OPTIMAL PRODUCT RELEASE TIME FOR A NEW HIGH-TECH STARTUP FIRM UNDER TECHNICAL UNCERTAINTY

MING-HUI WANG
School of Economic Mathematics, Southwestern University of Finance and Economics
Chengdu, Sichuan 611130, China

NAN-JING HUANG*
Department of Mathematics, Sichuan University
Chengdu, Sichuan 610064, China

DONAL O’REGAN
School of Mathematics, Statistics and Applied Mathematics, National University of Ireland
Galway H91 TK33, Ireland

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Abstract. Decision makers of new high-tech startup firms always want to choose an optimal time to launch their products which are under research and development (R&D) to obtain the maximum net income of these firms. However, existing models fail to consider the optimal release time of products for these new high-tech startup firms. In this paper, the optimal time to launch the product of the R&D project is assumed to be the first time when the product of the R&D project is released to the market. Based on this assumption, we develop a continuous-time model to find the optimal time at which the startup firm launches its product of the R&D project by considering the price of the similar product. Employing the methods of dynamic programming and variational inequalities, we also provide a closed form solution to our model. We also find that these high-tech startup firms prefer to delay their product release time when the price of the similar product is at a phase of rapid growth or the price has considerable uncertainty. Moreover, some numerical examples are provided to investigate the properties of our model.

1. Introduction. In the last decade, a growing number of new high-tech startup firms were established around the world in the manufacturing industry. For these new high-tech startup firms, they have to pay a lot of money for their laboratory equipment, experimental materials and processing equipments before starting their R&D projects, i.e., at the early stage, these firms have many expensive tangible assets with less or no intangible assets. In practice, this is a common case in, say for example, biotechnology companies, new material companies, pharmaceutical
companies and smartphone companies. They all need to obtain enough equipment to support their R&D projects at the beginning. In addition, some technical uncertainties exist in the R&D projects of these new high-tech startup firms.

Investors who invest in startup firms attempt to maximize the market value of investees so that they can get high returns in the future. According to previous models (see, e.g., [7], [9] and [20]), we see that there are two independent types of assets, tangible assets and intangible assets, that generate the market value of a firm. For these research-based high-technology startup firms, they already have enough tangible assets but have a lack of intangible assets. Thus, commercializing R&D projects to obtain the maximum value of these intangible is vitally important for both these startup firms and investors. However, new high-tech startup firms confront unfriendly market circumstances, such as capital inaccessibility, vicious competition with large companies and the inability of innovation commercialization, which present difficulties for these startup firms to launch their products and R&D projects. Many researchers have investigated these commercialization problems in both theoretical (see, e.g., [5], [10] and [18]) and practical (see, e.g., [14] and [19]) situations using different commercialization strategies. However, few researchers have investigated the best time to commercialize an R&D project, i.e., the optimal time to release the products of R&D project to the market. Startup firms are supposed to release their products of R&D projects prior to their completion at the optimal time so that they can inform target customers and advertise their products. In return, market reaction has an impact on the market value of startup firms which is important for these high-technology manufacturing startup firms. Positive response generates an increase in the market value of the firm, and vice versa. Therefore, it is of great significance to find the right time that the startup firms release their products of R&D projects so as to maximize their market value.

For example, Meizu Technology, one of the smartphone firms in China, released the R&D project of its new smartphone product named Note 9 in Jan. 11th 2019. This smartphone is very similar to Redmi Note 7 which was launched by Xiaomi in Jan. 3rd 2019. Xiaomi is a company listed on the Hong Kong Exchanges so that we can obtain the market reaction of Redmi Note 7 by observing its stock price changing. Unfortunately, according to the data from Yahoo (see https://sg.finance.yahoo.com/quote/1810.HK/ for the detailed stock price), we see that the shares of Xiaomi fell nearly 15% over a week after the launch conference of Redmi Note 7 and this shows that the market reaction was bad for Redmi Note 7. However, Meizu Technology chose to release the R&D project of Note 9 in Jan. 11th 2019. As a result, under the negative influence of Redmi Note 7, the market reaction of Note 9 was even poorer than Redmi Note 7. Looking at the data from Counterpoint (see https://www.counterpointresearch.com/china-smartphone-share/ for the detailed data), we see that the market share of Meizu Technology in Q1 2019 reduced 50% compared to Q4 2018. This was a catastrophe for Meizu Technology and also had a huge negative effect on the market value of Meizu Technology. However, if Meizu Technology chose an optimal time to release the R&D project of Note 9, then the bad influence from Xiaomi could be avoided and the market reaction of Note 9 would be better. As a result, it could make the market value of Meizu Technology increase.

On the other hand, uncertainties of R&D projects in research-based firms arise both internally and externally, some of which are market-irrelevant and cannot be hedged with financial assets. Huchzermeier and Loch [8] identified five common
causes of uncertainties in R&D projects, one of them comes from the technical uncertainty. Technical uncertainty relates to the physical difficulty of completing an R&D project and has important implications to new high-tech startup firms. A growing empirical literature has proposed potential solutions to deal with the technical uncertainty of R&D projects. In 1993, Pindyck [17] considered technical uncertainty and input cost uncertainty and gave a general model describing the expected cost of an R&D project. Kort [11] considered the effect of investment grants and spillover benefits to an R&D project and provided numerical examples based on Pindyck’s model in 1998. Then, Pennings and Lint [15] proposed a stochastic jump amplitude model to narrow the gap between the theory and the practice of decision making on investment in an R&D project with technical uncertainty. In 2010, Whalley [23] developed a model for entrepreneurs who invest in risky assets as well as R&D projects under various risk aversion. Pennings and Sereno [16] developed a real option model incorporating both the technical and economic uncertainties to evaluate a pharmaceutical R&D project in 2011. In 2018, Nishihara [13] developed a real options model for evaluating and optimizing an R&D project, which captured research duration, debt financing, and uncertainty of technology. Yu et al. [24] investigated an optimal contract design problem to analyze two different forms of firms under a multi-agent framework with knowledge-sharing which helps to deal with technical uncertainty. Recently, Wang and Huang [21] considered a two stage optimal consumption problem for an entrepreneur who invests into an R&D project with technical uncertainty and analyzed some properties of the related optimal investment time. Very recently, Wang and Huang [22] extended the classic optimal R&D investment model with technical uncertainty into regime-switching environment to help research based firms deal with different competitive environment and different macro-economic conditions which come from outside. In all, technical uncertainty is also important for research-based high-technology startup firms in the manufacturing industry. In this study, we mainly focus on the effects of technical uncertainties of R&D projects impacting on the investment cost of these research-based high-technology startup firms.

In this paper, we assume that there are some similar products (competition products) in the market. With all the reflections of the price of these products, market reaction of a brand new R&D project can be obtained. Thus, the problem of finding the optimal time to release the products of R&D project is tractable by employing the price of these achieved similar products. Based on the above assumption, we build a model for a new high-tech startup firm to obtain the optimal investment strategy on their R&D project and to find the optimal time to release their products of R&D projects to the market. Inspired by the model developed by Bensoussan et al. [2], we combine the market value model in [7] and the expected cost model of an R&D project in [17] to overcome these difficulties.

We divide the optimal release problem of a startup firm into two part. In the first part, we concentrate on determining the optimal investment to maximize the value of the R&D project. In the second part, both the optimal time of releasing the products of the R&D projects and the optimal investment cost have to be found by the startup firms. To solve the two problems, we obtain a Hamilton-Jacobi-Bellman (HJB) equation for the first part of the problem by standard dynamic programming. Then, we obtain a variational inequality (VI) for the second part of the problem by employing the method developed by [3]. Furthermore, we make some numerical simulations to study the properties of the solution to our model.
The rest of this paper is organized as follows. In section 2, we introduce some basic definitions and give some necessary assumptions. Then, we formulate a continuous-time model for a startup firm to determine the optimal investment cost for an R&D project at each point in time and to choose an optimal time to release the product of R&D project to the market. In section 3, we solve the optimal release problem via the method of the dynamic programming and provide a closed form solution for the VI. In section 4, we give some numerical simulations to investigate the quantitative properties concerned with our model. In section 5, we give a brief summary and conclusion.

2. Problem formulation. In this section, we construct a continuous-time model for a startup firm to determine the optimal investment for an R&D project at each point in time and to choose an optimal time to release the product of R&D project to the market before its successful completion. We also assume that no cash flows are generated from the R&D project until the market is informed.

The actual cost required to complete the R&D project for a startup firm, \( X(t) \), is unknown. Following the idea in [17], the evolution of \( X(t) \) can be governed as follows:

\[
\begin{align*}
    dX(t) &= -I(t)dt + \beta \sqrt{I(t)}X(t)dW_X(t), \\
    X(0) &= x > 0,
\end{align*}
\]

where \( W_X(t) \) is a standard Brownian motion, \( I(t) \) is the investment rate (investment cost at each point in time) of the startup firm throughout the entire R&D project and \( x \) indicates the investment cost at the beginning. Moreover, \( I(t) \) is constrained to be less than the maximum investment rate \( I^* \), i.e., \( I(t) \leq I^* \). We would like to point out that equation (1) involves the technical uncertainty of the R&D project while it does not contain the input cost uncertainty. Therefore, the eventual cost to complete the R&D project is only impacted by the rate of investment \( I(t) \) and the future information about the technical uncertainty. The excepted cost to complete the R&D project shifts only if the startup firm has investment behaviors, i.e., if \( I = 0 \), then the R&D project is abolished. On the other hand, the technical uncertainty cannot be hedged with other financial assets in the market. Hence, \( X(t) \) is independent of the other risky assets in the market.

Meanwhile, we assume that the similar product has been traded in the market. Following the idea of [12], we assume that the price of the similar product is exogenously determined, which follows a geometric Brownian motion process:

\[
\begin{align*}
    dY(t) &= \alpha Y(t)dt + \sigma Y(t)dW_Y(t), \\
    Y(0) &= y > 0,
\end{align*}
\]

where \( W_Y(t) \) is a standard Brownian motion, \( \alpha > 0 \) and \( \sigma > 0 \) are the yield and the volatility of the price of the similar product, respectively. Moreover, \( y \) is the initial price of the similar product. We note that \( W_X(t) \) and \( W_Y(t) \) stand for the technical uncertainty of the related R&D project and the volatility of the price of the similar product, respectively. The correlation between \( W_X(t) \) and \( W_Y(t) \) is low so that we

\text{1The evolution of the actual cost } X(t) \text{ which combined the technical uncertainty and input cost uncertainty in a single equation is governed as follows:}

\[
dX(t) = -I(t)dt + \beta \sqrt{I(t)}dW_X(t) + \phi X(t)dW_T(t),
\]

where \( \phi \) is a constant and \( W_T(t) \) is an other standard Brownian motion independent with \( W_X(t) \) (see [17]).
can assume that $W_X(t)$ and $W_Y(t)$ are independent. On the other hand, to simplify our model, we also need to assume that $W_X(t)$ and $W_Y(t)$ are independent. Thus, we can assume $X(t)$ and $Y(t)$ are independent here.

In this study, we follow the linear market value model introduced by Grilliches [7] to analyze the market value of a startup firm, in which the firm was considered to have two kinds of assets on its market value, $V$, and was formulated as follows

$$V(A, K) = q(A + K), \quad (3)$$

where $A$ is the value of the firm’s tangible assets (for example, plants and equipment), $K$ is the value of the firm’s knowledge assets (for example, patents, brands and R&D projects), and $q$ is the market valuation coefficient of the firm’s assets, which is given by

$$q = \exp(m + d + \gamma + u),$$

where $m$ is the permanent effects of the firm, $d$ is the industry effects of the firm, $u$ is the error term, and $\gamma$ is the individual disturbance. Since a startup firm affects the corresponding market and industry insignificantly, we can assume that $m = 0$ and $d = 0$. In addition, $u$ and $\gamma$ always play negative roles in the value of the firm. So we can assume that $u < 0$ and $\gamma < 0$. Consequently, equation (3) can be changed into

$$V = e^{(\gamma + u)}(A + K).$$

Taking logarithms of both sides of the above equation and subtracting the logarithm of $A$ results in

$$\ln \frac{V}{A} = u + \gamma + \ln(1 + \frac{K}{A}). \quad (4)$$

For new high-tech startup firms, a great amount of laboratory equipment and consumable items are needed. The value of the tangible assets for these firms are much higher than regular startup firms. It also leads to the ratio of knowledge assets to tangible assets being small for the firm, i.e., the value of $\frac{K}{A}$ is small. Thus, we have the following approximation

$$\ln(1 + \frac{K}{A}) \approx \frac{K}{A}.$$ 

Substituting it into (4), we have

$$\ln \frac{V}{A} \approx u + \gamma + \frac{K}{A}.$$ 

We note that the value of tangible assets can be estimated precisely, thereby $A$ can be considered as a known constant. Hence, the market value of a startup firm can be regarded as a function of $K$. With slight simplification, the startup firm’s market value function, $Q(K)$, can be considered as follows

$$Q(K) := \ln V(K) = \delta_1 K - \delta_2,$$

where $\delta_1$ is a positive constant related to $A$ and $\delta_2$ is a positive constant related to $u$ and $\gamma$.

Assuming that there is only one R&D project in the startup firm, we use the price of similar products being traded in the market to evaluate the knowledge assets of the startup firm. Since the knowledge assets are embedded in a whole R&D project, of which finished products resemble their similar products existing in the market, the value of the startup firm’s knowledge assets can be approximated by the price of similar products before its completion. From the definition of $V$, estimating the startup firm’s market value $V$ is equivalent to calculating the value
of the function $Q(Y)$. Without loss of generality, we assume that only one similar product is taken into account throughout the R&D project by the startup firm.\footnote{We can choose an appropriate one if there is more than one similar product in the market.} Clearly, the derivative of $Q(Y)$ with respect to $Y$ is $\frac{\partial Q}{\partial Y} = \delta$. The case of $\delta > 0$ means that the market value of the startup firm rises with an increase with $Y$. This is owed to the appreciation of the similar product’s price, which stimulates the growth of the value of the knowledge assets, thereby an increase in market value of the startup firm.

In general, the startup firms or the investors tend to reckon the net income of the R&D project at the beginning of it. The aim of the startup firm and investors is to find the optimal stopping time $\theta$ to maximize the expected market value and the net income of the R&D project while control the investment cost. Thus, the object function of the startup firm and investors is

$$J(x, y, \theta, I) = E\left[ \int_0^\theta -I(t)e^{-rt}dt + e^{-r\theta}[Q(Y(\theta)) + F(X(\theta))] \right],$$

where $r > 0$ is the discount factor, $\Xi$ denotes the set of stopping times valued in $[0, \infty)$. Moreover, the function $F(x)$ is the value of the following control problem

$$F(x) = \sup_{I \in O_F} E\left[ \int_0^T -I(t)e^{-rt}dt + e^{-rT}\tilde{V} \right], \quad (5)$$

where $\tilde{V}$ is the return of the R&D project which is assumed to be a constant, $T$ is the time at which the R&D project is successfully completed and $O_F$ is a set of admissible controls to be defined later. We would like to point out that $T$ here is considered as the average time of finishing these similar R&D projects which is a determined time not a random time. The firm can conveniently obtain the historical records of time length of finishing these similar R&D projects by other firms so that the average time of finishing these similar R&D projects can be also obtained by these historical records. Here, $F(x)$ is the net benefit function of the R&D project. In fact, the problem (5) is the classic optimal R&D investment problem, which was first developed by Pindyck in [17]. Therefore, the optimal release problem can be modeled as follows

$$J(x, y) = \sup_{(\theta, I) \in O_f} J(x, y, \theta, I), \quad (6)$$

where $O_f$ is a set of admissible control pairs which is defined later.

Now, consider a probability space $(\Omega, F, P)$ containing $W_X(t), W_Y(t)$ and $\Xi$. We also denote by $F_t$ the filtration generated by $W_X(t), W_Y(t)$ and $\Xi$.

**Definition 2.1.** An admissible stochastic control is an $F$-adapted control process $I(t)$ that satisfies $I(t) \in [0, I^*]$ and the local integrability condition

$$E\left[ \int_0^T I^2(s)ds \right] < +\infty, \quad \forall T < +\infty.$$
3. Solutions to the release problem. In this section, we mainly solve the optimal release problem. Initially, we solve the optimal R&D investment problem (5) by studying a related HJB equation. Then, in order to determine the optimal stopping time for the optimal release problem (6), we get a VI with a non-linear differential operator.

The optimal R&D investment problem (5) is an optimal stochastic control problem, whose value function \( F(x) \) is associated with the following HJB equation

\[
\begin{cases}
- rF + \sup_{I \in \mathcal{O}_F} \left\{ -I \frac{dF}{dx} + I \frac{\beta^2 x}{2} \frac{d^2F}{dx^2} - I \right\} = 0; \\
F(0) = \tilde{V}.
\end{cases}
\] (7)

The optimal investment rate for the R&D project can be obtained as follows

\[
\tilde{I}(t) = \begin{cases} 
I^*, & \text{if } \frac{\beta^2 x d^2F}{2 dx^2} - \frac{dF}{dx} - 1 \geq 0; \\
0, & \text{otherwise},
\end{cases}
\] (8)

which is a Bang-Bang control problem. According to (7) and (8), we know that there is a switching point \( X^* \) of the control \( \tilde{I}(t) \) at which \( F(X^*) = 0 \). Thus, we have the following equivalent optimal control

\[
\tilde{I}(t) = \begin{cases} 
I^*, & \text{if } X \leq X^*; \\
0, & \text{otherwise}.
\end{cases}
\] (9)

Here, \( X^* \) can also be found by the boundary condition

\[
\frac{\beta^2 X^*}{2} \frac{d^2F}{dx^2} - \frac{dF}{dx} X^* - 1 = 0.
\] (10)

The optimal investment strategy of the R&D project is either to invest at the maximum rate \( I^* \), when \( x \leq X^* \), or not to invest, \( \tilde{I} = 0 \), when \( x > X^* \). When \( x > X^* \), the expected cost to complete the R&D project is too high to afford for the investors or the startup firm. Hence, the threshold \( X^* \) should be found and considered as part of the solution to problem (5). Moreover, we need the so-called smooth pasting condition to ensure the continuity of \( F(X^*) \) at \( X^* \),

\[
\frac{dF(X^*)}{dX^*} = 0.
\] (11)

A family of solutions of \( F(x) \) for \( x \leq X^* \) can be obtained by solving (7), while a unique solution \( F(X) \) as well as \( X^* \), can be determined by conditions (10) and (11). Thus, we have to solve the following free boundary problems

\[
\begin{cases}
- rF + (-I^*) \frac{dF}{dx} + I^* \frac{\beta^2 x}{2} \frac{d^2F}{dx^2} - I^* = 0; \\
F(0) = \tilde{V}; \\
F(X^*) = 0; \\
\frac{dF(X^*)}{dX^*} = 0.
\end{cases}
\] (12)

Here, according to (7) and (8), \( F(x) = 0 \) and \( \frac{dF(x)}{dx} = 0 \) when \( x > X^* \). However, the problem (5) is a classic problem so that we omit the verification theorem here. Moreover, the free boundary problem (12) cannot obtain any closed-form solution so we give some numerical examples in section 4.
Now, we solve the optimal release problem (6). With the method of dynamic programming, the value function $J(x, y)$ is the solution of the following VI

$$
\begin{aligned}
&\begin{cases}
-rJ + \sup_{I \in \mathcal{O}} \left\{ (-I) \frac{\partial J}{\partial x} + I \frac{\beta^2}{2} x \frac{\partial^2 J}{\partial x^2} - I \right\} + \alpha y \frac{\partial J}{\partial y} + \frac{\sigma^2}{2} y^2 \frac{\partial^2 J}{\partial y^2} \leq 0, \\
J(x, y) \geq F(x) + Q(y),
\end{cases}
\end{aligned}
$$

(13)

Since the optimal investment rate $\tilde{I}$ is given by (9), the above VI (13) can be changed as follows

$$
\begin{aligned}
&\begin{cases}
-rJ + (-\tilde{I}) \frac{\partial J}{\partial x} + \tilde{I} \frac{\beta^2}{2} x \frac{\partial^2 J}{\partial x^2} - \tilde{I}^* + \alpha y \frac{\partial J}{\partial y} + \frac{\sigma^2}{2} y^2 \frac{\partial^2 J}{\partial y^2} \leq 0; \\
J(x, y) \geq F(x) + Q(y),
\end{cases}
\end{aligned}
$$

(14)

Here, the left hand of the first inequality involves the nonlinear second order differential operator associated with $J(x, y)$ and the optimal investment rate $\tilde{I}$ given by (9). From the definition of $F(x)$, we know that $F(x)$ also satisfies the boundary conditions (10) and (11). Therefore, we can look for a solution in the following form

$$
J(x, y) = F(x) + G(y),
$$

(15)

where $F(x)$ and $X^*$ can be obtained by (12). Substituting (12) and (15) into (14), respectively, (14) can be reduced to the following VI

$$
\begin{aligned}
&\begin{cases}
-rG + \alpha y \frac{dG}{dy} + \frac{\sigma^2}{2} y^2 \frac{d^2 G}{dy^2} \leq 0, \\
G(y) \geq Q(y),
\end{cases}
\end{aligned}
$$

(16)

By employing the method developed in [3] as well as some mild assumptions, we can obtain the following theorem which shows that (16) has a unique solution.

**Theorem 3.1.** If $r > \alpha > 0$, then

$$
G(y) = \begin{cases}
\frac{\delta_1}{(\lambda - 1)} \left( \frac{y}{\tilde{y}} \right)^\lambda, & y \leq \tilde{y}, \\
\delta_1 y - \delta_2, & y \geq \tilde{y}.
\end{cases}
$$

(17)
where
\[ \lambda = \frac{\left( \frac{\sigma^2}{2} - \alpha \right) + \sqrt{(\alpha - \frac{\sigma^2}{2})^2 + 2r\sigma^2}}{\sigma^2} \]
and
\[ \tilde{y} = \frac{\delta_2}{\delta_1} \left( \frac{\lambda}{\lambda - 1} \right). \] (18)

Proof. First, if \( r > \alpha \), then \( \lambda > 1 \) and \( \tilde{y} > 0 \). In order to complete the proof, we follow the method developed by [2]. Firstly, we need to verify the following two inequalities hold
\[ G(y) \geq \delta_1 y - \delta_2, \quad \text{if} \quad y \leq \tilde{y}, \] (19)
and
\[ -rG(y) + \alpha y \frac{dG(y)}{dy} + \frac{\sigma^2}{2} y^2 \frac{d^2G(y)}{dy^2} \leq 0, \quad \text{if} \quad y \geq \tilde{y}. \] (20)

For (19), let
\[ H(y) = G(y) - \delta_1 y + \delta_2. \]
Then, we have
\[ H'(y) = \frac{\delta_2 \lambda}{\lambda - 1} \frac{y^{\lambda-1}}{y^\lambda} - \delta_1 \]
and
\[ H''(y) = \lambda \delta_2 \frac{y^{\lambda-2}}{y^\lambda}. \]
If \( y \leq \tilde{y} \), then \( H'(y) < 0 \) and \( H''(y) > 0 \). Since \( H(y) > 0 \) on \((0, \tilde{y})\), we confirm that inequality (19) is true.

On the other hand, substituting (17) into (20), we have
\[ y \geq \frac{r\delta_2}{\delta_1(r-\alpha)} \quad \text{for} \quad y \geq \tilde{y}. \]
Since \( r > \alpha \), it follows that \( \tilde{y} \geq \frac{r\delta_2}{\delta_1(r-\alpha)} \), which implies (20). Thus, we have completed the proof. \( \square \)

Remark 3.1. Theorem 3.1 only considers the case of \( Y(t) > 0 \) as the value of the similar product cannot be negative. Moreover, (17) indicates that \( G(y) > 0 \) on \((0, +\infty)\). It reveals that the startup firm’s market value is positively affected by the value of the knowledge asset.

Theorem 3.2. The function \( J(x, y) \) which is given by (15) coincides with the value function (6).

Proof. From Theorem 3.1 and (12), we know that \( J(x, y) \) is a \( C^2(R^2) \) function. Moreover, from (9), we know that the optimal control \( \tilde{I}(t) \in O_F \). Let
\[ \theta(y) = \inf \{ t | Y(t) \geq \tilde{y} \}. \]
Then, we have
\[ E[\theta(y)] = -\frac{1}{\alpha} \ln(y), \quad y < \tilde{y}. \]
Thus, we can obtain $\theta(y) < +\infty$, a.s., which implies that the pair $(\theta(y), \tilde{t}(t)) \in \mathcal{O}_I$. Consider the function $f(t, x, y) = e^{-rt}J(x, y)$, and for any control $I \in \mathcal{F}_t$. By employing the Itô formula for $f(t, X(t), Y(t))$, we can obtain

$$df(t, X(t), Y(t)) = -re^{-rt}J(X(t), Y(t))dt + e^{-rt}\left[\left(-I\right)\frac{\partial J}{\partial X} + I\beta^2 J + \frac{\partial^2 J}{\partial X^2}\right]dt$$

$$+ e^{-rt}\left[\alpha Y(t)\frac{\partial J}{\partial Y} + \frac{\sigma^2}{2} Y(t)^2 \frac{\partial^2 J}{\partial Y^2}\right]dt$$

$$+ e^{-rt}\beta \sqrt{J(t)} \frac{\partial J}{\partial Y} dW_X(t) + e^{-rt}\sigma Y(t) \frac{\partial J}{\partial Y} dW_Y(t).$$

Then, by letting $\theta_n = \inf\{t|X(t) > n\}$, we can obtain

$$E[f(\theta(y) \wedge \theta_n, X(\theta(y) \wedge \theta_n), Y(\theta(y) \wedge \theta_n))]$$

$$= J(x, y) - E\left[\int_0^{\theta(y) \wedge \theta_n} re^{-rs}J(X(s), Y(s))ds\right]$$

$$+ E\left[\int_0^{\theta(y) \wedge \theta_n} e^{-rs}\left[\left(-I\right)\frac{\partial J}{\partial X} + I\beta^2 J + \frac{\partial^2 J}{\partial X^2}\right]ds\right]$$

$$+ E\left[\int_0^{\theta(y) \wedge \theta_n} e^{-rs}\beta \sqrt{J(s)} \frac{\partial J}{\partial Y} dW_X(s) + \int_0^{\theta(y) \wedge \theta_n} e^{-rs}\sigma Y(s) \frac{\partial J}{\partial Y} dW_Y(s)\right].$$

We note that $\frac{\partial J}{\partial X} = \frac{dF}{dx}$ which is bounded for every $s \in [0, \theta(y) \wedge \theta_n]$. This is because $\frac{dF}{dx}$ is a continuous function on $[0, X^*]$ and $\frac{dF}{dx} = 0$ on $[X^*, +\infty)$. Thus, we have

$$E\left[\int_0^{\theta(y) \wedge \theta_n} e^{-rs}\beta \sqrt{J(s)} \frac{\partial J}{\partial Y} dW_X(s)\right] = 0.$$

Since $Y(\theta(y) \wedge \theta_n) \leq \tilde{y}$, $\frac{\partial J}{\partial Y}$ is also bounded on $(0, \tilde{y})$ and $\frac{\partial J}{\partial Y} = \delta_1$ on $(\tilde{y}, +\infty]$. Thus,

$$E\left[\int_0^{\theta(y) \wedge \theta_n} e^{-rs}\sigma Y(s) \frac{\partial J}{\partial Y} dW_Y(s)\right] = 0.$$

Therefore, from (13), we can obtain that

$$J(x, y) \geq -E\left[\int_0^{\theta(y) \wedge \theta_n} e^{-rs}I(s)ds\right] + E[f(\theta(y) \wedge \theta_n, X(\theta(y) \wedge \theta_n), Y(\theta(y) \wedge \theta_n))].$$

Letting $n \to +\infty$, one has

$$J(x, y) \geq -E\left[\int_0^{\theta(y)} e^{-rs}I(s)ds\right] + E[f(\theta(y), X(\theta(y)), Y(\theta(y))].$$

Since

$$E[f(\theta(y), X(\theta(y)), Y(\theta(y)))] = E\left[e^{-\theta(y)}[Q(Y(\theta(y))] + F(X(\theta(y)))\right],$$

we can obtain that

$$J(x, y) \geq -E\left[\int_0^{\theta(y)} e^{-rs}I(s)ds\right] + E\left[e^{-\theta(y)}[Q(Y(\theta(y))] + F(X(\theta(y)))]\right].$$

(21)
In particular, inequality (21) becomes an equality if \( I = \tilde{I} \), and hence \( J(x, y) = J(x, y, \theta(y), \tilde{I}) \) which completes the proof.

By Theorem 3.1 and Theorem 3.2, the optimal stopping rule to achieve the supremum in the release problem (6) can be described by

\[
\theta(y) = \inf_t \{ t | Y(t) \geq \tilde{y} \},
\]

where, \( \tilde{y} \) is defined by (18). That is, when \( Y(t) \) first achieves \( \tilde{y} \), the product of R&D project of the startup firm should be released immediately.

**Remark 3.2.** From Theorem 3.2, we note that the optimal release time \( \theta \) does not depend on \( x \). This phenomenon shows that the value of the similar product has much more impact on the net income of the startup firm than the cost of the related R&D project.

**Remark 3.3.** From (22), we note that the optimal release time can be described as the price process \( Y(t) \) firstly achieves \( \tilde{y} \). In general, the price of a product can reflect the supply-demand relation of the product. Thus, the price threshold \( \tilde{y} \) may reflect a well-founded and stable supply-demand relationship of the product at time \( \theta(y) \). However, our model fails to analyze the detail of the supply-demand relationship of the product at \( \theta(y) \).

4. **Numerical simulations.** In this section, we carry out the numerical simulations to study the quantitative properties of solutions to our model. At first, we show how the investment cost \( x \) and the price of the similar product \( y \) impact on the startup firm’s net value \( J(x, y) \) in different scenarios. Then, we conduct a set of numerical experiments to examine how system parameters such as the maximum investment rate \( I^* \), the yield of the price of the similar product \( \alpha \) and the volatility of the similar product \( \sigma \) influence the maximum investment cost \( X^* \) and the threshold \( \tilde{y} \).

At first, we set four sets of parameters under different scenarios in Table 1. In Table 1, the value for the investment cost of the R&D project parameters such as \( \beta, I^* \), \( V \) are set in the spirit of the investment of R&D project with technical uncertainty literature (see, e.g., [11], [13] and [17]). The parameters relating to the price of the similar product such as \( \sigma, \alpha \) and \( r \) are set similarly to [6] and [12]. Moreover, we set the market value parameters \( \delta_1 \) and \( \delta_2 \) according to the example in [7].

| Parameters | Scenario 1 | Scenario 2 | Scenario 3 | Scenario 4 |
|------------|------------|------------|------------|------------|
| \( r \)    | 0.1        | 0.06       | 0.04       | 0.03       |
| \( \sigma \)| 0.8        | 0.5        | 0.3        | 0.1        |
| \( \alpha \)| 0.06       | 0.04       | 0.02       | 0.01       |
| \( \beta \)| 0.8        | 0.5        | 0.2        | 0.1        |
| \( \delta_1 \)| 3          | 1          | 0.5        | 0.1        |
| \( \delta_2 \)| 0.5        | 0.1        | 0.05       | 0.05       |
| \( V \)    | 80         | 40         | 20         | 10         |
| \( I^* \)  | 5          | 2          | 1          | 0.5        |

Figure 1 shows the startup firm’s net value \( J(x, y) \) as a function of \( x \) and \( y \) under four different scenarios, respectively. The related free boundary \( X^* \) and the
threshold $\tilde{y}$ are presented in Table 2. In Figure 1, an increase in $x$ results in a decrease in $J(x, y)$ in all four scenarios. The fundamental fact is that the larger expected cost, $x$, causes the less net value of the R&D project. Meanwhile, in all four scenarios, an increase in $y$ results in an increase in $J(x, y)$. This is because that the higher the value of the homogeneous product leads to the higher value of the knowledge asset. Consequently, the startup firm obtains the higher market value.

![Figure 1](image)

**Figure 1.** the behaviors of the startup firm’s net value $J(x, y)$ as a function of the expected cost of the R&D project $x$ and the price of the similar product $y$ in different scenarios.

| Scenario | $X^*$ | $\tilde{y}$ |
|----------|-------|-------------|
| 1        | 73.0839 | 1.8797 |
| 2        | 32.4965 | 0.9948 |
| 3        | 15.9099 | 0.4836 |
| 4        | 8.247  | 1         |

**Table 2.** Simulated results for $X^*$ and $\tilde{y}$.

In what follows, we explain how the value of $F(x)$ is affected by the maximum investment rate $I^*$ and $\beta$. In addition, all the other parameters here are set to the same ones in scenarios 2. Figure 2 indicates the net value of the R&D project $F(x)$ as a function of the expected cost $x$ in different maximum investment rate $I^*$ with $\beta = 0.5$. In Figure 2, an increase in $I^*$ results in an increase in $X^*$ as well as an increase in $F(x)$. This is due to the fact that the higher maximum investment rate $I^*$ leads to the higher investment cost and the earlier completion of the R&D project. As a result, the discount value of $F(x)$ becomes less. When $I^*$ takes different values, the corresponding boundaries $X^*$ are 32.4965, 40.5864 and
42.9594, respectively. Figure 2 also shows that the higher value of $F(x)$ results in the higher value of $X^*$.

Similarly, in Figure 3, an increase in $\beta$ results in an increase in both $X^*$ and $F(x)$. The technical uncertainty of the R&D project becomes higher as $\beta$ increase so that both the investment cost and the value of $F(x)$ become higher. When $\beta$ takes different values, the corresponding boundaries $X^*$ are 29.5129, 32.4965 and 39.5507, respectively.

Next, we examine the effect of both the yield and the volatility of the price of the similar product on the optimal releasing timing of the product of R&D project respectively. In addition, all the other parameters here are also set to the same ones in scenarios 2. Observing Figure 4, we can see that a bigger $\alpha$ leads to a high threshold $\tilde{y}$. It comes from the fact that the corresponding price of the similar product increases faster when $\alpha$ becomes bigger. The similar product has a positive market reaction if its price increases so that the startup firm should release the product of the R&D project later in case for more market value. Thus, the startup firm has a delayed release. Similarly, in Figure 5, an increase in $\sigma$ results in an increase in both $\tilde{y}$ and $g(y)$. The uncertainty of the price of the similar product becomes higher as $\sigma$ increase so that the startup firm willingly waits for
the uncertainty of the similar product’s price decrease. Therefore, the value of the threshold $\tilde{y}$ is at a high level here.

**Figure 4.** The behaviors of the value of $g(y)$ as a function of the price of the similar product $y$ in different $\alpha$ with $\alpha = 0.04$, $\alpha = 0.02$ and $\alpha = 0.01$, where the corresponding thresholds $\tilde{y}$ are 0.9984, 0.5344 and 0.4429, respectively.

**Figure 5.** The behaviors of the value of $g(y)$ as a function of the price of the similar product $y$ in different $\sigma$ with $\sigma = 0.3$, $\sigma = 0.5$ and $\sigma = 0.8$, where the corresponding thresholds $\tilde{y}$ are 0.5726, 0.9984 and 1.9849, respectively.

We examine the effect of both the $\delta_1$ and the $\delta_2$ of the market value function on the value of $g(y)$ and the threshold $\tilde{y}$, respectively. In addition, all the other parameters here are also set to the same ones in scenarios 2. From Figure 6, we can find that a bigger $\delta_1$ leads to a higher $g(y)$ but a lower threshold value $\tilde{y}$. This is due to the fact that the market value increases faster when $\delta_1$ becomes bigger. As a result, the startup firm needs to release the R&D project earlier. In Figure 7, the value of $g(y)$ is higher but the value of the threshold $\tilde{y}$ is lower when the value of $\delta_2$ is at a low level. When the value of $\delta_2$ increase, the market value of the startup firm would decrease. Consequently, the startup firm has a delayed release when the value of $\tilde{y}$ is at a high level.

Figure 8 explains how $\alpha$ impacts on the threshold $\tilde{y}$. In Figure 8, an increase in $\alpha$ results in an increase in $\tilde{y}$. The price of the similar product at a phase of rapid growth when the value of $\alpha$ becomes bigger. It indicates that the startup firm has a chance to obtain the higher market value in the future. Thus, our model encourages the startup firm to have a delayed release of the product of the R&D project when the price of the similar product increase rapidly. We would like to point out that
this is different from the classic real option models in [4] and the classic real option
game models in [1].

In Figure 9, an increase in $\sigma$ results in an increase in $\tilde{y}$. When the value of $\sigma$
increases, the uncertainty of the price of the similar product becomes larger so that
the startup firm have a delayed product of R&D project release.

5. Conclusions. In this study, a continuous-time model is developed to find both
the optimal R&D project release time and the optimal investment rate for a new
high-tech startup firm. We divide these problems into two part. By the method of
dynamic programming, we change the second problem and the first problem into
a VI with nonlinear operator and a linear equation, respectively. Then, both the
maximum net value and the optimal investment rate of the startup firm are given
for both the problems. We also obtain the threshold of the similar product’s price
for the release time of the product of the R&D project.

We discuss the impact of knowledge asset’s value and the expected cost to com-
plete the R&D project on the net value of the startup firm. There are two important
implications of our model. Firstly, if the price of the similar product becomes higher,
the knowledge asset’s value of the startup firm rises. The price of similar products
increase indicates that the market demand for such products is increasing. At this
time, the product which is developing in the startup is in short supply so that its
value increases and the value of the knowledge asset of the product simultaneously rise. Therefore, the knowledge asset value of the startup firm increases. This means that the similar product becomes popular, get a positive market reaction and have a higher degree of market demand or a lower degree of the market supply. It also suggests that the market encourages the startup firm to have a delayed release of the product of the R&D project when the price of the similar product is at a phase of rapid growth so that it could reach much higher market value in the future. Secondly, the understanding on these new high-tech products are different for different individuals so that the market has different reactions. It causes the uncertainty of the price of the similar product increase. As a result, the startup firm should be careful about its market behavior and have a delayed release to avoid the loss of market value. Moreover, if the expected cost of the R&D project becomes higher, then the market value of the startup firm would go to a low level. The profit of the R&D project would decrease as the expected cost of the R&D project becomes higher. Thus, the value of the startup firm would go to a low level.

We would like to point out that the optimal decision making model presented in this paper can be extended to other models wherein input cost uncertainty and the investment grants of a R&D project are relevant. We plan to address these models as we continue our research.

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E-mail address: wangmh@swufe.edu.cn
E-mail address: nanjinghuang@hotmail.com
E-mail address: donal.oregan@nuigalway.ie