NON-LOCALITY OF HYDRODYNAMIC AND MAGNETOHYDRODYNAMIC TURBULENCE

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ABSTRACT
We compare non-locality of interactions between different scales in hydrodynamic (HD) turbulence and magnetohydrodynamic (MHD) turbulence in a strongly magnetized medium. We use three-dimensional incompressible direct numerical simulations to evaluate non-locality of interactions. Our results show that non-locality in MHD turbulence is much more pronounced than that in HD turbulence. Roughly speaking, non-local interactions count for more than 10% of total interactions in our MHD simulation on a grid of $512^3$ points. However, there is no evidence that non-local interactions are important in our HD simulation with the same numerical resolution. We briefly discuss how non-locality affects the energy spectrum.

Key words: ISM: general – magnetohydrodynamics (MHD) – turbulence

1. INTRODUCTION
Turbulence is commonly observed in astrophysical fluids and in many cases such turbulence is accompanied by a strong magnetic field, which has a large impact on the dynamics of the turbulent cascade. Since turbulence influences many astrophysical processes (e.g., transport of mass and angular momentum, star formation, fragmentation of molecular clouds, heat and cosmic ray transport, magnetic reconnection, etc.), understanding the scaling properties of magnetohydrodynamic (MHD) turbulence is essential for theoretical astrophysics. For this reason, the literature regarding scaling relations of MHD turbulence is rich (see Goldreich & Sridhar 1995; Biskamp 2003, and references therein; see also Cho & Vishniac 2000b; Maron & Goldreich 2001; Müller et al. 2003; Müller & Grappin 2005; Boldyrev 2005, 2006; Beresnyak & Lazarian 2006; Mason et al. 2006; Gogoberidze 2007; Matthaeus et al. 2008).

In hydrodynamic (HD) turbulence, energy cascades down to smaller scales. Kinetic energy contained in an “eddy” is transferred to smaller eddies by shearing motions of other eddies (see, for example, Frisch 1995). Most theories on turbulence assume locality of interactions, which means that interactions between eddies of similar size dominate in such an energy cascade. In Fourier space, this means that a Fourier mode at a wavenumber $k = |\mathbf{k}|$, where $\mathbf{k}$ is the wavevector, interacts mainly with other modes having similar wavenumbers and transfers its energy to modes that have larger wavenumbers. Recently many researchers have investigated locality in HD turbulence (Mininni et al. 2008; Alexakis et al. 2007; Eyink & Aluie 2009; Aluie & Eyink 2009; see also Verma et al. 2005). In MHD turbulence with a strong mean field ($B_0$), locality is also generally assumed. However, in the MHD case, the nature of the energy cascade is slightly different. In the incompressible limit, any magnetic perturbation propagates along the magnetic field line. To first order, the speed of propagation is constant and equal to the Alfvén speed, $V_A = B_0/\sqrt{4\pi \rho}$, where $\rho$ is the density. Since wave packets are moving along the magnetic field line, there are two possible directions for propagation. If all the wave packets are moving in one direction, then they are stable to nonlinear order (Parker 1979). Therefore, in order to initiate turbulence, there must be opposite-traveling wave packets and the energy cascade occurs only when they collide. Therefore, in the MHD case, locality means that a wave packet (or “eddy”) transfers energy to smaller-scale wave packets by shearing motions of opposite-traveling wave packets of similar size.

There have been some discussions about non-locality in MHD turbulence with a strong mean field. For example, Alexakis (2007) theoretically studied a non-local model of MHD turbulence whereas Carati et al. (2006) briefly discussed non-local transfer of Elsässer energies. In their inspiring work, Beresnyak & Lazarian (2010) studied non-locality numerically and argued that “MHD turbulence is fairly non-local, at least less local than hydrodynamic turbulence” (see also Beresnyak & Lazarian 2009). They claimed that “(1) the lack of visible bottleneck effect in MHD turbulence, while it is clearly present in hydro turbulence, and (2) the dependence of kinetic and magnetic spectra on driving” support this idea. Teaca et al. (2009) calculated anisotropic energy transfer in Fourier space, but they did not pay much attention to the locality issue.

In this paper, we evaluate non-locality of HD and MHD turbulence quantitatively and present direct evidence that non-locality is clearly present in MHD turbulence. We consider only balanced strong MHD turbulence. Here balanced MHD turbulence means that the amplitudes of two opposite-traveling wave packets are almost equal. In Section 2, we describe our numerical setup. In Section 3, we present our results. In Section 4, we briefly discuss how non-locality affects the energy spectrum and give a summary of our work.

2. SIMULATIONS
We use a spectral code to solve the incompressible HD equation,
\[ \partial_t \mathbf{v} = -(\nabla \times \mathbf{v}) \times \mathbf{v} + \nu \nabla^2 \mathbf{v} + \mathbf{f} - \nabla P', \quad (1) \]
and the incompressible MHD equations,
\[ \partial_t \mathbf{v} = -(\nabla \times \mathbf{v}) \times \mathbf{v} + (\nabla \times \mathbf{B}) \times \mathbf{B} + \nu \nabla^2 \mathbf{v} + \mathbf{f} - \nabla P', \quad (2) \]
\[ \partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \nabla \times \mathbf{B} + \eta \nabla^2 \mathbf{B}, \quad (3) \]
\[ 1 \text{ When the mean field is weak or zero, the turbulence structure is very different (see, for example, Cho et al. 2009). There are many discussions about non-locality in this regime (see, for example, Alexakis et al. 2005b; Carati et al. 2006; Lessinnes et al. 2009; Yousef et al. 2007; Aluie & Eyink 2010).} \]
...occurs at $k_L$. $B$ components), and a periodic box of size $2\pi$, where $\mathbf{f}$ is a random forcing term, $P' \equiv P/\rho + v^2/2$, $P$ is pressure, $v$ is the velocity, and $\mathbf{B}$ is the magnetic field divided by $(4\pi\rho)^{1/2}$. (Note that we set $\rho = 1$ in our simulations.) Thus the field $\mathbf{B}$ is, in fact, the Alfvénic velocity. The velocity and the magnetic fields are divergence free: $\nabla \cdot \mathbf{v} = \nabla \cdot \mathbf{B} = 0$. The peak of energy injection occurs at $k_L \approx 2.5$, so the energy injection scale is $L \approx 2.5$. The amplitudes of the forcing components are tuned to ensure $v \approx 1$. The forcing function has the following form: $f(\mathbf{x}, t) = \sum_{j=1}^{22} f(\mathbf{k}_j) \exp(i\mathbf{k}_j \cdot \mathbf{x}) + \text{complex conjugate}$, where $f(\mathbf{k}_j)$ is a complex vector that is perpendicular to the wavevector $\mathbf{k}_j$. The 22 forcing components are distributed nearly isotropically in the range $2 \leq k \leq \sqrt{12}$, where $k = |\mathbf{k}|$. The phase of each forcing component fluctuates randomly, but it has a correlation time of approximately unity. The amplitude of each forcing component also fluctuates randomly. On average, each forcing component injects a similar amount of energy. In physical space this forcing $f(\mathbf{x}, t)$ corresponds to statistically homogeneous driving on large scales. Since we deal with incompressible turbulence, we drive only the solenoidal component of the velocity.

In the MHD simulation, the magnetic field consists of the uniform background field and a fluctuating field: $\mathbf{B} = \mathbf{B}_0 + \mathbf{b}$. The Alfvén velocity of the uniform background field, $\mathbf{B}_0$, is set to 0.8. At $t = 0$, the magnetic field has only the uniform component. We consider only the case where viscosity is equal to magnetic diffusivity: $\nu = \eta$. Details of the code can be found in Cho & Vishniac (2000a, 2000b). We list the results of our simulations in Table 1.

Figure 1(a) shows the time evolution of kinetic and magnetic energy densities. Figure 1(b) shows energy spectra at $t = 12$. The kinetic spectrum for the HD run (solid curve) is consistent with a Kolmogorov spectrum ($E(k) \propto k^{-5/3}$ for $k \in (2, 15)$) but shows a moderate increase of the slope for $k > 15$. The kinetic spectrum for the MHD run (dashed curve) is also consistent with a Kolmogorov one. However, the magnetic spectrum (dotted curve) is slightly shallower than a Kolmogorov one. Therefore, the spectrum of $b^2 + v^2$ (not shown) is slightly shallower than a Kolmogorov one.

### 3. RESULTS

#### 3.1. Shell-to-shell Energy Transfer

We can rewrite the MHD equations in Equations (2) and (3) using the Elsässer variables, $\mathbf{Z}^+ \equiv \mathbf{v} + \mathbf{B}$ and $\mathbf{Z}^- \equiv \mathbf{v} - \mathbf{B}$:

$$\partial_t \mathbf{Z}^+ = -\mathbf{Z}^- \cdot \nabla \mathbf{Z}^+ + \nu \nabla^2 \mathbf{Z}^+ + \mathbf{f} - \nabla P,$$

$$\partial_t \mathbf{Z}^- = -\mathbf{Z}^+ \cdot \nabla \mathbf{Z}^- + \nu \nabla^2 \mathbf{Z}^- + \mathbf{f} - \nabla P.$$  

The Elsässer variables denote the amplitudes of two opposite-traveling waves along the magnetic field line. The nonlinear term for $\partial_t \mathbf{Z}^+$, for example, states that energy transfer between $\mathbf{Z}^+$ modes is mediated by $\mathbf{Z}^-$ modes.

In the Fourier space, the nonlinear term in Equation (4), for example, becomes

$$N_k^+ \equiv -ik \cdot \sum_{p+q=k} Z_p^+ Z_q^-,$$

and the time derivative of $(1/2)|Z_k^+|^2$ is given by

$$Z_k^+ \cdot \partial_t Z_k^{+\ast} = Z_k^+ \cdot N_k^{+\ast} - v_k^2 |Z_k^+|^2,$$

where $^{\ast\ast}$ denotes the complex conjugate and we drop the forcing term because its role is limited in the inertial range. Energy transfer occurs only between $\mathbf{Z}_k^+$ and $\mathbf{Z}_q^-$ via shearing motions provided by $\mathbf{Z}_k^-$ modes. Without $\mathbf{Z}_-^-$ modes, $\mathbf{Z}_+^+$ modes alone do not interact with each other (see Section 3.3 for details).

If interactions are local in the Fourier space, we will have $p \sim q \sim k$. Since it is difficult to check the locality using individual

| Table 1 |
|-----------|
| Runs      | Resolution | $B_0$ | $\nu = \eta$ | $\langle v^2 \rangle$ | $\langle b^2 \rangle$ | $\langle L_{int} \rangle$ | $\langle \lambda_v \rangle$ | $\langle \lambda_b \rangle$ | $R_v$ | $R_b$ | $k_L$ |
| HD        | 512$^3$   | 128  | 0.0004     | 1.13            | 1.91            | 0.35            | 0.54            | 0.46            | 1.93 | 0.45 | 0.36 | 5090 | 920 | 220 |
| MHD       | 512$^3$   | 0.8  | 0.0004     | 0.54            | 0.46            | 1.93 | 0.45 | 0.36 | 3540 | 820 | 200 |

Notes. $B_0$ is the strength of the mean field, $\nu$ the viscosity, $\eta$ the magnetic diffusivity, $L_{int}(\equiv 2\pi \int_{kL} d\kappa (E_k k^{-4} d\kappa) )$ the integral scale of turbulence, $E_k(k)$ the energy spectrum for the velocity field, $\lambda_v(\equiv 2\pi \int_{kL} d\kappa (E_k k^{-4} d\kappa) )$ the Taylor micro-scale for the velocity field, $\lambda_b(\equiv 2\pi \int_{kL} d\kappa (E_k k^{-4} d\kappa) )$ the Taylor micro-scale for the magnetic field, $E_k(k)$ the magnetic energy spectrum, $R_v(\equiv V/L_{int})/\nu$ the Reynolds number, $V(\equiv \langle v^2 \rangle)^{1/2}$ the rms velocity, $R_b(\equiv \langle \lambda_b \rangle/\nu)$ the Taylor micro-scale based Reynolds number, $k_b(\equiv \langle (\epsilon)/\nu^3 \rangle)$ the viscous dissipation wavenumber, and $\epsilon$ the energy dissipation rate. In the MHD run, we use the total energy dissipation rate for $\epsilon$. The angle brackets, $\langle \cdots \rangle$, denote time averages.2

Notes. The wavenumbers used are 2 (3 components), $\sqrt{5}$ (12 components), 3 (3 components), and $\sqrt{12}$ (4 components). Therefore the peak of energy injection occurs at $k_L \approx 2.5$, and the energy injection scale is $L \approx 2.5$. The values of $B_0, \nu, \langle v^2 \rangle, \langle b^2 \rangle, \langle L_{int} \rangle, \langle \lambda_v \rangle, \langle \lambda_b \rangle, R_v, R_b, k_L$ for both HD and MHD runs are given in Table 1.
The energy transfer rate $T(k, p = all, q)$ (HD) and $T(k, p = 2&3, q)$ (HD) are shown in Figures 2(a) and (d), respectively. The contour diagrams in these figures are exactly anti-symmetric with respect to the $k$-line. The overall shape of the contour diagram for HD is consistent with the earlier findings (Alexakis et al. 2005a; Mininni et al. 2008). The value of $T(k, p = all, q)$ for HD is consistent with the earlier findings (Alexakis et al. 2005a; Mininni et al. 2008).

In the HD case, $Z^+$ modes in the $k$-shell gain energy from the $Z^-$ modes. The contour diagrams in Figure 2(c) and (f) (right panels) show the values of $T(k, p = all, q)$ in the HD case. This result is consistent with the concept of energy cascade: energy cascades down to smaller scales. Note that the values of $T(k, p = all, q)$ are very close to zero except near the $k = q$ line. Does this mean that locality is a good approximation?

Note that, when the outer scale of turbulence provides strong shear motions, $T(k, p = all, q)$ has positive peaks at $q = k - k_L$ and negative peaks at $q = k + k_L$. In our case, $k_L \sim 2.5$. Therefore, shearing motions of the outer scale can also produce diagrams similar to Figures 2(a) and (d). Indeed, when we plot $T(k, p = 2&3, q)$

$$T(k, p = 2&3, q) \equiv \sum_{p=2}^{p_{max}} T(k, p, q),$$

which is similar to $T(k, p = all, q)$ except for the fact that we sum from $p = 2$ to $p = 3$, the contour diagrams show similar features (Figures 2(b) and (e)). This result is consistent with earlier results for HD turbulence (e.g., Alexakis et al. 2005a). Note that, in Figures 2(a) and (d), the width of the contour lines near the $k = q$ line is narrower in the MHD case than in the HD case. This might mean that the effect of the outer scale is stronger in the MHD case than in the HD case.

Figures 2(c) and (f) show the values of $T(k_0, p = all, q)$ (solid) and $T(k_0, p = 2&3, q)$ (dotted) for the selected values of $k_0$. We take $k_0 = 6, 16,$ and $32$. The values for $k_0 = 16$ and $32$ are offset by $0.015$ and $0.03$, respectively, for clarity. In the HD case (upper panel), $T(k_0, p = all, q)$ and $T(k_0, p = 2&3, q)$ look different. However, in the MHD case (lower panel), $T(k_0, p = all, q)$ and $T(k_0, p = 2&3, q)$ look very similar, which might mean that the outer scale does play an important role in MHD energy cascade.
In order to evaluate the role of the outer scale in shell-to-shell energy transfer, we calculate the ratio

$$\frac{\sum_{q=0}^{k-1} T(k, p=2\&3, q) / \sum_{q=0}^{k-1} T(k, p=\text{all}, q)}{\sum_{q=0}^{k-1} T(k, p=\text{all}, q)}.$$ (10)

Figure 3(a) shows the ratios for HD and MHD. In the HD case (solid curve), the ratio is less than 0.5 for most values of k, which is consistent with Mininni et al. (2008). However, in the MHD case (dotted curve) the values are $\gtrsim 0.5$ for most values of k, which means that non-local interactions are indeed important for shell-to-shell energy transfer in the MHD case.

However, it is very important to note that the result in Figure 3(a) does not mean that non-local interactions are as strong as local interactions in the MHD cascade. The result in Figure 3(a) is only for a single shell.

In order to evaluate non-locality, we would better consider the effect of $p$-shells on a band of $k$-shells between $k_{\text{min}} = k/\alpha$ and $k_{\text{max}} = \alpha k$, where $\alpha$ is a constant. In this paper, we take $\alpha = \sqrt{2}$. The motivation for considering this quantity is that Fourier modes in $(k/\sqrt{2}, \sqrt{2}k)$ can define "eddies" on a scale $l \sim 1/k$. In Figure 3(b), we plot the ratio similar to that in Equation (10), but expressed in terms of

$$T(k_{\text{band}}, \ldots, \ldots) \equiv \sum_{k' = k_{\text{min}}}^{k_{\text{max}}} T(k', \ldots, \ldots),$$ (11)

where $k_{\text{min}} = k/\sqrt{2}$ and $k_{\text{max}} = \sqrt{2}k$. The summation for $q$ is from 0 to $k_{\text{min}} - 1$. The ratio for MHD (dotted) is non-negligible and still substantially larger than that for HD (solid). Therefore, we can conclude that non-locality is indeed present in MHD turbulence.5

3.2. More on Non-locality of MHD Turbulence

Figure 3(b) shows that $p = 2$ and $p = 3$ shells contribute more than 10% of the total energy flux. Then, which $p$-shell provides the strongest contribution to a $k$-band? In other words, which is the most shear-providing shell for a band of $k$-shells between $k/\sqrt{2}$ and $\sqrt{2}k$? To see this, we calculate the following quantity:

$$\sum_{q=0}^{24} T(k_{\text{band}}^{1\&2}, p, q) \equiv \sum_{q=0}^{24} \sum_{k'=25}^{50} T(k', p, q).$$ (12)

which is equal to the total energy transferred from all $q$-shells with $q \leq 24$ to the $k$-band between $k = 25$ and $k = 50$ by the shearing action of a $p$-shell. Figure 4 shows that each $p$-shell provides a similar contribution in the HD case (solid line). Therefore, non-locality does not seem to be important in HD turbulence. On the contrary, the $p = 2$ shell contributes most in the MHD case (dotted line).6 This is another piece of evidence that non-locality is clearly present in MHD turbulence.

3.3. Non-local Energy Transfer

So far, we have discussed non-local influence of the outer scale eddies. To be precise, we have considered the non-local effect of the outer scale $Z^{-}$ modes on the inertial range $Z^{+}$ modes.7 Then what can we say about the effect of outer scale $Z^{+}$ modes on inertial range $Z^{-}$ modes? Before answering this question, it is important to understand the distinctive roles of the $Z^{+}$ and $Z^{-}$ variables in the energy cascade of $Z^{+}$ variable. Let us consider the role of the $Z^{-}$ field in the evolution of $Z^{+}$. For simplicity, let us ignore the dissipation and the forcing terms. Suppose that the $Z^{-}$ field does not contain a fluctuating component (i.e., $Z^{-} = -B_{0}$ at $t = 0$ everywhere). Then, the nonlinear term in Equation (4) becomes $B_{0} \cdot \nabla Z^{+}$ and, therefore, any perturbation of the $Z^{+}$ field propagates in the opposite direction to the mean magnetic field without cascading energy. Therefore, to initiate turbulence, we need wave packets of $Z^{-}$ which propagate in the direction of the mean magnetic field. When wave packets of $Z^{+}$ and $Z^{-}$ collide, nonlinear energy cascade is possible. Even in this case the energy of each Els"asser variable ($\frac{1}{2} \int |Z^{+}|^{2} d^{3}x$ and $\frac{1}{2} \int |Z^{-}|^{2} d^{3}x$) is conserved. Let us,

5 We note that the ratio for MHD gradually decreases as $k$ increases. Although it is not very clear at this moment whether it will continue to drop when we have a very long inertial range, it is likely that the ratio will continue to drop as the non-local effects of the outer scale will ultimately vanish on very small scales. Nevertheless, non-locality is an important characteristic of MHD turbulence near the outer scale.

6 However, it is worth noting that the $p = 2$ shell accounts for only $\sim 17\%$ of the total energy transfer.

7 Of course, since the equations for Els"asser variables are symmetric (see Equations (4) and (5)), the non-local effect of the outer scale $Z^{-}$ modes on the inertial range $Z^{+}$ modes should be similar to the non-local effect of the outer scale $Z^{+}$ modes on the inertial range $Z^{-}$ modes.
for example, take the dot product of Equation (4) with $Z^+$. Then, we obtain

$$\partial_t \frac{1}{2} |Z^+|^2 = -Z^+ \cdot \nabla \frac{1}{2} |Z^+|^2 - \nabla \cdot (Z^+ P),$$  

(13)

where we drop the dissipation and forcing terms. Note that the first term on the right-hand side, which describes advection of the Elsässer energy $\frac{1}{2} |Z|^2$, is $-\nabla \cdot (\frac{1}{2} |Z| \nabla Z)$. If we use a periodic boundary condition or the Elsässer variables vanish at infinity, the terms on the right-hand side are zero and the Elsässer energy $\frac{1}{2} |Z|^2$ is conserved (see, for example, Maron & Goldreich 2001 for further discussion). It is evident that the $Z^+$ field provides advective effects and energy transfer occurs only between $Z^+$ modes. A very nice discussion about interactions between oppositely directed wave packets can be found in Lithwick et al. (2007; see their Figure 1). They discussed how “the oppositely directed wave packets distort one another” (Lithwick et al. 2007). In summary, when $Z^+$ modes interact with other $Z^+$ modes, actual energy transfer between the $Z^+$ modes does occur. On the other hand, when $Z^+$ modes interact with $Z^-$ modes, one type of Elsässer modes provides advection/shear to the other type of Elsässer modes and energy transfer between $Z^+$ and $Z^-$ does not occur.

Now it is time to clarify the meaning of non-locality. As we have discussed above, the nonlinear term for $\partial_t Z^+$ (i.e., $-Z^+ \cdot \nabla Z^+$), for example, contains both $Z^+$ and $Z^-$. Therefore non-locality has two meanings: (1) non-local effects of $Z^+$ and (2) non-local effects of $Z^-$. Since $Z^+$ modes do not lose or gain energy, the former type of non-locality does not involve energy transfer between the outer scale and small scales. In fact, when this kind of non-locality is present, energy transfer between adjacent shells is enhanced. The non-locality that we have discussed so far is this type of non-locality.

When the latter type of non-locality is present, there is direct energy transfer between different scales. In order to evaluate the energy transfer rate from the outer scale to a $k$-band between $k_{\min} = k/\sqrt{2}$ and $k_{\max} = \sqrt{2}k$, we calculate the ratio

$$\frac{T(k_{\text{band}}, p = \text{all}, q = 2\&3)}{\sum_{q=0}^{2\&3} T(k_{\text{band}}, p = \text{all}, q')},$$  

(14)

Figure 5(a) shows the ratios for HD and MHD. We can see that the ratios for this type of non-locality are smaller than those for the former type of non-locality (see Figure 3(b)). The ratio for the MHD case is higher than that for the HD case.

Figure 5(b) shows the values of

$$T(k_{\text{band}}, p = \text{all}, q) \equiv \sum_{k'=25}^{50} T(k', p = \text{all}, q)$$  

(15)

for HD and MHD. The values of $T(k_{\text{band}}, p = \text{all}, q = 2)$ and $T(k_{\text{band}}, p = \text{all}, q = 3)$ are not particularly larger than other values. It is clear from the figure that energy transfer from the outer scale ($q = 2$ and 3) to the $k$-band ($25 \leq k \leq 50$) is small. Therefore, non-local energy transfer from the outer scale to small scales may not be an important characteristic for both the HD and the MHD cases.

4. DISCUSSION AND SUMMARY

When shearing motions of the outer scale eddies influence energy transfer of inertial range eddies, the energy spectrum becomes flatter than the Kolmogorov one. Suppose that the shearing motions of the outer scale eddies completely dominate energy cascade. In this case, from

$$Z_i^2 / t_{\text{cas}} \sim Z_i^2 / (L/v_L) \propto Z_i^2 = \text{constant},$$  

(16)

can we easily show that the energy spectrum is $E(k) \propto k^{-1}$, where $Z_i$ is an Elsässer variable at scale $l$, $L$ the outer scale, and $v_L$ the rms velocity at the outer scale (see Equation (1) of Cho et al. 2003). If the shearing motions of the outer scale eddies do not completely dominate, we will have a spectrum between $k^{-1}$ and $k^{-5/3}$.

Indeed, in Figure 1 we observe that $E_s(k) + E_h(k)$ (hence the spectrum of $Z^*$ or $Z^-$) in MHD is flatter than the Kolmogorov spectrum. This is consistent with earlier numerical results (see, for example, Maron & Goldreich 2001; Müller et al. 2003).

In summary, we have found the following results.

1. We have developed a quantitative method to measure non-locality (see Figure 3(b) or Figure 4).
2. Our numerical calculations show that non-locality is more pronounced in MHD turbulence than in HD turbulence. This result confirms an earlier finding by Beresnyak & Lazarian (2010).
3. There are two forms of non-locality in MHD turbulence: non-local influence of shearing motions, which does not involve energy transfer between different scales, and non-local energy transfer between different scales (Section 3.3). In MHD, the former type of non-locality (i.e., non-local influence of outer scale shearing motions) is more important. It is not clear whether the latter type is important.
4. In MHD, non-locality is not negligible so that it might affect the dynamics of turbulent cascade.

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Figure 5. Non-local energy transfer. (a) Ratios of local to non-local direct energy transfer rates. The quantity shown here, $T(k, p = \text{all}, q = 2\&3) / T(k, \text{all}, 0 \leq q < k_{\text{min}})$, is the ratio of non-local energy transfer rate to total energy transfer rate. The numerator is the rate from $q = 2$ and $q = 3$ shells to $k$-bands between $k_{\min} = k/\sqrt{2}$ and $k_{\max} = \sqrt{2}k$ and the denominator is the total energy transfer rate from $q$-shells between 0 and $k_{\min} - 1$ to the same $k$-bands. The MHD case shows stronger non-locality. (b) The amount of energy transferred from a $q$-shell to a $k$-band ($25 \leq k \leq 50$) by the mediation of all $p$-shells. Solid line is for HD turbulence and dotted line for MHD turbulence. In both cases, non-locality is not conspicuous. Snapshot at $t = 12$. 

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