Solution of thermoelasticity problems for solids of revolution with transversal isotropic feature and a body force

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Abstract. The paper describes a method for determining a stress-strain state of transversally isotropic solids of revolution, which are simultaneously influenced by external load and a body force within a steady-state temperature field. The final condition of the body is determined by the combination of these 3 factors, which are also affecting each other. The boundary conditions method is applied to determine the final condition influenced by exposure to an external load and temperature; the inverse method is applied to determine the condition affected by and resulted from influence of the body force. Methodologies were developed to form the bases of internal and boundary conditions related by isomorphism, and the relations that define them were formulated. Problems were solved for a circular cylinder of rock and for a solid of revolution with nontrivial form. The results are presented graphically.

1. Introduction

Modern machine parts and construction elements are often made of polycrystalline metals and cermet as well as significantly anisotropic composite materials. Their load conditions are often impacted by various mechanical and thermal influences simultaneously. Continuous design optimisation and the desire to reduce the time and material costs associated with real-life testing have directed a significant amount of effort into improvement of predictive methods for stress-strain states influenced by mechanical and thermal load at the same time. Given the complex physical structure of those materials, the improvement of methods constitutes an important research problem.

Several research papers (i.e. [18], [19]) are dedicated to studying the thermomechanical processes of the finite deformation in an anisotropic environment. The papers explore the relationship between tension, deformation and the temperature using variational principles. The problems for hollow isotropic and anisotropic cylinders were solved in various formulations.

In case of small deformations of a strained elastic anisotropic solid, the relation between its deformations and temperatures is usually analysed using the Dyugamel-Neumann equation. The derivation of the description for such relationships from the thermomechanical perspective is described in Novatsky's monograph [10]. Scientific problems associate with thermoelasticity of anisotropic solids were analysed e.g. by B. E. Pobedri [14] and A. S. Kravchuk [6].

Determination of the temperature field defined by boundary temperatures values and heat fluxes for isotropic homogeneous and inhomogeneous bodies were examined using various methods in multiple works [12, 21, 22, 23, 24, 25].
Paper [9] illustrates generalized Fourier method which is used to solve an axisymmetric thermoelastic boundary problem for a transversally isotropic half-space with a spheroidal cavity.

Paper [16] considers the axisymmetric problem of static thermoelasticity for a transversally isotropic circular cylinder of a finite length. The basic equation for the problem is derived using a special stress function. It is proven that the operator within such function is symmetrical and positively definite, and hence the solution of the original equation is reduced to the problem of minimum functional.

Paper [6] describes an inverse method for determination of stress-stained condition for an elastic isotropic body for the load being continuous in a volume. The same method was used for the problem considering fictitious mass forces arising on the formation of fully parametric solutions for anisotropic bodies [20]. In the paper [5], the inverse method was used as a complimentary tool for definition of fully parametric solutions for an orthotropic plate with a degree of anisotropy.

Boundary axisymmetric problems falling in scope of the theory of elasticity were solved for bodies of revolution using the method of boundary states in [3], [4].

2. Problem formulation

We are considering an equilibrium state of a transversally isotropic body spatially limited by one or more coaxial surfaces of rotation and exposed to surface forces $R_n, Z_n$ (symmetrically distributed with respect to the axis of rotation), with mass forces $X$, located in the steady temperature field $T$ (Figure 1). We will define this task as the boundary problem of thermoelasticity.

![Figure 1. A transversely isotropic body of revolution](image)

The problem solution can be divided into three stages: first, we need to resolve the temperature problem for the internal state $\xi^0$ and define the elastic condition exposed to a body force $\xi^X$; then, the mechanical boundary problem with the given surface forces or displacements $\xi$ is solved; and finally, the resulting fields of mechanical characteristics are joined together as follows:

$$\Omega = \xi^0 + \xi^X + \xi.$$  \hspace{1cm} (1.1)

This will allow the analysis of strength and stiffness resulting from a cumulative impact of all factors affecting a deformable solid, making possible e.g. adjustment of the kinematics, external forces or temperatures during the designing phase. The problem discretization is justified by the complexity of the potential theory that covers all three forces (effects) of different physical nature.

The goal of this research is to apply the energy method for boundary conditions [11] in a group of static problems for anisotropic bodies within the thermoelasticity theory, to apply the inverse method [10] for the problems of definition of the mechanical characteristics of mass forces for anisotropic bodies, and to solve the classical boundary problem of the anisotropic elasticity theory. The solution presented in this paper will make it possible to estimate the influence of all factors and allow assessment of the stress levels and configurations of the deformed surfaces of solid bodies that operate in severe conditions.
3. Defining equations of elasticity theory for homogeneous transversally isotropic environment in cylindrical coordinates

Differential equations of equilibrium in a cylindrical coordinate system with axis $z$, $r$, $\theta$ in the presence of body forces are presented below [1]:

$$\frac{\partial \tau_z}{\partial z} + \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} + R = 0;$$

$$\frac{\partial \sigma_r}{\partial z} + \frac{\partial \tau_z}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{r\theta}}{r} + Z = 0; \tag{2.1}$$

$$\frac{\partial \tau_{r\theta}}{\partial z} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{r\theta}}{r} + Q = 0,$$

where $R$, $Z$, $Q$ – body forces.

Cauchy equation [1]:

$$\varepsilon_z = \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_z}{\partial r}; \varepsilon_r = \frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r + \sigma_\theta}{r};$$

$$\gamma_{r\theta} = \frac{\partial \tau_z}{\partial r} + \frac{\partial \tau_{r\theta}}{\partial z}; \gamma_{r\theta} = \frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta}; \gamma_{r\theta} = \frac{\partial \tau_{r\theta}}{\partial z} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta}.$$

General Hooke’s law [1]:

$$\varepsilon_z = \frac{1}{E_z} [\sigma_z - \nu_z (\sigma_r + \sigma_\theta)] + \alpha_z T;$$

$$\varepsilon_r = \frac{1}{E_r} (\sigma_r - \nu_r \sigma_\theta) - \frac{\nu_z}{E_z} \sigma_z + \alpha_r T; \tag{2.3}$$

$$\varepsilon_\theta = \frac{1}{E_r} (\sigma_\theta - \nu_\theta \sigma_r) - \frac{\nu_z}{E_z} \sigma_z + \alpha_\theta T;$$

$$\gamma_{r\theta} = \frac{1}{G_z} \tau_{r\theta}; \gamma_{r\theta} = \frac{1}{G_z} \tau_{r\theta}; \gamma_{r\theta} = \frac{1}{G_r} \tau_{r\theta} = \frac{2(1 + \nu_r)}{E_r} \tau_{r\theta},$$

where $E_z$ and $E_r$ are the moduli of elasticity, respectively, in the direction of $z$ axis and in the plane of isotropy,

$\nu_z$ - the Poisson's ratio characterizing the compression along $r$ as it depends from the extension along the axis $z$,

$\nu_r$ - the Poisson's ratio characterizing the transversal compression in the plane of isotropy against the extension in the same,

$G_z$ and $G_r$ - shear moduli in the plane of isotropy and the plane perpendicular to it,

$\alpha_z$ and $\alpha_r$ - thermal coefficients along the axes $z$ and $r$, respectively.

4. Axisymmetric problem of elastostatics for a transversally-isotropic body with the exclusion of temperature deformations

The work [1] describes the correlation between the spatial stress and strain conditions of an elastic transversally isotropic body (based on the method of integral overlays) and certain auxiliary two-dimensional conditions, the components of which depend on two coordinate variables $z$ and $y$.

The flatten deformation is used as a complementary condition. Such deformation arises in the cylinders that have elastic symmetry in the $zy$ plane (direction $\eta$) at each point of the surface [1]:

$$\sigma_z^{pl} = -\text{Re}[\gamma_z^1 \phi_1(z_1) + \gamma_z^2 \phi_2(z_2)];$$

$$\sigma_r^{pl} = \text{Re}[\phi_1(z_1) + \phi_2(z_2)];$$

$$\sigma_{r\theta}^{pl} = -\text{Re}[\gamma_r \phi_1(z_1) + \gamma_{r\theta} \phi_2(z_2)]; \tag{3.1}$$
\[ \sigma_{\eta}^{pl} = v_{z} \sigma_{\eta}^{y} + v_{r} \frac{E_r}{E_z} \sigma_{r}^{pl} ; \tau_{z\theta} = 0 ; \tau_{r\theta} = 0 ; \]
\[ u_z^{pl} = \text{Re} \{ p_1 \phi_1(\zeta_j) + p_2 \phi_2(\zeta_j) \} ; \]
\[ u_y^{pl} = \text{Re} \{ iq \phi_1(\zeta_j) + iq \phi_2(\zeta_j) \} , \]
wherein the constants \( q_1 \) and \( p_1 \) are determined by the elastic parameters of the material, \( \zeta_j = z / \gamma_j + iy \), and where \( \gamma_j \) - is the complex roots of a characteristic equation:
\[ \left[ 1 - v_z^2 \frac{E_r}{E_z} \right] \gamma_j^4 - \left[ \frac{E_z}{G_z} - 2v_z(1 + v_r) \gamma_j \right] \gamma_j^2 + (1 - v_z^2) \frac{E_z}{G_z} = 0 , \]
functions \( \phi_j(\zeta_j) \) are analytical in their variables.

The transition to the axisymmetric spatial state in cylindrical coordinates is defined by the following [3]:
\[ \sigma_z = \frac{1}{\pi} \int_{-r}^{r} \frac{\sigma_z^{pl}}{\sqrt{r^2 - y^2}} dy ; \sigma_y = \frac{1}{\pi} \int_{-r}^{r} \frac{\sigma_y^{pl}}{\sqrt{r^2 - y^2}} dy ; \sigma_{z\theta} = \sigma_{r\theta} ; \]
\[ \sigma_z - \sigma_y = \frac{1}{\pi} \int_{-r}^{r} \frac{(\sigma_y^{pl} - \sigma_z^{pl})(2y^2 - r^2)}{r^2 - y^2} dy ; \]
\[ \sigma_z + \sigma_y = \frac{1}{\pi} \int_{-r}^{r} \frac{\sigma_y^{pl} + \sigma_z^{pl}}{\sqrt{r^2 - y^2}} dy ; \]
\[ u = \frac{1}{\pi} \int_{-r}^{r} \frac{u_y^{pl}}{\sqrt{r^2 - y^2}} dy ; w = \frac{1}{\pi} \int_{-r}^{r} \frac{u_z^{pl}}{\sqrt{r^2 - y^2}} dy ; \ n = 0 . \]

5. Axisymmetric problem of elastostatics for a transversally-isotropic body with the exclusion of temperature deformations

The established temperature field \( T_0^{pl}(z, y) \) of the plane ancillary condition with no heat sources inside fulfills the heat conduction equation [1]:
\[ \left( k_z \frac{\partial^2}{\partial z^2} + k_r \frac{\partial^2}{\partial y^2} \right) T_0^{pl}(z, y) = 0 , \]
where \( k_z \) and \( k_r \) - thermal conductivity coefficients in the direction of and perpendicular to the axis of symmetry.
\[ T_0^{pl} = \frac{g_0}{E_z} \text{Re} \{ \phi_0'(\zeta_0) \} ; \zeta_0 = z / \gamma_0 + iy ; \gamma_0 = \sqrt{k_z / k_r} ; g_0 = \frac{(E_z - v_z^2E_r)(\gamma_0^2 - \gamma_0^2)(\gamma_0^2 - \gamma_0^2)}{\gamma_0^2(\alpha_z E_z + \alpha_y E_y) - \alpha_z E_z(1 + v_r)} . \]

Displacements and forces corresponding to the temperature field [1]:
\[ u_z^{pl} = \text{Re} \{ p_0 \phi_0(\zeta_0) \} ; u_y^{pl} = \text{Re} \{ iq_0 \phi_0(\zeta_0) \} ; u_{n\theta}^{pl} = 0 ; \]
\[ \sigma_z^{pl} = - \text{Re} \{ i \sigma_0^{pl} \phi_0(\zeta_0) \} ; \sigma_y^{pl} = \text{Re} \{ \sigma_0^{pl} \phi_0(\zeta_0) \} ; \sigma_{n\theta}^{pl} = - \text{Re} \{ \gamma_0 \phi_0(\zeta_0) \} ; \]
\[ \sigma_{z\theta} = \text{Re} \{ (1 - \epsilon_0) \phi_0(\zeta_0) \} ; \tau_{z\theta} = 0 ; \tau_{r\theta} = 0 , \]
where
\[ p_0 = \frac{\gamma_0}{E_z} \left[ \alpha_z + \alpha_y v_z \frac{E_y}{E_z} \right] g_0 - \gamma_0^2 \left( 1 - v_z^2 \frac{E_r}{E_z} \right) - v_z (1 - v_r) ; \]
\[
q_0 = -\frac{1 + v_r}{E} \left[ \alpha_r g_0 + (1 + v_r) \frac{E}{E_r} \right] \; \varepsilon_0 = 1 - v_r + \left( v_r^2 + \alpha_r g_0 \right) \frac{E_r}{E},
\]

\[\varphi_0(\xi_0)\] – some analytical function of the variable \(\xi_0\).

The transition to the spatial axisymmetric temperature condition is carried out according to the dependencies (3.2).

6. The main provisions of the boundary state method

The boundary state method (BSM) [11] is a new method for solving equations of mathematical physics. It has shown its effectiveness in solving boundary value problems of elasticity theory, both for isotropic and anisotropic environments, in solving problems of thermoelasticity, hydrodynamics of ideal fluids, and dynamics (oscillations) of isotropic bodies.

The method is based on the spaces of internal \(\Xi\) and boundary \(\bar{A}\) conditions:

\[\Xi = \{\xi_1, \xi_2, \xi_3, \ldots, \xi_k, \ldots\}; \quad \bar{A} = \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k, \ldots\}.\]

The internal state is determined by sets of the displacement vector components, strain and stress tenors:

\[\xi_k = \{u^k, e^k, \sigma^k\}. \quad (5.1)\]

The main difficulty in forming a solution in the BSM is the construction of the basis for internal conditions, which is based on a general or fundamental solution for the environment; it is also possible to apply any particular or special solutions.

The scalar product in the space \(\Xi\) of internal conditions is expressed through the internal energy of the body occupied by the area \(D\) the following is true:

\[\langle \xi_1, \xi_2 \rangle = \int_V \varepsilon^2 dV,\]

moreover, due to the commutativity of the medium conditions the following is valid:

\[\langle \xi_1, \xi_2 \rangle = \langle \xi_2, \xi_1 \rangle = \int_V \varepsilon^2 dV = \int_V \varepsilon^2 dV.\]

The boundary state is determined by the components of the displacement vector at the boundary points and surface forces:

\[\gamma_k = \{u^k, p^k\}; \quad p^k = \sigma^k n_j,\]

where \(n_j\) – the component normal to the boundary.

In the space of boundary conditions \(\bar{A}\), the scalar product expresses the application of external forces on the body surface \(S\), for example, for the 1st and 2nd conditions:

\[\langle \gamma_1, \gamma_2 \rangle = \int_S p^1 u^2 dS,\]

and by virtue of the possible movements principle the following is true:

\[\langle \gamma_1, \gamma_2 \rangle = \langle \gamma_2, \gamma_1 \rangle = \int_S p^1 u^2 dS = \int_S p^2 u^1 dS.\]

It has been proven that in the case of a smooth boundary both condition spaces will be a Hilbert space conjugated by isomorphism [11]. By definition, each element \(\xi_\xi \in \Xi\) corresponds to a single element \(\gamma_k \in \bar{A}\) by a one-to-one relationship: \(\xi_k \leftrightarrow \gamma_k\). Therefore, the search for the internal state can be reduced to the construction of an isomorphic boundary condition. The latter essentially depends on the boundary states. In context of the first and second basic problems of mechanics, the problem would be reduced to resolving a system of equations having Fourier coefficients as variables, which can be found by the expansion of desired internal and boundary conditions into a series of elements for the orthonormal basis as follows:
\[ \xi = \sum_{k=1}^{\infty} c_k \xi_k ; \quad \gamma = \sum_{k=1}^{\infty} c_k \gamma_k , \] (5.2)

or explicitly:

\[ p_i = \sum_{k=1}^{\infty} c_k p_i^k ; \quad u_i = \sum_{k=1}^{\infty} c_k u_i^k ; \quad \sigma_{ij} = \sum_{k=1}^{\infty} c_k \sigma_{ij}^k ; \quad \varepsilon_{ij} = \sum_{k=1}^{\infty} c_k \varepsilon_{ij}^k . \]

Fourier coefficients in the case of the first main problem with given forces on the boundary \( p \in \{ p_x, p_y, p_z \} \), are found to be as follows:

\[ c_k = (p, u^k) = \int_S (p_i u_i^k + p \theta v_i^k + p z w_i^k) dS , \] (5.3)

where \( u^k \in \{ u^k, v^k, w^k \} \) – displacement vector within the basis element \( \gamma_k = \{ u^k, p^k \} \).

Fourier coefficients, in the case of the second main problem with the displacements given on the boundary \( u \in \{ u, v, w \} \), are found to be as follows:

\[ c_k = (u, p^k) = \int_S (p_i^k + \theta p \theta i^k + \sigma z w i^k) dS , \]

where \( p^k \in \{ p_x^k, p_y^k, p_z^k \} \) – force vector in the basis element \( \gamma_k = \{ u^k, p^k \} \).

7. Solving the temperature problem

We will assume that the temperature field determining the temperature at any point, is known.

Determination of an anisotropic body’s mechanical state dependant on the temperature field is defined in a way structurally similar to the BSM. As we assume the internal state to be formed by the following sets \( \Xi_0 = \{ x^0, y^0, z^0 \} \)

\[ \xi_k = \{ u^0, \ v^0, \ w^0 \} \}

\[ \{ \varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}, \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz}, \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{xz}, \sigma_{yz} \} \}

The basis of the space \( \Xi_0 \) can be constructed by giving functions \( \varphi_0 \) in (8), (9) the following values: \( \varphi_0 = \xi_n^0, \ n = 1, 2, 3... \)

Orthonormalization of a basis for a space \( \Xi_0 \) is carried out in accordance with the recursive-matrix orthogonalization algorithm developed in [13], where the cross scalar products (for example, for the 1st and 2nd States) are expressed in the following way:

\[ \langle \xi_1^0, \xi_2^0 \rangle = \int_V T_{1,2}^0 dV . \]

The sought-after temperature state defined by a Fourier series:

\[ \xi_0^0 = \sum_{k=1}^{\infty} c_k \xi_0^k , \] (6.2)

where \( \xi_k^0 \) – the elements of the orthonormal basis of the internal state \( \Xi_0 \), \( c_k^0 \) – Fourier coefficients which are calculated as

\[ c_k^0 = \int_T T_{k}^0 dV , \] (6.3)

where \( T_k^0 \) is the temperature in the referenced element \( \xi_k^0 \), \( T \) – a specific temperature range. After the temperature range satisfying to the heat equation (4.1) is restored, the displacement vector, strain and stress tensors are determined according to (2.2), (2.3) corresponding to that temperature state.

8. The state influenced by the body mass

The work [7] describes the methodology for defining the stress-strain condition of isotropic bodies affected by non-conservative forces continuous in the volume.
A fundamental system of polynomials $y^a z^b$ is implement to construct the field of displacements from body forces for flattened auxiliary conditions, which can be placed at any position of the displacement vector $\mathbf{u}^X_{\mu\nu}(y, z)$, forming an acceptable elastic state:

$$\mathbf{u}^X_{\mu\nu} = \left\{ \left[ y^a z^b, 0 \right], \left[ 0, y^a z^b \right] \right\}.$$  

Further, according to (3.2), we determine the movement vector’s components $\mathbf{u}^X(r, z)$ for the spatial axisymmetric condition, and according to the sequence (2.1), (2.2), (2.3) corresponding body forces, strain and stress tensors are determined.

Searching for possible options within $\alpha + \beta \leq n$, $(n = 1, 2, 3 \ldots)$, we obtain a set of conditions containing a finite-dimensional basis, which makes it possible to decompose an arbitrary vector of continuous mass forces into a Fourier series' elements with an increase in the number $n$ to infinity.

Such basis allows the orthogonalization [13] in accordance with the scalar product (for example, for the 1st and 2nd states):

$$\left( \mathbf{X}^{(1)}, \mathbf{X}^{(2)} \right) = \int_V \mathbf{X}^{(1)} \cdot \mathbf{X}^{(2)} dV ; \quad \mathbf{X}^{(k)} = \left\{ R^{(k)}(r, z), Z^{(k)}(r, z) \right\}.$$  

Any continuous vector of volume forces can be represented as a Fourier series decomposed into elements of an orthonormal basis:

$$\mathbf{X} = \sum_{k=1}^{n} c^X k \mathbf{X}^{(k)}_{\mu\nu} , \quad c^X k = \left( \mathbf{X}, \mathbf{X}^{(k)}_{\mu\nu} \right) ,$$  

where $\mathbf{X} = \{ R, Z \}$ are the given body forces.

Each basic vector $\mathbf{X}^{(k)}$ has a corresponding displacement vector along with strain and stress tensors. The combination of these form an internal condition resultant from the action of body forces described by the following equation:

$$\xi^X = \sum_{k=1}^{n} c^X k \xi^X k ; \quad \mathbf{u}^X = \sum_{k=1}^{n} c^X k \mathbf{u}^{(k)} ; \quad e^X_{ij} = \sum_{k=1}^{n} c^X k e^{X(k)}_{ij} ; \quad \sigma^X_{ij} = \sum_{k=1}^{n} c^X k \sigma^{X(k)}_{ij} .$$  

9. **Solution of the boundary value problem of elasticity theory**

It’s possible to construct basis sets per (5.1) by generating possible variants for two analytic functions $\varphi_1(\zeta_1)$ and $\varphi_2(\zeta_2)$ in a plane auxiliary state (3.1). Representation of analytic functions for a plane limited area can be generalized as:

$$\varphi_j(\zeta_j) = \sum_{n=0}^{\infty} a_{n \mu} \zeta_j^n , \quad (j = 1, 2) .$$  

The basis for the interior state spaces in this case is formed by the following sets:

$$\begin{bmatrix} \varphi_1(\zeta_1) \\ \varphi_2(\zeta_2) \end{bmatrix} \in \left\{ \begin{bmatrix} \zeta_1^n \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \zeta_2^n \end{bmatrix}, \begin{bmatrix} 0 \\ i\zeta_2^n \end{bmatrix}, \ldots \right\} , \quad n=1, 2, \ldots .$$  

Mechanical characteristics of the plane auxiliary state can be determined according the equation (8.1) and followed by the transition to the three-dimensional state according to the dependencies as per (3.2).

The problem is then solved in spatial conditions as described in §5.

10. **Solution for the cylinder case**

The proposed method will be tested on the study of thermoelastic equilibrium of a transversally isotropic circular cylinder occupying the area $D = \{ (z, r) \mid 0 \leq r \leq 1, \ -2 \leq z \leq 2 \}$ made of large dark gray siltstone rock [8]. After the de-dimensionalization procedure, elastic characteristics of the material are as follows: $E_z = 6.21$; $E_r = 5.68$; $G_z = 2.55$; $\nu_z = 0.22$; $\nu_r = 0.22$. We select the
characteristics of a dolomite rock with similar structure as a basis for temperature characteristics. We apply extreme values of the linear temperature expansion coefficient \( k_z = 1.6 \), \( k_r = 6.5 \), as the thermal conductivity coefficients [17], [2]. Similarly, the coefficients of thermal expansion \( \alpha_z = 6.7 \), \( \alpha_r = 8.6 \) [15].

Let’s review the first basic problem of mechanics with external forces simulating all-round stretching:

\[
p_r = 1, p_z = 0, (r = 1, -2 \leq z \leq 2);
\]

\[
p_r = 0, p_z = -1, (z = -2, 0 \leq r \leq 1);
\]

\[
p_r = 0, p_z = 1, (z = 2, 0 \leq r \leq 1),
\]

and with a steady temperature field \( \dot{O} = z \). The cylinder is affected by body forces opposite to the axial direction \( Z = -z - 2 \) and simulating centrifugal inertia forces \( R = r \) \( (R = \omega^2 r, \omega^2 = 1, \omega \) - angular velocity of rotation relative to the \( z \) axis).

**Temperature problem.** The temperature function’s orthonormal basis per (6.1) is presented in Table 1 (5 elements are shown).

**Table 1.** An orthonormal basis set of temperature functions

| \( n \) | \( T^0 \) |
|-------|---------|
| \( \xi_1^0 \) | -0.70711 |
| \( \xi_2^0 \) | -0.61237 \( z \) |
| \( \xi_3^0 \) | 0.75375 + 0.07294 \( r^2 \) - 0.59266 \( z^2 \) |
| \( \xi_4^0 \) | 1.28853 \( z \) + 0.21475 \( r^2 \) \( z \) - 0.58163 \( z^3 \) |
| \( \xi_5^0 \) | -0.6645 - 0.21216 \( r^2 \) - 0.01278 \( r^4 \) + 1.7238 \( z^2 \) + 0.41552 \( r^2 \) \( z^2 \) - 0.56268 \( z^4 \) |

The Fourier Coefficients (6.3) \( c_k^0 \in \{0, -1.63299, 0, 0, 0, \ldots\} \). Solution per (6.2) provides a strict \( \xi^0 \):

\[
u^0 = -2.00736 \, rz; \ w^0 = -0.95057 \, r^2 + 7.72338 \, z; \ T^0 = z;
\]

\[
\sigma^0_r = -71.6334 \, z; \ \tau^0_r = -71.6334 \, z; \ \sigma^0_z = 19.9333 \, z; \ \tau^0_z = -9.96667 \, r; \ \tau^0_{\theta} = \tau^0_{\phi} = 0.
\]

**The body forces.** The orthonormal truncated basis of body forces \( X \) is presented in table 2 (10 elements are shown). When forming the basis, the first seven elements are set to zero \( (X = 0) \).

**Table 2.** Orthonormal basis set of body forces

| \( n \) | \( R \) | \( Z \) |
|-------|-------|-------|
| \( \xi_1^X \) | 0 | -0.70711 |
| \( \xi_2^X \) | 0 | -0.61237 \( z \) |
| \( \xi_3^X \) | \( -r \) | 0 |
| \( \xi_4^X \) | 0 | -0.79057 - 0.59293 \( z^2 \) |
| \( \xi_5^X \) | -0.86602 \( rz \) | 0 |
| \( \xi_6^X \) | 0 | 1.22474 - 2.4495 \( r^2 \) |
| \( \xi_7^X \) | 0 | 1.40312 \( z \) - 0.58463 \( z^3 \) |
| \( \xi_8^X \) | 1.11803 \( r \) - 0.83852 \( rz^2 \) | 0 |
| \( \xi_9^X \) | 0 | 1.06066 \( z \) - 2.1213 \( r^2 \) \( z \) |
| \( \xi_{10}^X \) | 2.82843 \( r \) - 4.24264 \( r^3 \) | 0 |
The Fourier Coefficients (7.1) \( c_k^{(0)} \in \{2.82843, 1.6323, -1.0, 0, \ldots\} \). The strict condition form of body forces is restored in line with (7.2) \( \xi \): 

\[
\begin{align*}
\sigma_r^x &= 0.56286 z - 1.00571 z^2; \\
\sigma_\theta^x &= 0.56286 z - 1.00571 z^2; \\
\sigma_z^x &= 2 z + 1.5 z^2; \\
\tau_{r\theta}^x &= -1 r_z; \\
\tau_{z\theta}^x &= \tau_{r\theta}^x = 0; \\
R^x &= r_z; \\
Z^x &= -2 - z.
\end{align*}
\]

The obtained stress expressions satisfy the equilibrium equations (2) taking into account the body forces.

**Boundary problem of elasticity theory.** It should be noted that the given forces should constitute a balanced system. The components of the displacement vector in orthonormal basis set (8.1) are presented in table 3 (6 elements are shown).

**Table 3.** An orthonormal basis set of the displacement vector component

| n  | \( u \)         | \( w \)         |
|----|-----------------|-----------------|
| \( \xi_1 \) | 0.13813 \( r \) | -0.25363 \( z \) |
| \( \xi_2 \) | -0.1235 \( r \) | -0.12721 \( z \) |
| \( \xi_3 \) | 0.1011 \( r_z \) | 0.04918 \( r^2 - 0.10496 z^2 \) |
| \( \xi_4 \) | -0.11683 \( r_z \) | 0.1565 \( r^2 - 0.03679 z^2 \) |
| \( \xi_5 \) | -0.09006 \( r - 0.02311 r^3 + 0.08488 r_z^2 \) | 0.18204 \( r + 0.0848 r_z - 0.05612 r_z^3 \) |
| \( \xi_6 \) | 0.19862 \( r - 0.02818 r^3 - 0.12783 r_z^2 \) | -0.11303 \( r + 0.23255 r_z^2 - 0.00081 r_z^3 \) |

The Fourier Coefficients (5.3) \( c_k \in \{0.04524, -0.74842, 0, 0, 0, \ldots\} \). The strict condition form of body forces is restored per (5.2) with the given forces as follows \( \xi \): 

\[
\begin{align*}
\sigma_r &= \sigma_\theta = \sigma_z = 1; \\
\tau_{r\theta} &= \tau_{z\theta} = \tau_{r\theta} = 0.
\end{align*}
\]

The integral solution of the boundary value problem of thermoelasticity (1.1) for a transtropic cylinder:

\[
\begin{align*}
| u | &= 0.09867 r - 2.00736 r_z - 0.19608 r_z^2; \\
| w | &= -0.95057 r^2 + 0.0837 z + 7.86266 z^2 + 0.106427 z^3; \\
\sigma_r &= 1 - 71.0706 z - 1.00571 z^2; \\
\sigma_\theta &= 1 - 71.0706 z - 1.00571 z^2; \\
\sigma_z &= 1 + 21.9333 z + 1.5 z^2; \\
\tau_{r\theta} &= -9.96667 r - r_z; \\
\tau_{r\theta} &= \tau_{r\theta} = 0.
\end{align*}
\]

**11. Solution of the computational problem for a solid of revolution**

Let us now consider a solid of revolution occupying an area with boundary conditions simulating inhomogeneous stretching in the direction \( r \), subject to mass forces \( X \), located in a steady-state field of temperatures \( T \) (Figure 2).

\[
\begin{align*}
p_r &= z \cdot 10^2, p_z = 0, (r = 2, \ 0 \leq z \leq 2); \\
p_r &= 0, p_z = 0, (r = 1, \ -1 \leq z \leq 0); \\
p_r &= 0, p_z = 0, (z = -1, \ 0 \leq r \leq 1); \\
p_r &= 0, p_z = 0, (z = 2, \ 0 \leq r \leq 2); \\
X &= 10 \cdot (r^2, z^2 + r); \\
\phi &= z^2.
\end{align*}
\]

**Figure 2.** Meridian section of the solid of revolution with boundary conditions
Temperature problem. An orthonormal basis with 20 elements was used to determine the elastic temperature field. Figure 3 is a graph illustrating the "saturation" of the Bessel sum (the left part of the Bessel inequality). This is an indirect evidence in favour of the solution.

The recovered temperature field is represented in Figure 4. It is compared with the field taken as precondition, which serves as an estimate for the accuracy of the result.

The outlines of the obtained stress tensor components are presented in Figure 5 (due to the axial symmetry, the region shown is \( \{(z, r) \mid 0 \leq r \leq 2, -1 \leq z \leq 2\} \)).
In determining the *elastic field of the mass forces*, a truncated orthonormal basis of 16 elements was used. The comparison of the given mass forces field of \( |X| = \sqrt{R^2 + Z^2} \) with the restored one is presented on Figure 6 (scale 1:10).

**Figure 6.** Contours: a) given field of mass forces; b) solution

Contours of other restored mechanical characteristics are presented on Figure 7 (scale 1:10).

**Figure 7.** Isolines of mechanical characteristics created by the body forces

*Boundary problem.* A basis with 55 elements was used in solving this problem. Verification was carried out by comparing the boundary forces obtained as a solution with those given (Figure 8). In the chart, the red marks signify the given values; while the blue solid line shows the restored values.
Figure 8. Verification of the boundary conditions for boundaries S1 and S3

Contours of the restored stress tensor components arising from the exposure to external forces are shown on figure 9.

Figure 9. Stress tensor component outlines from the boundary problem
The resulting condition is a sum of the impact of temperature, body forces, and external force (Figure 10).

![Figure 10. Isolines of stress tensor components of the resulting state](image)

Figure 10. Isolines of stress tensor components of the resulting state

Figure 11 illustrates the contour of deformed meridian section of the solid (deformations exaggerated).

![Figure 11. The outline of the deformed section](image)

Figure 11. The outline of the deformed section

The developed approach consisting of various methods is effective in determining the axisymmetric stress-strain state of transversally isotropic solids of revolution, integrating the impact of external and body forces as well as temperature.

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