Current conservation and ratio rules in magnetic metals with Coulomb repulsion

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December 2011

Abstract. From general considerations of spin-symmetry breaking associated with (anti-)ferromagnetism in metallic systems with Coulomb repulsion, we obtain interesting and simple all-order rules involving the ratios of the densities of states. These are exact for ferromagnetism under reasonable conditions, and nearly exact for anti-ferromagnetism. In the case of ferromagnetism, the comparison with the available experimental and theoretical numbers yields favourable results.

PACS. 11.40.-q Currents and their properties – 75.10.-b General theory and models of magnetic ordering

1 Introduction

1.1 Theoretical background

As is obvious and well known [1], magnetic order breaks SU(2) spin symmetry (hereafter called SU(2)_{\text{spin}}). This gives rise to gapless excitations in the form of Nambu–Goldstone modes which are magnons and, according to the Goldstone theorem, excitations with energy gap, which may be called the Higgs bosons.

These excitations behave like elementary fields, and their interaction is central to spin-current conservation but, at the same time, they comprise of electrons with which they interact: i.e., they are composite. This imposes severe constraints on the properties of these fields, which we aim to discuss and exploit in this paper.

The same situation, of Goldstone fields (i.e., both Goldstone and Higgs fields) that are themselves composite objects, arises notably in two problems in the context of high-energy physics. The first problem is that of axial symmetry breaking at low energy scales due to some interaction (N.B. not by the SU(3)_{C} strong interaction which is weakly interacting at those scales). The low-energy phenomenology would then not be dissimilar to that of the Standard Model, but with some constraints on quantities such as the Higgs-boson mass.

A new approach to these problems, due to Gribov, have appeared in refs. [2,3,4,5]. These involve the idea of super-criticality and a self-consistent treatment of the fermion and Goldstone fields in the presence of the super-critical interaction. We shall make use of, and extend, the methods presented therein, to the case of magnetism. As for the other approaches to these old problems, see, for example, ref. [6] for an old approach to the first problem, and ref. [7] for an overview of the various methods and techniques developed to handle the second problem.

1.2 Outline of the paper

Our work concerns systems of electrons (or holes) which interact under a generalized Coulomb exchange (i.e., exchange of a generic gapless photon). We consider the system in the spin-symmetry-broken phase that arise in ferromagnetism and anti-ferromagnetism.

Our aim is to obtain exact relations between quantities that characterize the spin-symmetry-broken phase using the Dyson–Schwinger equations. This is possible because of the presence of the Ward–Takahashi identities which arise because of the conservation of spin symmetry. It turns out that the form of the Coulomb interaction does not affect these relations. The Coulomb interaction does affect, for instance, the electronic self-energy, but these are incorporated in the relations in a general way.
Before presenting the full analytical framework, we start with the simple case of the discussion of ground-state stability in ferromagnetism, in sec. 2. This gives rise to an exact rule for ferromagnetism which involves the electronic densities of states. We discuss this case with illustrations and a physical interpretation.

The full framework, which employ current-conservation techniques, is developed in secs. 3 and 4. In sec. 3 the interaction is worked out and presented in the form of Feynman rules. In sec. 4, the parameters of the interaction is worked out and presented in the form of Feynman rules. In sec. 3, the techniques, is developed in secs. 3 and 4. In sec. 3, the interaction is worked out and presented in the form of Feynman rules. In sec. 4, the parameters of the interaction is worked out and presented in the form of Feynman rules.

Before we introduce the full framework, let us discuss the problem, which does not require the full formalism, and which nevertheless leads to a strikingly simple and useful rule. We shall expand the methods introduced here to build the formalism later.

2 The ferromagnetic ratio rule

Before we introduce the full framework, let us discuss the stability of the ferromagnetic ground state as an illustrative example. We do so because this is a relatively simple problem, which does not require the full formalism, and which nevertheless leads to a strikingly simple and useful ratio rule. We shall expand the methods introduced here to build the formalism later.

2.1 Description of the system

Let us write the electrons in the SU(2) spin doublet form:

\[ \psi_a \equiv \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}. \]  

The Lagrangian has the form:

\[ \mathcal{L} = \delta_{ab} \psi_a^* \left( \frac{i}{\partial t} - e(-i\nabla) - e\Gamma^\mu A_\mu \right) \psi_b + (\text{photon K.E.}) \]  

\[ (1) \]

\[ A_\mu \] is the electro-magnetic field, of which we shall retain only the electrostatic term \( A_0 \) later, as the contribution of the 3-vector potential is suppressed by the speed of light. The photon kinetic energy term may then be taken to be \( -(\nabla A_0)^2/2 \), which leads to an electrostatic \( 1/r \) interaction. \( e \) refers to the dispersion relation of the electron. \( e \) is the electro-magnetic charge, which is defined to be negative for electrons. \( \Gamma^\mu \) is the vertex function, whose time component is 1 for a Lagrangian of this form.

In the absence of magnetic order, the system is invariant under both the electromagnetic U(1)EM and the SU(2)spin rotations of \( \psi \), where the latter is represented by

\[ \Pi(\phi_i) \equiv \exp \left( i\sigma^i \phi_i / 2 \right). \]

\[ (3) \]

There are two conserved currents, which are orthogonal. The first is the U(1)EM current:

\[ J_{\text{EM}}^\mu = \psi_\uparrow^* \delta_{ab} \Gamma^\mu \psi_b. \]

\[ (4) \]

\( \mu \) refers to the time-space four-vector index \((=0,1,2,3)\).

The second is the SU(2)spin current. This is written as

\[ J_{\text{spin}}^\mu = \psi_\uparrow^* \sigma^i \Gamma^\mu \psi_b. \]

\[ (5) \]

Let us indicate the current diagrammatically by a cross. We see that current–current mixing, which we denote as \( I_{\text{mix}}^{\mu \nu} \) and whose lowest order term is given by

\[ \sigma^i_{ab} \mu \delta_{ab} \]

vanishes by symmetry for all \( i \) to all perturbative orders, and therefore the two currents are orthogonal.

When there is magnetic order, there arises, locally, a preferred orientation of spin, let us say along \( \downarrow \), and this breaks the SU(2)spin symmetry, viz:

\[ \text{SU}(2)_{\text{spin}} \rightarrow U(1)_z. \]

\[ (7) \]

As a result, there arises two Goldstone modes whose coupling is proportional to linear combinations of \( \sigma^1, \sigma^2 \), and a Higgs mode whose coupling is proportional to \( \sigma^3 \). The residual symmetry \( U(1)_z \) refers to the symmetry under rotation by the generator \( \sigma^3 \):

\[ U(\phi_3) \equiv \exp \left( i\sigma^3 \phi_3 / 2 \right) \equiv \text{diag} \left( e^{i\phi_3/2}, e^{-i\phi_3/2} \right). \]

The form of the effective theory will be discussed later.

2.2 U(1) current mixing

A result, which is almost trivial but possibly not previously discussed explicitly, is that after this symmetry breaking, the currents are no longer orthogonal. Equation \[ (6) \] is easily calculated. Of particular interest is the 0 \( \rightarrow \) 1 component of eqn. \[ (6) \] for \( i = 3 \) (i.e., the \( U(1)_z \) current: \( i = 1, 2 \) vanish) at zero external energy and momenta. The vertex function \( \Gamma^0 \) being equal to 1 when the electrons are Fermi-liquid-like for each spin orientation, we obtain

\[ I_{\text{mix}}^{00} = \lim_{q \rightarrow 0} \int \frac{d^{d+1}k}{(2\pi)^{d+1}} G_\uparrow(k)G_\uparrow(q+k) - G_\downarrow(k)G_\downarrow(q+k). \]

\[ (9) \]

Hence

\[ I_{\text{mix}}^{00} = g_{\uparrow}(\epsilon_F) - g_{\downarrow}(\epsilon_F). \]

\[ (10) \]

Note that \( I \) is defined with a negative sign, that is \( II = -A \), where \( A \) is the two-point amplitude. \( G \) are the electron Green’s functions. \( g(\epsilon_F) \) are the densities of states at Fermi energy.

As is well known, the Dyson–Schwinger all-order corrections to a fish diagram such as that indicated in eqn. \[ (8) \] is incorporated by replacing the Green’s functions by their
all-order counterparts and one or the other of the vertices (and not both) by their all-order counterpart. Equation (10) is an all-order expression in the sense that the Green’s functions are arbitrary. However, the vertex correction needs care. Let us consider the all-order correction to the photon vertex. At zero external energy and momenta, the time component of the all-order vertex is given by the energy derivative of \( G^{-1} \), by virtue of the Ward–Takahashi identity. It follows that, if the all-order Green’s function is given by

\[
G(E, k) = \frac{Z}{E - \epsilon(k) + i0 \text{sgn}(\epsilon(k) - \mu)}
\]

(11)
as is the case for the Fermi liquid, then the vertex is corrected by \( Z^{-1} \). On the other hand, there is \( Z^2 \) coming from \( G^2 \), and so the net result is proportional to \( Z \). \( Z \) being the correct renormalizing factor for the density of states, eqn. (10) is exact. It is not difficult to see that a more general form of \( G \) also admits this property:

\[
G(E, k) = \frac{Z(k)}{f(E - \epsilon(k) + i0 \text{sgn}(\epsilon(k) - \mu))}
\]

(12)

where \( f \) is any function, so long as the density of states is definable as the integral of \( G \). Thus it is not a necessary condition that the system is a Fermi liquid.

Returning to eqn. (10), in general, \( g_\uparrow \) and \( g_\downarrow \) are not equal at the Fermi surface, and therefore the currents mix.

It is worth noting here that the current mixing is zero in the case of anti-ferromagnetism, because the two sublattice contributions are equal and opposite.

Before proceeding, let us calculate the other fish diagrams. Both for the EM current and for the spin \( \text{U}(1)_z \) current, we obtain:

\[
P^{00}_{\text{EM}} = 2P^{00}_{z} = g_\uparrow(\epsilon_F) + g_\downarrow(\epsilon_F).
\]

(13)

Again, this is an exact result provided that the Green’s functions are of the form eqn. (12) and the densities of states can be defined as their integrals. Note that although we have retained the subscript ‘EM’ to refer to electromagnetism, in reality, we are analyzing electrostatics.

### 2.3 Derivation of the ratio rule

Let us now consider the stability of the ferromagnetic ground state. To do so, a primary condition is the vanishing of the tadpole:

\[
\sigma^3 \quad \delta_{ab}
\]

(14)
as is required by the condition that there are no terms that are linear in the Higgs field in the effective Lagrangian. In other words, the first derivative of free energy as a function of the magnetic order parameter must vanish when the ground state is stable. We will also need to check that the second derivative is positive. This means the term which is bilinear in the Higgs field, or the self-energy of the Higgs boson, is positive. The Higgs self-energy is the same as \( P^{00}_0 \) calculated earlier on, up to the square of a coupling constant. This is necessarily positive.

Equation (11) is calculated easily, and we obtain

\[
A^{\text{tadpole}}_z = \int \frac{d^{d+1}k}{(2\pi)^{d+1}} G_\uparrow(k) - G_\downarrow(k) = \rho_\uparrow - \rho_\downarrow.
\]

(15)

\( \rho \) refers to the total density of states of electrons. This is an exact expression since higher-order corrections to the tadpole are taken into account by making \( G \) all-order and vertex to be bare. This is always negative if \( \downarrow \) is the preferred orientation of spin.

It follows that eqn. (14) by itself is non-zero. However, the currents mix, and we must incorporate the contribution of the photon tadpole, multiplied by the Higgs–photon two-point function which has the same form as \( P^{00}_0 \) calculated in eqn. (10):

\[
\sigma^3_a \quad \delta_{ab} \quad \delta_{bc} \quad \sigma^3_c
\]

(16)

Now, to make this equation all-order, we must include the screening effect in the photon propagator, and this has the same form as \( P^{00}_0 \) calculated in eqn. (10). Altogether, we obtain

\[
A^{\text{tadpole}}_{\text{photon part}} = (\rho_\uparrow + \rho_\downarrow) e \times \frac{(g_\uparrow(\epsilon_F) - g_\downarrow(\epsilon_F)) e c}{(g_\uparrow(\epsilon_F) + g_\downarrow(\epsilon_F)) e^2}.
\]

(17)

The two contributions must vanish when added together. Although the charge \( e \) appears here, whether one takes the charge carriers to be electrons or holes is a matter of choice, so \( e \) can be taken as constant. Hence,

\[
\frac{g_\downarrow(\epsilon_F) - g_\uparrow(\epsilon_F)}{g_\uparrow(\epsilon_F) + g_\downarrow(\epsilon_F)} = \frac{\rho_\downarrow - \rho_\uparrow}{\rho_\uparrow + \rho_\downarrow}
\]

(18)
or,

\[
\frac{g_\uparrow(\epsilon_F)}{g_\downarrow(\epsilon_F)} = \frac{\rho_\uparrow}{\rho_\downarrow}
\]

(19)

This is our ferromagnetic ratio rule.

In eqn. (18), the left-hand side is often called the spin polarization \( P \), for example in the context of tunnel magnetoresistance. The right-hand side is the magnetic moment \( n_B = m/\mu_B \) divided by the number of carriers \( n \), i.e.,

\[
P = \frac{n_B}{n}
\]

(20)

Note that the definition of \( n \) is ambiguous. However, it is a measure of the number of electrons or holes that are actively involved in the formation of ferromagnetic order. As such, one would expect that its order is estimated by the number of carriers in the conduction band. If so, we obtain a simple rule of the thumb:

\[
\begin{cases}
g_\uparrow > g_\downarrow \quad (\text{electrons}), \\
g_\downarrow < g_\uparrow \quad (\text{holes}).
\end{cases}
\]

(21)
That is, the density of states of the majority-spin charge carriers is always greater. According to this rule, one expects the density of states to be rising for electrons and decreasing with energy for holes. This can be a useful rule of the thumb to establish whether a system with certain given density of states is likely to become a ferromagnet.

2.4 A diagrammatic derivation

The preceding derivation is formal and, we believe, complete, but it may appear baffling at the start that the sum of apparently only two contributions is sufficient to give all-order statements about the system.

Another way to derive the same result, in a more diagrammatic fashion, is to consider the mixing between screened photon and the Higgs boson. Let us denote the mixed states by \( \tilde{\gamma} \) and \( \tilde{h} \). Then the sum of the tadpoles

\[
\sigma_{\alpha\beta} = -\frac{\tilde{\gamma} \cdot \tilde{h}}{\epsilon}
\]

needs to be zero, whereas the sum of the tadpoles

\[
\delta_{\alpha\beta} = -\frac{\tilde{\gamma} \cdot \tilde{h}}{\epsilon}
\]

is non-zero, and gives a constant contribution to the energy levels of the states.

The sum over the former set of tadpoles can be expanded diagrammatically in terms of \( \tilde{\gamma} \) and \( \tilde{h} \) as:

\[
\sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} \cdots
\]

It will be seen that if the sum of the first two terms is zero, then the remaining contributions, which are proportional to the sum of first two terms, vanish automatically. Therefore it suffices to calculate the sum of the first two terms.

2.5 Examples

One way to understand eqn. (20) is as a definition of \( n \). This number can then be compared with the other estimates of the number of carriers such as by the Hall effect and with the nominal number of electrons or holes.

As a first example, in the case of half metals, and in the ideal case, the spin polarization would be perfect, i.e., \( P = 1 \). The magnetization will also be perfect, and eqn. (20) will be satisfied so long as we take \( n \) to be equal to \( n_B \).

Next, in the case of a Coulomb system whose renormalized dispersion relation is given exactly by \( \epsilon(k) = (\hbar k)^2/2m_e \), the density of states is given by:

\[
g(\epsilon) = \frac{1}{\pi^2} \sqrt{m_e^2 \epsilon^2 / 2}, \quad \rho = \frac{1}{3\pi^2} \sqrt{m_e^2 \epsilon^2 / 2}.
\]

We have defined \( \rho \) as the integral over whole of the occupied states. Equation (19) then admits the following solutions only:

\[
g_1 = g_1, \quad \text{or}, \quad g_2 = 0.
\]

That is, either the system is a half metal or there is no ferromagnetic order. This is consistent with the observation that group 1 elements are not ferromagnetic, and also with the observation that the Fermi gas system with quadratic (bare) dispersion relation is only weakly ferromagnetic [8].

Let us now consider the case of transition metals. In eqn. (20), \( n_B \) is the only quantity which is measured unambiguously and accurately. \( P \) is measurable, but different methods yield different results [9,10]. Furthermore, these experimental numbers do not match with the results of theoretical calculation [11].

In principle, theoretical numbers for \( g \) should be compared against theoretical numbers for \( P \), and experimental numbers for \( g \) should be compared against the experimental numbers for \( P \). Let us first look at the experimental numbers.

| Element | Carriers | \( P_T \) | \( P_C \) | \( n_B \) |
|---------|----------|----------|----------|---------|
| Fe      | electrons| 0.40     | 0.42     | 0.46    | 2.22    |
| Co      | holes    | 0.35     | 0.42     | 1.72    |
| Ni      | holes    | 0.23     | 0.43     | 0.465   | 0.606   |

Table 1. Experimental numbers for spin asymmetry \( P \), magnetic moment \( n_B \) per site, \( n \) calculated as the ratio of these two, the nominal number of charge carriers, and the number of charge carriers as measured using the Hall effect. Two values for \( P \) are taken from refs. [9] and [10], respectively. \( n_B \) is from ref. [12]. \( n_H \) is calculated from \( R_H \) listed in ref. [13].

In tab. 1 we summarize the experimental numbers for \( P \), \( n_B \) and \( n \). The calculated values of \( n \) are compared against the nominal number of charge carriers, i.e., the number of 4s and 3d electrons or holes, and with the number of carriers calculated from the Hall ratio \( R_H = -1/\rho_e \). We see that the the values of \( n \) do not seem to be in contradiction of the nominal number of charge carriers, in the sense that \( n < n_{\text{nominal}} \) for Fe and \( n \approx n_{\text{nominal}} \) for Ni and, arguably, Co. However, the variation in the experimental numbers is too great to make a concrete statement.

Let us now turn to the theoretical numbers.

In tab. 2 we show the numbers for \( g_1/g_\rho \) as estimated from ref. [11]. In the case of Fe, \( \rho \) could be estimated roughly by the eye as the ratio of the areas underneath the density-of-state curves. For Co and Ni, this was not possible because of the long tails in these curves. However, the relative size of \( g \) was consistent with the nature of the carriers. That is, the density of states was found to be greater for the majority spin. All three cases are thus not inconsistent with eqn. (19).
2.6 Physical interpretation

Let us discuss tadpole cancellation in a more intuitive fashion.

![Diagram](image)

\(\text{(a)} \quad \text{higgs} \quad \text{(b)} \quad \text{photon} \quad \text{(c)} \quad \text{(d)} \quad \text{sea} \quad \text{sea} \quad \text{sea} \quad \text{sea} \)

**Fig. 1.** The interaction of a conduction electron with the Fermi sea of electrons.

We also show the values of \(P\) calculated from \(g_\uparrow/g_\downarrow\). These are quite different from the experimental numbers introduced earlier. As a result, the numbers for \(\rho\) differ from before, if we use the same values of \(n_B\).

To summarize, it is difficult to check the ratio rules quantitatively, at the present level of accuracy.

### Table 2. Theoretical numbers for \(g_\uparrow/g_\downarrow\) and \(\rho_\uparrow/\rho_\downarrow\)

| Element | \(g_\uparrow/g_\downarrow\) | \(\rho_\uparrow/\rho_\downarrow\) | \(P\) | \(n = n_B/P\) |
|---------|----------------|-----------------|-----|-----------|
| Fe      | 2.1 ± 0.2      | 2.0 ± 0.04      | 0.63 ± 0.7 |
| Co      | 7.0 ± 1.5      | > 1             | 0.75 ± 0.04 | 2.3 ± 0.1 |
| Ni      | 10 ± 1.5       | > 1             | 0.82 ± 0.03 | 0.74 ± 0.03 |

**minority spin electrons, this interaction makes majority-spin electrons more energetically unfavourable. That is, Fig. (d), or the density fluctuation of spin, tends to suppress magnetic order.**

The exchange of the Higgs boson is not the only interaction between the conduction electrons and the sea, and in Fig. (d), we show the Coulomb exchange. This is always repulsive, and is of the same magnitude for both type of electrons, and so this diagram does not contribute to the formation or suppression of magnetic order.

**Figure (b) by itself is infinite since the photon propagator diverges at zero momentum transfer. This is, as usual, remedied by the screening effect which is shown in fig. (c).**

The screening effect, such as that shown in fig. (c), usually suppresses the charge. This is because a negative charge attracts positive charge, and this positive charge tends to cancel the negative charge.

However, the contribution of fig. (d) requires more thought. The sea electrons, which have negative charge, attract positive charge. When this positive charge has the same spin as the conduction electron, i.e., when the positive charge suppresses the electronic spin which is aligned with the spin of the conduction electron, the positive charge attracts this conduction electron. On the other hand, when the positive charge has opposite spin to that of the conduction electron, then the conduction electron is repelled.

Whether the interaction of fig. (d) tends to create magnetic order or suppress it depends on which type of spin is more likely to be excited, i.e., on the density of states at the Fermi surface. This is the meaning of the tadpole cancellation. In other words, the ferromagnetic ground state is stable when the interaction due to the fluctuation of spin density, which is mediated by the Higgs boson and which always suppresses the polarization of spin, is equal and opposite to the contribution due to the electrostatic polarization of fig. (d) which, depending on circumstances, can counteract it.

### 2.7 Comparison with the Hubbard model

When the Coulomb interaction is screened, the interaction becomes point-like in the limit of large screening, i.e.,

\[
\frac{1}{g_\uparrow(\epsilon_F) + g_\downarrow(\epsilon_F)} \rightarrow \hat{U}
\]

(27)

where \(\hat{U}\) is a constant which can be interpreted as the on-site Coulomb repulsion \(U\) up to a normalization. Let us now see what would happen if we were to start from a theory which treats the on-site Coulomb repulsion \(U\) as the starting point, such as the Hubbard model.

In this case, eqn. (14) is unchanged, but eqn. (16) is modified to take the following form:

\[
\sigma_{ab}^3 \hat{U}
\]

(28)

Here, as is usual in the Hubbard model, the spin going into the fish part must be opposite to the spin going into
the tadpole. That is,

$$A^{\text{tadpole}}_{\text{Hubbard}} = \hat{U} [g_T(\epsilon_F)\rho_\uparrow - g_L(\epsilon_F)\rho_\downarrow], \quad (29)$$

in the place of eqn. (17). This would lead to different consequences.

The origin of this discrepancy is clear. Expressed in terms of the screened Coulomb propagator, there are two contributions that go into eqn. (28). One is the genuine tadpole-like contribution of the form eqn. (16). The contribution of this term is given by

$$\hat{U} [g_T(\epsilon_F) - g_L(\epsilon_F)] (\rho_\uparrow + \rho_\downarrow), \quad (30)$$

so that this term has the same form as in eqn. (28). On the other hand, there is a second contribution, which is the self-energy correction:

$$\sigma_{\alpha\beta}^3 = \begin{array}{c}
\int d^4x \\
\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta} \partial^\gamma \Phi^{\delta} \\
\times \rho_\uparrow(\rho_\downarrow - \rho_\uparrow) \\
\end{array} , \quad (31)$$

The contribution due to this term is given by

$$\hat{U} [g_L(\epsilon_F)\rho_\uparrow - g_T(\epsilon_F)\rho_\downarrow]. \quad (32)$$

Adding together these two contributions yields eqn. (29).

The discrepancy comes because in our approach, the self-energy correction is absorbed in the all-order Green’s function, whereas in the Hubbard-model approach, this is not possible. In the Hubbard model, either both contributions are treated as a tadpole, or both contributions are treated as a self-energy correction. If the latter, one will have the condition that the simple tadpole, with the self-energy corrections, by itself vanishes. This condition requires $\rho_\uparrow = \rho_\downarrow$, and therefore we will never have a stable fermionic solution out of the Hubbard model.

This, in our opinion, is a limitation of the Hubbard model. The limitation is due to the inability to treat current–current mixing, which is the basis of our discussion in this section.

One may still argue that on-site Coulomb repulsion is present, physically. In other words, the effective screened Coulomb propagator, which ordinarily gives a divergent contribution at the origin up to a UV cut-off, is not really divergent but only large and finite at the origin.

If so, this may be thought of as a variant of the UV cut-off of the Coulomb propagator $D_{\text{photon}}$, which may be parametrized, for example, as

$$D_{\text{photon}}(k) = \frac{1}{-k^2 - ak^4}, \quad (33)$$

where $a$ is a parameter (positive or negative). Even if this is not permissible as a field theory, it is permissible as an UV (Pauli–Villars) regularization procedure. It will be seen that such a cut-off does not affect our argument at all, since our discussion involves zero momentum photons. The electron self-energy will be affected by the UV cut-off, but this does not affect our results explicitly.

### 3 Analysis of spin current conservation

Let us now move on to the formalism, which is required if we are to go beyond the tadpole-level analysis of the preceding section. We adapt Gribov’s analysis of axial current conservation [2,3] to the context of spin current conservation in systems with partial magnetic order.

To sum up in one phrase, our goal is to start from the Coulombic system, which is defined by eqn. (28), and solve it as exactly as possible using the Dyson–Schwinger equations, under a number of assumptions.

The major assumption is that of spontaneous symmetry breaking. If the spin symmetry is broken spontaneously, then the Goldstone theorem guarantees the presence of Goldstone and Higgs modes. These modes are, in terms of the initial Lagrangian, electronic excitations. However, in terms of the effective theory that appears at the end, they are elementary, and participate in the conservation of the spin current. This is the main property that allows us to solve the Dyson–Schwinger equations. The other assumptions, such as the linear or quadratic form of the magnon dispersion relation and the constancy of exchange energy which are sometimes required, are approximations, which we believe are viable, that can be lifted if one has the computational resources.

The resulting effective Lagrangian is found to have the following interaction term:

$$\mathcal{L}^{\text{eff}}_{\text{spin}} = \bar{\psi} \gamma \cdot \Phi \cdot \sigma \psi. \quad (34)$$

Here $\Phi$ is essentially the order-parameter field, but with a certain formal difference which we shall discuss. We would like to emphasize at this point that this equation is not our starting point. It is rather the end product of solving the Coulombic system by means of the Dyson–Schwinger equations, with the aid of the Goldstone theorem and the Ward–Takahashi identities.

Let us start by discussing current conservation. As discussed in the previous section, current is absolutely conserved when the spin symmetry is conserved.

$$\frac{\partial}{\partial x^{\mu}} J^{\mu, \text{spin}} = 0. \quad (35)$$

Current conservation is reflected in the following Ward–Takahashi identity:

$$\Gamma^{\mu}(q_1 - q_2)_\mu = G_{\lambda_1}^{-1}(q_1) - G_{\lambda_2}^{-1}(q_2). \quad (36)$$

$\Gamma^{\mu}$ is the vertex in the momentum space. $\lambda_1, \lambda_2$ refer to the spin states, but these are dummy indices here in the sense that $G_{\lambda}^{-1}(q)$ is independent of $\lambda$. Thus eqn. (36) holds for any combination of spin, and therefore current is conserved. $q$ are $d + 1$-vectors with components $(q_0, \mathbf{q})$. $q_0$ is the energy and $\mathbf{q}$ is the spatial momentum, with $\hbar = 1$.

The Ward–Takahashi identity is violated in the symmetry-broken phase, since there is now an energy difference $\Delta E$, which is the exchange energy, between the different spin states:

$$\Delta E = G_{\uparrow}^{-1}(q) - G_{\downarrow}^{-1}(q). \quad (37)$$
This is the definition for the ferromagnetic case, when \( \Delta E \) is positive if we take \( \downarrow \) to be the majority spin state. In the anti-ferromagnetic case, \( \Delta E \) is positive in one sub-lattice and negative in the other.

If \( G_{\lambda_1}^{-1} \) and \( G_{\lambda_2}^{-1} \) are both linear in energy, \( \Delta E \) is given by \( \epsilon_{\uparrow} - \epsilon_{\downarrow} \) and is constant up to a possible dependence on the spatial momentum \( \mathbf{q} \). In principle, \( \Delta E \) depends on \( \phi_{\downarrow} \) and \( \mathbf{q} \). In particular, at the threshold, \( G^{-1} \) would, in general, have a singular structure corresponding to the emission and absorption of the Goldstone boson \( \phi \) through the process \( e_\ast \rightarrow e\phi \). The results of the previous section are stable against such corrections, as we have discussed. However, the results of this section are more easily derived for constant \( \Delta E \), which corresponds to the case of the Fermi liquid whose exchange energy is constant.

After the symmetry violation, the currents \( J_{\mu} \) are no longer conserved, and the Ward–Takahashi identity is violated by

\[
\Gamma^{\mu}(q_1-q_2)_{\mu} \propto G_{\lambda_1}^{-1}(q_1)-G_{\lambda_2}^{-1}(q_2) \pm \Delta E \quad (\lambda_1 \neq \lambda_2). \tag{38}
\]

This \( \Delta E \) contribution is of the same form as the coupling of the Goldstone boson \( \phi \), and current conservation is restored by including the contribution of the Goldstone boson. This is the case even when \( \Delta E \) is not constant. Specifically, spin current conservation is restored for the vertex \( \tilde{\Gamma} \) which is modified by the inclusion of the Goldstone boson,

\[
\Gamma^\mu = \frac{\mu}{2} \chi \; + \; \frac{\mu}{2} \chi. \tag{39}
\]

As before, the crosses indicate the spin-current vertices, and the dashed line indicates the Goldstone boson. There is nothing strange in this result, since the Goldstone boson arose in the first place as the longitudinal component of the spin current. After taking away the longitudinal component, the remaining part is transverse and therefore satisfies the Ward–Takahashi identity. The current–magnon two-point function which appears in the second term consists of fermionic and bosonic loop. The latter contains magnons and the Higgs boson.

The Goldstone bosons \( \phi_1 \) and \( \phi_2 \) correspond to the SU(2) rotation perpendicular to the local orientation of spin (which is along \( z \)),

\[
U(\phi_1, \phi_2) = \exp \left[ if^{-1} \sum_{i=1}^{2} \phi_i \sigma_i \right], \tag{40}
\]

and they correspond physically to the magnons. \( f \) is the Goldstone boson form factor which, by virtue of eqn. (39), is calculated as the strength of the current–Goldstone-boson two-point amplitude.

### 3.1 The two-point function and the coupling with fermions

We have noted in eqn. (7) that there is a residual symmetry associated with \( U(1)_z \), which is conserved. The states can be classified according to the charges under this rotation group. First, we define the charge of \( \psi_\uparrow \) to be \( +1/2 \). The remaining charges follow automatically, and we obtain the values listed in tab. 3. These are necessarily conserved. Note that the \( U(1)_{\text{EM}} \) charges are \( e \) for the electron/hole fields and 0 for all others. These charges are also conserved.

| field | \( U(1)_z \) charge | \( U(1)_{\text{EM}} \) charge |
|-------|---------------------|-------------------------------|
| \( \psi_\uparrow \equiv \psi_\uparrow \) | +1/2 | \( e \) |
| \( \psi_\downarrow \equiv \psi_\downarrow \) | −1/2 | \( e \) |
| \( \phi_+ \equiv \phi_1 + i\phi_2 \) | +1 | 0 |
| \( \phi_- \equiv \phi_1 - i\phi_2 \) | −1 | 0 |
| \( h_0 \equiv h \) | 0 | 0 |

Table 3. The \( U(1)_z \) and \( U(1)_{\text{EM}} \) charges of the fields. \( e \) is positive for holes and negative for electrons.

In order that the Ward–Takahashi identity is satisfied by the vertex of eqn. (39), the following identity needs to be satisfied:

\[
\mu \chi = -f D_{\phi}^{-1}(q). \tag{41}
\]

Here, \( f \) is a constant of proportionality, and is the same quantity as that which appears in eqn. (39). \( q^\mu \) is the momentum flowing into the two-point function from the current (i.e., left to right). This present definition of \( f \) is more rigorous. Note that this also fixes the sign of \( D_{\phi} \). Our present definition corresponds to taking the couplings to be real and taking the sign of \( D_{\phi} \) to be opposite to that for scalar particles.

Given this definition of \( f \), we can determine the coupling constants with the fermions by the condition that eqn. (41) satisfies the Ward–Takahashi identity. We then obtain the Feynman rules that are given in figs. 2a and b.

![](image.png)

Fig. 2. The Feynman rules for the coupling of the magnons and the Higgs boson with the fermions.
For example, for the configuration of fig. 2a, eqn. (39) yields the following Ward–Takahashi identity:

$$q^\mu \Gamma_\mu + i^2 (-fD^{-1}_\phi(q))D_\phi(q)(f^{-1} \Delta E) = G_-^{-1} - G_+^{-1}. \quad (42)$$

Note that the same Feynman rules can be obtained, less rigorously, by considering the rotation associated with the Goldstone bosons, eqn. (40), and considering its coupling with the fermions in eqn. (2).

The vertices that involve the Higgs boson, which are shown in fig. 2c and d, cannot be fixed by this particular type of Ward–Takahashi identity. However, they can be fixed by considering the current insertion in the three-point amplitude, for example, as shown in fig. 3.

![Figure 3](image-url)

Fig. 3. The three diagrams whose sum must satisfy the Ward–Takahashi identity. The crosses correspond to the modified current vertex defined by eqn. (49).

The Ward–Takahashi identity applied to fig. 3 also allows us to determine the magnon–magnon–Higgs vertex. However, for doing so, we need to know the form of $D_\phi(q)$ and $D_\phi(q)$. Let us therefore calculate the current–magnon two-point function of eqn. (41).

For now, we calculate the fermionic loop, which is shown in fig. 2b. It should be noted that the end result of this calculation is independent of whether we consider $\phi_+ \text{ or } \phi_-$. Using the Feynman rule of fig. 2b, we obtain

$$iA^\mu_{\text{two-point}}(q) = i^4(-1) \int \frac{d^d+1k}{(2\pi)^{d+1}} \Gamma^\mu G_+(k)G_-(k-q)(f^{-1} \Delta E). \quad (43)$$

In particular, for the case $q \to 0$, we obtain the exact expression:

$$A^\mu_{\text{two-point}}(q \to 0) = \int \frac{d^d+1k}{(2\pi)^{d+1}} f^{-1} \Gamma^\mu [G-(k) - G+(k)],$$

where we made use of eqn. (67). For Fermi liquids, $1^\mu = 1$ and, by symmetry, the spatial components of this amplitude usually vanish at $q = 0$. We thus obtain

$$A^\mu_{\text{two-point}}(q \to 0) = f^{-1}(\rho_- - \rho_+). \quad (45)$$

Note that this vanishes for the case of anti-ferromagnetism where there is no global spin asymmetry.

When we compare eqn. (43) with eqn. (41), we see immediately that

$$f^2 = \rho_- - \rho_+ \quad \text{ferromagnetism},$$

and

$$\{-D^{-1}_\phi(q) = q^2/2m_\phi \quad \text{ferromagnetism},$$

$$\{-D^{-1}_\phi(q) = q^2/2m_\phi \quad \text{anti-ferromagnetism},$$

for small energy and momenta. The inclusion of the bosonic loop does not alter this conclusion. The unusual negative sign of $D$ reflects the fact that the magnons are pseudoscalar. That is, the fields are $i\phi$ rather than $\phi$ in our convention.

We need to calculate $f^2$ by other means, such as calculating the two-point amplitude for finite $q$, for anti-ferromagnetism. $u$ and $m_\phi$ are parameters which are in principle calculable by, for example, evaluating the finite $q$ case. However, the form of eqn. (43) implies $m_\phi \sim m_c$ and $u \sim v_F$.

There is actually a smarter method than to calculate the finite-$q$ case (which is cumbersome), but the full calculation, in the case of anti-ferromagnetism, requires our knowledge of the bosonic three-point functions. Let us therefore postpone the calculation of anti-ferromagnetic $f^2$ and other parameters for now.

Let us summarize the results of this section up to here. Firstly, we summarize the propagators and the two-point functions in fig. 5. The Higgs-boson Green’s functions are given in fig. 5, with a constant energy gap $\Delta_h$. $\Delta_h$ is defined as the energy gap for ferromagnetism and the energy gap squared for anti-ferromagnetism. This definition is convenient when we discuss the bosonic three-point functions. Note that it is an approximation to say that $\Delta_h$ is independent of momenta and energy. However, it becomes easier to implement current conservation in this manner. For completeness’s sake, we also list the screened photon Green’s function (with the approximation that the screening, $\Pi_{EM}$, is constant) and the Higgs–photon mixing. These are as given in the previous section.

Secondly, the fermionic vertices are as given before in fig. 2. We did not list the photonic vertex, but this is given by $e$. Note that the couplings given in fig. 2 can be summarized in the following compact form (c.f. eqn. (31)):

$$\mathcal{L}_{\text{eff}}^I = (f^{-1} \Delta E)\psi^\dagger \phi \cdot \sigma \psi,$$
\[
\begin{align*}
\phi_\pm & = \begin{cases} 
\phi_1 & \text{for } \phi_1, \\
\phi_2 & \text{for } \phi_2, 
\end{cases} \quad h = \begin{cases} 
q_1 & \text{for } q_1, \\
q_2 & \text{for } q_2, 
\end{cases} \\
F: \frac{-1}{q_0 - q_0^2 + \omega_0} & = \frac{+1}{q_0 - q_0^2 + \omega_0} \\
AF: \frac{-1}{q_0 - u^2 q_0^2 + \Delta h} & = \frac{+1}{q_0 - u^2 q_0^2 + \Delta h} \\
\end{align*}
\]

Fig. 5. The propagators and the two-point functions. \(H_{\text{EM}}\)

is given by the \(0 - 0\) component of eqn. (13), and \(\Pi_{\text{mixing}}\) is
given by the \(0 - 0\) component of eqn. (10).

where \(\Phi\) is defined by

\[
\Phi = (\phi_1, \phi_2, -v + h). \tag{49}
\]

\(\phi_1\) and \(\phi_2\) are as shown in tab. 3 and are given by

\[
\phi_1 = \frac{i}{2}(-\phi_+ + \phi_-), \quad \phi_2 = \frac{i}{2}(\phi_+ + \phi_-). \tag{50}
\]

\(v\) is a parameter, which has the interpretation as the vacuum expectation value of the \(\Phi\) field. In order that the energy difference between the two states that is given by eqn. (13) should agree with the actual energy difference \(\Delta E\), \(v\) needs to satisfy

\[
v = f/2. \tag{51}
\]

\(\Phi\) is essentially a magnetic order-parameter field. This differs from the more conventional form such as

\[
\mathcal{H}(\phi_1, \phi_2)(0, 0, v + h)^T, \tag{52}
\]

but they match in the limit of small fields, up to some differences in convention.

3.2 Bosonic vertices

Let us consider the Ward–Takahashi identity corresponding to the amplitude described by fig. 3.

We denote the initial state momentum to be \(q_1\) and the final state momenta to be \(q_2\) and \(q_3\). \(q_2\) is for the \(-1/2\) fermion. We denote the momentum which goes into the vertex by \(q\), so that \(q + q_1 = q_2 + q_3\).

It is not necessary that the fermions and the bosons are on shell, i.e., \(G^+_n(q_1)\) etc. need not be zero.

Upon contraction with \(q\), the first two diagrams yield

\[
q_w A^w_n = -i^2 f^{-1} \Delta E \left( 1 - G^{-1}_n(q_1) G_-(q + q_1) \right) \tag{53}
\]

and

\[
q_w A^w_n = i^2 f^{-1} \Delta E \left( G^{-1}_n(q_3) G_+(q_3 - q) - 1 \right). \tag{54}
\]

The amplitude as a whole satisfies the Ward identity if the third amplitude satisfies

\[
q_w A^w_n = 2i^2 f^{-1} \Delta E \left( 1 + D^{-1}_n(q_2) D_\phi(q_2 - q) \right). \tag{55}
\]

This requires vertices of the form shown in fig. 6.

\[
\begin{array}{cc}
(a) & (b) \\
\begin{array}{c}
\phi_1 \quad \phi_2 \\
\end{array} & \begin{array}{c}
h \\
\end{array} \\
\end{array}
\begin{array}{c}
F: \frac{-1}{q_0 - q_0^2 + \omega_0} \\
AF: \frac{-1}{q_0 - u^2 q_0^2 + \Delta h} \\
\end{array}
\begin{array}{c}
\phi_1 \quad \phi_2 \\
\end{array}
\begin{array}{c}
h \\
\end{array}
\]

Fig. 6. The bosonic three-point functions. In (b), \(q_1 + q_2\) is a short-hand notation for \((q_1 + q_2)\omega, u^2(q_1 + q_2))\).

Finally, we require the Ward–Takahashi identity for the sum of the three diagrams which are shown in fig. 7a–c. We choose the momenta to be \(h(q_1) \to h(q_2) + \phi_+(q_2)\), with \(q = q_2 + q_3 - q_1\) being the four-momentum flowing into the current. We obtain

\[
q_w A^w_n = -4i^2 f^{-1} \Delta h D^{-1}_n(q_1) D_\phi(q_1 + q) + 1, \tag{56}
\]

and

\[
q_w A^w_n = -4i^2 f^{-1} \Delta h D^{-1}_n(q_2) D_\phi(q_2 - q) + 1. \tag{57}
\]
Thus we require
\[ \mu \mathcal{A}_\mu^D = 8i^2 f^{-1} \Delta_k \] (58)
in order that the Ward–Takahashi identity is satisfied. Hence we obtain the Feynman rule shown in fig. 2.

4 Calculation of the parameters

In the preceding section, we worked out the form of the theory. Let us now work out the parameters.

In sec. 2 we used the condition of tadpole cancellation to work out a certain rule involving the ratios of density of states, that need to be satisfied in ferromagnetism.

In sec. 3 we presented the Feynman rules and were able to relate the current–magnon two-point function to the form of \( f^2 \) and the bosonic propagators.

We now generalize these results, and work out the four parameters, which are (1) \( f^2 \), (2) \( \Delta E \), (3) \( \Delta_k \) and (4) \( u \) or \( m_\sigma \).

Corresponding to these four unknowns, we have four equations, which involve: (1) tadpole cancellation, (2) the time component of the current–boson two-point function, (3) the space component of the same two-point function and (4) the Higgs-boson self-energy.

4.1 Tadpole cancellation

Let us start with the condition of tadpole cancellation.

We treated the ferromagnetic case in sec. 2. The antiferromagnetic case is calculated analogously, but the mechanism of cancellation is different. This time, we have the contribution of the magnon loop:

\[ i\mathcal{A}^\text{tadpole}_\text{magnon} = i^2 (2 f^{-1} \Delta_k) \int \frac{d^d+1k}{(2\pi)^{d+1}} D_\phi(k), \] (59)

which must be equal and opposite to the fermionic loop:

\[ i\mathcal{A}^\text{tadpole}_\text{fermion} = (-1)^{(d-1)/2} (f^{-1} \Delta E) \int \frac{d^d+1k}{(2\pi)^{d+1}} (G_+(k) - G_-(k)). \] (60)

Note that the sum over positive and negative \( \Delta E \) is implicit.

This gives us the following condition:

\[ -2\Delta_k \int \frac{d^d+1k}{(2\pi)^{d+1}} D_\phi(k) = \sum_{\text{sublattices}} [\rho_+ - \rho_-] \Delta E. \] (61)

Note that \( \Delta E \) is positive when \( \rho_- > \rho_+ \). That is, the right-hand side is negative. Let us introduce a more compact notation:

\[ 2\Delta_k \int D_\phi = \rho_M |\Delta E|. \] (62)

The convention is that \( \rho_M = \sum |\rho_+ - \rho_+| \) is positive. The integral of \( D_\phi \) is evaluated easily:

\[ \int D_\phi = \int \frac{d^d+1k}{(2\pi)^{d+1}} k_0^2 - u^2 k^2 + x \int \frac{d^d k}{2u(2\pi)^d} |k|. \] (63)

This is divergent at large momenta, and needs to be cut-off at \( |k| = K \), where \( K \sim \pi/a \). We then obtain

\[ \frac{\rho_M |\Delta E|}{2\Delta_k} = \frac{f K/4\pi u}{K^2/8\pi^2 u} \text{ (d = 2)}, \] \[ \frac{f K/4\pi u}{K^2/8\pi^2 u} \text{ (d = 3).} \] (64)

Note that the propagators are all-order, and this requires that \( \Delta E \) is bare, and that the vertex corrections are not included in eqn. (59). Whether \( \Delta E \) is stable against higher-order corrections depends on the size of the coupling \( f^{-1} \Delta E \) and the relative size of \( u \) compared with the electron velocity \( v \approx v_F \). The form of eqn. (59) corresponds to the all-order vertex, and therefore this equation, and eqn. (64) which follows from it, suffer from double counting. However, this ambiguity, that is due to double counting, cancels when we discuss the Higgs-boson self-energy later on.

4.2 Current–magnon two-point function

In sec. 3 we calculated the fermionic contribution to the current–magnon two-point function, which is shown in fig. 3. We obtained

\[ \mathcal{A}_\text{fermionic}^\mu = - \int \frac{d^d+1k}{(2\pi)^{d+1}} (f^{-1} \Delta E) / \Gamma^\mu G_+(k)G_-(k - q). \] (65)

For the simple case of \( \epsilon(k) = (\hbar k)^2/2m_e \), the vertex is given by \( /\Gamma^\mu = (1, (k - q/2)/m_e) \). The bosonic loop, which corresponds to fig. 4b, is written as

\[ \mathcal{A}_\text{bosonic}^\mu = \int \frac{d^d+1k}{(2\pi)^{d+1}} (2 f^{-1} \Delta E)(-2(2k - q)) D_h(k)D_\phi(k - q). \] (66)

This is for the anti-ferromagnetic case. Note that the case of ferromagnetism requires a further twist, as we have not yet included the higgs–screened-photon mixing.

Let us consider the limit of small external momentum, \( q \to 0 \). It is easy to see that the fermionic amplitude vanishes for anti-ferromagnetism. As for the bosonic amplitude, this vanishes for ferromagnetism because of the absence of negative energy states. The amplitude vanishes for anti-ferromagnetism also, but for a different reason, namely symmetry.

Let us, instead of trying to evaluate these integrals for arbitrary values of \( q \), make use of the Ward–Takahashi identities to replace the current–magnon two-point functions with the corresponding current–current two-point functions (c.f., ref. 2).

To do so, we first write down the current–current two-point functions as

\[ \Pi_\text{fermionic}^\mu = - \int \frac{d^d+1k}{(2\pi)^{d+1}} \Gamma^\mu G_+(k)G_-(k - q), \] (67)

and

\[ \Pi_\text{bosonic}^\mu = \int \frac{d^d+1k}{(2\pi)^{d+1}} (-2(q - 2k) \nu D_h(k)D_\phi(k - q). \] (68)
By virtue of the Ward–Takahashi identities, we obtain
\[ q_0 \Pi^{\mu \nu} - f A^\mu = C^{\mu \text{fermionic}} + C^{\mu \text{bosonic}}, \]  
(69)
where \( C^{\mu} \) are given by
\[ C^{\mu \text{fermionic}} = \int \frac{d^d k}{(2\pi)^{d+1}} \Gamma^{\mu}(k, q-k) \left( G_+(k) - G_-(k-q) \right), \]
(70)
and
\[ C^{\mu \text{bosonic}} = \int \frac{d^d k}{(2\pi)^{d+1}} \left(-2(2k-q)\nu\right) \left( D_h(k) + D_\phi(k-q) \right). \]
(71)

We now equate \( A \) with the Feynman rules of fig. 3, and take the derivative with respect to \( q_3 \) in the limit of small \( q \). For the fermionic case, we obtain
\[ \Pi^{\mu \lambda \text{fermionic}} \bigg|_{q \to 0} = \frac{f^2 m_\phi^{-1}}{2} \frac{\nu}{\nu^2 + \mu^2} \delta_{\mu \lambda} \]  
(72)
\[ I \] stands for the spatial identity matrix. The 0 – 0 component of this equation is zero on the right-hand side and in the second term of the left-hand side, whereas the first term on the left-hand side is non-zero:
\[ \Pi^{\mu \text{fermionic}} \bigg|_{q \to 0} = - \int \left[ G_+(k) + G_-(k) \right] \frac{\nu}{2 m_e} \frac{\nu}{\nu^2 + \mu^2}. \]
(73)
This happens because of the approximation \( D_\phi^{-1}(q) = q_0 - q^2 / 2 m_\phi \). There is, in principle, a \( q_0^2 \) term as well, the omission of which is inconsistent with the 0 term on the left-hand side. As for the spatial components, we obtain
\[ f^2 m_\phi^{-1} = \int \frac{d^d k}{(2\pi)^{d+1}} \left[ \left( G_+(k) + G_-(k) \right) \frac{\nu}{2 m_e} \frac{\nu}{\nu^2 + \mu^2} \right]. \]
(74)
Here, \( d/m_e \) refers to the second derivative of \( \epsilon(k) \), whereas \( v_\nu \) refers to the first derivative.

Let us introduce the following shorthand notation:
\[ f^2 m_\phi^{-1} = \left( \frac{\nu}{2 m_e} \frac{\nu}{\nu^2 + \mu^2} \right). \]
(75)
There is an obvious generalization to the case of spatial asymmetry. We expect this final result to be stable against higher-order corrections, because the renormalization factors due to the vertex correction and the Green’s functions cancel.

As discussed in sec. 2, \( \rho_- + \rho_+ \) is not well-defined. However, the ratio of \( \rho_- + \rho_+ \) against \( \rho_+ - \rho_- \) is well defined because of eqn. (18). Furthermore, \( f^2 \) is given by \( \rho_- - \rho_+ \).

As an order estimation, we can say that \( \nu_e \) can be taken to be almost constant near the Fermi surfaces. It would be a bad approximation to say that \( m_e \) is also constant, but we can introduce a quantity \( \frac{m_\phi}{\rho} \) to be the inverse of the average inverse fermion mass. We then obtain
\[ \frac{2 m_\phi}{\rho} \approx \frac{1}{P} - \frac{2 m_e^{-2}}{\Delta Ed}. \]
(76)
\( P \) is the spin asymmetry. \( m_\phi \) is necessarily positive, but \( \frac{m_\phi}{\rho} \) needs not be positive although we generally expect it to be. If \( \frac{m_\phi}{\rho} \) is positive, then the inequality reads
\[ P \leq \frac{\Delta Ed}{2 m_e v_F^2}. \]
(77)

Let us now turn to the anti-ferromagnetic case. Here we need both the fermionic and the bosonic loops. Corresponding to eqn. (72), we now have
\[ \Pi^{\mu \lambda} \bigg|_{q \to 0} + f^2 \frac{\nu}{\nu^2 + \mu^2} \delta_{\mu \lambda} = \frac{\partial}{\partial q_\lambda} C^{\mu \text{fermionic}} \bigg|_{q \to 0}. \]
(78)
The fermionic contributions are as given above. The bosonic contributions are given by
\[ \Pi^{\mu \lambda} \bigg|_{q \to 0} = -16 \int k^\mu k^\lambda D_h D_\phi, \]
(79)
and
\[ \frac{\partial}{\partial q_\lambda} C^{\mu \text{bosonic}} = 2 \frac{\nu}{\nu^2 + \mu^2} \delta_{\mu \lambda} D_h D_\phi. \]
(80)
using the same notation as in eqn. (62). The integral over \( D_\phi \) is given by eqn. (63), and is a positive quantity. The integral over \( D_h \) is given by
\[ \int D_h = - \frac{d^d k}{2(2\pi)^d} \frac{\nu}{\nu^2 + \mu^2} \]
(81)
and this is a negative quantity. Altogether, we obtain
\[ f^2 + 2 \int (D_h - D_\phi) - 16 \int k^2 D_h D_\phi = - \frac{\rho M}{\Delta Ed}, \]
(82)
and
\[ f^2 + 2 \int (D_h - D_\phi) - \frac{16}{d} \int k^2 D_h D_\phi \]
\[ = \frac{1}{d} \left( \frac{\nu}{\nu^2 + \mu^2} \right). \]
(83)

### 4.3 The Higgs-boson self-energy

We now come to the final condition, namely that the Higgs-boson excitation energy \( \Delta_h \) is given by the self-energy diagrams which are shown in fig. 8. The fermionic contribution is similar to \( \Pi^{00} \), which was calculated in sec. 2 and is given by
\[ -i \Pi^h_a = -i \int \frac{d^d k}{(2\pi)^d} \left( f^{-1} \Delta Ed \right)^2 \frac{\nu}{\nu^2 + \mu^2} \]
(84)
\[ \left( G_+(k) G_+(k-q) + G_-(k) G_-(k-q) \right). \]
now using eqn. (62), this reduces to

\[
H^h = (f^{-1}|\Delta E|^2 (g_-(\epsilon_F) + g_+(\epsilon_F)) \cdot \quad (85)
\]

Note that \(\Delta E\) refers to the bare quantity. This is because, firstly, \(\Delta E^2\) in eqn. (83) needs to be the product of the bare \(\Delta E\) and the renormalized \(\Delta E\). However, the renormalization of \(\Delta E\) gives rise to the renormalization factor \(Z^{-1}\) which is opposite to the renormalization factor \(Z\) for each propagator. It follows, therefore, that \(\Delta E\) in eqn. (85) actually refers to the bare quantity.

At \(q = 0\) (and at zero temperature), the contributions of fig. 5a and c are zero for ferromagnetism. Hence, for the case of ferromagnetism, we obtain

\[
\Delta_h = \frac{g_-(\epsilon_F) + g_+(\epsilon_F)}{\rho_- - \rho_+} (\Delta E)^2. \quad (86)
\]

If the density of states \(g\) is a linear function, then \(\Delta_h = 2\Delta E\) since \((\rho_- - \rho_+)\) is given by the area of a trapezium whose two parallel sides are \(g_-\) and \(g_+\), and whose height is \(\Delta E\). If not, and \(g\) is a convex function in between \(g_-\) and \(g_+\) as is the case for iron \[11\], \(\Delta_h\) will be less than \(2\Delta E\). This gives a useful estimate of the Higgs-boson excitation energy, which can be tested experimentally.

We should remember that the Higgs-screened-photon mixing cannot be neglected when the spin asymmetry \(P\) is large. The actual value of \(\Delta_h\) where the resonance occurs will be sensitive to the behaviour of the photonic modes (screened photon and plasmon).

Let us now turn to anti-ferromagnetism. The contribution of fig. 5b is given by

\[
-i\Pi^h_b = i \int \frac{d^d+1k}{(2\pi)^{d+1}} (2f^{-1}\Delta_h)^2 D_\phi(k) D_\phi(k - q). \quad (87)
\]

This is divergent, but is imaginary at zero temperature for \(q_0^2 - \omega^2 q^2 > 0\) (which is where the Higgs mode needs to exist). We therefore omit this contribution for now. The contribution of fig. 5c is given by

\[
-i\Pi^h_c = i \int \frac{d^d+1k}{(2\pi)^{d+1}} (-4f^{-2}\Delta_h) D_\phi(k). \quad (88)
\]

Now using eqn. (82), this reduces to

\[
\Pi^h_c = -4f^{-2}\Delta_h \int D_\phi(k) = -2f^{-2}\rho_m |\Delta E|. \quad (89)
\]

Hence,

\[
\Delta_h = \Pi^h_b + \Pi^h_c = 2f^{-2}|\Delta E| \left[\frac{1}{2} (g_-(\epsilon_F) + g_+(\epsilon)) |\Delta E|-\rho_m \right]. \quad (90)
\]

We thus obtain the anti-ferromagnetic ratio rule:

\[
\frac{g_-(\epsilon_F) + g_+(\epsilon_F)}{2\rho_M/|\Delta E|} > 1. \quad (91)
\]

This is satisfied if the density of states is a concave function in between the two Fermi energies.

Let us denote the concavity by \(\delta_c\), defined as:

\[
\delta_c = \frac{g_-(\epsilon_F) + g_+(\epsilon_F)}{2\rho_M/|\Delta E|} - 1. \quad (92)
\]

This then leads to

\[
\frac{2\rho_M/|\Delta E|}{\Delta_h} = \delta_c^{-1} f^2. \quad (93)
\]

By eqn. (64), we then obtain

\[
f^2 = \begin{cases} \delta_c K/\pi u & (d = 2), \\ \delta_c K^2/2\pi^2 u & (d = 3). \end{cases} \quad (94)
\]

Small \(\delta_c\) therefore leads to strong coupling \(f^{-1}\Delta E\). We expect physically that strong coupling tends to suppress magnetism, because the oscillations between the two spin states will become more frequent. Our results are not affected so long as the densities of states can be defined. However, \(\Delta E\) will receive a large correction through the electron self-energy.

4.4 Summary of results at zero temperature

Let us summarize our results.

For the case of ferromagnetism, the parameters \(\Delta E\), \(f^2\), \(m_\phi\) and \(\Delta_h\) are fixed by the following constraints:

\[
\rho_+ / \rho_- = g_+(\epsilon_F) / g_-(\epsilon_F), \quad (95)
\]

\[
f^2 = \rho_- - \rho_+, \quad (96)
\]

\[
f^2 m_\phi^{-1} = \left(\frac{\rho_+ + \rho_+}{2m_\phi} - \left(\rho_- - \rho_+\right) \frac{e^2}{\Delta Ed}\right), \quad (97)
\]

\[
f^2 \Delta_h = \left(g_-(\epsilon_F) + g_+(\epsilon_F)\right) (\Delta E)^2. \quad (98)
\]

Out of these equations, which are all non-perturbative, the first three are relations that only involve all-order quantities. In the last equation, \(\Delta E\) refers to the bare quantity. In eqn. (71), \(\Delta E\) is the all-order quantity, but is assumed to be more or less independent of energy and momenta (though generalization is possible).

For the case of anti-ferromagnetism, \(|\Delta E|\), \(f^2\), \(u\) and \(\Delta_h\) are fixed by eqns. (64), (82), (83) and (93). Equations (81) and (95) involve \(|\Delta E|\) as a bare quantity and \(\Delta_h\) in eqn. (93) is ambiguous. Equations (82) and (83) only involve all-order quantities, but are dependent on the UV cut-off, as is the case in eqn. (94).

We obtained the rule \(\delta_c > 0\), where \(\delta_c\) is a measure of concavity and is defined by eqn. (92).
4.5 Finite temperature analysis

Since our results involve diagrams that are evaluated for \( q = 0 \), it is, in principle, straightforward to generalize them to finite temperatures. However, the bosonic diagrams, which were zero in the case of ferromagnetism, become non-zero at finite temperatures, and therefore the resulting expressions are messy.

A full calculation is beyond the scope of this present analysis, but let us present two representative results.

First, for the case of anti-ferromagnetism, we have found that the bosonic loop of fig. 8 is real and diverges for finite \( T \):

\[
\Pi_b^0(T) = \left( \frac{2f^{-1}\Delta_h}{32T^3} \right) \int \frac{d^4k}{(2\pi)^4} \frac{d}{dx} \left( -\coth(x) \right) \bigg|_{x=\mu k/2T} \tag{99}
\]

This makes \( \Delta_h \) negative, and so magnetic order is forbidden. In our opinion, this implies that in anti-ferromagnetic metals, a genuine long-range order is not permitted, at least at finite temperatures.

Second, let us consider how the ratio rule of eqn. 158 is modified at finite temperatures. We now have the bosonic contribution which reads

\[
\mathcal{A}_{\text{boson}}(T) = -\left( 2f^{-1}\Delta_h \right) \int \frac{d^4k}{(2\pi)^4} \frac{1}{\exp(k^2/2m_\phi/T) - 1} \tag{100}
\]

This is evaluated easily using standard methods. For the case of three spatial dimensions, we obtain

\[
\mathcal{A}_{\text{boson}}(T) = -\left(2f^{-1}\Delta_h \zeta(3/2) \frac{m_\phi T}{2\pi} \right)^{3/2} \tag{101}
\]

Here, \( \zeta(3/2) = 2.612 \cdots \). This contribution should be equal and opposite to the fermionic contributions, which are given by

\[
\mathcal{A}_{\text{fermion}}(T) = (f^{-1}\Delta E) \left[ -(\rho_- - \rho_+)T + (\rho_+ + \rho_-)T P(T) \right]. \tag{102}
\]

Here \( \rho \) and \( P \) correspond to their finite-temperature counterparts:

\[
\rho_T = \int f((\epsilon - \mu)/T)g(\epsilon)d\epsilon, \tag{103}
\]

\[
g_T = -\int f'((\epsilon - \mu)/T)g(\epsilon)d\epsilon, \tag{104}
\]

where \( f \) is the Fermi distribution function. Hence

\[
-(\rho_- - \rho_+)T + (\rho_+ + \rho_-)T P(T) = 2\zeta(3/2) \left( \frac{m_\phi T}{2\pi} \right)^{3/2} \frac{\Delta_h}{\Delta E}. \tag{105}
\]

\( \Delta_h, m_\phi \) and \( \Delta E \) are also functions of temperature.

For small \( T \), we can assume that only the first term on the left-hand side depends significantly on \( T \), and that the parameters on the right-hand side can be taken as constants. This then implies that the magnetization goes down as \( T^{3/2} \), which is a well-known result. All of the parameters on the right-hand side are, in principle, measurable. This can then be tested experimentally.

5 Conclusions and Outlook

We presented a nonperturbative framework for treating magnetic order in metals, caused by a Coulomb interaction (or generalized Coulomb interaction).

We obtained interesting ‘ratio rules’ involving the densities of states for both ferromagnetic and anti-ferromagnetic cases. These involve all-order quantities (with the exception of \( |\Delta E| \)) and can therefore be compared directly with the experimental numbers, if they become available at greater precision.

We have seen that the shape of the density-of-states curve play an essential role in determining the possibility of magnetic ordering. The density of states must rise with energy, when the charge carriers are electrons, for ferromagnetism.

For anti-ferromagnetism, the density-of-states curve must be concave. However, we have seen that the radiative corrections, due to the magnons, at finite temperatures breaks long-range orders. In our understanding, this means that genuine long-range anti-ferromagnetic order is not possible, at least at finite temperatures. More work is required to elucidate the nature of the ground state.

The case of magnetic insulators is not covered by this work, in which the exchange energy \( \Delta E \) is considered to be more or less independent of \( k \).

Two cases require special attention, which we have not been able to discuss in much detail. The first is the case of strong coupling, which occurs when \( f \), or the vacuum-expectation value \( v \) of the magnetic order-parameter field, is small. Here, we expect that the radiative corrections suppress the magnetic order and that the system will favour the paramagnetic state. The second is the case of large magnon velocity \( u \), in comparison with the electron velocity \( v_F \), in the case of anti-ferromagnetism. Here, the response of the magnetic background becomes instantaneous towards the movement of the electron. We hope to be able to discuss this case in a separate publication.

The results of this work can be used to calculate arbitrary amplitudes, such as scattering amplitudes.

The methods presented in this work, being an adaptation of Gribov’s analysis of axial-current conservation, is of a general nature. However, we are presently unaware of other possible applications of the methods presented herein.

We thank I. Hase, S. Sharma, K. Yamaji and T. Yanagisawa for extensive and informative comments and discussions.

We have been informed by Dr. I. Hase that a phenomenological study of the correlation between densities of states of a material and its magnetic properties has previously been reported. However, we have not been able to locate this study.

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