STRONG PULLBACK ATTRACTORS FOR A NONCLASSICAL DIFFUSION EQUATION

XIAOLEI DONG
College of Information Science and Technology
Donghua University, Shanghai 201620, China

YUMING QIN*
1. Department of Mathematics
2. Institute for Nonlinear Science
Donghua University, Shanghai 201620, China

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1. Introduction. The general nonclassical diffusion equation describes some physical phenomena, for instance, non-Newtonian flows, soil mechanics and heat conduction theory. In this paper, we investigate the existence of pullback attractors in $H^2(\Omega) \cap H^1_0(\Omega)$ for the following nonclassical diffusion equations:

\[
\begin{cases}
\left\{ 
\begin{array}{ll}
    u_t - \mu \Delta u_t - \Delta u + g(u) = f(x, t), & x \in \Omega, \ t > \tau, \\
    u|_{\partial \Omega} = 0, & t > \tau, \\
    u|_{t=\tau} = u_{\tau}(x), & x \in \Omega,
\end{array}
\right.
\end{cases}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N \ (N \geq 5)$ with smooth boundary $\partial \Omega$, the nonlinearity $g(u)$ and the given external force term $f(t)$ satisfy some assumptions later, the parameter $\mu > 0$ be fixed. It is called the nonclassical diffusion equation when $\mu > 0$ and it is named the classical reaction-diffusion equation in the case $\mu = 0$.

To start with, let us briefly review some known results to the problem (1). During the last years, the existence and asymptotic behavior of the solutions to problem (1) have been investigated by many authors under the different assumption conditions. There have been some results [38, 31, 47, 32, 22, 1, 42, 46, 50, 35, 41, 45, 18] for problem (1) when the perturbation parameter $\mu > 0$ is a fixed constant. Authors in [38] investigated the existence of strong global attractors for a nonclassical diffusion equation with subcritical nonlinearity, and proved the global attractors $\mathcal{A}_\mu \to \mathcal{A}_0^+$ in the sense of Hausdorff semidistance in $H^1_0(\Omega)$ as $\mu \to 0^+$.

Authors in [22] studied...
the existence of exponential attractors for a three dimensional nonclassical diffusion equation in $H^2(\Omega) \cap H^1_0(\Omega)$. Sun and Yang [32] got the existence of global attractors for the autonomous nonclassical diffusion equation with critical nonlinearity by the operator decomposition method, and proved the existence of the exponential attractors of non-autonomous nonclassical diffusion equation by applying the criterion of the existence of exponential attractors for discrete process given in [13]. Anh and Bao [1] proved the existence of pullback $\mathcal{D}$ attractors and the upper semi-continuity of pullback $\mathcal{D}$ attractors in $H^1_0(\Omega)$ for the non-autonomous nonclassical diffusion equation by using the asymptotic a priori estimate method. Wang and Qin [42] studied the upper semi-continuous property of pullback attractors of nonclassical diffusion equations with non-autonomous perturbation when the nonlinearity $g$ satisfies the proper growth condition and the dissipation condition. Wu and Zhang [46] considered the long-time behaviour of the nonclassical diffusion equation with critical nonlinearity in $H^1_0(\Omega)$. Zhang and Ma [50] investigated the existence of exponential attractors of the nonclassical diffusion equation with critical nonlinearity and lower regular forcing term and got the finite fractal dimension of the global attractors. Under the conditions of nonlinearity $g$ satisfies the following Sobolev type growth and dissipation,

$$
\begin{align*}
g(u)u &\geq -\lambda u^2 - C, \\
g'(u) &\geq -\ell, \quad g(0) = 0, \\
|g'(u)| &\leq C(1 + |u|^\gamma), \\
\lim \inf_{|u| \to \infty} \frac{u g(u) - \kappa G(u)}{u^2} &\geq 0, \quad \text{for some } \kappa > 0, \\
\lim \inf_{|u| \to \infty} \frac{G(u)}{u^2} &\geq 0,
\end{align*}
$$

Toan [35] proved the existence of pullback attractors for a non-autonomous nonclassical diffusion equation in a non-cylindrical domain with the homogeneous Dirichlet boundary condition.

Recently, authors in [41] considered the upper semi-continuity of uniform attractors in $H^1_0(\Omega)$ for a non-autonomous nonclassical diffusion equation with critical nonlinearity. Wang, Zhu and Li [45] proved the regularity of pullback attractors for a three dimensional non-autonomous nonclassical diffusion equation with critical nonlinearity. Under a Sobolev type growth condition of the nonlinearity $g$, Lee and Toi [18] investigated the existence and upper semi-continuity of global attractors of nonclassical diffusion equation in a smooth bounded domain $\Omega$ with dynamic boundary condition. Wang and Hu [37] studied the continuity of exponential attractors in $H^1_0(\Omega)$ when the nonlinearity $g \in C^1$ satisfies the some suitable assumptions. To be more precise, our motivation in this paper is to use the method of [30] to prove the global existence and uniqueness of strong solutions in $H^2(\Omega) \cap H^1_0(\Omega)$ to the non-autonomous nonclassical diffusion equations, then we prove the existence of pullback attractors in $H^2(\Omega) \cap H^1_0(\Omega)$ by applying asymptotic a priori estimate method [44], which is still an open problem before the present paper solved it.

According the problem (1), we can observe that the equation contains the term $-\Delta u_t$, it is different from the original reaction diffusion equation. For instance, the solution of the usual reaction diffusion equation has some smoothing effect, in other words, if the initial data only belong to a weaker topology space, then the solution of the usual reaction diffusion equation will belong to a stronger topology space with higher regularity. However, for problem (1), both the initial data and the solution belong to the same space, but the solution has no higher regularity.
because of appearance of $-\Delta u_t$. To overcome these difficulties, motivated by [30], we prove the existence and uniqueness of strong solutions to problem (1) and use the asymptotic a priori estimate method [44] that has been developed recently to prove the existence of pullback attractors in $H^2(\Omega) \cap H^1_0(\Omega)$.

Besides, when the nonlinearity satisfies the arbitrary polynomial growth condition, Anh and Toan [3] investigated the existence of pullback attractors for a non-autonomous nonclassical diffusion equation in a non-cylindrical domain with the homogeneous Dirichlet boundary condition and got the upper semi-continuous of pullback attractors. Authors in [48] considered upper semi-continuity and regularity of attractors for a nonclassical diffusion equations by applying the operator decomposition method. In addition to the above results, the asymptotic behavior of solutions for a nonclassical diffusion equation with delay and memory has been extensively investigated by lots of authors in [6, 34, 53, 5, 17, 16, 2, 7, 8, 9, 10, 11, 12, 43, 39, 40, 49, 51] and references therein. Moreover, there have also some known results in [29, 21, 24, 54] for a nonclassical diffusion equation when $\mu(t)$ depends on time.

When the parameter $\mu = 0$, the problem (1) reduces to the problem of the usual reaction diffusion equations. To our best knowledge, many results have been obtained about the usual reaction diffusion equations under the different assumption conditions. The existence of attractors for reaction diffusion equations has been investigated in many literatures, see [55, 15, 52, 23, 25, 33, 36, 20, 19] and the references therein for the recent progress.

Finally, the rest of the paper is arranged as follows. In Section 2, we will give some definitions and lemmas will be used frequently and the main result Theorem 2.6. In Section 3, we prove the existence and uniqueness of strong solutions to problem (1) by using the method in [30]. In Section 4, we prove the existence of pullback absorbing sets and the asymptotic behavior of solutions for problem (1), then we complete the proof of Theorem 2.6.

Hereafter, let letter $C$ be a general positive constant, which may vary from line to line to each step.

2. Preliminaries and main result. In this section, we first introduce some notations on the function spaces and norms which will be used later to study the existence of pullback attractors.

As in [32], let

$$A = -\Delta,$$ with domain $D(A) = H^2(\Omega) \cap H^1_0(\Omega),$

and consider the family of Hilbert spaces $D(A^{s/2})$, $s \in \mathbb{R}$, with the standard inner products and norms, respectively,

$$(\cdot, \cdot)_{D(A^{s/2})} = (A^{s/2} \cdot, A^{s/2} \cdot) \text{ and } \| \cdot \|_{D(A^{s/2})} = |A^{s/2} \cdot|_2.$$

Let $H = L^2(\Omega)$, $V_1 = H^1_0(\Omega)$ and $V_2 = D(A) = H^2(\Omega) \cap H^1_0(\Omega)$. Denote by $(\cdot, \cdot)$ and $|\cdot|_2$ the inner product and norm of $H$, respectively. We denote by $(\langle \cdot, \cdot \rangle)$ and $\| \cdot \|_1$ the inner product and norm of $V_1$, respectively. Denote by $\| \cdot \|_2$ the norm of $V_2$.

To investigate problem (1), we need the following assumption conditions on $g$ (see [38]),
(i) The nonlinearity \( g \in C^1(\mathbb{R}) \) satisfies
\[
\begin{align*}
&g(u)u \geq -\lambda u^2 - C_\lambda, \\
g'(u) \geq -\ell, \quad g(0) = 0, \\
|g'(u)| \leq C(1 + |u|^\gamma),
\end{align*}
\]
where \( 0 \leq \gamma \leq \frac{2}{N-4} \), the constants \( \ell, \ C_\lambda > 0, \ 0 < \lambda < \frac{\lambda_1}{2} \), \( \lambda_1 > 0 \) is the first eigenvalue of \(-\Delta\) in \( \Omega \) with the homogeneous Dirichlet boundary condition, and \( G(u) = \int_0^u g(s)ds \) is the primitive function of \( g \).

(ii) We assume that the external \( f \in L^2_{loc}(\mathbb{R}, V_1) \) satisfies
\[
\int_{-\infty}^t e^\delta \| f \|^2 ds < +\infty, \text{ for any } t \in \mathbb{R},
\]
with \( \delta = \frac{\lambda_1}{4(1+\lambda_1\rho)} \).

(iii) The initial datum \( u_\tau \in H^2(\Omega) \cap H^1_0(\Omega) \).

**Definition 2.1.** ([20]) A two parameter family of mappings \( U(t, \tau) : X \to X, \ t \geq \tau, \ \tau \in \mathbb{R} \), is called to be a process if
1. \( U(t, \tau)x = x, \ \forall \tau \in \mathbb{R}, \ x \in X \),
2. \( U(t, \tau)U(s, \tau)x = U(t, \tau)U(s, \tau)x, \ t \geq s \geq \tau, \ \tau \in \mathbb{R}, \ x \in X \).

**Definition 2.2.** ([20]) A family of bounded sets \( \hat{B} = \{ B(t) : t \in \mathbb{R} \} \in \mathcal{D} \) is called pullback \( \mathcal{D} \)-absorbing for the process \( \{ U(t, \tau) \} \) if for any \( t \in \mathbb{R} \) and for any \( \hat{D} \in \mathcal{D} \), there exists \( \tau_0(t, \hat{D}) \leq t \) such that
\[
U(t, \tau)D(\tau) \subset B(t), \text{ for all } \tau \leq \tau_0(t, \hat{D}).
\]

Before the discussion of problem (1), we give the well-known condition (P) (see [44]) which will be devoted to the proof of the existence of pullback attractors.

**Definition 2.3.** ([44]) A process \( U \) defined on Banach space \( X \) is said to be satisfying condition (P) at \( \mathcal{B} = \{ B(t) \}_{t \in \mathbb{R}} \), if for any \( t \in \mathbb{R} \), any \( \varepsilon > 0 \), there exists a \( \tau_\varepsilon(= \tau(t, \mathcal{B}, \varepsilon)) \leq t \) and a finite dimensional subspace \( X_1 \) of \( X \) such that
(i) \( P(\bigcup_{s \leq \tau_\varepsilon} U(t, s)B(s)) \) is bounded; and
(ii) \( \| (I - P)(\bigcup_{s \leq \tau_\varepsilon} U(t, s)B(s)) \|_X \leq \varepsilon \), where \( P : X \to X_1 \) is a bounded projector.

**Lemma 2.4.** ([14]) Let \( u \in W^{1,p}(0, T; X) \) for some \( 1 \leq p \leq \infty \). Then
\[
u \in C([0, T]; X),
\]
where
\[
\| u \|_{W^{1,p}(0, T; X)} := \left( \int_0^T \| u \|_X^p + \| u' \|_X^p dt \right)^{1/p}, \quad (1 \leq p < +\infty);
\]
\[
\text{ess sup}_{0 \leq t \leq T} (\| u \|_X + \| u' \|_X), \quad (p = +\infty).
\]

**Lemma 2.5.** Let \( X \) be a Banach space and the process \( U \) is norm-to-weak continuous on \( X \). Then \( U \) has a pullback attractor \( \mathcal{A} = \{ A(t) \}_{t \in \mathbb{R}} \) in \( X \) provided that the following conditions hold true:
(i) \( U \) has a family of bounded absorbing sets \( \mathcal{B} = \{ B(t) \}_{t \in \mathbb{R}} \) in \( X \);
(ii) \( U \) satisfies condition (P) at \( \mathcal{B} \).

Now, we give the main result in this paper.

**Theorem 2.6.** Assume \( g \) and \( f \) satisfy the assumptions (i) – (ii), respectively. Let \( 0 \leq \gamma \leq \frac{2}{N-4}, \ \ell > 0, \ 0 < \lambda < \frac{\lambda_1}{2} \), then problem (1) possesses a pullback attractor in \( H^2(\Omega) \cap H^1_0(\Omega) \).

We will prove Theorem 2.6 by a series of lemmas in Sections 3-4.
3. Existence and uniqueness of solutions. We know that there have been some results \cite{1, 41, 42, 45} on the existences and properties of the pullback attractors for nonclassical diffusion equations (1) in $H^1_0(\Omega)$. We state the known result for the existence and uniqueness of weak solution to problem (1) as the following lemma.

Lemma 3.1. ([1]) For any $\tau \in \mathbb{R}$, $T > \tau$, for each $u_\tau \in H^1_0(\Omega)$, problem (1) has a unique weak solution $u = u(t; \tau) = u(t; u_\tau)$ with

$$u \in C([\tau, T]; H^1_0(\Omega)), \quad \frac{\partial u}{\partial t} \in L^2([\tau, T]; H^1_0(\Omega)).$$

We use the idea of \cite{30} to prove the higher regularity of solutions to problem (1) and show that if initial datum $u_\tau \in V_2$, then the solutions of problem (1) are in this space for $t \geq \tau$. Now, we show the existence and uniqueness of strong solutions to problem (1) as follows.

Lemma 3.2. Assume that $g$ satisfies (i) and $f \in L^2_{loc}(\mathbb{R}; V_1)$. Then for any initial datum $u_\tau \in V_2$ and for any $\tau \in \mathbb{R}$, $T > \tau$, problem (1) has a unique strong solution $u = u(t; \tau) = u(t; u_\tau)$ with

$$u \in C([\tau, T]; V_2).$$

Proof. Assume that $\lambda_i$ denote the eigenvalues of operator $A$ in $D(A)$, $(i = 1, 2, \ldots)$ and satisfy

$$0 < \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \to +\infty \text{ as } j \to +\infty.$$

Let $\omega_i$ be the eigenvector corresponding to the eigenvalue $\lambda_i$, $(i = 1, 2, \ldots)$. Then they consist of a group of orthogonal basis of $V_2$ and satisfy

$$A\omega_i = \lambda_i \omega_i, \quad \forall i \in \mathbb{N}.$$

(i) Step 1. We first give some estimates on the approximate solutions $u_n(t) = \sum_{i=1}^n u_{n,i} \omega_i$. Taking the inner product of

$$\frac{du_n}{dt} + \mu A\partial_t u_n + Au_n + g(u_n) = f_n(t) = \sum_{j=0}^n (f(t), \omega_j) \omega_j$$

with $u_n$ and using the assumption conditions (i) on $g$, we have

$$\frac{1}{2} \frac{d}{dt} \left( |u_n(t)|^2 + \mu \|u_n(t)\|^2 \right) + \|u_n(t)\|^2 - \lambda |u_n|^2 \leq (f_n(t), u_n(t)) + C\lambda |\Omega|,$$

where $|\Omega|$ is the Lebesgue measure of $\Omega$, $|\Omega| = \int_{\Omega} dx$.

Using the Young inequality, $\lambda_1 |u_n|^2 \leq \|u_n\|^2$ and $0 < \lambda < \frac{\lambda_1}{2}$, we obtain

$$|(f_n(t), u_n(t))| \leq \frac{1}{2\lambda} |f_n(t)|^2 + \frac{\lambda}{2} |u_n(t)|^2,$$

and
\[ \|u_n(t)\|_1^2 - \frac{3\lambda}{2}u_n(t)\|_2^2 \]
\[ = \frac{1}{4}\|u_n(t)\|_1^2 + \frac{3}{4}\|u_n(t)\|_2^2 - \frac{3\lambda}{2}u_n(t)\|_2^2 \]
\[ \geq \frac{1}{4}\|u_n(t)\|_1^2 + \left(\frac{3\lambda}{4} - \frac{3\lambda}{2}\right)u_n(t)\|_2^2 \]
\[ \geq \frac{1}{4}\|u_n(t)\|_1^2. \tag{5} \]

Inserting (4)-(5) into (3), we deduce
\[ \frac{d}{dt}\left(\|u_n(t)\|_2^2 + \mu\|u_n(t)\|_1^2\right) + \frac{1}{2}\|u_n(t)\|_1^2 \leq \frac{1}{\lambda}\|f_n(t)\|_2^2 + 2C\lambda\Omega. \tag{6} \]

Integrating the above inequality on \([\tau, t]\), we conclude
\[ \left(\|u_n(t)\|_2^2 + \mu\|u_n(t)\|_1^2\right) + \frac{1}{2}\int_{\tau}^{t}\|u_n(s)\|_1^2 ds \]
\[ \leq \left(\|u_n(\tau)\|_2^2 + \mu\|u_n(\tau)\|_1^2\right) + \frac{1}{\lambda}\int_{\tau}^{t}\|f_n(s)\|_2^2 ds + 2C\lambda\Omega(T - \tau), T > \tau. \tag{7} \]

Consequently, we have
\[ \{u_n\} \text{ is uniformly (w.r.t n) bounded in } L^\infty(\tau, T; V_1) \cap L^2(\tau, T; V_1). \]

(ii) Step 2. We prove some better estimates on the approximate solutions \(u_n(t)\) and their derivatives. To this end, multiplying (2) by \(Au_n\) and integrating the result over \(\Omega\), we conclude
\[ \frac{1}{2}\frac{d}{dt}\|u_n(t)\|_1^2 + \mu\frac{d}{dt}\|u_n(t)\|_2^2 + \|u_n(t)\|_2^2 + (g(u_n(t)), Au_n(t)) = (f_n(t), Au_n(t)). \tag{8} \]

By the assumption conditions (i) on \(g\), we have
\[ (g(u_n(t)), -\Delta u_n(t)) = \int_{\Omega} g'(u_n(t))\nabla u_n(t) \cdot \nabla u_n(t) dx \geq -\ell\|u_n(t)\|_1^2, \tag{9} \]
where we have used the facts \(u_n|_{\partial\Omega} = 0\) and \(g(0) = 0\).

Applying the Young inequality, we derive
\[ (f_n(t), Au_n(t)) \leq \frac{1}{2}\|f_n(t)\|_2^2 + \frac{1}{\ell}\|u_n(t)\|_1^2. \tag{10} \]

Inserting (9)-(10) into (8), we obtain
\[ \frac{1}{2}\frac{d}{dt}\|u_n(t)\|_1^2 + \mu\frac{d}{dt}\|u_n(t)\|_2^2 + \frac{1}{2}\|u_n(t)\|_2^2 \leq \|u_n(t)\|_1^2 + \frac{1}{\ell}\|u_n(t)\|_1^2 + \frac{1}{2}\|f_n(t)\|_2^2. \tag{11} \]

Integrating (11) on \([\tau, t]\) and using the result of Step 1, \(\tau \leq t \leq T\), we get
\[ \|u_n(t)\|_1^2 + \mu \|u_n(t)\|_2^2 + \int_\tau^t \|u_n(s)\|_2^2 ds \]

\[ \leq \|u_n(\tau)\|_1^2 + \mu \|u_n(\tau)\|_2^2 + 2\ell \int_\tau^t \|u_n(s)\|_1^2 ds + \int_\tau^t |f_n(s)|^2_2 ds \]

\[ \leq C \left( \|u_n(\tau)\|_1^2 + \mu \|u_n(\tau)\|_2^2 + \int_\tau^t |f_n(s)|^2_2 ds + T \right), \]  

(12)

thus \( \{u_n\} \) is uniformly \((w, r, t n)\) bounded in \( L^\infty(\tau, T; V_2) \cap L^2(\tau, T; V_2) \).

\( iii \) \textbf{Step 3}. Multiplying (2) by \( u_{nt} \) and integrating the result over \( \Omega \), and using the Cauchy inequality, it follows that

\[ \frac{d}{dt} \left( \|u_n(t)\|_1^2 + 2 \int_\Omega G(u_n(t)) \right) + |u_{nt}(t)|^2_2 + 2\mu \|u_{nt}(t)\|_1^2 \leq |f_n(t)|^2_2. \]  

(13)

Integrating (13) on \([\tau, t]\), we deduce

\[ \|u_n(t)\|_1^2 + \int_\tau^t (|u_{nt}(s)|^2_2 + 2\mu \|u_{nt}(s)\|_1^2) ds \]

\[ \leq \int_\tau^t |f_n(s)|^2_2 ds + \|u_n(\tau)\|_1^2 + 2 \int_\Omega G(u_n) dx - 2 \int_\Omega G(u_n(t)) dx. \]  

(14)

Combining with the following estimates:

\[ \int_\Omega G(u_n(\tau)) dx \leq \int_\Omega |G(u_n(\tau))| dx \leq C(1 + |u_n(\tau)|_{\gamma+\frac{2}{2}}^{\gamma+\frac{2}{2}}) \]

\[ \leq C(1 + \|u_n(\tau)\|_2^{2+2}), \]  

(15)

\[ \int_\Omega G(u_n(t)) dx \leq \int_\Omega |G(u_n(t))| dx \leq C(1 + |u_n(t)|_{\gamma+\frac{2}{2}}^{\gamma+\frac{2}{2}}) \]

\[ \leq C(1 + \|u_n(t)\|_2^{2+2}), \]  

(16)

which make use of \( \gamma + 2 \leq \frac{2N-6}{N-4} \) and \( D(A) \hookrightarrow L^{\frac{2N-6}{N-4}} \).

Inserting (15)-(16) into (14), we derive

\[ \|u_n(t)\|_1^2 + \int_\tau^t (|u_{nt}(s)|^2_2 + 2\mu \|u_{nt}(s)\|_1^2) ds \]

\[ \leq \int_\tau^t |f_n(s)|^2_2 ds + \|u_n(\tau)\|_1^2 + C(1 + \|u_n(\tau)\|_2^{\gamma+2} + \|u_n(t)\|_2^{\gamma+2}). \]  

(17)

Then it follows that \( \int_\tau^t (|u_{nt}(t)|^2_2 + \|u_{nt}(t)\|_1^2) dt \) is uniformly \((w, r, t n)\) bounded.

\( vii \) \textbf{Step 4}. Multiplying (2) by \( Au_{nt} \) and integrating the result over \( \Omega \), and using the Cauchy inequality, it follows that

\[ \frac{1}{2} \frac{d}{dt} \|u_n(t)\|_1^2 + \|u_{nt}(t)\|_1^2 + \mu \|u_{nt}(t)\|_2^2 = \int_\Omega f_n(t) Au_{nt}(t) dx - \int_\Omega g(u_n(t)) Au_{nt}(t) dx. \]  

(18)

Using the Cauchy inequality, we have
\[
\int_{\Omega} f_n(t)Au_n(t)\,dx \leq |f_n(t)|_2\|u_{nt}(t)\|_2 \leq \frac{1}{\mu}|f_n(t)|_2^2 + \frac{\mu}{4}\|u_{nt}(t)\|_2^2, \quad (19)
\]

\[
\int_{\Omega} g(u_n(t))Au_n(t)\,dx \leq \frac{1}{\mu}|g(u_n(t))|_2^2 + \frac{\mu}{4}\|u_{nt}(t)\|_2^2, \quad (20)
\]

\[
|g(u_n(t))|_2^2 \leq \int_{\Omega} |g(u_n(t))|^2\,dx \leq C(1 + |u_n(t)|_2^{2\gamma + 2}), \quad (21)
\]

where we have used the facts \(2\gamma + 2 \leq \frac{2N-4}{N-4}\) and \(D(A) \hookrightarrow L^{\frac{2N}{N-4}}\).

Inserting (19)-(21) into (18) and integrating the result on \([\tau, t]\), and using the result of Step 2, we get

\[
\|u_n\|_2^2 + \int_{\tau}^{t} (2\|u_{nt}(s)\|_1^2 + \mu\|u_{nt}(s)\|_2^2)\,ds
\]

\[
\leq \|u_n(\tau)\|_2^2 + \frac{2}{\mu} \int_{\tau}^{t} |f_n(s)|_2^2\,ds + C \int_{\tau}^{t} (1 + \|u_n(s)\|_2^{2\gamma + 2})\,ds
\]

\[
\leq C \left(\|u_n(\tau)\|_2^2 + \|u_n(\tau)\|_2^2 + \int_{\tau}^{t} |f_n(s)|_2^2\,ds + T\right). \quad (22)
\]

Then it follows that \(\int_{\tau}^{T} (\|u_{nt}(t)\|_1^2 + \|u_{nt}(t)\|_2^2)\,dt\) is uniformly \((w, r, t, n)\) bounded. Applying (12) and (21), we get

\[
\int_{\tau}^{T} |g(u_n(s))|_2^2\,ds \leq C \int_{\tau}^{T} (1 + \|u_n(s)\|_2^{2\gamma + 2})\,ds \leq C,
\]

implies

\[
\{g(u_n)\} \text{ is bounded in } L^2(\tau, T; L^2(\Omega)). \quad (23)
\]

Therefore, from (7), (12), (17), (22) and (23), we have

\[
\begin{align*}
& u_n \rightharpoonup u \text{ weakly in } L^\infty(\tau, T; V_2), \\
& u_n \rightharpoonup u \text{ weakly in } L^2(\tau, T; V_2), \\
& \partial_t u_n \rightharpoonup \partial_t u \text{ weakly in } L^2(\tau, T; V_2), \\
& g(u_n) \rightharpoonup \eta \text{ weakly in } L^2(\tau, T; L^2(\Omega)).
\end{align*}
\]

Using Theorem 8.1 in [30], we can deduce that \(u_n \to u\) strongly in \(L^2(\tau, T; H)\). Hence, \(u_n \rightharpoonup u\) a.e. in \(\Omega \times [\tau, T]\). Since \(g\) is continuous, it follows that \(g(u_n) \to g(u)\) a.e. in \(\Omega \times [\tau, T]\). Thanks to (23) and Lemma 8.3 in [30], one has

\[
g(u_n) \to g(u) \text{ weakly in } L^2(\tau, T; L^2(\Omega)),
\]

hence \(\eta = g(u)\).

However, the uniqueness follows from Lemma 3.1, since a strong solution is automatically a weak solution.

Moreover, using \(2\gamma + 2 \leq \frac{2N-4}{N-4}\), \(D(A) \hookrightarrow L^{\frac{2N}{N-4}}\) and \(u \in L^\infty(\tau, T; V_2) \cap L^2(\tau, T; V_2)\), we have

\[
\int_{\tau}^{T} |g(u(s))|_2^2\,ds \leq C \left(\int_{\tau}^{T} (\|u(s)\|_2^2 + \|u(s)\|_2^{2\gamma + 2})\,ds\right) \leq C.
\]
Then, \( v = u + \mu Au \in L^\infty(\tau, T; H) \) and \( v' = -Au - g(u) + f(t) \in L^2(\tau, T; H) \), from Lemma 2.4, we can get \( u \in C^0([\tau, T]; V_2) \). The proof is thus complete. \( \square \)

4. Pullback attractors in \( V_2 \). Thanks to Lemma 3.2, we can define a continuous process \( \{U(t, \tau)\}_{t \geq \tau} \) in \( V_2 \) by

\[
U(t, \tau)u_\tau = u(t), \ t \geq \tau, \tag{24}
\]

where \( u(t) \) is the unique strong solution of the problem (1) with \( f(t) = f \in L^2_{\text{loc}}(\mathbb{R}, V_1) \) and \( u(\tau) = u_\tau \in V_2 \).

4.1. Pullback Absorbing Balls in \( V_2 \). In this subsection, we will establish the existence of pullback absorbing sets for \( N \)-dimensional \((N \geq 5)\) nonclassical diffusion equation with Dirichlet boundary condition. Throughout this subsection, we always assume that the initial data belong to a bounded set of corresponding suitable space.

Next, we show the existence of pullback absorbing balls \( \{B(t)\} \) in \( V_2 \) for the nonclassical diffusion equation.

**Lemma 4.1.** Let \( g \) satisfy the assumption conditions (i), \( f \in L^2_{\text{loc}}(\mathbb{R}; V_1) \) satisfy

\[
\int_{-\infty}^{t} e^{\delta s} |f|_2^2 ds < +\infty, \text{ for any } t \in \mathbb{R}, \tag{25}
\]

then the process \( \{U(t, \tau)\}_{t \geq \tau} \) possesses a family of pullback absorbing balls \( \{B(t)\} \) in \( V_2 \) with the center zero and \( R(t) = C \left( 1 + e^{-\delta t} \int_{-\infty}^{t} e^{\delta s} |f(s)|_2^2 ds \right) \) with \( \delta = \frac{\lambda_1}{4(1+\lambda_1\mu)} \), for all \( \tau \leq \tau_B \leq t, \)

\[
B = \left\{ u \in V_2 : \|u\|_{V_2}^2 \leq R(t) \right\}. \tag{26}
\]

**Proof.** Multiplying (1) by \( u \) and integrating the result over \( \Omega \), we attain

\[
\frac{1}{2} \frac{d}{dt} (|u(t)|_2^2 + \mu \|u(t)\|_V^2) + \|u(t)\|_1^2 - \lambda \|u\|_2^2 \leq (f(t), u(t)) + C_\lambda |\Omega|, \tag{27}
\]

where \(|\Omega|\) is the Lebesgue measure of \( \Omega \).

Applying the Cauchy inequality, \( \lambda_1 \|u\|_2^2 \leq \|u\|_1^2 \) and \( 0 < \lambda < \frac{\lambda_1}{2} \), we derive

\[
(|f(t), u(t)| \leq \frac{1}{2\lambda} |f(t)|_2^2 + \frac{\lambda}{2} \|u(t)\|_2^2, \tag{28}
\]

and

\[
\|u(t)\|_1^2 - \frac{3\lambda}{2} \|u(t)\|_2^2 = \frac{1}{4} \|u(t)\|_1^2 + \frac{3}{4} \|u(t)\|_2^2 - \frac{3\lambda}{2} \|u(t)\|_2^2 \geq \frac{1}{4} \|u(t)\|_1^2 + \left( \frac{3\lambda_1}{4} - \frac{3\lambda}{2} \right) \|u(t)\|_2^2 \geq \frac{1}{4} \|u(t)\|_1^2. \tag{29}
\]

Inserting (28)-(29) into (27) and taking \( \delta = \frac{\lambda_1}{4(1+\lambda_1\mu)} \), we deduce

\[
\frac{d}{dt} (|u(t)|_2^2 + \mu \|u(t)\|_V^2) + \frac{1}{4} \|u(t)\|_1^2 + \delta (|u(t)|_2^2 + \mu \|u(t)\|_V^2) \leq \frac{1}{\lambda} |f(t)|_2^2 + 2C_\lambda |\Omega|. \tag{30}
\]
Utilizing the Gronwall lemma, we conclude

\[
e^{\delta t}(|u(t)|_2^2 + \mu \|u(t)\|_1^2) + \frac{1}{4} \int_{\tau}^{t} e^{\delta s} \|u(s)\|_1^2 ds \leq e^{\delta \tau}(|u(\tau)|_2^2 + \mu \|u(\tau)\|_1^2) + \left(\frac{2C_\lambda |\Omega|}{\delta} \right) e^{\delta t} + \frac{1}{\lambda} \int_{-\infty}^{t} e^{\delta s} |f(s)|_2^2 ds.
\]  

(31)

Let \( \tau < t - 1 \), from (31), we deduce that for all \( r \in [\tau, t - 1] \),

\[
e^{\delta(r+1)}(|u(r+1)|_2^2 + \mu \|u(r+1)\|_1^2) + \frac{1}{4} \int_{\tau}^{r+1} e^{\delta s} \|u(s)\|_1^2 ds \leq e^{\delta r}(|u(r)|_2^2 + \mu \|u(r)\|_1^2) + \left(\frac{2C_\lambda |\Omega|}{\delta} e^{\delta(r+1)}\right) + \frac{1}{\lambda} \int_{-\infty}^{t} e^{\delta s} |f(s)|_2^2 ds
\]

\[
\leq e^{\delta r}(|u(\tau)|_2^2 + \mu \|u(\tau)\|_1^2) + \left(\frac{2C_\lambda |\Omega|}{\delta} e^{\delta t}\right) + \frac{1}{\lambda} \int_{-\infty}^{t} e^{\delta s} |f(s)|_2^2 ds.
\]  

(32)

Multiplying (1) by \(-\Delta u\) and integrating the result over \( \Omega \), we derive

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \mu \frac{d}{dt} \|u(t)\|_2^2 + \|u\|_2^2 + (g(u(t)), -\Delta u(t)) = (f(t), -\Delta u(t)).
\]  

(33)

Using the assumption conditions (i) on \( g \), we have

\[
(g(u(t)), -\Delta u(t)) = \int_{\Omega} g'(u(t)) \nabla u(t) \cdot \nabla u(t) dx \geq -\ell \|u(t)\|_1^2,
\]

where we have used the facts \( u|_{\partial \Omega} = 0 \) and \( g(0) = 0 \).

Applying the Young inequality, we deduce

\[
(f(t), -\Delta u(t)) \leq \frac{1}{2} \|f(t)\|_2^2 + \frac{1}{2} \|u(t)\|_2^2.
\]  

(35)

By (33)-(35), we easily conclude that

\[
\frac{1}{2} \frac{d}{dt} \|u(t)\|_2^2 + \mu \frac{d}{dt} \|u(t)\|_2^2 + \|u\|_2^2 \leq \ell \|u(t)\|_1^2 + \frac{1}{2} \|f(t)\|_2^2.
\]  

(36)

which implies

\[
\frac{d}{dt} \|u(t)\|_2^2 + \mu \|u(t)\|_2^2 + \delta (\|u(t)\|_1^2 + \mu \|u(t)\|_2^2) + \frac{1}{2} \|u(t)\|_2^2 \leq 2\ell \|u(t)\|_1^2 + |f(t)|_2^2.
\]  

(37)

Multiplying (37) by \( e^{\delta t} \) and integrating the result over the interval \([\tau, t]\), and using (31), we obtain

\[
e^{\delta t}(|u(t)|_1^2 + \mu \|u(t)\|_2^2) + \frac{1}{2} \int_{\tau}^{t} e^{\delta s} \|u(s)\|_2^2 ds \leq e^{\delta \tau}(|u(\tau)|_2^2 + \mu \|u(\tau)\|_2^2) + 2\ell \int_{\tau}^{t} e^{\delta s} \|u(s)\|_1^2 ds + \int_{-\infty}^{t} e^{\delta s} |f(s)|_2^2 ds
\]

\[
\leq C \left(e^{\delta \tau}(|u(\tau)|_2^2 + \mu \|u(\tau)\|_2^2) + e^{\delta t} + \int_{-\infty}^{t} e^{\delta s} |f(s)|_2^2 ds\right).
\]  

(38)
Similarly to (32), let \( \tau < t - 1 \), by (38), we deduce that for all \( r \in [\tau, t - 1] \),
\[
e^{\delta(r+1)}(\|u(r + 1)\|_1^2 + \mu\|u(r + 1)\|_2^2) + \frac{1}{2} \int_{r}^{r+1} e^{\delta s} \|u(s)\|_2^2 ds \leq C(e^{\delta \tau}(\|u(r)\|_1^2 + \mu\|u(r)\|_2^2) + e^{\delta(r+1)} + \int_{-\infty}^{t} e^{\delta s}|f(s)|_2^2 ds) \leq C(e^{\delta \tau}(\|u(\tau)\|_1^2 + \mu\|u(\tau)\|_2^2) + e^{\delta \tau} + \int_{-\infty}^{t} e^{\delta s}|f(s)|_2^2 ds).
\]
From (36), we derive
\[
\frac{d}{dt}(\|u\|_1^2 + \mu\|u\|_2^2) \leq 2\epsilon(\|u\|_1^2 + \mu\|u\|_2^2) + |f|_2^2.
\]
Therefore,
\[
\frac{d}{dr}(e^{\delta r}(\|u\|_1^2 + \mu\|u\|_2^2)) = \delta(e^{\delta r}(\|u\|_1^2 + \mu\|u\|_2^2)) + e^{\delta r} \frac{d}{dr}(\|u\|_1^2 + \mu\|u\|_2^2) \leq C(e^{\delta r}(\|u\|_1^2 + \mu\|u\|_2^2) + e^{\delta r}|f|_2^2).
\]
In view of (32) and (39), applying the uniform Gronwall inequality [26, 27, 28], we conclude
\[
\|u\|_1^2 + \mu\|u\|_2^2 \leq C(e^{-\delta(t-\tau)}(\|u(\tau)\|_1^2 + \mu\|u(\tau)\|_2^2) + 1 + e^{-\delta t} \int_{-\infty}^{t} e^{\delta s}|f(s)|_2^2 ds)
\leq C(1 + e^{-\delta t} \int_{-\infty}^{t} e^{\delta s}|f(s)|_2^2 ds), \quad \tau \leq \tau_B \leq t.
\]
The proof is thus complete.

4.2. Asymptotically Compact in \( V_2 \). In this subsection, we will establish the existence of the pullback attractors in \( V_2 \) for problem (1). The main difficulty is to attain a higher regularity estimates to ensure the asymptotic compactness of process. To overcome this difficulty, we use the idea in [44] to solve it.

Lemma 4.2. Under the assumptions of Theorem 2.6, the process \( U(t, s) \) satisfies condition (P).

Proof. Let \( \{\omega_m\} \) be an orthogonal basis of \( H_m \) which consists of eigenvectors of \( A = -\Delta \). \( \lambda_m \) denote the corresponding eigenvalues, \( m = 1, 2, \ldots \) and \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_j \leq \cdots \to +\infty \) as \( j \to +\infty \). Then \( \{\omega_m\} \) is also an orthonormal basis of \( V_1 \) and \( V_2 \). We write \( H_m = \text{span}\{\omega_1, \cdots, \omega_m\} \) and \( P_m : V_2 \to H_m \) is the orthogonal projection.

For any \( u \in V_2 \), we decompose \( u \) into two parts,
\[
u = P_m u + (I - P_m)u \overset{\Delta}{=} v + w.
\]
Multiplying (1) by \(-\Delta w \) and integrating the resulting equation over \( \Omega \), we conclude
\[
\frac{1}{2} \frac{d}{dt}(\|u\|_1^2 + \mu\|w\|_2^2) + \|w\|_2^2 = (-g(u), -\Delta w) + (f(t), -\Delta w).
\]
Using the assumption conditions (i) on $g$, the Hölder inequality, Young inequality and $\lambda_{N+1}\|w\|_1^2 \leq \|w\|_2^2$, we deduce that
\[
\frac{1}{2} \frac{d}{dt}(\|w\|_1^2 + \mu\|w\|_2^2) + \|w\|_2^2 \\
\leq (1 + \|u\|_{L^\gamma})|\nabla u|_{L^\Delta} + \|w\|_1 + \|w\|_1 |f|_1 \\
\leq (1 + \|u\|_2^2)\|w\|_1 + \|w\|_1 |f|_1 \\
\leq \frac{R(t) + (R(t))^{\gamma+1}}{\lambda_{N+1}} + \frac{1}{2\lambda_{N+1}} |f(t)|_1^2 \\
\leq \frac{2(R(t))^{\gamma+1}}{\lambda_{N+1}} + \frac{1}{2\lambda_{N+1}} |f(t)|_1^2, \tau \leq \tau_B \leq t,
\]
where we used the embedding theorems $H^2(\Omega) \hookrightarrow L^{\gamma N}(\Omega)$ and $H^2(\Omega) \hookrightarrow W^{1,2(\Delta)}(\Omega)$.

Thus,
\[
\frac{1}{2} \frac{d}{dt}(\|w\|_1^2 + \mu\|w\|_2^2) + \frac{1}{2}\|w\|_2^2 \leq \frac{2(R(t))^{\gamma+1}}{\lambda_{N+1}} + \frac{1}{\lambda_{N+1}} |f(t)|_1^2, \tau \leq \tau_B \leq t. \quad (45)
\]

Utilizing the fact that $\lambda_1\|w\|_1^2 \leq \|w\|_2^2$ for $w \in V_2$, we have
\[
\frac{d}{dt}(\|w\|_1^2 + \mu\|w\|_2^2) + \delta(\|w\|_1^2 + \mu\|w\|_2^2) \\
\leq \frac{4(R(t))^{\gamma+1}}{\lambda_{N+1}} + \frac{2}{\lambda_{N+1}} |f(t)|_1^2, \tau \leq \tau_B \leq t, \quad (46)
\]
where we have used the fact $\delta(\|w\|_1^2 + \mu\|w\|_2^2) < \frac{\lambda_1}{\lambda_{N+1}}(\|w\|_1^2 + \mu\|w\|_2^2) < \|w\|_2^2$.

Applying the Gronwall inequality on $[\tau, t]$ to (46) (see [26, 27, 28]), we obtain
\[
\|w(t)\|_1^2 + \mu\|w(t)\|_2^2 \\
\leq e^{-\delta(t-\tau)}(\|w(\tau)\|_1^2 + \mu\|w(\tau)\|_2^2) + \frac{2}{\lambda_{N+1}} \int_{\tau}^{t} e^{\delta s} |f(s)|_1^2 ds \\
+ \frac{4}{\lambda_{N+1}} e^{-\delta t} \int_{\tau}^{t} e^{\delta s}(R(s))^{\gamma+1} ds \\
\leq e^{-\delta(t-\tau)}(\|w(\tau)\|_1^2 + \mu\|w(\tau)\|_2^2) + \frac{2}{\lambda_{N+1}} \int_{\tau}^{t} e^{\delta s} |f(s)|_1^2 ds \\
+ \frac{4(R(t))^{\gamma+1}}{\lambda_{N+1}}, \tau \leq s \leq t.
\]

Since $w_\tau \in B(0, R(t))$, we know that
\[
e^{-\delta(t-\tau)}(\|w(\tau)\|_1^2 + \mu\|w(\tau)\|_2^2) \to 0 \text{ as } \tau \to -\infty,
\]
and
\[
\frac{2}{\lambda_{N+1}} \int_{\tau}^{t} e^{\delta s} |f(s)|_1^2 ds + \frac{4(R(t))^{\gamma+1}}{\lambda_{N+1}}, \text{ as } N \to +\infty.
\]

Inserting (48)-(49) into (47), we conclude
\[
\|w(t)\|_1^2 + \mu\|w(t)\|_2^2 \leq C\epsilon, \tau \leq \tau_B \leq t.
\]
Hence the process $U$ defined by (24) satisfies the condition $(P)$ at $B$. The proof is thus complete.

**The proof of Theorem 2.6.** From Lemmas 3.2, 4.1 and 4.2, we deduce the existence of pullback attractors which completes the proof.

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E-mail address: xld0908@163.com
E-mail address: yuming.qin@hotmail.com
E-mail address: yuming@dhu.edu.cn