On the coordinate ring of spherical conjugacy classes

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Abstract

Let $G$ be a simple algebraic group over an algebraically closed field $k$ of characteristic zero and $O$ be a spherical conjugacy class of $G$. We determine the decomposition of the coordinate ring $k[O]$ of $O$ into simple $G$-modules.

1 Introduction

In [9] we proved the De Concini-Kac-Procesi conjecture on the quantized enveloping algebra $U_\varepsilon(g)$ (introduced in [14]) for simple $U_\varepsilon(g)$-modules over spherical conjugacy classes of $G$ (we recall that a conjugacy class $O$ in $G$ is called spherical if a Borel subgroup of $G$ has a dense orbit in $O$): our main tool was the representation theory of the quantized Borel subalgebra $B_\varepsilon$ introduced in [15].

To fix the notation, $G$ is a complex simple simply-connected algebraic group, $g$ its Lie algebra, $B$ a Borel subgroup of $G$, $T$ a maximal torus of $B$, $B^-$ the Borel subgroup opposite to $B$, $\{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots with respect to the choice of $(T, B)$. Let $W$ be the Weyl group of $G$ and let us denote by $s_i$ the reflection corresponding to the simple root $\alpha_i$: $\ell(w)$ is the length of the element $w \in W$ and $\text{rk}(1-w)$ is the rank of $1-w$ in the geometric representation of $W$.

The representation theory of $U_\varepsilon(g)$ is related to the stratification of $G$ given by conjugacy classes, while the representation theory of $B_\varepsilon$ is related to the stratification $\{X_w \mid w \in W\}$ of $B^-$, where $X_w = B^- \cap BwB$ for every $w \in W$ (each $X_w$ is an affine variety of dimension $n + \ell(w)$). We proved that for every spherical conjugacy class $O$ in $G$, there exists $w \in W$ such that $O \cap X_w \neq \emptyset$ and $\ell(w) + \text{rk}(1-w) = \dim O$: this then allows to prove the De Concini-Kac-Procesi conjecture for simple $U_\varepsilon(g)$-modules over elements in $O$. In fact we proved also a result in the opposite direction, giving therefore a characterization of spherical conjugacy classes in terms of the Weyl group ([9], Theorem 25):

let $O$ be a conjugacy class of $G$ and $w = w(O)$ be the unique element in $W$ such that $O \cap BwB$ is dense in $O$. Then $O$ is spherical if and only if $\dim O = \ell(w) + \text{rk}(1-w)$. 
Moreover \( w \) is always an involution (see [9], Remark 4, [10], Theorem 2.7). From this result we conjectured that, for a spherical \( \mathcal{O} \), the decomposition of the ring \( \mathbb{C}[\mathcal{O}] \) of regular functions on \( \mathcal{O} \) (to which we refer as to the coordinate ring of \( \mathcal{O} \)) as a \( G \)-module should be strictly related to \( w(\mathcal{O}) \). This is the motivation for the present paper.

We recall that \( \mathbb{C}[\mathcal{O}] \) is multiplicity-free, so that in order to obtain the decomposition of \( \mathbb{C}[\mathcal{O}] \) into simple components one has just to determine which simple modules occur in \( \mathbb{C}[\mathcal{O}] \):\

\[
\mathbb{C}[\mathcal{O}] \cong \bigoplus_{\lambda \in \lambda(\mathcal{O})} V(\lambda)
\]

where for each dominant weight \( \lambda \), \( V(\lambda) \) is the simple \( G \)-module of highest weight \( \lambda \) (if \( \lambda \in \lambda(\mathcal{O}) \) we say that \( \lambda \) occurs in \( \mathbb{C}[\mathcal{O}] \)).

The decomposition of the coordinate ring \( \mathbb{C}[X] \) for \( G \)-varieties \( X \) has been investigated by various authors. If \( \lambda \) is a non-zero highest weight, and \( v \in V(\lambda) \) is a non-zero highest weight vector, then \( \mathbb{C}[G.v] \) is isomorphic to \( \bigoplus_{n \geq 0} V(n\lambda) \) (44, Theorem 2). In particular this determines \( \mathbb{C}[\mathcal{O}] \) for the minimal unipotent orbit of \( G \). For a unipotent class in \( G \) (equivalently nilpotent orbit in \( g \)) McGovern ([30], Theorem 3.1) describes \( \mathbb{C}[\mathcal{O}] \) in terms of induced building blocks from a certain Levi subgroup of \( G \) (via sheaf cohomology on \( G/Q \), \( Q \) a parabolic subgroup of \( G \) associated to \( \mathcal{O} \)): it is then possible to obtain multiplicities of simple \( G \)-modules in \( \mathbb{C}[\mathcal{O}] \) as an alternating sum of certain partition functions. In the same paper the author gives a formula for \( \mathbb{C}[\hat{\mathcal{O}}] \), where \( \hat{\mathcal{O}} \) is the simply-connected cover of \( \mathcal{O} \) ([30], Theorem 4.1). Then in [31] there are tables for the sets of simple modules in \( \mathbb{C}[\hat{\mathcal{O}}] \) for spherical unipotent classes in the classical groups (and conjecturally in the exceptional groups). For type \( F_4 \) the monoid \( \lambda(\mathcal{O}) \) has been described in [7] for all spherical unipotent classes. For the maximal spherical unipotent class \( \mathcal{O} \) in \( E_8 \), it has been shown in [2], Theorem 1.1, that every simple \( G \)-module occurs in \( \mathbb{C}[\mathcal{O}] \) (so that \( \mathcal{O} \) is a model orbit). In [36], Panyushev gives tables for the sets of simple modules for (spherical) nilpotent orbits of height 2 (and conjecturally for height 3). In [28] the author describes explicitly the structure of principal model homogeneous spaces. For semisimple spherical classes, the description of \( \lambda(\mathcal{O}) \) may be deduced from the tables in [26]. See also [45], Théorème 3, where symmetric varieties are considered.

The main result of this paper is the following:

**Theorem.** Assume \( \mathcal{O} \) is a spherical conjugacy class in \( G \), and let \( w = w(\mathcal{O}) \). Then a dominant weight \( \lambda \) occurs in \( \mathbb{C}[\mathcal{O}] \) if and only if \( w(\lambda) = -\lambda \) and \( \lambda(S_\mathcal{O}) = 1 \).

Here \( S_\mathcal{O} \) is a certain (finite) elementary abelian 2-subgroup of \( T \) which we determine for every spherical conjugacy class, describing therefore explicitly \( \lambda(\mathcal{O}) \): see tables 1, \ldots, 26. In particular we completely solve the problem of determining the simple modules occurring in \( \mathbb{C}[\mathcal{O}] \) for unipotent classes ([22], 8.13, Remark 2), and obtain the decomposition of \( \mathbb{C}[\mathcal{O}] \) for conjugacy classes of mixed elements.
Our proof is based on the deformation result obtained by Brion in [4]. We have \( \mathbb{C}[O] = \mathbb{C}[G/H] = \mathbb{C}[G]^H \), where \( H \) is the centralizer of an element of \( O \) in \( G \). There exists a flat deformation of \( G/H \) to a quotient \( G/H_0 \), where \( H_0 \) contains the unipotent radical \( U^- \) of \( B^- \). We determine the decomposition of \( \mathbb{C}[G/H_0] \) into simple components (i.e. we determine \( \lambda(G/H_0) \)), relating the group \( H_0 \) with \( H \) via the theory of elementary embeddings ([29], [5]). We then prove the crucial fact that \( \lambda(O) \) is saturated ([34], §1.3), so that \( \mathbb{C}[G/H] = \mathbb{C}[G/H_0] \) as \( G \)-modules. We also determine the decomposition of the coordinate ring \( \mathbb{C}[\hat{O}] \) for the simply-connected cover \( \hat{O} \) of \( O \), and of \( \mathbb{C}[\overline{O}] \).

The paper is structured as follows. In Section 2 we introduce the notation. In Section 3 we recall some basic facts about spherical varieties and we prove the main theorem. In Section 4 we determine the group \( S_O \) for the spherical conjugacy classes in the various groups, determining therefore the monoid \( \lambda(O) \), and also \( \lambda(\hat{O}) \). In Section 5 we consider the coordinate ring \( \mathbb{C}[\overline{O}] \) of the closure of \( O \). It is well known that \( \mathbb{C}[\overline{O}] = \mathbb{C}[O] \) if and only if \( \overline{O} \) is normal: we list all cases in which the spherical conjugacy class \( O \) has normal closure and we determine \( \lambda(O) \) for the classes with non-normal closure. In section 6 we consider the case when \( G \) is not necessarily simply-connected.

All the results and proofs of this article remain valid for \( G \) a simple simply-connected algebraic group over an algebraically closed field \( k \) of characteristic zero.

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2 Preliminaries

We denote by \( \mathbb{C} \) the complex numbers, by \( \mathbb{R} \) the reals, by \( \mathbb{Z} \) the integers and by \( \mathbb{N} \) the natural numbers.

Let \( A = (a_{ij}) \) be a finite indecomposable Cartan matrix of rank \( n \). To \( A \) there is associated a root system \( \Phi \), a simple Lie algebra \( \mathfrak{g} \) and a simple simply-connected algebraic group \( G \) over \( \mathbb{C} \). We fix a maximal torus \( T \) of \( G \), and a Borel subgroup \( B \) containing \( T \): \( B^- \) is the Borel subgroup opposite to \( B \), \( U \) (respectively \( U^- \)) is the unipotent radical of \( B \) (respectively of \( B^- \)). If \( \chi \) is a character of \( T \), we still denote by \( \chi \) the character of \( B \) which extends \( \chi \). We denote by \( \mathfrak{h} \) the Lie algebra of \( T \). Then \( \Phi \) is the set of roots relative to \( T \), and \( B \) determines the set of positive roots \( \Phi^+ \), and the simple roots \( \Delta = \{\alpha_1, \ldots, \alpha_n\} \). We fix a total ordering on \( \Phi^+ \) compatible with the height function. We shall use the numbering and the description of the simple roots in terms of the canonical basis \( (e_1, \ldots, e_k) \) of an appropriate \( \mathbb{R}^k \) as in [3], Planches I-IX. For the exceptional groups, we shall write \( \beta = (m_1, \ldots, m_n) \) for \( \beta = m_1\alpha_1 + \ldots + m_n\alpha_n \).

If \( \gamma \) is a character of \( T \), we shall also denote by \( \gamma \) the corresponding linear form \( (d\gamma)_1 \) on \( \mathfrak{h} \). The real subspace of \( \mathfrak{h}^* \) spanned by the roots is a Euclidean space \( E \), endowed with the scalar
product \((\alpha_i, \alpha_j) = d_ia_{ij}\). Here \(\{d_1, \ldots, d_n\}\) are relatively prime positive integers such that if \(D\) is the diagonal matrix with entries \(d_1, \ldots, d_n\), then \(DA\) is symmetric. \(P\) is the weight lattice, \(P^+\) the monoid of dominant weights and \(W\) the Weyl group; \(s_i\) is the simple reflection associated to \(\alpha_i\), \(\{\omega_1, \ldots, \omega_p\}\) are the fundamental weights, \(w_0\) is the longest element of \(W\). In the expression 
\[\lambda = \sum_{i} k_i \omega_i\]
we always assume \(k_i\)'s and \(n_i\)'s in \(\mathbb{N}\). If \(V\) is a \(G\)-module, \(v \in V\), \(f \in V^*\), then the matrix coefficient \(c_{f,v} : G \to \mathbb{C}\) is defined by \(c_{f,v}(g) = f(g.v)\) for \(g \in G\). We consider the action of \(G \times G\) on \(\mathbb{C}[G]\)
\[
((g, g_1).f)(c) = f(g^{-1}cg_1)
\]
for \(c, g, g_1 \in G\), \(f \in \mathbb{C}[G]\). The algebraic version of the Peter-Weyl theorem gives the decomposition
\[
\mathbb{C}[G] = \bigoplus_{\lambda \in P^+} V(-w_0\lambda)^* \otimes V(-w_0\lambda)
\]
We put \(\Pi = \{1, \ldots, n\}\) and we fix a Chevalley basis \(\{h_i, i \in \Pi; e_\alpha, \alpha \in \Phi\}\) of \(g\). We shall denote by \(\hat{w}_i\), for \(i = 1, \ldots, n\), the elements in \(h\) defined by \(\alpha_j(\hat{w}_i) = \delta_{ij}\) (recall that \(\omega_j(h_i) = \delta_{ij}\)) for \(j = 1, \ldots, n\). As usual we put \(\langle x, y \rangle = \frac{2(x.y)}{(y,y)}\).

We use the notation \(x_\alpha(k), h_\alpha(z)\), for \(\alpha \in \Phi, k \in \mathbb{C}, z \in \mathbb{C}^*\) as in [43], [11]. For \(\alpha \in \Phi\) we put \(X_\alpha = \{x_\alpha(k) | k \in \mathbb{C}\}\), the root-subgroup corresponding to \(\alpha\), and \(H_\alpha = \{h_\alpha(z) | z \in \mathbb{C}^*\}\). For \(h \in h\) we put \(H_h = \exp \mathbb{C}h\). We identify \(W\) with \(N/T\), where \(N\) is the normalizer of \(T\); given an element \(w \in W\) we shall denote a representative of \(w\) in \(N\) by \(\hat{w}\). We choose the \(x_\alpha\)'s so that, for all \(\alpha \in \Phi, n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)\) lies in \(N\) and has image the reflection \(s_\alpha\) in \(W\).

Then
\[
x_\alpha(\xi)x_{-\alpha}(-\xi^{-1})x_\alpha(\xi) = h_\alpha(\xi)n_\alpha, \quad n_\alpha^2 = h_\alpha(-1)
\]
for every \(\xi \in \mathbb{C}^*, \alpha \in \Phi\) ([41], Proposition 11.2.1).

We put \(T^w = \{t \in T | wtw^{-1} = t\}, T_2 = \{t \in T | t^2 = 1\}\). In particular \(T^w = T_2\) if \(w = w_0 = -1\).

For algebraic groups we use the notation in [19], [12]. In particular, for \(J \subseteq \Pi, \Delta_J = \{\alpha_j | j \in J\}\), \(\Phi_J\) is the corresponding root system, \(W_J\) the Weyl group, \(P_J\) the standard parabolic subgroup of \(G\), \(L_J = T\langle X_\alpha | \alpha \in \Phi_J\rangle\) the standard Levi subgroup of \(P_J\). For \(z \in W\) we put \(U_z = U \cap z^{-1}U^{-1}z\). Then the unipotent radical \(R_{U_J}P_J\) of \(P_J\) is \(U_{w_0w_j}\), where \(w_j\) is the longest element of \(W_J\). Moreover \(U \cap L_J = U_{w_J}\) is a maximal unipotent subgroup of \(L_J\).

If \(\Psi\) is a subsystem of type \(X_r\) of \(\Phi\) and \(H\) is the subgroup generated by \(X_\alpha, \alpha \in \Psi\), we say that \(H\) is a \(X_r\)-subgroup of \(G\).

If \(X\) is an algebraic variety, we denote by \(\mathbb{C}[X]\) the ring of regular functions on \(X\). If \(X\) is a multiplicity-free \(G\)-variety, then we denote by \(\lambda(X)\) the set of dominant weights occurring in \(\mathbb{C}[X]\), i.e. \(\lambda \in P^+\) such that \(\mathbb{C}[X]\) contains (a copy of) \(V(\lambda)\). If \(x \in X\) we denote by \(G.x\)
the $G$-orbit of $x$ and by $G_x$ the isotropy subgroup of $x$ in $G$. If the homogeneous space $G/H$ is spherical, we say that $H$ is a spherical subgroup of $G$.

If $x$ is an element of a group $K$ and $H \leq K$, we shall also denote by $C(x)$ the centralizer of $x$ in $K$, and by $C_H(x)$ the centralizer of $x$ in $H$. If $x, y \in K$, then $x \sim y$ means that $x, y$ are conjugate in $K$. For unipotent classes in exceptional groups we use the notation in [12]. We use the description of centralizers of involutions as in [21].

## 3 The main theorem

Let $O$ be a spherical conjugacy class. Our aim is to determine $\lambda(O)$. For this purpose if $H$ is the centralizer of an element in $O$, we have $\mathbb{C}[O] = \mathbb{C}[G/H] = \mathbb{C}[G]^H$ and, from (2.1),

$$\mathbb{C}[G]^H = \bigoplus_{\lambda \in \lambda(O)} V(-w_0\lambda)^* \otimes u_\lambda$$

where $0 \neq u_\lambda \in V(-w_0\lambda)^H$ ([37], Theorem 3.12). We start by considering in general a spherical homogeneous space $G/H$. Without loss of generality we may assume $BH$ dense in $G$. By [4], Theorem 1, there exists a (flat) deformation of $G/H$ to a homogeneous (spherical) space $G/H_0$, where $H_0$ contains a maximal unipotent subgroup of $G$ (such an homogeneous space is called horospherical, and $H_0$ a horospherical contraction of $H$). An elementary embedding of $G/H$ is a pair $(X, x)$ where $X$ is a normal algebraic $G$-variety, $x \in X$ is such that $G.x$ is dense in $X$, $G_x = H$ and $X \setminus G.x$ is a $G$-orbit of codimension 1 ([6], 2.2). In [4] Brion constructs a $G \times \mathbb{C}^*$-variety and a flat $G \times \mathbb{C}^*$-morphism $p : Z \to \mathbb{C}$ (where $G$ acts trivially on $\mathbb{C}$ and $\mathbb{C}^*$ acts via homotheties) such that $p^{-1}(0) \cong G/H \times \mathbb{C}^*$ and $p^{-1}(0) \cong G/H_0$ ([4], Theoreme 1, [6] §3.11).

One may consider $Z$ as an elementary embedding $(Z, z)$ of $(G \times \mathbb{C}^*)/(H \times 1)$, with closed orbit $(G \times \mathbb{C}^*)/(H_0 \times \mathbb{C}^*)$; $H \times 1$ is the isotropy subgroup of $z$, $H_0 \times \mathbb{C}^*$ is the isotropy subgroup of an element in the closed orbit ([6], proof of Corollaire 3.7). Let $P = P_J$ be the parabolic subgroup associated to $H$, $P = \{g \in G \mid gBH = BH\}$, and let $L$ be a Levi subgroup (which we may assume equal to $L_J$, by taking an appropriate conjugate of $H$ instead of $H$) of $P$ adapted to $H$ ([6], 2.9): in particular

$$P \cap H = L \cap H \quad , \quad L' \leq H$$

Then $P \times \mathbb{C}^*$ is the parabolic subgroup of $G \times \mathbb{C}^*$ associated to $H \times 1$ and $L \times \mathbb{C}^*$ is a Levi subgroup adapted to $H \times 1$ ([6], Corollaire 3.7 and its proof).

By [6], Proposition 3.10, i), we have $H_0 \times \mathbb{C}^* = (R_u Q \times 1)(L \times \mathbb{C}^* \cap H_0 \times \mathbb{C}^*)$ where $Q$ is the opposite parabolic subgroup of $P$ with respect to $L$, so that

$$H_0 = (R_u Q)(L \cap H_0)$$
We show that $L \cap H = L \cap H_0$. Let $L = CL'$, where $C$ is the connected component of the centre of $L$. Then $L'$ is contained also in $H_0$, by [6], Théorème 3.6.

By [6], Proposition 3.4, $Z$ contains an open $P \times \mathbb{C}^*$-stable subset isomorphic to $R_u P \times W$ where $W$ is $L \times \mathbb{C}^*$-stable and meets the closed orbit, and $(W, z)$ is an elementary embedding of the torus $(C \times \mathbb{C}^*)/(C \cap H \times 1)$ ([5], proof of Lemme 4.2). Then $f = p_{[W} : W \rightarrow \mathbb{C}$ is a $(C \times \mathbb{C}^*)$-equivariant flat morphism such that $f^{-1}(\mathbb{C}^*) \cong C/C \cap H \times \mathbb{C}^*$ and $f^{-1}(0) \cong C/H_0 \cap C$. So the coordinate rings of these orbits are isomorphic $C$-modules and it follows that the isotropy groups of all points of $W$ are the same. In particular

\begin{equation}
C \cap H = C \cap H_0
\end{equation}

With the above notation we prove

**Theorem 3.1** Let $H$ be a spherical subgroup of $G$ such that $BH$ is dense in $G$ and $L = L_j$ is a Levi subgroup adapted to $H$. Then $H_0 = R_u Q (L \cap H) = (U^-, U_{w_j}, C \cap H)$.

**Proof.** By (3.5) we have

$L \cap H_0 = L' C \cap H_0 = L'(C \cap H_0) = L'(C \cap H) = L' C \cap H = L \cap H$

so that by (3.4) we conclude. \hfill \Box

**Definition 3.2** We put $\tilde{\lambda}(G/H) = \lambda(G/H_0)$.

Note that $\lambda(G/H) \leq \tilde{\lambda}(G/H)$ since $BH$ is dense in $G$, and more generally $Z \lambda(G/H) \cap P^+ \leq \tilde{\lambda}(G/H)$ ([34], part 2 of the proof of Proposition 1.5). Moreover

\begin{equation}
\lambda(G/H_0) = \{ \lambda \in P^+ \mid \lambda(T \cap H) = 1 \}
\end{equation}

since $\prod_{j \in J} H_{\alpha_j} \leq H$ and $X_{\alpha_j} v_{-\lambda} = v_{-\lambda}$ if $(\lambda, \alpha_j) = 0$ (here $v_{-\lambda}$ is a lowest weight vector of weight $-\lambda$ in $V(-\omega_0 \lambda)$). Also $B \cap H \leq P \cap H = L \cap H$, so that $B \cap H = U_{w_j}(T \cap H)$.

If $\lambda \in \tilde{\lambda}(G/H)$, then $F_{\lambda} : BH/H \rightarrow \mathbb{C}$, $b^{-1} H \mapsto \lambda(b)$ is a regular function on $BH/H$, and therefore a $B$-eigenvector of weight $\lambda$ in $\mathbb{C}(G/H)$. In case $G/H$ is quasi affine (as for conjugacy classes), then $Z \lambda(G/H) \cap P^+ = \tilde{\lambda}(G/H)$ since $\mathbb{C}(G/H) = \operatorname{Frac} \mathbb{C}[G/H]$, as in [34], Proposition 1.5. I do not know if $Z \lambda(G/H) \cap P^+ = \tilde{\lambda}(G/H)$ holds in general.

**Lemma 3.3** Suppose $F$ in $\operatorname{Frac} \mathbb{C}[G/H]$ is a $B$-eigenvector of weight $\lambda$ and $m \lambda$ lies in $\lambda(G/H)$ for a positive integer $m$. Then $F$ lies in $\mathbb{C}[G/H]$.

**Proof.** There exists a $B$-eigenvector $F_1 \in \mathbb{C}[G/H]$ of weight $m \lambda$. Then $F^m / F_1$ is invariant under $B$ (as its weight is 0). So $F^m / F_1$ is constant, as $G/H$ is spherical. In other words, $F^m$ is regular
on $G/H$. We conclude that $F$ is in $\mathbb{C}[G/H]$, since $\mathbb{C}[G/H]$ is integrally closed ([16], Lemma 1.8).

Let $\mathcal{O}$ be a spherical conjugacy class of $G$. We recall that $w = w(\mathcal{O})$ is the unique element (an involution) of $W$ such that $BwB \cap \mathcal{O}$ is (open) dense in $\mathcal{O}$. Let $v$ be the dense $B$-orbit in $\mathcal{O}$. Then $BG_y$ is dense in $G$ for any $y \in v$. The parabolic subgroup $P = P_j$ associated to $G_y$ coincides with $\{g \in G \mid g.v = v\}$. Moreover $v = \mathcal{O} \cap BwB$ ([9], Corollary 26), and it is affine, as an orbit of a soluble algebraic group.

We have $w = w_0w_j$, the subset $J$ is invariant under $\vartheta$, where $\vartheta$ is the symmetry of $\Pi$ induced by $-w_0$, and $w_0$ and $w_j$ act in the same way on $\Phi_J$ (see [10] the discussion at the end of section 3, Corollary 4.2, Remark 4.3 and Proposition 4.15).

Since all Levi subgroups of $P$ are conjugate under $R_yP$, we may choose $y \in v$ such that the standard Levi subgroup $L_j$ is adapted to $G_y$. For the rest of this section we fix such a $y$, and we put $H = G_y$, $P = P_j$, $L = L_j$. By Theorem 3.1, we have

\[(3.7) \quad H_0 = \langle U^-, U_{w_j}, C_y \rangle = \langle U^-, U_{w_j}, T_y \rangle\]

and $\tilde{\lambda}(\mathcal{O}) = \lambda(G/H_0)$.

We shall now relate $H$ with centralizers of elements in $v \cap wB$. By the Bruhat decomposition, $y$ is of the form $y = u\tilde{w}b$, where $u \in R_uP$ and $b \in B$. We put $x_1 = u^{-1}yu = \tilde{w}bu$. By [10], Corollary 4.13, $U_{w_j}(T^u)^o \leq C(x_1)$. Moreover, since $L' \leq C(y)$, by [10], Lemma 3.4, and commutation of $y$ with $X_{\pm\alpha_i}$ for $i \in J$, we get $L' \leq C(x_1)$ (see also the proof of [10], Proposition 4.15).

**Proposition 3.4** Let $x$ be in $\mathcal{O} \cap wB$. Then $T_x = T_y$ and $T \cap H^0 = T \cap C(x)^o$.

**Proof.** We observe that $C_{T^u}(x) \leq T$ by the Bruhat decomposition and $C_{T^u}(y) \leq T$, since $L$ is adapted to $C(y)$. Now $x_1 = u^{-1}yu = y^u$ implies

\[T_{x_1} = C_T(x_1) = C_{T^u}(x_1) \leq T \cap T^u = C_T(y) \leq C_{T^u}(y) \leq T \cap T^{u^{-1}} = C_T(u^{-1}) = C_T(u)\]

therefore if $t \in T_y$, then $t = t^u \in T_{x_1}$ and similarly if $t \in T_{x_1}$, then $t = t^{u^{-1}} \in T_y$. Hence $T_y = T_{x_1}$, and $T \cap C(y)^o = T \cap C(x_1)^o$. To conclude note that $\mathcal{O} \cap wB$ is the $T$-orbit of $x_1$. □

**Remark 3.5** In fact $C_L(x) = C_L(y)$ for every $x \in \mathcal{O} \cap wB$, since $L' \leq C(x)$.

**Remark 3.6** In general it is not true that $L_j$ is adapted to $C(x)$ for $x \in \mathcal{O} \cap wB$. For example if $\mathcal{O}$ is the minimal unipotent class, and $u$ is a non-identity element in $X_{-\beta}$, where $\beta$ is the highest root, then $C(u) \geq U^-$, so that there is a unique Levi subgroup of $P$ adapted to $C(u)$ ([6], Proposition 3.9), and this is $L_j$. Since $u \not\in wB$, there is no element $x \in wB$ such that $L_j$ is adapted to $C(x)$.
From Theorem 3.1 we get

**Corollary 3.7** Let \( O \) be a spherical conjugacy class, \( w = w(O) \) and \( x \) any element in \( O \cap wB \). Then \( H_0 = \langle U^-, U_{x,j}, T_x \rangle \), \( w = w_0w_j \).

By Proposition 3.4, we may put \( T_O = T_x \), for \( x \in O \cap wB \). Then \( T_O = T_y \) and \( (T^w)^o \leq T_O \leq T^w \) by [9], step 2 in the proof of Theorem 5.

We shall need the description of the monoid of weights \( \lambda \) such that \( w(\lambda) = -\lambda \). In the next lemma we consider more generally \( w \) of the form \( w = w_0w_j \), with \( J \) \( \vartheta \)-invariant.

**Lemma 3.8** Let \( J \subseteq \Pi \) be \( \vartheta \)-invariant and \( w = w_0w_j \). The dominant weight \( \lambda \) satisfies \( w(\lambda) = -\lambda \) if and only if \( \lambda = \sum_{i \in \Pi \setminus J} n_i \omega_i \) with \( n_{\vartheta(i)} = n_i \) for all \( i \in \Pi \setminus J \). Moreover \( w(\lambda) = -\lambda \) implies \( w_0(\lambda) = -\lambda \).

**Proof.** Let \( \lambda \in P^+ \), \( \lambda = \sum n_i \omega_i \), \( n_i \in \mathbb{N} \). For \( i \in \Pi \setminus J \) we have \( w_j(\omega_i) = \omega_i \), so that \( w(\omega_i) = -\omega_{\vartheta(i)} \).

It is clear that if \( \lambda = \sum_{i \in \Pi \setminus J} n_i \omega_i \) with \( n_i = n_{\vartheta(i)} \) for every \( i \in \Pi \setminus J \), then \( (w + 1)(\lambda) = 0 \).

On the other hand, assume \( w(\lambda) = -\lambda \). Then \( w_j(\lambda) = -w_0 \lambda \) and, by [20], Theorem 1.12 (a), we get \( -w_0 \lambda = \lambda \) and \( (\lambda, \alpha_j) = 0 \) for every \( j \in J \). Hence \( n_j = 0 \) for every \( j \in J \). Moreover, from \( \lambda = \sum_{i \in \Pi \setminus J} n_i \omega_i \) and \( -w_0 \lambda = \lambda \) it follows \( n_{\vartheta(i)} = n_i \) for all \( i \in \Pi \setminus J \).

**Remark 3.9** If \( S \) is a \( \vartheta \)-orbit in \( \Pi \setminus J \), and we put \( \omega_S = \sum_{i \in S} \omega_i \) then we have seen that \( \{ \omega_S \mid S \in (\Pi \setminus J)/\vartheta \} \) is a basis of the monoid \( \{ \lambda \in P^+ \mid w(\lambda) = -\lambda \} \), where \( (\Pi \setminus J)/\vartheta \) is the set of \( \vartheta \)-orbits in \( \Pi \setminus J \). If we also assume that \( w \) acts trivially on \( \Phi, \vartheta \) (as in the case of \( w = w(O) \)), then \( \{ \omega_S \mid S \in (\Pi \setminus J)/\vartheta \} \) is a basis of \( \ker(w + 1) \) in \( E \), and so a basis of the free abelian group \( \{ \lambda \in P \mid w(\lambda) = -\lambda \} \).

We describe \( \tilde{\lambda}(O) \). For this purpose we denote by \( S_O \) any supplement of \( (T^w)^o \) in \( T_O \) (i.e. \( S_O(T^w)^o = T_O \)). We also put \( P^+_w = \{ \lambda \in P^+ \mid w(\lambda) = -\lambda \} \). By Lemma 3.8 each element of \( P^+_w \) satisfies \( -w_0 \lambda = \lambda \), so that in particular any subset \( X \) of \( P^+_w \) is symmetric, i.e. \( -w_0(X) = X \) ([32], 4.2, [10], Theorem 4.17)).

**Theorem 3.10** Let \( O \) be a spherical conjugacy class, \( w = w(O) \) and let \( S_O \) be any supplement of \( (T^w)^o \) in \( T_O \). Then

\[
\tilde{\lambda}(O) = \{ \lambda \in P^+_w \mid \lambda(S_O) = 1 \}
\]

**Proof.** By (3.6), \( \tilde{\lambda}(O) = \{ \lambda \in P^+ \mid \lambda(T_O) = 1 \} \). Since \( (T^w)^o \leq T_O \), a necessary condition for \( \lambda \in P^+ \) to be in \( \tilde{\lambda}(O) \) is that \( \lambda(t t^w) = 1 \) for every \( t \in T \), as \( (T^w)^o = \{ t t^w \mid t \in T \} \). This condition is equivalent to \( (w + 1) \lambda = 0 \), so that \( \tilde{\lambda}(O) \leq P^+_w \). Let \( \lambda \in P^+_w \): then \( \lambda \in \tilde{\lambda}(O) \iff \lambda(S_O) = 1 \). \( \square \)
We shall prove the crucial fact that $\tilde{\lambda}(\mathcal{O}) = \lambda(\mathcal{O})$, so that the monoid $\lambda(\mathcal{O})$ is saturated (that is $\mathbb{Z}\lambda(\mathcal{O}) \cap P^+ = \lambda(\mathcal{O})$, [34], Definition 1.3). In the following, $x$ is a fixed element in $\mathcal{O} \cap wB$ and $\tilde{w}$ a representative of $w$ in $N$ such that $x = \tilde{w}u$, $u \in U$. If $u = \prod_{\alpha \in \Phi^+} x_\alpha(k_\alpha)$, and $i \in \Pi$, we say that $\alpha_i$ occurs in $x$ if $k_\alpha \neq 0$. This is independent of the chosen total ordering on $\Phi^+$.

For the closure $\tilde{\mathcal{O}}$ of $\mathcal{O}$ in $G$, the monoid $\lambda(\tilde{\mathcal{O}})$ of dominant weights occurring in $\mathbb{C}[\tilde{\mathcal{O}}]$ is a submonoid of $\lambda(\mathcal{O})$. We start with

**Proposition 3.11** Let $\lambda \in P^+$. Then $(1-w)\lambda$ lies in $\lambda(\tilde{\mathcal{O}})$.

**Proof.** Let $f \in V(\lambda)^*_{w\lambda}$, $v \in V(\lambda)_\lambda$ with $f(\tilde{w}.v) = 1$. Then $c_{f,v}(t^{-1}gt) = c_{t.f,v}(g) = ((1-w)\lambda)(t)c_{f,v}(g)$ for every $t \in T$, $g \in G$. For every $z$, $z_1 \in U$ we have

$$c_{f,v}(z_1xz) = f(z_1\tilde{w}uz.v) = f(z_1\tilde{w}.v) = f(\tilde{w}.v) = 1$$

since $z_1\tilde{w}.v = \tilde{w}.v + v_1$, where $v_1$ is a sum of weight vectors of weights strictly greater than $w\lambda$. Therefore for every $t \in T$, $z \in U$ we have

$$c_{f,v}(t^{-1}z^{-1}xzt) = ((1-w)\lambda)(t)$$

Since $B.x$ is dense in $\tilde{\mathcal{O}}$, by (3.8) the restriction of $c_{f,v}$ to $\tilde{\mathcal{O}}$ is a (non-zero) $B$-eigenvector of weight $(1-w)\lambda$ in $\mathbb{C}[\tilde{\mathcal{O}}]$. Hence $(1-w)\lambda \in \lambda(\tilde{\mathcal{O}})$.

**Corollary 3.12** Let $\lambda \in P^+_w$. Then $2\lambda$ lies in $\lambda(\tilde{\mathcal{O}})$.

**Corollary 3.13** Let $\lambda \in P^+$. Then $(1-w)\lambda \in \lambda(\mathcal{O})$. If moreover $\lambda \in P^+_w$, then $2\lambda$ lies in $\lambda(\mathcal{O})$.

**Proof.** This follows from the fact that $\lambda(\tilde{\mathcal{O}}) \leq \lambda(\mathcal{O})$.

We have shown that

$$2P^+_w \leq (1-w)P^+ \leq \lambda(\tilde{\mathcal{O}}) \leq \lambda(\mathcal{O}) \leq \tilde{\lambda}(\mathcal{O}) \leq P^+_w$$

We can prove that $\lambda(\mathcal{O})$ is saturated.

**Theorem 3.14** Let $\mathcal{O}$ be a spherical conjugacy class. Then $\lambda(\mathcal{O})$ is saturated.

**Proof.** Let $\lambda \in \tilde{\lambda}(\mathcal{O})$. We put $F(b^{-1}xb) = \lambda(b)$ for $b \in B$. We observed that $F$ is well-defined since $C_B(x) = T_xU_{w_j}$ and gives rise to a $B$-eigenvector of weight $\lambda$ in $\mathbb{C}(\mathcal{O})$. Since $\mathcal{O}$ is quasi affine, we conclude that $\lambda$ lies in $\lambda(\mathcal{O})$ by Theorem 3.10, Corollary 3.13 and Lemma 3.3.

Theorem 3.14 in particular proves Conjecture 5.12 (and 5.10 and 5.11) in [36].

To deal with $\lambda(\tilde{\mathcal{O}})$, in section 5 we shall make use of
Proposition 3.15 Let $\lambda \in P^+$, $i \in \Pi \setminus J$ be such that $\alpha_i$ occurs in $x$ and $(\lambda, \alpha_i) \neq 0$. Then $(1 - w)\lambda - \alpha_i \in \lambda(\O)$. 

Proof. Since $\langle \lambda, \alpha_i \rangle \neq 0$, $\lambda - \alpha_i$ is a weight of $V(\lambda)$. We construct two matrix coefficients. We fix a non-zero $v \in V(\lambda)_{\lambda - \alpha_i}$. By [43], Lemma 72, there exists a (unique) $v_\lambda \in V(\lambda)_\lambda$ such that $x_{\alpha_i}(k).v = v + kv_\lambda$ for every $k \in \C$. Then we choose $f \in V(\lambda)^*_{-\lambda}$ such that $f(\hat{w}.v_\lambda) = 1$.

Since $\alpha_i$ occurs in $x = \hat{w} u$, we have $u = x_{\alpha_i}(r)u'$, with $r \in \C^*$, $u' \in \prod_{\beta \in \Phi^+ \setminus \{\alpha_i\}} X_\beta$. Let $y, y_1 \in U$, and let $y = x_{\alpha_i}(k)y', y' \in \prod_{\beta \in \Phi^+ \setminus \{\alpha_i\}} X_\beta$, then

$$y_1^{-1}xy.v = y_1^{-1}\hat{w}.v + (k + r)y_1^{-1}\hat{w}.v_\lambda$$

The vector $\hat{w}.v$ has weight $\lambda - \alpha_i$, so that $y_1^{-1}\hat{w}.v$ is a sum of weight vectors of weight $\lambda - \alpha_i + \beta$, where $\beta$ is a sum of simple roots with non-negative coefficients. Assume $w_\lambda = w(\lambda - \alpha_i) + \beta$ for a certain $\beta$. Then $w(\alpha_i) = \beta$ would be positive, a contradiction since $i \in \Pi \setminus J$.

Hence $f(y_1^{-1}\hat{w}.v) = 0$. Similarly, $y_1^{-1}\hat{w}.v_\lambda = \hat{w}.v_\lambda + v'$, where $v'$ is a sum of weight vectors of weights greater than $w_\lambda$, hence $f(y_1^{-1}\hat{w}.v_\lambda) = f(\hat{w}.v_\lambda) = 1$, so that $c_{\hat{w},v}(y_1^{-1}xy) = k + r$.

The second matrix coefficient is defined dually. We fix a non-zero $f_1 \in V(\lambda)_{-\lambda - \alpha_i}$. There exists a (unique) $f_\lambda \in V(-w_\lambda \lambda)^*_\lambda$ such that $x_{\alpha_i}(k).f_1 = f_1 + k f_\lambda$ for every $k \in \C$.

Then we choose $v_1 \in V(-w_\lambda \lambda)_{-\lambda - \alpha_i}$ such that $f_\lambda(\hat{w}.v_1) = 1$. Let $z, z_1 \in U$, $z_1 = x_{\alpha_i}(k_1)z'$, $z' \in \prod_{\beta \in \Phi^+ \setminus \{\alpha_i\}} X_\beta$, then proceeding as before, we get $c_{f_1,v_1}(z_1^{-1}xz) = k_1$.

For $t \in T$, $z \in U$ we obtain

$$c_{f,v} - c_{f_1,v_1}(t^{-1}z^{-1}xz) = r ((1 - w)\lambda - \alpha_i)(t)$$

Since $B.x$ is dense in $\O$, by (3.10) the restriction of $c_{f,v} - c_{f_1,v_1}$ to $\O$ is a (non-zero) $B$-eigenvector of weight $(1 - w)\lambda - \alpha_i$ in $\C(\O)$. Hence $(1 - w)\lambda - \alpha_i \in \lambda(\O)$. \qed

Corollary 3.16 Let $i \in \Pi \setminus J$ be such that $\alpha_i$ occurs in $x$. Then $\omega_i + \omega_{\beta(i)} - \alpha_i$ lies in $\lambda(\O)$. 

Proof. This follows from Proposition 3.15 by taking $\lambda = \omega_i$. \qed

We can deal with other homogeneous spaces related to $\O$. The simply-connected cover (or the universal covering, as in [22], p. 107) $\hat{O}$ of $\O$ can be identified with $G/H^0$, since $G$ is simply-connected.

Corollary 3.17 Let $\O$ be a spherical conjugacy class, and let $S$ be a supplement of $(T^w)^0$ in $T \cap C(x)^0$. Then $\lambda(\hat{O}) = \{ \lambda \in P^+_w \mid \lambda(S) = 1 \}$ is saturated.

Proof. By [16], Corollary 2.2, $\hat{O}$ is quasi affine and, by [6], Proposition 5.1, 5.2, $L$ is adapted to $H^0$, so that $\hat{\lambda}(\hat{O}) = \hat{\lambda}(G/H^0) = \{ \lambda \in P^+_w \mid \lambda(S) = 1 \}$, since $(T^w)^0 \leq T \cap H^0$. Let $\lambda \in \hat{\lambda}(\hat{O})$; then $F_\lambda : BH^0/H^0 \to \C$, $b^{-1}H^0 \to \lambda(b)$ is a regular function on $BH^0/H^0$, and therefore a $B$-eigenvector of weight $\lambda$ in $\C(G/H^0)$. By Corollary 3.13, $2\lambda \in \lambda(G/H) \leq \lambda(G/H^0)$, and we conclude by Lemma 3.3 and Proposition 3.4. \qed
Corollary 3.18 Let \( K \) be a closed subgroup of \( G \) with \( H^o \leq K \leq N(H^o) \). Then \( \lambda(G/K) = \bar{\lambda}(G/K) \) (and \( \lambda(G/K) \) is saturated).

Proof. Since \( L \) is adapted to \( H \), we get \( N(H) = N(H^o) = H(C \cap N(H)) \) by [6], Corollaire 5.2, \( P \) is the parabolic subgroup corresponding to \( N(H) \) and \( L \) is adapted to \( N(H) \) (by the proof of [6], Proposition 5.2 a). Clearly the same holds for \( K \), since \( BH = BK \).

By Corollary 3.17, \( \lambda \in \lambda(G/H^o) \iff \lambda(T \cap H^o) = 1 \). We prove that \( \lambda \in \lambda(G/K) \iff \lambda(T \cap K) = 1 \). In one direction \( \lambda \in \lambda(G/K) \Rightarrow \lambda(T \cap K) = 1 \), since \( \lambda(G/K) \leq \bar{\lambda}(G/K) \). So assume \( \lambda(T \cap K) = 1 \). Then \( \lambda(T \cap H^o) = 1 \), so that \( \lambda \in \lambda(G/H^o) \), and in particular \( w_0 \lambda = -\lambda \). Let \( v \) be a non-zero vector in \( V(\lambda)^{H^o} \), and let \( v = v_{-\lambda} + v' \), with \( v_{-\lambda} \in V(\lambda)_{-\lambda} \), \( v' \in \sum_{m \geq -\lambda} V(\lambda)_m \); then \( v_{-\lambda} \neq 0 \), since \( BH^o \) is dense in \( G \).

Since \( V(\lambda)^{H^o} \) is 1-dimensional, there is a character \( \gamma \) of \( K \), trivial on \( H^o \), such that \( k.v = \gamma(k)v \) for \( k \in K \). Since \( K = H^o(T \cap K) \), \( v \) is \( K \)-invariant if and only if \( \gamma(T \cap K) = 1 \). But \( v_{-\lambda} \neq 0 \) implies \( \gamma(k) = -\lambda(k) \) for every \( k \in T \cap K \) so that \( v \) is \( K \)-invariant if and only if \( \lambda(T \cap K) = 1 \), and we are done.

Remark 3.19 In general \( K \) is not quasi affine: for instance the centralizer \( H \) of \( x_{-\beta}(1) \), \( \beta \) the highest root, contains \( U^- \), and \( T \leq N(H) \). Then \( N(H) \) is epimorphic, i.e. the minimal quasi affine subgroup of \( G \) containing \( N(H) \) is \( G \) ([16], p. 19, ex. 2). To our knowledge, it was known that \( \lambda(G/K) \) is saturated for symmetric varieties \( G/K \), due to the work of Vust, [45].

Proposition 3.20 We have

\[ H/H^o \cong T_y/T \cap H^o = T_x/T \cap C(x)^o \]

Proof. We have \( H = H^o(H \cap T) = H^oT_y \). Hence we get an epimorphism \( \pi : T_y \to H/H^o \), inducing an isomorphism \( \bar{\pi} : T_y/T \cap H^o \to H/H^o \), and we conclude by Proposition 3.4.

Corollary 3.21 If \( T^w \) is connected, then \( H \) is connected.

Proof. This follows from \( (T^w)^o \leq T \cap C(x)^o \leq T_x \leq T^w = (T^w)^o \) and Proposition 3.20.

Due to the fact that \( T \) is 2-divisible, we have the decomposition \( T = (T^w)^o(S^w)^o \) where \( S^w = \{ t \in T \mid t^w = t^{-1} \} \). Let \( t \in T^w \), \( t = sz \), with \( s \in (T^w)^o \), \( z \in (S^w)^o \). Then \( z = ts^{-1} \in T^w \cap (S^w)^o \leq T^w \cap S^w \leq T_2 \), the elementary abelian 2-subgroup of \( T \) of rank \( n \).

We note that \( (T^w)^o \cap (S^w)^o \) is finite, even though in general not trivial. Therefore \( z \in T_2 \), and \( T^w \leq (T^w)^o T_2 \). In particular we have

\[ T^w = (T^w)^o(T^w \cap (S^w)^o) = (T^w)^o(T^w \cap T_2) \]

and

\[ T_x = (T^w)^o(C(x) \cap (S^w)^o) = (T^w)^o(C(x) \cap T_2) \]
Moreover every subgroup $M$ of $T_2$ is a complemented group (i.e. for every subgroup $X$ of $M$ there exists a subgroup $Y$ such that $XY = M$ and $X \cap Y = 1$), hence we may find a subgroup $R$ of $T_2$ such that $T^w = (T^w) \circ R$. Then $T_x = (T^w) \circ (R \cap C(x))$ and $T \cap C(x) = (T^w) \circ (R \cap C(x))$. We put $S = R \cap C(x), S^\circ = R \cap C(x) \circ$. We have therefore proved

**Theorem 3.22** Let $O$ be a spherical conjugacy class, $w = w(O)$. Then

$$\lambda(O) = \{\lambda \in P_w^+ | \lambda(S) = 1\}, \quad \lambda(O) = \{\lambda \in P_w^+ | \lambda(S) = 1\}$$

\[\square\]

From Proposition 3.20 it follows that $H$ always splits over $H^\circ$: if $Y$ is a complement of $R \cap C(x)$ in $R \cap C(x)$, then $Y$ is a complement of $H^\circ$ in $H$.

## 4 Description of $\lambda(O)$ and $\lambda(\hat{O})$

From our discussion it is clear that to determine $\lambda(O)$ the most favourable case is when $T^w$ is connected, so that $T_x = T^w = (T^w) \circ$. In this case then $\lambda(O) = \lambda(\hat{O}) = P_w^+ = \{\sum_{i \in \Pi \setminus J} n_i \omega_i \mid n_{\theta(i)} = n_i\}$. We note that of course we have $Z(G) \leq T_x$, so that it is also straightforward to determine $\lambda(O)$ even when $T^w = (T^w) \circ Z(G)$, so that $T_x = T^w \circ$. In general it is quite cumbersome to determine $T_x$. Our strategy will be to determine $T^w$ as $T^w = (T^w) \circ \times R$, and then determine $R \cap C(x)$. To deal with unipotent classes, we shall usually start from the maximal one, (corresponding to $w_0$), and then deal with the remaining classes by an inductive procedure. In some cases we shall use an explicit form of an element $x$ (in $O \cap wB$), while in some other cases we shall determine $T \cap C(x)$ by analyzing the form of eventual involutions in $T_x \setminus Z(G)(T^w) \circ$. Note that when $T^w$ is connected (or $T^w = (T^w) \circ Z(G)$), it is not necessary to have an explicit description of $x \in O \cap wB$ (however in certain cases it will be necessary to have such a description in section 6).

We use the fact that if $G_1 \subset G_2$ are reductive algebraic groups and $u$ is a unipotent element in $G_1$ such that the conjugacy class of $u$ in $G_2$ is spherical, then the conjugacy class of $u$ in $G_1$ is spherical ([33], Corollary 2.3, Theorem 3.1).

The character group $X(T^w)$ is isomorphic to $P/ (1 - w)P$, since $P = X(T)$. Therefore $T^w$ is connected if and only if $P/ (1 - w)P$ is torsion free. We are reduced to calculate elementary divisors of the endomorphism $1 - w$ of $P$. We shall use the following results.

**Lemma 4.1** Assume the positive roots $\beta_i, \ldots, \beta_\ell$ are long and pairwise orthogonal. Then, for $\xi_1, \ldots, \xi_\ell \in \mathbb{C}^*$ and $g = x_{\beta_1}(-\xi_1^{-1}) \cdots x_{\beta_\ell}(-\xi_\ell^{-1})$ we have

$$gx_{-\beta_1}(\xi_1) \cdots x_{-\beta_\ell}(\xi_\ell) g^{-1} = n_{\beta_1} \cdots n_{\beta_\ell} h x_{\beta_1}(2\xi_1^{-1}) \cdots x_{\beta_\ell}(2\xi_\ell^{-1})$$

for a certain $h \in T$. 
**Decomposition of \( C[O] \)**

Proof. By (2.2) we have \( x_\alpha(-\xi^{-1})x_{-\alpha}(\xi)x_\alpha(\xi^{-1}) = n_\alpha h_\alpha(-\xi) x_\alpha(2\xi^{-1}) \). Hence we get the result with \( h = h_{\beta_1}(-\xi_1) \cdots h_{\beta_\ell}(-\xi_\ell) \).

\[ \square \]

**Proposition 4.2** Let \( \alpha \in \Phi \). Then \( T^{s_\alpha} \) is connected except in the following cases:

(i) \( G \) is of type \( A_1 \);

(ii) \( G \) is of type \( C_n \) and \( \alpha \) is long;

(iii) \( G \) is of type \( B_2 \) and \( \alpha \) is long.

In these cases we have \( T^{s_\alpha} = (T^{s_\alpha})^0 \times Z(G) \).

Proof. It is enough to determine in which cases the non-zero elementary divisor of \( 1 - s_i \) is not 1. Since \( (1 - s_i)\omega_j = \delta_{ij}\alpha_i \) and \( \alpha_i = \sum_k a_{ik}\omega_k \), this happens only for \( G \) of type \( A_1 \) and \( i = 1 \), \( C_n \) and \( i = n \), or \( B_2 \) and \( i = 1 \) ([18], pag. 59). In these cases the non-zero elementary divisor is 2, and \( T^{s_\alpha} = (T^{s_\alpha})^0 \times Z(G) \).

\[ \square \]

**Lemma 4.3** Let \( M \) be a connected algebraic group, \( S \) a torus of \( M \), \( g \) a semisimple element in \( C_M(S) \). Then \( \langle S, g \rangle \) is contained in a torus of \( M \).

Proof. See [18], Corollary 22.3 B.

\[ \square \]

**Lemma 4.4** Assume \( K \) is a connected spherical subgroup of \( G \) with no non-trivial characters. Then the monoid \( \lambda(G/K) \) is free.

Proof. We recall that we are assuming \( G \) simply-connected, so that by [16], Theorem 20.2, \( U \mathbb{C}[G/K] \) is a polynomial algebra. But \( U \mathbb{C}[G/K] \) is the monoid algebra of \( \lambda(G/K) \) and the monoid algebra is factorial if and only if \( \lambda(G/K) \) is free (see the proof of [32], Proposition 2).

\[ \square \]

**Lemma 4.5** Let \( V \) be a \( G \)-module, \( g \in G \), such that the image \( Q \) of the endomorphism \( p(g) \) of \( V \) is 1 dimensional for a certain polynomial \( p \). Assume \( M \leq C(g) \) has no non-trivial characters. Then \( M \) acts trivially on \( Q \).

Proof. This is clear.

Let \( S = \{ i, \vartheta(i) \} \) be a \( \vartheta \)-orbit in \( \Pi \setminus J \) consisting of 2 elements. We put \( H_S = \{ h_\alpha(z) h_{\alpha_{\vartheta(i)}}(z^{-1}) \mid z \in \mathbb{C}^* \} \). Let \( S_1 \) be the set of \( \vartheta \)-orbits in \( \Pi \setminus J \) consisting of 2 elements. Then, by Remark 3.9, \( \Delta_J \cup \{ \alpha_i - \alpha_{\vartheta(i)} \} \) is a basis of \( \ker(1 - w) \) and

\[
(4.11) \quad (T^w)^0 = \prod_{j \in J} H_{\alpha_j} \times \prod_{S \in S_1} H_S
\]

Let \( S = \{ i, \vartheta(i) \} \) be a \( \vartheta \)-orbit in \( \Pi \setminus J \) consisting of 2 elements. We put \( H_S = \{ h_\alpha(z) h_{\alpha_{\vartheta(i)}}(z^{-1}) \mid z \in \mathbb{C}^* \} \). Let \( S_1 \) be the set of \( \vartheta \)-orbits in \( \Pi \setminus J \) consisting of 2 elements. Then, by Remark 3.9, \( \Delta_J \cup \{ \alpha_i - \alpha_{\vartheta(i)} \} \) is a basis of \( \ker(1 - w) \) and

\[
(4.11) \quad (T^w)^0 = \prod_{j \in J} H_{\alpha_j} \times \prod_{S \in S_1} H_S
\]
We put \( \Psi_J = \{ \beta \in \Phi \mid w(\beta) = -\beta \} \). Then \( \Psi_J \) is a root system in \( \text{Im}(1 - w) \) ([40], Proposition 2), and \( w|_{\text{Im}(1-w)} = -1 \). If \( K = C((T^w)^\circ)' \), then \( K \) is semisimple with root system \( \Psi_J \) and maximal torus \( T(K) := T \cap K = (S^w)^\circ \).

For each spherical (non-central) conjugacy class \( \mathcal{O} \) we give the corresponding \( J \) and \( w \) as a product of commuting reflections using the tables in [9]. We give tables with corresponding \( \lambda(\mathcal{O}) \) and \( \lambda(\hat{\mathcal{O}}) \) (for semisimple classes we also give the type of the centralizer of elements in \( \mathcal{O} \)). In the cases when \( \lambda(\hat{\mathcal{O}}) = \lambda(\mathcal{O}) \), we leave a blank entry. For length reasons we shall give proofs only for some classes. In [9] for the classical groups we gave representative of semisimple conjugacy classes in \( \text{SL}(n) \), \( \text{Sp}(n) \) and \( \text{SO}(n) \). Here we shall give an expression in terms of \( \exp \). If \( g \) is in \( Z(G) \), then \( \mathcal{O}_g = \{g\} \), \( w = 1 \) and \( C[\mathcal{O}_g] = C \).

### 4.1 Type \( A_n \), \( n \geq 1 \).

Let \( m = \lfloor \frac{n+1}{2} \rfloor \), \( \beta_i = e_i - e_{n+2-i} \), for \( i = 1, \ldots, m \). For \( \ell = 1, \ldots, m-1 \) we put \( J_{\ell} = \{\ell+1, \ldots, n-\ell\} \), \( J_m = \emptyset \).

#### 4.1.1 Unipotent classes in \( A_n \).

If we denote by \( X_i \) the unipotent class \( (2^i, 1^{n+1-2i}) \), then

\[
X_\ell \longleftrightarrow J_\ell \longleftrightarrow s_{\beta_1} \cdots s_{\beta_\ell}
\]

for \( \ell = 1, \ldots, m \) (here \( w_0 = s_{\beta_1} \cdots s_{\beta_m} \)).

In this case \( T^w \) is almost always connected. There is only one case when it is not connected, namely when \( n \) is odd, \( n+1 = 2m \), and \( w = w_0 \). However in this case we have \( T^{w_0} = (T^w)^\circ Z(G) = (T^w)^\circ \times \{h_{\alpha_m}(-1)\} \).

We get

| \( \mathcal{O} \) | \( \lambda(\mathcal{O}) \) | \( \lambda(\hat{\mathcal{O}}) \) |
|-----------------|-----------------|-----------------|
| \( \ell = 1, \ldots, m-1 \) | \( \sum_{k=1}^\ell n_k (\omega_k + \omega_{n-k+1}) \) | \( \sum_{k=1}^{m-1} n_k (\omega_k + \omega_{n-k+1}) \) |
| \( X_m \) \( n = 2m \) | \( \sum_{k=1}^n n_k (\omega_k + \omega_{n-k+1}) \) | \( \sum_{k=1}^{m-1} n_k (\omega_k + \omega_{n-k+1}) + 2n_m \omega_m \) |
| \( X_m \) \( n+1 = 2m \) | \( \sum_{k=1}^{m-1} n_k (\omega_k + \omega_{n-k+1}) + 2n_m \omega_m \) | \( \sum_{k=1}^{m-1} n_k (\omega_k + \omega_{n-k+1}) + n_m \omega_m \) |

Table 1: \( \lambda(\mathcal{O}), \lambda(\hat{\mathcal{O}}) \) for unipotent classes in \( A_n \).

In particular \( \hat{X}_1 \) is a model homogeneous space for \( \text{SL}(2) \), and in fact the principal one, by [28], 3.3 (1).
4.1.2 Semisimple classes in $A_n$.

Following the notation in [9], Tables 1, 5 we get

$$T_1 A_{\ell-1} A_{n-\ell} \leftrightarrow J_{\ell} \leftrightarrow s_{\beta_1} \cdots s_{\beta_{\ell}}$$

for $\ell = 1, \ldots, m$.

We get

$$\lambda(O)$$

\begin{table}[h]
\begin{tabular}{|c|c|c|}
\hline
$O$ & $H$ & $\lambda(O)$ \\
\hline
$\exp(\zeta \omega_\ell)$ & $T_1 A_{\ell-1} A_{n-\ell}$ & $\sum_{k=1}^{\ell} n_k(\omega_k + \omega_{n-k+1})$ \\
$\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ & & \\
$\ell = 1, \ldots, m - 1$ & & \\
\hline
$\exp(\zeta \omega_m)$ & $T_1 A_{m-1} A_m$ & $\sum_{k=1}^{m} n_k(\omega_k + \omega_{n-k+1})$ \\
$\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ & & \\
$n = 2m$ & & \\
\hline
$\exp(\zeta \omega_m)$ & $T_1 A_{m-1} A_{m-1}$ & $\sum_{k=1}^{m-1} n_k(\omega_k + \omega_{n-k+1}) + 2n_m \omega_m$ \\
$\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ & & \\
$n + 1 = 2m$ & & \\
\hline
\end{tabular}
\caption{$\lambda(O)$ for semisimple classes in $A_n$.}
\end{table}

4.2 Type $C_n$, $n \geq 2$.

We have $\omega_\ell = e_1 + \cdots + e_\ell$ for $\ell = 1, \ldots, n$ and $Z(G) = \langle z \rangle$, where $z = \prod_{i=1}^{n+1} h_{\alpha_i}(-1)$.

4.2.1 Unipotent classes in $C_n$.

For $i = 1, \ldots, n$ we denote by $X_i$ the unipotent class $(2^i, 1^{2n-2i})$ and we put $\beta_i = 2e_i$, $J_i = \{i + 1, \ldots, n\}$ ($J_n = \emptyset$). Then

$$X_\ell \leftrightarrow J_\ell \leftrightarrow s_{\beta_1} \cdots s_{\beta_{\ell}}$$

for $\ell = 1, \ldots, n$ (here $w_0 = s_{\beta_1} \cdots s_{\beta_n}$).

**Lemma 4.6** Let $w = s_{\beta_1} \cdots s_{\beta_{\ell}}$ for $\ell = 1, \ldots, n$. Then

$$T^w = (T^w)^{\circ} \times R , \quad R = \langle h_{\alpha_1}(-1) \rangle \times \cdots \times \langle h_{\alpha_\ell}(-1) \rangle$$

**Proof.** For $\ell = 1, \ldots, n$ we have $(1 - w)P = Z(2\omega_1, \ldots, 2\omega_\ell)$. □

**Proposition 4.7** For $\ell = 1, \ldots, n$ we have

$$\lambda(X_\ell) = \{2n_1 \omega_1 + \cdots + 2n_\ell \omega_\ell \mid n_k \in \mathbb{N}\}$$
Proof. In [9] we exhibit the element \( x_{-\beta_1}(1) \cdots x_{-\beta_\ell}(1) \in \mathcal{O} \cap BwB \cap B^- \). By Lemma 4.1, we can choose
\[
x = n_{\beta_1} \cdots n_{\beta_\ell} h x_{\beta_1}(2) \cdots x_{\beta_\ell}(2) \in \mathcal{O} \cap wB
\]
for a certain \( h \in T \). Let now \( t \in R \). Then \( t \in C(x) \iff \beta_i(t) = 1 \) for \( i = 1, \ldots, \ell \). But \( \mathbb{Z}(\beta_1, \ldots, \beta_\ell) = \mathbb{Z}(2\omega_1, \ldots, 2\omega_\ell) \), so that \( R \leq T_x \), and \( T_x = T^w \). \( \square \)

Proposition 4.8 For \( \ell = 1, \ldots, n \) we have
\[
\lambda(\hat{X}_\ell) = \{ 2n_1\omega_1 + \cdots + 2n_{\ell-1}\omega_{\ell-1} + n_\ell \omega_\ell \mid n_k \in \mathbb{N} \}
\]
Proof. We have \( R \cap C(x)^0 = \{ h_{\alpha_1}(-1), \ldots, h_{\alpha_{\ell-1}}(-1) \} \). In fact, for \( i = 1, \ldots, \ell - 1 \)
\[
eq_\alpha - e_{-\alpha_i} \in C_g(\langle x_{\beta_1}(\xi) \cdots x_{\beta_\ell}(\xi) \rangle)
\]
for every \( \xi \in \mathbb{C} \), so that \( h_{\alpha_1}(-1) = \exp(\pi(e_{\alpha_i} - e_{-\alpha_i})) \in C(x)^0 \). On the other hand the reductive part of \( C(x) \) is of type \( Sp(2n - 2\ell) \times O(\ell) \), so that \( C(x)/C(x)^0 \) has order 2, and we are done. \( \square \)

Hence

| \( \mathcal{O} \) | \( \lambda(\mathcal{O}) \) | \( \lambda(\hat{\mathcal{O}}) \) |
|---|---|---|
| \( \hat{X}_\ell \) |  |  |
| \( \ell = 1, \ldots, n \) |  |  |
| \( \sum_{i=1}^\ell 2n_i\omega_i \) |  |  |
| \( \sum_{i=1}^{\ell-1} 2n_i\omega_i + n_\ell \omega_\ell \) |  |  |

Table 3: \( \lambda(\mathcal{O}), \lambda(\hat{\mathcal{O}}) \) for unipotent classes in \( C_n \).

4.2.2 Semisimple classes in \( C_n \).
Let \( p = \lfloor \frac{n}{2} \rfloor \). We put \( \gamma_\ell = e_{2\ell-1} + e_{2\ell} \), \( K_\ell = \{ 1, 3, \ldots, 2\ell - 1, 2\ell + 1, 2\ell + 2, \ldots, n \} \) for \( \ell = 1, \ldots, p \). Then, following the notation in [9], Tables 1, 5 we have
\[
C_{\ell}C_{n-\ell}, \quad \ell = 1, \ldots, p \quad \leftrightarrow \quad K_\ell \quad \leftrightarrow \quad s_{\gamma_1} \cdots s_{\gamma_\ell}
\]
\[
T_1C_{n-1} \quad \leftrightarrow \quad J_2 \quad \leftrightarrow \quad s_{\beta_1} s_{\beta_2}
\]
\[
T_1A_{n-1} \quad \leftrightarrow \quad \emptyset \quad \leftrightarrow \quad w_0
\]

We get

| \( \mathcal{O} \) | \( H \) | \( \lambda(\mathcal{O}) \) |
|---|---|---|
| \( \exp(\zeta \omega_n) \) \quad \zeta \in \mathbb{C} \setminus 2\pi i\mathbb{Z} \) | \( T_1A_{n-1} \) |  |
| \( \sum_{k=1}^n 2n_k\omega_k \) |  |  |
| \( \exp(\zeta \omega_1) \) \quad \zeta \in \mathbb{C} \setminus \pi i\mathbb{Z} \) | \( T_1C_{n-1} \) |  |
| \( 2n_1\omega_1 + n_2\omega_2 \) |  |  |
| \( \exp(\pi i \omega_\ell) \) \quad \ell = 1, \ldots, \lfloor \frac{n}{2} \rfloor \) | \( C_{\ell}C_{n-\ell} \) |  |
| \( \sum_{i=1}^\ell n_{2i} \omega_{2i} \) |  |  |

Table 4: \( \lambda(\mathcal{O}) \) for semisimple classes in \( C_n \).
4.2.3 Mixed classes in $C_n$.

We put $p = \left[\frac{n}{2}\right]$. From [9], Table 4, we get

$$\sigma_p x_{\lambda}(1)$$
$$\sigma_k x_{\alpha_n}(1), \quad k = 1, \ldots, p - 1 \quad \leftrightarrow \quad \varnothing \quad \leftrightarrow \quad \omega_0$$
$$\sigma_k x_{\beta_1}(1), \quad k = 1, \ldots, p \quad \leftrightarrow \quad J_{2k} \quad \leftrightarrow \quad s_{\beta_1} \cdots s_{\beta_{2k}}$$

Note that when $n$ is even, then $\sigma_p x_{\beta_1}(1) \sim z \sigma_p x_{\alpha_n}(1)$. We obtain

| $\sigma_p x_{\alpha_n}(1)$ | $\sigma_{\ell} x_{\alpha_n}(1)$ | $\sigma_{\ell} x_{\beta_1}(1)$ |
|----------------------------|---------------------------------|-----------------------------|
| $n_i \omega_i$ $\sum_{i=1}^{n}$ | $n_i \omega_i$ $\sum_{i=1}^{\ell+1}$ | $n_i \omega_i$ $\sum_{i=1}^{2\ell+1}$ |

Table 5: $\lambda(\mathcal{O}), \lambda(\hat{\mathcal{O}})$ for mixed classes in $C_n$.

In particular $\mathcal{O}_{\sigma_p x_{\alpha_n}(1)}$ is a model homogeneous space, and in fact the principal one, by [28], 3.3 (3).

To deal with types $D_n$ and $B_n$, we denote by $X_i$ the unipotent class which in $SO(s)$ has canonical form $(2^{2i}, 1^{s-4i})$, $i = 1, \ldots, \left[\frac{s}{4}\right]$ (for $s = 4m$, $i = m$ there are 2 classes of this form: $X_m$ and $X'_m$, the very even classes) and by $Z_i$ the unipotent class $(3, 2^{2(i-1)}, 1^{s-4i+1})$, $i = 1, \ldots, 1 + \left[\frac{s-3}{4}\right]$.

4.3 Type $D_n, n \geq 4$.

Let $m = \left[\frac{n}{2}\right]$. We have $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \ldots, n - 2$, $\omega_{n-1} = \frac{1}{2}(e_1 + \cdots + e_{n-1}) - \frac{1}{2}e_n$, $\omega_n = \frac{1}{2}(e_1 + \cdots + e_n)$. We put $\beta_i = e_{2i-1} + e_{2i}, \delta_i = e_{2i-1} - e_{2i}$ for $i = 1, \ldots, m$. For $\ell = 1, \ldots, m - 1$ we put $J_\ell = \{2\ell + 1, \ldots, n\}, J_m = \varnothing, K_\ell = J_\ell \cup \{1, 3, \ldots, 2\ell - 1\}$ for $\ell = 1, \ldots, m$.

4.3.1 Unipotent classes in $D_n, n$ even, $n = 2m$.

The center of $G$ is $(\prod_{i=1}^{m} h_{\alpha_{2i-1}}(-1), h_{\alpha_{n-1}}(-1)h_{\alpha_n}(-1))$. From [9] we get

$$Z_{\ell}, \quad \ell = 1, \ldots, m \quad \leftrightarrow \quad J_{\ell} \quad \leftrightarrow \quad s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell}$$
$$X_{\ell}, \quad \ell = 1, \ldots, m \quad \leftrightarrow \quad K_{\ell} \quad \leftrightarrow \quad s_{\beta_1} \cdots s_{\beta_\ell}$$
$$X_m' \quad \leftrightarrow \quad \{1, 3, \ldots, n - 3, n\} \quad \leftrightarrow \quad s_{\beta_1} \cdots s_{\beta_{m-1}} s_{\alpha_{n-1}}$$
We just point out that

\[ T_x = \begin{cases} Z(G) & \text{for } x \in Z_m \cap wB \\ (T^w)^\circ \times \langle \prod_{i=1}^{\ell} h_{n_{2i-1}}(-1) \rangle & \text{for } x \in Z_\ell \cap wB, \ell = 1, \ldots m - 1 \end{cases} \]

We get

\[
\begin{align*}
\lambda(O) & \lambda(\hat{O}) \\
X_\ell & \sum_{i=1}^{\ell} n_{2i} \omega_{2i} & \sum_{i=1}^{m-1} n_{2i} \omega_{2i} + n_n \omega_n \\
X_m & \sum_{i=1}^{m-1} n_{2i} \omega_{2i} + 2n_n \omega_n & \sum_{i=1}^{m-1} n_{2i} \omega_{2i} + n_n \omega_n \\
X'_m & \sum_{i=1}^{m-1} n_{2i} \omega_{2i} + 2n_{n-1} \omega_{n-1} & \sum_{i=1}^{m-1} n_{2i} \omega_{2i} + n_{n-1} \omega_{n-1} \\
Z_\ell & \sum_{i=1}^{2\ell} n_{i} \omega_i, \sum_{i=1}^{\ell} n_{2i-1} \text{ even} \quad \sum_{i=1}^{2\ell} n_{i} \omega_i \\
Z_m & \sum_{i=1}^{n} n_{i} \omega_i, \sum_{i=1}^{m} n_{2i-1} \text{ even}, n_{n-1} + n_n \text{ even} \quad \sum_{i=1}^{n} n_{i} \omega_i
\end{align*}
\]

Table 6: \( \lambda(O), \lambda(\hat{O}) \) for unipotent classes in \( D_n, n = 2m \).

In particular \( \hat{Z}_m \) is a model homogeneous space, and in fact the principal one, by [28], 3.3 (4).

### 4.3.2 Semisimple classes in \( D_n, n \text{ even } n = 2m \)

Following the notation in [9], Tables 1, 5 we have

\[
\begin{align*}
D_{\ell}D_{n-\ell} & \leftrightarrow J_\ell, \quad \ell = 1, \ldots, m \quad \leftrightarrow s_\beta_1 s_\delta_1 \cdots s_\beta_\ell s_\delta_\ell \\
T_1A_{n-1} & \leftrightarrow K_m, \quad \leftrightarrow s_\beta_1 \cdots s_\beta_m \\
(T_1A_{n-1})' & \leftrightarrow \{1, 3, \ldots, n - 3, n\} \quad \leftrightarrow s_\beta_1 \cdots s_{\beta_{m-1}} s_\alpha_{n-1}
\end{align*}
\]

There are two families of classes of semisimple elements with centralizer of type \( T_1A_{n-1} \): to distinguish them we wrote \( T_1A_{n-1} \) and \( (T_1A_{n-1})' \). We get
Decomposition of $\mathbb{C}[O]$

| $\mathcal{O}$ | $H$ | $\lambda(\mathcal{O})$ |
|---------------|-----|---------------------|
| $\exp(\zeta \omega_1)$ | $T_1D_{n-1}$ | $2n_1 \omega_1 + n_2 \omega_2$ |
| $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ | | |
| $\exp(\pi i \omega_\ell)$ | $D_\ell D_{n-\ell}$ | $\sum_{i=1}^{2\ell-1} 2n_i \omega_i + n_{2\ell} \omega_{2\ell}$ |
| $\ell = 2, \ldots, m - 1$ | | |
| $\exp(\pi i \omega_m)$ | $D_mD_m$ | $\sum_{i=1}^{n} 2n_i \omega_i$ |
| | | |
| $\exp(\zeta \omega_n)$ | $T_1A_{n-1}$ | $\sum_{i=1}^{m-1} n_{2i} \omega_{2i} + 2n_n \omega_n$ |
| $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ | | |
| $\exp(\zeta \omega_{n-1})$ | $(T_1A_{n-1})'$ | $\sum_{i=1}^{m-1} n_{2i} \omega_{2i} + 2n_{n-1} \omega_{n-1}$ |
| $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ | | |

Table 7: $\lambda(\mathcal{O})$ for semisimple classes in $D_n$, $n = 2m$.

4.3.3 Unipotent classes in $D_n$, $n$ odd, $n = 2m + 1$.

The center of $G$ is $\langle (\prod_{j=1}^{m} h_{\alpha_{2j-1}} (-1)) h_{\alpha_{n-1}} (i) h_{\alpha_n} (-i) \rangle$. From [9] we have

\[
Z_\ell, \quad \ell = 1, \ldots, m \quad \leftrightarrow \quad J_\ell \quad \leftrightarrow \quad s_{\beta_1}s_{\beta_1} \cdots s_{\beta_\ell}s_{\beta_\ell}
\]

\[
X_\ell, \quad \ell = 1, \ldots, m \quad \leftrightarrow \quad K_\ell \quad \leftrightarrow \quad s_{\beta_1} \cdots s_{\beta_\ell}
\]

We get

| $\mathcal{O}$ | $\lambda(\mathcal{O})$ | $\lambda(\mathcal{O})$ |
|---------------|----------------|----------------|
| $X_\ell$ | $\ell = 1, \ldots, m - 1$ | $\sum_{i=1}^{\ell} n_{2i} \omega_{2i}$ | |
| $X_m$ | | $\sum_{i=1}^{m-1} n_{2i} \omega_{2i} + n_{n-1} (\omega_{n-1} + \omega_n)$ | |
| $Z_\ell$ | $\ell = 1, \ldots, m - 1$ | $\sum_{i=1}^{2\ell} n_i \omega_i$, $\sum_{i=1}^{\ell} n_{2i-1}$ even | $\sum_{i=1}^{2\ell} n_i \omega_i$ |
| $Z_m$ | | $\sum_{i=1}^{n-2} n_i \omega_i + n_{n-1} (\omega_{n-1} + \omega_n)$, $\sum_{i=1}^{m} n_{2i-1}$ even | $\sum_{i=1}^{n-2} n_i \omega_i + n_{n-1} (\omega_{n-1} + \omega_n)$ |

Table 8: $\lambda(\mathcal{O})$, $\lambda(\hat{O})$ for unipotent classes in $D_n$, $n = 2m + 1$. 
4.3.4 Semisimple classes in $D_n$, $n$ odd, $n = 2m + 1$

$$D_\ell D_{n-\ell}, \quad \ell = 1, \ldots, m \quad \longleftrightarrow \quad J_\ell \quad \longleftrightarrow \quad s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell}$$

We obtain

| $\mathcal{O}$ | $H$ | $\lambda(\mathcal{O})$ |
|---------------|-----|------------------|
| $\exp(\zeta \omega_1)$, $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ | $T_1 D_{n-1}$ | $2n_1 \omega_1 + n_2 \omega_2$ |
| $\exp(\pi i \omega_\ell)$, $\ell = 2, \ldots, m - 1$ | $D_\ell D_{n-\ell}$ | $\sum_{i=1}^{2\ell-1} 2n_i \omega_i + n_2 \omega_{2\ell}$ |
| $\exp(\pi i \omega_m)$ | $D_m D_{m+1}$ | $\sum_{i=1}^{n-2} 2n_i \omega_i + n_{n-1} (\omega_{n-1} + \omega_n)$ |
| $\exp(\zeta \omega_n)$, $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ | $T_1 A_{n-1}$ | $\sum_{i=1}^{m-1} n_{2i} \omega_{2i} + n_{n-1} (\omega_{n-1} + \omega_n)$ |

Table 9: $\lambda(\mathcal{O})$ for semisimple classes in $D_n$, $n = 2m + 1$.

4.4 Type $B_n$, $n \geq 2$.

We put $m = \left\lceil \frac{n}{2} \right\rceil$. The center of $G$ is $\langle h_{\alpha_n} (-1) \rangle$. We have $\omega_i = e_1 + \cdots + e_i$ for $i = 1, \ldots, n - 1$, $\omega_n = \frac{1}{2} (e_1 + \cdots + e_n)$. We put $\gamma_\ell = e_\ell$, $M_\ell = \{ \ell + 1, \ldots, n \}$ for $\ell = 1, \ldots, n$ and $\beta_i = e_{2i-1} + e_{2i}$, $\delta_i = e_{2i-1} - e_{2i}$, $J_\ell = \{ 2\ell + 1, \ldots, n \}$, $K_\ell = J_\ell \cup \{ 1, 3, \ldots, 2\ell - 1 \}$ for $i = 1, \ldots, m$.

4.4.1 Unipotent classes in $B_n$, $n$ even, $n = 2m$.

$$Z_\ell, \quad \ell = 1, \ldots, m \quad \longleftrightarrow \quad J_\ell \quad \longleftrightarrow \quad s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell}$$

We obtain

| $\mathcal{O}$ | $\lambda(\mathcal{O})$ | $\lambda(\hat{\mathcal{O}})$ |
|---------------|------------------|------------------|
| $X_\ell$, $\ell = 1, \ldots, m - 1$ | $\sum_{i=1}^{\ell} n_{2i} \omega_{2i}$ | $\sum_{i=1}^{m} n_{2i} \omega_{2i}$ |
| $X_m$ | $\sum_{i=1}^{m-1} n_{2i} \omega_{2i} + 2n_{n} \omega_{n}$ | $\sum_{i=1}^{m} n_{2i} \omega_{2i}$ |
| $Z_\ell$, $\ell = 1, \ldots, m - 1$ | $\sum_{i=1}^{2\ell} n_i \omega_i$, $\sum_{i=1}^{\ell} n_{2i-1}$ even | $\sum_{i=1}^{2\ell} n_i \omega_i$ |
| $Z_m$ | $\sum_{i=1}^{n} n_i \omega_i$, $\sum_{i=1}^{m} n_{2i-1}$ even, $n_n$ even | $\sum_{i=1}^{n} n_i \omega_i$, $n_n$ even |

Table 10: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in $B_n$, $n = 2m$. 
Decomposition of $\mathbb{C}[O]$

4.4.2 Semisimple classes in $B_n$, $n$ even $n = 2m$

Following the notation in [9], Tables 1, 5 we have

$$D_\ell B_{n-\ell}, \ \ell = 1, \ldots, m \longleftrightarrow J_\ell \longleftrightarrow s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell}$$

$$D_\ell B_{n-\ell}, \ \ell = m + 1, \ldots, n \longleftrightarrow M_{2(n-\ell)+1} \longleftrightarrow s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_{2(n-\ell)+1}}$$

$$T_1 A_{n-1} \longleftrightarrow \emptyset \longleftrightarrow w_0$$

We obtain

| $\mathcal{O}$ | $H$ | $\lambda(\mathcal{O})$ |
|----------------|-----|-----------------------|
| $\exp(\zeta \omega_1)$, $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}, m \geq 2$ | $T_1 B_{n-1}$ | $2n_1 \omega_1 + n_2 \omega_2$ |
| $\exp(\zeta \omega_1)$, $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}, m = 1$ | $T_1 B_1$ | $2n_1 \omega_1 + 2n_2 \omega_2$ |
| $\exp(\pi i \omega_\ell)$, $\ell = 2, \ldots, m - 1$ | $D_\ell B_{n-\ell}$ | $\sum_{i=1}^{2\ell-1} 2n_i \omega_i + n_2 \omega_{2\ell}$ |
| $\exp(\pi i \omega_m)$ | $D_m B_m$ | $\sum_{i=1}^{n} 2n_i \omega_i$ |
| $\exp(\pi i \omega_\ell)$, $\ell = m + 1, \ldots, n$ | $D_\ell B_{n-\ell}$ | $\sum_{i=1}^{2(n-\ell)} 2n_i \omega_i + n_2 (n-\ell+1) \omega_{2(n-\ell)+1}$ |
| $\exp(\zeta \omega_n)$, $\zeta \in \mathbb{C} \setminus \pi i \mathbb{Z}$ | $T_1 A_{n-1}$ | $\sum_{i=1}^{n} n_i \omega_i$, $n$ even |

Table 11: $\lambda(\mathcal{O})$ for semisimple classes in $B_n$, $n = 2m$.

4.4.3 Mixed classes in $B_n$, $n$ even, $n = 2m$

From [9], Table 4, we get

$$\sigma_n x_{\beta_1}(1) \cdots x_{\beta_m}(1)$$

$$\sigma_n x_{\beta_1}(1) \cdots x_{\beta_\ell}(1), \ \ell = 1, \ldots, m - 1 \longleftrightarrow \emptyset \longleftrightarrow w_0$$

We obtain

| $\mathcal{O}$ | $\lambda(\mathcal{O})$ | $\lambda(\hat{\mathcal{O}})$ |
|----------------|-----------------------|------------------------|
| $\sigma_n x_{\beta_1}(1) \cdots x_{\beta_\ell}(1)$, $\ell = 1, \ldots, m - 1$ | $2\ell + 1$ | $\sum_{i=1}^{n} n_i \omega_i$ |
| $\sigma_n x_{\beta_1}(1) \cdots x_{\beta_m}(1)$ | $n$ | $\sum_{i=1}^{n} n_i \omega_i$, $n$ even |

Table 12: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for mixed classes in $B_n$, $n = 2m$.

In particular $\hat{\mathcal{O}}_{\sigma_n x_{\beta_1}(1) \cdots x_{\beta_m}(1)}$ is a model homogeneous space, and in fact the principal one, by [28], 3.3 (2).
4.4.4 Unipotent classes in $B_n$, $n$ odd, $n = 2m + 1$.

$Z_{\ell} \longleftarrow J_{\ell}, \; \ell = 1, \ldots, m \longleftarrow s_{\beta_1} s_{\delta_1} \cdots s_{\beta_{\ell}} s_{\delta_{\ell}}$

$Z_{m+1} \longleftarrow \varnothing \longleftarrow w_0 = s_{\beta_1} s_{\delta_1} \cdots s_{\beta_m} s_{\delta_m} s_{\alpha_n}$

$X_{\ell} \longleftarrow K_{\ell}, \; \ell = 1, \ldots, m \longleftarrow s_{\beta_1} \cdots s_{\beta_{\ell}}$

Lemma 4.9 Let $w = s_{\beta_1} \cdots s_{\beta_{\ell}}$ for $\ell = 1, \ldots, m$. Then $T^w$ is connected.

Proof. For $\ell = 1, \ldots, m$ we have $(1 - w)P = Z(\beta_1, \ldots, \beta_{\ell}) = Z(\omega_{2i} \mid i = 1, \ldots, \ell)$.

Lemma 4.10 Let $w = s_{\beta_1} \cdots s_{\beta_{\ell}} s_{\delta_1} \cdots s_{\delta_{\ell}}$ for $\ell = 1, \ldots, m$. Then

$$T^w = (T^w)^0 \times \langle h_{\alpha_1}(-1) \rangle \times \cdots \times \langle h_{\alpha_{2\ell - 1}}(-1) \rangle$$

Proof. For $\ell = 1, \ldots, m$ we have $(1 - w)P = Z(2\omega_1, \ldots, 2\omega_{2\ell - 1}, \omega_{2\ell})$.

For $\ell = 1$ we get $T^w = \langle h_{\alpha_1}(-1) \rangle \times (T^w)^0$. In [9] we exhibit the element $x_{-\beta_1(1)}x_{-\delta_1(1)} \in O \cap BwB \cap B^-$. We may therefore choose $x = n_{\beta_1} n_{\delta_1} h x_{\beta_1(2)} x_{\delta_1(2)}$ for a certain $h \in T$. Then $h_{\alpha_1}(-1) \in C(x)$, so that $T_x = T^w$.

Next we consider $Z_{m+1}$. We claim that $T_x = Z(G)$. Suppose for a contradiction that there is an involution $\sigma \in T_x \setminus Z(G)$. Then $x \in K = C(\sigma)$, and $K$ is the almost direct product $K_1 K_2$, of type $D_k B_{n-k}$, for some $k = 1, \ldots, n$. We get an orthogonal decomposition $E = E_1 \oplus E_2$ and a decomposition $x = x_1 x_2 \in K_1 K_2$. Then $-1 = w_0 = (w_1, w_2)$, where $w_i$ is the element of the Weyl group of $K_i$ corresponding to $x_i$ (the class of $x_i$ in $K_i$ is spherical). It follows that each $w_i = -1$, and $k$ is even. Then $x_1$ is in the class $Z_{k/2}$ of $K_1$ and $x_2$ in the class $Z_{m+1-k/2}$ of $K_2$. However, the product $x_1 x_2$ is not in the class $Z_{m+1}$ of $G$ (since in $x_1 x_2$ there are two rows with 3 boxes), a contradiction. Hence $T_x = Z(G)$.

We now deal with $Z_{\ell}$, $\ell = 2, \ldots, m$. Here $\Psi_{\ell}$ has basis $\{\alpha_1, \ldots, \alpha_{2\ell-1}, \gamma_{2\ell}\}$, and $C((T^w)^0)'$ is of type $B_{2\ell}$. From the construction in [9], proof of Theorem 2.11, we can find $x$ in the $D_{2\ell}$-subgroup $K$ of $C((T^w)^0)'$ generated by the long roots, that is the $D_{2\ell}$-subgroup with basis $\{\alpha_1, \ldots, \alpha_{2\ell-1}, \beta_{\ell}\}$. We have

$$Z(K) = Z(G) \times \langle \sigma \rangle, \; \sigma = \prod_{i=1}^{\ell} h_{\alpha_{2i-1}}(-1)$$

By Lemma 4.10, $T_x = (T^w)^0 \times (T_x \cap R)$, where $R = \langle h_{\alpha_1}(-1) \rangle \times \cdots \times \langle h_{\alpha_{2\ell-1}}(-1) \rangle \leq K$.

Since $x$ lies in the maximal spherical unipotent class of $D_{2\ell}$, from the result obtained for this class, we have $T_x \cap R = R \cap Z(K) = \langle \sigma \rangle$, hence $T_x = (T^w)^0 \times \langle \sigma \rangle$. We have proved

Proposition 4.11 For $\ell = 1, \ldots, m$ we have

$$\lambda(Z_{\ell}) = \left\{ \sum_{i=1}^{2\ell} n_i \omega_i \mid n_i \in \mathbb{N}, \sum_{i=1}^{\ell} n_{2i-1} \text{ even} \right\}$$
Moreover
\[ \lambda(Z_{m+1}) = \left\{ \sum_{i=1}^{n} n_i \omega_i \mid n_k \in \mathbb{N}, \ n_n \text{ even} \right\} \]

For the simply-connected cover we obtain

**Proposition 4.12** For \( \ell = 1, \ldots, m \) we have
\[ \lambda(\hat{Z}_\ell) = \left\{ \sum_{i=1}^{2\ell} n_i \omega_i \mid n_k \in \mathbb{N} \right\} \]

Moreover
\[ \lambda(\hat{Z}_{m+1}) = \left\{ \sum_{i=1}^{n} n_i \omega_i \mid n_k \in \mathbb{N} \right\} \]

**Proof.** Let \( u \in Z_\ell \), with \( \ell = 1, \ldots, m + 1 \). If \( C(u)^\circ = RC \) with \( R = R_u(C(u)) \), \( C \) connected reductive, then \( C \) is of type \( C_{\ell-1}D_{n-2\ell+1} \) ([12], §13.1). In particular \( C \) is semisimple since \( n - 2\ell + 1 \) is even. Hence \( \lambda(\hat{Z}_\ell) \) is free by Lemma 4.4.

For \( \ell = m + 1 \), we have \( Z(G) \not\subseteq C(x)^\circ \). In fact, we can take \( u = x_{\alpha_1}(1)x_{\alpha_3}(1) \cdots x_{\alpha_n}(1) \) in \( Z_{m+1} \). Then \( S = H_{\omega_2}H_{\omega_4} \cdots H_{\omega_{n-1}} \) is a maximal torus of \( C(u)^\circ \), and since \( Z(G) \cap S = \{1\} \), we get \( C(u) = C(u)^\circ \times Z(G) \) by Lemma 4.3. We are left to deal with \( \ell = 1 \). However for each \( \ell \), the image \( Q \) of \( (u - 1)^2 \) in \( V(\omega_1) \) (which is the natural module for \( B_n \)) has dimension 1, so \( C(u)^\circ \) acts trivially on \( Q \) by Lemma 4.5, and \( \omega_1 \in \lambda(\hat{Z}_\ell) \).

We summarize the results obtained in

| \( \mathcal{O} \) | \( \lambda(\mathcal{O}) \) | \( \lambda(\mathcal{O}) \) |
|---|---|---|
| \( X_\ell \) | \( \ell \) | \( \sum_{i=1}^{\ell} n_{2i} \omega_{2i} \) |
| \( Z_\ell \) | \( 2\ell \) | \( \sum_{i=1}^{\ell} n_{i} \omega_{i} \), \( \sum_{i=1}^{\ell-1} n_{2i} \) even | \( \sum_{i=1}^{\ell} n_{i} \omega_{i} \) |
| \( Z_{m+1} \) | \( \sum_{i=1}^{n} n_{i} \omega_{i} \), \( n_n \) even | \( \sum_{i=1}^{n} n_{i} \omega_{i} \) |

Table 13: \( \lambda(\mathcal{O}), \lambda(\hat{\mathcal{O}}) \) for unipotent classes in \( B_n, n = 2m + 1 \).

In particular \( \hat{Z}_{m+1} \) is a model homogeneous space, and in fact the principal one, by [28], 3.3 (2).

In section 5, we shall determine the decomposition of the coordinate ring of the closure of \( Z_{m+1} \). For this purpose we shall use the fact that if \( x \in Z_{m+1} \cap w_0B \), then \( \alpha_{n-1} \) occurs in \( x \) (see the discussion before Proposition 3.11). This can be checked by using the representative of \( Z_{m+1} \) in \( SO(2n+1) \) given in [9], proof of Theorem 12.
4.4.5 Semisimple classes in $B_n$, $n$ odd $n = 2m + 1$

Following the notation in [9], Tables 1, 5 we get

$$
\begin{align*}
D_\ell B_{n-\ell}, & \quad \ell = 1, \ldots, m \quad \leftrightarrow \quad J_\ell \quad \leftrightarrow \quad s_{\beta_1} s_{\delta_1} \cdots s_{\beta_\ell} s_{\delta_\ell} \\
D_\ell B_{n-\ell}, & \quad \ell = m + 1, \ldots, n \quad \leftrightarrow \quad M_{2(n-\ell)+1} \quad \leftrightarrow \quad s_{\gamma_1} s_{\gamma_2} \cdots s_{\gamma_{2(n-\ell)+1}} \\
T_1 A_{n-1} & \quad \leftrightarrow \quad \emptyset \quad \leftrightarrow \quad w_0 
\end{align*}
$$

and we obtain

| $\mathcal{O}$                          | $H$                          | $\lambda(\mathcal{O})$                                      |
|-----------------------------------------|------------------------------|-------------------------------------------------------------|
| $\exp(\zeta \omega_1)$, $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ | $T_1 B_{n-1}$               | $2n_1 \omega_1 + n_2 \omega_2$                             |
| $\exp(\pi i \omega_\ell)$, $\ell = 2, \ldots, m$ | $D_\ell B_{n-\ell}$         | $2^{\ell-1} \sum_{i=1}^{\ell} 2n_i \omega_i + n_2 \omega_\ell$ |
| $\exp(\pi i \omega_\ell)$, $\ell = m + 2, \ldots, n$ | $D_\ell B_{n-\ell}$         | $2^{n-\ell} \sum_{i=1}^{n-\ell} 2n_i \omega_i + n_2 \omega_\ell \omega_{n-\ell+1}$ |
| $\exp(\pi i \omega_{m+1})$             | $D_{m+1} B_{m}$             | $\sum_{i=1}^{m+1} 2n_i \omega_i$                           |
| $\exp(\zeta \omega_n)$, $\zeta \in \mathbb{C} \setminus \pi i \mathbb{Z}$ | $T_1 A_{n-1}$               | $\sum_{i=1}^{n} n_i \omega_i$, $n$ even                    |

Table 14: $\lambda(\mathcal{O})$ for semisimple classes in $B_n$, $n = 2m + 1$.

4.4.6 Mixed classes in $B_n$, $n$ odd, $n = 2m + 1$

From [9], Table 4, we get

$$
\sigma_n x_{\beta_1}(1) \cdots x_{\beta_\ell}(1), \quad \ell = 1, \ldots, m \quad \leftrightarrow \quad M_{2\ell+1} \quad \leftrightarrow \quad s_{\gamma_1} \cdots s_{\gamma_{2\ell+1}}
$$

and we obtain

| $\mathcal{O}$                          | $\lambda(\mathcal{O}) = \lambda(\mathcal{O})$                                      |
|-----------------------------------------|---------------------------------------------------------------------------------------------|
| $\sigma_n x_{\beta_1}(1) \cdots x_{\beta_\ell}(1)$, $\ell = 1, \ldots, m - 1$ | $2^{\ell+1} \sum_{i=1}^{\ell+1} n_i \omega_i$                                             |
| $\sigma_n x_{\beta_1}(1) \cdots x_{\beta_m}(1)$, $n$ even                                    | $\sum_{i=1}^{n} n_i \omega_i$, $n$ even                                                  |

Table 15: $\lambda(\mathcal{O}), \lambda(\mathcal{O})$ for mixed classes in $B_n$, $n = 2m + 1$. 
4.5 Type $E_6$.

We put

$\beta_1 = (1, 2, 2, 3, 2, 1), \quad \beta_2 = (1, 0, 1, 1, 1, 1)$
$\beta_3 = (0, 0, 1, 1, 1, 0), \quad \beta_4 = (0, 0, 0, 1, 0, 0)$

4.5.1 Unipotent classes in $E_6$.

$A_1 \leftrightarrow \{1, 3, 4, 5, 6\} \leftrightarrow s_{\beta_1}$
$2A_1 \leftrightarrow \{3, 4, 5\} \leftrightarrow s_{\beta_1}s_{\beta_2}$
$3A_1 \leftrightarrow \emptyset \leftrightarrow w_0 = s_{\beta_1} \cdots s_{\beta_4}$

We obtain

| $O$  | $n_2 \omega_2$ | $n_1(\omega_1 + \omega_6) + n_2 \omega_2$ | $n_1(\omega_1 + \omega_6) + n_3(\omega_3 + \omega_5) + n_2 \omega_2 + n_4 \omega_4$ |
|------|----------------|----------------------------------------|-----------------------------------------------|
| $A_1$|                |                                        |                                               |
| $2A_1$|                |                                        |                                               |
| $3A_1$|                |                                        |                                               |

Table 16: $\lambda(O), \lambda(\hat{O})$ for unipotent classes in $E_6$.

4.5.2 Semisimple classes in $E_6$

Following the notation in [9], Table 2, we have

$A_1 A_5 \leftrightarrow \emptyset \leftrightarrow w_0$
$D_5 T_1 \leftrightarrow \{3, 4, 5\} \leftrightarrow s_{\beta_1}s_{\beta_2}$

We obtain

| $O$  | $H$                  | $\lambda(O)$                                                |
|------|----------------------|-------------------------------------------------------------|
|      | $\exp(\pi i \omega_2)$ | $A_1 A_5 \quad n_1(\omega_1 + \omega_6) + n_3(\omega_3 + \omega_5) + 2n_2 \omega_2 + 2n_4 \omega_4$ |
|      | $\exp(\zeta \omega_1)$ | $D_5 T_1 \quad n_1(\omega_1 + \omega_6) + n_2 \omega_2$ |

| $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$ |

Table 17: $\lambda(O)$ for semisimple classes in $E_6$.

4.6 Type $E_7$.

Here $Z(G) = \langle h_{\alpha_2}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1) \rangle$. We put

$\beta_1 = (2, 2, 3, 4, 3, 2, 1), \quad \beta_2 = (0, 1, 1, 2, 2, 2, 1), \quad \beta_3 = (0, 1, 1, 2, 1, 0, 0), \quad \beta_4 = \alpha_7, \quad \beta_5 = \alpha_5, \quad \beta_6 = \alpha_3, \quad \beta_7 = \alpha_2$
4.6.1 Unipotent classes in $E_7$.

\[
\begin{align*}
A_1 & \iff \{2, 3, 4, 5, 6, 7\} \iff s_{\beta_1} \\
2A_1 & \iff \{2, 3, 4, 5, 7\} \iff s_{\beta_1} s_{\beta_2} \\
(3A_1)'' & \iff \{2, 3, 4, 5\} \iff s_{\beta_1} s_{\beta_2} s_{\beta_4} \\
(3A_1)' & \iff \{2, 5, 7\} \iff s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\beta_6} \\
4A_1 & \iff \emptyset \iff w_0 = s_{\beta_1} \cdots s_{\beta_7}
\end{align*}
\]

We obtain

\[
\begin{array}{|c|c|c|}
\hline
\mathcal{O} & \lambda(\mathcal{O}) & \lambda(\mathcal{O}) \\
\hline
A_1 & n_1 \omega_1 & \\
2A_1 & n_1 \omega_1 + n_6 \omega_6 & \\
(3A_1)'' & n_1 \omega_1 + n_6 \omega_6 + 2n_7 \omega_7 & n_1 \omega_1 + n_6 \omega_6 + n_7 \omega_7 \\
(3A_1)' & n_1 \omega_1 + n_3 \omega_3 + n_4 \omega_4 + n_6 \omega_6 & \\
4A_1 & \sum_{i=1}^7 n_i \omega_i, \ n_2 + n_5 + n_7 \text{ even} & \sum_{i=1}^7 n_i \omega_i \\
\hline
\end{array}
\]

Table 18: $\lambda(\mathcal{O}), \lambda(\hat{\mathcal{O}})$ for unipotent classes in $E_7$.

In particular the simply-connected cover of $4A_1$ is a model homogeneous space, and in fact the principal one, by [28], 3.3 (8).

**Remark 4.13** From our description, it follows that $C(x)$ is connected for the classes $A_1, 2A_1$ and $(3A_1)'$, while for $(3A_1)''$ and $4A_1$ we have $C(x) = C(x)^0 \times Z(G)$. This also follows from the tables in [1], where all unipotent classes are considered.

4.6.2 Semisimple classes in $E_7$

Following the notation in [9], Table 2, we have

\[
\begin{align*}
E_6T_1 & \iff \{2, 3, 4, 5\} \iff s_{\beta_1} s_{\beta_2} s_{\beta_4} \\
D_6A_1 & \iff \{2, 5, 7\} \iff s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\beta_6} \\
A_7 & \iff \emptyset \iff w_0
\end{align*}
\]

We obtain

\[
\begin{array}{|c|c|c|}
\hline
\mathcal{O} & H & \lambda(\mathcal{O}) \\
\hline
\exp(\zeta \omega_7) & E_6T_1 & n_1 \omega_1 + n_6 \omega_6 + 2n_7 \omega_7 \\
\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z} & \exp(\pi i \omega_1) & 2n_1 \omega_1 + 2n_3 \omega_3 + n_4 \omega_4 + n_6 \omega_6 \\
& D_6A_1 & \exp(\pi i \omega_2) & A_7 \\
& & \sum_{i=1}^7 2n_i \omega_i \\
\hline
\end{array}
\]

Table 19: $\lambda(\mathcal{O})$ for semisimple classes in $E_7$. 

We also obtain

\[
\begin{align*}
\lambda(O) & \oplus \lambda(\hat{O}) \\
\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z} & \lambda(\mathcal{O}) \\
\exp(\pi i \omega_1) & \lambda(\mathcal{O}) \\
\exp(\pi i \omega_2) & \lambda(\mathcal{O})
\end{align*}
\]
4.7 Type $E_8$.

We put

$$\beta_1 = (2, 3, 4, 6, 5, 4, 3, 2), \quad \beta_2 = (2, 2, 3, 4, 3, 2, 1, 0), \quad \beta_3 = (0, 1, 1, 2, 2, 1, 0),$$

$$\beta_4 = (0, 1, 2, 1, 0, 0, 0), \quad \beta_5 = \alpha_7, \quad \beta_6 = \alpha_5, \quad \beta_7 = \alpha_3, \quad \beta_8 = \alpha_2$$

4.7.1 Unipotent classes in $E_8$.


| Class | $\{1, 2, 3, 4, 5, 6, 7\}$ | $s_{\beta_1}$ |
|-------|-------------------------|----------------|
| $A_1$ | $\{2, 3, 4, 5, 6, 7\}$ | $s_{\beta_1}s_{\beta_2}$ |
| $2A_1$ | $\{2, 3, 4, 5\}$ | $s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5}$ |
| $3A_1$ | $\emptyset$ | $w_0 = s_{\beta_1}\cdots s_{\beta_8}$ |

We have

$$(1 - w)P = \begin{cases} \mathbb{Z}\langle\omega_8\rangle & \text{for } w = s_{\beta_1} \\ \mathbb{Z}\langle\omega_1, \omega_8\rangle & \text{for } w = s_{\beta_1}s_{\beta_2} \\ \mathbb{Z}\langle\omega_1, \omega_6, 2\omega_7, 2\omega_8\rangle & \text{for } w = s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_5} \end{cases}$$

Class $3A_1$. Here $\Psi_f$ has basis $\{\alpha_7, \alpha_8, \beta_2, \beta_3\}$, $K = C((T^w)^\circ)$ is of type $D_4$ and has center $\langle h_{\alpha_8} (-1) h_{\alpha_5} (-1), h_{\alpha_2} (-1) h_{\alpha_3} (-1) \rangle$ which is contained in $(T^w)^\circ$. Hence $T_x = (T^w)^\circ$.

Class $4A_1$. We claim that $T_x = 1$. Suppose for a contradiction there exists an involution $\sigma \in T_x$. Then $x \in K = C(\sigma)$. From the classification of involutions of $E_8$, it follows that $K$ is of type $D_8$ or $E_7A_1$. The class of $x$ in $K$ is spherical, and by the uniqueness of Bruhat decomposition, $x$ lies over the longest element of the Weyl group of $K$, which is $w_0$. By comparison of weighted Dynkin diagrams, the spherical unipotent class of $K$ over $w_0$ does not correspond to the class $4A_1$ of $E_8$, a contradiction.

We have shown that in all cases $T_x = (T^w)^\circ$, so that $C(x)$ is connected, as also follows from [12], p. 405. We have

| Class | $\lambda(\mathcal{O})$ |
|-------|------------------|
| $A_1$ | $n_8\omega_8$ |
| $2A_1$ | $n_1\omega_1 + n_8\omega_8$ |
| $3A_1$ | $n_1\omega_1 + n_6\omega_6 + n_7\omega_7 + n_8\omega_8$ |
| $4A_1$ | $\sum_{i=1}^{s} n_i\omega_i$ |

Table 20: $\lambda(\mathcal{O})$, $\lambda(\hat{\mathcal{O}})$ for unipotent classes in $E_8$.

In particular $4A_1$ is a model homogeneous space (see [2], Theorem 1.1), and in fact the principal one, by [28], 3.3 (9).
4.7.2 Semisimple classes in $E_8$.

Following the notation in [9], Table 2, we have

\[
A_1 E_7 \leftrightarrow \{2, 3, 4, 5\} \leftrightarrow s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\beta_4}
\]

\[
D_8 \leftrightarrow \emptyset \leftrightarrow w_0
\]

We obtain

| $O$          | $H$         | $\lambda(O)$                                      |
|--------------|-------------|---------------------------------------------------|
| $\exp(\pi i \omega_8)$ | $A_1 E_7$  | $n_1 \omega_1 + n_6 \omega_6 + 2n_7 \omega_7 + 2n_8 \omega_8$ |
| $\exp(\pi i \omega_1)$   | $D_8$       | $\sum_{i=1}^{8} 2n_i \omega_i$                   |

Table 21: $\lambda(O)$ for semisimple classes in $E_8$.

4.8 Type $F_4$.

We put

\[
\beta_1 = (2, 3, 4, 2), \quad \beta_2 = (0, 1, 2, 2), \\
\beta_3 = (0, 1, 2, 0), \quad \beta_4 = (0, 1, 0, 0)
\]

4.8.1 Unipotent classes in $F_4$.

\[
A_1 \leftrightarrow \{2, 3, 4\} \leftrightarrow s_{\beta_1}
\]

\[
A_1 \leftrightarrow \{2, 3\} \leftrightarrow s_{\beta_2}s_{\beta_3}
\]

\[
A_1 + \tilde{A}_1 \leftrightarrow \emptyset \leftrightarrow w_0 = s_{\beta_1} \cdots s_{\beta_4}
\]

We obtain

| $O$          | $\lambda(O)$                                      | $\lambda(O)$                                      |
|--------------|---------------------------------------------------|---------------------------------------------------|
| $A_1$        | $n_1 \omega_1$                                    |                                                   |
| $A_1$        | $n_1 \omega_1 + 2n_4 \omega_4$                    | $n_1 \omega_1 + n_4 \omega_4$                    |
| $A_1 + A_1$  | $n_1 \omega_1 + n_2 \omega_2 + 2n_3 \omega_3 + 2n_4 \omega_4$ |                                                   |

Table 22: $\lambda(O)$, $\lambda(\tilde{O})$ for unipotent classes in $F_4$. 
4.8.2 Semisimple classes in $F_4$.

Following the notation in [9], Table 2, we have

\[
\begin{align*}
A_1 C_3 & \longleftrightarrow \emptyset \longleftrightarrow w_0 \\
B_4 & \longleftrightarrow \{1, 2, 3\} \longleftrightarrow s_{\gamma_1}
\end{align*}
\]

where $\gamma_1$ is the highest short root $(1, 2, 3, 2)$.

We obtain

\[
\begin{array}{c|c|c}
\mathcal{O} & H & \lambda(\mathcal{O}) \\
\hline
\exp(\pi i \hat{\omega}_1) & A_1 C_3 & \sum_{i=1}^{4} 2n_i \omega_i \\
\exp(\pi i \hat{\omega}_4) & B_4 & n_4 \omega_4
\end{array}
\]

Table 23: $\lambda(\mathcal{O})$ for semisimple classes in $F_4$.

4.8.3 Mixed class in $F_4$.

We put $f_2 = \exp(\pi i \hat{\omega}_4)$. Then following [9], Table 4

\[
\mathcal{O}_{f_2x_{\beta_1}(1)} \longleftrightarrow \emptyset \longleftrightarrow w_0
\]

Assuming the existence of an involution in $T_x$ we get a contradiction, proving therefore that $T_x = 1$. Hence

\[
\begin{array}{c|c}
\mathcal{O} & \lambda(\mathcal{O}) = \lambda(\hat{\mathcal{O}}) \\
\hline
f_2x_{\beta_1}(1) & \sum_{i=1}^{4} n_i \omega_i
\end{array}
\]

Table 24: $\lambda(\mathcal{O})$ for the mixed class in $F_4$.

In particular $\mathcal{O}_{f_2x_{\beta_1}(1)}$ is a model homogeneous space, and in fact the principal one, by [28], 3.3 (6), see also [28] p. 300.

4.9 Type $G_2$.

We put $\beta_1 = (3, 2), \beta_2 = \alpha_1$. 
4.9.1 Unipotent classes in $G_2$.

\[
\begin{align*}
A_1 & \leftrightarrow \{1\} \leftrightarrow s_{\beta_1} \\
\tilde{A}_1 & \leftrightarrow \emptyset \leftrightarrow w_0 = s_{\beta_1}s_{\beta_2}
\end{align*}
\]

We get

| $\mathcal{O}$ | $\lambda(\mathcal{O}) = \lambda(\tilde{\mathcal{O}})$ |
|---------------|------------------|
| $A_1$         | $n_2\omega_2$    |
| $\tilde{A}_1$ | $n_1\omega_1 + n_2\omega_2$ |

Table 25: $\lambda(\mathcal{O})$, $\lambda(\tilde{\mathcal{O}})$ for unipotent classes in $G_2$.

In particular $\tilde{A}_1$ is a model homogeneous space, and in fact the principal one, by [28], 3.3 (5).

Using the embedding of $G$ into $SO(7)$, one can determine explicitly an $x \in \mathcal{O} \cap w_0 B$, where $\mathcal{O} = \tilde{A}_1$. Then one can check that both $\alpha_1$ and $\alpha_2$ occur in $x$ (see the discussion before Proposition 3.11). This fact will be used in section 5 to determine $C[\mathcal{O}]$.

4.9.2 Semisimple classes in $G_2$.

Following the notation in [9], Table 2, we have

\[
\begin{align*}
A_1 \tilde{A}_1 & \leftrightarrow \emptyset \leftrightarrow w_0 \\
A_2 & \leftrightarrow \{2\} \leftrightarrow s_{\gamma_1}
\end{align*}
\]

where $\gamma_1$ is the highest short root $(2,1)$.

The group $G_2$ has 1 class of involutions. However there is also a class of elements of order 3 which is spherical. We obtain

| $\mathcal{O}$ | $H$ | $\lambda(\mathcal{O})$ |
|---------------|-----|------------------|
| $\exp(\pi i \omega_2)$ | $A_1 \tilde{A}_1$ | $\sum_{i=1}^{2} 2n_i \omega_i$ |
| $\exp\left(\frac{2\pi i}{3} \omega_1\right)$ | $A_2$ | $n_1\omega_1$ |

Table 26: $\lambda(\mathcal{O})$ for semisimple classes in $G_2$. 
5 The coordinate ring of $\overline{\mathcal{O}}$

In this section we determine the decomposition of $\mathbb{C}[\overline{\mathcal{O}}]$ into simple $G$-modules, where $\overline{\mathcal{O}}$ is the closure of a spherical conjugacy class. Normality of conjugacy classes’ closures has been deeply investigated. For a survey on this topic, see [23], §8, [8], 7.9, Remark (iii). The first observation is that the problem is reduced to unipotent conjugacy classes in $G$ ([23], 8.1). In the following we are interested only in spherical conjugacy classes, and I recall the facts in this context. It is known that the closure of the minimal nilpotent orbit is always normal ([44], Theorem 2). Hesselink ([17]) proved normality for several small orbits in the classical cases and certain orbits for the exceptional cases: namely, following the notation in [12], $A_1$ and $2A_1$ in $E_6$, $A_1$, $2A_1$ and $(3A_1)''$ in $E_7$, $A_1$ and $2A_1$ in $E_8$, $A_1$ and $\tilde{A}_1$ in $F_4$, $A_1$ in $G_2$.

The classical groups have been considered in [24], [25]: for the special linear groups the closure of every conjugacy class is normal. For the symplectic and orthogonal groups there exist conjugacy classes with non-normal closure. However every spherical conjugacy class in the symplectic group has normal closure, since from the classification we know that the unipotent spherical conjugacy classes have only 2 columns (see also [17], §5, Criterion 2). For special orthogonal groups the results in [25] left open the cases of the very even unipotent classes. E. Sommers proved that these have normal closure in [39]. Taking into account the results in [25] and [39] it follows that every unipotent spherical conjugacy class in type $D_n$ and $B_n$ has normal closure except for the maximal class $Z_{m+1}$ in $B_n$, when $n = 2m + 1$, $m \geq 1$. From this and the classification of spherical conjugacy classes, it follows that every spherical conjugacy class has normal closure, except for the above mentioned class in $B_{2m+1}$.

For the exceptional groups, besides the results on the minimal orbit and Hesselink’s results, in [27] it is shown that the orbit $\tilde{A}_1$ in $G_2$ has a non-normal closure (see also [23]): here there is bijective normalization, contrary to the case of $Z_{m+1}$ in $B_{2m+1}$ where the closure is branched in codimension 2. In [7] the case of type $F_4$ is completely handled, and it follows that every spherical conjugacy class has normal closure. The same holds for $E_6$, as follows from [38] where every nilpotent orbit is considered. For the remaining nilpotent orbits in $E_7$ and $E_8$, in [8], 7.9, Remark (iii), A. Broer gives a list of orbits with normal closure. Among these there are all spherical nilpotent orbits in $E_7$ and $E_8$. We may therefore state

**Theorem 5.1** Let $\mathcal{O}$ be a spherical conjugacy class. Then $\overline{\mathcal{O}}$ is normal except for the class $Z_{m+1}$ in $B_{2m+1}$ ($m \geq 1$) and the class $\tilde{A}_1$ in $G_2$. $\Box$

**Remark 5.2** In [13], Example 4.4, Proposition 4.5, the authors prove normal closure for nilpotent orbits of height 2.

**Remark 5.3** In [35], 6.1, normality of $N^{\text{sph}}$ (the union of all spherical nilpotent orbits, which is in fact the closure of the unique maximal spherical nilpotent orbit) is discussed.
Remark 5.4 From (3.9) and Corollary 3.16 it is possible to prove normality of $\mathcal{O}$ in certain cases. For instance in type $C_n$ from Table 3 we get $\lambda(X_\ell) = 2P_{\alpha_\ell}^+$ for every unipotent class $X_\ell$. From (3.9) it follows that $\lambda(\mathcal{O}) = \lambda(\mathcal{O})$, so that $\mathcal{O}$ is normal.

We recall that in general $\mathbb{C}[\mathcal{O}]$ is the integral closure of $\mathbb{C}[\mathcal{O}]$ in its field of fractions and that $\mathbb{C}[\mathcal{O}] = \mathbb{C}[\mathcal{O}]$ if and only if $\mathcal{O}$ is normal ([22], Proposition and Corollary in 8.3). By Theorem 5.1, to describe the decomposition of $\mathcal{O}$ we are left to deal with $Z_{m+1}$ in $B_{2m+1}$ and with $\tilde{A}_1$ in $G_2$. We use the notation and the tables from section 4 for the cases $B_{2m+1}$ and $G_2$.

Theorem 5.5 Let $\mathcal{O} = Z_{m+1}$ in $B_n$, $n = 2m + 1$, $m \geq 1$. Then

$$\lambda(\mathcal{O}) = \left\{ \sum_{i=1}^{2m} n_i \omega_i \mid \sum_{i=1}^{m} n_{2i-1} \text{ even} \right\} \cup \left\{ \sum_{i=1}^{n} n_i \omega_i \mid n_n \text{ even, } n_n \geq 2 \right\}$$

Proof. Considering the $(G$-equivariant) restriction $\mathcal{O} : \mathbb{C}[\mathcal{O}] \rightarrow \mathbb{C}[Z_m] = \mathbb{C}[Z_m]$, we get

$$\left\{ \sum_{i=1}^{2m} n_i \omega_i \mid \sum_{i=1}^{m} n_{2i-1} \text{ even} \right\} \leq \lambda(\mathcal{O}) \text{.}$$

In particular for every even $j$, $\omega_j \in \lambda(\mathcal{O})$, and for every pair of odd $j$, $k$, with $1 \leq j \leq k < n$, $\omega_j + \omega_k \in \lambda(\mathcal{O})$. By Corollary 3.12, we have $2\omega_n \in \lambda(\mathcal{O})$. We show that $\omega_j + 2\omega_n \in \lambda(\mathcal{O})$ for every odd $j$, $j < n$. We have $\omega_{n-1} - \alpha_{n-1} = \omega_{n-2} + 2\omega_n$ and since $\alpha_{n-1}$ occurs in $x \in w_0 B \cap \mathcal{O}$, by Corollary 3.16, we get $\omega_{n-2} + 2\omega_n \in \lambda(\mathcal{O})$. Let $j$ be odd, $j < n - 2$. Then $\omega_j + 2\omega_n + 2\omega_{n-2} \in \lambda(\mathcal{O})$ since $\omega_{n-2} + 2\omega_n$ and $\omega_j + \omega_{n-2}$ are in $\lambda(\mathcal{O})$.

There exists $B$-eigenvectors $F, H$ in $\mathbb{C}[\mathcal{O}]$ of weights $\omega_j + 2\omega_n + 2\omega_{n-2}, 2\omega_{n-2}$ respectively. Then $F/H$ is a rational function on $\mathcal{O}$ of weight $\omega_j + 2\omega_n$ defined at least on $\mathcal{O}$, However $2\omega_{n-2}$ is also a weight in $\lambda(Z_m)$, so that $H$ is non-zero on the dense $B$-orbit $v$ in $Z_m$. Hence $F/H$ is defined on $v$, and it is zero on $v$, since if $F$ is zero on $Z_m, \omega_j + 2\omega_n + 2\omega_{n-2}$ not being in $\lambda(Z_m)$. It follows that $F/H$ is defined on $Z_m$, so that it is a regular function on $\mathcal{O} \cup Z_m$. By [25], Theorem 16.2, (iii), $F/H$ extends to $\mathcal{O}$, and $\omega_j + 2\omega_n$ lies in $\lambda(\mathcal{O})$. We have shown that

$$\lambda(\mathcal{O}) \geq \left\{ \sum_{i=1}^{2m} n_i \omega_i \mid \sum_{i=1}^{m} n_{2i-1} \text{ even} \right\} \cup \left\{ \sum_{i=1}^{n} n_i \omega_i \mid n_n \text{ even, } n_n \geq 2 \right\}$$

We prove that also the opposite inclusion holds. Assume $\lambda = \sum_{i=1}^{n} n_i \omega_i \in \lambda(\mathcal{O})$. Since $\lambda(\mathcal{O}) \leq \lambda(\mathcal{O})$, we have $n_n$ even. If $n_n \neq 0$ we are done. So assume $n_n = 0$. Let $y \in Z_{m+1} \cap U^- \cap B w_0 B$. We observe that $y_1 := \lim_{z \to 0} h_{\alpha_n}(z)^{-1} y h_{\alpha_n}(z)$ exists, and lies in $Z_m \cap U^- \cap B w B$, where $w = w(Z_m)$ (in [9] we give representatives for both classes in $SO(2n + 1)$, so that this may be checked directly). Now let $F : \mathcal{O} \rightarrow \mathbb{C}$ be a highest weight vector of weight $\lambda$, with $F(y) = 1$. Then $F(y_1) = 1$, since $h_{\alpha_n}(z) = 1$ for every $z \in \mathbb{C}^*$. Since $x_1 \in Z_m \cap w B$ lies in the $B$-orbit of $y_1$, we have $F(x_1) \neq 0$. But $\sigma = \prod_{i=1}^{n} h_{\alpha_{2i-1}}(-1) \in C(x_1)$, so that $F(x_1) = F(\sigma x_1 \sigma) = \lambda(\sigma) F(x_1)$ implies $\lambda(\sigma) = 1$, and we are done. \qed
Theorem 5.6 Let $O = \hat{A}_1$ in $G_2$. Then $\lambda(O)$ is the submonoid of $\lambda(O)$ generated by $2\omega_1, 3\omega_1, \omega_2$.

Proof. We know that $\omega_1 \in \lambda(O)$ and it follows from the proof of [27], Theorem 3.13, that $\omega_1 \not\in \lambda(O)$. We have

$$2\omega_1 - \alpha_1 = \omega_2, \quad 2\omega_2 - \alpha_2 = 3\omega_1$$

hence, by Corollary 3.12 and 3.16, we get $2\omega_1, 3\omega_1, \omega_2 \in \lambda(O)$, since both $\alpha_1, \alpha_2$ occur in $x \in w_2B \cap O$. Suppose for a contradiction that $\omega_1 + n\omega_2 \in \lambda(O)$ for a certain $n \in \mathbb{N}$. There exists $B$-eigenvectors $F$, $H$ in $C(O)$ of weights $\omega_1 + n\omega_2$, $n\omega_2$ respectively. Then $F/H$ is a rational function on $O$ of weight $\omega_1$ defined at least on $O$. However $n\omega_2$ is also a weight in $\lambda(A_1)$, so that $H$ is non-zero on the dense $B$-orbit $v$ in $A_1$. Hence $F/H$ is defined on $v$, and it is zero on $v$, since $F$ is zero on $A_1$, because $\omega_1 + n\omega_2$ is not in $\lambda(A_1)$. It follows that $F/H$ is defined on $A_1$. But $A_1$ has normal closure, so that $F/H$ is defined on the closure of $A_1$, and then on $O$, so that there is in $C(O)$ a $B$-eigenvector of weight $\omega_1$, a contradiction. $\square$

6 The general case

Let $G$ be as usual simply-connected, $D \leq Z(G), \overline{G} = G/D, \pi : G \to \overline{G}$ the canonical projection. Let $p \in G$ we put $g = \pi(g)$. We give a procedure to describe the coordinate ring of $O_p$, where $O_p$ is a spherical conjugacy class of $G$. Passing to $G$, we have to consider the quotient $G/\pi^{-1}(C_G(p))$. Let $p = sv$ be the Jordan-Chevalley decomposition of $p$, $w = w(O_p)$. We may assume $s \in T$. Let $W_{s,D} = \{w \in W \mid wsw^{-1} = zs, z \in D\}$, and $N_{s,D} \leq N$ such that $N_{s,D}/T = W_{s,D}$. Then $\pi^{-1}(C_G(p)) = C(v) \cap N_{s,D}C(s)$. Reasoning as in [42], Corollary II, 4.4, we have a homomorphism $\pi^{-1}(C_G(p)) \to D, g \mapsto [g, p]$ with kernel $C(p)$.

Let $y \in O_p \cap BwB$ be such that $L = L_j$ is adapted to $C(y)$. If $H = \pi^{-1}(C_G(p))$, then $\lambda(O_p) = \lambda(G/H) = \{\lambda \in P_w^+ \mid \lambda(T \cap H) = 1\}$ by Corollary 3.18. Let $x \in O_p \cap wB, x = wu$, with $u \in U$ and let $T_{x,D} = T \cap \pi^{-1}(C_G(p))$. By Proposition 3.4, we get $T \cap H = T_{x,D}$, hence

$$\lambda(O_p) = \{\lambda \in P_w^+ \mid \lambda(T_{x,D}) = 1\}$$

(6.12)

Let $T_{x,D} = \{t \in T \mid wt^{-1} = zt, z \in D\}$. From the Bruhat decomposition, we get $T_{x,D} \leq T_{x,D}$. Moreover since $w$ is an involution, for $t \in T_{x,D}$, we have $t = w^2t^{-2} = z^2t$, so that $z^2 = 1$. In particular $\pi^{-1}(C_G(p)) = N_{s,D_2}C(s), T_{x,D} = T_{x,D_2},$ where $D_2 = D \cap T_2$.

Let $t \in T$ and write $t = ab$, with $a \in (T^w)\circ, b \in (S^w)^\circ$. Then $wtw^{-1} = tz$ with $z \in D_2$ if and only if $z = b^2$. Since $(S^w)^\circ$ is connected, we get $T_{x,D} = T_{x,D\cap (S^w)^\circ}$ and

$$\pi^{-1}(C_G(p)) = \frac{T_{x,D}}{T_x} \approx D_2 \cap (S^w)^\circ$$

with $T_x = T^w \cap C(u), T_{x,D} = T_{x,D}^w \cap C(u)$. In particular, if $D_2 \cap (S^w)^\circ = 1$, then $\lambda(O_p) = \lambda(O_x)$. This equality means that $x$ is not conjugate to $zx$ for any $z \in D_2$, $z \neq 1$, and this may be
directly checked in many cases, for instance in type $A_n$ or $C_n$ (and of course always holds for $x$ unipotent). However, to deal with orthogonal groups and $E_7$, we determined explicitly the cases when $D_2 \cap (S^w)^0$ is non-trivial, and in each case we determined $T_{x,D}$ and therefore $\lambda(O_\pi)$.

Here we just observe that if $D_2 \cap (S^w)^0 \neq 1$, then $D_2 \cap (S^w)^0 \cong \mathbb{Z}/2\mathbb{Z}$, except possibly for $D = Z(G)$ in type $D_n$, $n = 2m$. It turns out that in this case for $\exp(\pi i \tilde{w}_m)$, we have $T_x = T_2$ and $T_{x,Z(G)}/T_x \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. More precisely

$$T_{x,Z(G)} = T_{Z(G)}^{w_0} = T_2 \langle h_{\alpha_{n-1}}(i)h_{\alpha_n}(i), \prod_{i=1}^{m} h_{\alpha_{2i-1}}(i) \rangle$$

so that in $G/Z(G) = PSO(2n)$, $n = 2m$,

$$\lambda(O_{\exp(\pi i \tilde{w}_m)}) = \left\{ \sum_{k=1}^{n} 2m_k \omega_k \mid m_k \in \mathbb{N}, m_{n-1} + m_n \text{ and } \sum_{i=1}^{m} m_{2i-1} \text{ even} \right\}$$

We add that for $SO(2n+1)$, $n \geq 1$ and $b_\lambda = \text{diag}(1, \lambda I_n, \lambda^{-1} I_n)$, $\lambda \neq \pm 1$, $O_{b_\lambda}$ is a model orbit, and in fact the principal one by [28], 3.3 $(2')$.

We conclude by presenting the results for $E_7$.

### 6.1 Type $E_7$, $D = Z(G)$

In this case $Z(G) = \langle z \rangle$, where $z = h_{\alpha_2}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1) = \exp(2\pi i \tilde{w}_2) = \exp(2\pi i \tilde{w}_7)$.

There are 3 elements of the Weyl group to be considered and only for $w = s_{\beta_1}s_{\beta_2}s_{\beta_4}$ and $w = w_0$ we have $z \in (S^w)^0$.

**Class of type $A_7$, $w = w_0$.** Here $x = n_{\beta_1} \cdots n_{\beta_7}$,

$$T_{Z(G)}^{w_0} = T_2 \langle \exp(\pi i \tilde{w}_2) \rangle = T_2 \langle h_{\alpha_2}(i)h_{\alpha_5}(i)h_{\alpha_7}(i) \rangle$$

since $\exp(\pi i \tilde{w}_2) \in (S^{w_0})^0 = T$ and $\exp(\pi i \tilde{w}_2)^2 = z$.

**Proposition 6.1** Let $G$ be of type $E_7$, $D = Z(G)$, then

$$\lambda(O_{\exp(\pi i \tilde{w}_2)}) = \left\{ \sum_{i=1}^{7} 2n_i \omega_i \mid n_2 + n_5 + n_7 \text{ even} \right\}$$

**Proof.** This follows from the fact that $T_{x,Z(G)} = T_{Z(G)}^{w_0}$. \hfill \Box

**Classes of type $E_6 T_1$, $w = s_{\beta_1}s_{\beta_2}s_{\beta_4}$, $T^w = (T^w)^0 \times \langle h_{\alpha_7}(-1) \rangle = (T^w)^0 \times Z(G)$.**

We have $T_{Z(G)}^w = T^w \langle \exp(\pi i \tilde{w}_7) \rangle = T^w \langle h_{\alpha_1}(-1)h_{\alpha_7}(i) \rangle$. If $\zeta \in \mathbb{C} \setminus 2\pi i \mathbb{Z}$, then

$$x_\zeta = n_{\beta_1}n_{\beta_2}n_{\alpha_7}h_{x_{\beta_1}}(\xi)x_{\beta_2}(\xi)x_{\alpha_7}(\xi) \in O_{\exp(\zeta \tilde{w}_7)} \cap n_{\beta_1}n_{\beta_2}n_{\alpha_7}B$$
Decomposition of $\mathbb{C}[\mathcal{O}]$

for a certain $h \in T$, with $\xi = \frac{1+e^\xi}{1-e^\xi}$, so that

$$T_{x, Z} = \begin{cases} T_Z^{w(G)} & \text{if } \zeta \in \pi \mathbb{Z} \setminus 2\pi \mathbb{Z} \\ T^w & \text{if } \zeta \in \mathbb{C} \setminus \pi \mathbb{Z} \end{cases}$$

since $\alpha_7(\exp(\pi i \bar{\omega}_7)) = -1$.

**Proposition 6.2** Let $G$ be of type $E_7$, $D = Z(G)$, then

$$\lambda(\mathcal{O}_{\exp(\zeta \omega_7)}) = \begin{cases} \{n_1 \omega_1 + n_6 \omega_6 + 2n_7 \omega_7 \mid n_1 + n_7 \text{ even}\} & \text{if } \zeta \in \pi \mathbb{Z} \setminus 2\pi \mathbb{Z} \\ \{n_1 \omega_1 + n_6 \omega_6 + 2n_7 \omega_7\} & \text{if } \zeta \in \mathbb{C} \setminus \pi \mathbb{Z} \end{cases}$$

**Addendum** In [9], Remark 5, we stated that if $\pi_1 : G \to G/U$ is the canonical projection, and $\mathcal{O}$ is a spherical conjugacy class, then $\pi_1|_{\mathcal{O}} : \mathcal{O} \to G/U$ has finite fibers. This is not correct, and one can only say that $\pi_1|_{\mathcal{O}}$ has generically finite fibers (if $w = w(\mathcal{O})$, and $g \in \mathcal{O} \cap BwB$, then $\pi_1^{-1}(gU)$ has $|T^w/T_x|$ elements, where $x \in \mathcal{O} \cap wB$).

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