Short range vs long range dependence. An hypothesis test based on Fractional Iterated Ornstein–Uhlenbeck processes

Juan Kalemkerian
Universidad de la República, Facultad de Ciencias

Andrés Sosa
Universidad de la República, Facultad de Ciencias Económicas y Administración

December 22, 2021

Abstract

In this work, which is based on the family of Fractional Iterated Ornstein–Uhlenbeck processes, we propose a new hypothesis test to contrast short memory versus long memory in time series. This family includes short memory and long memory processes, and has the ability to approximate a long memory processes by a short memory processes. Based on the asymptotic results of the estimators of its parameters, we will present the test and show how it can be implemented. Also, we will show a comparison with other tests widely used under both short memory and long memory scenarios. The main conclusion is that this new test is the one with best performance under the null hypothesis, and has the maximum power in some alternatives.

Keywords: long memory processes, hypothesis test, fractional Ornstein–Uhlenbeck processes

1 Introduction

It is commonly said that the sequence has long range dependence when we have a stationary and centered sequence of random variables $X_1, X_2, \ldots, X_n, \ldots$ such that $E(X_0X_n) \to 0$ and $\sum_{n=1}^{+\infty} |E(X_0X_n)| = +\infty$. In contrast, the sequence has short range dependence when it is satisfied that $\sum_{n=1}^{+\infty} |E(X_0X_n)| < +\infty$. Long memory was discovered by Hurst in his pioneering work ([10]), in which it shows its presence in a time series of variables (from several disciplines), such as rainfall, temperature, pressure, thickness of the rings of certain trees, sunspots and stock market phenomena. From that moment, long memory processes have been (and are still) studied extensively from both theoretical and practical points of view. Although there are alternative definitions of short range or long range dependence (see for instance [4], [8], [9], [20]), they are not all equivalent to
each other. However, the underlying idea in all cases is that when the autocorrelation function tends to zero quickly (slowly) we have a short (long) range process. It is worth noting that there are processes to model both short memory and long memory time series, even though it is clear that the long memory processes (such as ARFIMA process) are more complex models, are less intuitive and are more difficult to estimate.

In empirical applications, it is important to have an hypothesis test that can help us to decide if we are dealing with a short memory or long memory time series. Unfortunately, there are not many hypothesis tests to tackle these kinds of problems, while the existing tests have problems in their applicability. For example, they have a parameter in such way that the test can have a strong bias to reject the null hypothesis for some values of the parameter or a strong bias to non-reject the null hypothesis in the other cases. This fact is not surprising due to the complexity of the problem to be solved. Furthermore, even in those cases in which we take a value of the parameter where the test has a bias to non-reject the null hypothesis, we will show in Section 5 that there are several examples of short memory processes that are wrongly considered (with very high probability) as long memory processes. Consequently, we can conclude that the existing tests are not very reliable when working with real time series. This is an important problem because if we apply any of these hypothesis tests when we have a real time series, we cannot have much confidence that we are making the correct decision. From a simple generalisation of the R/S statistic proposed by Hurst ([10]) in 1991, Lo ([13]) develops a test that takes as the null hypothesis that the series has short memory dependence against the alternative that has long memory dependence. Although this is not the first hypothesis test that has been developed for this purpose (see for example [6]), it is a widely used test. From this work, different variants have been developed that have generated other hypothesis tests, such as Giraitis et al.’s ([7]) test that is based on the rescaled variance statistic, Lee and Schmidt’s ([19]) test, and others. A description of these and other tests can be found in [3].

Recently, a fractional iterated Ornstein–Uhlenbeck process was introduced in [17]. When the number of iterations in this family of processes is greater than or equal to 2, they are short memory processes. However, in some cases we can approximate (in a continuous way) the fractional Ornstein–Uhlenbeck long range process. In addition, if we use only one iteration, then we have a fractional Ornstein–Uhlenbeck process. This property can be use to design a hypothesis test to deal with this issue. In this work, we propose a new hypothesis test that is based on a fractional iterated Ornstein–Uhlenbeck process to test short range dependence against long range dependence in observed time series. In addition, we will make a comparison with other tests that are commonly used in practice using a wide spectrum of scenarios, covering both short memory and long memory time series.

The rest of this paper is organised as follows. In Section 2 we introduce the Fractional Iterated Ornstein–Uhlenbeck process, we explain the main properties that these processes satisfy and we also show the main tools for the approach of the proposed test. In Section 3 we provide the test approach, which is based on the properties given in Section 2. In Section 4 we explain the hypothesis test implementation. In Section 5 we
compare the performance of this new test against other existing tests in the literature. In this section, we also include a criterion to select the parameter before using each of the competitor tests. In Section 6, we show an application to real data. Our concluding remarks are given in Section 7.

2 Preliminaries

We start by defining the fractional iterated Ornstein–Uhlenbeck processes of order 2. We also summarise the main properties that will allow us to develop the idea of the hypothesis test. An introduction to this type of process, as well as its theoretical development, can be found in [14] and [17].

**Definition 1.** Given a fractional Brownian motion \( \{ B_H(t) \} \), a fractional Ornstein–Uhlenbeck process with parameters \( \sigma, \lambda > 0 \) and \( H \in (0, 1] \) is defined as \( \{ X_t \} \) where

\[
X_t = \sigma \int_{-\infty}^{t} e^{-\lambda(t-s)} dB_H(s) \text{ for all } t \in \mathbb{R}.
\]

In [5], it is proved that the process in Definition 1 is the only stationary solution of the stochastic equation

\[
dX_t = -\lambda X_t dt + \sigma dB_H(t).
\]

**Notation 1.** In this work, we use the notation \( \{ X_t \} \sim \text{FOU}(\lambda, \sigma, H) \) or \( \text{FOU}(\lambda, H) \) when \( \sigma = 1 \).

**Definition 2.** Given \( 0 \leq \lambda_1 < \lambda_2 \), a fractional Brownian motion \( \{ B_H(t) \} \) and the processes \( \{ X_t^{(i)} \} \) that satisfying \( X_t^{(i)} = \sigma \int_{-\infty}^{t} e^{-\lambda_i(t-s)} dB_H(s) \) for \( i = 1, 2 \). The fractional iterated Ornstein–Uhlenbeck process \( \{ X_t \} \) with parameters \( \lambda_1, \lambda_2, \sigma > 0 \) and \( H \in (0, 1] \) is

\[
X_t = \sigma \int_{-\infty}^{t} e^{-\lambda_2(t-s)} dX_t^{(1)} \text{ for all } t \in \mathbb{R}.
\]

**Notation 2.** Again, in this work we use the notation \( \{ X_t \} \sim \text{FOU}(\lambda_1, \lambda_2, \sigma, H) \) or \( \text{FOU}(\lambda_1, \lambda_2, H) \) when \( \sigma = 1 \).

**Remark 1.** It is important to note that in the case that \( \lambda_1 = 0 \), we have the classic Ornstein-Uhlenbeck process.

**Remark 2.** Observe that if we define the family of operators

\[
T_\lambda(y)(t) = \int_{-\infty}^{t} e^{-\lambda(t-s)} dy(s),
\]

therefore \( X_t^{(i)} = T_{\lambda_i}(B_H)(t) \) for \( i = 1, 2 \), and \( X_t = T_{\lambda_2}(X_t^{(1)})(t) = T_{\lambda_2}(T_{\lambda_1}(B_H))(t) \). This properties enables us to state that any \( \text{FOU}(\lambda_1, \lambda_2, H) \) is the composition of the operators \( T_{\lambda_1} \) and \( T_{\lambda_2} \) evaluated on a fractional Brownian motion with parameter \( H \).
Remark 3. Integrating by parts, we can obtain that any \( \{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda_1, \lambda_2, \sigma, H) \) can be expressed by

\[
X_t = \frac{\lambda_1}{\lambda_1 - \lambda_2} X_t^{(1)} + \frac{\lambda_2}{\lambda_2 - \lambda_1} X_t^{(2)} \quad \text{for all } t \in \mathbb{R}.
\]

thus, any \( \text{FOU}(\lambda_1, \lambda_2, H) \) is a linear combination of a \( \text{FOU}(\lambda_1, H) \) and \( \text{FOU}(\lambda_2, H) \) driven for the same fractional Brownian motion and therefore the composition is commutative: \( T_{\lambda_1} (T_{\lambda_2}) = T_{\lambda_2} (T_{\lambda_1}) \).

Remark 4. We can generalise and compose \( p \) times with the operators \( T_\lambda \), by the same or different values of \( \lambda \) and obtain the \( \text{FOU}(p) \) processes (according with notation given in \([17]\)) \((p\) iterations). However, for our purposes it is enough to work with the \( \text{FOU}(2) \) processes that is, \( \text{FOU}(\lambda_1, \lambda_2, \sigma, H) \) where \( 0 \leq \lambda_1 < \lambda_2 \).

Remark 5. Any \( \{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda_1, \lambda_2, \sigma, H) \) is a stationary centered Gaussian process.

In the memory process problem, it is important to know the autocorrelation function of the models. In the case of fractional Brownian motion \( \{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda_1, \lambda_2, \sigma, H) \) the autocovariance function is given by

\[
\mathbb{E}(X_0 X_t) = \frac{\sigma^2 H}{2} \left( \frac{\lambda_1^{2-2H}}{\lambda_1^2 - \lambda_2^2} f_H(\lambda_1 t) - \frac{\lambda_2^{2-2H}}{\lambda_1^2 - \lambda_2^2} f_H(\lambda_2 t) \right),
\]

where the function \( f_H \) is defined by

\[
f_H(t) = e^{-t} \left( \Gamma(2H) - \int_0^x e^{s(x-s)} ds \right) + e^{t} \left( \Gamma(2H) - \int_0^x e^{-s(x-s)} ds \right).
\]

For further details, the properties of the function \( f_H \) can be found in \([17]\). Between them, the asymptotic behavior of \( f_H \): if \( H \neq 1/2 \) we have that \( f_H(x) \sim 2(2H-1)x^{2H-2} \) as \( x \to +\infty \). In particular from \([1]\), we have that the autocovariance function of any \( \{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda, \sigma, H) \) is given by

\[
\mathbb{E}(X_0 X_t) = \frac{\sigma^2 H}{2\lambda^{2H}} f_H(\lambda t).
\]

Therefore, the equation \([2]\) allows us to conclude that any \( \text{FOU}(\lambda, \sigma, H) \) has short range dependence when \( H \leq 1/2 \) and long range dependence for \( H > 1/2 \).

Although from equation \([1]\) the autocovariance function of any \( \text{FOU}(\lambda_1, \lambda_2, \sigma, H) \) is a linear combination of the functions \( f_H(\lambda_1 t) \) and \( f_H(\lambda_2 t) \). In addition, when \( \lambda_1 > 0 \) any \( \text{FOU}(\lambda_1, \lambda_2, \sigma, H) \) process has short range dependence for any value of \( H \in (0, 1] \). However, if \( \lambda_1 = 0 \), then we have that a \( \text{FOU}(\lambda_2, \sigma, H) \) has short range dependence for \( H \leq 1/2 \) and long range dependence for \( H > 1/2 \).

When we have a \( \text{FOU}(\lambda_1, \lambda_2, \sigma, H) \) process observed in an equispaced sample of the interval \([0, T]\), under some conditions that imply \( T \to +\infty \) and \( T/n \to 0 \), the parameters \( \sigma, H \) can be estimated in a consistent way by a procedure proposed in the work \([13]\). In
Section 4 we summarise the explicit formulas to obtain \( \hat{H} \) and \( \hat{\sigma} \) and their asymptotic behavior. These results are the basis for the proposed hypothesis test. Moreover, it is satisfied that \( \hat{H} \) and \( \hat{\sigma} \) are asymptotically normal.

The structure of the spectral density is another interesting result of fractional iterated Ornstein–Uhlenbeck processes. It is possible to prove that for each process FOU(\( \lambda_1, \lambda_2, \sigma, H \)), the spectral density is given by

\[
 f^{(X)}(x) = \frac{\sigma^2 \Gamma(2H + 1) \sin((H\pi)|x|^{3-2H})}{2\pi(\lambda_1^2 + x^2)(\lambda_2^2 + x^2)}.
\]  

Regarding the parameters \( \lambda_1 \) and \( \lambda_2 \), taking advantage of the spectral density in the equation (3), when \( T \to +\infty \) can be estimated in a consistent way by a modified Whittle contrast if the process is observed in an equispaced sample of \([0, T]\) and has asymptotically joint Gaussian distribution. In Section 4 we summarise the procedure to estimate \( \hat{\lambda}_1 \) and \( \hat{\lambda}_2 \).

**Remark 6.** The interpretation of the \( H \) parameter in any FOU(\( \lambda_1, \lambda_2, \sigma, H \)) is related to the irregularity of the trajectories because \( 2H \) is the local Hölder index of the process. In the particular case in which \( \lambda_1 = 0 \), \( H \) is also a parameter that govern the memory of the process (i.e., long memory for \( H > 1/2 \) and short memory for \( H \leq 1/2 \)) in a similar way that it is interpreted in the fractional Gaussian noise or \( d \) parameter in ARFIMA model.

### 3 Statistical hypothesis testing

We assume that we have a sample \( X_{t_1}, X_{t_2}, ..., X_{t_n} \) of some centered stationary process \( \{X_t\}_{t \in \mathbb{R}} \) where \( 0 \leq t_1 < t_2 < ... < t_n \leq T \). Our objective is to detect if there is a short range or long range dependence in the time series. Therefore, we want to test

\[
 H_0 : \{X_t\}_{t \in \mathbb{R}} \text{ has short range dependence} \\
 H_1 : \{X_t\}_{t \in \mathbb{R}} \text{ has long range dependence}.
\]

We assume that the sample corresponds to \( \{X_t\}_{t \in \mathbb{R}} \sim \text{FOU}(\lambda_1, \lambda_2, \sigma, H) \) where \( 0 \leq \lambda_1 < \lambda_2, \sigma > 0, H \in (0, 1], \) and the observations are equispaced. Therefore, according to Section 2, we can express the hypotheses test in a parametric form as

\[
 H_0 : \ H \leq 1/2 \ or \ \lambda_1 > 0 \\
 H_1 : \ H > 1/2 \ and \ \lambda_1 = 0.
\]

Since that we have a consistent procedure to estimate \( H \) and \( \lambda_1 \) (see Section 4), it is natural to reject the null hypothesis when \( \hat{H} \geq k \) and \( \hat{\lambda}_1 \leq c \) where the values \( k \) and \( c \) are real constants such that

\[
 \sup_{H_0} \mathbb{P}\left( \{\hat{H} \geq k\} \cap \{\hat{\lambda}_1 \leq c\} \right) = \alpha
\]
where $\alpha$ is the signification level of the test. A simple way to obtain values of $k$ and $c$ such that the level of the test is less than or equal to $\alpha$, is to obtain $k$ such that $\sup_{H_0} P\left( \hat{H} \geq k \right) = \alpha$ and $c$ such that $\sup_{H_0} P\left( \hat{\lambda}_1 \leq c \right) = \alpha$.

The asymptotic distribution of $\hat{H}$ is normal and is independent of the values of $\lambda_1$ and $\lambda_2$, then

$$\sup_{H_0} P\left( \hat{H} \geq k \right) = \sup_{H \leq 1/2} P\left( \hat{H} \geq k \right)$$

and the supreme naturally is reached at $H = 1/2$. Therefore, it is easy to find $k$ such that $\sup_{H_0} P\left( \hat{H} \geq k \right) = \alpha$. To obtain $\sup_{H_0} P\left( \hat{\lambda}_1 \leq c \right)$, we observe that $P\left( \hat{\lambda}_1 \leq c \right) = g_c(\lambda_1, \lambda_2, \sigma, H)$, and it is natural to expect that the supreme is reached at the point (or points) where it is more difficult to decide whether $H_0$ or $H_1$ is true.

It is important to note that for small values of $c$, the function $g_c(\lambda_1, \lambda_2, \sigma, H)$ is increasing as an $H$–function for fixed values of $\sigma, \lambda_1$ and $\lambda_2$. For example, for each $H = 0.2, 0.3, 0.4, 0.5$ and $\lambda_1 = 0.3, \lambda_2 = 0.8, \sigma = 1$, we simulate 100 trajectories of FOU$(\lambda_1, \lambda_2, \sigma, H)$ in $[0, T]$ for $T = 100$ and sample size of $n = 1000$. In each case we have calculated $\hat{\lambda}_1$. In Figure 1, we show 100 values of $\hat{\lambda}_1$ ordered from the smallest to the largest. For other values of $\lambda_1, \lambda_2, \sigma$ the behavior is similar.

![Figure 1: 100 values of $\hat{\lambda}_1$ for an equispaced sample of length $n = 1000$ in $[0, 100]$ of a FOU$(\lambda_1 = 0.3, \lambda_2 = 0.8, \sigma = 1, H)$ process for $H = 0.2$ (black), $H = 0.3$ (blue), $H = 0.4$ (red) and $H = 0.5$ (brown).](image)

Then, $\sup_{H_0} g_c(\lambda_1, \lambda_2, \sigma, H) = g_c(\lambda_1, \lambda_2, \sigma, 1/2)$.
It should also be noted that the function $g_c(\lambda_1, \lambda_2, \sigma, 1/2)$ grows as $\lambda_2 - \lambda_1$ grows for moderately small $c$ values. For example when $\lambda_2 = 5, H = 0.5$ and each $\lambda_1 = 0, 1, 2, 3$ we have simulated 100 process of FOU($\lambda_1, \lambda_2, H$) in [0, $T$] for $T = 100$ and sample size of $n = 1000$. In each case we have calculated $\lambda_1$. In Figure 2 we show the 100 values of $\lambda_1$ ordered from the smallest to the largest. Again, the behavior is similar for other values of $\lambda_1, \lambda_2$.

![Figure 2: 100 values of $\hat{\lambda}_1$ for an equispaced sample of length $n = 1000$ in [0, 100] of a FOU($\lambda_1, \lambda_2 = 5, \sigma = 1, H = 0.5$) process for $\lambda_1 = 3$ (black), $\lambda_1 = 2$ (blue), $\lambda_1 = 1$ (red) and $\lambda_1 = 0$ (brown).](image)

Therefore

$$\sup_{H_0} P(\hat{\lambda}_1 \leq c) = \sup_{0 < \lambda_1 < \lambda_2 \leq \lambda} \sup_{H \leq 1/2} g_c(\lambda_1, \lambda_2, \sigma, H) = g_c(0, \tilde{\lambda}, \sigma, 1/2).$$

Taking into account that $\sigma$ does not appear in the hypotheses and that the behavior of $g_c$ as a function of $H$ and as a function of $\lambda_1, \lambda_2$ is the same for any value of $\sigma$, we propose to obtain an approximated value of $c$ using $\tilde{\sigma}$ instead of $\sigma$, which is simply performed

$$g_c(0, \tilde{\lambda}, \tilde{\sigma}, 1/2) = \alpha.$$

**Remark 7.** According with Remark 6, we can have a long memory process where the Hölder index is less than $1/2$. For this case, the FOU test naturally non rejects the null hypothesis. We observe examples of this situation in Section 5.
4 Implementation of the test

Given $X_1, X_2, ..., X_n$ observations of a stationary centered time series, to perform the test we need to consider the observations as an equispaced sample of FOU($\lambda_1, \lambda_2, \sigma, H$) in some interval $[0, T]$; that is, $X_{T/n}, X_{2T/n}, ..., X_T$. To perform a hypothesis test, we need to estimate the parameters $\sigma, H$ and $\lambda_1$ which depend on $T$ (except $H$ as we will look in the following subsection). Thus, we first need to know the value of $T$. In Subsection 4.2 we propose a criterion to select a suitable value of $T$.

4.1 Estimation of $\sigma, H$ and $\lambda_1$.

If we know the value of $T$, we can proceed to estimate the parameters $\sigma, H$ and $\lambda_1$. The estimation of $H$ is independent of the value of $T$, the estimation of $\sigma$ depends on $\hat{H}$ and $T$, and the estimation of $\lambda_1$ requires knowledge the values of $\hat{\sigma}, \hat{H}$ and $T$. More explicitly, the estimation of $\sigma, H$ and $\lambda_1$ is carried out in the following two steps.

Step 1. Estimation of $H$ and $\sigma$.

First, we need to select a filter $a = (a_0, a_1, ..., a_k)$ of length $k+1$ and order $L \geq 2$ (i.e., $\sum_{i=0}^{k} i^j a_i = 0$ for $j < L$ and $\sum_{i=0}^{k} i^L a_i = 0$). For example we can use $a = (-1, 2, -1)$ (filter of order 2) or the Daubechies filter of order 2. given by

$$a = (0.4829629131445, -0.8365163037378, 0.2241438680420, 0.1294095225512)/\sqrt{2},$$

In this way, the estimation of $H$ is given by

$$\hat{H} = \frac{1}{2} \log_2 \left( \frac{V_{n,a}^2}{V_{n,a}} \right), \quad (4)$$

where $a^2$ means the filter defined by $a^2 = (a_0, 0, a_1, 0, a_2, 0, ..., 0, a_k)$ of length $2k+1$ and order $L$ and $V_{n,a} := \frac{1}{n} \sum_{i=0}^{n-k} \left( \sum_{j=0}^{k} a_j X_{i+j} \right)^2$ is the quadratic variation of the sample associated to a filter $a$.

Second, once we have estimated $H$ the estimation of $\sigma$ is given by

$$\hat{\sigma} = \left( \frac{-2V_{n,a}}{\Delta_n^2 \sum_{i=0}^{k} \sum_{j=0}^{k} a_i a_j |i-j|^2 \hat{H}} \right)^{1/2}, \quad (5)$$

where $\Delta_n = \frac{T}{n}$.

Step 2. Estimation of $\lambda_1$.

If we define the function

$$U_T^{(n)}(\lambda_1, \lambda_2) = \frac{T}{n} \sum_{i=1}^{n} h_T^{(n)}(iT/n, \lambda_1, \lambda_2)$$

in which $h_T^{(n)}$ is
\[ h_T^{(n)}(x, \lambda_1, \lambda_2) = \frac{1}{2\pi} \left( \log f^{(X)}(x, \lambda_1, \lambda_2, \hat{\sigma}, \hat{H}) + \frac{I_T^{(n)}(x)}{f^{(X)}(x, \lambda_1, \lambda_2, \hat{\sigma}, \hat{H})} \right) w(x) \]

where \( f^{(X)}(x, \lambda_1, \lambda_2, \hat{\sigma}, \hat{H}) \) is the spectral density given in Section 3 (evaluated in \( H = \hat{H} \) and \( \sigma = \hat{\sigma} \)) and

\[ I_T^{(n)}(x) = \frac{T}{2\pi} \left| \frac{1}{n} \sum_{j=1}^{n} e^{\frac{i2\pi x}{n} X_{jT}} \right|^2 \]

is the discretization of the periodogram. The weight function is \( w(x) = \frac{|x|^c}{1+|x|^b} \) where \( a \) and \( b \) are parameters such that \( c \geq 4 \) and \( b \geq c + 3 \).

Finally, the estimation of the parameters \((\lambda_1, \lambda_2)\) are

\[ (\hat{\lambda}_1, \hat{\lambda}_2) = \arg \min_{(\lambda_1, \lambda_2) \in \Lambda} U_T^{(n)}(\lambda_1, \lambda_2) \quad (6) \]

where \( \Lambda \subset \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 : 0 < \lambda_1 < \lambda_2\} \) some compact set.

In [14] and [17] can be found the proof of the asymptotic results concerning these estimators.

### 4.2 Selection of a suitable value of \( T \)

The value of \( T \) gives us an idea about the unit measurement in which the observations are taking. Although in every case it is natural to take a certain value of \( T \) (e.g., if the observations are monthly and we have 120 observations, it is natural to take \( T = 120 \) months or \( T = 10 \) years) we can easily take any value of \( T \) and interpret it in terms of the original time measure of the data. Therefore, we can take advantage of this fact and choose a value of \( T \) according to certain criteria. To be in accordance with the asymptotic results we should choose a large value of \( T \) such that \( T/n \) is small.

Given a real time series, the decision of our test strongly depends on the choice of the parameter \( T \). For example in Table 1, we have simulated different ARMA(\( p, q \)) models where \( p, q \leq 1 \) and we observe that (depending of the selection of \( T \)) we can incorrectly reject the null hypothesis with an estimated probability higher or much higher than the significance level (10\% in Table 1). The same problem is also present in other existing tests in the literature (e.g., see [7] and [21]). This happens because they also have a parameter (instead \( T \)) that must be chosen before carrying out the test. We show this fact in the Section 5.

To deal with this problem, we propose to select a value of \( T \) such that the empirical probabilities of rejecting the null hypothesis are less than or equal to the significance level under a certain family of short memory process. Of course, it is impossible to consider every short memory process, hence we propose a value of \( T \) such that the empirical probabilities of rejecting the null hypothesis under every ARMA(\( p, q \)) process
for \( p, q \leq 1 \) and \( |\phi|, |\theta| \leq 0.8 \) will be at most the significance level. The reason to consider \( |\phi|, |\theta| \leq 0.8 \) is based on the fact that the process is close to non-invertibility and non-causality in the other cases. In all of the simulated cases (both short memory and long memory processes), it is observed that the empirical probabilities of rejecting the null hypothesis increases as \( T \) increases, see Figure 3. Therefore, to select the value of \( T \) to perform the test and to maximise the power of the test, we propose to select a maximum value of \( T \) such that the size of the test under every ARMA(\( p, q \)) where \( p, q \leq 1 \) will be less than or equal to the significance level. The reason why the ARMA(\( p, q \)) models are included where \( p, q \leq 1 \) is due to the broad practical utility of these models but nevertheless it is a bit more general than other studies that only include AR(1) models (see for instance [1]).

Table 1 shows that for sample size of \( n = 500 \) we must consider \( T \leq 0.06n \) as a possible value of \( T \) to carry out the test. It also shows us that the highest probabilities are obtained for the highest value of the parameter \( \phi \) and \( \theta \). In Figure 3, we show that for sample size of \( n = 500 \) and \( m = 1000 \) replications, the empirical probabilities of rejecting the null hypothesis under ARFIMA(\( 1, d, 1 \)) processes as a function of \( T \) when the AR and MA parameters are 0.8 and 0.8 respectively, for different values of \( d \). The horizontal black line shows the signification level (\( \alpha = 0.1 \)). It is observed that the power grows as the parameter \( d \) grows (i.e., the test works well). If \( d = 0 \) (short memory process), then the empirical probabilities of rejecting \( H_0 \) are greater than 0.1 for values of \( T > 0.06n \). Therefore, \( T \) grows as the empirical power grows and the optimal value is reached at \( T = 0.06n \) (for \( n = 500 \)). Following this idea, in Table 2 we show the proposed value of \( T \) to perform the test for different significance levels and sample sizes. It is observed that the fraction \( T/n \) is similar for every value of \( n \) between 500 and 5000.

5 Comparison with other tests

To compare the empirical power and the size of the proposed hypothesis test in this work (which we call the FOU test) with respect to other existing ones in the literature, we simulate different long and short memory processes and we analyze the test performance. We start by briefly describing each one of the tests that we use to make the comparison.

5.1 Other tests included in the comparison

1. The Lo test. This hypothesis test about short versus long range can be found in [18]. In this work, the author proposed the modified R/S statistic, which is defined as

\[
Q_n(q) := \frac{1}{\hat{\sigma}_n(q)} \left( \max_{1 \leq k \leq n} \sum_{1 \leq i \leq n}^k (X_i - \bar{X}_n) - \min_{1 \leq k \leq n} \sum_{1 \leq i \leq n}^k (X_i - \bar{X}_n) \right),
\]

where

\[
\hat{\sigma}_n^2(q) = S_n^2 + \frac{2}{n} \sum_{j=1}^q w_j(q) \sum_{i=j+1}^n (X_i - \bar{X}_n)(X_{i-j} - \bar{X}_n) \quad \text{being} \quad w_j(q) = 1 - \frac{j}{q+1}, \quad q < n,
\]
Figure 3: Empirical power as a function of $T/n$ from $m = 1000$ replications under $n = 500$ observations of an ARFIMA($1, d, 1$) model at significance level of $\alpha = 0.1$ for different values of $d$, where the parameters are $\phi = 0.8$ and $\theta = 0.8$. The horizontal line is the significance level of $\alpha = 0.1$. 
Table 1: Empirical probabilities, from 1000 simulations, to reject the short range dependence at significance level of 10% under several ARMA\((p,q)\) alternatives where \(p,q \leq 1\), for \(n = 500\) (sample size) and different values of \(T\).

| \(T\)               | 0.1\(n\) | 0.05\(n\) | 0.01\(n\) | 0.005\(n\) |
|---------------------|-----------|-----------|-----------|-----------|
| AR(1) : \(\phi = 0.2\) | 0.000     | 0.000     | 0.000     | 0.000     |
| AR(1) : \(\phi = 0.5\) | 0.000     | 0.000     | 0.000     | 0.000     |
| AR(1) : \(\phi = 0.8\) | 0.001     | 0.002     | 0.000     | 0.000     |
| MA(1) : \(\theta = 0.2\) | 0.000     | 0.000     | 0.000     | 0.000     |
| MA(1) : \(\theta = 0.5\) | 0.000     | 0.000     | 0.000     | 0.000     |
| MA(1) : \(\theta = 0.8\) | 0.010     | 0.062     | 0.086     | 0.000     |
| ARMA(1, 1) : \((\phi, \theta) = (0.4, 0.6)\) | 0.000     | 0.006     | 0.067     | 0.000     |
| ARMA(1, 1) : \((\phi, \theta) = (0.6, 0.6)\) | 0.288     | 0.017     | 0.024     | 0.000     |
| ARMA(1, 1) : \((\phi, \theta) = (0.6, 0.4)\) | 0.006     | 0.004     | 0.039     | 0.001     |
| ARMA(1, 1) : \((\phi, \theta) = (0.3, 0.8)\) | 0.000     | 0.010     | 0.059     | 0.000     |
| ARMA(1, 1) : \((\phi, \theta) = (0.5, 0.8)\) | 0.163     | 0.005     | 0.016     | 0.000     |
| ARMA(1, 1) : \((\phi, \theta) = (0.7, 0.8)\) | 0.987     | 0.690     | 0.033     | 0.003     |
| ARMA(1, 1) : \((\phi, \theta) = (0.8, 0.3)\) | 0.397     | 0.027     | 0.022     | 0.000     |
| ARMA(1, 1) : \((\phi, \theta) = (0.8, 0.5)\) | 0.943     | 0.517     | 0.002     | 0.000     |
| ARMA(1, 1) : \((\phi, \theta) = (0.8, 0.7)\) | 1.000     | 0.946     | 0.127     | 0.017     |

Table 2: Suitable value of the fraction \(T/n\) for different sample sizes and significance levels.

| \(\alpha/n\) | 500  | 1000 | 3000 | 5000 |
|---------------|------|------|------|------|
| 0.010         | 0.009| 0.045| 0.058| 0.062|
| 0.025         | 0.038| 0.050| 0.062| 0.065|
| 0.050         | 0.054| 0.056| 0.065| 0.068|
| 0.075         | 0.057| 0.060| 0.068| 0.069|
| 0.100         | 0.060| 0.063| 0.070| 0.072|
According with the suggestion in [12], we use $M_t$ tootic distribution under short memory and long memory dependence for the statistics adequately readjusted. It is important to highlight, that this work includes the asymptotically proved that $M_t$.

The Lobato-Robinson test is based on the statistic $Q_n$, we will call the Gromykov et al test as the $V/S$ test. The details and the asymptotic results for the case $0 < d < 1/2$ can be found in [7].

2. The $V/S$ test. In [7], the author proposed a rescaled variance test based on the $V/S$ statistic to test long range against short range. This work shows that the statistic has a simpler asymptotic than the modified rescaled range test (Lo’s test). The authors proposed the $M_n (q)$ statistic, which they called $V/S$ or rescaled variance statistics and is defined by

$$M_n(q) := \frac{1}{\tilde{\sigma}_n^2(q)n^2} \left[ \sum_{k=1}^n \left( \sum_{j=1}^k (X_j - \bar{X}_n) \right)^2 - \frac{1}{n} \left( \sum_{k=1}^n \sum_{j=1}^k (X_j - \bar{X}_n) \right)^2 \right].$$

The name $V/S$ comes from variance/S because the statistic $M_n(q)$ can be expressed as $M_n(q) = \frac{1}{\tilde{\sigma}_n^2(q)n^2} \left( \tilde{\sigma}_n^2(q) \right)^2$. Observe that $Q_n(q) = \frac{1}{\tilde{\sigma}_n^2(q)} (\max_{1 \leq k \leq n} S_k - \min_{1 \leq k \leq n} S_k^*)$, therefore the $M_n(q)$ statistic considers the sample variance of the values $S_k^*$ instead of the range of $S_k^*$. This work shows that the statistic adequately readjusted. It is important to highlight, that this work includes the asymptotic distribution under short memory and long memory dependence for the statistics $M_n(q)$ and $Q_n(q)$. Under some conditions of $q$ and the fourth cumulants the authors proved that $M_n$ converges to $F_{KS}(\sqrt{\pi}x)$, where $F_{KS}$ is the asymptotic distribution of the Kolmogorov-Smirnov statistic.

3. The Gromykov et al test. In [12], the authors proposed to split the original sample $X_1, X_2, ..., X_n$ into $m$ blocks of size $l$ and construct the periodogram of the entire sample given by $I_n(\lambda) = \frac{1}{2\pi n} \left| \sum_{j=1}^n X_j e^{ij\lambda} \right|^2$ and the periodogram of each block $I_{n,i}$ for $i = 1, 2, ..., m$ then work with

$$Q_{n,m}(s) = \sum_{j=1}^s \frac{I_n(\lambda_j)}{\frac{1}{m} \sum_{i=1}^m I_{n,i}(\lambda_j)}$$

as a test statistic, where $s$ is the number of Fourier frequencies to consider and $\lambda_j = \frac{2\pi j}{n}$.

If $\{X_t\}$ is a stationary linear process that is defined as $X_t = \sum_{j=0}^{+\infty} a_j \varepsilon_{t-j}$ where $\{\varepsilon_j\}$ are i.i.d. with zero mean and variance $\sigma^2 > 0$, is a short memory process in the sense that it has an spectral density of the form $f(\lambda) = |\lambda|^{-2d} g(\lambda)$ for $d = 0$ and $\sum_{j=1}^{+\infty} |a_j| < +\infty$, then for fixed $s$ the authors proved that $Q_{n,m}(s) \overset{d}{\to} Q(s)$ when $m \to +\infty$ being $m = o(n)$ where $Q(s)$ has Gamma distribution with parameters $(s, 1)$. The details and the asymptotic results for the case $0 < d < 1/2$ can be found in [7]. According with the suggestion in [12], we use $m = \sqrt{n}$ as a number of blocks. From now on, we will call the Gromykov et al test as the $Q$ test.

4. The Lobato–Robinson test. The Lobato-Robinson test is based on the statistic
defined by $LR = t^2$ where

$$t = \sqrt{m} \frac{\sum_{j=1}^{m} \nu_j I(\lambda_j)}{\sum_{j=1}^{m} I(\lambda_j)}$$

where $\nu_j = \log j - \frac{1}{m} \sum_{i=1}^{m} \log i$, $\lambda_j = \frac{2\pi j}{m}$ and $I(\lambda)$ is the periodogram. Under certain conditions of the number of blocks $m$, the authors proved that if the spectral density is twice bounded differentiable near 0 and $d = 0$, then the value $t$ converges in distribution of an standard normal distribution.

**Remark 8.** It is important to note that all of the considered tests have a parameter to select before applying it ($q$ in Lo and V/S, $m$ and $s$ in $Q$ test and $m$ in $LR$). The correct choice of the parameter is very difficult because if we select a value that is too small or too high, then each of the considered tests will have a bias to incorrectly reject the null hypothesis in some cases or incorrectly non-reject the null hypothesis in others.

To illustrate the affirmation given in Remark 8, we observe in Figure 4 the behavior of the empirical probability of rejecting the null hypothesis in the case in which the observed time series corresponds to an ARMA(1,1) model where $\theta = \phi = 0.8$. The empirical probability of rejecting the null hypothesis decreases as $q$ increases in Lo and V/S tests, decreases as $s$ increases in the Gromykov et al test and increases as $m$ increases in the LR test. The same behavior is repeated for any other simulated time series (short or long memory).

![Figure 4: Empirical probabilities for $m = 1000$ replications of reject the null hypothesis for an ARMA(1,1) model with parameters $\phi = \theta = 0.8$ at the 10% level of significance where the sample size is $n = 1000$.](image-url)
Table 3: Optimal value of $q$ for the Lo test and V/S test at the significance level of 10% (5%), optimal value of $s$ for the $Q$ test, and optimal value of $m$ for the LR test, in function of sample size $n$.

| $n$ | 500  | 1000 | 3000 | 5000 |
|-----|------|------|------|------|
| Lo  | 16(15)| 23(22)| 38(37)| 51(50)|
| V/S | 44(32)| 66(47)| 90(80)| 128(100)|
| $Q$  | 15(7) | 25(9) | 21(11) | 22(12) |
| LR  | 17(17) | 27(28) | 57(54) | 81(77) |

5.2 Short and long range processes considered in the comparison

To carry out a comparison between the FOU test and the other tests described in this subsection, we use the optimum value of $T$ according to Table 2. For the other tests (Lo, V/S, $Q$ and LR) we use the same criterion proposed in Subsection 4.2 and given in Table 3. In this way, all of the tests that we have considered have an empirical probability less than or equal to 0.1 of incorrectly rejecting the null hypothesis under the same family of short range dependence models, that is, under every ARMA(1,1) where $|\phi|, |\theta| \leq 0.8$. We have made the comparison looking the power under several long range alternatives and the size of the considered tests under several short range processes. In the comparison, we have considered ARFIMA, LARCH, FGN and FOU models. All these processes include (depending on the value of their parameters) both short and long range memory, whose definitions are outlined below.

1. A LARCH model is defined by $X_k = r_k^2$, where $r_k = \sigma_k \varepsilon_k$, $\sigma_k^2 = (\alpha + \sum_{j=1}^{+\infty} \beta_j r_{k-j})^2$ and $\{\varepsilon_k\}$ are white noise. Depending on the speed at which the coefficients tend to zero we can have a short or long memory process.

2. $\{X_t\}$ is an ARFIMA($p,d,q$) process when $(1 - B)^{-d} X_t$ is an ARMA($p,q$) process. If $d = 0$ we have an ARMA($p,q$) short memory process and when $0 < d < 1/2$ we have a long memory process.

3. The fractional Gaussian process (FGN($H$)) is defined as a stationary Gaussian centered process $\{X_t\}_{t \in \mathbb{N}}$ such that the autocovariance function is given by $\mathbb{E}(X_t X_0) = \frac{1}{2} \left( (t + 1)^{2H} - 2t^{2H} + (t - 1)^{2H} \right)$. It is known that when $H > 1/2$, we have a long memory process and if $H \leq 1/2$ then we have a short memory process.

4. The fractional Ornstein–Uhlenbeck process FOU($\lambda, \sigma, H$) was defined in Section 2. When $H \leq 1/2$ the process has short memory and when $H > 1/2$ long memory. When $H = 1/2$ we have the classical Ornstein–Uhlenbeck process.

5.3 Power comparison

In Table 4 we show the power at 10% of the tests under different ARFIMA($1,d,1$) alternatives for different values of $d > 0$ (i.e., long memory processes). The power of
each test was obtained from 1000 replications. Table 4 shows that the FOU test detects long memory when none of the parameters is close to zero. It is also observed that the performance of the test improves as both parameters increase their value and when \(d\) increases (as expected). When the value of both parameters is high, the FOU test has the best performance, while in almost all of the other cases, the LR test obtain the best results.

In Table 5 we have considered FGN(\(H\)) for different values of \(H > 1/2\) and the LARCH(0, \(d\), 0) long memory process where \(\beta_j = j^{d-1}\) and \(\alpha = 0.1\). For these families, we do not include the performance of the FOU test because the estimation of \(H\) is clearly less than 1/2, and therefore the test non-rejects the null hypothesis. This occurs due to what was observed in Remark 6 and Remark 7. The LR test obtains the best performance in all cases for a sample size of \(n = 5000\).

### 5.4 Size comparison

In this subsection, we have considered the LARCH(1, 0, 1) short memory process, where \(\phi = 0.1\), \(\theta = 0.2\), \(\alpha = 0.1\), \(\beta_j = \phi^{j-1}(\phi - \theta)\) and the \(\varepsilon_i\) are i.i.d. normal standard variables, FGN(\(H = 0.5\)) and Ornstein–Uhlenbeck process where \(\sigma = 1\) and \(\lambda = 0.8\). Concerning the ARMA process, we know that none of the hypothesis tests that we have considered fail to reject the null hypothesis for every ARMA(1,1) (in the sense that the percentage of reject the hypothesis of short memory process is not greater that the significance level) where \(|\theta|, |\phi| \leq 0.8\). For this reason we have considered ARMA(\(p,q\)) where \(p\) or \(q\) are greater than 1. We can draw important conclusions from Table 6. The V/S, Lo, Q and LR test work well under FGN, but they are terribly wrong in some cases of AR(2) alternatives and under Ornstein–Uhlenbeck processes, and the error get worse as the sample size increases. The FOU test is never wrong under the FGN and LARCH short memory models. Under the ARMA models the FOU test work well. Only under the Ornstein–Uhlenbeck processes observed in \([0,T]\) the empirical percentage of rejection of the null hypothesis in the FOU test is slightly greater than the size. In addition, this percentage increases as \(T\) decreases. This is to be expected because if \(T \to 0\), then the autocovariances of the process goes to \(V(X_0) > 0\), and therefore we get closer to a long memory process.

### 6 Application to real data

In this section we analyze an empirical application of a time series that has already been studied and modeled correctly with an ARFIMA long memory model in the work [16]. The dataset consists of weekly measurements of affluent energy generated by hydroelectric dams in Uruguay between the first week of 1909 and the last week of 2012. The time series has length 5408. Each observation corresponds to the weekly inflow energy generated by the three Uruguayan dams, measures in MWh. This time series is strongly related with the time series generated by the dam contributions and is also a good fit to a large memory model ([15]). After being seasonally adjusted and centered, this time
Table 4: Power comparison from 1000 simulations under several ARFIMA(1, d, 1) alternatives and different sample sizes (n) at level 10%. All calculations were performed using the values of the test parameters suggested in \cite{2} and \cite{3}.

| d   | n     | (0.3, 0.8) | (0.5, 0.8) | (0.7, 0.8) |
|-----|-------|------------|------------|------------|
|     | (500) | (100)      | (5000)     | (500)      | (1000)     | (5000)     |
|     | (500) | (100)      | (5000)     | (500)      | (1000)     | (5000)     |
|     | (500) | (100)      | (5000)     | (500)      | (1000)     | (5000)     |
| 0.1 | FOU   | 0.078      | 0.024      | 0.005      | 0.016      | 0.021      | 0.000      | 0.007      | 0.048      | 0.020      |
|     | V/S   | 0.155      | 0.175      | 0.270      | 0.189      | 0.207      | 0.294      | 0.181      | 0.209      | 0.277      |
|     | Lo    | 0.144      | 0.194      | 0.317      | 0.165      | 0.201      | 0.319      | 0.171      | 0.191      | 0.317      |
|     | Q     | 0.054      | 0.055      | 0.206      | 0.059      | 0.066      | 0.263      | 0.088      | 0.075      | 0.273      |
|     | LR    | 0.109      | 0.192      | **0.489**  | 0.114      | 0.205      | **0.540**  | 0.142      | **0.254**  | **0.542**  |
| 0.2 | FOU   | 0.079      | 0.029      | 0.020      | 0.015      | 0.020      | 0.000      | 0.291      | 0.318      | 0.330      |
|     | V/S   | 0.259      | 0.333      | 0.496      | 0.284      | 0.337      | 0.472      | 0.310      | 0.209      | 0.452      |
|     | Lo    | 0.280      | 0.358      | 0.608      | 0.257      | 0.358      | 0.585      | 0.303      | 0.392      | 0.587      |
|     | Q     | 0.062      | 0.068      | 0.587      | 0.050      | 0.073      | 0.586      | 0.108      | 0.092      | 0.615      |
|     | LR    | **0.300**  | **0.490**  | **0.940**  | **0.301**  | **0.521**  | **0.929**  | **0.330**  | **0.559**  | **0.925**  |
| 0.3 | FOU   | 0.082      | 0.038      | 0.015      | 0.071      | 0.037      | 0.018      | **0.723**  | **0.780**  | **0.895**  |
|     | V/S   | 0.390      | 0.464      | 0.689      | 0.396      | 0.493      | 0.667      | 0.394      | 0.487      | 0.693      |
|     | Lo    | 0.408      | 0.555      | 0.796      | 0.448      | 0.577      | 0.818      | 0.438      | 0.564      | 0.808      |
|     | Q     | 0.047      | 0.060      | 0.886      | 0.070      | 0.071      | 0.883      | 0.077      | 0.077      | 0.889      |
|     | LR    | **0.504**  | **0.748**  | **0.999**  | **0.540**  | **0.767**  | **0.998**  | 0.592      | 0.774      | **0.998**  |
| 0.4 | FOU   | 0.057      | 0.031      | 0.005      | 0.324      | 0.350      | 0.659      | **0.987**  | **0.998**  | **1.000**  |
|     | V/S   | 0.502      | 0.603      | 0.798      | 0.510      | 0.611      | 0.798      | 0.509      | 0.635      | 0.783      |
|     | Lo    | 0.525      | 0.688      | 0.907      | 0.551      | 0.698      | 0.924      | 0.571      | 0.678      | 0.903      |
|     | Q     | 0.060      | 0.044      | 0.982      | 0.051      | 0.044      | 0.989      | 0.043      | 0.047      | 0.985      |
|     | LR    | **0.702**  | **0.923**  | 1.000      | **0.732**  | **0.927**  | 1.000      | **0.770**  | **0.946**  | **0.999**  |
|     |       | (0.8, 0.3) | (0.8, 0.5) | (0.8, 0.7) | (0.8, 0.5) | (0.8, 0.7) | (0.8, 0.5) | (0.8, 0.7) | (0.8, 0.5) | (0.8, 0.7) |
**Table 5:** Power comparison from 1000 simulations under several fractional Gaussian noise (FGN) and LARCH(0, d, 0) alternatives and different sample sizes (n) at level 10%. All calculations were performed using the values of the test parameters suggested in 2 and 3.

| H | n | 500 | 1000 | 5000 | d | n | 500 | 1000 | 5000 |
|---|---|-----|------|------|---|---|-----|------|------|
| 0.6 | V/S | 0.122 | 0.154 | 0.248 | 0.1 | V/S | 0.044 | 0.066 | 0.099 |
|   | Lo | 0.147 | 0.208 | 0.312 |   | Lo | 0.013 | 0.029 | 0.040 |
|   | Q  | 0.020 | 0.049 | 0.200 |   | Q  | 0.126 | 0.146 | 0.212 |
|   | LR | 0.099 | 0.180 | 0.487 |   | LR | 0.075 | 0.122 | 0.271 |
| 0.7 | V/S | 0.210 | 0.266 | 0.423 | 0.2 | V/S | 0.041 | 0.063 | 0.107 |
|   | Lo | 0.273 | 0.345 | 0.602 |   | Lo | 0.009 | 0.033 | 0.062 |
|   | Q  | 0.050 | 0.054 | 0.552 |   | Q  | 0.138 | 0.158 | 0.207 |
|   | LR | 0.281 | 0.450 | 0.913 |   | LR | 0.084 | 0.153 | 0.354 |
| 0.8 | V/S | 0.285 | 0.381 | 0.640 | 0.3 | V/S | 0.053 | 0.058 | 0.151 |
|   | Lo | 0.408 | 0.541 | 0.776 |   | Lo | 0.017 | 0.034 | 0.073 |
|   | Q  | 0.051 | 0.057 | 0.866 |   | Q  | 0.141 | 0.151 | 0.215 |
|   | LR | 0.494 | 0.766 | 0.999 |   | LR | 0.088 | 0.152 | 0.381 |
| 0.9 | V/S | 0.387 | 0.494 | 0.736 | 0.4 | V/S | 0.050 | 0.058 | 0.116 |
|   | Lo | 0.532 | 0.685 | 0.903 |   | Lo | 0.026 | 0.031 | 0.066 |
|   | Q  | 0.041 | 0.042 | 0.980 |   | Q  | 0.144 | 0.148 | 0.242 |
|   | LR | 0.721 | 0.910 | 1.000 |   | LR | 0.095 | 0.186 | 0.383 |

**Table 6:** Size at level 10% of any considered test under several short range models. The empirical probabilities were calculated from 1000 replications. The ARMA(0.4, 0.55) means an AR(2) where $\phi = (0.4, 0.55)$. The OU case means an Ornstein–Uhlenbeck process where $\sigma = 1$ and $\lambda = 0.8$, observed in $[0, T]$ being $T = 100$, 50 and 10. All of the calculations were performed using the values of the test parameters suggested in 2 and 3.

|   | FOU | V/S | Lo | Q | LR |
|---|-----|-----|----|---|----|
| 1000 | 0.000 | 0.300 | 0.214 | 0.542 | 0.920 |
| 5000 | 0.000 | 0.305 | 0.120 | 0.056 | 0.892 |
| AR(0.4, 0.55) | 0.000 | 0.075 | 0.311 | 0.264 | 0.680 |
| FGN(H = 0.5) | 0.022 | 0.230 | 0.576 | 0.481 | 0.898 |
| OU(T = 100) | 0.053 | 0.168 | 0.959 | 0.915 | 0.997 |
| LO(T = 50) | 0.123 | 0.175 | 0.697 | 0.020 | 0.642 |
| LARCH(1, 0.1) | 0.000 | 0.000 | 0.099 | 0.029 | 0.212 |

18
Table 7: Parameter estimations for the affluent energy data series fitted to a FOU($\lambda_1, \lambda_2, \sigma, H$) model observed in $[0, T]$ for $T = 389.376$.

| $\lambda_1$ | $\lambda_2$ | $\hat{\sigma}$ | $\hat{H}$ | critical value at 10% | p-value |
|-------------|-------------|-----------------|----------|----------------------|---------|
| $10^{-14}$  | 5.8311      | 4.4593          | 0.7078   | 0.5033               | 0.0000  |

series has a good fit to an ARFIMA(3, $d$, 1) long memory process (see [16] for details). In Figure 5 we show the autocorrelation function of this time series before and after adjusted seasonally.

![Autocorrelation of the original time series (left) and seasonally adjusted time series (right).](image)

Figure 5: Autocorrelation of the original time series (left) and seasonally adjusted time series (right).

The main improvement of modelling this time series using long memory processes can be seen in the intensity curves (see the definition in [15]). The intensity curves can be used as a measure of persistence of droughts, especially the minimum curve, essential for energy planning. In [11] (Figure 14 and Figure 55) the intensity curves modeled by ARMA processes are shown, while in [15] (Figure 20) they are obtained by ARFIMA processes.

To perform the proposed test we have used (according to Table 2) $T = 0.072n = 389.376$. The FOU test clearly rejects the hypothesis of short memory dependence. In Table 7 we show the parameter estimation for this real time series fitted to a FOU($\lambda_1, \lambda_2, \sigma, H$) model. The estimated model is a FOU($\lambda_1 = 0, \lambda_2 = 5.8311, \sigma = 4.4593, H = 0.7078$) model, which corresponds to a long memory process. We also observe that the estimated value of $H = 0.7078$ is much higher than the critical value of the test. In this empirical application, the hypothesis test result is consistent with the previous results in [16].
7 Conclusions

In this work we present a new hypothesis test to contrast short memory versus long memory in time series, which is based on the Fractional Iterated Ornstein–Uhlenbeck processes. We present an implementation of the test and we carry out a simulation study that includes several families of processes of both short memory and long memory. We also compare the results with other hypothesis tests. In addition, we propose a suitable value of \( q \) to be used for the Lo test and V/S test, \( m, s \) for the Q test and \( m \) for LR test. With this election of the parameters, all of the considered tests maintain empirical probabilities of rejecting \( H_0 \) under every ARMA\((1,1)\) where \( |\phi|, |\theta| \leq 0.8 \), and every FGN and LARCH short memory process. Finally, we realise a real application in the time series of hydroelectric dams in Uruguay. A summary of the main conclusions that can be drawn from the simulation study follows:

- The FOU test has the best performance under the null hypothesis. There is more than one family of short memory processes that includes several examples where the other tests drive to a wrong decision, while the FOU test does not make a mistake in its decision.

- While the other tests can be wrong with a probability higher than desired under both the null hypothesis and the alternative hypothesis, the test proposed in this work is wrong with an excessively low probability under the null hypothesis. Therefore, when the null hypothesis is rejected in the FOU test, we can have greater confidence that the observed time series really does have a long memory.

- Under the ARFIMA\((1,d,1)\) for \( d > 0 \) where the parameters AR and MA are not very small, the FOU test has the best performance among all of the tests considered, in terms of getting the best power.

- In some cases, the FOU test in some cases is able to detect long memory processes when the other tests do not detect it.

Acknowledgements

We wish to thank Alejandro Cholaquidis for your help and support in the simulation study and José Rafael León for various rich conversations about this topic.

References

[1] Andrews, D. W. K. (1991). Heteroscedasticity and autocorrelation consistent covariance estimation. *Econometrica* 59, 817–858.

[2] Arratia, A., Cabaña, A. & Cabaña, E. (2016). A construction of Continuous time ARMA models by iterations of Ornstein-Uhlenbeck process, *SORT* Vol 40 (2) 267-302.
[3] Beran, J., Feng, Y., Ghosh, S., Kulik, R. (2013). Long-memory processes. Monographs on Statistics and Applied Probability (61). Springer.

[4] Brockwell, P. J. & Davies R. A. Time Series: Theory and Method 2nd ed. (1991). Springer, New York

[5] Cheridito, P., Kawaguchi, H. & Maejima, M., Fractional Ornstein-Uhlenbeck Processes. Electronic Journal of Probability, 8(3): 1-14,(2003).

[6] Geweke, J. & Porter-Hudak, S. (1983), The Estimation and Application of Long-Memory Time Series Models, Journal of Time Series Analysis 4, 221–238.

[7] Giraitis, L., Kokoszka, P., Leipus, R., Teysière, G. Rescaled variance and related tests for long memory in volatility and levels. Journal of Econometrics, (2003) 112, 265-294.

[8] Hasslett, J. & Raftery, E. Space-time modelling with long memory dependence: Assessing Ireland’s win power resource (1989). Journal of Applied Statistics, 38, 1-50.

[9] Hipel, K. W. & McLeod, A. I. Time Series Modelling of Water Resources and Environmental Systems (1994). Elsevier, New York.

[10] Hurst, H.E. Long-term storage capacity of reservoirs, Transactions of the American Society of Civil Engineers, Volume 116, 770-799 (1951).

[11] Graneri, J. R. Análisis de datos hidrológicos y procesos de memoria larga. https://hdl.handle.net/20.500.12008/24150 (2014).

[12] Gromykov, G., Haye, M. O. & Phillipe, A.A frequency-domain test for long range dependence Statistical Inference for Stochastic Processes (2018) 513-526.

[13] Istas, I. & Lang, G., Quadratic variations and estimation of the local Hölder index of a Gaussian process. Annals de l’Institute Henry Poincaré, 23(4): 407-436, (1997).

[14] Kalemkerian, J. Parameter Estimation for Discretely Observed Fractional Iterated Ornstein–Uhlenbeck Processes. (2020). arXiv:2004.10369

[15] Kalemkerian, J., Performance de distintos modelos Farima ajustados a la serie de aportes hidrológicos a las represas de Uruguay. Actas del X Clatse, Córdoba, (2012).

[16] Kalemkerian, J., Prediction using ARFIMA and FOU models of affluent energy. Memoria Investigaciones en Ingeniería, 15, 109-124, (2017).

[17] Kalemkerian, J., & León, J. R. Fractional iterated Ornstein-Uhlenbeck Processes, ALEA A Lat. Am. J. Probab. Math. Stat. (2019) 16 1105–1128.

[18] Lo, A. Long-term memory in stock market prices. Econometrica. (1991). 59, 1279–1313.

21
[19] Lee D., Schmidt P. On the power of the KPSS test of stationarity against fractionally integrated alternatives. *Journal of Econometrics* (1996). 73, 285–302.

[20] Palma, W. Long Memory Time Series: Theory and Methods (2007). John Wiley, Hoboken, NJ.

[21] Teverovsky, V., Taqqu, M. S., & Willinger, W. A critical look at Lo’s modified R/S statistic. *Journal of Statistical Planning and Inference*, (1999). 80, 211–227.