A GENERALIZATION OF TWISTOR LINES FOR COMPLEX TORI

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Abstract. In this work we generalize the classical notion of a twistor line in the period domain of compact complex tori, studied in [1]. We introduce two new types of the generalized lines, which are non-compact analytic curves in the period domain. We then study the analytic properties of the compactifications of the curves, preservation of the type (1,1) cohomology classes along the curves and the twistor path connectivity of the period domain by the curves of one of the new types.

Contents

1. Introduction 1
2. Proof of Theorem 1.1 5
3. Proof of Theorem 1.2 10
4. Proof of Theorem 1.3 15
References 18

1. Introduction

A manifold \( M \) is called hyperkähler with respect to a metric \( g \) (see [6, p. 548]) if there exist covariantly constant complex structures \( I, J \) and \( K \) which are isometries of the tangent bundle \( TM \) with respect to \( g \), satisfying the quaternionic relations

\[
I^2 = J^2 = K^2 = -1, \quad IJ = -JI = K.
\]

We call the ordered triple \( I, J, K \) a hyperkähler structure on \( M \) compatible with \( g \). A hyperkähler structure \( I, J, K \) gives rise to a sphere \( S^2 \) of complex structures on \( M \),

\[
S = S^2 = \{aI + bJ + cK|a^2 + b^2 + c^2 = 1\}.
\]

This sphere is called a twistor sphere or a twistor line.

The well known examples of compact hyperkähler manifolds are compact complex tori and irreducible holomorphic symplectic manifolds (IHS manifolds). We recall that an IHS manifold is a simply connected compact Kähler manifold \( M \) with \( H^0(M, \Omega_M^2) \) generated by an everywhere non-degenerate holomorphic 2-form \( \sigma \).

It is known that in the period domain of an IHS manifold any two periods can be connected by a path of twistor lines, see [3] or [4]. The twistor path connectivity of each of the two connected components of the period domain of complex tori was proved in [1].

In the present paper we generalize the notion of twistor lines to include certain non-compact analytic curves in the period domain of complex tori and study the
geometry of such curves, in particular, the behavior at infinity, the path connectivity problem and the preservation of Kähler classes along the curves.

Let us recall the construction of this period domain. Let $V_R$ be a real vector space of real dimension $4n$. A compact complex torus of complex dimension $2n$ considered as a real smooth manifold is the quotients $A = V_R/\Gamma$ of $V_R$ by a lattice $\Gamma$ with the complex structure given by an imaginary endomorphism $I: V_R \to V_R, I^2 = -Id$. Following $[\Pi]$ we denote the period domain of compact complex tori of complex dimension $2n$ by $\Compl$. It is the set of imaginary endomorphisms of $V_R$ and is diffeomorphic to the orbit $G I$, where $G = GL(V_R) = GL(4n, \mathbb{R})$ acts via the adjoint action, $g I := g \cdot I = g I g^{-1}$. This action naturally (pointwise) extends to the action on the set of twistor lines $S \subset \Compl$. The period domain $\Compl$ consists of two connected components, corresponding to two connected components of $G$. We have the embedding of $\Compl$ into the Grassmanian $Gr(2n, V_C)$ of $2n$-dimensional complex subspaces in $V_C = V_R \otimes \mathbb{C}$ given by $\Compl \ni I \mapsto (I I - i I) V_R \in Gr(2n, V_C)$, which maps $\Compl$ diffeomorphically onto an open subset of $Gr(2n, V_C)$. The complement of this open set is the real-analytic locus $\mathcal{L}_R = \{U \in Gr(2n, V_C) | U \cap V_R \neq \{0\}\}$ of $2n$-dimensional complex subspaces in $V_C$ having nontrivial intersection with $V_R$. This locus $\mathcal{L}_R$ is of codimension 1 in $Gr(2n, V_C)$ and it cuts $Gr(2n, V_C)$ into two pieces each of which is the corresponding component of $\Compl$. Further we will be dealing with a fixed connected component of $\Compl$ which we will denote also by $\Compl$.

Introduce the following 4-dimensional real algebras,

$$\mathbb{H}(\varepsilon) = \langle i, j | i^2 = -1, j^2 = \varepsilon, ij + ji = 0 \rangle, \varepsilon = -1, 0, 1.$$ 

The above introduced twistor spheres arise from embeddings of the algebra of quaternions $\mathbb{H} = \mathbb{H}(-1) \hookrightarrow End V_R$. The non-compact analogs arise from the embeddings $\mathbb{H}(\varepsilon) \hookrightarrow End V_R$ for $\varepsilon = 0, 1$. In the next subsection we explain why it is natural to consider these algebras along with $\mathbb{H}$.

1.1. Algebraic characterization of twistor lines in $\Compl$. In this subsection we give an algebraic characterization of the above introduced (compact) twistor lines, which is then generalized in order to define the non-compact analogs of twistor lines.

Let $I, J, K = I J \in End V_R$ be complex structures satisfying the quaternionic identities and

$$S = S(I, J) = \{a I + b J + c K | a^2 + b^2 + c^2 = 1\}$$

be the corresponding twistor sphere. The basis $I, J, K$ of the space $\mathbb{R}^3 = \langle I, J, K \rangle \subset End V_R$ is orthonormal with respect to the bilinear form $(u, v) = \frac{1}{4n} Tr(u v)$, which is non-degenerate on $\langle I, J, K \rangle$. Moreover, $u, v \in \langle I, J, K \rangle$ anticommute if and only if $u \perp v$. The sphere $S$ is a sphere of radius 1 in $\mathbb{R}^3 = \langle I, J, K \rangle$ centered at the origin. Let $J_1 \neq \pm J_2$ be some complex structures in $S$. Then the plane $\langle J_1, J_2 \rangle_R \subset End V_R$ intersects $S$ in the circle $S \cap \langle J_1, J_2 \rangle_R$, which contains a complex structure which anticommutes with $J_1$ (this can be seen by the means of the orthogonalization process applied to the pair $J_1, J_2$). Thus there exists a real number $\alpha$ such that $\alpha J_1 + J_2$ determines, after a scalar normalization, a complex structure in $S$ anticommuting with $J_1$, that is (even without the normalization),

$$(\alpha J_1 + J_2) J_1 + J_1 (\alpha J_1 + J_2) = 0,$$
which results in the relation

\[ J_1 J_2 + J_2 J_1 = 2\alpha \cdot Id. \]

The fact that \( \alpha J_1 + J_2 \) is proportional to a complex structure brings certain restrictions on \( \alpha \), that is,

\[ (\alpha J_1 + J_2)^2 = a \cdot Id, \quad a < 0, \]

then \( J = \frac{\alpha J_1 + J_2}{\sqrt{-a}} \) is the complex structure and \( J \) anticommutes with \( I = J_1 \). The left side \((\alpha J_1 + J_2)^2\) of the above equation is equal to

\[-(1 + \alpha^2)Id + \alpha \cdot (J_1 J_2 + J_2 J_1) = (\alpha^2 - 1)Id,\]

and thus the condition \( a = \alpha^2 - 1 < 0 \) is simply the condition \( |\alpha| < 1 \). Thus the necessary and sufficient condition that nonproportional \( J_1, J_2 \) belong to the same twistor sphere \( S = S(I, J) \) is that there exists \( \alpha \in \mathbb{R} \) such that

\[ J_1 J_2 + J_2 J_1 = 2\alpha \cdot Id, \quad |\alpha| < 1. \]

If we drop the restriction \( |\alpha| < 1 \) then the complex structure operators \( J_1, J_2 \) satisfying \( J_1^2 = J_2^2 = -Id, J_1 J_2 + J_2 J_1 = 2\alpha Id \) generate the 4-dimensional algebra \( \mathbb{H}(1) \subset End V_{\mathbb{R}} \) if \( |\alpha| > 1 \) and the algebra \( \mathbb{H}(0) \subset End V_{\mathbb{R}} \) if \( |\alpha| = 1 \). Indeed, if \( |\alpha| > 1 \) then the above calculations show that \( R = \frac{\alpha J_1 + J_2}{\sqrt{\alpha^2 - 1}} \) satisfies \( R^2 = Id \) and anticommutes with \( I = J_1 \), so that \( I \) and \( R \) generate a subalgebra of \( End V_{\mathbb{R}} \) isomorphic to \( \mathbb{H}(1) \).

If \( |\alpha| = 1 \) then \( N = \alpha J_1 + J_2 \) is a nilpotent operator, \( N^2 = 0 \), and \( N \) anticommutes with \( I = J_1 \) so that \( I \) and \( N \) generate a subalgebra isomorphic to \( \mathbb{H}(0) \).

Using the above introduced bilinear form \((\cdot, \cdot)\) we can universally express the above normalization process in all three cases as the orthogonalization process applied to \( J_1, J_2 \).

The image of the set of imaginary units (that is, elements, whose square is equal to \(-1\)), of the algebra \( \mathbb{H}(\varepsilon), \varepsilon = -1, 0, 1, \) under a faithful representation \( \mathbb{H}(\varepsilon) \to End V_{\mathbb{R}} \) is a subset in \( \text{Compl} \), which we call a (generalized) twistor line of type \( \mathbb{H}(\varepsilon) \). Certainly, we need to justify extending the terminology to the cases \( \varepsilon = 0, 1 \) by checking that thus defined subsets are indeed complex submanifolds in \( \text{Compl} \), which we will do later.

Consider first the case \( \mathbb{H}(1) \hookrightarrow End V_{\mathbb{R}} \). Denote the images of \( i \) and \( j \) under the embedding as \( I \) and \( R \), in order to emphasize that \( j \) acts as a reflection (or rotation by \( \pi \)) operator on \( V_{\mathbb{R}} \). Then \( I^2 = -Id, R^2 = Id, IR + RI = 0 \).

Let us describe the complex structure operators contained in the subspace \( \mathbb{R}^3 = \langle I, R, IR \rangle \subset End V_{\mathbb{R}} \), that is, the intersection \( \langle I, R, IR \rangle \cap \text{Compl} \). The combination \( xI + yR + zIR \) is a complex structure operator if and only if \((xI + yR + zIR)^2 = (-x^2 + y^2 + z^2)Id = -Id \), that is, \( x^2 - y^2 - z^2 = 1 \). Then the set

\[ S(I, R) = \{ xI + yR + zIR \, | \, x^2 - y^2 - z^2 = 1 \} \]

which is a two-sheeted hyperboloid consisting of complex structures, is a generalized twistor line of the type \( \mathbb{H}(1) \). Note that as the two connected components \( S(I, R)^+, S(I, R)^- \) of \( S(I, R) \) contain respectively \( I \) and \(-I \), they both must be contained in the same connected component \( \text{Compl} \) of the period domain, as we know that there exists a complex structure \( J, J^2 = -Id, \det J = 1 \), anticommuting with \( I \) so that \(-I = JIJ^{-1} \).
Next, let us consider the case $|\alpha| = 1$, that is $\mathbb{H}(0) \hookrightarrow End V_{\mathbb{R}}$. In this case we denote the images of the generators $i$ and $j$ of $\mathbb{H}(0)$ by $I$ and $N$, so that $I^2 = -Id, N^2 = 0, IN + NI = 0$. Then the set $S(I, N) = \langle I, N, IN \rangle \cap \text{Compl}$,

\[ S(I, N) = \{ \pm I + yN + zIN \mid y, z \in \mathbb{R} \} = \mathbb{R}^2 \cup \mathbb{R}^2, \]

is a generalized twistor line of type $\mathbb{H}(0)$. Indeed, the combination $(xI + yN + zIN)^2 = -x^2 Id$ is an imaginary unit if and only if $x = \pm 1$.

Again, as the connected components $S(I, N)^+, S(I, N)^-$ of $S(I, N)$ contain $I$ and $-I$, we have that the whole $S(I, N)$ is contained in a connected component of the period domain.

The group $G = GL(V_{\mathbb{R}})$ acts, via the adjoint action, on the set of generalized twistor lines, $gS(I, J) = S(gI, gJ)$ for $g \in G$, this action certainly preserves the type of the curves.

We denote the tangent cone at the point $p$ of a possibly singular complex manifold $M$ by $C_p M$.

1.2. The formulation of results. The following theorem gives an analytic description of the subsets $S(I, R)$ and $S(I, N)$ of $\text{Compl}$ and describes their behavior at the infinity $\mathcal{L}_R = \text{Gr}(2n, V_{\mathbb{C}}) \setminus \text{Compl}$.

**Theorem 1.1.** Each of the two sets $S(I, R) = S(I, R)^+ \cup S(I, R)^-$ and $S(I, N) = S(I, N)^+ \cup S(I, N)^-$ consists of two connected components, each of the components is a smooth complex 1-dimensional submanifold in $\text{Compl}$, diffeomorphic to an open 2-disk. Their analytic topology closures $\overline{S(I, R)}$ and $\overline{S(I, N)}$ in the Grassmanian $\text{Gr}(2n, V_{\mathbb{C}}) \supset \text{Compl}$ are complex-analytic curves.

The curve $\overline{S(I, R)}$ is a $\mathbb{P}^1 \subset \text{Gr}(2n, V_{\mathbb{C}})$, which is tangent to $\mathcal{L}_R$ along the real-analytic circle $S^1 = \overline{S(I, R)} \cap \mathcal{L}_R \subset \{ U \in \text{Gr}(2n, V_{\mathbb{C}}) \mid \dim U \cap V_{\mathbb{R}} = 2n \} \subset \mathcal{L}_R$, that is, for every $p \in S^1$ we have $T_p S(I, R) \subset C_p \mathcal{L}_R$.

The curve $\overline{S(I, N)}$ consists of two connected components $\overline{S(I, N)}^{\pm} = \overline{S(I, N)}^{\pm}$, each of which is a $\mathbb{P}^1 \subset \text{Gr}(2n, V_{\mathbb{C}})$ with exactly one point $p^{\pm} = \overline{S(I, N)}^{\pm} \cap \mathcal{L}_R$ at infinity. The points $p^{\pm}$ are singular points of $\mathcal{L}_R$ and the tangent planes $T_{p^{\pm}} S(I, N)$ intersect the respective tangent cones $C_{p^{\pm}} \mathcal{L}_R$ trivially.

Let $Hdg_S = \{ \Omega \in \text{Hom}(\wedge^2 V_{\mathbb{R}}, \mathbb{R}) \mid \lambda^* \Omega \lambda = \Omega, \lambda \in \mathcal{S} \}$ be the subspace of the alternating forms $\Omega$ on $V_{\mathbb{R}}$, determining cohomology classes staying of Hodge type $(1,1)$ along the line $S$. Note that as each generalized twistor line is central-symmetric, that is, for every $\lambda \in \mathcal{S}$ we have $-\lambda \in \mathcal{S}$, the subspace $Hdg_S$ does not change, if in its definition we replace $S$ with its connected component.

In [1] we considered “A toy example” of a compact twistor line $S$ in the period domain of complex tori of dimension 2. There we showed that the dimension of $Hdg_S$ is 3 and $Hdg_S$ does not contain any Kähler classes, or, in other words, following the notations of [1], $S$ is not contained in any locus $\text{Compl}_\Omega = \{ I \in \text{Compl} \mid I^t \Omega I = \Omega \} \subset \text{Compl}$, where the alternating 2-form $\Omega$ represents a Kähler class in $H^{1,1}(A, \mathbb{R})$, $\Omega$ and $I$ are written in a certain fixed basis of $V_{\mathbb{R}}$.

One may ask if for a generalized twistor line $S$ it is contained in any Kähler locus $\text{Compl}_\Omega$ or not. Here we answer this question for the general dimension case.
Theorem 1.2. For any twistor line of the type $\mathcal{H}(-1)$ the space $\text{Hdg}_S$ has the dimension $2n^2 + n$ and does not contain any K"ahler classes. All representations $\mathcal{H}(-1) \to \text{End} V_R$ are equivalent, or, which is the same, the adjoint action of $\text{GL}(V_R)$ on the set of compact twistor lines is transitive.

Let $S$ be a twistor line of the type $\mathcal{H}(1)$. Then $\dim \text{Hdg}_S = 2n^2 + n$. The subspace $\text{Hdg}_S$ contains two disjoint open cones of K"ahler classes, corresponding to each of the connected components of $S$. All faithful representations $\mathcal{H}(1) \to \text{End} V_R$ are equivalent, or, which is the same, the action of $\text{GL}(V_R)$ on the set of twistor lines of the type $\mathcal{H}(1)$ is transitive.

There are $n$ non-equivalent faithful representations $\mathcal{H}(0) \to \text{End} V_R$ parametrized by integers $1 \leq k \leq n$, equivalently, there are $n$ distinct $\text{GL}(V_R)$-orbits of lines of the type $\mathcal{H}(0)$ in $\text{Compl}$. For any non-compact twistor line $S$ of the type $\mathcal{H}(0)$ we have $\dim \text{Hdg}_S = k(k + 1) + (2n - k)^2$, for the respective parameter $k$, and $\text{Hdg}_S$ does not contain any K"ahler classes for either of the two connected components of $S$.

Now let us get to the problem of the twistor path connectivity of $\text{Compl}$. We say that $\text{Compl}$ is $\mathcal{H}(\varepsilon)$-connected if any two point in the same connected component of $\text{Compl}$ can be connected by a path of connected components of twistor lines of the type $\mathcal{H}(\varepsilon)$. The $\mathcal{H}(-1)$-connectivity of $\text{Compl}$ was proved in [1], here we present a proof that $\text{Compl}$ is $\mathcal{H}(1)$-connected.

In fact we will formulate and prove the connectivity for both compact lines and non-compact lines of the type $\mathcal{H}(1)$ in a uniform manner, so that we include the compact case in the formulation of the following theorem.

Theorem 1.3. The period domain $\text{Compl}$ is $\mathcal{H}(\varepsilon)$-connected for $\varepsilon = -1, 1$.

The idea of the proof is very similar to that in [1]. The methods that we are using do not seem to directly apply to the problem of $\mathcal{H}(0)$-connectivity, so this problem remains open.

Remark 1.4. Note that Theorem 1.2 and 1.3 do not imply the $\mathcal{H}(1)$-connectivity of the locus of polarized tori in $\text{Compl}$. Here these two theorems merely establish that the paths of lines along each of which some K"ahler classes survive can be used for connecting points, so that in that regard these lines are not more specific than the compact lines, along which, by Theorem 1.2 no K"ahler classes survive.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let us start with checking first that $S(I, R)$ and $S(I, N)$ are complex-analytic subspaces in $\text{Compl}$. In order to see this we note that the tangent space to $S(I, R)$ at $p = xI + yR + zIR$ is

$$T_pS(I, R) = \{uI + vR + wIR \mid xu - yv - zw = 0\}$$

and as the complex structure on $T\text{Compl} = T\text{Gr}(2n, V_C)_{\text{Compl}}$ acts on $T_p\text{Compl}$ by the left multiplication by $p$ (see, for example, [1]),

$$l_p : T_p\text{Compl} \to T_p\text{Compl}, a \mapsto p \cdot a,$$

it is enough to check that $l_p(T_pS(I, R)) = T_pS(I, R)$, that is, that $l_p(uI + vR + wIR) = (xI + yR + zIR)(uI + vR + wIR) = (-xu + yv + zw)I + (zw - xu)R + (xv - yu)IR = (zv - yw)I + (zu - xw)R + (xv - yu)IR$ is in $T_pS(I, R)$, which reduces
to checking that \( x(zv - yw) - y(zu - xw) - z(xy - yu) = 0 \). Thus \( S(I, R) \) is a complex analytic subset in \( \text{Compl} \).

Now, to verify the analyticity of \( S(I, N) \) we need to check that for \( p = \pm I + yN + zIN \) we need to check the invariance of the respective tangent space, \( l_p(T_p S(I, N)) = T_p S(I, N) \). Like previously \( (I + yN + zIN)(vN + wIN) = -wN + vIN \in T_p S(I, N) \) and similarly for \( p = -I + yN + zIN \).

Let us now study the points of the euclidean closures of \( \overline{S(I, R)}, S(I, N) \) at the infinity of the space \( \text{Compl} \), that is, at the locus \( \mathcal{L}_R \). Below we separately consider the cases of \( S(I, R) \) and \( S(I, N) \).

**Case of \( S(I, R) \).** Fix \( xI + yR + zIR \in S(I, R) \). We can always assume that \( z = 0 \) as \( (yR + zIR)^2 = y^2 + z^2 = x^2 - 1 \geq 0 \) and introducing \( R_1 = \frac{1}{\sqrt{y^2 + 1}}(yR + zIR) \) we get that \( xI + yR + zIR = xI + \sqrt{x^2 - 1}R_1 \in S(I, R_1) = S(I, R) \). Next we consider the real curve \( y \mapsto c(y) = yI + \sqrt{y^2 - 1}R \in S(I, R) \). The corresponding curve in the Grassmanian \( C: [1, \infty) \to Gr(2n, V_C) \) is given by \( y \mapsto (Id - ic(y))V_R \). Then we have the equality of the points of the Grassmanian

\[
C(y) = (Id - ic(y))V_R = \frac{1}{y}(Id - ic(y))V_R,
\]

so instead of \( Id - ic(y) \) we consider the operator

\[
\frac{1}{y}(Id - ic(y)) = \frac{1}{y}Id - i \left( I + \sqrt{y^2 - 1} \right) R
\]

and we see that \( \lim_{y \to \infty} \frac{1}{y}(Id - ic(y)) = -i(I + R) \in \text{End} V_C \). Let us clarify on what kind of an operator \( I + R \) is. It is clear that as \( I \) and \( R \) anticommute, \( (I + R)^2 = 0 \) and \( R \neq \pm Id \). Next, \( I + R = I(Id - IR) = (Id + IR)I \) and we see that \( I \) provides an isomorphism between eigenspaces \( \text{Ker} (Id - IR) \) and \( \text{Ker} (Id + IR) \) of \( IR \), which together with \( V_R = \text{Ker} (Id - IR) \oplus \text{Ker} (Id + IR) \) gives that \( \dim \text{Ker} (Id - IR) = \dim \text{Ker} (Id + IR) = 2n \). So we conclude that \( \text{rank}_R (I + R) = 2n \). As \( (I + R)^2 = 0 \) we see that \( \text{Im} (I + R) \subset \text{Ker} (I + R) \) and so we must have \( \text{Im} (I + R) = \text{Ker} (I + R) \).

The subspace \( (I + R)V_R \) is obviously not a complex subspace in \( V_C \). In order to see what the actual limit \( \lim_{y \to \infty} \frac{1}{y}(Id - ic(y))V_R \) is, we need to use that \( c(y) \) acts by multiplication by \( i \) on \( (Id - ic(y))V_R \). As \( \lim_{y \to \infty} \frac{1}{y}c(y)(Id - ic(y)) = (I + R) \), we have that \( \lim_{y \to \infty} (Id - ic(y))V_R = (I + R)V_R \oplus i(I + R)V_R \), which is a complex vector subspace of complex dimension \( 2n \) in \( V_C \), hence it belongs to \( \mathcal{L}_R \subset Gr(2n, V_C) \).

Next, replacing \( R \) with a general reflection operator \( R_1 \) as above, we get the whole circle of points in \( \mathcal{L}_R \) which can be considered as the points of (one of two connected components of) \( S(I, R) \) at infinity (it is easy to check that distinct \( R_1 \)'s indeed correspond to distinct points in \( \mathcal{L}_R \)).

Now let us apply the central symmetry \( xI + yR + zIR \mapsto -xI - yR - zIR \) to the curve \( c(y) \) so as to get the curve \( -c(y) = -yI - \sqrt{y^2 - 1}R \) which now belongs to another connected component of \( S(I, R) \). Then, similarly to the previous, \( \lim_{y \to \infty} \frac{1}{y}(Id - ic(y))V_R = (I + R)V_R \oplus i(I + R)V_R \), so that indeed \( \overline{S(I, R)} \) is connected (the two sheets of the hyperboloid glue together).
Next, let us prove the smoothness of the closure \( \overline{S(I, R)} \) and that \( \overline{S(I, R)} \subset Gr(2n, V_C) \) is tangent to \( L_R \), that is, at \( p \in S(I, R) \cap L_R \) we have \( T_pS(I, R) \subset T_pL_R \). To prove the tangency statement it is sufficient, therefore, to show that if we launch an arbitrary curve of the form \( (Id - ic(y))V_R \) from \( p = \lim_{y \to \infty} (Id - ic(y))V_R = (I + R)V_R \oplus i(I + R)V_R \) in the “backward direction”, the (nonzero) tangent vector to it will belong to \( T_pL_R \).

Let us start with proving the tangency statement and then we show the smoothness after. Introducing \( t = \frac{1}{y} \) we can rewrite the “reversed version” of our curve \( C \) as
\[
\tilde{C}: t \mapsto \left( Id - ic \left( \frac{1}{t} \right) \right) V_R = \left( Id - \frac{i}{t} \left( I + \sqrt{1 - t^2R} \right) \right) V_R = \\
\left( t \cdot Id - i(I + \sqrt{1 - t^2R}) \right) V_R.
\]

Then the tangent vector \( \tilde{C}'(0) \in Hom(p, V_C/p) \) is found as follows. Let \( U \subset V_R \) be a vector subspace such that \( V_R = Ker (I + R) \oplus U \). Then for every \( v \in (I + R)V_R \) there is a unique \( u \in U \) such that \( v = (I + R)u \) and clearly \( v = \lim_{t \to 0} i(t \cdot Id - i(I + \sqrt{1 - t^2R}))u, \)
where \( i(t \cdot Id - i(I + \sqrt{1 - t^2R}))u \in \tilde{C}(t) \). We launch a curve \( \Phi_t \in Hom(p, \tilde{C}(t)) \subset Hom(p, V_C) \) sending \( v \in (I + R)V_R \subset p \) to
\[
\Phi_t(v) = i(t \cdot Id - i(I + \sqrt{1 - t^2R}))u \in \tilde{C}(t)
\]
and \( iv \in i(I + R)V_R \subset p, v \in V_R \) to
\[
\Phi_t(iv) = -(t \cdot Id - i(I + \sqrt{1 - t^2R}))u \in \tilde{C}(t),
\]
\( \Phi_0 \) is the identity embedding \( p \mapsto V_C \). Let us differentiate \( \Phi_t \) at \( t = 0 \) and naturally descend the obtained homomorphism \( \varphi \in Hom(p, V_C) \) to a homomorphism \( \tilde{\varphi} \) in \( Hom(p, V_C/p) \). The homomorphism \( \varphi \) in \( Hom(p, V_C) \) is the mapping
\[
v \in (I + R)V_R \mapsto \frac{d}{dt} \left( i(t \cdot Id - i(I + \sqrt{1 - t^2R}))u \right) \bigg|_{t=0} = iu,
\]
\[
iv \in i(I + R)V_R \mapsto \frac{d}{dt} \left( -(t \cdot Id - i(I + \sqrt{1 - t^2R}))u \right) \bigg|_{t=0} = -u.
\]

As \( \varphi((I + R)V_R \oplus i(I + R)V_R) = U \oplus iU \in L_R \) then \( \tilde{\varphi} \in Hom(p, V_C/p) \) is a nonzero vector and it clearly belongs to \( T_pL_R \). In fact, the vector \( \tilde{\varphi} \) does not depend on the choice of our complement \( U \): if \( U_1 \) is another complement then for \( v \in (I + R)V_R \) we have \( \varphi_1(v) - \varphi(v) = i(u_1 - u) \in iKer(I + R) = Im (I + R) \) so that \( \tilde{\varphi}_1 = \tilde{\varphi} \) in \( Hom(p, V_C/p) \). This proves that \( \tilde{C}'(0) \in T_pL_R \).

The smoothness of \( S(I, R) \) at \( p \in S(I, R) \cap L_R \) is shown by considering an arbitrary curve together with its central-symmetric image \( t \mapsto \pm c(t) = \pm (x(t)I + y(t)R + z(t)IR) \in S(I, R) \) and the respective curves \( t \mapsto (Id \mp ic(t))V_R \subset Compl. \) Let us consider just the curve \( t \mapsto (Id - ic(t))V_R \). Let us write
\[
y(t) = \sqrt{(x(t))^2 - 1} \cos g(t), z(t) = \sqrt{(x(t))^2 - 1} \sin g(t)
\]
and assume that the function \( g(t) \) is continuously differentiable and \( g(0) = 0 \). Set \( b = g'(0) \). Now set \( x(t) = \frac{1}{t} \). Then \( (Id - ic(t))V_R = (tId - i(I + \sqrt{1 - t^2} \cos g(t)) \cdot R + \)
\[
\sqrt{1-t^2 \sin g(t) \cdot IR})V_R. \quad \text{Arguing as previously we get that}
\]
\[
\frac{d}{dt}(Id - ic(t))V_R \bigg|_{t=0} = \{(I + R)u \mapsto (b \cdot IR + i \cdot Id)u, i(I + R)u \mapsto (-Id + biIR)u\}. 
\]

All such tangent vectors and their nonzero scalar multiples form a subset in \( T_pS(I, R) \subset T_pGr(2n, V_C) = \text{Hom}(p, V_C/p) \) whose complement in \( T_pS(I, R) \) is the line \( \mathbb{R} \cdot \{(I + R)u \mapsto IRu, i(I + R)u \mapsto iIRu\} \), which is precisely the tangent line at \( p \) to the circle \( S(I, R) \cap L_R \). Considering the case of the curve \( t \mapsto (Id + ic(t))V_R \), which arises from the curve \( c(t) \subset S(I, R) \), central-symmetric to the previous one, we get the same plane with a removed line. Now it is clear that \( S(I, R) \) is smooth, in the differentiable sense, at the points in \( S(I, R) \cap L_R \).

As the tangent plane \( T_pS(I, R) \) is the limit of tangent planes \( T_qS(I, R) \) that are all invariant under the complex structure operator on \( T\text{Compl} \subset T\text{Gr}(2n, V_C) \), we conclude that \( T_pS(I, R) \) is also invariant under the complex structure operator, thus \( S(I, R) \) is indeed a smooth complex-analytic manifold. This completes the proof of the statement of the Theorem regarding \( S(I, R) \).

**Picture 1:** components \( S(I, R)^\pm \) glue into a sphere, tangent to \( L_R \) along a circle.

**The case of \( S(I, N) \).** As \( N^2 = 0 \) we certainly know that \( \text{Im} N \subset \text{Ker} N \), so that \( \dim \text{Im} N \leq \dim \text{Ker} N \), and, as \( \dim \text{Im} N + \dim \text{Ker} N = \dim V_R = 4n \), we must have \( \text{rank} N \leq 2n \), and in general it is possible to have a strict inequality. Now fixing \( \alpha, \beta \in \mathbb{R} \) and introducing \( c(y) = I + \alpha yN + \beta yIN \in S^+(I, N) \), where \( S^+(I, N) \) is one of two connected components of \( S(I, N) \) we have the limit \( p_+ = \lim_{y \to \infty} (Id - ic(y))V_R \), which is a (complex) \( 2n \)-subspace in \( V_C \), containing the (possibly proper) complex subspace
\[
(\alpha + \beta I)NV_R \oplus i(\alpha + \beta I)NV_R \subset p_+ \in L_R.
\]

Besides that, if we look at the vector subspace \( \text{Ker} N \subset V_R \), we can see that \( (Id - ic(y))\text{Ker} N = (Id - iI)\text{Ker} N \subset V_C \), so that it is the same complex subspace in \( V_C \) for all points \( c(y) = (Id - ic(y))V_R \) of our curve in \( \text{Gr}(2n, V_C) \). Now given that \( (\alpha + \beta I)N = N(\alpha - \beta I) \) and that for \( \alpha, \beta \) not both zero we have that \( \alpha \pm \beta I \) is an invertible operator, we have \( (\alpha + \beta I)NV_R = N(\alpha - \beta I)V_R = NV_R \). Making analogous calculations for another component of \( S(I, N) \) we can finally write our \( p_+ \) and \( p_- \) as
\[
p_\pm = (\text{Im} N \oplus i \cdot \text{Im} N) \oplus (Id \mp iI)\text{Ker} N.
\]
So we see that $S(I, N)$ consists of two connected components and $S(I, N) \cap L_R = \{p_+, p_-\}$.

Again, launching the curve $C(y)$ from the point $p_+$, at infinity and setting $y = \frac{1}{t}$, we get the curve

$$\tilde{C}(t) = C\left(\frac{1}{t}\right) = (t \cdot Id - i(t \cdot I + \alpha N + \beta IN))V_R.$$  

In order to find the tangent vector to such a curve at $p \in S(I, N) \cap L_R$ ($t = 0$), as a homomorphism $\tilde{\varphi} \in Hom(p_+, V_C/p_+)$ like in the previous case, we need to lift the curve $\tilde{C}(t)$ to the curve $\Phi_t \in Hom(p_+, \tilde{C}(t)) \subset Hom(p_+, V_C)$. For that note that $Ker N \subset V_R$ is $I$-invariant and let us choose an $I$-invariant subspace $U \subset V_R$ such that $V_R = Ker N \oplus U$. Then $N(\alpha - \beta I) : U \to Im N$ is an isomorphism and so for every $v \in Im N$ there exists a unique $u \in U$ such that $v = N(\alpha - \beta I)u$.

Next, define

$$\Phi_t(v) = i(t \cdot Id - i(t \cdot I + \alpha N + \beta IN))u,$$
$$\Phi_t(iv) = (t \cdot Id - i(t \cdot I + \alpha N + \beta IN))u$$

for $v \in Im N$, and $\Phi_t(v) = v$ for $v \in (Id - iI)Ker N$. Then for $v \in Im N$ we have $\Phi_0(v) = (\alpha N + \beta IN)u = N(\alpha - \beta I)u = v$, so that $\Phi_0$ is the identity embedding $p_+ \to V_C$.

Differentiating $\Phi_t$ at $t = 0$ we get the vector $\varphi_{\alpha+i\beta} \in Hom(p_+, V_C)$

$$v \mapsto i(Id - iI)u \mapsto (Id - iI)v \mapsto 0 \text{ for } v \in Ker N.$$ 

It is clear that the vectors $\varphi_{z_1}, \varphi_{z_2}, z_1, z_2 \in \mathbb{C} \setminus \{0\}, z_1 \neq -z_2$, defined with respect to the same $U$, satisfy $\varphi_{z_1} + \varphi_{z_2} = \varphi_{z_1z_2}$ and $a\varphi_z = \varphi_z$, $z \in \mathbb{C} \setminus \{0\}, a \in \mathbb{R} \setminus \{0\}$, so that our choice of $U$ provides a 2-plane in $Hom(p_+, V_C)$. Again, similarly to the previous case, the vectors $\tilde{\varphi}_{\alpha+i\beta} \in Hom(p_+, V_C/p_+)$ do not depend on the choice of the $I$-invariant complement $U$. Varying $\alpha, \beta$ we get the whole tangent 2-plane at $p_+$ and the intersection intersect $T_{p_+}S(I, N) \cap T_{p_+}L_R$ is zero.

As $\dim_R L_R = 8n^2 - 1$, that is, it is a codimension 1 locus in $Compl$, the tangent cone $T_{p_+}L_R$, intersecting the plane $T_{p_+}S(I, R)$ only at zero, cannot be an $8n^2 - 1$-dimensional vector subspace in the $8n^2$-dimensional vector space $T_{p_+}Compl$. All the observations are also similarly true for $p_-$.

**Picture 2:** Each component of the closure $\overline{S(I, N)}$ has precisely one point in $L_R$.  

\[ S(I, N)^+ \] \[ S(I, N)^- \] \[ S(I, N) \]
Now the proof of the theorem is complete. 

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Case of a compact $S$. The uniqueness of the representation equivalence classes for the algebra of quaternions, $\mathbb{H}(-1) = \mathbb{H} \to \text{End} V_\mathbb{K}$, is well known, it was discussed along with the transitivity of the $GL(V_\mathbb{K})$-action on twistor lines of the type $\mathbb{H}(-1)$ in [1].

Let $S = S(I, J)$ for anticommuting complex structures $I, J$. Let us fix a basis in $V_\mathbb{K}$ such that the complex structures $I, J$ have the following matrices in this basis

$$I = \begin{pmatrix} 0_{2n} & -1_{2n} \\ 1_{2n} & 0_{2n} \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} I_{2n} & 0_{2n} \\ 0_{2n} & -I_{2n} \end{pmatrix},$$

where we set $1_k$ to be the $k \times k$ identity matrix and $0_k$ the $k \times k$ zero matrix. Then setting $K = IJ$ we write the matrix of an arbitrary complex structure $\lambda = aI + bJ + cK \in S(I, J)$ in our basis as

$$\lambda = \begin{pmatrix} bI_{2n} & -a1_{2n} + cI_{2n} \\ a1_{2n} + cI_{2n} & -bI_{2n} \end{pmatrix}, a^2 + b^2 + c^2 = 1.$$

The matrices $Q$ corresponding to the classes in $Hdg_S \subset H^{1,1}(V_\mathbb{K}/\Gamma, \mathbb{R})$ (where we assume that the compact torus $V_\mathbb{K}/\Gamma$ is endowed with a complex structure) must satisfy $\lambda'Q\lambda = Q$ for all $\lambda \in S$, or, what is the same, they must satisfy $QI = IQ, QJ = JQ$. Such $Q$'s will automatically satisfy the first Riemann bilinear relation.

Let

$$Q = \begin{pmatrix} A & \; \; \; B \\ -B^t & \; \; \; D \end{pmatrix},$$

where $A, D$ are skew-symmetric $2n \times 2n$-matrices and $B$ an arbitrary $2n \times 2n$-matrix. Then the commutation relation $QI = IQ$ means that $D = A$ and $B^t = B$. The commutation relation $QJ = JQ$ means that if we write $A = \begin{pmatrix} A_1 & \; \; \; A_2 \\ -A_2^t & \; \; \; A_4 \end{pmatrix}$, where $A_1^t = -A_1, A_1' = -A_4, B = \begin{pmatrix} B_1 & \; \; \; B_2 \\ B_2^t & \; \; \; B_4 \end{pmatrix}$, then as

$$JQ = \begin{pmatrix} I_{2n} & \; \; \; 0_{2n} \\ 0_{2n} & \; \; \; -I_{2n} \end{pmatrix} \cdot \begin{pmatrix} A_1 & \; \; \; A_2 & B_1 & B_2 \\ -A_2 & \; \; \; A_4 & B_2^t & B_4 \\ -B_1 & \; \; \; -B_2 & A_1 & A_2 \\ -B_2 & \; \; \; -B_4 & -A_2^t & A_4 \end{pmatrix} = \begin{pmatrix} A_1' & \; \; \; -A_4 & -B_2^t & -B_4 \\ A_2 & \; \; \; A_1 & B_2 & B_1 \\ A_1 & \; \; \; A_2 & B_2 & B_1 \\ B_1 & \; \; \; B_2 & -A_4 & -A_2 \end{pmatrix},$$

and

$$QJ = \begin{pmatrix} A_1 & \; \; \; A_2 & B_1 & B_2 \\ -A_2^t & \; \; \; A_4 & B_2^t & B_4 \\ -B_1 & \; \; \; -B_2 & A_1 & A_2 \\ -B_2 & \; \; \; -B_4 & -A_2^t & A_4 \end{pmatrix} \cdot \begin{pmatrix} I_{2n} & \; \; \; 0_{2n} \\ 0_{2n} & \; \; \; -I_{2n} \end{pmatrix} = \begin{pmatrix} A_2 & \; \; \; -A_1 & -B_2 & -B_1 \\ A_4 & \; \; \; A_2^t & -B_4 & B_2^t \\ -B_2 & \; \; \; B_1 & -A_2 & A_1 \\ -B_4 & \; \; \; B_2 & -A_4 & -A_2^t \end{pmatrix},$$

we have that $A_2' = A_2, A_4 = A_1, B_2' = B_2, B_4 = -B_1$. Here we were able to get around without writing explicitly the first Riemann bilinear relation in the classical form, but in order to check the second one we need to write down the period matrix for a general point $\lambda \in S$. 


The dimension of all matrices \( Q \) satisfying the first bilinear relation is the sum of the dimensions of spaces of skew-symmetric \( n \times n \)-matrices \( A_1 \), of symmetric \( n \times n \)-matrices \( A_2, B_1, B_2 \), that is \( n(n-1)/2 + 3 \cdot n(n+1)/2 = n^2 + 2n^2 = 2n^2 + n \). This is the dimension of the subspace of classes \( Hdg_{S(I,J)} \) in \( H^{1,1}(V_\mathbb{R}/\Gamma, \mathbb{R}) \) which stay of type \((1,1)\) along \( S(I,J) \).

Let us now find the respective period matrices \( \Omega = (1_{2n}|Z) \) for the points of \( S \),

\[
(1_{2n}|Z) \cdot \lambda = (i1_{2n}|iZ),
\]

so that

\[
\begin{pmatrix} 0_n & -b1_n \\ b1_n & 0_n \end{pmatrix} + Z \cdot \begin{pmatrix} a1_n & -c1_n \\ c1_n & a1_n \end{pmatrix} = i1_{2n},
\]

which gives

\[
Z = \frac{1}{a^2 + c^2} \begin{pmatrix} (-bc + ai)1_n & (ab + ci)1_n \\ -(ab + ci)1_n & (-bc + ai)1_n \end{pmatrix}.
\]

The period matrices \( \Omega = (1_{2n}|Z) \) thus provide an affine chart containing all of \( S(I,J) \) except \( \pm J \) (the points of \( S \) for which \( a^2 + c^2 = 0 \)). Now following the Riemann bilinear notations in [5, Ch. 2.6] we set \( \tilde{\Omega} = \begin{pmatrix} \Omega \\ \overline{\Omega} \end{pmatrix} \), where \( \overline{\Omega} \) is obtained by complex conjugation of the entries of \( \Omega \), and we define the \( 4n \times 2n \)-matrix \( \tilde{\Pi} = (\Pi, \overline{\Pi}) \), such that \( \tilde{\Omega} \cdot \tilde{\Pi} = 1_{4n} \). Then setting \( \Pi = \begin{pmatrix} E \\ G \end{pmatrix} \) we get the defining equations for \( E \) and \( G \),

\[
E + ZG = 1_{2n}, \quad \overline{E} + Z\overline{G} = 0_{2n}.
\]

Subtracting from the first the complex conjugate of the second we get that \((Z - \overline{Z})G = 1_{2n} \), and so

\[
Z - \overline{Z} = 2Im Z = \frac{2i}{a^2 + c^2} \begin{pmatrix} a1_n & c1_n \\ -c1_n & a1_n \end{pmatrix}, \quad G = \frac{-i}{2} \begin{pmatrix} a1_n & -c1_n \\ c1_n & a1_n \end{pmatrix}.
\]

and then

\[
E = 1_{2n} - \frac{1}{a^2 + c^2} \begin{pmatrix} (-bc + ai)1_n & (ab + ci)1_n \\ -(ab + ci)1_n & (-bc + ai)1_n \end{pmatrix}, \quad G = \begin{pmatrix} (a1_n & -c1_n \\ c1_n & a1_n \end{pmatrix} = \begin{pmatrix} \frac{1}{2} a1_n & ib1_n \\ -ib1_n & 1_n \end{pmatrix}.
\]

Now we are ready to verify if the second Riemann bilinear relation \(-i\Pi^t Q \overline{\Pi} > 0 \) holds, scaling for convenience we write (where we replace entries of the kind \( x1_n \) with just \( x \) to keep the formulas compact) \( 4\Pi^t Q \overline{\Pi} = \begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix} \)
\[ A_1 + i(bA_2 + aB_1 + cB_2) \quad A_2 + i(-bA_1 - cB_1 + aB_2) \]
\[ -A_2 + i(bA_1 + aB_2 - cB_1) \quad A_1 + i(bA_2 - cB_2 - aB_1) \].

Now let us consider the upper-left $2n \times 2n$ block on the diagonal of $-i\Pi^t Q \Pi$ which is the hermitian matrix $\frac{1}{2}((bA_2 + aB_1 + cB_2) - iA_1)$. This matrix is positively or negatively definite if and only if its complex conjugate is such, so that if it is definite, then the real part $\frac{1}{2}(bA_2 + aB_1 + cB_2)$ is definite as well. For every $\lambda = aI + bJ + cK \in S(I, J)$ we always have that $-\lambda = -aI - bJ - cK \in S$, and the matrices $bA_2 + aB_1 + cB_2$ and $-bA_2 - aB_1 - cB_2$ cannot be both positively (or negatively) definite. This means that among classes in $Hdg_{S(I, J)}$ there are no Kähler classes. For a Kähler class $\Omega \in H^{1,1}(V_\mathbb{R}/\Gamma, \mathbb{R})$ the complex manifold $Compl_\Omega$ does not contain $S$, and so $Compl_\Omega$ and $S$ may only have finitely many points in common.

**Case of $S = S(I, R)$**. The anticommutation $IR = -RI$ tells us that $I$ establishes an isomorphism between the eigenspaces $Ker(R - Id)$ and $Ker(R + Id)$ of the operator $V_\mathbb{R} = Ker(R - Id) \oplus Ker(R + Id)$. Let $v_1, \ldots, v_{2n}$ be a basis of $Ker(R - Id)$, then $Iv_1, \ldots, Iv_{2n}$ is a basis of $Ker(R + Id)$.

Then we can write the matrices of $I$ and $R$ in the basis $v_1, v_2, \ldots, v_n$,

\[ I = \begin{pmatrix} 0_{2n} & -1_{2n} \\ 1_{2n} & 0_{2n} \end{pmatrix}, \quad R = \begin{pmatrix} 1_{2n} & 0_{2n} \\ 0_{2n} & -1_{2n} \end{pmatrix}. \]

This shows that all representation of the real 4-dimensional algebra $\mathbb{H}(1) = \langle i, r | i^2 = -1, r^2 = 1, ir + ri = 0 \rangle$ are equivalent, so that the group $GL(V_\mathbb{R})$ acts transitively on all twistor lines $S(I, R) \subset Compl \subset EndV_\mathbb{R}$. In fact the decomposition $V_\mathbb{R} = \langle v_1 + Iv_1, v_1 - Iv_1 \rangle \oplus \cdots \oplus \langle v_{2n} + Iv_{2n}, v_{2n} - Iv_{2n} \rangle$ breaks down the natural representation $i \mapsto I, r \mapsto R \in EndV_\mathbb{R}$ into the sum of irreducible 2-representations, and any two 2-representations of our algebra are isomorphic and faithful.

Like in the previous case the first bilinear relation for $Q$ holding over $S(I, R)$ is equivalent to the relations $\lambda'Q\lambda = Q$ for all $\lambda \in S(I, R)$. Note that, unlike in the previous case, the second generator $R$ of our algebra $\mathbb{H}(1)$ is not a complex structure, $R \notin S(I, R)$ and in the current case the relations $\lambda'Q\lambda = Q, \lambda \in S(I, R)$, are equivalent to the relations $IQ = QI, QR = RQ$. Indeed, setting $\lambda = I$ we get $IQ = QI$ and setting $\lambda = \sqrt{2}I + R$ we get the relation $(\sqrt{2}I + R)^t Q(\sqrt{2}I + R) = Q$. The latter, given that $(\sqrt{2}I + R)^t = -\sqrt{2}I + R = (\sqrt{2}I - R)^{-1}$, is equivalent to $Q(\sqrt{2}I + R) = (\sqrt{2}I - R)Q$, which together with the $I$-invariance of $Q$ implies that $QR = -RQ$. On the opposite, if $Q$ is $I$-invariant and $R$-antiinvariant, then $Q$ is $R_1$-antiinvariant for every $R_1 = bR + cIR$ and hence $\lambda$-invariant for every $\lambda = aI + \frac{\sqrt{2} - 1}{\sqrt{2} + 1}R_1 \in S(I, R)$.

The relations $IQ = QI, RQ = -QR$ tell us that the matrix $Q$ is an arbitrary matrix of the form

\[ Q = \begin{pmatrix} 0_{2n} & B \\ -B & 0_{2n} \end{pmatrix}, \]

where $B$ is any $2n \times 2n$ real symmetric matrix.

Now we need to write down the period matrices for the points $\lambda \in S(I, R)$ in order to formulate the second bilinear relation for $Q$. The matrix of the general $\lambda$ is given
by
\[ S(I, R) \ni \lambda = aI + bR + cIR = \begin{pmatrix} b1_{2n} & (c-a)1_{2n} \\ (a+c)1_{2n} & -b1_{2n} \end{pmatrix}, \quad a^2 - b^2 - c^2 = 1, \]
and we write down the relations defining the period matrices of points \( \lambda \in S(I, R) \), setting as earlier \( \Omega = (1_{2n}|Z) \),
\[
(1_{2n}|Z) \cdot \begin{pmatrix} b1_{2n} & (c-a)1_{2n} \\ (a+c)1_{2n} & -b1_{2n} \end{pmatrix} = (i1_{2n}|iZ).
\]
This gives us \( Z = \frac{-b+i}{a+c}1_{2n} \), and as \( a + c \neq 0 \) for \( a^2 - b^2 - c^2 = 1 \), our affine chart of matrices \( \Omega \) contains all of \( S(I, R) \). As earlier we write \( \tilde{\Omega} = \left( \frac{\Omega}{\Omega} \right) \) and \( \tilde{\Pi} = (\Pi, \Pi) \)
where \( \tilde{\Omega} \cdot \tilde{\Pi} = 1_{4n} \). Setting \( \Pi = \begin{pmatrix} E \\ G \end{pmatrix} \) and denoting \( u = \frac{-b+i}{a+c} \) we get
\[
\tilde{\Omega} \cdot \tilde{\Pi} = \left( \begin{pmatrix} 1_{2n} \\ u1_{2n} \end{pmatrix} \right) \cdot \begin{pmatrix} E \\ G \end{pmatrix} = 1_{4n}.
\]
Solving the latter matrix equation gives \( G = \frac{1}{u+\overline{a}}1_{2n} = -\frac{a+c}{2}i1_{2n} \) and \( E = 1_{2n} - uG = 1_{2n} - \frac{b+i}{a+c}(-\frac{a+c}{2}i)1_{2n} = (1 + \frac{-b+i}{2})1_{2n} = \frac{1-b+i}{2}1_{2n} \).

Setting now \( u_1 = \frac{1-bi}{2} \), \( u_2 = -\frac{a+c}{2}i \) we write the second bilinear relation as, \(-i\Pi^\dagger \overline{Q} \overline{\Pi} = \)
\[
-i(u_11_{2n}, u_21_{2n}) \begin{pmatrix} 0_{2n} & B \\ -B & 0_{2n} \end{pmatrix} \begin{pmatrix} \overline{u_1}1_{2n} \\ \overline{u_2}1_{2n} \end{pmatrix} = -i(-u_2B, u_1B) \begin{pmatrix} \overline{u_1}1_{2n} \\ \overline{u_2}1_{2n} \end{pmatrix} =
\]
\[
= -i(u_1\overline{u_2} - \overline{u_1}u_2)B = -\frac{i}{4}((1-bi)(a+c)i - (1+bi)(-(a+c)i))B =
\]
\[
= \frac{a+c}{4}(1-bi) + (1+bi))B = \frac{(a+c)}{2}B > 0.
\]

If we restrict to the upper sheet \( a = \sqrt{1+b^2+c^2} \) of our hyperboloid, we get
that \( B \) can be any positive definite matrix, and if we consider the lower sheet
\( a = -\sqrt{1+b^2+c^2} \) then \( B \) can be any negative definite matrix in order for the
relation \(-i\Pi^\dagger \overline{Q} \overline{\Pi} > 0 \) to hold.

**Remark 3.1.** Note that while \( S(I, R) \subset Gr(2n, V_\mathbb{C}) \) is a connected curve, the
components \( S(I, R)^+, S(I, R)^- \subset Compl \) do not share any Kähler class
staying of type (1,1) along both of them.

The dimension of all matrices \( Q \) satisfying the first bilinear relation is the dimension
of symmetric \( 2n \times 2n \)-matrices \( B \), that is, \( \frac{2n(2n+1)}{2} = n(2n+1) \). This is the dimension
of the subspace of classes \( Hdg_{S(I,R)} \) in \( H^{1,1}(V_\mathbb{R}/\Gamma, \mathbb{R}) \) which stay of type (1,1) along
\( S(I, R) \). The Kähler classes which stay of type (1,1) along one of the components
\( S(I, R) \) correspond to \( \pm B > 0 \) and form an open subset in \( Hdg_{S(I,R)} \).

**Case of** \( S(I, N) \). Again, we start with the representation theory of the real algebra
\( \mathbb{H}(0) = \{i, n\}^2 = -1, n^2 = 0, in + ni = 0 \).

As \( IN = -IN \) we have that both the kernel and the image of \( N \),
\( Im N \subset Ker N \subset V_\mathbb{R} \) are \( I \)-invariant subspaces. Next, choosing an
$I$-invariant complement $U$ for $\text{Ker} N$ in $V_R$, $V_R = \text{Ker} N \oplus U$, a basis $u_1, \ldots, u_k, Iu_1, \ldots, Iu_k$ of $U$, and an $I$-invariant complement $W$ for $\text{Im} N$ in $\text{Ker} N$ together with its basis of the form $w_1, \ldots, w_l, Iw_1, \ldots, Iw_l$, we get the basis $Nu_1, \ldots, Nu_k, Nu_1, \ldots, Nu_k$ of $\text{Im} N$, and we get the basis $Nu_1, \ldots, Nu_k, Nu_1, \ldots, w_l, Iw_1, \ldots, Iw_1, u_1, \ldots, u_k, Iu_1, \ldots, Iu_k$ of $V_R = \text{Im} N \oplus W \oplus U$, where $4k + 2l = \text{dim} V_R = 4n$.

In this basis $I$ and $N$ have the matrices

$$I = \begin{pmatrix}
0_k & 1_k & 0_{2k \times 2l} & 0_{2k} \\
-1_k & 0_k & 0_{2l \times 2k} & 0_{2k} \\
0_{2l \times 2k} & 0_l & -1_l & 0_{2l \times 2k} \\
0_{2k} & 0_{2k \times 2l} & 0_k & -1_k \\
\end{pmatrix},
N = \begin{pmatrix}
0_{2k} & 0_{2k \times 2l} & 1_{2k} \\
0_{2l \times 2k} & 0_{2l} & 0_{2l \times 2k} \\
0_{2k} & 0_{2k \times 2l} & 0_{2k} \\
\end{pmatrix},$$

which proves that the every representation of our algebra is reducible (but not completely reducible). All non-equivalent 4n-representations are parametrized by the values of nonnegative integers $k, l$ satisfying $4k + 2l = 4n$, that is, $l = 2(n - k)$, we have that there are $n$ such 4n-representations.

Now let $Q$ be a skew-symmetric $4n \times 4n$-matrix written in our basis. The first bilinear relation for $Q$ with respect to the periods in $S(I, N)$ can be written as

$$(I + aN + bIN)^t Q(I + aN + bIN) = Q$$

for all $a, b \in \mathbb{R}$.

Expanding as a polynomial in $a, b$, $(I + aN + bIN)^t Q(I + aN + bIN) = I^t QI + a(N^t QI + I^t QN) + b((IN)^t QI + I^t (IN) Q) + a^2 N^t QN + b^2 (IN)^t QN + (IN)^t QN = Q$ and equating the coefficients of the monomials to zero, we get the defining equations for $Q$. In particular, we have $I^t QI = Q$, which allows us to rewrite the $a$-coefficient as $N^t QI + I^t QN = N^t QI - QIN = N^t QI + QNI = (N^t Q + QN)I = 0$, that is, $N^t Q + QN = 0$. Similarly $b$-coefficient gives the same equality, $a^2$- and $b^2$-coefficients give the same equality $N^t QN = 0$, and $ab$-coefficient gives $N^t QIN + (IN)^t QN = N^t (QI + I^t Q)N = 0$, which holds automatically because already $QI + I^t Q = 0$ (or, because $N^t QIN + (IN)^t QN = -N^t QNI + N^t QNI = 0 + 0 = 0$).

Next, let

$$Q = \begin{pmatrix}
A & B & C \\
-B^t & D & E \\
-C^t & -E^t & F \\
\end{pmatrix},
A^t = -A, D^t = -D, F^t = -F,$$

where, as earlier, the blocks correspond to the decomposition $V_R = \text{Im} N \oplus W \oplus U$. Let us start with the equality $QN + N^t Q = 0$. Rewriting it as $QN = -N^t Q$ and evaluating both sides we get $QN =

$$
\begin{pmatrix}
A & B & C \\
-B^t & D & E \\
-C^t & -E^t & F \\
\end{pmatrix}
\begin{pmatrix}
0_{2k} & 0_{2k \times 2l} & 1_{2k} \\
0_{2l \times 2k} & 0_{2l} & 0_{2l \times 2k} \\
0_{2k} & 0_{2k \times 2l} & 0_{2k} \\
\end{pmatrix}
= \begin{pmatrix}
0_{2k} & 0_{2k \times 2l} & A \\
0_{2l \times 2k} & 0_{2l} & -B^t \\
0_{2k} & 0_{2k \times 2l} & -C^t \\
\end{pmatrix}.$$
and $N^t Q =$

$$
\begin{pmatrix}
0_{2k} & 0_{2k \times 2l} & 0_{2k} \\
0_{2l \times 2k} & 0_{2l} & 0_{2l \times 2k} \\
1_{2k} & 0_{2k \times 2l} & 0_{2k}
\end{pmatrix}
\begin{pmatrix}
A & B & C \\
-B^t & D & E \\
-C^t & -E^t & F
\end{pmatrix}
\begin{pmatrix}
0_{2k} & 0_{2k \times 2l} & 0_{2k} \\
0_{2l \times 2k} & 0_{2l} & 0_{2l \times 2k} \\
A & B & C
\end{pmatrix},
$$

so that $Q N = - N^t Q$ holds precisely when $A = 0, B = 0, C = C^t$. For such $Q$ we automatically have $N^t Q N = 0$ so that what is left to check is $I^t Q I = Q$, or, equivalently, $Q I = I Q$. Denoting the three blocks of $I$ on the diagonal by $I_1, I_2, -I_1$, we get

$$
I Q = \begin{pmatrix}
0_{2k} & 0_{2k \times 2l} & I_1 C \\
0_{2l \times 2k} & I_2 D & I_2 E \\
I_1 C & I_1 E^t & -I_1 F
\end{pmatrix},
Q I = \begin{pmatrix}
0_{2k} & 0_{2k \times 2l} & -C I_1 \\
0_{2l \times 2k} & D I_2 & -E I_1 \\
-C I_1 & -E^t I_2 & -F I_1
\end{pmatrix},
$$

so that $C$ is a symmetric $2k \times 2k$-matrix anticommuting with $I_1, D$ is a skew-symmetric $2l \times 2l$-matrix commuting with $I_2$, $F$ is a skew-symmetric $2k \times 2k$-matrix commuting with $I_1$ and $E$ is a $2l \times 2k$-matrix satisfying $I_2 E = -E I_1$. The vector spaces of such $C, D, F$ have the respective dimensions $k^2 + k, l^2$ and $k^2$. Writing $E = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{pmatrix}$, where all blocks have the same size $l \times k$, the equality $I_2 E = -E I_1$ gives $E_1 = E_2, E_3 = -E_2$, so that the dimension of such $E$’s is equal $2kl$. We recall that $4k + 2l = 4n$ so that $l = 2(n - k)$. Now the total dimension of the space of such matrices $Q$ is $k^2 + k + (k^2 + l^2 + 2kl) = k^2 + k + (k + l)^2 = k^2 + k + (2n - k)^2$. This is the dimension of the subspace $Hdg_S(I, N)$ of classes in $H^{1,1}(V_R / T, \mathbb{R})$ which stay of type $(1, 1)$ along $S(I, N)$. As the restriction of any such $Q$ to the $I$-invariant subspace $Im N$ is identically zero (and $N \neq 0$), we have that the hermitian forms representing $(1, 1)$-classes corresponding to our $Q$’s all have isotropic vectors, and thus cannot be positively definite, so that there are no Kähler classes surviving along any connected component of $S(I, N)$.

\[\square\]

4. PROOF OF THEOREM 1.3

Let $T$ be an element of $End V_R$, set $G_T$ to be the adjoint action stabilizer of $T$ in $G = GL(V_R), G_T = \{ g \in G \mid g T g^{-1} = g \}$. For $H(\varepsilon) \subset End V_R$ we set $G_{H(\varepsilon)} \subset G$ to be the adjoint action pointwise stabilizer of $H(\varepsilon)$.

The proof of Theorem 1.3 relies heavily on Proposition 4.5, whose proof requires the following lemma.

**Lemma 4.1.** Let $\varepsilon$ be any of $-1, 0, 1$. The algebra $H(\varepsilon)$ is generated, as a real algebra, by any two of its linearly independent elements from $\langle i, j, ij \rangle \subset H(\varepsilon)$.

**Proof.** Proof of Lemma 4.1 Set $k = ij$. Let $i_1 \neq \pm i_2$ be two elements in $\langle i, j, k \rangle \subset H(\varepsilon) = \langle i, j \mid i^2 = -1, j^2 = \varepsilon, ij + ji = 0 \rangle$. The proof relies on the use of the quadratic $H(\varepsilon)$-valued form $q$ on $H(\varepsilon)$, $q(x, y) = xy + yx$. The restriction $Q = q|_{\langle i, j, k \rangle}$ is an $\mathbb{R}$-valued form. The elements $x$ and $y$ in $\langle i, j, k \rangle$ anticommute if and only if $x \perp Q y$. For $\varepsilon = -1$ and $\varepsilon = 1$ the form $Q$ has respective signatures $(0, 3)$ and $(2, 1)$, so that it is nondegenerate and thus we can construct an 1-dimensional orthogonal complement to the plane $\langle i_1, i_2 \rangle$ in $\langle i, j, k \rangle$. If $u, v$ is an orthogonal basis of $\langle i_1, i_2 \rangle$ then $uv$ anticommutes with both $u$ and $v$ and so spans the orthogonal
is equal 8

In the case of \( \mathbb{H}(0) \) we cannot apply the same argument as \( Q = q|_{(i,j,k)} \) has now signature \((0,1,2)\), and so we do not apriori know if \( i_1i_2 \notin \langle i_1, i_2 \rangle \). In fact, we can give a direct argument here: multiplying by -1, if needed, we may assume that \( i_1 = i + a_1j + a_2k, i_2 = i + b_1j + b_2k \). Let us denote \( i' = i_1, j' = i_1 - i_2 \), then the elements \( 1, i', j', i'j' \) are easily seen to be linearly independent, so that \( \mathbb{H}(0) = \langle 1, i', j', i'j' \rangle = \langle i_1, i_2, i_1i_2 \rangle \).

As a consequence of Lemma 4.1 we get the following generalization of an analogous result in [1] stating that a twistor line is uniquely defined by any pair of its non-proportional points.

**Corollary 4.2.** Let \( S_1, S_2 \subset \text{Compl} \) be any two any generalized twistor lines. If the intersection of lines \( S_1 \cap S_2 \) contains non-proportional points, then \( S_1 = S_2 \).

Indeed, \( S_1 \subset \mathbb{H}(\varepsilon_1) \subset \text{End} V_\mathbb{R}, S_2 \subset \mathbb{H}(\varepsilon_2) \subset \text{End} V_\mathbb{R} \) and the imaginary units in \( S_1 \cap S_2 \) must be contained in the respective subspaces \( \langle i, j, k \rangle \) of each of the two algebras, so by Lemma 4.1 they generate each of \( \mathbb{H}(\varepsilon_1), \mathbb{H}(\varepsilon_2) \). Hence the latter subalgebras coincide, therefore \( S_1 = S_2 \).

**Corollary 4.3.** Let \( I_1, I_2 \) be any two linearly independent imaginary units in \( \mathbb{H}(\varepsilon) \subset \text{End} V_\mathbb{R} \), where \( \varepsilon \) is any of \(-1, 0, 1\). Then \( G_{I_1} \cap G_{I_2} = G_{\mathbb{H}(\varepsilon)} \).

Let \( I, J \in \text{End} V_\mathbb{R} \) satisfy \( I^2 = -Id, J^2 = \varepsilon Id, IJ + JI = 0 \) for \( \varepsilon \) equal 1 or -1. Set \( K = IJ \). One easily calculates that \( \dim_{\mathbb{R}} G_I = \dim_{\mathbb{R}} G_J = \dim_{\mathbb{R}} G_K = 8n^2 \) and \( \dim_{\mathbb{R}} G_{\mathbb{H}(\varepsilon)} = 4n^2 \).

**Lemma 4.4.** Let \( \varepsilon \) be any of \( 1, -1 \). We have the direct sum decomposition
\[ T_eG/T_eG_{\mathbb{H}(\varepsilon)} = T_eG_I/T_eG_{\mathbb{H}(\varepsilon)} \oplus T_eG_J/T_eG_{\mathbb{H}(\varepsilon)} \oplus T_eG_K/T_eG_{\mathbb{H}(\varepsilon)}. \]

**Proof.** First of all, the dimension of each of the three quotient spaces in the direct sum is equal \( 8n^2 - 4n^2 = 4n^2 \) so that the dimensions sum up to \( 12n^2 = \dim_{\mathbb{R}} T_eG/T_eG_{\mathbb{H}(\varepsilon)}. \)

Let us show that the sum of the subspaces is indeed direct. Let \( u \in T_eG_I, v \in T_eG_J, w \in T_eG_K \) and suppose \( u + v + w = 0 (mod T_eG_{\mathbb{H}(\varepsilon)}) \). Then applying the Lie bracket \([., I]\) to this equality and using that \( uI = Iu \) we get \( [v, I] + [w, I] \in T_eG_{\mathbb{H}(\varepsilon)} \).

Thus the sum \( [v, I] + [w, I] \) must commute with \( I \). By construction it also anticommutes with \( I \), so that indeed \( [v, I] + [w, I] = 0 \in T_eG \). Next, \( [v, I] \) anticommutes with \( I \) and, as \( v \) commutes with \( J \), \( [v, I] \) anticommutes with \( J \), so that finally \( [v, I] \) commutes with \( K \).

In a similar way we show that \( [w, I] \) commutes with \( J \), so that \( [v, I] = -[w, I] \) commutes with both \( K \) and \( J \). Now if \( J^2 = \pm Id \), that is \( J \) is an invertible element of our algebra, we have that \( [v, I] \) and \( [w, I] \) commute with \( I = -\varepsilon JK \), so that \( [v, I] = [w, I] = 0 \in T_eG \) and \( v \in T_eG_I \cap T_eG_J = T_eG_{\mathbb{H}(\varepsilon)} \), \( w \in T_eG_K \cap T_eG_I = T_eG_{\mathbb{H}(\varepsilon)}. \) Then \( u \in T_eG_{\mathbb{H}(\varepsilon)} \) as well. This proves that we have the stated direct sum decomposition.

**Proposition 4.5.** Let \( I_1, I_2, I_3 \) be complex structures belonging to the same twistor sphere \( S \) of type \( \mathbb{H}(\varepsilon) \), where \( \varepsilon = \pm 1 \). The submanifolds \( G_{I_1}/G_{\mathbb{H}(\varepsilon)} \supset G_{I_2}/G_{\mathbb{H}(\varepsilon)} \supset G_{I_3}/G_{\mathbb{H}(\varepsilon)} \) in \( G/G_{\mathbb{H}(\varepsilon)} \) intersect transversally (as a triple) at \( eG_{\mathbb{H}(\varepsilon)} \) if and only if \( I_1, I_2, I_3 \) are linearly independent as vectors in \( \text{End} V_\mathbb{R}. \)
Proof. The proof essentially uses the transversality proved in Lemma 4.4 that is, the fact that
\begin{equation}
\frac{T_e G}{T_e G_{\theta(e)}} = V_I \oplus V_J \oplus V_K,
\end{equation}
where we set \( V_I = T_e G_I / T_e G_{\theta(e)}, V_J = T_e G_J / T_e G_{\theta(e)}, V_K = T_e G_K / T_e G_{\theta(e)}, K = IJ. \)

Again, as previously, we have that \( \dim_{\mathbb{R}} V_1 + \dim_{\mathbb{R}} V_2 + \dim_{\mathbb{R}} V_3 = \dim_{\mathbb{R}} T_e G / T_e G_{\theta(e)}. \) Let \( S^2 = S(I, J) \) where \( I^2 = -Id, J^2 = \varepsilon Id, IJ = -JI. \) Let \( I_i = a_i I + b_i J + c_i K, i = 1, 2, 3. \) Suppose that for certain vectors \( X \in V_1, Y \in V_2 \) and \( Z \in V_3 \) we have \( X + Y + Z = 0. \) Let us decompose the vector \( X \) into the sum of its components in the respective subspaces of the decomposition (3), \( X = X_I + X_J + X_K, \) and do similarly for \( Y \) and \( Z. \) Then for \( X \) the commutation relation \( [X, I_1] = 0 \) can be written as
\[
a_1[X_J + X_K, I] + b_1[X_I + X_K, J] + c_1[X_I + X_J, K] = 0.
\]
Noting that in the above expression, for example, the term \([X_I, I]\) anticommutes with both \( I, J, \) hence commutes with \( K = IJ, \) and an analogous commutation holds for other terms as well (here it is important that \( J \) and \( K \) are invertible elements of our \( \mathbb{H}/(e), \) so that \( I = -\varepsilon JK) \), we can decompose the expression on the left side of the above equality with respect to (3),
\[
(b_1[X_K, J] + c_1[X_J, K]) + (a_1[X_K, I] + c_1[X_I, K]) + (a_1[X_J, I] + b_1[X_I, J]) = 0.
\]
From here we conclude that \( b_1[X_K, J] + c_1[X_J, K] = 0, a_1[X_K, I] + c_1[X_I, K] = 0 \) and \( a_1[X_J, I] + b_1[X_I, J] = 0. \) Assume, that \( a_1 \neq 0. \) Then we can use the last two equalities to express
\begin{equation}
[X_J, I] = -\frac{b_1}{a_1}[X_I, J], [X_K, I] = -\frac{c_1}{a_1}[X_I, K].
\end{equation}
Note that \([\cdot, J] : V_I \to V_K, [\cdot, K] : V_I \to V_J \) and \([\cdot, I] : V_I \to V_K \) are isomorphisms of the respective vector spaces. Let us denote these isomorphisms \( F_J, F_K, F_I \) respectively. Then equalities (4) allow us to write
\[
X_J = -\frac{b_1}{a_1} F_I^{-1} \circ F_J(X_I), X_K = -\frac{c_1}{a_1} F_I^{-1} \circ F_K(X_I),
\]
so that
\[
X = X_I + \left( -\frac{b_1}{a_1} F_I^{-1} \circ F_J(X_I) \right) + \left( -\frac{c_1}{a_1} F_I^{-1} \circ F_K(X_I) \right)
\]
Assuming that \( a_2, a_3 \neq 0 \) we can get similar expressions for \( Y \) and \( Z. \) Actually, for the case \( \varepsilon = 1 \) we automatically have that \( a_i \neq 0, 1 \leq i \leq 3, \) and for the case \( \varepsilon = -1 \) choosing the quaternionic triple \( I, J, K \) appropriately, we may assume that all \( a_i, i = 1, 2, 3, \) are nonzero. Using the above representation of \( V_J \) and \( V_K \)-components of \( X \) in terms of \( X_I, \) similarly for \( Y \) and \( Z, \) we can write the equality \( X + Y + Z = 0 \) component-wise
\[
\begin{pmatrix}
1 & 1 & 1 \\
-\frac{b_1}{a_1} & -\frac{b_2}{a_2} & -\frac{b_3}{a_3} \\
-\frac{c_1}{a_1} & -\frac{c_2}{a_2} & -\frac{c_3}{a_3}
\end{pmatrix}
\begin{pmatrix}
X_I \\
Y_I \\
Z_I
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
\]
This equation has a nontrivial solution if and only if the columns of the matrix, thus \( I_1, I_2, I_3, \) are linearly dependent. □
Proof of Theorem 1.3. Let $I_1, I_2, I_3$ be complex structures, linearly independent as elements in $\text{End} V$, belonging to the same connected component of a line $S(I, J)$, $I^2 = -Id, J^2 = \varepsilon, IJ = -JI$. Then by Corollary 1.3 we get that $G_{I_1} \cap G_{I_2} = G_I \cap G_J = G_{T(e)}$.

As in [1] we define the mapping $\Phi: G_{I_1} \times G_{I_2} \to \text{Compl}$ by $(g_1, g_2) \mapsto g_1 g_2 I_3 g_2^{-1} g_1^{-1} \in \text{Compl}$. Let us first show that near the identity element $(e, e) \in G_{I_1} \times G_{I_2}$ the mapping $\Phi$ is a submersion onto a neighborhood of $I_3$ in $\text{Compl}$. The differential $d_{(e,e)}\Phi: G_{I_1} \times G_{I_2} \to \text{Compl}$ factors through the quotient map $T_e G_{I_1} \oplus T_e G_{I_2} \to T_e G_{I_1}/T_e G_{T(e)} \oplus T_e G_{I_2}/T_e G_{T(e)}$, denote the resulting map $\tilde{d}_{(e,e)}\Phi: T_e G_{I_1}/T_e G_{T(e)} \oplus T_e G_{I_2}/T_e G_{T(e)} \to T_{I_3} \text{Compl}$. The dimensions of the domain and the target space of $\tilde{d}_{(e,e)}\Phi$ are equal, so in order to show that $\tilde{d}_{(e,e)}\Phi$ is a submersion near $(e, e) \in G_{I_1} \times G_{I_2}$ onto a neighborhood of $I_3$ in $\text{Compl}$ it is enough to show that $\tilde{d}_{(e,e)}\Phi$ is injective. For $X \in T_e G_{I_1}/T_e G_{T(e)}$ and $Y \in T_e G_{I_2}/T_e G_{T(e)}$ we have

$$d_{(e,e)}\Phi(X + Y) = \frac{d}{dt} \big|_{t=0} e^{tY} X e^{tY} e^{-tY} e^{-tX} = (X + Y) I_3 - I_3 (X + Y) = [X + Y, I_3] \in T_{I_3} \text{Compl}.$$ 

If we assume $[X + Y, I_3] = 0$ then $X + Y \in T_e G_{I_1}/T_e G_{T(e)}$ and so by Proposition 1.5 we have that $X = 0$ and $Y = 0$. This shows the required injectivity of $\tilde{d}_{(e,e)}\Phi$.

Now, locally around $I_3 \in \text{Compl}$ every point $I$ is in the image of $\Phi$, that is, $I = g_1 g_2 I_3 g_2^{-1} g_1^{-1} = g_1 g_2 I_3$ for $g_1 \in G_{I_1}, g_2 \in G_{I_2}$. Then the complex structures $g_1 g_2 I_2 = g_1 I_2, g_1 g_2 I_3$ span the line $g_1 g_2 S(I_2, I_3) = S(g_1 I_2, g_1 g_2 I_3) = g_1 S(I_2, g_2 I_3)$, the complex structures $g_1 I_1 = I_1, g_1 I_2$ span $g_1 S(I_1, I_2) = S(I_1, g_1 I_2)$. Now the consecutive lines $S = S(I_1, I_3) = S(I_1, I_2), g_1 S = S(I_1, I_2)$ and $g_1 g_2 S = S(g_1 I_2, g_1 g_2 I_3)$ form a path joining $I_3$ to $I = g_1 g_2 I_3$. Finally, passing to the global picture similarly to how it was done in [4], in every connected component of $\text{Compl}$ any two points can be joined by a path of connected components of generalized twistor lines.

References

[1] Buskin, N., Izadi, E., Twistor lines in the period domain of complex tori, [arXiv:1806.07831] [math.AG].
[2] Barth, W., Hulek, K., Peters, C., van de Ven, A., Compact complex surfaces, 2nd ed. 1995, XII, 436 p.
[3] Beauville, A., et al. Géométrie des surfaces $K3$: modules et périodes, Papers from the seminar held in Palaiseau, Astérisque 126 (1985), pp.1-193.
[4] Huybrechts, D., A global Torelli theorem for hyper-Kähler manifolds [after M. Verbitsky], Astérisque No. 348 (2012), pp. 375-403.
[5] Griffiths, P., Harris, J., Principles of complex algebraic geometry, John Wiley & Sons Inc., 1978, 813 p.
[6] Hitchin, N. J., Karlhede, A., Lindström, U., Roček, M., Hyper-Kähler metrics and supersymmetry, Comm. Math. Phys. Volume 108, Number 4 (1987), pp. 535-589.

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