Analyticity of Bounded Solutions of Analytic State-Dependent Delay Differential Equations

Qingwen Hu

aDepartment of Mathematical Sciences, The University of Texas at Dallas, Richardson, TX, 75080
9 July 2018

Abstract

We study the analyticity of bounded solutions of systems of analytic state-dependent delay differential equations. We obtain the analyticity of solutions by transforming the system of state-dependent delay equations into an abstract ordinary differential equation in a subspace of the sequence space $l^\infty(\mathbb{R}^{N+1})$ and prove the existence of complex extension of the bounded solutions. An example is given to illustrate the general results.

Key words: State-dependent delay, analyticity, bounded solutions

1 Introduction

The analyticity of bounded solutions of delay differential equations with constant delay such as the well-known Wright’s equation was established in work of Nussbaum [1]. It is natural to conjecture that this analyticity result holds true for many differential equations with state-dependent delay such as

$$\begin{aligned}
\dot{x}(t) &= f(x(t), x(t - \tau)), \\
\tau &= r(x(t)),
\end{aligned}$$

with analytic $f$ and $r$. In this paper, we solve this conjecture.

We should remark that the work of Mallet-Paret and Nussbaum [6] also presented some examples where bounded solutions are no-longer analytic, while Krisztin [3]

Email address: qingwen@utdallas.edu (Qingwen Hu).
showed that globally defined solutions of threshold type delay equations are analytic. Then an important theoretical problem is what would be the most general form of state-dependent delay differential equations for which the conjecture remains true for differential equations with state-dependent delay. We also notice that establishing the analyticity of bounded solutions such as periodic solutions is essential for describing the global dynamics of some state-dependent delay differential equations. For example, in [5] the nonexistence of a nonconstant $p$-periodic real-valued solution which is constant in a small interval in $\mathbb{R}$ was assumed in order to obtain the global continuation of periodic solutions of the following system

$$\begin{align*}
\dot{x}(t) &= f(x(t), x(t - \tau(t))), \\
\dot{\tau}(t) &= g(x(t), \tau(t)),
\end{align*}$$

(1.2)

with analytic $f$ and $g$. Specifically, it was needed to exclude the case where there is a nonconstant $p$-periodic solution for which

$$\tau(t) = \tau_0, \quad t \in I + kp, \quad k \in \mathbb{Z}$$

(1.3)

where $\tau_0 > 0$ is a constant and $I$ is an interval in $\mathbb{R}$ with length less than $p$. On the one hand, if there is such a periodic solution and if this solution is analytic on $\mathbb{R}$, then the delay $\tau$ must be a constant on the whole real line $\mathbb{R}$. On the other hand, under certain technical conditions, it can be ruled out the existence of such a periodic solution with constant delay by considering a cyclic system of ordinary differential equations (see [5] for more details) and hence these technical conditions can ensure the nonexistence of a nonconstant $p$-periodic solution for which $\tau$ remains to be a constant in a small interval in $\mathbb{R}$.

In this paper, we first note that bounded solutions of system (1.1) and system (1.2) and many others including those with “threshold delay” must satisfy the following differential equations with state-dependent delays

$$\begin{align*}
\dot{x}(t) &= f(x(t), x(t - \tau(t))), \\
\dot{\tau}(t) &= g(x(t), \eta(t)), \quad \eta \in \mathbb{R},
\end{align*}$$

(1.4)

where $\eta^0(t) = t$, $\eta(t) = t - \tau(t)$, $\eta^j(t) = \eta(\eta^{j-1}(t))$ for $j = 1, 2 \cdots, M$ with $M \in \mathbb{N}$, and we assume

\begin{itemize}
  \item[(A1)] The maps $f: U \times U \ni (\theta_1, \theta_2) \to f(\theta_1, \theta_2) \in \mathbb{C}^N$ and $g: U^M \times V \ni (\gamma_1, \gamma_2) \to g(\gamma_1, \gamma_2) \in \mathbb{C}$ are analytic with respect to $(\theta_1, \theta_2)$ and $(\gamma_1, \gamma_2)$, respectively, where $U \subseteq \mathbb{C}^N$, $V \subseteq \mathbb{C}$ are bounded open sets, $U^M = U \times U \times \cdots \times U$.
  \item[(A2)] There exist $l \in (0, 1)$ and $c > 1$ such that $|1 - g(\gamma_1, \gamma_2) - e^{\ell l}| < e^{-l}$ for all $(\gamma_1, \gamma_2) \in \overline{U^M \times V}$, where $\overline{U^M \times V}$ is the closure of $U^M \times V$.
\end{itemize}

(A1) is a natural assumption on the analyticity of $f$ and $g$ on their domains. (A2) is assuming that $g$ satisfies $l < |1 - g| < c$ which ensures that the mapping $\mathbb{R} \ni t \to t - \tau(t) \in \mathbb{R}$ is increasing with a bounded rate.
Let \((x, \tau) \in C(\mathbb{R}; \mathbb{R}^{N+1})\) be a bounded solution of system (1.4) and define the sequence \(((y_1, z_1), (y_2, z_2), \cdots)\) by

\[
(y_j(t), z_j(t)) = \left(\frac{1}{c^j} x(\eta^{j-1}(t)), \frac{1}{c^j} \tau(\eta^{j-1}(t))\right)
\]

for \(j \geq 1, j \in \mathbb{N}, t \in \mathbb{R}\).

The reason that we carry a term \(\frac{1}{c^j}\) will be clear by the end of this section. For \(j = 1\) we have for every \(t \in \mathbb{R}\),

\[
\frac{d}{dt} y_1(t) = \dot{x}(t) = \frac{1}{c} f(cy_1(t), c^2 y_2(t)), \quad \frac{d}{dt} z_1(t) = \dot{\tau}(t) = \frac{1}{c} g(cy_1(t), c^2 y_2(t), \cdots, c^{j+M-1} y_{j+M-1}(t), cz_1(t)).
\]

For \(j \geq 2, j \in \mathbb{N}\), we have for every \(t \in \mathbb{R}\),

\[
\frac{d}{dt} y_j(t) = \frac{1}{c^j} \dot{x}(\eta^{j-1}(t)) \prod_{i=0}^{j-2} \dot{\eta}(\eta^i(t))
\]

\[
= \dot{x}(\eta^{j-1}(t)) \frac{1}{c^j} \prod_{i=0}^{j-2} (1 - g(x(\eta^i(t)), x(\eta^{i+1}(t)), \cdots, x(\eta^{i+M-1}(t)), \tau(\eta^i(t)))
\]

\[
= \frac{f((c^j y_j(t), c^{j+1} y_{j+1}(t)))}{1 - g(c^j y_j(t), c^{j+1} y_{j+1}(t), \cdots, c^{j+M-1} y_{j+M-1}(t), c^j z_j(t))}
\]

\[
\times \frac{1}{c^j} \prod_{i=0}^{j-1} (1 - g(c^{i+1} y_{i+1}(t), c^{i+2} y_{i+2}(t), \cdots, c^{i+M} y_{i+M}(t), c^{i+1} z_{i+1}(t))),
\]

(1.7)

and

\[
\frac{d}{dt} z_j(t) = \frac{1}{c^j} \dot{\tau}(\eta^{j-1}(t)) \prod_{i=0}^{j-2} \dot{\eta}(\eta^i(t))
\]

\[
= \dot{\tau}(\eta^{j-1}(t)) \frac{1}{c^j} \prod_{i=0}^{j-2} (1 - g(x(\eta^i(t)), x(\eta^{i+1}(t)), \cdots, x(\eta^{i+M-1}(t)), \tau(\eta^i(t)))
\]

\[
= \frac{g(c^j y_j(t), c^{j+1} y_{j+1}(t), \cdots, c^{j+M-1} y_{j+M-1}(t), c^j z_j(t))}{1 - g(c^j y_j(t), c^{j+1} y_{j+1}(t), \cdots, c^{j+M-1} y_{j+M-1}(t), c^j z_j(t))}
\]

\[
\times \frac{1}{c^j} \prod_{i=0}^{j-1} (1 - g(c^{i+1} y_{i+1}(t), c^{i+2} y_{i+2}(t), \cdots, c^{i+M} y_{i+M}(t), c^{i+1} z_{i+1}(t))).
\]

(1.8)

Then the sequence \(((y_1, z_1), (y_2, z_2), \cdots), t \in \mathbb{R}\) satisfies a system of ordinary differential equations of (1.5), (1.6), (1.7) and (1.8). Namely, for every \(t \in \mathbb{R}\) and for \(j \geq 1, \cdots\),
we have

\[
\begin{align*}
\frac{dy_j(t)}{dt} &= \frac{f((c^j y_j(t), c^{j+1} y_{j+1}(t), \ldots, c^{j+M-1} y_{j+M-1}(t), c^j z_j(t)))}{1 - g(c^j y_j(t), c^{j+1} y_{j+1}(t), \ldots, c^{j+M-1} y_{j+M-1}(t), c^j z_j(t))} \\
\frac{dz_j(t)}{dt} &= \frac{g(c^j y_j(t), c^{j+1} y_{j+1}(t), \ldots, c^{j+M-1} y_{j+M-1}(t), c^j z_j(t))}{1 - g(c^j y_j(t), c^{j+1} y_{j+1}(t), \ldots, c^{j+M-1} y_{j+M-1}(t), c^j z_j(t))} \\
&\quad \times \prod_{i=0}^{j-1} (1 - g(c^i y_i(t), c^{i+1} y_{i+1}(t), \ldots, c^{j+M-1} y_{j+M-1}(t), c^j z_j(t))) \\
&\quad \times \prod_{i=0}^{j-1} (1 - g(c^i y_i(t), c^{i+1} y_{i+1}(t), \ldots, c^{j+M-1} y_{j+M-1}(t), c^j z_j(t))).
\end{align*}
\]

(1.9)

It remains to decide an appropriate space where \((y_1(t), z_1(t), (y_2, z_2), \ldots), t \in \mathbb{R}\)
lives in. With the arguments of \(f\) and \(g\) in the right hand side of (1.9), it turns out
that we can set \(w(t) = ((y_1(t), z_1(t)), (y_2(t), z_2(t)), \ldots)\) to be in the sequence space
\(l^c_\infty(\mathbb{R}^{N+1})\) defined by

\[
l^c_\infty(\mathbb{R}^{N+1}) = \{v = (v_1, v_2, \ldots, v_j, \ldots) \in l^\infty(\mathbb{R}^{N+1}) : \sup_{j \in \mathbb{N}} c^j|v_j| < +\infty\}, \quad (1.10)
\]

where we can find a subset such that the terms of \(f\) and \(g\) in system (1.9) are well-
defined. Besides, the product terms in system (1.9) need to be treated so that the
right hand side of system (1.9) always remains bounded as \(j \to \infty\). We address this
issue at Lemma 2.7.

With the above preparations we can represent system (1.9) by the following abstract
ordinary differential equation:

\[
\frac{d}{dt} w(t) = H(T w(t)), \quad (1.11)
\]

where the mapping \(T : l^c_\infty(\mathbb{R}^{N+1}) \to l^\infty(\mathbb{R}^{N+1})\) is defined by

\[
T(v_1, v_2, \ldots, v_j, \ldots) = (cv_1, c^2 v_2, \ldots, c^j v_j, \ldots),
\]

and \(H : l^\infty(\mathbb{R}^{N+1}) \to l^\infty(\mathbb{R}^{N+1})\) is defined by the right hand side of system (1.9).

To obtain the analyticity of bounded solutions \((x(t), \tau(t)), t \in \mathbb{R}\) of system (1.4), we
follow the idea of [1] to show that the solution \(w(t)\) to system (1.11) has a complex
extension and hence \((x(t), \tau(t)), t \in \mathbb{C}\) satisfies system (1.4) on the complex domain.
We remark that there are significant new challenges not present in [1] but in this
paper. First, the operator \(T\) is not a self mapping on \(l^c_\infty(\mathbb{C}^{N+1})\) and the range of \(H\)
is in \(l^\infty(\mathbb{R}^{N+1})\). This means that the right hand side of system (1.11) does not define
a vector field on \(l^c_\infty(\mathbb{C}^{N+1})\) while we are looking for solutions in \(l^c_\infty(\mathbb{C}^{N+1})\); Secondly,
when we transform system (1.11) into an integral form and consider the associated
fixed point problem on \(l^\infty(\mathbb{C}^{N+1})\) using the uniform contraction principle in Banach
spaces, we can not obtain a contractive mapping on $l^\infty(C^{N+1})$ unless we introduce a small perturbation. The problem is then reduced to show that the solution of the initial value problem associated with system (1.11) is the limit of that of the perturbed system.

We organize the remaining part of the paper as follows: in section 2, we will develop results on analyticity of $H$ in the right hand side of system (1.11) and some basic functional analysis necessary for proving the existence of complex extension of solutions to system (1.11), using the the uniform contraction principle in Banach spaces; We present the main results in section 3 and will illustrate this general result with an example in the last section.

2 Notations and Preliminary Results

Let $E$ be a complex Banach space, $D$ an open subset of the complex plane $\mathbb{C}$. A continuous mapping $u : D \ni t \mapsto u(t) \in E$ is called analytic if for every $t \in D$, $\lim_{t \to t_0} \frac{u(t) - u(t_0)}{t - t_0} = u'(t_0)$ exists. If $W$ is an open subset of $E$, $\tilde{E}$ is a complex Banach space, a continuous mapping $G : W \ni u \mapsto G(u) \in \tilde{E}$ is called analytic if for all $u_0 \in W$, and for all $h \in E$, the mapping $t \mapsto G(u_0 + th)$ is analytic in the neighbourhood of $0 \in \mathbb{C}$.

Let $K$ stand for the space of real numbers ($\mathbb{R}$) or complex numbers ($\mathbb{C}$). In the following, we develop some basic properties of the map $T$ and the spaces $l^\infty(K^{N+1})$ and $l^\infty(K^{N+1})$. We denote by $(v_j)_{j=1}^{\infty}$ the element $(v_1, v_2, \cdots, v_j, \cdots)$ in the sequence spaces.

**Lemma 2.1** Let $c > 1$ be a constant and $l^\infty_c(K^{N+1})$ be defined by

$$l^\infty_c(K^{N+1}) = \{ v = (v_j)_{j=1}^{\infty} \in l^\infty(K^{N+1}) : \sup_{j \in \mathbb{N}} c^j |v_j| < +\infty \}.$$  

Then $l^\infty_c(K^{N+1})$ is a Banach space under the norm $\| \cdot \|_{l^\infty_c(K^{N+1})}$ defined by

$$\| v \|_{l^\infty_c(K^{N+1})} = \sup_{j \in \mathbb{N}} c^j |v_j|.$$  

**Lemma 2.2** Let $m \in \mathbb{N}, m \geq 2$ be a constant and $l^\infty_m(K^{N+1})$ be defined by

$$l^\infty_m(K^{N+1}) = \{ v = (v_j)_{j=1}^{\infty} \in l^\infty(K^{N+1}) : \sup_{j \in \mathbb{N}} j^m |v_j| < +\infty \}.$$  

Then $l^\infty_m(K^{N+1})$ is a Banach space under the norm $\| \cdot \|_{l^\infty_m(K^{N+1})}$ defined by

$$\| v \|_{l^\infty_m(K^{N+1})} = \sup_{j \in \mathbb{N}} j^m |v_j|.$$  

Moreover, the embedding $I_m : l^\infty_m(K^{N+1}) \to l^\infty(K^{N+1})$ is compact.
such that for every $\varepsilon > 0$.

On the other hand, it follows from (2.1) that for every $\varepsilon > 0$.

Then we have

$$l^\infty(H)$$ is a Banach space, there exists $b^* \in l^\infty(K^{N+1})$ so that

$$\lim_{n \to +\infty} |b^n - b^*|_{l^\infty(K^{N+1})} = 0.$$ 

Then we have $v^* = (b_1^*, b_2^*, \ldots, b_j^*) \in l^\infty(K^{N+1})$ and

$$\lim_{n \to +\infty} |v^n - v^*|_{l^\infty(K^{N+1})} = \lim_{n \to +\infty} |b^n - b^*|_{l^\infty(K^{N+1})} = 0.$$ 

Next we show that the embedding $I_m : l^\infty(K^{N+1}) \to l^\infty(K^{N+1})$ is compact. For every $k \in \mathbb{N}$ we define the “cut-off” operator $H_k : l^\infty(K^{N+1}) \to l^\infty(K^{N+1})$ by

$$H_k(v_1, v_2, \ldots, v_k, \ldots) = (v_1, v_2, \ldots, v_k, 0, \ldots).$$ 

Then $H_k$ is compact since the dimension of the range is finite. Moreover we have

$$\| (I_m - H_k)(v_1, v_2, \ldots, v_j, \ldots) \|_{l^\infty(K^{N+1})} = \sup_{j \geq k+1} |v_j|,$$

which implies that $\|I_m - H_k\| \to 0$ as $k \to +\infty$ and hence $I_m$ is compact. \hfill $\blacksquare$

**Lemma 2.3** Let $c > 1$ be a constant. The closed unit ball of $l_c^\infty(K^{N+1})$ is closed under the norm $\| \cdot \|_{l^\infty(K^{N+1})}$.

**Proof** Let $B_c(1) = \{v \in l^\infty_c(K^{N+1}) : \|v\|_{l^\infty(K^{N+1})} \leq 1\}$. Let $\{v^n\}_{n=1}^{+\infty} \subset B_c(1)$ be a Cauchy sequence in the norm $\| \cdot \|_{l^\infty(K^{N+1})}$. Since $l^\infty_c(K^{N+1})$ is a subspace of the Banach space $(l^\infty(K^{N+1}), \| \cdot \|_{l^\infty(K^{N+1})})$. There exists $v^0 \in l^\infty(K^{N+1})$ such that

$$\lim_{n \to +\infty} \|v^n - v^0\|_{l^\infty(K^{N+1})} = 0.$$ \hspace{1cm} (2.1)

Now we show that $v^0 \in B_c(1)$. By way of contradiction, assume that $v^0 \notin B_c(1)$.

Then we distinguish the following two cases:

**Case 1.** $v^0 \notin l^\infty_c(K^{N+1})$. Then for every $K > 0$, there exists $j_0 \in \mathbb{N}$ such that $c^{j_0} |(v^0)_{j_0}| > K$. That is,

$$|(v^0)_{j_0}| > \frac{K}{c^{j_0}}.$$ \hspace{1cm} (2.2)

On the other hand, it follows from (2.1) that for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that for every $n > N_0$, we have $\sup_{j \in \mathbb{N}} |(v^0)_j - (v^n)_j| < \varepsilon$ which leads to $|(v^0)_j| - |(v^n)_j| < \varepsilon$, for every $j \in \mathbb{N}, n > N_0$. It follows that

$$|(v^n)_j| > |(v^0)_j| - \varepsilon,$$ for every $j \in \mathbb{N}, n > N_0$. \hspace{1cm} (2.3)

Choosing $j = j_0$ and $\epsilon = \frac{K}{2c^{j_0}}$ in (2.3), then by (2.1) and (2.2) we obtain that $|(v^n)_{j_0}| \geq |(v^0)_{j_0}| - \frac{K}{2c^{j_0}} > \frac{K}{2c^{j_0}}$, which leads to $|c^{j_0}(v^n)_{j_0}| > K/2$ for every $n > N_0$. That is,
Lemma 2.4

Let \( T \) be a compact operator such that for every \( N \in \mathbb{N} \), \( T \) has a compact inverse \( T \). On the other hand, it follows from (2.1) that for every \( v \), \( \| v \|_{l_c^\infty(\mathbb{K}^{N+1})} < 1 \), we know that

\[
\sup_{j \in \mathbb{N}} |(v^n)_j - (v)_j| < \varepsilon
\]

which leads to

\[
\|v^n\|_{l_c^\infty(\mathbb{K}^{N+1})} = \sup_{n > N} \|v^n\|_{l_c^\infty(\mathbb{K}^{N+1})} < \varepsilon.
\]

Then we have

\[
\|v^n\|_{l_c^\infty(\mathbb{K}^{N+1})} < \varepsilon.
\]

Note that \( \{v^n\}_{n=1}^{\infty} \subset B_c(1) \). Then by (2.5) we have

\[
1 \geq |(v^n)_j| - |(v)_j| + \varepsilon,
\]

for every \( j \in \mathbb{N}, n > N_0 \).

Choosing \( j = j_1, \varepsilon = \frac{s - 1}{2c^{j_1}} \) in (2.6) we obtain from (2.4) that

\[
\frac{1}{c^{j_1}} \geq |(v^n)_j| - |(v)_j| - \varepsilon = \frac{s}{c^{j_1}} - \frac{s - 1}{2c^{j_1}},
\]

for every \( n > N_0 \).

Then we have \( s < 1 \). This is a contradiction.

We remark that the unit sphere of \( l_c^\infty(\mathbb{K}^{N+1}) \) is not closed under the norm \( \| \cdot \|_{l_c^\infty(\mathbb{K}^{N+1})} \). In light of Lemma 2.3, we will equip bounded sets of \( l_c^\infty(\mathbb{K}^{N+1}) \) with the norm \( \| \cdot \|_{l_c^\infty(\mathbb{K}^{N+1})} \). The following three lemmas discuss the properties of a linear operator on \( l_c^\infty(\mathbb{K}^{N+1}) \) equipped with the norm \( \| \cdot \|_{l_c^\infty(\mathbb{K}^{N+1})} \).

**Lemma 2.4** Let \( c > 1 \) be a constant. The mapping \( T : (l_c^\infty(\mathbb{K}^{N+1}), \| \cdot \|_{l_c^\infty(\mathbb{K}^{N+1})}) \rightarrow (l_c^\infty(\mathbb{K}^{N+1}), \| \cdot \|_{l_c^\infty(\mathbb{K}^{N+1})}) \) defined by

\[
T(v_1, v_2, \ldots, v_j, \ldots) = (cv_1, c^2v_2, \ldots, c^jv_j, \ldots),
\]

has a compact inverse \( T^{-1} \) with norm \( \| T^{-1} \| = \frac{1}{c} \). Moreover, \( T \) is a closed operator.

**Proof** We first show that \( T^{-1} \) exists and is continuous. By definition of \( T \) and that \( c > 1 \), we know that \( T \) is 1-1 and onto. Therefore \( T^{-1} : (l_c^\infty(\mathbb{K}^{N+1}), \| \cdot \|_{l_c^\infty(\mathbb{K}^{N+1})}) \rightarrow (l_c^\infty(\mathbb{K}^{N+1}), \| \cdot \|_{l_c^\infty(\mathbb{K}^{N+1})}) \) exists and is given by

\[
T^{-1}(v_1, v_2, \ldots, v_j, \ldots) = (c^{-1}v_1, c^{-2}v_2, \ldots, c^{-j}v_j, \ldots).
\]

Moreover, we have

\[
\| T^{-1} \| = \sup_{v \in l_c^\infty(\mathbb{K}^{N+1})} \frac{\| T^{-1}v \|_{l_c^\infty(\mathbb{K}^{N+1})}}{\| v \|_{l_c^\infty(\mathbb{K}^{N+1})}} = \sup_{\| v \|_{l_c^\infty(\mathbb{K}^{N+1})} = 1} \| T^{-1}v \|_{l_c^\infty(\mathbb{K}^{N+1})} = \frac{1}{c}.
\]
Next we show that $T^{-1}$ is compact. For every $m \in \mathbb{N}$ we define an operator $H_m : (l^\infty(\mathbb{K}^{N+1}), \| \cdot \|_{l^\infty(\mathbb{K}^{N+1})}) \rightarrow (l^\infty(\mathbb{K}^{N+1}), \| \cdot \|_{l^\infty(\mathbb{K}^{N+1})})$ by
\[
H_m(v_1, v_2, \ldots, v_j, \ldots) = (c^{-1}v_1, c^{-2}v_2, \ldots, c^{-m}v_m, 0, \ldots).
\]
Then $H_m$ is compact since the dimension of the range is finite. Moreover we have
\[
\|(T^{-1} - H_m)(v_1, v_2, \ldots, v_j, \ldots)\|_{l^\infty(\mathbb{K}^{N+1})} = \sup_{j \geq m+1} e^{-j}\|(v_1, v_2, \ldots, v_j, \ldots)\|_{l^\infty(\mathbb{K}^{N+1})},
\]
which implies that $\|T^{-1} - H_m\| \rightarrow 0$ as $m \rightarrow +\infty$ and hence $T^{-1}$ is compact.

Next we show that $T$ is a closed operator. Let $\{v^n\}_{n=1}^{\infty} \subset l^\infty(\mathbb{K}^{N+1})$ be a convergent sequence such that $\lim_{n \rightarrow +\infty} \|v^n - v\|_{l^\infty(\mathbb{K}^{N+1})} = 0$ for some $v \in l^\infty(\mathbb{K}^{N+1})$, and such that $\lim_{n \rightarrow +\infty} \|Tv^n - u\|_{l^\infty(\mathbb{K}^{N+1})} = 0$ for some $u \in l^\infty(\mathbb{K}^{N+1})$. Then we have
\[
\|T^{-1}u - v\|_{l^\infty(\mathbb{K}^{N+1})} = \|T^{-1}u - v^n + v^n - v\|_{l^\infty(\mathbb{K}^{N+1})} = \|T^{-1}u - v^n\|_{l^\infty(\mathbb{K}^{N+1})} + \|v^n - v\|_{l^\infty(\mathbb{K}^{N+1})} \leq \|T^{-1}\| \cdot \|u - Tv^n\|_{l^\infty(\mathbb{K}^{N+1})} + \|v^n - v\|_{l^\infty(\mathbb{K}^{N+1})} \rightarrow 0 \text{ as } n \rightarrow +\infty.
\]
Therefore we have $T^{-1}u - v = 0$. That is, $Tv = u$. $T$ is closed. \qed

Denote by $\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))$ the space of bounded linear operators from $l^\infty(\mathbb{K}^{N+1})$ to $l^\infty(\mathbb{K}^{N+1})$. We have the following two lemmas which will be used when we deal with the integral forms of the relevant abstract ordinary differential equations.

**Lemma 2.5** Let the mapping $T : l^\infty(\mathbb{K}^{N+1}) \rightarrow l^\infty(\mathbb{K}^{N+1})$ be as in Lemma 2.4 and $\lambda \geq 0$. Then the mappings $I - T^{-1}$ and $\lambda I + T^{-1} : l^\infty(\mathbb{K}^{N+1}) \rightarrow l^\infty(\mathbb{K}^{N+1})$ are bounded linear operators with
\[
\|I - T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = 1,
\]
\[
\|\lambda I + T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = \lambda + \frac{1}{c}.
\]
Moreover, if $\lambda \in (0, 1 - 1/c)$ then
\[
\|(1 - \lambda)I - T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = 1 - \lambda.
\]

**Proof** Let $S(1) = \{v \in l^\infty(\mathbb{K}^{N+1}) : \sup_{j \in \mathbb{N}} |v_j| = 1\} \subset l^\infty(\mathbb{K}^{N+1})$. Note that
\[
\|I - T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = \sup_{v \in S(1)} \sup_{j \in \mathbb{N}} (1 - c^{-j})|v_j| \leq \sup_{v \in S(1)} \left( \sup_{j \in \mathbb{N}} |v_j| - \inf_{j \in \mathbb{N}} c^{-j}|v_j| \right) = 1.
\]
Taking $v_0 = \{\frac{j}{j+1} \bar{e}\}_{j=1}^\infty \in S(1)$ where $\bar{e}$ is a unit vector on the boundary of the unit ball of $\mathbb{K}^{N+1}$, we have
\[
\|I - T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = \sup_{v \in S(1)} \sup_{j \in \mathbb{N}} (1 - c^{-j})|v_j|
\geq \sup_{v = v_0} \left( \sup_{j \in \mathbb{N}} \frac{j}{j+1} (1 - c^{-j}) \right)
= 1.
\]
It follows that $\|I - T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = 1$. Moreover, we have
\[
\|\lambda I + T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = \sup_{v \in S(1)} \sup_{j \in \mathbb{N}} (\lambda + c^{-j})|v_j|
\leq \sup_{v \in S(1)} \left( \lambda \sup_{j \in \mathbb{N}} |v_j| + \sup_{j \in \mathbb{N}} c^{-j}|v_j| \right)
= \lambda + \frac{1}{c}.
\]
Taking $v'_0 = \{c^{-(j-1)} \bar{e}\}_{j=1}^\infty \in S(1)$, we have
\[
\|\lambda I + T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = \sup_{v \in S(1)} \sup_{j \in \mathbb{N}} (\lambda + c^{-j})|v_j|
\geq \sup_{v = v'_0} \left( \sup_{j \in \mathbb{N}} c^{-(j-1)} (\lambda + c^{-j}) \right)
= \lambda + \frac{1}{c}.
\]
It follows that $\|\lambda I + T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = \lambda + \frac{1}{c}$.

Finally, we show that $\|(1 - \lambda)I - T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = 1 - \lambda$. Note that we have $1 - \lambda - c^{-j} > 0$ for all $j \in \mathbb{N}$ since $\lambda \in (0, 1 - 1/c)$. Then on the one hand we have
\[
\|(1 - \lambda)I - T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = \sup_{v \in B(1)} \sup_{j \in \mathbb{N}} (1 - \lambda - c^{-j})|v_j|
\leq \sup_{j \in \mathbb{N}} (1 - \lambda - c^{-j})
= 1 - \lambda.
\]
On the other hand,
\[
\|(1 - \lambda)I - T^{-1}\|_{\mathcal{L}(l^\infty(\mathbb{K}^{N+1}); l^\infty(\mathbb{K}^{N+1}))} = \sup_{v \in B(1)} \sup_{j \in \mathbb{N}} (1 - \lambda - c^{-j})|v_j|
\geq \sup_{v = v'_0} \left( \sup_{j \in \mathbb{N}} (1 - \lambda - c^{-j})|v_j| \right)
= \sup_{j \in \mathbb{N}} (1 - \lambda - c^{-j}) \frac{j}{j+1}
= 1 - \lambda.
\]
It follows that \( \| (1 - \lambda)I - T^{-1} \|_{\mathcal{L}(l^\infty(K^{N+1});l^\infty(K^{N+1}))} = 1 - \lambda. \) \( \square \)

**Lemma 2.6** Let the mapping \( T : l^\infty(K^{N+1}) \rightarrow l^\infty(K^{N+1}) \) be as in Lemma 2.4. Then for every \( \lambda \geq 0 \), the mapping \((\lambda T + I)^{-1} : l^\infty(K^{N+1}) \rightarrow \ell_c(K^{N+1}) \subset l^\infty(K^{N+1})\) is continuous with norm
\[
\| (\lambda T + I)^{-1} \|_{\mathcal{L}(l^\infty(K^{N+1});l^\infty(K^{N+1}))} = \frac{1}{c\lambda + 1}.
\]

**Proof** We compute \( \| (\lambda T + I)^{-1} \|_{\mathcal{L}(l^\infty(K^{N+1});l^\infty(K^{N+1}))} \). Let \( S(1) = \{ v \in l^\infty(K^{N+1}) : \sup_{j \in \mathbb{N}} |v_j| = 1 \} \subset l^\infty(K^{N+1}). \) Note that
\[
\| (\lambda T + I)^{-1} \|_{\mathcal{L}(l^\infty(K^{N+1});l^\infty(K^{N+1}))} = \sup_{v \in S(1)} \sup_{j \in \mathbb{N}} \frac{|v_j|}{\lambda c^j + 1}
\]
\[
= \sup_{v \in S(1)} \sup_{j \in \mathbb{N}} \frac{1}{\lambda c^j + 1}
\]
\[
\leq \sup_{j \in \mathbb{N}} \frac{1}{\lambda c^j + 1}
\]
\[
= \frac{1}{c\lambda + 1}.
\]

Taking \( v_0 = \{ c^{-(j-1)}e_j \}_{j=1}^{\infty} \in B_c(1) \), we have
\[
\| (\lambda T + I)^{-1} \|_{\mathcal{L}(l^\infty(K^{N+1});l^\infty(K^{N+1}))} = \sup_{v \in S(1)} \sup_{j \in \mathbb{N}} \frac{|v_j|}{\lambda c^j + 1}
\]
\[
\geq \sup_{v \in S(1)} \sup_{j \in \mathbb{N}} \frac{1}{\lambda c^j + 1} |v_j|
\]
\[
\geq \sup_{j \in \mathbb{N}} \frac{1}{\lambda c^j + 1}
\]
\[
= \frac{1}{c\lambda + 1}.
\]

It follows that \( \| (\lambda T + I)^{-1} \|_{\mathcal{L}(l^\infty(K^{N+1});l^\infty(K^{N+1}))} = \frac{1}{c\lambda + 1}. \) \( \square \)

The following three lemmas address the well-posedness of system (1.11) and the analyticity of the map \( H \).

**Lemma 2.7** Assume (A1)-(A2). For every sequence \( \{(u_i, v_i)\}_{i=0}^{+\infty} \subset U \times V \), let \( \mu_i = (u_i, u_{i+1}, \ldots, u_{i+M-1}, v_i) \in U^M \times V \). Then we have
\[
\lim_{j \to +\infty} \frac{1}{c^j} \prod_{i=0}^{j-1} |1 - g(\mu_i)| = 0,
\]
Moreover, for every \( m \in \mathbb{N} \), we have
\[
\lim_{j \to +\infty} \frac{j^m}{c^j} \prod_{i=0}^{j-1} |1 - g(\mu_i)| = 0.
\]
Proof By (A2), we have $|1 - g(\gamma_1, \gamma_2)| < c$ and $|1 - g(\gamma_1, \gamma_2)|$ with $(\gamma_1, \gamma_2) \in \mathcal{U}^M \times \mathcal{V}$ has a supremum less than $c$. Let $s > 0$ be such that $c = e^s$. Then there exists $N_0 \geq 1$, $N_0 \in \mathbb{N}$ so that $|1 - g(\mu_i)| \leq e^{s(1 - \frac{1}{N_0})}$ for all $i \in \mathbb{N}$. Then for every $n \in \mathbb{N}$ we have
\[
|1 - g(\mu_i)| \leq e^{s(1 - \frac{1}{N_0})} \leq e^{s(1 - \frac{1}{N})} \text{ for all } i \geq nN_0.
\]
It follows that $\ln \left(\frac{|1 - g(\mu_i)|}{c}\right) \leq -\frac{ns}{i}$ for all $i \geq nN_0$. Then for $j > nN_0$ we have
\[
j^{-1} \sum_{i=0}^{j-1} \ln \left(\frac{|1 - g(\mu_i)|}{c}\right) = \sum_{i=0}^{nN_0-1} \ln \left(\frac{|1 - g(\mu_i)|}{c}\right) + \sum_{i=nN_0}^{j-1} \ln \left(\frac{|1 - g(\mu_i)|}{c}\right)
\leq \sum_{i=0}^{nN_0-1} \ln \left(\frac{|1 - g(\mu_i)|}{c}\right) + s \sum_{i=nN_0}^{j-1} \left(-\frac{n}{i}\right).
\tag{2.8}
\]
Let $c_0 = \sum_{i=0}^{nN_0-1} \ln \left(\frac{|1 - g(\mu_i)|}{c}\right)$. Then by (2.8) and (A2), we have
\[
0 < \frac{1}{c^j} \prod_{i=0}^{j-1} |1 - g(\mu_i)| = \exp \left(\sum_{i=0}^{j-1} \ln \left(\frac{|1 - g(\mu_i)|}{c}\right)\right)
\leq e^{c_0} \exp \left(s \sum_{i=nN_0}^{j-1} \left(-\frac{n}{i}\right)\right).
\tag{2.9}
\]
Taking limits as $j \to +\infty$ in (2.9) we have
\[
\lim_{j \to +\infty} \frac{1}{c^j} \prod_{i=0}^{j-1} |1 - g(\mu_i)| = 0.
\]
Choosing $n = m$ in the inequality (2.9), we have
\[
0 < \frac{j^m}{c^j} \prod_{i=0}^{j-1} |1 - g(\mu_i)| \leq j^me^{c_0} \exp \left(s \sum_{i=mN_0}^{j-1} \left(-\frac{m}{i}\right)\right)
= \exp \left(c_0 + \sum_{i=1}^{mN_0-1} \left(\frac{m}{i}\right) + \frac{m}{j}\right) \exp \left(mH_j\right)
= \exp \left(c_0 + \sum_{i=1}^{mN_0-1} \left(\frac{m}{i}\right) + \frac{m}{j}\right) \exp(m \ln j - mH_j),
\tag{2.10}
\]
where $H_j = 1 + \frac{1}{2} + \cdots + \frac{1}{j}$ and $\sum_{i=1}^{mN_0-1} \left(\frac{m}{i}\right)$ is regarded 0 if $mN_0 = 1$. We note that $\lim_{j \to +\infty} \ln j - H_j = -\gamma$ where $\gamma > 0$ is the Euler-Másccheroni constant. Taking supremum limits as $j \to +\infty$ in (2.10) we have
\[
0 < \limsup_{j \to +\infty} \frac{j^m}{c^j} \prod_{i=0}^{j-1} |1 - g(\mu_i)| \leq \exp \left(c_0 + \sum_{i=1}^{mN_0-1} \left(\frac{m}{i}\right) - m\gamma\right) < +\infty.
\]
Then we have
\[
\lim_{j \to +\infty} \frac{\partial^{m-1}}{\partial^j} \prod_{i=0}^{j-1} |1 - g(\mu_i)| \\
\leq \limsup_{j \to +\infty} \frac{\partial^m}{\partial^j} \prod_{i=0}^{j-1} |1 - g(\mu_i)| \lim_{j \to +\infty} \frac{1}{j} \\
= 0.
\]

Since \( m \in \mathbb{N} \) is arbitrary, it follows that \( \lim_{j \to +\infty} \frac{\partial^m}{\partial^j} \prod_{i=0}^{j-1} |1 - g(\mu_i)| = 0 \) for all \( m \in \mathbb{N} \).

Let \( l^\infty(U \times V) \) be the subset of \( l^\infty(\mathbb{K}^{N+1}) \) defined by
\[
l^\infty(U \times V) = \prod_{j=0}^{\infty} (U \times V).
\]

Note that \( l^\infty(U \times V) \) is not an open set of \( l^\infty(\mathbb{K}^{N+1}) \) if \( l^\infty(\mathbb{K}^{N+1}) \) is equipped with the product topology. However, we are concerned with the following set:
\[
A = \{ w = (w_0, w_1, \cdots) \in l^\infty(U \times V) : \{ w_j \}_{j=0}^{\infty} \subset Q_0 \text{ for some compact } Q_0 \subset U \times V \}. \tag{2.11}
\]

For every \( w = (w_0, w_1, \cdots) \in A \), we can find an open set \( P \) and a compact set \( Q \) such that \( \{ w_j \}_{j=0}^{\infty} \subset P \subset Q \subset U \times V \). Then \( w \in l^\infty(P) \subset A \subset l^\infty(U \times V) \). Namely, \( A \) is open under the box topology.

We also define the projections \( \chi_i : l^\infty(U \times V) \to U^M \times V \) with \( i \in \{0, 1, 2, \cdots \} \) by
\[
\chi_i(w) = (u_i, u_{i+1}, \cdots, u_{i+M-1}, v_i) \tag{2.12}
\]
for every \( w = ((u_i, v_i))_{i=1}^{\infty} \in l^\infty(U \times V) \).

**Lemma 2.8** Let \( A \) be defined at (2.11). Assume (A1 – A2). The mapping \( G \) defined by
\[
G : A \ni w = (w_0, w_1, w_2, \cdots, w_i, \cdots) \to G(w) = \left( \frac{1}{\partial^{j-1}} \prod_{i=0}^{j-1} (1 - g(\chi_i(w))) \right)_{j=1}^{+\infty},
\]
where \( w_i = (u_i, v_i) \in U \times V \), is continuous and analytic from \( A \subset l^\infty(U \times V) \) to \( l^\infty(\mathbb{C}^{N+1}) \).

**Proof** By Lemma 2.7 we know that \( G \) is a mapping from \( l^\infty(\mathbb{C}^{N+1}) \) to \( l^\infty(\mathbb{C}^{N+1}) \). Note that for every \( i, j \in \mathbb{N} \) with \( 0 \leq i \leq j - 1 \), we have
\[
\frac{\partial}{\partial \mu_i} \prod_{i=0}^{j-1} (1 - g(\mu_i)) = \frac{-\frac{\partial}{\partial \mu_i} g(\mu_i)}{(1 - g(\mu_i))} \prod_{i=0}^{j-1} (1 - g(\mu_i)). \tag{2.13}
\]
Let \((\mu_0, \mu_1, \ldots, \mu_{j-1})\) denote a column vector in \(\bigoplus_{i=0}^{j-1} \mathbb{C}^{MN+1}\). Then we have

\[
\frac{\partial}{\partial(\mu_0, \mu_1, \ldots, \mu_{j-1})} \prod_{i=0}^{j-1} (1 - g(\mu_i)) = \left( \prod_{i=0}^{j-1} (1 - g(\mu_i)) \right) \left( -\frac{\partial g(\mu_0)}{\partial \mu_0} \frac{1}{1 - g(\mu_0)}, -\frac{\partial g(\mu_1)}{\partial \mu_1} \frac{1}{1 - g(\mu_1)}, \ldots, -\frac{\partial g(\mu_{j-1})}{\partial \mu_{j-1}} \frac{1}{1 - g(\mu_{j-1})} \right),
\]

which is also regarded as a column vector in \(\bigoplus_{i=0}^{j-1} \mathbb{C}^{MN+1}\).

For every \(\epsilon > 0\), choose \(\delta = \epsilon\), for every \(w_1 = (w_{1i})\), \(w_2 = (w_{2i}) \in A\) with \(|w_1 - w_2|_{l^\infty(\mathbb{C}^{N+1})} < \delta\), by (2.13) and the Integral Mean Value Theorem, we have

\[
|G(w_1) - G(w_2)|_{l^\infty(\mathbb{C}^{N+1})} = \sup_{j \in \mathbb{N}} \frac{1}{c_j} \left| \prod_{i=0}^{j-1} (1 - g(\chi_i(w_1))) - \prod_{i=0}^{j-1} (1 - g(\chi_i(w_2))) \right| \\
\leq \sup_{j \in \mathbb{N}} \frac{1}{c_j} \left| \prod_{i=0}^{j-1} (1 - g(\bar{\chi}_i)) \right| \sum_{i=0}^{j-1} \frac{\partial g(\bar{\chi}_i)}{1 - g(\bar{\chi}_i)} (\chi_i(w_1) - \chi_i(w_2)) \\
= \sup_{j \in \mathbb{N}} \frac{j}{c_j} \prod_{i=0}^{j-1} |1 - g(\bar{\chi}_i)| \cdot M_0 \frac{l}{jM_0} |w_1 - w_2|_{l^\infty(\mathbb{C}^{N+1})} \\
= \frac{M_0 M_1}{l} \epsilon,
\]

where \(\bar{\chi}_i = \chi_i(w_1) + \theta(\chi_i(w_1) - \chi_i(w_2))\) for some \(\theta \in [0, 1]\). By (A2) we have \(l < |1 - g(\bar{\chi}_i)| < c\). By (A1), there exists \(M_0 > 0\) so that \(|\partial g(\bar{\chi}_i)| < M_0\). By Lemma 2.7, there exists \(M_1 > 0\) so that \(\sup_{j \in \mathbb{N}} \frac{j}{c_j} \prod_{i=0}^{j-1} |1 - g(\bar{\chi}_i)| < M_1\). It follows that

\[
|G(w_1) - G(w_2)|_{l^\infty(\mathbb{C}^{N+1})} \leq \frac{M_0 M_1}{l} \epsilon,
\]

which implies that \(G\) is continuous. Next, we show that for every \(w = (w_i) \in A \subset l^\infty(U \times V)\), and for all \(h = (h_j) \in l^\infty(\mathbb{C}^{N+1})\), the mapping \(\mathcal{G} : t \rightarrow G(w + th)\) is analytic in the neighborhood of \(0 \in \mathbb{C}\). Denote by \(\bar{G}h\) the sequence

\[
\left( \frac{1}{c_j} \left( \prod_{i=0}^{j-1} (1 - g(\chi_i(w))) \right) \sum_{i=0}^{j-1} \frac{-\partial g(\chi_i(w))}{1 - g(\chi_i(w))} \chi_i(h) \right)_{j=1}^\infty.
\]
Then by the same argument leading to (2.14), we know that \( Gh \in l^\infty(\mathbb{C}^{N+1}) \) and

\[
\left| \frac{G(w + th) - G(w)}{t} - Gh \right|_{l^\infty(\mathbb{C}^{N+1})}
\]

\[
= \sup_{j \in \mathbb{N}} \frac{1}{c^j} \left| \frac{1}{t} \left( \prod_{i=0}^{j-1} (1 - g(\chi_i(w + th))) - \prod_{i=0}^{j-1} (1 - g(\chi_i(w))) \right) \right.
- \left. \left( \prod_{i=0}^{j-1} (1 - g(\chi_i(w))) \right) \sum_{i=0}^{j-1} \frac{\frac{\partial}{\partial \chi_i} g(\chi_i(w)) \chi_i(h)}{1 - g(\chi_i)} \right|
\]

\[
\leq \sup_{j \in \mathbb{N}} \frac{1}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\tilde{\chi}_i)) \right) \left( \sum_{i=0}^{j-1} \frac{\frac{\partial}{\partial \chi_i} g(\tilde{\chi}_i) \chi_i(h)}{1 - g(\tilde{\chi}_i)} \right) \right|
+ \sup_{j \in \mathbb{N}} \frac{1}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\chi_i(w))) \right) \left( \sum_{i=0}^{j-1} \frac{\frac{\partial}{\partial \chi_i} g(\chi_i(w)) \chi_i(h)}{1 - g(\chi_i)} \right) \right |
\]

(2.15)

where \( \tilde{\chi}_i = \chi_i(w + t\theta h) \) for some \( \theta \in [0, 1] \). By applying the same argument leading to (2.14) on the first term of the last inequality of (2.15) and by Lemma 2.7, we have

\[
\lim \sup_{t \to 0} \frac{1}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\tilde{\chi}_i)) \right) \left( \sum_{i=0}^{j-1} \frac{\frac{\partial}{\partial \chi_i} g(\tilde{\chi}_i) \chi_i(h)}{1 - g(\tilde{\chi}_i)} \right) \right |
\leq \lim \sup_{t \to 0} \frac{1}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\tilde{\chi}_i)) \right) \left( \sum_{i=0}^{j-1} \frac{\frac{\partial}{\partial \chi_i} g(\tilde{\chi}_i) \chi_i(h)}{1 - g(\tilde{\chi}_i)} \right) \right |
\leq \lim \sup_{t \to 0} \frac{1}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\tilde{\chi}_i)) \right) \sum_{i=0}^{j-1} \frac{\frac{\partial}{\partial \chi_i} g(\tilde{\chi}_i)}{1 - g(\tilde{\chi}_i)} \theta t \chi_i(h) \right |
\]

\[
= \lim \sup_{t \to 0} \frac{1}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\tilde{\chi}_i)) \right) \frac{j M_0^2}{l^2} |\theta t|^2 \right |
\]

\[
= \lim \sup_{t \to 0} \frac{j^2}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\tilde{\chi}_i)) \right) \frac{M_0^2}{l^2} |\theta t|^2 \right |
\]

\[
= 0. \quad (2.16)
\]

where \( \tilde{\chi}_i = \chi_i(w + t\theta'h) \) for some \( \theta' \in [0, 1] \). By (A1), there exists \( M_2 > 0 \) so that \( |\frac{\partial^2}{\partial \chi_i^2} g(\chi_i(w))| < M_2 \) for every \( w \in A \) and \( i \in \mathbb{N} \). Then it follows from the Integral Mean Value Theorem that the second term of the last inequality of (2.15) satisfies
that

\[
\sup_{j \in \mathbb{N}} \frac{1}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\tilde{x}_i)) \right) \left( \sum_{i=0}^{j-1} -\frac{\partial}{\partial \chi_i} g(\chi_i) \chi_i(h) - \sum_{i=0}^{j-1} -\frac{\partial}{\partial \chi_i} g(\chi_i(w)) \chi_i(h) \right) \right| - \sum_{i=0}^{j-1} -\frac{\partial}{\partial \chi_i} g(\chi_i(w)) \chi_i(h)
\]

\[
+ \sum_{i=0}^{j-1} -\frac{\partial}{\partial \chi_i} g(\chi_i(w)) \chi_i(w) - \sum_{i=0}^{j-1} -\frac{\partial}{\partial \chi_i} g(\chi_i(w)) \chi_i(w)
\]

\[
\leq \sup_{j \in \mathbb{N}} \frac{1}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\tilde{x}_i)) \right) \left( \frac{jM_2 t}{l} |h|^2_{l^\infty(\mathbb{C}^{N+1})} \right) \right| + \left( -\frac{\partial}{\partial \chi_i} g(\chi_i(w)) \chi_i(h) \right) \sum_{i=0}^{j-1} \left( \frac{1}{1 - g(\tilde{x}_i)} - \frac{1}{1 - g(\chi_i(w))} \right)
\]

\[
= \sup_{j \in \mathbb{N}} \frac{1}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\tilde{x}_i)) \right) \left( \frac{jM_2 t}{l} |h|^2_{l^\infty(\mathbb{C}^{N+1})} \right) \right| + M_0 |h|_{l^\infty(\mathbb{C}^{N+1})} \sum_{i=0}^{j-1} \left( \frac{g(\tilde{x}_i) - g(\chi_i(w))}{1 - g(\tilde{x}_i)} \right)
\]

\[
\leq \sup_{j \in \mathbb{N}} \frac{1}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\tilde{x}_i)) \right) \left( \frac{jM_2 t}{l} |h|^2_{l^\infty(\mathbb{C}^{N+1})} \right) \right| + jM_2^2 |h|^2_{l^\infty(\mathbb{C}^{N+1})} t
\]

Then by Lemma 2.7 we have

\[
\lim_{t \to 0} \sup_{j \in \mathbb{N}} \frac{1}{c^j} \left| \left( \prod_{i=0}^{j-1} (1 - g(\tilde{x}_i)) \right) \left( \sum_{i=0}^{j-1} -\frac{\partial}{\partial \chi_i} g(\chi_i) \chi_i(h) - \sum_{i=0}^{j-1} -\frac{\partial}{\partial \chi_i} g(\chi_i(w)) \chi_i(h) \right) \right| = 0.
\] (2.17)

By (2.15), (2.16) and (2.17) we have

\[
\lim_{t \to 0} \left| \frac{G(w + th) - G(w) - \tilde{G}h}{t} \right|_{l^\infty(\mathbb{C}^{N+1})} = 0.
\]

Lemma 2.9 Assume (A1 – A2). Let the set A and the map G be as in Lemma 2.8. Define H : A ⊂ l^\infty(U \times V) → l^\infty(\mathbb{K}^{N+1}) by

\[
H(\theta) = (F_j(\theta)G_j(\theta))_{j=1}^{\infty} \in l^\infty(\mathbb{K}^{N+1}),
\]

where

\[
\theta = (\theta_1, \theta_2, \ldots, \theta_j, \ldots) = ((u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots) \in l^\infty(\mathbb{K}^{N+1}),
\]

\[
F_j(\theta) = \left( \frac{f(u_j, u_{j+1})}{1 - g(u_j, u_{j+1}, \ldots, u_{j+M-1}, v_j)} g(u_j, u_{j+1}, \ldots, u_{j+M-1}, v_j) \right)
\]

\[
\left( 1 - g(u_j, u_{j+1}, \ldots, u_{j+M-1}, v_j) \right)^2.
\]
for \( j \geq 1, j \in \mathbb{N} \). Then \( H : \bar{A} \rightarrow l^\infty(\mathbb{K}^{N+1}) \) is completely continuous and is analytic.

**Proof** According to Lemma 2.2, we only need to show that for every bounded set \( B \subset l^\infty(\mathbb{K}^{N+1}) \), \( H(B) \) is bounded in \( l_\infty^m(\mathbb{K}^{N+1}) \). By (A1)-(A2), we know that \( F(B) \) is bounded in \( l^\infty(\mathbb{K}^{N+1}) \). Then by Lemma 2.7, we have

\[
\lim_{j \to +\infty} j^m |F_j(\theta)G_j(\theta)| = 0.
\]

Therefore we have \( H(\theta) \in l_\infty^m(\mathbb{K}^{N+1}) \). By Lemma 2.2, \( H \) is completely continuous. Then by (A1)-(A2), \( F : l^\infty(\mathbb{C}^{N+1}) \ni \theta \rightarrow F(\theta) \in l^\infty(\mathbb{C}^{N+1}) \) is analytic. Then by the similar procedure for the proof of Lemma 2.8, we can show that for every \( w = (w_i) \in A \subset l^\infty(U \times V) \), and for all \( h = (h_i) \in l^\infty(\mathbb{C}^{N+1}) \), the mapping \( \mathcal{G} : t \rightarrow H(w + th) \) is analytic in the neighborhood of \( 0 \in \mathbb{C} \). \( \square \)

### 3 Main Results

Let \( \Omega \) be a bounded closed ball in \( \mathbb{K} \). We denote by \( C(\Omega; l^\infty(\mathbb{K}^{N+1})) \) the space of continuous functions \( u : \Omega \ni t \to u(t) \in l^\infty(\mathbb{K}^{N+1}) \) and denote by \( C^1(\Omega; l^\infty(\mathbb{K}^{N+1})) \) the space of continuously differentiable functions \( u : \Omega \ni t \to u(t) \in l^\infty(\mathbb{K}^{N+1}) \). Then it is clear that \( C(\Omega; l^\infty(\mathbb{K}^{N+1})) \) and \( C^1(\Omega; l^\infty(\mathbb{K}^{N+1})) \) are Banach spaces equipped, respectively, with the norms \( \|u\| = \max_{t \in \Omega} |u(t)|_{l^\infty(\mathbb{K}^{N+1})} \) and

\[
\|u\| = \max\{\max_{t \in \Omega} |u(t)|_{l^\infty(\mathbb{K}^{N+1})}, \max_{t \in \Omega} |u'(t)|_{l^\infty(\mathbb{K}^{N+1})}\}.
\]

**Theorem 3.1** Assume (A1 - A2). Let \( (x, \tau) \in \mathbb{R}^{N+1} \) be a bounded solution of system (1.4). Suppose that there exists a compact set \( Q \subset U \times V \) such that \( (x(t), \tau(t)) \in Q \) for all \( t \in \mathbb{R} \). Then \( (x, \tau) \) is analytic on \( \mathbb{R} \).

**Proof** We define \( ((y_j, z_j))_{j=1}^\infty \subset C(\mathbb{R}; l_c^\infty(\mathbb{R}^{N+1})) \) by

\[
(y_j(t), z_j(t)) = \left(\frac{1}{c^j} \eta_1^{j-1}(t), \frac{1}{c^j} \eta_2^{j-1}(t)\right) \quad \text{for } j \geq 1, j \in \mathbb{N}, t \in \mathbb{R}.
\]

Then by the derivation in Section 1, for every \( t \in \mathbb{R} \), \( ((y_j(t), z_j(t)))_{j=1}^\infty \subset l_c^\infty(\mathbb{R}^{N+1}) \) satisfies system (1.9).

Let

\[
F(\theta) = (F_1(\theta), F_2(\theta), \ldots, F_j(\theta), \ldots),
\]

where

\[
\theta = (\theta_1, \theta_2, \ldots, \theta_j, \ldots) = ((u_1, v_1), (u_2, v_2), \ldots, (u_j, v_j), \ldots) \in l^\infty(\mathbb{R}^{N+1}),
\]

\[
F_j(\theta) = \left(\frac{f(u_j, u_{j+1})}{1 - g(u_j, u_{j+1}, \ldots, u_{j+M-1}, v_j)}, \frac{g(u_j, u_{j+1}, \ldots, u_{j+M-1}, v_j)}{1 - g(u_j, u_{j+1}, \ldots, u_{j+M-1}, v_j)}\right),
\]

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for \( j \geq 1, j \in \mathbb{N} \).

Let \( T \) be as in Lemma 2.4, \( G \) in Lemma 2.8. Then \( w = ((y_1, z_1), (y_2, z_2), \cdots) \in C(\mathbb{R}; l^\infty_c(\mathbb{R}^{N+1})) \) is a solution of the following ordinary differential equation

\[
\frac{d}{dt} w(t) = H(Tw(t)), \tag{3.2}
\]

where \( H(T(w)) = (F_1(T(w))G_1(T(w)), F_2(T(w))G_2(T(w)), \cdots) \in l^\infty(\mathbb{R}^{N+1}) \) and \( G_j \) is the \( j \)-th coordinate of \( G \). Moreover, we notice that for every \( j \in \mathbb{N} \),

\[
\{(c^j y_j(t), c^j z_j(t)) : t \in \mathbb{R}\} = \{(x(t), y(t)) : t \in \mathbb{R}\} \subset Q.
\]

Then we have \( \{(c^j y_j(t), c^j z_j(t))_{j=1}^\infty : t \in \mathbb{R}\} \subset Q \) and

\[
Tw(t) = (c^j y_j(t), c^j z_j(t))_{j=1}^\infty \in A
\]

for every \( t \in \mathbb{R} \), where \( A \) is defined by (2.11). Let \( w_{t_0} = ((y_j(t_0), z_j(t_0)))_{j=1}^\infty \in l^\infty_c(\mathbb{R}^{N+1}), t_0 \in \mathbb{R} \). Then \( w(t) \) is a solution of the following initial value problem

\[
\begin{cases}
\frac{d}{dt} w(t) = H(Tw(t)), \\
w(t_0) = w_{t_0}.
\end{cases} \tag{3.3}
\]

To prove the existence of complex extension of \( w(t) \in l^\infty_c(\mathbb{R}^{N+1}) \), we put \( \nu(t) = Tw(t) \in l^\infty(\mathbb{C}^{N+1}) \) and consider equation (3.3) in \( l^\infty(\mathbb{C}^{N+1}) \). Then equation (3.3) is transformed into the following integral equation

\[
T^{-1} \nu(t) = w_{t_0} + \int_{t_0}^t H(\nu(s))ds, \tag{3.4}
\]

where the integral is taken along the linear path \( \xi \to t_0 + \xi(t-t_0), 0 \leq \xi \leq 1 \).

Denote by \( \Omega_h = \{t \in \mathbb{C} : |t-t_0| \leq h\} \) for some \( h > 0 \). To prove the existence and uniqueness of the solution using the Uniform Contraction Principle, we consider fixed point problem associated with the following mapping

\[
L_0(\nu)(t) = (I - T^{-1})\nu(t) + w_{t_0} + \int_{t_0}^t H(\nu(s))ds, \tag{3.5}
\]

on \( C(\Omega_h; A) \). However, by Lemma 2.5 \( L_0 \) is not contractive on \( C(\Omega_h; l^{\infty}(\mathbb{C}^{N+1})) \) in general, but its perturbation \( L : C(\Omega_h; l^{\infty}(\mathbb{C}^{N+1})) \times [0, 1] \to C(\Omega_h; l^{\infty}(\mathbb{C}^{N+1})) \) defined by

\[
L(\nu, \lambda)(t) = ((1 - \lambda)I - T^{-1})\nu(t) + (T^{-1} + \lambda I)\nu_{t_0} + \int_{t_0}^t H(\nu(s))ds, \tag{3.6}
\]

is contractive on \( C(\Omega_h; l^{\infty}(\mathbb{C}^{N+1})) \) for \( \lambda \in (0, 1 - 1/c) \) and some \( h > 0 \), where \( \nu_{t_0} = Tw_{t_0} \).
If there exists a \( \nu \in C(\Omega_h; A) \) such that \( L(\nu) = \nu \), then \( \nu \) is a solution of the following initial value problem:

\[
\begin{aligned}
(\lambda I + T^{-1}) \frac{d}{dt} \nu(t) &= H(\nu(t)), \\
\nu(t_0) &= Tw_{t_0}.
\end{aligned}
\]  

(3.7)

Writing (3.7) in integral form, we have

\[
T^{-1} \nu(t) = w_{t_0} + \int_{t_0}^t (\lambda T + I)^{-1} H(\nu(s)) \, ds.
\]  

(3.8)

By Lemma 2.9, \( H \) is analytic and \( L \) is an analytic mapping from \( A \subset l^\infty(\mathbb{C}^{N+1}) \) to \( l^\infty(\mathbb{C}^{N+1}) \).

We organize the remaining part of the proof as follows: We first show with claims 1 and 2 the existence and uniqueness of solutions \( \nu_\lambda \) of system (3.8) and with claim 3 \( \nu_\lambda \) satisfies that \( \lim_{\lambda \to 0^+} T^{-1} \nu_\lambda = w_0 \in C(\Omega_{h_0}; T^{-1}(\tilde{A})) \) for some \( h_0 > 0 \). Secondly, we show with claim 4 that the right hand side of system (3.8) is coordinate-wise convergent to that of system (3.4) with \( \lim_{n \to +\infty} H(\nu_{\lambda_n}(t)) = H(\nu_0(t)) \), where \( \nu_0 : \Omega_{h_0} \to \tilde{A} \) is a map such that \( H(\nu_0) \) is continuous in \( t \in \Omega_{h_0} \) and the sequence \( \{\lambda_n\}_{n=1}^\infty \) is in \( (0, 1 - \frac{1}{h_0}) \) with \( \lim_{n \to +\infty} \lambda_n = 0 \). Lastly, we show with claim 5 that \( H(Tw_0) = H(\nu_0) \) which implies that \( w_0 \) satisfies system (3.4) and hence it is the solution of the initial value problem (3.2).

Now we show the following

**Claim 1**: For every \( \lambda \in (0, 1) \), there exists \( h > 0 \) such that there exists one and only one point \( \nu_\lambda \in C(\Omega_h; \tilde{A}) \) such that \( L(\nu_\lambda, \lambda) = \nu_\lambda \) and \( \nu_\lambda \) is analytic and is differentiable with respect to \( \lambda \).

**Proof of Claim 1**: We only need to show that \( L \) is a contractive mapping in some closed neighborhood of \( w_{t_0} \) in \( C(\Omega_h; l^\infty(\mathbb{C}^{N+1})) \) where \( \Omega_h = \{t \in \mathbb{C} : |t - t_0| \leq h\} \) for some \( h > 0 \) to be determined. Denote by \( \| \cdot \|_C \) the supremum norm on the Banach space \( C(\Omega_h; l^\infty(\mathbb{C}^{N+1})) \). For every \( w_1, w_2 \in C(\Omega_h; l^\infty(\mathbb{C}^{N+1})) \) we have

\[
\begin{aligned}
\| L(w_1, \lambda) - L(w_2, \lambda) \|_C &= \max_{t \in \Omega_h} \| (1 - \lambda)I - T^{-1} w_1(t) + \int_{t_0}^t H(w_1(s)) \, ds \\
&\quad - (1 - \lambda)I - T^{-1} w_2(t) - \int_{t_0}^t H(w_2(s)) \, ds \|_{l^\infty(\mathbb{C}^{N+1})} \\
&\leq \max_{t \in \Omega_h} \| (1 - \lambda)I - T^{-1} \|_{l^\infty(\mathbb{C}^{N+1})} \| H(\nu_1(t)) - H(\nu_2(t)) \|_{l^\infty(\mathbb{C}^{N+1})} \\
&\quad + \max_{t \in \Omega_h} \int_{t_0}^t \| H(w_1(s)) - H(w_2(s)) \|_{l^\infty(\mathbb{C}^{N+1})} \, ds.
\end{aligned}
\]  

(3.9)
Since $H$ is analytic on $A$, there exist constants $\delta > 0$ and $l_0 > 0$ so that $|H(\nu_1) - H(\nu_2)|_{l_\infty(C^{N+1})} \leq l_0|\nu_1 - \nu_2|_{l_\infty(C^{N+1})}$ for every $\nu_1, \nu_2 \in A$ with $|\nu_1 - \nu(t_0)|_{l_\infty(C^{N+1})} \leq \delta, |\nu_2 - \nu(t_0)|_{l_\infty(C^{N+1})} \leq \delta$.

Let $X = \{\nu \in C(\Omega_h; \overline{A}) : \max_{t \in \Omega_h} |\nu(t) - \nu(t_0)|_{l_\infty(C^{N+1})} \leq \delta\}$. Then $X$ is a closed subset of the Banach space $C(\Omega_h; l_\infty(C^{N+1}))$. By (3.9) and by Lemma 2.5, we have

$$
\|L(w_1, \lambda) - L(w_2, \lambda)\|_{C} \leq (1 - \lambda)\|w_1 - w_2\|_{C} + l_0h\|w_1 - w_2\|_{C} = (1 - \lambda + l_0h)\|w_1 - w_2\|_{C},
$$

for every $w_1, w_2 \in X$. Therefore, if $h \in (0, \frac{\lambda}{l_0})$, then $1 - \lambda + l_0h \in (0, 1)$. Moreover, we choose $h > 0$ small enough so that

$$
\max_{t \in \Omega_h} \|L(\nu(t), \lambda) - \nu(t_0)\|_{l_\infty(C^{N+1})} = \max_{t \in \Omega_h} \left\|((1 - \lambda)I - T^{-1})(\nu(t) - \nu(t_0)) + \int_{t_0}^{t} H(\nu(s))ds\right\|_{l_\infty(C^{N+1})} \leq \delta.
$$

Then by the Uniform Contraction Principle in Banach spaces, we know that $L(\cdot, \lambda) : X \rightarrow X$ is a contractive mapping with a unique fixed point $\nu_{\lambda} \in C(\Omega_h; \overline{A})$ and $\nu_{\lambda}$ is analytic. Noticing that $L$ is linear in $\lambda$, $\nu_{\lambda}$ is differentiable with respect to $\lambda$. This completes the proof of Claim 1.

**Claim 2:** There exists $h_0 > 0$ and so that $\Omega_{h_0}$ is the common existence region of the fixed points $\nu_{\lambda}$ of $L(\nu, \lambda)$ for all $\lambda \in (0, 1 - 1/c)$.

**Proof of Claim 2:** Let $w_{\lambda} = T^{-1}\nu_{\lambda}, \nu_{\lambda} \in X$ where $X$ is as in Claim 1. Note that $\nu_{\lambda} \in C(\Omega_{h_0}; \overline{A})$ where $h_{\lambda} > 0$ is a constant depending on $\lambda$. Let $\overline{M} > 0$ be the supremum of $\|H(\nu)\|_{l_\infty(C^{N+1})}$ on $\overline{A}$. Let $0 < \beta \leq +\infty$ be such that $\{t \in \mathbb{C} : |t - t_0| < \beta\}$ is the maximal existence region of $\nu_{\lambda}(t)$ on $\overline{A}$. If $\beta = +\infty$, then $\nu_{\lambda}$ can be extended to the whole complex plane $\mathbb{C}$ with $\nu_{\lambda}(t) \in l_\infty(\overline{U} \times \overline{V})$ for all $t \in \mathbb{C}$. Otherwise, by Theorem 10.5.5 of [2], there exists $t_1 \in \{t \in \mathbb{C} : |t - t_0| < \beta\}$ so that $\nu_{\lambda}$ achieves value in the boundary of $A$. Let $B$ denote the boundary of $A$. Let $r$ be defined by

$$
r = \inf_{\nu \in B} \|T^{-1}(\nu - \nu_{t_0})\|_{l_\infty(C^{N+1})}.
$$

Now we show that $r > 0$. Suppose not. Note that by Lemma 2.4, $T^{-1}$ is compact and $B$ is closed and bounded in $l_\infty(C^{N+1})$. Therefore $r$ is the minimum norm of a compact set. There exists $\nu^* \in B$ such that $r = \|T^{-1}(\nu^* - \nu_{t_0})\|_{l_\infty(C^{N+1})} = 0$. Then we have $\nu_{t_0} = \nu(t_0) = \nu^* \in B$. This is a contradiction since $\nu(t_0)$ is in the interior of $A$. It follows that $r > 0$.

By Lemma 2.5, we know that $\lambda I + T^{-1} \in \mathcal{L}(l_\infty(C^{N+1}), l_\infty(C^{N+1}))$ has norm equal
to \( \lambda + \frac{1}{c} \). Then we have

\[
    r = \inf_{\nu \in B} \| T^{-1}(\nu - \nu(t_0)) \|_{L^\infty(\mathbb{C}^{N+1})}
    \leq \inf_{\nu \in B} \| (\lambda I + T^{-1})(\nu - \nu(t_0)) \|_{L^\infty(\mathbb{C}^{N+1})}.
\]

\[
    \leq \left\| (\lambda I + T^{-1})(\nu(t_1) - \nu(t_0)) \right\|_{L^\infty(\mathbb{C}^{N+1})}
    = \left\| \int_{t_0}^{t_1} (\lambda I + T^{-1})\nu'(s) ds \right\|_{L^\infty(\mathbb{C}^{N+1})}
    \leq \sup_{t \in \Omega_0} \int_{t_0}^{t} |H(\nu_\lambda(s))|_{L^\infty(\mathbb{C}^{N+1})} ds
    \leq \tilde{M}\beta.
\]

It follows that \( \beta \geq \frac{r}{\tilde{M}} \). Let \( h_0 = \frac{r}{\tilde{M}} \). Then \( \Omega_{h_0} \) is the common existence region of \( \nu_\lambda \) for all \( \lambda \in (0, 1 - 1/c) \). This completes the proof of Claim 2.

**Claim 3:** Let \( \nu_\lambda \), and \( h_0 \) be as in Claim 2. There exists an analytic function \( w_0 \in C(\Omega_{h_0}; T^{-1}(\bar{A})) \) so that \( \lim_{\lambda \to 0^+} \| T^{-1}\nu_\lambda - w_0 \|_{C(\Omega_{h_0}; L^\infty(\mathbb{C}^{N+1}))} = 0 \).

**Proof of Claim 3:** By Claim 2, we have \( w_\lambda = T^{-1}\nu_\lambda \in C(\Omega_{h_0}; T^{-1}(\bar{A})) \). Moreover, the uniformly bounded set \( \{w_\lambda : \lambda \in (0, 1 - 1/c)\} \) is compact in \( C(\Omega_{h_0}; T^{-1}(\bar{A})) \), by the Arzelá–Ascoli theorem, since for every \( \varepsilon > 0 \) there exists \( \tilde{\delta} = \frac{\varepsilon}{\tilde{M}} > 0 \) so that \( |t - t'| < \tilde{\delta} \) implies that

\[
    \| w_\lambda(t) - w_\lambda(t') \|_{L^\infty(\mathbb{C}^{N+1})} \leq \left\| \int_{t'}^t (\lambda T + I)^{-1}H(Tw_\lambda(s)) ds \right\|_{L^\infty(\mathbb{C}^{N+1})}
    \leq \tilde{M}\tilde{\delta}
    = \varepsilon,
\]

where \( \tilde{M} > 0 \) was defined in the proof of Claim 2, and Lemma 2.6 was applied to obtain the second inequality. Therefore, there exists \( w_0 \in C(\Omega_{h_0}; \bar{A}) \) so that

\[
    \lim_{\lambda \to 0} \| w_\lambda - w_0 \|_{C(\Omega_{h_0}; \bar{A})} = 0. \tag{3.10}
\]

Since \( \{w_\lambda\}_{\lambda \in (0, 1-1/c)} \) is a set of analytic functions in norm \( \| \cdot \|_{C(\Omega_{h_0}; L^\infty(\mathbb{C}^{N+1}))} \) and analytic in norm \( \| \cdot \|_{C(\Omega_{h_0}; L^\infty(\mathbb{C}^{N+1}))} \), \( w_0 \) is also analytic in norm \( \| \cdot \|_{C(\Omega_{h_0}; L^\infty(\mathbb{C}^{N+1}))} \).

Now we show that \( w_0 \in C(\Omega_{h_0}; T^{-1}(\bar{A})) \). First we show that \( w_0 \in C(\Omega_{h_0}; L^\infty(\mathbb{C}^{N+1})) \). Suppose that \( w_0 \not\in C(\Omega_{h_0}; L^\infty(\mathbb{C}^{N+1})) \). Then for every \( K > 0 \) there exists \( j_0 \in \mathbb{N} \) such that \( \sup_{t \in \Omega_{h_0}} \sup_{j \geq j_0} \| (w_0)_{j_0}(t) \| > K \). That is,

\[
    \sup_{t \in \Omega_{h_0}} \| (w_0)_{j_0}(t) \| > K. \tag{3.11}
\]

On the other hand, it follows from \( \lim_{\lambda \to 0^+} \| w_\lambda - w_0 \|_C = 0 \), that for every \( \varepsilon > 0 \),
there exists $\delta > 0$ such that for every $\lambda \in (0, \delta)$, we have

$$\sup_{t \in \Omega_{h_0}} \sup_{j \in \mathbb{N}} |(w_0)_j(t) - (w_\lambda)_j(t)| < \epsilon,$$

which leads to

$$\sup_{t \in \Omega_{h_0}} |(w_0)_j(t)| - \sup_{t \in \Omega_{h_0}} |(w_\lambda)_j(t)| < \epsilon, \text{ for every } j \in \mathbb{N}.$$ It follows that

$$\sup_{t \in \Omega_{h_0}} |(w_\lambda)_j(t)| > \sup_{t \in \Omega_{h_0}} |(w_0)_j(t)| - \epsilon, \text{ for every } j \in \mathbb{N}. \quad (3.12)$$

Choosing $j = j_0$ and $\epsilon = \frac{K}{2^{c_0}}$ in (3.12), then by (3.11) we obtain that

$$\sup_{t \in \Omega_{h_0}} |(w_\lambda)_j(t)| \geq \sup_{t \in \Omega_{h_0}} |(w_0)_j(t)| - \frac{K}{2^{c_0}},$$

which leads to $\sup_{t \in \Omega_{h_0}} |c_{j_0}(w_\lambda)_j(t)| > K/2$ for every $\lambda \in (0, \delta)$. That is, $w_\lambda \not\in C(\Omega_{h_0}; l^\infty_c(\mathbb{C}^{N+1}))$ as $\lambda \to 0$ and hence $\nu_\lambda = Tw_\lambda \not\in C(\Omega_{h_0}; l^\infty_c(\mathbb{C}^{N+1}))$. This is a contradiction and hence $w_0 \in C(\Omega_{h_0}; l^\infty_c(\mathbb{C}^{N+1}))$.

Next we show that $w_0 \in C(\Omega_{h_0}; T^\dagger(\bar{A}))$. Suppose not. Since $w_0 \in C(\Omega_{h_0}; l^\infty_c(\mathbb{C}^{N+1}))$, there exists $t^* \in \Omega_{h_0}$ so that $w_0(t^*) \in l^\infty_c(\mathbb{C}^{N+1}) \setminus T^\dagger(\bar{A})$. By (3.10) we have

$$\lim_{\lambda \to 0} \|w_\lambda(t^*) - w_0(t^*)\|_{l^\infty_c(\mathbb{C}^{N+1})} = 0. \quad (3.13)$$

Since $\{w_\lambda : \lambda \in (0, 1 - 1/c)\}$ is uniformly bounded in $C(\Omega_{h_0}; T^\dagger(\bar{A}))$, there exists a closed ball $B'$ in $T^\dagger(\bar{A})$ which contains the closure of $\{w_\lambda(t^*)\}_{\lambda \in (0, 1 - 1/c)}$. Then by Lemma 2.3 and by (3.13), we have $w_0(t^*) \in B' \subset T^\dagger(\bar{A})$ which is a contradiction. This completes the proof of Claim 3.

**Claim 4:** Let $h_0$ be as in Claim 2. There exists a map $\nu_0 : \Omega_{h_0} \to \bar{A}$ such that $H(\nu_0)$ is continuous and is such that for every $t \in \Omega_{h_0}$, there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1 - \frac{1}{c})$ with $\lim_{n \to +\infty} \lambda_n = 0$ such that $\lim_{n \to +\infty} H(\nu_{\lambda_n}(t)) = H(\nu_0(t))$.

**Proof of Claim 4:** Note that by Claim 1, $\nu_\lambda \in C(\Omega_{h_0}; T^\dagger(\bar{A}))$ is uniformly bounded with respect to $\lambda \in (0, 1 - \frac{1}{c})$. Since by Lemma 2.9 $H$ is completely continuous, for every $t \in \Omega_{h_0}$, the set

$$\left\{ H(\nu_\lambda(t)) : \lambda \in \left(0, 1 - \frac{1}{c}\right) \right\},$$

is pre-compact in $l^\infty_c(\mathbb{C}^{N+1})$. So there exists a sequence $\{\lambda_n\}_{n=1}^{\infty} \subset (0, 1 - \frac{1}{c})$ with $\lim_{n \to +\infty} \lambda_n = 0$ and $\nu_0(t) \in T^\dagger(\bar{A})$, where $T^\dagger(\bar{A})$ is compact, such that

$$\lim_{n \to +\infty} H(\nu_{\lambda_n}(t)) = \lim_{n \to +\infty} H(Tw_{\lambda_n}(t)) = H(\nu_0(t)). \quad (3.14)$$
Next we show that $H(\nu_0): \Omega_{h_0} \ni t \rightarrow H(\nu_0(t)) \in l^\infty(\mathbb{C}^{N+1})$ is continuous in $t \in \Omega_{h_0}$. Let $t \in \Omega_{h_0}$. By (3.14), for every $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for every $n > N_1$,

$$
\|H(\nu_{\lambda_n}(t)) - H(\nu_0(t))\|_{l^\infty(\mathbb{C}^{N+1})} < \frac{\epsilon}{3}. \tag{3.15}
$$

Since $H(\nu_{\lambda_n})$ is continuous, there exists $\delta > 0$ such that for every $t' \in \Omega_{h_0}$ with $|t - t'| < \delta$ we have

$$
\|H(\nu_{\lambda_n}(t)) - H(\nu_{\lambda_n}(t'))\|_{l^\infty(\mathbb{C}^{N+1})} < \frac{\epsilon}{3}. \tag{3.16}
$$

Taking subsequence of $\{\lambda_n\}$ if necessary, by (3.13) there exists $N'$ such that for every $n > N'$, we have

$$
\|H(\nu_{\lambda_n}(t')) - H(\nu_0(t'))\|_{l^\infty(\mathbb{C}^{N+1})} < \frac{\epsilon}{3}. \tag{3.17}
$$

By (3.15), (3.16) and (3.17) we have for $n > \max\{N_1, N'\}$,

$$
\|H(\nu_0(t)) - H(\nu_0(t'))\|_{l^\infty(\mathbb{C}^{N+1})} < \epsilon.
$$

That is $H(\nu_0)$ is continuous. This completes the proof of Claim 4.

**Claim 5:** Let $h_0$ be as in Claim 2, $w_0$ be as in Claim 3, $\nu_0$ be as in Claim 4. Then $H(Tw_0) = H(\nu_0)$ and $w_0$ is the solution of the initial value problem (3.3).

**Proof of Claim 5:** It follows from Claim 3 that $w_0$ is in $C(\Omega_{h_0}; T^{-1}(\bar{A}))$ and $w_0$ is the limit of $w_\lambda$ as $\lambda \rightarrow 0^+$ in the norm $\| \cdot \|_{C(\Omega_{h_0}; l^\infty(\mathbb{C}^{N+1}))}$. We first show that $Tw_\lambda$ converges to $Tw_0$ coordinate-wise. That is, for every $j \in \mathbb{N}$,

$$
\lim_{\lambda \rightarrow 0} \sup_{t \in \Omega_{h_0}} |(Tw_\lambda)_j(t) - (Tw_0)_j(t)| = 0. \tag{3.18}
$$

If not, there exists $j_0 \in \mathbb{N}$ and $\epsilon_0 > 0$ and a sequence $\{\lambda_n\}_{n=1}^\infty \subset (0, 1 - \frac{1}{c})$ converging to 0 such that

$$
\sup_{t \in \Omega_{h_0}} |(w_{\lambda_n})_{j_0}(t) - (w_0)_{j_0}(t)| \geq \frac{\epsilon_0}{c^{j_0}}, \text{ for all } n \in \mathbb{N},
$$

which leads to

$$
\sup_{t \in \Omega_{h_0}} \sup_{j \in \mathbb{N}} |(w_{\lambda_n})_j(t) - (w_0)_j(t)| \geq \sup_{t \in \Omega_{h_0}} |(w_{\lambda_n})_{j_0}(t) - (w_0)_{j_0}(t)| \geq \frac{\epsilon_0}{c^{j_0}}.
$$

for all $n \in \mathbb{N}$. This is a contradiction, since $w_0$ is the limit of $w_\lambda$ as $\lambda \rightarrow 0^+$ in the norm $\| \cdot \|_{C(\Omega_{h_0}; l^\infty(\mathbb{C}^{N+1}))}$.

Noticing that each coordinate of $H(Tw_\lambda)$ involves only finitely many coordinates of $Tw_\lambda$ and $H$ is analytic. $H(Tw_\lambda)$ converges to $H(Tw_0)$ coordinate-wise as $Tw_\lambda$
converges to $T w_0$ coordinate-wise with $\lambda \to 0^+$. By Claim 4, we have

$$H(\nu_0) = H(T w_0). \quad (3.19)$$

Noting that by Lemma 2.6 $(\lambda T + I)^{-1} \in \mathcal{L}(l^\infty(\mathbb{C}^{N+1}); l^\infty(\mathbb{C}^{N+1}))$ is bounded for every $\lambda \in [0, 1)$. Notice that $\nu_\lambda$, $\lambda \in (0, 1 - \frac{1}{\epsilon})$, satisfies $\mathbf{(3.8)}$. On the other hand, by Claim 3 we have

$$\lim_{\lambda \to 0^+} \|T^{-1} \nu_\lambda(t) - w_0(t)\|_{l^\infty(\mathbb{C}^{N+1})} = 0. \quad (3.20)$$

On the other hand, for every $j \in \mathbb{N}$ and $t \in \Omega_{h_0}$ we have

$$\left| \int_{t_0}^t \left[ (\lambda T + I)^{-1} H \right]_j (\nu_\lambda(s)) \, ds - \int_{t_0}^t H_j(\nu_0(s)) \, ds \right|$$

$$= \left| \int_{t_0}^t \left[ (\lambda T + I)^{-1} H \right]_j (\nu_\lambda(s)) - H_j(\nu_0(s)) \, ds \right|$$

$$= \left| \int_{t_0}^t \frac{1}{(\lambda c^j + 1)} H_j(\nu_\lambda(s)) - H_j(\nu_0(s)) \, ds \right|$$

$$\leq \left| \int_{t_0}^t \frac{1}{(\lambda c^j + 1)} (H_j(\nu_\lambda(s)) - H_j(\nu_0(s))) \, ds \right| + \left| \int_{t_0}^t \frac{\lambda c^j}{\lambda c^j + 1} H_j(\nu_0(s)) \, ds \right|$$

$$= \frac{1}{(\lambda c^j + 1)} \left| \int_{t_0}^t (H_j(\nu_\lambda(s)) - H_j(\nu_0(s))) \, ds \right| + \frac{\lambda c^j}{\lambda c^j + 1} \left| \int_{t_0}^t H_j(\nu_0(s)) \, ds \right|$$

$$= \frac{1}{(\lambda c^j + 1)} \left| \int_{t_0}^t (H_j(T w_\lambda(s)) - H_j(T w_0(s))) \, ds \right| + \frac{\lambda c^j}{\lambda c^j + 1} \left| \int_{t_0}^t H_j(\nu_0(s)) \, ds \right| \quad (3.21)$$

where $\xi \in \Omega_{h_0}$ and $H_j$ denotes the $j$-th coordinate of $H$. Since $H(T w_\lambda)$ converges to $H(T w_0)$ coordinate-wise as $T w_\lambda$ converges to $T w_0$ coordinate-wise with $\lambda \to 0^+$, uniformly with respect to $t \in \Omega_{h_0}$. Letting $\lambda \to 0^+$ in $\mathbf{(3.21)}$, we have for every $j \in \mathbb{N}$ and $t \in \Omega_{h_0}$,

$$\left| \int_{t_0}^t \left[ (\lambda T + I)^{-1} H \right]_j (\nu_\lambda(s)) \, ds - \int_{t_0}^t H_j(\nu_0(s)) \, ds \right| \to 0 \text{ as } \lambda \to 0^+. \quad (3.22)$$

By $\mathbf{(3.20)}$ and $\mathbf{(3.22)}$, we have for every $t \in \Omega_{h_0}$,

$$w_0(t) = w_{t_0} + \int_{t_0}^t H(\nu_0(s)) \, ds,$$

which combined with $\mathbf{(3.19)}$ gives

$$w_0(t) = w_{t_0} + \int_{t_0}^t H(T w_0(s)) \, ds.$$

That is, $w_0$ is a solution of the initial value problem $\mathbf{(3.3)}$. By analyticity of $w_0$, it is the unique solution of $\mathbf{(3.3)}$ which is the complex extension of the real-valued
solution \( w = ((y_1, z_1), (y_2, z_2), \ldots) \in C(\mathbb{R}; L^\infty_c(\mathbb{R}^{N+1})) \) at \( t = t_0 \in \mathbb{R} \). It follows that \( (x, \tau) = (cy_1, cz_1) \) is analytic at \( t_0 \). Since \( t_0 \in \mathbb{R} \) is arbitrary, \((x, \tau)\) is analytic on \( \mathbb{R} \). This completes the proof of Claim 5 and that of the theorem. \( \square \)

4 Example

In this section, we present an example from important applications. We now study the analyticity of periodic solutions for the following delay differential equations with adaptive delay:

\[
\begin{align*}
\dot{x}_1(t) &= -\mu x_1(t) + \sigma b(x_2(t - \tau(t))), \\
\dot{x}_2(t) &= -\mu x_2(t) + \sigma b(x_1(t - \tau(t))), \\
\dot{\tau}(t) &= 1 - h(x(t)) \cdot (1 + \tanh(\tau(t))),
\end{align*}
\]

(4.1)

where \( x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2 \), \( \tau(t) \in \mathbb{R} \), \( \tanh(\tau) = (e^{2\tau} - 1)/(e^{2\tau} + 1) \) and \( \mu > 0 \) is a constant. We make the following assumptions:

(\(\alpha_1\)) \( b: \mathbb{R} \rightarrow \mathbb{R} \) and \( h: \mathbb{R}^2 \rightarrow \mathbb{R} \) are continuously differentiable functions with \( b'(0) = -1 \);

(\(\alpha_2\)) There exist \( h_0 < h_1 \) in \((1/2, 1)\) such that \( h_1 > h(x) > h_0 \) for all \( x \in \mathbb{R}^2 \);

(\(\alpha_3\)) \( b \) is decreasing on \( \mathbb{R} \) and the map \( \mathbb{R} \ni y \rightarrow yb(y) \in \mathbb{R} \) is injective;

(\(\alpha_4\)) \( yb(y) < 0 \) for \( y \neq 0 \), and there exists a continuous function \( M: \mathbb{R} \ni \sigma \rightarrow M(\sigma) \in (0, +\infty) \) so that

\[
\frac{b(y)}{y} > -\frac{\mu}{2|\sigma|}, \quad \text{for } |y| \geq M(\sigma);
\]

(\(\alpha_5\)) \( h_0 > (1 + e^{-\pi})/2 \) and there exists \( \epsilon > 0 \) so that \( b \) and \( h \) have analytic complex extensions on

\[
U_0 \times V_0 = \{(p, q) \in \mathbb{C}^2 \times \mathbb{C} : \Re(p, q) \in \Omega_1, |\Im(p, q)| \leq \epsilon\}
\]

where \( \Omega_1 = (-M(\sigma), M(\sigma)) \times (-M(\sigma), M(\sigma)) \times \left(0, -\frac{\ln(2h_0 - 1)}{2}\right) \).

Lemma 4.1 ([5]) Assume (\(\alpha_1\))–(\(\alpha_4\)) hold. Then the range of every periodic solution \((x_1, x_2, \tau)\) of (4.1) with \( \sigma \in \mathbb{R} \) is contained in

\[
\Omega_1 = (-M(\sigma), M(\sigma)) \times (-M(\sigma), M(\sigma)) \times \left(0, -\frac{\ln(2h_0 - 1)}{2}\right).
\]

Theorem 4.2 Assume that (\(\alpha_1\))–(\(\alpha_5\)) hold. Then all the periodic solutions of (4.1) are analytic on \( \mathbb{R} \).

Proof By Lemma 4.1, the range of every periodic solution \((x_1, x_2, \tau)\) of (4.1) with \( \sigma \in \mathbb{R} \) is contained in \( \Omega_1 \). Now we apply Theorem 3.1. Let \( l = 1/2 \in (0, 1) \). For every
$(x(t), \tau(t)) = (x_1(t), x_2(t), \tau(t)) \in \overline{\Omega}_1, t \in \mathbb{R}$, we have

$$1 \leq 1 + \tanh \tau(t) \leq \frac{1}{h_0} < \frac{2}{1 + e^{-\pi}}$$

and hence by $(\alpha_2)$ and $(\alpha_3)$ we obtain

$$1 - (1 - h(x(t)) \cdot (1 + \tanh \tau(t))) - \frac{e + l}{2}$$

$$= h(x(t)) \cdot (1 + \tanh \tau(t)) - \frac{e + l}{2}$$

$$< 1 + \tanh \tau(t) - \frac{e + l}{2}$$

$$\leq \frac{2}{1 + e^{-\pi}} - \frac{e + l}{2}$$

$$< \frac{e - l}{2},$$

and

$$1 - (1 - h(x(t)) \cdot (1 + \tanh \tau(t))) - \frac{e + l}{2}$$

$$= h(x(t)) \cdot (1 + \tanh \tau(t)) - \frac{e + l}{2}$$

$$> \frac{1 + e^{-\pi}}{2} (1 + \tanh \tau(t)) - \frac{e + l}{2}$$

$$> \frac{2}{1 + e^{-\pi}} - \frac{e + l}{2}$$

$$> -\frac{e - l}{2}.$$

Therefore, we have $|1 - (1 - h(x(t)) \cdot (1 + \tanh \tau(t))) - \frac{e + l}{2}| < \frac{e - l}{2}$ for all $(x(t), \tau(t)) \in \overline{\Omega}_1$. Note that $1 > h_0 > (1 + e^{-\pi})/2$ and $(x(t), \tau(t)) \in \overline{\Omega}_1$ imply that $0 < \tau(t) < \frac{\pi}{2}$. And the complex extension of $1 + \tanh q$ is analytic for $|q| < \frac{\pi}{2}$, $q \in \mathbb{C}$. Then by $(\alpha_3)$ we can choose $\epsilon_0 \in (0, \epsilon)$ small enough so that $|1 - (1 - h(p) \cdot (1 + \tanh q)) - \frac{e + l}{2}| < \frac{\epsilon - l}{2}$ for all $(p, q) \in U \times V$ where

$$U \times V = \{(p, q) \in \mathbb{C}^2 \times \mathbb{C} : \Re(p, q) \in \Omega_1, |\Im(p, q)| < \epsilon_0\} \subset U_0 \times V_0.$$

Then by applying Theorem 3.1 on $U \times V$, analyticity of all the periodic solutions of (4.1) follows. \qed

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