Quadratic birational transformations in computer simulation of building structures

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Abstract. Quadratic transformations are a perspective way of form-making in architectural and engineering-construction design. A quadratic (nonlinear) transformation is based on the establishment of one-to-one correspondence between the algebraic n- and 2nd-order curves. In particular, lines correspond to the second-order curves, which are widely used in the design of gridwork building structures and utility lines. Quadratic transformations are used to model the profile of rotor blades of environmentally friendly wind farms. We developed algorithms for building corresponding points for the practical application of nonlinear transformations in the computer simulation of building structures. We have shown that the inversion relative to the circle is a central quadratic transformation with imaginary fundamental point, coinciding with the cyclic points of the plane. We considered an example of using the inversion for building the rotor blade profile of a wind power installation and a cam profile. Quadratic transformation algorithms are practically realized using computer graphics and specialized software tools.

1. Introduction
Modern means of computer graphics allow us to apply new methods of form-making in architectural and building design. One of these methods is the use of quadratic transformation in order to obtain lines and surfaces satisfying the given geometric conditions. Quadratic Cremona transformation \( \Omega \) of the plane is an almost everywhere one-to-one transformation, when lines turn to second-order curves [1-3]. Transformation \( \Omega \) can be given by two pairs of projective sheaves \( F_1(j_3, j_2, a_1) \sim F'_1(j'_2, j'_3, a'_1) \) or \( F_2(j_3, j_1, a_2) \sim F'_2(j'_1, j'_3, a'_2) \). The projective sheaves set the correspondence of plane fields \( \Pi \) and \( \Pi' \), when point \( A' = a'_1 \cap a'_2 \) of field \( \Pi' \) corresponds to point \( A = a_1 \cap a_2 \) of field \( \Pi \) (Figure 1). Lines \( j'_2, j'_1 \) intersecting at point \( F'_3 \) projectively correspond to common beam \( j_3 = F_1F_2 \) [4]. Quadratic transformation \( \Omega \) is used for building regular algebraic curves with predefined properties [5].
2. Properties of quadratic transformation

Property 1. Two pairs of projective sheaves $F_1 \sim F'_1$ and $F_2 \sim F'_2$ generate another pair of projective sheaves $F_3(j_1, j_2, a_3) \sim F'_3(j'_1, j'_2, a'_3)$. The proof follows from the consideration of projective correspondences in fields $\Pi$ and $\Pi'$ [6].

Property 2. If we specify projective coordinates with reference triangles $F_1F_2F_3$ and $F'_1F'_2F'_3$ in fields $\Pi$ and $\Pi'$ and take a pair of corresponding points $A, A'$ as unit points, quadratic mapping $\Omega$ will be as follows $x'=1/x$, $y'=1/y$. The proof follows from the invariance property of the complex ratio of the four points on the sides of the coordinate triangles. Making a mirror reflection of the plane of $\Pi'$ in the plane of $\Pi''$ relative to bisector line $y'=x'$, we obtain the quadratic correspondence of the coincident points of fields $\Pi$ and $\Pi''$ in the canonical form: $x''=1/y$, $y''=1/x$. These equations describe the central involutory transformation - symmetry relative to the single hyperbola $y=1/x$. The transformation centre coincides with the hyperbola centre.

Thus, quadratic transformation $\Omega$ with real $F$-points is equivalent to the symmetry relative to the single equilateral hyperbola. By the analogy with the inversion relative to the circle, the transformation $\Omega$ is an inversion relative to the hyperbola [7].

3. Parabola inversion

Conformal plane transformations [8] are used to study the airfoil flow. In particular, the inversion of the parabola relative to the circumference makes it possible to obtain a theoretical profile of the rotor blade of an environment-friendly wind farm, which is called Zhukovsky’s profile (Figure 2). In case of such mapping, the angle between the surface normal and the flow direction is preserved at each point of the profile. S.A. Chaplygin studied the aerodynamic properties of a wing with such cross-sectional shape (Figure 3) [9].

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When the center of transformation $\Omega$ is displaced relative to the center of the invariant circle, we obtain a private case of Hirsch transformation [10,11]. This transformation is not conformal, but it
allows us to build algebraic curves with the given dynamical properties. For example, a parabola can be transformed into the fourth-order closed algebraic curve ABCDA (Figure 4). Such curves are used for designing cam profiles [12].

4. Direct problem algorithm

Suppose that quadratic correspondence \( \Omega \) is given by three pairs of F-points and by a pair of corresponding points \( A \sim A' \). Two pairs of F-points can be imaginary conjugate. Let us draw a conic section \( e \) in field \( \Pi \) passing through \( F_1, F_2, A \), for which point \( F_3 \) and line \( j_3 \) are a pole and a polar.

We will draw the same conic section \( e' \) in field \( \Pi' \). Let us consider an auxiliary collinear correspondence \( \Delta \) of fields \( \Pi, \Pi' \), in which points \( F_2', F_1', F_3', A' \) of field \( \Pi' \) correspond to points \( F_1, F_2, F_3, A \) of field \( \Pi \) [13-18].

**Lemma 1.** Quadratic map \( \Omega \) and collineation \( \Delta \) set the same one-to-one correspondence between the points of conic sections \( e \) and \( e' \). Arbitrarily point \( B \) lying one corresponds to the same point \( B' \) in the maps of \( \Omega \) and \( \Delta \). The uniqueness is violated at F-points. The proof of Lemma 1 is based on the consideration of projective correspondences in the maps of \( \Omega \) and \( \Delta \).

**Lemma 2.** If in projective correspondence \( \Delta \) point \( B_\Delta \) of field \( \Pi' \) corresponds to point \( B \) of field \( \Pi \), in quadratic correspondence \( \Omega \) point \( B \) is mapped to point \( B' \) lying on line \( b_3'=F_3'B_\Delta \) and conjugate to point \( B_\Delta \) relative to curve \( e' \). In this case, point \( B_\Delta \) is mapped to point \( B_1 \) lying on \( b_3=F_3B \) and conjugate to point \( B \) relative to curve \( e \). The proof follows from Lemma 1 and from the consideration of the corresponding elements of transformation \( \Omega \).

It follows from Lemma 2 that quadratic map \( \Omega \) can be represented in the form \( \Omega=\Delta\gamma \), where \( \gamma \) is Hirsch transformation. Lemma 2 and its corollary fact allow us to formulate a universal algorithm of building the corresponding points in quadratic map \( \Omega \) set both by real and imaginary F-points.

**Action 1.** In field \( \Pi \), we draw conic section \( e \) passing through given point \( A \) and through a pair of points \( F_1,F_2 \), which can be either real or imaginary. Point \( F_3 \) and line \( j_3=F_1F_2 \) must be a pole and a polar relative to \( e \). Similarly, we find conic section \( e' \) on the plane of \( \Pi' \).

**Action 2.** Suppose that point \( B \) is given in the plane of \( \Pi \). It is required to find point \( B'=\Omega(B) \). We find point \( B_\Delta \) on the plane of \( \Pi \) corresponding to point \( B \) in collineation \( \Delta \). Then, in field \( \Pi' \), we mark point \( B' \) on line \( F_3'B_\Delta \) conjugated with point \( B_\Delta \) in the polar correspondence with nucleus of \( e' \). Point \( B' \) is the desired image of point \( B \) (Figure 5).

**Figure 5.** Direct problem solving.  
**Figure 6.** Auxiliary problem.
For a constructive implementation of the algorithm, it is necessary to solve an auxiliary problem: build a conic section passing through given point A and a pair of imaginary points F*1, F*2 on line j if pole F of line j is given relative to the desired curve.

**Solution of the auxiliary problem.** Suppose that imaginary F-points are given as double points of elliptic involution $\sigma$ on line j. The conjugate involution points are projected from Laguerre point L by an orthogonal sheaf of lines (Figure 6). Let us indicate a pair of corresponding points Q ≈ Q′ in involution $\sigma$. Line q=Q′F polarly corresponds to point Q relative to the required conic section e, therefore (AB1Q) = -1. Hence, we determine an additional point of curve e. If we take another pair of points corresponding to $\sigma$ on line j, we find one more point of curve e. Thus, you can build as many points of the desired conic section. The problem has a unique solution [18].

5. **Example of direct problem solution**

Quadratic correspondence $\Omega$ of fields $\Pi$, $\Pi'$ is given by a pair of corresponding points A~A' and triples of fundamental points (F*1, F*2, F3)~ (F*1', F*2', F3'). Imaginary conjugate F-points are given by elliptic involutions $\sigma(L, j3)$ and $\sigma'(L', j3')$ (Figure 7). It is required to build in field $\Pi'$ an image (homaloid) of line m lying in field $\Pi$.

According to the developed algorithm, we draw conic section e in the plane of $\Pi$ passing through given point A and imaginary points F*1, F*2. Point F3 and line j3 are a pole and a polar relative to e. We build the same on the plane of $\Pi'$ and obtain conical section e' (Figure 8).

![Figure 7. Initial data.](image)

![Figure 8. Building of m' homaloid.](image)

We find points P1~P1' on lines j and j' corresponding to $\Delta$ in the projectivity. An arbitrary line m of field $\Pi$ corresponds to line m'\$\Delta$ in $\Delta$. According to Lemma 1, in quadratic correspondence $\Omega$ and in linear correspondence $\Delta$ points M, N of the intersection of m and e correspond to points M', N' of the intersection of line m'\$\Delta$ and curve e'. Consequently, the desired image (homaloid) m' of line m must pass through points M', N' and through F-points F*1', F*2', F3'. According to Lemma 2, an arbitrary point of line m passes into the point of homaloid m' using linear mapping $\Delta$ and Hirsch transformation. For example, point B is linearly transformed into point B' in Hirsch transformation (see Figure 5).

6. **Inverse problem algorithm**

In plane fields $\Pi\neq\Pi'$ there are seven pairs of points A, B, C, D, E, R, T~ A', B', C', D', E', R', T', which are mutually corresponding in the quadratic birational mapping $\Omega$. It is required to find the fundamental points of mapping $\Omega$. 


Let us match the planes of \( \Pi, \Pi' \) and projectively transform one of the fields (for example, field \( \Pi \)) to match points \( A=A', B=B', C=C', D=D' \) [20]. These points determine a sheaf of conic sections \( \Psi \). Let us mark two pairs of given points that do not coincide with the base points of sheaf \( \Psi \), for example, \( E-E', R-R' \). Let us consider point \( K \) moving along an arbitrary conic section \( e \) of sheaf \( \Psi \) and make a double projection. First projection: points \( E \) and \( R \) are projected from the center of \( K \) to curve \( e \). We obtain point rows \( Q_E \) and \( Q_R \) on it. Second projection: point rows \( Q_E \) and \( Q_R \) are projected from points \( E', R' \) onto curve \( e \). We obtain point rows \( K'_E \) and \( K'_R \) on \( e \).

**Lemma 3.** Point rows \( K'_E \) and \( K'_R \) are projective. The proof follows from Pascal’s theorem. It should be also noted that rows \( K'_E, K'_R \) are projective to rows \( Q_E \) and \( Q_R \).

We find projectivity axis \( p \) of point rows \( K'_E, K'_R \) and mark points \( F'_1, F'_2 \) of the intersection of axis \( p \) with conic section \( e \). Point row \( K \) contains points \( F_1, F_2 \) corresponding to \( F'_1, F'_2 \).

Three pairs of points \( F_1=F'_1 \), \( F_2=F'_2 \), \( F_3=F'_3 \), where \( F_3=F'_3=EE' \cap RR' \), are \( F \)-points of transformation \( \Omega \) for six pairs of corresponding points \( A-A', B-B', C-C', D-D', E-E', R-R' \). An individual set of fundamental points corresponds to each curve of sheaf \( \Psi \) [19]. Consequently, if transformation \( \Omega \) is given by six pairs of corresponding points, the reconstruction problem has \( \infty^1 \) solutions.

**Lemma 4.** Bisecants \( F'_1F'_2 \) of all the conic sections of sheaf \( \Psi \) intersecting in some point \( ER' \), form a sheaf of lines projective to sheaf \( \Psi \). Similarly, bisecants \( F_1F_2 \) also form a sheaf with vertex \( ER \) projective to the sheaf of conic sections \( \Psi \) (without proof).

Based on lemmas 3 and 4 we form an algorithm of reconstructing quadratic birational transformation of coincident fields \( \Pi=\Pi' \) for seven pairs of corresponding points.

**Action 1.** We mark corresponding points \( E-E', R-R' \) on the coincident plane of \( \Pi=\Pi' \). Making a double projection from points \( E, R \), we find point rows \( K'_E \) and \( K'_R \) and the projectivity axes of these rows on the curves of sheaf \( \Psi \). The axes found form a sheaf with vertex \( ER' \). Each axis intersects the corresponding conic section at points \( F'_1, F'_2 \). Making a double projection from points \( E', R' \), we obtain a sheaf \( \{r_1, r_2, r_3, ...\} \) of the projectivity axes with vertex \( ER \). Each axis intersects the corresponding curve of sheaf \( \Psi \) at points \( F_1, F_2 \). The sheaves with vertices \( ER' \) and \( ER \) are projective to the sheaf of conic sections \( \Psi \) (Figure 9).

![Figure 9. Sheaf of projectivity axes.](image1)

![Figure 10. Building of F-points.](image2)

**Action 2.** We mark points \( E-E', T-T' \) on the coincident plane and repeat action 1. We obtain sheaves with vertices \( ET' \) and \( ET \) projective to the sheaf of conic sections \( \Psi \).
Action 3. We mark points R–R′, T–T′ on the coincident plane and repeat action 1. We obtain sheaves with vertices RT′ and RT projective to the sheaf of conic sections Ψ.

As a result of actions 1 ... 3, we obtained three pairs of sheaves with vertices ‘~ER, ET’~ET, RT’~RT. All the sheaves are projective to the sheaf of the conic sections, hence, projective to each other [20].

Action 4. We find second-order curve $g_E$ passing through the intersection points of the corresponding lines of sheaves ER and ET. We find second-order curve $g_T$ passing through the intersection points of the corresponding lines of sheaves ET and RT. Specialized software can be used to build second-order curves [16] or [21].

Action 5. Repeating action 4 for projective sheaves ER′~ET′ and ET′~RT′, we obtain curves $g_E'$ and $g_T'$.

Lemma 5. Curves $g_E$ and $g_T$ intersect in F-points $F_1, F_2, F_3$, and curves $g_E'$ and $g_T'$ intersect in F-points $F'_1, F'_2, F'_3$ of transformation $Ω$ (without proof).

It follows from Lemma 5 that the solution of the inverse problem is reduced to building the intersection points of two pairs of auxiliary conic sections $g_E, g_T$ and $g_E', g_T'$. It should be noted that auxiliary conic sections $g_E, g_T$ and $g_E', g_T'$ pass through points E, T, E′, T′, respectively. Curves $g_E$ and $g_T$ contain a common point ET, curves $g_E'$ and $g_T'$ contain a common point ET'.

One more pair of auxiliary conic sections $g_R$ and $g_R'$ can be used to check the obtained result. Curve $g_R$ is determined by the intersection points of the corresponding beams of projective sheaves ER and RT; curve $g_R'$ is determined by the intersection points of the corresponding beams of projective sheaves ER' and RT'. Auxiliary curves $g_E, g_T, g_R$ pass through each F-point (Figure 10). Similarly, auxiliary curves $g_E', g_T', g_R'$ pass through each F′-point.

7. Conclusion
We showed that every quadratic correspondence $Ω$ of plane fields $Π, Π'$ given both by real and imaginary F-points can be represented as a product of projective transformation and Hirsch transformation. We developed a universal algorithm that makes it possible to find the image of a line in quadratic correspondence of both real and imaginary F-points. The algorithm is used to build plane algebraic curves with given dynamic properties. We proposed a method for computer reconstruction of correspondence $Ω$ given by seven pairs of points. The obtained results help to develop the geometric theory of birational transformations and their practical application in computer geometric simulation.

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