KANTOROVICH TYPE INTEGRAL INEQUALITIES FOR TENSOR PRODUCT OF CONTINUOUS FIELDS OF HILBERT SPACE OPERATORS

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Abstract. This paper presents a number of Kantorovich type integral inequalities involving tensor products of continuous fields of bounded linear operators on a Hilbert space. Kantorovich type inequality in which the product is replaced by an operator mean is also considered. Such inequalities include discrete inequalities as special cases. Moreover, some generalizations of an additive Grüss integral inequality for operators are obtained.

1. Introduction

The classical Kantorovich inequality [12] asserts that for real numbers $a_i$ and $w_i$ such that $0 < a \leq a_i \leq b$ and $w_i \geq 0$ for all $1 \leq i \leq n$, we have

$$\left( \sum_{i=1}^{n} w_i a_i \right) \left( \sum_{i=1}^{n} \frac{w_i}{a_i} \right) \leq \frac{(a + b)^2}{4ab} \left( \sum_{i=1}^{n} w_i \right)^2.$$ (1.1)

This inequality can be viewed as a reverse weighted arithmetic-harmonic mean (AM-HM) inequality. The bound (1.1) is used for convergence analysis in numerical methods and statistics. Over the years, various generalizations, variations and refinements of this inequality in several settings have been investigated by many authors. This inequality has been proved to be equivalent to many inequalities, e.g. Cauchy-Schwarz-Bunyakovsky inequality and Wielant’s inequality; see also [8, 20]. In the literature, there is an integral version of Kantorovich inequality as follows. For a Riemann integrable function $f : [\alpha, \beta] \to \mathbb{R}$ with $a \leq f(x) \leq b$ for all $x \in [\alpha, \beta]$, we have (e.g. [1])

$$\int_{\alpha}^{\beta} f(x)^2 \, dx \leq \frac{(a + b)^2}{4ab} \left( \int_{\alpha}^{\beta} f(x) \, dx \right)^2.$$ (1.2)

This inequality is also called an additive version of Grüss inequality.

Many matrix versions of Kantorovich inequality were obtained in the literature, e.g. [3, 14, 15]. Let $\mathbb{M}_k$ be the algebra of $k$-by-$k$ complex matrices. Recall the Hadamard product of $A, B \in \mathbb{M}_k$ is defined to be the entrywise product:

$$A \circ B = [a_{ij} b_{ij}] \in \mathbb{M}_k.$$
A matrix analogue of this inequality involving Hadamard product is given in [16] as follows.

**Theorem 1.1** ([16], Theorem 2.2). For each \( i = 1, 2, \ldots, n \), let \( A_i \in \mathbb{M}_k \) be a positive definite matrix such that \( 0 < aI \leq A_i \leq bI \) and \( W_i \in \mathbb{M}_k \) a positive semidefinite matrix. Then

\[
\sum_{i=1}^{n} W_i^{\frac{1}{2}} A_i W_i^{\frac{1}{2}} \circ \sum_{i=1}^{n} W_i^{\frac{1}{2}} A_i^{-1} W_i^{\frac{1}{2}} \leq \frac{a^2 + b^2}{2ab} \left( \sum_{i=1}^{n} W_i \circ \sum_{i=1}^{n} W_i \right). \tag{1.3}
\]

Kantorovich inequality can be regarded as a reverse of the following Fiedler’s inequality:

\[
A \circ A^{-1} \geq I.
\]

for any positive definite matrix \( A \in \mathbb{M}_k \).

Operator versions of Kantorovich inequality was investigated, for instance, in [4, 5, 7, 17] and references therein. Kantorovich type inequality where the product is replaced by an operator mean was considered in [18, 19].

In this paper, we establish various integral inequalities of Kantorovich type for continuous field of Hilbert space operators. The inequalities (1.1) and (1.2) are generalized in many ways in terms of Bochner integrals on the Banach space of bounded linear operators. The products between two operators considered here are the Hilbert tensor product. Moreover, Kantorovich type inequalities involving Kubo-Ando operator means are obtained. Such integral inequalities include discrete inequalities as special cases. In particular, we get some generalizations of additive Grüss type inequality for operators.

This paper is organised as follows. We set up basic notations about continuous fields of operators and state the main assumption used throughout the paper in Section 2. Then Section 3 deals with Kantorovich type integral inequalities involving tensor product of continuous fields of operators. In Section 4, we first recall Kubo-Ando theory of operator means and then derive Kantorovich integral inequalities involving operator means. In the last section, we derive further operator integral inequalities, including additive Grüss inequality.

### 2. Continuous field of operators and its integralability

#### 2.1. Continuous field of operators

Throughout this paper, let \( \mathbb{H} \) be a complex Hilbert space. Denote by \( B(\mathbb{H}) \) the C*-algebra of bounded linear operators acting on \( \mathbb{H} \). The spectrum of \( A \in B(\mathbb{H}) \) is written as \( \text{Sp}(A) \). We shall write \( I \) for the identity operator on a Hilbert space; the space mentioned here should be clear from the context.

Let \( \Omega \) be a locally compact Hausdorff space endowed with a finite Radon measure \( \mu \). A family \( (A_t)_{t \in \Omega} \) of operators in \( B(\mathbb{H}) \) is said to be a continuous field of operators if the parametrization \( t \mapsto A_t \) is norm continuous on \( \Omega \). If, in addition, the norm function \( t \mapsto \|A_t\| \) is Lebesgue integrable on \( \Omega \), then we can form the Bochner integral of \( A_t \)’s as follows (see also [9]). Let \( \mathcal{P} \) be a partition of \( \Omega \) into disjoint Borel subsets and let \( \epsilon > 0 \) be a real number. For each operator \( A_t \) in
B(ℋ), we can approximate \( A_t \) by a net of operators in the form
\[
F_{\mathcal{P},\epsilon}(A_t) = \sum_{i=1}^{n} \mu(E_i) A_{t_i}
\]
where \( E_i \in \mathcal{P} \) and \( t_i \in E_i \subseteq \{t \in \Omega : \|A_t - A_{t_i}\| < \epsilon\} \) for each \( 1 \leq i \leq n \). Then the net \( F_{\mathcal{P},\epsilon}(A_t) \) converges uniformly to the Bochner integral
\[
\int_{\Omega} A_t \, d\mu(t).
\]
The set of continuous functions from \( \Omega \) to \( B(ℋ) \) becomes a C*-algebra under the pointwise operations and the C*-norm
\[
\|(A_t)_{t \in \Omega}\| = \sup_{t \in \Omega} \|A_t\|.
\]

2.2. Main hypothesis. Let \( (A_t)_{t \in \Omega} \) be a bounded continuous field of strictly positive operators in \( B(ℋ) \) such that
- the norm function \( t \mapsto \|A_t\| \) is Lebesgue integrable on \( \Omega \)
- \( \text{Sp}(A_t) \subseteq [a, b] \subseteq (0, \infty) \) for each \( t \in \Omega \).

Let \( (W_t)_{t \in \Omega} \) be a continuous field of positive operators in \( B(ℋ) \).

**Proposition 2.1.** Assume Main hypothesis. For any continuous function \( f : [a, b] \to \mathbb{R} \), we can form the Bochner integral
\[
\int_{\Omega} W_t^\frac{1}{2} f(A_t) W_t^\frac{1}{2} \, d\mu(t).
\]
In addition, if \( f([a, b]) \subseteq [0, \infty) \), then this operator is positive.

**Proof.** Since \( (\Omega, \mu) \) is a finite measure space, it suffices to prove the Lebesgue integrability of its norm function. Indeed, we have
\[
\int_{\Omega} \|W_t^\frac{1}{2} f(A_t) W_t^\frac{1}{2}\| \, d\mu(t) \leq \int_{\Omega} \|W_t^\frac{1}{2}\| \cdot \|f(A_t)\| \cdot \|W_t^\frac{1}{2}\| \, d\mu(t)
\]
\[
\leq \int_{\Omega} \|W_t\| \cdot \|f\|_{\infty} \, d\mu(t)
\]
\[
\leq \int_{\Omega} \sup_{t \in \Omega} \|W_t\| \cdot \|f\|_{\infty} \, d\mu(t)
\]
\[
= \mu(\Omega) \|f\|_{\infty} \sup_{t \in \Omega} \|W_t\|
\]
\[
< \infty.
\]
Suppose that \( f \) is positive on \( [a, b] \). Then \( f(A_t) \) is a positive operator for each \( t \in \Omega \). It follows that the integral is also positive. \( \square \)
3. Integral inequalities of Kantorovich type for tensor product of operators

In this section, we derive many integral inequalities of Kantorovich type for operators in which the product is given by the tensor product. Such inequalities includes discrete inequalities as special cases. In particular, we get a reverse of weighted AM-HM operator inequality.

3.1. Tensor products. For each fixed \( X \in B(\mathbb{H}) \), the map \( A \mapsto A \otimes X \) and the map \( A \mapsto X \otimes A \) are bounded linear operators from \( B(\mathbb{H}) \) to \( B(\mathbb{H} \otimes \mathbb{H}) \). It follows that

\[
\int_{\Omega} A_t d\mu(t) \otimes X = \int_{\Omega} (A_t \otimes X) d\mu(t). \tag{3.1}
\]

Moreover, these maps preserve positivity when the multiplier is a positive operator. For each \( A, B \in B(\mathbb{H}) \), we denote

\[
A \otimes_s B = \frac{1}{2}(A \otimes B + B \otimes A).
\]

Recall that the tensor power \( A \otimes^2 \) is defined to be \( A \otimes A \).

We start with the following estimation about tensor products.

**Lemma 3.1.** The minimum constant \( k \) for which the inequality

\[
A \otimes B^{-1} + A^{-1} \otimes B \leq kI. \tag{3.2}
\]

holds for all selfadjoint operators \( A, B \in B(\mathbb{H}) \) such that \( \text{Sp}(A), \text{Sp}(B) \subseteq [a, b] \subseteq (0, \infty) \) is determined by \( k = (a^2 + b^2)/(ab) \). Here, \( I \) denotes the identity on \( \mathbb{H} \otimes \mathbb{H} \).

**Proof.** First, note that the minimum constant \( k \) for which the scalar inequality

\[
\frac{x}{y} + \frac{y}{x} \leq k
\]

holds for all real numbers \( x, y \) such that \( x, y \in [a, b] \) is given by \( k = (a/b) + (b/a) \).

For selfadjoint operators \( A \) and \( B \) such that \( \text{Sp}(A), \text{Sp}(B) \subseteq [a, b] \subseteq (0, \infty) \), we have \( \|A\|, \|B\| \in [a, b] \) and hence

\[
\|A \otimes B^{-1} + A^{-1} \otimes B\| \leq \|A \otimes B^{-1}\| + \|A^{-1} \otimes B\|
\]

\[
= \|A\|\|B\|^{-1} + \|A\|^{-1}\|B\|
\]

\[
\leq \frac{a^2 + b^2}{ab}.
\]

Thus, we obtain the inequality (3.2). The constant \( (a^2 + b^2)/(ab) \) cannot be improved since the case \( A = aI_\mathbb{H} \) and \( B = bI_\mathbb{H} \) is reduced to the scalar case. \( \square \)

3.2. Kantorovich type integral inequalities. The following theorem is a Kantorovich type integral inequality.

**Theorem 3.2.** Under Main hypothesis, the following integral inequality holds

\[
\int_{\Omega} W_t^\frac{1}{2} A_t W_t^\frac{1}{2} d\mu(t) \otimes_s \int_{\Omega} W_t^\frac{1}{2} A_t^{-1} W_t^\frac{1}{2} d\mu(t) \leq \frac{a^2 + b^2}{2ab} \left( \int_{\Omega} W_t d\mu(t) \right)^\otimes_2. \tag{3.3}
\]

Moreover, the constant \( (a^2 + b^2)/(2ab) \) is best possible.
Proof. For convenience, let us denote
\[ X = \int_{\Omega} W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} \, d\mu(t) \quad \text{and} \quad Y = \int_{\Omega} W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}} \, d\mu(t). \]

It follows from the property (3.1) that
\[ X \otimes Y = \int_{\Omega} \left( \int_{\Omega} W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} \, d\mu(t) \right) \otimes W_r^{\frac{1}{2}} A_r^{-1} W_r^{\frac{1}{2}} \, d\mu(r) \]
\[ = \int_{\Omega^2} \left( W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r^{-1} W_r^{\frac{1}{2}} \right) \, d\mu(r) \, d\mu(t). \]

Similarly, we have
\[ Y \otimes X = \int_{\Omega^2} \left( W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r W_r^{\frac{1}{2}} \right) \, d\mu(r) \, d\mu(t). \]

It follows that
\[ 2 (X \otimes s Y) = \int_{\Omega^2} \left( W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r^{-1} W_r^{\frac{1}{2}} + W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}} \otimes W_r^{\frac{1}{2}} A_r W_r^{\frac{1}{2}} \right) \, d\mu(r) \, d\mu(t) \]
\[ = \int_{\Omega^2} (W_t \otimes W_r)^{\frac{1}{2}} (A_t \otimes A_r^{-1} + A_t^{-1} \otimes A_r) (W_t \otimes W_r)^{\frac{1}{2}} \, d\mu(r) \, d\mu(t). \]

By making use of Lemma 3.1 and the property (3.1), we obtain
\[ X \otimes_s Y \leq \frac{1}{2} \int_{\Omega} \int_{\Omega} \frac{a^2 + b^2}{ab} (W_t \otimes W_r) \, d\mu(r) \, d\mu(t) \]
\[ = \frac{a^2 + b^2}{2ab} \int_{\Omega} \left( \int_{\Omega} W_t \, d\mu(r) \right) \otimes W_t \, d\mu(t) \]
\[ = \frac{a^2 + b^2}{2ab} \int_{\Omega} W_t \, d\mu(t) \otimes \int_{\Omega} W_t \, d\mu(t). \]

Therefore, we arrive at (3.3). The best possibility of the constant \((a^2 + b^2)/(2ab)\) also comes from Lemma 3.1.

As a special case, we obtain a discrete version of the integral inequality (3.3) as follows.

**Corollary 3.3.** For each \(i = 1, 2, \ldots, n\), let \(A_i \in B(\mathbb{H})\) be a selfadjoint operator such that \(\text{Sp}(A_i) \subseteq [a, b] \subseteq (0, \infty)\) and let \(W_i\) be a positive operator in \(B(\mathbb{H})\). Then we have
\[ \sum_{i=1}^{n} W_i^{\frac{1}{2}} A_i W_i^{\frac{1}{2}} \otimes_s \sum_{i=1}^{n} W_i^{\frac{1}{2}} A_i^{-1} W_i^{\frac{1}{2}} \leq \frac{a^2 + b^2}{2ab} \left( \sum_{i=1}^{n} W_i \right)^{\otimes 2}. \]

**Proof.** Take \(\Omega = \{1, 2, \ldots, n\}\) and set \(\mu\) to be the counting measure in Theorem 3.2. \(\square\)

The next result is an integral inequality of Kantorovich type in which the weights are scalars.
Corollary 3.4. Assume Main hypothesis. For any continuous function \( w : \Omega \to [0, \infty) \), we have
\[
\int_\Omega w(t) A_t \, d\mu(t) \otimes_s \int_\Omega w(t) A_t^{-1} \, d\mu(t) \leq \frac{a^2 + b^2}{2ab} \left( \int_\Omega w(t) \, d\mu(t) \right)^2 I. \tag{3.5}
\]

Proof. Set \( W_t = w(t)I \) for each \( t \in \Omega \) in Theorem 3.2. \( \Box \)

The following result is a discrete version of the inequality \( (3.5) \).

Corollary 3.5. For each \( i = 1, 2, \ldots, n \), let \( A_i \in \mathcal{B}(\mathbb{H}) \) be a selfadjoint operator such that \( \text{Sp}(A_i) \subseteq [a, b] \subseteq (0, \infty) \) and let \( w_i \geq 0 \) be a constant. Then
\[
\left( \sum_{i=1}^n w_i A_i \right) \otimes_s \left( \sum_{i=1}^n w_i A_i^{-1} \right) \leq \frac{a^2 + b^2}{2ab} \left( \sum_{i=1}^n w_i \right)^2 I. \tag{3.6}
\]

Proof. Take \( \Omega = \{1, 2, \ldots, n\} \) and set \( \mu \) to be the counting measure in Corollary 3.4. \( \Box \)

From this corollary, when the weight \( w_i \) is \( 1/n \) for each \( i \), then
\[
\frac{1}{n} (A_1 + A_2 + \cdots + A_n) \otimes \frac{1}{n} (A_1^{-1} + A_2^{-1} + \cdots + A_n^{-1}) \leq \frac{a^2 + b^2}{2ab} I. \tag{3.7}
\]
Recall that the harmonic mean of \( A_1, A_2, \ldots, A_n \) is given by
\[n(A_1^{-1} + A_2^{-1} + \cdots + A_n^{-1})^{-1}.
Hence, Corollary 3.5 provides a reverse weighted AM-HM inequality.

4. Kantorovich integral inequalities involving operator means

In this section, we establish integral analogues of Kantorovich inequality involving operator means. To begin with, recall some fundamental facts in Kubo-Ando theory of operator means \([13]\); see also \([10, \text{Section 3}] \) and \([11, \text{Chapter 5}] \).

4.1. Preliminaries on operator means. An (operator) connection is a binary operation \( \sigma \) assigned to each pair of positive operators such that for all \( A, B, C, D \geq 0 \):

(M1) (joint) monotonicity: \( A \leq C, B \leq D \implies A \sigma B \leq C \sigma D \)
(M2) transformer inequality: \( C(A \sigma B)C \leq (CAC) \sigma (CBC) \)
(M3) (joint) continuity from above: for \( A_n, B_n \in \mathcal{B}(\mathbb{H})^+ \), if \( A_n \downarrow A \) and \( B_n \downarrow B \), then \( A_n \sigma B_n \downarrow A \sigma B \). Here, \( X_n \downarrow X \) indicates that \( (X_n) \) is a decreasing sequence converging strongly to \( X \).

Using (M2), every operator connection \( \sigma \) is invariant under congruence transformations in the sense that
\[
C(A \sigma B)C = (CAC) \sigma (CBC), \tag{4.1}
\]
for \( A, B \geq 0 \) and \( C > 0 \). Moreover, every connection \( \sigma \) satisfies
\[
(A + B) \sigma (C + D) \geq (A \sigma C) + (B \sigma D), \tag{4.2}
\]
for any \( A, B, C, D \geq 0 \).
An operator mean is a connection $\sigma$ with fixed point property $A \sigma A = A$ for all $A \geq 0$.

A major core of Kubo-Ando theory is the one-to-one correspondence between operator connections and operator monotone functions. Recall (e.g. [11, Chapter 4]) that a continuous function $f : [0, \infty) \to \mathbb{R}$ is said to be operator monotone if

$$A \leq B \implies f(A) \leq f(B)$$

holds for any positive operators $A$ and $B$.

**Proposition 4.1.** ([13, Theorem 3.4]) Given an operator connection $\sigma$, there is a unique operator monotone function $f : [0, \infty) \to [0, \infty)$ such that

$$f(A) = I \sigma A, \quad A \geq 0.$$  \hfill (4.3)

In fact, the map $\sigma \mapsto f$ is a bijection.

Such a function $f$ is called the representing function of $\sigma$.

**Lemma 4.2 ([2]).** Let $\sigma$ be an operator connection. Then for all positive operators $A$ and $B$ in $B(H)$, we have

$$\|A \sigma B\| \leq \|A\| \|\sigma\| \|B\|.$$  

Here, the connection $\sigma$ on the right hand side is the induced connection on $[0, \infty)$ defined by $(a \sigma b)I = aI \sigma bI$ for any $a, b \geq 0$.

We say that a function $f : [0, \infty) \to \mathbb{R}$ is super-multiplicative if $f(xy) \geq f(x)f(y)$ for all $x, y \geq 0$.

**Lemma 4.3.** Let $\sigma$ be an operator connection associated with an operator monotone function $f : [0, \infty) \to [0, \infty)$. If $f$ is super-multiplicative, then

$$(A \sigma C) \otimes_s (B \sigma D) \leq (A \otimes_s B) \sigma (C \otimes_s D)$$  \hfill (4.4)

for all positive operators $A, B, C, D$.

**Proof.** By a continuity argument using $(M1)$ and $(M3)$, we may assume that $A$ and $B$ are strictly positive. Putting $X = A^{-\frac{1}{2}}CA^{-\frac{1}{2}}$ and $Y = B^{-\frac{1}{2}}DB^{-\frac{1}{2}}$ yields

$$(A \sigma C) \otimes (B \sigma D) = (A \otimes B)^{\frac{1}{2}}[(I \sigma X) \otimes (I \sigma Y)](A \otimes B)^{\frac{1}{2}}$$

$$= (A \otimes B)^{\frac{1}{2}}f(X) \otimes f(Y)(A \otimes B)^{\frac{1}{2}}$$

$$\leq (A \otimes B)^{\frac{1}{2}}f(X \otimes Y)(A \otimes B)^{\frac{1}{2}}$$

$$= (A \otimes B)^{\frac{1}{2}}[I \sigma (X \otimes Y)](A \otimes B)^{\frac{1}{2}}$$

$$= (A \otimes B) \sigma (C \otimes D).$$

Here, we use the congruent invariance $(4.1)$ and the property $(4.3)$. Now,

$$(A \sigma C) \otimes (B \sigma D) + (B \sigma D) \otimes (A \sigma C)$$

$$\leq (A \otimes B) \sigma (C \otimes D) + (B \otimes A) \sigma (D \otimes C)$$

$$\leq [(A \otimes B) + (B \otimes A)] \sigma [(C \otimes D) + (D \otimes C)].$$

Hence, we obtain (4.4). \qed
4.2. Kantorovich type integral inequalities involving operator means.

The following result can be regarded as a Kantorovich type integral inequality concerning operator means.

**Theorem 4.4.** Assume Main hypothesis. Let \((B_t)_{t \in \Omega}\) be a bounded continuous field of strictly positive operators such that \(\text{Sp}(B_t) \subseteq [a, b]\) for each \(t \in \Omega\). Let \(\sigma\) be an operator mean with a super-multiplicative representing function. Then

\[
\int_{\Omega} W_t^{\frac{1}{2}}(A_t \sigma B_t)W_t^{\frac{1}{2}} d\mu(t) \lesssim \int_{\Omega} W_t^{\frac{1}{2}}(A_t^{-1} \sigma B_t^{-1})W_t^{\frac{1}{2}} d\mu(t)
\]

\[
\leq \frac{a^2 + b^2}{2ab} \left( \int_{\Omega} W_t d\mu(t) \right)^{\otimes 2}.
\]  \hspace{1cm} (4.5)

**Proof.** The function \(t \mapsto W_t^{\frac{1}{2}}(A_t \sigma B_t)W_t^{\frac{1}{2}}\) is Bochner integrable due to the norm estimate in Lemma 4.2. It follows that

\[
\int_{\Omega} W_t^{\frac{1}{2}}(A_t \sigma B_t)W_t^{\frac{1}{2}} d\mu(t) \lesssim \int_{\Omega} W_t^{\frac{1}{2}}(A_t^{-1} \sigma B_t^{-1})W_t^{\frac{1}{2}} d\mu(t)
\]

\[
\leq \int_{\Omega} \left( W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} \sigma W_t^{\frac{1}{2}} B_t W_t^{\frac{1}{2}} \right) d\mu(t) \lesssim \int_{\Omega} \left( W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}} \sigma W_t^{\frac{1}{2}} B_t^{-1} W_t^{\frac{1}{2}} \right) d\mu(t)
\]

(since \(\sigma\) satisfies the transformer inequality (M2))

\[
\leq \left[ \int_{\Omega} W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} d\mu(t) \right] \sigma \left[ \int_{\Omega} W_t^{\frac{1}{2}} B_t W_t^{\frac{1}{2}} d\mu(t) \right]
\]

\[
\lesssim \left[ \int_{\Omega} W_t^{\frac{1}{2}} A_t W_t^{\frac{1}{2}} d\mu(t) \right] \lesssim \left[ \int_{\Omega} W_t^{\frac{1}{2}} A_t^{-1} W_t^{\frac{1}{2}} d\mu(t) \right]
\]

\[
\sigma \left[ \int_{\Omega} W_t^{\frac{1}{2}} B_t W_t^{\frac{1}{2}} d\mu(t) \right] \lesssim \left[ \int_{\Omega} W_t^{\frac{1}{2}} B_t^{-1} W_t^{\frac{1}{2}} d\mu(t) \right]
\]

(by Lemma 4.3)

\[
\leq \frac{a^2 + b^2}{2ab} \left( \int_{\Omega} W_t d\mu(t) \right)^{\otimes 2} \sigma \frac{a^2 + b^2}{2ab} \left( \int_{\Omega} W_t d\mu(t) \right)^{\otimes 2}
\]

(by Theorem 3.3)

\[
= \frac{a^2 + b^2}{2ab} \left( \int_{\Omega} W_t d\mu(t) \right)^{\otimes 2}.
\]

(since \(\sigma\) satisfies the fixed point property).

The proof is complete. \(\square\)

Theorem 4.4 can be reduced to Theorem 3.2 by setting \(A_t = B_t\) for all \(t \in \Omega\).

The next result is discrete version of the inequality (4.5).

**Corollary 4.5.** For each \(i = 1, 2, \ldots, n\), let \(A_i, B_i \in B(\mathbb{H})\) be selfadjoint operators such that \(\text{Sp}(A_i), \text{Sp}(B_i) \subseteq [a, b] \subseteq (0, \infty)\) and let \(W_i\) be a positive operator in \(B(\mathbb{H})\). Then we have

\[
\sum_{i=1}^{n} W_i^{\frac{1}{2}}(A_i \sigma B_i)W_i^{\frac{1}{2}} \lesssim \sum_{i=1}^{n} W_i^{\frac{1}{2}}(A_i^{-1} \sigma B_i^{-1})W_i^{\frac{1}{2}} \leq \frac{a^2 + b^2}{2ab} \left( \sum_{i=1}^{n} W_i \right)^{\otimes 2}.
\]  \hspace{1cm} (4.6)
Proof. Take \( \Omega = \{1, 2, \ldots, n\} \) and set \( \mu \) to be the counting measure in Theorem 4.4.

5. Further operator integral inequalities

Theorem 3.3 can be extended in the following way:

**Theorem 5.1.** Assume Main hypothesis. Let \( f \) be a continuous real-valued function defined on \([a, b] \cup [1/b, 1/a]\) such that \( f(x)f(1/x) \leq 1 \) for all \( x \in [a, b] \). Suppose that \( f([a, b]) \subseteq [a, b] \) or \( f([a, b]) \subseteq [1/b, 1/a] \). Then

\[
\int_{\Omega} W^\frac{1}{2} f(A_t) W^\frac{1}{2} d\mu(t) \otimes_s \int_{\Omega} W^\frac{1}{2} f(A_{t}^{-1}) W^\frac{1}{2} d\mu(t) \leq \frac{a^2 + b^2}{2ab} \left( \int_{\Omega} W_t d\mu(t) \right)^2.
\]

(5.1)

**Proof.** Since \( \text{Sp}(A_t^{-1}) \subseteq [1/b, 1/a] \) for each \( t \), the function \( t \mapsto W^\frac{1}{2} f(A_{t}^{-1}) W^\frac{1}{2} \) is Bochner integrable by Proposition 2.1. The assumption also implies that \( f(A_{t}^{-1}) \leq f(A_t)^{-1} \) for each \( t \in \Omega \). The desired result now follows from Theorem 3.2. Note that the constant \((a^2 + b^2)/(2ab)\) is not affected.

Theorem 5.1 is reduced to Theorem 3.2 by setting \( f(x) = x \) or \( f(x) = 1/x \).

**Corollary 5.2.** Assume the hypothesis of Theorem 5.1. For any continuous function \( g : [a, b] \to [0, \infty) \), we have

\[
\int_{\Omega} f(A_t) g(A_t) d\mu(t) \otimes_s \int_{\Omega} f(A_{t}^{-1}) g(A_t) d\mu(t) \leq \frac{a^2 + b^2}{2ab} \left( \int_{\Omega} g(A_t) d\mu(t) \right)^2.
\]

(5.2)

**Proof.** Set \( W_t = g(A_t) \) for each \( t \in \Omega \) in Theorem 5.1. Then \((W_t)_{t \in \Omega}\) is a continuous field of positive operators.

The next result can be viewed as a generalization of Grüss inequality.

**Corollary 5.3.** Assume Main hypothesis. For any \( \lambda \in \mathbb{R} \), we have

\[
\int_{\Omega} A_{\lambda+1} d\mu(t) \otimes_s \int_{\Omega} A_{\lambda-1} d\mu(t) \leq \frac{a^2 + b^2}{2ab} \left( \int_{\Omega} A_{\lambda} d\mu(t) \right)^2.
\]

(5.3)

**Proof.** Put \( f(x) = x \) and \( g(x) = x^\lambda \) in Corollary 5.2.

The case \( \lambda = 1 \) and \( \mu(\Omega) = 1 \) in this corollary is a Grüss type integral inequality for tensor product of operators:

\[
\int_{\Omega} A_{t}^2 d\mu(t) \otimes_s I \leq \frac{a^2 + b^2}{2ab} \left( \int_{\Omega} A_{t} d\mu(t) \right)^2.
\]

(5.4)
Theorem 5.4. Assume Main hypothesis. Suppose that $1 \in [a, b]$. For any super-multiplicative operator monotone function $f : [0, \infty) \rightarrow [0, \infty)$ such that $f(1) = 1$, we have

$$\int_\Omega W_t^\frac{1}{2} f(A_t) W_t^\frac{1}{2} d\mu(t) \otimes_s \int_\Omega W_t^\frac{1}{2} f(A_t^{-1}) W_t^\frac{1}{2} d\mu(t) \leq \frac{a^2 + b^2}{2ab} \left( \int_\Omega W_t d\mu(t) \right)^{\otimes^2}.$$ (5.5)

Proof. By Proposition 4.1, there is an operator mean $\sigma$ such that $f(A_t) = I \sigma A_t$ for any $A_t \geq 0$. The desired result now follows from Theorem 4.4 by considering $I \sigma A_t$ instead of $A_t \sigma B_t$. □

Corollary 5.5. Assume Main hypothesis. Suppose that $1 \in [a, b]$. For any $\alpha \in [-1, 1]$ and a continuous function $g : [a, b] \rightarrow [0, \infty)$, we have

$$\int_\Omega A_t^\alpha g(A_t) d\mu(t) \otimes_s \int_\Omega A_t^{-\alpha} g(A_t) d\mu(t) \leq \frac{a^2 + b^2}{2ab} \left( \int_\Omega g(A_t) d\mu(t) \right)^{\otimes^2}. \quad (5.6)$$

Proof. Consider the operator monotone function $f(x) = x^\alpha$. Note that this function is super-multiplicative and satisfies $f(1) = 1$. The desired result now follows by replacing $W_t$ by $g(A_t)$ in Theorem 5.4. □

Under the hypothesis of Corollary 5.5, we have an interesting operator inequality. For each $\lambda \in \mathbb{R}$, putting $g(x) = x^\lambda$ in (5.6) yields

$$\int_\Omega A_t^{\lambda + \alpha} d\mu(t) \otimes_s \int_\Omega A_t^{-\lambda - \alpha} d\mu(t) \leq \frac{a^2 + b^2}{2ab} \left( \int_\Omega A_t^\lambda d\mu(t) \right)^{\otimes^2}. \quad (5.7)$$

Discrete versions for the inequalities in this section can be obtained by considering $\Omega$ to be a finite space equipped with the counting measure.

Acknowledgement. This research was supported by King Mongkut’s Institute of Technology Ladkrabang Research Fund grant no. KREF045710.

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