The Minimum Bends in a Polyline Drawing with Fixed Vertex Locations

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Abstract. We consider embeddings of planar graphs in $\mathbb{R}^2$ where vertices map to points and edges map to polylines. We refer to such an embedding as a polyline drawing, and ask how few bends are required to form such a drawing for an arbitrary planar graph. It has long been known that even when the vertex locations are completely fixed, a planar graph admits a polyline drawing where edges bend a total of $O(n^2)$ times. Our results show that this number of bends is optimal. In particular, we show that $\Omega(n^2)$ total bends is required to form a polyline drawing on any set of fixed vertex locations for almost all planar graphs. This result generalizes all previously known lower bounds, which only applied to convex point sets, and settles 2 open problems.

1 Introduction

A polyline drawing is an embedding of a planar graph in $\mathbb{R}^2$ where vertices map to points and edges map to polylines (piecewise linear curves). An example of such a drawing is depicted in Figure 1.

Applications of polyline drawings to VLSI circuit design and information visualization (see [1], [2]) have inspired the development of algorithms for constructing polyline drawings of arbitrary planar graphs. A particularly auspicious construction was shown in [3], where all edges map to straight-line segments and all vertices map into an $(n-2) \times (n-2)$ grid.

What if we further restrict the locations to which vertices map? Do all planar graphs admit polyline drawings if we fix the points to which vertices are mapped? The answer is yes but with a commensurate increase in the complexity of the polylines to which edges map. To what extent can this complexity be minimized? One prevailing way to measure this complexity is by counting the number of times the polylines bend (see [4], [5], [2], [6]).

When the vertex mapping is fixed, Pach and Wenger [6] showed that all planar graphs on $n$ vertices admit a polyline drawing with at most $O(n^2)$ bends. In the same paper, they showed that this result is optimal if the vertex mapping is to points in convex position. It was left as an open problem whether this lower bound applies to all fixed vertex mappings. We answer this open problem in Theorem 2 by extending the lower bound to all fixed vertex mappings. That is, for any fixed vertex mapping, we show that almost all planar graphs do not admit polyline drawings with $o(n^2)$ bends.
To prove our lower bound, we generalize the encoding techniques from [7], which were also limited to convex point sets. In particular, we show how to encode a planar graph more efficiently when it admits a polyline drawing with fewer bends (assuming a fixed vertex mapping). Very recently, Francke and Tóth [8] considered this same encoding problem. Their results gave an encoding bound of $O(\beta + n)$ bits when a polyline drawing with $\beta$ bends is admitted. Our results significantly improve this encoding bound to $n \lg(\beta/n) + O(n)$ bits.

Our encoding technique is described in Section 4. The approach exploits a relation between a graph’s separability and the numbers of bends with which it can be drawn. This separability is formalized by Lemma 3 in Section 3. The encoding technique also requires an efficient way to encode the path traced by a polyline, which is described in Section 2. Finally, the lower bound (Theorem 2) is presented in Section 5.

### 2 Convex Layer Diagrams

A **convex layer diagram** is an embedding of $l$ convex polygons in the plane such that the $i$-th convex polygon is strictly contained by the $(i + 1)$-th polygon. By convention, we refer to the outermost polygon as layer-1 and the innermost polygon as layer-$l$. As with any simple polygon, each layer defines a boundary consisting of layer edges and layer vertices. See Figure 2 for an example diagram.

Let $\Lambda$ be a fixed convex layer diagram drawn in $R^2$. A line segment is said to **cross** a layer edge if the two intersect at a point that is interior to both the edge and the line segment. A line segment is said to cross a layer boundary if it crosses one of its edges. The following lemma shows that the layer edges in...
A convex layer diagram with 4 layers.

A line segment crosses can be identified with an ordered pair of support vertices.

**Lemma 1.** Suppose that a line segment crosses \( k \geq 2 \) layer boundaries in \( \Lambda \). Then, the layer edges crossed by this line segment can be identified by a pair of support vertices \((v_i, v_j)\), each being incident to a crossed layer edge.

**Proof.** Let \( \Phi \) be the layer edges crossed by the line segment and let \( \Psi \) be the layer vertices incident to the edges in \( \Phi \). Extend the line segment to form a coincident line \( \gamma \). We will show how to transform \( \gamma \) via translations and rotations so that it intersects with two vertices in \( \Psi \), while preserving its edge intersections in \( \Phi \).

Start by perturbing \( \gamma \) by a sufficiently small rotation and translation to ensure that each vertex in \( \Psi \) is at a distinct distance from \( \gamma \) along the \( x \)-axis and that \( \gamma \) is aligned with neither the \( x \)-axis nor any edge in \( \Phi \).

We can then translate \( \gamma \) left until it first intersects a vertex \( v_i \) in \( \Psi \). Call this new line \( \gamma' \). Clearly, all edges in \( \Phi \) intersect \( \gamma' \). Rotate \( \gamma' \) counterclockwise about \( v_i \) until it intersects another vertex \( v_j \neq v_i \), and call this new line \( \gamma'' \). By the same argument \( \gamma'' \) must intersect all edges in \( \Phi \). See Figure 3 for an example of this process.

The line \( \gamma'' \) is uniquely determined by the line through \( v_i \) and \( v_j \). However, where \( \gamma'' \) intersects a layer vertex \( u \), there is ambiguity about which layer edge in \( \Lambda \) incident \( u \) had intersected with \( \gamma \). This ambiguity can be resolved by representing \( \gamma'' \) as the ordered pair \((v_i, v_j)\). Indeed, from knowing that \( v_i \) was used for rotation, we can rotate \( \gamma'' \) back clockwise by an sufficiently small angle. The resulting line can only intersect the one vertex \( v_i \) in \( \Psi \). By applying another sufficiently small translation to the right, the edges \( \Phi \) that \( \gamma \) originally intersected can be unambiguously determined.
3 Simple Polygon Separators

It is well known ([6], [9], [7]) that all planar graphs admit polyline drawings using a fixed vertex mapping such that the total number of bends is $O(n^2)$. We show that if a graph admits a polyline drawing with fewer bends, then its vertices are correspondingly more separable. This separability is formalized in Lemma 3.

To find the appropriate way of separating our graph $G$, we leverage the following weighted simple cycle separator theorem.

Lemma 2 ([10]). Let $G$ be a maximal planar graph with non-negative vertex weights adding to at most 1. Then, the vertex set of $G$ can be partitioned into 3 sets $V_1, V_2, C$ such that neither $V_1$ nor $V_2$ has total vertex weight exceeding $2/3$, no edge from $G$ joins a vertex in $V_1$ to a vertex in $V_2$, and the subgraph on $C$ is a simple cycle with at most $2\sqrt{n} + 7$ vertices.

Given a polyline drawing of a graph $G$, we can separate the vertices into two parts by superimposing a simple polygon $\Omega$. See Figure 4 for an example. The vertices that lie on or inside $\Omega$ define one part, and the vertices that lie outside $\Omega$ define the other. Let $V_1$ be the vertices on or inside $\Omega$, and let $V_2$ be the vertices outside $\Omega$. If $|V_1|$ and $|V_2|$ are both at most $2/3n$ and at most $\alpha$ edges in $G$ join a vertex in $V_1$ to a vertex in $V_2$, then $\Omega$ defines an $\alpha$-simple polygon separator.

Lemma 3. Let $G$ be a planar graph on $n$ vertices with bounded vertex degree $d$. Suppose that $G$ admits a polyline drawing with a total of $\beta$ bends. Then, there is an $\alpha$-simple polygon separator having at most $r$ edges, where $\alpha \leq dr$ and $r \leq 2\sqrt{n} + \beta + 10$. 

![Fig. 3. A line segment that crosses 6 layer edges is shown in (a). The transformed line segment is shown in (b) with the identifying support vertices.](image)
Fig. 4. A simple polygon separator superimposed on a graph. Two vertices lie outside and four lie on or inside the separator. Three edges join vertices on opposing sides of the separator.

Proof. Augment $G$ by adding $\beta + 3$ auxiliary vertices, one at each bend in its polyline drawing and an additional 3 vertices to form a triangle bounding the polyline drawing. We can further add edges to make a maximal planar graph $G'$ by triangulating this augmented polyline drawing, resulting in a drawing without any bends (a straight-line embedding). See Figure 5 for an example.

Observe that $G'$ has $n + \beta + 3$ total vertices. If we weight each auxiliary vertex 0 and each original vertex $1/n$, it follows from Theorem 2 that $G'$ contains a simple cycle separator $C$ with at most $2\sqrt{n + \beta + 3} + 7$ edges. Moreover, the number of original vertices lying inside this cycle and the number lying outside are both at most $2/3n$.

Since our drawing of $G'$ had no bends, this cycle corresponds precisely to a simple closed polygon $\Omega$ with $r$ edges, where $r$ is at most

$$2\sqrt{n + \beta + 3} + 7 \leq 2\sqrt{n + \beta} + 10.$$ 

The polyline of an edge in $G$ from a vertex on or inside $\Omega$ to a vertex outside $\Omega$ had to intersect with one of the vertices in $C$. Since at most $d$ edges in $G$ can intersect with any one vertex in $C$, it follows that $\Omega$ is an $\alpha$-simple polygon separator where $\alpha \leq dr$. 
4 Encoding Polyline Drawings

In this section, we describe a general method for encoding labeled planar graphs. An encoding is simply a sequence of bits (our choice for unit of information) from which the edges of the original graph can be unambiguously determined. Any information that is said to be fixed does not require encoding as it is assumed to be available at decoding time. The bounds given for our encoding algorithms are in terms of parameters that can be arbitrarily large (e.g. the number of vertices or edges in the graph being encoded).

Our encoding bounds provide the means for proving the lower bound of Theorem 2. However, a direct consequence of Theorem 1 resolves a conjecture from [8] in the positive. Namely, we show that there are at most $2^n \log(1+\beta/n) + O(n)$ $\beta$-bend polyline drawings on $n$ vertices whose mapping is fixed.

The core component of this encoding technique is encapsulated by the following lemma. The high-level idea of this lemma is as follows. First, we use Lemma 3 to partition a graph’s vertices into 2 roughly equal size parts $V_1, V_2$. This partition defines 3 subgraphs (edges between $V_1$ vertices, edges between $V_2$ vertices, and edges between a $V_1$ and a $V_2$ vertex). These 3 subgraphs are then encoded recursively. To complete the encoding, we just need to show how to encode the partition so that the original graph can be recovered from the 3 subgraphs. It turns out that the partition given by Lemma 3 can be encoded using only $O(\sqrt{\beta} + n \log(n/\sqrt{\beta} + n))$ bits. This additional information is sufficiently small to give the desired bound.
Lemma 4. Let $G$ be a planar graph on $n$ vertices where each vertex has degree at most 2. Suppose that $G$ admits a polyline drawing with a total of $\beta$ bends for which the vertex mapping is fixed (i.e. the point at which each vertex is drawn is fixed). Then, $G$ can be encoded using $n \lg(1 + \beta/n) + O(n)$ bits.

Proof. Define $\sigma = \beta + n$ (i.e. the total count of all vertices and bends), and let $m$ be the number of edges in $G$ (note that $m \leq n$). We will prove the stronger claim that $G$ can be encoded with $T(n, m, \sigma)$ bits, where

$$T(n, m, \sigma) \leq m \lg(\sigma/n) + 850n - 423\sqrt{\sigma} \lg(n/\sqrt{\sigma}),$$

by induction on $n$.

Since each vertex in $G$ has a degree at most 2, the edges in $G$ can be directed such that the outgoing degree of each vertex is at most 1. Thus, if we use $n$ bits to encode the vertices in $G$ with an outgoing edge, we can trivially encode all edges in $G$ by specifying at most one vertex for each edge. It follows that this encoding uses a total of $m \lg n + n$.

Since $\sqrt{\sigma} \lg(n/\sqrt{\sigma}) \leq n$ always, we have that

$$427n \leq 850n - 423\sqrt{\sigma} \lg(n/\sqrt{\sigma}).$$

Hence, this trivial encoding suffices for our base case as $m \lg n + n \leq 427n$ whenever $n \leq 2^{426}$. Suppose further that $\sigma \geq n^2/2^{426}$. In this case, we have the inequality

$$m \lg n + n \leq m \lg(\sigma/n) + 427n$$

and thus the trivial encoding still satisfies the desired bound.

Thus, we can proceed by way of induction assuming that the claim holds for smaller values of $n$ and that $n \geq 2^{426}$ and $\sigma < n^2/2^{426}$. Let $\Pi$ be the points to which the vertices in $G$ are mapped in the polyline drawing. We can form a convex layer diagram out of $\Pi$ as follows. Define layer 1 by the convex hull of $\Pi$. Remove the points in $\Pi$ that lie on its convex hull, and define layer 2 by the convex hull of the remaining points. We can repeat this process to define the $l$ layers of a convex layer diagram $\Lambda$.

The convex layer diagram $\Lambda$ depends only on the points $\Pi$ and is thus fixed. We can furthermore assume a fixed iteration order over its layer edges, starting from the outermost layer and iterating inward. To encode $G$, we will leverage this fixed structure of $\Lambda$.

Since $G$ has bounded degree 2, it follows by Lemma 3 that its polyline drawing has an $\alpha$-simple polygon separator $\Omega$ with at most $r$ edges, where $\alpha \leq 2r$ and $r \leq 2\sqrt{\sigma} + 10$. To simplify our encoding, we make modifications to $\Omega$. At each point in $\Pi$ that intersects with $\Omega$, we modify $\Omega$ in an $\epsilon$-neighborhood around this point so that it no longer intersects with the point. This modification can trivially be done by adding at most $2r$ edges (2 at each intersecting point in $\Pi$ of which there are at most $r$). We also split any edge in $\Omega$ that crosses a layer boundary more than once to ensure that each edge crosses each layer boundary at most once. Doing so adds at most $r$ additional edges (each edge can cross
a layer boundary at most twice by convexity and thus at most once split per original edge suffices).

Thus, we can assume that $\Omega$ has at most 4r edges and partitions the vertices in $G$ into two parts $V_1, V_2$, those mapping to points inside $\Omega$ and those mapping to points outside $\Omega$. We can further assume that $1/3n \leq |V_1| \leq |V_2| \leq 2/3n$ without loss of generality. We will show how to encode this vertex partition using at most $O(\sqrt{\sigma}\log(n/\sqrt{\sigma}))$ bits.

Each $\Omega$-edge crosses a (possibly empty) set of layer edges in $\Lambda$. If at most 1 layer edge is crossed by a given edge, define an auxiliary vertex at this crossing. Otherwise, define an auxiliary vertex at the first and last crossing. By Lemma 1, the entire set of layer edges crossed by a given $\Omega$-edge can be recovered if we further encode its support vertices. We introduce up to 2 additional auxiliary vertices at the crossings with the edges incident to the support vertices.

The total number of auxiliary vertices is at most 16r. Since there are at most $n$ layer edges, it follows that we can encode the number of auxiliary vertices that were added to each layer edge using $\log((n+16r)/16r) \leq 17r \log(n/r)$ bits. We can define a cycle $C$ that joins these auxiliary vertices in the order in which they intersect with $\Omega$. $C$ has fewer than $n$ vertices since $16r \leq 32\sqrt{n} + 160 < n$. Thus, if we adopt any fixed convention for positioning the auxiliary vertices along the layer edges, we can encode $C$ using $850(16r) = 13600r$ bits by the induction hypothesis. We can further annotate the edges in $C$ with an additional 16r bits to encode which vertices corresponded to support vertices. Thus, we have encoded the first and last intersections of each edge in $\Omega$ as well as the support vertices defining the interior intersections of this edge. We further know from $C$ the structure of $\Omega$ between layer boundaries and have thus encoded how to define a simple closed curve that is homotopic to $\Omega$ in $R^2 - \Pi$. This encoding allows us to unambiguously define the vertex partition $V_1, V_2$, and uses a total of $17r \log(n/r) + 13616r$ bits. Since $\sigma < n^2/2^{426}$, it follows that $\log(n/\sqrt{\sigma}) > 213$. Using both the constraints that $r \leq 2\sqrt{\sigma} + 10$ and $n \geq 2^{426}$, we can further show that $\log(n/r) \geq 426/2 - 2 = 212$. Thus, the number of bits used to encode $V_1, V_2$ is at most

$$17r \log(n/r) + 13616r \leq 82r \log(n/r).$$

Using the vertex partition $V_1, V_2$, we can partition the edges of $G$ into one of 3 subgraph $G_1, G_2, G_3$. $G_1$ is defined by the edges in $G$ between vertices in $V_1$, $G_2$ is defined by the edges in $G$ between vertices in $V_2$, and $G_3$ is defined by the edges between a vertex in $V_1$ and a vertex in $V_2$.

To complete the proof, we argue by way of induction as each of $G_1, G_2, G_3$ has fewer than $n$ vertices. Let $n_i$ and $m_i$ be the number of vertices and edges, respectively, in $G_i$ for $i = 1, 2, 3$. Similarly, define $\sigma_i = \beta_i + n_i$, where $\beta_i$ is the number of bends in $G_i$ for $i = 1, 2, 3$. We can thus complete the encoding for a total of

$$T(n, m, \sigma) \leq T(n_1, m_1, \sigma_1) + T(n_2, m_2, \sigma_2) + T(n_3, m_3, \sigma_3) + 82r \log(n/r)$$

bits.
Since \( V_1, V_2 \) were defined in terms of the \( \alpha \)-simple polygon separator \( \Omega \), it follows that \( n_1 + n_2 = n, \frac{1}{3} \leq n_1 \leq n_2 \leq 2/3n, n_3 \leq 4r, \) and \( m_1 + m_2 + m_3 = m \). Furthermore, we have the constraint that \( \beta_1 + \beta_2 + \beta_3 \leq \beta \). Subject to these constraints, our encoding size is thus

\[
T(n, m, \sigma) \leq m_1 \log(\frac{\sigma}{n_1}) + 850n_1 - 423\sqrt{\sigma_1} \log(n_1/\sqrt{\sigma_1}) \\
+ m_2 \log(\frac{\sigma_2}{n_2}) + 850n_2 - 423\sqrt{\sigma_2} \log(n_2/\sqrt{\sigma_2}) \\
+ m_3 \log(\frac{\sigma_3}{n_3}) + 850n_3 - 423\sqrt{\sigma_3} \log(n_3/\sqrt{\sigma_3}) \\
+ 82r \log(n/r)
\]

by induction. The remainder of the proof is to bound this expression. By using elementary calculus, one can show that this bound is largest when \( m_3 = 0 \) (Alternatively, one could redefine \( G_1 \) to include the edges of \( G_3 \), removing the need to encode \( G_3 \) by adding only an 850\( r \) term). Thus, our encoding size satisfies the inequality

\[
T(n, m, \sigma) \leq m_1 \log(\frac{\sigma}{n_1}) + 850n_1 - 423\sqrt{\sigma_1} \log(n_1/\sqrt{\sigma_1}) \\
+ m_2 \log(\frac{\sigma_2}{n_2}) + 850n_2 - 423\sqrt{\sigma_2} \log(n_2/\sqrt{\sigma_2}) \\
+ 82r \log(n/r)
\]

where \( m_1 + m_2 = m \) and \( \sigma_1 + \sigma_2 \leq \sigma \). Since each vertex in \( G_1 \) and \( G_2 \) has degree at most 2, it follows that \( m_1 \leq n_1 \) and \( m_2 \leq n_2 \). Thus, by our constraints on \( n_1 \) and \( n_2 \), we also have that \( 1/3m \leq m_1, m_2 \leq 2/3m \). Again, we can show using elementary calculus that this constrained inequality reaches its maximum when

\[
\frac{m_1}{m}, \frac{n_1}{n}, \frac{\sigma_1}{\sigma} = \frac{1}{3} \quad \text{and} \quad \frac{m_2}{m}, \frac{n_2}{n}, \frac{\sigma_2}{\sigma} = \frac{2}{3}
\]

(see [7] for more details on this derivation). Thus, it follows that

\[
T(n, m, \sigma) \leq m \log(\frac{\sigma}{n}) + 850n - 423\sqrt{\sigma} \log(n/\sqrt{\sigma}) \\
+ 82r \log(n/r) + 423\sqrt{\sigma} \log\sqrt{9/2} \\
- (\sqrt{1/3} + \sqrt{2/3} - 1)423\sqrt{\sigma} \log(n/\sqrt{\sigma})
\]

and since \( 82r \log(n/r) + 423\sqrt{\sigma} \log\sqrt{9/2} \leq 165\sqrt{\sigma} \log(n/\sqrt{\sigma}) \), it follows that the last two lines cancel and

\[
T(n, m, \sigma) \leq m \log(\frac{\sigma}{n}) + 850n_1 - 423\sqrt{\sigma} \log(n/\sqrt{\sigma})
\]

completing the proof.

We can in fact generalize our encoding to apply to any planar graph by a simple reduction to Lemma 4. This generalization gives the following theorem.

**Theorem 1.** Let \( G \) be a planar graph that admits a polyline drawing with a total of \( \beta \) bends under a fixed vertex mapping. Then, \( G \) can be encoded using \( n \log(1 + \beta/n) + O(n) \) bits.
Proof. Let’s assume that $G$ is connected, and let $T$ be a spanning tree of $G$. Suppose that we traverse the edges bounding the unique face of $T$. Since $T$ is a tree, we will traverse each edge exactly twice (both sides must be on this unique face). If we start the traversal along an arbitrary edge, and stop before traversing it a third time, we will visit a sequence of $2n$ vertices $S = v_1, v_2, \ldots, v_{2n}$.

Let $v_i$ be a vertex in $T$ and let $j$ be the first position at which it occurs in $S$ from the left. We define $\text{succ}(v_i)$ as the leftmost occurring vertex in $S$ whose first occurrence is at a position beyond $j$. We can then define a sequence of vertices $S' = v_i, \text{succ}(v_i), \text{succ}(\text{succ}(v_i)), \ldots$, terminating when a vertex has no successor.

This sequence of vertices corresponds to a path $P$ passing through each vertex in $T$ exactly once. See Figure 6 for an example of this construction.

![Fig. 6. The path (in red) defined by the traversal of the unique face of a tree (in black).](image)

From the polyline drawing of $G$ we can immediately construct a polyline drawing of $T$ having at most $\beta$ bends. We can thus construct a polyline drawing of $P$ using $2\beta + O(n)$ bends since the edges in $P$ correspond to a traversal along face of $T$. It follows from Lemma 4 that $P$ can be encoded using

$$n \log(1 + (2\beta + O(n))/n) + O(n) = n \log(1 + \beta/n) + O(n)$$

bits. From $P$ it is easy to recover $T$ with only an additional $2n$ bits. Indeed, the vertex sequence in $P$ (that is, $S'$) defines a subsequence of the face of $T$, the rest of which can be encoded by saying which vertices in $P$ to traverse back on.

Having encoded $T$, we only need to encode the remaining edges in $G$ that are not in $T$. The number of possible arrangements of these edges corresponds to the well studied Catalan numbers (i.e. they can be represented with a parenthesization of length $O(n)$). It follows that $G$ can be encoded with only $O(n)$ bits more than the encoding of $P$ for a total of $n \log(1 + \beta/n) + O(n)$ bits.
Finally, if $G$ in not connected, we can trivially make $G$ connected by adding edges without introducing more than $\beta$ additional bends, giving the same bound.

5 Lower Bound on the Number of Bends

Let $\Pi$ be a set of $n$ points in $\mathbb{R}^2$, and let $\pi : V \rightarrow \Pi$ be a fixed vertex mapping. The open question from [6] asked whether there exists a $\pi$ for which all planar graphs admit a polyline drawing with $o(n^2)$ total bends. The following theorem shows that this cannot be accomplished and that, for any $\pi$, the set of planar graphs requiring $\Omega(n^2)$ bends becomes dense as $n \rightarrow \infty$.

**Theorem 2.** Let $\pi : V \rightarrow \Pi$ be a fixed vertex mapping, and let $G = (V, E)$ be a planar graph sampled uniformly at random from the set of all planar graphs on $n$ vertices. Then, with high probability, all polyline drawings of $G$ using the vertex mapping $\pi$ have $\Omega(n^2)$ bends.

**Proof.** Suppose that $G$ is a planar graph sampled uniformly at random from the set of labeled planar graphs on $n$ vertices. The number of such graphs is known to be $\Theta(n^{-7/2}\gamma^{n!})$, where $\gamma \approx 27.22687$. Thus, for $n$ sufficiently large, it follows that at least $\lg n! - \Delta$ bits are required to encode $G$ with probability at least $1 - 2^{-\Delta}$.

Suppose that $G$ admits a polyline drawing using the vertex mapping $\pi$ having a total of $\beta$ bends. It follows from Theorem 1 that $G$ can be encoded with

$$n \log(1 + \beta/n) + O(n)$$

bits. Thus, the inequality

$$n \log(1 + \beta/n) + O(n) \geq \lg n! - \Delta$$

must hold with probability at least $1 - 2^{-\Delta}$. We can use Stirling’s approximation to show that $\lg n! \geq n \log n - n$ and thus

$$n \log(1 + \beta/n) + O(n) + \Delta \geq n \log n$$

or equivalently,

$$n \log \left( \frac{n + \beta}{n} \right) \geq n \log n$$

holds with probability at least $1 - 2^{-\Delta}$. If we divide through by $n$ and exponentiate both sides, it follows that

$$\frac{n + \beta}{n} \geq 2^{O(1) + \Delta/n} \geq n$$

or equivalently,

$$\beta \geq \frac{n^2}{2^{O(1) + \Delta/n}} - n$$

with probability at least $1 - 2^{-\Delta}$. In particular, we can choose $\Delta$ to be $n$, which shows that $\beta$ is $\Omega(n^2)$ with probability at least $1 - 2^{-n}$. 

6 Open Problems

We have shown that no fixed vertex mapping admits polyline drawings with \( o(n^2) \) total bends for all planar graphs. Moreover, the same result applies for paths, trees, outerplanar graphs, or any subset of planar graphs with at least \( n!/2^{\Omega(n)} \) elements on \( n \) labeled vertices.

On the other hand, using the techniques from [7] we can always construct a polyline drawing such that each edge bends at most 3\( n \) times each (in fact, 2\( n \) times each for a sufficiently random planar graph). Since these results are tight up to constant factors, we ask the following questions:

1. Can the constant 3 be improved? That is, assuming a fixed vertex mapping, do all planar graph admit polyline drawings having fewer than 3\( n \) bends? Do some fixed vertex mappings give provably better constant factors than others?

2. How tight of a constant can be shown in the lower bound? Using our approach, this lower bound constant directly maps to the constant of the \( O(n) \) term in the encoding of Theorem 1. To what extent can the \( O(n) \) term constant be reduced?

3. Can our encoding technique be used to prove other lower bounds in graph drawing? Are there other applications of this encoding technique outside of graph drawing?

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