su(1, 1) algebraic approach of the Dirac equation with Coulomb-type scalar and vector potentials in $D + 1$ dimensions

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Abstract – We study the Dirac equation with Coulomb-type vector and scalar potentials in $D + 1$ dimensions from an $su(1, 1)$ algebraic approach. The generators of this algebra are constructed by using the Schrödinger factorization. The theory of unitary representations for the $su(1, 1)$ Lie algebra allows us to obtain the energy spectrum and the supersymmetric ground state. For the cases where there exists either scalar or vector potential our results are reduced to those obtained by analytical techniques.

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Introduction. – As is well known the factorization methods are very important to study quantum systems [1–3], since they are the basis for obtaining the energy spectrum and the eigenfunctions in an algebraic way [3].

The fundamental ideas of factorization in quantum physics were settled by Dirac [1] and Schrödinger [2]. However, Infeld and Hull [3] were the first to introduce a systematic method to factorize and classify a large class of potentials. Moreover, for several problems it has been shown that factorization operators are directly related to the supersymmetric charges [4–7] introduced by Witten [8].

On the other hand, constants of motion for a given physical problem allow to simplify the corresponding Hamiltonian by reducing its degrees of freedom. Moreover, such conserved quantities are directly associated with symmetry groups and compact and non-compact Lie algebras [9]. Such algebras play a central role in studying many properties of quantum systems because, among other things, they are the basis for selection rules that forbid the existence of certain states and processes. There is not a unified way to obtain the explicit form of the generators of compact and non-compact algebras, but several to find them, as it is shown in [10,11]. However, in refs. [12–15] attempts to systematize the construction of the $su(1, 1)$ Lie algebra generators from factorization methods have been reported. Recently, in a series of papers it has been shown that the Schrödinger factorization operators can be used to construct the $su(1, 1)$ algebra generators for relativistic and non-relativistic central potential Hamiltonians [16–18]. These results enhance the importance of factorization methods in solving quantum systems.

The three-dimensional relativistic Kepler-Coulomb potential is one of the solvable problems in physics. This is due to the conservation of the total angular momentum and the Dirac and Lippmann-Johnson operators [19], which reduces to the Runge-Lenz vector in the non-relativistic limit. The first two are due to the existence of spin, and the latter gives account of the degeneracy in the eigenvalues of the Dirac operator of the energy spectrum. Symmetries and SUSY QM are intimately related. In fact, it has been shown that supersymmetry is generated by the Lippmann-Johnson operator [19]. In ref. [16] we have studied the relativistic Kepler-Coulomb problem from an algebraic approach by using the Schrödinger factorization to construct the $su(1, 1)$ algebra generators. This problem admits to be treated in other ways: analytical [20–25] and factorization

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methods [26–28], shape-invariance [26], SUSY QM for the first-[20,27] and second-order differential equations [28], two-variable realizations of the $su(1,1)$ Lie algebra [29] and using the Biedenharn-Temple operator [30].

In [31] Joseph studied the $1/r$ potential in $D + 1$ dimensions by means of the self-adjoint operators. Recently, the energy spectrum and the eigenfunctions of this problem were obtained by solving the confluent hypergeometric equation [32,33]. Moreover, in [34] the Johnson-Lippmann operator for this potential has been constructed and used to generate the SUSY charges.

The Dirac equation with Coulomb-type vector and scalar potentials in three dimensions has been solved by using SUSY QM [35] and by an analytical approach [36]. In [37], this problem has been solved in $D$-dimensions by constructing the angular eigenfunctions from the group theory and the radial equations solutions were expressed in terms of confluent hypergeometric functions.

In the present letter, we study the Dirac equation with Coulomb-type vector and scalar potentials in $D + 1$ dimensions from an $su(1,1)$ algebraic approach. We obtain the uncoupled second-order differential equations for bound states satisfied by the radial components and by applying the Schrödinger factorization we construct the corresponding $su(1,1)$ algebra generators. We use the theory of unitary representations to obtain the energy spectrum and the action of the $su(1,1)$ algebra generators on the radial eigenstates. Also, we find the Schrödinger and SUSY ground states. By particularizing our results to the cases where there exists either scalar or vector potential, or where both potentials are equal, we show that our treatment successfully reproduce the results obtained by analytical techniques. Finally, we give some concluding remarks.

The relativistic Dirac equation in $D + 1$ dimensions. – The Dirac equation in general dimensions is $H\Psi \equiv \left( \sum_{a=1}^{D} \alpha_a p_a + \beta (m + V_1(r) + V_2(r)) \right) \Psi = i \frac{\partial \Psi}{\partial t}$, with $\hbar = c = 1$, $m$ is the mass of the particle, $p_a = -i \partial_a = -i \frac{\partial}{\partial x^a}$, $1 \leq a \leq D$, $V_1$ and $V_2$ are the spherically symmetric scalar and vector potentials, respectively. Moreover, $\alpha_a$ and $\beta$ satisfy the relations $\alpha_a \alpha_a + \alpha_s \alpha_s = 2 \delta_{ab}, \alpha_a \beta + \beta \alpha_a = 0$ and $\alpha_s = 1$ [37]. In $D$ spatial dimensions the orbital angular momentum operators $L_{ab}$, the spinor operators $S_{ab}$ and the total angular momentum operators $J_{ab}$ are defined as:

$$L_{ab} = -L_{ba} = i \sum_{a,b} \alpha_a \frac{\partial}{\partial x^b} \alpha_b - i \sum_{a,b} \beta \frac{\partial}{\partial x^b} \beta_b,$$

$$S_{ab} = -S_{ba} = \frac{i}{2} \sum_{a,b} \alpha_a \alpha_b$$

and $J_{ab} = L_{ab} + S_{ab}$, respectively. Thus, $L^2 = \sum_{a \neq b} \sum_{a \neq b} \alpha_a \alpha_b$, $S^2 = \sum_{a \neq b} \sum_{a \neq b} \alpha_a \alpha_b$, $J^2 = \sum_{a \neq b} \sum_{a \neq b} \alpha_a \alpha_b$, with $1 \leq a \leq b \leq D$. Hence, for a spherically symmetric potential, the total angular momentum operator $J_{ab}$ and the spin-orbit operator $K_D = -\beta (J_J^2 - L^2 - S^2 + (D - 1)/2)$ commute with the Dirac Hamiltonian, $H$. For a given total angular momentum $j$, the eigenvalues of $K_D$ are $\kappa_D = \pm (j + (D - 2)/2)$, where the minus sign is for aligned spin $j = \ell + 1/2$, and the plus sign is for unaligned spin $j = \ell - 1/2$ [37].

We propose the wave function of the Dirac Hamiltonian $H$ to be of the form

$$\Psi(r,t) = r^{-\frac{D-1}{2}} \left( G^{(1)}_{\kappa_D}(r) Y^\ell_{jm}(\Omega_D) \right) e^{-iEt},$$

being $G^D_{\kappa_D}(r)$ the radial functions, $Y^\ell_{jm}(\Omega_D)$ and $Y^\ell_{jm}(\Omega_D)$ the hyperspherical harmonic functions coupled with the total angular momentum quantum number $j$ and $E$ the energy. Thus, the Dirac equation leads to the radial equation

$$\frac{dG^{(1)}_{\kappa_D}}{dr} + \frac{\kappa_D - \alpha m}{r} G^{(1)}_{\kappa_D} - \frac{\kappa_D + \alpha m}{r} V_1 G^{(1)}_{\kappa_D} = e^{-iEt}.$$  

In this work, we consider the Coulomb-type scalar and vector potentials $V_1 = -\frac{\alpha}{r}$, $V_2 = -\frac{\alpha}{r}$, with $\alpha$ and $\alpha'$ positive constants. In the following sections we study separately the cases $\alpha \neq \alpha'$ and $\alpha = \alpha'$.

Case $\alpha \neq \alpha'$. – By introducing the new variable

$$\rho = (\alpha \alpha' + \alpha m) r / \sqrt{\alpha^2 - \alpha'^2},$$

eq. (2) can be written as

$$A^+ F^{(2)}_{\kappa_D} = -\text{sgn}(\alpha - \alpha') \frac{1}{\gamma} \sqrt{\frac{\alpha + \alpha'}{\alpha' - \alpha}} \left( \kappa_D - \gamma + \alpha \alpha' \right) F^{(1)}_{\kappa_D},$$

where

$$A^\pm = \pm \frac{d}{dr} - \frac{\kappa_D}{\rho} + \frac{\sqrt{\alpha^2 - \alpha'^2}}{\gamma},$$

$$\left( F^{(1)}_{\kappa_D} \right) = \left[ \begin{array}{c} \kappa_D + \gamma & (\alpha - \alpha') \\ -\alpha + \alpha' & \kappa_D + \gamma \end{array} \right],$$

$$\left( F^{(2)}_{\kappa_D} \right) = \left[ \begin{array}{c} \kappa_D + \gamma & (\alpha + \alpha') \\ -\alpha + \alpha' & \kappa_D + \gamma \end{array} \right].$$

and

$$\text{sgn}(\alpha - \alpha') = \begin{cases} 1, & \text{if } \alpha > \alpha' \\ -1, & \text{if } \alpha < \alpha'. \end{cases}$$

From eqs. (4) and (5) we obtain the uncoupled second-order differential equations

$$-\frac{d^2}{dr^2} + \frac{\gamma (\gamma + 1)}{\rho^2} + \frac{\alpha^2 - \alpha'^2}{\gamma^2} - 2 \sqrt{\frac{\alpha^2 - \alpha'^2}{\gamma^2}} F^{(1,2)}_{\kappa_D} = \text{sgn}(\alpha - \alpha') \frac{1}{\gamma} \left( \kappa_D - \gamma \left( \frac{\alpha m + \alpha'}{\alpha \alpha' + \alpha m} \right) \right) F^{(1,2)}_{\kappa_D},$$

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apply the Schrödinger factorization \([2,18]\) to the left-hand side of eq. (9), as written
\[
\left( -\rho^2 \frac{d^2}{d\rho^2} + \xi^2 \rho^2 - 2 \sqrt{\alpha^2 - \alpha^2 |\rho|} \right) F^{(1)}_{k_D} = -\gamma (\gamma + 1) F^{(1)}_{k_D},
\]
(10)
with
\[
\xi^2 = \text{sgn}(\alpha - \alpha') \left[ \left( \frac{\alpha' E + \alpha m}{\alpha E + \alpha' m} \right)^2 - 1 \right].
\]

It must be noted that eq. (10) is formally obtained from eq. (11) by performing the change \(\gamma \rightarrow \gamma + 1\). Hence, by defining \(\psi_\gamma \equiv F^{(2)}_{k_D}\), it implies that \(F^{(1)}_{k_D} \propto \psi_{\gamma+1}\). Thus, the solution for the Dirac equation in spinorial form can be expressed as \(\Phi_{k_D} \equiv (F^{(1)}_{k_D} F^{(2)}_{k_D}) = (\psi_{\gamma+1})\).

The \(su(1,1)\) Lie algebra and the energy spectrum.

In order to construct the \(su(1,1)\) algebra generators, we apply the Schrödinger factorization \([2,18]\) to the left-hand side of eq. (11). Thus, we propose
\[
\left( \frac{d}{d\rho} + a\rho + b \right) \left( -\rho \frac{d}{d\rho} + cp + f \right) \psi_\gamma = g \psi_\gamma,
\]
(13)
where \(a, b, c, f, g\) are constants and \(g\) are fixed. This is because the change \(\gamma \rightarrow \gamma + 1\) leaves \(\gamma\) fixed. This is because the change \(\gamma \rightarrow \gamma + 1\) leaves \(\gamma\) fixed.

Expanding these expressions and comparing it with eq. (11) we obtain \(a = c = \pm \xi, f = 1 + b = \mp \sqrt{\alpha^2 - \alpha^2 |\rho|} \) and \(g = b(b+1) - \gamma(\gamma - 1)\). Therefore, the differential equation satisfied by the lower component of the spinor \(\Phi_{k_D}\) is factorized as
\[
(L_+ + 1) L_\pm \psi_\gamma = \left[ \left( \frac{\sqrt{\alpha^2 - \alpha^2 |\rho|}}{\alpha} \pm 1 \right) \frac{1}{2} - \left( \frac{\gamma - 1}{2} \right) \right] \psi_\gamma,
\]
(14)
where
\[
L_\pm = \mp \rho \frac{d}{d\rho} + \xi \rho - \sqrt{\alpha^2 - \alpha^2 |\rho|} \xi
\]
are the Schrödinger operators.

Also, we define the new pair of operators
\[
\Pi_\pm \equiv \mp \rho \frac{d}{d\rho} + \xi \rho - \Pi_3,
\]
(16)
where
\[
\Pi_3 \psi_\gamma \equiv \frac{1}{2\xi} \left( -\rho \frac{d^2}{d\rho^2} + \xi^2 \rho + \gamma (\gamma - 1) \right) \psi_\gamma = \psi_\gamma
\]
(17)
has been obtained from eq. (11).

A direct calculation shows that operators \(\Pi_\pm\) and \(\Pi_3\) close the \(su(1,1)\) Lie algebra
\[
[\Pi_\pm, \Pi_3] = \mp \Pi_\pm, \quad [\Pi_+, \Pi_-] = -2\Pi_3.
\]
(18)

Therefore, from eqs. (16) and (17) we find that the corresponding quadratic Casimir operator \(\Pi^2 = -\Pi_+ \Pi_- + (\Pi_3)^2 - \Pi_3\), satisfies the eigenvalue equation
\[
\Pi^2 \psi_\gamma = \gamma (\gamma - 1) \psi_\gamma,
\]
(19)
As we mentioned above, by performing the change \(\gamma \rightarrow \gamma + 1\) to \(\psi_\gamma\), we obtain the upper component of the spinor \(\Phi_{k_D}, \psi_{\gamma+1}\). Thus, the \(su(1,1)\) algebra generators for this component are
\[
\Gamma_3 \equiv \frac{1}{2\xi} \left( -\rho \frac{d^2}{d\rho^2} + \xi^2 \rho + \gamma (\gamma + 1) \right),
\]
(20)
\[
\Gamma_\pm \equiv \mp \rho \frac{d}{d\rho} + \xi \rho - \Gamma_3.
\]
(21)
In order to determine the properties of the operators \(\Pi_\pm\) and \(\Pi_3\), we introduce the inner product on the Hilbert space spanned by the radial eigenfunctions for the Dirac equation with scalar and vector potentials in \(D\) dimensions [38]
\[
(\phi, \zeta) \equiv \int_0^\infty \phi^*(\rho) \zeta(\rho) \rho^{-1} d\rho.
\]
(22)
Thus, the operator \(\Pi_3\) is Hermitian with respect to this scalar product. Moreover, using eqs. (16) and (22), it can be proved that operators \(\Pi_\pm\) are Hermitian conjugates, \(\Pi_\pm = \Pi_\mp^\dagger\).

The theory of unitary irreducible representations of the \(su(1,1)\) Lie algebra has been studied in several works [38,39] and it is based on the equations
\[
T_2 [\mu \nu] = \mu (\mu + 1) |\mu \nu],
\]
(23)
\[
T_3 [\mu \nu] = |\nu \mu],
\]
(24)
\[
T_\pm [\mu \nu] = (|\nu \mp \mu) (|\nu \pm \mu) + 1)^{1/2} |\mu \nu \pm 1],
\]
(25)
where \(T_2\) is the quadratic Casimir operator, \(\nu = \mu + q + 1, q = 0, 1, 2, \ldots\) and \(\mu > -1\). From eq. (25) it can be noted that \(T_+ (T_-)\) is the raising (lowering) operator for \(\nu\).

Thus, from eqs. (19) and (23), and (17) and (24), we find that
\[
\mu_\gamma = \gamma - 1,
\]
(26)
\[
\nu_\gamma = n_\gamma + \gamma = \sqrt{\alpha^2 - \alpha^2 |\rho|} \xi,
\]
(27)
respectively, with \(n_\gamma = 0, 1, 2, \ldots\).

Since the operators \(\Pi_\pm\) leave fixed the quantum number \(\mu_\gamma\), eq. (26) ensures that the values of \(\gamma\) remain fixed. This is because the change \(\nu_\gamma \rightarrow \nu_\gamma + 1\) induced by the operators \(\Pi_\pm\) on the basis vectors \(|\mu_\gamma \nu_\gamma\rangle\) implies
\( n_\gamma \to n_\gamma \pm 1 \). Thus, by setting |\( \nu_\gamma \rangle \rangle \to \psi_\gamma^n \) and from eqs. (25), (26) and (27), we find that

\[
\Pi \pm \psi_\gamma^n = C_{\pm}^{(n, \gamma, \gamma)} \psi_\gamma^n \pm \frac{1}{2},
\]

with

\[
C_{\pm}^{(n, \gamma, \gamma)} = [(n_\gamma + \gamma \mp \gamma \pm 1)(n_\gamma + \gamma \pm \gamma)]^{1/2}
\]

a real number.

By using eqs. (12) and (27) we find that the energy spectrum for the lower component of the spinor \( \Phi_{\gamma \rho} \), \( \psi_{\gamma + 1} \), is

\[
E_{\gamma + 1}/m = [-\alpha \mp \xi \sqrt{\zeta^2 + \alpha^2 - (\alpha')^2}]/(\alpha^2 + \zeta^2),
\]

where \( \zeta \equiv \gamma + n_{\gamma + 1} \).

If we perform the change \( \gamma \to \gamma + 1 \) in eq. (28), we find that the action of the ladder operators \( \Gamma_{\pm} \) on the functions \( \psi_{\gamma + 1} \) is

\[
\Gamma_{\pm} \psi_{\gamma + 1} = C_{\pm}^{(n, \gamma + 1, \gamma + 1)} \psi_{\gamma + 1} \pm \frac{1}{2},
\]

where the energy spectrum for the upper component of the spinor \( \Phi_{\gamma \rho} \) is

\[
E_{\gamma + 1}/m = [-\alpha \mp \xi \sqrt{\zeta^2 + \alpha^2 - (\alpha')^2}]/(\alpha^2 + \zeta^2),
\]

where \( \zeta \equiv \gamma + n_{\gamma + 1} + 1 \). However, since \( \psi_{\gamma + 1} \) and \( \psi_{\gamma + 1} \) are the components of the same spinor, they should have the same energy, \( E_\gamma = E_{\gamma + 1} \). Thus, we obtain \( n \equiv n_\gamma = n_{\gamma + 1} + 1 \), where \( n = 0, 1, 2, 3, \ldots \) is the radial quantum number.

Therefore, the energy spectrum for the Dirac Hamiltonian with scalar and vector potentials in \( D \) dimensions can be written as

\[
E = m \left[ \frac{-\alpha \mp \alpha \sqrt{\zeta^2 + \alpha^2 - (\alpha')^2}}{\alpha^2 + (\gamma + n)^2} \right]
\]

\[\pm \left[ \frac{\alpha \mp \alpha \sqrt{\zeta^2 + \alpha^2 - (\alpha')^2}}{\alpha^2 + (\gamma + n)^2} \right] - \frac{\alpha^2 - (\gamma + n)^2}{\alpha^2 + (\gamma + n)^2} \right],
\]

which is the same reported in [37], obtained from an analytical point of view. Moreover, this expression is reduced to that for the three-dimensional case [36].

In this way we have shown that \( \Phi_{\gamma \rho} \) is given by

\[
\Phi_{\gamma \rho}^n = \left( F_{0}^{1n} \right)_{\gamma \rho} = \left( \psi_{\gamma + 1}^{n-1} \right)_{\gamma \rho} \psi_{\gamma}^n.
\]

The relation between the components of the spinor via the radial quantum number \( n \), deduced from the theory of unitary representation can be observed from an analytical approach, as it is shown below.

**The Schrödinger and SUSY QM ground states.** From eq. (28), for \( n = 0 \) we find that only the state \( \psi_\gamma^0 \) is normalizable with respect to the inner product defined in expression (22) and satisfies the differential equation \( \Pi \psi_\gamma^0 = 0 \). Moreover, for \( n = 0 \) and from the definition of the radial quantum number, we find that \( n_{\gamma + 1} = -1 \). For this value, \( C_{(-1, \gamma + 1)}^{(n, \gamma, \gamma)} \) results in a complex number. From the theory of unitary representation and eq. (29), the function \( \psi_{\gamma + 1}^{-1} \) is non-normalizable [39]. Therefore, this function is not a physically acceptable solution and the spinor corresponding to \( n = 0 \) is

\[
\Phi_{\gamma \rho}^0 = \left( \begin{array}{c} 0 \\ \rho \gamma e^{-\sqrt{\alpha \gamma^2 - \alpha \gamma^2}} \end{array} \right),
\]

which we denote as the Schrödinger ground state.

In fact, the operators \( \Lambda^\pm \) in eqs. (4) and (5) are the SUSY operators, from which the partner Hamiltonians are given by \( H_+ = A^- A^+ \) and \( H_- = A^+ A^- \). The ground state for \( H_- \) satisfies the condition \( A^- \psi_{\gamma \rho}^0 \) SUSY = 0, that leads to the square-integrable eigenfunction \( \psi_{\gamma \rho}^0 \) SUSY = \( \rho \gamma e^{-\sqrt{\alpha \gamma^2 - \alpha \gamma^2}} \). On the other hand, the solution for the equation \( A^+ \psi_{\gamma + 1}^0 \) SUSY = 0 is \( \psi_{\gamma + 1}^0 \) SUSY = \( -\rho \gamma e^{-\sqrt{\alpha \gamma^2 - \alpha \gamma^2}} \). Since this function is not a square-integrable then, it is not a physically acceptable solution and it must be taken as the zero function. Hence, the SUSY ground-state is given by

\[
\Phi_{\gamma \rho}^\gamma \text{ SUSY} = \left( \begin{array}{c} 0 \\ \rho \gamma e^{-\sqrt{\alpha \gamma^2 - \alpha \gamma^2}} \end{array} \right).
\]

The above results lead us to the conclusion that the Schrödinger and SUSY ground states for the relativistic Dirac equation with Coulomb-type scalar and vector potentials are the same.

The explicit form of \( \Phi_{\gamma \rho}^n \), corresponding to higher-energy levels can be obtained from an analytical approach. In order to solve the differential equations (11) we propose

\[
F_{1}^{(2)}_{\gamma \rho} = \rho \gamma e^{-\xi \rho} \int \text{d}(\rho - \xi \rho) f(\rho),
\]

where \( f(\rho) \) must satisfy

\[
\left[ \frac{\text{d}^2}{\text{d}y^2} + (2\gamma - y) \frac{\text{d}}{\text{d}y} + \frac{\sqrt{\alpha^2 - \alpha^2}}{\xi^2} - \gamma \right] f(y/2) = 0,
\]

which is the same reported in [37], obtained from an analytical point of view. Moreover, this expression is reduced to that for the three-dimensional case [36].

In this way we have shown that \( \Phi_{\gamma \rho}^n \) is given by

\[
\Phi_{\gamma \rho}^n = \left( F_{1}^{(1)}_{\gamma \rho} \right) = \left( \psi_{\gamma + 1}^{n-1} \right)_{\gamma \rho} \psi_{\gamma}^n.
\]

The relation between the components of the spinor via the radial quantum number \( n \), deduced from the theory of unitary representation can be observed from an analytical approach, as it is shown below.

\[
\Pi \psi_{\gamma}^n \propto \psi_{\gamma + 1}^{n \pm 1},
\]

\[
\Pi \psi_{\gamma + 1}^{n - 1} \propto \psi_{\gamma + 1}^{n \pm 1}.
\]

It is worth to notice that eq. (35) is valid for any value of the radial quantum number except for \( n = 0 \). This is because the upper component of the spinor for \( n = 1 \) can not be obtained from the action of the operator \( \Gamma_+ \) on the upper component of the Schrödinger ground state, eq. (32).
Table 1: It is shown the expressions for γ and ξ² for particular values of α and α′ in D or three spatial dimensions.

| Cases | Spatial dimensions | Potential parameters | γ | ξ² |
|-------|-------------------|----------------------|---|----|
| I     | D                 | α ≠ 0, α′ = 0       | √κ_D² - α² | (m/E)² - 1 |
| II    | D                 | α′ ≠ 0, α = 0       | √κ_D² + α² | 1 - (E/m)² |
| III   | 3                 | α′ ≠ 0, α ≠ 0       | √κ₃² + α² - α² | sgn(α - α′) [α'E + α'm]² - 1 |
| IV    | 3                 | α ≠ 0, α′ = 0       | √κ₃² - α² | (m/E)² - 1 |
| V     | 3                 | α′ ≠ 0, α = 0       | √κ₃² + α² | 1 - (E/m)² |

Equations (17) and (35) allow us to show that the action of the Schrödinger operators on the states ψ_α grid is obtained from the operators Γ_D. Therefore, we conclude that the action of the Schrödinger operators on the components of the spinor Φ_D is to change only the radial quantum number n leaving fixed the Dirac quantum number κ_D = κ_D(γ).

Some particular cases of our general results obtained above are shown in table 1.

From an analytical point of view, case I was studied in [32], meanwhile cases III, IV and V were treated in [36]. On the other hand, case III was studied by SUSY QM in [35] and case IV was treated algebraically in [16]. It must be pointed out that our present results, which were obtained from an su(1,1) algebraic approach, are in full agreement with those reported in the last references.

**Case α = α′.** – Because of the change of variable given in (3), α = α′ is not an special case of the results of the previous section. For the present case, from the Dirac radial equation (2), we obtain the uncoupled second-order differential equations

\[
\left( -\rho^2 \frac{d^2}{d\rho^2} + \xi^2 \rho^2 - 2a\rho \right) F^{(1)}_{κ_D} = -κ_D (κ_D + 1) F^{(1)}_{κ_D},
\]

\[
\left( -\rho^2 \frac{d^2}{d\rho^2} + \xi^2 \rho^2 - 2a\rho \right) F^{(2)}_{κ_D} = -κ_D (κ_D - 1) F^{(2)}_{κ_D},
\]

where ξ² = (m-E)/m+E, ρ = (m+E)r, and

\[
\begin{pmatrix}
F^{(1)}_{κ_D} \\
F^{(2)}_{κ_D}
\end{pmatrix} = \begin{pmatrix}
κ_D & 0 \\
-α & κ_D
\end{pmatrix}
\begin{pmatrix}
F^{(1)}_{κ_D} \\
F^{(2)}_{κ_D}
\end{pmatrix}.
\]

Since eq. (36) is formally obtained from eq. (37) by performing the change κ_D → κ_D + 1 then, by defining ψ_{κ_D} = F^{(2)}_{κ_D}, we conclude that the solution to the Dirac equation in spinorial form is

\[
Φ_{κ_D} = \begin{pmatrix}
F^{(1)}_{κ_D} \\
F^{(2)}_{κ_D}
\end{pmatrix} = \begin{pmatrix}
ψ_{κ_D} \\
ψ_{κ_D + 1}
\end{pmatrix}.
\]

Proceeding in a similar way as in the case α ≠ α′, we obtain that the su(1,1) algebra generators for the upper and lower components of Φ_{κ_D} are

\[
χ_3 = χ_3(κ_D) \equiv \frac{1}{2ξ} \left( -ρ^2 \frac{d^2}{dρ^2} + ξ^2 ρ + κ_D(κ_D - 1) \right),
\]

and

\[
Λ_3 = χ_3(κ_D + 1), \quad Λ₋ = χ₋(κ_D + 1),
\]

respectively, with χ₃ψ_{κ_D} = (α/ξ)ψ_{κ_D} and Λ₃ψ_{κ_D+1} = (α/ξ)ψ_{κ_D+1}.

From the theory of unitary representations, eqs. (23), (24) and (25), we obtain that the corresponding energy spectrum is

\[
E = m \left[ 1 - \frac{2n^2}{α² + (n + |κ_D|)^²} \right].
\]

Also, we find that the Schrödinger and SUSY ground states are the same and are given by

\[
Ψ^{0}_{SU(2), SU(1)} = \begin{pmatrix}
0 \\
\rho^{κ_D} e^{-αρ/κ_D}
\end{pmatrix}, \quad \text{for} \quad κ_D > 0,
\]

and

\[
Ψ^{0}_{SU(2), SU(1)} = \begin{pmatrix}
0 \\
0
\end{pmatrix}, \quad \text{for} \quad κ_D < 0.
\]

These results are in accordance to those reported in [35] for three spatial dimensions.

**Concluding remarks.** – We studied the radial Dirac equations with Coulomb-type scalar and vector potentials in D + 1 dimensions from an su(1,1) algebraic approach. With the Schrödinger factorization we were able to construct two sets of su(1,1) algebra generators. We applied the theory of unitary representations for this algebra to find the general form of the spinor wave function, the energy spectrum and the SUSY ground state.
We proved that Schrödinger and SUSY ground states are the same. We showed that the action of the Schrödinger operators on the radial eigenstates is to change only the radial quantum number \( n \) leaving fixed the Dirac quantum number \( \kappa_D \). The non-compactness of the \( su(1,1) \) algebra reflects that, for a fixed number \( \kappa_D \), the quantum number \( n \) is bounded from below and unbounded from above. We tested our results by matching different cases where there exists either scalar or vector potentials in \( D \) or three spatial dimensions \([16,32,35–37]\). To our knowledge this is the first time where the Dirac equation in \( D \) dimensions with both scalar and vector potential has been treated by an \( su(1,1) \) algebraic approach.

Finally, our technique can be successfully applied to solve other relativistic problems like the Coulomb field with position-dependent mass \([40] \) or the Dirac-Morse \([41] \), which are works in progress.

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