Testing exogeneity in the functional linear regression model

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Abstract

We propose a novel test statistic for testing exogeneity in the functional linear regression model. In contrast to Hausman-type tests in finite dimensional linear regression setups, a direct extension to the functional linear regression model is not possible. Instead, we propose a test statistic based on the sum of the squared difference of projections of the two estimators for testing the null hypothesis of exogeneity in the functional linear regression model. We derive asymptotic normality under the null and consistency under general alternatives. Moreover, we prove bootstrap consistency results for residual-based bootstraps. In simulations, we investigate the finite sample performance of the proposed testing approach and illustrate the superiority of bootstrap-based approaches. In particular, the bootstrap approaches turn out to be much more robust with respect to the choice of the regularization parameter.

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1 Introduction

In functional linear regression models, goodness-of-fit tests are much more complicated to construct than e.g. in the multiple linear setting. This, among others stems to the fact that in functional linear regression models the $L_2$-distance of the slope function estimator to the true function has no proper limiting distribution. This was shown in Cardot et al. (2006) and Ruymgaart et al. (2000) for two estimators in the classical functional linear regression model under exogeneity. It turns out, that the lack of a proper limiting distribution also applies for other estimators using different model assumptions. This phenomenon inherent to functional data setups is probably one of the reasons why goodness-of-fit testing is generally not that widely developed for functional regression models yet. In particular, desirable counterparts of standard tests that are well-established in the multiple linear model are still missing in the functional linear setting.

In functional data settings, existing goodness-of-fit tests are described in Müller and Stadtmüller (2005), who use a suitable scalar product to transform the functions to a different space using the autocovariance operator to obtain a test statistic having a proper limiting distribution. Further approaches are given in Cuesta-Albertos et al. (2019) and García-Portugués et al. (2014), García-Portugués et al. (2020), who use random projections together with empirical process techniques.

In practice, one important model assumption is the exogeneity of the regressor. Especially in economics, this assumption is often violated such that the regressors are correlated with the error terms which leads to endogeneity issues. Estimating in such a model is an inverse problem. Neglecting endogeneity generally results in inconsistent estimators. Hence, it is important to test the data for exogeneity first. If the null hypothesis of exogeneity is rejected, different estimators such as e.g. instrumental variable (IV) estimators are required to achieve consistent estimation. See Johannes (2016), Florens et al. (2011) or Florens and Van Bellegem (2014), who consider such IV estimators in functional regression setups and derive asymptotic theory. While in the multiple linear regression model the Hausman test (see Hausman (1978) und Wu (1973)) is a standard and natural approach for testing exogeneity, this method cannot be transferred directly to the functional linear model since it is based on the $L_2$-distance of two slope function estimators due to the following proposition which transfers the results in Cardot et al. (2006) and Ruymgaart et al. (2000) to the present setting.

In the following, let $\hat{\beta}$ denote the estimator of the slope function in the exogeneous model described in Johannes (2013) and $\hat{\beta}_{IV}$, which is consistent under exogeneity, but inconsistent under endogeneity, and by $\hat{\beta}_{IV}$ the IV estimator in the endogeneous functional linear model given in Johannes (2016), which is consistent in both cases. Then, we get the following result.

**Proposition 1.1** In the functional linear regression model (2.1) defined below, even under
exogeneity, there is no random variable $Z$ with non-degenerate distribution, such that

$$s_n \| \hat{\beta}_{IV} - \beta \| \overset{D}{\to} Z$$

for some real sequence $(s_n)_{n \in \mathbb{N}}$ with $\lim_{n \to \infty} s_n = \infty$, where $\| \cdot \|$ denotes the norm of the Hilbert space.

The proof of this result mainly goes along the lines of the one in Cardot et al. (2006), see also Dorn (2021) for further details. This is why we just state the result here and use it as motivation for a different approach in the following. Motivated by the fact, that in contrast to the $L^2$-distance, the projection error typically has an asymptotic distribution (see e.g. Müller and Stadtmüller (2005) and Florens and Van Bellegem (2014)), we propose to use the sum of the squared difference of projections of the two estimators as test statistic.

The rest of the paper is organized as follows. In Section 2, we state the model assumptions, construct the test statistic and derive its asymptotic distribution. As the limiting distribution turns out to depend on unknown functional nuisance parameters, which are difficult to estimate, we propose residual-based bootstrap methods in Section 3 and prove their consistency. The finite sample performance of all discussed tests is investigated in Section 5. All longer proofs are deferred to the Appendix and additional auxiliary results to a supplement.

## 2 Model and test statistic

We consider the functional linear regression model

$$Y = \int_{[0,1]} X(t)\beta(t)dt + U = \langle \beta, X \rangle + U,$$  \hspace{1cm} (2.1)

where $Y$ is a real-valued random variable, $U$ is a real-valued error term with $E[U] = 0$ and $E[U^2] = \sigma^2 \in (0, \infty)$, $X$ is a functional random variable with values in $L^2([0,1])$ such that $\int_0^1 E|X(t)|^2 dt < \infty$. In this setup, the error variance $\sigma^2$ is unknown, and $\beta$ is an unknown function from the Sobolev space of periodically extendable square integrable functions denoted by

$$\mathcal{W}_\nu := \left\{ f \in L^2[0,1] : \| f \|^2_\nu := \sum_{k \in \mathbb{Z}} \gamma_k^2 |\langle f, \phi_k \rangle|^2 < \infty \right\},$$  \hspace{1cm} (2.2)

where $(\phi_k)_{k \in \mathbb{Z}}$ is the Fourier basis of $L^2([0,1])$, $\nu \in \mathbb{R}$ and $\gamma_k = 1 + |2\pi k|$, $k \in \mathbb{Z}$, see e.g. Tsybakov (2004). In the setup of (2.1), we will speak of exogeneity (and call $X$ an exogenous regressor), if

$$H_0 : \quad E[X(t)U] = 0 \text{ for all } t \in [0,1].$$  \hspace{1cm} (2.3)

Otherwise, we will speak of endogeneity (and call $X$ an endogeneous regressor), if

$$H_1 : \quad E[X(t)U] \neq 0 \text{ for at least one } t \in [0,1].$$  \hspace{1cm} (2.4)
For consistent estimation in the endogeneous case, we assume to additionally have a functional instrumental variable $W$ with values in $L_2([0,1])$ such that $\int_0^1 E|W(t)|^2 \, dt < \infty$ and $E[UW(t)] = 0$ for all $t \in [0,1]$. For the sake of simplicity, it is often assumed in the literature, that $E[X(t)] = E[W(t)] = 0$ holds for all $t \in [0,1]$. However, the general case can be handled along the same lines by centering with the sample mean in a first step and our results are stated for the general case. For estimating the cross-covariance operator, we also assume that $(X,W)$ is second-order stationary, see Johannes (2016).

Assumption 1 There exist functions $c_X, c_W, c_{WX} : [-1, 1] \rightarrow \mathbb{R}$, such that

\[
\begin{align*}
\text{Cov}(X(s), X(t)) &= c_X(t - s), \\
\text{Cov}(W(s), W(t)) &= c_W(t - s), \\
\text{Cov}(W(s), X(t)) &= c_{WX}(t - s),
\end{align*}
\]

for all $s, t \in [0,1]$, respectively, where $c_X$ is assumed to be continuous.

By imposing continuity of $c_X$, we have that whenever (2.4) holds for one $t \in [0,1]$, this immediately implies $E[X(t)U] \neq 0$ on some set with positive Lebesgue measure. This condition ensures, that the test statistic proposed in the following can be used to consistently test for the null hypothesis (2.3) against alternatives (2.4).

Note, that $c_X$ and $c_W$ define the kernels of the covariance operators $\Gamma_X$ of $X$ and $\Gamma_W$ of $W$, respectively, and $c_{WX}$ is the kernel of the cross covariance operator $\Gamma_{WX}$ of $X$ and $W$. The (joint) weak stationarity of $(X,W)$ ensures, that both covariance operators as well as the cross covariance operator have the same exponential system of eigenfunctions, which we denote by $(\phi_k)_{k \in \mathbb{N}}$. Hence, let $(x_k, \phi_k)_{k \in \mathbb{N}}$ be the eigensystem of $\Gamma_X$, $(w_k, \phi_k)_{k \in \mathbb{N}}$ the eigensystem of $\Gamma_W$ and $(c_k, \phi_k)_{k \in \mathbb{N}}$ the eigensystem of $\Gamma_{WX}$. Furthermore, denote $\lambda_k = \frac{|c_k|^2}{w_k}$.

Assumption 2 Throughout the article, we assume that all eigenvalues are strictly positive and that

\[
\sum_{k \in \mathbb{Z}} \left| E[Y\langle X, \phi_k \rangle^2 \right| x_k^2 < \infty.
\]

Furthermore, we denote by $\mu_X = \sum_{k \in \mathbb{Z}} \langle \mu_X, \phi_k \rangle \phi_k$ and $\mu_W = \sum_{k \in \mathbb{Z}} \langle \mu_W, \phi_k \rangle \phi_k$ the expectations of $X$ and $W$, respectively. Additionally, we assume that there exists some $0 < \tau < \infty$ such that

\[
\sup_{k \in \mathbb{Z}} \left| \frac{\lambda_k}{w_k} \right| \leq \tau.
\]

The last assumptions ensures, that the linear prediction of $X$ with respect to $W$ is well defined.
In principle, if they were available, IV estimation would be based on the optimal instrument \( \hat{W} \) defined by

\[
\hat{W} = \Gamma_{WX} \Gamma_w^{-1} W = \sum_{k \in \mathbb{Z}} \frac{c_k}{w_k} \langle W, \phi_k \rangle \phi_k
\]

and the eigenvalues \( (\lambda_k)_{k \in \mathbb{N}} \) of the corresponding cross covariance operator \( \Gamma_{WX} \). However, this is usual not the case and the optimal instrument respectively the corresponding eigenvalues of the cross covariance operator have to be estimated. Note, that \( \hat{W} \) could be exactly computed from \( X \) and \( W \), if the (cross) covariance operators were known and remark that \( \lambda_k = \frac{|c_k|^2}{w_k} \leq x_k \) for all \( k \in \mathbb{Z} \).

In the following, let \( \{(X_i, W_i, Y_i)\}_{i=1,...,n} \) be independent and identically distributed (i.i.d.) copies of \( (X, W, Y) \) and suppose \( (2.3) \) is valid. Then, we can consistently estimate the unknown slope function \( \beta \) due to Johannes (2013) and Johannes (2016) in two different ways. For this purpose, let \( (\alpha_n)_{n \in \mathbb{N}} \) be a sequence of regularization parameters such that \( \alpha_n > 0 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} \alpha_n = 0 \). To simplify notation, we will write \( \alpha \) for the regularization keeping in mind that it still depends on \( n \). Since the covariance operators and therefore the corresponding eigenvalues are unknown, they have to be estimated in a first step. Further, let \( \Gamma_{WX,n}, \Gamma_{X,n}, \Gamma_{W,n} : L_2([0,1]) \to L_2([0,1]) \) denote the empirical versions of \( \Gamma_{WX}, \Gamma_X \) and \( \Gamma_W \), respectively, defined by

\[
\Gamma_{WX,n}f := \frac{1}{n} \sum_{i=1}^n \langle W_i, f \rangle X_i, \quad \Gamma_{X,n}f := \frac{1}{n} \sum_{i=1}^n \langle X_i, f \rangle X_i, \quad \text{and} \quad \Gamma_{W,n}f := \frac{1}{n} \sum_{i=1}^n \langle W_i, f \rangle W_i
\]

for \( f \in L_2([0,1]) \). These estimators as well as the deduced estimators

\[
\hat{w}_k := \frac{1}{n} \sum_{i=1}^n |\langle W_i, \phi_k \rangle|^2, \quad \hat{x}_k := \frac{1}{n} \sum_{i=1}^n |\langle X_i, \phi_k \rangle|^2, \\
\hat{c}_k := \frac{1}{n} \sum_{i=1}^n \langle \phi_k, X_i \rangle \langle W_i, \phi_k \rangle, \quad \hat{\lambda}_k := \frac{|\hat{c}_k|^2}{\hat{w}_k} I\{\hat{w}_k \geq \alpha\}
\]

for the eigenvalues \( w_k, x_k, c_k \) and \( \lambda_k \), respectively, are consistent for all \( k \in \mathbb{Z} \). Hence, observations of the optimal linear instrument \( \hat{W} \) can be estimated by

\[
\hat{W}_{n,i} := \sum_{k \in \mathbb{Z}} \frac{\hat{c}_k}{\hat{w}_k} I\{\hat{w}_k \geq \alpha\} \langle W_i, \phi_k \rangle \phi_k, \quad i = 1, \ldots, n,
\]

and the corresponding cross covariance operator by

\[
\hat{\Gamma}_n := \frac{1}{n} \sum_{i=1}^n \langle \hat{W}_{n,i}, \cdot \rangle X_i = \frac{1}{n} \sum_{k \in \mathbb{Z}} \frac{\hat{c}_k}{\hat{w}_k} I\{\hat{w}_k \geq \alpha\} \sum_{i=1}^n \langle \cdot, X_i \rangle \langle W_i, \phi_k \rangle \phi_k. \quad (2.6)
\]

This allows to construct the IV-based estimator \( \hat{\beta}_{IV} \) of the slope function \( \beta \) defined by

\[
\hat{\beta}_{IV} := \sum_{k \in \mathbb{Z}} \frac{\hat{g}_k}{\hat{\lambda}_k} I\{\hat{\lambda}_k \geq \gamma_k^\alpha \} \phi_k = \frac{1}{n} \sum_{i=1}^n \langle W_i, \phi_k \rangle Y_i I\{\hat{\lambda}_k \geq \gamma_k^\alpha \} I\{\hat{w}_k \geq \alpha\} \phi_k, \quad (2.7)
\]
where
\[\hat{g}_k = \frac{1}{n} \sum_{i=1}^{n} Y_i \langle \tilde{W}_{n,i}, \phi_k \rangle.\]

As shown in \cite{Johannes2016} under Assumptions 1 and 2 the estimator \(\hat{\beta}_{IV}\) is consistent under the exogeneity assumption (2.3) as well as under endogeneity of (2.4). In contrast, again under Assumptions 1 and 2, the estimator
\[\hat{\beta} = \sum_{k \in \mathbb{Z}} \frac{1}{n} \sum_{i=1}^{n} \langle X_i, \phi_k \rangle Y_i \hat{x}_k I\{\hat{\lambda}_k \geq \alpha \gamma_k\} \phi_k\]  
(2.8)
is only consistent under the exogeneity assumption (2.3) (see Johannes (2013)) and inconsistent under endogeneity of (2.4). Note, that in comparison to the original definition in Johannes (2013), for \(\hat{\beta}\), we use the same indicator function \(I\{\hat{\lambda}_k \geq \alpha \gamma_k\}\) as in \(\hat{\beta}_{IV}\). It turned out, that the tests perform better if the same regularization is used in both estimators although it might not be the best choice for estimating \(\beta\) by \(\hat{\beta}\) under assumption (2.3).

Based on the two estimators (2.7) and (2.8), we construct the test statistic as
\[T_n = \frac{1}{n} \sum_{i=1}^{n} \left| \langle \hat{\beta}_{IV} - \hat{\beta}, X_i \rangle \right|^2 = \langle \hat{\beta}_{IV} - \hat{\beta}, \Gamma_{X,n} \left( \hat{\beta}_{IV} - \hat{\beta} \right) \rangle.\]  
(2.9)
The last representation above corresponds to the idea used in \cite{MuellerStadtmuller2005} to construct a goodness-of-fit test. The equivalence of both approaches can be seen by using the singular value decomposition for the estimators and for the covariance operator.

**Assumption 3** For the sequence of regularization parameters, we assume
\[\alpha_n = \alpha > 0 \quad \forall \ n \in \mathbb{N}, \quad \alpha = o(1) \quad \text{and} \quad \frac{1}{n \alpha^2} = o(1).\]

For the next results, different moment conditions for \(X, W, U\) are required. To simplify the notation, we introduce the following sets. In doing so, we assume, that all conditions on \(X\) and \(W\) mentioned above are fulfilled and define
\[\mathcal{F}^m_{\eta} := \left\{ (X, W) \left| \sup_{k \in \mathbb{Z}} \mathbb{E} \left[ \frac{|X, \phi_k|^m}{\sqrt{x_k}} \right] \leq \eta \quad \text{and} \quad \sup_{k \in \mathbb{Z}} \mathbb{E} \left[ \frac{|W, \phi_k|^m}{\sqrt{w_k}} \right] \leq \eta \right\};\]  
(2.10)
\[\mathcal{G}^m_{\eta} := \left\{ X \left| \Gamma_X > 0 \quad \text{and} \quad \sup_{k \in \mathbb{Z}} \mathbb{E} \left[ \frac{|X, \phi_k|^m}{\sqrt{x_k}} \right] \leq \eta \right\}.\]  
(2.11)

In the following, for an operator \(\Delta\), we denote by \(\Delta^\dagger\) the regularized inverse of the operator, that is
\[\Delta^\dagger = \sum_{k \in \mathbb{Z}} \frac{1}{\delta_k} I\{|\delta_k| > \alpha \gamma_k\} (\cdot, \phi_k) \phi_k\]
and we define
\[ t_n^2 := \| (\tilde{\Gamma}_{X,n}^\dagger - \Gamma_X^\dagger) \Gamma_X \|_{HS}^2 = \sum_{k \in \mathcal{K}_n} \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2, \tag{2.12} \]
where \( \| \cdot \|_{HS} \) denotes the Hilbert-Schmidt norm and we set
\[ \mathcal{K}_n := \{ k \in \mathbb{Z} \mid \lambda_k \geq \alpha \gamma_k \nu \}. \]

Now, we are in a position to state an asymptotic result for the test statistic.

**Theorem 2.1** In model (2.1), under Assumptions 1-3, let \( \{(X_i, W_i, Y_i)\}_{i=1}^n \) be i.i.d. copies of \((X, W, Y)\) with \((X, W) \in \mathcal{F}_\eta^{128}\) and \(E|U|^{128} \leq \eta < \infty\). Furthermore, let \( t_n \to \infty \) as \( n \to \infty \), and
\[ \frac{1}{t_n^4} \sum_{k \in \mathcal{K}_n} \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^4 = o(1), \quad \sum_{k \in \mathbb{Z}} |\langle \beta, \phi_k \rangle| \frac{x_k^3}{|c_k|^2} w_k < \infty, \quad \sum_{k \in \mathbb{Z}} \frac{x_k^2 w_k}{|c_k|^2} < \infty. \]
Then, under \( H_0 \), we have
\[ \frac{n}{t_n} (T_n - \mathfrak{B}_n - \mathfrak{R}_n) \overset{D}{\to} \mathcal{N}(0, \mathfrak{V}), \]
where
\[ \mathfrak{B}_n = \frac{n}{2t_n^2} (\langle \beta, \mu_X \rangle)^2 \sum_{k \in \mathbb{Z}} \left( \frac{\langle \mu_W, \phi_k \rangle}{c_k} - \frac{\langle \mu_X, \phi_k \rangle}{x_k} \right)^2 x_k I \{ \lambda_k \geq \alpha \gamma_k \nu \}, \]
\[ \mathfrak{R}_n = \frac{1}{n} \left( \sigma^2 + \sum_{m \in \mathbb{Z}} |\langle \beta, \phi_m \rangle|^2 x_m \right) \sum_{k \in \mathbb{Z}} \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) I \{ \lambda_k \geq \alpha \gamma_k \nu \}, \]
\[ \mathfrak{V} = \left( \sigma^2 + \sum_{m \in \mathbb{Z}} |\langle \beta, \phi_m \rangle|^2 x_m \right)^2. \]
Additionally, if \( X \) is centered, that is, \( E[X(t)] = 0 \) for all \( t \in [0, 1] \), we have \( \mu_X = 0 \) leading to \( \mathfrak{B}_n = 0 \).

**Proof.** For the sake of simplicity, we assume, that \( X \) is centered. If not, the additional bias term has to be taken into account as well as stated in the assertion of the theorem. We give a short overview of the proof. The used propositions and lemmas are stated and proven in the appendix. For the employed decomposition of the test statistic, we need several (modified) correlation operators of the instruments and \( X \). We define \( \mathcal{U}_n, \Delta_{W,n} : L_2([0, 1]) \to \mathbb{R} \) by
\[ \mathcal{U}_n f = \frac{1}{n} \sum_{i=1}^n (W_i \otimes U_i) f \quad \text{and} \quad \Delta_{W,n} f = \frac{1}{n} \sum_{i=1}^n (W_i \otimes Y_i) f, \]

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With the Lemma of Slutsky, the assertion follows.

A.5, A.2, A.3 and A.4 together with standard estimation techniques for the mixed terms.

\( V \) converges weakly due to Theorem A.1 to a normal distribution with mean 0 and variance

For the test statistic, the following decomposition holds

\[
\frac{n}{t_n} T_n = \frac{1}{t_n} \sum_{j=1}^{n} \langle T_{n,1} + T_{n,2} + T_{n,3}, X_j \rangle^2 + \frac{1}{t_n} \sum_{j=1}^{n} \langle T_{n,1} + T_{n,2} + T_{n,3}, X_j \rangle \langle X_j, R_n \rangle
\]

where

\[
T_{n,1} = \left( \tilde{\Gamma}_n^\dagger \tilde{U}_n - \Gamma_{X,n}^\dagger U_{X,n} \right) - \tilde{\Pi}_K \left( \tilde{\Gamma}_n^\dagger \tilde{U}_n - \Gamma_{X,n}^\dagger U_{X,n} \right)
\]

\[
T_{n,2} = \left( \tilde{\Gamma}_n^\dagger \tilde{U}_n - \Gamma_{X,n}^\dagger X_{n} \right) \beta - \tilde{\Pi}_K A_n
\]

\[
T_{n,3} = \tilde{\Pi}_K \left( \tilde{\Gamma}_n^\dagger \tilde{U}_n - \Gamma_{X,n}^\dagger U_{X,n} + A_n \right) - \left( \tilde{\Gamma}_n^\dagger \tilde{U}_n - \Gamma_{X,n}^\dagger U_{X,n} + A_n \right)
\]

\[
R_n = \tilde{\Gamma}_n^\dagger \tilde{U}_n - \Gamma_{X,n}^\dagger U_{X,n} + A_n
\]

and

\[
A_n = \frac{1}{n} \sum_{i=1}^{n} \sum_{k \in \mathbb{Z}} D_{i,k} I \{ \lambda_k \geq \alpha \gamma_k \} \sum_{m \in \mathbb{Z}, \ |m| \neq |k|} S_{i,m} \phi_k. \tag{2.14}
\]

When subtracting \( \mathfrak{R}_n \), the last term in (2.13) can be further decomposed to get

\[
\frac{1}{t_n} \sum_{j=1}^{n} |\langle R_n, X_j \rangle|^2 - \frac{n}{t_n} \mathfrak{R}_n = \frac{n}{t_n} R_{n,3} + \frac{n}{t_n} (R_{n,2} - \mathfrak{R}_n) + \frac{n}{t_n} (R_{n,1} + R_{n,4} + R_{n,5})
\]

where \( R_{n,i}, i = 1, \ldots, 5 \) are defined in the appendix. There, we will also see that

\[
\frac{n}{t_n} R_{n,3} = \frac{1}{nt_n} \sum_{k \in \mathbb{Z}} x_k I \{ \lambda_k \geq \alpha \gamma_k \} \sum_{i,j=1, i \neq j}^{n} D_{i,k} (\sigma U_i + \sum_{m \in \mathbb{Z}, \ |m| \neq |k|} S_{i,m}) D_{j,k} (\sigma U_j + \sum_{m \in \mathbb{Z}, \ |m| \neq |k|} S_{j,m})
\]

converges weakly due to Theorem A.1 to a normal distribution with mean 0 and variance \( \mathfrak{V} \), while all remaining terms are discussed to be asymptotically negligible using Propositions A.5, A.2, A.3, and A.4 together with standard estimation techniques for the mixed terms.

With the Lemma of Slutsky, the assertion follows. \( \square \)
To apply the above result for testing, the bias and variance term have to be estimated. To this end, note that $\sigma^2$ can be consistently estimated by

$$
\hat{\sigma}_n^2 = \frac{1}{n} \sum_{i=1}^{n} (Y_i - \langle \hat{\beta}_{IV}, X_i \rangle)^2
$$

(2.15)
due to the law of large numbers and since $\frac{1}{n} \sum_{i=1}^{n} (\beta - \hat{\beta}_{IV}, X_i)^2 = o_P(1)$ by similar calculations as in the derivation of the asymptotic distribution of $T_n$.

**Corollary 2.2** Suppose all assumptions of Theorem 2.1 hold. Then, under $H_0$, we have

$$
\frac{n \, T_n - \hat{\mathcal{B}}_n - \hat{\mathcal{R}}_n}{\sqrt{\hat{\mathcal{V}}_n}} \xrightarrow{d} \mathcal{N}(0,1),
$$

where $\hat{\sigma}_n^2$ is defined in (2.15) and

$$
\hat{t}_n^2 = \sum_{k \in \mathbb{Z}} \left( \frac{\hat{x}_k \hat{w}_k}{|\hat{c}_k|^2} - 1 \right)^2 I\{\hat{\lambda}_k \geq \alpha \gamma_k\},
$$

$$
\hat{\mathcal{B}}_n = \frac{n}{2 \hat{t}_n} \langle \hat{\beta}_{IV}, \hat{\mu}_X \rangle^2 \sum_{k \in \mathbb{Z}} \left( \frac{\langle \hat{\mu}_W, \hat{\phi}_k \rangle}{c_k} - \frac{\langle \hat{\mu}_X, \hat{\phi}_k \rangle}{x_k} \right)^2 \hat{x}_k I\{\hat{\lambda}_k \geq \alpha \gamma_k\},
$$

$$
\hat{\mathcal{R}}_n = \frac{1}{n} \left( \sigma_n^2 + \|\Gamma^{1/2}_{X,n} \hat{\beta}_{IV}\|_2^2 \right) \sum_{k \in \mathbb{Z}} \left( \frac{\hat{x}_k \hat{w}_k}{|\hat{c}_k|^2} - 1 \right) I\{\hat{\lambda}_k \geq \alpha \gamma_k\},
$$

$$
\hat{\mathcal{V}}_n = \left( \sigma_n^2 + \|\Gamma^{1/2}_{X,n} \hat{\beta}_{IV}\|_2^2 \right)^2.
$$

Using Corollary 2.2 it is possible to construct a test for the null hypothesis

$$
H_0 : \quad \mathbb{E}[X(t)U] = 0 \text{ for all } t \in [0,1]
$$

(2.16)
against

$$
H_1 : \quad \mathbb{E}[X(t)U] \neq 0 \text{ for at least one } t \in [0,1].
$$

(2.17)

For given size $\gamma \in (0,1)$, we can reject $H_0$ if

$$
\frac{n \, T_n - \hat{\mathcal{B}}_n - \hat{\mathcal{R}}_n}{\hat{t}_n \sqrt{\hat{\mathcal{V}}_n}} > u_{1-\gamma}
$$

(2.18)
where $u_{1-\gamma}$ denotes the $(1-\gamma)$-quantile of the standard normal distribution. That is, we get a one-sided test for $H_0$ against $H_1$. In the special case $\mu_X = 0$ we can neglect the additional bias term which avoids the use of its plug-in estimator such that the test has the simpler structure $I\{n(T_n - \hat{\mathcal{R}}_n)/\hat{t}_n \sqrt{\hat{\mathcal{V}}_n} > u_{1-\gamma}\}$. 

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Corollary 2.3 Suppose all assumptions of Corollary 2.2 hold. Then, under the alternative $H_1$, the test constructed in (2.18) is consistent.

Proof. We only consider the special case $\mu_X = 0$ here. The general case can be proven by similar arguments. Under $H_1$, $\hat{\beta}$ is not consistently estimating $\beta$ such that it converges in probability to $\beta + b$ for some $b \in L^2[0,1]$ with

$$b = \sigma \sum_{k \in \mathbb{Z}} \frac{E[U_1(X_1, \phi_k)]}{x_k} I\{\lambda_k \geq \alpha \gamma_k\} \phi_k(t),$$

which is in general not equal to $0 \in L^2[0,1]$ under endogeneity (by the continuity imposed in Assumption 1). Hence, we have

$$T_n = \frac{1}{n} \sum_{i=1}^{n} |\langle \hat{\beta}_{IV} - (\hat{\beta} - b), X_i \rangle|^2 - \frac{2}{n} \sum_{i=1}^{n} \langle \hat{\beta}_{IV} - (\hat{\beta} - b), X_i \rangle \langle b, X_i \rangle + \frac{1}{n} \sum_{i=1}^{n} |\langle b, X_i \rangle|^2$$

$$= \frac{1}{n} \sum_{i=1}^{n} |\langle \hat{\beta}_{IV} - (\hat{\beta} - b), X_i \rangle|^2 - O_p \left( \sqrt{\frac{t_n}{n}} \right) + O_p(1).$$

The standardized version of the first part converges in distribution to a standard normal distribution by similar arguments as in Theorem 2.1 and Corollary 2.2 while the sum of the remainder terms multiplied with $\frac{n}{t_n}$ goes to infinity for $n \to \infty$. Consequently, we have

$$P \left( \frac{n T_n - \hat{\mathbb{R}}_n}{\sqrt{\mathbb{V}_n}} > u_{1-\gamma} \right) \to 1$$

for $n \to \infty$. 

In practice, we do not know, if $\mu_X = 0$ such that a naive application of the asymptotic test without estimating $\mathbb{B}_n$ could result in wrong decisions. In addition, asymptotic tests based on plug-in methods as above usually exhibit a smaller power compared to other methods. This is due to the additional estimation step. The bootstrap version of the test discussed in the next section is expected to have better finite sample behavior, since it is not required to estimate the unknown bias and variance. This has additionally the effect, that we need not distinguish between the cases $\mu_X = 0$ and $\mu_X \neq 0$ which is a clear advantage of the bootstrap test.

3 Bootstrap Consistency

In this section, we use residual-based bootstrap procedures to estimate the distribution of

$$\frac{n}{t_n} (T_n - \mathbb{B}_n - \mathbb{R}_n)$$
under the null of exogeneity. To this end, we first estimate the residuals from the original data set and define

\[ \hat{U}_i = Y_i - \langle \hat{\beta}_{IV}, X_i \rangle, \quad i = 1, \ldots, n, \]

where we use the IV-based estimator, because it is consistent under the null hypothesis as well as under the alternative. However, using the classical estimator \( \hat{\beta} \) would also result in a proper bootstrap scheme to approximate the distribution of the test statistics under the null of exogeneity, since the independence of error and regressor in the bootstrap sample is achieved by the (fixed-design) bootstrap procedure itself. However, to get bootstrap data that mimics the true distribution under the null hypothesis of exogeneity given the original sample as close as possible, the IV-based estimator turns out to be more natural and performs better in simulations. In the sequel, different versions of residual-based bootstraps are considered. All bootstrap methods will follow these steps.

**Step 1.** Given i.i.d. observations \((X_i, W_i, Y_i), i = 1, \ldots, n\), we generate a bootstrap sample \((X_i, W_i, Y^*_i), i = 1, \ldots, n\), by

\[ Y^*_i = \langle \hat{\beta}_{IV}, X_i \rangle + U^*_i, \]

where the bootstrap errors \(U^*_i\) are generated from the residuals \(\hat{U}_1, \ldots, \hat{U}_n\) in such a way that the conditional independence of \(U^*_i\) and \((X_i, W_i)\) is ensured. A thorough discussion, which types of bootstrap are appropriate in this sense follows in the next subsection.

**Step 2.** From \((X_i, W_i, Y^*_i), i = 1, \ldots, n\), a bootstrap test statistic \(T^*_n\) is calculated.

**Step 3.** Repeat Steps 1. and 2. \(B\) times, where \(B\) is large, to get bootstrap realizations \(T^{*1}_n, \ldots, T^{*B}_n\) of the test statistic and denote by \(q_{1-\gamma}^* = T^*_{n, \lfloor B(1-\gamma) \rfloor}\) the corresponding empirical \((1-\gamma)\)-quantile.

As the bootstrap errors are generated such that conditional independence of \(U^*_i\) and \((X_i, W_i)\) is ensured, the bootstrap automatically adopts the exogeneity assumption. For the naive (Efron-type) residual bootstrap, this is trivially the case, because the bootstrap errors are drawn independently with replacement from the residuals, and for the wild bootstrap, since suitable bootstrap multiplier variables \(V_i\) will also be drawn independently from \(X_i\) and \(W_i\).

**Theorem 3.1** Under the assumptions of Theorem 2.1 let \(S_n = \{(X_i, W_i, Y_i)\}_{i=1, \ldots, n}\) be a set of i.i.d. copies of \((X, W, Y)\) with \((X, W) \in \mathcal{F}_{\eta}^{128}\) and \(E[U]^{128} \leq \eta < \infty\) and let \((t_n)_{n \in \mathbb{N}}\) from (2.12) fulfill \(\lim_{n \to \infty} t_n = \infty\). Additionally, suppose that

\[ \frac{1}{t_n^4} \sum_{k \in \mathcal{K}_n} \left( \frac{x_k^2 w_k}{|c_k|^2} - 1 \right)^4 = o(1), \quad \sum_{k \in \mathcal{K}_n} \left( x_k^2 E[|\beta - \hat{\beta}_{IV}, \phi_k|^4] \right)^{1/4} \frac{x_k^4 w_k^4}{|c_k|^{8}} = O(1), \]
\[\sum_{k \in \mathbb{Z}} \frac{x_k w_k^{1/2}}{|c_k|} < \infty, \quad \text{and} \quad \frac{1}{t_n} \sum_{k \in \mathbb{K}_n} \frac{x_k^{3/2} w_k}{|c_k|^2} = O(1)\]
hold. Then, under both \(H_0\) and \(H_1\), we have
\[
\sup_{t \in \mathbb{R}} \left| \mathbb{P} \left( \frac{n}{t_n} (T_n^* - B_n^* - R_n^*) \leq t \mid S_n \right) - \mathbb{P}_{H_0} \left( \frac{n}{t_n} (T_n - B_n - R_n) \leq t \right) \right| \overset{P}{\to} 0,
\]
where \(B_n^*\) and \(R_n^*\) denote the bootstrap versions of \(B_n\) and \(R_n\) defined in Theorem A.1 and \(P_{H_0}\) is the distribution of \(\frac{n}{t_n} (T_n - B_n - R_n)\) under \(H_0\).

Based on this result, we can again construct a one-sided test for the hypotheses (2.17) which rejects the null hypothesis if \(T_n > q_{1-\gamma}^*\) from Step 3 since \(T_n, T_n^* \geq 0\) and both asymptotically have the same bias and variance.

4 Generalization to other estimators and measuring goodness-of-fit

While the above results are stated for the spectral-cut-off estimators as proposed in Johannes (2013) and Johannes (2016), it is also possible to derive analogue results for other types of estimators like cut-off as in Müller and Stadtmüller (2005) or ones based on Tikhonov or ridge-type regularization. A quite general approach is given in Cardot et al. (2006) with a sequence of regularization functions \(f_n : [c_n, \infty) \to \mathbb{R}_0^+\) such that \(f_n\) is decreasing on \([c_n, 2z_1 - z_2]\) where \((z_j)_{j \in \mathbb{Z}}\) are the eigenvalues of the relevant covariance operator and \((c_n)_{n \in \mathbb{N}}\) is a decreasing sequence of positive values with \(c_n < z_1\). Furthermore \(\lim_{n \to \infty} \sup_{z \geq c_n} |zf_n(z) - 1| = o(1/\sqrt{n})\) and \(f_n\) is differentiable on \([c_n, \infty)\) which replaces Assumption 3. While the estimator \(\hat{\beta}\) from (2.8) above does not completely fit this situation it is not necessary to consider this modification if one is only interested in testing goodness-of-fit in an exogeneous model (2.1). For the sake of shorter notation we assume here \(\mu_X \equiv 0\). If \(\tilde{\beta}\) denotes the estimator proposed in Cardot et al. (2006) we obtain under Assumption 1 and the moment conditions in Theorem 2.1 the following result
\[
\frac{n}{s_n} \left( \frac{1}{n} \sum_{i=1}^{n} \left| \langle \tilde{\beta} - \beta, X_i \rangle \right|^2 - \mathfrak{R}_n \right) \overset{D}{\to} \mathcal{N}(0, \frak{M})
\]
with \(s_n = \sum_{k \in \mathbb{Z}} x_k^4 f_n^4(x_k)\), \(\frak{M}\) as in Theorem 2.1 and
\[
\mathfrak{R}_n = \frac{1}{n} \left( \sigma^2 + \sum_{m \in \mathbb{Z}} |\langle \beta, \phi_m \rangle|^2 x_m \right) \sum_{k \in \mathbb{Z}} f_n(x_k) = O \left( \frac{1}{\sqrt{n}} \right).
\]
If is straightforward to also generalize the instrumental variable estimator to other regularization schemes. We get an estimator
\[
\hat{\beta}_{IV} = \sum_{k \in \mathbb{Z}} \hat{g}_k f_n(\hat{\lambda}_k)
\]
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and, if we are willing to assume exogeneity here, derive by the same arguments as above

\[
\frac{n}{s_n,IV} \left( \frac{1}{n} \sum_{i=1}^{n} \left| \langle \tilde{\beta}_{IV} - \beta, X_i \rangle \right|^2 - \mathfrak{R}_n \right) \xrightarrow{D} N(0, \mathfrak{V})
\]

with \( s_n,IV = \sum_{k \in \mathbb{Z}} x_k^2 \lambda_k^2 f_n^4(\lambda_k) \), \( \mathfrak{V} \) as in Theorem 2.1 and

\[
\mathfrak{R}_n = \frac{1}{n} \mathfrak{V}^{1/2} \sum_{k \in \mathbb{Z}} x_k \lambda_k f_n(\lambda_k) = O \left( \frac{1}{\sqrt{n}} \right).
\]

The assumption of exogeneity is in this case not realistic because one would only use the instrumental variable estimator under endogeneity. Proving an analogue result under endogeneity is in principle possible but the proof differs in several points from the one presented here.

Using the estimators \( \tilde{\beta} \) and \( \tilde{\beta}_{IV} \) we can construct a test statistic similar to the one above. To this end we need a similar regularization scheme for both estimators. If we allow for a second argument in \( f_n(x_k, \lambda_k) \) the estimators involved in the test above can also be written with \( f_{n,1}(x_k, \lambda_k) = \frac{1}{x_k} I\{\lambda_k \geq \alpha \gamma_k^1\} \) for \( \tilde{\beta} \) and \( f_{n,2}(x_k, \lambda_k) = \frac{1}{\lambda_k} I\{\lambda_k \geq \alpha \gamma_k^2\} \) for \( \tilde{\beta}_{IV} \) and it is straightforward to generalize them at least to regularisation functions of type \( f_{n,1}(x_k, \lambda_k) = g_1(x_k, \lambda_k)f_n(\lambda_k) \) respectively \( f_{n,2}(x_k, \lambda_k) = g_2(x_k, \lambda_k)f_n(\lambda_k) \). Under Assumption 1, the moment assumptions of Theorem 2.1 and certain regularity conditions we derive under the null hypothesis

\[
\frac{n}{\bar{t}_n} (T_n - \mathfrak{B}_n - \mathfrak{R}_n) \xrightarrow{D} N(0, \mathfrak{V}),
\]

where

\[
\bar{t}_n = \sum_{k \in \mathbb{Z}} (\lambda_k g_1^2(x_k, \lambda_k) - 2\lambda_k g_1(x_k, \lambda_k)g_2(x_k, \lambda_k) + x_k g_1(x_k, \lambda_k))^2 \tilde{f}_n^2(\lambda_k),
\]

\( \mathfrak{V} \) as in Theorem 2.1 and

\[
\mathfrak{R}_n = \frac{1}{n} \mathfrak{V}^{1/2} \sum_{k \in \mathbb{Z}} (\lambda_k g_2^2(x_k, \lambda_k) - 2\lambda_k g_1(x_k, \lambda_k)g_2(x_k, \lambda_k) + x_k g_1(x_k, \lambda_k))^2 f_n(\lambda_k).
\]

For all results presented in this section it is again straightforward to derive empirical versions and bootstrap results.

5 Finite sample properties

In this section, we investigate the finite sample behavior of the tests proposed above under several degrees of endogeneity and for different slope functions. We generate our data from the model
\[
X(t) = \left( t + \frac{1}{2} \right) Z_1, \quad W(t) = \left( t + \frac{1}{2} \right) Z_2 + H
\]

and

\[
Y = \frac{1}{p+1} \sum_{l=0}^{p} X(l/p + 1) \cdot \beta(l/p) + \sigma \cdot \varepsilon,
\]

for some \( p \) large enough to approximate the integral sufficiently well. To control all correlations in the model, we generate i.i.d. copies of

\[
\begin{pmatrix}
Z_1 \\
Z_2 \\
U
\end{pmatrix}
\sim \mathcal{N}_3 \begin{pmatrix}
0 & 3 \nu \sqrt{6} & \rho \sqrt{3} \\
0 & 2 & 0 \\
\rho \sqrt{3} & 0 & 1
\end{pmatrix}
\]

with \( \text{corr}(Z_1, Z_2) = \nu \), \( \text{corr}(Z_1, U) = \rho \), see [Wong (1996)](http://dx.doi.org/10.1007/978-1-4419-8772-8_5). The random variable \( H \) is uniformly distributed on \( (-1/2, 1/2) \) and independent of \((Z_1, Z_2, \varepsilon)^\prime\). The parameter \( \rho \) controls the severity of endogeneity (if \( \rho = 0 \) we are in the exogenous case, i.e. under the null \( H_0 \)) and \( \nu \) the strength of the instrument \( W \). The standard deviation is assumed to be \( \sigma = 7/5 \).

In the following, as illustrated in Figure 1, we will use three different slope functions \( \beta_1, \beta_2 \) and \( \beta_3 \) defined by

\[
\begin{align*}
\beta_1(t) &= \sin(4\pi t) + \frac{1}{2} \sin(8\pi t) + \frac{1}{7} \sin(20\pi t), \\
\beta_2(t) &= \frac{2}{\pi} \arcsin(\cos(2\pi t)), \\
\beta_3(t) &= \sum_{n \in \mathbb{Z}} \int_{\mathbb{R}} r_n(s) k_{n,h}(t-s) \, ds,
\end{align*}
\]

where in \( \beta_3 \), \( r_n(t) = I\{n + \frac{1}{4}, n + \frac{3}{4}\}\) \( (t) \) and \( k_{n,h}(t) = \frac{1}{h} k_n\left(\frac{t}{h}\right) \) with

\[
k_n(t) = \frac{1}{C} \exp\left(-\frac{1}{1 - (t - 2n)^2}\right) I_{(-1+2n,2n+1)}(t)
\]

and \( C = \int_{\mathbb{R}} k_0(s) \, ds \). For all simulations, we generate 1000 Monte Carlo realizations and use \( B = 500 \) bootstrap replications.

Besides an Efron-type residual-based bootstrap, which draws the bootstrap errors \( U_i^* \), \( i = 1, \ldots, n \) independently with replacement from the residuals \( \hat{U}_1, \ldots, \hat{U}_n \), we consider also several versions of a residual-based wild bootstrap, where

\[
U_i^* = V_i \hat{U}_i, \quad i = 1, \ldots, n
\]

and the \( V_i \)'s are i.i.d. with \( \text{E}[V_i] = 0 \) and \( \text{E}[V_i^2] = 1 \) and independent of \((X_i, W_i, Y_i)_{i=1,\ldots,n}\). We consider different choices for the distribution of the \( V_i \)'s as commonly used in the literature,
see e.g. Mammen (1993)

\[ P \left( V_1 = \frac{- (\sqrt{5} - 1)}{2} \right) = \frac{\sqrt{5} + 1}{2\sqrt{5}}, \quad P \left( V_1 = \frac{\sqrt{5} + 1}{2} \right) = \frac{\sqrt{5} - 1}{2\sqrt{5}}, \quad (5.2) \]

\[ b) \quad P(V_1 = 1) = 0.5 = P(V_1 = -1), \quad (5.3) \]
\[ c) \quad V_1 \sim \mathcal{N}(0, 1). \quad (5.4) \]

In a first step we try to get an idea how to choose \( \alpha \) and, in a next step how to choose \( K_n \). To this end, we fix the degree of endogeneity with \( \rho = 0.4 \) and the strength of the instrument with \( \nu = 0.6 \). In Figure 2 the results for the asymptotic test using \( \beta_1 \) as slope parameter and different choices of \( \alpha \) are shown. We see that the best results are obtained for \( \alpha \) between 0.05 and 0.055. For smaller \( \alpha \), the test does not hold the prescribed level, while for larger \( \alpha \) the power is comparably small up to biased tests for \( \alpha \) larger than 0.07. Based on Figure 2 we can find a sequence of good choices for \( \alpha \) depending on the sample size varying from \( \alpha = 0.04 \) for \( n = 25 \) to \( \alpha = 0.053 \) for larger sample sizes up to 300. We see that the asymptotic test has only moderate power even for larger sample sizes. This is a well known effect with asymptotic tests using plug-in estimators.

The way out is typically a bootstrap-based test. The results for the residual-based bootstraps proposed in Section 3 and again \( \beta_1 \) are shown in Figure 3. It turns out, that the regularization parameter can be chosen considerably smaller than for the asymptotic test and the procedure is much more robust in choosing \( \alpha \). Nearly all tests hold the size of \( \gamma = 0.05 \) for larger sample sizes and the power increases with sample size for most choices of \( \alpha \) up to a value close to 1 already for \( n = 300 \). Again we can get an idea of choosing a good \( \alpha \) depending on the sample size which varies from \( \alpha = 0.01 \) for \( n = 25, 50 \) to \( \alpha = 0.0001 \) for \( n = 75, 100, 200 \) and 300.

Apparently all bootstrap procedures discussed in Section 3 perform comparably good which can be seen in Figure 4 for a choice of \( \alpha = 0.0001 \).
Comparing the performance of the bootstrap test for different slope functions, we discover that in all models the bootstrap test holds the size $\gamma = 0.05$ while we see in Table 1 that the power is similarly good for all settings with only slight disadvantages for the smoothed indicator function $\beta_3$. Finally, we inspect the influence of the degree of endogeneity and the strength of the instrument on the performance of the test. In Figure 5, we see that the power of the bootstrap test increases with increasing degree $\rho$ of endogeneity being already acceptable for $\rho = 0.3$. Figure 6 shows, that the performance of the test is highly dependent on the strength of the instrument. If the instrument is too weak, the power is too low and the test does not hold the size. It turns out, that for the setting with slope function $\beta_1$, $\rho = 0.4$ and $\alpha = 0.0001$, the bootstrap test performs best for a strength of the instrument around $\nu = 0.7$.

6 Concluding remarks

The underlying work is the first approach of testing for endogeneity in a functional regression setup by introducing a modified approach of the classical Hausman test in a multiple linear regression model. This modification is required, because the $L_2$-distance of two slope function estimators in functional linear regression models are shown to have no proper lim-
Figure 3: Empirical size and power of the bootstrap tests for several choices of $\alpha$. The gray solid line shows the target level $\gamma = 0.05$. The true slope function is $\beta_1$.

We prove asymptotic normality for the proposed modified Hausman-type test statistic, which allows for the construction of asymptotic tests for exogeneity. As the asymptotic test has several drawbacks such as many nuisance parameters, which are cumbersome to estimate, an additional bias term, which diverges when multiplied with the rate of convergence, and a high sensitivity to the choice of the regularization parameter, we propose suitable bootstrap versions of the test to approximate the null distribution. This avoids the additional estimation of nuisance parameters and turns out to be much more robust to the choice of the regularization parameter. This behavior is demonstrated in a detailed simulation study. Topics of ongoing work are the choice of the instrument, a data driven choice of the regularization parameter and the transfer to other regression models.

A Auxiliary Results for the Proof of Theorem 2.1

We assume for the sake of simplicity $E[X(t)] = E[W(t)] = 0$ for all $t \in [0, 1]$ and remember from Section 2 the decomposition of the test statistic with

$$R_{n,1} = \frac{1}{n^2} \sum_{k \in \mathbb{Z}} (\hat{x}_k - x_k) \left| \sum_{i=1}^{n} D_{i,k} I\{\lambda_k \geq \alpha \gamma_k^\nu\} \left(\sigma U_i + \sum_{m \in \mathbb{Z}, |m|\neq |k|} S_{i,m} \right) \right|^2$$

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Figure 4: Empirical power and size of the bootstrap tests for several choices of $\alpha$. The gray solid line shows the predetermined level $\gamma = 0.05$ of the test

\[ R_{n,2} = \frac{1}{n^2} \sum_{k \in \mathbb{Z}} x_k I\{\lambda_k \geq \alpha \gamma_k^\nu\} \sum_{i=1}^{n} |D_{i,k}|^2 \sigma U_i + \sum_{m \in \mathbb{Z}, |m| \neq |k|} S_{i,m}^2, \]

\[ R_{n,3} = \frac{1}{n^2} \sum_{k \in \mathbb{Z}} x_k I\{\lambda_k \geq \alpha \gamma_k^\nu\} \sum_{i,j=1, i \neq j}^{n} D_{i,k} \left( \sigma U_i + \sum_{m \in \mathbb{Z}, |m| \neq |k|} S_{i,m} \right) D_{j,k} \left( \sigma U_j + \sum_{m \in \mathbb{Z}, |m| \neq |k|} S_{j,m} \right), \]

\[ R_{n,4} = \frac{1}{n^3} \sum_{k,l \in \mathbb{Z}, |k| \neq |l|} \sum_{j=1}^{n} \langle \phi_k, X_j \rangle \langle X_j, \phi_l \rangle I\{\lambda_k \geq \alpha \gamma_k^\nu\} I\{\lambda_l \geq \alpha \gamma_l^\nu\} \]

\[ \times \sum_{i=1}^{n} D_{i,k} \left( \sigma U_i + \sum_{m \in \mathbb{Z}, |m| \neq |k|} S_{i,m} \right) D_{i,l} \left( \sigma U_i + \sum_{m \in \mathbb{Z}, |m| \neq |l|} S_{i,m} \right) \]

\[ R_{n,5} = \frac{1}{n^3} \sum_{k,l \in \mathbb{Z}, k \neq l} \sum_{j=1}^{n} \langle \phi_k, X_j \rangle \sum_{i_1, i_2=1, i_1 \neq i_2}^{n} D_{i_1,k} \left( \sigma U_{i_1} + \sum_{m \in \mathbb{Z}, |m| \neq |k|} S_{i_1,m} \right) D_{i_2,l} \left( \sigma U_{i_2} + \sum_{m \in \mathbb{Z}, |m| \neq |l|} S_{i_2,m} \right) \]

and define

\[ D_{i,k,n} = \frac{\langle W_i, \phi_k \rangle}{c_k} I\{\hat{w}_k \geq \alpha\} - \frac{1}{\bar{x}_k} \langle X_i, \phi_k \rangle, \]
Table 1: Empirical power of the bootstrap tests for slope functions defined in (5.1) using \( \rho = 0.4, \nu = 0.6 \) and \( \alpha = 0.0001 \).

| \( n \) | 25  | 50  | 75  | 100 | 125 | 150 | 175 | 200 | 225 | 250 | 275 | 300 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| \( \beta_1 \) | 0.111 | 0.507 | 0.773 | 0.901 | 0.960 | 0.992 | 0.997 | 0.998 | 1.000 | 1.000 | 1.000 | 1.000 |
| \( \beta_2 \) | 0.164 | 0.568 | 0.798 | 0.912 | 0.958 | 0.992 | 0.997 | 0.999 | 0.999 | 0.999 | 0.999 | 0.999 |
| \( \beta_3 \) | 0.255 | 0.560 | 0.733 | 0.853 | 0.904 | 0.961 | 0.978 | 0.990 | 0.993 | 0.994 | 0.997 | 0.998 |

The first result establishes the asymptotic distribution of the test statistic.

**Theorem A.1** Under the assumptions of Theorem 2.1, under the null hypothesis, and for \( (X, W) \in F_\eta \) and \( E[X(t)] = E[W(t)] = 0 \) for all \( t \in [0, 1] \), we have

\[
\frac{n R_n}{n} \xrightarrow{d} N(0, \mathfrak{R}).
\]

The remaining results are to show, that the remainder terms are negligible.

**Proposition A.2** Let \( (X, W) \in F_{128} \) and \( E[|U|] \leq \eta < \infty \). Under the assumptions of Theorem 2.1, we have

\[
\frac{1}{n} \sum_{j=1}^{n} |\langle T_{n,1}, X_j \rangle|^2 = o_P \left( \frac{1}{n} \right).
\]

**Proposition A.3** Under the assumptions of Theorem 2.1 and if \( (X, W) \in F_{64} \) and \( E[|U|] \leq \eta < \infty \), we have

\[
\frac{1}{n} \sum_{j=1}^{n} |\langle T_{n,2}, X_j \rangle|^2 = o_P \left( \frac{t_n}{n} \right).
\]

**Proposition A.4** Under the assumptions of Theorem 2.1 and if \( (X, W) \in F_{8} \) and \( E[|U|] \leq \eta < \infty \), we have

\[
\frac{1}{n} \sum_{j=1}^{n} |\langle T_{n,3}, X_j \rangle|^2 = o_P \left( \frac{t_n}{n} \right).
\]

**Proposition A.5** Under the assumptions of Theorem 2.1 and if \( E[|U|] \leq \eta < \infty \) and \( (X, W) \in F_{4} \), we have

\[
R_{n,1} = o_P \left( \frac{1}{n} \right), \quad R_{n,2} = \mathfrak{R}_n + o_P \left( \frac{t_n}{n} \right), \quad \mathfrak{R}_n = o \left( \frac{1}{\sqrt{n}} \right),
\]

\[
R_{n,4} = o_P \left( \frac{1}{n^{3/2}} \right), \quad R_{n,5} = o_P \left( \frac{1}{n} \right).
\]
B Auxiliary results

The results in this section are used at several places in the proofs. They follow from Lemma A.1 in Johannes (2016).

Lemma B.1 Let $X$ and $W$ have finite second moments and $m \in \mathbb{N}$. Then we have $\sum_{k \in \mathbb{Z}} x_k^{2m} < \infty$ and $\sum_{k \in \mathbb{Z}} x_k^{2m} w_k < \infty$. If additionally $X \in G^m_\eta$ and $\beta \in L_2([0,1])$, we have

$$E \left| \sum_{k \in \mathbb{Z}} \langle \beta, \phi_k \rangle \langle \phi_k, X \rangle \right|^{2m} < \infty.$$  

Lemma B.2 Let $p \in \mathbb{N}$ be fixed and suppose $(X,W) \in F^p_\eta$ and $E|U|^8p \leq \eta < \infty$. Then, there is a positive Konstant $C = C_p$ such that for $k \in \mathbb{Z}$, we have

$$E \left| I\{\hat{\lambda}_k \geq \alpha \gamma_k \} \right|^{p} \leq C_p \left\{ \frac{w_{k}^{p/2}}{|c_k|^p} + \frac{1}{x_k^{p/2}} \right\} \left( 1 + o(1) \right) \ (B.1)$$

and

$$E \left| I\{\hat{\lambda}_k \geq \alpha \gamma_k \} D_{i,k,n} \right|^{p} \leq C_p \left\{ \frac{w_{k}^{p/2}}{|c_k|^p} + \frac{1}{x_k^{p/2}} \right\} \left( 1 + o(1) \right) \ . \ (B.2)$$
Figure 6: Power and size of the bootstrap test for different strengths \( \nu \) of the instrument

C Proof of Theorem A.1

The proof follows by using a central limit theorem for martingal difference sequences with respect to \( (F_{n,j})_{n \in \mathbb{N}, 0 \leq j \leq n} \), where \( F_{n,j} = \sigma (X_1, W_1, Y_1, \ldots, X_j, W_j, Y_j) \) and \( F_{n,0} = \sigma (\emptyset, \Omega) \), see Hall and Heyde (1980) Theorem 3.2 and Corollary 3.1, for

\[
\frac{n}{2t_n} R_{n,3} = \sum_{j=2}^{n} \frac{1}{t_n} \sum_{k \in \mathbb{Z}} \mathcal{U}_{j,k} D_{j,k} \sum_{i=1}^{j-1} \mathcal{V}_{i,k} D_{i,k} x_k I \{ \lambda_k \geq \alpha \gamma_k^{\nu} \} = \sum_{j=2}^{n} Y_{n,j},
\]

where

\[
Y_{n,j} = \frac{1}{t_n} \sum_{k \in \mathbb{Z}} \mathcal{U}_{j,k} D_{j,k} Z_{n,j,k},
\]

and

\[
Z_{n,j,k} = \sum_{i=1}^{j-1} \mathcal{V}_{i,k} D_{i,k} x_k I \{ \lambda_k \geq \alpha \gamma_k^{\nu} \}.
\]

In a first step, we consider the conditional variance of the martingale difference scheme.

**Proposition C.1** Under the assumptions of Theorem 2.1 under the null hypothesis and for \( (X, W) \in F_n^1 \), we have

\[
\mathcal{V}_n := \sum_{j=2}^{n} \mathbb{E} \left[ Y_{n,j}^2 \mid F_{n,j-1} \right] \xrightarrow{P} \mathcal{V} \quad \text{as} \quad n \to \infty.
\]
Proof. Using that $\mathcal{V}_{j,k}D_{j,k}\overline{\mathcal{V}_{j,l}D_{j,l}}$ is independent of $(F_{n,j-1})_{j=1,...,n}$, we can decompose

$$\mathcal{W}_n = \frac{1}{t^2 n^2} \sum_{j=2}^{n} E \left( \sum_{k \in \mathbb{Z}} \mathcal{V}_{j,k}D_{j,k}Z_{n,j,k} \right)^2 \left| F_{n,j-1} \right|$$

$$= \frac{1}{t^2 n^2} \sum_{k \in \mathbb{Z}} x_k \left( \frac{x_kw_k}{|c_k|^2} - 1 \right) I\{\lambda_k \geq \alpha \gamma_k\} E|\mathcal{W}_{1,k}|^2 \left( \sum_{i=1}^{n-1} |\mathcal{V}_{i,k}D_{i,k}|^2 + \sum_{i,p=1, i\neq p}^{n-1} \mathcal{V}_{i,k}D_{i,k}\overline{\mathcal{V}_{p,k}D_{p,k}} \right)$$

$$= \mathcal{W}_{n,1} + \mathcal{W}_{n,2}.$$

We define

$$\mathcal{S}_n := \frac{1}{t^2 n^2} \sum_{k \in \mathbb{Z}} x_k \left( \frac{x_kw_k}{|c_k|^2} - 1 \right) I\{\lambda_k \geq \alpha \gamma_k\} \sum_{i=1}^{n-1} E|D_{i,k}|^2$$

and show

$$\mathcal{W}_{n,1} = \mathcal{S}_n + o(1)$$

by proving the corresponding $L_2$-convergence. Afterwards we show that $\mathcal{S}_n$ converges in probability to $\mathcal{W}$. Writing for $i \in \{1, \ldots, n\}$ and $k \in \mathbb{Z}$

$$|\mathcal{V}_{i,k}D_{i,k}|^2 E|\mathcal{W}_{1,k}|^2 - \mathcal{W} E|D_{i,k}|^2$$

$$= \mathcal{W}^{1/2} \left( |\mathcal{V}_{i,k}D_{i,k}|^2 - \mathcal{W}^{1/2} E|D_{i,k}|^2 \right) - |\mathcal{V}_{i,k}D_{i,k}|^2 |\langle \beta, \phi \rangle|^2 x_k.$$

and, observing that $\sigma^2 + \sum_{m \in \mathbb{Z}} |\langle \beta, \phi_m \rangle|^2 x_m \leq C_1$ for some constant $C_1 > 0$, we get

$$E (\mathcal{W}_{n,1} - \mathcal{S}_n)^2 \leq \mathcal{W}_{n,1} + \mathcal{W}_{n,2} + \mathcal{W}_{n,3}$$

with

$$\mathcal{W}_{n,1} = \frac{C}{t^4 n^2} \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_kw_k}{|c_k|^2} - 1 \right)^2 I\{\lambda_k \geq \alpha \gamma_k\} \left\{ \sum_{i=1}^{n-1} E \left( |\mathcal{V}_{i,k}D_{i,k}|^2 - \mathcal{W}^{1/2} E|D_{i,k}|^2 \right) \right\}$$

$$+ \sum_{i,p=1, i\neq p}^{n-1} E \left[ |\mathcal{V}_{i,k}D_{i,k}|^2 - \mathcal{W}^{1/2} E|D_{i,k}|^2 \right] E \left[ |\mathcal{V}_{p,k}D_{p,k}|^2 - \mathcal{W}^{1/2} E|D_{p,k}|^2 \right]$$

and

$$\mathcal{W}_{n,2} = \frac{C}{t^4 n^2} \sum_{k,l \in \mathbb{Z}, |k| \neq |l|} x_k \left( \frac{x_kw_k}{|c_k|^2} - 1 \right) I\{\lambda_k \geq \alpha \gamma_k\} x_l \left( \frac{x_lw_l}{|c_l|^2} - 1 \right) I\{\lambda_l \geq \alpha \gamma_l\}$$
\[
\begin{aligned}
&\left\{ \sum_{i=1}^{n-1} E\left[ (|\mathcal{U}_{i,k} D_{i,k}|^2 - \mathfrak{Y}^{1/2} E|D_{i,k}|^2) \left( |\mathcal{U}_{i,l} D_{i,l}|^2 - \mathfrak{Y}^{1/2} E|D_{i,l}|^2 \right) \right] \\
&\quad + \sum_{i,p=1, i \neq p}^{n-1} E\left[ (|\mathcal{U}_{i,k} D_{i,k}|^2 - \mathfrak{Y}^{1/2} E|D_{i,k}|^2) E\left[ (|\mathcal{U}_{i,l} D_{i,l}|^2 - \mathfrak{Y}^{1/2} E|D_{i,l}|^2) \right] \right]\right\} \\
&\quad \forall n, \frac{2}{t^4 n^2} \mathbb{E}\left( \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) |\langle \beta, \phi_k \rangle|^2 \mathbb{I}\{\lambda_k \geq \alpha \gamma_k \} \sum_{i=1}^{n-1} |\mathcal{U}_{i,k} D_{i,k}|^2 \right)^2.
\end{aligned}
\]

We have
\[
\mathbb{E}|\mathcal{U}_{j,k} D_{j,k}|^2 = \left( \sigma^2 + \sum_{m \in \mathbb{Z}, |m| \neq |k|} |\langle \beta, \phi_m \rangle|^2 x_m \right) \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right),
\]  
(C.1)

because $|\mathcal{U}_{j,k}|^2$ and $|D_{j,k}|^2$ are uncorrelated for all $k \in \mathbb{Z}$ and $j \in \{1, \ldots, n\}$. With Lemma B.1 and (E.6), for all $i \in \{1, \ldots, n\}$ and $k \in \mathcal{K}_n$, we have
\[
\mathbb{E}\left( (|\mathcal{U}_{i,k} D_{i,k}|^2 - \mathfrak{Y}^{1/2} E|D_{i,k}|^2) \right)^2 \leq C \left( \mathbb{E}|D_{1,k}|^4 - (\mathbb{E}|D_{1,k}|^2) \right) \leq C \mathbb{E}|D_{1,k}|^4
\leq \frac{C}{\alpha^2}
\]  
(C.2)

as well as
\[
\mathbb{E}\left[ (|\mathcal{U}_{i,k} D_{i,k}|^2 - \mathfrak{Y}^{1/2} E|D_{i,k}|^2) \right] = -\mathbb{E}|D_{1,k}|^2 |\langle \beta, \phi_k \rangle|^2 x_k
\]
\[
= - \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) |\langle \beta, \phi_k \rangle|^2.
\]  
(C.3)

For the mixed terms with $k, l \in \mathbb{Z}, |k| \neq |l|$ and $i \in \{1, \ldots, n\}$ and $\frac{w_k}{|c_k|^2} - \frac{1}{x_k} \geq 0$ for all $k \in \mathbb{Z}$, we get
\[
\mathbb{E}\left[ \left( |\mathcal{U}_{i,k} D_{i,k}|^2 - \mathfrak{Y}^{1/2} E|D_{i,k}|^2 \right) \left( |\mathcal{U}_{i,l} D_{i,l}|^2 - \mathfrak{Y}^{1/2} E|D_{i,l}|^2 \right) \right]
\leq \mathbb{E}\left[ |\mathcal{U}_{1,k} D_{1,k} | \mathcal{U}_{1,l} D_{1,l} \right] + \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) \left( \frac{w_l}{|c_l|^2} - \frac{1}{x_l} \right)
\leq C \left\{ \frac{1}{\alpha^2} |\langle \beta, \phi_k \rangle|^2 x_k |\langle \beta, \phi_l \rangle|^2 x_l + \frac{x_k}{\alpha} |\langle \beta, \phi_k \rangle|^2 \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right)
\right. \\
&\quad \left. + \frac{x_l}{\alpha} |\langle \beta, \phi_l \rangle|^2 \left( \frac{w_l}{|c_l|^2} - \frac{1}{x_l} \right) \right\}.
\]  
(C.4)

Using this, we have
\[
\mathbb{V}_{n,1} \leq \frac{C}{t^4 n^2} \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 \mathbb{I}\{\lambda_k \geq \alpha \gamma_k \} \left\{ \frac{n}{\alpha^2} + n^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 \right\} |\langle \beta, \phi_k \rangle|^4
\]
\[ \leq \frac{C}{t_n^4 n \alpha^2} \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 I\{\lambda_k \geq \alpha \gamma_k^\nu\} \]
\[ + \frac{C}{t_n^4} \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^4 |(\beta, \phi_k)|^4 I\{\lambda_k \geq \alpha \gamma_k^\nu\} \]
\[ = o \left( 1 + \frac{1}{t_n^2} \right) , \]

with some constant $C > 0$. With similar arguments we obtain
\[ \mathbb{V}_{n,2} = o \left( 1 + \frac{1}{t_n^2} + \frac{1}{\sqrt{nt_n}} \right) + o \left( \frac{1}{n} \right) . \]

and
\[ \mathbb{V}_{n,3} = o \left( 1 + \frac{1}{t_n^2} + \frac{1}{n} + \frac{1}{\sqrt{nt_n}} \right) . \]

which altogether results in
\[ \mathbb{V}_{n,1} = \mathcal{H}_n + o_P (1) . \]

The stochastic convergence of $\mathcal{H}_n$ follows by
\[ \mathcal{H}_n = \mathbb{V}_n - \frac{1}{t_n^2 n} \sum_{k \in \mathbb{Z}} \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 I\{\lambda_k \geq \alpha \gamma_k^\nu\} \rightarrow^P \mathbb{V} \]
for $n \to \infty$. For proving, that $\mathbb{V}_{n,2}$ converges stochastically to 0 we show again the corresponding $L_2$-convergence. To this end, we bound for all $i \in \{1, \ldots, n\}$ and $k \in \mathbb{Z}$ the term $E|\mathbb{V}_{1,k}|^2$ by a constant $C < \infty$ using the centeredness of $U$ and Lemma B.1, to obtain
\[ \mathbb{V}_{n,2} = o_P (1) . \]

The detailed arguments can be found in the supplementary material.

The second step is to show the conditional Lindeberg condition by verifying an unconditional Ljapunov condition.

Proposition C.2 Under the assumptions of Theorem 2.1 under the null hypothesis, and with $(X, W) \in \mathcal{F}_n^4$, we have
\[ \forall \varepsilon > 0 : \sum_{j=2}^{n} E \left[ Y_{n,j}^2 I\{|Y_{n,j}| > \varepsilon\} \mid \mathcal{F}_{n,j-1} \right] \rightarrow^P 0 \quad \text{as} \quad n \to \infty . \]
PROOF. It is shown in Ali et al. (2014) and Gänssler et al. (1978) that the conditional Lindeberg condition follows from the unconditional Ljapunov condition. We will show in the following, that

$$\sum_{j=2}^{n} E|Y_{n,j}|^4 = o(1)$$

and decompose

$$\sum_{j=2}^{n} E|Y_{n,j}|^4 = L_{n,1} + L_{n,2} + L_{n,3} + L_{n,4},$$

where

$$L_{n,1} = \frac{1}{t^4n^4} \sum_{j=2}^{n} \sum_{k \in \mathbb{Z}} E|\mathcal{W}_{j,k} D_{j,k} Z_{n,j,k}|^4,$$

$$L_{n,2} = \frac{1}{t^4n^4} \sum_{j=2}^{n} \sum_{k,l \in \mathbb{Z}, |k| \neq |l|} E|\mathcal{W}_{j,k} D_{j,k} Z_{n,j,k} \mathcal{W}_{j,l} D_{j,l} Z_{n,j,l}|^2,$$

$$L_{n,3} = \frac{1}{t^4n^4} \sum_{j=2}^{n} \sum_{k,l,q \in \mathbb{Z}, |k|, |l|, |p| \neq |q|, |k| \neq |l|} E \left[ |\mathcal{W}_{j,k} D_{j,k} Z_{n,j,k}|^2 \mathcal{W}_{j,l} D_{j,l} Z_{n,j,l} \mathcal{W}_{j,q} D_{j,q} Z_{n,j,q} \right],$$

$$L_{n,4} = \frac{1}{t^4n^4} \sum_{j=2}^{n} \sum_{k,l,p,q \in \mathbb{Z}, |k|, |l|, |p| \neq |q|, |k| \neq |l|} E \left[ \mathcal{W}_{j,k} D_{j,k} Z_{n,j,k} \mathcal{W}_{j,l} D_{j,l} Z_{n,j,l} \mathcal{W}_{j,p} D_{j,p} Z_{n,j,p} \mathcal{W}_{j,q} D_{j,q} Z_{n,j,q} \right].$$

For $L_{n,1}$ we use, that for all $k \in \mathbb{Z}, n \in \mathbb{N}, j \in \{1, \ldots, n\}$, $Z_{n,j,k}$ are stochastically independent of $\mathcal{W}_{j,k} D_{j,k}$ and $\mathcal{W}_{j,k}$ are uncorrelated with $D_{j,k}$. Furthermore, the fourth absolute moment of $\mathcal{W}_{j,k}$ is due to the centredness of $U$ and Lemma B.1 uniformly bounded. The fourth absolute moment of $D_{j,k}$ can be estimated using Assumption 3 and $(X,W) \in F_\eta^4$ as

$$E|D_{j,k}|^4 \leq C \left( \frac{E|W, \phi_k|^4}{|c_k|^4} + \frac{E|X, \phi_k|^4}{x_k^4} \right) \leq C \eta \left( \frac{w_k^2}{|c_k|^4} + \frac{1}{x_k^2} \right) \leq C \eta \frac{\alpha^2}{2}. \quad (C.6)$$

Again using similar arguments, we obtain

$$E|\mathcal{W}_{i,k} D_{i,k}|^2 = E|\mathcal{W}_{i,k}|^2 E|D_{i,k}|^2 \leq C \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right). \quad (C.7)$$

This results in

$$E \left| \sum_{i=1}^{j-1} \mathcal{W}_{i,k} D_{i,k} x_k I \{ \lambda_k \geq \alpha \gamma_k \} \right|^4$$

$$= x_k^4 I \{ \lambda_k \geq \alpha \gamma_k \} \left\{ \sum_{i=1}^{j-1} E|\mathcal{W}_{i,k}|^4 E|D_{i,k}|^4 + 2 \sum_{1 \leq i_1 < i_2 \leq j-1} E|\mathcal{W}_{i_1,k} D_{i_1,k}|^2 E|\mathcal{W}_{i_2,k} D_{i_2,k}|^2 \right\}$$

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\[ \leq \frac{Cn}{\alpha^2} x_k^4 I\{\lambda_k \geq \alpha \gamma_k^\nu\} + Cn^2 x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 I\{\lambda_k \geq \alpha \gamma_k^\nu\}. \] (C.8)

Putting these results together, we get

\[ L_{n,1} = \frac{1}{t_n^4 n^4} \sum_{j=2}^{n} \sum_{k \in \mathbb{Z}} E[\mathcal{W}_{j,k}^4] |E[D_{j,k}]|^4 |E[Z_{n,j,k}]|^4 \]
\[ \leq \frac{C}{t_n^4 n^4 \alpha^2} \sum_{j=2}^{n} \sum_{k \in \mathbb{Z}} E\left[ \sum_{i=1}^{j-1} \mathcal{W}_{i,k} D_{i,k} x_k I\{\lambda_k \geq \alpha \gamma_k^\nu\} \right]^4 \]
\[ \leq \frac{C}{t_n^4 n \alpha^2} \sum_{k \in \mathbb{Z}} x_k^2 I\{\lambda_k \geq \alpha \gamma_k^\nu\} \left( \frac{1}{n \alpha^2} x_k^2 + \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 \right) \]
\[ = o(1) \left( \sum_{k \in \mathbb{Z}} x_k^4 I\{\lambda_k \geq \alpha \gamma_k^\nu\} + \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 I\{\lambda_k \geq \alpha \gamma_k^\nu\} \right), \]

where the first series converges due to Lemma B.1 and the second series either also converges or, if not, can be bounded by \( Ct_n^2 \).

Considering \( L_{n,4} \), we use the stochastic independence of \( Z_{n,j,k} \) and \( \mathcal{W}_{j,l} D_{j,l} \) for all \( k, l \in \mathbb{Z} \), which results in

\[ E[\mathcal{W}_{j,k} D_{j,k} \mathcal{W}_{j,l} D_{j,l} Z_{n,j,l} \mathcal{W}_{j,p} D_{j,p} Z_{n,j,p} \mathcal{W}_{j,q} D_{j,q} Z_{n,j,q}] = E[\mathcal{W}_{j,k} D_{j,k} \mathcal{W}_{j,l} D_{j,l} \mathcal{W}_{j,p} D_{j,p} \mathcal{W}_{j,q} D_{j,q}] E[Z_{n,j,l} Z_{n,j,p} Z_{n,j,q}]. \]

The rest of the argument is just calculating the expectations using that for all \( j \in \{1, \ldots, n\}, D_{j,k}, D_{j,l}, D_{j,p} \) and \( D_{j,q} \) are uncorellated with \( S_{j,m} \) for all \( m \in \mathbb{Z} \setminus \{m \in \mathbb{Z} : |m| = |k|, |l|, |p|, |q|\} \) and stochastically independent of \( U_j \). Finally,

\[ E[S_{j,k} D_{j,k}] = \langle \beta, \phi_k \rangle E\left[ \langle \phi_k, X_j \rangle \left( \frac{\langle W_j, \phi_k \rangle}{c_k} - \langle X_j, \phi_k \rangle \right) \right] = \langle \beta, \phi_k \rangle \left( \frac{c_k - x_k}{c_k x_k} \right) = 0 \] (C.9)

and, in the same way, \( E[S_{j,k} D_{j,k}] = E[S_{j,k} D_{j,k}] = 0 \), which gives \( L_{n,4} = 0 \).

With similar arguments as above, which can be found in the supplementary material we get

\[ L_{n,2} = o \left( \frac{1}{t_n^4} + \frac{1}{t_n^2} + \frac{1}{t_n} \right) + O \left( \frac{1}{n} + \frac{1}{n^2} \right) = o(1), \]

and

\[ L_{n,3} = o \left( \frac{1}{t_n^2} \right). \]
All remainder terms can be estimated with similar techniques. We exemplarily show the idea for Proposition A.5 that is for $R_{n,2}$, in the supplementary material.

## D Proof of Theorem 3.1

Let $\Phi_V(\cdot)$ denote the distribution function of the normal distribution with mean zero and variance $\mathcal{V}$, $F_n$ the distribution function of $\frac{n}{t_n} (T_n - \mathcal{B}_n - R_n)$ and $F_{\mathcal{S}_n,n}$ the distribution function of the conditional distribution of $\frac{n}{t_n} (T_n^* - \mathcal{B}_n^* - R_n^*)$ given $\mathcal{S}_n$. By bounding

$$\sup_{t \in \mathbb{R}} \left| F_{\mathcal{S}_n,n}^*(t) - F_n(t) \right| \leq \sup_{t \in \mathbb{R}} \left| F_{\mathcal{S}_n,n}^*(t) - \Phi_V(t) \right| + \sup_{t \in \mathbb{R}} \left| F_n(t) - \Phi_V(t) \right| =: M_{1,n} + M_{2,n},$$

similar to the example in Section 29 of DasGupta (2008), it is enough to show the convergence of $M_{1,n}$ and $M_{2,n}$. Due to the continuity of $\phi_V$, the convergence of $M_{2,n}$ follows directly from Theorem 2.1 and Polya’s Theorem, as stated in Section 1.5.3 of Serfling (1980). Again, using Polya’s Theorem, it is enough to show for $M_{1,n}$, that for all $\varepsilon > 0$

$$\lim_{n \to \infty} \mathbb{P} \left( \left| F_{\mathcal{S}_n,n}^*(t) - \phi_V(t) \right| > \varepsilon \right) = 0. \quad (D.1)$$

For this we just immitate the proof of Theorem 2.1. Analogously to (2.13), we decompose

$$\frac{n}{t_n} T_n^* = \frac{1}{t_n} \sum_{j=1}^{n} \langle T_{n,1}^* + T_{n,2}^* + T_{n,3}^*, X_j \rangle^2 + \frac{1}{t_n} \sum_{j=1}^{n} \langle T_{n,1}^* + T_{n,2}^* + T_{n,3}^*, X_j \rangle \langle X_j, R_n^* \rangle$$

$$+ \frac{1}{t_n} \sum_{j=1}^{n} \langle X_j, T_{n,1}^* + T_{n,2}^* + T_{n,3}^*, R_n^* \rangle \langle R_n^*, X_j \rangle + \frac{1}{t_n} \sum_{j=1}^{n} \left| \langle R_n^*, X_j \rangle \right|^2,$$

where, similar to the proof of Theorem 2.1, we get

$$\frac{1}{t_n} \sum_{j=1}^{n} \left| \langle R_n^*, X_j \rangle \right|^2 - \frac{n}{t_n} \mathcal{R}_n = \frac{n}{t_n} R_{n,3}^* - \frac{n}{t_n} \mathcal{R}_n + \frac{n}{t_n} \left( R_{n,2}^* - \mathcal{R}_n \right) + \frac{n}{t_n} \left( R_{n,4}^* + R_{n,5}^* \right).$$

Then, $\frac{n}{t_n} (R_{n,3}^* - \mathcal{B}_n^* - \mathcal{R}_n^*)$ converges weakly in probability to $\mathcal{N}(0, \mathcal{V})$ along the lines of Theorem A.1. The remainder terms can be discussed to be negligible with the same arguments as for the remainder terms in Theorem 2.1.

## E Supplementary Material

### E.1 Proof of Proposition A.5

We give only the proof for $R_{n,2}$. We have

$$\frac{n^2}{t_n^2} \mathbb{E} |R_{n,2} - \mathcal{R}_n|^2$$

□
\[
\leq \frac{1}{t_n^2 n^2} \sum_{k \in \mathbb{Z}} x_k^2 I\{\lambda_k \geq \alpha \gamma_k^\nu\} E \left[ \sum_{i=1}^{n} \left( |D_{i,k} \mathcal{U}_{i,k}|^2 - \mathfrak{R}^{1/2} \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) \right) \right]^2 
\]
\[
+ \frac{1}{t_n^2 n^2} \sum_{k,l \in \mathbb{Z}, |k| \neq |l|} x_k I\{\lambda_k \geq \alpha \gamma_k^\nu\} x_l I\{\lambda_l \geq \alpha \gamma_l^\nu\} \sum_{i=1}^{n} E \left[ \left( |D_{i,k} \mathcal{U}_{i,k}|^2 - \mathfrak{R}^{1/2} \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) \right) \left( |D_{i,l} \mathcal{U}_{i,l}|^2 - \mathfrak{R}^{1/2} \left( \frac{w_l}{|c_l|^2} - \frac{1}{x_l} \right) \right) \right] 
\]
\[
+ \frac{1}{t_n^2 n^2} \sum_{k,l \in \mathbb{Z}, |k| \neq |l|} x_k I\{\lambda_k \geq \alpha \gamma_k^\nu\} x_l I\{\lambda_l \geq \alpha \gamma_l^\nu\} \sum_{i,p=1, i \neq p}^{n} E \left[ \left( |D_{i,k} \mathcal{U}_{i,k}|^2 - \mathfrak{R}^{1/2} \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) \right) \right] E \left[ \left( |D_{p,l} \mathcal{U}_{p,l}|^2 - \mathfrak{R}^{1/2} \left( \frac{w_l}{|c_l|^2} - \frac{1}{x_l} \right) \right) \right]. 
\]

The terms quadratic in \( k \in \mathbb{Z} \) can be estimated by Lemma B.1 and (E.6), while the other terms except the one coming from \( |\langle \beta, \phi_k \rangle|^2 x_k \) vanish

\[
E \left[ \sum_{i=1}^{n} \left( |D_{i,k} \mathcal{U}_{i,k}|^2 - \mathfrak{R}^{1/2} \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) \right) \right]^2 \leq C n^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) \left( 1 + \frac{1}{n} \right).
\]

Using the Cauchy-Schwarz inequality \((E.3)\), leads to

\[
E \left[ \left( |D_{i,k} \mathcal{U}_{i,k}|^2 - \mathfrak{R}^{1/2} \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) \right) \left( |D_{i,l} \mathcal{U}_{i,l}|^2 - \mathfrak{R}^{1/2} \left( \frac{w_l}{|c_l|^2} - \frac{1}{x_l} \right) \right) \right] \leq \frac{C}{\alpha^2}.
\]

The expectations with \( k,l \in \mathbb{Z}, |k| \neq |l| \) und \( i,p \in \{1, \ldots, n\}, i \neq p \) can be estimated by \((E.4)\). This finally yields

\[
\frac{n^2}{t_n^2} E|\mathcal{R}_{n,2} - \mathfrak{R}_n|^2
\]
\[
\leq \frac{1}{t_n^2 n^2} \sum_{k \in \mathbb{Z}} x_k^2 I\{\lambda_k \geq \alpha \gamma_k^\nu\} \left\{ \frac{C n}{\alpha^2} + C n^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) \left( 1 + \frac{1}{n} \right) \right\}
\]
\[
+ C \frac{t_n^2 n^2}{t_n^2 n^2} \sum_{k,l \in \mathbb{Z}, |k| \neq |l|} x_k I\{\lambda_k \geq \alpha \gamma_k^\nu\} x_l I\{\lambda_l \geq \alpha \gamma_l^\nu\} \left\{ \frac{n}{\alpha^2} + n(n-1) \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) \left( \frac{x_l w_l}{|c_l|^2} - 1 \right) \left( \frac{x_l w_l}{|c_l|^2} - 1 \right) \right\}
\]
\[
= o \left( 1 + \frac{1}{t_n^2} \right).
\]
The second part can be shown by using
\[ \frac{x_k w_k}{|c_k|^2} - 1 \leq \frac{1}{\alpha} (x_k - \lambda_k), \] (E.1)
for all \( k \in \mathcal{K}_n \) together with Lemma B.1

All the other parts of Proposition A.5 as well as Lemmas A.2-A.4 follow by very similar techniques. For details we refer to [5] in the main article.

### E.2 Details for the proof of Proposition C.1

Using that \( \mathcal{U}_{j,k} D_{j,k} \mathcal{D}_{j,l} \) is independent of \( (F_{n,j} - 1)_{j=1, \ldots, n} \), we can decompose
\[
\mathfrak{N}_n = \frac{1}{t_n^2 n^2} \sum_{j=2}^{n} \mathbb{E} \left[ \left| \sum_{k \in \mathbb{Z}} \mathcal{U}_{j,k} D_{j,k} Z_{n,j,k} \right|^2 \mid F_{n,j-1} \right]
\]
\[= \frac{1}{t_n^2 n^2} \sum_{k \in \mathbb{Z}} x_k \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) I\{\lambda_k \geq \alpha \gamma_k\} \mathbb{E} |\mathcal{U}_{1,k}|^2\]
\[\left( \sum_{i=1}^{n-1} |\mathcal{U}_{i,k} D_{i,k}|^2 + \sum_{\substack{i=p=1, \ i \neq p}}^{n-1} \mathcal{U}_{i,k} D_{i,k} \mathcal{U}_{p,k} \mathcal{D}_{p,k} \right)\]
\[= \mathfrak{N}_{n,1} + \mathfrak{N}_{n,2}.\]

We define
\[ \mathfrak{H}_n := \frac{\mathfrak{N}}{t_n^2 n^2} \sum_{k \in \mathbb{Z}} x_k \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) I\{\lambda_k \geq \alpha \gamma_k\} \sum_{i=1}^{n-1} \mathbb{E} |D_{i,k}|^2 \]
and show
\[ \mathfrak{N}_{n,1} = \mathfrak{H}_n + o(1) \]
by proving the corresponding \( L_2 \)-convergence. Afterwards we show that \( \mathfrak{H}_n \) converges in probability to \( \mathfrak{H} \). Writing for \( i \in \{1, \ldots, n\} \) and \( k \in \mathbb{Z} \)
\[ |\mathcal{U}_{i,k} D_{i,k}|^2 \mathbb{E} |\mathcal{U}_{1,k}|^2 - \mathbb{E} |D_{i,k}|^2 \]
\[= \mathfrak{N}^{1/2} \left[ |\mathcal{U}_{i,k} D_{i,k}|^2 - \mathfrak{N}^{1/2} \mathbb{E} |D_{i,k}|^2 \right] - |\mathcal{U}_{i,k} D_{i,k}|^2 |\langle \beta, \phi_k \rangle|^2 x_k. \]
and, observing that \( \sigma^2 + \sum_{m \in \mathbb{Z}} |\langle \beta, \phi_m \rangle|^2 x_m \leq C_1 \) for some constant \( C_1 > 0 \), we get
\[ \mathbb{E} (\mathfrak{N}_{n,1} - \mathfrak{H}_n)^2 \leq \mathfrak{N}_{n,1} + \mathfrak{N}_{n,2} \]
with
\[ \mathfrak{N}_{n,1} = \frac{C}{t_n^4 n^2} \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 I\{\lambda_k \geq \alpha \gamma_k\} \]
\[
\{ \sum_{i=1}^{n-1} E \left( |\mathcal{U}_{i,k}D_{i,k}|^2 - \mathfrak{V}^{1/2} E|D_{i,k}|^2 \right)^2 \\
+ \sum_{i,p=1, i\neq p}^{n} E \left[ |\mathcal{U}_{i,k}D_{i,k}|^2 - \mathfrak{V}^{1/2} E|D_{i,k}|^2 \right] E \left[ |\mathcal{U}_{p,k}D_{p,k}|^2 - \mathfrak{V}^{1/2} E|D_{i,k}|^2 \right] \} 
\]

\[
\forall_{n,2} = \frac{C}{t_n^2 n^2} \sum_{k,l \in \mathbb{Z}, |k| \neq |l| \in \mathbb{Z}} x_k \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) I \left\{ \lambda_k \geq \alpha \gamma_k^\nu \right\} x_l \left( \frac{x_l w_l}{|c_l|^2} - 1 \right) I \left\{ \lambda_l \geq \alpha \gamma_l^\nu \right\} 
\]

\[
\forall_{n,3} = \frac{2}{t_n^2 n^2} E \left( \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) \left| \langle \beta, \phi_k \rangle \right|^2 I \left\{ \lambda_k \geq \alpha \gamma_k^\nu \right\} \sum_{i=1}^{n-1} |\mathcal{U}_{i,k}D_{i,k}|^2 \right)^2 
\]

We have

\[
E |\mathcal{U}_{j,k} D_{j,k}|^2 = \left( \sigma^2 + \sum_{m \in \mathbb{Z}, |m| \neq |k|} \left| \langle \beta, \phi_m \rangle \right|^2 |x_m| \right) \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right), \quad (E.2)
\]

because $|\mathcal{U}_{j,k}|^2$ and $|D_{j,k}|^2$ are uncorrelated for all $k \in \mathbb{Z}$ and $j \in \{1, \ldots, n\}$. With Lemma B.1 and (E.6), for all $i \in \{1, \ldots, n\}$ and $k \in \mathcal{K}_n$, we have

\[
E \left( |\mathcal{U}_{i,k}D_{i,k}|^2 - \mathfrak{V}^{1/2} E|D_{i,k}|^2 \right)^2 \leq C \left( E|D_{1,k}|^4 - (E|D_{1,k}|^2)^2 \right) \leq CE|D_{1,k}|^4 
\]

\[
\leq \frac{C}{\alpha^2} \quad (E.3)
\]

as well as

\[
E \left[ |\mathcal{U}_{i,k}D_{i,k}|^2 - \mathfrak{V}^{1/2} E|D_{i,k}|^2 \right] = -E|D_{1,k}|^2 |\langle \beta, \phi_k \rangle|^2 x_k 
\]

\[
= - \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) |\langle \beta, \phi_k \rangle|^2. \quad (E.4)
\]

For the mixed terms with $k,l \in \mathbb{Z}, |k| \neq |l|$ and $i \in \{1, \ldots, n\}$ and $\frac{w_k}{|c_k|^2} - \frac{1}{x_k} \geq 0$ for all $k \in \mathbb{Z}$, we get

\[
E \left[ \left( |\mathcal{U}_{i,k}D_{i,k}|^2 - \mathfrak{V}^{1/2} E|D_{i,k}|^2 \right) \left( |\mathcal{U}_{i,l}D_{i,l}|^2 - \mathfrak{V}^{1/2} E|D_{i,l}|^2 \right) \right] 
\]

\[
\leq E \left[ |\mathcal{U}_{i,k}D_{i,k} \mathcal{U}_{i,l}D_{i,l}|^2 \right] + \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) \left( \frac{w_l}{|c_l|^2} - \frac{1}{x_l} \right) 
\]

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Using this, we have

\[
\mathbb{V}_{n,1} \leq C \left\{ \frac{1}{\alpha^2} |\beta, \phi_k|^2 x_k |\langle \beta, \phi_i \rangle|^2 x_i + \frac{x_i}{\alpha} |\langle \beta, \phi_i \rangle|^2 \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) \right. \\
+ \left. \frac{x_k}{\alpha} |\langle \beta, \phi_k \rangle|^2 \left( \frac{w_i}{|c_i|^2} - \frac{1}{x_i} \right) + \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) \left( \frac{w_i}{|c_i|^2} - \frac{1}{x_i} \right) \right\}.
\] (E.5)

Using similar arguments, we obtain

\[
\mathbb{V}_{n,2} \leq C \left\{ \frac{n}{\alpha} \left( \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) \langle \beta, \phi_k \rangle |\langle \beta, \phi_i \rangle|^2 |\langle \beta, \phi_k \rangle|^2 \right) \right. \\
+ \frac{1}{\alpha^2} \left( \sum_{i \in \mathbb{Z}} x_i^2 \left( \frac{x_i w_i}{|c_i|^2} - 1 \right) |\langle \beta, \phi_i \rangle|^2 |\langle \beta, \phi_i \rangle|^2 \right) \left( \frac{t_n^2}{n} \right) \left( \frac{t_n^2}{n} \right) \\
+ \frac{C}{t_n^4} \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) |\langle \beta, \phi_k \rangle|^4 \left( \frac{\alpha^2}{|c_k|^2} \right) \right\}.
\]

which can be further bounded using the Cauchy-Schwarz inequality to get

\[
\mathbb{V}_{n,2} = o \left( 1 + \frac{1}{t_n^2} + \frac{1}{\sqrt{n}t_n} \right) + o \left( \frac{1}{n} \right).
\]

Using similar arguments as for the first two terms, \( \mathbb{V}_{n,3} \) can also be bounded to get

\[
\mathbb{V}_{n,3} \leq C \left\{ \frac{1}{\alpha^2} \left( \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) |\langle \beta, \phi_k \rangle|^4 \left( \frac{\alpha^2}{|c_k|^2} \right) \right) \right. \\
+ \frac{C}{t_n^4} \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) |\langle \beta, \phi_k \rangle|^4 \left( \frac{\alpha^2}{|c_k|^2} \right) \left( \frac{t_n^2}{n} \right) \left( \frac{t_n^2}{n} \right) \\
+ \frac{C}{t_n^4} \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) |\langle \beta, \phi_k \rangle|^4 \left( \frac{\alpha^2}{|c_k|^2} \right) \left( \frac{t_n^2}{n} \right) \left( \frac{t_n^2}{n} \right) \\
+ \frac{C}{t_n^4} \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) |\langle \beta, \phi_k \rangle|^4 \left( \frac{\alpha^2}{|c_k|^2} \right) \left( \frac{t_n^2}{n} \right) \left( \frac{t_n^2}{n} \right) \right\}.
\]
 Altogether, we have
\[ \mathfrak{W}_{n,1} = \mathfrak{H}_n + o_P(1). \]

The stochastic convergence of \( \mathfrak{H}_n \) follows by
\[ \mathfrak{H}_n = \mathfrak{W} \left\{ \frac{n-1}{t_n^2} \sum_{k \in \mathbb{Z}} x_k \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 I\{\lambda_k \geq \alpha \gamma_k^\nu\} \right\} \]
for \( n \to \infty \). For proving, that \( \mathfrak{W}_{n,2} \) converges stochastically to 0 we show again the corresponding \( L_2 \)-convergence. To this end we bound for all \( i \in \{1, \ldots, n\} \) und \( k \in \mathbb{Z} \) the term \( E|\mathfrak{W}_{1,k}|^2 \) by a constant \( C < \infty \) using the centredness of \( U \) and Lemma B.1, to obtain
\[ E|\mathfrak{W}_{n,2}|^2 \leq \frac{C}{t_n^4} \left\{ \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 I\{\lambda_k \geq \alpha \gamma_k^\nu\} \right\} \sum_{i=1, i \neq p}^{n-1} \mathfrak{W}_{i,k} D_{i,k} \mathfrak{W}_{p,k} D_{p,k} \]
\[ + \sum_{k,l \in \mathbb{Z}, |k| \neq |l|} x_k \left( \frac{x_k w_k}{|c_k|^2} - 1 \right) I\{\lambda_k \geq \alpha \gamma_k^\nu\} x_l \left( \frac{x_l w_l}{|c_l|^2} - 1 \right) I\{\lambda_l \geq \alpha \gamma_l^\nu\} \]
\[ = \mathbb{E} \left[ \left( \sum_{i,p=1}^{n-1} \mathfrak{W}_{i,k} D_{i,k} \mathfrak{W}_{p,k} D_{p,k} \right) \left( \sum_{i,p=1}^{n-1} \mathfrak{W}_{i,l} D_{i,l} \mathfrak{W}_{p,l} D_{p,l} \right) \right]. \]

Since \( \mathfrak{W}_{i,k} D_{i,k} \) and \( \mathfrak{W}_{p,k} D_{p,k} \) are stochastically independent for \( p \neq i \), only the quadratic terms for \( k \in \mathbb{Z} \) are relevant
\[ \sum_{i,p=1}^{n-1} \mathbb{E} \left| \mathfrak{W}_{i,k} D_{i,k} \mathfrak{W}_{p,k} D_{p,k} \right|^2 = \sum_{i=p=1}^{n-1} \mathbb{E} \left| \mathfrak{W}_{i,k} D_{i,k} \right|^2 \mathbb{E} \left| \mathfrak{W}_{p,k} D_{p,k} \right|^2 \]
\[ = (n-1)(n-2) \left( \mathbb{E} |\mathfrak{W}_{1,k}|^2 \mathbb{E} |D_{1,k}|^2 \right)^2 \]
\[ \leq C n^2 \left( \frac{w_k}{|c_k|^2} - 1 \right)^2. \]

Under the assumptions of Theorem 2.1, this leads to
\[ E|\mathfrak{W}_{n,2}|^2 \leq \frac{C}{t_n^4} \sum_{k \in \mathbb{Z}} \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^4 I\{\lambda_k \geq \alpha \gamma_k^\nu\} = o(1), \]

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and therefore
\[ \mathcal{U}_{n,2} = o_P(1). \]

### E.3 Details for the proof of Proposition C.2

It is shown in [1] and [10] that the conditional Lindeberg condition follows from the unconditional Ljapunov condition. We will show in the following, that
\[ \sum_{j=2}^{n} E|Y_{n,j}|^4 = o(1) \]
and decompose
\[ \sum_{j=2}^{n} E|Y_{n,j}|^4 = L_{n,1} + L_{n,2} + L_{n,3} + L_{n,4}, \]
where
\[ L_{n,1} = \frac{1}{t n^4} \sum_{j=2}^{n} \sum_{k \in \mathbb{Z}} E|\mathcal{W}_{j,k} D_{j,k} Z_{n,j,k}|^4, \]
\[ L_{n,2} = \frac{1}{t n^4} \sum_{j=2}^{n} \sum_{k,l \in \mathbb{Z}, |k| \neq |l|} E|\mathcal{W}_{j,k} D_{j,k} Z_{n,j,k} \mathcal{W}_{j,l} D_{j,l} Z_{n,j,l}|^2, \]
\[ L_{n,3} = \frac{1}{t n^4} \sum_{j=2}^{n} \sum_{k,l,q \in \mathbb{Z}, |k|, |l|, |p| \neq |q|} E\left[|\mathcal{W}_{j,k} D_{j,k} Z_{n,j,k} |^2 \mathcal{W}_{j,l} D_{j,l} Z_{n,j,l} \mathcal{W}_{j,q} D_{j,q} Z_{n,j,q} \right], \]
\[ L_{n,4} = \frac{1}{t n^4} \sum_{j=2}^{n} \sum_{k,l,p,q \in \mathbb{Z}, |k|, |l|, |p| \neq |q|} E\left[|\mathcal{W}_{j,k} D_{j,k} Z_{n,j,k} \mathcal{W}_{j,l} D_{j,l} Z_{n,j,l} \mathcal{W}_{j,p} D_{j,p} Z_{n,j,p} \mathcal{W}_{j,q} D_{j,q} Z_{n,j,q} \right]. \]

For \( L_{n,1} \) we use, that for all \( k \in \mathbb{Z}, n \in \mathbb{N}, j \in \{1, \ldots, n\}, Z_{n,j,k} \) are stochastically independent of \( \mathcal{W}_{j,k} D_{j,k} \) and \( \mathcal{W}_{j,k} \) are uncorrelated with \( D_{j,k} \). Furthermore, the fourth absolute moment of \( \mathcal{W}_{j,k} \) is due to the centredness of \( U \) and Lemma B.1 uniformly bounded. The fourth absolute moment of \( D_{j,k} \) can be estimated using Assumption 3 and \((X, W) \in \mathcal{F}_\eta^4 \) as
\[ E|D_{j,k}|^4 \leq C \left( \frac{E|\langle W, \phi_k \rangle|^4}{|c_k|^4} + \frac{E|\langle X, \phi_k \rangle|^4}{x_k^4} \right) \leq C \eta \left( \frac{w_k^2}{|c_k|^4} + \frac{1}{x_k^2} \right) \leq C \eta \frac{1}{\alpha^2}. \] (E.6)

Again using similar arguments, we obtain
\[ E|\mathcal{W}_{i,k} D_{i,k}|^2 = E|\mathcal{W}_{i,k}|^2 E|D_{i,k}|^2 \leq C \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right). \] (E.7)

This results in
\[ E\left| \sum_{i=1}^{j-1} \mathcal{W}_{i,k} D_{i,k} x_k I\{\lambda_k \geq \alpha \gamma_k^\prime\} \right|^4 \]
and, in the same way, \( E \) { \lambda_k \geq \alpha \gamma_k \} \{ \sum_{i=1}^{j-1} E|\mathcal{U}_{i,k}|^4E|D_{i,k}|^4 + 2 \sum_{1 \leq i_1 \leq i_2 \leq j-1} E|\mathcal{U}_{i_1,k}D_{i_1,k}|^2E|\mathcal{U}_{i_2,k}D_{i_2,k}|^2 \}

\[ \leq \frac{C n}{\alpha^2} x_k^4 I\{\lambda_k \geq \alpha \gamma_k \} + C n^2 x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 I\{\lambda_k \geq \alpha \gamma_k \}. \] (E.8)

Putting these results together, for \( L_{n,1} \), we get

\[ L_{n,1} = \frac{1}{t_n^4 n^4} \sum_{j=2}^{n} \sum_{k \in \mathbb{Z}} E|\mathcal{U}_{j,k}|^4E|D_{j,k}|^4E|Z_{n,j,k}|^4 \]

\[ \leq \frac{C}{t_n^4 n^4 \alpha^2} \sum_{j=2}^{n} \sum_{k \in \mathbb{Z}} \left| \sum_{i=1}^{j-1} \mathcal{U}_{i,k}D_{i,k}x_k I\{\lambda_k \geq \alpha \gamma_k \} \right|^4 \]

\[ \leq \frac{C}{t_n^4 n \alpha^2} \sum_{k \in \mathbb{Z}} x_k^2 I\{\lambda_k \geq \alpha \gamma_k \} \left( \frac{1}{n \alpha^2} x_k^2 + \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 \right) \]

\[ = o(1) \frac{1}{n^4} \left( \sum_{k \in \mathbb{Z}} x_k^4 I\{\lambda_k \geq \alpha \gamma_k \} + \sum_{k \in \mathbb{Z}} x_k^2 \left( \frac{x_k w_k}{|c_k|^2} - 1 \right)^2 I\{\lambda_k \geq \alpha \gamma_k \} \right), \]

where the first series converges due to Lemma B.1 and the second series either also converges or, if not, can be bounded by \( Ct_n^2 \).

Considering \( L_{n,4} \), we use the stochastic independence of \( Z_{n,j,k} \) and \( \mathcal{U}_{j,l}D_{j,l} \) for all \( k, l \in \mathbb{Z} \), which results in

\[ E[\mathcal{U}_{j,k}D_{j,k}Z_{n,j,k} \mathcal{U}_{j,l}D_{j,l}Z_{n,j,l}] = E[\mathcal{U}_{j,k}D_{j,k} \mathcal{U}_{j,l}D_{j,l} Z_{n,j,l}] E[Z_{n,j,k}Z_{n,j,l}]. \]

The rest of the argumentation is just calculating the expectations using that for all \( j \in \{1, \ldots, n\} \), \( D_{j,k}, D_{j,l}, D_{j,p} \) and \( D_{j,q} \) are uncorrelated with \( S_{j,m} \) for all \( m \in \mathbb{Z} \setminus \{m \in \mathbb{Z} : |m| = |k|, |l|, |p|, |q| \} \) and stochastically independent of \( U_j \). Finally,

\[ E[S_{j,k} D_{j,k}] = \langle \beta, \phi_k \rangle E \left[ \langle \phi_k, X_j \rangle \left( \frac{W_j \phi_k}{c_k} - \frac{X_j \phi_k}{x_k} \right) \right] = \langle \beta, \phi_k \rangle \left( \frac{c_k}{c_k} - \frac{x_k}{x_k} \right) = 0 \] (E.9)

and, in the same way, \( E[S_{j,k} D_{j,k}] = E[S_{j,k} D_{j,k}] = 0 \), which gives \( L_{n,4} = 0 \).

With similar arguments as above, we get

\[ L_{n,2} = \frac{1}{t_n n^4} \sum_{j=2}^{n} \sum_{k,l \in \mathbb{Z}, k \neq l} E|\mathcal{U}_{j,k}D_{j,k} \mathcal{U}_{j,l}D_{j,l}|^2 E|Z_{n,j,k}Z_{n,j,l}|^2, \]

which can be further bounded by using

\[ E|S_{j,k} D_{j,k}|^2 \leq |\langle \beta, \phi_k \rangle|^2 \sqrt{E|\langle X, \phi_k \rangle|^4} E|D_{j,k}|^4 \]
\[
\leq \sqrt{n} |\langle \beta, \phi_k \rangle|^2 x_k \left( \frac{E|\langle W, \phi_k \rangle|^4}{|c_k|^4} + \frac{E|\langle X, \phi_k \rangle|^4}{x_k^4} \right)^{1/2}
\]

\[
\leq C |\langle \beta, \phi_k \rangle|^2 x_k \left( \frac{w_k^2}{|c_k|^2} + \frac{1}{x_k^2} \right)^{1/2} \leq \frac{C |\langle \beta, \phi_k \rangle|^2 x_k}{\alpha}
\]

and

\[
E[Z_{n,j,k}Z_{n,j,l}]^2 \\
\leq C x_k^2 x_l^2 I\{\lambda_k \geq \alpha \gamma_k^\nu\} I\{\lambda_l \geq \alpha \gamma_l^\nu\} (n-1) \left\{ \frac{C}{\alpha^2} |\langle \beta, \phi_k \rangle|^2 x_k |\langle \beta, \phi_l \rangle|^2 x_l + \frac{C |\langle \beta, \phi_l \rangle|^2 x_l}{\alpha} \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) + \frac{C |\langle \beta, \phi_l \rangle|^2 x_l}{\alpha} \left( \frac{w_l}{|c_l|^2} - \frac{1}{x_l} \right) \right\} + (n-2) \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) \left( \frac{w_l}{|c_l|^2} - \frac{1}{x_l} \right)
\]

This results in

\[
L_{n,2} \leq \frac{C}{t_n^4 n^2 \alpha^2} \left( \sum_{k \in \mathbb{Z}} |\langle \beta, \phi_k \rangle|^4 x_k^4 \right)^2 + \frac{C}{t_n^2 n^2 \alpha^2} \sum_{i \in \mathbb{Z}} |\langle \beta, \phi_i \rangle|^4 x_i^4 + \frac{C}{n^2}
\]

\[
+ \frac{C}{t_n^2 n^2 \alpha^2} \left( \sum_{k \in \mathbb{Z}} |\langle \beta, \phi_k \rangle|^2 x_k^2 \left( \frac{w_k x_k}{|c_k|^2} - 1 \right) I\{\lambda_k \geq \alpha \gamma_k^\nu\} \right)^2 \]

\[
+ \frac{C}{t_n^2 n^2 \alpha} \sum_{i \in \mathbb{Z}} |\langle \beta, \phi_i \rangle|^2 x_i^2 \left( \frac{w_i x_i}{|c_i|^2} - 1 \right) I\{\lambda_i \geq \alpha \gamma_i^\nu\} + \frac{C}{n}
\]

\[
\leq o \left( \frac{1}{t_n^4} + \frac{1}{t_n^2} \right) + O \left( \frac{1}{n} + \frac{1}{n^2} \right) + \frac{C}{t_n^2 n^2 \alpha} \sum_{k \in \mathbb{Z}} |\langle \beta, \phi_k \rangle|^4 x_k^4 + \frac{C}{t_n n \alpha} \sqrt{\sum_{k \in \mathbb{Z}} |\langle \beta, \phi_k \rangle|^4 x_k^4}
\]

\[
= o \left( \frac{1}{t_n^4} + \frac{1}{t_n^2} + \frac{1}{t_n} + \frac{1}{t_n \sqrt{n}} \right) + O \left( \frac{1}{n} + \frac{1}{n^2} \right)
\]

using the Hölder inequality and Lemma B.1.

For the summands in \(L_{n,3}\), we get

\[
E[|\mathcal{U}_{j,k} D_{j,k} Z_{n,j,k}^2 \mathcal{U}_{j,l} D_{j,l} Z_{n,j,l} \mathcal{U}_{j,q} D_{j,q} Z_{n,j,q}|] = E[|\mathcal{U}_{j,k} D_{j,k} |^2 \mathcal{U}_{j,l} D_{j,l} \mathcal{U}_{j,q} D_{j,q}] E[|Z_{n,j,k}|^2 Z_{n,j,l} Z_{n,j,q}].
\]

The first expectation is

\[
E[|\mathcal{U}_{j,k} D_{j,k}|^2 \mathcal{U}_{j,l} D_{j,l} \mathcal{U}_{j,q} D_{j,q}]
\]

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\[ = \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right) |\langle \beta, \phi_l \rangle|^2 |\langle \beta, \phi_q \rangle|^2 \]

\[ E \left[ |\langle X_j, \phi_l \rangle|^2 \left( \frac{\langle W_j, \phi_l \rangle}{c_l} - \frac{\langle X_j, \phi_l \rangle}{x_l} \right) \right] E \left[ |\langle X_j, \phi_q \rangle|^2 \left( \frac{\langle \phi_q, W_j \rangle}{c_q} - \frac{\langle \phi_q, X_j \rangle}{x_q} \right) \right], \]

while

\[ E\left[ |Z_{n,j,k}|^2 Z_{n,j,l} Z_{n,j,q} \right] = x_k^2 x_l x_q I\{\lambda_k \geq \alpha \gamma_k\} I\{\lambda_l \geq \alpha \gamma_l\} I\{\lambda_q \geq \alpha \gamma_q\} \sum_{i=1}^{j-1} E\left[ |W_{i,k}|^2 |W_{i,l}| D_{i,l} |W_{i,q}| D_{i,q} \right]. \]

Altogether, we have

\[ L_{n,3} \leq \frac{1}{t_n^2 n^2} \sum_{k,l,q \in \mathbb{Z}, |k|, |l|, |q| \neq |k|, |l|} \left( \frac{w_k}{|c_k|^2} - \frac{1}{x_k} \right)^2 |\langle \beta, \phi_l \rangle|^4 |\langle \beta, \phi_q \rangle|^4 \left( E \left[ |\langle X, \phi_l \rangle|^2 \left( \frac{\langle W, \phi_l \rangle}{c_l} - \frac{\langle X, \phi_l \rangle}{x_l} \right) \right] E \left[ |\langle X, \phi_q \rangle|^2 \left( \frac{\langle \phi_q, W \rangle}{c_q} - \frac{\langle \phi_q, X \rangle}{x_q} \right) \right] \right)^2. \]

The series can be bounded by \( t_n^2 \). Using the Hölder inequality for \( l \in \mathcal{K}_n \), we have

\[ \left( E \left[ |\langle X, \phi_l \rangle|^2 \left( \frac{\langle \phi_l, W \rangle}{c_l} - \frac{\langle X, \phi_l \rangle}{x_l} \right) \right] \right)^2 \leq E|\langle X, \phi_l \rangle|^4 E|D_{1,l}|^2 \leq \eta_n x_l^2 \left( \frac{w_l}{|c_l|^2} - \frac{1}{x_l} \right) \leq \frac{C}{\alpha^2 x_l^2}. \]

Finally, relying again on Assumption 3 and Lemma B.1, also \( L_{n,3} \) converges to 0 due to

\[ L_{n,3} \leq \frac{C}{t_n^2 n^2} \left( \sum_{k \in \mathbb{Z}} |\langle \beta, \phi_k \rangle|^4 x_k - \lambda_k \right)^{1/2} \leq \frac{C}{t_n^2 n^2 \alpha^2} \sum_{k \in \mathbb{Z}} |\langle \beta, \phi_k \rangle|^4 x_k (x_k - \lambda_k) = o \left( \frac{1}{t_n^2 n} \right). \]

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