AN ELEMENTARY AND CONSTRUCTIVE SOLUTION TO HILBERT’S 17TH PROBLEM FOR MATRICES

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Abstract. We give a short and elementary proof of a theorem of Procesi, Schacher and (independently) Gondard, Ribenboim that generalizes a famous result of Artin. Let be an \( n \times n \) symmetric matrix with entries in the polynomial ring \( \mathbb{R}[x_1, \ldots, x_m] \). The result is that if \( A \) is positive semidefinite for all substitutions \((x_1, \ldots, x_m) \in \mathbb{R}^m\), then \( A \) can be expressed as a sum of squares of symmetric matrices with entries in \( \mathbb{R}(x_1, \ldots, x_m) \). Moreover, our proof is constructive and gives explicit representations modulo the scalar case.

We shall give an elementary proof of the following theorem. Recall that a real matrix is positive semidefinite if it is symmetric with all nonnegative eigenvalues.

**Theorem 1.** Let \( A \) be a symmetric matrix with entries in \( \mathbb{R}[x_1, \ldots, x_m] \). If \( A \) is positive semidefinite for all substitutions \((x_1, \ldots, x_m) \in \mathbb{R}^m\), then \( A \) can be expressed as a sum of squares of symmetric matrices with entries in \( \mathbb{R}(x_1, \ldots, x_m) \).

This generalizes the following famous result of Artin on nonnegative polynomials; it is the starting point for a large body of work relating positivity and algebra.

**Theorem 2** (Artin). If \( f \in \mathbb{R}[x_1, \ldots, x_n] \) is nonnegative for all substitutions \((x_1, \ldots, x_n) \in \mathbb{R}^n\), then \( f \) is a sum of squares of rational functions in \( \mathbb{R}(x_1, \ldots, x_n) \).

Theorem 1 was originally proved in [3] and (within a general framework) in [7], although a formulation involving elements in a number field was already considered in [2]. Like Artin’s result, it guarantees algebraic certificates to (matrix) nonnegativity. However, the known proofs are nonconstructive, employing either model theory [3] or ultraproducts [7]. In contrast, we use only basic facts about real closed fields and linear algebra to give an explicit and elegant proof of Theorem 1.

Recall that a field \( F \) is real if \(-1\) is not a sum of squares in \( F \), and a real closed field \( R \) is a real field such that any algebraic extension of \( R \) that is real must be equal to \( R \). Real closed fields have a unique ordering, the nonnegative elements being the squares. For instance, \( \mathbb{R}(x_1, \ldots, x_m) \) is a real field and \( \mathbb{R} \) is real closed. A principal minor of a matrix is a determinant of a submatrix determined by the same row and column indices. The set of symmetric matrices over \( \mathbb{R} \) with all principal minors nonnegative coincides with the set of positive semidefinite matrices (see for example [4] p. 405), a fundamental relationship we exploit below. We will prove the following generalization of Theorem 1 to the setting of real fields.

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Proof of Theorem 3. Let $F$ be a real field and let $A$ be a symmetric matrix with entries in $F$. If the principal minors of $A$ can be expressed as sums of squares in $F$, then $A$ is a sum of squares of symmetric matrices with entries in $F$.

To see how Theorem 3 follows from Theorem 5 consider a principal minor $p(x_1, \ldots, x_m) \in \mathbb{R}[x_1, \ldots, x_m]$ of the matrix $A$. By assumption, it will be nonnegative for all substitutions $(x_1, \ldots, x_m) \in \mathbb{R}^m$, and therefore, Artin’s theorem implies that it is a sum of squares of rational functions. We may now invoke Theorem 3.

As another application, consider positive semidefinite matrices $A \in \mathbb{R}^{n \times n}$. Standard matrix theory allows one to write $A = B^2$ for a symmetric $B$ with entries that are algebraic numbers; however, Theorem 3 tells us that $A$ is actually a sum of squares of rational matrices. This follows since any nonnegative rational number $a/b = ab/b^2$ can be written as a sum of four rational squares by Lagrange’s theorem.

To prove Theorem 3 we begin with a lemma. For the basic theory of real closed fields (RCF) we will need, we refer the reader to [5, 6]. The main observation is that a symmetric matrix $A \in \mathbb{R}^{n \times n}$ has all nonnegative principal minors is diagonalizable over $R$ with nonnegative eigenvalues, just as is the case for $\mathbb{R}$.

Lemma 4. Suppose that $A$ satisfies the statement of Theorem 3. Then the minimal polynomial $p(t) \in F[t]$ of $A$ is of the form:

$$p(t) = \sum_{i=0}^{m} (-1)^{m-i} a_i t^i = t^m - a_{m-1} t^{m-1} + \cdots + (-1)^m a_0$$

for $a_i$ that are sums of squares of elements of $F$. Moreover, $a_1 \neq 0$.

Proof. Express the minimal polynomial of $A$ as in the statement of the theorem. We first make the following observation. Let $R$ be any real closure of $F$; this induces an ordering on $R$, in which the principal minors of $A$ are nonnegative (they are sums of squares). Since $A$ is diagonalizable over $R$ and has nonnegative eigenvalues, it follows that each $a_i \geq 0$ and also that $p(t)$ has no repeated roots.

Suppose now that some $a_i$ was not a sum of squares in $F$. Then there is an ordering of $F$ with $a_i$ negative. Let $R$ be a real closure of $F$ that extends the ordering on $F$. By above, $a_i$ is nonnegative, a contradiction. To verify the second claim, first notice that $t^2$ does not divide $p(t)$ so that $a_0$ and $a_1$ cannot both be 0. In a real closure of $F$, the coefficient $a_1$ is a sum of products of (nonnegative) roots of $p(t)$. It follows that if $a_1 = 0$, we have $(-1)^m a_0 = p(0) = 0$. Thus, $a_1 \neq 0$.

Proof of Theorem 3. Let $A$ be a symmetric matrix satisfying the hypotheses of the theorem. Also, let $p(t)$ be the minimal polynomial for $A$, which has the form prescribed by Lemma 4. For notational simplicity, we assume that $m$ is odd, although the argument is the same when $m$ is even. Since $p(A) = 0$, it follows that

$$(A^{m-1} + a_{m-2} A^{m-3} + \cdots + a_1 I) A = a_{m-1} A^{m-1} + a_{m-3} A^{m-3} + \cdots + a_0 I.$$ 

Set $B = A^{m-1} + \cdots + a_1 I$, which is invertible (since $a_1 \neq 0$, in any real closure of $F$, it is diagonalizable with strictly positive eigenvalues). Therefore, we have

$$A = B \cdot (a_{m-1} B^{-2} A^{m-1} + a_{m-3} B^{-2} A^{m-3} + \cdots + a_0 B^{-2}).$$

Since $B$ is a sum of squares and $B$ and $B^{-1}$ commute with $A$, the result follows.

Notice that our argument gives a commuting sum of squares representation, the existence of which was also observed in [7]. We close with two examples to illustrate the construction from our proof.
Example 5. The following symmetric matrix is always positive semidefinite:

\[
A = \begin{bmatrix}
1 & x_1 x_2 \\
x_1 x_2 & 1 + x_1^4 x_2^2 + x_1^2 x_2^4
\end{bmatrix}.
\]

However, it is not a sum of squares of matrix polynomials. To see this, let \( x = [1, -1]^T \) and suppose that \( A \) is a sum of polynomial squares; then so is the polynomial \( f(x_1, x_2) = x^T A x = 2 + x_1^2 x_2^2 + x_1^2 x_2^4 - 2x_1 x_2 \). Thus, we can express \( f = \sum_i p_i^2 \) for some polynomials \( p_i \) with \( \deg(p_i) \leq 3 \). Comparing coefficients, \( p_i \) cannot contain the monomials \( x_1, x_1^3, x_1^2, x_2, x_1 x_2, x_1 \) or \( x_2 \) so that we can write \( p_i = a_i + b_i x_1^2 x_2 + c_i x_1 x_2^2 \) for some \( a_i, b_i, c_i \in \mathbb{R} \). However, then we cannot produce the term \(-2x_1 x_2\) in \( f \), a contradiction. Similarly, \( \det(A) \) is not a sum of polynomial squares. It is, however, a sum of rational squares since \( (x_1^2 + x_2^2) \det(A) \) equals:

\[
\left( x_2 - \frac{1}{2}x_1^2 x_2 \right)^2 + \left( x_1 - \frac{1}{2}x_1^2 x_2 \right)^2 + 2 \left( x_1 x_2 - \frac{1}{2}x_1 x_2^3 - \frac{1}{2}x_1^2 x_2 \right)^2 + \\
\frac{3}{4} (x_1^2 x_2^4 + x_1^4 x_2^2) + \frac{1}{2} (x_1 x_2^3 + x_1^3 x_2)^2.
\]

Since \( A^2 - \text{tr}(A)A + \det(A)I = 0 \), we have the rational squares representation:

\[
A = \text{tr}(A) \left( (\text{tr}(A)^{-1}A)^2 + \det(A) (\text{tr}(A)^{-1}I)^2 \right). \quad \square
\]

Example 6. The following matrix is positive semidefinite for all substitutions:

\[
A = \begin{bmatrix}
x_1^2 + 2x_3^2 & -x_1 x_2 & -x_1 x_3 \\
-x_1 x_2 & x_2^2 + 2x_1 & -x_2 x_3 \\
-x_1 x_3 & -x_2 x_3 & x_3^2 + 2x_2
\end{bmatrix},
\]

but it is not a sum of polynomial squares \( \square \). Its minimal polynomial has coefficients

\[
a_2 = 3x_3^2 + 3x_2^2 + 3x_1^2, \quad a_1 = 2x_2^4 + 6x_1^2 x_2^2 + 6x_1^2 x_2^2 + 6x_1^4 + 2x_2^4 + 6x_2^2 x_3^2,
\]

\[
a_0 = 4x_1^2 x_2^3 + 4x_3^2 x_2^2 + 4x_1^2 x_3^2 + 4x_3^2 x_1^2 + 4x_3^2 x_1^2 x_2^2,
\]

which are all sums of squares. From formula \( \square \), we have

\[
A = (A^2 + a_1 I) \left( a_2 (A + a_1 A^{-1})^{-2} + a_0 (A^2 + a_1 I)^{-2} \right). \quad \square
\]

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