Volumes of Polytopes Without Triangulations

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Abstract

We introduce a new formalism for defining and computing the volumes of completely general polytopes in any dimension. The expressions that we obtain for these volumes are independent of any triangulation, and manifestly depend only on the vertices of the underlying polytope. As one application of this formalism, we obtain new expressions for tree-level, \( n \)-point NMHV amplitudes in \( N = 4 \) Super Yang-Mills (SYM) theory.
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1 Introduction

Our starting point is the BCFW representation of the $n$-point NMHV tree amplitude in $N = 4$ SYM theory, denoted by $M_{n}^{NMHV}$. This is, up to an overall MHV factor, given by [1]

$$M_{n}^{NMHV} = \frac{1}{2} \sum_{i,j}^{n} [\ast i(i+1)j(j+1)] \quad (1)$$

where $\ast$ is an arbitrary reference boundary and the sum on $n$ is cyclic (i.e., $n + 1 = 1$). Due to the additive properties of the $[abcde]$ objects, expression (1) is interpreted as the volume of a polytope given as a sum of volumes of simplices $[\ast i(i+1)j(j+1)]$. Accordingly, (1) corresponds to a particular triangulation of the underlying polytope. However, there are different ways to express $M_{n}^{NMHV}$ as a sum of $[abcde]$ objects, just as one can decompose a space into many different triangulations. Expressions of volumes as sums of $[abcde]$ objects are therefore not invariant, but rather dependent on an arbitrary choice of triangulation, some being more suitable for certain purposes than others.

In this note we develop a formalism for handling polytopes that allows us to express their volumes in a manner that makes no reference to any particular triangulation. As an application of this formalism, we show that $M_{n}^{NMHV}$ is most naturally written as

$$M_{n}^{NMHV} = \frac{1}{2 \cdot 4!} \sum_{i,j}^{n} F_{i(i+1)j(j+1)} \quad (2)$$

where the subscripts of the $F_{\ast}$ objects—which are yet to be defined, but which replace the $[abcde]$ objects as the “atomic” elements of our formalism—label the genuine (“physical”) vertices of the polytope. The above is an expression for $M_{n}^{NMHV}$ that is manifestly $\ast$-independent, manifestly depending only on vertices of the form $\{i, i+1, j, j+1\}$, and which is independent of any triangulation. In fact, as we will see, (2) contains all of the information needed to obtain any of the infinitely many triangulations of $M_{n}^{NMHV}$, as well as the volumes of any of the underlying polytope’s lower-dimensional bounding polytopes, in a simple, algorithmic, purely algebraic fashion.

We note that the polytopes that we define in this note are viewed as being embedded in $\mathbb{CP}^{n}$ (or $\mathbb{RP}^{n}$), and that the following discussion has yet to be generalized to the case of polytopes in the more general Grassmannian $G(k, n)$ with $k > 1$. Additionally, as will become clear, our discussion makes no reference as of yet to convexity (and therefore, to positivity) or even connectivity, but is instead completely general. Therefore, application of these ideas to recent work such as [2] or [3] is not yet clear, though we have reason to believe that such connections can be made. At the end of this note we mention the cohomological considerations that motivated these results and that may well be the key for generalizing these ideas to the aforementioned work.
2 Abstract Listed 2-Polytopes

We first present the entire story—from the definition of a polygon all the way through to the new $F_\ldots$ calculus for computing areas—in two dimensions where things are easiest to visualize. In this lowest of dimensions we will prove all statements rigorously. We then discuss the three- and four-dimensional stories in turn, leaving out almost all of the proofs as they are similar in nature to the two-dimensional case, differing almost solely in their algebraic complexity. We give the generalization of our definition of polytopes to arbitrary dimensions in Appendix C to show that such definitions are possible. However, as we are most interested in applying these ideas to the four-dimensional polytope corresponding to (1), we will not discuss higher dimensions in depth.

We quickly recall the basic facts about $\mathbb{CP}^n$ that we need before specializing in this and the next two sections, respectively, to the cases $n = 2, 3, 4$. Any $Z^\alpha \in \mathbb{CP}^n$ (with $n + 1$ homogeneous coordinates labeled by $\alpha = 0, 1, ..., n$) determines a unique linearly embedded $\mathbb{CP}^{n-1}$ in a dual $\mathbb{CP}^n$, whose elements $W_\alpha$ have $n + 1$ homogeneous coordinates labeled by lower indices. This correspondence between points and hyperplanes is realized via the usual linear homogenous pairing between variables and their duals: $W \cdot Z \equiv W_\alpha Z^\alpha = 0$. We assume an understanding of the intersection properties of $k$-planes in projective space, and details can be found in, for example, the appendix of [4].

Given three distinct $Z_1^\alpha, Z_2^\alpha, Z_3^\alpha \in \mathbb{CP}^2$, we obtain three distinct lines in the dual $\mathbb{CP}^2$, each pair intersecting in a unique point. Figure 1 depicts this scenario, with each line being labelled by its defining $Z_\alpha^\alpha$ in the dual space, and the vertices of the "triangle"\[^1\] being labelled by the pair of lines whose intersection defines it. This is the first of many important departures from standard polytope theory, where we label the $(n - 1)$-dimensional faces of our $n$-polytopes, as opposed to labeling the vertices. We then label the lower dimensional faces via the intersections of the $(n - 1)$-dimensional faces.

We aim to make rigorous the interpretation of the depiction in Figure 1 as a triangle, even though it does not properly reflect the topological qualities of the region defined by $Z_1^\alpha, Z_2^\alpha, Z_3^\alpha$. The key realization is that even though Figure 1 does not reflect much of the topological structure of our set up, it does correctly reflect how the projective lines intersect, and this is enough to define a meaningful notion of polygon. In particular, we choose to view polygons as abstract instructions for moving from one vertex to another along a well-defined edge (i.e., projective line). In Figure 1 it is clear that we can move from vertex $\{1, 2\}$ to $\{2, 3\}$ along the line 2. The fact that the line 2 is topologically an $S^2$ (being a linearly embedded $\mathbb{CP}^1$) is irrelevant.

The instructions “move from $\{1, 2\}$ to $\{2, 3\}$ along the line 2” can be unambiguously denoted by $\{1, 2\} \rightarrow \{2, 3\}$ where it is understood that we move along the line whose label is common to the two vertices. We denote the instructions $\{1, 2\} \rightarrow \{2, 3\}$ simply by $[1(2)3]$, making manifest both the starting and ending vertices as well as the line that joins them (in this case, 2). It is then clear how Figure 1 can be viewed as being one of

\[^1\]We use inverted commas here as we have yet to give a clear definition of a triangle.
two oppositely oriented triangles: one orientation corresponds to the instructions

\[
\{1, 2\} \to \{2, 3\} \to \{3, 1\} \to \{1, 2\} \left( = [1(2)3] + [2(3)1] + [3(1)2] \right) \tag{3}
\]

while the opposite orientation corresponds to the instructions

\[
\{2, 1\} \to \{1, 3\} \to \{3, 2\} \to \{2, 1\} \left( = [2(1)3] + [1(3)2] + [3(2)1] \right). \tag{4}
\]

The two different scenarios are depicted respectively on the left and right of Figure 2.

\footnote{We give the instructions using both the \(\{\cdot, \cdot\} \to \{\cdot, \cdot\}\) notation as well as the \([\cdot(\cdot)]\) notation, though we will begin to use only the latter.}
These ideas can be immediately and usefully generalized to define arbitrary polygons by noting that the quality of the “boundary of the boundary” vanishing—which is appropriate for any shape that we want to interpret as a “closed” object—is simply that the instructions in (3) and (4) end where they began (in addition to being well-defined at each step, namely that each sequential pair of vertices share a label). Thus, given any set of $N$ distinct elements $Z_1^\alpha, ..., Z_N^\alpha \in \mathbb{CP}^2$ defining a set of $N$ distinct lines in the dual space, we can view any set of instructions

$$\{i_1, i_2\} \rightarrow \{i_2, i_3\} \rightarrow \cdots \rightarrow \{i_n, i_1\} \rightarrow \{i_1, i_2\},$$

with each $i_k \in \{1, ..., N\}$, as defining a polygon. We say that such a set of abstract instructions is given by a cyclic list $l = (i_1, ..., i_n)$, where $n$ simply denotes the length of the list.

For example, with boundaries $\{1, 2, 3\}$, the triangle on the left in Figure 2 is defined by the list $l = (1, 2, 3)$, and with boundaries $\{1, 2, 3, 4, 5\}$, the “polygon” defined by the list $l = (3, 2, 1, 3, 4, 2, 5)$ is depicted in Figure 3. We emphasize that we still have yet to define what we mean by “polygon”, but it is clear from Figure 3 that we are building up to a definition that is not restricted to what we might want to call convex, or even connected polygons.

Many different lists correspond to what we will want to call the same polygon, as for example the cyclic permutation of any list defines equivalent instructions. However, the situation is more subtle than that, as for example the lists $l' = (321342425)$ and $l'' = (3425321)$ also define the same polygon as depicted in Figure 3. In order to

---

From here on we simply let $\{1, ..., N\}$ denote the set $\{Z_1^\alpha, ..., Z_N^\alpha\} \in \mathbb{CP}^n$ where $n = 2$ here and later in higher dimensions $n$ will be obvious from context.
introduce an equivalence of lists it is most useful to use the \([\cdot,\cdot]\) notation and introduce some more formal machinery.

We can place (and implicitly already have placed in, e.g., (3) and (4)) an additive structure on the formal objects \([i(j)k]\) in a natural way: 

\[-[k(j)i] \equiv [i(j)k] \quad \text{and} \quad [i(j)k] + [k(j)l] \equiv [i(j)l] \]

These definitions denote respectively the oriented nature of each edge (that edges with starting and ending vertices swapped are viewed as negatives of each other) and the fact that walking from point \(A\) to point \(B\) then from point \(B\) to point \(C\) (all along the same edge) is equivalent to walking from point \(A\) to point \(C\). In our notation this simply means that \((\{i,j\} \to \{j,k\}) + (\{j,k\} \to \{j,l\}) = \{i,j\} \to \{j,l\}\).

An edge set \(E\) is any formal sum of \([i(j)k]\) objects (called oriented edges), and if we define the boundary operator 

\[\partial[i(j)k] \equiv \{j,k\} - \{i,j\} \]

then we see that “the boundary of the boundary” of the “polygon” defined by \(E\) vanishes if and only if \(\partial E = 0\).

Given a cyclic list \(l = (i_1,\ldots,i_n)\), we define the edge set derived from \(l\), \(E(l)\), as

\[E(l) \equiv \sum_{j=1}^{n}[i_{(j-1)}i_ji_{(j+1)}], \quad \text{(6)}\]

where the sum on \(j\) is cyclic in the sense that \(i_{(n+1)} = i_1\). Thus, for \(l = (1,2,3)\), which determines the triangle on the left in Figure 2, we have

\[E(l = (123)) = [1(2)3] + [2(3)1] + [3(1)2], \quad \text{(7)}\]

as expected from (3), and for \(l = (3213425)\) defining the polygon in Figure 3 we have

\[E(l = (3213425)) = [3(2)1] + [2(1)3] + [1(3)4] + [3(4)2] + [4(2)5] + [2(5)3] + [5(3)2], \quad \text{(8)}\]

which are both readily checked to coincide with their respective figures. The equivalence relation amongst lists is now obvious: we say that \(l \sim l'\) if \(E(l) = E(l')\). One can readily check that the lists \(l, l', \text{ and } l''\) given above corresponding to Figure 3 are all equivalent.

It can be shown that, for any edge set \(E\), \(\partial E = 0 \iff E = E(l)\) for some cyclic list \(l\). We therefore make the following definition of abstract listed 2-polytope and can be sure that it allows for the most general type of polygon (i.e., possibly disconnected polygons with various components “wrapped around” many times, if desired).

**Definition 1.** An abstract listed 2-polytope \(P\) is equivalent to the following data.

i) A set \(S = \{1,\ldots,N\}\) of \(N\) distinct boundaries (lines) in the above sense.

ii) A cyclic list \(l = (i_1,\ldots,i_n)\) with each \(i_k \in S\).

### 2.1 Volumes of Abstract Listed 2-Polytopes

In line with [1], we say that the area \(A(l) = A((1,2,3))\) of the triangle defined by the list \(l = (1,2,3)\) is

\[A((1,2,3)) = \frac{1}{2} \frac{(123)^2}{(12P_0)(23P_0)(31P_0)} \equiv [123] \quad \text{(9)}\]
where $P_0^\alpha$ is a fixed reference boundary. We note that $[123]$ is totally antisymmetric in its arguments, in line with the swapping of orientations of the triangle that comes along with swapping any of the entries in the list $(1,2,3)$ (as in Figure 2). To generalize this notion of area to arbitrary polygons, we must note that there is a natural way to add two cyclic lists to get a cyclic list corresponding to the “superposition” of the two respective polygons. Namely, if $l_1 = (i_1, ..., i_n)$ and $l_2 = (j_1, ..., j_m)$ are two cyclic lists, then $l = (i_1, ..., i_n, i_1, j_1, ..., j_m, j_1)$ is defined to be the sum of $l_1$ and $l_2$, and it can be readily checked that

$$E(l) = E(l_1) + E(l_2).$$

(10)

It is also clear that the equivalence class of the sum of two lists depends only on the equivalence class of the two summands. For example, the list $l = (3,2,1,3,4,2,5)$ is in the same equivalence class as $l_1 + l_2$ where $l_1 = (3,2,1)$ and $l_2 = (3,4,2,5)$, by noting that $l_1 + l_2 \equiv (3,2,1,3,4,2,5,3) \sim (3,2,1,3,4,2,5)$.

The generalization of (9) to arbitrary cyclic lists is then clear once we impose the reasonable condition that

$$E(l_1) + E(l_2) = E(l) \Rightarrow A(l_1) + A(l_2) = A(l),$$

(11)

meaning that the area of the “superposition” of two listed 2-polytopes should be the sum of the areas of the individual polytopes. We then have that if $l = (i_1, ..., i_n)$ is a cyclic list, the area $A(l)$ of the corresponding 2-polytope is

$$A(l) = \sum_{k=1}^{n} [i_k i_{k+1} B]$$

(12)

where $B^\alpha$ is a fixed reference boundary, $A(l)$ is independent of our choice of $B^\alpha$, and the sum on $k$ is cyclic in the above sense. This can all be seen by noting that

$$l = (i_1, ..., i_n) \sim (i_1, i_2, B) + (i_2, i_3, B) + ... + (i_{(n-1)}, i_n, B) + (i_n, i_1, B)$$

(13)

for any $B^\alpha$.

Equation (12) gives the area of an arbitrary polygon in terms of a particular triangulation, and there are many different sums of $[ijk]$ objects that give an equivalent area for a given polygon. The formalism introduced in the next subsection will be completely oblivious to any choice of triangulation, and we simply use (12) to verify that our expression of the area of a polygon in terms of the $F_{ij}$ objects that we define is indeed valid. Once this is shown, though, we no longer need to concern ourselves with any particular triangulation.

2.2 2-Dimensional $F_{ij}$-Calculus

The key objects in the new formalism are objects of the form $F_{i_1,...,i_d}$ where $d$ is the dimension of the polytope under consideration. These are the objects that we will use
to express our volumes in a triangulation-independent manner, and which for reasons
that we will see should be viewed as the most basic, simple, and “atomic” objects of
the “volumes of polytopes” formalism, replacing the \([a_1...a_{d+1}]\) objects in this role. The
\(F_\alpha\) objects come from certain cohomological considerations that we will only allude to
in this note, leaving the details for a future note. In this note we will simply define
the \(F_\alpha\) objects as particular (and in general large) sums of the \([a_1...a_{d+1}]\) objects. The
simplicity of the resulting \(F_\alpha\)-calculus, along with its lack of reference to any arbitrary
choices (of, say, triangulations) makes unavoidable the shift in perspective from viewing
volumes of simplices as fundamental to viewing the \(F_\alpha\)’s as fundamental. Some of
the definitions we make here may appear arbitrary at first sight, this being purely a
consequence of our defining the \(F_i\) objects in terms of the \([a_1...a_{d+1}]\) objects (instead of vice-versa), and doing so without appealing
to the cohomological motivations for the definitions.

The success of the entire formalism relies almost solely on the fact that the \(F_\alpha\)
objects satisfy certain remarkable “cohomological” identities amongst themselves which
allow us to take the expression for a volume in terms of \(F_\alpha\)’s and algorithmically obtain
any triangulation we desire. We now turn to defining the \(F_\alpha\)’s in two dimensions and
seeing this cohomological identity explicitly.

We begin by quoting [1] and stating without proof the following:

\[
[124] + [234] + [314] = [123],
\]

(14)

for any boundaries \(Z^\alpha_1, Z^\alpha_2, Z^\alpha_3, Z^\alpha_4 \in \mathbb{CP}^2\). The natural interpretation of this result is
as the vanishing sum of the areas of four overlapping oriented triangles, and this was
indeed one of the starting points for the entire polytope picture as in, e.g. [5].

We now define the objects \(V[ij][kl]\) as follows:

\[
V[12][34] \equiv [123] - [124],
\]

(15)

and it is clear that each \(V[\cdot\cdot][\cdot\cdot]\) is antisymmetric under swapping the entries in a particular \(\cdot\cdot\). We note that by choosing \(B^\alpha\) to be 1, 2, 3, or 4 in \([12]\), one can show
that \(V[12][34] = A(l = (1, 4, 2, 3))\), so that each \(V[\cdot\cdot][\cdot\cdot]\) corresponds to the area of a
“quadrilateral”. Using \([14]\), one can show that each \(V[\cdot\cdot][\cdot\cdot]\) is also antisymmetric under swapping the \(\cdot\cdot\) brackets themselves, i.e., \(V[12][34] = -V[34][12]\). Thus all four entries
in \(V[\cdot\cdot][\cdot\cdot]\) are on equal footing, and we say that \(V[\cdot\cdot][\cdot\cdot]\) is totally antisymmetric to
mean both in the individual \(\cdot\cdot\) entries as well as in the \(\cdot\cdot\)’s themselves. It can also be
shown using \([14]\) that for any \(Z^\alpha_1, Z^\alpha_2, Z^\alpha_3, Z^\alpha_4, P^\alpha,\)

\[
V[12][3P] + V[12][P4] = V[12][34].
\]

(16)

This result reflects the fact that “slicing” a quadrilateral with a line allows one to express
the volume of the original quadrilateral as the sum of the two resulting quadrilaterals. Thus the \(V[\cdot\cdot][\cdot\cdot]\) objects have algebraic properties that make calculating with them
extremely easy, despite the fact that their “inner workings” are quite complex. In higher
dimensions the inner workings of the analogous objects are much more complex, but their algebraic properties are just as simple. This is the first departure from the standard way of computing these volumes—areas of quadrilaterals are in fact more fundamental than areas of triangles (though still less fundamental than the $F_{ij}$'s that we soon define).

We now define the main objects of our calculus and establish their most useful property.

**Definition 2.** Let $S = \{1, ..., N\}$ be a set of $N$ distinct boundaries, and let $Q^\alpha \in \mathbb{CP}^2$ be a fixed reference boundary. For each $i, j \in S$, we define

$$F_{ij} = \sum_{k \neq i, j}^N V[i][j][kQ].$$

Due to the antisymmetry in $i$ and $j$ of each $F_{ij}$, Definition 2 implicitly defines $\left(\begin{array}{c} N \\ 2 \end{array}\right)$ non-trivial functions. We also note that while each $F_{ij}$ depends on our choice of $Q^\alpha$, we suppress this dependence in our notation because we will soon see that the sums of $F_{ij}$'s that we will be interested in are independent of this choice. We now present the cohomological identity in two dimensions.

**Proposition 1.** Let $S = \{1, ..., N\}$ be a set of $N$ distinct boundaries and let $\{F_{ij}\}$ be as in Definition 2. Then for any $i, j, k \in S$,

$$\rho[iF_{jk}] = F_{ij} + F_{jk} + F_{ki} = 2[ijk].$$

Before proving this identity, we note a few of its important characteristics. First, it is very similar to the Čech cocycle condition as written in [6] for the first cohomology group (hence referring to it as the cohomological identity). This is no accident, but its detailed explanation will be given in a later note. Secondly, we note that the three terms on the left of (18) are individually highly “non-local” sums, since each $F_{ij}$ “knows about” all $N$ boundaries via its definition in Definition 2. Nonetheless, the particular combination in (18) reduces to the perfectly “local” expression of the volume of a triangle. Finally, we note that this result can be summarized by saying that a particular large sum of quadrilaterals (the largeness depending on the size of $N$) results in a two-fold covering of the triangle $[ijk]$. We now give the proof of this result.

**Proof of Proposition 1:**

$$F_{ij} + F_{jk} + F_{ki} = \sum_{l \neq i, j}^N V[ij][lQ] + \sum_{l \neq j, k}^N V[jk][lQ] + \sum_{l \neq k, i}^N V[ki][lQ]$$

$$= \sum_{l \neq i, j, k}^N (V[ij][lQ] + V[jk][lQ] + V[ki][lQ])$$

$$+ V[ij][kQ] + V[jk][iQ] - V[jQ][ki]$$

$$= 2[ijk]$$
Each individual $F_{ij}$ depends on our choice of reference boundary $Q^\alpha$ used to define it in (17), however the sums of the $F_{ij}$ objects that we will be interested in are independent of that choice. Namely, we have the following.

**Proposition 2.** Let $S = \{1, \ldots, N\}$ be a set of $N$ distinct boundaries, let $Q^\alpha$ and $Q'^\alpha$ be two reference boundaries, let $\{F_{ij}\}$ be as in Definition 2 using $Q^\alpha$, and let $\{F'_{ij}\}$ be as in Definition 2 using $Q'^\alpha$. Let $l = (i_1, \ldots, i_n)$ be a cyclic list with each $i_k \in S$, let $A = \sum_{k=1}^{n} F_{i_ki_{k+1}}$, and let $A' = \sum_{k=1}^{n} F'_{i_ki_{k+1}}$ where the sums on $k$ are cyclic. Then $A = A'$.

**Proof of Proposition 2.**

\[
A - A' = \sum_{k=1}^{n} (F_{i_ki_{k+1}} - F'_{i_ki_{k+1}})
\]

\[
= \sum_{k=1}^{n} \sum_{l \neq i_k, i_{k+1}} (V[i_ki_{k+1}][lQ] - V[i_ki_{k+1}][lQ'])
\]

\[
= \sum_{k=1}^{n} \sum_{l \neq i_k, i_{k+1}} V[i_ki_{k+1}][Q'Q]
\]

\[
= (N - 2) \sum_{k=1}^{n} V[i_ki_{k+1}][Q'Q]
\]

\[
= 0. \quad \square
\]

We now give the statement of the two-dimensional calculus.

**Theorem 1.** Let $S = \{1, \ldots, N\}$ be a set of $N$ boundaries, and let $l = (i_1, \ldots, i_n)$ be a cyclic list taking values in $S$. Let $A(l)$ be the area of the abstract listed 2-polytope defined by $l$, as in (12), with the reference boundary $B^\alpha$. Let $S' = S \cup \{B^\alpha\}$ be the set of boundaries $\{1, \ldots, N, B\}$, and let $\{F_{ij}\}$ be defined with respect to $S'$ as in Definition 2. Then $A(l) = \frac{1}{2} \sum_{k=1}^{n} F_{i_ki_{k+1}}$. In other words, the volume of the polygon defined by $l$ is equal to $\frac{1}{2}A$ as defined in Proposition 2.
Proof of Theorem \[1\]

\[
\frac{1}{2} \sum_{k=1}^{n} F_{ik} i_{k+1} = \frac{1}{2} \sum_{k=1}^{n} (F_{ik} i_{k+1} + F_{ik} B + F_{ik} B_{ik+1})
\]

\[
= \frac{1}{2} \sum_{k=1}^{n} (F_{ik} i_{k+1} + F_{ik} B + F_{ik} B)
\]

\[
= \frac{1}{2} \sum_{k=1}^{n} 2[i_{k+1} B]
\]

\[
= \sum_{k=1}^{n} [i_{k+1} B] = A(l)
\]

where in the first equality we simply added \(0 = f_{ik} B + f_{ik} B_{ik+1}\) to each term in the sum. In the second equality we took advantage of the fact that the sum on \(k\) is cyclic to relabel the index on the third term, and in the third equality we used (18). □

The two-dimensional calculus for calculating the area of an abstract listed 2-polytope is therefore the following. Take the cyclic list \(l = (i_1, \ldots, i_n)\) which defines our 2-polytope, and go down the list adding up \(F_{ik} i_{k+1}\) at each step—there is no need to think about any triangulation. In practice, we never have to make reference to the actual structure of the \(F_{ij}\)'s, because we can just use (18) to simplify our resulting sum of \(F_{ij}\)'s into a sum of areas of triangles, if we wish. By making different choices of simplification, we can recover any possible triangulation. Thus, our calculus does much more than simply recover the particular triangulation used in (12), and Theorem \[1\] simply says that any triangulation obtained from this sum of \(F_{ij}\)'s will correspond to the correct area. We also note that the sum \(\sum_k F_{ik} i_{k+1}\) is a sum over vertices, due to our definition of 2-polytopes in terms of cyclic lists, and if a vertex is “spurious”, as for example the \(\{3, 4\}\) vertex in the list \((1, 2, 3, 4, 3)\), then the corresponding \(F_{ij}\)’s immediately cancel. Thus, any \(F_{ij}\)’s that remain in the sum after cancellation at the \(F_{ij}\) level label “physical”, i.e. “genuine” vertices via its subscripts. Moreover, the expression of the area of a polygon in terms of \(F_{ij}\)’s is independent of any triangulation, and can be used together with (18) to algorithmically obtain any valid triangulation. Example \[1\] below illustrates some of the utility of the two-dimensional calculus.

We quickly note that in Theorem \[1\] we introduced the set of boundaries \(S' = \{1, \ldots, N, B\}\) solely to make the expression \(f_{ijB}\) well-defined. In practice, however, we only need to use \(f_{ij}\) objects whose subscripts take values in \(S = \{1, \ldots, N\}\). We can extend this set of boundaries however we wish since Proposition \[1\] which is the main result for our calculus, is independent of the set of boundaries we choose. This can be seen explicitly in the second equality of the proof of this proposition, where the sum vanishes regardless of what \(N\) is. Equivalently, we could also choose \(B\) to be within the

\[4\]By the phrase “canceling at the \(F_{ij}\) level” we simply mean making all cancellations that do not require “unravelling” the definition of the \(F_{ij}\)’s.
set $S$ itself, since $A(l)$ does not depend on our choice of $B^\alpha$. The example in Appendix A shows this fact in practice, as well as gives some geometric insight into what this calculus is all about.

**Example 1.** Consider the cyclic list $l = (1, 4, 2, 3)$, denoting the polygon depicted in Figure 4.

![Figure 4](image)

Here, $N = 4$ and so we construct the $\binom{4}{2}$ objects $\{F_{ij}\}$. Theorem 1 tells us that

$$A(l) = \frac{1}{2}(F_{14} + F_{42} + F_{23} + F_{31}).$$

(19)

By simplifying this expression in two different ways using (18), we get

$$A(l) = [123] - [124],$$

(20)

and as well as

$$A(l) = [314] + [423],$$

(21)

which we know from (14) to be equal. We have therefore recovered two different triangulations for the same 2-polytope. Suppose however that we were given the expression

$$(I) \equiv [145] + [425] + [236] + [316] + [156] + [265]$$

(22)

for some boundaries $\{1, ..., 6\}$. Using (18) in reverse, we can expand out each $[ijk]$ in terms of $F_{ij}$’s and make the obvious cancellations to get

$$(I) = \frac{1}{2}(F_{14} + F_{42} + F_{23} + F_{31} + F_{15} + F_{56} + F_{61} + F_{26} + F_{65} + F_{52})$$

$$= \frac{1}{2}(F_{14} + F_{42} + F_{23} + F_{31}) = A(l).$$

(23)
Thus, \((I)\) is seen to be both independent of the boundaries 5 and 6, as well as equal to \(A((1, 4, 2, 3))\), both of which are highly non-obvious facts when presented with (22). To go the other direction and recover the triangulation \((I)\) from (19) involves simply and repetitively adding zero in the form \(F_{ij} + F_{ji}\) in a straightforward way.

In the triangulations (20), (21), and (22), we have introduced spurious vertices into our triangulations. For example, in order to move from (19) to (20) we had to add and subtract \(F_{12}\), corresponding to the vertex \(\{1, 2\}\) which is not a part of the underlying polygon. An interesting class of triangulations are those that do not introduce any such spurious vertices, and the \(F\)-calculus can algorithmically recover these as well. For example, let us define \(L\) to be the unique line connecting vertex \(\{1, 4\}\) to vertex \(\{2, 3\}\), as in Figure 5.

Figure 5

Then the areas \([14L]\) and \([23L]\) of the two triangles defined by these three lines both vanish (for similar reasons to those discussed after (33) below), so we have

\[F_{14} + F_{4L} + F_{L1} = 2[14L] = 0\]  \hspace{1cm} (24)

and

\[F_{23} + F_{3L} + F_{L2} = 2[23L] = 0.\]  \hspace{1cm} (25)

In particular, \(F_{14} = F_{1L} + F_{L4}\) and \(F_{23} = F_{2L} + F_{L3}\), so that we can rewrite (19) as

\[A(l) = \frac{1}{2}(F_{1L} + F_{L4} + F_{42} + F_{2L} + F_{L3} + F_{31})\]  \hspace{1cm} (26)

We note that all of the vertices labelled by the subscripts of the \(F\)’s in (26) are still physical (i.e., not spurious) vertices. We can now use (18) (without introducing any new vertices) to find

\[A(l) = [L31] + [L42],\]  \hspace{1cm} (27)

as expected from Figure 5.

Thus, any triangulation of \(A(l)\) can be recovered from (19) by adding zero and/or introducing new boundaries in various ways, and then simplifying the resulting sum.
using (18) in various ways. The equality of any two triangulations can be checked in a similarly algorithmic way. It is also clear that we can recover the cyclic list itself from any triangulation of the underlying polygon (assuming we know what the genuine boundaries are, which is the case in e.g. BCFW recursion). Namely, given any triangulation, we express it in terms of the $F_\ldots$ objects using (18), and simplify the resulting sum until the subscripts of all of the $F_\ldots$’s take values in the given set of genuine boundaries. We will then always be left with the expression (19), which is readily seen to be obtained from the list $(1, 4, 2, 3)$. This process is clearly not limited to the particular list $l = (1, 4, 2, 3)$, but rather carries directly over for any list $l = (i_1, ..., i_n)$.

To summarize, the $F_\ldots$ calculus allows us to express the area of an arbitrarily complicated polygon in terms of objects whose subscripts label the genuine vertices, giving an expression that is independent of any triangulation. From this expression one can algorithmically obtain any triangulation by adding in spurious vertices and/or boundaries and using the cohomological identity (18). And, since a relevant cyclic list can be recovered from any particular triangulation, the benefits of the $F_\ldots$ calculus can be uncovered if one is given either a cyclic list or a particular triangulation. We now turn our attention to developing the analogous formalism in higher dimensions while preserving all of these benefits.

3 Abstract Listed 3-Polytopes

In order to develop the three-dimensional formalism we first seek to make precise what exactly we mean by “3-polytope”. We also want to be able to do this in terms of cyclic lists, as these are what our three-dimensional $F_\ldots$-calculus will be based on. To do this, we view a 3-polytope as being a set of 2-polytopes “glued together” in such a way as to make the “boundary of the boundary” of the polytope vanish.

We consider a set $S = \{1, ..., N\}$ of $N$ distinct boundaries $Z_i^\alpha \in \mathbb{CP}^3$, $1 \leq i \leq N$, where now each $Z_i^\alpha$ determines a 2-plane—or linearly embedded $\mathbb{CP}^2$—in the dual space. The intersection of any two distinct planes gives a line, and the intersection of any three gives a point. Accordingly, an oriented edge is now of the form $[i(jk)l]$ with $i, j, k, l \in S$, corresponding to the instructions $\{i, j, k\} \rightarrow \{j, k, l\}$, meaning “go from vertex $\{i, j, k\}$ to $\{j, k, l\}$ along the line defined by the intersection of the lines whose labels are common to the two vertices (in this case the line $j \cap k$)”. The same additive structure can be placed on these formal objects, where now we can only add two oriented edges when both of their parenthetical entries are the same. Thus, for example, $[1(23)4] + [1(34)5]$ is fully simplified whereas $[1(23)4] + [4(23)5] = [1(23)5]$. To see how cyclic lists come in, we motivate our discussion by considering Figure 6 which depicts the intersection structure of four boundaries $\{1, 2, 3, 4\}$ with a particular orientation. We clearly want to view this as an oriented 3-simplex.

We can describe this object using four cyclic lists, one for each face. Additionally, since the 2-polytope sitting on the $i^{th}$ face will, by definition, have the entry $i$ in each vertex, we need to change the instructions that our cyclic lists determine. Namely, with
Figure 6

\[ N \text{ boundaries in play, we get } N \text{ cyclic lists } \{ l_i = (j_{i1}, \ldots, j_{im_i}) \}_{1 \leq i \leq N} \text{ with each } j_{ik} \in S \text{ and where } n_i \text{ is simply the length of the } i^{th} \text{ list. Each list } l_i \text{—defining the 2-polytope on the } i^{th} \text{ face—defines the edge set } E_i \equiv E(l_i) \text{ as follows:} \]

\[
E(l_i) \equiv \sum_{k=1}^{n_i} [j_{i(k-1)}(ij_{ik})j_{i(k+1)}], \tag{28}
\]

with the sum on \( k \) cyclic. Thus, the four lists corresponding to Figure 6 are

\[
l_1 = (2, 3, 4) \tag{29}
\]
\[
l_2 = (1, 4, 3)
\]
\[
l_3 = (1, 2, 4)
\]
\[
l_4 = (1, 3, 2),
\]

and the edge set \( E_1 \), for example, is

\[
E_1 = [2(13)4] + [3(14)2] + [4(12)3], \tag{30}
\]

which is readily seen to agree with what we want to interpret as the "triangle" sitting on the boundary 1. We obtained the lists in (29) by orienting one of the boundary 2-faces and then orienting the rest in such a way that the "boundary of the boundary" vanished. Thus these four lists are not independent, but rather satisfy certain "gluing together" constraints. To obtain these constraints more generally, we introduce the notion of an edge set \( E_{i,s} \) which is the restriction of \( E_i \) to the boundary \( s \in S \) as follows:

\[
E_{i,s} \equiv \sum_{j_{ik}=s}^{n_i} [j_{i(k-1)}(ij_{ik})j_{i(k+1)}] = \sum_{j_{ik}=s}^{n_i} [j_{i(k-1)}(is)j_{i(k+1)}]. \tag{31}
\]

Thus, for example, with respect to the lists in (29) we have \( E_{1,2} = [4(12)3] \). Since the "boundary of the boundary" of a 3-polytope should be the sum of its oriented edges,
an abstract listed 3-polytope must be defined by lists \{l_i\} such that \(\sum_{i=1}^{N} E_i = 0\). However, since edges in \(E_{i;k}\) can only be cancelled by other edges in \(E_{i;k}\) or by edges in \(E_{k;i}\) (since the additive structure requires the parenthetical entries to be the same for simplification/cancellation to occur), and since we have

\[
\sum_{i=1}^{N} E_i = \sum_{i} \sum_{k \neq i} E_{i;k} = \frac{1}{2} \sum_{i} \sum_{k \neq i} (E_{i;k} + E_{k;i}),
\]

we are motivated to make the following definition.

**Definition 3.** An abstract listed 3-polytope \(P\) is equivalent to the following data.

i) A set \(S = \{1, ..., N\}\) of \(N\) distinct boundaries (2-planes) in the above sense.

ii) \(N\) cyclic lists \(\{l_i = (j_{i1}, ..., j_{im_i})\}_{1 \leq i \leq N}\) with each \(j_{ik} \in S\), such that for all \(i, k \in S\), \(E_{i;k} = -E_{k;i}\).

For example, with respect to the lists in (29), we have \(E_{2;1} = [3(12)4] = -[4(12)3] = -E_{1;2}\), and the rest can be checked explicitly though we know the conditions will all be satisfied since the lists were derived from Figure 6 in such a way as to guarantee this fact.

We note that abstract listed 3-polytopes can be just as “disconnected” and general as our 2-polytopes are. Each two face of a 3-polytope can be an arbitrarily complicated 2-polytope, and in particular we have no notion of convexity since the boundaries of a 3-polytope can be “disconnected”.

### 3.1 3D Volumes and the \(F_{ijk}\)-Calculus

For the sake of brevity we carry our discussion of volumes of 3-polytopes over from the two-dimensional case strictly by analogy, though more detailed treatments do exist. All proofs in this section are left out as they are similar in approach to the two-dimensional case, though we will briefly mention the main differences (when they exist) between the proofs in different dimensions. We begin by defining the volume of a 3-simplex:

\[
A_{3-\text{simplex}} = \frac{1}{6} \frac{\langle 1234 \rangle^3}{\langle 123P_0 \rangle \langle 234P_0 \rangle \langle 341P_0 \rangle \langle 412P_0 \rangle} \equiv [1234].
\]

We note—as it will be important later—that if the four 2-planes defined by \(Z_1^a, ..., Z_4^a\) intersect in a common point, then the \(\{Z_i^a\}_{1 \leq i \leq 4}\) are linearly dependent and so \([1234]\) vanishes\(^5\). The volume \(A(\{l_i\})\) of a general abstract listed 3-polytope defined by the lists \(\{l_i = (j_{i1}, ..., j_{im_i})\}\) is defined to be

\[
A(\{l_i\}) \equiv \frac{2}{3!} \sum_{i=1}^{N} \sum_{k=1}^{n_i} [i j_{ik} j_{i(k+1)}] B, \quad (34)
\]

\(^5\)This is in line with the interpretation of \([1234]\) as the volume of the simplex defined by four planes, since in this case the simplex collapses to a point with zero volume.
which can be shown to be independent of the reference boundary \( B^\alpha \) and where the sum on \( k \) (but not on \( i \)) is cyclic. This is simply a particular triangulation of our polytope, where the prefactor \( \frac{2}{3!} \) comes from the fact that, due to the constraints on the lists \( \{ l_i \} \), we are summing over each simplex once for every even permutation of \( i, j_k, j_{i(k+1)} \) in \( (34) \). We later use \( (34) \) to confirm that our \( F_{ijk} \)-calculus obtains the correct volume, though as in the two-dimensional case we will see that this calculus is completely independent of any particular triangulation.

We now introduce the three-dimensional analogues of the \( V[\cdot][\cdot][\cdot] \) objects. For any six boundaries \( 1, \ldots, 6 \), we define

\[
V[12][34][56] \equiv [1235] - [1236] - [1245] + [1246],
\]

and using the three-dimensional analogue of \( (14) \), one can show that \( V[\cdot][\cdot][\cdot] \) is fully antisymmetric both in its individual \([\cdot] \) entries as well as under swapping the \([\cdot] \)'s themselves. We define the brackets \( \{ \ldots \} \) to be one-half times the (non-normalized) antisymmetrization of the labels that are excluded from the vertical bars. For example,

\[
V[\{ij\}]\{k\}P\{QR\} \equiv V[\{ij\}]\{k\}P\{lQ\} + V[\{jk\}]\{iP\}\{lQ\} + V[\{ki\}]\{jP\}\{lQ\}.
\]

and

\[
V[\{ij\}]\{k|P]\{l|Q\} \equiv V[\{ij\}]\{k|P\}\{lQ\} - V[\{jk\}]\{iP\}\{lQ\} + V[\{kl\}]\{i|P\}\{jQ\} - V[\{li\}]\{j|P\}\{kQ\}.
\]

Then, given a set \( S = \{1, \ldots, N\} \) of \( N \) distinct boundaries and two fixed reference boundaries \( P^\alpha \) and \( Q^\alpha \), we define the following objects for each \( i, j, k \in S \):

\[
F_{ijk} \equiv \sum_{l \neq i, j, k}^{N} V[\{ij\}]\{k\}P\{lQ\}.
\]

Each \( F_{ijk} \) is clearly antisymmetric in its subscripts, and they can be straightforwardly shown to individually be independent of \( P^\alpha \) (though they are dependent on \( Q^\alpha \), as in the two-dimensional case). We then have the following cohomological identity.

\[
\rho[iF_{jkl}] \equiv F_{ijk} - F_{jkl} + F_{kli} - F_{lij} = 3![ijkl]
\]

and we again see the resemblance to the Čech cocycle condition now for the second cohomology group, the simplification of “non-local” terms to a purely local one, and the many-fold covering of the simplex.

As in the two-dimensional case, one can show that the sums of the \( F_{\ldots} \) objects that we will be interested in are guaranteed to be independent of the reference boundary \( Q^\alpha \) as well, and we also have the following result which gives the three-dimensional \( F_{ijk} \)-calculus. With \( A(\{l_i\}) \) being the volume of the abstract listed 3-polytope defined by the lists \( \{ l_i \} \) as in \( (34) \), one can show that

\[
A(\{l_i\}) = \frac{2}{(3!)^2} \sum_{i=1}^{N} \sum_{k=1}^{n_i} F_{ijk,j_{i(k+1)}},
\]

18
where the sum on $k$ (but not on $i$) is cyclic. The three-dimensional calculus is therefore to go along each cyclic list (one for each bounding plane) and add up the corresponding $F_{ijk}$ at each vertex. In other words, we simply go through the two-dimensional calculus on each face with the $F_{ij}$ objects replaced by $F_{ijk}$ objects in the appropriate way.

The proof of (40) is similar to the two-dimensional case, though relies heavily on the constraints imposed upon the lists by their defining a 3-polytope, and thus does involve some added care. For brevity, however, we leave this proof out since it is the utility of this formalism that we want to focus on. Namely, we again have an expression of the volume in terms of objects that label the genuine vertices of the underlying polytope, and we obtain this expression without any reference to a triangulation. We also have a cohomological identity that allows us to recover any triangulation that we want in the same manner as in two dimensions. Finally, we note that the same type of double sum is performed on the $F_{ijk}$’s in (40) as is performed on the $[abcd]$ objects in (34). This does not mean that the respective summands behave similarly, for proving the equivalence of these two sums is non-trivial, but it does mean that we gain all of the benefits of viewing the $F_{ijk}$’s as the atomic objects of the formalism without any added cost, and in particular without the cost of introducing more terms in the sums. We leave one three-dimensional example for Appendix B.

4 Abstract Listed 4-Polytopes

We now seek to define 4-polytopes in terms of cyclic lists, as we would then expect a useful $F_{ijkl}$-calculus to be obtained thereafter. The key observation in this regard is to view 4-polytopes as a set of 3-polytopes—one for each hyper-face—“glued-together” along their two-dimensional faces in such a way as to make the “boundary of the boundary” of the polytope vanish.

As usual, we suppose we have a set $S = \{1, ..., N\}$ of $N$ distinct elements $Z_{i1}, ..., Z_{iN} \in \mathbb{CP}^4$, defining $N$ distinct 3-planes (linearly embedded $\mathbb{CP}^3$’s) in the dual space, and we refer to $S$ as the set of boundaries. The intersection of any two distinct 3-planes determines a 2-plane (linearly embedded $\mathbb{CP}^2$), the intersection of any three distinct 3-planes determines a (complex projective) line denoted by $i \cap j \cap k$, and the intersection of any four distinct 3-planes determines a point denoted by $\{i, j, k, l\}$ and called a vertex. An oriented edge is now of the form $[i(jkl)m]$ and denotes the abstract instructions $\{i, j, k, l\} \rightarrow \{j, k, l, m\}$, meaning to go from the vertex $\{i, j, k, l\}$ to the vertex $\{j, k, l, m\}$ along the line defined by the intersection of the three 3-planes labelled by the three labels common to the two vertices.

We now say that an abstract listed 4-polytope $P$ is determined by $N^2$ cyclic lists $\{l_{ij} = (k_{i1}, ..., k_{ijn})\}$ where each $k_{ijl} \in S$ and $n_{ij}$ is the length of the list $l_{ij}$. We view the list $l_{ij}$ as the list defining the 2-polytope obtained by first restricting the 4-polytope $P$ to the three-dimensional polytope $P_i$ sitting on the $i^{th}$ face, and then restricting $P_i$ to the $j^{th}$ face to get the 2-polytope $P_{ij}$. The lists $l_{ij}$ labelled by the same boundary are
by convention empty.

We define the edge sets \( E_{ij} \equiv E(l_{ij}) \) in the usual way:

\[
E_{ij} = \sum_{l=1}^{n_{ij}} [k_{ij(l-1)}(ijk_{ij(l)})k_{ij(l+1)}],
\]

with the sum on \( l \) cyclic, so that \( E_{ij} \) corresponds to the edge set of the 2-polytope \( P_{ij} \).

We then have that the 3-polytope \( P_i \) is defined by the \( (N-1) \) lists \( \{l_{ij}\} \) with \( j \in S - \{i\} \), and accordingly by the edge sets \( E_{ij} \) with \( j \in S - \{i\} \). Namely, we simply fix the first subscript. Requiring that \( P_i \) is indeed a 3-polytope for each \( i \in S \), we simply carry over the 3-dimensional constraint which is that for all \( i,j,k \in S \),

\[
E_{ij};k = - E_{ik;j}.
\]

In order to view these 3-polytopes as being properly “glued” along their bounding 2-polytopes, we need to impose that for all \( i,j \in S \),

\[
E_{ij} = - E_{ji}.
\]

Combining these two constraints on the edge sets \( \{E_{ij}\} \), we are motivated to make the following definition of abstract listed 4-polytopes.

**Definition 4.** An abstract listed 4-polytope \( P \) is equivalent to the following data.

i) A set \( S = \{1,...,N\} \) of \( N \) distinct boundaries (3-planes) in the above sense.

ii) \( N^2 \) cyclic lists \( \{l_{ij} = (k_{ij1},...,k_{ijn_{ij}})\}_{1 \leq i,j \leq N} \) with each \( k_{ijl} \in S \), such that for all \( i,j,l \in S \),

\[
E_{ij;l} = (-1)^{[\sigma]} E_{\sigma(ij;l)} \text{ where } \sigma \in S_3 \text{ is any permutation of 3 objects}.
\]

At this point it is now clear how to extend our definition of abstract listed polytopes to any dimension, and we do so explicitly in Appendix C.

### 4.1 4D Volumes and the \( F_{ijkl} \)-Calculus

In what follows we suppose the cyclic lists \( \{l_{ij} = (k_{ij1},...,k_{ijn_{ij}})\} \) with \( k_{ijl} \in S = \{1,...,N\} \) define an abstract listed 4-polytope \( P \). We denote by \([12345]\) the volume of the 4-simplex bounded by the boundaries \( 1,...,5 \), which is given as the obvious analogue of \([33]\). We then define the volume \( A(\{l_{ij}\}) \) of \( P \), an arbitrary abstract listed 4-polytope, to be

\[
A(\{l_{ij}\}) = \frac{2}{4!} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{l=1}^{n_{ij}} [ijk_{ij(l+1)}j] B_{\alpha}
\]

for some reference boundary \( B^\alpha \) (though it can be shown that \( A(\{l_{ij}\}) \) is independent of our choice of \( B^\alpha \)) and where the sum on \( l \) (but not on \( i \) or \( j \)) is cyclic. We then define the four-dimensional analogue of the \( V[\cdot][\cdot][\cdot][\cdot] \) objects, where for any eight boundaries \( 1,...,8 \) we have

\[
V[12][34][56][78] = [12357] - [12358] - [12367] + [12368] - [12457] + [12458] + [12467] - [12468].
\]
and we define our $F_{ijkl}$ objects as follows:

$$F_{ijkl} = \sum_{m \neq i,j,k,l}^N V[\{ij\}|k|P_1||l|P_2||mQ]$$

for some reference boundaries $P_1^\alpha$, $P_2^\alpha$, and $Q^\alpha$. It can be shown that each $F_{ijkl}$ is individually independent of our choice of $P_1^\alpha$ and $P_2^\alpha$, but is dependent on our choice of $Q^\alpha$. It can also be shown, though, that the sum

$$\sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N F_{ijkl} n_{ij}$$

with the sum on $l$ (but not on $i$ or $j$) cyclic in the usual sense, is independent of our choice of $Q^\alpha$. Indeed, our four-dimensional calculus carries through just as it does in the lower dimensions with the following result, with $A(\{l_{ij}\})$ as in (42):

$$A(\{l_{ij}\}) = \frac{2}{(4!)^2} \sum_{i=1}^N \sum_{j=1}^N \sum_{l=1}^N F_{ijkl} n_{ij}$$

The proof of this result relies heavily on the constraints placed on the cyclic lists in order for them to form a genuine 4-polytope, but instead of focusing on the proof we will instead focus on the utility of this result. We first note that it is manifestly dependent only on the vertices of the underlying polytope, via the subscripts of each $F_{...}$ object, just as in the two- and three-dimensional cases. Moreover, using the following key result (whose proof we omit):

$$\rho[iF_{ijklm}] \equiv F_{ijkl} + F_{jklm} + F_{klmi} + F_{lmi} + F_{mijk} = 4![ijklm],$$

we can obtain any triangulation we desire by using (47) to simplify (46) in various ways, just as we could in the lower dimensions.

### 4.1.1 Lists from Triangulations

We have seen how we can express our volumes in a triangulation-independent and manifestly vertex-dependent way once we know the cyclic lists. However, since BCFW recursion gives us particular triangulations and not a tabulation of the cyclic lists defining a polytope, it is worthwhile to see how we can extract the lists from any particular triangulation. By doing this, we find that once we are given any particular triangulation, we can recover all of the information about the polytope—as well as any of its lower dimensional boundaries—in a completely algebraic and algorithmic way. This is all best seen via an example.

We consider a 4-simplex with the obvious triangulation being simply $[12345]$. Via (47), we see that

$$A(\{l_{ij}\}) = [12345] = \frac{1}{4!}(F_{1234} + F_{2345} + F_{3451} + F_{4512} + F_{5123}),$$

as in (48).
where we do not yet know the lists \( \{ l_{ij} \} \) defining the 4-simplex. We do know from (46), however, that (48) must be a sum over cyclic lists. Therefore, for example, by writing all of the \( F_{ijkl} \) objects in (48) with 1 as the left-most subscript and 2 as the second-to-left-most subscript (while keeping track of relative minus signs via the total antisymmetry of the \( F_{...} \)'s), we immediately read off that \( l_{12} = (3, 4, 5) \). The other lists can be read off similarly.

It is clear that if one is solely interested in the volume of a particular simplex, there is no need for obtaining all of the cyclic lists or for expanding the volume out as a sum of \( F_{...} \)'s—one simply would simply write \([abcde]\). However, the process of reading off cyclic lists from a particular triangulation allows for the full utility of the \( F_{...} \)-calculus for arbitrarily complex polytopes in a straightforward and algorithmic way. The \( F_{...} \) formalism has benefits that the formalism of calculating solely with \([abcde]\) objects lacks. Thus, even though the simplex itself appears more complicated in the \( F_{...} \) formalism, this is simply a product of the fact that the simplex is not the most fundamental object for calculating volumes of polytopes. It is in this sense that we are taking one step back in order to take more steps forward. In higher dimensions and/or for very complicated polytopes, the utility of the \( F_{...} \) formalism is most apparent.

4.1.2 Going to the Boundary

One of the benefits of using the cyclic list formalism in dimensions higher than two is that we can readily obtain the information (i.e., the cyclic lists and the volumes) of any of the lower dimensional boundary polytopes. For example, suppose we are given an abstract listed 4-polytope \( P \) defined by the lists \( \{ l_{ij} \} \) with entries in \( S = \{1, ..., N\} \). The boundaries \( \{ Z_i^a \} \) are all elements of \( \mathbb{CP}^4 \) and therefore have five homogeneous coordinates. Accordingly, we define the volume of a 3-simplex defined by the boundaries \( i,j,k,l \) restricted to the boundary \( I \in S \) as follows:

\[
[ijkl]_I \equiv \frac{1}{6} \frac{\langle ijklI \rangle^3}{\langle ijkIP_0 \rangle \langle jklIP_0 \rangle \langle kliIP_0 \rangle \langle lijIP_0 \rangle}.
\] (49)

The objects \( [ijkl]_I \) satisfy all of the same algebraic properties as the \( [ijkl] \) themselves, so our three-dimensional calculus carries directly through by defining the new objects \( \{ F_{ijk}^I \} \) where every \( [ijkl] \) is simply replaced by \( [ijkl]_I \). Then, if we want to calculate the volume of the 3-polytope sitting on the \( I^{th} \) boundary of \( P \), we simply apply the three-dimensional calculus to the lists \( \{ l_{ij} \} \) with \( j \in S - \{ I \} \) using the \( F_{ijk}^I \) objects. The same can be said about obtaining the area of any boundary 2-polytope by defining the area of a 2-simplex defined by the boundaries \( i,j,k \) and restricted to the \( I^{th} \) and then to the \( J^{th} \) face of \( P \) as

\[
[ijk]_{IJ} \equiv \frac{1}{2} \frac{\langle ijkJJ \rangle^2}{\langle ijJIP \rangle \langle jkJJP \rangle \langle kiJJP \rangle}.
\] (50)

The \( [ijk]_{IJ} \) objects satisfy all of the algebraic properties that the \( [ijk] \) objects satisfy, so by defining \( F_{ij}^{IJ} \) in the obvious way and applying the two-dimensional calculus to the
list $l_{i,j}$, we get the area of the polygon $P_{i,j}$. In the next section we see how all of the benefits of the $F$-formalism applies to the case of the polytope $M_n^{NMHV}$ discussed at the beginning of this note.

4.2 Applications to $M_n^{NMHV}$

Let us first use (47) to obtain a new, manifestly triangulation-independent, and solely vertex-dependent expression for $M_n^{NMHV}$. We write $0 = *$ in (1). Then

$$M_n^{NMHV} = \frac{1}{2} \sum_{i,j=1}^{n} [0i(i+1)j(j+1)]$$

$$= \frac{1}{2} \cdot \frac{4!}{4!} \sum_{i,j=1}^{n} (F_{0i(i+1)j} + F_{i(i+1)j(j+1)} + F_{i(i+1)j(j+1)}0i + F_{j(j+1)0i} + F_{j(i+1)0i})$$

$$= \frac{1}{2} \cdot \frac{4!}{4!} \sum_{i,j=1}^{n} F_{i(i+1)j(j+1)}$$

since any term with a $0$ subscript cancels after relabeling of the $i$’s and $j$’s, which we are permitted to do due to the cyclicity of the sum. Thus the dependence on the reference boundary $0$ manifestly drops out, while the fact that the underlying polytope only has vertices of the form $\{i, i+1, j, j+1\}$ remains manifest due to the fact that these are the only forms of the subscripts of the $F$’s. We can use (47) to obtain any valid triangulation we desire from (53), and conversely any triangulation obtained from (53) in this way is guaranteed to give the correct volume. We will see this explicitly when we apply this formalism to one of the bounding 3-polytopes of this 4-polytope below.

Since the general $n$-point $M_n^{NMHV}$ polytope would give rise to $n(n-1)$ non-trivial lists, let us focus on the $n = 6$ case and extract only those lists that describe the 3-polytope $P_2$ sitting on the boundary 2. We readily find from (53)

$$M_6^{NMHV} = \frac{1}{4!} (F_{1234} + F_{2345} + F_{2356} + F_{5612} + F_{6123} + F_{3456} + F_{4561} + F_{6134} + F_{1245})$$

from which we can read off the following lists:

$$l_{21} = (4, 3, 6, 5)$$
$$l_{22} = (1, 4, 5, 6)$$
$$l_{24} = (3, 1, 5)$$
$$l_{25} = (4, 1, 6, 3)$$
$$l_{26} = (5, 1, 3).$$
Using the three-dimensional calculus on these five lists with the $F_{ijk}$ objects, and denoting the volume of $P_2$ by $A_2(\{l_{2i}\})$, we find

$$A_2(\{l_{2i}\}) = \frac{1}{18} \sum_{i=1}^{6} \sum_{l=1}^{n_{2i}} F_{il(l+1)}^2$$

$$= \frac{1}{6} (F_{143}^2 + F_{136}^2 + F_{165}^2 + F_{154}^2 + F_{345}^2 + F_{356}^2).$$

(56)

We now introduce the spurious vertex $\{1, 3, 5\}$ (restricted to the boundary $2$) by adding and subtracting $F_{513}^2$ to (56) and then use (47) to find

$$A_2(\{l_{2i}\}) = [1365] - [1345],$$

(57)

which we note is in line with the results of [5]. We note that we did not need to know which spurious vertex to introduce. Rather, we could simply begin applying (47) to (56) one $F_{...}$ object at a time, picking up remainder terms at each step, where each remainder term that does not immediately cancel at the $F_{...}$ level tells us that we have introduced a spurious vertex. In this way the process of moving from a sum of $F_{...}$’s to a sum of volumes of simplices (possibly with spurious vertices) is purely algorithmic.

4.2.1 Triangulations Without Spurious Vertices

We can obtain infinitely many triangulations from (56) by allowing ourselves to introduce spurious boundaries as well as spurious vertices, by for example adding and subtracting $F_{127}^2$ where the boundary $7$ is some new reference boundary and then simplifying using (47). However, we can also be more careful and introduce spurious boundaries in such a way that we can triangulate our space without introducing spurious vertices. This was done in [1] for the bounding 3-polytope $P_2$ (both for the $n = 6$ as well as general $n$ cases) as well as in four dimensions for the full $M_{NMHV}$ polytope. The procedure in [1] relies heavily on geometric insight to “chop up” the various polytopes into simplices without introducing new vertices. Here, we focus solely on the $n = 6$ bounding 3-polytope $P_2$ and recover the particular spurious-vertex-free triangulation obtained in [1] in a purely algebraic fashion. In the process we allude to how this procedure can be generalized, thus reducing the problem of finding triangulations without spurious vertices to one of algorithmic algebra.

The general procedure is schematically as follows. We begin with a sum of $F_{...}$ objects denoting the physical vertices of the underlying polytope. Suppose there are $m$ such vertices. We pick one physical vertex arbitrarily—corresponding to a particular $F_{...}$ term—and “triangulate it away” by choosing three (in three dimensions) other $F_{...}$ objects whose subscripts share precisely two of the three labels of the original $F_{...}$. This corresponds to picking a total of four non-coplanar vertices, with the initially chosen vertex being connected by edges of the underlying polytope to the other three vertices. We then (if necessary) define the plane through these latter three vertices, add the volume of the simplex defined by this (possibly new) plane and the four vertices we have
chosen, and write out the remaining $F_{\ldots}$ terms. As we will see by example, this gives the original volume of the polytope with $m$ physical vertices as the sum of a simplex (with only physical vertices) and a polytope with $m - 1$ physical vertices in general depending on the plane just defined. The triangulation that one is left with is not unique as there are many arbitrary choices made along the way (as in, for example, which vertex to “triangulate away” at any given step), but it will be guaranteed to have no spurious vertices. Let us see how this works via an example.

Focusing on the bounding 3-polytope $P_2$ of $M_{6}^{NMHV}$, we suppress the 2 sub/superscript and simply note that all of the following takes place within the three-dimensional restriction to the boundary $Z_2$. From (56) we can arbitrarily choose to first triangulate away the vertex $\{1, 3, 6\}$. We also see that $\{1, 3, 4\}, \{1, 5, 6\}$, and $\{3, 5, 6\}$ are all physical vertices sharing precisely two labels with $F_{136}$. Accordingly, we define the plane $P_1^\alpha$ to be the plane through these three vertices:

$$P_1^\alpha \equiv \text{plane through } \{1, 3, 4\}, \{1, 5, 6\}, \{3, 5, 6\}.$$  \hspace{1cm} (58)

We then see that certain vertices are labelled in different equivalent ways, as for example $\{1, 3, 4\} = \{P_1, 1, 3\} = \{P_1, 3, 4\} = \{P_1, 1, 4\}$. From the discussion immediately following (33) and the definition of $P_1^\alpha$, we immediately find

$$[134P_1] = [156P_1] = [356P_1] = 0,$$  \hspace{1cm} (59)

which then gives, via (39), the following identities:

$$F_{143} = F_{P_1,14} + F_{P_1,43} + F_{P_1,31},$$

$$F_{165} = F_{P_1,16} + F_{P_1,65} + F_{P_1,51},$$

$$F_{563} = F_{P_1,56} + F_{P_1,63} + F_{P_1,35},$$

where in each line all three terms on the right hand side label the same vertex as the term on the left hand side. We then also have

$$F_{136} = 3![136P_1] + F_{P_1,13} + F_{P_1,36} + F_{P_1,61},$$  \hspace{1cm} (61)

and the vertices of $[136P_1]$ are all physical. We depict the geometry behind this algebra in Figure 7.

What we have done is use the vertex information given to us from (56) to eliminate the vertex $\{1, 3, 6\}$ by defining the plane through the end points of three edges connecting $\{1, 3, 6\}$ to other physical vertices (all of which being known from (56)). It is then no surprise that all of the vertices of $[136P_1]$ are physical.

By plugging (60) and (61) into (56), making the immediate cancellations, and recalling that we have suppressed the 2 label, we find

$$A_2(\{l_2\}) = [136P_1] + \frac{1}{6}((F_{P_1,14} + F_{P_1,43}) + (F_{P_1,65} + F_{P_1,51}) + (F_{P_1,56} + F_{P_1,35}) + F_{154} + F_{345})$$  \hspace{1cm} (62)\footnote{That is, unless this region of the polytope is “wrapped around” more than once, but for such cases one simply repeats this part of the procedure until the dependence on the initially chosen vertex is manifestly gone.}
where we have lumped two $F_{\ldots}$ terms together if they label the same (physical) vertex. Thus we see that we have now expressed the volume as the sum of a simplex with all physical vertices and a new polytope with only five physical vertices. The general idea, then, is to repeat this process until the “remainder” polytope has only four vertices and then employ the identities amongst the newly defined planes $P_1^\alpha$ to express this remainder polytope as the volume of a simplex (guaranteed to have only physical vertices). For completeness, we finish our current example.

We now define $P_2^\alpha$ as

$$P_2^\alpha \equiv \text{plane through } \{P_1, 1, 5\} = \{1, 5, 6\}, \{P_1, 1, 4\} = \{1, 3, 4\}, \{3, 4, 5\}. \quad (63)$$

Then by reading off the identities analogous to (60) for $F_1P_15, F_14P_1,$ and $F_345,$ and the result analogous to (61) for $F_{154}$ (“triangulating away” this vertex), putting it all together and employing one final use of (39), one finds

$$A_2(\{l_2i\}) = [136P_1]_2 + [154P_1]_2 + [35P_1P_2]_2, \quad (64)$$

where we have reinstated the subscript 2. The vertices of each simplex in (64) can be readily checked to all be physical vertices. The expression (64) agrees with what was found in [1] by “chopping up” the 3-polytope $P_2.$ In our formalism this triangulation (as well as any other) can be obtained from (56), or equivalently from any particular triangulation (by first reconstructing the cyclic lists), in a purely algebraic and algorithmic fashion. This example and the statement of the general procedure makes us believe that a general algorithm (for any polytope in any dimension) for moving from a sum of $F_{\ldots}$ objects to a sum of volumes of simplices with no spurious vertices should exist. This would be extremely useful both for higher dimensional polytopes as well as more complicated polytopes, where visualization and therefore geometric insight is hard to come by.

## 5 Summary and Outlook

In this note we have developed a formalism in which we define polytopes via their bounding hyper-planes and cyclic lists describing their bounding two-dimensional faces. These polytopes are completely general—there is no restriction to connectivity or convexity—and are simultaneously combinatorial in nature, as well as oriented. Additionally, we
found that by defining certain “non-local” sums of simplices \( F_{ij}, F_{ijk}, F_{ijkl}, \ldots \), we can express the volumes of abstract listed polytopes in a manner that depends only on the vertices of the underlying polytope and is therefore independent of any choice of triangulation. From this expression, any valid triangulation can be obtained simply, algebraically, and algorithmically using the “cohomological” identities \([18],[39],\) and \([47]\).

We also saw that by considering certain canonical subcollections of cyclic lists and defining canonical lower dimensional \( F \ldots \) objects with respect to certain boundaries, we can immediately calculate the volume of any lower dimensional bounding polytope. Finally, we saw that it is possible (and straightforward) to use any particular triangulation of a polytope to obtain the cyclic lists for the underlying polytope. Therefore, given any triangulation of a \( d \)-polytope, one can algorithmically obtain the volumes of any lower dimensional bounding polytopes by simply recovering the lists, restricting to the relevant lists, and using the relevant-dimensional \( F \ldots \)-calculus with these lists.

These considerations all took place within \( \mathbb{CP}^n \), which can be viewed as the simplest of all Grassmannia \( G(k,n) \). The most relevant extension of these ideas would be to Grassmannia with \( k > 1 \), though it is not obvious at this point how this can be done due to the loss of duality: viewing points in one copy of \( G(1,n) \) as hyper-planes in another. The most likely way to make contact with these more complex spaces will be in generalizing the cohomological descriptions of the \( F \ldots \) objects. The details of the cohomology underlying the definition of the \( F \ldots \)’s will be left for a future note. In short, though, these objects come from certain contour integrals of certain Čech cohomology classes of \( \mathbb{CP}^n \). The \( F_{ij} \) objects come from “\( \log \frac{Z_1 W}{Z_2 W} \) log(\( \frac{Z_3 W}{Z_4 W} \)”) representatives of \( \check{H}^1(\hat{U}, \mathcal{O}) \) where \( \hat{U} \) is a suitable submanifold of \( \mathbb{CP}^2 \), while the \( F_{ijk} \) objects come from “\( \log \log \log \)” representatives of \( \check{H}^2(\hat{U}, \mathcal{O}) \) with \( \hat{U} \subset \mathbb{CP}^3 \). Understanding this cohomological structure as well as its interaction with the base space (where the polytope lives) will likely be the key to understanding how this simple formalism generalizes to \( G(k,n) \).

Seeing how positivity and convexity makes an appearance in this formalism is also an important next step in order to make connection with the more recent work in [2]. We stress again that the current formalism makes no contact with these ideas—it does not require any constraints on the homogeneous coordinates nor the ordering of the “external data”. In this regard, this formalism is too general, but it is hoped that by seeing how these constraints appear in this formalism, we will be naturally directed towards how to most fruitfully lift these constraints again.

## Appendices

### A Example of \( F_{ij} \)-Calculus

We consider yet again the triangle, defined by the list \( l = (1,2,3) \), so that the set of boundaries is \( S = \{1,2,3\} \). To define our \( F_{ij} \)’s, we must introduce a reference boundary
$Q^\alpha$. In Figure 8 we give a possible picture of these choices.

With $S$ as above, i.e., not adding in another reference boundary $B^\alpha$ as in Theorem 1, we have

\[
F_{12} = V[12][3Q] \\
F_{23} = V[23][1Q] \\
F_{31} = V[31][2Q].
\]

(65) \hspace{2cm} (66) \hspace{2cm} (67)

Theorem 1 says that we should consider the sum

\[
\frac{1}{2}(F_{12} + F_{23} + F_{31})
\]

(68)

to get the area of the polygon defined by the list $l = (1, 2, 3)$. Let us see what each of these terms corresponds to. From the discussion following (15), we have that $F_{12}$ is the area of the polygon defined by the list $(1, Q, 2, 3)$. This polygon is depicted in Figure 9. We note that the actual area corresponding to $F_{12}$ is completely “non-local”, in the sense that the vertex $\{1, 2\}$ is not included in the resulting polygon. We now see that $F_{23}$ is the area of the polygon defined by the list $(2, Q, 3, 1)$, which is depicted in Figure 10. Finally, we see that $F_{31}$ is the area of the polygon defined by the list $(3, Q, 1, 2)$, depicted in Figure 11.

By “superposing” the three polygons depicted in Figures 9, 10 and cancelling the areas that have opposite orientations, we see that we are left precisely with 2 times the area of the triangle defined by the list $(123)$, and the dependence on $Q^\alpha$ drops out. Had we
included some other reference boundary $B^\alpha$ in $S$ to define $S' = S \cup \{B^\alpha\}$, then we would get new $F_{ij}$'s. Namely, we would have

\begin{align*}
F_{12} &= V[12][3Q] + V[12][BQ] \\
F_{23} &= V[23][1Q] + V[23][BQ] \\
F_{31} &= V[31][2Q] + V[31][BQ].
\end{align*}

(69) \hfill (70) \hfill (71)
and the corresponding pictures (the analogues of Figures 9-11) would be correspondingly more complex. However, we can see immediately that the sum

\[ F_{12} + F_{23} + F_{31} \] 

would be left unaffected, since

\[ V[12][BQ] + V[23][BQ] + V[31][BQ] = 0 \]

using (16). This is precisely the vanishing term in the second equality in the proof of Proposition 1 and this is precisely why we can add (but not subtract) as many boundaries to \( S \) as we would like, or as is convenient, without having to worry about the sum of \( F_{ij} \)'s that we are interested in being affected. Thus, we add \( B^\alpha \) to the set of boundaries \( S \) in Theorem 1 because considering the objects \( F_{iB} \) makes the proof simpler, but in practice we can deal only with \( S \) and define our \( F_{ij} \)'s with respect to it. We note that in this example the object \( F_{1B} \) (as well as \( F_{2B} \) and \( F_{3B} \)) can be defined using \( S' \), and we would have \( F_{1B} = V[1B][2Q] + V[1B][3Q] \) (and similarly for \( F_{2B} \) and \( F_{3B} \)), but since \( B \) does not make an appearance in the list \( l = (1, 2, 3) \) under consideration, we can disregard these objects.

B Abstract Listed 3-Cube

We consider a “cube”, depicted in Figure 12 and defined by six boundaries, so that \( S = \{1, 2, 3, 4, 5, 6\} \). We read off from the picture that the following six lists (one for
Each 2-face) are the lists that reflect the (oriented) combinatorial properties of the cube:

\[ l_1 = (4, 6, 3, 5) \]
\[ l_2 = (5, 3, 6, 4) \]
\[ l_3 = (1, 6, 2, 5) \]
\[ l_4 = (5, 2, 6, 1) \]
\[ l_5 = (1, 3, 2, 4) \]
\[ l_6 = (4, 2, 3, 1). \]

Equation \(40\) then tells us that (noting that \(n_i = 4\) for each \(i\))

\[ A = \frac{1}{18} \sum_{i=1}^{6} \sum_{k=1}^{4} F_{ijklj(k+1)} \]
\[ = \frac{1}{6} (F_{146} + F_{163} + F_{135} + F_{154} \]
\[ + F_{253} + F_{236} + F_{264} + F_{245}) \]
\[ = [1235] - [1236] - [1245] + [1246] \]
\[ = V[12][34][56]. \]

The triangulation \(77\) came about via one particular choice of simplifying \(76\) using \(39\). The expression \(76\) encodes all possible triangulations by making different choices of simplification using \(39\). Moreover, we see that the \(V[\cdot][\cdot][\cdot]\) expressions are interpreted as volumes of “3-D quadrilaterals” just as the two-dimensional \(V[\cdot][\cdot]\) objects are interpreted as areas of quadrilaterals.

### C  Abstract Listed \(d\)-Polytopes

Let \(d\) be the dimension of the polytope that we want to define and let \(S = \{1, \ldots, N\}\) be a set of \(N \geq d + 1\) distinct boundaries, i.e., a set \(\{Z_1^\alpha, \ldots, Z_N^\alpha\} \subset \mathbb{C}\mathbb{P}^d\). A vertex
in \(d\) dimensions is specified uniquely by the intersection of \(d\) distinct boundaries and is denoted by \(\{i_1, ..., i_d\}\) with \(\{i_k \in S\}\) pairwise distinct. A line is specified uniquely by the intersection of \((d - 1)\) boundaries, and so an oriented edge is denoted by \([j(i_1, ..., i_{d-1})k]\) with \(i_1, ..., i_{(d-1)}, j, k \in S\) and with \(\{i_1, ..., i_{(d-1)}\}\) pairwise distinct, and denotes the instruction \(\{j, i_1, ..., i_{(d-1)}\} \rightarrow \{k, i_1, ..., i_{(d-1)}\}\) along the line \(i_1 \cap ..., \cap i_{(d-1)}\). We place the usual additive structure on formal sums of oriented edges.

Our \(d\)-polytopes will then be specified by \(N^{d-2}\) cyclic lists

\[
\{l_{i_1, ..., i_{(d-2)}}\} = (j_{i_1, ..., i_{(d-2)}}, 1, ..., j_{i_1, ..., i_{(d-2)}}, n_{i_1, ..., i_{(d-2)}})
\]

with \(i_1, ..., i_{(d-2)} \in S\) and each \(j_{i_1, ..., i_{(d-2)}}, l \in S - \{i_1, ..., i_{(d-2)}\}\). Here, \(n_{i_1, ..., i_{(d-2)}}\) is just the length of the list \(l_{i_1, ..., i_{(d-2)}}\). The edge set \(E_{i_1, ..., i_{(d-2)}}\) derived from the cyclic list \(l_{i_1, ..., i_{(d-2)}}\) is defined to be

\[
E_{i_1, ..., i_{(d-2)}} = \sum_{l=1}^{n_{i_1, ..., i_{(d-2)}}} [j_{i_1, ..., i_{(d-2)}}, (l-1)(j_{i_1, ..., i_{(d-2)}}, l, i_1, ..., i_{(d-2)}), j_{i_1, ..., i_{(d-2)}}, (l+1)]
\]

where the sum is cyclic in the usual sense, and the edge set \(E_{i_1, ..., i_{(d-2)}}; s\) derived from the cyclic list \(l_{i_1, ..., i_{(d-2)}}\) with respect to \(s \in S\) is defined as

\[
E_{i_1, ..., i_{(d-2)}; s} = \sum_{l=1}^{n_{i_1, ..., i_{(d-2)}}} [j_{i_1, ..., i_{(d-2)}}, (l-1)(i_1, ..., i_{(d-2)}), j_{i_1, ..., i_{(d-2)}}, (l+1)]
\]

We can now easily generalize our definitions of abstract listed 2-, 3-, and 4-polytopes to a \(d\)-polytope for any \(d \geq 2\).

**Definition 5.** An abstract listed \(d\)-polytope with \(d \geq 2\) is equivalent to the following data:

i) A set \(S = \{1, ..., N\}\) of \(N\) distinct boundaries,

ii) A collection of \(N^{d-2}\) cyclic lists

\[
\{l_{i_1, ..., i_{(d-2)}}\} = (j_{i_1, ..., i_{(d-2)}}, 1, ..., j_{i_1, ..., i_{(d-2)}}, n_{i_1, ..., i_{(d-2)}})
\]

as defined above, such that for any \(i_1, ..., i_{(d-1)} \in S\),

\[
E_{i_1, ..., i_{(d-2)}; i_{(d-1)}} = (-1)^{\sigma}E_{\sigma(i_1, ..., i_{(d-2)}; i_{(d-1)})}
\]

where \(\sigma \in S_{(d-1)}\) is any permutation of \((d - 1)\) objects.
We note that this definition makes it clear that the restriction of a $d$-polytope to any number of boundaries (say, $p$ distinct boundaries) gives a $(d-p)$-polytope. Namely, if we are given a $d$-polytope, then the $(d-1)$-polytope $P_I$ obtained by restricting to the $I^{th}$ face is indeed a polytope, since then we have

$$E_{i_1,...,i_{(d-3)}i_{(d-2)}} = (-1)^{|\sigma|} E_{i_{\sigma(1)},...,i_{\sigma(d-3)}i_{\sigma(d-2)}}.$$  \hspace{0.5cm} (82)

The analogous statement can be said after restricting to the boundaries $I_1,...,I_p$ with $p \leq d - 2$.

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References

[1] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. Hodges, and J. Trnka, “A Note on Polytopes for Scattering Amplitudes,” [JHEP 1204 (2012) 081] arXiv:1012.6030 [hep-th]

[2] N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, A. B. Goncharov, A. Postnikov, et al., “Scattering Amplitudes and the Positive Grassmannian,” arXiv:1212.5605 [hep-th]

[3] N. Arkani-Hamed and J. Trnka, “The Amplituhedron,” arXiv:1312.2007 [hep-th]

[4] L. Hughston and T. Hurd, “A CP5 Calculus For Space-time Fields,” Phys.Rept. 100 (1983) 275–326

[5] A. Hodges, “Eliminating spurious poles from gauge-theoretic amplitudes,” JHEP 1305 (2013) 135 arXiv:0905.1473 [hep-th]

[6] S. Huggett and K. Tod, An Introduction to Twistor Theory. Cambridge University Press, 1994.