NORMAL FORMS OF REAL HYPERSURFACES WITH NONDEGENERATE LEVI FORM

WON K. PARK

ANY COMMENTS, SUGGESTIONS, ERRORS TO WONNIEPARK@POSTECH.AC.KR

Abstract. We present a proof of the existence and uniqueness theorem of a normalizing biholomorphic mapping to Chern-Moser normal form. The explicit form of the equation of a chain on a real hyperquadric is obtained. There exists a family of normal forms of real hypersurfaces including Chern-Moser normal form.

0. Introduction

Let $M$ be an analytic real hypersurface with nondegenerate Levi form in a complex manifold and $p$ be a point on $M$. Then it is known that there is a local coordinate system $z^1, z^2, \ldots, z^n, z^{n+1} \equiv w = u + iv$ with center at $p$, where $M$ is locally defined by the equation

$$v = \langle z, z \rangle + \sum_{s,t \geq 2} F_{st}(z, \bar{z}, u),$$

where

1. $\langle z, z \rangle \equiv z^1\bar{z}^1 + \cdots + z^e\bar{z}^e - z^{e+1}\bar{z}^{e+1} - \cdots - z^n\bar{z}^n$ for a positive integer $e$ in $\frac{n}{2} < e \leq n$,
2. $F_{st}(z, \bar{z}, u)$ is a real-analytic function of $z, u$ for each pair $(s, t) \in \mathbb{N}^2$, which satisfies

$$F_{st}(\mu z, \nu \bar{z}, u) = \mu^s \nu^t F_{st}(z, \bar{z}, u),$$

for all complex numbers $\mu, \nu$,
3. the functions $F_{22}, F_{23}, F_{33}$ satisfy the following conditions:

$$\Delta F_{22} = \Delta^2 F_{23} = \Delta^3 F_{33} = 0,$$

where

$$\Delta \equiv D_1^2 + \cdots + D_e^2 - D_{e+1} D_{e+1} - \cdots - D_n D_n,$$

$$D_k = \frac{\partial}{\partial z^k}, \quad \bar{D}_k = \frac{\partial}{\partial \bar{z}^k}, \quad k = 1, \ldots, n.$$

The local coordinate system (0.1) is called normal coordinate. The existence of a normal coordinate is a natural consequence of the following existence theorem of a normalizing biholomorphic mapping to Chern-Moser normal form.

E-MAIL: WONPKAR@EUCLID.POSTECH.AC.KR
MATHEMATICS SUBJECT CLASSIFICATION (1991): PRIMARY:32H99
KEY WORDS AND PHRASES: NORMAL FORMS, CHAINS, NORMALIZATIONS.
Theorem 0.1 (Chern-Moser). Let $M$ be an analytic real hypersurface with nondegenerate Levi form at the origin in $\mathbb{C}^{n+1}$ defined by the following equation:

$$v = F(z, \bar{z}, u), \quad F|_0 = dF|_0 = 0.$$ 

Then there is a biholomorphic mapping $\phi$ such that

$$\phi(M): v = \langle z, z \rangle + \sum_{s,t \geq 2} F^*_{st}(z, \bar{z}, u)$$

where

$$\Delta F^*_{22} = \Delta^2 F^*_{23} = \Delta^3 F^*_{33} = 0.$$ 

We shall modify the proof given by Chern and Moser on the existence theorem (cf. [CM]) in order that the proof itself yields the uniqueness theorem as well:

Theorem 0.2 (Chern-Moser). Let $M$ be a nondegenerate analytic real hypersurface in Theorem 0.1. Then the normalization $\phi = (f, g)$ in $\mathbb{C}^n \times \mathbb{C}$ is uniquely determined by the value

$$\frac{\partial f}{\partial z}|_0, \quad \frac{\partial f}{\partial w}|_0, \quad \Re \left( \frac{\partial g}{\partial w} \right)|_0, \quad \Re \left( \frac{\partial^2 g}{\partial w^2} \right)|_0.$$

An analytic curve on a nondegenerate real hypersurface $M$ is called a chain if it can be straightened by a normalization. We carefully examine the equation of a chain on $M$ so that, in particular, we obtain the explicit form of the equation of a chain $\gamma$ on a real hyperquadric:

$$\gamma: \left\{ \begin{array}{l} z = p(\mu) \\ w = \mu + i\langle p(\mu), p(\mu) \rangle \end{array} \right.$$ 

where the function $p(\mu)$ is a solution of the following ordinary differential equation (cf. [P])

$$p'' = \frac{2i(p'(p', p') (1 + 3i(p, p') - i\langle p', p' \rangle))}{(1 + i(p, p') - i\langle p', p' \rangle) (1 + 2i(p, p') - 2i\langle p', p' \rangle)}.$$ 

Normalizations of a real hypersurface $M$ to Chern-Moser normal form is parameterized by a finite dimensional group $H$ given by

$$\begin{pmatrix} \rho & 0 & 0 \\ -Ca & C & 0 \\ -r - i\langle a, a \rangle & 2ia^\dagger & 1 \end{pmatrix}$$

where

$$a^\dagger = (a^1, \ldots, a^r, -a^{r+1}, \ldots, -a^n).$$

The family of normalization of $M$ shall depend analytically on the parameters

$$C = \left( \frac{\partial f}{\partial z} \right)_0, \quad -Ca = \left( \frac{\partial f}{\partial w} \right)_0, \quad \rho = \Re \left( \frac{\partial g}{\partial w} \right)_0, \quad 2\rho r = \Re \left( \frac{\partial^2 g}{\partial w^2} \right)_0.$$ 

We shall show that there is a family of normal forms such that

$$v = \langle z, z \rangle + \sum_{s,t \geq 2} F_{st}(z, \bar{z}, u)$$

for $\alpha = 0$

$$v = -\frac{1}{2\alpha} \ln \{1 - 2\alpha \langle z, z \rangle\} + \sum_{s,t \geq 2} F_{st}(z, \bar{z}, u)$$

for $\alpha \neq 0$. 

where $\alpha \in \mathbb{R}$ and
\[
\Delta F_{22}(z, \overline{z}, u) = \Delta^2 F_{23}(z, \overline{z}, u) = 0 \\
\Delta^3 F_{33}(z, \overline{z}, u) = \beta \Delta^4 (F_{22}(z, \overline{z}, u))^2 \text{ for some } \beta \in \mathbb{R}.
\]
Normalization of a real hypersurface to any normal form among this family is parametrized by the group $H$.

1. Existence and Uniqueness Theorem

Let $M$ be an analytic real hypersurface defined near the origin by
\[v = F(z, \overline{z}, u), \quad F|_0 = 0\]
and $\Gamma : [-1, 1] \to M$ be an analytic real curve passing through the origin. Then the equation of $\Gamma$ is given as follows:
\[
\Gamma : \begin{cases} 
z = p(\mu), \\
w = q(\mu),
\end{cases}
\]
where $p(0) = q(0) = 0$. Since $\Gamma$ is obviously tangent to $M$ at the origin, we have the equality
\[
q'(0) = \left(1 + i \frac{\partial F}{\partial u}|_0\right) \Re q'(0) + i \left(\frac{\partial F}{\partial z^\alpha}|_0 p^{\alpha'}(0) + \frac{\partial F}{\partial \overline{z}^\alpha}|_0 \overline{p}^{\alpha'}(0)\right).
\]
Then $\Gamma$ is transversal to the complex tangent hyperplane of $M$ at the origin if and only if the following condition holds
\[
\Gamma, \frac{d}{dt} \notin \ker \partial \left\{\frac{w - \overline{w}}{2i} - F(z, \overline{z}, \frac{w + \overline{w}}{2})\right\}.
\]
Then the necessary and sufficient condition for the transversality is given by the inequality
\[
\frac{\partial F}{\partial z^\alpha}|_0 p^{\alpha'}(0) \neq \frac{1}{2i} \left(1 - i \frac{\partial F}{\partial u}|_0\right) q'(0).
\]
Thus, under the condition $F|_z|_0 = F|_\overline{z}|_0 = 0$, the transversality of $\Gamma$ to the complex tangent hyperplane of $M$ at the origin is equivalent to the inequality
\[
\Re q'(0) = \left\{1 + \left(\frac{\partial F}{\partial u}|_0\right)^2\right\}^{-1} q'(0) \neq 0.
\]
Hence we suppose that $M$ is an analytic real hypersurface defined by
\[v = F(z, \overline{z}, u),\]
where
\[F|_0 = F|_z|_0 = F|\overline{z}|_0 = 0.\]
Let $\Gamma$ be an analytic real curve on $M$ transversal to the complex tangent hyperplane at the origin. Then the equation of $\Gamma$ is uniquely given with a distinguished parameterization $\mu$ as follows:
\[
\Gamma : \begin{cases} 
z = p(\mu) \\
w = \mu + iF(p(\mu), \overline{p}(\mu), \mu)
\end{cases}.
\]
Lemma 1.1. Let $g(z, w)$ be a holomorphic function implicitly defined by the equations:

$$g(z, w) - g(0, w) = -2iF(p(w), \bar{p}(w), w) + 2iF \left( z + p(w), \bar{p}(w), w + \frac{1}{2}(g(z, w) - g(0, w)) \right),$$

(1.3)

$$g(0, w) = iF(p(w), \bar{p}(w), w).$$

Let $\phi$ be a biholomorphic mapping near the origin defined by

$$z = z^* + p(u^*),$$

$$w = w^* + g(z^*, w^*).$$

(1.4)

Then the mapping $\phi$ transforms the real hypersurface $M$ such that $M' \equiv \phi(M)$ is locally defined by an equation of the following form

$$v^* = \sum_{\min(s, t) \geq 1} F^*_s(z^*, \bar{z}^*, u^*)$$

and the curve $\Gamma$ on $M$ via the equation (1.2) is mapped on the $u$-curve, $z = v = 0$.

Note that the holomorphic function $g(z, w)$ is well defined because of the condition

$$F|_0 = F_z|_0 = F_{\bar{z}}|_0 = 0,$$

which implies

$$g|_0 = \frac{\partial g}{\partial z}|_0 = \Re \left( \frac{\partial g}{\partial w}|_0 \right) = 0.$$

Further, the mapping (1.4) is bijective at the origin. Hence the mapping (1.4) is biholomorphic near the origin for any analytic function $p(u)$ such that $p(0) = 0$.

Suppose that the transformed real hypersurface $M'$ is defined by

$$v^* = F^*(z^*, \bar{z}^*, u^*).$$

Then the mapping (1.4) yields the following equality:

$$g(z, u) - g(0, u) = 2iF \left( z + p(u), \bar{p}(u), u + \frac{1}{2}(g(z, u) + \overline{g(0, u)}) \right).$$

(1.5)

where

$$z = z^* + p(u^* + iv^*),$$

$$\bar{z} = \bar{z}^* + \bar{p}(u^* - iv^*),$$

$$u = u^* + \frac{1}{2}(g(z^*, w^* + iv^*) + \overline{g(z^*, u^* - iv^*)}).$$

Since $F$ and $F^*$ are real analytic, we can consider $z^*, \bar{z}^*, u^*$ as independent variables. Hence the condition of $F^*(z^*, 0, u^*) = v^* = 0$ is equivalent via the equality (1.5) to the following equality:

$$g(z, u) - \overline{g(0, u)} = 2iF \left( z + p(u), \bar{p}(u), u + \frac{1}{2}(g(z, u) + \overline{g(0, u)}) \right).$$

(1.6)

Taking $z = 0$ yields

$$g(0, u) - \overline{g(0, u)} = 2iF \left( p(u), \bar{p}(u), u + \frac{1}{2}(g(0, u) + \overline{g(0, u)}) \right).$$

(1.7)
Thus we easily see that

\[(1.8) \quad g(0, u) + \overline{g(0, u)} = 0\]

if and only if

\[g(0, u) = iF(p(u), \overline{p}(u), u).\]

Let \(\Gamma\) be a curve on \(M\) defined by a function \(p(u)\) via the equation \(1.2\). Then the mapping \((1.4)\) maps the curve \(\Gamma\) onto the \(u\)-curve in \(\mathbb{C}^{n+1}\) if and only if the condition \(1.8\) on \(g(z, w)\) is satisfied.

By requiring the condition \(1.8\), the equality \(1.6\) reduces to

\[g(z, u) - g(0, u) = -2iF(p(u), \overline{p}(u), u) + 2iF(z + p(u), \overline{p}(u), u + \frac{1}{2}\{g(z, u) - g(0, u)\}).\]

Thus the equalities \(1.6\) and \(1.7\) are satisfied by the function \(g(z, w)\) defined in the equalities \(1.3\). This completes the proof of Lemma 1.1.

Note that the mapping \((1.4)\) in Lemma 1.1 is completely determined by the analytic function \(p(u)\). From the equality \((1.3)\), we obtain the expansion of the holomorphic function \(g(z, w)\) as a power series of \(z\) up to order 3 inclusive as follows:

\[g(z, w) = iF(p(w), \overline{p}(w), w) + 2i(1 - iF')^{-1}\{F_\alpha z^\alpha + F_\alpha z^\alpha z^\beta + F_\alpha z^\alpha z^\beta z^\gamma\} - 2(1 - iF')^{-2}\{F_\alpha z^\alpha F_\beta z^\beta + F_\alpha z^\alpha F_\beta z^\beta F_\gamma z^\gamma + F_\alpha z^\alpha z^\beta F_\gamma z^\gamma\} - 2i(1 - iF')^{-3}\{F_\alpha z^\alpha (F_\beta z^\beta z^\gamma)^2 + 2F_\alpha z^\gamma F_\beta z^\gamma z^\gamma z^\gamma F'' + (F_\alpha z^\gamma)^2 F'' + (F_\alpha z^\gamma)^2 z^\beta z^\gamma z^\gamma F'' + 2(1 - iF')^{-4}\{3(F_\alpha z^\gamma)^2 F_\beta z^\beta F'' + (F_\alpha z^\gamma)^3 \cdot F'''\} + 4i(1 - iF')^{-5}(F_\alpha z^\gamma)^3 (F'')^2 + O(z^4)\]

\[(1.9)\]
where

\[
F_\alpha = \sum_{\alpha} \left( \frac{\partial F}{\partial z^\alpha} \right) (p(w), \overline{p}(w), w),
\]

\[
F' = \left( \frac{\partial F}{\partial u} \right) (p(w), \overline{p}(w), w),
\]

\[
F_{\alpha\beta} = \frac{1}{2} \sum_{\alpha,\beta} \left( \frac{\partial^2 F}{\partial z^\alpha \partial z^\beta} \right) (p(w), \overline{p}(w), w),
\]

\[
F'' = \left( \frac{\partial^2 F}{\partial u^2} \right) (p(w), \overline{p}(w), w),
\]

\[
F_{\alpha\beta\gamma} = \frac{1}{6} \sum_{\alpha,\beta,\gamma} \left( \frac{\partial^3 F}{\partial z^\alpha \partial z^\beta \partial z^\gamma} \right) (p(w), \overline{p}(w), w),
\]

\[
F'_{\alpha\beta} = \frac{1}{2} \sum_{\alpha,\beta} \left( \frac{\partial^3 F}{\partial z^\alpha \partial z^\beta \partial u} \right) (p(w), \overline{p}(w), w),
\]

\[
F'' = \left( \frac{\partial^3 F}{\partial u^3} \right) (p(w), \overline{p}(w), w).
\]

By Lemma 1.1, we have the following condition on the real hypersurface \( M' \):

\[ v = O(z^3). \]

Thus it suffices to obtain terms up to \( v^2 \) inclusive in order that we compute the functions

\[ F^s_t(z, \overline{z}, u) \]

of \( M' \) up to the type \((s, t), s + t \leq 5 \) inclusive.

We obtain the expansions of \( p^\alpha (u + iv) \) and \( \tilde{p}^\beta (u + iv) \) as power series of \( v \) as follows:

\[
p^\alpha (u + iv) = p^\alpha + p^\alpha' \cdot iv + p^\alpha'' \cdot \frac{(iv)^2}{2} + O(v^3)
\]

\[
\tilde{p}^\beta (u + iv) = \tilde{p}^\beta + \tilde{p}^\beta' \cdot iv + \tilde{p}^\beta'' \cdot \frac{(iv)^2}{2} + O(v^3).
\]
By using this expansion, we expand the holomorphic function \( g(z, w) \) as a power series of \( z \) and \( v \) in (1.9) as follows:

\[
g(z, w) = \sum_{k,l=0}^{\infty} g_k^{(l)}(z, u) \frac{(iv)^l}{l!}
\]

\[
= iF(p(u), \overline{\theta}(u), u) + g_0'(0, u)iv + g_0''(0, u) \frac{(iv)^2}{2}
\]

\[
+ g_1(z, u) + g_1'(z, u)iv + g_1''(z, u) \frac{(iv)^2}{2}
\]

\[
+ g_2(z, u) + g_2'(z, u)iv + g_3(z, u)
\]

\[
+ O(z^4) + O(z^3v) + O(z^2v^2) + O(zv^3) + O(v^3)
\]

where

\[
g_k^{(l)}(\mu z, u) = \mu^k g_k^{(l)}(z, u)
\]

for all complex number \( \mu \).

We easily see that the function \( g_k^{(l)}(z, u) \) depends analytically on the functions \( p(u) \) and \( \overline{\theta}(u) \), polynomially on the derivatives of \( p(u) \) and \( \overline{\theta}(u) \) up to order \( l \) inclusive such that the order sum of the derivatives in each term of \( g_k^{(l)}(z, u) \) is less than or equal to the integer \( l \). In low order terms, we obtain

\[
g_0(0, u) = iF(p(u), \overline{\theta}(u), u) = O(u)
\]

\[
g_0'(0, u) = iF_{p\alpha} + iF_{\overline{\theta}\beta} + iF' = O(1)
\]

\[
g_0''(0, u) = iF_{p\alpha\beta} + iF_{\overline{\theta}\beta\gamma} + 2iF_{\alpha\beta\gamma} + 2iF_{\overline{\theta}\alpha\beta\gamma} + 2iF_{\overline{\theta}\beta\gamma} + 2iF''
\]

and

\[
g_1(z, u) = 2i(1 - F')^{-1} F_{\alpha\beta z^\alpha} = O(zu)
\]

\[
g_1'(z, u) = 2i(1 - F')^{-1} \left\{ 2F_{\alpha\beta z^\alpha} p^{\beta\gamma} + F_{\overline{\theta}\alpha\beta z^\alpha} \overline{\theta}^{\beta\gamma} + F_{\alpha z^\alpha} \right\}
\]

\[
- 2(1 - F')^{-1} \left\{ F_{\alpha\beta} + F_{\overline{\theta}\alpha \beta}\overline{\theta}^{\beta} + 2F'' \right\} F_{\alpha z^\alpha}
\]

\[
= 4i F_{\alpha\beta} \mid_{\alpha \beta = 0} z^\alpha p^{\beta\gamma} + 2i F_{\alpha \beta} \mid_{\alpha \beta = 0} z^\alpha \overline{\theta}^{\beta\gamma} + O(zu)
\]

\[
g_2(z, u) = 2i(1 - F')^{-1} F_{\alpha\beta z^\alpha} z^\beta - 2(1 - F')^{-2} F_{\alpha z^\alpha} F'_{\beta} z^\beta
\]

\[
- 2i(1 - F')^{-3} (F_{\alpha z^\alpha})^2 F''
\]

\[
= 2i F_{\alpha\beta} \mid_{\alpha \beta = 0} z^\alpha z^\beta + O(z^2 u)
\]

The real hypersurface \( M' \) is defined by the following equation:

\[
v = F\left( z + p(u + iv), \overline{z} + \overline{p}(u - iv), u + \frac{1}{2} \left\{ g(z, u + iv) + \overline{g}(\overline{z}, u - iv) \right\} \right)
\]

(1.10) \( - \frac{1}{2i} \left\{ g(z, u + iv) - \overline{g}(\overline{z}, u - iv) \right\} . \)
We expand the right hand side of the equation (1.10) in low order terms of $v$ as follows:

$$
F\left(z + p(u + iv), \bar{z} + \bar{p}(u - iv), u + \frac{1}{2}\{g(z, u + iv) + \bar{g}(\bar{z}, u - iv)\}\right) - \frac{1}{2i}\{g(z, u + iv) - \bar{g}(\bar{z}, u - iv)\}
= A(z, \bar{z}, u) + vB(z, \bar{z}, u) + v^2C(z, \bar{z}, u) + O(v^3).
$$

Then we obtain

$$
A(z, \bar{z}, u) = F(z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) - \Im g(z, u)
$$

$$
B(z, \bar{z}, u) = iF_\alpha (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) p^\alpha (u)
- iF_\beta (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) p^\beta (u)
- F' (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) \Im g'(z, u)
- \Re g'(z, u)
$$

$$
C(z, \bar{z}, u) = -F_{\alpha \beta} (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) p^\alpha p^\beta (u)
+ F_{\alpha \beta} (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) p^\alpha p^\beta (u)
- F_{\alpha \bar{\beta}} (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) p^\alpha \bar{p}^\beta (u)
- iF'_{\alpha} (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) p^\alpha (u) \Im g'(z, u)
+ iF'_{\beta} (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) \bar{p}^\beta (u) \Im g'(z, u)
+ F'' (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) (\Im g'(z, u))^2
- \frac{1}{2} F'_{\alpha} (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) p^\alpha (u)
- \frac{1}{2} F'_{\bar{\beta}} (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) \bar{p}^\beta (u)
- \frac{1}{2} F'' (z + p(u), \bar{z} + \bar{p}(u), u + \Re g(z, u)) \Re g''(z, u)
+ \frac{1}{2} \Re g''(z, u)
$$

where

$$
g'(z, u) = \left(\frac{\partial g}{\partial w}\right)(z, u)
$$

$$
g''(z, u) = \left(\frac{\partial^2 g}{\partial w^2}\right)(z, u).
$$

We decompose the functions $A(z, \bar{z}, u), B(z, \bar{z}, u), C(z, \bar{z}, u)$ as follows:

$$
A(z, \bar{z}, u) = \sum_{\min(s, t) \geq 1} A_{st}(z, \bar{z}, u),
$$

$$
B(z, \bar{z}, u) = \sum_{\min(s, t) \geq 0} B_{st}(z, \bar{z}, u),
$$

$$
C(z, \bar{z}, u) = \sum_{\min(s, t) \geq 0} C_{st}(z, \bar{z}, u).
$$

We easily see the following facts:
(1) $A_{st}(z, \overline{z}, u)$ depends analytically on the functions $p(u)$ and $\overline{p}(u)$.

(2) $B_{st}(z, \overline{z}, u)$ depends analytically on the function $p(u)$ and $\overline{p}(u)$, at most linearly on the derivative $p'(u)$ and $\overline{p}'(u)$.

(3) $C_{st}(z, \overline{z}, u)$ depends analytically on the function $p(u)$ and $\overline{p}(u)$, at most quadratically on the derivative $p''(u)$ and $\overline{p}''(u)$, at most linearly on the derivative $p''(u)$ and $\overline{p}''(u)$ such that the derivative order sum of the derivatives of $p(u)$ and $\overline{p}(u)$ in each term is less than or equal to 2.

Lemma 1.2. The functions $C_{0t}(z, \overline{z}, u)$, $t \in \mathbb{N}$, do not depend on the derivative $p''(u)$ and $\overline{p}''(u)$.

This claim is easily verified by observing that the following functions

$$C(z, \overline{z}, u) \quad \text{and} \quad -\frac{1}{2} \left( \frac{\partial^2 A}{\partial u^2} \right)(z, \overline{z}, u)$$

depend in the same manner on the derivative $p''(u)$ and $\overline{p}''(u)$, because

$$A(z, \overline{z}, u + iv) = A(z, \overline{z}, u) + iv \left( \frac{\partial A}{\partial u} \right)(z, \overline{z}, u) - \frac{v^2}{2} \left( \frac{\partial^2 A}{\partial u^2} \right)(z, \overline{z}, u) + \cdots.$$ 

By the way, $A(z, 0, u) = 0$ is the defining equation of the function $g(z, u)$ in Lemma 1.1 with $g(0, u) = iF(p(u), \overline{p}(u), u)$. Thus we have the following identities

$$\left( \frac{\partial A}{\partial u} \right)(z, 0, u) = \left( \frac{\partial^2 A}{\partial u^2} \right)(z, 0, u) = \cdots = 0.$$

Note that the identity

$$\left( \frac{\partial^2 A}{\partial u^2} \right)(z, 0, u) = 0$$

gives the desired relation between the terms having $p''$ and $\overline{p}''$ and the terms not having $p''$ and $\overline{p}''$ so that we verify the function $C(z, 0, u)$ is independent of the derivative $p''(u)$ and $\overline{p}''(u)$. This proves the claim in Lemma 1.2.

Explicitly, we compute the expansion of the right hand side of the equation (1.10) in low order terms so that

$$v = A_{11}(z, \overline{z}, u) + A_{22}(z, \overline{z}, u) + A_{12}(z, \overline{z}, u) + A_{13}(z, \overline{z}, u) + A_{23}(z, \overline{z}, u)$$
$$+ A_{21}(z, \overline{z}, u) + A_{31}(z, \overline{z}, u) + A_{32}(z, \overline{z}, u)$$
$$+ v\{B_{00}(z, \overline{z}, u) + B_{11}(z, \overline{z}, u) + B_{01}(z, \overline{z}, u) + B_{02}(z, \overline{z}, u) + B_{12}(z, \overline{z}, u) + B_{10}(z, \overline{z}, u) + B_{20}(z, \overline{z}, u) + B_{21}(z, \overline{z}, u)\}$$
$$+ v^2\{C_{00}(z, \overline{z}, u) + C_{01}(z, \overline{z}, u) + C_{10}(z, \overline{z}, u)\}$$
$$+ O(z^4\overline{z}^4) + O(z^4\overline{z}^3) + O(z^3\overline{z}) + O(v^3)$$
$$+ \sum_{\min(s,t) \geq 1, s+t \geq 6} O(z^s\overline{z}^t) + \sum_{s+t \geq 4} O(vz^s\overline{z}^t) + \sum_{s+t \geq 2} O(v^2z^s\overline{z}^t).$$
where

\[
A_{11}(z, \zeta, u) = F_{\alpha \beta} z^\alpha \bar{z}^\beta - i (1 + i F')^{-1} F_{\alpha} z^\alpha \bar{F}_{\beta} \bar{z}^\beta \\
+ i (1 - i F')^{-1} F_{\alpha} z^\alpha \bar{F}_{\beta} \bar{z}^\beta \\
+ 2(1 + i F')^{-1} (1 - i F')^{-1} F_{\alpha} z^\alpha \bar{F}_{\beta} \bar{z}^\beta \\
= (z, z) + O(z u)
\]

\[
B_{00}(z, \zeta, u) = i (1 + i F')^{-1} F_{\alpha} p^\alpha - i (1 - i F')^{-1} F_{\beta} \bar{p}^\beta - (F')^2 \\
= O(1)
\]

\[
B_{01}(z, \zeta, u) = 2i F_{\alpha} p^\alpha z \bar{p} + 2 F'' F_{\alpha} \bar{p} + i (1 + i F') F_{\alpha} \bar{p} \\
+ 2i \left( F_{\alpha} p^\alpha + F_{\beta} \bar{p} \right) F'' (1 + i F')^{-1} F_{\beta} \bar{p}
\]

Then we obtain

\[
v = F_{11}^*(z, \zeta, u) + F_{12}^*(z, \zeta, u) + F_{13}^*(z, \zeta, u) + F_{23}^*(z, \zeta, u) \\
+ F_{21}^*(z, \zeta, u) + F_{31}^*(z, \zeta, u) + F_{32}^*(z, \zeta, u) \\
+ O(z u^4) + O(z^4 u) + \sum_{\min(s, t) \geq 1, s + t \geq 6} O(z^s u^t),
\]

where

\[
F_{11}^* = (1 - B_{00})^{-1} A_{11} \\
F_{12}^* = (1 - B_{00})^{-1} A_{12} + (1 - B_{00})^{-2} A_{11} B_{01} \\
F_{13}^* = (1 - B_{00})^{-1} A_{13} + (1 - B_{00})^{-2} (A_{11} B_{02} + A_{12} B_{01}) \\
+ (1 - B_{00})^{-3} A_{11} B_{01} \\
F_{22}^* = (1 - B_{00})^{-1} A_{22} + (1 - B_{00})^{-2} (A_{11} B_{11} + A_{12} B_{10} + A_{21} B_{01}) \\
+ (1 - B_{00})^{-3} (2 A_{11} B_{01} B_{10} + A_{11}^2 C_{00}) \\
F_{23}^* = (1 - B_{00})^{-1} A_{23} + (1 - B_{00})^{-2} (A_{11} B_{12} + A_{12} B_{11} + A_{21} B_{02} \\
+ A_{13} B_{10} + A_{22} B_{01}) \\
+ (1 - B_{00})^{-3} (2 A_{11} B_{01} B_{11} + 2 A_{11} B_{10} B_{02} + 2 A_{12} B_{01} B_{10} \\
+ A_{21} B_{01}^2 + A_{11}^2 C_{01} + 2 A_{11} A_{12} C_{00}) \\
+ 3(1 - B_{00})^{-4} (A_{11} B_{01}^2 B_{10} + A_{11}^2 B_{01} C_{00}).
\]

By Lemma 1.2, the functions $F_{22}^*, F_{23}^*$ does not depend on the derivative $p''$ and $\bar{p}'$, and the dependence of the coefficients in $F_{22}, F_{23}^*$ on the derivative $p'$ and $\bar{p}'$ is of the form:

\[
A_1(u, p, \bar{p}, p', \bar{p}') \frac{1}{(1 - B_{00})^3},
\]

and

\[
A_2(u, p, \bar{p}, p', \bar{p}') \frac{1}{(1 - B_{00})^4},
\]

where $A_1$ depends analytically on $u, p, \bar{p}$ and at most quadratically on $p', \bar{p}'$ and $A_2$ depends analytically on $u, p, \bar{p}$, at most cubically on $p', \bar{p}'$. 
For future reference, we analyze the terms containing the first order derivatives \( p', \overline{p}' \) in \( B_{00} \) and \( B_{01} \) so that
\[
O(up') + O(u\overline{p}') \quad \text{in} \quad B_{00}(z, \overline{z}, u)
\]
\[
2i(p', z) + O(\overline{u}p') + O(\overline{u}\overline{p}') \quad \text{in} \quad B_{01}(z, \overline{z}, u).
\]
Thus analyzing the terms containing the second order derivatives \( p'' \) and \( \overline{p}'' \) in
\[
\left( \frac{\partial F_{11}^*}{\partial u} \right)(z, \overline{z}, u) \quad \text{and} \quad \left( \frac{\partial F_{12}^*}{\partial u} \right)(z, \overline{z}, u),
\]
we obtain
\[
O(z\overline{u}p'') + O(z\overline{u}\overline{p}'') \quad \text{in} \quad \left( \frac{\partial F_{11}^*}{\partial u} \right)(z, \overline{z}, u)
\]
\[
2i(z, z)(p'', z) + O(z\overline{z}^2 u p'') + O(z\overline{z}^2 u \overline{p}'') \quad \text{in} \quad \left( \frac{\partial F_{12}^*}{\partial u} \right)(z, \overline{z}, u).
\]

Then we are ready to present a proof of the existence theorem for Chern-Moser normal form.

**Theorem 1.3 (Chern-Moser).** There is a biholomorphic mapping \( \phi \) which transforms \( M \) to a real hypersurface of the following form:
\[
v = \langle z, \overline{z} \rangle + \sum_{\min(s, t) \geq 2} F_{st}(z, \overline{z}, u),
\]
where
\[
\Delta^2 F_{23} = 0.
\]
Geometrically, there exists a unique analytic curve \( \Gamma \) on \( M \) which passes through the origin and is tangent to a vector transversal to the complex tangent hyperplane at the origin and which is mapped onto the \( u \)-curve by the biholomorphic mapping \( \phi \). Further, there exists a biholomorphic mapping \( \phi_1 \) which, in addition to (1.14), achieves the following conditions:
\[
\Delta F_{22} = \Delta^3 F_{33} = 0.
\]

Let \( M' \) be a real hypersurface obtained in Lemma 1.1 by the biholomorphic mapping (1.4), which is defined by the following equation:
\[
v = \sum_{\min(s, t) \geq 1} F_{st}^*(z, \overline{z}, u).
\]
Then there is a unique analytic function \( D(z, u) \) (cf. [CM]) such that
\[
F_{11}^*(z + D(z, u), \overline{z}, u) = \sum_{s \geq 1} F_{s1}^*(z, \overline{z}, u),
\]
and the function \( D(z, u) \) satisfies the condition
\[
D(0, u) = D_z(0, u) = 0.
\]
Thus \( D(z, u) \) depends analytically of \( u, p, \overline{p} \) and rationally of the derivative \( p', \overline{p}' \).

We decompose the function \( D(z, u) \) such that
\[
D(z, u) = \sum_{s \geq 2} D_s(z, u),
\]
where

\[ D_s(\mu z, u) = \mu^s D_s(z, u) \quad \text{for all } \mu \in \mathbb{C}. \]

Then the functions \( D_2(z, u), D_3(z, u) \) are given by

\[
A_{11} (D_2(z, u), \overline{z}, u) = A_{21} + (1 - B_{00})^{-1} A_{11} B_{10} \\
A_{11} (D_3(z, u), \overline{z}, u) = A_{31} + (1 - B_{00})^{-1} (A_{11} B_{20} + A_{21} B_{10}) \\
+ (1 - B_{00})^{-2} A_{11} B_{20}^2.
\]

Note that \( D_2(z, u), D_3(z, u) \) do not depend of the second order derivative \( p'', \overline{p''} \).

Then we obtain

\[
v = \sum_{\min(s, t) \geq 1} F^*_{st} (z, \overline{z}, u) = \sum_{\min(s, t) \geq 1} F^*_{11} \left( z, D(z, u), \overline{D(z, u)}, u \right) + \sum_{\min(s, t) \geq 2} G_{st} (z, \overline{z}, u).
\]

We notice

\[
G_{22} (z, \overline{z}, u) = F^*_{22} (z, \overline{z}, u) - F^*_{11} \left( D_2(z, u), \overline{D_2(z, u)}, u \right) \\
G_{23} (z, \overline{z}, u) = F^*_{23} (z, \overline{z}, u) - F^*_{11} \left( D_2(z, u), \overline{D_3(z, u)}, u \right).
\]

We easily see that the functions \( G_{22}, G_{23} \) depend on \( u, p, \overline{p}, p', \overline{p}' \) in the same form as respectively in (1.11) and (1.12).

We take an analytic function \( E(u) \) such that

\[
F^*_{11} (z, \overline{z}, u) = (E(u)z, E(u)\overline{z}), \quad \text{and} \quad E(0) = id_{n \times n}.
\]

Note that the function \( E(u) \) is determined up to the following relation:

\[
E_1(u) = U(u)E(u),
\]

where

\[
(U(u)z, U(u)\overline{z}) = (z, \overline{z}), \quad \text{and} \quad U(0) = id_{n \times n}.
\]

Then the biholomorphic mapping defined by the following equation:

\[
z^* = E(w) \{ z + D(z, w) \}, \\
w^* = w,
\]

transforms \( M' \) to a real hypersurface of the following form:

\[
v = (z, \overline{z}) + \sum_{\min(s, t) \geq 2} H_{st}(z, \overline{z}, u).
\]

By the way, we still obtain a real hypersurface in (1.16) by a biholomorphic mapping as follows:

\[
z^* = U(w)E(w) \{ z + D(z, w) \}, \\
w^* = w,
\]

where the holomorphic function \( U(w) \) satisfy the condition (1.15).
By using the expansion
\[
E(u) = E(w) - ivE'(w) + \cdots
\]
\[
U(u) = U(w) - ivU'(w) + \cdots,
\]
we obtain
\[
v = F^*_k \left( z + D(z, u), \bar{z} + D(z, w), u \right) + \sum_{\min(s, t) \geq 2} G_{st}(z, \bar{z}, u)
\]
\[
= \langle E(u)(z + D(z, u)), E(u)(z + D(z, w)) \rangle + \sum_{\min(s, t) \geq 2} G_{st}(z, \bar{z}, u)
\]
\[
= \langle U(u)E(u)(z + D(z, u)), U(u)E(u)(z + D(z, u)) \rangle + \sum_{\min(s, t) \geq 2} G_{st}(z, \bar{z}, u)
\]
\[
= \langle U(\bar{w})E(\bar{w})(z + D(z, w)), U(\bar{w})E(\bar{w})(z + D(z, w)) \rangle
\]
\[
- iv \langle \bar{U}(\bar{w})E(\bar{w})(z + D(z, w)), \bar{U}(\bar{w})E(\bar{w})(z + D(z, w)) \rangle
\]
\[
+ iv \{ \langle E'(\bar{w})(z + D(z, w)) + \bar{E}(\bar{w})D_u(z, w) \rangle, E(w)(z + D(z, w)) \} + iv \{ \langle \bar{E}'(\bar{w})(z + D(z, w)) + E(\bar{w})D_u(z, w) \rangle, E(w)(z + D(z, w)) \}
\]
\[
\sum_{\min(s, t) \geq 2} G_{st}(z, \bar{z}, u)
\]
\[
\sum_{\min(s, t) \geq 2} G_{st}(z, \bar{z}, u).
\]
\[
(1.18)
\]
where
\[
w = u + iv,
\]
\[
U'(u) = \frac{dU}{du}(u), \quad E'(u) = \frac{dE}{du}(u),
\]
\[
D_u(z, w) = \left( \frac{\partial D}{\partial u} \right)(z, w).
\]

By introducing a holomorphic variable \( z^\natural = z + D(z, w) \), we obtain from the equation (1.18):
\[
v = \langle U(\bar{w})E(\bar{w})z^\natural, U(\bar{w})E(\bar{w})z^\natural \rangle + G_{22}(z^\natural, \bar{z}^\natural, u)
\]
\[
- i \langle E(u)z^\natural, E(u)z^\natural \rangle \{ \langle U'(u)E(u)z^\natural, U(u)E(u)z^\natural \rangle
\]
\[
- \langle U(\bar{u})E(\bar{u})z^\natural, U'(\bar{u})E(u)z^\natural \rangle
\]
\[
- i \langle E(u)z^\natural, E(u)z^\natural \rangle \{ \langle E'(u)z^\natural, E(u)z^\natural \rangle - \langle E(u)z^\natural, E'(u)z^\natural \rangle \}
\]
\[
+ G_{23}^*(z^\natural, \bar{z}^\natural, u) + G_{32}^*(z^\natural, \bar{z}^\natural, u).
\]
\[
(1.19)
\]
\[
+ \sum_{\min(s, t) \geq 2, s + t \geq 6} G_{st}(z^\natural, \bar{z}^\natural, u).
\]
where
\[
G_{23}^*(z,\bar{z},u) = G_{23}(z,\bar{z},u) + i F_{11}^*(z,\bar{z},u) F_{11}^* \left( z, \left( \frac{\partial D_2}{\partial u} \right)(z,u), u \right) \\
- \sum \beta \left( \frac{\partial G_{22}}{\partial \bar{z}^\beta} \right)(z,\bar{z},u) D_2^2(z,u) \\
= G_{23}(z,\bar{z},u) + i F_{11}^*(z,\bar{z},u) \left( \frac{\partial F_{11}^*}{\partial u} \right) (z,\bar{z},u) \\
- i F_{11}^*(z,\bar{z},u) \left( \frac{\partial F_{11}^*}{\partial u} \right) \left( z, D_2(z,u), u \right) \\
- \sum \beta \left( \frac{\partial G_{22}}{\partial \bar{z}^\beta} \right)(z,\bar{z},u) D_2^2(z,u) \\
\] (1.20)

By the equalities (1.13), the dependence of the functions \( G_{23}^* \) on the second order derivatives \( p'' \) and \( \bar{p}'' \) is given as follows
\[
-2(z,z)^2(p'',z) + O(z^2\bar{z}^3up'') + O(z^2\bar{z}^3up'') \quad \text{in} \quad G_{23}^*(z,\bar{z},u).
\]
Notice that the function \( G_{23}^*(z,\bar{z},u) \) in (1.19) and (1.20) is independent of the function \( U(u) \).

Therefore after the biholomorphic mapping in (1.17), we obtain
\[
v = \langle z^*,z^* \rangle + H_{22}(z^*,\bar{z}^*,u) + H_{23}(z^*,\bar{z}^*,u) + H_{32}(z^*,\bar{z}^*,u) + \sum_{\min(s,t) \geq 2, s+t \geq 6} H_m(z^*,\bar{z}^*,u).
\]
where
\[
H_{22}(z,\bar{z},u) = G_{22} \left( E(u)^{-1}U(u)^{-1}z, \bar{E}(u)^{-1}U(u)^{-1}z, u \right) \\
- i(z,z) \{ \langle U'(u)U(u)^{-1}z, z \rangle - \langle z, U'(u)U(u)^{-1}z \rangle \} \\
- i(z,z) \{ \langle E'(u)E(u)^{-1}U(u)^{-1}z, U(u)^{-1}z \rangle - \langle U(u)^{-1}z, E'(u)E(u)^{-1}U(u)^{-1}z \rangle \}
\]
and the dependence of \( H_{23}(z,\bar{z},u) \) on \( p'',\bar{p}'' \) is as follows:
\[
H_{23}(z,\bar{z},0) = -2(z,z)^2(p''(0),z) + \bar{K}_{23}(z,\bar{z},0;p'(0),\bar{p}'(0)).
\]

By using the following identity
\[
\Delta^2 \{ (z,z)^2(p,z) \} = 2(n+1)(n+2)(p,z),
\]
the equation \( \Delta^2 H_{23} = 0 \) is a second order ordinary differential equation
\[
A_1 p'' + A_2 \bar{p}'' = B
\]
where
1. \( A_1, A_2 \) are \( n \times n \) matrix valued functions and \( B \) is \( \mathbb{C}^n \)-valued function,
2. \( A_1 = id_{n \times n} + O(u) \) and \( A_2 = O(u) \),
3. \( A_1, A_2, B \) depend analytically of \( u, p, \bar{p} \),
4. \( A_1, A_2 \) depend at most linearly of \( p, \bar{p} \),
5. \( B \) depends at most cubically of \( p, \bar{p} \).
Then we obtain
\[
p'' = Q(u, p, \overline{p}, p', \overline{p}')
\]
(1.21)
\[
\equiv \left( A_1 - A_2 \overline{A_1} A_2 \right)^{-1} \left( B - A_2 \overline{A_1} B \right)
\]
where the function \( Q \) depends rationally on the derivatives \( p', \overline{p}' \).

Therefore there exists a unique analytic curve \( \Gamma \) on \( M \) which passes through the origin and is tangent to a vector transversal to the complex tangent hyperplane at the origin and which is mapped by a biholomorphic mapping into the \( u \)-curve.

Since \( \langle U(u)z, U(u)z \rangle = \langle z, z \rangle \), we have identities
\[
\langle U'(u)U(u)^{-1} z, z \rangle + \langle z, U'(u)U(u)^{-1} z \rangle = 0
\]
\[
\text{Tr}(U'(u)U(u)^{-1}) + \overline{\text{Tr}(U'(u)U(u)^{-1})} = 0,
\]
where
\[
\text{Tr}(A) = \text{trace of } z \to Az,
\]
Then the equation \( \Delta H_{22} = 0 \) is given as follows:
\[
\langle U(u)^{-1}U'(u)z, z \rangle + \frac{1}{2(n + 2)} \langle z, z \rangle \text{Tr}(U(u)^{-1}U'(u))
\]
\[
= \frac{1}{2i(n + 2)} \Delta G_{22} \left( E(u)^{-1}z, E(u)^{-1}z, u \right)
\]
\[
- \frac{1}{2} \left\{ \langle E'(u)E(u)^{-1}z, z \rangle - \langle z, E'(u)E(u)^{-1}z \rangle \right\}
\]
(1.22)
\[
- \frac{1}{2(n + 2)} \langle z, z \rangle \left\{ \text{Tr}(E'(u)E(u)^{-1}) - \overline{\text{Tr}(E'(u)E(u)^{-1})} \right\}.
\]

By using the following identities
\[
\Delta \{ \langle z, z \rangle \langle Az, z \rangle \} = (n + 2) \langle Az, z \rangle + \text{Tr}(A) \langle z, z \rangle
\]
\[
\Delta^2 \{ \langle z, z \rangle \langle Az, z \rangle \} = 2(n + 1) \text{Tr}(A),
\]
we obtain
\[
\langle z, z \rangle \text{Tr}(U(u)^{-1}U'(u)) = \frac{1}{4i(n + 1)} \Delta^2 G_{22} \left( E(u)^{-1}z, E(u)^{-1}z, u \right)
\]
\[
- \frac{1}{2} \left\{ \text{Tr}(E'(u)E(u)^{-1}) - \overline{\text{Tr}(E'(u)E(u)^{-1})} \right\}.
\]
Thus the equation (1.22) is a first order ordinary differential equation of \( U(u) \) as follows:
\[
\langle U(u)^{-1}U'(u)z, z \rangle
\]
\[
= \frac{1}{2i(n + 2)} \Delta G_{22} \left( E(u)^{-1}z, E(u)^{-1}z, u \right)
\]
\[
- \frac{1}{8i(n + 1)(n + 2)} \langle z, z \rangle \Delta^2 G_{22} \left( E(u)^{-1}z, E(u)^{-1}z, u \right)
\]
\[
- \frac{1}{2} \left\{ \langle E'(u)E(u)^{-1}z, z \rangle - \langle z, E'(u)E(u)^{-1}z \rangle \right\}
\]
\[
- \frac{1}{4(n + 2)} \langle z, z \rangle \left\{ \text{Tr}(E'(u)E(u)^{-1}) - \overline{\text{Tr}(E'(u)E(u)^{-1})} \right\}.
\]
Hence by requiring
\[ U(0) = E(0) = \text{id}_{n \times n}, \]
there is a unique biholomorphic mapping
\[ z^* = U(w)E(w)\{z + D(z, w)\}, \]
\[ w^* = w, \]  
(1.23)
which transforms \( M' \) to a real hypersurface of the following form:
\[ v = \langle z, z \rangle + \sum_{\min(s, t) \geq 2} H_{st}(z, \bar{z}, u) \]
where
\[ \Delta_{H_{22}} = \Delta_{H_{23}} = 0. \]

We consider the following mappings
\[ \phi_1 : \begin{cases} z = z^* + p(w^*) \\ w = w^* + g(z^*, w^*) \end{cases} \]
\[ \phi_2 : \begin{cases} z^* = E(w)(z + D(z, w)) \\ w^* = w \end{cases} \]
\[ \phi_3 : \begin{cases} z^* = \sqrt{\text{sign}\{q'(0)\}}q'(w)Uz \\ w^* = q(w) \end{cases} \]
(1.25)
where \( p(w), g(z, w), E(w), D(z, w), q(w) \) are holomorphic functions satisfying
\[ \overline{q(0, u)} = -q(0, u), \quad \overline{q'(w)} = q'(w), \]
\[ p(0) = q(0) = 0, \quad \det q'(0) \neq 0, \quad \det U \neq 0 \]
\[ E(0) = \text{id}_{n \times n}, \quad D(0, w) = D_z(0, w) = 0. \]

We easily see by parameter counting that the mapping
\[ (\phi_1, \phi_2, \phi_3) \mapsto \phi_3 \circ \phi_2 \circ \phi_1 \]
is bijective. Hence a biholomorphic mapping \( \phi, \phi|_0 = 0 \), has a unique decomposition
\[ \phi = \phi_3 \circ \phi_2 \circ \phi_1. \]

Note that \( U(w) = E(w) = \text{id}_{n \times n} \) and \( D(z, w) = 0 \) in (1.23) whenever \( M \) is already in the form (1.24). Hence, from the decomposition (1.26), we easily see that any biholomorphic mapping, preserving the form (1.24) and the \( u \)-curve, is given by
\[ z^* = \sqrt{\text{sign}\{q'(0)\}}q'(w)Uz, \]
\[ w^* = q(w), \]  
(1.27)
where
\[ \overline{q(w)} = q(\bar{w}), \quad q(0) = 0, \quad q'(0) \neq 0, \]
\[ U \in GL(n; \mathbb{C}), \quad \langle Uz, U\bar{z} \rangle = \text{sign}\{q'(0)\}\langle z, \bar{z} \rangle. \]
The mapping in (1.27) transforms the real hypersurface defined by
\[ v^* = \langle z^*, z^* \rangle + H_{22}^*(z^*, z^*, u^*) + H_{23}^*(z^*, \bar{z}^*, u^*) + H_{32}^*(z^*, z^*, u^*) + H_{33}^*(z^*, z^*, u^*) \]
\[ + \sum_{\min(s,t) \geq 2, s+t \geq 7} H_s^*(z^*, z^*, u^*) \]
to a real hypersurface as follows:
\[ v = \langle z, z \rangle + q' H_{22}^*(Uz, \overline{Uz}, q(u)) \]
\[ + q' \sqrt{|q'|} \left( H_{23}^*(Uz, \overline{Uz}, q(u)) + H_{32}^*(Uz, \overline{Uz}, q(u)) \right) \]
\[ + sq'' H_{33}^*(Uz, \overline{Uz}, q(u)) + \left\{ \frac{1}{2} \left( \frac{q''}{q'} \right)^2 - \frac{q'''}{3q'} \right\} \langle z, z \rangle^3 \]
\[ + O(z^2 \bar{z}^2) + O(z^2 \bar{z}^4) \]
\[ = \langle z, z \rangle + H_{22}(z, \bar{z}, u) + H_{23}(z, \bar{z}, u) + H_{32}(z, \bar{z}, u) + H_{33}(z, \bar{z}, u) \]
\[ + O(z^2 \bar{z}^2) + O(z^2 \bar{z}^4). \]

Hence we obtain
\[ H_{22}(z, \bar{z}, u) = q' H_{22}^*(Uz, \overline{Uz}, q(u)) \]
\[ H_{23}(z, \bar{z}, u) = q' \sqrt{|q'|} H_{23}^*(Uz, \overline{Uz}, q(u)) \]
\[ H_{33}(z, \bar{z}, u) = sq'' H_{33}^*(Uz, \overline{Uz}, q(u)) + \left\{ \frac{1}{2} \left( \frac{q''}{q'} \right)^2 - \frac{q'''}{3q'} \right\} \langle z, z \rangle^3. \]
(1.28)

Note that \( \Delta H_{22}^* = \Delta^2 H_{22}^* = 0 \) whenever \( \Delta H_{22} = \Delta^2 H_{22} = 0 \).

We can achieve the condition \( \Delta^3 H_{33} = 0 \) by a third order ordinary differential equation as follows:
\[ \frac{q'''}{3q'} - \frac{1}{2} \left( \frac{q''}{q'} \right)^2 = \kappa(u), \]
(1.29)

where
\[ \kappa(u) = - \frac{1}{6n(n + 1)(n + 2)} \cdot \Delta^3 H_{33}(z, \bar{z}, u). \]
The differential equation in (1.29) determines a projective parameter on the \( u \)-curve. This completes the proof of Theorem 1.3.

**Theorem 1.4** (Chern-Moser). Let \( M \) be a nondegenerate analytic real hypersurface defined by the equation
\[ v = F(z, \bar{z}, u) \quad F|_0 = dF|_0 = 0. \]

Then a biholomorphic normalizing mapping of \( M, \phi = (f, g) \) in \( \mathbb{C}^n \times \mathbb{C} \), is uniquely determined by the value
\[ \left. \frac{\partial f}{\partial z} \right|_0, \quad \left. \frac{\partial f}{\partial w} \right|_0, \quad \Re \left( \left. \frac{\partial g}{\partial w} \right|_0 \right), \quad \Re \left( \left. \frac{\partial^2 f}{\partial w^2} \right|_0 \right). \]

As noted above, a biholomorphic mapping \( \phi \) satisfying \( \phi|_0 = 0 \) is uniquely decomposed to
\[ \phi = \phi_3 \circ \phi_2 \circ \phi_1 \]
where \( \phi_1, \phi_2, \phi_3 \) are biholomorphic mappings in (1.25) satisfying
\[
\phi_1|_0 = \phi_2|_0 = \phi_3|_0 = 0.
\]
Note that the mapping \((\phi_1, \phi_2, \phi_3) \mapsto \phi = \phi_3 \circ \phi_2 \circ \phi_1\) is bijective. We take \( \phi \) to be a normalizing biholomorphic mapping of \( M \). Then the uniqueness of \( \phi_1, \phi_2, \phi_3 \) up to the value 
\[
\frac{\partial f}{\partial z}|_0, \quad \frac{\partial f}{\partial w}|_0, \quad \mathbb{R} \left( \frac{\partial g}{\partial w}|_0 \right), \quad \mathbb{R} \left( \frac{\partial^2 g}{\partial w^2}|_0 \right)
\]
assures the uniqueness of the normalizing mapping \( \phi \). The uniqueness of \( \phi_1, \phi_2, \phi_3 \) each is verified in the proof of Theorem 1.3 through uniquely determining the holomorphic functions 
\[
p(w), \quad E(w), \quad q(w)
\]
via the ordinary differential equations (1.21), (1.22), (1.29) by the initial values
\[
p'(0), \quad E(0) \equiv U, \quad q'(0), \quad q''(0).
\]
We may have the following relations:
\[
\sqrt{|q'(0)|} U = \frac{\partial f}{\partial z}|_0, \quad -\sqrt{|q'(0)|} U p'(0) = \left(1 - i \frac{\partial F}{\partial z}|_0 \right)^{-1} \frac{\partial f}{\partial w}|_0
\]
\[
q'(0) = \mathbb{R} \left( \frac{\partial g}{\partial w}|_0 \right), \quad 2q'(0)q''(0) = \mathbb{R} \left\{ \left(1 - i \frac{\partial F}{\partial z}|_0 \right)^{-2} \frac{\partial^2 g}{\partial w^2}|_0 \right\}.
\]
For the case \( dF|_0 = 0 \) rather than \( F_z|_0 = F_w|_0 = 0 \), we have simpler relations:
\[
\sqrt{|q'(0)|} U = \frac{\partial f}{\partial z}|_0, \quad -\sqrt{|q'(0)|} U p'(0) = \frac{\partial f}{\partial w}|_0
\]
\[
q'(0) = \mathbb{R} \left( \frac{\partial g}{\partial w}|_0 \right), \quad 2q'(0)q''(0) = \mathbb{R} \left( \frac{\partial^2 g}{\partial w^2}|_0 \right)
\]
so that the values \( p'(0), E(0) \equiv U, q'(0), q''(0) \) are uniquely determined by
\[
\frac{\partial f}{\partial z}|_0, \quad \frac{\partial f}{\partial w}|_0, \quad \mathbb{R} \left( \frac{\partial g}{\partial w}|_0 \right), \quad \mathbb{R} \left( \frac{\partial^2 g}{\partial w^2}|_0 \right).
\]
This completes the proof of Theorem 1.4.

2. Chains and Orbit Parameters

1. Let \( M \) be a nondegenerate analytic real hypersurface. Then we may define a family of distinguished curves on \( M \) via Chern-Moser normal form, which are defined alternatively and identified to be the same by E. Cartan \([Ca]\) and Chern-Moser \([CM]\). Let \( \gamma : (0, 1) \to M \) be an open connected curve. Then the curve \( \gamma \) is called a chain if, for each point \( p \in \gamma \), there exist an open neighborhood \( U \) of the point \( p \) and a biholomorphic mapping \( \phi \) on \( U \) which translates the point \( p \) to the origin and transforms \( M \) to Chern-Moser normal form such that
\[
\phi \left( U \cap \gamma \right) \subset \left\{ z = v = 0 \right\}.
\]
By Theorem 1.3 and Theorem 1.4, a chain \( \gamma \) locally exists uniquely for each vector transversal to the complex tangent plane such that \( \gamma \) is tangential to the vector.
From the proof of Theorem 1.3, we have an ordinary differential equation which locally characterizes a chain $\gamma$, passing through the origin $0 \in M$. Suppose that
\[
\gamma : \begin{cases} 
z = p(u) \\
w = u + iF(p(u), \overline{p(u)}, u)
\end{cases}
\]
Then there exists an ordinary differential equation
\[
p'' = Q(u, p, \overline{p}, p', \overline{p'})
\]
such that the function $p(u)$ is a solution of the ordinary differential equation (2.1).

We take $M$ to be the real hyperquadric $v = \langle z, z \rangle$. Then the chain $\gamma$ is locally given by
\[
\gamma : \begin{cases} 
z = p(u) \\
w = u + i\langle p(u), p(u) \rangle
\end{cases}
\]
With $F(z, \overline{z}, u) = \langle z, z \rangle$, we obtain the equation $\Delta^2 F_{23} = 0$ as follows
\[
\begin{align*}
\left(1 - i\langle p', p \rangle + i\langle p, p' \rangle\right)p'' - ip' \langle p'' , p \rangle
\end{align*}
\]
Then we easily check that the equation of chains on a real hyperquadric is given by
\[
p'' = \frac{2ip'(p', p')(1 + 3i\langle p, p' \rangle - i\langle p', p \rangle)(1 + 2i\langle p, p' \rangle - 2i\langle p', p \rangle)}{(1 + i\langle p, p' \rangle - i\langle p', p \rangle)(1 + 2i\langle p, p' \rangle - 2i\langle p', p \rangle)}.
\]

II. The isotropy subgroup of the automorphism group of a real hyperquadric $v = \langle z, z \rangle$ consists of fractional linear mappings $\phi$ such that
\[
\phi = \phi_{\sigma} : \begin{cases} 
z^* = \frac{C(z-aw)}{1+2i\langle z,a \rangle - i\langle a,a \rangle} \\
w^* = \frac{\rho w}{1+2i\langle z,a \rangle - i\langle a,a \rangle}
\end{cases}
\]
where the constants $\sigma = (C, a, \rho, r)$ satisfy
\[
a \in \mathbb{C}^n, \quad \rho \neq 0, \quad \rho, r \in \mathbb{R}, \\
C \in GL(n; \mathbb{C}), \quad \langle Cz, Cz \rangle = \rho \langle z, z \rangle.
\]
Further, $\phi$ decomposes to
\[
\phi = \psi \circ \varphi,
\]
where
\[
\psi : \begin{cases} 
z^* = \frac{z-aw}{1+2i\langle z,a \rangle - i\langle a,a \rangle} \\
w^* = \frac{w}{1+2i\langle z,a \rangle - i\langle a,a \rangle}
\end{cases}
\]
and
\[
\varphi : \begin{cases} 
z^* = \frac{Cz}{1-\rho w} \\
w^* = \frac{w}{1-\rho w}
\end{cases}
\]
Hence the local automorphisms of a real hyperquadric is identified with a group $H$ of the following matrices:
\[
\begin{pmatrix}
\rho & 0 & 0 \\
-Ca & C & 0 \\
-\rho - i\langle a, a \rangle & 2ia^\dagger & 1
\end{pmatrix}
\]
where
\[
a^\dagger = \left(\overline{a}_1, \ldots, \overline{a}_n, -a_{n+1}, \ldots, -a_m\right).
\]
We easily verify
\[
\phi_{a_d}^*(v - \langle z, z \rangle) = (v - \langle z, z \rangle)\rho(1 + \delta)^{-1}(1 + \overline{\delta})^{-1},
\]
where
\[ 1 + \delta = 1 + 2i(z, a) - (r + i(a, a))w. \]

Hence the automorphisms \( \phi_\sigma \) are normalizations of a real hyperquadric. Further, by Theorem [1.4], each normalization of a real hyperquadric is necessarily an automorphism. Then a chain \( \gamma \) on a real hyperquadric is necessarily given by
\[
\gamma = \phi^{-1}(z = v = 0)
\]
so that the chain \( \gamma \) is just an intersection of a complex line.

By Theorem [1.4] each normalization \( N = (f, g) \) is uniquely determined by the initial value
\[
C, \ a, \ \rho, \ r
\]
such that
\[
\begin{align*}
    f(z, w) &= C(z - aw) + f^*(z, w) \\
    g(z, w) &= \rho(w + rw^2) + g^*(z, w)
\end{align*}
\]
where
\[
\begin{align*}
    f^*_0 &= df^*_0 = g^*_0 = dg^*_0 = \Re(g^*_{ww} |_0) = 0. \\
\end{align*}
\]

Hence the group \( H = \{(C, a, \rho, r)\} \) parameterizes the normalizations of a real hypersurface. Further, Theorem [1.3] and Theorem [1.4] together yields a family of polynomial identities(cf. [Pa1]). Then we have showed that the group \( H \) gives a group action via normalization on the class of normalized real hypersurfaces.

III. Suppose that \( M \) is an analytic real hypersurface defined near the origin by the following equation:
\[
v = \langle z, z \rangle + \sum_{\alpha, \beta} \left( \kappa_{\alpha \beta} z^\alpha \overline{z}^\beta + \kappa_{\beta \alpha} \overline{z}^\alpha z^\beta \right) + F(z, \bar{z}, u)
\]
where
\[
F(z, \bar{z}, u) = \sum_{k=3}^\infty F_k(z, z, u).
\]
Let \( N_\sigma \) be a normalization of \( M \) with the initial value \( \sigma = (C, a, \rho, r) \in H \) and let \( \phi_{\sigma'} = \varphi \circ \psi \) be a local automorphism of a real hyperquadric(cf. (2.2), (2.3)) with the initial value \( \sigma' = (C, a, \rho, r_0) \in H \), where
\[
r_0 = r - \Re(\kappa_{\alpha \beta} a^\alpha a^\beta). \]

Then there are two decompositions of \( N_\sigma \) as follows(cf. [CM]):
\[
\begin{align*}
    &N_\sigma = E \circ \phi_{\sigma'} = E \circ \varphi \circ \psi, \\
    &N_\sigma = \varphi \circ E \circ \psi,
\end{align*}
\]
where \( E \) is the normalization with the identity initial value.
Let $M$ be a nondegenerate real hypersurface and $N_\sigma$ be a normalization of $M$ with initial value $\sigma = (C, a, \rho, r) \in H$ such that $M' \equiv N_\sigma(M)$ is defined by the equation

$$v = \langle z, z \rangle + \sum_{\min(s, t) \geq 2} F^*_s(z, \bar{z}, u)$$

where

$$\Delta F^*_2 = \Delta^2 F^*_3 = \Delta^3 F^*_3 = 0.$$ 

Note that the mapping $\varphi$ is itself a normalization in the following decomposition:

$$N_\sigma = \varphi \circ E \circ \psi$$

(2.4)

where $N_{\sigma_1}$ is a normalization of $M$ with initial value $\sigma_1 = (id_{n \times n}, a, 1, 0) \in H$.

As a consequence of the decomposition (2.4), we notice that a normalization $N_\sigma$ is analytic of

$$z, \ w, \ C, \ \rho, \ r$$

near the point $z = w = r = 0$ and $C = id_{n \times n}, \rho = 1$. More precisely,

$$N_\sigma = \left( \frac{Cf(z, w)}{1 - rg(z, w)}, \frac{\rho g(z, w)}{1 - rg(z, w)} \right)$$

where

$$N_{\sigma_1} = (f(z, w), g(z, w)) \ \sigma_1 = (id_{n \times n}, a, 1, 0).$$

Notice that the size of convergence of the normalization $N_\sigma$ at the origin is determined by the value $a, r$.

Further, suppose that the transformed real hypersurface $N_\sigma(M)$ is defined by

$$v = \langle z, z \rangle + F^*(z, \bar{z}, u).$$

Then the function $F^*(z, \bar{z}, u)$ is real-analytic of

$$z, \ w, \ C, \ \rho, \ r$$

near the point $z = w = r = 0$ and $C = id_{n \times n}, \rho = 1$.

In fact, from the proof of Theorem 1.3, we obtain

**Theorem 2.1.** Let $M$ be a nondegenerate real-analytic real hypersurface defined by

$$v = F(z, \bar{z}, u) \ \ F|_0 = F_z|_0 = F_{\bar{z}}|_0 = 0.$$ 

Then $N_\sigma$ and $F^*(z, \bar{z}, u)$ are analytic of

$$z, \ w, \ C, \ a, \ \rho, \ r.$$ 

We have a natural group action by normalizations on the class of real hypersurfaces in normal form(cf. Pa1). Then, under a natural compact-open topology(cf. Na), we obtain

**Corollary 2.2.** The local automorphism group of a nondegenerate analytic real hypersurface is a Lie group.
3. Normal forms of real hypersurfaces

I. Let $M$ be a nondegenerate analytic real hypersurface defined by the equation

$$v = F^*_1(z, \bar{z}, u) + \sum_{s,t \geq 2} F^*_{st}(z, \bar{z}, u).$$

Let’s take a matrix valued function $E(u)$ satisfying

$$F^*_1(z, \bar{z}, u) = \langle E(u) z, E(u) z \rangle.$$

Then suppose that the biholomorphic mapping

$$\phi: \begin{cases} z^* = E(w) z \\ w^* = w \end{cases}$$

transforms $M$ to another real hypersurface $\phi(M)$ defined by

$$v = F^*_1(z, \bar{z}, u) + \sum_{s,t \geq 2} G_{st}(z, \bar{z}, u).$$

Then we have the following relation

$$F^*_1 \left( E(u)^{-1} E'(u) z, \bar{z}, u \right)$$

$$= -\frac{2i}{n+2} \cdot \text{tr} G_{22}(z, \bar{z}, u) + \frac{i}{(n+1)(n+2)} \cdot (\text{tr})^2 G_{22}(z, \bar{z}, u) \cdot F^*_1(z, \bar{z}, u)$$

$$+ \frac{2i}{n+2} \cdot \text{tr} F^*_{22} \left( E_1(u)^{-1} E(u) z, E_1(u)^{-1} E(u) z, u \right)$$

$$- \frac{i}{(n+1)(n+2)} \cdot (\text{tr})^2 F^*_{22}(z, \bar{z}, u) \cdot F^*_1(z, \bar{z}, u)$$

$$+ \frac{1}{2} \left( \frac{\partial F^*_1}{\partial u} \right)(z, \bar{z}, u).$$

where $E_1(u)$ is a given matrix valued function satisfying

$$F^*_1(z, \bar{z}, u) = \langle E_1(u) z, E_1(u) z \rangle.$$

Here we have constant solutions

$$E(u) = E(0)$$

whenever

$$\text{tr} F^*_{22}(z, \bar{z}, u) = \text{const}. F^*_1(z, \bar{z}, u) = \text{tr} G_{22}(z, \bar{z}, u)$$

and

$$\left( \frac{\partial F^*_1}{\partial u} \right)(z, \bar{z}, u) = 0.$$
Suppose that a real hypersurface $M$ is defined by the equation
\[
v = \langle z, z \rangle + \sum_{s, t \geq 2} G_{st} (z, \overline{z}, u) \quad \text{for } \alpha = 0
\]
\[
v = -\frac{1}{2\alpha} \ln \{1 - 2\alpha \langle z, z \rangle\} + \sum_{s, t \geq 2} G_{st} (z, \overline{z}, u) \quad \text{for } \alpha \neq 0
\]
where
\[
\Delta G_{22} (z, \overline{z}, u) = \Delta^2 G_{23} (z, \overline{z}, u) = 0.
\]
Let $\varphi$ be a biholomorphic mapping leaving the $u$-curve invariant and preserving the form (3.1). Then $\varphi$ is necessarily given by the following mapping(cf. [Pa2]):
\[
\varphi : \begin{cases} 
  z^* = \sqrt{\text{sign} \{ q'(0) \}} \langle z, U z \rangle \exp \frac{\alpha i}{2} (q(0) - w) \\
  w^* = q(w)
\end{cases}
\]
where $\alpha \in \mathbb{R}$ and
\[
\langle U z, U z \rangle = \text{sign} \{ q'(0) \} \langle z, z \rangle.
\]
Suppose that the biholomorphic mapping $\varphi$ transforms $M$ to a real hypersurface $\varphi(M)$ defined by
\[
v = -\frac{1}{2\alpha} \ln \{1 - 2\alpha \langle z, z \rangle\} + \sum_{s, t \geq 2} G^*_{st} (z, \overline{z}, u)
\]
where
\[
\Delta G^*_{22} (z, \overline{z}, u) = \Delta^2 G^*_{23} (z, \overline{z}, u) = 0.
\]
Then we have the following relation
\[
q'(u) G^*_{22} (U z, \overline{U z}, q(u)) = G_{22} (z, \overline{z}, u)
\]
\[
q'(u) \sqrt{|q'(u)|} \exp -\frac{\alpha i}{2} (q(u) - u) G^*_{23} (U z, \overline{U z}, q(u)) = G_{23}(z, \overline{z}, u)
\]
and
\[
\frac{q'''(u)}{3q'(u)} - \frac{1}{2} \left( \frac{q''(u)}{q'(u)} \right)^2 + \frac{\alpha^2}{6} (q'(u)^2 - 1) = \frac{1}{6(n+1)(n+2)} \left\{ q'(u)^2 \Delta^3 G^*_{33} (z, \overline{z}, q(u)) - \Delta^3 G_{33} (z, \overline{z}, u) \right\}.
\]
Notice that
\[
\frac{q'''}{3q} - \frac{1}{2} \left( \frac{q''}{q} \right)^2 + \frac{\alpha^2}{6} (q'^2 - 1) = 0
\]
whenever
\[
\Delta^3 G^*_{33} (z, \overline{z}, q) \ dq^2 = \Delta^3 G_{33} (z, \overline{z}, u) \ du^2.
\]
We want to restrict the mapping $\varphi$ so that the function $q(u)$ is a solution of the ordinary differential equation (3.2). The restriction on $\varphi$ has to be achieved by requiring an additional condition on the normal form (3.1).

We claim that the following choice works
\[
\Delta^3 G_{33} (z, \overline{z}, u) = \text{const.} \Delta^4 (G_{22} (z, \overline{z}, u))^2
\]
\[
\Delta^3 G^*_{33} (z, \overline{z}, q) = \text{const.} \Delta^4 (G^*_{22} (z, \overline{z}, q))^2.
\]
Because of the relation
\[ G^*_2 (z, \bar{z}, q) \, dq = G_{22} (z, \bar{z}, u) \, du, \]
the condition (3.3) gives
\[ q'(u)^2 \Delta^3 G^*_3 (z, \bar{z}, q(u)) = \Delta^3 G_{33} (z, \bar{z}, u) \]
which yields the ordinary differential equation (3.2).

Hence we define a normal form such that
\[ v = \langle z, z \rangle + \sum_{s, t \geq 2} G_{st} (z, \bar{z}, u) \text{ for } \alpha = 0 \]
\[ v = -\frac{1}{2\alpha} \ln \{1 - 2\alpha \langle z, z \rangle\} + \sum_{s, t \geq 2} G_{st} (z, \bar{z}, u) \text{ for } \alpha \neq 0 \]
where
\[ \Delta G_{22} (z, \bar{z}, u) = \Delta^2 G_{23} (z, \bar{z}, u) = 0 \]
\[ \Delta^3 G_{33} (z, \bar{z}, u) = \beta \Delta^4 (G_{22} (z, \bar{z}, u))^2 \text{ for some } \beta \in \mathbb{R}. \]

We easily see that all normalizations associated to any normal form above are uniquely determined by some constant initial values.

Chern-Moser normal form is given in the case of \( \alpha = \beta = 0 \) so that
\[ v = \langle z, z \rangle + \sum_{s, t \geq 2} G_{st} (z, \bar{z}, u) \]
where
\[ \Delta G_{22} = \Delta^2 G_{23} = \Delta^3 G_{33} = 0. \]

Moser-Vitushkin normal form is defined by taking \( \alpha \neq 0 \) and \( \beta = 0 \) so that
\[ v = -\frac{1}{2\alpha} \ln \{1 - 2\alpha \langle z, z \rangle\} + \sum_{s, t \geq 2} G_{st} (z, \bar{z}, u) \]
where
\[ \Delta G_{22} = \Delta^2 G_{23} = \Delta^3 G_{33} = 0. \]

We shall see each normal form has its own advantage in applications (cf. [Pa2]).

II. Burns and Shnider [BS] have reported that the geometric theory of Chern and Moser [CM] gives a projective parametrization on a chain which is different from the parametrization defined by Chern-Moser normal form (cf. [BFG]). From (1.28), we obtain
\[ \frac{q'''}{3q'} - \frac{1}{2} \left( \frac{q''}{q'} \right)^2 = \frac{1}{6n(n+1)(n+2)} \left\{ q'^2 \Delta^3 H^*_{33} (z, \bar{z}, q(u)) - \Delta^3 H_{33} (z, \bar{z}, u) \right\}. \]

Note that the solution \( q(u) \) in (3.4) is given by
\[ q(u) = \frac{au + b}{cu + d}; \quad ad - bc \neq 0, \quad a, b, c, d \in \mathbb{R} \]
whenever
\[ \varepsilon \equiv \Delta^3 H_{33} (z, \bar{z}, u) du^2 = \Delta^3 H^*_{33} (z, \bar{z}, q) dq^2. \]

The mapping (1.27) effects
\[ H_{2k} (z, \bar{z}, u) = q' \left| q' \right|^{k-2} H^*_{2k} (Uz, \bar{Uz}, q(u)) \text{ for } k \geq 2. \]
Thus there are many possible choices satisfying the equalities (3.5). To reduce the possible choices, we require that the function $\varepsilon$ is independent of a choice of a chain. Clearly there exists such a function $\varepsilon$, for instance, $\varepsilon = 0$ so that

$$\Delta^3 H_{33}^*(z, \overline{z}, q(u)) = \Delta^3 H_{33}(z, \overline{z}, u) = 0.$$  

The requirement is also satisfied by a function $\varepsilon$ if we define $\varepsilon$ as follows:

$$\varepsilon \equiv \Delta^3 H_{33} du^2$$

for a constant real number $c \in \mathbb{R}$. Thus we can define a normal form similar to Chern-Moser normal form except for replacing the condition $\Delta^3 H_{33} = 0$ with

$$(3.6) \sum N_{\alpha^\beta^\gamma^\delta^\rho} = \frac{1}{3} \sum N_{\gamma^\delta} N_{\alpha^\beta} \neq 1$$

where

$$H_{22}(z, \overline{z}, u) = \sum N_{\alpha^\beta^\gamma^\delta^\rho} z^{\alpha} \overline{z}^{\beta} \overline{z}^{\gamma} \overline{z}^{\delta}$$

$$H_{23}(z, \overline{z}, u) = \sum N_{\alpha^\beta^\gamma^\delta^\rho} z^{\alpha} \overline{z}^{\beta} \overline{z}^{\gamma} \overline{z}^{\delta} \overline{z}^{\rho}$$

$$(3.6) \sum N_{\alpha^\beta^\gamma^\delta^\rho} = \frac{1}{3} \sum N_{\gamma^\delta} N_{\alpha^\beta} \neq 1$$

Then the condition (3.6) gives on a chain the parametrization of the geometric theory of Chern and Moser (cf. [Fa]).

References

[BFG] M. Beals, C. Fefferman and R. Grossman. Strictly Pseudoconvex Domains in $\mathbb{C}^n$. Bull. Amer. Math. Soc., 8, pages 125-322, 1983

[BS] D. Burns Jr. and S. Shnider. Real hypersurfaces in complex manifolds. Several complex variables Proc. Sympos. Pure math. Vol. 30, Pt. 2, Amer. math. Soc., Providence, pages 141-168, 1977.

[Ca] E. Cartan. Sur la geometrie pseudo-conforme des hypersurfaces de l’espace de deux variables complexes. I. Ann. Math. Pura Appl. (4) 11; II. Ann. Scuola Norm. Sup. Pisa, (2) 1, pages I. 17-90; II. 333-354, 1932.

[CM] S. S. Chern and J. K. Moser. Real hypersurfaces in complex manifolds. Acta math. 133, pages 219-271, 1974.

[Fa] J. J. Faran. Segre families and real hypersurfaces. PhD thesis, University of California, Berkeley, 1978.

[Na] R. Narasimhan, Several complex variables. University of Chicago Press, Chicago and London, 1971

[Pa1] W. K. Park. Umbilic points and Real hyperquadrics. To appear.

[Pa2] W. K. Park. Analytic continuation of a biholomorphic mapping. To appear.

[Pi] S. I. Pinchuk. On the analytic continuation of holomorphic mappings. Math. USSR Sbornik 27, pages 375-392, 1975.