Bike-Sharing Systems under Markovian Environment

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Abstract

To reduce automobile exhaust pollution, traffic congestion and parking difficulties, bike-sharing systems are rapidly developed in many countries and more than 500 major cities in the world over the past decade. In this paper, we discuss a large-scale bike-sharing system under Markovian environment, and propose a mean-field matrix-analytic method in the study of bike-sharing systems through combining the mean-field theory with the time-inhomogeneous queues as well as the nonlinear QBD processes. Firstly, we establish an empirical measure process to express the states of this bike-sharing system. Secondly, we apply the mean-field theory to establishing a time-inhomogeneous $MAP(t)/MAP(t)/1/K + 2L + 1$ queue, and then to setting up a system of mean-field equations. Thirdly, we use the martingale limit theory to show the asymptotic independence of this bike-sharing system, and further analyze the limiting interchangeability as $N \to \infty$ and $t \to +\infty$. Based on this, we discuss and compute the fixed point in terms of a nonlinear QBD process. Finally, we analyze performance measures of this bike-sharing system, such as, the mean of stationary bike number at any station and the stationary probability of problematic stations. Furthermore, we use numerical examples to show how the performance measures depend on the key parameters of this bike-sharing system. We hope the methodology and results of this paper are applicable in the study of more general large-scale bike-sharing systems.

Keywords: Bike-sharing system; Markovian environment; mean-field theory; time-inhomogeneous queue; nonlinear QBD process; The mean-field matrix-analytic method; fixed point; probability of problematic stations.
1 Introduction

The bike-sharing systems are fast developing wide-spread adoption in major cities around the world, and are becoming a public mode of transportation devoted to short trips with mobility service of one-way use. Also, the bike-sharing systems are being regarded as a promising solution to many public transportation problems including automobile exhaust pollution, traffic congestion, parking difficulty, transportation noise and so forth. Larsen [37] reported that over 500 cities host advanced bike-sharing systems with a combined fleet of more than half a million bicycles up to April, 2013, also see Shaheen and Guzman [64] and Meddin and DeMaio [54] for more details. As a history overview of the bike-sharing systems, DeMaio [17] and Shaheen et al. [65] gave detailed comments and interpretations, and DeMaio [15] provided a valuable prospect in the 21st century. For the developing status of the bike-sharing systems launched in some countries or major cities, readers may refer to, such as, the United States by DeMaio and Gifford [16], France by Faye [18], the European countries by Janett and Hendrik [33], London by Lathia et al. [39], Montreal by Morency et al. [55], Beijing by Liu et al. [51], and a number of famous cities by Shu et al. [67]. While the synthesis of the literature for the bike-sharing systems was given by Fishman et al. [19] and Labadi et al. [36].

Some key issues in recent research on design, operations and optimization of the bike-sharing systems may be divided into two different classes. The first class is on the design issues, including number of stations, location of stations, number of bikes, types of bikes, design of parking spaces and so on. Readers may refer to Lin and Yang [49], Kumar and Bierlaire [35], Martinez et al. [53], Nair et al. [57] and Fricker and Gast [21]. The second class is to care for operations management and optimization, for instance, path scheduling, inventory management, bike redistribution, truck scheduling, price regulation, and applications of intelligent information technologies. Readers may refer to recent publications for details, among which are the bike repositioning (or redistribution) by Forma et al. [20], Vogel and Mattfeld [70], Benchimol et al. [4], Raviv et al. [60], Contardo et al. [11], Caggiani and Ottomanelli [8], Fricker et al. [22], Chemla et al. [9], Shu et al. [67], Fricker and Gast [21] and Labadi et al. [36]; the inventory management by Lin et al. [50], Raviv and Kolka [59] and Schuijbroek et al. [66]; the fleet management by George and Xia [30, 29], Godfrey and Powell [31], Nair and Miller-Hooks [56] and Guerriero et al. [32]; the price regulation by Waserhole and Jost [74], Waserhole et al. [75] and Fricker and
Gast [21]; the simulation models by Barth and Todd [2], Dell’Olio et al. [14] and Fricker and Gast [21]; and the data analysis by Froehlich and Oliver [24], Vogel et al. [71, 72], Borgnat et al. [5], Côme et al. [10] and Katzev [34].

1.1 Related works

A bike-sharing system has two classes of basic resources: Bikes and parking spaces, each of which should be available to demand of customers at any time or space. Since the bikes and packing spaces are finite and the number of bikes packed in each station is always random and dynamic, the factors frequently cause an undesirable phenomenon: Some stations contain more bikes but the others are seriously short of available bikes. In this situation, the dynamic unbalance of bikes distributed among the stations may have serious influence on customer satisfaction (or quality of service) through two specially problematic cases: A station is empty when a customer arrives at the station to rent a bike; while a station is full when a riding-bike customer arrives at the station to return her bike, where the empty or full station is called a problematic station. Notice that a crucial question for operational efficiency of the bike-sharing systems is the ability not only to meet the fluctuating demand for renting bikes at each station but also to provide enough vacant lockers to allow the renters to return the bikes at their destinations, thus the problematic stations reflect a common challenge faced by the bike-sharing systems in practice due to the stochastic and inhomogeneous nature of customer arrivals and of bike returns. Based on this, the customer satisfaction can increase if the probability of problematic stations is reduced, and so the quality of service in the bike sharing systems can be measured by means of the probability of problematic stations. Therefore, it is a key research of bike-sharing systems to provide effective methods or algorithms for computing the probability of problematic stations.

To compute the probability of problematic stations, queueing theory and Markov processes are two effective methods, but available works are still few up to now. On such a research line, the recent literature is classified into four basic types: (a) Queueing networks. Li et al. [47] described a more general large-scale bike-sharing system as a closed queueing network with two classes of customers, and provided an effective method for computing the stationary probability of problematic stations. Savin et al. [61] used a loss network as well as the admission control to discuss capacity allocation of a rental
model with two classes of customers, and studied the revenue management and fleet sizing decision. Adelman [1] applied a closed queueing network to propose an internal pricing mechanism for managing a fleet of service units, and also used a nonlinear flow model to discuss the price-based policy for the vehicle redistribution. George and Xia [30] used the closed queueing networks to study the vehicle rental systems, and determined the optimal number of parking spaces for each rental location. Zhang and Pavone [78] proposed a queueing network approach to analyze and control the mobility-on-demand systems. (b) Simple queues. To simply analyze one station or to establish an optimal condition, some simple queues are applied to discussing the bike-sharing systems. Leurent [40] used the $M/M/1/C$ queue to consider a vehicle-sharing system in which each station contains an expanded waiting room only for those problematic stations in which customers arriving at either a full station to return a bike or an empty station to rent a bike, and analyzed performance measures of an isolated station in this vehicle-sharing system. Schuijbroek et al. [66] computed the transient distribution of the $M/M/1/C$ queue, which is used to measure the level of service as an optimal condition in a mixed integer programming whose optimal solution deals with the inventory rebalancing and the vehicle routing. Raviv et al. [60] and Raviv and Kolka [59] provided an effective method for computing the transient distribution of a time-inhomogeneous $M(t)/M(t)/1/C$ queue, which is used to evaluate the expected number of bike shortages at any station.

To further develop the queueing analysis, the Markov decision processes, stochastic optimization and mean-field method are introduced to the study of bike-sharing systems. (c) Markov decision processes. To discuss the bike-sharing system, Waserhole and Jost [74], Waserhole and Jost [75, 76] and Waserhole et al. [77] used the simplified closed queuing networks to establish the Markov decision models, and computed the optimal policy by means of the fluid approximation which overcomes the state space explosion of multi-dimensional Markov decision processes. (d) Mean-field method. Fricker et al. [22] considered a space-inhomogeneous bike-sharing system with multiple clusters, and expressed the minimal proportion of problematic stations. Fricker and Gast [21] provided a detailed analysis for a space-homogeneous bike-sharing system in terms of the $M/M/1/K$ queue as well as some simple mean-field models, and crucially, they derived the closed-form solution to the minimal proportion of problematic stations. Fricker and Tibi [23] studied the central limit and local limit theorems for the independent (non-identically distributed) random variables, which support analysis of a generalized Jackson network with product-
form solution. Further, they used the limit theorems to give an outline of stationary asymptotic analysis for the locally space-homogeneous bike-sharing systems. Li et al. [44] provided a complete picture on how to jointly use the mean-field theory, the time-inhomogeneous queues and the nonlinear birth-death processes to analyze performance measures of the bike-sharing systems with walk times to rent a bike.

An aim of this paper is to apply the mean-field theory, the time-inhomogeneous queues and the nonlinear QBD processes to the study of bike-sharing systems, hence this motivates to propose a mean-field matrix-analytic method. For the mean-field method of stochastic networks, readers may refer to, such as, three survey papers by Sznitman [68], Benaim and Le Boudec [3] and Li [47]; and the mean-field limit by Vvedenskaya et al. [73], Turner [69], Graham [27, 28], Gast and Gaujal [25, 26], Li et al. [45, 46], Li and Lui [48] and Li et al. [41]. On the other hand, the QBD processes often provide a useful mathematical tool for studying stochastic models such as queueing systems, manufacturing systems, communication networks and healthcare systems. Readers may refer to Chapter 3 of Neuts [58], Latouche and Ramaswami [38], Li [42] and references therein.

1.2 Contributions of this paper

This paper makes three main contributions: Improving model descriptions from practical needs, numerically computing of the probability of problematic stations, and developing a mean-field matrix-analytic method. Each of these contributions is described as follows.

*Improving model descriptions from practical needs.* From practical needs in a large-scale bike-sharing system, this paper introduces two useful factors: A finite waiting room is added to each station; and using a Markovian environment to express inhomogeneous nature of customer arrivals and of bike returns. When there are sufficient bike-renting customers at some major cities (e.g. Beijing and Hangzhou), an adscititious waiting room designed at each station is always necessary and useful to improve the quality of service in the bike-sharing systems, and specifically to reduce the probability of problematic stations. Notice that such a waiting room was first proposed and discussed by Leurent [40] through applying the $M/M/1/C$ queue to performance analysis of only one isolated station. Differently, this paper designs the waiting room to each station so that this leads to a bike-sharing queueing network with many stations having waiting rooms, and analyzes this bike-sharing system with finite waiting rooms by means of the mean-field method.
Recall that the mean-field theory of bike-sharing systems was well established in Fricker et al. [22], Fricker and Gast [21] and Li et al. [44], while this paper specially generalizes it to propose a mean-field matrix-analytic method by which numerical computation is developed to be able to deal with more general bike-sharing systems in practice.

To our best knowledge, this paper is the first one applying the Markovian environment to the study of bike-sharing systems, where the Markovian environment is used to expresses space (or time) inhomogeneous nature of customer arrivals and of bike returns. Notice that the Markovian environment has extensively been discussed in queueing systems, manufacturing systems and communication networks, e.g., see Neuts [58], Li [42] and references therein. In general, the Markovian environment and its associated controlling Poisson processes can constitute a Markov-modulated Poisson process, or more generally, a Markovian arrival process, which are used to be able to express inhomogeneity and burstiness of some arrival flows. On the other hand, Li and Lui [48] and Li [47] are near to the idea and results of this paper in which the Markovian environment is related to both the block-structured mean-field theory and the nonlinear Markov processes.

**Numerically computing the probability of problematic stations.** Li et al. [47] indicated that each bike-sharing system can be related to a complicated closed queueing network so that it is always difficult and challenging to compute the probability of problematic stations. To this end, the mean-field method was developed in Fricker et al. [22], Fricker and Gast [21] and Li et al. [44] through simplifying some necessary assumptions (e.g., identical stations, and weak interactions); while those mean-filed results obtained therein are useful and valuable for design, operations and optimization of the bike-sharing systems. On this research line, the following two papers made basic contributions to expand the mean-field method to the block-structured cases. (1) Fricker and Tibi [23] used the local limit theorems to give an outline of stationary asymptotic analysis for the locally space-homogeneous bike-sharing systems. (2) This paper can numerically compute the probability of problematic stations for a block-structured bike-sharing system under Markovian environment.

**Developing a mean-field matrix-analytic method.** Notice that the non-Poisson arrival processes and non-exponential riding-bike times play an important role in the study of bike-sharing systems. To discuss a bike-sharing system with non-Poisson arrival processes and non-exponential riding-bike times, it is necessary to develop the mean-field matrix-analytic method, which is effective and efficient to deal with the bike-sharing systems with either the Markovian arrival processes or the phase type riding-bike times. As a
example, this paper analyzes the bike-sharing system under Markovian environment, and provides a clear picture of how to apply the mean-field matrix-analytic method through combining the mean-field theory with the time-inhomogeneous queues as well as the non-linear QBD processes according to the following three steps: (1) Setting up the system of mean-field equations by means of the mean-field theory as well as a time-inhomogeneous $MAP\left(t\right)/MAP\left(t\right)/1/K+2L+1$ queue; (2) using the martingale limit theory to demonstrate the asymptotic independence (or propagation of chaos) of this bike-sharing system, and further show the limiting interchangeability as $N \to \infty$ and $t \to +\infty$; and (3) numerically computing the fixed point which leads to performance analysis such as the mean of stationary bike number at any station, and the stationary probability of problematic stations. Also, some numerical examples are given for valuable observation on how the performance measures depend on some key parameters of this bike-sharing system. Therefore, the methodology and results of this paper, together with Li and Lui [48] and Li [47], gain new insights on applications of the mean-field matrix-analytic method to study nonlinear dynamics and interesting performance of more general bike-sharing systems in practice.

1.3 Outline of this paper

The remainder of this paper is organized as follows. In Section 2, we describe a large-scale bike-sharing system with $N$ identical stations under Markovian environment. Furthermore, an empirical measure process is introduced to express the states of this bike-sharing system. In Section 3, we use a probability-analytic method to set up a system of mean-field equations by means of the mean-field theory and a time-inhomogeneous $MAP\left(t\right)/MAP\left(t\right)/1/K+2L+1$ queue. In Section 4, we apply the martingale limit theory to demonstrating the asymptotic independence (or propagation of chaos) of the bike-sharing system. In Section 5, we discuss the fixed point of the system of limiting mean-field equations, and provide a nonlinear QBD process to compute the fixed point. Furthermore, we study the limiting interchangeability as $N \to \infty$ and $t \to +\infty$. In Section 6, we give four numerical examples to investigate the performance measures such as the mean of stationary bike number at any station, and the stationary probability of problematic stations. Some concluding remarks are given in Section 7.
2 Model Description

In this section, we describe a large-scale bike-sharing system with \( N \) identical stations under Markovian environment, in which operations mechanism, system parameters and model notation are listed with necessary interpretation. Furthermore, an empirical measure process is introduced to express the states of this bike-sharing system.

In a bike-sharing system, a customer first arrives at a station, rents a bike, and uses it for a while; then she returns the bike to a destination station. Once the customer finishes her trip and returns the bike to a station, she immediately leaves the bike-sharing system. Based on this, Li et al. [47] described such a practical bike-sharing system as a closed queueing network with two classes of customers, and computed the stationary probability of problematic stations by means of a more complicated expression in which the routing matrix and the associated relative arrival rates are formally given so that effective algorithms need to be developed in our future study.

As a better solvable way of the computation given in Li et al. [47], it is necessary and valuable to provide a simplified mean-field model of the bike-sharing system in which one main purpose is to simply show how the stationary probability of problematic stations depends on some simple and crucial parameters of the bike-sharing system. It is worth noting that this mean-field method has some basic advantages through establishing some useful relations among the simple and crucial parameters, e.g., see Fricker et al. [22], Fricker and Gast [21] and Li et al. [44] for more details.

The aim of this paper is to extend the mean-field method to be able to deal with more general block-structured bike-sharing systems in practice through binding the mean-field theory to the matrix-analytic method. To this end, we describe a large-scale bike-sharing system including operations mechanism, system parameters and model notation as follows:

(1) The \( N \) identical stations: To use the mean-field theory, we assume that the large-scale bike-sharing system contains \( N \) identical stations; and at the initial time \( t = 0 \), every station has \( C \) bikes and \( K \) parking places, in which \( 1 \leq C \leq K < \infty \). Every station continuously operates through either renting a bike or returning a bike, and so the number of bikes in any station can be regarded as a queueing process.

(2) Adding a finite waiting room at each station: To decrease the probability of problematic stations when customers are sufficient, we add a finite waiting room at each station. The finite waiting room have \( L \) waiting places, each of which is occupied by only
a customer when either she can not rent a bike from the station or she can not return her bike into the station. Based on this, designing and adding the finite waiting rooms contains two purposes: (a) When a customer arrives at an empty station where no bike can be rented, either she enters a waiting place in order to wait for a future bike with probability $\alpha \in [0, 1]$, or she immediately leaves the bike-sharing system with probability $1 - \alpha$. (b) When a riding-bike customer completes her trip and enters a full station where no empty parking space is seen, either she enters a waiting place in order to wait for a future parking space with probability $\beta \in [0, 1]$, or she directly rides her bike to another station to return the bike with probability $1 - \beta$. Notice that the riding-bike customer must return her bike to a station, and then she can leave the bike-sharing system, because each bike is an indispensable public equipment and cannot be lost.

(3) The Markovian environment: In this bike-sharing system, the arrival and travel processes are influenced (or controlled) by a Markovian environment, which is a continuous-time irreducible positive-recurrent Markov chain with an infinitesimal generator of size $m$ as follows:

$$
W = \begin{pmatrix}
    w_{1,1} & w_{1,2} & \cdots & w_{1,m} \\
    w_{2,1} & w_{2,2} & \cdots & w_{2,m} \\
    \vdots & \vdots & \ddots & \vdots \\
    w_{m,1} & w_{m,2} & \cdots & w_{m,m}
\end{pmatrix},
$$

where $w_{i,i} < 0$ for $1 \leq i \leq m$; $w_{i,j} \geq 0$ for $1 \leq i, j \leq m$ and $i \neq j$; $\sum_{j=1}^{m} w_{i,j} = 0$ for $1 \leq i \leq m$. At the same time, we denote by $\theta$ the stationary probability vector of the Markov chain $W$, that is, $\theta W = 0$ and $\theta e = 1$, where $e$ is a column vector of ones. Based on the Markov chain $W$, now we describe the arrival process and the travel times as follows:

(3.1) The arrival processes: If the Markovian environment is at State $j$, then the arrivals of customers at the bike-sharing system from the outside are a Poisson process with arrival rate $N\lambda_j$ for $1 \leq j \leq m$.

(3.2) The travel times: If the Markovian environment is at State $j$, then the travel time that a customer rides a bike from one station to another is exponential with travel rate $\mu_j$ for $1 \leq j \leq m$.

(4) The leaving principle: Once a customer finishes her trip and returns her bike to any station, she immediately leaves the bike-sharing system.
We assume that all the random variables defined above are independent of each other. When observing any station in the bike-sharing system, the finite waiting room and the Markovian environment play a key role in queueing analysis of this station. To explain this, the queueing structure of any station in this bike-sharing system is depicted in Figure 1.

Remark 1 On the one hand, the assumption of the $N$ identical stations is used to guarantee applicability of the mean-field theory (that is, the multi-dimensional Markov process is exchangeable). On the other hand, from a practical point of view, the stations in a major city are also designed as almost the same, for example, Hangzhou has 3000 stations, and each station contains about 30 bikes.

Remark 2 In some major cities, there are always sufficient bike-renting customers at the trips. To improve the quality of service (or to decrease the probability of problematic stations), an adscititious waiting room designed at each station of a bike-sharing system is always effective and useful. To do this, Leurent [40] first proposed such an idea of adscititious waiting rooms, and discussed the queueing process of only one isolated station.
by means of the $M/M/1/C$ queue. Differently from Leurent [40], this paper adds the waiting room to each station so that this leads to a bike-sharing queueing network with many stations having the finite waiting rooms, analyzes this bike-sharing queueing network in terms of the mean-field method, and compares performance measures of the bike-sharing systems with or without the finite waiting rooms through some numerical examples.

**Remark 3**

1. To our best knowledge, this paper is the first one introducing the Markovian environment to the study of bike-sharing systems. Notice that the Markovian environment can always be used to expresses space (or time) inhomogeneous nature of customer arrivals and of bike returns, e.g., see Neuts [58] and Li [42] for interpretation of a Markovian arrival process from communication networks and manufacturing systems. In fact, the space (or time) inhomogeneity of customer arrivals has a larger impact on performance measures of the bike-sharing systems. To explain this impact, this paper gives a detailed discussion by virtue of some numerical examples.

2. Under the Markovian environment, the mean-field analysis of the bike-sharing system is different from that in Fricker et al. [22], Fricker and Gast [21] and Li et al. [44]. Therefore, the mean-field theory needs to combine with the matrix-analytic method so that we propose a mean-field matrix-analytic method, which can deal with block-structure bike-sharing systems. Notice that this paper may be related to a near idea and the associate results given in Li and Lui [48] and Li [47] through the block-structured mean-field theory and the nonlinear Markov processes.

In the remainder of this section, we introduce an empirical measure process to express the states of this bike-sharing system.

Let $X_{n}^{(N)}(t)$ and $J(t)$ be the number of customers in Station $n$ and the state of the Markovian environment at time $t$, respectively. Then $X = \{(X_{1}^{(N)}(t), X_{2}^{(N)}(t), \ldots, X_{N}^{(N)}(t); J(t)) : t \geq 0\}$ is a $Nm$-dimensional Markov process, which is due to the assumptions on the Poisson arrivals, the exponential travel times and the Markovian environment. Notice that analysis of the $Nm$-dimensional Markov process is always difficult due to the “State Space Explosion”. For this, it is necessary to introduce an empirical measure process by means of the mean field theory.

For the Markov process $X = \left\{ \left( X_{1}^{(N)}(t), X_{2}^{(N)}(t), \ldots, X_{N}^{(N)}(t); J(t) \right) : t \geq 0 \right\}$, the
Empirical measure is defined as
\[ Y_{k,j}^{(N)}(t) = \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}\{X_{n}^{(N)}(t) = k, J(t) = j\}, \]
where \( \mathbf{1}\{\cdot\} \) is an indicative function. Obviously, \( Y_{k,j}^{(N)}(t) \) denotes the fraction of stations with \( k \) bikes and under the Markovian environment be at State \( j \) at time \( t \). It is easy to see that for \( -L \leq k \leq K + L \),
\[
0 \leq Y_{k,j}^{(N)}(t) \leq \sum_{j=1}^{m} Y_{k,j}^{(N)}(t) \leq \sum_{k=-L}^{K+L} \sum_{j=1}^{m} Y_{k,j}^{(N)}(t) = 1.
\]
Let
\[
Y_{k}^{(N)}(t) = \left( Y_{k,1}^{(N)}(t), Y_{k,2}^{(N)}(t), \ldots, Y_{k,m}^{(N)}(t) \right)
\]
and
\[
Y^{(N)}(t) = \left( Y_{-L}^{(N)}(t), Y_{-L+1}^{(N)}(t), \ldots, Y_{K+L-1}^{(N)}(t), Y_{K+L}^{(N)}(t) \right),
\]
which is a row vector of size \((K + 2L + 1) m\). Then \( \{Y^{(N)}(t) : t \geq 0\} \) is an empirical measure Markov process whose state space is given by \( \Omega = [0,1]^{(K+2L+1)m} \).

To consider the empirical measure Markov process \( \{Y^{(N)}(t) : t \geq 0\} \), we write
\[
y_{k,j}^{(N)}(t) = E \left[ Y_{k,j}^{(N)}(t) \right], \quad -L \leq k \leq K + L, 1 \leq j \leq m,
\]
(1) and
\[
Y_{k}^{(N)}(t) = \left( y_{k,1}^{(N)}(t), y_{k,2}^{(N)}(t), \ldots, y_{k,m}^{(N)}(t) \right),
\]
\[
y^{(N)}(t) = \left( y_{-L}^{(N)}(t), y_{-L+1}^{(N)}(t), \ldots, y_{K+L-1}^{(N)}(t), y_{K+L}^{(N)}(t) \right).
\]

3 Mean-Field Equations

In this section, we use a probability-analytic method to set up a system of mean-field equations according to two basic steps: (1) Using the mean-field theory to compute the instantaneous arrival rate and the instantaneous service rate of a single server queue under Markovian environment, hence we establish a time-inhomogeneous MAP \( t / MAP(t) / 1 / K + 2L + 1 \) queue. (2) Based on the time-inhomogeneous MAP \( t / MAP(t) / 1 / K + 2L + 1 \) queue, we set up the system of mean-field equations satisfied by the expected vector \( y^{(N)}(t) \).
3.1 A time-inhomogeneous MAP \((t) / MAP (t) / 1 / K + 2L + 1\) queue

In the bike-sharing system with \(N\) identical stations under Markovian environment, we define \(Q^{(N)} (t)\) as the number of bikes in any station at time \(t\). It is easy to see that if an outside customer arrives at a station and rents a bike therein, then \(Q^{(N)} (t)\) decreases one; while if a customer finishes her trip and returns a bike at the station, \(Q^{(N)} (t)\) increases one. Therefore, we can understand that the Markov process \(\{(Q^{(N)} (t), J (t)) : t \geq 0\}\) is a QBD process which is expressed by a time-inhomogeneous MAP \((t) / MAP (t) / 1 / K + 2L + 1\) queue, where MAP \((t)\) is an instantaneous Markov arrival process with a matrix descriptor \((C(t), D(t))\) of size \(m\), e.g., see Subsections 8.2.5 and 8.2.6 of Chapter 8 in Li [42] for more details.

To establish the \(MAP (t) / MAP (t) / 1 / K + 2L + 1\) queue, the following theorem provides expressions for the instantaneous arrival rate \(\xi^{(N)}_{l,j} (t)\) and the instantaneous service rate \(\eta^{(N)}_{k,j} (t)\) in this time-inhomogeneous queueing system. Notice that the two instantaneous rates are a key in our later study.

**Theorem 1** In the time-inhomogeneous MAP \((t) / MAP (t) / 1 / K + 2L + 1\) queue, the instantaneous service rate is given by

\[
\eta^{(N)}_{k,j} (t) = \begin{cases} 
\lambda_j, & 1 \leq k \leq K + L, \quad 1 \leq j \leq m \\
\lambda_j \alpha_l, & -(L - 1) \leq k \leq 0, \quad 1 \leq j \leq m.
\end{cases}
\]

At the same time, for \(1 \leq j \leq m\), the instantaneous arrival rate is given by

\[
\xi^{(N)}_{l,j} (t) = \begin{cases} 
\mu_j \frac{N}{C} \left[ C + (N - 1) \frac{K + L}{k=K} k_y^{(N)}_{k,j} (t) \\
+ \frac{K + L}{k=K} \frac{(1-\beta)y^{(N)}_{k,j} (t)}{1-(1-\beta)y^{(N)}_{k,j} (t)^2} + \frac{y^{(N)}_{k+L,j} (t)}{1-y^{(N)}_{k+L,j} (t)^2} \right], & -L \leq l \leq 0, \\
\mu_j \frac{N}{C} \left[ C - \frac{K + L}{k=K} k_y^{(N)}_{k,j} (t) \\
+ \frac{K + L}{k=K} \frac{(1-\beta)y^{(N)}_{k,j} (t)}{1-(1-\beta)y^{(N)}_{k,j} (t)^2} + \frac{y^{(N)}_{k+L,j} (t)}{1-y^{(N)}_{k+L,j} (t)^2} \right], & 1 \leq l \leq C - 1, \\
\beta \mu_j \frac{N}{C} \left[ (N - 1) \frac{K + L}{k=K} k_y^{(N)}_{k,j} (t) \\
+ \frac{K + L}{k=K} \frac{(1-\beta)y^{(N)}_{k,j} (t)}{1-(1-\beta)y^{(N)}_{k,j} (t)^2} + \frac{y^{(N)}_{k+L,j} (t)}{1-y^{(N)}_{k+L,j} (t)^2} \right], & C \leq l \leq K - 1, \\
\beta \mu_j \frac{N}{C} \left[ (N - 1) \frac{K + L}{k=K} k_y^{(N)}_{k,j} (t) \\
+ \frac{K + L}{k=K} \frac{(1-\beta)y^{(N)}_{k,j} (t)}{1-(1-\beta)y^{(N)}_{k,j} (t)^2} + \frac{y^{(N)}_{k+L,j} (t)}{1-y^{(N)}_{k+L,j} (t)^2} \right], & K \leq l \leq K + L - 1.
\end{cases}
\]
Proof: The proof of (2). When a customer arrives at a station, there exist two cases:

Case (a) If the station has at least one bike (that is, \(1 \leq k \leq K + L\)), then she immediately rents a bike and leaves the station. Hence \(\eta_{k,j}^{(N)}(t) = \lambda_j\) for \(1 \leq j \leq m\).

Case (b) If the station has no bike (that is, \((-L + 1) \leq k \leq 0\)), then she has two choices: she directly leaves this system with the probability \(1 - \alpha\); or she enters a waiting place with the probability \(\alpha\) in order to wait for returning a future bike. In this case, \(\eta_{k,j}^{(N)}(t) = \lambda_j \alpha\) for \(1 \leq j \leq m\).

Based on Cases (a) and (b), when the Markovian environment \(J(t) = j\), we have

\[
\eta_{k,j}^{(N)}(t) = \begin{cases} 
\lambda_j, & 1 \leq k \leq K + L, \quad 1 \leq j \leq m, \\
\lambda_j \alpha, & -(L - 1) \leq k \leq 0, \quad 1 \leq j \leq m.
\end{cases}
\]

The proof of (3). The computation of (3) is a bit complicated due to applications of the mean field theory. Notice that Figure 2 describes the state transitions of the process to return bikes at any station, and specifically, the bikes can not return to a full station so that they have to try to another station again with probability \(1 - \beta\). Based on this, we give the instantaneous arrival rate \(\xi_{l,j}^{(N)}(t)\) by means of a probability-analytic method as follows:

\[
\xi_{l,j}^{(N)}(t) = \frac{1}{N} \cdot \mu_j \cdot \text{the numbers of bikes ridden on all the roads}.
\]

Notice that the numbers of bikes ridden on all the roads contains two parts: (i) the numbers \(n_1\) of bikes ridden from a tagged station is given by

\[
n_1 = \begin{cases} 
C, & -L \leq l \leq 0, \\
C - l, & 1 \leq l \leq C - 1, \\
0, & C \leq l \leq K + L - 1,
\end{cases}
\]

and (ii) the numbers of bikes ridden from the other \(N - 1\) station is given by

\[
(N - 1) \left[\text{the average numbers of bikes ridden from any station} + \text{the average numbers of bikes which can not return to a full station with multiple retries}\right].
\]

Under the mean-field setting, the average numbers of bikes ridden from any station is given by \(C - \sum_{k=1}^{K+L} k \eta_{k,j}^{(N)}(t)\), while the average numbers of bikes which can not return to a full station with multiple retries is given a detailed computation in Case (a) below.

With the above analysis, our computation to derive the instantaneous arrival rate \(\xi_{l,j}^{(N)}(t)\) is divided into the following four cases.
Figure 2: The state transition relation of queueing process at any station

**Case (a):** When $-L \leq l \leq 0$, $1 \leq j \leq m$, we need to study three different classes for the initial distribution of bikes in any station. Note that in the last two classes, customers who arrive at a full station cannot return their bikes therein.

Class-1: The initial $C$ bikes in the station are all rented on the travels. Using the mean-field theory, we get that the average number of bikes rented on the travels from the other $N-1$ stations is given by

$$(N-1) \left[ C - \sum_{k=1}^{K+L} ky^{(N)}_{k,j} (t) \right],$$

where $\sum_{k=1}^{K+L} ky^{(N)}_{k,j} (t)$ is the average number of bikes parked in any station. Thus, the average number of bikes rented on the travels from the $N$ stations is given by

$$C + (N-1) \left[ C - \sum_{k=1}^{K+L} ky^{(N)}_{k,j} (t) \right].$$

Class-2: When a customer finishes her trip and arrives at a station in which there are $k$ bikes for $K \leq k \leq K+L-1$, she has to travel again in order to return the bike with the probability $1-\beta$. The average number of the re-travels is given by

$$\sum_{k=K}^{K+L-1} \left\{ (1-\beta) y^{(N)}_{k,j} (t) + 2 \left[ (1-\beta) y^{(N)}_{k,j} (t) \right]^2 + 3 \left[ (1-\beta) y^{(N)}_{k,j} (t) \right]^3 + \cdots \right\}$$

$$= \sum_{k=K}^{K+L-1} \frac{(1-\beta) y^{(N)}_{k,j} (t)}{1 - (1-\beta) y^{(N)}_{k,j} (t)},$$

where $n \left[ (1-\beta) y^{(N)}_{k,j} (t) \right]^n$ is the average number of re-travels for $n$ customers, and $x + 2x^2 + 3x^3 + \cdots = x/(1-x)^2$. 

15
Class-3: When a customer finishes her trip and arrives at a station in which there are $K + L$ bikes, she has to travel again in order to return the bike with the probability 1. The average number of the re-travels is given by

$$y_{K+L,j}^{(N)}(t) + 2\left[y_{K+L,j}^{(N)}(t)\right]^2 + 3\left[y_{K+L,j}^{(N)}(t)\right]^3 + \cdots = \frac{y_{K+L,j}^{(N)}(t)}{1 - y_{K+L,j}^{(N)}(t)^2}.$$  

Summarizing the above analysis, the instantaneous arrival rate is given by

$$\xi_{l,j}^{(N)}(t) = \frac{\mu_j}{N} \left\{ C + (N - 1) \left[ C - \sum_{k=K}^{K+L} k y_{k,j}^{(N)}(t) + \sum_{k=K}^{K+L-1} \frac{(1 - \beta) y_{k,j}^{(N)}(t)}{[1 - (1 - \beta) y_{k,j}^{(N)}(t)]^2} + \frac{y_{K+L,j}^{(N)}(t)}{[1 - y_{K+L,j}^{(N)}(t)]^2} \right] \right\}.$$  

Case (b): When $1 \leq l \leq C - 1$, the only difference of our derivation from Case (a) is that the initial $C$ bikes by the initial $C - l$ bikes in this station. Thus we get

$$\xi_{l,j}^{(N)}(t) = \frac{\mu_j}{N} \left\{ C - l + (N - 1) \left[ C - \sum_{k=K}^{K+L} k y_{k,j}^{(N)}(t) + \sum_{k=K}^{K+L-1} \frac{(1 - \beta) y_{k,j}^{(N)}(t)}{[1 - (1 - \beta) y_{k,j}^{(N)}(t)]^2} + \frac{y_{K+L,j}^{(N)}(t)}{[1 - y_{K+L,j}^{(N)}(t)]^2} \right] \right\}.$$  

Case (c): When $C \leq l \leq K - 1$, the only difference of our derivation from Case (a) is that the initial $C$ bikes in this station are all parked in the station. Hence we obtain

$$\xi_{l,j}^{(N)}(t) = \frac{\mu_j}{N} \left\{ (N - 1) \left[ C - \sum_{k=K}^{K+L} k y_{k,j}^{(N)}(t) + \sum_{k=K}^{K+L-1} \frac{(1 - \beta) y_{k,j}^{(N)}(t)}{[1 - (1 - \beta) y_{k,j}^{(N)}(t)]^2} + \frac{y_{K+L,j}^{(N)}(t)}{[1 - y_{K+L,j}^{(N)}(t)]^2} \right] \right\}.$$  

Case (d): When $K \leq l \leq K + L - 1$, the only difference of our derivation from Case (c) is that when a customer finishes her trip and arrives at this station, she enters the waiting places in order to wait for an empty parking place with the probability $\beta$. This gives

$$\xi_{l,j}^{(N)}(t) = \beta \frac{\mu_j}{N} \left\{ (N - 1) \left[ C - \sum_{k=K}^{K+L} k y_{k,j}^{(N)}(t) + \sum_{k=K}^{K+L-1} \frac{(1 - \beta) y_{k,j}^{(N)}(t)}{[1 - (1 - \beta) y_{k,j}^{(N)}(t)]^2} + \frac{y_{K+L,j}^{(N)}(t)}{[1 - y_{K+L,j}^{(N)}(t)]^2} \right] \right\}.$$  

Summarizing the above four cases, we obtain all the expressions given in (2). This completes the proof. ■

### 3.2 A system of mean-field equations

In the time-inhomogeneous $MAP(t)/MAP(t)/1$ queue, Theorem 1 gives the instantaneous arrival rate $\xi_{l,j}^{(N)}(t)$ and the instantaneous service rate $\eta_{k,j}^{(N)}(t)$. Based
Figure 3: The state transitions of the time-inhomogeneous QBD process

on this, we can describe the time-inhomogeneous QBD process \(\{(Q^{(N)}(t), J(t)) : t \geq 0\}\), where Figure 3 describes the state transitions of the time-inhomogeneous QBD process.

It follows from (1) that

\[y_{k,j}^{(N)}(t) = P\{Q^{(N)}(t) = k, J(t) = j\}, \quad -L \leq k \leq K + L, 1 \leq j \leq m.\]

Based on the instantaneous arrival rate \(\xi_{k,j}^{(N)}(t)\), we set that for \(-L \leq k \leq K + L - 1\)

\[
\Psi_k^{(N)}(t) = \begin{pmatrix}
0 & \xi_{k,1}^{(N)}(t) w_{1,1} & \cdots & \xi_{k,1}^{(N)}(t) w_{1,m} \\
\xi_{k,2}^{(N)}(t) w_{2,1} & 0 & \cdots & \xi_{k,2}^{(N)}(t) w_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{k,m}^{(N)}(t) w_{m,1} & \xi_{k,m}^{(N)}(t) w_{m,2} & \cdots & 0
\end{pmatrix},
\]

\[\hat{\Psi}_k^{(N)}(t) = \text{diag}(\xi_{k,1}^{(N)}(t) w_{1,1}, \xi_{k,2}^{(N)}(t) w_{2,2}, \ldots, \xi_{k,m}^{(N)}(t) w_{m,m}).\]

Similarly, it is easy to see from the instantaneous service rate \(\eta_{k,j}^{(N)}(t)\) that for \(-L + 1 \leq k \leq 0\)

\[
\Phi_k^{(N)}(t) = \begin{pmatrix}
0 & \lambda_1 \alpha w_{1,1} & \cdots & \lambda_1 \alpha w_{1,m} \\
\lambda_2 \alpha w_{2,1} & 0 & \cdots & \lambda_2 \alpha w_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_m \alpha w_{m,1} & \lambda_m \alpha w_{m,2} & \cdots & 0
\end{pmatrix} \overset{\text{Def}}{=} \Phi^{(N)}(\alpha),
\]

\[\hat{\Phi}_k^{(N)}(t) = \text{diag}(\lambda_1 \alpha w_{1,1}, \lambda_2 \alpha w_{2,2}, \ldots, \lambda_m \alpha w_{m,m}) \overset{\text{Def}}{=} \hat{\Phi}^{(N)}(\alpha);\]

and for \(1 \leq k \leq K + L\)

\[\Phi_k^{(N)}(t) = \Phi^{(N)}(1)\]

and

\[\hat{\Phi}_k^{(N)}(t) = \hat{\Phi}^{(N)}(1),\]

which are due to \(\alpha = 1.\)
For the time-inhomogeneous QBD process \( \{(Q^{(N)}(t), J(t)) : t \geq 0\} \), it follows from Figure 3 that the vector \( y^{(N)}(t) = (y^{(N)}_{-L}(t), y^{(N)}_{-L+1}(t), \ldots, y^{(N)}_{K+L-1}(t), y^{(N)}_{K+L}(t)) \) satisfies the system of mean-field (or ordinary differential) equations as follows:

\[
\frac{d}{dt} y^{(N)}_{-L}(t) = y^{(N)}_{-L}(t) \tilde{\Psi}_{-L}(t) + y^{(N)}_{-L+1}(t) \Phi_{-L+1}(t),
\]

for \(-L + 1 \leq k \leq K + L - 1\)

\[
\frac{d}{dt} y^{(N)}_{k}(t) = y^{(N)}_{k-1}(t) \Psi_{k-1}(t) + y^{(N)}_{k}(t) \left[ \tilde{\Psi}_{k}(t) + \tilde{\Phi}_{k}(t) \right] + y^{(N)}_{k+1}(t) \Phi_{k+1}(t),
\]

with the boundary condition

\[
\sum_{k=-L}^{K+L} y^{(N)}_{k}(t) e = 1,
\]

and with the initial condition

\[
y^{(N)}_{k}(0) = g_{k}, \quad -L \leq k \leq K + L,
\]

and

\[
g_{k} = (g_{k,1}, g_{k,2}, \ldots, g_{k,m}),
\]

\[
g = (g_{-L}, g_{-(L-1)}, \ldots, g_{K+L-1}, g_{K+L})
\]

is a probability vector of size \((K + 2L + 1)m\).

For convenience of description, we write the mean-field equations (4) to (8) into a matrix version as follows:

\[
\frac{d}{dt} y^{(N)}(t) = y^{(N)}(t) V y^{(N)}(t),
\]

with the boundary and initial conditions

\[
y^{(N)}(t) e = 1, \quad y^{(N)}(0) = g,
\]

where

\[
V y^{(N)}(t) = \begin{pmatrix}
A_{1,1} & A_{1,2} \\
A_{2,1} & A_{2,2} & A_{2,3} \\
A_{3,1} & A_{3,2} & A_{3,3} & A_{3,4} \\
A_{4,1} & A_{4,2} & A_{4,3} & A_{4,4}
\end{pmatrix},
\]
\[ A_{1,1} = \begin{pmatrix}
\hat{\Psi}^{(N)}_{-L} (t) & \Psi^{(N)}_{-L} (t) \\
\Phi^{(N)}_{-(L-1)} (t) & \hat{\Phi}^{(N)}_{-(L-1)} (t) + \hat{\Psi}^{(N)}_{-(L-1)} (t) & \Psi^{(N)}_{-(L-1)} (t) \\
& \ddots & \ddots & \ddots \\
& & \Phi^{(N)}_{-1} (t) & \hat{\Phi}^{(N)}_{-1} (t) + \hat{\Psi}^{(N)}_{-1} (t) & \Psi^{(N)}_{-1} (t) \\
& & & \hat{\Phi}^{(N)}_{0} (t) & \hat{\Psi}^{(N)}_{0} (t) + \hat{\Psi}^{(N)}_{0} (t)
\end{pmatrix}, \\
A_{1,2} = \begin{pmatrix}
\hat{\Phi}^{(N)}_{1} (t) + \hat{\Psi}^{(N)}_{1} (t) \\
\Phi^{(N)}_{2} (t) & \hat{\Phi}^{(N)}_{2} (t) + \hat{\Psi}^{(N)}_{2} (t) & \Psi^{(N)}_{2} (t) \\
& \ddots & \ddots & \ddots \\
& & \Phi^{(N)}_{L-2} (t) & \hat{\Phi}^{(N)}_{L-2} (t) + \hat{\Psi}^{(N)}_{L-2} (t) & \Psi^{(N)}_{L-2} (t) \\
& & & \hat{\Phi}^{(N)}_{L-1} (t) & \hat{\Psi}^{(N)}_{L-1} (t) + \hat{\Psi}^{(N)}_{L-1} (t)
\end{pmatrix}, \\
A_{2,3} = \begin{pmatrix}
\hat{\Phi}^{(N)}_{C} (t) + \hat{\Psi}^{(N)}_{C} (t) \\
\Phi^{(N)}_{C+1} (t) & \hat{\Phi}^{(N)}_{C+1} (t) + \hat{\Psi}^{(N)}_{C+1} (t) & \Psi^{(N)}_{C+1} (t) \\
& \ddots & \ddots & \ddots \\
& & \Phi^{(N)}_{K-2} (t) & \hat{\Phi}^{(N)}_{K-2} (t) + \hat{\Psi}^{(N)}_{K-2} (t) & \Psi^{(N)}_{K-2} (t) \\
& & & \hat{\Phi}^{(N)}_{K-1} (t) & \hat{\Psi}^{(N)}_{K-1} (t) + \hat{\Psi}^{(N)}_{K-1} (t)
\end{pmatrix}, \\
A_{3,4} = \begin{pmatrix}
\Phi^{(N)}_{K} (t) \\
\Psi^{(N)}_{K-1} (t)
\end{pmatrix}, \\
A_{4,3} = \begin{pmatrix}
\Phi^{(N)}_{K} (t)
\end{pmatrix}. \]
\[
A_{4,4} = \begin{pmatrix}
\hat{\Phi}_K^{(N)}(t) + \hat{\Psi}_K^{(N)}(t) & \Psi_K^{(N)}(t) & & \\
\hat{\Phi}_{K+1}^{(N)}(t) & \hat{\Phi}_{K+1}^{(N)}(t) + \hat{\Psi}_{K+1}^{(N)}(t) & \Psi_{K+1}^{(N)}(t) & \\
& \ddots & \ddots & \\
& & \hat{\Phi}_{K+L-1}^{(N)}(t) + \hat{\Psi}_{K+L-1}^{(N)}(t) & \Psi_{K+L-1}^{(N)}(t) \\
& & & \hat{\Phi}_{K+L}^{(N)}(t) \\
& & & \hat{\Phi}_{K+L}^{(N)}(t)
\end{pmatrix}.
\]

**Remark 4** To set up the system of mean-field equations, it is a key to observe two intuitive figures: **Figure 1** shows all the basic parameters of queueing process under Markovian environment when one isolated station is paid attention, and it is seen that the queueing process is related to the QBD process due to the role played by the Markovian environment. **Figure 3** can further exactly give expressions for the transition rates of the QBD process between two neighboring levels through considering the weak interaction among the \(N\) stations of the bike-sharing system in terms of the mean-field theory.

## 4 The Martingale Limits

In this section, we apply the martingale limit theory to showing the asymptotic independence of this bike-sharing system, that is, the sequence \(\{Y^{(N)}(t), t \geq 0\}\) of Markov processes asymptotically approaches a single trajectory identified by a solution to the system of limiting mean-field equations.

For the vector \(g = (g_{-L}, g_{-L+1}, \ldots, g_{K+L-1}, g_{K+L})\) where \(g_k = (g_{k,1}, g_{k,2}, \ldots, g_{k,m})\), we set
\[
\Omega_N = \{g : g \geq 0, \text{ ge } = 1, \text{ Ng is a vector of nonnegative integers}\}
\]
and
\[
\Omega = \{g : g \geq 0, \text{ ge } = 1\}.
\]

In the vector space \(\Omega\) (or \(\Omega_N\)), we take a metric
\[
\rho(g, g') = \max_{-L \leq k \leq K+L} \max_{1 \leq j \leq m} \{|g_{k,j} - g'_{k,j}|\}, \quad g, g' \in \Omega.
\]
Note that under the metric \(\rho(g, g')\), the vector space \(\Omega\) (or \(\Omega_N\)) is separable and compact.

Now we consider the Markov process \(\{Y^{(N)}(t), t \geq 0\}\) on state space \(\Omega_N\) for \(N = 1, 2, 3, \ldots\). Note that the stochastic evolution of this bike-sharing system is described as
the Markov process \( \{ Y^{(N)}(t), t \geq 0 \} \), and
\[
\frac{d}{dt} Y^{(N)}(t) = A_N f(Y^{(N)}(t)),
\]
where \( A_N \) acting on functions \( f : \Omega_N \to \mathbb{C}^1 \) is the generating operator of the Markov process \( \{ Y^{(N)}(t), t \geq 0 \} \), and
\[
A_N = A_N^{\text{renting}} + A_N^{\text{returning}} + A_N^{\text{environment}},
\]
where
\[
A_N^{\text{renting}} f(g) = N \sum_{j=1}^{m} \sum_{k=1}^{K+L} \lambda_j \sum_{l=0}^{C} \Theta(l) g_{k,j} \left[ f \left( g - \frac{e_{k,j}}{N} \right) - f(g) \right] + N \alpha \sum_{j=1}^{m} \sum_{k=-(L-1)}^{0} \sum_{l=0}^{C} \Theta(l) g_{k,j} \left[ f \left( g - \frac{e_{k,j}}{N} \right) - f(g) \right],
\]
\[
A_N^{\text{returning}} f(g) = \left\{ \sum_{j=1}^{m} \sum_{l=-L}^{0} \sum_{l=1}^{C-1} \mu_j g_{l,j} \Theta(0) + \sum_{j=1}^{m} \sum_{l=1}^{C-1} \mu_j g_{l,j} \Theta(l) \right\} \left[ f \left( g + \frac{e_{k,j}}{N} \right) - f(g) \right],
\]
\[
A_N^{\text{environment}} f(g) = N \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k=-L}^{K+L} g_{k,i} w_{i,j} \left[ f \left( g - \frac{e_{k,i}}{N} + \frac{e_{k,j}}{N} \right) - f(g) \right],
\]
and for \( 0 \leq l \leq C \)
\[
\Theta(l) = \left\{ C - l + (N - 1) \left[ C - \sum_{k=K}^{K+L} k g_{k,j} + \sum_{k=K}^{K+L-1} \frac{(1 - \beta) g_{k,j}}{1 - (1 - \beta) g_{k,j}^2} + \frac{g_{K+L,j}}{1 - g_{K+L,j}^2} \right] \right\}.
\]
When \( N \to \infty \), it is easy to check that
\[
N \left[ f \left( g + \frac{e_{k,j}}{N} \right) - f(g) \right] \to \frac{\partial}{\partial g_{k,j}} f(g),
\]
\[
N \left[ f \left( g - \frac{e_{k,j}}{N} \right) - f(g) \right] \to -\frac{\partial}{\partial g_{k,j}} f(g),
\]
\[
\left[ f \left( g - \frac{e_{k,i}}{N} + \frac{e_{k,j}}{N} \right) - f(g) \right] \to -\frac{\partial}{\partial g_{k,i}} f(g) + \frac{\partial}{\partial g_{k,j}} f(g),
\]
and for \( 0 \leq l \leq C \)
\[
\frac{1}{N} \Theta(l) = \frac{1}{N} \left\{ C - l + (N - 1) \left[ C - \sum_{k=K}^{K+L} k g_{k,j} + \sum_{k=K}^{K+L-1} \frac{(1 - \beta) g_{k,j}}{1 - (1 - \beta) g_{k,j}^2} + \frac{g_{K+L,j}}{1 - g_{K+L,j}^2} \right] \right\}
\to C - \sum_{k=K}^{K+L} k g_{k,j} + \sum_{k=K}^{K+L-1} \frac{(1 - \beta) g_{k,j}}{1 - (1 - \beta) g_{k,j}^2} + \frac{g_{K+L,j}}{1 - g_{K+L,j}^2} \overset{\text{def}}{=} \Theta.
\]
Let
\[
\mathbf{A} = \lim_{N \to \infty} \mathbf{A}_N, \quad \mathbf{A}_{\text{renting}} = \lim_{N \to \infty} \mathbf{A}_{\text{renting}}^N, \\
\mathbf{A}_{\text{returning}} = \lim_{N \to \infty} \mathbf{A}_{\text{returning}}^N, \quad \mathbf{A}_{\text{environment}} = \lim_{N \to \infty} \mathbf{A}_{\text{environment}}^N.
\]

Then
\[
\mathbf{A} f(\mathbf{g}) = -\sum_{j=1}^m \lambda_j \sum_{k=1}^{K+L} g_{k,j} \frac{\partial}{\partial g_{k,j}} f(\mathbf{g}) - \alpha \sum_{j=1}^m \sum_{k=-(L-1)}^0 g_{k,j} \frac{\partial}{\partial g_{k,j}} f(\mathbf{g})
\]
\[
+ \sum_{i=1}^m \sum_{j=1}^m \sum_{k=-L}^{K+L} g_{k,i} w_{i,j} \left[ -\frac{\partial}{\partial g_{k,i}} f(\mathbf{g}) + \frac{\partial}{\partial g_{k,j}} f(\mathbf{g}) \right]
\]
\[
+ \Theta \left( \sum_{j=1}^m \sum_{l=-L}^{K-1} \mu_{j,l} g_{l,j} + \beta \sum_{j=1}^m \sum_{l=K}^{K+L-1} \mu_{j,l} g_{l,j} \right) \frac{\partial}{\partial g_{k,j}} f(\mathbf{g}).
\]

(13)

Now, we discuss some convergence of the sequence \( \{ \mathbf{Y}^{(N)}(t) : t \geq 0 \} \) of Markov processes for \( N = 1, 2, 3, \ldots \), and our main purpose is to provide a basic support for our later study of various convergence involved. To this end, we consider the random vector \( \mathbf{Y}^{(N)}(t) \) with samples in \( \mathcal{P}(\mathbb{D}(R_+, \mathbf{N})) \), where \( R_+ = [0, +\infty) \), \( \mathbf{N} = ((k, j) : -L \leq k \leq K + L, 1 \leq j \leq m) \), \( \mathbb{D}(R_+, \mathbf{N}) \) is the Skorohod space, i.e., the set of mappings which are right continuous with left-hand limits (in short, Cadl\’ag), and \( \mathcal{P}(\cdot) \) is the set of probability measures defined in \( \mathbb{D}(R_+, \mathbf{N}) \). Notice that the convergence in the Skorohod topology means the convergence in distribution (or weak convergence) for the Skorohod topology on the space of trajectories. When the sequence \( \{ \mathbf{Y}^{(N)}(t), t \geq 0 \} \) of Markov processes converges in probability (or converges weakly), for the Skorohod topology, to a given probability vector \( \mathbf{Y}(t) \), we write the weak convergence as \( \mathbf{Y}(t) = \lim_{N \to \infty} \mathbf{Y}^{(N)}(t) \) or \( \mathbf{Y}^{(N)}(t) \Rightarrow \mathbf{Y}(t) \) for \( t \geq 0 \), as \( N \to \infty \).

Let \( \mathbf{Y}(t) = \lim_{N \to \infty} \mathbf{Y}^{(N)}(t) \). Then it is easy to see from (12) and (13) that the transition probabilities of the Markov process \( \{ \mathbf{Y}^{(N)}(t), t \geq 0 \} \) with generating operator \( \mathbf{A}_N \) uniformly converges on any finite time interval to the transition probabilities of the limiting Markov process \( \{ \mathbf{Y}(t), t \geq 0 \} \) with generating operator \( \mathbf{A} \).

Now we consider the limiting behavior of the sequence \( \{ \mathbf{Y}^{(N)}(t), t \geq 0 \} \) of Markov processes as \( N \to \infty \). To that end, we first give a system of limiting mean-field equations (14) to (15) below.

Set
\[
\mathbf{y}(t) = \lim_{N \to \infty} \mathbf{y}^{(N)}(t)
\]
and

\[ V_y(t) = \lim_{N \to \infty} V_y^{(N)}(t). \]

Then it follows from (9) and (10) that

\[
\frac{d}{dt} y(t) = y(t) V_y(t),
\]

(14)

\[ y(t) e = 1, \quad y(0) = g \in \Omega. \]

(15)

Note that the convergence in the Skorohod topology means the convergence in distribution for the Skorohod topology on the space of trajectories. The following theorem applies the martingale limit theory to discussing the weak convergence of the sequence \( \{ Y^{(N)}(t), t \geq 0 \} \) of Markov processes as \( N \) tends to infinity.

**Theorem 2** If \( Y^{(N)}(0) \) converges weakly to \( g \in \Omega \) as \( N \) tends to infinity, then the sequence \( \{ Y^{(N)}(t), t \geq 0 \} \) of Markov processes converges in the Skorohod topology to a solution \( y(t) \) to the system of limiting mean-field equations (14) to (15).

**Proof:** From the martingale characterization of the Markov jump process \( \{ Y^{(N)}(t), t \geq 0 \} \), it follows from Rogers and Williams [63, 62] that for \( -L \leq k \leq K + L \) and \( 1 \leq j \leq m \),

\[
M_{k,j}^{(N)}(t) = Y_{k,j}^{(N)}(t) - Y_{k,j}^{(N)}(0) - \int_0^t \sum_{\Lambda \in \Omega - \{ Y^{(N)}(s) \}} Q^{(N)}(Y^{(N)}(s), \Lambda) \left[ \Lambda_{k,j} - Y_{k,j}^{(N)}(s) \right] ds
\]

is a martingale with respect to the natural filtration associated to the Poisson processes involved in the renting and returning processes and in the Markovian environment, where \( Q^{(N)}(Y^{(N)}(s), \Lambda) \) is the \( Q \)-matrix of the Markov jump process \( \{ Y^{(N)}(t), t \geq 0 \} \) whose expression is given by means of the state change due to the renting and returning processes as well as the Markovian environment.

To express the \( Q \)-matrix \( Q^{(N)}(Y^{(N)}(s), \Lambda) \), we analyze three classes of state transitions as follows:

1. When a customer arrives at a station to rent a bike, the state transition rate is given by

\[
g_{k,j;k-1,j} = \begin{cases} 
\lambda_j, & 1 \leq k \leq K + L, \quad 1 \leq j \leq m, \\
\lambda_j \alpha, & -(L - 1) \leq k \leq 0, \quad 1 \leq j \leq m.
\end{cases}
\]
(2) When a customer returns her bike, the state transition rate is given by

$$q_{k,j,k+1,j}^{(N)}(t) = \left\{ \begin{array}{l}
\frac{\mu_i}{N} \left\{ C + (N - 1) \left[ C - \sum_{k=K}^{K+L} k y_{k,j}^{(N)}(t) \right] \\
+ \sum_{k=K}^{K+L-1} \left[ \frac{(1-\beta) y_{k,j}^{(N)}(t)}{1-(1-\beta) y_{k,j}^{(N)}(t)} \right] + \left[ \frac{y_{k,j}^{(N)}(t)}{1-y_{k,j}^{(N)}(t)} \right] \right\}, \quad -L \leq l \leq 0, \\
\frac{\mu_j}{N} \left\{ C - l + (N - 1) \left[ C - \sum_{k=K}^{K+L} k y_{k,j}^{(N)}(t) \right] \\
+ \sum_{k=K}^{K+L-1} \left[ \frac{(1-\beta) y_{k,j}^{(N)}(t)}{1-(1-\beta) y_{k,j}^{(N)}(t)} \right] + \left[ \frac{y_{k,j}^{(N)}(t)}{1-y_{k,j}^{(N)}(t)} \right] \right\}, \quad 1 \leq l \leq C - 1, \\
\frac{\beta \mu_j}{N} \left\{ (N - 1) \left[ C - \sum_{k=K}^{K+L} k y_{k,j}^{(N)}(t) \right] \\
+ \sum_{k=K}^{K+L-1} \left[ \frac{(1-\beta) y_{k,j}^{(N)}(t)}{1-(1-\beta) y_{k,j}^{(N)}(t)} \right] + \left[ \frac{y_{k,j}^{(N)}(t)}{1-y_{k,j}^{(N)}(t)} \right] \right\}, \quad C \leq l \leq K - 1, \\
\frac{\beta \mu_i}{N} \left\{ (N - 1) \left[ C - \sum_{k=K}^{K+L} k y_{k,j}^{(N)}(t) \right] \\
+ \sum_{k=K}^{K+L-1} \left[ \frac{(1-\beta) y_{k,j}^{(N)}(t)}{1-(1-\beta) y_{k,j}^{(N)}(t)} \right] + \left[ \frac{y_{k,j}^{(N)}(t)}{1-y_{k,j}^{(N)}(t)} \right] \right\}, \quad K \leq l \leq K + L - 1.
\end{array} \right.$$ 

(3) Due to the Markovian environment, the state transition rate is given by

$$q_{k,i,k,j} = w_{i,j}, \quad -L \leq k \leq K + L, i \neq j, 1 \leq i, j \leq m.$$ 

Based on the above three cases, the Q-matrix $Q^{(N)}(Y^{(N)}(s), \Lambda)$ is given by

$$Y_{k,j}^{(N)}(t) = M_{k,j}^{(N)}(t) + Y_{k,j}^{(N)}(0) + q_{k+1,j,k,j} \int_0^t Y_{k+1,j}^{(N)}(s) \, ds$$

$$+ \sum_{i \neq j} w_{i,j} \int_0^t Y_{i,j}^{(N)}(s) \, ds + \int_0^t q_{k-1,j,k,j} \int_0^t Y_{k-1,j}^{(N)}(s) \, ds \, ds$$

Using a similar method to Darling and Norris [12, 13], it is easy to see that if $Y^{(N)}(0)$ converges weakly to $g \in \Omega$ as $N$ tends to infinity, then the sequence $\{Y^{(N)}(t), t \geq 0\}$ of Markov processes is tight for the Skorohod topology, and any limit $Y(t)$ of $\{Y^{(N)}(t), t \geq 0\}$ asymptotically approaches a single trajectory identified by a solution $y(t)$ to the system of limiting mean-field equations (14) to (15). This completes the proof. 

5 A Nonlinear QBD Process

In this section, we discuss the fixed point of the system of limiting mean-field equations (14) to (15), and provide a nonlinear matrix-analytic method to compute the fixed point.
Furthermore, we study the limiting interchangeability of $y^{(N)}(t)$ as $N \to \infty$ and $t \to +\infty$, which is a key in performance evaluation of this bike-sharing system.

For the system of limiting mean-field equations (13) to (15), we have
\[
\frac{d}{dt} y(t) = y(t) V_y(t),
\]
and
\[
y(t)e = 1, \quad y(0) = g \in \Omega.
\]
A point $\pi \in \Omega$ is said to be a fixed point if $\lim_{t \to +\infty} \left[ \frac{d}{dt} y(t) \right] = 0$, or
\[
[y(t) V_y(t)]_{y(t)=\pi} = 0.
\]
In this case, we have
\[
\pi V_\pi = 0 \quad (16)
\]
and
\[
\pi e = 1. \quad (17)
\]

Now, we provide a nonlinear matrix-analytic method for computing the fixed point $\pi$ from the system of nonlinear equations: $\pi V_\pi = 0$ and $\pi e = 1$. To this end, it is necessary to explore the block structure of the system of nonlinear equations, hence this gives a nonlinear QBD process so that the $RG$-factorizations are applicable in our later analysis.

Let
\[
\xi_{k,j} = \lim_{t \to +\infty} \lim_{N \to \infty} \xi^{(N)}_{k,j}(t), \quad -L \leq k \leq K + L, 1 \leq j \leq m.
\]
Then
\[
\xi_{k,j} = \begin{cases} 
\mu_j \xi_{k,j}, & -L \leq k \leq K - 1, \quad 1 \leq j \leq m, \\
\beta \mu_j \xi_{k,j}, & K \leq k \leq K + L - 1, \quad 1 \leq j \leq m,
\end{cases}
\]
where
\[
\xi_{k,j} = C - \sum_{k=K}^{K+L} k \pi_{k,j} + \sum_{k=K}^{K+L-1} \frac{(1 - \beta) \pi_{k,j}}{[1 - (1 - \beta) \pi_{k,j}]^2} + \frac{\pi_{K+L,j}}{[1 - \pi_{K+L,j}]^2}.
\]
Thus we get
\[
\Psi_k = \begin{pmatrix} 
0 & \xi_{k,1} w_{1,1} & \cdots & \xi_{k,1} w_{1,m} \\
\xi_{k,2} w_{2,1} & 0 & \cdots & \xi_{k,2} w_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{k,m} w_{m,1} & \xi_{k,m} w_{m,2} & \cdots & 0
\end{pmatrix},
\]
\[
\hat{\Psi}_k = \text{diag} (\xi_{k,1} w_{1,1}, \xi_{k,2} w_{2,2}, \ldots, \xi_{k,m} w_{m,m}).
\]
for \(-L + 1 \leq k \leq 0\),

\[
\Phi_k = \begin{pmatrix}
0 & \lambda_1 \omega_{1,2} & \cdots & \lambda_1 \omega_{1,m} \\
\lambda_2 \omega_{2,1} & 0 & \cdots & \lambda_2 \omega_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_m \omega_{m,1} & \lambda_m \omega_{m,2} & \cdots & 0
\end{pmatrix}
\]

\[\text{Def } \Phi = \Phi(\alpha),\]

\[
\tilde{\Phi}_k = \text{diag}(\lambda_1 \omega_{1,1}, \lambda_2 \omega_{2,2}, \ldots, \lambda_m \omega_{m,m}) \text{ Def } \tilde{\Phi}(\alpha);
\]

for \(1 \leq k \leq K + L\),

\[
\Phi_k = \Phi(1)
\]

and

\[
\tilde{\Phi}_k = \tilde{\Phi}(1).
\]

It is easy to check from (11) that as \(N \to \infty\) and \(t \to +\infty\)

\[
V_{\pi} = \begin{pmatrix}
B_{1,1} & B_{1,2} \\
B_{2,1} & B_{2,2} & B_{2,3} \\
B_{3,2} & B_{3,3}
\end{pmatrix},
\]

(18)

where

\[
B_{1,1} = \begin{pmatrix}
\hat{\Psi}_L & \Phi_{(L-1)} & \Psi_{(L-1)} \\
\Phi_{(L-1)} & \hat{\Phi}_{(L-1)} + \hat{\Psi}_{(L-1)} & \Psi_{(L-1)} \\
& \ddots & \ddots & \ddots \\
& & \Phi_{(L-1)} & \hat{\Phi}_{(L-1)} + \hat{\Psi}_{(L-1)} & \Psi_{(L-1)}
\end{pmatrix},
\]

\[
B_{1,2} = \begin{pmatrix}
\Phi_1
\end{pmatrix},
\]

\[
B_{2,1} = \begin{pmatrix}
\Phi_1
\end{pmatrix},
\]

\[
B_{2,2} = \begin{pmatrix}
\Phi_1 & \Phi_1 & \Psi_1 \\
\Phi_2 & \Phi_2 + \Psi_2 & \Psi_2 \\
& \ddots & \ddots & \ddots \\
& & \Phi_{K-2} & \Phi_{K-2} + \Psi_{K-2} & \Psi_{K-2} \\
& & & \Phi_{K-1} & \Phi_{K-1} + \Psi_{K-1}
\end{pmatrix}
\]
Thus it follows from (18) that

\[ \begin{pmatrix}
\hat{\Phi}_K & \Psi_K \\
\hat{\Phi}_{K+1} & \hat{\Phi}_{K+1} + \hat{\Psi}_{K+1} & \Psi_{K+1} \\
\vdots & \ddots & \ddots & \ddots \\
\hat{\Phi}_{K+L-1} & \hat{\Phi}_{K+L-1} + \hat{\Psi}_{K+L-1} & \Psi_{K+L-1} & \Psi_{K+L} \\
\end{pmatrix} \]

Based on the nonlinear QBD process \( V_\pi \), we write

\[ \pi = (\pi_{-L}, \pi_{-L+1}, \ldots, \pi_{K+L-1}, \pi_{K+L}) \]

To use the \( RG \)-factorizations given in Section 1.3 of Chapter one in Li [42], we write

\[ R_{-L+1} (\pi) = -\Phi_{-L+1} \left( \hat{\Psi}_{-L} \right)^{-1}, \]

\[ R_{-L+2} (\pi) = -\Phi_{-L+2} \left\{ R_{-L+1} (\pi) \Psi_{-L} + \left[ \hat{\Phi}_{-L+1} + \hat{\Psi}_{-L+1} \right] \right\}^{-1}, \]

for \(-L + 2 \leq k \leq K + L,\)

\[ R_k (\pi) = -\Phi_k \left[ R_{k-1} (\pi) \Psi_{k-2} + \left( \hat{\Phi}_{k-1} + \hat{\Psi}_{k-1} \right) \right]^{-1}, \]

and the infinitesimal generator of the censored Markov chain to level \( K + L \) is given by

\[ \Xi_{K+L} = R_{K+L} (\pi) \Psi_{K+L-1} + \left( \hat{\Phi}_{K+L} + \hat{\Psi}_{K+L} \right). \]
Hence we obtain that for $k = -L$,

$$\pi_{-L} = \pi_{-L+1} R_{-L+1} (\pi),$$

and for $-L + 1 \leq k \leq K + L - 1$,

$$\pi_k = \pi_{k+1} R_{k+1} (\pi) = \pi_{K+L} R_{K+L} (\pi) R_{K+L-1} (\pi) R_{K+L-2} (\pi) \cdots R_{k+1} (\pi),$$

where the vector $\pi_{K+L}$ is a solution to the systems of nonlinear equations $\pi_{K+L} \Xi_{K+L} = 0$ and $\pi_{K+L} \left[ I + \sum_{k=-L}^{K+L-1} R_{K+L} (\pi) R_{K+L-1} (\pi) R_{K+L-2} (\pi) \cdots R_{k+1} (\pi) \right] e = 1$.

The following theorem is a summarization of the above analysis, and hence its proof is omitted here.

**Theorem 3** The fixed point $\pi$ is a solution to the systems of nonlinear equations

$$\pi = (\pi_{K+L} R_{K+L} (\pi) R_{K+L-1} (\pi) R_{K+L-2} (\pi) \cdots R_{-L+1} (\pi),$$

$$\pi_{K+L} R_{K+L} (\pi) R_{K+L-1} (\pi) R_{K+L-2} (\pi) \cdots R_{-L+2} (\pi),$$

$$\cdots, \pi_{K+L} R_{K+L} (\pi) R_{K+L-1} (\pi), \pi_{K+L} R_{K+L} (\pi), \pi_{K+L},$$

(19)

$$\pi_{K+L} \left[ R_{K+L} (\pi) \Psi_{K+L-1} + \left( \hat{\Phi}_{K+L} + \hat{\Psi}_{K+L} \right) \right] = 0$$

(20)

and

$$\pi_{K+L} \left[ I + \sum_{k=-L}^{K+L-1} R_{K+L} (\pi) R_{K+L-1} (\pi) R_{K+L-2} (\pi) \cdots R_{k+1} (\pi) \right] e = 1.$$  (21)

It is easy to see that although the systems of nonlinear equations: $\pi V_{\pi} = 0$ and $\pi e = 1$, are equivalent to the systems of nonlinear equations (19), (20) and (21), Theorem 3 can be used to design different algorithms to compute the fixed points $\pi$. Reader may refer to Li [47] and Li et al. [44] for more details.

In what follows we discuss the mean field limit of the empirical measure process associated to the bike-sharing system when the number $N$ of stations goes to infinity, and show that the fixed point is unique from the systems of nonlinear equations: $\pi V_{\pi} = 0$ and $\pi e = 1$. At the same time, this uniqueness of the fixed point indicates the asymptotic independence of the queueing processes describing the numbers of customers at the $N$ stations as $N \to \infty$, also known as the propagation of chaos.
For the unique fixed point $\pi$, we discuss the limiting interchangeability of the probability vector $\mathbf{y}^{(N)}(t, g)$ as $N \to \infty$ and $t \to +\infty$, where $\mathbf{y}^{(N)}(0, g) = g \in \Omega$. Notice that the limiting interchangeability is necessary in many practical applications when using the stationary probabilities of the limiting process to give an effective approximation for performance analysis of this bike-sharing system.

The following theorem gives the limit of the vector $\mathbf{y}(t, g)$ as $t \to +\infty$, that is,

$$\lim_{t \to +\infty} \mathbf{y}(t, g) = \lim_{t \to +\infty} \lim_{N \to \infty} \mathbf{y}^{(N)}(t, g).$$

**Theorem 4** For any $g \in \Omega$

$$\lim_{t \to +\infty} \mathbf{y}(t, g) = \pi.$$

Furthermore, there exists a unique probability measure $\varphi$ on $\Omega$, which is invariant under the map $g \mapsto \mathbf{y}(t, g)$, that is, for any continuous function $f : \Omega \to \mathbb{R}$ and $t > 0$

$$\int f(g) d\varphi(g) = \int f(\mathbf{y}(t, g)) d\varphi(g).$$

Also, $\varphi = \delta_{\pi}$ is the probability measure concentrated at the fixed point $\pi$.

**Proof:** It is seen from Theorem 2 that as $t \to +\infty$, the limit of $\mathbf{y}(t, g)$ exists on $\Omega$, and it is also a solution on $\Omega$ to the system of nonlinear equations (16) and (17). Since $\mathbf{y}(t, g)$ is the unique solution to the system of limiting mean-field equations (14) and (15), the vector $\lim_{t \to +\infty} \mathbf{y}(t, g)$ is also a solution to the system of nonlinear equations (16) and (17). Note that $\pi$ is the unique solution to the system of nonlinear equations (16) and (17), hence we obtain that $\lim_{t \to +\infty} \mathbf{y}(t, g) = \pi$. The second statement in this theorem can be immediately given by the probability measure of the limiting process $\{\mathbf{Y}(t), t \geq 0\}$ on state space $\Omega$. This completes the proof. \(\blacksquare\)

The following theorem indicates the weak convergence of the sequence $\{\varphi_N\}$ of stationary probability distributions for the sequence $\{\mathbf{Y}^{(N)}(t), t \geq 0\}$ of Markov processes to the probability measure concentrated at the fixed point $\pi$.

**Theorem 5** (1) For a fixed number $N = 1, 2, 3, \ldots$, the Markov process $\{\mathbf{Y}^{(N)}(t), t \geq 0\}$ is positive recurrent, and has a unique invariant distribution $\varphi_N$.

(2) $\{\varphi_N\}$ weakly converges to $\delta_{\pi}$, that is, for any continuous function $f : \Omega \to \mathbb{R}$

$$\lim_{N \to \infty} E_{\varphi_N} [f(g)] = f(\pi).$$
Proof: (1) From Theorem 3, this bike-sharing system of \(N\) identical stations is stable, hence this bike-sharing system has a unique invariant distribution \(\varphi_N\).

(2) Since \(\Omega\) is compact under the metric \(\rho(g, g')\), so it is the set \(\mathcal{P}(\Omega)\) of probability measures. Hence the sequence \(\{\varphi_N\}\) of invariant distributions has limiting points. A similar analysis to the proof of Theorem 5 in Martin and Suhov [52] shows that \(\{\varphi_N\}\) weakly converges to \(\delta_\pi\) and \(\lim_{N \to \infty} E_{\varphi_N}[f(g)] = f(\pi)\). This completes the proof.

Based on Theorems 4 and 5, we obtain a useful relation as follows:

\[
\lim_{t \to +\infty} \lim_{N \to \infty} y^{(N)}(t, g) = \lim_{N \to \infty} \lim_{t \to +\infty} y^{(N)}(t, g) = \pi.
\]

Therefore, we have

\[
\lim_{N \to \infty} y^{(N)}(t, g) = \pi,
\]

which justifies the interchange of the limits of \(N \to \infty\) and \(t \to +\infty\).

Finally, we further show the asymptotic independence (or propagation of chaos) of the queueing processes of this bike-sharing system as follows:

\[
\lim_{t \to +\infty} \lim_{N \to \infty} P \left\{ \begin{array}{l}
X_1^{(N)}(t) = n_1, J_1(t) = j_1; \ldots; X_k^{(N)}(t) = n_k, J_k(t) = j_k
\end{array} \right\} \\
= \lim_{N \to \infty} \lim_{t \to +\infty} P \left\{ \begin{array}{l}
X_1^{(N)}(t) = n_1, J_1(t) = j_1; \ldots; X_k^{(N)}(t) = n_k, J_k(t) = j_k
\end{array} \right\}
\]

\[
= \prod_{l=1}^{k} \pi_{n_l,j_l}
\]

and

\[
\lim_{N \to \infty} \lim_{t \to +\infty} \frac{1}{t} \int_0^t 1 \left\{ \begin{array}{l}
X_1^{(N)}(t) = n_1, J_1(t) = j_1; \ldots; X_k^{(N)}(t) = n_k, J_k(t) = j_k
\end{array} \right\} dt \\
= \lim_{t \to +\infty} \lim_{N \to \infty} \frac{1}{t} \int_0^t 1 \left\{ \begin{array}{l}
X_1^{(N)}(t) = n_1, J_1(t) = j_1; \ldots; X_k^{(N)}(t) = n_k, J_k(t) = j_k
\end{array} \right\} dt
\]

\[
= \prod_{l=1}^{k} \pi_{n_l,j_l} \quad \text{a.s.}
\]

In fact, the two types of limits may be used as an approximate computation for performance measures of this bike-sharing system, hence this demonstrates the key role played by the propagation of chaos.

The following remark provides a unified computational framework to deal with more general bike-sharing systems in practice in terms of the mean-field matrix-analytic method.
Remark 5 We write
\[ S = \{ \pi : \pi V_e = 0, \pi e = 1 \} . \]

It is possible that for a more general bike-sharing system, there exist multiple elements in the set \( S \). Notice that the elements of the set \( S \) must be isolated, thus if there exist multiple elements in the set \( S \), then
\[ S = \{ \pi^{(1)}, \pi^{(2)}, \pi^{(3)}, \ldots \}, \quad \pi^{(k)} \in \tilde{S}^{(k)} \text{ for } k \geq 1, \]
and
\[ G = \{ g^{(1)}, g^{(2)}, g^{(3)}, \ldots \}, \quad g^{(k)} \in \tilde{S}^{(k)} \text{ for } k \geq 1, \]
where \( \tilde{S}^{(k)} \) is a relative open set of \( \Omega \) for \( k \geq 1 \). If for each \( k \geq 1 \),
\[ Y^{(N)}(0) \Rightarrow g^{(k)} \in \tilde{S}^{(k)}, \text{ as } N \to \infty, \]
then
\[ Y^{(N)}(t) \Rightarrow y(t, g^{(k)}), \text{ as } N \to \infty, \quad (22) \]
and
\[ y(t, g^{(k)}) \to \pi^{(k)}, \text{ as } t \to +\infty, \quad (23) \]
where \( y(t, g^{(k)}) \) denotes that this solution \( y(t) \) depends on the initial vector \( g^{(k)} \in \tilde{S}^{(k)} \), and \( y(0, g^{(k)}) = g^{(k)} \). It is worthwhile to note that the initial vector \( g^{(k)} \in \tilde{S}^{(k)} \) with \( Y^{(N)}(0) \Rightarrow g^{(k)} \) as \( N \to \infty \) has a large impact on the limit of the solution \( y(t) \) as \( t \to +\infty \), thus a basic task for computing the multiple fixed points is to find a suitable collection of the relative open sets of \( \Omega \) as follows:
\[ \mathcal{F} = \{ \tilde{S}^{(1)}, \tilde{S}^{(2)}, \tilde{S}^{(3)}, \ldots \}, \]
which sufficiently corresponds to the set \( S = \{ \pi^{(1)}, \pi^{(2)}, \pi^{(3)}, \ldots \} \). Readers may refer to Li \[47\] for more details.

6 Numerical Analysis

In this section, we first use the fixed point to provide some interesting performance measures of this bike-sharing system, such as, the mean of stationary bike number at any
station, the stationary strong-probability of problematic stations, and the stationary weak-probability of problematic stations. Notice that these performance measures are well related to the two crucially practical factors: The finite waiting rooms and the Markovian environment. Then we use four numerical examples to demonstrate how the performance measures depend on the key parameters of this bike-sharing system. Therefore, this paper sets up numerical solution to more general bike-sharing systems by means of the nonlinear QBD processes, which gives a solvable way to deal with more real issues, such as, non-Poisson customer arrivals and non-exponential travel times.

6.1 Performance measures

Using the fixed point \( \pi = (\pi_{-L}, \pi_{-L+1}, \ldots, \pi_0, \pi_1, \ldots, \pi_{K+L-1}, \pi_{K+L}) \) where \( \pi_k = (\pi_k,1, \pi_k,2, \ldots, \pi_k,m) \) and \( \pi_k,e = \sum_{j=1}^{m} \pi_{k,j} \), we provide some interesting performance measures of this bike-sharing system from a practical point of view as follows:

(1) The mean of stationary bike number at any station

\[
E[Q] = \sum_{k=1}^{K+L} k \pi_k,e.
\]

(2-1) The average number of waiting places used by customers renting bikes at any station

\[
E[N_1] = \sum_{k=-L}^{-1} (-k) \pi_k,e.
\]

(2-2) The average number of waiting places used by customers returning bikes at any station

\[
E[N_2] = \sum_{k=K}^{K+L} k \pi_k,e.
\]

(2-3) The stationary expected total number of waiting places at any station

\[
E[N] = \sum_{k=-L}^{-1} (-k) \pi_k,e + \sum_{k=K}^{K+L} k \pi_k,e.
\]

(2-4) The stationary expected maximal number of waiting places at any station

\[
E[\tilde{N}] = \max \{E[N_1], E[N_2]\} = \max \left\{ \sum_{k=-L}^{-1} (-k) \pi_k,e, \sum_{k=K}^{K+L} k \pi_k,e \right\}.
\]
Although $E[N_1]$ and $E[N_2]$ can not exist simultaneously, the virtual value $E[N]$ and $E[\hat{N}]$ may be used to synthetically design the finite waiting rooms of this bike-sharing system.

(3) The stationary strong-probability of problematic stations

The strong-probability of problematic stations is a probability that there is both no bike ($-L \leq k \leq 0$) and no empty waiting places ($k = -L$) when renting a bike, or there is both no parking place ($K \leq k \leq K + L$) and no empty waiting places ($k = K + L$) when returning a bike. Thus the stationary strong-probability of problematic stations is given by

$$P_s = \pi_{-L}e + \pi_{K+L}e.$$ 

(4) The stationary weak-probability of problematic stations

The weak-probability of problematic stations is the probability that either there is no bike ($-L \leq k \leq 0$) when renting a bike, or there is no parking place ($K \leq k \leq K + L$) when returning a bike. Thus the stationary weak-probability of problematic stations is given by

$$P_w = \sum_{k=-L}^{0} \pi_k e + \sum_{k=K}^{K+L} \pi_k e.$$ 

6.2 Numerical examples

Now, we provide four numerical examples to show how the basic performance measures depend on the key parameters of this bike-sharing system.

Example one: Analysis of $E[Q]$

In this bike-sharing system, we take some basic parameters as follows:

$$K = 20, C = 10, L = 5, \lambda_1 = 10, \mu_1 = 20, \alpha = 0.5, \beta = 0.5, m = 2, w = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.$$ 

The left of figure 4 indicates how the mean $E[Q]$ of stationary bike number at any station depends on $\lambda_1 \in (10, 30)$ when $\mu_1 = 45, 50, 55$ and 60, respectively. It is seen that $E[Q]$ decreases as $\lambda_1$ increases but increases as $\mu_1$ increases. The right of figure 4 shows how $E[Q]$ depends on $\mu_1 \in (10, 30)$ when $\lambda_1 = 6, 7, 8$ and 9, respectively. It is seen that $E[Q]$ increases as $\mu_1$ increases but decreases as $\lambda_1$ increases. These numerical results may intuitively be understood as follows: The number of rented bikes increases as $\lambda_1$ increases,
thus $E[Q]$ decreases. At the same time, the number of returned bikes increases as $\mu_1$ increases. This shows that $E[Q]$ increases as $\mu_1$ increases.

**Example two: Analysis of $E[N]$**

In this bike-sharing system, we take the same parameters as those in Example one. The left of figure 5 shows how the stationary expected number $E[N]$ of waiting customers at any station depends on $\lambda_1 \in (15, 30)$ when $\mu_1 = 40, 45, 50$ and $55$, respectively. It is seen that $E[N]$ decreases with $\lambda_1$ increases but increases with $\mu_1$ increases. The right of figure 5 shows how $E[N]$ depends on $\mu_1 \in (15, 30)$ when $\lambda_1 = 6, 7, 8$ and $9$, respectively. It is seen that $E[N]$ increases as $\mu_1$ increases but decreases as $\lambda_1$ increases. Intuitively, the number of rented bikes increases as $\lambda_1$ increases, thus $E[N_1]$ increases but $E[N_2]$ decreases. On the other hand, the number of returned bikes increases as $\mu_1$ increases, so $E[N_2]$ increases but $E[N_1]$ decreases. Based on this, it is clear that $E[N_2]$ has more impact on $E[N]$ than $E[N_1]$.

**Example three: Analysis of the stationary strong-probability $p_s$**

In this bike-sharing system, we take the same parameters as those in Example one. The left of figure 6 shows how the stationary strong-probability $p_s$ depends on $\lambda_1 \in (15, 30)$ when $\mu_1 = 40, 45, 50$ and $55$, respectively. It is seen that $p_s$ decreases as $\lambda_1$ increases but increases as $\mu_1$ increases. The right of figure 4 shows how $p_s$ depends on $\mu_1 \in (15, 30)$ when $\lambda_1 = 6, 7, 8$ and $9$, respectively. It is seen that $p_s$ increases as $\mu_1$ increases but decreases as $\lambda_1$ increases. Intuitively, the number of rented bikes increases as $\lambda_1$ increases, thus $\pi_-Le$ increases but $\pi_{K+Le}$ decreases. On the other hand, the number of returned bikes
increases as \( \mu_1 \) increases, hence \( \pi_{K+L}e \) increases but \( \pi_{-L}e \) decreases. This demonstrates that \( \pi_{K+L}e \) has more impact on \( p_s \) than \( \pi_{-L}e \).

**Example four: Analysis of the stationary weak-probability \( p_w \)**

In this bike-sharing system, we take the same parameters as those in Example one. The left of Figure 7 shows how the stationary weak-probability \( p_w \) depends on \( \lambda_1 \in (15,30) \) when \( \mu_1 = 40, 45, 50 \) and \( 55 \), respectively. It is seen that \( p_w \) decreases as \( \lambda_1 \) increases but increases as \( \mu_1 \) increases. The right of figure 4 shows how \( p_w \) depends on \( \mu_1 \in (15,30) \) when \( \lambda_1 = 6, 7, 8 \) and \( 9 \), respectively. It is seen that \( p_w \) increases as \( \mu_1 \) increases but decreases as \( \lambda_1 \) increases. Intuitively, the number of rented bikes increases as \( \lambda_1 \) increases, thus \( \sum_{k=-L}^{0} \pi_k e \) increases but \( \sum_{k=K}^{K+L} \pi_k e \) decreases. On the other hand, the number
of returned bikes increases as \( \mu_1 \) increases, hence \( \sum_{k=K}^{K+L} \pi_k e \) increases but \( \sum_{k=-L}^{0} \pi_k e \) decreases. This indicates that \( \sum_{k=K}^{K+L} \pi_k e \) has more impact on \( p_w \) than \( \sum_{k=-L}^{0} \pi_k e \).

7 Concluding Remarks

In this paper, we describe a large-scale bike-sharing system under Markovian environment, and develop a mean-field matrix-analytic method through combining the mean-field theory with the time-inhomogeneous queues as well as the nonlinear QBD processes. Furthermore, we apply the martingale limit theory to showing the asymptotic independence (or propagation of chaos) of this bike-sharing system, and also study the limiting interchangeability as \( N \to \infty \) and \( t \to +\infty \). Based on this, we discuss the fixed point by means of a nonlinear QBD process so that we can give performance analysis of this bike-sharing system. From many practical applications, this bike-sharing system may be viewed as an important queueing model who is used to analyze some useful relation between system performance and organization structure. Notice that the mean-field matrix-analytic method is effective and efficient to helpfully design reasonable architecture of a bike-sharing system, such as, finding a better path scheduling, improving inventory management, redistributing the bikes among stations or clusters in terms of truck scheduling, price optimization, application of intelligent information technologies and so forth.

Note that this paper provides a clear way for how to use the mean-field matrix-analytic method to analyze performance measures of more general bike-sharing systems in practice.
through three key parts: (1) Setting up the system of mean-field equations, (2) necessary proofs of the asymptotic independence, and (3) performance computation of this bike-sharing system by means of the fixed point. Therefore, the results of this paper give a new highlight on understanding performance measures and operations management of bike-sharing systems. Along such a line, there are a number of interesting directions for potential future research, for example:

- analyzing the fixed point for more general bike-sharing systems in practice, and provide effective algorithms to deal with the nonlinear QBD processes;
- studying non-exponential travel times and non-Poisson customer arrivals in bike-sharing systems;
- introducing some better operations management, such as, redistribution of bikes by trucks, inventory management, applications of intelligent information system; and
- discussing large-scale bike-sharing systems with different clusters or/and under price optimization.

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