - 1/2)}{2\sqrt{\pi}} \sum_m \frac{\Gamma(\nu_m - s + 1)}{\Gamma(\nu_m + s)}.$$  \(130\)

For $s \to 0$ we have the asymptotic expansion [18]

$$\frac{\Gamma(\nu - s + 1)}{\Gamma(\nu + s)} = \nu^{1-2s}(1 + s \sum_{j=1}^\infty \frac{B_{2j}}{j}\nu^{-2j} + O(s^2))$$  \(131\)
with $B_j$ the Bernoulli numbers (for the Bernoulli numbers and polynomials we use the notation of [19]). In the limit $s \to 0$, $\nu_m = \frac{m+\delta}{\alpha}$ for $m \geq 0$ and $\nu_m = -\frac{m+\delta}{\alpha}$ for $m < 0$, taking into account that $\zeta(u, a)$ has a simple pole only at $u = 0$ with residue 1, we have

$$\Xi_K(s) \to -\frac{1}{2} \left\{ \sum_{n=0}^{\infty} \left[ \left( \frac{n+\delta}{\alpha} \right)^{1-2s} + s\left( \frac{n+\delta}{\alpha} \right)^{-1-2s} B_2 \right] \right. + \left. \sum_{n=1}^{\infty} \left[ \left( \frac{n-\delta}{\alpha} \right)^{1-2s} + s\left( \frac{n-\delta}{\alpha} \right)^{-1-2s} B_2 \right] \right\}$$

$$= -\frac{1}{2} \left[ \alpha^{2s-1} \zeta(2s-1, \delta) + B_2 s \alpha^{2s+1} \zeta(2s+1, \delta) + \alpha^{2s-1} \zeta(2s-1, 1-\delta) \right.$$

$$+ B_2 s \alpha^{2s+1} \zeta(2s+1, 1-\delta) \left]. \right.$$
we shall derive briefly the same result using a more conventional method, i.e. that of regularizing the conical singularity. The regulator we shall use is to replace the tip of the cone with a segment of a sphere (or of the Poincaré pseudo-sphere) connected smoothly to the remaining part of the cone. In such a scheme the eigenfunction problem can be solved exactly. The form of the eigensolutions on the cone will be fixed by taking the limit where the radius of the sphere (or pseudo-sphere) tends to zero, keeping the integrated curvature fixed.

A sphere of radius \( \frac{1}{2} \rho \) of constant curvature \( R = -2e^{-2\sigma} \Box \sigma = 8\rho^{-2} \) or the pseudosphere of constant curvature \( R = -8\rho^{-2} \) are described on the \( \omega \)-plane by the conformal factor \( e^{2\sigma} = (1 \pm u\bar{u})^{-2} \) with \( u = \omega/\rho \). Similarly a cone is described by the conformal factor \( e^{2\sigma} = c^2 (\omega\bar{\omega})^{\alpha-1} \). The radius at which the sphere connects to the cone will be denoted by \( r_0 = \rho v_0 \), \((r = |\omega| \text{ and } v = |u|)\). \( c \) is fixed by the matching condition

\[
(1 \pm \left( \frac{r_0}{\rho} \right)^2)^{-2} = c^2 (r_0^2)^{\alpha-1},
\]

from which we obtain \( c = \rho^{1-\alpha} \frac{v_0^{1-\alpha}}{1 \pm v_0^2} \). The integrated curvature on the segment of the sphere (pseudo-sphere) between \( r = 0 \) and \( r = r_0 \) is given by \( \pm 8\pi v_0^2/(1 \pm v_0^2) \). Imposing this quantity to be equal to \( 4\pi(1-\alpha) \) (i.e. to the curvature concentrated on the tip of the cone) fixes the value of \( r_0 = \rho v_0 \) to

\[
v_0^2 = \frac{1-\alpha}{1+\alpha} \quad \text{for the sphere} \quad (0 < \alpha < 1)
\]

and

\[
v_0^2 = \frac{\alpha - 1}{\alpha + 1} \quad \text{for the pseudo-sphere} \quad (1 < \alpha).
\]

From eq.(137) and eq.(138) we see that a segment of sphere or pseudo-sphere with given integrated curvature is described by a fixed value \( v_0 \), corresponding to \( r_0 = \rho v_0 \).

We shall now find the eigensolutions of \( L^\dagger L \) on the sphere or pseudo-sphere and then we shall connect them smoothly to those on the cone. Let us consider first the case of positive curvature. Using eqs.(8), (13), (14) the general eigenvalue equation

\[
e^{-2\sigma} L^\dagger e^{2\sigma} L e^{-2\sigma} \xi = \lambda^2 \xi
\]
becomes

\[
\left\{ x(1 + x)^2 \frac{d^2}{dx^2} + (1 + x)(3x + 1) \frac{d}{dx} - \frac{m^2(1 + x)^2}{4x} + m(1 + x) + 2 + \lambda^2 \rho^2 \right\} \xi^{(m)} = 0
\]  

(140)

where we have set \( \xi = e^{im\phi} \zeta^{(m)} \) and \( x = r^2 = \bar{\omega} \). Such an equation has three regular singular points and thus can be solved by standard methods [19]. We find

\[
m \geq 0 \quad \xi^{(m)} = \frac{v^m}{(1 + v^2)^2} \quad 2F_1(\gamma_1 + 2, 1 - \gamma_1; 1 + m; \frac{v^2}{1 + v^2})
\]  

(141)

\[
m < 0 \quad \xi^{(m)} = v^{-m} \quad 2F_1(\gamma_1, -1 - \gamma_1; 1 - m; \frac{v^2}{1 + v^2})
\]

where \( \gamma_1 = \frac{1}{2}(-1 + \sqrt{9 + 4(\rho \lambda)^2}) \). For \( \rho^2 = 0 \) they reduce to

\[
m \geq 0 \quad \xi^{(m)} = \frac{v^m}{(1 + v^2)^2}
\]  

(142)

\[
m < 0 \quad \xi^{(m)} = \frac{v^{-m}}{(1 + v^2)^2} \left[ (1 + v^2)^2 - \frac{2}{1 - m} v^2 (1 + v^2) + \frac{2}{(1 - m)(2 - m)} v^4 \right].
\]

On the pseudo-sphere we have

\[
m \geq 0 \quad \xi^{(m)} = \frac{v^m}{(1 - v^2)^2} \quad 2F_1(\gamma_2 + 2, 1 - \gamma_2; 1 + m; \frac{v^2}{v^2 - 1})
\]  

(143)

\[
m < 0 \quad \xi^{(m)} = v^{-m} \quad 2F_1(\gamma_2, -1 - \gamma_2; 1 - m; \frac{v^2}{v^2 - 1})
\]

where \( \gamma_2 = \frac{1}{2}(-1 + \sqrt{9 - 4(\rho \lambda)^2}) \). For \( \rho^2 = 0 \) we obtain

\[
m \geq 0 \quad \xi^{(m)} = \frac{v^m}{(1 - v^2)^2}
\]  

(144)

\[
m < 0 \quad \xi^{(m)} = \frac{v^{-m}}{(1 - v^2)^2} \left[ (1 - v^2)^2 + \frac{2}{1 - m} v^2 (1 - v^2) + \frac{2}{(1 - m)(2 - m)} v^4 \right].
\]

As we know from eq.(15) the general eigensolution on the cone for orbital angular momentum \( m \) has the form

\[
\xi^{(m)}_{\text{ext}} = v^{\alpha - 1} \left[ a(\rho) J_\gamma (2\rho \lambda v^\alpha) + b(\rho) J_{-\gamma} (2\rho \lambda v^\alpha) \right]
\]  

(145)
where $\gamma = \frac{m + \alpha - 1}{\alpha}$ and $p = \frac{\alpha_0}{\alpha(1 + \alpha_0)}$. The coefficients $a(\rho)$ and $b(\rho)$ are fixed by requiring the continuity of the logarithmic derivative of $e^{-2\sigma} \xi$ with respect to $\bar{\omega}$ at $|\omega| = r_0$. In fact from the structure of eigenvalue equation $e^{-2\sigma} \frac{\partial}{\partial \omega} e^{2\sigma} \frac{\partial}{\partial \bar{\omega}} e^{-2\sigma} \xi = -\lambda^2 \xi$, we see that failing to satisfy such a condition would produce a singular contribution at the matching point.

Let us see what is the form of the eigenfunctions on the cone fixed by the matching condition in the limit when the regulator $\rho$ tends to 0.

We consider first $m \geq 0$. For small $\rho$ the interior solution multiplied by the factor $e^{-2\sigma}$ becomes

$$e^{-2\sigma} \xi_{\text{int}} = u^m \left[ 1 + (\rho \lambda)^2 f \left( \frac{u\bar{u}}{1 \pm uu} \right) + O((\rho \lambda)^4) \right]$$

(146)

while the exterior solution multiplied by the conformal factor $e^{-2\sigma}$ becomes

$$(\rho \lambda)^{\gamma} a(\rho) u^m \left[ c_0 + c_1 (\rho \lambda)^2 (u\bar{u})^\alpha + O((\rho \lambda)^4) \right] + (\rho \lambda)^{\gamma} b(\rho) \bar{u}^{-m} \left[ (u\bar{u})^{1-\alpha} + O((\rho \lambda)^2) \right].$$

(147)

We notice that the lowest order in $\rho \lambda$ in the first term of eq.(147) has vanishing derivative with respect to $\bar{\omega}$. Thus the continuity of the logarithmic derivative for small $\rho \lambda$ takes the form

$$\frac{1}{\rho} k(\rho \lambda)^2 = \frac{1}{\rho} \frac{a(\rho)c_1 (\rho \lambda)^2 + b(\rho)c_2(\rho \lambda)^{-2\gamma}}{a(\rho)c_3 + c_4 b(\rho)(\rho \lambda)^{-2\gamma}}$$

(148)

which gives

$$\frac{b(\rho)}{a(\rho)} = \frac{(\rho \lambda)^{2+2\gamma}(c_1 - k c_3)}{k(\rho \lambda)^2 c_4 - c_2}.$$  

(149)

Thus for $m \geq 0$ we see that for $2 + 2\gamma > 0$, i.e. $\alpha > \frac{1}{2}$, $b(\rho)$ vanishes when the regulator is removed at constant integrated curvature.

Similarly one can deal with $m < 0$. In this case the derivative of the interior solution multiplied by the conformal factor tends to a finite limit for $(\rho \lambda)^2 \to 0$ and the analog of equation (149) is

$$\frac{a(\rho)}{b(\rho)} = \frac{(c_5 - k_1 c_7)(\rho \lambda)^{-2\gamma}}{k_1 c_8 - c_6 (\rho \lambda)^2}.$$  

(150)

Thus for $m < 0$ we have $a(\rho) \to 0$ for $\gamma < 0$, i.e. for $\alpha < 2$.

Thus for the opening of the cone $\alpha$ with $\frac{1}{2} < \alpha < 2$, as the regulator is removed, only the term $J_{\frac{m + \alpha - 1}{2}}$ survives for $m \geq 0$, while for $m < 0$ the surviving term is $J_{-\frac{m + \alpha - 1}{2}}$. This
is exactly the same result obtained in sect. 2.1 by imposing the Riemann–Roch relation.

D Integral representation of the heat kernel

We give here the expressions of the integrals appearing in eq.(54). Let us consider \([20, 10]\):

\[ K_{\alpha,\delta}(x, x'; t) = \frac{1}{4\pi t} e^{-\frac{|x-x'|^2}{4t}} + \frac{1}{16i\pi^2\alpha t} \int_{\Gamma} d\zeta e^{-\frac{1}{2}(\alpha^2 + x^2 - 2xx'\cos\zeta)} \frac{e^{\frac{i}{2\alpha}(\zeta + \phi' - \phi')(2\delta - 1)}}{\sin \frac{\zeta + \phi' - \phi}{2\alpha}} \] (151)

where \(\phi\) is the polar angle associated to the vector \(x\) and the where the integration contour \(\Gamma\) is composed of the two lines which go from \(-\pi - i\infty\) to \(-\pi + i\infty\) and from \(\pi + i\infty\) to \(\pi - i\infty\). We have \(K_{\alpha}(x, x'; t) = K_{\alpha,\alpha-1}(x, x'; t)\) and \(H_{\alpha}(x, x'; t) = K_{\alpha,2\alpha-1}(x, x'; t)\).

Evaluating the kernel \((151)\) at \(x' = x\) we obtain

\[ K_{\alpha,\delta}(x, x; t) = \frac{1}{4\pi t} + \frac{1}{16i\pi^2\alpha t} \int_{\Gamma} d\zeta e^{-\frac{1}{2}\alpha^2(1-\cos\zeta)} \frac{e^{\frac{i}{2\alpha}\zeta(2\delta - 1)}}{\sin \frac{\zeta}{2\alpha}} \] (152)

from which

\[ \lim_{\epsilon \to 0} \int d^2 x \log(\alpha |x|) K_{\alpha,\delta}(x, x; \epsilon) = (\log \alpha - \frac{\gamma_E}{2}) \left[ \frac{\delta(\delta - 1)}{2\alpha} + \frac{1 - \alpha^2}{12\alpha} \right] - \frac{1}{16i\pi} \int_{\Gamma} d\zeta \frac{e^{\frac{i}{2\alpha}\zeta(2\delta - 1)}}{\sin \frac{\zeta}{2\alpha}} \log(1 - \cos \zeta) \] (153)

It is easily checked that the last integral converges for \(|2\delta - 1| < 2\alpha + 1\). We notice that in our range \(\frac{1}{2} < \alpha < 2\), the above inequality is satisfied both for \(\delta = \alpha - 1\) and for \(\delta = 2\alpha - 1\). One can use the method of \([9]\) to write the integral of expression \((153)\) in \(\frac{\delta \alpha}{\alpha}\) as a single integral of real function.

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