ON ORIENTED PLANAR TREES WITH THREE LEAVES

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ABSTRACT. This elementary note proposes candidates for interesting continuous piecewise-smooth ‘Riemannian’ metrics on the moduli spaces of rooted geodesic trees embedded in the Poincaré disk. A related digression observes the existence of an apparently hitherto unrecognized abelian topological group structure on the real projective line. This is an early draft. The symbol $\ast$ indicates an illustration to be supplied in a later version.

0.1 Introduction A finite set of points on the boundary of the Poincaré unit disk can be shown to define a unique embedded geodesic not necessarily binary tree, with one point chosen as root and the rest as endpoints of its leaves. Alternatively, the dual $\mathcal{Z}(\mathbb{S})$ of such a spanning tree defines a decomposition of the disk into hyperbolic polygons, which can be imagined to be solutions of a geometric extremization problem analogous to Plateau’s $^1$, i.e. to find the pattern of cracks in a shattered windshield, given a collection of shocks or stresses at a finite number of boundary points. As a problem in analytic mechanics this seems to be a relatively intractable kind of free boundary problem, making a useful account of its literature problematic. The present sketch is concerned with the differential geometry of the space $\overline{M}_{0,n+1}(\mathbb{R})$ of such rooted geodesic hyperbolic trees, as providing an alternate approach to questions of this sort.

0.2 $\ast$ The simplest nontrivial case (rooted trees with three leaves) is classical: the cross-ratio $\rho = [x_0 : x_1 : x_2 : x_3]$ maps the space $\overline{M}_{0,4}(\mathbb{R})$ (of projective equivalence classes of four labelled points on the real projective line) diffeomorphically to the circle. When $\rho \in [0,1]$, a beautiful formula

$$\rho = \frac{1}{1 + e^{-\gamma}}$$

of Devadoss equates the cross-ratio to the logistic function of the signed hyperbolic length of the generically unique internal branch.

In general $\overline{M}_{0,n+1}(\mathbb{R})$ is tesselated by $\frac{1}{2}n!$ Stasheff associahedra, which can be regarded as moduli spaces for geodesic trees with leaves in fixed order.

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$^1$i.e. to find a minimal surface spanning a given closed space curve.
This formula then generalizes [3] (§ 6) to define a pseudometric which blows up along the associahedral faces. On the other hand the original construction of $\overline{M}_{0,n+1}(\mathbb{R})$ (as the real points of the Deligne-Mumford-Knudsen compactification of the stack of genus zero algebraic curves with marked points) makes it a smooth projective variety, which thus inherits a very nice Riemannian metric from its Kähler structure - which is however not unique, or necessarily well-behaved with respect to permutations of the marked points.

0.3 The purpose of this note is to suggest that pulling back (a variant of) the product metric along a generalized Albanese map

$\overline{M}_{0,n+1}(\mathbb{R}) \to \overline{M}_{0,3+1}(\mathbb{R})^{(n)}$

provides the moduli space with a continuous, though only piecewise smooth, $\Sigma_n$-equivariant metric; and, more generally, to suggest the relevance of the metric geometry of these spaces to the study of such generalized travelling salesman problems.

§ I The real projective line, reconsidered

1.1 The projective line $\mathbb{P}(\mathbb{F}_2) = \{0, 1, \infty\}$ over the field with two elements admits a transitive action of the symmetric group $\Sigma_3 \cong \text{Sl}_2(\mathbb{F}_2)$. The rational functions

$\tau_{01}(x) = 1 - x$, $\tau_{1\infty}(x) = (1 - x^{-1})^{-1}$

generate a group (under composition) isomorphic to $\Sigma_3$, satisfying the braid relation

$(\tau_{01} \circ \tau_{1\infty} \circ \tau_{01})(x) = (\tau_{1\infty} \circ \tau_{01} \circ \tau_{1\infty})(x) = x^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(x)$. The element

$\sigma(x) = (\tau_{1\infty} \circ \tau_{01})(x) = (1 - x)^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}(x)$

(sending $(01\infty)$ to $(\infty01)$) generates a cyclic subgroup $C_3 = \{1, \sigma, \sigma^2\}$ of $\Sigma_3$, with

$\tau_{01} \circ \tau_{1\infty} = \sigma^2(x) = 1 - x^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}(x)$;

thus $\tau_{01} \circ \sigma \circ \tau_{01} = \sigma^{-1}$. This lifts to an action of $C_3 \times C_2 \cong \Sigma_3$ on $\mathbb{C}$ by rational functions, which extends to a continuous (fixed-point free) action on the one-point compactification $\mathbb{R}_+ = \mathbb{R} \cup \infty = \mathbb{P}_1(\mathbb{R})$ of the real line. However $\sigma(x)$ is not differentiable at $x = 0$, and $\sigma'(x) = 0$ at $x = \infty$.

1.2 Let $(x_i) \in \mathbb{R}^4$, $0 \leq i \leq 3$, be a vector with all coordinates distinct, and define $x_{ij} = x_i - x_j \neq 0$ for $i \neq j$; then the cross-ratio

$[x_0 : x_1 : x_2 : x_3] = \frac{x_{01}}{x_{02}} \cdot \frac{x_{23}}{x_{13}}$. 

extends to the compactified quotient
\[
\text{Config}^4(\mathbb{R}) / \text{PGL}_2(\mathbb{R}) \subset \overline{M}_{0,4}(\mathbb{R}) \xrightarrow{\rho \equiv} \mathbb{P}_1(\mathbb{R})
\]
defining an isomorphism of the space of four points on the line, modulo projective equivalence, with the projective line itself. Thus the fractional linear transformation
\[
x \mapsto \frac{ax + b}{cx + d}
\]
defined by
\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \text{PGL}_2(\mathbb{R})
\]
(i.e \(ad - bc = \det T \neq 0\)) satisfies
\[
[A](x_0) : [A](x_1) : [A](x_2) : [A](x_3) = [x_0 : x_1 : x_2 : x_3];
\]
for example
\[
\rho = [0 : \rho : 1 : \infty] = [1 : \sigma(\rho) : \infty : 0] = [\infty : \sigma^2(\rho) : 0 : 1],
\]
e.g \(\rho = \sigma(\rho)^{-1}(\sigma(\rho) - 1)\).

**Remark** If \(\mathbb{P}_1(\mathbb{R})\) is ordered as usual (i.e with \(x \in \mathbb{R} \Rightarrow x < \infty\)), then for any quadruple \(x_0 < x_1 < x_2 < x_3\) there is an \(A \in \text{SL}_2(\mathbb{R})\) such that
\[
[A](x_0) = 0, \quad [A](x_2) = 1, \quad [A](x_3) = \infty, \quad \text{and} \quad [A](x_1) \in (0,1).
\]
If \(x_3 \neq \infty\), then
\[
A = (x_0 x_2 x_3)^{-1/2} \begin{bmatrix} x_2 x_3 & -x_1 x_3 \\ x_0 x_2 & -x_0 x_3 \end{bmatrix}
\]
is unique.

The group
\[
\Sigma_4 \cong \mathbb{F}_2^2 \rtimes \text{SL}_2(\mathbb{F}_2)
\]
of permutations of four things acts on \(\overline{M}_{0,4}(\mathbb{R})\), but the Klein subgroup \(\mathbb{F}_2^2\) generated by the transpositions
\[
(01)(23), \quad (02)(13), \quad (12)(30)
\]
leaves the cross-ratio invariant, so the action of \(\Sigma_4\) on \(\overline{M}_{0,4}(\mathbb{R})\) reduces to the action of \(\Sigma_3\) described above.

**1.3 ⋆** It will be convenient to decompose the projective line
\[
\mathbb{P}_1(\mathbb{R}) = [I] \cup [II] \cup [III]
\]

as the union of closed intervals
\[
[I] = [-\infty, 0], \quad [II] = [0, 1], \quad [III] = [1, \infty];
\]
thus \(\sigma\) maps \([I]\) bijectively to \([II]\), \([II]\) to \([III]\), and \([III]\) to \([I]\), preserving orientations, and is consistent on the boundary points
\[
[I] \cap [II] = \{0\}, \quad [II] \cap [III] = \{1\}, \quad [III] \cap [I] = \{\infty\}.
\]
Similarly, let \((X)\) denote the interior of \([X]\); then the (smooth) function
\[
k : \mathbb{R} - \{0, 1\} = (I) \cup (II) \cup (III) \to (0, 1)
\]
defined by \(k(x) = \sigma(x) = (1 - x)^{-1}\) if \(x \in (-\infty, 0) = (I)\),
\[= x \text{ if } x \in (0, 1) = (II), \text{ and} \]
\[= \sigma^{-1}(x) = 1 - x^{-1} \text{ if } x \in (-\infty, 0) = (III)\]
extends to a continuous function
\[
\kappa : \mathbb{R}_+ = \mathbb{P}_1(\mathbb{R}) \to (0, 1)_+ = \mathbb{R}/\mathbb{Z}
\]
sending \(\{0, 1, \infty\}\) to the compactification point \(0 = 1 \in (0, 1)_+\).

**Lemma** *The derivative \(\kappa'(x)\)*
\[
= (1 - x)^{-2}, \ x \in (I) \\
= 1, \ x \in (II) \\
= x^{-2}, \ x \in (III)
\]
exists and is continuous. Moreover,
\[
\int_{\mathbb{R}} \kappa'(x) \cdot dx = \int_{-\infty}^{0} (1 - x)^{-2} \cdot dx + \int_{0}^{1} 1 \cdot dx + \int_{1}^{\infty} x^{-2} \cdot dx = 1 + 1 + 1 = 3.
\]

**Proposition** \(\kappa \circ \sigma = \kappa\): thus \(\kappa\) identifies the quotient of \(\mathbb{P}_1(\mathbb{R})\) by \(C_3\) with \(\mathbb{R}/\mathbb{Z}\).

[For if \(x \in (I)\) then \(\sigma(x) \in (II)\) so \(\kappa(\sigma(x)) = \sigma(x) = \kappa(x)\); while if \(x \in (II)\) then \(\sigma(x) \in (III)\), so \(\kappa(\sigma(x)) = \sigma^{-1}(\sigma(x)) = x = \kappa(x)\). Finally, if \(x \in (III)\) then \(\sigma(x) \in (I)\) so
\[
\kappa(\sigma(x)) = (1 - \sigma(x))^{-1} = 1 - x^{-1} = \kappa(x).
\]

The resulting map is a three-fold cover, with multiplication by 3 as the induced homomorphism
\[
\pi_1(\mathbb{P}_1(\mathbb{R}), \infty) \cong \mathbb{Z} \to \mathbb{Z} \cong \pi_1(\mathbb{R}/\mathbb{Z}, 0).
\]
The action of \(\Sigma_3\) on \(\mathbb{P}_1(\mathbb{R})\) reduces to the orientation-reversing action of \(\Sigma_3/C_3 \cong \{\pm 1\}\) on \(\mathbb{P}_1(\mathbb{R})/C_3\) defined by \(\tau_{0,1}(x) = 1 - x\) on \(\mathbb{R}/\mathbb{Z}\) as in \(\S 1.2\). The composition
\[
\mathcal{M}_{0,3+1}(\mathbb{R}) \xrightarrow{\rho} \mathbb{P}_1(\mathbb{R}) \xrightarrow{\kappa} \mathbb{R}/\mathbb{Z}
\]
provides an interpretation the quotient space \(\mathcal{M}_{0,3+1}(\mathbb{R})/C_3\) as the space of configurations \(\{x_i, x_j, x_k, \infty\}\) of three cyclically ordered points on \(\mathbb{R}\). The one-form \(d \varphi \in \Omega^1(\mathbb{R}/\mathbb{Z})\) pulls back to a \(C_3\)-invariant one-form
\[
\omega = \kappa'(\rho)d\rho = \kappa^*(d\rho) \in \Omega^1(\mathcal{M}_{0,3+1}(\mathbb{R}))
\]
mapping to three times the fundamental class in \(H^1_{dr}(\mathcal{M}_{0,3+1}(\mathbb{R}))\).
1.4 ⋆ The inverse
\[ D(x) = -\log |1 - x^{-1}| = -\log |\sigma^{-1}(x)| : [0, 1] \to \mathbb{P}_1(\mathbb{R}) \]
of the logistic function \((1 + e^{-x})^{-1}\) extends to the closed interval by \(D(0) = D(1) = \infty\). Thus \(\kappa(x) = (D \circ \kappa)(x)\)
\[ = -\log |x| \text{ if } x \in [I], \]
\[ = -\log |\sigma^{-1}(x)| \text{ if } x \in [II], \]
\[ = -\log |\sigma^{-2}(x)| \text{ if } x \in [III] \]
defines a three-fold cover \(\kappa : (\mathbb{P}_1(\mathbb{R}), \{0, 1, \infty\}) \to (\mathbb{P}_1(\mathbb{R}), \infty)\) of the projective line. Its graph
\[ \mathbb{P}_1(\mathbb{R}) \ni x \mapsto (x, \kappa(x)) \in \mathbb{P}_1(\mathbb{R}) \times \mathbb{P}_1(\mathbb{R}) \]
has degree one along the first factor, and degree three along the second, defining a piecewise smooth helix wrapped around a torus.

**Proposition ⋆**
\[ \gamma = \kappa(\rho) : \overline{M}_{0,3+1}(\mathbb{R}) \to \mathbb{P}_1(\mathbb{R}) \]
extends Devadoss’s formula [2] (§6.1) for the oriented hyperbolic length of the generic internal edge of a hyperbolic rooted tree with three leaves. The square of the one-form
\[ d\gamma = \kappa'(\rho)d\rho \]
defines a pseudometric on \(\overline{M}_{0,3+1}(\mathbb{R})\) which blows up at \(\rho = 0, 1, \infty\).

1.5 The rational cohomology \(H^\ast(\overline{M}_{0,n+1}(\mathbb{R}), \mathbb{Q})\) of the moduli space of \(n+1\) ordered points on the line is calculated in [4] (§2.3), [7]; in particular
\[ H^1(\overline{M}_{0,n+1}(\mathbb{R}), \mathbb{Q}) \cong \Lambda^3 \mathfrak{h}_n \]
as \(\Sigma_{n+1}\)-modules, where \(\mathfrak{h}_n\) is the \(n\)-dimensional kernel of the trace homomorphism
\[ \mathbb{Q}^{n+1} \ni (v_0, \ldots, v_n) \mapsto \sum v_i \in \mathbb{Q} \]
(with \(\Sigma_{n+1}\) acting on the left by permuting coordinates). A subset \(S\) of \(\{1, \ldots, n\}\) of cardinality \(3 \leq |S| \leq n\) defines a forgetful morphism \(\overline{M}_{0,n+1}(\mathbb{R}) \to \overline{M}_{0,|S|+1}(\mathbb{R})\): thus a subset \(S = \{i < j < k\}\) defines a composition
\[ \overline{M}_{0,n+1}(\mathbb{R}) \xrightarrow{\nu_S} \overline{M}_{0,3+1}(\mathbb{R}) \xrightarrow{\kappa} \mathbb{R}/\mathbb{Z} . \]
and hence a one-form \(d\kappa_S \in \Omega^1(\overline{M}_{0,n+1}(\mathbb{R}))\). I will write \(d\kappa_S^\otimes 2 \in \Omega^\otimes 2(\overline{M}_{0,n+1}(\mathbb{R}))\) for the associated quadratic differential.

A basis \(v_i\) for \(\mathbb{Q}^{n+1}\) defines a basis
\[ \alpha_{ijk} = (v_i - v_0) \wedge (v_j - v_0) \wedge (v_k - v_0) \]
for \(\Lambda^3 \mathfrak{h}_n\); then
\[ \Lambda^3 \mathfrak{h}_n \ni \alpha_{ijk} \mapsto d\kappa_{ijk} \in \Omega^1(\overline{M}_{0,n+1}(\mathbb{R})) \]
is an injective homomorphism of $\Sigma_n$-modules.

**Claim** The average

$$ds_2^2 := \left(\frac{n}{3}\right)^{-1} \sum_{i<j<k} d\kappa_{ijk} \in \Omega^{\otimes 2}(\mathcal{M}_{0,n+1}(\mathbb{R}))$$

defines a continuous, piecewise smooth metric on the space of rooted hyperbolic $n$-leaved trees. The subgroup $\Sigma_n \subset \Sigma_{n+1}$ of permutations acts by isometries.

[Behind this lies the conjecture that the ‘Albanese’ map

$$\prod_{S \in \binom{n}{3}} \kappa_S : \mathcal{M}_{0,n+1}(\mathbb{R}) \to \mathcal{M}_{0,3+1}(\mathbb{R})^{\binom{n}{3}}$$

is an immersion.]

§II The Cayley transform, reconsidered

2.1 The fractional linear transformation

$$z = C(x) = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} \frac{x - i}{1 - ix} : \mathbb{P}_1(\mathbb{C}) \to \mathbb{P}_1(\mathbb{C})$$

restricts to stereographic projection

$$\mathbb{P}_1(\mathbb{R}) \supset \mathbb{R} \to \mathbb{T} \subset \mathbb{C}^\times \subset \mathbb{P}_1(\mathbb{C})$$

of the real line, sending $\pm \infty \to i$ and $\pm 1$ to $\pm 1$. Writing $t \mapsto e(t) = \exp(2\pi it)$ for the group isomorphism $\mathbb{R}/\mathbb{Z} \to \mathbb{T}$, we have

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} (e(t)) = i \tan \pi t ,$$

so (by the addition formula for the tangent function)

$$x = C^{-1}(e(t)) = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} (i \tan \pi t) = \begin{bmatrix} 1 - i & 1 + i \\ i - 1 & 1 - i \end{bmatrix} (i \tan \pi t)$$

$$= \frac{1 + \tan \pi t}{1 - \tan \pi t} = \tan \pi (t + \frac{1}{4})$$

defines a diffeomorphism (inverse to stereographic projection) of $\mathbb{T}$ with $\mathbb{R}_+ = \mathbb{P}_1(\mathbb{R})$. Similarly, $(2\pi iz)^{-1}dz \in \Omega^1(\mathbb{C}^\times)$ pulls back to $(\pi(1+x^2))^{-1}dx \in \Omega^1(\mathbb{R})$.

2.2 The action of $A \in \text{SL}_2(\mathbb{R})$ on $\mathbb{P}_1(\mathbb{R})$ defines an action of

$$\tilde{A} = C^{-1}AC = (\frac{1}{2})^2 \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

$$= \begin{bmatrix} u & v \\ \bar{v} & \bar{u} \end{bmatrix} \in \text{SU}(1,1) ,$$
where $|u|^2 - |v|^2 = 1$, with
\[
u = \frac{1}{2}((a + d) + i(c - b)), \quad v = \frac{1}{2}((b + c) + i(d - a))\]
on $\mathbb{C}$. Because the complex conjugate of
\[
\begin{bmatrix}
u \\ \bar{v}
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix} 
1 - 2i & +i \\
-1 & 1 + 2i
\end{bmatrix}, \quad \bar{v} = \frac{1}{2}
\begin{bmatrix}
1 + 2i \\ +i
\end{bmatrix},
\]
equals its inverse
\[
\begin{bmatrix}
u e(-t) + \bar{v} \\ ve(-t) + u
\end{bmatrix} = \frac{1}{2}
\begin{bmatrix} 
\bar{v} e(t) + \bar{u} \\
u e(t) + v
\end{bmatrix},
\]
this action takes the circle $T \subset \mathbb{C}^\times$ to itself.

Example ⋆
\[
\tilde{\sigma} = \frac{1}{2}
\begin{bmatrix} 
1 - 2i & +i \\
-1 & 1 + 2i
\end{bmatrix}, \quad \tilde{\sigma}^2 = \frac{1}{2}
\begin{bmatrix}
1 + 2i \\ +i
\end{bmatrix},
\]
so $\{1, \pm i\}$ is an orbit of $\tilde{C}_3$, and $\{-1, (4 \pm 3i)/5\}$ is another. Similarly,
\[
\tilde{\tau}_{01} = \frac{i}{2}
\begin{bmatrix}
-i \\ 1 - 2i
\end{bmatrix} + \frac{1}{2}
\begin{bmatrix} 
1 + 2i \\ i
\end{bmatrix}.
\]

2.3 The (renormalized) extension
\[
z \mapsto L(z) = iC(z) = \frac{1 + iz}{1 - iz} = \left[ \begin{array}{cc} i & 1 \\ -i & 1 \end{array} \right](z)
\]
of $C$ to a fractional linear transformation $L$ of the complex projective line $\mathbb{P}_1(\mathbb{C}) \cong \mathbb{C}_+\mathbb{P}_1(\mathbb{R})$ maps $\mathbb{P}_1(\mathbb{R})$ to $\mathbb{T}$.

Proposition
\[
L^{-1}(L(z_0) \cdot L(z_1)) = \frac{z_0 + z_1}{1 - z_0 z_1} := z_0 + L z_1
\]
restricts near 0 to the one-dimensional formal group law with $x \mapsto \tan x$ as its exponential.

Proof
\[
L^{-1}(z) = \left[ \begin{array}{cc} 1 & -1 \\ i & i \end{array} \right](z) = -i \frac{z - 1}{z + 1},
\]
so
\[
L^{-1}(L(z_0) \cdot L(z_1)) = L^{-1}\left[ \frac{1 + iz_0}{1 - iz_0}, \frac{1 + iz_1}{1 - iz_1} \right] =
\]
\[
\begin{align*}
(-i) \cdot \frac{(1 + iz_0)(1 + iz_1) - (1 - iz_0)(1 - iz_1)}{(1 + iz_0)(1 + iz_1) + (1 - iz_0)(1 - iz_1)} = \\
\begin{aligned}
(-i) \frac{2i(z_0 + z_1)}{2 - 2z_0 z_1} & = z_0 + L z_1
\end{aligned}
\end{align*}
\]
as claimed. $\Box$
Up to a Wick rotation $x \mapsto ix$, this is the formal group law defined by Weyl and Hirzebruch’s signature genus for oriented smooth manifolds. It is odd, in the sense that $[-1]_L(z) = -z$.

**Corollary** ($\mathbb{P}_1(\mathbb{R}) = \mathbb{R}_+, 0, +_L$) is a group, with $\infty$ as (the unique nontrivial) torsion point of order two.

In particular, if $x \in \mathbb{R}^\times$, then

$$x \mapsto x + L \infty = \lim_{w \to 0} \frac{x + w^{-1}}{1 - w^{-1}x} = \lim_{w \to 0} \frac{1 + wx}{w - x} = -x^{-1}.$$  

Similarly, $x +_L x = [2]_L(x) \to 0$ as $x \to \infty$, consistent with $\infty = [-1]_L(\infty)$, while

$$[3](w^{-1}) = \frac{3w^2 - 1}{w^3 - 3w} \to \infty$$  

as $w \to 0$, etc.

Note also that 1 is a 4-torsion point, i.e $1 +_L 1 = \infty$. More generally, the group $\mathbb{Q}/\mathbb{Z} \subset \mathbb{T}$ of torsion points for $+_L$ maps isomorphically to the set $\{\tan \pi x \mid x \in \mathbb{Q}\}$ of (cyclotomic) algebraic numbers.

**Note** that since $i +_L (-i)$ is undefined, this construction fails to define a composition operation on $\mathbb{P}_1(\mathbb{C})$.

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