Vanishing theorems for $L^2$ harmonic forms on complete Riemannian manifolds

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1 Introduction

It is an interesting problem in geometry and topology to find sufficient conditions for the space of harmonic $k$-forms to be trivial.

For compact manifolds, by Hodge theory, the space of harmonic $k$-forms is isomorphic to the $k$-th de Rham cohomology group. In particular, if the space of harmonic $k$-forms is trivial then the $k$-th de Rham cohomology group is trivial.

For complete Riemannian manifolds it is natural to consider $L^2$ harmonic forms. Even though the space of $L^2$ harmonic $k$-forms is not necessarily isomorphic to the $k$-th de Rham cohomology group, the theory of $L^2$ harmonic one-forms can be used to study the topology at infinity. Li and Tam proved that for a complete Riemannian manifold if the space of $L^2$ harmonic one-forms is trivial then the manifold has at most one non-parabolic end [16]. In particular, if the space of $L^2$ harmonic one-forms is trivial and there are no parabolic ends then the manifold is connected at infinity. It is well known that certain geometric conditions

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Abstract This paper contains some vanishing theorems for $L^2$ harmonic forms on complete Riemannian manifolds with a weighted Poincaré inequality and a certain lower bound of the curvature. The results are in the spirit of Li-Wang and Lam, but without assumptions of sign and growth rate of the weight function, so they can be applied to complete stable hypersurfaces.

Keywords Harmonic forms · Weighted Poincaré inequalities · Stable minimal hypersurfaces · Vanishing theorems

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imply the non-existence of parabolic ends, for example when the manifold satisfies a certain Sobolev inequality [15, Lemma 20.12]. This shows that the vanishing of $L^2$ harmonic one-forms is closely related to the topology at infinity of complete Riemannian manifolds.

For a complete Riemannian manifold $M$ recall that the first eigenvalue of the Laplacian is defined by

$$\lambda_1 (M) = \inf_{\phi \in C_c^\infty (M)} \frac{\int_M |\nabla \phi|^2}{\int_M \phi^2}.$$

Li and Wang proved the following vanishing theorem for $L^2$ harmonic one-forms on complete Riemannian manifolds with positive spectrum.

**Theorem 1** (Li-Wang [17]) If a complete Riemannian manifold $M^n$ has positive spectrum $\lambda_1 (M^n) > 0$ and the Ricci curvature satisfies

$$\text{Ric} \geq -a \lambda_1 (M^n)$$

for some $0 < a < \frac{n}{n-1}$, then the space of $L^2$ harmonic one-forms on $M^n$ is trivial.

The papers [13,17,18,20,22] contain several interesting results concerning the geometry and topology at infinity of complete Riemannian manifolds with positive spectrum.

For a complete Riemannian manifold $M$ and a continuous function $q$ on $M$ recall that $M$ satisfies a weighted Poincaré inequality with weight function $q$ if

$$\int_M q \phi^2 \leq \int_M |\nabla \phi|^2$$

for all functions $\phi$ in $C_c^\infty (M)$ [21]. Taking $q = \lambda_1 (M)$ in this definition recovers the first eigenvalue of the Laplacian.

Li and Wang’s theorem was recently generalized by Lam. He proved the following vanishing theorem for $L^2$ harmonic one-forms on complete Riemannian manifolds with a weighted Poincaré inequality.

**Theorem 2** (Lam [14]) Suppose a complete Riemannian manifold $M^n$ satisfies a weighted Poincaré inequality with weight function $q$ and the Ricci curvature satisfies

$$\text{Ric} \geq -aq$$

for some $0 < a < \frac{n}{n-1}$. Assume $q$ positive with growth rate

$$q(x) = O(\text{dist}(x, x_0)^{2-\alpha})$$

for some $0 < \alpha < 2$. Then the space of $L^2$ harmonic one-forms on $M^n$ is trivial.

Taking $q = \lambda_1 (M)$ in this theorem recovers Li and Wang’s theorem.

It is well known that a stable hypersurface $M^n$ in a Riemannian manifold $M^{n+1}$ satisfies a weighted Poincaré inequality with weight function

$$q = |A|^2 + \overline{\text{Ric}} (v,v),$$

where $A$ is the second fundamental form and $\overline{\text{Ric}} (v,v)$ is the Ricci curvature of $\overline{M}^{n+1}$ in the normal direction. Under certain natural conditions it is possible to show that the Ricci curvature of $M^n$ satisfies

$$\text{Ric} \geq -aq.$$
for some $0 < a < \frac{n}{n-1}$ (see Sect. 3). When Lam’s theorem is applied to $M^n$ the assumption of $q$ positive with growth rate $q(x) = O(\text{dist}(x, x_0)^{2-a})$ for some $0 < a < 2$ is not a natural condition. This example from hypersurface theory shows the importance of studying weighted Poincaré inequalities without assumptions of the weight function.

The following theorem improves Lam’s theorem by removing the assumptions of sign and growth rate of the weight function.

**Theorem 3** Suppose a complete non-compact Riemannian manifold $M^n$ satisfies a weighted Poincaré inequality with weight function $q$ and the Ricci curvature satisfies

$$\text{Ric} \geq -aq$$

for some $0 < a < \frac{n}{n-1}$. Then the space of $L^2$ harmonic one-forms on $M^n$ is trivial.

This theorem will be used to prove the vanishing of $L^2$ harmonic one-forms on complete stable minimal hypersurfaces in Riemannian manifolds with non-negative Bi-Ricci curvature (Theorem 8). When the Riemannian manifold has dimension at most 7 a similar result holds for hypersurfaces not necessarily minimal (Theorem 9).

The following theorem shows that Theorem 3 holds with a more general lower bound of the Ricci curvature when the first eigenvalue of the Laplacian satisfies a certain lower bound.

**Theorem 4** Suppose a complete Riemannian manifold $M^n$ satisfies a weighted Poincaré inequality with weight function $q$ and the Ricci curvature satisfies

$$\text{Ric} \geq -aq - b$$

for some $0 < a < \frac{n}{n-1}$ and $b > 0$. Assume the first eigenvalue of the Laplacian satisfies

$$\lambda_1(M^n) > \frac{b}{n-1-a}.$$  

Then the space of $L^2$ harmonic one-forms on $M^n$ is trivial.

This theorem can be used as following: with the same assumption of the Ricci curvature if the space of $L^2$ harmonic one-forms is non-trivial then the first eigenvalue of the Laplacian satisfies $\lambda_1(M^n) \leq \frac{b}{n-1-a}$.

This theorem will be used to prove the vanishing of $L^2$ harmonic one-forms on complete stable minimal hypersurfaces with a certain lower bound of the first eigenvalue of the Laplacian in Riemannian manifolds with $\text{BiRic} \geq -b$ (Theorem 10). When the Riemannian manifold has dimension at most 7 a similar result holds for hypersurfaces not necessarily minimal (Theorem 11). In particular, this gives an explicit upper bound of the first eigenvalue of the Laplacian of minimal and non-minimal stable hypersurfaces with a non-trivial space of $L^2$ harmonic one-forms in the hyperbolic space (Corollary 2 and Corollary 3).

This paper also contains the following rigidity result: in Theorem 4 with the same assumption of the Ricci curvature if $\lambda_1(M^n) = \frac{b}{n-1-a}$ and the space of $L^2$ harmonic one-forms is non-trivial then the universal cover of $M^n$ splits (Theorem 7).

The main difference between this paper and previous works is that the results here hold without assumptions of the weight function. Two important advances are Theorem 3 which improves Lam’s theorem and Theorem 4 which proves the vanishing of $L^2$ harmonic forms for more general lower bounds of the Ricci curvature. Another novelty is the possibility of applying results from the theory of Riemannian manifolds with a weighted Poincaré
inequality to the theory of stable hypersurfaces, which makes the proofs of many results short and clear.

As in previous works the proofs of the vanishing theorems of this paper begin with the well known Bochner-Weitzenböck formula. The key differences here are Lemma 1 and Lemma 2, which may be of independent interest. The proofs of the theorems for stable hypersurfaces use the following simple idea: a stable hypersurface satisfies a weighted Poincaré inequality, so to apply Theorem 3 or Theorem 4 to the hypersurface it suffices to show that the Ricci curvature of the hypersurface satisfies a certain lower bound.

Notice that Theorem 3 and Theorem 4 are actually corollaries of vanishing theorems for harmonic forms of any degree (Theorem 5 and Theorem 6 respectively).

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2 Vanishing of $L^2$ harmonic forms

For a Riemannian manifold $M$ the Hodge Laplacian is defined by

$$\Delta = -(dd^* + d^*d).$$

The space of $L^2$ harmonic $k$-forms is the set of all $k$-forms $\omega$ on $M$ such that

$$\Delta \omega = 0$$

and

$$\int_M |\omega|^2 < \infty.$$  

For a local orthonormal frame $e_1, \ldots, e_n$ on $M$ with dual coframe $e^1, \ldots, e^n$ the curvature operator acting on forms is defined by

$$\mathcal{R} = \sum_{i,j} e^i \wedge \iota_{e_j} R(e_j, e_i)$$

with $R(e_i, e_j) = \nabla_i \nabla_j - \nabla_j \nabla_i$. For a $k$-form $\omega$ on $M$ the following identity holds

$$\frac{1}{2} \Delta |\omega|^2 = |\nabla \omega|^2 + \langle \Delta \omega, \omega \rangle + \langle \mathcal{R} \omega, \omega \rangle.$$  \hspace{1cm} (2)

This identity is known as the Bochner-Weitzenböck formula [15, Lemma 3.4]. For a closed and co-closed $k$-form $\omega$ on $M^n$ the following inequality holds

$$|\nabla \omega|^2 \geq C_{n,k} |\nabla |\omega||^2$$  \hspace{1cm} (3)

with

$$C_{n,k} = \begin{cases} 
1 + \frac{1}{n-k}, & 1 \leq k \leq n/2, \\
1 + \frac{1}{k}, & n/2 \leq k \leq n-1.
\end{cases}$$

This inequality is known as the refined Kato inequality for $L^2$ harmonic forms [1,8].

The proof of Theorem 5 relies on the Bochner-Weitzenböck formula, the refined Kato inequality for $L^2$ harmonic forms and the following lemma, which may be of independent interest.
Lemma 1 Suppose a complete Riemannian manifold $M$ satisfies a weighted Poincaré inequality with weight function $q$. Assume a smooth function $f$ on $M$ satisfies
\[ f \Delta f \geq A |\nabla f|^2 - af^2 \] (4)
and
\[ \int_M f^2 < \infty \]
for some $0 < a < 1 + A$. Then $f$ is constant. Moreover, if $f$ is not identically zero then the volume of $M$ is finite and $q$ is identically zero.

Proof First we prove that $f$ is constant. Take a cutoff function $\phi$ on $M$ such that
\[ \phi = \begin{cases} 1 & \text{in } B_R, \\ 0 & \text{in } M \setminus B_{2R}, \\ 0 \leq \phi \leq 1 & \text{in } B_{2R} \setminus B_R \end{cases} \]
and
\[ |\nabla \phi| \leq \frac{2}{R} \text{ in } B_{2R} \setminus B_R. \]
Here $B_R$ is the open ball with center at a fixed point of $M$ and radius $R$. Multiplying inequality (4) by $\phi^2$ and integrating by parts gives
\[ (1 + A) \int_M |\nabla f|^2 \phi^2 \leq a \int_M qf^2 \phi^2 - 2 \int_M f \phi \langle \nabla f, \nabla \phi \rangle. \]
Putting $f \phi$ in the weighted Poincaré inequality (1) yields
\[ \int_M q (f \phi)^2 \leq \int_M f^2 |\nabla \phi|^2 + \int_M |\nabla f|^2 \phi^2 + 2 \int_M f \phi \langle \nabla f, \nabla \phi \rangle. \]
Combining these two inequalities gives
\[ (1 + A - a) \int_M |\nabla f|^2 \phi^2 \leq a \int_M f^2 |\nabla \phi|^2 + 2 (a - 1) \int_M f \phi \langle \nabla f, \nabla \phi \rangle. \]
Fix $\epsilon > 0$. By Cauchy–Schwarz and Young’s inequalities we have
\[ (1 + A - a - \epsilon |a - 1|) \int_M |\nabla f|^2 \phi^2 \leq \left( a + \frac{|a - 1|}{\epsilon} \right) \int_M f^2 |\nabla \phi|^2. \]
Note that $1 + A - a - \epsilon |a - 1| > 0$ for a sufficiently small $\epsilon > 0$. Since $\int_M f^2 < \infty$ sending $R \to \infty$ and using the monotone convergence theorem gives
\[ (1 + A - a - \epsilon |a - 1|) \int_M |\nabla f|^2 \leq 0. \]
This proves that $f$ is constant.

Now we prove that if $f$ is not identically zero then the volume of $M$ is finite and $q$ is identically zero. Assume $f \neq 0$. Then the volume of $M$ is finite because
\[ \text{vol} (M) = \frac{\int_M f^2}{f^2} < \infty. \]
Putting \( f \) in inequality (4) shows that \( aq f^2 \geq 0 \), which implies that \( q \geq 0 \). Putting \( \phi \) in the weighted Poincaré inequality (1) gives
\[
\int_M q\phi^2 \leq \frac{4 \text{vol}(M)}{R^2}.
\]
Sending \( R \to \infty \) and using the monotone convergence theorem yields
\[
\int_M q \leq 0.
\]
This proves that \( q \) is identically zero.

We prove the following vanishing theorem for \( \text{L}^2 \) harmonic \( k \)-forms on complete Riemannian manifolds with a weighted Poincaré inequality.

**Theorem 5** Suppose a complete Riemannian manifold \( M^n \) satisfies a weighted Poincaré inequality with weight function \( q \) and the curvature operator satisfies
\[
\langle R\omega, \omega \rangle \geq -aq |\omega|^2
\]
for all \( k \)-forms \( \omega \) on \( M^n \), where \( 0 < a < C_{n,k} \). Assume that at least one of the following conditions hold: (1) the volume of \( M^n \) is infinite; (2) \( q \) is not identically zero. Then the space of \( \text{L}^2 \) harmonic \( k \)-forms on \( M^n \) is trivial.

**Proof** Fix a \( \text{L}^2 \) harmonic \( k \)-form \( \omega \) on \( M^n \). By Bochner-Weitzenböck formula (2) we have
\[
|\omega| \Delta |\omega| = |\nabla \omega|^2 - |\nabla |\omega||^2 + \langle R\omega, \omega \rangle.
\]
Since \( \text{L}^2 \) harmonic forms are closed and co-closed it follows from the refined Kato inequality for \( \text{L}^2 \) harmonic forms (3) that
\[
\int_M \phi \Delta f \geq (C_{n,k} - 1) |\nabla f|^2 - aq f^2,
\]
where \( f = |\omega| \). Applying Lemma 1 with \( A = C_{n,k} - 1 \) shows that \( f \) is constant. If \( f \) is not identically zero it follows from Lemma 1 that the volume of \( M^n \) is finite and \( q \) is identically zero, a contradiction. This proves that \( \omega \) is identically zero. \( \square \)

Applying Theorem 5 with \( q = \lambda_1 (M^n) \) and using the fact that any complete manifold with positive spectrum has infinite volume proves the following corollary.

**Corollary 1** If a complete Riemannian manifold \( M^n \) has positive spectrum \( \lambda_1 (M^n) > 0 \) and the curvature operator satisfies
\[
\langle R\omega, \omega \rangle \geq -a\lambda_1 (M^n) |\omega|^2
\]
for all \( k \)-forms \( \omega \) on \( M^n \), where \( 0 < a < C_{n,k} \), then the space of \( \text{L}^2 \) harmonic \( k \)-forms on \( M^n \) is trivial.

Taking \( k = 1 \) in this corollary recovers Li and Wang’s theorem in the introduction because for one-forms \( \langle R\omega, \omega \rangle = \text{Ric} (\omega, \omega) \) and \( C_{n,1} = \frac{n}{n-1} \).

We can now prove Theorem 3.

**Proof** For one-forms \( \langle R\omega, \omega \rangle = \text{Ric} (\omega, \omega) \) and \( C_{n,1} = \frac{n}{n-1} \). Suppose the space of \( \text{L}^2 \) harmonic one-forms is non-trivial. Then by Theorem 5 the manifold has finite volume and non-negative Ricci curvature. However Yau proved that any complete non-compact Riemannian manifold with non-negative Ricci curvature has infinite volume [28]. This proves that the space of \( \text{L}^2 \) harmonic one-forms is trivial. \( \square \)
The proofs of Theorem 6 and Theorem 7 rely on the following lemma, which may be of independent interest.

**Lemma 2** Suppose a complete Riemannian manifold $M$ satisfies a weighted Poincaré inequality with weight function $q$. Assume a smooth function $f$ on $M$ satisfies

$$f \Delta f \geq A |\nabla f|^2 - aqf^2 - bf^2 \quad (5)$$

and

$$\int_M f^2 < \infty$$

for some $0 < a < 1 + A$ and $b > 0$. Then

$$\int_M |\nabla f|^2 \leq \frac{b}{1 + A - a} \int_M f^2. \quad (6)$$

Moreover, if equality holds in (6) then equality holds in (5).

**Proof** First we show inequality (6). Take a cutoff function $\phi$ on $M$ such that

$$\phi = \begin{cases} 1 & \text{in } BR, \\ 0 & \text{in } M \setminus B_{2R}, \\ 0 \leq \phi \leq 1 & \text{in } B_{2R} \setminus BR \end{cases}$$

and

$$|\nabla \phi| \leq \frac{2}{R} \text{ in } B_{2R} \setminus BR.$$ 

Here $BR$ is the open ball with center at a fixed point of $M$ and radius $R$. Multiplying inequality (5) by $\phi^2$ and integrating by parts gives

$$(1 + A) \int_M |\nabla f|^2 \phi^2 \leq a \int_M qf^2 \phi^2 + b \int_M f^2 \phi^2 - 2 \int_M f \phi \langle \nabla f, \nabla \phi \rangle.$$ 

Putting $f \phi$ in the weighted Poincaré inequality (1) yields

$$\int_M q (f \phi)^2 \leq \int_M f^2 |\nabla \phi|^2 + \int_M |\nabla f|^2 \phi^2 + 2 \int_M f \phi \langle \nabla f, \nabla \phi \rangle. \quad (7)$$

Combining these two inequalities gives

$$(1 + A - a) \int_M |\nabla f|^2 \phi^2 \leq b \int_M f^2 \phi^2 + a \int_M f^2 |\nabla \phi|^2 + 2(a - 1) \int_M f \phi \langle \nabla f, \nabla \phi \rangle.$$ 

Fix $\epsilon > 0$. By Cauchy–Schwarz and Young’s inequalities we have

$$(1 + A - a - \epsilon |a - 1|) \int_M |\nabla f|^2 \phi^2 \leq b \int_M f^2 \phi^2 + \left(a + \frac{|a - 1|}{\epsilon}\right) \int_M f^2 |\nabla \phi|^2.$$ 

Note that $1 + A - a - \epsilon |a - 1| > 0$ for all sufficiently small $\epsilon > 0$. Since $\int_M f^2 < \infty$ sending $R \to \infty$, using the monotone convergence theorem and then sending $\epsilon \to 0$ proves inequality (6).
Now we assume that equality holds in (6). Multiplying inequality (5) by $\phi^2$ and integrating gives

$$0 \leq \int_M (f \Delta f - A |\nabla f|^2 + aq f^2 + bf^2) \phi^2$$

$$= -(1 + A) \int_M |\nabla f|^2 \phi^2 + b \int_M f^2 \phi^2 - 2 \int_M f \phi \langle \nabla f, \nabla \phi \rangle + a \int_M q f^2 \phi^2.$$

Combining this with inequality (7) and using the Cauchy–Schwarz inequality yields

$$0 \leq \int_M (f \Delta f - A |\nabla f|^2 + aq f^2 + bf^2) \phi^2$$

$$\leq -(1 + A - a) \int_M |\nabla f|^2 \phi^2 + b \int_M f^2 \phi^2$$

$$+ a \int_M f^2 |\nabla \phi|^2 + 2 |a - 1| \left( \int_M f^2 |\nabla \phi|^2 \right)^{\frac{1}{2}} \left( \int_M |\nabla f|^2 \phi^2 \right)^{\frac{1}{2}}.$$

Since the quantity being integrated in (8) is non-negative and $\int_M f^2 < \infty$ and $\int_M |\nabla f|^2 < \infty$, sending $R \to \infty$ and using the monotone convergence theorem gives

$$0 \leq \int_M (f \Delta f - A |\nabla f|^2 + aq f^2 + bf^2)$$

$$\leq -(1 + A - a) \int_M |\nabla f|^2 + b \int_M f^2$$

$$= 0.$$

Since the quantity being integrated in (9) is non-negative it follows that equality holds in (5).

The proof of Theorem 5 also uses the following straightforward lemma.

**Lemma 3** If a smooth function $f$ on a complete Riemannian manifold $M$ satisfies

$$\int_M f^2 < \infty,$$

then

$$\lambda_1 (M) \int_M f^2 \leq \int_M |\nabla f|^2.$$

**Proof** Take a cutoff function $\phi$ on $M$ such that

$$\phi = \begin{cases} 1 & \text{in } B_R, \\ 0 & \text{in } M \setminus B_{2R}, \\ 0 \leq \phi \leq 1 & \text{in } B_{2R} \setminus B_R \end{cases}$$

and

$$|\nabla \phi| \leq \frac{2}{R} \text{ in } B_{2R} \setminus B_R.$$
Fix \( \epsilon > 0 \). Then
\[
\lambda_1(M) \int_M (f \phi)^2 \leq \int_M f^2|\nabla \phi|^2 + \int_M |\nabla f|^2 \phi^2 + 2 \int_M f \phi \langle \nabla f, \nabla \phi \rangle \\
\leq (1 + \epsilon) \int_M |\nabla f|^2 \phi^2 + \left(1 + \frac{1}{\epsilon}\right) \int_M f^2 |\nabla \phi|^2.
\]
Since \( \int_M f^2 < \infty \) sending \( R \to \infty \), using the monotone convergence theorem and then sending \( \epsilon \to 0 \) proves the result.

We prove the following vanishing theorem, which shows that Theorem 5 holds with a more general lower bound of the curvature operator when the first eigenvalue of the Laplacian satisfies a certain lower bound.

**Theorem 6** Suppose a complete Riemannian manifold \( M^n \) satisfies a weighted Poincaré inequality with weight function \( q \) and the curvature operator satisfies
\[
\langle R \omega, \omega \rangle \geq -aq |\omega|^2 - b |\omega|^2
\]
for all \( k \)-forms \( \omega \) on \( M^n \), where \( 0 < a < C_{n,k} \) and \( b > 0 \). Assume the first eigenvalue of the Laplacian satisfies
\[
\lambda_1(M^n) > \frac{b}{C_{n,k} - a}.
\]
Then the space of \( L^2 \) harmonic \( k \)-forms on \( M^n \) is trivial.

**Proof** Fix a \( L^2 \) harmonic \( k \)-form \( \omega \) on \( M^n \). As in the proof of Theorem 5 we have
\[
f \Delta f \geq (C_{n,k} - 1) |\nabla f|^2 - aq f^2 - bf^2,
\]
where \( f = |\omega| \). Applying Lemma 2 with \( A = C_{n,k} - 1 \) gives
\[
\int_{M^n} |\nabla f|^2 \leq \frac{b}{C_{n,k} - a} \int_{M^n} f^2.
\]
By Lemma 3 we have
\[
\lambda_1(M^n) \int_{M^n} f^2 \leq \int_{M^n} |\nabla f|^2.
\]
Combining the last two inequalities yields
\[
\lambda_1(M^n) \int_{M^n} f^2 \leq \frac{b}{C_{n,k} - a} \int_{M^n} f^2.
\]
If \( \omega \) is not identically zero then
\[
\lambda_1(M^n) \leq \frac{b}{C_{n,k} - a},
\]
a contradiction. This proves that \( \omega \) is identically zero.

Applying Theorem 6 to one-forms proves Theorem 4.

**Proof** For one-forms \( \langle R \omega, \omega \rangle = \text{Ric} (\omega, \omega) \) and \( C_{n,1} = \frac{n}{n-1} \).

To study the rigidity in Theorem 4 we use the following lemma.
Lemma 4 ([21, Lemma 4.1]) Let $M^n$ be a complete Riemannian manifold of dimension $n \geq 2$. Assume that the Ricci curvature of $M$ satisfies the lower bound

$$\text{Ric}_M(x) \geq -(n - 1) \tau(x)$$

for all $x \in M$. Suppose $f$ is a nonconstant harmonic function defined on $M$. Then the function $|\nabla f|$ must satisfy the differential inequality

$$\Delta |\nabla f| \geq -(n - 1) \tau |\nabla f| + \frac{|\nabla |\nabla f||^2}{(n - 1) |\nabla f|}$$

in the weak sense. Moreover, if equality holds, then $M$ is given by $M = \mathbb{R} \times N^{n-1}$ with the warped product metric

$$ds^2_M = dt^2 + \eta(t)^2 ds^2_N$$

for some positive function $\eta(t)$, and some manifold $N^{n-1}$. In this case, $\tau(t)$ is a function of $t$ alone satisfying

$$\eta''(t) \eta^{-1}(t) = \tau(t).$$

We prove the following rigidity theorem. The idea is to combine the lemma above with the equality conclusion of Lemma 2.

Theorem 7 Suppose a complete Riemannian manifold $M^n$ satisfies a weighted Poincaré inequality with weight function $q$ and the Ricci curvature satisfies

$$\text{Ric} \geq -aq - b$$

for some $0 < a < \frac{n}{n-1}$ and $b > 0$. Assume the first eigenvalue of the Laplacian satisfies

$$\lambda_1(M^n) = \frac{b}{\frac{n}{n-1} - a},$$

and the space of $L^2$ harmonic one-forms on $M^n$ is non-trivial. Then the universal cover of $M^n$ splits as $\tilde{M}^n = \mathbb{R} \times N^{n-1}$ with the warped product metric

$$g_{\tilde{M}^n} = dt^2 + \eta(t)^2 g_{N^{n-1}}$$

for some positive function $\eta(t)$ and some hypersurface $N^{n-1}$ in $\tilde{M}^n$. In this case, $q$ is a function of $t$ alone satisfying

$$\frac{\eta''(t)}{\eta(t)} = \frac{1}{n-1} (aq + b).$$

Proof Take a non-vanishing $L^2$ harmonic one-form $\omega$ on $M^n$. By the proof of Theorem 6 we have

$$f \Delta f \geq \frac{1}{n-1} |\nabla f|^2 - aqf^2 - bf^2$$

and

$$\int_{M^n} |\nabla f|^2 = \frac{b}{\frac{n}{n-1} - a} \int_{M^n} f^2,$$
where \( f = |\omega| \). By Lemma 2 we have
\[
f \Delta f = \frac{1}{n-1} |\nabla f|^2 - af^2 - bf^2.
\]
Lift the metric of \( M^n \) to the universal cover \( \tilde{M}^n \) and lift \( \omega \) to a harmonic one-form \( \tilde{\omega} \) on \( \tilde{M}^n \). Since \( \tilde{M}^n \) is simply connected, there is a smooth function \( h \) on \( \tilde{M}^n \) such that \( dh = \tilde{\omega} \). This shows that \( h \) is a non-constant harmonic function on \( \tilde{M}^n \) such that
\[
|dh| \Delta |dh| = \frac{1}{n-1} |\nabla |dh||^2 - (aq + b) |dh|^2.
\]
The conclusion follows from the lemma above. \( \square \)

**Remark 1** The results of this section can be improved using refined Kato inequalities for \( L^2 \) harmonic forms on Kähler manifolds ([14, Theorem 4.2] and [13, Theorem 3.1]): a \( L^2 \) harmonic one-form \( \omega \) on a complete Kähler manifold satisfies
\[
|\nabla \omega|^2 \geq 2 |\omega|^2.
\]
Using this inequality it is not difficult to show that Theorem 3 and Theorem 4 hold with “\( 0 < a < \frac{n}{n-1} \)” and “Riemannian manifold” replaced by “\( 0 < a < 2 \)” and “Kähler manifold” respectively.

### 3 Applications to stable hypersurfaces

For a hypersurface \( M^n \) in a Riemannian manifold \( \overline{M}^{n+1} \) the stability operator (or Jacobi operator) is defined by
\[
L = \Delta + |A|^2 + \overline{\text{Ric}} (\nu, \nu),
\]
where \( \Delta \) is the Laplacian of \( M^n \), \( A \) is the second fundamental form and \( \overline{\text{Ric}} (\nu, \nu) \) is the Ricci curvature of \( \overline{M}^{n+1} \) in the normal direction. The hypersurface \( M^n \) is called stable if the first eigenvalue of the stability operator is non-negative, in other words
\[
0 \leq \lambda_1 (L) = \inf_{\phi \in C_c^\infty (M^n)} \frac{\int_{M^n} (-L \phi) \phi}{\int_{M^n} \phi^2}.
\]
This shows that a stable hypersurface satisfies a weighted Poincaré inequality with weight function
\[
q = |A|^2 + \overline{\text{Ric}} (\nu, \nu).
\]
For orthonormal vector fields \( X \) and \( Y \) on \( \overline{M}^{n+1} \) the Bi-Ricci\( a \) curvature is defined by
\[
\overline{\text{BiRic}}^a (X, Y) = \overline{\text{Ric}} (X, X) + a \overline{\text{Ric}} (Y, Y) - \overline{K} (X, Y),
\]
where \( \overline{K} \) is the sectional curvature of \( \overline{M}^{n+1} \) and \( a \) is a constant. For \( a = 1 \) the Bi-Ricci\( a \) curvature is equal to the Bi-Ricci curvature defined by Shen and Ye [27]. Notice that if the sectional curvature is non-negative then the Bi-Ricci curvature is non-negative.

It is an interesting problem to study the geometry and topology of stable minimal hypersurfaces in Riemannian manifolds with a certain non-negative curvature. Fischer-Colbrie and Schoen classified complete stable minimal surfaces in three-dimensional Riemannian manifolds with non-negative sectional curvatures.
manifolds with non-negative scalar curvature [10]. Palmer proved that the space of $L^2$ harmonic one-forms on a complete stable minimal hypersurface in $\mathbb{R}^{n+1}$ is trivial [24]. Miyaoka and Tanno extended this result to hypersurfaces in Riemannian manifolds with non-negative sectional curvature and non-negative Bi-Ricci curvature respectively [23,26]. Cao, Shen and Zhu proved that a complete stable minimal hypersurface in $\mathbb{R}^{n+1}$ has only one end [3]. Li and Wang proved that for a complete stable minimal hypersurface properly immersed in a Riemannian manifold with non-negative sectional curvature either the hypersurface has only one end or it is totally geodesic with a certain decomposition [19]. Cheng proved that a complete stable minimal hypersurface in a Riemannian manifold of dimension at most 6 and positive Bi-Ricci curvature has only one end [5].

We prove the following vanishing theorem for $L^2$ harmonic one-forms on complete stable minimal hypersurfaces in Riemannian manifolds with non-negative Bi-Ricci curvature.

**Theorem 8** For a complete non-compact stable minimal hypersurface $M^n$ in a Riemannian manifold $\mathbb{M}^{n+1}$ with non-negative Bi-Ricci curvature for some $\frac{n-1}{n} \leq a < \frac{n}{n-1},$ the space of $L^2$ harmonic one-forms on $M^n$ is trivial.

**Proof** By assumption $M^n$ satisfies a weighted Poincaré inequality with weight function

$$q = |A|^2 + \text{BiRic}(\nu, \nu).$$

By Theorem 3 to complete the proof it suffices to show that the Ricci curvature of $M^n$ satisfies

$$\text{Ric} \geq -aq.$$

The proof is adapted from [19]. Take a local orthonormal frame $e_1, \ldots, e_n$ on $M^n$ diagonalizing the second fundamental form, in other words $A(e_i, e_j) = \lambda_i \delta_{ij}.$ By the Gauss equation for $i \neq j$ we have

$$K(e_i, e_j) = \text{K}(e_i, e_j) + \lambda_i \lambda_j.$$ 

Since $M^n$ is minimal we have

$$\lambda_1 + \sum_{i=2}^n \lambda_i = 0.$$

This shows that

$$\text{Ric}(e_1, e_1) = \text{BiRic}(e_1, e_1) - \text{K}(e_1, \nu) - \lambda_1^2.$$

This implies that

$$\text{Ric}(e_1, e_1) + aq = \text{BiRic}^{\text{af}}(e_1, v) + a |A|^2 - \lambda_1^2$$

$$\geq a \left( \lambda_1^2 + \sum_{i=2}^n \lambda_i^2 \right) - \lambda_1^2$$

$$\geq a \left( \lambda_1^2 + \frac{\sum_{i=2}^n \lambda_i^2}{n-1} \right) - \lambda_1^2$$

$$= \left( \frac{na}{n-1} - 1 \right) \lambda_1^2$$

$$\geq 0.$$ 

This proves the theorem. $\square$
Taking \( a = 1 \) in this theorem recovers Tanno’s theorem [26]. We hope to find a good example of a Riemannian manifold with non-negative Bi-Ricci\(^a\) curvature for some \( \frac{n-1}{n} \leq a < \frac{n}{n-1} \) but without non-negative Bi-Ricci curvature. Li and Wang’s theorem has a stronger conclusion but assuming the Riemannian manifold with non-negative sectional curvature and the hypersurface properly immersed [19]. Cheng’s theorem has a stronger conclusion but assuming the Riemannian manifold with positive Bi-Ricci\(^a\) curvature and \( 3 \leq n \leq 5 \) with \( \frac{2}{3} \leq a \leq 2 \) for \( n = 3 \) and \( \frac{n-1}{n} \leq a < \frac{4}{n-1} \) for \( n = 4, 5 \) [5].

There has been some interest in studying the topology at infinity of stable constant mean curvature hypersurfaces [4,6].

We prove the following vanishing theorem for \( L^2 \) harmonic one-forms on complete stable hypersurfaces, not necessarily minimal, in Riemannian manifolds with non-negative Bi-Ricci\(^a\) and dimension at most 7.

**Theorem 9** For a complete non-compact stable hypersurface \( M^n \) in a Riemannian manifold \( \mathbb{M}^{n+1} \) with \( 2 \leq n \leq 6 \) and non-negative Bi-Ricci\(^a\) curvature for some \( \frac{n-1}{2} \leq a < \frac{n}{n-1} \), the space of \( L^2 \) harmonic one-forms on \( M^n \) is trivial.

**Proof** Note that \( \frac{n-1}{2} < \frac{n}{n-1} \) for \( 2 \leq n \leq 6 \). As in Theorem 8 to complete the proof it suffices to show that the Ricci curvature of \( M^n \) satisfies

\[
\text{Ric} \geq -aq.
\]

As in Theorem 8 we have

\[
\text{Ric}(e_1, e_1) = \text{Ric}(e_1, e_1) - K(e_1, v) + \lambda_1 \sum_{i=2}^{n} \lambda_i.
\]

This shows that

\[
\text{Ric}(e_1, e_1) + aq = \text{BiRic}^a(e_1, e_1) + \lambda_1 \sum_{i=2}^{n} \lambda_i + a |A|^2
\]

\[
\geq \lambda_1 \sum_{i=2}^{n} \lambda_i + a \sum_{i=1}^{n} \lambda_i^2.
\]

If \( \lambda_1 = 0 \) then

\[
\text{Ric}(e_1, e_1) + aq \geq 0.
\]

If \( \lambda_1 \neq 0 \) then

\[
\text{Ric}(e_1, e_1) + aq \geq \lambda_1 \sum_{i=2}^{n} \lambda_i + a \sum_{i=1}^{n} \lambda_i^2
\]

\[
= \lambda_1^2 \left( \sum_{i=2}^{n} \left( \frac{\sqrt{a \lambda_i}}{\lambda_1} + \frac{1}{2\sqrt{a}} \right)^2 + a - \frac{n-1}{4a} \right)
\]

\[
\geq \lambda_1^2 \left( a - \frac{n-1}{4a} \right)
\]

\[
= 0.
\]

This proves the theorem. \( \square \)
Notice that the proof of the theorem implies following result: For a hypersurface $M^n$ in a Riemannian manifold $\overline{M}^{n+1}$ and $a \geq \frac{\sqrt{n-1}}{2}$ we have

$$\text{Ric} (e_1, e_1) + a \left( |A|^2 + \text{Ric} (v, v) \right) \geq \text{BiRic}^a (e_1, v).$$

After the paper was submitted the author became aware that this result was proved for hypersurfaces in Riemannian manifolds with non-negative sectional curvature by Kim and Yun [12] and Dung and Seo [9]. Our technique to find the lower bound of the Ricci curvature of the hypersurface is more elementary.

There has been some interest in finding estimates for eigenvalues of the Laplacian of minimal hypersurfaces in the hyperbolic space. Cheung and Leung proved that for a complete minimal hypersurface $M^n$ in the hyperbolic space $\mathbb{H}^{n+1}$ the first eigenvalue of the Laplacian of $M^n$ satisfies the lower bound $\lambda_1 (M^n) \geq \frac{1}{4} (n - 1)^2$ [7]. Candel proved that for a simply connected stable minimal surface $M^2$ in the three-dimensional hyperbolic space $\mathbb{H}^3$ the first eigenvalue of the Laplacian of $M^2$ satisfies the upper bound $\lambda_1 (M^2) \leq \frac{4}{3}$ [2]. Seo proved that for a complete stable minimal hypersurface $M^n$ in the hyperbolic space $\mathbb{H}^{n+1}$ if the norm of the second fundamental form belongs to $L^2$ then the first eigenvalue of the Laplacian of $M^n$ satisfies the upper bound $\lambda_1 (M^n) \leq n^2$ [25].

We prove the following vanishing theorem for $L^2$ harmonic one-forms on complete stable minimal hypersurfaces in Riemannian manifolds with $\text{BiRic}^a \geq -b$.

**Theorem 10** For a complete stable minimal hypersurface $M^n$ in a Riemannian manifold $\overline{M}^{n+1}$ with $\text{BiRic}^a \geq -b$ for some $\frac{n-1}{n} \leq a < \frac{n}{n-1}$ and $b > 0$, if the first eigenvalue of the Laplacian of $M^n$ satisfies

$$\lambda_1 (M^n) > \frac{b}{\frac{n}{n-1} - a},$$

then the space of $L^2$ harmonic one-forms on $M^n$ is trivial.

**Proof** By assumption $M^n$ satisfies a weighted Poincaré inequality with weight function

$$q = |A|^2 + \text{Ric} (v, v).$$

As in Theorem 8 the Ricci curvature of $M^n$ satisfies

$$\text{Ric} \geq -aq - b.$$

The conclusion follows from Theorem 4.

This theorem can be used to obtain an explicit upper bound of the first eigenvalue of the Laplacian of stable minimal hypersurfaces in Riemannian manifolds with constant negative sectional curvature.

**Corollary 2** For a complete stable minimal hypersurface $M^n$ in a Riemannian manifold $\overline{M}^{n+1}$ with constant sectional curvature $-\overline{K}^2 \neq 0$, if the space of $L^2$ harmonic one-forms on $M^n$ is non-trivial then the first eigenvalue of the Laplacian of $M^n$ satisfies

$$\lambda_1 (M^n) \leq \overline{K}^2 \left( \frac{2n (n - 1)^2}{2n - 1} \right).$$

In particular, this estimate holds when $M^n$ has at least two ends.
Proof The first conclusion follows by taking \( a = \frac{n-1}{n} \) and \( b = K^2 (n + an - 1) \) in Theorem 10. Now assume that \( M^n \) has at least two ends. Since \( \overline{M}^{n+1} \) has negative constant sectional curvature and \( M^n \) is minimal it follows from [11] that \( M^n \) satisfies a certain Sobolev inequality. Using Holder’s inequality gives a Sobolev inequality as in [15, Lemma20.12], which implies the non-existence of parabolic ends in \( M^n \). This shows that \( M^n \) has at least two non-parabolic ends. It follows from [16] that \( M^n \) admits a non-trivial \( L^2 \) harmonic one-form. The second conclusion follows from the first conclusion of the theorem.

Taking \( \overline{M}^{n+1} = \mathbb{H}^3 \) in this corollary recovers the upper bound obtained by Candel [2] but assuming the space of \( L^2 \) harmonic one-forms on \( M^2 \) non-trivial. Taking \( \overline{M}^{n+1} = \mathbb{H}^{n+1} \) in the corollary gives the upper bound \( \lambda_1 (M^n) \leq \frac{2n(n-1)^2}{2n-1} \), which is better than upper bound \( \lambda_1 (M^n) \leq n^2 \) obtained by Seo [25]. Notice that Seo’s theorem holds without assuming the space of \( L^2 \) harmonic one-forms non-trivial but assuming the norm of the second fundamental form \( L^2 \)-integrable.

We prove the following vanishing theorem for \( L^2 \) harmonic one-forms on complete stable hypersurfaces, not necessarily minimal, in Riemannian manifolds with \( \text{BiRic}^a \geq -b \) and dimension at most 7.

**Theorem 11** For a complete stable hypersurface \( M^n \) in a Riemannian manifold \( \overline{M}^{n+1} \) with \( 2 \leq n \leq 6 \) and \( \text{BiRic}^a \geq -b \) for some \( \frac{\sqrt{n-1}}{2} < a < \frac{n}{n-1} \) and \( b > 0 \), if the first eigenvalue of the Laplacian of \( M^n \) satisfies

\[
\lambda_1 (M^n) > \frac{b}{\frac{n}{n-1} - a}
\]

then the space of \( L^2 \) harmonic one-forms on \( M^n \) is trivial.

**Proof** Note that \( \frac{\sqrt{n-1}}{2} < \frac{n}{n-1} \) for \( 2 \leq n \leq 6 \). By assumption \( M^n \) satisfies a weighted Poincaré inequality with weight function

\[
q = |A|^2 + \overline{\text{Ric}} (v, v)
\]

As in Theorem 9 the Ricci curvature of \( M^n \) satisfies

\[
\text{Ric} \geq -aq - b.
\]

The conclusion follows from Theorem 4.

This theorem can be used to obtain an explicit upper bound of the first eigenvalue of the Laplacian of stable hypersurfaces, not necessarily minimal, in Riemannian manifolds with constant negative sectional curvature and dimension at most 7.

**Corollary 3** For a complete stable hypersurface \( M^n \) in a Riemannian manifold \( \overline{M}^{n+1} \) with constant sectional curvature \( -K^2 \neq 0 \) and dimension \( 2 \leq n \leq 6 \), if the space of \( L^2 \) harmonic one-forms on \( M^n \) is non-trivial then the first eigenvalue of the Laplacian of \( M^n \) satisfies

\[
\lambda_1 (M^n) \leq K^2 \left( \frac{n - 1 + \frac{\sqrt{n-1}}{2} \cdot n}{\frac{n}{n-1} - \frac{\sqrt{n-1}}{2}} \right).
\]

**Proof** The conclusion follows by taking \( a = \frac{\sqrt{n-1}}{2} \) and \( b = K^2 (n + an - 1) \) in Theorem 11.

\[\square\]
In particular, we obtain the following result.

**Corollary 4** For a complete stable surface $M^2$, not necessarily minimal, in the hyperbolic space $\mathbb{H}^3$, if the space of $L^2$ harmonic one-forms on $M^2$ is non-trivial then the first eigenvalue of the Laplacian of $M^2$ satisfies

$$\lambda_1 (M^2) \leq \frac{4}{3}.$$

This corollary extends the upper bound obtained by Candel [2] to non-minimal surfaces, but assuming the space of $L^2$ harmonic one-forms on $M^2$ non-trivial.

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