The S-matrix and ghost fields in quantum Yang-Mills gravity

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The S-matrix in quantum Yang-Mills gravity with translation gauge symmetry in flat space-time is investigated. We obtain the generating functional of Green’s functions, i.e., the vacuum-to-vacuum amplitude, for Yang-Mills gravity. The unitarity and gauge invariance of the S-matrix in a class of gauge conditions is preserved by massless ghost vector fields.

1 Introduction

The idea of a gravitational theory with space-time translational gauge symmetry is very interesting from the viewpoint of Yang-Mills theory. It has attracted many authors [1, 2, 3] because the translational symmetry implies the conservation of energy-momentum tensor, which is the source of the gravitational field. But most formulations were based on curved space-time and, hence, the problems of quantization and energy-momentum conservation remain unsolved. Recently, it has been shown that Yang-Mills gravity with translation gauge symmetry (or T(4) group) in flat 4-dimensional space-time leads to an ‘effective Riemannian metric tensor’ in the limit of geometric optics of wave equations.[3] Such a limiting effective metric tensor emerges in the Hamilton-Jacobi equation for light rays and classical particles. In this sense, classical general relativity based on Riemannian geometry in curved space-time may be interpreted as a classical manifestation of gauge fields with translation symmetry in flat space-time.
Thus, it appears as if light rays and classical particles in Yang-Mills gravity move in a ‘curved space-time with Riemannian geometry.’ However, the real underlying physical space-time of gauge fields and quantum particles is flat and, hence, has vanishing Riemann-Christoffel curvature tensor. This property of T(4) gauge field in the limit of geometric optics is essential for Yang-Mills gravity to be consistent with all known experiments.[3] Furthermore, the framework of flat space-time enables us to quantize Yang-Mills gravity with a well-defined and conserved energy-momentum tensor, just as the usual gauge theory. The graviton propagator is essential the same as that in the conventional theory. However, the graviton coupling in Yang-Mills gravity turns out to be much more simpler than that in Einstein gravity. These results and properties motivate further investigation of the Yang-Mills gravity.

In sharp contrast to the electrodynamics with Abelian group U(1), Yang-Mills gravity based on Abelian group T(4) of space-time translation symmetry needs ghost (or fictitious) particles to preserve the gauge invariance and unitarity of the S-matrix. The situation is similar to Yang-Mills theories with non-Abelian gauge groups. Yang-Mills gravity appears to be a natural generalization of the conserved charge associated with the U(1) group to the conserved energy-momentum tensor of the space-time translation group. Furthermore, it has a big difference from the usual gauge theories with internal gauge groups. Namely, the T(4) gauge field in Yang-Mills gravity is not a (Lorentz) vector field with dimensionless coupling constant. Rather, it is a symmetric tensor field, \( \phi_{\mu\nu} = \phi_{\nu\mu} \), whose coupling constant \( g \) has the dimension of length (in natural units, \( c = \hbar = 1 \)). This is due to the fact that the generators \( i\partial/\partial x^\mu \) of the space-time translation group has the dimension of \( 1/\text{length} \) and cannot be represented by dimensionless constant matrices. Such a tensor field \( \phi_{\mu\nu} \) may be termed ‘space-time gauge field.’

Yang-Mills gravity with the external space-time translational symmetry group is a generalization of Yang-Mills theory with internal gauge group. One can consider fiber bundle as the mathematical foundation of gauge theory with internal gauge group.[4] However, if the gauge group is external and non-compact, the corresponding fiber bundle is not as straightforward. One should be careful because one cannot take for granted that the same mathematical and physical properties for internal gauge groups hold in general for external symmetry groups.[5]
2 The Translation Gauge-Invariant Action and Gauge-Fixing Lagrangian

Yang-Mills gravity can be formulated in both inertial and non-inertial frames and in the presence of fermion fields.[1, 2] It is difficult to discuss quantum field theory and particle physics even in a simple non-inertial frame with a constant linear acceleration, where the accelerated transformation of space-time is smoothly connected to the Lorentz transformation in the limit of zero acceleration.[6, 7, 8] For simplicity, let us consider quantum Yang-Mills gravity in inertial frames with the Minkowski metric tensor $\eta^{\mu\nu} = (1, -1, -1, -1)$ and in the absence of fermions. The action $S_{pg}$ for pure gravity, involving space-time gauge fields $\phi_{\mu\nu}(x)$ and a gauge-fixing Lagrangian, is assumed to be[1]

$$S_{pg} = \int (L_\phi + L_\xi) dx,$$

(1)

$$L_\phi = \frac{1}{4g^2} \left( C^{\mu\nu\alpha}C_{\mu\nu\alpha} - 2C_{\mu\alpha}^\beta C^{\mu\beta}_\alpha \right),$$

(2)

$$C^{\mu\nu\alpha} = J^{\mu\sigma}\partial_\sigma J^{\nu\alpha} - J^{\nu\sigma}\partial_\sigma J^{\mu\alpha}, \quad J_{\mu\nu} = \eta_{\mu\nu} + g\phi_{\mu\nu} = J_{\nu\mu},$$

where $C^{\mu\nu\alpha}$ is the T(4) gauge curvature and $c = \hbar = 1$. We note that the Lagrangian $L_\phi$ changes only by a divergence under the translation gauge transformation, and the action functional $S_{\phi} = \int L_\phi dx$ is invariant under the space-time translation gauge transformation.[1] To quantize Yang-Mills gravity, it is necessary to include a gauge fixing Lagrangian $L_\xi$ in the action functional (1). For example, the gauge fixing Lagrangian enables us to have a well-defined graviton propagator (see eq. (23) below). The gauge-fixing Lagrangian $L_\xi$ is assumed to be

$$L_\xi = \frac{\xi}{2g^2} \left[ \partial^\mu J_{\mu\alpha} - \frac{1}{2} \partial_\alpha J \right] \left[ \partial_\nu J^{\nu\alpha} - \frac{1}{2} \partial^\alpha J \right],$$

(3)

$$J = J^\lambda_\lambda = \delta^\lambda_\lambda - g\phi, \quad \phi = \phi^\lambda_\lambda,$$

(4)

where $L_\xi$ involves an arbitrary gauge parameter $\xi$. The Lagrangian in (3) corresponds to a class of gauge conditions of the following form,

$$\frac{1}{2} \eta^{\rho\tau} \eta^{\nu\lambda} + \eta^{\tau\rho} \eta^{\mu\lambda} - \eta^{\mu\nu} \eta^{\rho\lambda} \partial_\lambda J_{\mu\nu} = \partial_\lambda J^{\rho\lambda} - \frac{1}{2} \partial^\rho J = Y^\rho,$$

(5)

where $Y^\rho$ is a suitable function of space-time.

The Lagrangian for pure gravity $L_{pg} = L_\phi + L_\xi$ can be expressed in terms of space-time gauge fields $\phi_{\mu\nu}$:

$$L_{pg} = L_2 + L_3 + L_4 + L_\xi,$$

(6)
where

\[ L_2 = \frac{1}{2} \left( \partial_\lambda \phi_{\alpha\beta} \partial_\lambda \phi^{\alpha\beta} - \partial_\lambda \phi_{\alpha\beta} \partial^\alpha \phi^{\lambda\beta} - \partial_\lambda \phi \partial^\lambda \phi \right) \right] + 2 \partial_\lambda \phi \partial^\beta \phi_{\lambda\beta} - \partial_\lambda \phi \phi_{\lambda\mu} \partial^\beta \phi^{\mu\beta}, \]  

\[ L_\xi = \frac{\xi}{2} \left[ (\partial_\lambda \phi^{\lambda\alpha}) \partial^\alpha \phi_{\rho\alpha} - (\partial_\lambda \phi^{\lambda\alpha}) \partial_\alpha \phi + \frac{1}{4} (\partial^\alpha \phi) \partial_\alpha \phi \right]. \]  

The Lagrangians \( L_2 \) and \( L_\xi \) involve quadratic tensor field and determine the propagator of the graviton in Yang-Mills gravity. The Lagrangians \( L_3 \) and \( L_4 \) correspond to the interactions of 3- and 4-gravitons respectively. They can also be obtained from the Lagrangian (2).

3 Gauge Conditions and Effective Lagrangian of Yang-Mills Gravity

To quantize a field with gauge symmetry in a covariant formulation, one has to impose a gauge condition. In Yang-Mills gravity, it is non-trivial to impose a gauge condition in general because the gauge condition does not hold for all time. We know that if one imposes a gauge condition in quantum electrodynamics (QED), the gauge condition satisfies a free field equation and, hence, hold for all times. However, this is true if and only if the gauge condition is linear. We have examined the problem of unitarity in QED if we imposed a quadratic gauge condition, we found that the gauge condition does not hold for all times. Roughly speaking, the longitudinal and time-like photons are no longer free particles, their interaction in the intermediate steps of a physical process will create extra unwanted amplitudes to upset gauge invariance and unitarity of the S-matrix in QED. Similar to the approach of Faddeev and Popov,[10, 11] QED with a quadratic gauge condition can be described by an effective Lagrangian which involves a ghost particle. This ghost particle produces extra amplitudes to cancel those of unphysical (longitudinal and time-like) photons, so that the gauge invariance and unitarity of S-matrix in QED are restored.[9] Similar mechanism of cancellation occurs in any theory with gauge symmetry or distorted gauge symmetry,[11, 12] and in Yang-Mills gravity.

We follow Faddeev and Popov’s approach to discuss how to fix a gauge for all times with the help of path integrals and derived the effective Lagrangian for quantum Yang-Mills gravity.[10, 11] From the action (1) with
the Lagrangian (2) and the gauge-fixing terms (3), we derived the Yang-Mills field equation

$$H^{\mu \nu} + \xi A^{\mu \nu} = 0,$$

where$$H^{\mu \nu} = \partial_\lambda (J^\lambda C_{\rho \mu \nu} - J_\rho C^{\alpha \beta} J^{\mu \nu} + C^{\mu \beta} J^{\nu \lambda})$$

$$-C^{\alpha \beta} \partial^\nu J_{\alpha \beta} + C^{\mu \beta} \partial^\nu J^{\alpha}_\beta - C^{\lambda \beta} \partial^\nu J^{\mu}_\lambda$$

and$$A^{\mu \nu} = \left[ \partial^\mu \left( J^\alpha_{\lambda \mu} - \frac{1}{2} \partial^\nu J^{\lambda} \right) - \frac{1}{2} \eta^{\mu \nu} \partial^\lambda \left( \partial^\sigma J_{\sigma \lambda} - \frac{1}{2} \partial^\lambda J \right) \right]_{(\mu \nu)},$$

where$$\left[ ... \right]_{(\mu \nu)}$$denotes that$$\mu$$and$$\nu$$in$$[...]$$should be made symmetric. The two terms in (11) are gauge-fixing terms, which are non-invariant under gauge transformations, similar to that in Einstein gravity.[13]

Let us consider a general class of the gauge conditions given in (5), where$$Y^\alpha (x)$$is a suitable function independent of the fields and the gauge function$$\Lambda^\alpha (x)$$. [1] With such a gauge condition, the vacuum-to-vacuum amplitude of the pure Yang-Mills gravity is given by

$$W(Y^\alpha) = \int d[\phi^{\rho \sigma}] \exp \left( i \int d^4 x (L_\phi + \phi^{\mu \nu} J^{\mu \nu}) \right) \times \det Q \prod_\alpha \delta \left( \partial_\lambda J^{\lambda \alpha} - \frac{1}{2} \partial^\alpha J^{\lambda} - Y^\alpha \right),$$

where$$j^{\mu \nu}$$are external sources and$$J^{\mu \nu}$$are the T(4) gauge transformations of$$J^{\mu \nu}$. [1]

$$J^{\mu \nu} = J^{\mu \nu} - \Lambda^{\lambda} \partial_\lambda J^{\mu \nu} + J^{\lambda \nu} \partial_\lambda \Lambda^\mu + J^{\mu \lambda} \partial_\lambda \Lambda^\nu.$$  

The delta function$$\delta (\partial_\lambda J^{\lambda \alpha} - \frac{1}{2} \partial^\alpha J^{\lambda})$$in the path integral (12) is to maintain the gauge condition for all times. [10, 12, 13] The functional determinant$$\det Q$$is defined by [11, 12]

$$\frac{1}{\det Q} = \int d[\Lambda^\rho (x)] \prod_\alpha \delta \left( \partial_\lambda J^{\lambda \alpha} (x) - \frac{1}{2} \partial^\alpha J^\lambda (x) - Y^\alpha (x) \right)$$

The matrix$$Q$$is obtained by considering the T(4) gauge transformation of the gauge condition$$Y^\alpha = \partial_\lambda J^{\lambda \alpha} - \frac{1}{2} \partial^\alpha J^\lambda$$. 

$$\partial_\lambda J^{\lambda \alpha} - \frac{1}{2} \partial^\alpha J^{\lambda} = \partial_\lambda J^{\lambda \alpha} - \frac{1}{2} \partial^\alpha J$$

$$+ \partial_\mu \left[ - (\partial_\lambda J^{\mu \alpha}) \Lambda^\lambda + J^{\lambda \alpha} \partial_\lambda \Lambda^\mu + J^{\mu \lambda} \partial_\lambda \Lambda^\alpha \right] + \frac{1}{2} \partial^\alpha [(\partial_\lambda J) \Lambda^\lambda].$$  

5
The equation of the ghost vector field $\chi^\alpha$ can be obtained from $\partial_\lambda J^{\lambda\alpha} - \frac{1}{2} \partial^\alpha J' - Y^\alpha = 0$ with the replacement $\Lambda^\alpha \rightarrow \chi^\alpha$. Thus, we have

$$\partial_\mu [-(\partial_\lambda J^{\mu\alpha}) \chi^\lambda + J^{\lambda\alpha} \partial_\lambda \chi^\mu + J^{\mu\lambda} \partial_\lambda \chi^\alpha] + \frac{1}{2} \partial^\alpha [\partial_\lambda J \chi^\lambda] = 0. \quad (16)$$

Since $J^{\mu\lambda} = \eta^{\mu\lambda} + g \phi^{\mu\lambda}$, equation (16) can be written as

$$\partial_\lambda \partial^\lambda \chi^\alpha + \partial^\alpha \partial_\mu \chi^\mu - g \partial_\mu [(\partial_\lambda \phi^{\mu\alpha}) \chi^\lambda] + g \partial_\mu (\phi^{\mu\lambda} \partial_\lambda \chi^\alpha) + g \frac{1}{2} \partial^\alpha [(\partial_\lambda \phi) \chi^\lambda] = 0. \quad (17)$$

This equation can be derived from the following Lagrangian for ghost fields $\chi'$ and $\chi$,

$$L'_{\chi} = \chi'_{\mu} Q_{\mu\nu} \chi^\nu = \frac{\xi}{2g^2} \left\{ \partial_\lambda J^{\lambda\alpha} - \frac{1}{2} \partial^\alpha J' - Y^\alpha = 0 \right\}$$

where we find that

$$Q_{\mu\nu} = \eta_{\mu\nu} \partial_\alpha \partial^\alpha + \partial_\mu \partial_\nu - g \eta_{\sigma\mu} (\partial_\rho \partial_\sigma \phi^{\sigma\rho}) + g \eta_{\sigma\mu} \phi^{\sigma\rho} \partial_\sigma \partial_\rho$$

$$+ g (\partial_\sigma \phi^{\sigma\rho}) \eta_{\mu\nu} \partial_\rho + g \eta_{\mu\nu} \phi^{\sigma\rho} \partial_\sigma \partial_\rho + g \frac{1}{2} (\partial_\mu \partial_\nu \phi) + g \frac{1}{2} (\partial_\nu \phi) \partial_\mu. \quad (20)$$

Since $W(Y^\alpha)$ is invariant under an infinitesimal change of $Y^\alpha(x)$ for all $Y^\alpha(x)$, we may write $W(Y^\alpha)$ in (12) as

$$W = \int W(Y^\alpha) \exp \left\{ -i \int d^4x \frac{\xi}{2g^2} Y^\alpha(x) Y_\alpha(x) \right\} d[Y^\alpha(x)]$$

$$= \int \exp \left\{ i \int d^4x \left[ L_{\phi} + \phi^{\mu\nu} j_{\mu\nu} \right] \right\}$$

$$= \int \exp \left\{ i \int d^4x \left[ L_{\phi} + \phi^{\mu\nu} j_{\mu\nu} - \frac{\xi}{2g^2} \left( \partial_\lambda J^{\lambda\alpha} - \frac{1}{2} \partial^\alpha J' \right) \right] \right\} \left( \partial_\sigma J^{\sigma\beta} - \frac{1}{2} \partial^{\beta} J \right) \eta_{\alpha\beta} \right\}$$

$$\frac{1}{2} \partial^\alpha [\partial_\lambda J \chi^\lambda] = 0.$$
where the ghost Lagrangian $L_{\chi'}$ is given by (19) and (20).

This effective Lagrangian $L_{eff}$ completely specifies the quantum Yang-Mills gravity, including the physical tensor gauge field, together with the unphysical vector-field $\partial_{\mu}\phi^\mu$. and the ghost vector-fields $\chi^\mu(x)$ and $\chi'^\mu(x)$. Thus, the Feynman rules of quantum Yang-Mills gravity can be derived from the effective Lagrangian (22).

4 Propagators of gravitons and ghost particles and their couplings

In general, the propagators of graviton and ghost particles and their couplings in Feynman rules depend on the specific form of gauge condition and the gauge parameters. We have chosen the gauge condition $\partial_{\mu}J^{\mu\nu} = (1/2)\partial^\nu J = Y^\nu$ with an arbitrary $\xi$ in the effective Lagrangian (22). From the free Lagrangians (7) and (8) for the tensor gauge field $\phi^{\mu\nu}$, one can obtain the graviton propagator,

$$G_{\alpha\beta,\rho\sigma} = i \left[ \frac{1}{k^2} (\eta_{\alpha\rho} \eta_{\beta\sigma} + \eta_{\beta\rho} \eta_{\alpha\sigma} - \eta_{\alpha\beta} \eta_{\rho\sigma} ) - \frac{1}{k^4} \left( 1 - \frac{2}{\xi} \right) \left( \eta_{\rho\alpha} k_{\beta} k_{\sigma} + \eta_{\rho\beta} k_{\sigma} k_{\alpha} + \eta_{\rho\sigma} k_{\alpha} k_{\beta} + \eta_{\alpha\beta} k_{\rho} k_{\sigma} \right) \right].$$

for arbitrary $\xi$. This is consistent with that obtained by Fradkin and Tyutin because their linearized field equations are the same.[13, 15, 16, 17]

On the other hand, if one chooses another gauge condition $\partial_{\lambda}J^{\rho\lambda} = Y^\rho$ with an arbitrary $\xi$, i.e., the gauge-fixing Lagrangian (8) is replaced by

$$L'_{\xi} = \frac{(\xi/2)(\partial_{\lambda}\phi^{\rho\lambda})\partial^\rho \phi_{\rho\lambda}}. \quad \text{We obtain a different graviton propagator} \quad G'_{\alpha\beta,\rho\sigma}:

$$

$$G'_{\alpha\beta,\rho\sigma} = i \left[ \frac{1}{k^2} (\eta_{\alpha\rho} \eta_{\beta\sigma} + \eta_{\beta\rho} \eta_{\alpha\sigma} - \eta_{\alpha\beta} \eta_{\rho\sigma} ) - \frac{1}{k^4} \left( 1 - \frac{2}{\xi} \right) \left( \eta_{\rho\alpha} k_{\beta} k_{\sigma} + \eta_{\rho\beta} k_{\sigma} k_{\alpha} + \eta_{\rho\sigma} k_{\alpha} k_{\beta} + \eta_{\alpha\beta} k_{\rho} k_{\sigma} \right) + \frac{1}{k^2} \left( \eta_{\rho\sigma} k_{\alpha} k_{\beta} + \eta_{\alpha\beta} k_{\rho} k_{\sigma} \right) + \frac{1}{k^6} \left( 1 - \frac{6}{\xi} \right) k_{\rho} k_{\sigma} k_{\alpha} k_{\beta} \right].$$

Similarly, the propagator of the ghost particle can be obtained from its free Lagrangian, i.e., (19) with (20) and $g = 0$. We obtain

$$G^{\mu\nu} = -\frac{i}{k^2} \left( \eta^{\mu\nu} - \frac{k^\mu k^\nu}{2k^2} \right).$$
The iǫ prescription for the Feynman propagators (23), (24) and (25) is understood.

In order to obtain the Feynman rule for ghost-ghost-graviton vertex, it is more convenient to use an equivalent form of the ghost Lagrangian:

\[ L_{\chi\chi} = -\partial_{\lambda} \chi_\alpha \partial^{\lambda} \chi^\alpha - \partial^\alpha \chi_\lambda \partial_\mu \chi^\mu + g(\partial_{\mu} \chi_\alpha)(\partial_{\lambda} \phi^{\mu\alpha})\chi^\lambda \]

\[ -g(\partial_{\mu} \chi_\alpha)\phi^{\alpha\lambda} \partial_\lambda \chi^\mu - g(\partial_{\mu} \chi_\alpha)\phi^{\mu\lambda} \partial_\lambda \chi^\alpha - \frac{1}{2}g(\partial^\alpha \chi_\lambda)(\partial_\lambda \phi)\chi^\alpha. \quad (26) \]

This equivalent form can be obtained by considering the action for the Lagrangian \( L_{\chi\chi} \) in (18) and using integration by parts. The Lagrangian (26) implies that the ghost-ghost-graviton vertex (denoted by \( \chi_\mu(p)\chi_\nu(q)\phi^{\alpha\beta}(k) \)) is

\[ ig \left[ \begin{array}{c} -p^\alpha k^\mu \eta^{\nu\beta} + p^\mu q^\alpha \eta^{\nu\beta} + p^\alpha q^\beta \eta^{\mu\nu} + \frac{1}{2}k^\mu p^\nu \eta^{\alpha\beta} \end{array} \right]_{(\alpha\beta)}, \quad (27) \]

where \([...](\alpha\beta)\) denotes that the indices \( \alpha \) and \( \beta \) in \([...]\) should be made symmetric. In the expression (27), all momenta are incoming to the vertex, \( p_\lambda + q_\lambda + k_\lambda = 0 \). All ghost-particle vertices are bilinear in the ghost particle, as shown in the ghost Lagrangian (19). As a result in the Feynman rules, the ghost particles appears, by definition of the physical subspace for the S-matrix, only in closed loops in the intermediate states of a physical process. Moreover, there is a factor of -1 for each ghost particle loop.[9, 11] Thus, the ghost particle resembles a fermion in the Feynman rules. The translation gauge invariance of Yang-Mills gravity implies that the observable results such as those obtained from the S-matrix should be independent of the gauge parameter \( \xi \) in the gauge-fixing term \( L_\xi \) given in (3).[11, 12]

5 Discussions

The theoretical structures of Yang-Mills gravity with external spacetime translation group have some similarities and some differences from the usual gauge field theories. For one thing, the generators of the spacetime translation group \( T(4) \) do not have the constant matrix representation. Consequently, the fiber bundle corresponding to Yang-Mills gravity is not straightforward. The \( T(4) \) gauge covariant derivative, \( \Delta^\nu = J^\nu_\lambda \partial_\lambda \), satisfies the Jacobi identity, \( [\Delta^\lambda, [\Delta^\nu, \Delta^\mu]] = [\Delta^\mu, [\Delta^\nu, \Delta^\lambda]] + [\Delta^\nu, [\Delta^\lambda, \Delta^\mu]] \equiv 0 \). But the corresponding Bianchi identity for the \( T(4) \) gauge curvature \( C^{\mu\nu\rho} \) differs from that in the usual gauge theory. We obtain a modified Bianchi identity:

\[ (\delta^\lambda_\alpha \Delta^\rho - \partial^\lambda J^\rho_\alpha)C^{\nu\mu\lambda} + (\delta^\lambda_\alpha \Delta^\mu - \partial^\lambda J^\mu_\alpha)C^{\nu\rho\lambda} \]

\[ + (\delta^\lambda_\alpha \Delta^\nu - \partial^\lambda J^\nu_\alpha)C^{\rho\mu\lambda} \equiv 0. \quad (28) \]
In usual gauge theories, a given gauge condition does not uniquely determine the effective Lagrangian for a theory. For example, there may be different ghost Lagrangians which can restore the gauge invariance and unitarity of the S-matrix in a theory. In some cases, when one uses the Lagrange multiplier, one obtains a different ghost Lagrangian, which can also cancel the unwanted amplitudes to restore unitarity. These properties appear to be true in a theory with a gauge symmetry.

In contrast to the previous works, Ning Wu recently attempted to give Einstein’s gravity a new interpretation based on a T(4) gauge symmetry. Namely, the conventional Lagrangian in the Hilbert-Einstein action is expressed in terms of the T(4) gauge curvature within the framework of flat space-time. Wu gave a formal proof that the quantum gravity based on the Hilbert-Einstein action with the T(4) gauge symmetry in flat space-time is renormalizable. In view of the profound difficulties in the quantization of Einstein’s gravity in curved space-time, it is desirable to have explicit calculations to substantiate a formal proof of renormalizability.

From the viewpoint of Feynman rules, there is a significant difference in the structure of interaction vertices between Yang-Mills gravity in flat space-time and other theories of gravity including Wu’s formulation of gravity. Namely, the maximum number of gravitons in a vertex is 4, as one can see in the total Lagrangian of Yang-Mills gravity. On the other hand, there exist vertices with arbitrary large numbers of gravitons in Einstein gravity and in other theories of gravity.

For an ordinary tensor field theory with a dimensional coupling constant and without a gauge symmetry, one would expect that the theory is not renormalizable based on power counting. Nevertheless, this argument may not be applicable to Yang-Mills gravity with T(4) gauge symmetry. This maximum 4-vertex for graviton coupling, together with the T(4) gauge symmetry and the conserved energy-momentum tensor, may be a gateway to the renormalizable quantum Yang-Mills gravity.

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Appendix. Ghost Particles and Unitarity of the S-matrix

Let us give some arguments and a proof for the unitarity of the S-matrix for pure gravity based on the effective Lagrangian (22). For a discussion of unitarity of the Yang-Mills gravity with the gauge condition in (5), one can write the ghost field $\chi^\mu$ in the following form,[13]

$$\chi^\mu(x) = \int d^4y D^\mu_\nu(x, y, \phi_{\alpha\beta}) \hat{\chi}^\nu(y)$$

(A1)

where $\chi^\mu$ satisfies equation (17) which can be written in the form

$$Q_{\mu\nu} \chi^\nu = 0,$$

(A2)

where $Q_{\mu\nu}$ is given in (20). Clearly, in the limit $g \to 0$, one has $J_{\mu\nu} \to \eta_{\mu\nu}$. Thus, the operator $Q_{\mu\nu}$ reduces to a non-singular differential operator in this limit,

$$Q_{\mu\nu} \to \eta_{\mu\nu} \partial_\lambda \partial_\nu \equiv Q^0_{\mu\lambda}.$$  

(A3)

This limiting property can be seen from (20). One can choose the function $D^\mu_\nu(x, y, \phi_{\alpha\beta})$ in equation (A1) to have the specific form

$$D^\mu_\nu = [Q^{-1}]^\mu\lambda Q^0_{\lambda\nu}$$  

(A4)

so that $\hat{\chi}^\mu$ satisfies the free field equation,

$$Q^0_{\lambda\mu} \hat{\chi}^\mu = [\partial^\sigma \partial_\sigma \eta_{\lambda\mu} + \partial_\lambda \partial_\mu] \hat{\chi}^\mu = 0.$$  

(A5)

The generating functional for connected Green’s functions in gauge invariant gravity can be defined after the gauge condition is specified.[13] Similarly, in Yang-Mills gravity the generating functional for connected Green’s functions (or the vacuum-to-vacuum amplitude) (21) can be written as[12, 26]

$$W^Y_\xi = \int d[\phi_{\alpha\beta}] \exp \left[ i \int d^4x \left( L_\phi + \frac{\xi}{2g^2} Y^\mu Y_\mu + \phi_{\mu\nu} j_{\mu\nu} \right) 

+ Tr \ln Q(Q^0)^{-1} \right],$$

(A6)

where the external sources $j_{\mu\nu}$ are arbitrary functions and $Y^\mu$ is given by equation (5). It follows from (A5) and (A6) that the S-matrix corresponding to the generating functional (A6) is unitary.[13, 11] The T(4) gauge field equations, $H^{\mu\nu} = 0$, for pure gravity hold in the physical subspace.
The last term in (A6) can be written in terms of vector-fermion ghost fields $\chi^\alpha(x)$ and $\chi'^\beta(x)$,[13, 26]

$$Tr \ln Q(\hat{Q}^0)^{-1} = \int d[\chi^\alpha, \chi'^\beta] \exp \left( i \int L_{\chi'\chi} d^4x \right), \quad (A7)$$

where the effective action $L_{\chi'\chi}$ is given by (19), which describes the ghost particles associated with the gauge specified in (5). Note that $\chi'^\mu$ is considered as an independent field. The quanta of the fields $\chi^\mu$ and $\chi'^\mu$ in the effective Lagrangian $L_{\chi'\chi}$ are the ghost (or fictitious) particles in Yang-Mills gravity. By definition of the physical states for the S-matrix, these ghost particles cannot exist in the external states. They can only appear in the intermediate steps of a physical process.[12, 14]

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