Reductions of nonlocal nonlinear Schrödinger equations to Painlevé type functions

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Abstract

In this paper, we take ODE reductions of the general nonlinear Schrödinger equation (NLS) AKNS system, and reduce them to Painlevé type equations. Specifically, the stationary solution is solved in terms of elliptic functions, and the similarity solution is solved in terms of the Painlevé IV transcendent. Since a number of newly proposed integrable ‘nonlocal’ NLS variants (the PT-symmetric nonlocal NLS, the reverse time NLS, and the reverse space-time NLS) are derivable as specific cases of this system, a consequence is that the nonlocal Painlevé type ODEs obtained from these nonlocal variants all reduce to previously known local equations.

1 Introduction

Integrable nonlinear partial differential equations (PDEs) play an important role in the study of nonlinear wave propagation, and have been studied extensively [3, 2, 26, 30]. Famous examples include the Korteweg-deVries equation, the sine-Gordon equation, and the nonlinear Schrödinger equation (NLS)

\[ iq_t(x, t) = q_{xx}(x, t) - 2\sigma q(x, t)^2 q^*(x, t), \quad \sigma = \pm 1, \]

where \( q \) is a complex valued function of \( x \) and \( t \), and \(^*\) denotes complex conjugation. Here \( \sigma = -1 \) corresponds to a focusing nonlinearity, and \( \sigma = 1 \) corresponds to a defocusing nonlinearity. Many of these equations constitute fundamental models for a wide range of physical phenomena - for instance, the NLS is used to model wave propagation in nonlinear optical fibres and waveguides [1], small-amplitude waves on the surface of deep inviscid water, Langmuir waves in hot plasmas, [25], and the mean-field regime of Bose-Einstein condensates [15].

Recently, there has been significant interest in newly discovered ‘nonlocal’ variants of previously studied integrable PDEs. In 2013, an integrable nonlocal NLS (NNLS) was proposed:

\[ iq_t(x, t) = q_{xx}(x, t) - 2\sigma q(x, t)^2 q^*(-x, t), \quad \sigma = \pm 1, \]

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where the ‘nonlocality’ refers to the simultaneous dependence on \( x \) and \(-x\) \([4]\). This inspired the discovery of many other integrable nonlocal equations \([5, 21]\). Among these, two further integrable NLS variants that are nonlocal in time were recovered: the reverse space-time NLS (RST NLS)

\[
i q_t(x,t) = q_{xx}(x,t) - 2\sigma q(x,t)^2 q(-x,-t), \quad \sigma = \pm 1,
\]

(3)

and the reverse time NLS (RT NLS)

\[
i q_t(x,t) = q_{xx}(x,t) - 2\sigma q(x,t)^2 q(x,-t), \quad \sigma = \pm 1.
\]

(4)

Each of these four equations are obtainable via the integrable AKNS system \([2]\)

\[
\begin{align*}
i q_t & = q_{xx} - 2q^2 r, \\
-ir_t & = r_{xx} - 2r^2 q,
\end{align*}
\]

(5a, 5b)

which establishes their integrability, since the same inverse scattering procedure will broadly apply to all of them, albeit with new symmetry relationships imposed \([5, 31]\). Under the symmetry reduction

\[
r(x,t) = \sigma q^*(x,t),
\]

(6)

the system (5) becomes equivalent to the classic NLS for \( q \). The nonlocal variants (2), (3), (4) are obtained in a similar manner from (5) via the reductions

\[
\begin{align*}
\text{NNLS} & : \quad r(x,t) = \sigma q^*(-x,t), \\
\text{RST NLS} & : \quad r(x,t) = \sigma q(-x,-t), \\
\text{RT NLS} & : \quad r(x,t) = \sigma q(x,-t).
\end{align*}
\]

(7-9)

Notably, each of NLS variants (1)-(4) can thus be viewed as a specific solution of the system (5).

The reverse space NNLS has been intensely studied since its introduction \([27, 20, 18, 19]\). In particular, an interesting property is that it exhibits so-called ‘parity-time (\(PT\)) symmetry’. That is, it can be viewed as a quantum mechanical (time-dependent) Schrödinger equation

\[
\begin{align*}
i q_t(x,t) & = q_{xx}(x,t) + V(x,t)q(x,t), \\
V(x,t) & = -2\sigma q(x,t)q^*(-x,t),
\end{align*}
\]

(10, 11)

where the self-induced potential \( V(x,t) \) satisfies the \(PT\)-symmetry condition \( V^*(-x,t) = V(x,t) \) \([4]\). This gives a connection to the field of \(PT\)-symmetric quantum mechanics, \(PT\)-symmetric optics, and other \(PT\)-symmetric physical applications which are currently the subject of much research activity \([6, 17, 22]\).

The general N-soliton behaviour of each of the three nonlocal NLS variants was investigated recently in \([31]\), which demonstrated new soliton behaviours for these equations, arising from novel configurations of eigenvalues in the spectral plane. The relation between nonlocal and local versions of integrable equations was studied in \([29]\), which demonstrated that in some cases it is possible to transform from the nonlocal to local case and vice versa. In terms of applications, the NNLS (2) was derived in a physical application of magnetics \([16]\), and more
generally, nonlocal space/time couplings in integrable equations may be used to model scenarios involving correlated events [24].

Another interesting aspect of integrable nonlinear PDEs is their relation to nonlinear ordinary differential equations (ODEs). It is well-known that reductions of integrable nonlinear PDEs yield (possibly after a transformation) ODEs of Painlevé type - which is to say the solutions of these ODEs do not have any movable singularities that are worse than poles [23]. Notable among Painlevé type ODEs are the Painlevé equations - a class of nonlinear second-order equations that define special functions called the Painlevé transcendent. These functions have many interesting analytic properties, and arise in a broad range of applications, like in the solutions of integrable PDEs such as the NLS, in statistical physics, random matrices, and quantum field theory [11, 12, 13]. Whilst extensive efforts have been made to classify Painlevé type ODEs, with complete results obtained in certain cases, classification in general remains an open problem [10].

In the case of the original NLS, it is known that the stationary solution \( q(x, t) = e^{i\lambda t}f(x) \) yields the ODE

\[-\lambda f = f'' - 2\sigma f^2 f^* \quad (12)\]

which is solved in terms of elliptic functions (the ODEs that define elliptic functions are of Painlevé type). Further, the similarity solution \( q(x, t) = \frac{1}{\sqrt{2t}}Q(z), z = \frac{x}{\sqrt{2t}} \) leads to

\[-iQ - izQ'' = Q'' - 2\sigma Q^2 Q^* \quad (13)\]

which is solved in terms of Painlevé IV [7, 9].

It was noted in [5] that taking analogous reductions for the nonlocal variants of NLS yield new Painlevé type equations that are nonlocal analogues of these previously known ones. In [28], the general stationary solution of the NNLS was investigated and reduced down to an elliptic function. However, to our best knowledge, the other reductions have not been studied.

In this brief communication, we deal with the AKNS system (5) in general and via integrating and transforming, obtain a superset of solutions for the 1) stationary solution case and 2) the similarity solution case. This solution set will also form a superset of the stationary and similarity solutions for each of the four NLS variants. We demonstrate that the general stationary case reduces to an equation solvable by known elliptic functions, and the general similarity solution reduces to Painlevé IV. That is, the broad outcomes that occur for the reductions of the standard NLS also hold for the other three nonlocal variants. Further, the nonlocal Painlevé type ODEs we obtain are solved in terms of previously known functions and so do not define any truly new functions.

2 General stationary solution

Consider the stationary solution

\[ q(x, t) = e^{i\lambda t}f(x), \quad (14) \]
where \( f(x) \) is a function \( \mathbb{R} \to \mathbb{C} \), and \( \lambda \in \mathbb{R} \) is a constant. Then \( q_i = i\lambda q_i \), and for each of the four symmetry reductions (6) to (9), we will have \( r_i = -i\lambda r_i \), which can be seen from

\[
\text{NLS:} \quad r(x,t) = \sigma q^*(x,t) = \sigma e^{-i\lambda t} f^*(x),
\]

\[
\text{NNLS:} \quad r(x,t) = \sigma q^* (-x,t) = \sigma e^{-i\lambda t} f^*(-x),
\]

\[
\text{RST NLS:} \quad r(x,t) = \sigma q(-x,-t) = \sigma e^{-i\lambda t} f(-x),
\]

\[
\text{RT NLS:} \quad r(x,t) = \sigma q(x,-t) = \sigma e^{-i\lambda t} f(x).
\]

Further, in each case (15a) to (15d), we can write \( r(x,t) := \sigma e^{-i\lambda t} g(x) \), where \( g(x) \) is related to \( f(x) \) through some combination of complex conjugation and the operation \( x \to -x \). For instance, in the case of the NNLS reduction (15b) we have \( g(x) = f^*(-x) \).

Hence, without loss of generality we can simplify the AKNS system (5) to the ODE system

\[
\begin{align*}
-\lambda f &= f'' - 2\sigma f^2 g, \quad (16a) \\
-\lambda g &= g'' - 2\sigma g^2 f,
\end{align*}
\]

which incorporates the stationary solutions of all four NLS variants as specific cases. In particular, for the nonlocal symmetry reductions, taking the stationary solution yields new Painlevé type equations which are nonlocal analogues of (12) - i.e. equation (16a) gives:

\[
\begin{align*}
\text{NNLS:} \quad -\lambda f(x) &= f''(x) - 2\sigma f(x)^2 f^*(-x), \quad (17a) \\
\text{RST NLS:} \quad -\lambda f(x) &= f''(x) - 2\sigma f(x)^2 f(-x). \quad (17b)
\end{align*}
\]

We observe that (17a) is the time-independent quantum mechanical Schrödinger equation with the \( PT \)-symmetric self-induced potential \(-2\sigma f(x) f^*(-x)\).

From here, we will neglect the relationship between \( f \) and \( g \), and solve for the ODE system (16) in general. We use (16a) to express \( g \) in terms of \( f \) and \( f'' \). Differentiating twice yields an expression for \( g'' \) in terms of \( f \) and derivatives of \( f \). Thus, we can replace all instances of \( g \) and \( g'' \) in (16b) to obtain a fourth-order ODE in \( f \)

\[
f^{(4)} f^2 - 2\lambda f^2 f'' - 3f f''' - 2\lambda f f'' - 4f^{(3)} f f' + 6f^2 f'' = 0. \quad (18)
\]

Noting the homogeneity of terms, we use the well-known substitution

\[
v(x) := \frac{f'(x)}{f(x)} \quad (19)
\]

which is valid for \( f \) not identically 0, to obtain an ODE of reduced order in the new function \( v \)

\[
v^{(3)} - 2\lambda v' - 6v^2 v' = 0. \quad (20)
\]

This can be directly integrated to obtain

\[
v'' - 2v^3 - 2\lambda v - a = 0. \quad (21)
\]

where \( a \in \mathbb{C} \) is a constant of integration. Multiplying by \( v' \), it can be integrated again to

\[
v'^2 = v^4 + 2\lambda v^2 + av + b, \quad (22)
\]

with \( a, b \in \mathbb{C} \) constants of integration. The general solution to this ODE is an elliptic function, which can be written as an elliptic integral of the first kind [8].
3 General similarity solution

Consider the similarity solution
\[ q(x, t) = \frac{1}{(2t)^{1/2}} Q\left(\frac{x}{(2t)^{1/2}}\right), \quad r(x, t) = \frac{1}{(2t)^{1/2}} R\left(\frac{x}{(2t)^{1/2}}\right). \] (23)

Analogously to the previous section, consider what happens for each specific reduction. The \(\mathcal{PT}\)-symmetric case behaves nicely:

\[
\text{NLS} \quad r(x, t) = \sigma q^*(x, t) \iff R(z) = \sigma Q^*(z), \quad \text{(24a)}
\]

\[
\text{NNLS} \quad r(x, t) = \sigma q^*(-x, t) \iff R(z) = \sigma Q^*(-z). \quad \text{(24b)}
\]

However, when time reversal is involved, we see

\[
\text{RT NLS} \quad r(x, t) = \sigma q(x, -t) \iff R(z) = \sigma \sqrt{-i} Q(\kappa z), \quad \text{(25a)}
\]

\[
\text{RST NLS} \quad r(x, t) = \sigma q(-x, -t) \iff R(z) = \sigma \sqrt{-i} Q(-\kappa z). \quad \text{(25b)}
\]

i.e., there is branch dependence, as noted in [5]. Let \(\kappa = \frac{1}{\sqrt{-1}} = \pm i\). Then the above two conditions become

\[
\text{RT NLS} \quad R(z) = \sigma \sqrt{-i} Q(\kappa z), \quad \text{(26a)}
\]

\[
\text{RST NLS} \quad R(z) = \sigma \sqrt{-i} Q(-\kappa z). \quad \text{(26b)}
\]

Substituting in, the AKNS system (5) reduces to the ODE system

\[
-iQ - izQ' = Q'' - 2Q^2 R, \quad \text{(27a)}
\]

\[
iR + izR' = R'' - 2R^2 Q. \quad \text{(27b)}
\]

In the cases of the three nonlocal symmetry reductions, we find nonlocal Painlevé equations

\[
\text{NNLS} \quad -iQ(z) - izQ'(z) = Q''(z) - 2\sigma Q(z)^2 Q^*(-z), \quad \text{(28a)}
\]

\[
\text{RT NLS} \quad -iQ(z) - izQ'(z) = Q''(z) - 2\sigma \kappa Q(z)^2 Q(\kappa z), \quad \text{(28b)}
\]

\[
\text{RST NLS} \quad -iQ(z) - izQ'(z) = Q''(z) - 2\sigma \kappa Q(z)^2 Q(-\kappa z). \quad \text{(28c)}
\]

The first one can be thought of as a \(\mathcal{PT}\)-symmetric Painlevé equation [5].

Again, we consider \(Q, R\) to be uncoupled and solve for (27) in general. Equation (27a) can be used to express \(R\) in terms of \(Q, Q', \) and \(Q''\). This can then be used to replace all instances of \(R\) and its derivatives in (27b) to obtain a fourth-order ODE in \(Q\)

\[
-6izQ'^3 - 6Q'^2 Q'' - Q^2 \left(Q^{(4)} + (z^2 - 2i)Q'' + 3zQ'ight) + Q \left(3Q'^2 + (z^2 + 2i)Q'^2 + Q' \left(4Q^{(3)} + 6izQ''\right)\right) - 2Q^3 = 0. \quad \text{(29)}
\]
Noting again the homogeneity, substitute
\[ V(z) := \frac{Q'(z)}{Q(z)} \]  
(30)
to obtain
\[ V^{(3)} + z^2 V' - 6izVV' - 6V^2 V' - 2iV' + 3zV - 4iV^2 + 2 = 0. \]  
(31)
Taking
\[ V(z) = Y(z) - \frac{i}{2} z \]  
(32)
transforms (31) to
\[ Y^{(3)} = 6Y^2 Y' + \left( \frac{z^2}{2} + 2i \right) \left( Y' + \frac{i}{2} \right) + iY^2 + zY \]  
(33)
which is sometimes known as Chazy VIII.a (see section 6.8 of [14]) with parameters \( \alpha = \frac{i}{2}, \beta = 0, \gamma = 2i \). The Chazy VIII.a equation has a general solution in terms of Painlevé IV: a final transformation
\[ Y(z) = i\sqrt{\alpha} w(\zeta) - \alpha z = \frac{1}{2} (1 + z)w \left( \frac{1}{2} (1 + i)z \right) - \frac{i}{2} z \]  
(34)
takes (33) to the canonical Painlevé IV
\[ w'' = \frac{1}{2w} w'^2 + \frac{3}{2} w^3 + 4zw^2 + 2(z^2 - A)w + \frac{B}{w} \]  
(35)
with parameters \( A = 1 \), and \( B \) arbitrary.

4 Conclusion

Two simple ODE reductions of the AKNS system associated with NLS type equations were reduced to well known Painlevé type ODEs through a series of integrations. This includes as specific cases a number of nonlocal and/or \( PT \)-symmetric Painlevé type ODEs that come from reductions of new nonlocal NLS variants.

The simple method we used to deal with ODEs with nonlocal dependency seems like it could be applicable in other cases. That is to say, a future step would be repeating the analysis for some other integrable PDE with ODE reductions known to be solved in terms of Painlevé type functions. One would take the nonlocal version of this PDE, apply an ODE reduction to get another nonlocal Painlevé type ODE, use symmetry to generate a system of equations, and finally recover a single local ODE which can be integrated repeatedly.

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