Magnetic focusing in atomic, nuclear and hadronic processes

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Abstract

Processes with oppositely charged spinor particles in initial and/or final states in homogeneous magnetic field B are subject to focusing effects in their relative motion, which yield the amplifying factors in probabilities growing as $eB$. In addition the increasing energy of some Landau levels influences the phase space. As a result some processes in the proper spin states can be enlarged as $\sim \frac{eB}{\kappa^2}$, where $\kappa^2$ is the characteristic 2d phase space factor available without magnetic field. Several examples, including neutron $\beta$ decay, positronium decay and $e^+e^-$ pair production, are quantitatively considered.

1 Introduction

The motion of charged particles in magnetic field (m.f.) is a standard topic of textbooks [1,2,3], and the behavior of atomic and nuclear systems in the framework of QED is extensively studied [4,5]. Recently also the hadronic systems in m.f. have attracted a lot of attention [6,7,8,9,10,11,12,13,14,15,16]. In particular, the role of m.f. in chiral symmetry breaking (CSB) was stressed both analytically [6] and on the lattice [7], see [8] for a review and references, and the corresponding phenomenon was coined magnetic catalysis.

The dynamical origin of magnetic catalysis in QCD was discovered recently in [17] and shown to be an example of a more general phenomenon – the magnetic focusing, which assembles together particles of opposite charges.
The study of relativistic QCD systems (quarks, gluons, hadrons) in m.f. has made it necessary to create and exploit the relativistic formalism, based on the path integrals with interaction as in Wilson loops, which allows to write down simple form Hamiltonians incorporating electromagnetic and strong interactions \[9\].

This formalism was used recently to calculate spectra of mesons in m.f. \[10, 11, 12\], including the Nambu-Goldstone mesons \[13\], and meson magnetic moments \[14\].

In the course of these studies it was found, that m.f. plays a very important role in “assembling” opposite charges near one another, i.e. the “focusing effect” to give giving contribution to the hyperfine (hf) splitting \(\sim |\psi(0)|^2\) growing as \(eB\) in hydrogen \[15\], as well as in relativistic \(q\bar{q}\) systems \[16\], making it necessary to introduce the smearing effect to consider hf as a perturbation. Moreover, it was found in \[17\], that the characteristic growth of quark condensate \(|\langle \bar{q}q \rangle|\) with \(eB\) is again due to the fact, that it is proportional to \(|\psi(0)|^2 \sim eB\), i.e. the focusing inside the \(q\bar{q}\) pair.

It is clear, that the focusing mechanism is of a general character and should show up in all cases, where such a factor \(|\psi(0)|^2\) for the wave function of relative coordinate of two oppositely charged particles appear. From the general scattering theory \[18\] it was shown, that this factor (for orbital momentum zero) always appears whenever the reaction has two strongly different ranges, \(r_{\text{ext}} \gg r_{\text{int}}\)

\[
dw = |\psi_{\text{ext}}(f)_{\text{int}}|^2 dw_{\text{int}} |\psi_{\text{ext}}(i)_{\text{int}}|^2
\]

where superscripts \(f, i\) refer to final, initial states. These effects of initial state interaction (ISI) or final state interaction (FSI) were carefully studied for the combination of Coulomb and nuclear forces \[19\].

In the case of m.f. two differences appear:

1. m.f. induce 2d discrete spectrum in external motion, hence the sum over spectrum should enter in (1) instead of a simple factor \(|\psi_{\text{ext}}(0)|^2\).

2. The masses of this discrete spectrum are generally growing with \(eB\) and strongly influence the available phase space, making in some cases the process impossible.

However for some lowest Landau levels (LLL), the energy is

\[
E_{n_{\perp}} = \sqrt{m^2 + (2n_{\perp} + 1 - \sigma)eB + p_z^2},
\]
with \( n_\perp = 0 \), where spin (magnetic moment) contribution \( \sigma = 1 \) exactly cancels the radial motion, yielding \( E_0 = \sqrt{m^2 + p_z^2} \). Thus one can gain in the resulting energy and phase space and one obtains for the creation of a pair the factor of growth

\[
\rho(eB) = \frac{w(eB)}{w(0)} = \frac{eB}{\kappa^2}
\]  

(3)

where \( \kappa^2 \) is the characteristic phase space, available for the perpendicular relative motion of two charges without m.f.

Our subsequent discussion in section 2 will be using the relativistic Hamiltonians in m.f. and the resulting eigenfunctions and energies, obtained in \([9, 10, 11, 12, 13, 14]\) and applicable in QED and QCD, augmented by the appropriate interaction terms. We shall proceed in section 3 with the simple example of \( e^+e^- \) production by \( \gamma \) in some reaction, and then comparing it with the pair creation in the constant electric field.

In section 4 we turn to the 3 body final state and consider the neutron \( \beta \) decay in m.f.

Other possible systems are discussed in section 5. In section 6 we give a short summary and prospectives. The appendix contains a short derivation of Eq. (1) using Jost solutions for external interaction.

## 2 Relativistic and nonrelativistic dynamics in strong magnetic field

Our final goal is to demonstrate how m.f. changes the relative motion of opposite charged particles and leads to the enhancement of the corresponding wave function at small distances, thus yielding the amplification factor for annihilation or production of such particles – the phenomenon of magnetic focusing.

To this end we are writing the relativistic Hamiltonian for a pair of particles with charges \( e_1 = -e_2 \equiv e \), which was already derived and exploited in \([10, 11, 12]\) and rederived in the framework of the new path integral representation in \([9]\). For particles with masses \( m_1, m_2 \) in m.f. \( B \) along \( z \) axis the Hamiltonian has the form (after the proper pseudomomentum factorization \([9, 10, 11]\)).
\[ H = \frac{P^2}{2(\omega_1 + \omega_2)} + \frac{\pi^2}{2\omega} + \frac{1}{2\omega} \frac{e^2}{4} (B \times \eta)^2 + \sum_{i=1,2} \frac{m_i^2 + \omega_i^2 - e_i \sigma_i B}{2\omega_i} + \hat{V}. \quad (4) \]

Here \( \eta = r_1 - r_2, \pi = \frac{\partial}{\partial r} \frac{\omega_1 k_1 - \omega_2 k_2}{\omega_1 + \omega_2}, \tilde{\omega} = \frac{\omega_1 \omega_2}{\omega_1 + \omega_2}, \) and \( \hat{V} \) is the sum of all interaction terms, including photon or gluon exchange \( V_{\text{Coulomb}} \equiv V_c(\eta), \) confining interaction \( V_{\text{conf}} \) for quarks, and spin-dependent and self-energy corrections, for details see [16].

The eigenfunction \( \Psi(\omega_1, \omega_2) \) and eigenvalues \( M_j \equiv M_{n_1, n_2}(\omega_1, \omega_2) \) depend on \( \omega_1, \omega_2 \) and the actual energy eigenvalue \( M_j^{(0)} \) is obtained from \( M_j(\omega_1, \omega_2) \) by the stationary value procedure,

\[ M_j^{(0)} = M_j(\omega_1^{(0)}, \omega_2^{(2)}), \quad \left. \frac{\partial M_j(\omega_1, \omega_2)}{\partial \omega_i} \right|_{\omega_i = \omega_i^{(0)}} = 0. \quad (5) \]

This scheme is discussed in detail in [9].

For nonrelativistic approximation the dominant terms are \( \frac{m_i^2 + \omega_i^2}{4\omega_i} \), which automatically give \( \omega_i^{(0)} = m_i \).

For strong m.f., when one can neglect \( \hat{V} \) in (4), i.e. for \( eB \gg \sigma \) in hadron systems and \( eB \gg (m_e \alpha)^2 \) in atomic systems, one immediately obtains the c.m. values of \( M_j^{(0)} \)

\[ M_j^{(0)} = \sqrt{m_1^2 + \pi_1^2 + eB(2n_1 + 1 - \sigma_{1z})} + \sqrt{m_2^2 + \pi_2^2 + eB(2n_1 + 1 + \sigma_{2z})}, \quad (6) \]

and the eigenfunction for \( n_\perp = 0 \) is (neglecting Coulomb interaction)

\[ \Psi(z, \eta_\perp) = e^{i\pi z} \frac{\varphi_{n_\perp}(\eta_\perp)}{\sqrt{L}}, \quad \varphi_{n_\perp}(\eta_\perp) = e^{-\frac{\pi^2}{2r_\perp^2}}, \quad r_\perp = \sqrt{\frac{2}{eB}}, \quad \varphi_0^2(0) = \frac{eB}{2\pi}. \quad (7) \]

Note, that \( \varphi_0^2(0) \) grows linearly with \( eB \), the property, which is the basic for the magnetic focusing phenomenon. As will be seen, another property is important: the energy eigenvalue \( M_{n_\perp=0}^{(0)}(\sigma_{1z} = 1, \sigma_{2z} = -1) \) does not grow with (and does not depend on) \( eB \). Thus the Lowest Landau Level (LLL) with \( n_\perp = 0 \) and \( \sigma_{1z} = -\sigma_{2z} = 1 \) (we shall call it the “zero level”) ensures the enhancement of probability of production or annihilation of the pair without increase of the spectrum, which would otherwise stop the process.
3 The electron-positron pair production in m.f.

Consider the process $A + B \rightarrow C + (e^+e^-)$, where $e^+e^-$ is produced by a virtual photon. The amplitude for the process without m.f. can be written as

$$\mathcal{M} = C_{\mu} \frac{1}{Q^2} (\bar{\psi} \gamma_{\mu} \psi), \quad \psi = \frac{u}{\sqrt{V_3}} e^{ik_+x}, \quad \bar{\psi} = \frac{\bar{u} e^{-ik_-x}}{\sqrt{V_3}}$$

and one should have in mind that both $e^+$ and $e^-$ are created at one point $x$. The probability can be written as

$$dw = \left| C_{\mu} \left( \bar{u} \gamma_{\mu} u \right) \right|^2 \frac{d^3k_+ d^3k_-}{(2\pi)^6} \delta(4)(Q - k^+ - k^-)(2\pi)^4 =$$

$$= \left| C_{\mu} \left( \bar{u} \gamma_{\mu} u \right) \right|^2 \frac{d^3k_+}{(2\pi)^2} \delta(Q_0 - 2\sqrt{(k^+)^2 + m_e^2}),$$

(9)

where we have assumed, that $Q = 0$.

We now turn to the case of nonzero m.f. Then the energy of $e^+e^-$ in m.f. $B$ can be written, using (6) as

$$E_{n_\perp}(\pi_z, B) = \sqrt{m_e^2 + \pi_z^2 + eB(2n_\perp + 1 - \sigma_{+z})} + \sqrt{m_e^2 + \pi_z^2 + eB(2n_\perp + 1 + \sigma_{-z})}$$

(10)

where $\pi_z = \frac{k_{+z} - k_{-z}}{2} = k_{+z}$, and $\sigma_{+z}$ and $\sigma_{-z}$ are doubled spin projection of $e^+$ and $e^-$ respectively, and we take into account, that $e_+ = -e_- \equiv e$.

For the probability one can write

$$dw = \left| C_{\mu} \left( \bar{u} \gamma_{\mu} u \right) \right|^2 \frac{d\pi_z}{2\pi} \sum_{n_\perp = 0} \delta(Q_0 - E_{n_\perp}(\pi_z)) \varphi_{n_\perp}^2(0),$$

(11)

and again $\varphi_0^2(0) = \frac{eB}{2\pi}$, as in (7).

It is important, that for the lowest energy state at $eB \rightarrow \infty$ both terms $(1 - \sigma_{1z})$ and $(1 + \sigma_{2z})$ should vanish, in which case the term with $n_\perp = 0$ always survives in (11), since $E_0$ does not contain $eB$. Looking at the structure $\bar{u} \gamma_{\mu} u$ and taking into account, that $u_+ = C\bar{u}$, $C = \gamma_2\gamma_4$, one comes to the conclusion, that for $\gamma_{\mu} = \gamma_3$ the cancellation mentioned above is possible, while for $\gamma_{\mu} = \gamma_1, \gamma_2, \gamma_4$ both $\sigma_{+z}$ and $\sigma_{-z}$ have the same sign.
Therefore for $eB \to \infty$ only the term with $\gamma_3$ survives and one obtains the behavior.

$$dw \cong \frac{eB}{2\pi} C_3 \left( \frac{\bar{u}\gamma_3 u}{Q^2} \right)^2 \frac{d\pi_z}{2\pi} \delta(Q_0 - E_0(\pi_z)), \quad (12)$$

which demonstrates the linear growth of the pair production with $eB$.

In the standard calculation of the $e^+e^-$ pair production in m.f. (see e.g. §91 of [3], and original papers [20] the probability was calculated using crossing relations with the process $e^- \to e^- + \gamma$ in m.f. and therefore the magnetic focusing effect was not taken into account.

It is interesting to apply our results to the process of $e^+e^-$ pair creation in the constant electric field (in Minkowski space-time).

In the proper-time formalism [21], see also chapter 4 of [4], the pair probability is

$$w(x) = \operatorname{Re} \operatorname{tr} \int_0^\infty ds \frac{d}{s} e^{-i(s(m^2-ic))} \langle x|\left(e^{iSD^2} - e^{i\sigma\partial^2}\right)|x\rangle \quad (13)$$

and $D_\mu = \partial_\mu - ieA_\mu(x), \ A_3(x) = -Et, t = x_0, \ A_\perp = \frac{1}{2}(B \times x), B$ is along $z$ axis.

According to our discussion above, only in this situation, when $E \parallel B$, the factor $\gamma_3$ in $A$ generates zero levels and the magnetic focusing produces the factor $eB/\pi$ for each pair without increasing effective mass of the $e^+e^-$ pair. The explicit form of the pair-production rate per volume per time was found in [22] for parallel $E$ and $B$ (remember $\gamma_3$ reasoning above)

$$w = \frac{(eE)(eB)}{(2\pi)^2} \coth \left( \frac{\pi B}{E} \right) \exp \left( -\frac{\pi m^2}{eE} \right), \quad (14)$$

where for $\pi B \gg E$ one can see a linear growth of $w$ with increasing $eB$. See more on pair creation in [4].

Now comparing (9) and (11) one finds the correspondence

$$\frac{d^2\pi_\perp}{(2\pi)^2} \to \sum_{n_\perp = 0}^{n_\perp(\text{max})} \varphi^2_{n_\perp}(0) \quad (15)$$

At this point one can compare our results with the statistical weight argument suggested in [11] [2], where the substitution should be performed with inclusion of magnetic field

$$\int \frac{d^3p}{(2\pi)^3} f(E) \to \int \sum_{n_\perp = 0}^{\infty} \frac{dp_z}{2\pi} f(E(B)) \frac{eB}{2\pi} \quad \text{ (16)}$$
and the sum over spin projections is assumed. One can see a close correspondence between (11) and (16), however (16) has to be attributed to any final particle, and only quasiclassical arguments are used for (16) in [1, 2].

It is interesting how the form (11) goes over into (9) when $eB \to 0$. To this end we write, generalizing $\varphi_{n\perp}(0)$ to $|\varphi_{n\perp}(r_{\text{int}})|^2$,

$$\varphi_{n\perp}(r_{\text{int}}) = \int \tilde{\varphi}_{n\perp}(\pi) \frac{d^2\pi}{(2\pi)^2} \exp(i\pi r_{\text{int}})$$

and

$$\sum_{n_{\perp}=0}^{n_{\perp}(\text{max})} |\varphi_{n\perp}(r_{\text{int}})|^2 = \sum_{n_{\perp}=0}^{n_{\perp}(\text{max})} \int \int \frac{d^2\pi}{(2\pi)^2} \frac{d^2\pi'}{(2\pi)^2} \tilde{\varphi}_{n\perp}(\pi) \tilde{\varphi}_{n\perp}^*(\pi') e^{i(\pi - \pi')r_{\text{int}}}.$$  

(17)

For large $n_{\perp}(\text{max}) \gg 1$ one can use the property of completeness of the set $\{\tilde{\varphi}_{n\perp}(\pi)\}$, which yields

$$\sum_{n_{\perp}=0}^{\infty} \tilde{\varphi}_{n\perp}(\pi) \tilde{\varphi}_{n\perp}^*(\pi') = (2\pi)^2 \delta^{(2)}(\pi - \pi'),$$

(18)

and

$$\sum_{n_{\perp}=0}^{n_{\perp}(\text{max})} |\varphi_{n\perp}(r_{\text{int}})|^2 \cong \int_0^{\pi(\text{max})} \frac{d^2\pi}{(2\pi)^2}.$$ 

(19)

In this way we are proving the correspondence (15), (13) for any $r_{\text{int}}$; note, however, that we have accounted only for the final state interaction (FSI) in m.f. and have shown, that it can be equivalently written in the form of a modified statistical weight. However, in the sum over $n_{\perp}$, the factor $|\varphi_{n\perp}(r_{\text{int}})|^2$ does not reduce to $\frac{eB}{2\pi}$, when $eB > \frac{1}{r_{\text{int}}}$.

4 Formalism for 3 body final states

In this section we consider an example of the neutron $\beta$ - decay, $n \to p + e^- + \bar{\nu}_e$, in the presence of the homogeneous magnetic field $B$ along $z$ axis.

Writing the matrix element as

$$\mathcal{M}_{ij} = G \cos \theta (\bar{\psi}_p(x)O^{(1)}(x)\psi_n(x))(\bar{\psi}_e(x)O^{(2)}(x)\psi_\nu),$$

(20)
where \( O^{(i)}_{\mu} = \gamma_{\mu}(1 - \alpha_i \gamma_5) \), and \( G \cos \theta \equiv \bar{G} = \frac{1.010^{-5}}{m_p^2} \), \( \alpha_1 = 1, 262, \alpha_2 = 0 \) and for \( B = 0 \) \( \psi_k(x) = \frac{ue^i \rho_k x}{\sqrt{\nu_3}} \), one obtains for the decay probability

\[
dw = \bar{G}^2 ((\bar{u}0_i u)(\bar{u}0_i u))^2 \left( \prod_{k=2,3,4} d^3 k \right) \frac{\delta^{(4)}(p_1 - p_2 - p_3 - p_4)}{(2\pi)^5}
\]

\[
= \bar{G}^2 ((\bar{u}0_i u)(\bar{u}0_i u))^2 \frac{(M_n - \varepsilon_2 - \varepsilon_3)\varepsilon_2 \varepsilon_3 d\varepsilon_2 d\varepsilon_3}{(4\pi)^3}
\]

(21)

and \( \varepsilon_i = \sqrt{P_i^2 + m_i^2}, \ i = 2, 3 \) refers to proton and electron respectively.

The integration in (21) proceeds inside the phase space of \( ep \) system.

We now want to apply the m.f. to our system and to this end we realize that it will act only on the relative coordinate \( \eta_\perp \) of (\( ep \)) system, perpendicular to the B direction, and we must separate out the \( \eta_\perp \) dependence both in the \( ep \) wave function and in the phase space integration one can consider (20) as a matrix element of the operator \( \hat{T} \) between initial and final states, see appendix, \( M_{if} = \langle \Psi_f^+ | \hat{T} | \Psi_i \rangle \), and both \( \Psi_f \) and \( \Psi_i \) enter at one space-time point in the limit of large \( W \) mass.

Therefore \( \Psi_f(x) = \psi_p(x)\psi_e(x) \rightarrow \Psi_{ep}(x) \) and the latter w.f. is defined by the m.f. Hamiltonian (\( H \)).

According to [5], the Hamiltonian for the neutral \( (ep) \) system can be written as in (\( H \)) with \( \omega_1 = \omega_p \approx m_p, \ \omega_2 = \omega_e \ll m_p \).

Noting, that \( \omega_1 \equiv \omega_p \approx m_p \gg \omega_2 \equiv \omega_e \), one obtains

\[
E_{ep}^{(0)} \equiv E_{ep}(\omega_p^{(0)}, \omega_e^{(0)}) \approx \frac{P^2}{2m_p} + \sqrt{m_p^2 + \pi_p^2 + eB(2n_\perp + 1 - \sigma_{p_\perp})} + \sqrt{m_e^2 + \pi_e^2 + eB(2n_\perp + 1 - \sigma_{e_\perp})}.
\]

(22)

Now the phase space can be rewritten as follows

\[
\frac{d^3 p_p d^3 p_e}{(2\pi)^6} = \frac{d^3 P d^3 \pi}{(2\pi)^6} = \frac{d^3 P d\pi_\perp d^2 \pi_\perp}{(2\pi)^2}
\]

(23)

and

\[
d\Phi \equiv \frac{d^3 p_p d^3 p_e d^3 P}{(2\pi)^5} \delta^{(4)}(p_n - p_p - p_e - p_\nu) = \frac{d^3 P d\pi_\perp d^2 \pi_\perp}{(2\pi)^6} \delta(m_n - |P| - E_{ep}^{(0)}) = \frac{P^2 d\pi_\perp}{2\pi^2 (1 + \frac{P}{m_p})} \frac{d^2 \pi_\perp}{(2\pi)^2}
\]

(24)
writing $E_{ep}^{(0)} \cong m_p + \varepsilon_e, \varepsilon_e \equiv \varepsilon$, one can express (21) in the form:

$$d\Phi = \frac{(\Delta m - \varepsilon)^2}{4\pi^3} d\pi \varpi d\varpi = (\Delta m - \varepsilon)^2 \frac{\sqrt{\varepsilon^2 - m_e^2}}{2\pi^3} \varepsilon d\varepsilon; \quad \Phi = \int d\Phi \cong \frac{1.63}{2\pi^2} m_e^5$$

(25)

$\Delta m = 1.29$ MeV, and finally

$$w_n = G^2 (1 + 3\alpha^2) \Phi; \quad \alpha \equiv g_A / g_V \approx 1.262 \quad (26)$$

In the presence of m.f. the values of $\pi_\perp$ are quantized in the Hamiltonian (4), so that one should replace as in (15), so that finally the decay probability becomes

$$dw = g^2 |(\bar{u}0_iu)(\bar{u}0_iu)|^2 \frac{P^2 d\pi_z}{4\pi^2 \left(1 + \frac{P}{m_p}\right)} \sum_{n_\perp=0}^{n_\perp \text{(max)}} \varphi_{n_\perp}^2(0)$$

(27)

and $n_\perp \text{(max)}$ is defined by the condition $m_n = P + E_{ep}^{(0)}(n_\perp)$.

As was shown in (19) for $eB \to 0$ one has the answer (21).

In the opposite limit, when $eB$ is large, $eB \gg m_e^2$ and $n_\perp \text{(max)} = 0$, one can read in (22) that $E_{ep}^{(0)}$ is

$$E_{ep}^{(0)}(eB \to \infty) \cong m_p + \sqrt{m_e^2 + \pi_z^2}, \quad \sigma_{ez} = -1 \quad (28)$$

and

$$\Delta m \equiv m_n - m_p = P + \sqrt{m_e^2 + \pi_z^2} \quad (29)$$

which defines allowable phase space for $(P, \pi_z)$. In this case for $n_\perp = 0$ one has the amplifying factor as in (21) $\varphi_{0}^2(0) = \frac{|eB|}{2\pi}$, which can be much larger, than $\int \pi_\perp \text{(max)} \frac{d^2 \pi}{[2\pi]^2} = \frac{\pi_\perp^2 \text{(max)}}{4\pi} \leq \frac{(\Delta m)^2 - m_e^2}{4\pi}$ for $eB \gg (\Delta m)^2 - m_e^2$, see [23, 24].

Inserting the values of $\Delta m = 2.53m_e$, one obtains the amplifying factor for $eB \gg m_e^2$, when only one term $n_\perp = 0$ should be kept in (26), $w_n(eB) = \frac{eB}{\kappa^2}$, and $\kappa^2 \approx m_e^2$. Exact calculation in [23] yields $w_n(eB)/w_n(0) = 0.77 \frac{eB}{m_e^2}$.

Note, however, that for small $\pi_\perp \text{(max)}$ and hence small $\kappa^2$, this ratio can be arbitrarily large.

For more discussion of this subject and additional references see [25].
5 Other interactions and other systems

One immediate application of the magnetic focusing is the hyperfine interaction in all systems. In hydrogen this effect was studied in [15] and it was shown, that in the standard form of the hyperfine shift

\[ \Delta E_{hf} = \frac{32\pi}{3} g_p\mu_B\mu_n |\Psi(0)|^2 \] (30)

the wave function of the ground state can be chosen as

\[ \Psi_0(\eta_\perp, z) = N \exp\left(-\frac{\eta_\perp^2 a^2}{2} - \frac{z^2 b^2}{2}\right) \] (31)

with \(|\Psi(0)|^2 = \frac{a^2 b}{\pi^2}\), and \(a, b\) are fitting parameters; for the free electron \(a = \frac{eB}{2mc}\), and for large \(eB > (m_e\alpha)^2, a^2 b\) grows with \(eB\) almost linearly, which gives the possibility to study this effect experimentally.

The same effect was found in relativistic hadronic systems in [16], where again the hf shift has the same form as in (25), namely

\[ \langle V_{hf} \rangle = \frac{8\pi\alpha_s(\sigma_1\sigma_2)}{9\omega_1\omega_2} |\Psi_{qq}(0)|^2 \] (32)

and one can show, that (32) when \(\omega_1, \omega_2\) are decreasing and \(\sigma_1\sigma_2 = -3\) leads to the absurd result of the negative PS masses at large \(eB\), contradicting the stability theorem, proved in [16], which tells, that energy eigenvalues in magnetic and vacuum Euclidean fields cannot be negative and hence \(V_{hf}\) cannot be treated perturbatively, when \(\omega_i \to 0\) and \(\langle V_{hf} \rangle \to \infty\), so that the relativistic smearing must be introduced, replacing \(\delta^{(3)}(\eta)\) in \(V_{hf}\) by \(\left(\frac{1}{\mu(\eta)}\right)^3 \exp(-\mu^2\eta^2), \mu \approx (1 \div 2)\) GeV.

In any case, the behavior (32) signifies the strong increase of the hyperfine splitting for hadrons in magnetic field, which is stabilized by the smearing [16].

The calculations of hadron masses, subject to strong hf interaction have been done in [10, 11, 12, 13]. In the case of \(\pi^0, \rho^0\) mesons the large \(eB\) asymptotics corresponds to \(\langle \sigma_1\sigma_2 \rangle = \langle -+ \rangle\) state for \(\pi^0\) and \(\langle ++ \rangle\) state for \(\rho^0\) and the strong hf interaction splits the masses, creating a deep minimum for the \(\pi^0\) mass. However, this calculation in [10] refers to the purely \(q\bar{q}\) components of \(\pi^0\), whereas the chiral dynamics in m.f. needs a special treatment, which
was performed in [12] and again showing a decreasing with \( eB \pi^0 \) mass, with a minimum.

The calculation in [13], was done basing on the unified theory of chiral dynamics with \( q\bar{q} \) degrees of freedom, developed earlier in [26]. In its turn in [17] it was shown, that the chiral condensate grows with m.f., first quadratically for \( eB < \sigma \) and then for \( eB \gg \sigma \) linearly with \( eB \). This behavior found in [17] agrees quantitatively with lattice data of [7], which contradict CPTh. In this way one can conclude, that the standard chiral theory can be applied only for \( eB \lesssim m^2_e \), as was noticed before in [27].

The subsequent analysis of Nambu-Goldstone (NG) mesons in m.f. was done in [13], where it was shown, that GMOR relations are valid for neutral NG mesons, but are violated for the charged ones, and the \( \pi^+ \) mass was calculated in agreement with lattice data from [28]. One should stress, that the phenomenon of “magnetic catalysis”, discussed in [6, 8] and implying the growth of chiral condensate \( \langle \bar{q}q \rangle \) in m.f. actually occurs due to magnetic focusing, as was shown in [17], since \( \langle \bar{q}q \rangle \sim \psi^2(0) \sim eB \).

We now turn to other processes, where the opposite charge particles appear in the initial state. One example is the \( \mu^- + p \rightarrow n + \nu \). In the case, when the process occurs from the ground state of (\( \mu^- p \)) atom, one can use the form (26) with \( r_{\perp}, r_z \) obtained in [15], which are reduced by the ratio \( \frac{m_e}{m_{\mu}}, \frac{m_{\mu}}{m_p} \).

Since the probability \( w_{\mu} \equiv w(\mu^- + p \rightarrow n + \nu) \) is proportional to \( |\Psi_{\mu p}(0)|^2 \), one can obtain the amplification coefficient

\[
\rho(B) \equiv \frac{w_{\mu}(B)}{w_{\mu}(0)} = \frac{v_+^2(0)r_z(0)}{r_+^2(B)r_z(B)}, \quad \rho(0) = 1. \tag{33}
\]

Using the data from the Fig.2 of [15], one easily obtains for \( H = \frac{eB}{m_p^2\alpha^2}, \ 0 \leq H \leq 2 \)

\[
\rho(H = 1) = \frac{1.33}{1^{21.15}} \approx 1.05; \quad \rho(H = 2) = 3.35.
\]

One can see, that in this case one needs much larger fields \( \left( \frac{m_{\mu}}{m_e} \right)^2 \) times larger to produce the same kind of effect, as in the electron case.

Let us turn now to the case of the positronium annihilation in m.f. Since the latter does not conserve the spin, but only total spin projection \( S_z \), it is convenient to discuss separately the case a) of \( S_z = 0 \) parapositronium and
orthopositronium, and b) $S_z \pm 1$ orthopositronium. In the first case, as in the $q\bar{q}$ case, both states are mixture of $\alpha = (\sigma_z = +1, \sigma_z = -1)$ $\equiv < + - |$ and $\beta = (- + |$, and parapositronium in strong m.f. tends to be a pure $\alpha$ state, whereas orthopositronium a pure $\beta$ state (see [10] for a similar discussion in the $q\bar{q}$ case). For the case b) $S_z = \pm 1$, in orthopositronium the corresponding ground state masses in strong m.f. grows as $\sim \sqrt{eB}$ with additional amplification due to the strong hf contribution. As a result the decay phase space increases with $eB$ and the decay probability in the case b) is growing both due to phase space and as in (33), where in $H \tilde{m}_\mu \rightarrow \frac{1}{2} m_e$.

The same happens in the $\beta$ state of orthopositronium, since the total energy in the $\beta$ state also grows as $\sim \sqrt{eB}$.

However, in the $\alpha$ state of parapositronium the mass is slightly decreasing, while the $|\psi(0)|^2$ factor is growing as $eB$, and we expect the same situation, as in the example of $(e^+e^-)$ pair creation in m.f., discussed in section 2. At this point it is necessary to stress also the difference between the two examples: for positronium annihilation the factor $\rho(B)$ is proportional to $|\psi(r_{an})|^2$, while the $(e^+e^-)$ pair creation occurs at one point and brings about factor $|\psi(0)|^2$. The resulting difference is expected to be of the order of unity for $eBr_{an}^2 < 1$.

However for large $eB > (m_e\alpha)^2$ one expects the behavior to be $w_{+-}(B) \approx \int |\psi_{+-}(r_{an})|^2 w^{(0)} d\tau$, where $w^{(0)} d\tau$ is the phase space integration (depending on $B$) with the two-photon annihilation amplitude $A^{(2)}$, $w^{(0)} = |A^{(2)}|^2$. Note, that both $< + - |$ and $< - + |$ states can be superpositions of $S = 1$ and $S = 0$ states. As a result we show the distributions $w_{+-}(B)$ and $w_{-+}(B)$ as functions of $eB$ and notice, that the linear growth for $eB \gg (m_e\alpha)^2$ is saturated for larger m.f., as was discussed previously.

In [29] the authors have used the gaussian form of positronium wave function and found the almost linear growth of two-photon annihilation with $eB \lesssim 10^{13}$ Gauss: $w(H = 1) = 3.35 \cdot 10^{12} s^{-1}$, $w(H = 3) = 12.3 \cdot 10^{12} s^{-1}$, $w(H = 1) = 3.35 \cdot 10^{12} s^{-1}$, $w(H + 10) = 5.09 \cdot 10^{13} s^{-1}$, $w(H = 44) = 11.8 \cdot 10^{13} s^{-1}$, where $H = \frac{B}{10^{12}}$ Gauss. One can see, that for $B = 10^{13}$ Gauss the growth of $w$ is weakening.

6 Summary and discussion

We have demonstrated, that the acting of m.f. on the system of two opposite charges in the initial or final state produces an amplifying factor, which can
grow linearly with $eB$ for $eB$ in the range $(eB)_{\text{min}} \lesssim eB \lesssim (eB)_{\text{max}}$.

In particular, for the $(e, -e)$ continuum production the relative probability $w(eB)/w(0) \simeq |\psi_{\text{cont}}(r_{\text{int}})|^2/\kappa^2$ can grow as $\frac{eB}{\kappa^2}$, where $\kappa^2$ is the effective phase space for the perpendicular motion in the $B = 0$ case. In this case $(eB)_{\text{min}}$ is also of the order of $\kappa^2$, while $(eB)_{\text{max}}$ is defined by the $(eB)_{\text{max}} \sim 1/r_{\text{int}}^2$, where $r_{\text{int}}$ refers to the range of the internal production process (e.g. $r_{\text{int}} \sim 1/m_W$ for neutron $\beta$-decay).

We have illustrated this behavior by the processes of the $e^+e^-$ pair creation and the neutron $\beta$-decay; in the last case this amplification was known and calculated by many authors, see e.g. [23], [24], [25] using the quasiclassical calculation of phase space (QPS) in m.f. as in [1] [2].

We have shown, that the FSI method of our paper gives the same result as QPS for $r_{\text{int}} = 0$ or for $eB \ll 1/r_{\text{int}}^2$; however for $eB \sim 1/r_{\text{int}}^2$ the QPS prediction overestimates the probability.

For the $(e, -e)$ bound state production the amplification factor is proportional to $|\psi_{\text{b.s.}}(r_{\text{int}})|^2$, and for $r_{\text{int}} = 0$ it grows linearly with $eB$ only for $eB \gg r_{\text{b.s.}}^2$, where $r_{\text{b.s.}}$ is the bound state radius at zero m.f., e.g. for positronium $r_{\text{b.s.}} = \frac{2}{m_e \alpha}$.

The same arguments can be applied in principle to the effect of constant m.f. in the initial state, as it is illustrated above in the paper by the example of the two-photon positron annihilation, see e.g. [29]. However in this case one should distinguish different physical situations; in astrophysics these processes are part of a set of reactions in m.f. at finite temperature and chemical potential, while in experiment one should take into account boundary conditions of the experimental device.

Incidentally, one should stress, that all above derivations disregarded the Coulomb potential role in ISI and FSI.

As it is known, the latter gives the factor for $(e, -e)$ system $|\psi_{\text{coul}}(0)|^2 = \frac{2\pi \eta}{\exp(-2\pi \eta)+1}$, $\eta = \frac{e^2}{v^2}$, which may bring an additional amplification of the magnetic focusing effect, discussed above. As it is, strictly speaking we can discuss the Coulomb amplification for $2\pi \eta \gg 1$ and $eB \ll \kappa^2$. However the explicit amplification factor in case, when both Coulomb interaction and m.f. are acting, is still not available and should be derived, using solutions when both interactions are present.

Finally, probably the most important conclusion of our paper is, that for $(e, -e)$ systems in continuum there occurs an amplification factor $\frac{eB}{\kappa^2}$, which does not depend on the masses of the charges and their sizes $R_i$ (provided $eB < 1/R_i^2$) and therefore can be important for stimulating dif-
ferent reactions in atomic, molecular, nuclear or hadronic reactions. At this point one should compare our results for the process of creation of the neutral system of two oppositely charged particles with a similar process, where single charged particles appear. In the latter case the particle is trapped in its motion across m.f. and cannot reach detector at the distance R cm, when m.f. is higher than approximately $1/R^2 \times 10^{-9}$ Gauss. This is in contrast with our neutral system, which has in continuum spectrum the radius $eB^{-1/2}$ and is freely moving across and along m.f., thus yielding the finite cross section. The author is grateful for useful discussions and suggestions to M.A. Andreichikov, B.M. Karnakov, B.O. Kerbikov, V.D. Orlovsky and M.I. Vysotsky.

Appendix

Initial and final interaction factors in process probability

One can use the formalism of to define the matrix element of a transition from the state $\beta$ to $\alpha$, generated by the interaction $V$ in the total hamiltonian

$$H = K + U + V$$

where $K$ is the kinetic term and $U$ and $V$ have different ranges $r_U, r_V$ correspondingly. To the first order in $V$ one can write:

$$f_{\beta\alpha} = (\varphi^-_\beta V \varphi^+_\alpha) + O(V^2)$$

where $\varphi^-_\beta, \varphi^+_\alpha$ are ingoing and outgoing solutions of the operator $K + U$,

$$\varphi^-_\beta = \chi_\beta + \frac{1}{E - K - i\varepsilon} U \varphi^-_\beta$$

and for $\varphi^+_\alpha$ one should replace $\beta \to \alpha, \varepsilon \to -\varepsilon$.

Introducing for the orbital momentum $l = 0$ the internal amplitude due to $V$ only as $f_{in}(E)$, one can write (see [18] for a detailed derivation)

$$f_{aa} = f_{in}(E)(\eta_{Jost})^{-2}$$
where $\eta_{Jost} = \eta(0)$, and $\eta(r)$ is the Jost solution of the external problem, satisfying the condition

$$\lim_{r \to \infty} \exp(-ikr)\eta(r) = 1,$$  (A 5)

while for the Coulomb type interaction one should replace $-ikr \to -ikr + \frac{\alpha}{e} \ln 2kr$.

$\eta_{Jost}$ can be expressed via the regular solution $\chi(r)$ of the external potential with the asymptotics

$$\chi_{ex}(r) \sim \frac{\sin(kr + \delta_{ext})e^{i\delta_{ext}}}{kr}, \quad r \to \infty,$$  (A 6)

and the connection is

$$\lim_{r \to 0} \chi(r) = \eta_{Jost}^{-1}.$$  (A 7)

and as a result one has (see [18]),

$$f_{\alpha\alpha}(E) = \frac{f_{in}(E)(\chi_{ex}(0))^2}{1 + f_{in}(E)g(E,0)},$$  (A 8)

where $g(e, r_{in})$ is expressed via Green’s functions

$$g(E, r) = G_{0}(r, r_{in}) - G_{ex}(r, r_{in})$$  (A 9)

In particular for the Coulomb external problem one finds [19]

$$\eta_{Jost} = \exp\left(\frac{\pi\gamma}{2}\right) / \Gamma(1 + i\gamma) = |\eta_{Jost}| \exp(-i\sigma_{0}),$$  (A 10)

$$\exp(2i\sigma_{0}) = \frac{\Gamma(1 + i\gamma)}{\Gamma(1 - i\gamma)},$$  (A 11)

$$|\eta_{Jost}|^{-2} \equiv C_{0}^{2} = \frac{2\pi\gamma}{\exp(2\pi\gamma) - 1},$$  (A 12)

where $\gamma = \frac{zae^{\mu}}{k}$, and $\gamma = -|\gamma|$ for opposite charge system. Note, however, that [A 4], [A 12] refer to the combination of the short range (nuclear) and Coulomb (external) interactions, and the latter acts both in the initial and final states ($\varphi_{\beta}$ and $\varphi_{\alpha}^{+}$ in (A 2)). In the more general case, when initial and final states are different (i.e. short range reaction with different particles and
interactions in the initial and final states), one should keep track of indices \( \beta \) and \( \alpha \), which are different, and \( \varphi^-_\beta \) and \( \varphi^+_{\alpha} \) satisfy equations of the form \[[A.3]\] for \( \varphi^-_\beta \) with \( U \equiv U_\beta, K = K_\beta \) and \( \varphi^+_{\alpha} \) satisfies

\[
\varphi^+_{\alpha} = \chi_{\alpha} + \frac{1}{E - K_{\alpha} + i\varepsilon} U_{\alpha} \varphi^+_{\alpha}.
\]

(A 13)

To clarify this situation we consider, as in \[18\], a simple model for the internal interaction

\[
V(\mathbf{r}, \mathbf{r}') = \lambda g_\beta(r)g_\alpha(r')
\]

(A 14)

and as a result the reaction amplitude becomes

\[
f_{\beta\alpha} = \lambda I_{\beta}I_{\alpha}, \quad I_{\beta} = \int \varphi^-_{*\beta} g_\beta d^3r.
\]

(A 15)

Taking into account different radii of internal and external motion, one can approximate

\[
I_{\beta} = \varphi^-_{*\beta}(0) \int g_\beta d^3r, \quad I_{\alpha} = \varphi^+_{\alpha}(0) \int g_\alpha d^3r
\]

(A 16)

a generalization of \( f_{\alpha\alpha} \) to the case of \( \alpha \neq \beta \) is (to the first order in \( f_{in} \))

\[
f_{\beta\alpha} = \left( \eta_{Jost}^{(\beta*)} \right)^{-1} f_{in}(E) \left( \eta_{Jost}^{(\alpha)} \right)^{-1}.
\]

(A 17)

we are especially interested in the case, when both Coulomb interaction and external magnetic field are present simultaneously. The first interaction is effective when \( \gamma = \left| \frac{Z_\alpha}{v} \right| \sim 1 \) and for \( \gamma \ll 1 \) one can retain only m.f. effects. In this case both \( \varphi^-_{\beta} \) and \( \varphi^+_{\alpha} \) are product of a plane wave in the \( z \) direction and bound state eigenfunction in the \((x, y)\) plane,

\[
\varphi^-_{\beta} = \varphi^+_{\alpha} = e^{ik_z z} \frac{1}{\sqrt{L}} \varphi_{n_\perp}(\mathbf{x}_\perp).
\]

(A 18)

References

[1] L.D.Landau and E.M.Lifshitz, Quantum Mechanics: Non-Relativistic Theory, Pergamon, New York, 1991, 3d. ed.

[2] L.D.Landau and E.M.Lifshitz, Statistical mechanics, Clarendon Press, Oxford, 1938.
[3] V.B.Berestetskii, E.M.Lifshitz and L.P.Pitaevskii, “Electrodynamics”, 2nd Ed., Pergamon, Oxford, 1982.

[4] C.Itzykson and J.-B.Zuber, Quantum Field Theory, (McGraw-Hill, New York, 1980).

[5] J.S.Schwinger, Particles, Sources and Fields, v.2, Addison-Wesley, 1988.

[6] V.Gusynin, V.Miransky and I.Shovkovy, Phys. Rev. Lett. 73, 3499 (1994).

[7] G.S.Bali, et.al., Phys. Rev. 86, 071502 (2012);
P.Buividovich, et.al., Phys. Lett. B 682, 484 (2010); M.D’Elia and F.Negro, Phys. Rev. D 83, 114028 (2011).

[8] I.Shovkovy, arXiv:1207.5081.

[9] Yu.A.Simonov, arXiv:13.03.4952.

[10] M.A.Andreichikov, B.O.Kerbikov and Yu.A.Simonov, arXiv:1210.0227, [hep-ph].

[11] M.A.Andreichikov, V.D.Orlovsky and Yu.A.Simonov, Phys. Rev. Lett. 110, 162002 (2013).

[12] M.A.Andreichikov, B.O.Kerbikov, V.D.Orlovsky and Yu.A.Simonov, arXiv:1304.2533; Phys. rev. D (in press).

[13] V.D.Orlovsky and Yu.A.Simonov, (in preparation).

[14] A.M.Badalian and Yu.A.Simonov, Phys. Rev. D 87, 074012 (2013).

[15] M.A.Andreichikov, B.O.Kerbikov and Yu.A.Simonov, arXiv:1304.2516.

[16] Yu.A.Simonov, arXiv:1304.0365.

[17] Yu.A.Simonov, arXiv:1212.3118.

[18] M.L.Goldberger and K.M.Watson, Collision Theory, Jhon Wiley and Sons, Inc., 1964; chapter 5; A.M.Badalian, L.P.Kok, M.I.Polikarpov and Yu.A.Simonov, Phys. Rept. 82, 32 (1982).
[19] G.Breit, E.Condon and R.Present, Phys. Rev. 50, 825 (1936). G.Breit, B.Thaxton and L.Eisenbud, Phys. Rev. 55, 1018 (1939); A.Sommerfeld, Atmobilau und Spektralinien (F.Vieweg and Sohn: 1921); G.Gamov, Zeit. Phys. 51, 204 91928); A.D.Sakharov, Sov. Phys. JETP 18, 631 (1948).

[20] A.I.Nikishov and V.I.Ritus, Sov. Phys. JETP. 19, 529 , 1191 (1964).

[21] J.S.Schwinger, Phys. Rev. 82, 664 (1951).

[22] A.I.Nikishov, Sov. Phys. JETP, 30, 660 (1970); Nucl. Phys. B21, 346 (1970).

[23] J.J.Matese and R.F.O’Connell, Phys. Rev. 180, 1289 (1969).

[24] L.Fassio-Canuto, Phys. Rev. 187, 2141 (1969).

[25] L.Korovina, Izv. Vuzov. Fizika 6, 86 (1964); I.Ternov, B.Lysov and L.Korovina, Mosc. U. Phys. Bull. 5, 58 (1965); V.Zakhartsev and Y.Loskutov, Mosc. U. Phys. Bull. 26 2, 24(1985); A.Studenikin, Sov. J. Astrophys. 28 (3), 639 (1988); Sov. J. Nucl. Phys. 49, 1031 91989); K.A.Kouzakov and A.I.Studenikin, arXiv: hep-ph/0412134.

[26] Yu. A.Simonov, Phys. At. Nucl. 60, 2069 (1997); ibid 67, 846 (2004); ibid 67, 1027 (2004); S.M.Fedorov and Yu.A.Simonov, JETP Lett. 78, 57 (2003); Yu.A.Simonov, Phys. Rev. D65, 094018 (2002).

[27] N.O.Agasian and I.Shushpanov, Phys. Lett. B 472, 143 (2000); JHEP 10110, 006 (2001);
N.O.Agasian, Phys. Lett. B 488, 39 (2000); Phys. Atom. Nucl. 64, 554 (2001).

[28] G.S.Bali,F.Bruckmann, G.Endrodi, et.al., Jorn. of High-Energy Phys., v.2012, 44; arXiv: 1111.4956.

[29] S.Carr and P.Sutherland, Astrophysics and space Science 58, 83 (1978).