Is sustainability of light-harvesting and waveguiding systems a quantum phenomenon?

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Abstract. It is shown that sustainability is a universal quantum-statistical phenomenon, which occurs during propagation of electromagnetic waves inside different dissipative media, such as waveguides, metamaterials or biological tissues. For illustrative purposes, we show a simple yet instructive example of environment-assisted excitonic energy transfer in photobiological complexes, such as photosynthetic reaction centers or centers of melanogenesis inside living organisms or organelles. We demonstrate that this transfer must be both quantum and sustainable to simultaneously endure continuous energy transfer and keep their internal structure from destruction or critical instability. Besides, the environment-assisted evolution of a sustainable type significantly lowers the entropy and improves the speed and capacity of energy transfer. As another example, we demonstrate how this phenomenon of sustainability can manifest itself in a large class of human-controlled electromagnetic systems, such as optical couplers and amplifiers.

1. Introduction

It is known that dissipation during electromagnetic (EM) waves’ propagation in nonconducting media inevitably results in energy and/or information loss. It is thus very important to learn the properties of these dissipative phenomena, in order to eliminate or at least control them, which seems to be inevitable for designing of a next generation of quantum-mechanical EM devices [1]. Besides, it allows to improve the sensitivity and non-invasivity of the known electromagnetic systems, such as radars and amplifiers.

The propagation of EM waves inside can be studied by means of the so-called Maxwell-Schrödinger (MS) analogy, which is the formal map between the Maxwell equations in nonconducting materials and the Schrödinger type equations [2, 3]. Moreover, this map serves as a first step towards formulating the quantum-statistical approach to the above-mentioned problems, which focuses on the density operator as a main value describing the evolution or distribution properties of the EM wave configurations. The distinctive feature of the density operator is that it must be adapted for the non-Hermitian Hamiltonians (NH), which abundantly occur in such configurations [4]. The interest to such an approach has been revived, quite recently, in the papers [5, 6, 7, 8, 9, 10, 11].

In the next section, we shall give a brief description of the Maxwell-Schrödinger analogy followed by the density operator approach. In subsequent sections we consider various applications of the formalism.
2. Basic formalism

We assume that an electromagnetic wave is propagating along $z$ direction in a medium, which is isotropic, linear, and nonconducting. Introducing the notations $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ and $\mathbf{H} = \mathbf{H}(\mathbf{r}, t)$ are electric and magnetic fields, $\times$ and $\cdot$ means the vector and scalar multiplication, $c = 1/\sqrt{\varepsilon_0\mu_0}$, and $\varepsilon_0$ and $\mu_0$ being the vacuum permittivity and permeability, respectively.

We also assume the harmonic time dependence for EM fields: $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(x, y, z)\exp(-i\omega t)$ and $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}(x, y, z)\exp(-i\omega t)$. Then one can decompose fields and differential operators into their transverse and longitudinal components with respect to the $z$ axis: $\mathbf{E} = \mathbf{E}_\perp + e_z E_z$, $\mathbf{H} = \mathbf{H}_\perp + e_z H_z$, $\nabla = \nabla_\perp + e_z \hat{\nabla}_z$. $\mathbf{e}_a$ being the basis vector of the corresponding axis. Values $\mathbf{E}_\perp$ and $\mathbf{H}_\perp$ are 2D vectors by their nature: $\mathbf{E}_\perp \cdot \mathbf{e}_z = \mathbf{H}_\perp \cdot \mathbf{e}_z = 0$. The Maxwell equations for the transverse component of EM fields can be written in a matrix differential equation form, which strongly resembles the Schrödinger equation. In units $c = 1$, we obtain

$$i \hbar \frac{\partial}{\partial z} \begin{pmatrix} \mathbf{E}_\perp \\ \mathbf{H}_\perp \end{pmatrix} = \mathcal{H}' \begin{pmatrix} \mathbf{E}_\perp \\ \mathbf{H}_\perp \end{pmatrix},$$

(1)

where

$$\mathcal{H}' = \mathbf{\hat{D}} = \mathbf{\hat{D}} \mathbf{\hat{L}} = \mathbf{\hat{L}}_e \begin{pmatrix} \mathbf{\hat{L}}_e & 0 \\ 0 & \mathbf{\hat{L}}_m \end{pmatrix} = \begin{pmatrix} 0 & -\mathbf{e}_z \times \mathbf{\hat{L}}_m \\ \mathbf{e}_z \times \mathbf{\hat{L}}_m & 0 \end{pmatrix},$$

(2)

where we introduced the differential operators $\mathbf{\hat{L}}_e = \varepsilon \omega - \omega^{-1} \mathbf{\nabla}_\perp \times \mu^{-1} \mathbf{\nabla}_\perp \times$ and $\mathbf{\hat{L}}_m = \mu \omega - \omega^{-1} \mathbf{\nabla}_\perp \times \varepsilon^{-1} \mathbf{\nabla}_\perp \times$, as well as the associated Hamilton matrix operator $\mathbf{\hat{D}} = \mathbf{\hat{D}}(\mathbf{\hat{L}}_e, \mathbf{\hat{L}}_m)$.

Furthermore, in order to reveal a presence of a Hilbert space here, let us introduce the norm

$$\mathcal{N}^2 \equiv \langle \mathbf{E}_\perp | \mathbf{E}_\perp \rangle + \langle \mathbf{H}_\perp | \mathbf{H}_\perp \rangle \equiv \int dx dy \left( |\mathbf{E}_\perp|^2 + |\mathbf{H}_\perp|^2 \right),$$

(3)

where the inner product is defined as an integral over the cross-section. Then one can employ the Dirac’s “braket” notation, where the ket vector, written as $|\Psi\rangle \equiv \frac{1}{\mathcal{N}} \langle \mathbf{E}_\perp | \mathbf{H}_\perp \rangle$ is automatically normalized to one, $\langle \Psi | \Phi \rangle = 1$.

In terms of this state vector, eq. (1) becomes

$$i \hbar \omega \frac{\partial}{\partial z} |\Psi\rangle = \mathcal{H}|\Psi\rangle,$$

(4)

where

$$\mathcal{H} = h \omega \left( \mathcal{H}' + \mathcal{H}_N \right) = h \omega \left( \mathbf{\hat{D}} - i \Gamma_N \mathbf{\hat{I}} \right),$$

(5)

and $\mathbf{\hat{I}}$ and $\mathbf{\hat{I}}$ are, respectively, an identity operator and rank-2 identity matrix, and we denoted $\Gamma_N = \frac{d}{dt} \ln |\mathcal{N}|$. The scale constant $\hbar$ is an effective Planck constant which is introduced for a purpose of keeping the dimensionality of our Schrödinger equation correct. From now on, we adopt units $\hbar = \hbar = c = 1$, which reflect an absence of a fundamental length scale in the Maxwell equations as such.

Formulæ above provide an exact mapping between EM wave propagation equations in media and equations of a Schrödinger type, which allows to employ a vast quantum mechanical machinery for studies of electromagnetic waves in media. It is however necessary to make an important generalization: transfer from the state vector approach to the quantum-statistical one. The latter is done by introducing the quantum density operator, also often dubbed as the density matrix, which expected to describe the distribution of EM wave’s momentum and energy along $z$ axis.
If a Hamiltonian is a non-Hermitian operator, then it can be separated into its self-adjoint and skew-adjoint components:
\[ \hat{\mathcal{H}} = \hat{\mathcal{H}}_+ + \hat{\mathcal{H}}_- = \hat{\mathcal{H}}_+ - i\hat{\Gamma}, \]
where \( \hat{\mathcal{H}}_\pm \equiv \frac{1}{2} \left( \hat{\mathcal{H}} \pm \hat{\mathcal{H}}^\dagger \right) = \pm \hat{\mathcal{H}}_\pm \), and \( \hat{\Gamma} \equiv i\hat{\mathcal{H}}_- = \hat{\Gamma}^\dagger \) is the decay operator; it is Hermitian by construction. In our case, we derive
\[ \hat{\mathcal{H}}_+ = \hat{\mathcal{H}}_+^\prime = \frac{1}{2} \left( \hat{\sigma}_2 \hat{D} + \hat{D}^\dagger \hat{\sigma}_2 \right), \]
\[ \hat{\Gamma} = \hat{\Gamma}^\prime + \Gamma_N \hat{I} = \frac{i}{2} \left( \hat{\sigma}_2 \hat{D} - \hat{D}^\dagger \hat{\sigma}_2 \right) + \Gamma_N \hat{I}. \]

Introducing a non-normalized density operator \( \hat{\Omega} \), we arrive at a master equation of a Liouvillian type \[ d/dz \hat{\Omega} = i \left( \hat{\Omega} \hat{\mathcal{H}}^\dagger - \hat{\mathcal{H}} \hat{\Omega} \right) = -i \left[ \hat{\mathcal{H}}_+ , \hat{\Omega} \right] - \{ \hat{\Gamma} , \hat{\Omega} \}, \]
where square (curly) brackets denote the commutator (anti-commutator).

One can easily check that the master equation does not preserve a trace of \( \hat{\Omega} \). Therefore, in the work \[5\] the additional, normalized, density operator
\[ \hat{\rho} = \hat{\Omega}/\text{tr} \hat{\Omega} \]
was introduced. Its physical meaning and rule will be discussed in what follows.

The Hamiltonian (5) can be in general very cumbersome and thus difficult for analytical studies. In some cases, however, it reduces to some simple Hamiltonian which captures the main features of a system in question. One such example is the excitonic Hamiltonian, which is popular in many areas of condensed-matter theory and photonics, including a theory of photosynthesis; it is studied in the next section. Another example is a coupled-mode theory, which is considered in the section 4.

3. Energy transfer in photobiological systems

In this section, we consider an example of the phenomenon of natural (not human-controlled) sustainability, which explains some striking features of photobiological systems. Examples of such systems are photosynthetic reaction centers inside living organisms, such as algae, or centers of melanogenesis inside organelles, such as cells of human skin. In essence, it is a photobiological complex is a system of a donor-acceptor type which captures an external photon, usually from visible or UV spectrum, and transfers its energy into a chemical center where this energy facilitates further biochemical reactions. Here we shall be interested only in the process of energy transfer itself under conditions of such a non-trivial environment and thermal effects, see ref. \[11\] and an extensive list of literature therein.

For donor-acceptor systems, the Hamiltonian (5) usually simplifies into simpler form, due to the effective degrees of freedom which emerge, one of examples being the exciton, a quasiparticle state of an electron and a hole, and its combinations with a quantum of EM wave, \textit{i.e.}, photon, and a quantum of acoustic oscillations, \textit{i.e.}, phonon. Let us consider a simplest model, a two-level donor-acceptor subsystem of an excitonic type. Its Hamiltonian can be written as
\[ \hat{H} = \frac{V}{2} \hat{\sigma}_1 + \frac{1}{2} (\varepsilon + i\Gamma) \hat{\sigma}_3 - \frac{i}{2} \hat{\Gamma} = \frac{1}{2} \left( \begin{array}{cc} \varepsilon & V \\ V & -\varepsilon - 2i\Gamma \end{array} \right), \]

3
where $\hat{\sigma}$’s are Pauli spin matrices, and $\varepsilon$ is a difference of the renormalized energy between levels. For weakly-coupled environments, one can impose that $|\Gamma| \ll |V| < \varepsilon$; for example, for quinon systems one can generally assume that $\varepsilon \sim 60$ ps$^{-1}$, $|V| \sim 20$ ps$^{-1}$ and $|\Gamma| \sim 1$ ps$^{-1}$.

Imposing a condition $\hat{\Omega}(0) = \hat{\rho}(0) = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix}$, $0 \leq p \leq 1$, an exact solution of a master equation can be found in the analytical form $\hat{\Omega} = \frac{1}{2} \hat{I} (S_{\Omega})_4 + \frac{1}{2} \sum_{a=1}^{3} (S_{\Omega})_a \hat{\sigma}_a$, where $(S_{\Omega})_j = \langle \sigma_j \rangle_{\Omega}$, $j = 1..4$, is the $j$th component of a Stokes vector computed with respect to the non-normalized density operator $\hat{\Omega}$: $S_{\Omega} = \left\{ \langle \hat{\sigma}_1 \rangle_{\Omega}, \langle \hat{\sigma}_2 \rangle_{\Omega}, \langle \hat{\sigma}_3 \rangle_{\Omega}, \langle \hat{I} \rangle_{\Omega} \right\}$, bearing in mind that $\langle \hat{I} \rangle_{\Omega} = \text{tr} \hat{\Omega}$.

Using the master equation (9), one can show that the Stokes vector components can be found as solutions of a set of differential equations

$$\frac{1}{\varepsilon} \frac{d}{dz} \begin{pmatrix} (S_{\Omega})_1 \\ (S_{\Omega})_2 \\ (S_{\Omega})_3 \\ (S_{\Omega})_4 \end{pmatrix} = \begin{pmatrix} -\Gamma_\varepsilon & -1 & 0 & 0 \\ 1 & -\Gamma_\varepsilon & -V_\varepsilon & 0 \\ 0 & V_\varepsilon & -\Gamma_\varepsilon & \Gamma_\varepsilon \\ 0 & 0 & \Gamma_\varepsilon & -\Gamma_\varepsilon \end{pmatrix} \begin{pmatrix} (S_{\Omega})_1 \\ (S_{\Omega})_2 \\ (S_{\Omega})_3 \\ (S_{\Omega})_4 \end{pmatrix}, \quad (12)$$

where $V_\varepsilon = V/\varepsilon$ and $\Gamma_\varepsilon = \Gamma/\varepsilon$. Alternatively, for illustrative purposes one can write equations for the normalized averages as:

$$\frac{1}{\varepsilon} \frac{d}{dz} \begin{pmatrix} (S_{\rho})_1 \\ (S_{\rho})_2 \\ (S_{\rho})_3 \\ (S_{\rho})_3 \end{pmatrix} = \begin{pmatrix} -\Gamma_\varepsilon (S_{\rho})_3 & -1 & 0 & 0 \\ 1 & -\Gamma_\varepsilon (S_{\rho})_3 & -V_\varepsilon & 0 \\ 0 & V_\varepsilon & -\Gamma_\varepsilon (S_{\rho})_3 & \Gamma_\varepsilon \\ 0 & 0 & \Gamma_\varepsilon & -\Gamma_\varepsilon \end{pmatrix} \begin{pmatrix} (S_{\rho})_1 \\ (S_{\rho})_2 \\ (S_{\rho})_3 \\ (S_{\rho})_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (13)$$

where $S_{\rho} = \left\{ \langle \hat{\sigma}_1 \rangle_{\rho}, \langle \hat{\sigma}_2 \rangle_{\rho}, \langle \hat{\sigma}_3 \rangle_{\rho} \right\}$. These equations demonstrate the above-mentioned nonlinear character of the sustainable evolution; besides, their nonlinearity is of a quadratic (Kerr) type which is quite common in optical systems.

The results of computations demonstrate the crucial role of the environment described by the parameter $\Gamma$ in the energy transport, despite it is weakly coupled [11]. They reveal that the energy transfer via the light-harvesting complex occurs much quicker for the sustainable evolution than for the non-sustainable. Furthermore, it is shown that the discharging of the donor level is more full for a sustainable dynamics than for non-sustainable. Therefore, sustainable complexes can transfer bigger portions of energy per system. Next, for non-sustainable dynamics, quantum coherence tends to zero exponentially with time, while it approaches a small yet nonzero constant for a sustainable evolution. Finally, one can show that for sustainable systems von Neumann entropy possesses smaller maximum values than for non-sustainable; it is vanishes substantially quicker as well in the sustainable case.

### 4. Coupled modes in optical waveguides

In this section, we discuss another example: a manifestation of the sustainability phenomenon in systems which are largely human-controlled, unlike those described in the previous section. The physical interpretation of the sustainable-type evolution (with the normalized density operator) is quite different from the previous section’s case, but the mathematical formalism is very similar.

The phenomenon of coupled modes occurs in a large class of optical waveguides including optical fibers, couplers and amplifiers. Its theory originates from studies of electron beam waves [12], and relies on a number of approximations which are not always obvious for optical waveguides but nevertheless working [13, 14, 15]. It begins with a simplification of the Maxwell equations by expanding the transverse electric and magnetic fields in series with respect to modes, referred as the modal expansion from now on.

According to this expansion, the complex-conjugated transverse fields are written in the form

$$E_\perp = \sum_q a_q \hat{E}_\perp q, \quad H_\perp = \sum_q a_q \hat{H}_\perp^* q, \quad (14)$$
where \( a_q = a_q(z) \) are expansion coefficients, the fields \( \mathbf{E}_{\perp q} = \mathbf{E}_{\perp q}(x, y) \) and \( \mathbf{H}_{\perp q} = \mathbf{H}_{\perp q}(x, y) \) are functions of the transverse coordinates only, and index \( q \) enumerates modes.

Substituting this ansatz into the Maxwell equations and invoking a number of simplifying assumptions [14], one eventually arrives at a set of differential equations for the expansion coefficients \( a_q \):

\[
\frac{d}{dz} a_p + i \beta_p a_p = i \sum_q C_{pq} a_q, \tag{15}
\]

where \( \beta_p \) is the \( p \)th modal propagation constant, \( C_{pq} \) is the coupling coefficient which is usually a cross-section integral of some function of fields and dielectric constants. In a case of a two modes’ coupling in a dimer, a two-waveguide coupler where the materials show symmetric effective gain and loss, this system can be further simplified. After some redefinitions, it acquires a form of a \( 2 \times 2 \)-matrix differential equation:

\[
-i \frac{d}{dz} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} n_1 & g \\ g & n_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \tag{16}
\]

where the complex-valued functions \( a_q = a_q(z) \) are the field amplitudes at each waveguide, and the parameters \( n_q \) and \( g \) are the effective refractive indices and waveguide coupling, respectively [16].

Furthermore, it is convenient to perform a phase transformation

\[
a_q = \exp \left( in_+ z \right) \xi_q, \tag{17}
\]

where \( n_+ = (n_1 + n_2)/2 \), in order to make the equation (16) more symmetric:

\[
i \frac{d}{dz} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \mathcal{H}' \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{18}
\]

where the matrix operator

\[
\mathcal{H}' = -g \hat{\sigma}_1 - ig \gamma \hat{\sigma}_3 = -g \begin{pmatrix} i \gamma & 1 \\ 1 & -i \gamma \end{pmatrix} \tag{19}
\]

is now traceless and contains only two real-valued parameters \( g \) and \( \gamma = -i n_- / g \); here, we have taken into account that for standard dimers the effective refractive indices of waveguides are equal, therefore \( n_- \) is purely imaginary. The operator (19) is known to have eigenvalues \( \epsilon'_\pm = \pm g \sqrt{1 - \gamma^2} \) which coalesce at the value \( \gamma_c = 1 \) which divides regions with real and complex eigenvalues in the parametric space. The operator (19) is known to obey symmetry with respect to the transformation which consists of the permutation of waveguides \( \pm i \gamma \rightarrow \mp i \gamma \) and propagation reversal \( z \rightarrow -z \) (for historical reasons, this symmetry is often called the \( PT \)-symmetry). However, this operator is not our system’s Hamiltonian yet; the latter is still to be derived in what follows.

Furthermore, repeating the arguments which led us from the equation (1) to (4), we can map the equation (18) onto the Schrödinger-type equation (4), where

\[
\mathcal{H}' = \mathcal{H}' + \mathcal{H}_N = -g \hat{\sigma}_1 - i g \gamma \hat{\sigma}_3 - i \Gamma_N \hat{I}
\]

\[
= -g \begin{pmatrix} i \gamma & 1 \\ 1 & -i \gamma \end{pmatrix} - i \Gamma_N \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{20}
\]

\[
|\Psi\rangle = \frac{1}{N} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \tag{21}
\]

where

\[
\hat{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{22}
\]
where the normalization factor was simplified after taking the cross-section integral and considering that $\mathcal{E}$’s do not depend on transverse coordinates:

$$\mathcal{N} = \sqrt{|\mathcal{E}_1|^2 + |\mathcal{E}_2|^2},$$

$$\Gamma_\mathcal{N} = \frac{1}{2g} \frac{d}{dz} \ln \left( |\mathcal{E}_1|^2 + |\mathcal{E}_2|^2 \right),$$

hence one can notice that the term $\Gamma_\mathcal{N}$ is a function of $z$, in general. It should be emphasized here that the Maxwell-Schrödinger analogy can be applied only if one works in terms of the normalized values $\bar{\mathcal{E}}_p = \mathcal{E}_p / \mathcal{N} = \mathcal{E}_p / \sqrt{|\mathcal{E}_1|^2 + |\mathcal{E}_2|^2}$, which is required for a conventional wave-mechanical interpretation of $|\Psi\rangle$ as a state vector and a ray in the Hilbert space which emerges. In the context of optical waveguides, it is equivalent to calibrating a measuring apparatus at the beginning of each measurement and thus avoiding divergent values.

The Hamiltonian (20) can be written in a decomposed form (6), where

$$\hat{\mathcal{H}}_+ = -g \hat{\sigma}_1 = - \left( \begin{array}{cc} 0 & g \\ 0 & 0 \end{array} \right) ,$$

$$\hat{\Gamma} = g \gamma \hat{\sigma}_3 + g \gamma_\mathcal{N} \hat{I} = g \left( \begin{array}{cc} \gamma_\mathcal{N} + \gamma & 0 \\ 0 & \gamma_\mathcal{N} - \gamma \end{array} \right) ,$$

where $\gamma_\mathcal{N} = \Gamma_\mathcal{N} / g = \frac{1}{2g} \frac{d}{dz} \ln \left( |\mathcal{E}_1|^2 + |\mathcal{E}_2|^2 \right)$ is a function of $z$, in general.

Notice that the total Hamiltonian (20) has complex eigenvalues $g \left( \pm \sqrt{1 - \gamma^2 - i \gamma_\mathcal{N}} \right)$ and it does not obey the above-mentioned PT symmetry, in general. In other words, a class of “PT-symmetric” Hamiltonians is a subset of a class of Hamiltonians which actually describes coupled modes in optical waveguides. Besides, the complex valued eigenvalues of (20) do not pose a problem here, because the master equation approach presumes that the subsystem’s Hamiltonian is still $\hat{\mathcal{H}}_+$ while the anti-Hermitian part of $\hat{\mathcal{H}}$ is regarded as a dissipator term in spirit of the other popular master equation approach, the Lindblad one.

Furthermore, using the master equation (9), one can show that the Stokes vector components can be found as solutions of a set of differential equations

$$\frac{1}{2g} \frac{d}{dz} \begin{pmatrix} \langle S_\Omega \rangle_1 \\ \langle S_\Omega \rangle_2 \\ \langle S_\Omega \rangle_3 \\ \langle S_\Omega \rangle_4 \end{pmatrix} = \begin{pmatrix} -\gamma_\mathcal{N} & 0 & 0 & 0 \\ 0 & -\gamma_\mathcal{N} & 1 & 0 \\ 0 & -1 & -\gamma_\mathcal{N} & -\gamma \\ 0 & 0 & -\gamma & -\gamma_\mathcal{N} \end{pmatrix} \begin{pmatrix} \langle S_\Omega \rangle_1 \\ \langle S_\Omega \rangle_2 \\ \langle S_\Omega \rangle_3 \\ \langle S_\Omega \rangle_4 \end{pmatrix} ,$$

where $\langle S_\Omega \rangle = \{ \langle \hat{\sigma}_1 \rangle_\Omega, \langle \hat{\sigma}_2 \rangle_\Omega, \langle \hat{\sigma}_3 \rangle_\Omega, \langle \hat{I} \rangle_\Omega \} = \text{tr} \hat{\Omega}$, as in the previous section.

Similarly to the previous section, one can also write equations for the normalized averages:

$$\frac{1}{2g} \frac{d}{dz} \begin{pmatrix} \langle S_\rho \rangle_1 \\ \langle S_\rho \rangle_2 \\ \langle S_\rho \rangle_3 \end{pmatrix} = \begin{pmatrix} \gamma \langle S_\rho \rangle_3 & 0 & 0 \\ 0 & \gamma \langle S_\rho \rangle_3 & 1 \\ 0 & -1 & \gamma \langle S_\rho \rangle_3 \end{pmatrix} \begin{pmatrix} \langle S_\rho \rangle_1 \\ \langle S_\rho \rangle_2 \\ \langle S_\rho \rangle_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix} ,$$

where $\langle S_\rho \rangle = \{ \langle \hat{\sigma}_1 \rangle_\rho, \langle \hat{\sigma}_2 \rangle_\rho, \langle \hat{\sigma}_3 \rangle_\rho \}$. Notice that the normalization term $\gamma_\mathcal{N}$ does not affect the sustainable evolution – but does affect the non-sustainable one. This effect is a anti-Hermitian analogue of invariance of Hermitian Hamiltonians with respect to a shift of energy levels by a constant value.

Equations (27) demonstrate the above-mentioned nonlinear character of the sustainable evolution and its difference from the non-sustainable one. They qualitatively resemble their counterparts from the previous section; in particular, their nonlinearity is also of a quadratic type.
5. Conclusion
In this paper, we addressed the phenomenon of sustainability from the viewpoint of modern statistical mechanics, both classical and quantum, using the methodology of a modern theory of open quantum systems, such as the density operator approach with non-Hermitian Hamiltonians. Our approach is applicable to a large class of electromagnetic waveguides as well as photonic and plasmonic materials and devices, both natural and human-controlled. Apart from deeper understanding of various complex phenomena occurring during EM wave propagation in media, our formalism offers a useful tool for creating different models. The latter can be used to derive different experimental observables, such as correlation functions, energy density, transmitted power, entropy, etc.

We described a simple example of environment-assisted excitonic energy transfer in photobiological complexes, such as photosynthetic reaction centers or centers of melanogenesis inside living organisms or organelles. We demonstrated that this transfer must be both quantum and sustainable to simultaneously endure continuous energy transfer and keep their internal structure from destruction or critical instability, lower the entropy and improve the speed and capacity of EMW energy transfer. As another example, we showed how the phenomenon of sustainability can manifest itself in a large class of human-controlled EM wave systems, such as optical couplers and amplifiers.

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