On the application of the Rodrigues displacement equation for the generation of finite helical axes for tibial motion

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Abstract
This work presents an alternate method for producing the finite helical axes for tibial motion. The method includes two procedures. In the first procedure, coordinates of the tibia in each specified position throughout full knee flexion to extension are measured in spatial Cartesian form. In the second procedure, the Rodrigues displacement equation is used to calculate finite helical axis parameters from the tibial displacements.

Key words: Knee motion, Tibial motion, Spatial motion, Screw motion, Screw axis, Finite helical axis, Rodrigues equation

1. Introduction

1.1 The Finite Helical Axis

Figure 1 illustrates an arbitrary tibial displacement of the human lower extremity. This displacement consists of the initial (position 1) and final (position j) positions of the tibia. This tibial displacement can be achieved by simultaneous rotation about and translation along an imaginary axis. Because a tibial displacement is finite and it can be achieved by a simultaneous rotation about and translation along an axis (parameters for screw motion or helical motion), the axis is called the finite helical axis. The finite helical axis can be useful as a kinematic landmark for knee motion and can reduce the number of skeletal landmarks required to describe kinematic motion (Blankevoort et. al., 1990, Hart et. al., 1991, Van Den Bogert et. al., 2008).

In an existing approach to generate finite helical axes, tibia and femur positions are first tracked (via a 6 DOF measurement system) over knee flexion and extension. Next, the tibial motion data are incorporated into finite rotation matrices and translation vectors. Using an algorithm, finite helical axis variables are calculated from the finite rotation matrices and translation vectors (Hart et. al., 1991). In another existing approach, the angular velocities of the tibia are measured (rather than positions) and are incorporated into velocity equations to calculate finite helical axes (Sheehan, 2007).
1.2 Additional Knee Axes and Axis Generation Methods

A review of published literature reveals that other types of knee axes have been defined (and corresponding construction methods developed) for other medical applications. The flexion-extension axis (FE) and the longitudinal rotation axis (LR) are types of knee axes commonly referred to with applications in prosthetic design, gait analysis, braces and reconstructive surgery (Hollister et. al., 1993, Churchill et. al., 1998, Coughlin et. al., 2003). Various approaches have been used to identify such axes. For example, a bone-mounted mechanical device called an “axis finder” was used to locate the FE and LR axes (Hollister et. al., 1993). These axes can also be calculated using tibia and femur position sensors in combination with computer-based optimization techniques (Churchill et. al., 1998 and Coughlin et. al., 2003). Utilizing a 3D CAD-based technique, a single flexion-extension axis was produced. This technique consists of fitting a cylinder in the posterior condyles of the femur and locating the center axis of the cylinder (Eckhoff et. al., 2001, Eckhoff et. al., 2003, Eckhoff et. al., 2005, Most et. al., 2004, Yin et. al., 2015). For this reason, the flexion-extension axis is also called the geometric center axis (GCA) or the cylinder axis (CA). Another approach for producing GCAs consists of calculating the optimum circle from arcs traced by the ankle center-point in knee flexure and extension (Asano et. al., 2005). The flexion-extension axis produced from this technique lies at center of the optimal circle. Another 3D CAD-based approach for producing the flexion-extension axis consists of bending the knee (in a virtual model) and locating points of pure translation in the bone (Yin et. al., 2015). The flexion-extension axis passes through these points. Additional clinical applications of flexion-extension axes include utilizing it as an aid for knee alignment, soft-tissue balancing in total knee arthroplasty as well as the position and tension of a graft in anterior cruciate ligament reconstruction. Classically, the transepicondylar axis (TEA), which is used for knee flexion-extension, can be determined via 3D CAD bone models by connecting the most prominent of the lateral femur which is a point within the lateral collateral ligament insertion site (the epicondyle) and the point representing the prominent or the sulcus point on the medial condyle (referred to as the clinical or surgical medial epicondyle respectively) (Most et. al., 2004, Yin et. al., 2015). The clinical applications for the TEA are similar to those for GCAs.

1.3 Scope of Current Work

In this work, an alternate method is presented for the generation of finite helical axes for tibial motion. This method includes an empirical approach for measuring the spatial Cartesian coordinates of the tibia throughout the full range of knee flexure and extension and an analytical approach for calculating finite helical axes. In the latter approach, the Rodrigues displacement equation is used to calculate finite helical axis variables from the tibial displacements.
2. Proposed Approach for Finite Helical Axis Generation

2.1 Measuring Tibial Positions

Figure 2 illustrates the workstation used in this work to measure tibial-femoral positions. The localizer measures and records position coordinates (in spatial Cartesian form) from trackers. In Figure 2, trackers are mechanically affixed directly to the tibia and femur bones. A direct tracker-to-bone connection (rather than tracker-to-skin connection), eliminates the bone position error introduced by the motion of leg skin tissue and muscle tissue. In this work, a cadaveric leg specimen was utilized.

Not shown in Figure 2 is the system of pulleys and cables used to support and actuate the cadaver leg from full flexion to full extension. A full description of this method for the kinematic acquisition of the femoral-tibial motion has been published (Belvedere et al., 2012). Such a system would not be needed if a living leg specimen is used since it is self-supported and self-actuated.

Both tibial and femoral motion are achieved in the position measurement system. As a result, position coordinates for the tibia and femur (with respect to the localizer) are measured. Taking the difference between the tibial and the femoral position coordinates produces position coordinates of the tibia with respect to a fixed femur (therefore \( \text{Tibia}_{\text{Femur}} = \text{Tibia}_{\text{Localizer}} - \text{Femur}_{\text{Localizer}} \)). The physical equivalent to producing \( \text{Tibia}_{\text{Femur}} \) would be to measure the tibial position coordinates with respect to a fully-immobilized femur (and affixing the localizer to the femur).

Equation (1) is a particular form of the Rodrigues displacement equation that accommodates rigid-body displacements defined by the spatial Cartesian coordinates of points \( p, q \) and \( r \) (Robotics, 2015). As illustrated in Figure 3, the screw parameters calculated from Equation (1) are the finite helical axis unit vector \( \mathbf{u} \), the axis point \( \mathbf{v} \), the axis rotation angle \( \phi \) and the axis translation magnitude \( s \).

Equation (2), which is formulated from Equation (1), becomes the finite helical axis unit vector when normalized (therefore \( \mathbf{u} = \frac{\mathbf{U}}{||\mathbf{U}||} \)) and the axis rotation angle can be expressed as \( \phi = 2\tan^{-1}(||\mathbf{U}||) \) (Robotics, 2015). Equations (3) and (4), which are also formulated from Equation (1), represent the axis point and the axis translation magnitude respectively (Robotics, 2015).

The equations for \( \mathbf{u}, \phi, \mathbf{v} \) and \( s \) were codified in a Matlab script to calculate an array of finite helical axis parameters for an array of tibial displacements. While some mathematical analysis software packages can accommodate the concise symbolic forms given for Equations (2) through (4), others may require the fully-expanded...
form of these equations. To produce scripts in Matlab to calculate $u$, $\phi$, $v$ and $s$, Equations (2) through (4) were first fully expanded symbolically in Matlab.

$$p_j - p_i = \tan(\theta/2) u \times (p_j + p_i - 2v) + su$$
$$q_j - q_i = \tan(\theta/2) u \times (q_j + q_i - 2v) + su$$
$$r_j - r_i = \tan(\theta/2) u \times (r_j + r_i - 2v) + su$$

$$U = \left[ \frac{(q_j - q_i) - (r_j - r_i)}{\tan(\theta/2)} \times \frac{(p_j - p_i) - (r_j - r_i)}{\tan(\theta/2)} \right]$$

$$v = \frac{1}{2} \left[ \frac{u \times (p_j - p_i)}{\tan(\theta/2)} \times \frac{u \times (r_j - r_i)}{\tan(\theta/2)} \right] u + p_j + p_i$$

$$s = u \cdot (p_j - p_i) = u \cdot (q_j - q_i) = u \cdot (r_j - r_i)$$

![Screw parameters and spatial rigid-body displacement parameters](image)

### 3. Finite Helical Axis Generation Example

#### 3.1 Problem Description and Solution

For this problem, a total of 48 tibial positions (also called *poses*) were measured and numbered sequentially from 1 to 48 using the workstation in Section 2.1. These positions represent tibial motion from a starting position of full knee flexion to an ending position of full knee extension. The flexion and extension positions are illustrated in Figure 4.

Table 1 includes the Cartesian coordinates for points $p$, $q$ and $r$ of 7 tibia positions (poses 1, 8, 16…48). These coordinates were produced by taking the difference of the tibial and femoral position coordinates measured using the system described in Section 2.1 (making the data in Table 1 identical to *TibiaFemur* in Section 2.1). Table 2 includes the finite helical axis variables calculated from Table 1 using the Rodrigues displacement equation (see Section 2.2). Because Equation (1) requires rigid-body displacements, the displacements were defined as the difference between the flexed tibial position (position #1) and the subsequent positions as the tibia was extended (see Table 2).
Fig. 4 Total tibial displacement range

Table 1 Measured tibial position coordinates

| Pose # | p [mm]         | q [mm]         | r [mm]          |
|--------|----------------|----------------|-----------------|
| 1      | 14.680,−154.706,99.948 | 14.805,−178.323,121.050 | −3.544,−170.047,138.409 |
| 8      | 15.300,−171.210,69.067  | 15.384,−198.891,84.456  | −3.610,−194.937,102.626 |
| 16     | 18.903,−178.168,−12.150  | 19.397,−209.835,−12.388  | 0.379,−215.587,5.269 |
| 24     | 20.841,−145.847,−85.202  | 21.453,−173.335,−100.920  | 1.911,−186.769,−88.912  |
| 32     | 25.900,−80.564,−135.789  | 27.553,−97.742,−162.345  | 9.340,−116.529,−157.674  |
| 40     | 33.549,−14.348,−150.303  | 36.251,−18.460,−181.590  | 16.995,−36.163,−186.314  |
| 48     | 40.683,3.600,−146.724  | 44.226,3.939,−178.194  | 23.201,−10.289,−186.071 |

Table 2 Calculated finite helical axis variables

| Pos1-Pos7 | u     | \( \phi \) [deg] | v [mm] | s [mm] |
|-----------|-------|------------------|-------|-------|
| 1-8       | 0.988,−0.131, 0.086 | 2.752 | −3.678,−21.849, 8.874 | 0.110 |
| 1-16      | 0.998,−0.047, 0.035 | 28.752 | −1.411,−20.395, 12.973 | 1.399 |
| 1-24      | 0.998,−0.060, 0.019 | 66.971 | −1.496,−20.714, 13.223 | 2.011 |
| 1-32      | 0.999,−0.009, 0.031 | 98.647 | −0.561,−16.544, 13.170 | 3.263 |
| 1-40      | 0.998,−0.058, 0.027 | 113.400 | −1.307,−17.183, 11.106 | 3.799 |
| 1-48      | 0.992,−0.126, 0.007 | 113.600 | −2.390,−18.213, 10.854 | 4.105 |

3.2 Resulting Finite Helical Axes

The six finite helical axis vectors \( \mathbf{u} \) included in Table 2 are illustrated in Figure 5. Because the tibia moves over the femoral condyle surfaces, the tibial finite helical axes pass through the femur (as evidenced by the location of the femoral condyle surfaces). In Figure 6, the finite helical axes for all 48 tibial positions are illustrated (with the eight individual examples highlighted in black).

It is important to note that because a purely analytical method is used to calculate the finite helical axis variables (see Section 2.2), the quality and accuracy of the axis variables determined is largely dependent on the accuracy of the instruments used to measure tibial positions (as well as the approaches used to obtain the kinematic data through the support and movement of the tibia in the case of a cadaveric leg).
Fig. 5 Isometric and top views of finite helical axes in Table 2
4. Discussion

The particular form of the Rodrigues displacement equation used in Section 2.2 offers the benefits of 1.) the analytical calculation of the finite helical axis parameters, 2.) having a compact form that is easily codifiable (particularly in today’s commercial mathematical analysis software) and 3.) having a form that directly incorporates spatial rigid-body Cartesian coordinates (which can be readily produced from a position measuring system). In comparison, in existing methods for finite helical axis generation, axis variables are calculated numerically from transformation matrices (using algorithms) and rigid-body motion is measured in 6 DOF or in terms of angular velocities (Blankevoort et. al., 1990, Hart et. al., 1991, Van Den Bogert et. al., 2008, Sheehan, 2007).

5. Conclusion

The two-part method presented in this work was used to produce finite helical axes from full knee flexion to full knee extension. Because the particular Rodrigues displacement equation form used in this work incorporates rigid-body Cartesian coordinates, tibial position coordinates (which are tracked by a position measurement system in Cartesian form) are used directly. Also, the particular Rodrigues displacement equation form used in this work enables the analytical calculation of all finite helical axis variables.

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