The Price of Anarchy in Bilateral Network Formation in an Adversary Model

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Abstract We study the bilateral version of the adversary network formation game introduced by the author in 2010. In bilateral network formation, a link is formed only if both endpoints agree on it and then both have to pay the link cost $\alpha > 0$ for it. In the adversary model, the cost of each player comprises this building cost plus the expected number of other players to which she will lose connection if one link is destroyed randomly according to a known probability distribution. Two adversaries are considered: one chooses the link to destroy uniformly at random from the set of all built links, the other instead concentrates the probability measure on the built links which cause a maximum number of player pairs to be separated when destroyed. Pairwise stability (PS) and pairwise Nash equilibrium (PNE) are used as equilibrium concepts. For the first adversary, we prove convexity of cost, hence PS and PNE coincide. The main result is an upper bound of 10 on the price of anarchy for this adversary, for link cost $\alpha > \frac{1}{2}$. New proof techniques are required to obtain this bound, compared to the version with unilateral link formation. For the second adversary, we also prove a bound tight up to constants, namely $\Theta(1 + \frac{n}{\alpha})$, where $n$ is the number of players.

Keywords Network formation · Bilateral link formation · Pairwise Nash equilibrium · Pairwise stability · Price of anarchy · Network robustness
1 Network Formation Games

Network formation games are strategic games with a certain structure based on players representing vertices in a graph; we will sometimes call a player also a “vertex”. The set of players is $V = [n] := \{1, \ldots, n\}$, $n \geq 3$, and the strategy space for each player is $\{0, 1\}^n$. A strategy profile $S = (S_v)_{v \in V} \in \{0, 1\}^{n \times n}$ determines an undirected, simple graph $G(S) = (V, E(S))$ by the application of a link formation rule. This rule is a parameter of the game. Two well-known rules are unilateral link formation (ULF) and bilateral link formation (BLF), the latter being the main topic of this work. Under ULF, the built graph is $G(S) = G^U(S) = (V, E^U(S))$, where

$$E^U(S) := \{\{v, w\} \in \binom{V}{2}; S_{vw} = 1 \lor S_{wv} = 1\}.$$  

Under BLF, the built graph is $G(S) = G^B(S) = (V, E^B(S))$, where

$$E^B(S) := \{\{v, w\} \in \binom{V}{2}; S_{vw} = 1 \land S_{wv} = 1\}.$$  

We omit the “U” and “B” superscripts from our notation if the link formation rule is clear from context or if we refer to no specific one.

If $S_{vw} = 1$ then this is interpreted as the request by player $v$ for a link to player $w$. Under ULF, this request is enough to have the link built. Under BLF, also the other player must request it, otherwise it is not built. Entries $S_{vw}$ are of no concern. Denote by $\overline{CE}(S) := \binom{V}{2} \setminus E(S)$ the links that are not there. To determine players’ costs, we need two more parameters: link cost $\alpha > 0$ and an indirect cost function $I_v$ for each player $v$, defined on all undirected graphs on $V$. Player $v$’s cost under strategy profile $S$ is then

$$C_v(S) := |S_v| \alpha + I_v(G(S)),$$

where $|S_v|$ denotes the number of 1s in $S_v$, i.e., $|S_v| = \sum_{w \in V} S_{vw}$. The first term, $|S_v| \alpha$, is called $v$’s building cost, and $I_v(G(S))$ is called her indirect cost; we also write $I_v(S) := I_v(G(S))$. An example for $I_v$ is $I_v(G) = \sum_{w \in V} \text{dist}_G(v, w)$ introduced by Fabrikant et al. [7], also known as the sum-distance model, but many others are conceivable. Oftentimes, $I_v(G) = \infty$ if $G$ is disconnected. We use total cost as the social cost, namely our social cost is $\text{SC}(S) := \sum_{v \in V} C_v(S)$. We call $\sum_{v \in V} |S_v| \alpha$ total building cost and $\sum_{v \in V} I_v(G(S))$ total indirect cost; hence social cost is total building cost plus total indirect cost. For fixed parameters $n$, $\alpha$, link formation rule, and indirect cost function, we call $S$ optimal if it has minimum social cost among all strategy profiles. Denote OPT that minimum social cost.

Strategy profile $S$ is called a Nash equilibrium (NE) if

$$C_v(S) \leq C_v(S_{-v}, X) \quad \forall v \in V \quad \forall X \in \{0, 1\}^n$$

where as usual $(S_{-v}, X)$ denotes the strategy profile resulting from $S$ by replacing $v$’s strategy by $X$. So a NE is characterized by no player having an incentive to deviate from her current strategy, assuming the strategies of the other players fixed. For $v, w \in V$ denote $S + vw$ the strategy profile $S'$ with $S'_{vw} = 1$ and otherwise identical to $S$. For $v, w \in V$ denote $S − vw$ the strategy profile $S'$ with $S'_{vw} = 0$ and otherwise
identical to $S$. Strategy profile $S$ is called a pairwise Nash equilibrium (PNE) if it is a NE and additionally

$$C_v(S + vw + wv) \leq C_v(S) \implies C_w(S + vw + wv) > C_w(S) \quad \forall \{v, w\} \in \mathcal{E}(S).$$

(1)

So each missing link requires the additional justification that it would be an impairment for at least one of its endpoints. Strategy profile $S$ is called pairwise stable (PS) if (1) holds and

$$C_v(S - vw) \geq C_v(S) \quad \forall \{v, w\} \in \mathcal{E}(S).$$

(2)

So pairwise stability (PS) is only concerned with single-link deviations. We call a strategy profile $S$ essential under ULF if $S_{vw} = 1$ implies $S_{wv} = 0$; we call it essential under BLF if $S_{vw} = 1$ implies $S_{wv} = 1$. Since a player has to pay $\alpha$ for each requested link, optima and equilibria of any of the three kinds (NE, PNE, and PS) are essential and we will thus limit our studies to essential strategy profiles in the following. With BLF, an essential strategy profile $S$ is completely determined by the built graph $G(S)$, and we call a graph $G$ PS if $G = G(S)$ for some PS strategy profile $S$. Likewise, we call a graph a PNE if it stems from a strategy profile being a PNE.

NE is well suited for ULF. However, it is less well suited for BLF since the strategy profile $S = 0$, i.e., where no player issues any requests, is a NE under BLF; indeed, due to the link formation rule, given $S = 0$ no player can make a change to her strategy that would have an effect on the graph built. Moreover, under BLF no player can unilaterally, i.e., by changing her strategy while strategies of all other players are maintained, build a link. Under ULF, links can be built unilaterally. Removal of links can happen unilaterally with ULF and BLF. We say that a player removes or sells a link present in the built graph if she changes her strategy so that this link will be no longer part of the built graph. Removing or selling a link will make her building cost smaller by the amount of $\alpha$. We say that players build, add, or buy a link not present in the built graph if they change their strategies so that this link will be part of the built graph. With BLF, this can only happen when the two endpoints of the link agree on it, and then building cost for each of them increases by $\alpha$.

Since this work is concerned with BLF, we will use PNE and PS. Clearly, if $S$ is a PNE then $S$ is PS. The converse holds if cost is convex on the set of PS strategy profiles, shown by Corbo and Parkes [5]. Convexity of cost relates removal of multiple links to removal of each of those links alone. This addresses the difference between PNE and PS: in the former, removal of multiple links has to be considered, whereas the latter is only concerned with removal of single links. Let $v \in V$ and $S$ a strategy profile. We call $C_v$ convex in $S$ if for all $k \in [n]$ and $\{w_1, \ldots, w_k\} \subseteq V$ we have $C_v(S - vw_1 - \ldots - vw_k) - C_v(S) \geq \sum_{i=1}^{k} (C_v(S - (v, w_i)) - C_v(S))$, or, equivalently,

$$I_v(S - vw_1 - \ldots - vw_k) - I_v(S) \geq \sum_{i=1}^{k} (I_v(S - v w_i) - I_v(S)).$$

(3)

We call $C_v$ convex on a set of strategy profiles $\mathcal{S} \subseteq \{0, 1\}^{n \times n}$, if it is convex in every $S \in \mathcal{S}$. We call $C_v$ convex if it is convex on the whole strategy space $\{0, 1\}^{n \times n}$. We say that cost is convex if $C_v$ is convex for each $v$. 

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For fixed parameters \( n, \alpha \), link formation rule, indirect cost function, and equilibrium concept (NE, PNE, PS), the price of anarchy is defined as

\[
\text{PoA} := \max_{S \text{ is an equilibrium}} \frac{\text{SC}(S)}{\text{OPT}}.
\]

This concept is originally due to Koutsoupias and Papadimitriou [13], and the name price of anarchy was shortly after coined by Papadimitriou [15].

## 2 Adversary Model

An adversary is a mapping assigning to each graph \( G = (V, E) \) a probability measure \( \Pr_G \) on the links \( E \) of \( G \). Given a connected graph \( G \), the relevance of a link \( e \) for a player \( v \) is the number of vertices that can, starting at \( v \), only be reached via \( e \). We denote the relevance of \( e \) for \( v \) by \( \text{rel}_G(e, v) \) and the sum of all relevances for a player by \( R_G(v) := \sum_{e \in E} \text{rel}_G(e, v) \). If \( P = (v, \ldots, w) \) is a path, then denote \( \text{rel}(P, v) := \sum_{e \in E(P)} \text{rel}(e, v) \) and \( \text{rel}(P, w) := \sum_{e \in E(P)} \text{rel}(e, w) \) the sum of relevances along \( P \) from the view of \( v \) or \( w \), respectively. A link in a connected graph is called a bridge if its removal destroys connectivity, or equivalently, if it is no part of any cycle. The relevance \( \text{rel}_G(e, v) \) is 0 iff \( e \) is not a bridge. Given a strategy profile \( S \) where \( G(S) \) is connected, we define the indirect cost of player \( v \) by

\[
I_v(S) := I_v(G(S)) := \sum_{e \in E(S)} \text{rel}_G(S)(e, v) \Pr_G(S)(\{e\}).
\]

When \( S \) is clear from context, we omit “\( S \)” and “\( G(S) \)” from the notation and we also write \( m := |E(S)| \) for the number of links. This indirect cost is the expected number of vertices to which \( v \) will lose connection when exactly one link is destroyed from \( G(S) \) randomly and according to the probability measure given by the adversary. For this indirect cost, we use the term disconnection cost in the following. We define disconnection cost to be \( \infty \) when \( G(S) \) is not connected.

Remark 1 If \( S \) is an optimum, a NE under ULF, a PNE under BLF, or PS under BLF, then \( G(S) \) is connected.

Proof Assume \( G(S) \) not being connected. Then disconnection cost for each player and also the total disconnection cost is \( \infty \) by definition. Since there are strategy profiles inducing graphs with finite total disconnection cost (for example a tree), \( S \) is not optimal. Under ULF, each player \( v \) can form a link from herself to each connected component of \( G(S) \), resulting in a connected graph with finite disconnection cost for \( v \). Hence \( S \) is not a NE. Moreover, an additional link cannot be an impairment for any of its too endpoints: either the graph remains disconnected with \( \infty \) as cost for everyone, or the graph becomes connected with a finite cost and hence an improvement for everyone. Hence (1) does not hold, so we neither have a PNE nor \( S \) is PS. \( \square \)

The separation \( \text{sep}(e) \) of a link \( e \) is the number of ordered vertex pairs that will be separated, i.e., pairs \((v, w)\) for which no \( v-w \) path will exist anymore, when \( e \) is destroyed. For a bridge \( e \), denote \( \nu(e) \) the number of vertices in the connected
component of $G - e$ that has a minimum number of vertices; note $\nu(e) \leq \lfloor \frac{n}{2} \rfloor$. If $e$ is not a bridge, we define $\nu(e) := 0$. Then $\text{sep}(e) = 2 \nu(e)(n - \nu(e)) \leq n^2$ and also $\text{sep}(e) = \sum_{v \in V} \text{rel}(e, v)$. If $e$ is a bridge, then $\text{sep}(e) \geq 2(n - 1)$. We can express the social cost in terms of separation (for connected $G(S)$):

$$SC(S) = 2m \alpha + \sum_{v \in V} \sum_{e \in E} \text{rel}(e, v) \Pr(\{|e|\}) = 2m \alpha + \sum_{e \in E} \text{sep}(e) \Pr(\{|e|\}).$$

Total building cost $2m \alpha$ as given here is right for BLF; for ULF it would be $mA$.

We will consider two different adversaries. One is called simple-minded and uses $\Pr(\{|e|\}) = \frac{1}{m}$ for each $e \in E$, i.e., it destroys one link uniformly at random. The other adversary is called smart and chooses the link to destroy uniformly at random from $E_{\max} := \{e \in E; \text{sep}(e) = \text{sep}\_{\max}\}$, where $\text{sep}\_{\max} := \max_{e \in E} \text{sep}(e)$, i.e., it chooses uniformly at random from the set of those links where each causes a maximum number of vertex pairs to be separated.\(^1\)

### 3 Previous and Related Work, Our Contribution and Outlook

Bilateral link formation follows a concept given by Myerson [14, p. 228] in a different context, quoting with added emphasis:

Now consider a link-formation process in which each player independently writes down a list of players with whom he wants to form a link, and the payoff allocation is the fair allocation above for the graph that contains a link for every pair of players that have named each other.

Jackson and Wolinsky [9] introduced the equilibrium concept of pairwise stability and discussed several variations of it, including what would later be known as PNE [9, p. 67]. Fabrikant et al. [7] initiated the quantitative study of the price of anarchy in a model that fits into the framework considered here, as per Sect. 1. They considered ULF and the sum-distance model, i.e., $I_v(G) = \sum_{w \in V} \text{dist}_G(v, w)$. Corbo and Parkes [5] initiated the study of the price of anarchy under BLF and discussed convexity; the latter was also addressed by Calvó-Armengol and Ilkiliç [3]. Since then, numerous results have been published; we refer to [11, Sect. 4] for a more comprehensive discussion. The adversary model was invented by the author in [10], and a constant bound on the price of anarchy for ULF for both the simple-minded and the smart adversary was published in [11]; improvements were given in [12]. Corresponding results for BLF were left to be established, and this is accomplished here.

The adversary model addresses robustness. This has been done before, e.g., in the symmetric connections model by Jackson and Wolinksy [9] and Baumann and Stiller [2] and extensions of it by Bala and Goyal [1] and Haller and Sarangi [8, 16]. But all of those models show substantial differences to the adversary model, the most prominent being that in the adversary model, failures of different links are not independent events (but mutually exclusive events) and that the adversary model

\(^1\) This does not automatically mean that the smart adversary poses the “bigger threat”, compared to the simple-minded one. The smart adversary is more focused on causing damage, but this also can make it more predictable, depending on the graph structure. However, in this work, we show that indeed, under the aspect of the price of anarchy, the smart adversary is worse than the simple-minded one.
allows the probability of failure to depend on the built graph. We again refer to [11, Sect. 4] for a more comprehensive discussion. An experimental robustness study of an extended version of the sum-distance model is given by Chun et al. [4]. Those publications document an interest in robustness aspects of networks formed by non-cooperative or weakly-cooperative agents. However, there is a lack of quantitative studies of the price of anarchy for robustness, since the focus for such studies has been mainly on models incorporating distances (initiated in [7]). The adversary model is a first step to systematically fill this gap. It provides a framework to model the effects on the agents’ behavior, who are expecting a more or less targeted attack on their infrastructure.

Generally, independent link failures (as in the symmetric connections model) describe the unavailability of links due to, e.g., deterioration, maintenance times, or influences affecting the whole infrastructure or large parts of it (e.g., natural disasters). Our adversary model, on the other hand, models the situation when faced with an entity that is malicious but only has limited means so that it can only destroy a limited number of links—the number being limited to 1 for now.

**Our Contribution** We prove existence of PNE and PS graphs in the adversary model under moderate assumptions on the adversary (for all \( \alpha > 0 \) and \( n \geq 3 \)). As the main result, we prove bounds on the price of anarchy, which are tight up to constants:

- for the simple-minded adversary an upper bound of 10 if \( \alpha > \frac{1}{2} \) and a lower bound of \( 2 - \varepsilon \) for any \( \varepsilon > 0 \) if \( n \) is large enough and \( \alpha \) appropriate depending on \( \varepsilon \);
- for the smart adversary an upper bound of \( 2 + \frac{3}{4} \frac{n}{\alpha} \) and a lower bound of \( (1 - \frac{1}{n}) + \frac{9}{8} \frac{n}{\alpha} \),

where for the lower bound we require \( n = 3n_0 - 2 \) for some integer \( n_0 \geq 3 \) and \( \alpha > 2 \).

All that is done for BLF and for PNE as well as PS. Moreover, we prove that the simple-minded adversary induces convex cost and thus PNE and PS coincide. The proof for the constant upper bound uses a diameter argument, similar to what has been done for ULF before, but here it is more complicated. The key idea is to show that long paths are “unique” in a certain sense (Proposition 12 and Lemma 13) in a PS graph.

**Open Problems** Extensions to other adversaries would be interesting, e.g., when \( \Pr(\{e\}) \) is proportional to \( \text{sep}(e) \). Allowing the adversary to destroy more than one link appears to provide a challenging task, since our fundamental tool for bounding the price of anarchy, the bridge tree (introduced in Sect. 6.1), appears inappropriate to capture relevant structure in that case.

Convexity of cost is also left open for the smart adversary. In [12], the author proved non-convexity, but only outside the set of PS graphs. It is moreover not known whether PS and PNE indeed diverge.

For the simple-minded adversary, the constant upper bound on the price of anarchy is only claimed for \( \alpha > \frac{1}{2} \). In fact, it holds whenever we can upper-bound the number of links in any equilibrium by \( 2(n - 1) \), which is the case when \( \alpha > \frac{1}{2} \). We also manage to go slightly below \( \frac{1}{2} \), showing \( O(1/\sqrt{\alpha}) \) if \( \alpha_0(n) \leq \alpha \leq \frac{1}{3} \), where \( \alpha_0(n) \approx \frac{1}{8} \) for large \( n \); a more precise statement is given in Theorem 2. We have no matching lower bound at this time, nor is the range of \( \alpha < \alpha_0(n) \) well understood. This should be considered in future work.
Technical Note  We use $\Theta(\cdot)$, $\Omega(\cdot)$, and $\Theta(\cdot)$ notation in a few places, but merely to provide intuition. All results are also given with concrete constants. When writing $\Theta(\cdot)$, we always mean an upper bound making no statement about a lower bound. When writing $\Omega(\cdot)$, we always mean a lower bound making no statement about an upper bound.

4 Simple Bounds

Since optima are connected, they have at least $n - 1$ links and hence social cost at least $2(n - 1)\alpha$. Using $\text{sep}(e) \leq n^2$ for all $e$, a rough bound on the price of anarchy follows for any adversary, where the social cost in the enumerator is for a worst-case equilibrium:

$$\text{PoA} \leq \frac{2m\alpha + \sum_{e \in E} \text{sep}(e) \Pr(e)}{2(n - 1)\alpha} \leq \frac{m}{n - 1} + \frac{n^2}{2(n - 1)\alpha}$$

$$= \frac{m}{n - 1} + \frac{n}{2\alpha} \left(1 + \frac{1}{n - 1}\right) \leq \frac{m}{n - 1} + \frac{3n}{4\alpha}.$$

The last step uses $n \geq 3$. It follows that, if $m \leq c_m(n - 1)$ in all equilibria of interest and for a constant $c_m$,\(^2\) then

$$\text{PoA} \leq c_m + \frac{3n}{4\alpha}.$$

If, additionally, $\alpha \geq n$, then

$$\text{PoA} \leq c_m + \frac{3}{4}.$$

We will see later that for our both adversaries, $c_m = 2$ holds provided that $\alpha > \frac{1}{2}$ (this requirement on $\alpha$ is necessary only for the simple-minded one). Hence for both adversaries if $\alpha \geq n$, we have a constant bound on the price of anarchy of $2 + \frac{3}{4} = 2.75$.

5 Optima and Equilibria

For optima, we repeat a proof from [11, Prop. 7.1] for ULF with minimal modifications, which adapt it to BLF. This result is for a general adversary, i.e., one that destroys one link according to some probability measure $\Pr$ on the links of the built graph.

**Proposition 2**  
(i) If $\alpha \leq n - 1$, the cycle is an optimum; it has social cost $2n\alpha$.
(ii) If $\alpha \geq n - 1$, a star is an optimum; it has social cost $2(n - 1)(\alpha + 1)$.

**Proof**  
An optimum can only be the cycle or a tree, because any graph containing a cycle has at least the building cost $n\alpha$ of the cycle, and the cycle has optimal disconnection cost. So an optimum is either the cycle, or it is cycle-free. Let $T$ be any tree. We have its disconnection cost:

\(^2\)In particular, $c_m$ is independent of $m$. The index serves the purpose of distinguishing it from other constants.
\[
\sum_{e \in E(T)} \text{sep}(e) \Pr(\{e\}) = 2 \sum_{e \in E(T)} \nu(e) (n - \nu(e)) \Pr(\{e\}) \geq 2 \cdot 1 (n - 1) \sum_{e \in E(T)} \Pr(\{e\}) = 2 (n - 1).
\]

We use that the function \( x \mapsto 2x (n - x) \) is (strictly) increasing on the interval \([1, \frac{n}{2}]\) for the lower bound. We conclude that the social cost of a tree is at least

\[
2 (n - 1) \alpha + 2 (n - 1) = 2 (n - 1) (\alpha + 1).
\]

(5)

Social cost of the cycle is \(2n\alpha\). So if \(\alpha \leq n - 1\), the cycle is better or as good as any tree, hence it is an optimum. If \(\alpha > n - 1\), then we look for a good tree. A star has social cost \(2(n - 1)(\alpha + 1)\), which matches the lower bound (5) and is hence optimal (and better than the cycle). \(\square\)

Note that, indeed, for \(\alpha = n - 1\), cycle and star co-exist as optima with social cost \(2n\alpha = 2n(n - 1) = 2(\alpha + 1)(n - 1)\).

Regarding equilibria, we prove existence for all adversaries where the probability of an edge only depends on its separation, that is, for which for all graphs \(G\) the following holds:

\[
\text{sep}(e) = \text{sep}(e') \implies \Pr(\{e\}) = \Pr(\{e'\}) \quad \forall e, e' \in E(G)
\]

We call such an adversary symmetric. Clearly, the simple-minded and the smart adversary are both symmetric. The proofs of the following two propositions are slight modifications of proofs from [11, Prop. 7.3 and 7.4].

**Proposition 3** A star is a PNE under BLF for \(\alpha > 2 - \frac{1}{n - 1}\) and \(n \geq 3\) if the adversary is symmetric.

**Proof** No edge can be sold, since this would make the graph disconnected. Regarding additional edges, we provide a simple argument, namely we show that each player’s disconnection cost is at most \(2 - \frac{1}{n - 1}\), so even if it could be brought down to 0, it would not be worth the cost of \(\alpha\). The center player has disconnection cost 1 (which is upper-bounded by \(2 - \frac{1}{n - 1}\)), since when the adversary strikes she will be disconnected from exactly one other player. All the edges have the same separation, hence the same probability, namely \(\frac{1}{n - 1}\). It follows that each of the outer players has disconnection cost

\[
\frac{(n - 1) + (n - 2)}{n - 1} = \frac{2(n - 1) - 1}{n - 1} = 2 - \frac{1}{n - 1}.
\]

\(\square\)

**Proposition 4** The cycle is a PNE under BLF for \(\alpha \leq \frac{n}{2}\) and \(n \geq 3\) if the adversary is symmetric.

**Proof** Any additional edge is an impairment since each player already has 0 disconnection cost. If a player removes her two incident edges, her disconnection cost rises
to \( \infty \), so this is no improvement. Hence we only have to consider when she removes one edge. Let \( \{v, w\} \) be an edge. We show that removing it does not improve cost for player \( v \). Removing the edge creates a path \( P \) with \( v \) at one endpoint. We show \( I_v(P) = \frac{n}{2} \), thus proving that saving \( \alpha \) on building cost does not give any improvement.

First consider that \( n \) is odd and hence \( P \) has an even number of edges, say \( e_1, \ldots, e_k, e'_k, \ldots, e'_1 \) for \( k = \lceil \frac{n-1}{2} \rceil \) and ordered by increasing distance from \( v \). For each \( i \) denote \( p_i := \Pr(\{e_i\}) \) and \( p'_i := \Pr(\{e'_i\}) \). By symmetry, we know \( p_i = p'_i \) for all \( i \), and thus \( \sum_{i=1}^{k} p_i = \frac{n}{2} \). It follows:

\[
I_v(P) = \sum_{i=1}^{k} p_i \cdot (n - i) + \sum_{i=1}^{k} p'_i \cdot i = n \sum_{i=1}^{k} p_i = \frac{n}{2}
\]

Now consider that \( n \) is even and hence \( P \) has an odd number of edges, say \( e_1, \ldots, e_k, e_0, e'_k, \ldots, e'_1 \) for \( k = \lfloor \frac{n}{2} \rfloor \). Choose notation as above, with \( p_0 := \Pr(\{e_0\}) \). By symmetry we have \( \sum_{i=1}^{k} p_i = \frac{1-p_0}{2} \). It follows:

\[
I_v(P) = \sum_{i=1}^{k} p_i \cdot (n - i) + p_0 \frac{n}{2} + \sum_{i=1}^{k} p'_i \cdot i = n \frac{1-p_0}{2} + p_0 \frac{n}{2} = \frac{n}{2}
\]

\( \square \)

Since for all \( \alpha > 0 \) and \( n \geq 3 \) we have \( 2 - \frac{1}{n-1} \leq \frac{n}{2} \), in total we get existence of PNE, and thus PS graphs, for the simple-minded and the smart adversary under BLF for all \( \alpha > 0 \) and all \( n \geq 3 \). Similar as for optima, there is a range where star and cycle co-exists (if \( n \geq 4 \)), this time as PNE.

### 6 Graph-Theoretic Preparations

Let \( G = (V, E) \) be a graph, by which we always mean an undirected, simple graph. A sequence of vertices \( A = (v_0, \ldots, v_k) \) is called a walk of length \( k \) if \( \{v_i, v_{i+1}\} \in E \) for all \( 0 \leq i < k \). The walk is called a path if all the vertices are distinct, i.e., if \( |\{v_0, \ldots, v_k\}| = k + 1 \). The walk is called a cycle if \( k \geq 3 \) and \( (v_0, \ldots, v_{k-1}) \) is a path and \( v_0 = v_k \). If \( A \) is a walk, denote \( V(A) = \{v_0, \ldots, v_k\} \) the vertices visited by \( A \) and \( E(A) = \{\{v_i, v_{i+1}\}; \ 0 \leq i < k\} \) the edges traversed by \( A \). Sometimes we write a walk in a way that gives names to the traversed edges, like \( A = (v_0, e_1, v_1, \ldots, e_k, v_k) \) where \( e_{i+1} = \{v_i, v_{i+1}\} \in E \) for all \( 0 \leq i < k \) and so \( E(A) = \{e_1, \ldots, e_k\} \).

#### 6.1 Islands and the Bridge Tree

Since the adversary only destroys one link, it can only cause damage if it chooses a bridge. Therefore, the bridge structure of the built graph is important. Let \( G = (V, E) \) be any connected graph. We call \( K \subseteq V \) a bridge-free connected component or an island if it is inclusion-maximal under the condition that the induced subgraph \( G[K] \)
does not contain any bridges of $G[K]$, or equivalently does not contain any bridges of $G$. It is easy to see that the set of all islands forms a partition of $V$ and that islands are connected only by bridges. The bridge tree $\tilde{G} = (\tilde{V}, \tilde{E})$ of $G$ is the following graph:

$$\tilde{V} := \{K \subseteq V ; K \text{ is an island}\}$$

$$\tilde{E} := \{(K, K') \in \binom{\tilde{V}}{2} ; \exists v \in K, w \in K' : \{v, w\} \in E\}$$

Another way of thinking of the bridge tree is that we successively contract each cycle to a new vertex which is adjacent to all vertices that had a neighbor in the contracted cycle. If for each $v \in V$ we denote $\kappa(v)$ the unique island with $v \in \kappa(v)$, then it is easy to see that $\{v, w\} \mapsto \{\kappa(v), \kappa(w)\}$ is a bijection between the bridges of $G$ and the links of $\tilde{G}$.

In the game theoretic situation, we look at the bridge tree of the built graph $G(S)$. When considering the effect of building additional links, we may treat vertices of the bridge tree as players. This is justified since links inside islands have 0 relevance. Hence for a strategy profile $S$ and $\{K, K'\} \in \binom{\tilde{V}}{2}$ the effect in disconnection cost of a new link between a player from $K$ and a player from $K'$ is specific to the pair $\{K, K'\}$ and not to the particular players. Note also that a link $e = \{v, w\}$ with $v, w$ in the same island not only has $\text{rel}(e, x) = 0$ for all $x \in V$, but also $\text{sep}(e) = 0$.

We make the convention that whenever we speak of the number of vertices in a subgraph $T$ of the bridge tree, we count $|v|$ for each vertex $v \in V(T)$, i.e., we count the vertices that are in the island $v$. An illustration is given in Fig. 1. We call a bridge tree PS if it stems from a PS graph.

### 6.2 Pseudo-Chord-Free Graphs

A link $e = \{u, v\}$ is called a chord if there is a cycle containing $u$ and $v$ but not traversing $e$, or equivalently, if there are two internally vertex-disjoint $u$-$v$-paths not traversing $e$. We call $e$ a pseudo-chord if there are two link-disjoint $u$-$v$-paths not traversing $e$. Clearly, a chord is also a pseudo-chord, but not vice versa in general. For the adversary model, note that selling a pseudo-chord will not change the relevance of any remaining link.

A bound on the number of links in a chord-free graph of $3n$ was proved by the author in [11] and improved to $2n$ in [12]. If also pseudo-chords are forbidden, a simpler proof can be given. We do so in the following and also improve the bound slightly.

**Proposition 5** A pseudo-chord-free and bridge-free graph on $n \geq 3$ vertices has at most $2n - 3$ links.

**Proof** Similar to a standard theorem in graph theory (see, e.g., the book by Diestel [6, p. 55], it can be shown that due to not containing bridges, the graph can be constructed starting with a cycle and then successively attaching paths or cycles of the form $(u, e_1, v_1, \ldots, v_k, e_{k+1}, w)$, where $u, w$ are vertices of the already constructed graph.
Fig. 1 Bridge tree construction. Vertices representing islands of more than 1 vertices have their number of vertices attached, here 4, 7, and 3, respectively. a A graph $G$, b The corresponding bridge tree $\tilde{G}$

and $v_1, \ldots, v_k, k \in \mathbb{N}_0$, are zero or more new vertices. All of the intermediate graphs are bridge-free as well.

Denote $t$ the number of steps in the construction (step 0 being starting with the cycle) and $n_i$ and $m_i$ the number of vertices and links, respectively, in the graph after step $i$, and $k_i$ the number of new vertices added in step $i$, for $i = 0, \ldots, t$. So in particular $n_0 = m_0$ is the length of the initial cycle, which is at least 3 by the definition of a cycle. We call $\delta_i := 2n_i - 3 - m_i$ the balance, so we have to ensure that $\delta_i \geq 0$. Clearly $\delta_0 = n_0 - 3 \geq 0$ since $n_0 \geq 3$.

In step $i \geq 1$ we add $k_i$ vertices and $k_i + 1$ links, so

$$\delta_{i+1} = 2n_{i+1} - 3 - m_{i+1} = 2(n_i + k_i) - 3 - (m_i + k_i + 1)$$

$$= 2n_i - 3 - m_i + k_i - 1 = \delta_i + k_i - 1.$$ 

Hence the balance does not decrease if $k_i \geq 1$.

Indeed, $k_i \geq 1$ is true for all $i \geq 1$: in each step, the constructed graph has no bridges and hence by Menger’s theorem between each two vertices there are two link-disjoint paths. Hence if $k_i = 0$, then a pseudo-chord would be added.  

□
Proposition 6 Let $G$ be a connected, pseudo-chord-free graph on $n \geq 3$ vertices and $r$ the number of its islands. Then $G$ has at most $2n - 2r - 1 \leq 2n - 3$ links.

Proof Denote $\{K_1, \ldots, K_r\}$ the islands of $G$. For each $i$ denote $n_i$ and $m_i$ the number of vertices and links in $K_i$, respectively. By Proposition 5, $m_i \leq 2n_i - 3$ for each $i$. We thus have $\sum_{i=1}^{r} m_i \leq 2n - 3r$. This only counts for links running inside of islands, it does not count for the links running between different islands. There are at most $r - 1$ of them since they correspond to links of the bridge tree, which has $r$ vertices and hence exactly $r - 1$ links. Hence the number of links in $G$ is bounded by $2n - 3r + (r - 1) = 2n - 2r - 1 \leq 2n - 3$, since $r \geq 1$. □

6.3 Multigraphs

We comment on the possibility of allowing multiple edges between two vertices. Since connectivity under removal of one edge is our topic, it is reasonable to allow up to two edges between two vertices. However, we stick to the simpler notion of graphs, instead of multigraphs, for two reasons. First, graphs are notationally simpler and have been the primary choice in the network formation literature. Second, it can be seen that none of our results substantially changes if we allow double-edges. We briefly give explanations for some places where this is not completely obvious.

The PNE existence results of Sect. 5 carry over. This is clear for the cycle, but for the star we have to look at the proof: it uses the simple argument that each player’s disconnection cost is below $\alpha$, so no additional edges can help.

The island decomposition works the same. We note that double-edges always run inside of an island, so the bridge tree is unaffected by the introduction of double-edges, and hence all of the proofs that only work with the bridge tree clearly carry over.

The definition of a pseudo-chord must be extended: if there is a double-edge between $v$ and $w$ and there exists a path from $v$ to $w$ not using either of those two edges, then each of those two edges is to be considered a pseudo-chord. The bounds for the number of edges in a pseudo-chord-free graph almost carry over; since in the proof of Proposition 5 we cannot assume the initial cycle to have at least 3, but only at least 2 vertices, the bound is only $2n - 2$. But this does not change any of the bounds on the price of anarchy that we derive from it. Note that we do not get an additional factor of 2 for the number of edges since just doubling an edge inside of a bridge-free graph will immediately create a pseudo-chord.

7 Simple-Minded Adversary

We have for the simple-minded adversary cost and social cost:

\[
C_v(S) = |S_v|\alpha + \frac{1}{m} \sum_{e \in E} \text{rel}(e, v) = |S_v|\alpha + \frac{1}{m} R(v)
\]

\[
SC(S) = 2m\alpha + \frac{1}{m} \sum_{v \in V} R(v) = 2m\alpha + \frac{1}{m} \sum_{e \in E} \text{sep}(e)
\]
7.1 Convexity of Cost

Convexity is interesting since it implies PNE and PS being equivalent. The proofs of the following remark and proposition are straightforward and purely graph theoretic.

**Remark 7** Let $G = (V, E)$ be a graph and $e = \{v, w\} \in E$ a non-bridge and $C$ any cycle with $e \in E(C)$. Then all bridges of $G - e$ that are non-bridges in $G$, are in $E(C)$.

**Proof** Let $f$ be a non-bridge in $G$ and a bridge in $G - e$. Then $G - e$ consists of two subgraphs $G_1$ and $G_2$ that are connected only by $f$. Since $f$ was no bridge before $e$ was removed, $e$ must also connect $G_1$ with $G_2$. Moreover, there are no other links between $G_1$ and $G_2$. It follows that any cycle that contains $e$ also contains $f$. \hfill $\Box$

**Proposition 8** Let $G = (V, E)$ be a connected graph, $v \in V$, $e = \{v, w\} \in E$, and $F \subseteq E \setminus \{e\}$ a set of links, each incident with $v$, so that $G' := G - F - e$ is still connected. Let $B_1$ be those links that are non-bridges in $G$ but bridges in $G - e$. Let $B_2$ be those links that are non-bridges in $G - F$ but bridges in $G - F - e$. Then $B_1 \subseteq B_2$.

**Proof** Since $G'$ is connected, there is a path $(v, e_1, v_1, \ldots, w) \in G'$. Then the cycle $C := (v, w, e_1, \ldots, v)$ is in $G - F$. By Remark 7, we have $B_1 \subseteq E(C)$. Hence all links in $B_1$ are on a cycle that is not destroyed by removal of $F$, so no link in $B_1$ is made a bridge by removal of $F$. It follows that $B_1 \subseteq B_2$. \hfill $\Box$

**Theorem 1** The simple-minded adversary induces convex cost.

**Proof** Let $v \in V$ and $w_1, \ldots, w_k \in V$ and $S$ be a strategy profile. We may assume that $S - vw_1 - \ldots - vw_k$ is still connected, since otherwise the left-hand-side of (3) is $\infty$. We show (3) proceeding by induction on $k$. The case $k = 1$ is clear. Let $k > 1$ and set $S' := S - vw_1 - \ldots - vw_{k-1}$. We show that switching from $S'$ to $S' - vw_k$ increases disconnection cost for $v$ at least as much as switching from $S$ to $S - vw_k$, i.e.,

$$I_v(S' - vw_k) - I_v(S') \geq I_v(S - vw_k) - I_v(S). \quad (7)$$

Denoting $R'_1 := R_{S' - vw_k}(v)$ and $R'_2 := R_{S'}(v)$ and $R_1 := R_{S - vw_k}(v)$ and $R_2 := R_S(v)$, we have:

\begin{align*}
(7) \iff \frac{R'_1}{m-k} - \frac{R'_2}{m-k+1} & \geq \frac{R_1}{m-1} - \frac{R_2}{m} \\
\frac{R'_1 - R'_2}{m-k+1} & \geq \frac{R_1}{m-1} - \frac{R_2}{m} - \frac{R'_1}{m} - \frac{(m-k)(m-k+1)}{m} \\
\frac{R'_1 - R'_2}{m-k+1} & \geq \frac{R_1}{m-1} - \frac{R_1}{m} - \frac{(m-k)(m-k+1)}{m} + \frac{R_1 - R_2}{m} \\
\frac{R'_1 - R'_2}{m-k+1} & \geq \frac{R_1}{(m-1)m} - \frac{R_1}{(m-k)(m-k+1)} + \frac{R_1 - R_2}{m} \quad \text{since } R'_1 \geq R_1
\end{align*}
\[ R'_1 - R'_2 \geq R_1 \left( \frac{1}{m-k} - \frac{1}{m-k(m-k+1)} \right) + \frac{R_1 - R_2}{m} \]

The inequality \( R'_1 \geq R_1 \) follows from the fact that removal of edges does not decrease the relevances of the remaining edges, and in this case all the removed edges have relevance 0 since their removal does not destroy connectivity. The above calculation shows that it suffices to prove \( R'_1 - R'_2 \geq R_1 - R_2 \).

When removing \( \{v, w_k\} \), relevance of zero or more links changes from 0 to a positive value; these are precisely those links which become bridges by the removal and which were no bridges before. No relevance is reduced by a removal.

Let \( B_1 \) be all those links that become bridges by the switch from \( S \) to \( S - vw_k \), and let \( B_2 \) be those that become bridges by the switch from \( S' \) to \( S' - vw_k \). Then \( B_1 \subseteq B_2 \) by Proposition 8. The increase in relevance from 0 to a positive value for \( e \in B_1 \) given \( S' \) is at least as high as when given \( S \). In other words, while \( \{v, w_1\}, \ldots, \{v, w_{k-1}\} \) are removed, the effect of all links in \( B_1 \) becoming bridges is saved until the removal of \( \{v, w_k\} \). We have shown that \( R'_1 - R'_2 \geq R_1 - R_2 \) and thus (7).

The proof is concluded by the following standard calculation:

\[
I_v(S - vw_1 - \ldots - vw_k) - I_v(S) \\
= I_v(S' - vw_k) - I_v(S') + I_v(S') - I_v(S) \\
\geq I_v(S - vw_k) - I_v(S) + \sum_{i=1}^{k-1} (I_v(S - vw_i) - I_v(S)) \\
\text{by (7)} \\
\geq I_v(S - vw_k) - I_v(S) + \sum_{i=1}^{k-1} (I_v(S - vw_i) - I_v(S)) \\
\text{by induction} \\
= \sum_{i=1}^{k} (I_v(S - vw_i) - I_v(S)).
\]

\[ \square \]

7.2 Bounding Total Building Cost

As a first step towards our bound on the price of anarchy, we bound the number of links and hence total building cost in a PS graph. Recall the bound on the number of links in pseudo-chord-free graphs in Sect. 6.2.

**Proposition 9** Let a pairwise stable graph \( G \) be given.

(i) If \( \alpha > \frac{1}{2} \), then \( G \) is pseudo-chord-free and hence has at most \( 2n - 3 \leq \mathcal{O}(n) \) links.

(ii) In general, \( G \) is pseudo-chord-free (with \( \leq 2n - 3 \) links) or has at most \( \frac{n}{\sqrt{2\alpha}} + 1 \) links.
Proof If \( G \) is bridge-free, selling a pseudo-chord is beneficial since disconnection cost 0 is maintained. So for both parts we assume that \( G \) contains bridges.

(i) The impairment in disconnection cost for a player \( v \) of selling a pseudo-chord is only due to the change in the denominator of the disconnection cost and is precisely \( \frac{1}{m(m-1)} R(v) \), which is upper-bounded by \( \frac{1}{2} \) since \( R(v) \leq \frac{n(n-1)}{2} \) and \( m \geq n \) \([11, Prop. 8.3]\). Hence if \( \alpha \) is larger than that, there is an incentive to sell the pseudo-chord.

(ii) Let \( G \) possess a pseudo-chord. This means that any of its two endpoints, say \( v \), deems it being no impairment to pay \( \alpha \) for this link, hence \( \frac{n(n-1)}{2} \geq \alpha \). It follows

\[
\frac{n^2}{2} \geq \frac{n(n-1)}{2} \geq R(v) \geq m(m-1) \alpha \geq (m-1)^2 \alpha ,
\]

hence \( \frac{n}{\sqrt{2\alpha}} + 1 \geq m \).

7.3 Bounding Total Separation

Convention: we will only work with the bridge tree in this section, so all vertices, links, and paths are in the bridge tree. The proof of the following remark is straightforward:

Remark 10 Let \( P = (v, \ldots, w) \) be a path (reminder: in the bridge tree). The benefit in disconnection cost of bypassing \( P \) can be lower-bounded:

\[
I_v(G) - I_v(G + \{v, w\}) \geq \frac{\sum_{e \in E(P)} \text{rel}(e, v)}{m+1} = \frac{\text{rel}(P, v)}{m+1} .
\]

The following bound on total separation holds for all connected graphs:

Lemma 11 (K. \([11, Cor. 8.12]\)) For each \( v \) we have \( R(v) \leq (n-1) \text{diam}(\tilde{G}) \), hence

\[
\sum_{e \in E} \text{sep}(e) < n^2 \text{diam}(\tilde{G}) .
\]

Since \( m \geq n - 1 \), it follows that total disconnection cost is bounded by

\[
\frac{n^2}{m} \text{diam}(\tilde{G}) \leq \frac{3}{2} n \text{diam}(\tilde{G}) .
\]

Under ULF, if \( S \) is a NE then \( \text{diam}(\tilde{G}(S)) \leq O(\alpha) \) \([11, Lem. 8.13]\), by which a constant bound is obtained on the price of anarchy. If we could prove the same for BLF and PS, we would obtain a constant bound as a result. However, an example shows that there is no hope for this; the diameter of a PS bridge tree can be \( \lfloor \sqrt{n} \rfloor \) \(\text{Sect. 7.4}\). The trick to handle this situation is to prove that essentially, there is only one such long path. Note that in non-PS graphs, total separation can be \( \Omega(n^3) \), e.g., if the graph is a path then total separation is \( \frac{1}{3} (n^3 - n) \). A generalized star with \( k \) rays of length \( \ell \) has total separation \( \Omega(\ell n^2) \), which is \( \Omega(n^{2+\varepsilon}) \) for \( \ell = \Theta(n^{\varepsilon}) \).

For a moment, orient the links of the bridge tree according to the rule that \( e = \{v, w\} \) is oriented \( (v, w) \) if \( \text{rel}(e, v) \leq \text{rel}(e, w) \), i.e., links point in the direction of fewer
vertices, ties broken arbitrarily. Due to cycle-freeness, there is a vertex $u$ having only out-links; and since each vertex has at most one in-link, this vertex $u$ is unique. We consider the bridge tree rooted at $u$ and discard the orientation, it has served its purpose. Denote $\text{lev}(v)$ the level of $v$ in the rooted bridge tree, in particular $\text{lev}(u) = 0$ and $\text{lev}(v) = 1$ for $v \in N(u)$. Each of the subtrees rooted at some $v \in N(u)$ is called a branch of the bridge tree.

A path $P = (v, \ldots, w)$ is called $v$-limited if $\text{rel}(P, v) \leq 2n\alpha$. It is called $w$-limited if $\text{rel}(P, w) \leq 2n\alpha$. A path is called a branch path if it traverses each level at most once, i.e., it runs straight up or down and in particular stays within one branch plus $u$. A branch path $P = (v, \ldots, w)$ with $\text{lev}(v) < \text{lev}(w)$ is called outward-limited if it is $v$-limited and is called inward-limited if it is $w$-limited. Denote $\alpha_0(n) := \frac{1}{8}(1 + \frac{1}{n-1})^2 \leq \frac{9}{32}$. We have:

**Proposition 12** Let $\alpha \geq \alpha_0(n)$.

(i) In a PS bridge tree, each path $P = (v, \ldots, w)$ is $v$-limited or $w$-limited (or both). In particular, each branch path is inward-limited or outward-limited (or both).

(ii) An inward-limited path has length at most $4\alpha \leq O(\alpha)$.

(iii) In a PS bridge tree, no more than one branch contains a path not being inward-limited.

**Proof** (i) Assume the contrary, i.e., $\text{rel}(P, v) > 2n\alpha$ and $\text{rel}(P, w) > 2n\alpha$. Then for each $x \in e := \{v, w\}$ by Remark 10 and Proposition 9 we have, using $\alpha \geq \alpha_0(n)$ for the second case (i.e., when $\alpha \leq \frac{1}{2}$),

$$I_x(G) - I_x(G + e) > \frac{2n\alpha}{m + 1} \geq \begin{cases} \frac{2n\alpha}{2n} = \alpha \\ \frac{2n\alpha}{\sqrt{2n}} + 2 \geq \alpha \end{cases},$$

a contradiction to PS.

(ii) Let $P = (v, \ldots, w)$ be inward-limited with $\text{lev}(v) < \text{lev}(w)$ and of length $\ell$. Let $N := n - n'$ with $n'$ the number of vertices in $P$’s branch; then by construction $N \geq \frac{n}{2}$. It follows $2n\alpha \geq \text{rel}(P, w) \geq N\ell \geq \frac{n}{2} \ell$, hence $4\alpha \geq \ell$. (This type of argument is the basis for $\text{diam}(\tilde{G}(S)) \leq O(\alpha)$ for ULF in [11, Lem. 8.13].)

(iii) Assuming the contrary, let $(v, \ldots, w)$ and $(x, \ldots, y)$ with $\text{lev}(v) < \text{lev}(w)$ and $\text{lev}(x) < \text{lev}(y)$ from different branches and both not inward-limited. Then

$$(w, \ldots, v, \ldots, u, \ldots, x, \ldots, y)$$

is not $w$-limited nor $y$-limited, contradicting (i). \qed

**Lemma 13** In a PS graph with $\alpha \geq \alpha_0(n)$, total separation is bounded by $12n^2\alpha$.

**Proof** Let $P = (u, \ldots, w)$ be a path of maximum length $\ell$ among all path starting at $u$. If $\ell \leq 4\alpha$, we are done by Lemma 11. Otherwise, if $\ell > 4\alpha$, by Proposition 12(ii), $P$ is not inward-limited. By Proposition 12(iii), branch paths in all other branches are inward-limited and hence of length at most $4\alpha$. \qed
We consider paths in $P$’s branch of the form $Q = (x, \ldots, y)$ with $x \in V(P)$ and $y \neq w$ a leaf. Denote $k := 4\alpha + 1$ if $4\alpha$ is integer and $k := \lceil 4\alpha \rceil$ otherwise. Call the final $k$ vertices of $P$ (before it ends with $w$) its lower part and the other vertices (starting with $u$) its upper part. If $Q$ attaches to the lower part of $P$, i.e., $x$ is in the lower part, then $|Q| \leq k - 1 \leq 4\alpha$ since $P$ has maximum length. If, on the other hand, $Q$ attaches to the upper part of $P$, then $A := (w, \ldots, x)$ has length at least $k > 4\alpha$. Denote $B := (w, \ldots, x, \ldots, y)$. If $Q$ is not inward-limited, then $B$ is not $y$-limited, by Proposition 12(i) it is hence $w$-limited. In particular, $A$ is $w$-limited, hence inward-limited and thus of length at most $4\alpha$, a contradiction. It follows that $Q$ is inward-limited and so $|Q| \leq 4\alpha$.

We bound sums of separation values. Since $P$ is not inward-limited, it is outward-limited, i.e., $u$-limited. Denote $P = (u = w_0, \ldots, w_\ell = w)$. It follows

$$\sum_{e \in E(P)} \text{sep}(e) = 2 \sum_{e = [w_i, w_{i+1}] \in E(P)} \text{rel}(e, w_i) \text{rel}(e, w_{i+1}) \leq 2 \sum_{e = [w_i, w_{i+1}] \in E(P)} \text{rel}(e, w) n = 2n \text{rel}(P, u) \leq 4n^2 \alpha \ .$$

(8)

If we bypass $P$, i.e., add $\{u, w\}$, then separation values for all $e \in E' := E \setminus E(P)$ are maintained. Hence we can compute $\sum_{e \in E'} \text{sep}(e)$ as if $P$ had been bypassed. But with the bypass, the diameter of the bridge tree is upper-bounded by $2 \cdot 4 \cdot \alpha = 8\alpha$ as per what we have established above. By Lemma 11 it follows $\sum_{e \in E'} \text{sep}(e) \leq 8n^2 \alpha$. Together with (8), the bound on total separation follows.

**Theorem 2** The price of anarchy for the simple-minded adversary with BLF and PNE or PS is upper-bounded by $1 + 9 = 10$ if $\alpha > \frac{1}{2}$ and by $\max\{1, \frac{1}{\sqrt{8\alpha}} + \frac{1}{6}\} + 9 \leq O(1/\sqrt{\alpha})$ if $a_0(n) \leq \alpha \leq \frac{1}{2}$.

**Proof** If $\alpha > n - 1$, then by Sect. 4 with $c_m = 2$, we have a bound of $2.75 \leq 10$.

Consider next $\alpha \leq n - 1$ and recall the optimum being $2n\alpha$. By Proposition 9, for total building cost we have the ratio of equilibrium to optimum bounded by $\frac{(2n - 3)\alpha}{2n\alpha} < 1$ if $\alpha > \frac{1}{2}$ and otherwise, if $\alpha \leq \frac{1}{2}$, the same bound or:

$$\frac{n \sqrt{\alpha}}{2na} + \alpha = \frac{1}{\sqrt{8\alpha}} + \frac{1}{2n} \leq \frac{1}{\sqrt{8\alpha}} + \frac{1}{6} .$$

For total disconnection cost, using $a_0(n) \leq \alpha$ and $m \geq n - 1$, we have the bound:

$$\frac{12n^2 \alpha}{m} = 6n \leq 6 \frac{n}{n-1} \leq 6 \frac{3}{2} = 9 .$$
7.4 Lower Bound

**Proposition 14** For $\alpha = 1$, the bridge tree can have diameter $\lfloor \sqrt{n} \rfloor$, even if the graph is PS.

**Proof** Consider a cycle with a path of length $\ell$ attached to it with one of its ends, as shown in Fig. 2. Diameter of the corresponding bridge tree is $\ell$. Let $n$ be the total number of vertices and assume

$$\frac{1}{n} \frac{\ell (\ell + 1)}{2} < \alpha \leq \frac{1}{n} \frac{(n - \ell - 1) (n - \ell)}{2}. \quad (9)$$

Because of the lower bound on $\alpha$, no vertex on the cycle wishes to connect to a vertex on the path, and also no vertex $v$ on the path wishes to connect to a vertex $w$ on the path if $w$ has greater distance to the cycle than $v$. Because of the upper bound on $\alpha$, it can also be shown easily that no two neighboring vertices on the cycle wish to sell the link between them. Trivially, no link on the path will be sold. Hence this graph is PS.

Now choose $n \geq 9$, $\ell := \lfloor \sqrt{n} \rfloor$, and $\alpha := 1$. Then

$$\frac{1}{n} \frac{\ell (\ell + 1)}{2} \leq \frac{1}{n} \frac{n + \sqrt{n}}{2} = \frac{1}{2} + \frac{1}{2 \sqrt{n}} < 1 = \alpha,$$

and, due to $n \geq 9$,

$$\frac{1}{n} \frac{(n - \ell - 1) (n - \ell)}{2} \geq \frac{1}{n} \frac{(n - \sqrt{n} - 1) (n - \sqrt{n})}{2} \geq \frac{1}{n} \frac{n^2 - 2n \sqrt{n} + n - n + \sqrt{n}}{2} \geq \frac{n - 2 \sqrt{n}}{2} = \sqrt{n} \frac{\sqrt{n} - 2}{2} \geq \sqrt{n} \frac{1}{2} \geq \frac{3}{2} > 1 = \alpha.$$

Hence (9) holds. Moreover, we have a diameter of $\ell = \lfloor \sqrt{n} \rfloor$ in the bridge tree. □
Theorem 3  For each \( \varepsilon > 0 \), there is \( n \) and \( \alpha \) such that the price of anarchy is lower-bounded by \( 2 - \varepsilon \).

Proof  We use the example of Proposition 14. Social cost is

\[
2n\alpha + \frac{1}{n} \sum_{k=1}^{\ell} (n-k)k + \frac{\ell (\ell + 1)}{n} \left( n - \frac{2\ell + 1}{3} \right).
\]

For a lower bound on the price of anarchy, we have to divide this by the optimum, which is \( 2n\alpha \) since \( \alpha \leq n - 1 \). So we choose \( \alpha \) as small as possible, i.e., \( \alpha = \frac{\ell (\ell + 1)}{2} \), hence \( 2n\alpha = \ell (\ell + 1) \). It follows a lower bound on the price of anarchy of:

\[
1 + \frac{1}{n} \left( n - \frac{2\ell + 1}{3} \right) = 1 + \left( 1 - \frac{2\ell + 1}{3n} \right) \geq 2 - \frac{2\sqrt{n} + 1}{3n},
\]

which tends to 2 as \( n \) tends to \( \infty \).

8 Smart Adversary

Recall that the smart adversary destroys one link uniformly at random from \( E_{\text{max}} = \{ e \in E : \text{sep}(e) = \text{sep}_{\text{max}} \} \). We call the links in \( E_{\text{max}} \) the critical links. Individual and social cost can be written as:

\[
C_v(S) = |S_v| \alpha + \frac{1}{|E_{\text{max}}|} \sum_{e \in E_{\text{max}}} \text{rel}(e, v)
\]

\[
SC(S) = 2m \alpha + \frac{1}{|E_{\text{max}}|} \sum_{e \in E_{\text{max}}} \text{sep}_{\text{max}} = 2m \alpha + \text{sep}_{\text{max}}
\]

Remark 15  A PNE or PS graph is pseudo-chord-free for the smart adversary.

Proof  Removing a pseudo-chord does not change the relevance of any link, nor does it change \( \text{sep}_{\text{max}} (= \max_e \text{sep}(e)) \), hence it does not change \( E_{\text{max}} \). A player will hence always opt to remove a pseudo-chord and so avoid the expense of \( \alpha \).

Given pseudo-chord-freeness, we have the upper bound of \( 2n - 3 \) on the number of links in a PNE or PS graph by Proposition 6. As seen in Sect. 4, this implies an upper bound of \( 2 + \frac{3}{4} \frac{n}{\alpha} \) on the price of anarchy for PNE and PS. It is tight up to constants as shown by the following example (Fig. 3).

Theorem 4  Let \( n = 3n_0 - 2 \) for some \( n_0 \in \mathbb{N} \) with \( n_0 \geq 3 \) and \( \alpha > 2 \), then the price of anarchy for PNE and PS with the smart adversary is lower-bounded by \( (1 - \frac{1}{n}) + \frac{2}{3} \frac{n}{\alpha} \).

Proof  If \( \alpha > n - 1 \), then the stated bound is less than 1, so the statement is trivial since the price of anarchy is always at least 1.
Fig. 3 Three stars of sizes \( n_0 \), \( n_0 - 1 \), and \( n_0 - 2 \); here \( n_0 = 5 \). The \( n_0 \) players in the star around \( u_1 \) would like to put the one critical link \( \{u_0, u_1\} \) on a cycle, if \( \alpha < n_0 \). Building, e.g., \( \{u_1, u_2\} \) would reduce their disconnection cost from \( n - n_0 \) to \( n_0 - 2 \), meaning an improvement of \( n_0 \). But no player from the stars around \( u_2 \) or \( u_3 \) is willing to cooperate.

Let \( \alpha \leq n - 1 \), so the optimum is \( 2n\alpha \). Consider three stars \( U_i, i = 1, 2, 3 \) with center vertices \( u_i, i = 1, 2, 3 \), and \( n_0, n_0 - 1 \), and \( n_0 - 2 \) vertices, respectively. Connect the stars via an additional vertex \( u_0 \) and additional links \( \{u_0, u_i\}, i = 1, 2, 3 \). This construction uses \( 3n_0 - 2 = n \) vertices. Then \( e_0 = \{u_0, u_1\} \) is the only critical link and \( n_0 = \frac{n + 2}{3} = \Theta(n) \). We have a total disconnection cost of \( 2\nu(e_0) (n - \nu(e_0)) = 2n_0 (n - n_0) \geq \frac{4}{9} n^2 \geq \Omega(n^2) \). It follows a ratio of social cost to the optimum of at least

\[
\frac{2(n - 1) \alpha + \frac{4}{9} n^2}{2n\alpha} = \left( 1 - \frac{1}{n} \right) + \frac{2}{9} \frac{n}{\alpha}
\]

We are left to show that this graph is a PNE, which implies PS. Clearly, no link can be sold, since that would make the graph disconnected. Therefore we only have to ensure that no link can be added that would be beneficial for one endpoint and at least no impairment for the other one, i.e., we have to show (1).

A link \( e \) that improves disconnection cost for some player has to put \( \{u_0, u_1\} \) on a cycle. If \( e \) connects a vertex in \( U_1 \) with a vertex in \( U_2 \), then \( \{u_0, u_3\} \) will become critical. For a vertex in \( U_2 \), this reduces disconnection cost from \( n_0 \) to \( n_0 - 2 \). So, since \( \alpha > 2 \), no vertex in \( U_2 \) agrees to build such a link.

A similar situation holds if \( e \) connects a vertex in \( U_1 \) with a vertex in \( U_3 + u_0 \) (including the possibility of putting a double-edge between \( u_0 \) and \( u_1 \)). It will result in \( \{u_0, u_2\} \) becoming critical. For a vertex in \( U_3 + u_0 \), this reduces disconnection cost from \( n_0 \) to \( n_0 - 1 \). So, since \( \alpha > 1 \), no vertex in \( U_3 + u_0 \) agrees to build such a link.

If we consider \( \alpha > 2 \) a constant, this theorem gives a lower bound of \( \Omega(n) \). For \( \alpha \geq \Omega(1) \) this is by (4) the worst that can happen for any adversary (note \( m \leq n^2 \)). This is particularly noteworthy since for ULF and NE, the smart adversary has a price of anarchy of \( \Theta(1) \) [11, Thm. 9.8].

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