Nonperturbative interaction effects in the thermodynamics of disordered wires

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We study nonperturbative interaction corrections to the thermodynamic quantities of multichannel disordered wires in the presence of the Coulomb interactions. Within the replica nonlinear $\sigma$-model (NL$\sigma$M) formalism, they arise from nonperturbative soliton saddle points of the NL$\sigma$M action. The problem is reduced to evaluating the partition function of a replicated classical one dimensional Coulomb gas. The state of the latter depends on two parameters: the number of transverse channels in the wire, $N_{ch}$, and the dimensionless conductance, $G(L_T)$, of a wire segment of length equal to the thermal diffusion length, $L_T$. At relatively high temperatures, $G(L_T) \gtrsim \ln N_{ch}$, the gas is dimerized, i.e. consists of bound neutral pairs. At lower temperatures, in $N_{ch} \gtrsim G(L_T) \gtrsim 1$, the pairs overlap and form a Coulomb plasma. The crossover between the two regimes occurs at a parametrically large conductance $G(L_T) \sim \ln N_{ch}$, and may be studied independently from the perturbative effects. Specializing to the high temperature regime, we obtain the leading nonperturbative correction to the wire heat capacity. Its ratio to the heat capacity for noninteracting electrons, $C_0$, is $\delta C/C \sim G^2(L_T)e^{-2G(L_T)}$.

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I. INTRODUCTION

The interplay between disorder and electron-electron interactions in conductors influences their low-temperature properties in an essential way. Depending on the disorder strength, the temperature and other system parameters a conductor may be either in the metallic or in the insulating regime. The manifestations of electron-electron interactions in the two regimes are quite different. In the insulating regime the charge in a given localized site is quantized in the metallic regime the charge in a given volume of the conductor can change continuously and charge discreteness effects are small. The two regimes can be distinguished by the value of the appropriately defined dimensionless conductance $G$, which is greater than unity in the metallic regime and smaller than unity in the insulating one.

The spatial extent of the solitons is given by the thermal diffusion length $L_T$, and their action is equal to $G(L_T)$. In contrast, the perturbative corrections are controlled by a single parameter, $G(L_T)$. For example, the leading perturbation theory correction to the heat capacity is $\delta C/\delta T \sim 1/G(L_T)$, where $C_0$ is the wire heat capacity in the noninteracting electron approximation.

Within the NL$\sigma$M formalism, the nonperturbative effects are described by soliton saddle points of the NL$\sigma$M action. The spatial extent of the solitons is given by the thermal diffusion length $L_T$, and their action is equal to $G(L_T)$. The nonperturbative contribution to the thermodynamic quantities is described by the partition function for a gas of these solitons. We map the problem onto a one dimensional replicated Coulomb gas. At high temperatures, $G(L_T) \gtrsim \ln N_{ch}$, the Coulomb gas is dimerized, i.e. consists of widely separated neutral pairs (dimers). In the temperature range $\ln N_{ch} \gtrsim G(L_T) \gtrsim 1$, the dimers are ionized and form a Coulomb plasma. Since the crossover between the two regimes occurs at a parametrically large conductance, $G(L_T) \sim \ln N_{ch}$, it can be studied independently from the perturbative effects. In this paper we specialize to the high temperature regime, leaving consideration of the crossover
to the low temperature one for future work.

The paper is organized as follows. In Sec. II we describe the NLσM for multichannel wires. In Sec. III we obtain the analytic solution for the saddle points of the NLσM action in the limit of the infinite number of channels \(N_{ch}\), and evaluate the functional integral over the fluctuations about the saddle points. In Sec. IV we obtain the leading nonperturbative correction to the thermodynamic quantities of the wire for \(N_{ch} \gg 1\). In Sec. V we summarize our results.

II. NONLINEAR \(\sigma\)-MODEL

We consider an infinitely long disordered wire with many transverse channels, \(N_{ch} \gg 1\). The disorder is assumed to be weak, so that the elastic mean free path \(l\) satisfies the condition \(k_F l \gg 1\), where \(k_F\) is the Fermi wave number. We consider the temperature \(T\) to be smaller than the Thouless energy for the transverse motion, \(E_T \equiv D/d^2\), where \(d\) is the transverse wire dimension, and \(D\) is the diffusion constant. In this regime the wire is described by the one-dimensional NLσM.

Thermodynamic properties of the system can be extracted from the averaged over disorder realizations replicated partition function, \(\langle Z^p \rangle = \langle \text{Tr} - e^{\mathcal{F}} \rangle\), with \(p\) being the number of replicas. We will be interested in the thermodynamic potential, which can be obtained using the replica trick:

\[
\langle \Omega \rangle = -T \langle \ln Z \rangle = -T \lim_{p \to 0} \frac{\langle Z^p \rangle - 1}{p}.
\]

In the diffusive regime the replicated partition function, \(\langle Z^p \rangle\), has a functional integral representation in terms of NLσM, describing the low-energy physics of the problem. The derivation of the NLσM action has become a standard procedure.\(^{18,21}\) Therefore, below we only present its final form, suitable for the problem under consideration. The NLσM action is a functional of two fields: the \(Q\)-matrix, parameterizing the diffusive degrees of freedom of electron motion, and electric potential \(V\). The former is a Hermitian matrix in the space of replicas and Matsubara frequencies, whose entries are \(4 \times 4\) matrices in the space \(S \otimes T\), given by the product of spin, \(S\), and time-reversal, \(T\), spaces.\(^{21,22}\) The slowly varying in space electric potential \(V_a\) is introduced to treat the long-range part of the Coulomb interaction in the replica \(a\). This part of the Coulomb interaction is of particular importance for the consideration below. It cannot be described by the Fermi-liquid interaction constants. Since the Fermi-liquid effects in disordered metals have been studied by Finkelstein\(^{18}\) and are not essential for the phenomena discussed in this paper, we ignore them in order to keep the presentation more transparent. Then the NLσM action can be written as

\[
\langle Z^p \rangle = \int \mathcal{D}[Q, V] e^{-S_Q - S_C},
\]

\[
S_Q = A \frac{\pi \nu}{2} \int dx \text{Tr} \left[ \frac{D}{4} (\nabla Q)^2 - (\hat{\xi} + \hat{V}) Q \right] + \frac{\lambda}{2} \int d\tau dx \sum_a V_a^2(x, \tau),
\]

\[
S_C = \frac{1}{2} \int d\tau dx dx' \sum_a V_a(x, \tau) K(x - x') V_a(x', \tau),
\]

where \(\text{Tr}\) denotes the trace over the replica, Matsubara and \(S \otimes T\) spaces, \(\nu\) is the density of states per spin at the Fermi level, and \(A\) is the wire cross section area. The matrices \(\hat{\xi}\) and \(\hat{V}\) have the following structure in the replica and \(S \otimes T\) spaces: \(\hat{\xi} = i\delta^{ab} \tau_3 \partial_x\), \(\hat{V} = \delta^{ab} \tau_0 V_a\), with \(\tau_i\)'s defined as \(\tau_i = t_i \otimes \sigma_0\), where \(\sigma_i, t_i\) are the Pauli matrices in the \(S\) and \(T\) spaces. The term \(S_Q\) defined in Eq. (2a), represents the part of the action that describes electrons moving in the presence of the auxiliary fields \(V_a\), whereas \(S_C\), defined in Eq. (2c), is the bare Coulomb action. The kernel \(K(x - x')\) describes the inverse effective Coulomb interaction in the wire. In particular, for a homogeneous wire in the absence of a nearby gate its Fourier transform is \(K(q) = 1/e^2 \ln \frac{q}{q_{Th}}\). We also assume that the external magnetic field is absent. The action (2) constitutes the NLσM.

The \(Q\)-matrix satisfies the nonlinear constraint \(Q^2 = \mathbb{1}\). It also satisfies the charge conjugation condition\(^{21}\):

\[
Q = CQ^T C^T, \quad C = \delta^{ab} \delta_{\varepsilon \varepsilon'} \otimes \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \delta^{ab} \delta_{\varepsilon \varepsilon'} \otimes (t_1 \otimes (-i\sigma_2)),
\]
where $a, b$ and $\varepsilon, \varepsilon'$ denote replica and Matsubara indices respectively, and the superscript $T$ denotes the transposition. In what follows we restrict ourselves to the case of strong spin-orbit scattering. In this case the $Q$-matrix belongs to the symplectic ensemble, and its matrix elements are unit matrices in the spin space.

To resolve the nonlinear constraint $Q^2 = \mathbb{1}$ we will use the exponential parameterization of the $Q$-matrix,

$$Q = e^{iW/2}Ae^{-iW/2}, \quad \Lambda_{ab}^{\varepsilon\varepsilon'} = \delta^{ab}\delta_{\varepsilon\varepsilon'}\tau_0\text{sgn}\varepsilon, \quad \{W,A\} = 0, \quad W = W^+, \quad (4)$$

where $\{A,B\}$ denotes the anticommutator of $A$ and $B$. The invariance of the $Q$-matrix with respect to the operation of charge conjugation, Eq. (3), and its hermiticity impose the following matrix structure on the rotation generators, $W$, in the $T$-space:

$$W_{\varepsilon\varepsilon'}^{ab} = \left( \begin{array}{cc} d & c \\ -c^* & -d^* \end{array} \right)_{\varepsilon\varepsilon'}^{ab}, \quad d^* = d^T = -\hat{c}. \quad (5)$$

The fields $(d,c)_{\varepsilon\varepsilon'}^{ab}$ represent the diffusion and cooperon degrees of freedom respectively, each being a unit matrix in the spin space.

The action in Eq. (2) is characterized by two parameters. The first is $G(L_T) = 4\pi\hbar\nu DA/L_T$, where $L_T = \sqrt{\hbar D/2\pi T}$ is the thermal diffusion length. It has the meaning of the dimensionless conductance of the wire segment of length $L_T$. The other one is the number of transverse channels in the wire, $N_{ch} = k_T^2 A/4\pi$. We consider a multichannel metallic wire, for which both parameters are large. From now on the Planck’s constant $\hbar$ is set to unity.

For large $G(L_T)$ we can evaluate the replicated partition function in the saddle point approximation. In this approximation the partition function is written as a sum of the contributions arising from all the saddle points:

$$\langle Z \rangle = \sum_{\text{saddle points}} e^{-S_{s.p.}} \int D[\delta Q, \delta V] e^{-\delta S[\delta Q, \delta V]}, \quad (6)$$

Here $S_{s.p.}$ denotes the NL$\sigma$M action evaluated at the saddle point, and $\delta Q, \delta V$ describe fluctuations of the $Q$-matrix and electric potentials $V_a$ around a particular saddle point. Finally $\delta S[\delta Q, \delta V]$ denotes the action change due to these fluctuations. In the next section we discuss the saddle points of the action (2) in the $G(L_T) = \text{const}, N_{ch} \to \infty$ limit, which is referred to below as the “$N_{ch} \to \infty$ limit” for brevity.

### III. SADDLE POINTS IN THE $N_{ch} \to \infty$ LIMIT

If the number of channels in the wire is sufficiently large, $e^2\nu A \gg 1$, one may neglect the Coulomb action, $S_C$, in Eq. (2) when looking for the saddle points. This corresponds to the charge neutrality limit, which can be seen by noting that formally such procedure corresponds to the limit $e \to \infty$, which clearly enforces electroneutrality. The saddle point equations in this limit are obtained by minimizing $S_Q$ in Eq. (2a) with respect to $V_a$ and $Q$, and read

$$D\nabla(Q\nabla Q) - [\hat{\varepsilon} + \hat{V}, Q] = 0, \quad (7a)$$

$$V_a - \frac{\pi}{4}\text{tr} Q^{aa}_{\tau \tau}(x) = 0, \quad (7b)$$

where $\text{tr}$ is the trace in the $S \otimes T$ space only. Equation (7a) is the Usadel equation, and Eq. (7b) represents the charge neutrality condition.

By direct substitution one can check that Eqs. (7) possess a set of stationary spatially uniform solutions, $Q_{\varepsilon\varepsilon'}^{ab} = \delta^{ab}\delta_{\varepsilon\varepsilon'}\tau_0\text{sgn}(\varepsilon + 2\pi T w_a)$, $V_a = 2\pi T w_a$, which are characterized by a set of integer winding numbers in each replica, $w_a$. All these solutions represent degenerate minima of the action (2a). The sum $4\sum_a w_a \equiv W$ (the factor 4 here arises from the $4 \times 4$ matrix structure of $Q_{\varepsilon\varepsilon'}^{ab}$ in the $S \otimes T$ space) defines the trace of the $Q$-matrix, $\text{Tr} Q = 2W$. The $Q$-matrices corresponding to the minima with different $w_a$, but the same $W$ can be transformed into each other via continuous rotations in the replica and Matsubara spaces, Eq. (4). Therefore, the NL$\sigma$M action contains soliton minima in which the $Q$-matrix and the potentials $V_a$ smoothly interpolate between their values in different uniform minima. Such solitons are similar to those first found in Ref. 8.

#### A. Single soliton solution

In this section we find an analytic solution to the saddle point equations (7) that correspond to a single soliton. To be specific, we construct a soliton that connects the following degenerate minima: $Q = \Lambda$, with all the winding
numbers \( w_n = 0 \) at \( x = -\infty \), and \( Q^{ab}_{\varepsilon \varepsilon'} = \delta^{ab} \delta_{\varepsilon \varepsilon'} \tau_0 \text{sgn}(\varepsilon + 2\pi T w_n) \) at \( x = \infty \), with \( w_{1,2} = \mp 1 \), all the other \( w_n \) being zero. This corresponds to a gradual change in the electric potential in replicas 1 and 2, \( V_{1,2} \), from zero at negative spatial infinity to \( \mp 2\pi T \) at positive infinity.

For such a soliton the generator \( W_0 \) parameterizing the saddle point \( Q \)-matrix via Eq. (4) corresponds to a rotation between Matsubara frequencies \( \pi T \) in replica 1, and \(-\pi T \) in replica 2. In this subspace \( W_0 \) has the following structure:

\[
W_0 = \begin{pmatrix} 0 & \hat{\lambda} \\ \hat{\lambda}^+ & 0 \end{pmatrix}, \quad \hat{\lambda} = \begin{pmatrix} \theta_d e^{i\phi} & \theta_c e^{i\chi} \\ -\theta_d e^{-i\phi} & -\theta_c e^{-i\chi} \end{pmatrix},
\]

(8)

where \( \theta_d, \theta_c, \phi \) and \( \chi \) are real parameters. In this equation the matrix element of \( W_0 \) in the upper-left-corner corresponds to \((W)_{11}^{\pi T,\pi T} \equiv 0 \), the one in the upper-right-corner to \((W)_{12}^{\pi T,-\pi T} = \hat{\lambda} \), and so on. All the other matrix elements of \( W_0 \) are zero.

Substituting the rotation generator (8) into Eq. (4) we obtain the matrix elements of the \( Q \)-matrix that participate in the rotation:

\[
\begin{pmatrix}
Q^{11}_{\pi T,\pi T} & Q^{12}_{\pi T,\pi T} \\
Q^{21}_{\pi T,-\pi T} & Q^{22}_{\pi T,-\pi T}
\end{pmatrix} = \begin{pmatrix}
\cos \theta_d \cos \theta_c & e^{i(\phi + \chi)} \sin \theta_d \sin \theta_c \\
e^{-i(\phi + \chi)} \sin \theta_d \sin \theta_c & \cos \theta_d \cos \theta_c
\end{pmatrix} \begin{pmatrix}
-\frac{ie^{i\phi} \sin \theta_d \cos \theta_c}{\cos \theta_d \cos \theta_c} & -\frac{-ie^{i\chi} \cos \theta_d \sin \theta_c}{\cos \theta_d \cos \theta_c} \\
\frac{ic \sin \theta_d \cos \theta_c}{\cos \theta_d \cos \theta_c} & \frac{-i \sin \theta_d \cos \theta_c}{\cos \theta_d \cos \theta_c}
\end{pmatrix} \begin{pmatrix}
\cos \theta_d \cos \theta_c & -\frac{-i \cos \theta_d \cos \theta_c}{\cos \theta_d \cos \theta_c} \\
\frac{ie^{-i\chi} \sin \theta_d \sin \theta_c}{\cos \theta_d \cos \theta_c} & \frac{ie^{i\phi} \sin \theta_d \sin \theta_c}{\cos \theta_d \cos \theta_c}
\end{pmatrix}.
\]

(9)

All the other matrix elements are those of the \( \Lambda \)-matrix.

The action for such a \( Q \)-matrix is independent of the angles \( \phi \) and \( \chi \) and depends only on \( (\nabla \phi)^2 \) and \( (\nabla \chi)^2 \) with positive coefficients. Therefore the action minimum corresponds to coordinate independent angles \( \phi \) and \( \chi \). It can be shown that the soliton solutions with the minimum action correspond to either \( \theta_d \neq 0, \theta_c = 0 \) (diffusonlike rotation) or \( \theta_d = 0, \theta_c \neq 0 \) (Cooperon-like rotation). In these cases substitution of Eq. (9) into (12) gives \( V_{1,2}(x) = \mp \pi T \{ 1 - \cos \theta_{d,c}(x) \} \) for the diffusonlike and Cooperon-like rotations respectively. Then Eq. (10) yields

\[
\nabla^2 \theta_{d,c} - \frac{1}{2L_T^2} \sin 2\theta_{d,c} = 0.
\]

(10)

The solution that corresponds to the sought soliton is

\[
\theta_{d,c}(x) = 2 \arctan \left( e^{(x-x_0)/L_T} \right) \equiv \theta_0(x - x_0),
\]

(11)

giving for the electric potentials

\[
V_{1,2}(x) = \mp \mathbb{V}^0(x - x_0) \equiv \mp \pi T \{ 1 + \tanh[(x - x_0)/L_T] \},
\]

(12)

which clearly satisfies \( V_{1,2}(x \to -\infty) = 0 \) and \( V_{1,2}(x \to \infty) = \mp 2\pi T \). Here \( x_0 \) denotes the soliton position.

Substituting the saddle point values of \( Q \) and \( V_{a} \), Eqs. (9) and (12), into the action (25), we obtain the action for a single soliton

\[
S_0 = G(L_T).
\]

(13)

We note that this action does not depend on the soliton position \( x_0 \), and the angles \( \phi \) and \( \chi \) in Eq. (9). However, for the diffusonlike (\( \theta_c = 0 \)) soliton the different values of the angle \( \chi \) correspond to the same \( Q \)-matrix, and similarly different values of \( \phi \) correspond to the same \( Q \)-matrix for the Cooperon-like (\( \theta_d = 0 \)) soliton. Therefore the action for the fluctuations about the soliton has only two zero modes. One is associated with a translation of the soliton (change in \( x_0 \)). The other corresponds to a rotation of the \( Q \)-matrix in the replica and Matsubara space caused by a uniform change in either \( \phi \) or \( \chi \), depending on whether we consider a diffusonlike or a Cooperon-like soliton. The presence of these zero modes needs to be borne in mind when integrating over the fluctuations about the soliton configurations.

B. Fluctuations around a single soliton

In this section we evaluate the single soliton contribution to the replicated partition function, Eq. (6), in the \( N_{ch} \to \infty \) limit. This requires evaluating the functional integral over the fluctuations of the \( Q \)-matrix and the potentials \( V_{a} \) around the single soliton saddle point.
As was explained at the end of Sec. III A, the fluctuation spectrum has two zero modes. We show below that all the other fluctuations are massive and integrate over them in the gaussian approximation. The resulting fluctuation determinant is convergent and is evaluated below. The integration over the zero modes is reduced to the integration over the soliton position and the rotation angle.

The translational zero mode represents a simultaneous spatial shift of the saddle point solution for the $Q$-matrix and the static (zero Matsubara frequency) component of the potentials $V_a$. In order to simplify the treatment of this zero mode we first integrate over the latter. This step involves no approximations since the action (2b) is quadratic in $V_a$. The resulting action depends only on the nonzero Matsubara components of $V_a$ and on the $Q$-matrix. In this representations the zero modes involve only the $Q$-matrix degrees of freedom, whereas all fluctuations of the nonzero Matsubara components of $V_a$ are massive. Then the single soliton contribution to the partition function, Eq. (6), in the $N_{ch} \rightarrow \infty$ limit can be written as $\exp[-G(L_T)] \Gamma_p$, where $\Gamma_p$ is the functional integral over the fluctuations about the soliton and is given by

$$\Gamma_p = \alpha^p \int \mathcal{D}[W, \delta V] e^{-S^{(2)}[W, \delta V]}.$$  

Here the fluctuations of the nonzero Matsubara components of the electric potential are denoted by $\delta V_a$, the matrix $W$ parameterizes the deviation of the $Q$-matrix from the saddle point, and $\alpha^p$ is the factor coming from integration over the static components of $V_a$. We will see later that in order to obtain the physical observables we will only need to evaluate $\Gamma_p$ at $p = 0$. Therefore, the value of $\alpha$ is of no importance. Finally, the quadratic fluctuation action $S^{(2)}[W, \delta V]$ is obtained by integrating over the fluctuations of the static component of $V_a$ in Eq. (2b), and expanding the resulting action to the second order in $W$. Its form depends on the $Q$-matrix parametrization.

In the remainder of this section we show that the fluctuation integral $\Gamma_p$ can be expressed as

$$\Gamma_p = \alpha^p G(L_T) \Upsilon_p \int \frac{dx_0}{L_T},  \tag{14}$$

where $x_0$ is the position of the soliton and $\Upsilon_p$ is a numerical factor independent of the system parameters. In order to evaluate the thermodynamic quantities we need only the $p = 0$ value of this quantity, which is calculated below, $\Upsilon \equiv \Upsilon_{p=0} \approx 8$.

In the remainder of the present section we derive Eq. (14). The presentation is organized as follows. In Sec. III B 1 we give the expression for the fluctuation action. In Sec. III B 2 we carry out the integration over the $Q$-matrix fluctuations. Section III B 3 deals with integration over the electric potential fluctuations. The reader not interested in the derivation of Eq. (14) may wish to proceed directly to Sec. IV, where we use it to evaluate nonperturbative corrections to the thermodynamic quantities.

1. Fluctuation action

We parameterize the deviations of the $Q$-matrix from the saddle point in terms of the matrix $W$, whose structure is described by Eq. (5), as follows:

$$Q = e^{iW_0/2} e^{iW/2} \Delta e^{-iW/2} e^{-iW_0/2}. \tag{15}$$

Here the matrix $W_0$ parameterizes the saddle point $Q$-matrix. For the soliton described in Sec. III A it is given by

$$W_0(x) = \begin{pmatrix} 0 & i\theta_0(x) \tau_i \\ -i\theta_0(x) \tau_i & 0 \end{pmatrix}. \tag{16}$$

Here $i = 0,1$ corresponds to diffusonlike and Cooperon-like rotations, $\theta_0$ is defined in Eq. (11), and we set $\phi = \chi = \pi/2$, and $x_0 = 0$ for convenience.

In the following we use dimensionless coordinate $\xi = x/L_T$, dimensionless fermionic Matsubara frequencies, $\epsilon = \pi/2\pi T$, and dimensionless Matsubara components of the electric potential, $V_\omega = \delta V_\omega/2\pi T$, where $\omega$ is an integer defining the bosonic Matsubara frequency, such that the latter is written as $2\pi T \omega$. In these variables the quadratic action in Eq. (13) can be written as

$$S^{(2)}[W, V] = S_{VV} + S_{WW} + S_{WW}.$$

Here $S_{VV}$ denotes the part of the action that is quadratic in the potentials $V_a$,

$$S_{VV} = \frac{G(L_T)}{2} \sum_a \sum_{\omega \neq 0} \int d \xi V_\omega^*(\xi) V_\omega^{-1}(\xi'), \tag{17a}$$
indices as the rotation generator,constituting it are identical. The term containing \( W \)independently. Moreover, with the exception of \( S_{12} \) in Eq. (18b), shows that the variables \( W \)for the diffuson-like soliton, i.e. we set \( \tau = \pi T \) and \( V \)to \( \pi T \). Here \( \langle \dots \rangle_w \) denotes the Hermitian conjugation in the \( S \otimes T \) space, i.e. corresponds to complex conjugation and transposition within a 4 \times 4 block, without interchanging replica or Matsubara indices. The diffuson-like soliton corresponds to \( \tau_i = \tau_0 \), and \( \tau_i = \tau_1 \) for the Cooperon-like one. To be specific, in what follows we consider the case of a diffuson-like soliton, i.e. we set \( \tau_i = \tau_0 \). In the Cooperon-like case the treatment exactly parallels the one presented below.

Introducing the notation

\[
\Gamma_W = \int D W \exp(-S_{WW}), \tag{19}
\]

and

\[
\Gamma_V = \int D [\mathcal{V}] e^{-S_{VV}} e^{-S_{WW}} = \int D [\mathcal{V}] e^{-S_{VV} + \frac{1}{2}(S_{WW})_w}, \tag{20}
\]

where \( \langle \dots \rangle_w \) denotes the gaussian average with respect to the action \( S_{WW} \), we can write Eq. (13) as

\[
\Gamma_p = \alpha_p \Gamma_W \Gamma_V. \tag{21}
\]

We evaluate quantities \( \Gamma_W \) and \( \Gamma_V \) in Sections III B 2 and III B 3.

2. Integration over \( W \)

We now evaluate the functional integral over the fluctuations of the \( Q \)-matrix, \( \Gamma_W \) in Eq. (19). Examination of the quadratic action in Eq. (18b), shows that the variables \( W_{<0,0^\prime>} \) with different replica or Matsubara indices fluctuate independently. Moreover, with the exception of \( W_{<0,0^\prime>} \) for each \( W_{<0,0^\prime>} \) the actions for the diffusons and Cooperons constituting it are identical. The term containing \( W_{<0,0^\prime>} \) is special because it has the same replica and Matsubara indices as the rotation generator \( W_0 \) parameterizing the saddle point. The fluctuations of the diffuson and Cooperon components of \( W_{<0,0^\prime>} \) are also independent, but their propagators are different. In particular, we will see that for a soliton represented by a diffuson-like rotation only diffuson part of \( W_{<0,0^\prime>} \) has zero modes, and vice versa for a Cooperon-like rotation.

In terms of the diffuson and Cooperon variables, see Eq. (5), the action (18b) can be written as
TABLE I: Potentials $U_{εε'}^{ab}$ appearing in the operators $\hat{L}_{εε'}^{ab}$, Eq. \[23\]. Each entry gives the potential specific to particular replica and Matsubara indices in terms of the potentials $v_{1,2}$ and $u$ defined in Eq. \[25\]. The Latin letters $(j, k)$ denote replica indices not equal to 1 or 2.

| $ε, ε'$ \( ab\) | $jk$ | 1j | 2j | j1 | j2 | 11 | 12 | 21 | 22 |
|-----------------|-----|----|----|----|----|----|----|----|----|
| $ε > πT, ε' < −πT$ | 0 | $v_2 - 1$ | $v_1$ | $v_2$ | $v_2 - 1$ | 0 | $2v_2 - 2$ | $2v_1$ | 0 |
| $ε = πT, ε' < −πT$ | 0 | $u$ | $v_1$ | $v_2$ | $v_2 - 1$ | $v_1 + u$ | $v_2 + u - 1$ | $2v_1$ | 0 |
| $ε > πT, ε' = −πT$ | 0 | $v_2 - 1$ | $v_1$ | $v_2$ | $u$ | 0 | $v_2 + u - 1$ | $2v_1$ | $v_1 + u$ |
| $ε = πT, ε' = −πT$ | 0 | $u$ | $v_1$ | $v_2$ | $u$ | $v_1 + u$ | Excluded | $2v_1$ | $v_1 + u$ |

where the primed sum means that the term with $a = 1, b = 2, ε = πT$ and $ε' = −πT$ is excluded, and $(d, c)_{πT, −πT}$. The operators $\hat{L}_{εε'}^{ab}, \hat{L}_{d,c}$ are all of the Schrödinger type and have the form,

$$\hat{L}_{d,c} = \frac{G(L_T)}{4} \left( \hat{L}_{ω=1} + u_{d,c}(ξ) \right),$$

$$\hat{L}_{εε'}^{ab} = \frac{G(L_T)}{4} \left( \hat{L}_{ε'ε} + U_{εε'}^{ab}(ξ) \right),$$

(23)

with the operator $\hat{L}_{ω}$ defined as

$$\hat{L}_{ω} = -\frac{d^2}{dξ^2} + ω,$$

(24)

with $ω$ and $ε$ being the appropriate dimensionless Matsubara frequencies. The potentials $u_{d,c}$ for $d_s, c_s$ are given by

$$u_{d}(ξ) = -\frac{2}{\cosh(ξ)}, \quad u_{c}(x) = -\frac{1}{\cosh(ξ)}.$$  

(25)

The potentials $U_{εε'}^{ab}$ depend on the replica and Matsubara indices involved and can be expressed in terms of the following potentials,

$$v_{1,2}(ξ) = \frac{1}{2}[1 ± \tanh(ξ)], \quad u(ξ) = −\frac{3}{4\cosh^2(ξ)}.$$  

(26)

The expressions for the potentials $U_{εε'}^{ab}$ in terms of $v_{1,2}(ξ)$ and $u(ξ)$ are summarized in Table I.

The operators $\hat{L}_{εε'}^{ab}$ and $\hat{L}_{c}$ are positive definite. The operator $\hat{L}_{d}$, Eq. \[23\], with the potential $u_{d}$, defined in Eq. \[26\], has one zero eigenvalue, with all the other ones being positive and separated by a finite gap. The integration over the zero modes requires a special consideration. We therefore defer the integration over the variables $d_s$ in $Γ_W$, Eq. \[19\], to the end of this section and begin by integrating over al the other variables first. To this end we introduce an auxiliary quantity $Γ'_W$ as

$$Γ_W = \frac{\int D[d_s] e^{-\frac{1}{4} \sum \int dξ d_s \hat{L}_{ω=1} d_s}}{\int D[d_s] e^{-\frac{1}{4} \sum \int dξ d_s \hat{L}_{ω=1} d_s}} Γ'_W \equiv Γ_d Γ'_W.$$  

(27)

Calculation of $Γ'_W$ reduces to evaluation of gaussian integrals. Since $(d, c)_{εε'}^{ab}$ are complex fields, the integration over each of them gives a factor of an inverse determinant of the corresponding operator in the quadratic action, Eq. \[22\], and we obtain the following expression for $Γ'_W$,

$$Γ'_W = \frac{\alpha^p}{\det \left( \frac{G(L_T)}{4} \hat{L}_{ω=1} \right) \det \hat{L}_c} \prod_{ε > 0, ε' < 0} \left( \det \hat{L}_{εε'}^{ab} \right)^{-2}.$$
The prime indicates that the product does not include the contribution from \(a = 1, b = 2, \varepsilon = \pi T, \varepsilon' = -\pi T\). The operators \(L_{e_{c}^{a}}^{b}\) in the expression for \(\Gamma_{W}'\) can be classified according to whether their replica indices correspond to the replicas participating in the soliton rotation. In particular, for \(a, b > 2\), the operators \(L_{e_{c}^{a}}^{b}\) are insensitive to the presence of a soliton. Denoting each of these operators as \(\hat{L}_{k}^{j}\), we see that the product over the replicas with \(a, b > 2\) contributes a factor \((\prod_{\varepsilon > 0, \varepsilon' < 0} \det \hat{L}_{k}^{j})^{-(p-2)^{2}}\) to the fluctuation determinant. Analogously, for \(a = 1, 2\) and \(b > 2\), we have \(p - 2\) identical operators \(L_{e_{c}^{a}}^{b}\) for each of \(a = 1\) and \(a = 2\), which we denote as \(\hat{L}_{1}^{j}\) and \(\hat{L}_{2}^{j}\) respectively. Finally, there are \(p - 2\) equal operators for \(a > 2\) and each of \(b = 1\) and \(b = 2\), denoted as \(\hat{L}_{k}^{j}\) and \(\hat{L}_{k}^{j}\). Using these observations we rewrite the previous equation as

\[
\Gamma_{W}' = \frac{\alpha^{p}}{\det \left( \frac{G(L_{\omega})}{4} L_{\omega = 1} \right) \det L_{c}} \left( \prod_{\varepsilon > 0, \varepsilon' < 0} \det \hat{L}_{k}^{j} \right)^{-(p-2)^{2}} \times \left( \prod_{\varepsilon > 0, \varepsilon' < 0} \det \hat{L}_{k}^{j} \right)^{-(p-2)}
\]

In the above expression the prime means that \(\hat{L}_{k}^{j}\) is excluded from the product. To compute the thermodynamic quantities we will need only the value of \(\Gamma_{W}'\) at \(p = 0\), for which we use the same notation,

\[
\Gamma_{W}' = \frac{\alpha^{p}}{\det \left( \frac{G(L_{\omega})}{4} L_{\omega = 1} \right) \det L_{c}} \left( \prod_{\varepsilon > 0, \varepsilon' < 0} \det \hat{L}_{k}^{j} \right)^{-(p-2)^{2}} \times \left( \prod_{\varepsilon > 0, \varepsilon' < 0} \det \hat{L}_{k}^{j} \right)^{2}
\]

Using Eq. (18), the definitions (25)–(26), and the identity \(\ln \det \hat{O} = \text{tr} \ln \hat{O}\) we can write for (28)

\[
\ln \Gamma_{W}' = 2 \sum_{\omega = 1}^{\infty} \left( 4 \omega \text{tr} \ln \frac{(L_{\omega} + v_{1})(L_{\omega} + v_{2})}{L_{\omega}(L_{\omega} + 1)} - \omega \text{tr} \ln \frac{(L_{\omega} + 2v_{1})(L_{\omega} + 2v_{2})}{L_{\omega}(L_{\omega} + 2)} + 4 \omega \ln \frac{L_{\omega} + u_{1}}{L_{\omega}} - 2 \omega \ln \frac{L_{\omega} + u_{1}}{L_{\omega}} \right)
\]

where \(\text{tr} \ln \) denotes the trace in the coordinate space, \(\text{tr} \ln \hat{O} = \int d\xi \hat{O}(\xi, \xi)\). The terms in Eq. (29) are grouped in such a way that each is finite at a given \(\omega\), i.e. does not diverge with the length of the system.

In Appendix A it is shown that each term in Eq. (29) can be evaluated using the formula

\[
\text{tr} \ln \frac{(L_{\omega} + U_{1})(L_{\omega} + U_{2})}{L_{\omega}(L_{\omega} + h)} = \ln tt' = \ln \sqrt{\frac{\omega + h}{\omega}},
\]

where the potentials \(U_{1}(\xi)\) and \(U_{2}(\xi)\) satisfy \(U_{1}(\xi) = U_{2}(-\xi), U_{1}(-\infty) = 0, U_{1}(\infty) = h\) (for the potentials from Eq. (29) the parameter \(h\) takes on the values 0, 1 or 2). The quantities \(t, t'\) describe the \(\xi \to \infty\) asymptotics of the two independent solutions of the equation

\[
\left[ L_{\omega} + U_{1}(\xi) \right] \psi = 0.
\]

Namely, if we find the two solutions, \(\psi_{1,2},\) whose asymptotics at \(\xi \to \pm \infty\) are given by \(\psi_{1}(\xi \to -\infty) \approx \exp(\sqrt{\omega + h} \xi)\) and \(\psi_{2}(\xi \to +\infty) \approx \exp(-\sqrt{\omega + h} \xi)\), the parameters \(t\) and \(t'\) are given by the coefficients in front of the growing exponentials in the asymptotics of these solutions at the opposite infinities,

\[
\psi_{1}(\xi) \approx t \exp(\sqrt{\omega + h} \xi), \quad \xi \to \infty; \quad \psi_{2}(\xi) \approx t' \exp(-\sqrt{\omega + h} \xi), \quad \xi \to -\infty.
\]
The last equality in Eq. (30) holds since $\sqrt{\omega + h} t = \sqrt{\omega} t'$, see Appendix A for details. The case of a potential vanishing at spatial infinities is recovered from Eq. (30) by setting $h = 0$, $U_1 = U_2 = U$, $t = t'$:

$$\text{tr}_1 \ln \frac{L_\omega + U}{L_\omega} = \ln t. \quad (33)$$

In order to find the parameters $t$ and $t'$ corresponding to the potentials in Eq. (20), we note that for each potential Eq. (31) has the general form

$$- \frac{d^2 \psi}{d\xi^2} + \left[ \omega - \frac{\alpha}{\cosh^2 \xi} + \frac{\beta}{2} (1 + \tanh \xi) \right] \psi = 0, \quad (34)$$

where the values of the parameters $\alpha, \beta$ depend on the specific potential. For example, the potentials $u(\xi)$ and $v_1(\xi)$ in Eq. (20) correspond to $\alpha = 3/4, \beta = 0$ and $\alpha = 0, \beta = 1$ respectively.

If one introduces the variable $z = (1 + \tanh \xi)/2$, and $y(z) = z^{-\sqrt{\omega}/2}(1 - z)^{-\sqrt{\omega+\beta}/2}\psi(z)$, the above equation reduces to the hypergeometric equation for $y(z)$:

$$z(1-z) \frac{d^2 y}{dz^2} + \left[ c - (a + b + 1)z \right] \frac{dy}{dz} - aby = 0, \quad (35)$$

where the parameters $a$, $b$, and $c$ are given by the following expressions:

$$c = 1 + \sqrt{\omega},$$
$$a = \frac{1}{2}(1 + \sqrt{\omega} + \sqrt{\omega + \beta} - \sqrt{1 + 4\alpha}),$$
$$b = \frac{1}{2}(1 + \sqrt{\omega} + \sqrt{\omega + \beta} + \sqrt{1 + 4\alpha}). \quad (36)$$

Using the properties of the hypergeometric functions $F(a,b,c,z)$ and switching back to the original variable $\xi = \arctanh(2z - 1)$ it is easy to show that the two independent solutions $\psi_{1,2}$ of Eq. (34) satisfying the desired asymptotics, $\psi_1(\xi \to -\infty) \to \exp(\sqrt{\omega} \xi)$ and $\psi_2(\xi \to \infty) \to \exp(-\sqrt{\omega + h} \xi)$, are given by

$$\psi_1(z) = z^{\sqrt{\omega}/2}(1 - z)^{\sqrt{\omega+\beta}/2} F(a,b,c,z),$$
$$\psi_2(z) = z^{-\sqrt{\omega}/2}(1 - z)^{-\sqrt{\omega+\beta}/2} F(a,b,a + b - c + 1,1 - z). \quad (37)$$

The asymptotic behavior of $\psi_1(\xi)$ at $\xi \to +\infty$ is

$$\psi_1(\xi \to +\infty) \approx \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)} e^{\sqrt{\omega+\beta} \xi}, \quad (38)$$

where $\Gamma(x)$ is the Euler gamma function. Comparing Eq. (35) with Eq. (32), we find that the value of the coefficient $t$ entering Eq. (30) is given by

$$t = \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)}, \quad (39)$$

with $a, b, c$ defined in Eq. (36). Using Eqs. (30), (33) and (39) and the identity $\Gamma(x + 1) = x\Gamma(x)$ we obtain for $\Gamma'_W$, Eq. (29):

$$\ln \Gamma'_W = 2 \sum_{\omega=1}^{\infty} \left( 4\omega \ln \left[ \frac{4\sqrt{\omega}\sqrt{\omega + 1}}{(\sqrt{\omega} + \sqrt{\omega + 1})^2} \right] - \omega \ln \left[ \frac{4\sqrt{\omega} \sqrt{\omega + 2}}{(\sqrt{\omega} + \sqrt{\omega + 2})^2} \right] \right) \approx \ln 0.08. \quad (40)$$
One can check that the sum over $\omega$ above converges, since the summand behaves like $\omega^{-3/2}$ for large $\omega$. The last equality was obtained by performing the summation numerically.

We now complete the evaluation of $\Gamma_W$ by computing the functional integral $\Gamma_d$, defined in Eq. (27). The operator $\hat{L}_d$ defined by Eqs. (24) and (29) has one zero eigenvalue. The corresponding eigenfunction is $1/\cosh \xi$. The fluctuations of $Rd_s$ and $\delta d_s$ in the numerator of Eq. (27) along this mode correspond to the rotational and translational zero modes of the soliton discussed at the end of Sec. II A.

Indeed, in the parametrization (15) a soliton displacement, $Q_0(\xi) \to Q_0(\xi - \xi_0)$, by a small amount $\xi_0 = x_0/L_T$ is described by the generator $\hat{W}_{\xi_0}$ that can be obtained from the condition

$$\delta Q \approx -ie^{iW_0/2} \Delta W_{\xi_0} e^{-iW_0/2} = -\frac{\partial Q_0}{\partial \xi} \xi_0,$$

where $Q_0$ is given by Eq. (15) with $W = 0$ and $W_0$ from Eq. (10) with $i = 0$. From this equation it follows that $W_{\xi_0}$ has the same structure as $W_0$, Eq. (8), with matrix $\hat{\lambda}$ replaced by $(W_{\xi_0})_0^{12}$, and $\eta$ defined as

$$(W_{\xi_0})_0^{12} = -\frac{\partial \theta_0(\xi)}{\partial \xi} \xi_0.$$

Comparing this expression with Eq. (41), we see that the soliton translation corresponds to the diffusion fluctuation of the form $d_s(\xi) = -i\frac{\partial \theta_0(\xi)}{\partial \xi} \xi_0 = -(i/\cosh \xi) \xi_0$. Along the same lines of reasoning it can be shown that the soliton rotation by the angle $\phi_0$, $\phi \to \pi/2 + \phi_0$ in Eq. (41), corresponds to $d_s(\xi) = (1/\cosh \xi) \phi_0$, and represents the other zero mode.

We separate the functional integral over $d_s$ in the numerator of Eq. (27) into a product of integrals over the zero and massive modes:

$$\Gamma_d = \int d\xi_0 \int d\phi_0 \left[ \frac{\Gamma(L_T)}{4} \int \frac{d\xi \xi^2}{\cosh \xi} \right] \int \frac{d\xi_0}{\sqrt{\hat{L}_d \cdot L_{\omega=1+u_d} d_s}}.$$

where $\hat{d}_s$ contains the massive modes only, and $J$ denotes the Jacobian for the change of variables $\{d_s\} \to \{\hat{d}_s, \xi_0, \phi_0\}$. The product of the Jacobian $J$ and the ratio of the functional integrals in this expression can be evaluated using the following trick. We introduce a regularized ratio $\Gamma_d(\eta)$ of the functional integrals over $d_s$ in the last equation by infinitesimally shifting the frequency $\omega$ from unity, $\omega \to 1 + \eta$, where $\eta$ is positive. As a result the zero modes acquire a finite mass and $\Gamma_d(\eta)$ can be written as

$$\Gamma_d(\eta) = \int d\xi_0 \int d\phi_0 \exp \left[ -\frac{\pi G(L_T)}{4} \int \frac{d\xi \xi^2}{\cosh \xi} (\xi_0^2 + \phi_0^2) \right] \int \frac{d\xi_0}{\sqrt{\hat{L}_d \cdot L_{\omega=1+u_d} d_s}}.$$

(41)

On the other hand this ratio of gaussian integrals can be calculated using Eq. (35), (37), and (36). In the limit of $\eta \to 0$ we obtain

$$\Gamma_d(\eta) = \det \begin{bmatrix} \hat{L}_{\omega=1+\eta} & \hat{L}_{\omega=1+\eta} + u_d \end{bmatrix} \frac{\Gamma(\sqrt{1+\eta}) \Gamma(\sqrt{1+\eta} + 2)}{\Gamma(\sqrt{1+\eta + 2})} \approx \frac{4}{\eta}.$$

(42)

To arrive at this expression set $\omega = 1 + \eta$ in Eq. (36) and used the fact that the potential $u_d$ corresponds to $\alpha = 2$ and $\beta = 0$. Integrating over $\xi_0$ and $\phi_0$ in Eq. (41) and comparing the result with Eq. (42) we conclude that $\Gamma_d$ can be written as

$$\Gamma_d = \frac{2 G(L_T)}{\pi} \int d\phi_0 \int d\xi_0.$$

Substituting this expression into Eq. (27) and integrating over $\phi_0$ we obtain the following expression for $\Gamma_W$,

$$\Gamma_W = 4 \Gamma_W G(L_T) \int d\xi_0,$$

(43)

with $\Gamma_W$ given by Eq. (40).
3. Integration over the electric potential fluctuations

We now turn to the evaluation of the functional integral over the potential fluctuations, $\Gamma_V$ in Eq. (20). The action for the potential fluctuations is obtained by evaluating the gaussian average, $\langle S_{WV}^2 \rangle_w$, in Eq. (20) with respect to the action $\delta_{WW}$ in Eq. (18). The result of this tedious, but straightforward calculation can be expressed in the form

$$S_{WV} - \frac{1}{2}\langle S_{WV}^2 \rangle_w = \frac{G(L)}{2} \sum_{\omega \neq 0} \int d\xi d\xi' \left[ \sum_{\alpha=1}^p V_\alpha(\xi) \Pi_\alpha(\xi - \xi') V_{a\omega}(\xi') - \sum_{a=1,2} V_{a\omega}(\xi) \delta_2(\xi, \xi') V_{a\omega}(\xi') \right]. \tag{44}$$

Here, in the $p-2$ replicas not participating in the soliton rotation, the dimensionless polarization operator $\Pi_0^\omega(\xi - \xi')$ is given by the usual expression,

$$\Pi_0^\omega(\xi - \xi') = \int \frac{dq}{2\pi} e^{iq(\xi - \xi')} \frac{q^2}{|\omega| + q^2}, \tag{45}$$

and in the remaining two replicas ($a = 1, 2$) the dimensionless polarization operators acquire a correction $\delta_2(\xi, \xi')$ due to the presence of a soliton,

$$\delta_2(\xi, \xi') = \cos \frac{\theta(\xi)}{2} G_1^\omega(\xi, \xi') \cos \frac{\theta(\xi')}{2} + \sin \frac{\theta(\xi)}{2} G_2^\omega(\xi, \xi') \sin \frac{\theta(\xi')}{2} - G_0^\omega(\xi - \xi'). \tag{46}$$

In the last equation we introduced the following Green’s functions:

$$G_0^\omega(\xi - \xi') = \hat{L}_{0|\omega|}^{-1},$$

$$G_{1,2}^\omega(\xi, \xi') = (\hat{L}_{|\omega|} + v_{1,2} + u)^{-1}, \tag{47}$$

where the operator $L_{\omega}$ and the potentials $v_{1}(\xi), v_{2}(\xi)$, and $u(\xi)$ are defined in Eqs. (24) and (26).

We note that the polarization operator in the presence of the soliton, Eq. (44), is diagonal in Matsubara frequencies. This is a consequence of the fact that the soliton saddle point is static. We also note that no inter-replica couplings between the potential fluctuations are generated.

As an important consistency check, let us prove that

$$\int_{-\infty}^{\infty} d\xi' \Pi_0^{\omega,0}(\xi, \xi') = 0, \tag{48}$$

which must hold due to particle number conservation. The polarization operator $\Pi_0^\omega$ automatically satisfies this property, as its Fourier transform is proportional to $q$. To prove that $\Pi_0^\omega$ satisfies the same condition, we note that $(-\frac{d^2}{d\xi^2} + v_1 + u) \cos \frac{\theta_0}{2} = 0$, and $(-\frac{d^2}{d\xi^2} + v_2 + u) \sin \frac{\theta_0}{2} = 0$. Thus we can write

$$\int_{-\infty}^{\infty} d\xi' \delta_2(\xi, \xi') = \int_{-\infty}^{\infty} d\xi' \cos \frac{\theta_0(\xi)}{2} G_1^\omega(\xi, \xi') \cos \frac{\theta_0(\xi')}{2} + \int_{-\infty}^{\infty} d\xi' \sin \frac{\theta_0(\xi)}{2} G_2^\omega(\xi, \xi') \sin \frac{\theta_0(\xi')}{2} - \int_{-\infty}^{\infty} d\xi' G_0^\omega(\xi - \xi') = \frac{1}{|\omega|} \cos^2 \frac{\theta_0}{2} + \frac{1}{|\omega|} \sin^2 \frac{\theta_0}{2} - \frac{1}{|\omega|} = 0, \tag{49}$$

as expected.

Performing the gaussian integral over $\nu$ in Eq. (20) and taking the number of replicas $p$ to zero we obtain

$$\Gamma_V = \prod_{\omega > 0} \frac{\det^2 \Pi_0^\omega}{\det (\Pi_0^\omega - \delta_2)} = e^{-2 \sum_{\omega > 0} \text{tr} \ln [1 - \Pi_0^\omega(\Pi_0^\omega)^{-1}].} \tag{50}$$

Due to the complicated form of $\delta_2$ evaluation of this quantity explicitly is a daunting task. In particular, the method of the previous section does not apply here because the polarization operators are not represented by Schrödinger-type operators. However, we note that the dimensionless polarization operators $\Pi_0^\omega$ and $\Pi_0^\omega - \delta_2$ are independent of the system parameters. Provided the sum over $\omega$ in Eq. (50) converges, it is clear that $\Gamma_V$ is a parameter-independent numerical factor. Below we prove that the sum over $\omega$ in the exponent of Eq. (50) does converge, and evaluate $\Gamma_V$ numerically.

To this end we obtain the large-$\omega$ asymptotics of the summand in Eq. (50). This can be done by expanding the logarithm in Eq. (50) to the first order in $\delta_2((\Pi_0^\omega)^{-1})$. 

\[
\text{tr}\ln[1 - \delta\Pi^\omega(\Pi_0^\omega)^{-1}] \approx -\text{tr}\delta\Pi^\omega(\Pi_0^\omega)^{-1} = - \int d\xi\, d\xi' \delta\Pi^\omega(\xi, \xi')(\Pi_0^\omega)^{-1}_\xi - \xi'.
\]

Using the Fourier transform of \(\Pi_0^\omega\) from Eq. (45), the above trace can be written as

\[
\text{tr}\delta\Pi^\omega(\Pi_0^\omega)^{-1} = \int \frac{dq}{2\pi} \frac{\omega + q^2}{q^2} \int d\xi\, d\xi' e^{-iq\xi} \delta\Pi^\omega(\xi, \xi') e^{iq\xi'}. 
\]

We note that each of the two terms in the expression for \(\delta\Pi^\omega\), Eq. (46), can be written as \(\psi_0(\xi)G(\xi, \xi')\psi_0(\xi')\), where \(G(\xi, \xi')\) is the resolvent of the operator \((\omega - \frac{d^2}{d\xi^2} + U)\). The phase factors in the last integral in Eq. (52) can be interpreted as a gauge transformation of the Green’s function \(\tilde{G}(\xi, \xi') = e^{-iq\xi}G(\xi, \xi')e^{iq\xi'} = (\omega + (\frac{1}{\xi} + q)^2 + U)^{-1}\). Therefore, the integral can be written as

\[
\int d\xi\, d\xi' e^{-iq\xi} \delta\Pi^\omega(\xi, \xi') e^{iq\xi'} = \int d\xi\, d\xi' \cos\frac{\theta_0(\xi)}{2}\left(\frac{1}{\omega + q^2 + 2q(\frac{1}{\xi} + u)}\right)\xi,\xi' \cos\frac{\theta_0(\xi')}{2} \\
+ \int d\xi\, d\xi' \sin\frac{\theta_0(\xi)}{2}\left(\frac{1}{\omega + q^2 + 2q(\frac{1}{\xi} + v)}\right)\xi,\xi' \sin\frac{\theta_0(\xi')}{2} \\
- \int d\xi\, \frac{1}{\omega + q^2}.
\]

Then we expand each of the first two kernels in the r.h.s. of Eq. (53) in powers of \([-\frac{d^2}{d\xi^2} + v, u + 2q(\frac{1}{\xi} + q)]\) / \((\omega + q^2)\) to the order that gives first nonvanishing contribution to the entire integral. The zeroth order term vanishes in the same way as it happened in Eq. (19), and so does the first order one being proportional to \(\cos\frac{\theta_0}{2}\cos\frac{\theta_0}{2} + \sin\frac{\theta_0}{2}\sin\frac{\theta_0}{2} \equiv \frac{d}{d\xi}(1/2) = 0\). Therefore, expansion to the second order gives the first nonzero contribution, and we arrive at

\[
\text{tr}\delta\Pi^\omega(K + \Pi_0^\omega)^{-1} \approx \int \frac{dq}{2\pi} \frac{\omega + q^2}{q^2} (-4) \frac{q^2}{q^2} \int d\xi\, \left\{\cos\frac{\theta_0}{2} \frac{d^2}{d\xi^2} \cos\frac{\theta_0}{2} + \sin\frac{\theta_0}{2} \frac{d^2}{d\xi^2} \sin\frac{\theta_0}{2}\right\}
\approx \frac{1}{2\omega^{3/2}}.
\]

Eq. (54) proves convergence of the sum over frequencies in Eq. (50). Therefore we do not have to introduce any additional regulators.

We can calculate \(\Gamma_V\) numerically by expanding the logarithm in Eq. (50) \(v\) in \(\delta\Pi^\omega(\Pi_0^\omega)^{-1}\) and calculating the corresponding traces. The explicit calculation shows that expansion to the third order yields a precision better than a percent, which is sufficient for our purposes. Proceeding this way we obtain \(\Gamma_V \approx 24\). Combining \(\Gamma_V\) with \(\Gamma_W\), expressed via \(\Gamma_W\) and \(\Gamma_d\) calculated in Eqs. (40) and (43), we obtain the final expression for \(\Upsilon\), determining the fluctuation integral \(\Gamma_{\rho=0}\), Eq. (14), needed to calculate the thermodynamics quantities:

\[
\Upsilon = 4\Gamma_W\Gamma_V \approx 8.
\]

IV. NONPERTURBATIVE CORRECTIONS TO THE THERMODYNAMIC QUANTITIES

In the previous section we found the soliton saddle points and showed that the functional integral over the fluctuations about a single soliton configuration can be expressed in terms of the integral over the soliton position, Eq. (14). In the present section we use these results to obtain nonperturbative corrections to the thermodynamic quantities at relatively high temperatures, \(G(L_T) \gg 1\). We begin by considering the \(N_{ch} \to \infty\) limit in Sec. IV A and turn to the case of large, but finite \(N_{ch}\) in Sec. IV B.
A. Infinite channel number

In the $N_{ch} \rightarrow \infty$ limit the Coulomb action (2c) vanishes. In this case the NL$\sigma$M action has infinitely many degenerate saddle points with spatially uniform potentials characterized by the winding numbers $w_a$, $V_a(x) = 2\pi T w_a$, with the usual saddle point, $Q = \Lambda$, corresponding to all $w_a = 0$. The single soliton solutions, studied in Sec. III A, represent exact inhomogeneous saddle points with a finite action $G(L_T)$ and correspond to a kink-like change of the electric potentials $V_a(x)$ by $\pm 2\pi T$, Eq. (12), in two of the replicas involved in the soliton rotation. The spatial size of the kink is given by the thermal diffusion length $L_T$. In the dilute soliton gas limit, which corresponds to $G(L_T) \gg 1$, multi-soliton saddle points can be viewed as sets of such kinks separated by distances much larger than $L_T$. In this case the action of a multi-soliton saddle point is given by the sum of single soliton actions. Similarly, the functional integral over the massive modes factors into a product of fluctuational determinants for each soliton. Thus in the dilute regime the soliton gas is noninteracting. Noting that the sum over the saddle points in Eq. (10) factorizes into a product of a sum over the uniform saddle points and the sum over the soliton configurations, we can easily find the multi-soliton contributions to the replicated partition function in the dilute soliton gas regime,

$$
\langle Z^p \rangle = Z_0^p \sum_{n=0}^{\infty} \frac{[2p(p-1)\alpha^p \Upsilon G(L_T)e^{-G(L_T)}]^n}{n!} \prod_{i=1}^{n} \int d\xi_0^{(i)} = Z_0^p \exp \left[ 2p(p-1)\alpha^p \Upsilon G(L_T)e^{-G(L_T)} L/L_T \right].
$$

(56)

Here $Z_0^p$ denotes the contribution of the homogeneous saddle points to the replicated partition function, $L/L_T$ is the dimensionless wire length, the factor of $p(p-1)$ arises from the number of ways the two replicas participating in the soliton rotation can be chosen from the $p$ replicas available, $\xi_0^{(i)}$ denotes the position of the $i$-th soliton, and the factor of two arises from taking the Cooperon-like and diffusonlike solitons into account. Substituting this result into Eq. (1) we obtain the leading nonperturbative correction to the average thermodynamic potential in the $N_{ch} \rightarrow \infty$ limit:

$$
\delta \Omega_\infty = 2\Upsilon G(L_T)e^{-G(L_T)} \frac{L}{L_T},
$$

(57)

where $\Upsilon$ is defined in Eq. (55).

The correction to the heat capacity can be obtained as $\delta C_\infty = -T \frac{\partial^2 \delta \Omega_\infty}{\partial T^2}$. Taking into account that the largest contribution comes from differentiating $G(L_T)$ in the exponential, we obtain the ratio of $\delta C_\infty$ to the heat capacity of noninteracting electrons, $C_0 = \frac{2e^2}{\pi^2} \nu ATL$,

$$
\frac{\delta C_\infty}{C_0} = -24\Upsilon G^2(L_T)e^{-G(L_T)}.
$$

(58)

The analysis above was restricted to the charge neutrality limit, $N_{ch} \rightarrow \infty$. In the next section we consider the case of a large, but finite number of channels in the wire. In this case the Coulomb action (2c) may not be neglected. Its presence significantly modifies the behavior of the soliton gas.

B. Finite channel number

For $N_{ch} \gg 1$ the influence of the Coulomb action (2c) on the soliton shape and on the massive fluctuations about the multi-soliton configurations is small and may be neglected. Therefore, each soliton configuration is still fully characterized by the soliton positions and the indices of the replicas participating in the soliton rotation. The Coulomb action for each configuration is given by the term $S_C$, Eq. (2c), evaluated for the specific potential profile $V_a(x)$ corresponding to such a configuration.

For a single soliton situated at $x_0$ the potential profile in the two replicas participating in the rotation is represented by a kink, $V_0(x - x_0)$, Eq. (12), in one of the replicas and an antikink, $-V_0(x - x_0)$, in the other one. Thus each soliton is characterized by its position $x_0^{(i)}$ and the index of the replica containing the kink, $a_+^{(i)}$, and the antikink, $a_-^{(i)}$. The potential profile for each soliton configuration is given by

$$
V_a(x) = \sum_\{i\} \left[ \delta_{a,a_+^{(i)}} - \delta_{a,a_-^{(i)}} \right] V_0(x - x_0^{(i)}).
$$

(59)

Using this representation, the replicated partition function, Eq. (10), can be written as

$$
\langle Z^p \rangle = Z_0^p \sum_{n=0}^{\infty} \frac{[2\alpha^p \Upsilon G(L_T)e^{-G(L_T)}]^n}{n!} \prod_{i=1}^{n} \int d\xi_0^{(i)} \sum_{a_{\pm}^{(i)}} \exp \left[ -S_C \left( \{\xi_0^{(i)}, a_{\pm}^{(i)}\} \right) \right],
$$

(60)
where \( S_C \left( \{ \xi^{(i)}, a^{(i)} \} \right) \) denotes the Coulomb action evaluated for a given soliton configuration \( \{ \xi^{(i)}, a^{(i)} \} \). Since the Coulomb action diverges for any uniform saddle point with \( w_a \neq 0 \), such saddle point are forbidden, and the factor \( \tilde{Z}_0 \), arising from the uniform saddle points, contains the contribution only from the usual saddle point, \( Q = \Lambda \), \( \{ w_a = 0 \} \).

Equation (60) is valid in the dilute soliton gas regime, \( G(L_T) \gg 1 \), in which the typical inter-soliton distance exceeds the thermal diffusion length \( L_T \). In the following we assume that at these length scales the Coulomb interaction is screened due to the presence of a nearby gate, so that its Fourier transform is given by \( K(q) \approx 1/e^2 \ln(\frac{d_g^2}{q^2}) \), where \( d_g \) is of the order of the distance to the gate. Then defining the kink density \( \rho_\alpha(\xi) \) in replica \( \alpha \),

\[
\rho_\alpha(\xi) = \sum_i \delta(\xi - \xi^{(i)}) \left[ \delta_{a,a^{(i)}} - \delta_{a_-,a^{(i)}} \right],
\]

we can express the Coulomb action in Eq. (60) in the dilute gas limit as

\[
\tilde{S}_C \left( \{ \xi^{(i)}, a^{(i)} \} \right) = -\frac{\pi v_F}{32e^2 \ln d_g} \frac{G(L_T)}{N_{ch}} \sum_{\alpha} \int d\xi d\xi' \rho_\alpha(\xi) \rho_\alpha(\xi') |\xi - \xi'|.
\]

Equations (60), (61), and (62) express the replicated partition function of the disordered wire as a partition function of a one dimensional replicated neutral gas of kinks and antikinks interacting via a linear potential. Importantly, the positive and negative charges in this gas occur only in pairs, such that the appearance of a positive charge in one replica is accompanied by the appearance of a negative charge in a different replica at the same spatial position. This problem can be mapped onto a one-dimensional replicated sine-Gordon model. Below we will not use this mapping, but work in the replicated kink gas representation.

Only the soliton configurations that correspond to a neutral kink gas in each replica give a nonvanishing contribution to the partition functions because all non-neutral configurations possess an infinite Coulomb action. The density of the kink gas is controlled by the fugacity, \( \Upsilon_p G(L_T)e^{-G(L_T)} \). Depending on its value the kink gas can be in two different regimes. At high temperatures, for \( G(L_T) \gtrsim \ln N_{ch} \), the gas is dimerized. In other words the kinks within each replica form a dilute gas of bound pairs of a kink and an antikink. At lower temperatures, \( \ln(N_{ch}) \gtrsim G(L_T) \gtrsim 1 \), the kink pairs overlap and form an ionized plasma. The dilute soliton gas approximation used to derive Eq. (60) is valid in both of these cases. We restrict our analysis below to the high temperature regime, \( G(L_T) > \ln N_{ch} \).

For \( G(L_T) \gg 1 \), Eq. (60) may be viewed as an expansion of the replicated partition function in the powers of the fugacity, \( \Upsilon_p G(L_T)e^{-G(L_T)} \). In the presence of the Coulomb action, the single soliton contribution to the partition function vanishes, since the corresponding Coulomb action is infinite. Therefore the leading term in this expansion is given by the contribution of two solitons which corresponds to two kink-antikink pairs in different replicas. We shall refer to this object as a dimer.

To evaluate the contribution of a single dimer into the replicated partition function we express the Coulomb action in terms of the kink-antikink separation within the dimer, \( \xi_{rel} \), and substitute the result into Eq. (60). Denoting the dimer center of mass coordinate by \( \xi_{cm} \), summing over all possible pairs of replicas that can accommodate the dimer, and recalling that each soliton can be either Cooperon-like or diffusonlike we obtain

\[
\langle Z^p \rangle = \tilde{Z}_0^p \left( 1 + 4p(p - 1)\alpha^p \Upsilon_p^2 G^2(L_T)e^{-2G(L_T)} \right) \frac{1}{2^p} \int_0^{L/L_T} d\xi_{cm} \int_{-\infty}^{\infty} d\xi_{rel} e^{-\xi_{rel}^2 |\xi_{rel}|L_T/L_N},
\]

where we introduced the notation

\[
L_N = \frac{8e^2 \ln d_g}{\pi v_F} \frac{N_{ch}}{G(L_T)} L_T,
\]

that has the meaning of the typical kink-antikink separation within each dimer. Since \( L_N \sim L_T/\sqrt{T_{rel}} \), where \( T_{rel} \) is the elastic mean free time, this length scale is much larger than \( L_T \) within the validity domain of the NLsM description. Therefore, the dilute soliton gas approximation is justified. Performing the integrals over \( \xi_{cm} \) and \( \xi_{rel} \) in Eq. (63) we get

\[
\langle Z^p \rangle = \tilde{Z}_0^p \left( 1 + 4p(p - 1)\alpha^p \Upsilon_p^2 \frac{L_N}{L_T} G^2(L_T)e^{-2G(L_T)} \right).
\]

This expression shows that the single dimer contribution to the partition function diverges as the length of the wire \( L \) goes to infinity. From the second term we infer that the spatial density of dimers is \( \tilde{c} \sim L_L^2 G^2(L_T)e^{-2G(L_T)} \). In the
regime \( G(L_T) \gtrsim \ln N_{ch} \), this density is smaller than \( 1/L_N \), and multisoliton configurations appear as a dilute gas of dimers.

Since the dimers in the dilute limit do not interact, the integration over all dimer configurations results in exponentiation of the correction arising from a single dimer, second term in Eq. (65).

\[
\langle Z^p \rangle = Z_0^p \left[ \sum_{n=0}^{\infty} \frac{1}{n!} \left( 4p(p-1)\alpha^2 G^2(L_T) e^{-2G(L_T)} \right)^n \right] = Z_0^p e^{4p(p-1)\alpha^2 G^2(L_T) e^{-2G(L_T)}}. \quad (66)
\]

Using Eq. (11) and the definition (64) we get the expression for the leading nonperturbative correction for the thermodynamic potential:

\[
\delta \Omega = \frac{32}{\pi} \Gamma^2 \frac{e^2}{v_F} \ln \left( \frac{d_q}{d} \right) N_{ch} G(L_T) e^{-2G(L_T)} \frac{L}{L_T} T, \quad (67)
\]

where \( \Gamma \) is defined in Eq. (55). Using this expression we obtain the ratio of the nonperturbative correction to the heat capacity to that of noninteracting electrons,

\[
\frac{\delta C}{C_0} = -\frac{384}{\pi} \frac{\Gamma^2}{v_F} \frac{e^2}{v_F} \ln \left( \frac{d_q}{d} \right) N_{ch} G^2(L_T) e^{-2G(L_T)}. \quad (68)
\]

Equations (67) and (68) are the main results of this paper. These results are drastically different from the expressions (57) and (58) obtained by taking the formal limit \( N_{ch} \to \infty \). We note that the corrections for the thermodynamic potential for infinite and finite \( N_{ch} \), Eqs. (67) and (67), become of the same order at \( G(L_T) \sim \ln N_{ch} \), when the dimer gas crosses over into the ionized regime.

V. SUMMARY

We studied nonperturbative interaction corrections to the thermodynamic quantities of a multichannel disordered wire. Within the replica NLxM formalism these corrections arise from soliton saddle points of the NLxM action. In the limit of infinite number of channels, \( N_{ch} \), in the wire we obtained the exact single soliton solution of the saddle point equations and evaluated the function integral over the fluctuation about the soliton configuration. We showed that for \( G(L_T) \gg 1 \) and \( N_{ch} \gg 1 \) nonperturbative corrections to the thermodynamic quantities of the system are described by a partition function for a dilute gas of solitons. The latter is equivalent to the partition function for a replicated classical one dimensional Coulomb gas. As the temperature is lowered, this gas undergoes a crossover from the dimerized regime of neutral soliton pairs at \( G(L_T) > \ln N_{ch} \) to the regime of ionized plasma for \( G(L_T) < \ln N_{ch} \). The crossover \( G(L_T) \sim \ln N_{ch} \gg 1 \) occurs at temperatures that are parametrically larger than those corresponding to the transition from weak to strong localization, \( G(L_T) \sim 1 \). This enables one to study this crossover separately from the perturbative effects. We specialized to the high temperature regime, \( G(L_T) \gtrsim \ln N_{ch} \) and obtained the leading nonperturbative correction to the specific heat (relative to that of noninteracting electrons), \( \delta C/C_0 \sim N_{ch} G^2(L_T) e^{-2G(L_T)} \), Eq. (65). We would like to emphasize that this correction is drastically different from the result obtained by taking the formal limit \( N_{ch} \to \infty \), Eq. (58), \( \delta C_\infty/C_0 \sim G^2(L_T) e^{-G(L_T)} \).

Although our treatment was specialized to the symplectic ensemble, we believe that the mapping of the nonperturbative corrections to the soliton gas and to the replicated Coulomb gas, described by Eqs. (63) holds for all three ensembles. Indeed, the existence of soliton minima is generic for all three ensembles. The mapping to the replicated classical Coulomb gas relies only on the fact that the functional integral over the fluctuations about a single soliton configuration can be reduced to the integral over the soliton position, Eq. (14). This, in turn, is a consequence of the fact that the integral over the massive modes converges, which we expect to be true for all ensembles.

The generalization of our formalism to the treatment of nonperturbative corrections to the transport characteristic is left for future work.

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APPENDIX A: DERIVATION OF Eq. (30)

In Sec. IIIB2 we encountered expressions containing determinants of Schrödinger-type operators of the form

\[ \ln D_0 = \text{tr}_\xi \ln \frac{\omega - \frac{d^2}{d\xi^2} + U}{\omega - \frac{d^2}{d\xi^2}}, \]  

\[ \ln D_h = \text{tr}_\xi \ln \frac{\left(\omega - \frac{d^2}{d\xi^2} + U_1\right)\left(\omega - \frac{d^2}{d\xi^2} + U_2\right)}{\left(\omega - \frac{d^2}{d\xi^2} + h\right)\left(\omega - \frac{d^2}{d\xi^2} + h(1 - \Theta)\right)}, \]  

where \( U(\xi) \) is a potential that vanishes at \( \xi \to \pm \infty \), and \( U_{1,2}(\xi) \) are step-like potentials, satisfying \( U_1(\xi) = U_2(-\xi) \), \( U_1(-\infty) = 0 \), \( U_1(\infty) = h \).

As explained in the text above Eq. (33), the trace in (A1a) can be obtained as a particular case of that in (A1b). Therefore, we concentrate our attention on the latter. We first rewrite Eq. (A1b) as

\[ \ln D_h = \text{tr}_\xi \ln \frac{\omega - \frac{d^2}{d\xi^2} + U_1}{\omega - \frac{d^2}{d\xi^2} + h\Theta} + \text{tr}_\xi \ln \frac{\omega - \frac{d^2}{d\xi^2} + U_2}{\omega - \frac{d^2}{d\xi^2} + h(1 - \Theta)} + \text{tr}_\xi \ln \frac{\left(\omega - \frac{d^2}{d\xi^2} + h\right)}{\left(\omega - \frac{d^2}{d\xi^2} + h(1 - \Theta)\right)}, \]  

where \( \Theta(\xi) \) is the step function. The third term does not depend on the potentials, and can be calculated explicitly, which is done at the end of this Appendix. The first two terms are equal since \( U_1(\xi) = U_2(-\xi) \). We denote each of them as \( \ln D_{h1} \), and proceed to calculate this quantity.

To compute

\[ \ln D_{h1} = \text{tr}_\xi \ln \frac{\omega - \frac{d^2}{d\xi^2} + U_1}{\omega - \frac{d^2}{d\xi^2} + h\Theta}, \]

we represent the potential \( U_1(\xi) \) as a sum \( U_1(\xi) = h\Theta(\xi) + v(\xi) \), where \( v(\xi) \) vanishes at spatial infinities, and express the variational derivative of \( \ln D_{h1} \) with respect to \( v(\xi) \) in terms of the Green’s function \( G(\xi, \xi') = \left(\omega - \frac{d^2}{d\xi^2} + U_1\right)^{-1} \bigg|_{\xi, \xi'} \),

\[ \frac{\delta \ln D_{h1}}{\delta v(\xi)} = \delta \text{tr}_\xi \ln G^{-1} = G(\xi, \xi), \]

The Green’s function \( G(\xi, \xi') \) can be found by solving the differential equation

\[ \left[ \omega - \frac{d^2}{d\xi^2} + U_1(\xi) \right] G(\xi, \xi') = \delta(\xi - \xi'), \]

with the boundary conditions that \( G(\xi, \xi') \) vanishes at spatial infinities \( \xi, \xi' \to \pm \infty \). It can be expressed\(^{222}\) in terms of the two independent solutions of the homogeneous equation

\[ \left[ \omega - \frac{d^2}{d\xi^2} + U_1(\xi) \right] \phi_i(\xi) = 0, \]

such that \( \phi_1(\xi \to -\infty) \to 0 \) and \( \phi_2(\xi \to +\infty) \to 0 \). In particular, at coinciding points we have

\[ G(\xi, \xi) = \frac{\phi_1(\xi)\phi_2(\xi)}{W[\phi_1(\xi), \phi_2(\xi)]}, \]

where \( W[\phi_1(\xi), \phi_2(\xi)] \) is the Wronskian of \( \phi_1(\xi) \) and \( \phi_2(\xi) \),

\[ W[\phi_1(\xi), \phi_2(\xi)] = \frac{d\phi_1(\xi)}{d\xi}\phi_2(\xi) - \phi_1(\xi)\frac{d\phi_2(\xi)}{d\xi}. \]

The Wronskian of the two independent solutions of Eq. (A6) does not depend on coordinate \( \xi \), and therefore may be expressed in terms of the \( \xi \to \pm \infty \) asymptotics of \( \phi_i(\xi) \). By appropriately normalizing the solutions, we can express the latter as

\[ \phi_1(\xi) = \begin{cases} e^{k\xi}, \xi \to -\infty, \\ t e^{k\xi}, \xi \to \infty \end{cases}, \quad \phi_2(\xi) = \begin{cases} t' e^{-k\xi}, \xi \to -\infty, \\ e^{-k\xi}, \xi \to \infty. \end{cases} \]
where \( k = \sqrt{\omega} \), \( \zeta = \sqrt{\omega + h} \), and \( t, t' \) depend on the specific form of the operator in Eq. (A9). Evaluating the Wronskian at \( \xi \to \pm \infty \) using the asymptotics (A9), we obtain

\[
W[\phi_1(\xi), \phi_2(\xi)] = 2\zeta t = 2kt'.
\]  

(A10)

Next we prove that

\[
\frac{\delta \ln D_{h1}}{\delta v(\xi)} = \frac{\delta \ln t}{\delta v(\xi)}.
\]  

(A11)

To this end we introduce an auxiliary construction

\[
\tilde{W}[\phi_1(\xi), \tilde{\phi}_2(\xi)] = \frac{d\phi_1(\xi)}{d\xi} \tilde{\phi}_2(\xi) - \phi_1(\xi) \frac{d\tilde{\phi}_2(\xi)}{d\xi}.
\]  

(A12)

Here \( \phi_1 \) and \( \tilde{\phi}_2 \) are solutions of Eq. (A6) with the same \( \omega \), but for two different potentials \( v(\xi) \) and \( \tilde{v}(\xi) \), both of which vanish at \( \xi \to \pm \infty \). The tilde denotes quantities corresponding to \( \tilde{v} \). We assume that \( \phi_1 \) and \( \tilde{\phi}_2 \) have the asymptotic form (A9), with \( \phi_2 \) characterized by \( \tilde{v} \).

Introducing \( \tilde{W}[\phi_1(\xi), \tilde{\phi}_2(\xi)] \), built out of the solutions of the same equation, the quantity \( \tilde{W}[\phi_1(\xi), \tilde{\phi}_2(\xi)] \) depends on the coordinate and satisfies the differential equation

\[
\frac{d\tilde{W}(\xi)}{d\xi} = \frac{d^2\phi_1(\xi)}{d\xi^2} \tilde{\phi}_2(\xi) - \phi_1(\xi) \frac{d^2\tilde{\phi}_2(\xi)}{d\xi^2} = [v(\xi) - \tilde{v}(\xi)] \phi_1 \tilde{\phi}_2,
\]  

(A13)

that follows directly from Eqs. (A6) for \( \phi_1 \) and \( \tilde{\phi}_2 \).

Integrating Eq. (A13) with respect to \( \xi \) from \(-\infty\) to \( \infty \) and using the asymptotic form of \( \phi_1 \) and \( \tilde{\phi}_2 \), Eq. (A9), we obtain

\[
\tilde{W}(\infty) - \tilde{W}(-\infty) = \int_{-\infty}^{\infty} d\xi (v - \tilde{v}) \phi_1 \tilde{\phi}_2 = 2\zeta t = 2kt'.
\]  

(A14)

Taking a variational derivative of this equation with respect to \( v(\xi) \) at \( v(\xi) = \tilde{v}(\xi) \) we obtain

\[
\phi_1(\xi)\tilde{\phi}_2(\xi) = 2\zeta \frac{\delta t}{\delta v(\xi)}.
\]  

(A15)

Plugging Eqs. (A10) and (A15) into Eq. (A7) for the Green’s function, and using Eq. (A4), we obtain Eq. (A11).

Integrating Eq. (A11) with respect to \( v \) from \( v(\xi) = 0 \) to its final value we obtain

\[
\ln D_{h1} = \ln \frac{t}{t_0},
\]  

(A16)

where \( t_0 \) is the coefficient in front of \( e^{\zeta} \) in asymptotic form (A9) of \( \phi_1 \) for \( v(\xi) = 0 \). The latter can be easily found from the continuity of the logarithmic derivative \( d\ln \phi_1(\xi)/d\xi \) at \( \xi = 0 \), and is given by \( t_0 = (1 + k/\zeta)/2 \).

Finally, the third term in Eq. (A2) can be calculated in the following manner. We denote this term by \( T_3(\omega) \), and introduce the Green’s functions \( g_0, g_h, \) and \( g^\pm \) that vanish at \( \xi, \xi' \to \pm \infty \) and satisfy the equations

\[
\left( \omega - \frac{d^2}{d\xi^2} \right) g_0 = \delta(\xi - \xi'), \quad \left( \omega - \frac{d^2}{d\xi^2} + h \right) g_h = \delta(\xi - \xi'),
\]

\[
\left( \omega - \frac{d^2}{d\xi^2} + h \Theta \right) g^+ = \delta(\xi - \xi'), \quad \left( \omega - \frac{d^2}{d\xi^2} + h(1 - \Theta) \right) g^- = \delta(\xi - \xi').
\]  

(A17)

Taking the derivative of \( T_3(\omega) \) with respect to \( \omega \) we obtain

\[
\frac{\partial T_3}{\partial \omega} = \int_{-\infty}^{\infty} d\xi [g^+(\xi, \xi) + g^-(\xi, \xi) - g_0(\xi, \xi) - g_h(\xi, \xi)].
\]  

(A18)

All Green’s functions entering this equation are easily calculated using the method of Wronskian, as was done above for a general potential. Specifically, we obtain the following expressions for the Green’s function at coinciding points:

\[
g_0(\xi, \xi) = \frac{1}{2k}, \quad g_h = \frac{1}{2\kappa}, \quad g^+ = g^-(\xi, -\xi'),
\]

\[
g^+(\xi, \xi) = \Theta(-\xi) \left( \frac{1}{2k} + \frac{k - \zeta}{2k(\kappa + \zeta)} e^{2k\xi} \right) + \Theta(\xi) \left( \frac{1}{2\zeta} + \frac{\zeta - k}{2\zeta(\kappa + \zeta)} e^{-2\zeta\xi} \right).
\]  

(A19)
Keeping in mind that $T_3(\omega \to \infty) \to 0$, we can express it as $T_3(\omega) = -\int_\omega^\infty d\omega' \frac{\partial T_3(\omega')}{\partial \omega'}$. Substituting the expressions (A17) for the Green’s functions into Eq. (A18), we obtain

$$T_3 = \ln \frac{\sqrt{\omega + h}}{\omega},$$  \hspace{1cm} \text{(A20)}

with $t_0$ defined below Eq. (A16).

Substituting Eqs. (A20) and (A16) into Eq. (A2) we obtain the final expression for trace in Eq. (A1b),

$$\text{tr}_\xi \ln \left( \omega - \frac{d^2}{d\omega^2} + U_1 \right) \left( \omega - \frac{d^2}{d\omega^2} + U_2 \right) = 2 \ln D_h + T_3 = \ln \sqrt{\omega + h} t^2 = \ln t t', \hspace{1cm} \text{(A21)}$$

where in the last expression we used $\sqrt{\omega + h} = \sqrt{\omega} t'$ to write a more symmetric expression, in which $t$ and $t'$ are defined in (A9). This proves Eq. 30.

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