Abstract

We propose new restarting strategies for accelerated gradient and accelerated coordinate descent methods. Our main contribution is to show that the restarted method has a geometric rate of convergence for any restarting frequency, and so it allows us to take profit of restarting even when we do not know the strong convexity coefficient. The scheme can be combined with adaptive restarting, leading to the first provable convergence for adaptive restarting schemes with accelerated gradient methods. Finally, we illustrate the properties of the algorithm on a regularized logistic regression problem and on a Lasso problem.

1 Introduction

1.1 Motivation

The proximal gradient method aims at minimizing composite convex functions of the form

\[ F(x) = f(x) + \psi(x), \quad x \in \mathbb{R}^n \]

where \( f \) is differentiable with Lipschitz gradient and \( \psi \) may be nonsmooth but has an easily computable proximal operator. For a mild additional computational cost, accelerated gradient methods transform the proximal gradient method, for which the optimality gap \( F(x_k) - F(x^*) \) decreases as \( O(1/k) \), into an algorithm with “optimal” \( O(1/k^2) \) complexity \([9]\). Accelerated variants include the dual accelerated proximal gradient \([10,12]\), the accelerated proximal gradient method (APG) \([18]\) and FISTA \([1]\). Gradient-type methods, also called first-order methods, are often used to solve large-scale problems because of their good scalability and easiness of implementation that facilitates parallel and distributed computations.

In the case when the nonsmooth function \( \psi \) is separable, which means that it writes as

\[ \psi(x) = \sum_i \psi^i(x^i), \quad x = (x^1, \ldots, x^n) \in \mathbb{R}^n, \]

coordinate descent methods are often considered thanks to the separability of the proximal operator of \( \psi \). These are optimization algorithms that update only one coordinate of the vector of variables at each iteration, hence using partial derivatives rather than the whole gradient. In \([11]\), Nesterov introduced the randomized coordinate descent method with an improved guarantee on the iteration complexity. He also gave an accelerated coordinate descent method for smooth functions. Lee and Sidford \([5]\) introduced an efficient implementation of the method and Fercoq and Richtárik \([4]\) developed the accelerated proximal coordinate descent method (APPROX) for the minimization of composite functions.

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When solving a strongly convex problem, classical (non-accelerated) gradient and coordinate descent methods automatically have a linear rate of convergence, i.e. \( F(x_k) - F(x_\star) \in O((1 - \mu)^k) \) for a problem dependent \( 0 < \mu < 1 \), whereas one needs to know explicitly the strong convexity parameter in order to set accelerated gradient and accelerated coordinate descent methods to have a linear rate of convergence, see for instance \([3, 6, 11, 12]\). Setting the algorithm with an incorrect parameter may result in a slower algorithm, sometimes even slower than if we had not tried to set an acceleration scheme \([13]\). This is a major drawback of the method because in general, the strong convexity parameter is difficult to estimate.

In the context of accelerated gradient method with unknown strong convexity parameter, Nesterov \([12]\) proposed a restarting scheme which adaptively approximate the strong convexity parameter. The similar idea was exploited by Lin and Xiao \([8]\) for sparse optimization. Nesterov \([12]\) also showed that, instead of deriving a new method designed to work better for strongly convex functions, one can restart the accelerated gradient method and get a linear convergence rate. However, the restarting frequency he proposed still depends explicitly on the strong convexity of the function and so O’Donoghue and Candes \([13]\) introduced some heuristics to adaptively restart the algorithm and obtain good results in practice.

1.2 Contributions

In this paper, we show that we can restart accelerated gradient and coordinate descent methods, including APG, FISTA and APPROX, at any frequency and get a linearly convergent algorithm. The rate depends on an estimate of the strong convexity and we show that for a wide range of this parameter, one obtains a faster rate than without acceleration. In particular, we do not require the estimate of the strong convexity coefficient to be smaller than the actual value. In this way, our result supports and explains the practical success of arbitrary periodic restart for accelerated gradient methods.

In order to obtain the improved theoretical rate, we need to define a novel point where the restart takes place, which is a convex combination of previous iterates. Our approach is radically different from the previous restarting schemes \([8, 12, 13]\), for which the evaluation of the gradient or the objective value is needed in order to verify the restarting condition. In particular, our approach can be extended to a restarted APPROX, which admits the same theoretical complexity bound as the accelerated coordinate descent methods for strongly convex functions \([7]\) and exhibits better performance in numerical experiments.

In Sections 2 and 3 we recall the main convergence results for accelerated gradient methods. In Section 4, we present our restarting rules: one for accelerated gradient and one for accelerated coordinate descent. Finally, we present numerical experiments on the lasso and logistic regression problem in Section 5.

2 Accelerated gradient schemes

2.1 Problem and assumptions

For simplicity we present the algorithm in coordinatewise form. The extension to blockwise setting follows naturally (see for instance \([4]\)). We consider the following optimization problem:

\[
\begin{align*}
\text{minimize} & \quad F(x) := f(x) + \psi(x) \\
\text{subject to} & \quad x = (x^1, \ldots, x^n) \in \mathbb{R}^n, 
\end{align*}
\]

where \( f : \mathbb{R}^n \to \mathbb{R} \) is a differentiable convex function and \( \psi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\} \) is a closed convex and separable function:

\[
\psi(x) = \sum_{i=1}^n \psi^i(x^i).
\]

Note that this implies that each function \( \psi^i : \mathbb{R} \to \mathbb{R} \) is closed and convex. Let \( x^\star \) denote a solution of (1). We further assume that for each positive vector \( v = (v_1, \ldots, v_n) \), there is a constant \( \mu_F(v) > 0 \) such that

\[
F(x) \geq F(x^\star) + \frac{\mu_F(v)}{2} \|x - x^\star\|_v^2,
\]
Algorithm 1 FISTA

1: Choose $x_0 \in \text{dom} \psi$. Set $\theta_0 = 1$ and $z_0 = x_0$.
2: for $k \geq 0$ do
3: $y_k = (1 - \theta_k)x_k + \theta_k z_k$
4: $z_{k+1} = \arg \min_{z \in \mathbb{R}^n} \left\{ \langle \nabla f(y_k), z - y_k \rangle + \frac{1}{2}\|z - y_k\|_v^2 + \psi(z) \right\}$
5: $x_{k+1} = y_k + \theta_k (z_{k+1} - y_k)$
6: $\theta_{k+1} = \frac{\theta_k^2 + 4\theta_k^2 - \theta_k^2}{2}$
7: end for

Algorithm 2 APG

Choose $x_0 \in \text{dom} \psi$. Set $\theta_0 = 1$ and $z_0 = x_0$.
for $k \geq 0$ do
$y_k = (1 - \theta_k)x_k + \theta_k z_k$
$z_{k+1} = \arg \min_{z \in \mathbb{R}^n} \left\{ \langle \nabla f(y_k), z - y_k \rangle + \frac{\theta_k}{2\tau}\|z - z_k\|_n^2 + \psi(z) \right\}$
$x_{k+1} = y_k + \theta_k (z_{k+1} - z_k)$
$\theta_{k+1} = \frac{\theta_k^2 + 4\theta_k^2 - \theta_k^2}{2}$
end for

Algorithm 3 APPROX

Choose $x_0 \in \text{dom} \psi$. Set $\theta_0 = \frac{2}{n}$ and $z_0 = x_0$.
for $k \geq 0$ do
$y_k = (1 - \theta_k)x_k + \theta_k z_k$
Randomly generate $S_k \sim \hat{S}$
for $i \in S_k$ do
$z_{k+1}^i = \arg \min_{z \in \mathbb{R}} \left\{ \langle \nabla_i f(y_k), z - y_k^i \rangle + \frac{\theta_k}{2\tau}\|z - y_k^i\|^2 + \psi(z) \right\}$
end for
$x_{k+1} = y_k + \frac{\theta_k}{\tau} (z_{k+1} - z_k)$
$\theta_{k+1} = \frac{\theta_k^2 + 4\theta_k^2 - \theta_k^2}{2}$
end for

where $\|\cdot\|_v$ denotes the weighted Euclidean norm in $\mathbb{R}^n$ defined by:

$$\|x\|_v^2 \overset{\text{def}}{=} \sum_{i=1}^n v_i(x_i)^2.$$
We have written the algorithms in a unified framework to emphasize their similarities. Practical implementations usually consider only two variables: \((x_k, y_k)\) for FISTA, \((y_k, z_k)\) for APG and \((z_k, w_k)\) where \(w_k = \theta_k^{-2} (x_k - z_k)\) for APPROX. One may also consider \(t_k = \theta_k^{-1}\) instead of \(\theta_k\).

The update in FISTA, APG or APPROX employs a positive vector \(v \in \mathbb{R}^n\). To guarantee the convergence of the algorithm, the positive vector \(v\) should satisfy the so-called expected separable overapproximation (ESO) assumption, developed in [3, 15] for the study of parallel coordinate descent methods.

**Assumption 1 (ESO).** We write \((f, \hat{S}) \sim \text{ESO}(v)\) if

\[
E \left[f(x + h_{[\hat{S}]})\right] \leq f(x) + \frac{\tau}{n} \left(\langle \nabla f(x), h \rangle + \frac{1}{2} \|h\|_v^2\right), \quad x, h \in \mathbb{R}^n.
\]

where for \(h = (h^1, \ldots, h^n) \in \mathbb{R}^n\) and \(S \subset [n]\), \(h_{[S]}\) is defined as:

\[h_{[S]} \defeq \sum_{i \in S} h^i e_i,\]

with \(e_i\) being the \(i\)th standard basis vectors in \(\mathbb{R}^n\).

We require that the positive vector \(v\) used in APPROX satisfy (3) with respect to the sampling \(\hat{S}\) used. Note that FISTA shares the same constant \(v\) with APG and APG can be seen as a special case of APPROX when \(\hat{S} = [n]\). Therefore the positive vector \(v\) used in FISTA and APG should then satisfy:

\[f(x + h) \leq f(x) + \langle \nabla f(x), h \rangle + \frac{1}{2} \|h\|_v^2, \quad x, h \in \mathbb{R}^n,
\]

which is nothing but a Lipschitz condition on the gradient of \(f\). In other words, the vector \(v\) used in FISTA and APG is just the Lipschitz constant of \(\nabla f\), given a diagonal scaling that may be chosen to improve the conditioning of the problem.

When in each step we update only one coordinate, we have \(\tau = 1\) and (3) reduces to:

\[
\frac{1}{n} \sum_{i=1}^n f(x + h^i e_i) \leq f(x) + \frac{1}{n} \left(\langle \nabla f(x), h \rangle + \frac{1}{2} \|h\|_v^2\right), \quad x, h \in \mathbb{R}^n.
\]

It is easy to see that in this case the vector \(v\) corresponds to the coordinate-wise Lipschitz constants of \(\nabla f\), see e.g. [11]. Explicit formulas for computing admissible \(v\) with respect to more general sampling \(\hat{S}\) can be found in [15, 3, 14].

### 3 Convergence results for accelerated gradients methods

In this section we review two basic convergence results of FISTA and APPROX, which will be used later to build restarted methods. We first recall the following properties on the sequence \(\{\theta_k\}\).

**Lemma 1.** The sequence \(\{\theta_k\}\) defined by \(\theta_0 \leq 1\) and \(\theta_{k+1} = \frac{\sqrt{\theta_k^2 + 4\theta_0^2} - \theta_0^2}{2}\) satisfies

\[
\frac{1}{k + 1/\theta_0} \leq \theta_k \leq \frac{2}{k + 2/\theta_0} \quad (5)
\]

\[
\frac{1 - \theta_{k+1}}{\theta_k^2} = \frac{1}{\theta_k^2}, \quad \forall k = 0, 1, \ldots \quad (6)
\]

\[
\theta_{k+1} \leq \theta_k, \quad \forall k = 0, 1, \ldots \quad (7)
\]
This is equivalent to \( k \) of Lemma 4.1 in [1] (Note that we have a shift of indices for our variable \( s \) (z)). Then, we rewrite the equality, we get that there exists \( \gamma \) such that

\[
\sum_{i=0}^{n_k} \gamma_k^i (z_i - x_*) = 0
\]

Proof. We give the proof for completeness. \( 0 \) holds because \( \theta_{k+1} = 1 \) is the unique positive square root to the polynomial \( P(X) = X^2 + \theta_k^2 X - \theta_k^2 \). \( 1 \) is a direct consequence of \( 2 \).

Let us prove \( 3 \) by induction. It is clear that \( \theta_0 \leq \frac{2}{\theta_0} \). Assume that \( \theta_k \leq \frac{2}{\theta_k} \). We know that \( P(\theta_{k+1}) = 0 \) and that \( P \) is an increasing function on \([0, +\infty)\). So we just need to show that \( P(\frac{2}{\theta_{k+1} + \theta_0}) \geq 0 \).

\[
P(\frac{2}{k+1 + 2/\theta_0}) \geq \frac{4}{(k+1 + 2/\theta_0)^2} + \frac{2}{k+1 + 2/\theta_0} \theta_k^2 - \theta_k^2
\]

As \( \theta_k \leq \frac{2}{\theta_k} \) and \( \frac{2}{k+1 + 2/\theta_0} - 1 \leq 0 \),

\[
P(\frac{2}{k+1 + 2/\theta_0}) \geq \frac{4}{(k+1 + 2/\theta_0)^2} + \frac{2}{(k+1 + 2/\theta_0)^2} - 1 \geq \frac{1}{(k+1 + 2/\theta_0)^2} \geq 0.
\]

For the other inequality, \( \frac{1}{k+1/\theta_0} \leq \theta_0 \). We now assume that \( \theta_k \geq \frac{1}{k+1/\theta_0} \) but that \( \theta_{k+1} < \frac{1}{k+1 + 1/\theta_0} \). Then, using \( 4 \) and the inequality we just proved we have

\[
(k+1 + 1/\theta_0)^2 < \frac{1}{\theta_{k+1}^2} \theta_k^2 + \frac{1}{\theta_{k+1}^2} \leq (k+1 + 1/\theta_0)^2 + (k+1 + 2/\theta_0).
\]

This is equivalent to

\[
2(k+1 + 1/\theta_0) + 1 < k+1 + 2/\theta_0
\]

which obviously does not hold for any \( k \geq 0 \). So \( \theta_{k+1} \geq \frac{1}{k+1 + 1/\theta_0} \).

Proposition 1. The iterates of FISTA satisfy for all \( k \geq 1 \),

\[
\frac{1}{\theta_{k-1}} (F(x_k) - F(x_*)) + \frac{1}{2} \| z_k - x_* \|_v^2 \leq \frac{1}{2} \| x_0 - x_* \|_v^2,
\]

and

\[
\frac{1}{2} \| z_k - x_* \|_v^2 \leq \frac{1}{2} \| x_0 - x_* \|_v^2.
\]

Proof. Since \( z_{k+1} = z_k + \theta_k^{-1} (x_{k+1} - y_k) = \theta_k^{-1} x_{k+1} - (\theta_k^{-1} - 1)x_k \), Inequality \( 5 \) is a simple consequence of Lemma 4.1 in [1] (Note that we have a shift of indices for our variables \( x_{k+1}, z_{k+1} \) vs \( x_k, u_k + x_* \) in [1]). For the second inequality, we first remark that as the left term in \( 5 \) is the sum of two nonnegative summands, each of them is smaller than the right hand side. Hence, for all \( k \),

\[
\frac{1}{2} \| z_k - x_* \|_v^2 \leq \frac{1}{2} \| x_0 - x_k \|_v^2.
\]

Then, we rewrite \( x_{i+1} = z_i + \theta_i^{-1} (x_{i+1} - y_i) \) as \( x_{i+1} = \theta_i z_{i+1} + (1 - \theta_i)x_i \). Recursively applying this convex equality, we get that there exists \( \gamma_k^i \geq 0 \) such that \( \sum_{i=0}^{n_k} \gamma_k^i = 1 \) and \( x_k = \sum_{i=0}^{n_k} \gamma_k^i z_i \).

\[
\frac{1}{2} \| x_k - x_* \|_v^2 = \frac{1}{2} \left\| \sum_{i=0}^{n_k} \gamma_k^i (z_i - x_*) \right\|_v^2 \leq \frac{1}{2} \sum_{i=0}^{n_k} \gamma_k^i \| z_i - x_* \|_v^2 \leq \frac{1}{2} \sum_{i=0}^{n_k} \gamma_k^i \| x_0 - x_k \|_v^2
\]

and we conclude using \( \sum_{i=0}^{n_k} \gamma_k^i = 1 \).

Proposition 2. The iterates of APPROX satisfy for all \( k \geq 1 \),

\[
\frac{1}{\theta_{k-1}^2} E[F(x_k) - F(x_*)] + \frac{1}{2 \theta_0^2} E[\| z_k - x_* \|_v^2] \leq \frac{1 - \theta_0}{\theta_0^2} (F(x_0) - F(x_*)) + \frac{1}{2 \theta_0^2} \| x_0 - x_* \|_v^2
\]
and
\[
\frac{1 - \theta_0}{\theta_0^2} E[F(x_k) - F(x^*)] + \frac{1}{2\theta_0^2} E[\|x_k - x^*\|_v^2] \leq \frac{1 - \theta_0}{\theta_0^2} (F(x_0) - F(x^*))
\]
\[
+ \frac{1}{2\theta_0^2} \|x_0 - x^*\|_v^2 - \sum_{i=0}^{k-1} \gamma^i_k E[F(x_i) - F(x^*)] - \left( \frac{1}{\theta_0 \theta_{k-1}} - \frac{1 - \theta_0}{\theta_0^2} \right) E[F(x_k) - F(x^*)]
\]
\[(11)\]
where \( \frac{1}{\theta_{i-1}} := \frac{1 - \theta_0}{\theta_0^2} \) and \( \gamma^i_k \) is defined recursively by setting \( \gamma_0^0 = 1, \gamma_0^1 = 0, \gamma_1^1 = 1 \) and for \( k \geq 1, \)
\[
\gamma_{k+1}^i = \begin{cases} 
(1 - \theta_k) \gamma_k^i, & i = 0, \ldots, k - 1, \\
\theta_k(1 - \frac{\gamma}{\theta_0} \theta_{k-1}) + \frac{\gamma}{\theta_k - \theta_k} - \theta_k, & i = k, \\
\frac{\gamma}{\theta_k}, & i = k + 1.
\end{cases}
\]
\[(12)\]

**Proof.** Inequality (10) is just Theorem 3 of [4]. For the second inequality, we first isolate \( E[\|z_k - x^*\|_v^2] \) in (10). For all \( k \geq 0, \)
\[
\frac{1 - \theta_0}{\theta_0^2} E[\|z_k - x^*\|_v^2] \leq \frac{1 - \theta_0}{\theta_0^2} (F(x_0) - F(x^*)) + \frac{1}{2\theta_0^2} \|x_0 - x^*\|_v^2 - \frac{1}{\theta_{k-1}^2} E[F(x_k) - F(x^*)].
\]
\[(13)\]
Then, we use the fact, proved in Lemma 2 of [4], that \( \gamma_k^i \geq 0, \sum_{i=0}^{k} \gamma_k^i = 1 \) and \( x_k = \sum_{i=0}^{k} \gamma_k^i z_i \). Therefore,
\[
\frac{1}{2\theta_0^2} E[\|x_k - x^*\|_v^2] = \frac{1}{2\theta_0^2} E \left[ \left\| \sum_{i=0}^{k} \gamma_k^i (z_i - x^*) \right\|_v^2 \right] \leq \frac{1}{2\theta_0^2} \sum_{i=0}^{k} \gamma_k^i E[\|z_i - x^*\|_v^2]
\]
\[(14)\]
Plugging (13) into (14) we get:
\[
\frac{1}{2\theta_0^2} E[\|x_k - x^*\|_v^2]
\]
\[
\leq \sum_{i=0}^{k} \gamma_k^i \left( \frac{1 - \theta_0}{\theta_0^2} (F(x_0) - F(x^*)) + \frac{1}{2\theta_0^2} \|x_0 - x^*\|_v^2 - \frac{1}{\theta_{k-1}^2} E[F(x_i) - F(x^*)] \right)
\]
\[
= \frac{1 - \theta_0}{\theta_0^2} (F(x_0) - F(x^*)) + \frac{1}{2\theta_0^2} \|x_0 - x^*\|_v^2 - \sum_{i=0}^{k-1} \gamma_k^i E[F(x_i) - F(x^*)] - \frac{\gamma_k^k}{\theta_{k-1}^2} E[F(x_k) - F(x^*)]
\]
and we deduce (11) using \( \gamma_k^k = \theta_{k-1} / \theta_0 \). \( \square \)

**Remark 2.** The strength of these propositions is that they are independent of the strong convexity parameter. Indeed, APG, FISTA and APPROX work for non-strongly convex minimization.

**Remark 3.** As APPROX generalizes APG, we have covered all three algorithms in the two propositions. A remarkable feature is that the result for FISTA and APG are exactly the same even though the algorithms are different.

### 4 Restarted gradient methods

The basic tool upon which we build our restarting rule is a contraction property. We first present two restarting rules that require a special condition in order to guarantee the linear convergence. Then we present new rules that are more complex but are always certified to give a linearly convergent algorithm.
4.1 Conditional restarting

The first rule is an extension of the “optimal fixed restart” of [12, 13] to FISTA and APPROX.

**Proposition 3** (Conditional restarting at $x_k$). Let $(x_k, z_k)$ be the iterates of FISTA or APPROX applied to (1). We have

$$\mathbb{E}[F(x_k) - F(x_*)] \leq \theta_k^2 - \frac{1}{\theta_0^2} (F(x_*) - F(x_0)),$$

Moreover, given $\alpha < 1$, if

$$k \geq \frac{2}{\theta_0} \left( \sqrt{\frac{1 + \mu_F(v)}{\alpha \mu_F(v)}} - 1 \right) + 1,$$

then $\mathbb{E}[F(x_k) - F(x_*)] \leq \alpha(F(x_0) - F(x_*)).$

**Proof.** By (8) and (10), the following holds for the iterates of FISTA ($\theta_0 = 1$) and APPROX:

$$\mathbb{E}[F(x_k) - F(x_*)] \leq \theta_k^2 - \frac{1}{\theta_0^2} (F(x_*) - F(x_0)) + \frac{1}{2\theta_0^2} \|x_0 - x_*\|^2_v.$$

Condition (15) is equivalent to:

$$\frac{4}{(k - 1 + 2/\theta_0)^2} \left( \frac{1}{\theta_0^2} + \frac{1}{\mu_F(v)\theta_0^2} \right) \leq \alpha,$$

and we have the contraction using (5).

**Remark 4.** Notice that the restarting rule (15) requires to know a lower bound on the strong convexity coefficient of $F$.

The next restarting rule is built upon a comparison condition and does not rely on any estimation of $\mu_F(v)$.

**Proposition 4** (Conditional restarting at $z_k$). Let $(x_k, z_k)$ be the iterates of FISTA or APG applied to (1). If $F(z_k) \leq F(x_k)$, then

$$\frac{1}{2} \|z_k - x_*\|^2_v \leq \frac{1}{2} \|z_0 - x_*\|^2_v.$$

**Proof.** By (8) and (10), the following holds for the iterates of FISTA and APG ($\theta_0 = 1$):

$$\frac{1}{\theta_k^2} (F(z_k) - F(x_*)) + \frac{1}{2} \|z_k - x_*\|^2_v \leq \frac{1}{2} \|x_0 - x_*\|^2_v.$$

By (2), we get

$$\left( \mu_F(v) + \frac{1}{2} \|z_k - x_*\|^2_v \leq \frac{1}{2} \|x_0 - x_*\|^2_v = \frac{1}{2} \|z_0 - x_*\|^2_v. $$

Here, we do not need to know $\mu_F(v)$ but we need to wait for $F(z_k)$ to be smaller than $F(x_k)$. This event does happen sometimes but there is no guarantee for it to happen when minimizing a given function. Hence we may wait for ever and never restart.
4.2 Unconditional restarting

In this section, we prove the main results of this paper: setting a convex combination of the past iterates as the restarting point leads to a linearly convergent restarted method.

We first show that for full-gradient accelerated methods, an arbitrary strict convex combination of the first iterates of APPROX works.

**Theorem 1** (Restarting for FISTA and APG). Let \((x_k, z_k)\) be the iterates of FISTA or APG applied to \(f\). Let \(\sigma \in [0, 1]\) and \(\bar{x}_k = (1 - \sigma)x_k + \sigma z_k\). We have

\[
\frac{1}{2} \|\bar{x}_k - x_*\|_v^2 \leq \frac{1}{2} \max \left\{ \sigma, 1 - \frac{\sigma \mu F(v)}{\theta_{k-1}^2} \right\} \|x_0 - x_*\|_v^2.
\]

**Proof.** By the definition of \(\bar{x}_k\):

\[
\frac{1}{2} \|\bar{x}_k - x_*\|_v^2 \leq \frac{1 - \sigma}{2} \|x_k - x_*\|_v^2 + \frac{\sigma}{2} \|z_k - x_*\|_v^2
\]

\[
= \left(1 - \sigma - \frac{\sigma \mu F(v)}{\theta_{k-1}^2}\right) \frac{1}{2} \|x_k - x_*\|_v^2 + \frac{\sigma}{\theta_{k-1}^2} \left(\frac{\mu F(v)}{2} \|x_k - x_*\|_v^2 + \frac{\theta_{k-1}^2}{2} \|z_k - x_*\|_v^2\right)
\]

\[
\leq \max \left\{ 0, 1 - \sigma - \frac{\sigma \mu F(v)}{\theta_{k-1}^2}\right\} \frac{1}{2} \|x_k - x_*\|_v^2 + \frac{\sigma}{\theta_{k-1}^2} \left(F(x_k) - F(x_*) + \frac{\theta_{k-1}^2}{2} \|z_k - x_*\|_v^2\right)
\]

Next we apply (8) and (9) (the same holds for APG by taking \(\theta_0 = 1\) in (10) and (11)):

\[
\frac{1}{2} \|\bar{x}_k - x_*\|_v^2 \leq \max \left\{ 0, 1 - \sigma - \frac{\sigma \mu F(v)}{\theta_{k-1}^2}\right\} \frac{1}{2} \|x_k - x_*\|_v^2 + \frac{\sigma}{2} \|x_0 - x_*\|_v^2
\]

\[
= \max \left\{ \sigma, 1 - \frac{\sigma \mu F(v)}{\theta_{k-1}^2}\right\} \frac{1}{2} \|x_0 - x_*\|_v^2.
\]

\[
\square
\]

For APPROX, we need a more complex restarting point.

**Theorem 2** (Restarting for APPROX). Let \(\gamma_i^k\) be the coefficients defined in (12) and

\[
\hat{x}_k = \sum_{i=0}^{k-1} \frac{\gamma_i^k}{\theta_i^2} x_i + \frac{1}{\theta_0 \theta_{k-1}} \left(\sum_{i=0}^{k-1} \frac{\gamma_i^k}{\theta_i^2} x_i + \frac{1}{\theta_0 \theta_{k-1}} \left(1 - \theta_0 \theta_{k-1} - \frac{1 - \theta_0}{\theta_0^2}\right) x_k\right)
\]

(18)

be a convex combination of the \(k\) first iterates of APPROX. Let \(\sigma \in [0, 1]\) and \(\bar{x}_k = \sigma x_k + (1 - \sigma)\hat{x}_k\). Denote

\[
\Delta(x) := \frac{1 - \theta_0}{\theta_0^2} (F(x) - F(x_*)) + \frac{1}{2 \theta_0^2} \|x - x_*\|_v^2
\]

and

\[
m_k(\mu) := \frac{\mu \theta_0^2}{1 + \mu (1 - \theta_0)} \left(\sum_{i=0}^{k-1} \frac{\gamma_i^k}{\theta_i^2} + \frac{1}{\theta_0 \theta_{k-1}} \left(1 - \theta_0 \theta_{k-1} - \frac{1 - \theta_0}{\theta_0^2}\right) \frac{1}{\theta_0^2}\right). \quad (19)
\]

We have

\[
E[\Delta(\bar{x}_k)] \leq \max \left\{ \sigma, 1 - \sigma m_k(\mu F(v)) \right\} \Delta(x_0).
\]


Proof. Note that by \[7\],
\[
\frac{1}{\theta_0 \theta_{k-1}} \geq \frac{1}{\theta_0^2} \geq \frac{1 - \theta_0}{\theta_0^2}.
\]
Hence \(\hat{x}_k\) is a convex combination of \(\{x_0, \ldots, x_k\}\). By \[11\] and the definition of \(\hat{x}_k\),
\[
\Delta(x_0) \geq E[\Delta(x)] + \sum_{i=0}^{k-1} \gamma^k_{i} E[F(x_i) - F(x)] + \left(\frac{1}{\theta_0 \theta_k - 1} - \frac{1 - \theta_0}{\theta_0^2}\right) E[F(x_k) - F(x)]
\]
\[
\geq E[\Delta(x)] + \left(\sum_{i=0}^{k-1} \gamma^k_{i} \frac{1}{\theta_0 \theta_k - 1} - \frac{1 - \theta_0}{\theta_0^2}\right) E[F(\hat{x}_k) - F(x)]
\]
In view of the strong convexity assumption \[2\],
\[
\Delta(x) = \frac{1 - \theta_0}{\theta_0^2} (F(x) - F(x_*)) + \frac{1}{2\theta_0^2} \|x - x_*\|^2 \leq \left(\frac{1 - \theta_0}{\theta_0^2} + \frac{1}{\mu_F(v) \theta_0^2}\right) (F(x) - F(x_*))
\]
Therefore,
\[
\Delta(x_0) \geq E[\Delta(x)] + \frac{\mu_F(v) \theta_0^2}{1 + \mu_F(v)(1 - \theta_0)} \left(\sum_{i=1}^{k-1} \gamma^k_{i} \frac{1}{\theta_0 \theta_k - 1} - \frac{1 - \theta_0}{\theta_0^2}\right) E[\Delta(\hat{x}_k)]
\]
\[
\overset{[10]}{\geq} E[\Delta(x)] + m_k(\mu_F(v)) E[\Delta(\hat{x}_k)]
\]
Moreover, using \[11\] again, we can easily see that \(E[\Delta(x_i)] \leq \Delta(x_0)\) for all \(i\) and thus \(E[\Delta(\hat{x}_k)] \leq \Delta(x_0)\).
Let us now consider \(\bar{x}_k = \sigma x_k + (1 - \sigma) \hat{x}_k\).
\[
E[\Delta(\bar{x}_k)] \leq \sigma E[\Delta(x)] + (1 - \sigma) E[\Delta(\hat{x}_k)]
\]
\[
= \sigma E[\Delta(x)] + \sigma m_k(\mu_F(v)) E[\Delta(\hat{x}_k)] + (1 - \sigma - \sigma m_k(\mu_F(v))) E[\Delta(\hat{x}_k)]
\]
\[
\leq \sigma (E[\Delta(x)] + m_k(\mu_F(v)) E[\Delta(\hat{x}_k)]) + \max (0, 1 - \sigma - \sigma m_k(\mu_F(v))) E[\Delta(\hat{x}_k)]
\]
\[
\leq \sigma \Delta(x_0) + \max (0, 1 - \sigma - \sigma m_k(\mu_F(v))) \Delta(x_0)
\]
\[
= \max (\sigma, 1 - \sigma m_k(\mu_F(v))) \Delta(x_0)
\]
\[
\square
\]
4.3 Restarted APPROX

We describe in Algorithm 4 the restarted APPROX method and give the convergence result in Theorem 3.

Remark 5. For this restarting rule to be useful, we need to be able to compute \(\hat{x}_k\) efficiently, in particular without computing \(x_i\) for \(i < k\). A way to do this is to use the variable \(w_i = \theta_{i-1}^{-2} (x_i - z_i)\), which is maintained up-to-date in the algorithm:
\[
\sum_{i=0}^{k-1} \frac{\gamma^k_{i}}{\theta_0^2} x_i = \sum_{i=0}^{k-1} \frac{\gamma^k_{i}}{\theta_0^2} z_i + \gamma^k_{k} w_i.
\]

Then, we can compute the sum using cumulative updates like in \[2\]. We develop this idea in Appendix A.

Theorem 3. Let us choose \(K \in \mathbb{N}\) and \(\sigma \in (0, 1)\) as we wish. Using the notation defined in Theorem 2, the iterates of Algorithm 4 satisfy for any \(k \geq K\)
\[
E[\Delta(x)] \leq \left(\max (\sigma, 1 - \sigma m_K(\mu_F(v)))^{1/K}\right)^{k-K} \Delta(x_0).
\]
Algorithm 4 APPROX+restart

Choose $x_0 \in \mathbb{R}^n$, set $z_0 = x_0$ and $\theta_0 = \frac{2}{\mu}$. Choose $\sigma \in (0, 1)$ and $K \in \mathbb{N}$.

for $k \geq 0$ do
  $y_k = (1 - \theta_k)x_k + \theta_k z_k$
  Generate a random set of coordinates $S_k \sim \hat{S}$
  for $i \in S_k$ do
    $z_{k+1}^i = \arg\min_{z \in \mathbb{R}} \{ \langle \nabla_i f(y_k), z - y_k^i \rangle + \frac{\theta_k n}{2\tau} \| z - z_k^i \|_v^2 + \psi(z) \}$
  end for
  $x_{k+1} = y_k + \frac{\theta_k}{\sqrt{\sigma^2 + 4\tau^2}} (z_{k+1} - z_k)$
  $\theta_{k+1} = \frac{\theta_k}{\sqrt{\sigma^2 + 4\tau^2}}$
  if $k \equiv 0 \mod K$ then
    $x_{k+1} \leftarrow \bar{x}_{k+1}$ (\bar{x}_{k+1} is defined in Theorem 2)
    $z_{k+1} \leftarrow \bar{x}_{k+1}$
    $\theta_{k+1} \leftarrow \theta_0$
  end if
end for

Proof. Let us write the Euclidean division $k = mK + r$ with $r \in [0, K - 1]$. Using (11) and Theorem 2

$$\Delta(x_k) \leq \Delta(x_{mK}) = \Delta(\bar{x}_{mK}) \leq \max(\sigma, 1 - \sigma mK(\mu_F(v))) \Delta(x_{(m-1)K}) \leq \max(\sigma, 1 - \sigma mK(\mu_F(v)))^m \Delta(x_0) = \left( \frac{\max(\sigma, 1 - \sigma mK(\mu_F(v)))}{\mu} \right)^{k-r} \Delta(x_0) \leq \left( \frac{\max(\sigma, 1 - \sigma mK(\mu_F(v)))}{\mu} \right)^{k-K} \Delta(x_0)$$

Theorem 3 shows that Algorithm 4 is linearly convergent with respect to arbitrary convex combination coefficient $\sigma \in (0, 1)$ and arbitrary restarting period $K \in \mathbb{N}$. This implies that we can always get linear convergence without any information on the strong convexity parameter of the objective function. The next proposition provides an estimation on the rate of convergence given a guess $\mu$ on the parameter $\mu_F(v)$ and a particular choice of $\sigma$ and $K$.

Proposition 5. Let $\mu \in (0, 1]$. Choose

$$K = \left\lceil \frac{2\sqrt{3}}{\theta_0} \sqrt{1 + \frac{1}{\mu} - \frac{2}{\theta_0} + 1} \right\rceil,$$  \hspace{1cm} (20)

and

$$\sigma = \frac{1}{1 + mK(\mu)}.$$  \hspace{1cm} (21)

The iterates of Algorithm 4 satisfy for any $k \geq K$

$$\mathbf{E}[\Delta(x_k)] \leq \left( 1 - \min \left( \frac{\mu F(v)}{\mu}, 1 \right) \frac{1 + \mu \theta_0}{2 + \mu} \right)^{\frac{k\theta_0}{2\sqrt{3}\sqrt{1 + \mu}}} \Delta(x_0).$$

Proof. This is a direct corollary of Proposition 6 by taking $\lambda = 1 + \mu$, presented in Appendix C
Let us have additional insight of Proposition 5.

**Corollary 1.** Denote $D(x_0) = \theta_0^2 \Delta(x_0) = (1 - \theta_0)(F(x_0) - F(x_*) + \frac{1}{2}\|x_0 - x_*\|_2^2$. Let $\mu \in (0, 1]$. Choose $K$ and $\sigma$ as in (20) and (21). Then for

$$k \geq \frac{n}{\tau} \left( 6\sqrt{\sigma} \max \left( \frac{1}{\sqrt{\mu}}, \frac{\sqrt{n}}{\mu_F(v)} \right) \log \left( \frac{D(x_0)}{\epsilon} \right) + 2\sqrt{3}\sqrt{1 + \frac{1}{\mu}} \right),$$

we have

$$(1 - \theta_0)(F(x_k) - F(x_*)) + \frac{1}{2}\|x_k - x_*\|_v^2 \leq \epsilon.$$

The proof of Corollary 1 is deferred to Appendix B. We therefore showed that for any strong convexity estimator $\mu \in (0, 1]$, the iteration complexity of Algorithm 4 is on the order of

$$O \left( \frac{n}{\tau} \max \left( \frac{1}{\sqrt{\mu}}, \frac{\sqrt{n}}{\mu_F(v)} \right) \log(1/\epsilon) \right) = \begin{cases} O \left( \frac{n}{\tau \sqrt{\mu}} \log(1/\epsilon) \right) & \text{if } \mu \leq \mu_F(v) \\
O \left( \frac{n}{\tau \mu} \log(1/\epsilon) \right) & \text{if } \mu > \mu_F(v) \end{cases}$$

where the $O$ notation hides logarithms of problem dependent constants and universal constants. Recall that for coordinate descent methods [11, 16], the iteration complexity bound is

$$O \left( \frac{n}{\tau \mu F(v)} \log(1/\epsilon) \right).$$

Therefore, if $\mu$ is an upper bound on $\mu_F(v)$, our iteration complexity bound improves over that of the randomized coordinate descent method [7, 17] by a factor of $\sqrt{\mu}$; if $\mu$ is a lower bound on $\mu_F(v)$ such that $\mu_F(v) \leq \sqrt{\mu} \leq \mu_F(v)$, we also obtain an improved complexity bound; only if $\mu < \mu^2_F$ is our bound worse. This observation will be illustrated in Section 5.

**Remark 6.** The theorems presented in this section can easily be combined with an adaptive restart strategy. We just need to define an interval $[K, \bar{K}]$ and allow the adaptive restart only if $k \in [K, \bar{K}]$. Then if $k = \bar{K}$ we force the restart. We obtain a linear convergence rate where the rate is given by the worst case in the interval.

5 Numerical experiments

5.1 Illustration of the theoretical bounds

We first illustrate the theoretical rate we have found. On Figure 1, we can see that restarted APPROX has a better rate of convergence than vanilla proximal coordinate descent for a wide range of estimates of the strong convexity. Indeed, in this example with $n = 10$ and $\mu_F(v) = 10^{-5}$, one can take $1.6 \times 10^{-9} \leq \mu \leq 0.04$. Note that this shows that even if the estimate $\mu$ is much larger than the true strong convexity coefficient $\mu_F(v)$, we already see an improved rate. Yet, of course the closer $\mu$ is to $\mu_F(v)$, the faster the algorithm will be.

On Figure 2, we fix the estimate of the strong convexity as $\mu = 10^{-3}$ and we plot the rate of convergence of the method for $\mu_F(v) \in [10^{-9}, 1]$.

5.2 Gradient methods

We then present experiments on accelerated gradient methods (the case $n = \tau$). We solve the $L^1$-regularised least squares problem (Lasso)

$$\min_{x \in \mathbb{R}^N} \frac{1}{2}\|Ax - b\|_2^2 + \lambda \|x\|_1.$$
Figure 1: Comparison of the rates of coordinate descent and restarted APPROX when $\tau = 1$, $n = 10$ and $\mu_F(v) = 10^{-5}$. Given an estimate $\mu$ of $\mu_F(v)$, we have chosen $\lambda = 1$ and $K$ and $\sigma$ as in Proposition 5. The blue solid line is the rate given by Theorem 2 and the red dash-dotted line is the simpler rate given in Proposition 5. We are plotting $1 - \rho$ in logarithmic scale for a better contrast.

Figure 2: Comparison of the rates of coordinate descent and restarted APPROX when $\tau = 1$, $n = 10$ and $\mu = 10^{-3}$ (this corresponds to a restart every $K \approx 10^7n$ iterations with $\sigma \approx 0.4$). For each possible value of $\mu_F(v)$, we have computed the rate as given in Proposition 5. With this choice of $\mu$, restarted APPROX has a better rate than coordinate descent as soon as $\mu_F(v) < 8 \cdot 10^{-3}$ and is about 5 times faster when $\mu_F(v)$ is small.
on the Iris dataset where \( A \in \mathbb{R}^{m \times n} \) is the design matrix and \( b \in \mathbb{R}^m \) is such that \( b_j = 1 \) if the label is “Iris-setosa”, \( b_j = -1 \) otherwise. We chose \( \lambda = \frac{\|A^Tb\|_{\infty}}{10} \). This dataset is rather small (\( n = 4 \) and \( m = 8124 \)). As we can see on Table 1, non-accelerated proximal gradient (underlined numbers) is faster than \( \|A^Tb\|_{\infty} \). Yet, accelerated gradient methods designed for strongly convex objectives may be faster than both vanilla proximal gradient and basic accelerated proximal gradient. The “true” strong convexity coefficient is around \( 5.3 \times 10^{-4} \): we can see that taking \( \mu_{\text{est}} \) close to this value leads indeed to a faster algorithm but that the algorithms are rather stable to approximations.

Dual APG is quite efficient on this dataset but as its restarting rule is based on a divergence detection scheme, some attention should be paid before using intermediate solutions. Also remark that when it is not restarted, APG exhibits the sublinear \( O(1/k^2) \) rate while FISTA seems to be take some profit of strong convexity even without restarting.

All restarting strategies seem to perform well. Note however that APG-\( \mu \) and FISTA-\( \mu \) are only proved to converge linearly when \( \mu \leq \mu_F(v) \) and that the adaptive restart of [13] is a heuristic restart. Indeed, with APG, the restart condition did not happen within the first 10,000 iterations, which shows that this adaptive restart is not always efficient.

The rule of Theorem 3 is more complex and seems to be slightly less efficient than the rule of Theorem 1. We still present this more complex rule because it is the only one that is proved to have a linear rate of convergence in the accelerated coordinate descent case \( n > \tau \).

| \( \mu_{\text{est}} \) | 1 | 0.1 | 0.01 | 0.001 | \( 10^{-4} \) | \( 10^{-5} \) | \( 10^{-6} \) | \( 10^{-8} \) |
|---------------------|---|----|-----|------|-------|-------|-------|-------|
| Dual APG with adaptive restart [12] | 447 | 398 | 265 | 156 | 162 | 163 | 163 | 163 |
| FISTA-\( \mu \) [13] | 751 | 352 | 170 | 173 | 264 | 291 | 277 | 277 |
| FISTA restarted: | | | | | | | | |
| at \( x \), Proposition 3 | 751 | 687 | 297 | 160 | 198 | 278 | 278 | 278 |
| at \( z \), Proposition 4 | 751 | 464 | 222 | 245 | 311 | 278 | 278 | 278 |
| as Theorem 1 | 633 | 274 | 168 | 211 | 278 | 278 | 278 | 278 |
| as Theorem 3 | 801 | 477 | 181 | 232 | 278 | 278 | 278 | 278 |
| if \( F(x_{k+1}) > F(x_k) \) [13] | | | | | | | 121 | |

| | \( 10^{-10} \) |
|---------------------|-------|
| APG-\( \mu \) [7] | 751 | 351 | 340 | 882 | 2580 | 7453 | >10000 | >10000 |
| APG restarted: | | | | | | | | |
| at \( x \), Proposition 3 | 751 | 684 | 297 | 189 | 311 | 894 | 1471 | 4488 |
| at \( z \), Proposition 4 | 751 | 463 | 221 | 232 | 281 | 460 | 1415 | 4473 |
| as Theorem 1 | 632 | 275 | 173 | 281 | 794 | 1310 | 3977 | >10000 |
| as Theorem 3 | 801 | 477 | 214 | 288 | 703 | 1166 | 3494 | >10000 |
| if \( F(x_{k+1}) > F(x_k) \) [13] | | | | | | | >10000 | |

Table 1: Number of iterations to reach \( F(x_k) - F(x^*) \leq 10^{-10} \) with various accelerated algorithms for the Lasso problem on the Iris dataset. In italics, no restart has taken place; underlined numbers means that the algorithm is equivalent to non-accelerated ISTA.

### 5.3 Coordinate descent

We solve the following logistic regression problem:

\[
\min_{x \in \mathbb{R}^N} \frac{\lambda_1}{2\|A^Tb\|_{\infty}} \sum_{j=1}^{m} \log(1 + \exp(b_j a_j^T x)) + \|x\|_1 + \frac{\lambda_2}{2} \|x\|^2
\]  

We consider

\[
f(x) = \frac{\lambda_1}{2\|A^Tb\|_{\infty}} \sum_{j=1}^{m} \log(1 + \exp(b_j a_j^T x)),
\]
and
\[\psi(x) = \|x\|_1 + \frac{\lambda_2}{2}\|x\|^2.\]

In particular, for serial sampling (\(\tau = 1\)), (3) is satisfied for
\[v_i = \frac{\lambda_1}{8\|A^\top b\|_\infty} \sum_{i=1}^m (b_j A_{ij})^2, \ i = 1, \ldots, n.\]  (23)

Then for the latter \(v\), (2) is satisfied for
\[\mu_F(v) = \mu_\psi := \lambda_2 \max_i v_i.\]  (24)

Even if the logistic objective is not strongly convex, we expect that the local curvature around the optimum is nonzero and so, that taking \(\mu > \mu_F(v) = \mu_\psi\) will be useful. We solve (22) for different values of \(\lambda_2\), using \(v\) and \(\mu_F(v) = \mu_\psi\) defined in (23) and (24).

We compare randomized coordinate descent (CD), APCG [7] and APPROX-restart (Algorithm 4) with \(K\) and \(\sigma\) given by Proposition 5 on the dataset rcv1. We run both APCG and APPROX-restart using four different values of \(\mu\): \(\mu_F(v)\), \(10\mu_F(v)\), \(100\mu_F(v)\) and \(1,000\mu_F(v)\). We stop the program when the duality gap is lower than \(10^{-10}\) or the running time is larger than 3,000s. The results are reported in Figure 3, where by \(\mu_F\) we refer to \(\mu_F(v)\) defined in (24).

Note that the convergence of APCG is only proved for \(\mu \leq \mu_F(v)\) in [7]. In our experiments, we observed numerical issues when running APCG for several cases when taking larger \(\mu\) (we were not able to compute the \(i\)th partial derivative at \(y_k = \rho_kw_k + z_k\) because \(\rho_k\) had reached the double precision float limit). Such cases can be identified in the plots if the line corresponding to APCG stops abruptly before the time limit (3000s) with a precision worse than \(10^{-10}\). On all the experiments, restarted APPROX is faster or much faster than APCG. Moreover, it is stable for any restarting frequency while APCG may fail if one is too optimistic when setting the strong convexity estimate.

### A Efficient implementation of the restart point

Computing \(\hat{x}_k\) using (18) will be inefficient for APPROX because it involves full-dimensional operations. We show in the following that a reformulation similar to the one in [5] can be done to avoid such expensive computations.

Define:
\[w_k = \theta_{k-1}^{-2}(x_k - z_k), \ \forall k \geq 1\]  (25)

We first recall from [4] that the update for \(\{w_k\}\) can be as efficient as for \(\{z_k\}\):
\[w_{k+1} = w_k - \frac{1 - \theta_k}{\theta_k^2}(z_{k+1} - z_k).\]

Define:
\[\alpha_i := \frac{\gamma_{i+1}^2}{\theta_i^2\theta_{i-1}^2}, \ \beta_i := \frac{\gamma_i^2}{\theta_i^2}, \ i = 0, 1, \ldots\]  (26)

It is easy to deduce from the recursive equation (6) that
\[\gamma_{k+1}^2 = \frac{\theta_k^2\gamma_{i+1}^2}{\theta_i^2}, \ \forall i = 0, \ldots, k.\]  (27)
Figure 3: Plots of results for dataset RCV1: APCG and APPROX-restart run with: $\mu = \mu F(v)$ (first row), $\mu = 10\mu F(v)$ (second row), $\mu = 100\mu F(v)$ (third row) and $\mu = 1000\mu F(v)$ (fourth row). Note that each column corresponds to the same problem, only the parametrization of the algorithms differ. APCG failed for $(\mu_\psi = 1/n, \mu = 1000\mu_\psi)$ and $(\mu_\psi = 0.1/n, \mu = 1000\mu_\psi)$. 
Consequently,
\[
\sum_{i=0}^{k} \frac{\gamma_i}{\theta_i} x_i = \sum_{i=0}^{k} \frac{\theta_i^2 \gamma_{i+1}}{\theta_i} x_i = \theta_k^2 \sum_{i=0}^{k} \alpha_i x_i.
\] (28)

Next we show that the sum vector (28) can be obtained efficiently using auxiliary coefficients and vectors. Define:

\[
a_{i+1} := \sum_{j=0}^{i} \alpha_j, \quad b_{i+1} := \sum_{j=0}^{i} \beta_j, \quad i = 0, 1, \ldots
\] (29)

\[
g_{i+1} := \sum_{j=0}^{i} a_{j+1}(z_{j+1} - z_j), \quad h_{i+1} := \sum_{j=0}^{i} b_{j+1}(w_{j+1} - w_j), \quad i = 0, 1, \ldots
\] (30)

Then clearly the update for \{g_k\} and \{h_k\} are also as efficient as for \{z_k\}:

\[
g_{k+1} = g_k + a_{k+1}(z_{k+1} - z_k), \quad h_{k+1} = h_k - \frac{b_{k+1}(1 - \frac{\theta_k}{\theta_k^2})}{\theta_k^2} (z_{k+1} - z_k).
\]

Moreover, it is easy to see that

\[
\sum_{i=0}^{k} \alpha_i x_i = \sum_{i=0}^{k} \alpha_i(z_i + \theta_i^2 w_i) = \sum_{i=0}^{k} \alpha_i z_i + \beta_i w_i
\]

\[
= -\sum_{i=0}^{k} a_{i+1}(z_{i+1} - z_i) - \sum_{i=0}^{k} b_{i+1}(w_{i+1} - w_i) + a_{k+1} z_{k+1} + b_{k+1} w_{k+1}
\]

\[
= -g_{k+1} - h_{k+1} + a_{k+1} z_{k+1} + b_{k+1} w_{k+1}
\] (31)

Hence,

\[
\bar{x}_{k+1} = \sigma x_{k+1} + (1 - \sigma) \bar{x}_{k+1}
\]

\[
= \sigma x_{k+1} + \frac{1 - \sigma}{\sum_{i=0}^{k} \gamma_i \theta_i} \left[ \sum_{i=0}^{k} \gamma_i x_i + \frac{n}{\tau} (\frac{1}{\theta_k} - \frac{n}{\tau} + 1) x_{k+1} \right]
\]

\[
\bar{x}_{k+1} = \sigma x_{k+1} + \frac{1 - \sigma}{\theta_k^2 \sum_{i=0}^{k} \alpha_i + \frac{n}{\tau} (\frac{1}{\theta_k} - \frac{n}{\tau} + 1)} \left[ \theta_k^2 \sum_{i=0}^{k} \alpha_i x_i + \frac{n}{\tau} (\frac{1}{\theta_k} - \frac{n}{\tau} + 1) x_{k+1} \right]
\]

\[
= \sigma x_{k+1} + \frac{1 - \sigma}{\theta_k^2 a_{k+1} + \frac{n}{\tau} (\frac{1}{\theta_k} - \frac{n}{\tau} + 1)} \left[ \theta_k^2 \sum_{i=0}^{k} \alpha_i x_i + \frac{n}{\tau} (\frac{1}{\theta_k} - \frac{n}{\tau} + 1) x_{k+1} \right]
\]

\[
\bar{x}_{k+1} = \sigma x_{k+1} + \frac{1 - \sigma}{\theta_k^2 a_{k+1} + \frac{n}{\tau} (\frac{1}{\theta_k} - \frac{n}{\tau} + 1)} \left[ \theta_k^2 \sum_{i=0}^{k} \alpha_i x_i + \frac{n}{\tau} (\frac{1}{\theta_k} - \frac{n}{\tau} + 1) x_{k+1} \right]
\]

Finally we plug in (25) and (31) to obtain:

\[
\bar{x}_{k+1} = z_{k+1} + \theta_k^2 w_{k+1} + \frac{(1 - \sigma) \theta_k^2 (-g_{k+1} - h_{k+1}) + (1 - \sigma)(\theta_k^2 b_{k+1} - \theta_k^4 a_{k+1}) w_{k+1}}{\theta_k^2 a_{k+1} + \frac{n}{\tau} (\frac{1}{\theta_k} - \frac{n}{\tau} + 1)}
\]

The above reasoning showed that Algorithm [5] is equivalent to Algorithm [4]. More importantly, note that full dimensional operations are avoided in Algorithm [5].
Algorithm 5 APPROX restart efficient equivalent

1: **Parameters**: Choose \( \hat{S}, x_0 \in \mathbb{R}^n, K \in \mathbb{N}, \sigma \in (0, 1). \)
2: **Initialization**: Set \( \tau = \mathbb{E}[|\hat{S}|], \theta_0 = \frac{\tau}{n}, r_0 = 0, a_0 = 0, b_0 = 0 \) and \( g_0 = 0 \), \( h_0 = 0 \), \( z_0 = x_0, u_0 = 0. \)
3: for \( k \geq 0 \) do
4: \( w_{k+1} \leftarrow w_k \)
5: \( z_{k+1} \leftarrow z_k \)
6: \( a_{k+1} = a_k + \frac{r_k(1-\theta_k)}{\theta_k^2} \)
7: \( b_{k+1} = b_k + \frac{r_k}{\theta_k^2} \)
8: Randomly generate \( S_k \sim \hat{S} \)
9: for \( i \in S_k \) do
10: \( t_i^k = \arg\min_{t \in \mathbb{R}} \{ \langle \nabla_i f(\theta_k^2 w_k + z_k), t \rangle + \frac{\mu \theta_k^2}{2\tau} |t|^2 + \psi^i(z_k^i + t) \} \)
11: \( z_{k+1}^i = z_k^i + t_i^k \)
12: \( w_{k+1}^i = w_k^i - \frac{1-\theta_k}{\theta_k^2} t_i^k \)
13: \( g_{k+1} = g_k + a_k + b_i^k t_i^k \)
14: \( h_{k+1}^i = h_k^i - \frac{b_{k+1}(1-\theta_k)}{\theta_k^2} t_i^k \)
end for
15: if \( k \equiv 0 \mod K \) then
16: \( z_{k+1} = z_{k+1} + \theta_k^2 w_{k+1} + \frac{(1-\sigma)^2(\theta_k^2 - \theta_{k+1}) + (1-\sigma)(\theta_k^2 \theta_{k+1} - \theta_k^2 \theta_{k+1})}{\theta_k^2 \theta_{k+1} + (1-\sigma) \theta_k^2 \theta_{k+1}} w_{k+1} \)
18: \( w_{k+1} = 0 \)
19: \( g_{k+1} = 0 \)
20: \( h_{k+1} = 0 \)
21: \( \theta_{k+1} = \theta_0, r_{k+1} = 0, a_{k+1} = 0, b_{k+1} = 0 \)
22: else
23: \( r_{k+1} = \frac{\sqrt{\theta_k^2 + 4\theta_k^2 - \theta_k^2}}{2} \)
24: \( r_{k+1} = \theta_{k+1} (1 - \frac{n}{\tau} \theta_k) + \frac{\mu}{\tau}(\theta_k - \theta_{k+1}) \)
end if
26: end for
27: OUTPUT : \( \theta_k^2 w_{k+1} + z_{k+1} \)

### B Additional insight on the rate of convergence

Define:

\[
\xi_k := \sum_{i=0}^{k} \frac{\gamma_k^i}{\theta_k^2 - 1}, \quad k = 1, 2, \ldots
\]

Then

\[
m_k(\mu) = \frac{\mu \theta_k^2}{1 + \mu (1-\theta_0)} \left( \xi_k - \frac{1 - \theta_0}{\theta_0^2} \right), \quad k = 1, 2, \ldots
\]

We first prove a simple recursive equation.

**Lemma 2.** For any \( k \geq 1 \) we have:

\[
\xi_{k+1} = (1 - \theta_k) \xi_k + \frac{1 + (n-1)\theta_k}{\theta_k}
\]

**Proof.** Let any \( k \geq 1 \). We first decompose the sum and use (12) to obtain:

\[
\xi_{k+1} = \sum_{i=0}^{k+1} \frac{\gamma_k^i}{\theta_k^2 - 1} = \sum_{i=0}^{k-1} \frac{\gamma_k^i}{\theta_k^2 - 1} + \frac{\gamma_k^k}{\theta_k^2 - 1} + \frac{\gamma_k^{k+1}}{\theta_k^2 - 1} = (1 - \theta_k) \sum_{i=0}^{k-1} \frac{\gamma_k^i}{\theta_k^2 - 1} + \frac{\gamma_k^k}{\theta_k^2 - 1} + \frac{\gamma_k^{k+1}}{\theta_k^2 - 1}.
\]
Then we get the recursive equation:

\[ \xi_{k+1} = (1 - \theta_k) \xi_k + \frac{\gamma_{k+1}}{\theta_{k+1}^2} - (1 - \theta_k) \frac{\gamma_k}{\theta_k^2} + \frac{\gamma_{k+1}}{\theta_{k+1}^2}, \]

which together with (6) and (12) yields (33).

**Lemma 3.** For any \( k \geq 1 \) we have

\[ \frac{1}{3 \theta_k^2} \leq \xi_k \leq \frac{1}{\theta_k^2}. \]  

(34)

**Proof.** We proceed by induction on \( k \). First for \( k = 1 \) we have:

\[ \xi_1 = \frac{1}{\theta_0^2} \frac{1 - \theta_1}{\theta_1^2} \leq \frac{1}{\theta_1^2}. \]

Now suppose that we have

\[ \xi_k \leq \frac{1}{\theta_k^2} \]

for some \( k \geq 1 \). Then

\[ \xi_{k+1} \leq \frac{1 - \theta_k}{\theta_k^2} + \frac{1}{\theta_k^2} \leq \frac{1 - \theta_{k+1}}{\theta_{k+1}^2} + \frac{1}{\theta_{k+1}^2} \]

\[ \leq \frac{1}{\theta_{k+1}^2}. \]

Thus we proved by recurrence the following upper bound:

\[ \xi_k \leq \frac{1}{\theta_k^2}, \quad \forall k = 1, 2, \ldots \]

Next we prove the lower bound again by induction on \( k \). Since \( \theta_k \leq 1 \), we first observe that

\[ \theta_{k+1} = \frac{\sqrt{\theta_k^4 + 4 \theta_k^2 - \theta_k^2}}{2} \leq \frac{\sqrt{5} - 1}{2} \leq \frac{1}{3}, \quad \forall k = 0, 1, 2, \ldots \]

Then we get:

\[ \frac{1}{\theta_k^2} = \frac{1 - \theta_{k+1}}{\theta_{k+1}^2} \geq \frac{1}{3 \theta_{k+1}^2}, \quad \forall k = 0, 1, 2, \ldots \]

(35)

In particular,

\[ \xi_1 = \frac{1}{\theta_0^2} \geq \frac{1}{3 \theta_1^2}. \]

Now suppose that we have

\[ \xi_k \geq \frac{1}{3 \theta_k^2} \]

for some \( k \geq 1 \). Then

\[ \xi_{k+1} \geq \frac{1 - \theta_k}{3 \theta_k^2} + \frac{1}{\theta_k^2} \geq \frac{1}{3 \theta_{k+1}^2} + \frac{1}{\theta_{k+1}^2} \]

Next we apply (35) to obtain:

\[ \xi_{k+1} \geq \frac{1}{3 \theta_{k+1}^2} - \frac{1}{3 \theta_{k+1}} + \frac{2}{3 \sqrt{3 \theta_{k+1}}} \geq \frac{1}{3 \theta_{k+1}^2} + \frac{1}{3 \theta_{k+1}} - \frac{1}{3 \theta_{k+1}} = \frac{1}{3 \theta_{k+1}^2}. \]

We thereby proved the lower bound for any \( k \).
We then deduce directly from (5) the following corollary.

**Corollary 2.** For any $k \geq 1$ we have

$$\frac{(k + 2/\theta_0)^2}{12} \leq \xi_k \leq (k + 1/\theta_0)^2.$$  \hfill (36)

**Lemma 4.** Let $\lambda \geq \mu$ and

$$K = \left\lceil \frac{2\sqrt{3}}{\theta_0} \sqrt{\frac{\lambda}{\mu} - \frac{2}{\theta_0}} \right\rceil.$$  \hfill (37)

Then the following inequalities hold:

$$\lambda \leq \mu \theta_0^2 \xi_K \leq 9\lambda,$$  \hfill (38)

$$K \leq \frac{2\sqrt{3}\sqrt{\lambda}}{\theta_0\sqrt{\mu}}.$$  \hfill (39)

**Proof.** Consider the interval

$$\left[ \frac{2\sqrt{3}}{\theta_0} \sqrt{\frac{\lambda}{\mu} - \frac{2}{\theta_0}}, \frac{2\sqrt{3}}{\theta_0} \sqrt{\frac{\lambda}{\mu} - \frac{1}{\theta_0}} \right].$$

We first observe that it is included in $\mathbb{R}_{>0}$ because $\lambda \geq \mu$. Moreover, the length of the interval is larger than 1. Therefore, $K$ defined by (37) satisfies:

$$\frac{2\sqrt{3}}{\theta_0} \sqrt{\frac{\lambda}{\mu} - \frac{2}{\theta_0}} \leq K \leq \frac{2\sqrt{3}\sqrt{\lambda}}{\theta_0\sqrt{\mu}}$$

Consequently we obtain (39) and

$$\frac{(K + 2/\theta_0)^2}{12} \geq \frac{\lambda}{\mu \theta_0^2}, \quad (K + 1/\theta_0)^2 \leq \frac{12\lambda}{\mu \theta_0^2},$$

which together with (36) implies (38). \hfill \square

**Proposition 6.** Let $\lambda \geq \mu$. Choose

$$K = \left\lceil \frac{2\sqrt{3}}{\theta_0} \sqrt{\frac{\lambda}{\mu} - \frac{2}{\theta_0}} \right\rceil,$$  \hfill (40)

and

$$\sigma = \frac{1}{1 + m_K(\mu)}.$$  \hfill (41)

Then the iterates of Algorithm 4 satisfy for any $k \geq K$

$$\mathbb{E}[\Delta(x_k)] \leq \left(1 - \min\left(\frac{\mu_F(v)}{\mu}, 1\right) \frac{\lambda - \mu(1 - \theta_0)}{\lambda + 1}\right)^{\frac{k\theta_0\sqrt{\lambda} - 1}{2\sqrt{3}\sqrt{\lambda}}} \Delta(x_0).$$
Proof. Let \( \{x_k\} \) be the iterates of Algorithm 4 using \( \sigma \) and \( K \) defined by (40) and (41). By Proposition 3 for any \( k \geq K \),

\[
E[\Delta(x_k)] \leq \left( \max (\sigma, 1 - \sigma m_K(\mu_F(v)))^{1/K} \right)^{k-K} \Delta(x_0).
\]

Thus we just need to prove:

\[
\left( \max (\sigma, 1 - \sigma m_K(\mu_F(v)))^{1/K} \right)^{k-K} \leq \left( 1 - \min \left( \frac{\mu_F(v)}{\mu}, 1 \right) \frac{\lambda - \mu(1 - \theta_0)}{\lambda + 1} \right)^{\frac{\theta_0}{2\lambda} \cdot \frac{\pi}{2} - 1}.
\]

We first note that if \( \mu > \mu_F(v) \), then

\[
m_K(\mu) > m_K(\mu_F(v)).
\]

Therefore,

\[
\max (\sigma, 1 - \sigma m_K(\mu_F(v))) = 1_{\mu \leq \mu_F(v)} \frac{1}{1 + m_K(\mu)} + 1_{\mu > \mu_F(v)} \frac{1 + m_K(\mu) - m_K(\mu_F(v))}{1 + m_K(\mu)}
\]

where

\[
1_{\mu \leq \mu_F(v)} = \begin{cases} 1 & \text{if } \mu \leq \mu_F(v) \\ 0 & \text{otherwise} \end{cases},
\]

and \( 1_{\mu > \mu_F(v)} = 1 - 1_{\mu \leq \mu_F(v)} \). Next we replace \( m_K(\mu) \) using (42) and rearrange the terms:

\[
\max \left( \sigma, 1 - \sigma m_K(\mu_F(v)) \right) = 1_{\mu \leq \mu_F(v)} \frac{1}{1 + \frac{\mu \theta_0^2(1 - \theta_0)}{1 + \mu(1 - \theta_0)} \left( \xi_K - \frac{1 - \theta_0}{\theta_0^2} \right)} + 1_{\mu > \mu_F(v)} \frac{1 + \left( \frac{\mu \theta_0^2}{1 + \mu(1 - \theta_0)} - \frac{\mu_F(v) \theta_0^2}{1 + \mu_F(v)(1 - \theta_0)} \right) \left( \xi_K - \frac{1 - \theta_0}{\theta_0^2} \right)}{1 + \frac{\mu \theta_0^2}{1 + \mu(1 - \theta_0)} \left( \xi_K - \frac{1 - \theta_0}{\theta_0^2} \right)}
\]

\[
= 1_{\mu \leq \mu_F(v)} \frac{1 + \mu(1 - \theta_0)}{1 + \mu \theta_0^2 \xi_K} + 1_{\mu > \mu_F(v)} \frac{1 + \mu(1 - \theta_0) + \left( \mu \theta_0^2 - \mu_F(v) \theta_0^2 \right) \left( \frac{1 + \mu(1 - \theta_0)}{1 + \mu_F(v)(1 - \theta_0)} \right) \left( \xi_K - \frac{1 - \theta_0}{\theta_0^2} \right)}{1 + \mu \theta_0^2 \xi_K}
\]

\[
= 1_{\mu \leq \mu_F(v)} \frac{1 + \mu(1 - \theta_0)}{1 + \mu \theta_0^2 \xi_K} + \frac{1 + \mu \theta_0^2 \xi_K - \left( \frac{1 + \mu(1 - \theta_0)}{1 + \mu_F(v)(1 - \theta_0)} \right) \mu_F(v) \theta_0^2 \left( \xi_K - \frac{1 - \theta_0}{\theta_0^2} \right)}{1 + \mu \theta_0^2 \xi_K}
\]

\[
= 1_{\mu \leq \mu_F(v)} \left( \frac{1 - \xi_K}{1 + \mu \theta_0^2 \xi_K} \right) + 1_{\mu > \mu_F(v)} \left( 1 - \frac{\mu \theta_0^2 \xi_K - \mu(1 - \theta_0)}{1 + \mu \theta_0^2 \xi_K} \right)
\]

\[
= 1 - \min \left( 1, \frac{\mu_F(v)}{\mu} \right) \frac{\mu \theta_0^2 \xi_K - \mu(1 - \theta_0)}{1 + \mu \theta_0^2 \xi_K} \leq 1 - \min \left( 1, \frac{\mu_F(v)}{\mu} \right) \frac{\lambda - \mu(1 - \theta_0)}{\lambda + 1}.
\]

\( (42) \)
Consequently,

\[
\left( \max (\sigma, 1 - \sigma m_K(\mu_F(v)))^{1/K} \right)^{k-K} \leq \left( 1 - \min \left( 1, \frac{\mu_F(v)}{\mu} \right) \frac{\lambda - \mu(1 - \theta_0)}{\lambda + 1} \right)^{k/k-1} \tag{43}
\]

Next we apply (39) and get:

\[
\left( 1 - \min \left( 1, \frac{\mu_F(v)}{\mu} \right) \frac{\lambda - \mu(1 - \theta_0)}{\lambda + 1} \right)^{\frac{k}{k-1}} \leq \left( 1 - \min \left( 1, \frac{\mu_F(v)}{\mu} \right) \frac{\lambda - \mu(1 - \theta_0)}{\lambda + 1} \right)^{\frac{\theta_0 \sqrt{\mu}}{2\sqrt{3}\sqrt{1 + \mu}}} \tag{44}
\]

Then by (43) and (44),

\[
\left( \max (\sigma, 1 - \sigma m_K(\mu_F(v)))^{1/K} \right)^{k-K} \leq \left( 1 - \min \left( 1, \frac{\mu_F(v)}{\mu} \right) \frac{\lambda - \mu(1 - \theta_0)}{\lambda + 1} \right)^{\frac{\theta_0 \sqrt{\mu}}{2\sqrt{3}\sqrt{1 + \mu}}} \]

which is the inequality we wanted to prove. \(\square\)

**proof of Corollary 1.** Taking \(\lambda = 1 + \mu\) in Proposition 6, we can see that we have the result if

\[
\mathbb{E}[\Delta(x_k)] \leq \left( 1 - \min \left( 1, \frac{\mu_F(v)}{\mu} \right) \frac{1 + \mu\theta_0}{2 + \mu} \right) \frac{\theta_0 \sqrt{\mu}}{2\sqrt{3}\sqrt{1 + \mu}} \Delta(x_0) \leq \epsilon.
\]

Passing to the logarithm leads to

\[
\log \left( 1 - \min \left( 1, \frac{\mu_F(v)}{\mu} \right) \frac{1 + \mu\theta_0}{2 + \mu} \right) \frac{\theta_0 \sqrt{\mu}}{2\sqrt{3}\sqrt{1 + \mu}} \Delta(x_0) \leq \log(\epsilon),
\]

which is equivalent to

\[
\log \left( \frac{\Delta(x_0)}{\epsilon} \right) \leq - \log \left( 1 - \min \left( 1, \frac{\mu_F(v)}{\mu} \right) \frac{1 + \mu\theta_0}{2 + \mu} \right) \frac{\theta_0 \sqrt{\mu}}{2\sqrt{3}\sqrt{1 + \mu}} - 1 \right).
\]

As \(-\log(1 - x) \geq x\), it is enough to have

\[
\log \left( \frac{\Delta(x_0)}{\epsilon} \right) \leq \min \left( 1, \frac{\mu_F(v)}{\mu} \right) \frac{1 + \mu\theta_0}{2 + \mu} \frac{\theta_0 \sqrt{\mu}}{2\sqrt{3}\sqrt{1 + \mu}} - 1 \right),
\]

which yields:

\[
k \geq \frac{2\sqrt{3} (2 + \mu) \sqrt{1 + \mu}}{\theta_0} \max \left( \frac{\mu_F(v)}{\mu}, \frac{1}{\sqrt{\mu}} \right) \log \left( \frac{\Delta(x_0)}{\epsilon} \right) + \frac{2\sqrt{2}}{\theta_0} \sqrt{1 + \frac{1}{\mu}}
\]

We get the corollary by noting that

\[
\frac{(2 + \mu) \sqrt{1 + \mu}}{1 + \mu\theta_0} \leq 3\sqrt{2}.
\]

\(\square\)

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