Abstract: We firstly generalize the fuzzy way-below relation on an $L$-poset, and consider its continuity by means of this relation. After that, we introduce a kind of stratified $L$-generalized convergence structure on an $L$-poset. In terms of that, $L$-fuzzy Scott topology and fuzzy Scott topology are considered, and the properties of fuzzy Scott topology are discussed in detail. At last, we investigate the Scott convergence of stratified $L$-filters on an $L$-poset, and show that an $L$-poset is continuous if and only if the Scott convergence on it coincides with the convergence with respect to the corresponding topological space.

Keywords: $L$-poset, Convergence structure, Scott topology, Stratified $L$-filter

MSC: 03E72, 06A06, 54A40

1 Introduction

Ordered structure and topological structure are two basic and crucial structures in Mathematics, closely related to each other. Many works have been done to compare and combine the two structures [1–6]. At the beginning, classical Scott convergence and Scott topology are, in view of the theory of continuous lattices, only defined for complete lattices. Not very soon, these definitions have been found to be very fruitful for dcpos [2]. Unfortunately, they are not fit for arbitrary partially ordered sets (posets), since the join of a directed subset is involved in the definition of Scott convergence, which may not exist in a poset. Regarding this, several alternative choices have been proposed to generalize the definition of Scott convergence in posets [1, 6–11], and the Scott topology related to Scott convergence has also been studied.

In recent years, quantitative domain theory has attracted many people because it provides a model of concurrent systems. Wagner’s $\Omega$-categories [12], Rutten’s generalized metric spaces [13] and Flagg’s continuity spaces [14] are examples of quantitative domain theory. Fan and Zhang [15, 16] studied quantitative domain via fuzzy set theory, where fuzzy partial order was clearly proposed. After analysis, it is easily seen that $\Omega$-categories could be regarded as a fuzzy preordered set in [15, 16], and a fuzzy partial ordered set (an $L$-poset) is equivalent to an $L$-ordered set introduced by Bělohlávek [17, 18]. Later on, Lai and Zhang [19, 20] studied complete and directed-complete $\Omega$-categories, and their continuity was also discussed. Following [15, 16, 19], Yao [21, 22] studied the continuity of fuzzy dcpos, and further extended the Scott convergence and Scott topology on classical dcpos to fuzzy setting. But the results in [22] do not adapt to fuzzy partially ordered sets as well as in the classical case, even the continuity needs to be modified. This provides sufficient motivations for this paper. We firstly redefine the fuzzy way-below relation on $L$-posets and reconsider the continuity. Then we introduce a kind of stratified $L$-generalized convergence structure.
on $L$-posets, and restudy fuzzy Scott topology associating with it. Finally, we establish the Scott convergence theory on $L$-posets, and prove that it is an effective tool to characterize the continuity.

The paper is organized as follows. In Section 2, we recall some necessary definitions and results needed later on. In Section 3, we give a fuzzy way-below relation on $L$-posets and based on that the continuity for $L$-posets is considered. In Section 4, we introduce a new stratified $L$-generalized convergence structure on $L$-posets, then study and characterize fuzzy Scott topology. In Section 5, the properties of Scott convergence are given, the description for continuous $L$-posets via Scott convergence is constructed. In the final section, we summarize the results and draw a conclusion.

2 Preliminaries

A complete residuated lattice [23] $L$ is a structure $(L, \ast, \rightarrow, \lor, \land, 0, 1)$, such that (1) $(L, \lor, \land, 0, 1)$ is a complete lattice with the greatest element 1 and the least element 0; (2) $(L, \ast, 1)$ is a commutative monoid with the identity 1 and $\ast$ is isotone at both arguments; (3) $(\ast, \rightarrow)$ is an adjoint pair, i.e., $x \ast y \leq z$ iff $x \leq y \rightarrow z$ for all $x, y, z \in L$.

Some basic properties of complete residuated lattices are collected here ([17, 23, 24]).

1. $1 \rightarrow a = a$;
2. $a \leq b$ iff $a \rightarrow b = 1$;
3. $(a \rightarrow (b \rightarrow c)) \leq a \rightarrow c$;
4. $(a \rightarrow (b \rightarrow c)) = b \rightarrow (a \rightarrow c)$ $(a \ast b) \rightarrow c$;
5. $a \leq b \rightarrow (a \ast b)$;
6. $a \ast (a \rightarrow b) \leq b$;
7. $a \ast (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \ast b_i)$, $a \ast (\bigwedge_{i \in I} b_i) \leq \bigwedge_{i \in I} (a \ast b_i)$;
8. $a \rightarrow (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \rightarrow b_i)$, $(\bigvee_{i \in I} a_i) \rightarrow b = \bigvee_{i \in I} (a_i \rightarrow b)$;
9. $a \rightarrow b \geq b$;
10. $b \rightarrow c \leq (a \rightarrow b) \rightarrow (a \rightarrow c)$, $b \rightarrow c \leq (c \rightarrow a) \rightarrow (b \rightarrow a)$;
11. $(a \rightarrow b) \rightarrow b \geq a$;
12. $(a \rightarrow b) \ast (c \rightarrow d) \leq (a \ast c) \ast (b \ast d)$.

Let $X$ be a nonempty set, $L^X$ denote the set of all $L$-subsets of $X$. For $A, B \in L^X$, define:

$(A \cap B)(x) = A(x) \land B(x)$, $(A \cup B)(x) = A(x) \lor B(x)$,
$(A \ast B)(x) = A(x) \ast B(x)$, $(A \rightarrow B)(x) = A(x) \rightarrow B(x)$.

Then $(L^X, \ast, \rightarrow, \lor, \land, 0, 1)$ is also a complete residuated lattice, and we never discriminate the constant value function $\bar{a}$ with $a$, e.g., $(a \ast A)(x) = a \ast A(x)$ and $(a \rightarrow A)(x) = a \rightarrow A(x)$ for every $x \in X$

A complete residuated lattice $L$ with $\ast = \land$ is just a complete Heyting algebra (or a frame). Throughout this paper, $L$ always denotes a complete Heyting algebra.

Fuzzy order was first introduced by Zadeh [25], from then on, different kinds of fuzzy order have been introduced and studied by different authors (the reader is referred to [4, 12, 15, 16, 26–28]). In this paper, we adopt the definition of fuzzy order in [15, 16].

**Definition 2.1.** A fuzzy (partial) order $e$ (also called an $L$-order) on $X$ is an $L$-relation satisfying:

1. $\forall x \in X, e(x, x) = 1$;
2. $\forall x, y, z \in X, e(x, y) \land e(y, z) \leq e(x, z)$;
3. $\forall x, y \in X, e(x, y) = e(y, x) = 1 \Rightarrow x = y$.

Then $(X, e)$ is called a fuzzy (partially) ordered set or an $L$-poset for simplicity.

**Remark 2.2.** In [17, 18], an $L$-preordered set is defined to be a triple $(X, R, \approx)$, where $\approx$ is an $L$-equality on $X$ and $R$ is an $L$-preorder on $X$ which is compatible with $\approx$. It is verified in [18, 22] that if $R$ is compatible with $\approx$, it must hold that $\approx = R \land R^P$. Thus, the $L$-equality $\approx$ is completely determined by $R$, so it can be omitted in the definition.
Example 2.3 ([15, 22, 26]).
(1) In a complete residuated lattice \( L \), define \( e_L : L \times L \to L \) by \( e_L(x, y) = x \to y \) for all \( x, y \in L \). Then \((L, e_L)\) is an \( L \)-poset.
(2) For all \( A, B \in L^X \), define \( \text{sub}(A, B) = \bigwedge_{x \in X} A(x) \to B(x) \), then \((L^X, \text{sub})\) is an \( L \)-poset where \( \text{sub} \) called the fuzzy inclusion order, and \( \text{sub}(A, B) \) is explained as the subsethood degree or fuzzy inclusion degree of \( A \) in \( B \).

Definition 2.4 ([16, 21, 22]). Let \((X, e)\) be an \( L \)-poset and \( z \in X \), \( A \in L^X \). Then
1. \( \forall^A \in L^X \) is defined by \( \forall x \in X \), \( A^x(x) = \bigwedge_{z \in X} (A(y) \to e(x, y)) \).
2. \( \exists^A \in L^X \) is defined by \( \exists x \in X \), \( A^x(x) = \bigvee_{y \in X} (A(y) \to e(x, y)) \).
3. \( \forall \in L^X \) is defined by \( \forall x \in X \), \( \forall(x) = \bigwedge_{y \in X} (A(y) \land e(x, y)) \), and \( \exists \in L^X \) is defined dually.

Moreover, \( A \) is called a lower \( L \)-set or fuzzy lower set if \( A(x) \land e(x, y) \leq A(y) \) for all \( x, y \in X \). \( A \) is called an upper \( L \)-set or fuzzy upper set if \( A(x) \land e(x, y) \leq A(y) \) for all \( x, y \in X \).

Lemma 2.5 ([16, 22]). Let \((X, e)\) be an \( L \)-poset. Then for all \( x, y \in X \),
\[
e(x, y) = \bigwedge_{z \in X} (e(z, x) \to e(z, y)) = \bigwedge_{z \in X} (e(y, z) \to e(x, z)).
\]

Definition 2.6 ([19, 21, 22]). Let \((X, e)\) be an \( L \)-poset and \( D \in L^X \). Then \( D \) is called a fuzzy directed set or directed \( L \)-set if
1. \( \forall x \in X \), \( D(x) \leq 1 \);
2. \( \forall x, y \in X \), \( D(x) \wedge D(y) \leq \bigvee_{z \in X} (D(z) \land e(x, z) \land e(y, z)) \).

A fuzzy directed set \( I \in L^X \) is called a fuzzy ideal if it is also a fuzzy lower set. The set of all fuzzy ideals of \((X, e)\) is denoted by \( I_L(X) \).

Definition 2.7 ([16, 19, 21, 22]). Let \((X, e)\) be an \( L \)-poset, \( A \in L^X \), \( x_0 \in X \). \( x_0 \) is called a fuzzy join (resp., fuzzy meet) of \( A \) denoted by \( \sqcup A \) (resp., \( \sqcap A \)) if
1. \( \forall x \in X \), \( A(x) \leq e(x, x_0) \) (resp., \( A(x) \leq e(x_0, x) \));
2. \( \forall y \in X \), \( \bigvee_{x \in X} A(x) \to e(x, y) \leq e(x_0, y) \) (resp., \( \bigwedge_{x \in X} A(x) \to e(y, x) \leq e(y, x_0) \)).

It is easy to see that the fuzzy join or the fuzzy meet is unique if it exists.

Theorem 2.8 ([19, 21, 26]). Let \((X, e)\) be an \( L \)-poset, \( A \in L^X \), \( x_0 \in X \). Then
1. \( x_0 = \sqcup A \iff \forall y \in X, e(x_0, y) = \bigwedge_{x \in X} (A(x) \to e(x, y)) \);
2. \( x_0 = \sqcap A \iff \forall y \in X, e(y, x_0) = \bigwedge_{x \in X} (A(x) \to e(y, x)) \).

Lemma 2.9. [20] Let \((X, e)\) be a complete \( L \)-lattice, and \( x \in X \), \( \{y_i\}_{i \in I} \subseteq X \). Then we have \( e(x, \bigwedge_{i \in I} y_i) = \bigwedge_{i \in I} e(x, y_i) \) and \( e(\bigvee_{i \in I} y_i, x) = \bigvee_{i \in I} e(y_i, x) \), where \( \bigwedge \) and \( \bigvee \) respectively, denote join and meet in the underlying poset \( \overline{X} = (X, \leq_e) \).

Proposition 2.10. [21, 22, 29] Let \((X, e)\) be an \( L \)-poset. Then
1. \( A^I(x) = \text{sub}(A, \uparrow x) \), \( A^\downarrow x = \text{sub}(A, \downarrow x) \);
2. \( e(\sqcup A, x) = \text{sub}(A, \sqcup x) \), \( e(x, \sqcap A) = \text{sub}(A, \sqcap x) \) with \( \sqcup A \) and \( \sqcap A \) exist;
3. \( e(x, y) \land \text{sub}(\uparrow y, A) \leq \text{sub}(\uparrow y, A) \);
4. \( \text{sub}(A, B) \leq e(\sqcup A, \sqcup B) \), \( \text{sub}(A, B) \leq e(\sqcap A, \sqcap B) \) with \( \sqcup A, \sqcap A, \sqcup B \) and \( \sqcap B \) exist;
5. If \( A \) is a lower \( L \)-set, then \( \text{sub}(\downarrow x, A) = A(x) \);
6. If \( A \) is an upper \( L \)-set, then \( \text{sub}(\uparrow x, A) = A(x) \).
Definition 2.11 ([21, 22, 26, 27]). Let $(X, e_X), (Y, e_Y)$ be $L$-posets and $f : X \to Y$ be a map. Then $f$ is said to be $L$-order-preserving or $L$-monotone (resp., $L$-antitone) if $ex(x, y) \leq ey(f(x), f(y))$ (resp., $ex(x, y) \geq ey(f(y), f(x))$) for all $x, y \in X$.

Definition 2.12 ([21, 22]). Let $(X, e_X), (Y, e_Y)$ be $L$-posets, $f : X \to Y$, $g : Y \to X$ be $L$-order-preserving maps. Then $(f, g)$ is called a fuzzy Galois connection between $X$ and $Y$ if $e_Y(f(x), y) = ex(x, g(y))$ for all $x \in X, y \in Y$, where $f$ is called the left adjoint of $g$ and dually $g$ the right adjoint of $f$.

3 The continuity of $L$-posets via fuzzy way-below relation

Way-below relation was first imported for investigating the continuity of complete lattices. It was also an effective tool to describe the continuity of dcpos. This observation had inspired several authors to study continuous posets [30–32]. Unfortunately, these works on posets were rather restrictive since the definition of the way-below relation only considers certain directed subsets, of which join exists. In view of this deficiency, Erné [33] introduced another tool to describe the continuity of dcpos. This observation had inspired several authors to study continuous posets via fuzzy way-below relation.

Proposition 3.2. Let $(X, e)$ be an $L$-poset. Then

1. $\forall x \in X, I \in \mathcal{I}_L(X), \coprod_{y \in X} e(x, y) \leq I(x)$.
2. $\forall x, y \in X, \bigwedge_{z \in X} e(x, z) \leq \downarrow y(x)$
3. $\forall x \in X, \downarrow x \leq \downarrow x$.
4. $\forall x, y, v \in X, e(u, x) \wedge \downarrow y(x) \wedge e(y, v) \leq \downarrow v(u)$.

Proof. (1). Refer to Proposition 5.7(1) in [21].

(2). $\downarrow y(x) = \bigwedge_{I \in \mathcal{I}_L(X)} I^{ul}(y) \rightarrow I(x)$.

(3). $\downarrow x = \bigwedge_{I \in \mathcal{I}_L(X)} I^{ul}(x) \rightarrow I \leq (\downarrow x)^{ul}(x) \rightarrow \downarrow x = \downarrow x$.

(4). For any $I \in \mathcal{I}_L(X)$, we have $e(u, x) \land (I^{ul}(y) \rightarrow I(x)) \land e(y, v) \land I^{ul}(v) \leq e(u, x) \land (I^{ul}(y) \rightarrow I(x)) \land I^{ul}(y)$.
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Then $e(u, x) \land (I^{ul}(y) \to I(x)) \land e(y, v) \leq I^{ul}(v) \to I(u)$.
Thus, $e(u, x) \land \downarrow y(x) \land e(y, v) = e(u, x) \land \bigwedge_{I \in \mathcal{I}_L(X)} (I^{ul}(y) \to I(x)) \land e(y, v) \leq \bigwedge_{I \in \mathcal{I}_L(X)} I^{ul}(v) \to I(u) = \downarrow v(u)$. \hfill \Box

**Proposition 3.3.** If $(X, e)$ is a continuous $L$-poset, then for all $x \in X$, $(\downarrow x)^u = \uparrow x$, and so $(\downarrow x)^{ul}(x) = 1$.

**Proof.** Suppose $x \in X$, then for every $y \in X$,

$$(\downarrow x)^u(y) = \bigwedge_{z \in X} (\downarrow x(z) \to e(z, y))$$

$${\text{sub}}(\downarrow x, \downarrow y)$$

$$= e(\uparrow \downarrow x, y)$$

$$= \uparrow x(y).$$

That is, $(\downarrow x)^u = \uparrow x$. Thus $(\downarrow x)^{ul}(x) = \downarrow x(x) = 1$. \hfill \Box

**Theorem 3.4.** If $(X, e)$ is a continuous $L$-poset, then $\downarrow y(x) = \bigvee_{z \in X} (\downarrow z(x) \land \downarrow y(z))$ for all $x, y \in X$.

**Proof.** At first, by Proposition 3.2(3)(4), we have $\bigvee_{z \in X} (\downarrow z(x) \land \downarrow y(z)) \leq \bigvee_{z \in X} (e(x, z) \land \downarrow y(z) \land e(y, y)) \leq \downarrow y(x)$. Next, define $A \in L^X$ by $A(x) = \bigvee_{z \in X} (\downarrow z(x) \land \downarrow y(z))$ for every $x \in X$. We only need to show that $\downarrow y(x) \leq A(x)$.

As shown in the proof of Theorem 5.9 in [22], we can see that $A \in \mathcal{I}_L(X)$.

Furthermore,

$$A^{ul}(y) = \bigwedge_{w \in X} (A^u(w) \to e(y, w))$$

$$= \bigwedge_{w \in X} \left( \bigwedge_{t \in X} (A(t) \to e(t, w)) \to e(y, w) \right)$$

$$= \bigwedge_{w \in X} \left( \bigwedge_{t \in X} \left( \bigvee_{z \in X} (\downarrow z(t) \land \downarrow y(z)) \to e(t, w) \right) \to e(y, w) \right)$$

$$= \bigwedge_{w \in X} \left( \bigwedge_{t \in X} \left( \bigwedge_{z \in X} (\downarrow z(t) \to e(t, w)) \to e(y, w) \right) \right)$$

$$= \bigwedge_{w \in X} \left( \bigwedge_{z \in X} (\downarrow y(z) \to \bigwedge_{t \in X} (\downarrow z(t) \to e(t, w)) \to e(y, w) \right)$$

$$= \bigwedge_{w \in X} \left( \bigwedge_{z \in X} \left( \downarrow y(z) \to sub(\downarrow z, \downarrow w) \right) \to e(y, w) \right)$$

$$= \bigwedge_{w \in X} \left( \bigwedge_{z \in X} \left( \downarrow y(z) \to e(\uparrow \downarrow z, w) \right) \to e(y, w) \right)$$

$$= \bigwedge_{w \in X} \left( \bigwedge_{z \in X} \left( \downarrow y(z) \to e(z, w) \right) \to e(y, w) \right)$$

$$= \bigwedge_{w \in X} \left( \downarrow (e(\uparrow \downarrow y, w) \to e(y, w)) \right)$$

$$= \bigwedge_{w \in X} (e(\uparrow \downarrow y, w) \to e(y, w))$$

$$= 1.$$

Therefore, $\downarrow y(x) = \bigwedge_{I \in \mathcal{I}_L(X)} I^{ul}(y) \to I(x) \leq A^{ul}(y) \to A(x) \equiv 1 \to A(x) = A(x)$, as needed. \hfill \Box
Theorem 3.5. Let \((X, e)\) be an \(L\)-poset, and \(\mathcal{I}_L^*(X) = \{I \in \mathcal{I}_L(X) : \sqcup I \text{ exists}\}\). Then \((X, e)\) is continuous iff \((\sqsubseteq, \sqcup)\) is a fuzzy Galois connection between \((X, e)\) and \((\mathcal{I}_L^*(X), \text{sub})\).

Proof. To show the necessity, assume \((X, e)\) is continuous. Clearly both \(\sqsubseteq\) and \(\sqcup\) are fuzzy order-preserving. \(\forall x \in X, I \in \mathcal{I}_L^*(X)\), then \(\text{sub}(\sqsubseteq x, I) \leq e(\sqcup \sqsubseteq x, \sqcup I) = e(x, \sqcup I)\). Furthermore,

\[
\text{sub}(\sqsubseteq x, I) = \bigwedge_{y \in X} (\sqsubseteq x(y) \rightarrow I(y)) \geq \bigwedge_{y \in X} (I^{ul}(x) \rightarrow I(y)) \\
= \bigwedge_{y \in X} (I^{ul}(x)) \\
= \bigwedge_{y \in X} (\text{sub}(I, \sqsubseteq y) \rightarrow e(x, y)) \\
= \bigwedge_{y \in X} (e(\sqcup I, y) \rightarrow e(x, y)) \\
= e(x, \sqcup I).
\]

Thus \(\text{sub}(\sqsubseteq x, I) = e(x, \sqcup I)\), and it implies that \((\sqsubseteq, \sqcup)\) forms a fuzzy Galois connection.

To show the sufficiency, suppose that \((\sqsubseteq, \sqcup)\) is a fuzzy Galois connection between \((X, e)\) and \((\mathcal{I}_L^*(X), \text{sub})\). Then for every \(x \in X\), we have \(\sqsubseteq x\) is directed, and \(e(x, \sqcup \sqsubseteq x) = \text{sub}(\sqsubseteq x, \sqsubseteq x) = 1\). On the other hand, \(e(\sqcup \sqsubseteq x, x) = \text{sub}(\sqsubseteq x, \sqsubseteq x) = 1\). So \(x = \sqcup \sqsubseteq x\), it implies that \((X, e)\) is continuous. \(\Box\)

4 Fuzzy Scott topology on \(L\)-posets

Classical Scott topology on complete lattices and dcpos is studied in [2]. After that, many works have been done to generalize that theory on posets [1, 6]. Recently, fuzzy Scott topology has been investigated on fuzzy ordered sets with the necessary condition that fuzzy joins of all directed fuzzy set exist (i.e. dcpos) [22]. In the absence of any sort of join, the previous result is invalid, so an additional consideration for fuzzy Scott topology on \(L\)-posets is needed. This is our motivation for this section.

Definition 4.1. [34, 36, 37] An \(L\)-filter on \(X\) is a map \(\mathcal{F} : L^X \rightarrow L\) satisfying that:

(LF1) \(\mathcal{F}(0) = 0\) and \(\mathcal{F}(1) = 1\);
(LF2) \(\forall A, B \in L^X,\mathcal{F}(A \land B) = \mathcal{F}(A) \land \mathcal{F}(B)\).

An \(L\)-filter is called stratified if it satisfies the following condition:

(SF) \(\forall a \in L,\mathcal{F}(a) \geq a\) or \(\mathcal{F}(a \land B) \geq a \land \mathcal{F}(B)\).

The set of all stratified \(L\)-filters on \(X\) will be denoted by \(\mathcal{F}^s_L(X)\).

Remark 4.2. The condition (LF2) in Definition 4.1 can be equivalently replaced by

\[
\forall A, B \in L^X, \mathcal{F}(A \rightarrow B) \leq \mathcal{F}(A) \rightarrow \mathcal{F}(B).
\]

Moreover, for any \(\mathcal{F} \in \mathcal{F}^s_L(X)\), we have \(\text{sub}(A, B) \leq \mathcal{F}(A \rightarrow B)\) for all \(A, B \in L^X\).

Example 4.3. [34, 35]
(1) For any \(x \in X\), define a map \([x] : L^X \rightarrow L\) as \([x](A) = A(x)\) for every \(A \in L^X\). Then \([x]\) is a stratified \(L\)-filter, called the principal \(L\)-filter of \(x\).
(2) Let \((X, \tau)\) be an L-fuzzy topological space and \(x \in X\). Define \(\mathcal{U}^X_\tau : L^X \to L\) by
\[
\forall A \in L^X, \mathcal{U}^X_\tau(A) = \bigvee_{B \leq A} B(x) \wedge \tau(B).
\]
Then \(\mathcal{U}^X_\tau\) is an L-filter, and it is stratified if \(\tau\) is enriched.

(3) Let \((X, \delta)\) be an L-topological space and \(x \in X\). Define \(\mathcal{U}^X_\delta : L^X \to L\) by \(\forall A \in L^X, \mathcal{U}^X_\delta(A) = A^\circ(x)\), where \(\circ\) is the L-interior operator of \((X, \delta)\). Then \(\mathcal{U}^X_\delta\) is an L-filter, and if \(\delta\) is stratified then so is \(\mathcal{U}^X_\delta\).

**Definition 4.4** ([34, 35, 38]). A stratified L-generalized convergence structure on \(X\) is a map \(R : \mathbb{F}^A_L(X) \times X \to L\) satisfying that
(1) \(\forall x \in X, R([x], x) = 1\);
(2) \(\forall x \in X, \forall \mathcal{F}, \mathcal{G} \in \mathbb{F}^A_L(X), \mathcal{F} \leq \mathcal{G} \Rightarrow R(\mathcal{F}, x) \leq R(\mathcal{G}, x)\);

If \(R\) is a stratified L-generalized convergence structure on \(X, x \in X\). Define \(\mathcal{U}^X_R : L^X \to L\) by
\[
\forall A \in L^X, \mathcal{U}^X_R(A) = \bigwedge_{\mathcal{F} \in \mathbb{F}^A_L(X)} (R(\mathcal{F}, x) \to \mathcal{F}(A)).
\]
Then we have the following theorem.

**Theorem 4.5** ([34]). Each stratified L-generalized convergence structure \(R\) on \(X\) induces an enriched L-fuzzy topology \(\tau_R\) on \(X\) given by
\[
\forall A \in L^X, \tau_R(A) = \bigwedge_{x \in X} \left( A(x) \to \bigwedge_{\mathcal{F} \in \mathbb{F}^A_L(X), \mathcal{F} \geq \mathcal{U}^X_R} \mathcal{F}(A) \right).
\]
and a stratified L-topology \(\delta_R = \{ A \in L^X : \tau_R(A) = 1 \}\).

Elicited by the well-known results, we aim to study topologies on an L-poset, then the consideration of a kind of convergence structures on it will be effective. To reach that goal we begin with the discussion of the lower bound of a stratified L-filter.

Let \((X, e)\) be an L-poset and \(\mathcal{F} \in \mathbb{F}^A_L(X)\). Define \(\mathcal{J}^l \in L^X\) by
\[
\forall x \in X, \mathcal{J}^l(x) = \bigvee_{A \in L^X} \mathcal{F}(A) \wedge A^l(x).
\]

**Proposition 4.6** ([22]). Let \((X, e)\) be an L-poset and \(\mathcal{F}, \mathcal{G} \in \mathbb{F}^A_L(X)\). Then
(1) \(\mathcal{J}^l(x) = \mathcal{F}(\mathcal{G}(x))\);
(2) \(\mathcal{F} \leq \mathcal{G} \Rightarrow \mathcal{J}^l \leq \mathcal{J}^l\);
(3) \(\forall x \in X, [x]^l \downarrow x\);

For a fuzzy ideal on \((X, e)\), define \(\mathcal{F}_I : L^X \to L\) by
\[
\forall A \in L^X, \mathcal{F}_I(A) = \bigvee_{x \in X} (I(x) \wedge \text{sub}(\mathcal{F}(x), A)).
\]
Then \(\mathcal{F}_I\) is a stratified L-filter on \(X\), and \(\mathcal{J}^l = I\) (refer to [22] for detail).

Let \((X, e)\) be an L-poset. Define a map \(S : \mathbb{F}^A_L(X) \times X \to L\) by
\[
\forall (\mathcal{F}, x) \in \mathbb{F}^A_L(X) \times X, S(\mathcal{F}, x) = \bigvee_{I \in \mathbb{I}_L(X)} \left( \text{sub}(I, \mathcal{F}^l) \wedge I^l(x) \right).
\]
It is easily seen that \(S\) is a stratified L-generalized convergence structure on \(X\), and \(S(\mathcal{F}, x)\) can be interpreted as the degree of \(\mathcal{F}\) Scott converges to \(x\). Moreover, we define \(\mathcal{U}^X_S : L^X \to L\) by
\[
\forall A \in L^X, \mathcal{U}^X_S(A) = \bigwedge_{\mathcal{F} \in \mathbb{F}^A_L(X)} (S(\mathcal{F}, x) \to \mathcal{F}(A)).
\]
Then $\mathcal{U}_S^X$ is a stratified $L$-filter.

By Theorem 4.5, there is an enriched $L$-fuzzy topology associated with $S$. We denote it as $\sigma_{LF}(X, e) \ (\sigma_{LF}(X)$ for short), that is,

$$\forall A \in L^X \, \sigma_{LF}(X, e)(A) = \bigwedge_{x \in X} \left( A(x) \Rightarrow \mathcal{U}_S^x(A) \right).$$

Furthermore, let

$$\sigma_L(X, e) = \{ A \in L^X : \sigma_{LF}(X, e)(A) = 1 \}.$$ 

Then $\sigma_L(X, e) \ (\sigma_L(X)$ for short) is a stratified $L$-topology called fuzzy Scott topology on $(X, e)$. We say an $L$-subset $A$ is fuzzy Scott open if $A \in \sigma_L(X)$.

**Proposition 4.7.** For $A \in L^X$, the following are equivalent:

1. $A$ is fuzzy Scott open;
2. $\forall x \in X$, $A(x) \leq \bigwedge_{I \in \mathcal{I}_L(X)} \left( I^{ul}(x) \Rightarrow \mathcal{F}_I(A) \right)$;
3. $A$ is an upper $L$-set, and $A(x) \leq \bigwedge_{I \in \mathcal{I}_L(X)} \left( I^{ul}(x) \Rightarrow \bigvee_{y \in X} (I(y) \land A(y)) \right)$ for all $x \in X$.

**Proof.** (1)$\Rightarrow$(2). Since $A$ is fuzzy Scott open, so for all $x \in X$, we have

$$A(x) \leq \bigwedge_{I \in \mathcal{I}_L(X)} \left( S(I, x) \Rightarrow \mathcal{F}(A) \right)$$

$$\leq \bigwedge_{I \in \mathcal{I}_L(X)} \left( S(I, x) \Rightarrow \mathcal{F}_I(A) \right)$$

$$\leq \bigwedge_{I \in \mathcal{I}_L(X)} \left( I^{ul}(x) \Rightarrow \mathcal{F}_I(A) \right).$$

(2)$\Rightarrow$(3). Above all, $A(x) \leq \bigwedge_{I \in \mathcal{I}_L(X)} \left( I^{ul}(x) \Rightarrow \mathcal{F}_I(A) \right)$ follows immediately from (2) and the fact that $\text{sub}(\uparrow x, A) \leq A(x)$. Next, for $\forall x, y \in X$,

$$A(x) \land e(x, y) \leq e(x, y) \land \bigwedge_{I \in \mathcal{I}_L(X)} \left( I^{ul}(x) \Rightarrow \mathcal{F}_I(A) \right)$$

$$\leq e(x, y) \land \left( (\downarrow x)^{ul}(x) \Rightarrow \mathcal{F}_{\downarrow x}(A) \right)$$

$$= e(x, y) \land \mathcal{F}_{\downarrow x}(A)$$

$$\leq e(x, y) \land \text{sub}(\uparrow x, A) \leq A(y).$$

It implies that $A$ is an upper $L$-set.

(3)$\Rightarrow$(1). For any $x \in X$ and $\mathcal{F} \in \mathcal{B}_L^{S}(X)$,

$$A(x) \land S(\mathcal{F}, x) = A(x) \land \bigvee_{I \in \mathcal{I}_L(X)} \left( \text{sub}(I, \mathcal{F}^I) \land I^{ul}(x) \right)$$

$$= \bigvee_{I \in \mathcal{I}_L(X)} \left( A(x) \land I^{ul}(x) \land \text{sub}(I, \mathcal{F}^I) \right)$$

$$\leq \bigvee_{I \in \mathcal{I}_L(X)} \bigvee_{y \in X} \left( A(y) \land I(y) \land \text{sub}(I, \mathcal{F}^I) \right)$$

$$\leq \bigvee_{y \in X} \left( A(y) \land \mathcal{F}^I(y) \right)$$

$$= \bigvee_{y \in X} \left( A(y) \land \mathcal{F}(\uparrow y) \right)$$

$$\leq \bigvee_{y \in X} \mathcal{F}(A(y) \land \uparrow y)$$

$$\leq \mathcal{F}(A).$$

So $A$ is fuzzy Scott open. \qed


Theorem 4.8. If \((X, e)\) is a continuous \(L\)-poset, and \(x \in X\), then \(\uparrow x\) is fuzzy Scott open where \(\uparrow x(y) = \downarrow y(x)\) for every \(y \in X\).

Proof. First, \(\uparrow x\) is an upper \(L\)-set obviously for \(\forall x \in X\).

For \(\forall y \in X, \forall I \in \mathcal{I}_L(X)\), we have \(\uparrow x(y) = \downarrow y(x) = \bigvee_{z \in X} (\downarrow z(x) \wedge \downarrow y(z))\). Since \(\downarrow y(z) = \bigwedge_{J \in \mathcal{I}(X)} (J^{ul}(y) \rightarrow J(z)) \leq I^{ul}(y) \rightarrow I(z)\), so
\[
\uparrow x(y) \leq \bigvee_{z \in X} \left( \uparrow x(z) \wedge \left( I^{ul}(y) \rightarrow I(z) \right) \right)
\leq \bigvee_{z \in X} \left( I^{ul}(y) \rightarrow \left( \uparrow x(z) \wedge I(z) \right) \right)
\leq I^{ul}(y) \rightarrow \bigvee_{z \in X} \left( \uparrow x(z) \wedge I(z) \right).
\]

As follows by Proposition 4.7(3), \(\uparrow x\) is fuzzy Scott open.

Theorem 4.9. If \((X, e)\) is a continuous \(L\)-poset, and \(A \subseteq L^X\), then \(A\) is fuzzy Scott open iff \(A\) is an upper \(L\)-set, and \(A(x) = \bigvee_{y \in X} (A(y) \wedge \downarrow x(y))\) for all \(x \in X\).

Proof. Necessity: Suppose \(A\) is fuzzy Scott open, then by Proposition 4.7, \(A\) is an upper \(L\)-set and for all \(x \in X\), we have
\[
A(x) \leq \left( \downarrow x \right)^{ul}(x) \rightarrow \mathcal{F}_{\downarrow x} \left( A \right)
= \mathcal{F}_{\downarrow x} \left( A \right)
= \bigvee_{y \in X} \left( \downarrow x(y) \wedge \text{sub}(\uparrow y, A) \right)
\leq \bigvee_{y \in X} \left( \downarrow x(y) \wedge A(y) \right).
\]

Clearly, \(A(x) = \bigvee_{y \in X} \left( A(y) \wedge \downarrow x(y) \right) \geq \bigvee_{y \in X} \left( A(y) \wedge \downarrow x(y) \right)\). Therefore, \(A(x) = \bigvee_{y \in X} \left( A(y) \wedge \downarrow x(y) \right)\).

Sufficiency: By Proposition 4.7, it suffices to show that for all \(x \in X\), \(I \in \mathcal{I}_L(X)\), \(A(x) \wedge I^{ul}(x) \leq \bigvee_{y \in X} \left( I(y) \wedge A(y) \right)\). In fact, by the conditions,
\[
A(x) \wedge I^{ul}(x) = \bigvee_{y \in X} \left( A(y) \wedge \downarrow x(y) \wedge I^{ul}(x) \right)
\leq \bigvee_{y \in X} \left( A(y) \wedge I^{ul}(x) \wedge (I^{ul}(x) \rightarrow I(y)) \right)
\leq \bigvee_{y \in X} \left( A(y) \wedge I(y) \right).
\]

Definition 4.10. Let \((X, e)\) be a \(L\)-poset, for any \(\lambda \in L\) and \(x \in X\), define \(\uparrow_{\lambda} x = \lambda \wedge \uparrow x\).

It is worth noting that \(\uparrow_{\lambda} x \in \sigma_L(X)\) in a continuous \(L\)-poset \((X, e)\) since \(\sigma_L(X)\) is stratified.

Theorem 4.11. Let \((X, e)\) be a continuous \(L\)-poset, then \(\{ \uparrow_{\lambda} x : \lambda \in L, x \in X \}\) is a basis of \(\sigma_L(X)\).

Proof. If \(A \in \sigma_L(X)\), then by the above theorem, for any \(x \in X\), \(A(x) = \bigvee_{y \in X} (A(y) \wedge \downarrow x(y)) = \bigvee_{y \in X} (A(y) \wedge \uparrow x(y)) = \bigvee_{y \in X} \uparrow_{A(y)} y(x)\). This implies that \(A = \bigvee_{y \in X} \uparrow_{A(y)} y\).

Theorem 4.12. Let \((X, e)\) be an \(L\)-poset, then for any \(x \in X\), \(\uparrow x^{\circ} \leq \uparrow x\) and \(\uparrow x^{\circ} = \uparrow x\) when \((X, e)\) is continuous, where \(\circ\) is the \(L\)-interior operator with respect to \(\sigma_L(X)\).
Proof. Suppose \( A \in \sigma_L(X) \) with \( A \leq \uparrow x \). Then by Proposition 4.7, for all \( y \in X \), we have
\[
A(y) \leq \bigwedge_{I \in \mathcal{I}_L(X)} \left( I^{ul}(y) \rightarrow \bigvee_{z \in Z} (I(z) \wedge A(z)) \right)
\]
\[
\leq \bigwedge_{I \in \mathcal{I}_L(X)} \left( I^{ul}(y) \rightarrow \bigvee_{z \in Z} (I(z) \wedge e(x, z)) \right)
\]
\[
\leq \bigwedge_{I \in \mathcal{I}_L(X)} (I^{ul}(y) \rightarrow I(x))
\]
\[
= \uparrow x(y).
\]
So \((\uparrow x)^0 = \bigvee \{ A \in \sigma_L(X) : A \leq \uparrow x \} \leq \uparrow x \). If \((X, e)\) is continuous, then \(\uparrow x \in \sigma_L(X)\) and \(\uparrow x \leq \uparrow x\). It implies \(\uparrow x \leq (\uparrow x)^0\), and so \((\uparrow x)^0 = \uparrow x\). \(\square\)

5 Scott convergence on \(L\)-posets

Usually, convergence theory cannot be ignored when considering topology. As shown before, on an \(L\)-poset, \(L\)-fuzzy Scott topology and fuzzy Scott topology naturally exist. A deeper problem arises in order to be compatible with the convergence under the related topology: how to define a fruitful convergence on an \(L\)-poset? This section will give the answer.

Definition 5.1. Let \((X, e)\) be an \(L\)-poset, \(x \in X\) and \(F\) a stratified \(L\)-filter on \(X\). Then we say \(F\) is Scott convergent to \(x\) if there exists \(I \in \mathcal{I}_L(X)\) such that \(I \leq F^L\) and \(I^{ul}(x) = 1\). We denote this by \(F \rightarrow_x \).

Proposition 5.2. Let \(I\) be a fuzzy ideal on an \(L\)-poset \((X, e)\). Then \(I^{ul}(x) \leq S(F_I, x)\) for all \(x \in X\).

Proof. By Proposition 3.4 in [38], we have \(F^L_I = I\). Thus for every \(x \in X\), \(S(F_I, x) = \bigvee_{I \in \mathcal{I}_L(X)} (\text{sub}(I, F^L_I) \wedge I^{ul}(x)) = \text{sub}(I, I) \wedge I^{ul}(x) = I^{ul}(x)\). \(\square\)

Corollary 5.3. If \((X, e)\) is a continuous \(L\)-poset, then for all \(x \in X\), \(S(F_{\uparrow x}, x) = 1\), i.e., \(F_{\uparrow x} \rightarrow_x \).

Proof. Since \((X, e)\) is continuous, then for all \(x \in X\), \(\downarrow x \in \mathcal{I}_L(X)\) and \(\uparrow \downarrow x = x\). Thus,
\[
(\downarrow x)^{ul}(x) = \bigwedge_{y \in X} \left( (\downarrow x)^{ul}(y) \rightarrow e(x, y) \right)
\]
\[
= \bigwedge_{y \in X} (\text{sub}(\downarrow x, y) \rightarrow e(x, y))
\]
\[
= \bigwedge_{y \in X} (e(\uparrow \downarrow x, y) \rightarrow e(x, y))
\]
\[
= 1.
\]
By Proposition 5.2, \(S(F_{\uparrow x}, x) \geq (\downarrow x)^{ul}(x) = 1\). \(\square\)

Recall that for an \(L\)-fuzzy topology \(\tau\) on \(X\), we call a stratified \(L\)-filter \(F\) is convergent to \(x \in X\), denoting \(F \rightarrow_\tau x\), if \(U^x_\tau \leq F\). Under an \(L\)-topology \(\delta\) on \(X\), a stratified \(L\)-filter \(F\) convergent to \(x \in X\), denoting \(F \rightarrow_\delta x\), if \(U^x_\delta \leq F\).

Proposition 5.4. Let \((X, e)\) be an \(L\)-poset, \(x \in X\) and \(F\) a stratified \(L\)-filter on \(X\). Then \(F \rightarrow_{\sigma_L(X)} x\) implies \(F \rightarrow_{\sigma_L(X)} x\).

Proof. By the definition, we only need to show that \(U^{x}_{\sigma_L(X)} \leq U^{x}_{\sigma_L(X)}\). In fact, for any \(A \in L^X\),
\[
U^{x}_{\sigma_L(X)}(A) = \bigvee \{ B(x) : B \leq A, \sigma_L(X)(B) = 1 \}
\]
\[ \leq \bigvee_{B \leq A} (B(x) \land \sigma_{LF}(X)(B)) \]

\[ = U_{\sigma_{LF}(X)}^X(A). \]

So \( U_{\sigma_{LF}(X)}^X \leq U_{\sigma_{LF}(X)}^X \), as needed. \( \square \)

**Proposition 5.5.** Let \((X, e)\) be an \(L\)-poset, then \((U_{\sigma_{LF}(X)}^X)^I \leq (U_{\sigma_{LF}(X)}^X)^I \leq \downarrow x\) for all \(x \in X\). If \((X, e)\) is continuous, then \((U_{\sigma_{LF}(X)}^X)^I = (U_{\sigma_{LF}(X)}^X)^I \leq \downarrow x\).

**Proof.** At first, \((U_{\sigma_{LF}(X)}^X)^I \leq (U_{\sigma_{LF}(X)}^X)^I \) just follows from the fact \(U_{\sigma_{LF}(X)}^X \leq U_{\sigma_{LF}(X)}^X\) for all \(x \in X\), which is shown in the above proposition.

Next, if \( y \in X \) and \( B \leq \uparrow y \), then for \( \forall I \in \mathcal{I}_L(X) \), we have

\[ B(x) \land \sigma_{LF}(X)(B) \land I^\text{ul}(x) = B(x) \land I^\text{ul}(x) \land \bigwedge_{(F, z) \in E(L)^X} (B(z) \land S(F, z) \rightarrow F(B)) \leq B(x) \land I^\text{ul}(x) \land \bigwedge_{z \in X} (B(z) \land I^\text{ul}(z) \rightarrow F_I(y)) \leq B(x) \land I^\text{ul}(x) \land \left( B(x) \land I^\text{ul}(x) \rightarrow I(y) \right) \leq I(y). \]

Hence, \( B(x) \land \sigma_{LF}(X)(B) \leq \bigwedge_{I \in \mathcal{I}_L(X)} (I^\text{ul}(x) \rightarrow I(y)) = \downarrow x(y) \). So \( (U_{\sigma_{LF}(X)}^X)^I(y) = U_{\sigma_{LF}(X)}^X(\uparrow y) = \bigwedge_{B \leq \uparrow y} (B(x) \land \sigma_{LF}(X)(B)) \leq \downarrow x(y) \) for every \( y \in X \), it implies \( (U_{\sigma_{LF}(X)}^X)^I \leq (U_{\sigma_{LF}(X)}^X)^I \leq \downarrow x \).

If \((X, e)\) is continuous, then \( \sigma_{LF}(X)(\uparrow y) = 1 \). Therefore, for every \( x \in X \), \( (U_{\sigma_{LF}(X)}^X)^I(y) = (U_{\sigma_{LF}(X)}^X)(\uparrow y) = \{ B(x) : B \leq \uparrow y, \sigma_{LF}(X)(B) = 1 \} \geq \uparrow y(x) = \downarrow x(y) \).

Thus, \( (U_{\sigma_{LF}(X)}^X)^I = (U_{\sigma_{LF}(X)}^X)^I = \downarrow x \). \( \square \)

**Proposition 5.6.** Let \((X, e)\) be an \(L\)-poset, \((F, x) \in \mathbb{F}^X_L(X) \times X\). Then \( S(F, x) \leq \text{sub}(U_{\sigma_{LF}(X)}^X, F)\). So \( F \rightarrow_s x \) implies \( F \rightarrow_{\sigma_{LF}(X)} x \).

**Proof.** For any \((F, x) \in \mathbb{F}^X_L(X) \times X\) and any \( A \in L^X\),

\[ U_{\sigma_{LF}(X)}^X(A) = \bigvee_{B \leq A} B(x) \land \sigma_{LF}(X)(B) \leq \bigvee_{B \leq A} B(x) \land (B(x) \land S(F, x) \rightarrow F(B)) \leq S(F, x) \rightarrow F(A) \]

Thus, \( \text{sub}(U_{\sigma_{LF}(X)}^X, F) = \bigwedge_{A \in L^X} \left( U_{\sigma_{LF}(X)}^X(A) \rightarrow F(A) \right) \geq S(F, x) \).

**Theorem 5.7.** Let \((X, e)\) be an \(L\)-poset, then the following are equivalent:

1. \((X, e)\) is continuous;
2. \( \forall F \in \mathbb{F}^X_L(X), F \rightarrow_s x \Leftrightarrow F \rightarrow_{\sigma_{LF}(X)} x \Leftrightarrow F \rightarrow_{\sigma_{LF}(X)} x \);
3. \( \forall x \in X, U_{\sigma_{LF}(X)}^X(x) \) is Scott convergent to \( x \);
4. \( \forall x \in X, U_{\sigma_{LF}(X)}^X(x) \) is Scott convergent to \( x \).

**Proof.** (1) \( \Rightarrow \) (2). For \( \forall F \in \mathbb{F}^X_L(X) \), clearly \( F \rightarrow_s x \Rightarrow F \rightarrow_{\sigma_{LF}(X)} x \Rightarrow F \rightarrow_{\sigma_{LF}(X)} x \) by Proposition 5.4 and Proposition 5.6, we only need to show \( F \rightarrow_{\sigma_{LF}(X)} x \Rightarrow F \rightarrow_s x \). If \( F \rightarrow_{\sigma_{LF}(X)} x \), then \( (U_{\sigma_{LF}(X)}^X)^I \leq F^I \). Since \((X, e)\) is continuous, then \( \downarrow x \in \mathcal{I}_L(X) \) and \( \downarrow x = (U_{\sigma_{LF}(X)}^X)^I \leq \downarrow x \). Thus \( S(F, x) \geq \text{sub}(\downarrow x, F^I) \land (\downarrow x)^{ul}(x) = 1 \). That is, \( F \rightarrow_s x \).

(2) \( \Rightarrow \) (3) is obvious.

(3) \( \Rightarrow \) (4) follows immediately from the fact \( U_{\sigma_{LF}(X)}^X \leq U_{\sigma_{LF}(X)}^X \).
(4)⇒(1). For all \( x \in X \), since \( \mathcal{U}^x_{\mathcal{I}_L(X)} \) is Scott convergent to \( x \), then there exists \( I \in \mathcal{I}_L(X) \) such that \( I \leq (\mathcal{U}^x_{\mathcal{I}_L(X)})^I \leq \downarrow x \) and \( I^\cup(x) = 1 \). If \( a, b \in X \), then

\[
\downarrow x(a) \land \downarrow x(b) = \bigwedge_{J \in \mathcal{I}_L(X)} (J^\cup(x) \to J(a)) \land \bigwedge_{J \in \mathcal{I}_L(X)} (J^\cup(x) \to J(b)) \\
\leq I(a) \land I(b) \\
\leq \bigvee_{c \in X} (I(c) \land e(a, c) \land e(b, c)) \\
\leq \bigvee_{c \in X} (\downarrow x(c) \land e(a, c) \land e(b, c)).
\]

Thus \( \downarrow x \) is directed by definition. Furthermore, for any \( y \in X \), we have

\[
\bigwedge_{z \in X} (\downarrow x(z) \to e(z, y)) \geq \bigwedge_{z \in X} (\downarrow x(z) \to e(z, y)) = e(x, y)
\]

and

\[
\bigwedge_{z \in X} (\downarrow x(z) \to e(z, y)) \leq \bigwedge_{z \in X} (I(z) \to e(z, y)) \\
= \text{sub}(I, \downarrow y) \land I^\cup(x) \\
\leq I^\cup(y) \land (I^\cup(y) \to e(x, y)) \\
\leq e(x, y).
\]

Hence, \( \bigwedge_{z \in X} (\downarrow x(z) \to e(z, y)) = e(x, y) \), and it implies that \( x = \uparrow \downarrow x \). So \((X, e)\) is continuous. \( \square \)

**Theorem 5.8.** Let \((X, e)\) be a continuous \(L\)-poset, \(x \in X\) and \(F\) a stratified \(L\)-filter on \(X\). Then \(F\) is Scott convergent to \(x\) iff \(\downarrow x \leq F^\cup\).

**Proof.** The sufficiency is obvious. To show the necessity, assume that \(F\) is Scott convergent to \(x\). There exists \(I \in \mathcal{I}_L(X)\) such that \(I \leq F^\cup\) and \(I^\cup(x) = 1\). For all \(y \in X\), \(\downarrow x(y) \leq I^\cup(x) \to I(y) \leq F^\cup(y)\). Thus \(\downarrow x \leq F^\cup\). \( \square \)

### 6 Conclusion

In this paper, we first extend the fuzzy way-below relation on fuzzy dcpos to fuzzy ordered sets without any additional conditions, and based on that, the continuity for \(L\)-posets is studied. Later on, we propose a kind of stratified \(L\)-generalized convergence structure, and then study fuzzy Scott topology. The Scott convergence theory on \(L\)-posets is established finally, and the continuity is well described by Scott convergence. That is, an \(L\)-poset is continuous if and only if Scott convergence coincides with convergence under either \(L\)-fuzzy Scott topology or fuzzy Scott topology. All the works will promote the development of quantitative domain theory.

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