Unifying the Strengths of Forces in Higher Dimensions

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We consider the embedding of the Standard Model fields in a \((4 + d)\)-dimensional theory while gravitons may propagate in \(d\) extra, compact dimensions. We study the modification of strengths of the gravitational and gauge interactions and, for various values of \(d\) and \(d'\), we determine the energy scale at which these strengths are unified. Special cases where the unification of strengths is characterized by the absence of any hierarchy problem are also presented.

It is widely believed that any fundamental theory capable of describing our world at higher energy scales always predicts the existence of extra, spatial, compact dimensions. Being motivated by attempts to lower the string unification scale \([3]\) or to lower the supersymmetry breaking scale \([5]\) or to lower the energy scale at which these strengths are unified. Special cases where the unification of strengths is characterized by the absence of any hierarchy problem are also presented.

The above theory and preserves Lorentz invariance, may be written as \([7]\)

\[
S_{4+d} = \int d^4x \, d^d z \left\{ -\frac{1}{4} \left( \partial_M \hat{A}_N^\alpha - \partial_N \hat{A}_M^\alpha + \frac{\hat{g}}{\sqrt{\Lambda^d}} C_{\alpha\beta\gamma} \hat{A}_M^\beta \hat{A}_N^\gamma \right)^2 - \bar{\Psi}_{L,R} \gamma^\mu \left( \partial_\mu - \frac{i \hat{g}}{\sqrt{\Lambda^d}} t^a \hat{A}_\mu^a \right) \Psi_{L,R} \delta(z) - \left( \partial_\mu - \frac{i \hat{g}}{\sqrt{\Lambda^d}} t^a \hat{A}_\mu^a \right) \phi \right|^2 - \hat{\mu}^2 \hat{\phi}^* \hat{\phi} - \frac{\hat{\lambda}}{2\Lambda^d} (\hat{\phi}^* \hat{\phi})^2 - \left( \frac{\hat{Y}_1}{\sqrt{\Lambda^d}} \bar{\Psi}_L \hat{\phi} \Psi_R + \frac{\hat{Y}_2}{\sqrt{\Lambda^d}} \bar{\Psi}_L \hat{\phi}_c \Psi_R + h.c. \right) \delta(z) \right\}, \tag{1}
\]

where \(\hat{g}, \hat{C}_{\alpha\beta\gamma}\) and \(t^a\) are the coupling constant, the structure constants and the generators, respectively, of the \(SU(N)\) gauge group while \(\hat{\mu}\) and \(\hat{\lambda}\) are the mass and coupling constant of the Higgs field. In order to render the coupling constants of the theory dimensionless in \((4 + d)\)-dimensions, an arbitrary energy scale \(\Lambda\) has been introduced. Note that, in the above, \(M,N = \{t, x_1, x_2, x_3, z_1, z_2, ..., z_d\}, \mu, \nu = \{t, x_1, x_2, x_3\}\) and the hat denotes \((4 + d)\)-dimensional quantities.

Next, we assume that the extra \(d\) dimensions are compactified over an internal manifold with the size of every compact dimension being \(2\ell\). Then, we can Fourier expand the \((4 + d)\)-dimensional vector and scalar fields along the compact dimensions in the following way

\[
\hat{\phi}(x,z) = \hat{\phi}^{(0)}(x) + \sum_{\vec{n}=1}^{\infty} \frac{\hat{\phi}^{(\vec{n})}(x)}{\sqrt{2}} \left( e^{i \frac{2\pi \vec{n} \cdot \vec{z}}{2\ell}} + e^{-i \frac{2\pi \vec{n} \cdot \vec{z}}{2\ell}} \right), \tag{2}
\]

where \(\vec{n} = \{n_1, n_2, ..., n_d\}\). By performing a Kaluza-Klein compactification, i.e. by using the above field expansion and integrating over the extra dimensions, the action \([4]\) reduces to an effective 4-dimensional theory. A prominent feature of this effective theory is its complexity due to extra terms involving the massive Kaluza-Klein (KK) excitations, \(\hat{\phi}^{(\vec{n})}(x)\), of all the fields propagating in the extra dimensions, apart from the usual, massless zero-modes, \(\hat{\phi}^{(0)}(x)\). Here, we are only interested in the part of the effective theory that contains the zero modes of the various fields and, more specifically, in the relations that hold between the \((4 + d)\) and 4-dimensional couplings and masses. These are found to be

*Our results apply also in the case where the fermions live in the bulk.
where we have used the following field redefinitions
\begin{align}
g = \frac{\hat{g}}{(2L\Lambda)^{d/2}}, \quad Y_{1,2} = \frac{\hat{Y}_{1,2}}{(2L\Lambda)^{d/2}}, \\
\mu^2 = \hat{\mu}^2, \quad \lambda = \frac{\hat{\lambda}}{(2\Lambda)^d},
\end{align}
where we have used the following field redefinitions
\begin{align}
A_\mu^a = (2L)^{d/2} \hat{A}_\mu^a, \quad \phi = (2L)^{d/2} \hat{\phi}
\end{align}
in order to obtain canonical kinetic terms in 4-dimensions. Note that the fermionic fields being always localized on the 4-dimensional boundary remain unchanged.

In the framework of the 4-dimensional effective theory, the gauge bosons and fermions acquire mass through the spontaneous symmetry breaking of the gauge group. Their masses are given in terms of the vacuum expectation value of the Higgs field which is also redefined according to eq. (\[\text{[3]}\]). The question, then, arises whether the mass spectrum of the 4-dimensional effective theory corresponds to a different one in the framework of the original (4+d)-dimensional theory. To answer this question, we consider the masses of the fermions which are given by the following expression
\begin{align}
m_{\Psi_{L,R}} = Y_{1,2} \frac{v}{\sqrt{2}} \frac{\hat{Y}_{1,2}}{\hat{\Lambda}^{d/2}} \frac{\hat{v}}{\sqrt{2}} = \hat{m}_{\Psi_{L,R}}.
\end{align}
Similar relations can also be written for the masses of the gauge bosons and the Higgs field. In all cases, the tree-level masses, that are generated via the Higgs mechanism in the 4-dimensional theory, remain unaltered when one embeds this theory in a higher-dimensional one.

However, this is not the case with the coupling constant \(\hat{g}\) that eventually determines the strength of the gauge interactions of the theory. The \(SU(N)\) gauge group of the (4+d)-dimensional theory could be replaced by the \(U(1) \times SU(2) \times SU(3)\) group with the gauge field \(\hat{A}_\mu\), the generators \(t^a\) and the coupling constant \(\hat{g}\) standing for each one of the corresponding quantities of the Standard Model group, i.e. \((\hat{B}_\mu, \hat{W}_\mu^a, \hat{G}_\mu^a)\), \((Y/2, \tau^a/2, \lambda^a/2)\) and \((\hat{g}_1, \hat{g}_2, \hat{g}_3)\), respectively. In that case, each coupling constant and gauge boson of the SM group is redefined according to eqs. (\[\text{[3]}\]) and (\[\text{[3]}\]), respectively.

By making use of the redefinition (\[\text{[3]}\]) for the gauge coupling constants, we find that the electric charge \(e\) changes as follows
\begin{align}
\hat{e} = \hat{g}_1 \cos \theta_W = (2L\Lambda)^{d/2} g_1 \cos \theta_W = (2L\Lambda)^{d/2} e, \quad (7)
\end{align}
where we have used the fact that the Weinberg angle \(\theta_W\) remains unaltered since its tangent is given by the ratio of the gauge couplings \(g_1\) and \(g_2\). The above rescaling of the electric charge inevitably affects the strength of the electromagnetic interactions in higher dimensions. The strength of a force is classically defined as the potential energy of the corresponding interaction between two identical particles, separated by a distance equal to the particle’s Compton wavelength, compared to the energy of the rest mass of each particle. As is well known, in 4-dimensions and for two particles with mass \(m\) and charge \(e\), the above rule gives the result
\begin{align}
\alpha_{EM} = \frac{E_{int}}{E_m} = \frac{1}{4\pi\varepsilon_0} \frac{e^2}{(m/hc)} = \frac{e^2}{4\pi\varepsilon_0} \simeq \frac{1}{137}, \quad (8)
\end{align}
where we have set \(c = \hbar = 1\). Generalizing the above definition of the strength of a force in \((4+d)\) dimensions, we obtain
\begin{align}
\hat{\alpha}_{EM} = \frac{E_{int}}{E_m} = \frac{1}{m} \frac{(\hat{e}^2/\Lambda^d)}{4\pi\varepsilon_0 (1/m)^{1+d}} = \alpha_{EM} \left(\frac{m}{M_x}\right)^d, \quad (9)
\end{align}
where we have used the fact that the masses of the particles do not change, according to (\[\text{[3]}\]), and where we have defined \(M_x \equiv (2L)^{-1}\). The above result clearly reveals that the strength of the electromagnetic force changes as the gauge bosons start “feeling” the extra dimensions. Moreover, the strength of the force strongly depends not only on the number and size of extra dimensions but also on the mass of the test particle, i.e. on the energy scale where the measurement takes place. It is also worth noting that the auxiliary energy scale \(\Lambda\) does not appear in the electromagnetic strength formula (\[\text{[3]}\]).

So far, we have not considered the gravitational interactions. We now assume that gravitons may propagate, apart from the usual 4 dimensions, in \(d' = \delta + d\) extra dimensions, where \(\delta\) is the number of transverse dimensions, with size \(2L'\), felt only by gravitons. In that case, the \((4+d')\)-dimensional action functional
\begin{align}
S_{4+d'} = -\frac{M_G^2}{16\pi} \int d^4x d^{d'}z \sqrt{G_{4+d'}} R_{4+d'} \quad (10)
\end{align}
reduces to an effective, 4-dimensional Einstein’s theory of gravity only when the following relation between the sizes of the extra dimensions and the energy scales of gravity, \(M_P\) and \(M_{GR}\), in 4 and \((4+d)\) dimensions, respectively,
\begin{align}
(2L')^\delta (2L)^d M_{GR}^{2+d'} = M_P^2 \quad (11)
\end{align}
holds. The strength of the gravitational interactions also changes when one introduces extra dimensions for the gravitons. Using the same rule as above, the strength of the gravitational interaction between two particles with mass \(m\), in 4 dimensions, is given by the expression
\begin{align}
\alpha_{GR} = \frac{E_{int}}{E_m} = \frac{1}{m} \frac{G_N m^2}{(h/mc)} = \left(\frac{m}{M_P}\right)^2, \quad (12)
\end{align}
where \(G_N = M_P^{-2}\) and natural units, \(\hbar = c = 1\), have been used again. The corresponding expression in \(4+d'\) dimensions takes the form
\begin{align}
\hat{\alpha}_{GR} = \frac{E_{int}}{E_m} = \frac{1}{\hat{m}} \frac{\hat{G}_N \hat{m}^2}{(1/\hat{m})^{1+d'}} = \left(\frac{m}{M_{GR}}\right)^{2+d'}, \quad (13)
\end{align}
where, now, $\hat{G}_N = 1/M_{GR}^{2+d}$. By choosing appropriately the sizes of the transverse and longitudinal, extra dimensions, the higher-dimensional gravity scale, $M_{GR}$, could be much lower than the 4-dimensional one, a fact which, subsequently, changes the strength of gravity.

The question we would like to address is the following: can the gravitational and electromagnetic force\footnote{Under the assumption that the electromagnetic, weak and strong forces “feel” the same number $d$ of extra dimensions, we may assume that their strengths remain comparable at every scale even in $(4+d)$-dimensions.} have comparable strengths in a world where both gravitons and gauge bosons feel extra dimensions? If so,

$$\hat{\alpha}_{EM} = \hat{\alpha}_{GR} \Rightarrow \alpha_{EM} \left( \frac{m_U}{M_x} \right)^d = \left( \frac{m_U}{M_{GR}} \right)^{2+d},$$

where $m_U$ denotes the scale where the unification of strengths takes place. Hereafter, for simplicity, we will take $\alpha_{EM} \approx 10^{-2}$. Regarding the values $\delta$ and $d$ of the extra dimensions, we may form four distinct categories:

(i) $\delta = d = 0$. In this case, $M_{GR} \equiv M_P$ and only at energies $m_U \approx 10^{18}$ GeV, i.e. close to the Planck scale, the strengths of the forces become equal.

(ii) $\delta \neq 0$, $d = 0$. Here, we assume that only the gravitational fields can feel extra dimensions. Then, from eq. (14), we may easily conclude that the unification of strengths takes place at a scale $m_U = (0.2 - 0.6)M_{GR}$, for $1 \leq \delta \leq 7$. This means that the strength of the gravitational force becomes comparable to that of the other forces only when the measurement takes place near the gravity scale, independently of where $M_{GR}$ lies. The contact with M-theory\footnote{Motivated by M-theory, we assume that $\delta, d \leq 7$.} is made for $\delta = 1$ and $M_{GR} \equiv (2L)^{-1} = 10^{12}$ GeV. Then, the unification takes place at $m_U \approx M_{GUT} \approx M_{GR} \approx 10^{16}$ GeV. In the case where $M_{GR} \approx 1$ TeV [8], the hierarchy problem is removed by bringing the gravitational scale down to the electroweak one, $M_W$. However, the energy gap between $M_W$ and the compactification scale $M_c$, which, for $\delta = 2$, amounts to 15 orders of magnitude, remains unexplained.

One could resolve the above problem by increasing the number $\delta$ of extra dimensions accompanied with a small increase in the value of the higher-dimensional gravity scale $M_{GR}$. For example, allowing the compactification scale, $M_c$, to be close to the electroweak one, i.e. $M_c = 10^{\pm1} M_W$, the gravity scale lies in the range $M_{GR} = (10^5 - 10^7)$ GeV, for $\delta \geq 6$. In this case, there is no energy gap between $M_c$ and $M_W$ while the gravity scale is still close to the electroweak scale (the ratio $M_{GR}/M_W$; in this case, is the same as the one of the top quark mass over the down quark mass, $m_t/m_d \approx 10^4$).

(iii) $\delta = 0$, $d \neq 0$. In this case, it is the $d$ extra, longitudinal dimensions that “open up” for the gauge and scalar fields, as well as for gravitons, at the scale $M_c$. Substituting eq. (11) into eq. (14) and solving for $m_U$, we find that the unification of strengths takes place only near the string scale, i.e. $m_U = M_s \approx 10^{18}$ GeV, independently of the number $d$ of extra dimensions. If we take the new gravity scale, $M_{GR}$, to be the string scale, then, from eq. (13), we are led to the following relation between $M_x$ and $M_{GR}$:

$$M_x = 10^{-2/d} M_{GR} = (0.01 - 0.5)M_{GR}, \quad (15)$$

for $1 \leq d \leq 7$. In this case, the compactification scale $M_x$ is 1 or 2 orders of magnitude smaller than the gravity scale. A quite interesting result arises for $d = 1$. Then, eq. (13) gives the result $M_x \approx 10^{16}$ GeV, which coincides with the scale of the unification of gauge couplings, $M_{GUT}$, in the minimal supersymmetric standard model (according to the power-law mechanism for the unification of gauge couplings [9]), the presence of the extra dimensions does not affect the unification scale $M_{GUT}$ where the compactification scale, $M_x$, is very close to the string scale, $M_s$. This arrangement is indeed necessary in order to prevent the gauge coupling $\hat{\alpha}_{EM}$ from acquiring an unacceptable small value near the unification scale, a goal which is also accomplished in our case. Although the hierarchy between the electroweak scale, $M_W$, and the gravity scale, $M_{GR}$, still remains, the existence of an extra, longitudinal dimension that “opens up” just above $M_{GUT}$ may provide a natural explanation for the energy gap between $M_{GUT}$ and $M_s$.

(iv) $\delta \neq 0$, $d \neq 0$. This is the most general case where both gravitational and gauge fields feel a number of extra dimensions. According to eq. (14), the unification of strengths, now, takes place at the energy scale

$$m_U^{2+\delta} = \alpha_{EM} M_P^{2+\delta} M_{GR}^{2+\delta}.$$  \hspace{1cm} (16)

We make the natural assumption that the ultimate unification of strengths occurs near the new gravity scale, i.e. $m_U = M_{GR}$. Then, the combination of eqs. (14) and (14) leads to the same relation (13) between the compactification scale $M_x$ and the gravity scale $M_{GR}$. On the other hand, the scale $M_c$, associated with the size of the transverse dimensions felt only by gravitons, can be found from the following expression:

$$M_c^{2+\delta} = \alpha_{EM}^{-1} M_{GR}^{2+\delta} M_P^{2+\delta}.$$  \hspace{1cm} (17)

Note that it is only the number of transverse dimensions $\delta$ that determines the compactification scale $M_c$, in terms
of $M_P$ and $M_{GR}$, exactly in the same way that it is only the number of longitudinal dimensions $d$ that determines the corresponding expression of $M_s$. If we, now, choose $M_{GR} = M_s \approx 10^{18}$ GeV and $d = 1$, the interesting result $M_s \simeq M_{GUT}$ of case (iii) arises once again. However, in order for this picture to be viable in the existence of extra, transverse dimensions for gravitons, the compactification scale $M_c$ should lie very close to the string scale for every value of $\delta$. Since $M_c, M_s \geq M_{GUT}$, the unification pattern of gauge couplings is not affected by the presence of the extra dimensions, either transverse or longitudinal, and the ultimate unification of all forces takes place exactly at the string scale. Moreover, the compactification scale $M_c$, being very close to $M_s$, does not introduce any new energy scale in the theory. The only other case where this problem is also resolved is when $M_c = 10^{4.1} M_W$, according to the analysis presented in case (ii). For $\delta = 6$ and $M_c = 10$ GeV, the gravity scale lies at the energy scale $2 \times 10^5$ GeV. Then, for $d = 1$, eq. (15) shows that the unification of strengths takes place only if $M_c$ is exactly two orders of magnitude smaller than $M_{GR}$, i.e. $M_c = 2$ TeV. Remarkably, this is in perfect agreement with the proposal of the existence of a single extra dimension for the gauge bosons with size at the TeV scale necessary for the explanation of the supersymmetry breaking scale $\delta$. The existence of an extra dimension of this size would lead to the unification of gauge couplings, through the power-law mechanism $\delta$, at an energy scale which is, approximately, one order of magnitude larger than $M_c$. The extra hypothesis, that gravitons may feel 6 extra, compact dimensions, brings the scale of gravity down to 200 TeV, thus, completing the picture of unification of forces and removing the hierarchy problem (by changing appropriately the values of $\delta$ and $M_c$, we may easily recover the case where $M_{GUT} = M_{GR} = M_s = 10$ TeV $\delta$).

Finally, let us stress that all the results presented above can be consistently embedded in a Type I string theory framework. Combining and rearranging eqs. (13) and (14), we can write our unification constraint in the form

$$\left(\frac{m_U}{M_p}\right)^2 = \hat{\alpha}_EM_U 2L/\delta \left(m_U 2L/\delta\right)^{-d},$$

which reduces to the Type I string constraint $M_s^2 \sim \left(\hat{\alpha}_{gauge} M_P V\sqrt{\delta}\right)^{-1}$ if we identify $m_U$ with $M_s$, and $\hat{\alpha}_E$ with $\hat{\alpha}_{gauge}$. The total compactified volume is now $V = (2L)^d (2L/\delta)^3$ and a T-duality transformation $m_U \leftrightarrow (m_U)^{-1}$ needs to be performed, where $r \equiv 2L, 2L'$.

In conclusion, in this letter, we have demonstrated that the addition of extra, compact dimensions for gravitons and gauge bosons modify the strengths of gravitational and gauge interactions and consequently the scale of their unification. When the gravitons propagate in $4 + \delta$ dimensions, the unification takes place only near the higher dimensional gravity scale, $M_{GR}$. If we choose $M_s \sim M_W$, no new scale is introduced in the theory while $M_{GR}$ turns out to be a few orders of magnitude larger than $M_W$. In the case where the gauge bosons feel $d$ extra dimensions, the unification occurs only near the string scale, $M_s$. If $M_{GR} \sim M_s$, $M_s$ turns out to be, for $d = 1$, of the order of the gauge coupling unification scale. Thus, the energy gap between $M_s$ and $M_{GUT}$ is attributed to the existence of one extra dimension for the gauge bosons which “opens up” just above $M_{GUT}$. An even more remarkable result arises in the case where both $\delta$ and $d$ are non-zero under the assumption that the gravitons can feel 6 extra dimensions with $M_c = 10$ GeV. Then, for $d = 1$, $M_s$ has exactly the right value to explain the size of the supersymmetry breaking scale with the gravity scale being only 2 orders of magnitude larger than $M_s$. In the framework of this 11-dimensional theory, the unification of all forces takes place at 200 TeV and the hierarchy problem is completely removed. It is worth noting that all the results derived from the above analysis, based on a non-renormalizable, effective field theory for gauge and gravitational interactions, can be consistently embedded in the framework of string theory unification.

The authors would like to thank K.R. Dienes, H.P. Nilles, R.G. Roberts, G.G. Ross and K. Tamvakis for fruitful discussions and comments. A.D is supported from Marie Curie Research Training Grants ERB-FMFBI-CT98-3438. P.K. is grateful to the Particle Physics Group at Rutherford Appleton Laboratory for the warm hospitality and financial support.

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