Necessary and sufficient conditions for boundedness of commutators of bilinear Hardy-Littlewood maximal function

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Abstract. Let \( M \) be the bilinear Hardy-Littlewood maximal function and \( \vec{b} = (b, b) \) be a collection of locally integrable functions. In this paper, the authors establish characterizations of the weighted BMO space in terms of several different commutators of bilinear Hardy-Littlewood maximal function, respectively; these commutators include the maximal iterated commutator \( M_{\Pi\vec{b}} \), the maximal linear commutator \( M_{\Sigma\vec{b}} \), the iterated commutator \([\Pi\vec{b}, M]\) and the linear commutator \([\Sigma\vec{b}, M]\).

§1 Introduction

A locally integrable function \( f \) is said to belong to BMO space if there exists a constant \( C > 0 \) such that for any cube \( Q \subset \mathbb{R}^n \),
\[
\frac{1}{|Q|} \int_Q |f(x) - f_Q|dx \leq C,
\]
where \( f_Q = \frac{1}{|Q|} \int_Q f(x)dx \) and the minimal constant \( C \) is defined by \( \|f\|_* \).

There are a number of classical results that demonstrate BMO functions are the right collections to do harmonic analysis on the boundedness of commutators. A well known result of Coifman, Rochberg and Weiss [7] states that the commutator
\[
[b, T](f) = bT(f) - T(bf)
\]
is bounded on some \( L^p, 1 < p < \infty \), if and only if \( b \in \text{BMO} \), where \( T \) be the classical Calderón-Zygmund operator. Chanillo [5] proved that if \( b \in \text{BMO} \), the commutator
\[
[b, I_\alpha]f(x) = b(x)I_\alpha f(x) - I_\alpha(bf)(x)
\]
is bounded from \( L^p \) to \( L^q \) with \( 1 < p < n/\alpha \) and \( 1/q = 1/p - \alpha/n \), where \( I_\alpha \) be a fractional
integral operator. Moreover, if \( n - \alpha \) is even, the reverse is also valid. A complete characterization of BMO via the commutator \([b, I_{\alpha}]\) was shown by Ding [8]. During the past thirty years, the theory was then extended and generalized to several directions. For instance, Bloom [3] investigated the characterization of BMO spaces in the weighted setting. In 1991, García-Cuerva, Harboure, Segovia and Torrea [11] showed that the maximal commutator
\[
M_b(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |b(x) - b(y)||f(y)|\,dy
\]
is bounded on \( L^p, 1 < p < \infty \), if and only if \( b \in \text{BMO} \). In 2000, Bastero, Milman and Ruiz [1] studied the necessary and sufficient conditions for the boundedness of \([b, M]\) on \( L^p \) spaces when \( 1 < p < \infty \). They showed that the commutator of Hardy-Littlewood maximal operator
\[
[b, M](f)(x) = b(x)M(f)(x) - M(bf)(x)
\]
is bounded on \( L^p, 1 < p < \infty \), if and only if \( b \in \text{BMO} \) with \( b^- \in L^\infty \), where \( b^-(x) = -\min\{b(x), 0\} \). In 2014, Zhang [24] considered the characterization of BMO via the commutator of the fractional maximal function on variable exponent Lebesgue spaces.

In the multilinear setting, the boundedness of commutators has been extensively studied already, as in Pérez and Torres’ [16], Tangs [19], Lerner, Ombrosi, Pérez, Torres, and Trujillo-González [12] and Chen and Xues [6], and Pérez, Pradolini, Torres, and Trujillo-González [15]. Specially, Chaffee and Torres [4], Wang, Pan and Jiang [20] and Zhang [23] contributed the theory of characterization of BMO spaces by considering the \textit{linear commutator} of Multilinear operators, respectively. In this paper, we will extend Zhang’s result to weighted case and we replace the linear commutators by \textit{iterated commutators}.

Our main results as follows.

\textbf{Theorem 1.1.} Let \( 1 < p_1, p_2 < \infty, \vec{b} = (b, b), 1/p = 1/p_1 + 1/p_2 \) and \( \omega \in A_1 \). Then the following are equivalent,

(A1) \( b \in \text{BMO}(\omega) \);

(A2) \( \mathcal{M}_{\Sigma} \) is bounded from \( L^{p_1}(\omega) \times L^{p_2}(\omega) \) to \( L^p(\omega^{1-p}) \);

(A3) \( \mathcal{M}_{\Pi} \) is bounded from \( L^{p_1}(\omega) \times L^{p_2}(\omega) \) to \( L^p(\omega^{1-2p}) \).

\textbf{Theorem 1.2.} Let \( 1 < p_1, p_2 < \infty, \vec{b} = (b, b), 1/p = 1/p_1 + 1/p_2 \) and \( \omega \in A_1 \). Then the following are equivalent,

(B1) \( b \in \text{BMO}(\omega) \) and \( b^-/\omega \in L^\infty \);

(B2) \( [\Sigma \vec{b}, \mathcal{M}] \) is bounded from \( L^{p_1}(\omega) \times L^{p_2}(\omega) \) to \( L^p(\omega^{1-p}) \);

(B3) \( [\Pi \vec{b}, \mathcal{M}] \) is bounded from \( L^{p_1}(\omega) \times L^{p_2}(\omega) \) to \( L^p(\omega^{1-2p}) \).
§2 Some preliminaries and notations

In 2009, Lerner, Ombrosi, Pérez, Torres and Trujillo-González [12] introduced the following multilinear maximal function that adapts to the multilinear Calderón-Zygmund theory. In this paper, we only consider the bilinear case. A similar argument also works for the multilinear cases.

**Definition 2.1.** For a collection of locally integrable functions $\vec{f} = (f_1, f_2)$, the bilinear maximal function $M$ is defined by

$$M(\vec{f})(x) = \sup_{Q \ni x} \frac{1}{|Q|^2} \int_Q \int_Q |f_i(y_i)| dy_i.$$

We now give the definitions of the maximal commutators and the commutators related to the bilinear maximal function $M$.

**Definition 2.2.** For two collections of locally integrable functions $\vec{f} = (f_1, f_2)$ and $\vec{b} = (b_1, b_2)$, the maximal linear commutator $M_{\Sigma \vec{b}}$ is defined by

$$M_{\Sigma \vec{b}}(\vec{f})(x) = \sum_{i=1}^{2} M_{b_i}(\vec{f})(x),$$

where

$$M_{b_i}(\vec{f})(x) = \frac{1}{|Q|^2} \int_Q \int_Q |b_i(y) - b_i(x)| f_i(y) dy_1 dy_2.$$

The maximal iterated commutator $M_{\Pi \vec{b}}$ is defined by

$$M_{\Pi \vec{b}}(\vec{f})(x) = \frac{1}{|Q|^2} \int_Q \int_Q \prod_{i=1}^{2} |b_i(y) - b_i(x)| f_i(y) dy_1 dy_2.$$

The linear commutator of $M$ is defined by

$$[\Sigma \vec{b}, M](\vec{f})(x) = [b_1, M]^{(1)}(\vec{f})(x) + [b_2, M]^{(2)}(\vec{f})(x),$$

where

$$[b_1, M]^{(1)}(\vec{f})(x) = b_1(x)M(\vec{f})(x) - M(b_1 f_1, f_2)(x)$$

and

$$[b_2, M]^{(2)}(\vec{f})(x) = b_2(x)M(\vec{f})(x) - M(f_1, b_2 f_2)(x).$$

The iterated commutator of $M$ is defined by

$$[\Pi \vec{b}, M](\vec{f})(x) = b_1(x)b_2(x)M(\vec{f})(x) - b_1(x)M(f_1, b_2 f_2)(x) - b_2(x)M(b_1 f_1, f_2)(x) + M(b_1 f_1, b_2 f_2)(x).$$

We now recall the definition of $A_p$ weight introduced by Muckenhoup [13].

**Definition 2.3.** For $1 < p < \infty$ and a nonnegative locally integrable function $\omega$ on $\mathbb{R}^n$, $\omega$ is in the Muckenhoup $A_p$ class if it satisfies the condition

$$\sup_{Q} \left( \frac{1}{|Q|} \int_Q \omega(x) dx \right) \left( \frac{1}{|Q|} \int_Q \omega(x)^{-\frac{1}{p-1}} dx \right)^{p-1} < \infty.$$
And a weight function \( \omega \) belongs to the class \( A_1 \) if there exists \( C > 0 \) such that for every cube \( Q \),
\[
\frac{1}{|Q|} \int_Q \omega(x) dx \leq C \inf_{x \in Q} \omega(x).
\]
We write \( A_{\infty} = \bigcup_{1 \leq p < \infty} A_p \).

**Definition 2.4.** Let \( 1 \leq p < \infty \). Given a a nonnegative locally integrable function \( \omega \), the weighted \( BMO \) space \( BMO^p(\omega) \) is defined be the set of all functions \( f \in L^1_{loc}(\mathbb{R}^n) \) such that
\[
\|f\|_{BMO^p(\omega)} := \sup_Q \left( \frac{1}{w(Q)} \int_Q |f(y) - f_Q|^p \omega(y)^{1-p} dy \right)^{1/p} < \infty,
\]
where the supremum is taken over all cubes \( Q \subset \mathbb{R}^n \) and \( \omega(Q) = \int_Q \omega(x) dx \). We write \( BMO^1(\omega) = BMO(\omega) \) simple.

**Remark** For \( 1 \leq p < \infty \) and \( \omega \in A_1 \), García-Cuerva [10] proved that \( BMO(\omega) = BMO^p(\omega) \) with equivalence of the corresponding norms.

Standard real analysis tools as the weighted maximal function \( M_\omega(f) \), the sharp maximal function \( M^s(f) \) carries over to this context, namely,
\[
M_\omega(f)(x) = \sup_{Q \ni x} \frac{1}{\omega(Q)} \int_Q |f(y)| \omega(y) dy;
\]
\[
M^s(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - c_i| dy \cong \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy.
\]
A variant of weighted maximal function and sharp maximal operator \( M_{\omega,s}(f)(x) = (M_\omega(f^s))^{1/s} \) and \( M^s_{\omega}(f)(x) = (M^s(f^s)(x))^{1/s} \), which will become the main tool in our scheme.

**§3 Main lemmas**

To prove Theorem 1.1 and Theorem 1.2 we need the following results.

**Lemma 3.1.** Let \( \omega \in A_1 \), \( \vec{b} = (b, b) \) and \( b \in BMO(\omega) \). Then
\[
M^2_{\frac{1}{2}}(M_{\vec{B}}(\vec{f}))(x) \lesssim \|b\|^2_{BMO(\omega)} \omega(x)^2 M(M(\vec{f}))(x))
\]
\[
+ \|b\|^2_{BMO(\omega)} \omega(x)^2 \prod_{i=1}^2 M_{\omega,s}(f_i)(x)
\]
\[
+ \sum_{i=1}^2 \|b\|^2_{BMO(\omega)} (M_{\omega}(b_i^i)(\vec{f}))(x),
\]
for any \( 1 < s < \infty \) and bounded compact supported functions \( f_1, f_2 \).

**Proof.** First of all, we give the definition of the following auxiliary maximal function, which has been studied in [17] and [18] for the linear case. Let \( \varphi(x) \geq 0 \) be a smooth function such that \( \varphi(t) = e^{-2n} \varphi(\frac{t}{2}) \), \( |\varphi'(t)| \lesssim t^{-1} \) and \( \chi_{[0,1]}(t) \leq \varphi(t) \leq \chi_{[0,2]}(t) \).

Let
\[
\Phi(f, g)(x) = \sup_{\epsilon > 0} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \varphi_{\epsilon}(|x - y_1| + |x - y_2|) \prod_{i=1}^2 |f_i(y_i)| dy_1 dy_2,
\]
and
\[
\Phi_{\Pi B}(f_1, f_2)(x) = \sup_{\epsilon > 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi_{\epsilon}(|x - y_1| + |x - y_2|) \prod_{i=1}^{2} |b(x) - b(y_i)||f_i(y_i)|dy_1dy_2.
\]

We first show that
\[
\Phi(f_1, f_2)(x) \approx \mathcal{M}(f_1, f_2)(x).
\]
In fact, let \(B_\epsilon = \{ y \in \mathbb{R}^n : |x - y| \leq \epsilon \} \). It is easy to see that
\[
B_\epsilon \times B_\epsilon \subset \{(y_1, y_2) : |x - y_1| + |x - y_2| \leq \epsilon \} \subset B_\epsilon \times B_\epsilon.
\]
The bounded compact supported condition of \(\varphi\) gives
\[
\Phi(f_1, f_2)(x) = \sup_{\epsilon > 0} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \varphi_{\epsilon}(|x - y_1| + |x - y_2|) |f_1(y_1)||f_2(y_2)|dy_1dy_2
\]
\[
\leq \sup_{\epsilon > 0} \frac{1}{\epsilon^{2n}} \int_{B_\epsilon} \int_{B_\epsilon} \varphi\left(\frac{|x - y_1| + |x - y_2|}{\epsilon}\right) |f_1(y_1)||f_2(y_2)|dy_1dy_2
\]
\[
\lesssim \mathcal{M}(f_1, f_2)(x)
\]
and
\[
\Phi(f_1, f_2)(x) \geq \sup_{\epsilon > 0} \frac{1}{\epsilon^{2n}} \int_{B_\epsilon} \int_{B_\epsilon} \varphi\left(\frac{|x - y_1| + |x - y_2|}{\epsilon}\right) |f_1(y_1)||f_2(y_2)|dy_1dy_2
\]
\[
\gtrsim \mathcal{M}(f_1, f_2)(x).
\]

We can also obtain that \(\Phi_{\Pi B}(f_1, f_2)(x) \approx \mathcal{M}_{\Pi B}(f_1, f_2)(x)\).

Now, we shall estimate the sharp maximal function of the auxiliary maximal function. Let \(Q\) be a cube and \(x \in Q\). Then, for any \(z \in Q\) we have
\[
|\Phi_{\Pi B}(f_1, f_2)(z) - c_Q| \leq |b(z) - b_Q|^2 \Phi(f_1, f_2)(z)
\]
\[
+ |b(z) - b_Q| \Phi(f_1, (b - b_Q)f_2)(z)
\]
\[
+ |b(z) - b_Q| \Phi((b - b_Q)f_1, f_2)(z)
\]
\[
+ \Phi((b - b_Q)f_1, (b - b_Q)f_2)(z) - c_Q
\]
\[
=: A_Q^1(z) + A_Q^2(z) + A_Q^3(z) + A_Q^4(z),
\]
where \(c_Q = (\Phi((b - b_Q)f_1^\infty, (b - b_Q)f_2^\infty))_Q\) and \(f_i^\infty\) will be defined later.

Therefore,
\[
\left(\frac{1}{|Q|} \int_{Q} |\Phi_{\Pi B}(f_1, f_2)(z)|^{\delta} - |c_Q|^\delta|dz\right)^{1/\delta} \lesssim \left(\frac{1}{|Q|} \int_{Q} |\Phi_{\Pi B}(f_1, f_2)(z) - c_Q|^\delta|dz\right)^{1/\delta}
\]
\[
\lesssim \sum_{j=1}^{4} A_j,
\]
where \(A_j = \left(\frac{1}{|Q|} \int_{Q} (A_Q^j(z))^\delta|dz\right)^{1/\delta}, j = 1, 2, 3, 4\) and taking \(\delta = 1/3\).

Let us consider first the term \(A_1\). By averaging \(A_1^Q\) over \(Q\), we get
\[
A_1 = \left(\frac{1}{|Q|} \int_{Q} \left(\left|b(z) - b_Q\right|^2 \Phi(f_1, f_2)(z)\right)^{1/3} |dz\right)^3
\]
\[
\lesssim \|b\|_{\text{BMO}(\omega)}^2 |\omega(Q)|^2 \frac{1}{|Q|} \int_{Q} M(f_1, f_2)(z)dz
\]
\[
\lesssim \|b\|_{\text{BMO}(\omega)}^2 \omega(x)^2 M(\mathcal{M}(f_1, f_2))(x).
\]
Let us consider next the term $A_2$. We write
\[
A_2^Q(z) = |b(z) - b_Q|\Phi(f_1, |b - b_Q|f_2)(z)
\]
\[
\leq |b(z) - b_Q|\Phi(f_1, (|b(z) - b_Q| + |b(z) - b|)f_2)(z)
\]
\[
\leq |b(z) - b_Q|^2\mathcal{M}(f_1, f_2)(z) + |b(z) - b_Q|\mathcal{M}_b^2(f_1, f_2)(z)
\]
\[
=: A_{21}^Q(z) + A_{22}^Q(z).
\]

For $A_{21}^Q(z)$, the fact that $\Phi(f_1, f_2)(z) \lesssim \mathcal{M}(f_1, f_2)(z)$ gives
\[
A_{21} := \left( \frac{1}{|Q|} \int_Q (A_{21}^Q(z))^\delta \, dz \right)^{1/\delta} 
\]
\[
\lesssim \left( \frac{\omega(|Q|)}{|Q|} \|b\|_{\text{BMO}(\omega)} \right)^2 \frac{1}{|Q|} \int_Q \mathcal{M}(f_1, f_2)(z) \, dz
\]
\[
\lesssim \|b\|^2_{\text{BMO}(\omega)} \omega(x)^2 M(\mathcal{M}(f_1, f_2))(x).
\]

For $A_{22}^Q(z)$,
\[
A_{22} := \left( \frac{1}{|Q|} \int_Q (A_{22}^Q(z))^\delta \, dz \right)^{1/\delta}
\]
\[
\lesssim \omega(x)\|b\|_{\text{BMO}(\omega)} \frac{1}{|Q|} \left( \int_Q |\mathcal{M}_b^2(f_1, f_2)(z)|^{1/2} \right)^2
\]
\[
\lesssim \omega(x)\|b\|_{\text{BMO}(\omega)} M_{1/2}(\mathcal{M}_b^2(f_1, f_2))(x).
\]

The same process also follows that
\[
A_3 \lesssim \|b\|^2_{\text{BMO}(\omega)} \omega(x)^2 M(\mathcal{M}(f_1, f_2))(x) + \omega(x)\|b\|_{\text{BMO}(\omega)} M_{1/2}(\mathcal{M}_b^2(f_1, f_2))(x).
\]

To estimate $A_4$, we split $f_j$ to $f_j = f_j^0 + f_j^\infty$ with $f_j^0 = f_j 1_{2Q}$. We write
\[
A_4^Q \leq |\Phi((b - b_Q)f_j^0, (b - b_Q)f_j^\infty)(z)|
\]
\[
+ |\Phi((b - b_Q)f_j^0, (b - b_Q)f_j^\infty)(z)|
\]
\[
+ |\Phi((b - b_Q)f_j^\infty, (b - b_Q)f_j^\infty)(z)|
\]
\[
+ |\Phi((b - b_Q)f_j^\infty, (b - b_Q)f_j^\infty)(z) - c_Q|
\]
\[
=: A_{41}^Q(z) + A_{42}^Q(z) + A_{43}^Q(z) + A_{44}^Q(z).
\]

Then
\[
A_4 \leq \left( \frac{1}{|Q|} \int_Q (\sum_{j=1}^4 A_{4j}^Q(z))^\delta \, dz \right)^{1/\delta}
\]
\[
\lesssim \sum_{j=1}^4 \left( \frac{1}{|Q|} \int_Q (A_{4j}^Q(z))^\delta \, dz \right)^{1/\delta}
\]
\[
\lesssim \sum_{j=1}^4 A_{4j}.
\]
By Kolmogorov inequality and the fact that $\mathcal{M}$ is bounded from $L^1 \times L^1$ to $L^{1/2, \infty}$, we have

$$A_{41} \leq \frac{C}{|Q|^2} \|\Phi ((b - b_Q) f_1^0, (b - b_Q) f_2^0)\|_{L^{1/2, \infty}}$$

$$\leq \frac{C}{|Q|^2} \|\mathcal{M}((b - b_Q) f_1^0, (b - b_Q) f_2^0)\|_{L^{1/2, \infty}}$$

$$\leq \frac{C}{|Q|^2} \prod_{i=1}^2 \int_{2Q} |b(y_i) - b_Q| |f_i(y_i)| dy_i$$

$$\leq \frac{C}{|Q|^2} \prod_{i=1}^2 \left( \int_{2Q} |b(y_i) - b_Q|^s |\omega(y_i)| dy_i \right)^{1/s'} \left( \int_{2Q} |f_i(y_i)|^s |\omega(y_i)| dy_i \right)^{1/s}$$

$$\lesssim \prod_{i=1}^2 \|b\|_{\text{BMO}^s(\omega)(x) M_{\omega,s}(f_i)(x)}.$$ 

For $A_{42}$, it is easy to see that

$$\varphi_s (|z - y_1| + |z - y_2|) \lesssim \frac{1}{(|z - y_1| + |z - y_2|)^{2n}},$$

then

$$A_{42} \lesssim \frac{1}{|Q|^2} \int_Q \int_{R^n \setminus 2Q} \int_{R^n \setminus 2Q} \frac{|b(y_1) - b_Q| |f_1(y_1)| |b(y_2) - b_Q| |f_2(y_2)|}{(|z - y_1| + |z - y_2|)^{2n}} dy_1 dy_2 dz$$

$$\lesssim \int_Q \int_{R^n \setminus 2Q} \frac{|b(y_1) - b_Q| |f_1(y_1)| |b(y_2) - b_Q| |f_2(y_2)|}{(|z - y_1| + |z - y_2|)^{2n}} dy_1 dy_2 dz$$

$$\lesssim \frac{1}{|Q|^2} \int_Q \frac{|b(y_1) - b_Q| |f_1(y_1)|}{(|z - y_1| + |z - y_2|)^{2n}} \int_{R^n \setminus 2Q} \frac{|b(y_2) - b_Q| |f_2(y_2)|}{|z - y_2|^{2n}} dy_2 dz$$

$$\lesssim \|b\|_{\text{BMO}^s(\omega)(x) M_{\omega,s}(f_1)(x)} \sum_{k=1}^\infty \frac{2^{-kn}}{|2^k Q|} \int_{2^k Q} |b(y_2) - b_Q| |f_2(y_2)| dy_2$$

$$\lesssim \|b\|_{\text{BMO}^s(\omega)(x) M_{\omega,s}(f_1)(x)} \sum_{k=1}^\infty \frac{2^{-kn}}{|2^k Q|}$$

$$\times \left[ \int_{2^k Q} |b(y_1) - m_{2^k Q}(b)| |f_2(y_2)| dy_2 + \int_{2^k Q} |m_{2^k Q}(b) - b_Q| |f_2(y_2)| dy_2 \right]$$

$$\lesssim \|b\|_{\text{BMO}^s(\omega)(x) M_{\omega,s}(f_1)(x)} \sum_{k=1}^\infty \frac{2^{-kn}}{|2^k Q|}$$

$$\times \left[ \|b\|_{\text{BMO}^s(\omega)(x) M_{\omega,s}(f_2)(x)} + k \|b\|_{\text{BMO}(\omega)(x) M(f_2)(x)} \right]$$

$$\lesssim \prod_{i=1}^2 \|b\|_{\text{BMO}^s(\omega)(x) M_{\omega,s}(f_i)(x)}.$$ 

Similarly, for $A_{43}$, we have

$$A_{43} \lesssim \prod_{i=1}^2 \|b\|_{\text{BMO}^s(\omega)(x) M_{\omega,s}(f_i)(x)}.$$
For $|z - z'| \leq \frac{1}{2} \max\{|z - y_1|, |z - y_2|\}$,
\[
|\varphi_s(|z - y_1| + |z - y_2|) - \varphi_s(|z' - y_1| + |z' - y_2|)| \lesssim \frac{|z - z'|}{(|z - y_1| + |z - y_2|)^{2n+1}}.
\]
Therefore,
\[
\Phi((b(z) - b)f_1^\infty, (b(z) - b)f_2^\infty)(z) - \Phi((b(z) - b)f_1^\infty, (b(z) - b)f_2^\infty)(z') \lesssim \sup_{c>0} \int_{\mathbb{R}^n \setminus 2Q} \int_{\mathbb{R}^n \setminus 2Q} |\varphi_s(|z - y_1| + |z - y_2|) - \varphi_s(|z' - y_1| + |z' - y_2|)|
\times \prod_{i=1}^2 |b(y_i) - b_Q| |f_i(y_i)| dy_1 dy_2
\lesssim \prod_{i=1}^2 \int_{\mathbb{R}^n \setminus 2Q} \frac{|z - z'|^{\epsilon_1}}{|z - y_i|^{n+\epsilon_1}} |b(y_i) - b_Q| |f_i(y_i)| dy_i
\lesssim \prod_{i=1}^2 \sum_{k=1}^{2N} \frac{2^{k\epsilon_1}}{|2^k Q|} \int_{2kQ} |b(y_i) - b_Q| |f_i(y_i)| dy_i
\lesssim \prod_{i=1}^2 \|b\|_{\text{BMO}(\omega)}^{\epsilon_1} M_{\omega,s}(f_i)(x),
\]
where $\epsilon_1, \epsilon_2 > 0$ with $\epsilon_1 + \epsilon_2 = 1$.

Collecting our estimates, we have shown that
\[
M_{\frac{n}{2}}^{2} (M_{\text{HOM}}(\bar{f})) (x) \lesssim \|b\|_{\text{BMO}(\omega)}^{2} M(M(\bar{f}))(x)
+ \|b\|_{\text{BMO}(\omega)}^{2} M_{\omega,s}(f_1)(x)
+ \sum_{i=1}^{2} \|b\|_{\text{BMO}(\omega)} M_{\omega,s}(f_i)(x) M(f_2)(x),
\]
for any $1 < s < \infty$ and bounded compact supported functions $f_1, f_2$. \hfill \Box

**Lemma 3.2.** Let $\omega \in A_1$, $\bar{b} = (b, b)$ and $b \in \text{BMO}(\omega)$. Then there exist a constant $C$ such that
\[
M_{\frac{n}{2}}^{2} (M_b^{(i)}(\bar{f})) (x) \lesssim \|b\|_{\text{BMO}(\omega)} M(M(\bar{f}))(x)
+ \|b\|_{\text{BMO}(\omega)} M_{\omega,s}(f_1)(x) M(f_2)(x),
\]
for any $1 < s < \infty$ and bounded compact supported functions $f_1, f_2$.

**Proof.** Let $Q$ be a cube and $x \in Q$. Then, for $z \in Q$ we have
\[
|\Phi_b^{(1)}(f_1, f_2)(z) - c_Q| \leq |b(z) - b_Q| \Phi(f_1, f_2)(z)
+ |\Phi((b - b_Q)f_1, f_2)(z) - c_Q|
=: B_1^Q(z) + B_2^Q(z).
\]
Therefore,
\[
\left( \frac{1}{|Q|} \int_Q \left| \Phi_{\xi}^{(1)}(f_1, f_2)(z) \right|^{1/2} - |c_Q|^{1/2} \right)^2 \lesssim \left( \frac{1}{|Q|} \int_Q |\Phi_{\xi}(f_1, f_2)(z) - c_Q|^{1/2} \right)^2
\]
\[
\lesssim \sum_{j=1}^{2} B_j,
\]
where \(B_j = \left( \frac{1}{|Q|} \int_Q (B_j^Q(z))^2 dz \right)^{1/2}, j = 1, 2\).

Let us consider first the term \(B_1\). By averaging \(B_1^Q\) over \(Q\), we get
\[
B_1 = \left( \frac{1}{|Q|} \int_Q \left( |b(z) - b_Q| \Phi(f_1, f_2)(z) \right) \right)^{1/2} dz
\]
\[
\lesssim \|b\|_{BMO(\omega)} \frac{\omega(Q)}{|Q|} \int_Q |\mathcal{M}(f_1, f_2)(z) dz
\]
\[
\lesssim \|b\|_{BMO(\omega)} \omega(x) M(\mathcal{M}(f_1, f_2)) (x).
\]
Let us consider next the term \(B_2\). We split \(f_j\) to \(f_j = f_j^1 + f_j^\infty\) with \(f_j^0 = f_j \chi_{2Q}\). We write
\[
B_2^Q \leq |\Phi((b - b_Q)f_j^1, f_j^0)(z)| + |\Phi((b - b_Q)f_j^0, f_j^1)(z)|
\]
\[
+ |\Phi((b - b_Q)f_j^\infty, f_j^0)(z)| + |\Phi((b - b_Q)f_j^0, f_j^\infty)(z) - c_Q|
\]
\[
=: B_{21}^Q(z) + B_{22}^Q(z) + B_{23}^Q(z) + B_{24}^Q(z).
\]
By Kolmogorov inequality and the fact that \(\mathcal{M}\) is bounded from \(L^1 \times L^1\) to \(L^{1/2, \infty}\), we have
\[
B_{21} \leq \frac{C}{|Q|^2} \left\| \Phi((b - b_Q)f_j^1, f_j^0) \right\|_{L^{1/2, \infty}}
\]
\[
\lesssim \frac{1}{|Q|^2} \left\| \mathcal{M}(b - b_Q)f_j^0, f_j^1 \right\|_{L^{1/2, \infty}}
\]
\[
\lesssim \frac{1}{|Q|^2} \int_{2Q} \left| b(y_1) - b_Q \right| |f_1(y_1)| dy_1 \int_{2Q} |f_2(y_2)| dy_2
\]
\[
\lesssim \|b\|_{BMO(\omega)} \omega(x) M(\mathcal{M}(f_1, f_2))(x).
\]
For \(B_{22}\),
\[
B_{22} \leq \frac{1}{|Q|} \int_Q \int_{2Q} \int_{R^n \setminus 2Q} \left| b(y_1) - b_Q \right| |f_1(y_1)| |f_2(y_2)| dy_1 dy_2 dz
\]
\[
\lesssim \frac{1}{|Q|} \int_Q \int_{2Q} \int_{R^n \setminus 2Q} \left| b(y_1) - b_Q \right| |f_1(y_1)| |f_2(y_2)| dy_1 dy_2 dz
\]
\[
\lesssim \frac{1}{|Q|} \int_{2Q} \left| b(y_1) - b_Q \right| |f_1(y_1)| dy_1 \int_{2Q} \int_{R^n \setminus 2Q} \left| f_2(y_2) \right| \left| \frac{dz}{|z - y_2|^{2n}} \right| dy_2
\]
\[
\lesssim \|b\|_{BMO(\omega)} \omega(x) M(\mathcal{M}(f_1, f_2))(x) \sum_{k=1}^{\infty} \left( \frac{2^{kn}}{|Q|} \right) \int_{2^k Q} |f_2(y)| dy_2
\]
\[
\lesssim \|b\|_{BMO(\omega)} \omega(x) M(\mathcal{M}(f_1, f_2))(x).
\]
For $B_{23}$, we have

$$B_{23} \lesssim \frac{1}{|Q|} \int_Q \int_{R^n \setminus 2Q} \int_{2Q} \frac{|b(y_1) - b_Q||f_1(y_1)||f_2(y_2)|}{|z - y_1 + |z - y_2|^{2n}} dy_1 dy_2 dz$$

$$\lesssim \frac{1}{|Q|} \int_Q \int_{R^n \setminus 2Q} \int_{2Q} \frac{|b(y_1) - b_Q||f_1(y_1)||f_2(y_2)|}{|z - y_1 + |z - y_2|^{2n}} dy_1 dy_2 dz$$

$$\lesssim \frac{1}{|Q|} \int_Q \int_{R^n \setminus 2Q} \frac{|b(y_1) - b_Q||f_1(y_1)|}{|z - y_1|^{2n}} dy_1 dz \int_{2Q} |f_2(y_2)| dy_2$$

$$\lesssim \|b\|_{\text{BMO}'} \omega(x) M_{\omega,s}(f_1)(x) M(f_2)(x).$$

Concerning the last estimate for $B_{24}$, for any $z' \in Q$ and $y_1, y_2 \in \mathbb{R}^n \setminus 2Q$, we have

$$|\Phi((b - b_Q)f_1^\infty, f_2^\infty)(z) - \Phi((b - b_Q)f_1^\infty, f_2^\infty)(z')|$$

$$\lesssim \sup_{z > 0} \int_{\mathbb{R}^n \setminus 2Q} \int_{\mathbb{R}^n \setminus 2Q} |\varphi_c(|z - y_1| + |z - y_2|) - \varphi_c(|z' - y_1| + |z' - y_2|)|$$

$$\times |b(y_1) - b_Q||f_1(y_1)||f_2(y_2)| dy_1 dy_2$$

$$\lesssim \int_{\mathbb{R}^n \setminus 2Q} \frac{|b(y_1) - b_Q||f_1(y_1)|}{|z - y_1|^{2n+1}} dy_1 \int_{\mathbb{R}^n \setminus 2Q} \frac{|f_2(y_2)|}{|z - y_2|^{2n}} dy_2$$

$$\lesssim \sum_{k=2}^{\infty} \frac{2^{-k}\epsilon_1}{|2^k Q|^2} \int_{2^k Q} |b(y_1) - b_Q||f_1(y_1)| dy_1 \sum_{i=2}^{\infty} \frac{2^{-k}\epsilon_2}{|2^k Q|^2} \int_{2^k Q} |f_2(y_2)| dy_2$$

$$\lesssim \|b\|_{\text{BMO}'} \omega(x) M_{\omega,s}(f_1)(x) M(f_2)(x).$$

where $\epsilon_1, \epsilon_2 > 0$ with $\epsilon_1 + \epsilon_2 = 1$. Taking the mean over $Q$ for $z$ and $z'$ respectively, we obtain

$$B_{24} \lesssim \frac{1}{|Q|} \int_Q \int_Q \frac{1}{|Q|} |\Phi((b - b_Q)f_1^\infty, f_2^\infty)(z) - c_Q| dz$$

$$\lesssim \frac{1}{|Q|} \int_Q \int_Q \frac{1}{|Q|} |\Phi((b - b_Q)f_1^\infty, f_2^\infty)(z) - \Phi((b - b_Q)f_1^\infty, f_2^\infty)(z')| dz dz'$$

$$\lesssim \|b\|_{\text{BMO}'} \omega(x) M_{\omega,s}(f_1)(x) M(f_2)(x).$$

Collecting our estimates, we have shown that

$$M_2^s(M_b^{-1}(f))(x) \lesssim \|b\|_{\text{BMO}'} \omega(x) M_{\omega}(f)(x)$$

for any $1 < s < \infty$ and bounded compact supported functions $f_1, f_2$.

Similarly, we have

**Lemma 3.3.** Let $\omega \in A_1$, $\bar{b} = (b, b)$ and $b \in \text{BMO}(\omega)$. Then there exist a constant $C$ such that

$$M_2^s(M_b^{-1}(f))(x) \lesssim \|b\|_{\text{BMO}(\omega)} \omega(x) M_{\omega}(f)(x)$$

for any $1 < s < \infty$ and bounded compact supported functions $f_1, f_2$.

**Lemma 3.4.** Let $\omega \in A_1$ and $0 < p < \infty$. Then $\omega^{-p} \in A_\infty$.

**Proof.** If $0 < p \leq 1$, then $1 - p \in [0, 1)$. It is easy to see that $\omega^{-p} \in A_1 \subset A_\infty$.

If $1 < p < \infty$, it follows from $\omega \in A_1 \subset A_p$ that $\omega^{-p} \in A_{p'} \subset A_\infty$.
Lemma 3.5. Let \( \omega \in A_1 \), \( 1 < s < p_1, p_2 < \infty \) and \( 1/p = 1/p_1 + 1/p_2 \). Then both \( \mathcal{M}(\vec{f}) \) and \( \prod_{i=1}^{\sharp} M_{\omega,s}(f_i) \) are bounded from \( L^{p_1}(\omega) \times L^{p_2}(\omega) \) to \( L^p(\omega) \).

Proof. From the fact that \( M(f)(x) \lesssim M_{\omega,s}(f)(x) \) and \( M_{\omega,s}(f)(x) \) is bounded on \( L^p(\omega) \) for \( 1 < s < p_1, p_2 < \infty \), it is easy to obtain that both \( \mathcal{M}(\vec{f}) \) and \( \prod_{i=1}^{\sharp} M_{\omega,s}(f_i) \) are bounded from \( L^{p_1}(\omega) \times L^{p_2}(\omega) \) to \( L^p(\omega) \).

The following relationships between \( M_4 \) and \( M^\sharp \) to be used is a version of the classical ones due to Fefferman and Stein \[9\].

Lemma 3.6. Let \( 0 < p, \delta < \infty \) and \( \omega \in A_\infty \). There exist a positive \( C \) such that
\[
\int_{\mathbb{R}^n} (M_4 f(x))^p \omega(x) dx \leq C \int_{\mathbb{R}^n} (M^\sharp f(x))^p \omega(x) dx,
\]
for any smooth function \( f \) for which the left-hand side is finite.

Lemma 3.7. Let \( Q_0 \) be any fixed cube and \( b \) be a locally integral function. Then, for any \( x \in Q_0 \), we get
\[
\mathcal{M}(\chi_{Q_0}, \chi_{Q_0})(x) = 1; \quad (1) \\
\mathcal{M}(b \chi_{Q_0}, \chi_{Q_0})(x) = \mathcal{M}(\chi_{Q_0}, b \chi_{Q_0})(x) = M_{Q_0}(b)(x); \quad (2) \\
\mathcal{M}(b \chi_{Q_0}, b \chi_{Q_0})(x) = M_{Q_0}^2(b)(x), \quad (3)
\]
where \( M_{Q_0}(b)(x) = \sup_{Q_0 \ni Q \ni x} \frac{1}{|Q|} \int_Q |b(y)|dy \).

Proof. We only give the proof of (3) and the proof of (1), (2) are similar. For any \( x \in Q_0 \), we have
\[
M_{Q_0}^2(b)(x) = \left( \sup_{Q_0 \ni Q \ni x} \frac{1}{|Q|} \int_Q |b(y)|dy \right)^2 = \sup_{Q_0 \ni Q \ni x} \frac{1}{|Q|} \int_Q |b(y)| \chi_{Q_0}(y)dy \int_Q |b(y)| \chi_{Q_0}(y)dy \leq \mathcal{M}(b \chi_{Q_0}, b \chi_{Q_0})(x).
\]

On the other hand, for any cube \( Q \subset \mathbb{R}^n \), we can construct a cube \( Q_1 \) such that
\[
Q_0 \ni Q_1 \ni Q_0 \ni Q \ni x
\]
and \( |Q_1| \leq |Q| \). Therefore,
\[
\frac{1}{|Q|} \int_{Q \cap Q_0} |b(y)|dy \leq \frac{1}{|Q_1|} \int_{Q_1} |b(y)|dy \leq M_{Q_0}(b)(x).
\]
Thus,
\[
\mathcal{M}(b \chi_{Q_0}, b \chi_{Q_0})(x) = \sup_{Q_0 \ni Q \ni x} \left( \frac{1}{|Q|} \int_Q |b(y)| \chi_{Q_0}(y)dy \right)^2 \leq M_{Q_0}^2(b)(x),
\]
then (3) is proved.

§4 Proofs of Theorem 1.1 and Theorem 1.2

Proof of Theorem 1.1 (A1) \( \Rightarrow \) (A2): It is enough to prove Theorem 1.1 for \( f_1, f_2 \) being bounded functions with compact support. We observe that to use the Fefferman-Stein inequality, one needs to verify that certain terms in the left-hand side of the inequalities are finite.
Applying a similar argument as in [12, pp.32-33], the boundedness properties of $\mathcal{M}$ and Fatou’s lemma, one gets the desired result.

Since Lemma 3.4 and $\omega \in A_1$, then $\omega^{1-p} \in A_{\infty}$. By Lemma 3.2 and Lemma 3.3 with $1 < s < \min\{p_1, p_2\}$, from a standard argument that we can obtain
\[
\|M_{\Sigma B}(\tilde{f})\|_{L^p(\omega^{1-p})} \lesssim \|M_{\frac{1}{2}}(M_{\Sigma B}(\tilde{f}))\|_{L^p(\omega^{1-p})} \|M_{\frac{1}{2}}(M_{\Sigma B}(\tilde{f}))\|_{L^p(\omega^{1-p})}
\lesssim \|b\|_{\text{BMO}(\omega)} \left(\|M(\tilde{f})\|_{L^p(\omega)} + \|\prod_{i=1}^2 M_{\omega,s}(f_i)\|_{L^p(\omega)}\right)
\lesssim \|b\|_{\text{BMO}(\omega)} \prod_{i=1}^2 \|f_i\|_{L^p(\omega)}.
\]

(A2) $\Rightarrow$ (A1): Let $Q$ be any fixed cube. Suppose that $M_{\Sigma B}$ is bounded from $L^{p_1}(\omega) \times L^{p_2}(\omega)$ into $L^p(\omega^{1-p})$, then
\[
\|M_{\Sigma B}(\chi_Q, \chi_Q)\|_{L^p(\omega^{1-p})} \lesssim \|\chi_Q\|_{L^{p_1}(\omega)} \|\chi_Q\|_{L^{p_2}(\omega)} \lesssim \omega(Q)^{\frac{1}{p}},
\]
which implies that
\[
\frac{2}{\omega(Q)} \int_Q |b(x) - b_Q| dx = \frac{1}{\omega(Q)} \int_Q |Q|^{-\frac{1}{2}} \int_Q |b(x) - b(y_1)| \chi_Q(y_1) \chi_Q(y_2) dy_1 dy_2 dx + \frac{1}{\omega(Q)} \int_Q |Q|^{-\frac{1}{2}} \int_Q |b(x) - b(y_2)| \chi_Q(y_1) \chi_Q(y_2) dy_1 dy_2 dx
\lesssim \frac{1}{\omega(Q)} \int_Q M_{\Sigma B}(\chi_Q, \chi_Q)(x) dx \lesssim \frac{1}{\omega(Q)} \left( \int_Q |M_{\Sigma B}(\chi_Q, \chi_Q)(x)|^p \omega(x)^{1-p} dx \right)^{1/p} \left( \int_Q \omega(x)^{1/p} dx \right)^{1/p'} \lesssim \frac{1}{\omega(Q)^{1/p}} \|M_{\Sigma B}(\chi_Q, \chi_Q)\|_{L^p(\omega^{1-p})} \lesssim \|M_{\Sigma B}\|_{L^{p_1}(\omega) \times L^{p_2}(\omega) \rightarrow L^p(\omega^{1-p})}.
\]
Thus showing that $b \in \text{BMO}(\omega)$.

(A1) $\Rightarrow$ (A3): Since $\omega \in A_1$, Lemma 3.4 implies that $\omega^{1-2p} \in A_{\infty}$. From Lemma 3.1, Lemma 3.2 and Lemma 3.3 with $1 < s < \min\{p_1, p_2\}$, we get
\[
\|M_{\Pi B}(\tilde{f})\|_{L^p(\omega^{1-2p})} \lesssim \|M_{\frac{1}{2}}(M_{\Pi B}(\tilde{f}))\|_{L^p(\omega^{1-2p})} \lesssim \|M_{\frac{1}{2}}(M_{\Pi B}(\tilde{f}))\|_{L^p(\omega^{1-2p})}
\lesssim \|b\|_{\text{BMO}(\omega)} \left(\|M(\tilde{f})\|_{L^p(\omega)} + \|\prod_{i=1}^2 M_{\omega,s}(f_i)\|_{L^p(\omega)}\right)
+ \sum_{i=1}^2 \|b\|_{\text{BMO}(\omega)} \|M_{\frac{1}{2}}(M_{\Pi B}^{(i)}(\tilde{f}))(x)\|_{L^p(\omega^{1-p})}
\lesssim \|b\|_{\text{BMO}(\omega)} \prod_{i=1}^2 \|f_i\|_{L^p(\omega)}.
\]
Therefore, by Theorem 1.1 we have
\[
\frac{1}{\omega(Q)} \int_Q |b(x) - b_Q| \omega(x)^{-1} dx 
\leq \frac{1}{\omega(Q)} \int_Q |\omega(x)|^{-1} \int_Q |b(y_1) - b(y_2)| dy_1 dy_2 dx 
\leq \frac{1}{\omega(Q)} \int_Q M_{\Pi b}(\chi Q, \chi Q)(x) \omega(x)^{-1} dx 
\leq \frac{1}{\omega(Q)} \int_Q |M_{\Pi b}(\chi Q, \chi Q)(x)|^p \omega(x)^{1-2p} dx \left( \int_Q \omega(x) dx \right)^{1/p'} 
\leq \frac{1}{\omega(Q)^{1/p}} \| M_{\Pi b}(\chi Q, \chi Q) \|_{L^p(\omega^{-2})} 
\leq \| M_{\Pi b} \|_{L^p(\omega) \times L^2(\omega) \rightarrow L^p(\omega^{-2})}.
\]

Thus we complete the proof of Theorem 1.1.

---

**Proof of Theorem 1.2.** (B1) \( \Rightarrow \) (B2): By the definition of \( \mathcal{M}(\vec{f}) \), we have
\[
M(b f_1, f_2)(x) = M(|b| f_1, f_2)(x), \quad M(f_1, b f_2)(x) = M(f_1, |b| f_2)(x).
\]
Then
\[
|b, M|[1] \vec{f}(x) = M(|b| f_1, f_2)(x) - M(b f_1, f_2)(x) - b x M \vec{f}(x) + M(|b| f_1, f_2)(x) 
\leq b^{-1}(x) M \vec{f}(x).
\]
Similarly, we also have
\[
|b, M|[2] \vec{f}(x) = M(|b| f_1, f_2)(x) \leq b^{-1}(x) M \vec{f}(x).
\]
Since \(|a - c| \leq |a| - |c|\) for any real numbers \(a\) and \(c\), there holds
\[
|b, M|[i] \vec{f}(x) \leq M_b[i] \vec{f}(x),
\]
for \(i = 1, 2\). This shows that
\[
|\Sigma \vec{b}, M|[1] \vec{f}(x) \leq M_{\Sigma \vec{b}} \vec{f}(x) + b^{-1}(x) M \vec{f}(x).
\]
Applying (4) and Theorem 1.1 we have
\[
\| |\Sigma \vec{b}, M|[1] \vec{f}(x) \|_{L^p(\omega^{-p})} \leq \| M_{\Sigma \vec{b}} \vec{f}(x) \|_{L^p(\omega^{-p})} + \| b^{-1} M \vec{f}(x) \|_{L^p(\omega^{-p})} 
\leq (\| b^{-1}/\omega \|_{L^\infty} + \| b \|_{BMO(\omega)}) \| f_1 \|_{L^p(\omega)} \| f_2 \|_{L^2(\omega)}.
\]

Therefore, \(b \in \text{BMO}(\omega)\) with \(b^{-1}/\omega \in L^\infty\) implies that \([\Sigma \vec{b}, M]\) is bounded from \(L^p(\omega) \times L^2(\omega)\) to \(L^p(\omega^{-p})\).

---

(B2) \( \Rightarrow \) (B1): Let \(Q_0\) be any fixed cube. By Lemma 3.7, for any \(x \in Q_0\),
\[
b(x) = b(x) M(\chi_{Q_0}, \chi_{Q_0})(x),
M_{Q_0}(b)(x) = M(b \chi_{Q_0}, \chi_{Q_0})(x) = M(\chi_{Q_0}, b \chi_{Q_0})(x),
\]
and
\[
M_{Q_0}(b)(x) = b(x) M(\chi_{Q_0}, \chi_{Q_0})(x) = M_{Q_0}(b)(x).
\]
Then,
\[
\frac{2}{\omega(Q_0)} \int_{Q_0} |b(x) - M_{Q_0}(b)(x)| \, dx
\]
\[=
\frac{2}{\omega(Q_0)} \int_{Q_0} |b(x)M(\chi_{Q_0}, \chi_{Q_0})(x) - M(b\chi_{Q_0}, \chi_{Q_0})(x)| \, dx
\]
\[=
\frac{1}{\omega(Q_0)} \int_{Q_0} |b(x)M(\chi_{Q_0}, \chi_{Q_0})(x) - M(b\chi_{Q_0}, \chi_{Q_0})(x) + b(x)M(\chi_{Q_0}, \chi_{Q_0})(x) - M(\chi_{Q_0}, b\chi_{Q_0})(x)| \, dx
\]
\[\lesssim \frac{1}{\omega(Q_0)} \int_{Q_0} \|\Sigma B, M\| \chi_{Q_0}(x) \, dx
\]
\[\lesssim \frac{1}{\omega(Q_0)} \left( \int_{Q_0} \|\Sigma B, M\| \chi_{Q_0}(x) \right)^p \omega(x)^{1-p} \, dx \right)^{1/p} \cdot \left( \int_{Q_0} \omega(x) \, dx \right)^{1/p'}
\]
\[\lesssim \frac{1}{\omega(Q_0)} \|\Sigma B, M\| \chi_{Q_0}(x) \|_{L^p(\omega ^{1-p})} \]
\[\lesssim \frac{1}{\omega(Q_0)} \|\Sigma B, M\| \chi_{Q_0}(x) \|_{L^p(\omega ^{1-p})}.
\]
Now, we have all the ingredients to prove \( b \in \text{BMO}(\omega) \) and \( b^-/\omega \in L^\infty \).

In order to show that \( b^-/\omega \in L^\infty \), observe that for any \( x \in Q_0 \), \( M_{Q_0}(b)(x) \geq |b(x)| \).
Therefore,
\[
0 \leq b^-(x) \leq M_{Q_0}(b)(x) - b^+(x) + b^- (x) = M_{Q_0}(b)(x) - b(x),
\]
which gives
\[
\frac{1}{|Q_0|} \int_{Q_0} \frac{b^-(x)}{\omega(x)} \, dx \leq \frac{1}{|Q_0|} \int_{Q_0} \frac{b^- (x)}{\omega(x)} \, dx \cdot \frac{1}{\inf_{x \in Q_0} \omega(x)}
\]
\[\leq \frac{1}{|Q_0|} \int_{Q_0} |b(x) - M_{Q_0}(b)(x)| \, dx \cdot \frac{|Q_0|}{\omega(Q_0)}
\]
\[\leq \|\Sigma B, M\| \|_{L^{p_1}(\omega) \times L^{p_2}(\omega) \rightarrow L^p(\omega ^{1-p})},
\]
this yields that
\[
(b^-/\omega)_{Q_0} \leq \|\Sigma B, M\| \|_{L^{p_1}(\omega) \times L^{p_2}(\omega) \rightarrow L^p(\omega ^{1-p})}.
\]
Thus, the boundedness of \( b^-/\omega \) follows from Lebesgue’s differentiation theorem.

\((B1) \Rightarrow (B3)\): Let \( \vec{B} = (|b|, b) \) and \( \vec{B} = (|b|, b) \). Then
\[
\left| \Pi \vec{B}, M \right| (f_1, f_2)(x) - \left| \Pi \vec{B}, M \right| (f_1, f_2)(x)
\]
\[\lesssim \left| b(x)b(x)M(\vec{f})(x) - b(x)M(\vec{f})(x) - |b(x)|b(x)M(\vec{f})(x) + |b(x)|M(\vec{f})(x) - b(x)M(\vec{f})(x) \right|
\]
\[\lesssim b^- (x) \| \vec{B}, M \|^2 (f_1, f_2)(x).
\]
Similarly, we also have
\[
\left| [\Pi_b, \mathcal{M}] (f_1, f_2)(x) - [\Pi_b, \mathcal{M}] (f_1, f_2)(x) \right|
\leq \left| b(x) |b(x)| \mathcal{M}(\tilde{f})(x) - |b(x)| \mathcal{M}(bf_1, f_2)(x) \right|
- |b(x)| b(x) \mathcal{M}(\tilde{f})(x) + |b(x)| \mathcal{M}(bf_1, f_2)(x)
\leq b^{-}(x) \left| [b, \mathcal{M}] (f_1, f_2)(x) \right|
\]

Noting that
\[
||[\Pi_b, \mathcal{M}](\tilde{f})(x)|| \leq \mathcal{M}_{\Pi_b}(\tilde{f})(x),
\]
which yields that
\[
||[\Pi_b, \mathcal{M}](\tilde{f})(x)|| \leq \mathcal{M}_{\Pi_b}(\tilde{f})(x) + b^{-}(x) \mathcal{M}_{\Sigma_B}(\tilde{f})(x) + (b^{-}(x))^2 \mathcal{M}(\tilde{f})(x).
\]
It follows from Theorem 1.1 and \(b^{-}/\omega \in L^\infty\) that
\[
||[\Pi_b, \mathcal{M}](\tilde{f})(x)||_{L^p(\omega^{1-2p})}
\leq \mathcal{M}_{\Pi_b}(\tilde{f})(x) + ||b^{-} \mathcal{M}_{\Sigma_B}(\tilde{f})||_{L^p(\omega^{1-2p})} + ||b^{-} \mathcal{M}(\tilde{f})||_{L^p(\omega^{1-2p})}
\leq ||b||_{BMO(\omega)} ||f_1||_{L^{p_1}(\omega)} ||f_2||_{L^{p_2}(\omega)} + ||b^{-}/\omega||_{L^\infty} ||\mathcal{M}_{\Sigma_B}(\tilde{f})||_{L^p(\omega^{1-2p})}
\leq (||b^{-}/\omega||_{L^\infty} + ||b||_{BMO(\omega)})^2 ||f_1||_{L^{p_1}(\omega)} ||f_2||_{L^{p_2}(\omega)}
\]
this leads to our results.

\(B3 \Rightarrow B1\): Let \(Q_0\) be any fixed cube. By Lemma 3.5, for any \(x \in Q_0\),
\[
b(x)^2 = b(x)^2 \mathcal{M}(\chi_{Q_0}, \chi_{Q_0})(x),
b(x)M_{Q_0}(b)(x) = b(x)\mathcal{M}(b\chi_{Q_0}, \chi_{Q_0})(x) = b(x)\mathcal{M}(\chi_{Q_0}, b\chi_{Q_0})(x),
\]
Then,
\[
\frac{1}{\omega(Q_0)} \int_{Q_0} |b(x) - M_{Q_0}(b)(x)|^2 \omega(x)^{-1} dx
\]
\[
= \frac{1}{\omega(Q_0)} \int_{Q_0} (b(x)^2 - 2b(x)M_{Q_0}(b)(x) + M_{Q_0}^2(b)(x)) \omega(x)^{-1} dx
\]
\[
= \frac{1}{\omega(Q_0)} \int_{Q_0} [\Pi_b, \mathcal{M}](\chi_{Q_0}, \chi_{Q_0})(x) \omega(x)^{-1} dx
\]
\[
\leq \frac{1}{\omega(Q_0)} \left( \int_{Q_0} \left( [\Pi_b, \mathcal{M}](\chi_{Q_0}, \chi_{Q_0})(x) \right)^p \omega(x)^{1-2p} dx \right)^{1/p} \left( \int_{Q_0} \omega(x) dx \right)^{1/p'}
\]
\[
= \frac{1}{\omega(Q_0)^{1/p}} \left\| [\Pi_b, \mathcal{M}](\chi_{Q_0}, \chi_{Q_0}) \right\|_{L^p(\omega^{1-2p})}
\]
\[
\leq \left\| [\Pi_b, \mathcal{M}] \right\|_{L^{p_1}(\omega) \times L^{p_2}(\omega) \to L^p(\omega^{1-2p})}.
\]
Now, we have all the ingredients to prove $b \in \text{BMO}(\omega)$ and $b^{-}/\omega \in L^\infty$. 
\[
\frac{1}{\omega(Q_0)} \int_{Q_0} |b(x) - b_{Q_0}|^2 \omega(x)^{-1} dx
\]
\[
\lesssim \frac{1}{\omega(Q_0)} \int_{Q_0} |b(x) - M_{Q_0}(b)(x)|^2 \omega(x)^{-1} dx
\]
\[
+ \frac{1}{\omega(Q_0)} \int_{Q_0} |b_{Q_0} - M_{Q_0}(b)(x)|^2 \omega(x)^{-1} dx
\]
\[
\lesssim \frac{1}{\omega(Q_0)} \int_{Q_0} |b(x) - M_{Q_0}(b)(x)|^2 \omega(x)^{-1} dx
\]
\[
\lesssim \|[[\vec{B}, \mathcal{M}]]_{L^p(\omega) \times L^p(\omega) \to L^p(\omega^{1-2p})}.
\]
which implies that $b \in \text{BMO}^2(\omega)$; that is, $b \in \text{BMO}(\omega)$.

For any $x \in Q_0$, we have
\[
\frac{1}{|Q_0|} \int_{Q_0} \frac{b^-(x)}{\omega(x)} dx \lesssim \frac{1}{|Q_0|} \int_{Q_0} b^-(x) dx \cdot \frac{1}{\inf_{x \in Q_0} \omega(x)}
\]
\[
\lesssim \frac{1}{|Q_0|} \int_{Q_0} |b(x) - M_{Q_0}(b)(x)| dx \cdot \frac{|Q_0|}{\omega(Q_0)}
\]
\[
\lesssim \left( \frac{1}{\omega(Q_0)} \int_{Q_0} |b(x) - M_{Q_0}(b)(x)|^2 \omega(x)^{-1} dx \right)^{1/2}
\]
\[
\lesssim \|[[\vec{B}, \mathcal{M}]]_{L^p(\omega) \times L^p(\omega) \to L^p(\omega^{1-2p})},
\]
which yields 
\[
(b^{-}/\omega)_{Q_0} \lesssim \|[[\vec{B}, \mathcal{M}]]_{L^p(\omega) \times L^p(\omega) \to L^p(\omega^{1-2p})}.
\]
Thus, the boundedness of $b^{-}/\omega$ follows from Lebesgue’s differentiation theorem.

The proof of Theorem 1.2 is complete. \hfill \Box

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