ABSENCE OF REPLICA SYMMETRY BREAKING IN FINITE FIFTH MOMENT RANDOM FIELD ISING MODEL

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Abstract. This work is concerned with the theory of the Random Field Ising Model on the hypercubic lattice, in the presence of a independent disorder with finite fifth moment. We showed the absence of replica symmetry in any dimensions, at any temperature and field strength, almost surely.

1. Introduction

The Random Field Ising Model (RFIM) is probably one of the simplest non-trivial models in Statistical Mechanics that belongs to a class of disordered spin models in which the disorder, so-called quenched random magnetic field, is coupled to the order parameter of the system. This model is under intensive investigation both experimentally and theoretically and until now a lot has been studied on various aspects, especially the study of the existence of phase transition. The earliest attempt to address the question of phase transition in the RFIM goes back to Imry and Ma (1975). They proposed an extension of the famous Peierls argument to study phase transition in this model. Following their arguments for $d \leq 2$ the uniqueness of the Gibbs states would be expected, while for $d \geq 3$ this model should have phase transition. Remarkable progress on this problem in dimensions $d \geq 3$ was made by Imbrie and Kupiainen (1987-1988) using the renormalization group. Subsequently the case $d \leq 2$ was solved by Aizenman and Wehr (1989-1990). Recently, there are many studies about the properties of the RFIM, such as decay of correlations, phase transitions (see, e.g., and the replica symmetry breaking (see, e.g., ).

There are several methods to study disordered systems like the Sherrington-Kirkpatrick model and the RFIM - the cavity and replica methods being among them. The replica method enables us to calculate the disorder-averaged value of $\ln Z$, where $Z$ denotes the partition function, in simpler ways, calculating the disorder average $Z^n$. In other words, it uses $n$ copies or replicas of the system (see ). This method works with $n$ as an integer number but later, by using an analytic continuation to the real numbers, analysis was carried out for $n$ tending to 0. Using this method the solution is given in terms of its replicas and is known

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as the replica-symmetric solution. The direct use of the replica method can result in non-physical conclusions. To avoid that, the replica symmetry breaking scheme (see [30, 36]) is used. In order to use this scheme we first need to know if the model has this property. Taking this into consideration, it is common for disorder systems like spin glasses to analyze the replica symmetry breaking scheme. There are also many contributions from theoretical physics viewpoint to the study of the RFIM, see for example [31]. Chatterjee [15], using the definition of Parisi [38], showed that this model does not have the replica symmetry breaking. Later, this same property was proven for the transverse and longitudinal RFIM in [25].

Among the main contributions of this paper is the introduction of a class of independent disorders with finite 5-th moment where similar results of [15] can be recovered by a generalized Gaussian integration by parts and Guirlanda-Guerra identities (see [11, 21] for a literature review). In other words, we show that the absence of replica symmetry breaking is still valid for the model proposed in this paper. Most of the works in the RFIM assumes Gaussianity of the disorders (see [2, 12, 15, 16]). In [16], at the end of its introduction, the author warns about the difficulty to adapt Gaussian methods and results for models in the presence of disorders with distribution that is not necessarily Gaussian. On the other hand, we emphasize that, combining the results of this paper with the main theorem of Panchenko (2011) [35] we implicitly established the Parisi ultrametricity property (see [35, 37]) in our RFIM.

It is known that the presence of disorder in condensed matter systems give rises to new phases and phase transitions possibly related with the multiplicity of metastable states. A system is said to be in a spin glass phase if and only if the ferromagnetic susceptibility is finite, while the spin glass susceptibility is infinite. A long-standing debate on the presence or absence of (elusive) spin glass phase in several systems, such as the RFIM and the Ginzburg-Landau model (or the so-called $\phi^4$-theory), has been the subject of research for many years. In [28] and [27], the authors stated that the RFIM and the Ginzburg-Landau model, respectively, with non-negative interactions and arbitrary disorder on an arbitrary lattice does not have a spin glass phase. That is, they argued that on both ferromagnetic systems the spin glass susceptibility is always upper-bounded by the ferromagnetic susceptibility, consequently excluding the possibility of a spin glass phase. In 2015, Chatterjee [15] gave a rigorous mathematical proof supporting part of the findings claimed in [28], in the random (Gaussian) field Ising model on the hypercubic lattice. Here, we extended the Chatterjee’s results to a large class of not necessarily Gaussian random fields, thus giving a further partial support in favor of the absence of a spin glass phase.

This paper is organized as follows. In Section 2, we begin by presenting the RFIM and setting up some basic definitions. Furthermore, in this section an extension of the main result of [15] in a more general setting is stated. In Section 3, an outline of the proof of this extension is given and in Section 4 the proof itself is presented,
in details (see Theorem 1). We end this paper with the proof of our main tool in Appendix (see Proposition 4.14).

2. The model

Given \( n \geq 1 \), let \( V_n = Z^d \cap [1, n]^d, d \geq 1 \), be a finite subset of vertices of \( d \)-dimensional hypercubic lattice with cardinality denoted by \( |V_n| \). The (random) Gibbs measure of the RFIM on the set of spin configurations \( \{-1, 1\}^{V_n} \) is given by

\[
G_n(\{\sigma\}) = \frac{1}{Z_n} \exp \left( -H_n(\sigma) \right),
\]

where \( H_n \) is a Hamiltonian on \( \{-1, 1\}^{V_n} \) given by

\[
-H_n(\sigma) := \beta \sum_{\langle xy \rangle} \sigma_x \sigma_y + h \sum_x g_x \sigma_x.
\]

Here, \( \langle xy \rangle \) below the first sum means that we are summing over \( x, y \) that are neighbors, \( \beta \) and \( h \) are positive parameters, called inverse temperature and field strength, respectively. The partition function \( Z_n = Z_n(\beta, h) \) enters the definition of \( G_n \) as a normalizing factor and the \( g_x \)'s are independent random variables (that collectively are called the disorder) of form

\[
g_x := h_x \zeta_x, \quad \forall x \in Z^d, \quad \sup_{x \in Z^d} |h_x| \leq 1,
\]

where \( (\zeta_x) \) is an arbitrary disorder with the following properties: the \( \zeta_x \)'s are independent identically distributed (i.i.d.) real-valued random variables with zero-mean and unit-variance such that \( \zeta_x^5 \) is integrable. Furthermore, the sequence \( (h_x) \) is a non-zero non-invariant magnetic external field such that

\[
\sum_{x \in V_n} |h_x| = o(|V_n|), \quad \text{as } n \to \infty.
\]

One can take, for example, \( h_x = h^* \|x\|^{-\alpha} \), for \( x \neq 0 \) and \( h_0 = h^* \), with \( \alpha > 0 \), \( h^* \in (0, 1) \) fixed, where \( \|x - y\| \) denotes the distance between \( x \) and \( y \) on the hypercubic lattice. Indeed, if \( \alpha > d \), \( (h_x) \) is summable then (1) follows trivially; also when \( \alpha \leq d \), the external field is not summable and in this case the condition (1) is obtained by counting over the sizes of \( \|x\| \), i.e. doing \( \|x\| = r \), with \( r = 1, 2, 3, \ldots \), and next by using the Stolz-Cesàro Theorem. The study of the classical nearest neighbor ferromagnetic Ising model in the presence of positive power-law decay external fields with power \( \alpha \) appeared recently in several works, see, e.g., [8, 9, 18].

Remark 2.1. In the reference [5], Example 3, the authors considered the RFIM where the field strength \( h \) is a small perturbation (some sort of mean field model). In our setting the field strength remains unchanged with respect to the volume.

Remark 2.2. Taking \( h_x = \pm 1 \) and \( \zeta_x \sim N(0, 1) \) for all \( x \), in (3), the condition (4) is not satisfied, but this one is not required by the Gaussian integration by parts. Then, in this paper we recovered the results of [15] for the RFIM in a general environment.
Remark 2.3. The ideal would be to consider the condition: there exist constants $c_1, d_1 > 0$ such that

$$c_1 \leq \liminf_{n \to \infty} \frac{1}{|V_n|} \sum_{x \in V_n} |h_x| \quad \text{and} \quad \limsup_{n \to \infty} \frac{1}{|V_n|} \sum_{x \in V_n} |h_x| \leq d_1,$$

instead the condition [14]. Since the techniques used in this paper include a generalized Gaussian integration by parts, we emphasize that unfortunately we can not weaken [14] by condition [5]. Note also that the magnetic fields $(h_x)$ considered here satisfy the inequality of the right side of [5] with $d_1 = 1$, but not the condition of the left side.

2.1. Some definitions. For a function $f : (\{-1, 1\}^{V_n})^m \to \mathbb{R}$, $m \geq 1$, we define

$$\langle f \rangle := \int f(\sigma^1, \ldots, \sigma^m) \, dG_n(\sigma^1) \cdots dG_n(\sigma^m)$$

$$= \frac{1}{Z_n^m} \sum_{\sigma^1, \ldots, \sigma^m} f(\sigma^1, \ldots, \sigma^m) \exp \left( \beta \sum_{x} \sum_{y} \sigma_x^y + h \sum_{x} g_x \sigma_x^2 \right).$$

Let $\langle \cdot \rangle_{g=u}$ be the Gibbs expectation defined by setting $g_x$ in $\langle \cdot \rangle$ to be $u_x$, for each $x \in V_n$. The randomness of the $g_x$’s will be represented by the non-Gaussian measure $\gamma$ on $\mathbb{R}^{d^d}$. Following the notation of Talagrand [39], we write

$$\nu(f) := \mathbb{E}(f) = \int \langle f \rangle_{g=u} \, d\gamma(u),$$

averaging over disorder realizations.

If $\sigma^1, \sigma^2, \ldots$ are i.i.d. configurations under Gibbs measure [11], known as replicas, the generalized overlap (with reference to [14]) between two replicas $\sigma^l, \sigma^s$ is defined as

$$R_{l,s} = R_{l,s}(\sigma^l, \sigma^s) := \frac{1}{|V_n|} \sum_{x \in V_n} (\mathbb{E}g_x^2)^{1-\delta_{l,s}} \sigma_x^l \sigma_x^s, \quad \forall l, s,$$

where $\delta$ is the Kronecker delta function and $\mathbb{E}(g_x g_y) = \delta_{x,y} h_x h_y$. Without loss of generality we are assuming that the deterministic constant $R_{l,l}$ is equal to 1 for all $\sigma^l$. Note that $|R_{l,s}| \leq 1$ (by Schwartz’ inequality) and that the infinite random array $R = (R_{l,s})_{l,s \geq 1}$ is symmetric, non-negative definite, weakly exchangeable (that is, $(R_{l,s})_{1 \leq l, s \leq m}$ and $(R_{\rho(l), \rho(s)})_{1 \leq l, s \leq m}$ have the same distribution, for any permutation $\rho : \{1, \ldots, m\} \to \{1, \ldots, m\}$ and for any $m \geq 1$). The array $R$ is said to satisfy the Ghirlanda-Guerra identities (see [11]) if for any $m \geq 2$ and any bounded measurable function $f = f((R_{l,s})_{1 \leq l, s \leq m})$,

$$\nu(f R_{1,m+1}) - \frac{1}{m} \nu(f) \nu(R_{1,2}) - \frac{1}{m} \sum_{s=2}^{m} \nu(f R_{1,s}) \to 0,$$

at almost all $$(\beta, h).$$
For each \((\beta, h) \in (0, \infty)^2\), let

\[
F_n = F_n(\beta, h) := \log Z_n, \quad \psi_n = \psi_n(\beta, h) := \frac{F_n}{|V_n|}, \quad p_n = p_n(\beta, h) := E\psi_n,
\]

where \(\psi_n\) is proportional to the free energy and \(p_n\) is the value expected of \(\psi_n\). It is well-known that in the thermodynamic limit, \(p_n = p(\beta, h) := \lim_{n \to \infty} p_n\) is well defined (see Lemma 2.1 in [15]), \(p\) is a convex function of \(h\) for every fixed \(\beta\) and the same is true for \(F_n, \psi_n\) and \(p_n\) (see Lemma 2.2 in [15]). Then, it is natural to introduce the set

\[
\mathcal{A} := \{(\beta, h) \in (0, \infty)^2 : \frac{\partial p}{\partial h}(\beta, h) \neq \frac{\partial p}{\partial h}(\beta, h)\}.
\]

It is known that, the set \(\mathcal{A}\) has zero Lebesgue measure and moreover this set is countable (see Lemma 2.3 in [15]).

2.2. The main result. The system is said to exhibit replica symmetry breaking (as stated in [38]) if the limiting distribution of the random variable \(R_{1,2}\), when \(n \to \infty\), denoted by \(p(q)\) for each \(q\), has more than one point in its support. In the present paper, the next main theorem shows that this does not happen for the RFIM. That is, we will show that for any \((\beta, h) \not\in \mathcal{A}\) there exists a constant \(q_{\beta, h} \in [-1, 1]\) such that \(p(q)\) is a point distribution concentrated on \(q_{\beta, h}\). A sufficient condition to prove this is to verify convergence in quadratic mean,

\[
\nu((R_{1,2} - q_{\beta, h})^2) \to 0.
\]

**Theorem 1** (Lack of Replica Symmetry Breaking in the RFIM). For any \((\beta, h) \not\in \mathcal{A}\) the infinite volume limit: \(\lim_{n \to \infty} \nu(R_{1,2})\), exists, and the variance of the overlap \(R_{1,2}\) vanishes

\[
\nu((R_{1,2} - \nu(R_{1,2}))^2) \to 0.
\]

That is, the overlap \(R_{1,2}\) is self-averaging in the RFIM defined by (1). Furthermore, there exists a constant \(q_{\beta, h} \in [-1, 1]\) such that (11) is satisfied.

The rest of this paper is devoted to the proof of Theorem 1.

3. Outline of the proof

The key results that structure the content of the main theorem of this paper are the following:

(I) The FKG property of the RFIM: if \(f\) and \(g\) are two monotone increasing functions on the configuration space \((-1, 1)^{V_n}\), then \(\langle fg \rangle \geq \langle f \rangle \langle g \rangle\). The proof follows by verifying the FKG lattice condition [20] for any realization of the disorder.

(II) A generalized Gaussian integration by parts: if \(f = f((R_{i,s})_{1 \leq i, s \leq m}) : \mathbb{R}^{m(m-1)/2} \to [-1, 1]\) is a bounded measurable function of the overlaps that not change
with \( n \), using Taylor’s Theorem (see Propositions 4.13 and 4.14 in Appendix) and the condition (11), we show that

\[
\sum_x \mathbb{E} g_x \langle \sigma_x^1 f \rangle - \sum_x \mathbb{E} g_x^2 \mathbb{E} \frac{\partial^2 \langle \sigma_x f \rangle}{\partial g_x \partial \nu} = o(|V_n|),
\]

(12)

\[
\sum_{x,y} \mathbb{E} g_x g_y (F_n - \mathbb{E} F_n) - \sum_{x,y} \mathbb{E} g_x^2 \mathbb{E} g_y^2 \mathbb{E} \frac{\partial^2 F_n}{\partial g_x \partial g_y} = o(|V_n|^2),
\]

(13)

\[
\sum_{x,y} \mathbb{E} g_x g_y \langle \sigma_x \sigma_y \rangle - \sum_{x,y} \mathbb{E} g_x^2 \mathbb{E} g_y^2 \mathbb{E} \frac{\partial^2 \langle \sigma_x \sigma_y \rangle}{\partial g_x \partial g_y} = o(|V_n|^2),
\]

(14)

in the limit \( n \to \infty \), where \( F_n \) is as in (9) and \( \langle \sigma_x \sigma_y \rangle := \langle \sigma_x \sigma_y \rangle - \langle \sigma_x \rangle \langle \sigma_y \rangle \) denotes the truncated two-point correlation for the Gibbs measure in finite volume defined by (1).

(III) \( \mathbb{E} \left( \langle R^{1,2}_n \rangle - \langle R^{1,2}_n \rangle \right) = o(1) \) and \( \nu \left( (m(\sigma) - \langle m(\sigma) \rangle)^2 \right) = o(1) \), as \( n \to \infty \) (see Lemma 4.3), where \( R^{1,2}_n \) is as in (7) and \( m(\sigma) := \sum_x \sigma_x / |V_n| \) defines the magnetization: the proof of this item follows by using FKG inequality (Item (I)) and the generalized Gaussian integration by parts (Item (III)).

(IV) \( \nu \left( |\Delta_n - \nu(\Delta_n)\rangle\right) = o(1) \), as \( n \to \infty \), \( \forall (\beta, h) \in \mathcal{A} \) (see Lemma 4.9), where \( \Delta_n = \sum_{x \in V_n} \sigma_x / |V_n| \) following the same steps as Lemma 2.7 in [15], we obtain that \( \forall (\beta, h) \in \mathcal{A} \), \( \nu(\Delta_n) \to \frac{\partial \nu}{\partial h}(\beta, h) \) and \( \mathbb{E} \langle |\Delta_n - \nu(\Delta_n)\rangle \rangle \to 0 \).

Combining these two limits with (14) the proof of this item follows.

(V) The Ghirlanda-Guerra identities (see Lemma 4.12): for any bounded measurable function \( f : \mathbb{R}^{m(m-1)/2} \to [-1, 1] \), using (12), we show that

\[
\nu(\Delta_n(\sigma^1) f) - h \nu \left( \left( \sum_{s=1}^m R_{1,s} - m R_{1,m+1} \right) f \right) = O(|V_n|).
\]

(15)

Equivalently, the difference \( \nu(\Delta_n(\sigma^1) f) - \nu(\Delta_n) \nu(f) \) approximates

\[
h \nu \left( \left( \nu(R_{1,2}) + \sum_{s=2}^m R_{1,s} - m R_{1,m+1} \right) f \right),
\]

as \( n \to \infty \). We emphasize that the last expression is exactly equal to zero under the Gaussian disorder by the Gaussian integration by parts. Combining the last approximation with the limit \( \mathbb{E} \langle |\Delta_n - \nu(\Delta_n)\rangle \rangle \to 0 \) given in (IV), we conclude that the Ghirlanda-Guerra identities (8) are satisfied.

Finally, using properties of symmetry of overlaps \( R_{i,s} \), Items (V) and (III), it follows that the variance of \( R_{1,2} \), with respect to \( \nu \), converges to zero as \( n \to \infty \). Then it is enough to prove that the limit of \( \nu(R_{1,2}) \), as \( n \to \infty \), exists. Indeed, taking \( f = m = 1 \) in (15) and using the limit \( \nu(\Delta_n) \to \frac{\partial \nu}{\partial h}(\beta, h) \) given in (IV), we obtain that \( \nu(\Delta_n) = h(1 - \nu(R_{1,2})) + o(|V_n|) \to \frac{\partial \nu}{\partial h}(\beta, h) \) and the proof of theorem follows by taking \( q_{\beta, h} := 1 - \frac{1}{h} \frac{\partial \nu}{\partial h}(\beta, h) \).
4. Proof

Before presenting the proof of the main theorem of this paper, we state and prove some preliminaries results (propositions and lemmas).

The proof of the next result was inspired in the arguments of the proof of Lemma 2.5 in the preprint [26].

Lemma 4.1. There are a function $\theta : \mathbb{N} \to \mathbb{R}$ and a constant $C > 0$ such that

$$\text{Var}(F_n) \leq (C h |V_n| + \theta(n)) h.$$ 

Proof. For all $s \in [0,1]$ we consider a new random field $G = (G_x)$ given by

$$G_x = G_{x,s} := \sqrt{s} g_x + \sqrt{1-s} g^*_x, \quad \forall x \in V_n,$$

where $g = (g_x)$ and $g^* = (g^*_x)$ consist of independent random variables. We define also a generating function

$$\gamma_n(s) := \mathbb{E} [\mathbb{E}^* F_n(G)]^2, \quad \text{with } F_n(G) = \log Z_n(G),$$

where $\mathbb{E}$ and $\mathbb{E}^*$ denote expectation over $g$ and $g^*$, respectively.

A straightforward computation shows that

$$\frac{d\gamma_n}{ds}(s) = \sum_x \mathbb{E} \left[ \frac{\partial}{\partial s} \mathbb{E}^* F_n(G) \mathbb{E}^* \frac{\partial F_n(G)}{\partial G_x} - \mathbb{E}^* F_n(G) \mathbb{E}^* \frac{g^*}{\sqrt{1-s}} \frac{\partial F_n(G)}{\partial g^*_x} \right].$$

Let $f(G)|_{g_x = u}$ be the function defined by setting $g_x$ in $f(G)$ to be $u$ for all Borel measurable function $f$ depending of the disorder $G$, and

$$F_x(u) := \mathbb{E}^* F_n(G) \mathbb{E}^* \frac{\partial F_n(G)}{\partial G_x} \bigg|_{g_x = u} \quad \text{and} \quad F^*_x(v) := \frac{\partial F_n(G)}{\partial g^*_x} \bigg|_{g^*_x = v},$$

for each $u, v \in \mathbb{R}$. Furthermore, let $\langle \cdot \rangle_G$ be the Gibbs expectation $\langle \cdot \rangle$ defined by setting the disorder $G$ instead of one $g$, and

$$f_x(u) := \mathbb{E} F_x(u) \quad \text{and} \quad f^*_x(v) := \mathbb{E} F^*_x(v).$$

Since

$$\frac{\partial F_n(G)}{\partial g_x} = h \langle \sigma_x \rangle_G, \quad \frac{\partial^2 F_n(G)}{\partial g_x^2} = h^2 (\langle \sigma_x^2 \rangle_G - \langle \sigma_x \rangle_G^2),$$

$$\frac{\partial^3 F_n(G)}{\partial g_x^3} = -2h^3 \langle \sigma_x \rangle_G (\langle \sigma_x^2 \rangle_G - \langle \sigma_x \rangle_G^2),$$

$$\frac{\partial^4 F_n(G)}{\partial g_x^4} = 4h^4 (\langle \sigma_x \rangle_G^2 - \frac{1}{2}) (\langle \sigma_x^2 \rangle_G - \langle \sigma_x \rangle_G^2),$$

the real-valued functions $f_x (u)$ and $f^*_x (v)$ have bounded continuous third-order derivatives. Then, by Proposition 4.13 in Appendix,

$$\mathbb{E} \left[ \frac{\partial}{\partial s} \mathbb{E}^* F_n(G) \mathbb{E}^* \frac{\partial F_n(G)}{\partial g_x} \bigg|_{g_x = u} \right] = h^2 \mathbb{E} \left[ \frac{1}{\sqrt{s}} \frac{\partial}{\partial g_x} \mathbb{E}^* F_n(G) \mathbb{E}^* \frac{\partial F_n(G)}{\partial g_x} \bigg|_{g_x = u} \right] + \frac{1}{\sqrt{s}} \gamma^2_{g_x} (f_x);$$

$$\mathbb{E}^* \frac{g^*}{\sqrt{1-s}} \frac{\partial F_n(G)}{\partial g^*_x} = h^2 \mathbb{E}^* \frac{\partial}{\partial g^*_x} \mathbb{E}^* \frac{\partial F_n(G)}{\partial g^*_x} + \frac{1}{\sqrt{1-s}} \gamma^2_{g^*_x} (f^*_x),$$

$\Box$
where $\gamma^2_{g_x}(f_x) := \mathbb{E}(g_x \int_0^{g_x} (g_x - u) \frac{d^2 f_x}{du^2}(u) \, du) - h_x^2 \mathbb{E}(\int_0^{g_x} (g_x - u) \frac{d^3 f_x}{du^3}(u) \, du)$. Substituting the last two identities in (10), we have

$$\frac{d\gamma_n}{ds}(s) = \sum_x h_x^2 \mathbb{E}\left[ \frac{1}{\sqrt{s}} \frac{\partial}{\partial g_x} \mathbb{E}^* F_n(G) \mathbb{E}^* \frac{\partial F_n(G)}{\partial g_x} - \frac{1}{\sqrt{1-s}} \mathbb{E}^* F_n(G) \mathbb{E}^* \frac{\partial F_n(G)}{\partial g_x} \right]$$

(18) $$+ h\theta(n, s),$$

where $\theta(n, s) := \sum_x \left[ \frac{1}{\sqrt{s}} \gamma^2_{g_x}(f_x) - \frac{1}{\sqrt{1-s}} \gamma^2_{g_x}(f_x') \right]$.

On the other hand, using that $\frac{\partial F_n(G)}{\partial g_x} = \sqrt{s} \frac{\partial F_n(G)}{\partial g_x}$ and $\frac{\partial F_n(G)}{\partial g_x} = \sqrt{1-s} \frac{\partial F_n(G)}{\partial g_x}$, it follows that

$$\frac{1}{\sqrt{s}} \frac{\partial}{\partial g_x} \mathbb{E}^* F_n(G) \mathbb{E}^* \frac{\partial F_n(G)}{\partial g_x} - \mathbb{E}^* F_n(G) \mathbb{E}^* \frac{1}{\sqrt{1-s}} \frac{\partial F_n(G)}{\partial g_x} = \left( \mathbb{E}^* \frac{\partial F_n(G)}{\partial g_x} \right)^2.$$

Then, for each $s \in (0, 1)$, (18) can be rewritten as

$$\frac{d\gamma_n}{ds}(s) = \sum_x h_x^2 \mathbb{E}\left[ \mathbb{E}^* \frac{\partial F_n(G)}{\partial g_x} \right]^2 + h\theta(n, s).$$

Jensen’s inequality, the item (17) and the inequality $h_x^2 \leq 1$ for all $x$ give

$$\frac{d\gamma_n}{ds}(s) \leq C_2 h^2 |V_n| + h\theta(n, s).$$

Integrating the above inequality from 0 to 1 and using the relation $\text{Var}(F_n) = \gamma_n(1) - \gamma_n(0) = \int_0^1 \frac{d\gamma_n}{ds}(s) \, ds$,

$$\text{Var}(F_n) \leq C_2 h^2 |V_n| + h \int_0^1 \theta(n, s) \, ds.$$

Finally, taking $\theta(n) := \int_0^1 \theta(n, s) \, ds$ the proof follows. \hfill \square

The proof of the next proposition makes essential use of Lemma 4.1.

**Proposition 4.2.** For any $n$ and any $(\beta, h) \in (0, \infty)^2$, there exists an application $\ell : V_n \times V_n \to \mathbb{R}$ such that

$$\sum_{x, y \in V_n} \left( h_x h_y \mathbb{E} \frac{\partial^2 F_n}{\partial g_x \partial g_y} + \ell(x, y) \right)^2 - \sum_{x \in V_n} \left( h_x^2 + \ell(x, x) \right)^2 \leq 2(Ch|V_n| + \theta(n))h,$$

with $C$ and $\theta(\cdot)$ the same constant and function as in Lemma 4.1, respectively.

**Proof.** Let $F := F_n - \mathbb{E} F_n$. Let $W$ denote the set of all unordered pairs $\{x, y\}$, where $x, y \in V_n$ and $x \neq y$. For each $e = \{x, y\} \in W$, let $s_e := \zeta_x \zeta_y$, $c_e := \mathbb{E} s_e F$ and $S := \sum_{e \in W} c_e s_e$, where $\zeta_x$ is as in (3). Since $\zeta_x$ has zero-mean and unit-variance for all $x$, the $\zeta_e$ in $W$, $\mathbb{E} s_e s_{e'} = 0$, and for any $e$, $\mathbb{E} s_e^2 = 1$. Thus, $\mathbb{E} S^2 = \sum_{e \in W} c_e^2 = \sum_{e \in W} \mathbb{E} s_e^2 F = \mathbb{E} S F$. Consequently, $\mathbb{E} F^2 = \mathbb{E}(F - S)^2 + \mathbb{E} S^2$. Then, by Lemma 4.1

$$\sum_{e \in W} c_e^2 = \mathbb{E} S^2 \leq \mathbb{E} F^2 = \text{Var}(F_n) \leq (Ch|V_n| + \theta(n))h.$$

(20)
Let \( F_{x,y}(u,v) \) be the function defined by setting \( g_x \) and \( g_y \) in \( F \) to be \( u \) and \( v \) respectively, for any \( e = \{x,y\} \). Applying a generalized Gaussian integration by parts (see Proposition 4.14 in Appendix), with \( f_{x,y}(u,v) := \mathbb{E} F_{x,y}(u,v) \), we have

\[
(21) \quad h_x h_y c_e = \mathbb{E} g_x g_y F = h_x^2 h_y^2 \mathbb{E} \frac{\partial^2 f_{x,y}}{\partial g_x \partial g_y} + \gamma^2_{g_x,g_y}(f_{x,y}),
\]

where

\[
(22) \quad \gamma^2_{g_x,g_y}(f_{x,y}) := \mathbb{E} \left( g_x g_y \int_0^{g_x} \int_0^{g_y} (g_x - u) \frac{\partial^3 f_{x,y}}{\partial u^2 \partial v} \, du \, dv \right) - h_x^2 h_y^2 \mathbb{E} \left( \int_0^{g_x} \int_0^{g_y} (g_x - u) \frac{\partial^5 f_{x,y}}{\partial u^4 \partial v^2} \, du \, dv \right) - h_x^2 h_y^2 \mathbb{E} \left( \int_0^{g_x} (g_x - u) \frac{\partial^4 f_{x,y}(u,0)}{\partial u^2 \partial v^2} \, du + \int_0^{g_y} (g_y - v) \frac{\partial^4 f_{x,y}(0,v)}{\partial u \partial v^3} \, dv \right).
\]

Combining (21) and (22), it follows that

\[
\sum_{x,y} \left( h_x h_y \mathbb{E} \frac{\partial^2 f_{x,y}}{\partial g_x \partial g_y} + (h_x h_y)^{-1} \gamma^2_{g_x,g_y}(f_{x,y}) \right)^2
\leq \sum_{x \neq y} \left( h_x h_y \mathbb{E} \frac{\partial^2 f_{x,y}}{\partial g_x \partial g_y} + (h_x h_y)^{-1} \gamma^2_{g_x,g_y}(f_{x,y}) \right)^2 + \sum_x \left( h_x^2 + h_x^{-2} \gamma^2_{g_x,g_x}(f_{x,x}) \right)^2
\leq 2(C h |V_n| + \theta(n)) h + \sum_x \left( h_x^2 + h_x^{-2} \gamma^2_{g_x,g_x}(f_{x,x}) \right)^2.
\]

Thus, taking \( \ell(x,y) := (h_x h_y)^{-1} \gamma^2_{g_x,g_y}(f_{x,y}) \) the proof of proposition follows. \( \square \)

The FKG property for the RFIM and the Proposition 4.2 have an important role in the proof of the following result.

**Proposition 4.3.** For any \( n \) and any \( (\beta, h) \in (0, \infty)^2 \) there exists a sequence \( \alpha_n := \alpha_n(h) = o(1) \) such that

\[
\mathbb{E} \left( \langle R^2_{1,2} \rangle - \langle R_{1,2} \rangle^2 \right) \leq \frac{4}{h^2} \sqrt{\left( \frac{\alpha_n}{|V_n|^2} + \frac{\theta(n)}{|V_n|^2} \right) h + \frac{1}{|V_n|^2} \sum_x \ell^2(x,x) - \frac{2\theta_1(n)}{h^2|V_n|^2}},
\]

where \( \theta(\cdot) \) and \( \ell(\cdot, \cdot) \) are as in Lemma 4.1 and Proposition 4.2, respectively, and \( \theta_1(n) := \sum_{x,y \in V_n} h_x h_y \ell(x,y) \).

**Proof.** It is well-known that

\[
(23) \quad \frac{1}{h^2} \frac{\partial^2 F}{\partial g_x \partial g_y} = \langle \sigma_x; \sigma_y \rangle,
\]
where \( \langle \sigma_x; \sigma_y \rangle \) is the truncated two-point correlation. By FKG inequality, \( \mathbb{E}\langle \sigma_x; \sigma_y \rangle \geq 0 \) for each \( x, y \). Then,

\[
\mathbb{E} \left( \langle R_{1,2}^2 \rangle - \langle R_{1,2} \rangle^2 \right) = \frac{1}{|V_n|^2} \sum_{x,y} h_x h_y \mathbb{E} \langle \sigma_x; \sigma_y \rangle (\langle \sigma_x \sigma_y \rangle + \langle \sigma_x \rangle \langle \sigma_y \rangle)
\]

\[
\leq \frac{2}{h^2 |V_n|^2} \sum_{x,y} h_x h_y \mathbb{E} \frac{\partial^2 F}{\partial g_x \partial g_y},
\]

where in the inequality we use (23) and the upper bound \( \langle \sigma_x \sigma_y \rangle + \langle \sigma_x \rangle \langle \sigma_y \rangle \leq 2 \). Seeing that \( |h_x| \leq 1 \) for all \( x \), by the Cauchy-Schwarz inequality the right-hand side term of the above inequality is at most

\[
\frac{2}{h^2 |V_n|^2} \left( \sqrt{\sum_{x,y} h_x^2 h_y^2 \sum_{x,y} (h_x h_y \mathbb{E} \frac{\partial^2 F}{\partial g_x \partial g_y} + \ell(x,y))^2} - \theta_1(n) \right),
\]

where \( \ell(x,y) = (h_x h_y)^{-1} \gamma^2_{g_x,g_y}(f_{x,y}) \). By Proposition 4.2 and by Minkowski’s Inequality, the expression of left side of above difference is at most

\[
\frac{4}{h^2} \sqrt{\left( \frac{Ch + h^{-1}}{|V_n|} + \frac{\theta(n)}{|V_n|^2} \right) h + \frac{1}{|V_n|^2} \sum_x \ell^2(x,x)}.
\]

Therefore, taking \( \alpha_n := (Ch + h^{-1})/|V_n| \), the proof of proposition is complete. \( \square \)

**Remark 4.4.** As mentioned before, the following result was also proved in [3], Example 3, in the case that the field strength \( h \) is a small perturbation with a decay ratio which similar but different from ours.

The inequality provided by Proposition 4.3 will allow us the next key lemma.

**Lemma 4.5.** For any \((\beta, h) \in (0, \infty)^2\),

\[
\mathbb{E} \left( \langle R_{1,2}^2 \rangle - \langle R_{1,2} \rangle^2 \right) \to 0.
\]

That is, in mean, the variance of the overlap \( R_{1,2} \), with respect to the Gibbs measure \( \mathbb{P} \), converges to zero in the thermodynamic limit.

**Proof.** By Proposition 4.3 it is enough to prove that \( \theta(n) = o(|V_n|^2) \), \( \theta_1(n) = o(|V_n|^2) \) and \( \sum_x \ell^2(x,x) = o(|V_n|^2) \). That is,

\[
(24) \quad \frac{1}{|V_n|^2} \sum_x (h_x^2 \gamma^2_{g_x,g_x}(f_{x,x}))^2 \to 0 \quad \text{and} \quad \frac{1}{|V_n|^2} \sum_{x,y} \gamma^2_{g_x,g_y}(f_{x,y}) \to 0,
\]

\[
(25) \quad \frac{1}{|V_n|^2} \int_0^1 \sum_x \left[ \frac{1}{\sqrt{s}} \gamma^2_{g_x}(f_x) - \frac{1}{\sqrt{1-s}} \gamma^2_{g_x}(f_x) \right] ds \to 0,
\]

where \( \gamma^2_{g_x,g_y}(f_{x,y}) \) is as in (22). Let us give the details for the convergences in (24). The other case (23) is analogous.

Since \( \left| \frac{\partial^{i+j} f_{x,u}(u,v)}{\partial u^i \partial v^j} \right| \leq C_{ij} h^{i+j} \), for some constant \( C_{ij} > 0 \) depending on the indices \( i, j \), it follows that

\[
(26) \quad \left| \int_0^g \int_0^g (g_x - u) \frac{\partial^2 f_{x,u}(u,v)}{\partial u^i \partial v^j} \, du \, dv \right| \leq \frac{C_{ij} h^{|i+j|} |g_x|}{2} \zeta_x^2 |\zeta_y|
\]
and by Items (45)–(47) of Proposition 4.13 that

\[
\left| g_x g_y \int_0^x \int_0^y (g_x - u) \frac{\partial^2 f_{x,y}(u,v)}{\partial u^2} \, du \, dv \right| \leq \frac{C_3 h^3}{h_x h_y^2} |\zeta_x|^3 |\zeta_y|^2, \tag{27}
\]

\[
\left| \int_0^x (g_x - u) \frac{\partial^2 f_{x,y}(u,0)}{\partial u^3} \, du \right| \leq \frac{C_3 h^4 h_y^2}{2} |\zeta_x|^2, \tag{28}
\]

\[
\left| \int_0^y (g_y - v) \frac{\partial^4 f_{x,y}(0,v)}{\partial u^3 \partial v} \, dv \right| \leq \frac{C_3 h^4 h_y^2}{2} |\zeta_y|^2. \tag{29}
\]

Using the definition of \(\gamma_{g_x,g_y}^2(f_{x,y})\) in (22) and taking \(x = y\) in the inequalities (26)–(29), it follows that

\[
\limsup_{n \to \infty} \frac{1}{|V_n|^2} \sum_{x} \left( h_x^{-2} \gamma_{g_x,g_x}^2(f_{x,y}) \right)^2 \leq \lim_{n \to \infty} \frac{C}{|V_n|^2} \sum_{x} h_x^2 (E|\zeta_x|^5)^2 = 0,
\]

where \(C := (C_{21} + C_{32} h^2 + C_{31} h + C_{13}) h^6 / 4.\) The last equality follows from (4) and of the assumption that the \(\zeta_x\)'s are identically distributed and satisfy \(E|\zeta_x|^5 < \infty.\) Therefore, the limit of the left side of (24) follows.

Similarly, using the definition of \(\gamma_{g_x,g_y}^2(f_{x,y})\) in (22) and the inequalities (26)–(29), it is proved that

\[
\limsup_{n \to \infty} \frac{1}{|V_n|^2} \sum_{x,y} |\gamma_{g_x,g_y}^2(f_{x,y})| \leq \lim_{n \to \infty} \frac{\tilde{C}}{|V_n|^2} \sum_{x,y} h_x^2 h_y^2 (E|\zeta_x|^3 + E|\zeta_y| + 2) = 0,
\]

where \(\tilde{C} := \max\{C_{21}, C_{32} h^2, C_{31} h, C_{13} h\} h^3 / 2.\) Again, the last equality follows by hypothesis (4) and of the assumption that the \(\zeta_x\)'s are identically distributed and satisfy \(E|\zeta_x|^3 < \infty.\) Then the limit of the right side of (21) is valid and the proof of lemma is complete. \(\square\)

The proof of Proposition 4.3 plays an important role in the proof of the following result.

**Proposition 4.6.** For any \(n\) and any \((\beta, h) \in (0, \infty)^2,\)

\[
\frac{1}{4h^2} \sum_{x,y \in V_n} h_x^2 h_y^2 \mathbb{E} \frac{\partial^2 F_n}{\partial g_x \partial g_y^2} \leq 2|V_n| \sqrt{(\alpha_n |V_n|^2 + \theta(n)) h + \sum_x \ell^2(x,x) - \theta_1(n)},
\]

with \(\alpha_n\) as in Proposition 4.3, \(\theta(\cdot)\) as in Lemma 4.1, \(\ell(\cdot, \cdot)\) as in Proposition 4.2 and \(\theta_1(n) = \sum_{x,y \in V_n} h_x h_y \ell(x,y).\)

**Proof.** As a sub-product of the proof of Proposition 4.3 we have

\[
\sum_{x,y} h_x^2 h_y^2 \mathbb{E} \frac{\partial^2 F_n}{\partial g_x \partial g_y} \leq 2|V_n| \sqrt{(\alpha_n |V_n|^2 + \theta(n)) h + \sum_x \ell^2(x,x) - \theta_1(n)},
\]

with \(\ell(x,y) = (h_x h_y)^{-1} \gamma_{g_x,g_y}^2(f_{x,y}).\)
On the other hand, we claim that
\[
\frac{\partial^4 F_n}{\partial g_x^2 \partial g_y^2} \leq 4h^2 \frac{\partial^2 F_n}{\partial g_x \partial g_y}.
\]
Indeed, a straightforward computation shows that \(\frac{\partial \langle \sigma_x \sigma_y \rangle}{\partial g_x} = -2h \langle \sigma_x \rangle \langle \sigma_y \rangle\). Then, using the identity (23), we have
\[
\frac{\partial^4 F_n}{\partial g_x^2 \partial g_y^2} = 4h^2 \left( \langle \sigma_x \rangle \langle \sigma_y \rangle - \frac{1}{2} \right) \frac{\partial^2 F_n}{\partial g_x \partial g_y}.
\]
Since \(\langle \sigma_x \rangle \leq 1\) for all \(x\), and the derivative \(\frac{\partial^2 F_n}{\partial g_x \partial g_y}\) is non-negative, the claim follows.

Finally, combining (30) and (31) the proof of proposition follows. \(\square\)

Next, define
\[
\Delta_n = \Delta_n(\sigma) := \frac{1}{|V_n|} \sum_{x \in V_n} g_x \sigma_x.
\]
That is, \(\Delta_n\) is the part of the energy due to the disorder.

**Remark 4.7.** Note that the absolute value of covariance between \(\Delta_n(\sigma^i)\) and \(\Delta_n(\sigma^s)\) doesn’t grow faster than the absolute value of overlap \(R_{i,s}\) between two replicas \(\sigma^i, \sigma^s\).

**Remark 4.8.** Since \(\langle \Delta_n \rangle = \frac{\partial \psi_n}{\partial h}, \nu(\Delta_n) = \frac{\partial p_n}{\partial h}\), \text{Var}(F_n) \leq (C h |V_n| + \theta(n))h\) (see Lemma 4.11) and \(p = \lim_{n \to \infty} p_n\) exists and is finite for all \((\beta, h)\), by convexity arguments of the function \(h \mapsto \psi_n(\beta, h)\) it follows that: for any \((\beta, h) \in \mathcal{A}^c\),
\[
\nu(\Delta_n) \to \frac{\partial p}{\partial h}(\beta, h), \quad \mathbb{E}\langle \Delta_n \rangle - \nu(\Delta_n) \to 0.
\]
For more details see Lemma 2.7 in [15].

**Lemma 4.9.** For any \((\beta, h) \in \mathcal{A}^c\),
\[
\nu(\{\Delta_n - \nu(\Delta_n)\}) \to 0.
\]

**Proof.** Let \(\langle \cdot \rangle_{g_x=u, g_y=v}\) be the Gibbs expectation defined by setting \(g_x\) and \(g_y\) in \(\langle \cdot \rangle\) to be \(u\) and \(v\) respectively, and \(F_{x,y}^*(u, v) := \langle \sigma_x; \sigma_y \rangle_{g_x=u, g_y=v}\). A generalized Gaussian integration by parts (see Proposition 4.14 in Appendix), with \(f_{x,y}^*(u, v) = \mathbb{E}F_{x,y}^*(u, v)\), gives
\[
\mathbb{E}g_x g_y f_{x,y}^* = h_x^2 h_y^2 \mathbb{E} \frac{\partial^2 f_{x,y}^*}{\partial g_x \partial g_y} + \gamma_{g_x g_y}^2 (f_{x,y}^*),
\]
where $\gamma_{g_x,g_y}(f_x^*, y)$ is defined analogously as in (22) with $f_x^*, y$ instead of $f_{x,y}$. Dividing this equality by $|V_n|^2$ and summing over all $x, y \in V_n$, and using (23), we obtain

\begin{equation}
\mathbb{E}\left(\langle \Delta_n^2 \rangle - \langle \Delta_n \rangle^2 \right) = \frac{1}{h^2|V_n|^2} \sum_{x,y} \left( h_x^2 h_y^2 \mathbb{E} \frac{\partial^4 F_n}{\partial g_x^2 \partial g_y^2} + h^2 \gamma_{g_x,g_y}^2(f_x^*) \right).
\end{equation}

By Proposition 4.6, the expression (32) is at most

\begin{equation}
8 \sqrt{\left( \alpha_n + \frac{\theta(n)}{|V_n|^2} \right) h + \frac{1}{|V_n|^2} \sum_x \ell^2(x, x) - \frac{4\theta_1(n)}{|V_n|^2} + \frac{1}{|V_n|^2} \sum_{x,y} \gamma_{g_x,g_y}^2(f_x^*)},
\end{equation}

where $\ell(x, y) = (h_x h_y)^{-1} \gamma_{g_x,g_y}^2(F)$. That is,

\begin{equation}
\mathbb{E}\left(\langle \Delta_n^2 \rangle - \langle \Delta_n \rangle^2 \right) \leq 8 \sqrt{\left( \alpha_n + \frac{\theta(n)}{|V_n|^2} \right) h + \frac{1}{|V_n|^2} \sum_x \ell^2(x, x) - \frac{4\theta_1(n)}{|V_n|^2} + \frac{1}{|V_n|^2} \sum_{x,y} \gamma_{g_x,g_y}^2(f_x^*)}. \tag{33}
\end{equation}

Items (24) and (25) show that $\theta(n) = o(|V_n|^2)$, $\theta_1(n) = o(|V_n|^2)$ and $\sum_x \ell^2(x, x) = o(|V_n|^2)$. Analogously to the proof of Items (24)-(25), using Items (45)-(47) of Proposition 4.13 in Appendix, it is verified that $\sum_{x,y} \gamma_{g_x,g_y}^2(f_{x,y}) = o(|V_n|^2)$. Therefore, since $\alpha_n = o(1)$ (see Proposition 4.13), $\mathbb{E}\left(\langle \Delta_n^2 \rangle - \langle \Delta_n \rangle^2 \right) = o(1)$.

Combining (33) with the inequality

\begin{equation}
\nu(|\Delta_n - \langle \Delta_n \rangle|) \leq \sqrt{\mathbb{E}\left(\langle \Delta_n^2 \rangle - \langle \Delta_n \rangle^2 \right)},
\end{equation}

and after using the convergences mentioned above one finds that $\nu(|\Delta_n - \langle \Delta_n \rangle|)$ converges to 0 as $n \to \infty$. Since $(\beta, h) \in \mathcal{A}^c$, the proof follows by Remark 4.18.

\textbf{ Remark 4.10.} In [3], [34] and [33], by using different techniques, the authors proved the following general result

\begin{equation}
\nu\left(|H_n| - \nu\left(H_n\right)\right) \to 0,
\end{equation}

where $H_n$ is the Hamiltonian (2) of the RFIM. Under an ergodic or mixing hypothesis a straightforward computation shows that this result implies our Lemma 4.9.

\textbf{ Remark 4.11.} Taking $h_x = \pm 1$ and $\zeta_x \sim N(0, 1)$ for all $x$, in Lemma 4.9, we recovered the proof of Lemma 2.9 in [12] using classic inequalities and the essential inequality (31) instead of using the Hermite polynomials, as was done in Lemma 2.8 of [13].

Take any integer $m \geq 2$ and let $\sigma^1, \ldots, \sigma^m, \sigma^{m+1}$ denote $m + 1$ spin configurations drawn independently from the Gibbs measure. Let $R_{l,s}$ the overlap between $\sigma^l$ and $\sigma^s$ defined in (7), with $l, s = 1, \ldots, m + 1$. Let $f : \mathbb{R}^{m(m-1)/2} \to [-1, 1]$ be a bounded measurable function of these overlaps that do not change with $n$. 

\textbf{ Remark 4.12.}
Lemma 4.12 (Ghirlanda-Guerra identities). Consider the RFIM defined by the Gibbs measure in (1). Then, the identity (3) is satisfied at almost all \((\beta, h)\). That is, if \(f\) is as above,
\[
\nu(fR_{1,m+1}) = \frac{1}{m} \nu(f)\nu(R_{1,2}) - \frac{1}{m} \sum_{s=2}^{m} \nu(fR_{1,s}) \not\to 0,
\]
for each \((\beta, h)\) in \(A^c\).

Proof. Let \(\langle \cdot \rangle_{g_x=u}\) be the Gibbs expectation defined by setting \(g_x\) in \(\langle \cdot \rangle\) to be \(u\) and \(F_x(u) := \langle \sigma_x^1 f \rangle_{g_x=u}\). Using (30), a straightforward calculus show that
\[
\frac{\partial^j F_x(u)}{\partial u^j} = h^j \left( \sigma_x^1 \cdot \left( \sum_{s=1}^{m} \sigma_x^s - m\sigma_x^{m+1} \right) \right)_{g_x=u}, \quad j = 1, 2, \ldots.
\]
A generalized Gaussian integration by parts (see Proposition 4.13 in Appendix), with \(f_x(u) := E F_x(u)\), gives
\[
E g_x f_x - h^2 \frac{d f_x}{d g_x} = \gamma^2_{g_x}(f_x),
\]
where \(\gamma^2_{g_x}(f_x) = E (g_x f_x - u) \frac{d f_x}{d u} du - h^2 E \left( \int_0^{g_x} (g_x - u) \frac{d^2 f_x}{d u^2} du \right)\). Dividing the above equality by \(|V_n|\) and summing over all \(x \in V_n\), and using (34) with \(j = 1\), we have
\[
\nu(\Delta_n(\sigma^1)f) - h \nu \left( \sum_{s=1}^{m} R_{1,s} - mR_{1,m+1} \right) \frac{f}{|V_n|} \sum_{x \in V_n} \gamma^2_{g_x}(f_x).
\]
Since \(|\frac{\partial^j F_x(u)}{\partial u^j}| \leq (2mh)^j \|f\|_\infty\), it follows that \(\left| \int_0^{g_x} (g_x - u) \frac{d^2 f_x}{d u^2} du \right| \leq 4(mh)^3 \|f\|_\infty h^2_x \zeta^2_x\)
and, by Item 43 of Proposition 4.13 that
\[
\left| g_x f_x - u \right| \left| \frac{d^2 f_x}{d u^2} \right| \|f\|_\infty \|h_x\|^3 |\zeta_x|^3.
\]
Then,
\[
\limsup_{n \to \infty} \sup_{\|f\|_\infty} \left| \frac{1}{|V_n|} \sum_{x \in V_n} \gamma^2_{g_x}(F_x) \right| \leq \lim_{n \to \infty} \frac{(1+mh)(2mh)^3 \|f\|_\infty}{|V_n|} \sum_{x \in V_n} \|h_x\|^3 |\zeta_x|^3 = 0.
\]
Here, the last equality follows from (14) and of the assumption that the \(\zeta_x\)'s are identically distributed and satisfy \(E|\zeta_x|^3 < \infty\). Therefore, in (35), follows that
\[
\lim_{n \to \infty} \sup_{\|f\|_\infty} \left| \nu(\Delta_n(\sigma^1)f) - h \nu \left( \sum_{s=1}^{m} R_{1,s} - mR_{1,m+1} \right) \right| = 0.
\]
Since \(\nu(|\Delta_n - \nu(\Delta_n)|) \not\to 0\) for any \((\beta, h) \in A^c\) (see Lemma 4.9), it is well-known (see e.g. [39], Section 2.12) that (36) is sufficient to guarantee the validity of the Ghirlanda-Guerra identities (8). The proof of lemma is complete. \(\square\)
Proof of Theorem 1. The proof follows the same path as in [15] and we present it for the sake of completeness. Let \( q_{\beta,h,n} := \nu(R_{1,2}) \). Taking \( f = 1 \) and \( m = 1 \) in (35) we obtain
\[
\nu(\Delta_n(\sigma^1)) = h(1 - q_{\beta,h,n}) + \frac{1}{|V_n|} \sum_{x} \gamma_{g_x}^2(f_x), \quad \frac{1}{|V_n|} \sum_{x} \gamma_{g_x}^2(f_x) \to 0.
\]
On the other hand, choosing \( m = 2 \) and \( f = R_{1,2} \) in Lemma 4.12 gives
\[
\nu(R_{1,2}R_{1,3}) - \frac{1}{2} q_{\beta,h,n}^2 - \frac{1}{2} \nu(R_{1,2}^2) \to 0.
\]
Choosing \( m = 3 \) and \( f = R_{2,3} \) gives
\[
\nu(R_{2,3}R_{1,4}) - \frac{1}{3} q_{\beta,h,n}^2 - \frac{1}{3} \sum_{s=0}^{3} \nu(R_{2,3}R_{1,s}) \to 0.
\]
By symmetry between replicas, \( \nu(R_{2,3}R_{1,2}) = \nu(R_{2,3}R_{1,3}) = \nu(R_{1,2}R_{1,3}) \), then, we can multiply (38) by 2/3 and add to (39) to get
\[
\frac{2}{3} \nu(R_{1,2}^2) - q_{\beta,h,n}^2 - \mathbb{E}\left( (R_{1,2}^2) - (R_{2,3}R_{1,4}) \right) \to 0.
\]
Seeing that the sequence \( (\sigma^i) \) is an independent sequence under Gibbs’ measure, \( \langle R_{2,3}R_{1,4} \rangle = \frac{1}{|V_n|} \sum_{x,y \in V_n} h_x^2 h_y^2 \langle \sigma_x^2 \sigma_y^3 \sigma_x^1 \sigma_y^4 \rangle = \langle R_{1,2}^4 \rangle \). Combining this with (40) and after using Lemma 4.15 we have \( \nu(R_{1,2}^2) - q_{\beta,h,n}^2 \to 0 \). Therefore,
\[
\nu((R_{1,2} - q_{\beta,h,n})^2) = \nu((R_{1,2} - \nu(R_{1,2}))^2) = \nu(R_{1,2}^2) - q_{\beta,h,n}^2 \to 0.
\]
Note that by (37) and by Remark 4.8 \( q_{\beta,h} := \lim_{n \to \infty} q_{\beta,h,n} = 1 - \frac{1}{n} \frac{\partial}{\partial h} \nu(\beta,h) \) exists. Therefore, taking \( n \to \infty \) in the inequality
\[
\nu((R_{1,2} - q_{\beta,h})^2) \leq 2 \nu((R_{1,2} - q_{\beta,h,n})^2) + 2(q_{\beta,h,n} - q_{\beta,h})^2
\]
and using (41), the proof of Theorem 1 follows. \( \square \)

Appendix

The proof of the next result appears in Chen (2019) [17], Proposition 6.1.

Proposition 4.13. Let \( Y \) be a real-valued random variable with zero-mean and finite-variance \( \sigma^2 \), with \( \sigma > 0 \). For any function \( f : \mathbb{R} \to \mathbb{R} \) with a bounded continuous third-order derivative, we have
\[
\mathbb{E} Y f(Y) = \sigma^2 \mathbb{E} f'(Y) + \gamma_Y^2(f),
\]
where
\[
\gamma_Y^2(f) := \mathbb{E} \left( Y \int_0^Y (Y - u) f''(u) \, du \right) - \sigma^2 \mathbb{E} \left( \int_0^Y (Y - u) f'''(u) \, du \right),
\]
with
\[
\left| Y \int_0^Y (Y - u) f''(u) \, du \right| \leq |Y| \int_0^Y \min \{ 2 \| f' \|_\infty, \| f'' \|_\infty \} \, du.
\]
The next result is new and can be seen as a generalization of Proposition 4.13 for the bivariate case. In order to lighten the notation we will write \( \partial_{i,j} f \) to denote the partial derivative of order \( i \) and \( j \) for the first and second component respectively.

**Proposition 4.14** (A generalized Gaussian integration by parts). Let \( X \) and \( Y \) be two independent real-valued random variables with zero-mean and finite-variances \( \sigma_X^2 \) and \( \sigma_Y^2 \) respectively, with \( \sigma_X, \sigma_Y \) both positive. For any function \( f : \mathbb{R}^2 \to \mathbb{R} \) with a bounded continuous fifth-order derivative, we have

\[
\begin{align*}
\mathbb{E}_{XY} f(X,Y) &= \sigma_X^2 \sigma_Y^2 \mathbb{E}_{XY} \partial_{1,1} f(X,Y) + \gamma_{X,Y}^2(f), \\
\end{align*}
\]

where
\[
\gamma_{X,Y}^2(f) := \mathbb{E} \left( XY \int_0^X \int_0^Y (X-u) \partial_{2,1} f(u,v) \, du \, dv \right) - \sigma_X^2 \sigma_Y^2 \mathbb{E} \left( \int_0^X \int_0^Y (X-u) \partial_{3,2} f(u,v) \, du \, dv \right) - \sigma_X^2 \sigma_Y^2 \mathbb{E} \left( \int_0^X (X-u) \partial_{3,1} f(u,0) \, du + \int_0^Y (Y-v) \partial_{1,3} f(0,v) \, dv \right).
\]

Furthermore,
\[
\begin{align*}
&\left| \int_0^X \int_0^Y (X-u) \partial_{2,1} f(u,v) \, du \, dv \right| \leq |X||Y| \int_0^X \int_0^Y \min \left\{ 2 \| \partial_{2,1} f \|_\infty, \| \partial_{3,1} f \|_\infty \right\} \, du \, dv, \\
&\left| \int_0^X (X-u) \partial_{3,1} f(u,0) \, du \right| \leq \int_0^X \min \left\{ 2 \| \partial_{2,1} f \|_\infty, \| \partial_{3,1} f \|_\infty \right\} \, du, \\
&\left| \int_0^Y (Y-v) \partial_{1,3} f(0,v) \, dv \right| \leq \int_0^Y \min \left\{ 2 \| \partial_{1,2} f \|_\infty, \| \partial_{1,3} f \|_\infty \right\} \, dv.
\end{align*}
\]

**Proof.** Taylor's Theorem for multivariate functions gives,

\[
XY f(X,Y) = XY f(0,0) + X^2 Y \partial_{1,0} f(0,0) + XY^2 \partial_{0,1} f(0,0) + \frac{X^2 Y}{2} \partial_{2,0} f(0,0) + \frac{XY^2}{2} \partial_{0,2} f(0,0) + X^2 Y^2 \partial_{1,1} f(0,0) \\
+ XY \int_0^X \int_0^Y \frac{(X-u)^2}{2} \partial_{3,0} f(u,0) \, du + XY \int_0^Y \frac{(Y-v)^2}{2} \partial_{0,3} f(0,v) \, dv \\
+ X^2 Y \int_0^Y (Y-v) \partial_{1,2} f(0,v) \, dv + XY \int_0^Y \int_0^X (X-u) \partial_{2,1} f(u,v) \, du \, dv \\
+ \sigma_X^2 \sigma_Y^2 \left( \partial_{1,1} f(X,Y) - \partial_{1,1} f(0,0) - X \partial_{2,1} f(0,Y) - Y \partial_{1,2} f(0,0) \\
- \int_0^X (X-u) \partial_{3,1} f(u,0) \, du - \int_0^Y (Y-v) \partial_{1,3} f(0,v) \, dv + R(X,Y) \right),
\]

where
\[
R(X,Y) := -\partial_{1,1} f(X,Y) + \partial_{1,1} f(0,0) + X \partial_{2,1} f(0,Y) + Y \partial_{1,2} f(0,0) \\
+ \int_0^X (X-u) \partial_{3,1} f(u,0) \, du + \int_0^Y (Y-v) \partial_{1,3} f(0,v) \, dv.
\]
A simple observation shows that
\[
R(X,Y) = \int_0^X (X - u) (\partial_{3,1} f(u,Y) - \partial_{3,1} f(u,0)) \, du
\]
\[(48)\]
\[- \int_0^X \int_0^Y (X - u) \partial_{3,2} f(u,v) \, dudv.\]

Since \(X, Y\) are independent random variables, \(\mathbb{E}X = \mathbb{E}Y = 0\) and \(\mathbb{E}X^2 = \sigma_X^2, \mathbb{E}Y^2 = \sigma_Y^2\), one finds that
\[
\mathbb{E}XY f(X,Y)
= \sigma_X^2 \sigma_Y^2 \mathbb{E} \partial_{1,1} f(X,Y) + \sigma_X^2 \sigma_Y^2 \mathbb{E} R(X,Y)
+ \mathbb{E}\left( XY \int_0^X \int_0^Y (X - u) \partial_{2,1} f(u,v) \, dudv \right)
- \sigma_X^2 \sigma_Y^2 \mathbb{E}\left( \int_0^X (X - u) \partial_{3,1} f(u,0) \, du + \int_0^Y (Y - v) \partial_{1,3} f(0,v) \, dv \right).
\]

Then, using (48) the proof of (44) follows. On the other hand, the Items (45), (46) and (47) follow by combining each of the following identities
\[
\int_0^X \int_0^Y (X - u) \partial_{2,1} f(u,v) \, dudv = \int_0^X \int_0^Y (\partial_{1,1} f(u,v) - \partial_{1,1} f(0,v)) \, dudv,
\]
\[
\int_0^X (X - u) \partial_{3,1} f(u,0) \, du = \int_0^X (\partial_{2,1} f(u,0) - \partial_{2,1} f(0,0)) \, du,
\]
\[
\int_0^Y (Y - v) \partial_{1,3} f(0,v) \, dv = \int_0^Y (\partial_{1,2} f(0,v) - \partial_{1,2} f(0,0)) \, dv,
\]
with the Mean-Value Theorem.

\[\square\]

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