The spectrum of the Schrödinger Hamiltonian for trapped particles in a cylinder with a topological defect perturbed by attractive delta interactions

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Abstract

In this paper we use the technique used in [1]-[8] to deal with delta interactions in a rigorous way in a curved spacetime represented by a cosmic string along the z axis. This mathematical machinery is applied in order to study the discrete spectrum of a point-mass particle confined in an infinitely long cylinder with a conical defect on the z axis and perturbed by delta interactions. We derive a suitable approximate formula for the total energy. As a consequence, we found the existence of a mixing of states with positive or zero energy with the ones with negative energy (bound states). This mixture depends on the radius R of the trapping cylinder. The number of quantum bound states

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is an increasing function of the radius $R$. It is also interesting to note the presence of states with zero total energy (quasi free states). Apart from the gravitational background, the model presented in this paper can be of interest, for example, in the context of nanophysics and graphene modeling. In particular, the graphene with double layer can be modeled within our model, with the double layer given by delta perturbations and the string on the $z$–axis denoting topological defects of the graphene.

**Keywords**: Schrödinger Hamiltonian; delta interactions; trapped particles; topological defects: nanophysics.

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1 Introduction

Since the final formulation of general relativity, a great deal of theoretical work (see for example [9] and references therein) has been done in order to unify general relativity and quantum mechanics (loop string, causal sets, non-commutative geometry...). Unfortunately, a shared quantum gravity theory is still lacking. Technical difficulties are related to the covariance of general relativity under coordinate transformations and to the extreme difficulty to define localized energies in general relativity.

In practice, the main issue is due to the lack of an experimental signature as a guidance for theoretical physicists. To this purpose, it is important to study possible detectable effects induced by gravity on given quantum systems. As an example, the effects induced by some specific general relativistic background on a hydrogen atom has been investigated in [10] [11], where a shift effect on the spectrum arises depending on the particular chosen background. Similar shift effects can also be found in [12] [13]. Moreover, the experiments performed in [14] confirmed the effects induced by a gravitational field on the phase difference between two neutron beams. Another phenomenon is due to the interaction gravity-quantum system, for instance neutrino oscillations [15]. Another intriguing line of research is related to the well-known Aharonov-Bohm effect [16] where a charged particle can interact with an electromagnetic field also in those regions with vanishing field. In general relativity similar phenomena are well known in presence of topological defects generated by a cosmic string [17]. In this framework, the metric outside the string is locally flat and, as a consequence, a particle at a fixed position is not influenced by a gravitational field. Nevertheless,
the topology generated by a string along the $z$-axis is not the one of a Minkowskian spacetime, but rather it shows a conical defect. This defect produces many effects on traveling test particles, for example, gravitational lensing [18], pair production [19] and the gravitational counterpart of the Aharonov-Bohm effect [20].

The study of the effects arising from an infinitely long cosmic string on a non-relativistic quantum system has been further analysed in [21], where, shift effects given by a cosmic string arise in particular by considering a quantum system with harmonic and Coulombian potential. In this paper we continue the investigation carried out in [21] considering a quantum system perturbed by delta interactions. As shown in [1]-[8], the spectrum of the Schrödinger Hamiltonian perturbed by spatially symmetric point interactions can show interesting shift effects after using the suitable mathematical treatment.

It is thus interesting, in line of the reasonings above, to investigate the possible physical consequences of quantum shift effects present in the papers [1]-[8] in a curved background and in particular in the presence of topological (gravitational) defects induced by a cosmic string. This line of research can also be of great interest in the context of graphene modeling, where the topological defects in a curved background represented by a cosmic string can be used to depict defects present on the graphene [22]. The structure of the paper is the following. In section 2 we present the Hamiltonian of the model to be studied. Section 3 is devoted to the study of the radial part of the Schrödinger equation with Dirichlet boundary conditions. In section 4 we apply the mathematical tools presented in [4] to study the spectrum of a mass-particle confined in a infinitely long cylinder with a conical topological defect. Section 5 is devoted to the conclusions and final remarks.

2 The model

To start with, we consider the exterior metric of an infinitely long static string along the $z$ axis. Its energy-momentum tensor is nothing else but $T_{\alpha\beta} = \mu \delta(x) \delta(y) diag(-1, 0, 0, 1)$, where we used Cartesian coordinates $(t, x, y, z)$ with metric signature $(-, +, +, +)$ and $\mu$ is a constant linear mass density of the string. We introduce the deficit angle $B$ with $B = 1 - 4G\mu/c^2$. In cylindrical coordinates $(t, \rho, \phi, z)$ the line element of a cosmic string is

$$ds^2 = -dt^2 + d\rho^2 + B^2 \rho^2 d\phi^2 + dz^2.$$  \hspace{1cm} (1)

For positive values of the string density $\mu$ with $B \in [0, 1]$, we have the so-called deficit angle. In fact, by performing the coordinate transformation
\( \phi \rightarrow B\phi \) we have formally a Minkowskian metric with \( \phi \) in the domain \( \phi \in [0, 2\pi B] \). For \( \mu < 0 \) we have the so-called surplus-deficit angle metric with \( B > 1 \). Such a situation is exotic in the context of general relativity since a negative energy density does not satisfy the weak energy condition (i.e. the positivity of the energy density for any timelike experimenter). Nevertheless, negative energies can be allowed in a quantum field theory context where fluctuations can induce negative energies (Casimir effect). Strings with negative (attractive) energy density can be considered in the framework of graphene modeling (see for example [22]).

In a static background \([21]\) we can write the Schrödinger equation using the Laplace-Beltrami Laplacian \( \nabla^2_B \). By denoting with \( g \) the determinant of the spatial metric, obtained at \( t = \text{constant} \), with \( g_{ij} \) the spatial metric \((\{i, j\} = \{1, 2, 3\})\) we have \( \nabla^2_B = g^{-\frac{1}{2}} \partial_i \left( g^{ij} \sqrt{g} \partial^j \right) \). The Schrödinger equation for the wave function \( \psi = \psi(t, x^i) \) of a quantum particle with rest mass \( M \) interacting with a static external potential \( V_{\text{ext}} = V(x^i) \) becomes:

\[
\frac{i\hbar}{\partial t} \psi = -\frac{\hbar^2}{2M} \nabla^2_B \psi + V_{\text{ext}} \psi, \tag{2}
\]

with the Hamiltonian \( H \) given by \( H = \frac{\hbar^2}{2M} + V_{\text{ext}} \). We can now specify the equation (2) with the metric (1). For \( \nabla^2_B \) we have

\[
\nabla^2_B = \nabla^2_B = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{B^2 \rho^2} \frac{\partial^2}{\partial \phi^2}. \tag{3}
\]

The Laplacian (3), can be perturbed by delta interaction potentials \( V_{\text{ext}} \).

In order to use the mathematical machinery in [4], we must specify the expressions for the external potential \( V_{\text{ext}} \). To obtain a discrete spectrum, the L-B operator (3) must be perturbed by an attractive potential. We work in a Lorentzian manifold with 3 + 1 dimensions. Hence, the static external potential generally depends on the three spatial coordinates \( \{\rho, z, \phi\} \).

We consider the case with potentials looking like \( V_{\text{ext}} \sim -\lambda \delta(z \pm z_0) \) with \( \lambda > 0 \). In this case, the support of the delta is provided by the planes \( z = z_0 \).

Due to the static nature of the metric (2), we can consider the stationary form of (2), with \( H\psi(\rho, z, \phi) = E\psi(\rho, z, \phi) \) and \( H = H_0 + V_{\text{ext}} \), \( H_0 \) being the free Schrödinger Hamiltonian. After setting the geometrized units with \( G = c = h = 1 \), and by posing, without loss of generality, \( M = 1/2 \) \(^1\), we

\(^1\)The mass choice \( M = 1/2 \) is in agreement with the one present in the paper [4]. Obviously this choice does not represent a loss of generality since we can always take the transformations \( E \rightarrow E/2 \) and \( \lambda \rightarrow \lambda/2 \) and the usual choice \( M = 1 \) is regained.
have
\[-\left[\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + \frac{1}{B^2 \rho^2} \frac{\partial^2}{\partial \phi^2}\right] \psi + V_{ext}(\rho, z) \psi = \hbar \frac{\partial}{\partial t} \psi.\] (4)

Let us set
\[\psi(t, \rho, z, \phi) = ke^{\left(-\frac{Et}{\hbar}\right)} \psi(\rho, z, \phi),\] (5)
where $k \in \mathbb{R}^+$ is the normalization constant.

First of all, we consider the following expression for the perturbed potential $V_{ext}$:
\[V_{ext}(\rho, z) = V_{ext}(z) = -\lambda \left[\delta(z - z_0) + \delta(z + z_0)\right].\] (6)

The potential (6) represents a delta perturbation with support on the planes $z = \pm z_0$.

By setting $\psi(\rho, z, \phi) = \psi(\rho) \xi(z) \zeta(\phi)$ with $\zeta(\phi) \sim e^{iaB\phi}$, $a \in \mathbb{R}$ (see next section for more details), we have
\[-\left[\frac{d^2 \psi(\rho)}{d \rho^2} + \frac{1}{\rho} \frac{d \psi(\rho)}{d \rho} - \frac{a^2}{\rho^2} \psi(\rho)\right] = E_B \psi(\rho),\] (7)
\[-\frac{d^2 \xi(z)}{d z^2} - \lambda \left[\delta(z - z_0) + \delta(z + z_0)\right] \psi(z), = E_z \psi(z),\] (8)
where
\[E = E_B + E_z.\] (9)

In the general relativity context, the condition $\mu > 0$ is required in order to have a non-exotic matter content for the string. However, the case $\mu < 0$ can be of interest, for example, in the modeling of graphene [22]. The generalized eigenvalues $E_B$, related to the wavefunction $\psi(\rho)$, give rise to a continuous spectrum. We can also consider interesting cases where the particle is 'trapped' inside an infinitely long cylinder of radius $\rho_0$, so that we can introduce the infinite potential wall $V$ with $V = 0$ for $\rho \in (0, \rho_0)$ and $V = \infty$ for $\rho \geq \rho_0$.

In practice, a 'huge' wall trapping the particle is introduced and as a consequence, the particle is trapped within the cylinder of radius $\rho_0$. This modeling could be of interest to study the behavior of electrons inside a cylindrical conductor with a topological defect along the $z$-axis, and perturbed by delta attractive potentials. In this case the Dirichlet boundary condition $\psi(\rho = \rho_0 = R) = 0$ must be imposed. Moreover, note that by

\[\text{A similar technique has been used in [21] in order to obtain the spectrum of trapped gravitons by linearization of Einstein's equations [23].}\]
adopting the further Dirichlet condition $\psi(z = q) = 0, q > z_0$, we have a model representing physically a mass-particle trapped inside a finite cylinder with a topological defect and delta interactions on the parallel planes $z = \pm z_0$, however, it will not be investigated in this paper.

For instance, the electrons in a finite metal or in the quantum-confined Stark effect modulator [26]. Moreover, quantum dots [27] are very small semiconductors on the scale of nanometers. In these physical situations, electrons show quantum confinement and are thus trapped inside the dot: this could be modeled in our framework by setting $\rho_0$ of the order of nanometers.

Concerning the wave function $\psi(\rho, z, \phi) = k\psi(\rho)\xi(z)\zeta(\phi)$, the normalization requires:

$$k^2 \int_0^{2\pi} d\phi \int_0^{\rho_0} \rho d\rho \int_{-\infty}^{+\infty} \psi^2(\rho, z, \phi)dz = 1. \quad (10)$$

Note that the exact value of $k$ plays no role in result of this paper and we omit the calculation of the integral (10).

3 Radial spectrum of a point-mass particle with a topological defect

As is well known, the Schrödinger equation (4) can be separated by using cylindrical coordinates, as shown in (5), (7) and (8). The equation with $\psi(z)$ will be analyzed in the next section together with the explicit expressions for the spectrum of the ground state and of the first excited state. Concerning the $\zeta(\phi)$ component, from (4)-(5) we get:

$$\frac{d^2\zeta(\phi)}{d\phi^2} + a^2B^2\zeta(\phi) = 0. \quad (11)$$

The solution of (11) is

$$\zeta(\phi) \sim e^{iaB\phi} \quad (12)$$

with $a \in \mathbb{R}$. In order to guarantee the continuity of the wave function, we have to impose $\zeta(0) = \zeta(2\pi)$. This condition is not trivial in presence of a topological defect ($B \neq 1$). In fact, the effective range of the angular part of the spacetime is $\phi \in [0, 2\pi B]$ and the surfaces $\phi = 0$ and $\phi = 2\pi B$ must be connected, in order to have a well-defined manifold with metric (1). For $B < 1$, the range of $\phi$ is less than $2\pi$, while for $B > 1$ we have a surplus angle with $\phi$ greater than $2\pi$. Then we obtain

$$1 = e^{2\pi B a}, \quad (13)$$
so that, due to Euler’s relation, we get

\[ a = \frac{n}{B}, \quad n \in \mathbb{Z}. \tag{14} \]

From expression (14), for the radial equation (7), we obtain

\[ \rho^2 \frac{d^2 \psi(\rho)}{d\rho^2} + \rho \frac{d\psi(\rho)}{d\rho} + \left( E_B \rho^2 - \frac{n^2}{B^2} \right) \psi(\rho) = 0. \tag{15} \]

Equation (15) can be recast in a more suitable form with the position \( q = b \rho \), where \( b = \sqrt{E_B} \) \( (E_B \geq 0) \); hence we can write

\[ q^2 \frac{d^2 \psi(q)}{dq^2} + q \frac{d\psi(q)}{dq} + \left( q^2 - \frac{n^2}{B^2} \right) \psi(q) = 0. \tag{16} \]

For \( B = 1 \), (16) is nothing else but Bessel’s equation whose solutions can be expressed in terms of Bessel functions \( J_n(s) \) with \( n \in \mathbb{N} \). In our case, the presence of the topological defect makes the usual solutions \( J_n \) no more suitable. Instead, we need the Bessel’s functions with fractional index \( \nu \), \( \nu = n/B \).

4 Joint spectrum of a point-mass particle trapped inside a cylinder perturbed by delta interactions

By using the mathematical tools presented in [4], we can exploit the following equations in order to obtain the ground state energy and first bound state respectively:

\[ \frac{2 \left| E_z^{(0)} \right|^{1/2}}{1 + e^{-2\alpha_0 \left| E_z^{(0)} \right|^{1/2}}} = \lambda, \tag{17} \]

\[ \frac{2 \left| E_z^{(1)} \right|^{1/2}}{1 - e^{-2\alpha_0 \left| E_z^{(1)} \right|^{1/2}}} = \lambda. \tag{18} \]

In obtaining (17) and (18), we used the mass normalization \( M = 1/2 \) together with \( G = c = \hbar = 1 \). By using the transformations

\[ \lambda \to \frac{2M\lambda}{\hbar^2}, \quad E_z \to \frac{2ME_z}{\hbar^2}, \tag{19} \]
(17) and (18) become respectively:

$$\frac{\hbar |2M\mathcal{E}_z^{(0)}|^{1/2}}{1 + e^{-\frac{2z_0}{\hbar} |2M\mathcal{E}_z^{(0)}|^{1/2}}} = M\bar{\lambda}, \quad (20)$$

$$\frac{\hbar |2M\mathcal{E}_z^{(1)}|^{1/2}}{1 - e^{-\frac{2z_0}{\hbar} |2M\mathcal{E}_z^{(1)}|^{1/2}}} = M\bar{\lambda}, \quad (21)$$

A first study of the equations (20)-(21) can be performed using the same technique present in [4]. After introducing the adimensional variable $\xi$ with $\xi = \frac{z_0}{\hbar} |2M\mathcal{E}_z^{(0)}|^{1/2}$, we obtain the equivalent equation

$$F(\xi) = \frac{\xi}{1 + e^{-2\xi}} = \frac{z_0 M\bar{\lambda}}{\hbar^2}. \quad (22)$$

Since $\frac{dF}{d\xi} > 0$, $\forall \xi \geq 0$, we can invert (22) with inverse $F^{-1}(\xi)$, $\forall \xi \geq 0$. In particular, we are interested in the limiting case with $z_0 \to 0^+$ and $z_0 \to +\infty$.

In the former case, we have

$$\mathcal{E}_z^{(0)} \to -\frac{2M\bar{\lambda}^2}{\hbar^2}, \quad \text{for} \quad z_0 \to 0^+, \quad (23)$$

while in the latter case we get

$$\mathcal{E}_z^{(0)} \to -\frac{M\bar{\lambda}^2}{2\hbar^2}, \quad \text{for} \quad z_0 \to +\infty. \quad (24)$$

Moreover, the derivative $\frac{d\mathcal{E}_z^{(0)}}{dz_0}$ can be easily computed by using implicit differentiation with the result $\frac{d\mathcal{E}_z^{(0)}}{dz_0} > 0$. Hence, $\mathcal{E}_z^{(0)}$ is an increasing function of $z_0$ for any fixed $\bar{\lambda}$ and $M$.

Summarizing, we have $\mathcal{E}_z^{(0)} \in (-\frac{2M\bar{\lambda}^2}{\hbar^2}, -\frac{M\bar{\lambda}^2}{2\hbar^2})$.

A similar study can be carried out for equation (21). An important difference with respect to (20) is that the counterpart of (22), namely

$$G(\xi) = \frac{\xi}{1 - e^{-2\xi}} = \frac{z_0 M\bar{\lambda}}{\hbar^2}, \quad (25)$$

has a removable singularity for $\mathcal{E}_z^{(1)} \to 0$ and (see [4] for more details) the inverse function $G^{-1}(\xi)$ is defined only on $[1, +\infty)$. Thus, the following existence condition, present in [1] with $M = 1/2$,

$$2Mz_0\bar{\lambda} > \hbar^2 \quad (26)$$
must be satisfied. It is interesting to point out that, for a given choice of the parameters \( \{\lambda, z_0\} \), the existence condition (26) imposes a minimal mass in order to be fulfilled. This means that particles with mass violating the (26) cannot be in the excited states, hence they can only live in the ground state.

In a similar manner as for (23) and (24), we obtain \( E_z^{(1)} \rightarrow 0 \) for \( z_0 \rightarrow 0^+ \), and \( E_z^{(1)} \rightarrow -\frac{M^2 \lambda^2}{2\hbar^2} \) for \( z_0 \rightarrow +\infty \). Due to (26), the bound state energy in the limit for \( z_0 \rightarrow +\infty \), can be obtained at a given finite value for \( \lambda \) only for massless particles with \( M \sim 1/z_0 \). Moreover, we have \( \frac{dE_z^{(1)}}{dz_0} < 0 \) and \( E_z^{(1)} \) is a decreasing function of \( z_0 \).

Summarizing, for the excited state we have \( E_z^{(1)} \in (-\frac{M^2 \lambda^2}{2\hbar^2}, 0) \). For \( z_0 \rightarrow +\infty \) we obtain \( E_z^{(1)} \rightarrow E_z^{(0)} \) with \( E_z^{(1)} > E_z^{(0)} \).

In order to get the total spectrum of the trapped particle, we must study the solutions of (16). The aforementioned equation is nothing else but the one giving the Bessel’s functions \( \psi(q) \sim J_\nu(q) \) of first kind with non-integer order \( \nu = n/B \). The expression for the solution of (16) is given in terms of the series expansion

\[
J_\nu(q) = \left( \frac{q}{2} \right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j! (j+\nu+1)} \left( \frac{q}{2} \right)^{2j},
\]

with \( \Gamma(s) = \int_0^\infty e^{-t} t^{s-1} dt \). For \( \nu \notin \mathbb{N} \), the functions \( J_\nu(q) \) and \( J_{-\nu}(q) \) are linearly independent. We choose the regular solutions \( J_\nu(q) \) with \( \nu = n/B \geq 0 \) and hence we set \( n \in \mathbb{N} \). In order to impose the Dirichelet boundary condition \( \psi(\rho = \rho_0 = R) = 0 \rightarrow J_\nu(\rho_0) = 0 \), we need an explicit expression for \( J_\nu(q) \). From (27), the following expansions for \( q << 1 \) and \( q >> 1 \), with \( \nu \geq 0 \) can be obtained [28]:

\[
J_\nu(q) \rightarrow \frac{1}{\Gamma(\nu+1)} \left( \frac{q}{2} \right)^2, \text{ as } q << 1, \quad (28)
\]

\[
J_\nu(q) \rightarrow \sqrt{\frac{2}{\pi q}} \cos \left( q - \frac{\pi \nu}{2} - \frac{\pi}{4} \right), \text{ as } q >> 1.
\]

The transition from the behavior for small \( q \) and large \( q \) does happen when \( q \simeq \nu \). In practice, [29] is a sufficient approximation also for \( q \geq \nu \). As a consequence of these results, we can obtain suitable approximations for the zeros \( q_{\nu m} \) of the Bessel’s functions with \( J_\nu(q_{\nu m}) = 0 \) and \( m \in \mathbb{N} \). In fact, for any \( \nu \leq 2 \), zeros are approximatively separated by \( \pi \) (see [28]):

\[
E_{Bnm} \simeq \frac{\hbar^2 (\alpha_\nu + m\pi)^2}{2MR^2}, \text{ } m \in \mathbb{N}.
\]
For example, choosing $\alpha_0 = 2.405, \alpha_1 = 3.832, \alpha_2 = 5.136, \alpha_\nu \in [2.405, 5.136]$ for $\nu \in [0, 2]$. For $\nu > 2$, a suitable approximation for the roots of the equation $J_\nu(q(R)) = 0$ is given by:

$$q_{nm} = \frac{\pi n}{2B} + \pi m + \frac{3\pi}{4}, \quad (31)$$

with

$$E_{Bnm} = E_{\nu m} \simeq \frac{\pi^2 \hbar^2}{2MR^2} \left( \frac{n}{2B} + m + \frac{3}{4} \right)^2. \quad (32)$$

Note that, for $B = 1$ or $\nu \in \mathbb{N}$, the Bessel’s expressions with integer order are regained, while for $\nu = n + 1/2$ we obtain the spherical Bessel’s functions. In any case, (32) gives rise to an acceptable approximation also for the quantum states $\nu \{0, 1, 2\}$ and, as a consequence, the expression (32) provides a sufficient expression for our purposes. Hence, as a consequence of the results on the admissible values for $E^{(0)}_z$ and $E^{(1)}_z$, we get:

$$E_{t nm} = E_{\nu m} - \frac{H(z_0, \lambda) M \lambda^2}{\hbar^2}, \quad (33)$$

with $H(z_0, \lambda) \in (1/2, 2)$ for the ground state energy $E^{(0)}_z$ and $H(z_0, \lambda) \in (0,1/2)$ for the excited state energy $E^{(1)}_z$. The spectrum given by (33) is obviously discrete with quantum numbers $\{n, m\} \in \mathbb{N}$ and physical parameters $\{R, M, \lambda, z_0, B\}$. The bound states are the ones with $E_{t nm} < 0$. It is physically relevant to study possible configurations separating bound states with $E_{t nm} < 0$ from the ones with $E_{t nm} > 0$. To this purpose, note that, at any fixed quantum state with quantum numbers $\{n, m\}$ and with any choice of the parameters $\{M, \lambda, z_0, B\}$, there exists a radius $R_{nm}$ for the trapping cylinder such that for $R = R_{nm}$ we obtain $E_{t nm} = 0$:

$$R_{nm} = \frac{\pi \hbar^2}{M \lambda \sqrt{2H(z_0, \lambda)}} \left( \frac{n}{2B} + m + \frac{3}{4} \right). \quad (34)$$

Equality (34) shows interesting physical consequences. By fixing the physical parameters $\{M, \lambda, z_0, B\}$ and the integers $\{n, m\}$, with $\{n, m\} = \{\pi, \bar{m}\}$, it is always possible to 'build' the trapping cylinder with radius $R_{nm} = \overline{R}_{nm}$. From (33) we deduce that particles in quantum states, with quantum numbers $\{n, m\}$, satisfying the inequality

$$\left( \frac{n}{2B} + m \right) < \left( \frac{\pi}{2B} + \bar{m} \right), \quad (35)$$

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are in bound states with $E_{tnm} < 0$, while the ones satisfying the opposite inequality are in states with $E_{tnm} > 0$ and finally, the ones with quantum numbers $\{n, m\} = \{\overline{n}, \overline{m}\}$, have $E_{tnm} = 0$. In practice, apart from the case with $\{n, m\} = \{0, 0\}$, where $E_{tnm} > 0$, $\forall\{n, m\} \neq \{0, 0\}$, in a more general framework, with $\{n, m\} \neq \{0, 0\}$, we can observe states with $E_{tnm} \geq 0$ and bound states with $E_{tnm} < 0$.

Concerning the role of $B$, depicting the topological defect, it is worth pointing out that from (34), by fixing the all other physical parameters, $R_{nm}$ is a monotonic decreasing function of $B$. Hence, for $B \in (0, 1)$, suitable with positive string-density tension in a general relativistic context, the value of $R_{nm}$ is greater than the one in the case with $B \in (1, +\infty)$, with a negative string-density tension, suitable in the framework of nanophysics, electrons in a finite metal, semiconductors or graphene modeling. This fact is in agreement with physical intuition, since we expect that a negative string-density tension favors bound states with $E_{tnm} < 0$.

Our model could be also applied in the context of graphene modeling [22]. In fact, by placing the two planes $z = \pm z_0$, in such a way that $z_0 << 1$, we can mimic a graphene with double layer, with the string, where $B > 1$ denoting a possible defect.

In any case, this paper represents the first step, in order to study nanophysics systems in terms of curved backgrounds used in general relativity.

5 Conclusions and final remarks

In this note we studied the discrete spectrum arising for a mass particle trapped in an infinitely long cylinder with delta-interactions along the planes $z = \pm z_0$, with a topological defect (cosmic string) along the $z-$axis. In this framework, the paper provides a first attempt to apply the technique in [1]-[8] to a curved background with the Laplace-Beltrami operator. The physical effects shown by a cosmic string are very similar to the well-known Aharonov-Bohm effect [16] in quantum mechanics, since the string determines a deviation on the trajectory of a particle, despite the locally flat character of the manifold.

With the value of the deficit angle $B \in (0, 1)$, we have a string with a positive energy density tension; hence, this kind of modeling could be of interest in the context of general relativity and cosmology, where topological defects are expected to play a role in the primordial history of our universe (primordial inflation). Conversely, for $B \in (1, +\infty)$, we observe an energy momentum tensor $T_{\mu\nu}$, with a negative energy-density tension on the $z-$axis. A neg-
ative tension for the string can be used in the framework of [22]. In this article we have placed our attention on a model representing a mass particle trapped inside an infinitely long cylinder with a topological defect depicted by a string. The resulting Hamiltonian has been perturbed by delta potentials lying on the parallel planes $z = \pm z_0$, so that it is also separable with respect to the cylindrical coordinates $\{\rho, z, \phi\}$. Concerning the treatment of the part of the Hamiltonian depending on the $z$ coordinate, we used the machinery developed in [1],[2] and in particular, we have adapted the results present in [4], to write down the spectrum for the two bound states, due to the delta perturbations at $z = \pm z_0$. For the spectrum resulting from the Hamiltonian depending on the coordinates $\rho,z$, we have exploited the Bessel’s functions $J_\nu(q)$, with non-integer order, further we have obtained a suitable approximating formula for the zeros of $J_\nu(q)$. The joint spectrum has thus been studied. An interesting consequence of our model is the general presence of a mixing between discrete states with positive total energy, together with bound states and ones with zero energy. As pointed out at the end of section 4, such a model can be of interest, for example in the framework of nanophysics and physics of graphene. Specifically, the model presented in this article could provide a way to model double-layer graphene, given the great interest it has drawn in the last two decades. Moreover, we can consider particles with a given mass $M$. Following the result of section 4, and in particular due to (34), by fixing the radius $R_{nm}$ to $R_{nm} = R_{00}$, and by physical transformations, we are able to recast bosons in the state $\{n,m\} = \{0,0\}$. In this case, the total energy is zero. This situation can be observed also in a generic excited state provided by:

$$\left(\frac{n}{2B} + m\right) = \left(\frac{\pi}{2B} + \overline{m}\right).$$

(36)

In some sense, this configuration could represent an analogous realization of the Bose-Einstein condensation (BEC), also for excited states as proposed in [29], by using a generalization of the Gibbs distribution [30].

The link between our approach and nanophysics will be matter of further investigations, in particular by adding different potentials to the model.

As a final consideration, our model could be also of interest to mimic, in some sense, a black hole. In this regard, note that, the part of the spectrum due to the hard wall at $\rho = R$, is very similar to the one obtained in [24] depicting gravitons inside the black hole.
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