New constructions of quaternary bent functions

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Abstract In this paper, a new construction of quaternary bent functions from quaternary quadratic forms over Galois rings of characteristic 4 is proposed. Based on this construction, several new classes of quaternary bent functions are obtained, and as a consequence, several new classes of quadratic binary bent and semi-bent functions in polynomial forms are derived. This work generalizes the recent work of N. Li, X. Tang and T. Helleseth.

Keywords Galois ring · Teichmüller set · Quaternary quadratic form · Quaternary bent function · Bent and semi-bent function

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1 Introduction

Let $F_{2^n}$ be the finite field with $2^n$ elements, where $n$ is a positive integer. Any map from $F_{2^n}$ to the integer residue ring $Z_q$ is called an $n$-variable $Z_q$-valued Boolean function. Particularly, $Z_2$-valued Boolean functions are just the usual binary Boolean functions and $Z_4$-valued Boolean functions are also known as quaternary Boolean functions.

For an $n$-variable $Z_q$-valued Boolean function $f$, its Fourier transform at any $a \in F_{2^n}$ is defined as

$$\hat{f}(a) = \sum_{x \in F_{2^n}} \zeta_q^{f(x)} (-1)^{\text{tr}_1^n(ax)},$$

where $\zeta_q$ is a $q$-th complex primitive root of unity and “$\text{tr}_1^n(\cdot)$” denotes the trace function from $F_{2^n}$ to $F_2$, i.e. $\text{tr}(x) = \sum_{i=0}^{n-1} x_{2^i}$. $f$ is called a $Z_q$-valued (generalized) bent function if its Fourier transform has a constant magnitude, or more precisely, $|\hat{f}(a)| = 2^{n/2}$ for any $a \in F_{2^n}$. In the case $q = 2$, the Fourier
transform of $f$ is always known as its Walsh transform and $f$ is just the so-called bent function if it is a $\mathbb{Z}_2$-valued bent function.

$\mathbb{Z}_q$-valued bent functions were introduced by Schmidt in [13] when seeking for good codes for multi-code code-division multiple access (MC-CDMA) systems. In fact, a $\mathbb{Z}_q$-valued bent function corresponds to a code that can reduce the peak-to-average power ratio (PAPR) [5] in such systems to the lowest possible value (called a constant-amplitude code). As a result, constructions of $\mathbb{Z}_q$-valued bent functions will promise useful objects in communication systems.

Generally speaking, $q$ is often chosen to be a power of 2 in applications and the simplest case is $q = 2$. But there are a main drawback with binary bent functions that they only exist for even number of variables. To avoid this drawback, a lot of attention has been paid to quaternary bent functions as they exist for both even and odd number of variables, and constructions of them have been extensively studied.

The main technique to construct quaternary bent functions is to construct certain trace forms of Galois rings of characteristic 4 and consider their limitations to the Teichmüller sets since the Teichmüller sets are isomorphic to the finite fields under certain multiplications and additions. For example, in his Ph.D thesis [11], Schmidt considered quaternary Boolean functions of the form

$$Q(x) = \varepsilon + \text{Tr}_1^n(ax + 2bx^3), \ x \in \mathbb{T},$$

where $\varepsilon \in \mathbb{Z}_4$, $a, b \in \text{GR}(4,n)$ and “$\text{Tr}_1^n(\cdot)$” represents the trace function from $\text{GR}(4,n)$ to $\mathbb{Z}_4$ (we denote the Galois ring with $4^n$ elements by $\text{GR}(4,n)$ and its Teichmüller set by $\mathbb{T}$). He reduced the conditions under which $Q$ was a quaternary bent function to the existence of roots of certain cubic equations over $\mathbb{F}_{2^n}$. Very recently, Li et al. studied quaternary Boolean functions of the form

$$Q(x) = \text{Tr}_1^n \left( x + 2 \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} c_i x^{1+2^i} \right), \ x \in \mathbb{T},$$

where $k$ is any positive integer and $c_i \in \mathbb{Z}_2$, $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$. It was proved that $Q$ is quaternary bent if and only if $\text{gcd}(c(x^k), x^n - 1) = 1$ for

$$c(x) = 1 + \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} c_i (x^i + x^{n-i}) \in \mathbb{F}_2[x].$$

Moreover, several classes of $c(x)$ satisfying such a condition were constructed and thus several new classes of quaternary bent function were obtained.

In this paper, we devote to generalizing Li et al.’s work. We consider quaternary Boolean functions of the form

$$Q(x) = \text{Tr}_1^n \left( \alpha x + 2 \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} c_i \beta x^{1+2^i} \right), \ x \in \mathbb{T},$$

where $n = em$, $\alpha, \beta \in \mathbb{T}$ and $c_i \in \mathbb{Z}_2$, $1 \leq i \leq \lfloor \frac{n-1}{2} \rfloor$. For some special choices of $\alpha$ and $\beta$, we can derive the conditions for $Q$ to be bent. Furthermore, we explicitly construct several classes of coefficients sets $\{c_i\}$ meeting this condition, some of which can be implied by Li et al.’s constructions.
On the other hand, by virtue of the connections between quaternary bent functions and binary bent and semi-bent functions deduced by Stănică et al. [15], we further derive new classes of binary bent or semi-bent functions, respectively, of the form

\[ f_Q(x) = p(\alpha x) + \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} c_i \text{tr}_1^n(\beta x^{1+2^{2i}}), \quad x \in \mathbb{F}_{2^n}, \]

according as \( n \) is even or odd, respectively, where

\[ p(x) = \begin{cases} \sum_{i=1}^{\frac{n-1}{2}} \text{tr}_1^n(x^{1+2^i}) + \text{tr}_1^{n/2}(x^{1+2^{n/2}}) & \text{if } n \text{ is even,} \\ \sum_{i=1}^{\frac{n+1}{2}} \text{tr}_1^n(x^{1+2^i}) & \text{if } n \text{ is odd.} \end{cases} \]

They are all quadratic bent or semi-bent functions in polynomial forms [1].

The rest of the paper is organized as follows. In Section 2, we recall some necessary backgrounds on bent functions, Galois rings and quadratic forms over them. In Section 3, our constructions of quaternary bent functions are proposed, based on which some new classes of bent and semi-bent functions are derived in Section 4. Concluding remarks are given in Section 5.

2 Preliminaries

2.1 Quaternary bent functions and binary (semi-)bent functions

Recall that an \( n \)-variable quaternary Boolean function \( f \) is called bent if \( |\hat{f}(a)| = 2^{n/2} \) for any \( a \in \mathbb{F}_{2^n} \), where

\[ \hat{f}(a) = \sum_{x \in \mathbb{F}_{2^n}} f(x)(-1)^{\text{tr}_1^n(ax)}, \]

\( i = \sqrt{-1} \). An \( n \)-variable binary Boolean function \( g \) is called bent [10] if \( |\hat{g}(a)| = 2^{n/2} \) for any \( a \in \mathbb{F}_{2^n} \), where

\[ \hat{g}(a) = \sum_{x \in \mathbb{F}_{2^n}} (-1)^{g(x)+\text{tr}_1^n(ax)}. \]

Besides, \( g \) is called semi-bent if \( |\hat{g}(a)| \in \{0, 2^{(n+2)/2}\} \) [2]. It is easy to derive that binary bent functions only exist for even \( n \), while quaternary bent functions exist for both even and odd \( n \).

As every element of \( \mathbb{Z}_4 \) has a 2-adic expansion, there exist two binary Boolean functions \( f_0 \) and \( f_1 \) such that \( f = f_0 + 2f_1 \) for any quaternary Boolean function \( f \). Under this representation, connections between quaternary bent functions and binary bent and semi-bent functions are obtained by Stănică et al. recently.

**Theorem 1** ([15]) Let \( f(x) \) be an \( n \)-variable quaternary Boolean function and \( f(x) = f_0(x) + 2f_1(x) \) be its 2-adic expansion. Denote \( \Phi(f)(y,z) = f_0(y)z + f_1(y) \) with \( y \in \mathbb{F}_{2^n} \) and \( z \in \mathbb{F}_2 \), which can be viewed as an \((n+1)\)-variable Boolean function over \( \mathbb{F}_{2^n} \times \mathbb{F}_2 \). Then

\( 1) |\hat{f}(a)|^2 = (|\hat{f}_0(a)|^2 + |\hat{f}_1(a)|^2) / 2 \) for any \( a \in \mathbb{F}_{2^n} \);
(2) \( f(x) \) is quaternary bent if and only if \( f_1(x) \) and \( f_0(x) + f_1(x) \) are both binary bent or semi-bent, respectively, according as \( n \) is even or odd, respectively.

(3) \( \phi(f)(y,z) \) is binary bent or semi-bent, respectively, according as \( n \) is odd or even, respectively, if \( f(x) \) is quaternary bent.

Remark 1 The binary Boolean function \( \phi(f) \) in Theorem [1] is often called the Gray image of the quaternary Boolean function \( f \).

2.2 Galois ring of characteristic 4

A Galois ring is a Galois extension of an integer residue ring with a prime power moduli, and this prime power is called the characteristic of it. Since we will only focus on the quaternary case, we just recall some basic results of Galois rings of characteristic 4. For their proofs, we refer to [9].

The Galois ring \( GR(4,n) \) with \( 4^n \) elements is an \( n \)-th Galois extension of \( \mathbb{Z}_4 \). In fact, the Galois theory of ring extensions are much like that of field extensions. More precisely, to obtain \( GR(4,n) \), we can add (formal) roots of a monic basic irreducible polynomial of degree \( n \) over \( \mathbb{Z}_4 \). Here a basic irreducible polynomial over \( \mathbb{Z}_4 \) is a polynomial whose modulo 2 reduction is a primitive polynomial over \( \mathbb{F}_2 \). Assume \( \xi \) is a root of this monic basic irreducible polynomial of order \( 2^n - 1 \) (i.e. \( \xi^{2^n-1} = 1 \)), then \( GR(4,n) \cong \mathbb{Z}_4[\xi] \). The set \( \mathbb{T} = \{0, 1, \xi, \ldots, \xi^{2^n-2}\} \) is called the Teichm"uller set of \( GR(4,n) \). It is just the roots set of the equation \( x^{2^n} - x = 0 \) in \( GR(4,n) \). It is obvious that \( \mathbb{T}^* = \mathbb{T}\setminus\{0\} \) forms a cyclic group under the multiplication of \( GR(4,n) \), which is isomorphic to \( \mathbb{F}_{2^n}^* \), the multiplicative group of the finite field \( \mathbb{F}_{2^n} \). However, \( \mathbb{T} \) does not form a group under the addition of \( GR(4,n) \) since it is not closed under this operation. To make \( \mathbb{T} \) into an additive group, we can introduce a new operation \( \oplus \) defined by

\[
x \oplus y = x + y + 2\sqrt{xy}
\]

for any \( x, y \in \mathbb{T} \) (here \( \sqrt{xy} \) denotes \( (xy)^{2^n-1} \)). It can be proved that under the multiplication of \( GR(4,n) \) and the addition \( \oplus \), \( \mathbb{T} \) forms a field which is isomorphic to the finite field \( \mathbb{F}_{2^n} \). Besides, if we denote by \( \mu \) the modulo 2 reduction map, we have \( \mu(\mathbb{T}) = \mathbb{F}_{2^n} \).

Every element \( z \in GR(4,n) \) can be uniquely represented as \( z = x + 2y \) where \( x, y \in \mathbb{T} \). Under this representation, the trace function from \( GR(4,n) \) to \( \mathbb{Z}_4 \) is defined as

\[
\text{Tr}^n_i(z) = \sum_{j=0}^{n-1} (x^j + y^j).
\]

The trace function over \( GR(4,n) \) and that over \( \mathbb{F}_{2^n} \) are related via the map \( \mu \) as follows:

(1) \( \mu \left( \text{Tr}^n_i(z) \right) = \text{tr}^n_i(\mu(z)) \);

(2) \( \text{Tr}^n_i(2z) = 2\text{Tr}^n_i(z) = 2\text{tr}^n_i(\mu(z)) \).

2.3 Quaternary quadratic form

A quaternary quadratic form \( F \) is a mapping from \( \mathbb{T} \) to \( \mathbb{Z}_4 \) satisfying \( F(0) = 0 \) and

\[
F(x \oplus y) = F(x) + F(y) + 2B(x,y),
\]

where \( B(x,y) \) is a quadratic function in \( \mathbb{T} \). The following properties hold:

(1) If \( f(x) \) is binary quadratic, then \( \phi(f)(y,z) \) is binary bent or semi-bent according as \( n \) is odd or even, respectively.

(2) \( \phi(f)(y,z) \) is quaternary bent if and only if \( f(\xi) \) is binary bent or semi-bent, respectively, according as \( n \) is even or odd, respectively.
where $B$ is a symmetric bilinear form on $\mathbb{T}$, i.e. $B$ can induce a map $\mathbb{T} \times \mathbb{T} \to \mathbb{Z}_4$ such that $B(x, y) = B(y, x)$ and $B(x \oplus y, z) = B(x, z) + B(y, z)$. $B$ is often called the associate bilinear form of $F$. The rank of $F$ is defined as the codimension of the radical space of $B$, $\text{rad}(B)$, over $\mathbb{F}_2$, where

$$\text{rad}(B) = \{ x \in \mathbb{T} \mid B(x, y) = 0 \text{ for any } y \in \mathbb{T} \}$$

(it is easy to see that $\text{rad}(B)$ is a vector space over $\mathbb{F}_2$).

Clearly a quaternary quadratic form can be viewed as a quaternary Boolean function. Direct computations show that its Fourier transform can be completely determined by its rank. Details of computing exponential sums over Galois rings can be found in [12,14]. We just involve a main result here.

**Theorem 2** ([14]) A quaternary quadratic form $F$ is quaternary bent if and only if it is of full rank, or equivalently, $\text{rad}(B) = \{0\}$ where $B$ is the associate bilinear form of $F$.

### 3 New constructions of quaternary bent functions

In the rest part of the paper, we assume $n = em$, $e \geq 1$ and fix a positive integer $k$. Denote by $\mathbb{T}_e$ the set of nonzero elements in $\mathbb{T}$ satisfying $x^{2^e} = x$, where $\mathbb{T}$ is the Teichmüller set of $\text{GR}(4,n)$. In the following, we study bentness of a special quaternary quadratic form in $n$ variables.

**Theorem 3** Assume $\alpha, \beta \in \mathbb{T}_e$ and $\beta = \alpha^2$. Let

$$Q(x) = \text{Tr}_e^n (\alpha x + 2 \sum_{i=1}^{\lceil \frac{m-1}{2} \rceil} c_i x + 1 + 2^{x+k})$$

where $c_i \in \mathbb{Z}_2$, $1 \leq i \leq \lceil \frac{m-1}{2} \rceil$. Then $Q(x)$ is a quaternary bent function if and only if $\gcd(c(x^k), x^m - 1) = 1$ where

$$c(x) = 1 + \sum_{i=1}^{\lceil \frac{m-1}{2} \rceil} c_i x^i + x^{m-i} \in \mathbb{F}_2[x].$$

It is obvious that when $e = 1$ and $\alpha = 1$, Theorem 3 coincides with [7, Theorem 1]. Hence the construction of quaternary bent functions in Theorem 3 can be viewed as a generalization of that in [7] as long as $n$ is not a prime.

To prove Theorem 3 we need the following lemmas.

**Lemma 1** ([8]) Let $L(x) = \sum_{i=0}^{n-1} c_i x^i$ and $l(x) = \sum_{i=0}^{n-1} c_i x^i$ both be polynomials over $\mathbb{F}_2$. Then $L(x)$ has only one root in $\mathbb{F}_{2^n}$ if and only if $\gcd(l(x), x^n - 1) = 1$.

**Lemma 2** Let $p(x), q(x) \in \mathbb{F}_2[x]$ and $s$ be any fixed positive integer. Then $\gcd(p(x), q(x)) = 1$ if and only if $\gcd(p(x^s), q(x^s)) = 1$.

**Proof** If $\gcd(p(x), q(x)) = 1$, there exist $a(x)$ and $b(x)$ over $\mathbb{F}_2$ such that $a(x)p(x) + b(x)q(x) = 1$ according to Bézout’s identity. Then we have $a(x^s)p(x^s) + b(x^s)q(x^s) = 1$, which implies $\gcd(p(x^s), q(x^s)) = 1$. On the contrary, assume $\gcd(p(x^s), q(x^s)) = 1$. If $\gcd(p(x), q(x)) = d(x)$ with $\text{deg} d > 0$, there exist $a(x)$ and $b(x)$ over $\mathbb{F}_2$ such that $a(x)p(x) + b(x)q(x) = d(x)$ and thus $a(x^s)p(x^s) + b(x^s)q(x^s) = d(x^s)$, which contradicts the fact $\gcd(p(x^s), q(x^s)) = 1$. \qed
**Proof of Theorem 3**

Obviously $Q(x)$ is a quaternary quadratic form. The bilinear form $B$ associated to it satisfies

$$2B(x, y) = Q(x \oplus y) - Q(x) - Q(y)$$

$$= \text{Tr}_1^q \left( \alpha (x + y + 2\sqrt{xy}) + 2 \sum_{i=1}^{m-1} c_i \beta (x + y + 2\sqrt{xy})^{1+2^k i} \right)$$

$$- \text{Tr}_1^q \left( \alpha x + 2 \sum_{i=1}^{m-1} c_i \bar{\beta} x^{1+2^k i} \right) - \text{Tr}_1^q \left( \alpha y + 2 \sum_{i=1}^{m-1} c_i \bar{\beta} y^{1+2^k i} \right)$$

$$= 2\text{tr}_1^q \left( \alpha \sqrt{xy} \right) + 2\text{tr}_1^q \left( \sum_{i=1}^{m-1} c_i \bar{\beta} (\bar{x} + \bar{y})^{1+2^k i} \right)$$

$$- 2\text{tr}_1^q \left( \sum_{i=1}^{m-1} c_i \bar{\beta} x^{1+2^k i} \right) - 2\text{tr}_1^q \left( \sum_{i=1}^{m-1} c_i \bar{\beta} \bar{y}^{1+2^k i} \right)$$

$$= 2\text{tr}_1^q (\bar{x} \bar{y}) + 2\text{tr}_1^q \left( \sum_{i=1}^{m-1} c_i \bar{\beta} (\bar{x}^{2^k i} + \bar{x}^{2^k (m-i)} \bar{y}) \right)$$

(here we distinguish $\bar{x}$ with $\mu(x)$ for any $x \in \mathbb{T}$ for simplicity). Since $\bar{\alpha}^2 = \bar{\beta}$ and for $1 \leq i \leq \left \lfloor \frac{m-1}{2} \right \rfloor$, \[ \text{tr}_1^q \left( c_i \bar{\beta} \bar{x}^{2^k i} \right) = \text{tr}_1^q \left( (c_i \bar{\beta}) \bar{x}^{2^k i} \bar{y} \right) = \text{tr}_1^q \left( c_i \bar{\beta} x^{2^k (m-i)} \bar{y} \right) \]

as $\beta \in \mathbb{T}_e^*$, we have

$$2B(x, y) = 2\text{tr}_1^q \left( \bar{\beta} \bar{y} \left[ \bar{x} + \sum_{i=1}^{m-1} c_i \left( \bar{x}^{2^k i} + \bar{x}^{2^k (m-i)} \right) \right] \right).$$

Hence from Theorem 2 we know that $Q(x)$ is quaternary bent if and only if the polynomial

$$L(x) = x + \sum_{i=1}^{m-1} c_i \left( x^{2^k i} + x^{2^k (m-i)} \right)$$

has only one root $0$ in $\mathbb{F}_{2^n}$. By Lemma 1, this is equivalent to

$$\gcd \left( 1 + \sum_{i=1}^{m-1} c_i \left( x^{2^k i} + x^{2^k (m-i)} \right), x^n - 1 \right) = \gcd \left( 1 + \sum_{i=1}^{m-1} c_i \left( x^{2^k i} + x^{2^k (m-i)} \right), x^m - 1 \right) = 1,$$

which is further equivalent to

$$\gcd \left( 1 + \sum_{i=1}^{m-1} c_i \left( x^{2^k i} + x^{2^k (m-i)} \right), x^m - 1 \right) = 1$$
Proposition 2

We call a set of elements \( \mathcal{C} = \{c_i \in \mathbb{F}_2 \mid 1 \leq i \leq \left\lfloor \frac{m-1}{2} \right\rfloor \} \) a QBF-set w.r.t. \((n,k)\) if the polynomial \(c(x)\) in the form \((2)\) defined by it satisfies \(\gcd(c(x^n), x^m - 1) = 1\). By Theorem \((3)\) explicit constructions of QBF-sets will promise new classes of quaternary bent functions in the form \((2)\). In the remaining part of this section, we devote to finding several constructions of QBF-sets.

Firstly, from the constructions of quaternary bent functions proposed in \([7]\), we can directly obtain some new classes of QBF-sets. The results, whose detailed proofs can be found in \([7, Corollary 1,2,3,4,5]\), are summarized in the following proposition.

**Proposition 1**

(1) Assume \(m \geq 7\) and let \(\mathcal{C} = \{2,3\}\). Then \(\mathcal{C}\) is a QBF-set w.r.t. \((n,k)\) if and only if \(\gcd(m,3k) = \gcd(m,k)\);

(2) Let \(t\) be a positive integer with \(t < (m+1)/4\). Let \(\mathcal{C} = \{2i+1 \mid 0 \leq i \leq t\}\). Then \(\mathcal{C}\) is a QBF-set w.r.t. \((n,k)\) if and only if \(\gcd(m,2t+3)k = \gcd(m,2t+1)k = \gcd(m,k)\);

(3) Let \(t\) be a positive integer with \(t < (m+3)/4\). Let \(\mathcal{C} = \{2i \mid 1 \leq i \leq t\} \cup \{1\}\). Then \(\mathcal{C}\) is a QBF-set w.r.t. \((n,k)\) if and only if \(\gcd(m,2t+3)k = \gcd(m,2t-1)k = \gcd(m,k)\);

(4) Let \(t\) be a positive integer with \(t < m/2\). Let \(\mathcal{C} = \{i \mid 1 \leq i \leq t\}\). Then \(\mathcal{C}\) is a QBF-set w.r.t. \((n,k)\) if and only if \(\gcd(m,2t+1)k = \gcd(m,k)\);

(5) Assume \(m \geq 13\) and let \(\mathcal{C} = \{2,5,6\}\). Then \(\mathcal{C}\) is a QBF-set w.r.t. \((n,k)\) if and only if \(\gcd(m,5k) = \gcd(m,k)\);

(6) Assume \(m \geq 11\) and let \(\mathcal{C} = \{1,4,5\}\). Then \(\mathcal{C}\) is a QBF-set w.r.t. \((n,k)\) if and only if \(\gcd(m,3k) = \gcd(m,7k) = \gcd(m,k)\);

(7) Assume \(m \geq 13\) and let \(\mathcal{C} = \{3,5,6\}\). Then \(\mathcal{C}\) is a QBF-set w.r.t. \((n,k)\) if and only if \(\gcd(m,3k) = \gcd(m,5k) = \gcd(m,7k) = \gcd(m,k)\).

In addition, we can construct some new classes of QBF-sets.

**Proposition 2**

Let \(t, s\) be positive integers with \(s < t < \left\lfloor \frac{m-1}{2} \right\rfloor\). Let \(\mathcal{C} = \{t-s,s,t+s\}\). Then \(\mathcal{C}\) is a QBF-set w.r.t. \((n,k)\) if and only if \(\gcd(m,3tk) = \gcd(m,tk)\) and \(\gcd(m,3sk) = \gcd(m,sk)\).

**Proof**

Since \(c(x) = 1 + x^t-x^s + x^t + x^{t+s} + x^{m-(t-s)} + x^{m-s} + x^{m-t} + x^{m-(t+s)}\),

we have

\[
\gcd(c(x^k), x^m - 1) = \gcd(1 + x^{k(t-s)} + x^{ks} + x^{k(t+s)} + x^{m-k(t-s)} + x^{m-ks} + x^{m-kt} + x^{m-k(t+s)}, x^m - 1)
\]

\[
= \gcd(x^{k(t+s)} + x^{2kt} + x^{kt+2ks} + x^{2kt+ks} + x^{2k(t+s)} + x^{2ks} + x^{k(t-s)} + x^{ks} + 1, x^m - 1)
\]

\[
= \gcd((1 + x^{kt} + x^{2kt})(1 + x^{ks} + x^{2ks}), x^m - 1)
\]

\[
= \gcd \left( x^{3kt} - 1, \frac{x^{3ks} - 1}{x^{kt} - 1}, x^m - 1 \right).
\]

Thus \(\gcd(c(x^k), x^m - 1) = 1\) if and only if \(\gcd(m,3tk) = \gcd(m,tk)\) and \(\gcd(m,3sk) = \gcd(m,sk)\). □

**Proposition 3**

Assume \(m \geq 11\) and let \(\mathcal{C} = \{1,3,4,5\}\). Then \(\mathcal{C}\) is a QBF-set w.r.t. \((n,k)\) if and only if \(\gcd(m,3k) = \gcd(m,k)\).
Proof Since
\[ c(x) = 1 + x + x^3 + x^4 + x^5 + x^{m-5} + x^{m-4} + x^{m-3} + x^{m-1}, \]
we have
\[ \gcd(c(x^k), x^{m-1}) = \gcd(1 + x^k + x^{3k} + x^{4k} + x^{5k} + x^{m-5k} + x^{m-4k} + x^{m-3k} + x^{m-k}, x^{m-1}) \]
\[ = \gcd(x^{5k} + x^{6k} + x^{9k} + x^{10k} + 1 + x^k + x^{2k} + x^{4k}, x^{m-1}) \]
\[ = \gcd((1 + x^k + x^{2k})^3, x^{m-1}) \]
\[ = \gcd \left( \frac{x^{3k} - 1}{x^k - 1}, x^{m-1} \right). \]
Thus \( \gcd(c(x^k), x^{m-1}) = 1 \) if and only if \( \gcd(m, 3k) = \gcd(m, k) \). \( \square \)

**Proposition 4** Assume \( m \geq 13 \) and let \( \mathcal{C} = \{1, 2, 5, 6\} \). Then \( \mathcal{C} \) is a QBF-set w.r.t. \( (n, k) \) if and only if \( \gcd(m, 3k) = \gcd(m, k) \).

Proof Since
\[ c(x) = 1 + x + x^2 + x^5 + x^{m-6} + x^{m-5} + x^{m-2} + x^{m-1}, \]
we have
\[ \gcd(c(x^k), x^{m-1}) = \gcd(1 + x^k + x^{2k} + x^{5k} + x^{6k} + x^{m-6k} + x^{m-5k} + x^{m-2k} + x^{m-k}, x^{m-1}) \]
\[ = \gcd(x^{6k} + x^{7k} + x^{8k} + x^{11k} + x^{12k} + 1 + x^k + x^{4k} + x^{5k}, x^{m-1}) \]
\[ = \gcd((1 + x^{3k} + x^{6k})(1 + x^k + x^{2k})^3, x^{m-1}) \]
\[ = \gcd \left( \frac{x^{9k} - 1}{x^k - 1} \cdot \left( \frac{x^{3k} - 1}{x^k - 1} \right)^3, x^{m-1} \right). \]
Thus \( \gcd(c(x^k), x^{m-1}) = 1 \) if and only if \( \gcd(m, 3k) = \gcd(m, k) \) and \( \gcd(m, 9k) = \gcd(m, 3k) \). But \( \gcd(m, 3k) = \gcd(m, k) \) implies that \( \gcd(m, 9k) = \gcd(m, 3k) \), so this condition is enough. \( \square \)

**Proposition 5** Assume \( m \geq 13 \) and let \( \mathcal{C} = \{2, 3, 6\} \). Then \( \mathcal{C} \) is a QBF-set w.r.t. \( (n, k) \) if and only if \( \gcd(m, 3k) = \gcd(m, 7k) = \gcd(m, k) \).

Proof Since
\[ c(x) = 1 + x^2 + x^3 + x^6 + x^{m-6} + x^{m-3} + x^{m-2}, \]
we have
\[ \gcd(c(x^k), x^{m-1}) = \gcd(1 + x^{2k} + x^{3k} + x^{6k} + x^{m-6k} + x^{m-3k} + x^{m-2k}, x^{m-1}) \]
\[ = \gcd(x^{6k} + x^{8k} + x^{9k} + x^{12k} + 1 + x^{3k} + x^{4k}, x^{m-1}) \]
\[ = \gcd((1 + x^k + x^{2k})^3(1 + x^k + x^{2k} + x^{3k} + x^{4k} + x^{5k} + x^{6k}), x^{m-1}) \]
\[ = \gcd \left( \left( \frac{x^{3k} - 1}{x^k - 1} \right)^3 \cdot \frac{x^{7k} - 1}{x^k - 1}, x^{m-1} \right). \]
Thus \( \gcd(c(x^k), x^{m-1}) = 1 \) if and only if \( \gcd(m, 3k) = \gcd(m, 7k) = \gcd(m, k) \). \( \square \)
Proposition 6 Assume \( m \geq 15 \) and let \( \mathcal{C} = \{2, 3, 4, 7\} \). Then \( \mathcal{C} \) is a QBF-set w.r.t. \((n,k)\) if and only if \( \gcd(m, 3k) = \gcd(m, 5k) = \gcd(m, k) \).

Proof Since

\[ c(x) = 1 + x^2 + x^3 + x^4 + x^7 + x^{m-7} + x^{m-4} + x^{m-3} + x^{m-2}, \]

we have

\[ \gcd(c(x^k), x^m - 1) = \gcd(1 + x^{2k} + x^{3k} + x^{4k} + x^{7k} + x^{m-7k} + x^{m-4k} + x^{m-3k} + x^{m-2k}, x^m - 1) \]
\[ = \gcd(x^{7k} + x^{9k} + x^{10k} + x^{14k} + 1 + x^{3k} + x^{4k} + x^{5k}, x^m - 1) \]
\[ = \gcd((1 + x^k + x^{2k})(1 + x^k + x^{2k} + x^{3k} + x^{4k})^3, x^m - 1) \]
\[ = \gcd \left( \frac{x^{3k} - 1}{x^k - 1}, (\frac{x^{5k} - 1}{x^k - 1})^3, x^m - 1 \right) \]

Thus \( \gcd(c(x^k), x^m - 1) = 1 \) if and only if \( \gcd(m, 3k) = \gcd(m, 5k) = \gcd(m, k) \). \( \square \)

Proposition 7 Assume \( m \geq 15 \) and let \( \mathcal{C} = \{1, 2, 3, 7\} \). Then \( \mathcal{C} \) is a QBF-set w.r.t. \((n,k)\) if and only if \( \gcd(m, 3k) = \gcd(m, 5k) = \gcd(m, 7k) = \gcd(m, k) \).

Proof Since

\[ c(x) = 1 + x + x^2 + x^3 + x^7 + x^{m-7} + x^{m-3} + x^{m-2} + x^{m-1}, \]

we have

\[ \gcd(c(x^k), x^m - 1) = \gcd(1 + x^k + x^{2k} + x^{3k} + x^{7k} + x^{m-7k} + x^{m-3k} + x^{m-2k} + x^{m-k}, x^m - 1) \]
\[ = \gcd(x^{7k} + x^{8k} + x^{10k} + x^{14k} + 1 + x^{4k} + x^{6k}, x^m - 1) \]
\[ = \gcd((1 + x^k + x^{2k})^2(1 + x^k + x^{2k} + x^{3k} + x^{4k})(1 + x^k + x^{2k} + x^{3k} + x^{4k} + x^{5k} + x^{6k}), x^m - 1) \]
\[ = \gcd \left( \left( \frac{x^{3k} - 1}{x^k - 1} \right)^2, \frac{x^{5k} - 1}{x^k - 1}, \frac{x^{7k} - 1}{x^k - 1}, x^m - 1 \right) \]

Thus \( \gcd(c(x^k), x^m - 1) = 1 \) if and only if \( \gcd(m, 3k) = \gcd(m, 5k) = \gcd(m, 7k) = \gcd(m, k) \). \( \square \)

Proposition 8 Assume \( m \geq 17 \) and let \( \mathcal{C} = \{1, 3, 5, 6, 8\} \). Then \( \mathcal{C} \) is a QBF-set w.r.t. \((n,k)\) if and only if \( \gcd(m, 3k) = \gcd(m, 7k) = \gcd(m, k) \).

Proof Since

\[ c(x) = 1 + x + x^3 + x^5 + x^6 + x^8 + x^{m-8} + x^{m-6} + x^{m-5} + x^{m-3} + x^{m-1}, \]

we have

\[ \gcd(c(x^k), x^m - 1) = \gcd(1 + x^k + x^{3k} + x^{5k} + x^{6k} + x^{8k} + x^{m-8k} + x^{m-6k} + x^{m-5k} + x^{m-3k} + x^{m-k}, x^m - 1) \]
\[ = \gcd(x^{8k} + x^{9k} + x^{11k} + x^{13k} + x^{14k} + x^{16k} + 1 + x^{2k} + x^{3k} + x^{5k} + x^{6k}, x^m - 1) \]
\[ = \gcd((1 + x^k + x^{2k})^5(1 + x^k + x^{2k} + x^{3k} + x^{4k} + x^{5k} + x^{6k}), x^m - 1) \]
\[ = \gcd \left( \left( \frac{x^k - 1}{x^k - 1} \right)^5 \cdot \frac{x^7 - 1}{x^5 - 1}, x^m - 1 \right) . \]

Thus \( \gcd(c(x^k), x^m - 1) = 1 \) if and only if \( \gcd(m, 3k) = \gcd(m, 7k) = \gcd(m, k) \). \( \square \)

**Proposition 9** Let \( t \) be a positive integer with \( t < (m-3)/4 \). Let \( \mathcal{C} = \{2i+1 \mid 1 \leq i \leq t \} \cup \{2t+2\} \). Then \( \mathcal{C} \) is a QBF-set w.r.t. \( (n, k) \) if and only if \( \gcd(m, (2t+3)k) = \gcd(m, (2t+1)k) = \gcd(m, 3k) = \gcd(m, k) \).

**Proof** Since

\[ c(x) = 1 + \sum_{i=1}^{t} \left(x^{2i+1} + x^{m-(2i+1)} + x^{2i+2} + x^{m-(2i+2)}\right), \]

we have

\[ \gcd(c(x^k), x^m - 1) = \gcd \left( 1 + \sum_{i=1}^{t} \left(x^{(2i+1)k} + x^{m-(2i+1)k} + x^{(2i+2)k} + x^{m-(2i+2)k}, x^m - 1 \right) \right) \]

\[ = \gcd \left( x^{(2t+2)k} + x^{(2t+2)k} \sum_{i=1}^{t} x^{(2i+1)k} + \sum_{i=1}^{t} x^{(2i-1)k} + x^{(4t+4)k} + 1, x^m - 1 \right) \]

\[ = \gcd \left( (1 + x^k + x^{2k}) \left( 1 + \sum_{i=1}^{2t} x^i \right) \left( \sum_{i=1}^{2t+2} x^i \right) \right), x^m - 1 \right) \]

\[ = \gcd \left( \frac{x^{3k} - 1}{x^k - 1} \cdot \frac{x^{(2t+1)k} - 1}{x^k - 1} \cdot \frac{x^{(2t+3)k} - 1}{x^k - 1}, x^m - 1 \right) \].

Thus \( \gcd(c(x^k), x^m - 1) = 1 \) if and only if \( \gcd(m, (2t+3)k) = \gcd(m, (2t+1)k) = \gcd(m, 3k) = \gcd(m, k) \). \( \square \)

**Proposition 10** Let \( t \) be a positive integer with \( t < (m-1)/4 \). Let \( \mathcal{C} = \{2i \mid 2 \leq i \leq t \} \cup \{1, 2t+1\} \). Then \( \mathcal{C} \) is a QBF-set w.r.t. \( (n, k) \) if and only if \( \gcd(m, (2t+3)k) = \gcd(m, (2t-1)k) = \gcd(m, 3k) = \gcd(m, k) \).

**Proof** Since

\[ c(x) = 1 + x + \sum_{i=2}^{t} \left(x^{2i} + x^{m-2i} + x^{2i+1} + x^{m-(2i+1)} + x^{m-1}\right), \]

we have

\[ \gcd(c(x^k), x^m - 1) = \gcd \left( 1 + x^k + \sum_{i=2}^{t} \left(x^{2i} + x^{m-2i} + x^{2i+1} + x^{m-(2i+1)} + x^{m-1}, x^m - 1 \right) \right) \]

\[ = \gcd \left( x^{(2t+1)k} + x^{(2t+2)k} + x^{(2t+1)k} \sum_{i=2}^{t} x^{2i} + \sum_{i=1}^{t} x^{(2i-1)k} + x^{(4t+2)k} + 1 + x^{2t+2k}, x^m - 1 \right) \]

\[ = \gcd \left( (1 + x^k + x^{2k}) \left( 1 + \sum_{i=1}^{2t} x^i \right) \left( \sum_{i=1}^{2t+2} x^i \right), x^m - 1 \right) \]
New constructions of quaternary bent functions

\[ \gcd \left( \frac{x^{3k} - 1}{x^3 - 1} \cdot \frac{x^{(2r-1)k} - 1}{x^r - 1} \cdot \frac{x^{(2r+3)k} - 1}{x^r - 1} \cdot x^m - 1 \right). \]

Thus \( \gcd(c(x^t), x^m - 1) = 1 \) if and only if \( \gcd(m, (2t+3)k) = \gcd(m, (2t-1)k) = \gcd(m, 3k) = \gcd(m, k) \). \( \square \)

By virtue of the QBF-sets characterized in Proposition 1 to 10, we can obtain many new classes of quaternary bent functions.

4 New classes of quadratic bent and semi-bent function in polynomial forms

According to the connections between quaternary bent functions and binary bent and semi-bent functions indicated in Theorem 1, we can derive some new classes of binary bent and semi-bent functions based on the classes of quaternary bent functions constructed in Section 3.

Firstly, we recall a result concerning 2-adic expansion of the trace function over \( \text{GR}(4, n) \).

Lemma 3 (3) \( \text{Tr}_1^n(x) = \text{tr}_1^n(\bar{x}) + 2p(\bar{x}) \),

where \( p(x) \) is defined by (1).

From Lemma 3 we know that \( \text{Tr}_1^n(\lambda x) = \text{tr}_1^n(\bar{\lambda} \bar{x}) + 2p(\bar{\lambda} \bar{x}) \) for any \( \lambda \in \mathbb{T} \). Hence the quaternary Boolean function defined by (2) can be expressed as

\[ Q(x) = \text{tr}_1^n(\bar{\alpha} \bar{x}) + 2 \left[ p(\bar{\alpha} \bar{x}) + \sum_{i=1}^{\lfloor m/2 \rfloor} c_i \text{tr}_1^n(\bar{\beta} x^{1+2^k i}) \right]. \]

From Theorem 1(2), we obtain the following result, which generalizes [7, Theorem 3].

**Theorem 4** Assume \( \alpha, \beta \in \mathbb{F}_{2^e}^* \) and \( \beta = \alpha^2 \). Let

\[ f_Q(x) = p(\alpha x) + \sum_{i=1}^{\lfloor m/2 \rfloor} c_i \text{tr}_1^n(\beta x^{1+2^k i}), \quad x \in \mathbb{F}_{2^n}, \tag{4} \]

where \( c_i \in \mathbb{Z}_2, \quad 1 \leq i \leq \lfloor m/2 \rfloor \). Then \( f_Q(x) \) is a bent function or a semi-bent function, respectively, according as \( n \) is even or odd, respectively, if and only if \( \gcd(c(x^t), x^m - 1) = 1 \) where \( c(x) \) is defined by (3).

It is clear that the Boolean function \( f_Q(x) \) in Theorem 4 is a quadratic Boolean function in the so-called polynomial form (1). In fact, constructions of quadratic bent and semi-bent functions in polynomial forms are extensively studied by several authors (see, for example, [6,16,14]). From Theorem 4 we can directly obtain many new classes of quadratic bent and semi-bent functions in polynomial forms by choosing the sets of coefficients \( \{c_i\} \) to be QBF-sets characterized by, say, Proposition 1 to 10. However, it is not a simple matter to get them via studying quadratic forms over finite fields, which is the standard approach to study quadratic binary Boolean functions.
5 Conclusions

In this paper, we propose a new construction of quaternary bent functions from quadratic forms over Galois rings of characteristic 4, which generalize some previous work. By characterizing the so-called QBF-sets, we can explicitly construct several new classes of quaternary bent functions. They furthermore derive several new classes of binary bent and semi-bent functions which are quadratic ones in polynomials forms.

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