Emerging locality of network influence

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Many complex systems exhibit a natural hierarchy in which elements can be ranked according to a notion of “influence”. Examples include the position of prey and predators in a food chain (so-called trophic levels), or of manufactured goods in a production chain (leading to the notion of upstreamness). Finding the most “influential” nodes is key to understand the functioning and robustness of networked systems. The influence a node exerts on its neighborhood is an intrinsically non-local concept: it depends self-consistently on the influence exerted by all other nodes on their respective neighborhoods. Therefore, the complete and accurate knowledge of the interactions between constituents is ordinarily required for its computation. Using a low-rank approximation, we show instead that in a variety of contexts, only local information about the neighborhoods of nodes is enough to reliably estimate how influential they are, without the need to infer or reconstruct the whole map of interactions. We show that our framework is successful in approximating with high accuracy different incarnations of influence in systems as diverse as the WWW PageRank, trophic levels of ecosystems, input-output tables of complex economies, and centrality measures of social networks. We also discuss the implications of this “emerging locality” on the approximate calculation of non-linear network observables.

INTRODUCTION

Understanding how complex systems are internally organized and function collectively is a central question in many areas of science, with applications as diverse as hubs detection in epidemiology \cite{1,2} and the understanding of protein-protein interaction networks in biological applications \cite{3}, to the study of systemic risk in financial networks \cite{4,5}. One of the most relevant issues is how constituents – for example, nodes in a complex networks – can be efficiently ranked according to some meaningful local or global metric \cite{6}. The simplest local observable is the degree \( k_i \) of node \( i \), which is a useful – albeit simplistic – way to assess how “important” a node is based only on the number and strength of its immediate contacts. Global measures, such as different types of centrality, take a more holistic view of a node in the context of how fast information can travel through it from and towards the outer regions of the network. The downside is that they are usually more difficult to compute accurately, and are very sensitive to minute details of how the network is organized \cite{7}. Moreover, there is now quite some evidence that many local and global centrality measures are typically correlated with the local degree \cite{8,9,10,11} despite their premise of global character.

In the context of quintessentially global measures, a prominent way to classify and rank nodes is based on an umbrella notion that we will refer to as influence, which can be defined self-consistently across the whole network. The idea is that a node \( i \)’s influence should be slightly larger – on average – than that of the nodes “influenced” by \( i \), i.e. the nodes \( j \) in its neighborhood \( \partial i \). In formulae,

\[ D_i = 1 + \langle D_{j \in \partial i} \rangle . \quad (1) \]

In theoretical ecology, for instance, species forming a complex ecosystem can be assigned a natural “influence” (trophic) level in the food-chain hierarchy, precisely computed using \cite{12}: apex predators are high-up in the food chain because they either eat top-level predators themselves, or because they consume a large number of medium- or low-level species. The notion of influence as defined in \cite{12} appears however in a much broader range of incarnations: the PageRank algorithm used by Google \cite{13} classifies web pages as very relevant if they are linked to by other very relevant addresses, or by very many other pages. In economics, industrial sectors of different countries can be ranked based on their position and importance in the production chain using so-called upstreamness metrics, which measure the distance of a product from final consumption \cite{14,15,16}. The Katz centrality \( K_i \) of node \( i \) – which is a weighted sum of paths traveling through \( i \) \cite{17} – can be itself cast in the form of \cite{12}, as well as the Mean First Passage Time of a walker from a source to a target node \cite{13,18,19}. These examples further support the notion that the “influence” framework is in principle a very general and intuitive way to address the ranking problem on networks from a global perspective.

Influence is indeed an intrinsically non-local quantity, which depends on the full set of interactions in the network. Rearranging slightly \cite{12}, we may rewrite the influence vector \( \mathbf{D} \) of the \( N \) nodes as

\[ \mathbf{D} = (\mathbb{1} - \mathbf{A})^{-1} \mathbb{1} , \quad (2) \]

where \( \mathbb{1} \) is the identity matrix, \( 1 \) is a vector of all ones, and \( \mathbf{A} \) is the interaction matrix: depending on the circumstances, it may simply be the weighted adjacency matrix of the network (denoted by \( \mathbf{A} \) in the following), or a simple function of it that encodes pairwise interactions between constituents.
The matrix $G(A) = (I - A)^{-1}$ appearing in (2) is referred to as the resolvent of $A$. In many interesting examples – where some form of dissipation or external contribution is involved – the matrix $A$ is non-negative and sub-stochastic ($0 \leq \sum_j a_{ij} \leq 1$): each entry may represent the fraction of calories a species contributes to the diet of another in an ecosystem, or the fraction of money an industrial sector injects into the economic cycle.

The seemingly harmless (2) has however three important drawbacks: (i) it requires the inversion of a possibly large and ill-conditioned matrix, which makes a numerical approach computationally intensive and prone to inaccuracies [21], (ii) the nonlinear relation between the influence and the interaction matrix $A$ makes it difficult to infer the functional dependence of the former on network parameters (e.g. the mean degree) from the knowledge of the latter – unless the matrix possess symmetries or a specific structure, and (iii) it requires the complete and accurate knowledge of all the pairwise interactions strengths in the network (encoded in $A$), which may be not realistically achieved in many real-life settings.

For instance, reconstructing the food web interaction tables is a notoriously difficult task [22], which can be accomplished with some acceptable accuracy only for very small systems, comprising not more than a handful of species [23]. In economic studies, the compilation of Input/Output tables of developed countries – unraveling the complex web of interconnections between industrial and financial sectors – is routinely plagued with all sorts of sampling and surveying errors [24] [25]. Calculating the ranking of constituents (e.g., webpages) in large systems is computationally expensive and the classical PageRank algorithm is difficult to directly implement and optimize in distributed computation frameworks such as Spark [26], therefore fostering the quest for faster approximate routes [27]. Having a tool to efficiently and accurately rank constituents of a system we know very little about would therefore come in very handy.

In this paper, we indeed show that – in a variety of important contexts – the influence attributes of a node $i$ in a complex network can be determined with high accuracy using only local information. In particular, the detailed knowledge of the complete state of the network is often irrelevant. This emerging locality of network influence stems from the observation that a non-negative and sub-stochastic interaction matrix $A$ often displays a “large” Perron-Frobenius eigenvalue, which is well-separated from the bulk of all the others. When this happens, $A$ can be faithfully approximated by a rank-1 matrix $\hat{A}$, which retains only some local information stored in $A$.

As long as the network of interactions is “not too sparse”, this approximation dramatically reduces the amount of detailed information needed to rank constituents as fast as possible, without compromising accuracy. Moreover, it makes it possible to accurately approximate all sorts of non-linear functions of the interaction matrix, as we demonstrate below in the case of exponential and power-law centrality measures.

**RANK-1 APPROXIMATION FOR INFLUENCE**

Consider the resolvent $G(A) = (I - A)^{-1}$ of the interaction matrix $A$ of a system of interest, and imagine that a detailed knowledge of all the entries of $A$ is not available. We are only able to accurately estimate $2N$ constants $r = (r_1, \ldots, r_N)$ and $c = (c_1, \ldots, c_N)$, namely the sums of the $N$ rows and columns of $A$. This scenario is extremely common: in economics, often only aggregate information is available about sectorial outputs, or inter-banking exposure [28], and the same happens in the study of ecosystems where often only the overall “energy balance” of each species can be accurately estimated [29].

In this scenario, we may construct a simple rank-1 approximation $\hat{A}$ for the matrix $A$ as follows

$$\hat{A} = \frac{1}{N} gg^T = \begin{pmatrix} \frac{g_1 g_1}{N} & \cdots & \frac{g_N g_N}{N} \\ \vdots & \ddots & \vdots \\ \frac{g_1 g_N}{N} & \cdots & \frac{g_N g_N}{N} \end{pmatrix},$$

where the entries of the column vectors $g = (g_1, \ldots, g_N)$ and $q = (q_1, \ldots, q_N)$ can be determined by solving a set of equations imposing the constraint that $A$ and $\hat{A}$ share the same row and column sums

$$r_i = \sum_j A_{ij} \equiv \sum_k g_k \frac{q_i}{N} = \bar{g} q_i ,$$

$$c_j = \sum_i A_{ij} \equiv \sum_k g_k \frac{q_j}{N} = \bar{g} q_j .$$

This yields eventually the unique matrix (see Supporting Information)

$$\hat{A} = \frac{1}{mN} re^T ,$$

with $m = \frac{1}{N} \sum_{ij} A_{ij} = \frac{1}{N} \sum_j c_j = \frac{1}{N} \sum_i r_i$. The rank-1 matrix $\hat{A}$ in (6) is nothing but the Maximum Entropy reconstructed matrix (see e.g. [30] [31]) subject to the row and column constraints in (4) and (5) - see Supporting Information. In the case when $A$ is the unweighted adjacency matrix of a network, replacing $\hat{A}$ with its rank-1 counterpart $\hat{A}$ corresponds to the annealed network approximation [32] [33]. Also, the same result could be interpreted as a first-order expansion of the standard fitness model [34] [35].

If the only information we have is about row sums, then the corresponding rank-1 matrix is

$$\hat{A} = \begin{pmatrix} \frac{r_1}{N} & \cdots & \frac{r_1}{N} \\ \vdots & \ddots & \vdots \\ \frac{r_N}{N} & \cdots & \frac{r_N}{N} \end{pmatrix} .$$

Approximating the resolvent - or Green’s function - with a low-rank matrix to speed up computations is a customary tool in hard-core condensed matter [36] [37] and fluid dynamics [38], while being far less common in the context of complexity studies. It is indeed well known [39] that matrix inversion scales as $N^3$ in the worst case scenario of a structure-less dense
transformation (Maslov-Sneppen [47, 48]) that preserve the degree sequence while drastically altering the topology, and we show that indeed the Katz centrality of nodes is preserved in the not-too-sparse regime.

It is interesting to compare the rank-1 approximation $G(\hat{A})$ for the resolvent $G(A)$ in (9) with an alternative approach, which is possibly the simplest approximation/reconstruction scheme one could adopt. The resolvent $G(A)$ can be formally expanded in a power series as

$$G(A) = \mathbb{I} + A + A^2 + \ldots,$$

(11)

which admits an interesting interpretation e.g., in the economical context: the first contribution accounts for the direct increase in output of all industrial sectors that is necessary to meet an increase in final demand. The second contribution accounts for the increase in output that is needed to meet the increment in input required by all sectors to meet the increase in final demand. This chain of $k$-th order effects is encoded in the $k$-th term of the expansion, and unravels the technological interdependence of the productive system within an economy. The crudest approximation to the resolvent consists in truncating the above expansion at a given finite order. In Fig. 1 we show that this approximation is usually inferior to the rank-1 formula (9) in “spectrally-gapped” cases [19], unless a large number of terms is retained. Moreover, a direct construction of the “best” $k$-th order approximant:

$$G_k(A) = \mathbb{I} + \sum_{\ell=1}^{k} A^{\ell}$$

(12)

starting from the knowledge of row and column sums alone is not an obvious task.

In the next section, we spell out in more detail a few incarnations of the influence relation (2), before putting the approximate formula (9) to the test on synthetic and empirical networks.

**NOTABLE EXAMPLES OF INFLUENCE**

In social networks, the influence of nodes can be assessed via the Katz centrality [17], which takes into account the total number of walks between a pair of agents. The Katz centrality computes the relative influence of a node within a network by measuring the number of its immediate neighbors (first degree nodes) and also all other nodes in the network that connect to the node under consideration through these immediate neighbors. Connections made with distant neighbors are, however, penalized by an attenuation factor $\alpha < 1$. In formulae, the Katz centrality of the $i$-th node of a network of size $N$ described by the $(0,1)$-adjacency matrix $\hat{A}$ is given by

$$K_i = \sum_{\ell=1}^{\infty} \sum_{j=1}^{N} \alpha^\ell (\hat{A}^\ell)_{ij}.$$

(13)

It measures the total number of paths of given length reaching $i$ from any other node, where paths of length $\ell$ are given a

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1. The spectral gap is defined as $\lambda_1 = \max\{\lambda_2, \ldots, \lambda_{N-1}\}$, with $\lambda_1$ real and $\lambda_1 < 1$ is the Perron-Frobenius eigenvalue.

2. The out-strength $k_i^{(out)}$ of node $i$ is equal to $\sum_j \hat{A}_{ij}$, whereas the in-strength $k_i^{(in)}$ is equal to $\sum_j \hat{A}_{ji}$. If the network is unweighted, the strengths reduce to the out-degree and in-degree, respectively.
The trophic level of species $i$ is defined as

$$\mathcal{T}_i = 1 + \sum_j S_{ij} \mathcal{T}_j,$$

or if expressed in the typical “influence” form

$$\mathcal{T}(A) = (I - A)^{-1} \mathbf{1},$$

with $A \equiv S$. This definition indeed implies that $\mathcal{T}_i = 1$ – the lowest possible value – is reserved to species $i$ such that $A_{ij} = 0$ for all $j$, i.e. those who occupy the lowest level of the food chain.

Finally, in the context of complex economies and so-called input-output models [54], measures of influence are used to determine the role and importance of industrial sectors within an economy [44]. We can define the so-called upstreamness [12,15] as follows:

$$\mathcal{L}(A) = (I - A)^{-1} \mathbf{1},$$

where the matrix $A_{ij}$ is the dollar amount of sector $i$’s output that is needed to produce one dollar of $j$’s output. The upstreamness of an industry, or financial sector, is used to assess their relative position in the value production chain [15], and their relation to economic growth [16].

We are now ready to put the approximate, local formulae [9] and [10] for each node’s influence to the test: in the next section, we assess how well the formulae perform on network synthetic data (in the context of Katz centrality), as well as on different incarnations of the “influence” framework in the context of economics, social networks, relevance of web pages, and ecology – as described above.

**TEST OF THE FORMULA ON DATA**

In this section, we will test how the rank-1 formula with single [10] and double constraints [9] performs on synthetic network data and on four sources of empirical data, considering different incarnations of the influence ranking. Tests include computing the Katz centrality on synthetic Erdős-Rényi (ER) and Scale Free (SF) networks, measuring the so-called upstreamness, i.e., inter-sectoral dependencies in complex economies, Katz centrality of users of social networks, relevance of web-pages using PageRank, and trophic levels of species within an ecosystem.

**Katz centrality of synthetic networks.** We first test the rank-1 formula on synthetic data. We consider two types of networks in Fig. 2 (i) directed Erdős-Rényi (ER) random graphs, with $p$ being the probability of having an edge between any two nodes of the graph and (ii) scale-free (SF) graphs with a power-law degree distribution.

We use $\alpha = 0.3/\bar{d}$, where $\bar{d}$ is the maximum eigenvalue. For Erdős-Rényi graphs, $p = \langle k \rangle/(N - 1)$, with $\langle k \rangle$ the mean degree, which can be also used as a measure of sparsity for Scale Free graphs. A scale free graph is defined by the degree distribution $P(k) \propto k^{-\gamma}$ (here we use $\gamma = 2.2$), and can be discounted weight by a factor $\alpha^\ell$. Eq. (13) can be cast in the “influence” form [2] as

$$\mathcal{K}(A) = (I - A)^{-1} \mathbf{1} - \mathbf{1},$$

with $A = \alpha \tilde{A}$.

Similarly, the PageRank algorithm used by Google [12] returns rankings $\mathcal{R}$ of $N$ websites based on the assumption that more “important” websites are likely to receive more links from other websites. Given a web environment composed of $N$ sites, one forms the re-scaled adjacency matrix $\tilde{A}$ is such a way that $\tilde{A}_{ij}$ is the ratio between number of links pointing from page $j$ to page $i$ to the total number of outbound links of page $j$. Then one defines the damping factor $d$ as the probability that an imaginary surfer who is randomly clicking on links will keep doing so at any given step (typically, $d \simeq 0.85$). Given these ingredients, the ranking of pages follows the formula

$$\mathcal{R}(A) = \frac{1 - d}{N} (I - A)^{-1} \mathbf{1},$$

where $A = d \tilde{A}$.

Influence measures can also be introduced in the context of hierarchies in complex ecosystems. In an ecosystem composed of $N$ species, one introduces the interaction matrix $S_{ij}$ that represents the fraction of biomass transferred from the species $j$ to the species $i$, often described as the fraction of $j$ in the “diet” of $i$. The trophic levels of species are the relative positions that they occupy in the ecosystem [22,49,53], i.e., apex predators have a larger trophic level than phytoplankton.

![Katz Centrality ER](image-url)
constructed using the Bianconi-Barabási algorithm \([55, 56]\) . We define the following metric
\[
\sigma = \text{std} \left( \left( \frac{\kappa_i^{(\text{num})}}{\kappa_i^{(\text{approx})}} - 1 \right) \right),
\]
where \(\kappa_i^{(\text{num})}\) is the Katz centrality of the \(i\)-th node of the \(m\)-th instance \((m = 1, \ldots, M)\), evaluated by numerical inversion \([14]\), \(\kappa_i^{(\text{approx})}\) is the approximate Katz centrality of the same node specializing the rank-1 formula \([9]\), and the standard deviation \(\text{std}(\cdot)\) is taken across nodes of the same instance. In Fig. 2 we plot \(\sigma\) as a function of \(p\) for ER and SF graphs, and its average \(\langle \sigma \rangle\) over \(M = 3600\) instances, where a lower \(\sigma\) clearly indicates a better agreement between the exact and approximate formulae.

We see that for ER graphs, the agreement between the rank-1 approximation of the Katz ranking and the actual value evaluated via matrix inversion is almost perfect, unless the graph is very sparse. For scale-free graphs the agreement is still good, but the noise is again larger for sparser networks. This is in agreement with a similar observation in the context of Mean First Passage Time for random walks on networks \([19]\) that the accuracy of the approximate formula \([9]\) depends on the spectral gap of the interaction matrix, which makes it not suitable for “too sparse” networks.

We now test the fact that the Katz centrality of a node in not-too-sparse networks should be largely determined by the node’s strength (or degree) by considering a class of simple degree-preserving transformations that dramatically alter the topology of a network. The Maslov-Sneppen rule \([17, 43]\), shown in Fig. 3 works as follows: given two pairs of nodes \(i\) and \(j\) and \(i'\) and \(j'\) of a directed graph, such that the links \(i \rightarrow j\) and \(i' \rightarrow j'\) do exist, one performs the rewiring \(i \rightarrow j'\) and \(i' \rightarrow j\). Clearly, both the out-strengths of \(i\) and \(i'\) are unchanged, and so are the in-strengths of \(j\) and \(j'\). We generate \(M = 100\) Erdős-Rényi graphs of size \(N = 100\) with edge probability \(p\), and perform 100 dynamical (time) steps \(t\). At each time step, we select two pairs of edges at random, and we rewrite them, as depicted in the inset of Fig. 3. For each network \(\ell\) in the sample, we calculate the Katz centrality \(\kappa_i^{(\ell)}(t)\) \((\alpha = 0.3)\) of each node \(i\). Then, as a function of \(p\) and for every sample \(\ell\) and every node, we calculate the discrete time derivative \(\partial_t \kappa_i^{(\ell)}(t) = \kappa_i^{(\ell)}(t) - \kappa_i^{(\ell)}(t - 1)\). Then, we calculate \(\langle \partial_t \kappa_i^{(\ell)}(t) \rangle = \frac{1}{NM(T - 1)} \sum_{t=1}^{T-1} \sum_{i=1}^{N} \sum_{\ell=1}^{M} \partial_t \kappa_i^{(\ell)}(t)\), which is the average change in the centrality value across nodes, time, and samples. This average clearly depends only on the edge probability \(p\). In Fig. 3 we see that the average derivative is close to zero across the full range of \(p\), while the fluctuations – assessed by the standard deviation across instances and time – also become negligible after \(p \approx 0.3\). This confirms that the Katz centrality is strongly correlated to the degree sequence of the network, and even more so the denser the network is.

Katz centrality of social network data. The Katz centrality \([13]\) is also calculated on the widely used Ego-Facebook dataset \([57]\), comprising the matrix of interactions of 4039 users. We chose the value of the damping factor \(\alpha = 2.87 \times 10^{-4}\), which is the largest admissible value for which (i) the series in \([13]\) converges, and (ii) the matrix \(\alpha \cdot \hat{A}\) is substochastic. Each point in Fig. 4 - panel (a) - represents the centrality value of one of the users. We compare the empirical Katz centrality against the theoretical rank-1 approximation with single (green dots, see \([10]\)) and double (red dots, see \([9]\)) constraints.
Economic dataset. The upstreamness \( \{18\} \) is computed using the data from the National Input-Output Tables (NIOT) of the World Input-Output Database (WIOD - 2013 release) \( \{58\} \). The NIOT dataset represents the flow of money between 35 industrial sectors for 29 world economies, and it includes the years 1995 – 2011. The full list of countries and economic sectors as well as a scheme of the structure of the input-output tables can be found in \( \{44\} \)\( \{58\} \).

The matrix \( A \) from which we calculate the upstreamness is computed from the full-input-output table by each country normalising the entries of the I-O table by the total output (the row sum of the matrix plus the demand of external entities) \( \{44\} \). Each point in Fig. \( \{4\} \), panel (b) - represents the average upstreamness (over all sectors) of a given country in a given year (the plot thus includes 39 countries \( \times 17 \) years \( = 663 \) data points).

PageRank. The PageRank \( \{15\} \) is evaluated on the \( \{\text{wbs}\} \) Stanford dataset, a collection of 9914 indexed web-pages of the World Wide Web \( \{59\} \)\( \{60\} \) dating from 2001, for a value of the damping parameter equal to \( d = 0.3 \) (Panel (c), Fig. \( \{4\} \)).

Trophic Levels. The trophic levels (defined in \( \{17\} \)) are computed for the 51 species that are part of the St. Marks food web \( \{61\} \) (panel (d), Fig. \( \{4\} \)). Since on average the size of ecosystems does not normally exceed \( \sim 200 \) species, to match the empirical data with the rank-1 approximation we have also constructed the corresponding stochastic niche model in order to generate and reproduce the typical trophic diameter of the food web, as it is customary in this type of analysis \( \{23\} \)\( \{62\} \)\( \{64\} \). We used a “cascade”, or niche model, extracted from the empirical food web \( A \), not only reproducing the mean and the variance of the empirical data, but rather estimating the full distribution \( \{65\} \). Each black dot in Fig. \( \{4\} \), panel (d) corresponds to one of the \( T_k(A) \) values \( (k = 1, \ldots, 51 \) and \( s = 1, \ldots, 100) \) generated by 100 instances of the random model with \( N = 51 \) generative species each – plotted against the corresponding theoretical value estimated using \( \{67\} \) and \( \{10\} \). While there are sample to sample fluctuations, binning (purple dots in Fig. \( \{4\} \), panel (d)) reveals that the average trophic levels nicely follow the theoretical approximations. More generally, we observe an excellent agreement between our approximate formula – which does not require any matrix inversion, nor the full details of the pairwise interaction matrix – and the empirical data in all cases, especially using the doubly-constrained version of the influence, Eq. \( \{9\} \).

BEYOND INFLUENCE

The rank-1 approximation for the resolvent allows us to get analytical access to a number of other network observables that can be written as functions \( F(A) \) of the interaction matrix \( A \). The key identity is the Dunford-Taylor formula \( \{66\} \), with which we can write a matrix function as the following integral

\[
F(A) = \frac{1}{2\pi i} \int_{\Gamma} dz (zI - A)^{-1} F(z) ,
\]

assuming that \( F \) is analytic, and the domain \( \Gamma \) encircles the spectrum of \( A \).

Approximating \( A \) with \( \hat{A} \) as in \( \{5\} \) and replacing the resolvent \( G(A) \) with \( G(\hat{A}) \) as defined in \( \{5\} \), we obtain the following approximation (see Supporting Information for the derivation)

\[
F(A) \approx F(0) I + \frac{F(\xi) - F(0)}{mN\xi} r e^{T} ,
\]

where \( \xi = \sum r_i c_i / (mN) \).

For instance, many graph invariants, such as topological indices, can be written in the form of a matrix exponential. A non-local quantity of interest is the so-called graph communicability \( \{67\} \) given by

\[
c_{ij} = (e^{\hat{A}})_{ij}
\]

where \( e^{\hat{A}} = \sum_{k=0}^{\infty} \frac{\hat{A}^k}{k!} \) is the exponential of the adjacency matrix. This measure accounts for all possible channels of communication between node \( i \) and \( j \), giving more weight to the shorter paths connecting them on the network represented by \( \hat{A} \). The communicability of a node \( i \) is given by \( c_i = \sum_j c_{ij} \), and the total communicability of the graph is defined as the Estrada index \( EE(\hat{A}) = \sum c_i \). Using \( \{21\} \) we can also provide a natural explanation for the observed correlation between the strengths of nodes (or simply their degrees for unweighted graphs) and the communicability of the network, observed in \( \{68\} \). In particular, we can approximate the communicability matrix as follows

\[
e^{\beta \hat{A}} \approx I + \exp \left\{ \frac{\beta k_{\text{out}} k_{\text{in}}}{\sum_{ij} k_{\text{out}} k_{\text{in}}} - 1 \right\}
\]

where \( k_{\text{out}} \) and \( k_{\text{in}} \) are the column vectors of out- and in-strengths of the network, and \( \beta \) denotes the dot product. For uncorrelated networks, the average value of the communicability over an ensemble of networks can therefore be computed – at least in principle – using

\[
\langle f (k_{\text{in}}, k_{\text{out}}) \rangle = \sum_{m,n} f(m,n) \prod_{j=1}^{N} p(m_j, n_j) ,
\]

where \( p(k_{\text{in}}, k_{\text{out}}) \) is the joint distribution of in- and out-strengths of the nodes.

In order to see how well our approximate formula performs as a function of the connectivity of the graph, we define the metric (analog of \( \{19\} \))

\[
\sigma = \text{std} \left( \frac{c_{i\text{num}}}{c_{i\text{approx}}} - 1 \right) ,
\]

where \( c_{i\text{num}} \) is the communicability of the \( i \)-th node of the \( m \)-th instance \( (m = 1, \ldots, M) \) of an Erdős-Rényi graph of size \( N = 100 \), evaluated using \( \{22\} \), while \( c_{i\text{approx}} \) is the approximate communicability of the same node obtained from
and the standard deviation \( \text{std}(\cdot) \) is taken across nodes of the same instance. In Fig. 5 we plot \( \sigma \) as a function of the edge probability \( p \), where a lower \( \sigma \) clearly indicates a better agreement between the exact and approximate formulae. The thickness of the red curve corresponds to the spread of \( \sigma \) across \( M = 100 \) instances, whereas the black dashed curve corresponds to the average \( \sigma \) over such instances. As expected, the denser the graph the better the agreement with our rank-1 approximation. In the inset of Fig. 5 we show a scatter plot of the communicability of the \( i \)-th node computed using our approximate formula \(23\) against empirical data for different edge probabilities \( p \).

These results can be easily generalized to further incarnations of exponential measures. These includes for example the generalized Estrada index defined as \( \text{GE}(\hat{A}) = \text{Tr}(e^{\beta\hat{A}}) = \sum_i \lambda_i(e^{\beta A}) \), where \( \text{Tr} \) is the matrix trace, originally introduced to measure the degree of folding of a protein and later extended to characterize topological properties of graphs \[67\][69][73].

Evaluating these types of observable with high accuracy requires non-trivial computational resources for large graphs, as they normally scales as \( N^3 \) with the number of nodes \[71\][72]. In addition, there are not many exact or even approximate formulae in the case of dense graphs, exception made for very specific topologies for which the graph eigenvalues are exactly known. Results for dense weighted graphs are even scarcer. In this context, accurate formulae derived from a simple rank-1 approximation may be potentially quite useful.

Our rank-1 approximation using the Dunford-Taylor identity \[21\] can also be applied to further nonlinear observables in the context of centrality measures. A straightforward extension is for example the application to centrality measures of the form \( F(A) = A^k \). Using \[21\] we obtain \( \langle A^k \rangle_{ij} \approx \frac{k-1}{mN} (rc^T)_{ij} \). A notable example for this class of centrality measure is the so-called diffusion centrality \[74\][75], which is defined in terms of a weighted sum of matrix powers and used in simple models of information diffusion to measure the expected number of times a node receives a given piece of information.
Our findings follow from the observation that the matrix $\hat{A}$ that best approximates a given interaction matrix $A$ (in the Maximum Entropy sense), and shares with $A$ the row/column sums, has rank-1. This provides an immediate route – via the Sherman-Morrison formula $[3]$ – to approximate the resolvent matrix $G(A) = (\mathbf{1} - A)^{-1}$, which appears in the general definition of “influence” (see [2]) and thus in all its particular incarnations described above. The emerging locality that stems from this approximation – and its universal validity across fields and disparate applications – is particularly appealing for various reasons:

- In ecology/economy – among many other fields – a complete and accurate control over the interactions between species/productive sectors is plagued with a wealth of theoretical and practical problems. Our approach offers an avenue to forgo the full knowledge of the interaction matrix and determine the set of influence levels with high accuracy using only limited and local information about the activity of each player.

- Centrality measures are known to correlate with the degree of the corresponding nodes. At least in the context of “spectrally gapped” networks, our approach makes this connection clear and transparent, allowing for a more straightforward analytical treatment.

- Being the resolvent $G(A)$ the building block of any analytic matrix function via the Dunford-Taylor formula $[20]$, the rank-1 approximation $G(\hat{A})$ for the resolvent actually carries over to a number of non-linear network observables beyond the influence. We have presented examples for exponential and power-law centrality measures, and provided evidence that our rank-1 approximation can be very accurate, and more analytically tractable than its original version, at least in the not-too-sparse regime.

In future studies, it will be important to (i) characterize in a more quantitative way the accuracy of the approximation as a function of the spectral gap, (ii) study more thoroughly the effect of graph heterogeneity on the quality of the approximation, and (iii) explore rank-$k$ (with $k > 1$) systematic approximations for the resolvent.

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**SUPPORTING INFORMATION**

**Uniqueness of rank-1 approximation**

We show here that the rank-1 matrix $\hat{A}$ in [3] with prescribed row and column sums ($r$ and $c$) is unique. Since $\hat{A}$ is rank-1, we must have $g = \lambda r$ and $q = \mu c$. Imposing the conditions on the row and column sums (see e.g. [4]), we are unable to fix the constants $\lambda$ and $\mu$ separately, but their product $\lambda \mu$ must be equal to $1/m$. Computing $\hat{A} = (1/N)gq^T = \lambda \mu (1/N)rc^T = (1/mN)rc^T$ as in [6],

\[
\hat{A} \text{ can be written as the Maximum Entropy matrix with prescribed row and column sums.}
\]

Giving the entries $0 \leq \hat{A}_{ij} \leq 1$ a probabilistic interpretation, we ask what the assignment of $\hat{A}_{ij}$ should be that maximizes the entropy

\[
S = -\sum_{i,j} \hat{A}_{ij} \ln \hat{A}_{ij}
\]

subject to the constraints $r_i = \sum_j \hat{A}_{ij}$ and $c_j = \sum_i \hat{A}_{ij}$. Constructing the Lagrangian

\[
\mathcal{L} = \sum_{i,j} \hat{A}_{ij} \ln \hat{A}_{ij} - \lambda_0 \left( \sum_{i,j} \hat{A}_{ij} - N \mu \right)
\]

\[
-\sum_i \lambda_i \left( \sum_j \hat{A}_{ij} - r_i \right) - \sum_j \mu_j \left( \sum_i \hat{A}_{ij} - c_j \right)
\]

and computing its stationary points, we obtain the equations

\[
\ln \hat{A}_{ij} + 1 - \lambda_0 - \lambda_i - \mu_j = 0 \Rightarrow \hat{A}_{ij} = \frac{1}{Z} \phi_i \psi_j
\]

where we set $1/Z = \exp(\lambda_0 - 1)$, $\exp(\lambda_i) = \phi_i$ and $\exp(\mu_j) = \psi_j$. Therefore, we see that the MaxEnt matrix $\hat{A}$ with prescribed row and column sums is rank-1. Imposing the row and column sums constraints (or simply appealing to uniqueness), we finally obtain again $\hat{A} = (mN)^{-1}rc^T$. 

Generalized nonlinear formula

First, we use the following representation of the matrix $F(A)$

$$F(A) = \frac{1}{2\pi i} \oint_{\Lambda} dz F(z)(zI - A)^{-1}$$

$$= \frac{1}{2\pi i} \oint_{\Lambda} dz \frac{F(z)}{z} \left(1 - \frac{1}{z}A\right)^{-1},$$

(29)

where the integration is over a curve $\Lambda$ in the complex plane, which contains all eigenvalues of $A$.

We now apply the approximate expression for the resolvent

$$F(A) \approx \frac{1}{2\pi i} \oint_{\Lambda} dz \frac{F(z)}{z} \left[I + \frac{1}{mNz - \sum_{i=1}^{N} r_i c_i} r e^T\right],$$

(30)

where we recall the definitions $r_i = \sum_j A_{ij}$, $c_j = \sum_i A_{ij}$, and $m = \sum_i r_i = \sum_j c_j$. It is easy to show that $\xi = \sum_i r_i c_i/(mN)$ is included within the curve $\Lambda$. Therefore, using

$$\frac{1}{z(z - \xi)} = \frac{1}{\xi} \left(\frac{1}{z - \xi} - \frac{1}{z}\right),$$

(31)

and applying Cauchy’s residue theorem to the singular points $z = \xi$ and $z = 0$, we obtain the identities

$$\frac{1}{2\pi i} \oint_{\Lambda} dz \frac{F(z)}{z} = F(0),$$

(32)

$$\frac{1}{2\pi i} \oint_{\Lambda} dz \frac{F(z)}{z(z - \xi)} = \frac{F(\xi)}{\xi} - \frac{F(0)}{\xi}.$$  

(33)

We thus obtain the formula (21) of the main text

$$F(A) \approx F(0)I + \frac{F(\xi) - F(0)}{mN\xi} r e^T.$$  

(34)