Geometric Ergodicity of the MUCOGARCH(1,1) Process

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Abstract

For the multivariate COGARCH(1,1) volatility process we show sufficient conditions for the existence of a unique stationary distribution, for the geometric ergodicity and for the finiteness of moments of the stationary distribution. One of the conditions demands a sufficiently fast exponential decay of the MUCOGARCH(1,1) volatility process. Furthermore, we show easily applicable sufficient conditions for the needed irreducibility of the volatility process living in the cone of positive semidefinite matrices, if the driving Lévy process is a compound Poisson process.

AMS Subject Classification 2010: Primary: 60J25 Secondary: 60G10, 60G51

Keywords: Feller process, Foster-Lyapunov drift condition, Harris recurrence, irreducibility, Lévy process, multivariate stochastic volatility model

1 Introduction

General autoregressive conditionally heteroscedastic (GARCH) time series models, as introduced in [13], are of high interest for financial economics. They capture many typical features of observed financial data, the so-called stylized facts (see [34]). A continuous time extension, which captures the same stylized facts as the discrete time GARCH model, but can also be used for irregularly-spaced and high-frequency data, is the COGARCH process, see e.g. [16, 38, 39]. The use in financial modelling is studied e.g. in [7, 40, 53] and the statistical estimation in [12, 55, 44], for example. Furthermore, an asymmetric variant is proposed in [9] and an extension allowing for more flexibility in the autocovariance function in [8].

To model and understand the behavior of several interrelated time series as well as to price derivatives on several underlyings or to assess the risk of a portfolio multivariate models for financial markets are needed. The fluctuations of the volatilities and correlations over time call for employing stochastic volatility models which in a multivariate set-up means that one has to specify a latent process for the instantaneous covariance matrix. Thus, one needs to consider appropriate stochastic processes in the cone of positive semi-definite matrices. Many popular multivariate stochastic volatility models in continuous time, which in many financial applications is preferable to modelling in discrete time, are of an affine type, thus falling into the framework of [20]. Popular examples include the Wishart (see e.g. [2, 18, 32]) and the

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Ornstein-Uhlenbeck type stochastic volatility model (see [10, 33, 52], for example), which also has been enhanced to the supOU model allowing for possible long memory (cf. [1, 63]). Thus they have two driving sources of randomness and their tail-behavior is typically equivalent to the one of the driving noise (see [28, 51]). A very nice feature of GARCH models is that they have only one source of randomness and their structure ensures heavily-tailed stationary behavior even for very light tailed driving noises ([5, 29]). In discrete time one of the most general multivariate GARCH versions (see [6, 30] for an overview) is the BEKK model, defined in [25], and the MUCOGARCH(1,1) process introduced and studied in [62], is the continuous time analogue, which we are investigating further in this paper.

The existence and uniqueness of a stationary solution as well as the convergence to the stationary solution is of high interest and importance. Geometric ergodicity ensures fast convergence to the stationary regime in simulations and paves the way for statistical inference. By the same argument as in [47, Proof of Theorem 4.3, Step 2] geometric ergodicity and the existence of some $p$-moments of the stationary distribution provide exponential $\beta$-mixing for Markov processes. This in turn can be used to show a central limit theorem for the process, see for instance [22], and so allows to prove for example asymptotic normality of estimators (see e.g. [12, 35] in the context of univariate COGARCH(1,1) processes).

In many applications involving time series (multivariate) ARMA-GARCH models (see e.g. [19, 31, 43]) turn out to be adequate and geometric ergodicity is again key to understand the asymptotic behaviour of statistical estimators. In continuous time a promising analogue currently investigated in [55] seems to be a (multivariate) CARMA process (see e.g. [17, 27, 46]) driven by a (multivariate) COGARCH process. The present paper also lays important foundations for the analysis of such models.

For the univariate COGARCH process geometric ergodicity was shown by [26] and discussed it for the BEKK GARCH process. In [62] for the MUCOGARCH process the existence of a stationary distribution is shown by tightness arguments, but the paper failed to establish uniqueness or convergence to the stationary distribution. In this paper we deduce under the assumption of irreducibility sufficient conditions for the uniqueness of the stationary distribution, the convergence to it with an exponential rate and some finite $p$-moment of the stationary distribution of the MUCOGARCH volatility process $Y$. To show this we use the theory of Markov process, see e.g. [23, 48]. A further result of this theory is, that our volatility process is positive Harris recurrent. If the driving Lévy process is a compound Poisson process, we show easily applicable conditions ensuring irreducibility of the volatility process in the cone of positive semidefinite matrices.

Like in the discrete time BEKK case the non-linear structure of the SDE will prohibit us from using well-established results for random recurrence equations like in the one-dimensional case and due to the rank one jumps establishing irreducibility is a very tricky issue. To obtain the latter [15] actually used techniques from algebraic geometry (see also [14]) whereas we use a direct probabilistic approach playing the question back to the well-understood existence of a density for a Wishart distribution. However, we restrict ourselves to processes of order (1,1) while in the discrete time BEKK case general orders were considered. The reason is that on the one hand it turns out that order (1,1) GARCH processes are sufficient in most applications and on the other hand multivariate COGARCH(p,q) processes can be defined in principle ([61, Section 6.6]) but no reasonable conditions on the possible parameters are known. Already in the univariate case these conditions are quite involved (cf. [16, Section 5], [64]). On the other hand we can look at the finiteness of an arbitrary $p$-th moment (of the volatility process) and use drift conditions related to it, whereas [15] only looked at the first...
moment for the BEKK case.

After a brief summary of some preliminaries and notations, the remainder of the paper is organized as follows: In Section 3 we recall the definition of the MUCO GARCH(1,1) process and some of its properties of relevance later on. In Section 4 we present our main results: sufficient conditions ensuring the geometric ergodicity of the volatility process $Y$ and the irreducibility of $Y$. Furthermore, we compare the conditions for geometric ergodicity to previously known conditions for (first order) stationarity. Section 5 first gives a brief repetition of the Markov theory we use and the proofs of our results are developed.

2 Preliminaries

Throughout we assume that all random variables and processes are defined on a given filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in T})$ with $T = \mathbb{N}$ in the discrete time case and $T = \mathbb{R}^+$ in the continuous one. Moreover, in the continuous time setting we assume the usual conditions (complete, right continuous filtration) to be satisfied. For Markov processes in discrete and continuous time we refer to [50] and respectively [2, 24]. The definition of transition semigroups, as we use it, can be found for instance in [24].

We only repeat the definition of weak Feller processes since we distinguish between $C_b$- and $C_0$-Feller processes in this paper. By $C_b(U)$ we denote the set of all continuous and bounded functions $f : U \rightarrow \mathbb{R}$ and by $C_0(U)$ those continuous functions, which vanish at infinity.

**Definition 2.1** (Stochastic continuity and Feller processes). Let $(\mathbb{P}_t)_{t \in \mathbb{R}^+}$ be the transition semigroup of a time homogeneous Markov process $\Phi$ with topological state space $U$.

(i) $(\mathbb{P}_t)_{t \in \mathbb{R}^+}$ or $\Phi$ is called **stochastically continuous** if

$$\lim_{t \to 0} \mathbb{P}_t(x, \mathcal{N}(x)) = 1$$  \hspace{1cm} (2.1)

for all $x \in U$ and open neighborhoods $\mathcal{N}(x)$ of $x$.

(ii) $(\mathbb{P}_t)_{t \in \mathbb{R}^+}$ or $\Phi$ is a (weak) $C_b$-Feller semigroup or process if it is stochastically continuous and

$$\mathbb{P}_t(C_b(U)) \subseteq C_b(U) \text{ for all } t \geq 0.$$  \hspace{1cm} (2.2)

(iii) If in (ii) we have instead of (2.2)

$$\mathbb{P}_t(C_0(U)) \subseteq C_0(U) \text{ for all } t \geq 0$$  \hspace{1cm} (2.3)

we call the semigroup or the process (weak) $C_0$-Feller.

2.1 Notation

The set of real $m \times n$ matrices is denoted by $M_{m,n}(\mathbb{R})$ or only by $M_n(\mathbb{R})$ if $m = n$. For the invertible $n \times n$ matrices we write $GL_n(\mathbb{R})$. Its linear subspace of symmetric matrices we denote by $S_n$, by $S_n^+$ the closed cone of $S_n$ of positive semidefinite matrices and the open cone of positive definite matrices by $S_n^{++}$. Further we denote by $I_n$ the $n \times n$ identity matrix.

We introduce the natural ordering on $S_n$ and denote it by $\preceq$, that is for $A, B \in S_n$, it holds $A \preceq B \iff B - A \in S_n^+$. The tensor (Kronecker) product of two matrices $A, B$ is written as $A \otimes B$. vec denotes the well-known vectorization operator that maps the $n \times n$ matrices to $\mathbb{R}^{n^2}$ by stacking the columns of the matrices below another. For more information regarding
the tensor product and vec operator we refer to [11]. The spectrum of a matrix is denoted by $\sigma(\cdot)$ and the spectral radius by $\rho(\cdot)$. $\text{Re}(x)$ is the real part of a complex number and in relation to the spectrum, $\text{Re}(\sigma(\cdot))$, means the real part of the set. Finally, $A^\top$ is the transpose of a matrix $A \in M_{m,n}(\mathbb{R})$.

With $\|\cdot\|_2$ we denote both the Euclidean norm for vectors and the corresponding operator norm for matrices and with $\|\cdot\|_F$ the Frobenius norm for matrices.

Furthermore, we employ an intuitive notation with respect to the (stochastic) integration with matrix-valued integrators referring to any of the standard texts (e.g. [54]) for a comprehensive treatment of the theory of stochastic integration. If $(X_t)_{t \in \mathbb{R}^+}$ is a semi-martingale in $\mathbb{R}^m$ and $(Y_t)_{t \in \mathbb{R}^+}$ one in $\mathbb{R}^n$ then the quadratic variation $([X,Y]_t)_{t \in \mathbb{R}^+}$ is defined as the finite variation process in $M_{m,n}(\mathbb{R})$ with components $[X,Y]_{ij,t} = [X_i,Y_j]_t$ for $t \in \mathbb{R}^+$ and $i = 1, \ldots, m$, $j = 1, \ldots, n$.

For a Lévy process $L$ in $\mathbb{R}^n$ with jump measure $\mu_L$ the discontinuous part of the quadratic variation is $[L,L]_t^d := \int_0^t \int_{\mathbb{R}^n} xx^\top \mu_L(ds,dx) = \sum_{0 \leq s \leq t} (\Delta L_s)(\Delta L_s)^\top$.

As usual we denote by $\mathcal{M}_1(\mathbb{R}^n)$, the set of all probability measures on the Borel-$\sigma$-algebra of $\mathbb{R}^n$.

3 Multivariate COGARCH(1,1) process

In this section we repeat the definition of the MUCOGARCH(1,1) process and some known properties. All definitions and properties in this section are taken from [62].

3.1 Definition of the multivariate COGARCH(1,1) process

**Definition 3.1** ([MUCOGARCH(1,1), [62, Definition 3.1]. Let $L$ be an $\mathbb{R}^d$-valued Lévy process, $A, B \in M_d(\mathbb{R})$ and $C \in S^+_{d}$. The **MUCOGARCH(1,1) process** $G = (G_t)_{t \geq 0}$ is defined as the solution of

$$dG_t = V_t^{\frac{1}{2}} dL_t$$

$$V_t = Y_t + C$$

$$dY_t = (B Y_{t-} + Y_{t-} B^\top) dt + AV_t^{\frac{1}{2}} d[L,L]_t^{\frac{1}{2}} V_t^{\frac{1}{2}} A^\top,$$

with initial values $G_0 \in \mathbb{R}^d$ and $Y_0 \in S^+_{d}$.

The process $Y = (Y_t)_{t \geq 0}$ is called **MUCOGARCH(1,1) volatility process**.

Since we only consider MUCOGARCH(1,1) processes, we often simply write MUCOGARCH. Equations (3.2) and (3.3) directly give us an SDE for the covariance matrix process $V$:

$$dV_t = (B(V_{t-} - C) + (V_{t-} - C)B^\top) dt + AV_t^{\frac{1}{2}} d[L,L]_t^{\frac{1}{2}} V_t^{\frac{1}{2}} A^\top.$$  

(3.4)

Provided $\sigma(B) \subset (-\infty, 0) + i\mathbb{R}$, we see that $V$, as long as no jumps occur, returns to the level $C$ at an exponential rate determined by $B$. Since all jumps are positive semidefinite, $C$ is not a mean level but a lower bound.
An equivalent representation is obtained by using the vec operator:

\[
\begin{align*}
    dG_t &= V_{t-}^{1/2}dL_t, \quad V_t = C + Y_t \\
    d\text{vec}(Y_t) &= (B \otimes I + I \otimes B)\text{vec}(Y_{t-})dt + (A \otimes A)(V_{t-}^{1/2} \otimes V_{t-}^{1/2})d\text{vec}([L,L]^d_t) \\
    d\text{vec}(V_t) &= (B \otimes I + I \otimes B)(\text{vec}(V_{t-}) - \text{vec}(C))dt + (A \otimes A)(V_{t-}^{1/2} \otimes V_{t-}^{1/2})d\text{vec}([L,L]^d_t).
\end{align*}
\]

To have the MUCOGARCH process well-defined, we have to know that a unique solution of the SDE system exists and the solution of \( Y \) and \( V \) does not leave the set \( S^+_d \). In the following we always understand that our processes live on \( S^+_d \) (not on \( M_d(\mathbb{R}) \)) in vec representation on \( \text{vec}(S_d) \). We can identify with \( \mathbb{R}^{d(d+1)/2} \). Since \( S^+_d \) is an open subset of \( S_d \), we now are in the most natural setting for SDEs and we get:

**Theorem 3.2** ([62], Theorem 3.2). Let \( A, B \in M_d(\mathbb{R}) \), \( C \in S^+_d \) and \( L \) be a \( d \)-dimensional Lévy process. Then the SDE \((\ref{3.3})\) with initial value \( Y_0 \in S^+_d \) has a unique positive semi-definite solution \((Y_t)_{t \in \mathbb{R}^+}\). The solution \((Y_t)_{t \in \mathbb{R}^+}\) is locally bounded and of finite variation. Moreover, it satisfies \( Y_t \geq e^{Bt}Y_0e^{B^Tt} \) for all \( t \in \mathbb{R}^+ \).

Further we have the following representation for the solution \( Y_t \):

**Theorem 3.3** ([62], Theorem 3.6). The MUCOGARCH(1,1) volatility process \( Y \) satisfies

\[
Y_t = e^{Bt}Y_0e^{B^Tt} + \int_0^t e^{B(t-s)}A(C + Y_{s-})^{1/2}d[L,L]^d_s(C + Y_{s-})^{1/2}A^+e^{B^T(t-s)}\tag{3.5}
\]

for all \( t \in \mathbb{R}^+ \).

### 3.2 Properties of the MUCOGARCH volatility process \( Y \).

**Theorem 3.4** (Markov property, [62], Theorem 4.4). The MUCOGARCH process \((G,Y)\) as well as its volatility process \( Y \) alone are temporally homogeneous strong Markov processes on \( \mathbb{R}^d \times S^+_d \) and \( S^+_d \), respectively, and they have the weak \( C^0 \)-Feller property.

Some sufficient conditions for the existence of a stationary distribution are already known. For this we need some notations from [62]: Assume now that \( B \) is diagonalizable and let \( S \in GL_d(\mathbb{C}) \) be such that \( S^{-1}BS \) is diagonal. Then we define the norm \( \| \cdot \|_{B,S} \) on \( M_d(\mathbb{R}) \) by \( \|X\|_{B,S} := \|(S^{-1} \otimes S^{-1})X(S \otimes S)\|_2 \) for \( X \in M_d(\mathbb{R}) \). It should be noted that \( \| \cdot \|_{B,S} \) depends both on \( B \) and on the choice of the matrix \( S \) diagonalizing \( B \). Observe that \( \| \cdot \|_{B,S} \) is an operator norm, namely the one associated to the norm \( \|x\|_{B,S} := \|(S^{-1} \otimes S^{-1})x\|_2 \) on \( \mathbb{R}^{d^2} \). Besides, \( \| \cdot \|_{B,S} \) actually is simply the norm \( \| \cdot \|_2 \) provided \( S \) is a unitary matrix.

**Theorem 3.5** ([62], Theorem 4.5). Let \( B \in M_d(\mathbb{R}) \) be diagonalizable with \( S \in GL_d(\mathbb{C}) \) such that \( S^{-1}BS \) is diagonal. Furthermore, let \( L \) be a \( d \)-dimensional Lévy process with non-zero Lévy measure \( \nu_L \), define

\[
\lambda := \max(Re(\sigma(B))), \quad K_{2,B} := \max_{X \in S^+_d} \left( \frac{\|X\|_2}{\|\text{vec}(X)\|_{B,S}} \right) \quad \text{and} \quad \alpha_1 := \|S\|^2_2\|S^{-1}\|^2_2K_{2,B}\|A \otimes A\|_{B,S}.
\]
Assume that
\[
\int_{\mathbb{R}^d} \log \left(1 + \alpha_1 \|\text{vec}(yy^\top)\|_{B,S}\right) \nu_L(dy) < -2\lambda. \quad (3.6)
\]

Then there exists a stationary distribution \(\mu \in M_1(S^+_+\times)\) for the MUCOGARCH(1,1) volatility process \(Y\) such that
\[
\int_{\mathbb{R}^d} \left(1 + \alpha_1 \|\text{vec}(yy^\top)\|_{B,S}\right)^{k-1} \nu_L(dy) < -2\lambda k \quad (3.7)
\]
for some \(k \in \mathbb{N}\) implies that \(\int_{\mathbb{R}^d} \|x\|^k \mu(dx) < \infty\), i.e. the \(k\)-th moment of the stationary distribution is finite.

**Remark 3.6.** From [38, Lemma 4.1(d)] it follows that, if \((3.7)\) is satisfied for \(k \in \mathbb{N}\) then it is also satisfied for all \(\tilde{k} \in \mathbb{N}, \tilde{k} \leq k\).

### 4 Geometric ergodicity of the MUCOGARCH volatility process \(Y\)

In Theorem 3.5 the existence of a stationary distribution is shown, but neither the uniqueness nor that it is a limiting distribution. This is the subject of our main theorem. Furthermore, we can show, that the convergence in total variation is exponentially fast, therefore our volatility process is geometrically ergodic.

**Theorem 4.1** (Geometric ergodicity). Let \(Y\) be a MUCOGARCH volatility process with \(\sigma(B) \subset (-\infty, 0) + i\mathbb{R}\). Assume

(i) \(Y\) is \(\mu\)-irreducible with the support of \(\mu\) having non-empty interior and aperiodic,

(ii) there exists a \(p \in [1, \infty)\) such that
\[
\int_{\mathbb{R}^d} \left(2^{p-1} \left(1 + \|A \otimes A\|_2 \|\text{vec}(yy^\top)\|_2\right)^{p} - 1\right) \nu_L(dy) + m_Bp < 0 \quad (4.1)
\]
with \(m_B := \max\{x^T((B \otimes I) + (I \otimes B))x : x \in \mathbb{R}^d, \|x\|_2 = 1\}\) the maximum of the real numerical range,

(iii) \(E(\|L_1\|_2^p) < \infty\).

Then a unique stationary distribution for the MUCOGARCH volatility process \(Y\) exists, \(Y\) is positive Harris recurrent, geometrically ergodic and the stationary distribution has a finite \(p\)-th moment.

This result holds also for \(p \in (0,1)\) with the additional assumption, that the \(p\)-variation of \(L\) exists.

**Theorem 4.2.** Let \(Y\) be a MUCOGARCH volatility process with \(\sigma(B) \subset (-\infty, 0) + i\mathbb{R}\). Assume

(i) \(Y\) is \(\mu\)-irreducible with the support of \(\mu\) having non-empty interior and aperiodic,
(ii) there exists a \( p \in (0, 1) \) such that
\[
\int_{\mathbb{R}^d} \left( \left( 1 + \|A \otimes A\|_2 \|\text{vec}(yy^\top)\|_2 \right)^p - 1 \right) \nu_L(dy) + m_B p < 0 \tag{4.2}
\]

(iii) the \( 2p \) moments and variation exist, i.e.
\[
\int_{\mathbb{R}^d} \|y\|^{2p} \nu_L(dy) < \infty. \tag{4.3}
\]

Then a unique stationary distribution for the MUCOGARCH volatility process \( Y \) exists, \( Y \) is positive Harris recurrent, geometrically ergodic and the stationary distribution has a finite \( p \)-th moment.

These results raise the questions how to easily compute the numerical radius (at least in special cases) and how the conditions compare to previously known stationarity conditions.

Remark 4.3. The real numerical range of a matrix \( A \in \mathbb{R}^{n \times n} \) is defined by
\[
\Psi_1(A) \triangleq \{ x^\top Ax : x \in \mathbb{R}^n \text{ and } x^\top x = 1 \},
\]
see e.g. [11, Fact 8.14.8]. Some facts are:
- \( \Psi_1(A) = [\lambda_{\min}(\frac{1}{2}(A + A^\top)), \lambda_{\max}(\frac{1}{2}(A + A^\top))] \),
- if \( A \) is symmetric, then \( \Psi_1(A) = [\lambda_{\min}(A), \lambda_{\max}(A)] \).

Hence if \( B \) and thus \( (B \otimes I) + (I \otimes B) \) is symmetric \( m_B = \lambda_{\max}((B \otimes I) + (I \otimes B)) = 2\lambda_{\max}(B) \) (see [37, Theorem 4.4.5]). From [62, Remark 4.6] we thus see that conditions (4.1) and (3.7) are identical for a symmetric \( B \) and \( p = k = 1 \). For a symmetric \( B \) and \( p > 1 \) that are very similar only that an additional factor of \( 2^{p-1} \) appears inside the integral in the conditions ensuring geometric ergodicity. So the previously known conditions for the existence of a stationary distribution are somewhat weaker than our conditions for geometric ergodicity, but the difference does not appear dramatic. It should be also noted that in contrast to [62] we do not need to restrict ourselves to \( B \) being diagonalizable and integer moments \( p \).

Remark 4.4. According to [62, Theorem 4.20, its proof and Remark 4.9] the conditions for asymptotic first-order stationarity are:

(i) there exists a constant \( \sigma_L \in \mathbb{R}^+ \) such that \( \int_{\mathbb{R}^d} xx^\top \nu_L(dx) = \sigma_L I_d \).

(ii) \( \sigma(B) \subset (-\infty, 0) + i\mathbb{R} \)

(iii) \( \sigma \left( ((B \otimes I) + (I \otimes B)) + (A \otimes A) \int_{\mathbb{R}^d} xx^\top \nu_L(dx) \right) \subset (-\infty, 0) + i\mathbb{R} \). We use the notation \( \tilde{\mathcal{B}} := ((B \otimes I) + (I \otimes B)) + (A \otimes A) \int_{\mathbb{R}^d} xx^\top \nu_L(dx) \).

In our Theorem 4.1 for \( p = 1 \) the condition (4.1) reduces to
\[
\int_{\mathbb{R}^d} \|A \otimes A\|_2 \|\text{vec}(yy^\top)\|_2 \nu_L(dy) + m_B < 0. \tag{4.4}
\]

Assuming \( B \) and thereby \( \mathcal{B} \) is symmetric, our condition (4.4) immediately implies (iii) \( \sigma(\tilde{\mathcal{B}}) \subset (-\infty, 0) + i\mathbb{R} \).
To show this let $\lambda = \max\{\Re(\mu) : \mu \in \sigma(\tilde{B})\}$. By a corollary of the Bauer-Fike theorem, see [34, Corollary 6.3.4], there exists a $\mu \in \sigma(B)$ such that

$$
\lambda - \mu \leq |\lambda - \mu| \leq ||(A \otimes A)\int_{\mathbb{R}^d} xx^T \nu_L(dx)||_2.
$$

Observe that

$$
||(A \otimes A)\int_{\mathbb{R}^d} xx^T \nu_L(dx)||_2 \leq ||A \otimes A||_2 \int_{\mathbb{R}^d} ||xx^T||_2 \nu_L(dy) \leq ||A \otimes A||_2 \int_{\mathbb{R}^d} \|\text{vec}(xx^T)\|_2 \nu_L(dy).
$$

Thus

$$
\lambda \leq \mu + ||A \otimes A||_2 \int_{\mathbb{R}^d} \|\text{vec}(xx^T)\|_2 \nu_L(dy)
$$

and since $\mu \leq \lambda_{\max}(B)$ this provides $\lambda \leq \lambda_{\max}(B) + ||A \otimes A||_2 \int_{\mathbb{R}^d} \|\text{vec}(xx^T)\|_2 \nu_L(dy) < 0$ by [4.4] and so $\sigma(\tilde{B}) \subset (-\infty, 0) + i\mathbb{R}$.

So also the known first order stationarity condition turns out to be implied by our conditions for geometric ergodicity for $B$ being symmetric. Due to the need to link the numerical range and the eigenvalues it seems not really feasible to compare these conditions if $B$ is not symmetric.

**Proposition 4.5** (Improved Feller Property and Borel right process). The MUCOGARCH volatility process $Y$ is non-explosive and a weak $C_0$-Feller process. Thereby it is a Borel right Markov process.

We now establish conditions for the irreducibility and aperiodicity of $Y$. For this we have to assume that our driving Lévy process $L$ is of compound Poisson type.

**Theorem 4.6** (Irreducibility and Aperiodicity). Let $Y$ be a MUCOGARCH volatility process driven by a compound Poisson process $L$ and with $A \in GL_d(\mathbb{R})$. If the jump distribution of $L$ has a non-trivial absolutely continuous component equivalent to the Lebesgue measure on $\mathbb{R}^d$, then $Y$ is irreducible with respect to the Lebesgue measure on $S^+_d$ and aperiodic.

We soften the conditions on the jump distribution of the Compound Poisson process:

**Corollary 4.7.** Let $Y$ be a MUCOGARCH volatility process driven by a compound Poisson process $L$ and with $A \in GL_d(\mathbb{R})$. If the jump distribution of $L$ has a non-trivial absolutely continuous component and the density of the component w.r.t. the Lebesgue measure on $\mathbb{R}^d$ is strictly positive in a neighborhood of zero, then $Y$ is irreducible w.r.t. the Lebesgue measure restricted to an open neighborhood of zero in $S^+_d$ and aperiodic.

**Remark 4.8.** If the driving Lévy process is a general Lévy process, whose Lévy measure has a non-trivial absolutely continuous component and the density of the component w.r.t. the Lebesgue measure on $S^+_d$ is strictly positive in a neighborhood of zero, we can show that $Y$ is open-set irreducible w.r.t. the Lebesgue measure restricted to an open neighborhood of zero in $S^+_d$. For strong Feller processes open-set irreducibility provides irreducibility, but the strong Feller property is to the best of our knowledge hard to establish for Lévy-driven SDEs. A classical way to show irreducibility is by using density or support theorems based on the Malliavin Calculus, see e.g. [21, 42, 60]. But they all require, that the coefficients of the SDE have bounded derivatives, which is not the case for the MUCOGARCH volatility process. So finding criteria for irreducibility in the infinite activity case appears to be a very challenging question beyond the scope of the present paper.

Note that the condition that the Lévy measure has an absolutely continuous component with a support containing zero is the obvious analogue on the condition on the noise in [15].
5 Proofs

To prove our results we use the stability concepts for Markov processes of [48, 49].

5.1 Markov processes and ergodicity

In the first subsection we give a short introduction to the definitions and results for general continuous time Markov processes. Mostly we follow the notations and definitions of [23, Section 3].

We consider a continuous time Markov process \( \Phi = (\Phi_t)_{t \geq 0} \) on a topological space \( X \) with transition probabilities \( \mathbb{P}^t(x, A) = \mathbb{P}_x(\Phi_t \in A) \) for \( x \in X, A \in \mathcal{B}(X) \).

To define non-explosivity, we consider a fixed family \( \{O_n | n \in \mathbb{Z}_+\} \) of open precompact sets, i.e. the closure of \( O_n \) is a compact subset of \( X \), for which \( O_n \nearrow X \) as \( n \to \infty \). With \( T_m \) we denote the first entrance time to \( O_m \) and by \( \xi \) the exit time for the process, defined as

\[
\xi = \lim_{m \to \infty} T_m.
\]

**Definition 5.1** (Non-explosivity, [49, Chapter 1.2]). We call the process \( \Phi \) non-explosive if \( \mathbb{P}_x(\xi = \infty) = 1 \) for all \( x \in X \).

Since the definition of a Borel right process is only a technical one and there exists a result, that every \( C_0 \)-Feller process is a Borel right process (see Section 5.2, Proof of Proposition 4.5), we skip this definition.

We additionally assume that \( \Phi \) is a non-explosive Borel right process on a locally compact, separable metric space \((X, \mathcal{B}(X))\), with \( \mathcal{B}(X) \) the Borel \( \sigma \)-field on \( X \). For the definitions and details of the existence and structure we refer to [59]. The operator \( \mathbb{P}^t \) from the transition semigroup acts on a bounded measurable function \( f \) via

\[
\mathbb{P}^t f(x) = \int_X \mathbb{P}^t(x, dy) f(y) \quad (5.1)
\]

and on a \( \sigma \)-finite measure \( \mu \) on \( X \) via

\[
\mu \mathbb{P}^t(A) = \int_X \mu(dy) \mathbb{P}^t(y, A). \quad (5.2)
\]

**Definition 5.2** (Invariant measure, [23, Chapter 3]). A \( \sigma \)-finite measure \( \pi \) on \( \mathcal{B}(X) \) with the property

\[
\pi = \pi \mathbb{P}^t, \quad \forall t \geq 0
\]

is called invariant.

Notation: By \( \pi \) we always denote the unique invariant measure of \( \Phi \), if it exists.

**Definition 5.3** (Exponential ergodicity, [23, Chapter 3]). \( \Phi \) is called exponentially ergodic, if an invariant measure \( \pi \) exists and satisfies for all \( x \in X \)

\[
\|\mathbb{P}^t(x, .) - \pi\|_{TV} \leq M(x) \rho^t, \quad \forall t \geq 0
\]

(5.4)

for some finite \( M(x) \), some \( \rho < 1 \) and where \( \| . \|_{TV} \) denotes the total variation norm. If this convergence holds for the \( f \)-norm \( \| \mu \|_{f} := \sup_{|g| \leq f} | \int \mu(dy) g(y) | \) (for any signed measure \( \mu \)), where \( f \) is a measurable function from the state space \( X \) to \([1, \infty)\), we call the process \( f \)-exponentially ergodic.
A seemingly stronger formulation of \( V \)-exponential ergodicity is \( V \)-uniform ergodicity: We require that \( M(x) = V(x) \cdot D \) with some finite constant \( D \).

**Definition 5.4** (\( V \)-uniform ergodicity, [23, Chapter 3]). \( \Phi \) is called \( V \)-uniformly ergodic, if a measurable function \( V : X \rightarrow [1, \infty) \) exists such that for all \( x \in X \)

\[
\|P^t(x,.) - \pi\|_V \leq V(x) D \rho^t, \quad t \geq 0 \tag{5.5}
\]

holds for some \( D < \infty, \rho < 1 \).

To prove ergodicity we need the definitions of irreducibility and aperiodicity.

**Definition 5.5** ([23, Chapter 3]). For any \( \sigma \)-finite measure \( \mu \) on \( B(X) \) we call the process \( \Phi \) \( \mu \)-irreducible if for any \( B \in B(X) \) with \( \mu(B) > 0 \)

\[
E_x(\eta_B) > 0, \forall x \in X \tag{5.6}
\]

holds, where \( \eta_B := \int_0^\infty \mathbb{1}_{\{\Phi_t \in B\}} dt \) is the occupation time.

This is obviously the same condition as

\[
\int_0^\infty P^t(x,B) dt > 0, \forall x \in X. \tag{5.7}
\]

If \( \Phi \) is \( \mu \)-irreducible, there exists a maximal irreducibility measure \( \psi \) such that every other irreducibility measure \( \nu \) is absolutely continuous with respect to \( \psi \). We write \( B^+(X) \) for the collection of all measurable subsets \( A \in B(X) \) with \( \psi(A) > 0 \).

**Remark 5.6.** In [63, Proposition 1.1] it was shown, that if the discrete-time \( h \)-skeleton of a process, the \( \mathbb{P}^h \)-chain, is \( \psi \)-irreducible for some \( h > 0 \), then it holds for the continuous time process. If the \( \mathbb{P}^h \)-chain is \( \psi \)-irreducible for every \( h > 0 \), we call the process simultaneously \( \psi \)-irreducible.

One probabilistic form of stability is the concept of Harris recurrence.

**Definition 5.7** (Harris recurrence, [48, Chapter 2.2]).

(i) \( \Phi \) is called Harris recurrent, if either

- \( \mathbb{P}_x(\eta_A = \infty) = 1 \) whenever \( \phi(A) > 0 \) for some \( \sigma \)-finite measure \( \phi \), or
- \( \mathbb{P}_x(\tau_A < \infty) = 1 \) whenever \( \mu(A) > 0 \) for some \( \sigma \)-finite measure \( \mu \). \( \tau_A := \inf\{t \geq 0 : \Phi_t \in A\} \) is the first hitting time of \( A \).

(ii) Suppose that \( \Phi \) is Harris recurrent with finite invariant measure \( \pi \), then \( \Phi \) is called positive Harris recurrent.

To define the class of subsets of \( X \) called petite sets, we suppose that \( a \) is a probability distribution on \( \mathbb{R}_+ \). We define the Markov transition function \( K_a \) for the chain sampled by \( a \) as

\[
K_a(x,A) := \int_0^\infty P^t(x,A) a(dt), \forall x \in X, A \in B. \tag{5.8}
\]
Definition 5.8 (Petite and small sets, [23, Chapter 3]). A nonempty set $C \in \mathcal{B}$ is called $\nu_a$-petite, if $\nu_a$ is a nontrivial measure on $\mathcal{B}(X)$ and $a$ is a sampling distribution on $(0, \infty)$ satisfying
\[
K_a(x, \cdot) \geq \nu_a(\cdot), \quad \forall x \in C.
\] (5.9)

When the sampling distribution $a$ is degenerate, i.e. a single point mass, we call the set $C$ small.

Remark 5.9. Like in the discrete time Markov chain theory the set $C$ is small, if there exists an $m > 0$ and a nontrivial measure $\nu_m$ on $\mathcal{B}(X)$ such that for all $x \in C, B \in \mathcal{B}(X)$
\[
\mathbb{P}^m(x, B) \geq \nu_m(B)
\] (5.10)
holds.

For discrete time chains there exist a well known concept of periodicity, see for example [54, Chapter 5.4]. For continuous time processes this definition is not adaptable, since there are no fixed time steps. But a similar concept is the definition of aperiodicity for continuous time Markov processes as introduced in [23].

Definition 5.10 ([23, Chapter 3]). A $\psi$-irreducible Markov process is called aperiodic if for some small set $C \in \mathcal{B}^+(X)$ there exists a $T$ such that $\mathbb{P}^t(x, C) > 0$ for all $t \geq T$ and all $x \in C$.

Proposition 5.11. When $\Phi$ is simultaneously $\psi$-irreducible then we know from [64, Proposition 1.2] that every skeleton chain is aperiodic in the sense of a discrete time Markov chain.

For discrete time Markov processes there exist conditions such that every compact set is petite and every petite set is small:

Proposition 5.12 ([50], Theorem 6.0.1 and 5.5.7).
(i) If $\Phi$, a discrete time Markov process, is a $\Psi$-irreducible Feller chain with $\text{supp}(\Psi)$ having non-empty interior, every compact set is petite.
(ii) If $\Phi$ is irreducible and aperiodic, then every petite set is small.

Remark 5.13. Proposition 5.12(i) is also true for continuous time Markov processes, see [48].

To introduce the Foster-Lyapunov criterion for ergodicity we need the concept of the extended generator of a Markov process.

Definition 5.14 (Extended generator, [23, Chapter 4]). $\mathcal{D}(\mathcal{A})$ denotes the set of all functions $f : X \times \mathbb{R}_+ \to \mathbb{R}$ for which a function $g : X \times \mathbb{R}_+ \to \mathbb{R}$ exists, such that $\forall x \in X, t > 0$
\[
\mathbb{E}_x(f(\Phi_t, t)) = f(x, 0) + \mathbb{E}_x \left( \int_0^t g(\Phi_s, s) ds \right),
\] (5.11)
\[
\mathbb{E}_x \left( \int_0^t |g(\Phi_s, s)| ds \right) < \infty
\] (5.12)
holds. We write $\mathcal{A}f := g$ and call $\mathcal{A}$ the extended generator of $\Phi$. $\mathcal{D}(\mathcal{A})$ is called the domain of $\mathcal{A}$.
The next theorem from [23] gives for an irreducible and aperiodic Markov process a sufficient criterion to be $V$-uniformly ergodic. This is a modification of the Foster-Lyapunov drift criterion of [49].

**Theorem 5.15** ([23], Theorem 5.2). Let $(\Phi_t)_{t \geq 0}$ be a $\mu$-irreducible and aperiodic Markov process. If there exist constants $b, c > 0$ and a petite set $C$ in $\mathcal{B}(X)$ as well as a measurable function $V : X \to [1, \infty)$ such that

$$AV \leq -bV + c1_C,$$

(5.13)

where $A$ is the extended generator, then $(\Phi_t)_{t \geq 0}$ is $V$-uniformly ergodic.

## 5.2 Proofs of Section 4

We prove the geometric ergodicity of the MUCOGARCH volatility process by using Theorem 5.15. We first prove Proposition 4.5 to ensure that we are in the setting of Theorem 5.15.

### 5.2.1 Proof of Proposition 4.5

(i) $C_0$-Feller property: Let $f \in C_0(S^+_d)$. We have to show that $\forall t \geq 0$

$$\mathbb{P}_t f(x) \to 0, \quad \text{for } x \to \infty,$$

(5.14)

where we understand $x \to \infty$ in the sense of $\|x\|_2 \to \infty$. Since $\mathbb{P}_t f(x) = \mathbb{E}(f(Y_t(x)))$, where $x$ denotes the starting point $Y_0 = x$, it is enough to show that $Y_t$ goes to infinity for $x \to \infty$:

$$\|Y_t\|_2 \geq \|e^{Bt}Y_0e^{B^Tt}\|_2$$

$$= \|e^{Bt}\|_2^2\|x\|_2 \to \infty \text{ for } \|x\|_2 \to \infty.$$

(ii) That a $C_0$-Feller process is a Borel right process, follows from [43]: Combining the definition of strongly continuous contraction semigroups, Theorem 4.1.1 and the fact, that Feller processes by their definition are Borel right.

(iii) The non-explosivity property is shown in the proof of Theorem 6.3.7 in [61]. □

**Remark 5.16.** For $\mathbb{R}^d$-valued solutions to Lévy-driven stochastic differential equations

$$dX_t = \sigma(X_{t-})dL_t$$

[41] gives necessary and sufficient conditions for the rich Feller property, which includes the $C_0$-Feller property, if $\sigma$ is continuous and of linear growth. But since our direct proof is quite short, we prefer it instead of adapting the result of [41] to our state space.

### 5.2.2 Proof of Theorem 4.1 and 4.2: Geometric ergodicity

The first and main part of the proof is to show the geometric ergodicity. After that we show the positive Harris recurrence, which essentially is a consequence of the proof of the geometric ergodicity. To prove the geometric ergodicity, and hence the existence and uniqueness of a stationary distribution, of the MUCOGARCH volatility process, it is enough to show that the Foster-Lyapunov drift condition of Theorem 5.15 holds. The remaining conditions of Theorem 5.15 are proved in Proposition 4.9 or given by the assumptions.
The proof is structured as follows: First we deduce the extended generator of our process. Then we have to choose a test function, according to the assumptions of finite \( p \)-th moments resp. finite \( p \)-variation we choose \( u(x) = \|x\|_p^p + 1 \). From here on we distinguish the two cases \( p \geq 1 \) and \( p \in (0,1) \). First we consider the case \( p \geq 1 \) as in Theorem 4.1 and deduce the Foster-Lyapunov drift condition. Then the second case \( p \in (0,1) \), Theorem 4.2 follows by a slight variation in the arguments. In the last step we show that our chosen test function belongs to the domain of the extended generator.

**Deduction of the extended generator:**

We calculate the extended generator via the stochastic symbol, see [58, Chapter 6.1].

To get the stochastic symbol of the MUCOGARCH volatility process, we need the characteristic exponent, see [57, Theorem 3.1]. For this we look at the MUCOGARCH volatility process in vec representation

\[
dvec(Y_t) = (B \otimes I + I \otimes B)\text{vec}(Y_t^-)dt + (A \otimes A)(V_{t-} \otimes V_{t-}^{\frac{1}{2}})dvec[L,L]_t^d
\]

\[
= \left((A \otimes A)(V_{t-} \otimes V_{t-}^{\frac{1}{2}}), (B \otimes I + I \otimes B)\text{vec}(Y_t^-)\right) \cdot \left(dvec[L,L]_t^d \over dt \right).
\]

Hence we need the characteristic exponent of \( \left(\text{vec}[L,L]_t^d \over t \right) \).

Let \( \gamma, A_L, \mu \) be the characteristic triplet of our given Lévy process \( L \). The truncation function regarding the characteristic triplet of \( L \) is not relevant, as for the characteristic triplet of \( \text{vec}([L,L]^d) \) we can choose the truncation function \( \chi \equiv 0 \) since \( \text{vec}([L,L]^d) \) has finite variation. Then

\[
\begin{bmatrix}
0_{d^2} \\
0_{d^2+1, d^2+1}, \nu_{\text{vec}} \\
0_{\nu_{\text{vec}} = \rho}
\end{bmatrix}
\]

is the characteristic triplet of \( \left(\text{vec}[L,L]_t^d \over t \right) \) w.r.t. truncation function \( \chi \equiv 0 \), where \( \nu_{\text{vec}} \) is defined by \( \nu_{\text{vec}}(B) = \int_{\mathbb{R}^d} \mathbb{1}_{\{x \in \mathbb{R}^d: \text{vec}(x) \in B\}}(yy^\top)\mu(dy) \). By \( 0_{d^2} \) we denote the \( d^2 \)-dimensional vector consisting only of zeros and \( 0_{d^2+1, d^2+1} \) is the \((d^2 + 1) \times (d^2 + 1)\) zero matrix. The triplet is obtained by using the formulas for the characteristic triplet under linear transformations, see [56, Proposition 11.10].

Now we can obtain the symbol from the characteristic exponent

\[
p(x, \xi) = \psi(\Phi(x)^\top \xi)
\]

\[
= i(\Phi(x)^\top \xi)^\top \left(0_{d^2} \over 1\right) + \int_{\mathbb{R}^{d^2+1}} \left(e^{i\langle \Phi(x)^\top \xi, y \rangle} - 1\right) \varphi(dy).
\]

Note that \( \Phi(x) = ((A \otimes A)((C + x)_{\frac{1}{2}} \otimes (C + x)_{\frac{1}{2}}), (B \otimes I + I \otimes B)\text{vec}(x)) \) is of the form \((a(x), \ b(x))\) with \( a(x) \in \mathbb{R}^{d^2} \times d^2 \) and \( b(x) \in \mathbb{R}^{d^2} \). For \( \xi \in \mathbb{R}^{d^2} \) it holds that

\[
\Phi(x)^\top \xi = (a(x)^\top \xi, b(x)^\top \xi)^\top
\]

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and with this
\[
\left\langle \Phi(x)\xi,\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = b(x)\xi.
\]

Thus
\[
p(x,\xi) = -ib(x)^{\top}\xi + 0 - \int_{\mathbb{R}^{d+1}} \left( e^{i\xi^{\top}\Phi(x)y} - 1 \right) \tilde{\nu}(dy)
\]
is the stochastic symbol for the MUCOGRARCH(1,1) volatility process, see [57, Theorem 3.1] and the extended generator is given by
\[
Au(x) = b(x)^{\top}\nabla u(x) + \int_{\mathbb{R}^{d+1}} \left( u(x + \Phi(x)y) - u(x) \right) \tilde{\nu}(dy)
=: Du(x) + J u(x),
\]
see [58, Chapter 6.1]. We abbreviate the first addend, the drift part, with \( Du(x) \) and the second, the jump part, with \( J u(x) \).

**Foster-Lyapunov drift inequality:**
Below we consider the drift part \( D \) and the jump part \( J \) separately. For both parts we deduce some upper bounds, which we can bring together and get the upper bound, which is required in the Foster-Lyapunov drift condition.

As test function we choose \( u(x) = \|x\|_2^p + 1 \), thus \( u(x) \geq 1 \). Note that the gradient of \( u \) is given by \( \nabla u(x) = p\|x\|^{p-2}x \).

Further observe that the state space of \( Y \) and with that the domain of \( u \) is \( \mathbb{S}_d^+ \). For \( p \in (0,2) \) the gradient of \( u \) has a singularity in 0, but in the generator we look at \( b(x)^{\top}\nabla u(x) \), which is continuous in 0.

Moreover by the definition of \( \tilde{\nu} \)
\[
\int_{\mathbb{R}^{d+1}} \left( e^{i\xi^{\top}\Phi(x)y} - 1 \right) \tilde{\nu}(dy) = \int_{\mathbb{R}^{d}} \left( e^{i\xi^{\top}A(x)y} - 1 \right) \nu_{vec}(dy).
\]

Since the inequalities distinguish for \( p < 1 \) and \( p \geq 1 \) we now look at both cases separately:

**I** The case \( p \geq 1 \), Theorem 4.1

\[
J u(x) = \int_{\mathbb{R}^{d}} \left[ u \left( x + \left( A \otimes A \right) \left( \text{vec}^{-1}(x) + C \right)^{\frac{1}{2}} \otimes \left( \text{vec}^{-1}(x) + C \right)^{\frac{1}{2}} \right) \right]^{\top} z - u(x) \right] \nu_{vec}(dz)
= \int_{\mathbb{R}^{d}} \left\| x + \left( A \otimes A \right) \left( \text{vec}^{-1}(x) + C \right)^{\frac{1}{2}} \otimes \left( \text{vec}^{-1}(x) + C \right)^{\frac{1}{2}} \right\|_2 \nu_{vec}(dz)
\leq \int_{\mathbb{R}^{d}} \left\| x \right\|_2 + \left\| \left( A \otimes A \right) \left( \text{vec}^{-1}(x) + C \right)^{\frac{1}{2}} \otimes \left( \text{vec}^{-1}(x) + C \right)^{\frac{1}{2}} \right\|_2 \nu_{vec}(dz)
\leq \left\| x \right\|_2 + \left\| \left( A \otimes A \right) \left( \text{vec}^{-1}(x) + C \right)^{\frac{1}{2}} \otimes \left( \text{vec}^{-1}(x) + C \right)^{\frac{1}{2}} \right\|_2 \nu_{vec}(dz)
\]

Since \( \left\| \text{vec}^{-1}(x) \right\|_2 \leq \left\| \text{vec}^{-1}(x) \right\|_F = \left\| x \right\|_2 \) we have
\[
\left\| \left( A \otimes A \right) \left( \text{vec}^{-1}(x) + C \right)^{\frac{1}{2}} \otimes \left( \text{vec}^{-1}(x) + C \right)^{\frac{1}{2}} \right\|_2 \nu_{vec}(dz)
\leq \left\| A \otimes A \right\|_2 \left\| \text{vec}^{-1}(x) + C \right\|_2 \left\| z \right\|_2
\leq \left\| A \otimes A \right\|_2 \left\| x \right\|_2 \left\| z \right\|_2 + \left\| A \otimes A \right\|_2 \left\| C \right\|_2 \left\| z \right\|_2
\]
(5.16)
and with the inequality \(\|x + y\|^p \leq 2^{p-1} (\|x\|^p + \|y\|^p)\) this yields

\[
\mathcal{J} u(x) \leq \|x\|_2^p \int_{\mathbb{R}^d^2} \left(2^{p-1} \left(1 + \|A \otimes A\|_2 \|z\|_2\right)^p - 1\right) \nu_{\text{vec}}(dz) + d
\]

(5.17)

\[
\leq (-m BP - c) \|x\|^p_2 + d,
\]

for some constant \(c > 0\) and \(d := 2^{p-1} \|A \otimes A\|_2^p \|C\|_2^p \int_{\mathbb{R}^d^2} \|z\|^p_2 \nu_{\text{vec}}(dz)\), which is finite by assumption (iii) and \(\nu_{\text{vec}}\) being a Lévy measure.

For the drift part we get

\[
\mathcal{D} u(x) = \left(\left((B \otimes I) + (I \otimes B)\right) x\right)^\top \|x\|_2^{p-2}
\]

\[
= x^\top \left((B \otimes I) + (I \otimes B)\right) x \|x\|_2^{p-2}
\]

\[
\leq m BP \|x\|^2_2.
\]

Summarizing we have

\[
\mathcal{A} u(x) \leq -c \|x\|^p_2 + d.
\]

(5.18)

For \(\|x\|_2 > k\) and \(k\) big enough there exists \(0 < c_1 < c\) such that

(5.18) \leq -c_1 (\|x\|^p_2 + 1).

For \(\|x\|_2 \leq k\) we have in (5.18):

\[-c \|x\|^p_2 + d \leq -c_1 \|x\|^p_2 + d = -c_1 (\|x\|^p_2 + 1) + e,
\]

with \(e := c_1 + d > 0\). Altogether we have

\[
\mathcal{A} u(x) \leq -c_1 u(x) + e \mathbb{1}_{D_k},
\]

(5.19)

where \(D_k := \{x : \|x\|_2 \leq k\}\) is a compact set. By Proposition 5.12(i) this is also a petite set. Therefore the Foster-Lyapunov drift condition is proved.

(II) The case \(p \in (0, 1)\), Theorem 4.2

In the deduction of the bound (5.19) the assumption \(p \geq 1\) was only used for the inequality (5.17). For \(p \in (0, 1)\) we use instead of the inequality based on convexity that

\[(x + y)^p \leq x^p + y^p, \quad x, y \geq 0\]

holds by a generalization of the Binomial theorem. Together with assumption (4.2) we get that the jump part is bounded by

\[
\mathcal{J} u(x) \leq \|x\|_2^p \int_{\mathbb{R}^d^2} \left(1 + \|A \otimes A\|_2 \|z\|_2\right)^p - 1\right) \nu_{\text{vec}}(dz) + d
\]

(5.20)

\[
\leq (-m BP - c) \|x\|^p_2 + d,
\]

for some constant \(c > 0\) and \(d := \|A \otimes A\|^p_2 \|C\|^p_2 \int_{\mathbb{R}^d^2} \|z\|^p_2 \nu_{\text{vec}}(dz)\).

Together with the bound of the drift part, which does not change for \(p \in (0, 1)\), this yields the Foster-Lyapunov inequality (5.18) with new constants.
Test function belongs to the domain:

Further we have to show that our chosen \( u(x) = \|x\|^p_2 + 1 \) is in the domain of \( \mathcal{A} \). For this we have to show that the two conditions \( (5.11) \) and \( (5.12) \) are fulfilled. The first condition holds by the theory of the stochastic symbol we used, see \([58]\). For the second one we have to show that

\[
\mathbb{E}_x \int_0^t |\mathcal{A}u(\text{vec}(Y_s))| ds < \infty
\]

holds for all \( t > 0 \) and all starting points \( \text{vec}(Y_0) = x \in \text{vec}(\mathbb{S}^+_d) \).

To show this we first deduce a bound for \( |\mathcal{A}u(x)| \), then we bound \( \mathbb{E}_x(u(\text{vec}(Y_t))) \) and finally we can bring this together to get \( (5.12) \).

Using the triangle inequality we split \( |\mathcal{A}u(x)| \) again in the drift part and the jump part. For the absolute value of the jump part we can use the upper bounds \( (5.17) \) resp. \( (5.20) \) since the jumps are non-negative: To show this first notice that

\[
\|A + B\|_F = (\text{Tr}(A + B)(A + B)^\top)^{\frac{1}{2}} = \left(\text{Tr}(AA^\top + BB^\top) + 2\text{Tr}(AB^\top)\right)^{\frac{1}{2}} \\
\geq \left(\text{Tr}(AA^\top)\right)^{\frac{1}{2}} = \|A\|_F
\]

if \( A, B \in \mathbb{S}^+_d \) since the cone of positive semidefinite matrices is self dual with the inner product defined by the trace: \( \langle A, B \rangle = \text{Tr}(AB) \). Thereby we get \( \|x + \Phi(x)y\|_2 = \|\text{vec}^{-1}(x + \Phi(x)y)\|_F \geq \|\text{vec}^{-1}(x)\|_F = \|x\|_2 \), which means \( u(x + \Phi(x)y) - u(x) \geq 0 \).

The absolute value of the drift part is bounded as follows

\[
|\mathcal{D}u(x)| = \left|\left(((B \otimes I) + (I \otimes B)) x\right)^\top xp \|x\|^{p-2}_2\right| \\
= \left|x^\top ((B \otimes I) + (I \otimes B)) x\right| |p| \|x\|^{p-2}_2 \\
\leq |m_{\text{min},B}| |p| \|x\|^{p}_2,
\]

where \( m_{\text{min},B} \) is the minimum of the real numerical range of \( (B \otimes I) + (I \otimes B) \).

Adding both parts together we get

\[
|\mathcal{A}(u(x))| \leq |m_{\text{min},B}| |p| \|x\|^{p}_2 + (-m_Bp)\|x\|^{p}_2 + d \leq c_2u(x) \tag{5.21}
\]

for some constant \( c_2 > 0 \). Notice that this works for \( p \in (0, 1) \) and \( p \geq 1 \) in the same way, only the arising constant \( d \) and thus also \( c_2 \) differs.

From equation \( (5.21) \) we use

\[
\mathcal{A}u(x) \leq c_2u(x) \quad \forall x \in \mathbb{R}^d
\]

to get

\[
\mathbb{E}_x(u(\text{vec}(Y_t))) = u(x) + \mathbb{E}_x \left[ \int_0^t \mathcal{A}u(\text{vec}(Y_s)) ds \right] \\
\leq u(x) + \mathbb{E}_x \left[ \int_0^t (c_2u(\text{vec}(Y_s))) ds \right].
\]

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Applying Gronwall’s inequality (see for example [54, Chapter V, Theorem 68]) we get
\[
\mathbb{E}_x(u(\text{vec}(Y_t))) \leq u(x)e^{ct^2}\tag{5.22}
\]
for all \(t > 0\).

Overall we have
\[
\mathbb{E}_x\left(\int_0^t |A(\text{vec}(Y_s))|ds\right) \leq \mathbb{E}_x\left(\int_0^t c_2u(\text{vec}(Y_s))ds\right)
= c_2\int_0^t \mathbb{E}_x(u(\text{vec}(Y_s)))ds
\leq c_2\int_0^t (u(x)e^{ct^2})ds < \infty.
\]

To show the positive Harris recurrence of the volatility process \(Y\) and the finiteness of the \(p\)-moments of the stationary distribution we use the skeleton chains. In [23, Theorem 5.1] it is shown, that the Foster-Lyapunov condition for the extended generator, as we have shown, implies a Foster-Lyapunov drift condition for the skeleton chains. Further observe that petite sets are small, since by the assumption of irreducibility we can use the same arguments as in the upcoming proof of Theorem 4.6. With that we can apply [15, Theorem 3.12] and get the positive Harris recurrence for every skeleton chain and the finiteness of the \(p\)-moments of the stationary distribution. By definition the positive Harris recurrence for every skeleton chain implies it also for the volatility process \(Y\). \(\blacksquare\)

5.2.3 Proof of Theorem 4.6: Irreducibility and Aperiodicity

Let \(\nu\) be the Lévy measure of \(L\). We have \(\nu = \nu_{\text{ac}} + \check{\nu}\), where \(\nu_{\text{ac}}\) is the absolute continuous component and \(\nu(\mathbb{R}) < \infty\). Moreover we can split \(L\) into the corresponding processes \(L_{\text{ac}} + \tilde{L}\), where \(L_{\text{ac}}\) and \(\tilde{L}\) are independent. We set \(B_T = \{\omega \in \Omega \mid \tilde{L}_t = 0 \forall t \in [0, T]\}\). Then for any event \(A\) it holds that
\[
P(A \cap B_T) > 0 \iff P(A|B_T) > 0
\]
and \(P(A \cap B_T) > 0 \Rightarrow P(A) > 0\). So in the following we assume w.l.o.g. \(\tilde{L} = 0\) as otherwise the below arguments and the independence of \(L_{\text{ac}}\) and \(\tilde{L}\) imply that \(P(Y_t \in A \mid Y_0 = x) > 0\) results from \(P(Y_t \in A \mid Y_0 = x, B_t) > 0\).

**I) Irreducibility:**

As it was noted in Remark 5.6 to prove the irreducibility of the MUCOARCH volatility process it is enough to show it for a skeleton chain.

Let \(\delta > 0\) and set \(t_k := k\delta, \forall k \in \mathbb{N}_0\). We consider the skeleton chain
\[
Y_{tn} = e^{B_{tn}}Y_{t_0}e^{B^\top t_n} + \int_{t_{n-1}}^{t_n} e^{B(t_n-s)}A(C + Y_{s-})^{1/2}d[L, L]^d_{s}(C + Y_{s-})^{1/2}A^\top e^{B^\top(t_n-s)}.
\]

To show irreducibility w.r.t. \(\lambda_{a_d^+}\) we have to show that for any \(A \in \mathcal{B}(S_d^+)\) with \(\lambda_{a_d^+}(A) > 0\) there exists an \(l\) such that
\[
P(Y_{tl} \in A|Y_0 = y_0) > 0 \tag{5.23}
\]

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for all \( y_0 \in \mathbb{S}^+_d \). With

\[
\mathbb{P}(Y_t \in A|Y_{0} = y_0) \\
\geq \mathbb{P}(Y_t \in A, \text{ exactly one jump in every time interval } (t_0, t_1), \cdots, (t_{l-1}, t_l)|Y_{0} = y_0) \\
= \mathbb{P}(Y_t \in A|Y_{0} = y_0, \text{ exactly one jump in every time interval } (t_0, t_1), \cdots, (t_{l-1}, t_l)) \\
\cdot \mathbb{P}(\text{ exactly one jump in every time interval } (t_0, t_1), \cdots, (t_{l-1}, t_l)),
\]

(5.24)

and the fact that the last factor is strictly positive we can w.l.o.g. assume, that we have exactly one jump in every time interval \((t_k, t_{k+1})\) \(\forall k = 0, \cdots, l - 1\).

We denote by \( \tau_k \) the jump time of our Lévy process in \((t_k, t_{k+1})\). With the assumption, that we only have one jump on every time interval, the skeleton chain can be represented by the sum of the jumps, where \( L_t = \sum_{i=1}^{N_t} X_i \) is the used representation for the Compound Poisson Process \( L \). We fix the number of time steps \( l \geq d \) and get:

\[
Y_{t_l} = e^{B_{t_l}Y_0}e^{B^\top t_l} \\
+ \sum_{i=1}^{l} e^{B(t_{l-i}-\tau_i)}A(C + Y_{t_{i-1}})^{\frac{1}{2}}X_i (C + Y_{t_{i-1}})^{\frac{1}{2}}A^\top e^{B^\top (t_{i-1}-\tau_i)}.
\]

(5.25)

First we show that the sum of jumps in \((5.25)\) has a positive density on \( \mathbb{S}^+_d \). Note that every single jump is of rank one, but due to the discrete time multivariate GARCH model, the BEKK GARCH model, see \([15]\), we see, that with enough jumps we get a positive density and reach every set “above” \( e^{B(t_l)Y_0}e^{B^\top t_l} \). (Above means in the sense of the order we introduced for symmetric matrices.)

We define

\[
Z_{i}^{(l)} := e^{B(t_{l}-\tau_i)}A(C + Y_{t_{i-1}})^{\frac{1}{2}}X_i
\]

(5.26)

and with \((5.25)\) we have

\[
Z_{i}^{(l)} = e^{B(t_{l}-\tau_i)}A(C + e^{B(t_{l-1})}\sum_{j=1}^{i} e^{B(t_{j-1}-t_j)}Z_{j}^{(l)}Z_{j}^{(l)\top}e^{B^\top (t_{j-1}-t_j)})^{\frac{1}{2}}X_i.
\]

\(X_1, X_2, \cdots\) are the jump heights of the Compound Poisson process and due to our assumption they are iid, absolutely continuous w.r.t. Lebesgue measure on \( \mathbb{R}^d \) with a strictly positive density. We see immediately that

\[
Z_{1}^{(l)}|Y_0, \tau_{1}
\]

is absolutely continuous with a strictly positive density \( f_{Z_{1}^{(l)}|Y_0, \tau_{1}} \). Iteratively we get that every

\[
Z_{i}^{(l)}|Y_0, Z_{1}^{(l)}, \cdots, Z_{i-1}^{(l), \tau_{1}}
\]

is absolute continuous with a strictly positive density \( f_{Z_{i}^{(l)}|Y_0, Z_{1}^{(l)}, \cdots, Z_{i-1}^{(l), \tau_{1}}} \) for all \( i = 2, \ldots, l \).

We denote with \( f_{Z^{(l)}|Y_0, \tau_{1}, \cdots, \tau_{l}} \) the density of \( Z^{(l)} = (Z_{1}^{(l)}, \cdots, Z_{l}^{(l)})^\top \) given \( Y_0, \tau_1, \ldots, \tau_l \). Note that given \( Z_{j}^{(l)} \), \( j < i \), \( Z_{i}^{(l)} \) is independent of \( \tau_{j} \). By the rules for conditional densities we get

\[
f_{Z^{(l)}|Y_0, \tau_{1}, \cdots, \tau_{l}} = f_{Z_{1}^{(l)}|Y_0, \tau_{1}} \cdot f_{Z_{2}^{(l)}|Y_0, Z_{1}^{(l), \tau_{2}} \cdots f_{Z_{i}^{(l)}|Y_0, Z_{1}^{(l)}, \cdots, Z_{i-1}^{(l), \tau_{i}}}
\]

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is strictly positive on $\mathbb{R}^d$. Thus an equivalent measure $Q$, $Q \sim P$, exists such that $Z^{(i)}_t \mid Y_0, \tau_1, \ldots, \tau_i, \ldots, Z^{(i)}_t \mid Y_0, \tau_1, \ldots, \tau_i$ are iid normally distributed. In [1] it is shown, that for $l \geq d$

$$
\Gamma := \sum_{i=1}^{l} Z^{(i)}_t \mid Y_0, \tau_1, \ldots, \tau_i \cdot Z^{(i)\top}_t \mid Y_0, \tau_1, \ldots, \tau_i \quad (5.28)
$$

has a strictly positive density under $Q$ on $\mathbb{S}^+_d$ w.r.t. the Lebesgue measure on $\mathbb{S}^+_d$. But since $Q$ and $P$ are equivalent, $\Gamma$ has also a strictly positive density under $P$ w.r.t. Lebesgue measure on $\mathbb{S}^+_d$.

This yields

$$
P(Y_{t_i} \in A \mid Y_0 = y_0) = \int_{\mathbb{S}^+_d} P(e^{Bt_i} Y_0 e^{B\top t_i} + \Gamma \in A \mid Y_0 = y_0, \tau_1 = k_1, \ldots, \tau_l = k_l) d\mu(\tau_1, \ldots, \tau_l) > 0 \quad (5.29)
$$

if $P(e^{Bt_i} Y_0 e^{B\top t_i} + \Gamma \in A \mid Y_0 = y_0, \tau_1 = k_1, \ldots, \tau_l = k_l) > 0$. Here we use that $d\mu(\tau_1, \ldots, \tau_l)$ is not trivial, since $\tau_1, \ldots, \tau_l$, are the jump times of a Compound Poisson Process. Above we have shown that $P(e^{Bt_i} Y_0 e^{B\top t_i} + \Gamma \in A \mid Y_0 = y_0, \tau_1 = k_1, \ldots, \tau_l = k_l) > 0$ if

$$
\lambda_{\mathbb{S}^+_d} \left( A \cap \{ x \in \mathbb{S}^+_d \mid x \geq e^{Bt_i} Y_0 e^{B\top t_i} \} \right) > 0.
$$

Further note that we assumed $\sigma(B) \subset (-\infty, 0) + i\mathbb{R}$ and so $e^{Bt_i} Y_0 e^{B\top t_i}$ converges to $0$ for $t \to \infty$. Thus we can choose $l$ big enough such that $\lambda_{\mathbb{S}^+_d} \left( A \cap \{ x \in \mathbb{S}^+_d \mid x \geq e^{Bt_i} Y_0 e^{B\top t_i} \} \right) > 0$ for every $A \in \mathbb{S}^+_d$ with $\lambda_{\mathbb{S}^+_d}(A) > 0$. Since for irreducibility it is enough to have one time point $t_l$ such that $P(Y_{t_l} \in A \mid Y_0 = y_0) > 0$ holds, the proof is completed.

**II) Aperiodicity:**

Since we can show the irreducibility for every skeleton chain, we have simultaneously irreducibility, see Remark 5.6 and then by Proposition 5.11 we know that every skeleton chain is aperiodic. Using Proposition 5.12 we know for every skeleton chain, that every compact set is also small.

We define the set

$$
C := \{ x \in \mathbb{S}^+_d \mid \| x \|_2 \leq K \}, \quad (5.29)
$$

with a constant $K > 0$. Obviously $C$ is a compact set and thus a small set for every skeleton chain. But then by Remark 5.9 it is also small for the continuous time Markov Process $(Y_t)_{t \geq 0}$. To show aperiodicity for $(Y_t)_{t \geq 0}$ in the sense of Definition 5.11 we prove that there exists a $T > 0$ such that

$$
P^d(x, C) > 0 \quad (5.30)
$$

holds for all $x \in C$ and all $t \geq T$.

Using

$$
P^d(x, C) \geq P(x, C \cap \{ \text{no jump up to time } t \}) \quad (5.31)
$$

we consider $Y_t$ under the condition "no jump up to time $t". With $Y_0 = x \in C$ we have

$$
Y_t = e^{Bt} x e^{B\top t} \quad (5.32)
$$

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Since \( \lambda = \max(\text{Re}(\sigma(B))) < 0 \) there exists \( \delta > 0 \), and \( C \geq 1 \) such that \( \| e^{Bt} \|_2 \leq C e^{-\delta t} \) and hence we have
\[
\| e^{Bt} x e^{B^\top t} \|_2 \leq C e^{-2\delta t} \| x \|_2 \\
\leq C e^{-2\delta t} K \\
\leq K
\]
for all \( t \geq \frac{\ln(C)}{2\delta} \). Hence \( Y_t \in C \) for all \( t \geq \frac{\ln(C)}{2\delta} \) and thus (5.30) holds.

5.2.4 Proof of Corollary 4.7

The proof is similar to that of Theorem 4.6 with the difference that we now assume for the jump sizes \( X_t \) that they have a density, which is strictly positive in a neighborhood of zero, e.g. \( \exists k > 0 \) such that every \( X_t \) has a strictly positive density on \( \{ x \in \mathbb{R}^d : \| x \| \leq k \} \). We use the same notation as in the previous proof, but we disregard the indices \( (l) \) to simplify it. By the definition of \( Z_t \) and the same iteration as in the first case we show, that \( Z := (\tilde{Z}_1, \ldots, \tilde{Z}_l)(Y_0, \tau_1, \ldots, \tau_l) \) has a strictly positive density on a suitable null-environment.

To show that we introduce the concept of modulus of injectivity. For \( A \in M_\mathbb{R}(\mathbb{R}) \) we define
\[
j(A) := \min_{x \in \mathbb{R}^d} \frac{\| A x \|_2}{\| x \|_2}
\]
as the modulus of injectivity, which has the following properties:
\[
0 \leq j(A) \leq \| A \|_2 \quad \text{and} \quad \| A x \|_2 \geq j(A) \| x \|_2 \quad \text{as well as for} \quad A, B \in M_\mathbb{R}(\mathbb{R}) \quad j(A \cdot B) \geq j(A) \cdot j(B).
\]

With that we get for \( Z_1 \)
\[
\| Z_1 \|_2 = \| e^{B(t_1 - \tau_1)} A(C + Y_0)^{\frac{1}{2}} X_1 \|_2 \\
\geq j(e^{B(t_1 - \tau_1)} A(C + Y_0)^{\frac{1}{2}}) \| X_1 \|_2 \\
\geq j(e^{B(t_1 - \tau_1)} j(A) j((C + Y_0)^{\frac{1}{2}}) \| X_1 \|_2 \\
\geq j(e^{B(t_1 - \tau_1)} j(A) j(C^{\frac{1}{2}}) \| X_1 \|_2
\]
and thus \( Z_1 | Y_0, \tau_1 \) has a strictly positive density on \( \{ x \in \mathbb{R}^d : \| x \| \leq \hat{k} \} \), where \( \hat{k} := j(e^{B(t_1 - \tau_1)} j(A) j(C^{\frac{1}{2}}) k \). Iteratively get that every \( Z_l | Y_0, \tau_1, \ldots, Z_{l-1} \) has a strictly positive density on \( \{ x \in \mathbb{R}^d : \| x \| \leq \hat{k} \} \) and as in the first case this shows that \( Z := (\tilde{Z}_1, \ldots, \tilde{Z}_l)(Y_0, \tau_1, \ldots, \tau_l) \) has a strictly positive density on \( \{ x = (x_1, \ldots, x_l)^\top \in \mathbb{R}^{dl} : \| x_i \| \leq \hat{k} \ \forall i = 1, \ldots, l \} \).

We fix an \( \hat{k} \), \( 0 < \hat{k} < \hat{k} \) and set \( \hat{K} := \{ x = (x_1, \ldots, x_l)^\top \in \mathbb{R}^{dl} : \| x_i \| \leq k \ \forall i = 1, \ldots, l \} \). Now we can construct random variables \( \check{Z}_i, \ i = 1, \ldots, l \), such that
\[
\check{Z} := (\check{Z}_1, \ldots, \check{Z}_l)(Y_0, \tau_1, \ldots, \tau_l) \quad \text{has a strictly positive density on} \quad \mathbb{R}^{dl}
\]
and
\[
\mathbb{1}_{\check{K}} \cdot \check{Z} | Y_0, \tau_1, \ldots, \tau_l \quad \overset{D}{=} \quad \mathbb{1}_{\check{K}} \cdot Z | Y_0, \tau_1, \ldots, \tau_l. \quad (5.33)
\]

Due to the first case we now can choose a measure \( Q \) such that the \( \check{Z}_i | Y_0, \tau_1, \ldots, \tau_l, Z_1, \ldots, Z_{l-1} \) are iid normal distributed and the random variable
\[
\check{\Gamma} := \sum_{i=1}^l \check{Z}_i | Y_0, \tau_1, \ldots, \tau_l, Z_1, \ldots, Z_{l-1} \check{Z}_i^\top | Y_0, \tau_1, \ldots, \tau_l, Z_1, \ldots, Z_{l-1}
\]
has a strictly positive density on \( S_{dl}^+ \).

With the equivalence of \( \mathbb{P} \) and \( \mathbb{Q} \) also \( \mathbb{P}(\check{\Gamma} \in A) > 0 \) for every \( A \in \mathcal{B}(S_{dl}^+) \).
Further we define $\mathcal{E} := \{ x \in \mathbb{S}_d^+ : x = \sum_{i=1}^l z_i z_i^\top, z_i \in \mathbb{R}^d, \|x\|_2 \leq \hat{k}\}$ and $\mathcal{K} := \{ x \in \mathbb{S}_d^+ : \exists z_1, \ldots, z_l \in \mathbb{R}^d \text{ such that } x = \sum_{i=1}^l z_i z_i^\top \text{ and } \|z_i\| \leq \hat{k} \forall i = 1, \ldots, l\}$. Let $x = \sum_{i=1}^l z_i z_i^\top \in \mathcal{E}$. Then $x = \sum_{i=1}^l z_i z_i^\top \geq \sum_{j=1}^l z_j z_j^\top$ for all $j = 1, \ldots, l$ and thus $\|z_j z_j^\top\|_2 = \|z_j\|_2 \leq \hat{k}$, which means that $x \in \mathcal{K}$ and thereby $\mathcal{E} \subseteq \mathcal{K}$.

Now let $A \in \mathcal{B}(\mathbb{S}_d^+)$ and note that $1 \mathcal{K} \cdot \hat{\Gamma} \overset{\mathcal{P}}{=} 1 \mathcal{K} \cdot \Gamma$. Finally we get

\[
\mathbb{P}(\Gamma \in A \cap \mathcal{E}) = \mathbb{P}(\Gamma \in A \cap \mathcal{E} \cap \mathcal{K}) = \mathbb{P}(\hat{\Gamma} \in A \cap \mathcal{E}) > 0
\]

if $\lambda_{\mathbb{S}_d^+}(A \cap \mathcal{E}) > 0$.

With the same conditioning argument as in the proof of Theorem 4.6 and again using the fact that by assumption there always exists a $l$ such that $\lambda_{\mathbb{S}_d^+}(A \cap \mathcal{E} \cap \{ x \in \mathbb{S}_d^+ : x \geq e^B t_0 e^{B^\top t_1} \}) > 0$ if $\lambda_{\mathbb{S}_d^+}(A \cap \mathcal{E}) > 0$, we get irreducibility w.r.t. the measure $\lambda_{\mathbb{S}_d^+ \cap \mathcal{E}}$ defined by $\lambda_{\mathbb{S}_d^+ \cap \mathcal{E}}(B) := \lambda_{\mathbb{S}_d^+}(B \cap \mathcal{E})$ for all $B \in \mathcal{B}(\mathbb{S}_d^+)$. Aperiodicity follows as in the proof of Theorem 4.6, since we only used the compound Poisson structure of $L$ and not the assumption on the jump distribution. □

Acknowledgements

The second author gratefully acknowledges support by the DFG Graduiertenkolleg 1100.

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