FRACTIONAL PARTS OF POLYNOMIALS OVER THE PRIMES. II

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To Glyn Harman on his sixtieth birthday.

Abstract. Let \( \| \ldots \| \) denote distance from the integers. Let \( \alpha, \beta, \gamma \) be real numbers with \( \alpha \) irrational. We show that the inequality
\[ \| \alpha p^2 + \beta p + \gamma \| < p^{-37/210} \]
has infinitely many solutions in primes \( p \), sharpening a result due to Harman (1996) in the case \( \beta = 0 \) and Baker (2017) in the general case.

1. Introduction

Let \( f_k(x) = \alpha x^k + \cdots + \beta x + \gamma \) be a polynomial of degree \( k > 1 \) with irrational leading coefficient. Inequalities of the form
\[ \| f_k(p) \| < p^{-\rho_k} \]
for infinitely many primes \( p \) were studied by Vinogradov [11]; see [1] for the strongest available results. The present paper gives a new result for \( k = 2 \).

Theorem 1. Let \( \rho_2 = 37/210 = 0.1761 \ldots \); then \((\mathbb{I})\) holds for infinitely many primes \( p \).

The values \( \rho_2 = 1/8 - \varepsilon, 3/20, 2/13 = 0.1538 \ldots \) were given by Ghosh [6], Baker and Harman [2], and Harman [7] in the case \( \beta = 0 \); Baker [1] extended the last result to general \( \beta \).
As in [1], [2], [7] we use the Harman sieve. We make progress in the present paper by giving new bounds for sums of the shape
\[
\sum_{\ell \leq L} c_\ell \sum_{R < r \leq 2R} a_r \sum_{\frac{N}{2} < sr \leq N} b_s e(\ell g(rs))
\]
where \(g\) is the approximating polynomial to \(f_2\) in [1]. Type I sums (in which \(b_s \equiv 1\)) are treated in Section 2, and general (Type II) sums in Section 3. In the Type I case, \((a_r)_{R < r \leq 2R}\) is restricted to convolutions of shorter sequences; a lemma of Birch and Davenport [4] on Diophantine approximation plays a key role. For Type II sums, a subsidiary task is the study of the average behavior, as \(n\) varies, of the number of solutions of
\[
\ell_1 y_1^2 - \ell_2 y_2^2 = n \quad (\ell_1, \ell_2 \leq L; \, y_1, y_2 \in [Y, 2Y]).
\]

One sum that eludes the Harman sieve takes the form
\[
\sum_{(2N)^{1/2} \leq p < (2N)^{1/2} + \frac{31}{6}} \frac{1}{pp' \in \left[\frac{N}{2}, N\right], \|g(pp')\| \leq \delta}
\]
(where \(\rho = \frac{37}{210} + \varepsilon, \sigma = \rho - \frac{1}{6} \) and \(\delta = \frac{1}{2} N^{-\rho+\varepsilon}\)). We bound it above using the form of the linear sieve given by Iwaniec [10]. This sum is treated in Section 5; the sums accessible via the Harman sieve are in Section 4. Section 6 contains the sieve decomposition of \(S(A, (2N)^{1/2})\) (defined below), and the calculations leading to Theorem 1. Integrals that appear here and in earlier drafts were calculated by Andreas Wein-gartner; thanks, Andreas, for your generosity.

The following notations will be used:

\[
\|\theta\| = \min_{n \in \mathbb{Z}} |\theta - n|.
\]

\[
|\mathcal{E}| = \sum_{n \in \mathcal{E}} 1 \quad (\mathcal{E} \subset [1, N]).
\]

\(\chi_{\mathcal{E}} = \) indicator function of \(\mathcal{E}\).

\(C\ldots\) absolute constant, not the same at each occurrence.

\(\lambda\ldots\) real number with \(|\lambda| \leq C\), not the same at each occurrence.

\(\varepsilon\ldots\) sufficiently small positive number; \(\eta = \varepsilon^9\).
\[
\gg, \ll \ldots \text{indicate implied constants that may depend on } \varepsilon.
\]

\[
y \sim Y \ldots \text{indicates } Y < y \leq 2Y.
\]

\[
e(\theta) = e^{2\pi i \theta}.
\]

\[
\frac{a}{q} \ldots \text{fraction in lowest terms with } |\alpha - \frac{a}{q}| < \frac{1}{q^2}, \text{with } q \text{ sufficiently large}; \ g(x) = \frac{a}{q} x^2 + \beta x + \gamma.
\]

We choose \( N \) so that
\[
L_1^{1/2} N \ll q \ll L_1^{1/2} N \quad \text{with} \quad L_1 = 2N^{\rho - \frac{\varepsilon}{2}}
\]
and write
\[
L = N^\rho - \varepsilon/3.
\]

Here \( \rho \) will ultimately be \( \frac{37}{410} + \varepsilon \); earlier in the paper we restrict \( \rho \) somewhat less. We do suppose \( \rho > \frac{1}{6} \), and write
\[
\sigma = \rho - 1/6.
\]

We reserve the symbols \( p, p_1, p', \ldots \) for prime numbers.

Let \( \delta = L_1^{-1} \) and
\[
\mathcal{A} = \left\{ n : n \sim \frac{N}{2}, \|g(n)\| < \delta \right\}, \quad \mathcal{B} = \left\{ n : n \sim \frac{N}{2} \right\}.
\]

We write \( I(m) \) for an arbitrary subinterval of \( \left( \frac{N}{2m}, \frac{N}{m} \right] \).

2. Type I sums

The object of this section is to prove

**Theorem 2.** Let \( \frac{1}{6} < \rho < \frac{2}{11} \). Let \( V \geq 1, W \geq 1, \)
\[
V^3W^2 \ll N^{2-3\rho}, VW^3 \ll N^{3-15\rho/2}, VW \ll N^{2-8\rho}.
\]

Let
\[
T := \sum_{\ell=1}^{L} \sum_{v \sim V} \sum_{w \sim W} \left| \sum_{n \in I(vw)} e(\ell g(vwn)) \right|.
\]

Then
\[
T \ll N^{1-10\eta}.
\]

We require several lemmas.
Lemma 1. Let $1 \leq Y \ll N^{1-2\rho}$. Let $S$ be the set of $y \in (Y, 4Y]$ with $\langle y, q \rangle \leq N^\rho$ and

\begin{equation}
T(y) := \sum_{\ell=1}^{L} \left| \sum_{n \in I(y)} e(\ell g(yn)) \right| > N^{1-10\eta}Y^{-1}.
\end{equation}

There are a set $S^* \subset S$ and positive numbers $S, Z$ with the following properties.

(i) We have

\begin{equation}
\sum_{y \in S} T(y) \ll L^{1/2}N^{10\eta}Z^{1/2} |S^*|;
\end{equation}

(ii) for $y \in S^*$, we have

\begin{equation}
\left| \frac{sa y^2}{q} - u \right| < Z^{-1}
\end{equation}

for some $s = s(y)$ in $\mathbb{N}, u = u(y)$ in $\mathbb{Z}, (u, q) = 1$, and

\begin{equation}
s \sim S \ll L^2N^\eta;
\end{equation}

(iii) $Z$ satisfies

\begin{equation}
(N/Y)^2L^{-1}N^{-\eta} \ll Z \ll LS^{-1}(N/Y)^2.
\end{equation}

Proof. This can readily be extracted from the proof of [11 Lemma 8], with $5\eta$ in place of $\eta$. □

Lemma 2. Let $\theta$ be a real number and suppose there exist $R$ distinct integer pairs $x, z$ satisfying

\begin{equation}
|\theta x - z| < \zeta, \ 0 < |x| < X,
\end{equation}

where $R \geq 24\zeta X > 0$. Then all integer pairs $x, z$ satisfying (2.6) have the same ratio $z/x$.

Proof. Birch and Davenport [4]. □

Lemma 3. Suppose that $s \geq 1, D \geq 1, sD^2 < q, Z \geq 2$. The number of solutions $y \in (Y, 4Y]$, with $\langle y, q \rangle \leq D$, of the inequality

\[ \left\| \frac{sa y^2}{q} \right\| < \frac{1}{Z} \]
is
\[ \ll N^{\eta} \left( \frac{Y + q^{1/2}}{Z^{1/2}} \right). \]

**Proof.** [1, Lemma 3]. \( \square \)

**Proof of Theorem 2.** In the notation of Lemma 1 with \( Y = VW \), let \( \mathcal{C} \) be the set of pairs \((v, w)\) for which \( v \sim V, w \sim W, \, vw \in S^* \). As in [1], proof of Lemma 8, it suffices to prove that
\[ \sum_{(v, w) \in \mathcal{C}} T(v, w) \ll N^{1 - 10\eta}. \] (2.7)

In view of the Type I result obtained in [1, Lemma 8], we may suppose that
\[ VW > N^{1 - 5\rho/2 - \eta}. \]

We note that \[ |\mathcal{C}| \ll \frac{N^{2\eta}S(VW + q^{1/2})}{Z^{1/2}} \] as a consequence of Lemma 3; indeed
\[ |\mathcal{C}| \ll \frac{N^{2\eta}SVW}{Z^{1/2}}, \] (2.8)

since \( VW > N^{1 - 5\rho/2 - \eta} > N^{1/2 + \rho/4} \gg q^{1/2} \) for \( \varepsilon \) sufficiently small.

Suppose for a moment that
\[ S < N^{1 - 23\eta}(VW)^{-1}L^{-1/2}; \] (2.9)

then (2.2) gives (with a divisor argument)
\[ \sum_{(v, w) \in \mathcal{C}} T(v, w) \ll L^{1/2}N^{11\eta}Z^{1/2}|\mathcal{C}| \ll N^{1 - 10\eta} \] (from (2.8), (2.9)), giving (2.7). So we may suppose that
\[ S \geq N^{1 - 23\eta}(VW)^{-1}L^{-1/2}. \]

It now follows from (2.5) that
\[ Z \leq \left( \frac{N}{VW} \right)^2 L \frac{VWL^{1/2}}{N^{1 - 23\eta}} = \frac{N^{1 + 23\eta}}{VW} L^{3/2}. \] (2.10)
Now let 
\[ \mathcal{C}(w) = \{ v \sim V : (v, w) \in \mathcal{C} \} \]
and for \( K \geq 1 \), let
\[ \mathcal{E}(K) = \{ w \sim W : K \leq |\mathcal{C}(w)| < 2K \}. \]
Then
\[ |\mathcal{C}| = \sum_{w \sim W} |\mathcal{C}(w)| \leq \sum_{K=2^n \in [1,V]} 2K|\mathcal{E}(K)|. \]
We choose \( K, 1 \leq K \leq V \), so that
\[ (2.11) \quad |\mathcal{C}| \ll \log N|\mathcal{E}(K)|K. \]
Suppose for a moment that 
\[ L^{1/2}Z^{1/2}KW < N^{1-2\eta}. \]
Then arguing as above,
\[ \sum_{(v,w) \in \mathcal{C}} T(v, w) \ll L^{1/2}N^{11\eta}Z^{1/2}\log N|\mathcal{E}(K)|K \]
\[ \ll N^{12\eta}L^{1/2}Z^{1/2}WK \ll N^{1-10\eta}. \]
Thus we may suppose that
\[ (2.12) \quad K \geq \frac{N^{1-2\eta}}{L^{1/2}Z^{1/2}W}. \]
For the next stage of the argument, let \( w \) be a fixed integer in \( \mathcal{E}(K) \).
We apply Lemma 2, taking
\[ \theta = \frac{w^2 a}{q}, \quad X = 8SV^2, \quad \zeta = \frac{1}{Z}, \]
since (by (2.3) and the definition of \( \mathcal{C} \))
\[ |(sv^2)\theta - u| < \frac{1}{Z}, \quad s = s(v, w), u = u(v, w) \]
for every \( v \) in \( \mathcal{C}(w) \). By a divisor argument, the number of distinct \( sv^2 \) as \( v \) varies over \( \mathcal{C}(w) \) is \( \gg KN^{-n} \). Thus in the notation of Lemma 2

\[
 R \gg KN^{-n} \gg X\zeta N = \frac{8SV^2N^n}{Z}.
\]

To see this,

\[
 KN^{-n}(8SV^2N^n/Z)^{-1} \gg \frac{N^{1-24\eta}}{Z^{1/2}L^{1/2}W} \frac{Z}{SV^2}
\]

(from (2.12))

\[
 \gg \frac{N^{1-25\eta}Z^{1/2}}{L^{5/2}V^2W} \gg \frac{N^{1-26\eta}(N/VW)}{L^{3}V^2W}
\]

(from (2.14))

\[
 \gg 1
\]

from the hypothesis of the theorem.

Accordingly, all \( \frac{u}{sv^2} \) with \( v \in \mathcal{C}(w) \) can be written in the form

\[
 (2.13) \quad \frac{u}{sv^2} = \frac{t}{r}, \quad (t, r) = 1, \quad t = t(w), \quad r = r(w)
\]

for a certain \( t \in \mathbb{Z}, r \in \mathbb{N} \) independent of \( v \).

We record a lower bound for \( K \) that does not contain \( Z \). From (2.12), (2.10), we have

\[
 (2.14) \quad K \geq \frac{N^{1-22\eta}}{L^{1/2}W} \frac{(VW)^{1/2}}{L^{3/4}N^{1/2+12\eta}} = N^{1/2-34\eta}V^{1/2}W^{-1/2}L^{-5/4}.
\]

We now select a divisor \( z \) of \( r \) such that the set

\[
 \mathcal{C}(w, z) = \left\{ v \in \mathcal{C}(w) : \frac{r}{\langle s, r \rangle} = z \right\}
\]

satisfies

\[
 (2.15) \quad KN^{-n} \ll |\mathcal{C}(w, z)| \leq 2K.
\]

For each \( v \) in \( \mathcal{C}(w, z) \), we have

\[
 z \mid v^2.
\]
It is convenient to write \( z = bc^2 \) where \( b \) is square-free, and define \( k = v^2/bc^2 \). Then

\[ bc^2k = v^2, \ k = bd^2 \text{ (where } d \in \mathbb{N}, \ bcd = v. \)

This leads to the upper bound

\[ |C(w, z)| \ll \frac{V}{bc}, \]

from which we infer, using (2.15), that

(2.16) \[ bc \ll \frac{VN^\eta}{K}. \]

Now we re-examine our rational approximation

\[ \left| \frac{sv^2w^2}{q} - w \right| < Z^{-1} \]

(see (2.3) and the definition of \( C \)). We observe that

(2.17) \[ \left| \frac{rw^2a}{q} - t \right| = \left| \frac{sv^2w^2a}{q} - u \right| \leq \frac{z}{V^2} \frac{1}{Z} = \frac{bc^2}{V^2} \frac{1}{Z} \ll \frac{N^{2\eta}}{K^2Z} \]

for \( v \in C(w, z) \), using (2.16).

Let \( \mathcal{F} \) be the set of natural numbers \( b_0c_0^2m, \ (b_0, c_0, m) \in \mathbb{N}^3, \ b_0c_0 < \frac{VN^{2\eta}}{K}, \ m < L^2N^\eta. \) The number of possibilities for \( b_0c_0^2 \) here is \( \ll \frac{VN^{3\eta}}{K} \), since \( b_0c_0^2 \) is a divisor of \((b_0c_0)^2\). Hence

(2.18) \[ |\mathcal{F}| \ll \frac{VL^2N^{4\eta}}{K}. \]

All the integers \( r \) occurring in (2.17) are in \( \mathcal{F} \). Hence

(2.19) \[ |\mathcal{E}(K)| = \sum_{r \in \mathcal{F}} \sum_{w \in \mathcal{E}(K), r(w) = r} 1 \ll |\mathcal{F}| \frac{N^{2\eta}(W + q^{1/2})}{KZ^{1/2}}, \]
on bounding the number of \( w \) in \( \mathcal{E}(K) \) with \( r(w) = r \) via Lemma 3. Combining (2.18), (2.19), and recalling (2.11),

\[
|\mathcal{E}(K)| \ll \frac{VL^2 N^{6\eta}}{K} \left( W + q^{1/2} \right) KZ^{1/2},
\]

\[
\sum_{(v,w) \in \mathcal{C}} T(vw) \ll L^{1/2} N^{11\eta} Z^{1/2}|\mathcal{C}|
\]

\[
\ll L^{1/2} N^{12\eta} Z^{1/2} |\mathcal{E}(K)|K
\]

\[
\ll \frac{L^{5/2} N^{18\eta} V(W + q^{1/2})}{K}.
\]

We now use the lower bound (2.14) for \( K \) and obtain

\[
\sum_{(v,w) \in \mathcal{C}} T(vw) \ll L^{15/4} (VW)^{1/2} N^{-\frac{1}{2}} + 66\eta} (W + q^{1/2}).
\]

Now (2.7) follows on applying the bounds for \( VW^3 \) and \( VW \) in the hypothesis of Theorem 2. This completes the proof of Theorem 2. \( \square \)

3. TYPE II SUMS

**Lemma 4.** Let \( Y \geq 1 \). For \( n \in \mathbb{N} \), let \( R(n) \) denote the number of quadruples \((\ell_1, \ell_2, y_1, y_2)\) with \( 1 \leq \ell_i \leq L, y_i \sim Y \) such that

\[
(3.1) \quad \ell_1 y_1^2 - \ell_2 y_2^2 = n.
\]

Then

\[
(3.2) \quad R(n) \ll (LY)^{1+\eta},
\]

\[
(3.3) \quad \sum_{n \in \mathbb{Z}} R(n)^2 \ll L^{3+\eta} Y^{2+\eta}.
\]

**Proof.** For (3.2), fix \( \ell_2 \) and \( y_2 \); then the equation

\[
\ell_1 y_1^2 = \ell_2 y_2^2 + n
\]

(with \( \ell_1 \neq 0, y_1 \neq 0 \)) determines \( \ell_1, y_1 \) up to \( O((LY)^{\eta}) \) possibilities.

For (3.3), we observe that

\[
\sum_{n \in \mathbb{Z}} R(n)^2
\]
is the number of tuples \( \ell_1, \ell_2, \ell_3, \ell_4, y_1, y_2, y_3, y_4 \) with \( 1 \leq \ell_i \leq L \), \( y_i \sim Y \) and
\[
\ell_1 y_1^2 - \ell_2 y_2^2 = \ell_3 y_3^2 - \ell_4 y_4^2.
\]
This may be expressed as an integral:
\[
(3.4) \quad \sum_{n \in \mathbb{Z}} R(n)^2 = \int_0^1 \left| \sum_{\ell \leq L} \sum_{y \sim Y} e(\ell y^2 t) \right|^2 \left| \sum_{\ell_0 \leq L} \sum_{y_0 \sim Y} e(\ell_0 y_0^2) \right|^2 \, dt \leq LV
\]
by the Cauchy-Schwarz inequality, where
\[
V = \sum_{1 \leq \ell \leq L} \int_0^1 \left| \sum_{y \sim Y} e(\ell y^2 t) \right|^2 \sum_{\ell_0 \leq L} \sum_{y_0 \sim Y} e(\ell_0 y_0^2 t)^2 \, dt
\]
Now \( V \) is the number of solutions of
\[
(3.5) \quad \ell (y_1^2 - y_2^2) = \ell_3 y_3^2 - \ell_4 y_4^2
\]
with \( 1 \leq \ell, \ell_3, \ell_4 \leq L \) and \( y_i \sim Y (1 \leq i \leq 4) \).
We first consider \( V_1 \), the number of solutions of (3.5) with
\[
(3.6) \quad \ell_3 y_3^2 = \ell_4 y_4^2.
\]
If (3.5) and (3.6) hold, then \( y_1 = y_2 \). There are \( O((LY)^{1+\eta}) \) possibilities for \( \ell_3, \ell_4, y_3, y_4 \) and for each of these, at most \( LY \) possibilities for \( \ell, y_1, y_2 \). Thus
\[
V_1 \ll (LY)^{2+\eta}.
\]
Now consider \( V_2 \), the number of solutions of (3.5) for which (3.6) is violated. There are \( O(L^2 Y^2) \) possibilities for \( \ell_3, y_3, \ell_4, y_4 \). For each of these, there are \( O((LY)^{\eta}) \) possibilities for \( \ell, y_1 - y_2 \) and \( y_1 + y_2 \), hence \( O((LY)^{\eta}) \) possibilities for \( \ell, y_1, y_2 \). Thus
\[
(3.7) \quad V_2 \ll (LY)^{2+\eta}, V \ll (LY)^{2+\eta}.
\]
Now (3.3) now follows on combining (3.4) and (3.7).
\[\square\]

**Theorem 3.** For \( \frac{1}{6} < \rho < \frac{1}{5} \), and \( N^\rho \ll Y \ll N^{1-4\rho} \), \( |c_\ell| \leq 1 \), \( |a_x| \leq 1 \), \( |b_y| \leq 1 \), we have
\[
\sum_{\ell \leq L} c_\ell \sum_{y \sim Y} b_y \sum_{x \in I(y)} a_x e(\ell g(xy)) \ll N^{1-10\eta}.
\]
Proof. Just as in \cite{[1]} proof of Lemma 9 we need only show that
\begin{equation}
S' := \sum_{\ell \leq L} c_{\ell} \sum_{y \sim Y} b_y \sum_{x \leq \frac{N}{Y}} a_x e(\ell g(xy)) \ll N^{1-11\eta}.
\end{equation}

Again arguing as in that proof,
\begin{equation}
|S'|^2 \leq \frac{N}{Y} \sum_{\ell \in \mathcal{L}} |S(\ell)|,
\end{equation}

where \( \ell = (\ell_1, \ell_2, y_1, y_2) \), \( \mathcal{L} = \{ \ell : \ell_1, \ell_2 \leq L, y_1, y_2 \sim Y \} \) and
\[
S(\ell) := \sum_{x \leq \frac{N}{Y}} e(\ell_1 g(xy_1) - \ell_2 g(xy_2)).
\]

The contribution to the right-hand side of (3.9) from those \( \ell \) with \( \ell_1 y_1^2 = \ell_2 y_2^2 \) is
\[
\ll \frac{N}{Y} \cdot LY \cdot \frac{N^{1+\eta}}{Y}
\]
(by a divisor argument)
\[
\ll N^{2-22\eta}
\]
since \( Y \gg N^\rho \). The contribution from those \( \ell \) with
\[
|S(\ell)| \leq \frac{N^{1-22\eta}}{Y} L^{-2}
\]
is
\[
\leq \frac{N}{Y} \cdot L^2 Y^2 \cdot \frac{N^{1-22\eta}}{Y} L^{-2} = N^{2-22\eta}.
\]

It remains to consider \( \mathcal{M} \), the set of \( \ell \) in \( \mathcal{L} \) with
\( \ell_1 y_1^2 > \ell_2 y_2^2 \)
and
\[
|S(\ell)| > \frac{N^{1-22\eta}}{Y} L^{-2}.
\]
We apply [1, Lemma 5] with $M = 1$, $X = NY^{-1}$, $P = \frac{N^{3-2\eta}}{Y^2}$. We require
\[ P \geq X^{\frac{1}{2} + \eta} \]
which holds since
\[ PX^{-\frac{1}{2} - \eta} \geq N^{\frac{1}{2} - 2\eta} L^{-2} Y^{-1/2} \]
\[ \geq N^{\frac{1}{2} - 2\eta} L^{-2} N^{-1/2 + 2\rho} \geq 1. \]
Thus for each $\ell \in \mathcal{M}$ there exists a natural number $s$,
\begin{equation}
(3.10) \quad s = s(\ell) \leq L^4 N^\eta, \quad \left| s(\ell_1 y_1^2 - \ell_2 y_2) \frac{a}{q} - u_2 \right| = \frac{1}{Z(\ell)} \quad \left| s(\ell_1 \alpha_1 y_1 - \ell_2 \alpha_1 y_2) - u_1 \right| = \frac{1}{W(\ell)}
\end{equation}
with $u_1, u_2 \in \mathbb{Z}$, $(u_2, s) = 1$,
\[ Z(\ell) \geq \left( \frac{N}{Y} \right)^2 L^{-4} N^{-\eta}, W(\ell) \geq \frac{N}{Y} L^{-4} N^{-\eta}. \]
Now $s \leq N / Y$ and, writing $\gamma_2 = (\ell_1 y_1^2 - \ell_2 y_2^2) a / q$, $\gamma_1 = (\ell_1 y_1 - \ell_2 y_2) \alpha_1$,
\[ |s \gamma_j - u_j| \leq (2k^2)^{-1} \left( \frac{N}{Y} \right)^{1-j} (j = 1, 2). \]
Thus we can appeal to [1, Lemma 7] with $k = 2$ and $L$ replaced by 1.
Let
\[ \beta_j = \gamma_j - \frac{u_j}{s}, F(x) = \sum_{j=1}^{2} \beta_j x^j, \quad G(x) = \sum_{j=1}^{2} u_j x^j. \]
\[ S(s, G) = \sum_{u=1}^{s} e \left( \frac{G(u)}{s} \right). \]
We obtain
\begin{equation}
(3.11) \quad S(\ell) = \sum_{x \leq \frac{N}{Y}} e(\gamma_2 x^2 + \gamma_1 x) = s^{-1} S(s, G) \int_{0}^{\frac{N}{Y}} e(F(z))dz + O(N^{3\eta} L^2). \quad \text{(3.11)}
\end{equation}
Now $N^3 L^2$ is of smaller order than $\frac{N^{1-2\eta}}{Y} L^2$, so that
\begin{equation}
(3.12) \quad |s^{-1} S(s, G) \int_0^{N/Y} e(F(z)) dz| \gg \frac{N^{1-2\eta}}{Y} L^2.
\end{equation}

Moreover, by standard bounds, the left-hand side of (3.12) is
\begin{equation}
(3.13) \quad \ll S^{-1/2} \min(N/Y, (SZ(\ell))^{1/2}).
\end{equation}

We now use a standard splitting-up argument to choose a subset $Q$ of $M$ such that
\begin{equation}
(3.14) \quad s(\ell) \sim S, \quad \left| s(\ell_1 y_1^2 - \ell_2 y_2^2) \frac{a}{q} - u_2 \right| < \frac{1}{Z} \quad (\ell \in Q),
\end{equation}
where $(N/Y)^2 L^{-4} N^{-\eta} \leq Z \leq Z_0$, with
\[\frac{N}{Y} = (SZ_0)^{1/2},\]
and moreover
\[S \leq L^4 N^\eta, \quad Z \geq \left(\frac{N}{Y}\right)^2 L^{-4} N^{-\eta},\]
while
\[\sum_{\ell \in M} |S(\ell)| \leq (\log N)^2 \sum_{\ell \in Q} |S(\ell)|.\]

Compare e.g. the argument in [1, proof of Lemma 8]. In order to obtain (3.8) it remains to show that
\[\frac{N}{Y} \sum_{\ell \in Q} |S(\ell)| \ll N^{2-23\eta}.\]

Using (3.11)–(3.14), we find that
\[|S(\ell)| \ll Z^{1/2} \quad (\ell \in Q)\]
and we must show that
\begin{equation}
(3.15) \quad Z^{1/2} |Q| \ll Y N^{1-23\eta}.
\end{equation}

For each $\ell$ in $Q$ there is an $s \sim S$ with
\[s(\ell_1 y_1^2 - \ell_2 y_2^2) \in C,\]
where
\[ \mathcal{C} = \left\{ n : 1 \leq n \leq 2SLY^2, \ |na \ (\text{mod } q)| < \frac{q}{Z} \right\}. \]

Clearly
\[ |\mathcal{C}| \ll \left( \frac{SLY^2}{q} + 1 \right) \left( \frac{q}{Z} + 1 \right) = \frac{SLY^2}{Z} + \frac{SLY^2}{q} + \frac{q}{Z} + 1. \quad (3.16) \]

Given \( m \in \mathcal{C} \), let \( h_1(m), \ldots, h_j(m), j = j(m) \ll N^n \), be the divisors \( h \) of \( m \) with \( h \sim S \). Each element of \( Q \) satisfies
\[ (\ell_1y_1^2 - \ell_2y_2^2)h_i(m) = m \]
for some \( m \in \mathcal{C} \) and some \( i, 1 \leq i \leq j(m) \). Let
\[ \mathcal{K} = \{ m/h_i(m) : 1 \leq i \leq j(m), m \in \mathcal{C} \}. \]

Then
\[ |\mathcal{K}| \ll N^n |\mathcal{C}|, \quad (3.17) \]
while
\[ |Q| = \sum_{\substack{\ell \in \mathcal{M} \\ell_1y_1^2 - \ell_2y_2^2 \in \mathcal{K}}} 1 \]
\[ \leq \sum_{n \in \mathcal{K}} \sum_{\substack{\ell \in \mathcal{M} \\ell_1y_1^2 - \ell_2y_2^2 = n}} 1 \]
\[ = \sum_{n \in \mathcal{K}} R(n) \]
in the notation of Lemma 4. Applying Cauchy’s inequality,
\[ |Q| \leq |\mathcal{K}|^{1/2} \left( \sum_{n \geq 1} R(n)^2 \right)^{1/2} \]
\[ \ll (L^n Y^2)^{1/2} |\mathcal{C}|^{1/2} N^n, \]
by Lemma 4 and (3.17).
Alternatively, (3.2) yields

\begin{equation}
|Q| \leq \left( \max_{n \in \mathcal{K}} R(n) \right) |\mathcal{K}| \ll LY|C|N^{2\eta}.
\end{equation}

We now find that, depending on the value of $Y$, either (3.18) or (3.19) yields the desired bound (3.15). Suppose first that

\begin{equation}
Y > N^{3\rho/2}S^{-1/2}.
\end{equation}

In view of (3.16), (3.18), we need to verify the four bounds

\begin{align}
Z^{1/2}(L^3Y^2)^{1/2} \left( \frac{SLY^2}{Z} \right)^{1/2} &\ll YN^{1-24\eta}, \\
Z^{1/2}(L^2Y^2)^{1/2} \left( \frac{SLY^2}{q} \right)^{1/2} &\ll YN^{1-24\eta}, \\
Z^{1/2}(L^3Y^2)^{1/2} \left( \frac{q}{Z} \right)^{1/2} &\ll YN^{1-24\eta}, \\
Z^{1/2}(L^3Y^2)^{1/2} &\ll YN^{1-24\eta}.
\end{align}

First of all, (3.21) holds since

\[ L^2Y^2S^{1/2}(YN^{1-24\eta})^{-1} \ll YN^{4\rho-1} \ll 1. \]

Next, (3.22) holds since

\[ Z^{1/2}L^2S^{1/2}Y^{1/2}q^{-1/2}(YN^{1-24\eta})^{-1} \ll S^{-1/2}NY^{-1}L^2YS^{1/2}N^{-\frac{3}{4}+\frac{1}{2}+24\eta} \ll N^7\rho/4-1/2 \ll 1. \]

Next, (3.23) holds since

\[ L^{3/2}Yq^{1/2}(YN^{1-24\eta})^{-1} \ll N^7\rho/4-1/2 \ll 1. \]

Finally, (3.24) holds since

\[ Z^{1/2}L^{3/2}Y(N^{1-24\eta})^{-1} \ll N^3\rho/2S^{-1/2}Y^{-1} \ll 1 \]

from (3.21).
Now suppose that
\[(3.25)\]
\[Y \ll (N^2-4\rho-26\eta S^{-1})^{1/3},\]
we employ (3.16) and (3.19). We need to verify the bounds
\[(3.26)\]
\[Z^{1/2}LY(SLY^2/Z) \ll YN^{1-25\eta},\]
\[(3.27)\]
\[Z^{1/2}LY(SLY^2/q) \ll YN^{1-25\eta},\]
\[(3.28)\]
\[Z^{1/2}LY q/Z \ll YN^{1-25\eta},\]
\[(3.29)\]
\[Z^{1/2}LY \ll YN^{1-25\eta}.
\]

First of all, (3.26) holds since
\[
L^2SY^3Z^{-1/2}(YN^{1-25\eta})^{-1}
\leq L^4SY^3N^{26\eta-2} \ll 1
\]
from (3.25). Next, (3.27) holds since
\[
Z^{1/2}L^2SY^3q^{-1}(YN^{1-25\eta})^{-1}
\ll \frac{N}{Y} S^{1/2}L^2Y^2N^{-2-\frac{\eta}{2}+25\eta}
\ll YN^{-1+7\rho/2} \ll 1.
\]

Next, (3.28) holds since
\[
Z^{-1/2}LYq(YN^{1-25\eta})^{-1} \ll YN^{-1+7\rho/2+26\eta} \ll 1.
\]

Finally, (3.29) holds since
\[
Z^{1/2}LY(YN^{1-25\eta})^{-1} \ll \frac{N^{25\eta}L}{Y} \ll 1.
\]

In order to complete the proof, we show that the ranges of $Y$ in (3.20) and (3.25) overlap. We have
\[
N^{3\rho/2} S^{-1/2} < (N^2-4\rho-26\eta S^{-1})^{1/3},
\]
that is
\[
S^{1/6} > N^{(17\rho-4+52\eta)/6},
\]
since $\rho < 1/5$. This completes the proof of (3.15), and Theorem 3 follows. □

4. Asymptotic formulae via Harman sieve and generalized Vaughan identity

In the present section and the next, we suppose that

\[
120^{-1} < \sigma \leq 102^{-1}.
\]

We write

\[
b = \frac{1}{6} - 5\sigma, \quad f = \frac{4}{3} - 4\sigma, \quad z = N^b \quad \text{and} \quad P(s) = \prod_{p < s} p \quad (s > 1).
\]

For a finite set $\mathcal{E} \subset \mathbb{N}$, let

\[
\mathcal{E}_d = \{m : dm \in \mathcal{E}\},
\]

\[
S(\mathcal{E}, w) = |\{m \in \mathcal{E} : (m, P(w)) = 1\}|.
\]

As in [II], our claim in Theorem [II] is a corollary of the lower bound

\[
(4.2) \quad S(\mathcal{A}, (2N)^{1/2}) > \frac{1}{200} \frac{\delta N}{\log N}
\]

for $\rho = \rho_2$.

We introduce some ‘comparison’ results for the pair $S(\mathcal{A}, w)$ and $2\delta S(\mathcal{B}, w)$, and similar pairs, that will be needed in Section 6. First of all, we have

\[
(4.3) \quad \sum_{p_1 \sim P_1} \cdots \sum_{p_t \sim P_t} \sum_{p'_1 \sim Q_1} \cdots \sum_{p'_t \sim Q_t} \left( \sum_{p_1 \cdots p_t p'_1 \cdots p'_t \in \mathcal{A}} 1 - 2\delta \sum_{p_1 \cdots p_t p'_1 \cdots p'_t \in \mathcal{B}} 1 \right)
\]

whenever $\tau$ is a positive constant and some subproduct $R$ of $P_1 \ldots P_t Q_1 \ldots Q_t$ satisfies

\[
(4.4) \quad N^\rho \ll R \ll N^f.
\]
This is a consequence of Theorem 2; compare the discussion in Section 3.2 and 3.5. Additional inequalities such as \( p_j \leq K \) may be included in the summation in (4.3) without affecting its validity, as explained in Section 3.2.

The following lemma is essentially the special case \( M = 2X^\alpha, S = 1 \) of Lemma 14, and is a variant of Theorem 3.1.

**Lemma 5.** Let \( w \) be a complex function with support in \([1, N], |w(n)| \leq N^{1/\eta} (n \geq 1)\). Let \( 0 < \theta < \theta + \psi < 1/2 \). Let

\[
S(r, v) := \sum_{(n, P(v))=1} w(rn).
\]

Suppose that, for some \( Y > 1 \) we have, (for any coefficients \( a_m, |a_m| \leq 1, c_n, |c_n| \leq 1 \) and \( b_n, |b_n| \leq \tau(n) \))

(4.5) \[
\sum_{m \leq 2N^\alpha} a_m \sum_{n} w(mn) \ll Y,
\]

(4.6) \[
\sum_{N^\theta \leq h \leq N^{\theta+\psi}} c_h \sum_{n} b_n w(mn) \ll Y.
\]

Let \( u_r (r < N^\theta) \) be complex numbers with \( |u_r| \leq 1, u_r = 0 \) for \( (r, P(N^\eta)) > 1 \). Then

(4.7) \[
\sum_{r < N^\theta} u_r S(r, N^\psi) \ll Y (\log N)^3.
\]

We can deduce the following ‘bilinear’ lemma.

**Lemma 6.** Let \( w, \theta, \psi, S(r, v) \) be as in Lemma 5. Suppose that we have the hypothesis (4.6) and in addition, for some \( T \in [1, N] \),

(4.8) \[
\sum_{m \leq 2N^\theta} a_m \sum_{t \leq T} c_t \sum_{n} w(mtn) \ll Y
\]

for any \( a_m, c_t \) with \( |a_m| \leq 1, |c_t| \leq 1 \). Then for any \( u_r (r < N^\theta), v_t (t \leq T) \) with \( |u_r| \leq 1, |c_t| \leq 1 \), \( u_r = 0 \) for \( (r, P(N^\eta)) > 1 \), we have

(4.9) \[
\sum_{r \leq R} u_r \sum_{t \leq T} v_t S(rt, N^\psi) \ll Y N^{2\eta}.
\]

**Proof.** We apply Lemma 4 with \( w \) replaced by \( w^* \),

\[
w^*(n) = \sum_{t \leq T} v_t w(nt),
\]
so that \( S(r, v) \) is replaced by

\[
S^*(r, v) = \sum_{(n, P(v)) = 1} \sum_{t \leq T} v_t w(nt).
\]

From (4.8), (4.9) the hypotheses of Lemma 5 are satisfied with \( Y \) replaced by \( Y N_\eta \): for example,

\[
\sum_{N^\# \leq m \leq N^\# + \psi} a_m \sum_n b_n w^*(mn) = \sum_{N^\# \leq m \leq N^\# + \psi} a_m \sum_{t \leq T} \sum_n b_n v_t w(mtn) \ll Y N_\eta
\]

(we may group the product \( mt \) as a single variable and apply (4.6)). The conclusion (4.7) with \( S \) replaced with \( S^* \) gives the desired bound (4.9). \( \square \)

We now apply Lemma 6 with \( w(n) = \chi_A(n) - 2\delta \chi_B(n) \), \( \theta = \rho \), \( \theta + \psi = f \), where \( T = (2N)^\nu \) and the non-negative number \( \nu \) satisfies

\[
3\rho + 2\nu \leq \frac{3}{2} - 3\sigma, \tag{4.10}
\]

\[
\rho + 3\nu \leq \frac{7}{4} - \frac{15\sigma}{2}, \tag{4.11}
\]

\[
\rho + \nu \leq \frac{2}{3} - 8\sigma. \tag{4.12}
\]

Lemma 7. Suppose that (4.10)–(4.12) hold. Then

\[
(4.13) \quad \sum_{r < N^\rho} u_r \sum_{t \leq (2N)^\rho} v_t (S(A_{rt}, z) - 2\delta S(B_{rt}, z)) \ll \delta N^{1-\eta},
\]

whenever \( |u_r| \leq 1 \), \( (r, P(N^\eta)) = 1 \) for \( u_r \neq 0 \) and \( |v_t| \leq 1 \).

Proof. We take \( Y = \delta N^{1-3\eta} \). The hypothesis (4.8) is a consequence of Theorem 2 because of (4.10)–(4.12). The hypothesis (4.6) is a consequence of Theorem 2. Now the conclusion (4.9) may be written in the form (4.13). \( \square \)
Lemma 8. Let \(0 < g \leq \frac{1}{6} - 2\sigma, \frac{1}{3} - 4\sigma < \gamma < \frac{2}{3} - 8\sigma\). Let \(\rho_1 \geq \cdots \geq \rho_t \geq 0\) with \(\rho_1 + \cdots + \rho_t = \gamma, \rho_1 \leq \gamma - g\). There is a set \(C \subset \{1, \ldots, t\}\) with \(\sum_{i \in C} \rho_i \in \left[g, \frac{1}{3} - 4\sigma\right]\).

Proof. Suppose that no such \(C\) exists. Now suppose first that \(\rho_1 \leq g\). Since \(2g \leq \frac{1}{3} - 4\sigma\) we can prove successively that \(\rho_1 + \rho_2, \ldots, \rho_1 + \cdots + \rho_t\) are in \([0, g]\). This is absurd.

Thus we must have \(\rho_1 > \frac{1}{3} - 4\sigma\). But now \(\rho_2 + \cdots + \rho_t = \gamma - \rho_1 < \gamma - (\frac{1}{3} - 4\sigma) < \frac{1}{3} - 4\sigma\), and \(\rho_2 + \cdots + \rho_t \geq g\) since \(\rho_1 \leq \gamma - g\). This is absurd. \(\square\)

Lemma 9. Let \(F\) be a complex function on \([1, N]\). The sum
\[
\sum_{k \leq N} \Lambda(k) F(k)
\]
may be decomposed into at most \(C(\log N)^8\) sums of the form
\[
\sum_{n_i \in I_i; n_1 \ldots n_8 \leq N} (\log n_1) \mu(n_5) \ldots \mu(n_8) F(n_1 \ldots n_8)
\]
where \(I_i = (N_i, 2N_i], \prod N_i < N\) and \(2N_i \leq N^{1/4}\) if \(i > 4\).

Proof. This is a case of Heath-Brown’s ‘generalized Vaughan identity’ \([9]\). \(\square\)

Lemma 10. Let
\[
(2N)^{3/8 + 33\sigma/4} \leq Q < Q' \leq (2N)^{\frac{1}{2}}, Q' \leq 2Q.
\]
We have
\[
\sum_{Q \leq p < Q'} (S(A_p, z) - 2\delta S(B_p, z)) \ll \delta N^{1-\eta}.
\]

Proof. Arguing as in the proof of (4.3), it will suffice to show that
\[
\sum_{Q \leq p < Q'} \sum_{\ell \leq L} c_{\ell} \sum_{\substack{N \leq n \leq 2N \\\frac{N}{2} < r \leq 2N \\\gcd(n, P(z)) = 1}} e(\ell g(pm)) \ll N^{1-2\eta}
\]
for \(|c_{\ell}| \leq 1\). By a partial summation argument, it suffices to obtain
\[
\sum_{Q \leq m < Q'} \Lambda(m) \sum_{\ell \leq L} c_{\ell} \sum_{\substack{N \leq n \leq 2N \\\frac{N}{2} < r \leq 2N \\\gcd(n, P(z)) = 1}} e(\ell g(mn)) \ll N^{1-2\eta}.
\]

Applying Lemma \([9]\) we need only show that
\begin{align*}
\sum_{Q \leq n_1 \ldots n_8 \leq Q' \forall i} (\log n_1) \mu(n_5) \ldots \mu(n_8) & \sum_{\ell \leq L} c_{\ell} \sum_{N_{\ell} \leq N} e(\ell g(n_1 \ldots n_8)) \ll N^{1-3\eta}\end{align*}

whenever \( \prod_i N_i < N \) and \( 2N_i \leq N^{1/4} \) for \( i > 4 \). Thus \( Q \ll N_1 \ldots N_8 \ll Q \). We write \( N_1 \ldots N_8 = N^\gamma \).

We may assume in view of Theorem 3 that no subproduct \( X \) of \( N_1, \ldots, N_k \) satisfies
\begin{equation}
N^\rho \ll X \ll N^f.
\end{equation}

We now reorder \( N_1, \ldots, N_k \) as \( (N_1 \ldots N_k)^{\rho_j} \) \((1 \leq j \leq k)\) with \( \rho_1 \geq \cdots \geq \rho_k \geq 0 \). Let \( g = \min(3\gamma - \frac{5}{4} - \frac{15\sigma}{2}, \gamma - \frac{1}{3} - 8\sigma) \), so that \( 0 < g \leq \gamma - \frac{1}{3} - 8\sigma \) and
\begin{equation*}
g = \begin{cases}
3\gamma - \frac{5}{4} - \frac{15\sigma}{2} & (\gamma \leq \frac{11}{24} - \frac{\sigma}{4}) \\
\gamma - \frac{1}{3} - 8\sigma & (\gamma > \frac{11}{24} - \frac{\sigma}{4})
\end{cases}
\end{equation*}

We divide the argument into two cases.

**Case 1.** We have \( \rho_1 \geq \gamma - g > \frac{1}{4} \). Thus \( i \leq 4 \) and (after a partial summation if necessary) we can apply Theorem 2 to the sum in (4.15) with
\begin{equation*}
V \ll N^{\gamma - \rho_1}, \ W \ll N^{1-\gamma}.
\end{equation*}

We verify the hypotheses of Theorem 2. First,
\begin{align*}
3(\gamma - \rho_1) + 2(1 - \gamma) & \leq 3g + 2 - 2\gamma \leq \frac{3}{2} - 3\sigma = 2 - 3\rho,
\end{align*}

since
\begin{equation*}
g \leq \gamma - \frac{1}{3} - 8\sigma \leq \frac{1}{3} \left(2\gamma - 3\sigma - \frac{1}{2}\right).
\end{equation*}

Next,
\begin{align*}
(\gamma - \rho_1) + 3(1 - \gamma) & \leq g + 3 - 3\gamma \leq \frac{7}{4} - \frac{15\sigma}{2} = 3 - \frac{15\rho}{2}
\end{align*}

and
\begin{align*}
(\gamma - \rho_1) + (1 - \gamma) & \leq g + 1 - \gamma \leq \frac{2}{3} - 8\sigma = 2 - 8\rho
\end{align*}
from the definition of $g$. Now (4.15) follows from Theorem 2.

Case 2. We have $\rho_1 < \gamma - g$. By Lemma 8 and the absence of a product $X$ satisfying (4.17), there is a subsum $u = \sum_{i \in C} \rho_i$ such that

$$g \leq u < \frac{1}{6} + \sigma.$$ 

We are now in a position to apply Lemma 6 with $\theta = \rho$, $\theta + \psi = f$,

$$w(n) = \begin{cases} 
\sum_{\ell \leq L} c_{\ell} e(\ell g(n)) & \left(\frac{N}{2} < n \leq N\right) \\
0 & \text{(otherwise)},
\end{cases}$$

and

$$T \ll N^{\gamma - u}.$$ 

We need to verify (4.6), (4.8). Clearly (4.8) is a consequence of Theorem 3. As for (4.8), we need to verify the hypotheses of Theorem 2 with $V \leq 2N^{1/6 + \sigma}$, $W \ll N^{\gamma - u}$. Suppose first that $\gamma > \frac{11}{24} - \frac{\sigma}{4}$. Then $\gamma - u \leq \frac{1}{3} + 8\sigma$,

$$3 \left(\frac{1}{6} + \sigma\right) + 2 \left(\frac{1}{3} + 8\sigma\right) = \frac{7}{6} + 19\sigma < \frac{3}{2} - 3\sigma;$$

$$\left(\frac{1}{6} + \sigma\right) + 3 \left(\frac{1}{3} + 8\sigma\right) = \frac{7}{6} + 25\sigma < \frac{7}{4} - \frac{15\sigma}{2};$$

$$\left(\frac{1}{6} + \sigma\right) + \left(\frac{1}{3} + 8\sigma\right) = \frac{1}{2} + 9\sigma \leq \frac{2}{3} - 8\sigma.$$ 

Now suppose that $\gamma \leq \frac{11}{24} - \frac{\sigma}{4}$. Then $\gamma - u \leq \frac{5}{4} + \frac{15\sigma}{2} - 2\gamma$,

$$3 \left(\frac{1}{6} + \sigma\right) + 2 \left(\frac{5}{4} + \frac{15\sigma}{2} - 2\gamma\right) = 3 + 18\sigma - 4\gamma$$

$$\leq 3 + 18\sigma - 4 \left(\frac{3}{8} + \frac{33\sigma}{4}\right) < \frac{3}{2} - 3\sigma,$$
\[
\left( \frac{1}{6} + \sigma \right) + 3 \left( \frac{5}{4} + \frac{15\sigma}{2} - 2\gamma \right) = \frac{47}{12} + \frac{47\sigma}{2} - 6\gamma \\
\leq \frac{47}{12} + \frac{47\sigma}{2} - 6 \left( \frac{3}{8} + \frac{33\sigma}{4} \right) < \frac{7}{4} - \frac{15\sigma}{2}
\]
and
\[
\left( \frac{1}{6} + \sigma \right) + \left( \frac{5}{4} + \frac{15\sigma}{2} - 2\gamma \right) = \frac{17}{12} + \frac{17\sigma}{2} - 2\gamma \leq \frac{2}{3} - 8\sigma.
\]
Thus Theorem 2 yields the desired estimate (4.8). Now the lemma follows from Lemma 6.

There is a short interval in which we can use Lemma 9 directly to obtain a conclusion stronger than (4.15).

Lemma 11. We have
\[
\sum_{Q \leq p < Q'} \left( S(A_p, p) - 2\delta S(B_p, p) \right) \ll \delta N^{1-\eta}
\]
whenever
\[
(2N)^{1/3 + 8\sigma} \leq Q < Q' \leq (2N)^{\frac{1}{2} - 9\sigma}, Q' \leq 2Q.
\]

Proof. In this range of \( Q \) we have
\[
\sum_{Q \leq p < Q'} S(A_p, p) = \sum_{Q \leq p < Q'} 1;
\]
similarly with \( B \) in place of \( A \). We apply Lemma 9 to decompose the sum over \( p' \) (with \( \Lambda(m) \) in place of \( p' \)). Clearly it will be enough to show that, for \( |c_\ell| \leq 1 \),
\[
(4.18) \sum_{Q \leq p < Q'} \sum_{\frac{N}{2} < n_1 \ldots n_8 \leq N} (\log n_1) \mu(n_5) \ldots \mu(n_8) \sum_{\ell \leq L} c_\ell e(\ell g(p n_1 \ldots n_8)) \ll N^{1-3\eta},
\]
where \( N \ll Q \prod_{i=1}^{8} N_i \ll N \) and \( N_i \ll (N/Q)^{1/4} \) for \( i \geq 5 \). As in the preceding proof we may suppose that no subproduct \( X \) of \( N_1 \ldots N_8 \)
satisfies (4.17). Writing \( Q = N^\gamma \), we have
\[
N^{\frac{1}{6} - \gamma} \ll N^{\frac{1}{6} - 4\sigma}.
\]
Thus it is clear by a ‘reflection’ argument that no such \( X \) can satisfy
\[
N^{\frac{1}{6} + \sigma} \ll X \ll N^{1 - \gamma - (\frac{1}{6} - \sigma)} = N^{\frac{5}{6} - \gamma - \sigma}.
\]
Moreover,
\[
\frac{5}{6} - \gamma - \sigma > \frac{5}{6} - \left( \frac{1}{2} - 9\sigma \right) - \sigma > 2 \left( \frac{1}{6} + \sigma \right).
\]
It follows that
\[
\prod_{N_i \leq N^{\frac{1}{6} + \sigma}} N_i \leq N^{\frac{1}{6} + \sigma}.
\]
There cannot be two indices \( i \) with \( N_i > N^{\frac{5}{6} - \gamma - \sigma} \), since
\[
\frac{5}{3} - 2\gamma - 2\sigma > 1 - \gamma.
\]
Hence there is a \( j \) with
\[
N_j > N^{\frac{5}{6} - \gamma - \sigma}, \quad N^{1 - \gamma} / N_j < N^{\frac{1}{6} + \sigma}.
\]
We are now in a position to apply Theorem 2 with
\[
V \ll N^{\frac{1}{6} + \sigma}, \quad W \ll N^\gamma.
\]
We make the usual verification:
\[
3 \left( \frac{1}{6} + \sigma \right) + 2\gamma \leq 3 \left( \frac{1}{6} + \sigma \right) + 2 \left( \frac{1}{2} - 9\sigma \right) < \frac{3}{2} - 3\sigma;
\]
\[
\left( \frac{1}{6} + \sigma \right) + 3\gamma \leq \frac{1}{6} + \sigma + 3 \left( \frac{1}{2} - 9\sigma \right) < \frac{7}{4} - \frac{15\sigma}{2};
\]
\[
\left( \frac{1}{6} + \sigma \right) + \gamma \leq \frac{1}{6} + \sigma + \frac{1}{2} - 9\sigma = \frac{2}{3} - 8\sigma.
\]
Thus Theorem 2 yields (4.18). This completes the proof of Lemma 11. \( \square \)
5. Application of the linear sieve

In order to obtain an upper bound for

\[(5.1) \sum_{P \leq p < P'} S(A_p, p)\]

whenever

\[(5.2) (2N)^{\frac{1}{2} - 9\sigma} \leq P \leq P' \leq (2N)^{\frac{3}{2} + \frac{33\sigma}{4}}, P' \leq 2P,\]

we apply Theorem 4 of Iwaniec [10], which we state in a form sufficient for our needs. The quantity estimated will actually exceed that in (5.1), which will be exploited in Section 6.

Let $\mathcal{E}$ be a set of integers in $[1, N]$. Fix an approximation $X$ to $|\mathcal{E}|$ and write

\[r(\mathcal{E}, d) = |\mathcal{E}_d| - \frac{X}{d}.\]

Let $F(s)$ be the upper bound function for the linear sieve [10, p. 309]. In the following lemma, let $D \geq Z \geq 2$. Let

\[\mathcal{G} = \{D^{\varepsilon(1+\eta)^n} : n \geq 0\},\]

\[\mathcal{H} = \{D = (D_1, \ldots, D_r) : r \geq 1, D_\ell \in \mathcal{G} \text{ for } 1 \leq \ell \leq r, D_r \leq \cdots \leq D_1 < D^{1/2}\},\]

\[\mathcal{D}^+ = \left\{D \in \mathcal{H} : D_1 \cdots D_\ell D_\ell^3 < D \text{ for } 0 \leq \ell \leq \frac{r - 1}{2}\right\}.\]

**Lemma 12.** With the above notations, we have

\[S(\mathcal{E}, Z) \leq V(Z)X \left\{F \left(\frac{\log D}{\log Z}\right) + E\right\} + R^+_1 + R^+,
\]

where

\[E < C(\varepsilon + \varepsilon^{-8} (\log D)^{-1/3}),\]

\[R^+_1 = \sum_{\substack{d < D^\varepsilon \ni \phi^+_d(D^\varepsilon) \cdot r(A, d),}}\]

\[R^+ = \sum_{D \in \mathcal{D}^+} \sum_{\substack{d < D^\varepsilon \ni \Lambda^+_d(\varepsilon, D) H_d(\mathcal{E}, Z, \varepsilon, D),}}\]
with some coefficients $\phi_d^+(D^r)$ and $\Lambda_d^+(\epsilon, D)$ bounded by 1 in absolute value. Here

$$H_d(\mathcal{E}, Z, \epsilon, D) = \sum_{D_i \leq p_i < D_i^{1+\eta}, p_i < Z} r(\mathcal{E}, dp_1 \ldots p_r).$$

In our application, we shall take $\mathcal{E} = A_p$ for any $p \in [P, P')$, and $X = 2\delta|B_p|$. We write $P = N^\gamma$. We shall take $D = N^{4\sigma - 3\gamma - 2\epsilon}$ and $Z = N^{2\sigma + \frac{\gamma}{2}}$. It is easily verified that $\frac{4}{3} \leq Z \leq D^{1/2}$, so that

$$V(Z)F \left( \frac{\log D}{\log Z} \right) = \frac{2}{\log D} \left( 1 + \lambda \epsilon \right).$$

We apply Lemma 11 and sum over $p$, obtaining (on noting that $P > D^4 Z$)

$$\sum_{P \leq p < P'} S(A_p, Z) \leq V(Z) \left\{ F \left( \frac{\log D}{\log Z} \right) + C\epsilon \right\} \sum_{P \leq p < P'} 2\delta|B_p| + E_1 + E_2.$$

Here

$$E_1 = \sum_{\substack{d < D^a \\text{d} | D^2}} \sum_{P \leq p < P'} r(A, pd),$$

$$E_2 = \sum_{\substack{d < D^a \\text{d} | D^2}} \sum_{P \leq p < P'} \sum_{D \in D^+} \Lambda_d^+(\epsilon, D) \sum_{\substack{D_i \leq p_i < D_i^{1+\eta} \\text{p_i < Z} \\text{1 \leq i \leq r}}} r(A, pdp_1 \ldots p_r).$$

We shall show that

$$E_2 \ll \delta N^{1-\eta};$$

the proof that $E_1 \ll \delta N^{1-\eta}$ is similar but simpler.

Reducing the task of bounding $Z$ to estimating exponential sums as in previous sections, it suffices to prove that for $|c_\ell| \leq 1$, and a fixed
For the final stage of our work we take $\rho = \frac{37}{210} + \varepsilon$ so that $\sigma = \frac{1}{105} + \varepsilon$.

In this section, each $S_j$ ($j \geq 0$) that occurs takes the form

$$S_j = \sum_{1 \leq r \leq 8} \sum_{p_1 \cdots p_r, p'_1 \cdots p'_{r'} \in \mathcal{A}} 1.$$
where the asterisk indicates a restriction of \( p_1 \ldots p_r \) and \( p'_1 \ldots p'_t \) to certain subsets of \([1, N]\) depending on \( j \); \( S'_j \) is obtained from \( S_j \) on replacing \( \mathcal{A} \) by \( \mathcal{B} \). For some of these values of \( j \) we write

\[ S_j = K_j + D_j \]

where \( K_j \) is defined by the following additional condition of summation within \( S_j \): a subproduct \( R \) of \( p_1 \ldots p_r \ p'_1 \ldots p'_t \) satisfies

\[ N^\rho \ll R \ll N^\gamma. \]

We split up \( S'_j \) as \( S'_j = K'_j + D'_j \) in the same way. As noted in Section 5,

\[ S_j \geq K_j \geq 2\delta K'_j (1 - C\varepsilon) \]

\[ \geq 2\delta K'_j - \frac{C\delta \varepsilon N}{\log N} \]

\[ \geq 2\delta S'_j - 2\delta D'_j - \frac{C\delta \varepsilon N}{\log N} \]

whenever (6.1) is used.

Conversion of sums into integrals, with an acceptable error, in the following is along familiar lines (see [5]). Concerning Buchstab’s function \( \omega(t) \), we note that

\[ \omega(t) \leq \kappa \text{ for } t \geq \frac{1}{\kappa} \]

provided that \( \kappa \geq 0.5672 \), and that

\[ \omega(t) \geq e^{-\gamma} - 2.1 \times 10^{-8} \ (t \geq 6); \]

see Cheer and Goldston [5].

Let \( S_0 = S(\mathcal{A}, (2N)^{1/2}) \). Using Buchstab’s identity and writing \( p_i = (2N)^{a_i} \), we have

\[ S_0 = S_1 - \sum_{j=2}^{7} S_j \]

where
\[ S_1 = S(A, z), \]
\[ S_j = \sum_{\alpha_1 \in I_j} S(A_{p_1, p_1}) \quad (2 \leq j \leq 7), \]
with \( I_2 = \left[ \frac{1}{6} - 5\sigma, \frac{1}{6} + \sigma \right], I_3 = \left( \frac{1}{6} + \sigma, \frac{1}{3} - 4\sigma \right), I_4 = \left[ \frac{1}{3} - 4\sigma, \frac{1}{3} + 8\sigma \right), \]
\( I_5 = \left[ \frac{1}{3} + 8\sigma, \frac{1}{2} - 9\sigma \right), I_6 = \left[ \frac{1}{2} - 9\sigma, \frac{3}{8} + \frac{3\sigma}{4} \right) \) and \( I_7 = \left[ \frac{3}{8} + \frac{3\sigma}{4}, \frac{1}{2} \right]. \)

Recalling (4.3) and Lemmas 7 (with \( \nu = 0 \)) and 11, we have

(6.7) \[ S_j = 2\delta S_j'(1 + \lambda \varepsilon) \]
for \( j = 1, 3, 5 \). For \( j = 2 \), we apply Buchstab’s identity three further times to obtain

(6.8) \[ S_2 = S_8 - S_9 + S_{10} - S_{11} \]
where

\[ S_8 = \sum_{b \leq \alpha_1 < \rho} S(A_{p_1}, z), \quad S_9 = \sum_{b \leq \alpha_2 \leq \alpha_1 < \rho} S(A_{p_1 p_2}, z), \]
\[ S_{10} = \sum_{b \leq \alpha_3 \leq \alpha_2 \leq \alpha_1 < \rho} S(A_{p_1 p_2 p_3}, z) \]
and

\[ S_{11} = \sum_{b \leq \alpha_4 \leq \alpha_3 \leq \alpha_2 \leq \alpha_1 < \rho} S(A_{p_1 p_2 p_3 p_4}, p_4). \]

We can apply Lemma 7 to \( S_8, S_9, S_{10} \) to obtain (6.7): for example, in \( S_{10} \),

\[ 3\rho + 2(\alpha_1 + \alpha_2) \leq 7\rho \leq \frac{3}{2} - 2\sigma \]
\[ \rho + 3(\alpha_1 + \alpha_2) \leq 7\rho \leq \frac{7}{4} - \frac{15\sigma}{2}, \]
\[ \rho + (\alpha_1 + \alpha_2) \leq 3\rho \leq \frac{2}{3} - 8\sigma. \]

We treat \( S_{11} \) via (6.3).
For $j = 4$ we apply Buchstab’s identity once. Iterating once more for part of the sum over $p_1, p_2$, this gives

(6.9) \[ S_4 = S_{12} - S_{13} - S_{14} + S_{15}, \]

where

\[
S_{12} = \sum_{\alpha_1 \in I_4} S(A_{p_1}, z), \quad S_{13} = \sum_{\alpha_1 \in I_4, \alpha_2 \in [\frac{f}{2}, \alpha_1]} S(A_{p_1 p_2}, p_2),
\]
\[
S_{14} = \sum_{\alpha_1 \in I_4, \alpha_2 \in [b, \frac{f}{2})} S(A_{p_1 p_2}, z), \quad S_{15} = \sum_{\alpha_1 \in I_4, b \leq \alpha_3 \leq \alpha_2 < f/2} S(A_{p_1 p_2 p_3}, p_3).
\]

We have (6.7) for $S_{12}, S_{14}$ since Lemma 9 is applicable. For example, for $S_{14}$ we have

\[
3\rho + 2\alpha_2 \leq \frac{1}{2} + 3\sigma + \frac{2}{3} + 16\sigma < \frac{3}{2} - 3\sigma,
\]
\[
\rho + 3\alpha_2 \leq \frac{1}{6} + \sigma + 1 + 24\sigma < \frac{7}{4} - \frac{15\sigma}{2},
\]
\[
\rho + \alpha_2 \leq \frac{1}{6} + \sigma + \frac{1}{3} + 8\sigma < \frac{2}{3} - 8\sigma.
\]

We have (6.7) also for $S_{15}$, this time using (4.3), since $\rho \leq 2b \leq \alpha_2 + \alpha_3 < f$ in $S_{15}$. For $S_{13}$, we use the lower bound (6.3).

We also apply Buchstab once more to $S_7$,

(6.10) \[ S_7 = S_{16} - S_{17}, \]

where

\[
S_{16} = \sum_{\alpha_1 \in I_7} S(A_{p_1}, z)
\]

satisfies (6.7) by Lemma 10 and

\[
S_{17} = \sum_{\alpha_1 \in I_7, b \leq \alpha_2 < \alpha_1} S(A_{p_1 p_2}, p_2)
\]
is bounded below as in (6.3).

For $S_6$, we proceed differently. We have

(6.11) \[ S_6 + S_{18} = \sum_{\alpha_1 \in I_6} S(A_{p_1}, Z) \]
where
\[ S_{18} = \left\{ p_1 p'_1 \ldots p'_r \in \left( \frac{N}{2}, N \right] : \alpha_1 \in I_6, r \geq 2, Z \leq p'_1 \leq \cdots \leq p'_r \right\} \]
is treated as in [6.3].

By (5.7),
\[ \sum_{\alpha_1 \in I_6} S(A_{p_1}, Z) \leq \frac{\delta N (1 + C\varepsilon)}{\log N} \int_{I_7} \frac{d\alpha_1}{\alpha_1} \frac{2}{\frac{2}{3} - 8\sigma - \alpha_1}. \]
Moreover,
\[ S'_6 + S'_{18} = \frac{\delta N}{\log N} (1 + \lambda\varepsilon) \int_{I_7} \frac{d\alpha_1}{\alpha_1} \frac{2}{b} \omega \left( \frac{1 - \alpha_1}{b/2} \right). \]
Combining (6.11)–(6.13) with \( j = 18 \),
\[ S_6 \leq 2\delta S'_6 + \frac{\delta N}{\log N} \int_{I_7} \frac{d\alpha_1}{\alpha_1} \frac{2}{\frac{2}{3} - 8\sigma - \alpha_1} \left( \frac{2}{\frac{2}{3} - 8\sigma - \alpha_1} - \frac{2}{b} \omega \left( \frac{1 - \alpha_1}{b/2} \right) \right) \]
\[ + 2\delta D'_{18} + \frac{C\varepsilon\delta N}{\log N}. \]

Our sieve decomposition, obtained by combining (6.6), (6.8), (6.9) and (6.10), is
\[ S_0 = S_1 - S_3 - S_5 - (S_8 - S_9 + S_{10} - S_{11}) - (S_{12} - S_{13} - S_{14} + S_{15}) - (S_{16} - S_{17}) - S_6 \]
and also holds if \( S_j \) is replaced by \( S'_j \). Combining all applications of (6.7) and (6.3) with (6.14), we end up with
\[ S_0 \geq S'_0 - 2\delta (D'_{11} + D'_{13} + D'_{17} + D'_{18}) \]
\[ \quad - \frac{\delta N}{\log N} \int_{I_7} \frac{d\alpha_1}{\alpha_1} \frac{2}{\frac{2}{3} - 8\sigma - \alpha_1} \left( \frac{2}{\frac{2}{3} - 8\sigma - \alpha_1} - \frac{2}{b} \omega \left( \frac{1 - \alpha_1}{b/2} \right) \right) \]
\[ \quad - \frac{C\varepsilon\delta N}{\log N}. \]

Our next task is to evaluate \( D'_{11}, D'_{13}, D'_{17}, \) and \( D'_{18} \) with sufficient accuracy. Using \( 2b > c \) we find that in \( D'_{11}, \alpha_3 + \alpha_4 \not\in [c, f] \) implies
$\alpha_3 + \alpha_4 > f$. With a little thought we find that

$$D'_1 \leq \sum_{b \leq \alpha_4 \leq \alpha_3 \leq \alpha_2 \leq \alpha_1 < \rho, \alpha_3 + \alpha_4 > f} \frac{1}{p_1p_2p_3p_4 \in (\frac{N}{2}, N], p|n_4 \rightarrow p \geq p_4}$$

$$\leq (1 + \varepsilon) \frac{N}{2 \log N} J_1,$$

where

$$J_1 = \int \int \int \int_{b \leq w \leq z \leq y \leq x < \rho, z + w > f} \frac{dx}{x} \frac{dy}{y} \frac{dz}{z} \frac{dw}{w} \omega \left( \frac{1 - x - y - z - w}{w} \right).$$

It is simplest to replace the $\omega$ factor by 0.59775 using (6.4). Carrying out the $x$ and $w$ integrations (and using $z > f/2$) leads to

$$J_1 \leq 0.59775 \int \int \int \int_{\frac{f}{2} < z \leq y \leq \rho} \frac{dy}{y} \frac{dz}{z} \left( \frac{1}{f - z} - \frac{1}{z} \right) \log \frac{\rho}{y} < 0.000691.$$

For $D'_{13}$, $D'_{17}$ we write the numbers $p_1, p_2, p'_1, \ldots, p'_t$ that appear in $D'_j$ as

$$(2N)^{\alpha_1}, (2N)^{\alpha_2}, (2N)^{\beta_1}, \ldots, (2N)^{\beta_k}, (2N)^{\gamma_1}, \ldots, (2N)^{\gamma_\ell}$$

where $k \geq 0, \ell \geq 0, k + \ell \geq 1$,

$$\alpha_2 \leq \beta_1 \leq \cdots \leq \beta_k < \rho,$$

$$f < \gamma_1 \leq \cdots \leq \gamma_\ell$$

and

$$\alpha_1 + \alpha_2 + \beta_1 + \cdots + \beta_k + \gamma_1 + \cdots + \gamma_\ell = 1 + \lambda \varepsilon.$$

We now consider $D'_{13}$. First we treat the contribution $D^{(1)}$ from $\alpha_2 > f$. Thus $k = 0$. We cannot have $\ell \geq 2$, since

$$4 \left( \frac{1}{3} - 4\sigma \right) > 1.$$
If we consider $\ell = 1$, we have $\alpha_1 + 2\alpha_2 < 1$ and obtain

$$D^{(1)} = (1 + \lambda \varepsilon) \frac{\delta N}{\log N} J_2,$$

where

$$J_2 = \int_{\frac{1}{3}-4\sigma}^{\frac{1}{3}+8\sigma} \int_{\frac{1}{3}-4\sigma}^{\min\left(x, \frac{1}{3}-y\right)} \frac{dy \, dx}{xy(1-x-y)} < 0.059343. \quad (6.19)$$

Let $D^{(2)}$ be the contribution to $D_{13}$ from $\alpha_2 < \rho$. We have $\alpha_2 + \beta_1 \geq f$, $\beta_1 \geq f/2$, by an argument used above. We cannot have $\ell \geq 2$ since

$$3 \left( \frac{1}{3} - 4\sigma \right) + \frac{1}{6} - 2\sigma = \frac{7}{6} - 14\sigma > 1. \quad (6.20)$$

If $\ell = 1$, we must have $k \leq 1$ (use (6.20) again). If $\ell = 0$, we must have $k \leq 3$ similarly; however, $k > 2$, since

$$\frac{1}{3} + 8\sigma + 3 \left( \frac{1}{6} + \sigma \right) < 1.$$ 

The three remaining cases lead to

$$D^{(2)} = (1 + \lambda \varepsilon)(J_3 + J_4 + J_5),$$

where

$$J_3 = \int_{\frac{1}{3}-4\sigma}^{\frac{1}{3}+8\sigma} \int_{\frac{1}{3}-2\sigma}^{\frac{1}{5}+\sigma} \frac{dy \, dx}{xy(1-x-y)} < 0.118914 \quad (\ell = 1, k = 0), \quad (6.22)$$

$$J_4 = \int_{\frac{1}{3}-4\sigma}^{\frac{1}{3}+8\sigma} \int_{\frac{1}{6}-2\sigma}^{\frac{1}{6}+\sigma} \int_{\max\left(y, \frac{1}{3}-4\sigma-y\right)}^{\frac{1}{5}+\sigma} \frac{dz \, dy \, dx}{xyz(1-x-y-z)} < 0.027404 \quad (\ell = 1, k = 1), \quad (6.23)$$

$$J_5 = \int_{\frac{1}{3}-4\sigma}^{\frac{1}{3}+8\sigma} \int_{\frac{1}{6}-2\sigma}^{\frac{1}{6}+\sigma} \int_{\max\left(y, \frac{1}{3}-4\sigma-y\right)}^{\frac{1}{5}+\sigma} \int_{\frac{1}{5}-(x+y+z)}^{(1-x-y-z)/2} \frac{dw \, dz \, dy \, dx}{xyzw(1-x-y-z-w)} < 0.000245 \quad (\ell = 0, k = 3). \quad (6.24)$$
(The integral was bounded above by an integral in \(x, y, z\), using \(1 - x - y - z - w \geq w\), before carrying out the \(w\) integration.)

In \(D_{17}'\), we cannot have \(\alpha_2 \geq \rho\), since then

\[
\alpha_2 < \frac{1}{2} (1 - \alpha_1) < \frac{1}{2} \left( \frac{5}{8} - \frac{33\sigma}{4} \right) < \frac{1}{3} - 4\sigma,
\]

which is absurd. Thus \(\alpha_2 < \rho\); we cannot have \(\ell \geq 2\) or \(\ell = 1, k \geq 1\) or \(\ell = 0, k \geq 3\) since

\[
\frac{3}{8} + \frac{33\sigma}{4} + 2 \left( \frac{1}{3} - 4\sigma \right) > 1.
\]

For \(\ell = 0\), we have \(k > 1\) since

\[
\frac{1}{2} + 2 \left( \frac{1}{6} + \sigma \right) < 1.
\]

Thus

\[
(6.25) \quad D'_{17} = (1 + \lambda \varepsilon) \frac{N}{2 \log N} (J_6 + J_7),
\]

where

\[
J_6 = \int_{\frac{3}{8} + \frac{33\sigma}{4}}^{\frac{1}{2}} \int_{\frac{1}{6} - 5\sigma}^{\frac{1}{2} + \sigma} dx \int_{\frac{1}{6} - 5\sigma}^{\frac{1}{2} + \sigma} dy \int_{\frac{1}{6} - 5\sigma}^{\frac{1}{2} + \sigma} \frac{dydx}{xy(1 - x - y)} < 0.060205 \quad (\ell = 1, k = 0);
\]

\[
J_7 = \int_{\frac{1}{6} + 2\sigma}^{\frac{1}{2} + \sigma} \int_{\frac{1}{6} + 2\sigma}^{\frac{1}{2} + \sigma} \int_{\frac{1}{6} + 2\sigma}^{\frac{1}{2} + \sigma} \int_{\frac{1}{6} + 2\sigma}^{\frac{1}{2} + \sigma} dx \int_{\frac{1}{6} + 2\sigma}^{\frac{1}{2} + \sigma} dy \int_{\frac{1}{6} + 2\sigma}^{\frac{1}{2} + \sigma} \frac{dzdydx}{xyz(1 - x - y - z)} < 0.000237
\]

\[\quad (\ell = 0, k = 2).\]

In \(D_{18}'\), we write the numbers that appear as \(p_1, p'_1, \ldots, p'_r\) as

\[
(2N)^{\alpha_1}, (2N)^{\beta_1} \leq \cdots \leq (2N)^{\beta_k}, (2N)^{\gamma_1} \leq \cdots \leq (2N)^{\gamma_k}
\]

where \(k + \ell = r \geq 2, k \geq 0, \ell \geq 0, \beta_k < \rho, \gamma_1 > f\),

\[
\alpha_1 + \beta_1 + \cdots + \beta_k + \gamma_1 + \cdots + \gamma_k = 1 + \lambda_9 \varepsilon.
\]

We observe that

\[
\frac{1}{2} - 9\sigma + 2 \left( \frac{1}{3} - 4\sigma \right) > 1.
\]
Thus we cannot have $\ell \geq 2$ or $\ell = 1$, $k \geq 2$ or $\ell = 0$, $k \geq 4$. We cannot have $\ell = 0$, $k \leq 3$ since

$$\frac{3}{8} + \frac{33\sigma}{4} + 3 \left(\frac{1}{6} + \sigma\right) < 1.$$ 

Since $\ell + k \geq 2$, the only remaining possibility is $\ell = 1$, $k = 1$, so that

\begin{equation}
D_1' = (1 + \lambda \varepsilon) \frac{N}{2 \log N} J_8
\end{equation}

where

$$J_8 = \int_{\frac{1}{2} - 9\sigma}^{\frac{1}{2} + \sigma} \int_{\frac{1}{8} + \sigma}^{\frac{1}{6} + \sigma} \frac{dydx}{xy(1 - x - y)}.$$ 

Combining this with (6.14) we obtain

\begin{equation}
S_6 \leq 2\delta S_6' + \frac{\delta N}{\log N} J_9,
\end{equation}

with

\begin{equation}
J_9 = \int_T dx \left(\frac{2}{3} - 8\sigma - x\right) - \frac{2}{b} \omega \left(\frac{1 - \alpha_1}{b/2}\right) + \int_{\rho/2}^{\rho} \frac{dy}{y(1 - x - y)} < 0.727494
\end{equation}

(we obtain this using (6.5)). Since $S_0' = (1 + \lambda \varepsilon) \frac{N}{2 \log N}$, we only need to add up our integrals to obtain Theorem 1 from (6.15)–(6.28):

$$J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7 + J_9 < 0.994569 < 1 - \frac{1}{200}.$$ 

\section*{References}

[1] R. C. Baker, Fractional parts of polynomials over the primes, *Mathematika*, to appear (2017). Available at arXiv:1606.05779.

[2] R. C. Baker and G. Harman, On the distribution of $\alpha p^k$ modulo one, *Mathematika* 48 (1991), 170–184.

[3] R. C. Baker and A. Weingartner, A ternary Diophantine inequality over primes, *Acta Arith.* 162 (2014), 159–196.

[4] B. J. Birch and H. Davenport, On a theorem of Davenport and Heilbronn, *Acta Math.* 100 (1958), 259–279.

[5] A. Y. Cheer and D. A. Goldston, A differential-delay equation arising from the sieve of Eratosthenes, *Math. Comp.* 55 (1990), 129–141.
[6] A. Ghosh, The distribution of $\alpha p^2$ modulo 1, *Proc. London Math Soc.* **42** (1981), 252–269.

[7] G. Harman, On the distribution of $\alpha p$ modulo one II, *Proc. London Math. Soc.* **72** (1996), 241–260.

[8] G. Harman, *Prime-detecting Sieves*, Princeton University Press, Princeton, N.J., 2007.

[9] D. R. Heath-Brown, Prime numbers in short intervals and a generalized Vaughan identity, *Can. J. Math.* **34** (1982), 1365–1377.

[10] H. Iwaniec, A new form of the error term in the linear sieve, *Acta Arith.* **37** (1980), 307–320.

[11] I. M. Vinogradov, On the estimate of trigonometric sums with prime numbers, *Izvestiya Akad. Nauk. SSSR Ser. Mat.* **12** (1948), 225–248.

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