Localized states and topological properties in one-dimensional periodic wave systems

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Localized states at the boundaries of periodic system have attracted great interests since the discovery of the topological insulators. Topological edge or surface states have been predicted and observed in two-dimensional and three-dimensional systems. Recently there is trend to search topologically protected localized states in one-dimensional (1D) periodic systems. Contrary to the previous works, we show the edge states and interface states are universal phenomena in 1D periodic systems, which need not the topological protection. By mapping the 1D system to 2D quantum Hall system we demonstrate the universal existence of the edge states. Similarly the existence of interface states are proved by mapping the interface of two 1D system to the interface of two quantum Hall system with different Chern numbers. Our work provide a new perceptive on the topological phenomena in 1D periodic system. The universality of the edge states or interface states enable us to find such states in all kinds of 1D wave systems, macroscopic and microscopic, classical and quantum. For example we show such edge states and interface states can be observed in the circular periodic systems.

The researches on the topological phenomena in 1D periodic systems can be broadly classified into two categories. In the first authors try to use concepts such as Zak phase to characterize the topology of an 1D system and they regard the existence of the edge or interface states as a
hallmark of topological non-triviality of the 1D system. However, as we show in the sup-
porting information(SI) most of the so-called topological invariants depends on how we describe the system and most of the so-called topologically distinct phases are just same bulk system with different choices of prime cells. The so-called topological edge states are shown to be boundary-condition-dependent and they reflect no topological non-triviality. So we feel there is an urgent need to stop this trend of researches.

Another group of authors study the topological phenomena by introducing variable parameters to 1D systems. In this case the 1D system can be used to simulate the higher dimensional system. For example, we consider a 1D system has one such parameter $k_y$ and the Hamiltonian of the bulk 1D system is periodic respect to $k_y$. When $k_y$ continuously varies by one period the bulk Hamiltonian will return to itself. If we identify $k_y$ to one component of the wave vector of the 2D system, the 1D system will be mapped into a 2D system. Then topological concept such as Chern number or $Z_2$ invariant can be used to study the topological phenomena in the 1D system. If the topological invariant is non-trivial there will be gapless edge state spectrum as functions of $k_y$.

The key to understand the universality of the localized states is to realize that every 1D system has the parameter for 2D mapping. The parameter is the distance $d$ that 1D system is translated. If we consider the Bloch waves as functions of both 1D wave vector $k_x$ and $d$, they form 2D bands. The Chern number of the 2D bands can be obtained by studying the variation of the centers of the 1D Wannier functions with $d$. The Wannier functions are translated with the system. When the system is translated by one period the bulk system return to itself while the centers of Wannier functions vary by one period. So the Chern numbers of all the bands are one. Thus we show all
the 1D periodic systems can be mapped into quantum Hall systems. According to the bulk-edge correspondence of the 2D system there are always gapless edge state spectrum as functions of the parameter \(d\) in energy or frequency gaps for a given boundary condition\(^{26}\).

As a first example, we consider the simplest tight-binding model

\[
\hat{H} = \sum_n \epsilon_n \hat{c}_n^\dagger \hat{c}_n + \sum_n (t \hat{c}_{n+1}^\dagger \hat{c}_n + \text{H.C.})
\]

It has been long recognized this model can exhibit localized eigenstate in discussing the lattice vibration\(^{27}\). Obviously, it is a topologically trivial system. There is no localized state at the ends of the system when open boundary condition is used. However, the open boundary condition is rather artificial which means the onsite potential goes abruptly to infinity outside the region we considered. Here we use an onsite potential which goes to infinity continuously as is shown in Fig.1(a). We assume the onsite potential serves as the boundary condition and it keeps fixed when the sites are translated. If we consider the Bloch wave as function both of quasi-momentum \(k_x\) and \(d\), the Chern number of the bulk band is 1 as is discussed above (the Chern number can also be calculated from the definition as is shown in SI). The spectrum as functions of \(d\) is plotted in Fig.1 (b), which shows there is one branch of gapless edge states at each end the system. Here gapless means the edge states spectrums continuously connect the band top and the positive infinity. Since we use a topological argument, the specific form the onsite potential is irrelevant. What is important is that we can always find edge state by manipulating the boundary condition in this system. The conclusion can be generalized to all tight-binding models. In this way we show that edge states can be find in all arbitrary tight-binding systems.

The same argument can be used to study the continuous models. The eigenvalue equation of non-
interacting quantum particles in a periodic potential can be expressed as

\[ -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi(x) + V(x)\psi(x) = \epsilon\psi(x) \]  

(2)

The equation can be used to study electrons or cold atoms in 1D systems. Classical wave in 1D periodic medium can be studied by using a similar eigenvalue equation

\[ \frac{d^2}{dx^2} \psi(x) + \omega^2 \rho(x) \psi(x) = 0 \]  

(3)

For example, in a periodic string \( \rho(x) \) is in proportion to density of the string and in a periodic 1D photonic crystal \( \rho(x) \) is related to the index of refraction. Both of the equations are Hill’s type, so we can use theory of Hill’s equation to study the localized states. It can be proved that there is one and only one edge state in each energy or frequency gap when the length of the system a multiple of the period and the same homogeneous boundary condition is used at the two ends of the system. We also show to observe the edge state spectrum and wave function we do not need a long system: a one-period-long system is enough for this purpose. (See the SI for details)

The topological argument for the existence of edge states in continuous systems is illustrated in Fig. 2 with a periodic string. Three type of boundary conditions are used in Fig. 2. The results in Fig. 2 b-d show that in nth bulk gap there are n branches of edge states at each end of the string because the Chern number of the bands equals 1. Again, due to the topological nature of the argument, specific boundary condition is irrelevant as far as the existence of the gapless edge states is concerned.

The universal existence of the interface states with the continuous model can be demonstrated by using a similar topological argument. We can design a continuous cyclic process in which the two continuous systems both are translated by \( n_1 \) and \( n_2 \) periods respectively, where \( n_1 \) and \( n_2 \) are two
integers. The connection points of the two systems are fixed and arbitrary connecting conditions can be used in this process as long as they are same in the initial and final states. Then the two 1D systems are mapped into two 2D quantum Hall systems. The Chern numbers of each bulk band of the two 2D systems are $n_1$ and $n_2$. The existence of interface states of between the two 1D system can be obtained by studying the 2D interface states. Because the band structure is independent of the translation, an overlapping bulk band gap of the two 1D system will also be a overlapping gap of the two 2D systems. If the Chern numbers of the bulk band below the overlapping gap are different for the two systems, which can always be achieved by different choices of $n_1$ and $n_2$, there will be gapless interface states in the gap. The number of branches of the interface states are determined by the difference of the Chern numbers as is illustrated in Fig.3. So by choosing a suitable way to connect two periodic system we can always find interface states between two 1D systems.

Such localized states in 1D systems are not limited to the above cases. To further illustrate the universality of those states we show the such edge states in different quantum wave guides as is shown in Fig.4.(a)-(c). Similar mathematical methods can be used to study the periodic electromagnetic wave guides, water waves in 1D periodical water tanks and membranes etc. To observe such localized states, the system is not necessarily linear. For example, such localized states can be realized in circular periodic systems. As is shown in Fig.4 (d)-(g) the edge and interface states present in a circular quantum well whose potential is periodic in the angular direction. The universality of the localized states in the circular systems can be demonstrated with a similar topological argument as in the linear case.
The discovery of the university of the localized states in 1D systems not only helps to clarify the topological concepts in 1D systems, but also leads to a variety of applications. First it enables us to find all kinds of such localized states without worrying about the topological property of the 1D periodic system. Compared with microscopic quantum systems such states can be more easily observed in macroscopic classical systems, such as classical strings, membranes, water tanks, waveguides, resonant cavities, phononic and photonic crystals. Some of the quantum systems such as polymers may also be used to study such localized states.

Second, simulation of the 2D topological property with 1D system becomes a trivial matter as demonstrated above: any 1D system can be used to observe the gapless edge states. In fact by using classical 1D system, such as string, membrane, electromagnetic wave guides or water tanks the winding number of gapless edge states can be easily observed. Because the bulk wave functions can be easily observed in such systems, e.g. in a string, the Chern number of the bulk bands can be obtained the bulk wave functions. The bulk-edge correspondence can be confirmed by comparing the bulk Chern number and the winding number for the edge states.

Third, our theory make the observation of the Thouless pumping\textsuperscript{25} a trivial matter. Instead of modulating the potential carefully we can just translate the entire periodic system by one period. In this process the integer number of charges are transported if there are integer numbers of charge in one prime cell as are in Ref\textsuperscript{29,30}. As is demonstrated above this charge pump is no less topological than these in Ref. because it can also be characterized by a Chern number. As for the classical analogy of quantum pumping process that pumps edge state from one end of system to another\textsuperscript{22}, we can more easily and accurately realize it by using classical coupled oscillators or periodic strings as is
Another possible application of the localized states is remote stimulation. The classical localized vibration are limited a small space, the magnitude of the vibration decrease exponentially outside of this region. However, we can used vibration source that is far away to this region to stimulate the localized states. We have observed this phenomenon in a classical periodic string which is shown in the supplementary Fig.16. Because of the universality of the localized states we predict that the remote stimulations can be observed in all kinds of 1D periodic mechanical and electromagnetic systems and they can find applications in various engineering problems.

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Supplementary Information for Localized states and topological properties in one-dimensional periodic wave systems

It’s shocking to see that the same 1D periodic system is considered as two different topological phases just because different choices of prime cells are chosen or different boundary conditions are used in so many literatures. It is no less shocking to see that Zak phase is considered as topological invariant by so many authors. So here we give an elementary theory on such properties.

1 Features of the topological property in 1D system

To determine whether a property in 1D system is topological or not we discuss some features of the topological property. First, a topological property should not change under continuous change. For example topological property of a system should not change only because we translate the system by a certain distance. From another point of view, the topological property should not depend on how we describe the system. If a variable is a topological invariant it should not depend on the coordinate system we use, e.g. it should be independent of the where the origin of the coordinate system is. If two observers want to determine whether one system is topological or not and if they reach opposite conclusions just because they place the origins of the coordinate systems at different places, we can safely conclude the property they study is not topological at all. Unfortunately, as we will see in section 6 most of the so-called topological properties in previous works on topology of 1D system fall into this category.

Second, a topological property of the bands is a bulk property and thus it should not depend on the boundary condition of the system. The bulk bands can be obtained by using periodic boundary
condition or by assuming the system is infinite. So the specific boundary condition should not affect the bulk property of the system. For example the bulk band structure should not change only because we delete one site from or add one site to the system. However, we will show in many literatures the bulk band properties such as Zak pase is considered to depend on the choices of the boundary conditions. It equivalent to say that the bulk system will immediately collapse to a topological or trivial state when we observe the boundary of the system and before that the bulk property is unknown. This is certainly not the case.

Finally, if the existence of some property reflect the topological non-triviality of the bulk bands, it should be absent when the system is considered to be topologically trivial. There is a lots works that claim the mere existence edge states is a consequence of the topological non-triviality of the bulk system. Logically, this means that edge state should not exist when one system is considered as topologically trivial. However, we will this is not the case in the main text.

2 Zak phase in continuous system

Zak phase has been widely used to characterize the topological property of the 1D system. We will use both the continuous model and tight-binding model to show the Zak phase and other related properties are not topological invariant even if when there is inversion symmetry. In a continuous model, we show the Zak phase depend on the choice of the origin of the coordinate and is independent of the choice of the prime cell. For a given band with Bloch function \( \psi_k(x) = \exp(ikx)u_k(x) \), the Zak phase can be defined as \( \varphi_{\text{Zak}} = i \int_0^{2\pi/a} \int_{\text{cell}} u^*_k(x) \frac{\partial}{\partial k} u_k(x) dx dk \), where \( a \) is the lattice constant and \( u_{2\pi/a}(x) = \exp(-i2\pi/ax)u_0(x) \). The integration respects to \( x \) is on a
prime cell and the $u_k(x)$ is normalized by $\int_{\text{cell}} u_k^* (x) u_k(x) = 1$. If we choose a different origin of the coordinate system $x$ becomes $x' - d$. Bloch function takes the form $\psi_k (x') = \exp[ik(x'-d)] u_k(x'-d) = \exp(ikx') \exp(-ikd) u_k(x'-d)$. In the new coordinate $u'_k(x') = \exp(-ikd) u_k(x'-d)$ satisfies $u'_{2\pi/a}(x') = \exp(-i2\pi/ax') u'_0(x')$. So the Zak phase becomes

$$\varphi'_{\text{Zak}} = i \int_0^{2\pi/a} \int_{\text{cell}} u'_k(x') \frac{\partial}{\partial k} u_k(x') dx' dk = d2\pi/a + \varphi_{\text{Zak}}$$  \hspace{1cm} (4)

Although here we get a similar result to that in supplementary information in Ref\textsuperscript{19}, our conclusion is just the opposite: the Zak phase depends on the choice of the coordinate system but has nothing to do with the choice of the origin of prime(unit) cell. The prime cell is a volume of space that, when translated through all the vectors in the Bravais lattice fill all of the space without overlapping itself or leaving voids\textsuperscript{31}. There are infinite ways of choosing the prime cell. Two choices of prime cells can always be made identical by breaking one prime cell to parts and translating the different parts by different Bravais lattices. Because $u_k(x)$ is a periodic function of the Bravais lattice, when the a volume is translated by a Bravais lattices the integration of $u_k^*(x) \frac{\partial}{\partial k} u_k(x)$ on that volume will not change. Thus the integration of $u_k^*(x) \frac{\partial}{\partial k} u_k(x)$ is independent of the choice of the prime cell, so is the Zak phase.

The $\varphi_{\text{Zak}}$ is gauge dependent, e.g. under a gauge transformation $u_k(x) \rightarrow \exp(-if(k)) u_k(x)$, where $f(k)$ is a continuous function and $f(2\pi/a) - f(0) = n2\pi$ with $n$ as an integer, $\varphi_{\text{Zak}}$ with be transformed to $\varphi_{\text{Zak}} + n2\pi$. The nondeterminacy of the Zak phase have well physical explanation: $\varphi_{\text{Zak}} a/2\pi$ corresponds to the position of a Wannier function, which is clearly coordinate-dependent and defined modulo $a$.

In Ref\textsuperscript{17} it’s shown that the center of the Wannier functions in a 1D inversion symmetric system...
can only locate at the inversion centers. Even if we limit our choice of the origin of the coordinate
systems to the inversion centers, there are still two choices in one prime cell. In this case the Zak
phase can still assume two values 0 and $\pi$ according to the choices of the origins of coordinate
systems. So the Zak phase is not a topological invariant even if the system has the inversion
symmetry.

3 Zak phase and other properties in tight-binding system

For the tight-binding models, because the index $n$, which denotes the prime cells, is not accurately
corresponds to the position of the orbits, the Zak phase does depend on the choice of the prime
cell. All the orbits have the same index $n$ belong to the same prime cell. We can always regroup
orbits into a different choice of prime cell by choosing different indexes for the orbits. We consider
a general tight-binding model

$$\hat{H} = \sum_{mnij} t_{ij}(m - n) \hat{C}_{mi}^{\dagger} \hat{C}_{nj}$$

(5)

where index $m$ and $n$ are the lattice indexes, $i$ and $j$ denote the orbits for given prime cells. $t_{ij}(m - n) = t_{ji}^*(n - m)$ are the hopping amplitudes. We can make a different choice of the prime cell by
the transformation $\hat{B}_{n'j} = \hat{C}_{n'+\delta_{ij}} = \hat{C}_{nj}$. Where $\delta_{ij}$ are arbitrary integers and unless all $\delta_{ij}$ are all
equal, orbits with same $n'$ will become a different type of prime cell. After the transformation the
Hamiltonian becomes

$$\hat{H} = \sum_{m'n'ij} t'_{ij}(m' - n') \hat{B}^{\dagger}_{m'i} \hat{B}_{n'j}$$

(6)

where $t'_{ij}(m' - n') = t_{ij}(m' - n' + \delta_i - \delta_j)$. We can transform Hamiltonian to the momentum space
by transformations $\hat{C}_{nj} = 1/\sqrt{N} \sum_k \hat{C}_{kj} \exp[ik(n + n_d)]$ and $\hat{B}_{n'j} = 1/\sqrt{N} \sum_k \hat{B}_{kj} \exp[ik(n' +
\[ n_d \in [0, 1) \] is an arbitrary real number, which indicates the origin of coordinate system.

When we use the lattice waves as basis vectors it is implied that the \( n \)th site is located at \( n+n_d \) (Here we assume lattice constant is 1). So the index \( n \) can not specify the actual position of an orbit. We will show the Zak phases in the tight-binding models also depend on the origin of coordinate system by showing that they are functions of \( n_d \). The Hamiltonian becomes

\[
\hat{H} = \sum_{kij} \hat{C}_{ki}^\dagger T(k)_{ij} \hat{C}_{kj} = \sum_{kij} \hat{B}_{ki}^\dagger T'(k)_{ij} \hat{B}_{kj}
\]  

(7)

where \( T(k)_{ij} = \sum_n t_{ij}(n) \exp(-ikn) \) and \( T'(k)_{ij} = \exp[ik(\delta_i - \delta_j)]T(k)_{ij} \). \( T(k) \) and \( T'(k) \) can be considered as \( M \times M \) matrixes, where \( M \) is the number of orbits within one prime cell. The two matrixes are related by a transformation \( T = U(k)T'U(k)^\dagger \), where \( U(k) \) is a diagonal matrix with \( U_{ii} = \exp(ik\delta_i) \). The eigenvectors \( |v_\mu(k)\rangle \) of \( T(k) \) are chosen to satisfy \( |v_\mu(0)\rangle = |v_\mu(2\pi)\rangle \). The eigenvectors of \( T'(k) \) can be chosen as \( |v'_\mu(k)\rangle = U|v_\mu(k)\rangle \), which also satisfy \( |v'_\mu(0)\rangle = |v_\mu'(2\pi)\rangle \).

To construct Wannier functions the Bloch waves \( |\psi_\mu(k)\rangle = \exp[ik(n+n_d)]|u_\mu(k)\rangle \) must satisfy \( \psi_\mu(0) = \psi_\mu(2\pi) \). So \( |u_\mu(k)\rangle \) can be chosen as \( \exp(-ikn_d)|v_\mu(k)\rangle \) in calculating the Zak phase.

Thus, the Zak phase can expressed as

\[
\varphi_{Zak} = i \int_0^{2\pi} \langle u_\mu(k) | \frac{\partial}{\partial k} | u_\mu(k) \rangle \, dk = n_d 2\pi + i \int_0^{2\pi} \langle v_\mu(k) | \frac{\partial}{\partial k} | v_\mu(k) \rangle \, dk
\]  

(8)

Because \( |v_\mu(k)\rangle \) is independent of \( n_d \), \( \varphi_{Zak} \) is a function of \( n_d \), which can be explained by the fact that Zak phase corresponds to the position of the Wannier function. The Zak phase also depends on the choice the prime cell in the tight-binding model, e.g.

\[
\varphi'_{Zak} = n_d 2\pi + i \int_0^{2\pi} \langle v'_\mu(k) | \frac{\partial}{\partial k} | v'_\mu(k) \rangle \, dk = \varphi_{Zak} + i \int_0^{2\pi} \langle v_\mu(k) | U(k)^\dagger \frac{\partial}{\partial k} U(k) | v_\mu(k) \rangle \, dk
\]

\[
= \varphi_{Zak} - \sum_i \delta_i \int_0^{2\pi} |\langle i | v_\mu(k) \rangle|^2 \, dk = \varphi_{Zak} - 2\pi \sum_i \delta_i v_i
\]  

(9)
where $\langle i | v_\mu(k) \rangle$ is the $i$th component of $|v_\mu(k)\rangle$ and $v_i = \int_0^{2\pi} |\langle i | v_\mu(k) \rangle|^2 \, dk / (2\pi)$. Even when $n_d$ is given, e.g. $n_d = 0$, the Zak phase is still dependent on the choice of prime cell. By applying the Weyl’s equidistribution theorem\cite{32} we find that Zak phase can only take finitely many values (modulo $2\pi$) when all of $v_i$ are rational numbers. If at least one of $v_i$ irrational, Zak phase can take infinite number of values (modulo $2\pi$) and these values are equidistributed in $[0, 2\pi)$. So again we show the Zak phase is not a topological invariant. Various winding properties of the Hamiltonian and the Bloch waves have been used to characterize the topological phase of the 1D systems. However, as is shown above that $T(k)_{ij}$ depends on the choice of prime cell and $|u_\mu(k)\rangle$ depends on the choices of both the prime cell and the origin of the coordinate system. Besides, winding property of $|u_\mu(k)\rangle$ is gauge-dependent. So they can not be considered as topological invariants.

Now we can explain why the Zak phases are mistaken for topological invariants in so many literatures. First, in most of the works the origin of the coordinate is chosen as to make $n_d = 0$ with the exception of Ref.1, where $n_d = 1/2$ is used and thus $\phi_{Zak}$ can assume $\pm \pi/2$ instead of $0$ or $\pi$ as in most of the literatures (However, they erroneously believe this is due to the choice of unit cell). Second, only one form of prime cell is used when the Zak phases are calculated in most of previous works. With this choice of unit cell they calculate the Zak phase and winding number of the Hamiltonian of different systems. However, as we will see there is no natural unique choice of the prime cell. For example, with the SSH model, even if we limited ourself to the choice that only two adjacent sites can be grouped to prime cell we still have two choices. Unfortunately, in most of the literatures studying the SSH model, the two choices of prime cell for the same system are considered as two topologically distinct phases. There is even some authors who are not aware
of the gauge dependency of the Zak phase and consider $\varphi_{Zak} = 0$ and $\varphi_{Zak} = 2\pi$ as topologically distinct.

4 The SSH model

As an example now we apply our theory to the widely used SSH model. For example in Ref.\textsuperscript{19} the SSH model takes the form

$$\hat{H} = -\sum_n (J\hat{a}_n^\dagger\hat{b}_n + J'\hat{a}_n^\dagger\hat{b}_{n-1} + \text{h.c.})$$  \hspace{1cm} (10)

In our formulation it equivalents to the situation that only non-vanishing hopping amplitudes are $t_{12}(0) = t_{21}(0) = -J$ and $t_{12}(1) = t_{21}(1) = -J'$. In the momentum space

$$T(k) = \begin{pmatrix} 0 & -J - J' \exp(-ik) \\ -J - J' \exp(ik) & 0 \end{pmatrix}$$  \hspace{1cm} (11)

A choice of the eigenvectors $|v_\mu(k)\rangle$ that satisfies $|v_\mu(0)\rangle = |v_\mu(2\pi)\rangle$ is

$$|v_1(k)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \exp(i\theta_k) \end{pmatrix}$$  \hspace{1cm} \hspace{1cm}  $$|v_2(k)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\exp(i\theta_k) \end{pmatrix}$$  \hspace{1cm} (12)

where $\theta_k$ is the argument of $J + J' \exp(ik)$. Then $|u_\mu(k)\rangle$ can be expressed as

$$|u_1(k)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp(-ikn_d) \\ \exp[i(\theta_k - kn_d)] \end{pmatrix}$$  \hspace{1cm}  $$|u_2(k)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} \exp(-ikn_d) \\ -\exp[i(\theta_k - kn_d)] \end{pmatrix}$$  \hspace{1cm} (13)

The Zak phases for the two bands are identical $\varphi_{Zak} = n_d 2\pi$. With a different choice of the prime cell the non-vanishing hopping amplitudes are $t'_{12}(\delta_2 - \delta_1) = t'_{21}(\delta_1 - \delta_2) = -J$ and
\[ t_{12}'(1 + \delta_2 - \delta_1) = t_{21}'(-1 + \delta_1 - \delta_2) = -J \]

\[
T'(k) = \begin{pmatrix}
0 & [-J - J' \exp(-ik)] \exp[i(\delta_1 - \delta_2)] \\
[-J - J' \exp(ik)] \exp[i(\delta_2 - \delta_1)] & 0
\end{pmatrix}
\]

(14)

The eigenvectors can be chosen as

\[
|v_1'(k)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix}
\exp(i\delta_1 k) \\
\exp[i(\theta_k + \delta_2 k)]
\end{pmatrix}
\]

\[
|v_2'(k)\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix}
\exp(i\delta_1 k) \\
-\exp[i(\theta_k + \delta_2 k)]
\end{pmatrix}
\]

(15)

The Zak phases for the two bands \( \varphi_{Zak} = d2\pi - 2\pi(\delta_1/2 + \delta_2/2) \). Because \( v_1 = v_2 = 1/2 \) the Zak phase can take two possible values \( n_d2\pi \) and \( (n_d + 1/2)2\pi \) modulo \( 2\pi \) for a given \( n_d \). In most of the literatures \( n_d = 0 \) is implied, which is a convenient choice but not a necessary choice as is demonstrated above. Even with the choice \( n_d = 0 \) and the inversion symmetry of the SSH, the Zak phase of a given system can still assume two values 0 and \( \pi \). So using \( \varphi_{Zak} = 0 \) and \( \varphi_{Zak} = \pi \) to distinguish two different topological phases of SSH model is impossible.

5 Calculation of Chern number with most simple tight-binding model

Because the discontinuity of \( |u_{\mu}(k)\rangle \) in the Brillouin zone we calculate the Chern number of this tight binding model by transforming it to a continuous model. We assume each site represents an orbit \( f(x - n - n_d) \), where \( f(x) \) is a normalized real even function confined within \( |x| < 1/2 \). Then the Bloch function can be chosen as \( \psi_{kxnd}(x) = \sum_n e^{ikx} f(x - n - n_d) = e^{ikx} u_{kxnd}(x) \), where \( u_{kxnd}(x) = \sum_n e^{-ikx} f(x - n - n_d) \). We can use \( u_{kxnd}(x) \) to calculate the Berry connection

\[
A_{k_x} = i \int_0^1 u^*_{kx_{nd}}(x) \frac{\partial}{\partial k_x} u_{kx_{nd}}(x) dx = i \int_{n_d - 1/2}^{n_d + 1/2} u^*_{kx_{nd}}(x) \frac{\partial}{\partial k_x} u_{kx_{nd}}(x) dx
\]

\[
= \int_{-1/2}^{+1/2} f(x)(x + n_d) f(x) dx = n_d
\]

(16)
\[ A_{n_d} = i \int_0^1 u^*_k u_{d}(x) \frac{\partial}{\partial n_d} u_{k,n_d}(x) dx = i \int_{n_d-1/2}^{n_d+1/2} u^*_k u_{d}(x) \frac{\partial}{\partial n_d} u_{k,n_d}(x) dx \]

Then we get the Berry curvature

\[ F(k_x, n_d) = \frac{\partial}{\partial n_d} A_{k_x} - \frac{\partial}{\partial k_x} A_{n_d} = 1 \tag{18} \]

The Chern number of the bulk band is

\[ C = \frac{1}{2\pi} \int \int F(k_x, n_d) dk_x dn_d = 1 \tag{19} \]

6 Comments on the works on 1D topological property

In this section we comment on several works on the 1D topological properties. Because there are so many works on this subject, our selection of the literatures in this section is not aimed at inclusive. We only choose those we deem as most typical and influential. We can give a simple criterion to test whether a 1D periodic system is topological or not. If a system simultaneously satisfies the two conditions: a) it exhibit no superconductivity. b) it has no variable parameter or the parameter is fixed, then this 1D system is not topological. Another simple rule is that all the so-called Zak phase related topological properties are topologically trivial. We will demonstrate these systems are not topological by showing the so-claimed topological invariant is not invariant and the so-called topologically distinct systems are actually topologically equivalent or completely identical.

In Ref.12 both the continuous model and the tight-binding model are used to study the topological property of the 1D systems. The authors of Ref.1 claim that they have measured the difference
of Zak phases of two topologically distinct systems. To show these two systems are actually topologically equivalent we first study the continuous model. In Ref. the cases for lattice potential \( V(x) = V_1 \sin^2(k_1 x + \phi/2) + V_s \sin^2(2k_1 x + \pi/2) \) are claimed to be topologically distinct. However, simple derivation shows these two cases are not only topologically equivalent: they correspond to the same lattice potential with a translation of \( d = \pm \lambda_1/4 = \pm \pi/(2k_1) \) e.g. if we translate one of the systems by \( d \) the two systems will become identical. Equivalently, if the origin of the coordinate of one phase is shifted by \( d \) the lattice potential of this system will be identical to the other phase.

\[
V(x') = V(x - d) = V_1 \sin^2(k_1 x' + 0/2) + V_s \sin^2(2k_1 x' + \pi/2) \\
= V_1 \sin^2[k_1(x - d) + 0/2] + V_s \sin^2[2k_1(x - d) + \pi/2] \\
= V_1 \sin^2(k_1 x + \pi/2) + V_s \sin^2(2k_1 x + \pi/2)
\] (20)

If two researchers want to determine whether this system is topological or not by measuring the Berry phase. They will get completely opposite results if one use coordinate \( x \) and the other use coordinate \( x' \).

With the SSH tight-binding model \( \hat{H} = -\sum_n (J\hat{a}_n^{\dagger}\hat{b}_n + J'\hat{a}_n^{\dagger}\hat{b}_{n-1} + h.c.) \) the two cases \( J > J' \) and \( J' > J \) are considered as two different topological phases. However, if we use a different choice of prime cell, e.g. \( \delta_1 = 1 \), then the non-vanishing hopping amplitudes become \( t'_{21}(1) = t'_{12}(-1) = -J \) and \( t_{12}(0) = t_{21}(0) = -J' \). In the later choice the roles of \( J' \) and \( J \) are switched. The Zak phases in these two choices differ by \( \pi \). So again the two so-called topologically distinct phases are only same system with different ways to describe it.

The winding property of the pseudo-spin vector is also used to characterize the topological phases
However, as we demonstrated above, the vector $|u_{\mu}(k)\rangle$ depends on the choices of origin of the coordinate and prime cell. With a different choice of $n_d$ (in ref.\cite{19} they choose $n_d = 1/2$) or a different choice of prime cell the winding property will change, so it is not a topological property.

In the supplementary information they try to give a recipe to determine whether the system is topologically trivial or non-trivial by observing the existence of the edge state of the system. First, we point out that what they study is not two bulk systems, but same system with different choice of prime cell. So their conclusion is equivalent to say that we can not know whether a system is topological without looking at the boundary of this system and when the boundaries of the system is observed the system will immediately collapse to the topological or trivial states depending on how the system is terminated as is shown in Fig.5. If it is true, we can no longer consider the so-called topological property a bulk property because it is completely determined by the boundary conditions. Further more their conclusion that boundary condition determines bulk topology is obviously erroneous as is illustrated in Fig.6. They also claim that the presence and the absence of the edge states still hold when a power law potential $(r/r_0)^{\gamma}$ is present. In fact their conclusion is just a coincidence. If their conclusion is true it should not depend on the magnitude of the potential.

Our simulation shows that in the so-called topological trivial case when potential is $0.55(r/r_0)^{\gamma}$ there will be edge states as is shown in ADDfigure. Similarly in the so-called topological case the edge states disappear from the band gap when the trap potential is $0.6(r/r_0)^{\gamma}$. This again confirm our conclusion the existence of the edge states depends on how we manipulate the edge instead of the bulk band topology.
In Ref.\cite{8} a so-called topological zero-energy modes are proposed in off-diagonal AAH model. When \( b = 1/2 \) this off-diagonal AAH model \( \hat{H} = \sum_{n=1}^{N-1} (t[1 + \lambda \cos(n\pi + \varphi_\lambda)] \hat{c}_{n+1}^\dagger \hat{c}_n + h.c.) \) is nothing but a SSH model in disguise: if we let \( J = t[1 - \lambda \cos(\varphi_\lambda)] \), \( J' = t[1 + \lambda \cos(\varphi_\lambda)] \), \( \hat{a}_n = \hat{c}_{2n} \) and \( \hat{b}_n = \hat{c}_{2n-1} \), the Hamiltonian will become a SSH type \( \hat{H} = \sum_n (J\hat{a}_n^\dagger \hat{b}_n + J'\hat{b}_{n+1}^\dagger \hat{a}_n + h.c.). \) \( J > J' \) when \( -\pi/2 < \varphi_\lambda < \pi/2 \) and \( J' > J \) when \( \pi/2 < \varphi_\lambda < 3\pi/2 \). These two cases are topologically equivalent with the same argument given above.

In Ref.\cite{8} they try to use a \( Z_2 \) index of the Majorana chain to characterize this system. However, this topological index only relevant when there is superconductivity as is shown in Ref.\cite{33}. We can easily explain why, although the Hamiltonian in Ref.\cite{8} has a similar form to that in Ref.\cite{33} it fails to exhibit topological nontrivial phase. The fact that \( \hat{c}_j \) belongs to one physical site and can not be cut apart lies at the heart of the argument that makes the Majorana Fermion topological in Ref.\cite{33}. In contract, in Ref.2 each \( \hat{\gamma}_n (\hat{\tau}_n) \) belongs to one independent physical site \( n \), which means they represent independent freedoms. So the topological invariant in Ref.\cite{33} can not be applied to discuss the SSH model.

To show the edge states in the off-diagonal model do not signify topological non-triviality we can add one site to the end the model and cut off the first one. The conclusion on the existence or not of edge states in the system is just opposite to that in Ref.\cite{8} as is shown in Fig.7. This shows that the so-called topological distinct states are nothing but same system with different boundary conditions. Similar mistake is made in Ref.\cite{13}, we show that the edge states depend on the boundary condition in Fig.8.
In Ref.\textsuperscript{34} the following tight-binding model

\[ \hat{H} = \sum_j \hat{C}_j^\dagger \begin{pmatrix} -t_s & -t_{sp} \\ t_{sp} & t_s \end{pmatrix} \hat{C}_{j+1} + \text{h.c.} + \sum_j \hat{C}_j^\dagger (\mu + \Delta_y \sigma_y) \hat{C}_j \]  

(21)

is studied. Beside Zak phase they also use the winding number of the Hamiltonian to characterize the topology of the 1D system. Such winding numbers of the Hamiltonian appear in many literatures. However, the winding number is also not a topological invariant because it depends on the choice of the unit cells. For example, in this case use a different choice of unit cell \( \tilde{h}(k) \) becomes

\[ \tilde{h}(k) = ([2t_{tp} \sin(k)] + \Delta_y \sin[(\delta_1 - \delta_2)k], [2t_{tp} \sin(k)] + \Delta_y \cos[(\delta_1 - \delta_2)k], -(t_s + t_p) \cos(k)) \]  

(22)

We can see that the winding property of the Hamiltonian depends on \( \delta_1 - \delta_2 \) as is shown in Fig.9, so it can not be considered as an topological invariant. It can be calculated that the Zak phase

\[ \varphi_{Zak}' = n_d 2\pi - 2\pi (\delta_1/2 + \delta_2/2) \text{ modulo } 2\pi. \]  

When \( n_d = 0 \), as is assumed in most of the works, the Zak phase can always take 0 and \( \pi \) in both the so-called topological case and the trivial case. So it is impossible to discriminate two topological distinct cases by using the Berry phases. To show the edge state is not a topological property we can calculate the spectrum in the so-called trivial case when \( \Delta_y > 2t_{tp} \) with a different open boundary condition from that is used in Ref.\textsuperscript{34} We find there are same type of edge states as the so-called topological case as is shown in Fig.10. This again shows the triviality of the edge states in 1D system.

In Ref.\textsuperscript{7} they use the SSH model to discuss the topological property of 1D system and the condition for existence of the edge states. Here their argument suffer from the same flaws discussed above. In addition to the Zak phase, they use the winding number of the element of the Hamiltonian to classify the topological property of SSH model. As is shown the above the element of Hamilto-
nian depends on the choice of prime cell and so does the winding numbers as is shown in Fig.11. Similar winding numbers have been used in several papers to characterize the so-called topological phases.

In Ref. they seem to take the absolute value Zak phase seriously and even the \( \varphi_{Zak} = 0 \) and \( \varphi_{Zak} = 2\pi \) cases are deemed as topologically distinct. However, as is shown above, \( \varphi_{Zak} = 0 \) and \( \varphi_{Zak} = 2\pi \) are identical because the Zak phases are defined modulo \( 2\pi \). The edge existence of the edge states also depends on boundary conditions as is shown in Fig.12.

In Ref. Aubry-Andre model is used to study the topological property of the 1D system. They claim the appearance of the edge states indicating the bulk state is topological insulator characterized by non-zero Chern number. However, as we show in the main text the mere existence of the edge states is not a topological property. In this case the gaplessness of the edge states spectrum as functions of \( \delta \) is topological. This is because the model can be mapped into a 2D model when \( \delta \) is identified with \( -k_x \) as is shown in Ref. Then the gaplessness of edge states is protected by the Chern number of the 2D system. However, they try to relate the Chern number of 1D system to the some density plateaus for fixed \( \delta \) and claim the Chern number can be observed with fixed \( \delta \).

This is clearly a mistake because the density plateaus have nothing to do with the Chern number of bulk system and the they have misused the the Streda formula. In fact the density plateaus reflect nothing but the band structure of the 1D periodic system can still be approximately applicable when there is a slowly changing potential present in this system. As we shown Fig.13, every 1D system can exhibit this density plateaus and they have nothing to do with the topological property of the 1D system.
In Refs.\textsuperscript{11} and \textsuperscript{5} interface states between two 1D periodical systems are proposed and observed. They do find something that is truly topological in inversion symmetric 1D system: the relative Zak phase of two bands, which is independent of the coordinate system we use and can only assume 0 or $\pi$ (modulo $2\pi$). However, here we still can not conclude that the existence of the interface states indicate the topological difference of the two systems. First, they use the total Zak phase of the bands to explain of the existence of the interface states. As we have shown the total Zak phase depends on the coordinate system and is not a topological invariant. With a given coordinate system the relative total Zak phase is still dependent of the relative displacement of the two systems. So the relative total Zak phase also is not a topological invariant. Second, as we have shown in the main text, there is interface between two identical systems if they are connected in a suitable way. Obviously, there is no topological difference between the two identical systems. So the existence do not reflect topological difference of the two connecting 1D systems.

There are some authors use the quantum walk to study the topological property of 1D systems. However, their discussions on the topological property suffer from the same fatal flaws as discussed above: the topological invariant they defined are not invariant and the systems they considered as topologically distinct are actually topologically equivalent or completely identical as is illustrated in Fig.14 and Fig.15.
7 Edge states with Hill’s equation

In the main text we use topological argument to show the triviality of the localized states in 1D periodic system. To illustrate the ubiquitous of the edge states in 1D system we use the theory of Hill’s equation\(^{28}\) to show we can always find edge state in a 1D continuous system.

\[
\frac{d^2}{dx^2} \psi(x) + \lambda \rho(x, k_y) \psi(x) = 0
\]  

(23)

where \(\rho(x + a, k_y) = \rho(x, k_y)\). This eigenequation of the periodic string is an equation of Hill’s type\(^7\). We can transform S.1 to a Hill’s equation with standard form by Liouville transformation if \(\rho(x, k_y)\) is positive and twice differentiable\(^2\). The transformed equation is an eigenequation for quantum particle in one-dimensional periodic potentials. So the problem of a periodic string is mathematically equivalent to a one-dimensional quantum problem. In the following discussions we will not discriminate the classical and quantum systems.

To discuss the edge states we first state some of the results about Hill’s equation without proof\(^7\).

Let \(y_1(x)\) and \(y_2(x)\) be two solutions of Hill’s equation with same \(\lambda\) satisfying the initial condition:

\[
y_1(0) = 1, \quad y'_1(0) = 0, \quad y_2(0) = 0, \quad y'_2(0) = 1
\]  

(24)

A solution of S.1 \(y(x)\) satisfies

\[
\begin{pmatrix}
  y(x) \\
  y'(x)
\end{pmatrix} =
\begin{pmatrix}
  y_1(x) & y_2(x) \\
  y'_1(x) & y'_2(x)
\end{pmatrix}
\begin{pmatrix}
  y(0) \\
  y'(0)
\end{pmatrix}
\]  

(25)
We define $T(x) = \begin{pmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{pmatrix}$ and let $x = na + x_r$, where $n$ is an integer and $0 \leq x_r < a$. Then
\begin{equation}
\begin{pmatrix} y(x) \\ y'(x) \end{pmatrix} = T(x_r)T(a)^n \begin{pmatrix} y(0) \\ y'(0) \end{pmatrix} = T(x_r)\begin{pmatrix} y(na) \\ y'(na) \end{pmatrix} \tag{26}
\end{equation}

The discriminant of Hill’s equation is defined by $\Delta(\lambda) = y_1(a) + y_2'(a)$, which is the trace of $T(a)$. Let $\lambda_n$ denote the zeros of $\Delta(\lambda) - 2$ and $\lambda'_n$ the zeros of $\Delta(\lambda) + 2$. They interlace as follow:
\begin{equation}
\lambda_0 < \lambda'_1 \leq \lambda'_2 < \lambda_1 \leq \lambda_2 < \lambda'_3 \leq \lambda'_4 \ldots \tag{27}
\end{equation}

In the intervals $(\lambda_0, \lambda'_1), (\lambda'_2, \lambda_1), (\lambda_2, \lambda'_3), \ldots, |\Delta(\lambda)| < 2$ and all solutions are stable, so they are called the stability intervals. The intervals $[-\infty, \lambda_0], [\lambda'_1, \lambda'_2], [\lambda_1, \lambda_2], \ldots$ are referred to as the instability intervals in which $|\Delta(\lambda)| \geq 2$. When we study quantum particles in periodic potentials or a string with periodic density, the intervals $[\lambda_0, \lambda'_1], [\lambda'_2, \lambda_1], [\lambda_2, \lambda'_3], \ldots$ are the energy or frequency bands. In these bands the eigenfunctions of S.1 can be chosen as Bloch waves $u_{k_x k_y}(x)e^{i k_x x}$. $\lambda_0, \lambda_2, \lambda_4, \ldots$ and $\lambda'_0, \lambda'_2, \lambda'_4, \ldots$ correspond to bottoms of the bands located at $k_x = 0$ and $k_x = \pi/a$ respectively. While $\lambda_1, \lambda_3, \lambda_5, \ldots$ and $\lambda'_1, \lambda'_3, \lambda'_5, \ldots$ correspond to tops of the bands located at $k_x = 0$ and $k_x = \pi/a$ respectively.

The characteristic equation of matrix $T(a)$ is $\rho^2 - \Delta(\lambda) \rho + 1 = 0$. When $|\Delta(\lambda)| \neq 2$ there are two eigenvectors $\chi_1$ and $\chi_2$ corresponding to the eigenvalues $\rho_1$ and $\rho_2$. Then
\begin{equation}
\begin{pmatrix} y(na) \\ y'(na) \end{pmatrix} = c_1 \rho_1^n \chi_1 + c_2 \rho_2^n \chi_2 \tag{28}
\end{equation}

where $c_1$ and $c_2$ are constants. If $|\Delta(\lambda)| \leq 2$, $|\rho_1| = |\rho_2| = 1$. We can see from S.4 and S.6 the solution $y(x)$ won’t be localized at the edge, so there is no edge state in the bulk bands.
Now we discuss the edge states in a one-dimensional system with length of $Na$, where $N$ is an integer. We assume the system satisfies a homogeneous boundary condition

$$\cos \alpha y(0) + \sin \alpha y'(0) = 0$$

$$\cos \alpha y(Na) + \sin \alpha y'(Na) = 0$$  \hspace{1cm} (29)$$

where $\alpha$ is a real number. This boundary condition is more general than the fixed boundary condition $\alpha = 0$, which has been used in most of the works discussing edge states. The situation we consider here corresponds to the commensurate case in ref\textsuperscript{26}, the incommensurate case will be discussed elsewhere.

In the bulk gaps $|\Delta(\lambda)| > 2$, so the eigenvalues of $T(a)$ are two different real numbers that satisfy $\rho_1 \rho_2 = 1$. We assume $|\rho_1| < 1$ and $|\rho_2| > 1$. If a solution satisfies S.7, $[y(0), y'(0)]^T$ and $[y(Na), y'(Na)]^T$ are linearly dependent, which means $c_1 \chi_1 + c_2 \chi_2$ and $c_1 \rho_1^{Na} \chi_1 + c_2 \rho_2^{Na} \chi_2$ must also be linearly dependent. However this won’t happen unless one of $c_1$ and $c_2$ is zero because $\rho_1^{Na} \neq \rho_2^{Na}$ and $\chi_1$ and $\chi_2$ are linearly independent. So $(y(0), y'(0))^T$ of an edge state solution satisfying S.7 must be an eigenvector of $T(a)$.

On the other hand, if $(y(0), y'(0))^T$ is an eigenvector of $T(a)$ with $\lambda$ in the gap $y(x)$ must satisfy S.7 for some real $\alpha$ because $(y(Na), y'(Na))^T = T(a)^N (y(0), y'(0))^T = \rho^N (y(0), y'(0))^T$ and $\rho^N$ is a real number. $(y(0), y'(0))^T = c_1 \chi_1$ and $(y(0), y'(0))^T = c_2 \chi_2$ indicate that the edge states locate at the left and right end of the one-dimensional system respectively. So $y(x)$ being a wave function of edge state satisfying S.7 is equivalent to $(y(0), y'(0))^T$ being an eigenvector of $T(a)$ with $\lambda$ in the gap.
If \((y(0), y'(0))^T\) is an eigenvector of \(T(a)\) with real eigenvalue, then \(y(x)\) will satisfy the condition

\[
\begin{align*}
\cos \alpha y(0) + \sin \alpha y'(0) &= 0 \\
\cos \alpha y(a) + \sin \alpha y'(a) &= 0
\end{align*}
\]

for some real \(\alpha\). That is each edge state corresponds to a solution of a Sturm-Liouville problem within a period \([0, a]\).

Now we show for a given \(\alpha\) each solution of the Sturm-Liouville problem corresponds to an edge state. From Ref.\(^{35}\) we can see \((y(0), y'(0))^T\) of each solution of the Sturm-Liouville problem must be an eigenvector of \(T(a)\) with eigenvalue \(y(a)/y(0)\) or \(y'(a)/y'(0)\). Because \(y(a)/y(0)\) or \(y'(a)/y'(0)\) of the Sturm-Liouville problem is real, each solution of the Sturm-Liouville problem must determine an edge state except for the rare cases with \(y(a)/y(0) = \pm 1\) or \(y'(a)/y'(0) = \pm 1\), in which the solutions correspond to bulk Bloch waves at the top or bottom of the band. For convenience these Bloch waves are also classified as edge states.

In Ref.\(^{35}\) it is proved that there is one and only one solution of the Sturm-Liouville problem in each band gap for the cases \(\alpha = 0\) and \(\alpha = \pi/2\). The same arguments in Ref.\(^{35}\) can be used to show that this conclusion is true for arbitrary real \(\alpha\). So for a given \(\alpha\) there is one and only one edge state satisfying (30) in one gap. Thus we conclude that the edge state problem of the periodic system satisfying (30) is equivalent to a Sturm-Liouville problem in \(x \in [0, a]\) satisfying S.8. So the frequency or energy of an edge state can be obtained from a Sturm-Liouville problem.

By comparing \(|y(a)|\) with \(|y(0)|\) or \(|y'(a)|\) with \(|y'(0)|\) we can predict at which end the edge state is located. We also conclude that to observe the edge state or edge state spectrum we need only a one-period-long system. In this system all the bulk states disappear and only the edge states
remains. The conclusion facilitate our observation of the edge states greatly.
Figure 1 | Edge states in the tight-binding model

(a) Schematic view of a tight-binding system. The red line denotes the onsite potential $\epsilon_0$. $\epsilon_0 = 2$ in the bulk. $\epsilon_0 = 1 + a/x$ and $\epsilon_0 = 1 + a/(Na - x)$ at the two ends respectively. Where $a$ is the lattice constant and $N$ is the number of sites. Here $N = 6$. (b) Edge state spectrum of this model. Solid(dashed) line denotes the edge states at the left(right) end. (c) Density distribution of the wave functions of the states denoted by dots in (b).
Figure 2 | Bulk and edge states of the periodic string. a, Eigenfrequencies of the bulk frequency bands. Here we choose \( \rho(x, k_y) = m_0 + m_1 \sin(2\pi(x - d)/a) \), where \( m_0 = 1 \), \( m_1 = 0.5 \) and \( a = 1 \). \( \omega \) of bulk bands is independent of \( d \). b-d, Eigenfrequencies of the edge states as functions of \( d \). Boundary condition of string is \( \cos\alpha\psi(0) + \sin\alpha\psi(0) = \cos\alpha\psi(Na) + \sin\alpha\psi(Na) = 0 \). In b-d \( \alpha = 0, \pi/4 \) and \( \pi/2 \) respectively. Solid(dashed) lines denote frequencies of edge states at left(right) end of the string. Colored areas denote the frequencies of the bulk bands. e and f Amplitudes of the edge state wave functions as functions of \( d \). e and f correspond to edge states in the lowest and second lowest gaps in b, where the fixed boundary condition is used.
Figure 3 | Interface states between two 1D periodic quantum system. a Schematic diagram of the interface states of two quantum Hall system with opposite Chern number $\pm 1$. There are two branches of interface states in the same direction. b, The bulk bands of the two periodic systems with periods $a_1 = 0.8$ and $a_2 = 1.2$. The periodic potentials are $V_1(x) = 10 \cos(2\pi x/a_1 - k_y)$ and $V_1(x) = 15 \cos(2\pi x/a_2 + k_y)$. $\hbar = 2m = 1$ is used in the calculation. c Interface state spectrum as function of $k_y$. When $k_y$ varies $2\pi$ the two systems are both translated by one period in opposite direction. So the Chern number of the bulk bands is $1$ and $-1$ in system $1$ and $2$. d Interface state wave function correspond to the dot in c. The vertical line indicate the position of the interface.
Figure 4 | Edge states periodic quantum waveguide and circular quantum well

a, Schematic diagram of a quantum wave guide. The cross section of the waveguide is square with side length $1 + 0.5 \cos(2\pi(x - d)/a)$, where $a = 1$ is the lattice length. Two vertical impenetrable planes located at $x = 0$ and $x = 10a$ serve as edges. b, Bulk spectrum of the Bloch waves of the wave guide. It is obtained by finding the Bloch type of solution of the eigenequation $-\nabla^2 \psi = \epsilon \psi$ with $\psi = 0$ at surface of the waveguide. c, Edge state spectrum of the wave guide as function of $d$. At the two vertical planes $\psi = 0$. d, Edge state spectrum of a circular quantum well. It is obtained by solving the eigenequation $(-\nabla^2 + V(r, \phi))\psi = \epsilon \psi$ with $\psi = 0$ at $r = r_0$. Where $r_0 = 5$ is the radius of the quantum well. $\psi = 0$ at $\phi = 0$ and $\phi = 2\pi$ serves as fixed boundary condition. $V(r, \phi) = -0.5r^2(1 + 0.5 \cos(14\phi - ky))$ is the angular periodic potential. e-g, Density distribution of the wave functions denoted by dots in d.
Figure 5 | An illustration of the SSH model in Ref.\cite{19}. If two observers study the same SSH model but choose different prime cells as is shown a and b. With the choice in a(b) they will find $\varphi_{Zak} = 0 (\varphi_{Zak} = \pi)$. So their conclusions on the topological property of the bulk system is opposite. So they decide to resolve the problem by observing the ends of the system and the system will immediate collapse to a trivial(topological) phase if the boundary condition is that shown in c(d). However, if the boundary condition is that shown in e they won’t be able to decide whether the system is topological or not with the theory in Ref.\cite{19}. 

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Figure 6 | SSH model in a power law potential. a and b (e and f) are the so-called topological (trivial) case in Ref. However, if we continuously change the amplitude of the trapping potential and when the potential becomes $0.6(r/r_0)^{0.55}(r/r_0)^{0.55}$ the spectrum and the density become those in c and d (g and h), which are almost identical to the so-called trivial (topological) case. The results show that the existence of the edge states depends on the amplitude of the potential, so it is not a topological property.
Figure 7 | Spectrum of off-diagonal AAH model in Ref. 8. a) Illustration of off-diagonal AAH model with $b = 1/2$ in REF.1, which is shown to be equivalent to the SSH model. Where $J = t[1 - \lambda \cos(\varphi \lambda)]$ and $J' = t[1 + \lambda \cos(\varphi \lambda)]$. b) Spectrum of this model as is shown in Ref. c) The same bulk model as in a with one site added to the end and the first site deleted. d) Spectrum of the model in c. We see the edge state property are opposite in the same bulk system with different choice of boundary conditions, which shows the edge states are not a bulk property.
Figure 8 | Spectrum of the model in Ref.13. 

a. The spectrum of the model in Ref.4 with open boundary condition, where $T = 3$, $\lambda = 0.6$. 

b. Spectrum of same model in a with one site added to the end and the first site deleted. 

c. Spectrum of same model in a with two sites added to the end and the first two sites deleted. So again we show the existence of the edge states depends on the boundary condition.
Figure 9 | Winding property of the Hamiltonian in Ref. [34] with different choices of prime cells

a The case considered in Ref. 4. b and c correspond to the same model with different choices of prime cells. $\delta_1 - \delta_2$ equals 1 and 2 in b and c respectively. We see this winding property depends on the choice of prime cells, so it can not be topological.
Figure 10 | Spectrum of the model in Ref.\textsuperscript{34}. \textbf{a} The spectrum of the model in Ref.\textsuperscript{34} with open boundary condition, where $t_s = t_p = t_{sp}$ corresponding to the so-called topological case in ref.4. \textbf{b} Spectrum of the so-called trivial case in Ref.4, where $t_s = t_p = t_{sp}$ and $\Delta y = 4t_{sp}$. \textbf{c} Spectrum of same model in \textbf{b} with the first and last sites deleted. We can see the spectrum has the same form as the so-called topological case in \textbf{a}. So the existence of the edge states is just a boundary property, which doesn’t reflect the topology of the bulk band.
Figure 11 | Winding properties of the Hamiltonian in Ref. with different choices of prime cells. \( a-c(d-f) \) The so-called topological (trivial) case in Ref, where \( \delta_1 - \delta_2 = 0, 1 \) and \(-1\) in \( a-c(d-f) \) respectively. We can see with different choices of prime cells the winding property of the so-called different cases can be made identical.
Figure 12 | Spectrum of the model with $\varphi_{Zak} = 2\pi$ in Ref. [16]  

- **a** The spectrum of the model in Ref. [16] with open boundary condition, where $\delta t = -0.5t, \lambda = 3t$ and $\lambda' = 9t$ corresponding to the so called topological case with $Z = 2$ in Ref. [16].

- **b** Spectrum of same model in **a** with the first and last sites deleted.

- **c** Spectrum of same model in **a** with the last site deleted. Again we show the existence of the edge states is just a boundary property.
Figure 13 | Density plateau of a trivial tight-binding model in harmonious trap
To illustrate the Density plateau in Ref.\textsuperscript{12} is not a topological property we use a trivial tight-binding model
\[ \hat{H} = \sum_{n_i} \epsilon_i \hat{C}_{n_i}^\dagger \hat{C}_{n_i} + \sum_{n_i}(t \hat{C}_{n+1}^\dagger \hat{C}_{n_i} + H.C.) \]
in a confining potential \[ V_H(n - n_0)^2 \], where \( i = 1, 2, 3, 4 \) and 5 corresponding to five orbits in one prime cell, \( \epsilon_i = i \) and \( V_H = 0.00002 \). The colored bands above correspond to the area between the band bottoms and band tops in the presence of confining potential. The band bottoms and tops are calculated by summing the band bottoms and tops in the free case and the potential. Here we assume the band structure are preserved in the confining potential. The horizontal line corresponds to the Fermi level. We can see the density plateau only reflect the number of the fully occupied bands and have no topological indications. Every multi-band model with band gap in slowly changing potential can exhibit such plateaus.
Figure 14 | Winding property of the wave function in Ref. 20 with different choices of prime cells. We consider the so-called topological case with $\nu' = 3/2\nu$ in ref.1. The red, green and blue circles correspond to $\delta_1 - \delta_2 = 0, 1$ and $-1$ respectively. We can see the winding number of the same system depends on the choice of prime cell, so it is not a topological property.
Figure 15 | An illustration to show the so-called topological property in Ref.\textsuperscript{20} depends on the observers. In Ref.\textsuperscript{20} the only instruction to choose the prime cell is that the decaying site $B$ is on the left $A$. Two observers study the same system from two sides of the system will reach contradictory conclusions about the so-called topological property of the system, so this property can not be topological at all. \textbf{a} and \textbf{b} correspond to the views from $O_1$ and $O_2$. Clearly one observer will find the system topological and the other will find it trivial.
Figure 16 | An experimental result of remote stimulation with a periodic string. We measure the amplitudes of the wave function of an edge states in a periodic string. The experimental data are compared with the simulation result. The periodicity of the density of the string is created by wrapping silver paper to the string. The edge states vibration is stimulate by a magnet located at the right end of the string, where the vibration of the string is invisible to naked eyes.