CONCENTRATION FOR INTEGRABLE DIRECTED POLYMER MODELS

CHRISTIAN NOACK AND PHILIPPE SOSOE

Abstract. In this paper, we consider four integrable models of directed polymers for which the free energy is known to exhibit KPZ fluctuations. A common framework for the analysis of these models was introduced in [7].

We derive estimates for the central moments of the partition function, of any order, on the near-optimal scale $N^{1/3+\epsilon}$, using the iterative method we applied to the semi-discrete polymer in [17]. Among the innovations exploiting the invariant structure, we develop formulas for correlations between functions of the free energy and the boundary weights that replace the Gaussian integration by parts appearing in our previous paper [17].

1. Introduction

In this paper, we consider four models for 1+1 dimensional integrable polymers in random environment, and study the higher moments of the centered free energy: the log-gamma polymer, introduced by Seppäläinen [22]; the strict-weak polymer, which was simultaneously introduced and analyzed by Corwin-Seppäläinen-Shen [10] and O’Connell-Ortmann [19]; the beta random walk of Barraquand and Corwin [4]; and the inverse beta model introduced by Thiery and Le Doussal [24].

The models in question are distinguished because they each possess algebraic structure that has enabled the verification of several predictions regarding their fluctuations. These include upper and lower bounds for the variance of the free energy, of order $O(N^{2/3})$ (see [22] for log-gamma and [9] for the three other models) as well as asymptotic Tracy-Widom distribution (see [5] for the log-gamma polymer, the original papers [10, 19, 4] for the strict-weak polymer and beta random walk models, as well a formal argument for the inverse beta model in [24]). Results of this type are characteristic of the KPZ universality class [8], and are expected to hold in a more general setting where the integrable structure is not available, but proving this is out of reach using current methods.

We note that the techniques used to prove asymptotic Tracy-Widom distribution by relating the free energy to a Fredholm determinant are markedly different from those that have been used to obtain variance bounds starting with the work of Seppäläinen [22]. The ideas in that work have their origins in earlier work of Seppäläinen and co-authors on fluctuations of one-dimensional interacting particle systems [2, 6, 3, 1]. Despite their power, the Bethe ansatz methods used to obtain the asymptotic distribution are not easily adapted to estimating the size of the central moments.

Here, we build on our previous paper [17] on the O’Connell-Yor polymer, a semi-discrete 1+1 dimensional polymer model introduced in [20], to obtain bounds of nearly optimal order for all the central moments of the free energy in the stationary version of each of the four models mentioned above. Our main result, Theorem 1, states that for each $k \geq 1$, the $k$th central moment of the free energy in a system of size $O(N^2)$ is bounded by $O(N^{k/3+\epsilon})$, where the implicit constant depends on $\epsilon$.

P.S.’s research was partially supported by NSF grant DMS 1811093. C.N. was supported by NSF RTG grant 1645643 while this research was carried out.
The proof in [17] proceeded by deriving a pair of inequalities which appear related to the physicists’ KPZ scaling relations and which enable an iterative proof of the bound for the order of fluctuations of the free energy by successive improvements starting from the trivial $O(N^{1/2})$ bound. A crucial idea was the repeated application of Gaussian integration by parts to relate cross-terms involving the "boundary Brownian motion" component of the free energy and the free energy itself to quenched cumulants of the first vertical jump of the polymers paths. This tool is not available in the discrete models we consider here. Nevertheless, we develop a substitute for it by introducing a sequence of polynomials which play a role analogous to that of Hermite polynomials for the O’Connell-Yor polymer, and allow us to derive formulas for the cumulants of the partition function in terms of quenched cumulants of the time of the first jump. Here, the Mellin transform framework introduced in [7] plays a central role. See Section 4.4.

1.1. The polymer model. To each edge $e$ of the $\mathbb{Z}^2_+$ lattice we assign a positive random weight. The superscripts 1 and 2 are used to denote horizontal and vertical edge weights, respectively. For $z \in \mathbb{N}^2$, let $Y^1_z$ and $Y^2_z$ denote the horizontal and vertical incoming edge weights, see Figure 1. We assume that the collection of pairs $\{(Y^1_z, Y^2_z)\}_{z \in \mathbb{N}^2}$ is independent and identically distributed with common distribution $(Y^1, Y^2)$, but do not insist that $Y^1$ is independent of $Y^2$. $(Y^1_z, Y^2_z)_{z \in \mathbb{N}^2}$ are the bulk weights. For $x \in \mathbb{N} \times \{0\}$, let $R^1_x$ denote the horizontal incoming edge weight, and for $y \in \{0\} \times \mathbb{N}$, let $R^2_y$ denote the vertical incoming edge weight. We take the collections $\{R^1_x\}_{x \in \mathbb{N} \times \{0\}}$ and $\{R^2_y\}_{y \in \{0\} \times \mathbb{N}}$ to be independent and identically distributed, with common distributions $R^1$ and $R^2$. We refer to these as the horizontal and vertical boundary weights, respectively. We further assume that the horizontal boundary weights, the vertical boundary weights, and the bulk weights are independent of each other. This assignment of edge weights is illustrated in Figure 1.

We call
\begin{equation}
\omega = \{R^1_x, R^2_y, (Y^1_z, Y^2_z) : x \in \mathbb{N} \times \{0\}, y \in \{0\} \times \mathbb{N}, z \in \mathbb{N}^2\}
\end{equation}
the polymer environment. We use $\mathbb{P}$ and $\mathbb{E}$ to denote the probability measure and corresponding expectation of the polymer environment.

![Figure 1. Assignment of edge weights.](image-url)
for an example of an up-right path. Let $\Pi_{m,n}$ denote the collection of all such paths. We identify paths $x_*= (x_0,x_1,\ldots,x_{m+n})$ either with their sequence of vertices or with their sequence of edges $(e_1,\ldots,e_{m+n})$, where $e_i = \{x_{i-1},x_i\}$, as convenient. Define the quenched polymer measure on $\Pi_{m,n}$,

$$Q_{m,n}(x_*) := \frac{1}{Z_{m,n}} \prod_{i=1}^{m+n} \omega_{e_i},$$

where $\omega_{e_i}$ is the weight associated to the edge $e_i$ and

$$Z_{m,n} := \sum_{x_*\in\Pi_{m,n}} \prod_{i=1}^{m+n} \omega_{e_i},$$

is the associated partition function. At the origin, define $Z_{0,0} := 1$. Taking the expectation $E$ of the quenched measure with respect to the edge weights gives the annealed measure on $\Pi_{m,n}$

(2) $P_{m,n}(x_*) := E[Q_{m,n}(x_*)].$

The annealed expectation will be denoted by $E_{m,n}.$

We specify the edge weight distributions for the four stationary polymer models. The notation $X \sim \text{Ga}(\alpha,\beta)$ is used to denote that a random variable is gamma$(\alpha,\beta)$ distributed, i.e. has density

$$\frac{1}{\Gamma(\alpha)^{\beta}}x^{\alpha-1}e^{-\beta x}$$

supported on $(0,\infty)$, where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ is the gamma function. $X \sim \text{Be}(\alpha,\beta)$ is used to say that $X$ is beta$(\alpha,\beta)$ distributed, i.e. has density

$$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1},$$

supported on $(0,1)$. We then use $X \sim \text{Ga}^{-1}(\alpha,\beta)$ and $X \sim \text{Be}^{-1}(\alpha,\beta)$ to denote that $X^{-1} \sim \text{Ga}(\alpha,\beta)$ and $X^{-1} \sim \text{Be}(\alpha,\beta)$, respectively. We also use $X \sim (\text{Be}^{-1}(\alpha,\beta) - 1)$ to denote that $X + 1 \sim \text{Be}^{-1}(\alpha,\beta)$.

Each of the four models we consider is obtained by choosing the distribution of the boundary and bulk weights according to one of the four following specifications:
• **Inverse-gamma (IG):** This is also known as the log-gamma model. Assume $\mu > \theta > 0$, $\beta > 0$ and

\[
R^1 \sim \text{Ga}^{-1}(\mu - \theta, \beta) \quad R^2 \sim \text{Ga}^{-1}(\theta, \beta) \\
(Y^1, Y^2) = (X, X) \quad \text{where} \quad X \sim \text{Ga}^{-1}(\mu, \beta).
\]

(3)

• **Gamma (G):** This is also known as the strict-weak model. Assume $\theta$, $\mu$, $\beta > 0$ and

\[
R^1 \sim \text{Ga}(\mu + \theta, \beta) \quad R^2 \sim \text{Be}^{-1}(\theta, \mu) \\
(Y^1, Y^2) = (X, 1) \quad \text{where} \quad X \sim \text{Ga}(\mu, \beta).
\]

(4)

• **Beta (B):** Assume $\theta$, $\mu$, $\beta > 0$ and

\[
R^1 \sim \text{Be}(\mu + \theta, \beta) \quad R^2 \sim \text{Be}^{-1}(\theta, \mu) \\
(Y^1, Y^2) = (X, 1 - X) \quad \text{where} \quad X \sim \text{Be}(\mu, \beta).
\]

(5)

• **Inverse-beta (IB):** Assume $\mu > \theta > 0$, $\beta > 0$ and

\[
R^1 \sim \text{Be}^{-1}(\mu - \theta, \beta) \quad R^2 \sim (\text{Be}^{-1}(\theta, \beta + \mu - \theta) - 1) \\
(Y^1, Y^2) = (X, X - 1) \quad \text{where} \quad X \sim \text{Be}^{-1}(\mu, \beta).
\]

(6)

Note that each of these four choices in fact generates a family of models by choosing different values of the parameters $\mu, \theta, \beta$. The name of each model refers to the distribution of the bulk weights. We call these models the four basic beta-gamma models.

1.2. Main Result. Having defined the models we will consider, we are now ready to state our main result. Given a path $x \in \Pi_{m,n}$, define the exit points of the path from the horizontal and vertical axes by

\[
t_1 := \max\{i : (i, 0) \in x\} \quad \text{and} \quad t_2 := \max\{j : (0, j) \in x\}.
\]

**Theorem 1.** Assume that the polymer environment has edge weight distributions $R^1, R^2, (Y^1, Y^2)$ as in one of (3) through (6), and let $(m, n) = (m_N, n_N)^N_{N=1}$ be a sequence such that

\[
|m_N - N \text{Var}[\log R^2]| \leq \gamma N^{2/3} \quad \text{and} \quad |n_N - N \text{Var}[\log R^1]| \leq \gamma N^{2/3}
\]

for some fixed $\gamma > 0$. Then for every $\epsilon > 0$ and $p > 0$, there exists a constant $C = C(\epsilon, p) > 0$ such that for any $N \in \mathbb{N}$,

\[
E[|\log Z_{m,n}|^p] \leq CN^{\frac{p}{2} + \epsilon} \quad \text{and} \quad E_m, n[(t_j)^p] \leq CN^{\frac{p}{2} + \epsilon} \quad \text{for both} \quad j = 1, 2.
\]

(9)

(10)

Note that by [7, Theorem 1.2], we have

\[
E[|\log Z_{m,n}|^2] \geq cN^{\frac{1}{2}}
\]

in the regime considered in [11]. By Jensen’s inequality this implies

\[
E[|\log Z_{m,n}|^p] \geq c^p N^{\frac{p}{2}},
\]

so that the bound we obtain is indeed near-optimal. We have not quantified the dependence of the constants on $\epsilon$ and $p$. In particular, the estimates in the Theorem are obtained by an iterative process, where the number of iterations depends on $\epsilon$. With our current method, the implicit constants grow without bounds as the number of iterations increases. It would be of great interest
to prove the Theorem with $\epsilon = 0$, but it is not clear to us whether this can be easily achieved by extending our current methods. We leave these questions to further work.

We also obtain exact formulas for the cumulants of the free energy, see Corollary 4.8.

1.3. Outline of the paper. In Section 2, after establishing some basic notation, we recall the Mellin transform framework introduced in [7], where it was noticed that the four basic beta-gamma models can be treated simultaneously.

In Section 3, we recall the "down-right" property shared by the four basic beta-gamma models. This is a consequence of the stronger Burke property, and implies in particular that the free energy can be written as the sum of two i.i.d. sums of order $O(N)$ see [13]. Understanding the fluctuations of the free energy becomes equivalent to understanding the correlation between these two sums, or equivalently the correlation between one of them and the free energy. This is manifested in the expansion for the cumulants of the free energy appearing in Lemma 3.1.

In Section 4, we develop a formula of "integration by parts" type which expresses certain correlations appearing in the expansion for the cumulant in terms of derivatives of expectations of moments of the free energy with respect to the parameter in the boundary weights. See Lemma 4.1. We use this to obtain formulas relating the the cumulants of the free energy to expectations of productions of quenched cumulants of $t_1$, the first jump in the system. See Corollary 4.8.

In Section 5, we prove our main result. The key idea here is that the formulas obtained in Section 4 allow one to get improved estimates (compared to the trivial $O(N^{1/2})$ bound) for the moments of the free energy given estimates for the annealed moments of $t_1$. See Lemma 5.7. Conversely, an inequality due essentially to Seppäläinen [22] relates the moments of $t_1$ to moments of the centered free energy. Iterating through these two inequalities a finite number of times, we can obtain bounds that are arbitrarily close to order $N^{1/3}$.

1.4. Acknowledgements. We would like to thank the anonymous referees for a detailed reading of an earlier manuscript and several helpful suggestions and corrections.

2. Preliminaries and notation

2.1. Notation. We let $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{Z} = \{0, 1, \ldots\}$, while $\mathbb{R}$ denotes the real numbers.

Let $|x|$ denote the greatest integer less than or equal to $x$. Let $\vee$ and $\wedge$ denote maximum and minimum, respectively:

$$a \vee b = \max\{a, b\},$$

$$a \wedge b = \min\{a, b\}.$$ 

Given a real valued function $f$, let $\text{supp}(f) = \{x : f(x) \neq 0\}$ denote the support of the function $f$ (note that we do not insist on taking the closure of this set).

Given a random variable $X$ with finite expectation, we let

$$\overline{X} = X - \mathbb{E}[X].$$

The symbol $\otimes$ is used to denote (independent) product distribution.

2.2. The Mellin transform framework. Here we introduce a framework, developed in [7], which allows us to treat the four basic beta-gamma models simultaneously.

Given a function $f : (0, \infty) \rightarrow [0, \infty)$, write $M_f$ for its Mellin transform

$$M_f(a) := \int_0^\infty x^{a-1} f(x) dx$$

for $a > 0$.
for any $a \in \mathbb{R}$ such that the integral converges. For more information about the Mellin transform and its relation to other classical integral transforms see [25], especially Sections 1.5, 1.29 and 7.7; note however that we use only elementary properties of the transform in this work. Define

$$D(M_f) := \text{interior}(\{a \in \mathbb{R} : 0 < M_f(a) < \infty\}).$$

**Definition 2.1.** Given a function $f : (0, \infty) \to [0, \infty)$ such that $D(M_f)$ is non-empty, we define a family of densities on $(0, \infty)$ parametrized by $a \in D(M_f)$:

$$\rho_{f,a}(x) := M_f(a)^{-1} x^{a-1} f(x).$$

We write $X \sim m_f(a)$ to denote that the random variable $X$ has density $\rho_{f,a}$.

**Remark 2.** If $f : (0, \infty) \to [0, \infty)$ is such that $D(M_f)$ is non-empty, then $M_f$ is $C^\infty$ throughout $D(M_f)$. Furthermore, if $X \sim m_f(a)$, then

1. $\log X$ has finite exponential moments. That is, since $D(M_f)$ is open, there exists some $\epsilon > 0$ such that

$$\mathbb{E}[e^{\epsilon \log X}] \leq \mathbb{E}[X^\epsilon] + \mathbb{E}[X^{-\epsilon}] = \frac{M_f(a + \epsilon) + M_f(a - \epsilon)}{M_f(a)} < \infty.$$

2. For all $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial a^k} M_f(a) = M_f(a) \mathbb{E}[(\log X)^k].$$

3. The $k$th cumulant of $\log X$, which we denote by $\kappa_k(\log X)$ following the notation introduced in [16], equals $\psi_k^f(a)$, where

$$\psi_k^f(a) := \frac{\partial^{k+1}}{\partial a^{k+1}} \log M_f(a) \text{ for } k \in \mathbb{Z}_+. $$

### 2.3. The four basic beta-gamma models are Mellin-type.

The random variables appearing in each of the four basic beta-gamma models have densities of the form (11), for various choices of $f$, which we specify here. In the table below, we assume $b > 0$ and $a \in D(M_f)$.

| $f(x)$ | $m_f(a)$ |
|--------|----------|
| $e^{-bx}$ | $\text{Ga}(a, b)$ |
| $e^{-b/x}$ | $\text{Ga}^{-1}(-a, b)$ |
| $(1 - x)^{b-1} 1_{\{0 < x < 1\}}$ | $\text{Be}(a, b)$ |
| $(1 - x^{-1})^{k-1} 1_{\{x > 1\}}$ | $\text{Be}^{-1}(-a, b)$ |
| $\left(\frac{x}{x+1}\right)^b$ | $\text{Be}^{-1}(-a, b+a) - 1$ |

To express the distribution of the polymer environment in each of the four models given in (3) through (6) within the above framework, we let

$$\rho_{f_1}^1, \rho_{f_2}^2, (X) \sim m_{f_1}(a_1) \otimes m_{f_2}(a_2) \otimes m_{f_1}(a_3),$$

where the functions $f^1$, $f^2$ and parameters $a_j$, $j = 1, 2, 3$ are given in the table in Figure 3. Recall that in each of the models, $(Y^1, Y^2)$ are given in terms of $X$. For Figure 3 we assume $\mu, \beta > 0$. 


The following definition is a weaker form of the Burke property, see \cite[Theorem 3.3]{22}.

\begin{align*}
\text{Proposition 3.1.} \quad & R_k \text{ and } R_{k+1} \text{ are independent and each } R_k, R_{k+1} \text{ satisfies } \\
& \text{for each } k \in \mathbb{Z} \text{ we associate the random variable } \\
& \Lambda_{\{\pi_{k-1}, \pi_k\}} := \begin{cases} 
R_{\pi_k}^1 & \text{if } \{\pi_{k-1}, \pi_k\} \text{ is horizontal,} \\
R_{\pi_k-1}^2 & \text{if } \{\pi_{k-1}, \pi_k\} \text{ is vertical.}
\end{cases}
\end{align*}

The following definition is a weaker form of the Burke property, see \cite[Theorem 3.3]{22}.

\begin{definition}
Say the polymer model has the down-right property if for all down-right paths \( \pi = \{\pi_k\}_{k \in \mathbb{Z}} \), the random variables
\[ \Lambda(\pi) := \{\Lambda_{\{\pi_{k-1}, \pi_k\}} : k \in \mathbb{Z}\} \]
are independent and each \( R_{\pi_k}^1 \) and \( R_{\pi_k}^2 \) appearing in the collection are respectively distributed as \( R^1 \) and \( R^2 \).
\end{definition}

\begin{proposition}
Each of the four basic beta-gamma models, \cite{3} through \cite{6}, possesses the down-right property.
\end{proposition}

3.1. Consequences of the down-right property. The free energy has two useful expressions.

\begin{align*}
\log Z_{m,n} &= W + N = S + E \\
W_n &= \sum_{j=1}^n \log R_{0,j}^2, \quad E_n := \sum_{j=1}^n \log R_{m,j}^2, \quad N_m := \sum_{i=1}^m \log R_{i,n}^1, \quad S_m := \sum_{i=1}^m \log R_{i,0}^1.
\end{align*}
Notice that \( W_n = \log Z_{0,n} \) and \( S_m = \log Z_{m,0} \). If the model possesses the down-right property (see Definition 3.1), then \( W_n, E_n, N_m, S_m \) are each sums of i.i.d. random variables. The subscripts \( n \) and \( m \) on \( W_n, E_n, N_m, S_m \) indicate the length of the sums.

Recall that if the random variables \( X_1, \ldots, X_k \) have finite exponential moments, then their joint cumulant is defined by

\[
\kappa(X_1, \ldots, X_k) := \frac{\partial^k}{\partial \xi_1 \cdots \partial \xi_k} \log \mathbb{E}[e^{\sum_{j=1}^k \xi_j X_j}] \Big|_{\xi_i = 0}
\]

Alternatively, the joint cumulant can be written as a combination of products of expectations of the underlying random variables:

\[
\kappa(X_1, \ldots, X_k) = \sum_{\pi \in \mathcal{P}} ((\pi| - 1)!(-1)^{m-n} \prod_{B \in \pi} \mathbb{E} \left[ \prod_{i \in B} X_i \right])
\]

where \( \mathcal{P} \) ranges over partitions \( \pi \) of \( \{1, \ldots, k\} \) and \( |A| \) stands for the size of the set \( A \). In the case where \( X_1 = X_2 = \cdots = X_k = X \), the joint cumulant reduces to the \( k \)-th cumulant of \( X \) which we denote by \( \kappa_k(X) \). The identity (16) is classical and indeed is often used as the definition of joint cumulants. For the reader’s convenience, we check its equivalence with the definition (15) in Appendix A

**Lemma 3.1.** Assume the polymer environment is such that \( |\log R^1|, |\log R^2|, |\log Y^1|, \) and \( |\log Y^2| \) all have finite exponential moments. Let \( N_m, E_n, S_m \) and \( W_n \) be as in (14). Then, for any positive integer \( k \),

\[
\kappa_k(\log Z_{m,n}) = \kappa_k(E_n) - (-1)^k \kappa_k(S_m) - \sum_{j=1}^{k-1} \binom{k}{j} (-1)^{k-j} \kappa(\log Z_{m,n}, \ldots, \log Z_{m,n}, S_m, \ldots, S_m) \text{ and } \kappa_k(\log Z_{m,n}) = \kappa_k(E_n) - (-1)^k \kappa_k(W_n) - \sum_{j=1}^{k-1} \binom{k}{j} (-1)^{k-j} \kappa(\log Z_{m,n}, \ldots, \log Z_{m,n}, W_n, \ldots, W_n).
\]

Moreover, if the polymer model also possesses the down-right property, then

\[
\kappa_k(E_n) = \kappa_k(W_n) = n \kappa_k(R^1) \quad \text{and} \quad \kappa_k(N_m) = \kappa_k(S_m) = n \kappa_k(R^2).
\]

**Proof.** By Lemma C.1 \( \log Z_{m,n}, N_m, S_m, E_n, W_n \) all have finite exponential moments, so their cumulants and joint cumulants exist. By (13), \( E_n = \log Z_{m,n} - S_n \). Since the joint cumulant is multi-linear,

\[
\kappa_k(E_n) = \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} \kappa(\log Z_{m,n}, \ldots, \log Z_{m,n}, S_m, \ldots, S_m).
\]

The \( j = k \) term in the summand is \( \kappa_k(\log Z_{m,n}) \) and the \( j = 0 \) term is \( (-1)^k \kappa_k(S_m) \). Rearranging yields equation (17). To obtain equation (18), apply the same argument with \( (N_m, W_n) \) in place of \( (E_n, S_m) \). For the last part of the Lemma use the fact that \( \kappa_k(X + Y) = \kappa_k(X) + \kappa_k(Y) \) if \( X \) and \( Y \) are independent. \( \square \)
Remark 4. Each of the four basic beta-gamma models satisfy the moment conditions of Lemma 3.1.

4. Formulas for the central moments

In the next two sections, we give an exact formula for the terms appearing in the summands on the right-hand side of (17). The same arguments can be used to obtain analogous estimates for (18), but as will be apparent in Section 5, the estimates in (17) will be sufficient to obtain Theorem 1.

4.1. Integration by parts type formula. Referring to the notation in Section 2.2, fix an integer \( r \geq 1 \) and let \( f_k : (0, \infty) \to [0, \infty) \) for \( k = 1, \ldots, r \) and \( a, a_0, a_1 \) be real numbers such that \( a_0 < a < a_1 \) and \( [a_0, a_1] \subset \cap_{k=1}^r D(M_f) \). Consider a collection of independent random variables \( \{X_k\}_{k=1}^r \) where \( X_k \sim m_f(a) \) for all \( 1 \leq k \leq r \), and let \( \mathbb{E}^a \) correspond to the expectation over these random variables. Finally, define

\[
T := \sum_{k=1}^r \log X_k.
\]

We introduce a sequence \( \{p_n(t, a; r)\}_{n \geq 0} \) of \( n \)-th degree polynomials in \( t \) defined recursively by

\[
p_0(t, a; r) = 1
\]

\[
p_n(t, a; r) = \frac{\partial}{\partial a} p_{n-1}(t, a; r) + p_{n-1}(t, a; r)(t - \mathbb{E}^a[T]) \quad \text{for} \quad n \geq 1.
\]

Note that the dependence of \( p_n(\cdot, \cdot; r) \) on \( r \) is polynomial, a fact we use explicitly later in our argument (see Proposition 5.1).

The following lemma extends Lemma B.2 from [7].

**Lemma 4.1.** \( A : \mathbb{R}^r \to \mathbb{R} \) be a measurable function such that \( \mathbb{E}^a[A(X_1, \cdots, X_r)^2] < \infty \) for all \( a \in [a_0, a_1] \). Then

\[
\frac{\partial^n}{\partial a^n} \mathbb{E}^a[A(X_1, \cdots, X_r)] = \mathbb{E}^a[A(X_1, \cdots, X_r)p_n(T, a; r)].
\]

**Proof.** The joint density of \( (\log X_1, \log X_2, \ldots, \log X_r) \) is given by

\[
g(x_1, \ldots, x_r) = \frac{e^{\sum_{k=1}^r x_k}}{\prod_{k=1}^r M_f(a)} \prod_{k=1}^r f_k(e^{x_k}).
\]

Thus the density of \( T = \sum_{k=1}^r \log X_k \) is

\[
h_a(t) = \frac{e^{at}}{\prod_{k=1}^r M_f(a)} \int_{\mathbb{R}^{r-1}} f_1(e^{x_1}) f_2(e^{x_2-x_1}) \cdots f_r(e^{t-x_{r-1}}) dx_1, \ldots, x_{r-1}.
\]

Therefore, the joint density of \( (\log X_1, \log X_2, \ldots, \log X_r) \) given that \( T = t \) is given by

\[
\frac{g(x_1, \ldots, x_r) \mathbb{I}_{\sum_{k=1}^r x_k = t}}{h_a(t)} = \frac{\prod_{k=1}^r f_k(e^{x_k})}{\int_{\mathbb{R}^{r-1}} f_1(e^{x_1}) f_2(e^{x_2-x_1}) \cdots f_r(e^{t-x_{r-1}}) dx_1, \ldots, x_{r-1}},
\]

which has no \( a \)-dependence. Recursion (19) and \( \frac{\partial}{\partial a} h_a(t) = h_a(t) (t - \mathbb{E}^a[T]) \) inductively imply

\[
\frac{\partial^n}{\partial a^n} h_a(t) = h_a(t) p_n(t, a; r) \quad \text{for all} \quad n \in \mathbb{Z}_+.
\]
By (22) and (23),
\[
\frac{\partial^n}{\partial a^n} \mathbb{E}^a[A(X_1, \ldots, X_k)] = \frac{\partial^n}{\partial a^n} \int_{\mathbb{R}} \mathbb{E}^a[A(X_1, \ldots, X_k)|T = t]h_a(t)\,dt \\
= \int_{\mathbb{R}} \mathbb{E}^a[A(X_1, \ldots, X_k)|T = t] \frac{\partial^n}{\partial a^n} h_a(t)\,dt \\
= \int_{\mathbb{R}} \mathbb{E}^a[A(X_1, \ldots, X_k)|T = t]h_a(t)p_n(t, a; r)\,dt \\
= \mathbb{E}^a\left(A(X_1, \ldots, X_k)p_n(T, a; r)\right).
\]

The interchanging of the n-th derivative and the integral will be justified by the bound:
\[
(24) \quad \int_{\mathbb{R}} \mathbb{E}[|A(\{X_k\}_{k=1}^\infty)| \mid T = t] \sup_{a \in [a_0, a_1]} \left| \frac{\partial^n}{\partial a^n} h_a(t) \right| \,dt < \infty.
\]
To obtain this bound first notice that recursion (19) implies that \(p_n(t, a)\) are degree \(n\) polynomials in \(t\) with coefficients that are smooth in \(a\). Thus, by (23), there exist constants \(0 < C_n < \infty\) independent of \(t\) such that
\[
(25) \quad \sup_{a \in [a_0, a_1]} \left| \frac{\partial^n}{\partial a^n} h_a(t) \right| \leq C_n (1 + |t|)^n \sup_{a \in [a_0, a_1]} h_a(t).
\]

Once we show that there is a constant \(C\) depending only on \(a_0\) and \(a_1\) such that
\[
(26) \quad \sup_{a \in [a_0, a_1]} h_a(t) \leq h_{a_0}(t) + Ch_{a_1}(t) \quad \text{for all } s \in \mathbb{R},
\]
(25) will give the bound (24) since
\[
\int_{\mathbb{R}} \mathbb{E}[|A(\{X_k\}_{k=1}^\infty)| | T = t](1 + |t|)^n h_a(t)\,dt = \mathbb{E}^{a_1}[|A(\{X_k\}_{k=1}^\infty)|(1 + |T|)^n] \\
\leq \mathbb{E}^{a_1}[(A(\{X_k\}_{k=1}^\infty))^2]\mathbb{E}^{a_1}[(1 + |T|)^{2n}]^\frac{1}{2}.
\]
This is finite since \(\mathbb{E}^{a_1}[A(\{X_k\}_{k=1}^\infty)^2] < \infty\) by assumption, and \(T\) is a sum of independent random variables with finite exponential moments. All that is left to do is verify the bound (26). To accomplish this, notice that equation (21) implies
\[
(27) \quad \frac{\partial}{\partial a} \log h_a(t) = t - \mathbb{E}^a[T].
\]
Since \(\mathbb{E}^a[T] = \sum_{k=1}^\infty \psi^h_0(a, a)\), \(a \mapsto \mathbb{E}^a[T]\) is an increasing function (recall that \(\frac{d}{da} \psi^h_0(a) = \psi^h_1(a) = \text{Var}[X_k] > 0\)). Therefore, for all \(t \leq \mathbb{E}^{a_0}[T]\), the function \(a \mapsto h_a(t)\) is non-increasing on \([a_0, a_1]\) which gives
\[
\sup_{a \in [a_0, a_1]} h_a(t) \leq h_{a_0}(t) \quad \text{for all } t \leq \mathbb{E}^{a_0}[T].
\]
On the other hand, if \(t > \mathbb{E}^{a_0}[T]\), then
\[
(28) \quad \frac{\partial}{\partial a} \log \left(h_a(t) \exp \left(a(\mathbb{E}^{a_1}[T] - \mathbb{E}^{a_0}[T])\right)\right) = t - \mathbb{E}^a[T] + \mathbb{E}^{a_1}[T] - \mathbb{E}^{a_0}[T] > 0
\]
for all \(a \in [a_0, a_1]\). Thus, for all \(t > \mathbb{E}^{a_0}[T]\),
\[
a \mapsto h_a(t) \exp \left(a(\mathbb{E}^{a_1}[T] - \mathbb{E}^{a_0}[T])\right)
\]
is increasing on the interval \([a_0, a_1]\). Therefore
\[
\sup_{a \in [a_0, a_1]} h_a(t) \leq C_3 h_{a_1}(t) \text{ for all } t \in \mathbb{E}^{a_0}[T]
\]
where \(C = \exp \left( (a_1 - a_0)(\mathbb{E}^{a_1}[T] - \mathbb{E}^{a_0}[T]) \right)\). Combining (27) and (28) gives the desired result. \(\square\)

**Lemma 4.2.** Assume the polymer environment satisfies \(R^1_{i,0} \sim m_f(a)\) for all \(i \geq 1\). Let \(k \geq 2\) and \(1 \leq j \leq k\). For \(r \geq 1\), let
\[
S_r := \sum_{i=1}^{r} R^1_{i,0}.
\]
Then,
\[
\kappa(\log Z_{m,n}, \ldots, \log Z_{m,n}, S_r, \ldots, S_r) = \kappa(\log Z_{m,n}, \ldots, \log Z_{m,n}, S_r, \ldots, S_r)
\]
\[
= \sum_{\pi \in \mathcal{P}} (-1)^{|\pi| - 1} \prod_{B \in \pi} \left[ \log Z_{m,n}^{(|B| \cap \{1, \ldots, j\})} p_{|B| \cap \{1, \ldots, j\}}(S_r, a; r) \right] + r\psi_{k-j}(a)
\]
where \(\mathcal{P}\) ranges over partitions \(\pi\) of \(\{1, \ldots, k\}\) such that no block \(B \in \pi\) is contained in \(\{j+1, \ldots, k\}\), and \(p_k(s, a)\) are polynomials recursively defined by (19) in the case \(f_j = f\) for all \(j\).

**Proof.** Introduce the function
\[
g(s, a; r) := e^{as - r \log M_f(a)}.
\]
Note that
\[
\frac{\partial}{\partial a} g(s, a; r) = (s - r \psi'_0(a))g(s, a; r),
\]
so
\[
\frac{\partial}{\partial a} \left( g(s, a; r)p_{k-1}(s, a; r) \right) = g(s, a; r)\frac{\partial}{\partial a} p_{k-1}(s, a; r) + g(s, a; r)p_{k-1}(s, a; r)(s - r \psi'_0(a))
\]
\[
= g(s, a; r)p_{k}(s, a; r).
\]
Rearranging, this gives
\[
p_k(s, a; r) = \frac{1}{g(s, a; r)} \frac{\partial}{\partial a} \left( g(s, a; r)p_{k-1}(s, a; r) \right)
\]
\[
= \ldots
\]
\[
= \frac{1}{g(s, a; r)} \frac{\partial^l}{\partial a^l} \left( g(s, a; r)p_{k-l}(s, a; r) \right).
\]
Letting \(l = k\), we have
\[
p_k(s, a; r) = \frac{\partial^k}{\partial a^k} \left( g(s, a; r) \right)
\]
By Taylor expansion around \(a\) for \(\lambda\) sufficiently small, we have
\[
g(s, a + \lambda; r) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{\partial^k}{\partial a^k} g(s, a; r),
\]
which yields the generating function for the polynomials $p_k(s, a; r)$:

$$
\frac{g(s, a + \lambda; r)}{g(s, a; r)} = e^{\lambda s} \left( \frac{M_f(a)}{M_f(a + \lambda)} \right)^r = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} p_k(s, a; r).
$$

(30)

Using this, we have a formula for joint cumulants, as follows. Recall

$$
\kappa(A, \ldots, A, S_r, \ldots, S_r) = \frac{\partial^{j \text{ times}}}{\partial \xi_1 \cdots \partial k} \log \mathbb{E}[e^{(\xi_1 + \cdots + \xi_k)A} e^{(\xi_{j+1} + \cdots + \xi_k)S_r}] |_{\xi_i = 0}.
$$

Inserting the generating function (30), we have

$$
e^{(\xi_{j+1} + \cdots + \xi_k)S_r} = \left( \frac{M_f(a + \xi_{j+1} + \cdots + \xi_k)}{M_f(a)} \right)^r \sum_{l=0}^{\infty} \frac{(\xi_{j+1} + \cdots + \xi_k)^l}{l!} p_l(S_r, a; r).
$$

Taking expectations, then logarithms, we have

$$
\log \mathbb{E}[e^{(\xi_{j+1} + \cdots + \xi_k)S_r}] = \log \mathbb{E}[e^{(\xi_1 + \cdots + \xi_k)A} \sum_{l=0}^{\infty} \frac{(\xi_{j+1} + \cdots + \xi_k)^l}{l!} p_l(S_r, a; r)]
$$

$$
- r \log M_f(a) + r \log M_f(a + \xi_{j+1} + \cdots + \xi_k).
$$

Setting $A = \log Z_{m,n}$ and taking derivates with respect to the $\xi_i$’s, then evaluating them at zero gives (29).

The only part of the statement that still requires comment is the assertion that partitions $P$ with a block contained in $\{j + 1, \ldots, k\}$ make a zero contribution. This is because

$$
\mathbb{E}[p_n(S_r, a; r)] = 0
$$

for $n \geq 1$, as follows from (29) with $A = 1$.

\[\square\]

4.2. Coupling of polymer environments. In order to compare polymer environments with different parameters, we use a coupling to express the boundary weights as functions of i.i.d. uniform$(0, 1)$ random variables.

Recall the notation from Section 2.2. Suppose $f : (0, \infty) \to [0, \infty)$ is a smooth function on its support, supp$(f)$ is open, and $D(M_f)$ is non-empty. Define $F^f : D(M_f) \times (0, \infty) \to [0, 1]$ by

$$
F^f(a, x) := \frac{1}{M_f(a)} \int_0^x y^{a-1} f(y) dy.
$$

For fixed $a \in D(M_f)$, $x \mapsto F^f(a, x)$ is the cdf of a random variable with $X \sim m_f(a)$. Note that $F^f(a, \cdot)$ is a bijection between supp$(f)$ and $(0, 1)$. For $a \in D(M_f)$, let $H^f(a, \cdot)$ be the inverse of $F^f(a, \cdot)$ defined on $(0, 1)$. By the implicit function theorem, $H^f : D(M_f) \times (0, 1) \to \text{supp}(f)$ is a smooth function in both of its variables satisfying

$$
\frac{\partial}{\partial a} \log H^f(a, x) = L^f(a, H^f(a, x)),
$$

(31)

where

$$
L^f(a, x) := \frac{1}{x^a f(x)} \int_0^x y^{a-1} (\psi_f(a) - \log y) f(y) dy.
$$

Another expression for $L^f$ shows that it is a strictly positive function:

$$
L^f(a, x) = -\frac{1}{x^a f(x)} \text{Cov}(\log X, 1_{\{X \leq x\}}) > 0,
$$

(32)
because the right side of (32) is the negative of the correlation between an increasing and a decreasing function of $X$. Since $f$ is smooth, $L^f$ is smooth as a function on $D(M_f) \times \text{supp}(f)$. Note that if $\eta$ is a uniform$(0,1)$ distributed random variable, then $H^f(a, \eta) \sim m_f(a)$ for every $a \in D(M_f)$. This gives us a useful coupling as follows.

Fix $m, n \in \mathbb{N}$ and an environment $\omega$ on the square with lower-left corner $(0,0)$ and upper-right corner $(m,n)$. Assume the random variables attached to the southern boundary $R^1_{i,0}$ all have $m_f(a)$ distributions. Fix $1 \leq r \leq m$ and let $\{\eta_i\}_{i=1}^r$ be i.i.d. uniform$(0,1)$ distributed random variables which are also independent of the original environment $\omega$. Now create a new environment $\tilde{\omega}$ by replacing $R^1_{i,0}$ in the original environment along the southern boundary by

$$
\tilde{R}^1_{i,0}(b) := H^f(b, \eta_i)
$$

only for $i = 1, \ldots, r$. When $b = a$ the new environment is equal in distribution to the old one:

$$
\tilde{\omega} \overset{d}{=} \omega, \quad b = a.
$$

Write

$$
Z_{m,n}(b) := Z^\omega_{m,n} = \sum_{x \in \Pi_{m,n}} \prod_{i=1}^{m+n} \tilde{\omega}(x_{i-1},x_i) = \sum_{x \in \Pi_{m,n}} \prod_{i=1}^{t_1(x) \wedge r} \tilde{R}^1_{i,0} \prod_{i=t_1(x) \wedge r+1}^{m+n} \omega(x_{i-1},x_i),
$$

where $t_1(x) := \max\{i \geq 0 : x_i = (i,0)\}$, i.e. the exit time from the southern boundary. By equation (31),

$$
\frac{\partial}{\partial b} \bigg|_{b=a} \log Z_{m,n}(b) = \tilde{E}^a_{m,n}\left[ \sum_{i=1}^{t_1(x) \wedge r} L^f(a, \tilde{R}^1_{i,0}) \right],
$$

Therefore

$$
\frac{\partial}{\partial b} \bigg|_{b=a} \log Z_{m,n}(b) = \tilde{E}^a_{m,n}\left[ \sum_{i=1}^{t_1(x) \wedge r} L^f(a, \tilde{R}^1_{i,0}) \right],
$$

where $\tilde{E}^a_{m,n}[F(x.)]$ is defined to be the expectation of the function $F$ of up-right paths $x$. under the quenched probability measure

$$
\tilde{G}^b_{m,n}(x.):= \frac{1}{Z_{m,n}(b)} \prod_{i=1}^{m+n} \tilde{\omega}(x_{i-1},x_i) = \frac{1}{Z_{m,n}(b)} \prod_{i=1}^{t_1(x) \wedge r} \tilde{R}^1_{i,0}(b) \prod_{i=t_1(x) \wedge r+1}^{m+n} \omega(x_{i-1},x_i).
$$

We will use $\hat{E}$ and $\hat{P}$ to denote annealed expectation and probability for the measure in (35).

By (33) and (34),

$$
\frac{\partial}{\partial b} \bigg|_{b=a} \log G^b_{m,n}(x.) = \sum_{i=1}^{t_1(x) \wedge r} L^f(a, \tilde{R}^1_{i,0}) - \hat{E}^a_{m,n}\left[ \sum_{i=1}^{t_1(x) \wedge r} L^f(a, \tilde{R}^1_{i,0}) \right].
$$

4.3. Derivatives with respect to boundary parameters. We now use equation (36) to provide a recursion which will help determine $\frac{\partial}{\partial b} \bigg|_{b=a} \log Z_{m,n}(b)$.

**Definition 4.1.** Suppose $G(a,x) : D(M_f) \times \text{supp}(f) \to \mathbb{R}$ is a $C^1$ function. The action of the operator $\hat{\partial}$ on $G$ is given by

$$
\hat{\partial} G(a,x) := \frac{\partial G}{\partial a}(a,x) + x L^f(a,x) \frac{\partial G}{\partial x}(a,x).
$$
Lemma 4.3. Let $G(a, x) : D(M_f) \times \text{supp}(f) \rightarrow \mathbb{R}$ be a $C^k$ function. Then the function $\tilde{G} : D(M_f) \times \text{supp}(f) \rightarrow \mathbb{R}$ is $C^{k-1}$ and satisfies the two equations:

$$\tilde{G}(a, \tilde{R}_{i,0}(a)) = \frac{\partial}{\partial b} \bigg|_{b=a} (G(b, \tilde{R}_{i,0}(b))).$$

(38)

\[
\frac{\partial}{\partial b} \bigg|_{b=a} \tilde{E}_{m,n} \left[ \sum_{i=1}^{t_1(x) \wedge r} G(b, \tilde{R}_{i,0}) \right] = \tilde{E}_{m,n} \left[ \sum_{i=1}^{t_1(x) \wedge r} \tilde{G}(a, \tilde{R}_{i,0}) \right] \\
+ \tilde{Cov}_{m,n} \left( \sum_{i=1}^{t_1(x) \wedge r} G(a, \tilde{R}_{i,0}(a)) \right) \sum_{i=1}^{t_1(x) \wedge r} L'(a, \tilde{R}_{i,0}(a)),
\]

where $\tilde{Cov}_{m,n}$ stands for the covariance under $\tilde{E}_{m,n}$.

Proof. By equation (33), we have

\[
\frac{\partial}{\partial b} \bigg|_{b=a} G(b, \tilde{R}_{i,0}(b)) = \frac{\partial}{\partial a} (a, \tilde{R}_{i,0}(a)) + \frac{\partial}{\partial b} \bigg|_{b=a} \tilde{R}_{i,0} \frac{\partial}{\partial x} (a, \tilde{R}_{i,0}).
\]

\[
\tilde{E}_{m,n} \left[ \sum_{i=1}^{t_1(x) \wedge r} G(a, \tilde{R}_{i,0}) \right] = \sum_{x \in \Pi_{m,n}} \left( \sum_{i=1}^{t_1(x) \wedge r} G(a, \tilde{R}_{i,0}) \right) \tilde{Q}_{m,n}(x).
\]

Therefore,

(39)

\[
\frac{\partial}{\partial b} \bigg|_{b=a} \tilde{E}_{m,n} \left[ \sum_{i=1}^{t_1(x) \wedge r} G(a, \tilde{R}_{i,0}) \right] = \sum_{x \in \Pi_{m,n}} \frac{\partial}{\partial b} \bigg|_{b=a} \left( \sum_{i=1}^{t_1(x) \wedge r} G(a, \tilde{R}_{i,0}) \right) \tilde{Q}_{m,n}(x).
\]

(40)

\[
+ \sum_{x \in \Pi_{m,n}} \left( \sum_{i=1}^{t_1(x) \wedge r} G(a, \tilde{R}_{i,0}) \right) \frac{\partial}{\partial b} \bigg|_{b=a} \tilde{Q}_{m,n}(x).
\]
By the first part of the lemma, the right-hand side of (39) equals $\bar{E}_{m,n}^a \left[ \sum_{i=1}^{t_1(x) \wedge R} \left( \hat{\delta} G(a, \tilde{R}_{i,0}^1) \right) \right]$. Using equation (36), multi-linearity of the joint cumulant gives

$$\sum_{x \in \Pi_{m,n}} \left( \sum_{i=1}^{t_1(x) \wedge R} G(a, \tilde{R}_{i,0}^1) \right) \left( \sum_{i=1}^{t_1(x) \wedge R} L^f(a, \tilde{R}_{i,0}^1) - \bar{E}_{m,n}^a \left[ \sum_{i=1}^{t_1(x) \wedge R} L^f(a, \tilde{R}_{i,0}^1) \right] \right) \tilde{Q}_{m,n}(x)$$

$$= \bar{E}_{m,n}^a \left[ \sum_{i=1}^{t_1(x) \wedge R} G(a, \tilde{R}_{i,0}^1) \right] \left( \sum_{i=1}^{t_1(x) \wedge R} L^f(a, \tilde{R}_{i,0}^1) \right)$$

$$= \bar{E}_{m,n}^a \left[ \sum_{i=1}^{t_1(x) \wedge R} G(a, \tilde{R}_{i,0}^1) \right] \tilde{E}_{m,n}^a \left[ \sum_{i=1}^{t_1(x) \wedge R} L^f(a, \tilde{R}_{i,0}^1) \right]$$

$$= \tilde{\text{Cov}}_{m,n}^a \left( \sum_{i=1}^{t_1(x) \wedge R} G(a, \tilde{R}_{i,0}^1), \sum_{i=1}^{t_1(x) \wedge R} L^f(a, \tilde{R}_{i,0}^1) \right).$$

Letting $\tilde{\kappa}_k^a(X_1, \ldots, X_k)$ denote the quenched joint cumulant of $k$ random variables with respect to $\bar{E}_{m,n}^a$, repeated application of (38) and the chain rule give:

**Lemma 4.4.** For $g_1, \ldots, g_k \in C^\infty(D(M_f) \times \text{supp}(f))$,

$$\frac{\partial}{\partial a_k} \tilde{\kappa}_k^a \left( \sum_{i=1}^{t_1(x) \wedge R} g_1(a, \tilde{R}_{i,0}^1), \sum_{i=1}^{t_1(x) \wedge R} g_2(a, \tilde{R}_{i,0}^1), \ldots, \sum_{i=1}^{t_1(x) \wedge R} g_k(a, \tilde{R}_{i,0}^1) \right)$$

$$= \sum_{\delta_{i+\cdots+\delta_k=1}} \tilde{\kappa}_k^a \left( \sum_{i=1}^{t_1(x) \wedge R} \tilde{\delta}^{\delta_1} g_1(a, \tilde{R}_{i,0}^1), \sum_{i=1}^{t_1(x) \wedge R} \tilde{\delta}^{\delta_2} g_2(a, \tilde{R}_{i,0}^1), \ldots, \sum_{i=1}^{t_1(x) \wedge R} \tilde{\delta}^{\delta_k} g_k(a, \tilde{R}_{i,0}^1) \right)$$

$$+ \tilde{\kappa}_{k+1}^a \left( \sum_{i=1}^{t_1(x) \wedge R} g_1(a, \tilde{R}_{i,0}^1), \sum_{i=1}^{t_1(x) \wedge R} g_2(a, \tilde{R}_{i,0}^1), \ldots, \sum_{i=1}^{t_1(x) \wedge R} g_k(a, \tilde{R}_{i,0}^1), \sum_{i=1}^{t_1(x) \wedge R} L^f(a, \tilde{R}_{i,0}^1) \right).$$

**Proof.** Write $X_l(a) := \sum_{i=1}^{t_1(x) \wedge R} g_l(a, \tilde{R}_{i,0}^1)$ for $l = 1, \ldots, k$. We will perturb the parameter in the environment and the parameters in the arguments of the cumulant separately. To this end, define

$$F(a, h) := \tilde{\kappa}_k^h(X_1(a), X_2(a), \ldots, X_k(a)).$$

Using equation (33) and multi-linearity of the joint cumulant gives

$$\frac{\partial}{\partial a} F(a, h) = \sum_{\delta_{i+\cdots+\delta_k=1}} \tilde{\delta}^{\delta_1} X_1(a), \tilde{\delta}^{\delta_2} X_2(a), \ldots, \tilde{\delta}^{\delta_k} X_k(a).$$

Now define $Y(h) := \sum_{i=1}^{t_1(x) \wedge R} L^f(h, H^f(h, \eta_i))$. Then

$$\frac{\partial}{\partial h} \bar{E}_{m,n}^h \left[ e^{\sum_{i=1}^{t_1(x) \wedge R} X_i(a)} \right] = \bar{E}_{m,n}^h \left[ e^{\sum_{i=1}^{t_1(x) \wedge R} X_i(a)} \bar{Y}(h) \right].$$
where the centering is respect to \( \hat{E}_{m,n}^h \). Thus
\[
\frac{\partial}{\partial h} F(a,h) = \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} \frac{\partial}{\partial h} \hat{E}_{m,n}^h \sum_{i=1}^k \xi_i X_i(a) \bigg|_{\xi_1=\cdots=\xi_k=0} = \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_{k+1}} \hat{E}_{m,n}^h \sum_{i=1}^k \xi_i X_i(a) + \xi_{k+1} Y(h) \bigg|_{\xi_1=\cdots=\xi_k=\xi_{k+1}=0} = \tilde{\kappa}_{k+1} \left( \sum_{i=1}^{t_1 \wedge \tau} g_1(a, \tilde{R}_{i,0}^1), \sum_{i=1}^{t_1 \wedge \tau} g_2(a, \tilde{R}_{i,0}^1), \ldots, \sum_{i=1}^{t_1 \wedge \tau} g_k(a, \tilde{R}_{i,0}^1), Y(h) \right).
\]

Combining (4.3) and (41) gives
\[
\frac{\partial}{\partial a} F(a,a) = \left( \frac{\partial F}{\partial a}(a,h) + \frac{\partial F}{\partial b}(a,h) \right) \bigg|_{h=a}
\]
yielding the desired result. \( \square \)

Equation (43) and a repeated application of the previous lemma now give

**Corollary 4.5.** Let \( k \geq 1 \). There are non-negative constants \( C_{k,j,\bar{\ell}} \) indexed by \( 1 \leq j \leq k \) and \( j \)-tuples \( \bar{\ell} = (\ell_1, \ldots, \ell_j) \) with \( \ell_i \geq 0 \) and \( \sum_{1 \leq i \leq j} \ell_i = k - j \) such that
\[
\left. \frac{\partial^k}{\partial b^k} \log Z_{m,n}(b) \right|_{b=a} = \sum_{j=1}^k \sum_{\bar{\ell}} C_{k,j,\bar{\ell}} \tilde{\kappa}_j^\ell \left( \sum_{i=1}^{t_1 \wedge \tau} \tilde{\delta}^{\ell_1} L^i(a, \tilde{R}_{i,0}^1), \ldots, \sum_{i=1}^{t_1 \wedge \tau} \tilde{\delta}^{\ell_j} L^j(a, \tilde{R}_{i,0}^1) \right).
\]

Here \( \tilde{\delta}^\ell \) is the operator defined in (37), iterated \( \ell \) times. The constants satisfy
\[
\sum_{j=1}^k \sum_{\bar{\ell}} C_{k,j,\bar{\ell}} \leq k!.
\]

Note that \( C_{k,k,(0,\ldots,0)} = 1 \), so that the the \( j = k \)-th term in the sum is the \( k \)-th quenched cumulant of \( \sum_{i=1}^{t_1 \wedge \tau} L^i(a, \tilde{R}_{i,0}^1) \).

**Proof.** For \( k = 1 \), we have
\[
\left. \frac{\partial}{\partial b} \log Z_{m,n}(b) \right|_{b=a} = \tilde{\kappa}_j^a \left( \sum_{i=1}^{t_1 \wedge \tau} L^i(a, \tilde{R}_{i,0}^1) \right).
\]

This has the required form
\[
\tilde{\kappa}_j^a \left( \sum_{i=1}^{t_1 \wedge \tau} \tilde{\delta}^{\ell_1} L^i(a, \tilde{R}_{i,0}^1), \ldots, \sum_{i=1}^{t_1 \wedge \tau} \tilde{\delta}^{\ell_j} L^j(a, \tilde{R}_{i,0}^1) \right)
\]
with \( j = 1 \) and \( \bar{\ell} = (0) \). Using Lemma 4.4, differentiating any term of the form (43) produces \( j + 1 \) terms of the same form where one of the \( \ell_i \) or \( j \) are increased by one. Thus, differentiating (42) \( k - 1 \) times, we obtain a sum of at most \( k! \) terms, all of the form (43). \( \square \)
4.4. Application of integration by parts. We now use the results obtained thus far to provide exact expressions for the cumulants of the free energy \( \log Z_{m,n} \) (see Corollary 4.8).

We now impose the following assumption on \( f \) to have control of the moments of \( \partial^k L^f \).

**Hypothesis 1.** Assume \( f \) has non-empty, open support \( \text{supp}(f) \), non-empty \( D(M_f) \), \( f \) is smooth on its support, and for every \( k \in \mathbb{Z}_+ \), there exist \( C_k \in C(D(M_f)) \), continuous functions on \( D(M_f) \), such that

\[
|\partial^k L^f(a,x)| \leq C_k(a) \left( 1 + |\log x|^{k+1} \right) \quad \text{for all } (a,x) \in D(M_f) \times \text{supp}(f).
\]

The following theorem says that all functions appearing in the four models \( \hat{\omega}, \hat{\omega}^\prime, \omega, \omega^\prime \) satisfy this hypothesis.

**Theorem 5.** Let \( f \) be one of the functions appearing in Table 2.3, meaning \( f \) corresponds to a gamma, inverse-gamma, beta, inverse-beta, or beta-prime distribution. Then \( f \) satisfies Hypothesis 1.

The proof of Theorem 5 is relegated to the appendix.

**Lemma 4.6.** Assume \( f \) satisfies Hypothesis 1. Let \( [a_0,a_1] \subset D(M_f) \), \( \eta \sim \text{uniform}(0,1) \), and \( k \in \mathbb{N} \). Then the random variable

\[
\sup_{a \in [a_0,a_1]} |\partial^k L^f(a,H^f(a,\eta))|
\]

has finite moments of all orders.

**Proof.** Put \( Y := \sup_{a \in [a_0,a_1]} |\partial^k L^f(a,H^f(a,\eta))| \). By Hypothesis 1 there exists a constant \( C > 0 \) such that

\[
|\partial^k L^f(a,H^f(a,\eta))| \leq C \left( 1 + |\log H^f(a,\eta)|^{k+1} \right)
\]

holds for all \( a \in [a_0,a_1] \). Since \( H^f(a_j,\eta) \sim m_f(a_j) \) for \( j = 0,1 \), and \( a_0,a_1 \in D(M_f) \), both \( \log H^f(a_0,\eta) \) and \( \log H^f(a_1,\eta) \) have finite exponential moments. Thus \( Y \) has finite moments of all orders.

Now write \( \hat{\pi}, \hat{\pi}^\prime \) for the probability measure and expectation corresponding to the environment \( \hat{\omega} \), as defined in Section 4.2, and \( \pi, \pi^\prime \) for probability measure and expectation corresponding to the environment \( \omega \). Write

\[
\tilde{\sigma}_0(r) := \log Z_{m,n}(a) - \hat{E}[\log Z_{m,n}(a)] \quad \text{and} \quad \tilde{\sigma}_k(r) := \frac{\partial^k}{\partial b^k} \Big|_{b=a} \log Z_{m,n}(b) \quad \text{for } k \in \mathbb{N}.
\]

Similarly, define

\[
\sigma_0(r) := \log Z_{m,n} - E[\log Z_{m,n}]
\]

\[
\sigma_k(r) := \sum_{j=1}^{k} \sum_{\ell_1+\cdots+\ell_j=k-j} C_{k,j;\ell_1,\cdots,\ell_j} \left( \sum_{i=1}^{\ell_1} \partial^\ell_i L^f(a,R_{i,0}^1), \cdots, \sum_{i=1}^{\ell_j} \partial^\ell_j L^f(a,R_{i,0}^1) \right), \quad k \in \mathbb{N},
\]

where \( C_{k,j;\ell_1,\cdots,\ell_j} \) is the coefficient of \( \partial^k \) in the Taylor expansion of \( \log Z_{m,n} \).
where \( \kappa_k^Q(X_1, \ldots, X_k) \) denotes the joint-cumulant of the random variable \( X_1, \ldots, X_k \) with respect to the quenched measure \( Q_{m,n} \) and \( C_{k,j,\ell} \) are the constants appearing in Corollary 4.5. By Corollary 4.5 \( \sigma_k(r) \overset{d}{=} \tilde{\sigma}_k^Q(r) \). Recall that our environment \( \tilde{\omega} \) has only changed the southern boundary random variables between the origin and the point \((r, 0)\), so \( \log Z_{m,n}(b) \) only depends on \( a \) through \( R_{i,0}^1 \), \( r < i \leq m \).

**Lemma 4.7.** If \( f \) satisfies Hypothesis 1 and \( S_r = \sum_{i=1}^r \log R_{i,0}^1 \), then for any \( j, k \geq 0 \),

\[
\mathbb{E}[(\log Z_{m,n})^j p_k(S_r, a; r)] = \sum_{\ell_i \geq 0} k! \prod_{i=1}^j \mathbb{E} \left[ \prod_{i=1}^j \sigma_{\ell_i}(r) \right].
\]

**Proof.** Write \( g(a) := \tilde{\mathbb{E}}^a[\log Z_{m,n}] \). Then the left-hand side of equation (46) is equal to

\[
\mathbb{E}[\log Z_{m,n} - g(b)]^j p_k(S_r, a; r)]|_{b = a}.
\]

Fix \( b \in D(M_f) \) and let \( \mathcal{F} \) be the sigma-algebra generated by the random variables \( R_{1,0}^1, \ldots, R_{r,0}^1 \). Then there exists a measurable function function \( A : \mathbb{R}^r \to \mathbb{R} \) such that \( A(R_{1,0}^1, \ldots, R_{r,0}^1) = \mathbb{E}[(\log Z_{m,n} - g(b))^j] | \mathcal{F} \) almost surely. By Lemma A.1 from [7], \( A \in L^2(\mathcal{F}) \). Since \( S_r \in \mathcal{F} \), Lemma 4.1 gives

\[
\mathbb{E}[(\log Z_{m,n} - g(b))^j p_k(S_r, a; r))] = \mathbb{E}^a[A(R_{1,0}^1, \ldots, R_{r,0}^1)]
\]

\[
= \frac{\partial^k}{\partial a^k} \mathbb{E}^a[A(R_{1,0}^1, \ldots, R_{r,0}^1)]
\]

\[
= \frac{\partial^k}{\partial a^k} \mathbb{E}[(\log Z_{m,n} - g(b))^j],
\]

where \( \mathbb{E}^a \) emphasizes that we are only taking expectations over \( \{R_{i,0}^1\}_{i=1}^r \). Now fix \( a_0 \) and \( a_1 \) such that \( a \in [a_0, a_1] \subset D(M_f) \). Using Corollary 4.5 Lemma 4.6 and \( t_1 \leq m \), we see that

\[
\tilde{\mathbb{E}} \left[ \sup_{a \in [a_0, a_1]} \left| \frac{\partial^k}{\partial a^k} (\log Z_{m,n}(a) - g(b))^j \right| \right] < \infty.
\]

Thus

\[
\frac{\partial^k}{\partial a^k} \tilde{\mathbb{E}} \left[ (\log Z_{m,n}(a) - g(b))^j \right] = \tilde{\mathbb{E}} \left[ \frac{\partial^k}{\partial a^k} (\log Z_{m,n}(a) - g(b))^j \right]
\]

\[
= \sum_{\ell_i \geq 0} k! \prod_{i=1}^j \tilde{\mathbb{E}} \left[ \prod_{i=1}^j \tilde{\sigma}_{\ell_i}(r) (\log Z_{m,n}(a) - g(b)) \right].
\]

Therefore

\[
\left. \frac{\partial^k}{\partial a^k} \right|_{b = a} = \sum_{\ell_i \geq 0} k! \prod_{i=1}^j \frac{\prod_{i=1}^j}{\ell_1! \cdots \ell_j!} \tilde{\mathbb{E}} \left[ \prod_{i=1}^j \tilde{\sigma}_{\ell_i}(r) \right]
\]

\[
= \sum_{\ell_i \geq 0} k! \prod_{i=1}^j \frac{\prod_{i=1}^j}{\ell_1! \cdots \ell_j!} \left[ \prod_{i=1}^j \sigma_{\ell_i}(r) \right] .
\]

\[ \Box \]
Corollary 4.8. When \( r = m \) and \( k \) is even,
\[
\kappa_k(\log Z_{m,n}) = \kappa_k(E_n) - \kappa_k(S_m) - \sum_{j=1}^{k-1} \binom{k}{j} (-1)^j \kappa(\log Z_{m,n}, \ldots, \log Z_{m,n}, S_m, \ldots, S_m) \text{ and }
\]
\[
= n\kappa_k(\log R^2) - m\kappa_k(\log R^1)
\]
\[
+ \sum_{\pi \in \mathcal{P}} (|\pi| - 1)!(-1)^{|\pi|} \left[ \prod_{B \in \pi} \mathbb{E} \left[ (\log Z_{m,n})^{a_{\pi,B} p_B} (S_m, a; m) \right] \right],
\]
where \( a_{j,B} = |B \cap \{1, \ldots, j\}| \), \( b_{j,B} = |B \cap \{j+1, \ldots, k\}| = |B| - a_{j,B} \). Moreover,
\[
\mathbb{E}[\left( \log Z_{m,n} \right)^p k(S_m, a; m))] = \sum_{\ell_1 + \cdots + \ell_j = k} \frac{k!}{\ell_1! \cdots \ell_j!} \mathbb{E} \left[ \prod_{i=1}^j \sigma_i(t_1) \right].
\]

Note that in the case \( k = 2 \), the formula in the previous corollary coincides with the variance representation in Seppäläinen [22, Theorem 3.7].

5. Estimates for the central moments

Lemma 5.1. Let \( 0 < r \) and put \( S_n = \sum_{i=1}^n g(a, R_{i,0}) \) where \( g(a, R_{i,0}) \) has finite moments of all orders. Recall the notation [2] for the annealed expectation with respect to the polymer environment. Then, for all \( k \in 2\mathbb{N} \) there exist finite constants \( C_k = C_k(a) > 0 \) which are locally bounded in \( a \), such that
\[
E_{m,n}[\left( S_{t_1} - S_{t_1 \wedge r} \right)^k] \leq C_k \left( E_{m,n}[\left( t_1 - t_1 \wedge r \right)^k] + 1 \right) \text{ for all } (m, n) \in \mathbb{N}^2.
\]

Here the centering is with respect to the annealed measure \( E_{m,n} \).

Proof.

\[
E_{m,n} \left[ \left( S_{t_1} - S_{1 \wedge r} \right)^k \right] = E_{m,n} \left[ \sum_{l>\tau} 1_{\{t_1=l\}} \left( S_l - S_{l \wedge r} \right)^k \right]
\]
\[
+ (-1)^k P_{m,n}(t_1 \leq r) E_{m,n}[S_{t_1} - S_{t_1 \wedge r}]^k.
\]

We now treat (49) and (50) separately.

\[
(49) = E \left[ \sum_{l>\tau} \left( S_l - S_{l \wedge r} \right)^k Q_{m,n}(t_1 = l) \right]
\]
\[
\leq E \left[ \sum_{l>\tau} \left( S_l - S_{l \wedge r} \right)^k 1_{\{S_l - S_{l \wedge r} > l - r\}} \right]
\]
\[
+ E \left[ \sum_{l>\tau} (l - r)^k Q_{m,n}(t_1 = l) \right] 
\]
\[
\leq E \left[ \sum_{l=1}^\infty \left( S_l \right)^k 1_{\{S_l > l\}} \right] \quad \text{by reindexing}
\]
\[
+ E_{m,n} \left[ (t_1 - t_1 \wedge r)^k \right].
\]
The last inequality follows from stationarity. Since $S_t$ is an i.i.d. sum of mean zero random variables which have finite moments of all orders,
\[
\mathbb{E} \left[ (S_t)^k \mathbb{1}_{|S_t| > l} \right] \leq \mathbb{E} \left[ (S_t)^{2k} \right]^{\frac{1}{4}} \mathbb{P} \left( |S_t| > l \right)^{\frac{1}{4}} \leq C_k l^{\frac{k}{2}} \mathbb{E} \left[ \frac{(S_t)}{l} \right]^{\frac{k}{2}} \leq C_k l^{-k}
\]
which is summable over $l$.

For equation (50), we repeat the proof in [22, Lemma 4.2]:
\[
E_{m,n}[S_{t_1} - S_{t_1 \wedge r}] = E_{m,n}[(t_1 - t_1 \wedge r)]E[g(a, R_i, 0)] + \mathbb{E} \left[ \sum_{i=\tau+1}^{t_1} g(a, R_i, 0) \right]
\]
(51)
\[
= C E_{m,n}[(t_1 - t_1 \wedge r)] + \sum_{k=1}^{m} \mathbb{E}[Q_{m,n}(t_1 = k) S_k]
\]
\[
= C(E_{m,n}[(t_1 - t_1 \wedge r)] + 1) + \sum_{k=1}^{m} \mathbb{E}[1_{|S_k| > k}] S_k
\]
\[
\leq C(E_{m,n}[(t_1 - t_1 \wedge r)] + 1).
\]
Taking the $k$-th powers and using Jensen’s inequality completes the proof.

Given a random variable $X$ and $p \in [1, \infty)$, we write
\[
\|X\|_{p,\mathbb{E}} := \mathbb{E}[|X|^p]^{\frac{1}{p}},
\]
\[
\|X\|_{p,E_{m,n}} := E_{m,n}[|X|^p]^{\frac{1}{p}}
\]
for the $p$-th norm with respect to the regular expectation $\mathbb{E}$ and the annealed expectation $E_{m,n}$. When $m, n$ is understood we write $E = E_{m,n}$.

**Lemma 5.2.** For every even integer $k \geq 2$ there exists a constant $C_k$ such that whenever $\{X_i\}_{i=1}^k$ are random variables with finite annealed moments, then
\[
\|\kappa^Q_k(X_1, \ldots, X_k)\|_{p,\mathbb{E}} \leq C_k \prod_{i=1}^k \|X_i\|_{p,k,E}
\]
where the centering on the right-hand side is with respect to the annealed measure $E$.

**Proof.** $E_{m,n}[X_i]$ are constants and therefore
\[
\kappa^Q_k(X_1, \ldots, X_k) = \kappa^Q_k(\overline{X}_1, \ldots, \overline{X}_k).
\]
\[
\left| E^Q \left[ \prod_{i \in B} |X_{i}| \right]^p \right| \leq \prod_{i \in B} E^Q \left[ |X_i|^{p|B|} \right]^{\frac{1}{p|B|}} \leq \prod_{i \in B} E^Q \left[ |X_i|^{p|B|} \right]^{\frac{1}{p}}.
\]
Using Hölder’s generalized inequality again,
\[
\mathbb{E} \left[ \prod_{i=1}^k E^Q[|X_i|^{p|B|}]^{\frac{1}{p}} \right] \leq \prod_{i=1}^k \left( \mathbb{E} \left[ E^Q[|X_i|^{p|B|}] \right]^{\frac{1}{p}} \right)^{\frac{1}{p}}.
\]
Taking the $p$-th root and plugging this into (16) yields the desired result with $C_k = (k - 1)!2^k$. 

\[\Box\]
The following allows us to control moments of $\sigma_k(r)$ in terms of annealed moments of the exit time $t_1$.

**Lemma 5.3.** For any $k \in 2\mathbb{N}$, $p \in [1, \infty)$, there exist positive constants $C(k, p)$ such that the following two conditions hold for all $r, M \in \mathbb{N}$:

\[
\|\sigma_k(r)\|_{p, E} \leq C(k, p) \left(1 + \|t_1\|^k\right) \frac{\|t_1\|^M}{rM},
\]

(52)

\[
\|\sigma_k(m) - \sigma_k(r)\|_{p, E} \leq C(k, p) \left(1 + \|t_1\|^k\right) \frac{\|t_1\|^M}{rM}.
\]

(53)

**Proof.** For $\ell, m \in \mathbb{N}$ define

\[
\sum_{i=1}^{m} \sigma_{\ell}(m) := \sum_{i=1}^{m} \sigma_{\ell}(m).
\]

Taking $L_p$ norms of (46), gives

\[
\|\sigma_k(r)\|_{p, E} \leq \sum_{j=1}^{k} \sum_{\ell_i \geq 0, \ell_i = -k-j} C_{k, j, \ell} \|\kappa_j^0(X_{\ell_1}(t_1 \land r), \ldots, X_{\ell_j}(t_1 \land r))\|_{p, E},
\]

and

\[
\|\sigma_k(m) - \sigma_k(r)\|_{p, E} \leq \sum_{j=1}^{k} \sum_{\ell_i \geq 0, \ell_i = -k-j} C_{k, j, \ell} \|\kappa_j^0(X_{\ell_1}(t_1), \ldots, X_{\ell_j}(t_1)) - \kappa_j^0(X_{\ell_1}(t_1 \land r), \ldots, X_{\ell_j}(t_1 \land r))\|_{p, E}.
\]

(55)

Lemma 5.2, Lemma 5.1, equation (54), and Jensen’s inequality give (52). By (55) and a telescoping argument, to obtain (53) it suffices to bound $\|\kappa_j^0(Y_1, \ldots, Y_j)\|_{p, E}$ where

\[
Y_i = \begin{cases} X_i(t_1 \land r) & \text{for } 1 \leq i < s, \\ X_i(t_1) - X_i(t_1 \land r) & \text{for } i = s, \\ X_i(t_1) & \text{for } s < i \leq j, \end{cases}
\]

and $s \in \{1, \ldots, j\}$ is fixed. By Lemma 5.1 and Jensen’s inequality, for $i \neq s$,

\[
\|Y_i\|_{p, E} \leq \|Y_i\|_{p, k, E} \leq C_{2p, k} \left(1 + \|t_1\|_{p, k, E}\right).
\]

(56)

By Jensen’s inequality, the Cauchy-Schwarz inequality, Lemma 5.1 and Markov’s inequality,

\[
\|Y_s\|_{p, E} \leq \|Y_s\|_{p, k, E} = \left[\mathbb{E}[X_s(t_1) - X_s(t_1 \land r)]\right]^{1/2}
\]

\[
\leq \left[\mathbb{E}[\chi_s(t_1 - X_s(t_1 \land r))]\right]^{1/2}
\]

\[
\leq \left[\mathbb{E}[\chi_s(t_1 - X_s(t_1 \land r))]\right]^{1/2}
\]

\[
\leq C_{2p, k} \left(1 + \|t_1 - t_1 \land r\|_{2p, E}\right) \frac{\|t_1\|^M}{rM}.
\]

(54)
In the third to last inequality we again used a slight modification of [22] Lemma 4.2 as in [51]. Another application of Jensen’s inequality along with (56) gives (53).

For the following lemma, recall the notation $\mathbb{P}^{(a_1,a_2)}$ and $\mathbb{E}^{(a_1,a_2)}$ defined in Section 2.3.

**Lemma 5.4.** Assume the polymer environment is as in [12] and the sequence $(m, n) = (m_N, n_N)_{N=1}^\infty$ satisfies

$$|m - N\psi_1^2(a_2)| \vee |n - N\psi_1^1(a_1)| \leq \gamma N^{1/2}$$

where $\gamma$ is some positive constant. Then there exist finite positive constants $C_1, C_2, C_3, \delta, \delta_1, b$ (uniformly bounded in $(a_1, a_2)$) such that for all $N \in \mathbb{N}$ the following two bounds hold simultaneously for $j = 1, 2$: for all $C_1N^{1/2} \leq u \leq \delta N$,

$$\mathbb{P}^{(a_1,a_2)}[Q_{m,n}(t_j \geq u) \geq e^{-\frac{su^2}{2}}] \leq C_2 \left( \frac{N^k}{u^2} \left( \mathbb{E}^{(a_1,a_2)}[(\log Z_{m,n})^k] + \mathbb{E}^{(a_1(\lambda_j), a_2(\lambda_j))}[(\log Z_{m,n})^k] \right) \right)$$

where $a_1(\lambda) := a_1 + \lambda$, $a_2(\lambda) = a_2 - \lambda$, $\lambda_1 = \frac{bN}{n}$, and $\lambda_2 = -\frac{bN}{n}$, while for $u \geq \delta N$,

$$\mathbb{P}^{(a_1,a_2)}[Q_{m,n}(t_j \geq u) \geq e^{-\delta u}] \leq 2e^{-C_3}.$$

**Proof.** Follow the proof of Proposition 4.3 in [7] verbatim up to the displayed inequality

$$(4.8) \leq \mathbb{P} \left[ \log Z_{m,n}(a_1(\lambda_j), a_2(\lambda_j)) - \log Z_{m,n}(a_1, a_2) \geq C_{m}u^2 \right],$$

where (4.8) refers to the corresponding equation in [7].

Now rather than bounding by the second moment, bound by the $k$-th moment to get

$$(4.8) \leq \left( \frac{N}{C_{m}u^2} \right)^k \mathbb{E} \left[ \left( \log Z_{m,n}(a_1(\lambda_j), a_2(\lambda_j)) - \log Z_{m,n}(a_1, a_2) \right)^k \right].$$

$$\leq C_2 \frac{N^k}{u^2k} \left( \mathbb{E}^{(a_1,a_2)}[(\log Z_{m,n})^k] + \mathbb{E}^{(a_1(\lambda_j), a_2(\lambda_j))}[(\log Z_{m,n})^k] \right).$$

The proof of the second part is just as in Proposition 4.3 of [7].

**Corollary 5.5.** Let $k \geq 2$. Suppose there exist $\delta, \epsilon_0 > 0$ such that $[a_1 - \epsilon_0, a_1 + \epsilon_0] \times [a_2 - \epsilon_0, a_2 + \epsilon_0] \subseteq D(M_{f_1}) \times D(M_{f_2})$ and the following holds for every $N \in \mathbb{N}$ and every $\lambda \in [-\epsilon_0, \epsilon_0]$:

$$(57) \quad \mathbb{E}^{(a_1(\lambda), a_2(\lambda))} \left[ \left( \log Z_{m,n}(\lambda) \right)^k \right] \leq CN^{(\frac{4}{k})k+\delta k}$$

where $a_1(\lambda) = a_1 - \lambda$, and $a_2(\lambda) = a_2 + \lambda$. Then, for all $\epsilon > 0$ there exists a positive constant $C' = C'(\epsilon, k, a_1, a_2)$ such that the following bound holds for every $N \in \mathbb{N}$ and every $\lambda \in [-\frac{a_0}{2}, \frac{a_0}{2}]$:

$$\mathbb{E}^{(a_1(\lambda), a_2(\lambda))} \left[ (t_j)^{2k} \right] \leq C' N^{(\frac{4}{k})k+\delta k+\epsilon}$$

for both $j = 1, 2$.

Here $\mathbb{E}^{(a_1,a_2)}$ denotes the annealed expectations with respect to the measure on paths in the environment [12].

**Proof.** We apply Lemma 5.4 and use the same notation as in that lemma for constants. Fix $\lambda_0 \in [-\frac{a_0}{2}, \frac{a_0}{2}]$ and put $(\tilde{a}_1, \tilde{a}_2) = (a_1(\lambda_0), a_2(\lambda_0)) \in D(M_{f_1}) \times D(M_{f_2})$. Note that $\tilde{a}_1 + \tilde{a}_2 = \lambda_0$. Then

$$(57) \quad \mathbb{E}^{(a_1(\lambda), a_2(\lambda))} \left[ \left( \log Z_{m,n}(\lambda) \right)^k \right] \leq CN^{(\frac{4}{k})k+\delta k}$$
\[ a_1 + a_2 = a_3 \text{ (see (12)). So by Lemma 5.4 and (37) there exist positive constants } N_0 = N_0(\epsilon_0) \in \mathbb{N}, \ C_1 = C_1(\epsilon_0), \delta = \delta(\epsilon_0) \text{ such that for all } N \geq N_0,
\]
\[
E^{(a_1, a_2)} [(t_j)^{2k}] \leq \left( C_1 N^{\frac{\delta}{2}} \right)^{2k} + (2k)(\epsilon N)^{\epsilon} \int_{C_1 N^{\frac{\delta}{2}}}^{C_1 N^{\frac{\delta}{2}}} u^{2k-1-\epsilon} P^{(a_1, a_2)}(t_j > u) du + C'(\delta, \delta_1, C_3, N_0)
\]
\[
\leq \left( C_1 N^{\frac{\delta}{2}} \right)^{2k} + (2k)(\delta N)^{\epsilon} C C_2 N^{(\frac{\delta}{4})k+\delta k} \int_{C_1 N^{\frac{\delta}{2}}}^{C_1 N^{\frac{\delta}{2}}} u^{-1-\epsilon} du
\]
\[
\leq C(\epsilon, k, \epsilon_0) N^{(\frac{\delta}{4}+\delta k+\epsilon)}.
\]

\[
\Box
\]

**Lemma 5.6.** Assume the polymer environment is distributed as in (12) and the sequence \((m, n) = (m_N, n_N)\) satisfies

\[
|m - N \psi^{f_2} (a_2)| \lor |n - N \psi^{f_1} (a_1)| \leq \gamma N^{\frac{\delta}{2}}
\]

where \(\gamma\) is some positive constant. Further, suppose there exist positive constants \(\delta, \epsilon_0, \{C_k\}_{k=1}^\infty\) such that \([a_1 - \epsilon_0, a_1 + \epsilon_0] \times [a_2 - \epsilon_0, a_2 + \epsilon_0] \subset D(M_f1) \times D(M_f2)\) and the following hold for every \(k, N \in \mathbb{N}\) and every \(\lambda \in [-\epsilon_0, \epsilon_0]:\)

\[
E^{(a_1(\lambda), a_2(\lambda))} \left[ \left( \log Z_{m,n} \right)^k \right] \leq C_k N^{(\frac{\delta}{4}+\delta k)}.
\]

Then for all \(\epsilon > 0, M > 0\), there exist positive constants \(\{C_{j,l} = C_{j,l}(a_1, a_2, \epsilon, \delta, M)\}_{j,l=1}^{2r}\) (locally bounded in \(a_1, a_2\)) such that for all \(N \in \mathbb{N}\) we have the following:

\[
\left| E \left[ \left( \log Z_{m,n} \right)^j p_l(S_m, a_1; m) \right] - \left( \log Z_{m,n} \right)^j p_l(S_\tau, a_1; \tau) \right| \leq C_{j,l} N^{-M},
\]

where \(S_\tau = \sum_{i=1}^r \log R_{i,0}^1\) and \(\tau = N^{(\frac{\delta}{4}+\epsilon+r)}\).

**Proof.** By (46), for all \(0 \leq r \leq m,\)

\[
E \left[ \left( \log Z_{m,n} \right)^j p_k(S_\tau, a_1; \tau) \right] = \sum_{\ell_1 + \cdots + \ell_j = k} \frac{k!}{l_1! \cdots l_j!} E \left[ \prod_{i=1}^j \sigma_{\ell_i}(r) \right]
\]

where \(\sigma_0(r) = \log Z_{m,n}\). It will therefore suffice to compare \(\sigma_r(m)\) with \(\sigma_r(r)\). Specifically, for fixed \(\ell_1, \ldots, \ell_j\), such that \(\sum_{i=1}^j \ell_i = k\), we wish to estimate

\[
E \left[ \prod_{i=1}^j \sigma_{\ell_i}(m) - \prod_{i=1}^j \sigma_{\ell_i}(r) \right].
\]

By a telescoping argument it suffices to bound

\[
E \left[ \sigma_{\ell_a}(m) \prod_{\ell \in I_1} \sigma_{\ell_i}(m) \prod_{\ell \in I_2} \sigma(r) \right] - E \left[ \sigma_{\ell_a}(r) \prod_{\ell \in I_1} \sigma_{\ell_i}(m) \prod_{\ell \in I_2} \sigma(r) \right]
\]

where \(a \in \{1, 2, \ldots, j\}\) is such that \(\ell_a \neq 0, I_1 = \{1, \ldots, a-1\}, I_2 = \{a+1, \ldots, j\},\) and \(\sum_{i=1}^j \ell_i = k\).

By the generalized Hölder inequality this is bounded by

\[
\| \sigma_{\ell_a}(m) - \sigma_{\ell_a}(r) \|_{2,E} \prod_{\ell \in I_1} \| \sigma_{\ell_i}(m) \|_{2(j-1),E} \prod_{\ell \in I_2} \| \sigma_{\ell_i}(r) \|_{2(j-1),E}
\]
Let \( I_0 = \{ 1 \leq i \leq j : \ell_i = 0 \} \). By Lemma 5.3 for any \( r, M \in \mathbb{N} \),
\[
[59] \leq C(\ell, 2) \left( 1 + \|(t_1)^\ell\|_{4,E} \right) \frac{\|(t_1)^M \|_{4,E}}{r^M} \log Z_{m,n} \prod_{i \in I_0} C(\ell_i, 2(j-1)E) \left( 1 + \|(t_1)^\ell\|_{2(j-1)E} \right).
\]
Using the assumption [58], by Corollary 5.5 for any \( \epsilon > 0 \) there exists a constant \( C(\epsilon, p) \geq 1 \) (uniformly bounded in \( (a_1, a_2) \)) such that
\[
\|\log Z_{m,n}\|_{p,E} \leq C(p, \epsilon) N^{(\frac{1}{2}+\delta)} \quad \text{and} \quad \|\ell_1\|_{p,E} \leq C(\ell, p, \epsilon) N^{(\frac{1}{2}+\delta)\epsilon + \frac{\epsilon}{\delta}}.
\]
This implies the existence of positive constants \( C' = C'(k, j, M) \) such that for all \( M \in \mathbb{N} \) and all \( N \in \mathbb{N} \),
\[
[60] \leq C' N^{(\frac{2}{3}+\frac{4}{3})j\ell_a + \frac{2}{3} \cdot N^{(\frac{2}{3}+\frac{4}{3})M + \frac{\epsilon}{\delta} r, -M} \cdot N^{(\frac{1}{2}+\delta)\epsilon_l} \prod_{i \in I_0} N^{(\frac{2}{3}+\frac{4}{3})}.
\]
Choosing
\[
r = |\tau| \geq N^{\frac{2}{3}+\frac{4}{3}+\epsilon} - 1,
\]
we obtain the bound
\[
C' N^{(\frac{2}{3}+\delta)(j-1) + (\frac{2}{3}+\frac{4}{3})k + 2\epsilon - M_0 \epsilon} \leq -K.
\]

Before proceeding to Lemma 5.7, we note the following property of the polynomials \( p_n(T, a; r) \) introduced in (19):

**Proposition 5.1.** For each \( n \),
\[
p_n(t, a; r) = \sum_{j=0}^{n} c_j(a)(t - r\psi_0(a))^{a_j b_j},
\]
where \( c_j(a) \) are independent of \( r \) and \( 0 \leq a_j, b_j \leq n \) are integers with
\[
\frac{a_j}{2} + b_j = \frac{n}{2}.
\]
In particular, if \( T = \sum_{k=1}^{r} \log X_k \) where \( X_k \sim m_f(a) \), then we have for integers \( b, k \geq 0 \)
\[
\mathbb{E}[|p_n(T, a; r)|^k] \leq C_{b, k} r^{kb/2}.
\]
**Proof.** The result is clearly true for \( p_0(T, a; r) \). Next, we note that if \( a_j, b_j \) satisfy (61), then
\[
\begin{align*}
\frac{\partial}{\partial a}(t - r\psi_0(a))^{a_j b_j} &= (t - r\psi_0(a))^{a_j b_j} \cdot (t - r\psi_0(r)) \\
&= -a_j \psi_1(a)(t - r\psi_0(a))^{a_j - 1} b_j + (t - r\psi_0(a))^{a_j} b_j.
\end{align*}
\]
Noting that
\[
\frac{a_j - 1}{2} + b_j + 1 = \frac{a_j + 1}{2} + b_j = \frac{n + 1}{2},
\]
the claim follows by induction from the definition (19). \(\square\)
Lemma 5.7. With the same assumptions as in Lemma 5.6, for all $k \in \mathbb{N}$ there exist positive constants $C_k = C_k(a_1, a_2)$ (locally bounded) such that for all even $k \geq 2$,

$$E[(\log Z_{m,n})^k] \leq C_k N^{(\frac{1}{3} + \frac{\delta}{2})k} \quad \text{for all } N \in \mathbb{N}. \tag{63}$$

Proof. The proof is by induction on $k$. For $k = 2$, (63) holds with $\delta = 0$. Assuming the estimate for even exponents less than $k$, we use the first expression in Corollary 4.8 to express the cumulant $\kappa_k(\log Z_{m,n})$ as a sum of terms of the form

$$\prod_{B \in \pi} E\left[(\log Z_{m,n})^{a_{j,B} B_j,B} (S_m, a_1; m)\right], \tag{64}$$

where $\pi$ is a partition of $\{1, \ldots, k\}$ into $|\pi|$ blocks $B$, and $a_{j,B} + B_j,B = |B|$. Using equation (46) and Lemma 5.6 with $K > 2k$, we have, for $\tau = \eta^{2/3+\delta/2+\epsilon}$,

$$\prod_{B \in \pi} E\left[(\log Z_{m,n})^{a_{j,B} B_j,B} (S_m, a_1; m)\right] = \prod_{B \in \pi} E\left[(\log Z_{m,n})^{a_{j,B} B_j,B} (S \tau, a_1; \tau)\right] + O(n^{-k}). \tag{65}$$

Taking absolute values and applying Hölder’s inequality,

$$|E[(\log Z_{m,n})_{a_{j,B} B_j,B} (S \tau, a_1; \tau)]| \leq E[(\log Z_{m,n})^{a_{j,B} B_j,B}]^{a_j} E[(p_{B_j,B} (S \tau, a_1; \tau))^{b_j,B}]^{b_j,B}, \tag{66}$$

where $\frac{a_j}{a_j} + \frac{b_j}{b_j} = 1$. The last inequality follows from equation (62) in Proposition 5.1. Taking the product over $B \in \pi$, we have, up to a constant factor, the bound:

$$n^{(1/3+\delta/2+\epsilon)} E[(\log Z_{m,n})^{a_j}]^{a_j} \tag{67}$$

where

$$a_j := \sum_B a_{j,B} \quad \text{and} \quad b_j := \sum_B b_{j,B},$$

so $\frac{a_j}{a_j} + \frac{b_j}{b_j} = 1$. Note that for $1 \leq j \leq k - 1$, we have $a_j \leq k - 1$. Applying Young’s inequality $xy \leq \frac{1}{p} x^p + \frac{1}{q} y^q$ to (65), we find that for $\eta > 0$, any term of the form (64) is bounded by

$$\eta E[(\log Z_{m,n})^{a_j}] + C(\eta) n^{(1/3+\delta/2+\epsilon/2)k} + O(n^{-k}).$$

Combining this with Corollary 4.8, we have

$$\kappa_k(\log Z_{m,n}) = C(k) \eta E[(\log Z_{m,n})^{a_j}] + C(k) C(\eta) n^{(1/3+\delta/2+\epsilon/2)k} + O(n^{-k}). \tag{66}$$

Writing

$$\kappa_k(\log Z_{m,n}) = E[(\log Z_{m,n})^{a_j}] + \sum_{|\alpha| = k} \sum_{0 \leq \alpha_i < k} c_\alpha \prod_i E[(\log Z_{m,n})^{\alpha_i}], \tag{67}$$

where the sum is over multi-indices $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\sum_\alpha \alpha_i = k$. If some $\alpha_i = k - 1$, then the product must equal zero. Therefore, by the induction assumption, all terms in the sum on the right of (67) are of order $\eta^{(1/3+\delta/3)k}$. Choosing $\eta$ sufficiently small in (66) and absorbing $\epsilon/2$ into $\delta/4$, we obtain the result. \qed
5.1. Finishing the argument. Combining Corollary 5.5 and Lemma 5.6 we obtain the following:

**Lemma 5.8.** Assume the polymer environment is distributed as in (12) and the sequence \((m, n) = (m_N, n_N)_{N=1}^\infty\) satisfies
\[
|m - N\psi_1^2(a_2)| \lor |n - N\psi_1^f(a_1)| \leq \gamma N^{\frac{3}{2}}
\]
where \(\gamma\) is some positive constant. Further, suppose there exist positive constants \(\delta, \epsilon_0, C(k)\) for \(k \in \{2, 4, \ldots\}\) such that \([a_1 - \epsilon_0, a_1 + \epsilon_0] \times [a_2 - \epsilon_0, a_2 + \epsilon_0] \subseteq D(M_{f_1}) \times D(M_{f_2})\) and the following hold for any even \(k\) and any \(\lambda \in [\epsilon_0, \epsilon_0]\):
\[
\mathbb{E}^{(a_1(\lambda), a_2(\lambda))} \left[ \left( \log \frac{Z_{m, n}}{N} \right)^k \right] \leq C(k)N^{(\frac{3}{4} + \delta)k}.
\]
Then there exist constants \(C'(k) > 0\) for \(k \in \{2, 4, \ldots\}\) such that for any even \(k\) and any \(\lambda \in [\epsilon_0, \epsilon_0]\):
\[
\mathbb{E}^{(a_1(\lambda), a_2(\lambda))} \left[ \left( \log \frac{Z_{m, n}}{N} \right)^k \right] \leq C'(k)N^{(\frac{3}{4} + \delta)k}.
\]

Theorem 1 will follow from repeated application of Lemma 5.8 once we prove the following:

**Proposition 5.2.** Assume the polymer environment is distributed as in (12) and the sequence \((m, n) = (m_N, n_N)_{N=1}^\infty\) satisfies
\[
|m - N\psi_1^2(a_2)| \lor |n - N\psi_1^f(a_1)| \leq \gamma N^{\frac{3}{2}}
\]
where \(\gamma\) is some positive constant. Then there exists positive constants \(\epsilon_0\) and \(C(k)\) for \(k \in \{2, 4, \ldots\}\) such that \([a_1 - \epsilon_0, a_1 + \epsilon_0] \times [a_2 - \epsilon_0, a_2 + \epsilon_0] \subseteq D(M_{f_1}) \times D(M_{f_2})\) and the following hold for any even \(k\) and any \(\lambda \in [\epsilon_0, \epsilon_0]\):
\[
\mathbb{E}^{(a_1(\lambda), a_2(\lambda))} \left[ \left( \log \frac{Z_{m, n}}{N} \right)^k \right] \leq C(k)N^{(\frac{3}{4} + \delta)k}.
\]

**Proof.** Since \((a_1, a_2) \in D(M_{f_1}) \times D(M_{f_2})\), there exists a positive constant \(\epsilon_0\) such that \([a_1 - \epsilon_0, a_1 + \epsilon_0] \times [a_2 - \epsilon_0, a_2 + \epsilon_0] \subseteq D(M_{f_1}) \times D(M_{f_2})\). With notation as in Subsection 3.1, if we define \(A := \log Z_{m, n}\), then \(A = S_m + E_n\). Thus, for even \(k\),
\[
\mathbb{E}[A^k] \leq 2^{k-1}(\mathbb{E}[S_m^k] + \mathbb{E}[E_n^k]).
\]
By Proposition 3.1 all four models described by (12) have the down-right property. So by the discussion in Subsection 3.1, all random variables whose common distributions continuously depends on \(a_1\) and \(a_2\) respectively. Moreover, by Remark 2 all random inside of the summations have finite exponential moments. Therefore, for every \(k \in \{2, 4, \ldots\}\) there exists a positive constant \(C_k = C_k(a_1, a_2)\), which is continuous in \((a_1, a_2)\), such that
\[
\mathbb{E}[S_m^k] \leq C_k m^{k/2} \quad \text{for all} \quad m \geq 1
\]
and
\[
\mathbb{E}[E_n^k] \leq C_k n^{k/2} \quad \text{for all} \quad n \geq 1.
\]
Using equations (70) and (68) now yields the desired result. □

**Proof of Theorem 1.** The four basic beta-gamma models (3)-(6) can all be described by equation (12). So let \(\epsilon > 0\) and \((a_1, a_2) \in D(M_{f_1}) \times D(M_{f_2})\). Fix even integers \(k\), \(M\) such that \(p \leq k\) and
\[
\frac{(1/6)}{3^M} \leq \epsilon.
\]
By Jensen's inequality, it suffices to show the bounds (9) and (10) hold with $p$ replaced by $k$. Now apply Proposition 5.2 followed by $M$ consecutive applications of Lemma 5.8 to obtain the bound (9). Finally, apply Corollary 5.5 to both $t_1$ and $t_2$ to obtain the bound (10).

**Appendix A. Combinatorial formula for cumulants**

Here we derive the formula (10) for the joint cumulants. This identity is classical and appears on Wikipedia under *Cumulants* [20], but we could not locate a suitable proof to cite.

We will prove the following by induction. Denote
\[ Z := \mathbb{E}[e^{\sum_{i=1}^n \xi_i X_i}], \]
\[ E[\cdot] := \frac{1}{Z} \mathbb{E}[e^{\sum_{i=1}^n \xi_i X_i} \cdot]. \]

Note that for $k \leq n$
\[ \kappa_k(X_1, \ldots, X_k) = \hat{\partial}_{\xi_1} \cdots \hat{\partial}_{\xi_k} \log Z|_{\xi_1=\ldots=\xi_n=0}. \]

We will show by induction that
\[ \partial_{\xi_1} \cdots \partial_{\xi_k} \log Z = \sum_{\pi \in \mathcal{P}(1,\ldots,k)} ((|\pi| - 1)!(-1)^{|\pi| - 1} \prod_{B \in \pi} E[\prod_{i \in B} X_i]). \]

**Proof.** Note that the result holds for $k = 1$. Indeed, this case
\[ \partial_{\xi_1} \log Z = E[X_1]. \]

Assume the result for $k \leq n - 1$. We prove the result for $k + 1$. Differentiating (71), we obtain for each $\pi \in \mathcal{P}(1,\ldots,k)$ appearing in the sum (71):
\[ \partial_{\xi_{k+1}} \prod_{B \in \pi} E[\prod_{i \in B} X_i] = \sum_{B' \in \pi} \partial_{\xi_{k+1}} \prod_{B' \in \pi} E[\prod_{i \in B'} X_i] \prod_{j \in B' \setminus B} E[\prod_{i \in B \setminus j} X_i]. \]

For the derivative, we have
\[ \partial_{\xi_{k+1}} E[\prod_{j \in B'} X_j] = \partial_{\xi_{k+1}} \frac{1}{Z} \mathbb{E}[e^{\sum_{i=1}^n \xi_i X_j} \prod_{j \in B'} X_j] \]
\[ = E[X_{k+1} \prod_{j \in B'} X_j] - E[X_{k+1}] E[\prod_{j \in B'} X_j]. \]

Thus, we have
\[ \partial_{\xi_{k+1}} \prod_{B \in \pi} E[\prod_{i \in B} X_i] = \sum_{B' \in \pi} E[X_{k+1} \prod_{j \in B'} X_j] \prod_{j \in B' \setminus B} E[\prod_{i \in B \setminus j} X_i] \]
\[ - \sum_{B' \in \pi} E[X_{k+1}] E[\prod_{j \in B'} X_j] \prod_{j \in B' \setminus B} E[\prod_{i \in B \setminus j} X_i] \]
\[ = \sum_{B' \in \pi} E[X_{k+1} \prod_{j \in B'} X_j] \prod_{j \in B' \setminus B} E[\prod_{i \in B \setminus j} X_i] \]
\[ - |\pi| E[X_{k+1}] \prod_{B \in \pi} E[\prod_{i \in B} X_i]. \]

The first term corresponds to adding a factor $X_{k+1}$ to a single $B$ block of the partition $\pi$ and the second term corresponds to adding a 1-term block $\{k+1\}$ to $\pi$. Summing (72) over $\pi \in \mathcal{P}(1,\ldots,k)$,
we obtain
\[ \hat{c}_{\xi_{k+1}} \sum_{\pi \in \mathcal{P}(1, \ldots, k)} (|\pi| - 1)!(-1)^{|\pi| - 1} \prod_{B \in \pi} E[\prod_{i \in B} X_i] \]
\[ = \sum_{\pi \in \mathcal{P}(1, \ldots, k)} (|\pi| - 1)!(-1)^{|\pi| - 1} \sum_{B' \in \pi} E[X_{k+1} | \prod_{j \in B'} X_j \prod_{B \neq B'} E[\prod_{i \in B} X_i] \]
\[ - \sum_{\pi \in \mathcal{P}(1, \ldots, k)} |\pi|!(|\pi| - 1)^{\mathcal{P}(1, \ldots, k)} - 1 \prod_{B \in \pi} E[\prod_{i \in B} X_i] \]
\[ = \sum_{\pi \in \mathcal{P}(1, \ldots, k+1)} (|\pi| - 1)!(-1)^{|\pi| - 1} \prod_{B \in \pi} E[\prod_{i \in B} X_i]. \]

To verify the final step, note that any partition of \{1, \ldots, k+1\} which contains \{k+1\} as a single element block induces a partition \(\pi\) of \{1, \ldots, k\} from the remaining blocks with \(|\pi| = |\tilde{\pi}| - 1\); otherwise, if \(\{k+1\}\) does not appear as block in \(\tilde{\pi}\), the partition can be obtained from some \(\pi \in \mathcal{P}(1, \ldots, k)\) by adding \(k+1\) to one of the \(|\pi|\) blocks without changing the number of blocks. \(\square\)

**APPENDIX B. PROOF OF THEOREM**

The next lemma says that it suffices to verify Hypothesis 1 for \(f(x) = e^{-bx}\), \(f(x) = (1 - x)^{b-1}\mathbb{1}_{0<x<1}\), and \(f(x) = \left(\frac{x}{1+x}\right)^b\) where \(b > 0\).

For \(A \subset \mathbb{R}\) write \(-A = \{-a : a \in A\}\) and \(A^{-1} = \{a^{-1} : a \in A\}\) assuming that \(0 \notin A\).

**Lemma B.1.** If the function \(f\) satisfies Hypothesis 1 then so does the function \(g(x) := f(\frac{1}{x})\) for \(x \in (0, \infty)\), with the same constants \(C_j(a)\).

**Proof.** Recall the notation in Sections 2.2 and 4.2. Clearly \(\text{supp}(g) = \text{supp}(f)^{-1}\) and \(D(M_f) = -D(M_g)\). As in the proof of [17, Lemma A.1], one can verify that:

\[ F^g(a, x) = 1 - F^f(-a, \frac{1}{x}) \quad \text{for } (a, x) \in D(M_g) \times \text{supp}(g) \]
\[ L^g(a, x) = L^f(-a, \frac{1}{x}) \quad \text{for } (a, x) \in D(M_g) \times \text{supp}(g) \]
\[ H^g(a, p) = \frac{1}{H^f(-a, 1-p)} \quad \text{for } (a, p) \in D(M_g) \times (0, 1). \]

Combining the last two equalities gives
\( L^g(a, H^g(a, p)) = L^f(-a, H^f(-a, 1-p)) \quad \text{for } (a, p) \in D(M_g) \times (0, 1). \)

Recall also the definition of the derivative \(\hat{\partial}\) in (37). We write \(\hat{\partial}_f\) and \(\hat{\partial}_g\) to denote the dependence on the underlying function. Recall that
\[ \hat{\partial}_g^k L^g(a, H^g(a, p)) = \frac{\hat{\partial}_a^k}{\hat{\partial}^k} L^g(a, H^g(a, p)) \quad \text{for all } (a, p) \in D(M_g) \times \text{supp}(g). \]

Applying \(\frac{\hat{\partial}_a^k}{\hat{\partial}^k}\) to equation (74) gives
\[ \frac{\hat{\partial}_a^k}{\hat{\partial}^k} L^g(a, H^g(a, p)) = (-1)^k \frac{\hat{\partial}_b^k}{\hat{\partial}^k} \left( L^f(b, H^f(b, 1-p)) \right) \bigg|_{b=a} \]
\[ = (-1)^k \hat{\partial}_f^k L^f(-a, H^f(-a, 1-p)), \]
Lemma B.2. Assume \( x = H^g(a, p) \) and using equation (73), we get
\[
\tilde{\partial}^k h^g(a, x) = (-1)^k \tilde{\partial}^k f^g(-a, 1 - p) \quad \text{for all } (a, p) \in D(M_g) \times (0, 1).
\]

Making the substitution \( x = H^g(a, p) \) and using equation (73), we get
\[
\tilde{\partial}^k h^g(a, x) = (-1)^k \tilde{\partial}^k f^g(-a, 1) \quad \text{for all } (a, x) \in D(M_g) \times \text{supp}(g).
\]

Taking absolute values and using the fact that \(|\log x| = |\log \frac{1}{x}| \) completes the proof. \(\square\)

Write \( C^\infty(A) \) for the set of smooth functions defined on a set \( A \). For a fixed \( f \) with non-empty \( D(M_f) \) and which is smooth on its open support, define the linear transformations \( T \) and \( S \) on \( C^\infty(D(M_f) \times \text{supp}(f)) \) by
\[
T(h)(a, x) := \frac{1}{x^a f(x)} \int_0^x h(a, y) y^{a-1} f(y) dy
\]
\[
S(h)(a, x) := \frac{\partial h}{\partial a}(a, x) + h(a, x) \log x
\]
for \( h \in C^\infty(D(M_f) \times \text{supp}(f)) \) and \( (a, x) \in D(M_f) \times \text{supp}(f) \). Notice that when \( h(a, x) = \psi^f_0(a) - \log x \), \( T(h) = \tilde{L}^f \). Notice that \( \tilde{\partial}^k \) in (37) is also a linear transformation on \( C^\infty(D(M_f) \times \text{supp}(f)) \).

The following lemma gives a useful recursion for \( \tilde{\partial}^k L^f \):

**Lemma B.2.** Assume \( f : (0, \infty) \to [0, \infty) \) has non-empty \( D(M_f) \), open \( \text{supp}(f) \), and satisfies \( f \in C^\infty(\text{supp}(f)) \). If \( h \in C^\infty(D(M_f) \times \text{supp}(f)) \), then for \( (a, x) \in D(M_f) \times \text{supp}(f) \), and
\[
(\tilde{\partial} T(h))(a, x) = T \circ S(h)(a, x) - \left[ \left( a + x \frac{f'(x)}{f(x)} \right) L^f(a, x) + \log x \right] T(h)(a, x) + h(a, x) L^f(a, x).
\]

Moreover, if there exists an integer \( k \geq 1 \) and a constant \( C = C(a_0, a_1) > 0 \) such that
\[
\sup_{a \in [a_0, a_1]} \left| \frac{\partial h}{\partial a}(a, x) \right| \leq C \left( 1 + |\log x|^k \right) \quad \text{for all } x \in \mathbb{R},
\]
then
\[
\int_0^\infty h(a, y) y^{a-1} f(y) dy = 0 \Rightarrow \int_0^\infty S(h)(a, y) y^{a-1} f(y) dy = 0.
\]

**Proof.** A computation yields
\[
\frac{\partial T(h)}{\partial a}(a, x) = -\log x \cdot T(h)(a, x) + T \circ S(h)(a, x)
\]
\[
\frac{\partial T(h)}{\partial x}(a, x) = \left( -\frac{a}{x} - \frac{f'(x)}{f(x)} \right) T(h)(a, x) + \frac{h(a, x)}{x},
\]
which gives the first part. For the second part, by Remark 2 in Section 2.2, \(|\log X|\) has finite exponential moments. We can therefore exchange the derivative with the integral in the expression
\[
\frac{\partial}{\partial a} \int_0^\infty h(a, y) y^{a-1} f(y) dy.
\]
\(\square\)
For $a \in D(M_f)$ and $x > 0$, recursively define

$$
\begin{align*}
& h_1(a, x) := \psi'_0(a) - \log x \quad \text{and} \\
& h_n(a, x) := S(h_{n-1})(a, x) \quad \text{for } n \geq 2.
\end{align*}
$$

Then $h_n(a, x)$ is an $n$-th degree polynomial in $\log x$ with coefficients that are smooth in $a$. Thus, there exist constants $C_n > 0$ for $n = 1, 2, \ldots$ such that

$$
\sup_{a \in [a_0, a_1]} \left| \frac{\partial h_n}{\partial a} (a, x) \right| \leq C_n (1 + |\log x|^n) \quad \text{for all } x > 0.
$$

By the second part of Lemma B.2

$$
\int_0^\infty h_n(a, x)y^{a-1}f(y)dy = 0 \quad \text{for all } n \in \mathbb{N} \text{ and } a \in (a_0, a_1).
$$

The functions $h_n$ will serve as a basis generating all functions obtainable from $L^f$ through repeated application of the operation $\hat{\partial}$. To proceed we define some algebraic structures.

Given a subset $F \subset C^\infty(D(M_f) \times \text{supp}(f))$, define $\mathcal{A}(F) \subset C^\infty(D(M_f) \times \text{supp}(f))$ to be the algebra generated by $F$ over the ring $C^\infty(D(M_f))$. More specifically, $g \in \mathcal{A}(F) \Leftrightarrow$ there exist $c_1, \ldots, c_n \in C^\infty(D(M_f))$ and $g_1, \ldots, g_n \in C^\infty(D(M_f) \times \text{supp}(f))$ such that

$$
g(a, x) = \sum_{i=1}^n c_i(a)g_i(a, x) \quad \text{for all } (a, x) \in D(M_f) \times \text{supp}(f).
$$

Now let

$$
r(x) := \frac{x f'(x)}{f(x)}
$$

for $x \in \text{supp}(f)$ and put

$$
F := \{ \log x, T(h_n)(a, x), r(x)T(h_n)(a, x) : n \in \mathbb{N} \}.
$$

Define the degree function $\text{deg} : F \to \mathbb{Z}_+$ by

$$
\text{deg}(\log x) = 1 \quad \text{and } \text{deg}(T(h_n)) = \text{deg}(rT(h_n)) = n \quad \text{for } n \in \mathbb{N}.
$$

Extend the degree function to $\mathcal{A}(F)$ by defining

$$
\text{deg}(c) := 0, \quad \text{deg}(g \cdot h) := \text{deg}(g) + \text{deg}(h), \quad \text{and } \text{deg}(g + h) := \max(\text{deg}(g), \text{deg}(h))
$$

for $c \in C^\infty(D(M_f))$, and non-zero $f, g \in C^\infty(D(M_f) \times \text{supp}(f))$. Note that this turns $\text{deg}$ into an algebra homomorphism from $\mathcal{A}(F)$ into $\mathbb{Z}_+$ with the $(+, \max)$ algebra. For $n \in \mathbb{N}$, let $A_n := \{ g \in \mathcal{A}(F) : \text{deg}(g) \leq n \}$. Note that $A_n$ is linear.

**Lemma B.3.** Suppose $\text{supp}(f)$ is non-empty and open, $D(M_f)$ is non-empty, $f$ is smooth on its support, and the following two statements hold for every $n \in \mathbb{N}$:

1. $\hat{\partial}r \cdot T(h_n) \in A_{n+1}$, and
2. there exists some $C_n \in C(D(M_f))$ such that

$$
(1 + |r(x)|) |T(h_n)(a, x)| \leq C_n(a) (1 + |\log x|^n) \quad \text{for all } (a, x) \in D(M_f) \times \text{supp}(f).
$$

Then $f$ satisfies Hypothesis \[1\]
Proof. We first claim that

\[ \text{(79)} \quad \tilde{\partial} \text{ maps } A_n \to A_{n+1} \text{ for all } n \in \mathbb{N}. \]

To see this, notice that \( \tilde{\partial} \) satisfies a product rule:

\[ \tilde{\partial}(g \cdot h) = (\tilde{\partial}g) \cdot h + g \cdot (\tilde{\partial}h), \]

and it maps \( C^\infty(D(M_f)) \to C^\infty(D(M_f)) \). Thus, to show \( \text{(79)} \), it suffices to show \( \tilde{\partial}(\log x) \in A_1 \) and for all \( n \in \mathbb{N} \), \( \tilde{\partial}(T(h_n)) \) and \( \tilde{\partial}(r \cdot T(h_n)) \) are in \( A_{n+1} \).

Clearly, \( \tilde{\partial}(\log x) = L^f = T(h_1) \in F \) has degree 1 by definition, so it is in \( A_1 \). By Lemma B.2

\[ \tilde{\partial}(T(h_n)) = T(h_{n+1}) - [(a - r(x))L^f + \log x] T(h_n) + h_n \cdot L^f \in A_{n+1} \]

since \( T(h_{n+1}) \in A_{n+1}, T(h_n) \in A_n, \) and \( r \cdot L^f, L^f, \log x \in A_1 \). Additionally, using \( r \cdot T(h_{n+1}) \in A_{n+1} \) and \( r \cdot T(h_n) \in A_n \) we see that \( r \cdot \tilde{\partial}(T(h_n)) \in A_{n+1} \) as well. By assumption, \( \tilde{\partial}(r) \cdot T(h_n) \in A_{n+1} \), so the product rule implies

\[ \tilde{\partial}(r \cdot T(h_n)) \in A_{n+1}, \]

which completes \( \text{(79)} \).

Now define

\[ B := \left\{ g \in A(F) : \text{ there exists } c \in C(D(M_f)) \text{ for which } |g(a,x)| \leq c(a) \left(1 + |\log x|^{|\deg(g)|}\right) \right\}. \]

\( B \) is a sub-algebra of \( A(F) \), which clearly contains \( \log x \). By assumption 2. in the statement of the Lemma,

\[ \text{(81)} \quad F \subset B, \]

which implies \( B = A(F) \). Now \( L^f \in A_1 \) and \( \text{(79)} \) implies \( (\tilde{\partial})^n L^f \in A_{n+1} \subset B \) which completes the proof. \( \square \)

We now prove Theorem 5.

Proof of Theorem 5. By Lemma B.1 it suffices to consider only the functions \( f(x) = e^{-bx} \), \( f(x) = (1 - x)^{b-1} \cdot \mathbb{1}_{(0 \leq x < 1)} \), and \( f(x) = \left( \frac{x}{1+x} \right)^b \) for \( b > 0 \). We check that the assumptions of Lemma B.3 are satisfied in these three cases.

First note that since \( h_n(a,x) \) is an \( n \)-th degree polynomial in \( \log x \) with coefficients that are smooth in \( a \), for every \( n \in \mathbb{N} \) there exists some \( \tilde{C}_n \in C(D(M_f)) \) such that

\[ |h_n(a,x)| \leq \tilde{C}_n(a) \left(1 + |\log x|^n\right) \text{ for all } (a,x) \in D(M_f) \times \text{supp}(f). \]

Moreover, by \( \text{(77)} \),

\[ \text{(83)} \quad \int_0^\infty h_n(a,y)y^{a-1} f(y) dy = \int_x^\infty h_n(a,y)y^{a-1} f(y) dy \text{ for all } x \geq 0. \]

Case 1: \( f(x) = e^{-bx} \). Here \( \text{supp}(f) = (0, \infty) = D(M_f) \), \( f \) is clearly smooth on \( (0, \infty) \), and \( r(x) = -bx \). Notice that

\[ \tilde{\partial} r = -bx L^f = r \cdot L^f \]
lies in $A_1$ by definition. So clearly the first assumption of Lemma B.3 holds. We now check the second assumption of Lemma B.3 is satisfied. When $0 < x \leq 1$, using (82) and $a > 0$,

$$|T(h_j)(a, x)| \leq \frac{e^{bx}}{x^a} \int_0^x |h_n(a, y)| y^{a-1} e^{-y} dy \leq \frac{e^{b\tilde{C}_j(a)}}{x^a} \int_0^x \left( 1 + |\log y|^j \right) y^{a-1} dy \leq C_j(a) \left( 1 + |\log x|^j \right).$$

When $x > 1$, using (82), (83), followed by the substitution $y \mapsto \frac{y}{x} - 1$, and finally an application of integration by parts yields

$$|T(h_j)(a, x)| \leq \frac{e^{bx} \tilde{C}_j(a)}{x^a} \int_x^\infty \left( 1 + |\log y|^j \right) y^{a-1} e^{-by} dy$$

$$= \tilde{C}_j(a) \int_0^\infty \left( 1 + \log(y + 1) + \log x^j \right) y^{a-1} e^{-by} dy$$

$$= \frac{\tilde{C}_j(a)}{bx} \left( 1 + \log 2^j + |\log x|^j \right) + O\left( \frac{1}{x^2} \right)$$

$$\leq \frac{\tilde{C}_j(a)}{bx} \left( 1 + |\log x|^j \right),$$

where we increased $\tilde{C}_j(a)$ in the last step if necessary. Combining these two bounds yields the desired result, completing the proof for Case 1.

**Case 2**: $f(x) = (1 - x)^{b-1} \mathbb{1}_{(0 < x < 1)}$. Here $\text{supp}(f) = (0, 1)$, $D(M_f) = (0, \infty)$, $f$ is clearly smooth on $(0, 1)$, and $r(x) = -(b - 1) \frac{x}{1-x}$. To see that the first assumption in Lemma B.3 holds, notice that

$$\tilde{r} = -(b - 1) \frac{x}{(1 - x)^2} L^f = r \cdot \left( 1 + \frac{r}{1 - b} \right) L^f.$$ 

Thus $\tilde{r} \cdot T(h_j) = (r \cdot L^f) \cdot T(h_j) + \frac{1}{1 - r} (r \cdot L^f) \cdot (r \cdot T(h_j)) \in A_{j+1}$ since $r \cdot L^f \in A_1$, and $r \cdot T(h_j) \in A_j$ by definition. We now check the second assumption of Lemma B.3 is satisfied. By (82), we have the bounds

$$|h_j(a, y)y^{a-1} f(y)| \leq \begin{cases} \tilde{C}_j(a) (1 + |\log y|^j) y^{a-1} & \text{if } 0 < y < \frac{1}{2} \\ \tilde{C}_j(a) (1 - y)^{b-1} & \text{if } \frac{1}{2} \leq y < 1. \end{cases}$$

Since $a > 0$, for $0 < x < \frac{1}{2}$,

$$|T(h_j)(a, x)| \leq \frac{2a\tilde{C}_j(a)}{x^a} \int_0^x (1 + |\log y|^j) y^{a-1} dy \leq \tilde{C}_j(a) \left( 1 + |\log x|^j \right)$$

Similarly, using equation (83), for $\frac{1}{2} \leq x < 1$,

$$|T(h_j)(a, x)| \leq \frac{2a\tilde{C}_j(a)}{(1 - x)^{b-1}} \int_x^1 (1 - y)^{b-1} dy \leq \tilde{C}_j(a)(1 - x).$$

where we increased $\tilde{C}_j(a)$ if necessary. Thus, for all $0 < x < 1$, $|r(x)| \leq |b - 1| \frac{1}{1-x}$ implies

$$|T(h_j)(a, x)| \vee |r(x)T(h_j)(a, x)| \leq C_j(a) \left( 1 + |\log x|^j \right).$$

This completes the proof for case 2.
Case 3: $f(x) = \left( \frac{x}{1+x} \right)^b$. Here $\supp(f) = (0, \infty)$, $D(M_f) = (-b, 0)$, $f$ is clearly smooth on $(0, \infty)$, and $r(x) = b \frac{x}{1+x}$. To see the first assumption of Lemma B.3 is satisfied, notice that

$\tilde{\vartheta} r = -b \frac{x}{(1+x)^2} L^f = -r \cdot \left( 1 - \frac{r}{b} \right) L^f$.

Thus $\tilde{\vartheta} r \cdot T(h_j) = -(r \cdot L^f) \cdot T(h_j) + \frac{1}{b} (r \cdot L^f) \cdot (r \cdot T(h_j)) \in A_{j+1}$ since $r \cdot L^f \in A_1$, and $r \cdot T(h_j) \in A_j$ by definition. We now check the second assumption of Lemma B.3. By (82), we have the bounds

$|h_j(a, y) y^{a-1} f(y)| \leq \begin{cases} \check{C}_j(a) (1 + |\log y|^j) y^{a+b-1} & \text{if } 0 < y < 1 \\ \check{C}_j(a) y^{a-1} & \text{if } 1 \leq y < \infty. \end{cases}$

Since $a + b > 0$, for $0 < x < 1$,

$$|T(h_j)(a, x)| \leq 2^b \frac{\check{C}_j(a)}{x^{a+b}} \int_0^x (1 + |\log y|^j) y^{a+b-1} dy \leq \check{C}_j(a) (1 + |\log x|^j)$$

Similarly, using equation (83), for $1 \leq x < \infty$,

$$|T(h_j)(a, x)| \leq 2^b \frac{\check{C}_j(a)}{x^{a}} \int_x^\infty y^{a-1} dy \leq \check{C}_j(a).$$

where we increased $\check{C}_j(a)$ if necessary. Thus, for all $0 < x < \infty$, $|r(x)| < |b|$ implies

$$|T(h_j)(a, x)| \vee |r(x)T(h_j)(a, x)| \leq C_j(a) (1 + |\log x|^j).$$

This completes the proof for case 3.

\[ \square \]

Appendix C. Finite exponential moments for the free energy

**Lemma C.1.** Assume the polymer environment is such that $|\log R^1|$, $|\log R^2|$, $|\log Y^1|$, and $|\log Y^2|$ all have finite exponential moments. Then,

$|\log Z_{m,n}|$ has finite exponential moments for all $(m, n) \in \mathbb{Z}_+^2$.

**Proof.** Since $\log Z_{0,0} = 0$, $\log Z_{k,0} = \sum_{i=1}^k R_{i,0}^1$, and $\log Z_{0,k} = \sum_{j=1}^k R_{0,j}^2$, $\log Z_x$ has finite exponential moments for any $x \in \mathbb{Z}_+^2 \setminus \mathbb{N}^2$. When $x \in \mathbb{N}^2$, the recursion (19) implies that

$$\log Y^1_x + \log Z_x^{-a_1} \land (\log Y^2_x + \log Z_x^{-a_2} \leq \log Z_x - \log 2 \leq \log Y^1_x + \log Z_x^{-a_1} \lor (\log Y^2_x + \log Z_x^{-a_2}$$

Thus

$$|\log Z_x - \log 2| \leq |\log Y^1_x + \log Z_x^{-a_1}| \lor |\log Y^2_x + \log Z_x^{-a_2}.$$ 

Since $|\log Y^1_x|$ and $|\log Y^2_x|$ have finite exponential moments, and inductive argument finishes the proof. \[ \square \]

**References**

[1] M. Balázs, E. Cator, T. Seppäläinen, *Cube Root Fluctuations for the Corner Growth Model Associated to the Exclusion Process*, Electron. J. of Probab. 11, 2006.

[2] Balázs, M., and Seppäläinen, T. *Order of current variance and diffusivity in the asymmetric simple exclusion process*. Ann. of Math., 171(2), 2010.

[3] Balázs, M., Quastel, J., and Seppäläinen, T. *Fluctuation exponent of the KPZ/stochastic Burgers equation*. J. Amer. Math. Soc., 24(3), 2011.

[4] G. Barraquand and I. Corwin, *Random-walk in beta-distributed random environment*, Probab. Theory Related Fields 167(3-4), 2017.
A. Borodin, I. Corwin, P. Ferrari, Free energy fluctuations for directed polymers in random media in $1+1$ dimension, Comm. Pure and Appl. Math. 67(7), 2014.

[6] Balázs, M., Komjáthy, J., and Seppäläinen, T. Microscopic concavity and fluctuation bounds in a class of deposition processes. Ann. Inst. Henri Poincaré Probab. Stat. 48(1), 2012.

[7] H. Chaumont and C. Noack, Fluctuation exponents for stationary exactly solvable lattice polymer models via a Mellin transform framework, ALEA Lat. Am. J. Probab. 15, 2018.

[8] I. Corwin, The Kardar-Parisi-Zhang equation and its universality class, Random Matrices Theory and Appl. 1(01), 2012.

[9] I. Corwin and M. Nica, Intermediate disorder limits for multi-layer semi-discrete directed polymers, Electron. J. Probab. 22, 2017.

[10] I. Corwin, T. Seppäläinen, and H. Shen, The strict-weak lattice polymer, J. Stat. Phys. 160(4), 2015.

T. Imamura and T. Sasamoto, Determinantal Structures in the O’Connell-Yor Directed Random Polymer Model, J. Stat. Phys. 163(4), 2016.

[12] C. Janjigian, Large deviations of the free energy in the O’Connell-Yor polymer, J. Stat. Phys. 160(4), 2015.

M. Jara and G. Moreno-Flores, Stationary Directed Polymers and Energy Solutions of the Burgers Equation, arXiv preprint 1908.06591, 2019.

[14] J. Moriarty and N. O’Connell, On the free energy of a directed polymer in a Brownian environment. Markov Process. and Related Fields 13(2), 2007.

[15] N. O’Connell, M. Yor. Brownian Analogues of Burke’s Theorem, Stochastic Process. Appl. 96(2), 2001.

[16] T. Seppäläinen, Bounds for scaling exponents in a one-dimensional directed polymer with boundary conditions, Ann. Probab. 40(1), 2009.

[17] T. Seppäläinen, B. Valkó, B. Valkó, Bounds for scaling exponents in a $1+1$ dimensional polymer in a Brownian environment, ALEA Lat. Am. J. Probab. 7, 2020.

[18] T. Thiery and P. Le Doussal, On integrable directed polymer models on the square lattice, J. Phys. A 48(46), 2015.

[19] Wikipedia contributors, Cumulant, Wikipedia, The Free Encyclopedia, 5 December 2020, 06:57 UTC,

https://en.wikipedia.org/w/index.php?title=Cumulant&oldid=992433669.

Department of Mathematics, Cornell University. Malott Hall, Ithaca NY 14850, USA.
Email address: noack@cornell.edu

Department of Mathematics, Cornell University. Malott Hall, Ithaca NY 14850, USA.
Email address: psosoe@math.cornell.edu