Higher Spin Conformal Currents in Minkowski Space

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Abstract

Using unfolded formulation of free equations for massless fields of all spins we obtain explicit form of gauge invariant higher-spin conformal conserved charges bilinear in 4d massless fields of arbitrary spins.

1 Introduction

In this note we give an explicit form of gauge invariant higher-spin (HS) conserved currents built of 4d massless fields of all spins. To the best of our knowledge a realization of the conformal HS currents built of massless fields of all spins in the 4d Minkowski space has not been yet available in the literature in the full generality, although some particular examples of conformal HS currents built of massless fields of lower spins $s \leq 1$ were considered. In particular, $x$-independent HS conformal currents built of massless scalar, spinor and Maxwell field were found in [1] and $x$-dependent HS currents built of massless scalar and spinor were found in [2]. We extend these results in two directions: we allow (i) the constituent fields to carry any spin and (ii) explicit dependence on the space-time coordinates.

Our construction is based on the unfolded formulation of dynamical equations in the form of zero-curvature equations [3] and is analogous to the construction of HS currents [4] in the generalized space-time with matrix coordinates [5, 6, 7, 8].
2 Unfolded Massless Field Equations in 4d Minkowski space

In [3, 7] it was shown that the equations for (field strengths of) massless fields of all spins in Minkowski space can be concisely formulated in the unfolded form

$$\frac{\partial}{\partial x^{ab}} C(w, \bar{w}|x) + \frac{\partial^2}{\partial w^a \partial \bar{w}^b} C(w, \bar{w}|x) = 0. \quad (2.1)$$

Here $w^a$ and $\bar{w}^b$ are auxiliary commuting conjugated two-component spinor coordinates $(a, b = 1, 2$ and $\dot{a}, \dot{b} = \dot{1}, \dot{2})$ and $x^{ab}$ are Minkowski coordinates in two-component spinor notations. The two-component indices are raised and lowered as follows

$$A^a = \varepsilon^{ab} A_b, \quad A_a = \varepsilon_{ba} A^b, \quad \varepsilon_{ab} = -\varepsilon_{ba}, \quad \varepsilon_{12} = 1 \quad (2.2)$$

and analogously for dotted indices. The relationship with the tensor notations is based on

$$V^{\dot{a}b} = A^{\nu} \sigma^{\dot{a}b}_{\nu}, \quad (2.3)$$

where $\sigma^{\dot{a}b}_{\nu}$ $(\nu = 0, 1, 2, 3)$ are four Hermitian $2 \times 2$ matrices.

The meaning of the equation (2.1) is as follows. The fields $C(w, \bar{w}|x)$ are assumed to be expandable in power series in $w^a$ and $\bar{w}^\dot{a}$

$$C(w, \bar{w}|x) = \sum_{m,n=0}^{\infty} C_{a_1...a_n \dot{a}_1...\dot{a}_m}(x) w^{a_1} \ldots w^{a_n} \bar{w}^{\dot{a}_1} \ldots \bar{w}^{\dot{a}_m}. \quad (2.4)$$

The operator

$$N_{w,\bar{w}} = w^a \frac{\partial}{\partial w^a} - \bar{w}^\dot{a} \frac{\partial}{\partial \bar{w}^\dot{a}}$$

commutes with $\frac{\partial^2}{\partial w^a \partial \bar{w}^b}$. Solutions of the equation (2.1) with fixed eigenvalues of $N_{w,\bar{w}}$ form invariant subspaces which describe fields of different helicities $h$

$$N_{w,\bar{w}} C(w, \bar{w}|x) = 2hC(w, \bar{w}|x). \quad (2.5)$$

The meaning of the fields $C(w, \bar{w}|x)$ is as follows [7]. The holomorphic fields

$$C(w, 0|x) = \sum_{2s=0}^{\infty} C_{a_1...a_{2s}}(x) w^{a_1} \ldots w^{a_{2s}}$$

and their complex conjugates$^1$

$$C(0, \bar{w}|x) = \sum_{2s=0}^{\infty} C_{\dot{a}_1...\dot{a}_{2s}}(x) \bar{w}^{\dot{a}_1} \ldots \bar{w}^{\dot{a}_{2s}}$$

$^1w^a$ is complex conjugated to $\bar{w}^\dot{a}$
describe, respectively, selfdual (positive helicity) and antiselfdual (negative helicity) gauge invariant on-mass-shell nontrivial combinations of derivatives of massless gauge fields of all spins \( s = 0, 1/2, 1, \ldots \infty \), where

\[
\frac{w^a}{\partial w^a}C(w, 0|x) = 2sC(w, 0|x)
\]

and

\[
\frac{\bar{w}^a}{\partial \bar{w}^a}C(0, \bar{w}|x) = 2sC(0, \bar{w}|x).
\]

These include scalar \((s = 0)\)

\[
c(x) = C(0, 0|x),
\]

spinor \((s = 1/2)\)

\[
c_a(x) = \frac{\partial}{\partial w^a}C(w, 0|x)|_{w=0}, \quad \bar{c}_a(x) = \frac{\partial}{\partial \bar{w}^a}C(0, \bar{w}|x)|_{\bar{w}=0},
\]

Maxwell tensor \((s = 1)\)

\[
c_{ab}(x) = \frac{1}{2} \frac{\partial^2}{\partial w^a \partial w^b}C(w, 0|x)|_{w=0}, \quad \bar{c}_{ab}(x) = \frac{1}{2} \frac{\partial^2}{\partial \bar{w}^a \partial \bar{w}^b}C(0, \bar{w}|x)|_{\bar{w}=0},
\]

Rarita-Schwinger field strength \((s = 3/2)\)

\[
c_{abc}(x) = \frac{1}{3!} \frac{\partial^3}{\partial w^a \partial w^b \partial w^c}C(w, 0|x)|_{w=0}, \quad \bar{c}_{abc}(x) = \frac{1}{3!} \frac{\partial^3}{\partial \bar{w}^a \partial \bar{w}^b \partial \bar{w}^c}C(0, \bar{w}|x)|_{\bar{w}=0},
\]

Weyl tensor \((s = 2)\)

\[
c_{abcd}(x) = \frac{1}{4!} \frac{\partial^4}{\partial w^a \partial w^b \partial w^c \partial w^d}C(w, 0|x)|_{w=0}, \quad \bar{c}_{abcd}(x) = \frac{1}{4!} \frac{\partial^4}{\partial \bar{w}^a \partial \bar{w}^b \partial \bar{w}^c \partial \bar{w}^d}C(0, \bar{w}|x)|_{\bar{w}=0},
\]

etc.

The primary fields are those contained in \(C(w, 0|x)\) and their complex conjugates \(C(0, \bar{w}|x)\). These are lowest order gauge invariant combinations of derivatives of massless gauge fields, which turn out to be of order \([s]\) for a spin \(s\) field and were considered by many authors (see e.g. [9]). The descendants are described by those components of \(C(w, \bar{w}|x)\) that depend both on \(w\) and on \(\bar{w}\) and therefore are expressed in terms of derivatives of the primary fields by (2.1).

The dynamical HS field equations are the following consequences of (2.1)

\[
\frac{\partial}{\partial x^a b} \frac{\partial}{\partial w^a}C(w, 0|x) = 0, \quad \frac{\partial}{\partial x^a b} \frac{\partial}{\partial \bar{w}^a}C(0, \bar{w}|x) = 0 \quad (2.6)
\]
for \( s \neq 0 \) and the massless Klein-Gordon equation
\[
\frac{\partial^2}{\partial x^a \partial x^b} c(x) = 0
\]  
(2.7)

for \( s = 0 \) scalar (for \( s > 0 \) it is a consequence of (2.6)).

Given function \( C(w, \bar{w}|0) \) of the spinors \( w^a \) and \( \bar{w}^\dot{a} \) it uniquely reconstructs a solution of the equation (2.1) by
\[
C(w, \bar{w}|x) = \exp \left( -x^{ab} \frac{\partial^2}{\partial w^a \partial \bar{w}^b} \right) C(w, \bar{w}|0) .
\]

Other way around, given solution of the equations (2.1) the full dependence on \( w \) and \( \bar{w} \) is reconstructed as follows. The Taylor expansion gives
\[
C(w, \bar{w}|x) = \exp \left( w^a \frac{\partial}{\partial v^a} + \bar{w}^\dot{a} \frac{\partial}{\partial \bar{v}^\dot{a}} \right) C(v, \bar{v}|x) \bigg|_{v=\bar{v}=0} .
\]

For a given helicity \( h \geq 0 \) we obtain
\[
C(w, \bar{w}|x) = \frac{1}{(2h)!} \left( w^b \frac{\partial}{\partial v^b} \right)^{2h} F_h \left( w^a \frac{\partial}{\partial v^a} \bar{w}^\dot{a} \frac{\partial}{\partial \bar{v}^\dot{a}} \right) C(v, \bar{v}|x) \bigg|_{v=\bar{v}=0} ,
\]
where the function
\[
F_h(r) = \sum_{n=0}^{\infty} \frac{(2h)!}{n!(2h+n)!} (r)^n
\]
is related to the regular Bessel functions \( I_k(x) \) (see, e.g., [10]) as follows
\[
\frac{r^h}{(2h)!} F_h(r) = I_{2h}(2r^{\frac{1}{2}}).
\]

Now, using again equation (2.1), we obtain for a field with a positive helicity \( h \)
\[
C^h(w, \bar{w}|x) = F_h(-w^a \bar{w}^\dot{b} \partial_{ab}) C^h(w, 0|x) ,
\]  
(2.8)

where \( \partial_{a\dot{a}} = \frac{\partial}{\partial x^a \partial \bar{x}^\dot{a}} \). Analogously, one obtains for negative helicities \( h < 0 \)
\[
C^h(w, \bar{w}|x) = F_{|h|}(-w^a \bar{w}^\dot{b} \partial_{ab}) C^h(0, \bar{w}|x) .
\]  
(2.9)

This reconstructs the dependence on \( w \) and \( \bar{w} \).

### 3 Higher-Spin Conformal Currents

From the equation (2.1) it follows [7] that the field equations for massless fields of all spins are \( sp(8) \) symmetric with \( sp(8) \) being a maximal finite-dimensional subalgebra of the infinite-dimensional HS symmetry. This symmetry is conformal because \( sp(8) \)
contains the $4d$ conformal algebra $su(2,2)$ as a subalgebra. The infinite set of
conformal HS symmetries suggests the existence of conserved HS currents.

The HS charges in Minkowski space should have the form

$$Q(\eta) = \int_{\Sigma^3} \Omega^3(\eta),$$

(3.1)

where \(\eta\) denotes the HS symmetry parameters, \(\Omega^3(\eta)\) is a on-mass-shell closed 3-
form dual to the conserved current, and \(\Sigma^3\) is an arbitrary 3-dimensional surface
in the Minkowski space-time usually identified with the space surface \(R^3\), i.e. the
Cauchy surface for the problem.

Using the unfolded form of the massless field equations it is easy to write down
explicit formulae for the conserved HS charges in $4d$ Minkowski space. Let us con-
sider the following 3-form in Minkowski space

$$\Omega^3(\eta) =$$

\begin{equation}
\begin{aligned}
&dx_{\eta \bar{\alpha}} \wedge dx^{\bar{a}} \wedge dx_{\bar{a}} \eta_{\bar{b}_1 \ldots \bar{b}_{i+1} \ldots \bar{b}_t} \alpha^{\bar{a}_1 \ldots \alpha_s} x^{\bar{b}_1 \bar{\epsilon}_1} \ldots x^{\bar{b}_t \bar{\epsilon}_t} \bar{x}^{\bar{c} \bar{b}_1 \ldots \bar{b}_{i+1} \ldots \bar{b}_t} \alpha^{\bar{a}_1 \ldots \alpha_s} x^{\bar{b}_1 \bar{\epsilon}_1} \ldots x^{\bar{b}_t \bar{\epsilon}_t} T_{\bar{c} \bar{a}_1 \ldots \alpha_s \bar{a}_1 \ldots \alpha_s} x^{\bar{b}_1 \bar{\epsilon}_1} \ldots x^{\bar{b}_t \bar{\epsilon}_t},
\end{aligned}
\end{equation}

(3.2)

where \(\eta_{\bar{b}_1 \ldots \bar{b}_t} \alpha^{\bar{a}_1 \ldots \alpha_s}\) are arbitrary HS symmetry parameters symmetric in lower and
upper indices, and the generalized stress tensor \(T_{\alpha_1 \ldots \alpha_n}\) is also symmetric. We use
notation with four-component Greek indices being equivalent to a pair of dotted
and undotted two-component indices, e.g., \(\alpha = a, \bar{a}\).

The form (3.2) is closed

$$d \Omega^3(\eta) = 0$$

(3.3)

provided that the generalized stress tensor \(T_{\alpha_1 \ldots \alpha_n}(x)\) satisfies the conservation con-
dition

$$\frac{\partial}{\partial x^{\bar{b}_b}} T^{\bar{b}_b}_{\alpha_1 \ldots \alpha_{n-2}}(x) = 0.$$  

(3.4)

Indeed, taking into account that the generalized stress tensor \(T_{\alpha_1 \ldots \alpha_n}(x)\) is symmetric
in its indices, (3.3) is a simple consequence of the fact that

$$dx_{a\bar{a}} \wedge dx^{\bar{a}} \wedge dx_c \wedge dx_{\hat{\beta} \bar{a}} \wedge dx_{\beta \bar{a}} \wedge dx_{\hat{\beta} \bar{a}} \wedge dx_{\beta \bar{a}} = \frac{1}{4} dx_{a\bar{a}} \wedge dx^{\bar{a}} \wedge dx^{\bar{a}} \wedge dx_{\beta \bar{a}} \wedge dx_{\hat{\beta} \bar{a}} \wedge dx_{\hat{\beta} \bar{a}} \wedge dx_{\beta \bar{a}} \frac{\partial}{\partial x^{\bar{c} \bar{c}}}.$$  

For the case with an equal number of dotted and undotted indices among the
indices \(\alpha\) in (3.4), it amounts to the usual conservation condition for traceless symmetric tensors which is well-known to be related to conformal HS symmetries [1].

The equation (3.4) tells us however that, in the general case, all irreducible tensors of the $4d$ Lorentz algebra may appear as generalized HS conserved stress tensors except for those described by purely dotted (i.e., selfdual) or purely undotted (i.e., antiselfdual) components. Note that, in the tensorial language, generalized stress
tensors of integer spins are described by various traceless Lorentz tensors that have
symmetry properties of Young tableaux with at least two rows. The components of
\(T_{\alpha_1 \ldots \alpha_n}(x)\) that do not contribute to the conserved charge are described by various
two-row rectangular Young tableaux.
The key observation is that the generalized stress tensor

\[ T^{k l}_{a_1 \ldots a_n}(x) = \frac{\partial}{\partial y^{a_1}} \cdots \frac{\partial}{\partial y^{a_n}} (C^k(y|x)C^l(iy|x)) \bigg|_{y=0}, \]

(3.5)

where \( y^a = (w^a, \bar{w}^{\dot{a}}) \), satisfies the conservation condition (3.4) provided that the field \( C^k(y|x^{a\dot{a}}) \) satisfies the 4\(d\) unfolded equation (2.1). Indeed, from (2.1) it follows

\[
\frac{\partial}{\partial x^{bb}} \frac{\partial}{\partial \bar{w}^b} \frac{\partial}{\partial \bar{w}^b} (C^k(y|x)C^l(iy|x)) = -\frac{\partial}{\partial \bar{w}^b} \frac{\partial}{\partial \bar{w}^b} \left( C^k(y|x) \frac{\partial}{\partial \bar{w}^b} C^l(iy|x) - \frac{\partial}{\partial w^b} \frac{\partial}{\partial \bar{w}^b} C^k(y|x) C^l(iy|x) \right) = 0.
\]

Note that the conserved currents built of HS fields according to (3.5) contain higher derivatives. This is in agreement with the analysis of [11] as well as with the general property of HS theories that their interactions contain higher derivatives [12, 13].

4 Examples

In this section we consider some examples of conserved currents resulting from the general construction.

In terms of two-component fields, the dynamical equations (2.6), (2.7) on the (anti)selfdual components \( c_{a_1 a_2 \ldots a_2 s}(x) \) and \( \bar{c}_{\dot{a}_1 \dot{a}_2 \ldots \dot{a}_2 s}(x) \) read

\[ \partial^{a_1 \dot{a}_1} c_{a_1 a_2 \ldots a_2 s} = 0, \quad \partial^{a_1 \dot{a}_1} \bar{c}_{\dot{a}_1 \dot{a}_2 \ldots \dot{a}_2 s} = 0. \]

(4.1)

These equations imply that space-time derivatives of the field strengths are symmetric in dotted and undotted indices separately.

The straightforward substitution of (2.8), (2.9) into (3.5) gives the generalized stress tensor that contains \( p \) derivatives acting on spin-\( s \) selfdual and spin-\( s' \) antiselfdual constituent fields

\[ T^{s, s', p}[k, l]_{a(2s+p), \dot{a}(2s'+p)} = \sum_{j=0}^{j=p} (-)^j \frac{(2s)! (2s+p)! (2s')! (2s'+p)!}{(2s+j)! (p-j)! (2s'+p-j)!} \partial_{a\dot{a}(j)} c^k_{a(2s)} \partial_{\dot{a}\dot{a}(p-j)} \bar{c}^l_{\dot{a}(2s')}, \]

(4.2)

where we use notation

\[ \partial_{a\dot{a}(k)} \equiv \frac{\partial^k}{\partial x^{a\dot{a}} \ldots \partial x^{a\dot{a}}}, \]

(4.3)

and the indices denoted by the same letter are assumed to be symmetrized (with the convention that the symmetrization is a projector, i.e. the repeated symmetrization leaves a symmetrized tensor unchanged). Analogously one can construct selfdual-selfdual \( T^{s, s', p}[k, l]_{a(2s+2s'+p), \dot{a}(p)} \) and antiselfdual-antiselfdual \( T^{s, s', p}[k, l]_{a(p), \dot{a}(2s+2s'+p)} \) generalized stress tensors.
The generalized irreducible angular momentum tensors obtained from (3.2) have the form

\[ M^{s,s',p|m,m',n|k,l} = T^{s,s',p|k,l}_{a(2s+p-m-m'-n),a(2s'+p+m-m'-n)} x^{b(m)} \hat{a}(m) x^{b(m')} \hat{a}(m') x^{c(i(n)} , \]

with \( x^{b(m)} \hat{a}(m) \equiv x^b_\hat{a} \ldots x^b_\hat{a} \).

For the particular case of fields of equal spins, we obtain the generalized stress tensors

\[ T^{s,s,p|k,l}_{a(2s+p),a(2s+p)} = \sum_{j=0}^{j=p} \frac{(-1)^j ((2s)!)^2 ((2s+p)!)^2}{(2s+j)!(p-j)!(2s+p-j)!j!} \partial_{a\hat{a}(j)c_a(2s)}(x) \partial_{a\hat{a}(p-j)c_{\hat{a}(2s)}}(x) \]

and \( T^{s,s,p|k,l}_{a(4s+p),a(p)}, T^{s,s,p|k,l}_{a(p),a(4s+p)} \) corresponding to rank 2s + p symmetric traceless tensors \( T^{s,s,p|k,l}_{\mu(2s+p)} \) in the tensor notation.

Let us consider some lower-spin examples. A spin-0 massless scalar field \( \phi^k(x) \) satisfies the Klein-Gordon equation

\[ \partial^\mu \partial_\mu \phi^k(x) = 0, \]

equivalent to (2.7). The HS totally symmetric conserved currents built of higher derivatives of the scalar field \([1, 2]\) are

\[ J^{k,l}_{a\hat{a}} = \partial_{a\hat{a}} \phi^k \phi^l - \partial_{a\hat{a}} \phi^k \phi^l \]

and the improved stress tensor

\[ T^{0,0,2|k,l}_{aa,\ddot{a}\ddot{a}} = \phi^k \partial_{aa} \partial_{a\ddot{a}} \phi^l - 4 \partial_{a\ddot{a}} \phi^k \partial_{aa} \phi^l + \partial_{a\ddot{a}} \partial_{aa} \phi^k \phi^l , \]

which is symmetric in the (discarded) color indices. In tensor notation, these have the form

\[ J^{k,l}_{\mu} = \phi^k \partial_\mu \phi^l - \phi^l \partial_\mu \phi^k \]

and

\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \phi \partial_\mu \phi \partial_\nu \phi - \frac{1}{4} \eta_{\mu\nu} \partial_\lambda \phi \partial_\lambda \phi . \]

A spin-\( \frac{1}{2} \) field \( \psi(x) \) satisfying the massless Dirac equation \( \gamma_\mu \partial^\mu \psi(x) = 0 \) is described by \( c_a(x) \) and \( \bar{c}_\ddot{a}(x) \) that satisfy (4.1). Neglecting the color indices, the electric current \( T^{2,1}_{a,\ddot{a}} \) and stress tensor \( T^{2,1}_{a,\ddot{a}} \) are

\[ T^{2,1}_{a,\ddot{a}} = c_a \bar{c}_\ddot{a} , \]
A supercurrent, which mixes spin-0 and spin-$\frac{1}{2}$ fields, is given by $T^{\frac{1}{2},0,1}_{\alpha(2),\dot{\alpha}(2)}$, 

$$T^{\frac{1}{2},0,1}_{\alpha(2),\dot{\alpha}(2)} = 2 (c_{\alpha} \partial_{\alpha \dot{\alpha}} \bar{c}_{\dot{\alpha}} - \partial_{\alpha \dot{\alpha}} c_{\alpha} \bar{c}_{\dot{\alpha}}). \quad (4.14)$$

and its complex conjugate.

A massless spin-1 field, can be described by a gauge invariant field strength satisfying the Maxwell equations

$$\partial^\mu F_{\mu \nu} = 0, \quad \partial_{[\rho} F_{\mu \nu]} = 0. \quad (4.16)$$

In terms of two-component spinors, $F_{\mu \nu}$ is described by $c_{\alpha \dot{\alpha}}$ and $\bar{c}_{\dot{\alpha} \alpha}$, while the Maxwell equations have the form (4.1).

The stress tensor

$$T_{\mu \nu} = -F_{\mu}^{\sigma} F_{\sigma \nu} + \frac{1}{4} \eta_{\mu \nu} F^2 \quad (4.17)$$

is described by $T^{1,1,0}_{\mu \nu \lambda}$

$$T^{1,1,0}_{\alpha \dot{\alpha} \alpha \dot{\alpha}} = 4 c_{\alpha \dot{\alpha}} \bar{c}_{\dot{\alpha} \alpha}. \quad (4.18)$$

Analogously to the scalar field case, there exist totally symmetric HS conserved currents built of higher derivatives of the spin-1 field strength [1]. These are the generalized stress tensors $T^{1,1,p}_{\mu \nu \lambda}$.

With the aid of the stress tensor $T_{\mu \nu}$ one can construct an angular momentum tensor

$$M_{\mu \nu \lambda} = T_{\mu \nu \lambda} x_{\lambda} - T_{\mu \lambda} x_{\nu}. \quad (4.19)$$

in the case of spin-0 the corresponding spinor-tensors are $M^{0,0,2,1,0,0}_{\alpha \dot{\alpha}(3)}$, $M^{0,0,2,0,1,0}_{\alpha(3) \dot{\alpha}}$ and $M^{0,0,2,0,0,1}_{\alpha \dot{\alpha}}$.

A massless spin-2 field describes the linearized gravity. The linearized gauge invariant combinations of derivatives of a linearized metric tensor are given by the linearized Riemann tensor. Its trace part is zero by virtue of Einstein equations. The nonzero traceless part is called Weyl tensor $H_{\mu \nu \lambda \rho}$. As a consequence of Einstein equations, the Weyl tensor satisfies differential restrictions by virtue of Bianchi identities. In terms of two-component spinors Weyl tensor is described by the self-dual component $c_{abcd}(x)$ and antiselfdual component $\bar{c}_{abcd}(x)$. The consequences of Einstein equations have the form (4.1).

It is well-known that there is a conserved current called Bel-Robinson tensor [14], [15] that is bilinear in Weyl tensor. In terms of tensors, it has the form

$$T_{\mu \nu \lambda} = H_{\mu \sigma \nu \eta} H_{\lambda}^{\sigma \rho} \eta + * H_{\mu \sigma \nu \eta} * H_{\lambda}^{\sigma \rho} \eta, \quad (4.20)$$

where the Hodge star $*$ denotes the dualization by virtue of the Levi-Civita tensor $\varepsilon_{\mu \nu \lambda \rho}$. In terms of two component spinors the Bel-Robinson tensor is described by $T^{2,2,0}_{a(4),\dot{a}(4)}$, having the simple form

$$T^{2,2,0}_{a(4),\dot{a}(4)} = (4!)^2 c_{a(4)} \bar{c}_{a(4)}. \quad (4.21)$$

Higher symmetric generalized strength tensors built of the Weyl tensor are given by the formula (4.6) with $s = 2$. 

8
5 Conclusion

Although the obtained list of conserved currents is infinite it does not reproduce some of the expected symmetry generators and as such is incomplete. This is not surprising because even ordinary conserved currents like stress tensor and electric charge for higher spins are not in the presented class of gauge invariant currents. Indeed, it is well-known that the energy-momentum conservation in gravity is described in terms of gauge non-invariant pseudo-tensor [17] which, however gives rise to gauge invariant total energy and momentum conservation laws at the level of free fields. The same happens for all higher spins [11, 18, 19]. The reason is simply that the gauge invariant tensors $C(w, \bar{w}|x)$ contain at least $s$ derivatives of a spin-$s$ gauge potential and therefore can only appear in the conserved currents which themselves carry sufficiently high spins.

The system of higher spin fields of all spins is $sp(8)$ invariant. [5, 7]. The pattern of the symmetry parameters and corresponding conserved currents in two-component spinor notations is as follows: generalized translations have symmetry parameters $\eta^{\dot{a}a}$, $\eta^{\dot{a}a}$, $\eta^{\dot{a}a}$ and conserved currents $T_{a\dot{a}a}^{k\dot{a}}$, $T_{aa\dot{a}}^{k\dot{a}}$: $T_{a\dot{a}a}^{k\dot{a}}$. Generalized Lorentz boosts and dilatation have symmetry parameters $\eta^{\dot{a}a}$, $\eta^{\dot{a}a}$, $\eta^{\dot{a}a}$, $\eta^{\dot{a}a}$ and conserved currents $T_{a\dot{a}a\dot{a}}^{k\dot{a}ab}$, $T_{a\dot{a}a\dot{a}}^{k\dot{a}ab}$, $T_{a\dot{a}a\dot{a}}^{k\dot{a}ab}$, $T_{a\dot{a}a\dot{a}}^{k\dot{a}ab}$. Generalized special conformal transformations have symmetry parameters $\eta^{bb}$, $\eta^{bb}$, $\eta^{bb}$ and conserved currents $T_{a\dot{a}ab}^{k\dot{a}ab}$, $T_{a\dot{a}ab}^{k\dot{a}ab}$, $T_{a\dot{a}ab}^{k\dot{a}ab}$.

The list of generators of this type (which includes the generators of the usual conformal algebra $su(2, 2) \subset sp(8)$) that can be constructed in terms of invariant higher spin tensors is quite short

$$T_{a\dot{a}a}^{k\dot{a}} = c^k \partial_{a\dot{a}} \partial_{a\dot{a}} c^l - 4 \partial_{a\dot{a}} c^k \partial_{a\dot{a}} c^l + \partial_{a\dot{a}} \partial_{a\dot{a}} c^k c^l + 4 c_{a\dot{a}} c_{a\dot{a}}; 4 \partial_{a\dot{a}} c_{a\dot{a}} c^l; 2 \left(c_a \partial_{a\dot{a}} c_a - \partial_{a\dot{a}} c_{a\dot{a}} \right), \quad (5.1)$$

$$T_{a\dot{a}a\dot{a}}^{k\dot{a}ab} = 2 \left(c_a \partial_{a\dot{a}} c_a - \partial_{a\dot{a}} c_{a\dot{a}} \right); \quad 6 c_{a\dot{a}} c_{a\dot{a}} c^l, \quad (5.2)$$

$$T_{a\dot{a}a\dot{a}}^{k\dot{a}ab} = 6 c_{a\dot{a}} c_{a\dot{a}} c^k; \quad 6 c_{a\dot{a}} \partial_{a\dot{a}} c^k - 2 \partial_{a\dot{a}} c_{a\dot{a}} c^k, \quad (5.3)$$

and obviously incomplete because the $sp(8)$ symmetry mixes fields of all spins while the generators (5.1), (5.2) and (5.3) do not contain higher spin fields at all. It remains to be investigated whether it is possible to complete the list of higher spin conserved currents presented in this paper by the higher spin pseudotensors which may not be gauge invariant but allow for the construction of invariant conserved charges.

Finally let us note that the formula for conformal HS currents presented in this paper is analogous to the formula of [4, 16] for conserved HS currents in the ten-dimensional space-time $M_4$ suggested for the description of 4d massless HS fields in [5, 7, 8]. In fact, the expression (3.5) for $T_{a_1...a_n}^{k\dot{a}}(y|x^{a\dot{a}})$ is the reduction to the Minkowski space of the generalized stress tensor [4] $T_{a_1...a_n}^{k\dot{a}}(y|X^{a\dot{a}})$, where $X^{a\dot{a}}$ are symmetric matrix coordinates of $M_4$. The conservation condition (3.4) is also the reduction of the conservation condition in $M_4$. The explicit relationship between the two constructions, which requires an appropriate integration over a noncontractible cycle in $M_4$ remains to be elaborated, however.
Acknowledgments

The work was supported in part by grants RFBR No. 05-02-17654, LSS No. 1578.2003-2 and INTAS No. 03-51-6346.

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