BIRATIONALLY NONEQUIVALENT LINEAR ACTIONS;
CAYLEY DEGREES OF SIMPLE ALGEBRAIC GROUPS;
AND SINGULARITIES OF TWO-DIMENSIONAL QUOTIENTS

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Below all algebraic varieties are taken over an algebraically closed field $k$ of characteristic zero.

1. BIRATIONALLY NONEQUIVALENT LINEAR ACTIONS

Let $G$ be a reductive algebraic group. In 1992 P. Katsylo published the following

Conjecture 1.1. ([Ka]) Let $V$ and $W$ be finite dimensional algebraic $G$-modules with trivial stabilizers of points in general position. Then the following properties are equivalent:

(i) $\dim V = \dim W$;
(ii) there exists a $G$-equivariant birational map $V \dasharrow W$.

In [Ka] Conjecture 1.1 was proved for $G = \text{SL}_2$, $\text{PSL}_2$, and the symmetric groups $S_n$, $n \leq 4$. However E. Tevelev observed (unpublished) that Conjecture 1.1 fails for one-dimensional spaces and $G = \mathbb{Z}/n$, $n \neq 2, 3, 4, 6$; the same observation was independently made in [RY]. In 2000 new counterexamples to Conjecture 1.1 have been found in [RY], where a birational classification of finite dimensional $G$-modules for diagonalizable $G$ has been obtained. Being sceptical about Conjecture 1.1, in 1993 I suggested to consider $W = V^*$, the dual module of $V$:

Problem 1.1. Are there a connected semisimple group $G$ and a finite dimensional algebraic $G$-module $V$ with trivial stabilizers of points in general position such that $V$ and $V^*$ are not birationally $G$-isomorphic?

This problem was communicated to some people, see, e.g., [RV]. So far it is still open.

It is well known that, if $G$ is connected, then $V^*$ is $V$ “twisted” by an automorphism of $G$. This naturally leads to the following generalization. Let $H$ be an algebraic group acting on an algebraic variety $X$,

$$H \times X \to X, \quad (h,x) \mapsto h \cdot x.$$ 

Let $\sigma: H \to H$, $h \mapsto \sigma h$, be an automorphism of $H$. Consider the following new action of $H$ on $X$:

$$H \times X \to X, \quad (h,x) \mapsto \sigma h \cdot x.$$ 

Then the new $H$-variety appearing in this way is denoted by $\sigma X$ and called $X$ “twisted” by $\sigma$. Problem 1.1 can be now generalized as follows:

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Problem 1.2. Are there a connected semisimple group $G$, a finite dimensional algebraic $G$-module $V$ with trivial stabilizers of points in general position, and an automorphism $\sigma$ of $G$ such that $V$ and $\sigma V$ are not birationally $G$-isomorphic?

Note that if in Problem 1.2 one replaces $V$ by a $G$-variety $X$, then the answer is positive (see [RV] where this is proved for $G = \mathrm{PGL}_n$).

It is clear that one should consider only outer automorphisms $\sigma$ in Problem 1.2. Also, it is easily seen that if $H$ is special in the sense of J.-P. Serre, [S], then $X$ and $\sigma X$ are always birationally $G$-isomorphic, cf. [RV]. In particular, answering Problems 1.1 and 1.2 for simple $G$, one should consider only

\[
\mathrm{SL}_n/\mu_d \text{ where } d \neq 1, \text{ and the groups of types } \mathrm{D}_l (l \text{ is odd in Problem 1.1}), \mathrm{E}_6.
\]

Problem 1.3. Let $G = \mathrm{SL}_d/\mu_d$. Let $V$ be the $d$-th symmetric power of $k^d$ endowed with the natural action of $G$. Let $\sigma$ be the automorphism of $G$ induced by the automorphism $g \mapsto (g^T)^{-1}$ of $\mathrm{SL}_d$. Are $V$ and $\sigma V$ birationally $G$-isomorphic?

Note that for $d > 3$ in Problem 1.3, stabilizers of points in general position in $V$ are trivial, [P].

2. Cayley degrees of simple algebraic groups

Let $G$ be a connected reductive algebraic group and let $\text{Lie } G$ be its Lie algebra. Consider the action of $G$ on $\text{Lie } G$ via the adjoint representation and on $G$ by conjugation.

Definition 2.1. ([LPR$_1$]) $G$ is called Cayley group if $G$ and $\text{Lie } G$ are birationally $G$-isomorphic.

All simple Cayley groups have been classified in [LPR$_1$, Theorem 1.31]: they are precisely the groups from the list

\[
\mathrm{SL}_n, n \leq 3; \quad \mathrm{SO}_n, n \neq 2, 4; \quad \mathrm{Sp}_{2n}; \quad \mathrm{PGL}_n.
\]

For every $G$, by [LPR$_1$, Prop. 10.5] there always exists a dominant $G$-equivariant rational map $G \rightarrow \rightarrow \text{Lie } G$. So the following number is well defined:

Definition 2.2. ([LPR$_1$]) The Cayley degree $\text{Cay}(G)$ of $G$ is the minimum of degrees of dominant rational $G$-equivariant maps $G \rightarrow \rightarrow \text{Lie } G$.

So $G$ is Cayley if and only if $\text{Cay}(G) = 1$. In general, $\text{Cay}(G)$ “measures” how far $G$ is from being Cayley.

Problem 2.1. ([LPR$_1$]) Find the Cayley degrees of connected simple algebraic groups.

In [LPR$_1$, LPR$_2$]) it is proved that

\[
\text{Cay}(\mathrm{SL}_n) \leq n - 2, \text{ for } n \geq 3; \quad \text{Cay}(\mathrm{SL}_n/\mu_d) \leq n/d;
\]

\[
\text{Cay}(\mathrm{Spin}_n) = \begin{cases} 2 & \text{for } n \geq 6, \\ 1 & \text{for } n \leq 5; \end{cases}
\]

\[
\text{Cay}(\mathrm{G}_2) = 2; \quad \text{Cay}(\mathrm{G}_2 \times \mathrm{G}_m^2) = 1.
\]

In particular, this implies that $\text{Cay}(\mathrm{SL}_4) = 2$ and $2 \leq \text{Cay}(\mathrm{SL}_5) \leq 3$.

Problem 2.2. Find $\text{Cay}(\mathrm{SL}_5)$. 
At the moment no examples of groups whose Cayley degree is bigger than 2 are known.

**Problem 2.3.** Is there $G$ such that $\text{Cay}(G) > 2$? Is there a simple such $G$?

More generally,

**Problem 2.4.** Given a $d \in \mathbb{N}$, is there $G$ such that $\text{Cay}(G) > d$? Is there a simple such $G$?

### 3. Singularities of two-dimensional quotients

Using a result of [KR], it was recently proved in [G$_2$] that if a complex reductive algebraic group $G$ acts algebraically on $\mathbb{C}^n$ and the categorical quotient $\mathbb{C}^n//G$ is two-dimensional, then $\mathbb{C}^n//G$ is isomorphic to $\mathbb{C}^2/\Gamma$, where $\Gamma$ is a finite group acting algebraically on $\mathbb{C}^2$. This theorem can be considered as a generalization of C. T. C. Wall’s conjecture for the linear action of $G$ on $\mathbb{C}^n$ proved in [G$_1$].

This result, discussed in Koras’ talk at this Workshop, prompted the following question.

**Problem 3.1.** (M. Miyanishi) What are the groups $\Gamma$ occuring in the above situation?

I conjecture that the following holds.

**Conjecture 3.1.** If $G$ is connected, then $\Gamma$ is cyclic.

Note that it was conjectured in [P$_1$] and proved in [Ke] that if in the above situation the group $G$ is connected semisimple and the action of $G$ on $\mathbb{C}^n$ is linear, then $\Gamma$ is trivial.

**References**

[1] R. V. Gurjar, *On a conjecture of C. T. C. Wall*, J. Math. Kyoto Univ. 31 (1991), 1121–1124.

[2] R. Gurjar, *Two-dimensional quotients of $\mathbb{C}^n$ are isomorphic to $\mathbb{C}^2/\Gamma$*, Transform. Groups 12 (2007), no. 1, 117–125.

[3] P. Katsylo, *On the birational classification of linear representations*, preprint MPI-92-1 (1992), Max-Planck-Inst. für Mathematik.

[4] G. Kempf, *Some quotient surfaces are smooth* 27 (1980), 285–299.

[5] M. Koras, P. Russell, *Contractible affine surfaces with a quotient singularity*, Transform. Groups 12 (2007), no. 2, to appear.

[6] N. Lemire, V. L. Popov, Z. Reichstein, *Cayley groups*, J. American Mathematical Society 19 (2006), 921–967.

[7] N. Lemire, V. L. Popov, Z. Reichstein, *On the Cayley degree of an algebraic group*, in: Actas del XVI Coloquio Latinoamericano de Algebra, Revista Matematica Iberoamericana, to appear, arXiv: math.AG/0608473.

[8] A. M. Popov, *Irreducible simple linear Lie groups with finite standard subgroups of general position*, Funct. Anal. Appl. 9 (1975), 346–347.

[9] V. L. Popov, *Representations with a free module of covariants*, Funct. Anal. Appl. 10 (1976), 242–244.

[10] Z. Reichstein, A. Vistoli, *Birational isomorphisms between twisted group actions*, math.AG/0504306.

[11] Z. Reichstein, B. Youssin, *A birational invariant for algebraic group actions*, Pacific J. Math. 204 (2002), no. 1, 223–246.

[12] J.-P. Serre, *Espaces fibrés algébriques*, in: Séminaire Chevalley “Anneaux de Chou”, exposé n°1, ENS, Paris, 1958, pp. 1-01–1-37.