A NOTE ON SOME SUB-GAUSSIAN RANDOM VARIABLES

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ABSTRACT. In [8] the author of this paper continued the research on the complex-valued discrete random variables $X_l(m, N)$ ($0 \leq l \leq N - 1$, $1 \leq M \leq N$) recently introduced and studied in [24]. Here we extend our results by considering $X_l(m, N)$ as sub-Gaussian random variables. Our investigation is motivated by the known fact that the so-called Restricted Isometry Property (RIP) introduced in [4] holds with high probability for any matrix generated by a sub-Gaussian random variable. Notice that sensing matrices with the RIP play a crucial role in Theory of compressive sensing.

Our main results concern the proofs of the lower and upper bound estimates of the expected values of the random variables $|X_l(m, N)|$, $|U_l(m, N)|$ and $|V_l(m, N)|$, where $U_l(m, N)$ and $U_l(m, N)$ are the real and the imaginary part of $X_l(m, N)$, respectively. These estimates are also given in terms of related sub-Gaussian norm $\| \cdot \|_{\psi_2}$ considered in [28]. Moreover, we prove a refinement of the mentioned upper bound estimates for the real and the imaginary part of $X_l(m, N)$.

1. INTRODUCTION AND PRELIMINARY RESULTS

The recent paper [24] by L.J. Stanković, S. Stanković and M. Amin provides a statistical analysis for efficient detection of signal components when missing data samples are present (cf. [25], [17, Section 2], [20] and [22]). This analysis is closely related to compressive sensing type problems. For more information on the development of compressive sensing (also known as compressed sensing, compressive sampling, or sparse recovery), see [6], [7], [19, Chapter 10] and [21]. For an excellent survey on this topic with applications and related references see [26] (also see [15]). Notice that in the statistical methodology presented in [24] a class of complex-valued discrete random variables (denoted in [8] as $X_l(m, N)$ with $0 \leq l \leq N - 1$ and $1 \leq M \leq N$), plays a crucial role.

Following [8], the random variable $X_l(m, N)$ can be defined as follows.

Definition 1.1. ([8, Definition 1.2]) Let $N$, $l$ and $m$ be arbitrary nonnegative integers such that $0 \leq l \leq N - 1$ and $1 \leq m \leq N$. Let $\Phi(l, N)$ be a multiset defined as

\[
\Phi(l, N) = \{e^{-j2\pi n/N} : n = 1, 2, \ldots, N\}.
\]
Define the discrete complex-valued random variable \( X_l(m,N) = X_l(m) \) as

\[
\text{Prob} \left( X_l(m,N) = \sum_{i=1}^{m} e^{-j2\pi i/N} \right) = \frac{1}{\binom{N}{m}} \cdot \left| \left\{ \{t_1,t_2,\ldots,t_m\} \subset \{1,2,\ldots,N\} : \sum_{i=1}^{m} e^{-j2\pi i/N} = \sum_{i=1}^{m} e^{-j2\pi n_i/N} \right\} \right|,
\]

where \( \{n_1,n_2,\ldots,n_m\} \) is an arbitrary fixed subset of \( \{1,2,\ldots,N\} \) such that \( 1 \leq n_1 < n_2 < \cdots < n_m \leq N \); moreover, \( q(n_1,n_2,\ldots,n_m) \) is the cardinality of a collection of all subsets \( \{t_1,t_2,\ldots,t_m\} \) of the set \( \{1,2,\ldots,N\} \) such that \( \sum_{i=1}^{m} e^{-j2\pi t_i/N} = \sum_{i=1}^{m} e^{-j2\pi n_i/N} \).

Let us recall that by (2) is well defined the random variable \( X_l(m,N) \) taking into account the general additive property of probabilility function \( \text{Prob}(\cdot) \) and the fact that there are \( \binom{N}{m} \) index sets \( T \subset \{1,2,\ldots,N\} \) with \( m \) elements.

As noticed in [8 Definition 1.2'], the random variable \( X_l(m,N) \) can be formally expressed as a sum

\[
X_l(m,N) = \sum_{n \in S} e^{-j2\pi n/N},
\]

where the summation ranges over any subset \( S \) of size \( m \) (the so-called \( m \)-element subset) without replacement from the set \( \{1,2,\ldots,N\} \). Notice that the number of these subsets \( S \) of \( \{1,2,\ldots,N\} \) is \( \binom{N}{m} \), and the probability of each value of \( X_l(m,N) \) is assumed to be equal \( 1/\binom{N}{m} \).

As usually, throughout our considerations we use the term “multiset” (often written as “set”) to mean “a totality having possible multiplicities”; so that two (multi)sets will be counted as equal if and only if they have the same elements with identical multiplicities.

Here as always in the sequel, we will denote by \( \mathbb{E}[X] \) and \( \text{Var}[X] \) the expected value and the variance of any complex-valued (or real-valued) random variable \( X \). Moreover, for any random variable \( X_l(m,N) \) from Definition 1.1 we shall write

\[
X_l(m,N) = U_l(m,N) + jV_l(m,N),
\]

where \( U_l(m,N) \) is the real part and \( V_l(m,N) \) is the imaginary part of \( X_l(m,N) \). Of course, \( U_l(m,N) \) and \( V_l(m,N) \) can be considered as the real-valued random variables associated with \( X_l(m,N) \). If \( l \geq 1 \), then it was proved in [24] (also see [8] (18) of Theorem 2.4) that

\[
\mathbb{E}[X_l(m,N)] = \mathbb{E}[U_l(m,N)] = \mathbb{E}[U_l(m,N)] = 0,
\]

Furthermore, it was proved in [24] (also see [8] (19) of Theorem 2.4) that

\[
\text{Var}[X_l(m,N)] = \mathbb{E}[[X_l(m,N)]^2] = \frac{m(N-m)}{N-1},
\]
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whenever $1 \leq l \leq N - 1$ and $1 \leq m \leq N$. Moreover, if in addition, we suppose that $N \neq 2l$, then [8] (23) of Corollary 2.6]

$$E[(U_l(m, N))^2] = E[(V_l(m, N))^2] = \frac{m(N - m)}{2(N - 1)}. \tag{6}$$

It was also proved in [8, Theorem 2.8] that if $l \neq 0$, then for every positive integer $k$ that is not divisible by $N/\gcd(N, l)$ ($\gcd(N, l)$ denotes the greatest common divisor of $N$ and $l$), the $k$th moment $\mu_k := E[X_l(m, N)]$ of the random variable $X_l(m, N)$ is equal to zero. In general case, $\mu_k = |X_l(m, N)|$ is a real number [8, Proposition 2.10].

Notice that (1) for $l = 0$ implies that

$$\Phi(0, N) = \{1, \ldots, 1\}.$$

Moreover, it is obvious that the multiset $\Phi(l, N)$ given by (1) is in fact the set consisting of $N$ (distinct) elements if and only if $l$ and $N$ are relatively prime positive integers (for related discussion, see [11]).

Recall that by using an Elementary Number Theory approach to some compressive sensing problems, different classes of random variables $X_l(m, N)$ are considered and compared in [11]. Furthermore, in order to establish a probabilistic approach to Welch bound on the coherence of a matrix over the field $\mathbb{C}$ (or $\mathbb{R}$), a generalization of the random variable $X_l(m, N)$ is defined and studied in [10]. For more information on the coherence of a matrix and related Welch bound, see [7, Chapter 5, Theorem 5.7] (also see [23], [18] and [29]).

Notice also that a Bernoulli probability model, similar to the distribution $\tilde{X}_l(m, N)$ defined below, was often used in the famous paper [3] by Cand`es, Romberg and Tao. Accordingly, we believe that for some further probabilistic studies of sparse signal recovery, it can be of interest the complex-valued discrete random variable $\tilde{X}_l(m, N)$ defined in [9]. Namely, for nonnegative integers $N, l$ and $m$ such that $0 \leq l \leq N - 1$ and $1 \leq m \leq N$, in [9] it was studied in some sense analogous random variable $\tilde{X}_l(m, N)$ to the random variable $X_l(m, N)$, defined as a sum

$$\tilde{X}_l(m, N) = \sum_{n=1}^{N} \exp\left(-\frac{2jnl\pi}{N}\right) B_n,$$

where $B_n$ ($n = 1, \ldots, N$) are independent identically distributed Bernoulli random variables (binomial distributions) taking only the values 0 and 1 with probability $0$ and $m/N$, respectively, i.e.,

$$B_n = \begin{cases} 0 & \text{with probability } 1 - \frac{m}{N} \\ 1 & \text{with probability } \frac{m}{N}. \end{cases}$$

Clearly, the range of the random variable $\tilde{X}_l(m, N)$ consists of all possible $2^N - 1$ sums of the elements of the (multi)set $\{e^{-2j2n\pi/N} : n = 1, 2, \ldots, N\}$.

If $l \geq 1$, then it is proved in [9] Proposition 2.1] that

$$E[\tilde{X}_l(m, N)] = E[\tilde{U}_l(m, N)] = E[\tilde{V}_l(m, N)] = 0. \tag{7}$$
Furthermore, it is proved in [9] Proposition 2.1 that

\[ \text{Var}[\tilde{X}_l(m, N)] = \frac{m(N - m)}{N}. \]  

If in addition we suppose that \( N \neq 2l \), then [9, Proposition 2.1]

\[ \text{Var}[\tilde{U}_l(m, N)] = \text{Var}[\tilde{V}_l(m, N)] = \frac{m(N - m)}{2N}. \]

**Remark 1.2.** From (4) and (7) it follows that for each \( l \geq 1 \) \( X_l(m, N) \) and \( \tilde{X}_l(m, N) \) are zero-mean random variables. From the expressions (5) and (8) it follows that

\[ \frac{\text{Var}[X_l(m, N)]}{\text{Var}[\tilde{X}_l(m, N)]} = \frac{N}{N-1}, \]

i.e.,

\[ \frac{\sigma[X_l(m, N)]}{\sigma[\tilde{X}_l(m, N)]} = \sqrt{\frac{N}{N-1}}. \]

Furthermore, if \( N \neq 2l \), then from (6) and (9) of [8] Theorem 2.4] we find that the proportions (10) and (11) are also valid after replacing \( X_l(m, N) \) by \( U_l(m, N) \) (resp. \( V_l(m, N) \)) and \( \tilde{X}_l(m, N) \) by \( \tilde{U}_l(m, N) \) (resp. \( \tilde{V}_l(m, N) \)).

Notice that in Statistics the uncorrected sample variance or sometimes the variance of the sample (observed values) \( \{x_1, x_2, \ldots, x_N\} \) with the arithmetic mean value \( \bar{x} \), is defined as

\[ s_N = \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2. \]

If the biased sample variance (the second central moment of the sample, which is a downward-biased estimate of the population variance) is used to compute an estimate of the population standard deviation, the result is equal to \( s_N \) given by the above formula.

An unbiased estimator of the variance is given by applying Bessel’s correction, using \( N - 1 \) instead of \( N \) to yield the unbiased sample variance, denoted by \( \bar{s}_N^2 \) and defined as

\[ \bar{s}_N^2 = \frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2. \]

From (12) and (13) we see that the proportion (10) can be extended as

\[ \frac{\text{Var}[X_l(m, N)]}{\text{Var}[\tilde{X}_l(m, N)]} = \frac{s_N^2}{\bar{s}_N^2} = \frac{N}{N-1}. \]

The above proportion suggests the fact that probably in some statistical sense between the random variables \( X_l(m, N) \) and \( \tilde{X}_l(m, N) \) there exists a “connection of type unbiased sample variance - biased sample variance”. Moreover, the values \( N/(N-1) \) should be influenced by the fact that \( \tilde{X}_l(m, N) \) is a sum of \( N \) independent random variables, while the random variable \( X_l(m, N) \) is defined on the set \( \Phi(l, N) \) consisting
of \( N \) (not necessarily distinct) elements that are “not independent” in the sense that their sum is equal to zero.

Notice that the random variables \( X_l(m, N) \) and \( \tilde{X}_l(m, N) \) and their real and imaginary parts are bounded random variables. Therefore, all these random variables are sub-Gaussian (see Section 2). In Section 2, we give the assertions concerning the lower and upper bound estimates of the expected values of the random variables \(|X_l(m, N)|\), \(|U_l(m, N)|\) and \(|V_l(m, N)|\). These estimates are also given in terms of related sub-Gaussian norm \( \| \cdot \|_{\psi_2} \) considered in [28]. Moreover, we formulate a refinement of the all mentioned upper bound estimates concerning the random variables \(|U_l(m, N)|\) and \(|V_l(m, N)|\). Proofs of all these estimates are given in Section 3.

2. The main results

Theorem 2.1. Let \( N \geq 2 \), \( l \) and \( m \) be nonnegative integers such that \( 0 \leq l \leq N - 1 \) and \( 1 \leq m \leq N \). Then the following probability estimates are satisfied:

(i) \( e^{\frac{N-m}{m(N-1)}} \leq \mathbb{E} \left[ \exp \left( \frac{|X_l(m, N)|^2}{m^2} \right) \right] \leq e; \)

(ii) \( e^{\frac{N-m}{m(N-1)}} \leq \mathbb{E} \left[ \exp \left( \frac{|U_l(m, N)|^2}{m^2} \right) \right] \leq e; \)

(iii) \( e^{\frac{N-m}{m(N-1)}} \leq \mathbb{E} \left[ \exp \left( \frac{|V_l(m, N)|^2}{m^2} \right) \right] \leq e \quad \text{if} \quad l \geq 1. \)

Notice that the estimates on the right hand side of (i), (ii) and (iii) of Theorem 2.1 are rough because of the fact they are directly obtained by using only the fact that the random variables \(|X_l(m, N)|\), \(|U_l(m, N)|\) and \(|V_l(m, N)|\) are upper bounded by the constant \( m \). Accordingly, if \( l \geq 1 \), then the equality in each of these inequalities holds if and only if \( N = 1 \), i.e., when \( X_l(m, N) \), \( U_l(m, N) \) and \( V_l(m, N) \) are constant random variables identically equal to one. We believe that for non-constant cases, these inequalities should be significantly improved.

Theorem 2.1 can be reformulated as follows.

Theorem 2.2. Let \( N \geq 2 \), \( l \) and \( m \) be nonnegative integers such that \( 0 \leq l \leq N - 1 \) and \( 1 \leq m \leq N \). Then the following probability estimates are satisfied:

(i) \( e^{\frac{m(N-m)}{(N-1)}} \leq \mathbb{E} \left[ \exp \left( \frac{|X_l(m, N)|^2}{m^2} \right) \right] \leq e^{m^2}; \)

(ii) \( e^{\frac{m(N-m)}{(N-1)}} \leq \mathbb{E} \left[ \exp \left( \frac{|U_l(m, N)|^2}{m^2} \right) \right] \leq e^{m^2}; \)

(iii) \( e^{\frac{m(N-m)}{(N-1)}} \leq \mathbb{E} \left[ \exp \left( \frac{|V_l(m, N)|^2}{m^2} \right) \right] \leq e^{m^2} \quad \text{if} \quad l \geq 1. \)

Let us recall that a real-valued random variable \( X \) is sub-Gaussian if its distribution is dominated by a normal distribution. More precisely, a real-valued random variable \( X \) is sub-Gaussian if there holds

\[ \text{Prob}(|X| > t) \leq \exp \left( 1 - \frac{t^2}{C^2} \right) \quad \text{for all} \quad t \geq 0, \]

where \( C > 0 \) is a real constant that does not depends on \( t \).

A systematic introduction into sub-Gaussian random variables can be found in [27], Lemma 5.5 in Subsection 5.2.3 and Subsection 5.2.5; here we briefly mention the basic definitions. Notice that the Restricted Isometry Property (RIP) holds with high probability for any matrix generated by a sub-Gaussian random variable (see [5], [16]).
Moreover, a relationship between the concepts of coherence and RIP of a matrix was established in [1] and [2]. Namely, in these papers it is proved that a matrix $A$ with the coherence $\mu(A)$ satisfies the RIP with the sparsity order $k \leq \frac{1}{\mu(A)} + 1$. Therefore, it is desirable to give explicit construction of matrices with small coherence in compressed sensing.

One of several equivalent ways to define this rigorously is to require the Orlicz norm $\|X\|_{\psi_2}$ defined as

$$\|X\|_{\psi_2} := \inf\{K > 0 : \mathbb{E} \left[ \psi_2 \left( \frac{|X|}{K} \right) \right] \leq 1\}$$

to be finite, for the Orlicz function $\psi_2(x) = \exp(x^2) - 1$. The class of sub-Gaussian random variables on a given probability space is thus a normed space endowed with Orlicz norm $\| \cdot \|_{\psi_2}$. This definition is in spirit topological in view of the fact that the classical Orlicz norm is used for the definition of many topological vector spaces. For more details on the Orlicz function and related topological vector spaces, see [14]. Recall that in Real and Complex Analysis many function spaces are endowed with the topology induced by an Orlicz norm (see [12, Chapter 7] and [13]).

Obviously, (cf. [28, Definitions 2.5.6 and Example 2.7.13]) the above Orlicz norm $\| \cdot \|_{\psi_2}$ is exactly the sub-Gaussian norm $\| \cdot \|'_{\psi_2}$ which is for the sub-Gaussian real-valued random variable $X$ defined as

$$\|X\|'_{\psi_2} = \inf\{K > 0 : \mathbb{E} \left[ \exp \left( \frac{X^2}{K^2} \right) \right] \leq 2\}.$$

Accordingly, in the sequel we shall write $\| \cdot \|_{\psi_2}$ instead of $\| \cdot \|'_{\psi_2}$.

In view of the mentioned facts, a random variable $X$ is sub-Gaussian if and only if

$$\mathbb{E} \left[ \exp \left( \frac{X^2}{\psi} \right) \right] \leq 2$$

for some real constant $\psi > 0$. Hence, any bounded real-valued random variable $X$ is sub-Gaussian, and clearly, there holds

$$\|X\|_{\psi_2} \leq \frac{1}{\sqrt{\ln 2}} \|X\|_{\infty} \approx 1.20112 \|X\|_{\infty},$$

where $\| \cdot \|_{\infty}$ is the usual supremum norm. Moreover, if $X$ is a centered normal random variable with variance $\sigma^2$, then $X$ is sub-Gaussian with $\|X\|_{\psi_2} \leq C\sigma$, where $C$ is an absolute constant [27, Subsection 5.2.4].

Another definition of the sub-Gaussian norm $\|X\|'_{\psi_2}$ for the sub-Gaussian random variable $X$ was given in [27, Definition 5.7] as

$$\|X\|'_{\psi_2} = \sup_{p \geq 1} \left( p^{-1/2} \left( \mathbb{E}[|X|^p] \right)^{1/p} \right).$$

Obviously, there holds

$$\|X\|'_{\psi_2} \leq \|X\|_{\infty}.$$

In particular, $X_i(m, N)$, $U_i(m, N)$ and $V_i(m, N)$ are sub-Gaussian random variables. Clearly, in terms of the sub-Gaussian norm $\| \cdot \|_{\psi_2}$ Theorem 2.2 can be reformulated as follows.
Proposition 2.3. Let $N \geq 1$, $l$ and $m$ be nonnegative integers such that $0 \leq l \leq N - 1$ and $1 \leq m \leq N$. Then $|X_l(m, N)|$, $U_l(m, N)$ and $V_l(m, N)$ are sub-Gaussian random variables. Moreover, there holds
\begin{align*}
\text{(i)} & \quad \sqrt{\frac{m(N-m)}{(N-1)\ln 2}} \leq ||X_l(m, N)||_{\psi^2} \leq \frac{m}{\sqrt{\ln 2}}; \\
\text{(ii)} & \quad \sqrt{\frac{m(N-m)}{2(N-1)\ln 2}} \leq ||U_l(m, N)||_{\psi^2} \leq \frac{m}{\sqrt{\ln 2}}; \\
\text{(iii)} & \quad \sqrt{\frac{m(N-m)}{2(N-1)\ln 2}} \leq ||V_l(m, N)||_{\psi^2} \leq \frac{m}{\sqrt{\ln 2}} \quad \text{if} \quad l \geq 1.
\end{align*}

The upper bound $m/\sqrt{\ln 2}$ on the right hand side of the estimates (ii) and (iii) of Proposition 2.3 can be improved for a large class of random variables $U_l(m, N)$ and $V_l(m, N)$. This is given by the following result.

Proposition 2.4. Let $N \geq 2$, $l$ and $m$ be positive integers such that $1 \leq l \leq N - 1$ and $1 \leq m \leq N$. If $N$ and $l$ are relatively prime, then
\begin{align*}
\text{(14)} & \quad ||U_l(m, N)||_{\psi^2} \leq \frac{\sin \frac{m\pi}{N}}{\sqrt{\ln 2} \sin \frac{\pi}{N}} \\
\text{(15)} & \quad ||V_l(m, N)||_{\psi^2} \leq \begin{cases} 
\frac{\sin \frac{m\pi}{N} \sin \frac{2(N-1)\pi}{4}}{\sqrt{\ln 2} \sin \frac{\pi}{N}} & \text{if } m \text{ is even} \\
\frac{\sin \frac{m\pi}{N} \sin \frac{2(N+1)\pi}{4N}}{\sqrt{\ln 2} \sin \frac{\pi}{N}} & \text{if } m \text{ is odd}
\end{cases}
\end{align*}

Remark 2.5. Notice that if $m \sim cN$ for some constant $c$ with $0 < c \leq 1/2$ and all sufficiently large values of $N$, then $\sin(\pi/N) \approx \pi/N$ and thus, the upper bound on the right hand side of estimates (14) and (15) is
\[ \sim N \sin(c\pi)/(\pi\sqrt{\ln 2}) = 0.382329N \sin(c\pi). \]

On the other hand, from (ii) and (iii) of Proposition 2.3 we see that for such a value $m$, the lower bound on the left hand side of the estimates (ii) and (iii) is
\[ \sim \sqrt{c(1-c)N/(2\ln 2)}. \]

For example, if $m \sim N/2$ (i.e., $c = 1/2$), then these upper and lower bounds are approximately equal to $0.382329N$ and $0.424661\sqrt{N}$, respectively.

From the estimates (14), (15) and proof of Proposition 2.4, taking into account that $|X_l(m, N)| = \sqrt{(U_l(m, N))^2 + (V_l(m, N))^2}$, it follows immediately the following result.

Proposition 2.6. Let $N \geq 2$, $l$ and $m$ be positive integers such that $1 \leq l \leq N - 1$ and $1 \leq m \leq N$. If $N$ and $l$ are relatively prime, then
\[ ||X_l(m, N)||_{\psi^2} \leq \frac{\sqrt{2} \sin \frac{m\pi}{N}}{\sqrt{2} \sin \frac{\pi}{N}}. \]
3. Proofs of the results

Proof of Theorem 2.1. First notice that for \( l = 0 \) and all \( m \) with \( 1 \leq m \leq N \), \( |X_0(m, N)|, U_0(m, N) \) and \( V_0(m, N) \) are constant random variables with

\[
\text{Prob}(|X_0(m, N)| = m) = \text{Prob}(U_0(m, N) = m) = \text{Prob}(V_0(m, N) = 0) = 1.
\]

Therefore, both double inequalities (i) and (ii) are satisfied.

Now suppose that \( 1 \leq l \leq N - 1 \). Since the random variables \(|X_l(m, N)|^2\), \((U_l(m, N))^2\) and \((V_l(m, N))^2\) are obviously bounded below by the constant \( m^2 \), the inequalities on the right hand side of (i), (ii) and (iii) are trivially satisfied.

Notice that

\[
E \left[ \exp \left( \frac{|X_l(m, N)|^2}{m^2} \right) \right] = \frac{1}{\binom{N}{m}} \sum_{\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, N\}} \exp \left( \frac{w_{i_1} + w_{i_2} + \cdots + w_{i_m}}{m^2} \right) \frac{(w_{i_1} + w_{i_2} + \cdots + w_{i_m})}{m^2},
\]

where the summation ranges over all \( \binom{N}{m} \) subsets \( \{i_1, i_2, \ldots, i_m\} \) of \( \{1, 2, \ldots, N\} \) with \( 1 \leq i_1 < i_2 < \cdots < i_m \leq N \). Notice that

\[
A_{\{i_1, i_2, \ldots, i_m\}} := \exp \left( \frac{w_{i_1} + w_{i_2} + \cdots + w_{i_m}}{m^2} \right) \frac{(w_{i_1} + w_{i_2} + \cdots + w_{i_m})}{m^2}
\]

are positive real numbers for each subset \( \{i_1, i_2, \ldots, i_m\} \) of \( \{1, 2, \ldots, N\} \) with \( 1 \leq i_1 < i_2 < \cdots < i_m \leq N \). Then applying to these numbers the classical arithmetic-geometric mean inequality \( \left( \sum_k a_k \right) / n \geq \sqrt[n]{\prod_k a_k} \) \((n \in \mathbb{N}, a_1, \ldots, a_n \in \mathbb{R}^+)\), and using the expression (16), we find that the right hand side of this expression is

\[
\geq \binom{N}{m} \sqrt[m^2]{\sum_{\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, N\}} \left( \frac{w_{i_1} + w_{i_2} + \cdots + w_{i_m}}{m^2} \right) \left( \frac{(w_{i_1} + w_{i_2} + \cdots + w_{i_m})}{m^2} \right)}
\]

\[
= \binom{N}{m} \sqrt[m^2]{\frac{1}{m} \sum_{\{i_1, i_2, \ldots, i_m\} \subset \{1, 2, \ldots, N\}} \left( \frac{N - m}{m(N - 1)} \right)} = \exp \left( \frac{N - m}{m(N - 1)} \right).
\]

This proves the left hand side of the inequality (i) of Theorem 2.1.

Finally, notice that the left hand sides of inequalities (ii) and (iii) of Theorem 2.1 can be proved in the same manner as that of (i), using in the final step the first and the second equality of the expression (6), respectively. Hence, these proofs can be omitted, and proof of Theorem 2.1 is completed.

\[\square \]

Proof of Theorem 2.2. Proof of Theorem 2.2 is completely similar to those of Theorem 2.1 and hence, may be omitted.

\[\square \]

Proof of Proposition 2.3. The first assertion is an immediate consequence of inequalities on the right hand sides of (i), (ii) and (iii) of Theorem 2.1. The inequalities on the right hand side of (i), (ii) and (iii) are also immediate consequences of the inequalities on the right hand sides of (i), (ii) and (iii) of Theorem 2.1, respectively. Finally, the inequalities on the left hand side of (i), (ii) and (iii) can be proved in the same manner as those of (i) of Theorem 2.1.

\[\square \]
Proof of Proposition 2.4. Since by the assumption, \( N \) and \( l \) are relatively prime positive integers, then the multiset \( \Phi(l, N) \) defined by (1) consists of \( N \) distinct elements, and it can be written as

\[
\Phi(l, N) = \{1, w, w^2, \ldots, w^{N-1}\},
\]

where \( w = \exp\left(\frac{2\pi j}{N}\right) \) is the primitive \( N \)th root of unity. Then the ranges (the sets of all values) of the random variables \( U_l(m, N) \) and \( V_l(m, N) \) are respectively given by

\[
\begin{align*}
(18) & \quad \mathcal{R}(U_l(m, N)) = \left\{ \cos \frac{2k_1\pi}{N} + \cos \frac{2k_2\pi}{N} + \cdots + \cos \frac{2k_m\pi}{N} : 0 \leq k_1 < k_2 < \cdots < k_m \leq N - 1 \right\} \\
(19) & \quad \mathcal{R}(V_l(m, N)) = \left\{ \sin \frac{2k_1\pi}{N} + \sin \frac{2k_2\pi}{N} + \cdots + \sin \frac{2k_m\pi}{N} : 0 \leq k_1 < k_2 < \cdots < k_m \leq N - 1 \right\}.
\end{align*}
\]

In the whole proof \( M_1 \) and \( M_2 \) will always denote the maximal value and the minimal value of considered random variable \( U_l(m, N) \) or \( V_l(m, N) \), respectively. In order to obtain the upper bounds for \( \|U_l(m, N)\|_\infty \) and \( \|V_l(m, N)\|_\infty \), in view of the antisymmetric property of random variables \( U_l(m, N) \) and \( V_l(m, N) \) given in [8, Proposition 2.1], without loss of generality, in the whole proof we can suppose that \( m \leq \lfloor N/2 \rfloor \) (\( \lfloor x \rfloor \) denotes the greatest integer not exceeding a real number \( x \)).

Proof of the inequality (14). As noticed in Section 2, every bounded random variable \( X \) is sub-Gaussian, and there holds

\[
(20) \quad \|X\|_{\psi_2} \leq \frac{1}{\sqrt{\ln 2}} \|X\|_\infty,
\]

where \( \| \cdot \|_\infty \) is the usual supremum norm.

We will consider the cases when a positive integer \( m \) is odd and when \( m \) is even.

The first case: \( m \) is an odd positive integer. Put \( m = 2s + 1 \) with integer \( s \geq 0 \). If \( s = 0 \) then \( m = 1 \), and hence,

\[
\mathcal{R}(U_l(1, N)) = \left\{ 1, \cos \frac{2\pi}{N}, \ldots, \cos \frac{2(N-1)\pi}{N} \right\}.
\]

Therefore, \( \|U_l(m, N)\|_\infty \leq 1 \), which together with (20) yields

\[
\|X\|_{\psi_2} \leq \frac{1}{\sqrt{\ln 2}}.
\]

Notice that the above inequality coincides with (14) for \( m = 1 \).

Now suppose that \( s \geq 1 \), i.e., \( m \geq 3 \). Since by the above assumption, \( m \leq \lfloor N/2 \rfloor \), it follows that \( s \leq \lfloor N/2 \rfloor / 2 - 1 \leq N/4 - 1 \), and hence, we have

\[
(21) \quad \cos \frac{2k\pi}{N} > 0 \quad \text{for all} \quad k = 1, 2, \ldots, s.
\]

Since the function \( f(x) := \cos x \) is decreasing on the segment \([0, \pi]\) and since \( \cos x = \cos(2\pi - x) \), in view of (18) and (21), we conclude that the random variable \( U_l(m, N) \)
attains its maximal value equals to

\[ M_1 = 1 + \sum_{k=1}^{s} \cos \frac{2k\pi}{N} + \sum_{k=1}^{s} \cos \frac{2(N-k)\pi}{N} = 1 + 2 \sum_{k=1}^{s} \cos \frac{2k\pi}{N}. \]  

Since \( \cos \frac{2k\pi}{N} = \Re \left( \exp \left(2k\pi j/N \right) \right) = \Re(w^k) \), from (22) we obtain

\[ M_1 = 1 + 2 \sum_{k=1}^{s} \Re(w^k) = 1 + 2\Re \left( \sum_{k=1}^{s} w^k \right) \\
= 1 + 2\Re \left( \frac{w - w^{s+1}}{1-w} \right) = 1 + 2\Re \left( \frac{w - w^{s+1}}{1-w} \cdot \frac{1-w}{1-w} \right) \\
= 1 + 2 \cdot \Re \left( \frac{w - 1 - w^{s+1} + w^s}{1 - 2\Re(w) + |w|^2} \right) = 1 + 2 \cdot \frac{\cos \frac{2\pi}{N} - 1 - \cos \frac{2(s+1)\pi}{N} + \cos \frac{2s\pi}{N}}{2 - 2 \cos \frac{2\pi}{N}} \\
(23) \quad = \frac{\cos \frac{2s\pi}{N} - \cos \frac{2(s+1)\pi}{N}}{1 - \cos \frac{2\pi}{N}} \\
(by \ using \ the \ identities \ \cos \alpha - \cos \beta = 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta - \alpha}{2} \ \text{and} \ \ 1 - \cos 2\alpha = 2 \sin^2 \alpha) \\
= \frac{2 \sin \frac{(2s+1)\pi}{N} \sin \frac{\pi}{N}}{2 \sin^2 \frac{\pi}{N}} = \frac{\sin \frac{(2s+1)\pi}{N}}{\sin \frac{\pi}{N}} = \frac{\sin \frac{m\pi}{N}}{\sin \frac{\pi}{N}}. \]

In order to determine the minimal value \( M_2 \) of the random variable \( U_l(m, N) \), we will consider the following two subcases:

The first subcase: \( N \) is an even positive integer. Take \( N = 2n \) with \( n \in \mathbb{N} \). Then by using the same argument applied for determining the above maximal value \( M_1 \) of \( U_l(m, N) \), (22) and (23), we obtain

\[ M_2 = \cos \frac{2n\pi}{2n} + \sum_{t=n-s}^{n-1} \cos \frac{2t\pi}{2n} + \sum_{t=n+1}^{n+s} \cos \frac{2t\pi}{2n} \\
= -1 + \sum_{k=1}^{s} \cos \frac{2(n-k)\pi}{2n} + \sum_{k=1}^{s} \cos \frac{2(n+k)\pi}{2n} \\
= -1 - \sum_{k=1}^{s} \cos \frac{2k\pi}{2n} - \sum_{k=1}^{s} \cos \frac{2k\pi}{2n} \quad (\text{the change} \ 2n = N) \\
= -M_1 = -\frac{\sin \frac{m\pi}{N}}{\sin \frac{\pi}{N}}. \]
The second subcase: \( N \) is an odd positive integer. Take \( N = 2n + 1 \) with \( n \in \mathbb{N} \). Then similarly as above, we find that

\[
M_2 = \cos \frac{2(n - s) \pi}{2n + 1} + \sum_{t=n-s+1}^{n} \cos \frac{2t \pi}{2n + 1} + \sum_{t=n+1}^{n+s} \cos \frac{2t \pi}{2n + 1}
\]

\[
= - \cos \left( \pi - \frac{2(n - s) \pi}{2n + 1} \right) - \sum_{t=n-s+1}^{n} \cos \left( \pi - \frac{2t \pi}{2n + 1} \right) - \sum_{t=n+1}^{n+s} \cos \left( \frac{2t \pi}{2n + 1} - \pi \right)
\]

\[
= - \cos \left( \frac{(2s + 1) \pi}{2n + 1} \right) - \sum_{t=n-s+1}^{n} \cos \left( \frac{2n + 1 - 2t \pi}{2n + 1} \right) - \sum_{t=n+1}^{n+s} \cos \left( \frac{2t \pi}{2n + 1} - \pi \right)
\]

\[
= - \cos \left( \frac{(2s + 1) \pi}{2n + 1} \right) - \sum_{k=1}^{s} \cos \left( \frac{(2k - 1) \pi}{2n + 1} \right) - \sum_{k=1}^{s} \cos \left( \frac{(2k - 1) \pi}{2n + 1} \right)
\]

\[
= - \cos \left( \frac{(2s + 1) \pi}{2n + 1} \right) - 2 \sum_{k=1}^{s} \cos \left( \frac{(2k - 1) \pi}{2n + 1} \right).
\]
If we put $\xi = \exp(j\pi/(2n+1))$, then $\cos \frac{t\pi}{2n+1} = \Re(\xi^t)$ for each $t \in \mathbb{N}$, and hence, from (25) we get

$$M_2 = -\cos \frac{(2s+1)\pi}{2n+1} - 2 \sum_{k=1}^{s} \Re(\xi^{2k-1}) = -\cos \frac{(2s+1)\pi}{2n+1} - 2 \Re \left( \sum_{k=1}^{s} \xi^{2k-1} \right)$$

$$= -\cos \frac{(2s+1)\pi}{2n+1} - 2 \Re \left( \frac{\xi - \xi^{2s+1}}{1 - \xi^2} \right)$$

$$= -\cos \frac{(2s+1)\pi}{2n+1} - 2 \Re \left( \frac{\xi - \xi^{2s+1}}{1 - \xi^2} \cdot \frac{1 - \xi^2}{1 - \xi^2} \right)$$

$$= -\cos \frac{(2s+1)\pi}{2n+1} - 2 \Re(\xi^{2s-1} - \xi^{2s+1})$$

**(26)**

$$= -\cos \frac{(2s+1)\pi}{2n+1} - \cos \frac{(2s+1)\pi}{2n+1} - \cos \frac{(2s-1)\pi}{2n+1}$$

(by using the identity $\cos \alpha - \cos \beta = 2 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta - \alpha}{2}$ and $1 - \cos 2\alpha = 2 \sin^2 \alpha$)

$$= -\cos \frac{(2s+1)\pi}{2n+1} - 2 \sin \frac{2s\pi}{2n+1} \sin \frac{\pi}{2n+1}$$

$$= -\cos \frac{(2s+1)\pi}{2n+1} - \frac{\sin \frac{\pi}{2n+1}}{\sin \frac{2\pi}{2n+1}} + \sin \frac{2s\pi}{2n+1}$$

(by using the identity $\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$)

$$= -\sin \frac{\pi}{2n+1} \cos \frac{(2s+1)\pi}{2n+1} + \sin \frac{(2s+1)\pi}{2n+1} \cos \sin \frac{\pi}{2n+1} - \cos \frac{(2s+1)\pi}{2n+1} \sin \frac{\pi}{2n+1}$$

$$= -\sin \frac{(2s+1)\pi}{2n+1} \cos \frac{\pi}{2n+1} - \sin \frac{(2s+1)\pi}{2n+1} \cos \frac{\pi}{2n+1}$$

$$= -\sin \frac{(2s+1)\pi}{2n+1} \cos \frac{\pi}{2n+1} - \sin \frac{(2s+1)\pi}{2n+1} \cos \frac{\pi}{2n+1}$$

From (23), (24) and (26) we see that $|M_2| \leq M_1$ for every odd integer $m \geq 3$, and hence for such a $m$ we have

$$\|U_l(m, N)\|_{\infty} = \max \{M_1, |M_2|\} = \frac{\sin \frac{m\pi}{N}}{\sqrt{\ln 2 \sin \frac{\pi}{N}}}.$$

From (20) and (27) we immediately obtain

$$\|U_l(m, N)\|_{\psi_2} \leq \frac{\sin \frac{m\pi}{N}}{\sqrt{\ln 2 \sin \frac{\pi}{N}}},$$

as asserted.

**The second case:** $m$ is an even positive integer. Take $m = 2s$ with integer $s \geq 1$. Then by using the same argument applied in the first case, similarly as in the first case,
we find that the random variable $U_l(m, N)$ attains its maximal value equals to

$$M_1 = 1 + \cos \frac{2s\pi}{N} + \sum_{k=1}^{s-1} \cos \frac{2k\pi}{N} + \sum_{k=1}^{s-1} \cos \frac{2(N-k)\pi}{N}$$

$$= 1 + \cos \frac{2s\pi}{N} + 2 \sum_{k=1}^{s-1} \cos \frac{2k\pi}{N}$$

$$= \sin \frac{\pi}{N} + \sin \frac{\pi}{N} \cos \frac{2s\pi}{N} + \sin \frac{(2s-1)\pi}{N} - \sin \frac{\pi}{N}$$

$$= \frac{\sin \frac{\pi}{N} \cos \frac{2s\pi}{N} + \sin \frac{2s\pi}{N} \cos \frac{\pi}{N} - \sin \frac{2s\pi}{N} \cos \frac{\pi}{N}}{\sin \frac{\pi}{N}}$$

$$= \frac{\sin \frac{4s\pi}{N} \cos \frac{\pi}{N}}{\sin \frac{\pi}{N}} = \frac{\sin \frac{m\pi}{N} \cos \frac{\pi}{N}}{\sin \frac{\pi}{N}}.$$

(29)

If $N = 2n$ ($n \in \mathbb{N}$) is an even positive integer, then proceeding in the same manner as in the above first subcase (see (24)), we obtain that the minimal value of the random variable $U_l(m, N)$ is equal to

$$M_2 = \cos \frac{2n\pi}{2n} + 2 \sum_{k=n-s+1}^{n-1} \cos \frac{2k\pi}{2n} + \cos \frac{2(n-s)\pi}{2n}$$

$$= -1 + \frac{2 \sin \frac{(s-1)\pi}{2n} \cos \frac{(2n-s)\pi}{2n}}{\sin \frac{\pi}{2n}} + \cos \left( \pi - \frac{s\pi}{n} \right)$$

$$= -1 - \frac{2 \sin \frac{(s-1)\pi}{2n} \cos \frac{s\pi}{2n}}{\sin \frac{\pi}{2n}} - \cos \frac{s\pi}{n}$$

$$= -\sin \frac{\pi}{2n} - \sin \frac{(2s-1)\pi}{2n} + \sin \frac{\pi}{2n} - \sin \frac{\pi}{2n} \cos \frac{s\pi}{n}$$

$$= -\sin \frac{\pi}{2n} + \sin \frac{\pi}{2n} \cos \frac{s\pi}{n} - \sin \frac{\pi}{2n} \cos \frac{s\pi}{n}$$

$$= -\sin \frac{\pi}{2n} + \sin \frac{2s\pi}{2n} \cos \frac{\pi}{2n} + \sin \frac{\pi}{2n} \cos \frac{s\pi}{n} - \sin \frac{\pi}{2n} \cos \frac{s\pi}{n}$$

$$= -\sin \frac{2s\pi}{2n} \cos \frac{\pi}{2n} + \sin \frac{\pi}{2n} \cos \frac{s\pi}{n} - \sin \frac{\pi}{2n} \cos \frac{s\pi}{n}$$

$$= -\sin \frac{2s\pi}{2n} \cos \frac{\pi}{2n} = -\sin \frac{m\pi}{N} \cos \frac{\pi}{N}.$$

(30)

If $N = 2n + 1$ ($n \in \mathbb{N}$) is an odd positive integer, then similarly as in the previous cases, we obtain that the minimal value of the random variable $U_l(m, N)$ is equal to

$$M_2 = \sum_{k=n-s+1}^{n+s} \cos \frac{2k\pi}{N} = \frac{\sin \frac{2s\pi}{2n+1} \cos \frac{(2n+1)\pi}{2n+1}}{\sin \frac{\pi}{2n+1}}$$

$$= -\sin \frac{\pi}{N}.$$

(31)
From (29), (30) and (31) we see that for each even integer \( m \geq 2 \),
\[
\|X_l(m, N)\|_\infty = \max\{M_1, |M_2|\} \leq \frac{\sin \frac{m\pi}{N}}{\sin \frac{\pi}{N}},
\]
which in view of the inequality (20) yields
\[
\|U_l(m, N)\|_{\psi^2} \leq \frac{\sin \frac{m\pi}{N}}{\sqrt{\ln 2} \sin \frac{\pi}{N}}.
\]
Therefore, proof of the inequality (14) is completed.

**Proof of the inequality (15).** In order to prove the inequality (15), we proceed similarly as in the case of \( U_l(m, N) \). Since \( \sin \frac{2k\pi}{N} = \Im(\exp(2k\pi j/N)) = \Im(w^k) \), proceeding by the analogous way as in (23) (replacing \( \Re(\cdot) \) by \( \Im(\cdot) \)), we obtain the following known identity:
\[
(32) \quad \sum_{k=t}^{t+q} \sin \frac{2k\pi}{N} = \frac{\sin \frac{(q+1)\pi}{N} - \sin \frac{(2t+q)\pi}{N}}{\sin \frac{\pi}{N}},
\]
where \( t \geq 1 \) and \( q \geq 0 \) are nonnegative integers. Using the identity (32) and considering the cases when \( m \) is odd and \( m \) is even both divided into the following four subcases: \( N \equiv 0 \pmod{4}, N \equiv 1 \pmod{4}, N \equiv 2 \pmod{4} \) and \( N \equiv 3 \pmod{4} \), we can arrive at the estimate given by (15) by considering the following four cases.

**The first case:** \( m \) is an even positive integer and \( N \equiv 1 \pmod{4} \). Put \( m = 2s \) and \( N = 4n + 1 \) for some integers \( s \geq 1 \) and \( n \geq 1 \). Then it is easy to see that
\[
M_1 = \sum_{k=n-s+1}^{n+s} \sin \frac{2k\pi}{4n + 1},
\]
which by using the identity (32) immediately yields
\[
(33) \quad M_1 = \frac{\sin \frac{2s\pi}{4n + 1} \sin \frac{(2n+1)\pi}{4n + 1}}{\sin \frac{\pi}{4n + 1}} = \frac{\sin \frac{m\pi}{N} \sin \frac{(2|N|/4)+1)\pi}{N}}{\sin \frac{\pi}{N}}.
\]

Similarly, we have
\[
M_2 = \sum_{k=3n-s+1}^{3n+s} \sin \frac{2k\pi}{4n + 1},
\]
whence by using the identity (32) it follows that
\[
(34) \quad M_2 = \frac{\sin \frac{2s\pi}{4n + 1} \sin \frac{(6n+1)\pi}{4n + 1}}{\sin \frac{\pi}{4n + 1}} = -\frac{\sin \frac{2s\pi}{4n + 1} \sin \frac{(2n+1)\pi}{4n + 1}}{\sin \frac{\pi}{4n + 1}} = -\frac{\sin \frac{m\pi}{N} \sin \frac{(2|N|/4)+1)\pi}{N}}{\sin \frac{\pi}{N}}.
\]

From (33) and (34) we immediately obtain
\[
(35) \quad \|V_l(m, N)\|_\infty = \max \{M_1, |M_2|\} = \frac{\sin \frac{m\pi}{N} \sin \frac{(2|N|/4)+1)\pi}{N}}{\sin \frac{\pi}{N}}.
\]

**The second case:** \( m \) is an even positive integer and \( N \equiv 3 \pmod{4} \). Put \( m = 2s \) and \( N = 4n + 3 \) for some integers \( s \geq 1 \) and \( n \geq 1 \). Then as in the first case, it is easy
to see that
\[ M_1 = \sum_{k=n-s+1}^{n+s} \sin \frac{2k\pi}{4n+1}, \]
which by using the identity (32) immediately yields
\[ M_1 = \frac{\sin 2s\pi}{\sin \frac{\pi}{4n+1}} \frac{\sin \frac{(2n+1)\pi}{4n+1}}{\sin \frac{\pi}{4n+1}} = \frac{\sin \frac{m\pi}{N}}{\sin \frac{\pi}{N}}. \quad (36) \]

Similarly, we have
\[ M_2 = \sum_{k=3n-s+3}^{3n+s+2} \sin \frac{2k\pi}{4n+3}, \]
whence by using the identity (32), it follows that
\[ M_2 = \frac{\sin 3s\pi}{\sin \frac{\pi}{4n+3}} \frac{\sin \frac{(6n+5)\pi}{4n+3}}{\sin \frac{\pi}{4n+3}} = \frac{\sin \frac{m\pi}{N}}{\sin \frac{\pi}{N}}. \quad (37) \]

The equalities (36) and (37) imply that
\[ \|V_l(m, N)\|_\infty = \max \{M_1, |M_2|\} = \frac{\sin \frac{m\pi}{N}}{\sin \frac{\pi}{N}}. \quad (38) \]

*The third case: m and N are even positive integers.* Put \( m = 2s \) and \( N = 2n \) for some integers \( s \geq 1 \) and \( n \geq 1 \). Then it is easy to check that
\[ M_1 = \sum_{k=[n/2]-s+1}^{[n/2]+s} \sin \frac{2k\pi}{2n}, \]
which by applying the identity (32) and some basic trigonometric identities to both cases \( N \equiv 0 \pmod{4} \) and \( N \equiv 2 \pmod{4} \), we immediately obtain
\[ M_1 = \frac{\sin \frac{m\pi}{N}}{\sin \frac{\pi}{N}}. \quad (39) \]

Similarly, we find that
\[ M_2 = \sum_{k=[3n/2]-s+1}^{[3n/2]+s} \sin \frac{2k\pi}{2n}, \]
whence by applying the identity (32) and some basic trigonometric identities we get
\[ M_2 = -\frac{\sin \frac{2\pi}{2n}}{\sin \frac{\pi}{2n}} \frac{\sin \frac{(3n+1)\pi}{2n}}{\sin \frac{\pi}{2n}} = \frac{\sin \frac{m\pi}{N}}{\sin \frac{\pi}{N}}. \quad (40) \]

The equalities (39) and (40) imply that
\[ \|V_l(m, N)\|_\infty = \max \{M_1, |M_2|\} = \frac{\sin \frac{m\pi}{N}}{\sin \frac{\pi}{N}}. \quad (41) \]
The fourth case: \( m \geq 1 \) is an odd positive integer. If we take \( m = 2s + 1 \) with some integer \( s \geq 0 \), then by considering the all four subcases \( N \pmod{4} \), we can easily arrive to the equality

\[
M_1 = \sum_{k=\lceil(N+1)/4\rceil-s}^{\lfloor(N+1)/4\rceil+s} \sin \frac{2k\pi}{N},
\]

which by applying the identity (32) and some basic trigonometric identities, immediately yields

\[
M_1 = \frac{\sin \frac{m\pi}{N} \sin \frac{2(N+1)/4}{N}}{\sin \frac{\pi}{N}}. \tag{42}
\]

Similarly, we find that

\[
M_2 = \sum_{k=\lceil(3N+1)/4\rceil-s}^{\lfloor(3N+1)/4\rceil+s} \sin \frac{2k\pi}{N},
\]

which by applying the identity (32) and some basic trigonometric identities, immediately gives

\[
M_2 = \frac{\sin \frac{m\pi}{N} \sin \frac{2(3N+1)/4}{N}}{\sin \frac{\pi}{N}}. \tag{43}
\]

If \( N \) is even, then \( 2\lceil(3N + 1)/4\rceil - 2\lfloor(N + 1)/4\rfloor = N \), and thus, from (42) and (43) we have that \( M_2 = -M_1 \). If \( N \) is odd, then \( 2\lceil(3N + 1)/4\rceil + 2\lfloor(N + 1)/4\rfloor = 2N \), and so, from (42) and (43) we also have that \( M_2 = -M_1 \). Therefore, for each \( N \geq 2 \) there holds

\[
\|V_l(m, N)\|_\infty = \max \{M_1, |M_2|\} = M_1 = \frac{\sin \frac{m\pi}{N} \sin \frac{2(N+1)/4}{N}}{\sin \frac{\pi}{N}}. \tag{44}
\]

Finally, (20) and the equalities (35), (38), (41) and (44) immediately yield the equality (15). This completes proof of Proposition 2.4. \( \Box \)

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