Heat content, torsional rigidity and generalized Hardy inequalities for complete Riemannian manifolds

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Abstract

We obtain upper bounds on the heat content and on the torsional rigidity of a complete Riemannian manifold $M$, assuming a generalized Hardy inequality for the Dirichlet Laplacian on $M$.

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1 Introduction

Let $M$ be a complete $C^\infty$ Riemannian manifold with boundary $\partial M$, and let $-\Delta$ be the Dirichlet Laplacian on $M$. In this paper we investigate the heat content of $M$ if $M$ has uniform initial temperature one, while $\partial M$ is kept at temperature zero for all time $t$. Let $u : M \times [0, \infty) \to \mathbb{R}$ be the unique weak solution of

$$\Delta u = \frac{\partial u}{\partial t}, \quad x \in M, \ t > 0, \quad (1)$$

$$u(x, 0) = 1, \quad x \in M. \quad (2)$$

The heat content $Q_M(t)$ is defined by

$$Q_M(t) = \int_M u(x; t) dx. \quad (3)$$

It is well known that if $M$ is compact and $\partial M$ is $C^\infty$ then there exists an asymptotic series for the heat content of the form

$$Q_M(t) = \sum_{j=0}^J b_j t^{j/2} + O(t^{(J+1)/2}), \quad t \to 0, \quad (4)$$

where $J \in \mathbb{N}$, and where the coefficients are locally computable invariants of $M$. In particular,

$$b_0 = \int_M 1 dx = \text{vol}(M), \quad (5)$$

$$b_1 = -\frac{2}{\pi^{1/2}} \int_{\partial M} 1 dy = -\frac{2}{\pi^{1/2}} \text{area}(\partial M). \quad (6)$$

For details we refer to [4, 5].

In this paper we are concerned with the non-classical situation, where $M$ is complete but non-compact. We shall be concerned with the setting where either $M$ itself has infinite volume or where $M$ has finite volume and $\partial M$ has infinite area. We recall the following [3].

Let $M$ be the closure of an open set $M_0$ in Euclidean space with boundary $\partial M = M \setminus M_0$. For $x \in M_0$ we define the distance in direction $u, \ |u| = 1$, by

$$d_u(x) = \min \{|t| : x + tu \in \partial M\}, \quad (7)$$

and the mean distance function $\rho : M_0 \to [0, \infty)$ by

$$\frac{1}{\rho^2(x)} = \frac{1}{\text{area}(S_{m-1})} \int_{S_{m-1}} \frac{du}{d_u^2(x)}, \quad (8)$$
where $S_{m-1}$ is the unit sphere in $\mathbb{R}^m$. Theorem 3.4 in [3] asserts that if
\begin{equation}
\rho^2(x) = o(\log(1 + |x|^2))^{-1}, |x| \to \infty, x \in M_0
\end{equation}
then $M$ has finite heat content for all positive $t$. However, no upper bounds for the heat content in terms of $t$ were obtained in this general situation.

In this paper we obtain bounds for the heat content for a wide class of $C^\infty$ complete Riemannian manifolds assuming a generalized Hardy inequality.

Let $\delta : M \to [0, \infty)$ denote the distance to the boundary function on $M$:
\begin{equation}
\delta(x) = \min\{d(x, y) : y \in \partial M\},
\end{equation}
where $d(x, y)$ is the Riemannian distance between $x$ and $y$. We say that $-\Delta$ satisfies a generalized Hardy inequality if there exist constants $c > 0$ and $\gamma \in (0, 2]$ such that
\begin{equation}
-\Delta \geq \frac{c}{\delta^\gamma},
\end{equation}
in the sense of quadratic forms.

**Theorem 1.** Let $M$ be a complete $C^\infty$ Riemannian manifold, and suppose that (11) holds for some $c > 0$ and $\gamma \in (0, 2]$. Suppose there exists $\beta \in (0, 2\gamma]$ such that
\begin{equation}
\int_M \delta^\beta(x) dx < \infty.
\end{equation}
Then for all $t > 0$
\begin{equation}
Q_M(t) \leq \left(\frac{(\beta + \gamma)^2}{2e\beta\gamma c}\right)^{\beta/\gamma} \left(\int_M \delta^\beta(x) dx\right) t^{-\beta/\gamma}.
\end{equation}

**Theorem 2.** Let $M$ be a complete $C^\infty$ Riemannian manifold with finite volume, and suppose that (11) holds for some $c > 0$ and $\gamma \in (0, 2]$. Then for all $t > 0$
\begin{equation}
Q_M(t) \leq \text{vol}(M) - 4^{-1} \int_{\{x \in M : \delta(x) < (2ct)^{1/\gamma}\}} 1 \, dx.
\end{equation}

The Hardy inequality (11) guarantees that the boundary $\partial M$ is not too thin, and that sufficient cooling of $M_0$ near $\partial M$ will take place. Condition (12) in Theorem 1 guarantees that $M$ does not have to much measure away from $\partial M$. Both the validity and applications of inequalities like (11) to spectral theory have been investigated in depth [8-11, 13].

**Remark 3.** If $M_0$ is simply connected in $\mathbb{R}^2$ then (11) holds with $\gamma = 2$ and $c = 1/16$. If $M_0$ is convex in $\mathbb{R}^m$ then (11) holds with $\gamma = 2$ and $c = 1/4$ [8, 9].
For open sets $M_0$ in $\mathbb{R}^m$ it was shown (Theorem 1.5.3 in [8]) that

$$-\Delta \geq \frac{m}{4\rho^2},$$

(15)
in the sense of quadratic forms. The proofs of Theorems 1 and 2 together with (15) give the following.

**Corollary 4.** Let $M$ be the closure of an open set $M_0$ in Euclidean space with boundary $\partial M = M \setminus M_0$. Suppose there exists $\beta \in (0, 4]$ such that

$$\int_M \rho^{\beta}(x)dx < \infty.$$  

(16)

Then for all $t > 0$

$$Q_M(t) \leq \left(\frac{(\beta + 2)^2}{e \beta m}\right)^{\beta/2} \left(\int_M \rho^{\beta}(x)dx\right) t^{-\beta/2}.$$  

(17)

**Corollary 5.** Let $M$ be the closure of an open set $M_0$ in Euclidean space with boundary $\partial M = M \setminus M_0$, and with finite volume. Then for all $t > 0$

$$Q_M(t) \leq \text{vol}(M) - 4^{-1} \int_{\{x \in M : \rho(x) < (mt/2)^{1/2}\}} 1 \, dx,$$  

(18)

Let $p_M(x, y; t), x \in M, y \in M, t > 0$ be the Dirichlet heat kernel for $M$. We say that $M$ has finite torsional rigidity $P_M$ if

$$P_M = \int_0^\infty \int_M \int_M p_M(x, y; t) \, dx \, dy \, dt < \infty.$$  

(19)

Let $M_0$ be an open subset of $\mathbb{R}^m$. It is well known that if $M_0$ has finite volume then $M$ has finite torsional rigidity. However, the converse is not true. In [1] we showed that if $M_0 \subset \mathbb{R}^m$ satisfies (11) for $\gamma = 2$ and some $c > 0$, then $P_M$ is finite if and only if (12) holds with $\beta = 2$. Since the solution of (1-2) with Dirichlet boundary conditions on $\partial M$ is given by

$$u(x; t) = \int_M p_M(x, y; t)dy,$$  

(20)

it follows that

$$Q_M(t) = \int_M \int_M p_M(x, y; t) \, dx \, dy,$$  

(21)

and

$$P_M = \int_0^\infty Q_M(t) \, dt.$$  

(22)

Theorem 1 gives the following.
**Corollary 6.** Suppose $M$ is a complete $C^\infty$ Riemannian manifold, and suppose that (11) holds for some $c > 0$ and $\gamma \in (0, 2]$. Suppose there exists $\varepsilon > 0$ such that (12) holds for all $\beta \in (\gamma - \varepsilon, \gamma + \varepsilon)$. Then $M$ has finite torsional rigidity.

The proof of Corollary 6 is elementary. We bound $Q_M(t)$ for small $t$ by (13) with $\beta = \gamma - \varepsilon/2$, and for large $t$ by (13) with $\beta = \gamma + \varepsilon/2$. Similarly one can show that for open sets in $\mathbb{R}^m$, $P_M$ is finite if (16) holds for all $\beta$ in some neighbourhood $\beta = 2$. The following result is an improvement.

**Theorem 7.** Let $M$ be the closure of an open set $M_0$ in $\mathbb{R}^m$ with boundary $\partial M = M \setminus M_0$. Suppose that $\int_M \rho^2(x)dx < \infty$. Then

$$P_M \leq \frac{4}{m} \int_M \rho^2(x)dx.$$  

(23)

In Lemma 2.6 of [3] it was shown that if $M_0 \subset \mathbb{R}^m$ is such that $Q_M(t)$ is finite for all $t > 0$ then trace $(e^{t\Delta})$ is finite for all $t > 0$. In the more general setting of complete $C^\infty$ Riemannian manifolds we have the following.

**Corollary 8.** Let $M$ be a complete $C^\infty$ Riemannian manifold, and suppose that (11) holds for some $c > 0$ and $\gamma \in (0, 2]$. Suppose (12) holds for some $\beta \in (0, 2\gamma]$, and suppose there exists a function $g : (0, \infty) \to (0, \infty)$ such that

$$p_M(x, x; t) \leq g(t), \quad x \in M.$$  

(24)

Then for all $t > 0$

$$\text{trace}(e^{t\Delta}) = \int_M p_M(x, x; t)dx \leq g(t/2)Q_M(t/2) < \infty.$$  

(25)

Sufficient conditions on the geometry of $M$ which guarantee the uniform bound (24) were obtained by several authors (Section 6 in [12] and the references therein).

We conclude this introduction with an example to show that Theorems 1 and 2 are close to being sharp.

**Example 9.** Let $M(\alpha) \subset \mathbb{R}^2$ be given by

$$M(\alpha) = \{(\xi_1, \xi_2) : \xi_1 \geq 0, |\xi_2| \leq (\xi_1 + 1)^{-\alpha}\},$$  

(26)

where $\alpha > 0$ is a constant.
Since $M(\alpha)$ is simply connected, we have by Remark 3 that (11) holds with $\gamma = 2$ and $c = \frac{1}{16}$. $M(\alpha)$ has infinite volume if and only if $\alpha \leq 1$. Estimate (12) holds in this case if and only if $\beta > \frac{1}{\alpha}$. We can choose $\beta \in (0, 4]$ if and only if $\alpha > \frac{1}{5}$. We conclude by Theorem 1 that for $\frac{1}{5} < \alpha \leq 1$ and any $\varepsilon > 0$

$$Q_{M(\alpha)}(t) \leq K_1 t^{(\alpha - 1)/(2\alpha) - \varepsilon}, \quad t > 0,$$

where $K_1$ is a finite positive constant depending on $\alpha$ and $\varepsilon$ respectively.

Theorem 2 gives that for $\alpha > 1$

$$\liminf_{t \to 0} (\text{vol}(M(\alpha) - Q_{M(\alpha)}(t)))^{(1-\alpha)/(2\alpha)} \geq K_2,$$

where $K_2$ is a strictly positive constant depending on $\alpha$. The precise asymptotic behaviour of $Q_{M(\alpha)}(t)$ as $t \to 0$ has been computed in [2]. The results in [2] show for example that (27) holds for all $0 < \alpha < 1$ with $\varepsilon = 0$.

## 2 Proof of Theorem 1

Let $\Delta$ be the Dirichlet Laplacian acting in $L^2(M)$, and let $u : M \times [0, \infty) \to \mathbb{R}$ be the unique weak solution of (1) with initial condition

$$u(x; 0) = f(x), \quad x \in M,$$

where $f : M \to [0, \infty)$ is bounded and measurable. Then

$$u = e^{t\Delta} f,$$

with integral representation

$$u(x; t) = \int_M p_M(x, y; t) f(y) dy.$$  

We let $\varepsilon > 0$, and choose $\{f_\varepsilon : \varepsilon > 0\}$ to be a family of $C^\infty$ functions on $M$ such that $0 \leq f_\varepsilon \leq 1$, $f_\varepsilon$ is monotone increasing as $\varepsilon \to 0$, and

$$f_\varepsilon(x) = \begin{cases} 0, & \delta(x) < \varepsilon, \\ 1, & 2\varepsilon \leq \delta(x). \end{cases}$$

It follows by the maximum principle (Section 2.4 in [12]) that

$$\int_M p_M(x, y; t) dy \leq 1.$$
By Fubini’s theorem and (33) we conclude that the unique solution \( u_\varepsilon \) satisfies
\[
\int_M u_\varepsilon(x; t) \, dx = \int_M \int_M p(x, y; t) f_\varepsilon(y) \, dx \, dy
\leq \int_M f_\varepsilon(y) \, dy \leq \int_{\{x \in M : \delta(x) \geq \varepsilon\}} 1 \, dx
\leq \int_{\{x \in M : \delta^\beta(x) \geq \varepsilon^\beta\}} \varepsilon^{-\beta} \delta^\beta(x) \leq \varepsilon^{-\beta} \int_M \delta^\beta(x) \, dx < \infty.
\]  
(34)

Let \( p \geq 3/2 \). By the maximum principle we have that \( 0 \leq u_\varepsilon \leq 1 \). Hence \( u_\varepsilon^{2p-2} \leq u_\varepsilon \), and
\[
\int_M u_\varepsilon^{2p-2}(x; t) \, dx \leq \int_M u_\varepsilon(x; t) \, dx.
\]  
(35)

Hence, by Fubini’s theorem, Cauchy-Schwarz’s inequality and (35)
\[
\left| -\frac{d}{dt} \int_M u_\varepsilon^p(x; t) \, dx \right| = p \left| \int_M u_\varepsilon^{p-1}(x; t) \frac{\partial u_\varepsilon}{\partial t}(x; t) \, dx \right|
\leq p \left\{ \int_M u_\varepsilon^{2p-2}(x; t) \, dx \right\}^{1/2} \left\{ \int_M \left( \frac{\partial u_\varepsilon}{\partial t}(x; t) \right)^2 \, dx \right\}^{1/2}
\leq p \left\{ \int_M u_\varepsilon(x; t) \, dx \right\}^{1/2} \left\{ \int_M \left( \frac{\partial u_\varepsilon}{\partial t}(x; t) \right)^2 \, dx \right\}^{1/2}.
\]  
(36)

But
\[
\int_M \left( \frac{\partial u_\varepsilon}{\partial t}(x; t) \right)^2 \, dx = \langle \Delta e^{t\Delta} f_\varepsilon, \, \Delta e^{t\Delta} f_\varepsilon \rangle
= \langle (\Delta e^{t\Delta/2})e^{t\Delta/2} f_\varepsilon, (\Delta e^{t\Delta/2})e^{t\Delta/2} f_\varepsilon \rangle,
\]  
(37)

and since \( \Delta e^{t\Delta/2} \) is a bounded operator, bounded by \( 2/(et) \), we have that
\[
\int_M \left( \frac{\partial u_\varepsilon}{\partial t}(x; t) \right)^2 \, dx \leq \frac{4}{e^2 t^2} \langle e^{t\Delta/2} f_\varepsilon, e^{t\Delta/2} f_\varepsilon \rangle
\leq \frac{4}{e^2 t^2} \langle e^{t\Delta} f_\varepsilon, f_\varepsilon \rangle
\leq \frac{4}{e^2 t^2} \langle e^{t\Delta} f_\varepsilon, 1 \rangle
= \frac{4}{e^2 t^2} \int_M u_\varepsilon(x; t) \, dx.
\]  
(38)
By \((36-38)\) we conclude that we have the estimate
\[
- \frac{d}{dt} \int_M u_\varepsilon^p(x; t) \, dx \le \frac{2p}{ct} \int_M u_\varepsilon(x; t) \, dx.
\] (39)

We use Fubini's theorem, integration by parts, and the generalized Hardy inequality \((11)\) to obtain
\[
- \frac{d}{dt} \int_M u_\varepsilon^p(x; t) \, dx = - p \int_M u_\varepsilon^{p-1}(x; t) \Delta u_\varepsilon(x; t) \, dx
\]
\[
= p (p - 1) \int_M u_\varepsilon^{p-2}(x; t) (\nabla u_\varepsilon(x; t))^2 \, dx
\]
\[
= \frac{4(p - 1)}{p} \int_M |\nabla u_\varepsilon^{p/2}(x; t)|^2 \, dx
\]
\[
\ge \frac{4(p - 1)c}{p} \int_M u_\varepsilon^p(x; t) \delta^{-\gamma}(x) \, dx.
\] (40)

Combining the estimates of \((39)\) and \((40)\) yields
\[
\int_M u_\varepsilon(x; t) \ge \frac{2(p - 1)ect}{p^2} \int_M u_\varepsilon^p(x; t) \delta^{-\gamma}(x) \, dx.
\] (41)

By \((34)\) the left hand side of \((41)\) is finite and this implies that the right hand side of \((41)\) is finite as well. By Hölder’s inequality
\[
\int_M u_\varepsilon(x; t) \, dx \le \left\{ \int_M u_\varepsilon^p(x; t) \delta^{-\gamma}(x) \, dx \right\}^{1/p} \left\{ \int_M \delta(x)^{\gamma/(p-1)} \, dx \right\}^{(p-1)/p}.
\] (42)

Since \(\beta \in (0, 2\gamma]\) by the hypothesis in Theorem \(\|\) the choice
\[
p = 1 + \frac{\gamma}{\beta}
\] (43)

guarantees that \(p \geq 3/2\). Then \(\delta^{\gamma/(p-1)}\) is integrable by \((12)\). By \((41)\) and \((42)\)
\[
\int_M u_\varepsilon^p(x; t) \delta^{-\gamma}(x) \, dx \le \left( \frac{p^2}{2(p - 1)ect} \right)^{p/(p-1)} \int_M \delta(x)^{\gamma/(p-1)} \, dx.
\] (44)

Substitution of \((44)\) into \((12)\) then results into
\[
\int_M u_\varepsilon(x; t) \, dx \le \left( \frac{p^2}{2(p - 1)ect} \right)^{1/(p-1)} \int_M \delta(x)^{\gamma/(p-1)} \, dx,
\] (45)
and a further substitution of (43) into (45) gives
\[ \int_M u_\varepsilon(x; t) \, dx \leq \left( \frac{(\beta + \gamma)^2}{2 \varepsilon^{2\gamma \beta c t}} \right)^{\beta/\gamma} \int_M \delta^\beta(x) \, dx. \] (46)

Since the right hand side of (46) is independent of \( \varepsilon \) we have by Fatou’s lemma
\[ Q_M(t) = \int_M \left( \lim_{\varepsilon \to 0} u_\varepsilon(x; t) \right) \, dx \]
\[ \leq \liminf_{\varepsilon \to 0} \int_M u_\varepsilon(x; t) \leq \left( \frac{(\beta + \gamma)^2}{2 \varepsilon^{2\gamma \beta c t}} \right)^{\beta/\gamma} \int_M \delta^\beta(x) \, dx. \] (47)

\[ \square \]

3 Proof of Theorem 2

Let \( f = 1 \) in (29), and note that
\[ Q_M(2t) = \langle e^{2\Delta} 1, 1 \rangle = \langle e^{t\Delta} 1, e^{t\Delta} 1 \rangle = \int_M u^2(x; t) \, dx. \] (48)

Hence (40) for \( p = 2 \) gives
\[ -\frac{d}{dt} Q_M(2t) \geq 2c \int_M u^2(x; t) \delta^{-\gamma}(x) \, dx. \] (49)

Integrating this inequality with respect to \( t \) over \([0, t]\) yields by Fubini’s theorem
\[ \text{vol}(M) - Q_M(2t) \geq 2c \int_M dx \int_0^t u^2(x; \tau) \delta^{-\gamma}(x) \, d\tau. \] (50)

It follows, by the maximum principle, that \( u(x; \tau) \geq u(x; 2t) \) for all \( 0 \leq \tau \leq t \). Hence
\[ \text{vol}(M) - Q_M(2t) \geq 2ct \int_M u^2(x; 2t) \delta^{-\gamma}(x) \, dx. \] (51)

Let \( u = 1 - v \). We use the definition given in (3) and we use equation (51) to see that for any \( \varepsilon > 0 \)
\[ \int_M v(x; t) \, dx \geq ct \int_M (1 - v(x; t))^2 \delta^{-\gamma}(x) \, dx \]
\[ \geq ct \int_{\{x \in M: \delta(x) < \varepsilon\}} (1 - v(x; t))^2 \delta^{-\gamma}(x) \, dx \]
\[ \geq cte^{-\gamma} \int_{\{x \in M: \delta(x) < \varepsilon\}} (1 - 2v(x; t)) \, dx. \] (52)
It follows that
\[(1 + 2ct\varepsilon^{-\gamma}) \int_M v(x; t) dx \geq c t \varepsilon^{-\gamma} \int_{\{x \in M: \delta(x) < \varepsilon\}} 1 \, dx. \tag{53}\]

The choice
\[\varepsilon = (2ct)^{1/\gamma} \tag{54}\]

in (53) completes the proof of Theorem 2.

To prove Corollaries 4 and 5 respectively we note that (15) holds for open sets \(M_0 \subset \mathbb{R}^m\). We follow the proofs of Theorems 1 and 2 respectively by replacing \(\delta\) by \(\rho\), \(c\) by \(m/4\) and \(\gamma\) by 2 throughout.

To prove Corollary 8 we note that by (5.2) in Section 5.1 of [12] and (24)
\[p_{M}(x, y; t) \leq \left(\frac{p_{M}(x, x; t)p_{M}(y, y; t)}{p_{M}(x, y; t/2)}\right)^{1/2} \leq g(t). \tag{55}\]

Hence
\[
\int_M p_{M}(x, x; t) \, dx = \int_M \int_M p_{M}^2(x, y; t/2) \, dxdy \\
\leq g(t/2) \int_M \int_M p_{M}(x, y; t/2) \, dxdy \leq g(t/2)Q_{M}(t/2) < \infty. \tag{56}\]

4 Proof of Theorem 7

The proof of Theorem 7 is based on a couple of lemmas which are of independent interest, and which are related to results on the expected life time of \(h\)-conditioned Brownian motion [6, 7].

**Lemma 10.** Let \(M_0\) be an open set in \(\mathbb{R}^m\), \(m \geq 3\). Suppose that for all \(\varepsilon > 0\)
\[\text{vol}\{x \in M_0 : \delta(x) > \varepsilon\} < \infty. \tag{57}\]

Let \(\{f_{\varepsilon} : \varepsilon > 0\}\) be as in the proof of Theorem 4. Then
\[-\Delta w = f_{\varepsilon}, \tag{58}\]
has a unique, weak, bounded and non-negative solution \(w_{\varepsilon}\) with
\[\|w_{\varepsilon}\|_\infty \leq \frac{m}{4\pi(m - 2)} \text{vol}\{x \in M : \delta(x) > \varepsilon\}^{2/m}. \tag{59}\]
Proof. Put
\[ M_\varepsilon = \{ x \in M : \delta(x) > \varepsilon \}. \quad (60) \]
Since \( \delta \leq \rho \), (69) implies that \( \int_M \delta^2(x)dx < \infty \). Then the last three inequalities in the right hand side of (34) for \( \beta = 2 \) imply \( \text{vol}(M_\varepsilon) < \infty \) for \( \varepsilon > 0 \).

Since \((-\Delta)^{-1}\) has integral kernel
\[ \int_0^\infty p_M(x, y; t)dt \]
we have that
\[ w_\varepsilon(x) = \int_M \int_0^\infty p_M(x, y; t) dt \ f_\varepsilon(y)dy \]
\[ = \int_{M_\varepsilon} \int_0^\infty p_M(x, y; t) dt \ f_\varepsilon(y)dy \quad (62) \]
\[ \leq \int_{M_\varepsilon} \int_0^\infty p_M(x, y; t) dt dy. \]

By positivity of the Dirichlet heat Kernel
\[ \int_{M_\varepsilon} p_M(x, y; t)dy \leq \int_M p_M(x, y; t)dy \leq 1. \quad (63) \]
Hence for \( t_0 > 0 \) we have by Fubini’s theorem
\[ \int_{M_\varepsilon} \int_0^{t_0} p_M(x, y; t) dt dy \leq t_0. \quad (64) \]
Moreover, by monotonicity of the Dirichlet heat Kernel
\[ p_M(x, y; t) \leq (4\pi t)^{-m/2}. \quad (65) \]
Hence for \( m \geq 3 \)
\[ \int_{M_\varepsilon} \int_0^\infty p_M(x, y; t) dt dy \leq (4\pi)^{-m/2} \frac{2}{m - 2} t_0^{1-m/2} \text{vol}(M_\varepsilon). \quad (66) \]
By (61) and (66) we conclude that
\[ w_\varepsilon(x) \leq t_0 + (4\pi)^{-m/2} \frac{2}{m - 2} t_0^{1-m/2} \text{vol}(M_\varepsilon). \quad (67) \]
We minimize the right hand side of (67) by setting
\[ t_0 = (4\pi)^{-1} (\text{vol}(M_\varepsilon))^{2/m}, \quad (68) \]
and (59) now follows. \( \square \)
Lemma 11. Let $M_0$ be an open set in $\mathbb{R}^2$. Suppose that

$$\int_M \rho^2(x)dx < \infty. \quad (69)$$

Let $\{f_\varepsilon : \varepsilon > 0\}$ be as in the Proof of Theorem 7. Then (58) has a unique, weak, bounded and non-negative solution $w_\varepsilon$ with

$$||w_\varepsilon||_\infty \leq \left( \frac{8}{\pi} \text{vol}(M_\varepsilon) \int_M \rho^2(x)dx \right)^{1/2}. \quad (70)$$

Proof. Let

$$\lambda = \inf \text{spec}(-\Delta). \quad (71)$$

Then by (15) we have for any smooth function $u$ with compact support in $M_0$, and with $||u||_2 = 1$,

$$1 = ||u||_2^2 \leq \left( \int_M \left( \frac{u(x)}{\rho(x)} \right)^2 dx \int_M \rho^2(x)dx \right)^{1/2} \leq \left( 2 \int_M |\nabla u(x)|^2dx \int_M \rho^2(x)dx \right)^{1/2}. \quad (72)$$

Taking the infimum over all such $u$ we obtain by (69) and (72)

$$\lambda \geq \frac{1}{2} \left( \int_M \rho^2(x)dx \right)^{-1} > 0. \quad (73)$$

To prove Lemma 11 we have by domain monotonicity

$$p_M(x, x; t) \leq e^{-t\lambda/2}p_M(x, x; t/2) \leq (2\pi t)^{-1}e^{-t\lambda/2}, \quad (74)$$

and hence by (53)

$$p_M(x, y; t) \leq (2\pi t)^{-1}e^{-t\lambda/2}. \quad (75)$$

As in the proof of Lemma 10 we have the estimates (62-64). Estimate (66) is replaced, using (75), by

$$\int_{M_\varepsilon} \int_{t_0}^\infty p_M(x, y; t)dtdy \leq \int_{M_\varepsilon} 1 dy \int_{t_0}^\infty (2\pi t)^{-1}e^{-t\lambda/2}dt \quad (76)$$

$$= (\pi t_0 \lambda)^{-1} \text{vol}(M_\varepsilon).$$

By (62, 64) and (76)

$$w_\varepsilon(x) \leq t_0 + (\pi t_0 \lambda)^{-1} \text{vol}(M_\varepsilon). \quad (77)$$

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We minimize the right hand side of (77) by setting
\[ t_0 = \left( \frac{\text{vol}(M_\varepsilon)}{(\pi \lambda)} \right)^{1/2}. \]
We then substitute the lower bound for \( \lambda \) in (73) to obtain (70).

We can now give the proof of Theorem 7. We choose the family \( \{ f_\varepsilon : \varepsilon > 0 \} \) as in the proof of the Theorem 1. Since we have assumed that \( \int_M \rho^2(x)dx < \infty \) the support of \( f_\varepsilon \) has finite volume. Since \( 0 \leq f_\varepsilon \leq 1 \) and since \( ||w_\varepsilon||_{\infty} < \infty \) (by Lemmas 10 and 11) we have that \( w_\varepsilon f_\varepsilon \) is integrable on \( M \). By (58), an integration by parts and (15) we conclude that
\[
\infty > \int_M w_\varepsilon(x)f_\varepsilon(x)dx \geq -\int_M w_\varepsilon(x)\Delta w_\varepsilon(x)dx = \int_M |\nabla w_\varepsilon(x)|^2dx \\
\geq \frac{m}{4} \int_M w_\varepsilon^2(x)\rho^{-2}(x)dx.
\]
(78)
By assumption \( \int_M \rho^2(x)dx < \infty \). Hence by Hölder’s inequality
\[
\int_M w_\varepsilon(x)f_\varepsilon(x)dx \leq \left( \int_M \frac{w_\varepsilon^2(x)f_\varepsilon^2(x)}{\rho^2(x)}dx \int_M \rho^2(x)dx \right)^{1/2} \\
\leq \left( \int_M w_\varepsilon^2(x)\rho^{-2}(x)dx \int_M \rho^2(x)dx \right)^{1/2}
\]
(79)
By (78) and (79)
\[
\int_M w_\varepsilon^2(x)\rho^{-2}(x)dx \leq \frac{16}{m^2} \int_M \rho^2(x)dx,
\]
(80)
and by (79) and (80)
\[
\int_M w_\varepsilon(x)f_\varepsilon(x)dx \leq \frac{4}{m} \int_M \rho^2(x)dx.
\]
(81)
By the first equality in (62), (81) and Fubini’s theorem
\[
\int_0^\infty \int_M \int_M p_M(x, y; t)f_\varepsilon(x)f_\varepsilon(y)dx dy dt \leq \frac{4}{m} \int_M \rho^2(x)dx.
\]
(82)
Since \( f_\varepsilon \) is increasing as \( \varepsilon \to 0 \), we have by the monotone convergence theorem
\[
\int_0^\infty \int_M \int_M p_M(x, y; t)dx dy dt \leq \frac{4}{m} \int_M \rho^2(x)dx.
\]
(83)
But this is the conclusion of Theorem 7 by definition (22).
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