Deriving Telescope Mueller Matrices Using Daytime Sky Polarization Observations

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ABSTRACT. Telescopes often modify the input polarization of a source so that the measured circular or linear output state of the optical signal can be significantly different from the input. This mixing, or polarization “cross talk,” is defined by the optical system Mueller matrix. We describe an efficient method here for recovering the input polarization state of the light and the full $4 \times 4$ Mueller matrix of the telescope with an accuracy of a few percent without external masks or telescope hardware modification. Observations of the bright, highly polarized daytime sky using the Haleakala 3.7 m AEOS telescope and a coudé spectropolarimeter demonstrate the technique.

Online material: color figures

1. INTRODUCTION

Spectropolarimetry is a powerful tool that may be enhanced by efficient techniques for telescope-detector polarization calibration. Unfortunately, it is common for modern altitude-azimuth telescopes with multiple mirror reflections to scramble the input polarization of the light before it reaches the analyzer of a coudé, Nasmyth, or Gregorian polarimeter. Solar astronomers, in particular, have developed a repertoire of techniques for calibrating telescopes (Elmore et al. 1992, 2010; Kuhn et al. 1993; Giro et al. 2003; Socas-Navarro 2005a, 2005b; Socas-Navarro et al. 2005, 2006; Selbing 2005; Ichimoto et al. 2008; Keller & Snik 2009). Our experience with the Advanced Electro-Optical System (AEOS) 3.67 m telescope is that linearly polarized light can even, in some circumstances, be transformed to nearly 100% circularly polarized light. There are seven highly oblique reflections before polarization is analyzed using our high-resolution visible and infrared spectropolarimeter (HiVIS) (e.g., Hodapp et al. 2000; Thornton et al. 2003; Harrington et al. 2006, 2009, 2010; Harrington & Kuhn 2007, 2008, 2009a, 2009b).

Effective calibration of the telescope and instrument minimizes these systematic errors and enhances the polarimetric precision of an observation. Moving optical elements, detector effects such as nonuniform sensitivity or nonlinearity, atmospheric seeing, and transparency variations can all induce polarization errors that are often larger than the photon noise. Strategies to minimize, stabilize, and calibrate the instrument and telescope often involve use of nonmoving optics and emphasize rapid chopping and common-path differential techniques. Calibration techniques include observing unpolarized and polarized standard stars and use of polarized optical calibration units that are designed to mimic the nominal illumination of the telescope optics during target observations.

The new technique presented here is designed to yield the full Mueller matrix calibration of many telescope-polarimeter systems with an accuracy of a few percent. It requires only two or more observations of the linearly polarized sky due to solar Rayleigh scattering. The daytime sky is normally easily observable, highly polarized, and is a relatively well characterized calibration source. It is observable without using precious dark observing time and requires no hardware calibration modifications of the telescope. As long as multiple linear-polarization input states are detected at each telescope pointing, the full telescope Mueller matrix can be recovered with an accuracy of a few percent. This is realized by observing the solar-illuminated sky with the Sun at different zenith and azimuth positions. This article illustrates the technique using the Haleakala Air Force AEOS 3.7 m telescope on Maui.

1.1. Polarization

The following discussion of polarization formalism closely follows Collet (1992) and Clarke (2009). In the Stokes formalism, the polarization state of light is denoted as a 4-vector:

$$S_i = [I, Q, U, V]^T.$$  \hfill (1)

In this formalism, $I$ represents the total intensity, $Q$ and $U$ are the linearly polarized intensity along polarization position angles $0^\circ$ and $45^\circ$ in the plane perpendicular to the light beam, and $V$ is the right-handed circularly polarized intensity. Note that according to this definition, linear-polarization along angles $90^\circ$ and $135^\circ$ will be denoted as $-Q$ and $-U$, respectively. The typical convention for astronomical polarimetry is for the $+Q$ electric field vibration direction to be aligned...
north-south, while $+U$ has the electric field vibration direction aligned to north-east and south-west.

The normalized Stokes parameters are denoted with lowercase and are defined as

$$[1, q, u, v]^T = [I, Q, U, V]^T / I.$$  \hspace{1cm} (2)  

The degree of polarization can be defined as a ratio of polarized light to the total intensity of the beam:

$$P = \frac{\sqrt{Q^2 + U^2 + V^2}}{I} = \sqrt{q^2 + u^2 + v^2}.$$  \hspace{1cm} (3)  

For details on polarization of light and stellar spectropolarimetry, see Collet (1992) and Clarke (2009).

To describe how polarized light propagates through any optical system, the Mueller matrix is constructed, which specifies how the incident polarization state is transferred to the output polarization state. The Mueller matrix is a $4 \times 4$ set of transfer coefficients that when multiplied by the input Stokes vector ($\mathbf{S}_{\text{input}}$) gives the output Stokes vector ($\mathbf{S}_{\text{output}}$):

$$\mathbf{S}_{\text{output}} = \mathbf{M}_{ij} \mathbf{S}_{\text{input}}.$$  \hspace{1cm} (4)  

If the Mueller matrix for a system is known, then one inverts the matrix and deprojects a set of measurements to recover the inputs. One can represent the individual Mueller matrix terms as describing how one incident state transfers to another. In this work we will use the notation

$$\mathbf{M}_{ij} = \begin{pmatrix} I & Q & U & V \\ I & Q & Q & U & V \\ U & Q & U & V \\ I & V & U & V \end{pmatrix}.$$  \hspace{1cm} (5)  

### 1.2. Deriving Telescope Mueller Matrix Elements

Typical calibration schemes on smaller telescopes often use fixed polarizing filters placed over the telescope aperture to provide known input states that are detected and yield terms of the Mueller matrix. This approach has been used for solar telescopes (Socas-Navarro 2005a, 2005b; Socas-Navarro et al. 2005, 2006). Another approach also used in the solar observations has been to image known sources and use spectropolarimetric data and Zeman effect symmetry properties to isolate and determine terms in the Mueller matrix (Kuhn et al. 1993; Elmore et al. 2010). Night-time observations can use unpolarized and polarized standard stars to measure polarization properties of telescopes (e.g., Hsu & Breger 1982; Schmidt et al. 1992; Gil-Hutton & Benavidez 2003; Fossati et al. 2007). Many studies have either measured and calibrated telescopes, measured mirror properties, or attempted to design instruments with minimal polarimetric defects (e.g., Sánchez Almeida et al. 1991; Giro et al. 2003; Patat & Romaniello 2006; Tinbergen 2007; Joos et al. 2008; van Harten et al. 2009; Roelfsema et al. 2010).

### 1.3. The Polared Sky as a Calibration

The observed polarization of the daytime sky is a useful calibration source. Scattered sunlight is bright, highly (linearly) polarized, and typically illuminates the telescope optics more realistically than calibration screens. A single-scattering atomic Rayleigh calculation (e.g., Coulson 1980, 1998) is often adequate to describe the skylight polarization. More realistic modeling, involving multiple scattering and aerosol scattering, is also available using industry-standard atmospheric radiative transfer software packages such as MODTRAN (e.g., Fetrow et al. 2002; Berk et al. 2006).

There are many atmospheric and geometric considerations that determine the skylight polarization at a particular observatory site. The linear-polarization amplitude angle can depend on the solar elevation, atmospheric aerosol content, aerosol vertical distribution, aerosol scattering phase function, wavelength of the observation, and secondary sources of illumination (e.g., Horváth et al. 2002a, 2002b; Lee 1998; Liu & Voss 1997; Suhai & Horváth 2004; Gál et al. 2001; Vermeulen et al. 2000; Pomozi et al. 2001; Cronin et al. 2005, 2006; Hegedüs et al. 2007). Anisotropic scattered sunlight from reflections off land or water can be highly polarized and temporally variable (Litvinov et al. 2010; Peltoniemi et al. 2009; He et al. 2010; Salinas & Liew 2007; Ota et al. 2010; Kisselev & Bulgarelli 2004). Aerosol particle optical properties and vertical distributions also vary (e.g., Wu & Jin 1997; Shukurov & Shukurov 2006; Vermeulen et al. 2000; Ougolnikov & Maslov 2002, 2005a, 2005b, 2009a, 2009b; Ugonokov et al. 2004). The polarization can change across atmospheric absorption bands or can be influenced by other scattering mechanisms (e.g., Boesche et al. 2006; Zeng et al. 2008; Aben et al. 1999, 2001).

Deviations from a single-scattering Rayleigh model grow as the aerosol, cloud, ground, or sea-surface scattering sources affect the telescope line of sight. Clear, cloudless, low-aerosol conditions should yield high linear-polarization amplitudes and small deviations in the polarization direction from a Rayleigh model. Observations generally support this conclusion (Pust & Shaw 2005, 2006a, 2006b, 2007, 2008, 2009; Shaw et al. 2010).

The geometry of our Rayleigh sky model is shown in Figure 1. The geometrical inputs are the observer’s location (latitude, longitude, and elevation) and local time. The solar location and relevant angles from the telescope pointing are computed from the spherical geometry in Figure 1. The maximum degree of polarization ($\delta_{\text{max}}$) in this model occurs at a scattering angle ($\gamma$) of $90^\circ$. The Rayleigh sky model predicts the degree of polarization ($\delta$) at any telescope pointing as

$$\delta = \frac{\delta_{\text{max}} \sin^2 \gamma}{1 + \cos^2 \gamma}.$$  \hspace{1cm} (6)  

Since the angle of polarization in the Rayleigh sky model is always perpendicular to the scattering plane, one can derive the angle of polarization with respect to the altitude and azimuth.
axes of the telescope. The law of cosines for the scattering plane
breaks down at the zenith, where $\theta$ is 0° and one must simply
calculate the difference in azimuth between the telescope
pointing and the solar azimuth to find the orientation of the
polarization. An example of the model we compared with
our observations is shown in Figure 2 for Haleakala on 2010
January 27.

One set of all-sky polarization measurements obtained at the
Mauna Loa Observatory, a nearby site, and at very similar alti-
tude to Haleakala (Dahlberg et al. 2009), is particularly relevant.
The maximum degree of polarization ($\delta_{\text{max}}$) changed throughout
the day in their observations from a value of $\sim 60\%$ with the Sun
at an elevation of 30° to 85% with the Sun setting. Although this
is a significant change in the maximum degree of polarization ($\delta_{\text{max}}$), presumably due to multiple scattering, the polarization
direction at all telescope pointings is expected to be close to the
Rayleigh model (e.g., Dahlberg et al. 2009; Suhai & Horváth
2004; Pust & Shaw 2009; Pomozi et al. 2001). In our subse-
quent analysis we will use the Rayleigh model to define the in-
put polarization angle. We believe our assumed input angle
is accurate over most of the sky to less than 5° (e.g., Suhai &
Horváth 2004).

1.4. Using the Sky for Full Stokes Calibration

Despite the fact that the Rayleigh sky only provides linearly
polarized light to the telescope, with reasonable assumptions,
we can recover the full Mueller matrix. We find that the tele-
scope and spectrograph only weakly polarize unpolarized input
light, and the optical system has only a weak depolarization ef-
flect on polarized input light. Thus, the first row and column of
the system Mueller matrix can be approximated by the corre-
sponding row and column of the identity matrix. Direct mea-
surements of these terms using stellar observations with AEOS
confirm that over the visible spectrum these Mueller matrix
terms are less than 5%. With this assumption the telescope sys-
tem scrambles the input polarization simply by rotating the
input four-element Stokes vector in the 3-space defined along
the $+Q$ direction, and $+U$ is defined as E-field vibrations along
the $+Q$, $+U$, and $+V$ polarization axes to yield the output mea-
sured state.

In general, only two measurements with different input linear-polarization orientations are sufficient to determine the
telescope polarization properties. In practice, we define a
least-squares problem that takes several input polarization mea-
surements of the sky at different times, but with identical optical
configurations, to derive telescope properties.

2. POLARIZED SKY OBSERVATIONS

To demonstrate this technique, we collected sky observations
with AEOS using the new low spectral resolution mode for our
coûde spectrograph we call LoVIS. This new spectrograph has
been characterized to verify the expected performance using
standard stars, calibration optics, and various tests outlined in
Appendix C. The LoVIS system polarization properties most
relevant to this work include the telescope and instrument-induced polarization, depolarization, and the polarimetric response of LoVIS. These are briefly described subsequently and in Appendix C.

The measured degree of polarization for the many LoVIS sky observations we have obtained shows that there is relatively low instrument depolarization. For example, on 2010 January 27 UT we have 20 sky polarization observations with the telescope pointed at the zenith taken over 3 hr, ending just after sunset. The measured degree of polarization averaged over all wavelengths is shown in Figure 3 as the solid line. Following Dahlberg et al. (2009, Fig. 7), we assume the maximum degree of polarization ($\delta_{\text{max}}$) for the Rayleigh sky as roughly 65% when the solar elevation is 30°, 75% at 15°, and 85% at 0°. If we apply this simple linear relation for $\delta_{\text{max}}$ and compute the Rayleigh sky polarization ($\delta$) at the zenith for our observations, we get the dashed curve in Figure 3. The figure also shows the scattering angle ($\gamma$) between the zenith telescope pointing and the Sun.

The computed degree of polarization changes with wavelength for all 20 observations on January 27 is shown in Figure 4. The average polarization shown in Figure 3 was removed from the spectra, leaving only residual chromatic changes. There is a mild trend with wavelength for the measured polarization to decrease by about 10% from 5500 Å to 7500 Å. The atmospheric oxygen absorption band shows a mild increase in polarization, and there is also a substantial change in the chromatic trend seen in the final observation taken just after sunset, as shown by the dashed line in Figure 4, consistent with studies of the occultation and changing illumination (e.g., Ougolnikov et al. 2003, 2004).

Given that the AEOS telescope and LoVIS spectropolarimeter induce polarization of less than 3% and the depolarization is of the same order, we can use a $3 \times 3$ Mueller matrix approximation. In this $3 \times 3$ $QUV$ space, the degree of polarization is preserved for any input state, and the $3 \times 3$ Mueller matrix would be approximated as a rotation matrix inside the Poincaré sphere.

3. Telescope Polarization as Rotation

To model the $3 \times 3$ Mueller matrix as a rotation matrix, we will use Euler angles. We scaled all our measured Stokes vectors to unit length in order to remove the residual effects from changes in the sky degree of polarization, telescope-induced polarization, and depolarization and to put our measurements on the Poincaré sphere. We denote the three Euler angles as $\alpha$, $\beta$, and $\gamma$ and use a shorthand notation for sines and cosines, where $\cos(\gamma)$ is shortened to $c_\gamma$. We specify the rotation matrix ($R_{ij}$) using the $ZXZ$ convention as

$$R_{ij} = \begin{pmatrix} c_\gamma & s_\gamma & 0 \\ -s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\beta & s_\beta \\ 0 & -s_\beta & c_\beta \end{pmatrix} \begin{pmatrix} c_\alpha & s_\alpha & 0 \\ -s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

With this definition for the rotation matrix, we solve for the Euler angles, assuming a fully linearly polarized daytime sky as calibration input. If we denote the measured Stokes parameters, $S_i$, as $(q_m, u_m, v_m)$ and the input sky Stokes parameters, $S_i$, as $(q_0, u_0, v_0)$, we have

$$S_i = R_{ij}S_j.$$
As \((q_i, u_i, 0)\), then the \(3 \times 3\) \(QUV\) Mueller matrix elements at each wavelength are

\[
S_i = \begin{pmatrix}
q_m \\
u_m \\
v_m
\end{pmatrix} = M_j R_j = \begin{pmatrix}
QQ & UQ & VQ \\
QU & UU & VU \\
QV & UV & VV
\end{pmatrix} \begin{pmatrix}
q_i \\
u_i \\
0
\end{pmatrix}. \tag{8}
\]

This set of equations has six variables and only five known quantities. We have no \(V\) input to constrain the \(VQ, UV,\) and \(VV\) terms. Nevertheless, two input and output vectors on the Poincaré sphere are sufficient to fully specify the full \(3 \times 3\) rotation matrix. Thus, we use the fact that the sky polarization changes orientation with time and take measurements at identical telescope pointings separated by enough time for the solar sky illumination to change. This yields an overconstrained solvable problem for all six linear-polarization terms in the Mueller matrix.

When using this rotation-matrix approximation for the telescope Mueller matrix, the Rayleigh sky input Stokes parameters multiply each term of the rotation matrix to give a system of equations for the three Euler angles (\(\alpha, \beta,\) and \(\gamma\)). This system of equations can be solved using a normal nonlinear least-squares minimization by searching the \((\alpha, \beta, \gamma)\) space for minima in squared error. This direct solution of this set of equations using standard minimization routines is subject to several ambiguities that affect convergence using standard minimization routines. For example, Euler angles are symmetric under the exchange of \(\alpha, \beta,\) and \(\gamma\) with \(-180^\circ + \alpha, -\beta\) and \(-180^\circ + \gamma\). Solutions are also identical with multiples of \(2\pi\). As a convenient approach to this least-squares problem, we used a different set of equations in a two-step solution. We first perform simple least-squares solution directly for the six linear-polarization Mueller matrix elements. In the second step we fit a rotation matrix to the six Mueller matrix elements at each wavelength. As an example, Figure 5 shows the Mueller matrix elements derived with the 2010 January 27 observations. The details of our methods for deriving Euler angles and an example of how one could plan sky calibration observations are outlined in Appendices A and B.

### 3.1. Euler Angle Solution Properties

The Euler angles are well-behaved smooth functions of both wavelength and telescope pointing. For instance, the Euler angle variations with wavelength calculated from 2010 January 27 observations at a telescope pointing of at 90° altitude and 225° azimuth are shown in Figure 6. The change with wavelength is dominated by \(\beta\), while smaller variation is seen in \(\alpha\) and \(\gamma\). For clarity in the figure, the Euler angle values at 6560 Å have been removed so that all curves overlap. Similar curves are seen for other telescope pointings.

The Euler angles are also smooth functions of telescope pointing. We have observations on multiple days taken on a grid of eight azimuths and three different altitudes. Figure 7 shows the derived Euler angles at 6560 Å as a functions of azimuth. There are three different curves corresponding to the three different telescope altitudes. We find that \(\gamma\) effectively absorbs the changing azimuth, while \(\alpha\) strongly varies with altitude. Since there is substantial telescope-induced rotation, these curves show significant deviations from a purely geometrical rotation.

The derived Euler angles in Figure 6 are quite repeatable when derived from calibration observations taken on different days. For instance, we have two sets of sky observations taken

**Figure 5.**—Six Mueller matrix elements vs. wavelength at a telescope pointing of 90° altitude and 225° azimuth. These were derived from three sky measurements on 2010 January 27 using the two-step solution. The \(UV\) term grows with increasing wavelength, while the \(QU\) term decreases with wavelength. The other Mueller matrix elements remain near 0 or 1 and show smaller chromatic variation.

**Figure 6.**—Variation in Euler angles with wavelength at a telescope pointing of 90° altitude and 225° azimuth. The value for each Euler angle at 6560 Å have been removed for clarity \((\alpha = 176.5°, \beta = 81.6°,\) and \(\gamma = 98.8°)\). The change with wavelength is dominated by \(\beta\), which corresponds to the changes in the \(QU\) and \(UV\) Mueller matrix elements from Fig. 5. The statistical noise is smaller than the width of the line.
2009 December 10 and 11, each consisting of three observation sets spaced roughly 1 hr apart. The Euler angles derived on different days agree to within 0.5°, giving an estimate of the systematic error limits. The Mueller matrix elements derived with observations from different days agree to better than 0.01. The changes in $\beta$ with telescope azimuth shown in Figure 7 are well above the systematic noise of roughly 0.5° and are repeatable even when using many different combinations of calibration data on both days. Similar trends for Euler angle variation with telescope pointing are seen at other wavelengths.

### 3.2. Calibrating Spectropolarimetric Observations

Once telescope calibrations have been derived, observations subsequently taken with the instrument at the same telescope pointing can be derotated. The calibrated polarization measurements are then oriented in a reference frame on the sky that is corrected for all geometric, telescope, and instrument-induced rotation. We have done many experiments where sky observations on some days are used to calibrate sky observations taken on other days to examine the stability and repeatability of the calibrations.

As an example, Figure 8 shows six individual sky spectropolarimetric measurements taken on 2009 December 10 and 11 at different times of the afternoon. All observations were taken at a telescope pointing of azimuth 021° and altitude 70°. The input linear-polarization orientation changes with time, and there is significant chromatism and cross talk observed in all Stokes parameters. The three observations from each day are taken at similar times, and the observed polarization spectra are similar.

In order to illustrate a general method of telescope calibration, imagine a scenario where we derive a set of calibrations using observations from one day and then use this telescope calibration to derotate observations taken on another day at an identical telescope pointing. We will treat the 2009 December 10 observations as a calibration set and derive Euler angles for all telescope pointings and wavelengths. With these Euler angles in hand, we will then treat observations taken on 2009 December 11 as our science targets. The derotated science observations will be compared against the Rayleigh sky prediction. This scenario is exactly what a typical observer would perform during routine operations. The difference between the derotated December 11 observations and the expected sky input Stokes parameters are shown in Figure 9. The residual error is typically less than 0.01 for any Stokes parameter, showing that the derotated observations match the expected sky model with very high accuracy.

We get consistent results for Euler angles using many different combinations of observations taken between 2009 August and 2010 March. Over this time period there were changes to the optical configuration, such as remounting of the retarders, switching gratings, and changing LoVIS/HiVIS modes, as would be expected during normal observing operations. If one derives a residual rotation angle between the derotated measurements and the predicted sky, the residual rotation angles are almost always less than 3° when using any calibration set against any other observation set. If one uses calibration data where there are no optical changes between calibration and observation, this resulting angular error is much smaller, typically less than 0.5° rotation. This allows us to estimate systematic errors.

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*Fig. 7.* Variation in derived Euler angles at 6560 Å as a function of telescope pointing. Top: $\alpha$ for all telescope azimuths at elevations of 90° (solid line), 75° (dashed line), and 55° (dot-dashed line). Middle: $\beta$. Bottom: $\gamma$ for all azimuths at the same elevations as $\alpha$ using the same line scheme. This figure clearly shows that $\alpha$ changes the strongest with elevation, while $\gamma$ shows changes with azimuth. There is not much change seen in $\beta$.

*Fig. 8.* Measured spectropolarimetry with LoVIS while the telescope is fixed at altitude 70° and azimuth 021° scaled to 100% polarization. Three observations were taken on 2009 December (solid lines) and another three observations were taken at similar times on 2009 December 11 (dashed lines). The statistical noise is smaller than the width of the line.
error from optical configuration rotation changes as a few degrees of residual rotation. When observing as calibrated, the errors are under 1° QUV rotation.

4. DISCUSSION

We have demonstrated a simple method for deriving telescope polarization properties and Mueller matrix elements. By using observations of the bright, highly polarized daytime sky and taking observations at multiple times using identical optical configurations, one can extract the polarization response of the telescope.

APPENDIX A

SOLUTIONS FOR EULER ANGLES

There are a number of methods available to solve for Euler angles and Mueller matrix elements given a set of sky polarization observations. Suppose that we have measurements at different times but at identical telescope pointings. Multiplying out terms in equation (7) for the ZXZ convention, we get

\[
\mathbf{R}_{ij} = \begin{pmatrix}
 c_\alpha c_\gamma - s_\alpha c_\beta s_\gamma & s_\alpha c_\gamma + c_\alpha c_\beta s_\gamma & s_\beta s_\gamma \\
 -c_\alpha s_\gamma - s_\alpha c_\beta c_\gamma & -s_\alpha s_\gamma + c_\alpha c_\beta c_\gamma & s_\beta c_\gamma \\
 s_\alpha s_\beta & -c_\alpha s_\beta & c_\beta 
\end{pmatrix}.
\]  \tag{A1}

Then equating Mueller matrix elements to rotation-matrix elements, we could write the system of equations for the three Euler angles as

\[
q_m = q_{i1}(+c_\alpha c_\gamma - c_\beta s_\gamma) + u_{i1}(+c_\alpha s_\gamma + c_\beta c_\gamma),
\]

\[
u_m = u_{i1}(-c_\beta s_\gamma - c_\alpha s_\gamma) + u_{i1}(+c_\alpha c_\beta c_\gamma - s_\gamma),
\]

\[
v_m = q_{i2}(+s_\alpha s_\beta) + u_{i2}(-c_\alpha s_\beta),
\]

\[
q_m = q_{i2}(+c_\alpha c_\gamma - c_\beta s_\gamma) + u_{i2}(+c_\alpha s_\gamma + c_\beta c_\gamma),
\]

\[
u_m = u_{i2}(-c_\beta s_\gamma - c_\alpha s_\gamma) + u_{i2}(+c_\alpha c_\beta c_\gamma - s_\gamma),
\]

\[
v_m = q_{i2}(+s_\alpha s_\beta) + u_{i2}(-c_\alpha s_\beta).
\]  \tag{A2}

This system of equations can be solved using a normal nonlinear least-squares minimization by searching the \((\alpha, \beta, \gamma)\) space for
minima in squared error. With the measured Stokes vector \((\mathbf{S}_i)\), the Rayleigh sky input vector \((\mathbf{R}_i)\), and a rotation matrix \((\mathbf{R}_{ij})\), we define the error \((\epsilon)\) as

\[
\epsilon^2(\alpha, \beta, \gamma) = \sum_{i=1}^{3} \sum_{j=1}^{3} [\mathbf{S}_i - \mathbf{R}_i \mathbf{R}_{ij} (\alpha, \beta, \gamma)]^2. \tag{A3}
\]

For \(n\) measurements, this gives us \(3 \times n\) terms in this least-squares problem. The Euler angles give identical rotation-matrix elements under the exchange of \(\alpha, \beta, \gamma\) with \(-180^\circ + \alpha, -\beta\) and \(-180^\circ + \gamma\), as well as with additional multiples of \(2\pi\). This direct solution is easily solvable in principle, but the ambiguities make implementing this solution with existing software languages more cumbersome than is necessary.

As an alternative method to the direct least-squares solution for Euler angles, we can do a two-step process that gives the same result but is much easier to implement. First, we solve a system of equations for the Mueller matrix elements directly that are not subject to rotational ambiguity. With the Mueller matrix elements in hand, we can then perform a simple fit of a rotation-matrix elements to the derived Mueller matrix terms. This two-step process allows us to have accurate starting guesses to speed up the minimization process, resolve Euler angle ambiguities, and estimate error propagation.

When deriving the Mueller matrix elements of the telescope, one must take care that the actual derived matrices are physical. For instance, there are various matrix properties and quantities that one can derive to test the physicality of the matrix (Givens & Kostinski 1993; Kostinski et al. 1993; Takakura & Stoll 2009). Noise and systematic uncertainty might give overpolarizing or unphysical Mueller matrices such as the preceding element that is greater than 1. By simply using Mueller matrix elements to fit a rotation matrix, we avoid overpolarizing.

Provided that the arrays are indexed properly, the normal solution for Mueller matrix elements can be computed via the normal least-squares method. We rearrange the time-varying Rayleigh sky inputs to \((\mathbf{R}_{ij})\) for \(i\) independent observations and \(j\) input Stokes parameters. The measured Stokes parameters \((\mathbf{S}_i)\) become individual column vectors. The unknown Mueller matrix elements are also arranged as a column vector by output Stokes parameter \((\mathbf{M}_j)\). If we write measured Stokes parameters as \((q_{m1}, u_{m1}, v_{m1})\) and the Rayleigh input Stokes parameters as \((q_r, u_r)\), we can explicitly write a set of equations for just two Mueller matrix elements:

\[
\mathbf{S}_i = \begin{pmatrix} q_{m1} \\ q_{m2} \\ q_{m3} \end{pmatrix} = \mathbf{R}_{ij} \mathbf{M}_j = \begin{pmatrix} q_{r1} \\ q_{r2} \\ q_{r3} \end{pmatrix} \begin{pmatrix} u_{r1} \\ u_{r2} \\ u_{r3} \end{pmatrix} = \begin{pmatrix} QQ \\ UQ \\ UU \end{pmatrix}. \tag{A4}
\]

We have three such equations for each set of Mueller matrix elements. With this, we can express the residual error \((\epsilon_i)\) for each incident Stokes parameter \((\mathbf{S}_i)\) with an implied sum over \(j\) as

\[
\epsilon_i = \mathbf{S}_i - \mathbf{R}_i \mathbf{R}_{ij} \mathbf{M}_j. \tag{A5}
\]

The normal solution of an overspecified system of equations is easily derived in a least-squares sense using matrix notation. Using the total error \(E\) as the sum of all residuals for \(m\) independent observations, we get

\[
E = \sum_{i=1}^{m} \epsilon_i^2. \tag{A6}
\]

We solve the least-squares system for the unknown Mueller matrix element \((\mathbf{M}_j)\) by minimizing the error with respect to each equation. The partial derivative of equation (A5) for \(\epsilon_i\) with respect to \(\mathbf{M}_j\) is just the sky input elements \(\mathbf{R}_{ij}\). Taking the partial with respect to each input Stokes parameter, we get

\[
\frac{\partial E}{\partial \mathbf{M}_j} = 2 \sum_i \epsilon_i \frac{\partial \epsilon_i}{\partial \mathbf{M}_j} = -2 \sum_i \mathbf{R}_{ij} (\mathbf{S}_i - \sum_k \mathbf{R}_{ik} \mathbf{M}_k) = 0. \tag{A7}
\]

We have inserted a dummy sum over the index \(k\). Multiplying out the terms and rearranging gives us the normal equations,

\[
\sum_i \sum_k \mathbf{R}_{ij} \mathbf{R}_{ik} \mathbf{M}_k = \sum_i \mathbf{R}_{ij} \mathbf{S}_i, \tag{A8}
\]

which, when written in matrix notation, is the familiar solution of a system of equations via the normal method:

\[
\mathbf{M} = \frac{\mathbf{R}^T \mathbf{S}}{\mathbf{R}^T \mathbf{R}}. \tag{A9}
\]

This simple equation is very easy to implement with a few lines of code, provided observation times are chosen to give a range of input states for a well-conditioned inversion. The noise properties and inversion characteristics of this equation can be calculated and optimized in advance of observations. We can write the matrix \(\mathbf{A}\) with an implied sum over \(i\) observations for each term as

\[
\mathbf{A} = \mathbf{R}^T \mathbf{R} = \begin{pmatrix} q_r q_r & q_r u_r \\ q_r u_r & u_r u_r \end{pmatrix}. \tag{A10}
\]

The solution to the equations for the three sets of Mueller matrix elements can be written as

\[
\mathbf{M}_j = \begin{pmatrix} QQ \\ UQ \\ UU \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} q_r q_{m1} \\ q_r u_{m1} \\ u_r q_{m1} \\ u_r u_{m1} \end{pmatrix}. \tag{A11}
\]

\[
\mathbf{M}_j = \begin{pmatrix} QQ \\ UQ \\ UU \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} q_r u_{m1} \\ u_r q_{m1} \\ u_r u_{m1} \end{pmatrix}. \tag{A12}
\]
\[ \mathbf{M}_j = \begin{pmatrix} QV \\ UV \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} q_r v_m \\ u_r v_m \end{pmatrix}. \]  
(A13)

As an example, if we compute the inverse of \( \mathbf{A} \) and multiply out \( \mathbf{A}^{-1} \) for the \( QQ \) term, we can write

\[ QQ = \frac{(q_r q_m) (u_r u_r) - (u_r q_m) (q_r u_r)}{(q_r q_r) (u_r u_r) - (q_r u_r) (q_r u_r)}. \]  
(A14)

In this manner, we can easily implement the usual matrix formalism with a time series of daytime sky observations to measure six Mueller matrix elements.

With the Mueller matrix terms in hand, solving for accurate initial Euler angle guesses is straightforward. As an example, by using \( QV \) and \( UV \) derived previously and equating to rotation terms from Eq. (A1), one can solve for \( \alpha \) and \( \beta \) directly:

\[ \tan(\alpha) = -\frac{QV}{UV}, \]  
(A15)

\[ \sin(\beta) = \frac{QV}{\sin(\alpha)} = -\frac{UV}{\cos(\alpha)}. \]  
(A16)

If one writes \( c_r \), as \( x \) then uses coefficients \( a = QQ/s_a s_\beta \) and \( b = c_\alpha/s_\alpha s_\beta \), one can numerically solve a quadratic for and estimate of \( \gamma \):

\[ 0 = (b^2 + 1)x^2 - (2ab)x + (a^2 - 1). \]  
(A17)

The sign ambiguity in solving for \( \gamma = \cos^{-1}(x) \) is resolved by comparing the sign of the computed Mueller matrix elements with those of the rotation matrix derived with these estimated Euler angles. The estimate shown here only utilized \( QQ, QV, \) and \( UV \), but provided a guess for minimization routines typically accurate to better than 2° in our data set.

In our two-step method, we next solved for the best-fit Euler angles by doing a normal nonlinear least-squares minimization by searching the \( (\alpha, \beta, \gamma) \) space around our initial guesses for minima in the summed squared error:

\[ E(\alpha, \beta, \gamma) = \sum_{i=1}^{3} \sum_{j=1}^{3} [M_{ij} - R_{ij}(\alpha, \beta, \gamma)]^2. \]  
(A18)

By using the built-in IDL routines POWELL or AMOEBA, we can construct the error function and find the minima. Both of these minimization routines require estimated starting points, but give the same best-fit Euler angles to better than one part per million. Note that by choosing Euler angle guesses in the proper quadrants, the derived rotation matrices avoid the ambiguity of \( \alpha, \beta, \) and \( \gamma \) with \( -180^\circ + \alpha, -\beta \) and \( -180^\circ + \gamma \). The differences between rotation-matrix elements and Mueller matrix elements for our observations are typically much less than 0.1. The two-step solution gives us the same Euler angles as the direct least-squares solution, which is typically better than 0.2° and allows for computationally inexpensive and easy implementation of the least-squares minimization.

In order to verify this method, we computed the error functions \( E \) for both the direct solution and the two-step solution and verified the minima with a direct search of the entire Euler angle space. We created a grid of Euler angles from \( -180^\circ \) to \( 180^\circ \) (using 2° increments) and computed the error \( E \) as in equation (A3). Figure 10 shows the derived errors as functions of \( \beta \) and \( \gamma \) for four different values of \( \alpha \). Each panel is shaded to show the square error \( E \) increasing from dark to light. The Euler angles found using the two-step method are identical to those found with the direct solve within the crude 2° sampling we chose for this brute-force search. The error functions are always continuous, with only two minima in the search volume, showing that convergence is easily achieved using any minimization scheme.

**APPENDIX B**

**PLANNING SKY OBSERVATIONS USING TIME DEPENDENCE FOR MODULATION**

In order to use this technique efficiently, one must consider the noise propagation when inverting a sequence of sky observations to derive telescope Mueller matrix properties. There is an analogy between using the time-dependent Rayleigh sky to measure polarization properties of the telescope and the retardancy chosen to create an efficient modulation scheme for...
polarization measurements. The underlying mathematics are the same, and one can easily derive the requirements of the observations needed to derive high-accuracy telescope Mueller matrix measurements. Effectively, one is attempting to derive properties of a matrix through what different groups call demodulations, inversions, or deprojections. Here, we outline the analogy between polarimetric modulation and calculating noise properties with the Rayleigh sky to determine a quality observing sequence for deriving telescope Mueller matrices.

Polarimeters modulate the incoming polarization state via retardance amplitude and orientation changes. This retardance modulation is translated into varying intensities using an analyzer such as a polarizer, polarizing beam splitter, or crystal blocks such as Wollaston prisms or Savart plates. These modulation schemes can vary widely. For example, Compaïn et al. (1999), del Toro Iniesta & Collados (2000), De Martino et al. (2003), Nagaraju et al. (2007), and Tomczyk et al. (2010) overview optimal schemes and error propagation, and they outline schemes to maximize or balance polarimetric efficiency and create polychromatic systems. There have been many implementations of achromatic and polychromatic designs in both stellar and solar communities (e.g., Gisler et al. 2003; Hanaoka 2004; Xu et al. 2006). In the notation of these studies, the instrument modulates the incoming polarization information into a series of measured intensities \( I_i \) for \( i \) independent observations via the modulation matrix \( O_{ij} \) for \( j \) input Stokes parameters \( S_j \):

\[
I_i = O_{ij}S_j. \tag{B1}
\]

This is exactly analogous to our situation, where we have changed the matrix indices to be \( i \) independent Stokes parameter measurements for \( j \) different sky input Stokes parameters:

\[
S_i = R_{ij}M_j. \tag{B2}
\]

In most nighttime polarimeters, instruments choose a modulation matrix that separates and measures individual parameters of the Stokes vector:

\[
O_{ij} = \begin{pmatrix}
1 & +1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
1 & 0 & +1 & 0 \\
1 & 0 & -1 & 0 \\
1 & 0 & 0 & +1 \\
1 & 0 & 0 & -1
\end{pmatrix}. \tag{B3}
\]

Other instruments choose only four measurements: the minimum number of exposures required to measure the Stokes vector. In these schemes, one can balance the efficiency of the measurement to minimize the noise on each Stokes parameter:

\[
O = \begin{pmatrix}
1 & +1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
1 & 0 & +1 & 0 \\
1 & 0 & 0 & +1
\end{pmatrix}. \tag{B4}
\]

One recovers the input Stokes vector from a series of intensity measurements by inverting the modulation matrix \( O \), provided that it is a square, and multiplying this inverse by the measured intensities. If the matrix is not square, one can simply solve the overspecified system of equations via the normal least-squares formalism:

\[
S = O^T I. \tag{B5}
\]

Even for nonsquare matrices, we can define a demodulation matrix that captures the transfer properties of the modulation scheme:

\[
D_{ij} = [O^T O]^{-1}O^T. \tag{B6}
\]

In our case, the Rayleigh sky input parameters become the modulation matrix \( O_{ij} = R_{ij} \), and the formalism for noise propagation developed in many studies, such as in del Toro Iniesta & Collados (2000), apply.

As an example, we will use the 2010 January 27 Mueller matrices derived in Figure 5. The sky polarization rotated by about 33° during this period, and the three calculated Rayleigh sky inputs scaled to 100% polarization are

\[
R_{ij} = \begin{pmatrix}
q_{r_1} & u_{r_1} \\
q_{r_2} & u_{r_2} \\
q_{r_3} & u_{r_3}
\end{pmatrix} = \begin{pmatrix}
+0.980 & +0.201 \\
+0.847 & +0.534 \\
+0.716 & +0.698
\end{pmatrix}. \tag{B7}
\]

If each measurement has the same noise \( \sigma \) and there are \( n \) total measurements, then the noise on each demodulated parameter \( \sigma_i \) becomes

\[
\sigma_i^2 = n\sigma^2 \sum_{j=1}^{n} D_{ij}^2. \tag{B8}
\]

And the efficiency of the observation becomes

\[
e_i = \left(n \sum_{j=1}^{n} D_{ij}^2\right)^{-\frac{1}{2}}. \tag{B9}
\]

For instance, the normal modulation sequence of equation (B4) used by most nighttime spectropolarimeters gives \( n = 6 \) and \( e_i^2 = [1, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}] \). The efficiency-balanced scheme gives the same relative efficiencies as the normal modulation scheme, but uses only four exposures instead of six. In the case of our January 27 observations considering only \( qu \) terms, our demodulation matrix is
\[
D_{ij} = [R^T R]^{-1} R^T = \begin{pmatrix} +1.23 & +0.16 & -0.48 \\ -1.49 & +0.43 & +1.54 \end{pmatrix}.
\] (B10)

With this demodulation matrix the efficiency for \( u \) is only 60% worse than \( g \), and we have efficiencies of \( e_i = [0.43, 0.26] \) computed with equation (B9). The Mueller matrix derived from these Rayleigh sky observations will have similar noise properties between \( Q \) and \( U \) terms.

One must take care with this technique to build up observations over a wide range of solar locations so that the inversion is well conditioned. The path of the Sun throughout the day will create regions of little input sky Stokes vector rotation, causing a poorly constrained inversion with high condition number. For instance, at our location in the tropics the Sun rises and sets without changing azimuth until it rises quite high in the sky. We are constrained to observing in early morning and late evening with the dome walls raised, since we may not expose the telescope to the Sun. This causes input vectors at east-west pointings to be mostly \( q \)-oriented, with little rotation over many hours. Observations at other times of the year or at higher solar elevations are required to have a well-conditioned inversion. One can easily build up the expected sky input polarizations at a given observing site with the Rayleigh sky polarization equations. Then it is straightforward to determine the modulation matrix and noise propagation for a planned observing sequence to ensure a well-measured telescope matrix with a good signal-to-noise ratio.

**APPENDIX C**

**LOVIS SPECTROPOLARIMETER CHARACTERIZATION**

In order to expand the capabilities of our spectropolarimeter, we have adapted the optics for use at low spectral resolution. The polarimeter unit for the spectrograph is a Savart plate, providing dual-beam analyzing mounted behind the entrance slit with either two rotating achromatic retarders or two liquid-crystal variable retarders (LCVRs) outlined in Harrington & Kuhn (2008) and Harrington et al. (2010). We have modified the HiVIS echelle housing to allow a flat mirror to be mounted, bypassing the echelle without disrupting the optical path. We call this new mode LoVIS.

With a flat mirror replacing the echelle, the only dispersive element in the spectrograph is the cross-disperser working in first order. Only a single order is imaged on the detector, making the illuminated region of the device much smaller. Figure 11 shows a raw data frame from a spectrophotometric standard star (HR7940) taken with LoVIS in spectropolarimetric mode using the apogee detector. There is a single spectral order seen, with orthogonally polarized spectra being displaced roughly 70 pixels spatially by the Savart plate. Wavelength increases from right to left, with the H\( \alpha \) line seen near spectral pixel 1400 and the 6870 Å atmospheric band seen near pixel 900. We have two separate cross-dispersers available on the rotation stage blazed for 6050 Å and 8600 Å (Hodapp et al. 2000; Thornton et al. 2003).

The new LoVIS mode gives us much higher sensitivity and allows for much faster calibrations at spectral resolutions of 1000 to 3000, depending on the slit. The 1.5” slit had spectral resolutions of 850 and 990 derived from thorium-argon (ThAr) lines at 5700 Å and 7000 Å. These lines were spectrally sampled with roughly 9.5 pixels per Gaussian full width at half-maximum at each wavelength. The IDL reduction scripts we wrote for HiVIS outlined in Harrington & Kuhn (2008) were adapted to this new low-resolution data. Figure 12 shows the extracted spectrum from the raw data of Figure 11.

This change in resolution allows us to observe much fainter targets or to observe sources at a much higher cadence. When observing the daytime sky with AEOS and LoVIS, we can achieve a spectropolarimetric precision of 0.1% with a 5 s exposure time using the 1.5” slit. The polarimetric images sequences finish in 1 minute, and the bulk of the overhead is from rotating the achromatic wave plates in the typical six-exposure sequence.

To calibrate this new polarimetric mode and the telescope polarization, we used LoVIS to gather a wide range of sky and calibration observations from 2009 August to 2010 February. All observations were taken with the apogee detector.

![Example LoVIS data when observing a spectrophotometric standard star on 2009 September 4. Wavelength increases from right to left. The Na D lines, H\( \alpha \), and the atmospheric \( \lambda \) band can be seen by eye near spectral pixels 2300, 1400, and 900, respectively. See the electronic edition of the PASP for a color version of this figure.](Image)
described in Harrington et al. (2010). We have sky observations taken solely at the zenith, as well as observations on a grid of altitudes and azimuths. On several occasions we did observations for 3 or 4 hr continuously before sunset to accumulate a wide range of polarization inputs at multiple pointings.

In order to decouple the LoVIS and AEOS polarization measurements, we did a number of tests with our polarization calibration unit mounted at the slit, as well as at the entrance port to our instrument at the calibration lamp unit. We did tests both without and with the image rotator inserted in the path. This image rotator is the source of most of the cross talk in the LoVIS instrument. Removing it makes the polarization properties of LoVIS much more benign, as shown in Harrington et al. (2010, Figs. 15 and 16). There is a transmissive window that separates the coudé rooms from the central optical path to the telescope. This window is directly after the last fold mirror of the telescope, m7, and can be set to either BK7 or Infrasil. This window can also have a polarimetric effect on the beam, and both windows were also tested. Effectively, we find that the polarization properties derived for HiVIS are similar to LoVIS and that we can efficiently reproduce polarization measurements with both rotating achromatic wave plates and LCVR modulators.

We also measured the induced polarization of the entire system when illuminated by a point-source utilizing unpolarized standard stars. As an example, we show observations of many unpolarized standard stars at many different pointings observed with LoVIS on 2009 September 5. Each Stokes parameter is shown in percent. The induced polarization is at the few percent level, with some dependence on wavelength.

In summary, the polarization properties of LoVIS are similar to HiVIS. The induced polarization is below a few percent. The polarization calibration optics give very pure inputs that are reproduced with both achromatic wave plate and LCVR modulators. This new low-resolution mode works efficiently and allows us to obtain calibration observations at a much faster cadence.

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Fig. 12.—Example extracted spectrum of the spectrophotometric standard star HR7950 taken on 2009 September 4. The Hα, Na D lines, and the atmospheric A band are identified.

Fig. 13.—Telescope-induced polarization. We measured quv spectra for many different unpolarized standard stars at many different pointings observed with LoVIS on 2009 September 5. Each Stokes parameter is shown in percent. The induced polarization is at the few percent level, with some dependence on wavelength.
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