Numerical evaluation of spherical GJMS determinants for even dimensions

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The functional determinants of the GJMS scalar operators, $P_{2k}$, on even–dimensional spheres are computed via Barnes multiple gamma functions relying on the numerical availability of the digamma function. For the critical $k = d/2$ case, it is necessary to calculate the Stirling moduli. The results are presented as graphs and show a series of extrema in the effective action ($\sim -\log \det$) as $k$ is varied in the reals. For odd dimensions these extrema occur at integer $k$.

The multiplicative anomalies are given as odd polynomials in $k$ and it is emphasised that that the Dirichlet–to–Robin factorisation, $P_{2l+1}$, $l \in \mathbb{Z}$, gives the same results as $P_{2k}$ if $k = l + 1/2$.

The formula for the Stirling moduli in terms of derivatives of the Riemann $\zeta$–function is rederived by an improved method.

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1. Introduction

In a recent note I gave a simple quadrature for the (log of) the functional determinant of the scalar GJMS operator on odd spheres. In this communication, I discuss the case of even spheres using a different method, which applies also to the odd case, so allowing a check.

Critical operators present a slightly complicated analysis requiring the separate computation of Barnes’ ‘Stirling moduli’ and so I begin with the subcritical situation which corresponds to the restriction \( k < d/2 \) where \( k \) is the level of the GJMS operator, \( P_{2k} \). On the \( d \)-sphere this operator takes the product form, [1],

\[
P_{2k}(d) = \prod_{j=0}^{k-1} \left( B^2 - \alpha_j^2 \right), \quad \alpha_j = j + 1/2,
\]

where \( B \equiv \sqrt{P_2 + 1/4} \) with \( P_2 = -\Delta_2 + ((d - 1)^2 - 1)/4 \) the Yamabe–Penrose operator (sometimes denoted by \( Y_d \)) on the sphere.

There is an alternative, analytic form,

\[
P_{2k}(d) = \frac{\Gamma(B + 1/2 + k)}{\Gamma(B + 1/2 - k)},
\]

continuing \( k \) off the integers.

The critical value of \( k \) is the one for which a zero mode first appears as \( k \) is increased, at which point the operator becomes the analogue of the minimally coupled scalar Laplacian.

2. The subcritical cases, \( k < d/2 \)

The basic formula has been given in [2] in terms of the multiple gamma function in one higher dimension,

\[
\log \det P_{2k} = \log \frac{\Gamma_{d+1}(d/2 + k) \Gamma_{d+1}(d/2 + k + 1)}{\Gamma_{d+1}(d/2 - k) \Gamma_{d+1}(d/2 - k + 1)} - 2M(d, (d - 1)/2, k). \tag{3}
\]

The first part of this formula follows, after a little algebra, from the unquestioning use of the product nature of \( P_{2k} \). The second part corrects this expectation, and constitutes a multiplicative anomaly. There seems to be no way of determining what this is except by direct calculation, as in [2] where an evaluation is detailed. I take the opportunity of correcting the expression displayed there.
3. The multiplicative anomaly

The anomaly, $M$, can be thought of as composed of two parts, one coming
from the factorisation of each bracket in (3) and the other from the product of
these brackets. I find,

$$M(d, a, k) = M_1(d, a, k) + M_2(d, a, k),$$

where,

$$M_1(d, a, k) = - \sum_{r=1}^{u} \frac{1}{r} \left( \sum_{j=0}^{k-1} \alpha_j^{2r} \right) H_k(r) N_{2r}(d, a),$$

and

$$M_2(d, a, k) = \frac{1}{2k} \sum_{r=1}^{u} \frac{1}{r} \sum_{t=1}^{u-r} \frac{1}{t} \sum_{i<j=0}^{k-1} \alpha_i^{2r} \alpha_j^{2t} N_{2r+2t}(d, a).$$

The upper limit $u$ equals $d/2$ for even dimensions, and $(d-1)/2$ for odd.

In these formulae $H_k$ is related to the harmonic series $H(r) = \sum_{n=1}^{r} 1/n$ by

$$H_k(r) = H(2r - 1) - \frac{1}{2k} H(r - 1),$$

and $N$ is the residue at the pole of the Barnes $\zeta$-function,

$$\zeta_d(s + r, a) \to \frac{N_r(d, a)}{s} + R_r(d, a) \quad \text{as } s \to 0,$$

where $1 \leq r \leq d$. The parameter $a$ occurs in the eigenvalues and equals $(d \pm 1)/2$
for Neumann/Dirichlet conditions on the rim of the $d$–hemisphere. The full sphere
is obtained by combining these two contributions. To complete the picture, $N$ is
given by a generalised Bernoulli polynomial,

$$N_r(d, a) = \frac{1}{(r-1)!(d-r)!} B_{d-r}(d)(a),$$

easily computed.

From a property of the Bernoulli polynomials, the multiplicative anomaly is
the same for Neumann and Dirichlet conditions, so in this case I denote both by

$$M(d, k) = M(d, (d \pm 1)/2, k).$$

For a given dimension, $d$, $M$ reduces to an odd polynomial in $k$ which vanishes
at $k = \pm 1/2$, both desired since reversing $k$ inverts the operator and $k = 1/2$
formally gives just one factor. I give some examples of the factored quantity $\mathcal{M}(d, k) \equiv M(d, k)/k(1 - 4k^2)$

\[
\begin{align*}
\mathcal{M}(4, k) &= \frac{1}{4320} (28k^2 - 33) \\
\mathcal{M}(6, k) &= \frac{1}{907200} (174k^4 - 1017k^2 + 955) \\
\mathcal{M}(8, k) &= \frac{1}{1524096000} (4656k^6 - 74556k^4 + 304600k^2 - 262185) \\
\mathcal{M}(10, k) &= \frac{9236k^8 - 311180k^6 + 3236824k^4 - 11339733k^2 + 9322110}{301771008000}.
\end{align*}
\]

(5)

For odd dimensions, the Neumann and Dirichlet anomalies are equal and opposite so adding for the full sphere gives zero. I list a few cases of the $\mathcal{M}(d, k)$,

\[
\begin{align*}
\mathcal{M}(3, k) &= \pm \frac{1}{24} \\
\mathcal{M}(5, k) &= \pm \frac{1}{4320} (14k^2 - 39) \\
\mathcal{M}(7, k) &= \pm \frac{1}{14515200} (1392k^4 - 14016k^2 + 28745) \\
\mathcal{M}(9, k) &= \pm \frac{1}{3048192000} (4656k^6 - 111096k^4 + 749695k^2 - 1355760).
\end{align*}
\]

(6)

I mention the curious fact that subtracting these Neumann and Dirichlet anomalies produces the Dirac anomalies on spheres of one dimension lower, up to the spin degeneracy factor.

4. The Dirichlet–to–Robin factorisation

When $k$ is a half-integer ($k = l + 1/2$) the intertwinor form (2) gives the alternative factorisation

\[
P_{2l+1}(d) = B \prod_{j=1}^{l} (B^2 - j^2)
= B \prod_{j=1}^{l} (B + j)(B - j),
\]

(7)

which is a well known Dirichlet–to–Robin boundary pseudo–operator, the simplest example being $P_1 = B$.

It can be shown by calculation that $\log \det P_{2l+1}$ produces the same function of $k$ as $\log \det P_{2k}$ if $k = l + 1/2$. This equality reinforces my continuation of $k$ into the reals. It will be considered further in a communication in preparation.
5. Computational method

The method consists of integrating the multiple digamma function, $\psi_d$, defined by,

$$\psi_d(z) = \frac{\partial}{\partial z} \log \Gamma_d(z),$$

because $\psi_d$ is expressible in terms of the ordinary digamma function which is available numerically.\(^2\) This results in the combination appearing in (3),

$$\log \frac{\Gamma_d(z_2)}{\Gamma_d(z_1)} = \int_{z_1}^{z_2} dz \psi_d(z), \quad (8)$$

with

$$\psi_d(z) = \frac{(-1)^{d-1}}{(d-1)!} \left( B_{d-1}^{(d)}(z) \psi(z) + Q_d(z) \right), \quad (9)$$

where the polynomial $Q$ is given by,

$$Q_d(z) = -(-1)^{d-1} \sum_{n=1}^{d-1} \frac{(-1)^n}{n} B_{d-n-1}^{(d-n)}(d-z) B_n^{(n)}(z).$$

The expression for $\psi_d$, (9), follows, [2], on iteration of recursion relations given by Barnes, [6]. (See also Onodera, [7].) Again I present just one example,

$$\psi_4(z) = -\frac{z(z-1)(z-2)}{6} \psi(z) + \frac{22z^3 - 114z^2 + 167z - 60}{72}. \quad (9)$$

It is then a simple matter to accurately evaluate the expression (3) since there are no poles of the $\psi$–function in the integration intervals and the multiplicative anomaly is explicit. I exhibit the results in the form of graphs which are more expressive than just numbers. The plots show minus the logdet, which, for convenience only, I refer to as the effective action.\(^3\)

I have treated $k$ as a continuous parameter which I can sensibly do in view of the polynomial nature of the multiplicative anomaly and the appearance of $k$ in just the integration limits. See remarks in section 4. (Equivalently, the multiple $\Gamma$–functions in (3) are functions of their arguments.)

Figures 1 to 3 plot the effective action for dimensions 2, 4, 6, 8, 10 and 12 against the scaled variable $2k/d$ while Figure 4 shows that for $d = 12$ against a portion

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\(^2\) The same numerical method is adopted by Adamchik, [3], for the double $\Gamma$–function, $G$. See also Kamela and Burgess, [4], and Basar and Dunne, [5], for example.

\(^3\) For ordinary scalar field theory there should be a factor of 1/2.
of the $k$ range. The curves show a series of extrema of which the deepest one by far is that close to the critical value $k = d/2$, where the effective action diverges positively.
6. The critical case, $k = d/2$

Allowing for the effects of the zero mode occurring when $k = d/2$ changes the expression for the log det to,

$$
\log \det P_d = \log \frac{(d-1)! \Gamma_{d+1}(d) \Gamma_{d+1}(d+1)}{\Gamma_{d+1}(1)} - 2M(d, d/2)
$$

$$
= \log \frac{\Gamma_{d+1}(d) \Gamma_{d+1}(d+1)}{\Gamma_{d+1}(1) \Gamma_{d+1}(1)} + \log \Gamma_{d+1}(1) + \log(d-1)! - 2M(d, d/2). \quad (10)
$$
The expression has been arranged so that the first term can be computed as before in terms of $\psi$, but at the expense of introducing the term $\log \Gamma_{d+1}(1) = \log \rho_d$, where $\rho_d$ is a $d$-ple Stirling modular form, [6].

It is thus seen that, in this approach, the values of the $\rho_d$ are required and I treat this as an independent calculation discussed in the next section.

Using the values there, I give some numbers for critical effective actions $\equiv -\log \text{dets} \equiv F(d)$, and a graph. I find,

$$F(2) = -1.16168458, \quad F(4) = -3.65377559, \quad F(6) = -7.00372497$$

$$F(8) = 10.96429058, \quad F(10) = -15.39808452, \quad F(12) = -20.21727752.$$  

The value of $F(2)$ agrees with the standard result for the (minimal) Laplacian on the two–sphere, i.e. $4\zeta'_R(-1) - 1/2$, going back to 1979.

![Fig.5. GJMS critical eff. action](image)

7. The Stirling moduli

The way I have chosen to compute the moduli, $\rho_r$, is via Barnes’ Binet–type formula which I copy here, [6] p.411,

$$\log \rho_r = -\int_0^{\infty} \frac{dz}{z} \left[ \frac{1}{(1 - e^{-z})^r} - \sum_{s=1}^{r-1} (-1)^s \frac{z^{s-r}}{s!} B_s^{(r)} \right] - 1 + \left( 1 - \frac{(-1)^r B_r^{(r)}}{r!} \right) e^{-z},$$

(11)

This could be regarded as a Weierstrass regularised expression. There seems to be an error in the formula in [6] which I have corrected.
where the $B^{(r)}_s$ are generalised Bernoulli numbers. This allows an adequate numerical treatment after dividing the integration range into three – a small $z$ polynomial part, a large $z$, asymptotic part and an intermediate region where the exact integrand has to be used. The Bernoulli numbers can be calculated by recursion and stored for fast recall.

I list a few values of the moduli

$$
\begin{align*}
\rho_2 &= 2.95754543, \quad \rho_3 = 3.26184986, \quad \rho_4 = 3.49650066 \\
\rho_5 &= 3.68915038, \quad \rho_6 = 3.85333102, \quad \rho_7 = 3.99679300 \\
\rho_8 &= 4.12443095, \quad \rho_9 = 4.23955124, \quad \rho_{10} = 4.34450055,
\end{align*}
$$

and plot a graph, Fig.6.

This is a somewhat workaday technique. A more elegant, but particular, formula results from expressing the relevant Barnes $\zeta$–function as a sum of Hurwitz $\zeta$–functions and is outlined in the following section.
8. More on moduli

I present some basic information and manipulation that warrants further exposure, although the results, if not the methods, are well known to workers in the field. I use Barnes’ original notation, except for the Bernoulli polynomials.

In the case that all the parameters are unity, Barnes, [8] p.431, showed that his \( \zeta \)–function could be expressed as a sum of Hurwitz functions.\(^5\) Thus, [8],

\[
\zeta_d(s,a) = \sum_{r=1}^{d} \frac{(-1)^{d-r}}{(r-1)!(d-r)!} B_{d-r}^{(d)}(a) \zeta_R(s + 1 - r, a). \tag{12}
\]

The definition of the multiple \( \Gamma \) is,

\[
\zeta'_d(0,a) = \log \frac{\Gamma_d(a)}{\rho_d}
\]

with the normalisation modulus, \( \rho_d \), given by,

\[
\log \rho_d = - \lim_{a \to 0} \left( \zeta'_d(0,a) + \log a \right).
\]

Further, differentiating the recursion for \( \Gamma_d \), it follows that \( \rho_d = \Gamma_{d+1}(1) \) so,

\[
\zeta'_d(0,1) = \log \frac{\rho_{d-1}}{\rho_d}, \tag{13}
\]

and therefore,

\[
\sum_{r=1}^{d} \zeta'_r(0,1) = - \log(\rho_d/\rho_0), \tag{14}
\]

which together with (12), constitutes a means of computing \( \rho_d \) since \( \rho_0 = 1 \) (following from the ‘trivial’ \( \zeta \)–function, \( \zeta_0(s,a) = a^{-s} \)). Hence,

\[
\log \rho_d = \sum_{r=0}^{d} A_r^{(d)} \zeta_R'(-r), \tag{15}
\]

in terms of the derivatives of the Riemann \( \zeta \)–function, which are available to any accuracy in some computer languages. In contrast, the expression (11), although

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\(^5\) This follows essentially by expanding the degeneracy, a binomial coefficient. On p.433, Barnes, with some foresight, remarks ‘It is evident that such algebra is capable of almost indefinite development’.

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an integral, employs only standard functions and it can be adapted easily to the
case when the parameters are not all unity.

Although the result is known, in various forms, I will derive the coefficients in
(15), from (12), the coefficients in which, at \( a = 1 \), are Stirling numbers, but I do
not use this. In fact there is no need to compute the sum (14) by brute force as I
will now show.

Moving to the required quantity, immediately from (12),

\[
\zeta_r'(0, 1) = \sum_{k=1}^{r} \frac{(-1)^{d-k}}{(k-1)!(r-k)!} B^{(r)}_{r-k}(1) \zeta_R'(1-k)
\]

\[
= \sum_{k=1}^{r} \frac{(-1)^{r-k}}{(k-1)!(r-k)!} \left( \frac{r}{k} B^{(r+1)}_{r-k}(1) + \frac{(r-1)(r-k)}{k} B^{(r)}_{r-k-1}(1) \right) \zeta_R'(1-k)
\]

\[
= \sum_{k=1}^{r} \frac{(-1)^{k}}{k!} \left( \frac{(-1)^{r}}{(r-k)!} B^{(r+1)}_{r-k}(1) - \frac{(-1)^{r-1}(r-1)}{(r-1-k)!} B^{(r)}_{r-k-1}(1) \right) \zeta_R'(1-k)
\]

(16)

where the recursion, [9] p.186,\(^6\)

\[
B^{(n+1)}_{\nu}(x) = \frac{n-\nu}{n} B^{(n)}_{\nu}(x) + (x-n) \frac{\nu}{n} B^{(n)}_{\nu-1}(x),
\]

has been employed.

Note now that the second term in brackets in (16) is the negative of the first
one, after setting \( r \to r - 1 \). On trivial summation of (14) by cancellation, or
otherwise, there results,

\[
\log \rho_d = d \sum_{k=1}^{d} \frac{(-1)^{d-k+1}}{k!} \frac{1}{(d-k)!} B^{(d+1)}_{d-k}(1) \zeta_R'(1-k).
\]

(17)

I now say, for comparison, that Stirling numbers, \( s \), can be introduced by the
relation,

\[
s(d, k) = \binom{d}{k} B^{(d+1)}_{d-k}(1),
\]

and (17) reads,

\[
\log \rho_d = \frac{1}{(d-1)!} \sum_{k=1}^{d} (-1)^{d-k+1} s(d, k) \zeta_R'(1-k).
\]

(18)

\(^6\) At \( x = 1 \) this is the recursion for Stirling numbers.
This result is derived by Kanemitsu, Kumagai and Yoshimoto, [10], in a more roundabout way. It tidies up Proposition 2.4 of Vardi, [11], who seems to have been the first to pursue the path leading to (15).

The easier identity,

$$\log \Gamma_d(1) = \frac{1}{(d-1)!} \sum_{k=1}^{d} (-1)^{d-k} s(d-1, k-1) \zeta_R'(1-k),$$

which is the first line of (16) (really due to Barnes), is given by Quine and Choi, [12], for example.

9. Odd dimensions

For odd dimensions, the same expression, (3), holds with zero multiplicative anomaly and the numerical evaluation proceeds as in section 3. As an example, Figure 7 plots the interpolation provided by (3), for \(d=9\), of the values computed in [13] for integer \(k\) by a different process. It shows extrema at these integers. This can be proved analytically.

![Fig. 7. GJMS eff. act, d=9](image)

10. Conclusion and remarks

The functional determinants of the GJMS scalar operators, \(P_{2k}\), on even–dimensional spheres have been computed via Barnes multiple gamma functions in terms of the digamma function. For the critical \(k = d/2\) case, it is necessary to
obtain the Stirling moduli. The results are presented as graphs and show a series of extrema in the ‘effective action’ ($\sim - \log \det P_{2k}$) as $k$ is varied in the reals. For odd dimensions these extrema occur at integer $k$. This is modified by the existence of a multiplicative anomaly for even dimensions.

I wish to draw attention to the fact that the continuations, in $k$, of $\log \det P_{2k}$, where $P_{2k}$ is given by (2), provided by the two (distinct) factorisations, (1) and (7), are identical.

The values confirm an expectation that the determinant tends to unity as the dimension increases. For odd dimensions, the explicit integral given in [13] proves immediately that this is so. Møller, [14], has shown, by a more complicated analysis, that it holds for odd and even dimensions for the simple Laplacian and the Dirac operator. These cases were discussed by Bär and Schopka, [15], who made the numerical observation which can also be seen in earlier calculations, [16], [12].

It is possible very easily to extend the calculations of the scalar GJMS operator to a Dirac version where the $B$ operator in (1) is replaced by the square root of the iterated (‘squared’) Dirac operator, and this will be detailed at another time. Product, higher derivative, higher spin propagation operators, and their determinants and anomalies, have been recently investigated by Tseytlin, [17]. Useful relevant analysis is also given by Aros and Diaz, [18].

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