Copositive certificates of non-negativity for polynomials on semialgebraic sets *

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Abstract

A certificate of non-negativity is a way to write a given function so that its non-negativity becomes evident. Certificates of non-negativity are fundamental tools in optimization, and they underlie powerful algorithmic techniques for various types of optimization problems. We propose certificates of non-negativity of polynomials based on copositive polynomials. The certificates we obtain are valid for generic semialgebraic sets and have a fixed small degree, while commonly used sums-of-squares (SOS) certificates are guaranteed to be valid only for compact semialgebraic sets and could have large degree. Optimization over the cone of copositive polynomials is not tractable, but this cone has been well studied. The main benefit of our copositive certificates of non-negativity is their ability to translate results known exclusively for copositive polynomials to more general semialgebraic sets. In particular, we show how to use copositive polynomials to construct structured (e.g., sparse) certificates of non-negativity, even for unstructured semialgebraic sets. Last but not least, copositive certificates can be used to obtain not only hierarchies of tractable lower bounds, but also hierarchies of tractable upper bounds for polynomial optimization problems.

Keywords: Certificates of Non-negativity; Copositive Polynomials; Polynomial Optimization; Sparsity.

1 Introduction

Certificates of non-negativity are fundamental tools in optimization, and they underlie powerful algorithmic techniques for various types of optimization problems. Commonly used certificates of non-negativity of polynomials on basic semialgebraic sets include the classical Pólya’s Positivstellensatz [29], the more modern Schmüdgen’s Positivstellensatz [77], and Putinar’s Positivstellensatz [68]. Herein, we use the terms Positivstellensatz and certificate of non-negativity interchangeably.

To illustrate the concept of a certificate of non-negativity, let \( p, h_1, ..., h_m \) be polynomials. Assume we would like to know whether \( p \) is non-negative on the set \( S = \{ x \in \mathbb{R}^n : h_1(x) \geq \} \)

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0, ..., $h_m(x) \geq 0 \}$. If there exist a polynomial $F(x, u)$ non-negative for all $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ such that $p(x) = F(x, h_1(x), \ldots, h_m(x))$, then we are sure that $p$ is non-negative on $S$. We call such $F$ a certificate of non-negativity for $p$. For instance, one could have $F(x, u) = \sigma_0(x) + \sum_{i=1}^m \sigma_i(x)u_i$, where $\sigma_0, \ldots, \sigma_m$ are sums-of-squares (SOS) polynomials \[\text{[7]}\]. From Putinar’s Positivstellensatz \[\text{[68]}\], it is known that the latter certificate exists for $p$ on $S$ if the quadratic module generated by $h_1, \ldots, h_m$ is Archimedean and $p(x) > 0$ for all $x \in S$.

In this paper we study certificates of non-negativity based on copositivity. Polynomials that are non-negative on the non-negative orthant are called copositive polynomials \[\text{[see, e.g. 9]}\]. More specifically, one can show that $p$ is non-negative on $S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \}$ by demonstrating that for some $k \geq 0$

\[(1 + e^Ty + e^Tz)^kp(y - z) = F(y, z, h_1(y - z), \ldots, h_m(y - z)), \text{ where } F(y, z, u) \text{ is copositive.} \quad (1)\]

Such $F$ is called a copositive certificate of non-negativity of $p$ on $S$. For any $x \in S$, taking $x^+ = \max(x, 0)$ and $x^- = -\min(x, 0)$, where the maximum and minimum are taken component-wise, we have that $x^+, x^- \geq 0$ and therefore,

\[p(x) = p(x^+ - x^-) = F(x^+, x^-, h_1(x^+ - x^-), \ldots, h_m(x^+ - x^-))(1 + e^Tx^+ + e^Tx^-)^{-k} = F(x^+, x^-, h_1(x), \ldots, h_m(x))(1 + e^T|x|)^{-k} \geq 0,\]

as $F$ is copositive. Above, and throughout the article, we use $e$ to denote the vector of all-ones of appropriate dimension, and for $x \in \mathbb{R}^n$, $|x|$ stands for the component-wise absolute value of $x$ (i.e., $|x| = |x_i|, i = 1, \ldots, n$). In Theorems \[\text{[1]}\] and \[\text{[3]}\] we prove the existence of copositive certificates under mild assumptions which hold generically. In particular, no compactness or similar properties are assumed.

One essential property of the copositive certificates of non-negativity we propose is that the degree of $F$ in \[\text{[1]}\] is known a priori. Namely, this degree is bounded by the maximum of the degree of $p$ and twice the degree of the polynomials defining the set $S$. As a consequence, questions on the non-negativity of polynomials on generic basic semialgebraic sets reduce to finding a copositive polynomial satisfying \[\text{[1]}\] of small and, more importantly, known degree. This result is in line with recent results by Huq \[\text{[32]}\] on small copositive extended formulations for some combinatorial problems.

Optimization over the cone of copositive polynomials is hard \[\text{[54]}\]; however, this cone has been well studied. In particular, there exists a plenty of tractable approximations to it \[\text{[see, e.g. 12, 34, 40, 51, 60, 89]}\], as well as several certificates of copositivity \[\text{[for instance, 21, 29]}\]. The main benefit of our copositive certificates of non-negativity is their ability to translate results known exclusively for copositive polynomials to more general semialgebraic sets (see Section \[\text{[1.2]}\] for a more detailed explanation of our contributions).

### 1.1 Certificates of non-negativity and polynomial optimization

Classically, certificates of non-negativity based on SOS and non-negative coefficients (SOS-certificates), have been used to solve/approximate polynomial optimization (PO) problems \[\text{[43, 62, 81]}\]. PO encompass a wide variety of optimization problems including combinatorial and some non-convex optimization problems. Pólya’s, Schmüdgen’s, and Putinar’s Positivstellensatzen are examples of SOS certificates, and their applications in PO are illustrated in recent works \[\text{[e.g., 18, 30, 37, 43, 47, 61, 62, 64, among numerous others]}\]. Searching for a given SOS-certificate of non-negativity of
a fixed degree translates into solving a number of linear matrix inequalities (LMI). As the degree of the SOS-certificate is not known a priori, this method constructs a hierarchy of LMI approximations to the underlying problem. That is, optimization problems with a linear objective and LMI constraints [see 7]. LMI problems usually have the form of a linear program (LP), second-order cone program (SOCP) or semidefinite program (SDP), which can be solved to a given precision using interior-point methods [see 71]. The main drawbacks of using SOS certificates are the exponential growth of the LMI hierarchies in terms of the certificate’s degree and the lack of SOS-certificates for many interesting cases. To guarantee the existence of SOS-certificates, usually some form of compactness is needed [see 15, for a detailed analysis].

To deal with the fast growing size of SOS certificates, one could use subsets of SOS polynomials whose LMI reformulations do not result in full dimensional SDPs. For instance, in certain cases the structure of the problem allows arguing that sparse SOS certificates can be used as not all monomials have to be present in the certificates. This approach results in smaller convergent approximations to PO problems over some compact sets [examples are presented in 35, 44, 84, 87]. For non-structured problems, one could use scaled diagonally dominant sums-of-squares (SDSOS) instead of classical SOS. SDSOS are a type of SOS which result in LP or SOCP relaxations of PO problems. Such relaxations are computationally cheaper than SDPs and provide valid bounds [1] on PO problems. However these bounds are either not proven to converge or require the use of additional methods to ensure convergence [2].

Another way to deal with the flaws of SOS certificates would be to replace SOS in the expressions of certificates with different non-negative polynomials. Some existing examples include hyperbolic polynomials and non-negative circuit polynomials. The set of hyperbolic polynomials contains the set of SOS polynomials as a strict subset. Hence replacing SOS with hyperbolic polynomials provides hyperbolic programming relaxations of PO [76], which could potentially result in stronger bounds or faster convergence compared to classical SOS relaxations. Hyperbolic programs can be solved using interior-point methods, but efficient hyperbolic solvers are still under development, and the hyperbolic cone is not yet fully understood [72]. Non-negative circuit polynomials form neither a subset nor a superset of the cone of SOS polynomials. The relation between the two sets of polynomials depends on the degree and the number of variables [33]. Certificates based on non-negative circuit polynomials result in geometric programming relaxations of PO problems [24, 85] which converge under certain Archimedean conditions.

Given the key role that compactness plays for SOS certificates and their alternatives, a question that has attracted much research attention is which certificates exist on non-compact sets. In particular, Marshall [53], Powers [66] derive certificates of non-negativity for the case in which the underlying domain is a cylinder with a compact cross-section. Nguyen and Powers [50] derive certificates of non-negativity for the case in which the underlying domain is a strip or a half-strip. For more general settings, Demmel et al. [19], Marshall [52], Nie et al. [58], Vui and So’n [83], Wang [86] provide certificates of non-negativity, based on Putinar’s and Schmüdgen’s Positivstellensatzen, that do not require the underlying set to be compact. The latter certificates exploit gradient, Jacobian and KKT ideals. More recently, Jeyakumar et al. [36] have provided certificates of non-negativity for non-compact semialgebraic sets if a certain modification of the set is compact. Following the results in [36], Jeyakumar et al. [35] provide a certificate of non-negativity, based on Putinar’s Positivstellensatz, for coercive polynomials over possibly unbounded semialgebraic sets. Also recently, Guo et al. [26] have derived conditions under which Schmüdgen’s Positivstellensatz can be used to certify the non-negativity of a polynomial on a possibly unbounded convex set. Two other examples of research in this direction that are related to the results in this paper, are the works of Putinar and Vasilescu [70] and Dickinson and Povh [22].
1.2 Contribution

Now we give more details about our contributions.

Existence of copositive certificates of non-negativity

A common assumption for the existence of SOS-certificates of non-negativity for a polynomial \( p \) is the positivity of \( p \). As we are interested in certifying the non-negativity of a polynomial on a given set \( S \) that might be unbounded, we request \( p \) to be not only positive on \( S \), but also “strongly positive” on \( S \) (see Definition 2). In Theorem 3 we show that, given polynomials \( h_1, \ldots, h_m \) of degree at most \( d \) such that \( S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \} \) is non-empty, for all \( p \) strongly positive on \( S \), we always have a copositive certificate \( F \) as given by (1) of degree \( \max\{2d, \deg(p)\} \).

In particular, \( k = \max\{2d - \deg(p), 0\} \) in (1).

As we are interested in certifying the copositivity of \( F \), and certificates of copositivity usually exist for the interior of the cone of copositive polynomials, we show that for the compact case we can construct copositive certificates that lie in this interior (see Theorem 3). We also provide several equivalent characterizations of the interior of the cone of copositive polynomials (see Corollary 2).

In Section 3.2 we show that the strong positivity condition is generic since it is implied by a particular generic algebraic condition on \( S \) considered in [26, 27, 57].

Structure-rich certificates of non-negativity

The copositive approach we propose allows constructing a certificate of non-negativity from any certificate of copositivity (and any certificate of non-negativity on the non-negative orthant or standard simplex, in particular). This provides a universal procedure to obtain new certificates with desired properties on generic basic semi-algebraic sets. To illustrate this approach, in Section 4 we construct two new certificates of non-negativity on compact sets which do not require full-dimensional SOS polynomials. The special structure of these certificates provides computational advantages when compared to classical SOS-based certificates. Notice that, even though we focus on SOS, our methods could be used to obtain certificates of non-negativity based on circuit, hyperbolic polynomials and/or any general type of certificate of non-negativity on \( \mathbb{R}^n \) (see Corollary 5).

Besides the new certificates, we also obtain an elementary proof of the seminal theorem by Handelman [28] and an alternative proof of Schmüdgen’s Positivstellensatz [77] which shortcuts the proof by Schweighofer [79].

Applications to polynomial optimization

Our contribution to PO is twofold. On the one hand, our certificates allow us to apply to generic basic semi-algebraic sets a variety of results which are valid only for optimization over the non-negative orthant [see, e.g., 12, 21, 49, 51]. In particular, we can use both inner and outer approximations to the cone of copositive polynomials to obtain LMI hierarchies of upper and lower bounds for generic PO problems (see, Section 5).

This is in contrast with commonly used LMI hierarchies which only provide lower bounds for (minimization) PO problems [see, 3, 7]. On the other hand, under our assumptions a PO problem can be reformulated as an optimization problem over copositive polynomials of a fixed degree. This result connects copositive optimization and PO in general and advance the ongoing research on copositive reformulations of optimization problems. This line of research started with the work by Bomze et al. [10] showing that (potentially non-convex) standard quadratic optimization problems can be reformulated as copositive optimization problems. Further, Burer [13], Arima et al.
Proof. Let \( p \) we can write \( \| p \| = \max \{ |p| : x \in S \} \). We denote by \( C \) as the set of polynomials non-negative on \( S \). In this paper we usually deal with \( \text{int} P(d) \) (respectively \( \text{int} P(d) \)) the subset of \( \mathbb{R}[x] \) of polynomials of degree not larger than (resp. equal to) \( d \). For \( p \in \mathbb{R}[x] \), with degree \( \deg p = d \), let \( C_{d,a} \) denote the multinomial coefficient \( C_{d,a} := \frac{d!}{(d-\alpha_1)\alpha_1! \cdots (d-\alpha_m)\alpha_m!} n_d^a := \{ \alpha \in \mathbb{N}^m : e^\alpha \leq d \} \). Then, given \( p(x) \in \mathbb{R}[x] \) with \( \deg p = d \), we can write \( p(x) = \sum_{\alpha \in \mathbb{N}^d} C_{d,a} p_{a} x^\alpha \) for some \( p_{a} \in \mathbb{R} \), where \( x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \). We define \( \| p \| = \max \{|p_{a}| : \alpha \in \mathbb{N}^m, e^\alpha \leq d \} \).

**Lemma 1.** Let \( p \in \mathbb{R}[x] \). For any \( x \in \mathbb{R}^n \) we have

\[
p(x) \leq \| p \|(1 + e^T|x|)^{\deg p}.
\]

**Proof.** Given \( p \in \mathbb{R}[x] \) with \( \deg p = d \), and \( x \in \mathbb{R}^n \) we have

\[
p(x) \leq \sum_{\alpha \in \mathbb{N}^d} C_{d,a} |p_{a}| |x|^\alpha \leq \| p \| \sum_{\alpha \in \mathbb{N}^d} C_{d,a} |x|^\alpha \leq \| p \|(1 + e^T|x|)^{d}.
\]

For any \( S \subseteq \mathbb{R}^n \), we define

\[
\mathcal{P}(S) = \{ p \in \mathbb{R}[x] : p(x) \geq 0 \text{ for all } x \in S \},
\]

as the set of polynomials non-negative on \( S \). Similarly, we define

\[
\mathcal{P}^+(S) = \{ p \in \mathbb{R}[x] : p(x) > 0 \text{ for all } x \in S \},
\]

as the set of polynomials positive on \( S \). Furthermore, let \( \mathcal{P}_d(S) := \mathcal{P}(S) \cap \mathbb{R}_d[x] \) (resp. \( \mathcal{P}_d^+(S) := \mathcal{P}^+(S) \cap \mathbb{R}_d[x] \)) denote the set of polynomials of degree at most \( d \) that are non-negative (resp. positive) on \( S \). In this paper we usually deal with \( \text{int} \mathcal{P}_d(S) \) the interior of \( \mathcal{P}_d(S) \). Since \( \mathbb{R}_d[x] \) is a finite-dimensional vector space and \( \mathcal{P}_d(S) \) is convex, the interior and the algebraic interior of \( \mathcal{P}_d(S) \) coincide [see, e.g., 31, Chapter 17]. This fact is formally stated in Lemma 2.

**Lemma 2.** Let \( S \subseteq \mathbb{R}^n \). Then

\[
\text{int} \mathcal{P}_d(S) = \{ p \in \mathcal{P}_d(S) : \text{for all } q \in \mathbb{R}_d[x] \text{ there exists } \varepsilon > 0 \text{ such that } p - \varepsilon q \in \mathcal{P}_d(S) \}.
\]

Central to our discussion are copositive polynomials \( \mathbf{9} \) and sum-of-squares polynomials (SOS) \( \mathbf{7} \). A polynomial is copositive if it is non-negative on the non-negative orthant. Formally, a polynomial \( p \in \mathbb{R}_d[x] \) is copositive if \( p \in \mathcal{P}_d(\mathbb{R}^n_+) \). A polynomial \( p \in \mathbb{R}_{2d}[x] \) is SOS if \( p(x) = \sum_{l \leq l \in \mathbb{N}} q^l(x)^2 \) for some \( q^1, \ldots, q^l \in \mathbb{R}_d[x], l \in \mathbb{N} \).

We call a set of the form \( S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \} \) where \( h_1, \ldots, h_m \in \mathbb{R}[x] \) a basic semialgebraic set.
2.1 Strong positivity

One of the assumptions for the existence of classical SOS-certificates of non-negativity of \( p \) on \( S \) like the ones derived by Schmüdgen [77], Putinar [68], and Handelman [28], is the positivity of \( p \) on \( S \). In these classical theorems the assumptions also imply compactness of the semialgebraic set \( S \). Notice that

\[
S \text{ compact } \Rightarrow \text{int} \mathcal{P}_d(S) = \mathcal{P}_d^+(S).
\]  

(2)

When \( S \) is not compact, \( \mathcal{P}_d^+(S) \supset \text{int} \mathcal{P}_d(S) \). For example, the polynomial \( p(x) := 1 \) belongs to \( \text{int} \mathcal{P}_d(S) \) only when \( S \) is compact. We are interested in certifying the non-negativity of a polynomial \( p \) on a given basic semialgebraic set \( S \) that might be unbounded (i.e., not compact). In the existing results over non-compact sets, the positivity on \( S \) alone is not enough. Usually, assumptions on the behaviour of \( p \) at infinity; that is, the behaviour of \( p \) on the “directions” in which \( S \) becomes unbounded, are necessary [see, e.g., 69, 73]. Our certificates are not an exception to this rule, they exist for a subset of \( \text{int} \mathcal{P}_d(S) \) with a certain behavior at infinity which we describe next.

Given a polynomial \( p \in \mathbb{R}_d[x] \), let \( \tilde{p}(x) \) denote the homogeneous component of \( p(x) \) of the highest total degree. That is, \( \tilde{p}(x) \) is obtained by dropping from \( p(x) \) all the terms whose total degree is less than \( \text{deg} p \). Notice that \( \tilde{p}(x) \) determines the behavior of \( p \) at infinity. Namely, if \( \tilde{p}(y) > 0 \) for some \( y \in \mathbb{R}^n \), then there is \( t_0 \in \mathbb{R} \) such that \( p(ty) > 0 \) for all \( t > t_0 \), since the homogeneous component of the highest degree will eventually dominate the behavior of \( p \). Similarly if \( \tilde{p}(y) < 0 \), \( p \) will become eventually negative in the \( y \) direction. However, if \( \tilde{p}(y) = 0 \), we do not know how \( p(ty) \) behaves when \( t \) goes to infinity.

**Definition 1.** Let \( h_1, \ldots , h_m, g_1, \ldots , g_r \in \mathbb{R}[x] \) and let \( S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots , h_m(x) \geq 0, g_1(x) = 0, \ldots , g_r(x) = 0 \} \). We denote by \( \tilde{S} \) the following set

\[
\tilde{S} = \{ x \in \mathbb{R}^n : \tilde{h}_1(x) \geq 0, \ldots , \tilde{h}_m(x) \geq 0, \tilde{g}_1(x) = 0, \ldots , \tilde{g}_r(x) = 0 \}.
\]  

(3)

**Remark 1.** Note that from Definition 1 it follows that if \( S' = S \cap \mathbb{R}^n_+ \), then \( \tilde{S}' = \tilde{S} \cap \mathbb{R}^n_+ \), a fact that we will use throughout the article.

**Definition 2** (Strong positivity). We say that \( p \) is strongly positive on \( S \) when

\[
p \in \mathcal{P}_d^+(S) \text{ and } \tilde{p} \in \mathcal{P}_d^+(\tilde{S} \setminus \{0\}).
\]  

(4)

Strong positivity has been used in [69, Thm. 4.2] and [22, Property 3.5]. In particular, strong positivity on \( S \) is sufficient for the certificates of non-negativity in [22] to exist. Theorem 3 shows that for any semialgebraic set \( S \), copositive certificates of non-negativity exists for polynomials that are strongly positive on \( S \). Strongly positive polynomials belong to \( \text{int} \mathcal{P}_d(S) \), as formally stated in Proposition 3.

**Proposition 1.** Let \( S \) be a basic semialgebraic set. Then,

\[
\left\{ p \in \mathbb{R}_d[x] : p \in \mathcal{P}_d^+(S), \tilde{p} \in \mathcal{P}_d^+(\tilde{S} \setminus \{0\}) \right\} \subseteq \text{int} \mathcal{P}_d(S)
\]

**Proof.** The inclusion follows from Proposition 1 and Lemma 1(ii) in Section 3.2 \( \square \)

3 Copositive certificates of non-negativity

In this section, we prove our main results, namely, the existence of copositive certificates of non-negativity of the form 1 for all polynomials that are strongly positive on a basic semialgebraic set.
Let $p, g_1, \ldots, g_m \in \mathbb{R}_d[x]$ be such that $g_1, \ldots, g_m \in \mathcal{P}(\mathbb{R}^n_+)$, and $S = \{x \in \mathbb{R}^n_+ : g_1(x) = 0, \ldots, g_m(x) = 0\}$ be non-empty. Let $p \in \mathbb{R}_d[x]$ be such that $p \in \mathcal{P}^+(S)$ and $\tilde{p} \in \mathcal{P}^+(\tilde{S} \setminus \{0\})$. Then there exists $F \in \mathcal{P}_d(\mathbb{R}^n_+)$ and $\alpha_j \in \mathbb{R}_{d-deg}g_j[x]$ for $j = 1, \ldots, m$ such that

$$p(x) = F(x) + \sum_{j=1}^m \alpha_j(x)g_j(x).$$

For ease of presentation, in what follows we often assume that $S \subseteq \mathbb{R}^n_+$. However, as shown in Section 3.1 this assumption can be made without loss of generality for compact sets and can be removed after doubling the number of variables for non-compact sets. Now, we prove the existence of copositive certificates under the extra assumption $S \subseteq \mathbb{R}^n_+$.

**Theorem 1.** Let $h_1, \ldots, h_m \in \mathbb{R}[x]$, and $S = \{x \in \mathbb{R}^n_+ : h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}$ be non-empty. Let $d_{\text{max}} = \max\{\deg h_1, \ldots, \deg h_m, \lfloor \frac{\deg p}{2}\rfloor\}$. Assume that $p \in \mathcal{P}_{2d_{\text{max}}}^+(S)$ and $\tilde{p} \in \mathcal{P}_{2d_{\text{max}}}^+(\tilde{S} \setminus \{0\})$. Then there exists $F \in \mathcal{P}_{2d_{\text{max}}}^+(\mathbb{R}^{n+m}_+)$ such that

$$(1 + e^\top x)^{2d_{\text{max}}-\deg p}p(x) = F(x, h_1(x), \ldots, h_m(x)).$$

**Proof.** Let $d_j = \deg h_j$, $j \in \{1, \ldots, m\}$. Define $g_j : \mathbb{R}^{n+m} \to \mathbb{R}$ as

$$g_j(x, u) := \left((1 + e^\top x)^{d_{\text{max}}-d_j}h_j(x) - u_j^{d_{\text{max}}}\right)^2$$

for $j = 1, \ldots, m$. Let

$$U = \{(x, u) \in \mathbb{R}^{n+m}_+ : g_1(x, u) = 0, \ldots, g_m(x, u) = 0\},$$

and let $q(x) := (1 + e^\top x)^{2d_{\text{max}}-\deg p}p(x)$. We apply Proposition 2 to $U$ and $q$. To do this, we first check that the assumptions of the proposition hold. First, note that $S$ non-empty implies $U$ non-empty. Also, for any $(x, u) \in U$ we have $x \in S$ and thus $q(x) > 0$; that is, $q \in \mathcal{P}_{2d_{\text{max}}}^+(U)$. Moreover, let $(z, v) \in \tilde{U}$. We have that $\tilde{g}_j(z, v) = (\tilde{h}_j(z)(e^\top z)^{d_{\text{max}}-d_j} - v_j^{d_{\text{max}}})^2$, $j = 1, \ldots, m$. Hence, if $z = 0$, then $v = 0$. If $z \neq 0$, then $\tilde{h}_j(z)(e^\top z)^{d_{\text{max}}-d_j} = v_j^{d_{\text{max}}} \geq 0$ for $j = 1, \ldots, m$. Therefore $z \in \tilde{S}$, which implies $\tilde{q}(z) = (e^\top z)^{2d_{\text{max}}-\deg p}\tilde{p}(z) > 0$, since $\tilde{p} \in \mathcal{P}_{2d_{\text{max}}}^+(\tilde{S} \setminus \{0\})$. Hence $\tilde{q} \in \mathcal{P}_{2d_{\text{max}}}^+(\tilde{U} \setminus \{0\})$.

Proposition 2 implies that there is $G \in \mathcal{P}_{2d_{\text{max}}}^+(\mathbb{R}^{n+m}_+)$ and $\alpha_j \in \mathbb{R}$ such that

$$q(x) = G(x, u) + \sum_{j=1}^m \alpha_j g_j(x, u) = G(x, u) + \sum_{j=1}^m \alpha_j \left((\tilde{h}_j(x)(1 + e^\top x)^{d_{\text{max}}-d_j} - u_j^{d_{\text{max}}})^2\right).$$

Since the right-hand side of the representation depends on $x$ and $u$ while the left-hand side depends on $x$ only, $u$ has to cancel out on the right-hand side. Since $\alpha_j \in \mathbb{R}$ and $g_j(x, u)$ depends on $u_j$ only, the monomials with with $u_1, \ldots, u_m$ in the expression marked by * do not cancel out with each other. Thus all these monomials have to cancel out with monomials of $G$. Moreover, $G$ cannot contain any other monomials with $u_1, \ldots, u_j$. Therefore in all monomials in $G$ the degrees of $u_j$ are $d_{\text{max}}$ or $2d_{\text{max}}$, for all $j \in \{1, \ldots, m\}$. 

Kuryatnikova [42] considers the particular case in which the semialgebraic set of interest is defined by equality constraints only. Proposition 2 is the stepping stone to our copositive certificates of non-negativity.
Now, taking \( u_j = (1 + e^T x)^{d_{\max} - d_j} h_j(x) \) for all \( j \in \{1, \ldots, m\} \), we obtain
\[
(1 + e^T x)^{2d_{\max} - \deg p} p(x) = F(x, h_1(x)(1 + e^T x)^{d_{\max} - d_1}, \ldots, h_m(x)(1 + e^T x)^{d_{\max} - d_m}),
\]
where \( F(x, u_1, \ldots, u_m) := G(x, u_1^{1/d_{\max}}, \ldots, u_m^{1/d_{\max}}) \) is a polynomial. To finish, notice that \( G \in \mathcal{P}_{2d_{\max}}(\mathbb{R}_+^{n+m}) \) implies \( F \in \mathcal{P}_{2d_{\max}}(\mathbb{R}_+^{n+m}) \).

Next, we show a stronger version of Theorem 1 for compact sets. Namely, for compact sets the pre-multiplier \((1 + e^T x)^{2d_{\max} - \deg p}\) can be omitted, and the copositive certificate \( F \) belongs to the interior of the cone of copositive polynomials.

**Theorem 2.** Let \( h_1, \ldots, h_m \in \mathbb{R}[x] \), and let \( S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \} \) be non-empty. Define \( d_{\max} = \max \{ \deg h_1, \ldots, \deg h_m, \frac{\deg p}{2} \} \). Let \( M > 0 \) be such that \( S \subseteq \{ x \in \mathbb{R}^n : e^T x \leq M \} \). If \( p \in \mathcal{P}_{2d_{\max}}^+(S) \), then there exists \( F \in \text{int} \mathcal{P}_{2d_{\max}}(\mathbb{R}_+^{n+m+1}) \) such that
\[
p(x) = F\left( x, h_1(x), \ldots, h_m(x), M - e^T x + \sum_{j=1}^m ((1 + M)^{d_j} ||h_j|| - h_j(x)) \right).
\]

**Proof.** Since \( S \) is bounded, \( S \) is compact. Since \( p \in \mathcal{P}_{2d_{\max}}^+(S) = \text{int} \mathcal{P}_{2d_{\max}}(S) \) (recall (2)) and \( S \) is compact, there exists \( \varepsilon > 0 \) such that \( q(x) = p(x) - \varepsilon(1 + e^T x)^{2d_{\max}} \in \mathcal{P}_{2d_{\max}}^+(S) \). Let \( d_j = \deg h_j \), \( j \in \{1, \ldots, m\} \). Define \( g_j : \mathbb{R}^{n+m} \to \mathbb{R} \) as \( g_j(x, u) := (h_j(x) - u_j)^2 \) for \( j = 1, \ldots, m \). Also, let \( M = \sum_{j=1}^m (1 + M)^{d_j} ||h_j|| \) and
\[
U := \left\{ (x, u, v) \in \mathbb{R}_+^{n+m+1} : g_j(x, u) = 0, (\hat{M} + M - e^T x - e^T u - v)^2 = 0 \right\}.
\]
We apply Proposition 2 to \( U \) and \( q \). To do this, we first check that the assumptions of the proposition hold. Let \( x \in S \). For \( j = 1, \ldots, m \) let \( u_j = h_j(x) \geq 0 \), from Lemma 1. Let \( v = \hat{M} + M - e^T x - e^T u \geq 0 \), from the assumption on \( M \). Thus \( U \) is non empty as \( (x, u, v) \in U \). For any \( (x, u, v) \in U \), we have \( x \in S \) and thus \( q(x) > 0 \); that is, \( q \in \mathcal{P}_{2d_{\max}}^+(U) \). Moreover, \( (x, u, v) \in U \) implies \( (x, u, v) \in \mathbb{R}_+^{n+m+1} \) and \( -e^T u - e^T x = v \). Therefore \( \hat{U} = \{0\} \). Hence \( \hat{q} \in \mathcal{P}_{2d_{\max}}^+(\hat{U} \setminus \{0\}) \). Thus, Proposition 2 implies that there is \( G \in \text{int} \mathcal{P}_{2d_{\max}}(\mathbb{R}_+^{n+m+1}) \), \( \alpha_j \in \mathcal{P}_{2d_{\max} - d_j}(x, u, v) \), for all \( j \in \{1, \ldots, m\} \), and \( \beta \in \mathbb{R}_{2d_{\max} - 2}[x, u, v] \) such that
\[
q(x) = G(x, u, v) + \sum_{j=1}^m \alpha_j(x, u, v)g_j(x, u) + \beta(x, u, v)(\hat{M} + M - e^T x - e^T u - v)^2.
\]
Now, for any given \( x \), take \( u_j = h_j(x) \) for \( j \in \{1, \ldots, m\} \), and \( v = \hat{M} + M - e^T x - e^T u \) to obtain
\[
p(x) = G\left( x, h_1(x), \ldots, h_m(x), M - e^T x + \sum_{j=1}^m ((1 + M)^{d_j} ||h_j|| - h_j(x)) \right) + \varepsilon(1 + e^T x)^{2d_{\max}}
\]
\[
= F\left( x, h_1(x), \ldots, h_m(x), M - e^T x + \sum_{j=1}^m ((1 + M)^{d_j} ||h_j|| - h_j(x)) \right),
\]
where \( F(x, u, v) = G(x, u, v) + \varepsilon(1 + e^T x)^{2d_{\max}} \). Using Lemma 2, \( G \in \text{int} \mathcal{P}_{2d_{\max}}(\mathbb{R}_+^{n+m+1}) \), and \((1 + e^T x)^{2d_{\max}} \in \mathcal{P}_{2d_{\max}}(\mathbb{R}_+^{n+m+1})\), we obtain \( F \in \text{int} \mathcal{P}_{2d_{\max}}(\mathbb{R}_+^{n+m+1}) \).
Notice that (as mentioned in the proof above) under the assumptions of Theorem 2 we have that $M - e^T x + \sum_{j=1}^m \left( (1 + M)^d h_j - h_j(x) \right) \geq 0$ for all $x \in S$, by Lemma 1. Therefore the representation of $p$ we obtain in this theorem is clearly non-negative on $S$ and defines a copositive certificate of non-negativity of $p$ on $S$. Since $F$ in Theorem 2 lies in the interior of the cone of copositive polynomials, we can use the existing certificates of copositivity to obtain new certificates of non-negativity on compact sets (see Section 3).

We would like to emphasize the differences between Theorem 2 and Schmüdgen’s Positivstellensatz (Theorem 3). Let $S = \{ x \in \mathbb{R}_n^+ : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \}$ be compact and let $p \in \mathcal{P}^+(S)$. Schmüdgen’s Positivstellensatz shows that $p(x) = F(x, h_1(x), \ldots, h_m(x))$ for some $F \in \mathbb{R}[x, u]$ such that $F(x, u) = \sum_{\alpha \in \{0,1\}^m} \sigma_\alpha(x)u^\alpha$, where $\sigma_\alpha$ is an SOS polynomial for all $\alpha \in \{0,1\}^m$. Such $F$ is clearly copositive, however, the degree bounds for $\sigma_\alpha$ can be high and are nontrivial to compute. On the contrary, Theorem 2 guarantees a representation $p(x) = F(x, h_1(x), \ldots, h_m(x), M - e^T x + \sum_{j=1}^m \left( (1 + M)^d j h_j - h_j(x) \right) )$ of degree $2d_{\text{max}}$, where $d_{\text{max}} = \max\{ \deg h_1, \ldots, \deg h_m, \left\lceil \frac{\deg p}{2} \right\rceil \}$. Notice that the situation is similar when comparing Theorem 2 with Putinar’s Positivstellensatz (presented in [3]): the degree bounds for the latter certificate are exponential in the degree of $p$ and the number of variables [3].

### 3.1 Removing the condition $S \subseteq \mathbb{R}_n^+$

In Theorem 1 we require that the basic semialgebraic set $S$ is a subset of the non-negative orthant. In general, the condition can be dropped after doubling the number of variables; that is, by using the common substitution $x_i = y_i - z_i$, with $y_i, z_i \in \mathbb{R}_+$ for each $i \in \{1, \ldots, n\}$. Next, in Lemma 3 we show how to do this, while maintaining the validity of the other assumptions of Theorem 1.

**Lemma 3.** Let $h_1, \ldots, h_m \in \mathbb{R}[x]$, and let $S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \}$ be non-empty. Assume $p \in \mathcal{P}^+(S)$ and $\tilde{p} \in \mathcal{P}^+(\tilde{S} \setminus \{0\})$. Define

$$T := \{(y, z) \in \mathbb{R}_+^{2n} : h_1(y - z) \geq 0, \ldots, h_m(y - z) \geq 0\} = \{(y, z) \in \mathbb{R}_+^{2n} : y - z \in S\},$$

then $T$ is non-empty, $p(y - z) \in \mathcal{P}^+(T)$ and $\tilde{p}(y - z) \in \mathcal{P}^+(\tilde{T} \setminus \{0\})$.

**Proof.** The statement follows after noticing that $x \in S$ implies $(\max\{0, x\}, -\min\{0, x\}) \in T$ and $\tilde{T} = \{(y, z) \in \mathbb{R}_+^{2n} : y - z \in \tilde{S}\}$. \hfill $\square$

**Theorem 3.** Let $h_1, \ldots, h_m \in \mathbb{R}[x]$, and let $S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \}$ be non-empty. Denote $d_{\text{max}} = \max\{ \deg h_1, \ldots, \deg h_m, \left\lceil \frac{\deg p}{2} \right\rceil \}$. Assume that $p \in \mathcal{P}_{2d_{\text{max}}}^+(S)$ and $\tilde{p} \in \mathcal{P}_{2d_{\text{max}}}^+(\tilde{S} \setminus \{0\})$. Then there is $F \in \mathcal{P}_{2d_{\text{max}}}^+(\mathbb{R}^{2n})$ such that

$$(1 + e^T y + e^T z)^{2d_{\text{max}} - \deg p} p(y - z) = F(y, z, h_1(y - z), \ldots, h_m(y - z)).$$

**Proof.** Define $T := \{(y, z) \in \mathbb{R}_+^{2n} : h_1(y - z) \geq 0, \ldots, h_m(y - z) \geq 0\} = \{(y, z) \in \mathbb{R}_+^{2n} : y - z \in S\}$. By Lemma 3, the conditions of Theorem 1 are satisfied for the polynomial $p(y - z) \in \mathbb{R}[y, z]$ and the set $T \subseteq \mathbb{R}^{2n}$. Thus the result follows after applying Theorem 1 to $p(y - z) \in \mathbb{R}[y, z]$ and $T \subseteq \mathbb{R}_+^{2n}$. \hfill $\square$

For compact semialgebraic sets $S \subseteq \mathbb{R}^n$ that do not belong to the non-negative orthant, doubling the number of the variables is not needed since we can translate the set to the non-negative orthant. Similar to Lemma 3, the conditions of Theorem 1 will be maintained after applying the translation. We use this fact in Section 3 to reformulate PO problems over compact sets (see the proof of Corollary 3).
3.2 Genericity of strong positivity

We say that a property holds generically on a given set if it holds on a dense open subset of this set. In this section we show that the strong positivity condition holds generically. First we introduce some additional definitions. We define the homogenization of \( p \in \mathbb{R}[x] \) [see, e.g., 73] as the polynomial

\[
p^h(x_0, x) = p \left( \frac{x_0}{x}, \ldots, \frac{x_0}{x} \right) x_0^{\deg(p)}.
\]

(5)

Notice that by construction,

\[
p(x) = p^h(1, x) \text{ and } \tilde{p}(x) = p^h(0, x).
\]

(6)

**Definition 3.** Let \( h_1, \ldots, h_m \in \mathbb{R}[x] \) and let \( S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \} \). We denote by \( S^h \) the following set

\[
S^h = \left\{ (x_0, x) \in \mathbb{R}^{n+1} : x_0 \geq 0, \ h_1^h(x_0, x) \geq 0, \ldots, h_m^h(x_0, x) \geq 0 \right\}.
\]

(7)

**Definition 4** (Guo et al. 26, 27 and Nie 57). The semialgebraic set \( S \) is called closed at infinity if

\[
\text{cl}(\text{cone}(\{1\} \times S)) = S^h
\]

(8)

Closedness at infinity is one of the sufficient conditions for hierarchies of relaxations to PO problems proposed in [26, 27, 57] to converge to the optimal value [see, e.g., 57, Thm 2.5, condition (d)]. In [27, 57], this condition is shown to hold generically.

**Proposition 3** (Nie 57, Sec. 3, Guo et al. 27, Sec. 2.2). Generically, a basic semialgebraic set \( S \) is closed at infinity; that is, generically \( \text{cl}(\text{cone}(\{1\} \times S)) = S^h \).

To connect closedness at infinity with strong positivity, we introduce the horizon cone of \( S \subseteq \mathbb{R}^n \),

\[
S^\infty := \{ x : (0, x) \in \text{cl}(\text{cone}(\{1\} \times S)) \}.
\]

The notation stems from the work by Peña et al. 65 who use an alternative definition of \( S^\infty \) to obtain completely positive reformulations of equality constrained PO problems.

**Lemma 4.** Let \( S \subseteq \mathbb{R}^n \). Then

(i) If \( p \in \mathbb{R}[x] \) is bounded on \( S \) from below, then \( \tilde{p} \in \mathcal{P}_{\deg p}(S^\infty) \).

(ii) If \( S \) is a basic semialgebraic set, then \( S^\infty \subseteq \tilde{S} \).

**Proof.** Statement (ii) follows from (i). Statement (i) is proved in 65, Lemma 1. \( \square \)

Using the horizon cone and Lemma 4 we can characterize \( \text{int} \mathcal{P}_d(S) \) for unbounded \( S \).

**Proposition 4.** Let \( S \subseteq \mathbb{R}^n \) be unbounded. Then \( \text{int} \mathcal{P}_d(S) = \{ p \in \mathbb{R}_{=d}[x] : p \in \mathcal{P}_d^+(S^\infty \setminus \{0\}) \} \).

**Proof.** Let \( p \in \text{int} \mathcal{P}_d(S) \), then it follows that \( p \in \mathbb{R}_{=d}[x] \) and \( p \in \mathcal{P}_d^+(S) \). To show that \( \tilde{p} \in \mathcal{P}_d^+(S^\infty \setminus \{0\}) \), let \( y \in S^\infty, y \neq 0 \). Without loss of generality, \( y_1 > 0 \). Then, for some \( \varepsilon > 0 \) the polynomial \( q(x) = p(x) - \varepsilon x_1^d \in \mathcal{P}_d(S) \). From Lemma 4(i), \( \tilde{q} \in \mathcal{P}_d(S^\infty) \), therefore \( \tilde{p}(y) \geq \varepsilon y_1^d > 0 \). Thus, \( \text{int} \mathcal{P}_d(S) \subseteq \{ p \in \mathbb{R}_{=d}[x] : p \in \mathcal{P}_d^+(S), \tilde{p} \in \mathcal{P}_d^+(S^\infty \setminus \{0\}) \} \). To show that \( \text{int} \mathcal{P}_d(S) \supseteq \{ p \in
\[ \mathbb{R}_{=d}[x] : p \in \mathcal{P}_{d}^{+}(S), \tilde{p} \in \mathcal{P}_{d}^{+}(S^{\infty} \setminus \{0\}) \], let \( p \in \mathbb{R}_{=d}[x] \) such that \( p \in \mathcal{P}_{d}^{+}(S) \) and \( \tilde{p} \in \mathcal{P}_{d}^{+}(S^{\infty} \setminus \{0\}) \). For the sake of contradiction, assume \( p \notin \text{int} \mathcal{P}_{d}(S) \). Then there exists \( q \in \mathbb{R}_{=d}[x] \) such that for \( k = 1, 2, \ldots \), there exists \( x^{k} \in S \) such that

\[ p(x^{k}) - \frac{1}{k}q(x^{k}) < 0. \]

The sequence \( x^{k}, k = 1, \ldots \) is unbounded. Define \( \lambda^{k} := \frac{1}{\|x^{k}\|^{2}}, k = 1, \ldots \) so that \( \lim_{k \to \infty} \lambda^{k} = 0 \). The sequence \( \lambda^{k}x^{k}, k = 1, \ldots \) is bounded and thus has a convergent subsequence with a limit \( y \in S' := \{ y \in S^{\infty} : \|y\| = 1 \} \). We have then, for all \( \varepsilon \),

\[ 0 > \lim_{k \to \infty} (\lambda^{k})^{d}(p(x^{k}) - \varepsilon q(x^{k})) = \begin{cases} \tilde{p}(y), & \text{if } \deg q < d \\ \tilde{p}(y) - \varepsilon \tilde{q}(y), & \text{if } \deg q = d. \end{cases} \]

But \( \tilde{p} \in \mathcal{P}_{d}^{+}(S') \) and \( S' \) is compact. Thus for some \( \varepsilon > 0 \) small enough we obtain a contradiction. \( \square \)

Lemma 4(ii) and Proposition 4 together imply that every polynomial of degree \( d \) that is strongly positive on \( S \) is in \( \text{int} \mathcal{P}_{d}(S) \) (as stated in Proposition 4).

**Corollary 1.** Let \( d > 0 \). Generically, given a basic semialgebraic set \( S \) and a polynomial \( p \in \mathcal{P}_{d}(S) \), we have that \( p \) is strongly positive on \( S \).

**Proof.** From Proposition 4, generically \( \text{cl}(\text{cone}(\{1\} \times S)) = S^{h} \). Hence \( \tilde{S} = \{ x : (0, x) \in S^{h} \} = S^{\infty} \). If \( S \) is compact, then \( \tilde{S} = S^{\infty} = \{0\} \). Otherwise, using Proposition 4

\[ \text{int} \mathcal{P}_{d}(S) = \left\{ p \in \mathbb{R}_{=d}[x] : p \in \mathcal{P}_{d}^{+}(S), \tilde{p} \in \mathcal{P}_{d}^{+}(\tilde{S} \setminus \{0\}) \right\}. \]

Since \( p \in \mathcal{P}_{d}(S) \), generically \( p \in \text{int} \mathcal{P}_{d}(S) \), and thus \( p \) is strongly positive. \( \square \)

Since we are especially interested in copositive polynomials, we next look at the interior of the cone of copositive polynomials of degree at most \( d \). Proposition 4 implies the following characterizations of the interior of \( \mathbb{R}^{n}_{=} \).

**Corollary 2.** For any \( p \in \mathbb{R}_{=d}[x] \) the following statements are equivalent

(i) \( p \in \text{int} \mathcal{P}_{d}^{+}(\mathbb{R}^{n}_{+}) \).

(ii) \( \deg p = d, p \in \mathcal{P}_{d}^{+}(\mathbb{R}^{n}_{+}) \) and \( \tilde{p} \in \mathcal{P}_{d}^{+}(\mathbb{R}^{n}_{+} \setminus \{0\}) \).

(iii) \( \deg p = d \) and \( p^{h} \in \mathcal{P}_{d}^{+}(\mathbb{R}^{n+1}_{+} \setminus \{0\}) \).

**Proof.** Statement (ii) follows from Proposition 4. Statement (iii) follows from statement (ii) and (6). \( \square \)

### 3.3 Examples

By Proposition 4, if \( S \) is closed at infinity, then generically a non-negative polynomial on \( S \) is strongly positive on \( S \). Next, we present some examples of sets that are closed or not closed at infinity and show several sufficient conditions for closedness at infinity. One could expect that this condition is always satisfied for compact sets or for sets generated by one constraint. However, Example 4 shows that both statements are false.
Example 1 (Violation of closedness at infinity for a compact set generated by one constraint). Let \( h(x_1, x_2) = -x_1^4 - x_2^2 + 1 \). And let \( S = \{(x_1, x_2) \in \mathbb{R}^2 : h(x_1, x_2) \geq 0\} \). Notice that \((0, 0) \in S\), so \( S \) is non-empty. Also, \( S \subseteq [-1,1]^2 \) and thus compact since it is a bounded basic closed semialgebraic set. We have \((0, 0, 1) \in S^h\), but we claim that \((x_0, x_1, x_2) \in \text{cl} \left( \text{cone}(\{1\} \times S) \right) \) and \( x_0 = 0 \) implies \( x_2 = 0 \). This is because for any \((x_0, x_1, x_2) \in \text{cone}(\{1\} \times S) \) we have \( x_1^4 + x_2^2x_0^2 \leq x_0^4 \) which implies \(|x_2| \leq x_0\).

Example 2 (Violation of closedness at infinity for an unbounded set). Let \( h_1(x) = x_1, h_2(x) = x_2, h_3(x) = (x_1x_2 + 1)(x_1 - x_2) \) and let

\[
S = \{ x \in \mathbb{R}^2 : h_1(x) \geq 0, h_2(x) \geq 0, h_3(x) \geq 0 \}.
\]

For any \( t \geq 0 \) we have that \( (0, 0, t) \in S^h \). On the other hand, \( x = (x_0, x_1, x_2) \in \text{cone}(\{1\} \times S) \setminus \{0\} \) we have \( x_1 \geq 0, x_2 \geq 0 \) and \( (x_1x_2 + x_0^2)(x_1 - x_2) \geq 0 \), that is \( x_1 \leq x_2 \). Thus \((0, 0, t) \notin \text{cl} \left( \text{cone}(\{1\} \times S) \right) \) for \( t > 0 \).

Now, we turn our attention to sufficient conditions for closedness at infinity. As this property holds generically (see Proposition 3), it is not a surprise that there are several families of semialgebraic sets for which closedness at infinity is straightforward to verify.

Proposition 5 (Proposition 4.22. in [42]). Let \( h_1, \ldots, h_m \in \mathbb{R}[x] \) and \( S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \} \). If any of the following conditions hold, then \( \text{cl} \left( \text{cone}(\{1\} \times S) \right) = S^h \).

(i) \( h_m(x) = N - \|x\|^2 \) for some \( N > 0 \).
(ii) \( h_1, \ldots, h_m \) are homogeneous.
(iii) \( h_i(x) = q^i_1(x) \cdots q^i_{k_i}(x) \) for some \( k_i > 0 \) and \( q^i_1, \ldots, q^i_{k_i} \in \mathbb{R}[1,x] \). Notice that in this case \( S \) is a union of polyhedra.
(iv) \( n \geq 2 \) and \( S = \{ x \in \mathbb{R}^n : (x_n - \sum_{i=1}^{n-1} x_i^2 - b)q(x) \geq 0, x_n \geq 0 \} \), where \( b \in \mathbb{R} \) and \( q \in \mathbb{R}[x] \) is such that \( \hat{q} \in \mathcal{P}^+(\mathbb{R}^n \setminus \{0\}) \).

An important question in algebraic geometry and in optimization is when the non-negativity of a polynomial on a set can or cannot be certified using the quadratic module [see 7]. Putinar [68] answers this question affirmatively when the quadratic module is archimedean. Putinar’s Positivstellensatz [68] underlines LMI approximations of PO problems with compact or “compactifiable” feasible sets [see, e.g., 36, 43, 44, 17] since one could add the norm-constraint \( N - \|x\|^2 \geq 0 \) to the description of such \( S \). In our next example, we show that copositive certificates of non-negativity could exist on the sets where the certificates based on the quadratic module do not exist.

We say that \( \{h_1(x), \ldots, h_m(x)\} \) satisfies the strong moment property (SMP) if the quadratic module generated by them contains \( \text{int} \mathcal{P}(S) \), where \( S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \} \) [see 17, Prop. 2.6, cond. (1)]. Equivalently, SMP means that every \( p \in \text{int} \mathcal{P}(S) \) can be written in the form

\[
p(x) = \sigma_0(x) + \sum_{j=1}^m \sigma_j(x)h_j(x), \text{ where } \sigma_j \text{ is SOS polynomial for all } j = 0, \ldots, m.
\]

Definition 5 (Tentacles). Given a compact set \( K \subseteq \mathbb{R}^n \) with nonempty interior, a tentacle of \( K \) in direction \( z \) is the set \( T_{K,z} := \{ (\lambda^1 x_1, \ldots, \lambda^n x_n) : \lambda \geq 1, x = (x_1, \ldots, x_n) \in K \} \).

Netzer [55] shows that if \( S \) contains tentacles of a certain type, this set does not satisfy the SMP.
Example 3 (A set that violates the SMP but is closed at infinity). Let \( n \geq 2 \) and consider the set
\[
S = \left\{ x \in \mathbb{R}^n : \left( x_n - \sum_{i=1}^{n-1} x_i^2 \right) \left( 2 \sum_{i=1}^{n-1} x_i^2 - x_n \right) \geq 0, x_n \geq 0 \right\}.
\]
Figure 1 shows this set for \( n = 2 \).

Figure 1: Illustration of the set \( S = \{ x \in \mathbb{R}^2 : (x_2 - x_1^2)(2x_1^2 - x_2) \geq 0, x_2 \geq 0 \} \) (in gray).

This example is similar to one presented in [55, Section 6]. Let 
\[z = (1, \ldots, 1, 2)^\top \]
and
\[K = \{ x \in \mathbb{R}^n : |x_n - n + \frac{1}{10n}| \leq \frac{1}{10n} \text{ and } |x_i - 1| \leq \frac{1}{10n} \text{ for } i = 1, \ldots, n-1 \}.
\]
We claim that the tentacle \( T_{K,z} \subseteq S \). From [55, Thm. 5.4] we obtain that \( \{(x_n - \sum_{i=1}^{n-1} x_i^2)(2 \sum_{i=1}^{n-1} x_i^2 - x_n)\} \) does not satisfy the SMP. Thus, for some \( d > 0 \) there is \( p \in \text{int} \mathcal{P}_d(S) \) for which no certificate of non-negativity of the form (9) exists. On the other hand, from Proposition 5(iv) the closedness at infinity condition holds. Hence Theorem 3 implies that for all \( p \in \text{int} \mathcal{P}_d(S) \) a copositive certificate of non-negativity exists.

To prove the claim, first notice that if \( x \in S \), then for every \( \lambda > 0 \), \( (\lambda x_1, \ldots, \lambda x_{n-1}, \lambda^2 x_n) \in S \). Thus, it is enough to show that \( K \subseteq S \). Since \( n \geq 2 \), for \( x \in K \) we have
\[
x_n - \sum_{i=1}^{n-1} x_i^2 \geq n - \frac{1}{2} - \frac{1}{10n} - (n - 1) \left( 1 + \frac{1}{10n} \right)^2 = \frac{3}{10} + \frac{1 + 9n}{100n^2} > 0,
\]
\[
2 \sum_{i=1}^{n-1} x_i^2 - x_n \geq 2(n - 1) \left( 1 - \frac{1}{10n} \right)^2 - n + \frac{1}{2} - \frac{1}{10n} = n - \frac{19}{10} + \frac{16n - 1}{50n^2} > 0.
\]

4 LP-based and sparse certificates of non-negativity on compact sets

Using Theorems 1, 2, and 3 one can construct, from any certificate of copositivity, a corresponding certificate of non-negativity for any given semialgebraic set \( S \) and any strongly positive polynomial on \( S \). In this section we use two certificates of copositivity to illustrate this approach and obtain new certificates of non-negativity.

Our first example (see Corollary 4) is based on the celebrated Pólya’s certificate of copositivity. Applying this certificate in optimization leads to LP approximations of PO problems. More importantly, the certificate can be strengthened so that instead of non-negative constant polynomials...
(resulting in LP approximations) one can use any set of non-negative polynomials with non-zero constant terms, such as SOS, scaled diagonally dominant SOS (SDSOS) \[^4\]}, hyperbolic polynomials \[^6\], non-negative circuit polynomials \[^3\], etc. (see Corollary \[^5\]). As a result, one obtains convergent LMI hierarchies of approximations to PO problems in practice provide stronger bounds than the mentioned LP hierarchies [see, e.g., \[^40\], \[^41\]].

Our second example is a sparse certificate of non-negativity for generic semialgebraic sets. More precisely, we present an SOS-based certificate where all but two SOS polynomials are univariate which results in the use of lower dimensional SDP constraints in LMI approximations of PO problems (see Corollary \[^3\]). To obtain this result, we propose a new sparse certificate of copositivity in Theorem \[^5\].

In both examples we use certificates of copositivity which are guaranteed to exist for polynomials in \(\mathbb{P}_d(\mathbb{R}_+^{n})\). For this reason we limit ourselves to compact sets, in order to take advantage of Theorem \[^2\] which guarantees the existence of copositive certificate lying in the interior of the cone of copositive polynomials. Also, for ease of presentation we consider semialgebraic sets \(S \subset \mathbb{R}^n_+\) since any compact set can be translated to the non-negative orthant. Corollary \[^5\] in Section \[^5\] shows how to apply the results in this section for general compact sets.

The certificates we obtain in this section use rational polynomial expressions to certify the non-negativity of a polynomial on a set \(S = \{x \in \mathbb{R}^n_+ : h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}\). That is, the certificates are (basically) of the form \(G(h_1(x), \ldots, h_m(x))p(x) = F(h_1(x), \ldots, h_m(x))\), where \(F, G\) are copositive polynomials. The existence of such rational certificates is in general guaranteed by the Krivine-Stengle Positivstellensatz \[^3\], \[^82\]. However, the problem of finding such certificates is not tractable in general because the denominator \(G\) is unknown [see, e.g., \[^36\], for more details]. The rational certificates of non-negativity introduced in this section have fixed denominators. Hence, these certificates result in tractable lower bound approximations to PO problems. We present examples of such approximations in Section \[^5\].

### 4.1 LP certificates

Our first illustration of constructing new certificates of non-negativity using certificates of copositivity is based on Pólya’s certificate of copositivity [see, e.g., \[^29\]].

**Theorem 4** (Pólya’s Positivstellensatz \[^29\], Sec. 2.2). Let \(F \in \mathbb{R}[x]\) be a homogeneous polynomial such that \(F \in \mathbb{P}^+(\mathbb{R}^n_+ \setminus \{0\})\). Then for some \(r > 0\) all the coefficients of \((e^r x)^t F(x)\) are non-negative.

**Corollary 3.** Let \(d > 0\) and \(F \in \mathbb{R}_d[x]\) be such that \(F \in \mathbb{P}_d(\mathbb{R}^n_+).\) Then for some \(r > 0\) all the coefficients of \((1 + e^r x)^t F(x)\) are non-negative.

**Proof.** The result follows from Corollary \[^2\](iii) by applying Theorem \[^4\] to \(F^h\).

Combining Theorem \[^2\] and Corollary \[^3\] we obtain the new certificate of non-negativity stated in Corollary \[^4\] below.

For ease of presentation, we use the following notation. As before, for \(n > 0\) and \(d \geq 0\) we define \(\mathbb{N}_d^n = \{\alpha \in \mathbb{N}^n : e^\top \alpha \leq d\}\). Given polynomials \(h_1(x), \ldots, h_m(x)\) and \(\alpha \in \mathbb{N}_d^n\) we use \(h(x)_{\alpha} := \prod_{j=1}^n h_j(x_{\alpha})\). In particular \(x^\alpha = \prod_{j=1}^n x_j^{\alpha_j}\). Also, we use \(h(x)\) to arrange the polynomials \(h_1(x), \ldots, h_m(x)\) in an array; that is, \(h(x) := [h_1(x), \ldots, h_m(x)]^\top\).

**Corollary 4.** Let \(p, h_1, \ldots, h_m \in \mathbb{R}[x]\) and \(S = \{x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}\) be non-empty. Also, let \(M > 0\) be such that \(S \subseteq \{x \in \mathbb{R}^n_+ : e^\top x \leq M\}\), and let any \(a \in \mathbb{R}^n_+\) and \(b \in \mathbb{R}^n_+\) be given.
Denote $d_{\text{max}} = \max\{\deg h_1, \ldots, \deg h_m, \lceil \frac{\deg p}{2} \rceil\}$. If $p \in \mathcal{P}_{d_{\text{max}}^+}(S)$ then there exists $r \geq 0$ and $c_{\alpha, \beta, \eta} \geq 0$ for $(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{d_{\text{max}}(2d_{\text{max}}+r)}$ such that

$$
(1 + a^T x + b^T h(x))^r p(x) = \sum_{(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{d_{\text{max}}(2d_{\text{max}}+r)}} c_{\alpha, \beta, \gamma} x^\alpha h(x)^\beta (M - e^T x)^\gamma,
$$

(10)

Proof. Let $d_j = \deg h_j$. Denote $g_j(x) = (1 + M)^{d_j} \|h_j\| - h_j(x)$. By Theorem 2 there is $F \in \text{int} \mathcal{P}_{d_{\text{max}}^+}(\mathbb{R}^{n+m+1}_+)$ such that

$$
p(x) = F(x, h(x), M - e^T x + e^T g(x)).
$$

(11)

Let $\dot{M} = 1 + M + \sum_{j=1}^m (1 + M)^{d_j} \|h_j\|$. $\dot{a} = Ma + e$, $\dot{b}_j = Mb + e$. By construction $\dot{a} > 0$, $\dot{b} > 0$. Denote $\dot{x}_i = x_i \dot{a}_i$ for $i \in \{1, \ldots, n\}$ and $\dot{u}_j = u_j \dot{b}_j$ for $j \in \{1, \ldots, m\}$. Then

$$F(\dot{x}, \dot{u}, v) \in \text{int} \mathcal{P}_{d_{\text{max}}^+}(\mathbb{R}^{n+m+1}_+).
$$

By Theorem 3 we obtain that there is $r \geq 0$ and $k_{\alpha, \beta, \gamma} \geq 0$ for $(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{d_{\text{max}}+r}$ such that

$$
(1 + e^T \dot{x} + e^T \dot{u} + v)^r F(\dot{x}, \dot{u}, v) = \sum_{(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{d_{\text{max}}+r}} k_{\alpha, \beta, \gamma} x^\alpha \dot{u}^\beta v^\gamma.
$$

Thus,

$$
(1 + \dot{a}^T x + \dot{b}^T u + v)^r F(x, u, v) = \sum_{(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{d_{\text{max}}+r}} \hat{k}_{\alpha, \beta, \gamma} x^\alpha u^\beta v^\gamma,
$$

where $\hat{k}_{\alpha, \beta, \gamma} \geq 0$ for all $(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{d_{\text{max}}+r}$. Using (11) we obtain that

$$
(1 + \dot{a}^T x + \dot{b}^T h(x) + M - e^T x + e^T g(x))^r p(x) = \sum_{(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{d_{\text{max}}+r}} \hat{k}_{\alpha, \beta, \gamma} x^\alpha h(x)^\beta (M - e^T x + e^T g(x))^\gamma.
$$

(12)

To finish the proof, notice that

$$
1 + \dot{a}^T x + \dot{b}^T h(x) + M - e^T x + e^T g(x) = 1 + \left(Ma + e\right)^T x + \sum_{j=1}^m (Mb_j + 1) h_j(x)
$$

$$
+ \sum_{j=1}^m \left((1 + M)^{d_j} \|h_j\| - h_j(x)\right) + M - e^T x
$$

$$
= \dot{M}(1 + a^T x + b^T h(x)),
$$

which, up to a positive constant multiplier, is equivalent to the left-hand side factor of $p(x)$ in (10).

Also, for each $j = 1, \ldots, m$ we have

$$
g_j(x) = (1 + M)^{d_j} \|h_j\| - h_j(x)
$$

$$
= \|h_j\|((1 + M)^{d_j} - (1 + e^T x)^{d_j}) + \sum_{\alpha \in \mathbb{N}^n_{d_j}} C_{d_j, \alpha}(\|h_j\| - (h_j)_\alpha)x^\alpha
$$

$$
= \|h_j\|(M - e^T x)\sum_{k=0}^{d_j-1} (M + 1)^{d_j-k-1}(1 + e^T x)^k + \sum_{\alpha \in \mathbb{N}^n_{d_j}} C_{\deg h_j, \alpha}(\|h_j\| - (h_j)_\alpha)x^\alpha.
$$

After replacing the expression for $g_j(x), j = 1, \ldots, m$ above into $e^T g(x)$ in the right hand side of (12), the right-hand side of (12) is equivalent to the right-hand side of (10). \qed
Remark 2. The choice of the vectors \(a, b\) in Corollary 4 is free, and we can set \(a = 0, b = 0\) to eliminate the pre-multiplier in front of \(p(x)\) in (10).

For every \(r \in \mathbb{N}\), the only unknowns in the certificate from Corollary 4 are the non-negative constants \(c_{\alpha, \beta, \gamma} \geq 0\) for all \((\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{\max(2d_{\max}+r)}\). The representation (10) is linear in these constants. As we show in Section 3, we can use the hierarchy (10) for every \(r \in \mathbb{N}\), to obtain LP lower bounds for PO problems over compact semialgebraic sets. Setting \(a = e, b = 0, \gamma = 0\) in (10) results in the certificate by Dickinson and Povh [22], which is guaranteed to exist under the strong positivity assumption \(p \in \mathcal{P}^+(S)\) and \(\tilde{p} \in \mathcal{P}^+(\tilde{S} \setminus \{0\})\).

Recently, it has been a topic of great interest to replace SOS-based certificates by certificates based on other types of non-negative polynomials. The idea is to provide alternative certificates that can lead to LMI relaxation bounds that are computationally cheaper to compute, but still provide quality bounds for the PO problem. This is typically done by replacing the full dimensional SOS polynomials on non-negative certificates based on Putinar’s Positivistellensatz by SDSOS [1] (which result in second-order cone programming (SOCP) relaxations), hyperbolic polynomials [76] (which results hyperbolic programming relaxations) and non-negative circuit polynomials [24, 85] (which result in geometric programming relaxations). Since these alternative sets of polynomials are not necessarily supersets of SOS polynomials, the resulting LMI hierarchies of bounds on PO problems can require additional assumptions to converge [see, for instance, 24], are not proven to converge [e.g., 1] or require the use of additional methods to ensure convergence [see e.g., 2]. In contrast, all earlier mentioned subsets of non-negative polynomials can be used to strengthen the LP-based certificates from Corollary 4 to obtain alternative (and potentially tighter) convergent LMI hierarchies of bounds for PO problems with compact feasible sets.

**Corollary 5.** Let \(p, h_1, \ldots, h_m \in \mathbb{R}[x]\) and \(S = \{x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}\) be non-empty. Also, let \(M\) be such that \(S \subseteq \{x \in \mathbb{R}^n_+ : e^\top x \leq M\}\), and let any \(a \in \mathbb{R}^n_+\) and \(b \in \mathbb{R}^n_+\) be given. Denote \(d_{\max} = \max\{\deg h_1, \ldots, \deg h_m, \lceil \frac{\deg p}{2} \rceil\}\). Let \(d \geq 0\) and let \(K \subset \mathcal{P}_d(\mathbb{R}^n)\) be such that \(\mathbb{R}_+ \subseteq K\). If \(p \in \mathcal{P}_{2d_{\max}}^+(S)\) then there exists \(r \geq 0\) and \(c_{\alpha, \beta, \gamma} \in K\) for \((\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{\max(2d_{\max}+r)}\) such that

\[
(1 + a^\top x + b^\top h(x))^r p(x) = \sum_{(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{\max(2d_{\max}+r)}} c_{\alpha, \beta, \gamma} x^\alpha h(x)^\beta (M - e^\top x)^\gamma,
\]

(13)

The computational benefits of the certificates arising from Corollary 5 have been explored by Kuang et al. [40, 41], who consider setting \(K\) to be: \(\mathbb{R}_+\), as well as the cone of quadratic diagonally dominant SOS (DSOS), SDSOS, and SOS polynomials. The authors conclude that PO-hierarchies based on certificate (13) can be more computationally efficient compared to the broadly used Lasserre’s hierarchies [43, 50], as well as the SDSOS hierarchies in [1].

A final noteworthy characteristic of the LP certificates proposed in this section is that the unknown \(r\) could be small, even without considering the additional strengthenings, beyond the use of non-negative constant polynomials, in Corollary 5. That is, the polynomials involved in the certificate (10) can have low degrees. This implies that the LP one has to solve to find a certificate is not too large, as illustrated by Example 4. This situation is in contrast to the existing limited research on LP certificates of non-negativity of polynomials [see, e.g., 18, 38, 48, 75, for noteworthy examples].

**Example 4** (Low degree convergence). We show that the polyhedral hierarchy (10) could convergence for small \(r\) by considering an instance of the Celis-Dennis-Tapia (CDT) problem [see 14]. This classical problem is concerned with the non-negativity of a quadratic polynomial on the intersection
of two ellipses. Recent advances on this problem have been made thanks to the use of polynomial optimization techniques [see 11]. Specifically, for \( n \geq 3 \) consider the polynomial \( q \in \mathbb{R}[x] \): 

\[
q(x) := -2x_1 + 8x_1 \sum_{i=1}^{n} x_i.
\]

Note that \( q \) is not a copositive polynomial; that is, \( q \not\in \mathcal{P}(\mathbb{R}^n_{+}) \). In particular, 

\[
q(x_1, 0, \ldots, 0) < 0 \text{ for } 0 < x_1 < 1/4.
\]

However, we can use Corollary 4 to certify that \( q \in \mathcal{P}(\mathcal{B}_c \cap \mathcal{B}_{c/2}) \), where 

\[
\mathcal{B}_c = \{ x \in \mathbb{R}^n : b_c(x) := 1 - \|x - c\|^2 \geq 0 \},
\]

is the unitary ball centered at \( c \in \mathbb{R}^n \). In particular notice that 

\[
(1 + e^\top x + b_c(x) + b_{c/2}(x)) q(x) = 8x_1(e^\top x)(b_c(x) + b_{c/2}(x)) + x_1 \left( \left( \frac{5}{2}n - 6 \right) + 8(e^\top x)^2 + 4 \sum_{i=2}^{n} x_i^2 \right),
\]

for \( n \geq 3 \). After expanding the right hand side, the expression above has the form (10) with \( r = 1 \). In particular, this certifies that \( q \) is non-negative on the \( \mathcal{B}_c \cap \mathcal{B}_{c/2} \).

4.2 Sparse certificates

As another illustration of the power of our approach, we construct sparse SOS-certificates of non-negativity of polynomials \( \sigma_0 \) and \( \sigma_1 \), and bivariate homogeneous SOS polynomials \( \hat{\sigma}_0, \ldots, \hat{\sigma}_n \) such that

\[
(1 + e^\top x)^r F(x) = \sigma_0(x) + \sigma_1(x) \sum_{0 \leq i \leq j \leq n} x_i x_j + (1 + e^\top x) \sum_{i=0}^{n} \hat{\sigma}_i (x_i, 1 + e^\top x) x_i,
\]

(14)

where \( x_0 := 1 \).

Combining Theorem [5] and Theorem [2] we obtain a sparse certificate of non-negativity on compact sets.

Corollary 6. Let \( h_1, \ldots, h_m \in \mathbb{R}[x] \), and \( S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \} \) be non-empty. Let \( M > 0 \) be such that \( S \subseteq \{ x \in \mathbb{R}^n : e^\top x \leq M \} \). Denote \( X = (x_1, \ldots, x_n, h_1(x), \ldots, h_m(x), M - e^\top x + \sum_{j=1}^{m} (1 + M)^{dj} \| h_j \| - h_j(x)) \). If \( p \in \mathcal{P}^+(S) \), then there exist \( n \)-variate SOS polynomials \( \sigma_0 \) and \( \sigma_1 \), and univariate SOS polynomials \( \hat{\sigma}_1, \ldots, \hat{\sigma}_{n+m+1} \) such that 

\[
p(x) = \sigma_0(x) + \sigma_1(x) \left( \left( 1 + M + \sum_{j=1}^{m} (1 + M)^{dj} \| h_j \| \right)^2 - 1 - \sum_{i=1}^{n+m+1} X_i^2 \right) + \sum_{i=1}^{n+m+1} \hat{\sigma}_i (X_i) X_i.
\]

(15)
Proof. Denote $X_0 = 1$. From Theorem 5 with $d = \deg p$, and Theorem 2 we obtain that there are $r \geq 0$ and $n$-variate SOS polynomials $\sigma_0$ and $\sigma_1$, and homogeneous bivariate SOS polynomials $\hat{\sigma}_0, \ldots, \hat{\sigma}_{n+m+1}$ such that
\[
(1 + e^t X)^r p(x) = \sigma_0(x) + \sigma_1(x) \sum_{0 \leq i, j \leq n+m+1} X_i X_j + (1 + e^t X) \sum_{i=0}^{n+m+1} \hat{\sigma}_i (X_i, 1 + e^t X) X_i.
\]
Using $X_0 + e^t X = 1 + e^t X = 1 + M + \sum_{j=1}^{m} (1 + M)^d_j \|h_j\|$ and
\[
\sum_{0 \leq i, j \leq n+m+1} 2X_i X_j = (X_0 + e^t X)^2 - \sum_{i=0}^{n+m+1} X_i^2,
\]
we obtain (15), up to a positive constant multiplier.

The certificates constructed in Theorem 5 and Corollary 6 are sparse in the sense that the SOS polynomial multipliers $\hat{\sigma}_i$, $i = 1, \ldots, n+m+1$ are all sparse. Indeed while $\sigma_0$ and $\sigma_1$ are full SOS, each $\hat{\sigma}_i$ is univariate. A univariate SOS of degree $d$ can be represented using a $(d + 1) \times (d + 1)$ SDP matrix which is much smaller than the one needed to represent a multivariate SOS of the same degree.

The rest of this section shows the proof of Theorem 5.

**Lemma 5.** Let $S \subset \mathbb{R}^n$ be non-empty and compact, and let $p \in \mathbb{R}[x]$. Then $p \in \mathcal{P}^+(S \cap \mathbb{R}^n_+)$ if and only if
\[
p(x) = q(x) + \sum_{i=1}^{n} x_i \sigma_i(x_i),
\]
where $\sigma_1, \ldots, \sigma_n$ are univariate SOS polynomials and $q \in \mathcal{P}^+(S)$.

**Proof.** If $S \subset \mathbb{R}^n_+$, then the result is straightforward, thus further in the proof we assume that $S \not\subset \mathbb{R}^n_+$. Without loss of generality, there exists $k \leq n$ such that $\{x \in S, x_i < 0\} \neq \emptyset$ for all $i \in \{1, \ldots, k\}$, and $\{x \in S, x_i < 0\} = \emptyset$ for all $i \in \{k+1, \ldots, n\}$. Since $p \in \mathcal{P}^+(S \cap \mathbb{R}^n_+)$, there exists $\varepsilon > 0$ such that $x \in S$ and $x > -\varepsilon$ implies $p(x) \geq 0$. Also, let $M > 0$ be such that $x \in S$ implies $x < M$. Let $p_{\min}^0 = \min \{p(x) : x \in S \cap \mathbb{R}^n_+\}$, and let $p_{\min}^i = \min \{p(x) : x \in S, x_i \leq -\varepsilon\}$ for $i \in \{1, \ldots, k\}$. Consider the function $f_i(x) = a_i x_i e^{-b_i x_i}$ for some $a_i > 0$, $b_i > 0$. For any $x \in \mathbb{R}^n$ and $i \in \{1, \ldots, n\}$, we have that $f_i(x)$ is positive for $x_i > 0$ and negative for $x_i < 0$. For any $i \in \{1, \ldots, k\}$, we can tailor $a_i$ and $b_i$ so that $\max \{f_i(x) : x \in S : x_i \leq -\varepsilon\} \leq -\varepsilon a_i < \frac{p_{\min}^i}{n}$ and $\max \{f_i(x) : x \in S \cap \mathbb{R}^n_+\} \leq a_i M e^{-b_i M} < \frac{p_{\min}^i}{n}$. For $i \in \{k+1, \ldots, n\}$, we let $f_i(x) = 0$. Defining $f(x) = \sum_{i=1}^{n} f_i(x)$ we obtain $p(x) > f(x)$ for all $x \in S$.

Let $i \in \{1, \ldots, k\}$. We show that $f_i(x)$ can be approximated as closely as desired by $x_i \sigma_i(x_i)$, where $\sigma_i$ is a univariate SOS, which implies $p(x) = q(x) + \sum_{i=1}^{n} x_i \sigma_i(x_i)$ where $q(x) \geq 0$ for all $x \in S$.

For any $l \geq 0$ consider the Taylor approximation of $e^t$ with $2l$ terms:
\[
T_l(t) = \sum_{j=0}^{2l} \frac{t^j}{j!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^{2l}}{(2l)!}.
\]
Since the Taylor series converges uniformly on bounded intervals, by growing $l$, one can approximate $f_i(x)$ to any desired accuracy by $a_i x_i T_l(-b_i x_i)$. Hence it is enough to show that $T_l(t)$ is an SOS,
or equivalently, given that $T_i$ is a univariate polynomial, that $T_i(t) \geq 0$ for all $t$ [see, e.g., 73]. We prove the non-negativity of $T_i$ by contradiction. Assume $T_i$ is nonnegative. Then it must have a zero as $T_i(0) = 1$. Let $t^*$ be the largest zero of $T_i$. Then $t^* < 0$ and $T'_i(t^*) > 0$. But for any $t$, $T'_i(t) = \sum_{j=0}^{2r-1} t^j = T_i(t) - \frac{t^{2r}}{(2r)!}$. Thus $0 < T'_i(t^*) = -\frac{t^{2r}}{(2r)!} < 0$, which is a contradiction. \[ \square\]

**Remark 3.** From the proof of Lemma 5 it follows that one could use the same SOS polynomial $\sigma_i = \sigma$ for $i = 1, \ldots, n$ in (18).

Now, we use the representation from Lemma 5 and some of the known certificates of non-negativity on compact sets [63, 77] to prove Theorem 5.

**Proof of Theorem 5.** Let $\Delta^n$ be the standard simplex in $\mathbb{R}^n$: $\Delta^n = \{x \in \mathbb{R}^n_+: e^\top x = 1\}$. Denote the unit ball in $\mathbb{R}^n$ by $B^n = \{x \in \mathbb{R}^n_+: \|x\|^2 \leq 1\}$. We can write

$$\Delta^{n+1} = \{(x_0, x) \in \mathbb{R}^{n+1}_+: (x_0 + e^\top x - 1)^2 = 0\} \cap B^{n+1}.$$  

Since $F \in \mathcal{P}_d(\mathbb{R}^n_+)$, we have by Corollary 2(iii) that $F^h \in \mathcal{P}_d^+(\mathbb{R}^{n+1}_+ \setminus \{0\}) \in \mathcal{P}^+(\Delta^{n+1})$. Also, from [63, Corollary 2],

$$(x_0 + e^\top x)^2 F^h(x_0, x) = g(x_0, x) + h(x_0, x)(x_0 + e^\top x - 1)^2,$$  

where $h \in \mathcal{P}_d(x_0, x)$ and $q \in \mathcal{P}_{d+2}(\mathbb{R}^{n+1}_+ \cap B^{n+1})$.

Hence by Lemma 5

$$g(x_0, x) = g(x_0, x) + \sum_{i=0}^{n} x_i \hat{\sigma}_i(x_i),$$  

where $g(x_0, x) \in \mathcal{P}(B^{n+1})$ and $\hat{\sigma}_0, \ldots, \hat{\sigma}_n$ are univariate SOS polynomials. By Schm"udgen’s Positivstellensatz [77] (Theorem 8), we obtain

$$g(x_0, x) = \sigma_0(x_0, x) + \sigma_1(x_0, x) \left(1 - \sum_{i=0}^{n} x_i^2\right)$$  

where $\sigma_0, \sigma_1$ are SOS polynomials. Now, we use the substitution $(x_0, x) \rightarrow \left(\frac{1}{1+e^\top x}, \frac{x}{1+e^\top x}\right)$, and (17)-(19) to obtain:

$$F^h \left(\frac{1}{1+e^\top x}, \frac{x}{1+e^\top x}\right) = \sigma'_0\left(\frac{1}{1+e^\top x}, \frac{x}{1+e^\top x}\right) + \sigma'_1\left(\frac{1}{1+e^\top x}, \frac{x}{1+e^\top x}\right) \sum_{n \geq j > 0} x_i^j x_j \sum_{i=0}^{n} \hat{\sigma}_i'\left(\frac{x}{1+e^\top x}\right) \frac{x}{1+e^\top x}.$$  

Note that: (i) from [5], it follows that $F^h(1+e^\top x, 1+e^\top x) = (1+e^\top x)^{-d} F(x)$; (ii) for any even large enough $M \in \mathbb{N}$, we have that if $\sigma(x_0, x)$ is a SOS polynomial, then $(1+e^\top x)^M \sigma\left(\frac{1}{1+e^\top x}, \frac{x}{1+e^\top x}\right)$ is a SOS polynomial in $n$ variables; and (iii) if $\sigma(x_i)$ is a SOS polynomial for any $i = 1, \ldots, n$, then $(1+e^\top x)^M \sigma\left(\frac{x_i}{1+e^\top x}\right)$ is a bivariate homogeneous SOS polynomial in $x_i$ and $(1+e^\top x)$. Hence the theorem follows by multiplying (20) by $(1+e^\top x)^M$ for an even large enough $M \in \mathbb{N}$. \[ \square\]

5. **Copositive certificates of non-negativity in polynomial optimization**

In the spirit of the seminal work of Lasserre [43] and a large body of literature in PO, now we present a convex reformulation of PO problems using Theorem 3. More precisely, we reformulate a
PO problem as an equivalent linear optimization problem over the cone of copositive polynomials of a known fixed degree. An advantage of using the copositive reformulation is that it allows constructing both inner and outer LMI hierarchies of approximations. Thus, it is possible to obtain arbitrarily close upper and lower bounds to the underlying PO problem. Consider the following standard PO problem:

$$
\lambda^* = \inf_x \{ p(x) : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \}.
$$

(21)

**Theorem 6.** Let $p, h_1, \ldots, h_m \in \mathbb{R}[x]$ and $S = \{x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}$ be non-empty. Denote $d_{\text{max}} = \max\{\deg h_1, \ldots, \deg h_m, \lceil \deg p/2 \rceil\}$. Then $\lambda^*_{\text{cop}} \leq \lambda^*$, where

$$
\lambda^*_{\text{cop}} = \sup_{\lambda, F} \lambda
$$

s.t. $$(1 + e^\top y + e^\top z)^{2d_{\text{max}} - \deg p(y - z) - \lambda} = F(y, z, h_1(y - z), \ldots, h_m(y - z)).$$

$F \in \mathcal{P}_{2d_{\text{max}}} (\mathbb{R}^{2n+m})$.

If $S \subseteq \mathbb{R}^n_+$, then one can set $z = 0$ (i.e., $z$ can be eliminated), and $F \in \mathcal{P}_{2d_{\text{max}}} (\mathbb{R}^{n+m})$ in (22). If $\tilde{p} \in \mathcal{P}_{2d_{\text{max}}}^+(\tilde{S} \setminus \{0\})$, then $\lambda^*_{\text{cop}} = \lambda^*$.

**Proof.** If $(\lambda, F)$ is a feasible solution to (22), then $F$ is a copositive certificate of non-negativity for $p(x) - \lambda$ on $S$, that is $\lambda \leq \lambda^*$. Thus $\lambda^*_{\text{cop}} \leq \lambda^*$. Assume $\tilde{p} \in \mathcal{P}^+(\tilde{S} \setminus \{0\})$. If $p$ is unbounded from below on $S$, then (21) is infeasible and its optimal value is $\lambda^* = -\infty$. It then follows from $\lambda^*_{\text{cop}} \leq \lambda^*$ that $\lambda^*_{\text{cop}} = -\infty$ also. Assume therefore that $p$ is bounded from below. Consider any $\lambda < \lambda^*$. Then we have $q := p - \lambda \in \mathcal{P}_{2d_{\text{max}}}^+(S)$, and $\tilde{q} = \tilde{p} \in \mathcal{P}_{2d_{\text{max}}}^+(\tilde{S} \setminus \{0\})$. Hence the result follows by applying Theorem 3 to $q$ and $S$. Finally, if $S \subseteq \mathbb{R}^n_+$, then we can set $z = 0$ by Theorem 1. \hfill \Box

**Corollary 7.** Generically, the defining polynomials of a basic semialgebraic set $S$ and a polynomial $p$ are such that if we minimize $p$ on $S$, then $\lambda^*_{\text{cop}} = \lambda^*$.

**Proof.** By genericity of closedness at infinity, we have $\text{cl}(\text{cone}(\{1\} \times S)) = S^h$ from Proposition 8. Hence $\tilde{S} = \{x : (0, x) \in S^h\} = S^\infty$. If $p$ is unbounded on $S$ from below, then $\lambda^* = -\infty = \lambda^*_{\text{cop}}$. If $p$ is bounded on $S$ from below, then $\tilde{p} \in \mathcal{P}(S^\infty)$ by Lemma 4. Hence $\tilde{p} \in \mathcal{P}(\tilde{S})$. Therefore generically $p \in \mathbb{R}[x]$ is either unbounded from below on $S$ or has $\tilde{p} \in \mathcal{P}^+(\tilde{S} \setminus \{0\})$. We conclude by applying Theorem 6 to the problem of minimizing $p$ on $S$. \hfill \Box

To numerically use problem (22), one can replace the condition $F \in \mathcal{P}_{d_{\text{max}}} (\mathbb{R}^{2n+m})$ by any certificate of copositivity. Possible choices are Pólya’s Positivstellensatz [29, Sec. 2.2], the certificate of copositivity we propose in Theorem 5, or the certificate of copositivity by Dickinson and Povh [21, Thm. 2.4]. One could also construct convergent inner hierarchies for the cone of copositive tensors based on the method by Bundfuss and Dür [12].

On compact sets, we obtain stronger results. Namely, certificates (13) and (15) provide convergent lower bounds for (21). As an example, we present the usage of certificate (13) below.

**Corollary 8.** Let $h_1, \ldots, h_m \in \mathbb{R}[x]$, and $S = \{x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}$ be non-empty. Define $d_{\text{max}} = \max\{\deg h_1, \ldots, \deg h_m, \lceil \deg p/2 \rceil\}$. Let $M > 0$ be such that $S \subseteq \{x \in \mathbb{R}^n : |x| \leq Me\}$. Let $d \geq 0$ and let $\mathcal{K} \subseteq \mathcal{P}_d (\mathbb{R}^n)$ be such that $\mathbb{R}_+ \subseteq \mathcal{K}$. For any $r \in \mathbb{N}$, define

$$
\lambda^r = \sup_{\lambda, (\alpha, \beta, \gamma)} \lambda
$$

s.t. $$(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{d_{\text{max}}(2d_{\text{max}}+r)} \quad c_{\alpha, \beta, \gamma} (y)^{\beta}(2nM - e^\top y)^{\gamma},$$

$$
c_{\alpha, \beta, \gamma} \in \mathcal{K} \text{ for } (\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}_{d_{\text{max}}(2d_{\text{max}}+r)}.
$$

(23)
Then \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda^* \), and \( \lim_{r \to \infty} \lambda^r = \lambda^* \).

Proof. Let \( y := x + M e \). Then \( y \in \mathbb{R}^n_+ \) and
\[
e^t y \leq e^t x + nM \leq e^t |x| + nM \leq 2nM.
\]
Define \( T := \{ y \in \mathbb{R}^n_+ : h_1(y - Me) \geq 0, \ldots, h_m(y - Me) \geq 0 \} = \{ y \in \mathbb{R}^n_+ : y - Me \in S \} \). Since \( S \) is compact and non-empty, \( T \) is compact and non-empty. Moreover, \( p(y - Me) \in \mathcal{P}_d^{d_{\lambda}}(T) \). Hence the conditions of Corollary 5 are satisfied for \( p(y - Me) \in \mathbb{R}[y] \) on \( T \subseteq \mathbb{R}^n_+ \). First,
\[
\lambda^r \leq \inf \{ p(y - Me) : y \in T \} = \inf \{ p(y - Me) : y - Me \in S \} = \inf \{ p(x) : x \in S \} = \lambda^*.
\]
Now, notice that if \((\lambda, (c_{\alpha, \beta, \gamma}))\) is feasible for problem (23) with \( r \), then it is also feasible for problem (23) with \( r + 1 \). Hence \( \lambda^r \) is non-decreasing in \( r \). To prove the convergence, it is left to show that for any \( k > 0 \) there is \( r \) such that \( \lambda^r \geq \lambda^* - \frac{1}{k} \). Compactness of \( T \) implies that \( p(y - Me) \) is bounded on \( T \) from below. Consider any \( \lambda^* - \frac{1}{k} < \lambda < \lambda^* \). Then we have \( q := p(y - Me) - \lambda \in \mathcal{P}_d^{d_{\lambda}}(T) \). Hence, by Corollary 5 with \( a = 0 \), \( b = 0 \), there is \( r \) and \((c_{\alpha, \beta, \gamma})\) such that \((\lambda, (c_{\alpha, \beta, \gamma}))\) is feasible for problem (23) with \( r \). Therefore \( \lambda^r \geq \lambda \geq \lambda^* - \frac{1}{k} \).

To obtain more information on \( \lambda^r \) for general sets, we can additionally construct upper bounds on \( \lambda^* \) by applying to problem (22) outer – instead of inner – approximations to the cone of copositive polynomials. We do not present this approach here since the resulting bound is an upper bound on \( \lambda^* \) only when (22) is a reformulation of (21), for instance, when \( \tilde{p} \in \mathcal{P}^+(\tilde{S} \setminus \{0\}) \). Instead, in Proposition 6 we construct an upper bound on \( \lambda^* \) that is always valid.

**Proposition 6.** Let \( p, h_1, \ldots, h_m \in \mathbb{R}[x] \) and \( S = \{ x \in \mathbb{R}^n_+ : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \} \) be non-empty. Define \( d_{\lambda} = \max \{ \deg h_1, \ldots, \deg h_m, \frac{\deg p}{2} \} \). Let \( \lambda^* \) be the objective value of (21) and let \( \lambda_{lb} \leq \lambda^* \) and \( \varepsilon > 0 \) be given. Define
\[
\lambda_{\varepsilon} = \sup_{\lambda \in \mathbb{R}_+} \lambda \quad \text{s.t.} \quad (1 + e^t y + e^t z)^{2d_{\lambda}} - \deg p(y - z) - \lambda + \varepsilon (1 + e^t y + e^t z)^{\deg p} = F(y, z, h_1(y - z), \ldots, h_m(y - z), p(y - z) - \lambda_{lb}),
\]
\[
F \in \mathcal{P}_d^{d_{\lambda}}(\mathbb{R}^n_+ + m + 1).
\]
Then \( \lambda_{\varepsilon} \geq \lambda^* \) and \( \lim_{\varepsilon \to 0^+} \lambda_{\varepsilon} = \lambda^* \).

Proof. Let \( S_{lb} := \{ x \in S : p(x) \geq \lambda_{lb} \} \) and \( T_{lb} := \{ (y, z) \in \mathbb{R}^n_+ : y - z \in S_{lb} \} \). We have \( \tilde{S}_{lb} = \{ x \in \tilde{S} : \tilde{p}(x) \geq 0 \} \) and \( \tilde{T}_{lb} := \{ (y, z) \in \mathbb{R}^n_+ : y - z \in \tilde{S}_{lb} \} \). Let \( q(y, z) = (1 + e^t y + e^t z)^{\deg p} \). Notice that \( S_{lb} \) is non-empty, which implies that \( T_{lb} \) is non-empty. Also, \( \tilde{p}(x) \in \mathcal{P}(\tilde{S}_{lb}) \), which implies that \( \tilde{p}(y - z) \in \mathcal{P}(\tilde{T}_{lb}) \). Since \( \tilde{q}(y, z) \in \mathcal{P}_d^+(\mathbb{R}^n_+ \setminus \{0\}) \), it follows that \( \tilde{p}(y - z) + \varepsilon \tilde{q}(y, z) \in \mathcal{P}_d^+(\tilde{T}_{lb} \setminus \{0\}) \). Then, using Theorem 3 we obtain that
\[
\lambda_{\varepsilon} = \inf \{ p(y - z) + \varepsilon q(y, z) : (y, z) \in T_{lb} \} \geq \inf \{ p(y - z) : (y, z) \in T_{lb} \}
\]
\[
= \inf \{ p(x) : x \in S_{lb} \} = \inf \{ p(x) : x \in S \} = \lambda^* \quad \text{(26)}
\]
To show the convergence, notice that \( \lambda_{\varepsilon} \) is non-increasing in \( \varepsilon \), and thus it is enough to show that for any \( k > 0 \) there is \( \varepsilon \) such that \( \lambda_{\varepsilon} < \lambda^* + \frac{1}{k} \). Fix \( k \), let \( x^k \in S \) be such that \( p(x^k) < \lambda^* + \frac{1}{k} \). Define \( y^k = \max(x, 0), z^k = -\min(x, 0), \) and let \( e^k := \frac{1}{2kq(y^k, z^k)} \), we have \( \lambda_{e^k} \leq p(y^k - z^k) + \frac{\tilde{q}(y^k, z^k)}{2kq(y^k, z^k)} = p(x^k) + \frac{1}{2k} < \lambda^* + \frac{1}{k} \).
To numerically use the upper bound $\lambda_*$, we can use any outer approximation to the set of copositive polynomials. Some examples are the simplicial partitions approach by Bundfuss and Dür [12], the simplex discretization approach by Yildirim [89] and the moment matrices approach by Lasserre [46, 49].

Proposition 6 illustrates how the copositive certificates of non-negativity proposed in Theorems 1 and 3 can be used, in contrast to the use of classical certificates of non-negativity [see, e.g., 43], to obtain not only lower but also upper bounds for PO problems. This allows to obtain realistic estimates of how far the convergent lower bounds from Corollary 8 are from the optimal value $\lambda^*$.

Besides improving estimates for $\lambda^*$, the proposed construction of bounds extends the range of applications of results specific for copositive polynomials (such as Pólya’s theorem or the results from [12, 20, 46, 89]) to general basic semialgebraic sets.

6 Relationship to Handelman’s and Schmüdgen’s Positivstellensätze

In this section we obtain Handelman’s Positivstellensatz [28] and Schmüdgen’s [77] Positivstellensatz using Corollary 4. Besides the cited classical proofs by Handelman and Schmüdgen, there are a few other proofs [6, 67, 79] for the first theorem and [6, 78, 79] for the second one. The alternative proofs exploit tools from various fields of mathematics, but mainly abstract algebra. Our proofs are different in the sense that we use Corollary 4 and standard optimization tools, with minimum use of algebraic tools. Our approach to Schmüdgen’s theorem partially follows the approach of Schweighofer [79]. Both our proof and the proof in [79] exploit a result by Berr and Wörmann [6] (Proposition 8). In both our approach and the proof of Schweighofer [79], the polynomial $p(x)$ is associated with some copositive polynomial $F(x, h_1(x), \ldots, h_m(x))$. This polynomial is homogenized, and Pólya’s theorem (Theorem 4) is applied to it. However, the ways in which the existence of $F(x, h_1(x), \ldots, h_m(x))$ is established are different: our reasoning goes through Corollary 4, while Schweighofer [79] uses tools from algebraic geometry.

We apply our approach first to prove Handelman’s Positivstellensatz [28].

**Theorem 7** (Handelman’s Positivstellensatz [28]). Let $S = \{ x : Ax \leq b \}$ be a non-empty polytope. Let $p \in \mathcal{P}^+(S)$. Then

$$p(x) = \sum_{\alpha \in \mathbb{N}^m} c_\alpha (b - Ax)^\alpha,$$

for some $c_\alpha \geq 0$ for all $\alpha \in \mathbb{N}^m$.

For our alternative proof, we use the following version of Farkas’ lemma.

**Proposition 7** (Ziegler [90, Proposition 1.9]). Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^n$ be such that $S = \{ x : Ax \leq b \}$ is non-empty. If $c_0 + c^\top x \in \mathcal{P}_1(S)$, then there exist $u, u_0 \geq 0$ such that $c_0 + c^\top x = u^\top (b - Ax) + u_0$.

**Proof of Theorem 7** Let $\hat{x}_i = \min_{x \in S} x_i$. We use the translation $x \to y - \hat{x}$. Define $S' = \{ y \in \mathbb{R}^n : A(y - \hat{x}) \leq b \} \subseteq \mathbb{R}^n_+$ so that $S' \subseteq \mathbb{R}^n_+$. Clearly, $S'$ is non empty. Also, as $S$ is compact, $S'$ is compact and there is $M > 0$ such that $S' \subseteq \{ y \in \mathbb{R}^n_+ : e^\top y \leq M \}$. From Corollary 4 after letting $a = 0$ and $b = 0$, it follows that as $p(y - \hat{x}) \in \mathcal{P}^+(S')$, there exists $d \geq 0$ and $c_{\alpha, \beta, \gamma} \geq 0$ for $(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}$ such that

$$p(y - \hat{x}) = \sum_{(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}} c_{\alpha, \beta, \gamma} y^\alpha (A(y - \hat{x}) - b)^\beta (M - e^\top y)^\gamma.$$
Now, substitute back $y \rightarrow x + \hat{x}$. We have $x_i + \hat{x}_i \in \mathcal{P}_1(S)$ for all $i = 1, \ldots, n$, and $M - e^T (x + \hat{x}) \in \mathcal{P}_1(S)$. The result then follows by using Proposition 7 to replace $x_i + \hat{x}_i$ for all $i = 1, \ldots, n$, and $M - e^T (x + \hat{x})$ in the representation above, respectively by expressions of the form \( u_j, u_j^i \geq 0 \), $i = 1, \ldots, n + 1$.

Now we prove Schmüdgen’s Positivstellensatz.

**Theorem 8** (Schmüdgen’s Positivstellensatz [71]). Let $h_1(x), \ldots, h_m(x) \in \mathbb{R}[x]$ be such that $S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \}$ is non-empty and compact, and let $p \in \mathcal{P}^+(S)$. Then there is $r \geq 0$ such that

\[
p = \sum_{\alpha \in \{0,1\}^m} \sigma_\alpha h^\alpha,
\]

for some SOS polynomials $\sigma_\alpha$ of degree $r - \deg(h^\alpha)$ for all $\alpha \in \{0,1\}^m$.

The approach to prove Theorem 8 is the same used to prove Theorem 7. First we use a weaker result that allows us to add redundant constraints to the semialgebraic set $S$ that can then be written in terms of the original constraints defining $S$. For that we use a result by Berr and Wörmann [6].

**Proposition 8** (Berr and Wörmann [6], Schweighofer [79]). Let $h_1(x), \ldots, h_m(x) \in \mathbb{R}[x]$ be such that $S = \{ x \in \mathbb{R}^n : h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \}$ is non-empty and compact. Then for every polynomial $p \in \mathbb{R}[x]$ there exists $t \in \mathbb{R}_+$ such that $t + p$ and $t - p$ have a representation of the form (27).

The proposition is weaker than Schmüdgen’s Positivstellensatz: it holds for every $p \in \mathbb{R}[x]$ and does not require positivity of $p$ on $S$. Intuitively, since $S$ is bounded, one can make the minimum of $t \pm p$ as large as desired by growing $t$. Theorem 8 shows that for $p > 0$ on $S$, there is representation of the form (27) for $t + p$, where $t = 0$.

**Proof of Theorem 8** For $i = 1, \ldots, n$, apply Proposition 8 on $S$ to obtain $\hat{x} \in \mathbb{R}^n_+$ such that for $i = 1, \ldots, n$,

\[
x_i + \hat{x}_i = \sum_{\alpha \in \{0,1\}^m} \sigma_\alpha^i (x) h(x)^\alpha.
\]

and $M > 0$ such that

\[
M - e^T (x + \hat{x}) = \sum_{\alpha \in \{0,1\}^m} \hat{\sigma}_\alpha (x) \hat{h}(x)^\alpha.
\]

Next, apply the translation $x \rightarrow y - \hat{x}$ obtaining $S' = \{ y \in \mathbb{R}^n_+ : y - \hat{x} \in S \}$. Then we have that $S' \subseteq \mathbb{R}^n_+$ and non empty.

From Corollary 4 after letting $a = 0$ and $b = 0$, it follows that as $p(y - \hat{x}) \in \mathcal{P}^+(S')$, there exists $d \geq 0$ and $c_{\alpha,\beta,\gamma} \geq 0$ for $(\alpha, \beta, \gamma) \in \mathbb{N}^{n+m+1}$ such that

\[
p(y - \hat{x}) = \sum_{(\alpha,\beta,\gamma) \in \mathbb{N}^{n+m+1}} c_{\alpha,\beta,\gamma} y^\alpha h(y - \hat{x})^\beta (M - e^T y)^\gamma,
\]

The result then follows after replacing $y$ by $x + \hat{x}$, substituting representations (28) and (29) into (30), expanding, and using the fact that the product of SOS polynomials is a SOS polynomial.
7 Concluding remarks

In this paper we propose copositive certificates of non-negativity of polynomials over semialgebraic sets. We show that under some mild assumptions, such a copositive certificate of small and known degree exists on a given basic closed semialgebraic set, not necessarily compact (see Theorems 1 and 3). Moreover, these assumptions hold generically. Certifying copositivity is an NP-hard problem. However, one can use existing outer and inner approximations to the set of copositive polynomials. These approximations, in combination with the copositive certificates we propose, deliver new results about the non-negativity of polynomials over generic semialgebraic sets. In particular, we obtain LMI hierarchies of upper and lower bounds on polynomial optimization problems and derive new structured certificates of non-negativity on compact sets.

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