ELLIPTIC ASYMPTOTIC REPRESENTATION OF THE FIFTH PAINLEVÉ TRANSCENDENTS (CORRECTED VERSION)

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Abstract. For the fifth Painlevé transcendents an asymptotic representation by the Jacobi sn-function is presented in cheese-like strips along generic directions near the point at infinity. Its elliptic main part depends on a single integration constant, which is the phase shift and is parametrised by monodromy data for the associated isomonodromy deformation. In addition, under a certain supposition, the error term is also expressed by an explicit asymptotic formula, whose leading term is written in terms of integrals of the sn-function and the \( \vartheta \)-function, and contains the other integration constant. This paper contains corrections of the Stokes graph and of the related results in the early version.

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1. Introduction

The fifth Painlevé equation

\[
(P_V) \frac{d^2 y}{dx^2} = \left( \frac{1}{2y} + \frac{1}{y-1} \right) \left( \frac{dy}{dx} \right)^2 - \frac{1}{y} \frac{dy}{dx}
\]

\[
+ \frac{(y-1)^2}{8x^2} \left( (\theta_0 - \theta_1 + \theta_\infty)^2 y - \frac{(\theta_0 - \theta_1 - \theta_\infty)^2}{y} \right) + \left( 1 - \theta_0 - \theta_1 \right) \frac{y}{x} - \frac{y(y+1)}{2(y-1)}
\]

with \( \theta_0, \theta_1, \theta_\infty \in \mathbb{C} \) defines nonlinear special functions, which are meromorphic on the universal covering space of \( \mathbb{C} \setminus \{0\} \). A general solution is expressed by a convergent series in a spiral domain around \( x = 0 \), and admits asymptotic representations as \( x \to \infty \) along the real and the imaginary axes (cf. e.g. \cite{37}, \cite{36}). Using the isomonodromy property and the WKB-analysis, for a generic case of (P_V) Andreev and Kitaev \cite{5} obtained families of solutions near \( x = 0 \) and \( x = \infty \) on the positive real axis, and connection formulas for these solutions. Along the imaginary axis asymptotic solutions and the associated monodromy data have been studied by \cite{6}, \cite{38}. Furthermore Lisovyy et al. \cite{29} gave a connection formula for the tau-function \( \tau_V(x) \) between \( x = 0 \) and \( x = i\infty \) and the ratios of multipliers of \( \tau_V(x) \) as \( x \to 0, +\infty, i\infty \).

Near \( x = \infty \) along directions other than the real or imaginary axis, a general solution of (P_V) behaves quite differently. In generic directions it is known that, for solutions of the Painlevé equations (P_1), \ldots, (P_IV) except truncated or classical ones, the Boutroux ansatz holds \cite{8}, \cite{9}. Elliptic asymptotic representations have been studied for (P_1),
and the \( \vartheta \) asymptotic formula, whose leading term is written in terms of integrals of the \( \text{sn} \)-function. The other integration constant is hidden in the error term, and on a single integration constant parametrised by the monodromy data that appears as the phase shift of it. The \( \vartheta \) function in the expression has the modulus \( A \).

Under a certain supposition we express the error term by an explicit representation for solutions of (P-I) (defined in Section 2) remain invariant under a small change of \( x \). We apply the WKB-analysis to calculate the monodromy data, from which the elliptic expression follows.

In this paper we show the Boutroux ansatz for the fifth Painlevé equation (P-V), that is, present an elliptic asymptotic representation for a general solution of (P-V), which is given by the Jacobi \( \text{sn} \)-function, along generic directions near the point at infinity (Theorem 2.1). In deriving our results we employ the isomonodromy property described as follows: for the complex parameter \( x \) and \( \vartheta \) term under a certain supposition. Its leading term is written in terms of integrals of the \( \text{sn} \)-function and the \( \vartheta \) function. The \( \text{sn} \)-function and the \( \vartheta \) function are, present an elliptic asymptotic representation for a general solution of (P-V) (Iwaki’s recent work [17] by the topological recursion is remarkable.

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In the early version of [39] the Stokes graph is incorrect, and it affects the phase shifts of asymptotic solutions in [39, Theorems 2.1 and 2.2]. This paper contains the amended Stokes graph and the subsequent modifications including the corrected phase shifts in these theorems (Corrigendum of [39]).

This paper is organised as follows. Section 2 describes our main results on the elliptic expression by the Jacobi sn-function for $0 < |\phi| < \pi/2$ with $\phi = \arg x$ (Theorem 2.1), on the error term with the explicit leading term (Theorem 2.3 and Corollary 2.4), and on the elliptic expression for $0 < |\phi \mp \pi| < \pi/2$ (Theorem 2.2). Theorems 2.1 and 2.2 contain the corrected phase shifts. Section 3 is devoted to basic facts necessary in proving the main results: parametrisation of $y(x)$ by the monodromy data; turning points and Stokes curves for the symmetric linear system (3.4) with $\lambda = e^{i\phi}(2\xi - 1)$; and a WKB-solution in the canonical domain and local solutions around turning points. In the calculation of the monodromy in Section 4 we use the Stokes matrix and a connection matrix along a path passing through turning points. In other words, it is possible to replace the calculation of the monodromy matrices by that of suitable connection matrices.

Our discussion follows the justification scheme of Kitaev [26], and the WKB analysis is carried out under the supposition (4.3). The monodromy matrices are expressed in terms of integrals related to the characteristic roots and the WKB-solution. In Section 5 these integrals are represented by the elliptic integrals of the first and the second kind, and the $\vartheta$-function (Propositions 5.3 and 5.4 and Corollary 5.4). In Section 6 using the propositions in Section 5 with the justification scheme, we obtain the elliptic main part of $y(x)$ and an asymptotic expression of $B_\phi(t)$. Solving equations (6.7) and (6.8) equivalent to (P_V) we derive the explicit asymptotic formula of the error term. Section 7 discusses an amendment to the arguments along the early line, which suggests the proper Stokes graph. Several properties of the Boutroux equations and the trajectory of its solution are important in our arguments, which are discussed in the final section.

Throughout this paper we use the following symbols:

1. $\sigma_1, \sigma_2, \sigma_3$ are the Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};
\]

2. for complex-valued functions $f$ and $g$, we write $f \ll g$ or $g \gg f$ if $f = O(|g|)$, and write $f \asymp g$ if $g \ll f \ll g$.

2. Main results

To state our main theorems, we make necessary preparations. Let system (1.1) admit the isomonodromy property with respect to a fundamental matrix solution of the form

\[
\Xi(\xi) = \Xi(x, \xi) = (I + O(\xi^{-1})) \exp \left( \frac{1}{2} (x\xi - \theta_\infty \log \xi) \sigma_3 \right)
\]
as $\xi \to \infty$ through the sector $|\arg(x\xi) - \pi/2| < \pi$. Let $M^0, M^1, M^\infty \in SL_2(\mathbb{C})$ be the monodromy matrices given by the analytic continuations of $\Xi(\xi)$ along loops $l_0, l_1$. 

\[
\Xi(\xi) = \Xi(x, \xi) = (I + O(\xi^{-1})) \exp \left( \frac{1}{2} (x\xi - \theta_\infty \log \xi) \sigma_3 \right)
\]
\( l_\infty \in \pi_1(P^1(\mathbb{C}) \setminus \{0,1,\infty\}) \) as in Figure 2.1 defined for \(-\pi/2 < \arg x < \pi/2\), which start from the point \(p_{st}\) satisfying 100 < \(|p_{st}| < \infty\) and \(\arg(xp_{st}) = \pi/2\) and surround, respectively, \(\xi = 0\), \(\xi = 1\) and \(\xi = \infty\) anticlockwise. It is easy to see \(M^\infty M^1 M^0 = I\).

Denote by \(\Xi_1(\xi)\) and \(\Xi_2(\xi)\) the matrix solutions of system (1.11) admitting the same asymptotic representations as (2.1) as \(\xi \to \infty\) through the sectors \(|\arg(x\xi + \pi/2)| < \pi\) and \(|\arg(x\xi - 3\pi/2)| < \pi\), respectively. Then the Stokes matrices are defined by

\[
(2.2) \quad \Xi(\xi) = \Xi_1(\xi)S_1, \quad \Xi_2(\xi) = \Xi(\xi)S_2, \quad S_1 = \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 1 & s_2 \\ 0 & 1 \end{pmatrix}.
\]

These monodromy data \(M^0, M^1, S_1, S_2\) are as given in \([5, (2.3), (2.12)]\). As will be explained in Section 8.11 if \(M^0, M^1 \neq \pm I\), solutions of (P) are labelled and parametrised by entries of \(M^0, M^1\), or of \(S_1, S_2\).

For a given number \(\phi \in \mathbb{R}\), the Boutroux equations

\[
(2.3) \quad \text{Re} e^{i\phi} \int_a \sqrt{\frac{A_\phi - z^2}{1 - z^2}} \, dz = \text{Re} e^{i\phi} \int_b \sqrt{\frac{A_\phi - z^2}{1 - z^2}} \, dz = 0
\]

admit a unique solution \(A_\phi \in \mathbb{C}\) having the properties (cf. Proposition 8.11 and Figure 8.2 (a)):

(i) 0 \leq \text{Re} A_\phi \leq 1 for \(\phi \in \mathbb{R}\), and \(A_0 = 0, A_{\pm \pi/2} = 1\);
(ii) \(A_{-\phi} = \overline{A_\phi}, A_{\phi \pm \pi} = A_\phi\) for \(\phi \in \mathbb{R}\);
(iii) for 0 \leq \phi \leq \pi/2, \text{Im} A_\phi \geq 0, and, for \(-\pi/2 \leq \phi \leq 0, \text{Im} A_\phi \leq 0\).

By (i) we fix \(\text{Re} A_\phi^{1/2} \geq 0\), and then \((A_\phi)_{1/2} = \overline{A_\phi^{1/2}}\). Here \(a\) and \(b\) denote basic cycles as in Figure 2.2 on the elliptic curve \(\Pi = \Pi^*_1 \cup \Pi^*_2\) given by \(w(A_\phi, z) = \sqrt{(1 - z^2)(A_\phi - z^2)}\) such that \(\Pi^*_1\) and \(\Pi^*_2\) are glued along the cuts \([-1, -A_\phi^{1/2}]\) and \([A_\phi^{1/2}, 1]\). The branches of these algebraic functions are fixed in such a way that \(z^{-2} \sqrt{(A_\phi - z^2)(1 - z^2)} \to -1\) and \(\sqrt{(A_\phi - z^2)/(1 - z^2)} \to 1\) as \(z \to \infty\) on the upper sheet \(\Pi^*_1\). Write

\[
\Omega_{a,b} = \int_{a,b} \frac{dz}{w(A_\phi, z)}, \quad \xi_{a,b} = \int_{a,b} \sqrt{\frac{A_\phi - z^2}{1 - z^2}} \, dz,
\]

and let \(\text{sn}(u; k)\) denote the Jacobi sn-function with modulus \(k\).
2.1. Elliptic representation. We have

**Theorem 2.1.** Suppose that $0 < |\phi| < \pi/2$. Let $y(x)$ be the solution of $(P_V)$ corresponding to $M^0 = (m_{ij}^0)$, $M^1 = (m_{ij}^1)$ such that $m_{11}^0 m_{11}^1 m_{21}^0 m_{12}^1 \neq 0$. Then

$$\frac{y(x) + 1}{y(x) - 1} = A_{\phi}^{1/2} \text{sn}((x - x_0)/2 + \Delta(x); A_{\phi}^{1/2}),$$

$$\frac{y'(x)^2 - y(x)^2}{y(x)(y(x) - 1)^2} = \frac{1}{4}(1 - A_{\phi}) + O(x^{-1})$$

(y = dy/dx) with $\Delta(x) = O(x^{-2/9+\varepsilon})$ as $x = e^{i\phi}t \to \infty$ through the cheese-like strip

$$S(\phi, t_\infty, \kappa_0, \delta_0) = \{x = e^{i\phi}t; \text{Re}t > t_\infty, |\text{Im}t| < \kappa_0\} \cup \{|x - \rho| < \delta_0\},$$

$$\mathcal{P}_0 = \{\rho; \text{sn}((\rho - x_0)/2; A_{\phi}^{1/2}) = \infty\} = \{x_0 + \Omega_a \mathbb{Z} + \Omega_b(2\mathbb{Z} + 1)\},$$

$0 < \varepsilon < 2/9$ being arbitrary, $\kappa_0 > 0$ a given number, $\delta_0 > 0$ a given small number, and $t_\infty = t_\infty(\kappa_0, \delta_0)$ a large number depending on $(\kappa_0, \delta_0)$. Furthermore, $x_0$ is such that $x_0 \in S(\phi, t_\infty, \kappa_0, \delta_0)$ and that

$$x_0 = \frac{-1}{\pi i} \left(\Omega_b \log(m_{21}^0 m_{12}^1) + \Omega_a \log m_{\phi}\right) - \left(\frac{1}{2}\Omega_a + \Omega_b\right)(\theta_\infty + 1) \mod 2\Omega_a \mathbb{Z} + 2\Omega_b \mathbb{Z},$$

where $m_{\phi} = m_{11}^0$ if $-\pi/2 < \phi < 0$, and $e^{-\pi\theta_\infty}(m_{11}^1)^{-1}$ if $0 < \phi < \pi/2$.

**Remark 2.1.** For the truncated or classical solutions, the condition $m_{11}^0 m_{11}^1 m_{21}^0 m_{12}^1 \neq 0$ is not fulfilled (cf. [3] §5, [2], [4]).

**Remark 2.2.** Among the singular values $y = 0, 1, \infty$ of $(P_V)$, the neighbourhoods of only 1-points are excluded from the cheese-like domain of the elliptic expression of $y(x)$.

The solution $y(x)$ given in Theorem 2.1 is labelled by $M^{0,1} = (m_{ij}^{0,1})$ satisfying $m_{11}^0 m_{11}^1 m_{21}^0 m_{12}^1 \neq 0$. Denote this by $y(x) = y(M^0, M^1; x)$. Note that $\Omega_a$ and $\Omega_b$ are determined by $A_{\phi}$, which does not depend on $M^{0,1}$.

**Theorem 2.2.** Let $0 < |\phi - \pi| < \pi/2$. Write $\tilde{M}^0 = S_2^{-1} M^0 S_2$, $\tilde{M}^1 = S_2^{-1} M^1 S_2$ with $\tilde{M}^0 = (\tilde{m}_{ij}^0)$, $\tilde{M}^1 = (\tilde{m}_{ij}^1)$. If $\tilde{m}_{12}^0 \tilde{m}_{21}^0 \tilde{m}_{22}^0 \tilde{m}_{12}^1 \neq 0$, then $y(\tilde{M}^0, \tilde{M}^1; x)$ admits an elliptic representation as in Theorem 2.1 with the substitutions $A_{\phi} \mapsto A_{\phi-\pi}$, $m_{21}^0 m_{12}^1 \mapsto (\tilde{m}_{12}^0 \tilde{m}_{21}^0)^{-1}$, $m_{\phi} \mapsto \tilde{m}_{\phi}$, where $\tilde{m}_{\phi} = e^{i\phi}(\tilde{m}_{22}^0)^{-1}$ if $\pi/2 < \phi < \pi$, and $\tilde{m}_{22}^1$ if $\pi < \phi < 3\pi/2$.
Remark 2.3. Let $0 < |\phi \mp 2p\pi| < \pi/2$ or $|\phi \mp 2p\pi - \pi| < \pi/2$ with $p = 1, 2, 3, \ldots$. Set

$$U_p = \begin{cases} S_2S_3 \cdots S_{2p}S_{2p+1}, & \text{if } p > 0, \\ S_1^{-1}S_0^{-1} \cdots S_{2p-1}^{-1}S_{2p+2}, & \text{if } p < 0, \end{cases}$$

$$\check{U}_p = U_pS_{2p+2} \text{ if } p > 0, \quad \check{U}_p = U_pS_{2p+1}^{-1} \text{ if } p < 0,$$

and

$$M_p^0 = ((m_p^0)_{ij}) = U_p^{-1}M_0U_p, \quad M_p^1 = ((m_p^1)_{ij}) = U_p^{-1}M_1U_p,$$

$$\check{M}_p^0 = ((\check{m}_p^0)_{ij}) = \check{U}_p^{-1}M_0\check{U}_p, \quad \check{M}_p^1 = ((\check{m}_p^1)_{ij}) = \check{U}_p^{-1}M_1\check{U}_p,$$

where $S_{k+2} = e^{i\pi\theta_\infty \sigma_3}S_3 e^{-i\pi\theta_\infty \sigma_3}$ for $k \in \mathbb{Z}^{[5]} (2.5)$. Then $y(M_0, M_1; x)$ admits an elliptic representation as in Theorem 2.1 with the following substitutions (see the proof of Theorem 2.2 in Section 6.6):

1. for $0 < |\phi \mp 2p\pi| < \pi/2$, $A_\phi \mapsto A_{\phi \mp 2p\pi}$, $m_{21}^0 m_{12}^0 \mapsto (m_p^0)_{21}(m_p^1)_{12}$, $m_\phi \mapsto m_\phi = m_\phi |_{m_{ij}^0 \mapsto (m_{ij}^0)_{ij}}$; and

2. for $0 < |\phi \mp 2p\pi - \pi| < \pi/2$, $A_\phi \mapsto A_{\phi \mp 2p\pi - \pi}$, $m_{21}^0 m_{12}^0 \mapsto ((\check{m}_p^0)_{12}(\check{m}_p^1)_{21})^{-1}$, $m_\phi \mapsto \check{m}_\phi = m_\phi |_{m_{ij}^0 \mapsto (m_{ij}^0)_{ij}}$.

2.2. Error term $\Delta(x)$. The elliptic expression above apparently contains the single integration constant $x_0$, and the other one is hidden in the error term $\Delta(x) \ll x^{-2/9+\varepsilon}$. To give a conjectural representation of $\Delta(x)$ we describe some necessary functions and constants.

For the same monodromy data $M^0, M^1$ as in Theorem 2.1 let $b(x)$ related to $y(x)$ be defined by

$$a_\phi = A_\phi + \frac{B_\phi(t)}{t} = A_\phi + \frac{b(x)}{x} \text{ with } x = e^{i\phi}t, \quad i.e. \ b(e^{i\phi}t) = e^{i\phi}B_\phi(t)$$

(cf. (3.10), (6.9)). Set

$$\psi_0(x) = A_\phi^{1/2}sn((x-x_0)/2; A_\phi^{1/2}),$$

$$b_0(x) = \beta_0 - \frac{2\varepsilon_a}{\Omega_a} x - \frac{8}{\Omega_a} \vartheta' \left( \frac{1}{2\Omega_a}(x-x_0), \tau_0 \right), \quad \tau_0 = \frac{\Omega_b}{\Omega_a},$$

$$\beta_0 = -\frac{8}{\Omega_a} (\log(m_{21}^0 m_{12}^0) + \pi i(\theta_\infty + 1)),$$

where

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 + 2\pi i n z}, \quad \text{Im } \tau > 0$$

with $\vartheta'(z, \tau) = (d/dz)\vartheta(z, \tau)$ is the $\vartheta$-function (cf. Section 5.2). Then $\psi_0(x)$ solves

$$2\psi_0' = \sqrt{P(\psi_0)} = w(A_\phi, \psi_0), \quad P(\psi) := (1 - \psi^2)(A_\phi - \psi^2),$$

and $b_0(x)$ fulfills $b_0(e^{i\phi}t) - e^{i\phi}B_\phi(t) \ll t^{-2/9+\varepsilon}$ in $S(\phi, t_\infty, \kappa_0, \delta_0)$ (Proposition 5.6 and Corollary 6.1). Furthermore

$$b_0'(x) = 2(\psi_0(x) - A_\phi) + 4\psi_0'(x)$$
Then \( \int_{\tau}^{S} (\chi_0(x) = b(x) - b_0(x) - b_0(x)h(x) \text{ are represented by} \)

\[
\chi_0(x) = \int_{\infty}^{x} (\psi_0^2 - A_\phi)(F_1(\psi_0, b_0)^2 - 2F_2(\psi_0)) \frac{d\xi}{\xi^2} + O(x^{-2}),
\]

\[
h(x) = -\int_{\infty}^{x} F_1(\psi_0, b_0 + \chi_0) \frac{d\xi}{\xi} + \int_{\infty}^{x} \left( F_2(\psi_0) - \frac{3}{2}F_1(\psi_0, b_0)^2 \right) \frac{d\xi}{\xi^2} + O(x^{-2})
\]

\[
= -\int_{\infty}^{x} F_1(\psi_0, b_0) \frac{d\xi}{\xi} + \int_{\infty}^{x} \left( F_2(\psi_0) - \frac{3}{2}F_1(\psi_0, b_0)^2 \right) \frac{d\xi}{\xi^2}
\]

\[
+ \frac{1}{2} \int_{\infty}^{x} \frac{1}{A_\phi} \psi_0 \int_{\infty}^{\xi} (\psi_0^2 - A_\phi)(F_1(\psi_0, b_0)^2 - 2F_2(\psi_0)) \frac{d\xi_1}{\xi_1^2} \frac{d\xi}{\xi} + O(x^{-2})
\]

in \( \hat{S}(\phi, t, \kappa, \delta_0) \), where

\[
F_1(\psi, b) = \frac{4(\theta_0 + \theta_1)\psi - b}{2(A_\phi - \psi^2)}, \quad F_2(\psi) = \frac{2(2(\theta_0 - \theta_1)\theta_\infty + (\theta_0 - \theta_1)^2 + \theta_\infty^2)}{(1 - \psi^2)(A_\phi - \psi^2)}
\]

and \( \int_{\infty}^{x} d\xi \) denotes the integral such that

\[
\lim_{x_n \to \infty} \int_{x_n}^{x} d\xi \quad \text{with} \quad x_n \in \hat{S}(\phi, t, \kappa, \delta_0).
\]

Remark 2.4. For (P1) or (P11) the relation between the error term \( h(x) \) and \( b(x) - b_0(x) \) in the \( \tau \)-function is referred to by Kitaev [28, p. 121].

Corollary 2.4. Let

\[
F_0(\psi_0) = \frac{2(\theta_0 + \theta_1)\psi_0}{A_\phi - \psi_0^2}, \quad G_0(\psi_0, b_0) = \frac{b_0}{2(A_\phi - \psi_0^2)}, \quad F_1(\psi_0, b_0) = F_0(\psi_0) - G_0(\psi_0, b_0).
\]

Then

\[
\chi_0(x) = -\frac{1}{2} \int_{\infty}^{x} b_0(G_0(\psi_0, b_0) - 2F_0(\psi_0)) \frac{d\xi}{\xi^2} - 4(2(\theta_0^2 + \theta_1^2) + \theta_\infty^2)x^{-1} + O(x^{-2}),
\]

\[
h(x) = -\int_{\infty}^{x} F_1(\psi_0, b_0) \frac{d\xi}{\xi} - \frac{3}{2} \int_{\infty}^{\xi} G_0(\psi_0, b_0)(G_0(\psi_0, b_0) - 2F_0(\psi_0)) \frac{d\xi}{\xi^2}
\]

\[
- \frac{1}{4} \int_{\infty}^{x} \frac{1}{A_\phi - \psi_0^2} \int_{\infty}^{\xi} b_0(G_0(\psi_0, b_0) - 2F_0(\psi_0)) \frac{d\xi_1}{\xi_1^2} \frac{d\xi}{\xi}
\]
Furthermore, these functions are written in the form
\[ x\chi_0(x) = \omega_0(x, \beta_0) + O(x^{-1}), \quad x\chi(h(x)) = \Omega_0(x, \beta_0) + O(x^{-1}) \]
with \( \omega_0(x, \beta_0) \) at most linear in \( \beta_0 \) and
\[ \Omega_0(x, \beta_0) = \frac{\beta_0^2}{8A_\phi(1 - A_\phi)} + \Omega_{01}(x)\beta_0 + \Omega_{02}(x), \]
\( \omega_0(x, \beta_0), \Omega_{01}(x), \Omega_{02}(x) \) being bounded in \( \tilde{S}(\phi, t_\infty, \kappa_0, \delta_0) \).

Remark 2.5. The corollary above implies the error term \( \Delta(x) = h(x)/2 \) depends on the integration constant \( \beta_0 \).

3. Basic facts

3.1. Parametrisation of \( y(x) \) by the monodromy data. Note that
\[
M^1 M^0 = S_1^{-1} e^{-\pi i \theta_\infty \sigma_3} S_2^{-1}
\]  
[3, (2.8), (2.13)]. For the monodromy matrices \( M^0, M^1 \), let \( \mathcal{M} \) be the algebraic variety consisting of \( (M^0, M^1) \in SL_2(\mathbb{C})^2 \) such that
\[
\operatorname{tr} M^0 = 2 \cos \pi \theta_0, \quad \operatorname{tr} M^1 = 2 \cos \pi \theta_1, \quad (M^1 M^0)_{11} = e^{-\pi i \theta_\infty},
\]
which is called the manifold of monodromy data. By [3,2] \( \dim_{\mathbb{C}} \mathcal{M} = 3 \), and a generic point \( (M^0, M^1) \in \mathcal{M} \) may be represented by three parameters \( q_0, q_1, r \), say, as follows:
\[
M^0 = \begin{pmatrix}
\cos \pi \theta_0 - q_0 & r \rho(q_0, q_1)^{-1}(\cos^2 \pi \theta_0 - q_0^2 - 1) \\
r^{-1} \rho(q_0, q_1) & \cos \pi \theta_0 + q_0
\end{pmatrix},
\]
\[
M^1 = \begin{pmatrix}
\cos \pi \theta_1 - q_1 & r \\
r^{-1}(\cos^2 \pi \theta_1 - q_1^2 - 1) & \cos \pi \theta_1 + q_1
\end{pmatrix}
\]
with \( \rho(q_0, q_1) = e^{-\pi i \theta_\infty} - (\cos \pi \theta_0 - q_0)(\cos \pi \theta_1 - q_1) \). By using (3.1) the Stokes multipliers \( s_1, s_2 \) (respectively, the entries \( m_{ij}^0, m_{ij}^1 \)) are written in terms of \( m_{ij}^0, m_{ij}^1 \) (respectively, \( s_1, s_2 \)). For the matrices above,
\[
e^{-\pi i \theta_\infty}(\cos \pi \theta_1 - q_1) - (\cos \pi \theta_0 + q_0) = (e^{-\pi i \theta_\infty} - (\cos \pi \theta_0 - q_0)(\cos \pi \theta_1 - q_1)) r^{-1} s_2,
\]
\[
e^{-\pi i \theta_\infty}(\cos \pi \theta_0 - q_0) - (\cos \pi \theta_1 + q_1) = r s_1,
\]
and, conversely, \( q_0, q_1 \) are algebraic in \( r s_1, r^{-1} s_2 \).

For a diagonal matrix \( d_0^a \) with \( d_0 \in \mathbb{C} \setminus \{0\} \), the gauge transformation \( \Xi = d_0^a \tilde{\Xi} d_0^{-a} \) changes \([1,1] \) into a system with \( (y, z, d_0^a u) \) in place of \( (y, z, u) \), and the monodromy matrices for the matrix solution \( \tilde{\Xi}(\xi) \) of the same asymptotic form as of \([2,1] \) become \( d_0^{-a} M^0 d_0^a, d_0^{-a} M^1 d_0^a \). By this fact combined with the surjectivity of the Riemann-Hilbert correspondence [4, 33] and the uniqueness [5] Propositions 2.1 and 2.2] (see also [12] Proposition 5.9 and Theorem 5.5], [13], [5] §§3, 4, 5], [2], [3] and [4]) we have
Proposition 3.1. Let \( \mathcal{Y}(P_V) \) be the family of solutions of \((P_V)\). For \((M^0, M^1), (\tilde{M}^0, \tilde{M}^1) \in \mathcal{M}\), write \((M^0, M^1) \sim (\tilde{M}^0, \tilde{M}^1)\) if there exists \(d_0 \in \mathbb{C} \setminus \{0\}\) such that \((\tilde{M}^0, \tilde{M}^1) = d_0^{-\sigma_3}(M^0, M^1)d_0^{\sigma_3}\). Let \(\varphi : \mathcal{Y}(P_V) \to \mathcal{M}\). Then we have the canonical bijection

\[ \varphi^* : \mathcal{Y}^*(P_V) = \mathcal{Y}(P_V) \setminus \mathcal{Y}_0(P_V) \to \mathcal{M}^* = (\mathcal{M} \setminus \mathcal{M}_0)/\sim, \]

where

\[ \mathcal{M}_0 = \{(M^0, M^1) \in \mathcal{M} ; \; M^0 = \pm I \text{ or } M^1 = \pm I\}, \]

\[ \mathcal{Y}_0(P_V) = \{y \in \mathcal{Y}(P_V) ; \; \varphi(y) \in \mathcal{M}_0\} \]

and \(\dim_{\mathbb{C}} \mathcal{M}^* = 2\).

Thus the solutions in \(\mathcal{Y}^*(P_V)\) are parametrised by \((M^0, M^1) \in \mathcal{M}^*\), essentially by two parameters.

3.2. Symmetric linear system. To consider \(y(x) \in \mathcal{Y}^*(P_V)\) along a ray \(\arg x = \phi\) with \(|\phi| < \pi/2\), and to convert (1.1) to a symmetric form, set

\[ x = e^{i\phi}t, \quad t > 0, \quad \xi = (e^{-i\phi}\lambda + 1)/2. \]

Then by the gauge transformation \(Y = \exp(-\varpi(t, \phi)\sigma_3)\Xi\) with \(\varpi(t, \phi) = e^{i\phi}t/4 + (\theta_\infty/2)(i\phi + \log 2)\) system (1.1) is taken to

\[ \frac{dY}{d\lambda} = tB(t, \lambda)Y \]

with \(B(t, \lambda) = \hat{u}^{\sigma_3/2}(b_3\sigma_3 + b_2\sigma_2 + b_1\sigma_1)\hat{u}^{-\sigma_3/2}\). Here \(\hat{u} = u \exp(-2\varpi(t, \phi))\) and

\[ b_3 = b_3(t, \lambda) = \frac{1}{4} + t^{-1}\left(\frac{a_{11}^{11}}{\lambda + e^{i\phi}} + \frac{a_{11}^{21}}{\lambda - e^{i\phi}}\right), \]

\[ b_2 = b_2(t, \lambda) = \frac{it^{-1}}{2}\left(\frac{a_{12}^{12} - a_{21}^{21}}{\lambda + e^{i\phi}} + \frac{a_{12}^{21} - a_{21}^{12}}{\lambda - e^{i\phi}}\right), \]

\[ b_1 = b_1(t, \lambda) = \frac{t^{-1}}{2}\left(\frac{a_{12}^{12} + a_{21}^{21}}{\lambda + e^{i\phi}} + \frac{a_{12}^{21} + a_{21}^{12}}{\lambda - e^{i\phi}}\right), \]

\(a_{ij}^{ij}\) and \(a_{ij}^{ij}\) being the entries of \(A_0|_{u=1}, A_1|_{u=1}\), that is,

\[ a_{11}^{11} = \frac{3 + \theta_0}{2}, \quad a_{12}^{12} = -(3 + \theta_0), \]

\[ a_{01}^{21} = 3, \quad a_{02}^{22} = -a_{11}^{11}, \]

\[ a_{11}^{11} = -\frac{3}{2} - \frac{\theta_0 + \theta_\infty}{2}, \quad a_{12}^{12} = y\left(3 + \frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right), \]

\[ a_{21}^{21} = -\frac{1}{y}\left(3 + \frac{\theta_0 + \theta_1 + \theta_\infty}{2}\right), \quad a_{22}^{22} = -a_{11}^{11}. \]

Note that

\[(\lambda^2 - e^{2i\phi})b_3 = \frac{1}{4}(\lambda^2 - e^{2i\phi}) - 2e^{i\phi}t^{-1}\frac{3}{2} - \frac{1}{2}(\theta_\infty\lambda + (2\theta_0 + \theta_\infty)e^{i\phi})t^{-1},\]

\[y(\lambda^2 - e^{2i\phi})(b_1 + ib_2) = ((y - 1)(\lambda + e^{i\phi}) - 2e^{i\phi}y)t^{-1}\frac{3}{2} - \frac{1}{2}(\theta_0 + \theta_1 + \theta_\infty)(\lambda + e^{i\phi})t^{-1},\]
(\lambda^2 - e^{2i\phi})(b_1 - ib_2) = ((y - 1)(\lambda + e^{i\phi}) + 2e^{i\phi})t^{-1}\hat{3}
+ \left(\frac{y}{2}(\theta_0 - \theta_1 + \theta_\infty)(\lambda + e^{i\phi}) - \theta_0(\lambda - e^{i\phi})\right)t^{-1}.

Let loops \( \hat{l}_0, \hat{l}_1 \) and a point \( \hat{p}_{st} \) in the \( \lambda \)-plane be the images of \( l_0, l_1 \) and \( p_{st} \) under (3.3). The loops \( \hat{l}_0, \hat{l}_1 \) start from \( \hat{p}_{st} \) and surround \( \lambda = -e^{i\phi}, \lambda = e^{i\phi} \), respectively; and \( \arg \hat{p}_{st} = \pi/2 \) (cf. Figure 3.1).

![Figure 3.1. Loops \( \hat{l}_0 \) and \( \hat{l}_1 \) on the \( \lambda \)-plane](image)

Then (3.4) admits the matrix solution \( Y(t, \lambda) = \exp(-\varpi(t, \phi)\sigma_3)\Xi(e^{i\phi}t, (e^{-i\phi}\lambda+1)/2) \) (cf. (2.1)) with the properties:

(i) \( Y(t, \lambda) \) has the asymptotic representation

\[ Y(t, \lambda) = (I + O(\lambda^{-1})) \exp\left(\frac{1}{4}(t\lambda - 2\theta_\infty \log \lambda)\sigma_3\right) \]

as \( \lambda \to \infty \) through the sector \( |\arg \lambda - \pi/2| < \pi \), the branch of \( \log \lambda \) being taken in such a way that \( \text{Im}(\log \lambda) \to \pi/2 \) as \( \lambda \to \infty \) through this sector;

(ii) the isomonodromy deformation yields the same monodromy data \( M^0, M^1, S_1, S_2 \) as in Section 2, where \( M^0 \) and \( M^1 \) are given by the analytic continuation of \( Y(t, \lambda) \) along \( \hat{l}_0 \) and \( \hat{l}_1 \), respectively, and \( S_1 \) and \( S_2 \) are defined by \( Y_1(t, \lambda) \) and \( Y_2(t, \lambda) \) having the same asymptotic representation as (3.6) in the sectors \( |\arg \lambda+\pi/2| < \pi \) and \( |\arg \lambda-3\pi/2| < \pi \), respectively;

(iii) system (3.4) has the isomonodromy property if and only if \( (y, \hat{\phi}, \hat{u}) \) with \( \hat{u} = u \exp(-2\varpi(t, \phi)) \) satisfies

\[ t_{\hat{y}} = e^{i\phi}y - 2\hat{3}(y - 1)^2 - \frac{(y - 1)}{2}\left((\theta_0 - \theta_1 + \theta_\infty)y - (3\theta_0 + \theta_1 + \theta_\infty)\right), \]

\[ t_{\hat{3}} = \hat{3}\left(3 + \frac{1}{2}(\theta_0 - \theta_1 + \theta_\infty)\right) - \frac{1}{y}\left(\hat{3} + \frac{1}{2}(\theta_0 + \theta_1 + \theta_\infty)\right)\],

\[ \frac{t_{\hat{u}}}{u} = -2\hat{3} - \theta_0 + y\left(3 + \frac{1}{2}(\theta_0 - \theta_1 + \theta_\infty)\right) + \frac{1}{y}\left(\hat{3} + \frac{1}{2}(\theta_0 + \theta_1 + \theta_\infty)\right) \]

\( (y_t = dy/dt) \), and then \( y(e^{i\phi}t) \in \mathcal{Y}^*(\mathcal{P}_V) \) is parametrised by \( (M^0, M^1) \in \mathcal{M}^* \).

Remark 3.1. In what follows we denote \( y(e^{i\phi}t) \) by \( y(t) \) for brevity, and set

\[ \hat{3} = -\frac{(y - e^{i\phi}y)t}{2(y - 1)^2} - \frac{1}{4}(\theta_0 - \theta_1 + \theta_\infty) + \frac{\theta_0 + \theta_1}{2(y - 1)}, \]

(3.8)
which is the first equation of (3.7).

Remark 3.2. Let $s$ be a substitution given by $e^{i\phi} \mapsto -e^{i\phi}$, $y \mapsto y^{-1}$, $(\theta_0, \theta_1) \mapsto (\theta_1, \theta_0)$. It is easy to see that $s(\mathfrak{z}) = -\mathfrak{z} - (\theta_0 + \theta_1 + \theta_\infty)/2$. Then system (3.4) is invariant under the extension of $s$: $(s, Y \mapsto y^{1/2}Y)$.

3.3. Characteristic roots, turning points and Stokes curves. In the remaining part of this section, Sections 4 and 5 we are concerned with the direct monodromy problem for system (3.4), that is, calculation of the monodromy. Then in this section, Sections 4 and 5 we are concerned with the direct monodromy problem for system (3.4), that is, calculation of the monodromy. Then let us suppose that $y, y^*$ and $u$ are arbitrary complex parameters.

To calculate the monodromy data for system (3.4) we need to know the characteristic roots of $\mathcal{B}(t, \lambda)$ and their turning points. The characteristic roots $\pm \mu = \pm \mu(t, \lambda)$ are given by

$$
\mu^2 = -\det \mathcal{B}(t, \lambda) = b_0^2 + (-ib_2 + b_1)(ib_2 + b_1) = b_1^2 + b_2^2 + b_3^2.
$$

Using (3.8), we obtain

$$
(3.9) \quad 4(e^{2i\phi} - \lambda^2)\mu^2 = \frac{1}{4}(e^{2i\phi} - \lambda^2)(e^{2i\phi}a_\phi - \lambda^2 + 4\theta_\infty t^{-1}\lambda) + 2(\theta_1^2 - \theta_0^2)e^{i\phi}t^{-2}\lambda + 2(\theta_0^2 + \theta_1^2)e^{2i\phi}t^{-2},
$$

where $a_\phi = a_\phi(t)$ is given by

$$
(3.10) \quad a_\phi = 1 - \frac{4(e^{-2i\phi}(y^*)^2 - y^2)}{y(y - 1)^2} + 4e^{-i\phi}(\theta_0 + \theta_1)\frac{y + 1}{y - 1}(-t^{-1})
$$

$$
+ e^{-2i\phi}\frac{y - 1}{y}((\theta_0 - \theta_1 + \theta_\infty)^2y - (\theta_0 - \theta_1 - \theta_\infty)^2)t^{-1}.
$$

as $t \to \infty$ outside the exceptional set $\bigcup_{\sigma \in Z(1)} \{ |e^{i\phi}t - \sigma| < \delta_1 \}$, $Z(1) = \{ \sigma; y(\sigma) = 1 \}$.

Note that this is also valid in the cheese-like strip containing the ray arg $x = \phi$:

$$
S'(\phi, t'_{\infty}, \kappa_0, \delta_1) = \{ e^{i\phi}t; \quad \text{Re} t > t'_{\infty}, |\text{Im} t| < \kappa_0 \} \setminus \bigcup_{\sigma \in Z(1)} \{ |e^{i\phi}t - \sigma| < \delta_1 \},
$$

$t'_{\infty}$ and $\kappa_0$ ($> 2\delta_1$) being given numbers. By (3.10), the turning points, that is, the zeros of $\mu$ are given by

**Proposition 3.2.** For each $t$, let the square roots of $a_\phi = a_\phi(t)$ be denoted by $\pm a_\phi^{1/2}$, where $\text{Re} a_\phi^{1/2} \geq 0$ and $\text{Im} a_\phi^{1/2} > 0$ if $\text{Re} a_\phi^{1/2} = 0$. Then, for $|\phi| < \pi/2$, the turning points are

$$
\lambda_1 = e^{i\phi}a_\phi^{1/2} + 2\theta_\infty t^{-1} + O(t^{-2}), \quad \lambda_2 = -e^{i\phi}a_\phi^{1/2} + 2\theta_\infty t^{-1} + O(t^{-2}),
$$

$$
\lambda_1^0 = e^{i\phi} + O(t^{-2}), \quad \lambda_2^0 = -e^{i\phi} + O(t^{-2})
$$

as $t \to \infty$, and these are simple. Furthermore

$$
\mu^2 = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_1^0)(\lambda - \lambda_2^0)}{16(\lambda - e^{i\phi}/2(\lambda + e^{i\phi})^2)}.
$$
Remark 3.3. On the positive real axis all the solutions of \((P_{V})\) corresponding to the monodromy data such that \(m_{01}^{1} m_{21}^{0} m_{11}^{1} m_{12}^{1} \neq 0\) are given by \([5, \text{Theorems 3.1 and 4.1}].\) By using the expressions of these solutions it is easy to verify \(a_{0}(t) = a_{\phi}(t) |_{\phi=0} \ll t^{-\varepsilon}\) as \(t \to \infty\) (for solutions of \([5, \text{Theorem 4.1}].\)) \(t \to \infty\) along a suitable path avoiding poles).

Then \(\Re a_{\phi}(t)^{1/2} \ll |t^{-\varepsilon} + |\phi|\) uniformly in \(t\) for sufficiently small \(|\phi|\), which implies that, as long as \(m_{01}^{0} m_{21}^{0} m_{11}^{1} m_{12}^{1} \neq 0\), every corresponding solution fulfills \(0 \leq \Re a_{\phi}(t)^{1/2} < 1\) if \(|\phi|\) is sufficiently small. On the other hand, for a general solution in \([38, \text{Theorem 2.18}]\) along the imaginary axis, \(a_{\pi/2}(t) = 1 + O(t^{-\varepsilon}).\)

Remark 3.4. To the monodromy data such that \(m_{01}^{0} m_{21}^{0} m_{11}^{1} m_{12}^{1} = 0\) correspond truncated solutions in sectors containing the positive real axis \([2,1]\). Now, for these solutions, let us consider \(a_{\phi}(t) = (y(t), y_{t}(t))\). Then we have \(a_{\phi}(t) \ll t^{-1}\) for \(\phi\) in some intervals containing \(\phi = 0\). In the case \(m_{11}^{1} = 0\) the solution \(y(x) \sim -1 + e^{x} e^{-t}e^{ix/2}\) in \(0 \leq \arg x \leq \pi\) \([4, \text{Proposition 5}].\) \([5, \text{Corollary 5.2}]\) satisfies \(a_{\phi}(t) \ll t^{-1}\) for \(0 \leq \phi \leq \pi\).

If \(m_{01}^{1} = m_{11}^{1} = 0\), then \(y(x) \sim -1 + 4(\theta_{0} + \theta_{1} - 1)e^{-x} \approx |\arg x| < \pi\) \([4, \text{Proposition 2}].\) \([5, \text{Corollary 5.3}]\) satisfies \(a_{\phi}(t) \ll t^{-2}\) for \(|\phi| < \pi\). For \(m_{11}^{1} = 0\) or \(m_{21}^{0} = 0\), truncated solutions such that \(a_{\phi}(t) \ll t^{-1}\) for \(|\phi| < \pi/2\) are given by \([2,1]\).

By \([39, \text{Theorem 3.9}]\) the characteristic root \(\mu = \mu(t, \lambda)\) is written in the form

\[
(3.11) \quad \mu = \frac{1}{4} \sqrt{e^{2i\phi}a_{\phi} - \lambda^2} + \frac{\theta_{\infty} \lambda t^{-1}}{2(e^{2i\phi} - \lambda^2)} + g_{2}(t, \lambda)t^{-2}
\]

as \(t \to \infty\). Here \(g_{2}(t, \lambda)\) has branch points at \(\lambda_{01,2}, \lambda_{1,2}, \pm e^{i\phi}\) and \(\pm e^{i\phi} a_{\phi}^{1/2}\), but it fulfills \(g_{2}(t, \lambda) \ll |\lambda^{2} - e^{2i\phi}a_{\phi}|^{-3/2} + |\lambda^{2} - e^{2i\phi}|^{-5/2} \ll 1\). The algebraic function \(\mu(t, \lambda)\) is given on the Riemann surface consisting of two copies of \(\lambda\)-plane \(\mathbb{P}_{\pm}\) and \(\mathbb{P}_{-}\) glued along the cuts \([\lambda_{1}, \lambda_{0}^{1}], [\lambda_{0}^{1}, \lambda_{2}^{1}]\) (cf. Figure 3.2 (a)). Note that, in \((3.11)\), each square root is fixed in such a way that

\[
a_{\phi}^{-1/2} \sqrt{e^{2i\phi}a_{\phi} - \lambda^2}, \quad e^{-2i\phi}a_{\phi}^{-1/2} \sqrt{(e^{2i\phi}a_{\phi} - \lambda^2)(e^{2i\phi} - \lambda^2)} \rightarrow \pm 1
\]

as \(\lambda \to 0\) on \(\mathbb{P}_{\pm}\), \(a_{\phi}^{1/2}\) being as in Proposition 3.2.

A Stokes curve is defined by

\[
\Re \int_{\lambda_{*}}^{\lambda} \mu(t, \tau)d\tau = 0,
\]

where \(\lambda_{*}\) is a turning point \([11]\). This curve connects \(\lambda_{*}\) to another turning point, \(\pm e^{i\phi}\) or \(\infty\).

In what follows, the Stokes graph is discussed under the supposition \(m_{11}^{0} m_{21}^{0} m_{11}^{1} m_{12}^{1} \neq 0\). As long as the turning points do not coalesce, in our case the proper Stokes graph reflects the Boutroux equations (see Section 7). As in Remark 3.3 if \(m_{11}^{0} m_{21}^{0} m_{11}^{1} m_{12}^{1} \neq 0\), then \(0 \leq \Re a_{\phi}^{1/2} < 1\) at least for sufficiently small \(|\phi|\). Suppose that \(0 < |\phi| < \pi/2\).

Let us consider the limit Stokes graph. Set \(\mathbb{P}_{\pm}^{\infty} = \lim_{t \to \infty} \mathbb{P}_{\pm} \cup \mathbb{P}_{-}\), which is a two-sheeted Riemann surface glued along the cuts \([-e^{i\phi}, \lambda_{2}^{1}], [\lambda_{1}, e^{i\phi}]\). Then this limit
Stokes graph on $\mathbb{P}^\infty_+$ is considered to be as in Figure 3.2 (b), (c) (cf. Proposition 8.17), in which, if $0 < \phi < \pi/2$, the Stokes curves connect $\lambda_1$ to $e^{i\phi}$, $\lambda_2$ and $i\infty$, and $\lambda_2$ to $-e^{i\phi}$ and $-i\infty$.

![Stokes graph diagram](image)

(a) $[\lambda_1, \lambda_1^0]$, $[\lambda_1^0, \lambda_2]$

(b) $-\pi/2 < \phi < 0$

(c) $0 < \phi < \pi/2$

**Figure 3.2.** Cuts and the limit Stokes graph on $\mathbb{P}_+$

An unbounded domain $D \subset \mathbb{P}_+ \cup \mathbb{P}_-$ is called a canonical domain if, for each $\lambda \in D$, there exist contours $C_\pm(\lambda) \subset D$ ending at $\lambda$ such that

$$\text{Re} \int_{\lambda_-}^\lambda \mu(\tau) d\tau \to -\infty \quad \text{(respectively, } \text{Re} \int_{\lambda_+}^\lambda \mu(\tau) d\tau \to +\infty)$$

as $\lambda_- \to \infty$ along $C_-(\lambda)$ (respectively, as $\lambda_+ \to \infty$ along $C_+(\lambda)$) (see [11], [12, p. 242]). The interior of a canonical domain contains exactly one Stokes curve, and its boundary consists of Stokes curves.

### 3.4. WKB-solution

The following is a WKB-solution of (3.4) in a canonical domain.

**Proposition 3.3.** In the canonical domain whose interior contains a Stokes curve issuing from the turning point $\lambda_1$ or $\lambda_2$, system (3.4) with $\hat{u} \equiv 1$ admits a solution expressed by

$$\Psi_{\text{WKB}}(\lambda) = T(I + O(t^{-\delta})) \exp \left( \int_{\lambda_0}^\lambda \Lambda(\tau) d\tau \right)$$

outside suitable neighbourhoods of zeros of $b_1 \pm ib_2$ as long as $|\lambda \pm e^{i\phi}| \gg t^{-2+2\delta}$, $|\lambda - \lambda_1|$, $|\lambda - \lambda_0^{\pm}| \gg t^{-2/3+(2/3)\delta}$ ($i = 1, 2$), $0 < \delta < 1$ being arbitrary. Here $\tilde{\lambda}_i$ is a base point near $\lambda_1$ or $\lambda_2$, and $\Lambda(\tau)$ and $T$ are given by

$$\Lambda(\lambda) = t\mu\sigma_3 - \text{diag}T^{-1}T_{\lambda}, \quad T = \begin{pmatrix} 1 & b_3 - \mu \\ \mu - b_3 & b_1 + ib_2 \end{pmatrix}.$$

**Remark 3.5.** In this proposition

$$\text{diag}T^{-1}T_{\lambda} = \frac{1}{2\mu(\mu + b_3)} (i(b_1b_2' - b_1'b_2)\sigma_3 + (b_3\mu' - b_3'\mu)I) \quad (b_i' = \partial b_i/\partial \lambda)$$

$$= \frac{1}{4} \left( 1 - \frac{b_3}{\mu} \right) \partial \log \frac{b_1 + ib_2}{b_1 - ib_2} \sigma_3 + \frac{1}{2} \partial \log \frac{\mu}{\mu + b_3} I.$$
Proof. By \( Y = T \hat{Y} \) system (3.4) with \( \hat{u} \equiv 1 \) becomes
\[
(3.12) \quad \hat{Y}_\lambda = (t \mu \sigma_3 - T^{-1} T_\lambda) \hat{Y}.
\]
To remove \( R = T^{-1} T_\lambda - \text{diag} T^{-1} T_\lambda \) set
\[
T_1 = \frac{1}{2t \mu} \begin{pmatrix} 0 & R_{12} \\ -R_{21} & 0 \end{pmatrix}, \quad R_{12} = \frac{\mu + b_3}{2 \mu} \frac{\partial}{\partial \lambda} \left( \frac{b_3 - \mu}{b_1 + ib_2} \right), \quad R_{21} = \frac{\mu + b_3}{2 \mu} \frac{\partial}{\partial \lambda} \left( \frac{\mu - b_3}{b_1 - ib_2} \right),
\]
which fulfills \( [t \mu \sigma_3, T_1] = R \). Now we would like to find \( X_1 \) such that the transformation
\[
\hat{Y} = (I + T_1)(I + X_1) Z
\]
takes (3.12) to
\[
Z_\lambda = \Lambda Z = (t \mu \sigma_3 - \text{diag} T^{-1} T_\lambda) Z,
\]
that is,
\[
(T_1)\lambda (I + X_1) + (I + T_1)(X_1) \lambda + (I + T_1)(I + X_1) \Lambda = (\Lambda - R)(I + T_1)(I + X_1).
\]
It follows that
\[
(3.13) \quad (X_1)\lambda = [\Lambda, X_1] + (I + T_1)^{-1} Q(I + X_1)
\]
with \( Q = -(T_1)\lambda - R(I+T_1) + [\Lambda, T_1] = -(T_1)\lambda - T^{-1} T_\lambda T_1 + T_1 \text{diag} T^{-1} T_\lambda \). Near \( \lambda = \mp e^{i \phi} \), we have \( b_3, |b_1 \pm ib_2| \ll |\lambda \pm e^{i \phi}|^{-1} \), \( b_1 b_2 - b_1' b_2' \ll |\lambda \pm e^{i \phi}|^{-2} \), \( \mu \asymp |\lambda \pm e^{i \phi}|^{-1/2} \), and hence \( ||R|| \ll |\lambda \pm e^{i \phi}|^{-1} \asymp \mu^2 \), \( ||\text{diag} T^{-1} T_\lambda|| \ll |\lambda \pm e^{i \phi}|^{-1} \), \( ||T_1|| \ll t^{-1} |\lambda \pm e^{i \phi}|^{-1/2} \).
Near \( \lambda = \lambda_i \) (i = 1, 2) we have \( b_3, |b_1 b_2 - b_1' b_2'| \ll 1, \mu \asymp |\lambda - \lambda_i|^{1/2} \), and hence \( ||R|| \ll |\lambda - \lambda_i|^{-1} \), \( ||\text{diag} T^{-1} T_\lambda|| \ll |\lambda - \lambda_i|^{-3/2} \), \( ||T_1|| \ll t^{-1} |\lambda - \lambda_i|^{-3/2} \). Therefore \( ||Q|| \ll t^{-1} |\lambda \pm e^{i \phi}|^{-3/2} \) near \( \lambda = \mp e^{i \phi} \), and \( ||Q|| \ll t^{-1} |\lambda - \lambda_i|^{-5/2} \) near \( \lambda = \lambda_i \).
Furthermore near \( \lambda = \infty \), observe that \( \mu = 1/4 + O(\lambda^{-2}) \) on \( \mathbb{P}_+ \) and \( b_3 = 1/4 + O(\lambda^{-2}) \). Then \( \pm (b_3 - \mu)/(b_1 \pm ib_2) = -(b_1 \mp ib_2)/(\mu + b_3) \ll \lambda^{-1} \), which means \( T = I + O(\lambda^{-1}) \).
It is easy to see that \( ||T_1|| \ll t^{-1} \lambda^{-2} \), \( ||Q|| \ll t^{-1} \lambda^{-3} \) near \( \lambda = \infty \). Near \( \infty \) on \( \mathbb{P}_- \), \( ||T_1|| \ll t^{-1} \lambda^{-1} \), \( ||Q|| \ll t^{-1} \lambda^{-2} \), since \( \mu + 1/4 \ll t^{-1} \lambda^{-1} \). From (3.13) along \( C_\pm(\lambda) \) (cf. Section 3.3) it follows that
\[
X_1(\lambda) = \int_{C(\lambda)} \exp \left( \int_{\xi}^{\lambda} \Lambda(\tau) d\tau \right) (I + T_1(\xi))^{-1} Q(\xi)(I + X_1(\xi)) \exp \left( -\int_{\xi}^{\lambda} \Lambda(\tau) d\tau \right) d\xi,
\]
where \( C(\lambda) \) is a set of contours ending at \( \lambda \) chosen suitably for each entry according to the multiplier caused by \( \exp(\int_{\xi}^{\lambda} \Lambda(\tau) d\tau) \) and its reciprocal. Using this equation we may show that
\[
||(I + T_1)(I + X_1) - I|| \ll t^{-1} (|\lambda \pm e^{i \phi}|^{-1/2} + |\lambda - \lambda_1|^{-3/2} + |\lambda - \lambda_2|^{-3/2} + 1)
\]
by the same argument as in the proof of [12, Theorem 7.2]. \qed
3.5. Local solutions around turning points. For \( t = 1 \) or \( 2 \), if \( |\lambda - \lambda_i| \ll t^{-2/3} \), the WKB-solution given above fails in expressing the asymptotic behaviour. In the neighbourhood of \( \lambda_i \) system (3.4) is reduced to

\[
\frac{dW}{d\zeta} = \begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} W.
\]

Then we have the following local solution around the turning point.

**Proposition 3.4.** For each turning point \( \lambda_i \) \((i = 1, 2)\) write \( c_k = b_k(\lambda_i) \), \( c'_k = (b_k)_\lambda(\lambda_i) \) \((k = 1, 2, 3)\), and suppose that \( c_k, c'_k \) are bounded and \( c_1 \pm ic_2 \neq 0 \). Let \( \lambda - \lambda_i = (2\kappa)^{-1/3}t^{-2/3}(\zeta + \zeta_0) \) with \( \kappa = c_1c'_1 + c_2c'_2 + c_3c'_3, \ |\zeta_0| \ll t^{-1/3} \). Then system (3.4) with \( \hat{\vartheta} \equiv 1 \) admits a matrix solution given by

\[
\Phi_i(\lambda) = T_i(I + O(t^{-\delta'})) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} W(\zeta), \quad T_i = \begin{pmatrix} 1 & -\frac{c_3}{c_1 + ic_2} \\ c_3 & 1 \end{pmatrix}
\]

as long as \( |\zeta| \ll t^{(2/3 - \delta')/3} \), that is, \( |\lambda - \lambda_i| \ll t^{-2/3 + (2/3 - \delta')/3} \), \( 0 < \delta' < 2/3 \) being arbitrary. Here \( \hat{\vartheta} = 2(2\kappa)^{-1/3}(t, \lambda)_{t, \lambda}^{1/3} \), and \( W(\zeta) \) is a matrix solution of system (3.14) admitting the canonical matrix solutions \( W_\nu(\zeta) (\nu = 0, \pm 1, \pm 2, \ldots) \) such that

\[
W_\nu(\zeta) = \zeta^{-(1/4)\sigma_3}(\sigma_3 + \sigma_1)(I + O(\zeta^{-3/2}))\exp((2/3)\zeta^{3/2}\sigma_3)
\]

as \( \zeta \to \infty \) through the sector \( \Sigma_\nu : |\arg \zeta - (2\nu - 1)\pi/3| < 2\pi/3 \), and that \( W_{\nu+1}(\zeta) = W_\nu(\zeta)G_\nu \) with

\[
G_1 = \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}, \quad G_{\nu+1} = \sigma_1 G_\nu \sigma_1.
\]

**Remark 3.6.** Putting \( \lambda - \lambda_i = (2\kappa)^{-1/3}\omega^{2j}t^{-2/3}(\zeta + \zeta_0), \ \omega = e^{2\pi i/3}, \ j \in \{0, \pm 1\} \), we have the expression of \( \Phi_i(\lambda) \) with \( \hat{\vartheta} = 2(2\kappa)^{-1/3}\omega^{2j}(t, \lambda)_{t, \lambda}^{1/3} \).

**Proof.** Since \( \mu^2 = b_1^2 + b_2^2 + b_3^2 \), we have \( c_1^2 + c_2^2 + c_3^2 = \mu(\lambda_i)^2 = 0 \). Write \( B(t, \lambda) = B_0(t) + B_1(t, \lambda) \) with

\[
B_0(t) = B(t, \lambda_i) = \begin{pmatrix} c_3 \\ c_1 - ic_2 \\ c_1 + ic_2 \end{pmatrix}, \quad B_1(t, \lambda) = \begin{pmatrix} \delta_3 \\ \delta_1 - i\delta_2 \\ \delta_1 + i\delta_2 \end{pmatrix},
\]

where \( \delta_k = b_k - c_k \) \((k = 1, 2, 3)\). Observing

\[
T_i^{-1}B_0(t)T_i = \begin{pmatrix} 0 & 2(c_1 - ic_2) \\ 0 & 0 \end{pmatrix},
\]

\[
T_i^{-1}B_1(t, \lambda)T_i = \begin{pmatrix} ic_3^{-1}(c_2\delta_1 - c_1\delta_2) \\ (c_1 - ic_2)^{-1}(c_1\delta_1 + c_2\delta_2 + c_3\delta_3) \\ -ic_3^{-1}(c_2\delta_1 - c_1\delta_2) \end{pmatrix}
\]

we have

\[
T_i^{-1}B(t, \lambda)T_i = \begin{pmatrix} 0 & 2(c_1 - ic_2) \\ 0 & 0 \end{pmatrix}
\]
\[
+ \left( i c_3^{-1}(c_1 c_2 - c_1 c_2' - c_2 c_3) \quad (c_1 + ic_2)^{-1}(c_1 c_1' + c_2 c_2' - c_3 c_3') \quad -ic_3^{-1}(c_1 c_2 - c_1 c_2') \right) \eta + O(\eta^2)
\]
with \( \eta = \lambda - \lambda_i \). By

\[
Y = T_t(I + Ln)Z, \quad L = \begin{pmatrix} q & 0 \\ p & -q \end{pmatrix}, \quad p = -\frac{i(c_1 c_2 - c_1 c_2')}{2c_3(c_1 - ic_2)}, \quad q = -\frac{c_1 c_1' + c_2 c_2' - c_3 c_3'}{4c_3}
\]

system (3.14) is changed into

\[
\frac{dZ}{d\eta} = iB_t(t, \eta)Z,
\]

\[
B_t(t, \eta) = \begin{pmatrix} 0 & 2(c_1 - ic_2) \\ \kappa(c_1 - ic_2)^{-1} \eta & 0 \end{pmatrix} + \begin{pmatrix} -q & 0 \\ -p & q \end{pmatrix} t^{-1} + O(\eta(t^{-1} + |\eta|)).
\]

Set \( \eta = \beta z \) and \( \hat{t} = 2(c_1 - ic_2)\beta t \) with \( \beta = (2\kappa)^{-1/3} t^{-2/3} \). Then

\[
\frac{dZ}{dz} = \left( \begin{pmatrix} -q\beta & \hat{t} \\ \hat{t}^{-1}z - p\beta & q\beta \end{pmatrix} + O(\hat{t}^{-3}|z|^2 + |\hat{t}^{-4}z|) \right) Z.
\]

The further change of variables

\[
Z = \begin{pmatrix} 1 & 0 \\ 0 & \hat{t}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ q\beta & 1 \end{pmatrix} V, \quad \zeta = z - \zeta_0, \quad \zeta_0 = p\beta \hat{t} - q^2 \beta^2 \ll t^{-1/3}
\]
yields

\[
(3.15) \quad \frac{dV}{d\zeta} = \left( \begin{pmatrix} 0 & 1 \\ \zeta & 0 \end{pmatrix} + \Delta(t, \zeta) \right) V, \quad \Delta(t, \zeta) \ll |t^{-2/3}\zeta^2| + |t^{-1}\zeta|.
\]

As a model equation of this let us consider system (3.14) possessing the matrix solutions

\[
W(\zeta) = \begin{pmatrix} Bi(\zeta) & Ai(\zeta) \\ Bi_\zeta(\zeta) & Ai_\zeta(\zeta) \end{pmatrix},
\]

and satisfying the properties in the proposition. Here

\[
Ai(\zeta) \sim \zeta^{-1/4} \exp\left(-\frac{2}{3}\zeta^{3/2}\right) \quad \text{as} \; \zeta \to \infty \text{ through } |\arg \zeta| < \pi,
\]

\[
Bi(\zeta) = \omega^{-1/4} Ai(\omega^{-1}\zeta) \sim \zeta^{-1/4} \exp\left(\frac{2}{3}\zeta^{3/2}\right) \quad \text{as} \; \zeta \to \infty \text{ through } |\arg \zeta - 2\pi/3| < \pi,
\]

(see [11], [10]). Let \( V(\zeta) = (I + X(\zeta))W(\zeta) \) reduces (3.15) to (3.14). Note that \( W(\zeta)W(\zeta)^{-1} = W_\nu(\zeta)W_\nu(\zeta)^{-1} \). Then, in \( \Sigma_\nu \) (\( \nu = 0, \pm 1 \)), \( X(\zeta) \) fulfills

\[
X(\zeta) = \int_{C_\nu(\zeta)} W_\nu(\zeta)W_\nu(\zeta)^{-1}\Delta(t, \xi)(I + X(\xi))W_\nu(\xi)W_\nu(\xi)^{-1}d\xi,
\]

where \( C_\nu(\zeta) \) is a set of contours ending at \( \zeta \) chosen suitably for each term according to the multiplier caused by \( W_\nu(\zeta)W_\nu(\zeta)^{-1} \) and its reciprocal. From this equation we derive

\[
\|X(\zeta)\| \ll |t^{-2/3}\zeta^3| + |t^{-1}\zeta^2| \ll t^{-\delta} \quad \text{as long as } |\zeta| \ll t^{2/9 - \delta/3} \text{ with } 0 < \delta < 2/3. \quad \square
\]
4. Monodromy matrices

We would like to find the monodromy matrices $M^0$, $M^1$ with respect to the matrix solution (3.6). Let $M^0_*$ and $M^1_*$ be the monodromy matrices in the case where (3.6) solves (3.4) with $\hat{u} \equiv 1$. Since the gauge transformation $Y = \hat{u}^{\sigma_s/2}Y_*\hat{u}^{-\sigma_s/2}$ reduces (3.4) to the system with $\hat{u} \equiv 1$, the monodromy matrices $M^0$, $M^1$ in the case of general $\hat{u}$ are given by

$$M^0 = \hat{u}^{\sigma_s/2}M^0_*\hat{u}^{-\sigma_s/2}, \quad M^1 = \hat{u}^{\sigma_s/2}M^1_*\hat{u}^{-\sigma_s/2}.$$

In this section we calculate $M^0_*$ and $M^1_*$ by using connection matrices and $S^*_1$, $S^*_2$ such that

$$S_1 = \hat{u}^{\sigma_s/2}S^*_1\hat{u}^{-\sigma_s/2}, \quad S_2 = \hat{u}^{\sigma_s/2}S^*_2\hat{u}^{-\sigma_s/2}.$$

4.1. Stokes graph. Recall that the characteristic root $\mu(\lambda)$ is considered on the Riemann surface $\mathbb{P}_+ \cup \mathbb{P}_-$ glued along the cuts $[\lambda_1, \lambda_0^0]$ and $[\lambda_0^0, \lambda_2]$. For $-\pi/2 < \phi < 0$, let $c_2^\infty$, $\hat{c}_1^\infty$, $c_0$ be Stokes curves on $\mathbb{P}_+^\infty$ connecting $\lambda_2$ to $i\infty$, $\lambda_1$ to $-i\infty$, $\lambda_2$ to $\lambda_1$, respectively, and, for $0 < \phi < \pi/2$, $c_1^\infty$, $\hat{c}_2^\infty$, $c_0$ connecting $\lambda_1$ to $i\infty$, $\lambda_2$ to $-i\infty$, $\lambda_2$ to $\lambda_1$, respectively. When $t \to \infty$, the turning points $\lambda_{1,2}^0$ tend to $\pm e^{i\phi}$. The limit Stokes graph on $\mathbb{P}_+$ is as in Figure 4.1 (a), (b) (cf. Proposition 8.17).

![Figure 4.1. Limit Stokes graph](image)

4.2. Connection matrix. For (3.4) with $\hat{u} \equiv 1$, the Stokes matrix $S^*_2$ is given by $Y_2 = YS^*_2$ (cf. (2.2)). Set $(i\infty, -i\infty) = (e^{\pi i/2}\infty, e^{3\pi i/2}\infty)$. Then $l_{2}^\infty = (-c_2^\infty) \cup \hat{c}_2^\infty$ is a path joining $e^{\pi i/2}\infty$ to $e^{3\pi i/2}\infty$ along this Stokes curves. Let $\Gamma^\infty_{2}$ be a connection matrix such that $Y_2\Gamma^\infty_{2} = Y$ given by the analytic continuation along $l_{2}^\infty$. Then the analytic continuation of $Y$ along the loop $l_0$ (Figure 2.1) is $Y(\Gamma^\infty_{2})^{-1}(S^*_2)^{-1} = YM^0_*$, which implies

$$\Gamma^\infty_{2}M^0_* = (S^*_2)^{-1}.$$  

Setting $(i\infty, -i\infty) = (e^{\pi i/2}\infty, e^{-\pi i/2}\infty)$, we similarly have

$$M^1_*\Gamma^\infty_{1} = (S^*_1)^{-1},$$
where $\Gamma_{1}^{\infty}$ is a connection matrix such that $Y \Gamma_{1}^{\infty} = Y_{1}$ along $l_{1}^{\infty} = (-c_{1}^{\infty}) \cup \hat{c}_{1}^{\infty}$ joining $e^{\pi i/2 \infty}$ to $e^{-\pi i/2 \infty}$. The connection matrices $\Gamma_{1}^{\infty}, \Gamma_{2}^{\infty}$ are constructed by combining matching procedures.

Let us explain the calculation of $\Gamma_{2}^{\infty}$ in the case $0 < \phi < \pi/2$. To discuss according to the justification scheme of Kitaev [26], suppose that $a_{\phi} = a_{\phi}(t)$ is given by (3.10) with $(y, y^{*}) = (y(t), y^{*}(t))$ not necessarily solving $(P_{V})$, and that

$$a_{\phi}(t) = A_{\phi} + t^{-1}B_{\phi}(t), \quad B_{\phi}(t) = O(1)$$

for $t \in S_{s}(\phi, t^{\prime}_{\infty}, \kappa_{0}, \delta_{1})$ with given $\kappa_{0}$, given small $\delta_{1}$ and sufficiently large $t^{\prime}_{\infty}$. Here $A_{\phi}$ is a unique solution of the Boutroux equation (2.3), and $S_{s}(\phi, t^{\prime}_{\infty}, \kappa_{0}, \delta_{1}) = \{ t ; \text{Re} t > t^{\prime}_{\infty}, |\text{Im} t| < \kappa_{0}, |y^{*}(t)| + |y(t)| - 1 - 1 < \delta_{1}^{-1} \}$.

Let the limit Stokes graph be as in Figure 4.1 (b). In what follows we use the following notation:

1. (1) for the WKB-solution of Proposition 3.3 write $\Lambda(\tau)$ in the component-wise form $\Lambda(\tau) = \Lambda_{3}(\tau) + \Lambda_{1}(\tau)$ with $\Lambda_{3}(\tau) \in \mathbb{C}\sigma_{3}, \Lambda_{1}(\tau) \in \mathbb{C}I$;

2. $c_{0} = (c_{1} - ic_{2})/c_{3}$ and $d_{0} = (d_{1} - id_{2})/d_{3}$, where $c_{k} = b_{k}(\lambda_{2}), d_{k} = b_{k}(\lambda_{1})$ for $k = 1, 2, 3$.

3. (3) for large $t, \lambda_{1}, \lambda_{2}, c_{1}^{\infty}, \hat{c}_{2}^{\infty}, c_{0}$ also denote turning points and curves approaching the limiting ones in Figure 4.1 (ii).

In Propositions 3.3 and 3.4 set $\delta = \delta^{\prime} = 2/9 - \varepsilon, 0 < \varepsilon < 2/9$ being arbitrary. Then both propositions apply to the annulus

$$A_{\varepsilon} : \quad |t|^{-2/3+(2/3)(2/9-\varepsilon)} \ll |\lambda - \lambda_{1}| \ll |t|^{-2/3+(2/3)(2/9+\varepsilon)/2} \quad (t = 1, 2).$$

In what follows we write

$$\delta = 2/9 - \varepsilon.$$

Consider the Stokes graph on the plane $\mathbb{C}\setminus [e^{i\delta}, +\infty]$ containing the sector $0 < \arg \lambda < 2\pi, |\lambda| > 2022$. The connection matrix along $c_{1}^{\infty} \cup c_{0} \cup \hat{c}_{2}^{\infty}$ consists of the following matrices $\Gamma_{a}, \ldots, \Gamma_{1}$.

(a) Let $\Psi_{\infty}(\lambda)$ be the WKB solution along $c_{1}^{\infty}$ with a base point $\tilde{\lambda}_{1} \in c_{1}^{\infty}, |\tilde{\lambda}_{1} - \lambda_{1}| \asymp t^{-1}$ near $\lambda_{1}$, and set $Y(\lambda) = \Psi_{\infty}(\lambda)\Gamma_{a}$. Then we have

$$\Gamma_{a} = \exp \left( - \int_{\lambda_{1}}^{\lambda} \Lambda(\tau) d\tau \right) T^{-1}(I + O(t^{\delta} + |\lambda|^{-1})) \exp \left( \frac{1}{4}(t\lambda - 2\theta_{\infty} \log \lambda)\sigma_{3} \right)$$

$$= C_{3}(\tilde{\lambda}_{1})c_{1}(\tilde{\lambda}_{1})(I + O(t^{\delta})) \exp \left( - \lim_{\lambda \to \infty} \int_{\lambda_{1}}^{\lambda} \Lambda_{3}(\tau) d\tau - \frac{1}{4}(t\lambda - 2\theta_{\infty} \log \lambda)\sigma_{3} \right)$$

with $C_{3}(\tilde{\lambda}_{1}) = \exp(\int_{\lambda_{1}}^{\tilde{\lambda}_{1}} \Lambda_{3}(\tau) d\tau), c_{1}(\tilde{\lambda}_{1}) = \exp(- \int_{\lambda_{1}}^{\tilde{\lambda}_{1}} \Lambda_{1}(\tau) d\tau)$.

(b) For $\Psi_{\infty}(\lambda)$ and $\Phi_{1}(\lambda)$ (cf. Proposition 3.4) in the annulus $A_{\varepsilon}$ set $\Psi_{\infty}(\lambda) = \Phi_{1}(\lambda)\Gamma_{a}$ along $c_{1}^{\infty}$. By Remark 3.6 we may suppose that the curve $(2\kappa)^{1/3}(\lambda - \tilde{\lambda}_{1}) = t^{-2/3}(\zeta + O(t^{-1/3}))$ with $\lambda \in c_{1}^{\infty}$ enters the sector $\Sigma_{1} : |\arg \zeta - \pi/3| < 2\pi/3$, and
that \( \Sigma_1 \) does not intersect the cut \([e^{i\phi}, \lambda_1]\). Write \( K^{-1} = 2(2\kappa)^{-1/3}(d_1 - id_2) \). Then, by Propositions \( \text{3.3} \) and \( \text{3.4} \)

\[
\Gamma_b = \Phi_1(\lambda)^{-1}\Psi_\infty(\lambda)
\]

\[
= W(\zeta)^{-1}\left(\begin{array}{cc} 1 & 0 \\ 0 & Kt^{-1/3} \end{array}\right)^{-1}(I + O(t^{-\delta}))(I + O(t^{-\delta}))^{-1}
\]

\[
\times \left(\begin{array}{cc} 1 & (b_3 - \mu)/(b_1 + ib_2) \\ (\mu - b_3)/(b_1 - ib_2) & 1 \end{array}\right)(I + O(t^{-\delta}))(I + O(t^{-\delta}))^{-1}
\]

\[
= W(\zeta)^{-1}\left(\begin{array}{cc} 1 & d_3/(d_1 + id_2) \\ \mu t^{1/3}/(2K(d_1 - id_2)) & 1 \end{array}\right)(I + O(t^{-\delta}))(I + O(t^{-\delta}))^{-1}
\]

for \( \lambda \in \mathcal{A}_c \cap \mathfrak{c}_\infty^\infty \), where \((\mu - b_3)/(b_1 - ib_2) = (\mu - d_3)/(d_1 + id_2) = 0(\eta), \eta = \lambda - \lambda_1 \).

Using

\[
\mu = (b_1^2 + b_2^2 + b_3^2)^{1/2} = (2(d_1d_1' + d_2d_2' + d_3d_3')(\eta + O(\eta^2)))^{1/2}
\]

\[
= (2\kappa)^{1/2}\eta^{1/2}(1 + O(\eta)) = (2\kappa)^{1/3}t^{-1/3}\zeta^{1/2}(1 + O(\eta))
\]

\[= 2K(d_1 - id_2)t^{-1/3}\zeta^{1/2}(1 + O(\eta)),
\]

we have

\[
\Gamma_b = \exp\left(\int_{\lambda_1}^{\lambda} \Lambda(\tau)d\tau - \frac{2}{3}\zeta^{3/2}\sigma_3\right)\zeta^{-1/4}(1 + O(t^{-\delta}))(1 + O(t^{-\delta}))^{-1}
\]

By Remark \( \text{3.5} \), \( \Lambda_3(\lambda) = [(2\kappa)^{1/2}\eta^{1/2}(1 + O(\eta)) + O(\eta^{-1/2})]\sigma_3 \) and \( \Lambda_I(\lambda) = [-\log \eta_\eta/4 + O(\eta^{-1/2})]I \) for \( \eta = \lambda - \lambda_1, \lambda \in \mathcal{A}_c \cap \mathfrak{c}_\infty^\infty \). Hence

\[
\Gamma_b = (I + O(t^{-\delta}))\exp\left(\int_{\lambda_1}^{\lambda} \Lambda_3(\tau)d\tau + O(\eta^{1/2})\right)(\zeta_1)^{-1/4}(1 + O(t^{-\delta}))(1 + O(t^{-\delta}))^{-1}
\]

with suitably chosen \( \zeta_1 \approx \eta_1 = \lambda_1 - \lambda_1 \). Since \( \eta^{1/2} \ll t^{-1/3+1/27+\varepsilon} \ll t^{-\delta} \),

\[
\Gamma_b = (\zeta_1)^{-1/4}(I + O(t^{-\delta}))\tilde{C}_3(\tilde{\lambda}_1)^{-1}(1 + O(t^{-\delta}))^{-1}
\]

(c) Let \( \Phi_{1-}(\lambda) \) be the solution by Proposition \( \text{3.4} \) near \( c_0 \) in an annulus and set \( \Phi_1(\lambda) = \Phi_{1-}(\lambda)G_c \). Then, by Proposition \( \text{3.3} \)

\[
G_c = \Phi_{1-}(\lambda)^{-1}\Phi_1(\lambda) = (\Phi_1(\lambda)G_1)^{-1}\Phi_1(\lambda) = G_1^{-1} = \left(\begin{array}{cc} 1 & i \\ 0 & 1 \end{array}\right).
\]

(d) Let \( \Psi_\infty(\lambda) \) be the WKB solution along \( c_0 \) with the base point \( \tilde{\lambda}_1 \in c_0 \) near \( \lambda_1 \), and set \( \Phi_{1-}(\lambda) = \Psi_\infty(\lambda)\Gamma_d \). By the same argument as in step (b), we have

\[
\Gamma_d = (\zeta_1)^{-1/4}(I + O(t^{-\delta}))\tilde{C}_3(\tilde{\lambda}_1)^{-1}(1 + O(t^{-\delta}))^{-1}
\]

with \( \zeta_1 \approx \tilde{\lambda}_1 - \lambda_1, \tilde{C}_3(\tilde{\lambda}_1) = \exp(\int_{\lambda_1}^{\tilde{\lambda}_1} \Lambda(\tau)d\tau) \).
(e) Let \( \Psi_{\infty 2}(\lambda) \) be the WKB solution along \( \mathbf{c}_0 \) with the base point \( \lambda_2 \in \mathbf{c}_0 \) near \( \lambda_2 \), and set \( \Psi_{\infty 1}(\lambda) = \Psi_{\infty 2}(\lambda)\Gamma_e \). Then

\[
\Gamma_e = (I + O(t^{-\delta})) \tilde{C}_3(\tilde{\lambda}_1' I) \tilde{C}_2(\tilde{\lambda}_2') \tilde{c}_I(\tilde{\lambda}_1', \lambda_2) \exp\left(-\int_{\lambda_2}^{\lambda_1} \Lambda_3(\tau) d\tau\right)
\]

with \( \tilde{c}_I(\tilde{\lambda}_1', \lambda_2') = \exp\left(\int_{\tilde{\lambda}_1'}^{\lambda_2'} \Lambda_1(\tau) d\tau\right) \), \( \tilde{C}_3(\lambda_2') = \exp\left(\int_{\lambda_2}^{\lambda_2'} \Lambda_3(\tau) d\tau\right) \).

(f) Let \( \Phi_2(\lambda) \) be the solution by Proposition 3.4 along \( \mathbf{c}_0 \) near \( \lambda_2 \), and set \( \Psi_{\infty 2}(\lambda) = \Phi_2(\lambda)\Gamma_i \). Then

\[
\Gamma_i = (\zeta_2')^{1/4} (I + O(t^{-\delta})) \tilde{C}_3(\lambda_2')^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -c_0^{-1} \end{pmatrix}
\]

with \( \zeta_2' \approx \lambda_2' - \lambda_2 \).

(g) Let \( \Phi_{2-}(\lambda) \) be the solution by Proposition 3.4 near \( \tilde{c}_2^\infty \) in the annulus \( \mathcal{A}_c \) around \( \lambda_2 \), and set \( \Phi_2(\lambda) = \Phi_{2-}(\lambda)G_g \). Then

\[
G_g = \Phi_{2-}(\lambda)^{-1} \Phi_2(\lambda) = (\Phi_2(\lambda)G_0^{-1})^{-1} \Phi_2(\lambda) = G_0 = \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix}
\]

(h) For the WKB-solution \( \hat{\Psi}_{\infty}(\lambda) \) along \( \tilde{c}_2^\infty \) set \( \hat{\Phi}_{2-}(\lambda) = \hat{\Psi}_{\infty}(\lambda)\Gamma_h \) for \( \lambda \in \mathcal{A}_c \cap \tilde{c}_2^\infty \). Then

\[
\Gamma_h = (\tilde{\zeta}_2')^{-1/4} (I + O(t^{-\delta})) \tilde{C}_3(\lambda_2') \begin{pmatrix} 1 & 0 \\ 0 & -c_0^{-1} \end{pmatrix}
\]

with \( \tilde{\zeta}_2' \approx \lambda_2' - \lambda_2, \tilde{C}_3(\lambda_2') = \exp\left(\int_{\lambda_2}^{\lambda_2'} \Lambda_3(\tau) d\tau\right) \).

(i) Set \( \hat{\Psi}_{\infty}(\lambda) = Y_2(t, \lambda)\Gamma_i \). Then

\[
\hat{\Gamma}_\infty = \tilde{C}_3(\lambda_2')^{-1} \tilde{c}_I(\lambda_2') (I + O(t^{-\delta}))
\]

\[
\times \exp\left(\lim_{\lambda \to \infty} \left(\int_{\lambda_2}^{\lambda} \Lambda_3(\tau) d\tau - \frac{1}{4} (t \lambda - 2 \theta_\infty \log \lambda) \sigma_3\right)\right)
\]

with \( \tilde{c}_I(\lambda_2') = \exp\left(\int_{\lambda_2}^{\lambda_2'} \Lambda_1(\tau) d\tau\right) \).

Then, collecting the matrices above and using a symmetric property about the scalar part, we have

\[
\Gamma_{\infty 2} = \Gamma_e \Gamma_i G_g \Gamma_\infty \Gamma_1 G_c \Gamma_a
\]

\[
= (I + O(t^{-\delta})) \exp(\tilde{J}_2 \sigma_3) \begin{pmatrix} 1 & 0 \\ 0 & -c_0^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -c_0 \end{pmatrix}
\]

\[
\times \exp(-J_0 \sigma_3) \begin{pmatrix} 1 & 0 \\ 0 & -d_0^{-1} \end{pmatrix} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -d_0 \end{pmatrix} \mathcal{D}_0 \exp(-J_1 \sigma_3)
\]

\[
= (I + O(t^{-\delta})) \begin{pmatrix} e^{J_2 - J_0 - J_1} & -id_0 e^{J_2 - J_0 + J_1} \\ ic_0 e^{J_2 - J_0 - J_1} & e^{J_1 - J_2} (e^{J_0 + c_0^{-1} d_0} e^{-J_0}) \end{pmatrix}
\]
Here

\[ J_j = \lim_{\lambda \to \infty} \left( \int_{\lambda_j}^{\lambda} \Lambda_3(\tau) d\tau - \frac{1}{4}(t\lambda - 2\theta_\infty \log \lambda)\sigma_3 \right) \quad (j = 1, 2), \]

\[ \hat{J}_2 = \lim_{\lambda \to \infty} \left( \int_{\lambda_2}^{\lambda} \Lambda_3(\tau) d\tau - \frac{1}{4}(t\lambda - 2\theta_\infty \log \lambda)\sigma_3 \right), \quad J_0 = J_2 - J_1 = \int_{\lambda_2}^{\lambda_1} \Lambda_3(\tau) d\tau. \]

The matrix \( \Gamma_{\infty 1} \) is calculated by using the same Stokes graph of Figure 4.1 on the plane \( \mathbb{C} \setminus [-\infty, -e^{i\phi}] \) containing the sector \(-\pi < \arg \lambda < \pi, \| \lambda \| > 2022\). Note that, in this case, the curve \( \hat{c}_2^\infty \) tends to \( e^{-\pi i/2\infty} \), and denote it by \( \hat{c}_2^\infty \). The calculation of \( \Gamma_{\infty 1} \) begins with setting \( Y_1(\lambda) = \hat{\Psi}_\infty^*(\lambda) \Gamma_a^* \), where \( \hat{\Psi}_\infty^*(\lambda) \) is the WKB solution along \( \hat{c}_2^\infty \) tending to \( e^{-\pi i/2\infty} \). Repeating step by step matchings along \( \hat{c}_2^\infty \cup c_0 \cup c_1^\infty \), we have

\[ \Gamma_{\infty 1} = (I + O(t^{-\delta})) \exp(J_1\sigma_3) \left( \begin{array}{cc} 1 & 0 \\ 0 & -d_0^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & -i \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -d_0 \end{array} \right) \]

\[ \times \exp(J_0\sigma_3) \left( \begin{array}{cc} 1 & 0 \\ 0 & -c_0^{-1} \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ i & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & -c_0 \end{array} \right) \exp(-\hat{J}_2\sigma_3) \]

\[ = (I + O(t^{-\delta})) \left( \begin{array}{ccc} e^{J_1^*-J_2}(e^{J_0} + c_0^{-1}d_0e^{-J_0}) & -id_0e^{J_0} & id_0e^{J_0} \\ i^{-1}e^{J_0} - J_0 - J_1 & e^{J_0} - J_0 - J_1 \end{array} \right) \]

with

\[ \hat{J}_2 = \lim_{\lambda \to \infty} \left( \int_{\lambda_2}^{\lambda} \Lambda_3(\tau) d\tau - \frac{1}{4}(t\lambda - 2\theta_\infty \log \lambda)\sigma_3 \right). \]

Thus we have \( \Gamma_{\infty 2} \) and \( \Gamma_{\infty 1} \) for \( 0 < \phi < \pi/2 \). In the case \(-\pi/2 < \phi < 0\), using the Stokes graph of Figure 4.1(a), we have similarly

\[ \Gamma_{\infty 2} = (I + O(t^{-\delta})) \left( \begin{array}{ccc} e^{J_1^*-J_2}(e^{J_0} + c_0^{-1}d_0e^{-J_0}) & id_0e^{J_0} & -id_0e^{J_0} \\ i^{-1}e^{J_0} - J_0 - J_1 & e^{J_0} - J_0 - J_1 \end{array} \right), \]

\[ \Gamma_{\infty 1} = (I + O(t^{-\delta})) \left( \begin{array}{ccc} e^{J_2-J_0-J_1} \quad id_0e^{J_0} + J_1 \\ -i^{-1}e^{J_0} - J_0 - J_1 \end{array} \right) \]

where

\[ \hat{J}_1 = \lim_{\lambda \to \infty} \left( \int_{\lambda_1}^{\lambda} \Lambda_3(\tau) d\tau - \frac{1}{4}(t\lambda - 2\theta_\infty \log \lambda)\sigma_3 \right) \]

\[ \hat{J}_1^* = \lim_{\lambda \to \infty} \left( \int_{\lambda_1}^{\lambda} \Lambda_3(\tau) d\tau - \frac{1}{4}(t\lambda - 2\theta_\infty \log \lambda)\sigma_3 \right) \]

with a curve \( \hat{c}_1^* \) tending to \( e^{3\pi i/2\infty} \).

Let \( M^0 = (m^0_j) \) and \( M^1 = (m^1_j) \), and suppose that \( 0 < \phi < \pi/2 \). The relations \( \Gamma_{\infty 2}^*M^0 = (S^2)^{-1}, \Gamma_{\infty 1}M^0 = (S^1)^{-1} \) yield

\[ e^{J_2-J_0-J_1}m_{11}^0 - id_0e^{J_2-J_0-J_1}m_{21}^0 = 1, \]

\[ ic_0^{-1}e^{J_2-J_0-J_1}m_{11}^0 + (e^{J_0} + c_0^{-1}d_0e^{-J_0})e^{-J_2+J_1}m_{21}^0 = 0, \]
\[ \begin{align*}
\lambda &= \lambda(z) = e^{i\phi}z. \\
\mu &= \mu(z) = \frac{1}{4} \sqrt{\frac{a_{\phi} - z^2}{1 - z^2}} + \frac{e^{-i\phi} \theta_{\infty} z}{2w} t^{-1} + \tilde{g}_2(t, z) t^{-2},
\end{align*} \]

and \( \tilde{g}_2(t, z) \ll 1 \) if \( |z^2 - 1|^{-1}, |z^2 - a_{\phi}|^{-1} \ll 1 \). Let \( z(F) = z(F_+) \cup z(F_-) \) be the image of \( F = F_+ \cup F_- \) under the map \( [5.1] \), and set \( (\Pi, \Pi_+, \Pi_-) = (z(F), z(F_+), z(F_-))|_\ast \), where the modification \( \ast \) denotes the replacement of the cuts \( [z_1, z_0] \mapsto [a_{\phi}^{1/2}, 1], [z_0, z_2] \mapsto [-1, -a_{\phi}^{1/2}] \). Then \( \Pi \) is the elliptic curve defined by \( w(z) \), which consists of two copies of \( z \)-plane glued along the cuts \( [-1, -a_{\phi}^{1/2}] \) and \( [a_{\phi}^{1/2}, 1] \). In \( [5.2] \) each square root is such that \( a_{\phi}^{-1/2} \sqrt{(a_{\phi} - z^2)/(1 - z^2)} \), \( a_{\phi}^{-1/2} w(z) \to \pm 1 \) as \( z \to 0 \) on \( \Pi_\pm \). The characteristic

\[ \begin{align*}
&\text{id}_0 e^{J_{\delta} - J_0 + J_1} m_{11}^1 + e^{J_{\delta} - J_0 - J_1} \hat{u}^{-1} m_{12}^1 = 0, \\
&(e^{J_0} + c_0^{-1} d_0 e^{-J_0}) e^{-J_0 + J_1} m_{11}^1 - i c_0^{-1} e^{-J_0 - J_1} \hat{u}^{-1} m_{12}^1 = 1,
\end{align*} \]

from which it follows that

\[ \begin{align*}
\hat{u} m_{21}^0 &= -i c_0^{-1} e^{-J_0 - J_1} \left( 1 + O(t^{-\delta}) \right), \\
m_{11}^0 &= e^{J_0 + J_1 - J_2} \left( 1 + c_0^{-1} d_0 e^{-2J_0} \right) \left( 1 + O(t^{-\delta}) \right), \\
\hat{u}^{-1} m_{12}^1 &= -i d_0 e^{-J_0 + J_1 + J_2} \left( 1 + O(t^{-\delta}) \right), \\
m_{11}^1 &= e^{-J_0 - J_1 + J_2} \left( 1 + O(t^{-\delta}) \right).
\end{align*} \]

Therefore entries of \( M^0 \) and \( M^1 \) satisfy

\[ \begin{align*}
m_{11}^1 m_{11}^1 &= -(1 + c_0 d_0^{-1} e^{2J_0}) \left( 1 + O(t^{-\delta}) \right), \\
m_{11}^0 m_{11}^0 &= e^{J_2 - J_1} \left( 1 + c_0^{-1} d_0 e^{-2J_0} \right) \left( 1 + O(t^{-\delta}) \right).
\end{align*} \]

In the case \( -\pi/2 < \phi < 0 \), we have the first equality above and \( m_{11}^0 = e^{J_2 - J_1} \left( 1 + O(t^{-\delta}) \right) \). Note that \( m_{11}^0 m_{11}^1 + m_{21}^0 m_{12}^1 = e^{-\pi i \theta_\infty} \) follows from \( M_1 M_0 = S_1^{-1} e^{-\pi i \theta_\infty} S_2^{-1} \). Then, for \( 0 < |\phi| < \pi/2 \),

\[ m_\phi = (1 + O(t^{-\delta}))[J_2 - J_1], \]

with \( m_\phi \) as in Theorem \([2.1]\).
root \( \mu = \mu(z) \) itself is an algebraic function on \( z(\mathbb{P}) \) such that \( a^{1/2}_\phi \mu(z) \to \pm 1/4 \) as \( z \to 0 \) on \( z(\mathbb{P}_\pm) \).

Let \( a \) and \( b \) be the cycles on \( \Pi \) descried as in Figure 5.1. We remark that \( a \) and \( b \) may also be regarded as those on \( z(\mathbb{P}) \), if \( t \) is sufficiently large and if the distance between \( a \cup b \) and \( \{ \pm 1 \} \cup \{ \pm a^{1/2}_\phi \} \) is \( \gg 1 \). We use the same symbols \( a \) and \( b \) as in Figure 2.2 provided that \( A^{1/2}_\phi = \lim_{t \to \infty} a^{1/2}_\phi(t) \), which will not cause confusions.

### 5.1. Expressions in terms of elliptic integrals

Let \( J_\mu^t \) and \( \tilde{J}_\mu^t \) be such that \( J_\mu^t \sigma_3 = J_\mu^t \sigma_3|_{\lambda_3(\tau) \to t\mu(\tau)\sigma_3} \) and \( \tilde{J}_\mu^t \sigma_3 = \tilde{J}_\mu^t \sigma_3|_{\lambda_3(\tau) \to t\mu(\tau)\sigma_3} \), respectively. By Proposition 5.1 and by use of \( \int_{\lambda_3} t(\mu(\tau) - \frac{1}{4}(t - 2t_\infty\tau^{-1}))d\tau \), we have

\[
J_1^t - J_2^t = \int_{\lambda_1}^{\lambda_2} t\mu(\tau)d\tau = e^{i\phi} t \int_{z_1}^{z_2} \mu(\lambda(z))dz
\]

\[
= -e^{i\phi} t \int_{-a^{1/2}_\phi}^{a^{1/2}_\phi} \mu(\lambda(z))dz + t (I_+ + I_-),
\]

where

\[
|I_\pm| \ll \left| \int_{z_1}^{z_2} \mu(\lambda(z))dz \right| \ll t^{-3/2}.
\]

Hence

\[
= -\frac{e^{i\phi}}{2} \int_a \mu(\lambda(z))dz + O(t^{-1/2})
\]

\[
= -\frac{e^{i\phi}}{8} \int_a \left( \sqrt{\frac{a_\phi - z^2}{1 - z^2}} + \frac{2e^{-i\phi}t\theta\tau^{-1}z}{w} \right)dz - \frac{e^{i\phi}}{2} t^{-1} \int_a \tilde{g}_2(t, z)dz + O(t^{-1/2})
\]

\[
= -\frac{e^{i\phi}}{8} \int_a \sqrt{\frac{a_\phi - z^2}{1 - z^2}} \ dz + O(t^{-1/2}).
\]

Furthermore we have

\[
J_2^t - \tilde{J}_2^t = e^{i\phi} t \int_b \mu(\lambda(z))dz
\]

\[
= \frac{e^{i\phi}}{4} \int_b \left( \sqrt{\frac{a_\phi - z^2}{1 - z^2}} + \frac{2e^{-i\phi}t\theta\tau^{-1}z}{w} \right)dz + e^{i\phi} t^{-1} \int_b \tilde{g}_2(t, z)dz
\]
\[ = \frac{e^{i\phi}}{4} t \int_b \sqrt{\frac{a_\phi - z^2}{1 - z^2}} \, dz - \frac{\theta_\infty \pi i}{2} + O(t^{-1}). \]

Observe that \( \sqrt{(a_\phi - z^2)/(1 - z^2)} = (1/w)(a_\phi - z^2) = -(w/z)' + a_\phi/w - z^{-2}a_\phi/w, \) we have

**Proposition 5.2.**

\[ J_1^\mu - J_2^\mu = -\frac{e^{i\phi}}{8} t \int_a \sqrt{\frac{a_\phi - z^2}{1 - z^2}} \, dz + O(t^{-1/2}) \]

\[ = -\frac{e^{i\phi}}{8} a_\phi t \int_a \left( \frac{1}{w} - \frac{1}{z^2 w} \right) \, dz + O(t^{-1/2}), \]

\[ J_2^\mu - \hat{J}_2^\mu = \frac{e^{i\phi}}{4} t \int_b \sqrt{\frac{a_\phi - z^2}{1 - z^2}} \, dz - \frac{\theta_\infty \pi i}{2} + O(t^{-1}) \]

\[ = \frac{e^{i\phi}}{4} a_\phi t \int_b \left( \frac{1}{w} - \frac{1}{z^2 w} \right) \, dz - \frac{\theta_\infty \pi i}{2} + O(t^{-1}). \]

To calculate \( J_{1,2}, \hat{J}_2 \) it is necessary to know integrals of

\[ \text{diag} T^{-1} T_{\lambda} |_{\sigma_3} = \frac{e^{-i\phi}}{4} \left( 1 - \frac{b_3}{\mu} \right) \frac{d}{dz} \log \frac{b_1 + ib_2}{b_1 - ib_2} = \frac{i e^{-i\phi}(b_1 b'_2 - b'_1 b_2)}{2\mu(\mu + b_3)}. \]

By (5.1), \( b_k \) are written in the form

\[ (z^2 - 1)b_3 = \frac{1}{4}(z^2 - 1 - 4\mathfrak{g}_0) + O(t^{-1}), \]

\[ y(z^2 - 1)(b_1 + ib_2) = \frac{1}{2}(y - 1)\mathfrak{g}_0(z + 1) - y\mathfrak{g}_0 + O(t^{-1}), \]

\[ (z^2 - 1)(b_1 - ib_2) = \frac{1}{2}(y - 1)\mathfrak{g}_0(z + 1) + \mathfrak{g}_0 + O(t^{-1}) \]

with \( \mathfrak{g}_0 = -(e^{-i\phi}y^* - y)(y - 1)^{-2}. \) Let \( z_\pm \) be such that

\[ b_1(z_+) + ib_2(z_+) = 0, \quad b_1(z_-) - ib_2(z_-) = 0. \]

It is easy to see that

\[ z_+ = \frac{y + 1}{y - 1} + O(t^{-1}), \quad z_- = -\frac{y + 1}{y - 1} + O(t^{-1}), \]

and that \( \mu(z_\pm)^2 = b_3(z_\pm)^2. \) Furthermore, by (5.3), \( \| \text{diag} T^{-1} T_{\lambda} \| \ll |z|^{-1/2} \) near \( z = \pm 1, \) \( \text{diag} T^{-1} T_{\lambda} \) has poles at \( z_\pm \in \Pi_+, \) and is holomorphic around \( z_\pm \in \Pi_+. \) From (3.9) combined with (5.2) it follows that

\[ (1 - z^2)\mu = \frac{1}{4} w(z) \left( 1 + \frac{2 e^{-i\phi}\theta_\infty t^{-1}}{a_\phi - z^2} + O(t^{-2}) \right). \]

Note that \( b_3(z_\pm) = e^{-i\phi}y^{-1}y^*/4 + O(t^{-1}) \) and \( \mu(z_\pm)^2 = e^{-2i\phi}y^{-2}(y^*)^2/16 + O(t^{-1}). \) When \( z_\pm \in \Pi_-, \)

\[ ((z_\pm)^2 - 1)b_3(z_\pm) = -((z_\pm)^2 - 1)\mu(z_\pm) = \frac{1}{4} w(z_\pm)(1 + O(t^{-1})). \]
The relations
\[
(z^2 - 1)b_3 \left( \frac{1}{z - z^+_1} - \frac{1}{z - z^-_1} \right) = \frac{1}{4}(z_+ - z_-) + \frac{((z_+)^2 - 1)b_3(z_+)}{z - z_+} - \frac{((z_-)^2 - 1)b_3(z_-)}{z - z_-}
\]
\begin{align*}
&= \frac{1}{4}(z_+ - z_- + \frac{w(z_+)}{z - z_+} - \frac{w(z_-)}{z - z_-}) + O(t^{-1})
\end{align*}
and
\[
\text{diag} T^{-1} \lambda|_{\sigma_3} = \frac{e^{-i\phi}}{4} \left( 1 - \frac{b_3}{\mu} \right) \left( \frac{1}{z - z_+} - \frac{1}{z - z_-} \right)
\]
yield
\[
\text{diag} T^{-1} \lambda|_{\sigma_3} = \frac{e^{-i\phi}}{4} \left( \frac{1}{z - z_+} - \frac{1}{z - z_-} \right) + \left( z_+ - z_- + \frac{w(z_+)}{z - z_+} - \frac{w(z_-)}{z - z_-} \right) \frac{1}{w(z)} + O(t^{-1}).
\]
Hence we have
\[
\left( \int_{e_1^\infty} - \int_{e_1^2} \right) \text{diag} T^{-1} \lambda|_{\sigma_3} d\tau = \int_{\lambda_1}^{\lambda_2} \text{diag} T^{-1} \lambda|_{\sigma_3} d\tau = e^{i\phi} \int_{\lambda_1}^{\lambda_2} \text{diag} T^{-1} \lambda|_{\sigma_3} dz
\]
\begin{align*}
&= \frac{1}{4} \log \frac{(z_1 - z_+)(z_1 - z_-)}{(z_2 - z_+)(z_2 - z_-)} \\
&\quad - \frac{1}{8} \int_a \left( \frac{z_+ - z_-}{w} + \frac{w(z_+)}{(z - z_+)} w - \frac{w(z_-)}{(z - z_-)} w \right) dz + \pi i r_a + O(t^{-1}),
\end{align*}
and
\[
\left( \int_{e_2^\infty} - \int_{e_2^\infty} \right) \text{diag} T^{-1} \lambda|_{\sigma_3} d\tau = e^{i\phi} \int_b \text{diag} T^{-1} \lambda|_{\sigma_3} dz
\]
\begin{align*}
&= \frac{1}{4} \int_b \left( \frac{z_+ - z_-}{w} + \frac{w(z_+)}{(z - z_+)} w - \frac{w(z_-)}{(z - z_-)} w \right) dz + 2\pi i r_b + O(t^{-1}),
\end{align*}
where \(\pi i r_a, 2\pi i r_b\) with \(r_a, r_b = 0, 1\) are the contributions from the poles \(z\) in deforming the contours. Since \((b_1(z) - ib_2(z))/(b_1(z) + ib_2(z)) = y(z - z_-)/(z - z_+),\)
\[
c_0^2 = \frac{c_1 - i c_2}{c_1 + i c_2} = -\frac{y(z_2 - z_-)}{z_2 - z_+}, \quad d_0^2 = -\frac{y(z_1 - z_-)}{z_1 - z_+}.
\]
These combined with Proposition 5.2 and (4.1) yield

**Proposition 5.3.** Set
\[
W_0(z) = \frac{e^{i\phi}}{4} a_\phi \left( \frac{1}{w} - \frac{1}{z^2 w} \right), \quad W_1(z) = \frac{1}{4} \left( \frac{z_+ - z_-}{w} + \frac{w(z_+)}{(z - z_+)} w - \frac{w(z_-)}{(z - z_-)} w \right).
\]

Then
\[
J_2 - \bar{J}_2 = \frac{e^{i\phi}}{4} t \int_b \sqrt{\frac{a_\phi - z^2}{1 - z^2}} d\tau - \int_b W_1(z) dz - \frac{\theta_{\infty} \pi i}{2} - 2\pi i r_b + O(t^{-1})
\]
\[
= \int_b \left( tW_0(z) - W_1(z) \right) dz - \frac{\theta_{\infty} \pi i}{2} - 2\pi i r_b + O(t^{-1}),
\]
\[
2(J_1 - J_2) + \log(c_0^{-1} d_0) = \frac{e^{i\phi}}{4} t \int_a \sqrt{\frac{a_\phi - z^2}{1 - z^2}} d\tau + \int_a W_1(z) dz + 2\pi i r_a + O(t^{-1/2})
\]
\[
= \int_a \left( tW_0(z) - W_1(z) \right) dz + 2\pi i r_a + O(t^{-1/2}).
\]
Corollary 5.4. We have
\[
\log m_{\psi} = \frac{e^{i\phi}}{4} t \int_{b} \sqrt{\frac{a_{\psi} - z^2}{1 - z^2}} dz - \int_{b} W_{1}(z) dz - \frac{\theta_{\infty} \pi i}{2} + O(t^{-\delta})
\]
\[
= \int_{b} (tW_{0}(z) - W_{1}(z)) dz - \frac{\theta_{\infty} \pi i}{2} + O(t^{-\delta}),
\]
\[
\log(m_{21}m_{12}^{0}) = -\frac{e^{i\phi}}{4} t \int_{a} \sqrt{\frac{a_{\psi} - z^2}{1 - z^2}} dz + \int_{a} W_{1}(z) dz - (\theta_{\infty} + 1) \pi i + O(t^{-\delta})
\]
\[
= - \int_{a} (tW_{0}(z) - W_{1}(z)) dz - (\theta_{\infty} + 1) \pi i + O(t^{-\delta}).
\]

5.2. Expressions in terms of the \(\vartheta\)-function. Under the supposition \(a_{\psi} \neq 0, 1\), write
\[
\tilde{\text{sn}}(u) = a_{\psi}^{1/2} \text{sn}(u; a_{\psi}^{1/2}).
\]
Then \(z = \tilde{\text{sn}}(u)\) satisfies \((dz/du)^2 = w(z)^2 = (1 - z^2)(a_{\psi} - z^2)\). Setting \(z = \tilde{\text{sn}}(u)\) we have, for a given \(z_0\) on the elliptic curve \(\Pi = \Pi_{+} \cup \Pi_{-}\),
\[
\frac{dz}{(z - z_0) w(z)} = \frac{du}{\tilde{\text{sn}}(u) - \tilde{\text{sn}}(u_0)}, \quad z_0 = \tilde{\text{sn}}(u_0).
\]
Let \(u_0^{\pm}\) be such that \(z_0^{(\pm)} = \tilde{\text{sn}}(u_0^{\pm})\), where \(z_0^{(+) = (z_0, w(z_0^{(+)})}, z_0^{(-)} = (z_0, -w(z_0^{(+)})\).
Since \(\tilde{\text{sn}}(u_0^{\pm}) = \pm w(z_0^{(+)})\),
\[
\frac{1}{\tilde{\text{sn}}(u) - z_0} = \frac{1}{w(z_0^{(+)})} (\zeta(u - u_0^{+}) - \zeta(u - u_0^{-})) - \frac{1}{w(z_0^{(+)})} \left( \frac{w'(z_0^{(+)})}{2} - \zeta(u_0^{+} - u_0^{-}) \right)
\]
\[
= \frac{1}{w(z_0^{(+)})} \frac{d}{du} \log \frac{\sigma(u - u_0^{+})}{\sigma(u - u_0^{-})} - \frac{1}{w(z_0^{(+)})} \left( \frac{w'(z_0^{(+)})}{2} - \frac{\sigma'(u_0^{+} - u_0^{-})}{\sigma(u_0^{+} - u_0^{-})} \right).
\]
This function may be written in terms of the \(\vartheta\)-function
\[
\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i z n^2 + 2 \pi i n z}, \quad \text{Im} \tau > 0
\]
coinciding with \(\vartheta_3\) of Jacobi and having the properties:
\[
\vartheta(z \pm 1, \tau) = \vartheta(z, \tau), \quad \vartheta(-z, \tau) = \vartheta(z, \tau), \quad \vartheta(z \pm \tau, \tau) = e^{-\pi i (\tau \pm 2z)} \vartheta(z, \tau).
\]
Let us write the fundamental periods of \(\Pi\) as
\[
\omega_a = \omega_a(t) = \int_{a} \frac{dz}{w}, \quad \omega_b = \omega_b(t) = \int_{b} \frac{dz}{w}.
\]
Then \(\tilde{\text{sn}}(u)\) with the modulus \(k = a_{\psi}^{1/2}\) has the periods \(\omega_a = 4K, \omega_b = 2iK'\), and
\[
\frac{d}{du} \log \frac{\sigma(u - u_0^{+})}{\sigma(u - u_0^{-})} = \frac{2\zeta(\omega_a/2)}{\omega_a} (u_0^{+} - u_0^{-}) du + d \log \frac{\vartheta(F(z_0^{(+)}, z + \nu, \tau)}{\vartheta(F(z_0^{(-)}, z + \nu, \tau)}
\]
\[
\frac{\sigma'}{\sigma} (u_0^{+} - u_0^{-}) = - \frac{2\zeta(\omega_a/2)}{\omega_a} (u_0^{+} - u_0^{-}) + \frac{i\pi}{\omega_a} + \frac{1}{\omega_a} \vartheta(F(z_0^{(-)}, z_0^{(+)}) + \nu, \tau)
\]
(cf. [14], [11]), where
\[
(5.7) \quad \tau = \frac{\omega_b}{\omega_a}, \quad \nu = \frac{1}{2} (1 + \tau), \quad F(z_*, z) = \frac{1}{\omega_a} \int_{z_*}^{z} \frac{dz}{w(z)} = \frac{1}{\omega_a} (u - u_*)
\]
with \( z = \text{sn}(u) \), \( z_* = \text{sn}(u_*) \). Thus we have

\[
\frac{dz}{(z - z_0)w(z)} = \frac{1}{w(z_0^+)} d\log \frac{\vartheta(F(z_0^+, z) + \nu, \tau)}{\vartheta(F(z_0^-, z) + \nu, \tau)}
- \frac{1}{w(z_0^+)} \left( \frac{w'(z_0^+)}{2} - \frac{1}{\omega_a} \left( i\pi + \frac{\vartheta'}{\vartheta}(F(z_0^-, z_0^+) + \nu, \tau) \right) \right) dz,
\]

which yields

(5.8)
\[
\int_a \frac{dz}{(z - z_0)w(z)} = -\frac{w'(z_0^+)}{2w(z_0^+)} \omega_a + \frac{1}{w(z_0^+)} \left( i\pi + \frac{\vartheta'}{\vartheta}(F(z_0^-, z_0^+) + \nu, \tau) \right),
\]

(5.9)
\[
\int_b \frac{dz}{(z - z_0)w(z)} = \frac{2\pi i}{w(z_0^+)} F(z_0^-, z_0^+) + \tau \int_a \frac{dz}{(z - z_0)w(z)}.
\]

The integrals
\[
\int_a \frac{dz}{z^2w(z)} = \frac{1}{2}(1 + a_{\phi}^{-1}) \omega_a - \frac{2}{a_{\phi} \omega_a} \left( \frac{\vartheta'}{\vartheta} - \left( \frac{\vartheta'}{\vartheta} \right)^2 \right),
\]
\[
\int_b \frac{dz}{z^2w(z)} = \frac{4\pi i}{a_{\phi} \omega_a} + \tau \int_a \frac{dz}{z^2w(z)}
\]
follow from \((\partial/\partial z_0)\) (5.8), \((\partial/\partial z_0)\) (5.9) with \( z_0 = 0 \). Furthermore, using the relation
\[
\left( \frac{z_0}{2} - \frac{w'(z_0)}{4} \right) \omega_a + \frac{1}{2} \frac{\vartheta'}{\vartheta}(F(z_0^-, z_0^+) + \nu, \tau) + \frac{\pi i}{2} = \frac{\vartheta'}{\vartheta} \left( \frac{1}{2} F(z_0^-, z_0^+) - \frac{1}{4}, \tau \right)
\]
derived by comparing the residues of poles \( z_0 = \pm 1, \pm a_{\phi}^{1/2} \) and \( \infty^+ (\in \Pi_\pm) \) on \( \Pi \) (cf. [23, p.513], [28, (3.5)])], we obtain from (5.8) that
\[
\int_a \frac{w(z_0)dz}{(z - z_0)w(z)} = -z_0 \omega_a + 2 \frac{\vartheta'}{\vartheta} \left( \frac{1}{2} F(z_0^-, z_0^+) - \frac{1}{4}, \tau \right).
\]

From these relations with \( z_0 = z_+, z_- \) (cf. [5.5]) it follows that

**Proposition 5.5.** For \( W_0(z) \) and \( W_1(z) \) in Proposition 5.3

\[
\int_a W_0(z)dz = \frac{e^{i\phi}}{8} \left( (a_{\phi} - 1) \omega_a + \frac{4}{\omega_a} \left( \frac{\vartheta''}{\vartheta} - \left( \frac{\vartheta'}{\vartheta} \right)^2 \right)(\tau/2, \tau) \right),
\]
\[
\int_b W_0(z)dz - \tau \int_a W_0(z)dz = -\frac{e^{i\phi} \pi i}{\omega_a},
\]
\[
\int_a W_1(z)dz = \frac{\vartheta'}{\vartheta} \left( \frac{1}{2} F(z_+^-, z_+^+) - \frac{1}{4}, \tau \right) + O(t^{-1}),
\]
\[
\int_b W_1(z)dz - \tau \int_a W_1(z)dz = \pi i F(z_+^-, z_+^+) + O(t^{-1}),
\]

where \( z_+ = (y + 1)/(y - 1) + O(t^{-1}) \), \( z_+^+ = (z_+, w(z_+^+)) \), \( z_- = (z_+, -w(z_+^+)) \).
5.3. Expression of $B_\phi(t)$. Recall that $a_\phi(t) = A_\phi + t^{-1}B_\phi(t)$, $B_\phi(t) = O(1)$ in the domain $S_*(\phi, t_\infty', \kappa_0, \delta_1)$, where $A_\phi$ is a unique solution of the Boutroux equation (2.3). The quantity $B_\phi(t)$ is written in terms of

$$
\Omega_{a,b} = \int_{a,b} \frac{dz}{w(A_\phi, z)}, \quad \mathcal{E}_{a,b} = \int_{a,b} \sqrt{\frac{A_\phi - z^2}{1 - z^2}} dz,
$$

where $w(A_\phi, z) = \sqrt{(A_\phi - z^2)(1 - z^2)}$, and $a, b$ are two cycles on $\Pi^*$. Observing that

$$
\sqrt{\frac{a_\phi - z^2}{1 - z^2}} - \sqrt{\frac{A_\phi - z^2}{1 - z^2}} = \frac{1}{\sqrt{1 - z^2}} \sqrt{(a_\phi - z^2 - A_\phi - z^2)} = \frac{t^{-1}B_\phi}{2w(A_\phi, z)} (1 + O(t^{-1}B_\phi)),
$$

and combining this with Corollary 5.4 and Proposition 5.5, we obtain

$$
\frac{e^{i\phi}}{4} \left( t\mathcal{E}_a + \frac{\Omega_a}{2} B_\phi(1 + O(t^{-1}B_\phi)) \right) = \int_a W_1(z) dz - \log(m_{21}^0 m_{12}^1) - \pi i(\theta_\infty + 1) + O(t^{-\delta})
$$

with

$$
\int_a W_1(z) dz = \frac{\gamma'}{\gamma} \left( \frac{1}{2} \hat{F}(z_+^-, z_+^+) - \frac{1}{4}, \tau \right) + O(t^{-1}).
$$

**Proposition 5.6.** In $S'((\phi, t_\infty', \kappa_0, \delta_1)$, $B_\phi(t)$ is bounded, and

$$
\frac{e^{i\phi}}{4} \left( t\mathcal{E}_b + \frac{\Omega_b}{2} B_\phi(1 + O(t^{-1}B_\phi)) \right) = \int_b W_1(z) dz + \frac{\theta_\infty}{2} \pi i + \log m_0 + O(t^{-\delta})
$$

with

$$
\int_b W_1(z) dz = \frac{\gamma'}{\gamma} \left( \frac{1}{2} \hat{F}(z_+^-, z_+^+) + \frac{\hat{\tau}}{4}, \hat{\tau} \right) + O(t^{-1}),
$$

in which $\hat{F}$ denotes $F$ corresponding to $\hat{\tau} = (-\omega_1)/\omega_b$. Since $\Re \int_{a,b} W_1(z) dz$ are bounded in $S_*(\phi, t_\infty', \kappa_0, \delta_1)$, the Boutroux equations (2.3) with $A_\phi$ are equivalent to the boundedness of $\Re (e^{i\phi} \Omega_{a,b} B_\phi)$, namely, the boundedness of $B_\phi(t)$.

**Proposition 5.7.** We have

$$
\int_{z_+^{(+)}} \frac{dz}{w(A_\phi, z)} = \int_{z_-^{(+)}} \frac{dz}{w(z)} + O(t^{-1}).
$$

To show this proposition, we note the following lemma, which is verified by combining

$$
3w(A_\phi, z) = (zw(A_\phi, z))' + (A_\phi + 1) \sqrt{\frac{A_\phi - z^2}{1 - z^2} - A_\phi (A_\phi - 1)} \frac{1}{w(A_\phi, z)}
$$

with

$$
J_a\Omega_b - J_b\Omega_a = \frac{4}{3}(1 + A_\phi)\pi i, \quad J_{a,b} = \int_{a,b} w(A_\phi, z) dz.
$$

The derivation of the last equality is similar to that of Legendre’s relation [11], [11].

**Lemma 5.8.** $\mathcal{E}_a\Omega_b - \mathcal{E}_b\Omega_a = 4\pi i$. 

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Proof. By the boundedness of $B_\phi(t)$, $\omega_{a,b} = \Omega_{a,b} + O(t^{-1})$ for $a, b \subset \Pi \cap \Pi^*$. From Proposition 5.5 and Corollary 5.4 it follows that

$$
\left( \int_b^a - t \int_a^b \right) (tW_0(z) - W_1(t))dz = -\frac{e^{i\phi} i}{\omega_a} t - \pi i F(z^+, z^-) + O(t^{-1})
$$

$$
= -\frac{e^{i\phi} i}{\Omega_a} t - 2\pi i \left( p_+(t) + \frac{\Omega_b}{\Omega_a} q_+(t) \right) + O(1) \ll 1,
$$

$p_+(t), q_+(t) \in \mathbb{Z}$. Write $e^{i\phi} t E_{a}/8 + \pi i q_+(t) = X_i, e^{i\phi} t E_{b}/8 - \pi i p_+(t) = Y_i$, where, by (2.3), $X, Y \in \mathbb{R}$. Then, by Lemma 5.8

$$
\frac{-e^{i\phi} i}{\Omega_a} t - \frac{e^{i\phi}}{4} \left( \frac{\Omega_b}{\Omega_a} E_a \right) t - 2 \left( \Omega_b \Omega_a \right) X - Y \right) i = -2 \left( \frac{\Omega_b}{\Omega_a} X - Y \right) i \ll 1
$$

with $\text{Im} \left( \frac{\Omega_b}{\Omega_a} \right) > 0$. This implies $|X|, |Y| \ll 1$, and hence

$$
(5.10) \quad \pi i q_+(t) = -e^{i\phi} E_a t/8 + O(1), \quad \pi i p_+(t) = e^{i\phi} E_b t/8 + O(1)
$$

as $t \to \infty$. We would like to evaluate

$$
\Upsilon = \left| \int_{z_+}^{z_-} \left( \frac{1}{w(z)} - \frac{1}{w(A_\phi, z)} \right) dz \right|
$$

in which the integrand is

$$
\frac{1}{w(z)} - \frac{1}{w(A_\phi, z)} = \frac{-B_\phi(t)t^{-1}}{2(A_\phi - z^2)w(A_\phi, z)} + O(t^{-2}).
$$

Observe that the contour $[z^+_-, z^+_+]$ on $\Pi \cap \Pi^*$ may be decomposed into $2p_+(t)a \cup 2q_+(t)b \cup a_+ \cup a_-$, where the length of $a_\pm$ is $\ll 1$. Using (5.10) and Lemma 5.8 we have

$$
\Upsilon \ll \int_{z_-}^{z_+} \frac{B_\phi(t)t^{-1}}{(A_\phi - z^2)w(A_\phi, z)} dz + O(t^{-1}) \ll |B_\phi t^{-1}| |p_+(t)a + q_+(t)b| + O(t^{-1})
$$

$$
\ll |E_{b} j_a - E_{a} j_b| + O(t^{-1}) = \left| \frac{\partial}{\partial A_\phi} (E_a \Omega_b - E_b \Omega_a) \right| + O(t^{-1}) \ll t^{-1}
$$

with

$$
j_{a,b} = \int_{a,b} \frac{dz}{(A_\phi - z^2)w(A_\phi, z)},
$$

which completes the proof of the proposition. \qed

6. PROOFS OF THEOREMS 2.1, 2.3 AND 2.2

Let $y(t)$ be a function satisfying (4.23), and let $(M^0, M^1) \in SL_2(\mathbb{C})^2$ be such that $m_{11}^0 m_{41}^1 m_{21}^0 m_{12}^1 \neq 0$. Suppose that $0 < |\phi| < \pi/2$. 

6.1. Proof of Theorem 2.1. Note that

$$F(z_+^+, z_-^+) = \frac{1}{\omega_\alpha} \int_{z_-^+}^{z_+^+} \frac{dz}{w(z)} = \frac{2}{\omega_\alpha} \int_{0^+}^{z_+^+} \frac{dz}{w(z)} - \frac{1}{2} + O(t^{-1})$$

on II. By Propositions 5.5, 5.7 and Corollary 5.4

$$\omega_\alpha \left( \int_{-\tau}^{0} (tW_0(z) - W_1(z))dz \right)$$

= \omega_\alpha \left( \log m_\phi + \log m_\phi + \pi i(\theta_\infty + 1) + \frac{\pi i\theta_\infty}{2} + O(t^{-\delta}) \right)

= \Omega_\alpha \left( \log m_\phi + \Omega_\beta (\log m_\phi + \pi i(\theta_\infty + 1)) + \frac{\pi i\theta_\infty}{2} + O(t^{-\delta}) \right)

= -e^{i\phi} t - \pi i\omega_\alpha F(z_+^+, z_-^+) + O(t^{-\delta})

= -e^{i\phi} t - 2\pi i \int_{0^+}^{z_+^+} \frac{dz}{w(z)} + \frac{\pi i}{2} \omega_\alpha + O(t^{-\delta})

= -e^{i\phi} t - 2\pi i \int_{0^+}^{z_+^+} \frac{dz}{w(A_\phi, z)} + \frac{\pi i}{2} \Omega_\alpha + O(t^{-\delta})

with z_+^+ = (y(t) + 1)/(y(t) - 1). From (5.2) with \omega_{\alpha, b} = \Omega_{\alpha, b} + O(t^{-1}), it follows that

$$\int_{0^+}^{z_+^+} \frac{dz}{w(A_\phi, z)} = -\frac{1}{2} (e^{i\phi} t - \tilde{x}_0) + O(t^{-\delta}),$$

where

$$\tilde{x}_0 = x_0 + \Omega_\alpha,$$

$$x_0 \equiv \frac{-1}{\pi i} \left( \Omega_\alpha \log(m_0 + m_1^2) + \Omega_\alpha \log m_\phi \right) - (\Omega_\alpha/2 + \Omega_\beta)(\theta_\infty + 1) \mod 2\Omega_\alpha Z + 2\Omega_\beta Z.$$

This gives

$$\frac{y(t) + 1}{y(t) - 1} = A_{\phi}^{1/2} \sin((e^{i\phi} t - x_0)/2 + O(t^{-\delta}); A_{\phi}^{1/2})$$

as \( t \to \infty \) through \( S_\epsilon(\phi, t'_\infty, \epsilon_0, \delta_1) \). Thus we obtain the desired asymptotic form.

Let \( W_1^*(z) \) be the result of replacement of \( w(z) \) with \( w(A_\phi, z) \) in \( W_1(z) \), that is,

$$W_1^*(z) = \frac{1}{4} \left( \frac{z_+ - z_-}{w(A_\phi, z)} + \frac{w(A_\phi, z_+)}{(z - z_+)w(A_\phi, z) - \frac{w(A_\phi, z_-)}{(z - z_-)w(A_\phi, z)} \right),$$

which differs from the early \( W_1(z) \) by \( O(t^{-1}) \) along \( a \), since \( B_\phi(t) \ll 1 \). Then, by Proposition 5.6

$$\int_a W_1(z) = \int_a W_1^*(z)dz + O(t^{-1}) = \frac{y'}{y} \left( \frac{1}{2} \frac{F^*(z_+^-, z_-^+)}{\Omega_\alpha} - \frac{1}{4} \frac{\Omega_\beta}{\Omega_\alpha} \right) + O(t^{-1})$$

with

$$F^*(z_+^-, z_-^+) = \frac{1}{\Omega_\alpha} \int_{z_-^+}^{z_+^+} \frac{dz}{w(A_\phi, z)} = \frac{2}{\Omega_\alpha} \int_{0^+}^{z_+^+} \frac{dz}{w(A_\phi, z)} - \frac{1}{2} + O(t^{-1}).$$

By the same argument as in the derivation of the formula in Proposition 5.6 we have
Corollary 6.1. In $S_\ast(\phi, t_\ast, \kappa_0, \delta_0)$,
\[
\frac{e^{i\phi}}{4} \left( t\mathcal{E}_a + \frac{\Omega_a}{2} B_\phi \right) = -\frac{\partial^\prime}{\partial t} \left( \frac{1}{2\Omega_a} (e^{i\phi} t - x_0), \frac{\Omega_b}{\Omega_a} \right) - \log(m_{21}^0 m_{12}^1) - \pi i(\theta_\infty + 1) + O(t^{-\delta}).
\]

Justification

The justification of (6.3) as a solution of (P\textsubscript{V}) is made along the argument in [28, pp. 105–106, pp. 120–121]. Let $\mathcal{M} = (m, m_{21}^0 m_{12}^1)$ be a point such that $m_{11}^0 m_{11}^1 m_{12}^0 m_{12}^1 \neq 0$ on the monodromy manifold for system (3.4). Relation (6.3) and Corollary 6.1 provide the leading terms $y_{\ast\ast} = y_{\ast\ast}(\mathcal{M}, t)$ and $(B_\phi)_{\ast\ast} = (B_\phi)_{\ast\ast}(\mathcal{M}, t)$ without $O(t^{-\delta})$. Viewing (4.3) we set $2y_{\ast\ast} = 2y_{\ast\ast}(\mathcal{M}, t) = \sqrt{\varphi(t, y_{\ast\ast}, (B_\phi)_{\ast\ast})}$ with
\[
\varphi(t, y, B) = e^{2\pi i} y(y + (1 - A_\phi)(y - 1)^2) + e^{i\phi} y(y - 1)(4(\theta_0 + \theta_1)(y + 1) - e^{i\phi}(y - 1)B) t^{-1} + (y - 1)^3 ((\theta_0 - \theta_1 + \theta_\infty)^2 y - (\theta_0 - \theta_1 - \theta_\infty)^2) t^{-2}.
\]

where the branch of the square root is chosen in such a way that the leading term of $r^\ast_{\ast\ast}$ is compatible with $(d/dt)y_{\ast\ast}$. Then $(y_{\ast\ast}, y_{\ast\ast}) = (y_{\ast\ast}(\mathcal{M}, t), y_{\ast\ast}(\mathcal{M}, t))$ satisfies (4.3) with $B_\phi(t) = (B_\phi)_{\ast\ast}(\mathcal{M}, t)$ in $\tilde{S}(\phi, t_\infty, \kappa_0, \delta_0)$, and the relation $\mathcal{M} = \mathcal{M}(t, y_{\ast\ast}, y_{\ast\ast})$ [28, (2.28)]

where $\mathcal{M}(t, y, y^\ast)$ is a collection of explicit functions $y(t), B_\phi(t)$ resulting from the WKB procedure. Here $\tilde{S}(\phi, t_\infty, \kappa_0, \delta_0) = \{t \mid \text{Re} t > t_\infty, |\text{Im} t| < \kappa_0\} \setminus \bigcup_{\sigma \in Z_{\pm 1}} \{t - e^{-i\phi} \sigma \in \delta_0\}$ with $Z_{\pm 1} = x_0 + \Omega_a(Z + \frac{i}{2}) + 2\Omega_b Z, Z_\infty = x_0 + \Omega_a Z + 2\Omega_b(Z + \frac{i}{2})$, $Z_0 = \{e^{i\phi} t \mid y_{\ast\ast}(t) = 0, \infty\}$. The monodromy data $\mathcal{M}_{\ast\ast}(t)$ for system (3.4) with $B(t, \lambda)$ containing $(y_{\ast\ast}, y_{\ast\ast})$ is given by $\mathcal{M}_{\ast\ast}(t) = \mathcal{M}(t, y_{\ast\ast}, y_{\ast\ast}) + O(t^{-\delta})$ [28, (2.27)] as a result of the repeated WKB procedure. Thus, for $|t| \geq T_0$, we have $\|\mathcal{M}_{\ast\ast}(t) - \mathcal{M}\| \leq C t^{-\delta}$ valid uniformly in a neighbourhood of $\mathcal{M}$, where $C$ and $T_0$ are independent of $\mathcal{M}$ [28, (2.26)].

Then the justification scheme [26] applies to our case (see also [12, Theorem 5.5]). This justification combined with the maximal modulus principle in each neighbourhood of $\sigma \in Z_0 \cup Z_{\pm 1}$ leads us Theorem 2.1 in $S(\rho, t_\infty, \kappa_0, \delta_0)$.

6.2. System equivalent to (P\textsubscript{V}). To prove Theorem 2.3 we need a system equivalent to (P\textsubscript{V}). Multiplying both sides of (P\textsubscript{V}) by $2(dy/dx)y^{-1}(y - 1)^{-2}$, we write (P\textsubscript{V}) in the form
\[
\frac{d}{dx} L = -2x^{-1} L - \frac{2x^{-1} y}{(y - 1)^2} + 2(1 - \theta_0 - \theta_1) x^{-2} y^{-1}.
\]

where
\[
L = L(x) := \frac{(y')^2}{y(y - 1)^2} - \frac{y}{(y - 1)^2} + 2(1 - \theta_0 - \theta_1) x^{-1} y^{-1} - \frac{x^{-2}}{4} \left((\theta_0 - \theta_1 + \theta_\infty)^2 y + (\theta_0 - \theta_1 - \theta_\infty)^2 \frac{1}{y}\right).
\]

In (6.5) set
\[
\psi = \frac{y + 1}{y - 1}, \quad x = e^{i\phi} t
\]
The quantity $a$ with $\psi$ to $\psi$ and (6.8) is also written in the form (6.9) from (6.6) and (6.8) it follows that another approach to the first equation admits the solution $a$ expressed by the Jacobi sn-function with $\Omega$ and (6.10) and (6.11), we have $B(6.12) \left( \theta = 1 \right)$ \( \psi \). Let us seek a function $b_0(x)$ for (6.3). Put $u = \left( x - x_0 \right)/2$. Then this becomes (6.12) \( (b_0) = 4(\psi_0)(A_0 - \psi_0) = 4(\psi_0)(A_0 + 4A_0(\sin^2 u - 1)). \)
Comparison of double poles of doubly periodic functions yields

\[(\psi_0)_u + A_\phi (\sn^2 u - 1) + \frac{2}{\Omega_a} \frac{d}{du} \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a}, \tau_0 \right) \right) \equiv c_0 \in \mathbb{C} \]

with \(\tau_0 = \Omega_b / \Omega_a\). Integrating this with (6.12) along \([0, u]\) and putting \(u = 2K = \Omega_a / 2\), we have

\[b_0(x) = b_0(x_0) - \frac{2\varepsilon_a}{\Omega_a} (x - x_0) - \frac{8}{\Omega_a} \vartheta' \left( \frac{1}{2\Omega_a} (x - x_0), \tau_0 \right),\]

since \(2c_0 = -2\varepsilon_a / \Omega_a\) follows from

\[A_\phi \int_0^K (\sn^2 u - 1) du = -\int_0^{A_\phi^{1/2}} \sqrt{A_\phi - z^2} dz = -\frac{\varepsilon_a}{4} (z = A_\phi^{1/2} \sn u).\]

This is consistent with Corollary 6.1 if

\[b_0(x_0) = \beta_0 - \frac{2\varepsilon_a}{\Omega_a} x_0 = -\frac{8}{\Omega_a} (\log(m_2^0 m_1^0) + \pi i (\theta_\infty + 1)) - \frac{2\varepsilon_a}{\Omega_a} x_0.\]

Therefore \(b_0(x)\) satisfies

\[b_0'(x) = 2(\psi_0(x)^2 - A_\phi) + 4\psi_0'(x)\]

and \(b_0(e^{i\phi} t) - e^{i\phi} B_\phi(t) \ll t^{-2/3+\varepsilon}\) in \(S(\phi, t_\infty, \kappa_0, \delta_0)\).

### 6.4. Proof of Theorem 2.3

In this section we suppose that

\[\Delta(x) = h(x)/2 \ll x^{-1} \text{ in } S(\phi, t_\infty, \kappa_0, \delta_0).\]

By Theorem 2.1 \((\psi(x), b(x))\) with

\[\psi(x) = A_\phi^{1/2} \sn((x - x_0)/2 + h(x)/2; A_\phi^{1/2})\]

associated with the monodromy data \(M^0, M^1\) as in Theorem 2.1 fulfills (6.10) and (6.11).

Under (6.15), \(\psi(x) - \psi_0(x) \ll h(x) \ll x^{-1}\). Note that

\[2\psi' = (1 + h')A_\phi^{1/2} \sn(u; A_\phi^{1/2}) \big|_{u = (x - x_0)/2 + h/2} = (1 + h')\sqrt{1 - \psi^2} (A_\phi - \psi^2).\]

Since \(b(x) \ll 1\), from (6.11) it follows that

\[h' = -F_1(\psi, b)x^{-1} + \left( F_2(\psi) - \frac{1}{2} F_1(\psi, b)^2 \right) x^{-2} + O(x^{-3})\]

in \(\tilde{S}(\phi, t_\infty, \kappa_0, \delta_0)\) with \(F_1(\psi, b), F_2(\psi)\) as in Theorem 2.3. Using

\[\psi = \psi_0 + \psi'_0 h + O(h^2), \quad \psi'_0 = \sqrt{P(\psi_0)} = \sqrt{(1 - \psi_0^2)(A_\phi - \psi_0^2)},\]

we immediately obtain

\[h' = -F_1(\psi_0, b)x^{-1} + \left( F_2(\psi_0) - \frac{1}{2} F_1(\psi_0, b)^2 \right) x^{-2} - (F_1)_\psi(\psi_0, b)\psi'_0 h x^{-1} + O(x^{-3}) = -F_1(\psi_0, b)x^{-1} + O(x^{-2})\]

in \(\tilde{S}(\phi, t_\infty, \kappa_0, \delta_0)\).

### Proposition 6.2

**Under supposition** (6.15), \(b(x) - b_0(x) \ll x^{-1}\) in \(\tilde{S}(\phi, t_\infty, \kappa_0, \delta_0)\).
Proof. From (6.11) it follows that, for \( x < x_0^{(\nu)} \in S(\phi, t_\infty, \kappa_0, \delta_0) \cap (x_0 + 2\Omega_aZ + 2\Omega_bZ), \)

\[
\begin{align*}
b(x) - b(x_0^{(\nu)}) &= \int_{x_0^{(\nu)}}^{x} (4\psi' - 2(A_\phi - \psi^2))d\xi + \int_{x_0^{(\nu)}}^{x} (4(\theta_0 + \theta_1)\psi - b)\xi^{-1}d\xi \\
&= 4\psi_0(x) + O(h(x_0^{(\nu)})) - 2\int_{x_0^{(\nu)}}^{x} (A_\phi - \psi_0^2 - (\psi_0')^2)h)d\xi \\
&\quad + \int_{x_0^{(\nu)}}^{x} (4(\theta_0 + \theta_1)\psi - b)\xi^{-1}d\xi + O(x^{-1}) \\
&= \int_{x_0^{(\nu)}}^{x} (4\psi' - 2(A_\phi - \psi_0^2))d\xi - 2\int_{x_0^{(\nu)}}^{x} \psi_0^2h'd\xi \\
&\quad + \int_{x_0^{(\nu)}}^{x} 2(A_\phi - \psi_0^2)F_1(\psi, b_0)\xi^{-1}d\xi + O(x^{-1}),
\end{align*}
\]

since \( \psi_0(x_0^{(\nu)}) = 0. \) By (6.12)

\[
b_0(x) = b_0(x_0^{(\nu)}) + \int_{x_0^{(\nu)}}^{x} (4\psi' - 2(A_\phi - \psi_0^2))d\xi.
\]

Then, by (6.16), in \( \tilde{S}(\phi, t_\infty, \kappa_0, \delta_0), \)

\[
b(x) - b_0(x) - (b(x_0^{(\nu)}) - b_0(x_0^{(\nu)})) = -2\int_{x_0^{(\nu)}}^{x} \psi_0^2h'd\xi - \int_{x_0^{(\nu)}}^{x} 2(A_\phi - \psi_0^2)h'd\xi + O(x^{-1})
\]

\[
= -2A_\phi h(x) + O(x^{-1}).
\]

Passing to the limit \( x_0^{(\nu)} \to \infty \) through \( \tilde{S}(\phi, t_\infty, \kappa_0, \delta_0), \) we have \( b(x) - b_0(x) \ll x^{-1}, \)
which completes the proof. \( \square \)

By Proposition 6.2, (6.16) is newly rewritten in the form

\[
(6.17) \quad h' = -\left(F_1(\psi, b)x^{-1} + \left(F_2(\psi) - \frac{1}{2}F_1(\psi, b_0)^2\right)x^{-2}
\right.
\]

\[-(F_1)\psi(\psi, b_0)\psi'hx^{-1} + O(x^{-3})
\]

\[-F_1(\psi, b_0)x^{-1} + O(x^{-2})
\]

in \( \tilde{S}(\phi, t_\infty, \kappa_0, \delta_0). \) From (6.17), for \( x, x_n \in \tilde{S}(\phi, t_\infty, \kappa_0, \delta_0) \cap \{|x| > |x_0|\} \) such the \( |x| < |x_n|, \) we derive

\[
h(x) = h(x_n) - \int_{x_n}^{x} F_1(\psi, b_0)\frac{d\xi}{\xi} + \int_{x_n}^{x} \left(F_2(\psi) - \frac{1}{2}F_1(\psi, b_0)^2\right)\frac{d\xi}{\xi^2} - I_0 + O(x^{-2}),
\]

in which, by (6.17) and integration by parts,

\[
I_0 = \int_{x_n}^{x} (F_1(\psi, b_0)\psi'h)\frac{d\xi}{\xi} = \int_{x_n}^{x} (F_1(\psi, 0))\xi - \frac{1}{2} \left(\frac{1}{A_\phi - \psi_0^2}\right) b_0 h\frac{d\xi}{\xi}
\]

\[
= \int_{x_n}^{x} \left(F_1(\psi, 0)F_1(\psi, b_0) - \frac{b_0F_1(\psi, b_0) - b_0b_0}{2(A_\phi - \psi_0^2)}\right)\frac{d\xi}{\xi^2} + O(x^{-2})
\]

\[
= \int_{x_n}^{x} F_1(\psi, b_0)^2\frac{d\xi}{\xi^2} + \frac{1}{2} \int_{x_n}^{x} \frac{b_0h}{A_\phi - \psi_0^2} \frac{d\xi}{\xi} + O(x^{-2}).
\]
This implies
\begin{equation}
(6.18) \quad h(x) = -\int_{x}^{\infty} F_{1}(\psi_{0}, b_{0})\frac{d\xi}{\xi} + \int_{x}^{\infty} \left( F_{2}(\psi_{0}) - \frac{3}{2} F_{1}(\psi_{0}, b_{0})^{2} \right)\frac{d\xi}{\xi^2}
\end{equation}
\begin{equation*}
- \frac{1}{2} \int_{x}^{\infty} \frac{b_{0} h}{A_{\phi} - \psi_{0}^{2}}\frac{d\xi}{\xi} + O(x^{-2})
\end{equation*}
in \( \hat{S}(\phi, t_{\infty}, \kappa_{0}, \delta_{0}) \). Here \( \int_{x}^{\infty} F_{1}(\psi_{0}, b_{0})\xi^{-1}d\xi = \int_{x}^{\infty} F_{1}(\psi_{0}, b_{0})\xi^{-1}d\xi + O(x^{-1}) \ll x^{-1} \) (cf. Section 6.5). By (6.11) and (6.14), the quantity \( \chi(x) = b(x) - b_{0}(x) \) is given by
\begin{equation*}
\chi'(x) = 2(\psi^{2} - \psi_{0}^{2}) + 4(\psi' - \psi_{0}') + (4(\theta_{0} + \theta_{1})\psi - b_{0} - \chi)x^{-1}
\end{equation*}
\begin{equation*}
= 2(2\psi_{0}\psi'_{0}h + ((\psi_{0}')^{2} + \psi_{0}\psi''_{0})h^{2}) + 4(\psi - \psi_{0}')
\end{equation*}
\begin{equation*}
+ (4(\theta_{0} + \theta_{1})(\psi_{0} + \psi'_{0}h) - b_{0} - \chi)x^{-1} + O(x^{-3}),
\end{equation*}
and hence
\begin{equation*}
\chi(x) - \chi(x_{n}) = 4(\psi'_{0}h - \psi_{0}'(x_{n})h(x_{n})) + 2 \int_{x_{n}}^{x} (2\psi_{0}\psi'_{0}h + ((\psi_{0}')^{2} + \psi_{0}\psi''_{0})h^{2})d\xi
\end{equation*}
\begin{equation*}
+ 4(\theta_{0} + \theta_{1}) \int_{x_{n}}^{x} (\psi_{0} + \psi'_{0}h)\frac{d\xi}{\xi} - \int_{x_{n}}^{x} (b_{0} + \chi)\frac{d\xi}{\xi} + O(x^{-2})
\end{equation*}
with \( \chi(x_{n}), h(x_{n}) \ll x_{n}^{-1} \). Here, by (6.17) and integration by parts
\begin{equation*}
2 \int_{x_{n}}^{x} ((\psi_{0}')^{2} + \psi_{0}\psi''_{0})h^{2}d\xi = \int_{x_{n}}^{x} (\psi_{0}')^{2}h^{2}d\xi = -2 \int_{x_{n}}^{x} (\psi_{0}')h'hd\xi + O(x^{-2})
\end{equation*}
\begin{equation*}
= 2 \int_{x_{n}}^{x} (\psi_{0}')^{2}F_{1}(\psi_{0}, b_{0})h\frac{d\xi}{\xi} + O(x^{-2})
\end{equation*}
\begin{equation*}
= 2 \int_{x_{n}}^{x} \psi_{0}^{2}(F_{1}(\psi_{0}, b_{0})^{2} - F_{1}(\psi_{0}, b_{0})\xi_{0})\frac{d\xi}{\xi^{2}} + O(x^{-2}),
\end{equation*}
\begin{equation*}
\int_{x_{n}}^{x} \psi_{0}'h\frac{d\xi}{\xi} = \int_{x_{n}}^{x} \psi_{0}F_{1}(\psi_{0}, b_{0})\frac{d\xi}{\xi^{2}} + O(x^{-2}),
\end{equation*}
\begin{equation*}
2 \int_{x_{n}}^{x} \psi_{0}\psi'_{0}hd\xi = \psi_{0}^{2}h + \int_{x_{n}}^{x} \psi_{0}^{2}F_{1}(\psi_{0}, b_{0}) - \frac{\chi}{2(A_{\phi} - \psi_{0}^{2})} + (F_{1})(\psi_{0}, b_{0})\psi'_{0}h\frac{d\xi}{\xi}
\end{equation*}
\begin{equation*}
- \int_{x_{n}}^{x} \psi_{0}^{2}(F_{2}(\psi_{0}) - \frac{1}{2} F_{1}(\psi_{0}, b_{0})^{2})\frac{d\xi}{\xi^{2}} + O(|x^{-2}| + |x_{n}^{-1}|).
\end{equation*}
Insertion of these yields
\begin{equation*}
\chi(x) = (4\psi'_{0} + 2\psi_{0}^{2})h + 2 \int_{x_{n}}^{x} \psi_{0}^{2}((F_{1})(\psi_{0}, b_{0})\psi'_{0} - F_{1}(\psi_{0}, b_{0})\xi_{0})\frac{d\xi}{\xi}
\end{equation*}
\begin{equation*}
+ I_{1} + I_{2} + O(|x^{-2}| + |x_{n}^{-1}|),
\end{equation*}
\begin{equation*}
I_{1} = \int_{x_{n}}^{x} \psi_{0}^{2}(3F_{1}(\psi_{0}, b_{0})^{2} - 2F_{2}(\psi_{0}))\frac{d\xi}{\xi^{2}}
\end{equation*}
\begin{equation*}
+ 4(\theta_{0} + \theta_{1}) \int_{x_{n}}^{x} \psi_{0}F_{1}(\psi_{0}, b_{0})\frac{d\xi}{\xi^{2}} - \int_{x_{n}}^{x} \frac{A_{\phi} \chi}{A_{\phi} - \psi_{0}^{2}}\frac{d\xi}{\xi},
\end{equation*}
\begin{equation*}
I_{2} = \int_{x_{n}}^{x} (2\psi_{0}^{2}F_{1}(\psi_{0}, b_{0}) + 4(\theta_{0} + \theta_{1})\psi_{0} - b_{0})\frac{d\xi}{\xi}.
\end{equation*}
Since \(4(\theta_0 + \theta_1)\psi_0 - b_0 = 2(A_\phi - \psi_0^2)F_1(\psi_0, b_0) = 2(A_\phi - \psi_0^2)F_1(\psi_0, b) + \chi\), by (6.17)

\[
I_2 = 2A_\phi \int_{x_n}^x F_1(\psi_0, b_0) d\xi
\]

\[
= -2A_\phi h + 2A_\phi \int_{x_n}^x \left( \frac{\chi}{2(A_\phi - \psi_0^2)} - (F_1(\psi_0, b_0)\psi_0' h) \right) d\xi
\]

\[
+ 2A_\phi \int_{x_n}^x \left(F_2(\psi_0) - \frac{1}{2}F_1(\psi_0, b_0)^2\right) \frac{d\xi}{\xi^2} + O(|x^{-2}| + |x_n^{-1}|),
\]

passing to the limit \(x_n \to \infty\), we have

\[
\chi(x) = b_0'(x) h + 2 \int_{-\infty}^x \left((\psi_0^2 - A_\phi)(F_1(\psi_0, b_0)\psi_0' - \psi_0^2 F_1(\psi_0, b_0)) h \frac{d\xi}{\xi}
\]

\[
+ \int_{-\infty}^x \left(2\psi_0^2 F_1(\psi_0, b_0)^2 + (\psi_0^2 - A_\phi)(F_1(\psi_0, b_0)^2 - 2F_2(\psi_0)) \right) \frac{d\xi}{\xi^2}
\]

\[
+ 4(\theta_0 + \theta_1) \int_{-\infty}^x \psi_0 F_1(\psi_0, b_0) \frac{d\xi}{\xi^2} + O(x^{-2}).
\]

Furthermore, observing that

\[
\int_{-\infty}^x (\psi_0^2 - A_\phi)(F_1(\psi_0, b_0)\psi_0' h \frac{d\xi}{\xi} = \int_{-\infty}^x \left(\psi_0^2 - A_\phi\right) (F_1(\psi_0, b_0)) h \frac{d\xi}{\xi}
\]

\[
= \int_{-\infty}^x (\psi_0^2 - A_\phi) F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} - \int_{-\infty}^x \left(\psi_0^2 F_1(\psi_0, b_0) + b_0' h \right) \frac{d\xi}{\xi} + O(x^{-2})
\]

with

\[
- \int_{-\infty}^x \psi_0^2 F_1(\psi_0, b_0) h \frac{d\xi}{\xi} = \int_{-\infty}^x \psi_0^2 F_1(\psi_0, b_0) h \frac{d\xi}{\xi} - \int_{-\infty}^x \psi_0 F_1(\psi_0, b_0)^2 \frac{d\xi}{\xi^2} + O(x^{-2}),
\]

we arrive at

\[
\chi(x) = b_0'(x) h + \int_{-\infty}^x (\psi_0^2 - A_\phi)(F_1(\psi_0, b_0)^2 - 2F_2(\psi_0)) \frac{d\xi}{\xi^2} + O(x^{-2}).
\]

From this the representation of \(\chi_0(x) = \chi(x) - b_0'(x) h\) immediately follows, and insertion of \(\chi_0(x)\) into (6.18) leads us the conclusion of Theorem 2.3.

6.5. Proof of Corollary 2.4 In examining \(h(x)\) and \(\chi_0(x)\) the following primitive functions are useful.

**Lemma 6.3.** Let \(\nu_0 = (1 + \tau_0)/2\) with \(\tau_0 = \Omega_b/\Omega_a\). Then, for \(\text{sn} u = \text{sn}(u; A_{\phi}^{1/2})\),

\[
u_0 = \int_0^u \frac{du}{1 - \text{sn}^2 u} = \frac{1}{(A_{\phi} - 1)\Omega_a}
\]

\[
\times \left( E_a u + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} - \frac{1}{4} + \nu_0, \tau_0 \right) + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} + \frac{1}{4} + \nu_0, \tau_0 \right) + 2\pi i \right),
\]

\[
v_0 = \int_0^u \frac{\text{sn} u du}{1 - \text{sn}^2 u} = \frac{1}{(A_{\phi} - 1)\Omega_a} \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} - \frac{1}{4} + \nu_0, \tau_0 \right) - \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} + \frac{1}{4} + \nu_0, \tau_0 \right) \right)
\]

\[
- \frac{\vartheta'}{\vartheta} \left( -\frac{1}{4} + \nu_0, \tau_0 \right) + \frac{\vartheta'}{\vartheta} \left( \frac{1}{4} + \nu_0, \tau_0 \right) \right),
\]
Setting or a sum of $\beta$ with

$$
\int_0^u \frac{du}{1 - A_\phi \text{sn}^2 u} = \frac{1}{(1 - A_\phi)\Omega_a} \\
\times \left( (\mathcal{E}_a + (1 - A_\phi)\Omega_a)u + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} - \frac{1}{4}, \tau_0 \right) \right),
$$

$$
v_1 = \int_0^u \frac{\text{sn} u du}{1 - A_\phi \text{sn}^2 u} = \frac{1}{A_\phi^{1/2}(1 - A_\phi)\Omega_a} \\
\times \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} + \frac{1}{4}, \tau_0 \right) \right) - \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} - \frac{1}{4}, \tau_0 \right) - \frac{\vartheta'}{\vartheta} \left( \frac{1}{4}, \tau_0 \right) + \frac{\vartheta'}{\vartheta} \left( -\frac{1}{4}, \tau_0 \right),
$$

$$
v_2 = \int_0^u \frac{\text{sn} u du}{(1 - \text{sn}^2 u)^2} = \frac{1}{(A_\phi - 1)^2\Omega_a} \\
\times \left( \frac{2}{3}(2A_\phi - 1) \left( \mathcal{E}_a u + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} - \frac{1}{4} + \nu_0, \tau_0 \right) \right) \right) \\
- \frac{A_\phi}{3} (A_\phi - 1)\Omega_a u - \frac{1}{6} \left( \frac{d}{du} \right) \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} - \frac{1}{4} + \nu_0, \tau_0 \right) \right) + \frac{\vartheta'}{\vartheta} \left( \frac{1}{4} + \nu_0, \tau_0 \right),
$$

$$
u_2 = \int_0^u \frac{\text{sn} u du}{(1 - \text{sn}^2 u)^2} = -\frac{1}{6(\Omega_a - 1)^2\Omega_a} \left( \left( \frac{d}{du} \right)^2 + 1 - 5A_\phi \right) \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} - \frac{1}{4} + \nu_0, \tau_0 \right) \right) \\
- \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} + \frac{1}{4}, \nu_0, \tau_0 \right) - \frac{\vartheta'}{\vartheta} \left( \frac{1}{4} + \nu_0, \tau_0 \right) + \frac{\vartheta'}{\vartheta} \left( \frac{1}{4} + \nu_0, \tau_0 \right).
$$

**Proof.** Using $(\text{sn}^2 u - 1)^{-1} = -(\text{cn}^2 u)^{-1}$, $\text{cn}^2 (u + 2K \pm K^\prime) = (1 - k^2)\text{sn}^2 u (1 + O(u^2))$, we have

$$(k^2 - 1)(\text{sn}^2 u - 1)^{-1} - k^2 \text{sn}^2 (u - K + iK^\prime) = -k^2.$$

Setting

$$k^2(\text{sn}^2 (u - K + iK^\prime) - 1) + \frac{1}{\Omega_a} \frac{d}{du} \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} - \frac{1}{4} + \nu_0, \tau_0 \right) \right) + \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} + \frac{1}{4} + \nu_0, \tau_0 \right) \equiv c_0,$$

and integrating on $[-iK^\prime, -iK^\prime + K]$, we have $c_0 = -\mathcal{E}_a/\Omega_a$, which implies the equality $u_0$. By using $\pm \text{sn} (u + 2K \pm K) = 1 + (1/2)(k^2 - 1)u^2 + \cdots$ set

$$
\frac{\text{sn} u}{\text{sn}^2 u - 1} + \frac{1}{(k^2 - 1)\Omega_a} \frac{d}{du} \left( \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} - \frac{1}{4} + \nu_0, \tau_0 \right) \right) - \frac{\vartheta'}{\vartheta} \left( \frac{u}{\Omega_a} + \frac{1}{4} + \nu_0, \tau_0 \right) \equiv c_0.
$$

Integrating on $[0, 4K]$ to see $c_0 = 0$, we obtain $v_1$. By using $\text{dn}^{-2} u$ for $u_1$, $v_1$ and $(k^2 - 1)^2(\text{sn}^2 - 1)^{-2} = k^4 \text{cn}^4 (u - K + iK^\prime)$ for $u_2$, $v_2$, the remaining integrals are derived in an analogous manner.

**Remark 6.1.** Each of the primitive functions $u_0$, $v_0$, $v_1$, $v_2$ is a doubly periodic function or a sum of $b_0(2u + x_0 + c_0)$. Indeed,

$$u_0 = -(8(A_\phi - 1))^{-1}(\beta(2u + x_0 + \Omega_a(2\nu_0 - 1/2)) + \beta(2u + x_0 - \Omega_a(2\nu_0 - 1/2)) + c_1$$

with $\beta(x) = b_0(x) - \beta_0$, and is bounded for $x = x_0 + 2u \in \tilde{S}(\phi, t_\infty, \kappa_0, \delta_0)$. The functions $u_1$ and $u_2$ are sums of $c_2u$ and such bounded functions.

Now we are ready to verify Corollary 24. Note that

$$\chi_0(x) = -\frac{1}{2} \int_0^x b_0(G_0(0, 0) - 2F_0(0)) \frac{d\xi}{\xi^2}$$
6.3 and Remark 6.1

\[ F_0(\psi_0) = 2(\theta_0 + \theta_1)\psi(\phi^2 - \psi_0^2)^{-1} \]

and \( G_0(\psi_0, b_0) = b_0(2(\phi^2 - \psi_0^2)^{-1}) \). By Lemma 6.2 and Remark 6.1 for \( u = (x - x_0)/2 \) and \( \psi_0 \) with \( k = \phi^2/2, 4K = \Omega_a, 2K = \Omega_b, \)

\[ \int_0^u F_2(\psi_0) du = \frac{2}{2} \int_{\phi_0 - 1}^1 \left( 1 - \psi_0^2 - \psi_0^2 \right)(2(\theta_0 - \theta_1)\theta_\infty \psi_0 + (\theta_0 - \theta_1)^2 + \theta_\infty^2) du \]

where \( \text{bdd} \) denotes a function bounded in \( \tilde{S}(\phi, t_\infty, \kappa_0, \delta_1) \), and, by integration by parts,

\[ \int_{\phi_0}^{\infty} F_2(\psi_0) \frac{d\xi}{\xi^2} = \frac{2((\theta_0 - \theta_1)^2 + \theta_\infty^2)}{2} x^{-1} + O(x^{-2}). \]

Similarly,

\[ \int_{\phi_0}^{\infty} F_0(\psi_0)^2 \frac{d\xi}{\xi^2} = \frac{4(\theta_0 + \theta_1)^2}{3(\phi_0 - 1)} x^{-1} + O(x^{-2}), \]

\[ \int_{\phi_0}^{\infty} (\psi_0^2 - \phi_0^2) F_2(\psi_0) \frac{d\xi}{\xi^2} = 2((\theta_0 - \theta_1)^2 + \theta_\infty^2)x^{-1} + O(x^{-2}), \]

\[ \int_{\phi_0}^{\infty} (\psi_0^2 - \phi_0^2) F_0(\psi_0)^2 \frac{d\xi}{\xi^2} = -4(\theta_0 + \theta_1)^2 x^{-1} + O(x^{-2}). \]

Using these estimates, we may rewrite the expressions of \( h(x) \) and \( \chi_0(x) \) as in the corollary.

The coefficient of \( \beta_0^2 \) in \( \chi_0(x) \) is

\[ -\frac{1}{4} \int_{\phi_0}^{\infty} \frac{1}{(\phi_0 - \phi_0^2)^2} \frac{d\xi}{\xi^2} = -\frac{1}{4} U(x) x^{-2} + \frac{1}{2} \int_{\phi_0}^{\infty} U(x) \frac{d\xi}{\xi^2} \ll x^{-2}, \]

where \( U(x) \) is a primitive function of \((\phi_0 - \phi_0^2)^{-1}\) bounded in \( \tilde{S}(\phi, t_\infty, \kappa_0, \delta_0) \). By Lemma 6.3 and Remark 6.1,

\[ \int_0^u \frac{du}{(\phi_0 - \phi_0^2)^2} = \frac{u}{3A_0(\phi_0 - 1)} + \text{bdd}, \]

and hence the coefficient of \( \beta_0^2 \) in \( \phi(x) \) is \((8A_0(1 - A_0))^{-1}\). Thus Corollary 2.4 is proved.

6.6. Proof of Theorem 2.2. Suppose that \( 0 < |\phi - \pi| < \pi/2 \), i.e. \( \pi/2 < \phi < 3\pi/2 \). Recall that \( \mu(\lambda) \) is on the Riemann surface \( \mathbb{P}^+ \cup \mathbb{P}^- \) glued along the cuts \([\lambda_0, \lambda_1] \) and \([\lambda_2, \lambda_2^2] \). Note that \( A_0 = A_0 - \pi \) by Lemma 8.14. Let \( S(\pi/2, 3\pi/2) \) on \( \mathbb{P}^+ \) be the Stokes graph as described in Figure 6.1(a), (b), in which \( \xi_3^\infty \) joins \( \lambda_1 \) or \( \lambda_2 \) to \( i\infty \), and \( c_3^\infty \) joins \( \lambda_1 \) or \( \lambda_2 \) to \(-i\infty \). The anticlockwise \( \pi \)-radian rotation of the Stokes graph \( S(-\pi/2, \pi/2) \) for \( 0 < |\phi| < \pi/2 \) as in Figure 4.1 results in \( S(\pi/2, 3\pi/2) \). The curve \( c_3^\infty \) corresponds to \( c_1^\infty \) or \( c_2^\infty \). Let the loops \( \hat{l}_0, \hat{l}_1 \) be the results of the same rotation of \( l_0, l_1 \) in Figure 3.1.
The loops \( \tilde{l}_0, \tilde{l}_1 \) and the starting point \( \tilde{p}_{st} \) are as in Figure 6.1 (c), and \( \arg(\tilde{p}_{st}) = 3\pi/2 \).

Let \( \tilde{M}^0 \) and \( \tilde{M}^1 \) be the monodromy matrices defined by the analytic continuation of \( Y_2(t, \lambda) \) along the loops \( \tilde{l}_0, \tilde{l}_1 \), respectively. Recalling that \( Y_2(t, \lambda) = Y(t, \lambda)S_2 \), and that the analytic continuation of \( Y(t, \lambda) \) along \( l_0, l_1 \) are \( Y(t, \lambda)M^0, Y(t, \lambda)M^1 \), respectively, we have \( S_2^{-1}M^0S_2 = \tilde{M}^0, S_2^{-1}M^1S_2 = \tilde{M}^1 \).

In the calculation of \( \tilde{M}^0 \) and \( \tilde{M}^1 \) this Stokes graph is used. Suppose that \( \pi < \phi < 3\pi/2 \). Let \( Y_3(t, \lambda) \) be the matrix solution admitting the same asymptotic representation as (3.6) in the sector \( |\arg \lambda - 5\pi/2| < \pi \). Denote by \( \Gamma_3 \) a connection matrix such that \( Y_2 = Y_3\Gamma_3 \) along \( c_3^\infty \cup c_0 \cup c_3^\infty \) joining \( e^{5\pi i/2} \) to \( e^{3\pi i/2} \). The Stokes matrix \( S_3 = e^{\pi i \theta_{\infty} \sigma_3}S_1e^{-\pi i \theta_{\infty} \sigma_3} \) is given by \( Y_3 = Y_2S_3 \). Then \( \Gamma_3\tilde{M}^0 = S_3^{-1} \). The WKB analysis with the Stokes graph in Figure 6.1 (b) on \( \mathbb{C} \setminus [-\infty, e^{i\phi}] \) leads us to

\[
\Gamma_3 = (I + O(t^{-\delta})) \begin{pmatrix}
J_3 & -iC_0 e^{J_0 + J_3 + J_4} \\
i\delta_0^{-1}e^{J_0 - J_3 - J_4} & e^{-J_3 + J_4}
\end{pmatrix},
\]

where \( J_0 = \int_{\lambda_2}^{\lambda_1} \Lambda_3(\tau)d\tau \),

\[
J_3 = \lim_{\lambda \to \infty} \int_{\lambda_1}^{\lambda} \Lambda_3(\tau)d\tau - \frac{1}{4}(t\lambda - 2\theta_{\infty} \log \lambda)\sigma_3,
\]

\[
\tilde{J}_3 = \lim_{\lambda \to \infty} \int_{\lambda}^{\lambda_1} \Lambda_3(\tau)d\tau - \frac{1}{4}(t\lambda - 2\theta_{\infty} \log \lambda)\sigma_3,
\]

Recall \( \Gamma_{\infty} \) such that \( Y_2 \Gamma_{\infty} = Y \) along a path joining \( e^{3\pi i/2} \) to \( e^{\pi i/2} \) on the right-hand side of \( e^{i\phi} \) in Figure 6.1 (b). Then \( \tilde{M}^1 \Gamma_{\infty} = S_2^{-1} \), and, by the use of the Stokes graph with \( c_2^\infty \) in place of \( \hat{c}_3^\infty \),

\[
\Gamma_{\infty} = (I + O(t^{-\delta})) \begin{pmatrix}
J_3 & -iC_0 e^{J_0 + J_3 + J_4} \\
i\delta_0^{-1}e^{J_0 - J_3 - J_4} & e^{-J_3 + J_4}
\end{pmatrix}
\]

with \( J_2 \) as in Section 4. Note that \( \tilde{m}_{12}^0 \tilde{m}_{21}^1 + \tilde{m}_{22}^0 \tilde{m}_{22}^1 = e^{\pi i \theta_{\infty}} \) follows from \( \tilde{M}^1 \tilde{M}^0 = S_2^{-1}S_1^{-1}e^{-\pi i \theta_{\infty} \sigma_3} \). From the relations \( \Gamma_3 M^0 = S_3^{-1} \) and \( \tilde{M}^1 \Gamma_{\infty} = S_2^{-1} \), we derive

\[
e^{J_3 - J_1} (1 + O(t^{-\delta})) = \tilde{m}_{12}^1, \quad -c_0^{-1}d_0 e^{2J_1 - 2J_2}(1 + O(t^{-\delta})) = e^{\pi i \theta_{\infty}} (\tilde{m}_{12}^0 \tilde{m}_{21}^1)^{-1}.
\]
Since $J_3 - J_1$ corresponds to the cycle $b$, this implies the conclusion for $\pi < \phi < 3\pi/2$. In the case $\pi/2 < \phi < \pi$, denoting $J_{3|\lambda_1 \to \lambda_2}$ and $J_{3|\lambda_2 \to \lambda_1}$ by the same symbols $J_3$ and $\hat{J}_3$, respectively, we have, by using the Stokes graph in Figure 6.1 (a),

$$
\Gamma_3 = (I + O(t^{-\delta})) \begin{pmatrix}
e^{-j_0 + \hat{J}_3 - J_3} & i c_0 e^{-j_0 + \hat{J}_3 + J_3} \\
-id_0^{-1} e^{-j_0 - J_3 - J_3} & e^{-j_3 + J_3} (e^{-j_0} + c_0 d_0^{-1} e^{j_0})
\end{pmatrix},
$$

$$
\Gamma_{\infty}^{2} = (I + O(t^{-\delta})) \begin{pmatrix}e^{j_3 - \hat{J}_1 - (e^{-j_0} + c_0 d_0^{-1} e^{j_0})} & -ic_0 e^{j_0 + \hat{J}_3 + J_3} \\
id_0^{-1} e^{-j_3 - \hat{J}_1} & e^{-j_3 + J_3 + J_0}
\end{pmatrix}.
$$

Similarly we have

$$
e^{j_2 - \hat{J}_3} (1 + O(t^{-\delta})) = \frac{\hat{m}_{22}^0}{\hat{m}_{12} m_{12}^0 + \hat{m}_{22}^0 \hat{m}_{12}^0},
$$

$$\quad -c_0^{-1} d_0 e^{2j_1 - 2j_2} (1 + O(t^{-\delta})) = \frac{e^{\pi i \theta_{\infty}}}{\hat{m}_{12} m_{12}^0},$$

from which the conclusion of the theorem follows.

For general angles $\phi$ such that $|\phi - k\pi| < \pi/2$ ($k \in \mathbb{Z}$), denote by $\hat{S}_k^0$, $\hat{S}_k^1$ and $\hat{S}(k\pi - \pi/2, k\pi + \pi/2)$ the $k\pi$-rotation of $\hat{S}_0$, $\hat{S}_1$, and $S(-\pi/2, \pi/2)$, respectively. Let $\hat{Y}^p(t, \lambda)$ be the matrix solution of (3.4) admitting the same asymptotic representation as (3.4) in the sector $|\arg \lambda - 2\pi n - \pi/2| < \pi$, and let $M_p^0$, $M_p^1$ be the monodromy matrices given by the analytic continuation of $\hat{Y}^p(t, \lambda)$ along $\hat{S}_k^0$, $\hat{S}_k^1$, respectively. Especially, $\hat{Y}^0(t, \lambda) = Y(t, \lambda)$, $\hat{S}_0 = \hat{S}_1$, $\hat{S}_0 = \hat{S}_1$, $M_p^0 = M_0$, $M_p^1 = M_1$. Then using $M_p^0$, $M_p^1$ we may reduce the general case to the one $0 < |\phi| < \pi/2$ or $0 < |\phi - \pi| < \pi/2$, to which Theorem 2.1 or 2.2 apply (cf. Remark 2.3).

7. On an Amendment to the Early Proof of Theorem 2.1

To prove Theorem 2.1 along the early line in [39], the following modification of the arguments is considered. Suppose that $y(t)$ in $B(t, \lambda)$ of (3.4) solves (P_V) and is labelled by the monodromy data $M_0^0$, $M_1^1$ such that $m_{11} m_{12}^0 m_{12}^1 m_{12}^0 \neq 0$. For a given sequence $\{e^{i \phi_n t_n}\} \subset S(\phi, t_n, \kappa_0, \delta_1)$, by the boundedness of $a_\phi(t)$, we may suppose that $a_\phi(t_n) \to a^\infty$ as $t_n \to +\infty$, $\phi_n \to \phi$ for some $a^\infty \in \mathbb{C}$. Then, it is likely that the point-wise limit Stokes graph with the limit turning points $\lambda_1^\infty$, $\lambda_2^\infty$ as $t_n \to +\infty$ falls on one of the following:

(C1) $(\lambda_1^\infty, \lambda_2^\infty) = (0, 0)$ or $(e^{i \phi}, -e^{i \phi})$;

(C2) the limit Stokes graph as in Figure 4.1 (a) or 39 Figure 4.1, in which $\lambda_1^\infty$ and $\lambda_2^\infty$ are not connected;

(C3) the limit Stokes graph as in Figure 4.1 (b) or Figure 4.1 (a), (b), in which $\lambda_1^\infty$ and $\lambda_2^\infty$ are connected.

The case (C1) is treated in [2] and [5], in deriving asymptotic solutions on the positive real axis, i.e. $\lambda = 0$, and then $m_{11}^0 m_{11}^0 m_{12}^0 m_{12}^0 = 0$. In our case $0 < |\phi| < \pi/2$, if we could derive $m_{11}^0 m_{11}^0 m_{12}^0 m_{12}^0 = 0$ by examining $M_0^0$, $M_1^1$ for $e^{i \phi t}$ in every neighbourhood of $e^{i \phi_n t_n}$, then the case (C1) is excluded. In the case (C2), in every neighbourhood of $e^{i \phi_n t_n}$, using the Stokes graph in [39] Figure 4.1, we calculated $\Gamma_{\infty}^{1}$ and $\Gamma_{\infty}^{2}$ in [39] Section 4], and in the proof of [39] Proposition 5.6] it is shown that $a^\infty = A_\phi$.
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Figure 7.1. Stokes graphs

solves the Boutroux equations. This implies \( \Re e^{i\phi} \int_{z_1}^{z_2} \sqrt{(A_\phi - z^2)/(1 - z^2)}dz = 0 \) with \( e^{i\phi} \lambda_2^\infty = z_j \) (\( j = 1, 2 \)), that is, \( \lambda_1^\infty \) and \( \lambda_2^\infty \) are connected, contradicting the supposition. Therefore (C3) is the only possible case. Using the Stokes graph of (C3), we may carry out the calculation in Section 4 and prove (4.4) and Corollary 5.4. Then we may show the following corresponding to [39, Proposition 5.6].

Claim. Let \( 0 < |\phi| < \pi/2 \). There exists \( A_\phi \in \mathbb{C} \setminus \{0, 1\} \) such that \( a_\phi(t) \to \lambda_\phi \) as \( e^{i\phi}t \to \infty \) through \( S'(\phi, t', \kappa_0, \delta_1) \) and that

\[
\Re e^{i\phi} \int \sqrt{A_\phi - z^2} \frac{dz}{1 - z^2} = \Re e^{i\phi} \int \sqrt{A_\phi - z^2} \frac{dz}{1 - z^2} = 0.
\]

If this is established by fixing ambiguous parts of the argument above, Theorem 2.1 may be derived by the same procedure as in [39, Section 6.1].

8. Modulus \( A_\phi \) and the Boutroux equations

We examine a solution \( A \in \mathbb{C} \) of the Boutroux equations. Let the branch of \( A^{1/2} \) (\( \neq 0 \)) be fixed in such a way that \( \Re A^{1/2} \geq 0 \), and \( \Im A^{1/2} > 0 \) if \( \Re A^{1/2} = 0 \). In accordance with [31, Appendix I] set

\[
I_a(A) = \int_a \sqrt{\frac{A - z^2}{1 - z^2}} dz, \quad I_b(A) = \int_b \sqrt{\frac{A - z^2}{1 - z^2}} dz, \quad I(A) = \frac{I_a(A)}{I_b(A)},
\]

in which the cycles \( a \) and \( b \) are as in Figure 2.2 with \( A_\phi = A \).

Lemma 8.1. Let \( A \in \mathbb{C} \). Then \( I(A) \in \mathbb{R} \) if and only if, for some \( \phi \in \mathbb{R} \), \( A \) solves the Boutroux equations (BE)\( _\phi \) : \( \Re e^{i\phi}I_a(A) = \Re e^{i\phi}I_b(A) = 0 \).

Proof. Suppose that \( I(A) = \rho \in \mathbb{R} \), and write \( I_a(A) = u + iv, \) \( I_b(A) = U + iV \) with \( u, v, U, V \in \mathbb{R} \). Then \( u = \rho U, \) \( v = \rho V \), and hence \( u/v = U/V = \tan \phi \) for some \( \phi \in \mathbb{R} \), which implies \( \Re e^{i\phi}I_a(A) = \Re e^{i\phi}I_b(A) = 0 \).

By Lemma 5.8,

\[
I'(A) = \frac{1}{2I_b(A)^2} (\omega_a(A)I_b(A) - \omega_b(A)I_a(A)) = -\frac{2\pi i}{I_b(A)^2},
\]

\[
\omega_{a,b}(A) = \int_{a,b} \frac{dz}{\sqrt{(A - z^2)(1 - z^2)}}.
\]
Lemma 8.2. The map $\mathcal{I}(A)$ is conformal on $\mathbb{C}$ as long as $I_b(A) \neq 0, \infty$.

Near $A = \infty$, observing that

$$I_a(A) = 4A^{1/2} \int_0^1 \frac{1 - z^2/A}{1 - z^2} \, dz = 2\pi A^{1/2}(1 + O(A^{-1})),$$

and that

$$I_b(A) = 2A^{1/2} \int_1^{A^{1/2}} \frac{1}{\sqrt{z^2 - 1}} + \frac{-z^2/A}{\sqrt{z^2 - 1}(1 + \sqrt{1 - z^2/A})} \, dz$$

$$= -iA^{1/2} \log A \left(1 + O(|\log A|^{-1})\right),$$

we have $\text{Im}\left(1/I(A)\right) = -(2\pi)^{-1} \log |A| (1 + o(1))$ as $A \to \infty$, which implies

Lemma 8.3. The set

$$\mathcal{R}(BE) = \mathcal{I}^{-1}(\mathbb{R}) = \{A \in \mathbb{C} : A \text{ solves } (BE)_\phi \text{ for some } \phi \in \mathbb{R}\}$$

is bounded.

Let us observe the dependence of $A \in \mathcal{R}(BE)$ on $\phi$ or $t = \tan \phi$.

Since $I_a(0) = 0$, $I_b(0) = 2i$, $A = 0$ solves $(BE)_{\phi=0}$. Conversely we may give the uniqueness lemma, which is crucial in discussing $(BE)_\phi$. This is proved by an argument similar to that in [27, §7]).

Lemma 8.4. If $A$ solves $(BE)_{\phi=0}$, then $A = 0$.

Proof. Suppose that $\text{Re } I_a(A) = \text{Re } I_b(A) = 0$. Then $I_b(A)$ is pure imaginary, and $I_b(A) = -I_b(A) = -I_b(\overline{A}) = I_b(\overline{A})$, that is, $I_b(A) - I_b(\overline{A}) = 0$.

Figure 8.1. Cycle $b$

(a) $0 \leq \text{Re } A^{1/2} < 1$

(b) $\text{Re } A^{1/2} = 1$

(c) $\text{Re } A^{1/2} > 1$

(a) Case where $0 \leq \text{Re } A^{1/2} < 1$: Write $A^{1/2} = \alpha + i\beta$ with $0 \leq \alpha < 1$, say, $\beta \geq 0$. Then the cycle $b$ may be deformed in such a way that $b$ surrounds anticlockwise the cuts
The function $\sqrt{(A - z^2)(1 - z^2)}$ (respectively, $\sqrt{(A - z^2)(1 - z^2)}$) may be treated on the plane with the cuts $[-1, -\alpha] \cup [-\alpha, -\alpha - i\beta]$ (respectively, $[-1, -\alpha] \cup [-\alpha, -\alpha + i\beta]$). We have

$$I_b(A) - I_b(\overline{A}) = \int_b \left( \sqrt{\frac{A - z^2}{1 - z^2}} - \sqrt{\frac{\overline{A} - z^2}{1 - z^2}} \right) dz = (A - \overline{A})I_b(A, \overline{A}) = 0,$$

where

$$I_b(A, \overline{A}) = \int_b \frac{dz}{\sqrt{1 - z^2}(\sqrt{A - z^2} + \sqrt{A - z^2})}.$$

To show $A \in \mathbb{R}$, suppose the contrary $A - \overline{A} \neq 0$. Dividing $b$ into five parts, we have

$$I_b(A, \overline{A}) = I_0^\beta + \tilde{I}_0^\beta + H_{-\alpha}^{-1} + \tilde{J}_0^{-\beta} + J_0^{-\beta},$$

in which

$$I_0^\beta = \int_0^\beta \frac{idt}{\sqrt{1 - (-\alpha + it)^2}(\sqrt{A - (-\alpha + it)^2} + \sqrt{A - (-\alpha + it)^2})},$$

$$\tilde{I}_0^\beta = \int_0^\beta \frac{idt}{\sqrt{1 - (-\alpha + it)^2}(\sqrt{A - (-\alpha + it)^2} - \sqrt{A - (-\alpha + it)^2})},$$

$$H_{-\alpha}^{-1} = \int_{-\alpha}^{-1} \frac{2dt}{\sqrt{1 - t^2}(\sqrt{A - t^2} - \sqrt{A - t^2})},$$

$$\tilde{J}_0^{-\beta} = \int_0^{-\beta} \frac{idt}{\sqrt{1 - (-\alpha + it)^2}(\sqrt{A - (-\alpha + it)^2} - \sqrt{A - (-\alpha + it)^2})},$$

$$J_0^{-\beta} = \int_0^{-\beta} \frac{idt}{\sqrt{1 - (-\alpha + it)^2}(\sqrt{A - (-\alpha + it)^2} - \sqrt{A - (-\alpha + it)^2})}.$$

Then

$$(I_0^\beta + \tilde{I}_0^\beta + \tilde{J}_0^{-\beta} + J_0^{-\beta}) = \frac{2i}{A - \overline{A}} \int_0^\beta \left( \frac{A - (\alpha + it)^2}{1 - (\alpha + it)^2} - \frac{A - (\alpha + it)^2}{1 - (\alpha + it)^2} \right) dt \in i\mathbb{R},$$

(for the branch of $\sqrt{(A - z^2)/(1 - z^2)}$ see Section 5). The remaining integral $H_{-\alpha}^{-1}$ is

$$-\frac{1}{2}H_{-\alpha}^{-1} = \frac{i}{A - \overline{A}} \int_\alpha^1 \frac{\sqrt{t^2 - (\alpha + i\beta)^2} + \sqrt{t^2 - (\alpha - i\beta)^2}}{\sqrt{1 - t^2}} dt$$

$$= \frac{2i}{A - \overline{A}} \int_\alpha^1 \text{Re} \frac{\sqrt{t^2 - \alpha^2 + \beta^2 - 2i\alpha\beta}}{\sqrt{1 - t^2}} dt$$

$$= \frac{\sqrt{2}i}{A - \overline{A}} \int_\alpha^1 \frac{\sqrt{t^2 - \alpha^2 + \beta^2 + \sqrt{(t^2 - \alpha^2 + \beta^2)^2 + 4\alpha^2\beta^2}}}{\sqrt{1 - t^2}} dt \in \mathbb{R} \setminus \{0\}.$$
(b) Case where $\text{Re} A^{1/2} = 1$: Write $A^{1/2} = 1 + i\beta$, say $\beta \geq 0$ (cf. Figure 8.1 (b)). Then

$$I_b(A) = \int_b^0 \sqrt{\frac{A - z^2}{1 - z^2}} dz = 2i \int_{-\beta}^0 \sqrt{\frac{(-1 - i\beta)^2 - (-1 + it)^2}{1 - (-1 + it)^2}} dt$$

$$= 2i \int_{0}^{\beta} \sqrt{\frac{t^2 - \beta^2 - 2(t - \beta)i}{t^2 - 2ti}} dt = -2 \int_{0}^{\beta} \sqrt{\frac{\beta - t}{t(4 + t^2)} \sqrt{t^2 + \beta t + 4 + 2\beta i}} dt$$

with

$$\text{Re} \sqrt{t^2 + \beta t + 4 + 2\beta i} = \sqrt{t^2 + \beta t + 4 + \sqrt{(t^2 + \beta t + 4)^2 + 4\beta^2}} \geq 2\sqrt{2},$$

which implies $\text{Re} I_b(A) \neq 0$.

(c) Case where $\text{Re} A^{1/2} > 1$: It is shown that $\text{Re} I_b(A) = 0$ implies $A \in \mathbb{R}$ by an argument similar to that in the case (a).

Thus in every case we have shown $A \in \mathbb{R}$ or $\text{Re} I_b(A) \neq 0$. We may examine $I_a(A)$ and $I_b(A)$ for each $A \in \mathbb{R}$ to conclude that $\text{Re} I_a(A) = \text{Re} I_b(A) = 0$ if and only if $A = 0$. This completes the proof. \hfill \square

**Corollary 8.5.** For every $A \in \mathbb{C}$, $(I_a(A), I_b(A)) \neq (0, 0)$.

**Corollary 8.6.** If $\text{Re} I_b(A) = 0$, then $A = 0$.

Since $I_a(1) = 4$, $I_b(1) = 0$, the number $A = 1$ solves $(\text{BE})_{\phi = \pm \pi/2}$. Observe that $\text{Re} iI_b(A) = 0$ implies $I_b(A) = -I_b(A)$. Then, similarly we have

**Lemma 8.7.** If $A$ solves $(\text{BE})_{\phi = \pm \pi/2}$, then $A = 1$.

**Corollary 8.8.** If $\text{Re} iI_b(A) = 0$, then $A = 1$.

**Lemma 8.9.** If $|\phi|$ is sufficiently small, equations $(\text{BE})_{\phi}$ admit a solution $A_\phi = x(\phi) + iy(\phi)$ such that

$$x(\phi) = -\frac{4\phi^2}{\log \phi} (1 + o(1)), \quad y(\phi) = -\frac{4\phi}{\log \phi} (1 + o(1)),$$

which is unique around $A = 0$.

**Proof.** Suppose that $|A|$ is small and $\text{Re} A^{1/2} \geq 0$. Then

$$I_a(A) = \int_a^1 \sqrt{\frac{A - z^2}{1 - z^2}} dz = 2 \int_{A^{1/2}}^{A^{1/2}} \sqrt{\frac{A - z^2}{1 - z^2}} dz = 2A \int_{-1}^{1} \sqrt{\frac{1 - t^2}{1 - At^2}} dt = \pi A + O(A^2),$$

$$I_b(A) = \int_{b}^{1} \frac{zdz}{\sqrt{1 - z^2}} - A \int_{A^{1/2}}^{1} \frac{dz}{\sqrt{1 - z^2}(z + \sqrt{z^2 - A})}$$

$$= \frac{i}{2} (4 + A \log A + O(A)).$$
From $\text{Re} e^{i\phi} I_a(A_\phi) = \text{Re} e^{i\phi} I_b(A_\phi) = 0$, that is,

$$\text{Re} ((A_\phi + O(A_\phi^2))(\cos \phi + i \sin \phi)) = \text{Re} (i(4 + A_\phi \log A_\phi + O(A_\phi))(\cos \phi + i \sin \phi)) = 0$$

with $A_\phi = x(\phi) + iy(\phi)$, the conclusion follows. \hfill \Box

Similarly we have

**Lemma 8.10.** If $|\phi \mp \pi/2|$ is sufficiently small, equations (BE)$_\phi$ admit a solution $A_\phi = x(\phi) + iy(\phi)$ such that

$$x(\phi) = 1 + \frac{4\hat{\phi}_\pm}{\log \phi_\pm}(1 + o(1)), \quad y(\phi) = \frac{4\hat{\phi}_\pm}{\log \phi_\pm}(1 + o(1))$$

with $\phi = \pm \pi/2 + \hat{\phi}_\pm$.

**Lemma 8.11.** Suppose that $0 < |\phi_0| < \pi/2$ and that $A(\phi_0)$ solves (BE)$_{\phi=\phi_0}$. Then there exists a curve $\Gamma(\phi_0)$ given by $A = A(\phi_0, \phi)$ for $|\phi| \leq \pi/2$, where $A(\phi_0, \phi)$ has the properties:

(i) $A(\phi_0, \phi_0) = A(\phi_0)$, $A(\phi_0, 0) = 0$, $A(\phi_0, \pm \pi/2) = 1$;
(ii) $A(\phi_0, \phi)$ is continuous in $\phi$ for $|\phi| \leq \pi/2$ and smooth for $0 < |\phi| < \pi/2$;
(iii) $A(\phi_0, \phi)$ solves (BE)$_\phi$ for $|\phi| \leq \pi/2$.

**Proof.** Set

$$A = x + iy, \quad I_a(A) = u(A) + iv(A), \quad I_b(A) = U(A) + iV(A)$$

with $x, y, u(A), v(A), U(A), V(A) \in \mathbb{R}$. Then $A$ solves (BE)$_\phi$ if and only if

$$\text{Re} e^{i\phi} I_a(A) = u(A) \cos \phi - v(A) \sin \phi = \text{Re} e^{i\phi} I_b(A) = U(A) \cos \phi - V(A) \sin \phi = 0,$$

that is,

(8.1) $$u(A) - v(A)t = U(A) - V(A)t = 0 \quad \text{with} \quad t = \tan \phi.$$

By the Cauchy-Riemann relations the Jacobian for (8.1) is

(8.2) $$J(v, V; t; A) = \det \begin{pmatrix} v_y - tv_x & -v_x - tv_y \\ V_y - tV_x & -V_x - tV_y \end{pmatrix} = (1 + t^2)(v_x V_y - v_y V_x)$$

$$= -\frac{i}{8}(1 + t^2)(\omega_a(A)\overline{\omega_b(A)} - \overline{\omega_a(A)}\omega_b(A)) = -\frac{1}{4}(1 + t^2)\text{Im} \frac{\omega_b(A)}{\omega_a(A)}$$

$$= -\frac{1}{4}(1 + t^2)|\omega_a(A)|^2\text{Im} \frac{\omega_b(A)}{\omega_a(A)} < 0, \quad \neq \infty,$$

provided that $A \neq 0, 1$. By supposition, since $A(\phi_0) \neq 0, 1$, there exists a function $A_0(\phi)$ with the properties:

(a) $A_0(\phi_0) = A(\phi_0)$;
(b) $A_0(\phi)$ is smooth for $|\phi - \phi_0| < \varepsilon_*$, $\varepsilon_*$ being sufficiently small;
(c) $A_0(\phi)$ solves (8.1), i.e. (BE)$_\phi$ for $|\phi - \phi_0| < \varepsilon_*$ and is a unique solution in a small neighbourhood of $A(\phi_0)$. 
Let us consider the case $0 < \phi_0 < \pi/2$. Denote by $\mathcal{F}(\phi_0)$ the family of functions $\hat{\nu}_i(\phi)$ with the properties:

(a) $\hat{\nu}_i(\phi_0) = A(\phi_0)$;

(b) $\hat{\nu}_i(\phi)$ is smooth for $\phi_0 - \varepsilon_* < \phi < \phi_0 + \varepsilon_*$;

(c) $\hat{\nu}_i(\phi)$ solves (8.1) for $\phi_0 - \varepsilon_* < \phi < \phi_0$.

Then $A_0(\phi) \in \mathcal{F}(\phi_0)$. For any $\hat{\nu}_i(\phi), \hat{\nu}_i(\phi) \in \mathcal{F}(\phi_0)$ with $\hat{\nu}_i > \hat{\nu}_i$, $\hat{\nu}_i(\phi) \equiv \hat{\nu}_i(\phi)$ holds if $\phi_0 - \varepsilon_* < \phi < \phi_0$. Let $\hat{\nu}_i(\phi)$ be the maximal extension of all $\hat{\nu}_i(\phi) \in \mathcal{F}(\phi_0)$, and set $\hat{\nu}_i = \sup_i \hat{\nu}_i$. Then $\hat{\nu}_i(\phi)$ solves (8.1) and is smooth for $\phi_0 - \varepsilon_* < \phi < \phi_0$.

Suppose that $\phi_0 < \pi/2$. Since $\hat{\nu}_i(\phi_0) \neq 0, 1$, the Jacobian $J(\nu, \tan \phi; \nu(\phi_0))$ does not vanish, which implies that $\hat{\nu}_i(\phi)$ may be extended beyond $\phi_0$. This is a contradiction. Hence we have $\phi_0 = \pi/2$, and by Lemma 8.11, $\hat{\nu}_i(\pi/2) = 1$. Similarly we may construct a lower extension $\hat{\nu}_i(\phi)$ for $0 < \phi < \phi_0 + \varepsilon_*$ satisfying $\hat{\nu}_i(0) = 0$, and then have the extension $A^-(\phi_0, \phi)$ for $0 < \phi < \pi/2$. The case $-\pi/2 < \phi_0 < 0$ may be treated in the same way to obtain $A^-(\phi_0, \phi)$ for $-\pi/2 < \phi < 0$. For a given $\phi_0$ satisfying, say, $0 < \phi_0 < \pi/2$, combining $A^+(\phi_0, \phi)$ with $A^-(\phi_0, \phi)$, where $\phi_0 < \phi < 0$ close to $0$ is such that $A_\phi$ is a solution given in Lemma 8.9 we obtain the desired extension $A(\phi_0, \phi)$ for $|\phi| \leq \pi/2$. Thus the lemma is proved.

**Corollary 8.12.** Under the same supposition as in Lemma 8.11, $(d/d\phi)A(\phi_0, \phi) \neq 0$ for $0 < |\phi| < \pi/2$. Furthermore, $(d/d\phi)\mathcal{I}(\nu(\phi_0, \phi)) < 0$ or $< 0$ for $0 < \phi < \pi/2$, and so for $-\pi/2 < \phi < 0$.

**Proof.** From (8.1) it follows that

$$J(v, \tan \nu; \nu(\phi_0, \phi)) \left( x'(t) \quad y'(t) \right) = \frac{v(\nu(\phi_0, \phi))}{V(\nu(\phi_0, \phi))},$$

where $A(\phi_0, \nu) = x(t) + iy(t)$. By Corollary 8.5 and (8.2), $(x'(t), y'(t)) \neq (0, 0)$ if $0 < |\phi| < \pi/2$, i.e. $t \in \mathbb{R} \setminus \{0\}$, and then $(d/d\nu)A(\phi_0, \phi) = (x'(t) + iy'(t))/\cos^2 \phi \neq 0$. Since $\mathcal{I}(A(\phi_0, \phi)) \in \mathbb{R}$ by Lemma 8.11, we have

$$\frac{d}{d\phi}\mathcal{I}(A(\phi_0, \phi)) = \frac{d}{d\phi}A(\phi_0, \phi) \frac{-2\pi i}{\mathcal{I}(A(\phi_0, \phi))^2} \in \mathbb{R} \setminus \{0\}$$

for $0 < |\phi| < \pi/2$, from which the conclusion follows.

**Proposition 8.13.** For each $\phi_0$ such that $|\phi_0| \leq \pi/2$, equations (BE)$_{\phi=\phi_0}$ admit a unique solution $A_{\phi_0} \in \mathbb{C}$.

**Proof.** Let $\hat{\phi}_0$ be so close to $0$ that $A_{\hat{\phi}_0}$ is a solution given in Lemma 8.9 Lemma 8.11 with $\phi_0 = \hat{\phi}_0$ provides a curve $\Gamma(\hat{\phi}_0)$ containing a solution of (BE)$_{\phi=\phi_0}$ for each fixed $\phi_0$.

It remains to show the uniqueness of a solution for $\phi_0 \neq 0, \pm \pi/2$. Suppose that $A_{\phi_0}$ and $A_{\phi_0}'$ solve (BE)$_{\phi=\phi_0}$. Then, by Lemmas 8.2 and 8.11, there exist curves $\Gamma(\phi_0)$ and $\Gamma'(\phi_0)$ such that $\Gamma(\phi_0) \ni 0, A_{\phi_0}$, $\Gamma'(\phi_0) \ni 0, A_{\phi_0}'$. Then, by (8.2) (or the conformality of Lemma 8.2), we have $\Gamma(\phi_0) = \Gamma'(\phi_0) \ni A_{\phi_0} = A_{\phi_0}'$, which completes the proof.

By the uniqueness above we easily have
Lemma 8.14. For $\phi \in \mathbb{R}$, $(BE)_\phi$ admit a unique solution $A_\phi$, which satisfies

$$A_{\phi \pm \pi} = A_\phi, \quad A_{-\phi} = \overline{A_\phi}.$$ 

Lemma 8.15. Each $A_\phi$ given in Lemma 8.14 satisfies $0 \leq \text{Re} A_\phi \leq 1$. For $0 < \phi < \pi/2$ (respectively, $-\pi/2 < \phi < 0$), $(d/d\phi)\text{Re} A_\phi > 0$ (respectively, $< 0$).

Proof. Let $A_\phi = x(t) + iy(t)$, $t = \tan \phi$. Then, by Corollary 8.12

$$(d/dt) \mathcal{I}(A_\phi) = (x'(t) + iy'(t))(-2\pi i)\mathcal{I}_b(A_\phi)^{-2} \in \mathbb{R} \setminus \{0\}$$

for $0 < |\phi| < \pi/2$. This yields $x'(t)(U_*^2 - V_*^2) - 2y'(t)U_*V_* = 0$, where $I_b(A_\phi)^{-1} = U_* + iV_*$. Suppose that, $x'(t_0) = 0$ and $0 < \text{Re} A_{\phi_0} < 1$, for some $t_0 = \tan \phi_0 \neq 0, \pm \infty$. Since $y'(t_0) \neq 0$, $U_*V_* = 0$. If $U_* = 0$, then $\text{Re} I_b(A_{\phi_0}) = 0$, and hence $A_{\phi_0} = 0$, i.e. $\phi_0 = 0$ by Corollary 8.6. If $V_* = 0$, then $\text{Re} i I_b(A_{\phi_0}) = 0$, and hence $A_{\phi_0} = 1$, i.e. $\phi_0 = \pm \pi/2$ by Corollary 8.8. Thus we have shown that $x'(t) > 0$ or $x'(t) < 0$ for $0 < |\phi| < \pi/2$, $t = \tan \phi$, which implies $0 \leq \text{Re} A_\phi \leq 1$. \hfill $\square$

Remark 8.1. In the proof above, it is easy to see that $y'(t) = 0$ occurs if and only if $U_* = \pm V_*$, that is, $\phi = \pm \pi/4, \pm 3\pi/4$.

By Lemmas 8.9, 8.11, 8.15, Proposition 8.13 and Remark 8.1, we have

Proposition 8.16. There exists a Jordan closed curve $\Gamma_0 = \{A_\phi; \ |\phi| \leq \pi/2\}$ with the properties:

(i) $A_0 = 0$, $A_{\pm \pi/2} = 1$;
(ii) $A_\phi$ is smooth for $0 < |\phi| < \pi/2$;
(iii) for every $\phi, |\phi| \leq \pi/2$, $A_\phi$ solves $(BE)_\phi$.

By the properties above the trajectory of $\Gamma_0$ of $A_\phi$ for $|\phi| \leq \pi/2$ is as in Figure 8.2 (a).

![Figure 8.2](image)

Thus we have
Proposition 8.17. (1) For $|\phi| \leq \pi/2$, the Boutroux equations \((BE)_\phi\) have a unique solution $A_\phi$ with the properties:

(i) $A_0 = 0$, $A_{\pm\pi/2} = 1$;
(ii) $A_\phi$ is smooth in $\phi$ such that $0 < |\phi| < \pi/2$;
(iii) for $0 < \phi < \pi/2$ (respectively, $-\pi/2 < \phi < 0$), $x(t) = \text{Re} A_\phi$, $t = \tan \phi$ satisfies $x'(t) > 0$ (respectively, $x'(t) < 0$), and $y(t) = \text{Im} A_\phi$ satisfies $y'(t) = 0$ if and only if $\phi = \pm \pi/4, \pm 3\pi/4$;
(iv) $0 < \text{Re} A_\phi < 1$ for $0 < |\phi| < \pi/2$, and $\text{Im} A_\phi > 0$ for $0 < \phi < \pi/2$ and $0 < \phi < 0$ for $-\pi/2 < \phi < 0$.

(2) For $\phi \in \mathbb{R}$, $A_\phi$ may be extended by using the relations $A_{-\phi} = \overline{A_\phi}$, $A_{\phi \pm \pi} = A_\phi$.

Remark 8.2. It is easily verified that $0 < \text{Re} A^{1/2}_\phi < 1$ for $0 < |\phi| < \pi/2$, if the trajectory $\Gamma_0 = \{A_\phi; |\phi| \leq \pi/2\}$ is contained in $P_0 = \{x + iy; y^2 \leq 4(1-x)\}$. For some $\varepsilon_1 \geq \varepsilon_0 > 0$, by Lemma 8.10 the set $P_0$ contains the arc $\{A_\phi; |\phi - \pi/2| < \varepsilon_0\}$, and $\{A_\phi; |\phi - \pi/2 - \varepsilon_1|\}$ as well. It is likely that the trajectory of $A^{1/2}_\phi$ is as in Figure 8.2 (b). It remains, however, the possibility of the existence of $A_\phi \in \Gamma_0$ such that $\text{Re} A^{1/2}_\phi > 1$ and $|\text{Im} A^{1/2}_\phi| > \varepsilon_2 > 0$. To deny this possibility it is necessary to show that, say, $\text{Im} \mathcal{I}(A) \neq 0$ on $\{A; \text{Re} A^{1/2} = 1\} \setminus \{A = 1\} = \partial P_0 \setminus \{(1,0)\}$.

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