Pointwise Amenable, Non-Amenable Banach Algebras

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Abstract. It is shown that a pointwise amenable Banach algebra need not be amenable. This positively answer a question raised by Dales, Ghahramani and Loy.

Keywords: amenable, pointwise amenable, Connes amenable, character amenable.

MSC 2010: Primary: 46H25; Secondary: 43A07.

1. Introduction

Let $A$ be a Banach algebra, and let $E$ be a Banach $A$-bimodule. A derivation is a bounded linear map $D : A \to E$ satisfying $D(ab) = Da \cdot b + a \cdot Db \ (a, b \in A)$. A Banach algebra $A$ is amenable if for any Banach $A$-bimodule $E$, any derivation $D : A \to E^*$ is inner, that is, there exists $\eta \in E^*$ with $D(a) = ad_\eta(a) = a \cdot \eta - \eta \cdot a$, $a \in A$. This powerful notion introduced by B. E. Johnson [5]. The pointwise variant of amenability introduced by H. G. Dales, F. Ghahramani and R. J. Loy, and appeared formally in [2]. A Banach algebra $A$ is pointwise amenable at $a \in A$ if for any Banach $A$-bimodule $E$, any derivation $D : A \to E^*$ is pointwise inner at $a$, that is, there exists $\eta \in E^*$ with $D(a) = ad_\eta(a)$. Further, $A$ is pointwise amenable if it is pointwise amenable at each $a \in A$.

For a Banach algebra $A$, recall that the projective tensor product $A \hat{\otimes} A$ is a Banach $A$-bimodule in a natural way and the bounded linear map $\pi : A \hat{\otimes} A \to A$ defined by $\pi(a \otimes b) = ab$, $(a, b \in A)$ is a Banach $A$-bimodule homomorphism.

Obviously, every amenable Banach algebra is pointwise amenable. For the converse, however, it has been open so far if there is a pointwise amenable Banach algebra which is not already amenable [2] Problem 5).

In this note, we give an illuminating example to show that the problem has an affirmative answer.

2. The Results

Let $V$ be a Banach space, and let $f \in V^*$ (the dual space of $V$) be a non-zero element such that $||f|| \leq 1$. Then $V$ equipped with the product defined by $ab :=$
$f(a)b$ for $a, b \in \mathcal{V}$, is a Banach algebra denoted by $\mathcal{V}_f$. In general, $\mathcal{V}_f$ is a non-commutative, non-unital Banach algebra without right approximate identity. One may see [3] [8] for more details and properties on this type of algebras.

A small variation of standard argument in [6] gives the following characterization of pointwise amenability.

**Proposition 2.1.** A Banach algebra $\mathfrak{A}$ is pointwise amenable if and only if for each $a \in \mathfrak{A}$ there exists a net $(m_\alpha)_{\alpha} \subseteq \mathfrak{A} \hat{\otimes} \mathfrak{A}$ (depending on $a$) such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $a\pi(m_\alpha) \to a$.

**Theorem 2.2.** Let $\mathcal{V}$ be a Banach space, and let $f \in \mathcal{V}^*$ be a non-zero element such that $\|f\| \leq 1$. Then $\mathcal{V}_f$ is pointwise amenable, but not amenable.

**Proof.** For $a \in \mathcal{V}_f$, we consider $m := \sum_{n=1}^{\infty} b_n \otimes c_n \in \mathcal{V}_f \hat{\otimes} \mathcal{V}_f$ with $\sum_{n=1}^{\infty} f(b_n) = \frac{1}{f(a)}$ and $c_n := a$ for all $n$. Then

$$a \cdot m = \sum_{n=1}^{\infty} ab_n \otimes c_n = f(a) \sum_{n=1}^{\infty} b_n \otimes c_n = f(a)m$$

and similarly $m \cdot a = f(a)m$, so that $a \cdot m = m \cdot a$. Next, $\pi(m) = \sum_{n=1}^{\infty} b_n c_n = (\sum_{n=1}^{\infty} f(b_n))a = \frac{1}{f(a)}a$. Hence

$$a\pi(m) = \frac{1}{f(a)}a^2 = \frac{1}{f(a)}f(a)a = a.$$ 

Therefore, $\mathcal{V}_f$ is pointwise amenable by Proposition 2.1. Because of the lack of a bounded approximate identity, $\mathcal{V}_f$ is not amenable.

**Remark 2.3.** Similar to [2], we may define pointwise contractible Banach algebras (see for instance [1] for the definition of contractible Banach algebras). Then Theorem 2.2 says that $\mathcal{V}_f$ is pointwise contractible. Notice that $\mathcal{V}_f$ is not contractible, because it has no identity.

Let $\mathfrak{A}$ be a Banach algebra. A Banach $\mathfrak{A}$-bimodule $E$ is *dual* if there is a closed submodule $E_*$ of $E^*$ such that $E = (E_*)^*$. A Banach algebra $\mathfrak{A}$ is *dual* if it is a dual Banach space such that multiplication is separately continuous in the $w^*$-topology. For a dual Banach algebra $\mathfrak{A}$, a dual Banach $\mathfrak{A}$-bimodule $E$ is *normal* if the module actions of $\mathfrak{A}$ on $E$ are $w^*$-continuous. The notion of Connes amenability for dual Banach algebras, which is another modification of the notion of amenability systematically introduced by V. Runde [9]. A dual Banach algebra $\mathfrak{A}$ is *Connes amenable* if every $w^*$-continuous derivation from $\mathfrak{A}$ into a normal, dual Banach $\mathfrak{A}$-bimodule is inner. The pointwise variant of Connes amenability introduced in [11]. A dual Banach algebra $\mathfrak{A}$ is *pointwise Connes amenable at* $a \in \mathfrak{A}$ if for every normal, dual Banach $\mathfrak{A}$-bimodule $E$, every $w^*$-continuous derivation $D : \mathfrak{A} \to E$ is
pointwise inner at $a$. Moreover, $\mathfrak{A}$ is \textit{pointwise Connes amenable} if it is pointwise Connes amenable at each $a \in \mathfrak{A}$.

\textbf{Theorem 2.4.} Let $\mathcal{V}$ be a dual Banach space, and let $f \in \mathcal{V}^*$ be a $w^*$-continuous non-zero element such that $||f|| \leq 1$. Then $\mathcal{V}_f$ is a pointwise Connes amenable, non-Connes amenable dual Banach algebra.

\textit{Proof.} Since $f$ is $w^*$-continuous, the multiplication on $\mathcal{V}_f$ is separately $w^*$-continuous, and so $\mathcal{V}_f$ is a dual Banach algebra. As $\mathcal{V}_f$ is pointwise amenable by Theorem 2.2, it is automatically pointwise Connes amenable. Since $\mathcal{V}_f$ does not have an identity, it is not Connes amenable by \cite{9} Proposition 4.1. \qed

The following is \cite{11} Theorem 2.6.

\textbf{Theorem 2.5.} Every commutative, pointwise Connes amenable dual Banach algebra has an identity.

\textbf{Remark 2.6.} The fact that $\mathcal{V}_f$ is non-commutative pointwise Connes amenable but non-unital, shows that the commutativity in Theorem 2.5 can not be dropped.

\textbf{Proposition 2.7.} Let $\mathcal{V}$ be a Banach space, and let $f \in \mathcal{V}^*$ be a non-zero element such that $||f|| \leq 1$. Then $\mathcal{V}_f$ is not approximately amenable.

\textit{Proof.} As $\mathcal{V}_f$ has no right approximate identity, it is not approximately amenable by \cite{4} Lemma 2.2. \qed

The following is a result of F. Ghahramani, see \cite{2} Theorem 1.5.4.

\textbf{Theorem 2.8.} Every pointwise amenable, commutative Banach algebra is approximately amenable.

\textbf{Remark 2.9.} By Theorem 2.2 and Proposition 2.7, $\mathcal{V}_f$ is a pointwise amenable, non-approximately amenable Banach algebra which is not commutative. So, without commutativity Theorem 2.8 is not true.

We conclude by looking at character amenability of $\mathcal{V}_f$. For a Banach algebra $\mathfrak{A}$, we write $\Delta(\mathfrak{A})$ for the set of all continuous homomorphisms from $\mathfrak{A}$ onto $\mathbb{C}$. From \cite{7}, we recall that a Banach algebra $\mathfrak{A}$ is $\varphi$-\textit{amenable}, $\varphi \in \Delta(\mathfrak{A})$, if there exists a bounded linear functional $m$ on $\mathfrak{A}^*$ satisfying $m(\varphi) = 1$ and $m(g \cdot a) = \varphi(a)m(g)$, \((a \in \mathfrak{A}, g \in \mathfrak{A}^*)\). Further, $\mathfrak{A}$ is \textit{character amenable} if it is $\varphi$-amenable for all $\varphi \in \Delta(\mathfrak{A})$ \cite{10}.

\textbf{Proposition 2.10.} Let $\mathcal{V}$ be a Banach space, and let $f \in \mathcal{V}^*$ be a non-zero element such that $||f|| \leq 1$. Then $\mathcal{V}_f$ is character amenable.
Proof. Clearly $f$ is a continuous homomorphisms from $V_f$ onto $\mathbb{C}$. Choose $u \in V_f$ with $f(u) = 1$, and set $u_\alpha := u$ for each $\alpha$. Then $au_\alpha - f(a)u_\alpha = 0$ for all $a \in V_f$. Hence, $V_f$ is $f$-amenable by [7, Theorem 1.4]. It follows from $\Delta(V_f) = \{f\}$ that $V_f$ is character amenable. 

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