Abstract

The problem of time in canonical quantum gravity is related to the fact that the canonical description is based on the prior choice of a spacelike foliation, hence making a reference to a spacetime metric. However, the metric is expected to be a dynamical, fluctuating quantity in quantum gravity. We show how this problem can be solved in the histories formulation of general relativity. We implement the 3+1 decomposition using metric-dependent foliations which remain spacelike with respect to all possible Lorentzian metrics. This allows us to find an explicit relation of covariant and canonical quantities which preserves the spacetime character of the canonical description. In this new construction, we also have a coexistence of the spacetime diffeomorphisms group, Diff(M), and the Dirac algebra of constraints.
1 Introduction

1.1 On the quantisation of a Diff(M)-invariant theory of gravitation.

One of the major approaches to the quantisation of gravity is the canonical one, either in its original form—involving geometrodynamic variables [1]—or in terms of the loop variables (see [2] for a review), introduced via the connection formulation of canonical general relativity [3]. The canonical connection formulation of the general quantisation programme starts from the Hamiltonian description of general relativity and seeks to implement some version of the general theory of canonically quantising systems with first class constraints.

Canonical quantisation involves the identification of a Hilbert space on which the canonical commutation relations—or some other appropriate algebraic structure—can be implemented, thereby defining the kinematical variables of quantum gravity. The Hilbert space is chosen to allow the representation of the constraints of the Hamiltonian description in terms of self-adjoint operators, preserving the classical Dirac algebra of constraints. Then, one has to find the zero eigenspace of the constraint operators, in order to define the physical Hilbert space. This is the scope of the original Dirac quantisation of constrained systems: variations are usually employed in the case of gravity (or special models), because the constraint operators are not expected to have a discrete spectrum.

In the canonical quantisation scheme, much of the research has focused on the technical problems of constructing the Hilbert space, writing the constraint operators, and finding their spectrum. However, the canonical formalism encounters serious problems even at the first stage of implementation. In particular, the fact that general relativity is a generally covariant theory raises grave doubts about the conceptual adequacy of the canonical method of quantisation—at least in its present form.

This last remark is highlighted by the fact that the equations of general relativity are covariant with respect to the action of the diffeomorphisms group Diff(M), of the spacetime manifold M. Although the invariance under the action of Diff(M) is particularly relevant to the notion of an ‘observable’, it does not pose great difficulties in the classical theory, since once the equations of motion are solved the Lorentzian metric on M can be used to
implement concepts like causality and spacelike separation.

However, in quantum theory such notions are lost, because the geometry of spacetime is expected to be subject to quantum fluctuations. This creates problems even at the first step of the quantisation procedure, namely the definition of the canonical commutation relations. The canonical commutation relations are defined on a ‘spacelike’ surface: however, a surface is spacelike with respect to some particular spacetime metric $g$, which is itself a quantum observable that is expected to fluctuate.

The prior definability of the canonical commutation relations is not merely a mathematical requirement. In a generic quantum field theory the canonical commutation relations implement the principle of microcausality: namely that field observables that are defined in spacelike separated regions commute. However, if the notion of spacelikeness is also dynamical, it is not clear in what way this relation will persist.

Moreover, a spacelike foliation is necessary for the implementation of the $3+1$ decomposition and the definition of the Hamiltonian. Again we are faced with the question of how to reconcile the requirement of spacelikeness with the expectation that different metrics will take part in the quantum description. This problem needs to be addressed if the canonical quantum theory is to have a spacetime character, i.e. if the quantum true degrees of freedom are to correspond to a Lorentzian four-metric.

Even more, one may question whether the predictions of the resulting quantum theories are independent of the choice of foliation. The Hilbert space of the quantum theory, which it is constructed canonically, is not straightforwardly compatible with the Diff($M$) symmetry. In the canonical theory, the symmetry group is the one generated by the Dirac algebra of constraints, which is mathematically distinct from the Diff($M$) group. In effect, different choices of foliation lead a priori to different quantum theories, and there is absolutely no guarantee that these quantum theories are unitarily equivalent (or physically equivalent in some other generalised sense). The canonical description cannot provide an answer to these questions, because once the foliation is employed for the 3+1 decomposition, its effect is lost, and there is no way of relating the predictions corresponding to different foliations.

These are serious problems, which challenge the validity of the canonical approach towards the description of a generally covariant theory of quantum gravity.
Finally, the problem which is perhaps most well known, is the problem of time (for a review see [4, 5]). The Hamiltonian of general relativity is a combination of the first class constraints, hence it vanishes on the reduced state space. It is expected also to vanish on the physical Hilbert space constructed in the quantisation scheme. This means that there is no notion of time evolution in the space of true degrees of freedom. More than that, the notion of time as causal ordering seems to be lost.

In contrast, the tensorial expressions of the equations of motion are Diff(M)-invariant in the Lagrangian formalism. It is not surprising then that Dirac[17] attempted to write a Lagrangian quantum action functional analogue for general relativity; his results led to the well known path integrals techniques. However, path integrals cannot be well defined for general relativity. Moreover, path-integral techniques do not provide a full description of the quantum theory and need to be supplemented with the introduction of Hilbert space objects—and hence a canonical description—in order to make physical predictions.

It seems very natural, therefore, to wish for a theory that combines the virtues of both formalisms: the Lagrangian, and the Hamiltonian.

1.2 On the dual spacetime-canonical nature of histories formalism.

The consistent-histories approach to quantum theory was initiated by Griffiths, Omnès[6], Gell-Mann and Hartle [7]. A history is defined as a sequence of time-ordered propositions about properties of a physical system, each of which can be represented, as usual, by a projection operator. In normal quantum theory it is not possible to assign a probability measure to the set of all histories; however, when a certain 'decoherence condition' is satisfied by a set of histories, the elements of this set can be given probabilities. The probability information of the theory is encoded in the decoherence functional—a complex function of pairs of histories.

Isham and Linden proposed that propositions about the histories of a system should be represented by projection operators on a new, ‘history’ Hilbert space [8, 9, 10]. An important way of understanding the history Hilbert space \( \mathcal{V} \) is in terms of the representations of the ‘history group’—in elementary systems this is the history analogue of the canonical group.
For example, for the simple case of a point particle moving on a line, the history group for a continuous time parameter \( t \) is described by the history commutation relations

\[
[x_t, x_{t'}] = 0 = [p_t, p_{t'}] \quad (1.1)
\]
\[
[x_t, p_{t'}] = i\hbar \delta(t - t'), \quad (1.2)
\]

where the spectral projectors of the (Schrödinger picture) operators \( x_t \) and \( p_t \) represent the values of position and momentum respectively at time \( t \).

This particular history algebra is equivalent to the algebra of a 1+1-dimensional quantum field theory, and hence techniques from quantum field theory (for example, for handling the problem of the existence of many inequivalent representations of the algebra in Eqs. (1.1–1.2)) can be used in the study of the history Hilbert space. This was done successfully in [11], where we showed that the physically appropriate representation can be uniquely constructed by demanding the existence of a time-averaged Hamiltonian operator \( H := \int dt \, H_t \).

A significant result emerged from the study of continuous-time transformations. Namely, that there exist two distinct generators of time transformation [12]. One refers to time as an ordering parameter (\( t \)-label in Eqs. (1.1–1.2)), which is related to the causal structure and the kinematics of a theory. The other generator refers to time as it appears in the implementation of dynamical laws (the label \( s \) in the ‘history Heisenberg picture’ operator \( x_t(s) := e^{isH} x_t e^{-isH} \)), and it is related to the Hamiltonian evolution and the dynamics of a theory.

Most importantly: for any specific physical system these two transformations are intertwined by the definition of the action operator—a quantum analogue of the classical action functional. Hence, the definition of these two distinct operators implementing time transformations signifies the distinction between the kinematics and the dynamics of the theory. This distinction and the corresponding definitions are also valid for classical histories.

One of the most important consequences of the histories approach is that a combined spacetime-canonical formalism emerges. The richer temporal structure of a history theory allows the simultaneous description of both spacetime and canonical objects.

In a preliminary study [13], we presented a history version of general relativity, which demonstrates a new relation between the group structures,
associated to the Lagrangian and Hamiltonian approaches. In particular, we showed that in this histories version of canonical general relativity there exists a representation of the spacetime diffeomorphisms group $\text{Diff}(M)$, together with a history analogue of the Dirac algebra of constraints.

However, various important issues arose. First, the history canonical algebra depends on the choice of a Lorentzian foliation. Hence, a natural question is the degree to which physical results depend upon this choice. For each choice, the solutions to the equations of motion enable us to construct different 4-metrics. If different descriptions are to be equivalent, two distinct 4-metrics should be related by a spacetime diffeomorphism. We should therefore establish that the action of the spacetime diffeomorphisms group intertwines between the constructions associated with the different choices of the foliation. This involves the comparison of history state space associated with arbitrary choices of foliation.

Second, and perhaps more important, we need to question the notion of a spacelike foliation itself. Since the spacetime causal structure is a dynamical object, the notion of a foliation being spacelike has meaning only after the solution to the classical equations of motion has been selected. However, in the histories description we do not use just a single solution of the classical equations of motion (indeed, many of the possible histories are not solutions at all), and in these circumstances the notion of a ‘spacelike’ foliation loses its meaning.

These are some of the deepest issues not only of history theories but of any canonical approach to gravity. Hence, they inevitably require a significant reworking of the theory. In the present paper, we successfully address these issues by focusing on a crucial point of the histories formalism: the connection between the covariant and the canonical description of histories general relativity.

The key idea of this new construction is the introduction of the notion of a metric-dependent foliation. Specifically, we choose foliations that are functionals of the four-metric $g$, and that are required to be spacelike with respect to $g$. In particular, under the action of a spacetime diffeomorphism, a foliation that is spacelike will preserve this character as both the embedding and the metric will transform together. This is a simple idea, but it works very successfully for the histories formalism, and allows us to include all different choices of foliation in studying the foliation dependence of the history canonical algebra.
The plan of the paper is as follows. In section 2 we write a brief summary of the histories formalism. In section 3.1 we present the spacetime description of histories general relativity, and we write the representation of the spacetime diffeomorphisms $\text{Diff}(M)$. In section 3.2 we give a detailed presentation of the relation between the spacetime and the canonical descriptions. We emphasise the difference between foliations that do not have a metric-dependence, and those that do, and we use the latter to construct the explicit relation between the covariant history space $\Pi^{\text{cov}}$ and the canonical history space $\Pi^{\text{can}}$. Next, we write the symplectic forms for both descriptions, and we show that they are equivalent.

In section 3.3, we present the canonical description of histories general relativity. We write the Dirac algebra of constraints, and we show that in the histories theory of metric-dependent foliations, there exist representations of both the group of spacetime diffeomorphisms and of the Dirac algebra of constraints.

## 2 Background

**Temporal Structure of HPO histories theory.** Although the histories programme originated from the consistent histories theory, it was developed with an emphasis on the ‘temporal’ logic of the theory [8]. However, the HPO (‘Histories Projection Operator’) theory takes a completely different turn once the concept of time is introduced in a new way [12].

In the consistent histories formalism, a history $\alpha = (\hat{\alpha}_{t_1}, \hat{\alpha}_{t_2}, \ldots, \hat{\alpha}_{t_n})$ is defined to be a collection of projection operators $\hat{\alpha}_{t_i}$, $i = 1, 2, \ldots, n$, each of which represents a property of the system at the single time $t_i$. Therefore, the emphasis is placed on histories, rather than properties at a single time, which in turn gives rise to the possibility of generalized histories with novel concepts of time.

The HPO approach, places particular emphasis on temporal logic. This is achieved by representing the history $\alpha$ as the operator $\hat{\alpha} := \hat{\alpha}_{t_1} \otimes \hat{\alpha}_{t_2} \otimes \cdots \otimes \hat{\alpha}_{t_n}$ which is a genuine projection operator on the tensor product $\otimes_{i=1}^{n} \mathcal{H}_{t_i}$ of copies of the standard Hilbert space $\mathcal{H}$. Note that to use this construction in any type of field theory requires an extension to a continuous time label, and hence to an appropriate definition of the continuous tensor product $\otimes_{t \in \mathbb{R}} \mathcal{H}_t$ [10, 14].
A central feature of the histories theory is the development of the novel temporal structure [12], namely the existence of two distinct types of time transformation.

**Classical Histories** The space of classical histories $\Pi = \{ \gamma | \gamma : \mathbb{R} \rightarrow \Gamma \}$ is the set of all smooth paths on the classical state space $\Gamma$. It can be equipped with a natural symplectic structure, which gives rise to Poisson brackets. For the simple case of a particle on a line, we have

\[
\{ x_t, x_{t'} \}_\Pi = 0 \quad (2.1)
\]
\[
\{ p_t, p_{t'} \}_\Pi = 0 \quad (2.2)
\]
\[
\{ x_t, p_{t'} \}_\Pi = \delta(t - t') \quad (2.3)
\]

where

\[
x_t : \Pi \rightarrow \mathbb{R} \quad (2.4)
\]
\[
\gamma \mapsto x_t(\gamma) := x(\gamma(t)) \quad (2.5)
\]
and similarly for $p_t$.

The classical analogue of the Liouville operator is defined as

\[
V(\gamma) := \int_{-\infty}^{\infty} dt p_t \dot{x}_t, \quad (2.6)
\]
and the Hamiltonian (i.e., time-averaged energy) function $H$ is defined as

\[
H(\gamma) := \int_{-\infty}^{\infty} dt H_t(x_t, p_t) \quad (2.7)
\]
where $H_t$ is the Hamiltonian that is associated with the copy $\Gamma_t$ of the normal classical state space with the same time label $t$.

The temporal structure leads to the histories analogue of the classical equations of motion

\[
\{ F, V \}_\Pi (\gamma_{cl}) = \{ F, H \}_\Pi (\gamma_{cl}) \quad (2.8)
\]
where $F$ is any function on $\Pi$, and where the path $\gamma_{cl}$ is a solution of the equations of motion.
A crucial result is that the history equivalent of the classical equations of motion is given by the following condition that holds for all functions $F$ on $\Pi$ when $\gamma_{cl}$ is a classical solution:

$$\{F, S\}_{\Pi} (\gamma_{cl}) = 0,$$

(2.9)

where

$$S(\gamma) := \int_{-\infty}^{\infty} dt \left( p_t \dot{x}_t - H_t(x_t, p_t) \right) = V(\gamma) - H(\gamma)$$

(2.10)

is the classical analogue of the action operator. This is the history analogue of the least action principle [12].

The temporal structure of HPO histories enables us to treat parameterised systems in such a way that the problem of time does not arise [15]. Indeed, histories keep their intrinsic temporality after the implementation of the constraint: thus there is no uncertainty about the temporal-ordering properties of the physical system.

**Quantum histories.** In the corresponding quantum theory, the Hamiltonian operator $H$ and the ‘Liouville’ operator $V$ are the generators of the two types of time transformation [12]. Specifically, the Hamiltonian $H$ is the generator of the unitary time evolution with respect to the ‘internal’ time label $s$; this has no effect on the time label $t$. On the other hand, the Liouville operator $V$—defined in analogy to the kinematical part of the classical action functional—generates time translations along the $t$-time axis without affecting the $s$-label.

We can define the action operator $S$ as a quantum analogue of the classical action functional Eq. (2.10). It transpires that the action operator $S$ generates both types of time transformation, and in this sense it is the generator of physical time translations in the histories formalism.

The time transformations generated by the action operator $S$ resemble the canonical transformations generated by the Hamilton-Jacobi action functional. Indeed, there is an interesting relation between the definition of $S$ and the well-known work by Dirac on the Lagrangian theory for quantum mechanics [17, 12]. In particular, motivated by the fact that—contrary to the Hamiltonian method—the Lagrangian method can be expressed relativistically (on account of the action function being a relativistic invariant), Dirac tried to take over the general ideas of the classical Lagrangian theory, albeit not the equations of the Lagrangian theory *per se*. 
3 Histories General Relativity

3.1 Spacetime description of histories general relativity theory

In order to apply the histories theory to general relativity we start with the description of spacetime quantities. We consider a four-dimensional manifold $M$, which has the topology $\mathbb{R} \times \Sigma$. The history space is defined as $\Pi^{\text{cov}} = T^*\text{LRiem}(M)$, where $\text{LRiem}(M)$ is the space of all Lorentzian four-metrics $g_{\mu\nu}$, and $T^*\text{LRiem}(M)$ is its cotangent bundle. Hence, the history space $\Pi^{\text{cov}}$ for general relativity is the space of all histories $(g_{\mu\nu}, \pi^{\mu\nu})$.

The history space $\Pi^{\text{cov}}$ is equipped with the symplectic form

$$\Omega = \int d^4X \, \delta \pi^{\mu\nu}(X) \wedge \delta g_{\mu\nu}(X), \quad (3.1)$$

where $X$ is a point in the spacetime $M$, and where $g_{\mu\nu}(X)$ is a four-metric that belongs to the space of Lorentzian metrics $\text{LRiem}(M)$, and $\pi^{\mu\nu}(X)$ is the conjugate variable.

The symplectic structure Eq. (3.1) generates the following covariant Poisson brackets algebra, on the history space $\Pi^{\text{cov}}$

$$\{g_{\mu\nu}(X), g_{\alpha\beta}(X')\} = 0 \quad (3.2)$$
$$\{\pi^{\mu\nu}(X), \pi^{\alpha\beta}(X')\} = 0 \quad (3.3)$$
$$\{g_{\mu\nu}(X), \pi^{\alpha\beta}(X')\} = \delta_{(\mu\nu)}^{\alpha\beta} \delta^4(X, X') \quad (3.4)$$

where $\delta_{(\mu\nu)}^{\alpha\beta} := \frac{1}{2}(\delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} + \delta_{\mu}^{\beta} \delta_{\nu}^{\alpha})$.

3.1.1 The representation of the group $\text{Diff}(M)$

The critical observation now is that we can write a representation of the spacetime diffeomorphisms group $\text{Diff}(M)$ on the history space $\Pi^{\text{cov}}$.

In previous applications of the histories formalism we defined the Liouville function $V$ as the generator of time translations with respect to the external time $t$ that appears as a kinematical ordering parameter [12, 15, 16]. In the present case we define, by analogy, the Liouville function $V_W$ associated with any vector field $W$ on $M$ as

$$V_W := \int d^4X \, \pi^{\mu\nu}(X) \mathcal{L}_W g_{\mu\nu}(X) \quad (3.5)$$
where $\mathcal{L}_W$ denotes the Lie derivative with respect to $W$. This is the analogue of the expression that is used in the normal canonical theory for the representations of spatial diffeomorphisms.

The fundamental result is that these generalised Liouville functions $V_W$, defined for any vector field $W$ as in Eq. (3.5), satisfy the Lie algebra of the \textit{spacetime diffeomorphisms group} $\text{Diff}(M)$:

$$\{ V_{W_1}, V_{W_2} \} = V_{[W_1,W_2]},$$

where $[W_1,W_2]$ is the Lie bracket between vector fields $W_1$ and $W_2$ on the manifold $M$.

We note here that the functional $V_W$ acts upon the basic variables of the theory as an infinitesimal diffeomorphism:

$$\{ g_{\mu\nu}(X), V_W \} = \mathcal{L}_W g_{\mu\nu}(X),$$

$$\{ \pi^{\mu\nu}(X), V_W \} = \mathcal{L}_W \pi^{\mu\nu}(X).$$

### 3.2 Relation between spacetime and canonical descriptions

Next, we show how the histories Dirac algebra of constraints appears in the history space $\Pi^{\text{cov}}$. For this reason we study the relation of the covariant (spacetime) description with its canonical (evolutionary) counterpart.

We introduce a $3+1$ foliation of the spacetime $M$, which is spacelike with respect to a Lorentzian metric $g$, in order to construct a $3+1$ description of the theory. However, a key feature of the present construction is that this foliation is required to be \textit{four-metric dependent} in order to address the key issue of requiring all the different choices of foliation to be spacelike. We will then show the relation between the covariant history space $\Pi^{\text{cov}}$ and its canonical counterpart $\Pi^{\text{can}}$.

In the appendix A we have collected some mathematical definitions that are necessary for the understanding of the connection between the covariant, and the $3+1$, formulations of the theory.

#### 3.2.1 Foliations not depending on four-metric $g$

We consider the spacetime manifold $M = \mathbb{R} \times \Sigma$ and the space $\text{Fol}(M)$ of all foliations of $M$ that are spacelike with respect to at least one Lorentzian
metric. For each specific Lorentzian metric $g$ we choose a spacelike foliation, $\mathcal{E} : \mathbb{R} \times \Sigma \to M$ with an associated family of spacelike embeddings $\mathcal{E}_t : \Sigma \to M$, $t \in \mathbb{R}$. We then define the pull-back of $g_{\mu\nu}$ to $\mathbb{R} \times \Sigma$ as $\mathcal{E}^* g$. We also wish to pull-back the conjugate variable $\pi^{\alpha\beta}$ to $\mathbb{R} \times \Sigma$. For this purpose, we lower the indices and define the field $\pi_{\alpha\beta}(X) = g_{\gamma\alpha}(X) g_{\zeta\beta}(X) \pi^{\gamma\zeta}(X)$, which is a $(0, 2)$ tensor that can be pull-backed on $\Sigma$ in the usual way.

Using the Poisson bracket equations (3.2–3.4), we get the relations

$$\{ g_{\mu\nu}(X), g_{\alpha\beta}(X') \} = 0 \quad (3.9)$$
$$\{ \pi_{\mu\nu}(X), \pi_{\alpha\beta}(X') \} = 0 \quad (3.10)$$
$$\{ g_{\mu\nu}(X), \pi_{\alpha\beta}(X') \} = g_{(\mu\alpha} g_{\nu)\beta}(X) \delta^{4}(X, X') \quad (3.11)$$

where $g_{(\mu\alpha} g_{\nu)\beta}(X) := \frac{1}{2}(g_{\mu\alpha}(X) g_{\nu\beta}(X) + g_{\nu\alpha}(X) g_{\mu\beta}(X))$.

We define the deformation vector Eq. (A.6), that is uniquely selected by the choice of this one-parameter family of embeddings of $\Sigma$ in $M$. This family of embeddings also allows the selection of a coordinate system common to all the embedded three-surfaces in the sense that the coordinate defined on the reference three-surface $\Sigma$ is shared by all of them.

Next we define the spatial parts of the pull-back of $g_{\mu\nu}(X)$ to $\mathbb{R} \times \Sigma$ by $\mathcal{E}$ as

$$h_{ij}(t, x) := \mathcal{E}_i^\mu(t, x) \mathcal{E}_j^\nu(t, x) g_{\mu\nu}(\mathcal{E}(t, x)) \quad (3.12)$$

where $\mathcal{E}_i^\mu(t, x) := \partial_i(\mathcal{E}^\mu(t, x))$.

The choice of a foliation $\mathcal{E}$, spacelike with respect to a specific metric $g$, uniquely defines the lapse function $N$ and shift vector $N_i$ of the $3 + 1$ decomposition of the four-metric, as

$$N(t, x) = \dot{\mathcal{E}}^\mu n_\mu(\mathcal{E}(t, x)) \quad (3.13)$$
$$N_i(t, x) = \mathcal{E}_i^\mu \mathcal{E}_j^\nu \dot{\mathcal{E}}^\nu g_{\mu\nu}(\mathcal{E}(t, x)) \quad (3.14)$$

where $n_\mu$ is the unit, timelike vector, normal to the foliation, and

$$g^{\mu\nu} = -\frac{1}{N^2} \dot{\mathcal{E}}^\mu \dot{\mathcal{E}}^\nu + (\mathcal{E}_i^\mu \mathcal{E}_j^\nu + \mathcal{E}_i^\nu \mathcal{E}_j^\mu) \frac{N_i}{N^2} + (h^{ij} - \frac{N_i N_j}{N^2}) \mathcal{E}_i^\mu \mathcal{E}_j^\nu. \quad (3.15)$$
3.2.2 Foliations depending on four-metric $g$

We have showed so far that, for a fixed metric $g$ we can choose the foliation to be spacelike in the sense that $t \mapsto h_{ij}(t, \underline{x})$ is a path in the space of Riemannian metrics on $\Sigma$. For an appropriate topology on $LRiem(M)$, this spacelike character will be maintained for some open neighborhood of the Lorentzian metric $g$. However, this foliation fails to be spacelike for most other Lorentzian metrics on $M$. This feature is not important at the level of the classical theory, because we only consider the four-metric, which is the solution of the equations of motion; however it can be expected to be a non-trivial issue in the quantum theory.

When we consider a fixed foliation for all four-metrics, there will be metrics $g \in LRiem(M)$, for which some of the pullbacks $\mathcal{E}_i^* g$, $t \in \mathbb{R}$, will not be a Riemannian three-metric on $\Sigma$. In other words, the pull-back space $\mathcal{E}_i^* LRiem(M)$ does not coincide with the space of Riemannian metrics $Riem(\Sigma)$, which is the space of the canonical description of general relativity.

We want to place special emphasis on this point: it reflects one of the major conceptual problems of the canonical description of gravity, and it is directly related to the problem of time in quantum gravity. As explained in the Introduction, a general relativity canonical description involves the choice of a specific spacelike foliation; however, in a theory where the metric is a non-deterministic dynamical variable—as it is expected in quantum gravity—the notion of ‘spacelike’ has no a priori meaning.

In order to address this important issue we introduce the notion of a metric-dependent foliation.

To this end, for each $g \in LRiem(M)$ we choose a spacelike foliation $\mathcal{E}[g]$. For a given Lorentzian metric $g$, we use the foliation $\mathcal{E}[g]$ to split $g$ with respect to the Riemannian three-metric $h_{ij}$, the lapse function $N$ and the shift vector $N^i$ as

$$h_{ij}(t, \underline{x}; g) := \mathcal{E}_i^\mu(t, \underline{x}; g) \mathcal{E}_j^\nu(t, \underline{x}; g) g_{\mu\nu}(\mathcal{E}(t, \underline{x}; g)) \quad (3.16)$$

$$N_i(t, \underline{x}; g) := \mathcal{E}_i^\mu(t, \underline{x}; g) \mathcal{E}_j^\nu(t, \underline{x}; g) g_{\mu\nu}(\mathcal{E}(t, \underline{x}; g)) \quad (3.17)$$

$$-N^2(t, \underline{x}; g) := \mathcal{E}_i^\mu(t, \underline{x}; g) \mathcal{E}_j^\nu(t, \underline{x}; g) g_{\mu\nu}(\mathcal{E}(t, \underline{x}; g)) - N_i N^i(t, \underline{x}) (3.18)$$
3.2.3 The relation between the covariant history space $\Pi^{\text{cov}}$ and the history space of the canonical description $\Pi^{\text{can}}$

We have showed that the history space of the spacetime description of histories general relativity $\Pi^{\text{cov}} = T^*LRiem(M)$ is equipped with the symplectic structure characterised by the symplectic form Eq. (3.1). In order to relate $\Pi^{\text{cov}}$ with its canonical counterpart, it suffices to write its symplectic form $\Omega$ in terms of the canonical variables $h_{ij}$, $N_i$ and $N$—that enter in the $3+1$ decomposition of the Lorentzian metric $g$—and their corresponding conjugate momenta.

To this end, starting from Eq. (3.15), we have

$$\delta g^{\mu\nu} = \left[ \frac{1}{N^2} \delta (\dot{\xi}^\mu \dot{\xi}^\nu) + \delta \left( \frac{N_i}{N^2} (\dot{\xi}^\mu \mathcal{E}^\nu_i + \mathcal{E}^\mu_i \dot{\xi}^\nu) + \delta h^{ij} \mathcal{E}^\mu_i \mathcal{E}^\nu_j - \delta \left( \frac{N_i N_j}{N^2} \right) \mathcal{E}^\mu_i \mathcal{E}^\nu_j \right] \right] \delta g^{\rho\sigma} \text{ (3.19)}$$

We note that the first bracket of the right-hand side of Eq. (3.19) corresponds to the variation of $g$ (in terms of $\delta N_i$, $\delta N$, and $\delta h^{ij}$) that would occur if the foliation was not metric-dependent; we will denote this term as $A^{\mu\nu}$. The terms within the second bracket arise from the fact that the foliation is metric dependent. These extra terms can be written in the form $(B^{\mu\nu}_{\rho\sigma} + C^{\mu\nu}_{\rho\sigma} + D^{\mu\nu}_{\rho\sigma}) \delta g^{\rho\sigma}$ where

$$B^{\mu\nu}_{\rho\sigma}(X, X') \equiv \left( \frac{2}{N^2} \dot{\xi}^\nu(X; g) + \frac{2 N_i}{N^2} \mathcal{E}^\nu_i(X; g) \right) \frac{\delta \dot{\xi}^\mu(X; g)}{\delta g^{\rho\sigma}(X')} \text{ (3.20)}$$

$$C^{\mu\nu}_{\rho\sigma}(X, X') \equiv \left( \frac{2 N_i}{N^2} \dot{\xi}^\nu(X; g) + (h^{ij} - \frac{N_i N_j}{N^2}) \mathcal{E}^\nu_j(X; g) \right) \frac{\delta \mathcal{E}^\mu_i(X; g)}{\delta g^{\rho\sigma}(X')} \text{ (3.21)}$$

$$D^{\mu\nu}_{\rho\sigma}(X, X') \equiv \left( \frac{N_i N_j}{N^2} \mathcal{E}^\nu_j(X; g) \right) \frac{\delta \mathcal{E}^\mu_i(X; g)}{\delta g^{\rho\sigma}(X')} \text{ (3.22)}$$

so that we have

$$\delta g^{\mu\nu} = A^{\mu\nu} + (B^{\mu\nu}_{\rho\sigma} + C^{\mu\nu}_{\rho\sigma} + D^{\mu\nu}_{\rho\sigma}) \delta g^{\rho\sigma} \text{ (3.23)}$$

which can be rewritten as

$$A^{\mu\nu} = E^{\mu\nu}_{\rho\sigma} \delta g^{\rho\sigma} \text{ (3.24)}$$

where

$$E^{\mu\nu}_{\rho\sigma}(X, X') = (1 - B - C - D)^{\mu\nu}_{\rho\sigma}(X, X'). \text{ (3.25)}$$
For Eq. 3.24 to be meaningful, it is necessary that the inverse of the tensor \( E_{\mu\nu} \) exists. This holds for a metric-independent \( E \) since then \( B = C = D = 0 \), and the condition will continue to be satisfied for small values of the functional derivative \( \delta E^\mu / \delta g^{\rho\sigma} \). There is a prima facie possibility that \( E \) could become non-invertible for sufficiently large values of \( \delta E^\mu / \delta g^{\rho\sigma} \), but this is part of the general question of the overall global structure of the history symplectic space and is not something with which we shall concern ourselves in the present paper.

With this proviso, detailed calculations show that the symplectic form \( \Omega \) can be written in the equivalent canonical form, with respect to a chosen element, \( E \), of \( \text{Fol}_g(M) \), as

\[
\Omega = \int d^4X \delta \pi^{\mu\nu} \wedge \delta g_{\mu\nu} = -\int d^4X \delta \tilde{\pi}^{ij} \wedge \delta h_{ij} + \delta \tilde{p} \wedge \delta N + \delta \tilde{p}_i \wedge \delta N^i, \tag{3.26}
\]

where

\[
\tilde{\pi}^{ij} := K(t, x)(E^{-1})_{\mu\nu} h^{\mu k} h^{\nu l} E^\mu_{,k} E^\nu_{,l} \tag{3.27}
\]

\[
\tilde{p} := -K(t, x)(E^{-1})_{\mu\nu} \frac{2}{N^3} (\dot{E}^\mu \dot{E}^\nu - 2 N^i (E^\mu_{,i} \dot{E}^\nu + \dot{E}^\mu E^\nu_{,i}) + N^i N^j E^\mu_{,i} E^\nu_{,j}) \tag{3.28}
\]

\[
\tilde{p}_i := -K(t, x)(E^{-1})_{\mu\nu} (E^\mu_{,i} \dot{E}^\nu + \dot{E}^\mu E^\nu_{,i} - N^j (E^\mu_{,i} E^\nu_{,j} + E^\mu_{,j} E^\nu_{,i})). \tag{3.29}
\]

Here \( K(t, x) \) is the determinant of the transformation from the \( X \) to the \( (t, x) \) variables. Given that the volume form reads

\[
\sqrt{-g} \, dX^0 \wedge dX^1 \wedge dX^2 \wedge dX^3 = N \sqrt{\tilde{h}} \, dt \wedge dx^1 \wedge dx^2 \wedge dx^3, \tag{3.30}
\]

we identify \( K(t, x) \) as

\[
K(t, x) = \frac{N(t, x) \sqrt{\tilde{h}(t, x)}}{\sqrt{-g(E(t, x))}}. \tag{3.31}
\]

Here \( \tilde{h} \) is the determinant of the matrix \( h_{ij} \). Note that it is a density of weight 1 with respect to time as well as spatial variables and this renders \( \tilde{\pi}^{ij}, \tilde{p}_i, \tilde{p} \) densities of weight 1 with respect to time.

\[1\]It is customary in the canonical description to consider the lapse function as a density.
In terms of the normal vector \( n^\mu = \frac{1}{\tilde{N}}(\tilde{\mathcal{E}}^\mu - N^j\mathcal{E}^\mu_i) \), the momenta \( \tilde{p} \) and \( \tilde{p}_i \) are

\[
\tilde{p} := -K(t, \underline{x}) \frac{2}{\tilde{N}}(E^{-1\top} \pi)^{\mu\nu} n_\mu n_\nu \quad \text{(3.32)}
\]
\[
\tilde{p}_i := -K(t, \underline{x}) (E^{-1\top} \pi)^{\mu\nu} (n^\mu \dot{E}^\nu_i + n^\nu \dot{E}_i^\mu) \quad \text{(3.33)}
\]

In the special case of a metric-independent foliation, we recover the familiar definitions

\[
\tilde{\pi}^{ij} := K(t, \underline{x}) \pi^{\mu\nu} h^{ik} h^{jl} \mathcal{E}_k^\mu \mathcal{E}_l^\nu \quad \text{(3.34)}
\]
\[
\tilde{p} := -K(t, \underline{x}) \pi^{\mu\nu} 2\frac{\tilde{N}}{\tilde{N}} n^\mu n^\nu \quad \text{(3.35)}
\]
\[
\tilde{p}_i := -K(t, \underline{x}) (\pi^{\mu\nu} n^\mu \dot{E}^\nu_i + n^\nu \dot{E}_i^\mu) \quad \text{(3.36)}
\]

To formulate the final step of the connection between the covariant and the canonical histories spaces, we recall that, in the histories formalism, the basic element is a history, which is a path \( t \mapsto \Gamma \). The objects \( \tilde{p}(t, \underline{x}) \) and \( \tilde{p}_i(t, \underline{x}) \) are densities with respect to reparameterisations of the \( t \) label, hence the association \( t \mapsto \tilde{p}(t, \underline{x}) \) does not correspond to a path in the space of scalar fields on \( \Sigma \). For this reason, we can use as history canonical variables the objects \( \pi^{ij}(t, \underline{x}), p(t, \underline{x}) \) and \( p_i(t, \underline{x}) \), that are scalar functions with respect to the time variable \( t \). Hence, we define the scalar histories quantities

\[
\pi^{ij}(t, \underline{x}) := \alpha(t)\tilde{\pi}^{ij}(t, \underline{x}) \quad \text{(3.37)}
\]
\[
p_i(t, \underline{x}) := \alpha(t)\tilde{p}_i(t, \underline{x}) \quad \text{(3.38)}
\]
\[
p(t, \underline{x}) := \alpha(t)\tilde{p}(t, \underline{x}) \quad \text{(3.39)}
\]

where \( \tilde{N} \) is defined from Eq. (3.18), and where \( \alpha(t) \) is some strictly positive scalar density of weight \(-1\) in the variable \( t^2 \).

However, by its definition (3.17) the lapse function is a scalar function on \( M \). The reason it is considered as density with respect to time is equation (3.35). The determinant \( \sqrt{h} \) is, strictly speaking, a density of weight 1 with respect to time, even though it is defined by means of the spatial metric \( h_{ij} \). However, in the canonical treatment the time-density nature of \( \sqrt{h} \) is ignored and for this reason the lapse function is implicitly considered as containing the weight of the temporal density.

\(^2\)The three-metric \( h_{ij} \), and the conjugate lapse function \( p \) and the conjugate shift vector \( N_i \) are scalar functions with respect to time.
Finally, the symplectic form $\Omega$ can be written in its equivalent *histories* canonical form as

$$\Omega = \int d^4X \delta \pi^{\mu\nu} \wedge \delta g_{\mu\nu} = -\int d^4X \delta \pi^{\mu\nu} \wedge \delta g^{\mu\nu}$$  \hspace{1cm} (3.40)

$$= \int d^3x \frac{dt}{\alpha(t)} (\delta \pi_{ij} \wedge \delta h_{ij} + \delta p \wedge \delta N + \delta p_i \wedge \delta N^i),$$

Hence, the covariant histories space $\Pi^{\text{cov}}$ is equivalent to the canonical histories space $P^{\text{can}} = \times_t (T^* \text{Riem}(\Sigma_t) \times T^* \text{Vec}(\Sigma_t) \times T^* \text{C}^\infty(\Sigma_t))$, where $\text{Riem}(\Sigma_t)$ is the space of all Riemannian three-metrics on the surface $\Sigma_t$, $\text{Vec}(\Sigma_t)$ is the space of all vector fields on $\Sigma_t$, and $\text{C}^\infty(\Sigma_t)$ is the space of all smooth scalar functions on $\Sigma_t$.

It is important to stress, once more, that this equivalence is only possible because of the introduction of the metric-dependent foliation. In its absence, the canonical histories do not correspond, in general, to genuine spacetime quantities, namely *Lorentzian* metrics.

### 3.3 Canonical description of histories general relativity theory

In the previous section, we presented in detail the connection between the covariant and the canonical description of histories general relativity. In particular, we explained the relation between the respective histories spaces $\Pi^{\text{cov}}$ and $\Pi^{\text{can}}$, and we properly defined the histories variables of the canonical description, in relation to the $3+1$ decomposition of the Lorentzian metric $g$ with respect to a metric-dependent foliation $\mathcal{E}(X; g)$.

In this section, we will present in detail the canonical treatment of the theory, and we will write explicitly the representation of the Dirac algebra of constraints.

#### 3.3.1 Canonical treatment: basic structure

The history space $\Pi^{\text{can}}$ of the canonical description is a suitable subset of the Cartesian product $\times_t \Gamma_t$ of copies of the classical general relativity state space $\Gamma = \Gamma(\Sigma)$, labelled by a parameter $t$, with $t \in \mathbb{R}$. Here $\Sigma$ is a fixed three-manifold.
In particular, we have showed above that \( \Gamma(\Sigma) = T^*\text{Riem}(\Sigma) \times T^*\text{Vec}(\Sigma) \times T^*C^\infty(\Sigma) \), where \( \text{Riem}(\Sigma) \) is the space of Riemannian metrics on \( \Sigma \); i.e., an element of \( \Gamma(\Sigma) \) is a pair \((h_{ij}, \pi_{kl}, N^i, p_i, N, p)\). A history is defined to be any smooth map \( t \mapsto (h_{ij}(t, \bar{\mathbf{x}}), \pi_{kl}(t, \bar{\mathbf{x}}), N^i(t, \bar{\mathbf{x}}), p_i(t, \bar{\mathbf{x}}), N(t, \bar{\mathbf{x}}), p(t, \bar{\mathbf{x}})) \).

The history version of the canonical Poisson brackets can be derived from the covariant Poisson brackets Eqs. (3.2)–(3.4) as

\[
\{h_{ij}(t, \bar{\mathbf{x}}), h_{kl}(t', \bar{\mathbf{x}}')\} = 0 \quad (3.41)
\]

\[
\{\pi_{ij}(t, \bar{\mathbf{x}}), \pi_{kl}(t', \bar{\mathbf{x}}')\} = 0 \quad (3.42)
\]

\[
\{h_{ij}(t, \bar{\mathbf{x}}), \pi_{kl}(t', \bar{\mathbf{x}}')\} = \delta_{(ij)}^{kl} \alpha(t) \delta(t, t') \delta^3(\bar{\mathbf{x}}, \bar{\mathbf{x}}') \quad (3.43)
\]

\[
\{N^i(t, \bar{\mathbf{x}}), N^j(t', \bar{\mathbf{x}}')\} = \alpha(t) \delta(t, t') \delta^3(\bar{\mathbf{x}}, \bar{\mathbf{x}}') \quad (3.44)
\]

\[
\{p_i(t, \bar{\mathbf{x}}), p_j(t', \bar{\mathbf{x}}')\} = 0 \quad (3.45)
\]

\[
\{N^i(t, \bar{\mathbf{x}}), N^j(t', \bar{\mathbf{x}}')\} = 0 \quad (3.46)
\]

\[
\{p_i(t, \bar{\mathbf{x}}), p_j(t', \bar{\mathbf{x}}')\} = 0, \quad (3.47)
\]

where we have defined \( \delta_{(ij)}^{kl} := \frac{1}{2} (\delta_i^k \delta_j^l + \delta_i^l \delta_j^k) \), and where \( \alpha(t) \) has been defined earlier. All quantities \( N, N^i, p \) and \( p_i \) have vanishing Poisson brackets with \( \pi^{ij} \) and \( h_{ij} \).

### 3.3.2 The Dirac algebra of constraints

The construction above leads naturally to a one-parameter family of Dirac super-hamiltonians \( t \mapsto \mathcal{H}_\perp(t, \bar{\mathbf{x}}) \) and super-momenta \( t \mapsto \mathcal{H}_i(t, \bar{\mathbf{x}}) \). In the standard canonical approach to general relativity[1, 5, 4], the super-hamiltonian and super-momenta are

\[
\mathcal{H}_\perp = \kappa^2 \hbar^{-1/2} (\pi^{ij} \pi_{ij} - \frac{1}{2} (\pi_i^i)^2) - \kappa^{-2} \hbar^{1/2} R \quad (3.50)
\]

\[
\mathcal{H}_i = -2 \nabla_j \pi^{ij}, \quad (3.51)
\]

where \( \kappa^2 = 8\pi G/c^2 \) and \( \nabla \) denotes the spatial covariant derivative. We note that both these quantities are spatial scalar densities, hence they can be smeared with scalar quantities.
The history analogue of these expressions is

\[
\mathcal{H}_\perp(t, \varpi) := \kappa^2 \alpha^{-1}(t) h^{-1/2}(t, \varpi)(\pi^{ij}(t, \varpi)\pi_{ij}(t, \varpi) - \frac{1}{2}(\pi_i^i)^2(t, \varpi)) - \\
\kappa^{-2} h^{1/2}(t, \varpi)\alpha(t) R(t, \varpi) 
\]

(3.52)

\[
\mathcal{H}^i(t, \varpi) := -2\nabla_j \pi^{ij}(t, \varpi).
\]

(3.53)

We have introduced the weight \( \alpha(t) \) in order to render the determinant \( h \) a density of weight zero with respect to time.

For each choice of the weight function \( \alpha \), these quantities on \( \mathbb{R} \times \Sigma \) satisfy the history version of the Dirac algebra

\[
\{ \mathcal{H}_i(t, \varpi), \mathcal{H}_j(t', \varpi') \} = -\mathcal{H}_j(t, \varpi) \delta(t, t') \alpha(t') \partial^j \delta^3(\varpi, \varpi') \\
+ \mathcal{H}_i(t, \varpi) \delta(t, t') \alpha(t') \partial^i \delta^3(\varpi, \varpi')
\]

(3.54)

\[
\{ \mathcal{H}_i(t, \varpi), \mathcal{H}_\perp(t', \varpi') \} = \mathcal{H}_\perp(t, \varpi) \delta(t, t') \alpha(t') \partial^i \delta^3(\varpi, \varpi')
\]

(3.55)

\[
\{ \mathcal{H}_\perp(t, \varpi), \mathcal{H}_\perp(t', \varpi') \} = h^{ij}(t, \varpi) \mathcal{H}_i(t, \varpi) \delta(t, t') \alpha(t') \partial^j \delta^3(\varpi, \varpi') \\
- h^{ij}(t', \varpi') \mathcal{H}_i(t', \varpi') \delta(t, t') \alpha(t') \partial^j \delta^3(\varpi, \varpi').
\]

(3.56)

Note, that when we introduce back the variables \( \tilde{\pi}^{ij} \) and \( \tilde{h} \) that are densities of weight 1 with respect to time, the dependence on \( \alpha(t) \) drops out from the expressions for the constraints.

The smeared form of the super-hamiltonian \( \mathcal{H}_\perp(t, \varpi) \) and the super-momentum \( \mathcal{H}_i(t, \varpi) \) history quantities are defined using as their smearing functions, respectively, a scalar function \( L \), and a spatial vector field \( \vec{L} \) as follows:

\[
\mathcal{H}(L) := \int d^3 \varpi \int dt \alpha(t)^{-1} L(t, \varpi) \mathcal{H}_\perp(t, \varpi)
\]

(3.57)

\[
\mathcal{H}(\vec{L}) := \int d^3 \varpi \int dt \alpha(t)^{-1} L^i(t, \varpi) \mathcal{H}_i(t, \varpi).
\]

(3.58)

The smeared form of this history version of the Dirac algebra is

\[
\{ \mathcal{H}(\vec{L}), \mathcal{H}(\vec{L'}) \} = \mathcal{H}(\vec{L}, \vec{L'})
\]

(3.59)

\[
\{ \mathcal{H}(\vec{L}), \mathcal{H}(L) \} = \mathcal{H}(L \vec{L})
\]

(3.60)

\[
\{ \mathcal{H}(L), \mathcal{H}(L') \} = \mathcal{H}(L L')
\]

(3.61)

where in Eq. (3.61) we have \( K^i := h^{ij}(L \partial_j L' - L' \partial_j L) \), with \( i = 1, 2, 3 \).
Hence, because the generators $H_{\perp}(t, \vec{x})$ and $H^i(t, \vec{x})$ of the history Dirac algebra Eqs. (3.54–3.56) trivially commute with the variables $N, N^i, p$ and $p_i$ of the history algebra Eqs. (3.41–3.49), we recover exactly the history version of the Dirac algebra. Therefore, on the history space $\Pi^{\text{cov}} = \Pi^{\text{can}}$ we have a representation of the Dirac algebra together with a representation of the spacetime diffeomorphisms group $\text{Diff}(M)$.

This result is different from the one we obtained in an earlier paper [13], in the sense that the 3+1 decomposition here is obtained by means of the metric-dependent foliation. It was not $a$ priori evident that the results would stay the same. The conclusion is that the structure of the canonical theory is not affected by the introduction of the metric-dependent foliation, but the latter is crucial if the canonical theory is to preserve the spacetime character of the theory, namely the Lorentzian nature of the spacetime metric.

4 Conclusions

In this paper, we have placed special emphasis on the dual nature of the histories theory, that allows the comparison of spacelike and canonical objects, as well as the explicit study of the different choices of foliation.

We showed that the histories theory preserves the spacetime character of the canonical description of general relativity. The introduction of the metric-dependent foliation solved the problem of the loss of the spacelike character of the foliation associated to the 3+1 decomposition. This allows the derivation of the exact relation between the spacelike (covariant) and the canonical descriptions of the histories general relativity theory.

We concluded with the rather significant result that a representation of both the group of spacetime diffeomorphisms and the Dirac algebra of constraints co-exists in the metric-dependent description of histories theory.

The detailed study of the symmetries of the theory, the construction of the reduced state space, and the histories treatment of the problem of time, are studied in a continuation of this work [18], which culminates in an explicit demonstration of the $\text{Diff}(M)$-invariance of canonical general relativity.

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A Foliations: some definitions

A foliation of a four-manifold $M$ of topology $\Sigma \times \mathbb{R}$ by three-surfaces $\Sigma$ is defined as a map

$$\mathcal{E} : \Sigma \times \mathbb{R} \mapsto M$$

$$(x, t) \rightarrow \mathcal{E}(x, t) := \mathcal{E}_t(x).$$

(A.1)

(A.2)

Associated to such a foliation is an one-parameter family of embeddings

$$\mathcal{E}_t : \Sigma \mapsto M$$

$$x \rightarrow \mathcal{E}_t(x).$$

(A.3)

(A.4)

The submanifolds $\Sigma_t$ of $M$ defined as $\Sigma_t = \mathcal{E}_t(\Sigma)$, for each $t$ are known as the leaves of the foliation $\mathcal{E}$. The choice of a foliation allows the selection of a coordinate system common to all $\Sigma_t$ three-surfaces, in the sense that the coordinate defined on the reference three-surface $\Sigma$ is shared by all $\Sigma_t$ three-surfaces.

For a coordinate system $x^i$ on $\Sigma$, where $i = 1, 2, 3$, the three vector fields $\mathcal{E}^i$, tangent to the foliation, are defined as

$$\mathcal{E}^i = \frac{\partial}{\partial x^i} \mathcal{E}^{-1}(X).$$

(A.5)

Transverse to the leaves of the foliation is the deformation vector, which is defined as

$$t^\mu(X) = \frac{\partial}{\partial t} \mathcal{E}^{-1}(X).$$

(A.6)

The vector fields $\mathcal{E}^i$ and $t^\mu$ form a coordinate basis, so they satisfy

$$[t, \mathcal{E}_i]^\mu = 0$$

$$[\mathcal{E}_i, \mathcal{E}_j]^\mu = 0.$$
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