ON HARDY–LITTLEWOOD-TYPE
AND HAUSDORFF–YOUNG-TYPE INEQUALITIES
FOR GENERALIZED GEGENBAUER EXPANSIONS

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Abstract. Using well-known techniques, we establish Hardy–Littlewood-type and Hausdorff–Young-type inequalities for generalized Gegenbauer expansions and their unification.

Key words and phrases: orthogonal polynomials, Jacobi polynomials, Gegenbauer polynomials, generalized Gegenbauer polynomials, Hardy–Littlewood-type inequalities, Hausdorff–Young-type inequalities

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1. Introduction and preliminaries

In this section, we introduce some classes of orthogonal polynomials on [−1, 1], including the so-called generalized Gegenbauer polynomials. For a background and more details on the orthogonal polynomials, the reader is referred to [126, 10].

Let \( \alpha, \beta > -1 \). The Jacobi polynomials, denoted by \( P_{n}^{(\alpha,\beta)}(\cdot) \), where \( n = 0, 1, \ldots \), are orthogonal with respect to the Jacobi weight function \( w_{\alpha,\beta}(t) = (1-t)^{\alpha}(1+t)^{\beta} \) on \([-1,1]\), namely,

\[
\int_{-1}^{1} P_{n}^{(\alpha,\beta)}(t) P_{m}^{(\alpha,\beta)}(t) w_{\alpha,\beta}(t) \, dt = \begin{cases} 
2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1) \\
(2n+\alpha+\beta+1) \Gamma(n+1) \Gamma(n+\alpha+\beta+1) 
\end{cases}, \quad n = m, \\
0, \quad n \neq m.
\]

Here, as usual, \( \Gamma \) is the gamma function.

For \( \lambda > -\frac{1}{2}, \mu \geq 0 \), and \( n = 0, 1, \ldots \), the generalized Gegenbauer polynomials \( C_{n}^{(\lambda,\mu)}(\cdot) \) are defined by

\[
C_{2n}^{(\lambda,\mu)}(t) = a_{2n}^{(\lambda,\mu)} P_{n}^{(\lambda-1/2,\mu-1/2)}(2t^{2} - 1), \quad a_{2n}^{(\lambda,\mu)} = \frac{(\lambda+\mu)_{n}}{(\mu+\frac{1}{2})_{n}},
\]

\[
C_{2n+1}^{(\lambda,\mu)}(t) = a_{2n+1}^{(\lambda,\mu)} t P_{n}^{(\lambda-1/2,\mu+1/2)}(2t^{2} - 1), \quad a_{2n+1}^{(\lambda,\mu)} = \frac{(\lambda+\mu)_{n+1}}{(\mu+\frac{1}{2})_{n+1}},
\]

where \((\lambda)_{n}\) denotes the Pochhammer symbol given by

\[
(\lambda)_{0} = 1, \quad (\lambda)_{n} = \lambda(\lambda+1) \cdots (\lambda+n-1) \quad \text{for} \quad n = 1, 2, \ldots
\]

They are orthogonal with respect to the weight function

\[
v_{\lambda,\mu}(t) = |t|^{2\mu}(1-t^{2})^{\lambda-1/2}, \quad t \in [-1,1].
\]

For \( \mu = 0 \), these polynomials, denoted by \( C_{n}^{(\lambda)}(\cdot) \), are called the Gegenbauer polynomials:

\[
C_{n}^{(\lambda)}(t) = C_{n}^{(\lambda,0)}(t) = \frac{(2\lambda)_{n}}{(\lambda+\frac{1}{2})_{n}} P_{n}^{(\lambda-1/2,\lambda-1/2)}(t).
\]
For \( \lambda > -\frac{1}{2} \), \( \mu > 0 \), and \( n = 0, 1, \ldots \), we have the following connection:

\[
C_n^{(\lambda,\mu)}(t) = c_\mu \int_{-1}^{1} C_n^{\lambda+\mu}(tx)(1 + x)(1 - x^2)^{\mu-1} \, dx, \quad c_\mu^{-1} = 2 \int_{0}^{1} (1 - x^2)^{\mu-1} \, dx.
\]

Denote by \( \{\tilde{C}_n^{(\lambda,\mu)}(\cdot)\}_{n=0}^{\infty} \) the sequence of orthonormal generalized Gegenbauer polynomials. It is easily verified that these polynomials are given by the following formulae:

\[
\tilde{C}_2n^{(\lambda,\mu)}(t) = \tilde{a}_{2n}^{(\lambda,\mu)} P_n^{(\lambda-1/2,\mu-1/2)}(2t^2 - 1),
\]

\[
\tilde{\alpha}^{(\lambda,\mu)}_{2n} = \left( \frac{(2n + \lambda + \mu)\Gamma(n + 1)\Gamma(n + \lambda + \mu)}{\Gamma(n + \lambda + \frac{1}{2})\Gamma(n + \mu + \frac{1}{2})} \right)^{1/2},
\]

\[
\tilde{C}_{2n+1}^{(\lambda,\mu)}(t) = \tilde{a}_{2n+1}^{(\lambda,\mu)} t P_n^{(\lambda-1/2,\mu+1/2)}(2t^2 - 1),
\]

\[
\tilde{\alpha}^{(\lambda,\mu)}_{2n+1} = \left( \frac{(2n + \lambda + \mu + 1)\Gamma(n + 1)\Gamma(n + \lambda + \mu + 1)}{\Gamma(n + \lambda + \frac{3}{2})\Gamma(n + \mu + \frac{3}{2})} \right)^{1/2}.
\]

The generalized Gegenbauer polynomials play an important role in Dunkl harmonic analysis (see, for example, [2, 6]). So, the study of these polynomials and their applications is very natural.

The notation \( f(n) \asymp g(n) \), \( n \to \infty \), means that there exist positive constants \( C_1, C_2 \), and a positive integer \( n_0 \) such that \( 0 \leq C_1 g(n) \leq f(n) \leq C_2 g(n) \) for all \( n \geq n_0 \). For brevity, we will omit “\( n \to \infty \)” in the asymptotic notation.

Define the uniform norm of a continuous function \( f \) on \([-1, 1]\) by

\[
\|f\|_\infty = \max_{-1 \leq t \leq 1} |f(t)|.
\]

The maximum of two real numbers \( x \) and \( y \) is denoted by \( \max(x, y) \).

In [11], we prove the following result.

**Theorem 1.** Let \( \lambda > -\frac{1}{2}, \mu > 0 \). Then

\[
\|\tilde{C}_n^{(\lambda,\mu)}\|_\infty \asymp n^{\max(\lambda,\mu)}.
\]

Given \( 1 \leq p \leq \infty \), we denote by \( L_p(v_{\lambda,\mu}) \) the space of complex-valued Lebesgue measurable functions \( f \) on \([-1, 1]\) with finite norm

\[
\|f\|_{L_p(v_{\lambda,\mu})} = \left( \int_{-1}^{1} |f(t)|^p v_{\lambda,\mu}(t) \, dt \right)^{1/p}, \quad 1 \leq p < \infty,
\]

\[
\|f\|_{L_\infty} = \sup_{x \in [-1, 1]} |f(x)|, \quad p = \infty.
\]

For a function \( f \in L_p(v_{\lambda,\mu}), 1 \leq p \leq \infty \), the generalized Gegenbauer expansion is defined by

\[
f(t) \sim \sum_{n=0}^{\infty} \hat{f}_n \tilde{C}_n^{(\lambda,\mu)}(t), \quad \text{where} \quad \hat{f}_n = \int_{-1}^{1} f(t) \tilde{C}_n^{(\lambda,\mu)}(t) v_{\lambda,\mu}(t) \, dt.
\]

For \( 1 < p < \infty \), we denote by \( p' \) the conjugate exponent to \( p \), that is, \( \frac{1}{p} + \frac{1}{p'} = 1 \).

The aim of this paper is to establish Hardy–Littlewood-type and Hausdorff–Young-type inequalities for generalized Gegenbauer expansions in Sections 2 and 3, respectively. Also, we give their unification in Section 4.
2. Hardy–Littlewood-type inequalities for generalized Gegenbauer expansions

The analogue of the Hardy–Littlewood inequality is given in the following theorem, which can be deduced as a corollary from [9, Theorems 3.2 and 3.6] (for (2) and (4), respectively). Nevertheless, for convenience we give a direct proof of the theorem, based on Theorem 1 and our settings.

**Theorem 2.** (a) If $1 < p \leq 2$ and $f \in L_p(v_{\lambda,\mu})$, then

$$\left\{ \sum_{n=0}^{\infty} \left( (n+1) \left( \frac{1}{p} - \frac{1}{2} \right)(\max(\lambda,\mu)+1) |\hat{f}_n| \right)^p \right\}^{1/p} \leq A_p \|f\|_{L_p(v_{\lambda,\mu})}. \quad (2)$$

(b) If $2 \leq q < \infty$ and $\phi$ is a function on non-negative integers satisfying

$$\sum_{n=0}^{\infty} \left( (n+1) \left( \frac{1}{q} - \frac{1}{2} \right)(\max(\lambda,\mu)+1) |\phi(n)| \right)^q < \infty, \quad (3)$$

then the algebraic polynomials

$$\Phi_N(t) = \sum_{n=0}^{N} \phi(n) \tilde{C}^\lambda \mu_n(t)$$

converge in $L_q(v_{\lambda,\mu})$ to a function $f$ satisfying $\hat{f}_n = \phi(n), n = 0, 1, \ldots$, and

$$\|f\|_{L_q(v_{\lambda,\mu})} \leq A_q \left\{ \sum_{n=0}^{\infty} \left( (n+1) \left( \frac{1}{q} - \frac{1}{2} \right)(\max(\lambda,\mu)+1) |\phi(n)| \right)^q \right\}^{1/q}. \quad (4)$$

**Proof.** Let $\sigma = \max(\lambda,\mu) + 1$.

(a) To prove (2), we note that for $p = 2$ the Parseval identity implies equality in (2) with $A_2 = 1$. Consider (2) as the transformation from $L_p(v_{\lambda,\mu})$ into the sequence $\{ (n+1)^{\frac{1}{q} - \frac{1}{2}}(\max(\lambda,\mu)+1) |\phi(n)| \}_{n=0}^{\infty}$ in the $\ell_p$ norm with the weight $\{ (n+1)^{-2\sigma} \}_{n=0}^{\infty}$ and show that this transformation is of weak type $(1,1)$. We have

$$m\{ n: (n+1)^{\sigma} |\hat{f}_n| > t \} = \sum_{(n+1)^{\sigma} |\hat{f}_n| > t} (n+1)^{-2\sigma} \equiv I_t.$$ 

By Theorem 1, $|\hat{f}_n| \leq C_1 \|f\|_{L_1(v_{\lambda,\mu})} (n+1)^{\sigma-1}$ and consequently

$$I_t \leq \sum_{(n+1)^{\sigma} |\hat{f}_n| > t} (n+1)^{-2\sigma}, \quad A = C_2 \left( \frac{t}{\|f\|_{L_1(v_{\lambda,\mu})}} \right)^{\frac{1}{2\sigma-1}}.$$ 

Hence, using the easily verified inequality

$$\sum_{(n+1)^{\sigma} > \tilde{A}} (1+n)^{-\delta} \leq 2^{\delta-1} \tilde{A}^{-\delta+1}, \quad \tilde{A} > 0, \quad \delta \geq 2,$$
we observe that, for \( \tilde{A} = A \) and \( \delta = 2\sigma \),

\[
I_t \leq C_3 \frac{\|f\|_{L_1(v_{\lambda,\mu})}}{t}.
\]

The last estimate is a weak \((1,1)\) estimate which, using the Marcinkiewicz interpolation theorem, implies \((2)\).

(b) We have \(1 < q' \leq 2\). For brevity, write \( \psi_n \) in place of \( \left( (n+1) \left( \frac{1}{q} - \frac{1}{q'} \right) \sigma \left| \phi(n) \right| \right)^q \). Suppose that \( g \in L_{q'}(v_{\lambda,\mu}) \) and that \( N < N' \) are positive integers. Applying Hölder’s inequality and (a), we find that

\[
\left| \int_{-1}^{1} \Phi_N(t) \ g(t) \ v_{\lambda,\mu}(t) \ dt \right| = \left| \sum_{n=0}^{N} \phi(n) \hat{g}_n \right| = \left| \sum_{n=0}^{N} \left( n + 1 \right) \left( \frac{1}{q'} - \frac{1}{q} \right) \sigma \left| \phi(n) \right| \left( n + 1 \right) \left( \frac{1}{q} - \frac{1}{q'} \right) \sigma \hat{g}_n \right| \leq \left\{ \sum_{n=0}^{N} \psi_n \right\}^{1/q} \left\{ \sum_{n=0}^{N} \left( n + 1 \right) \left( \frac{1}{q'} - \frac{1}{q} \right) \sigma |\hat{g}_n| \right\}^{q'/q'} \leq \left\{ \sum_{n=0}^{N} \psi_n \right\}^{1/q} A_{q'} \|g\|_{L_{q'}(v_{\lambda,\mu})}. \tag{5}
\]

Similarly,

\[
\left| \int_{-1}^{1} \left( \Phi_N(t) - \Phi_{N'}(t) \right) \ g(t) \ v_{\lambda,\mu}(t) \ dt \right| \leq \left\{ \sum_{n=N+1}^{N'} \psi_n \right\}^{1/q} A_{q'} \|g\|_{L_{q'}(v_{\lambda,\mu})}. \tag{6}
\]

Hence, by \([7, \text{Theorem (12.13)}]\), the inequalities \((5)\) and \((6)\) lead respectively to the estimates

\[
\|\Phi_N\|_{L_q(v_{\lambda,\mu})} \leq \left\{ \sum_{n=0}^{N} \psi_n \right\}^{1/q} A_{q'} \tag{7}
\]

and

\[
\|\Phi_N - \Phi_{N'}\|_{L_q(v_{\lambda,\mu})} \leq \left\{ \sum_{n=N+1}^{N'} \psi_n \right\}^{1/q} A_{q'}.
\]

The last inequality combined with \((3)\) show that the sequence \( \{\Phi_N\}_{N=1}^{\infty} \) is a Cauchy sequence in \( L_q(v_{\lambda,\mu}) \) and therefore convergent in \( L_q(v_{\lambda,\mu}) \); let \( f \) be its limit. Then, by mean convergence,

\[
\hat{f}_n = \lim_{N \to \infty} \left( \hat{\Phi}_N \right)_n, \quad n = 0, 1, \ldots,
\]

which is easily seen to equal \( \phi(n) \). Moreover, the defining relation

\[
f = \lim_{N \to \infty} \Phi_N \quad \text{in} \quad L_q(v_{\lambda,\mu})
\]

and the inequality \((7)\) show that \((1)\) holds and so complete the proof. \( \square \)
To prove the following result, we use the Riesz–Thorin interpolation theorem.

**Theorem 3.** (a) If $1 < p \leq 2$ and $f \in L_p(v_\lambda,\mu)$, then

$$\left\{ \sum_{n=0}^{\infty} \left( n + 1 \right) \left( \frac{1}{q'} - \frac{1}{q} \right) \max(\lambda,\mu) |\hat{f}_n| \right\}^{1/p'} \leq B_p \|f\|_{L_p(v_\lambda,\mu)}.$$  \hfill (8)

(b) If $2 \leq q < \infty$ and $\phi$ is a function on non-negative integers satisfying

$$\sum_{n=0}^{\infty} \left( n + 1 \right) \left( \frac{1}{q'} - \frac{1}{q} \right) \max(\lambda,\mu) |\phi(n)| < \infty,$$

then the algebraic polynomials

$$\Phi_N(t) = \sum_{n=0}^{N} \phi(n) \tilde{C}_n^{(\lambda,\mu)}(t)$$

converge in $L_q(v_\lambda,\mu)$ to a function $f$ satisfying $\hat{f}_n = \phi(n)$, $n = 0, 1, \ldots$, and

$$\|f\|_{L_q(v_\lambda,\mu)} \leq B_q \left\{ \sum_{n=0}^{\infty} \left( n + 1 \right) \left( \frac{1}{q'} - \frac{1}{q} \right) \max(\lambda,\mu) |\phi(n)| \right\}^{q'/q}.$$  \hfill (10)

**Proof.** Let $\sigma = \max(\lambda,\mu)$.

(a) Note that for $p = 2$ the Parseval identity implies equality in (8) with $B_2 = 1$. We now consider (8) as the transformation from $L_p(v_\lambda,\mu)$ into the sequence $\{ (n+1)^{-\sigma} \hat{f}_n \}_{n=0}^{\infty}$ in the $\ell_p$ norm with the weight $\{ (n+1)^{2\sigma} \}_{n=0}^{\infty}$ and show that this transformation is of strong type $(1, \infty)$. Using Theorem 1, we get

$$\sup_{n=0,1,\ldots} \left\{ (n+1)^{-\sigma} |\hat{f}_n| \right\} \leq B_1 \|f\|_{L_1(v_\lambda,\mu)}.$$ 

Thus, applying the Riesz–Thorin theorem, we deduce (8) with $B_p = B_1^{\frac{1}{p} - \frac{1}{q'}}$.

(b) The proof of this part is closely related to the proof of part (b) of Theorem 2. One can obtain this proof. We give it here for completeness.

We have $1 < q' \leq 2$. For brevity, write $\psi_n$ in place of $\left( (n+1)^{\frac{1}{q'} - \frac{1}{q}} |\phi(n)| \right)^{q'}$. Suppose that $g \in L_{q'}(v_\lambda,\mu)$ and that $N < N'$ are positive integers. Applying Hölder’s inequality and
(a), we find that
\[ \left| \int_{-1}^{1} \Phi_N(t) g(t) v_{\lambda,\mu}(t) \, dt \right| = \left| \sum_{n=0}^{N} \phi(n) \hat{g}_n \right| = \left| \sum_{n=0}^{N} (n+1) \left( \alpha^{\beta} \right)^{\sigma} \phi(n) (n+1) \left( \alpha^{\beta} \right)^{\sigma} \hat{g}_n \right| \leq \left\{ \sum_{n=0}^{N} \psi_n \right\}^{1/q} \left\{ \sum_{n=0}^{N} \left( (n+1) \left( \alpha^{\beta} \right)^{\sigma} |\hat{g}_n| \right)^{q} \right\}^{1/q} \leq \left\{ \sum_{n=0}^{N} \psi_n \right\}^{1/q} B_q g_{L_q(v_{\lambda,\mu})}. \]
(11)

Similarly,
\[ \left| \int_{-1}^{1} (\Phi_N(t) - \Phi_{N'}(t)) g(t) v_{\lambda,\mu}(t) \, dt \right| \leq \left\{ \sum_{n=N+1}^{N'} \psi_n \right\}^{1/q} B_q g_{L_q(v_{\lambda,\mu})}. \]
(12)

Hence, by [7, Theorem (12.13)], the inequalities (11) and (12) lead respectively to the estimates
\[ \| \Phi_N \|_{L_q(v_{\lambda,\mu})} \leq \left\{ \sum_{n=0}^{N} \psi_n \right\}^{1/q} B_q. \]
(13)

and
\[ \| \Phi_N - \Phi_{N'} \|_{L_q(v_{\lambda,\mu})} \leq \left\{ \sum_{n=N+1}^{N'} \psi_n \right\}^{1/q} B_q. \]

The last inequality combined with (7) show that the sequence \( \{ \Phi_N \}_{N=1}^{\infty} \) is a Cauchy sequence in \( L_q(v_{\lambda,\mu}) \) and therefore convergent in \( L_q(v_{\lambda,\mu}) \); let \( f \) be its limit. Then, by mean convergence,
\[ \hat{f}_n = \lim_{N \to \infty} \left( \Phi_N \right)_n, \quad n = 0, 1, \ldots, \]
which is easily seen to equal \( \phi(n) \). Moreover, the defining relation
\[ f = \lim_{N \to \infty} \Phi_N \quad \text{in} \quad L_q(v_{\lambda,\mu}) \]
and the inequality (13) show that (10) holds and so complete the proof. \( \square \)

4. Unification of the Hardy–Littlewood-type and the Hausdorff–Young-type inequalities

Theorem 5 contains the Hardy–Littlewood-type and the Hausdorff–Young-type inequalities for the expansions by orthonormal polynomials with respect to the weight function \( v_{\lambda,\mu} \) (see (11)). To prove it, we need Stein’s modification of the Riesz–Thorin interpolation theorem (see [8, Theorem 2, p. 485]) given below.
Theorem 4 (Stein). Suppose \( \nu_1 \) and \( \nu_2 \) are \( \sigma \)-finite measures on \( M \) and \( S \), respectively, and \( T \) is a linear operator defined on \( \nu_1 \)-measurable functions on \( M \) to \( \nu_2 \)-measurable functions on \( S \). Let \( 1 \leq r_0, r_1, s_0, s_1 \leq \infty \) and
\[
\frac{1}{r} = \frac{1}{r_0} + \frac{1}{r_1}, \quad \frac{1}{s} = \frac{1}{s_0} + \frac{1}{s_1}, \text{ where } 0 \leq t \leq 1.
\]
Suppose further that
\[
\|(Tg) \cdot v_i\|_{L_{s_i}(S,\nu_2)} \leq L_i \|g \cdot u_i\|_{L_{r_i}(M,\nu_1)}, \quad i = 0, 1,
\]
where \( u_i \) and \( v_i \) are non-negative weight functions. Let \( u = u_0^{1-t} \cdot u_1^t \), \( v = v_0^{1-t} \cdot v_1^t \).

Then
\[
\|(Tg) \cdot v\|_{L_s(S,\nu_2)} \leq L \|g \cdot u\|_{L_r(M,\nu_1)}
\]
with \( L = L_0^{1-t} \cdot L_1^t \).

Theorem 5. Let \( \sigma = \max(\lambda, \mu) \).

(a) If \( 1 < p \leq 2 \), \( f \in L_p(v_{\lambda,\mu}) \), and \( p \leq s \leq p' \), then
\[
\left\{ \sum_{n=0}^{\infty} \left( (n+1) \left( \frac{1}{p} - \frac{1}{s} \right) + \left( \frac{1}{p} - \frac{1}{t} \right) (\sigma + 1) \right) |\hat{f}_n| \right\}^{1/s} \leq C_p(s) \|f\|_{L_p(v_{\lambda,\mu})}.
\]
(b) If \( 2 \leq q < \infty \), \( q' \leq r \leq q \), and \( \phi \) is a function on non-negative integers satisfying
\[
\sum_{n=0}^{\infty} \left( (n+1) \left( \frac{1}{q} - \frac{1}{r} \right) + \left( \frac{1}{q} - \frac{1}{\sigma} \right) (\sigma + 1) |\phi(n)| \right)^{q'} < \infty,
\]
then the algebraic polynomials
\[
\Phi_N(t) = \sum_{n=0}^{N} \phi(n) \tilde{C}_n^{(\lambda,\mu)}(t)
\]
converge in \( L_q(v_{\lambda,\mu}) \) to a function \( f \) satisfying \( \hat{f}_n = \phi(n), n = 0, 1, \ldots, \) and
\[
\|f\|_{L_q(v_{\lambda,\mu})} \leq C_q(r) \left\{ \sum_{n=0}^{\infty} \left( (n+1) \left( \frac{1}{q} - \frac{1}{r} \right) + \left( \frac{1}{q} - \frac{1}{\sigma} \right) (\sigma + 1) |\phi(n)| \right) \right\}^{1/r'}.
\]

Proof. (a) This part was proved for \( s = p \) (with \( C_p(p) = A_p \)) and \( s = p' \) (with \( C_p(p') = B_p \)) in Theorems 2 and 8 respectively. So for \( p = 2 \), we obtain the equality in (14) with \( C_2(2) = 1 \).

Consider now the case that \( 1 < p < 2 \). To prove (14), we set in Theorem 4 \( M = [-1, 1], \nu_1 \) the Lebesgue measure, \( S = \{n\}_{n=0}^{\infty}, \nu_2 \) the counting measure, \( g = f, Tg = \{\hat{f}_n\}_{n=0}^{\infty}, \)
\[
r = r_0 = r_1 = p, u = u_0 = u_1 = v_{\lambda,\mu}, s_0 = s_1 = p', v_0 = \{(n+1) \left( \frac{1}{p'} - \frac{1}{p} \right) \}_{n=0}^{\infty},
\]
\[
u_1 = \{(n+1) \left( \frac{1}{p'} - \frac{1}{p} \right) (\sigma + 1) \}_{n=0}^{\infty}, L_0 = B_p, L_1 = A_p, \quad \text{and} \quad \frac{1}{s} = \frac{1-t}{p'} + \frac{t}{p}.
\]
As \( \frac{1}{p} + \frac{1}{p'} = 1 \), \( \frac{1}{s} - \frac{1}{p'} = (1-t) \left( \frac{1}{p'} - \frac{1}{p} \right) \), \( \frac{1}{p'} - \frac{1}{s} = t \left( \frac{1}{p'} - \frac{1}{p} \right) \), the proof of (14) is concluded.

Because of
\[
1-t = \frac{1}{s} - \frac{1}{p'}, \quad t = \frac{1}{p'} - \frac{1}{s},
\]
it is clear that \( C_p(s) = B_p^{1-t} A_p^t \).

(b) Taking into account the previously given proofs (see parts (b) and (b) in Theorems 2 and 8 respectively), the proof is obvious and left to the reader. \( \square \)
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