THE GOLOMB TOPOLOGY OF POLYNOMIAL RINGS, II

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Abstract. We study the interplay of the Golomb topology and the algebraic structure in polynomial rings $K[X]$ over a field $K$. In particular, we focus on infinite fields $K$ of positive characteristic such that the set of irreducible polynomials of $K[X]$ is dense in the Golomb space $G(K[X])$. We show that, in this case, the characteristic of $K$ is a topological invariant, and that any self-homeomorphism of $G(K[X])$ is the composition of multiplication by a unit and a ring automorphism of $K[X]$.

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1. Introduction. Let $R$ be an integral domain, i.e., a commutative unitary ring without zero-divisors. The Golomb space $G(R)$ on $R$ is the topological space having $R^* := R \setminus \{0\}$ as its base space and whose topology (the Golomb topology) is generated by the cosets $a + I$, where $a \in R$ and $I$ is an ideal such that $a$ and $I$ are coprime, i.e., $\langle a, I \rangle = R$. This construction was originally considered on the set $\mathbb{N}$ of natural numbers by Brown [4] and Golomb [7, 8], as part of a series of coset topologies [9], and subsequently extended to arbitrary rings in [5] (following ideas introduced in [1]) with particular focus on what happens when $R$ is a Dedekind domain with infinitely many maximal ideals. In this case, $G(R)$ is a Hausdorff space that is not regular, and is a connected space that is disconnected at each of its points.

An interesting question is how much the topological structure of $G(R)$ reflects the algebraic structure of $R$: for example, it is an open question whether the fact that $G(R)$ and $G(S)$ are homeomorphic implies that $R$ and $S$ are isomorphic (as rings). Relatedly, one can ask if there are self-homeomorphisms of $G(R)$ besides the ones arising from the algebraic structure, i.e., multiplication by units and automorphisms of $R$ (and their compositions).

These problems were studied in [10] for $R = \mathbb{Z}$, showing that the only self-homeomorphisms are the trivial ones (the identity and the multiplication by $-1$) [10, Theorem 7.7] and that $G(\mathbb{Z}) \simeq G(R)$ cannot happen when $R \neq \mathbb{Z}$ is contained in the algebraic closure of $\mathbb{Q}$ [10, Theorem 7.8]; a variant of its method showed that $\mathbb{N}$, with the Golomb topology, is rigid, i.e., it does not have any nontrivial self-homeomorphism [2]. As a second case, [11] studied the space $G(R)$ when $R = K[X]$ is a polynomial ring over a field $K$, showing that several algebraic properties...
of $K$ (for example, having positive or zero characteristic, being algebraically or separably closed) imply different properties on the Golomb space. In particular, it was shown that if $K, K'$ are fields of positive characteristic that are algebraic over their base field, then $G(K[X]) \simeq G(K'[X])$ imply $K \simeq K'$ when one of them is algebraically closed [11, Theorem 5.11 and Corollary 7.2] and when they have the same characteristic [11, Theorem 7.5].

In this paper, we improve these results in two ways. In Section 2, we show that, if the set of irreducible polynomials of $K[X]$ is dense in $G(K[X])$, then the characteristic of $K$ can be detected from the Golomb topology; that is, if $K, K'$ satisfy these hypothesis and $G(K[X]) \simeq G(K'[X])$, then the characteristic of $K$ and $K'$ are equal (Theorem 2.6). As a consequence, we show that if $G(K[X]) \simeq G(K'[X])$ and $K$ is algebraic over $\mathbb{F}_p$, then $K$ and $K'$ must be isomorphic (Theorem 2.7). In Section 3, we concentrate on self-homeomorphisms of the Golomb space $G(K[X])$ and show that (under the above density hypothesis, and the condition that the characteristic of $K$ is positive) all such self-homeomorphisms are algebraic in nature, being compositions of a multiplication by a unit and a ring automorphism of $K[X]$ (Theorem 3.9). The key to both results is a variant of the proofs of [1, Lemma 5.10] and [11, Theorem 6.7], which allows to prove that, under appropriate hypothesis, any homeomorphism of Golomb spaces respects powers, i.e., $h(a^n) = h(a)^n$ for every $a \in G(R)$, $n \in \mathbb{N}$.

Throughout the paper, $K$ is a field, and if $R$ is an integral domain then $G(R)$ denotes the Golomb space as defined above. Given a set $X \subseteq R$, we set $X^\bullet := X \setminus \{0\}$. By a “polynomial ring”, we always mean a polynomial ring in one variable. We denote by $U(R)$ the set of the units of $R$, so that $U(K[X]) = K^\bullet$.

If $h : G(R) \rightarrow G(S)$ is a homeomorphism, then $h$ sends units to units [5, Theorem 13] and prime ideals to prime ideals (i.e., if $P$ is a prime ideal of $R$, then $h(P^\bullet) \cup \{0\}$ is a prime ideal of $S$; equivalently, $h(P^\bullet) = Q^\bullet$ for some prime ideal $Q$ of $S$) [11, Theorem 3.6].

If $x$ is a prime element of $R$, we set $\text{pow}(x) := \{ux^n \mid u \in U(R), n \in \mathbb{N}\}$. If $h : G(R) \rightarrow G(S)$ is a homeomorphism, then $h(\text{pow}(x)) = \text{pow}(h(x))$ (see [11, Section 2]).

If $P$ is a prime ideal of $R$, the $P$-topology on $R \setminus P$ is just the $P$-adic topology, i.e., the topology having the sets $a + P^n$, for $n \in \mathbb{N}$, as a local basis for $a$. The $P$-topology can be recovered from the Golomb topology, and thus, for any prime ideal $P$ of $R$, a homeomorphism $h : G(R) \rightarrow G(S)$ restricts to a homeomorphism between $R \setminus P$ endowed with the $P$-topology and $S \setminus Q$ endowed with the $Q$-topology (where $h(P^\bullet) = Q^\bullet$) [10, Theorem 4.3].

We say that $R$ is Dirichlet if the set of irreducible elements of $R$ is dense, with respect to the Golomb topology. If $R = K[X]$ is Dirichlet, where $K$ is a field, we say for brevity that $K$ itself is Dirichlet. This happens, for example, when $K$ is pseudo-algebraically closed and admits separable irreducible polynomials of arbitrary large degree [3, Theorem A]; in particular, it happens when $K$ is an algebraic extension of a finite field that is not algebraically closed [6, Corollary 11.2.4].

2. Detecting the characteristic. In this section, we show how to detect the characteristic of a polynomial ring from its Golomb topology.
Given an element $t \in R^\bullet$, we denote by $\sigma_t$ the multiplication by $t$, i.e.,

$$
\sigma_t : G(R) \rightarrow G(R),
y \mapsto ty.
$$

It is easy to see that the map $\sigma_t$ is always continuous, and that it is a homeomorphism if $t$ is a unit of $R$.

If $R$ is a polynomial ring over a Dirichlet field, and $h : G(R) \rightarrow G(R)$ is a self-homeomorphism such that $h(P^\bullet) = P^\bullet$ for every prime ideal $P$, then $h$ must be equal to $\sigma_u$ for some unit $u$ [11, Proposition 7.4]; this result can be improved in the following way.

**Proposition 2.1.** Let $R, S$ be polynomials rings over Dirichlet fields. If $h : G(R) \rightarrow G(S)$ is a homeomorphism such that $h(1) = 1$, then $h(uz) = h(u)h(z)$ for every $u \in U(R)$ and $z \in R^\bullet$.

**Proof.** Let $u$ be a unit of $R$. Consider the map $h' := h^{-1} \circ \sigma_{h(u)} \circ h$, that is,

$$
h'(z) = h^{-1}(h(u)h(z))
$$

for every $z \in R^\bullet$. Since $u$ is a unit, so is $h(u)$, and thus $\sigma_{h(u)}$ is a self-homeomorphism of $G(S)$; therefore, $h'$ is a self-homeomorphism of $G(R)$.

Let now $P$ be a prime ideal of $R$; then, $h(P^\bullet) = Q^\bullet$ for some prime ideal $Q$ of $S$. For every $z \in P$, we have $h(u)h(z) \in Q$, and thus $h'(z) = h^{-1}(h(u)h(z)) \in P$. Hence, $h'(P) \subseteq P$, i.e., $h$ fixes every prime ideal of $R$; by [11, Proposition 7.4], $h' = \sigma_t$ for some unit $t$ of $R$. However, since $h(1) = 1$,

$$
h'(1) = h^{-1}(h(u)h(1)) = h^{-1}(h(u)) = u,
$$

and it must be $t = u$, i.e., $h'(z) = uz$ for every $z \in R^\bullet$; it follows that $h(uz) = h(u)h(z)$ for every $z \in R^\bullet$. \qed

We now want to study the relationship between a homeomorphism of Golomb topologies and powers of elements. For an element $z \in R^\bullet$, we denote by $z^N$ the set of powers of $z$, i.e.,

$$
z^N := \{ z^n \mid n \in \mathbb{N} \}.
$$

**Lemma 2.2.** Let $R, S$ be polynomial rings over Dirichlet fields, and let $h : G(R) \rightarrow G(S)$ be a homeomorphism such that $h(1) = 1$. The set

$$
X_h := \{ z \in R \mid z \text{ is irreducible and } h(z^N) = h(z)^N \}
$$

is dense in $G(R)$.

**Proof.** Let $Q$ be a prime ideal of $S$, and let $P$ be the prime ideal of $R$ such that $h(P^\bullet) = Q^\bullet$. Then, $h$ restricts to a homeomorphism between the $P$-topology on $R \setminus P$ and the $Q$-topology on $S \setminus Q$; it follows that for every $m \in \mathbb{N}$ there is an $n$ such that $h(1 + P^m) \subseteq 1 + Q^m$. Let $z$ be any irreducible element in $1 + P^n$ (which exists since $R$ is Dirichlet); then, for every $t \in \mathbb{N}$, we have $z^t \in 1 + P^n$ too, and
thus also $h(z^t) \in 1 + Q^n$. Moreover, $z^t \in \text{pow}(z)$, and thus there are a unit $u$ of $S$ and an integer $\alpha(t)$ such that $h(z^t) = uh(z)^{\alpha(t)}$. Since $U(S) \rightarrow S/Q$ is injective, it follows that $u = 1$ for every $t$. Hence, $h(z^N) = h(z)^N$.

Let now $\Omega = c + I$ be a subbasic open set of $G(R)$, and let $d + J$ be a subbasic open set contained in $h(\Omega)$. Let $Q$ be a prime ideal of $S$ that is coprime with $J$; then, the prime $P$ such that $h(P^\bullet) = Q^\bullet$ is coprime with $I$. Since $R$ is Dirichlet, $(c + I) \cap (1 + P^n)$ contains irreducible elements for every $n$; in particular, if $n$ is large enough, every irreducible polynomial in $1 + P^n$ is in $X_h$, and thus $X_h \cap (c + I)$ is nonempty. Thus $X_h$ is dense. □

The following proof follows the one of [11, Theorem 6.8], which in turn was based on the one of [1, Lemma 5.10].

**Lemma 2.3.** Let $R, S$ be polynomial rings over Dirichlet fields, and let $h : G(R) \rightarrow G(S)$ be a homeomorphism such that $h(1) = 1$. For every $a \in R$, we have $h(a^N) = h(a)^N$.

**Proof.** If $a$ is a unit, the statement follows from Proposition 2.1. Take any $a \in R$, $a \notin U(R)$, and let $b := h(a)$. We first claim that $h(a^N) \subseteq b^N$. Take any $n \in \mathbb{N}$ and let $f : G(R) \rightarrow G(R)$ be defined as $f(y) = y^n$, and let $\phi := h \circ f \circ h^{-1}$. Then, $\phi : G(S) \rightarrow G(S)$ is continuous (since $f$ is continuous) and $\phi(P^\bullet) \subseteq P^\bullet$ for every prime ideal $P$ of $R$. Let

$$c := \phi(b) = h(f(a)) = h(a^n),$$

and suppose that $c \notin b^N$. Take an integer $k$ such that $\deg b^k > \deg h(a^n) = \deg c$; then, since $b$ and $b^k - 1$ are coprime, so are $c$ and $b^k - 1$ (since $c$ and $b$ belong to the same prime ideals, by [10, Theorem 3.6]), and thus $\Omega := c + (b^k - 1)S$ is an open set. Since $\phi$ is open, there is a neighborhood $b + dS$ of $b$ such that $\phi(b + dS) \subseteq \Omega$.

By Lemma 2.2, we can find an irreducible polynomial $z \in h^{-1}(b + d(b^k - 1)S)$ such that $h(z^N) = h(z)^N$; setting $y := h(z)$, we have

$$\phi(y) = (h \circ f \circ h^{-1})(h(z)) = h(z^n) = h(z)^l$$

for some integer $l$. Thus, we have

$$\phi(y) \in \phi(b + dS) \subseteq \Omega = c + (b^k - 1)S$$

and

$$\phi(y) = h(z)^l \in (b + d(b^k - 1)S)^l \subseteq b^l + (b^k - 1)S;$$

hence, $c \equiv b^l \mod (b^k - 1)S$. If $l = ik + j$, we have $b^j \equiv b^l \mod (b^k - 1)S$; hence, we have $c \equiv b^j \mod (b^k - 1)S$ for some $0 \leq j < k$. However, by hypothesis, $c \neq b^j$, and the degree of both $c$ and $b^j$ is less than the degree of $b^k - 1$; this is a contradiction, and thus we must have $h(a^N) \subseteq b^N$.

The opposite inclusion is obtained using the homeomorphism $h^{-1}$. Thus, $h(a^N) = b^N = h(a)^N$, as claimed. □
Lemma 2.4. Let $R$ be an integral domain. If $U(R)$ is infinite, then its exponent is infinite.

Proof. Suppose that the exponent of $U(R)$ is finite, say equal to $n$. Since $R$ is a domain, there can be at most $n$ units of order divisible by $n$ (the roots of $X^n - 1 = 0$); however, this is impossible since $U(R)$ is infinite. Hence, the exponent is infinite. □

Proposition 2.5. Let $R, S$ be polynomial rings over infinite Dirichlet fields, and let $h : G(R) \to G(S)$ be a homeomorphism such that $h(1) = 1$. For every $a \in R$ and every $n \in \mathbb{N}$, we have $h(a^n) = h(a)^n$.

Proof. If $a$ is a unit the claim follows from Proposition 2.1. Take $a \in R \setminus U(R)$ and $n \in \mathbb{N}$; by Lemma 2.3, we have $h(a^n) = h(a)^s$ for some $s \in \mathbb{N}$. By hypothesis, $R$ has infinitely many units, and thus by Lemma 2.4 there is a unit $u$ whose order is larger than $n$ and $s$; then, using Proposition 2.1 we have

$$h((ua)^n) = (h(ua))^t = (h(u)h(a))^t = h(u)^t h(a)^t$$

for some $t$ and

$$h((ua)^n) = h(u^n a^n) = h(u^n)h(a^n) = h(u)^n h(a)^s.$$

The equality

$$h(u)^t h(a)^t = h(u)^n h(a)^s$$

can only hold if $n = t = s$; in particular, $n = s$ and $h(a^n) = h(a)^n$, as claimed. □

Theorem 2.6. Let $K, K'$ be infinite Dirichlet fields. If $G(K[X])$ and $G(K'[X])$ are homeomorphic, then they have the same characteristic.

Proof. If one of $K$ and $K'$ has characteristic 0, the claim follows from [11, Corollary 4.2]. Suppose thus that $\text{char } K = p$ and $\text{char } K' = q$, with $p, q > 0$.

Let $h : G(K[X]) \to G(K'[X])$ be a homeomorphism such that $h(1) = 1$, and let $a$ be an irreducible element. Let $f$ be a factor of $a - 1$. By [11, Proposition 5.5], the sequence $a^{p^n}$ converges to 1 in the $(f)$-topology, and thus also $h(a^{p^n})$ converges to 1 in the $h((f))$-topology. By Proposition 2.5, $h(a^{p^n}) = h(a)^{p^n}$. By [11, Proposition 5.5], it follows that $v_q(p^n) \to \infty$, where $v_q$ is the $q$-adic valuation; thus, it must be $q = p$. The claim is proved. □

We note that the proof of the previous theorem actually needs only one irreducible element $a$ such that $h(a^n) = h(a)^n$ for every $n$.

Theorem 2.7. Let $K, K'$ be fields. If $K$ is algebraic over $\mathbb{F}_p$ and $G(K[X]) \simeq G(K'[X])$, then $K \simeq K'$. 

Proof. If \( K \) is finite then \( |U(K[X])| = |K| - 1 \). Since the set of units is preserved by homeomorphisms of the Golomb topology, \( K \) and \( K' \) must have the same cardinality, and thus they are isomorphic. Suppose \( K, K' \) are infinite.

By [11, Corollary 4.2], \( K' \) has positive characteristic, and by [11, Corollary 7.2] \( K' \) must be algebraic over its base field \( \mathbb{F}_q \).

If \( K \) is algebraically closed, then the set \( G_1(K[X]) := \{ z \in K[X] \mid z \text{ is contained in a unique prime ideal} \} \) is not dense in \( G(K[X]) \) [11, Proposition 5.2(a)]; conversely, if \( K \) is not algebraically closed then \( G_1(K[X]) \) is dense since it contains the irreducible polynomials. The same holds for \( K' \). Since a homeomorphism of Golomb topologies sends \( G_1(K[X]) \) to \( G_1(K'[X]) \) [10, Theorem 3.6(b)], it follows that if \( K \) is algebraically closed then so is \( K' \), and in this case \( p = q \) by [11, Theorem 5.11]; hence \( K \cong K' \).

If \( K \) and \( K' \) are not algebraically closed, they are pseudo-algebraically closed fields containing separable irreducible polynomials of arbitrarily large degree [3, Theorem A]; by Theorem 2.6, it follows that \( p = q \). By [11, Theorem 7.5], it follows that \( K \cong K' \). \( \square \)

3. Self-homeomorphisms and automorphisms. In this section, we concentrate on the study of self-homeomorphisms of the Golomb space \( G(K[X]) \). We shall work under the following assumptions:

**Hypothesis 3.1.**

- \( K \) is an infinite field of characteristic \( p > 0 \);
- \( R := K[X] \);
- \( R \) is Dirichlet, i.e., the set of irreducible polynomials is dense in \( G(R) \);
- \( h : G(R) \rightarrow G(R) \) is a self-homeomorphism such that \( h(1) = 1 \).

In particular, under these hypothesis, we have \( h(a^n) = h(a)^n \) for every \( a \in R \) and \( n \in \mathbb{N} \), by Proposition 2.5.

Given a set \( X \subseteq R \), we define the \( p \)-radical of \( X \) as

\[
\text{rad}_p(X) := \{ f \in R \mid f^{p^n} \in X \text{ for every large } n \}.
\]

This construction is invariant under \( h \), in the following sense.

**Lemma 3.2.** Assume Hypothesis 3.1. For every \( X \subseteq R^* \), we have \( h(\text{rad}_p(X)) = \text{rad}_p(h(X)) \).

**Proof.** If \( f \in \text{rad}_p(X) \), then \( f^{p^n} \in X \) for every \( n \geq N \), and thus \( h(f^{p^n}) \in h(X) \). By Proposition 2.5, it follows that \( h(f)^{p^n} \in h(X) \) for \( n \geq N \), and thus \( h(f) \in \text{rad}_p(h(X)) \). Conversely, if \( g = h(f) \in \text{rad}_p(h(X)) \), then \( g^{p^n} = h(f)^{p^n} = h(f^{p^n}) \) belongs to \( h(X) \) for large \( n \), and thus applying \( h^{-1} \) we have \( f^{p^n} \in X \) for large \( n \). Thus \( f \in \text{rad}_p(X) \) and \( g \in h(\text{rad}_p(X)) \). \( \square \)
Lemma 3.3. Let $P$ a prime ideal of a ring $R$ of characteristic $p$. For every $m \in \mathbb{N}^+$, we have $\text{rad}_p(1 + P^m) = 1 + P$.

Proof. If $t \in 1 + P$, then $t = 1 + x$ with $x \in P$; if $p^n \geq m$, then $t^{p^n} = (1 + x)^{p^n} = 1 + x^{p^n} \in 1 + P^m$, and thus $t \in \text{rad}_p(1 + P^m)$. Conversely, if $t \in \text{rad}_p(1 + P^m)$ then $t^{p^n} \in 1 + P^m$ for some $m$; in particular, $t^{p^n} \in 1 + P$, and thus $t^{p^n} - 1 = (t - 1)^{p^n} \in P$. Hence $t - 1 \in P$, i.e., $t \in 1 + P$. \hfill $\square$

Proposition 3.4. Assume Hypothesis 3.1, and let $P, Q$ be prime ideals of $R$ with $h(P^*) = Q^*$. Then, $h(1 + P) = 1 + Q$.

Proof. Since $h(1) = 1$, $h(1 + P)$ is open in the $Q$-topology, and thus $1 + Q^m \subseteq h(1 + P)$ for some $m$. Therefore,

$$1 + Q = \text{rad}_p(1 + Q^m) \subseteq \text{rad}_p(h(1 + P)) = h(\text{rad}_p(1 + P)) = h(1 + P)$$

using Lemmas 3.2 and 3.3. Applying the same reasoning to $h^{-1}$ gives $1 + P \subseteq h^{-1}(1 + Q)$, i.e., $h(1 + P) \subseteq 1 + Q$. Hence, $h(1 + P) = 1 + Q$, as claimed. \hfill $\square$

Corollary 3.5. Assume Hypothesis 3.1, and let $P, Q$ be prime ideals of $R$ with $h(P^*) = Q^*$. If $u \in K^*$, then $h(u + P) = h(u) + Q$. In particular, if $f \equiv u \mod P$ for some unit $u$, then $h(f) \equiv h(u) \mod Q$.

Proof. By Proposition 2.1, we have $u + P = u(1 + P)$; by Proposition 3.4, it follows that

$$h(u + P) = h(u)h(1 + P) = h(u)(1 + Q) = h(u) + Q,$$

as claimed. The “in particular” statement follows from the fact that $f \equiv u \mod P$ is equivalent to $f \in u + P$. \hfill $\square$

Corollary 3.6. Assume Hypothesis 3.1. If $f$ is a linear polynomial, so is $h(f)$.

Proof. Since $f$ is linear, $(f)$ is prime and every polynomial $g \notin (f)$ is equivalent to a unit modulo $(f)$. By Corollary 3.5, it follows that every $g \notin (h(f))$ is equivalent to a unit modulo $(h(f))$; hence, $h(f)$ must be linear too. \hfill $\square$

In particular, by Corollary 3.6, $h(X)$ is a linear polynomial; it follows that there is an automorphism $\sigma$ of $R$ sending $h(X)$ to $X$. Since $\sigma$ restricts to a self-homeomorphism of $G(R)$, passing from $h$ to $H := \sigma \circ h$ we obtain a self-homeomorphism of $G(R)$ that fixes both 1 and $X$. Thus, it is not restrictive to assume also that $h(X) = X$, as we do in the remaining part of the section.

Lemma 3.7. Assume Hypothesis 3.1 and suppose $h(X) = X$. For every $u \in K^*$, we have $h(X + u) = X + h(u)$.
Proof. We have
\[
\begin{align*}
X + u &\equiv u \mod (X) \\
X &\equiv -u \mod (X + u).
\end{align*}
\]
Applying \(h\) to both equivalences, using \(h(X) = X\) and Corollary 3.5, we have
\[
\begin{align*}
h(X + u) &\equiv h(u) \mod (X) \\
X &\equiv -h(u) \mod (h(X + u)).
\end{align*}
\]
The first equation implies that \(X\) divides \(h(X + u) - h(u)\); since \(h(X + u)\) is linear, it follows that \(h(X + u) = vX + h(u)\) for some \(v \in K^*\). The second equation implies that \(vX + h(u)\) divides \(X + h(u)\); the only possibility is \(v = 1\), i.e., \(h(X + u) = X + h(u)\). The claim is proved.

\[\qed\]

Lemma 3.8. Assume Hypothesis 3.1 and suppose \(h(X) = X\). Then, the map
\[H : K \to K,\]
\[
a \mapsto \begin{cases} 
  h(a) & \text{if } a \neq 0 \\
  0 & \text{if } a = 0
\end{cases}
\]
is an automorphism of \(K\).

Proof. Let \(a, b \in K\). If \(a = 0\) or \(b = 0\) then clearly \(H(ab) = H(a)H(b)\) and \(H(a + b) = H(a) + H(b)\).

Suppose \(a \neq 0 \neq b\). Then, \(H = h\) on these values, and \(h(ab) = h(a)h(b)\) by Proposition 2.1. Furthermore,
\[
X + a + b \equiv b \mod (X + a);
\]
applying \(h\) and using Lemma 3.7, we have
\[
X + h(a + b) \equiv h(b) \mod (X + h(a)),
\]
that is, \(X + h(a)\) divides \(X + h(a + b) - h(b)\). Thus, it must be \(X + h(a) = X + h(a + b) - h(b)\) and \(h(a) = h(a + b) - h(b)\), that is, \(h(a) + h(b) = h(a + b)\). It follows that \(H\) is an automorphism of \(K\), as claimed.

\[\qed\]

Theorem 3.9. Let \(K\) be an infinite Dirichlet field of positive characteristic, and let \(h\) be a self-homeomorphism of \(G(K[X])\). Then, there are a unit \(u \in K^*\) and an automorphism \(\sigma\) of \(K[X]\) such that \(h(f) = u\sigma(f)\) for every \(f \in K[X]^*\).

Proof. By [5, Theorem 13], \(h(1) = u\) is a unit of \(R\); since the multiplication \(\sigma_u\) is a self-homeomorphism of \(G(R)\), the map \(h_1 := \sigma_u^{-1} \circ h\) is a self-homeomorphism such that \(h_1(1) = 1\).

By Corollary 3.6, \(h_1(X)\) is a linear polynomial, and thus there is an automorphism \(\sigma_1\) of \(R\) such that \(h_1(X) = \sigma_1(X)\). Thus, \(h_2 := \sigma_1^{-1} \circ h_1\) is again a
self-homeomorphism of $G(R)$, and furthermore both 1 and $X$ are fixed points of $h_2$.

By Lemma 3.8, the restriction of $h_2$ to $K$ is an automorphism $H$; this map can be extended to an automorphism $\sigma_2$ of $R$ by setting $\sigma_2 \left( \sum_i a_i X^i \right) = \sum_i H(a_i) X^i$. Thus, $h_3 := \sigma_2^{-1} \circ h_2$ is a self-homeomorphism of $G(R)$ such that $h_3(u) = u$ for every $u \in K^\bullet$ and $h_3(X) = X$. By Lemma 3.7, it follows that $h_3(X + u) = X + u$ for every $u \in K^\bullet$.

Let now $f \in R$, and take any $t \in K$. Then,

$$
\begin{cases}
  f \equiv f(t) \mod (X - t) \\
  h_3(f) \equiv h_3(f)(t) \mod (X - t).
\end{cases}
$$

Since $h_3((X - t)) = (X - t)$, by Corollary 3.5 the first equivalence also implies $h_3(f) \equiv h_3(f(t)) \mod (X - t)$. Hence, $h_3(f)(t) = h_3(f(t)) = f(t)$. Since this happens for every $t \in K$, and $K$ is infinite, it follows that $f = h_3(f)$, that is, $h_3$ is the identity on $G(R)$.

Going back to the definition,

$$
    h_3 = \sigma_2^{-1} \circ h_2 = \sigma_2^{-1} \circ \sigma_1^{-1} \circ h_1 = \sigma_2^{-1} \circ \sigma_1^{-1} \circ \sigma_1^{-1} \circ h,
$$

i.e., $h = \sigma_1 \circ \sigma_2$. Since $\sigma_u(x) = ux$ for every $x$, setting $\sigma := \sigma_1 \circ \sigma_2$ (which is still an automorphism of $R$), we obtain $h(f) = u\sigma(f)$ for every $f \in G(R)$, as claimed. \hfill \Box

\section*{References}

1. Taras Banakh, Jerzy Miaskanekowski, and Slawomir Turek, On continuous self-maps and homeomorphisms of the Golomb space, *Comment. Math. Univ. Carolin.* 59(4) (2018), 423–442.

2. Taras Banakh, Dario Spirito, and Slawomir Turek, The Golomb space is topologically rigid, submitted.

3. Lior Bary-Soroker, Dirichlet’s theorem for polynomial rings, *Proc. Amer. Math. Soc.* 137(1) (2009), 73–83.

4. Morton Brown, A countable connected Hausdorff space, In: *The April meeting in New York*, L.W. Cohen, editor, Vol. 4, pp. 330–371; *Bull. Amer. Math. Soc.*, (1953), Abstract 423.

5. Pete L. Clark, Noah Lebowitz-Lockard, and Paul Pollack, A note on Golomb topologies, *Quaest. Math.* 42(1) (2019), 73–86.

6. Michael D. Fried and Moshe Jarden, *Field arithmetic*, Vol. 11 of Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge. A Series of Modern Surveys in Mathematics, (Results in Mathematics and Related Areas, 3rd Series, A Series of Modern Surveys in Mathematics), second edition, Springer-Verlag, Berlin, 2005.

7. Solomon W. Golomb, A connected topology for the integers, *Amer. Math. Monthly* 66 (1959), 663–665.

8. , Arithmetica topologica, In: *General Topology and its Relations to Modern Analysis and Algebra (Proc. Sympos., Prague, 1961)*, pp. 179–186, Academic Press, New York; Publ. House Czech. Acad. Sci., Prague, 1962.
9. John Knopfmacher and Stefan Porubsky, Topologies related to arithmetical properties of integral domains, *Exposition. Math.* 15(2) (1997), 131–148.

10. Dario Spirito, The Golomb topology on a Dedekind domain and the group of units of its quotients, *Topology Appl.* 273 (2020), 107101.

11. ____________, The Golomb topology of polynomial rings, *Quaestiones Mathematicae*, 44(4) (2021), 447-468.

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