On the vector-valued Littlewood-Paley-Rubio de Francia inequality

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Abstract

The paper studies Banach spaces satisfying the Littlewood-Paley-Rubio de Francia property \( LPR_p \), \( 2 \leq p < \infty \). The paper shows that every Banach lattice whose 2-concavification is a UMD Banach lattice has this property. The paper also shows that every space having \( LPR_q \) also has \( LPR_p \) with \( q \leq p < \infty \).

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1 Introduction

Let \( X \) be a Banach space and \( L^p(\mathbb{R}; X) \) be the Bochner space of \( p \)-integrable \( X \)-valued functions on \( \mathbb{R} \). If \( X = \mathbb{C} \), we abbreviate \( L^p(\mathbb{R}; X) = L^p(\mathbb{R}) \), \( 1 \leq p < \infty \). For every \( f \in L^1(\mathbb{R}; X) \), \( \hat{f} \) stands for the Fourier transform. If \( I \subseteq \mathbb{R} \) is an interval, then \( S_I \) is the Riesz projection adjusted to the interval \( I \), i.e.,

\[
S_I f(t) = \int_I \hat{f}(s) e^{2\pi ist} ds.
\]

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The following remarkable inequality was proved by J.L. Rubio de Francia in [9].

For every $2 \leq p < \infty$, there is a constant $c_p$ such that for every collection of pairwise disjoint intervals $(I_j)_{j=1}^\infty$, the following estimate holds

$$\left\| \left( \sum_{j=1}^\infty |S_{I_j}f|^2 \right)^{1/2} \right\|_{L^p(\mathbb{R})} \leq c_p \|f\|_{L^p(\mathbb{R})}, \quad \forall f \in L^p(\mathbb{R}).$$  \hspace{1cm} (1)

In this note, we shall discuss the version of the theorem above when functions take values in a Banach space $X$. Let $(\varepsilon_k)_{k \geq 1}$ be the system of Rademacher functions on $[0, 1]$. The space $\text{Rad}(X)$ is the closure in $L^p([0, 1]; X)$, $1 \leq p < \infty$ of all $X$-valued functions of the form

$$g(\omega) = \sum_{k=1}^n \varepsilon_k(\omega) x_k, \quad x_k \in X, \quad n \geq 1.$$  

The above definition is independent of $1 \leq p < \infty$. It follows from the Khintchine-Kahane inequality (see [6]). In fact, the above fact is a consequence of a, so-called, contraction principle. It states that, for every sequence of elements $(x_j)_{j=1}^\infty \subseteq X$ and sequence of complex numbers $(\alpha_j)_{j=1}^\infty$ such that $|\alpha_j| \leq 1$, $j \geq 1$, the following inequality holds

$$\left\| \sum_{j=1}^\infty \alpha_j \varepsilon_j x_j \right\|_{L^p(\mathbb{R}; \text{Rad}(X))} \leq c_p \left\| \sum_{j=1}^\infty \varepsilon_j x_j \right\|_{L^p(\mathbb{R}; \text{Rad}(X))}.$$  

We shall employ this principle on numerous occasions in this paper.

Following [1], we shall call $X$ a space with the $LPR_p$ property with $2 \leq p < \infty$, if there exists a constant $c > 0$ such that for any collection of pairwise disjoint intervals $(I_j)_{j=1}^\infty$ we have that

$$\left\| \sum_{j=1}^\infty \varepsilon_j S_{I_j}f \right\|_{L^p(\mathbb{R}; \text{Rad}(X))} \leq c \|f\|_{L^p(\mathbb{R}; X)}, \quad \forall f \in L^p(\mathbb{R}; X).$$ \hspace{1cm} (2)

It was proved in [5] that every space with the $LPR_p$ property is necessarily UMD and of type 2. It is an open problem whether the converse is true. It is also unknown whether $LPR_p$ is independent of $p$. Note that Rubio de Francia’s inequality says that $C$ has the $LPR_p$ property for every $2 \leq p < \infty$. By
the Khintchine inequality and the Fubini theorem we see that any $L^p$-space with $2 \leq p < \infty$ has the LPR$_p$ property. Using interpolation, we deduce that a Lorentz space $L^{p,r}$ has the LPR$_q$ property for some indices $p, r$ and $q$. However, until recently there were no non-trivial examples of spaces with LPR$_p$ found.

If $X$ is a Banach lattice, estimate (2) admits a pleasant form as in the scalar case:

$$
\left\| \left( \sum_{j=1}^{\infty} |S_j f|^2 \right)^{\frac{1}{2}} \right\|_{L^p[\mathbb{R};X]} \leq c \left\| f \right\|_{L^p[\mathbb{R};X]}, \quad \forall \ f \in L^p[\mathbb{R};X].
$$

We shall show that if the 2-concavification $X_{(2)}$ of $X$ is a UMD Banach lattice, then (3) holds for all $2 < p < \infty$, so $X$ is a space with the LPR$_p$ property. Recall that $X_{(2)}$ is the lattice defined by the following quasi-norm

$$
\left\| f \right\|_{X_{(2)}} = \left\| \left| f \right|^{\frac{1}{2}} \right\|^2_X.
$$

The space $X_{(2)}$ is a Banach lattice if and only if $X$ is 2-convex, i.e.,

$$
\left\| \left( \sum_{j=1}^{n} |f_j|^2 \right)^{\frac{1}{2}} \right\|_{X} \leq \left( \sum_{j=1}^{n} \left\| f_j \right\|^2_X \right)^{\frac{1}{2}}.
$$

We refer to [6] for more information on Banach lattices.

We shall also show that if $X$ is a Banach space (not necessarily a lattice) with the LPR$_q$ property for some $q$, then $X$ has the LPR$_p$ property for every $p \geq q$.

## 2 Dyadic decomposition

For every interval $I \subseteq \mathbb{R}$, let $2I$ be the interval of double length and the same centre as $I$. Let $\mathcal{I} = (I_j)_{j=1}^{\infty}$ be a collection of pairwise disjoint intervals. We set $2\mathcal{I} = (2I_j)_{j=1}^{\infty}$. The collection $\mathcal{I}$ is called well-distributed if there is a number $d$ such that each element of $2\mathcal{I}$ intersects at most $d$ other elements of $2\mathcal{I}$.

In this section, we fix a pairwise disjoint collection of intervals $(I_j)_{j=1}^{\infty}$ and we break each interval $I_j, j \geq 1$ into a number of smaller dyadic subintervals such that the new collection is well-distributed. This construction was employed in a number of earlier papers.
We start with two elementary remarks on estimate (2) or (3). Firstly, it suffices to consider a finite sequence \((I_j)\) of disjoint finite intervals. Secondly, by dilation, we may assume \(|I_j| \geq 4\) for all \(j\). Thus all sums on \(j\) and \(k\) in what follows are finite. Fix \(j \geq 1\). Let \(I_j = (a_j, b_j)\). Let \(n_j = \max\{n \in \mathbb{N} : 2^{n+1} \leq b_j - a_j + 4\}\). We first split \(I_j\) into two subintervals with equal length

\[I_a^j = (a_j, a_j + b_j/2)\quad \text{and}\quad I_b^j = (a_j + b_j/2, b_j).\]

Then we decompose \(I_a^j\) and \(I_b^j\) into relative dyadic subintervals as follows:

\[I_a^j = \bigcup_{k=1}^{n_j} (a_{j,k}, a_{j,k+1}]\quad \text{and}\quad I_b^j = \bigcup_{k=1}^{n_j} (b_{j,k}, b_{j,k+1}],\]

where

\[a_{j,k} = a_j - 2 + 2^k, \quad 1 \leq k \leq n_j\quad \text{and}\quad a_{j,n_j+1} = \frac{a_j + b_j}{2};\]

\[b_{j,k} = b_j + 2 - 2^k, \quad 1 \leq k \leq n_j\quad \text{and}\quad b_{j,n_j+1} = \frac{a_j + b_j}{2}.\]

Let

\[I_{a,j,k}^j = (a_{j,k}, a_{j,k+1}], \quad I_{b,j,k}^j = (b_{j,k+1}, b_{j,k}]\]

for \(1 \leq k \leq n_j\) and let \(I_{a,j,k}^j, I_{b,j,k}^j\) be the empty set for the other \(k\)’s. Also put

\[\tilde{I}_{a,j,n_j}^j = (a_j - 2 + 2^{n_j}, a_j - 2 + 2^{n_j+1}]\quad \text{and}\quad \tilde{I}_{b,j,n_j}^j = (b_j + 2 - 2^{n_j+1}, b_j + 2 - 2^{n_j}].\]

**Lemma 1.** A Banach space \(X\) has the \(LPR_p\) property if there is a constant \(c > 0\) such that

\[\max_{u=a,b} \left\| \sum_{j=1}^{\infty} \varepsilon_j \sum_{k=1}^{n_j} \varepsilon_k^u S_{I_{j,k}^u} f \right\|_{L^p(\mathbb{R}; X)} \leq c \|f\|_{L^p(\mathbb{R}; \operatorname{Rad}_2(X))}, \quad \forall f \in L^p(\mathbb{R}; X), \quad (4)\]

where \(\operatorname{Rad}_2(X) = \operatorname{Rad}(\operatorname{Rad}'(X))\) and \(\operatorname{Rad}'(X)\) is the space with respect to another copy of the Rademacher system \((\varepsilon_k^u)_{k \geq 1}\).

Observe that if (4) holds, for every family of intervals \((I_j)_{j=1}^{\infty}\), then \(X\) is a UMD space. Indeed, (4) implies that

\[\left\| S_{I_{j,k}^u} f \right\|_{L^p(\mathbb{R}, X)} \leq c \|f\|_{L^p(\mathbb{R}, X)}, \quad u = a, b, \quad j \geq 1, \quad 1 \leq k \leq n_j.\]
That is, by adjusting the choice of intervals, it implies that every projection $S_I$ is bounded on $L^p(\mathbb{R},X)$ and

$$\sup_{I \subseteq \mathbb{R}} \|S_I\|_{L^p(\mathbb{R},X) \to L^p(\mathbb{R},X)} < +\infty.$$  

The latter is equivalent to the fact that $X$ is UMD (see [3]).

Proof. Let $f \in L^p(\mathbb{R};X)$. Then

$$\left\| \sum_{j=1}^{\infty} \varepsilon_j S_{I_j} f \right\|_{L^p(\mathbb{R};\text{Rad}(X))} \leq \left\| \sum_{j=1}^{\infty} \varepsilon_j S_{I^a_j k} f \right\|_{L^p(\mathbb{R};\text{Rad}(X))} + \left\| \sum_{j=1}^{\infty} \varepsilon_j S_{I^b_j} f \right\|_{L^p(\mathbb{R};\text{Rad}(X))}.$$  

Using the subintervals $I^a_{j,k}$ and the contraction principle, we write

$$\left\| \sum_{j=1}^{\infty} \varepsilon_j S_{I^a_j} f \right\|_{L^p(\mathbb{R};\text{Rad}(X))} = \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j S_{I^a_{j,k}} f \right\|_{L^p(\mathbb{R};\text{Rad}(X))} \sim \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \exp(-2\pi i a_j \cdot) S_{I^a_{j,k}} f \right\|_{L^p(\mathbb{R};\text{Rad}(X))}.$$  

Note that

$$\exp(-2\pi i a_j \cdot) S_{I^a_{j,k}} f = S_{I^a_{j,k} - a_j} \left[ \exp(-2\pi i a_j \cdot) f \right]$$

and

$$I^a_{j,k} - a_j = (2^k - 2, 2^{k+1} - 2], \quad k < n_j; \quad I^a_{j,n_j} - a_j \subseteq (2^{n_j} - 2, 2^{n_j+1} - 2].$$

Recall that $X$ is a UMD space. Therefore, applying Bourgain’s Fourier multiplier theorem (see [3]) to the function

$$\sum_{j=1}^{n_j} \sum_{k=1}^{k_{n_j}} \varepsilon_j \exp(-2\pi i a_j \cdot) S_{I^a_{j,k}} f \in L^p(\mathbb{R};\text{Rad}(X)),$$
we obtain (the contraction principle being used in the last step)

\[
\left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \exp(-2\pi i a_j \cdot) S_{j,k} f \right\|_{L^p(\mathbb{R}; \text{Rad}(X))} \lesssim \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon_k \exp(-2\pi i a_j \cdot) S_{j,k} f \right\|_{L^p(\mathbb{R}; \text{Rad}_2(X))}.
\]

Similarly,

\[
\left\| \sum_{j=1}^{\infty} \varepsilon_j S_{j} f \right\|_{L^p(\mathbb{R}; \text{Rad}(X))} \lesssim \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon_k S_{j,k} f \right\|_{L^p(\mathbb{R}; \text{Rad}_2(X))}.
\]

Combining the preceding estimates, we get

\[
\left\| \sum_{j=1}^{\infty} \varepsilon_j S_{j} f \right\|_{L^p(\mathbb{R}; \text{Rad}(X))} \leq c_p \left[ \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon_k S_{j,k} f \right\|_{L^p(\mathbb{R}; \text{Rad}_2(X))} + \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon_k S_{j,k} f \right\|_{L^p(\mathbb{R}; \text{Rad}_2(X))} \right].
\]

Let us observe that, if \( X \) is a UMD space, the argument in the proof above shows that

\[
\left\| \sum_{j=1}^{\infty} \varepsilon_j S_{j} f \right\|_{L^p(\mathbb{R}; \text{Rad}(X))} \lesssim \max_{u=a,b} \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon_k S_{j,k} f \right\|_{L^p(\mathbb{R}; \text{Rad}_2(X))}.
\]

Moreover, the argument can be reversed to show the opposite estimate (see the proof of (5) below.) This observation is summarised in the following remark.

**Remark 2.**  i) If \( X \) is a UMD space, then

\[
\left\| \sum_{j=1}^{\infty} \varepsilon_j S_{j} f \right\|_{L^p(\mathbb{R}; \text{Rad}(X))} \lesssim \max_{u=a,b} \left\| \sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon_k S_{j,k} f \right\|_{L^p(\mathbb{R}; \text{Rad}_2(X))}.
\]

\[6\]
ii) If $\mathcal{I} = (I_j)_{j \geq 1}$ is a collection of pairwise disjoint intervals and $\mathcal{I}_u = \left(I_{j,k}^u\right)_{j \geq 1, 1 \leq k \leq n_j}$, $u = a, b$, then both collections $\mathcal{I}_a$ and $\mathcal{I}_b$ are well-distributed.

iii) If $X$ is a Banach lattice then it has the $\alpha$-property (see [7]). That is,

$$
\left\| \sum_{j,k=1}^{\infty} \varepsilon_j \varepsilon_k' x_{jk} \right\|_{\mathrm{Rad}_2(X)} \sim \left\| \sum_{j,k=1}^{\infty} \varepsilon_j \varepsilon_k x_{jk} \right\|_{\mathrm{Rad}(X)}
$$

where $(\varepsilon_{jk})$ is an independent family of Rademacher functions.

iv) The above two observations imply that if $X$ is a Banach lattice, then it has the $\text{LPR}_p$ property if and only if estimate (2) holds for every well-distributed collection of intervals $\mathcal{I}$.

3 LPR-estimate for Banach lattices

**Theorem 3.** If $X$ is a Banach lattice such that $X_{(2)}$ is a UMD Banach space, then $X$ has the $\text{LPR}_p$ property for every $2 < p < \infty$.

We shall need the following remark for the proof.

**Remark 4.** If $X$ is UMD and $1 < p < \infty$, then the family $\{S_I\}_{I \subseteq \mathbb{I}}$ is $R$-bounded (see [4]), i.e.,

$$
\left\| \sum_{I \subseteq \mathbb{I}} \varepsilon_I S_I f_I \right\|_{L^p(R; \mathrm{Rad}(X))} \leq c_X \left\| \sum_{I \subseteq \mathbb{I}} \varepsilon_I f_I \right\|_{L^p(R; \mathrm{Rad}(X))}.
$$

**Proof of Theorem 3** The proof directly employs the pointwise estimate of [9]. We assume, that $X$ is a Köthe function space on a measure space $(\Omega, \mu)$.

Let $f \in L^1_{\text{loc}}(\mathbb{R}; X)$. Let $M(f)$ be the Hardy-Littlewood maximal function of $f$, i.e.,

$$
M(f)(t) = \sup_{I \ni t} \frac{1}{|I|} \int_I |f(s)| \, ds
$$

and

$$
M_2(f) = \left[ M |f|^2 \right]^{\frac{1}{2}}.
$$
Let
\[ f^\sharp(t) = \sup_{I \subseteq \mathbb{R}} \frac{1}{|I|} \int_I |f(s) - f_I| \, ds, \quad f_I = \frac{1}{|I|} \int_I f(s) \, ds. \]

Note that \( M(f) \) is a function of two variables \((t, \omega)\): for each fixed \( \omega \), \( M(f)(\cdot, \omega) \) is the usual Hardy-Littlewood maximal function of \( f(\cdot, \omega) \). The same remark applies to \( M_2(f) \) and \( f^\sharp \). For \( f \) sufficiently nice (which will be assumed in the sequel), all these functions are well-defined.

Observe that due to Remark 2 we have only to show estimate (2) for a well-distributed family of intervals. Let us fix a family of pairwise disjoint intervals \( I \) and let us assume that \( I \) is well-distributed. Fix a Schwartz function \( \psi(t) \) whose Fourier transform satisfies
\[ \chi[-1/2, 1/2] \leq \hat{\psi} \leq \chi[-1,1]. \]

If \( I \in I \), then we set
\[ \psi_I(t) = |I| \exp(2\pi ic_I t) \psi(|I| t), \]
where \( c_I \) is the centre of \( I \). The Fourier transform of \( \psi_I \) is adapted to \( I \), i.e.
\[ \chi_I \leq \hat{\psi}_I \leq \chi_{2I}. \]

In particular,
\[ S_I(f) = \psi_I * S_I(f). \]

Consequently, from the Khintchine inequality and Remark 3,
\[ \left\| \left( \sum_{I \in I} |S_I(f)|^2 \right)^{1/2} \right\|_{L_p(\mathbb{R}, X)} \leq c_p \| G(f) \|_{L_p(\mathbb{R}, X)}, \quad 1 < p < \infty, \]
where
\[ G(f) = \left( \sum_{I \in I} |\psi_I * f|^2 \right)^{1/2}, \quad f \in L^1(\mathbb{R}; X). \]

Thus, to finish the proof, we need to show that
\[ \|G(f)\|_{L_p(\mathbb{R}, X)} \leq c_p \|f\|_{L_p(\mathbb{R}, X)}, \quad 2 < p < \infty. \]
It was shown in [11] that $G(f(\cdot, \omega))^2$ is almost everywhere dominated by $M_2(f(\cdot, \omega))$, i.e.,

$$G(f(\cdot, \omega))^2 \leq c M_2(f(\cdot, \omega)), \quad \text{a.e. } \omega \in \Omega,$$

for some universal $c > 0$. Since

$$G(f(t, \omega)) = G(f(\cdot, \omega))(t) \quad \text{and} \quad M_2(f(t, \omega)) = M_2(f(\cdot, \omega))(t), \quad t \in \mathbb{R}, \ \omega \in \Omega,$$

we clearly have that

$$G(f)^2 \leq c M_2(f).$$

Therefore,

$$\|G(f)^2\|_{L^p(\mathbb{R}; X)} \leq c \|M_2(f)\|_{L^p(\mathbb{R}; X)}.$$  

It remains to prove

$$\|G(f)\|_{L^p(\mathbb{R}; X)} \leq C \|G(f)^2\|_{L^p(\mathbb{R}; X)} \quad \text{and} \quad \|M_2(f)\|_{L^p(\mathbb{R}; X)} \leq C \|f\|_{L^p(\mathbb{R}; X)}.$$  

The second inequality above immediately follows from Bourgain’s maximal inequality for UMD lattices (applied to $X(2)$ here, see [10, Theorem 3]):

$$\|M_2(f)\|_{L^p(\mathbb{R}; X)} = \|M(|f|^2)\|_{L^p(\mathbb{R}; X)} \leq C \|f\|_{L^p(\mathbb{R}; X)}.$$  

It remains to show the first one. To this end we shall prove the following inequality (for a general $f$ instead of $G(f)$)

$$\|f\|_{L^p(\mathbb{R}; X)} \leq C \|f^2\|_{L^p(\mathbb{R}; X)}.$$  

This is again an immediate consequence of the following classical duality inequality (see [12, p. 146])

$$\left| \int_{\mathbb{R}} u v \right| \leq C \int_{\mathbb{R}} u^2 \mathcal{M}(v)$$

for any $u \in L^p(\mathbb{R})$ and $v \in L^{p'}(\mathbb{R})$, where $\mathcal{M}(v)$ denotes the grand maximal function of $v$. Note that $\mathcal{M}(v) \leq CM(v)$. Now let $g \in L^{p'}(\mathbb{R}; X^*)$ be a nice function. We then have

$$\left| \int_{\mathbb{R} \times \Omega} f g \right| \leq C \int_{\mathbb{R} \times \Omega} f^2 \mathcal{M}(g) \leq C \|f^2\|_{L^p(\mathbb{R}; X)} \|\mathcal{M}(g)\|_{L^{p'}(\mathbb{R}; X^*)} \leq C \|f^2\|_{L^p(\mathbb{R}; X)} \|g\|_{L^{p'}(\mathbb{R}; X^*)}.$$  

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where we have used again Bourgain’s maximal inequality for $g$ (noting that $X^*$ is also a UMD lattice). Therefore, taking supremum over all $g$ in the unit ball of $L^p'(\mathbb{R}; X^*)$, we deduce the desired inequality, so prove the theorem.

Finally, observe that the proof above operates with individual functions. This, coupled with the UMD property of $X$, implies that $X$ can always be assumed separable and it can always be equipped with a weak unit.

\[\square\]

4 LPR property for general Banach spaces

Let $X$ be a Banach space (not necessarily a lattice). We shall prove the following theorem.

**Theorem 5.** If $X$ has the LPR$_q$ for some $2 \leq q < \infty$, then $X$ has the LPR$_p$ for any $q \leq p < \infty$.

The proof of the theorem requires some lemmas.

**Lemma 6.** Assume that $X$ has the LPR$_q$ property. Let $(I_j)_{j \geq 1}$ be a finite sequence of mutually disjoint intervals of $\mathbb{R}$ and $(I_{j,k})_{k=1}^{n_j}$ be a finite family of mutually disjoint subintervals of $I_j$ for each $j \geq 1$. Assume that the relative position of $I_{j,k}$ in $I_j$ is independent of $j$, i.e., $I_{j,k} - a_j = I_{j',k} - a'_j$ whenever both $I_{j,k}$ and $I_{j',k}$ are present (i.e., $k \leq \min\{n_j, n_{j'}\}$), where $a_j$ is the left endpoint of $I_j$. Then

\[
\sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon'_k S_{I_{j,k}} f \sim \sum_{k=1}^{\infty} \varepsilon'_k \sum_{j : n_j \geq k} \varepsilon_j S_{I_{j,k} - a_j} (\exp(-2\pi i a_j \cdot) f)
\]

**Proof.** We first assume that $\bigcup_{k=1}^{n_j} I_{j,k} = I_j$ for each $j \geq 1$. Note that

\[
S_{I_{j,k}} f = \exp(2\pi i a_j \cdot) S_{I_{j,k} - a_j} (\exp(-2\pi i a_j \cdot) f).
\]

Thus, by the contraction principle,

\[
\sum_{j=1}^{\infty} \sum_{k=1}^{n_j} \varepsilon_j \varepsilon'_k S_{I_{j,k}} f \sim \sum_{k=1}^{\infty} \varepsilon'_k \sum_{j : n_j \geq k} \varepsilon_j S_{I_{j,k} - a_j} (\exp(-2\pi i a_j \cdot) f)
\]
Since $X$ has the LPR$_q$ property, so does $\text{Rad}(X)$. Let us apply this property of $\text{Rad}(X)$ to the intervals $\left(\tilde{I}_k\right)_{k \geq 1}$, where $\tilde{I}_k = I_{j,k} - a_j$, for some $j$ such that $n_j \geq k$ (for any such $j$ the interval $I_{j,k} - a_j$ is independent of $j$ by the assumptions of the lemma). We apply this property to the function

$$\sum_{k=1}^{\infty} \sum_{j: \ n_j \geq k} \varepsilon_j S_{I_{j,k} - a_j} (\exp(-2\pi ia_j) f) = \sum_{k=1}^{\infty} S_{\tilde{I}_k} \left[ \sum_{j: \ n_j \geq k} \varepsilon_j (\exp(-2\pi ia_j f)) \right].$$

We obtain

$$\left\| \sum_{k=1}^{\infty} \varepsilon_k' \sum_{j: \ n_j \geq k} \varepsilon_j S_{I_{j,k} - a_j} (\exp(-2\pi ia_j f)) \right\|_q$$

$$\leq c \left\| \sum_{k=1}^{\infty} \sum_{j: \ n_j \geq k} \varepsilon_j S_{I_{j,k} - a_j} (\exp(-2\pi ia_j f)) \right\|_q \sim c \left\| \sum_{j=1}^{n_j} \sum_{k=1}^{n_j} \varepsilon_j S_{I_{j,k} f} \right\|_q$$

$$= c \left\| \sum_{j=1}^{\infty} \varepsilon_j S_{I_{j}} f \right\|_q \leq c \| f \|_q. \quad (5)$$

Assume now that $\bigcup_{k=1}^{n_j} I_{j,k} \neq I_j$ for some $j$. In this case, consider the family of intervals $\left(\tilde{I}_k\right)_{k=1}^{\infty}$ introduced above. Observe that every $\tilde{I}_k \subseteq [0, +\infty)$. Observe also that the the right ends of the intervals $(I_j - a_j)_{j \geq 1}$, that is the points $b_j - a_j$ do not belong to the union $\bigcup_{k=1}^{\infty} \tilde{I}_k$. Let $\left(\tilde{I}_\ell\right)_{\ell=1}^{\infty}$ be the family of disjoint intervals such that

$$\bigcup_{\ell=1}^{\infty} \tilde{I}_\ell = [0, +\infty) \setminus \bigcup_{k=1}^{\infty} \tilde{I}_k$$

and such that neither of the points $(b_j - a_j)_{j=1}^{\infty}$ is inner for some $\tilde{I}_\ell$. Let also $m_j$ be the maximum number such that the intervals $\tilde{I}_\ell$ with $\ell \leq m_j$ are all to the left of the point $b_j - a_j$. Set $I_{j,\ell} = \tilde{I}_\ell + a_j$. Then,

$$I_j = \bigcup_{k=1}^{n_j} I_{j,k} + \bigcup_{\ell=1}^{m_j} I_{j,\ell}. $$

It is clear that the relative position of $(I_{j,k})_{k=1}^{n_j} \cup (I_{j,\ell})_{\ell=1}^{m_j}$ in $I_j$ is again independent of $j$.

Before we proceed, let us re-index the intervals $(I_{j,k})_{k=1}^{n_j}$ and $(I_{j,\ell})_{\ell=1}^{m_j}$ into a family $(I_{j,s})_{s=1}^{m_j+n_j}$ as follows. We arrange these intervals from left to right.
within $I_j$ and index them sequentially from 1 up to $n_j + m_j$. Moreover, let $K_j \subseteq [1, n_j + m_j]$ be the subset corresponding to the first family of intervals and $L_j \subseteq [1, n_j + m_j]$ be the subset of indices corresponding to the second family of intervals. Observe that, if $K = \bigcup_{j=1}^{\infty} K_j$ and $L = \bigcup_{j=1}^{\infty} L_j$, then, for every $j$, $K_j = K \cap [1, n_j + m_j]$ and, similarly, $L_j = L \cap [1, n_j + m_j]$. Thus by the previous part we get

$$\left\| \sum_{s=1}^{n_j + m_j} \epsilon_j \epsilon_s S_{I_j,s} f \right\|_q \leq c_q \|f\|_q.$$

Observe also that

$$\sum_{j=1}^{\infty} \sum_{s=1}^{n_j + m_j} \epsilon_j \epsilon_s S_{I_j,s} f = \sum_{j=1}^{\infty} \sum_{s=1}^{n_j + m_j} \epsilon_j \epsilon_s S_{I_j,s} f = \sum_{s=1}^{\infty} \sum_{j: n_j + m_j \geq s} \epsilon_j \epsilon_s S_{I_j,s} f + \sum_{s \in L} \sum_{j: n_j + m_j \geq s} \epsilon_j \epsilon_s S_{I_j,s} f$$

Thus, by taking projection onto the subspace spanned by $\{\epsilon_s\}_{s \in K}$, we continue

$$\left\| \sum_{s \in K: n_j + m_j \geq s} \epsilon_j \epsilon_s S_{I_j,s} f \right\|_q \leq c_q \|f\|_q.$$ 

Finally, we observe that

$$\sum_{s \in K: n_j + m_j \geq s} \epsilon_j \epsilon_s S_{I_j,s} f = \sum_{j=1}^{n_j} \sum_{k=1}^{m_j} \epsilon_j \epsilon_k S_{I_{j,k}} f.$$ 

Hence the lemma is proved.

The following lemma is interesting in its own right. We shall only need its first part.

**Lemma 7.** Let $Y$ be a Banach space. Let $(\Sigma, \nu)$ be a measure space and $(h_j) \subset L^2(\Sigma)$ a finite sequence.

i) If $Y$ is of cotype 2 and there exists a constant $c$ such that

$$\| \sum_{j} \alpha_j h_j \|_2 \leq c \left( \sum_{j} |\alpha_j|^2 \right)^{1/2}, \quad \forall \alpha_j \in \mathbb{C},$$

then

$$\| \sum_{j} h_j a_j \|_{L^2(\Sigma; Y)} \leq c \| \sum_{j} \epsilon_j a_j \|_{\text{Rad}(Y)}, \quad \forall a_j \in Y.$$
ii) If $Y$ is of type 2 and there exists a constant $c$ such that
\[
\left( \sum_j |\alpha_j|^2 \right)^{1/2} \leq c \left\| \sum_j \alpha_j h_j \right\|_2, \quad \forall \alpha_j \in \mathbb{C},
\]
then
\[
\left\| \sum_j \varepsilon_j a_j \right\|_{\text{Rad}(Y)} \leq c' \left\| \sum_j h_j a_j \right\|_{L^2(\Sigma; Y)}, \quad \forall a_j \in Y.
\]

Proof. i) Let $(a_j) \subset Y$ be a finite sequence. Consider the operator $u : \ell^2 \to Y$ defined by
\[
u(\alpha) = \sum_j \alpha_j a_j, \quad \forall \alpha = (\alpha_j) \in \ell^2.
\]
It is well known (see \[8, Lemma 3.8 and Theorem 3.9\]) that
\[
\pi_2(u) \leq c_0 \left\| \sum \varepsilon_j a_j \right\|_{\text{Rad}(Y)},
\]
where $c_0$ is a constant depending only on the cotype 2 constant of $Y$. Let $h(\sigma) = (h_j(\sigma))_j$ for $\sigma \in \Sigma$. Then by the assumption on $(h_j)$ we get
\[
\left\| \sum_j h_j a_j \right\|_{L^2(\Sigma; Y)} =
\begin{align*}
&\pi_2(u) \sup \left\{ \left( \int_{\Sigma} \left| \sum_j \varepsilon_j h_j(s)^2 ds \right|^{1/2} : \xi \in \ell^2, \left\| \xi \right\|_2 \leq 1 \right\} \\
&\leq c' \left\| \sum \varepsilon_j a_j \right\|_{\text{Rad}(Y)}.
\end{align*}
\]

ii) Let $H$ be the linear span of $(h_j)$ in $L^2(\Sigma)$. Let $h^*_j$ be the functional on $H$ such that $h^*_j(h_k) = \delta_{j,k}$. We extend $h^*_j$ to the whole $L^2(\Sigma)$ by setting $h^*_j = 0$ on $H^\perp$. Then $h^*_j \in L^2(\Sigma)$ and the assumption implies that
\[
\left\| \sum_j \beta_j h^*_j \right\|_2 \leq c' \left\| \sum_j |\beta_j|^2 \right\|^{1/2}, \quad \forall \beta_j \in \mathbb{C}.
\]
Now let $(a^*_j) \subset Y^*$ be a finite sequence. Applying i) to $Y^*$ and $(h^*_j)$ we obtain
\[
\left| \sum_j \langle a^*_j, a_j \rangle \right| = \left| \sum_j h^*_j a^*_j, \sum_j h_j a_j \right| \\
\leq \left\| \sum_j h^*_j a^*_j \right\|_{L^2(\Sigma; Y^*)} \left\| \sum_j h_j a_j \right\|_{L^2(\Sigma; Y)} \\
\leq c' \left\| \sum \varepsilon_j a^*_j \right\|_{\text{Rad}(Y^*)} \left\| \sum_j h_j a_j \right\|_{L^2(\Sigma; Y)}.
\]
Taking the supremum over \((a_j^\ast) \subset Y^*\) such that \(\| \sum \varepsilon_j a_j^\ast \|_{\text{Rad}(Y^*)} \leq 1\), we get the assertion. 

Now we proceed to the proof of Theorem 5. It is divided into several steps.

**The singular integral operator** \(T\). Let \((I_j)_j\) be a family of disjoint finite intervals and \(\psi\) be a Schwartz function as in Sections 2 and 3. We keep the notation introduced there. We now set up an appropriate singular integral operator corresponding to (4). It suffices to consider the family \((I_{a,j,k})_{j,k}\), \((I_{b,j,k})_{j,k}\) being treated similarly. Henceforth, we shall denote \(I_{a,j,k}\) simply by \(I_{j,k}\). Let \(c_{j,k} = a_{j,k} + 2^{k-1}\) for \(1 \leq k \leq n_j\). Note that \(c_{j,k}\) is the centre of \(I_{j,k}\) if \(k < n_j\) and of \(\tilde{I}_{j,k}\) if \(k = n_j\). Define

\[ \psi_{j,k}(x) = 2^k \exp(2\pi ic_{j,k} x) \psi(2^k x) \]

so that the Fourier transform of \(\psi_{j,k}\) is adapted to \(I_{j,k}\), i.e.

\[ \chi_{I_{j,k}} \leq \hat{\psi}_{j,k} \leq 2\chi_{2I_{j,k}} \text{ for } k < n_j \quad \text{and} \quad \chi_{\tilde{I}_{j,n_j}} \leq \hat{\psi}_{j,n_j} \leq 2\chi_{2\tilde{I}_{j,n_j}}. \] (6)

We should emphasise that our choice of \(c_{j,k}\) is different from that of Rubio de Francia (in [9]) which is \(c_{j,k} = n_{j,k} 2^k\) for some integer \(n_{j,k}\). Rubio de Francia’s choice makes his calculations easier than ours in the scalar-valued case. The sole reason for our choice of \(c_{j,k}\) is that \(c_{j,k}\) splits into a sum of two terms depending on \(j\) and \(k\) separately. Namely, \(c_{j,k} = a_j - 2 + 2^k + 2^{k-1}\). By (6),

\[ S_{I_{j,k}} f = S_{I_{j,k}} \psi_{j,k} * f. \]

We then deduce, by the splitting property and Remark 4

\[ \| \sum_{j,k} \varepsilon_j \varepsilon_k S_{I_{j,k}} f \|_p \leq C_p \| \sum_{j,k} \varepsilon_j \varepsilon_k \psi_{j,k} * f \|_p. \]

Now write

\[ \psi_{j,k} * f(x) = \int 2^k \psi(2^k (x - y)) \exp(2\pi i c_{j,k}(x - y)) f(y) dy \]

\[ = \exp(2\pi i c_{j,k} x) \int 2^k \psi(2^k (x - y)) \exp(-2\pi i c_{j,k} y) f(y) dy \]

\[ = \exp(2\pi i c_{j,k} x) \int K_{j,k}(x, y) f(y) dy, \]

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where
\[ K_{j,k}(x, y) = 2^k \psi(2^k(x-y)) \exp(-2\pi ic_{j,k} y). \tag{7} \]

Using the splitting property of the \(c_{j,k}\) mentioned previously and the contraction principle, for every \(x \in \mathbb{R}\) we have
\[
\left\| \sum_{j,k} \varepsilon_j \varepsilon'_k \psi_{j,k} * f(x) \right\|_{\text{Rad}_2(X)} \\
= \left\| \sum_{j,k} \varepsilon_j \varepsilon'_k \exp(2\pi ic_{j,k} x) \int K_{j,k}(x, y) f(y) dy \right\|_{\text{Rad}_2(X)} \\
\sim \left\| \sum_{j,k} \varepsilon_j \varepsilon'_k \int K_{j,k}(x, y) f(y) dy \right\|_{\text{Rad}_2(X)}.
\]

Thus we are led to introducing the vector-valued kernel \(K\):
\[
K(x, y) = \sum_{j,k} \varepsilon_j \varepsilon'_k K_{j,k}(x, y) \in L^2(\Omega), \quad x, y \in \mathbb{R}. \tag{8}
\]

\(K\) is also viewed as a kernel taking values in \(B(X, \text{Rad}_2(X))\) by multiplication. Let \(T\) be the associated singular integral operator:
\[
T(f)(x) = \int K(x, y) f(y) dy, \quad f \in L^p(\mathbb{R}; X).
\]

By the discussion above, inequality (4) is reduced to the boundedness of \(T\) from \(L^p(\mathbb{R}; X)\) to \(L^p(\mathbb{R}; \text{Rad}_2(X))\):
\[
\left\| T(f) \right\|_p \leq c_p \left\| f \right\|_p, \quad \forall f \in L^p(\mathbb{R}; X). \tag{9}
\]

The \(L^q\) boundedness of \(T\). We have the following.

**Lemma 8.** \(T\) is bounded from \(L^q(\mathbb{R}; X)\) to \(L^q(\mathbb{R}; \text{Rad}_2(X))\).

**Proof.** Let \(f \in L^q(\mathbb{R}; X)\). By the previous discussion we have
\[
\left\| Tf \right\|_q \sim \left\| \sum_{j,k} \varepsilon_j \varepsilon'_k \psi_{j,k} * f \right\|_q.
\]

By (6)
\[
\sum_{j,k} \varepsilon_j \varepsilon'_k \psi_{j,k} * f = \sum_{j,k} \varepsilon_j \varepsilon'_k \psi_{j,k} * (S_{2t_{j,k}} f).
\]
Note that for each \( j \) the last interval \( I_{j,n_j} \) above should be the dyadic interval \( \tilde{I}_{j,n_j} \). We claim that

\[
\left\| \sum_{j,k} \varepsilon_j \varepsilon_k' \psi_{j,k} * g_{j,k} \right\|_q \leq c \left\| \sum_{j,k} \varepsilon_j \varepsilon_k' g_{j,k} \right\|_q, \quad \forall \ g_{j,k} \in L^q(\mathbb{R}; X).
\]

Indeed, using the splitting property of the \( c_{j,k} \) we have

\[
\left\| \sum_{j,k} \varepsilon_j \varepsilon_k' \psi_{j,k} * g_{j,k} \right\|_q \sim \left\| \sum_{j,k} \varepsilon_j \varepsilon_k' \tilde{\psi}_{j,k} * \tilde{g}_{j,k} \right\|_q,
\]

where

\[
\tilde{\psi}_{j,k}(x) = 2^k \psi(2^k x) \quad \text{and} \quad \tilde{g}_{j,k}(x) = \exp(-2\pi ic_{j,k} x)g_{j,k}(x).
\]

For \( x \in \mathbb{R} \) define the operator \( N(x) : \text{Rad}_2(X) \to \text{Rad}_2(X) \) by

\[
N(x)(\sum_{j,k} \varepsilon_j \varepsilon_k' a_{j,k}) = \sum_{j,k} \varepsilon_j \varepsilon_k' \tilde{\psi}_{j,k}(x)a_{j,k}.
\]

It is obvious that \( N : \mathbb{R} \to B(\text{Rad}_2(X)) \) is a smooth function and

\[
\sum_{j,k} \varepsilon_j \varepsilon_k' \tilde{\psi}_{j,k} * \tilde{g}_{j,k} = N * \tilde{g} \quad \text{with} \quad \tilde{g} = \sum_{j,k} \varepsilon_j \varepsilon_k' \tilde{g}_{j,k}.
\]

It is also easy to check that \( N \) satisfies [11, Theorem 3.4]. Since \( \text{Rad}_2(X) \) is a UMD space, it follows from [11] that the convolution operator with \( N \) is bounded on \( L^q(\mathbb{R}; \text{Rad}_2(X)) \). Thus

\[
\left\| \sum_{j,k} \varepsilon_j \varepsilon_k' \tilde{\psi}_{j,k} * \tilde{g}_{j,k} \right\|_q \leq c \left\| \sum_{j,k} \varepsilon_j \varepsilon_k' \tilde{g}_{j,k} \right\|_q.
\]

Using again the splitting property of the \( c_{j,k} \) and going back to the \( g_{j,k} \), we prove the claim. Consequently, we have

\[
\left\| T(f) \right\|_q \leq c \left\| \sum_{j,k} \varepsilon_j \varepsilon_k' S_{2I_{j,k}} f \right\|_q.
\]

We split the family \( \{2I_{j,k}\} \) into three subfamilies \( \{2I_{j,3k+\ell}\} \) of disjoint intervals with \( \ell \in \{0, 1, 2\} \). Accordingly, we have

\[
\left\| T(f) \right\|_q \leq c \sum_{\ell=0}^2 \left\| \sum_{j,k} \varepsilon_j \varepsilon_k' S_{2I_{j,3k+\ell}} f \right\|_q.
\]
Each subfamily \( \{2I_{j,3k+\ell}\}_{j,k} \) satisfies the condition of Lemma 6. Hence
\[
\left\| \sum_{j,k} \varepsilon_j \varepsilon'_k S_{2I_{j,3k+\ell}} f \right\|_q \leq c \left\| f \right\|_q.
\]
Thus the lemma is proved. \( \square \)

An estimate on the kernel \( K \). This subsection contains the key estimate on the kernel \( K \) defined in (8). Fix \( x, z \in \mathbb{R} \) and an integer \( m \geq 1 \). Let
\[
I_m(x,z) = \{ y \in \mathbb{R} : 2^m|x-z| < |y-z| \leq 2^{m+1}|x-z| \}.
\]

**Lemma 9.** If \( X^* \) is of cotype 2 and if \( (\lambda_{j,k}) \subset X^* \), then
\[
\int_{I_m(x,z)} \left\| \sum_{j,k} [K_{j,k}(x,y) - K_{j,k}(z,y)] \lambda_{j,k} \right\|^2_{X^*} dy \leq c \left\| \sum_{j,k} \varepsilon_j \varepsilon'_k \lambda_{j,k} \right\|^2_{\text{Rad}(X^*)}.
\]

**Proof.** Let \( (\lambda_{j,k}) \subset X^* \) such that
\[
\left\| \sum_{j,k} \varepsilon_j \varepsilon'_k \lambda_{j,k} \right\|_{\text{Rad}(X^*)} \leq 1.
\]
By the definition of \( K_{j,k} \) in (7), we have
\[
\sum_{j,k} [K_{j,k}(x,y) - K_{j,k}(z,y)] \lambda_{j,k} = \sum_k \mu_k 2^k \left[ \psi(2^k(x-y)) - \psi(2^k(z-y)) \right] q_k(y),
\]
where
\[
\mu_k = \left\| \sum_j \varepsilon_j \lambda_{j,k} \right\|_{\text{Rad}(X^*)} \quad \text{and} \quad q_k(y) = \mu_k^{-1} \sum_j \lambda_{j,k} \exp(-2\pi i c_{j,k} y).
\]
Since \( \text{Rad}(X^*) \) is of cotype 2,
\[
\sum_k \mu_k^2 \leq c \left\| \sum_k \varepsilon'_k \sum_j \varepsilon_j \lambda_{j,k} \right\|^2_{\text{Rad}(\text{Rad}(X^*))} \leq c.
\]
Thus
\[
\int_{I_m(x,z)} \left\| \sum_{j,k} [K_{j,k}(x,y) - K_{j,k}(z,y)] \lambda_{j,k} \right\|^2_{X^*} dy \\
\leq \sum_k 2^{2k} \sup_{y \in I_m(x,z)} \left| \psi(2^k(x-y)) - \psi(2^k(z-y)) \right|^2 \int_{I_m(x,z)} \left\| q_k(y) \right\|^2_{X^*} dy.
\]

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Note that for fixed $k$

$$|c_{j,k} - c_{j',k}| \geq 2^k, \quad \forall j \neq j'.$$

(10)

Now we appeal to the following classical inequality on Dirichlet series with small gaps. Let $(\gamma_j)$ be a finite sequence of real numbers such that

$$\gamma_{j+1} - \gamma_j \geq 1, \quad \forall j \geq 1.$$

Then, by [13, Ch. V, Theorem 9.9], for any interval $I \subset \mathbb{R}$ and any sequence $(\alpha_j) \subset \mathbb{C}$

$$\int_I \left| \sum_j \alpha_j \exp(2\pi i \gamma_j y) \right|^2 \, dy \leq c \max(|I|, 1) \sum_j |\alpha_j|^2,$$

where $c$ is an absolute constant. Applying this to the function $q_k$, using Lemma 7 and (10), we find

$$\int_{I_m(x,z)} \|q_k\|_{X^*}^2 \, dy \leq c 2^{-k} \max(2^k |I_m(x,z)|, 1) \mu_k^{-2} \|\sum_j \varepsilon_j \lambda_{j,k}\|_{\text{Rad}(X^*)}^2
= c \max(2^m |x - z|, 2^{-k}).$$

Let

$$k_0 = \min \{ k \in \mathbb{N} : 2^{-k} \leq 2^m |x - z| \} \quad \text{and} \quad k_1 = \min \{ k \in \mathbb{N} : 2^{-k} \leq 2^{2m/3} |x - z| \}.$$ 

Note that $k_0 \leq k_1$. For $k \leq k_1$ we have

$$|\psi(2^k(x - y)) - \psi(2^k(z - y))| \leq c 2^k |x - z|.$$ 

Recall that $\psi$ is a Schwartz function, in particular $|x|^2 |\psi(x)| \leq c$. Thus, for $k \geq k_1$, we have

$$|\psi(2^k(x - y)) - \psi(2^k(z - y))| \leq c 2^{-2k}|y - z|^{-2} \leq c 2^{-2k - 2m}|x - z|^{-2},$$

where the second estimate comes from the fact that $y \in I_m(x,z)$. Let

$$\alpha_k = 2^{2k} \sup_{y \in I_m(x,z)} |\psi(2^k(x - y)) - \psi(2^k(z - y))|^2 \int_{I_m(x,z)} \|q_k(y)\|_{X^*}^2 \, dy.$$
Combining the preceding inequalities, we deduce the following estimates on $\alpha_k$:

$$
\begin{align*}
\alpha_k &\leq c 2^{2k} 2^k |x-z|^{2^k} = c 2^{3k} |x-z|^2 \quad \text{for} \quad k \leq k_0; \\
\alpha_k &\leq c 2^{2k} 2^k |x-z|^{2^m} = c 2^{4k} 2^m |x-z|^3 \quad \text{for} \quad k_0 < k < k_1; \\
\alpha_k &\leq c 2^{2k} (2^k + m |x-z|)^{-4} 2^m |x-z| = c 2^{-2k} 2^{-3m} |x-z|^{-3} \quad \text{for} \quad k \geq k_1.
\end{align*}
$$

Therefore,

$$
\int_{I_m(x,z)} \left\| \sum_{j,k} [K_{j,k}(x,y) - K_{i,k}(z,y)] \lambda_{j,k} \right\|^2_{X^*} dy 
\leq \sum_{1 \leq k \leq k_0} \alpha_k + \sum_{k_0 < k < k_1} \alpha_k + \sum_{k \geq k_1} \alpha_k 
\leq c \left[ 2^{3k_0} |x-z|^2 + 2^{4k_1} 2^m |x-z|^3 + 2^{-2k_1} 2^{-3m} |x-z|^{-3} \right] 
\leq c 2^{-5m/3} |x-z|^{-1}.
$$

This is the desired estimate for the kernel $K$. \hfill \Box

The $L^\infty$-BMO boundedness. Recall that $T$ is the singular integral operator associated with the kernel $K$.

Lemma 10. The operator $T$ is bounded from $L^\infty(\mathbb{R}; X)$ to $\text{BMO}(\mathbb{R}; \text{Rad}_2(X))$.

Proof. Recall that

$$
\|g\|_{\text{BMO}(\mathbb{R}; X)} \leq 2 \sup_{I \subseteq \mathbb{R}} \frac{1}{\|I\|} \int_I \|g(x) - b_I\|_X \, dx,
$$

where $\{b_I\}_{I \subseteq \mathbb{R}} \subseteq X$ is any family of elements of $X$ assigned to each interval $I \subseteq \mathbb{R}$. Fix a function $f \in L^\infty(\mathbb{R}; X)$ with $\|f\|_\infty \leq 1$ and an interval $I \subseteq \mathbb{R}$. Let $z$ be the centre of $I$ and let $b_I = \int_{2I} K(z,y) f(y) \, dy$. Then, for $x \in I$,

$$
Tf(x) - b_I = \int_{(2I)^c} [K(x,y) - K(z,y)] f(y) \, dy + \int_{2I} K(x,y) f(y) \, dy.
$$

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Thus
\[
\frac{1}{|I|} \int_I \| T f(x) - b_I \|_{\text{Rad}_2(X)} \, dx \\
\leq \frac{1}{|I|} \int_I \| \int_{(2t)^c} [K(x, y) - K(z, y)] f(y) \, dy \|_{\text{Rad}_2(X)} \, dx \\
+ \frac{1}{|I|} \int_I \| \int_{2t} K(x, y) f(y) \, dy \|_{\text{Rad}_2(X)} \, dx \\
\overset{\text{def}}{=} A + B.
\]

By Lemma \text{9}, we have
\[
B \leq |I|^{-1/q} \| T(f \chi_{2t}) \|_q \leq c.
\]

To estimate \( A \), fix \( x \in I \). Choose \((\lambda_{j, k}) \subset X^*\) such that
\[
\| \sum_j \varepsilon_j \varepsilon_k' \lambda_{j, k} \|_{\text{Rad}_2(X^*)} \leq 1.
\]
and
\[
\| \int_{(2t)^c} [K(x, y) - K(z, y)] f(y) \, dy \|_{\text{Rad}_2(X)} \\
\sim \sum_{j, k} \langle \lambda_{j, k}, \int_{(2t)^c} [K_{j, k}(x, y) - K_{j, k}(z, y)] f(y) \, dy \rangle
\]
Then by Lemma \text{9} we find
\[
\| \int_{(2t)^c} [K(x, y) - K(z, y)] f(y) \, dy \|_{\text{Rad}_2(X)} \\
\leq \int_{(2t)^c} \| \sum_{j, k} [K_{j, k}(x, y) - K_{j, k}(z, y)] \lambda_{j, k} \|_{X^*} \, dy \\
\leq \sum_{m=1}^{\infty} |I_m(x, z)|^{1/2} \left( \int_{I_m(x, z)} \| \sum_{j, k} [K_{j, k}(x, y) - K_{j, k}(z, y)] \lambda_{j, k} \|_{X^*}^2 \, dy \right)^{1/2} \\
\leq c \sum_{m=1}^{\infty} (2^m |x - z|)^{1/2} (2^{5m/3} |x - z|)^{-1/2} \\
\leq \sum_{m=1}^{\infty} c 2^{-m/3} \leq c.
\]
Therefore, \( A \leq c \). Thus \( T \) is bounded from \( L^\infty(\mathbb{R}; X) \) to \( \text{BMO}(\mathbb{R}; \text{Rad}_2(X)) \). \( \square \)
Combining the result of Lemma 10 and Lemma 8 and applying interpolation (see [2]), we immediately see that the operator $T$ is bounded from $L^p(\mathbb{R}; X)$ to $L^q(\mathbb{R}; \text{Rad}_2(X))$ for every $q < p < \infty$. Thus Theorem 5 is proved.

**Remark 11.** Let

$$T(f)^+(x) = \sup_{x \in I} \frac{1}{|I|} \int_I \|T(f)(y) - T(f)_I\|_{\text{Rad}_2(X)} dy$$

and

$$M_q(f)(x) = \sup_{x \in I} \left( \frac{1}{|I|} \int_I \|f(y)\|_X^q dy \right)^{\frac{1}{q}}.$$ 

Under the assumption of Theorem 5 one can show the following pointwise estimate:

$$T(f)^+ \leq c M_q(f).$$

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