Quantum Algorithms for Systems of Linear Equations

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Abstract

This article reviews the 2008 quantum algorithm for linear systems of equations due to Harrow, Hassidim and Lloyd, as well as some of the followup and related work. It was submitted to the Springer Encyclopedia of Algorithms.

1 Problem Definition

The problem is to find a vector \( x \in \mathbb{C}^N \) such that \( Ax = b \), for some given inputs \( A \in \mathbb{C}^{N \times N} \) and \( b \in \mathbb{C}^N \). Several variants are also possible, such as rectangular matrices \( A \), including overdetermined and underdetermined systems of equations.

Unlike in the classical case, the output of this algorithm is a quantum state on \( \log(N) \) qubits whose amplitudes are proportional to the entries of \( x \), along with a classical estimate of \( \|x\| = \sqrt{\sum_i |x_i|^2} \). Similarly the input \( b \) is given as a quantum state. The matrix \( A \) is specified implicitly as a row-computable matrix. Specifying the input and output in this way makes it possible to find \( x \) in time sublinear, or even polylogarithmic, in \( N \). The next section has more discussion of the relation of this algorithm to classical linear systems solvers.

2 Key Results

Suppose that:

- \( A \in \mathbb{C}^{N \times N} \) is Hermitian, has all eigenvalues in the range \([-1, -1/\kappa] \cup [1/\kappa, 1]\) for some known \( \kappa \geq 1 \) and has \( \leq s \) nonzero entries per row. The parameter \( \kappa \) is called the condition number (defined more generally to be the ratio of the largest to the smallest singular value) and \( s \) is the sparsity.
- There is a quantum algorithm running in time \( T_A \) that takes an input \( i \in [N] \) and outputs the nonzero entries of the \( i \)th row, together with their location.
- Assume that \( \|b\| = 1 \) and that there is a corresponding quantum state to produce the state \( |b\rangle \) that runs in time \( T_B \).

Define \( \tilde{x}' := A^{-1} |b\rangle \) and \( x = \tilde{x}' / \|\tilde{x}'\| \).

We use the notation \( x \) to refer to the vector as a mathematical object and \( |x\rangle \) to refer to the corresponding quantum state on \( \log(N) \) qubits. For a variable \( T \) let \( \tilde{O}(T) \) denote a quantity upper bounded by \( T \cdot \text{poly} \log(T) \). The norm of a vector \( \|x\| \) is the usual Euclidean norm \( \sqrt{\sum_i |x_i|^2} \), while for a matrix \( \|A\| \) is the operator norm \( \max_{\|x\|=1} \|Ax\| \), or equivalently the largest singular value of \( A \).

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2.1 Quantum Algorithm for Linear Systems

The main result is that $|x|$ and $||x'||$ can be produced, both up to error $\epsilon$, in time $\text{poly}(\kappa, s, \epsilon^{-1}, \log(N), T_A, T_B)$. More precisely, the following run-times are known:

$$\tilde{O}(\kappa T_B + \log(N)s^2\kappa^2T_A/\epsilon)$$

$$\tilde{O}(\kappa T_B + \log(N)s^2\kappa T_A/\epsilon^3)$$

A key subroutine is Hamiltonian simulation, and the run-times in are based on the recent improvements in this component due to .

3 Hardness results and comparison to classical algorithms

These algorithms are analogous to classical algorithms for solving linear systems of equations, but do not achieve exactly the same thing. Most classical algorithms output the entire vector $x$ as a list of $N$ numbers while the quantum algorithms output the state $|x\rangle$, i.e. a superposition on $\log(N)$ qubits whose $N$ amplitudes equal $x$. This allows potentially faster algorithms but for some tasks will be weaker. This resembles the difference between the Quantum Fourier Transform and the classical Fast Fourier Transform.

To compare the classical and quantum complexities for this problem, it is necessary to examine the precise variant of linear-system solving that is performed by quantum algorithms. It may be that better classical algorithms could exist for this problem. One possibility is that all quantum algorithms could be simulated more quickly by classical algorithms. It turns out that in a certain sense this is the only possibility. This is because the linear systems problem is BQP-complete; i.e. solving large sparse well-conditioned linear systems of equations is equivalent in power to general purpose quantum computing.

To make this precise, define $\text{LinearSystemSample}(N, \kappa, \epsilon, T_A)$ to be the problem of producing a sample $i \in [N]$ from a distribution $p$ satisfying $\sum_{i=1}^{N} |p_i - |x_i|^2| \leq \epsilon$, where $x = x' / ||x'||$, $x' = A^{-1}b$ and $b = e_1$ (i.e. one in the first entry and zero elsewhere). Additionally the eigenvalues of $A$ should have absolute value between $1/\kappa$ and 1, and there should exist a classical algorithm for computing the entries of a row of $A$ that runs in time $T_A$. This problem differs slightly from the version described above, but only in ways that make it easier, so that it still makes sense to talk about a matching hardness result.

**Theorem 1.** Consider a quantum circuit on $n$ qubits that applies two-qubit unitary gates $U_1, \ldots, U_T$ to the $|0\rangle^\otimes n$ state and concludes by outputting the result of measuring the first qubit. It is possible to simulate this measurement outcome up to error $\epsilon$ by reducing to $\text{LinearSystemSample}(N, \kappa, \epsilon/2, T_A)$ with $N = O(2^nT/\epsilon)$, $\kappa = O(T/\epsilon)$ and $T_A = \text{poly log}(N)$.

In other words, $\text{LinearSystemSample}$ is at least as hard to solve as any quantum computation of the appropriate size. This result is nearly tight. In other words, when combined with the algorithm of the the relation between $N, \kappa$ (for linear system solving) and $n, T$ (for quantum circuits) is known to be nearly optimal, while the correct $\epsilon$ dependence is known up to a polynomial factor.

Theorem can also rule out classical algorithms for $\text{LinearSystemSample}(N, \kappa, \epsilon, T_A)$. Known algorithms for the problem (assuming for simplicity that $A$ is $s$-sparse) run in time $\text{poly}(N)\text{poly log}(\kappa/\epsilon) + N T_A$ (direct solvers), $N \text{poly}(\kappa)\text{poly log}(1/\epsilon)T_A$ (iterative methods), or even $s^{\kappa \text{ln}(1/\epsilon)} \text{poly log}(N)$ (direct expansion of $x \approx \sum_{n \leq \kappa \text{ln}(1/\epsilon)} (I - A)^n b$, assuming $A$ is positive semidefinite). Depending on the parameters $N, \kappa, \epsilon, s$, a different one of these may be optimal. And from Theorem it follows (a) that any nontrivial improvement in these algorithms would imply a general improvement in the ability of classical computers to simulate quantum mechanics; and (b) that such improvement is impossible for algorithms that use the function describing $A$ in a black-box manner (i.e. as an oracle).

4 Applications and extensions

Linear system solving is usually a subroutine in a larger algorithm, and the following algorithms apply it to a variety of settings. Complexity analyses can be found in the cited papers, but since hardness results are
4.1 Machine learning [6]

A widely-used application of linear systems of equations is to performing least-squares estimation of a model. In this problem, we are given a matrix $A \in \mathbb{R}^{n \times p}$ with $n \geq p$ (for an overdetermined model) along with a vector $b \in \mathbb{R}^n$ and we wish to compute $\arg \min_{x \in \mathbb{R}^p} \| Ax - b \|$. If $A$ is well-conditioned, sparse and implicitly specified, then the state $|x\rangle$ can be found quickly [6], and from this features of $x$ can be extracted by measurement.

4.2 Differential equations [2]

Consider the differential equation
\[
\dot{x}(t) = A(t)x(t) + b(t) \quad x(t) \in \mathbb{R}^N.
\] (2)

One of the simplest ways to solve this is to discretize time to take values $t_1 < \ldots < t_m$ and approximate
\[
x(t_{i+1}) \approx x(t_i) + (A(t_i)x(t_i) + b(t_i))(t_{i+1} - t_i).
\] (3)

By treating $(x(t_1), \ldots, x(t_m))$ as a single vector of size $Nm$ we can find this vector as a solution of the linear system of equations specified by (3). More sophisticated higher-order solvers can also be made quantum; see [2] for details.

4.3 Boundary-value problems [4]

The solution to PDEs can also be expressed in terms of the solution to a linear system of equations. For example, in Poisson’s equation we are given a function $Q : \mathbb{R}^3 \rightarrow \mathbb{R}$ and want to find $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $-\nabla^2 u = Q$. By defining $x$ and $b$ to be discretized versions of $u, Q$, this PDE becomes an equation of the form $Ax = b$. One challenge is that if $A$ is the finite-difference operator (i.e. discretized second derivative) for an $L \times L \times L$ box then its condition number will scale as $L^2$. Since the total number of points is $O(L^3)$, this means the quantum algorithm cannot achieve a substantial speedup. Classically this condition number is typically reduced by using preconditioners. A method for using preconditioners with the quantum linear system solver was presented in [4], along with an application to an electromagnetic scattering problem. The resulting complexity is still not known.

References

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