A Global ILC Approach in Pixel Space over Large Angular Scales of the Sky Using CMB Covariance Matrix

Vipin Sudevan and Rajib Saha

Physics Department, Indian Institute of Science Education and Research Bhopal, Bhopal, M.P, 462066, India

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Abstract

We propose a new internal linear combination (ILC) method in the pixel space, applicable on large angular scales of the sky, to estimate a foreground-minimized cosmic microwave background (CMB) temperature anisotropy map by incorporating prior knowledge about the theoretical CMB covariance matrix. The usual ILC method in pixel space, on the contrary, does not use any information about the underlying CMB covariance matrix. The new approach complements the usual pixel space ILC technique specifically at low-multipole regions, using global information available from the theoretical CMB covariance matrix and from the data. Since we apply our method over the large scale on the sky containing low multipoles, we perform foreground minimization globally. We apply our methods on low-resolution Planck and WMAP foreground-contaminated CMB maps and validate the methodology by performing detailed Monte Carlo simulations. Our cleaned CMB map and its power spectrum have significantly less error than those obtained following the usual ILC technique at low resolution that does not use CMB covariance information. Another very important advantage of our method is that the cleaned power spectrum does not have any negative bias at the low multipoles because of effective suppression of CMB–foreground chance correlations on large angular scales of the sky. Our cleaned CMB map and its power spectrum match well with those estimated by other research groups.

Key words: cosmic background radiation – cosmology: observations – diffuse radiation

1. Introduction

For reconstruction of cosmic microwave background (CMB) signal from multifrequency observations an important method is internal linear combination (ILC; Tegmark & Efstathiou 1996; Bennett et al. 2003; Tegmark et al. 2003; Eriksen et al. 2004; Saha et al. 2006; Hinshaw et al. 2007). To obtain a foreground-minimized CMB map, the ILC method requires neither to explicitly model the frequency spectra of individual foreground components nor to model the foreground amplitudes (at some reference frequency) in terms of so-called foreground templates. The only assumption one makes related to foregrounds is that each of them has a frequency spectrum that is different from the frequency spectrum of the CMB component, which is assumed to be that of a blackbody in nature (Mather et al. 1994; Fixsen et al. 1996). The basic idea behind the ILC method is to linearly superpose the available foreground-contaminated CMB maps using certain amplitude terms, a set of weights, to estimate a foreground-minimized CMB map. The weights are obtained by minimizing the variance of the cleaned map and can be computed analytically by using a simple formula. In spite of being simple to design, ILC is a powerful technique to reconstruct a cleaned CMB map. It is, therefore, necessary—for a few important reasons—to investigate the performance of the usual ILC method in some hitherto-unexplored cases. First, while estimating the weights, the usual ILC method in pixel space does not take into account the covariance structure of the CMB maps. In other words, it does not use the fact that the final cleaned map, if it is perfectly cleaned off all foregrounds and detector noise is negligible, should have a covariance structure consistent with the underlying theoretical model. Second, some of the maximum likelihood methods (Eriksen et al. 2007, 2008a, 2008b; Gold et al. 2011; Planck Collaboration et al. 2016a, 2016c) for component separation, however, use CMB and detector noise covariance matrices to reconstruct CMB and all foreground components. It is therefore natural to ask the question, can we generalize the usual ILC method in pixel space to incorporate CMB covariance information also?

In the present work we seek to find a solution to the above problem and generalize the pixel space ILC method by taking into account prior information of the theoretical covariance matrix of the CMB maps. Therefore, instead of minimizing simple variance of the cleaned map, we propose to estimate the weights by minimizing the reduced variance of the cleaned map, the reduced variance being defined by the CMB covariance-weighted variance of the cleaned map, which is explained in Section 2. Since storage space into the computer disks of such a full pixel space covariance matrix increases rapidly with the HEALPix pixel resolution parameter $N_{\text{side}}^2 \sim N_{\text{side}}^4$, in the current work we use low pixel resolution maps. Further, to focus largely on the low multipoles, we smooth the input $N_{\text{side}} = 16$ maps by a Gaussian window function of FWHM = 9°. At this smoothing the input maps contain approximately 2.5 pixels per beam width, which implies that these maps are properly bandwidth limited. The larger beam smoothing also reduces detector noise contributions at different pixels.

Our method at low resolution bears an interesting complementarity in its approach when compared with the usual pixel space ILC method at high resolution, which does not use the CMB covariance matrix. Since the level of foreground contamination and its spectral properties vary with the sky positions, in a high-resolution analysis of the usual ILC method one performs foreground cleaning individually over several smaller regions of the sky, in such a way that the foreground spectral properties and the level of foreground contaminations in each region remain approximately constant. Because of the low pixel resolution (and large smoothing on the low pixel...
resolution maps) of this work, we chose either to perform foreground removal over the entire sky or to divide the sky into a small number of regions. In the second approach we divide the sky into two regions and clean them individually in a total of two iterations. Our aim is to use as much large sky fraction as possible during foreground removal and information about the CMB theoretical covariance matrix from the corresponding large fraction, so that our method becomes a global method of foreground minimization. Thus, the method may be seen as dual to the usual high-resolution ILC method, wherein the former uses global information from the covariance matrix and the data to estimate the foreground-minimized CMB map, while the latter relies on the local information of foreground properties.

By performing detailed Monte Carlo simulations, we find that the new ILC method of this work has significantly fewer reconstruction errors in cleaned maps and power spectrum than the usual ILC method in pixel space, over large angular scales of the sky. The cleaned power spectrum of our method does not have the negative bias at the low multipole region that is present in the usual ILC method. The negative bias in the usual ILC method is caused by chance correlations between CMB and foreground components for Stokes $Q$ and $P$ polarizations in the presence of a varying spectral index of the synchrotron component. The iterative harmonic space ILC algorithm was applied on high-resolution Planck and WMAP data, and one of its limitations, arising as a result of foreground leakage, was first discovered and remedied by Sudevan et al. (2017). Basak & Delabrouille (2012) and Basak & Delabrouille (2013) implement a needlet space ILC algorithm to incorporate localization of foreground emission in both pixel space and its “Fourier” space. A variant of the ILC technique by minimizing a measure of non-Gaussianity was implemented on temperature and polarization data by Saha (2011) and Perkayastha & Saha (2017), respectively. Eriksen et al. (2007, 2008a, 2008b) propose Gibbs sampling for component separation. Gold et al. (2011) use the Markov Chain Monte Carlo method to jointly estimate CMB and foreground components from WMAP data.

We organize our paper as follows. In Section 2 we discuss the formalism of the new method. We describe the method of computing the theoretical CMB covariance matrix in Section 3 and comment on its singular nature in Section 4. In Section 5 we describe in detail our foreground minimization approaches on Planck and WMAP low-resolution maps. We discuss the cleaned maps and CMB angular power spectra obtained from data in Section 6. We validate our foreground minimization methods by performing Monte Carlo simulations in Section 7. In Section 8 we show the advantage of the new ILC approach in pixel space over the usual ILC approach for analysis over large angular scales on the sky. We investigate the role of CMB–foreground chance correlation in not-so-efficient foreground removal by the usual ILC methods at low resolution in Section 9, and we comment that, using the CMB covariance matrix in our new method, we effectively suppress such chance correlations, which leads to improved foreground minimization. Finally, we conclude in Section 10.

2. Formalism

Assume that we have $n$ full-sky foreground-contaminated CMB maps, $X_i$ at a frequency $\nu_i$, with $i = 1, 2, ..., n$ at some beam and pixel resolution in thermodynamic temperature units. We assume that the mean temperature corresponding to each frequency $\nu_i$ has already been subtracted from each $X_i$. $y$ represents the cleaned CMB map obtained by linear combination of $n$ input maps $X_i$ with weight factor $w_i$, i.e.,

$$y = \sum_{i=1}^{n} w_i X_i. \quad (1)$$

Here each $X_i$ and $y$ are $N \times 1$ vectors describing the full-sky HEALPix$^1$ map with $N$ pixels for a pixel resolution parameter $N_{\text{side}}$ ($N = 12 N_{\text{side}}^2$), smoothed by Gaussian beam of certain FWHM. Instead of minimizing cleaned map variance $y'^T y'$ like the usual pixel space ILC method, we propose a more general approach by incorporating the prior information about the theoretical CMB covariance matrix. We minimize

$$\sigma^2 = y^T C^{-1} y, \quad (2)$$

where $C$ represents the CMB theoretical covariance matrix, which, as discussed in Section 4, may not always be invertible. $C^{-1}$ represents the Moore–Penrose generalized inverse (Moore 1920; Penrose 1955) of matrix $C$. Using Equation (1), we can write Equation (2) as

$$\sigma^2 = W A W^T, \quad (3)$$

where $W = (w_1, w_2, w_3, ..., w_n)$ is a $1 \times n$ row vector of weight factors of different frequency maps and $A$ is an $n \times n$ matrix with its elements $A_{ij}$ satisfying

$$A_{ij} = X_i^T C^{-1} X_j. \quad (4)$$

Since the spectral distribution of CMB photons is that of a blackbody to a very good approximation, CMB anisotropy in thermodynamic temperature units is independent of frequency bands. To reconstruct CMB anisotropies without introducing any multiplicative bias in its amplitude, we constrain the weights for all frequency bands to sum to unity, i.e., $\sum_{i=1}^{n} w_i = 1$. The choice of weights that minimize the variance given by Equation (2) subject to the above constraint is obtained following Lagrange’s multiplier approach (see, e.g., Saha et al. 2008; see also Tegmark & Efstathiou 1996; Tegmark et al. 2003; Saha et al. 2006),

$$W = \frac{e A^T}{e A^T e}, \quad (5)$$

where $A^\dagger$ represents the Moore–Penrose generalized inverse of matrix $A$ and $e = (1, 1, ..., 1)$ is a $1 \times n$ row vector representing the shape vector of CMB in thermodynamic temperature units.

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1. Hierarchical Equal Area isoLatitude Pixelization of sphere; see, e.g., Górski et al. (2005).
3. Computing the CMB Covariance Matrix

To compute elements of the CMB covariance matrix, \( C \) we assume the principle of statistical isotropy of CMB anisotropy. Under this assumption, the elements \( C_{ij} \) of matrix \( C \), at the chosen beam and pixel resolution, are given by

\[
C_{ij} = \sum_{\ell = 2}^{\ell_{\text{max}}} \frac{2\ell + 1}{4\pi} C_\ell B^2_\ell P_\ell(\cos(\gamma_{ij})) P^2_\ell, \tag{6}
\]

where \( C_\ell \) is the fiducial CMB angular power spectrum (Planck Collaboration et al. 2016d), \( B_\ell \) represents the beam transfer function, \( P_\ell \) denotes Legendre polynomials, and \( P_\ell \) is the pixel window function for the given \( N_{\text{side}} \) parameter. The cosine of the angle \( \gamma_{ij} \) is obtained following

\[
\cos(\gamma_{ij}) = \cos(\theta_i)\cos(\theta_j) + \sin(\theta_i)\sin(\theta_j)\cos(\phi_i - \phi_j), \tag{7}
\]

where \((\theta_i, \phi_i)\) and \((\theta_j, \phi_j)\) are spherical polar angles of \(i\)th and \(j\)th pixels of the map, respectively. Under the assumption of statistical isotropy, \( C \) is independent of any particular choice of coordinate system (e.g., galactic, ecliptic, or any Euler rotated version of these coordinate systems) in which the input maps may be provided. We note, however, that the assumption of statistical isotropy is not a necessity in our method. If needed, we can also use a covariance matrix compatible with the statistically anisotropic model, which may be caused, for example, by a nontrivial primordial power spectrum (Ghosh et al. 2016; Contreras et al. 2017).

4. Is \( C \) Singular?

As is the case for this work, rank, \( r \), of \( C \) is less than its dimension \( N \). The rank of \( C \) is simply equal to the effective number of independent \( a_{lm} \) modes (real and imaginary) that are used in Equation (6) to generate each element of the theoretical covariance matrix. A quick calculation shows that \( r = (\ell_{\text{max}} + 1)(\ell_{\text{max}} + 2) - (\ell_{\text{max}} + 1) - 4 \), when the summation over multipoles in Equation (6) extends up to \( \ell = \ell_{\text{max}} \). Since we use \( N_{\text{side}} = 16 \) HEALPix maps in our analysis, \( \ell_{\text{max}} = 2 \times N_{\text{side}} = 32 \) for us, implying \( r = 1085 \), which is less than the dimension of \( C \), which is \( N = 3072 \). Since \( C \) is singular, we use its generalized inverse in Equation (2).

5. Methodology

5.1. Input Maps and Data Processing

We use Planck 2015 released LFI 30, 44, and 70 GHz and HFI 100, 143, 217, and 353 GHz frequency maps along with the WMAP 9yr difference assembly (DA) maps in our analysis. For each of these maps we convert them to spherical harmonic space up to \( \ell_{\text{max}} = 32 \) and smooth the resulting \( a_{lm} \) coefficients by the ratio \( B^0_\ell P^0_\ell / B^1_\ell P^1_\ell \), where \( B^0_\ell \) and \( P^0_\ell \) represent the beam and pixel window functions of the original maps, whereas \( B^1_\ell \) and \( P^1_\ell \) represent the corresponding window functions for the \( N_{\text{side}} = 16 \) maps. We take \( B^0_\ell \) corresponding to a Gaussian beam of FWHM = 9°. We convert the smoothed spherical harmonic coefficients to \( N_{\text{side}} = 16 \) maps using the HEALPix-supplied facility synfast. For each of the WMAP Q, V, and W bands we average all the DA maps for any given frequency band. We convert all these maps into \( \mu K \) (thermodynamic temperature units and subtract the corresponding mean temperature from each frequency map. This results in a total of 12 input maps for foreground removal at \( N_{\text{side}} = 16 \).

5.2. Method 1

Since we are interested in a global method of foreground minimization, our aim is to use as much sky region as possible to estimate the weights. In the first method, we therefore estimate the weights using information obtained from the entire sky. We first estimate the full-sky CMB theoretical covariance matrix using Equation (6). We obtain \( C^1 \) using the singular value decomposition of \( C^1 \) and applying a cutoff of \( 1.0 \times 10^{-7} \) on the singular values. We show different square blocks across the diagonal of the \( C \) matrix estimated for the entire sky in Figure 1. Nondiagonal elements of this matrix show significant coupling between different pixel pairs for a pure CMB map and justify using Equation (2) for minimization instead of ignoring such correlations as is done in the usual pixel space ILC approach. Using \( C^1 \), we obtain weights for foreground removal using Equations (4) and (5). The cleaned map obtained using these weights is discussed in Section 6.

5.3. Method 2

Since, when compared with the expected level of CMB temperature anisotropy, the region near the galactic plane is more strongly contaminated by the foregrounds than the outside region, it is desirable to perform foreground removal separately on the sky region away from the plane and inside the plane. Moreover, the spectral properties of the foregrounds vary with sky positions, specifically near the galactic plane. The WMAP science team produces the internal linear combination map at \( N_{\text{side}} = 512 \) by dividing the galactic plane into 11 different regions (Bennett et al. 2013). The sky region outside the plane was cleaned in a single iteration. The work of this paper, however, intends to use global information from the theoretical CMB covariance matrix and data. Keeping in mind such dual requirements, we divide the sky into two regions and clean each as described below. The reason why we divide the sky into a smaller number of regions than the usual ILC approach in pixel space at high resolution is that we are interested in low-resolution maps focusing on the low multipoles. The lack of structures on small scales in the input maps ensures that the sky regions need not be too small.

5.3.1. Sky Division

To identify the region near the galactic plane that contains strong foreground emissions, we take Planck 353 and 70 GHz frequency maps at \( N_{\text{side}} = 2048 \). We downgrade these maps to \( N_{\text{side}} = 256 \) and smooth them by the ratio of window functions of a Gaussian beam of FWHM = 6° and the original beam functions of the \( N_{\text{side}} = 2048 \) maps at their native resolutions. We subtract the resulting reduced resolution 70 GHz map from 353 GHz map at \( N_{\text{side}} = 256 \). The difference map contains strong emissions from thermal dust at 353 GHz. We identify pixels of the difference map with values \( \geq 5000 \mu K \) and assign a value of unity to them and zero to the rest. We downgrade this binary map at \( N_{\text{side}} = 16 \). Finally, we reassign all nonzero pixels of the downgraded map a value of zero and the rest a value of unity. This sky region defined by the zero pixel values contains strong thermal dust emissions. The region complementary to this strong thermal dust emission survives after
application of the ThDust5000 mask. The sky region removed by this mask is shown in deep blue in Figure 2.

5.3.2. Foreground Cleaning

Based on the discussions of the previous sections, we perform the foreground cleaning following the second method in the following three steps:

1. We estimate the covariance matrix \( \tilde{C} \) applicable for the sky region defined by the ThDust5000 mask. This is done by using Equation (6) for all the pixel pairs \((i,j)\) that survive after application of the mask. We estimate \( \tilde{C}^{-1} \) following the same procedure as described in Section 5.2.
2. We use this generalized inverse of the partial sky CMB covariance matrix in Equation (4) to obtain elements of the partial sky matrix \( A \) using this partial sky matrix in Equation (5), we obtain the weights corresponding to the ThDust5000 sky region. Using these weights, we obtain the cleaned ThDust5000 sky region.
3. Now we replace the ThDust5000 sky region of all foreground-contaminated input maps by the cleaned region obtained above. The resulting 12 maps of the galactic regions are yet to be cleaned and strongly contaminated by the foregrounds. To clean the galactic region, we repeat steps 1 and 2 above over the full sky. The cleaned map obtained at this point is the full-sky cleaned map obtained by Method 2.

6. Results

6.1. Cleaned Maps

Using the first method, the weights for different WMAP and Planck channels become \(-0.093, 0.226, 0.424, -0.392, -0.859, -0.105, 0.195, 0.390, 0.890, 0.906, -0.607, \) and \(0.0245\) in increasing order of frequency of the 12 input maps from 23 to 353 GHz. We use these weights to linearly combine the 12 input maps to estimate the cleaned CMB map at \( N_{\text{side}} = 16 \) and at a Gaussian beam resolution of FWHM = \(9^\circ\) (henceforth we call this cleaned map CMap1). We show CMap1 in the top panel of Figure 3. Visually CMap1 does not contain any foreground residuals. We compare this map with other foreground-minimized CMB maps, each of which is obtained by employing a different algorithm at higher beam and pixel resolutions, as reported in the literature. The COMMANDER CMB map was obtained following joint estimation of CMB and all foreground components, the NILC CMB map was obtained by employing an internal linear combination algorithm in the needlet space, and the SMICA CMB map was obtained by using the spectral matching technique (see, e.g., Planck Collaboration et al. 2016b for detailed discussion about these maps). The WMAP science team produced a CMB map by using the usual ILC approach in pixel space (Hinshaw et al. 2007; Gold et al. 2011). We downgrade these high-resolution maps at \( N_{\text{side}} = 16 \) and bring them to a common beam resolution of \(9^\circ\). We show the difference of CMap1 from the resulting COMMANDER and NILC maps in the middle left and right panels of Figure 3, respectively. The bottom left and right panels show differences of CMap1 from the SMICA and WMAP ILC maps, respectively. Since monopole and dipoles are not of any cosmological interest, we have removed any residual dipole and monopole from all four difference maps shown in this figure. Clearly, our cleaned CMB map matches well with these cleaned CMB maps in the higher galactic plane. Along the galactic plane we find some differences. However, as one can easily make out, such a difference along the galactic plane exists for any pair of all five low-resolution CMB maps discussed in this section.

Following the second method, we recover a cleaned map (CMap2) similar to CMap1. The weights for the sky region that survived after application of the ThDust5000 mask are \(-0.066, 0.083, 0.500, -0.306, -0.562, -0.757, 0.021, 0.917, 0.876, 0.948, -0.684, 0.031\), respectively, for different frequencies increasing from 23 to 353 GHz (see, e.g., step 2 of Section 5.3.2). The corresponding weights for the full-sky (step 3 of Method 2) linear combination are \(-0.084, 0.240, 0.414, -0.399, -0.994, 0.100, 0.217, 0.277, 0.913, 0.896, -0.604\, and
A common feature of the weights for both these regions is that strongly contaminated frequency maps (e.g., K1 band or 353 GHz) get low (negative or positive) weights to cancel out foregrounds from all frequencies. The CMap2 matches closely with CMap1. We show the difference CMap2 - CMap1 in Figure 4. Clearly, Method 2 has slightly fewer foreground residuals along both sides of the galactic plane at the expense of some additional detector noise residuals along the ecliptic plane. We compare the full-sky power spectra of CMap1 and CMap2 along with other CMB spectra in Section 6.2.

6.2. Power Spectrum

We show the CMB angular power spectra after corrections of beam and pixel effects obtained from the full sky of CMap1 and CMap2 in the top panel of Figure 5. The theoretical CMB angular power spectrum is shown as a red line to guide the eye. The error bars show the reconstruction error in the power spectrum obtained from Method 2 and agree well with the cosmic-variance-induced errors (see, e.g., Section 7). The bottom panel of this figure shows the difference of the spectra of these two maps. As we see from this figure, both spectra match very well with each other. Such a close match is also expected from the very small difference between the two maps.
cleaned maps as shown in Figure 4. These results suggest that our new ILC approach is very weakly dependent on the sky divisions. This justifies following a global approach of foreground cleaning on large angular scales on the sky, as is done in this work. However, since Method 2 simultaneously follows a global approach and performs foreground removal in an iterative fashion, we treat the CMB angular power spectrum of CMap2 as the main power spectrum of this work estimated using low-resolution Planck and WMAP maps.

We compare the full-sky CMB angular power spectrum obtained from CMap2 with the corresponding spectra obtained from the COMMANDER, NILC, SMICA, and WMAP ILC maps. We show these spectra in the top left and top right panels of Figure 6. Also shown in these two panels is the CMB theoretical angular power spectrum obtained from Planck 2015 results. The bottom panels of this figure show the difference of angular power spectra of this work with the other spectra of the corresponding top panels. As we see from this figure, the CMB angular power spectrum from CMap2 matches closely with the angular spectra of these cleaned maps. A similar result was obtained considering the CMB angular power spectrum from CMap1 also. It is noteworthy that power spectra of CMap2 and the NILC map agree excellently for the entire multipole range \(2 \leq \ell \leq 32\).

7. Monte Carlo Simulations

7.1. Input CMB, Foreground, and Noise Maps

We validate the methodology for the first and second methods by performing detailed Monte Carlo simulations of the entire foreground removal and power spectrum estimation procedures. For this purpose, we first generate foreground maps at different WMAP and Planck frequency bands of this work. The free-free, synchrotron, and thermal dust emissions at different frequencies are first obtained at \(N_{\text{side}} = 256\) and beam resolution \(1^\circ\) following the procedure as described in Sudevan et al. (2017).\(^2\) We then downgrade the pixel resolution of each component map to \(N_{\text{side}} = 16\) and smooth each one by a Gaussian beam function of FWHM \(= \sqrt{540^2 - 60^2} = 536/66\) so as to bring all component maps for all frequency maps to the common resolution of \(9^\circ\). We generate CMB temperature anisotropy maps at \(N_{\text{side}} = 16\) and FWHM = \(9^\circ\) by using the theoretical CMB power spectrum, consistent with cosmological parameters obtained by Planck Collaboration et al. (2016d). The procedure to generate the detector noise maps remains similar to Sudevan et al. (2017). Following the same procedure as described by these authors, we first generate noise maps at \(N_{\text{side}} = 512\) (for WMAP DA maps) or 1024 and 2048 (for Planck frequency maps). We then convert these maps to spherical harmonic space up to \(\ell_{\text{max}} = 32\) and multiply the resulting spherical harmonic coefficients by the ratio of the window function corresponding to FWHM = \(9^\circ\) and the native beam window function of each WMAP DA (or Planck) frequency bands. For each of the WMAP \(Q, V,\) and \(W\) bands, we average the DA noise maps to generate a single noise map corresponding to the given frequency band. We generate a set of 200 noise maps for each of the 12 frequency maps of our analysis. Each of these noise maps has uncorrelated noise properties. We add the CMB, foreground, and noise maps generated above to obtain a set of frequency maps that represent realistic observations of WMAP and Planck missions at \(N_{\text{side}} = 16\) and FWHM = \(9^\circ\). We generate a total of 200 such sets of input frequency maps for Monte Carlo simulations.

7.2. Results

7.2.1. Reconstruction Error in Cleaned Maps

If the input CMB map for the \(i\)th Monte Carlo simulation is denoted by \(T_i(p)\), where \(p\) denotes the pixel index, and the corresponding foreground-minimized CMB map is denoted by \(T'_i(p)\), the map representing the reconstruction error for the particular simulation is then given by \(\Delta T_i(p) = T'_i(p) - T_i(p)\). We estimate the standard deviation map using all 200 error maps for each of our two methods of this work. The error maps for Methods 1 and 2 are shown, respectively, in the top and bottom panels of Figure 7. As seen from this figure, using the iterative method reduces the reconstruction error in the northern and southern hemispheres toward the galactic center region. Also seen from this figure is a lower reconstruction error near the north polar spur region. The average variance per pixel over full sky for Method 1 (estimated from the top panel Figure 7) is \(6.41 \mu K^2\), compared to a value of \(5.25 \mu K^2\) for Method 2 (bottom panel). Corresponding average variances for the ThDust5000 mask region are \(1.75\) and \(0.97 \mu K^2\), respectively.

\(^2\) Unlike the work of Sudevan et al. (2017), in the current work we use a spatially constant spectral index \((\beta_s = -3.00)\) for the synchrotron component for all WMAP and Planck frequencies.
For the galactic region not covered by the thermal dust mask the average variances become 22.84 and 20.36 μK respectively, for Method 1 and Method 2. We conclude that both methods work with comparable efficiencies; however, the second method performs better than the first method in terms of foreground removal.

7.2.2. CMB Angular Power Spectrum

Using 200 foreground-cleaned maps obtained from Monte Carlo simulations of foreground removal and subsequent CMB angular power spectrum estimation over the complete sky region, we assess reconstruction error in cleaned CMB power spectra obtained using Method 1 and Method 2. In the top panel of Figure 8 we plot the mean CMB angular power spectrum (green points) obtained following Method 2, along with the standard deviation of the cleaned power spectrum for any one of the simulations. The mean foreground-cleaned power spectrum agrees well with the theoretical CMB power spectrum (red line), which is used to generate random (and isotropic) CMB realizations. The cosmic variance error limit is shown by the colored band around the theoretical CMB power spectrum. The close match of cosmic variance and the reconstruction error on the cleaned power spectrum at each multipole implies that the recovered angular power spectrum is only cosmic variance limited, and reconstruction error due to foreground residuals (plus any error induced by detector noise) is a subdominant source of contamination on the angular scales chosen in this work. In the middle panel of Figure 8 we closely investigate any reconstruction biases that may exist in the foreground-cleaned power spectrum of Method 2 by plotting

Figure 6. Top panels: CMB angular power spectrum obtained from CMap2 of this work, with the same estimated from other foreground-cleaned CMB maps as mentioned. Bottom panels: difference of the CMap2 spectrum with those obtained from the other foreground-cleaned maps of the top panels. The zero level is shown by the red dashed line.

Figure 7. Top panel: standard deviation map obtained from the difference of the foreground-minimized CMB map and the corresponding randomly generated input CMB map using 200 Monte Carlo simulations of foreground minimization following Method 1 as described in Section 5.2. Bottom panel: standard deviation map obtained for the 200 Monte Carlo simulations of Method 2 (see, e.g., Section 5.3). All units are in μK thermodynamic temperature. The reconstruction errors for these two methods are discussed in Section 7.2.
the difference between the foreground-minimized mean CMB power spectrum and the CMB theoretical power spectrum. The error bar at each multipole plotted in this panel is applicable for the mean CMB angular power spectrum, and therefore they are obtained by scaling the corresponding reconstruction error of the top panel by $1/\sqrt{N_{\text{sim}}}$, where the number of simulations $N_{\text{sim}} = 200$. For all the multipoles except $\ell = 29$ the significance of any difference between the mean cleaned spectrum and the theoretical CMB spectrum is less than $2\sigma$. For $\ell = 29$ the significance of deviation is $2.8\sigma$. This shows that the power spectrum obtained from Method 2 has no significant bias that may arise as a result of imperfect foreground residuals.

The bottom panel of Figure 8 shows the difference between mean CMB power spectra obtained from Method 1 and Method 2. The error bars of this plot are computed from foreground-cleaned maps of Method 1, and they are applicable for the mean power spectrum. Clearly, mean spectra obtained by the two methods of this work agree very well with each other. Both methods produce comparable error bars as well.

8. Advantage of the Global ILC Method at Low Resolution

The global ILC method has two very important advantages over the usual ILC method in pixel space that does not take into account prior information about the CMB theoretical covariance matrix. First, the globally cleaned CMB map has less reconstruction error at each pixel. Second, the usual ILC approach (without using the covariance information) at low resolution leads to a bias in the power spectrum that remains absent in the proposed methods of this work. The cause of these limitations in the usual ILC approach in a low-resolution analysis is a chance correlation between the CMB and foreground (and detector noise) components, which cannot be ignored over large scales of the sky. In this section we discuss the advantages of our approach.

Using the simulated frequency maps at $N_{\text{side}} = 16^\circ$ and $9^\circ$ resolution (e.g., Section 7.1), we perform 200 Monte Carlo simulations over the complete sky region using the usual ILC approach, wherein no CMB covariance matrix is used. The error map in CMB reconstruction is then computed in the same fashion as discussed in Section 7.2.1. The standard deviation map is plotted in Figure 9, which indicates a strong residual, not only on the galactic plane but also in higher galactic latitudes. Unlike the small variance per pixel reported in Section 7.2.1, the average variance per pixel for Figure 9 is large (89.17 $\mu$K$^2$). This clearly demonstrates the first advantage, i.e., a sharp decrease in reconstruction error of the cleaned CMB map, when we incorporate prior information about theoretical covariance of the CMB component.

The larger reconstruction error in cleaned maps in the usual ILC approach causes a significant bias in the power spectrum, which is a quadratic function of the data. We show the mean power spectrum computed from 200 Monte Carlo simulations of the usual ILC approach over the entire sky in green in the top panel of Figure 10, along with the Planck 2015 theoretical power spectrum, which is used to generate the input CMB power spectrum obtained from Method 2 of this work, along with the theoretical CMB power spectrum (red line). The error bar computed from cosmic variance estimated from the theoretical power spectrum is shown by the filled region. The reconstruction error on the cleaned CMB power spectrum obtained from any one of the simulations is shown in green.
maps. Clearly, a positive bias exists owing to imperfect foreground residuals in the cleaned spectrum starting from multipole \( \ell = 8 \). Another interesting feature of the top panel is the existence of a negative bias for \( \ell \leq 5 \). Such a negative bias is expected and was first reported by Saha et al. (2006) and is discussed extensively in Saha et al. (2008) (see also Sudevan et al. 2017 for such bias in a high-resolution analysis) for multipole space ILC methods. In fact, observing the error pattern of Figure 9, it is likely that a positive bias due to residual foregrounds exists even at low multipoles \( \ell \leq 8 \) on the top of the additional negative bias in this multipole range. The bottom panel of Figure 10 compares the reconstruction error in the cleaned power spectrum with the error due to cosmic variance alone. Starting from multipole \( \ell \sim 10 \), we see that the error in the usual ILC power spectrum becomes larger than the cosmic-variance-induced error. Interestingly, due to the existence of negative bias at the low multipoles, the error in the cleaned spectrum becomes biased low for \( \ell \leq 4 \). The bias existing in the cleaned power spectrum of the usual ILC approach at low resolution, along with the larger error in the reconstructed power spectrum from this approach, justifies our second point of advantage (discussed at the beginning of the current section) of the new approach described in this article.

9. Role of CMB–Foreground (or CMB–Noise) Chance Correlation

Having discussed in the previous section the advantages of the global ILC method of this work, we now focus on the cause of excess residuals in the usual ILC method when applied to low-resolution maps. If we apply the usual ILC method on the input maps described in Section 2, the variance of the cleaned map becomes

\[
\hat{\sigma}^2 = W\hat{C}W^T,
\]

where \( \hat{C} \) is an \( n \times n \) matrix representing the covariance between different input frequency maps (from which mean temperature anisotropies corresponding to each frequency are already subtracted). Similar to Equation (5), the set of weights that minimizes variance of the cleaned map subject to the constraint that CMB is preserved is given by (Tegmark & Efstathiou 1996; Tegmark et al. 2003; Saha et al. 2008)

\[
W = \frac{\mathbf{e}C^\dagger}{\mathbf{e}\hat{C}\mathbf{e}^\dagger}.
\]

The data covariance matrix \( \hat{C} \) follows \( \hat{C} = \sigma^2 C e^T A \), where \( \sigma^2 \) represents the variance of the CMB component, which is independent of the frequency, \( \hat{C}_f \) is the mixed covariance matrix representing the chance correlation between the CMB and all foreground components for a given realization of CMB (e.g., pure CMB signal in our universe), and finally, \( C_f \) is the foreground covariance matrix.\(^5\) Following Saha et al. (2008), we note that \( e^T \in \mathbb{C} \hat{C}_f + C_f \), so that the generalized Sherman–Morrison formula for the Moore–Penrose generalized inverse of rank one update becomes

\[
\hat{A} = \hat{A} + \frac{1}{\lambda} fg^T,
\]

where \( \lambda = 1 + e\hat{A}^\dagger e^T, f = \hat{A}^\dagger e^T, \) and \( g = \hat{A}^\dagger e^T \). Using Equation (10), we obtain

\[
W = \frac{\mathbf{e}\hat{A}^\dagger}{\mathbf{e}\hat{A}^\dagger \mathbf{e}^\dagger}.
\]

Using Equation (11), we conclude that the weights are independent of the exact level of CMB variance \( \sigma^2 \) for the particular random realization. This is expected since the weights in the usual ILC method, in principle, should only be determined by the foregrounds as long as CMB follows a blackbody distribution. One may interpret Equation (11) as the usual ILC weights minimizing the part of the variance in the cleaned map that arises as a result of CMB–foreground chance correlation and foreground components. Since \( \hat{A} = \hat{C}_f + C_f \), we see from Equation (11) that in practice the ILC weights depend not only on the foreground covariance matrix \( C_f \) but

\(\text{Figure 10. Top panel: mean CMB power spectrum (in green) obtained from 200 Monte Carlo simulations of the usual ILC approach over the entire sky on low-resolution maps as discussed in Section 8, along with the theoretical CMB angular power spectrum (red line) consistent with Planck 2015 results. The filled color band shows the cosmic variance excursion limit of the observed CMB angular power spectrum. The green error bars show the reconstruction error in the cleaned power spectrum at different multipoles. The bottom panel closely compares the reconstruction error bars with the cosmic-variance-induced errors. Residuals in the cleaned maps cause a larger than cosmic variance error starting from } \ell \sim 10. \text{ For low multipoles, } \ell \leq 4 \text{ reconstruction error becomes less than cosmic-variance-induced error owing to a chance correlation of CMB with foregrounds (and detector noise).} \)
also on the CMB–foreground chance-correlation matrix \( \hat{C}_f \).

What would happen if in Equation (11) we could replace \( \hat{A} \) by \( C_f \)? We note that such a choice is not possible for analysis of the real data since the covariance matrix for the foregrounds is not known exactly a priori. However, in Monte Carlo simulations we can always assume that \( C_f \) is known. This will be the situation when weights are not affected by the chance-correlation matrix. If we know the true foreground covariance matrix accurately, in the usual ILC procedure one will just minimize the part of the variance in the cleaned map that arises as a result of foreground components. Clearly, this is

\[
\sigma_f^2 = W C_f W^T.
\]

Minimizing \( \sigma_f^2 \) subject to the constraint that CMB is preserved gives

\[
W = \frac{e C_f e}{e C_f e^T}.
\] (12)

We perform detailed Monte Carlo simulations of foreground minimization at low resolution following the usual ILC method, with simulated WMAP and Planck observations to investigate the difference in the cleaned maps obtained by two different ways. First, the weights are determined following Equation (11), and second, they are determined by Equation (12). In the first case we recover results that are similar to those shown in Figures 9 and 10. This implies that in the presence of CMB–foreground chance correlations the usual ILC method performs a poor foreground subtraction on large scales on the sky. In the second case, when the chance-correlation matrix is absent, the method performs foreground removal very well. The standard deviation map computed from the difference of cleaned CMB maps and the corresponding input CMB maps, for this case, is shown in Figure 11. The standard deviation map is consistent with a detector noise pattern without any visible signature of residual foregrounds. The mean pixel variance of this map is only 0.14 \( \mu K^2 \), indicating greatly improved foreground subtraction compared to the case when CMB–foreground chance correlation is present. We reemphasize that, although we can use Equation (12) for the case of Monte Carlo simulations where the input foreground models are known, in practice we cannot use this equation to estimate ILC weights since the foreground covariance matrix \( C_f \) is unknown for the observed sky. We use Equations (12) and (11) in Monte Carlo simulations to establish that the CMB–foreground chance correlations cause significant residuals in the usual ILC method. The global ILC method that proposes to use CMB covariance information thus becomes a greatly beneficial method, improving performance of the usual ILC method without any need to know \( C_f \).

Apart from comparing the pixel reconstruction error maps (e.g., Figures 9 and 11) or the power spectra of cleaned maps, there is another way in which we see that using the theoretical CMB covariance matrix helps to greatly improve the usual ILC results. In Figure 12 \((x, y)\) coordinates of any blue point are given respectively by value of weight for a particular frequency band obtained using the usual ILC method and the corresponding value of the weight using Equation (12) while cleaning a given set of input frequency maps. The \(y\)-coordinate of the yellow points is the same as the blue point for the same set of input frequency maps; however, the \(x\)-coordinate of the yellow points represents weights for the global ILC method using information about the theoretical CMB covariance matrix. The blue points show significantly larger dispersion along the horizontal axes for all frequency bands compared to the corresponding dispersion of the yellow points. The new method of this work efficiently reduces the larger dispersion of weights of the usual ILC method and produces better foreground-minimized CMB maps at low resolution. The \(y\)-coordinates of all points of this figure show some level of fluctuations, even if we use Equation (12) to estimate the weights that represent the \(y\)-coordinates. This is because, apart from the foregrounds, \( C_f \) contains a small level of detector noise. The \(x\)-coordinate of the vertical line of each plot shows the value of the weight when no detector noise is present in \( C_f \). Each of these values remains the same for different Monte Carlo simulations and represents weights that will be necessary to remove foregrounds in an ideal noiseless experiment. We finally note that, using the CMB theoretical covariance matrix in Equation (2), we efficiently suppress CMB large-angle covariances, which leads to significantly smaller dispersion of weights because of smaller CMB–foreground chance correlation. The small dispersion of our weights results in a greatly improved foreground minimization compared to the usual ILC method on large scales of the sky.

10. Discussions and Conclusion

We have developed a new ILC method for foreground minimization in pixel space for application on large angular scales on the sky using prior information about the theoretical CMB covariance matrix. We apply the methodology on low-resolution WMAP and Planck frequency maps and show that the cleaned CMB temperature anisotropy map obtained by us matches very well with those obtained by other science groups of Planck and WMAP. This shows that results of CMB maps and its power spectrum are robust with respect to a variety of analysis pipeline. We validate the methodology of our foreground removal by detailed Monte Carlo simulations. Usage of this new approach has several benefits over naive application of the usual ILC approach in pixel space over large scales of the sky.

1. First, the new approach generates a cleaned CMB map that has significantly lower reconstruction error due to foreground residuals. The power spectrum from the cleaned map also has the lower reconstruction error for
our case; the standard deviations of the CMB angular power spectrum estimated from the Monte Carlo simulations agree with those estimated from the cosmic variance alone.

2. Second, the CMB angular power spectrum obtained from our cleaned maps does not have any visible signature of negative bias at the low-multipole region, which is seen to be present for pixel space application of the usual ILC method over large scales on the sky. Such a negative bias is also reported in the harmonic space ILC method by Saha et al. (2006), and its properties and origin were investigated in detail by Saha et al. (2008). The negative bias arises as a result of a chance correlation between CMB and foreground components on a particular realization of the sky. Using inverse weight of the CMB theoretical covariance matrix in Equation (2), we effectively get rid of such chance correlations and the resulting negative biases in the cleaned CMB angular power spectrum at low multipoles.

The new method complements the usual ILC approach in pixel space, which so far has been applied on high-resolution maps by incorporating local information available from input frequency maps to better remove foregrounds, the spectral properties of which vary with the sky positions. On the very large scales the spectral properties of foregrounds are expected to vary by a small amount over the entire sky. We show that, on the large scale, it is sufficient to perform ILC foreground removal by dividing the sky merely into two regions, provided that we use the prior information available from CMB covariance matrix globally on the sky. Although we have assumed a theoretical CMB covariance matrix consistent with assumption of statistical isotropy of CMB in Equation (6), in principle, one can also use a covariance matrix in our method, which is not statistically isotropic. This brings about a possibility to open up a new avenue to incorporate such additional information in our method that may be a signature of a nontrivial primordial power spectrum (Ghosh et al. 2016; Contreras et al. 2017). Taking into account the global nature of our low-resolution analysis and the local nature of the high-resolution analysis of the usual ILC method, we now consider the pixel space ILC method in a general perspective that incorporates a very comprehensive duality in its nature. We hope that our method will be useful to analyze low-resolution polarization maps from Planck or future-generation CMB missions.

To find the pixel space CMB covariance matrix, we use a fiducial CMB theoretical angular power spectrum, which we take to be consistent with the Planck 2015 results. Although this reduces errors, it also causes an informative dependence of the cleaned power spectrum on the a priori chosen CMB theoretical power spectrum, and hence the cleaned power spectrum can be biased if there is a bias in our knowledge of prior. For cosmological analysis of the cleaned power spectrum, therefore, proper care must be taken by taking into account possible bias in the prior. To avoid bias on our cleaned power spectrum due to the choice of a biased prior, an important future project will be to use the Gibbs sampling technique to sample both the cleaned map and the theoretical power spectrum given the observed data.

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Figure 12. Scatter plot obtained from Monte Carlo simulations showing lower dispersion (along horizontal axis) of weights (yellow) when we follow the global ILC method with prior information from the theoretical CMB covariance matrix on large scales on the sky for different WMAP and Planck frequency bands. The y-coordinates of blue or yellow points represent weights obtained from Monte Carlo simulations using Equation (12). The larger dispersion of blue points along the horizontal axes causes larger reconstruction error in cleaned maps for the usual ILC methods at low resolution. The vertical lines represent values of weights obtained using Equation (12) but without any detector noise in the simulations.
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**ORCID iDs**

Rajib Saha @ https://orcid.org/0000-0002-4444-1081

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The Astrophysical Journal, 867:74 (12pp), 2018 November 1