Relativistic corrections to static properties of the proton

D. Bedoya Fierro, N. G. Kelkar and M. Nowakowski

Departamento de Física, Universidad de los Andes,
Cra.1E No.18A-10, Santafe de Bogota, Colombia

Abstract

A new method to relate the proton electromagnetic form factors in momentum space to the corresponding charge and magnetization densities with the inclusion of relativistic corrections is presented by extending the standard Breit equation to higher orders in its $1/c^2$ expansion. Applying a Lorentz boost to the relativistically corrected form factors $\tilde{G}_{E,M}(q^2)$ in the Breit frame, moments of the charge and magnetization distributions are evaluated. The proton charge radius thus determined is found to be smaller and hence in better agreement with recent spectroscopy results.

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Measurements of the ground state properties of the most basic element of the constituents of stable matter, namely, the proton, has intrigued physicists since the sixties until now. One such measurement involves the extraction of elastic electromagnetic form factors of the proton from electron proton scattering cross sections [1]. In the non-relativistic limit, the Fourier transform of the so called Sachs form factors $G_E(q^2)$ and $G_M(q^2)$ [2] describe the charge distribution $\rho_C(r)$ and magnetization current distribution $\rho_M(r)$ in the nucleon respectively [3]. An experimental determination of $G_E(q^2)$ from electron proton scattering can thus enable one to determine the charge radius of the proton (see [4] and references therein). However, in spite of the experimental and theoretical efforts over decades, there remain discrepancies [3, 5] between different experiments as well as theory and experiment. For example, the proton radius determined from muonic hydrogen spectroscopy [6] was much smaller than the average value obtained from electron proton scattering. The shrunk proton gave rise to explanations ranging from the charge density being poorly constrained by data [7] to those involving large extra dimensions and non-identical protons [8]. Given the uncertainty involved in a basic property such as the size of the proton, a re-examination of the connection of the electromagnetic form factors to the nucleon properties is timely.

Here, we present a new approach to relate the form factors in momentum space to their coordinate space counterparts (the charge and magnetization densities) with the inclusion of relativistic corrections. These form factors can be used to evaluate the relativistic corrections to the static properties such as the radius of the proton. This is achieved by extending [9, 10] the standard Breit equation [11, 12] (which involves an expansion of the amplitude to order $1/c^2$) to higher orders. The proton electric potential $V_p(r)$ in this equation is used to find the density $\rho_C(r)$ via the Poisson equation, $\nabla^2 V_p = -\rho_C$. The hyperfine interaction terms in the Breit equation are shown to be related to the magnetization density $\rho_M(r)$. An interesting outcome of the calculation is that the relativistic corrections to the charge form factor depend on the magnetic form factor and vice versa. In addition to the above, the effect of applying a Lorentz boost to the form factors in the Breit frame is investigated. The two effects, namely, the relativistic corrections to $G_E(q^2)$ and the Lorentz boost taken together lead to a smaller proton radius which is in agreement with the results of [6].

In order to make the approach clear let us begin with the standard Breit equation for the Hamiltonian $H_B$ [9] which results from the $1/c^2$ expansion of the elastic electron proton transition matrix element $M_{fi}$. This amplitude can be written as, $M_{fi} =$
$w_{S_e}^\dagger w_{S_p}^\dagger \hat{H}_B(p_e, p_p; \sigma_e, \sigma_p; q) w_{S_e} w_{S_p}$, where $w_{S_e, S_p}$ are two component spinors. In the diagonal case, $S_e' = S_e, S_p' = S_p$ and we write $M_{fi} = M_{fi}(p_e, p_p; \xi_e, \xi_p; q)$, where we used $w_{S_e}^\dagger \sigma_w S = \text{Tr}[\rho \sigma] = \xi$ with $\rho$ being the spin density matrix. Let us now rearrange terms from the Breit Hamiltonian $H_B$ such that, $H_B = eV_p(q) + \mu_e \cdot B(q) + \ldots$, where, $\mu_e = -(e/2m_e)\sigma_e$.

$V_p(q)$ is the potential part remaining after separating all the $\sigma_i$ operator dependent and differential operator $p_i$ dependent terms. Here $i$ is either $e$ or $p$. In addition, we choose $V_p(q)$ not to contain the electron mass as the electric proton potential should not depend on the probe. These restrictions allow $V_p(q)$ to be interpreted as a proton electric potential in momentum space. For the standard Breit equation at lowest order in $1/c^2$, with form factors, this indeed leads to [10],

$$V_p(q) = \left[ F_1\left( \frac{1}{q^2} \right) - F_2\left( \frac{1}{4m_e^2c^2} \right) \right] = \left[ \frac{G_E(q^2)}{q^2} \right]. \quad (1)$$

The Fourier transform of $V_p(q)$ is then the electric potential $V_p(r) = \int e^{-iq \cdot r} \left( G_E(q^2)/q^2 \right) d^3q$. The Laplacian of $V_p(r)$, namely, $\nabla^2 V_p(r) = -\int e^{-iq \cdot r} G_E(q^2) d^3q$ taken together with $\nabla^2 V_p(r) = -\rho_c(r)$ then brings us to the standard definition of the proton charge density $\rho_c(r) = \int e^{-iq \cdot r} G_E(q^2) d^3q$. Applying similar restrictions to the magnetic field in the second term in $H_B$, i.e., the magnetic field of the proton $B(q)$ should not contain any electron mass or operator $p_i$ dependence, the terms which remain (apart from the Coulomb term) are those corresponding to the hyperfine interaction. The hyperfine interaction potential with form factors [10] is given as,

$$V(q)_{hfs} = \alpha \left[ \frac{(\sigma_e \cdot \sigma_p)}{4m_e m_p c^2} - \frac{(\sigma_e \cdot q) \cdot (\sigma_p \cdot q)}{4m_e m_p c^2 q^2} \right] G_M(q^2) = \mu_e \cdot B(q), \quad (2)$$

with $\mu_e = -(e/2m_e)\sigma_e$ as defined earlier. The magnetic field of the proton is thus,

$$B(q) = \left[ \frac{q(\sigma_p \cdot q) - \sigma_p q^2}{2m_p c^2 q^2} \right] G_M(q^2). \quad (3)$$

Taking the Fourier transform of $B(q)$ and using the static Maxwell equation $\nabla \times B(r) = j(r)$, we can identify

$$j(r) = \int d^3q e^{-iq \cdot r} \frac{G_M(q^2) (q \times \sigma)}{2m_p c^2}, \quad (4)$$

thus implying $j(r) \propto \nabla \times M$ with $M = \rho_M(r)\sigma_p$ which defines the proton magnetization distribution $\rho_M(r) = \int e^{iq \cdot r} G_M(q^2) d^3q$. In the diagonal case, by the replacement of $\sigma$ by
the polarization vector $\xi$, we can conclude that polarized protons will have a magnetic field of the form, $B(r) \propto \nabla^2(\xi \cdot \nabla)\rho_\mu - \xi \rho_\mu$. All the conclusions drawn above and derived at the lowest order in the relativistic expansion are, of course, valid if we include relativistic corrections. In this case we should replace $G_{E,M}$ by the modified $\tilde{G}_{E,M}$ (to be discussed later) where $\tilde{G}_{E,M}$ contain the relativistic corrections and are strictly speaking valid in the Breit frame.

The procedure to obtain the Breit potential at higher orders using the electron-proton scattering amplitude is exactly the same as that described in [9, 10] except that the proton and electron wave functions which were written in [9, 10] using the non-relativistic approximation with corrections up to order $1/c^2$ are now replaced by those containing relativistic corrections up to order $1/c^6$. This is done by using the Foldy Wouthuysen transformation [13], $\Psi_{FW} = U\Psi_D$, where

$$U = \sqrt{\frac{(E + mc^2)}{2E}} \left(1 + \frac{\beta \alpha \cdot \mathbf{p}c}{E + mc^2}\right),$$

$$H_D \Psi_D = E \Psi_D,$$

$$E = \sqrt{\frac{p^2c^2 + m^2c^4}{2}},$$

$$H_{FW} \Psi_{FW} = \beta E \Psi_{FW} \text{ and } \alpha, \beta \text{ the usual Dirac matrices.}$$

It then follows that

$$\Psi_{FW} = \left[ E(1 + \beta)/\sqrt{2E(E + mc^2)} \right] \Psi_D,$$

where, $\Psi_{FW}$ contains both the positive and negative energy solutions. The upper and lower components $\Psi_{FW}^+$ and $\Psi_{FW}^-$ of $\Psi_{FW}$ can be shown to be related to the Dirac upper and lower components $\phi_\mu$ and $\chi_\mu$ respectively as

$$\Psi_{FW}^+ = \sqrt{\frac{2E}{E + mc^2}} \begin{pmatrix} \phi_\mu \\ 0 \end{pmatrix}, \quad \Psi_{FW}^- = \sqrt{\frac{2E}{E + mc^2}} \begin{pmatrix} 0 \\ \chi_\mu \end{pmatrix} .$$

The relativistic energy $E$ of the particle includes also its rest energy $mc^2$ which must be excluded in arriving at a non-relativistic approximation. We must therefore replace $\Psi$ (FW or D) by $\Psi'$ defined as $\Psi = \Psi' e^{-imc^2t/\hbar}$. This leads to a relation between the upper and lower components $\phi'$ and $\chi'$ of $\Psi'$ given as, $\chi' = (1/2mc) [1 + (E_S/2mc^2)] \sigma \cdot \mathbf{p} \phi'$, where $E_S$ is the energy eigenvalue in the Schrödinger equation. Identifying the upper component $\Psi_{FW}^+$ of $\Psi_{FW}$ with the non-relativistic Schrödinger spinor $\psi$, we get, $w \sqrt{(E + mc^2)/2E} = \phi'$. Finally, expanding $E = (p^2c^2 + m^2c^4)^{1/2}$ and replacing for $\phi'$ in terms of $w$ in $\chi'$, we obtain the spinor to be used in the calculation of the amplitude $M_{fi} = e^2(\tilde{u}'_\mu \Gamma'^\mu \psi \psi) D_{\mu\nu}(q^2)(\tilde{u}'\nu \Gamma'^\nu \psi \psi)$ as

$$u_i = \sqrt{2m_i} \left(1 - \frac{p^2}{8m_i^2c^2} + \frac{\lambda_0 p^4}{m_i^4c^4} + \frac{\lambda_3 p^6}{m_i^6c^6} \right) w_i$$

or

$$u_i = \sqrt{2m_i} \left(1 - \frac{\lambda_2 p^2}{m_i^2c^2} + \frac{\lambda_3 p^4}{m_i^4c^4} \right) \frac{\sigma \cdot \mathbf{p} \cdot \psi}{2m_i c} w_i ,$$

$$u_i = \sqrt{2m_i} \left(1 - \frac{\lambda_2 p^2}{m_i^2c^2} + \frac{\lambda_3 p^4}{m_i^4c^4} \right) \frac{\sigma \cdot \mathbf{p} \cdot \psi}{2m_i c} w_i ,$$

$$u_i = \sqrt{2m_i} \left(1 - \frac{\lambda_2 p^2}{m_i^2c^2} + \frac{\lambda_3 p^4}{m_i^4c^4} \right) \frac{\sigma \cdot \mathbf{p} \cdot \psi}{2m_i c} w_i .$$
The above equation can be rewritten as

\[ \int \approx 1 + (q \frac{\sigma_{\mu\nu} q_{\nu}}{2mc^2}) F_{2}^{i} \] 

and due to the alternating sign in (8), the first four terms in the curly bracket in (8) can be approximated as

\[ 1 + \frac{q^2}{8mc^2} + \frac{3q^4}{128m_p^2c^4} + \frac{13q^6}{1024m_p^4c^6} \]

hence the Breit equation with form factors is evaluated just as in [9, 10]. Note that the energy transfer at the vertices is chosen to be zero, i.e., \( q^2 = \omega^2/c^2 - q^2 \) is replaced by \( q^2 = -q^2 \). Formally, this is achieved by going to the Breit frame. This is in keeping with the quasistatic approach wherein we are going to relate the proton potential obtained from the Breit equation to the charge density via the Poisson equation. The higher order Breit equation with form factors thus obtained is very lengthy and will be given elsewhere. Here we shall discuss the parts relevant for obtaining the relativistic corrections to the charge and magnetic form factors.

The proton electric potential \( \tilde{V}_p(q) \) with relativistic corrections is obtained from the higher order Breit equation in the same manner as explained before for the standard Breit equation. Dropping all terms involving the spin and momentum operators as well as those containing the electron mass, what remains in the higher order Breit equation is

\[
\tilde{V}_p(q) = \frac{G_E(q^2)}{q^2} \left\{ 1 - \frac{q^2}{8m_p^2c^2} + \frac{3q^4}{128m_p^4c^4} - \frac{13q^6}{1024m_p^6c^6} + \frac{G_M(q^2)}{G_E(q^2)} \frac{q^6}{16m_p^4c^4} \left[ 1 - \frac{7q^2}{8m_p^2c^2} + \frac{87q^4}{128m_p^4c^4} \right] \right\}.
\]

The above equation can be rewritten as \( \tilde{V}_p(q) = \tilde{G}_E(q^2)/q^2 \), such that, \( -\nabla^2 \tilde{V}_p(r) = \int e^{-iqr} \tilde{G}_E(q^2) d^3q = \tilde{\rho}_e(r) \). The modified magnetic form factor \( \tilde{G}_M(q^2) \) is obtained by examining the hyperfine interaction terms as mentioned before, however, in the higher order Breit equation. Noting that the terms of order \( 1/c^6 \) and higher are of decreasing importance and due to the alternating sign in (8), the first four terms in the curly bracket in (8) can be approximated as \( [1 + (q^2/4m_p^2)]^{-1/2} \). The expressions for \( \tilde{G}_{E,M} \) can thus be summarized in an expansion effectively as

\[
\tilde{G}_E(q^2) \simeq G_E(q^2) \left( 1 + \frac{q^2}{4m_p^2c^2} \right)^{-1/2} + \frac{G_M(q^2)q^4}{16m_p^4c^4} \left( 1 + \frac{aq^2}{4m_p^2c^2} \right)^{-b}
\]

\[
\tilde{G}_M(q^2) \simeq G_M(q^2) \left( 1 + \frac{q^2}{4m_p^2c^2} \right)^{-1/2} - \frac{G_E(q^2)q^2}{4m_p^2c^2} \left( 1 + \frac{aq^2}{4m_p^2c^2} \right)^{-b},
\]

with \( a = 19/7 \) and \( b = 49/38 \). It is interesting that the modified form factors \( \tilde{G}_E \) and \( \tilde{G}_M \) depend on both the \( G_E \) and \( G_M \) Sachs form factors and have relativistic corrections of a similar form with the same exponents \( a \) and \( b \). Note also that the exponent –1/2 in
the first terms is approximate (in contrast to the exact \( [1 + (q^2/4m_p^2)]^{-1/2} \) in [15, 16]) At order \( 1/c^2 \), the expression for \( \tilde{G}_E(q^2) \approx G_E(q^2)(1 - q^2/8m_p^2c^2) \) is independent of \( G_M \) as in [15–17], however, the magnetic form factor at order \( 1/c^2 \) reduces to \( \tilde{G}_M(q^2) \approx G_M(q^2)(1 - q^2/8m_p^2c^2) - G_E(q^2)q^2/4m_p^2c^2 \) and contains apart from the Darwin term \( q^2/8m_p^2c^2 \) a term dependent on \( G_E \).

Since we chose the energy transfer in the evaluation of the electron - proton scattering amplitude, \( \omega = 0 \), the above form factors are similar to those usually given in the so-called Breit frame. Thus the above corrections are relativistic corrections to the Breit frame form factors. An additional important relativistic correction arises due to the Lorentz contraction of the spatial distributions in the Breit frame. The latter has been discussed at length in [18] where the author proposes the use of the Fourier transform of \( G_{E,M}(q^2) = G_{E,M}(q^2)[1 + (q^2/4m_p^2)]^{\lambda_{E,M}} \), rather than the Fourier transform of \( G_{E,M}(q^2) \) in order to determine the density distributions of the nucleon. With \( \lambda_{E,M} \) being model dependent constants, they eventually appear as parameters in the determination of the proton radius and other moments. The author in [18] fitted the form factor data to obtain \( \lambda_E = \lambda_M = 2 \) in agreement with some [19] while in contrast with other predictions [20] of \( \lambda_E = 0 \) and \( \lambda_E = \lambda_M = 1 \) based on soliton and cluster models. The relativistic corrections and the Lorentz boost lead to the following corrections

\[
\langle \tilde{r}_{E}^2 \rangle^L = \langle r_{E}^2 \rangle + \frac{3}{4m_p^2c^2} (1 - 2\lambda_E) \tag{10}
\]

\[
\langle \tilde{r}_{E}^4 \rangle^L = \langle r_{E}^4 \rangle - \frac{5}{m_p^2c^2} \langle r_{E}^2 \rangle \left( \lambda_E - \frac{1}{2} \right) + \frac{15}{4m_p^2c^2} (\lambda_E^2 - 2\lambda_E + 2\mu_p) + \frac{45}{16m_p^2c^4} \]

to the moments \( \langle \tilde{r}_{E}^2 \rangle^L = -6[d\tilde{G}_E^L/dq^2]_{q^2=0} \) and \( \langle \tilde{r}_{E}^4 \rangle^L = 60[d^2\tilde{G}_E^L/d(q^2)^2]_{q^2=0} \) (with \( \tilde{G}_E^L(q^2) = \tilde{G}_E(q^2) [1 + (q^2/4m_p^2c^2)]^{\lambda_E} \)) of the proton charge distribution. The magnetic radius with relativistic and Lorentz boost corrections is given by,

\[
\langle \tilde{r}_{M}^2 \rangle^L = \langle r_{M}^2 \rangle + \frac{3}{4m_p^2c^2} \left[ 1 + \frac{2}{\mu_p} - 2\lambda_M \right] \tag{11}
\]

The relativistic corrections alone arising from [9] can be found by setting \( \lambda_{E,M} = 0 \).

The effect of the Lorentz boost in general is to reduce the radius and the fourth moment of the proton charge as compared to that obtained from \( G_E(q^2) \) in the Breit frame. The relativistic corrections introduced with the use of the higher order Breit potential can be
TABLE I: Corrections to the proton charge radius $r_p = \langle r_E^2 \rangle^{1/2}$ in fm. $\tilde{r}_p$ in the last two columns gives the proton radius with the relativistic as well as the Lorentz boost corrections included. The fourth moments $r^4 = \langle r_E^4 \rangle$ of the proton charge distribution with corrections are given in the brackets (in fm$^4$).

|                | $r_p$  | $\tilde{r}_p$ | $r_p^{L,\lambda_E=1}$ | $r_p^{L,\lambda_E=2}$ | $\tilde{r}_p^{L,\lambda_E=1}$ | $\tilde{r}_p^{L,\lambda_E=2}$ |
|----------------|--------|---------------|-------------------------|-------------------------|--------------------------------|--------------------------------|
| Dipole [1]     | 0.811  | 0.831         | 0.769                   | 0.725                   | 0.790                          | 0.747                          |
|                | (1.083) | (1.202)       | (0.937)                 | (0.806)                 | (1.049)                        | (0.911)                        |
| [21] Fit I     | 0.893  | 0.912         | 0.855                   | 0.816                   | 0.875                          | 0.836                          |
|                | (1.824) | (1.959)       | (1.648)                 | (1.486)                 | (1.775)                        | (1.606)                        |
| [21] Fit II    | 0.866  | 0.885         | 0.827                   | 0.786                   | 0.847                          | 0.807                          |
|                | (1.623) | (1.792)       | (1.496)                 | (1.345)                 | (1.618)                        | (1.460)                        |
| [22]           | 0.858  | 0.877         | 0.819                   | 0.777                   | 0.839                          | 0.798                          |
|                | (1.488) | (1.616)       | (1.325)                 | (1.177)                 | (1.446)                        | (1.290)                        |

seen in general to increase the radius of the proton. However, a fortuitous combination of the two effects, brings the proton radius closer to $r_p = 0.84087(39)$ fm obtained from precise Lamb shift measurements [6]. For a Lorentz boost with $\lambda = 1$ such an agreement is favoured by Fit II in [21] which gives $\tilde{r}_p = 0.847$ fm and $\tilde{r}_p = 0.839$ fm obtained from the bump-tail parametrization (with parameters from Table II in [22]). Indeed, if we apply the Lorentz boost with $\lambda = 1$ to a recent radius of $r_p = 0.8795$ fm deduced by Bernauer et al. [23] we obtain $r_p^{L} = 0.841$ fm which is once again close to the spectroscopy result. The reason for applying only the Lorentz boost and not the entire relativistic corrections is the fact that Bernauer et al. include in their analysis, the so called “Coulomb corrections” [15], which are somewhat similar to the relativistic corrections (only at order $1/c^2$) of the present work. The proton magnetic radius, $r_M = 0.87$ fm [24], with relativistic and Lorentz boost corrections changes to $r_M = 0.865$ fm. Finally, we must emphasize that the proton is characterized fully by all its moments and not just the radius. The corrections in Eqs (9) introduce a significant change in $\langle r^4 \rangle$ too (see Table II). The fourth moments were shown in [25] to govern the size of the third Zemach moments of the proton charge distribution.
The relations between charge/magnetization densities and the form factors are necessarily of non-relativistic nature. In other words, relativistic corrections can be computed and the standard relation between the Sachs form factors \( G_E \) and \( G_M \) and the densities is valid only at the lowest order of the non-relativistic expansion. To compute the relativistic corrections in a consistent way we employed the higher order Breit equation in which, for instance, terms independent of the probe, spin and momentum operators should correspond to the proton electric potential in momentum space. Using the Poisson equation, this potential gives us the relativistically modified charge density. A similar procedure can be found for the magnetization density. Both results are valid in the Breit frame. Hence using a Lorentz transformation suggested in the literature, we can bring them to the rest frame of the proton and calculate the modified moments of the proton charge and magnetization densities. An interesting outcome of the manipulations, i.e., including relativistic corrections and the Lorentz transformation is that the proton radius from scattering experiments comes closer to the result obtained from atomic spectroscopy, thus partly resolving the so-called proton puzzle.

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