Relativistic Wigner Function, Charge Variable and Structure of Position Operator *

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Abstract

The relativistic phase-space representation by means of the usual position and momentum operators for a class of observables with Weyl symbols independent of charge variable (i.e. with any combination of position and momentum) is proposed. The dynamical equation coincides with its analogue in the non-local theory (generalization of the Newton-Wigner position operator approach) under conditions when particles creation is impossible. Differences reveal themselves in specific constraints on possible initial conditions.

I. INTRODUCTION

Attempts to generalize the Weyl-Wigner-Moyal (WWM) formalism for relativistic case lead to a number of problems. The first problem is that Weyl rule does not include time as a dynamical variable, and the scalar product in Hilbert space of states is formulated not for functions square integrable over the whole space-time but in the three-dimensional space or on a space-like hyper-surface only. It leads to appearance of non Lorentz invariant formulation for the star-product and, as a consequence, to non Lorentz invariant quantum Liouville equation as well. We suppose that in the contemporary approaches this problem

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can be resolved in the framework of Tomonaga - Schwinger description of quantum field theory [1,2].

Next essential problem in relativistic WWM formalism is absence of a well-defined position operator. Indeed, even for a free particle the usual position operator is not one-particle one. Furthermore, there are some problems with semi-classical limit here. They could be resolved by introduction of one-particle well defined Newton - Wigner position operator [3], however one meets difficulties related to Lorentz invariance as well, but at the level of the wave equation. We propose formulation of the WWM formalism by means of the usual position operator but for such class of observables that their Weyl symbols are independent of the charge variable (we denominate them as the charge-invariant observables). It is possible to introduce the usual (not matrix-valued) Wigner function here. This object includes four components: two correspond to particle and antiparticle (even part of Wigner function), and two more are interference terms (odd part of Wigner function, that is to make no effect on expected values of observables because of the charge superselection rule ). Evolution equation (in the case when one-particle interpretation of relativistic quantum mechanics is possible, i.e. for a free particle and a particle in a static magnetic field) for odd part turns it in classical limit to zero, and equation for even part coincides with its analogue in the Newton - Wigner position operator [4] and in the non-local theory [3,5,6] approaches. The difference reveals itself in peculiarities of constraints on initial conditions [7].

We consider how expected values of the charge-invariant observables distinguish between in two approaches to definition of position operator. We find it very important, all the more that there are suppositions that it can be related to the existence of the preferred frame in the Universe [8]. In that case its introduction is related to attempts of correct tachyons description and, as a consequence, to a possible explanation of instant quantum correlations in relativistic case.

II. SYMBOLS, DEFINITIONS AND REPRESENTATION

Here we use the Feshbach-Villars formalism for the Klein - Gordon equation [9]. If one applies the standard change of wave function

$$\psi = \frac{1}{\sqrt{2}}(\varphi + \chi)$$

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{mc^2}{\sqrt{2}}(\varphi - \chi),$$ (1)

this equations can be written in the form of the usual Srödinger equation for two-component wave function with the Hamiltonian:

$$\hat{H} = (\tau_3 + i\tau_2)\left(\frac{\hat{p} - e\hat{A}(x)}{2m}\right)^2 + \tau_3mc^2 + e\varphi(x).$$ (2)

The transform matrix to the Feshbach-Villars (energy) representation has the following form:

$$U(\hat{p}) = \frac{1}{2\sqrt{mc^2E(\hat{p})}}[(E(\hat{p}) + mc^2) + (E(\hat{p}) - mc^2)\tau_1]$$

$$U(\hat{n}) = \frac{1}{2\sqrt{mc^2E(\hat{n})}}[(E(\hat{n}) + mc^2) + (E(\hat{n}) - mc^2)\tau_1].$$ (3)
Where $E(p)$ and $E(n)$ are the expressions for the energy of a free particle and the spectrum of relativistic rotator:

$$
E(p) = \sqrt{m^2c^4 + c^2p^2},
E(n) = \sqrt{m^2c^4 + 2mc^2\hbar \omega(n + \frac{1}{2})}.
$$

(4)

Following combination of the matrix (3) is frequently used:

$$
R(s_1, s_2) = U(s_1)U^{-1}(s_2) = \varepsilon(s_1, s_2) + \chi(s_1, s_2)\tau_1,
$$

(5)

here $s = p, n$. It contains even and odd parts and is expressed via the energy (4):

$$
\varepsilon(s_1, s_2) = \frac{E(s_1) + E(s_2)}{2\sqrt{E(s_1)E(s_2)}}
$$

$$
\chi(s_1, s_2) = \frac{E(s_1) - E(s_2)}{2\sqrt{E(s_1)E(s_2)}}.
$$

(6)

### III. MATRIX-VALUED PHASE-SPACE REPRESENTATION FOR SCALAR CHARGED PARTICLES

The matrix-valued Weyl transform of an operator $\hat{A}_{\alpha\beta}$ is determined in the following way:

$$
\hat{A}_{\alpha\beta} = \sum_{\gamma = \pm 1}^{+\infty} \int_{-\infty}^{+\infty} A_{\gamma}(p, q)\hat{W}_{\alpha}(p, q)d\rho d\sigma.
$$

(7)

$\hat{W}_{\alpha}(p, q)$ is the operator of quasi-probability density:

$$
\hat{W}_{\alpha}(p, q) = \frac{1}{(2\pi\hbar)^d} \int_{-\infty}^{+\infty} \left| q + \frac{Q}{2} \right| \delta_{\alpha\beta} e^{+\hbar Qp} dQ \left| q - \frac{Q}{2} \right|.
$$

(8)

Inverse transformation has the following form:

$$
A_{\alpha}(p, q) = \int_{-\infty}^{+\infty} \left| p + \frac{Q}{2} \right| \hat{A}_{\alpha}(q) e^{-\hbar Qp} dQ.
$$

(9)

The matrix-valued Moyal bracket can be determined in the following form:

$$
\{ \hat{A}(p, q), \hat{B}(p, q) \}_M = \frac{1}{i\hbar} \left( \hat{A}(p, q) \star \hat{B}(p, q) - \hat{B}(p, q) \star \hat{A}(p, q) \right).
$$

(10)

The matrix-valued Weyl transformation of the density matrix leads to the matrix-valued Wigner function:

$$
W_{\alpha}(p, q) = \frac{1}{(2\pi\hbar)^d} \int_{-\infty}^{+\infty} \psi_{\alpha}^{*}(q + \frac{Q}{2})\psi_{\beta}(q - \frac{Q}{2}) e^{+\hbar Qp} dQ.
$$

(11)
After standard transformations, similar to the usual WWM formalism, we obtain the quantum Liouville equation

$$\partial_t \hat{W} = \{ \hat{H}, \hat{W} \}_M.$$  \hspace{1cm} (12)

**Why this formalism is not Lorentz invariant?** Possible explanation in the framework of the Tomonaga-Schwinger approach to quantum field theory.

The fact is that average values calculated in this approach coincide with ones in the usual (Schrödinger) representation of quantum mechanics that is Lorentz invariant. Nevertheless, the scalar product is determined here with functions that are square integrable in a certain space-like hyper-surface. One can relate this hyper-surface to the measuring device frame. In other words, the wavefunction collapse occurs in a frame in which the equation (12) is written. The absence of Lorentz invariance is a consequence of the fact that the Weyl rule does not include time as an independent dynamical variable. Note, that equation (12) can be written with four-dimensional Lorentz invariant symbols only, but to do this we have to involve into consideration a certain time-like unit vector in a way similar to the Tomonaga-Schwinger approach to quantum field theory. It is the four-velocity of the frame where wavefunction reduction occurs (the measuring device frame) relative to the second static observer (watching observer). In our approach this process obeys the relativity of simultaneity. As a result, at a certain instant, in a static frame (attached to the watching observer), a state with a part of the wavefunction before measurement on one hand and a part of the wave function after measurement on another hand, is realized.

Hence we find to be very important and of fundamental value the fact that quantum mechanics in the Wigner representation necessarily includes the four-velocity of the measuring device frame in explicit form.

**IV. CHARGE-INvariant OBSERVABLES IN THE FESHBACH-VILLARS REPRESENTATION. FREE PARTICLE**

Let the matrix-valued Weyl symbol be proportional to the identity matrix:

$$A^\alpha_\beta (p, q) = A(p, q) \delta^\alpha_\beta. \hspace{1cm} (13)$$

Such symbols do not depend on the charge variable, so that the class of dynamical variables, which corresponds to these we denominate here, are more brief, as a class of charge-invariant observables. Most of dynamical variables that we consider in relativistic (non-quantum) mechanics belong to this class. The reason for this is the absence of a dependence on the charge variable in classical mechanics.

The Weyl transformation for charge-invariant observables in the Feshbach-Villars representation has the form:

$$\hat{A}^{\text{FV}}_\alpha^\beta = \frac{1}{(2\pi \hbar)^d} \int_{-\infty}^{+\infty} p + \frac{P}{2} A(p, q) R^\alpha_\beta (p + \frac{P}{2}, p - \frac{P}{2}) e^{-iPq} dP dP dq \left| p - \frac{P}{2} \right|. \hspace{1cm} (14)$$

Unlike the Newton-Wigner coordinate approach and the non-relativistic case there is a matrix-valued function (5) here.
Consequences from (14) are expressions for even $[\hat{A}]$ and odd $\{\hat{A}\}$ parts of the operator of a charge-invariant observable in terms of its Weyl symbol:

$$[\hat{A}^{\text{FV}}]_\alpha^\beta = \frac{1}{(2\pi\hbar)^d} \int_{-\infty}^{+\infty} \left| p + \frac{P}{2} \right| A(p, q) \varepsilon(p + \frac{P}{2}, p - \frac{P}{2}) \delta_\alpha^\beta e^{-\frac{i\pi P q}{\hbar}} dP dp dq \left( p - \frac{P}{2} \right), \quad (15)$$

$$\{\hat{A}^{\text{FV}}\}_\alpha^\beta = \frac{1}{(2\pi\hbar)^d} \int_{-\infty}^{+\infty} \left| p + \frac{P}{2} \right| A(p, q) \chi(p + \frac{P}{2}, p - \frac{P}{2}) \tau_1^\beta \delta_\alpha^\beta e^{-\frac{i\pi P q}{\hbar}} dP dp dq \left( p - \frac{P}{2} \right). \quad (16)$$

Then, one can obtain a formula that reconstructs the Weyl symbol of operator, with even and odd parts, in the Feshbach - Villars representation:

$$A(p, q) \delta_\alpha^\beta = \sum_{\gamma = \pm 1} \int_{-\infty}^{+\infty} R^{-1, \gamma}(p - \frac{P}{2}, p + \frac{P}{2}) \left( p + \frac{P}{2} \right) \hat{A}^{\text{FV}} \alpha^\gamma \left| p - \frac{P}{2} \right| e^{\frac{i\pi P q}{\hbar}} dP, \quad (17)$$

$$A(p, q) \delta_\alpha^\beta = \int_{-\infty}^{+\infty} \varepsilon^{-1}(p - \frac{P}{2}, p + \frac{P}{2}) \left( p + \frac{P}{2} \right) \hat{A}^{\text{FV}} \alpha^\beta \left| p - \frac{P}{2} \right| e^{\frac{i\pi P q}{\hbar}} dP, \quad (18)$$

$$A(p, q) \delta_\alpha^\beta = \sum_{\gamma = \pm 1} \int_{-\infty}^{+\infty} \chi^{-1}(p - \frac{P}{2}, p + \frac{P}{2}) \tau_1^\gamma \left( p + \frac{P}{2} \right) \hat{A}^{\text{FV}} \alpha^\gamma \left| p - \frac{P}{2} \right| e^{\frac{i\pi P q}{\hbar}} dP. \quad (19)$$

Comparing (18) and (19) we conclude that matrix elements (integral kernels) of even and odd parts of the operator of an arbitrary charge-invariant variable are uniquely related to each other due to the Weyl rule:

$$\langle p_1 | \{\hat{A}^{\text{FV}}\}_\alpha^\beta | p_2 \rangle = \frac{E(p_1) - E(p_2)}{E(p_1) + E(p_2)} \sum_{\gamma = \pm 1} \tau_1^\gamma \langle p_1 | \hat{A}^{\text{FV}} \alpha^\gamma | p_2 \rangle. \quad (20)$$

A consequence from this expression is the fact that odd part of an operator independent on position is zero. If one uses as $\hat{A}$, for example, the scalar potential of an electric field, expression (20) establishes a quantitative relationship between the effects of motion of a particle in an electric field and its interaction with polarizable vacuum (trembling motion, Zitterbewegung).

**V. CHARGE-IN Variant OBSERVABLES IN THE FESHBACH-VILLARS REPRESENTATION. CONSTANT MAGNETIC FIELD**

We will provide a similar consideration for a particle in a constant magnetic field in the energy representation. We introduce a hermitian generalization of the Wigner function:
\[ W_{nm}(p, q) = \frac{1}{(2\pi\hbar)^2} \int_{-\infty}^{+\infty} \varphi_m^*(p + \frac{P}{2}) \varphi_n(p - \frac{P}{2}) e^{-\frac{i}{\hbar} \mathcal{P} q} dP, \]  

(21)

where \( \varphi_n(p) \) is the momentum part of the eigenfunction of the Hamiltonian.

The Weyl transformation for charge-invariant observables in the energy representation has the form:

\[ A_{nm\alpha\beta}^{E} = R_{\alpha\beta}^{\gamma}(m, n) \int_{-\infty}^{+\infty} A(p, q) W_{nm}(p, q) dp dq. \]  

(22)

Unlike the non-local theory and the non-relativistic case there is a matrix-valued function here.

Consequences from (22) are expressions for even \( \hat{A} \) and odd \( \{\hat{A}\} \) parts of the operator of a charge-invariant observable in terms of its Weyl symbol:

\[ [A_{nm\alpha\beta}^{E}] = \varepsilon(m, n) \delta_{\alpha\beta} \int_{-\infty}^{+\infty} A(p, q) W_{nm}(p, q) dp dq, \]  

(23)

\[ \{A_{nm\alpha\beta}^{E}\} = \chi(m, n) \tau_{1\alpha\beta} \int_{-\infty}^{+\infty} A(p, q) W_{nm}(p, q) dp dq, \]  

(24)

Then, one can obtain a formula that reconstructs the Weyl symbol of operator, with even and odd parts, in the energy representation:

\[ A(p, q) \delta_{\alpha\beta} = \sum_{\gamma = \pm 1}^{\infty} R_{\gamma\beta}^{\alpha}(m, n) A_{nm\alpha\gamma}^{E} W_{mn}(p, q), \]  

(25)

\[ A(p, q) \delta_{\alpha\beta} = \sum_{mn}^{\infty} \varepsilon_{\alpha\beta}^{\gamma} A_{nm\alpha\gamma}^{E} W_{mn}(p, q), \]  

(26)

\[ A(p, q) \delta_{\alpha\beta} = \sum_{\gamma = \pm 1}^{\infty} \chi_{\gamma\beta}^{\alpha} \{A_{nm\alpha\gamma}^{E}\} W_{mn}(p, q). \]  

(27)

Comparing (26) and (27) we conclude that matrix elements (integral kernels) of even and odd parts of the operator of an arbitrary charge-invariant variable are uniquely related to each other due to the Weyl rule:

\[ \{\hat{A}^{FV}\}_{nm\alpha\beta} = \frac{E(m) - E(n)}{E(m) + E(n)} \sum_{\gamma = \pm 1}^{\infty} \tau_{1\gamma\beta}^{\alpha} \{\hat{A}^{FV}\}_{nm\alpha\gamma}. \]  

(28)
VI. WIGNER FUNCTION AND QUANTUM LIOUVILLE EQUATION FOR CHARGE-INVARIANT OBSERVABLES. FREE PARTICLE

It is easy to see from (9) that it is possible to introduce the usual Wigner function for the charge-invariant observables in such a way that their average values are determined by the formula:

\[ \bar{A} = \int_{-\infty}^{+\infty} A(p, q)W(p, q)dpdq. \]  

The Wigner function can be determined as the sum of the following components:

\[ W_{\alpha}^{\beta}(p, q) = \langle \psi_{\beta}| \hat{W}_{FV}^{\alpha, \beta}(p, q)|\psi_{\alpha} \rangle. \]  

And, in fact, is the average value of the operator of quasi-probability density in the Feshbach-Villars representation:

\[ \hat{W}_{FV}^{\alpha, \beta}(p, q) = \frac{1}{(2\pi\hbar)^d} \int \left| p + \frac{P}{2} \right\rangle \left( \begin{array}{c} \alpha \\beta \end{array} \right) R_{\alpha}^{\beta}(p + \frac{P}{2}, p - \frac{P}{2}) e^{-\frac{i}{\hbar}Pq}dP \left\langle p - \frac{P}{2} \right| \]  

Substituting (31) into (30), we obtain for the Wigner function components the following expressions:

\[ W_{\alpha}^{\alpha}(p, q) = \frac{1}{(2\pi\hbar)^d} \int \varepsilon(p + \frac{P}{2}, p - \frac{P}{2}) \psi_{\alpha}^{*}(p + \frac{P}{2}) \psi_{\alpha}(p - \frac{P}{2}) e^{-\frac{i}{\hbar}Pq}dP, \]  

\[ W_{\alpha}^{-\alpha}(p, q) = \frac{1}{(2\pi\hbar)^d} \int \chi(p + \frac{P}{2}, p - \frac{P}{2}) \psi_{\alpha}^{*}(p + \frac{P}{2}) \psi^{-\alpha}(p - \frac{P}{2}) e^{-\frac{i}{\hbar}Pq}dP. \]

Even components of the Wigner function (32) correspond to a charge definite state. The value of odd components (33) for such a state is zero. The expression (32) differs from analogous one for a non-relativistic Wigner function and relativistic one determined using the Newton-Wigner position operator by the function \( \varepsilon(p_1, p_2) \) under the integral sign (see (1)).

The following equations can be obtained in a standard way, through differentiating Wigner function components with respect to time:

\[ \partial_t W_{\alpha}^{\alpha}(p, q, t) = \alpha \frac{2}{\hbar} E(p) \sin \left\{ -\frac{\hbar}{2} \left( \overleftarrow{\partial_{\alpha}} \overrightarrow{\partial_{\alpha}} \right) \right\} W_{\alpha}^{\alpha}(p, q, t), \]  

\[ \partial_t W_{\alpha}^{-\alpha}(p, q, t) = i\alpha \frac{2}{\hbar} E(p) \cos \left\{ \frac{\hbar}{2} \left( \overleftarrow{\partial_{\alpha}} \overrightarrow{\partial_{\alpha}} \right) \right\} W_{\alpha}^{-\alpha}(p, q, t). \]  

Nevertheless, the Wigner function components are not independent, i.e. specific constraints are imposed on solutions of the system (34), (35):
\[
(E(p_1) - E(p_2))^2 \int_{-\infty}^{+\infty} W_+^+(q_1)W_-^-(q_1) e^{i\frac{\bar{\hbar}}{\bar{\hbar}}(q_1 - q_2)(p_1 - p_2)} dq_1 dq_2 = \\
= (E(p_1) + E(p_2))^2 \int_{-\infty}^{+\infty} W_+^-(q_1)W_-^+(q_1) e^{i\frac{\bar{\hbar}}{\bar{\hbar}}(q_1 + q_2)(p_1 - p_2)} dq_1 dq_2,
\]

(36)

\[
W^{*\alpha}(p, q) = W^\alpha(p, q),
\]

(37)

\[
W^{-\alpha}(p, q) = W^{-\alpha}(p, q).
\]

(38)

It is essential that the equation for even components of the Wigner function coincides with the analogous expression obtained for the formalism where the Newton-Wigner position operator is used. Hence, the dynamics of quasi-distribution functions for charge-definite or mixed on the charge variable states is identical in both cases.

VII. WIGNER FUNCTION AND QUANTUM LIOUVILLE EQUATION FOR CHARGE-INVARIENT OBSERVABLES. CONSTANT MAGNETIC FIELD

The Wigner function components in the case of a particle in a constant magnetic field can be determined in two ways: via the wavefunction in the energy representation

\[
W^{\alpha}(p, q) = \sum_{nm} \varepsilon(m, n) W_{nm}(p, q) C^*_m C^\alpha_n,
\]

(39)

\[
W^{-\alpha}(p, q) = \sum_{nm} \chi(m, n) W_{nm}(p, q) C^*_m C^{-\alpha}_n,
\]

(40)

and via the wavefunction in the representation of the non-local theory

\[
W^{\alpha}(p, q) = \frac{1}{(2\pi\bar{\hbar})^d} \int \varepsilon(p + \frac{P}{2}, p_1; p - \frac{P}{2}, p_2) \psi^{\alpha}(p_1) \psi^{\alpha}(p_2) e^{-i\bar{\hbar} Pq} dP dp_1 dp_2,
\]

(41)

\[
W^{-\alpha}(p, q) = \frac{1}{(2\pi\bar{\hbar})^d} \int \chi(p + \frac{P}{2}, p_1; p - \frac{P}{2}, p_2) \psi^{-\alpha}(p_1) \psi^{-\alpha}(p_2) e^{-i\bar{\hbar} Pq} dP dp_1 dp_2.
\]

(42)

Here we have introduced the following generalized functions:

\[
\varepsilon(p', p_1; p''', p_2) = \sum_{nm} \varepsilon(m, n) \varphi^*_m(p') \varphi_m(p_1) \varphi_n(p'') \varphi^*_n(p_2),
\]

(43)

\[
\chi(p', p_1; p''', p_2) = \sum_{nm} \chi(m, n) \varphi^*_m(p') \varphi_m(p_1) \varphi_n(p'') \varphi^*_n(p_2).
\]

(44)

The evolution equations can be obtained in a standard way, through applying the well-known expressions for a Hermitian generalization of the Wigner function:
Here we have introduced the following generalized functions:

\[ \psi \]

It should be noticed that the square root is defined here by means of the star-product behavior, not related to the complicated charge structure of the position operator, and it leads to the appearance of specific peculiarities in a relativistic quantum particle analog of the expression (36) has the following form:

\[ \sqrt{E(p, q) = \sqrt{m^2 c^4 + e^2 (p - eA(q))^2}}. \]  

Here we have introduced the following effective Hamiltonian:

\[ E(p, q) = \sqrt{m^2 c^4 + e^2 (p - eA(q))^2}. \]

It should be noticed that the square root is defined here by means of the star-product \( \star \equiv e^{\frac{\Phi}{2}}(\frac{\partial_q \overrightarrow{p}}{-\overrightarrow{p}} - \frac{\partial_p \overrightarrow{q}}{-\overrightarrow{q}}) \). This is a common feature for the both usual and non-local theories and it leads to the appearance of specific peculiarities in a relativistic quantum particle behavior, not related to the complicated charge structure of the position operator.

Similar to the free particle case, one can find specific constraints on solutions of the system (15), (16). The expressions (37) and (38) are left without modification, and the analog of the expression (39) has the following form:

\[ \begin{align*}
\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} & \varepsilon^{-1}(p_1, p_2, p_3)W_{++}(\frac{1}{2}(p_1 + p_2), q_1) e^{\frac{i}{\hbar}(p_1 - p_2)q} dp_1 dp_2 dq \\
\times & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \chi^{-1}(p_1, p_2, p_3)W_{--}(\frac{1}{2}(p_1 + p_2), q_2) e^{\frac{i}{\hbar}(p_1 - p_2)q} dp_1 dp_2 dq \\
= & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \chi^{-1}(p_1, p_2, p_3)W_{--}(\frac{1}{2}(p_1 + p_2), q_1) e^{\frac{i}{\hbar}(p_1 - p_2)q} dp_1 dp_2 dq \\
\times & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \chi^{-1}(p_1, p_2, p_3)W_{--}(\frac{1}{2}(p_1 + p_2), q_2) e^{\frac{i}{\hbar}(p_1 - p_2)q} dp_1 dp_2 dq.
\end{align*} \]  

Here we have introduced the following generalized functions:

\[ \varepsilon^{-1}(p_1, p_2, p_3) = \sum_{nm} \varepsilon^{-1}(m, n) \varphi^*_m(p_1) \varphi_m(p_2) \varphi_n(p_3), \]  

\[ \chi^{-1}(p_1, p_2, p_3) = \sum_{nm} \chi^{-1}(m, n) \varphi^*_m(p_1) \varphi_m(p_2) \varphi_n(p_3). \]

**VIII. Statistical Properties of the Wigner Function for Charge-Invariant Observables. Free Particle**

Constraint on the initial conditions of the Wigner function is the general peculiarity of the approach described here because equations are identical in both cases (for charge definite states). In this Section we show how some theorems and properties differ from their analogues in the usual WWM formalism and in approach where the Newton - Wigner position operator is used.
• **The property of normality.** Even part of the Wigner function \( \mathcal{W}^{\alpha}(p) \) is normalized in the whole phase space, and integral of the odd part \( \mathcal{W}^{-\alpha}(p) \) is zero.

• **The compatibility of the Wigner function** (32) with distributions in the coordinate and momentum spaces for a charge-definite state.

\[
\mathcal{W}^{\alpha}(p) = \psi^{\ast}_{\alpha}(p)\psi^{\alpha}(p),
\]

\[
\mathcal{W}^{-\alpha}(q) = \psi^{\ast}_{\alpha}(q)\varepsilon\left(i\hbar \overleftarrow{\partial}_q, i\hbar \overrightarrow{\partial}_q\right)\psi^{\alpha}(q),
\]

where \( \psi_{\alpha}(q) \) is the wavefunction in the representation of the Newton-Wigner coordinate.

• **\( n \)-th moment of the coordinate** can be written as follows:

\[
\langle q^n \rangle = \int_{-\infty}^{+\infty} \left\{ \psi_{\alpha}^{\ast}(p) \left[i\hbar \overleftarrow{\partial}_p\right]^n \psi^{\alpha}(p) \varepsilon(p, p') \right\}_{p'=p} dp.
\]

The first moment (average coordinate) has a value similar to one in the Newton-Wigner coordinate approach. Differences manifest themselves in higher moments.

• **Criterion of pure state.** For the functions \( \mathcal{W}^{\alpha}(p, q) \) and \( \mathcal{W}^{-\alpha}(p, q) \) to be even and odd components of the Wigner function for charge-invariant observables, it is necessary and sufficient that equalities (36), (37), (38) hold true, and the following conditions are satisfied:

\[
\frac{\partial^2}{\partial p_1 \partial p_2} \ln \mathcal{W}^{\alpha}(\frac{1}{2}(p_1 + p_2), q)e^{\frac{\hbar}{2i}(p_1 - p_2)q} dq = -\frac{c^4 p_1 p_2}{E(p_1)E(p_2)(E(p_1) + E(p_2))^2},
\]

\[
\frac{\partial^2}{\partial p_1 \partial p_2} \ln \mathcal{W}^{-\alpha}(\frac{1}{2}(p_1 + p_2), q)e^{\frac{\hbar}{2i}(p_1 - p_2)q} dq = -\frac{c^4 p_1 p_2}{E(p_1)E(p_2)(E(p_1) - E(p_2))^2}.
\]

• **Criterion of pure and mixed charge-definite state.**

\[
\int_{-\infty}^{+\infty} \mathcal{W}^{\alpha}(p, q)e^{-2 \left(p + \frac{\hbar}{2i} \overleftarrow{\partial}_q, p + \frac{\hbar}{2i} \overrightarrow{\partial}_q\right) \mathcal{W}^{\alpha}(p, q)} dp dq \leq \frac{1}{(2\pi\hbar)^d}.
\]

For a pure state this inequality turns into an equality.
IX. STATISTICAL PROPERTIES OF THE WIGNER FUNCTION FOR CHARGE-INVARIIANT OBSERVABLES. CONSTANT MAGNETIC FIELD

• The property of normality. Even part of the Wigner function (39), (41) is normalized in the whole phase space, and integral of the odd part (40), (42) is zero.

• The compatibility of the Wigner function (39), (41) with distributions in the coordinate and momentum spaces for a charge-definite state.

\[
W_\alpha^\alpha(p) = \psi_\alpha^*(p)\varepsilon\left(\hat{n}, \frac{\hat{p}}{\hbar}\right)\psi_\alpha(p), \quad (57)
\]

\[
W_{\alpha}^{(-\alpha)}(q) = \psi_\alpha^*(q)\varepsilon\left(-\hat{n}, \frac{-\hat{p}}{\hbar}\right)\psi_\alpha(q), \quad (58)
\]

Here \(\psi_\alpha(q)\), \(\psi_\alpha(p)\) is the wavefunction in representation of the non-local theory.

• \(n\)-th moment of the coordinate and momentum can be written as follows:

\[
\langle q^s \rangle = \sum_{nm} q^s \varepsilon_{mn} C^\alpha_{mn} C_n^\alpha, \quad (59)
\]

\[
\langle p^s \rangle = \sum_{nm} p^s \varepsilon_{mn} C^\alpha_{mn} C_n^\alpha. \quad (60)
\]

The expressions for the first moments (average coordinate and momentum) are not similar to those in the non-local theory.

• Criterion of pure state. For the functions \(W_\alpha^\alpha(p,q)\) and \(W_{\alpha}^{(-\alpha)}(p,q)\) to be even and odd components of the Wigner function for charge-invariant observables, it is necessary and sufficient that equalities (37), (38), (48) hold true, and the following conditions are satisfied:

\[
\frac{\partial^2}{\partial p_1 \partial p_2} \ln \int_{-\infty}^{+\infty} \varepsilon^{-1}(p', p_1; p'', p_2) W_\alpha^\alpha\left(\frac{1}{2}(p' + p''), q\right) e^{i\pi(p' - p'')q} dq dp' dp'' = 0, \quad (61)
\]

\[
\frac{\partial^2}{\partial p_1 \partial p_2} \ln \int_{-\infty}^{+\infty} \chi^{-1}(p', p_1; p'', p_2) W_{\alpha}^{(-\alpha)}\left(\frac{1}{2}(p' + p''), q\right) e^{i\pi(p' - p'')q} dq dp' dp'' = 0. \quad (62)
\]

X. CONCLUSIONS

1. The usual (not Lorentz invariant) Weyl rule makes it possible to introduce Wigner function that is not Lorentz invariant, but all expected values calculated with it coincide with ones calculated with Lorentz invariant wave function. This results in fact that quantum mechanics in the Wigner formulation contains with necessity a measuring device frame.
2. Phase space for a scalar charged particle is not only limited by three couples of the momenta and coordinates. The charge dimension exists as well. However, in the approach presented here we leave the operator nature of such variables without modification. As a result, the matrix-valued Wigner function is the density matrix in charge space with standard rules of expected values calculation.

3. If we limit our consideration only to such elements of dynamical algebra that do not depend on variables of the charge space, it is possible to introduce the usual Wigner function. This object differs from the Wigner function for non-relativistic particle and from Wigner function in the Newton - Wigner position operator approach as well. Under conditions when creation of particles is impossible, the evolution equations coincide in the both cases. Differences reveals themselves in the constraint on possible initial conditions of the Wigner function.

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