Finite-Level Quantization Procedures for Construction and Decoding of Polar Codes
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Abstract—We consider finite-level, symmetric quantization procedures for construction and decoding of polar codes. Whether polarization occurs in the presence of quantization is not known in general. In [1], it is shown that a simple three-level quantization procedure polarizes and a calculation method is proposed to obtain a lower bound for achievable rates. We find an improved calculation method for achievable rates and also the exact asymptotic behavior of the block error probability for the simple case. We then prove that certain D-level quantization schemes polarize and give a lower bound on achievable rates. Furthermore, we show that a broad class of quantization procedures result in a weaker form of the polarization phenomenon.

I. INTRODUCTION

Polar codes are the first class of channel codes that achieve capacity for Binary-input Memoryless Symmetric (BMS) channels with low encoding and decoding complexities [2]. As the name suggests, polar codes are based on a polarization phenomenon, which we now describe briefly: Given two identical and independent instances of a BMS channel $W : \mathcal{X} \rightarrow \mathcal{Y}$, create two synthetic channels $W^+ : \mathcal{X} \rightarrow \mathcal{Y}^2$ and $W^- : \mathcal{X} \rightarrow \mathcal{Y}^2 \times \mathcal{X}$ with the polar transform introduced in [2]. Arıkan has shown that the mutual information of $W^+$ is greater than the mutual information of $W^-$ and their average is equal to that of $W$. This means that from a BMS channel $W$, its ‘worse’ and ‘better’ versions are synthesized while the average mutual information is preserved. Recursive application of the above construction allows one to synthesize channels $W^n_s$ for all $s_n \in \{+, -\}^n$ in $n$ steps. Arıkan has also shown that a fraction of synthetic channels eventually become ‘perfect’ whereas the other fraction eventually become ‘useless’. In other words, they eventually polarize. Together with the fact that the average mutual information remains same at each step and the error probability of perfect channels behave as $O(2^{-2n/2})$ (cf. [3]), this shows the capacity achieving property of polar codes.

Arıkan has introduced the Successive Cancellation Decoder (SCD) in [2], which estimates the input sequence by calculating the individual log-likelihood ratios (LLR) for each bit, exploiting the recursive structure. The basis of code construction is to send the information bits through synthetic channels that are close to perfect. Identifying these almost perfect channels can in principle be done with a density evolution algorithm [4]. We exploit the inherent symmetry of BMS channels and assume all-zero sequence is sent throughout this manuscript. Under this assumption and supposing that the channel output is $Y$, the update equations for LLRs are given by

$$L^- = L \oplus L', \quad L^+ = L + L'$$

where $L \triangleq \ln \left( \frac{W(Y|0)}{W(Y|1)} \right)$, $a \oplus b \triangleq \ln \left( \frac{e^{a+b}+1}{e^{a-b}+1} \right)$ and $L'$ is an identical and independent copy of $L$. Similar to the creation of synthetic channels, one can calculate the distribution of any $L^n_s$, $s_n \in \{+, -\}^n$. Note that the distribution of $L^n_s$ is equivalent to the channel transition transition probabilities of $W^n_s$ given all-zero input.

Now, we state two challenges about code construction and decoder implementation:

1) In general, equations (1) suggest that the support size of LLRs grow exponentially in block length. To overcome this problem, special degradation procedures or approximations are proposed (e.g., see [5], [6]).

2) LLRs are real numbers, therefore implementation of a real-time SCD has to include an inherent quantization scheme depending on the required precision (cf. [7]). In [1], robustness of polarization with respect to a specific family of quantization schemes was examined and the authors have shown that even a simple 3-level quantization scheme polarizes.

We refer the reader to the partial list ([8]–[12]) for other studies on these considerations. To the best of our knowledge, little is known about polarization for finite-level quantization schemes other than the three-level case just mentioned. We have found that a weaker polarization phenomenon compared to that in [2] takes place under some constraints.

The main results of this manuscript are:

(i) For the three-level quantization scheme in [1], an improved calculation method for the lower bound for achievable rates is obtained.

(ii) The exact asymptotic behavior of block error probability for the same three-quantized decoder is found to be $O(2^{-\sqrt{N \phi}})$, where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $N = 2^n$ is the block length.

(iii) A broad family of finite-level quantization procedures weakly polarize. The family will be defined in Section III.

II. NOTATION

The random variables are denoted with uppercase letters whereas their realizations are denoted with lowercase letters (e.g., $X_n$ and $x_n$). Sets and events are denoted with script-style letters (e.g., $\mathcal{A}_n$, $\mathcal{G}_n$). As a special case, $\Pi_\mathcal{A}$ denotes the set of all probability distributions on $\mathbb{R}$. $|\mathcal{A}|$ denotes the cardinality of a set $\mathcal{A}$. Vectors and sequences are denoted by boldface letters. If the length is known, it is added as a subscript (e.g., $s_n$). If the length is not known or has no importance, we drop the subscript (e.g., $s$). $\mathbb{1}_\mathcal{A}$ denotes the indicator function for a set $\mathcal{A}$.
We abbreviate the following operations: \( a \land b \equiv \min\{a, b\} \), \( a \lor b \equiv \max\{a, b\} \), \( \text{sign}(x) \equiv 1_{\{x > 0\}} - 1_{\{x < 0\}} \). \( h(x) \equiv -x \log x - (1 - x) \log(1 - x) \) is the binary entropy function defined for \( x \in [0, 1] \). All the logarithms are in base 2 unless we use the notation \( \ln \) for natural logarithm.

III. STATIC AND DYNAMIC QUANTIZATION PROCEDURES

We define a family of symmetric quantization procedures to unify the approaches in [1], [5] and [6].

**Definition 1** (D-quantization family and admissible quantization procedures). For a finite \( D \in \mathbb{N} \), a D-quantization family \( Q^{(D)} \) is a family of odd, increasing step functions which can take at most \( D \) values. Moreover, the members are right continuous on \( \mathbb{R}_+ \), and left continuous on \( \mathbb{R}_- \). We also define the family of admissible quantization procedures as \( Q \equiv \bigcup_{D \geq 1} Q^{(D)} \).

Restriction to odd functions provides symmetry. This is necessary to preserve the property that the set of BMS channels are invariant under polar transforms with quantization schemes.

Note that Definition 1 implies that for all \( Q \in \mathcal{Q} \), \( Q(0) = 0 \). Hence, one can always take \( D \) as an odd number. Furthermore, for any member of \( \mathcal{Q} \), the quantization intervals in \( \mathbb{R}_+ \) together with their images contain all the information needed for its behavior in \( \mathbb{R} \). Taking into account the above, we have the following definition of static and dynamic quantization procedures.

**Definition 2** (D-static and D-dynamic quantization). A D-static quantization \( Q^{(D)} : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) is a member of \( Q^{(D)} \), where the right limits of quantization intervals in \( \mathbb{R}_+ \) and their images are given in the parameter \( \beta(\mathbb{P}) \), \( \mathbb{P} \in \mathbb{P}_\beta \). \( \beta(\mathbb{P}) \) is a set of 2-tuples with \( |\beta| = \frac{D-1}{2} \) that depends on the distribution \( \mathbb{P} \). A D-static quantization is a D-dynamic quantization with \( \beta \) being same for all \( \mathbb{P} \in \mathbb{P}_\beta \).

We give a simple example of a D-static quantization procedure.

**Example 1.** Given \( \alpha_1, \alpha_2, \gamma_1, \gamma_2 \in \mathbb{R}, 0 < \alpha_1 < \alpha_2 \) and \( 0 < \gamma_1 < \gamma_2 \), let \( \beta = \{(\alpha_1, \gamma_1), (\alpha_2, \gamma_2)\} \). \( Q^{(S)}(x) \) is depicted in Figure 1:

![Graphical representation of \( Q^{(S)}(x) \).](image)

A special case is when \( \alpha_1 = 0 \). Then, \( Q_\beta(0) = 0 \) and \( Q_\beta(x) = \gamma_1 \) for \( 0 < x < \alpha_2 \). Observe that \( Q_\beta \) is not continuous at zero for this case.

We sometimes drop the superscript \( (D) \) if the number of quantization levels \( D \) is known or trivial. For dynamic quantization procedures, the notation \( \beta(Y) \) is equivalent to \( \beta(\mathbb{P}) \) if a random variable \( Y \) with distribution \( \mathbb{P} \) is to be quantized.

\( \mathcal{Q} \) contains a broad class of practical quantization procedures. Observe that any quantization scheme similar to those in [1] belongs to \( Q^{(D)} \). Furthermore, it is immediate from Definition 2 that \( Q \circ Q' \in \mathcal{Q} \) for all \( Q, Q' \in \mathcal{Q} \). This implies that the greedy quantization procedures in [5] and [6] are dynamic quantization procedures which belong to \( \mathcal{Q} \) with the additional condition that zero is an absorbing support, namely, any combination of the zero support with some nonzero support should map to zero. We also emphasize that the widely used approximation (cf. [13])

\[
a \ominus b \equiv (|a| \land |b|)\text{sign}(ab) \approx a \ominus b
\]

results in a dynamic quantization procedure under some conditions.

**Lemma 1.** Consider a discrete random variable \( L \) and its identical and independent copy \( L' \) that takes values in the finite set \( \mathcal{L} = \{d_1, \ldots, d_n\} \) for some \( n \in \mathbb{N} \). Take the symmetrized set \( \hat{\mathcal{L}} \equiv \mathcal{L} \cup (-\mathcal{L}), \) where \( -\mathcal{L} = \{-d_1, \ldots, -d_n\} \). Suppose the non-negative elements of \( \hat{\mathcal{L}} \) are ordered as \( \alpha_1 \leq \ldots \leq \alpha_n \) for some \( m \). If \( \alpha_{i+1} > \ln(e^{\alpha_i} + \sqrt{e^{2\alpha_i} - 1}) \) for all \( 1 \leq i \leq m - 1 \), there exists a dynamic quantization procedure \( Q_\beta(L) \) such that \( L \ominus L' = Q_\beta(L) \ominus Q_\beta(L') \).

**Proof:** The random variable \( L \ominus L' \) takes values in the set \( \hat{\mathcal{L}} \equiv \mathcal{L} \cup (-\mathcal{L}) \). Suppose \( \alpha_i > \ln(e^{\alpha_i-1} + \sqrt{e^{2\alpha_i-1} - 1}) \) for all \( 1 \leq i \leq m - 1 \) and \( \alpha_0 = 0 \). Then, one can show \( \alpha_{i-1} < \alpha_i \ominus \alpha_{i+1} < \alpha_i \ominus \alpha_m < \alpha_i \) for all \( i \in [m] \). Take the dynamic quantization procedure \( Q_\beta(L) \) with

\[
\beta(L) = \bigcup_{i=1}^{m} \{(\alpha_i \ominus \alpha_i, \alpha_i)\}.
\]

With the above selection, \( Q_\beta(L) \left( \bigcup_{i,j \geq 1} \{(\alpha_i \ominus \alpha_j)\} \right) = \alpha_i = \bigcup_{i,j \geq 1} \{(\alpha_i \ominus \alpha_j)\} \). In other words, every \( \alpha_i \ominus \alpha_j \) is mapped to \( \alpha_i \ominus \alpha_j \). Since this true for all \( 1 \leq i, j \leq m \), \( L \ominus L' \) \( \land 0 = (Q_\beta(L) \ominus Q_\beta(L')) \bigcup 0 \). The proof for the negative support follows similarly.

IV. THREE-QUANTIZED CASE

In this section, we study the same three-level quantization procedure from [1]. We briefly explain the findings in [1] with an improvement on calculation of the lower bound for the fraction of perfect channels. We also find the exact asymptotic behavior of the block error probability.

Consider a BMS channel \( W \), whose output \( Y \) takes values from the set \( \{-\lambda, 0, \lambda\} \). If the initial channel has support size larger than three, it can be quantized with any desired procedure until we obtain a channel with three outputs. The static quantization procedure we consider throughout this section is \( Q^{(S)} \), \( \beta = \{(0, 1)\} \). Verbally, \( Q_\beta \) results in only propagating the signs of the quantized random variables. The quantized channel output, \( Y^{*n} = Q_\beta(Y^{*n-1} + s_n) \), \( s_n \in \{+, -\} \), \( n \geq 1 \) with \( Y^{*n-1} \) defined according to (1); has therefore three parameters, namely \( p^{*n} \equiv \Pr(Y^{*n} = 1) \), \( m^{*n} \equiv \Pr(Y^{*n} = -1) \) and \( e^{*n} \equiv \Pr(Y^{*n} = 0) = 1 - p^{*n} - m^{*n} \). Without loss
of generality, we assume $p \geq m$. Otherwise, one can negate the channel output to fulfill this condition. These parameters completely describe the distribution of $Y^{sn}$. Referring to (1), iterations of $(p, m, z)$ under $Q_\beta$ are given by
\begin{align}
p^+ &= p^2 + 2pz \\
m^+ &= m^2 + 2mz \\
z^+ &= z^2 + 2mp
\end{align}
It is possible to calculate $(p^s, m^s, z^s)$ for any $s \in \{+, -\}$ with the above transformations. Note that these transformations preserve $p^s \geq m^s$.

A. Feasible Region for $Y^s$

Our purpose is to track these parameters for the statistic $Y^s$. At first sight, it may seem that $p^s$ and $m^s$ can take any value in the set $R_3 \triangleq \{(p, m) : p + m \leq 1, p \geq m \geq 0\}$. However, this is not the case. It is known that $Y^s$ has gone through $+$ transformation once, there are some restrictions on the feasible region for its parameters.

**Lemma 2.** Define the limiting curve as the $(p, m)$ pairs with the following parametric equations:
\begin{align}
p^*(t) &= \sqrt{4t^4 - 3t^2} \\
m^*(t) &= 1 - 3t + \frac{3}{2}t^2 + \frac{p^*(t)}{2}, \quad t \in [0, 1].
\end{align}
Let
\[ R_3^+ \triangleq R_3 \cap \left( \bigcup_{t \in [0,1]} \{(p, m) : 0 \leq m \leq m^*(t), p = p^*(t)\} \right). \]
Then, for any $s_n \in \{+, -\}^n$, $n \geq 1$
(i) It is sufficient that $s_n$ contains at least one $+$ to ensure $(p^{s_n}, m^{s_n}) \in R_3^+$.
(ii) If $(p^{s^n}, m^{s^n}) \in R_3^+$, then $(p^{s^{n+1}}, m^{s^{n+1}}) \in R_3^+$ for $s \in \{+, -\}$. In words, once $(p^{s^n}, m^{s^n})$ is driven under the limiting curve, it remains there.

Proof of Lemma 2 is omitted due to space limitations. We refer to [14] for a complete proof.

B. Polarization of Quantized Statistics

With a similar approach to those in [1] and [2], parameters of quantized statistics can be examined in a probabilistic setting. The setting is described below:

Fix $\Omega \triangleq \{+, -\}$ and let $S_n = (S_1, S_2, \ldots, S_n)$ be a sequence of $n$ random variables where each $S_k$ is independently and uniformly distributed on $\{+, -\}$. Define the natural filtration $\mathcal{F}_n \triangleq \sigma(S_n)$. Also define $\mathcal{F} \triangleq \sigma(S_n)$. These ingredients completely define the probability space with filtration $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ and for a quantized statistic obtained in $n$ polarization steps, any of its parameter becomes an $\mathcal{F}_n$-measurable random variable, namely $P_n \triangleq p^{S_n}$, $Z_n \triangleq s^{S_n}$ and $M_n \triangleq m^{S_n}$. Also note that any function of $D_n \triangleq (P_n, Z_n, M_n)$ becomes random.

The quantized statistic $Y^{s_n}$ can also be represented as a ‘quantized’ or ‘degraded’ synthetic BMS channel $\hat{W}^{s_n}$ with
\[ \hat{W}^{s_n}(y|0) = \begin{cases} P_n, & y = 1 \\ Z_n, & y = 0 \\ M_n, & y = -1 \end{cases} \]

It is known that any bounded submartingale or supermartingale converges almost surely (see, e.g., [15]). Therefore, if a function of $D_n$ is a submartingale or a supermartingale, it may give information on whether polarization occurs. From this perspective, we list some consequences of the quantization procedure $Q_\beta$ in terms of probabilistic arguments. One can verify that $P_n, M_n, Z_n$ themselves exhibit submartingale/supermartingale properties [1]. Moreover, the mutual information of $\hat{W}^{s_n}$,
\[ I(\hat{W}^{s_n}) \triangleq (p^{s_n} + m^{s_n}) \left( 1 - h \left( \frac{m^{s_n}}{p^{s_n} + m^{s_n}} \right) \right) \]
is a supermartingale. This property follows simply from data processing inequality as the average mutual information is preserved without quantization.

**Lemma 3** ([1], Lemma 4). The random variables $P_n, Z_n, M_n$ converge almost surely. Moreover, $Z_\infty \triangleq \lim_{n \to \infty} Z_n = 0$ or 1, $P_\infty \triangleq \lim_{n \to \infty} P_n = 0$ or 1 and $M_\infty \triangleq \lim_{n \to \infty} M_n = 0$ almost surely. Namely, $Y^{s_n}$ polarizes.

Lemma 3 simply follows from the fact that $Z_n$ is a submartingale and $M_n$ supermartingale.

Knowing that the quantized statistics polarize, we elaborate on the question of what fraction of these statistics carry lossless information. We note that it is very hard to obtain an exact expression for this fraction. Let $\gamma$ quantify the fraction of the lossless statistics. Lower and upper bounds for $\gamma$ can be obtained from the submartingale and supermartingale properties of some functions $f(D_n)$ with $f(1, 0, 0) = 1$ and $f(0, 1, 0) = 0$. Suppose $f(D_n)$ is a bounded submartingale (supermartingale), i.e., it satisfies $\frac{f'(d_n) + f'(d_\infty)}{2} \leq f(d), \forall d \in R_3$ — this condition can be verified numerically as it is equivalent to a maximization problem over a compact region. Then $\gamma = \mathbb{E}[f(D\infty)] \leq \mathbb{E}[f(p, z, m)]$, which shows that $f$ is useful to obtain an lower (upper) bound on $\gamma$. In [1] it is shown that $I(W)^2 \leq \gamma \leq I(W)$ as $I(W^{s_n})$ is a supermartingale and $I(W^{s_n})^2$ submartingale. In addition, we have numerically found that $I(1.24(W^{s_n}))$ is submartingale if $(p^{s_n}, m^{s_n})$ belongs to $R_3^+$. Hence, we have the following improved lower bound for $\gamma$.

**Lemma 4.** If the original $(p, m) \in R_3^+$, then $I(1.24(W))$ is a lower bound for $\gamma$. If not, then $\frac{1}{2}I(1.24(W^+)) + \frac{1}{2}I^2(W^-)$ is a lower bound for $\gamma$. More precisely, define
\[ F_0(W) \triangleq \begin{cases} I(1.24(W)), & (p, m) \in R_3^+ \\ \frac{1}{2}I(1.24(W^+)) + \frac{1}{2}I^2(W^-), & \text{else} \end{cases} \]
Then, $F_0(W) \leq \gamma$.

**Corollary 1.** $F$ can be improved by increasing the number of polarization steps. Namely, define
\[ F_n(W) \triangleq \begin{cases} \frac{1}{2^n} \sum_{s_n \in \{+, -\}^n} I(1.24(W^{s_n})), & (p, m) \in R_3^+ \\ \frac{1}{2^n} \sum_{s_n \in \{+, -\}^n \setminus \{0\}} I(1.24(W^{s_n}) + I^2(W^-))^n), & \text{else} \end{cases} \]
Then, $F_0(W) \leq F_1(W) \leq \gamma$. 

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The proposed method for calculation of the lower bound in [1] relies on the fact that $\gamma$ is bounded from above and below as $\mathbb{E}[I^2(W_{S^n})] \leq \gamma \leq \mathbb{E}[I(W_{S^n})]$, and $\mathbb{E}[I(W_{S^n})] - \mathbb{E}[I^2(W_{S^n})] \leq \delta$ for some $\delta > 0$ and large enough $n$. Therefore, one can obtain a confidence interval of $\delta$ for large $n$. Since $\mathbb{E}[I(W_{S^n})] - F_n(W)$ decreases faster, the same confidence interval $\delta$ can be achieved with smaller $n$ compared to the first method. This results in an improved calculation of lower bounds.

C. Rate of Polarization

From the previous section, we know that the quantized statistics polarize. However, it is required that the error probability $P_e(W_{S^n}) \triangleq M_n + \frac{1}{2}Z_n$ of each perfect statistic decays fast enough, i.e., $o(2^{-n})$, to ensure reliable communication under the aforementioned quantization procedure. For the unquantized case, it is found in [3] that the Bhattacharyya parameter $Z_0(W_{S^n})$, which is an upper bound to the error probability, decays as $O(2^{-2n/2})$ and in [1], it is shown that $Z_0(W_{S^n}) \leq 2\sqrt{P_n/\lambda n} + Z_n$ decays as $O(2^{-2n})$, $\alpha < \log \frac{1.5}{2}$ under $Q_\beta$ according to the previously given probabilistic setting. Since $(P_n, M_n) \in \mathcal{R}_\beta$ and thus $M_n \leq Z_n$ eventually, this also implies $Z_n$ and $M_n$ decay at least with the same rate. However, one cannot compare the decay rates of $Z_n$ and $M_n$ only knowing the decay rate of $Z_0(W_{S^n})$. If $M_n$ decays much faster than $Z_n$, it is possible that the code constructed with $Q_\beta$ can be concatenated with an erasure-only code as an outer code for large $n$. Unfortunately, this is not the case. To show this, we present the following lemma and theorem, whose complete proofs are given in [14].

**Lemma 5.** For all $\epsilon_r > 0$,
\[
\lim_{n \to \infty} \mathbb{P} \left( \left| \frac{\log M_n}{\log Z_n} - \phi \right| \leq \epsilon_r \right) = \gamma, \quad \phi = 1 + \frac{\sqrt{5}}{2}.
\]

The proof follows from the fact that the iterations of $\frac{\log M_n}{\log Z_n}$ converge to the mapping $x \to \frac{1}{2} + \frac{1}{x}$ with probability close to $\gamma$. Therefore, it converges to the fixed point $\phi$ in Lemma 5 suggests that with probability close to $\gamma$, $M_n$ and $Z_n$ decay with same rate. With the next theorem, we obtain the exact rate.

**Theorem 1.** In limit, the random processes $Z_n$ and $M_n$ roughly behave as $O(2^{-2n})$, $\alpha = \log \frac{2}{5}$ with probability close to $\gamma$. That is, for any $\delta, \delta' > 0$,
\[
\lim_{n \to \infty} \mathbb{P} \left( 2^{-2n\log \frac{\phi - \delta}{\phi}} \leq Z_n \leq 2^{-2n\log \frac{\phi - \delta'}{\phi}} \right) = \gamma
\]
and
\[
\lim_{n \to \infty} \mathbb{P} \left( 2^{-2n\log \frac{\phi - \delta'}{\phi}} \leq M_n \leq 2^{-2n\log \frac{\phi - \delta}{\phi}} \right) = \gamma.
\]

The proof for Theorem 1 follows straightforwardly from Lemma 5 as it provides that $z^{e+\epsilon} \leq z^\phi \leq 3z^{e-\epsilon}$ with probability close to $\gamma$. Using the same machinery in [3], one can prove the case for $Z_n$. The case for $M_n$ follows similarly.

Lemma 5 and Theorem 1 imply that $Z_n$ and $M_n$ decay at the same rate. Consequently, concatenation with an erasure-only code does not improve the error probability. Also note that the rate of polarization for this particular three-quantized case is bounded away from $O(2^{-2n^{2/2}})$, which shows that longer codes are required to ensure reliable communication compared to the unquantized case.

V. D-QUANTIZED CASE

In this section, we consider static and dynamic quantization procedures $Q_\beta^{(D)}$, where $D = 2d + 1$ is an odd number by definition. Note that $|\beta| = d$. Similar to the three-level case, we start with a BMS channel $W$ whose output $Y$ takes values in the set $\{0, \pm \lambda_1, \ldots, \pm \lambda_d\}$, $\lambda_i > 0$, $1 \leq i \leq d$. Define the parameters of the quantized statistic $Y^{w+}$ as $p_{i}^{w+}$, $m_{i}^{w+}$ and $z_{i}^{w+}$ in a similar fashion to that in Section IV and assume $p_{i} \geq m_{i}$. Also define $p_{i}^{w-} = \sum_{i=1}^{d} p_{i}^{w+}$ and $m_{i}^{w-} = \sum_{i=1}^{d} m_{i}^{w+}$. In general, it appears to be hard to obtain good lower bounds on the achievable rates for quantization procedures with output size greater than three. However, we have found that there are non-trivial D-static and D-dynamic quantization procedures that result in the same dynamics as the simple three-quantized case. We formally define these procedures below.

**Definition 3** (Proper quantization procedures). A quantization procedure $Q_\beta^{(D)}$ is proper if $\beta(P)_{i} \neq \beta(P)_{j}$ for all $i \neq j \in [d]$ and $P \in \mathcal{P}$. In words, $\beta$ consists of distinct elements.

Note that if a quantization procedure is not proper, then it is equivalent to another quantization procedure with $|\beta| < d$.

**Lemma 6.** There exists
(i) a pair of proper D-static quantization procedures $Q_\beta^{+}$, $Q_\beta^{-}$ with $Y^{+} = Q_\beta^{+}(Y + Y')$, $Y^{-} = Q_\beta^{-}(Y \oplus Y')$ that results in the same dynamics as the three-quantized case,
(ii) a single proper D-static quantization procedure $Q_\beta$ that results in the same dynamics as the three-quantized case.

**Proof Sketch:**
(i) Take any $\alpha_1, \ldots, \alpha_d \in \mathbb{R}^+$ and $\beta^+ = \cup_{i=1}^{d} \{(\alpha_i, \alpha_i)\}$, $\beta^- = \cup_{i=1}^{d} \{(\alpha_i \oplus \alpha_i, \alpha_i)\}$ such that $0 < \alpha_1 < \alpha_i < 2\alpha_1$, $2 \leq i \leq d$.
(ii) Take $\beta = \cup_{i=1}^{d} \{(\alpha_i \oplus \alpha_i, \alpha_i)\}$ such that $0 < \alpha_1 < \alpha_i < 2(\alpha_1 \oplus \alpha_i)$, $2 \leq i \leq d$.

Under these assumptions, one can verify that the resulting dynamics for both cases become the same as those in the formerly discussed three-quantized case.

Lemma 6 shows that with a pair of two proper D-static quantization procedures, or with a single proper D-static quantization procedure, the system performance can be made equivalent to that in the simple three-quantized case. This also implies that there are proper D-dynamic quantization schemes with the same performance. Based on this fact, some lower bounds on the achievable rates can be derived for D-quantization families.

**Lemma 7.** Consider the function $F_n$ defined in Corollary 1 for an $n \geq 0$. Then, the following claims hold:

(i) With a pair of proper D-static quantization procedures $Q_\beta^{+}$ and $Q_\beta^{-}$, one can achieve rates greater than
\[
R_{\nu,2}^{(D)}(W) = \max_{\alpha_1, \alpha_2, \alpha_3 \geq 2, \nu} \frac{F_n(W^+) + F_n(W^-)}{2},
\]
where $\beta^+ = \bigcup_{i=1}^{d} \{(\alpha_i, \alpha_i)\}$ and $\beta^- = \bigcup_{i=1}^{d} \{(\alpha_i \oplus \alpha_i, \alpha_i)\}$.

(ii) With a single proper $D$-static quantization procedure $Q_\beta$, one can achieve rates greater than

$$R_s^{(D)}(W) \triangleq \max_{0 < \alpha_1 < \cdots < \alpha_d \leq \lambda_d} \frac{F_n(W^+) + F_n(W^-)}{2},$$

where $\beta = \bigcup_{i=1}^{d} \{(\alpha_i \oplus \alpha_i, \alpha_i)\}$.

(iii) With a proper $D$-dynamic quantization procedure $Q_\beta$, one can achieve rates greater than

$$R_s^{(D)}(W) \triangleq \max_{Q_\beta(Y \oplus Y') \in \mathcal{Q}(\beta)} \frac{F_n(W^+) + F_n(W^-)}{2},$$

where $Y^+ = Q_\beta(Y \oplus Y')(Y + Y')$ and $Y^- = Q_\beta(Y \oplus Y')(Y - Y')$. In other words, quantize $Y + Y'$ and $Y \oplus Y'$ in the best possible way to maximize the objective function.

Proof: For (i) and (ii), take the procedures described in Lemma 6. Since the evolution of the parameters are same as the three-quantized case after one polarization step, we use the same lower bound. The last inequalities are added to make the region compact. For (iii), we see that at any step, a proper dynamic quantization exists to ensure that the parameters evolve similarly to the three-quantized case. Quantization at first step is optimized to get a better lower bound.

It is important to note that the special quantization schemes considered in the proof of Lemma 6 ensure that the quantized statistics polarize as the resulting dynamics are equivalent to that in three-level case. At first glance, it is not obvious that the statistics polarize for any admissible quantization procedure. Surprisingly, the quantized statistics polarize in a weaker manner under any admissible static or dynamic quantization procedure.

Theorem 2. Consider the probabilistic setting in Section IV-B and define $P_{i,n} \triangleq p_{i,n} s_{i,n}$, $M_{i,n} \triangleq m_{i,n} s_{i,n}$ for all $i \in \{1, \ldots, d\}$. Then, for all static or dynamic quantization procedures in $Q_\beta$, $Z_n$ converges to 0 or 1 almost surely and for any $i$, $P_{i,n} M_{i,n}$ converges to 0 in probability.

Proof: We use the abbreviations $X_n \xrightarrow{a.s.} c$ and $X_n \xrightarrow{p} c$ to denote that $X_n$ converges to $c \in \mathbb{R}$ almost surely or in probability respectively. For every static or dynamic quantization procedure $Q_\beta$ in $Q$, it is known that $Q_\beta(0) = 0$. This implies that if $Y = 0$ or $Y' = 0$ then $Y^- = Q_\beta(Y \oplus Y') = 0$ and if $Y, Y' = 0$ or $Y = -Y'$ then $Y^+ = Q_\beta(Y + Y') = 0$. One thus obtains

$$z^+ \geq 2z + z^2, \quad z^- \geq z^2 + 2 \sum_{i=1}^{d} p_i m_i,$$

Therefore, $Z_n$ is a bounded submartingale as $\mathbb{E}[Z_{n+1}|\mathcal{F}_n] \geq Z_n + \sum_{i=1}^{d} P_{i,n} M_{i,n}$. Considering the $+$ transformation and following the same steps in [2], we obtain

$$\mathbb{E}[Z_{n+1} - Z_n] \geq \frac{1}{2} \mathbb{E}[Z_n - Z_n] \geq \frac{1}{2} \mathbb{E}[Z_n - Z_n^2].$$

Since $\lim_n \mathbb{E}[Z_{n+1} - Z_n] = 0$ and $Z_n$ converges almost surely, $Z_n \xrightarrow{a.s.} 0$ or 1. Studying the $+$ transformation instead, we obtain

$$\mathbb{E}[Z_n^+ - Z_n] = \mathbb{E} \left[ Z_n^2 - Z_n + 2 \sum_{i=0}^{d} P_{i,n} M_{i,n} + J_n \right],$$

where $J_n$ is an $\mathcal{F}_n$-measurable non-negative remainder term. The right hand side also goes to zero as $n$ tends to infinity. This implies that $Z_n^2 - Z_n + 2 \sum_{i=0}^{d} P_{i,n} M_{i,n} + J_n \to 0$. It is well-known that if $X_n \to x$ and $Y_n \to y$ for some constants $x$ and $y$, then $X_n + Y_n \to x + y$. From this fact, we conclude that $\sum_{i=0}^{d} P_{i,n} M_{i,n} + J_n \to 0$ as well. Since all $P_{i,n} M_{i,n}$’s and $J_n$ are non-negative random variables, we have $P_{i,n} M_{i,n} \to 0$ for all $i \in \{1, \ldots, d\}$.

Theorem 2 has significance in practice as it implies Tal-Vardy construction in [5] under the assumption that zero is an absorbing support, any quantization scheme as in [1] and many other schemes weakly polarize. The weak polarization implies that for sufficiently large $n$, some fraction of synthetic channels meet the condition that $W^{s_n}(y|0)$ and $W^{s_n}(y|1)$ have almost non-overlapping supports. If one is allowed to remap the supports and change the quantization procedure once at some $n$, one can show that the quantized statistics can be forced to polarize strongly.

Lemma 8. Assume $Z_{\infty} = 0$ with probability $\gamma_Z > 0$, i.e., a non-zero fraction $\gamma_Z$ of quantized statistics tend to output 0 with zero probability. Given $\epsilon, \delta > 0$ and $\beta \leq \gamma_Z$, one can ensure that the quantized statistics polarize and at least $(\gamma_Z - \delta)(1 - \epsilon - 2\sqrt{\delta \epsilon})^2$ fraction of the statistics will eventually become perfect by remapping of supports and changing the procedure to the simple three-quantized case after some $n_0(\delta, \epsilon)$.

Proof: Given $\epsilon, \delta$, Theorem 2 implies the existence of an $n_0$ such that

$$\mathbb{P}(Z_n \leq \epsilon, P_{i,n} M_{i,n} \leq \epsilon, i \in \{1, \ldots, d\}) \geq \gamma_Z - \delta, \quad n \geq n_0.$$

We consider $s_n \in \{+, -\}^n$ such that the condition in the above event holds. For such $s_n$, $p_i^{s_n} \land m_i^{s_n} \leq \sqrt{\epsilon}$ for all $i \in \{1, \ldots, d\}$. At $n_0$, we remap the support such that $m_i^{s_n} \leftarrow p_i^{s_n} \land m_i^{s_n}$ and we switch to the simple three-level quantization procedure $Q_\beta, \beta = \{(0, 1)\}$. This ensures that $m_i^{s_n} \leq \sqrt{\epsilon}$. Under these conditions the Bhattacharyya parameters are bounded as $Z_k(W^{s_n}) \leq z^{s_n} + 2p^{s_n} m^{s_n} \leq \epsilon + 2\sqrt{\delta \epsilon}$. For BMS channels, it is known that $I(W) \geq 1 - Z_k(W)$, thus $I(W^{s_n}) \geq 1 - \epsilon - 2\sqrt{\delta \epsilon}$. Observe that the specific three-quantized case polarizes strongly. Now we use the simple lower bound $I(W^2)$ to show that at least $(\gamma_Z - \delta)(1 - \epsilon - 2\sqrt{\delta \epsilon})^2$ fraction of channels will eventually become perfect.

The three-quantized case assures that the block error probability behaves roughly as $O(2\sqrt{\delta \epsilon})$. Together with Lemma 8, it implies that one achieves reliable communication at rates arbitrarily close to $\gamma_Z$ by constructing and decoding polar codes with $D$-level quantization procedures, if it is allowed to change the procedure and remap the supports once at an arbitrary $n$. 

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