ALGEBRAIC LOGIC, VARIETIES OF ALGEBRAS AND ALGEBRAIC VARIETIES

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Abstract. The aim of the paper is discussion of connections between the three kinds of objects named in the title. In a sense, it is a survey of such connections; however, some new directions are also considered. This relates, especially, to sections 3, 4 and 5, where we consider a field that could be understood as an universal algebraic geometry. This geometry is parallel to universal algebra.

In the monograph [51] algebraic logic was used for building up a model of a database. Later on, the structures arising there turned out to be useful for solving several problems from algebra. This is the position which the present paper is written from.

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INTRODUCTION

Essentially, the paper consists of two parts. The first part includes §1 and §2. In the first section, preliminary information on algebraic logic is given (see also [51]), while the second one contains a survey of certain applications.

All the rest makes up the second part. Its main subject is equations and identities in arbitrary algebraic structures. The very notion of an equation is treated very widely, and a solution of an equation is a point regarded as an algebra homomorphism. Here the connections existing between algebraic logic and universal algebra work. The investigation is carried out after the pattern of algebraic geometry, and geometry is considered on three levels. These are the equational logic level (§3 and §4), and the levels of quantifier-free logic and first-order logic (§5). Everywhere, what we have in mind is, in fact, Θ-logic, where Θ is some variety of algebras.

On the equational level, a certain general statement of the Hilbert Nullstellensatz (i.e. theorem on zeros) is given; it is applicable in all cases and admits one more view on the classical theorem. This general formulation has also a linkage with the notion of geometric equivalence of algebras. For every algebra $G \in \Theta$ a category $K_G$ of algebraic varieties related with $G$ is defined. It is contravariantly embedded in the category of algebras from Θ. If algebras $G_1$ and $G_2$ are geometrically equivalent, then the categories $K_{G_1}$ and $K_{G_2}$ are also equivalent.

What can be said in general about the algebras $G_1$ and $G_2$ for which the categories $K_{G_1}$ and $K_{G_2}$ are equivalent? The solution of this problem we consider in the separate paper.

It seems to us that the material contained in the second part is only the beginning of a vast theme. The part is two-aimed. On the one hand, it is the wish for seeing how far the ideas of algebraic geometry are applicable in universal algebra, and on the other one—the needs and interests of the algebra itself, desire to look at its problems at a new "viewing angle".

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§1. ALGEBRAIC LOGIC

1. Some problems. Let $\Theta$ be a variety of algebras, $W$ a free algebra of countable rank in $\Theta$, $u$ a formula of the first-order calculus specialized in $\Theta$. It will be clear from the following four examples what we mean by a "specialized" formula.

1. $w \equiv w'$,
2. $w_1 \equiv w'_1 \lor \cdots \lor w_n \equiv w'_n$,.....
3. \( w_1 \equiv w'_1 \land \cdots \land w_n \equiv w'_n \rightarrow w \equiv w' \),
4. \( w_1 \equiv w'_1 \lor \cdots \lor w_n \equiv w'_n \lor v_1 \not= v'_1 \lor \cdots \lor v_m \not= v'_m \).

All \( w \) and \( v \) are elements of \( W \), and if we speak of many-sorted algebras (which also are possible), then the sign \( \equiv \) connects elements of the same sort. \( \Theta \) can be the variety of all groups, semigroups, quasigroups, rings, associative or Lie rings, automata, etc.

Let further \( G \) be an algebra from \( \Theta \). On \( G \), also some relations could be given. How should one understand that a formula \( u \) of the corresponding language is satisfied in \( G \), or more generally, what is to be meant by the value of \( u \) in the algebra or model \( G \)?

If \( u \) is \( w \equiv w' \), the value of \( u \) in \( G \) is the set of those homomorphisms \( \mu : W \rightarrow G \) for which the equality \( w^\mu = w'^\mu \) holds in \( G \). If it is the whole set \( \text{Hom}(W,G) \), then \( u \) is said to be valid in \( G \), or an identity of the algebra \( G \).

We have a similar situation also in the next example. The value of \( u \) in \( G \) is the subset of \( \text{Hom}(W,G) \) consisting of those \( \mu : W \rightarrow G \) for which at last one of the equalities \( w^\mu_1 = w'^\mu_1, \ldots, w^\mu_n = w'^\mu_n \) holds in \( G \). If it is so for all \( \mu \), then \( u \) is a pseudoidentity of \( G \).

The third example is connected with quasi-identities, and the fourth one—with universal formulas.

For any \( u \), the value of \( u \) in \( G \) is defined inductively, and it is a subset of \( \text{Hom}(W,G) \) both for algebras and models.

We have defined the value of a formula semantically, and the problem now is to formalize this semantics using any natural structure. So, it is convenient to turn to algebraic logic.

Let us consider the following class of problems. Let \( T \) be some set of axioms (formulas in a given specialized language), and let \( K = T' \) be the class of models it specifies. All formulas from \( T \) are valid in these models. The class \( K \) is axiomatic, and if \( K' = T'' \) is the set of formulas valid in algebras of \( K \), then \( T'' \) is the closure of \( T \). How could we get all formulas from \( T'' \) syntactically if we proceed from \( T' \)? This problem is well solved in logic, but its solution in algebraic logic is more natural.

It is of some interest to speak of formulas of a specified kind, for example, identities, quasi-identities, pseudoidentities, universal formulas, etc., and to consider the respective closures. Here we deal with algebras without additional relations. The problem is easily solved for identities by using the free algebra \( W \), but in other cases it is natural to turn our attention to algebraic logic.

We can also speak of the closure of a class. A set of identities specifies a variety. The closure of an arbitrary class of algebras from \( \Theta \) up to the variety (a subvariety of \( \Theta \)) is constructed in accordance with the Birkhoff theorem. There are similar theorems for pseudovarieties, which are defined by pseudoidentities, for quasivarieties and universal classes.

The system of formulas valid in \( G \) is an important characteristic of an algebra \( G \in \Theta \). It is especially true of identities. Varieties are also of interest because each variety has free algebras, and each variety isolates the corresponding verbal congruence on every algebra \( G \).

All subvarieties of \( \Theta \) are controlled by the free algebra \( W \). For other axiomatic classes, an algebra \( U = U(\Theta) \) constructed from the corresponding specialized first-order calculus using \( W \) plays the role of such controlling object. Of course, varieties could also be controlled by this algebra.

2. \( \Theta \)-logic. \( \Theta \)-logic is built up on the ground of some variety of algebras \( \Theta \). Since we have in mind many-sorted algebras as well, let us recall some related concepts (see [10, 22, 8, 6]).

First of all, we fix a set of sorts \( \Gamma \). Correspondingly, we consider a many-sorted set \( G = (G_i, i \in \Gamma) \). Each \( G_i \) is the domain of the sort \( i \). Denote by \( \Omega \) a set of operation symbols. Each \( \omega \in \Omega \) has a definite
type $\tau = \tau(\omega) = (i_1, \ldots, i_n; j)$, where $i, j \in \Gamma$. Such a symbol $\omega$ is realized in $G$ as an operation, i.e. a map $\omega: G_{i_1} \times \cdots \times G_{i_n} \to G_j$.

In each algebra $G$, all symbols $\omega \in \Omega$ are assumed to be realized this way. So, $G$ is an $\Omega$-algebra.

For given $\Gamma$ and $\Omega$, an **algebra morphism** $G \to G'$ has the form

$$\mu = (\mu_i, i \in \Gamma): G = (G_i, i \in \Gamma) \to G' = (G'_i, i \in \Gamma),$$

where each $\mu_i: G_i \to G'_i$ is a map of sets. **Algebra homomorphisms** are morphisms that preserve operations. For $\mu = (\mu_i, i \in \Gamma)$ and $\omega \in \Omega$ this means that if $\tau(\omega) = (i_1, \ldots, i_n; j)$ and $a_1 \in G_{i_1}, \ldots, a_n \in G_{i_n}$, then

$$(a_1 \cdots a_n \omega)^{\mu_i} = a_1^{\mu_{i_1}} \cdots a_n^{\mu_{i_n}} \omega.$$

The multiplication of such many-sorted maps is defined componentwise, and the product of homomorphisms is a homomorphism. Sometimes we write $a^\mu$ instead of $a^\mu_i$. If all $\mu_i$ are surjections (injections), then $\mu$ is also a surjection (injection). If all $\mu_i$ are bijections, then $\mu$ is a bijection, and $\mu^{-1} = (\mu_i^{-1}, i \in \Gamma)$. A bijective homomorphism $\mu$ is an isomorphism.

A **kernel** of a homomorphism $\mu: G \to G'$ has the form $\rho = (\rho_i, i \in \Gamma)$, where each $\rho_i$ is the kernel equivalence of the map $\mu_i: G_i \to G'_i$. A **many-sorted equivalence** $\rho$ is a congruence if it preserves all operations $\omega \in \Omega$. This means that if $\tau(\omega) = (i_1, \ldots, i_n; j)$ and $a_1, a'_1 \in G_{i_1}, \ldots, a_n, a'_n \in G_{i_n}$, then

$$a_1\rho_{i_1}a'_1, \ldots, a_n\rho_{i_n}a'_n \Rightarrow (a_1 \cdots a_n\omega)\rho_j(a'_1 \cdots a'_n\omega).$$

If $\rho$ is a congruence of $G$, then one can consider the quotient algebra $G/\rho = (G_i/\rho_i, i \in \Gamma)$. The notions of a subalgebra and a Cartesian product of algebras are defined in a natural way. If, for example, $G^\alpha = (G^\alpha_i, i \in \Gamma)$ are $\Omega$-algebras, $\alpha \in I$, then $\prod G^\alpha = (\prod G^\alpha_i, i \in \Gamma)$, and if $a_1 \in \prod G^\alpha_{i_1}, \ldots, a_n \in \prod G^\alpha_{i_n}$ then $(a_1 \cdots a_n)\omega(\alpha) = a_1(\alpha) \cdots a_n(\alpha)\omega$, provided $\tau(\omega) = (i_1, \ldots, i_n; j)$.

Now let us make some notes on varieties of algebras.

Varieties of algebras are specified by identities. Let us consider the many-sorted case in detail. Let $X = (X_i, i \in \Gamma)$ be a many-sorted set with all sets $X_i$ countable. Starting from the set of operation symbols $\Omega$, we can construct the terms over $X$. Denote the system of terms by $W = (W_i, i \in \Gamma)$. The inductive definition of $W$ runs as follows. Every set $X_i$ is included in $W_i$, the set of terms of the sort $i$. If $\omega \in \Omega$, $\tau(\omega) = (i_1, \ldots, i_n; j)$ and $w_1, \ldots, w_n$ are terms of the sorts $i_1, \ldots, i_n$ respectively, then $w_1 \cdots w_n\omega$ is a term of the sort $j$. If there are nullary operation symbols (with $n = 0$) of the sort $i \in \Omega$, then they also belong to $W_i$. An algebra $W$ is, naturally, an $\Omega$-algebra, and it is called the **absolutely free $\Omega$-algebra**.

An identity is a formula of the kind $w \equiv w'$, where $w$ and $w'$ are terms from $W$ of the same sort, say $i$. Such a formula is valid in an $\Omega$-algebra $G = (G_i, i \in \Gamma)$, if $w^{\mu_i} = w'^{\mu_i}$ for every homomorphism $\mu: W \to G$.

A set of identities determines a **variety of $\Omega$-algebras**, i.e. the class of algebras satisfying all identities from a given set. In every variety $\Theta$ the set $X = (X_i, i \in \Gamma)$ picks out a free algebra, which is a quotient algebra of the absolutely free algebra $W$.

The Birkhoff’s theorem holds also in the many-sorted case.

A **variety $\Theta$** is a variety if and only if it is closed under Cartesian products, subalgebras and homomorphic images.

Any variety $\Theta$ can be taken for the initial variety, and one can classify various subvarieties and other axiomatizable classes in $\Theta$. If $X = (X_i, i \in \Gamma)$ is a many-sorted set, then the free algebra in $\Theta$ associated with $X$ is also denoted by $W = (W_i, i \in \Gamma)$. Subvarieties of $\Theta$ are defined by identities $w \equiv w'$, where
w and \( w' \) are terms of the same sort in this new \( W \). Fully characteristic congruences of \( W \) correspond to closed sets of identities.

For an arbitrary algebra \( G = (G_i, i \in \Gamma) \) from the variety \( \Theta \), we can consider the set of homomorphisms \( \text{Hom}(W, G) \).

Now we pass to \( \Theta \)-logic.

We again consider the many-sorted case with the set of sorts \( \Gamma \). Let \( X \) be the set of variables, with the stratification map \( n : X \to \Gamma \). This map is surjective and divides \( X \) into sets \( X_i, i \in \Gamma \). Each \( X_i \) consists of the variables of the sort \( i \), where the sort of \( x \) is \( i = n(x) \). So, we have a many-sorted set \( X = (X_i, i \in \Gamma) \).

We fix the set of operation symbols \( \Omega \) (so a certain \( \Gamma \)-type \( \tau \) corresponds to each \( \omega \in \Omega \)), and select the variety \( \Theta \) of \( \Omega \)-algebras. With \( \Theta \) we associate a logic which is called \( \Theta \)-logic. Let \( W = (W_i, i \in \Gamma) \) be the free over \( X \) algebra in \( \Theta \). We also fix the set of relation symbols \( \Phi \); each \( \phi \in \Phi \) has a type \( \tau = \tau(\phi) = (i_1, \ldots, i_n) \), realized in the algebra \( G = (G_i, i \in \Gamma) \) as a subset of the Cartesian product \( G_{i_1} \times \cdots \times G_{i_n} \).

Now we can construct the set of formulas of \( \Theta \)-logic. First, we define the elementary formulas. These are of the form

\[ \varphi(w_1, \ldots, w_n), \]

where \( \tau(\varphi) = (i_1, \ldots, i_n) \), \( w_1 \in W_{i_1}, \ldots, w_n \in W_{i_n} \). Denote the set of all elementary formulas by \( \Phi W \).

We let \( L = \{\lor, \land, \neg, \exists x\}_{x \in X} \) be the signature of logical symbols, and we construct the absolutely free algebra over \( \Phi W \) in this signature. The corresponding formula algebra is denoted by \( L\Phi W \).

The set \( L\Phi W \) has a part whose elements are logical axioms.

They are the usual axioms of calculus with functional symbols. (Warning: the axiom set can be not effective in general.) The terms of this calculus are \( \Theta \)-terms. The rules of inference are standard:

1. *Modus ponens*: \( u \) and \( u \to v \) imply \( v \),
2. *Generalization*: \( u \) implies \( \forall xu \).

Here \( u, v \in L\Phi W, u \to v \) stands for \( \neg u \lor v \), and \( \forall xu \) means \( \neg \exists x \neg u \).

Formulas together with axioms and rules of inference constitute the (first-order) \( \Theta \)-logic. In particular, we can speak about the logic of group theory, the logic of ring theory, etc.

A set of formulas is said to be *closed* if it contains all the axioms and is invariant with respect to the rules of inference.

With each \( \varphi \in \Phi \) of type \( \tau = (i_1, \ldots, i_n) \) we associate a set of mutually distinct variables \( x_{i_1}^\varphi, \ldots, x_{i_n}^\varphi \). Then the formula \( \varphi(x_{i_1}^\varphi, \ldots, x_{i_n}^\varphi) \) is called a basic one. The variables occurring in basic formulas are called *attributes*. The set of attributes is denoted by \( X_0 \), the set of basic formulas is denoted by \( \Phi X_0 \). It is a small part of the set of elementary formulas \( \Phi W \).

For applications, logic should also contain equalities. An *equality* is a formula of the kind \( w \equiv w' \), where \( w \) and \( w' \) are elements of \( W \) of the same sort. Informally, such a formula can be regarded either as an equation or as an identity. In the *equality logic* the set of logical axioms is extended by specific equality axioms. Any equality is considered to be an additional elementary formula. The absolutely free formula algebra can be constructed in equality logic, too. We use the same notation \( L\Phi W \) for it.
3. Algebraic $\Theta$ logic. Normally, quantifiers are logical symbols, but they also can be defined to be operations of a Boolean algebra:

An existential quantifier $\exists$ on a Boolean algebra $H$ is a map $\exists: H \to H$ subject to the following conditions:

1. $\exists 0 = 0$,
2. $a \leq \exists a$,
3. $\exists (a \land \exists b) = \exists a \land \exists b$,

where $a, b \in H$ and 0 is the zero element of $H$.

Every quantifier is a closure operator on $H$, and two existential quantifiers can be non-permutable. Furthermore, an existential quantifier is additive: $\exists (a \lor b) = \exists a \lor \exists b$. The set of all $a \in H$ with $\exists a = a$ is a subalgebra of the Boolean algebra $H$. A universal quantifier $\forall: H \to H$ is defined dually:

1. $\forall 1 = 1$,
2. $\forall a \leq a$,
3. $\forall (a \lor \forall b) = \forall a \lor \forall b$.

Analogously we have: $\forall (a \land b) = \forall a \land \forall b$.

There is a well-known correspondence between the two species of quantifiers which allows to switch back and forth from $\exists$ to $\forall$.

In algebraic logic, algebraic structures of logic are constructed and studied. For example, with the classical propositional calculus associated are Boolean algebras, and with the intuitionistic propositional calculus Heyting algebras are connected.

There are three approaches to algebraization of first-order logic, namely, the Tarski’s cylindric algebras [26], the Halmos’ polyadic algebras [24] and the categorical approach by Lawvere [33, 34]. These approaches are based on deep analysis of calculus (as a rule, we shall use the word “calculus” for the first-order $\Theta$-logic). There are also algebraic equivalents of nonclassical first-order logics [17]. Similar constructions are developed for other logics [12]. See also [6, 9, 7, 8, 10, 11, 66, 67, 68, 62, 63].

We consider the respective algebraizations of $\Theta$-logic. This generalization is necessary for databases with the data type $\Theta$, and for algebra itself as well. We confine ourselves with Halmos algebras.

Let us proceed from a fixed scheme consisting of a set $X = (X_i, \ i \in \Gamma)$, a variety $\Theta$, and an algebra $W$ free in $\Theta$ over $X$. Since the latter is uniquely determined by $\Theta$ and $X$, we do not write it out. We also take into account the semigroup $\text{End}W$, whose elements are considered to be additional operators.

3.1. Definition. Suppose $(X, \Theta)$ is a scheme. An algebra $H$ is a Halmos algebra in this scheme if

1.1. $H$ is a Boolean algebra.
1.2. The semigroup $\text{End}W$ acts on $H$ as a semigroup of Boolean endomorphisms.
1.3. Action of quantifiers of the form $\exists (Y), Y \subset X$, is defined.

These actions are connected by the following conditions:

2.1. $\exists (\emptyset)$ acts trivially.
2.2. $\exists (Y_1 \cup Y_2) = \exists (Y_1) \exists (Y_2)$.
2.3. $s_1 \exists (Y) = s_2 \exists (Y)$ if $s_1, s_2 \in \text{End}W$ and $s_1(x) = s_2(x)$ for $x \in X \setminus Y$.
2.4. $\exists (Y)s = s \exists (s^{-1}Y)$ for $s \in \text{End}W$ if the following conditions are fulfilled:
   1) $s(x_1) = s(x_2) \in Y$ implies $x_1 = x_2$,
   2) If $x \notin s^{-1}Y$, then $\Delta s(x) \cap Y = \emptyset$. 


Here, $s^{-1}Y = \{ x, \ s(x) \in Y \}$ and $\Delta s(x)$ is the support of the element $w = s(x) \in W$, i.e., the set of all $x \in X$ involved in the (representation of the) element $w$. For a precise definition of a support for Halmos algebras, see [31].

All Halmos algebras in the given scheme form a variety denoted by HA$_\Theta$. We will deal with such Halmos algebras and occasionally will call them Halmos algebras specialized in $\Theta$. In the next three subsections we shall present examples of Halmos algebras.

3.2. Given $G \in \Theta$, consider Hom($W, G$). Denote by $M_G$ the set of all subsets of Hom($W, G$), i.e. $M_G = \text{Sub}(\text{Hom}(W, G))$. In fact, $M_G$ is a Boolean algebra. If $A \in M_G$, $\mu \in \text{Hom}(W, G)$ and $s \in \text{End} W$, then define $\mu s$ by the rule $\mu s(x) = \mu(s(x))$. An action of the semigroup End $W$ on $M_G$ is defined by

$$
\mu \in s A \iff \mu s \in A.
$$

Where $Y \subset X$, we let $\mu \in \exists(Y) A$ if there is $\nu: W \rightarrow G$ in $A$ such that $\mu(x) = \nu(x)$ for every $x \in X \setminus Y$. This defines the action of a quantifier on $M_G$. All the axioms of Halmos algebra are fulfilled in $M_G$, and so this is the first example of a HA$_\Theta$-algebra.

Let $H$ be a Halmos algebra and $h \in H$. We denote by $\Delta h$ the support of $h$:

$$
\Delta h = \{ x \in X, \exists x h \neq h \}.
$$

If $\Delta h$ is finite, then the element $h$ is said to be finitely supported. In the previous item we needed supports of elements of the algebra $W$, while here we deal with supports in a Halmos algebra. All finitely supported elements of $H$ constitute a subalgebra called the locally finite part of $H$. The algebra $H$ is locally finite if all of its elements have finite supports.

3.3. Denote by $V_G$ the locally finite part of the Halmos algebra $M_G$. Here $A \in V_G$ if, for some finite subset $Y \subset X$ and elements $\mu, \nu \in \text{Hom}(W, D),$

$$
\mu \in A \iff \nu \in A
$$

whenever $\mu(x) = \nu(x)$ for all $x \in Y$. In other words, belonging of a row to the set $A$ is checked on a finite part of $X$.

3.4. Now let us consider our main example: the HA$_\Theta$-algebra of first order calculus. Suppose that all $X_i$ in $X = (X_i, \ i \in \Gamma)$ are infinite. Besides that, the set $\Phi$ of relation symbols is added to the scheme, and we again have the formula algebra L$\Phi W$.

Variables occur in formulas, and an occurrence of a variable is either free or bound. We define the action of an element $s \in \text{End} W$ on the set L$\Phi W$ as follows. If $u$ is a formula and the variables $x_1, \ldots, x_n$ occur freely in it, then we substitute them by $sx_1, \ldots, sx_n$, respectively, in all free occurrences of them. So we get $su$. For example, it follows from this definition that if $u = \varphi(w_1, \ldots, w_n)$ is an elementary formula, then $su = \varphi(sw_1, \ldots, sw_n)$. However, the definition does not provide a representation of the semigroup End$W$ in the set L$\Phi W$. Simple examples show that the condition $(s_1 \cdot s_2)u = s_1(s_2u)$ is not fulfilled. Let us define an equivalence $\rho$ on the set L$\Phi W$ by the rule: $u \rho v$ if $u$ and $v$ only differ in the names of bound variables. Take the quotient set L$\Phi W/\rho = \overline{L\Phi W}$, and call this passage factorization by renaming bound variables. We call elements of $\overline{L\Phi W}$ formulas, too, but they are regarded up to renaming bound variables. It is easy to see that the equivalence $\rho$ is compatible with the signature, but is not $\neg$-with the action of elements from End$W$. Therefore, all operations from the set $L = \{ \forall, \land, \lor, \exists x, x \in X \}$ are defined on $\overline{L\Phi W}$, but action of elements from End$W$ has to be defined separately. This is carried out as follows.

Let $u$ be a formula, $x_1, \ldots, x_n$ be the list of all of its free variables, and take $sx_1, \ldots, sx_n$, $s \in \text{End} W$. We say that $u$ is open for $s$ if there are no bound variables in $u$ belonging to any of the sets $\Delta sx_1, \ldots, \Delta sx_n$. For each $u$, we denote by $\bar{u}$ the corresponding class of equivalent elements. For $s \in \text{End} W$ we always can
find some formula \( u' \) in the class \( \bar{u} \) which is open for \( s \). Then we set \( s\bar{u} = \bar{su'} \). It is easily understood that if we have another formula \( u'' \) in \( \bar{u} \) which is open for \( s \), then \( su'psu'' \) and \( su'' = \bar{su}' \). Hence the definition of \( s\bar{u} \) is correct. This rule gives the representation of the semigroup \( \text{End}W \) as a semigroup of transformations of the formula set \( L\Phi W \).

From now on we proceed from the set of formulas \( L\Phi W \). Axioms and rules of inference are related to this set, too. As before, they are standard.

Now we pass to Lindenbaum-Tarski algebra. We have an equivalence \( \tau \) which is defined as follows: \( \bar{u}r\bar{v} \) if the formula \( \bar{u} \rightarrow \bar{v} \wedge (\bar{v} \rightarrow \bar{u}) \) is derivable. It can be verified that \( \tau \) is congruence on \( L\Phi W \), and this \( \tau \) is also compatible with the action of the semigroup \( \text{End}W \).

Denote by \( U \) the result of factorization of \( L\Phi W \) by \( \tau \). Define an equivalence \( \eta \) on \( L\Phi W \) by the rule: \( u\eta v \Leftrightarrow \bar{u}r\bar{v} \). Since \( \rho \subset \tau \), the set \( U = L\Phi W/\tau \) can be identified with \( L\Phi W/\eta \). It is important to emphasize that the equivalence \( \eta \) can also be defined by means of Lindenbaum-Tarski scheme, and \( U \) is the Lindenbaum-Tarski algebra. It can be proved that:

1. \( U \) is a Boolean algebra with respect to the operations \( \vee, \wedge, \neg; \)
2. The semigroup \( \text{End}W \) acts on \( U \) as a semigroup of endomorphisms of this algebra;
3. The operations \( \exists x \) are pairwise permutable. This allows us to define in \( U \) quantifiers \( \exists(Y) \) for all \( Y \subset X \).

All the above leads to the following result:

**3.5. Theorem.** The algebra \( U \) with the indicated operations is an algebra in \( \text{HA}_\Theta \).

This is a syntactical approach to the definition of Halmos algebra of first-order \( \Theta \)-logic. Such an approach is realized by Z. Diskin [12]. There is also a semantical approach which is described in [51]. Both of them give the same result. Finally, we can use the verbal congruence of the variety \( \text{HA}_\Theta \), and obtain once more the same algebra \( U \). Elements of \( U \) are the formulas of \( \Theta \)-logic, now considered up to the equivalence just defined.

Now we are going to discuss homomorphisms of Halmos algebras. We start with a very important property of the algebra \( U \), and note first of all that this algebra is locally finite. Take the basic set \( \Phi X_0 \) in the formula algebra \( L\Phi W \), and let \( U_0 \) be the corresponding basic set in \( U \). The set \( U_0 \) generates the algebra \( U \).

**3.6. Theorem.** Let \( H \) be an arbitrary Halmos algebra, and let \( \zeta: \Phi X_0 \rightarrow H \) be a map such that, for every \( u = \varphi(x_1, \ldots, x_n) \in \Phi X_0 \), \( \Delta\zeta(u) \subset \{x_1, \ldots, x_n\} \). Such \( \zeta \) gives another map \( \zeta: U_0 \rightarrow H \), and the latter one is uniquely extended to a homomorphism \( \zeta: U \rightarrow H \).

Given a model \( (G, \Phi, f), G \in \Theta \), consider the particular case when \( H = V_G \). Define \( \zeta = \hat{f}: \Phi X_0 \rightarrow V_G \) by the rule: if \( u = \varphi(x_1, \ldots, x_n) \in \Phi X_0 \), then

\[
\hat{f}(u) = \{\mu, \mu \in \text{Hom}(W, G), (\mu(x_1), \ldots, \mu(x_n)) \in f(\varphi)\}.
\]

Then \( \Delta\hat{f}(u) = \{x_1, \ldots, x_n\} \), and we have a homomorphism

\[
\hat{f}: U \rightarrow V_G.
\]

It follows from its definition that if \( u = \varphi(w_1, \ldots, w_n) \) is an elementary formula, then

\[
\hat{f}(u) = \{\mu, \mu \in \text{Hom}(W, D), (\mu(w_1), \ldots, \mu(w_n)) \in f(\varphi)\}.
\]

So, for every model \( (G, \Phi, f), G \in \Theta \), we have the canonical homomorphism \( \hat{f}: U \rightarrow V_G \), and every homomorphism \( U \rightarrow V_G \) proves to be of this sort. Now we can say that, for every \( u \in U \), \( f(u) \) is the value of \( u \) in the model \( G \). \( f(u) = 1 \) means that \( u \) is valid in \( G \).
We add a few words about kernels of homomorphisms in the variety HA_φ. If σ: H → H' is a homomorphism in HA_φ, then we have two kernels: the coinage of the zero and the coinage of the unit. The coinage of the zero is an ideal, and the coinage of the unit is a filter. A subset T of H is a filter if

1. a ∧ b ∈ T if a and b belong to T,
2. a ∨ b ∈ T if a ∈ T and b ∈ H,
3. ∀(Y)a ∈ T if a ∈ T, Y ⊂ X.

The definition of an ideal is dual. For every filter T, the quotient algebra H/F is at the same time also the algebra H/F, F being the ideal defined by the rule: h ∈ F if and only if h ∈ T.

If T is a subset of H, then the filter of H generated by T consists of elements of the form

∀(X)a_1 ∧ · · · ∧ ∀(X)a_n ∧ b, a_i ∈ T, b ∈ H.

Every filter is closed under existential quantifiers, while every ideal is closed under universal quantifiers.

Besides, it can be proved that both ideals and filters are closed under the action of the semigroup End_W.}

3.7. Theorem. A set T ⊂ H is a filter of H if and only if it satisfies the conditions

1. 1 ∈ T,
2. if a ∈ T, a → b ∈ T, then b ∈ T,
3. if a ∈ T, then ∀(Y)a ∈ T, Y ⊂ X.

Here, a → b = ~a ∨ b. We will consider two rules of inference in Halmos algebras:

1. from a and a → b, infer b,
2. from a, infer ∀(Y)a, Y ⊂ X.

If H is locally finite, then the second rule can be replaced by

2'. from a, infer ∀xa, x ∈ X.

For every set T, one can consider the set of elements which are inferred from T.

3.8. Theorem. If T is a subset of H containing the unit, then the filter generated by T is the set of all those h ∈ H which are inferred from T.

The notion of inferability in Halmos algebras agrees with that for formulas in logic. The notion of a filter of a Halmos algebra corresponds to the notion of a closed set of formulas.

Given a model (G, Φ, f), we have ̃f: U → V_G. We can consider the corresponding filter Ker ̃f as the elementary theory (θ-theory) of the model.

3.9. Theorem. Let T be a nonempty subset of U, T' = K be an axiomatizable class of models defined by the set T, and T'' = K' be the axiom set of K in U (the closure of T). Then T'' is the filter of U generated by T.

We now make some remarks on equalities in a Halmos algebra.

An equality in a Halmos algebra H is a new nullary operation w ≡ w', where w, w' ∈ W and are both of the same sort.

3.10. Definition. An algebra H ∈ HA_φ is an algebra with equalities if all the operations w ≡ w' are defined as elements of H and the following axioms hold:

1. s(w ≡ w') = (sw ≡ sw'), s ∈ End_W,
2. (w ≡ w) = 1, w ∈ W,
3. (w_1 ≡ w'_1) ∧ · · · ∧ (w_n ≡ w'_n) < w_1 · · · w_nω ≡ w'_1 · · · w'_nω if ω ∈ Ω is of an appropriate type,
4. \( s_w^x a \land (w \equiv w') < s_w^x a, \quad a \in H, \) where \( s_w^x \) takes \( w \) into an element \( w \) of the same sort and leaves every \( y \neq x \) fixed.

In the algebra \( V_G \), equalities are defined by the rule: \( w \equiv w' \) is the set of all those \( \mu: W \rightarrow G \) for which \( w^\mu = w'^\mu \) in \( G \). When considering the algebra \( U \) with equalities, the symbol \( \equiv \) is supposed to be added to \( \Phi \), and the initial axioms of \( \Theta \)-logic are supplemented by the standard axioms of equality. In this case the set \( \Phi \) may be empty. Such an algebra \( U \) arises from \( \Theta \)-logic with equalities.

Since equalities are regarded as nullary operations, any subalgebra of an algebra with equalities should contain all the elements \( w \equiv w' \). The same remark concerns homomorphisms between algebras with equalities. If \( H \) is an algebra with equalities, then so is \( H/T \), where \( T \) is a filter.

When \( \Phi \) is empty, the algebra \( U \) with equalities is the object that controls all axiomatizable classes of algebras in \( \Theta \). Relaying on \( U \), we can solve the problems mentioned before. The algebra \( U \) with \( \Phi \) nonempty plays the same role for models.

We make here some further remarks.

Let \( G = (G_i, \ i \in \Gamma) \) be an algebra in the variety \( \Theta \). In order to investigate the elementary theories of the models with given \( \Phi \) and realized on \( G \), it will be useful to introduce this algebra into the language and the algebra of the corresponding calculus. We assume that \( G \) is specified by generators and defining relations.

We fix the scheme of calculus, which includes the mapping \( n: X \rightarrow \Gamma \), variety \( \Theta \) with the set of operation symbols \( \Omega \), and the set of relation symbols \( \Phi \).

We denote the set of generators of \( G \) by \( M = \{ M_i, \ i \in \Gamma \} \) and the set of defining relations by \( \tau \). To each \( a \in M \), we attach a variable \( y = y_a \). This way with each \( M_i \) a set of variables \( Y_i \) is associated, and we obtain a many-sorted set \( Y = (Y_i, \ i \in \Gamma) \). Let \( W_G \) be the free algebra in \( \Theta \) over \( Y \). The correspondence \( y_a \mapsto a \) yields an epimorphism \( \nu: W_G \rightarrow G \). The kernel \( \text{Ker} \nu = \rho \) is generated by \( \tau \), and then we have an isomorphism \( W_G/\rho \rightarrow \Gamma \).

For every \( a \in M_i \) and all \( i \in \Gamma \), we add to \( \Omega \) a symbol of a nullary operation \( \omega_a \). Let \( \Omega' \) be the new set of operation symbols, and let \( \Theta' \) be the variety of \( \Omega' \)-algebras defined by the identities of the variety \( \Theta \) and defining relations of \( G \). Here, if \( w(y_{a_1}, \ldots, y_{a_n}) = w'(y_{a_1}, \ldots, y_{a_n}) \) is a defining relation, then we must rewrite it in the form
\[
w(\omega_{a_1}, \ldots, \omega_{a_n}) = w'(\omega_{a_1}, \ldots, \omega_{a_n}).
\]
We have no variables here, and this equality is also an identity.

Let \( W \) be the free algebra in \( \Theta \) over \( X = (X_i, \ i \in \Gamma) \), and let \( W' \) be the free algebra in \( \Theta' \), also over \( X \).

All \( \omega_a \) are elements of \( W' \). Suppose that \( G' \) is a subalgebra of \( W' \) generated by these elements. It can be proved that there exists a canonical isomorphism \( \nu: G' \rightarrow G \) and that \( W' \) is the free product of \( W \) and \( G' \) in \( \Theta \), i.e. \( W' = W * G' \).

In the old scheme we had a Halmos algebra \( U \), and the new scheme gives rise to the algebra \( U' \). There is a canonical injection \( U \rightarrow U' \). The kernel of it admits a good description in the case \( G \) is finitely defined in \( \Theta \).

The algebra \( G \) can be taken to be an algebra in the variety \( \Theta' \), and we can identify the homomorphism sets \( \text{Hom}(W, G) \) and \( \text{Hom}(W', G) \). Simultaneously, we can identify algebras \( V_G \) and \( V_{G'} \). But \( V_G \) is in the old scheme with the semigroup \( \text{End}W \), and \( V_{G'} \) in the new scheme with the semigroup \( \text{End}W' \).

Finally, take a homomorphism \( \mu: W \rightarrow G \), which we identify with \( \mu: W' \rightarrow G \), and set \( s = \nu^{-1} \mu: W' \rightarrow G' \), where \( \nu: G' \rightarrow G \) is the foregoing canonical isomorphism. Because \( G' \) is a subalgebra of
characteristic congruence on $W$, $s \in \text{End}W'$. Take now some $\tau: W' \to G$ and $x \in X$. We then have $\mu(x) = \nu \nu^{-1} \mu(x) = \nu s(x)$. Since $s(x) \in G'$ is a constant, $\nu$ and $\tau$ act equally on $s(x)$, and $\mu(x) = \tau s(x)$. It is so for all $x \in X$; therefore, $\mu = \tau s$ for each $\tau$.

Given a model $(G, \Phi, f), G \in \Theta$, where $G$ is also considered as an algebra in $\Theta'$, we have the homomorphism $\hat{f}: U' \to V_G$. We consider every $u \in U$ as an element in $U'$, and the value of $u$ in $G$ is $\hat{f}(u)$. In order to calculate this value, we must answer the question when $\mu \in \hat{f}(u)$.

Together with $u$, we also consider the element $su$. We can interpret it as the result of substitution of the row $\mu$ in the formula $u$.

3.11. Theorem. $\mu$ belongs to $\hat{f}(u)$ if and only if $\hat{f}(su) = 1$.

Proof. Let $\mu \in \hat{f}(u)$. Take an arbitrary $\tau$, $\tau s = \mu$. Then $\tau s \in \hat{f}(u)$ and $\tau \in s \hat{f}(u) = \hat{f}(su)$. Since $\tau$ is arbitrary, we have $\hat{f}(su) = 1$.

Now let $\hat{f}(su) = 1$. Then $\mu \in \hat{f}(su) = s \hat{f}(u)$, and $\mu s = \mu \in \hat{f}(u)$. □

The equality $\hat{f}(su) = 1$ means that $su \in \text{Ker} \hat{f}$, where $\text{Ker} \hat{f}$ is the elementary theory of the given model.

We see that, in the extended language, the value of an arbitrary formula in a model can be calculated from the elementary theory.

§2. Some applications

1. Closures of formula systems. We begin with closures of systems consisting either of identities or pseudoidentities. The discussion will be based on the Halmos algebra of calculus, which we denote by $U$.

Take a scheme $X, \Gamma, n: X \to \Gamma, \Omega, \Theta$, all $X_i$ in $X$ being infinite. $U$ is a Halmos algebra with equalities in this scheme, $W = (W_i, i \in \Gamma)$ is the free algebra over $X$ in $\Theta$. The set of relation symbols $\Phi$ is empty. It is well known that every set of identities can be presented in the algebra $W$. Then its closure is a fully characteristic congruence on $W$, generated by the set. The same set of identities, and its closure, can also be presented in the algebra $U$. We identify each identity $w \equiv w'$ with the respective equality element of $U$.

Let $T$ be a set of equalities of $U$, which are treated as identities specifying some variety $K$. From the result given in terms of algebra $W$ we can conclude that the set $T$ is closed if the following conditions are fulfilled:

1. $(w \equiv w) = 1 \in T$ and $T$ is closed under the semigroup $\text{End}W$,
2. if $w \equiv w' \in T$ and $w'' \equiv w''' \in T$, then $w \equiv w'' \in T$,
3. if $\omega \in \Omega$ and the type of $\omega$ is $(i_1, \ldots, i_n; j)$, then it follows from $w_1 \equiv w_1' \in T, \ldots, w_n \equiv w_n' \in T$, $w_k, w_k' \in W_{i_k}$, that $w_1 \cdots w_n \omega \equiv w_1' \cdots w_n' \omega \in T$.

As we know, the formulas derivable from $T$ form the filter generated by $T$. This filter necessary contains all identities of $K$. Any of them is derivable from $T$, but, in general, the derivation can contain not only equalities. However, we may use in the derivation only the rules listed above: the list is known to be complete for equational formulas.

Now a few remarks on pseudoidentities follow. Here, $T$ is a set of formulas—again, elements from $U$—of the type

\[ w_1 \equiv w_1' \lor \cdots \lor w_n \equiv w_n'. \]
Let $T$ be a closed set. This means that if $K$ is a pseudovariety in $\Theta$ defined by $T$, then all pseudoidentities of algebras from $K$ are in $T$. It is obvious that

1. $1 \in T$ and $T$ is closed under the semigroup $\text{End} W$,
2. if $u \in T$ and $v$ is a pseudoidentity in $U$, then $u \vee v \in T$.

We shall formulate one more condition which also has to be fulfilled. Suppose we are given pseudoidentities $u_1, \ldots, u_r$ from $U$, where $u_k$ is

$$(w^k_1 \equiv w^{k'}_1) \lor \cdots \lor (w^k_n \equiv w^{k'}_n), \quad k = 1, \ldots, r.$$  

Then we define a new set of pseudoidentities, which we denote by $u_1 \circ \cdots \circ u_r$, as follows. First, we set $W \times W$ to be the union of all $W_i \times W_i$, $i \in \Gamma$. Next, we take, for every $u_k$, the subset $u^k_i$ of $W \times W$ consisting of all pairs $(w^k_i, w^{k'}_i)$, $i = 1, \ldots, n$, and make up the Cartesian product $V = u^1_1 \times \cdots \times u^r_r$.

If $p = (p_1, \ldots, p_r) \in V$, then we denote by $\rho(p)$ the congruence on $V$ generated by all $p_1, \ldots, p_r$, $\rho(p) = (\rho_i, \ 1 \leq i \leq \Gamma)$. Also let $\psi$ be a function on $V$ such that $\psi(p)$ is a pair $(w, w')$ contained in some $\rho_i, \ 1 \leq i \leq \Gamma$, from $\rho(p)$. We denote by $u = u(\psi)$ the pseudoidentity $\bigwedge (w \equiv w')$ where the disjunction is taken over all $p \in V$. The set $u_1 \circ \cdots \circ u_r$ consists of all such $u(\psi)$ for all $\psi$. Now we can write out the third condition.

3. If $u_1, \ldots, u_r \in T$, then $u_1 \circ \cdots \circ u_r$ is a subset of $T$.

Let us verify that this condition is satisfied if $T$ is closed. Assume that $u_1, \ldots, u_r \in T$ and that $u = u(\psi) \in u_1 \circ \cdots \circ u_r$. Take a homomorphism $\mu: W \to G$, $G \in K$. Then for every $u_k, \ 1 \leq k \leq r$, and $w \in W_i$, we find $w^k_i = e^k_i$ so, that $w^k_{i, \mu} = e^k_{i, \mu}$ in $G$. Denote $(w^k_i, w^{k'}_i)$ by $p_k$ and take $p = (p_1, \ldots, p_r)$. All $p_k$ are in $\text{Ker} \mu$, hence $\rho(p) \subset \text{Ker} \mu$. By the definition $\psi(p) = (w, w')$ also lies in $\text{Ker} \mu$, and we have $w^\mu = w'^\mu$ in $G$. This means that the pseudoidentity $u = u(\psi)$ is valid in $G$.

The converse also can be proved, and we have

1.1. Theorem. The set of pseudoidentities $T$ is closed if and only if $T$ satisfies the conditions 1, 2, 3.

We may consider the conditions 1, 2, 3 as rules of inference, so that we can construct the closure $T'$ of every $T$.

This result belongs to A. Kushkuley. My participation in obtaining it is not great: I only consider the many-sorted case, and put it in terms of Halmos algebras. A. Kushkuley and S. Rosenberg (the former is now in the USA, and the latter is in Jerusalem, both of them are from Riga) have also obtained a generalization, which we are going to consider.

We shall deal with closures of universal formulas. The universal formulas we consider are formulas of type

$$(w_1 \equiv w'_1) \lor \cdots \lor (w_n \equiv w'_n) \lor (v_1 \neq v'_1) \lor \cdots \lor (v_m \neq v'_m), \quad w, v \in W.$$  

A set of such formulas describes an universal class of algebras.

Given a set $T$ of universal formulas from $U$, we must construct the closure for $T$.

For every universal formula $u$ of the above kind, we take the set $u^+$ consisting of all pairs $(w_1, w'_1), \ldots, (w_n, w'_n)$, and let $u^-$ be the set of pairs $(v_1, v'_1), \ldots, (v_m, v'_m)$. The union of $u^+$ and $u^-$ is denoted by $u^\circ$.

As before, for every set of universal formulas $u_1, \ldots, u_r$, we construct a new set $u_1 \circ \cdots \circ u_r$ of such formulas. At first, we take the Cartesian product $u^1_1 \times \cdots \times u^r_r = V$. If $p = (p_1, \ldots, p_r) \in V$, then $p^+$ is the set of "positive" pairs, used in the notation of $p$, and $p^-$ is the set of the "negative" pairs. If $M$ is a
subset of $W \times W$, then $\rho(M)$ is the congruence of $W$ generated by $M$. Now let $u \in u_1 \circ \cdots \circ u_r$ if for every $p \in V$

$$\rho(u^- \cup p^+) \cap (u^+ \cup p^-) \neq \emptyset.$$  

This definition generalizes that given for pseudoidentities.

Now let us show that if each of $u_1, \ldots, u_r$ is valid in $G$ and $u \in u_1 \circ \cdots \circ u_r$, then $u$ is also valid in $G$.

Given a homomorphism $\mu: W \to G$, we construct an element $p = (p_1, \ldots, p_n) \in V$ depending on $\mu$. Take any $u_k$:

$$(w_1^k \equiv w_1^{k'}) \lor \cdots \lor (w_n^k \equiv w_n^{k'}) \lor (u_1^k \not\equiv u_1^{k'}) \lor \cdots \lor (u_n^k \not\equiv u_n^{k'}).$$

$k = 1, \ldots, r$. As $u_k$ is valid in $G$, we have, for a given $\mu$, some pair $(w_1^k, w_1^{k'})$ with $w_i^{k\mu} = w_i^{k\mu'}$ or some other $(v_j^k, v_j^{k'})$ with $v_j^{k\mu} \not= v_j^{k\mu'}$. We take any one of them to be $p_k$, and this way we construct $p = (p_1, \ldots, p_r)$. For every pair $(w, w')$ in $p^+$, we have $w^\mu = w'^\mu$, and for every $(v, v')$ in $p^-\setminus v^\mu \not= v'^\mu$. Assume now that the pair $(w, w')$ belongs to the intersection

$$\rho(u^- \cup p^+) \cap (u^+ \cup p^-).$$

All pairs in $p^+$ belong to $\text{Ker} \mu$. If this is true also of all pairs in $u^-$, then $\rho(\rho(u^- \cup p^+) \subset \text{Ker} \mu$. Then $(w, w')$ also is in $\text{Ker} \mu$, $w^\mu = w'^\mu$ and $(w, w') \not\in p^-$. We have $(w, w') \in u^+$. If some $(w, w')$ from $u^-$ is not in $\text{Ker} \mu$, then $w^\mu \not= w'^\mu$. In all cases, the homomorphism $\mu$ belongs to the value of the formula $u$ in $G$. This holds for every $\mu$, and so $u$ is valid in $G$.

Now we can formulate the main result.

1.2. Theorem. \[32\] The set $T$ of universal formulas is closed if and only if the following conditions are satisfied:

1. $1 \in T$, and $T$ is closed under the action of the semigroup $\text{End} W$.
2. If $u \in T$ and $v$ is universal in $U$, then $u \lor v \in T$.
3. If $u_1, \ldots, u_r \in T$, then all $u \in u_1 \circ \cdots \circ u_r$ also belong to $T$.

Theorem 1.1 is a particular case of this theorem; we only must everywhere delete ”negative” parts. The closure of every set of universal formulas can be constructed in virtue of Theorem 1.2.

Now we shall consider closures of sets of quasi-identities. A quasi-identity is an element $u \in U$ of the type

$$(w_1 \equiv w_1') \land \cdots \land (w_n \equiv w_n') \to (w \equiv w').$$

This is a universal formula of a particular kind. We are interested in the question how to obtain the closure of a set of quasi-identities. The problem was investigated by R. Quackenbush. \[38\] We translate his result in terms of the algebra $U$. We rewrite $u$ in the form $u_0 \to (w \equiv w')$, where $u_0$ is $(w_1 \equiv w_1') \land \cdots \land (w_n \equiv w_n')$.

1.3. Theorem. The set of quasi-identities $T$ is closed if and only if it satisfies the following conditions:

1. $1 \in T$, and the set $T$ is invariant under the semigroup $\text{End} W$.
2. If $u$ is $(w_1 \equiv w_1') \land \cdots \land (w_i \equiv w_i') \land \cdots \land (w_n \equiv w_n')$, then $u \to (w_i \equiv w_i') \in T$.
3. If $u_0 \to (w \equiv w') \in T$ and $u_0 \to (w' \equiv w''') \in T$, then $u_0 \to (w \equiv w''') \in T$.
4. If $\omega \in \Omega$ and the type of $\omega$ is $(i_1, \ldots, i_n; j)$, and if $u_0 \to (w_k \equiv w_k') \in T$, $k = 1, \ldots, n$, $w_k, w_k' \in W_{i_k}$, then $u_0 \to (w_1 \cdots w_n \equiv w_1' \cdots w_n') \in T$.
5. If $u_0 \to (w_i \equiv w_i') \in T$, $i = 1, \ldots, n$, and if $(w_1 \equiv w_1') \land \cdots \land (w_n \equiv w_n') \to (w \equiv w') \in T$, then $u_0 \to (w \equiv w') \in T$. 

Necessity of this is obvious, and sufficiency in this theorem and in 1.1 and 1.2 is based on the following scheme. Let the set \( T \) satisfy the conditions of the theorem, and assume that the formula \( u \in U \) is not in \( T \). Then an algebra \( G \) satisfying \( T \), but not \( u \), can be found. The main problem is to construct such \( G \).

In \([58]\) the closure problem is dealt with for universal formulas as well. The result obtained there differs from that of Kushkuley and Rosenberg. (Both results were discovered at the same time.) Moreover, implicative classes are discussed in \([58]\), and the rules of inference include also this one: from \( u_0 \to 0 \), infer \( u_0 \to (w \equiv w') \) for every \( w \) and \( w' \) of the same sort. On this question, see also \([30, 64, 27]\).

2. Quasigroups. We shall consider here a known problem in the quasigroup theory.

A quasigroup is a group without associativity and, of course, without unit. If the unit is added, then we have a loop. More precisely, a quasigroup \( Q \) is a set with one binary operation of multiplication, and equations \( ax = b \) and \( ya = b \) are solved in \( Q \) uniquely. We introduce two additional operations: \( x = a \backslash b \) and \( y = b / a \). So the class of all quasigroups is a variety with the specifying identities

\[
x(x \backslash y) = y, \quad (x/y)y = x, \quad (xy)/y = x, \quad x \backslash (xy) = y.
\]

Adding a nullary operation 1 with identities \( 1x = x \) and \( x1 = x \), we get the variety of loops. The variety of groups arises when we add the associativity requirement. This is the definition of the variety of groups in the quasigroup signature. So, the notion of a quasigroup, as well as that of a semigroup, generalizes the notion of a group, but they do this in different ways.

Quasigroups have arisen from some problems of geometry. Loops also have applications in algebraic geometry, but these applications are not like to those of groups. In particular, we cannot speak about representations of quasigroups as quasigroups of permutations. In the theory of quasigroups, along with homomorphisms, the homotopies are used. A homotopy is a triplet

\[
\mu = (\mu_1, \mu_2, \mu_3) : Q \rightarrow Q',
\]

such that

\[
x^{\mu_1} y^{\mu_2} = (xy)^{\mu_3}.
\]

We can also speak of the category of quasigroups with homotopies as morphisms. If all maps \( \mu_1, \mu_2 \) and \( \mu_3 \) are bijective, then \( \mu \) is an isotopy. In geometric applications, quasigroups are considered up to isotopies.

In the group theory the notion of an isotopy is not of interest. For groups, an isotopy reduces to an isomorphism, and a homotopy—to a homomorphism. On the other hand, a quasigroup isotopic to a group may be not a group.

If we take the class \( K \) of all quasigroups isotopic to groups, then \( K \) is a variety closed under isotopy. It differs from the variety of all quasigroups.

We have the following general result.

2.1. Theorem. Let \( \Theta \) be a variety of groups, and \( \Theta' \) be the class of quasigroups isotopic to groups from \( \Theta \). Then:

1. \( \Theta' \) is a variety of quasigroups,

and

2. \( \Theta' \) is invariant under isotopy.

Long ago the geometrical applications prompted the following problem, which was stated, as it seems, by V.D. Belousow. Under what conditions a variety of quasigroups is closed under isotopies? Every variety is closed under isomorphisms, but not every one is closed under isotopies—for example, any variety
of groups. Moreover, if $\Theta$ is a variety of quasigroups and $\Theta'$ is the class of quasigroups isotopic to the quasigroups from $\Theta$, then the class $\Theta'$ is closed under isotopies, but it may be not a variety.

This problem was solved some years ago by A.A. Gvaramia during his postdoc in Riga, and even in a more general setting—for an arbitrary axiomatizable class of quasigroups. The solution was given using Halmos algebras [22, 23].

I bring here the sketch of the solution. First, together with the category of quasigroups with homotopies, we also take the category of the three-sorted quasigroups. Its objects have the form $A = (A_1, A_2, A_3)$, and there is defined an operation $*: A_1 \times A_2 \to A_3$ such that every pair of elements in $a_1 \cdot a_2 = a_3$ uniquely determines the third one. As in quasigroups, we have the inverse operations $*^{-1} = \backslash: A_1 \times A_3 \to A_2$ and $^{-1}*: = /: A_3 \times A_2 \to A_1$. We call such objects also invertible automata.

Morphisms in this category are the homomorphisms

$$
\mu = (\mu_1, \mu_2, \mu_3): A = (A_1, A_2, A_3) \to A' = (A'_1, A'_2, A'_3).
$$

They are coordinated with all three operations. We connect a regular automaton at $Q = (Q, Q, Q)$ with every quasigroup $Q$, and every homotopy $\mu = (\mu_1, \mu_2, \mu_3): Q \to Q'$ gives a homomorphism

$$
\text{at } \mu = (\mu_1, \mu_2, \mu_3): \text{ at } Q \to \text{ at } Q'.
$$

So we can consider the category of quasigroups with homotopies as a subcategory of the category of invertible automata.

It is easily proved that, for every automaton $A = (A_1, A_2, A_3)$, there is a quasigroup $Q$ with an isomorphism

$$
\mu = (\mu_1, \mu_2, \mu_3): A \to \text{ at } Q.
$$

Note that if $\mu = (\mu_1, \mu_2, \mu_3): Q \to Q'$ is a homotopy, then the image of $\mu$ in $Q'$ is a three-sorted subquasigroup of the quasigroup $Q'$. It is obvious that if $X$ is a variety of invertible automata, and $qX$ consists of all quasigroups $Q$ such that at $Q \in X$, then $qX$ is a variety of quasigroups, closed under isotopies.

Let, on the other hand, $\Theta$ be a class of quasigroups and at $\Theta$ consist of automata each of which is isomorphic to some at $Q$ with $Q \in \Theta$. So we have:

1. If $X$ is an abstract class of automata, then at $(qX) = X$,
2. If $\Theta$ is closed under isotopy, then $q(\Theta) = \Theta$.

This leads to a connection between varieties of quasigroups closed under isotopies, and varieties of invertible automata.

We are interested in identities and arbitrary formulas which define classes closed under isotopies.

Let $F = F(X)$ be the free quasigroup over the set $X$, and let $X = X_1 \cup X_2 \cup X_3$ be some partition of $X$. We have also the triple $X = (X_1, X_2, X_3)$, and it generates an automaton $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ in $F$. All intersections $\Phi_i \cap \Phi_j, i \neq j$, prove to be empty, and $\Phi$ is the free automaton over $X = (X_1, X_2, X_3)$.

Let now $\Theta$ be the variety of all quasigroups and $\Theta'$—the variety of all invertible automata. We can consider calculi in these varieties, and hence we have Halmos algebras $U$ and $U'$ over $X$ and $X = (X_1, X_2, X_3)$ respectively. The above notes allow us to consider an injection $U' \to U$, and so we can take in $U$ some three-sorted formulas which at the same time are thought of as one-sorted. We call these formulas special.

Given a quasigroup $Q$, we have the canonical homomorphisms

$$
f_Q: U \to V_Q
$$
where all \(w, v\) identities, pseudoidentities and so on. In general, universal formulas have the form 
\[
f_\mathcal{Q}(u) = 1 \in \mathcal{Q} \Leftrightarrow f_{\mathcal{Q}}(u) = 1 \in \mathcal{Q}.
\]

Assume now that \(u\) is valid in \(Q\) and that \(Q'\) is isotopic to \(Q\). Then \(u\) is valid also in at \(Q\) and at \(Q'\). Hence, \(u\) is valid in \(Q'\). So the class of all special formulas is closed under isotopies. The converse is proved by some additional reasoning, also in terms of Halmos algebras. We thus have

2.2. Theorem. A formula is closed under isotopies if and only if it is equivalent to a special one. The same holds for sets of formulas.

Note also that, for every formula, some special derived formula can be taken, and this leads to a construction of basis of special formulas.

Let us make some remarks about the derived formulas.

Set \(X = \{(u), \{v\}, X'\}\), and let \(\Phi = (\Phi_1, \Phi_2, \Phi_3)\) be an automaton over \(X\). Introduce an operation \(\circ\) on \(\Phi_3\) by the rule:
\[
f_1 \circ f_2 = (f_1/v)(f_2).
\]

Then \(\Phi_3\) becomes a quasigroup. Let \(F = F(X)\) be the free quasigroup over \(X = \{u\} \cup \{v\} \cup X'\), and \(F(X')\)–the free quasigroup over \(X'\). The identity map \(X' \to X'\) gives a homomorphism \(F(X') \to \Phi_3\), where \(\Phi_3\) is a quasigroup with respect to the operation \(\circ\). For every \(w \in F(X')\), the corresponding \(\bar{w} \in \Phi_3\) is an automata element. The identity \(w \equiv w'\) gives a derived automata identity \(\bar{w} \equiv \bar{w}'\), and likewise for arbitrary formulas. For example, the automata identity corresponding to the identity \((xy)z = x(yz)\) is the one
\[
((x/v)(u/y)/v)(u/z) \equiv (x/v)(u/((y/v)(u/z))),
\]
which specifies the variety of quasigroups isotopic to groups. If \(\Theta\) is the variety of groups which satisfy some set of identities \((w_\alpha \equiv w'_\alpha, \alpha \in I)\), then the set \(\bar{w}_\alpha \equiv \bar{w}'_\alpha, \alpha \in I\) determines the variety \(\Theta'\) of quasigroups isotopic to groups from \(\Theta\). The main result yields characteristic conditions for varieties, quasivarieties, pseudovarieties, universal classes, etc. of quasigroups closed under isotopies.

3. Algebraic logic in group representations. A representation is considered to be a pair \(\rho = (V, G)\), where \(V\) is a \(K\)-module, \(K\) is a commutative ring with unit, and \(G\) is a group acting on \(V\). Let \(\mu: G \to \text{Aut} V\) be the corresponding homomorphism. Varieties of representations are studied in [55].

Halmos algebras can be applied in the situation when \(\Theta\) is the variety of representations over a given ring \(K\). Let \(X\) be an infinite set of variable that run over \(V\), and \(Y\) be an infinite set of variables that run over the acting group \(G\). We then have \(F = F(X)\) and \(W = (XKF, F)\).

There are two types of elementary formulas:

1. \(x_1 \circ u_1 + \cdots + x_n \circ u_n \equiv 0, \quad u_i \in KF,\)
2. \(f \equiv 1, \quad f \in F.\)

We call the formulas of the first type the action formulas. An action formula is constructed from the elementary action formulas by means of Boolean operations and quantifiers with variables from \(X\): we do not quantify variables ranging over the group. In particular, we can speak of action identities, quasidentities, pseudoidentities and so on. In general, universal formulas have the form
\[
(w_1 \equiv w'_1) \lor \cdots \lor (w_n \equiv w'_n) \lor (v_1 \not\equiv v'_1) \lor \cdots \lor (v_m \not\equiv v'_m),
\]
where all \(w, v \in W\).
We have mentioned in [55] that every saturated variety of representations can be defined by action identities. This is true also of quasivarieties and, possibly, of pseudovarieties and universal classes of representations as well.

3.1. Proposition. Let \( u \) be an action formula, and let \( \rho = (V, G), \bar{\rho} = (V, \bar{G}) \) be a representation and the corresponding faithful representation, respectively. Then the formula \( u \) is valid in \( \rho \) if and only if it is valid in \( \bar{\rho} \).

The proof is direct, by using the homomorphisms
\[
f_\rho: U \to V_\rho \quad \text{and} \quad f_{\bar{\rho}}: U \to V_{\bar{\rho}}.
\]

3.2. Proposition. Let \( u \) be an action formula, \( \rho = (V, G) \) be a representation and \( \rho' = (V, H) \) be its subrepresentation, where \( H \) is a subgroup of \( G \). Then \( u \) is valid in \( \rho' \) whenever it is valid in \( \rho \).

The next propositions follows from the two preceding ones.

3.3. Proposition. Let \( T \) be a set of action formulas and \( T' \)—a class of representations defined by \( T \), \( T' = X \). Then \( X \) is saturated and right hereditary.

Recall that a class \( X \) is called saturated under the condition that the representation \( \rho = (V, G) \) belongs to \( X \) if and only if the representation \( \bar{\rho} = (V, \bar{G}) \) belongs to \( X \). This situation can also be described as follows. Representations \((V, G)\) and \((V', G')\) are said to be similar if the respective faithful representations are isomorphic. An abstract class \( X \) of representations is saturated if and only if \( X \) is invariant under passing to similar representations.

Right hereditary means here that \((V, G) \in X \) implies \((V, H) \in X \) if \( H \) is a subgroup of \( G \) and \((V, H) \) is a subrepresentation of \((V, G)\). Proposition 3.3 gives sufficient conditions for determining a class by action formulas. The following question is on necessary and sufficient conditions.

3.4. Problem. Is it true that an abstract class of representations can be determined by a set \( T \) of action formulas if and only if \( \mathfrak{X} \) is axiomatizable, saturated and right hereditary?

This question, as it seems to us, does not appear to be difficult. We must use the known conditions of axiomatizability, and pass to the class of all representations of the free group \( F \) of countable rank in the given \( \mathfrak{X} \).

It may also happen that the property of right hereditariness follows from the first two conditions.

A set of formulas \( T \subset U \) is said to be saturated if the class of representations \( \mathfrak{X} = T' \) is a saturated class.

3.5. Problem. Is it true that \( T \) is saturated if and only if it is equivalent to some set \( T_1 \) of action formulas?

This problem is related with the preceding one. Two sets, \( T \) and \( T_1 \) are equivalent when the classes \( T' \) and \( T'_1 \) coincide. Syntactically this means that both \( T \) and \( T_1 \) generate the same filter.

Independently we can speak of equivalence of sets of identities, quasi-identities, pseudoidentities, and universal formulas. For conditions of such equivalences in terms of the algebra \( U \), see the subsection 2.1 above. Hence, Problem 3.5 can be specified having in mind such sets of formulas. We can also consider characterizations of a single saturated formula.

Let us conclude with the following result. Given a class of representations \( \mathfrak{X} \), we denote by \( \tilde{\mathfrak{X}} \) the class of groups admitting a faithful representation in \( \mathfrak{X} \).

3.6. Proposition. If \( \mathfrak{X} \) is a universal and saturated class of representations over \( K \), then the class of groups \( \tilde{\mathfrak{X}} \) is a universal class.
This means that such $\mathfrak{X}$ admits a description by universal formulas in the group theory logic. In particular, the class $\mathfrak{X}$, for any pseudovariety of representations, is characterized by universal formulas in the group theory logic.

4. Databases. Constructing a database model presupposes that given are a data algebra $G = (G_i, i \in \Gamma)$, a set of relation symbols $\Phi$, and a set of states $F$. Every $f \in F$ is a function that realizes every $\varphi \in \Phi$ as a relation on $G$. For every $f \in F$, the triple $(G, \Phi, f)$ is a model. We must also take some scheme relatively to which databases are to be considered. For this purpose we take the scheme in which Halmos algebras were defined. In particular, $G \in \Theta$, and every $\varphi \in \Phi$ has a type $\tau = (i_1, ..., i_n)$, $i \in \Gamma$. In this scheme the triple $(G, \Phi, F)$ presents a database, but this is not yet a database model. We call the triple a passive database.

Database receives queries and produce replies to them. The queries are written as formulas, i.e., elements of algebra $L\Phi W$. The same query can be written out in different equivalent ways. This equivalence is the same which we got by the rule of Lindenbaum-Tarski. Hence, we must consider a query as a class of equivalent formulas, and then the algebra of queries is the Halmos algebra $U$. The algebra of replies is also a Halmos algebra. It is the algebra $V_G$ constructed for $G$ in the given scheme. To every $f \in F$ a homomorphism $f: U \to V_G$ corresponds. $f(u)$ is the value of a formula $u$ in $G$, and at the same time this is the reply to the query $u$ in the state $f$. We also write $f(u) = f \ast u$. If $u$ is a formula in $L\Phi W$, and $\bar{u}$ is the corresponding element in $U$, then $f \ast u = f \ast \bar{u}$. So we have a database $(F, U, V_G)$ with an operation $\ast$. The algebra $U$ here does not depend on $G$ and $F$, it depends only on $\Phi$ and the scheme. In this sense, $U$ is the universal query algebra. This algebra can be compressed, and $V_G$ can be reduced. First of all, take a subalgebra $R$ of $V_G$ generated by all $f \ast u$, $f \in F$, $u \in U$. This gives us $(F, U, R)$. The next step consists in specifying the filter $T$ in $U$ by the rule: $u \in T$ if $f \ast u = 1$ for all $f \in F$. Let $Q = U/T$; this way we obtain a reduced database $(F, Q, R)$. If here $u \in U$, and $q = \bar{u}$ is the corresponding element in $Q$, then $f \ast q = f \ast \bar{u} = f \ast u$. The database $(F, Q, R)$ is constructed in the given scheme from the passive database $(G, \Phi, F)$. We call $(F, Q, R)$ an active database, or an algebraic model of a database.

Using the model $(F, Q, R)$, in which Halmos algebras play an essential role, we can solve various database problems. However, $HA_{\Phi}$ is very hard to be used for computer applications. That is why we must return, in our final conclusions, to passive databases.

Let us denote by $F_G$ the system of all possible states of the collection $\Phi$ in the algebra $G$. Then we arrive at the universal database $$(F_G, U, V_G).$$ If $\delta: G \to G'$ is a surjective homomorphism, then it produces an injective database homomorphism $$\delta_*: (F_G, U, V_G) \to (F_{G'}, U, V_{G'}).$$ For all $f \in F_{G'}$ and $u \in U$, $$(f \ast u)^\delta_* = f^{\delta_*} \ast u.$$ See §51 §53.

All this will be in use in §5.

§3. Algebraic varieties and varieties of algebras

1. Basic concepts. The present and next section relate to the level of equational logic, and they are not immediately connected with algebraic logic. In the following we, however, shall move to the universal logic level, and constructions related to algebraic logic, and even to databases, find essential applications there.
At present, we are interested in equations and identities over arbitrary algebraic structures. Here, algebraic varieties correlated with arbitrary varieties of algebras are considered.

We first remind some matters well-known in algebraic geometry. Let $P$ be a field and $K$ its extension. We consider the ring of polynomials $R = P[x_1, \ldots, x_n]$, and take the affine point space $K(n)$. If $T$ is a collection of polynomials from $R$, then it is attached the algebraic variety $T'$, which is now treated as an affine space. This shows that $T'$ is always an ideal of $R$.

Now we can take up the general viewpoint. We proceed from the variety $\Theta$ of all commutative and associative algebras with unit over the field $P$. $R$ is the free algebra of this variety over the set of variables $X = \{x_1, \ldots, x_n\}$; the field $K$ can also be considered as an algebra from $\Theta$. Every point $(a_1, \ldots, a_n) = a \in K(n)$ specifies a mapping $\mu: X \to K$, $\mu(x_i) = a_i$, $i = 1, 2, \ldots, n$. The mapping determines an algebra homomorphism $\mu: R \to K$. Therefore, we may identify the space $K(n)$ with the homomorphism set $\text{Hom}(R, K)$. Here, the "point" $\mu$ is a root of the polynomial $\varphi = \varphi(x_1, \ldots, x_n)$ if $\varphi \in \text{Ker}\mu$. An algebraic variety is now treated as a subset of $\text{Hom}(R, K)$, and the set $\text{Hom}(R, K)$ can be thought of as an affine space.

Let us rewrite the Galois correspondence considered above in these new terms. It is easily seen that

$$T' = \{\mu, T \subseteq \text{Ker}\mu\},$$

$$A' = \bigcap_{\mu \in A} \text{Ker}\mu.$$

This shows that $A'$ is always an ideal.

Now we can take up the general viewpoint.

Assume that $\Theta$ is any variety of algebras of the signature $\Omega$. The algebras may be many-sorted; then $\Gamma$ stands for the set of sorts. Let $X$ be a set of variables, and let $W = W(X)$ be the algebra from $\Theta$ free over $X$. Take an algebra $G \in \Theta$ and consider the set $\text{Hom}(W, G)$, which is now treated as an affine space. We are going to define a Galois correspondence between binary relations $T$ on $W$ and subsets of $\text{Hom}(W, G)$. $T$ is the set of pairs $(w, w')$ with $w$ and $w'$ of the same sort, and $wT w'$, as usually, means that $(w, w') \in T$. The equation $w = w'$ is related with every pair $(w, w')$, and one can also consider the formula $w \equiv w'$ (as an element of the Halmos algebra $U$).
Every $\mu = (\mu_i, \ i \in T) \colon W = (W_i, \ i \in \Gamma) \to G = (G_i, \ i \in \Gamma)$ has the kernel $\text{Ker}_\mu = (\text{Ker}_{\mu_i}, \ i \in \Gamma)$, where $\text{Ker}_{\mu_i}$ is the kernel equivalence of the mapping $\mu_i$, i.e. the set of pairs $(w, w')$, $w, w' \in W_i$, with $w^{\mu_i} = w'^{\mu_i}$ or, what is the same, $w^{\mu} = w'^{\mu}$. We also consider the kernel $\text{Ker}_{\mu}$ as the union of all $\text{Ker}_{\mu_i}$, $i \in \Gamma$. On the other hand, $\text{Ker}_{\mu}$ is a congruence of $W$.

Now let $A$ be any subset of $\text{Hom}(W, G)$. We set

$$A' = T = \bigcap_{\mu \in A} \text{Ker}_\mu.$$  

If $T$ is a binary relation on $W$, then $T' = A$ is defined by the rule

$$A = \{\mu, \ T \subset \text{Ker}_\mu\}.$$  

$A = T'$ is the "algebraic variety" in $\text{Hom}(W, G)$ specified by the collection $T$, while $T = A'$ is always a congruence of $W$. So we have got a Galois correspondence. Every set $A$ can be closed up to an algebraic variety $A''$, and every $T$ – up to an congruence $T''$. Where $T$ is a congruence, links between $T$ and $T''$ are revealed by an appropriate "Hilbert’s Nullstellensatz".

Clearly, the intersection of a collection of algebraic varieties is a variety again, and if $A$ is a subset of $\text{Hom}(W, G)$, then the closure $A''$ is the least variety including $A$. In the present general situation, however, the union of two varieties can fail to be a variety, and the closure of a sum of sets generally differs from the sum of the closures. Later on, we shall generalize the notion of an algebraic variety and improve this shortcoming.

As noticed above, a pair $(w, w')$ can be regarded as an equation $w = w'$. Then the statement $(w, w') \in \text{Ker}_\mu$ means that the point $\mu$ satisfies the equation $T = A'$ is a congruence of $W$. So we get a Galois correspondence. Every set $A$ can be closed up to an algebraic variety $A''$, and every $T$ – up to a congruence $T''$. Where $T$ is a congruence, links between $T$ and $T''$ are revealed by an appropriate "Hilbert’s Nullstellensatz".

A binary relation $T$ specifies the variety of algebras $G \in \Theta$ what the identity $w \equiv w'$ is valid in for every $(w, w') \in T$. This is a variety in $\Theta$. The same $T$, for every particular $G \in \Theta$, specifies an algebraic variety thought of as a subset of $\text{Hom}(W, G)$. This is the connection between varieties of algebras and algebraic varieties.

Let $T(G)$ stands for the verbal congruence of all identities of any algebra $G \in \Theta$ in $W$. For an arbitrary collection $T$, the equality $T' = \text{Hom}(W, G)$ means that $T \subset T(G)$.

Let us discuss another specific situation.

Recall that the unit congruence is one in which any two elements of the same sort are identified. The zero congruence presupposes that elements are equivalent only if they are equal. The zero congruence is included in any $T$, and any $T$ is included in the unit congruence. Let $T_0$ stands for the zero congruence and $T_1$– for the unit congruence on $W$. Then, obviously, $T_0 = \text{Hom}(W, G)$. Moreover, $\mu \in T_1$ if $T_1 \subset \text{Ker}_\mu$, i.e. if $\text{Ker}_\mu$ is the unit congruence. Such $\mu$ need not exists, i.e. $T_1$ may be the empty set. Clearly, $T'$ can be empty for other $T$ as well.

Now we single out the case when $G$ has a one-element subalgebra $H$. If $G$ is many-sorted, then the subalgebra is of the form $H = (H_i, \ i \in \Gamma)$ with all $H_i$ singletones. $H$ determines a homomorphism $\mu_0 \colon W \to G$, and $\text{Ker}_{\mu_0} = T_1$. Here, $T_1 = \{\mu_0\}$. If there is no such $H$, then $T_1$ is empty. If a one-element $H$ exists, then, for every $T$, $T \subset \text{Ker}_{\mu_0}$, and $T'$ is non-empty, $\mu_0 \in T'$.

We further observe that if $A$ is the empty algebraic variety, then $A'$ is defined to equal $T_1$, and if $A = \text{Hom}(W, G)$, then $A' = T(G)$.

We consider separately the case when $G$ has a one-element subalgebra with the homomorphism $\mu_0$, and $A = \{\mu_0\}$. It is obvious that then $A' = T_1$. 

We see that the idea of an algebraic variety, originally linked with algebraic geometry, can be carried over to arbitrary varieties of algebras.

Let us mention the following obvious relationships. Suppose that $\alpha$ runs over some set $I$. Then

1. $(\bigcup A_\alpha)' = \bigcap A_\alpha'$.  
2. $(\bigcup T_\alpha)' = \bigcap T_\alpha'$.  
3. $\bigcup T_\alpha' \subseteq (\bigcap T_\alpha)'$.  
4. $\bigcup A_\alpha' \subseteq (\bigcap A_\alpha)'$.

We also note, finally, that all constructions here are carried out with respect to a certain set of variables $X$. This set may be either finite or infinite. In the next section, of concern to us will be, among other things, the question what happens under changes of $X$.

2. Hilbert’s Nullstellensatz. First of all, we comment on the structure of the ”general solution” of an equation system $T$. We look for solutions in some algebra $H \in \Theta$, and assume that a surjective homomorphism $\mu_0 : W \to G$ with kernel $T$ is given. In particular, it can be the natural homomorphism $\mu_0 : W \to W/T$.

We consider the set $\text{Hom}(G, H)$, and let $\mu_0\text{Hom}(G, H)$ be the set of all products $\mu_0\nu$, $\nu \in \text{Hom}(G, H)$. Surjectivity of $\mu_0$ implies that $\mu_0\nu_1 = \mu_0\nu_2$ if and only if $\nu_1 = \nu_2$. Clearly, $\mu_0\text{Hom}(G, H)$ is a subset of $\text{Hom}(W, H)$.

2.1. Proposition. For any $T$, $T_H' = T' = \mu_0\text{Hom}(G, H)$.

Proof. We use the commutative diagram

$$
\begin{array}{ccc}
W & \xrightarrow{\mu} & H \\
\downarrow{\mu_0} & & \downarrow{\nu} \\
G & & 
\end{array}
$$

with $\nu$ uniquely determined by $\mu \in T' = A$. By the condition, $T = \text{Ker}\mu_0 \subseteq \text{Ker}\mu$, and this implies that such $\nu$ ever exists. Therefore $T' \subseteq \mu_0\text{Hom}(G, H)$.

We now take any $\mu = \mu_0\nu$ and assume that $wTw'$. Then $w^{\mu_0} = w'^{\mu_0}$ and $w^\mu = w'^\mu$, $(w, w') \in \text{Ker}\mu$. Hence, $T \subseteq \text{Ker}\mu$, $\mu \in T' = A$. This gives the converse inclusion $\mu_0\text{Hom}(G, H) \subseteq T'$.

Hence, the general solution of the equation system $T$, where $T$ is a congruence, can be presented as follows:

$$A = T' = \mu_0\text{Hom}(W/T, H).$$

Let, furthermore, $G$ and $H$ be two algebras from $\Theta$. We consider $\text{Hom}(G, H)$ and set

$$(H - \text{Ker})(G) = \bigcap_{\nu} \text{Ker}\nu,$$

where the intersection is over all $\nu : G \to H$. So $(H - \text{Ker})(G)$ is a congruence on $G$ depending on $H$.

Assume again that we are given a surjective homomorphism $\mu_0 : W \to G$ with kernel $T$. Then the following theorem holds.
2.2. Theorem. Let $\mu_0^{-1}$ means "the inverse image under $\mu_0". Then

$$T''_H = \mu_0^{-1}(H - \text{Ker})(G).$$

In particular, the next theorem can be regarded as general Hilbert theorem on zeros.

2.3. Theorem. For every congruence $T$ on $W$,

$$T''_H = \mu_0^{-1}(H - \text{Ker})(W/T).$$

**Proof of Theorem 2.2.** Let $\tau = (H - \text{Ker})(G)$. We consider the composition homomorphism

$$W \xrightarrow{\mu_0} G \xrightarrow{\mu} G/\tau,$$

where $\mu_1$ is the natural homomorphism, and set $\overline{T} = \text{Ker}\mu_0\mu_1$. Then $w\overline{T}w' = w^\mu\text{Ker}\mu_1w'^{\mu_0}$, i.e. $w^{\mu_0}\tau w'^{\mu_0}$. So $\overline{T} = \mu_0^{-1}(\tau)$. We shall verify that $\overline{T} = T''_H$.

Assume that $w\overline{T}w'$. By the definition of the congruence $\tau$, $(w^{\mu_0}, w'^{\mu_0}) \in \text{Ker}\nu$ and $w^{\mu_0}\nu = w'^{\mu_0}\nu$ for every $\nu: G \to H$. By Proposition 2.1, $\mu_0\nu$ is an element of $T''_H = A$, and $(w, w') \in \text{Ker}\mu_0\nu$. Therefore, $(w, w') \in T''_H$, and we have make sure that $\overline{T} \subset T''_H$.

Now assume that $wT''_Hw'$. Then $w^{\mu_0}\nu = w'^{\mu_0}\nu$ for every $\nu: G \to H$, and $(w^{\mu_0}, w'^{\mu_0}) \in \bigcap_{\nu} \text{Ker}\nu = \tau$. This implies that $w^{\mu_0}\mu_1 = w'^{\mu_0}\mu_1$ and $w\overline{T}w'$. So $T''_H \subset \overline{T}$.

Thus, $\overline{T} = T''_H$, and Theorem 2.2, as well as Theorem 2.3, are proved.

We now shall derive the classical Hilbert theorem from the general theorem 2.3. Two general facts of commutative algebra will be used; they actually are related to the Hilbert theorem.

The first one says that if $G$ is a finitely generated associative and commutative algebra, then its Jacobson radical $\text{Rad}G$ is, at the same time, the null-radical that coincides with the set of nilpotent elements of $G$.

The other fact that we need consists in the following: if $T$ is a proper ideal of the ring $R = P[x_1, \ldots, x_n]$, and if $K$ is an algebraically closed extension of the field $P$, then there is a homomorphism $\mu: R \to K$ for which $T \subset \text{Ker}\mu$. A property like this could serve as a general definition of algebraic closeness of arbitrary universal algebras.

We now check that, under these conditions, the following equality holds:

$$(K - \text{Ker})(R/T) = \text{Rad}(R/T).$$

The radical on the right is the intersection of the maximal ideals. Suppose that $T_0/T$ is a maximal ideal of $R/T$. Then $T_0$ is a maximal ideal of $R$, and there is a homomorphism $\mu: R \to K$ for which $T_0 \subset \text{Ker}\mu$. It follows from the maximality condition that $T_0 = \text{Ker}\mu$. Since $T \subset \text{Ker}\mu$, the homomorphism $\mu$ induces another homomorphism $\nu: R/T \to K$, and here $T_0/T = \text{Ker}\nu$. Therefore, every maximal ideal of $R/T$ is realized as the kernel of some $\nu$. This means that the inclusion

$$\text{Rad}(R/T) \supset (K - \text{Ker})(R/T)$$

holds. Every element of $\text{Rad}(R/T)$ is nilpotent, and every nilpotent element of $R/T$ belongs to the kernel of any $\nu: R/T \to K$. Hence the converse conclusion.

The Hilbert theorem now is an obvious consequence of the equality just proved and Theorem 2.3.

Other applications of Theorem 2.3 will be discussed in what follows. In any particular case all reduces to calculating the corresponding $(H - \text{Ker})(W/T)$.  

We note, furthermore, that triviality of the kernel \((H - \text{Ker})(G)\) means that the algebra \(G\) has a full system of representations in \(H\). This, in its turn, means that the congruence \(T\) in \(W\) is closed, \(T = T''\), if and only if the algebra \(W/T\) has a full system of representations in \(H\).

Let us give one more variant of Hilbert theorem.

2.4. Theorem. For any congruence \(T\) in a free algebra \(W\) and any algebra \(G \in \Theta\) the corresponding closure \(T_G''\) is the intersection of all congruences \(\tau\) in \(W\), containing \(T\) and such that there is an injection \(W/\tau \to G\).

In particular, if \(\Theta\) is a variety of all groups and \(G = F\) is a free group, then the group \(W/T''\) is approximated by free groups.

3. Verbal varieties. We shall consider the particular case when \(T\) is a fully invariant, or verbal, congruence on \(W = W(X)\). For every algebra \(G \in \Theta\), we call the respective algebraic variety \(T'\) a verbal variety. We are interested in \(T''\) in this case.

The congruence \(T\) determines a variety of algebras \(\Theta_T\), which is a subvariety of \(\Theta\). Given an algebra \(G\), we consider all its subalgebras \(H\) in \(\Theta_T\). For these, equations from \(T\) become identities, \(T_H' = \text{Hom}(W,H)\).

We denote the system of all the subalgebras by \(\Theta_T(G)\). This object is, to a certain extent, the dual of the verbal congruence on \(G\) relatively to \(\Theta_T\). We also denote by \(\bar{T}(G)\) the congruence composed of the identities of the class \(\Theta_T(G)\) in the free algebra \(W\); we shall call it the congruence of identities (of \(\Theta_T(G)\)). For any \(T\), the congruence \(\bar{T}(G)\) is verbal.

3.1. Theorem. If \(T\) is a verbal congruence, then \(T_G'' = \bar{T}(G)\). Specifically, \(T_G''\) is also a verbal congruence.

Proof. Let us compute the kernel \((G - \text{Ker})(W/T)\).

First of all, we observe that the algebra \(W/T\) is free in \(\Theta_T\) over \(X\).

Furthermore, we make the following general remark. Let \(\Theta\) be a variety of \(\Omega\)-algebras, \(W(X)\) be the free algebra in \(\Theta\) over \(X\), and \(G\) be an \(\Omega\)-algebra not necessary from \(\Theta\). We shall examine the kernel \((G - \text{Ker})(W(X))\).

To this end, we select in \(G\) all the subalgebras \(H\) belonging to \(\Theta\), and denote the system of these subalgebras by \(\Theta(G)\). Let \(T(G)\) be the congruence of all identities of \(\Theta(G)\) in \(W(X)\). Then \((G - \text{Ker})(W(X)) = T(G)\).

Let us demonstrate this. We denote by \(M\) the system of homomorphisms \(\nu: W(X) \to H\), where \(H\) is a subalgebra of \(G\) from \(\Theta(G)\). The intersection of all kernels \(\text{Ker} \nu\) over all \(\nu \in M\) is \(T(G)\). Every \(\nu \in M\) is at the same time a homomorphism \(\nu: W(X) \to G\). On the other hand, \(\nu: W(X) \to G\), where \(\text{im} \nu = H\) is a subalgebra in \(\Theta(G)\), and we also have \(\nu: W(X) \to H\). Therefore, we may identify the sets \(M\) and \(\text{Hom}(W(X),G)\). This leads to the needed equality.

We apply it in the situation with \(\Theta_T\) taken for \(\Theta\). Then the kernel \((G - \text{Ker})(W/T)\) is the congruence of identities of the system \(\Theta_T(G)\) in \(W/T\). But in this case the full inverse image \(T''_G = \mu_0^{-1}(G - \text{Ker})(W/T)\) is the congruence of the identities of \(\Theta_T(G)\) in \(W(X)\). It follows that \(T''_G = \bar{T}(G)\).

There is an example.

Suppose that the initial variety \(\Theta\) is the variety of groups, the set \(X\) is infinite, \(F(X)\) is the corresponding free group, and \(F^1(X) = T\) is its commutant. Take a group \(G\) and consider two cases: \(G\) is of
finite exponent and the exponent of \( G \) is infinite. The commutant \( T \) determines the variety of commutative groups, and the same variety is generated, in the second case, by the commutative subgroups of \( G \). For this reason the commutant is closed in the second case, \( T = T'' \). In the first case the commutative subgroups of \( G \) generate the variety of commutative groups of exponent \( n \). Consequently, \( T'' \) is a verbal congruence generated by the commutant and the element \( x^n \).

We make one more useful remark concerning verbal varieties.

3.2. Proposition. Suppose that \( T \) is a verbal congruence on \( W(X) \) and that \( G \) is an algebra from \( \Theta \). Then

\[
A = T' = \bigcup_H \text{Hom}(W, H),
\]

where the union is taken over all \( H \in \Theta_T(G) \).

**Proof.** Let \( \mu \in A \). Then \( T \subset \text{Ker}\mu \). As \( W/T \in \Theta_T \), also \( W/\text{Ker}\mu \in \Theta_T \). But then \( H = \text{im}\mu \in \Theta_T(G) \), \( \mu \in \text{Hom}(W, H) \).

Notice that here, and below, if \( H \) is a subalgebra of \( G \), then \( \text{Hom}(W, H) \) is treated as \( \text{Hom}(W, G) \). Moreover, \( T'H = T' \cap \text{Hom}(W, H) \) for every \( T \).

Now assume that \( H \in \Theta_T(G) \). This means that \( T'H = \text{Hom}(W, H) \), and then \( \text{Hom}(W, H) \) is included into \( T' = A \).

Theorem 3.1 also is an easy consequence of the above remark.

4. Geometric equivalence of algebras.

4.1. Definition. Algebras \( G_1 \) and \( G_2 \) from \( \Theta \) are said to be geometrically equivalent if

\[
T''_{G_1} = T''_{G_2}
\]

for every \( T \) from \( W(X) \).

It is easily understood that this condition is equivalent to the following one: any congruence \( T \) on \( W(X) \) is \( G_1 \)-closed if and only if it is \( G_2 \)-closed: \( T''_{G_1} = T \) iff \( T''_{G_2} = T \).

Indeed, the equivalence of \( G_1 \) and \( G_2 \) implies the latter condition. Assume, on the other hand, that this condition is fulfilled for every \( T \). Then \( T \subset T''_{G_1} \) and, furthermore, \( T''_{G_2} \subset T''_{G_1} \). The converse inclusion is proved in the same way.

For every \( G \in \Theta \), let \( M_G \) stands for the system of all algebraic varieties in \( \text{Hom}(W, G) \). By \( \text{Cl}_G(W) \) we denote the system of all \( G \)-closed congruences on \( W = W(X) \). There is a natural bijection between the sets \( M_G \) and \( \text{Cl}_G(W) \).

Equivalence of the algebras \( G_1 \) and \( G_2 \) means that the sets \( \text{Cl}_{G_1}(W) \) and \( \text{Cl}_{G_2}(W) \) coincide. Now it is clear that the equivalence determines a canonic bijection \( M_{G_1} \rightarrow M_{G_2} \). We once more stress that the definition of equivalence is related to specific \( X \). Therefore, the question is, in fact, on \( X \)-equivalence.

Clearly, if \( G_1 \) and \( G_2 \) are isomorphic, then they are equivalent relatively to every \( X \).

The main problem is to learn to recognize equivalence of algebras by their properties. For example, can two groups be equivalent if one of them is commutative while the other is not? In the classical geometry two algebraically closed fields \( K_1 \) and \( K_2 \), if they both are extensions of the same \( P \), are equivalent. As we have already mentioned, the corresponding problem for fields that are not algebraically closed, is still open. We can only note, for example, that every \( K \) is equivalent to every of its ultarpowers (cf. §5).
It follows from Theorem 2.3 that algebras $G_1$ and $G_2$ are equivalent if and only if
\[(G_1 - \text{Ker})(W/T) = (G_2 - \text{Ker})(W/T)\]
for every $T$.

Let us take, for example, the variety of vector spaces over a given field $P$ for $\Theta$. If $G$ and $H$ are vector spaces in $\Theta$, then $(H - \text{Ker})(G) = 0$, whence, in this case, all $T$ are closed for every $G$, and any two spaces are equivalent.

**4.2. Problem.** Assume that $K$ is a commutative ring with unit and that $\Theta$ is the variety of $K$-modules. Under what conditions are two modules $G_1$ and $G_2$ equivalent?

Of course, the problem has to be considered for several particular $K$, e.g. $K = \mathbb{Z}$. What two commutative groups are equivalent? The general problem also depends on the choice of the set of variables $X$.

Let again the variety $\Theta$ be arbitrary, and let $X$ be fixed.

**4.3. Theorem.** If algebras $G_1$ and $G_2$ are $X$-equivalent, then they have the same identities on $W(X)$.

*Proof.* We shall apply Theorem 3.1. Let $T = T(G_1)$ be the congruence of all identities of $G_1$ on $W = W(X)$. Then $T'_{G_1} = \text{Hom}(W,G_1)$, and $T''_{G_1} = T(G_1) = T$. Now take $T''_{G_2} = \tilde{T}(G_2)$, and let $\Theta_T(G_2)$ be the system of all subalgebras $H$ of $G$ belonging to the variety $\Theta_T$. $T(G_2)$ is the congruence of all identities of $\Theta_T(G_2)$. If $G_1$ and $G_2$ are equivalent, then $T = \tilde{T}(G_2)$. We have $\tilde{T}(G_2) \supset T(G_1)$, whence $T(G_1) \supset T(G_2)$. Likewise, $T(G_1) \subset T(G_2)$ and, consequently, $T(G_1) = T(G_2)$. \(\square\)

For $X$ infinite, the equality $T(G_1) = T(G_2)$ implies that $\text{Var}(G_1) = \text{Var}(G_2)$; this means that the algebras $T(G_1)$ and $T(G_2)$ have the same equational theory and the same equational logic.

We shall mention some consequences for $X$ infinite.

First of all, we observe that if $G_1$ and $G_2$ are finite simple groups, then $\text{Var}(G_1) = \text{Var}(G_2)$ only if $G_1$ and $G_2$ are isomorphic. We therefore conclude that two finite simple groups are equivalent if and only if they are isomorphic.

Moreover, we now can say that a commutative group is equivalent to no non-commutative group.

It is naturally, in the case $X$ is infinite, to wonder whether equivalence of algebras $G_1$ and $G_2$ implies that they have the same universal theory. The example with vector spaces demonstrates that it is in general not so. It is not so also under arbitrary $\Theta$. This follows form the proposition below.

We take any $\Theta$ and also fix arbitrary $X$.

**4.4. Proposition.** For every algebra $G \in \Theta$ and every set $I$, the algebras $G$ and $G^I$ are equivalent.

*Proof.* We confine ourselves to one-sorted algebras.

Let us take any algebra $A \in \Theta$, and show that the equality
\[(G - \text{Ker})(A) = (G^I - \text{Ker})(A)\]
ever holds. We denote its left-hand part by $\tau_1$ and the right-hand one – by $\tau_2$. Suppose that $a\tau_1 a'$ for $a$ and $a'$ from $A$. This means that, for every $\nu: A \to G$, $a'' = a'''$. We need to verify that $a\tau_2 a'$. To this end, take arbitrary $\mu: A \to G^I$. We then have to prove that $a'' = a'''$, i.e. that $a''(\alpha) = a''(\alpha)$ for every $\alpha \in I$. 

Let us consider projections $\pi_\alpha: G^I \to G$. Then $\mu \pi_\alpha = \nu_\alpha: A \to G$. Moreover, $a^\mu(\alpha) = a^{\mu \pi_\alpha}$. What remains to show is that $a^\mu \pi_\alpha = a^{\mu \pi_\alpha}$ or, equivalently, that $a^\nu = a^{\nu_\alpha}$ for every $\alpha \in I$. Clearly, this is so when $a \tau_1 a'$. Therefore, $a \tau_1 a'$ implies $a \tau_2 a'$.

Conversely, suppose that $a \tau_2 a'$. Given $\nu: A \to G$, we construct $\mu: A \to G^I$ by setting $a^\mu(\alpha) = a^\nu$ for every $\alpha$. For all $a \in A$, the elements $a^\mu$ are constants, and $a^\mu = a^{\mu}$. If $\alpha \in I$, then $a^\mu(\alpha) = a^\nu = a^{\mu}(\alpha) = a^\nu$. So $a \tau_1 a'$, and $a \tau_2 a'$ implies $a \tau_1 a'$. We have arrived at $\tau_1 = \tau_2$. 

Now, we take a congruence $T$ on $W$ and let $A$ be the algebra $W/T$. Then 

$$(G - \ker)(W/T) = (G^I - \ker)(W/T).$$

This means that the algebras $G$ and $G^I$ are equivalent.

Generally, equivalent algebras $G$ and $G^I$ may have different pseudoidentities. Much more, distinct are the universal theories of $G$ and $G^I$.

The concept of algebra equivalence can also be defined on the universal logic level. As we shall see, in this case the equivalence of algebras implies that they have the same universal theories.

In conclusion of the subsection we note that if two algebras $G_1$ and $G_2$ are $X$-equivalent and if neither of them has a one-element subalgebra, then, for any proper congruence $T$ on $W(X)$, the varieties $A = T'_G$, and $B = T'_G$, are either both empty or both nonempty.

Indeed, assume that $G_1$ and $G_2$ are equivalent and that $A$ is empty. Then $T'_G = A'$ is the unit congruence $T_1$. But then $T_1 = T'_G = B'$ and $B = B'' = T'_G$ are empty varieties.

The converse does not hold: if for any proper congruence $T$ on $W(X)$ the varieties $A = T'_G$, and $B = T'_G$, are both empty, or both nonempty, then this does not mean that $G_1$ and $G_2$ are equivalent.

5. Generalized equations. In this subsection, an algebra $G$ from the variety $\Theta$ is assumed to be fixed, and constants occurring in the equations considered are supposed to belong $G$.

Such equations are connected with the passage to another variety $\Theta' = \Theta^G$ depending on $G$. We begin with defining the category $\Theta'$. Its objects are pairs $(H, h)$ where $H$ is an algebra from $\Theta$ and $h: G \to H$ is a homomorphism in $\Theta$. The representation–homomorphism $h$ makes elements of $G$ constants in $H$. The pairs $(H, h)$ are termed $G$-algebras, or algebras over $G$; compare with associative algebras over a field $P$.

If $(H_1, h_1)$ and $(H_2, h_2)$ are two $G$-algebras, then a homomorphism $\alpha: H_1 \to H_2$ is a $G$-algebra (homo)morphism in case the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{h_1} & H_1 \\
\downarrow h_2 & & \downarrow \alpha \\
H_2 & & 
\end{array}
\]

commutes.

The category of $G$-algebras can be presented as a variety if one specifies the algebra $G$ by generators and relations. Generators are nullary operations added to the collection of primitive operations $\Omega$, while defining relations are added to the collection of identities specifying $\Theta$; cf. §3 and §2 here. This way we obtain a new variety which, generally, depends on the particular presentation of $G$ by generators and
relations. However, all these varieties are equivalent as categories, and all the categories are, in turn, equivalent to the category of $G$-algebras. In what follows, the category $\Theta'$ is regarded as a variety.

If $X$ is a set, possibly, many-sorted, and $W(X) = W$ is the free algebra in $\Theta$ over $X$, then the free over the same $X$ algebra $W'$ in $\Theta'$ can be presented as the free (in $\Theta$) product $G \ast W$ with the homomorphism $h_0: G \to G \ast W$ determined by the corresponding projection.

Where $(H, h)$ is a $G$-algebra, every mapping $\mu: X \to H$ induces a homomorphism $\mu: W \to H$. Together with $h: G \to H$, it gives $\mu: G \ast W \to H$. This latter $\mu$ is a $G$-algebra homomorphism; commutativity of the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{h_0} & G \ast W \\
\downarrow{h} & & \downarrow{\mu} \\
H & & 
\end{array}
\]

follows from the definition of a free product.

We still note that the algebra $G$ can be regarded as a $G$-algebra if we proceed from the identity homomorphism $G \to G$.

Now a generalized equation has the form $w = w'$, where $w$ and $w'$ are elements of $W'$ of the same sort. The coefficients of the equation are also from $G$. Such equations are resolved in $G$-algebras $H$, in particular, in the $G$-algebra $G$. Clearly, all the constructions considered above are applicable to these equations. The initial variety here is $\Theta'$.

Of special interest is the situation when the homomorphism $h: G \to H$ in a $G$-algebra $H$ is injective. We shall prove, in this context, the following well-known result.

Let us agree to say that a $G$-algebra $(H, h)$ is faithful if $h: G \to H$ is an injection.

5.1. Proposition. Suppose that $T$ is a congruence on the $G$-algebra $G \ast W$, and let $\alpha: G \ast W \to G \ast W/T$ be the natural homomorphism. The system of equations $T$ has a solution in a faithful $G$-algebra $H$ if and only if the homomorphism $h_0 \alpha: G \to (G \ast W)/T$ is an injection.

Proof. Assume that $h = h_0 \alpha$ is an injection. Then we take $H = (G \ast W)/T$ and consider the $G$-algebra $(H, h)$, which is faithful. The point $\mu = \alpha: G \ast W \to H$ is a solution of the system of equations $T$.

Now suppose that $\mu: G \ast W \to H$ is a solution of the system $T$ in a faithful $G$-algebra $H$ with an injection $h: G \to H$. By the definition of a homomorphism in $\Theta'$, we have a commutative diagram

\[
\begin{array}{ccc}
G \ast W & \xrightarrow{\mu} & H \\
\downarrow{h_0} & & \downarrow{h} \\
G & & 
\end{array}
\]

Since $\mu$ is a solution of $T$, the inclusion $T \subset \text{Ker} \mu$ holds, and this gives one more commutative diagram
Comparing the diagrams, we observe that $h_0 \alpha / h = h$. Therefore, since $h$ is an injection, so is $h_0 \alpha$.

As to the variety $\Theta'$, we note that if $G$ is not a one-element algebra, then no faithful $G$-algebra has a one-element subalgebra. For this reason, some system $T$ may fail to have a solution in any such an algebra at all, and the corresponding $T'$ may be empty. If $\Theta$ is the variety of all groups, then here all $T'$ are nonempty, but the situation changes for $G$-groups.

Along with generalized equations, generalized identities can be considered. The literature on generalized equations and identities is quite extensive [61, 46, 47, 36, 37, 57].

We shall make some remarks on the closure of a point. We shall make it apparent that if equations admit solutions in a $G$-algebra $G$, then every point $\mu: G * W \to G$ coincides with its closure.

We proceed from the projections $h_0: G \to G * W$ and $h_1: W \to G * W$. For every point $\nu: G * W \to G$, we also have $h_0 \nu = \varepsilon: G \to G$.

For each $x \in X$, we take $x^{h_1}$ and $x^{h_1 \mu h_0}$. These elements both belong to $G * W$ so that the equation $x^{h_1} = x^{h_1 \mu h_0}$ makes sense. We denote the system of such equations for all $x \in X$ by $T$. The equality $x^{h_1 \mu h_0} = x^{h_1 \mu}$ holds for every $\nu$, so if $\nu = \mu$, then $x^{h_1 \mu} = x^{h_1 \mu h_0 \mu}$. Therefore, $T \subseteq \text{Ker}\mu$. If $\text{Ker}\mu \subseteq \text{Ker}\nu$, then it follows that $x^{h_1 \nu} = x^{h_1 \mu} = x^{h_1 \nu h_0}$ holds. This means that $\mu$ and $\nu$ agree on $W$. Moreover, $g^{h_0 \mu} = g = g^{h_0 \nu}$ for every $g \in G$. Consequently, $\mu$ and $\nu$ agree on $G$. But then $\mu = \nu$. Thus, the closure of the point $\mu$ only consists on $\mu$ itself.

We return to the situation of $G$-algebras over given $\Theta$. Let $T$ be a congruence in a free algebra $G * W$. The solution is considered in $G$-algebra $G$. 

5.2. Proposition. A congruence $T''_G$ is the intersection of all congruences $\tau$ in $G * W$, containing $T$, such that $G * W / \tau$ and $G$ are isomorphic $G$-algebras.

The Proposition follows from 2.4 and from that $G$-algebra $G$ has no proper subalgebras.

6. Algebra and topology in connection with varieties. We assume in this subsection that $\Theta$ and $X$ are fixed, take $W = W(X)$, and consider algebraic varieties in the "affine space" $\text{Hom}(W, G)$, $G \in \Theta$.

Suppose that $T$ is a binary relation on $W(X)$ and $A = T'$ is the corresponding algebraic variety. We couple with $A$ the algebra $W/A' = W / T''$ in $\Theta$. By the definition, $A' = T''$ is the intersection of all $\text{Ker}\mu$, $\mu \in A$. So $W/A'$ is approximated by algebras $W / \text{Ker}\mu$ or, what is the same, by algebras $\text{im}\mu$, $\mu \in A$, which are subalgebras of $G$.

We now shall look at the algebra $W/A'$ from another point of view.

We shall deal with mappings (functions) $\alpha: A \to G$, where $A$ is an algebraic variety in $\text{Hom}(W, G)$, $G \in \Theta$. Such a function is said to be regular if there is $w \in W$ satisfying $\alpha(\mu) = w^{\mu}$ for every point $\mu$. If many-sorted algebras are considered, then the element $w$ has a definite sort, and the question is on a regular function of this sort.

The function $\alpha$ can also be given by another element $w'$ of the same sort. Then $w^{\mu} = w'^{\mu}$ for every $\mu \in A$. This means that $(w, w') \in A'$. 
For each $\mu \in A$, $A' \subset \text{Ker}\mu$; consequently, we also have a homomorphism $\mu : W/A' \to G$. Given $w \in W$, we denote by $\bar{w}$ the corresponding element of $W/A'$. Then $\alpha(\mu) = \bar{w}\mu$. Therefore, elements of $W/A'$ can be regarded as regular functions of kind $A \to G$.

The algebra $W/A'$ itself is termed the algebra of regular functions on the variety $A$ with values in $G$. All this is in agreement with the presentation of $W/A'$ as a subdirect product of the algebras $W/\text{Ker}\mu$, $\mu \in A$. The algebra $W/A'$ is also called the co-ordinate algebra of the variety $\Theta$.

We call, furthermore, an algebra $H \in \Theta$ $G$-exact, $G \in \Theta$, if $(G - \text{Ker})(H)$ is the trivial congruence on $H$. Let $\mu_0 : W(X) \to H$ be a surjective homomorphism for such $H$ with $\text{Ker}\mu_0 = T$. Then, obviously, the algebra $H$ can be presented as the co-ordinate algebra of the variety $A = T'$ in $\text{Hom}(W(X), G)$.

On the other hand, every algebra $W/A'$ is $G$-exact.

The algebra $W/A'$ is an important invariant of the variety $A$. Below, we shall introduce, for $G$ fixed, the notion of isomorphism of two varieties. It will be proved that varieties $A$ and $B$ are isomorphic if and only if isomorphic are respective algebras $W/A'$ and $W/B'$. This is a generalization of a classical theorem.

Varieties can be classified from the viewpoint of properties of their algebras, e.g. according to identities of the algebras.

In particular, varieties $A$ and $B$ could be called similar if

$$\text{Var}(W/A') = \text{Var}(W/B').$$

Now assume that $A$ is a verbal variety specified by a verbal congruence $T$. Then $A' = T''$ is also a verbal congruence, and $W/A'$ is a free over $X$ algebra of the corresponding subvariety of $\Theta$.

We now return to the question on equivalence of two algebras $G_1$ and $G_2$ from $\Theta$. If $T$ is a congruence on $W$, then we also have $A = T'_{G_1}$, and $B = T'_{G_2}$. If $G_1$ and $G_2$ are equivalent, then $A'$ and $B'$ coincide, and so do also the algebras $W/A'$ and $W/B'$. This is one more argument in favour of the notion of equivalence we discuss.

Regretfully, we cannot speak of isomorphism of the varieties $A$ and $B$, for they are related to distinct $G_1$ and $G_2$. Nevertheless, there must be something in common for $A$ and $B$; at least, in the situation of the classical geometry. Cf. also §5.

Now we shall comment on the topology on $\text{Hom}(W, G)$ connected with algebraic varieties. We already have noticed that the sum of two algebraic varieties need not be an algebraic variety. Because of this, in order to obtain a topology on $\text{Hom}(W, G)$ we regard algebraic varieties and finite unions of them to be the closed sets.

These sets are described by systems of pseudoequations. For more detail, see §5. The corresponding topology is considered as a Zariski topology on $\text{Hom}(W, G)$. It is not clear to us what it can offer in the general situation under consideration. However, one useful consideration concerning the closure of a point can be made. We already discussed a particular situation of this sort; now the overall picture will be sketched.

Let us take a point $\mu \in \text{Hom}(W, G)$. The closure of $\{\mu\}$ is $\{\mu\}''$. As $\{\mu\}' = T$ is $\text{Ker}\mu$, we conclude that $\{\mu\}'' = \{\nu, T = \text{Ker}\mu \subset \text{Ker}\nu\}$. The set $\text{Hom}(W, G)$ can be equipped with a pseudo-ordering relation by setting $\mu \leq \nu$ if $\text{Ker}\mu \subset \text{Ker}\nu$. Then the closure of the point $\mu$ is the set of the points $\nu$ with $\mu \leq \nu$. Points $\mu$ and $\nu$ are equivalent if $\mu \leq \nu$ and $\nu \leq \mu$, i.e. if $\text{Ker}\mu = \text{Ker}\nu$.

Clearly, equivalence of $\mu$ and $\nu$ also means the closures of these points coincide. All this is well-known in the classical situation.
As we know, two semigroups – EndW and EndG – are acting on Hom(W, G). Let us see how the actions conform with algebraic varieties.

First of all, recall that an action of a semigroup is defined by way of multiplying morphisms. If \( s \in S = \text{End}W \) and \( \mu \in \text{Hom}(W, G) \), then \( \mu s \) is given by the rule \( (\mu s)(x) = \mu(s(x)) \). If \( \sigma \in \text{End}G \), then, for \( \sigma \mu, (\sigma \mu)(x) = \sigma(\mu(x)) \). We here apply morphisms from the left. If \( A \) is a subset of \( \text{Hom}(W, G) \), then

\[
\mu \in sA \iff \mu s \in A, \quad \mu \in A\sigma \iff \sigma \mu \in A.
\]

It is easily seen that the algebraic variety \( T' = A \) for every \( T \) on \( W \) is always invariant under the action of \( \text{End}G \): if \( \mu \in A \), then \( \sigma \mu \in A \). This way, on each \( A \) the action structure of \( \text{End}G \) is defined. In particular, if \( \sigma \in \text{Aut}G \), then the points \( \mu \) and \( \sigma \mu \) are equivalent: they determine the same closure.

If \( T \) is a binary relation on \( W \), then we define \( sT, s \in \text{End}W \), to be the new binary relation determined by the rule

\[
w(sT)w' \quad \text{if there are } w_1 \text{ and } w'_1 \text{ such that } w_1 = w, \quad w'_1 = w' \text{ and } w_1 Tw'_1.
\]

**6.1. Proposition.** 1. For each \( s \in \text{End}W \) and \( T \),

\[
(sT)' = sT'.
\]

2. For each \( A \subset \text{Hom}(W, G) \) and \( s \in \text{Aut}W \),

\[
(sA)' = sA'.
\]

These equalities will be verified in §5 in a more general context.

We note two consequences of the proposition.

1. If \( A = T' \) is an algebraic variety, then so is the set \( sA \) for every \( s \in \text{End}W \). In other words, the system \( M_G \) of all algebraic varieties in \( \text{Hom}(W, G) \) is invariant with respect to the action of the semigroup \( \text{End}W \).

2. If \( T = A' \) is a closed congruence on \( W \), then so is the congruence \( sT \) for every automorphism \( s \) of \( W \): if \( T'' = T \), then \( (sT)' = sT' \). The system \( \text{Cl}_G(W) \) is invariant with respect to action of the group \( \text{Aut}W \).

We shall see below that if \( s \) is an automorphism, then it determines an isomorphism between the varieties \( A \) and \( sA \).

The next proposition, which reveals connections with closures, is also related to Proposition 6.1.

**6.2. Proposition.** Assume that \( s \in \text{Aut}W \). Then, for all \( T \) and \( A \),

1. \( (sT)' = sT'' \).
2. \( (sA)' = sA'' \).

**Proof.** Of course, \( (sT)' = sT' \). We apply once more: \( (sT)' = (sT)' = sT'' \). Likewise in the second case: \( (sA)' = sA' \) and, further, \( (sA)' = (sA)' = sA'' \). \( \square \)

Finally, we comment on the lattice of varieties for \( G \) fixed. In the classical situation all varieties make up a lattice, which is a sublattice of the distributive lattice of all subsets of \( \text{Hom}(W, G) \). The lattice of varieties appears also in the general case. Then \( A \cdot B = A \cap B \) and \( A + B = (A \cup B)' = (A' \cap B')' \). What can we say about this lattice? Are there any connections with congruence lattice of \( W \)? What about two algebras \( G_1 \) and \( G_2 \) when the respective variety lattices are isomorphic? We have not examined these questions.
7. Relation to the $\Theta$-structure of algebras. This subsection is concerned with the subject of the preceding one. We here equip the set $\text{Hom}(W, G)$ with the structure of the variety $\Theta$. This can only be done in the case of one-sorted algebras, for the set $\text{Hom}(W, G)$ is always one-sorted. So let $\Theta$ be a variety of one-sorted $\Omega$-algebras.

Let $X$ be the set of variables, and assume that $G$ is an algebra from $\Theta$. Then $G^X$ is also an algebra in $\Theta$. The $\Theta$-structure of $G^X$ can be transferred to $\text{Hom}(W, G)$.

If $\omega \in \Omega$ is an $n$-ary operation, then for homomorphisms $\mu_1, \ldots, \mu_n \in \text{Hom}(W, G)$

$$x^{\mu_1 \cdots \mu_n} = x^{\mu_1} \cdots x^{\mu_n}. $$

But we cannot be sure that

$$w^{\mu_1 \cdots \mu_n} = w^{\mu_1} \cdots w^{\mu_n}. $$

for arbitrary $w \in W$. This is so only under some specific conditions, which are discussed below.

Let $\omega_1$ be an $n$-ary operation and $\omega_2$-an $m$-ary operation from $\Omega$, none of them nullary, and consider a matrix $(x_{ij})$, $i = 1, \ldots, m$, $j = 1, \ldots, n$, consisting of variables. Set

$$w_1 = (x_{11} \cdots x_{1n}\omega_1) \cdots (x_{m1} \cdots x_{mn}\omega_1)\omega_2,$$

$$w_2 = (x_{11} \cdots x_{1m}\omega_2) \cdots (x_{1n} \cdots x_{mn}\omega_2)\omega_1.$$ 

The formula $w_1 \equiv w_2$ is a kind of a commutation law for the operations $\omega_1$ and $\omega_2$. If $\omega_1 = 0_\alpha$ is a nullary operation, and $\omega_2 = \omega$ is arbitrary, then their commutation means that

$$0_\alpha \cdots 0_\alpha\omega = 0_\alpha.$$

Commutation of two nullary operations $0_\alpha$ and $0_\beta$ means that $0_\alpha = 0_\beta$.

The commutation law can be applied to coinciding operations, too. For example, in a group this law, when applied to multiplication, means that the group is Abelian. This need not be so in a semigroup.

An algebra $G \in \Theta$ is said to be commutative if the commutation law holds in it for every two operations, including the case of equal operations.

7.1. Proposition. If $G$ is commutative, then

$$w^{\mu_1 \cdots \mu_n\omega} = w^{\mu_1} \cdots w^{\mu_n}\omega$$

for every operation $\omega \in \Omega$, all $\mu_1, \ldots, \mu_n$, and every $w \in W$.

Now assume that $G$ is commutative and that $T$ is a set of formulas of kind $w \equiv w'$.

7.2. Proposition. The algebraic variety $A = T'$ is a subalgebra of $\text{Hom}(W, G)$.

Proof. We have to find out whether the set $A$ is closed under the operations from $\Omega$. Let $\omega \in \Omega$ be an $n$-ary operation, and let $\mu_1, \ldots, \mu_n \in A$. We shall check that $\mu_1 \cdots \mu_n\omega \in A$. Suppose that $w \equiv w' \in T$; then

$$w^{\mu_1 \cdots \mu_n\omega} = w^{\mu_1} \cdots w^{\mu_n}\omega = w^{\mu_1} \cdots w^{\mu_n}\omega = w^{\mu_1 \cdots \mu_n\omega}, \quad \mu_1 \cdots \mu_n\omega \in A.$$ 

We can use all this as follows.

Assume that $\Omega_0$ is a subset of $\Omega$. Any algebra $G \in \Theta$ can be regarded as an $\Omega_0$-algebra. Considered this way, it may turn out to be commutative, though, in general, it may be non-commutative as well. We may choose $\Omega_0$ in several ways. In doing so, we also can apply the above considerations and conclude for the respective $T$'s that $A = T'$ is an $\Omega_0$-closed variety.
Let us draw some consequences of this for the classical situation. The operation system \( \Omega \) consists here of addition, multiplication, zero, unit and scalars. If we take \( \Omega_0 \) to contain addition, scalars and the zero, then the corresponding algebras are vector spaces, and they are commutative. The related equations are of the form
\[
\alpha_1 x_1 + \cdots + \alpha_n x_n = 0.
\]
The corresponding varieties are \( \Omega_0 \)-algebras.

In another case \( \Omega_0 \) consists of multiplication and the unit. This also is a commutative collection. The equations take either the form
\[
x_1^{n_1} \cdots x_k^{n_k} = x_1^{m_1} \cdots x_i^{m_i}
\]
of the form \( x_1^{n_1} \cdots x_k^{n_k} = 1 \).

For the corresponding \( T \)'s, the varieties \( A = T' \) are invariant under \( \Omega_0 \).

In particular, the parabola \( y^2 = x \) is closed under multiplication, while the parabola \( y^2 = 2px \) does not possess this property. The scalars do not commute with multiplication. This is true also of hyperbolas \( xy = 1 \) and \( xy = a \). The zero commutes with addition and multiplication, but the unit does not commute with addition; also the zero and the unit do not commute.

The general theory developed here is, of course, applicable when \( \Theta \) is the variety of \( K \)-modules, where \( K \) is a commutative ring with unit. In this case, the algebraic varieties in \( \text{Hom}(W,G) \) for every \( G \in \Theta \) are submodules. However, not every submodule is a variety.

Let us comment on the latter observation. Suppose that \( W = KX \) is the free module over \( X = \{x_1, \ldots, x_n\} \), that \( G \) is some other module and that \( T \) is a submodule of \( KX \). According to the general theory, the corresponding algebraic variety \( A = T' \) is of the form \( A = \mu_0 \text{Hom}(KX/T, G) \), where \( \mu_0 \) is the natural homomorphism. We restrict below the discussion to the simple case when \( K \) is a field, and assume that the space \( KX/T \) is \( m \)-dimensional, \( m < n \). Under a suitable enumeration of elements in \( X \), \( KX/T \) admits a basis consisting of elements \( x_1^{\mu_0}, \ldots, x_m^{\mu_0} \). With the homomorphism \( \mu_0 \), the \( (n \times m) \)-matrix of the kind
\[
\mu_0 = \begin{pmatrix}
1 & & 0 \\
& \ddots & \\
0 & & 1 \\
\vdots & & \ddots \\
\vdots & & \vdots \\
\end{pmatrix}
\]
is related.

Now assume that \( G \) is \( k \)-dimensional. Then elements of \( \text{Hom}(KX/T, G) \) are presented by matrices
\[
\nu = \begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1k} \\
\vdots & \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mk}
\end{pmatrix}
\]
Elements of the variety \( A \) are composed of \( (n \times k) \)-matrices of the kind
\[
\mu_0 \nu = \begin{pmatrix}
\alpha_{11} & \cdots & \alpha_{1k} \\
& \ddots & \vdots \\
\alpha_{m1} & \cdots & \alpha_{mk} \\
\vdots & \ddots & \vdots \\
\vdots & & \ddots \\
\end{pmatrix}
\]
The upper \((m \times k)\)-part of such a matrix is quite arbitrary, for \(\nu\) is arbitrary. The lower part depends of the fixed matrix \(\mu_0\). Now it is clear that there are subspaces in \(\text{Hom}(KX,G)\) that are not algebraic varieties.

8. Additional remarks. We begin with some notes concerning the case \(\Theta\) is the variety of groups. First of all, we make a simple observation. The set \(X\) is assumed to be fixed.

We denote by \(F = F(X)\) the free group over \(X\). Suppose that \(G\) is a torsion-free group and that \(T\) is a normal subgroup of \(F\). The kernel of every homomorphism \(\nu: F/T \rightarrow G\) contains elements of finite order from \(F/T\). For this reason, the kernel \((G - \text{Ker})(F/T)\) contains all elements of finite order from \(F/T\). This also means that all \(\varphi \in F\) with \(\varphi^n \in T\) for some \(n\) belong to \(T'\).

If \(F/T\) is a nilpotent group, then all such \(\varphi\)'s make up a normal subgroup of \(F\), which is naturally denoted by \(\sqrt{T}\). Now \(T'' \supset \sqrt{T}\). In a number of cases even the equality \(T'' = \sqrt{T}\) holds. This is something like the Hilbert theorem.

We shall further discuss geometric equivalence of groups.

8.1. Proposition. Suppose that \(G_1\) and \(G_2\) are equivalent groups. If \(G_1\) is torsion-free, then so is \(G_2\).

**Proof.** Assume that \(G_1\) is torsion-free and that \(G_2\) has a cyclic subgroup \(H\) of order \(n\). For \(T\), take the verbal subgroup of \(F\) over the variety of groups of exponent \(n\). Then \((G_1 - \text{Ker})(F/T) = F/T\). Now let \(\nu: F/T \rightarrow H\) be a non-trivial homomorphism. It also is an element of \(\text{Hom}(F/T,G_2)\). Since the kernel \(\text{Ker}\nu\) differs from \(F/T\), we infer that \((G_2 - \text{Ker})(F/T) \neq F/T\). Therefore, \(G_1\) and \(G_2\) are not equivalent, and this contradicts the supposition. 

If both \(G_1\) and \(G_2\) are periodic and if they are equivalent, then they must have the same exponent.

The next question seems to be simple. Suppose that \(G_1\) and \(G_2\) are equivalent and \(G_1\) is periodic. Is also \(G_2\) periodic?

We now shall consider separately the case when \(X\) only consists of one element \(x\). We shall show that, under this assumption, any two torsion-free groups are equivalent.

Let \(G_1\) and \(G_2\) be such groups, \(F = F(X)\) be an infinite cyclic group, \(T\) be a subgroup of \(F\). If \(T\) only consists of the unit, then \((G_1 - \text{Ker})(F/T) = (G_2 - \text{Ker})(F/T)\), and both these kernels only consist of the unit. If \(T\) still contains something else, then \(F/T\) is finite, and \((G_1 - \text{Ker})(F/T) = (G_2 - \text{Ker})(F/T) = F/T\).

This conclusion does not remain valid if \(X = \{x, y\}\). If \(G_1\) is a commutative, and \(G_2\) is a non-commutative group, both torsion-free, then they are not equivalent.

Already in the classical algebraic geometry it can be proved that the equality \(\text{Var}(G_1) = \text{Var}(G_2)\) does not imply equivalence of \(G_1\) and \(G_2\). It is easily seen that the some holds for groups. Let us demonstrate this.

Assume we are given a surjective group homomorphism \(\delta: G \rightarrow H\), where \(G\) is torsion-free and \(H\) is periodic. Let \(G_1 = G\) and \(G_2 = G \times H\). Then \(G_1\) and \(G_2\) are not equivalent, but \(\text{Var}(G_1) = \text{Var}(G_2)\).

The following problem admits a simple solution. Find all groups \(G\) for which all invariant subgroups \(T\) of \(F(X)\) are closed.

Now we pass to the general situation. It is not difficult to observe that \(T\) is a congruence on \(W = W(X)\), then \(T = T''\) for some \(G \in \Theta\). We can take the algebra \(W/T\) for \(G\). In this connection, we note one more problem which is rather ambiguous.
Let $T$ and $T_1$ be two congruences on $W(X)$ with $T \subset T_1$. What can be said concerning existence of $G \in \Theta$ such that $T'_G = T_1$?

Such a group $G$ does not exist, for instance, if the congruence $T$ is verbal and $T_1$ is not. If both congruences are fully characteristic, then the problem can be solved in a simple manner. Indeed, the following holds:

$$(W/T_1 - \operatorname{Ker})(W/T) = T_1/T.$$  

Now if $G = W/T_1$, then $T'_G = T_1$.

We now make a remark that also concerns with an arbitrary $\Theta$. Assume that $G$ and $H$ are algebras from $\Theta$. We treat $\operatorname{Hom}(G, H)$ as the set of representations of $G$ into $H$. We shall couple with it a variety of representations, which will be determined up to isomorphism of algebraic varieties.

Let the algebra $G$ be specified by generators and defining relations. Let, furthermore, $X$ be the set of its generators, and $T$ be the congruence on $W(X)$ generated by the relations. Then there is a surjective homomorphism $\mu_0 : W \to G$ with the kernel $T = \operatorname{Ker}\mu_0$. The corresponding algebraic variety $A = T'$ is specified in $\operatorname{Hom}(W, H)$. Then $A = \mu_0 \operatorname{Hom}(G, H)$ is the variety of representations of $G$ in $H$ we are interested in. As we shall see, passage to another system of generators and relations leads to isomorphism of algebraic varieties.

Turning back to groups, let us consider two groups $G$ and $H = \operatorname{Aut}(V)$ where $V$ is a module over some $K$. Then the question is of the variety of linear representations of the given $G$ in a linear group $H$. It is an algebraic variety in $\operatorname{Hom}(W, H)$; cf. [30]. One can consider various subvarieties of it and relate them with classification problem for representations. Also, the problem of geometrical equivalence of the groups $\operatorname{Aut} V_1$ and $\operatorname{Aut} V_2$ naturally arises here, $V_1$ and $V_2$ being various modules (over the same $K$, or not).

One also can speak about geometric equivalence of two representations on the basis of the variety $\Theta$ of all representations over a given ring $K$; see [33]. In particular, two irreducible and faithful representations of finite groups over the same field are geometrically equivalent if and only if they are isomorphic. Algebraic varieties related to linear representations motivate various interesting ideas. This is a separate subject.

Now again we shall make some general observations. The variety $\Theta$ is arbitrary, and fixed are the free algebra $W = W(X)$ and $G \in \Theta$. For every $\mu : W \to G$, the collection of the elements $x^\mu$ with $x \in X$ is a generating set of $\operatorname{im}\mu$. If, furthermore, $T$ is a congruence on $W$ and $\mu \in A = T'$, then $T \subset \operatorname{Ker}\mu$; this means that $T$ is induced in the system of defining relations of the algebra $\operatorname{im}\mu$. Therefore, we have information about the generators and relations of algebras of the kind $\operatorname{im}\mu$ for all $\mu \in A = T'$.

For example, if the question is of groups and if $T$ contains all commutators $[x, y]$ for $x, y \in X$, then all subgroups of kind $\operatorname{im}\mu, \mu \in A = T'$, of any $G$ are commutative.

In conclusion of the section, we note that the theory we deal with here was stimulated, in considerable extent, by investigations of equations in groups. These investigations, in they turn, are connected with geometrical algebra; see [19-20-31]. See also [49] as a survey of works on geometrical algebra, in particular, of the works of E. Rips and Z. Sela.

The geometric approach clears some ways for seeking solutions. Generally the aims of algebraic geometry are wider. We have in mind both introducing geometric concepts in algebraic structures and algebraic interpretation of the arising geometric structures. With respect to this, geometric algebra and algebraic geometry are close to each other; however, they are oriented to different geometric structures. But speaking generally, geometry and algebra in either field are heavily intertwined. Our interests are focused chiefly on algebra. It is difficult to perceive that general algebraic varieties could be well-connected with substantial geometry.
§4. VARYING THE VARIABLES SET, THE BASE VARIETY AND THE BASE ALGEBRA

1. Changing $X$. We count the variety fixed, and change the set of variables, $X$. So, we deal with $W(X)$ and $W(Y)$.

1.1. Theorem. If $X \subset Y$ and algebras $G_1$ and $G_2$ are $Y$-equivalent, then they are also $X$-equivalent.

Proof. We treat $W(X)$ as a subalgebra of $W(Y)$, and take some $G \in \Theta$. Then every $\mu : W(Y) \to G$ induces $\nu : W(X) \to G$. On the other hand, there are several mappings $\mu : W(Y) \to G$ inducing a given $\nu : W(X) \to G$. We shall write $\nu = \mu^\alpha$.

We shall show that if $A$ is an algebraic variety in $\text{Hom}(W, G)$, then its full inverse $\alpha$-image $B$ is an algebraic variety in $\text{Hom}(W(Y), G)$. Suppose that $A = T'$, where $T$ is a binary relation on $W(X)$. It can be considered as a relation on $W(Y)$ as well. We shall write $T = T_X$ and $T = T_Y$, respectively, in this connection. Let us check that $T'_Y = B$.

First observe that if $\nu = \mu^\alpha$, then $\text{Ker} \nu = \text{Ker} \mu \cap W(X)$.

Now let $\mu \in B$. Then $\mu^\alpha = \nu \in A$ and $T = T_X \subset \text{Ker} \nu = W(X) \cap \text{Ker} \mu$. But then $T = T_Y \subset \text{Ker} \mu$ and $\mu \in T'_Y$. Conversely, let $\mu \in T'_Y$. Then $T = T_Y \subset \text{Ker} \mu \cap W(X) = \text{Ker} \nu$, where $\nu = \mu^\alpha$, and, further, $\nu \in T' = T'_Y = A$ and $\mu \in B$.

Now we make some remarks on congruences.

If $T$ is a congruence on $W(Y)$, then we have a congruence $W(X) \cap T$ on $W(X)$. Being a binary relation, it generates a congruence on $W(Y)$. The latter one is included in $T$ but does not, generally, coincide with $T$.

Assume now that $A$ is a variety in $\text{Hom}(W, G)$, $A' = T_X$ and $B$ is the full inverse image of $A$, and let $T_Y = B'$. We shall verify that $T_Y \cap W(X) = T_X$.

Clearly, $B' = T_Y = \bigcap_{\mu \in B} \text{Ker} \mu$. Furthermore,

$$T_Y \cap W(X) = \bigcap_{\mu \in B} \text{Ker} \mu \cap W(X) = \bigcap_{\mu \in B} (\text{Ker} \mu \cap W(X)) = \bigcap_{\nu \in A} \text{Ker} \nu = T_X.$$ 

Now suppose that $T = T_X$ is a closed congruence on $W_X$. For it, we shall construct a closed congruence $T_Y$ in $W(Y)$ so, that $T_Y \cap W(X) = T_X$. Take $A = T_X$ and let $B$ be the full preimage of $A$. Then take $B' = T_Y$. It is a closed congruence, and $A' = T_X$. Now indeed, $T_Y \cap W(X) = T_X$.

Now, we return to the theorem. Assume $G_1$ and $G_2$ are $Y$-equivalent. This means that the congruence $T_Y$ is $G_1$-closed if and only if it is $G_2$-closed. We shall check the same for $X$. Take $T_X$ in $W(X)$ and assume that this congruence is $G_1$-closed. Suppose that $T_Y$ is $G_1$-closed congruence on $W(Y)$ such that $T_Y \cap W(X) = T_X$. The congruence $T_Y$ is $G_2$-closed. We have to prove that $T_X$ is $G_2$-closed, too. To proceed, some additional remarks are needed.

Let again $G$ be any algebra, and let $B$ be an algebraic variety in $\text{Hom}(W(Y), G)$ determined by some closed $T_Y = B'$. We denote by $A$ the $\alpha$-image of $B$ in $\text{Hom}(W, G)$, and check that then $A' = B' \cap W(X)$. Suppose that $(w, w') \in A'$. Both $w$ and $w'$ are elements of $W(X)$, and $(w, w') \in \text{Ker} \nu$ for every $\nu \in A$. Now if $\mu \in B$, then $\mu^\alpha = \nu \in A$. This means that $w^\nu = w'^\nu$, $w'^\mu = w'^\mu$ and, therefore, $(w, w') \in \text{Ker} \mu$. Since this holds for every $\mu \in B$, we conclude that $(w, w') \in \bigcap_{\mu \in B} \text{Ker} \mu = B'$. Therefore, $(w, w') \in B' \cap W(X)$.

If, conversely, $(w, w') \in B' \cap W(X)$, then $w'^\mu = w'^\mu$ for every $\mu \in B$. Take any $\nu \in A$ of the kind $\nu = \mu^\alpha$. Since $w$ and $w'$ belong to $W(X)$, we obtain that $w^\nu = w'^\mu = w'^\mu$ and $(w, w') \in \text{Ker} \nu$. This holds for every $\nu \in A$, thereby $(w, w') \in A'$. 

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In particular, if $G = G_2$, then $T_Y \cap W(X) = T_X = A'$ for some suitable $A$. This means that the congruence $T_X$ is $G_2$-closed. As to $A$, this set need not be an algebraic variety.

We have demonstrated that if a congruence $T_X$ is $G_1$-closed, then it is $G_2$-closed, and the converse also holds. So, $G_1$ and $G_2$ are $X$-equivalent. The proof of the theorem is completed. \hfill \Box

As we saw, the converse is not true: $X$-equivalence does not imply $Y$-equivalence.

1.2. Problem. Given finite $X$ and $Y$ with $X \subset Y$, find algebras $G_1$ and $G_2$ that are $X$-equivalent but not $Y$-equivalent. Is this always possible?

1.3. Problem. Is it true or not that $X$-equivalence of $G_1$ and $G_2$ for every finite $X$ implies their $Y$-equivalence for $Y$ enumerable?

Let us add a remark which will be used later. Let $X \subset Y$, and $T = T_X$ be a binary relation in $W(X)$. If we consider $T$ as $T_Y$ in $W(Y)$, then $T''_Y \cap W(X) = T''_X$.

Indeed, if $T'_X = A$, then $T''_Y = B$ is a full coimage of $A$. By the definition, $T''_X = A'$, $T''_Y = B'$, and, as we have seen, $A' = B' \cap W(X)$.

Now, we pass to another important subject. We count the variety $\Theta$ fixed, and take $X$ and $Y$ either distinct or coinciding. We then have the algebras $W(X)$ and $W(Y)$, respectively, and suppose the algebra $G$ to be given. We are going to co-ordinate the varieties in $\text{Hom}(W(Y), G)$ and in $\text{Hom}(W, G)$.

We consider the set $\text{Hom}(W(Y), W(X))$, which becomes the semigroup $\text{EndW}$ when $X$ and $Y$ coincide.

For every $s: W(Y) \to W(X)$ and every $\nu: W(X) \to G$, we have $\mu = \nu s: W(Y) \to G$. This gives us the mapping

$$\tilde{s}: \text{Hom}(W, G) \to \text{Hom}(W(Y), G).$$

If, furthermore, $A = T'_1$ is an algebraic variety in $\text{Hom}(W, G)$ and $B = T''_2$ is an algebraic variety in $\text{Hom}(W(Y), G)$, and if $T_1$ and $T_2$ are congruences on $W(X)$ and $W(Y)$, respectively, then $\tilde{s}$ determines a morphism $s: A \to B$ if and only if $\nu s \in B$ for every $\nu \in A$.

1.4. Proposition. The element $s \in \text{Hom}(W(Y), W(X))$ determines a morphism $s: A \to B$ if and only if $w T''_2 w'$ implies $s(w) T''_1 s(w')$.

Proof. Assume that $\nu s \in B$ for every $\nu \in A$ and that $w T''_2 w'$. We need to prove that $s(w) T''_1 s(w')$. We have $T''_1 = A'$ and $T''_2 = B'$. Moreover, $A' = \bigcap_{\nu \in A} \text{Ker} \nu$. We shall check that, for all $\nu \in A$, $(s(w), s(w')) \in \text{Ker} \nu$ or, in other words, $\nu s(w) = \nu s(w')$. By the definition, $T''_2 = B' = \bigcap_{\mu \in B} \text{Ker} \mu$. Hence, $w T''_2 w'$ means that $w^\mu = w'^\mu$. In particular, this is true of $\mu = \nu s$, and then $\nu s(w) = \nu s(w')$.

To prove the converse, we assume that $w T''_2 w'$ implies $s(w) T''_1 s(w')$ and that $\nu \in A$ is given. We shall check that $\nu s \in B$ or, equivalently, $\nu s(w) = \nu s(w')$ whenever $w T''_2 w'$. Suppose the latter condition is fulfilled. Then also $s(w) T''_1 s(w')$. Now, if $\nu \in A$, then $\nu s(w) = \nu s(w')$, $\nu s \in B$. \hfill \Box

This proposition has the following application.

1.5. Proposition. To every morphism $s: A \to B$, there is an algebra homomorphism $s': W(Y)/B' \to W(X)/A'$. The converse also holds: every algebra homomorphism $s'$ induces a morphism $s$ between the respective algebraic varieties.

Proof. Assume we are given a morphism $s: A \to B$, $s \in \text{Hom}(W(Y), W(X))$. The homomorphism $s: W(Y) \to W(X)$ and the natural homomorphism $s_0: W(X) \to W(X)/T''_1$ give $s \sigma_0: W(Y) \to W(X)/T''_1$. By Proposition 1.4, the congruence $T''_2$ is included in the
kernel of the homomorphism $s\sigma_0$. Because of this, also the homomorphism $\sigma: W(Y) / T_2'' \rightarrow W(X) / T_1''$ is defined. It remains to observe that $T_2'' = B'$ and $T_1'' = A'$. We proceed here from $A = T_1'$ and $B = T_2'$.

We pass to the final part of the proposition. Assume we are given a homomorphism $\sigma: W(Y) / T_2'' \rightarrow W(X) / T_1''$. There is a related commuting diagram

$$
\begin{array}{ccc}
W(Y) & \xrightarrow{s} & W(X) \\
\sigma_1 \downarrow & & \downarrow \sigma_0 \\
W(Y) / T_2'' & \xrightarrow{\sigma} & W(X) / T_1''
\end{array}
$$

where $\sigma_1$ and $\sigma_0$ are the natural homomorphisms, and $s$ also is specified in a natural way.

Now assume that $wT_2''w'$. This means that $w^{\sigma_1} = w'^{\sigma_1}$. But then $w^{\sigma_1} = w'^{\sigma_1}$ and $w^{\sigma_0} = w'^{\sigma_0}$, whence $s(w)T_1's(w')$. Now it follows from Proposition 1.4 that we have a morphism $s: A \rightarrow B$. \hfill \Box

We now pass to examples.

Assume we are given $s: W(Y) \rightarrow W(X)$, with the corresponding $\hat{s}: \text{Hom}(W, G) \rightarrow \text{Hom}(W(Y), G)$, $G \in \Theta$. For every subset $B$ of $\text{Hom}(W(Y), G)$, we define $sB = A$, a subset of $\text{Hom}(W, G)$, by the rule: $\mu \in A = sB \Leftrightarrow \mu s \in B$. Moreover, where $T$ is a binary relation on $W(Y)$, we define the binary relation $sT$ on $W(X)$, as above, by the rule: $w sT w'$ if there are $w_1$ and $w_1'$ in $W(Y)$ such that $w_1 = w$, $w_1' = w'$ and $w_1Tw_1'$. Again, $(sT)' = sT$.

Let us prove this. Assume that $\mu \in (sT)'$, i.e., $\mu s \in T'$, $T \subset \text{Ker}\mu$. Let $w_1Tw_1'$. We have to see whether $(s(w_1))'' = s(w_1')''$. Take $s(w_1) = w$, $s(w_1') = w'$. Then $w sT w'$. Since $sT \subset \text{Ker}\mu$, $w'' = w''$. Thus, $s(w_1)'' = s(w_1')''$. We obtain that $\mu \in sT'$.

Conversely, assume that $\mu \in sT'$. We shall check that $\mu \in (sT)'$. Let $w sT w'$. We have to make sure that $w'' = w''$. By the condition, $w_1, w_1' \in W(Y)$ with $s(w_1) = w$, $s(w_1') = w'$ and $w_1Tw_1'$. Since $\mu \in sT'$, $\mu s \in T'$ and $s(w_1)'' = s(w_1')''$. This gives us $w'' = w''$, $\mu \in (sT)'$.

In particular, if $B = T'$ is an algebraic variety in $\text{Hom}(W(Y), G)$, then $A = sB = sT' = (sT)'$ is an algebraic variety in $\text{Hom}(W, G)$.

Moreover, if $\mu \in A$, then $\mu s \in B$, and $s: A \rightarrow B$ is a morphism. In general, $s: A \rightarrow B$ is a morphism if $A \subset sB$.

We further consider a particular situation. Let $Y$ be a subset of $X$, and take for $s$ the corresponding injection $s: W(Y) \rightarrow W(X)$. Then

$$
\hat{s}: \text{Hom}(W, G) \rightarrow \text{Hom}(W(Y), G)
$$

is the projection which we have already used. If $B$ is an algebraic variety in $\text{Hom}(W(Y), G)$, then $sB = A$ is the corresponding full preimage. We have the morphism $s: A \rightarrow B$. Cf. the proof of Theorem 1.1.

**1.6.** Definition. A morphism $s: A \rightarrow B$, $s \in \text{Hom}(W(Y), W(X))$ is an algebraic variety isomorphism if it has the inverse morphism $s': B \rightarrow A$. 
Here $s' \in \text{Hom}(W(X), W(Y))$, and, for every $\nu \in A$ and $\mu \in B$, $\nu s s' = \nu$ and $\mu s s' = \mu$. If there is such $s$, then the varieties $A$ and $B$ are isomorphic.

1.7. Theorem. Varieties $A$ and $B$ are isomorphic if and only if isomorphic are the respective algebras $W(X)/A'$ and $W(Y)/B'$.

proof. Assume that $A$ and $B$ are isomorphic and that $s$ and $s'$ are the respective morphisms. We also have $s: \text{Hom}(W(Y)/A, W(X))$ and, simultaneously, the homomorphisms $\sigma: \text{hom}(W(Y)/A, W(X))$ and $\sigma': \text{hom}(W(Y)/A, W(X))$. Let us check that they are isomorphic to each other.

We need to see whether $s' \circ (w_{1})_{\mu} \circ w_{1}$ for every $w \in W(Y)$ and $s' \circ (w_{1})_{\mu} \circ w_{1}$ for every $w_{1} \in W(X)$.

The condition $s' \circ (w_{1})_{\mu} \circ w_{1}$ means that $(s' \circ (w_{1}))_{\mu} = w_{1}$ for every $\mu \in B$. In other notation this means that $\mu s' \circ (w_{1}) = \mu \circ (w_{1})$. Since $\mu s' = \mu$, the equality holds. Likewise, the other condition is also fulfilled. Therefore, $\sigma$ is an algebra isomorphism.

Now assume that $\sigma: \text{hom}(W(Y)/A, W(X))$ is an algebra isomorphism and $\sigma': \text{hom}(W(Y)/A, W(X))$ is the inverse isomorphism. Let us consider the commuting diagrams

\[
\begin{array}{cccc}
W(Y) & \xrightarrow{s} & W(X) & \xrightarrow{s'} & W(Y) \\
\downarrow{\sigma_1} & & \downarrow{\sigma_0} & & \downarrow{\sigma_1} \\
W(Y)/A_1 & \xrightarrow{\sigma} & W(X)/A_1 & \xrightarrow{\sigma'} & W(Y)/A_1 \\
\end{array}
\]

We shall prove that the morphisms $s: A \rightarrow B$ and $s': B \rightarrow A$ are mutually inverse--i.e. that $\nu s s' = \nu$ and $\mu s s' = \mu$ for all $\nu \in A$ and $\mu \in B$. Take any $w \in W(X)$, and check that $\nu s s' = \nu (w)$. Clearly, $\sigma_1 s(w) = \sigma \sigma_0 (w)$. Apply $\sigma$; then $\sigma_1 s(w) = \sigma \sigma_0 w$. This gives $s' \circ (w_{1})_{\mu} \circ w_{1}$. But then $\nu s s' = \nu (w)$ and, furthermore, $s s' = \nu$ for every $\nu \in A$. Likewise, $\mu s s' = \mu$.

1.8. Proposition. If $s: A \rightarrow B$ is an isomorphism, then it is a bijection between $A$ and $B$.

proof. Assume that $\nu_1 s = \nu_2 s \in B$ for $\nu_1, \nu_2 \in A$. We apply the inverse $s'$; then $\nu_1 s s' = \nu_2 s s'$ and $\nu_1 = \nu_2$. Now assume that $\mu \in B$. Then $\mu = \mu s s' \in A$, $\mu s' = \nu$ and $\nu s = \mu$. 

If $s: A \rightarrow B$ is a morphism, then $A \subset sB$. The converse also holds. $A \subset sB$ means that $s: A \rightarrow B$ is a morphism. It may seem that, if $s$ is an isomorphism, then $A = sB$; however, it is not the case. If $\nu \in sB$, then $w \in B$ and $\nu s s' \in A$. But we cannot claim that $\nu s s' = \nu$, for we do not know whether, and cannot conclude that, $\nu \in A$.

Theorem 1.7 holds also in the case $X = Y$, and then $s$ and $s'$ are elements of $\text{End}(W)$. In particular, $s \in \text{Aut}(W)$ and $s^{-1} = s'$, then, for any $B$, we take $A = sB$, and $s: A \rightarrow B$ is a variety isomorphism with the inverse $s' = s^{-1}$. Here, $W = \text{hom}(W, W)$, and some algebra $G \in \Theta$ is presumed to be given. We cannot claim that every isomorphism, for $X$ fixed, is determined by an automorphism.

It was already noticed, and Theorem 1.7 confirms this, that varieties can be distinguished on the level of the respective algebras, and using properties of the algebras. The corresponding theorem is well-known in the classical situation; here, varieties are distinguished geometrically, too.
We now give an application of the theorem proved.

Assume that \( G \) and \( H \) are two algebras from \( \Theta \), and consider the set \( \text{Hom}(G, H) \). Suppose \( G \) is specified by generators and relations. Let \( X \) be the set of generators and \( W(X) \) the corresponding free algebra. We have the canonical surjective homomorphism \( \mu_0 : W(X) \to G \). Its kernel, \( \text{Ker} \mu_0 = T \), is regarded as a defining relation. We now pass to \( \text{Hom}(W(X), H) \) and consider here the algebraic variety \( A \) determined by the congruence \( T, A = T' \). As we know, \( A = \mu_0 \text{Hom}(G, H) \). We take, furthermore, \( \tau = (H - \text{Ker})(G) \), and consider the composition homomorphism

\[
W(X) \xrightarrow{\mu_0} G \xrightarrow{\mu_1} G/\tau,
\]

where \( \mu_1 \) is the natural homomorphism. Then \( T'' = \text{Ker} \mu_0 \mu_1 = \mu_0^{-1}(\tau) \), and \( W(X)/T'' \) is an algebra isomorphic to \( G/\tau \).

Now suppose that the algebra \( G \) is specified by generators and relations in two different ways. Let \( Y \) be the new system of generators, let \( W(Y) \) be the free algebra corresponding to it, and let \( \mu_1 : W(Y) \to G \) be the canonical homomorphism with the kernel \( T_1 = \text{Ker} \mu_1 \). We pass to \( \text{Hom}(W(Y), H) \) and take here \( B = T'_1 \). Just as above, \( T''_1 \) is the kernel of the composition homomorphism \( W(Y) \xrightarrow{\mu_1} G \xrightarrow{\mu'_1} G/\tau \), and the algebra \( W(Y)/T''_1 \) is isomorphic to \( G/\tau \). The algebras \( W(X)/T'' \) and \( W(Y)/T''_1 \) are isomorphic; hence, so are the varieties \( A \) and \( B \).

We have proved

1.9. Proposition. Suppose that, in \( \Theta \), given are algebras \( G \) and \( H \), and that \( G \) is specified by generators and relations in different ways. Then the respective algebraic varieties related with \( H \) are isomorphic.

It also follows from the above that if \( T_1 \) is a congruence on \( W(X) \) and \( T_2 \) is a congruence on \( W(Y) \), and if the algebras \( W(X)/T_1 \) and \( W(Y)/T_2 \) are isomorphic, then, for a given \( G \), the algebras \( W(X)/T_1'' \) and \( W(Y)/T_2'' \) are isomorphic. Isomorphic are also the varieties \( T_1' \) and \( T_2' \).

2. Changing \( \Theta \). Let us consider the situation when some subvariety \( \Theta_0 \) containing an algebra \( G \) is picked out of the given \( \Theta \). We are interested in connections between geometries for \( G \) relatively to \( \Theta \) and \( \Theta_0 \). We count the set of variables \( X \) fixed, and denote the corresponding free algebras over \( X \) by \( W \) and \( W_0 \), respectively. To the variety \( \Theta_0 \) there is the verbal congruence \( T_0 \) on \( W \), and we have the natural epimorphism \( \mu_0 : W \to W_0 \) with the kernel \( T_0 \).

Assume, furthermore, that \( A_0 \) is a subset of \( \text{Hom}(W_0, G) \), and take \( A = \mu_0 A_0 = \{ \mu = \mu_0 \nu, \ \nu \in A_0 \} \). Of course, the passage \( \nu \mapsto \mu \) determines a bijection \( A_0 \to A \).

Every element \( \mu \in \text{Hom}(W, G) \) is uniquely presented in the form \( \mu = \mu_0 \nu, \ \nu \in \text{Hom}(W_0, G) \); therefore, if \( A \) is a subset of \( \text{Hom}(W, G) \), then \( A = \mu_0 A_0, \ A_0 \subset \text{Hom}(W_0, G) \).

We aim to demonstrate that algebraic varieties here are linked with algebraic varieties.

Every algebraic variety \( A \) can be presented as \( A = T' \), where \( T \) is a congruence on \( W \) including the congruence \( T_0 \). \( T \) leads to the congruence \( T/T_0 \) on \( W_0 \). The passage \( T \mapsto T/T_0 \) is a bijection between the congruences on \( W_0 \) and those on \( W \) including \( T_0 \). Moreover, \( wTw' \) holds if and only if so does \( w^\mu (T/T_0) w'^\mu \).

2.1. Proposition. The following relationship always holds:

\[
T' = \mu_0(T/T_0)'.
\]

Proof. Let \( \mu \in A = T', \mu = \mu_0 \nu \). We have to check that \( \nu \in A_0 = (T/T_0)' \). Assume that \( w^\mu (T/T_0) w'^\mu \). Then \( wTw' \) and \( w^\mu = w'^\mu \) as well. This gives \( w^{\mu_0 \nu} = w^{\mu_0 \nu}, (w^{\mu_0 \nu})^\nu = (w^{\mu_0})^\nu \) and \( \nu \in A_0 \).

Conversely, if \( \nu \in A_0 \) and \( wTw' \) holds, then also \( w^\mu_0 (T/T_0) w'^\mu_0 \) and \( w^{\mu_0 \nu} = w^{\mu_0 \nu}, \ w^\mu = w'^\mu, \ T \subset \text{Ker} \mu, \ \mu \in A \). The proposition is proved. \( \square \)
Proof. Let $T = A'$. By the definition, $T = \bigcap_{w \in A} \ker \mu$. Since always $T_0 \subset \ker \mu$, we conclude that $T_0 \subset T$; so the quotient $T/T_0$ makes sense. We need to check that $A'_0 = T/T_0$. Assume that $w^{\mu_0}(T/T_0) w^{\mu_0}$. Then also $wT^T w'$, and for every $\mu \in A$ such that $\mu = \mu_0\nu$ with $\nu \in A_0$, we have $w^\mu = w^\mu$, $w^{\mu_0\nu} = w^{\mu_0\nu}$ and $(w^{\mu_0}, w^{\mu_0}) \in \ker \nu$. This holds for any $\nu \in A_0$; therefore, $(w^{\mu_0}, w^{\mu_0}) \in A'_0$.

If, conversely, $(w^{\mu_0}, w^{\mu_0}) \in A'_0$ holds, then $w^{\mu_0\nu} = w^{\mu_0\nu}$ for every $\nu \in A_1$. But then, for every $\mu \in A$, $w^\mu = w^\mu_0 \ (w, w') \in T$. From here, $(w^{\mu_0}, w^{\mu_0}) \in T/T_0$. □

We now want to link $A''$ with $A'_0$, and $T''$ with $(T/T_0)'''$.

2.3. Proposition. For $T \supset T_0$,

\[ T''/T_0 = (T/T_0)''' \]

Proof. Let $A = T'$, and assume that $A = \mu_0 A_0$. Then $A_0 = (T/T_0)'$, $T'' = A'$ and $A'/T_0 = A'_0$. Therefrom, $T''/T_0 = (T/T_0)'''$. □

2.4. Proposition. If $A = \mu_0 A_0$, then

\[ A'' = \mu_0 A'_0 \]

Proof. Clearly, $A'_0 = A'/T_0$. Let $A' = T$; then $A'' = T' = \mu_0(T/T_0)' = \mu_0(A'/T_0)' = \mu_0(A'_0)' = \mu_0 A'_0$. □

We now want to prove that the property of two algebras to be equivalent do not depend geometrically from the variety which they belong to.

2.5. Proposition. Suppose that $G_1$ and $G_2$ are two algebras in $\Theta$ and that they both belong to a subvariety $\Theta_0$. For given $X$, the algebras are equivalent in $\Theta$ if and only if they are equivalent in $\Theta_0$.

Proof. We proceed from the free algebras $W = W(X)$ and $W_0 = W_0(X)$. Assume that algebras $G_1$ and $G_2$ are equivalent in $\Theta_0$. We need to check that they are equivalent in $\Theta$. It suffices to consider a congruence $T$ on $W$ that includes $T_0$. Then $(T/T_0)''_{G_1} = (T/T_0)''_{G_2}$. It follows that $T''_{G_1}/T_0 = T''_{G_2}/T_0$. Hence, $T''_{G_1} = T''_{G_2}$, and $G_1$ and $G_2$ are equivalent in $\Theta$.

Now assume that $G_1$ and $G_2$ are equivalent in $\Theta$. We shall check their equivalence in $\Theta_0$.

We present any congruence on $W_0$ as $T/T_0$, where $T$ is a congruence on $W$ including $T_0$. We have to prove that

\[ (T/T_0)''_{G_1} = (T/T_0)''_{G_2} \]

By the assumption, $T''_{G_1} = T''_{G_2}$, and then $T''_{G_1}/T_0 = T''_{G_2}/T_0$. The needed equality now follows; hence, $G_1$ and $G_2$ are equivalent in $\Theta$. □
We could make use of this observation, for example, as follows. Let \( \Theta \) be the variety of groups, let \( G_1 \) and \( G_2 \) be commutative groups, and suppose we want to know if they are equivalent. It is sufficient to proceed from the variety \( \Theta_0 \) of commutative groups.

Under the assumption that \( X \) is infinite, the test on equivalence runs, in general, as follows. Given \( G_1 \) and \( G_2 \), we find \( \text{Var}G_1 \) and \( \text{Var}G_2 \). If the varieties are distinct, then \( G_1 \) and \( G_2 \) are not equivalent. Otherwise, let \( \Theta_0 = \text{Var}G_1 = \text{Var}G_2 \). The further checking is fulfilled in \( \Theta_0 \).

Our concern is, furthermore, with the following problem. Given are \( \Theta \) and \( G \). We look for conditions under which an algebraic variety \( A = T' \) can be specified by a finite \( T \) or, alternatively, when are all \( A \) for a given \( G \) finitely based. The answer depends, generally, on \( \Theta \). For this reason, it would be better to put the question still another way. Suppose that \( \Theta_0 = \text{Var}G \). Then every \( A \) can be presented as \( A = \mu A_0 \), where \( A_0 \) is a variety in the corresponding \( \text{Hom}(W_0,G) \). \( A \) being finitely based means here that this \( A_0 \) is finitely based.

There are several problems that deserve attention.

3. Changing the algebra \( G \). We are now interested in the following problem: the congruence \( T \) on \( W(X) = W \) being fixed, what are the connections between algebraic varieties for the algebra \( G \), its subalgebras, and its homomorphic images.

It was already noted that if \( \text{Hom}(W,H) \) is considered to be a subset of \( \text{Hom}(W,G) \) whenever \( H \) is a subalgebra of \( G \), then for every \( T \), \( T''_H = T''_G \cap \text{Hom}(W,H) \). Now, what are connections between \( T''_H \) and \( T''_G \)?

3.1. Proposition. Suppose that \( T'_G = T'_H \cup B \) and that the intersection of \( B \) and \( T'_H \) is empty. Then

\[
T''_G = T''_H \cap B'.
\]

Proof. Clearly, \( T''_H = \bigcap_{\nu \in T'_H} \text{Ker} \nu, T''_G = \bigcap_{\mu \in T'_G} \text{Ker} \mu \), and what is needed follows. \( \square \)

In particular, \( T''_G \supset T''_H \), and if \( T''_H = T \), then \( T''_G = T \).

Obviously, every algebraic variety \( A \) in \( \text{Hom}(W,H) \) can be presented as \( A = B \cap \text{Hom}(W,H) \), where \( B \) is a variety in \( \text{Hom}(W,G) \). If \( A \) is not a variety, then \( A'' = (A')_G \cap \text{Hom}(W,H) \).

Now assume that we are given a surjective homomorphism \( \delta: G \to H \) and a congruence \( T \) on \( W \). Then we also have \( \delta: \text{Hom}(W,G) \to \text{Hom}(W,H) \), \( \delta(\mu) = \mu \delta, \mu \in \text{Hom}(W,G) \). This mapping is surjective.

Take \( T' = A \) and \( T' = B \). Also, the mappings \( \delta \) and \( \delta^* \) will be needed. We remind their definitions. The first mapping takes a subset of \( \text{Hom}(W,H) \) into its \( \delta \)-preimage in \( \text{Hom}(W,G) \), while the other one acts into the opposite direction and gives the images. We shall concern with the sets \( \delta(A) \) and \( \delta^*(B) \).

We immediately obtain the inclusions \( B \subseteq \delta(A) \) and \( \delta^*(B) \subseteq A \), and they both are strict. A natural question arises concerning the varieties \( \langle \delta(A) \rangle_H \) and \( \langle \delta^*(B) \rangle_H \). Since \( A \) is a variety, \( \langle \delta^*(B) \rangle_H \subseteq A \). Generally, this inclusion is also strict. Probably, there is nothing of interest that could be added to it in the general case.

The main difficulty is that there is no method which could enable one to build up, from a given congruence \( T \) and a homomorphism \( \delta: G \to H \), new congruences only depending on \( T \) and \( \delta \) and making it possible to compute the corresponding varieties.
4. The category of varieties. Our aim here is to precise the definition of the category of algebraic varieties relatively to given $\Theta$ and $G \in \Theta$. We shall vary the set of variables $X$ and deal with various algebras $W(X)$. The question is on equational varieties.

Assume that $A = T_1'$ and $B = T_2'$ are varieties in Hom($W(X), G$) and Hom($W(Y), G$), respectively. The morphism $\alpha: A \rightarrow B$ is now interpreted as a regular mapping. This is a mapping for which there is a homomorphism $s: W(Y) \rightarrow W(X)$ such that $\alpha(\mu) = \mu s$ for all $\mu \in A$. This $s$ need not to be uniquely determined by $\alpha$.

**4.1. Proposition.** The equality $\mu s = \mu s'$ holds for all $\mu \in A$ if and only if $s$ and $s'$ induce the same homomorphism

$$\sigma: W(Y)/B' \rightarrow W(X)/A'. $$

**Proof.** Assume that $\mu s = \mu s' \in B$. Then we have the diagrams

$$\begin{array}{ccc}
W(Y) & \xrightarrow{s} & W(X) \\
\sigma_1 & & \sigma_0 \\
\downarrow & & \downarrow \\
W(Y)/T'_2 & \xrightarrow{s} & W(X)/T'_1 \\
\sigma_1 & & \sigma_0 \\
\downarrow & & \downarrow \\
W(Y)/T''_2 & \xrightarrow{s'} & W(X)/T''_1 \\
\end{array}$$

We shall prove that, under the assumption, $\tilde{s} = \tilde{s}'$. Suppose that $w$ is any element of $W(Y)$. Let us take $w^{s_1 s} = w^{s_0}$ and $w^{s_0} = w^{s'} s_0$, and check that these elements coincide. This means that $(w'^{s_1})^{(s_0)} = (w'^{s_0})^{(s_0)}$ whenever $\mu \in A$. Since $\mu s = \mu s'$, the latter equality holds. We have proved a half of the proposition.

To prove the converse, assume that $\tilde{s} = \tilde{s}'$. Then $w^{s_1 s} = w^{s_0} = w^{s_0} = w^{s'} s_0$, and $(w'^{s_1})^{(s_0)} = (w'^{s_0})^{(s_0)}$. Thus $(\mu s)(w) = (\mu s')(w)$ for every $\mu \in A$. This goes for any $w \in W(Y)$, whence $\mu s = \mu s'$. $\square$

We see that the homomorphism $\sigma: W(Y)/B' \rightarrow W(X)/A'$ corresponds to any regular mapping $\alpha: A \rightarrow B$ in a one-to-one manner.

Now assume that we are given regular mappings $\alpha: A \rightarrow B$ and $\beta: B \rightarrow C$, $\alpha(\mu) = \mu s_1$, $\mu \in A$ and $\beta(\nu) = \nu s_2$, $\nu \in B$. Then $\beta(\alpha(\mu)) = (\alpha \beta)(\mu) = \mu s_1 s_2$. So $\alpha \beta$ is a regular mapping determined by the product $s_1 s_2$. The unit map $\varepsilon: A \rightarrow A$ is given by the unit of End$W(X)$.

In this way we arrive at the category of algebraic varieties, which we denote by $K_G$. The objects of the category are the algebraic varieties, and morphisms of $K_G$ are regular mappings. The passage from every $A$ to the respective the algebra $W(X)/A'$ in $\Theta$ is a contravariant functor from $K_G$ to the category $\Theta$.

The category $K_G$ provides the endomorphism semigroup End$A$ and the automorphism group Aut $A$ of every object $A$. The group Aut $A$ is anti-isomorphic to Aut$(W/A')$, and End$A$ is anti-isomorphic to End$(W/A')$.

Now suppose that two algebras, $G_1$ and $G_2$, are given, and take the related categories $K_1 = K_{G_1}$ and $K_2 = K_{G_2}$.
Let $A$ be a variety in $K_1$ with a definite set of variables $X$, and let $A = T^ω_{G_1}$. We assume that the congruence $T$ is closed under $G_1$, $T^ω_{G_1} = T$, and set $F(A) = T^ω_{G_1}$. Furthermore, we take a morphism $α: A \to B$ from $K_1$ and suppose that it is produced by some $s: W(Y) \to W(X)$. We also suppose that the variety $B$ is related with $Y$: $B = T^α_{G_1}$. The congruence $T_1$ here is also assumed to be $G_1$-closed. Since $α: A \to B$ is a morphism, $T_1$ and $T$ are connected as follows: $wT_1w'$ implies $s(w)Ts(w')$. Now let $μ ∈ F(A) = T^ω_{G_2}$. Then $μs(w) = μs(w')$ if $wT_1w'$. This means that $μs ∈ T^α_{G_2} = F(B)$. Thus, at the same time, we have the morphism

$$s: F(A) \to F(B).$$

This way, a functor $F: K_1 \to K_2$ is defined.

In what follows, we deal with the situation when all the sets $X$ are finite.

4.2. Theorem. If the algebras $G_1$ and $G_2$ are equivalent with respect to all $X$, then the categories $K_1$ and $K_2$ are equivalent.

Proof. The function $F: K_1 \to K_2$ is defined just as above. Given any object $A$, we take $A' = T$, and also $A'' = A = T^ω_{G_1}$ and $F(A) = T^ω_{G_2}$. Here, the congruence $T$ is both $G_1$-closed and $G_2$-closed.

Now if we are given a morphism $α: A \to B$ determined by some $s: W(Y) \to W(X)$, then $s$ yields also the morphism $F(α): F(A) \to F(B)$.

Likewise, $F': K_2 \to K_1$ is constructed.

Eventually, this way we obtain an equivalence between categories. Indeed, let $A$ be an object from $K_1$. Then $A = T^α_{G_1} = T'$, where $T$ is a simultaneously $G_1$-closed and $G_2$-closed congruence. Also, $F(A) = T^ω_{G_2}$ and, further, $F'(F(A)) = F'((T^ω_{G_2})) = T^ω_{G_1} = T' = A$. Now assume that the morphism $α: A \to B$ is given by some $s: W(Y) \to W(X)$. This $s$ yields the morphisms $F(α): F(A) \to F(B)$ and $F'F(α): A \to B$. Since $s$ gives both $α$ and $F'F(α)$, we conclude that $F'F(α) = α$. 

4.3. Problem. Is the converse true, i.e. does equivalence of the categories $K_1$ and $K_2$ imply that the respective algebras are equivalent? Is there any necessity for introducing a new notion of equivalence for algebras via equivalence of these categories?

§5. ALGEBRAIC LOGIC AND ALGEBRAIC VARIETIES

1. Basic concepts. We will generalize here the notion of an algebraic variety. Need for such a generalization already appeared when the sum of two varieties was dealt with. The sum cannot be given by means of equational logic, and we are going to generalize the very notion of an equation, and that of a solution as well.

At given variety $Θ$, the corresponding Halmos algebra $U$ is considered instead of the free algebra $W = W(X)$; naturally, we regard the set of variables, $X$, to be infinite. Of course, $U$ is an algebra with equalities. As to the collection $Φ$ of relation symbols, it may be either empty or nonempty. If $Φ$ is empty, we speak merely of algebras, while in the case $Φ$ is nonempty we are dealing with models. A model is of the form $(G, Φ, f)$ where $G ∈ Θ$ and $f$ is a state realizing $Φ$ in $G$. Every such a model determines a homomorphism $f: U \to V_G$. If $u ∈ U$, we also write $f(u) = f * u$ and consider this element of $V$ as a subset of $\text{Hom}(W, G)$, i.e. of the same ”affine space". If $Φ$ is empty, we write $f = f_G$.

We now consider a formula $u$ as an equation; a formula of kind $w ≡ w'$ is an equation of a special form. The point $μ ∈ \text{Hom}(W, G)$ is a solution of the ”equation" $u$ in a model $(G, Φ, f)$ if $μ ∈ f * u$. This definition conforms well with what we saw in the case of equations of kind $w ≡ w'$, and such a generalization is useful also in classical geometry.
We now can consider \( f \ast u \) as the algebraic variety related with the model given and determined by the formula \( u \). In a database, \( f \ast u \) is the answer to the query \( u \) at the state \( f \). Therefore, the answer to a query can be treated as an algebraic variety.

In general, such a generalized algebraic variety in \( \text{Hom}(W, G) \) is given by a collection \( T \) of formulas from \( U \). In a database context \( T \) can be thought to be a system on queries. The common reply to the system is the corresponding variety. Again, a Galois correspondence can be established between such collections \( T \) and subsets of \( \text{Hom}(W, G) \), and it is in agreement with what was said earlier. The connection is set up as follows.

If \( T \) is a set of formulas (elements of \( U \)) and \((G, \Phi, f)\) is a model, \( G \in \Theta \), then we let
\[
T' = \bigcap_{u \in T} (f \ast u).
\]
Here, all the \( f \ast u \) are subsets of \( \text{Hom}(W, G) \), and \( T' \) is a subset of \( \text{Hom}(W, G) \).

If, on the other hand, \( A \) is a subset of \( \text{Hom}(W, G) \), then
\[
A' = \{ u, \ A \subset f \ast u \}.
\]

As mentioned above, this definition agrees with what appears in the case of equational logic. Let us explain this point.

First of all, we remind that if \( w \equiv w' \) is an equality, then \( \mu \in f_G \ast (w \equiv w') \iff w^\mu = w'^\mu \). Hence,
\[
\mu \in f_G \ast (w \equiv w') \iff (w, w') \in \text{Ker} \mu.
\]
Now assume that all the formulas \( u \in T \) are equalities. We shall check that
\[
\bigcap_{w \equiv w' \in T} f \ast (w \equiv w') = \{ \mu, \ T \subset \text{Ker} \mu \}.
\]
If \( \mu \in \bigcap f \ast (w \equiv w') \), then \((w, w') \in \text{Ker} \mu \) for every \( w \equiv w' \in T \), and \( T \subset \text{Ker} \mu \). If, conversely, \( T \subset \text{Ker} \mu \), then \( w \equiv w' \in T \) implies \( w^\mu = w'^\mu \), and \( \mu \in f \ast (w \equiv w') \). This holds for all \( w \equiv w' \in T \).

Now assume that, for a given \( w \equiv w' \), \( A \) is a subset of \( f \ast (w \equiv w') \). This means that \( w^\mu = w'^\mu \) for every \( \mu \in A \), i.e. \( (w, w') \in \bigcap_{\mu \in A} \text{Ker} \mu \). If, on the other hand, \((w, w') \in \bigcap_{\mu \in A} \text{Ker} \mu \), then, for every \( \mu \in A \), \( w^\mu = w'^\mu \), \( \mu \in f \ast (w \equiv w') \), and furthermore, \( A \subset f \ast (w \equiv w') \).

It also is easily seen that the passages \( T \to A = T' \) and \( A \to T = A' \) really give a Galois correspondence. Obviously, \( T_1 \subset T_2 \) implies \( T'_2 \subset T'_1 \) and \( A_1 \subset A_2 \) implies \( A'_2 \subset A'_1 \). Let us verify that \( A \subset A'' \) and \( T \subset T'' \).

We know that \( A'' = (A')' = \bigcap_{u \in A} f \ast u \). Let \( \mu \in A \), and let \( u \in A' \), \( A \subset f \ast u \). Then \( \mu \in f \ast u \), and this is the case for all \( u \in A \) and \( \mu \in A'' \).

Furthermore, \( T'' = (T')' = \{ u, \ T' \subset f \ast u \} \). Here, \( T' = \bigcap_{u \in T} f \ast u \), and then \( T' \subset f \ast u \) for every \( u \in T \). Hence, \( u \in T \) implies \( u \in T'' \).

We now call a subset \( A \subset \text{Hom}(W, G) \) an algebraic variety for the model \((G, \Phi, f)\) if \( A = T' \) for some \( T \subset U \). \( A \) is an algebraic variety if an only if \( A = A'' \).

It follows immediately from the definitions that, for every \( A \subset \text{Hom}(W, G) \), we formerly had a congruence \( A' = T \) of the free algebra \( W \) that depended on the algebra \( G \). Here we have a Boolean filter \( A' = T \) of the Halmos algebra.
$U$ that depends, in general, on the model $(G,\Phi,f)$. Clearly, the congruence $T$ can also be presented in the Halmos algebra $U$.

Let us consider separately the case when $A$ is all the set $\text{Hom}(W,G)$. Then the congruence $T = A'$ is the equational theory $T(G)$ of the algebra $G$. If, furthermore, $T = A' = \{u, A \subset f * u\}$, then for $A = \text{Hom}(W,G)$ we obtain that $f * u = \text{Hom}(W,G)$, and $T = \text{Ker} \hat{f}$, $T = T(G,\Phi,f)$ is the elementary theory of the model under consideration. Here $T$ is a filter of the Halmos algebra $U$. A question naturally arises, for which $A$ the respective $T = A'$ is a filter of $U$. The condition looks as $A \subset f * u \Rightarrow A \subset f * \forall(x)u$, and the right hand inclusion is equivalent to $A \subset \forall(x)(f * u)$.

1.1. Proposition. If for some set $A$ the corresponding $T = A'$ is a filter of the Halmos algebra $U$, then either $T$ is an improper filter or $T = T(G,\Phi,f)$.

Proof. Assume first that $A$ is empty. Then $A \subset f * u$ for all $u$, and $T = A' = U$ is an improper filter, which contains no model.

Now assume that $A$ is nonempty, $A \subset f * u$, and $T = A'$ is a filter. Then $\forall(x)u \in T$ and $A \subset \forall(x)(f * u)$. If $f * u$ is a proper subset of $\text{Hom}(W,G)$, then the set $\forall(x)(f * u)$ is empty. This contradicts the assumption. Consequently, $u \in \text{Ker} \hat{f}$ for every $u \in T$, and $T$ is included in the elementary theory of the model $(G,\Phi,f)$. On the other hand, for every $A$, the elementary theory of the model is included in $A' = T$. Thus, if $A$ is nonempty and $A' = T$ is a filter, then $T = T(G,\Phi,f)$.  

We already know that it is the case if $A = \text{Hom}(W,G)$. So, one thing more that we have to comprehend is wether it is possible when $A$ is a proper subset of $\text{Hom}(W,G)$.

Now assume that $T$ is a filter of the Halmos algebra $U$.

1.2. Proposition. Either $T'$ is empty or $T' = \text{Hom}(W,G)$.

Proof. Clearly, $T' = \bigcap_{u \in T} f * u$. Hence, if $f * u$ is empty for some $u$, then so is $T'$. If $f * u$ is a proper subset of $\text{Hom}(W,G)$, then $\forall(x)(f * u)$ is empty, and $f * \forall(x)u$ is empty as well. But $\forall(x)u \in T$, hence, $T'$ is empty. Consequently, if $T'$ is nonempty, then $f * u = \text{Hom}(W,G)$ for all $u \in T$, and $T' = \text{Hom}(W,G)$.  

Now let $A$ be nonempty subset of $\text{Hom}(W,G)$ such that $T = A'$ is a filter. Then $A'' = \text{Hom}(W,G)$. In particular, if $A$ is a proper algebraic variety, then $T = A'$ is not a filter of the Halmos algebra $U$.

Also, the following notes are self-evident.

If $T$ only contains the zero, then the set $T'$ is empty. If the unit is the single element of $T$, then $T' = \text{Hom}(W,G)$. For $T$ empty, again $T' = \text{Hom}(W,G)$, and if $T = U$, then $T'$ is empty. The case when $T$ consists of equalities was considered earlier.

Clearly, the intersection of varieties is a variety, and for every $A$, $A''$ is the intersection off all varieties including $A$.

1.3. Proposition. The sum of a finite number of the varieties is also a variety.

Proof. It suffices to deal with two summands. Suppose that $A = T'_1$ and $B = T'_2$, and let $T = T_1 \lor T_2$ be the set of formulas $u \lor v$ with $u \in T_1$ and $v \in T_2$. Then

$$T' = \bigcap_{u \in T_1, v \in T_2} f * (u \lor v) = \bigcap_{u \in T_1 \lor v \in T_2} ((f * u) \lor (f * v)) = \bigcap_{u \in T_1} (f * u) \lor \bigcap_{v \in T_2} (f * v) = T'_1 \lor T'_2 = A \lor B.$$
So, $A \cup B = T'$ is an algebraic variety. \hfill \square

We note in addition that if $A = T'$ and $T$ is finite, say, $T = \{u_1, \ldots, u_n \}$, then $A$ can be given by the formula $u = u_1 \land \cdots \land u_n$. In this case, the set $\neg A$ is also a variety, and is determined by the formula $\neg u$. In general, the complement of a variety need not be a variety itself. This, particularly, means that the system of all varieties for a given model $(G, \Phi, f)$ may be not a subalgebra of the Boolean algebra $\mathfrak{M}_G$ of all subsets of $\text{Hom}(W; G)$.

On the other hand, the following proposition holds.

1.4. Proposition. If $A = T'$ is an algebraic variety, then for every $s \in \text{End}_W$, the set $sA$ is also an algebraic variety.

Proof. Assume that $sT = \{su, \ u \in T\}$. We shall verify that 
\[(sT)' = sT'.\]
Assume $\mu \in sT'$. Then $\mu s \in T' = \bigcap_{u \in T}(f * u)$. For every $u \in T$, $\mu s \in f * u$ and $\mu s = f * su$. Here, $su$ is any element of $sT$, and $\mu \in (sT)'$. Now assume that $\mu \in (sT)'$, $\mu \in \bigcap_{u \in T}(f * su)$. Then for any $u \in T$, $\mu \in f * su = f(su)$ and $\mu s \in f * u$. Since it is so for all $u \in T$, we conclude that $\mu s \in T'$ and $\mu \in sT'$.

In particular, $sA$ is a variety determined by $sT$.

Assume now that $A = T'$ and that $Y$ is a subset of $X$. Let $\exists(Y)T$ stand for $\{\exists(Y)u, \ u \in T\}$. We shall verify the inclusion 
\[\exists(Y)T' \subset (\exists(Y)T').\]
Given $\mu \in \exists(Y)T'$, we choose $\nu \in T'$ so that $\mu(x) = \nu(x)$ for all $x \in Y$. For every $u \in T$, we have $\nu \in f * u$, and then $\mu \in \exists(Y)(f * u) = f * \exists(Y)u$. Since $\exists(Y)u$ is an element of $\exists(Y)T$, we conclude that $\mu \in (\exists(Y)T)'$. The needed inclusion is proved.

The converse inclusion is, in general, not true. However, if the set $T$ is finite and $A = T'$, then $\exists(Y)A$ is a variety for every $Y \subseteq X$. Indeed, if $T$ is finite, then $A = f * u$ where $u$ is a conjunction of elements of $T$. Then $\exists(Y)A = f * \exists(Y)u$.

Apparently, it is not, in general, the case that for any variety $A$ the set $\exists(Y)A$ is again a variety.

1.5. Proposition. Suppose that $A$ is a subset of $\text{Hom}(W, G)$ and that $s \in \text{Aut} W$. Then 
\[(sA)' = sA'.\]
Proof. Assume that $u \in (sA)'$. Then $sA \subseteq f * u$, $A \subseteq s^{-1}(f * u) = f * s^{-1}u$, $s^{-1}u \in A'$. Hence, $u \in sA'$. Conversely, assume that $u \in sA'$, i.e. $u = sv$ with $v \in A'$. Then $A \subseteq f * v$, $sA \subseteq s(f * v) = f * sv = f * u$, whence $u \in (sA)'$.

We note here that the inclusion $sA' \subseteq sA$ holds for arbitrary $s \in \text{End}_W$.

It follows from the proposition that if $T = A'$ is a closed collection of formulas, then so is $sT$ for every $s \in \text{Aut} W$.

Moreover, the two preceding propositions imply that, for every $s \in \text{Aut} W$ and all $A$ and $T$,
\[
(sA)'' = sA'', \quad (sT)''' = sT'''.
\]
Let again $Y \subseteq X$, and let $A$ be arbitrary. Then we have the inclusion 
\[\exists(Y)A' \subset (\exists(Y)A)'\].
Indeed, suppose that $u \in \exists(Y)A'$, $u = \exists(Y)v$, $v \in A'$, $A \subset f \ast v$. Then $\exists(Y)A \subset \exists(Y)(f \ast v) = f \ast \exists(Y)v = f \ast u$. This gives $u \in (\exists(Y)A)'$.

The converse inclusion does not hold.

$Y$ may be the whole set $X$. All formulas in $\exists(X)A'$ are closed. If $A$ is nonempty, then $\exists(X)A = \text{Hom}(W,G)$ and $(\exists(Y)A)'$ is a filter of $U$. It does not consist of closed formulas only.

Let us present some remarks concerning the action of semigroup $\text{End}G$. A more general situation involving the endomorphism semigroup of the model $(G, \Phi, f)$ could be considered. We, however, shall restrict ourselves to $\text{End}G$, and the case in point will then be empty $\Phi$, i.e. we shall only deal with algebras. Let us consider two instances.

1. Suppose $\delta \in \text{Aut}G$. There is an automorphism $\delta_*$ of the Halmos algebra $\mathfrak{M}_G$ which corresponds to this element. For every formula $u \in U$,

$$\delta_*(f_G \ast u) = f_G \ast u.$$ 

It follows here from that $\delta_*(A) = A$ for any algebraic variety $A$. This also means that $\mu \delta \in A$ whenever $\mu \in A$, i.e. every algebraic variety is invariant under the group $\text{Aut}G$.

It is not, in general, the case with arbitrary endomorphism, and the argument does not work also when $\Phi$ is nonempty.

2. Now suppose that $\delta \in \text{End}G$ and that an universal formula $u$ is of kind

$$w_1 \equiv w'_1 \lor \cdots \lor w_n \equiv w'_n,$$

i.e. is a pseudoindentity regarded as a pseudoequation. In this case the variety $f_G \ast u$ is invariant relatively to $\delta$. Any variety specified by a collection of pseudoindentities is also invariant under $\text{End}G$.

2. Generalized varieties, topology and other subjects. We shall make some notes on topology on $\text{Hom}(W,G)$ related to generalized varieties we are considering here. The model $(G, \Phi, f)$ is supposed to be fixed.

A formula $u \in U$ is said to be positive if it can be built up from the basic ones without using negations. All the other operations from the Halmos algebra signature are admitted.

We shall deal with varieties in $\text{Hom}(W,G)$ determined by collections $T$ of positive formulas. Such varieties might be termed positive. The intersection of any collection of positive varieties is again a positive variety, and so is the sum of a finite number of positive varieties. This induces a topology on $\text{Hom}(W,G)$ with positive varieties as closed sets. This topology is more refined than the defined above Zariski topology.

As to the Zariski topology, its closed sets here are exactly the algebraic varieties determined by collections of formulas which we consider to be pseudoindentities. This is a particular case of positive formulas. It is naturally here to proceed from an empty $\Phi$ and consider algebras $G \in \Theta$. In the classical case, pseudoindentities reduce to indentities.

In the rest we shall change the notation related to Galois correspondences, having in mind a subsequent comparison of them.

As in §3, we set for every $A \subset \text{Hom}(W,G)$:

$$T = A' = \bigcap_{\mu \in A} \text{Ker}\mu,$$

but we now treat $T$ as a set of equalities of the Halmos algebra $U$. Moreover, let

$$T = A^\lor = \{u \in U, A \subset f \ast u\}.$$
We presuppose here that \((G, \Phi, f)\) is a model.

In addition, we denote by \(U_0\) the set of all equalities of \(U\). Then, as we know,
\[
A' = A^\vee \cap U_0 \text{ if } \Phi \text{ is empty.}
\]

If, furthermore, \(T\) is a collection of formulas, then
\[
A = T^\vee = \bigcap_{u \in T} (f * u).
\]

If \(T\) is a set of equalities, then \(T^\vee\) coincides with \(T' = \{ \mu, \ T \subset \text{Ker} \mu \}\).

We are now interested in relationships between \(A''\) and \(A^\vee\), and between \(T''\) and \(T^\vee\), \(\Phi = \emptyset\). We always have \(A \subset A^\vee \subset A''\) and \(T \subset T'' \subset T^\vee\), provided \(T\) is a collection of equalities. Immediately,
\[
A'' = (A')' = (A^\vee \cap U_0)^\vee \supset A^\vee,
\]
and, if \(T\) is a collection of equalities, then
\[
T'' = (T')' = (T^\vee)' = (T^\vee)\bigcap U_0 = T^\vee \cap U_0.
\]

We have previously introduced the notion of geometric equivalence of two algebras \(G_1\) and \(G_2\) from \(\Theta\). Here we shall call this kind of geometric equivalence weak, and define the strong geometric equivalence by the condition: for every collection \(T\) of formulas
\[
T_{G_1}^{\vee \vee} = T_{G_2}^{\vee \vee}.
\]

The motivation for this distinction is provided by the following proposition.

2.1. Proposition. If two algebras, \(G_1\) and \(G_2\), are strongly equivalent, then they are also weakly equivalent.

Proof. Assume that \(G_1\) and \(G_2\) are strongly equivalent, and let \(T\) be a collection of equalities. Then
\[
T''_{G_1} = T_{G_1}^{\vee \vee} \cap U_0 = T_{G_2}^{\vee \vee} \cap U_0 = T''_{G_2},
\]
and \(G_1\) and \(G_2\) are weakly equivalent. \(\square\)

We now shall consider the case \(A\), the set of points, only consists of one point \(\mu\): \(A = \{ \mu \}\), and find \(A^\vee\) and \(A^{\vee \vee}\)-closures of the point.

First of all, we shall show that \(A^\vee\) is an ultrafilter of the Boolean algebra \(U\). We know that \(A^\vee\) is a filter, and we have to check that either \(u \in A^\vee\) or \(\neg u \in A^\vee\) for every \(u \in U\).

Clearly,
\[
(f * u) \cup \neg(f * u) = \text{Hom}(W, G).
\]

If \(\mu \in f * u\), then \(u \in A^\vee\). If \(\mu \in \neg(f * u) = f * \neg u\), then \(\neg u \in A^\vee\). Therefore, the filter \(A^\vee\) is an ultrafilter for any singleton \(A\).

We move to \(A^{\vee \vee} = \bigcap_{u \in A^\vee} (f * u)\).

As \(A^{\vee \vee} \subset A''\), for every \(\nu \in A^{\vee \vee}\), \(\text{Ker} \mu \subset \text{Ker} \nu\). Now assume that we have passed from the variety \(\Theta\) to the variety \(\Theta''\) of \(G\)-algebras and, consequently, from the Halmos algebra \(U\) to \(U''\). Then the point \(\mu\) is weakly closed: \(A = A''\) implies \(A^{\vee \vee} = A\).

Let us, furthermore, consider the following question: given a filter of Halmos algebra \(U\), what can we say about its closure \(T^{\vee \vee}\)?

2.2. Proposition. If \(T\) is a filter, then either \(T^{\vee \vee} = U\) or \(T^{\vee \vee} = T(G, \Phi, f)\).
Proof. Assume that $T$ is a filter of the Halmos algebra $U$. Accordingly to Proposition 1.2, the variety $T^\vee$ is either empty or equal to $\text{Hom}(W,G)$. In the first case $T^\vee = U$, in the second - $T^\vee$ is $T(G,\Phi,f)$ and, furthermore, $T \subset T(G,\Phi,f)$. So if the inclusion does not hold, then $T^\vee = U$. 

Hence, if the question is about bijection between algebraic varieties for a given model $(G,\Phi,f)$ and closed collections $T$ of formulas, the unique filter occurring here is the elementary theory $T(G,\Phi,f)$.

2.3. Problem. Assume that $H$ is a subalgebra of $G$ and $(H,\Phi,f_H)$ is the model induced by $(G,\Phi,f)$. Consider $\text{Hom}(W,H)$ as a subset $A$ of $\text{Hom}(W,G)$. Investigate the connection between the Boolean filter $A^\vee$ and the filter (elementary theory) $T(H,\Phi,f_H)$. Compare here $A^\vee$ with $T^\vee(H,\Phi,f_H)$ and find $T^\vee(H,\Phi,f_H)$.

This is a general problem, but it can also be considered in connection with various special situations.

Let us make, finally, the following obvious note:

if $T$ is an ultrafilter of a Boolean algebra $U$, then it is closed, $T^\vee = T$, if and only if

\[ \bigcap_{u \in T} (f \ast u) \] as a nonempty set.

3. Passage to submodels. We are considering models $(G,\Phi,f)$, $G \in \Theta$. A submodel of such a model looks like $(H,\Phi,f_H)$. Here, $H$ is a subalgebra of $G$ and $f_H$ is the restriction of $f$ to $H$.

We shall specify some details. The algebras are presented in the form $G = (G_i, i \in \Gamma)$ and $H = (H_i, i \in \Gamma)$, respectively. For every $i \in \Gamma$, $H_i$ is a subset of $G_i$. Now assume that $\varphi$ is a relation symbol from $\Phi$ of type $\tau = (i_1,\ldots,i_n)$. There are Cartesian products

\[ G_{i_1} \times \cdots \times G_{i_n} \quad \text{and} \quad H_{i_1} \times \cdots \times H_{i_n}, \]

and the latter is considered to be a subset of the former one. So,

\[ f_H(\varphi) = f(\varphi) \cap (H_{i_1} \times \cdots \times H_{i_n}). \]

This defines $f_H$ as a restriction of the function $f$. As usually, we also treat $\text{Hom}(W,H)$ as a subset of $\text{Hom}(W,G)$. Then, for every formula $u \in U$, $f_H \ast u$ is a subset of $\text{Hom}(W,H)$, and a subset of $\text{Hom}(W,G)$ as well.

We now shall demonstrate that the following holds for $u \in U$ a basic formula:

\[ f_H \ast u = (f \ast u) \cap \text{Hom}(W,H). \]

Assume that $u = \varphi(x_1,\ldots,x_n)$, $\varphi \in \Phi$, $\tau = \tau(\varphi) = (i_1,\ldots,i_n)$. Then $\mu \in f \ast u$ if and only if

\[ (\mu(x_1),\ldots,\mu(x_n)) \in f(\varphi). \]

At the same time, $\mu \in f_H \ast u$ for $\mu : W \to H$ if and only if

\[ (\mu(x_1),\ldots,\mu(x_n)) \in f_H(\varphi) = f(\varphi) \cap (H_{i_1} \times \cdots \times H_{i_n}). \]

Now let $\mu \in f_H \ast u$. Then $(\mu(x_1),\ldots,\mu(x_n)) \in f_H(\varphi) = f(\varphi) \cap (H_{i_1} \times \cdots \times H_{i_n})$. Here $(\mu(x_1),\ldots,\mu(x_n)) \in f(\varphi)$ implies $\mu \in f \ast u$. By the definition, also $\mu \in \text{Hom}(W,H)$. Hence, $\mu \in (f \ast u) \cap \text{Hom}(W,H)$.

Let, conversely, $\mu \in ((f \ast u) \cap \text{Hom}(W,H))$. Then $(\mu(x_1),\ldots,\mu(x_n)) \in f(\varphi)$. Moreover, $\mu \in \text{Hom}(W,H)$ implies that $(\mu(x_1),\ldots,\mu(x_n)) \in H_{i_1} \times \cdots \times H_{i_n}$. Therefore, $(\mu(x_1),\ldots,\mu(x_n)) \in f(\varphi) \cap (H_{i_1} \times \cdots \times H_{i_n}) = f_H(\varphi)$, and we come to $\mu \in f_H \ast u$. We see that for a basic formula $u$ the equality

\[ f_H \ast u = (f \ast u) \cap \text{Hom}(W,H) \]

holds.
We shall call this equality the fundamental equality for the formula \( u \). We also shall deal with the inclusion

\[
f_H \ast u \subset (f \ast u) \cap \text{Hom}(W,H),
\]
called the fundamental inclusion (for \( u \)).

A formula \( u \) is said to be open, or quantifier-free, if it is built up from basic formulas without using quantifiers.

### 3.1. Proposition

The fundamental equality holds for all open formulas. The fundamental inclusion holds for all positive formulas.

**Proof.** Both the equality and the inclusion are fulfilled for basic formulas, and we shall proceed by induction. Let \( M_0 \) be the set of all formulas \( u \) for which the fundamental equality holds, and let \( M_1 \) be the set of those \( u \) with the fundamental inclusion.

We shall show that \( M_0 \) is invariant relatively to Boolean operations and the action of the semigroups \( \text{End}W \). This will mean that all open formulas belong to \( M_0 \). We shall also show that \( M_1 \) is invariant relatively to \( \lor \) and \( \land \), and relatively to \( \text{End}W \) and quantifiers. Consequently, all positive formulas belong to \( M_1 \).

Assume that \( u \in M_1 \) and \( s \in \text{End}W \). We have to prove that \( su \in M_1 \). Let \( \mu \in f_H \ast su \). Then \( \mu \in s(f_H \ast u) \), \( \mu s \in f_H \ast u \subset (f \ast u) \cap \text{Hom}(W,H) \), \( \mu \in (f \ast su) \cap \text{Hom}(W,H) \). So \( su \in M_1 \).

Now assume that \( u \in M_0 \); we shall verify that \( su \in M_0 \). As \( u \in M_1 \), the fundamental inclusion holds for \( su \). Let \( \mu \in (f \ast su) \cap \text{Hom}(W,H) \). Then \( \mu s \in (f \ast u) \cap \text{Hom}(W,H) = f_H \ast u \), \( \mu \in f_H \ast su \). For \( su \), the converse, \( su \in M_0 \), holds. Now let \( u \in M_1 \) and \( Y \subset X \). We have to check that \( \exists(Y)u \in M_1 \).

Let \( \mu \in f_H \ast \exists(Y)u = \exists(Y)(f_H \ast u) \). We can select \( \nu \in f_H \ast u \) so, that \( \mu(x) = \nu(x) \) for \( x \not\in Y \). Then \( \nu \in f \ast u \); therefore, \( \mu \in \exists(Y)(f \ast u) = f \ast \exists(Y)u \), \( \mu \in (f \ast \exists(Y)u) \cap \text{Hom}(W,H) \). Consequently, \( \exists(Y)u \in M_1 \).

Assume, furthermore, \( u \in M_0 \); we shall check that \( \neg u \in M_0 \). Let \( \mu \in f_H \ast \neg u = \neg(f_H \ast u) \). Here \( \mu : W \rightarrow H \) does not belong to \( f_H \ast u = (f \ast u) \cap \text{Hom}(W,H) \). Thus, \( \mu \in \neg(f \ast u) = f \ast \neg u \), \( \mu \in (f \ast \neg u) \cap \text{Hom}(W,H) \).

To verify the converse, let \( \mu \in (f \ast \neg u) \cap \text{Hom}(W,H) \). Then \( \mu \in \neg(f \ast u) \) and \( \mu \) does not belong to \( f \ast u \). Accordingly, \( \mu \) does not belong to \( (f \ast u) \cap \text{Hom}(W,H) = f_H \ast u \). It follows that \( \mu \in \neg(f_H \ast u) = f_H \ast \neg u \). We conclude that \( \neg u \in M_0 \).

Now assume that \( u_1, u_2 \in M_1 \). We shall show that \( u_1 \land u_2 \) and \( u_1 \lor u_2 \) also belong to \( M_1 \). We have:

\[
f_H \ast (u_1 \land u_2) =
(f_H \ast u_1) \cap (f_H \ast u_2) \subset ((f \ast u_1) \cap \text{Hom}(W,H)) \cap ((f \ast u_2) \cap \text{Hom}(W,H)) =
(f \ast u_1) \cap (f \ast u_2) \cap \text{Hom}(W,H) = f \ast (u_1 \land u_2) \cap \text{Hom}(W,H),
\]
and \( u_1 \land u_2 \in M_1 \). Likewise,

\[
f \ast (u_1 \lor u_2) \cap \text{Hom}(W,H) = ((f \ast u_1) \cup (f \ast u_2)) \cap \text{Hom}(W,H) =
((f \ast u_1) \cap \text{Hom}(W,H)) \cup ((f \ast u_2) \cap \text{Hom}(W,H)) \supset (f_H \ast u_1) \cup (f_H \ast u_2) =
f_H \ast (u_1 \lor u_2),
\]
and \( u_1 \lor u_2 \in M_1 \).

This way we can also demonstrate that \( u_1 \lor u_2 \in M_0 \) and \( u_1 \land u_2 \in M_0 \) if \( u_1, u_2 \in M_0 \). In both cases, we write equalities instead of inclusions, and what is needed follows.

The proposition is proved. \( \square \)
As to positive formulas, we note that if \( u \) is positive, then \( \neg u \) is negative and, moreover, if \( u \in M_1 \), then \( \neg u \) may do not belong to \( M_1 \). Indeed, the inclusion \( f_H * u \subseteq f * u \) changes to \( f_H * \neg u \subseteq f * \neg u \).

Let us make some notes regarding to the set \( M_0 \) in connection with quantifiers \( \exists(Y) \). We shall show that \( M_0 \) does not possess the invariance property relatively to such quantifiers.

Suppose that an algebra \( G \in \Theta \) and a subalgebra \( H \subseteq G \) are given. Let \( u \in M_0 \) be so selected that

1. \( f_H * u = \emptyset \),
2. there is \( \nu \in f * u \) such that \( \nu(x) \in H \) for some \( x \in H \).

Let \( Y = X \setminus \{x\} \). Then \( \exists(Y) f_H * u = f_H * \exists(Y) u = \emptyset \). We choose \( \mu \in \exists(Y)(f * u) = f * \exists(Y) u \) so that

\[
\mu(x) = \nu(x).
\]

Then \( \mu \in \exists(Y)(f * u) = f * \exists(Y) u \) and \( \mu \in f * \exists(Y) u \cap \text{Hom}(W,H) \). The fundamental equality is not fulfilled here, and \( \exists(Y) u \notin M_0 \).

More specifically, assume that \( \Theta \) is the variety of groups, \( G \in \Theta \) and \( H \) is the unit subgroup. Take the formula \( x \neq y \) for \( u \). Then \( u \) is open, and \( u \in M_0 \). Moreover, \( f_H * u = \emptyset \). If \( G \) is not trivial, then there is the needed \( \nu \) such that \( \nu(x) = 1 \) and \( \nu(y) \neq 1 \) for \( y \neq x \). Here \( f = f_G \), and \( \mu = \mu_0 \) is the trivial homomorphism. The necessary conditions are all fulfilled.

4. Geometry on the level of quantifier-free logic. We have considered the levels of equational logic and pseudoequational logic, as well as that of first-order logic. Now the initial logic is the logic of open formulas. We shall also make a note regarding the case of positive formulas

The Galois correspondence is assigned between subsets of \( \text{Hom}(W,G) \) (the model \((G,\Phi,f)\) is supposed to be fixed) and collections \( T \) of open formulas in a Halmos algebra \( U \). Just as above, we denote the connection by \( \triangledown \). If \( A \) is a subset of \( \text{Hom}(W,G) \), then \( T = A^{\triangledown} \) is the set of open formulas \( u \) with \( A \subseteq f * u \). If \( T \) is a set of open formulas, then \( A = T^{\triangledown} = \bigcap_{u \in T} (f * u) \). \( A^{\triangledown} \) and \( T^{\triangledown} \) are also to be considered in this sense.

Let us set \( A = \text{Hom}(W,G) \) and find \( A^{\triangledown} \). If \( u \) is an open formula with \( A \subseteq f * u \), then \( u \) belongs to the open theory of the model \((G,\Phi,f)\) which we denote here by \( T(G,\Phi,f) \). On the other hand, \( T(G,\Phi,f) \subseteq A^{\triangledown} \) for every \( A \). Therefore, \( A^{\triangledown} = T(G,\Phi,f) \) in the case under consideration.

**4.1.** Proposition. Suppose that \( A = \text{Hom}(W,H) \), where \( H \) is a subalgebra of \( G \). Then

\[
A^{\triangledown} = T(H,\Phi,f_H).
\]

**Proof.** Let \( u \in A^{\triangledown} \), \( A \subseteq f * u \). Then

\[
f_H * u = (f * u) \cap \text{Hom}(W,H) = \text{Hom}(W,H).
\]

Therefore, \( u \in T(H,\Phi,f_H) \) and, consequently, \( A^{\triangledown} \subseteq T(H,\Phi,f_H) \). On the other hand, if \( u \in T(H,\Phi,f) \), then \( f_H * u = \text{Hom}(W,H) \subseteq f * u, u \in A^{\triangledown} \). This gives us the converse inclusion. \( \square \)

**4.2.** Proposition. Under the same conditions, in positive formula logic

\[
T(H,\Phi,f_H) \subseteq A^{\triangledown}.
\]

**Proof.** Let \( u \) be a positive formula and \( f_H * u = \text{Hom}(W,H) \). Then \( A = \text{Hom}(W,H) \subseteq f * u, u \in A^{\triangledown} \). \( \square \)

Now we pass to the main results.

If \( T \) is a collection of open formulas, then \( \Theta_T \) is the class of all those models \((G,\Phi,f)\) which all formulas from \( T \) are valid in. Here, \( G \in \Theta \), the collection \( \Phi \) is fixed for the Halmos algebra \( U \), \( \Theta_T \) is an axiomatic class.

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We again choose a model \((G, \Phi, f)\). In the following theorem, which is an analogue of Theorem 3.1 from §3, the passages \(T \mapsto A = T^\forall\) and \(A \mapsto A^\forall = T\) (only open formulas from \(U\) are taken into account) are considered with respect to this model.

4.3. Theorem. Suppose that the set \(T\) is invariant relatively to the action of the semigroup \(\text{End} W\). Then

1. An element \(\mu \in \text{Hom}(W, G)\) belongs to the variety \(A = T^\forall\) if and only if the model \((H, \Phi, f_H)\) with \(H = \text{im} \mu\) belongs to \(\Theta_T\),
2. the closure \(T^{\forall\forall}\) coincides with the open theory of all \(\Theta_T\)-submodels of the model \((G, \Phi, f)\).

Proof. Assume that \(\mu \in A\) and \(H = \text{Im} \mu\) and consider the model \((H, \Phi, f_H)\). We have to demonstrate that, for every \(u \in T\), \(f_H \ast u = \text{Hom}(W, H)\). Let \(\nu \in \text{Hom}(W, H)\). Then for some \(s \in \text{End} W\) the diagram

\[
\begin{array}{ccc}
W & \xrightarrow{s} & W \\
\downarrow{\nu} & & \downarrow{\mu} \\
H & & \\
\end{array}
\]

commutes, and \(\nu = \mu s\). The statement \(\nu = \mu s \in f_H \ast u\) is equivalent to \(\mu \in f_H \ast su\).

Since the formula \(su\) is open, the later statement is equivalent to \(\mu \in f \ast su\). It follows from the conditions that \(\mu \in f \ast su\) holds, and then so do the statements \(\mu \in f_H \ast su\) and \(\nu = \mu s \in f_H \ast u\). This is the case for every \(\nu \in \text{Hom}(W, H)\), and \(f_H \ast u = \text{Hom}(W, H)\). Here \(u\) is an element of \(T\), so the model \((H, \Phi, f_H)\) belongs to \(\Theta_T\).

Now let us check the converse. Assume that, for \(\text{im} \mu = H\), the model \((H, \Phi, f_H)\) belongs to \(\Theta_T\). We have to derive that \(\mu \in A = T^\forall\).

For all \(u \in T\), \(f_H \ast u = \text{Hom}(W, H)\). As \(\mu \in \text{Hom}(W, H)\), we conclude that \(\mu \in f_H \ast u\). Consequently, \(\mu \in f \ast u\). This argument remains valid for every \(u \in T\) and \(\mu \in T^{\forall\forall} = A\).

Note that this converse condition does not depend on the assumption that \(T\) is invariant under the action of the semigroup \(\text{End} W\). We now observe that

\[A = \bigcup \text{Hom}(W, H),\]

where the union is taken over all subalgebras \(H \subseteq G\) such that the model \((H, \Phi, f_H)\) belongs to \(\Theta_T\).

Indeed, if \(\mu \in A\), then \(H = \text{im} \mu\) determines a model in \(\Theta_T\), and \(\mu \in \text{Hom}(W, H)\). On the other hand, if \(\mu \in \text{Hom}(W, H)\) with \((H, \Phi, f_H) \in \Theta_T\), then \(\mu \in f_H \ast u\), \(u \in T\) and, furthermore, \(\mu \in f \ast u\). This is so for every \(u\), and \(\mu \in A\).

The corresponding open theory will be denoted by \(\overline{T}\). We have to show that \(T^{\forall\forall} = \overline{T}\).

Let \(u \in \overline{T}\). This means that, for every model \((H, \Phi, f_H)\) from \(\Theta_T\) with \(H \subseteq G\), \(f_H \ast u = \text{Hom}(W, H)\). Now take \(\mu \in A\). Then \(\mu \in \text{Hom}(W, H) = f_H \ast u\) for an appropriate \(H\). Moreover, \(\mu \in f \ast u\). This takes place for every \(\mu \in A\), consequently \(A \subseteq f \ast u\), \(u \in T^{\forall\forall} = A^{\forall}\).

Let, conversely, \(u \in T^{\forall\forall}\). Take \(H \subseteq G\) so that \((H, \Phi, f_H) \in \Theta_T\); we have to verify that \(f_H \ast u = \text{Hom}(W, H)\). Whenever \(\mu \in \text{Hom}(W, H)\), \(\mu \in A\) and \(\mu \in f \ast u\); hence, \(\mu \in f_H \ast u\). This gives the inclusion
Hom(W, H) \subset f_H \ast u; the converse follows immediately from definitions. Therefore, \( f_H \ast u = \text{Hom}(W, H) \), indeed. All this remains valid for every appropriate \( H \) and \( u \in \overline{T} \).

The proof is completed. \( \square \)

It is readily seen that \( f \ast u = \text{Hom}(W, G) \) always implies \( f \ast su = s(f \ast u) = \text{Hom}(W, G) \). Therefore, we can maintain that \( su \in \overline{T} \) if \( u \in \overline{T} \), i.e. the class \( \overline{T} = T^\vee \), as well as \( T \), is invariant relatively to action of End\( W \).

Just as for algebras, we now shall define the notion of equivalence for models. The question is about the geometric equivalence, and it is now considered in the class of open formulas. We say that the models \( \Phi(H, f_1) \) and \( \Phi(G, f_2) \) are equivalent if the respective closures \( T^\vee \) of every collection \( T \subset U \) of open formulas for the models are equal.

One easily realizes that isomorphic models are equivalent. This seems obvious, but we shall advance a formal proof using some useful considerations.

The model isomorphism

\[
\delta: (G_1, \Phi, f_1) \to (G_2, \Phi, f_2)
\]

is, first of all, an isomorphism \( \delta: G_1 \to G_2 \) between algebras from \( \Theta \). It induces a Halmos algebra isomorphism \( \delta_*: \mathcal{M}_{G_2} \to \mathcal{M}_{G_1} \) and a bijection \( \delta_*: F_{G_2} \to F_{G_1} \) between realization systems, each realization \( f \) belonging to a database state, and in this case

\[
(f \ast u)^{\delta_*} = f^{\delta_*} \ast u, \quad f \in F_{G_2}, \ u \in U.
\]

\( \delta \) is a model isomorphism if and only if \( f_2^{\delta_*} = f_1 \).

Now suppose that \( T \subset U \) is a system of formulas, \( A = T^\vee \) in the first model and \( B = T^\vee \) in the second one. What are relations between \( A \) and \( B \)? We have \( A = \bigcap_{u \in T} f_1 \ast u \), \( B = \bigcap_{u \in T} f_2 \ast u \), and \( f_1 \ast u = f_2 \ast u = (f_2 \ast u)^{\delta_*} \). Hence \( A = B^{\delta_*} \), and \( \mu: W \to G_1 \) belongs to \( A \) if and only if \( \mu \delta = \nu: W \to G_2 \) belongs to \( B \).

Now

\[
T_1^{\vee \vee} = A^\vee = \{ u \in U, \ A \subset f_1 \ast u \},
\]

\[
T_2^{\vee \vee} = B^\vee = \{ u \in U, \ B \subset f_2 \ast u \}.
\]

If \( u \in T_1^{\vee \vee} \), then \( A \subset f_1 \ast u \), \( B^{\delta_*} \subset f_2^{\delta_*} \ast u = (f_2 \ast u)^{\delta_*} \), and \( B \subset f_2 \ast u, \ u \in T_2^{\vee \vee} \).

The converse is proved analogously, and \( T_1^{\vee \vee} = T_2^{\vee \vee} \). If we have isomorphic models, this holds for every \( T \).

We can confine ourselves here to systems of open formulas.

Just as before, the question is how to learn to recognize equivalence of two models. The next theorem is similar to Theorem 4.3 of §3.

4.4. Theorem. Suppose that models \( (G_1, \Phi, f_1) \) and \( (G_2, \Phi, f_2) \) are equivalent in the open formula geometry. Then their open theories coincide.

Proof. First of all, we shall prove the well-known fact that if \( \mathcal{X} \) is the class of models defined by a collection \( T \) of open formulas, then, for every model \( (G, \Phi, f) \in \mathcal{X} \), all submodels of it also are in \( \mathcal{X} \).

Assume that \( (H, \Phi, f_H) \) is a submodel. We have to verify that \( f_H \ast u = \text{Hom}(W, H) \) for every formula \( u \in T \). Since \( u \) is open, \( f_H \ast u = (f \ast u) \cap \text{Hom}(W, H) \). But \( f \ast u = \text{Hom}(W, G) \); so, \( f_H \ast u = \text{Hom}(W, H) \).
Now assume that $T_1$ is an open theory of the first model and $X_1$ is the class of models defined by formulas from $T_1$. Similarly, we take $T_2$ and $X_2$ for the second model. $T_1^\vee$ for the former model is $\text{Hom}(W,G)$, and $T_2^\vee$ for it is $T_1$. The same holds for the latter one; we count the models as equivalent.

Further, let $(H_\alpha, \Phi, f_\alpha)$ be the family of all those submodels of $(G_2, \Phi, f_2)$ which all formulas from $T_1$ are valid in. Then $T_1$ is the open theory of the class of these models. All models $(H_\alpha, \Phi, f_\alpha)$ belong to $X_2$. As they also belong to $X_1$, and even generate $X_1$, we obtain that $X_1 \subset X_2$. Just in the same way we conclude that $X_2 \subset X_1$. Hence the theorem.

The theorem can also be applied in the case the set $\Phi$ is empty and the question is about algebras. In this case open formulas are treated as universal, and we then speak of universal theories and universal classes of algebras.

Therefore, if two algebras are geometrically equivalent in universal logic, then they have the same universal theory. □

5. Halmos algebras and Boolean algebras of varieties. Conclusion. Assume that $H$ is a Halmos algebra and $T$ is its Boolean filter. Let $T^-$ stand for the subset of $T$ consisting of all $h$ from $T$ such that $\forall (X) h \in T$. Here, $X$ is the set of variables. It is well-known that $T^-$ is a filter of the Halmos algebra $H$. It also is known that if $T$ is an ultrafilter, then $T^-$ is a maximal filter of $H$, and the algebra $H/T^-$ is simple.

We also note that if $T_\alpha$, $\alpha \in I$, is a collection of Boolean filters of $H$, then

$$(\bigcap_\alpha T_\alpha)^- = \bigcap_\alpha (T_\alpha^-).$$

Now let $H$ be the Halmos algebra $U$. For some algebra $G \in \Theta$ and set $A \subset \text{Hom}(W,G)$, we considered in §3 the algebra $W/A'$ which can be presented as a subdirect product of all $W/\ker \mu$ with $\mu \in A$.

Assume now that a model $(G, \Phi, f)$ is considered. We shall relate a Halmos algebra to the same set $A$. Take the Boolean filter $T = A^\vee$ of the Halmos algebra $U$ and pass to $T^- = A^\vee$. The corresponding Halmos algebra is $U/A^\vee$. If, furthermore, $A = \{\mu\}$ is a singleton, then set $A^\vee = T_\mu$. As $A^\vee$ is an ultrafilter, $T_\mu$ is a maximal filter in $U$, $T_\mu \in \text{Spec } U$.

It is easily seen that $A^\vee = \bigcap_{\mu \in A} T_\mu$ for $A$ arbitrary. Indeed, we always have $A^\vee = \bigcap_{\mu \in A} \{\mu\}^\vee$. If we apply the operation $\neg$, we obtain

$$A^\vee = (\bigcap_{\mu \in A} \{\mu\}^\vee)^\neg = \bigcap_{\mu \in A} (\{\mu\}^\vee)^\neg = \bigcap_{\mu \in A} T_\mu.$$

Thus we come to

**5.1. Theorem.** For every $A$, the Halmos algebra $U/A^\vee$ is a subdirect product of all the simple algebras $U/T_\mu$ with $\mu \in A$.

One could take for $A$ the algebraic variety defined by a collection of formulas $T \subset U$: $A = T^\vee$. Then $A^\vee = T^\vee$ and $A^\vee = T^\vee$.

We do not know, however, what role the algebra $U/A^\vee$ plays for the variety $A$. This is one among a lot of questions the answer to which is to be get known.

It is not improbable that the relation algebras, a product of the categorial approach to algebraic logic (see, e.g. [22]), naturally shall find their applications in this theory along with Halmos algebras.

We note, furthermore, that to any variety $A$ a Boolean algebra $U/A^\vee$ can be related, and bring forward the following proposition in this connection.
5.2. Proposition. Every variety morphism \( s: A \to B \) induces a Boolean homomorphism 

\[ \tilde{s}: U/B^\vee \to U/A^\vee. \]

**Proof.** We first make some preliminary notes. If \( A \) is a subset of \( \text{Hom}(W, G) \), then \( As \) stands for the set of all \( \mu s, \mu \in A \). An endomorphism \( s \in \text{End}W \) determines a morphism \( A \to B \) if and only if \( As \subseteq B \).

Given a collection \( T \) of elements of \( U \), we define the set \( Ts \) by the rule \( u \in Ts \iff su \in T \). Then the following equality holds:

\[ (As)^\vee = A^\vee s. \]

Let us verify it.

Suppose that \( \in (As)^\vee \). Then \( As \subseteq f \ast u \). Furthermore, \( \mu s \in f \ast u \) whenever \( \mu \in A \). Hence, \( u \in s(f \ast u) = f \ast su, A \subseteq f \ast su, su \in A^\vee, u \in A^\vee s \).

Now let \( u \in A^\vee s, su \in A^\vee, A \subseteq f \ast su \). Then for every \( \mu \in A, \mu \in s(f \ast u) \) and \( \mu s \in f \ast u \). Therefore, \( As \subseteq f \ast u \) and \( u \in As \).

Now assume that we are given a morphism \( s: A \to B \). Then \( As \subseteq B \) and \( (As)^\vee = A^\vee s \subseteq B^\vee \). This means that \( su \in A^\vee \) if \( u \in B^\vee \).

Let us consider the composition Boolean homomorphism

\[ U \xrightarrow{\tilde{s}} U/B^\vee \xrightarrow{\tilde{\sigma}_0} U/A^\vee. \]

Since \( u \in B^\vee \) implies \( su \in A^\vee \), the filter \( B^\vee \) is included into the kernel of the homomorphism \( s\sigma_0 \). This means that \( s\sigma_0 \) induces the homomorphism \( \tilde{\sigma} \) we look for. \( \square \)

So the proposition is proved. It admits conversion. Suppose that \( B^\vee \) is included in the kernel of \( s\sigma_0 \). This means that, for every \( u \in B^\vee \), the element \( su \) belongs to \( A^\vee \), i.e. \( sB^\vee \subseteq A^\vee \). Applying \( \vee \), we get \( A^\vee = A \subseteq (sB^\vee)^\vee = sB^{\vee \vee} = sB \). What does this mean is that \( s \) specifies a morphism \( A \to B \). If, in particular, the varieties \( A \) and \( B \) are isomorphic, then so are the algebras \( U/B^\vee \) and \( U/A^\vee \).

Let us verify this. Suppose that \( \mu ss' = \mu \) for every \( \mu \in A \). We have to check that, for every \( u \in U \), the elements \( ss'u \) and \( u \) are equivalent modulo the filter \( A^\vee \), i.e.

\[ (ss'u \to u) \land (u \to ss'u) \in A^\vee \]

or, what amounts to the same,

\[ A \subseteq f \ast (ss'u \to u) \cap f \ast (u \to ss'u). \]

Take \( \mu \in A \) and let \( \mu \in f \ast ss'u \). Then \( \mu ss' = \mu \in f \ast u \). Therefore, \( A \subseteq f \ast (ss'u \to u) \). Analogously, if \( \mu = \mu ss' \in f \ast u \), then \( \mu \in ss'(f \ast u) = f \ast ss'u \). So \( A \subseteq f \ast (u \to ss'u) \). Thus, \( ss'u \) and \( u \) are equivalent.

In the same fashion we check that \( \nu = \nu ss' \) for \( \nu \in B \) implies equivalence of \( ss'u \) and \( u \) modulo \( B^\vee \). All this eventually means that the respective homomorphism \( \tilde{s}: U/B^\vee \to U/A^\vee \) and \( \tilde{s}' : U/A^\vee \to U/B^\vee \) are mutually inverse.

The converse is not generally true, for the endomorphisms of the Boolean algebra \( U \) by no means are exhausted by elements of the semigroup \( \text{End}W \). But we can, of course, confine ourselves to the isomorphisms between \( U/B^\vee \) and \( U/A^\vee \), induced by endomorphisms from \( \text{End}W \).

Our further remarks are also related with the notion of an algebraic variety morphism. The question is of varieties specified by subsets of the Halmos algebra \( U \).

If \( A \) and \( B \) are varieties in \( \text{Hom}(W, G) \) with a model \((G, \Phi, f)\), then the morphism \( A \to B \) is determined by some \( s \in \text{End}W \) such that \( \nu s \in B \) for every \( \nu \in A \). If, for example, \( A = sB \), then we have \( s: A \to B \).
Let us consider separately the case when the set $T$ is invariant under $s$, i.e. $sT \subseteq T$. Assume that $A = T^{\vee}$. Then $(sT)^{\vee} = sT^{\vee} = sA \supset T^{\vee} = A$. This means that the variety $A$ is invariant under $s$, $\nu \in A$ implies $\nu s \in A$, and we have a morphism $s : A \to A$.

If $T$ is invariant under $\text{End}W$, then so is $A = T^{\vee}$ as well, and the semigroup $\text{End}W$ acts on $A$ as an semigroup of endomorphisms of the variety.

If, on the other hand, the variety $A = T^{\vee}$ is invariant relatively to the action of $\text{End}W$, then always $sA \supset A = T^{\vee}$ and $T^{\vee} \supset (sA)^{\vee}$. However, we cannot present $(sA^{\vee})$ as $sA^{\vee} = sT^{\vee}$, so we cannot claim that the collection $T^{\vee} \cup$ is invariant relatively to the action of $\text{End}W$.

Still if the variety $A = T^{\vee}$ is invariant relatively to $\text{Aut}^{} W$, then the filter $T^{\vee \vee}$ also has the same feature.

One can consider, for any variety $A$, the semigroup $\text{End}A$ of those $s \in \text{End}W$ acting on $A$. It is a subsemigroup of $\text{End}W$ and characterizes the given $A$. $\text{Aut}^{} A$ is the symmetry group of the variety $A$. If $A$ and $B$ are isomorphic, then $\text{Aut}^{} A$ and $\text{Aut}^{} B$ are conjugated in a way. For $\Phi$ empty, and on the equational level, the groups $\text{Aut}^{} A$ and $\text{Aut}^{} (W/A')$ are well-connected; they are canonically anti-isomorphic.

As to the group $\text{Aut}^{} A$, we have to note in addition that, in fact, it is not a subgroup of $\text{Aut}^{} W$. Not every automorphism of the variety $A$ is induced by some $s \in \text{Aut}^{} W$. The subset of those $s \in \text{Aut}^{} W$ under which $A$ is invariant is a subgroup of $\text{Aut}^{} W$ and, at the same time, of $\text{Aut}^{} A$. We denote the subgroup by $\text{Aut}_0^{} A$. The initial group for all the groups $\text{Aut}_0^{} A$ was $\text{Aut}^{} W$; it acts on $\text{Hom}(W,G)$ and is well-coordinated with several topologies that have been studied. Other groups related to topologies on $\text{Hom}(W,G)$ could be taken instead of $\text{Aut}^{} W$. They might be either the topologies discussed above or any other ones. Along these lines, the geometry of varieties can be enriched.

When $\Phi$ is empty, every algebraic variety $A$ is also invariant relatively to the action of $\text{Aut}^{} G$. Actions of $\text{Aut}^{} G$ and $\text{Aut}^{} A$ on the variety $A$ determine the geometry of it in many respects.

We further make some conclusive remarks. We have discussed the notion of algebraic closeness of an algebra $G \in \Theta$ in the subsection devoted to the Hilbert theorem on zeros. The algebra $G$ is said to be algebraically closed if, for every finite $X$ and proper congruence $T$ on $W(X)$, the variety $T' = A$ in $\text{Hom}(W,G)$ is nonempty (equivalently, for every $T$ the set $\text{Hom}(W/T,G)$ is nonempty).

If $G$ has a one-element subalgebra, then we must regard $T'$ be nontrivial rather than nonempty. It seems likely that such a version of the definition of algebraic closeness for algebras deserves a special discussion.

Also, it is natural to speak of algebraic closeness for some pregiven $X$.

We now can claim that if $G_1$ and $G_2$ are algebras without one-element subalgebras, and if one of them is algebraically closed while the other is not, then the algebras are not equivalent. Indeed, suppose that $G_1$ is algebraically closed and $G_2$ is not. Then there is a proper congruence $T$ on $W(X)$ for which the variety $B = T_{G_2}$ is empty. The variety $A = T_{G_1}$ for the same $T$ is not empty. It follows therefrom, as we saw in §3, that $G_1$ and $G_2$ are not equivalent.

So, we generally cannot claim that algebraic closeness of algebras $G_1$ and $G_2$ implies their equivalence.

The claim is justified in the case of classical geometry, but not always. The algebras $G_1$ and $G_2$ can both be algebraically closed and nevertheless have distinct systems of identities. Then they are not equivalent.

Of some interest is the situation when $G_1$ and $G_2$ are algebraically closed and do have the same identities.
In the context in question, if an algebra $G$ is algebraically closed, then every algebra, containing $G$, has the same property. In particular, we consider extension $K$ of the field $P$ as algebraically closed if $K$ contains algebraic closure of the field $P$.

Now, we return to geometric equivalence of algebras. We shall assume that there is a nullary operation $0$ among the ground operations of the variety $\Theta$ which singles out of every $G \in \Theta$ a one-element subalgebra.

Let us begin with few general facts and then apply them to commutative groups.

5.3. Proposition. Suppose that $I$ is a set and every $\alpha \in I$ is assigned an algebra $H_{\alpha} = \Theta$. Then, for every $G \in \Theta$,

$$\bigcap_{\alpha} (H_{\alpha} - \text{Ker})(G) = \text{Ker}(G).$$

Proof. $\prod_{\alpha}$ is the Cartesian multiplication. Let $\tau_1$ stands for the left hand congruence and $\tau_2$–for the right hand one. We shall use standard arguments which already were in use above. Assume that $g_1$ and $g_2$ are two elements of $G$, and that $g_1 \tau_2 g_2$. This means that $g_1^\nu = g_2^\nu$ for all $\alpha \in I$ and $\nu: G \to H_{\alpha}$. Let us take $\mu: G \to \prod_{\alpha} H_{\alpha}$ and verify that $g_1^\mu = g_2^\mu$.

The equality means that $g_1^\mu(\alpha) = g_2^\mu(\alpha)$ for every $\alpha \in I$. We use the projections $\pi_{\alpha}: \prod_{\alpha} H_{\alpha} \to H_{\alpha}$ and denote $\mu \pi_{\alpha} = \nu_{\alpha}$. Then $g_1^\mu(\alpha) = g_2^\mu(\alpha)$, i.e. $g_1^\mu = g_2^\mu$ and, further, $g_1 \tau_1 g_2$.

Assume that, conversely, $g_1 \tau_1 g_2$. Given $\alpha \in I$ and $\nu: G \to H_{\alpha}$, we define $\mu$ by the rule: $g^\mu(\alpha) = g^\nu$, and $g^\mu(\beta)$ is the zero if $\beta \neq \alpha$. Then $\mu: G \to \prod_{\alpha} H_{\alpha}$ and $g_1^\mu = g_2^\mu$. But then $g_1^\mu(\alpha) = g_2^\mu(\alpha) = g_2^\nu$. Therefore, $g_1 \tau_2 g_2$.

Apart from Cartesian products $\prod_{\alpha}$, we shall also discuss direct products $\prod_{\alpha}^0$. Here $\prod_{\alpha}^0 H_{\alpha}$ consists of all those $\alpha \in \prod_{\alpha} H_{\alpha}$ that take the zero as the value almost everywhere. This is a subalgebra of $\prod_{\alpha} H_{\alpha}$.

Clearly, if $H_1$ is a subalgebra of $H_2$, then

$$(H_1 - \text{Ker})(G) \supset (H_2 - \text{Ker})(G).$$

5.4. Proposition. Always

$$\bigcap_{\alpha}^0 (H_{\alpha} - \text{Ker})(G) = \bigcap_{\alpha} (H_{\alpha} - \text{Ker})(G).$$

Proof. Of course,

$$\bigcap_{\alpha}^0 (H_{\alpha} - \text{Ker})(G) \supset \bigcap_{\alpha} (H_{\alpha} - \text{Ker})(G) = \bigcap_{\alpha} (H_{\alpha} - \text{Ker})(G).$$

On the other hand,

$$(H_{\alpha} - \text{Ker})(G) \supset \bigcap_{\alpha}^0 (H_{\alpha} - \text{Ker})(G),$$

for $H_{\alpha}$ is a subalgebra of the direct product. This is so for any $\alpha \in I$, so that

$$\prod_{\alpha}^0 ((H_{\alpha} - \text{Ker})(G)) \supset \bigcap_{\alpha}^0 (H_{\alpha} - \text{Ker})(G).$$

These two inclusions justify the needed equality.

5.5. Proposition. Suppose that algebras $H_{\alpha}$ and $H_{\alpha}'$, $\alpha \in I$, are geometrically equivalent. Then $\prod_{\alpha} H_{\alpha}$ and $\prod_{\alpha} H_{\alpha}'$ are also equivalent.
Proof. For an arbitrary $G = W/T$,

$$(\prod \alpha H_\alpha - \text{Ker})(G) = \bigcap \alpha (H_\alpha - \text{Ker})(G) = \bigcap \alpha (H'_\alpha - \text{Ker}(G)) = (\prod \alpha H'_\alpha - \text{Ker})(G).$$

□

This gives what is desired.

5.6. Proposition. Suppose that every algebra $H_\alpha$ in $\prod \alpha H_\alpha$ can be regarded as a subalgebra of one of them, call it $H$. Then $H$ and $\prod \alpha H_\alpha$ are geometrically equivalent.

Proof. $(\prod \alpha H_\alpha - \text{Ker})(G) = \bigcap \alpha (H_\alpha - \text{Ker})(G) = (H - \text{Ker})(G)$, for always $(H_\alpha - \text{Ker})(G) \supset (H - \text{Ker})(G)$. □

Now we pass to commutative groups.

5.7. Theorem. Two commutative groups, $A_1$ and $A_2$, with finite exponents are geometrically equivalent if and only if their exponents coincide or, what amounts to the same, $\text{Var} A_1 = \text{Var} A_2$.

Proof. In one direction the assertion follows from a general result in §3. We shall prove the converse.

Assume first that the groups $A_1$ and $A_2$ are primary with respect to the same $p$, and that $p^n$ is their common exponent. Both groups are direct products of cyclic ones. All these cyclic groups in $A_1$ are subgroups of some group $A^0_1$ of order $p^n$. In the same manner we isolate a group $A^0_2$ of the same order. Then

$$(A_1 - \text{Ker})(G) = (A^0_1 - \text{Ker})(G), \quad (A_2 - \text{Ker})(G) = (A^0_2 - \text{Ker})(G).$$

Since $A^0_1$ and $A^0_2$ are isomorphic,

$$(A_1 - \text{Ker})(G) = (A_2 - \text{Ker})(G),$$

so $A_1$ and $A_2$ are equivalent.

We pass to the general case. Let $A_1 = \prod \alpha A_{1,p}$ and $A_2 = \prod \alpha A_{2,p}$ be decompositions into Sylow groups. If $m$ is the common exponent of $A_1$ and $A_2$, $m = p_1^{n_1} \cdots p_k^{n_k}$, then in both cases $p$ runs over the collection $p_1, \cdots, p_k$, and the groups $A_{1,p}$ and $A_{2,p}$ are of the same exponent. Hence, they are equivalent, and so are the groups $A_1$ and $A_2$. □

It is easy to prove also the following.

1. Any two commutative torsion-free groups are $X$-equivalent for every finite $X$.

2. Two mixed finitely generated commutative groups are equivalent if and only if their periodic parts are of the same exponent.

We now turn attention to some problem that concerns the classical situation.

Consider a field $P$ and two its extensions $K_1$ and $K_2$. If both $K_1$ and $K_2$ are algebraically closed, then they are geometrically equivalent. Then they have the same equational theories. Actually, it is known that even their elementary theories coincide.

The question is whether or not $K_1$ and $K_2$ are geometrically equivalent on the universal logic level if they are algebraically closed.

Let us remind that we have proved, on the equational level, equivalence of every algebra to every of its Cartesian powers. In parallel, the following problem could be noticed: Whether or not every model
(algebra) is equivalent, already on the level of open logic, to every of its ultrapowers. We now shall
discuss a partial solution of this problem, and afterwards shall point to some applications to fields.

We begin with some preliminary notes.

Assume we are given a model \((G, \Phi, f)\), \(G \in \Theta\), a set \(I\) and an ultrafilter \(D\) on it. Let \((\bar{G}, \Phi, \bar{f})\) be the
ultrapower of the model with respect to \(D\). Here, \(\bar{G}\) is the ultrpower of the algebra \(G\), \(\bar{G} = G^I / D\), and \(\bar{f}\) is defined in a special way (see, for example, [51]). We take, furthermore, \(\text{Hom}(W, G)\) and \(\text{Hom}(W, \bar{G})\). For every formula \(u \in U\), \(f^* u \subset \text{Hom}(W, G)\) and \(\bar{f}^* u \subset \text{Hom}(W, \bar{G})\). If \(\mu: W \rightarrow \bar{G}\) and if \(\xi: G^I \rightarrow \bar{G}\) is the natural homomorphism, then we obtain the commutative diagram

\[
\begin{array}{ccc}
W & \xrightarrow{\mu} & \bar{G} \\
\downarrow{\nu} & & \downarrow{\xi} \\
G^I & & \\
\end{array}
\]

Here, in general, \(\nu\) is not uniquely determined by \(\mu\). We also consider the projections \(\pi_\alpha: G^I \rightarrow G\), and the compositions \(\nu_\alpha = \nu \pi_\alpha: W \rightarrow G\).

The following rule holds (cf. [51]):

\[\mu \in \bar{f}^* u \iff \{\alpha \in I, \nu_\alpha \in f^* u\} \in D.\]

It does not depend on the choice of \(\nu\).

5.8. Theorem. Let \(T\) be a finite subset of \(U\). Then \(T^{\check{\check{\vee}}}\) for the model \((G, \Phi, f)\) coincides with \(T^{\check{\check{\vee}}}\) for the model \((\bar{G}, \Phi, \bar{f})\).

**Proof.** We take \(T^{\check{\check{\vee}}} = B\) for the initial model and \(T^{\check{\check{\vee}}} = A\) for the ultrapower, and set \(T^{\check{\check{\vee}}} = B^\vee\) and \(T^{\check{\check{\vee}}} = A^\vee\). We then have to prove that \(A^\vee = B^\vee\).

First let \(u \in B^\vee, B \subset f^* u\). We shall verify that \(u \in A^\vee, A \subset \bar{f}^* u\), i.e. that \(\mu \in \bar{f}^* u\) whenever \(\mu \in A\). The conclusion \(\mu \in \bar{f}^* u\) means that,

\[\{\alpha \in I, \nu_\alpha \in f^* u\} \in D.\]

The condition \(\mu \in A\) means that \(\mu \in \bar{f}^* v\) for every \(v \in T\), i.e.

\[\{\beta \in I, \nu_\beta \in f^* v\} = I_{\mu, v} \in D.\]

Now set \(I_\mu = \bigcap_{v \in T} I_{\mu, v}\); then \(I_\mu \in D\). By the definition,

\[\alpha \in I_\mu \Rightarrow \nu_\alpha \in B = \bigcap_{v \in T} f^* v.\]

Moreover,

\[\{\alpha \in I, \nu_\alpha \in B\} \in D.\]

By the choice of \(B, B \subset f^* u\), consequently, \(\nu_\alpha \in B\) implies \(\nu_\alpha \in f^* u\). Also

\[\{\alpha \in I, \nu_\alpha \in f^* u\} \in D.\]

But then \(\mu \in \bar{f}^* u\). Hence, \(B^\vee \subset A^\vee\).
To prove the converse inclusion, finiteness of $T$ is not needed. Let $u \in A^\forall$. This means that $A \subset \tilde{f} \ast u$. We are going to demonstrate that $B \subset f \ast u$. Let $\nu_0 \in B$, $\nu_0 : W \to G$. We also take a constant $\nu : W \to G^I$ so that $\nu_\alpha = \nu \pi_\alpha = \nu_0$ for every $\alpha$. Finally, let $\mu = \nu \xi$.

We shall need to know that $\mu \in A$, i.e. that, for every $v \in T$, $\mu \in \tilde{f} \ast u$ or, equivalentially,
\[ \{ \alpha \in I, \nu_\alpha \in f \ast v \} \in D. \]
As $\nu_\alpha = \nu_0$ and $\nu_0 \in B \subset f \ast v$, we conclude that
\[ \{ \alpha \in I, \nu_\alpha \in f \ast v \} = I \in D. \]
Therefore, $\mu \in A$ and, furthermore, $\mu \in \tilde{f} \ast u$, i.e.
\[ \{ \alpha \in I, \nu_\alpha = \nu_0 \in f \ast u \} \in D. \]
This set is not empty, and $\nu_0 \in f \ast u$. This is so for every $\nu_0 \in B$, and then $B \subset f \ast u$, $u \in B^\forall$.

The theorem is proved. \hfill \Box

It can be applied in the classical situation. If $K$ is a field that is an extension of a field $P$, then each of its ultrapowers $\tilde{K}$ is also a field extending $P$. Therefore, $T_K^\forall = T_{\tilde{K}}^\forall$ for every finite collection $T$.

Moreover, we now may say that $K$ and $\tilde{K}$ are geometrically equivalent on the equational level and for any finite $Y \subset X$. Indeed, if a finite $Y$ is selected, then it suffices to confine ourselves to finite collections $T$ of polynomials. Now $T_K^\forall = T_{\tilde{K}}^\forall$ implies $T''_K = T''_{\tilde{K}}$.

Here, we consider the collection $T$ as a set of formulas in the Halmos algebra $U$, and write $T = T_X$. In Section 4, the equality $T''_{X,K} \cap W(Y) = T''_Y$, $Y \subset X$ was true for every algebra $G \in \Theta$. In the situation when $T = T_Y$ and $W(Y) = P[Y]$ we have
\[ T''_{Y,K} = T''_{X,K} \cap P[Y] = T''_{X,K} \cap P[Y] = T''_{Y,\tilde{K}}. \]
This means that $K$ and $\tilde{K}$ are $Y$-equivalent for any finite $Y$.

The subsequent Proposition and Problem also relate to the classical case.

5.9. Proposition. Two finite extensions $K_1$ and $K_2$ of the field $P$ are equivalent if and only if they are isomorphic.

5.10. Problem. Are every two really closed extensions of the field $P$ always equivalent?

The proof of Proposition 5.11 is similar to that of Proposition 5.9.

5.11. Proposition. Let $\Theta$ be the variety of all associative algebras over the field $P$. Then finite dimensional simple algebras $G_1$ and $G_2$ in $\Theta$ are equivalent if and only if they are isomorphic.

The similar is true for simple Lie algebras.

Finally, we formulate a general problem concerning group representations.

Suppose that $V$ is a $K$-module, $K$–a commutative ring with unit, and $\text{Aut } V$–the automorphism group of $V$. Select $X$ and let $F = F(X)$ be the group of free over $X$. Consider the set of representations $\text{Hom}(F, \text{Aut } V)$. Also, select a variety $\mathfrak{X}$ of group representations over $K$. Single out a subset $A$ of $\text{Hom}(F, \text{Aut } V)$ according to the rule:
\[ \mu \in A \iff \text{the representation of } (V, \text{im } \mu) \text{ belongs to } \mathfrak{X}. \]
What can be said about $A'$, $A^y$, $A''$ and $A^{yy}$? Here, of interest are both questions—what is in common for all $X$, and what is the state of affairs for several concrete $X$, e.g. for $X = S^n$ (see $55$).

In this problem, the corresponding $\mu$ is characterized by properties of the action. If $X$ is the variety of groups, and if we consider group properties, then we know the answer: $A = T'$, where $T'$ is the collection of identities of $X$. Therefore, in this case $A = \{ \mu, \text{ im } \mu \in X \}$ is an algebraic variety.

We have considered in the paper a certain general scheme. As to really deep and interesting investigations, they have to be related to several special varieties $\Theta$. Along with classical varieties $\Theta$, it is natural to admit the variety of modules over a fixed $K$. For myself—I would like to select varieties of interesting representations of groups over $K$.

A lot of problems related to solving equations in groups use group representations by groups of tree automorphisms rather than linear representations.

**References**

[1] H. Andr´eka, T. Gergely, I. N´emety, On universal algebraic constructions of logic, Studia Logica, 36(1977), 9–47.
[2] H. Andr´eka, I. N´emety, I. Sain, Abstract model theory approach to algebraic logic, Preprint, 1984 (or 1992).
[3] G. Birkhoff, J. Lipson, Heterogeneous algebras, J. Comb. Theory, 8(1970), no. 1, 115–133.
[4] W.J. Blok, D. Pigozzi, Algebraizable logics, Memoirs AMS, 77(1989), no. 396.
[5] S.L. Bloom, E.G. Wagner, Many-sorted theories and their algebras with some applications to data types, In: Algebraic methods in semantics, Cambridge Univ. Press (1985), 133–168.
[6] J. Cirulis, Cylindric and relational algebras (Russian), Preprint, Riga, (1986).
[7] J. Cirulis, An algebraization of first order logic with terms, In: Algebraic Logic (Proc. Int. Conf. Budapest 1988), North-Holland, Amsterdam et al., (1991), 125–146.
[8] J. Cirulis, Superdiagonals of universal algebras, Acta Univ. Latviensis, 576(1992), 29–36.
[9] J. Cirulis, An axiomatic approach to relational algebras, manuscript, Riga, University of Latvia, December, (1993).
[10] J. Cirulis, Abstract algebras of finitary relations: several non-traditional axiomatizations, Acta Univ. Latviensis, 595(1994), 23–48.
[11] J. Cirulis, Corrections to the paper “An algebraization of the first order logic with terms, Acta Univ. Latviensis, 595(1994), 49–52.
[12] Z. Diskin, Polyadic algebras for the nonclassical logics, Riga, Manuscript (1991), unpublished.
[13] Z. Diskin, A unifying approach to algebraization of different logical systems, Frame Inform Systems, Riga, Report FIS/DBDL 9403 (1994).
[14] Z. Diskin, When is a semantically defined logic algebraizable, Acta Univ. Latviensis, 595(1994), 57–82.
[15] N. Feldman, Cylindric algebras with terms, J. Symb. Logic, 55(1990), 854–866.
[16] J. Galler, Cylindric and polyadic algebras, Proc. AMS, 8(1957), 176–183.
[17] J. Georgescu, Modal Polyadic Algebras, Bull. Math. Soc., Sci. Math. Roumanie, (1979), v.23 (71), no.1, 49–64.
[18] R. Goldblatt, Topoi—the Categorical Analysis of Logic, North-Holland, New York, Oxford, 1979.
[19] M. Gromov, Infinite groups as geometric objects, Proc. ICM, Warszaw, 1989.
[20] M. Gromov, Hyperbolic groups, In: Essays in Group Theory, Springer (1987), 75–265.
[21] N. Gupta, A solution of the dimension subgroup problem, Journ. of Algebra, 138(1991), no. 2, 479–490.
[22] A. Gvaramia, Classes of quasigroups closed under isotopy, DAN USSR, 282(1985), no. 5, 1047–1051.
[23] A. Gvaramia, B. Plotkin, The homotopies of quasigroups and universal algebra, In: Universal Algebra and Quasigroup Theory, Helderman Verlag, Berlin, (1992), 89–99.
[24] P.R. Halmos, Algebraic Logic, Chelcea, N.Y., 1962.
[25] R. Hartshorne, Algebraic Geometry, Springer, New York e.a., 1977.
[26] L. Henkin, J.D. Monk, A. Tarski, Cylindric Algebras, North-Holland Publ. Co. (1971, 1985).
[27] H. Henkel, Fully invariant algebraic closure systems of congruences and quasivarieties of algebras, Coll. Math. Soc. J. Bolyai, 43(1983), Szeged(Hungary), 189–206.
[28] P.I. Higgins, Algebras with a scheme of operators, Math. Nachr., 27(1963), no. 1–2, 115–132.
[29] P.T. Johnstone, Topos Theory, Acad. Press, London e.a., 1977.
[30] L. Kelly, Complete rules of inference for universal sentences, Studia Sci. Math. Hungar., 19(1984), 347–361.
[31] A. Kock, G. Reyes, Doctrines in Categorical logic, In: Handbook of Math. Logic, North-Holland, Amsterdam e.a., 1977.
[32] A. Kushkulei, S. Rosenberg, A syntactical characterization of universal classes, Preprint, Riga, 1988.
[33] F.W. Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad. Sci., **21**(1963), 1–23.
[34] F.W. Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad. Sci., **50**(1963), 869–872.
[35] F.W. Lawvere, *Some algebraic problems in the context of functorial semantics of algebraic theories*, Rep. Midwest Category Seminar II, **13**(1968), 41–61.
[36] F. Levin, *Solutions of equations over groups*, Bull. Amer. Math. Soc., **68**(1962), 603–604.
[37] R. Lyndon, *Equations in free groups*, Trans. Amer. Math. Soc., **96**(1960), 445–457.
[38] S. MacLane, *Categories for the Working Mathematician*, Springer, 1971.
[39] M. Makkai, G. Reyes, *First Order Categorical Logic*, LNM, **611**(1977), Springer.
[40] A.I. Maltsev, *Model correspondences (Russian)*, Izv. AN SSSR (ser. Math.), **23**(1959), no. 3, 313–336.
[41] A.I. Maltsev, *Algebraic Systems*, North Holland, 1973.
[42] E.G. Manes, *Algebraic Theories*, Springer, 1976.
[43] G. Mashevitsky, *An example of a semigroup without finite basis of pseudoidentities*, Negev, preprint.
[44] I. Nemeti, *Algebraizations of quantifier logics (overview)*, Studia Logica, **50**(1991), no. 3/4, 485–569.
[45] A. Nerode, *Some lectures on modal logic*, In: Logic, Algebra and Computations, Springer (1991), 281–335.
[46] B.H. Neumann, *Adjunction of elements to groups*, J. London Math. Soc., **18**(1943), 4–11.
[47] B.H. Neumann, *A note on algebraically closed groups*, J. London Math. Soc., **27**(1952), 247–249.
[48] A. Olshanski, *Quasidentities in finite groups (Russian)*, Sib. Math. Journ., **15**(1974), no. 6, 1409–1413.
[49] F. Paulin, *Actions de Groupes sur les arbres*, Seminaire Bourbaki, 48 eme annee, 1995-96, no.808, 1–31.
[50] V. Platonov, A. Rapintchuk, *Algebraic Groups and Number Theory*, Nauka, Moscow, 1991.
[51] B.I. Plotkin, *Universal Algebra, Algebraic Logic, and Databases*, Kluwer Acad. Publ., 1994.
[52] B.I. Plotkin, *Algebraic logic, varieties of algebras, and algebraic varieties*, Jerusalem, Preprint, 1995.
[53] B.I. Plotkin, *Algorithm, categories, and databases*, Handbook of algebra, Elsevier, to appear.
[54] B. Plotkin, A. Kushkulei, *Identities of regular representations of groups (Russian)*, Preprint, Riga, (1980).
[55] B. Plotkin, E. Vovsi, *Varieties of Representations of Groups (Russian)*, Riga, Zinatne, 1983.
[56] B. Plotkin, L. Greenglaz, A. Gvaramia, *Algebraic Structures in Automata and Databases Theory*, World Scientific, Singapore e.a., 1992.
[57] M. Prishchepcov, *On small length equations over torsion-free groups*, Int. J. Algebra and Comput., **4**(1994), no. 4, 575–589.
[58] R. Quackenbush, *Completeness theorems for universal and implicational logics of algebras via congruences*, Proc. AMS, **103**(1988), no. 4, 1015–1021.
[59] E. Rips, *On the fourth integer dimensional subgroup*, Israel Journal of Math., **12**(1972), no. 4, 342–346.
[60] E. Rips, *Subgroups of small cancellation groups*, Bull. London. Math. Soc., **14**(1982), 45–47.
[61] E. Rips, Z. Sela, *Canonical representatives and equations in hyperbolic groups*, preprint IHES.
[62] S. Rosenberg, *About varieties of Halmos algebras (Russian)*, Latv. Math. Ezhegodnik, **32**(1988), 85–89.
[63] S.M. Rosenberg, *Specialized relational and Halmos algebras (Russian)*, Latv. Mat. Ezhegodnik, **34**(1993), 219–229.
[64] A. Selman, *Completeness of calculi for axiomatically defined classes of algebras*, Algebra Universalis, **2**(1972), 20–32.
[65] I. Shaparevich, *The Foundations of Algebraic Geometry*, Nauka, Moscow, 1972.
[66] N.D. Volkov, *Halmos algebras and relational algebras (Russian)*, Latv. Mat. Ezhegodnik, **30**(1986), 110–123.
[67] N.D. Volkov, *Transfer from relational algebras to Halmos algebras*, In: Algebra and Discrete Math., Latv. State Univ., Riga (1986), 43–53.
[68] N.D. Volkov, *On equivalence between categories of Halmos algebras and of relational algebras*, Latv. Mat. Ezhegodnik, **34**(1993), 171–180.

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