A NOTE ON SINGULAR TIME OF MEAN CURVATURE FLOW

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ABSTRACT. We show that mean curvature flow of a compact submanifold in a complete Riemannian manifold cannot form singularity at time infinity if the ambient Riemannian manifold has bounded geometry and satisfies certain curvature and volume growth conditions.

1. INTRODUCTION

Mean curvature flow develops singularities if the second fundamental forms of the time dependent immersions become unbounded. It is well known that mean curvature flow of any closed manifold in the Euclidean space develops singularities in finite time. This follows from a maximum principle and barrier argument. In this note, we show, using integral estimates, that mean curvature flow cannot form singularity at \( t = \infty \) for a class of ambient Riemannian manifolds. More precisely, we prove

**Theorem 1.1.** Let \( \Sigma^n \) be a compact manifold and let \( M^m \) be a complete Riemannian manifold with bounded geometry. Suppose that \( F(t) : \Sigma \to M \) satisfies mean curvature flow for \( t \in [0, T) \) and \( T \) is the first singular time. If \( (M, g) \) is Ricci parallel with nonnegative sectional curvature, and its volume growth satisfies

\[
\text{Vol}(B_p(R)) \geq c R^{m-n+\epsilon}
\]

for \( R > R_0 \), where \( \epsilon, c, R_0 \) are fixed positive constants and \( B_p(R) \) is the geodesic ball at \( p \in M \), then \( T \) has to be finite. In particular, if \( (M, g) \) is analytic, then either the mean curvature flow \( F : \Sigma \to M \) develops a finite time singularity, or it converges to a compact minimal submanifold in \( (M, g) \).

A rescaling process is usually applied when singularities are forming. A sequence of rescaled flows may, however, move to infinity in \( \mathbb{R}^m \) and fail to form a limit. Particularly, this may happen at type-II singularities if one scales the flow by the maximum length of the second fundamental forms at a sequence of times approaching to the first blow up time. To get compactness, one may consider the geometric limits for mean curvature flows, in the sense of Cheeger-Gromov, as Hamilton did for the Ricci flow [6].

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For mean curvature flow, the lower bound on the injectivity radius follows, unlike the Ricci flow, from the bound of the second fundamental form $A$. The curvature of a submanifold in the flow is also bounded if $A$ is bounded, by the Gauss equation. The smoothness estimate for mean curvature flows [3] ensures that all higher derivatives of the second fundamental form are bounded when $A$ is bounded. These enable one to construct a limiting mean curvature flow for a sequence of rescaled flows with a uniform bound on the second fundamental forms. We present a detailed analysis on constructing ancient solutions at any singularity and eternal solutions at a type II singularity, although the result (cf. Theorem 2.4) is known, as the results and the arguments will be used in proving Theorem 1.1.

Combining the geometric limit construction with Hamilton’s monotonicity formula [4] for mean curvature flow and Li-Yau’s heat kernel estimates [9], we find an upper bound for the first singular time $T$ in terms of volume, and then the volume growth condition imposed on $M$ rules out formation of singularity at infinity.

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2. Geometric limit along mean curvature flow

Geometric limits of Riemannian manifolds and geometric limits along the Ricci flow are well developed, see [6], [10]. Since we will use the geometric limits in the formation of singularities along the mean curvature flow in an essential way, we include some basic facts for completeness.

In this section we do not need to assume the Riemannian manifolds be compact.

**Definition 1.** Let $(M_k, g_k, x_k)$ be a based complete Riemannian manifold for each positive integer $k$. A geometric limit of the sequence $\{M_k, g_k, x_k\}$ is a based complete Riemannian manifold $(M_\infty, g_\infty, x_\infty)$ such that:

1. there exists an increasing sequence of connected open subsets $U_k$ of $M_\infty$ exhausting the manifold $M_\infty$, namely $M_\infty = \cup U_k$ and $U_k$ satisfy (a) $\overline{U_k}$ is compact, (b) $\overline{U_k} \subset U_{k+1}$, (c) $x_\infty \subset U_k$, for all $k$.
2. for each $k$ there exists a smooth embedding $\varphi_k : (U_k, x_\infty) \to (M_k, x_k)$ such that $\varphi_k(x_\infty) = x_k$ and $\lim_{k \to \infty} \varphi_k^* g_k = g_\infty$.

where the limit is in the uniform $C^\infty$ topology on compact subsets of $M_\infty$.

Similarly, we can define a geometric limit of a sequence of immersions.

**Definition 2.** Let $F_k : (\Sigma_k, x_k) \to (N, g, p)$ be a sequence of immersions with $F_k(x_k) = p \in N$, where $(N, g)$ is a fixed Riemannian manifold. A geometric limit of the sequence is an immersion $F_\infty : (\Sigma_\infty, x_\infty) \to (N, g, p)$ such that:
(1) there exists an increasing sequence of connected open subsets $U_k$ of $\Sigma_\infty$, which exhaust the manifold $\Sigma_\infty$, namely $\Sigma_\infty = \bigcup U_k$, and satisfy the following for all $k$: (a) $\overline{U}_k$ is compact, (b) $\overline{U}_k \subset U_{k+1}$, (c) $x_\infty \in \overline{U}_k$.

(2) for each $k$ there exists a smooth embedding $\varphi_k : (U_k, x_\infty) \to (\Sigma_k, x_k)$ such that $\varphi(x_\infty) = x_k$ and
\[
\lim_{k \to \infty} F_k \circ \varphi_k = F_\infty,
\]
where the limit is in the uniform $C^\infty$ topology on compact subsets of $\Sigma_\infty$. In particular, $(\Sigma_\infty, x_\infty, F_\infty^*g)$ is a geometric limit of the sequence $(\Sigma_k, x_k, F_k^*g)$ as Riemannian manifolds.

A basic fact of geometric limit of Riemannian manifolds is the following, which is the $C^\infty$ version of the classical Cheeger-Gromov compactness.

**Theorem 2.1. (Cheeger-Gromov)** Let $(M_k, g_k, x_k)$ be a sequence of connected and based Riemannian manifolds. Suppose that

(1) for every $R < \infty$, the ball $B(x_k, R)$ has compact closure in $M_k$ for all $k$ sufficiently large;

(2) for each integer $l \geq 0$ and each $R < \infty$, there is a constant $C = C(l, R)$ such that
\[
\|\nabla^l Rm(g_k)\| \leq C
\]
on $B(x_k, R)$ for all $k$ sufficiently large;

(3) there is a constant $\delta > 0$ such that $\inj_{(M_k, g_k)}(x_k) \geq \delta$ for all $k$ sufficiently large.

Then after passing to a subsequence there is a geometric limit $\{M_\infty, g_\infty, x_\infty\}$ which is a complete Riemannian manifold.

The proof of the above theorem is quite standard (cf. [10]). For evolution equations such as the Ricci flow or the mean curvature flow, estimates on the higher derivatives of the curvature are the consequence of the curvature bound, by the smooth estimate. So the key assumption is that the curvature bound and the injective radius bound. For an immersion, however, a lower bound on injectivity radius follows from an upper bound on the second fundamental form $A$.

A Riemannian manifold $(M, g)$ has bounded geometry if the injectivity radius, the curvatures and the derivatives of the curvatures are uniformly bounded.

**Proposition 2.2.** Suppose that
\[
F : \Sigma \to (M, g)
\]
is an immersion where the ambient space $(M, g)$ is a fixed smooth Riemannian manifold with bounded geometry. Suppose that for each $l \geq 0$, there exists a constant $C = C(l)$ such that $|\nabla^l A| \leq C$, where $\nabla$ is the covariant derivative of $(\Sigma, F^*g)$. Then the injectivity radius of $(\Sigma, F^*g)$ is uniformly bounded from below by a positive constant.
Proof. We argue by contradiction. Suppose that there exist a sequence of immersions

$$F_i : (\Sigma_i, x_i) \to (M, g, F_i(x_i))$$

with second fundamental forms $A_i$ and all their higher derivatives bounded by constants independent of $i$, but the injectivity radius $\iota_i$ at $x_i \in (\Sigma_i, F_i^*g)$ goes to zero. Consider the sequence

$$F_i : (\Sigma_i, x_i) \to \left(M, \frac{1}{\iota_i^2}g, F_i(x_i)\right).$$

Then $(\Sigma_i, F_i^*(\iota_i^{-2}g), x_i)$ is a sequence of Riemannian manifolds with bounded curvature and all higher derivatives of the curvature are also bounded, by Gauss equation for submanifolds and that $(M, \iota_i^{-2}g)$ has bounded geometry and $|\nabla^i A_i| \leq C(i)$. The injective radius of $(\Sigma_i, F_i^*(\iota_i^{-2}g))$ at $x_i$ is 1. Hence $(\Sigma_i, g_i, x_i)$, where $g_i = F_i^*(\iota_i^{-2}g)$, converges in $C^\infty$ topology in the sense of Cheeger-Gromov to a geometric limit $(\Sigma_\infty, g_\infty, x_\infty)$, by Theorem 2.1. So there exists an exhausting sequence of relatively compact open subsets $U_i$ of $\Sigma_\infty$ and a sequence of $C^\infty$ embeddings $\varphi_i$ such that $\varphi_i^*g_i \to g_\infty$ in $C^\infty$ topology on every compact subset of $\Sigma_\infty$. In particular the injectivity radius at $x_\infty$ is equal to 1. To see this, note that the injectivity radius is equal to the minimum of the conjugate radius and half of the shortest geodesic loop. In our case, since the curvature of $g_i$ goes to zero when $i \to \infty$, the conjugate radius goes to infinity. Hence, there is a geodesic loop $l_i$ through $x_i$ in $(\Sigma_i, g_i)$ with length 2. Then $\varphi_i^{-1}(l_i)$ is a sequence of loops through $x_\infty$ with length converging to 2. It follows that the injective radius at $x_\infty$ is less than or equal to 1. It is clear that the injectivity radius at $x_\infty$ cannot be strictly less than 1 as the injectivity radius of $(\Sigma_i, g_i)$ at $x_i$ is 1.

Note that $(M, \iota_i^{-2}g, F_i(x_i))$ converges to the standard Euclidean space $(\mathbb{R}^m, dx^2, 0)$ in $C^\infty$ topology on every compact subset. Namely, there exists an exhausting relatively compact open subsets $V_i$ of $\mathbb{R}^m$ and a sequence of $C^\infty$ embeddings $\phi_i$ such that $\phi_i^*(\iota_i^{-2}g)$ converges to $dx^2$ on every compact subset of $\mathbb{R}^m$ with $\phi_i(0) = F_i(x_i)$. Consider the immersions

$$\tilde{F}_i = \phi_i^{-1} \circ F_i \circ \varphi_i : (\Sigma_\infty, x_\infty) \to (\mathbb{R}^m, 0).$$

The second fundamental forms $\tilde{A}_i$ of $\tilde{F}_i$ are uniformly bounded and all their higher derivatives are bounded as well (actually all go to zero), independent of $i$. Hence $\tilde{F}_i$ converges in $C^\infty$ topology on compact sets, as a geometric limit, to an immersion

$$F_\infty : (\Sigma_\infty, x_\infty) \to (\mathbb{R}^m, 0).$$

In particular, we have $F_\infty^*dx^2 = g_\infty$. This statement is known, however, we include a proof here for completeness. First note that the injectivity radius at $x_\infty$ is 1, consider the geodesic ball $B_1(x_\infty)$ in $(\Sigma_\infty, g_\infty)$, which we can identify with the standard Euclidean ball $B_1(0) \subset \mathbb{R}^m$ through the exponential map $\exp_{x_\infty}$. Consider the sequence
of immersions
\[ \tilde{F}_i \circ \exp_{x_\infty} : B_1(0) \to \mathbb{R}^m, \quad \tilde{F}_i(0) = 0. \]
We know that the second fundamental forms of the immersions \( \tilde{F}_i \circ \exp_{x_\infty} \) are uniformly bounded, this means that the Hessian of the mappings \( \tilde{F}_i \circ \exp_{x_\infty} \) are uniformly bounded. Also all the higher derivatives of the second fundamental forms, therefore of the mappings \( \tilde{F}_i \circ \exp_{x_\infty} \), are uniformly bounded. It follows that \( \tilde{F}_i \circ \exp_{x_\infty} \) converges in \( B_{1/2}(0) \) to a smooth map \( \tilde{F}_\infty : B_{1/2}(0) \to \mathbb{R}^m \) by Arezella-Ascoli Theorem. We can construct
\[ F_\infty = \tilde{F}_\infty \circ \exp^{-1}_{x_\infty} \]
in \( B_{1/2}(x_\infty) \). To show that we have a limit map \( F_\infty \) on whole manifold \( \Sigma_\infty \), we use the geodesic balls to cover the manifold \( \Sigma_\infty \). Note that for any \( y \in \Sigma_\infty \), the injectivity radius is bounded from below by \( d(x_\infty, y) \) since the curvature is uniformly bounded. The argument then follows from the standard argument of geometric limit of Riemannian manifolds by diagonal process. The reader can refer to [6], [10] for full details of the argument.

The second fundamental form \( \tilde{A}_\infty \) of the complete submanifold \( F_\infty(\Sigma_\infty) \) is zero because
\[ |\tilde{A}_i|_{g_i}^2 = \epsilon_i^{-2} |A_i|_{F_i^*g_i}^2 \to 0, \text{ as } i \to \infty. \]
This implies that \( (\Sigma_\infty, x_\infty) \) is a smoothly immersed totally geodesic submanifold of \( \mathbb{R}^m \), hence \( (\Sigma_\infty, g_\infty) \) has to be Euclidean itself. But this contradicts that injectivity radius of \( x_\infty \) is equal to 1.

By the smoothness property of the mean curvature flow [3] (the proof holds for general codimension) and Proposition 2.2, we can get a compactness property along the mean curvature flow with bounded second fundamental form, similar to the result of Hamilton [6] in the Ricci flow.

**Theorem 2.3.** Fix \(-\infty \leq T' \leq 0 \leq T \leq \infty \) with \( T' < T \). Let \( \{\Sigma_k, F_k, x_k\} \) be a sequence of based mean curvature flows with
\[ F_k(t) : \Sigma_k \to \mathbb{R}^m, \quad F_k(x_k, 0) = 0. \]
Suppose that the lengths of the second fundamental forms \( A_k \) of \( F_k \) are uniformly bounded above by a constant \( C \) independent of \( k \) and time \( t \). Then there exists a subsequence of \( \{\Sigma_k, F_k, x_k\} \) which converges to a mean curvature flow \( \{\Sigma_\infty, F_\infty(t), (x_\infty, 0)\} \) as a geometric limit, where
\[ F_\infty(t) : \Sigma_\infty \to \mathbb{R}^m, \quad F_\infty(x_\infty, 0) = 0, \quad t \in (T', T), \]
and \( (\Sigma_\infty, F_\infty(t)^*(dx^2)) \) is a complete Riemannian manifold.

**Proof.** The proof essentially follows Hamilton’s argument [6] for the Ricci flow. By Proposition 2.2, the injective radius of \( (\Sigma_k, F_k(t)^*(dx^2)) \) at any point is uniformly bounded, independent of \( t \) and \( k \). Along the mean curvature flow, the smoothness
estimate holds, hence all the higher derivatives of the second fundamental forms \( A_k \) of \( F_k \) are uniformly bounded because \( |A_k| \) are uniformly bounded above. \( F_k \) satisfies the mean curvature flow equation, it follows that all the derivatives of \( F_k \) with respect to time \( t \) are also uniformly bounded. Consider the Riemannian manifolds \((\Sigma_k, F_k(0)^*(dx^2), x_k)\). By the assumption, this sequence has uniformly bounded injective radius and uniformly bounded curvature and their higher derivatives. It follows that it sub-converges to a complete Riemannian manifold \((\Sigma_\infty, g_\infty, x_\infty)\). For any fixed time \( T' \leq t_1 < 0 < t_2 \leq T \) and a fixed constant \( R \), take a geodesic ball \( B_{2R}(x_\infty) \subset (\Sigma_\infty, g_\infty, x_\infty) \). For \( k \) sufficient large, we can find an embedding \( \phi_k : B_{2R}(x_\infty) \to \Sigma_k \) such that \( \phi_k(x_\infty) = x_k \). Define

\[
\tilde{F}_k^R(t) = F_k(t) \circ \phi_k : B_{2R}(x_\infty) \to \mathbb{R}^m.
\]

Note \( \phi_i \) is time independent. Consider the sequence of immersions

\[
\tilde{F}_k^R : B_{2R}(x_\infty) \times [t_1, t_2] \to \mathbb{R}^m
\]

with \( \tilde{F}_k^R(x_\infty, 0) = 0 \). For simplicity, we can assume that \( 2R \) is less than the injective radius and then by using the exponential map, we can identify \( B_{2R}(x_\infty) \) with the Euclidean ball, as we did in Proposition 2.2. It follows that all derivatives of \( \tilde{F}_k^R \), as an mapping from \( B_{2R}(x_\infty) \times [t_1, t_2] \) to \( \mathbb{R}^m \) with \( \tilde{F}_k^R(x_\infty, 0) = 0 \), are bounded. By the classical Asscoli theorem, it sub-converges to a smooth mapping

\[
F_\infty^R : B_R(x_\infty) \times [t_1, t_2] \to \mathbb{R}^m.
\]

If \( 2R \) is larger than the injective radius, one applies the covering argument as in [6] to show the convergence in the geodesic ball \( B_R(x_\infty) \). Now we let \( R \to \infty, t_1 \to T', t_2 \to T \) and apply a standard diagonal sequence argument to obtain a limiting immersion

\[
F_\infty : \Sigma_\infty \times (T', T) \to \mathbb{R}^m.
\]

It is clear that \( F_\infty(x_\infty, 0) = 0 \) and \( F_\infty \) satisfies the mean curvature flow equation. \( \square \)

Let \( F(t) : \Sigma^n \to (M, g) \) be a smooth mean curvature flow solution of a compact manifold \( \Sigma \) in a smooth Riemannian manifold \( (M, g) \). We can use the results above to form a geometric limit along the mean curvature flow by re-scaling process.

**Theorem 2.4.** Let \( (M, g) \) be a complete Riemannian manifold with bounded geometry. If \( T \) is the first singular time of the mean curvature flow \( F(t) : \Sigma \to (M, g) \). Then there exists \( (x_i, t_i) \) and \( A_i \to \infty \) such that

\[
F_i(x, s) = F \left( x, \frac{s}{A_i^2} + t_i \right) \to (M, A_i^2 g)
\]

is a sequence of mean curvature flow solutions, and it sub-converges to an ancient mean curvature flow solution for \( s \in (-\infty, 0] \), \( F_\infty(s) : \Sigma_\infty \to \mathbb{R}^m \) with \( |A_\infty(x, s)| \leq \)
$|A_\infty(x_\infty, 0)| = 1, F_\infty(x_\infty, 0) = 0$. If the singularity is of type II, $F_\infty(s)$ can be constructed as an eternal solution. In particular, when $(M, g) = (\mathbb{R}^n, dx^2)$, $\Sigma_\infty(t)$ has at most Euclidean volume growth.

Proof. Suppose that $T$ is the first singular time. For $t < T$, denote

$$A(t) = \max_p |A(p, t)|.$$ 

There exist $(x_i, t_i)$ such that $t_i \to T$ and

$$A_i = \max_{t \leq t_i} A(t) = |A(x_i, t_i)| \to \infty.$$ 

Consider the sequence of flows defined by

$$F_i(x, s) = F\left(x, \frac{s}{A_i^2} + t_i\right) : \Sigma \to (M, A_i^2 g)$$

for $(x, s) \in \Sigma \times [-A_i^2 t_i, 0]$. It is clear that $F_i(s)$ is still a mean curvature flow solution with $|A_i(s)| \leq 1$. Set the marked points to be $q_i = F\left(x_i, \frac{s}{A_i^2} + t_i\right)$. It is clear that $(M, A_i^2 g, q_i)$ sub-converges to the standard Euclidean space $(\mathbb{R}^m, dx^2, 0)$ when $i \to \infty$ since $(M, g)$ has bounded geometry.

At $s = 0$, the sequence of Riemannian manifolds $(\Sigma, F_i(0)^*(A_i^2 g), x_i)$ sub-converges smoothly to a Riemannian manifold $(\Sigma_\infty, g_\infty, x_\infty)$ in the Cheeger-Gromov sense. So there exists a sequence of exhausting relatively compact open subsets $U_i$ of $\Sigma_\infty$ and there is a sequence of $C^\infty$ embedding

$$\varphi_i : U_i \to (\Sigma, F_i(0)^*(A_i^2 g)), \quad \varphi_i(x_\infty) = x_i.$$ 

There also exists a sequence of exhausting relatively compact open subsets $V_i$ of $\mathbb{R}^m$ and a sequence of $C^\infty$ embeddings

$$\phi_i : V_i \to (M, A_i^2 g), \quad \phi_i(0) = F_i(x_i, 0)$$

such that $\phi_i^*(A_i^2 g)$ converges to $dx^2$ on every compact subset of $\mathbb{R}^m$.

For any fixed $s_0 \in (-\infty, 0)$, we can take $V_i$ such that for any $s \in [s_0, 0]$, $\phi_i^{-1} \circ F_i(s) \circ \varphi_i(U_i) \subset V_i$, and then we define

$$\tilde{F}_i^R(x, s) = \phi_i^{-1} \circ F_i(x, s) \circ \varphi_i : U_i \to V_i.$$ 

For any fixed $R$, by taking $i$ sufficiently large we may assume that $U_i$ contains the geodesic ball $B_{2R}(x_\infty)$ in $(\Sigma_\infty, g_\infty, x_\infty)$. It is clear that

$$\tilde{F}_i^R : B_{2R}(x_\infty) \subset (\Sigma_\infty, g_\infty, x_\infty) \to (\mathbb{R}^m, \phi_i^*(A_i^2 g), 0)$$

is a mean curvature flow solution with bounded second fundamental form. By the smoothness property of mean curvature flow, all the higher derivatives of $\tilde{F}_i$ and the derivatives with respect to time are also bounded. Note that the chosen subsequence $\phi_i^*(A_i^2 g)$ converges to the Euclidean metric $dx^2$ on $\mathbb{R}^m$ when $i$ goes to infinity. Hence the ambient metrics are all equivalent on any fixed compact subset, especially on
Then we apply the argument in Theorem 2.3 to conclude that the sequence \( \tilde{F}_i \) sub-converges to an immersion

\[ F_\infty^R : B_R(x_\infty) \times [s_0, 0] \to (\mathbb{R}^m, dx^2) \]

with bounded second fundamental form and its higher derivatives in \( B_R(x_\infty) \times [s_0, 0] \).

Now letting \( R \to \infty \), we obtain a limiting immersion

\[ F_\infty : \Sigma_\infty \times [s_0, 0] \to (\mathbb{R}^m, dx^2). \]

Since \( \phi_i \) converges to an isometric embedding independent of time, it is clear \( F_\infty(s) \) still satisfies the mean curvature flow equation. Taking \( s_0 \to -\infty \), we can get an ancient solution of mean curvature flow

\[ F_\infty : \Sigma_\infty \times (-\infty, 0] \to \mathbb{R}^m. \]

It is clear that \( F_\infty(x_\infty, 0) = 0 \), and \( |A_\infty(x, s)| \leq |A_\infty(x_\infty, 0)| = 1 \).

If the mean curvature flow develops a type II singularity at \( T \), one can follow Hamilton’s work on Ricci flow [7] to construct an eternal solution along the mean curvature flow.

The statement that \( \Sigma_\infty(t) \) has at most Euclidean volume growth holds in more general setting. See section 3 for more details of the proof. \( \square \)

When \( n = 2 \) and \( H_\infty \equiv 0 \), we have the following

**Proposition 2.5.** Let \( F(t) : \Sigma \to \mathbb{R}^m \) be a smooth mean curvature flow of a compact 2-dimensional surface \( \Sigma \) on \( [0, T) \). Let \( \Sigma_\infty(s) \) be the geometric limit of \( F(t) \) as in Theorem 2.4. If \( H_\infty \equiv 0 \), then \( \Sigma_\infty \) has finite total curvature.

**Proof.** A complete minimal surface in \( \mathbb{R}^m \) for arbitrary \( m \geq 3 \) has finite total curvature if and only if it is of finite topological type and has quadratic area growth [2]. A surface is of finite topological type if it has finite genus and finitely many ends.

First, by Theorem 2.4 the area growth of \( \Sigma_\infty \) is at most quadratic.

Second, we show that \( \Sigma_\infty \) has finite genus. Since \( \Sigma_\infty \) is a geometric limit of \( \Sigma \) after suitable blowing up, there exists a sequence of exhausting open subsets \( U_k \) of \( \Sigma_\infty \) and a sequence of embeddings \( \varphi_k \) such that

\[ \varphi_k : U_k \to \varphi_k(U_k) \subset \Sigma \]

is a diffeomorphism for each \( k \). Then the genus of \( U_k \) is less than or equal to that of \( \Sigma \). Therefore \( \Sigma_\infty \) has only finite genus since the sequence \( \{U_k\} \) exhausts \( \Sigma_\infty \).

Finally, we claim that \( \Sigma_\infty \) has only finitely many ends. For any fixed \( p \in \Sigma_\infty \), consider \( \Sigma_\infty \setminus B_R(p) \), where \( B_R(p) \) is Euclidean ball in \( \mathbb{R}^m \). Let \( n_R \) denote the number of the disjoint components in \( \Sigma_\infty \setminus B_R(p) \). If \( \Sigma_\infty \) has infinite many ends, then \( n_R \to \infty \) when \( R \to \infty \). On each component, we can pick up a point \( x_i \) such that the Euclidean distance \( d(x_i, p) = 2R \) for \( i = 1, \cdots, n_R \). We know

\[ B_R(x_i) \cap B_R(x_j) = \emptyset \]
if \( i \neq j \). Now consider \( \Sigma_\infty \cap B_{3R}(p) \), then we know that
\[
\Sigma_\infty \cap \bigcup_{k=0}^{n_R} B_R(x_k) \subset \Sigma_\infty \cap B_{3R}(p),
\]
where \( x_0 = p \). From the monotonicity formula on the area ratio for minimal surfaces in \( \mathbb{R}^m \),
\[
\text{area}(\Sigma_\infty \cap B_R(x_i)) \geq \pi R^2
\]
for \( i = 0, 1, \ldots, n_R \). It follows that
\[
\text{area}(\Sigma_\infty \cap B_{3R}(p)) \geq (n_R + 1)\pi R^2.
\]
However, we know that \( \Sigma_\infty \) has at most Euclidean volume growth, it follows that
\[
\text{area}(\Sigma \cap B_{3R}(p)) \leq C(3R)^2.
\]
But this contradicts with \( n_R \to \infty \). It follows that \( \Sigma_\infty \) has finite many ends.
Therefore, we have shown, by \([2]\), that \( \Sigma_\infty \) has finite total curvature
\[
\int_{\Sigma_\infty} K = 2\pi l < \infty
\]
where \( K \) is the Gauss curvature of \( \Sigma_\infty \), \( l \) is an nonnegative integer, and the equality is a classical result of Osserman on complete minimal surfaces with finite total curvature, and \( \Sigma_\infty \) is conformally diffeomorphic to a closed Riemann surface punctured in a finite number of points \([11]\). \( \square \)

3. Finite time singularity

Now we assume that \((M, g)\) is Ricci parallel with non-negative sectional curvatures. Hamilton \([4]\) derived a monotonicity formula for the mean curvature flow with non-flat ambient space as follows. Note that this coincides to the renowned monotonicity formula derived by Huisken when \((M, g)\) is Euclidean \([8]\). Let \( k \) be a solution of the backward heat equation in \([0, T]\),
\[
\frac{\partial}{\partial t} k = -\Delta_M k.
\]
Hamilton calculated that
\[
\frac{d}{dt} (T-t)^{(m-n)/2} \int_{\Sigma(t)} k d\mu = -(T-t)^{(m-n)/2} \int_{\Sigma(t)} \left( H - \frac{Dk}{k} \right)^2 d\mu
\]
\[
-(T-t)^{(m-n)/2} \int_{\Sigma(t)} g^{\alpha\beta} \left( D_\alpha D_\beta k - \frac{D_\alpha k D_\beta k}{k} - \frac{1}{2(T-t)k g_{\alpha\beta}} \right) d\mu.
\]
If $M$ is Ricci parallel with non-negative positive sectional curvatures, Hamilton can show that the matrix in the last integral is non-negative by the Harnack inequality [5]. It follows that

$$(T - t)^{\frac{m-n}{2}} \int_{\Sigma(t)} kd\mu$$

is non-increasing. To use the monotonicity formula of Hamilton in an effective way, we need some properties about the positive solution of the backward heat equation, which are proved by Li-Yau [9]. One can also find the proof in Schoen-Yau [12].

**Lemma 3.1.** Let $M$ be a complete manifold with non-negative Ricci curvature. Let $u$ be a positive solution of the backward heat equation in $[0, T)$

$$\frac{\partial}{\partial t} u = -\Delta_M u$$

with $\int_M u \equiv 1$ and $u(T)$ is the $\delta$ function centered at $p$. Then we have

1. $c(T - t)^{m/2} \leq u(x, t)$ when $t \to T$ and $d(x, p) \leq \sqrt{T-t}$.
2. For $t > 0$ and $x \in M$, we have

$$u(x, T - t) \leq C(\delta, m)Vol^{-1/2}(\sqrt{T})Vol^{-1/2}(\sqrt{t}) \exp \left( -\frac{d^2(x, p)}{4 + \delta t} \right).$$

Now we are in the position to prove Theorem 1.1.

**Proof.** Keep the same notations as in Theorem 2.4. Suppose that $T = \infty$ is the first singular time. This means that the flow exists for all time and as time approaches infinity the second fundamental form becomes unbounded. Denote $A(t) = \max_p |A(p, t)|$. There exist $(x_i, t_i)$ such that $t_i \to \infty$ and $A_i = \max_{t \leq t_i} A(t) = |A(x_i, t_i)| \to \infty$ as $i \to \infty$. Denote $p_i = F(x_i, t_i)$. By Theorem 2.4 we can construct an ancient solution of mean curvature flow

$$F_\infty(s) : \Sigma_\infty \to \mathbb{R}^m$$

for $s \in (-\infty, 0]$, where $F_\infty(x_\infty, 0) = 0$.

Now we consider the volume growth of $\Sigma_\infty(0)$. For any $R$ fixed, we have

$$(3.1) \quad \int_{\Sigma_\infty(0) \cap B_R(0)} d\mu_\infty(0) = \lim_{i \to \infty} \int_{\Sigma_i(0) \cap B_R^i(p_i)} d\mu_i(0),$$

where $B_R(0)$ is the radius $R$ ball in $\mathbb{R}^m$, while $B_R^i(p_i)$ is the radius $R$ ball in $(M, A_i^2 g)$. It is clear from the rescaling process that

$$(3.2) \quad \int_{\Sigma_i(0) \cap B_R^i(p_i)} d\mu_i(0) = \int_{\Sigma_i(0) \cap B_R^i(p_i)} A_i^{n} d\mu(t_i),$$
where $B^i_{\frac{R}{A_i}}(p_i)$ is the radius $\frac{R}{A_i}$ ball in $(M, g)$. By Lemma 3.1 (1), we have

$$\int_{\Sigma(t_i) \cap B^i_{\frac{R}{A_i}}(p_i)} A^i \, d\mu(t_i) = R^n \int_{\Sigma(t_i) \cap B^i_{\frac{R}{A_i}}(p_i)} \frac{1}{R^n/A^i} \, d\mu(t_i)$$

$$= R^n (T_i - t_i)^{(m-n)/2} \int_{\Sigma(t_i) \cap B^i_{\frac{R}{A_i}}(p_i)} \frac{1}{(T_i - t_i)^{m/2}} \, d\mu(t_i)$$

$$\leq CR^n (T_i - t_i)^{(m-n)/2} \int_{\Sigma(t_i)} k_i(x, t_i) \, d\mu(t_0),$$

(3.3)

where $T_i = \frac{R_i^2}{A_i} + t_i$ and $k_i(x, t)$ for $t \in [0, T_i]$ is the solution of the backward heat equation

$$\frac{\partial}{\partial t} k_i = -\Delta_M k_i$$

with $\int_M k_i \equiv 1$ and $k_i(x, T_i)$ is the delta function centered at $F(x_i, t_i)$. Namely, on $(M, g)$ and for each $i$, the function $k_i(x, T_i - t)$ is the fundamental solution of the forward heat equation with singularity at the point $F(x_i, t_i)$ in $M$, for $T_i - t \in [0, \infty)$.

By Hamilton’s monotonicity formula for mean curvature flow, we know that

$$(T_i - t_i)^{(m-n)/2} \int_{\Sigma(t_i)} k_i(x, t_i) \, d\mu(t_i) \leq T_i^{(m-n)/2} \int_{\Sigma(0)} k_i(x, 0) \, d\mu(0).$$

(3.4)

Note by Lemma 3.1 (2), we have that

$$(T_i - t_i)^{(m-n)/2} \int_{\Sigma(t_i)} k_i(x, t_i) \, d\mu(t_i) \leq CVol^1_x(\sqrt{\frac{T_i}{T}})Vol^1_{F(x_i, t_i)}(\sqrt{T_i}).$$

(3.5)

If $(M, g)$ satisfies the volume growth condition (1.1), by (3.4) and (3.5), we get that

$$(T_i - t_i)^{(m-n)/2} \int_{\Sigma(t_i)} k_i(x, t_i) \, d\mu(t_i) \leq CVol(\Sigma_0)T_i^{-\epsilon}.$$

(3.6)

Since $T_i \to \infty$ when $i \to \infty$, by (3.1) – (3.6), we get that

$$\int_{\Sigma_{\infty}(0) \cap B_R(0)} d\mu_{\infty}(0) \leq CT_i^{-\epsilon} Vol(\Sigma_0) R^n.$$

(3.7)

Since $\epsilon > 0$, the right hand side of (3.7) goes to zero as $i \to \infty$, but this is impossible. It follows that if $T$ is a singular time it must be finite. In other words, if the mean curvature flow solution $F(t)$ exists for all time, the second fundamental form is uniformly bounded. In particular, if $(M, g)$ is analytic and there is no finite time singularity, the mean curvature flow $F(t) : \Sigma \to M$ converges to a compact minimal submanifold along the flow [13].

When $T$ is finite, one can bound $k_i(x, 0)$ in (3.5) in terms of $T$ since $(M, g)$ has bounded geometry. It follows from (3.1) – (3.5) that $\Sigma_{\infty}(0)$ has at most Euclidean
volume growth. Note this does not need \((M, g)\) satisfies volume growth condition \((1.1)\). The proof for any \(\Sigma_\infty(s), s \in (-\infty, 0]\) is similar. \(\square\)

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