THE GORENSTEIN PROJECTIVE MODULES ARE PRECOVERING

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Abstract. The Gorenstein projective modules are proved to form a precovering class in the module category of a ring which has a dualizing complex.

0. Introduction

This paper proves over a wide class of rings that the Gorenstein projective modules form a precovering class in the module category. Let me explain this statement. There are two terms of mystery, “Gorenstein projective modules” and “precovering class”; I will explain the latter first.

Precovering classes are also known by the name of contravariantly finite classes. In a module category, a class \(G\) of modules is precovering if it satisfies the following: For each module \(M\), there exists a homomorphism \(G \to M\) with \(G\) in \(G\), such that if \(\tilde{G} \to M\) is any homomorphism with \(\tilde{G}\) in \(G\) then the dotted arrow exists to make the following diagram commutative,

\[
\begin{array}{ccc}
G & \Rightarrow & M \\
\downarrow & & \downarrow \\
\tilde{G} & \Rightarrow & M
\end{array}
\]

The homomorphism \(G \to M\) is called a \(G\)-precover of \(M\). By taking the kernel of \(G \to M\), taking a \(G\)-precover, and repeating, I can construct a \(G\)-resolution of \(M\),

\[
\cdots \to G_2 \to G_1 \to G_0 \to M \to 0,
\]

which becomes exact when I apply the functor \(\text{Hom}(G, -)\) with \(G\) in \(G\). The name “precovering class” is due to [4]. If \(G\) is precovering,
then I can do homological algebra using $G$-resolutions instead of projective resolutions. There is a large literature on this so-called “relative homological algebra”; an example is the recent [9].

Gorenstein projective modules are modules which have the form $G = \text{Ker}(E^1 \rightarrow E^2)$, where $E$ is a complex of projective modules which is exact and satisfies that $\text{Hom}(E, Q)$ is exact for each projective module $Q$. The first complete statement of this definition seems to be in [3], but the idea goes back to [1]. The point is that one can do homological algebra with Gorenstein projective modules instead of projective modules, and that such “Gorenstein homological algebra” does for Gorenstein rings what ordinary homological algebra does for rings of finite global dimension. The archetypal result is that a noetherian local commutative ring is Gorenstein if and only if each finitely generated module has a bounded Gorenstein projective resolution. An extensive theory of Gorenstein homological algebra has been developed; part of it is already in [1], a comprehensive source which was up to date when it was written is [3], and for some recent work see [9] and [10].

A weakness of the existing literature on Gorenstein homological algebra is that it has only been known in special cases that the Gorenstein projective modules form a precovering class. The state of the art appears to be [10, prop. 2.18] which only proves the precovering property over Gorenstein rings.

The traditional remedy for this weakness has been to relax the conditions imposed on the Gorenstein projective resolutions used in the theory, which are then required just to be exact rather than obtained from successive precovers. This is formalized in the theory of resolving classes, but suffers from the serious shortcoming that it does not permit the definition of relative derived functors.

However, this paper removes the weakness by proving that the Gorenstein projective modules do form a precovering class over a ring which has a dualizing complex. For the sake of simplicity, the main part of the paper, sections [1] to [3] proves this result over a noetherian commutative ring. However, as I will show in section [4] the proofs really apply to much more general (non-commutative) algebras with dualizing complexes.

The idea of the proof is taken from [12], and is, to my knowledge, different from that used in other papers on precovering classes. Rather than attack the problem directly, I pass to $K(\text{Pro}A)$, the homotopy category of complexes of projective modules over the ring $A$.

Inside it sits the subcategory $E(A)$ of complexes $E$ which are exact and have $\text{Hom}(E, Q)$ exact for each projective module $Q$. Crucially, this subcategory can be characterized as the kernel of a homological
functor which respects small coproducts (see the proof of proposition 2.2), and this enables me to use Bousfield localization to see that the inclusion functor $E(A) \to K(\text{Pro} A)$ has a right adjoint $K(\text{Pro} A) \to E(A)$ (proposition 2.2).

This implies that $E(A)$ is a precovering class in $K(\text{Pro} A)$ (proposition 2.4). However, the Gorenstein projective modules are the modules of the form $\text{Ker}(E^1 \to E^2)$ for $E$ in $E(A)$, and it turns out that the result on $E(A)$ descends to give that the Gorenstein projective modules form a precovering class in the module category (lemma 3.1 and theorem 3.2).

Let me mention some related work: First, several previous papers have investigated whether the Gorenstein projective modules form a precovering class. As mentioned, I believe the state of the art to be [10, prop. 2.18]. Secondly, in addition to defining Gorenstein projective modules, the paper [6] also defined Gorenstein flat and Gorenstein injective modules, and [5] and [7] proved for large classes of rings that the Gorenstein flat modules form a precovering class and that the Gorenstein injective modules form a preenveloping class (the dual notion to precovering). Hence the present paper is a natural complement to [5] and [7].

Notation. Let me close the introduction by setting up a minimum of notation. In sections 1, 2, and 3 (but not in section 4), the following two setups are in force.

**Setup 0.1.** Let $A$ be a noetherian commutative ring with a dualizing complex $D$. That is,

(i) The cohomology of $D$ is bounded and finitely generated over $A$.

(ii) The injective dimension $\text{id}_A D$ is finite.

(iii) The canonical morphism $A \to \text{RHom}_A(D, D)$ in the derived category $D(A)$ is an isomorphism.

**Setup 0.2.** Let $D \to I$ be an injective resolution so that $I$ is a bounded complex.

See [8, chp. V] for background on dualizing complexes.

**Definition 0.3.** By $E(A)$ is denoted the class of complexes $E$ of $A$-modules so that $E$ consists of projective modules, is exact, and has $\text{Hom}_A(E, Q)$ exact for each projective $A$-module $Q$.

I will view $E(A)$ as a full subcategory of $K(\text{Pro} A)$, the homotopy category of complexes of projective $A$-modules.
Definition 0.4. An \( A \)-module is called Gorenstein projective if it has the form \( \text{Ker}(E^1 \to E^2) \) for some \( E \in \mathcal{E}(A) \).

Observe that each projective \( A \)-module \( Q \) is Gorenstein projective, since \( Q \) is equal to \( \text{Ker}(Q \to 0) \), and since \( Q \to 0 \) is part of the complex \( \cdots \to 0 \to Q \to 0 \to \cdots \) which is null homotopic and hence in \( \mathcal{E}(A) \).

Remark 0.5. Since \( A \) has a dualizing complex, it has finite Krull dimension by [8, cor. V.5.2], so by [10, Seconde partie, cor. (3.2.7)], each flat \( A \)-module has finite projective dimension.

1. A Lemma

The following lemma uses \( I \), the bounded injective resolution of \( D \) from setup 0.2.

Lemma 1.1. Let \( P \) be a complex of projective \( A \)-modules. Then

\[
\text{Hom}_A(P, Q) \text{ is exact for each projective } A\text{-module } Q \iff I \otimes_A P \text{ is exact.}
\]

Proof. \( \Rightarrow \) Suppose that \( \text{Hom}(P, Q) \) is exact for each projective module \( Q \). To see that \( I \otimes P \) is an exact complex, it is enough to see that

\[
\text{Hom}(I \otimes P, J) \cong \text{Hom}(P, \text{Hom}(I, J))
\]

is exact for each injective module \( J \).

But \( \text{Hom}(I, J) \) is a bounded complex of flat modules, so is finitely built from flat modules in the homotopy category of complexes of \( A \)-modules, \( \mathcal{K} \text{(Mod } A) \), so it is enough to see that \( \text{Hom}(P, F) \) is exact for each flat module \( F \).

Since \( F \) has finite projective dimension by remark 0.5, there is a projective resolution \( \tilde{P} \xrightarrow{\sim} F \) with \( \tilde{P} \) bounded. Since \( P \) consists of projective modules and both \( \tilde{P} \) and \( F \) are bounded, this induces a quasi-isomorphism

\[
\text{Hom}(P, \tilde{P}) \approx \text{Hom}(P, F).
\]

So it is enough to see that \( \text{Hom}(P, \tilde{P}) \) is exact.

But \( \tilde{P} \) is a bounded complex of projective modules, so is finitely built from projective modules, so it is enough to see that \( \text{Hom}(P, Q) \) is exact for each projective module \( Q \). And this holds by assumption.

\( \Leftarrow \) Suppose that \( I \otimes P \) is an exact complex. I must show that \( \text{Hom}(P, Q) \) is exact for each projective module \( Q \).
First observe that by [2, thm. (3.2)], there is an isomorphism
\[ Q \simto R\text{Hom}(D, D \otimes^L Q). \]

Of course, I can replace \(D\) by \(I\) to get
\[ Q \simto R\text{Hom}(I, I \otimes^L Q). \]

Here \(I \otimes^L Q \cong I \otimes Q\) because \(Q\) is projective. Moreover, \(I \otimes Q\) is a bounded complex of injective modules so \(R\text{Hom}(I, I \otimes^L Q) \cong R\text{Hom}(I, I \otimes Q) \cong \text{Hom}(I, I \otimes Q)\). So the above isomorphism in the derived category is represented by the chain map
\[ Q \longrightarrow \text{Hom}(I, I \otimes Q) \]
which must accordingly be a quasi-isomorphism.

Completing to a distinguished triangle in \(K(\text{Mod}\ A)\) gives
\[ Q \longrightarrow \text{Hom}(I, I \otimes Q) \longrightarrow C \longrightarrow \]
where \(C\) is exact. Here \(I\) and \(I \otimes Q\) are bounded, so \(\text{Hom}(I, I \otimes Q)\) is bounded. As the same is true for \(Q\), the complex \(C\) is also bounded.

Now, the distinguished triangle gives another distinguished triangle
\[ \text{Hom}(P, Q) \longrightarrow \text{Hom}(P, \text{Hom}(I, I \otimes Q)) \longrightarrow \text{Hom}(P, C) \longrightarrow . \]

Here \(\text{Hom}(P, C)\) is exact because \(P\) is a complex of projective modules while \(C\) is a bounded exact complex. So to see that \(\text{Hom}(P, Q)\) is exact as desired, it is enough to see that \(\text{Hom}(P, \text{Hom}(I, I \otimes Q))\) is exact.

However,
\[ \text{Hom}(P, \text{Hom}(I, I \otimes Q)) \cong \text{Hom}(I \otimes P, I \otimes Q). \]

And this is exact because \(I \otimes P\) is exact by assumption while \(I \otimes Q\) is a bounded complex of injective modules. \(\square\)

2. Complexes

**Lemma 2.1.** The triangulated category \(K(\text{Pro}\ A)\) is compactly generated.

*Proof.* The ring \(A\) is noetherian and hence coherent, and by remark [0.5] each flat \(A\)-module has finite projective dimension. So the proposition holds by [13, thm. 2.4]. \(\square\)

Combining lemmas [1.1] and [2.1] gives the following key result.

**Proposition 2.2.** The inclusion functor \(e_* : E(A) \longrightarrow K(\text{Pro}\ A)\) has a right-adjoint \(e^! : K(\text{Pro}\ A) \longrightarrow E(A)\).
Proof. Consider the functor

$$k(-) = H^0((A \oplus I) \otimes_A -) : K(\text{Pro} A) \rightarrow \text{Ab}$$

from the homotopy category of complexes of projective $A$-modules to the category of abelian groups. This is clearly a homological functor respecting set indexed coproducts. Moreover,

$$k(\Sigma^i P) \cong H^i(P) \oplus H^i(I \otimes P),$$

where $\Sigma^i$ denotes $i$'th suspension, so for $P$ to satisfy $k(\Sigma^i P) = 0$ for each $i$ means

$$H^i(P) = 0$$

and

$$H^i(I \otimes P) = 0$$

for each $i$. Using lemma 1.1, this shows

$$\{ P \in K(\text{Pro} A) \mid k(\Sigma^i P) = 0 \text{ for each } i \} = E(A).$$

That is, $E(A)$ is the kernel of the homological functor $k$.

One consequence of this is that $E(A)$ is closed under set indexed coproducts. Hence [12, lem. 3.5] says that for $e_*$ to have a right-adjoint is the same as for the Verdier quotient $K(\text{Pro} A)/E(A)$ to satisfy that each Hom set is in fact a set (as opposed to a class).

Now, the category $K(\text{Pro} A)$ is compactly generated by lemma 2.1. By [15, lem. 4.5.13] with $\beta = \aleph_0$, this even implies that there is only a set of isomorphism classes of compact objects in $K(\text{Pro} A)$. Hence the version of Bousfield localization given in [12, thm. 4.1] applies to the functor $k$ on $K(\text{Pro} A)$, and gives that $K(\text{Pro} A)$ modulo the kernel of $k$ satisfies that each Hom is a set. That is, $K(\text{Pro} A)/E(A)$ satisfies that each Hom is a set, as desired. \qed

The following elementary result holds by [12, prop. 4.10].

Lemma 2.3. Let $e_* : E \rightarrow K$ be the inclusion of a full subcategory, and suppose that $e^* : K \rightarrow E$ is a right adjoint to $e_*$. Then $E$ is a precovering class in $K$.

Combining proposition 2.2 and lemma 2.3 shows the following.

Proposition 2.4. The class $E(A)$ is precovering in $K(\text{Pro} A)$. 

3. Modules

Lemma 3.1. Let $M$ be an $A$-module. There exists a homomorphism

$$G \xrightarrow{g} M$$

where $G$ is a Gorenstein projective $A$-module, such that if

$$\tilde{G} \xrightarrow{\tilde{g}} M$$

is any homomorphism with $\tilde{G}$ a Gorenstein projective $A$-module then there exists a homomorphism

$$\tilde{G} \xrightarrow{\tilde{\gamma}} G$$

so that $\tilde{g} - g\tilde{\gamma}$ factors through a projective $A$-module.

Proof. Let $P \xrightarrow{\sim} M$ be a projective resolution. Then $P$ is in $K(\text{Pro} A)$; let $E \xrightarrow{\varepsilon} P$ be an $E(A)$-precover which exists by proposition 2.4. This gives

$$\cdots \rightarrow E^{-1} \xrightarrow{e^{-1}} E^0 \xrightarrow{e^0} E^1 \xrightarrow{G} E^2 \xrightarrow{} \cdots$$

$$\cdots \rightarrow P^{-1} \xrightarrow{} P^0 \xrightarrow{} 0 \xrightarrow{} 0 \xrightarrow{} \cdots,$$

where $G = \text{Ker}(E^1 \rightarrow E^2)$ is Gorenstein projective.

Now let $\tilde{G} \xrightarrow{\tilde{g}} M$ be a homomorphism with $\tilde{G}$ Gorenstein projective. Pick $\tilde{E}$ in $E(A)$ so that $\tilde{G} = \text{Ker}(\tilde{E}^1 \rightarrow \tilde{E}^2)$. Clearly $\tilde{g}$ extends to a chain map $\tilde{E} \xrightarrow{\tilde{\varepsilon}} P$ so that $\tilde{g}$ and $\tilde{\varepsilon}$ fit together in

$$\cdots \rightarrow \tilde{E}^{-1} \xrightarrow{\tilde{e}^{-1}} \tilde{E}^0 \xrightarrow{\tilde{e}^0} \tilde{E}^1 \xrightarrow{\tilde{G}} \tilde{E}^2 \xrightarrow{} \cdots$$

$$\cdots \rightarrow P^{-1} \xrightarrow{} P^0 \xrightarrow{} 0 \xrightarrow{} 0 \xrightarrow{} \cdots,$$
Since $E \xrightarrow{e} P$ is an $E(A)$-precover, there now exists a chain map $\tilde{E} \xrightarrow{\tilde{e}} E$ so that

$$
\begin{array}{ccc}
\tilde{E} & \xrightarrow{\tilde{e}} & E \\
\uparrow{\tilde{e}} & & \uparrow{e} \\
\downarrow{\tilde{e}} & & \\
P \xrightarrow{\tilde{e}} & & \\
\end{array}
$$

is commutative in $K(\text{Pro}A)$.

The chain map $\tilde{e}$ induces a homomorphism $\tilde{G} \xrightarrow{\tilde{g}} G$ so that $\tilde{e}$ and $\tilde{g}$ fit together in

$$
\begin{array}{cccccccc}
\cdots & \xrightarrow{\tilde{e}^{-1}} & \tilde{E}^{-1} & \xrightarrow{\tilde{e}^0} & \tilde{E}^0 & \xrightarrow{\tilde{e}^1} & \tilde{E}^1 & \xrightarrow{\tilde{e}^2} & \cdots \\
\downarrow{\tilde{e}^{-1}} & & \downarrow{\tilde{e}^0} & & \downarrow{\tilde{g}} & & \downarrow{\tilde{e}^1} & & \downarrow{\tilde{e}^2} \\
\cdots & \xrightarrow{\tilde{e}^{-1}} & E^{-1} & \xrightarrow{\tilde{e}^0} & E^0 & \xrightarrow{\tilde{e}^1} & E^1 & \xrightarrow{\tilde{e}^2} & \cdots \\
\downarrow{\tilde{g}} & & \downarrow{\tilde{g}} & & \downarrow{g} & & \downarrow{\tilde{g}} & & \\
G & & & & & & G & & \\
\end{array}
$$

So now there are homomorphisms

$$
\begin{array}{ccc}
\tilde{G} & \xrightarrow{\tilde{g}} & M. \\
\uparrow{\tilde{g}} & & \\
\downarrow{\tilde{g}} & & \\
G & \xrightarrow{g} & M. \\
\end{array}
$$

If diagram (1) were commutative as a diagram of chain maps, then diagram (2) would be commutative as a diagram of modules. As it is, diagram (1) is only commutative in $K(\text{Pro}A)$, that is, it is commutative up to chain homotopy. It is not hard to see that hence, in diagram (2), the difference $\tilde{g} - g\tilde{g}$; while not necessarily zero, must factor through the module $\tilde{E}^1$. That is, $\tilde{g} - g\tilde{g}$ factors through a projective module. □

**Theorem 3.2.** Recall setup 0.1. In this situation, the Gorenstein pro-projective modules form a precovering class in the module category of $A$.

**Proof.** Let $M$ be a module. Pick a homomorphism $G \xrightarrow{\tilde{g}} M$ with the property described in lemma 3.1, and pick a surjection $Q \longrightarrow M$ where $Q$ is projective. It is easy to see from lemma 3.1 that the induced
homomorphism $G \oplus Q \to M$ is a precover with respect to the class of Gorenstein projective modules.

4. Non-commutative algebras

The purpose of this short section is to point out that the above results apply much more generally than to noetherian commutative rings with dualizing complexes. The following setups replace the setups from the introduction.

Setup 4.1. Let $A$ be a left-coherent and right-noetherian $k$-algebra over the field $k$ so that there exists a left-noetherian $k$-algebra $B$ and a dualizing complex $BD_A$. That is, $D$ is a complex of $B$-left-$A$-right-modules, and

(i) The cohomology of $D$ is bounded and finitely generated both over $B$ and over $A^{\text{op}}$.

(ii) The injective dimensions $\text{id}_B D$ and $\text{id}_{A^{\text{op}}} D$ are finite.

(iii) The canonical morphisms

\[ A \to \text{RHom}_B(D, D) \quad \text{and} \quad B \to \text{RHom}_{A^{\text{op}}}(D, D) \]

in the derived categories $D(A \otimes_k A^{\text{op}})$ and $D(B \otimes_k B^{\text{op}})$ are isomorphisms.

Setup 4.2. Let $D \to I$ be an injective resolution of $D$ over $B \otimes_k A^{\text{op}}$. Below, I will replace $I$ by a bounded truncation. This may ruin the property that $I$ is an injective resolution over $B \otimes_k A^{\text{op}}$, but because $\text{id}_B D$ and $\text{id}_{A^{\text{op}}} D$ are finite, I can still suppose that $I$ consists of modules which are injective both over $B$ and over $A^{\text{op}}$.

The above definition of dualizing complexes over non-commutative algebras is due to [18, def. 1.1].

With setups 0.1 and 0.2 replaced by setups 4.1 and 4.2, let me inspect the rest of the paper. As the ground ring $A$ is now non-commutative, I must replace “module” by “left-module” throughout. Remark 0.5 also needs to be replaced by the following.

Remark 4.3. The results of [11] apply under setup 4.1, and so each flat $A$-left-module has finite projective dimension.

After this, the proof of lemma 1.1 goes through if one keeps track of left and right structures throughout. The proofs of lemma 2.1 and proposition 2.2 also still work, and along with lemma 2.3 this still implies proposition 2.4. And finally, the proofs of lemma 3.1 and theorem 3.2 still go through.

So theorem 3.2 remains valid. Let me formulate this in full.
Theorem 4.4. Recall setup 4.1. In this situation, the Gorenstein projective modules form a precovering class in the category of $A$-left-modules.

Corollary 4.5. Let $R$ be a noetherian $k$-algebra and suppose that one of the following holds.

(i) $R$ is a complete semi-local PI algebra.
(ii) $R$ has a filtration $F$ so that the associated graded algebra $\text{gr}^F R$ is connected and noetherian, and either PI, graded FBN, or with enough normal elements.

Then the Gorenstein projective modules form a precovering class in the category of $A$-left-modules.

Proof. The algebra $R$ can be used as $A$ in setup 4.1 and theorem 4.4 because $B$ and $D$ exist. In case (i) this is by [17, cor. 0.2], and in case (ii) by [18, cor. 6.9]. □

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