Evaluation of the stress-strain state of a twisted rod made from an anisotropic material in the shear direction at creep

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Abstract. Torsion of circular samples cut out in the longitudinal direction of a plate from a transversally-isotropic alloy with reduced resistance to creep in a direction at angle of 45° to the direction of the plate normal is simulated. To verify the calculations in the finite-element program, a lower and upper estimate of the torsion angle was obtained based on the principles of minimum of the total power and minimum of the additional dissipation power.

1. Introduction
Modern materials have the properties of anisotropy. That creates certain difficulties when processing and in predicting service life. The work [1] gives the results of experiments on creep at constant stresses on tension and compression of samples cut out in directions: longitudinal (along the rental), transverse plate normal and at an angle of 45° to the normal direction of the plate. These data indicate that the direction at an angle of 45° to the plate normal is the weakest direction for some alloys (V95pchT2, 1161, AK4-1).

The investigation of the problems of rods torsion in the conditions of anisotropic creep considered in [2, 3] is continued in the present paper. These works contain a description of the ANSYS Beam189 and Solid45 elements testing for calculation of circular rod. Since obtaining an analytical solution under anisotropic creep conditions even for a rod of circular cross-section is problematic, in order to estimate the reliability of finite element modeling results, the present work considers the simplest upper and lower bounds for the rate of torsion angle.

2. Theory
Considering torsion of a rod of arbitrary cross section with free ends, the creep strain rates are related to the stresses by the relations [2–4]:

\[
\dot{\gamma}_{23} = \eta_{23} = W_{12} + \theta x_1 = 2T^{-1}A_{23}\sigma_{23},
\]

\[
\dot{\gamma}_{13} = \eta_{13} = W_{13} - \theta x_2 = 2T^{-1}A_{31}\sigma_{13},
\]

where \(T = \left(2A_{23}\sigma_{23}^2 + 2A_{31}\sigma_{13}^2\right)^{0.5}\); \(W(x_1, x_2)\) is function corresponding to the displacement of cross-section points (warping) in the \(x_3\) direction along the rod and arising from the torsion; \(\theta\) is rate of torsion angle per unit length, \(n\) is a creep constant of the material. The coefficients of the quadratic form \(A_j\) are defined as follows [5]:

\[2A_{23} = 4B_{23}^m - A_{22} - A_{33},\quad 2A_{31} = 4B_{31}^m - A_{33} - A_{11},\quad 2A_{21} = 4B_{21}^m - A_{11} - A_{22}\]
\[2A_1 = B_{22}^m + B_{33}^m - B_{11}^m, \quad 2A_2 = B_{11}^m + B_{22}^m - B_{33}^m, \quad 2A_3 = B_{11}^m + B_{22}^m - B_{33}^m, \]

Here \( B_{11}, B_{22}, B_{33} \) are constants of one-dimensional creep in three main directions (1 - longitudinal, 2 - normal to the plate, 3 - transverse); \( B_{12}, B_{23}, B_{31} \) are similar constants in three directions along the axes in the coordinate system obtained by rotating the original coordinate system by \( 45^\circ \); \( m = 2(n + 1) \).

In the case of a transversally-isotropic alloy with a reduced resistance to creep at an angle of \( 45^\circ \) to the direction of the plate normal, we have

\[B_{11} = B_{22} = B_{33} = B_{31} = B_{12} = B_{23} = B_A.\]

From (1), the stress components are

\[\sigma_{23} = \frac{\eta_{23}}{2A_3}G^{(1-n)/n}, \quad \sigma_{13} = \frac{\eta_{13}}{2A_1}G^{(1-n)/n},\]

where \( G = \left(\frac{\eta_{13}}{2A_1} + \frac{\eta_{23}}{2A_2}\right)^{0.5}, \quad G = T^n.\)

The equation of equilibrium for a rod with free ends [4]

\[\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0.\]

Boundary conditions on the contour of the cross section are as follows

\[\sigma_{13}n_1 + \sigma_{23}n_2 = 0.\]

The compatibility condition for creep strain rates is transformed to the form

\[\frac{\partial}{\partial x_2}\left(2A_3 T^{n-1}\sigma_{13}\right) - \frac{\partial}{\partial x_1}\left(2A_2 T^{n-1}\sigma_{23}\right) = -2\theta .\]

Torque is

\[M = \iint_S (\sigma_{23} x_1 - \sigma_{13} x_2) dx_1 dx_2.\]

Considering torsion of a solid circular rod of isotropic material, the warping of section is missing and the solution can be obtained analytically. In the case of an arbitrary cross-section rod of isotropic material, various approximate methods are considered in [4], including the method of combining solutions of the elastic and ideal-plastic problems. Finite-element discretization and finite-difference approximation methods for solving the torsion problems of isotropic rods of arbitrary cross section are considered in [6–8]. A detailed analysis of torsion of anisotropic elastic rods is given in [9]. In particular, an analytic solution of Saint-Venant for the elliptical cross-section, a method for finding a solution by means of a functions of a complex variable, as well as approximate methods of solving by means of energy methods are considered. A cyclic creep torsion of thick-walled tubular specimens was analyzed in [10] by taking the creep-induced anisotropy into account.

The solution of torsion problem can be obtained by converting permitting ratios to the differential equation with respect to function of warping \( W(x_1, x_2) \) [6–7], or with respect to function of stresses \( F(x_1, x_2) \), such that \( \sigma_{13} = \hat{\partial} F/\hat{\partial} x_2, \quad \sigma_{23} = -\hat{\partial} F/\hat{\partial} x_1 \) [4, 9]. Then torque (8) is equal to

\[M = \iint_S \left(\frac{W_2 + \theta x_1}{2A_3} x_1 - \frac{W_1 - \theta x_2}{2A_3} x_2\right) \left(\frac{(W_2 + \theta x_1)^2}{2A_2} + \frac{(W_1 - \theta x_2)^2}{2A_1}\right)^{1/2} dx_1 dx_2\]

or

\[M = 2\iint_S F dx_1 dx_2,\]

where \( k_1 = (1-n)/n \). In both cases, for the rods of an arbitrary cross-section, the solution of the system (1)–(8) can be obtained using finite element method or finite-difference approximation. Approximate estimates can be used to verify these solutions.
For a lower estimate of the rate of torsion angle, we use the condition of minimum of the total power [4]:

\[
I_1 = \int_S \left( \frac{n}{n+1} G k_2 - \theta \left( \frac{x_1 f_2}{2A_{23}} - \frac{x_2 f_3}{2A_{31}} \right) G \right) S dx_1 dx_2 = \text{min},
\]

(11)

\[
k_2 = (n+1)/n.\]

Suppose that the function of warping similar to the case of anisotropic elasticity [9]

\[
W(x_1, x_2) = \theta W_0(x_1, x_2) = \theta c_1 x_1 x_2,
\]

(12)

where \( c_1 \) is a constant. Function (12) satisfies the equation of compatibility (7). Taking into account (1), (12) expression (11) is transformed to

\[
I_1 = \theta^{k_2} \int_S \left( \frac{1}{k_2} \left( \frac{x_1^2 (c_1 + 1)^2}{2A_{23}} + \frac{x_2^2 (c_1 - 1)^2}{2A_{31}} \right)^{\frac{k_2}{2}} - \left( \frac{x_1^2 (c_1 + 1)}{2A_{23}} - \frac{x_2^2 (c_1 - 1)}{2A_{31}} \right) \left( \frac{x_1^2 (c_1 + 1)^2}{2A_{23}} + \frac{x_2^2 (c_1 - 1)^2}{2A_{31}} \right)^{\frac{k_1/2}{2}} \right) dx_1 dx_2.
\]

(13)

Finding the constant \( c_1 \) at which the integral (13) achieves the minimum, we find the the rate of torsion angle from (9)

\[
\theta = \left( \frac{M}{R_1} \right)^n,
\]

(14)

where \( R_1 = 4 \int_0 a f(x_2) \left( \frac{x_1^2 (c_1 + 1)}{2A_{23}} - \frac{x_2^2 (c_1 - 1)}{2A_{31}} \right) \left( \frac{x_1^2 (c_1 + 1)^2}{2A_{23}} + \frac{x_2^2 (c_1 - 1)^2}{2A_{31}} \right)^{\frac{k_1/2}{2}} dx_1 dx_2,
\]

\[
f(x_2) = (a^2 - x_2^2)^{0.5},\]

\( a \) is the radius of the rod cross section.

For the upper estimate of the rate of torsion angle, we use the condition of minimum of the additional dissipation power [4, 9]:

\[
I_2 = \int_S \left( \frac{1}{n+1} T^{n+1} - 2\theta T \right) dx_1 dx_2 = \text{min}.
\]

(15)

Suppose that the stresses function has the form

\[
F(x_1, x_2) = c_2 T^{1/n} F_0(x_1, x_2) = c_2 T^{1/n} \left( 1 - \left( \frac{x_1^2 + x_2^2}{a^2} \right)^{k_2/2} \right),
\]

(16)

where \( c_2 \) is a constant. Function (15) up to a multiplier is the solution of torsion problem of circular rod made of an isotropic material under conditions of steady-state creep. This function satisfies the equation of equilibrium and boundary conditions. Substituting (16) into (15), from the equation \( dI_2 / dc_2 = 0 \) we find the constant at which (15) reaches a minimum:

\[
c_2 = \left( \frac{2 \int_S F_0 dx_1 dx_2}{\int_S (2A_{23} F_{0,1}^2 + 2A_{31} F_{0,2}^2)^{(n+1)/2} dx_1 dx_2} \right)^{1/n},
\]

(17)

Then taking (16), (17) into account, we find the the rate of torsion angle from (10)

\[
\theta = \left( \frac{M}{c_2 R_2} \right)^n,
\]

(18)

where \( R_2 = 8 \int_0^a f(x_2) F_0 dx_1 dx_2 \).
3. Results and discussion

Based on the experiments carried out for the V95pchT2 alloy (almost pure alloy V95 treated in mode T2) at $T = 180\degree$C, the weakest direction at an angle of 45° to the normal direction of the plate under creep conditions was determined and the following coefficients were obtained for the creep strain rates on tension [1]: $B_0 = 6.3 \cdot 10^{-31}$ MPa$^n$ s$^{-1}$ for longitudinal, transverse and the plate normal directions; $B_\perp = 3.9 \cdot 10^{-30}$ MPa$^n$ s$^{-1}$ for direction at the angle of 45° to the plate normal. The creep constant $n = 10$ is the same for all directions and does not depend on the sign of the applied load.

The calculations are performed with parameters: the radius of the rod $a = 0.02$ m, the length $L = 0.1$ m, torque $M = 3500$ Nm, the Poisson's ratio $\nu = 0.4$, Young's modulus $E = 55000$ MPa, the time of torsion $t = 600$ s. The longitudinal direction of the plate coincides with the direction of the $x_3$ axis of the rod. The direction of the normal to the plate coincides with the direction of the $x_2$ axis.

Table 1 for comparison includes the full angle of torsion $\varphi = Lt\theta$ (radians):

- calculated in ANSYS using the Beam189 and Solid45 (angle estimated by the displacement) elements without taking into account the initial elastic component;
- the approximate estimates (14), (18);
- the analytical solutions for a rod cut out from an isotropic material ($B_H = B_0$) [4]

$$\theta = \left(\sqrt{3}\right)^{n+1} \frac{B_0}{a} \left(\frac{3+1/n}{2\pi a^3 M}\right)^n, \quad (19)$$

- the analytical solutions for a rod cut out in the normal direction of a plate of transversally-isotropic alloy (3) [2–3]

$$\theta = \left(\frac{2A_{12}}{a}\right)^{(n+1)/2} \left(\frac{3+1/n}{2\pi a^3 M}\right)^n. \quad (20)$$

| type of symmetry       | direction of rod cut | angle (formula) | Beam189 | Solid45 |
|------------------------|----------------------|-----------------|---------|---------|
| isotropic              | any                  | 0.18 (19)       | 0.20    | 0.20    |
| transversally-isotropic| longitudinal         | 0.54 (14)/0.8 (18) | 0.53    | 0.72    |
| transversally-isotropic| normal               | 1.77 (20)       | 1.77    | 1.87    |

Work [3] gives the torsion angle for the mesh of Solid45 elements: 10 divisions along the radius, 40 along the boundary of the cross section (circle), 10 along the axis of the rod, a time step of 5 s in the steady-state creep. Work [2] gives the torsion angle for the mesh of Beam189 elements: 10; 32; 10; 5. Table 1 shows the angle values for the smaller mesh of Solid45 elements and time step: 16; 64; 20; 1. A mesh of Beam189 elements is the same, but the time step in the steady-state creep is reduced to 1 s. To apply torque load on one of the ends of the solid model, Shell181 elements were used, which creates a small edge effect and gives an uneven twist angle. Therefore, the angle was calculated from the displacements of the points near the end, where the shell elements are absent.

Table 2 shows the values of the maximum stresses calculated by different methods for a rod cut out longitudinally from a transversally-isotropic plate. If $c_1 = 0$ in (14), then there is no warping.

For a rod cut out from a material under the assumption of isotropy, the stresses are $\sigma_{13} = \sigma_{23} = 217.7$ MPa for Solid45 and $\sigma_{13} = \sigma_{23} = 215.6$ MPa for the Beam189. For a rod cut out in the normal direction of a plate of transversally-isotropic material, stresses are $\sigma_{13} = \sigma_{23} = 216.1$ MPa for Solid45 and $\sigma_{13} = \sigma_{23} = 215.4$ MPa for the Beam189.
Table 2. The maximum stresses $\sigma_{13}, \sigma_{23}$ and full angle of torsion $\phi$ for a rod cut out longitudinally from a transversally-isotropic plate.

| value          | formula (14), $I_1 = \min$ | formula (14), $c_1 = 0$ | formula (18), $I_2 = \min$ | Beam189 | Solid45 |
|----------------|-----------------------------|--------------------------|-----------------------------|---------|---------|
| $\sigma_{13}$ (MPa) | 238.2                      | 240.1                   | 215.9                      | 239.6   | 238.3   |
| $\sigma_{23}$ (MPa) | 194.3                      | 190.4                   | 215.9                      | 190.2   | 201.9   |
| $\phi$ (radian)  | 0.54                       | 0.51                    | 0.8                        | 0.53    | 0.72    |

Figure 1a and Figure 1b show the isolines of the stresses $\sigma_{13}$ and $\sigma_{23}$ at the torsion of rod cut out in the longitudinal direction $x_3$ (coincide with Z) made of an alloy with the reduced resistance to creep strain in the direction 45° to the normal direction of plate $x_2$ (coincide with Y).

Figure 1. Stresses isolines (MPa) of a rod cut out in the longitudinal direction from a transversally-isotropic plate calculated by the Beam189 element $\sigma_{13}$ (a), $\sigma_{23}$ (b).

It should be noted that the calculations using Solid45 for small mesh takes considerable time. Given this, the methods proposed can be useful for a preliminary analysis of loads.

4. Conclusion
The torsion of circular rods cut out in the longitudinal direction of a plate from a transversally isotropic alloy with reduced resistance to creep deformation in the direction at an angle of 45° to the direction of the plate normal is considered. The obtained lower and upper estimates of the torsion angle rate based on the principles of minimum of the total power and minimum of the additional dissipation power describe the results of numerical simulation with a good accuracy. The developed technique can be applied to the problems of torsion of rods of arbitrary cross section for predicting loads and estimating of stresses.

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