Thermodynamics for radiating shells in anti-de Sitter space-time

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Abstract

A thermodynamical description for the quasi-static collapse of radiating, self-gravitating spherical shells of matter in anti-de Sitter space-time is obtained. It is shown that the specific heat at constant area and other thermodynamical quantities may diverge before a black hole has eventually formed. This suggests the possibility of a phase transition occurring along the collapse process. The differences with respect to the asymptotically flat case are also highlighted.

Key words: Radiating thin shells, Anti-de Sitter space-time, Thermodynamics

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1 Introduction

The collapse of spherically symmetric, self-gravitating thin shells has been widely studied as a simplified model for the process of black hole formation [1,2,3,4]. In Ref. [1] it was shown that the black hole entropy, expressed in terms of the area of the horizon, can be interpreted as the entropy of a shell of matter that contracts reversibly from infinity to its event horizon. A thermodynamical formalism was then introduced in order to describe the contraction of the shell. In Refs. [2,3] the quasi-static collapse of a non-radiating dust shell was investigated in the perspective of applying the AdS-CFT correspondence [5] to the gravitational collapse, as a first step with the aim of obtaining a unitary description for the black hole formation and evaporation.

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In these notes, we will examine the collapse of a *radiating* spherical shell of matter in anti-de Sitter (AdS) space-time. The inclusion of radiation in the model may in fact help in the understanding of the black hole formation, as suggested in Ref. [2]. The collapse is assumed to be a *quasi-static* process, in the sense that the shell contraction velocity is sufficiently small so that the system (shell-matter) can be described as evolving through a succession of equilibrium states. This assumption allows us to introduce a thermodynamical formalism (see Refs. [1,6] for the case of an asymptotically flat space-time) to describe the process. The properties of the system depend on the equation of state, that is a relation between the thermodynamically independent quantities. In order to obtain some explicit results we shall consider the case of a power-law dependence of the shell temperature (introduced as usual through the second principle of thermodynamics) on the horizon radius. In Appendix A the particular choice corresponding to Hawking temperature is then considered.

We use units for which \( \hbar = c = k_B = 1 \), with \( k_B \) the Boltzmann constant.

2 Thermodynamics

The spherically symmetric space-time we consider is divided into an inner region and an outer one by a thin massive spherical shell. The inner region can be expressed in static coordinates as

\[
\int_1^2 = -f_i(r) \, dt^2 + \frac{dr^2}{f_i(r)} + r^2 \, d\Omega^2 ,
\]

and will be taken to be described by a Schwarzschild metric, so that

\[
f_i(r) = 1 - \frac{2m}{r} ,
\]

where \( m \) is a constant ADM mass. The outer region, because of the radiation emitted by the shell, is described by a Vaidya-AdS space-time

\[
\int_3^2 = -\frac{1}{f_o(r, t)} \left[ \left( \frac{\partial_t M(r, t)}{\partial_r M(r, t)} \right)^2 \, dt^2 - dr^2 \right] + r^2 \, d\Omega^2 ,
\]

with

\[
f_o(r, t) = 1 - \frac{2M(r, t)}{r} + \frac{r^2}{\ell^2}
\]
where \( M(r,t) \) is the Bondi mass and its dependence on the time \( t \) is related to the amount of radiation (energy) flowing out of the shell, \( \partial_t M \) and \( \partial_r M \) are the partial derivatives of \( M(r,t) \) with respect to \( t \) and \( r \) respectively, and \( \ell \) is the AdS radius.

We shall obtain the thermodynamical description for the evolution of a thin shell, by assuming that the collapse can be described as a sequence of equilibrium states. Israel’s junction equations [7] for a static thin shell located at radius \( r = R \), allow us to relate the proper mass of the shell \( E \) to the inner and outer metrics through the equation

\[
E(R, M) = 4\pi R^2 \rho = R \left( \sqrt{f_i(R)} - \sqrt{f_o(R)} \right) ,
\]

where \( \rho \) is the surface energy density, and to evaluate the surface tension, denoted by \( P \), as

\[
P(R, M) \equiv \frac{\partial E}{\partial A} = \frac{1}{8\pi R} \left[ \sqrt{f_i(R)} - \sqrt{f_o(R)} + \frac{1}{\sqrt{f_i(R)}} \frac{m}{R} - \frac{1}{\sqrt{f_o(R)}} \left( \frac{M}{R} + \frac{R^2}{\ell^2} \right) \right] ,
\]

where \( A = 4\pi R^2 \) is the shell area. The continuity equation for the matter has to be taken as a constraint, and can be expressed in the form

\[
\frac{dL}{d\tau} = \frac{1}{\sqrt{f_o(R)}} \frac{dM}{d\tau} ,
\]

where \( L \) is the shell luminosity and \( \tau \) is the proper time of an observer sitting on the shell.

In order to obtain a description of the collapse process in a thermodynamical language one has to set up a correspondence between the mechanical properties of the shell, such as its tension and proper mass, and thermodynamical quantities such as the pressure, the internal energy, the temperature or the entropy. The proper mass of the shell is naturally identified with its internal energy (see Refs. [1,6]) and the surface tension with the thermodynamical pressure. This means that the shell, considered as a thermodynamical system, is characterized by an internal energy \( E(R, M) \) and a pressure \( P(R, M) \). We note that in our formalism the Schwarzschild mass \( M \) and the radius \( R \) are taken to be the dynamical independent variables, whereas \( m \) and \( \ell \) are taken to be fixed parameters. Therefore, we shall often find it convenient to use \( M \) instead of the horizon radius \( R_h \), the latter being defined by \( f_o(R_h) = 0 \), that is
\[ M = \frac{R_h}{2} \left(1 + \frac{R_h^2}{\ell^2}\right). \] (8)

One may now introduce the first law of thermodynamics, associated with energy conservation, by defining the infinitesimal heat flow \( \delta Q \) by

\[ \delta Q = dE - P dA. \] (9)

On using the explicit expressions for the pressure and the internal energy one finds

\[ \delta Q = \frac{dM}{\sqrt{f_o(R)}}. \] (10)

This expression agrees with the luminosity of a collapsing shell as described in Eq. (7). The above Eq. (9) can thus be re-interpreted as the continuity equation for the matter on the shell, and as such it confirms that the definitions of the internal energy (5) and pressure (6) are correct.

It is now possible to introduce a temperature \( T \) through the second principle of thermodynamics, that is the existence of the entropy as the exact differential

\[ dS = \frac{\delta Q}{T}. \] (11)

The temperature appears as an integrating factor which must satisfy the integrability condition

\[ \frac{\partial}{\partial R} \left(T \sqrt{f_o(R)}\right)^{-1} = 0, \] (12)

whose general solution is

\[ T = \frac{B_h(R_h)}{\sqrt{f_o(R)}}, \] (13)

where \( B_h = B_h(R_h) \) is an arbitrary function of the horizon radius \( R_h \), leading to

\[ dS = \left(1 + 3 \frac{R_h^2}{\ell^2}\right) \frac{dR_h}{2 B_h}. \] (14)
We note that the temperature exhibits the usual Tolman radial dependence. Once the temperature is fixed one may evaluate the specific heat at constant radius \( C_R \)

\[
C_R \equiv T \left( \frac{\partial S}{\partial T} \right)_R = T \left( \frac{\partial S}{\partial R_h} \right)_R \left( \frac{\partial T}{\partial R_h} \right)_R^{-1} = \left[ \frac{2 \ell^2 B'_h}{\ell^2 + 3 R_h^2} + \frac{B_h}{R f_o(R)} \right]^{-1},
\]

(15)

where \( B'_h = dB_h/dR_h \). The above expression shows a possible singularity for \( R \) satisfying

\[
(R - R_h) \left( \ell^2 + R^2 + R R_h + R_h^2 \right) = -\frac{\ell^2 + 3 R_h^2}{(\ln B_h^2)'}. \]

(16)

The specific heat at constant tension takes the form

\[
C_P = T \left( \frac{\partial S}{\partial T} \right)_P = \frac{T}{2 B_h} \left( 1 + 3 \frac{R_h^2}{\ell^2} \right) \left[ \left( \frac{\partial T}{\partial R_h} \right)_R - \left( \frac{\partial T}{\partial P} \right)_{R_h} \left( \frac{\partial P}{\partial R_h} \right)_R \right]^{-1},
\]

(17)

whose explicit expression we omit for the sake of brevity. Other thermodynamical quantities of interest, related to the second derivative of the Gibbs potential \[8\], are the change in area with respect to the temperature for fixed tension \((\partial A/\partial T)_P\) and with respect to the tension for fixed temperature \((\partial A/\partial P)_T\).

All such quantities show a singular behavior if there exists an \( R \) satisfying

\[
\frac{3 R_h}{2 R} \left( 1 + \frac{R_h^2}{\ell^2} \right) \left[ 1 - \frac{R_h}{2 R} \left( 1 + \frac{R_h^2}{\ell^2} \right) - \frac{R_h^2}{\ell^2} \right]
+ \left[ \frac{f_o(R)}{f_i(R)} \right]^{3/2} \left( 1 - \frac{3 m}{R} + \frac{3 m^2}{R^2} \right)
= 1 - \frac{R_h^2}{4 R^2} \left[ 1 + \frac{R_h^2}{\ell^2} + \frac{2 R^3}{R_h \ell^2} \right] \left[ 1 + f_o(R) \frac{\ell^2}{\ell^2 + 3 R_h^2} \right]^{-1}.
\]

(18)

In order to have an explicit expression for the specific heats and to proceed further in our investigation, we need an equation of state, that is an expression for the function \( B_h \). Let us examine a rather general case assuming a
power-law dependence of the function $B_h$ on the horizon radius, leading to the temperature

$$T = \frac{1}{\sqrt{f_o(R)}} \frac{1}{4\pi R_h^a},$$

with $a$ a constant. We can now determine the specific heat at constant area

$$C_R = -4\pi f_o(R) R_h^{a+1} \left(1 + 3 \frac{R_h^2}{\ell^2}\right) \left(2a f_o(R) + R_h \frac{\partial f_o(R)}{\partial R_h}\right)^{-1}. \quad (20)$$

This implies that $C_R$ diverges for

$$0 = 2a f_o(R) + R_h \frac{\partial f_o(R)}{\partial R_h} \quad \Rightarrow \quad 2a \left(1 + \frac{R^2}{\ell^2}\right) - R_h \left[(1 + 2a) + (3 + 2a) \frac{R_h^2}{\ell^2}\right]. \quad (21)$$

Let us examine the above equation. One finds that $C_R$ has at most one singularity at $R = R_\ell$ for

$$a > 0 \quad \text{with } R_\ell > R_h, \quad \forall R_h \geq 0 \quad (22)$$

and

$$-3/2 \leq a \leq -1/2 \quad \text{with } 0 \leq R_\ell < R_h. \quad \text{with } 0 \leq R_\ell < R_h. \quad (23)$$

The behaviour of the specific heat at constant tension $C_P$ is singular for $a > -3/2$, as is shown (along with $C_R$) in Fig. 1, for any value of $\ell$, and the radius for which the singularity appears decreases as $a$ increases.

It is now interesting to compare the above singular behaviours with that of the asymptotically flat case, which is obtained for $\ell \to \infty$. For instance, Eq. (21) simplifies considerably and one finds that the singularity moves to
Fig. 1. Behaviour of the specific heats at constant area $C_R$ (solid line) and constant tension $C_P$ (dashed line) for $m = 0$, $a = 1$ and $R_h = 1$: (a) in AdS with $\ell = 10$; and (b) in asymptotically flat space ($\ell \to \infty$).

$$R_\infty = \left( 1 + \frac{1}{2a} \right) R_h ,$$

which is a physical (i.e. positive) radius for $a > -1/2$ and is larger than $R_h$ for $a > 0$. Let us note that the singularity (23), which occurs inside the shell horizon in AdS, is now replaced by an again hidden (inside the horizon) one for $-1/2 < a < 0$.

3 Conclusions

We have analysed the thermodynamical behaviour for the collapse of a radiating shell in an AdS space-time, under the assumption that the evolution consists of a succession of equilibrium states, that is the process is quasi-static. On identifying the internal energy and surface tension of the shell, we were able to evaluate the specific heats at constant area and tension and other related thermodynamical quantities when the temperature is given by a power law of the horizon radius as in Eq. (19). Their behavior may suggest the existence of a phase transition before the shell reaches its Schwarzschild radius. Of course, in a realistic case, the shell ADM mass and horizon could change quickly in time and the adiabaticity (quasi-staticity) of the process may be lost. There is however evidence for cases in which the shell naturally emits radiation (having a Hawking temperature) in such a way that its contraction velocity remains small [9] and the quasi-static approximation can therefore be applied.

The case of the Hawking temperature is not of the form (19) and is analyzed in Appendix A. A very interesting feature of the model is then the appearance of a threshold value for the AdS parameter $\ell \simeq 7 R_h / 4$, which leads to two very different behaviours for the specific heats at constant area and tension, as shown in Fig. A.1.
We feel that these singularities in thermodynamically quantities such as specific heats may be of relevance and deserve further investigation.

A Hawking temperature

Let us examine the case in which the temperature is that of a black hole with horizon radius $R_h$ [10], implying

$$B_h = \frac{1}{4\pi R_h} \left( 1 + 3 \frac{R_h^2}{\ell^2} \right), \quad (A.1)$$

which seems to be the most natural choice if we assume that at the end of the collapse the system behaves as if a black hole were being formed (for an analysis supporting the naturalness of this choice see Refs. [6,11,12]). On substituting for $B_h$ in Eq. (16) one obtains the equation

$$R + \frac{R^3}{\ell^2} = \frac{R_h}{2} \left( 3 + 2 \frac{R_h^2}{\ell^2} + 3 \frac{R_h^4}{\ell^4} \right), \quad (A.2)$$

which determines the singularity of the specific heat at constant area. Let us note that for $\ell \to \infty$, the singularity for the specific heat at constant radius is located at $R = 3 M$ as in the asymptotically flat case [see Eq. (24) and Ref. [6]].

In order to examine the singularities of $C_R$ and $C_P$ for a general value of $\ell$ one must study Eqs. (16) and (18). This analysis shows that for $\ell > \ell_0 \simeq 7R_h/4$, $C_R$ has a singularity and changes sign for a finite radius, as shown in Fig. A.1. As $\ell$ approaches $\ell_0$ from above, the singularity moves to arbitrarily large values of $R$. On the other hand the specific heat at constant tension $C_P$, shows a singularity at the horizon $R_h$ and at a finite radius. The singularity moves to arbitrarily large radii as $\ell \to \ell_0$ and $\ell \to \infty$. The singularity of $C_P$ always occurs for a radius greater than that for which $C_R$ is singular.

For the case $\ell < \ell_0$, $C_R$ does not show any singularity and remains positive for any radius of the shell, becoming zero at the horizon radius, whereas $C_P$ is regular everywhere except at the horizon, as is shown in Fig. A.1. It is worth noting that the singularity in $C_P$ is not present in the purely Schwarzschild case, thus it is a peculiar feature for the AdS space-time.

We finally note that for the choice of a Hawking temperature, the entropy, following Eqs. (14) and (A.1), is given by
Fig. A.1. Behaviour of the specific heats at constant area $C_R$ (solid line) and constant tension $C_P$ (dashed line) for $m = 0$ and $R_h = 1$: (a) $\ell = 2R_h > \ell_0$; and (b) $\ell = R_h < \ell_0$.

\[ S = \int \frac{\delta Q}{T} = \pi R_h^2 = \frac{1}{4} \text{ (horizon area)} \quad \text{(A.3)} \]

This expression will exhibit a simple additive property, in the sense that the entropy of two non-interacting (well separated) shells will just be the sum of the two entropies, as expected for usual thermodynamical systems [12]. Such an additive property also rules out any integration constant in the Eq. (A.3).

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