Existence and Destruction of the Kantorovich Main Continuous Solutions of Nonlinear Integral Equations

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Introduction

Consider the nonlinear Volterra integral equation of the second kind

\[ x(t) = \int_0^t K(t, s, x(s)) \, ds, \quad 0 < t < T < \infty, \quad x(0) = 0. \]  

(1)

Definition \cite[1]{[1]}, c.467] Continuous function \( x(t) \), satisfying the equation (1), we name Kantorovich main solution if the sequence

\[ x_n = \int_0^t K(t, s, x_{n-1}(s)) \, ds, \quad x_0(t) = 0 \]

converges to function \( x(t) \) \( \forall t \in (0, T) \). If in addition \( \lim_{t \to T} |x(t)| = +\infty \), then solution has blow-up point in the point \( T \).

Let us find the guaranteed interval \([0, T]\) where exists the main solution such as the blow-up may occur if one continue solution onto \([T, +\infty)\). Beside, one must find the positive continuous function \( \hat{x}(t) \), defined on \([0, T]\) such as for the main solution \( x(t) \) the following a priory estimate \( |x(t)| \leq \hat{x}(t) \) if fulfilled for \( t \in [0, T] \).

In this paragraph we employ the classical approach by L.V.Kantorovich \cite{1} (see chapter 12). For the majorizing equations construction we will use the algorithm proposed in \cite{2}.

Let us introduce the following conditions

A. Let function \( K(t, s, x) \) be defined, continuous and differentiable wrt \( x \) in \( D = \{0 < s < t < T, T \leq \infty, |x| < \infty\} \).  

B. Let we can construct functions \( m(s), \gamma(x) \) which are continuous, positive and monotonically increasing functions defined for \( 0 < s < \infty, 0 < x < \infty \), such as in \( D \) for any \( t \) from \([0, \infty)\) the following inequalities are fulfilled

\[ |K(t, s, x)| \leq m(s)\gamma(|x|), \]
\[ |K'(t, s, x)| \leq m(s)\gamma'(|x|). \]
The case of $\gamma(0) = 0$ we exclude since in that case equation (1) has only trivial solution. Such solution is the main according to Kantorovich definition. Below the functions $m(s), \gamma(x)$ are assumed positive, monotone increasing, and $\gamma(x)$ is assumed convex wrt $x$.

1 Integral Majorizing Equation

Let us introduce majorizing integral equation

$$\hat{x}(t) = \int_0^t m(s)\gamma(\hat{x}(s))\,ds,$$  \hspace{1cm} (2)

which is equivalent to the Cauchy problem for the differential equation with separable variables:

$$\begin{cases}
\frac{d\hat{x}}{dt} = m(t)\gamma(\hat{x}(t)) \\
\hat{x}|_{t=0} = 0.
\end{cases}$$  \hspace{1cm} (3)

Thus, the solution $\hat{x}(t)$ of integral equation (2) satisfies the equation $\Phi(x) = M(t)$, where $\Phi(x) = \int_0^x \frac{dx}{\gamma(x)}$, $M(t) = \int_0^t m(t)\,dt$. Because of monotone increasing positive continuous function $\Phi(x)$ exists inverse mapping $\Phi^{-1}$ with define area $[0, \infty)$, if $\lim_{x \to \infty} \int_0^x \frac{dx}{\gamma(x)} = +\infty$, and with define area $[0, l)$, if $\lim_{x \to \infty} \int_0^x \frac{dx}{\gamma(x)} = l$. Thus in the first case the Cauchy problem has unique positive solution $\hat{x}(t) \in C^{(1)}_{[0,\infty)}$, and in $C^{(1)}_{[0,l)}$ in the second case.

**Remark** Solution $\hat{x}(t)$ can be constructed with successive approximations as solution of $\Phi(x) - M(t) = 0$. Indeed, equation $\Phi(t) - M(t) = 0$ defines $\hat{x}(t)$ as explicit continuous function $\hat{x}(t) \to 0$ for $t \to 0$, since

$$\frac{d}{dx} (\Phi(x) - M(t)) = \frac{1}{\gamma(x)} \neq 0.$$ 

Hence the solution $\hat{x}(t)$ can be constructed with successive approximations:

$$x_n(t) = x_{n-1}(t) - \gamma(0) [\Phi(x_{n-1}(t)) - M(t)], \quad x_0 = 0$$

on the small interval $[0, \Delta], \Delta > 0$. 

Constructed solution can be continued on the whole domain of $\Phi^{-1}$ by repeated application of the implicit theorem application.

Let $\gamma(x)$ be polynomial with positive coefficients and $\gamma(0) \neq 0$. In such case (ref. [7], p.344) function $\Phi(x)$ can be explicitly constructed in terms of logarithms, arctangents and rational functions, which allows us in basic cases to construct $\Phi^{-1}$ and to explicitly build $\hat{x}(t)$.

In general case in order to build $\hat{x}(t) = \Phi^{-1}(M(t))$, satisfying equation (2), one may employ the following Lemma.

Lemma 1. If
$$\lim_{x \to +\infty} \int_0^x \frac{dx}{\gamma(x)} = +\infty,$$
then majorizing equation (2) has for $t \in [0, \infty)$ continuous solution $\hat{x}(t)$. Moreover, the sequence
$$x_n(t) = \int_0^t m(s) \gamma(x_{n-1}(s)) \, ds, \quad x_0(t) = 0 \quad (4)$$
converges for $t \in [0, \infty)$ to function $\hat{x}(t)$.

**Proof.** Existence of the solution $\hat{x}(t)$ of equation (2) follows from above proved solution existence of equivalent Cauchy problem (3). Herewith the sequence $\{x_n(t)\}$ will be monotone increasing and bounded above since $\hat{x}(t)$ satisfies the equation (4). Hence $\{x_n(t)\}$ has limit. Thus, the sequence $\{x_n(t)\}$ is fundamental in the space $C[0,T_1]$ for $T_1 < \infty$. The space $C[0,T_1]$ is complete one and convergence is uniform for $T_1 < \infty$.

Lemma 2. Let
$$\lim_{x \to +\infty} \int_0^x \frac{dx}{\gamma(x)} = l.$$ Let us introduce the interval $[0,T_1)$, where $T_1 > 0$ in uniquely defined from the condition $\int_0^{T_1} m(s) \, ds = l$. Then Cauchy problem (3) has positive solution $\hat{x}(t) \in C[0,T_1)$, sequence $\{x_n(t)\}$ (4) converges to $\hat{x}(t)$ as $n \to \infty$, $\lim_{t \to T_1} \hat{x}(t) = \infty$.

Proof of existence of the desired function $\hat{x}(t)$ on the interval $[0,T_1)$ follows from proved existence of inverse mapping $\Phi^{-1} : [0,l] \to [0,\infty)$.

Since $l = \int_0^{\infty} \frac{dx}{\gamma(x)} = \int_0^{T_1} m(s) \, ds$, based on Lemma 2 we have $\lim_{t \to T_1} \hat{x}(t) = +\infty$.

**Theorem 1.** Let conditions (A), (B) and
$$\lim_{x \to +\infty} \int_0^x \frac{dx}{\gamma(x)} = +\infty$$
be fulfilled. Then integral equation (1) has main solution $x(t)$, defined for $t \in [0, \infty)$ and the following a priori estimate $|x(t)| \leq \hat{x}(t)$ is valid, where $\hat{x}(t)$ is solution of the Cauchy problem (3).
Proof. Let us introduce the sequence

\[ x_n(t) = \int_0^t K(t, s, x_{n-1}(s)) \, ds, \quad x_0(t) \equiv 0, \]

\[ \hat{x}_n(t) = \int_0^t m(s) \gamma(\hat{x}_{n-1}(s)) \, ds, \quad \hat{x}_0(t) \equiv 0. \]

Then because of [2] and due to the Theorem conditions we have the following inequality

\[ |x_n + p(t) - x_n(t)| \leq \hat{x}_n + p(t) - \hat{x}_n(t), \quad |x_n(t)| \leq \hat{x}_n(t), \quad \text{for } t \in [0, T_1], T_1 < \infty. \]

Because of Lemma 1 the positive monotonic increasing sequence \( \hat{x}_n(t) \) is the Cauchy sequence in the norm of the space \( C_{[0, T_1]} \), i.e. \( \max_{0 \leq t \leq T_1} (x_{n+p}(t) - x_n(t)) \leq \varepsilon \) for \( n \geq N(\varepsilon) \) and for arbitrary \( p \).

Hence, \( \max_{0 \leq t \leq T_1} |x_{n+p}(t) - x_n(t)| \leq \varepsilon \) for \( n \geq N(\varepsilon) \), \( \forall p \). Hence the sequence \( \{x_n(t)\} \) is in the sphere \( S(0, \hat{x}(t)) \) and its the Cauchy sequence. Because of completeness of the space \( C_{[0, T_1]} \)

\[ \exists \lim_{n \to \infty} x_n(t) = x(t). \]

Since \( K(t, s, x) \) is continuous wrt \( x \) then function \( x(t) \) satisfies the condition (1). The theorem is proved.

The classic Hartman-Wintner theorem on the Cauchy problem solution continuation on semi-axis [8] follows from the proved theorem.

Corollary 1 Let \( \gamma(x) = a + bx \) in the condition (B) be linear function, \( a \geq 0, b > 0 \). Then for main solution of equation (1) the following a priori estimate

\[ |x(t)| \leq a \int_0^t m(z) \exp \left( b \int_z^t m(s) \, ds \right) \, dz \]

is fulfilled for \( 0 \leq t < \infty \).

In order to prove this corollary it is enough to verify that the Cauchy problem \( \frac{dx}{dt} = m(t)(a + bx(t)), \quad x|_{t=0} = 0 \) has the solution

\[ \hat{x}(t) = a \int_0^t m(z) \exp \left( b \int_z^t m(s) \, ds \right) \, dz \]

for \( 0 \leq t < \infty \).
**Theorem 2.** Let conditions (A) and (B) be fulfilled. Let \( \lim_{x \to \infty} \int_0^x \frac{dx}{\gamma(x)} = l \). Introduce the interval \([0, T_1]\), where \( T_1 > 0 \) is uniquely defined from the equality \( \int_0^{T_1} m(s) \, ds = l \). Then the integral equation (1) in \( C[0, T_1) \) has the main solution \( x(t) \). For \( t \in [0, T_1) \) its a priori estimate \( |x(t)| \leq \hat{x}(t) \), is fulfilled, where \( \hat{x}(t) \) is the solution of the Cauchy problem (3), \( \lim_{t \to T_1} \hat{x}(t) = +\infty \).

Proof follows from the Theorem 1 taking into account the Lemma 2 results.

**Corollary 2.** (Alternative global solvability of equation (1)) Let conditions of Theorem 2 be fulfilled. Then either solution of the equation (1) can be continued on the whole semi-axis or on \([T_1, +\infty)\) or there the blow-up point exists.

Corollary 2 refines and generalizes the V. I. Yudovich theorem on global resolvability of the Cauchy problem (ref. Theorem 1 in [8], p. 19).

**Corollary 3.** Let conditions of the Theorem 2 be fulfilled, and in addition let \( \gamma(x) = ax^2 + bx + c \), where \( 4ac - b^2 > 0, a > 0, b > 0, c \geq 0 \). Then main solution \( x(t) \) of the equation (1) exists on \([0, T_1]\), where positive \( T_1 \) is defined from condition

\[
\int_0^{T_1} m(s) \, ds = \frac{2}{\sqrt{4ac - b^2}} \left( \frac{\pi}{2} - \arctg \frac{b}{\sqrt{4ac - b^2}} \right).
\]

For \( t \in [0, T_1) \) we have the estimate \( |x(t)| \leq \hat{x}(t) \), where function \( \hat{x}(t) \) is defined by formula

\[
\hat{x}(t) = \frac{\sqrt{4ac - b^2}}{2a} \left[ \arctg \frac{b}{\sqrt{4ac - b^2}} + \frac{\sqrt{4ac - b^2}}{2a} \int_0^t m(s) \, ds \right] - \frac{b}{2a}.
\]

**Proof.** Solution to the corresponding Cauchy problem (3) can be constructed easily when Corollary 3 conditions are fulfilled. Indeed, (see [6], p.36), \( \Phi(x) = \int_0^x \frac{dx}{ax^2 + bx + c} = \frac{2}{\sqrt{4ac - b^2}} \left( \arctg \frac{2ax + b}{\sqrt{4ac - b^2}} - \arctg \frac{b}{\sqrt{4ac - b^2}} \right) \),
\( M(t) = \int_0^t m(s) \, ds \). As a result, the equality \( \int_0^\infty \frac{dx}{\gamma(x)} = \int_0^{T_1} m(s) \, ds \), which serves to find \( T_1 \), means that
\[
\frac{2}{\sqrt{4ac-b^2}} \left( \frac{\pi}{2} - \arctan \frac{b}{\sqrt{4ac-b^2}} \right) = \int_0^{T_1} m(s) \, ds. \quad (6)
\]
From equation \( \Phi(x) = M(t) \) it follows that majorizing function \( \hat{x}(t) \), on \([0, T_1]\), must be constructed with formula (5).

**Corollary 3** can be employed to find \( \hat{x}(t) \) in the problem of extending the solution of the equation (1). Indeed, let conditions of Theorem 2 be fulfilled and function \( K \) depends on parameter \( \lambda \in \mathbb{R}^n, ||\lambda|| < \delta \) (i.e. \( K = K(t, s, x, \lambda) \)). Then in condition B function \( \gamma(||x||, ||\lambda||) \) will depend on this parameter’s norm.

Let \( \gamma(||x||, 0) = 0 \). Then equation (1) has trivial solution for \( \lambda = 0 \). If \( \lambda \neq 0 \) equation (1) has nontrivial main solution. Next result allows us to estimate the interval where exists main solution to the equation (1) for \( 0 < ||\lambda|| < \delta \).

**Corollary 4** Let conditions of the Theorem 2 are fulfilled and let \( m(s) = 1, \gamma(x, \lambda) = a(||\lambda||)x^2 + b(||\lambda||)x + c(||\lambda||) \). Let \( a(||\lambda||), b(||\lambda||), \) and \( c(||\lambda||) \) are positive infinitesimal functions for \( ||\lambda|| \to 0 \). Suppose that in a punctured neighborhood \( 0 < ||\lambda|| < \delta \) the following inequalities are fulfilled:
\[
4a(||\lambda||)c(||\lambda||) - b^2(||\lambda||) = \Delta(||\lambda||) > 0,
\]

\[
\sup_{||\lambda|| < \delta} \arctan \frac{b(||\lambda||)}{\sqrt{\Delta(||\lambda||)}} = \sigma, \quad \sigma < \frac{\pi}{2}.
\]

Then for \( 0 < ||\lambda|| < \delta \) equation (1) has the main solution \( x(t, \lambda) \), defined on the intervals \([0, T(||\lambda||)], T(||\lambda||) = \frac{2}{\sqrt{\Delta(||\lambda||)}} \left( \frac{\pi}{2} - \sigma \right) \). Therefore \( \lim_{||\lambda|| \to 0} T(||\lambda||) = +\infty \) and the following a proper estimate for the main solution is fulfilled: \( |x(t, \lambda)| \leq \hat{x}(t, \lambda) \). Here the positive majorizing function \( \hat{x}(t, \lambda) \) is defined on the interval \([0, T(||\lambda||)]\) using the formula (5) where coefficients are defined as follows: \( a = a(||\lambda||), b = b(||\lambda||), \) \( c = c(||\lambda||), \lim_{t \to T(||\lambda||)} \hat{x}(t, \lambda) = +\infty \).

### 2 Algebraic majorants

In order to estimate the guaranteed closed interval \([0, T]\) for existence of main solution of the equation (1) and its norm estimation \( C_{[0,T]} \) the
algebraic majorants are useful.
Indeed, let condition (A) be fulfilled and let in addition the following condition be fulfilled:

( ) Let there exists continuous, differentiable and convex wrt $r$, positive and monotonic increasing function $M(\rho, s, r)$, defined for $\rho \geq 0, s \geq 0, r > 0$ such as in the area $D$ for $t \in [0, \rho], |x| \leq r$ the following inequalities are fulfilled

$$\left| \int_{0}^{t} K(t, s, x(s)) \, ds \right| \leq \int_{0}^{\rho} M(\rho, s, r) \, ds,$$

$$\left| \int_{0}^{t} K'(t, s, x(s)) \, ds \right| \leq \int_{0}^{\rho} M'(\rho, s, r) \, ds.$$

Let us introduce function

$$M(\rho, r) = \int_{0}^{\rho} M(\rho, s, r) \, ds$$

and it’s positive derivative

$$M'(\rho, r) = \int_{0}^{\rho} M'(\rho, s, r) \, ds.$$

**Lemma 3.** System

$$\begin{array}{l}
    r = M(r, \rho), \\
    1 = M'_r(r, \rho)
\end{array} \quad (7)$$

has unique positive solution $r^*, \rho^*$. Moreover, for any $\rho \in [0, \rho^*]$ the equation $r = M(r, \rho)$ has main solution $r(\rho)$, i.e. monotonic increasing sequence $r_n = M(r_{n-1}, \rho), r_0 = 0$, converges to the solution of equation $r = M(r, \rho)$ for any $\rho \in [0, \rho^*]$.

Proof is geometrically obvious (here readers may refer to [9], p. 218), if on the plane $(y, r)$ one consider the graphs of the curves $y = M(r, \rho)$ for various $\rho$ and bisection $y = r$. Line $y = r$ tangents the curve $y = M(r, \rho^*)$ in the point $(r^*, \rho^*)$.

**Theorem 3** Let conditions (A) and (C) be fulfilled, $(r^*, \rho^*)$ is positive solution of the system (7). Then main solution $x(t)$ of the equation
exists in $C_{[0, r^*]}$ and the following estimate is fulfilled

$$\max_{0 \leq t \leq r^*} |x(t)| \leq r^*.$$  

**Proof.** Let us introduce two sequences

$$x_n(t) = \int_0^t K(t, s, x_{n-1}(s)) \, ds,$$

$$r_n = M(r_{n-1}, r^*),$$

$x_0(t) = 0, r_0 = 0$. Then the following inequality $||x_{n+p} - x_n||_{C_{[0, r^*]}} \leq r_{n+p} - r_n$ is valid for $n \geq N(\varepsilon)$ and for any $p$. Therefore since due to the Lemma 3 we have $\lim_{n \to 0} r_n = r^*$, then $r_{n+p} - r_n \leq \varepsilon$ for $n \geq N(\varepsilon)$ $p$. Therefore the sequence $\{x_n(t)\}$, for $x_0 = 0$ remains fundamental, $||x_n|| \leq r^*$ and the theorem is proved.

Let us consider the following example:

$$x(t) = \int_0^t (K_2(t, s)x^2(s) + K_1(t, s)x(s) + K_0(t, s)) \, s^2 \, ds.$$  

Let

$$\sup_{0 \leq s \leq t < \infty, i = 1, 2, 3} |K_i(t, s)| \leq 1.$$  

The corresponding majorant algebraic system

$$\begin{cases} r = \frac{r^3}{3}(1 + r + r^2), \\
1 = \frac{r^3}{3}(1 + 2r) \end{cases}$$

has the following solution: $r^* = 1, \rho^* = 1$. Therefore, based on Theorem 3 the integral equation has the main solution $x(t) \in C_{[0,1]}$, $||x|| \leq 1$. The integral majorant $x(t) = \int_0^t (x(s)^2 + x(s)) \, ds + \frac{r^3}{3}$ and Corollary 3 gives us more complete information regarding the solution. Indeed, the integral equation has continuous solution $x(t)$ on the interval $[0, 1.5365]$ and the following estimate is fulfilled $|x(t)| \leq 0.8660 \tan (0.5236 + 0.2886t^3) - 0.5$ for $0 \leq t < 1.5365$. 

Figure 1: Majorant of the solution. Solution $|x(t)|$ belongs to the gray zone

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