MOMENT BOUNDS FOR NON-LINEAR FUNCTIONALS OF THE PERIODOGRAM

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ABSTRACT. In this paper, we prove the validity of the Edgeworth expansion of the Discrete Fourier transforms of some linear time series. This result is applied to approach moments of non linear functionals of the periodogram. As an illustration, we give an expression of the mean square error of the Geweke and Porter-Hudak estimator of the long memory parameter. We prove that this estimator is rate optimal, extending the result of Giraitis, Robinson, and Samarov [1997] from Gaussian to linear processes.

Keywords : linear processes, discrete Fourier transform, periodogram, long range dependence, Geweke and Porter-Hudak (GPH) estimator.

1. Introduction

Many estimators in time series analysis involve non-linear functionals of the periodogram. Examples include the estimation of the innovation variance [Chen and Hannan, 1980, Lee, Cho, Kim, and Park, 1995, Deo and Chen, 2000, Ginovian, 2003], log-periodogram regression [Taniguchi, 1979, 1991, Shimotsu and Phillips, 2002], robust non-parametric estimation of the spectral density [von Sachs, 1994, Janas and von Sachs, 1995]. Non-linear functionals of the periodogram also play a predominant role in the analysis of long-memory time-series: one of the much widely used estimator of the memory parameter is based on the regression of the log-periodogram ordinates on the log-frequency [Geweke and Porter-Hudak, 1983, see also Robinson 1995b, Moulines and Soulier 1999].

The statistical analysis of such functionals has proved to be a very challenging problem, due to the intricate dependence structure of periodogram ordinates. The first attempts to study these statistics were made under the additional assumption that the underlying process is Gaussian. Because the Fourier transform coefficients are in this case also Gaussian, one may then apply results on non-linear transforms of Gaussian random variables; see for example Taqqu [1977], Taniguchi [1980] and Arcones [1994].

These techniques do not extend to non-Gaussian processes. A first step to weaken this assumption was taken by Chen and Hannan [1980] who proved the consistency of an additive functional of the log-periodogram of a linear stationary process, with an application to the estimation of the innovation variance. These techniques were based on the so-called Bartlett [1955] expansion; this technique was later improved by Fay, Moulines, and Soulier [2002] who proved a central limit theorems for these functionals. It used by Velasco [2000] to establish the weak consistency of the log-periodogram regression estimate of the long memory parameter for long range dependent linear time series. Edgeworth expansions are used to estimates moments of the functional of the unobservable periodogram of the innovation sequence. Remainder
terms can be bounded in probability. The Bartlett expansion is indeed useful to establish limit theorems but does not in general allow to determine the moments of these functionals.

An alternative approach has been considered by [von Sachs 1994], [Janas and von Sachs 1995]. These authors prove the mean-square consistency of general additive functional of non-linear transforms of the (tapered) periodogram, using Edgeworth expansions of the discrete Fourier transform of the observed time series itself. [Janas and von Sachs 1995] apply these results to prove the mean-square consistency of an Huberized (peak insensitive) non-parametric spectral estimator. These results rely on the Edgeworth expansion of a triangular array of strongly mixing process with geometrically mixing coefficient established by [Götze and Hipp 1983]. The mixing conditions herein are rather stringent, and thus the conclusions reached by [Janas and von Sachs 1995] are proved under a set of restrictive assumptions, precluding for instance their use in a long-memory context.

The main objective of this paper is to develop a method allowing to compute the moments of functionals of non-linear transforms of the (possibly tapered) periodogram of a linear process. These results are based on Edgeworth expansion of a (possibly infinite) triangular array of i.i.d. random variables obtained earlier in [Fay, Moulines, and Soulier 2004] and recalled in Appendix A. The linearity of the process is then crucial. Our results cover both short-memory and long-memory processes.

The remaining of the paper is organized as follows. In Section 2 we give the assumptions on the linear structure of the time series and define the cumbersome notations related to Edgeworth expansions. In Section 3 we formulate the validity of Edgeworth expansions and moment bounds under short memory set of hypotheses. As an application, we derive the mean-square consistency of additive functionals of non-linear transform of the periodogram for a short-memory linear time-series. In Section 4 we follow the same lines but in a long-range dependence framework, and apply the moment bounds we obtained to control the mean-square error of the Geweke and Porter-Hudak [1983] estimator of the fractional difference parameter for a non-Gaussian linear long-memory process. This extends the rate optimality property of the Geweke and Porter-Hudak (hereafter, GPH) estimator obtained earlier by [Giraitis, Robinson, and Samarov 1997] for Gaussian processes. A small Monte-Carlo experiment is run to confirm our results for finite-sample observations. Proofs are postponed to the appendices.

2. Notations and assumptions

Assume that \( X = (X_t)_{t \in \mathbb{Z}} \) is a covariance stationary process that have a spectral density \( f \). For any integer \( r \geq 0 \), we define the tapered discrete Fourier transform (DFT) and periodogram of order \( r \) as

\[
d_{r,n}(\lambda) \stackrel{\text{def}}{=} (2\pi n a_r)^{-1/2} \sum_{t=1}^{n} h_{t,n} X_t e^{it\lambda}, \quad I_{r,n}(\lambda) \stackrel{\text{def}}{=} |d_{r,n}(\lambda)|^2
\]

where \( h_{t,n} \stackrel{\text{def}}{=} 1 - e^{2\pi it/n} \) is the data taper introduced in [Hurvich and Chen 2000] and \( a_r \stackrel{\text{def}}{=} n^{-1} \sum_{t=1}^{n} |h_{t,n}|^{2r} = \left( \frac{2r}{r} \right) \) is a normalization factor. Denote \( d_{r,n,k} = d_{r,n}(\lambda_k) \) and \( I_{r,n,k} = I_{r,n}(\lambda_k) \) the tapered DFT and tapered periodogram evaluated at the Fourier frequencies
\[ \lambda_k \overset{\text{def}}{=} \frac{2 \pi k}{n}, \quad k = 1, \ldots, [(n - 1)/2]. \]
Define for \( r \in \mathbb{N} \), \( D_{r,n}(\lambda) \) the normalized kernel function
\[ D_{r,n}(\lambda) \overset{\text{def}}{=} (na_r)^{-1/2} \sum_{t=1}^{n} h_{t,n}^r \exp(\imath t \lambda) = (na_r)^{-1/2} \sum_{k=0}^{r} \binom{r}{k} (-1)^k D_n(\lambda + \lambda_k) \quad (2.2) \]
where \( D_n(\lambda) \overset{\text{def}}{=} \sum_{t=1}^{n} e^{-\imath \lambda t} \) denotes the non-symmetric Dirichlet kernel. The latter relation implies that \( D_{r,n}(\lambda_k) = 0 \) for \( k \in \{1, \ldots, \tilde{n}\} \), with \( \tilde{n} \overset{\text{def}}{=} [(n - 2r - 1)/2] \), so that the tapered Fourier transform is invariant to shift in the mean. As shown in Hurvich and Chen [2000], the decay rate of the kernel in the frequency domain increases with the kernel order, namely
\[ \forall \lambda \in [-3\pi/2, 3\pi/2], \quad |D_{r,n}(\lambda)| \leq \frac{Cn^{1/2}}{(1 + n|\lambda|)^{r+1}} \quad (2.3) \]
This property means that higher order kernels are more effective to control frequency leakage. If \( X \) is a white noise and \( r = 0 \), the DFT ordinates at different Fourier frequencies are uncorrelated. This property is lost by tapering. More precisely, for \( 1 \leq k \neq \bar{r} \leq \tilde{n} \), \( \mathbb{E}[d_{r,n,k}d_{r,n,j}] = 0 \), and \( \mathbb{E}[d_{r,n,k}d_{\bar{r},n,j}] \overset{\text{def}}{=} (2\pi)^{-1}\zeta_r(k - j) \) where \( \bar{z} \) denotes the complex conjugate of \( z \) and \( \zeta_r \) defined in [3.6].

Many statistical applications (see the references given in the Introduction) require to study weighted sums of non-linear functionals of the periodogram ordinates
\[ T_n(X, \phi) = \sum_{k=1}^{K} \beta_{n,k} \phi \left( \frac{I_{r,n,k}}{f(\lambda_k)} \right), \quad (2.4) \]
where \( (\beta_{n,k})_{k \in \{1, \ldots, K\}} \) is a triangular array of real numbers. If \( X \) is a Gaussian white noise, then \( (I_{r,n,k}) \) are i.i.d and the moments of the sum \( T_n(X, \phi) \) can be calculated explicitly. In any other case, the random variables \( (I_{r,n,k})_{k \in \{1, \ldots, K\}} \) are not independent, and the calculation of the moments of \( T_n(X, \phi) \) is a difficult problem. The only attempt to solve it has been made by Janas and von Sachs [1995], who proposed a technique to compute moment of order 1 and 2. As already outlined, their results are based on mixing conditions, precluding their use for long-memory processes.

Remark. Sometimes the periodogram ordinates are averaged along blocks of adjacent frequencies. This technique is known as pooling and is appropriate to reduce asymptotic variance of the estimators of non linear functionals of the periodogram (see Robinson [1995b,a]). For simplicity, we will not present any explicit result or application with the pooled periodogram, but the Edgeworth expansion results that follow allow to derive moment bounds on functionals of tapered and pooled periodogram as well.

In this contribution, we focus on non-Gaussian strict sense linear processes, i.e. it is assumed that
\[ X_t = \sum_{j \in \mathbb{Z}} \psi_j Z_{t-j}, \quad \sum_{j \in \mathbb{Z}} \psi_j^2 < \infty , \quad (2.5) \]
where \( (Z_j)_{j \in \mathbb{Z}} \) is a sequence of i.i.d random variables such that \( \mathbb{E}[Z_1] = 0, \mathbb{E}[Z_1^2] = 1 \). In addition, for some \( s \geq 3, p \geq 1 \) and \( p' \geq 0 \),

(A1) \( \mathbb{E}[|Z_1|^s] < \infty \) and \( \int_{\mathbb{R}} |t|^{p'} |\mathbb{E}[e^{i t Z_1}]|^{p'} \, dt < \infty \).
Remark. Apart from a classical moment condition, (A11) suppose that the distribution of the i.i.d. noise is smooth; for example, lattice distributions are forbidden. This condition is stronger than the usual Cramér condition. It ensures that the distributions of the Fourier coefficients of $Z$ are eventually continuous. We need this continuity to bound moments of singular functionals of the periodogram. Note that this condition could be dispensed with, were concerned with smooth functionals.

Define $\psi(\lambda) = \sum_{j \in \mathbb{Z}} \psi_j e^{ij\lambda}$ (the convergence holds in $L^2([\pi, \pi], dx)$) the transfer function of the linear filter $(\psi_j)_{j \in \mathbb{Z}}$ and $f(\lambda) = (2\pi)^{-1/2} |\psi(\lambda)|^2$ the spectral density of the process $X$. For an integer $k \in \{1, \ldots, n\}$ such that $f(\lambda_k) \neq 0$, define the normalized DFT $\omega_{r,n,k} \defeq \sqrt{2\pi} \, d_{r,n,k}/|\psi(\lambda_k)|$. Let $k_1 < k_2 < \ldots < k_u$ be an ordered $u$-tuple of such integers in the range $1, \ldots, n$ and write $k = (k_1, \ldots, k_u)$. Define (the reference to $r$ is suppressed in the notation)

$$S_n(k) \defeq [\text{Re}(\omega_{r,n,k_1}), \text{Im}(\omega_{r,n,k_1}), \ldots, \text{Re}(\omega_{r,n,k_u}), \text{Im}(\omega_{r,n,k_u})].$$

(2.6)

With those definitions,

$$I_{n,k,r} = f(\lambda_k) |\omega_{r,n,k}|^2 = f(\lambda_k) |S_n(k)|^2.$$

(2.7)

Since $X$ admits the linear representation (2.5), $S_n(k)$ can be further expressed as a $2u$-dimensional infinite triangular array in the variables $(Z_t)_{t \in \mathbb{Z}}$. Precisely

$$S_n(k) = \sum_{j \in \mathbb{Z}} U_{n,j}(k) Z_j,$$

(2.8)

with

$$U_{n,j}(k) \defeq (na_r)^{-1/2} F_n^{-1}(k) \sum_{t=1}^n \psi_{t-j} C_{n,t}(k),$$

(2.9)

$$C_{n,t}(k) \defeq \sum_{p=0}^r (-1)^{p} \binom{r}{p} \left( \cos(t\lambda_{k_1+p}), \sin(t\lambda_{k_1+p}), \ldots, \cos(t\lambda_{k_u+p}), \sin(t\lambda_{k_u+p}) \right)'$$

and

$$F_n(k) \defeq \text{diag}(|\psi(\lambda_{k_1})|, |\psi(\lambda_{k_1})|, \ldots, |\psi(\lambda_{k_u})|, |\psi(\lambda_{k_u})|).$$

To formulate our results, some notations related to Edgeworth expansions are required, which we take from the monograph of Bhattacharya and Rao [1976]. For $u$ a positive integer, $\nu = (\nu_1, \ldots, \nu_u) \in \mathbb{N}^u$ and $z = (z_1, \ldots, z_u) \in \mathbb{C}^u$, denote $|\nu| = \sum_{i=1}^u \nu_i$, $\nu! = \nu_1! \nu_2! \cdots \nu_u!$ and $z^\nu = z_1^{\nu_1} z_2^{\nu_2} \cdots z_u^{\nu_u}$. If $1 \leq |\nu| \leq s$, denote $\chi_{n,\nu}(k)$ the cumulants of $S_n(k)$. Then $\chi_{n,\nu}(k) = \kappa_{|\nu|} \sum_{j \in \mathbb{Z}} U_{n,j}(k) U_{n,j}(k)'$ where $\kappa_r$ denotes the $r$-th cumulant of $Z_1$, $r \leq s$. Let $\chi_n(k) \defeq \text{cov}[S_n(k)] = \sum_{j \in \mathbb{Z}} U_{n,j}(k) U_{n,j}(k)'$.

Let $\chi = \{\chi_\nu; \nu \in \mathbb{N}^u\}$ be a set of real numbers. For any integer $r \geq 2$ and $z \in \mathbb{C}^u$, define $\lambda_r(z) \defeq r! \sum_{|\nu|=r} \frac{\chi_{\nu} z^\nu}{\nu!}$. The polynomials $\hat{P}_r(z, \chi)$ are formally defined for $r \geq 1$ by the identities

$$1 + \sum_{r=1}^\infty \hat{P}_r(z, \chi) t^r = \exp\left\{ \sum_{r=3}^\infty \frac{\lambda_r(z)}{r!} t^{r-2} \right\} = 1 + \sum_{m=1}^\infty \frac{1}{m!} \left( \sum_{r=3}^\infty \frac{\chi_r(z)}{r!} t^{r-2} \right)^m,$$

and we set $\hat{P}_0 \equiv 0$. Denote $\varphi_V$ the density of a Gaussian r.v in $\mathbb{R}^u$ with zero mean and non-singular covariance matrix $V$. Define $P_r : \mathbb{R}^u \mapsto \mathbb{R}$ by $P_r(x, V, \chi) = \left[ \hat{P}_r(-D, \chi) \right] \varphi_V(x).$
where, for any polynomial \( P(z) = \sum a_\nu z^\nu \), \( P(-D) \) is interpreted as a polynomial in the differentiation operator \( D \), \( P(-D) = \sum a_\nu (-1)^\nu D^\nu \), with \( D^\nu = \frac{\partial^|\nu|}{\partial x_1^{\nu_1} \ldots \partial x_u^{\nu_u}} \), \( \nu = (\nu_1, \ldots, \nu_u) \in \mathbb{N}^u \). By construction \( P_r \) and \( \tilde{P}_r \) do not depend on the coefficient \( \chi_\nu \) if \( |\nu| > r + 2 \), and \( \tilde{P}_r (it, \chi) e^{-t^2Vt/2} \) is the Fourier transform of \( P_r (x, V, \chi) \). Let \( \xi_\Gamma \) be a centered \( \alpha \)-dimensional Gaussian vector with covariance matrix \( \Gamma \) and \( g : \mathbb{R}^\alpha \rightarrow \mathbb{R} \) a measurable mapping. Define \( N_s (g) = \int_{\mathbb{R}^\alpha} (1 + ||x||^2)^{-1} |g(x)| \, dx \) and \( ||g||_2^2 = \mathbb{E}[g^2(\xi_\Gamma)] \). The Hermite rank of \( g \), \( ||g||_2^2 < \infty \), with respect to \( \Gamma \) is defined as the smallest integer \( \tau \) such that there exists a polynomial \( P \) of degree \( \tau \) with \( \mathbb{E}[g(\xi_\Gamma) P(\xi_\Gamma)] \neq 0 \). We denote \( \tau (g, \Gamma) \) the (positive) Hermite rank of \( g - \mathbb{E}[g(\xi_\Gamma)] \) with respect to \( \Gamma \).

3. Moment bounds: short memory case

In this section we consider short-range dependent processes. For any reals \( \alpha, \delta > 0 \) and \( \beta < \infty \), denote by \( G(\alpha, \beta, \delta) \) the set of real sequences \((\psi_j)_{j \in \mathbb{Z}}\) such that

\[
|\psi_0| + \sum_{j \in \mathbb{Z}} j^{1/2+\delta}|\psi_j| \leq \beta, \quad (3.1)
\]

\[
\alpha \leq \inf_{\lambda \in [-\pi, \pi]} |\psi(\lambda)|. \quad (3.2)
\]

**Theorem 1.** Assume \((\psi_j)_{j \in \mathbb{Z}} \in G(\alpha, \beta, \delta)\) with some integer \( s \geq 3 \), \( p \geq 1 \) and \( p' = 0 \) and assume that \((\psi_j)_{j \in \mathbb{Z}} \in G(\alpha, \beta, \delta)\) for some \( \alpha, \delta > 0 \) and \( \beta < \infty \). Then, there exists constants \( C \) and \( N \) (depending only on \( s, p, \alpha, \beta, \delta, u \) and the distribution of \( Z_0 \)) such that, for all \( n \geq N \), and all \( u \)-tuple \( k \) of distinct integers, the distribution of \( S_n (k) \) has a density \( q_{n,k} \) with respect to Lebesgue’s measure on \( \mathbb{R}^{2u} \) and

\[
\sup_{x \in \mathbb{R}^{2u}} (1 + ||x||^s) q_{n,k}(x) - \sum_{r=0}^{s-3} \int_{\mathbb{R}^{2u}} P_r (x, V_n (k), \{\chi_{n,\nu}(k)\}) \, dx \leq C n^{-(s-2)/2} . \quad (3.3)
\]

Several interesting consequences can be derived from this result. A straightforward integration of the expansion \((3.3)\) yields the following corollary which gives an Edgeworth expansion of some moment \( \mathbb{E}[g(S_n (k))] \) around the centered Gaussian distribution with covariance matrix \( V_n (k) \).

**Corollary 2.** There exists a constant \( C \) and an integer \( N \) (depending only on \( s, p, \alpha, \beta, \delta, u \) and the distribution of \( Z_0 \)) such that, for any \( u \)-tuple of distinct integers \( k \), \( n \geq N \) and measurable function \( g \) satisfying \( N_s (g) < \infty \),

\[
\left| \mathbb{E}[g(S_n (k))] - \sum_{r=0}^{s-3} \int_{\mathbb{R}^{2u}} g(x) P_r (x, V_n (k), \{\chi_{n,\nu}(k)\}) \, dx \right| \leq C N_s (g) n^{-(s-2)/2}. \quad (3.4)
\]

One can also use Theorem 1 to develop the same moment around the limiting Gaussian distribution of \( S_n \). Recalling that \( \omega_{r,n,k} = a_r^{-1/2} \sum_{s=0}^{r} \binom{r}{s} (-1)^s \omega_{0,n,k+s} \), we have

\[
\lim_{n \rightarrow \infty} V_n (k) = V(k) \quad \text{under short memory conditions},
\]

where \( V(k) \) is the \( 2u \times 2u \) matrix defined component-wise by

\[
[V(k)]_{2i-1,2j-1} = [V(k)]_{2i,2j} = \frac{1}{2} \varsigma_r (k_i - k_j), \quad [V(k)]_{2i-1,2j} = [V(k)]_{2i,2j-1} = 0, \quad (3.5)
\]
Recalling (2.7), products of functionals of the periodogram are included in this particular case. Better bounds are obtained by considering frequencies $\varpi$ that the asymptotic decorrelation is achieved, $V$ mappings $g$ expansions and approximating the terms appearing in these expansions. We shall consider $E$ for $i,j$ $n$ $GILLES FÄY$ for $i,j$ $n$ $[1980]$ in the Gaussian case and Janas and von Sachs $[1995]$ for non-Gaussian linear process. Note that

$$\left|E\left[g(S_n(k))\right] - \int_{\mathbb{R}^{2u}} g(x)\varphi_{V(k)}(x) \, dx\right| \leq C \left\{ n^{-1/2}N_3(g) + n^{-\tau}(g,V(k))/2 \right\}.$$  (3.7)

For some functions $g$, it is possible to sharpen this result by considering higher-order ($s > 3$) expansions and approximating the terms appearing in these expansions. We shall consider mappings $g : \mathbb{R}^{2u} \to \mathbb{R}$ such that

$$g(x_1, \ldots, x_{2u}) = \prod_{j=1}^{u} g_j(x_{2j-1}, x_{2j})$$

with $g_j(x, y) = g_j(y, x) = g_j(-x, y)$, $j = 1, \ldots, u$.  (3.8)

Recalling (2.7), products of functionals of the periodogram are included in this particular case. Better bounds are obtained by considering frequencies $k_1, \ldots, k_u$ separated by $r$, so that the asymptotic decorrelation is achieved, $V(k) = 1/2I_{2u}$ as in the $r = 0$ case. Under those conditions, the $O(n^{-1/2})$ of Corollary 3 can be improved to $O(n^{-1})$.

**Corollary 4.** Under the hypothesis that $s \geq 4$, there exists a constant $C$ and $N$ (depending only on $s, p, \alpha, \beta, \delta, u$ and the distribution of $Z_0$), such that for all measurable function $g$ satisfying (2.7) and such that $N_3(g) < \infty$, all $u$-tuple of ordered integers $k$ such that $k_i < k_{i+1} - r$, and any $n \geq N$,

$$\left|E\left[g(S_n(k))\right] - \int_{\mathbb{R}^{2u}} g(x)\varphi_{I_{2u}/2}(x) \, dx\right| \leq C \left\{ n^{-(s-2)/2}N_3(g) + n^{-1}(1 + \|x\|^s)g(x)\right\}_{1/2}.  \right\}.$$  (3.9)

The proofs of Corollaries 3 and 4 are postponed to the Appendix E.

**Remark.** Pushing to higher orders $s \geq 4$ in Corollary 4 is sometimes necessary to have $N_s(g) < \infty$ (see the applications below). But it does not improve the $O(n^{-1})$ bound.

To illustrate the results above, we compute bounds for the mean-square error of plug-in estimators of non-linear functionals of the spectral density $\Lambda(f) = \int_0^\pi w(\lambda)G(f(\lambda)) \, d\lambda$ where $w$ is a function of bounded variation and $G$ is a function such that there exists a function $H$ satisfying, for any $x > 0$, $\int_0^\pi |H(xv)|e^{-u} \, dv < \infty$ and $\int_{v>0} H(xv)e^{-u} \, dv = G(x)$, i.e. $H$ is the inverse Laplace transform of the function $t \mapsto G(1/t)/t$. We consider the following estimator

$$\hat{\Lambda}_n = (\pi/\bar{n}) \sum_{k=1}^{\bar{n}} w(\lambda_k)H(I_{n,k})$$

and put $\Lambda_n = (\pi/\bar{n}) \sum_{k=1}^{\bar{n}} w(\lambda_k)G(f(\lambda_k))$. Here, $r = 0$ and $I_{n,k} \defeq I_{0,n,k}$ is the ordinary periodogram. We assume that the approximation error $\Lambda_n - \Lambda$ is negligible in comparison with the mean-square error $E(\hat{\Lambda}_n - \Lambda_n)^2$. These functionals have been studied in Taniguchi [1980] in the Gaussian case and Janas and von Sachs [1995] for non-Gaussian linear process,
under rather stringent assumptions [see also Deo and Chen, 2000, and the references therein]. The moment bounds we have established allow to extend Janas and von Sachs’ result, by relaxing the conditions on the dependence (from $|\psi_j| < C\rho^{|j|}$ for some $\rho \in (0, 1)$) to $\sum_{j \in \mathbb{Z}} |j|^{1/2} |\psi_j| < \infty$.

**Proposition 5.** Let $(X_t)_{t \in \mathbb{Z}}$ be sequence satisfying the assumptions of Theorem 7 with some $s \geq 4$. Put $H_1(x_1, x_2) = H(x_1^2 + x_2^2)$, $H_2(x_1, x_2, x_3, x_4) = H_1(x_1, x_2)H_1(x_3, x_4)$ and assume that $N_3(H_1^2) < \infty$ and $N_5(H_2) < \infty$. Then, uniformly in $f \in G(\alpha, \beta, \delta)$

$$\mathbb{E}[(\hat{A}_n - A_n)^2] \leq Cn^{-1}.$$ 

**Sketch of the proof.** Applying Corollary 3 to the function $g_{k,f}(x_1, x_2) = H[f(\lambda_k)(x_1^2 + x_2^2)]$ and Corollary 4 to $g_{k,j,f}(x_1, x_2, x_3, x_4) = H[f(\lambda_j)(x_1^2 + x_2^2)]H[f(\lambda_j)(x_3^2 + x_4^2)]$ yield asymptotic expansions for the moments $\mathbb{E}[H^2(I_{n,k})]$ and $\mathbb{E}[H(I_{n,k})H(I_{n,j})]$, which are sufficient to derive the result. The uniformity of the constant $C$ follows from the existence of bounds on $N_3(g_{k,f})$ and $N_4(g_{k,j,f})$ which are uniform in $\psi \in G(\alpha, \beta, \delta).$ 

4. Moment bounds : Long memory case

4.1. Assumptions and main results. We consider two sets of assumptions, depending on available information on the behavior of the spectral density outside a neighborhood of the zero frequency. Recall that a real valued function $\phi$ defined in a neighborhood of zero is regularly varying at zero with index $\rho \in \mathbb{R}$ if, for all $x$ and all $t > 0$, $\lim_{x \to 0} \phi(tx)/\phi(x) = t^\rho$. If $\rho = 0$, the function $\phi$ is said slowly varying at zero. Let $\vartheta \in (0, \pi)$, $0 < \delta < 1/2$, $\Delta < \delta$. We say that the linear filter $(\psi_j)_{j \in \mathbb{Z}}$ belongs to the set $\mathcal{F}(\vartheta, \delta, \Delta, \mu)$ if $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$ and if there exists $d \in [\Delta, \delta]$ such that $\psi(\lambda)$ is regularly varying at zero with index $-d$ and that

$$\frac{\int_{0}^{\pi} \lambda^{2d} |\psi(\lambda)|^2 \, d\lambda}{\min_{0 \leq |\lambda| \leq \vartheta} \lambda^{2d} |\psi(\lambda)|^2} \leq \mu,$$

$$\forall j \geq 0, \quad \frac{|\psi_j| + \sum_{|\lambda| \geq j} |\psi_{\lambda+1} - \psi_{\lambda}|}{\min_{0 \leq \lambda \leq \vartheta} \lambda^{\delta} |\psi(\lambda)|} \leq \mu(1 + j)^{d-1},$$

An example is provided by $\psi(\lambda) \overset{\text{def}}{=} (1 - e^{i\lambda})^{-d}$ the transfer function of the causal fractional integration filter, $\psi_t = \Gamma(t + d)/\Gamma(d)\Gamma(t + 1)$, $t \geq 0$.

**Local-to-zero assumptions.** We first consider local-to-zero assumptions for which nothing is required outside a neighborhood of the zero frequency, apart from integrability of the spectral density (see Robinson 1995b). For $\beta > 0$, we say that the sequence $(\psi_j)_{j \in \mathbb{Z}}$ belongs to the set $\mathcal{F}_{\text{local}}(\vartheta, \beta, \delta, \Delta, \mu)$ if $(\psi_j)_{j \in \mathbb{Z}} \in \mathcal{F}(\vartheta, \delta, \Delta, \mu)$ and

$$\forall \lambda \in (0, \vartheta), \quad \frac{|\psi^*(\lambda) - \psi^*(0)|}{\min_{\lambda \in [0, \vartheta]} |\psi^*(\lambda)|} \leq \mu\lambda^\beta$$

with $\psi^*(\lambda) = (1 - e^{i\lambda})^d\psi(\lambda)$ where $d$ is the index of regular variation of $\psi$. This class is quite general and includes the impulse response of FARIMA filters [see Doukhan, Oppenheim, and Taqqu, 2002, and the references therein] but also processes whose spectral density may exhibit singularity outside the zero frequency, such as the Gegenbauer’s processes. As seen below, under local-to-zero assumptions, the validity of the Edgeworth expansion can only be established for the DFT coefficients in a degenerating neighborhood of zero frequency. This is enough for, say, semi-parametric estimation of the long-memory index by the GPH method.
Global assumptions. In some situations, it is possible to formulate regularity assumptions over full the frequency range [−π, π] or a subset of it. These assumptions allow to prove the validity of the Edgeworth expansion for all the frequency ordinate. We say that the sequence \((\psi_j)\) belongs to the set \(\mathcal{F}_{\text{global}}(\vartheta, \beta, \delta, \Delta, \mu)\) if \((\psi_j) \in \mathcal{F}_{\text{local}}(\vartheta, \beta, \delta, \Delta, \mu)\) and if in addition, for all \((\lambda, \lambda') \in (0, \vartheta) \times (0, \vartheta)\),

\[
|\psi^*(\lambda) - \psi^*(\lambda')| \leq \mu |\psi^*(\lambda)| \vee |\psi^*(\lambda')| |\lambda - \lambda'| \tag{4.4}
\]

Under those assumptions and as in the short-memory case, we are able to prove the validity of the Edgeworth expansion for the DFT’s (Theorem 6) and deduce some moment bounds (Corollaries 7, 9 and 10). In comparison with short memory results, note that tapering \((\text{Corollaries 7, 9 and 10})\) of the Edgeworth expansion for the DFT’s (Theorem 6) and deduce some moment bounds \(\mathcal{F}\) under long-range dependence and for fixed \(k\), the limiting covariance matrix of \(S_n(k)\) fully depends on \(k\) and not only on \((k_2 - k_1, \ldots, k_0 - k_{n-1})\). This behavior at “very-low frequencies” as been studied for instance by Hurvich and Beltrao [1993]. However, one can control the covariance of the standardized DFT coefficients and then the difference \(V_n(k) - V(k)\) thanks to the following lemma.

Theorem 6. Assume \((\text{A1})\) with some integer \(s \geq 3\), \(p \geq 1\) and \(p' \geq s\). Let \(r\) be a positive integer and \(\beta, \delta, \Delta, \mu, \vartheta\) be constants such that \(0 < \delta < 1/2\), \(-r + 1/2 < \Delta \leq 0\), \(\mu > 0\) and \(\vartheta < 0\). Let \((m_n)_{n \geq 0}\) be a non-decreasing sequence. Assume either

\[
(\psi_j)_{j \in \mathbb{Z}} \in \mathcal{F}_{\text{local}}(\vartheta, \beta, \delta, \Delta, \mu) \quad \text{and} \quad \lim_{n \to \infty} \left( \frac{1}{m_n} + \frac{m_n}{n} \right) = 0 \tag{4.5}
\]

or

\[
(\psi_j)_{j \in \mathbb{Z}} \in \mathcal{F}_{\text{global}}(\vartheta, \beta, \delta, \Delta, \mu) \quad \text{and} \quad m_n \leq \vartheta n. \tag{4.6}
\]

Then there exist a constant \(C\) and positive integers \(K_0\), \(N_0\) which depends only on \(\vartheta, \beta, \delta, \Delta, \mu\), the distribution of \(Z_1\) and the sequence \((m_n)\), such that for any \(n \geq N_0\) and \(k = (k_1, \ldots, k_u)\) of integers in the range \(\{K_0, \ldots, m_n\}\), the distribution of \(S_n(k)\) has a density \(q_{n,k}\) with respect to Lebesgue measure on \(\mathbb{R}^{2u}\) which satisfies

\[
\sup_{x \in \mathbb{R}^{2u}} (1 + \|x\|^s) |q_{n,k}(x) - \sum_{r=0}^{s-3} P_r(x, V_n(k), \{\chi_{n,\nu}(k)\})| \leq C n^{-(s-2)/2}. \tag{4.7}
\]

If \(u = 1\), one can take \(K_0 = 1\).

Integrating some function \(g\) against the density \(q_{n,k}\) and using (4.7) yields the following corollary.

Corollary 7. Under the assumptions of Theorem 6, there exists a constant \(C\) and an integer \(N\) depending only on \(\vartheta, \beta, \delta, \Delta, \mu, u, r\) and such that, for all \(u\)-tuple of distinct integers \(k\) satisfying \(K_0 \leq \min(k)\), \(\max(k) \leq m_n\) and any \(n \geq N\), and all measurable function \(g\) such that \(N_s(g) < \infty\),

\[
\left| \mathbb{E}[g(S_n(k))] - \sum_{r=0}^{s-3} \int_{\mathbb{R}^{2u}} g(x) P_r(x, V_n(k), \{\chi_{n,\nu}(k)\}) \, dx \right| \leq C N_s(g) n^{-(s-2)/2}. \tag{4.8}
\]

Similarly to the short-memory case, one could approximate \(\mathbb{E}[g(S_n(k))]\) using the limiting distribution of \(S_n(k)\) in place of the Gaussian approximation as Corollary 7. Under long-range dependence and for fixed \(k\), the limiting covariance matrix of \(S_n(k)\) fully depends on \(k\) and not only on \((k_2 - k_1, \ldots, k_0 - k_{n-1})\). This behavior at “very-low frequencies” as been studied for instance by Hurvich and Beltrao [1993]. However, one can control the covariance of the standardized DFT coefficients and then the difference \(V_n(k) - V(k)\) thanks to the following lemma.
4.2.1. Theoretical results. A very widely used estimator of the memory parameter $d$ was introduced by Geweke and Porter-Hudak [1983]. It is obtained from the linear regression of the log-periodogram of the observations using the logarithm of the frequencies as explanatory variable. In contrast with the Whittle estimator, the GPH is defined explicitly in terms of the log-periodogram ordinates, see Eq. (4.11) below. Many theoretical work has been achieved on this estimator, in stationary or non-stationary contexts (see Faÿ, Moulines, Rouna, and Taqqu [2008] for a survey of the main results). For instance, Giraitis et al. [1997] proved that the GPH of Gaussian $X$ is rate optimal for the quadratic risk and over some classes of spectral densities that is included in our $\mathcal{F}_{\text{local}}$. To compute the risk of the GPH estimator, one need to compute or approximate moments of the log-periodogram. The log-periodogram is a non-smooth function of the Fourier transform of the observation, which are Gaussian if $X$ is Gaussian. The proof by Giraitis et al. [1997] relies on moment bounds of non-linear function of Gaussian variables [see Arcones, 1994, Soulier, 2001]; this technique does not extend
naturally to non-Gaussian time series. Here, we shall apply the Edgeworth approximations obtained in preceding section to extend this result to the case of strong sense linear process. For the sake of simplicity of exposition, we only consider a taper of order \( r = 1 \) and write \( I_k = I_{1,n,k} \). The GPH estimator is obtained by an ordinary least square regression of \( \log(\lambda_k) \) on \( \log(2\sin(\lambda_k/2)) \) [see Geweke and Porter-Hudak [1983], Robinson [1995]]. With the frequency spacing and taper order \( r, \) one regresses on every \( r + 1 \) frequency. For \( r = 1 \) it writes

\[
(\hat{d}_m, \hat{C}) = \arg \min_{d',C'} \left\{ \sum_{k=1}^{m} \left[ \log(I_{2k+1}) + 2d' \log(2\sin(\lambda_{2k+1}/2)) - C' \right]^2 \right\},
\]

where \( m = m(n) \) is a bandwidth parameter. Explicitly

\[
\hat{d}_m = s_m^{-2} \sum_{k=1}^{m} \nu_k \log(I_{2k+1}), \quad (4.11)
\]

with \( \nu_k = -2 \left( \log(2\sin(\lambda_{2k+1}/2)) - \frac{1}{m} \sum_{j=1}^{m} \log(2\sin(\lambda_j/2)) \right) \) and \( s_m^2 = \sum_{k=1}^{m} \nu_k^2 \). We consider the mean square error (MSE) of the GPH estimator. Theorem 11 gives a bound on the MSE which is uniform over a class of long-range dependent linear processes, from which rate optimality can be deduced.

**Theorem 11.** Under the assumptions of Theorem \( 4 \) with \( s \geq 5 \) and conditions \( 4.5 \), there exists a constant \( C \) which depends only on \( \beta, \delta, \Delta, \vartheta, \mu \) and the distribution of \( Z_1 \) such that

\[
E[(\hat{d}_m - d)^2] \leq C \left\{ \left( \frac{m^2}{n} \right)^{2\beta} + \frac{1}{m} \right\}.
\]

With \( m \) proportional to \( n^{2\beta/(2\beta+1)} \), \( E[(\hat{d}_m - d)^2] \leq Cn^{-2\beta/(2\beta+1)}. \)

**Remark.** The condition \( s \geq 5 \) seems slightly stronger than necessary for bounding the MSE of \( \hat{d} \). But it is allows the function \( h(x_1, \ldots, x_4) = g(x_1, x_2)g(x_3, x_4) \) with \( g(x) = \log(||x||^2) - \bar{\eta} \) to have finite \( N_s(h) \) norm (see Corollary 4 and the remark that follows.

### 4.2.2. Monte Carlo results.
Assuming more stringent global condition on the regularity of the spectral density allows one to evaluate the bias term in the decomposition of the mean squared error. For comparison, using the specific set of assumptions Hurvich, Deo, and Brodsky [1998], we can prove that the leading terms in the MSE are of the form \( am^4/n^4 + b/m \) for bandwidth such that \( \lim_{m \to \infty} 1/m + m \log(m)/n = 0 \). The constant \( a \) and \( b \) can be made uniform in the class of spectral densities under consideration. It shows that the MSE of the GPH estimator is asymptotically insensitive to the distribution of the innovation as soon as this distribution satisfies some moment and regularity conditions. Finite sample implications of this statement is illustrated here by the results of a Monte Carlo study. For sample sizes \( n = 250, 500, 1000, 2500, 5000 \), we have simulated 1000 realizations of a FARIMA(1, \( d \), 0) processes defined by

\[
(1 - B)^{0.3} (1 - 0.3B) X_t = Z_t
\]

where \( B \) is the back-shift operator and \( (Z_t)_{t \in \mathbb{Z}} \) is a zero mean unit variance i.i.d sequence with the following marginal distributions (a) Gaussian (b) Laplacian (c) zero-mean (shifted) Pareto, with

\[
\mathbb{P}(Z_0 \leq u) = (1 - (u + 7/6)^{-7})1_{u \geq -7/6}.
\]
Whereas it is possible to simulate exactly a Gaussian FARIMA \((p, d, q)\) process (e.g. computing the covariance structure and using Levinson-Durbin algorithm), there is no general way to do it for non Gaussian processes. In the Monte-Carlo experiment, the process \((X_t)\) is obtained using a truncated MA(\(\infty\)) representation. For each realization of each process, we evaluate the squared error \((\hat{d_m} - d)\) and define the Monte Carlo MSE as the average of those errors. We have focused on the sensitivity with respect to the distribution of \(Z\) of the bandwidth \(m\) which is optimal in the MSE sense. Figure 1 and Table 1 show that for sample size \(n = 250\) the MSE is minimized at \(m = 37\) or 38 which means that the optimal bandwidth is about the same for those three linear processes. Figure 2 represents the box-and-whiskers plot of the GPH estimator for two different sample sizes and the three models we are concerned with. Here again, the sensitivity with respect to the distribution of the driving noise is hardly discernible. In Table 1 we displayed the value of the bias and of the mean square error of the GPH at this estimated optimal bandwidth.

**Figure 1.** Comparisons of the MSE versus the bandwidth for the FARIMA processes (a),(b) and (c). Sample size \(n = 250\)

**Figure 2.** Box-plot of the GPH estimator for processes (a),(b) and (c), sample size \(n = 250, 2500\)

**Appendix A. Edgeworth expansion for triangular arrays**

In this section we recall the theorem established in [Faï, Moulines, and Soulier 2004]. Let \((Z_t)_{t \in \mathbb{Z}}\) be an i.i.d sequence and \((U_{n,j})_{j \in \mathbb{Z}, n \in \mathbb{N}}\) an array of vectors in \(\mathbb{R}^u\), where \(u\) is an integer.
Define \( S_n = \sum_{j \in \mathbb{Z}} U_{n,j} Z_j \) and let \( V_n = \sum_{j \in \mathbb{Z}} U_{n,j} U'_{n,j} \). For \( \nu \in \mathbb{N}^s \), \( 2 \leq |\nu| \leq s \), denote \( \chi_{n,\nu} \) the cumulants of \( S_n \). Then \( \chi_{n,\nu} = \kappa_r \sum_{j \in \mathbb{Z}} U_{n,j}^{\nu} \), where \( \kappa_r \) denotes the \( r \)-th cumulant of \( Z_1 \), \( r \leq s \). Consider the following assumptions.

(B1) There exist positive constants \( v_* \) and \( v^* \) such that

\[
 v_* \leq \lim \inf_{n} v_{\min}[V_n] \leq \lim \sup_{n} v_{\max}[V_n] \leq v^* 
\]

where \( v_{\min}[V_n] \) (resp. \( v_{\max}[V_n] \)) is the smallest (resp. the largest) eigenvalue of \( V_n \).

(B2) There exist positive constants \( \eta, c_0 \), a sequence \( (M_n)_{n \in \mathbb{N}} \) of positive numbers, and a sequence \( (J_n)_{n \in \mathbb{N}} \) of subsets of \( \mathbb{Z} \), such that, for all \( n \geq 0 \)

\[
 \sup_{j \in \mathbb{Z}} \|U_{n,j}\| \leq M_n \quad \lim_{n \to \infty} M_n = 0 \quad \text{card}(J_n) \leq c_0 M_n^{-2} \quad \text{and} \quad \sum_{j \in J_n} \|U_{n,j}\|^2 \geq \eta. \tag{A.1}
\]

(B3) There exist \( \zeta \geq 1 \) and a sequence \( (M_n)_{n \in \mathbb{N}} \) satisfying \( (A.1) \) such that

\[
 \sup_{n \geq 0} M_n^\zeta \sum_{j \in \mathbb{Z}} \|U_{n,j}\| < \infty. \tag{A.2}
\]

Theorem 12. (Faÿ, Moulines, and Soulier, 2004). Let \( s \geq 3 \), and \( p' \geq 0 \) be integers and \( p \geq 1 \) be a real number. Assume \( (A.1) \) \((s,p,p')\), \( (B.1) \) and \( (B.2) \). Assume in addition either \( (B.3) \) or \( p' \geq s \) in \( (A.1) \) \((s,p,p')\). Then, there exist a constant \( C \) and an integer \( N \) (depending only on the distribution of \( Z_1 \), and the constants appearing in the assumptions) such that, for all
Then by (B.4), there exists a linear combination of \( a \) and \( c \) with respect to \( \psi \).

\[
\sup_{x \in \mathbb{R}^n} (1 + \|x\|^s) |q_n(x) - \sum_{r=0}^{s-3} P_r(x, V_n, \chi_{n, r})| \leq C \sum_{j \in \mathbb{Z}} \|U_{n,j}\|^s \quad (A.4)
\]

**Appendix B. Proof of Theorem [1]**

The proof consists in checking that assumptions (B.1), (B.2) and (B.3) hold uniformly with respect to \( \psi \in G(\alpha, \beta, \delta) \) and \( k \) for \( U_{n,j} \)'s of the form \( \psi_{n,j} \). To prove (B.1), write \( V_n(k) = V(k) + W_n(k) \), with \( V(k) \) defined in (3.5). Define \( \|W\|_1 = \max_{1 \leq i \leq v} \sum_{j=1}^{v} |w_{i,j}| \) for any matrix \( W = (w_{i,j})_{1 \leq i,j \leq v} \). Similarly to Hannan [1960, p. 54], we have under (3.1)

\[
\|W_n(k)\|_1 \leq C(\alpha, \beta)n^{-1}. \quad (B.1)
\]

The matrices \( V(k) \) have the following algebraic property.

**Lemma 13.** There exist two positive constants \( v_* \) and \( v^* \) such that

\[
2v_* \leq \inf v_{\min}[V(k)] \leq \sup v_{\max}[V(k)] \leq 2v^* \quad (B.2)
\]

where the infimum and supremum are taken over all the \( u \)-tuples of distinct integers in \( \mathbb{N}^u \).

**Proof.** Noting that trace\([V(k)] = u,\]

\[
v_{\max}[V(k)] \leq \text{trace}[V(k)]/2u = 1/2. \quad (B.3)
\]

Take \( v^* = 1/4 \). Recall that \( k_1 < \cdots < k_u \). Note that, for any \( n \geq 2k_u + 2r + 1 \), \( V(k) \) is the covariance matrix of

\[
\sqrt{2\pi}(c_{r,n,k_1}, s_{r,n,k_1}, \ldots, c_{r,n,k_u}, s_{r,n,k_u})
\]

with \( c_{r,n,k} = (2\pi a_r n)^{-1/2} \sum_{t=1}^{n} h_{t,n} r Y_t \cos(t \lambda_k) \) and \( s_{r,n,k} = (2\pi a_r n)^{-1/2} \sum_{t=1}^{n} h_{t,n} r Y_t \sin(t \lambda_k) \) the sine and cosine transform of a unit-variance zero-mean Gaussian white noise \( (Y_n)_{n \in \mathbb{Z}} \).

Recall that

\[
c_{r,n,k} = a_r^{-1/2} \sum_{l=0}^{r} (-1)^l \binom{r}{l} c_{0,n,k+l} \quad \text{and} \quad s_{r,n,k} = a_r^{-1/2} \sum_{l=0}^{r} (-1)^l \binom{r}{l} s_{0,n,k+l}. \quad (B.4)
\]

The random variables \( c_{0,n,k} \) and \( s_{0,n,k}, k = 1, \ldots, [(n-1)/2] \) are centered i.i.d Gaussian with variance \( 1/4\pi \). Assume that \( V(k) \) is not invertible. It yields that for some \( 2u \)-tuple of reals \((\alpha_1, \beta_1, \ldots, \alpha_u, \beta_u) \neq (0, 0, \cdots, 0, 0),\)

\[
\sum_{j=1}^{u} (\alpha_j c_{r,n,k_j} + \beta_j s_{r,n,k_j})^2 = 0.
\]

Then by (B.4), there exists a linear combination of \( c_{0,n,k} \)'s and \( s_{0,n,k} \)'s that is equal to zero. \( c_{0,n,k_u+r} \) and \( s_{0,n,k_u+r} \) appear in this combination with coefficients \( a_r^{-1/2}(-1)^r \alpha_u \) and \( a_r^{-1/2}(-1)^r \beta_u \), respectively. It follows from the independence and non-degeneracy of the \( c_{0,n,k} \)'s and \( s_{0,n,k} \)'s that \( \alpha_u = \beta_u = 0 \). Iterating the argument yields the contradiction \( \alpha_u = \beta_u = \alpha_{u-1} = \beta_{u-1} = \cdots = \alpha_1 = \beta_1 = 0 \). Thus for any \( u \)-tuple \( k \) of distinct integers

\[
v_{\min}[V(k)] > 0. \quad (B.5)
\]
It remains to prove that $v_{\min}[V(k)]$ is bounded away from zero uniformly in $k$. Define

$$K_u = \{k = (k_1, \cdots, k_u) \in \mathbb{N}^u, 1 \leq u' \leq u, 0 < k_{i+1} - k_i \leq r\}.$$  

Note now that by (3.5), $v_{\min}[V(k)]$ is a function of the vector $(k_2 - k_1, k_3 - k_2, \cdots, k_u - k_{u-1})$ thus taking finitely many different values on $K_u$. From the this remark and (3.5),

$$v_1 \overset{\text{def}}{=} \inf_{k \in K_u} v_{\min}[V(k)] > 0$$  

(B.6)

since the infimum is taken on a finite set of positive values. Consider now a $u$-tuple $k$ that does not belong to $K_u$; In this case, for some $i \in \{1, \cdots, u - 1\}$, $k_{i+1} - k_i > r$, and then $k$ may be partitioned as $L \geq 2$ blocks of indexes $(k_1, \ldots, k_L)$ such that all the $k_i$’s belong to $K_u$ and, for all $i \in \{1, \cdots, L - 1\}$, $\min k_{i+1} - \max k_i > r$. Let $l_i$ denotes the length of the block $k_i, i = 1, \cdots, L$. By this construction and (3.5), the matrix $V(k)$ has a block-diagonal structure

$$V(k) = \begin{pmatrix}
V(k_1) & 0 \\
0 & \ddots \\
& & V(k_L)
\end{pmatrix}.$$  

Using (B.6),

$$v_{\min}[V(k)](v_{\max}[V(k)])^{2u-1} \geq \det[V(k)] = \prod_{i=1}^{L} \det[V(k_i)] \geq \prod_{i=1}^{L} v_1^{2l_i} = v_1^{2u}.$$  

(B.7)

We conclude from (B.7) and (B.3) that

$$v_{\min}[V(k)] \geq v_1^{2u} > 0.$$  

(B.8)

(B.2) follows from (B.6) and (B.8) with $v_* = \frac{1}{2} \min (v_1, v_2)$. □

Proof of Theorem 1. By (B.1) and Lemma 13 (B1) holds with $v_*$ and $v^*$ of Lemma 13 for some $N_0, n \geq N_0$ and uniformly in $k, \alpha$ and $\beta$. With (B.3),

$$\sum_{j \in \mathbb{Z}} \|U_{n,j}(k)\|^2 - u = |\text{trace}[V_n(k)] - u| \leq C(\alpha, \beta)n^{-1/2}.$$  

(B.9)

Prove now that (B.2) is verified. Since $f$ is bounded away from zero and $\sum_{j \in \mathbb{Z}} |\psi_j| \leq \beta < \infty$, (A.1) and (A.2) are verified with $M_n \overset{\text{def}}{=} C(r)\beta \alpha^{-1/2} n^{-1/2}$. Put $J_n = \{j, |j| < 2n\}$. Then $\text{card}(J_n) \leq c_0 M_n^{-2}$ for some $c_0$ depending only on $r, \alpha, \beta$ and

$$\sum_{|j| \in \mathbb{Z} \backslash J_n} \|U_{n,j}(k)\|^2 \leq C(r)\alpha^{-2} n^{-1} \sum_{|j| \geq 2n} \left( \sum_{t=1}^{n} |\psi_{t+j}| \right)^2 \leq C(r)\alpha^{-2} \sum_{|j| \geq 2n} \sum_{t=1}^{n} |\psi_{t+j}| \leq C(r)\alpha^{-2} \sum_{|j| \geq n} |j| \psi_j^2.$$  

(B.10)

Under (A.1), $|\psi_j| \leq \beta |j|^{-1/2-\delta}$ so that

$$\sum_{|j| \geq n} |j| \psi_j^2 \leq \beta n^{-2\delta} \sum_{|j| \geq n} |j|^{1/2+\delta} |\psi_j| \leq \beta^2 n^{-2\delta}.$$  

(B.11)
For any $\epsilon > 0$ and large enough $n$, $\sum_{j | j \geq n} \|U_{n,j}(k)\|^2 \leq \epsilon$ uniformly in $k$ and $\psi \in \mathcal{G}(\alpha, \beta, \delta)$. (A.3) follows from (B.9), (B.10) and (B.11). Finally,

$$\sum_{j \in \mathbb{Z}} \|U_{n,j}(k)\| \leq C(r)\alpha^{-1/2} n^{-1/2} \sum_{t=1}^{n} |\psi_{t+j}| = C(r)\alpha^{-1/2} n^{1/2} \sum_{j \in \mathbb{Z}} |\psi_j|$$

so that (B.3) holds with $\zeta = 1$. \hfill \Box

**Appendix C. Proof of Lemma 8**

The proof is an adaptation of Lang and Soulier [2002] to fit our need of uniformity of the bounds with respect to the function $\psi$ whether it belongs to $\mathcal{F}_{\text{global}}$ or $\mathcal{F}_{\text{local}}$ only. For sake of brevity, the proof is omitted and we refer the interested reader to their paper. It derives from their more general analytical lemma that we recall here.

**Lemma 14 [Lang and Soulier [2002]].** Let $q \in \mathbb{N}$, $K \geq 1$, $\vartheta \in (0, \pi]$. Let $\psi$ be an integrable function on $[-\pi, \pi]$, such that for all $x \in (0, \vartheta) \setminus \{0\}$, $\psi(-x) = \overline{\psi}(x)$ and

$$|\psi(x) - \psi(y)| \leq K \frac{|\psi(x)| + |\psi(y)|}{x \wedge y} |x - y|, \text{ for all } (x, y) \in (0, \vartheta) \times (0, \vartheta) \quad (C.1)$$

Assume that $|\psi|$ is regularly varying at zero with index $\rho$ and that

$$\psi(x) = x^\rho c(x) \exp\left\{ \int_x^\vartheta \frac{g(s)}{s} ds \right\}$$

with (i) $\lim_{x \to 0} g(x) = 0$; (ii) $\lim_{x \to 0} c(x)$ exists in $(0, \infty)$. Let $D_n$ be such that for $x \in [-\pi, \pi]$,

$$|D_n(x)| \leq C \frac{n^{1/2}}{(1 + n|x|)^{q+1}}. \quad (C.2)$$

- If $\rho \in (-1, 2q + 1)$, there exists a constant $C$ such that, for all $n \geq 1$ and all $k$ such that $0 < x_k \leq \vartheta/2$,

  $$\left| \int_{-\pi}^{\pi} \left( \frac{\psi(x)}{\psi(x_k)} - 1 \right) |D_n(x_k - x)|^2 dx \right| \leq C \log^{\nu(q)}(k)^k.$$ \quad (C.3)

  with $\nu(0) = 1$ and $\nu(q) = 0$ if $q \geq 1$.

- If $\rho \in (-1/2, q + 1/2)$, there exists a constant $C$ such that, for all $n \geq 1$ and all integers $j$, such that $0 < x_k \neq x_j \leq \vartheta/2$,

  $$\left| \int_{-\pi}^{\pi} \left( \frac{\psi(x)}{\psi(x_k)} - 1 \right) D_n(x_k - x) D_n(x - x_j) dx \right|$$

  $$\leq \left\{ \begin{array}{ll}
    C(1 + |\psi(x)/\psi(x_k)|) |k - j|^{-q(j \vee k)} & \leq C(jk)^{-1/2} \quad (q > 0); \\
    C(jk)^{-1/2} \log(j \vee k) & \quad (q = 0).
  \end{array} \right. \quad (C.4)$$

- For any $\beta > 0$, if $\rho \in (-1, 2q + 1)$, for any integer $k$ such that $0 < x_k \leq \vartheta/2$,

  $$\left| \int_{-\pi}^{\pi} \frac{\psi(x)}{\psi(x_k)} |D_n(x_k - x)|^2 dx \right| \leq C \left\{ k^{-2q-1} + (k/n)^\beta \right\}. \quad (C.5)$$
Applying Lemma 8, we obtain
\[
\frac{\lambda_\beta}{\psi(x_k)} D_n(x_k - x) D_n(x_j - x) \, dx
\]
There exist integers
\[
F
\]
As in Appendix B, we put \(\tilde{\varphi}\). Proof. Using (2.3) and (4.1), we get
\[
\text{where} \quad \tilde{\varphi}, \beta, \delta, \Delta
\]
For (D.2), it remains to prove that for any integer \(k\), \(1 \leq k \leq m_n\), \(v_\ast \leq v_\min[V_n(k)]\) ; (D.2)
\[
\text{(3) for all u-tuple k of distinct integers, } K_0 \leq k \leq m_n,
\]
\[
v_\ast \leq v_\min[V_n(k)].
\]
Proof. As in Appendix [D] we put \(V_n(k) = V(k) + W_n(k)\) where \(V(k)\) is defined in (3.5). Applying Lemma 8 we obtain
\[
\|W_n(k)\|_1 \leq C(\vartheta, \beta, \delta, \Delta, \mu)
\]
\[
\text{(D.1) follows immediately. The proof of (D.3) follows by picking } N_0, K_0 \text{ large enough.}
\]
\[
\text{For (D.2), it remains to prove that for any integer } k, 1 \leq k \leq K_0, \ V_n(k) \text{ converges to a positive definite matrix } \tilde{V}(k)\text{ and that this convergence is uniform w.r.t to } \psi, \text{ for } \psi \in \mathcal{F}_{0, \text{local}}(\vartheta, \beta, \delta, \Delta, \mu) \text{ or } \psi \in \mathcal{F}_{0, \text{global}}(\vartheta, \beta, \delta, \Delta, \mu). \text{ What follows is an adaptation of [Touditsky, Moulines, and Soulier, 2001, Lemma 7.3].}\n\]
\[
\mathbb{E}[\omega_{r,n,k}^2] = \frac{1}{f(\lambda_k)} 
\left( \int_{|\lambda| \leq \vartheta \pi} + \int_{|\lambda| > \vartheta \pi} \right) |D_{r,n}(\lambda - \lambda_k)|^2 f(\lambda) \, d\lambda =: A_1 + A_2
\]
where \(D_{r,n}\) is defined in (2.2). For \(n \geq 4\pi K_0 / \vartheta\), \(1 \leq k \leq K_0\) and \(|\lambda| \geq \vartheta \pi, |n(\lambda - \lambda_k)| \geq n\vartheta / 2\). Using (2.3) and (4.1), we get
\[
A_2 \leq \frac{C \lambda_k^2}{\lambda_k f(\lambda_k)} n^{-2r - 1} \int_{|\lambda| > \vartheta \pi} \lambda^{2d} f(\lambda) \, d\lambda \leq C n^{-2r}
\]
By change of variable,
\[
A_1 = \frac{n^{2d} |1 - e^{-i\lambda_k}|^{2d}}{\int_{|\lambda| \leq \vartheta \pi} n^{-1/2} D_{r,n}(\lambda / n - \lambda_k) |2 n^{-2d} |1 - e^{-i\lambda / n}|^{-2d} f_\ast(\lambda / n) \, d\lambda.}
\]
Write \( \lim_{n \to \infty} n^{-1/2} D_{r,n}(\lambda/n) = \frac{1}{\sqrt{2\pi}r} \int_0^1 (1 - e^{2\pi is})^r e^{-i\lambda s} ds =: \hat{h}_r(\lambda). \) By Riemann approximation, it can be seen that \( |n^{-1/2} D_{r,n}(\lambda/n) - \hat{h}_r(\lambda)| \leq C(1 + |\lambda|)/n. \) Note also that \( |\hat{h}_r(\lambda)| \leq C|\lambda|^{-r-1}. \) Then
\[
\left| A_1 - \frac{n^{2d}|1 - e^{i\lambda_k}|^{2d}}{f^*(\lambda_k)} \times \int_{-n^\theta}^{n^\theta} |\hat{h}_r(\lambda - 2\pi k)|^2 n^{-2d}|1 - e^{i\lambda/n}|^{-2d} f^*(\lambda/n) d\lambda \right|
\leq C k^{2d} n^{-r} \leq C n^{-r}. \tag{D.7}
\]
Here and in the following, \( C \) is a generic constant which depends only on \( \vartheta, \beta, \delta, \Delta, \mu, r \) and \( K_0. \) For \( |\lambda| \leq n^\vartheta, \) using (4.5),
\[
\frac{f^*(0)}{f^*(\lambda_k)} \left| n^{-2d}|1 - e^{i\lambda/n}|^{-2d} - |\lambda|^{-2d} \right| + \frac{\left| f^*(\lambda/n) - f^*(0) \right|}{f^*(\lambda_k)} n^{-2d}|1 - e^{i\lambda/n}|^{-2d}
\leq C \left\{ \left| n^{-2d}|1 - e^{i\lambda/n}|^{-2d} - |\lambda|^{-2d} \right| + n^{-2d}|1 - e^{i\lambda/n}|^{-2d} \left| \frac{\lambda}{n} \right|^{\beta'} \right\}. \tag{D.8}
\]
with \( \beta' = \beta \wedge 1. \) For \( x \in [-\pi, \pi], \frac{2}{\vartheta} |x| \leq |e^{ix} - 1| = 2 \sin \frac{\pi x}{2} \leq |x| \) and \( ||e^{ix} - 1| - |x|| \leq x^2/2. \) Also, for any \( \nu \in \mathbb{R}, x > 0, y > 0, |x^\nu - y^\nu| \leq |\nu|(x^{\nu-1} \vee y^{\nu-1})|x - y|. \) Using those relations, write, for \( \lambda \in [-n\pi, n\pi],
\[
\left| n^{-2d}|1 - e^{i\lambda/n}|^{-2d} - |\lambda|^{-2d} \right| \leq C n^{-2d} \left| \frac{\lambda}{n} \right|^{-2d-1} \left| 1 - e^{i\lambda/n} \right| - \left| \frac{\lambda}{n} \right|
\leq C n^{-1} |\lambda|^{-2d+1}
\]
Then
\[
\int_{-n^\theta}^{n^\theta} |\hat{h}_r(2\pi k - \lambda)|^2 n^{-2d}|1 - e^{i\lambda/n}|^{-2d} - |\lambda|^{-2d} d\lambda
\leq C n^{-1} \int_{-n^\theta}^{n^\theta} |\hat{h}_r(2\pi k - \lambda)|^2 |\lambda|^{-2d+1} d\lambda
\leq C n^{-1} \int_{-n^\theta}^{n^\theta} |\hat{h}_r(2\pi k - \lambda)|^2 |\lambda|^{2r} d\lambda \leq C n^{-1} \tag{D.9}
\]
and
\[
\int_{-n^\theta}^{n^\theta} |\hat{h}_r(2\pi k - \lambda)|^2 n^{-2d}|1 - e^{i\lambda/n}|^{-2d} \left( \frac{\lambda}{n} \right)^{\beta'} d\lambda
\leq n^{-\beta'} \int_{-n^\theta}^{n^\theta} |\hat{h}_r(2\pi k - \lambda)|^2 |\lambda|^{-2d+\beta'} d\lambda
\leq C n^{-\beta'}. \tag{D.10}
\]
Gathering (D.3), (D.6), (D.7), (D.8), (D.9), (D.10) yields
\[
\left| E[|\omega_{r,n,k}|^2] - \frac{(2\pi)^{2d} k^2 f^*(0)}{f^*(\lambda_k)} \int_{-\infty}^{+\infty} |\hat{h}_r(\lambda - 2\pi k)|^2 |\lambda|^{-2d} d\lambda \right| \leq C n^{-\beta'}
\]
Similar arguments leads to
\[
\left| E[\omega_{r,n,k}^2] - \frac{(2\pi)^{2d} k^2 f^*(0)}{f^*(\lambda_k)} \int_{-\infty}^{+\infty} h_r(\lambda - 2\pi k) \hat{h}_r(\lambda + 2\pi k) |\lambda|^{-2d} d\lambda \right| \leq C n^{-\beta'}/n.
Defining the scalar product \((u, v)_d = \int_{\mathbb{R}} u(\lambda)v(\lambda)|\lambda|^{-2d}d\lambda\), then det \(V_n(k)\) is uniformly approximated by the Gram determinant of the functions \(\hat{h}_r(\lambda-2k\pi)\) and \(\hat{h}_r(\lambda+2k\pi)\) associated with the product \((\cdot, \cdot)_d\) and then is a continuous function of \(\eta_k(d) := \lim_{n \to \infty} E[|\omega_{r,n,k}|^2]\) and \(\eta'_k(d) := \lim_{n \to \infty} E[|\omega'_{r,n,k}|^2]\). The whole set of functions \(\hat{h}_r(\lambda+2j\pi), j \in \mathbb{Z}\) is linearly independent, so that those determinant are positive. Using continuity of \(\eta_k\) and \(\eta'_k\) w.r.t. \(d\), the infimum on the compact set \([-\Delta, \delta]\) and the minimum over \(k = 1, \ldots, K_0\) is positive too, which concludes the proof. \(\square\)

**Lemma 16.** There exists a constant \(C\) (depending only on \(\vartheta, \beta, \delta, \Delta, \mu, r\)) such that for all \(k \in \{1, \ldots, \tilde{n}\}\),

\[
\frac{1}{\sqrt{n}f(\lambda_k)} \left| \sum_{t=1}^{n} h_{t,n}^r \psi_{t+j} e^{it\lambda_k} \right| \leq C n^{-1/2}.
\]

(D.11)

**Proof.** The main tool of the proof is the bound (2.3) and the technique are the same as the one used in the proof of Lemma 8. Decompose

\[
|\psi(\lambda_k)|^{-1} \frac{1}{\sqrt{2\pi a_{n}} n} \sum_{t=1}^{n} h_{t,n}^r \psi_{t+j} e^{it\lambda_k} = |\psi(\lambda_k)|^{-1} \int_{-\pi}^{\pi} \psi(\lambda) e^{ij\lambda} D_{n,r}(\lambda_k - \lambda) d\lambda
\]

into

\[
A_1 = |\psi(\lambda_k)|^{-1} \left( \int_{-\pi}^{-\vartheta} + \int_{\vartheta}^{\pi} \right) \psi(\lambda) e^{ij\lambda} D_{n,r}(\lambda_k - \lambda) d\lambda,
\]

\[
A_2 = |\psi(\lambda_k)|^{-1} \psi(0) \int_{-\vartheta}^{\vartheta} (1 - e^{i\lambda})^{-d} e^{ij\lambda} D_{n,r}(\lambda_k - \lambda) d\lambda,
\]

\[
A_3 = |\psi(\lambda_k)|^{-1} \int_{-\vartheta}^{\vartheta} (1 - e^{i\lambda})^{-d} (\psi'(\lambda) - \psi(0)) e^{ij\lambda} D_{n,r}(\lambda_k - \lambda) d\lambda.
\]

By Eq. (2.3), if \(|\lambda| \in [\vartheta, \pi]\), \(|D_{n,r}(\lambda_k - \lambda)| \leq C n^{-1/2-r}\). Note that \(n^{-1} \lambda_k^d = n^{-1} \lambda_k^{-1} \lambda_k^{d+1} \leq 1/(2\pi k)\). (4.11) implies that \(|A_1| \leq C n^{1/2-r} \leq C n^{-1/2}\). Consider \(A_2\). Since \(\int_{-\pi}^{\pi} D_{n,r}(\lambda) d\lambda = 0\),

\[
A_2 = \int_{-\vartheta}^{\vartheta} \Delta(\lambda, \lambda_k) D_{n,r}(\lambda_k - \lambda) d\lambda, \quad \Delta(\lambda, \lambda_k) = \left( (1 - e^{i\lambda})^{-d} - (1 - e^{i\lambda_k})^{-d} \right) e^{ij\lambda} |\psi(\lambda_k)|^{-1}.
\]

Decompose this integral on the intervals \([-\vartheta, -\lambda_k/2], [-\lambda_k/2, \lambda_k/2], [\lambda_k/2, 2\lambda_k]\) and \([2\lambda_k, \vartheta]\). If \(\lambda \in [\lambda_k/2, \lambda_k/2]\), then \(|D_{n,r}(\lambda_k - \lambda)| \leq C \sqrt{n}k^{-r-1}\) and \(|\Delta(\lambda, \lambda_k)| \leq C (|\lambda|^{-d} \lambda_k^{d} + 1)\). Hence:

\[
\left| \int_{-\lambda_k/2}^{\lambda_k/2} \Delta(\lambda, \lambda_k) D_{n,r}(\lambda_k - \lambda) d\lambda \right| \leq C k^{-r} n^{-1/2}.
\]

If \(\lambda \in [\lambda_k/2, 2\lambda_k]\), then \(|\Delta(\lambda, \lambda_k)| \leq C \left( \lambda_k^{-1} |\lambda - \lambda_k| + 1 \right)\). Since \(\int_{-\lambda_k/2}^{\lambda_k/2} (1 + n|\lambda|)^{-r-1} d\lambda \leq C n^{-1}\), we have

\[
\left| \int_{\lambda_k/2}^{2\lambda_k} \Delta(\lambda, \lambda_k) D_{n,r}(\lambda_k - \lambda) d\lambda \right| \leq C n^{-1/2}.
\]
Lemma 17. There exists a constant $C$ (depending only on $\vartheta$, $\beta$, $\delta$, $\Delta$, $\mu, r$) such that for all $k \in \{1, \ldots, \tilde{n}\}$,

$$\frac{1}{\sqrt{n} f(\lambda_k)} \left| \sum_{i=1}^{n} h_{t,n}^{r} \psi_{t+j} e^{it\lambda_k} \right| \leq Cn^{-1/2} \lambda_k^{d-1} (1 + |j|)^{d-1} \leq Cn^{-1/2}((1 + |j|)/n)^{d-1}. \quad (D.12)$$

Proof. By applying the definition of the weights $h_{t,n}^{r}$ and summation by parts, we have:

$$\sum_{i=1}^{n} h_{t,n}^{r} \psi_{t+j} e^{it\lambda_k} = \sum_{p=0}^{r} (-1)^{p} \binom{r}{p} \sum_{t=1}^{n} \psi_{t+j} e^{it(\lambda + \lambda_p)},$$

$$\sum_{i=1}^{n} \psi_{t+j} e^{it\lambda_k} = \sum_{t=1}^{n-1} \left\{ \left( \sum_{u=1}^{t} e^{iu\lambda_k} \right) (\psi_{t+j} - \psi_{t+j+1}) + \left( \sum_{u=1}^{n} e^{iu\lambda_k} \right) \psi_{n+j} \right\}.$$

For all $y \in (0, \pi)$ and all $\ell \in \mathbb{N}^*$, $\left| \sum_{y=1}^{\ell} e^{iuy} \right| \leq 2/y$. The proof follows from condition $\Delta.$

Proceed now with the proof of Theorem 6. If $|j| \geq n$, then $((1 + |j|)/n)^{d-1} \leq 1$. Hence by Lemma 16, for some constant $C$ which depends only on $\beta$, $\delta$, $\Delta$, $\vartheta$, $\mu, r$ and the distribution of $Z_1$,

$$\forall j, n, k, \quad M_{n,j} \overset{\text{def}}{=} Cn^{-1/2} \left( 1 \wedge ((1 + |j|)/n)^{d-1} \right) \geq \|U_{n,j}(k)\|.$$

Note that

$$M_n \overset{\text{def}}{=} \sup_{j \in \mathbb{Z}} M_{n,j} = Cn^{-1/2}.$$
Then (A.1) and (A.2) hold uniformly in $k$. By Lemma 15, Eq. (D.3) or (D.2), we have
\[ \sum_j \|U_{n,j}(k)\|^2 = \text{trace}[V_n(k)] \geq v_\ast > 0. \]

Finally, define for any $\gamma \geq 1$ the set $J_n = \{ j \in \mathbb{Z}, |j| \leq \gamma n \}$. Then $\text{card}(J_n) \leq c_0 M_n^{-2}$ and
\[ \frac{\sum_{j \in \mathbb{Z}} \|U_{n,j}(k)\|^2}{\sum_{j \in \mathbb{Z}} \|U_{n,j}(k)\|^2} \leq C'(v_\ast)^{-1} n^{1-2\delta} \sum_{|j| \geq \gamma n} j^{2\delta-2} \leq C'(v_\ast)^{-1} n^{1-2\delta-1}. \]

Choosing $\gamma$ large enough yields (A.3) uniformly.

**Appendix E. Proofs of Corollaries 3, 4, 9 and 10**

**Proof of Corollary 3.** By the triangle inequality, the LHS of inequality (3.7) is bounded by
\[ \left| \mathbb{E}[g(S_n(k))] - \int_{\mathbb{R}^{2u}} g(x) \varphi_{V_n(k)}(x) \, dx \right| + \int_{\mathbb{R}^{2u}} g(x) \{ \varphi_{V_n(k)}(x) - \varphi_{V(k)}(x) \} \, dx. \]

By Corollary 2 with $s = 3$, the first term of the previous display is bounded by $Cn^{-1/2} N_3(g)$. For $A$ a matrix, denote $\rho(A)$ its spectral radius. Denote $I_a$ the $a$-dimensional identity matrix. To bound the second term, note that $\rho(V_n(k) - V(k)) \leq C(\alpha, \beta)n^{-1}$ by (E.1) and that $\tau(g, V(k)) \geq 1$ by definition, then apply the following lemma which is an easy adaptation of Soulier [2001, Theorem 2.1].

**Lemma 18.** Let $\Gamma$ be a $u$-dimensional positive matrix. There exists $\epsilon > 0$ and a constant $C$ such that, for all symmetric positive matrix $\Gamma'$ verifying $\rho(\Gamma'^{-1} - \Gamma^{-1}) < \epsilon$, and for all measurable functions $g$ on $\mathbb{R}^u$ satisfying $\|g\|^2 < \infty$, we have
\[ \left| \int_{\mathbb{R}^u} g(x) \{ \varphi_{\Gamma'}(x) - \varphi_{\Gamma}(x) \} \, dx \right| \leq C \rho^{\tau(g, \Gamma')/2}(\Gamma' - \Gamma) \|g\|_\Gamma. \]

**Proof of Corollary 4.** The LHS of (3.9) is bounded by $A_1 + A_2 + A_3 + A_4$ with
\[ A_1 = \left| \mathbb{E}[g(S_n(k))] - \int_{\mathbb{R}^{2u}} g(x) \sum_{r=0}^{s-3} P_r(x, V_n(k), \{\chi_{n,\nu}(k)\}) \right|, \]
\[ A_2 = \left| \int_{\mathbb{R}^{2u}} g(x) \{ \varphi_{V_n(k)}(x) - \varphi_{I_{2u}/2}(x) \} \, dx \right|, \]
\[ A_3 = \left| \int_{\mathbb{R}^{2u}} g(x) P_1(x, V_n(k), \{\chi_{n,\nu}(k)\}) \, dx \right|, \]
\[ A_4 = \left| \int_{\mathbb{R}^{2u}} g(x) \sum_{r=2}^{s-3} P_r(x, V_n(k), \{\chi_{n,\nu}(k)\}) \, dx \right|. \]

$A_4 = 0$ if $s = 4$. Using (3.8), we get $\tau(g, I_{2u}/2) = 2$. It follows, as in the proof of Corollary 3 that $A_2$ is bounded by $Cn^{-1}\|g\|_{V(k)}$, whereas $A_1$ is bounded by $Cn^{-(s-2)/2} N_s(g)$. Write shortly $P_r(x, V_n(k), \{\chi_{n,\nu}(k)\}) = R_r(x)\varphi_{V_n(k)}$, where $R_r$ is a polynomial of order $r + 2$ (the dependence w.r.t $V_n(k)$ and $\{\chi_{n,\nu}(k)\}$ is omitted in this notation). Also note that
\[ |\chi_{n,\nu}(k)| \leq |\kappa_{|\nu|}| \sum_{j \in \mathbb{Z}} \|U_{n,j}(k)\|^{\nu} \leq |\kappa_{|\nu|}| M_n^{\nu-2} \left( \sum_{j \in \mathbb{Z}} \|U_{n,j}(k)\|^2 \right) \leq |\kappa_{|\nu|}| M_n^{\nu-2} \text{trace}[V_n(k)] \leq C|\kappa_{|\nu|}| M_n^{\nu-2} \quad \text{(E.1)} \]
where $M_n \leq C(\alpha, \beta)n^{-1/2}$. Then, the coefficients of $R_r$ are $O(n^{-r/2})$ uniformly in $k$ and $\psi$ since they involve $\lambda_{k,\nu}(k)$’s with $|\nu| = r$ and elements of $V_n^{-1}(k)$ [for details, see Bhattacharya and Radhakrishna 1976]. Let $F_n(k) \equiv (V_n^{-1}(k) - V^{-1}(k))/2$ and write
\[
\int g(x)\varphi_{V_n(k)}(x)R_r(x)\, dx = \left| \frac{\det V(k)}{\det V_n(k)} \right|^{1/2} \int g(x)R_r(x)\exp\{-x'F_n(k)x\}\varphi_{V(k)}(x)\, dx
\] (E.2)
By (E.1), $\|F_n(k)\|_1 \leq Cn^{-1}$ and $|\det(V_n(k))|^{-1/2} - |\det(V(k))|^{-1/2}| \leq Cn^{-1}$ uniformly so that $A_4 \leq Cn^{-1}(1 + \|x\|^4)g(x)\|2u/3$. We can derive this way that $A_3 \leq Cn^{-1/2}$ which is not enough. Improving this bound requires some care and uses the symmetries of $g$. Actually, $R_1$ is a sum of polynomials which are odd with respect to one or three components. Write
\[
\exp\{-x'F_n(k)x\} - 1 + x'F_n(k)x \leq Cn^{-2}\|x\|^4 \exp\{Cn^{-1}\|x\|^2\}
\] (E.3)
and notice that $\{1 - x'F_n(k)x\}R_1(x)$ is a sum of polynomials of the form $\prod r_i(x_{2i-1}, x_{2i})$, each of them being odd with respect to at least one variable. Consider a typical term odd with respect to $x_1$, say. Using (E.5)
\[
\int_{\mathbb{R}^{2u}} g(x)\prod r_i(x_{2i-1}, x_{2i})\varphi_{I_{2u}/2}(x)\, dx = \int_{\mathbb{R}^2} g_1(x_1, x_2)r_1(x_1, x_2)\varphi_{I_2}(x_1, x_2)\, dx_1\, dx_2
\]
\[
\times \int_{\mathbb{R}^{2u-2}} \prod g_i(x_{2i-1}, x_{2i})r_i(x_{2i-1}, x_{2i})\varphi_{I_{2u-2}/2}(x)\, dx_3\cdots\, dx_{2u-2} = 0,
\]
since the first integral vanishes. Hence, $\int_{\mathbb{R}^4} h(x)R_1(x)(x'F_n(k)x)\varphi_{I_4}(x)\, dx = 0$. Gathering (E.1), (E.2) and (E.3), $A_3 \leq Cn^{-2}$.

**Proofs of Corollaries 3 and 4**. As those corollaries are the counterparts of Corollaries 3 and 4 in a long memory context, we only give the necessary adaptations from the preceding proofs. From Lemma 8, $\rho(V_n(k) - V(k)) \leq Cp(k, k, n, \beta)$, $\|F_n(k)\|_1 \leq Cp(k, j, n, \beta)$ and
\[
|\det(V_n(k))^{-1/2} - |\det(V(k))^{-1/2}| \leq Cp(k, j, n, \beta).
\]
The LHS of (E.3) is now bounded by $p^2(k, j, n, \beta)\|x\|^4 \exp\{Cp(k, j, n, \beta)\|x\|^2\}$. The term $A_3$ is then bounded by
\[
Cn^{-1/2}p^2(k, j, n, \beta) \int_{\mathbb{R}^4} \|x\|^5 h(x)\exp\{-\|x\|^2(1 + Cp(k, j, n, \beta))\}\, dx.
\]
If $m = o(n)$ and $K_1 > 2C$, then for large enough $n$ and $K_1 \leq k < j - r \leq m - r$, the integral is uniformly bounded. Thus $A_3 \leq Cn^{-1/2}p^2(k, j, n, \beta)$ whereas $A_1 \leq Cp^2(k, j, n, \beta)$.

**Appendix F. Proof of Theorem 11**

In the sequel, $C$ denotes a constant which depends only on $\beta$, $\delta$, $\vartheta$, $\mu$ and the distribution of $Z_1$ and whose value may change upon each appearance. Note first that $|\nu_k| = O(\log(k))$, $s_n^2/m \to C > 0$ (see for instance, Robinson 1995), or Hurvich, Deo, and Brodsky 1998). Define $f^*(\lambda) = |1 - e^{-i\lambda}|^{-2d} f(\lambda)$ and $L(\lambda) = \log(f^*(\lambda)/f^*(0))$. Since $\psi \in F(\vartheta, \beta, \delta, \Delta, \mu)$, there exists a constant $C$ such that
\[
\forall k \in \{1, \cdots, m\}, \quad |L(\lambda_k)| \leq C|\lambda_k|^\beta.
\] (F.1)
Let $\overline{\eta}$ denote $\mathbb{E}(\log \|Y\|^2)$ where $Y$ is a centered Gaussian random vector with covariance matrix $I_2/2$. Define $\eta_k = \log(I_k/f(\lambda_k)) - \overline{\eta}$, $1 \leq k \leq m$. With these notations and since $\sum_{k=1}^{m} \nu_k = 0$, (4.11) yields

$$d_m = d + s_m^{-2} \sum_{k=1}^{m} \nu_k \eta_k + s_m^{-2} \sum_{k=1}^{m} \nu_k L(\lambda_k) =: d + W_m + b_m. \tag{F.2}$$

The mean-square error of the GPH writes $\mathbb{E}((d_m - d)^2) = EW_m^2 + 2b_m \mathbb{E}W_m + b_m^2$. Applying (F.1) and the Cauchy-Schwartz inequality, we have

$$|b_m| \leq Cs_m^{-2} \sum_{k=1}^{m} \nu_k |\lambda_k|^\beta \leq C(m/n)^\beta. \tag{F.3}$$

Thus, to prove Theorem 11 we only need to show that $\mathbb{E}[W_m^2] \leq Cm^{-1}$. We now compute $\mathbb{E}[W_m^2]$. Let $\ell = \ell(m)$ be a non decreasing sequence of integers such that $1 \leq \ell \leq m$ and define $W_{1,m} = s_m^{-2} \sum_{k=1}^{\ell} \nu_k \eta_k$ and $W_{2,m} = W_m - W_{1,m}$. We first give a bound for $\mathbb{E}[W_{1,m}^2]$. Note that

$$\mathbb{E}[W_{1,m}^2] \leq \ell s_m^{-4} \sum_{k=1}^{\ell} \nu_k^2 \mathbb{E}[\eta_k^2]. \tag{F.4}$$

For $x \in \mathbb{R}^2$, define $g(x) = \log(||x||^2) - \overline{\eta}$. Then $\eta_k = g(S_{n,k})$ and $N_3(g^2) < \infty$. For $(x_1,\ldots,x_4) \in \mathbb{R}^2$, define $h(x_1,\ldots,x_4) = g(x_1,x_2)g(x_3,x_4)$. Then $\eta_k \eta_j = h(S_{n,k,j})$, $h$ has property (B.8) and

$$N_5(h) = \int_{\mathbb{R}^4} \frac{g(x_1,x_2)g(x_3,x_4)}{1 + ||x||^5} \leq 4(N_5/2(g)^2 \, dx < \infty$$

where we have used $4(1+(a^2+b^2)^{s/2}) \geq (1+|a|^{s/2})(1+|b|^{s/2})$. Note that $N_4(h) = N_2(g) = +\infty$, which motivates the expansion up to order $s = 5$. Let $\sigma^2 \overset{\text{def}}{=} \text{var}(\log \|Y\|^2) = \pi^2/6$. Applying Corollaries 9 and 10 respectively to the functions $g, h$, we get for some integer $l_0$ and any $k, j$ such that $l_0 \leq k < j \leq m$,

$$|\mathbb{E}[\eta_k^2] - \sigma^2| \leq C(\beta, \delta, \vartheta, \mu) \left\{ k^{-1} + (k/n)^\beta + n^{-1/2} \right\}. \tag{F.5}$$

$$|\mathbb{E}[\eta_k \eta_j]| \leq C(\beta, \delta, \vartheta, \mu) \left\{ k^{-2} + (j/n)^{2\beta} + n^{-1} \right\}. \tag{F.6}$$

(F.4) and (F.5) yield $\mathbb{E}[W_{1,m}^2] \leq C\ell^2 m^{-2}$. We now bound $\mathbb{E}[W_{2,m}^2]$: \n
$$\mathbb{E}[W_{2,m}^2] = s_m^{-4} \sum_{k=\ell+1}^{m} \nu_k^2 \mathbb{E}[\eta_k^2] + 2s_m^{-4} \sum_{\ell<k<j\leq m} \nu_k \nu_j \mathbb{E}[\eta_k \eta_j].$$

Using (F.5) and (F.6), we obtain

$$\mathbb{E}[W_{2,m}^2] - s_m^{-2} \sigma^2 \leq C(\beta, \delta, \vartheta, \mu) s_m^{-4} \sum_{k=\ell+1}^{m} \nu_k^2 \left\{ k^{-1} + (k/n)^\beta + n^{-1/2} \right\} + C(\beta, \delta, \vartheta, \mu) s_m^{-4} \sum_{\ell<k<j\leq m} \nu_k \nu_j \left\{ k^{-2} + (j/n)^{2\beta} + n^{-1} \right\} = C(\beta, \delta, \vartheta, \mu) s_m^{-2} \left\{ 1 + O \left( \ell^{-1/2} + m^{1/2} l^{-3/2} + m^{2\beta+1} n^{-2\beta} + m/n \right) \right\}. \tag{F.7}$$
Choosing $\ell \leq m$ such that $\ell^{2} = o(m)$ and $m = o(\ell^{3})$ (for instance $\ell = [m^{\eta}]$ with $1/3 < \eta < 1/2$) yields $E[W_{m}^{2}] = O(m^{-1})$. This bound and (F.3) conclude the proof of Theorem 11.

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