Algebraic geometry

Diagonal property of the symmetric product of a smooth curve

Propriété de la diagonale pour les produits symétriques d'une courbe lisse

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Let \( C \) be an irreducible smooth projective curve defined over an algebraically closed field. We prove that the symmetric product \( \text{Sym}^d(C) \) has the diagonal property for all \( d \geq 1 \). For any positive integers \( n \) and \( r \), let \( Q_{O^r}^{\text{nr}}(nr) \) be the Quot scheme parameterizing all the torsion quotients of \( O^r \) of degree \( nr \). We prove that \( Q_{O^r}^{\text{nr}}(nr) \) has the weak-point property.

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1. Introduction

In [8], Pragacz, Srinivas and Pati introduced the diagonal and (weak) point properties of a variety, which we recall. Let \( X \) be a variety of dimension \( d \) over an algebraically closed field \( k \). It is said to have the diagonal property if there is a vector bundle \( E \to X \times X \) of rank \( d \), and a section \( s \in H^0(X \times X, E) \), such that the zero scheme of \( s \) is the diagonal in \( X \times X \). The variety \( X \) is said to have the weak point property if there is a vector bundle \( F \) on \( X \) of rank \( d \), and a section \( t \in H^0(X, F) \), such that the zero scheme of \( t \) is a (reduced) point of \( X \). The diagonal property implies the weak-point property because the restriction of the above section \( s \) to \( X \times \{x_0\} \) vanishes exactly on \( x_0 \).

These properties were extensively studied in [8] and [5]. In particular, it was shown that:
• they impose strong conditions on the variety,
• on the other hand there are many example of varieties with these properties.

Here we investigate these conditions for some varieties associated with a smooth projective curve.

Let $C$ be an irreducible smooth projective curve over $k$. For any positive integer $d$, let $\text{Sym}^d(C)$ be the quotient of $C^d$ for the natural action of the group of permutations of $\{1, \ldots, d\}$. It is a smooth projective variety of dimension $d$. We prove the following (Theorem 3.1).

**Theorem 1.1.** The variety $\text{Sym}^d(C)$ has the diagonal property.

Theorem 1 in [8, p. 1236] contains several examples of surfaces satisfying the diagonal property. We note that the surface $\text{Sym}^2(C)$ is not among them.

For positive integers $n$ and $d$, let $Q_{\mathcal{O}_C^\oplus}(d)$ be the Quot scheme parameterizing the torsion quotients of $O_C^\oplus$ of degree $d$. Quot schemes were constructed in [6] (see [7] for an exposition on [6]). The variety $Q_{\mathcal{O}_C^\oplus}(d)$ is smooth projective, and its dimension is $nd$. Note that $Q_{\mathcal{O}_C}(d) = \text{Sym}^d(C)$. These varieties $Q_{\mathcal{O}_C^\oplus}(d)$ are extensively studied in algebraic geometry and mathematical physics (see [3,2,1,4] and references therein).

We prove the following (Theorem 2.2).

**Theorem 1.2.** If $d$ is a multiple of $n$, then the variety $Q_{\mathcal{O}_C^\oplus}(d)$ has the weak-point property.

2. Quot scheme and the weak-point property

We continue with the notation of the introduction.

For a locally free coherent sheaf $E$ of rank $n$ on $C$, let $Q_E(d)$ be the Quot scheme parameterizing all torsion quotients of $E$ of degree $d$. Equivalently, $Q_E(d)$ parametrizes all coherent subsheaves of $E$ of rank $n$ and degree $\deg(E) - d$. Note that any coherent subsheaf of $E$ is locally free because any torsion-free coherent sheaf on a smooth curve is locally free. This $Q_E(d)$ is an irreducible smooth projective variety of dimension $nd$.

There is a natural morphism

$$\varphi' : Q_E(d) \to Q_{n \times E}(d)$$

that sends any subsheaf $S \subset E$ of rank $n$ and degree $\deg(E) - d$ to the subsheaf $\bigwedge^n S \subset \bigwedge^n E$. Next note that $Q_{n \times E}(d)$ is identified with the symmetric product $\text{Sym}^d(C)$ by sending any subsheaf $S' \subset \bigwedge^n E$ to the scheme theoretic support of the quotient sheaf $(\bigwedge^n E)/'$. Let

$$\varphi : Q_E(d) \to \text{Sym}^d(C)$$

be the composition of $\varphi'$ with this identification of $Q_{n \times E}(d)$ with $\text{Sym}^d(C)$. It should be mentioned that for a subsheaf $S \subset E$ of rank $n$ and degree $\deg(E) - d$, the image $\varphi(S) \in \text{Sym}^d(C)$ does not, in general, coincide with the scheme theoretic support of the quotient sheaf $E/S$.

The symmetric product $\text{Sym}^d(C)$ is the moduli space of effective divisors of degree $d$ on $C$. Let

$$D \subset Y := C \times \text{Sym}^d(C)$$

be the universal divisor. So the fiber of $D$ over a point $a \in \text{Sym}^d(C)$ is the zero dimensional subscheme of $C$ of length $d$ defined by $a$. Let

$$D = (\text{Id}_C \times \varphi)^{-1}(D) \subset C \times Q_E(d)$$

be the inverse image of $D$, where $\varphi$ is constructed in (2.1).

**Remark 2.1.** Let $L$ be a line bundle on $C$. For $E$ as above, if $S \subset E$ is a subsheaf of rank $n$ and degree $\deg(E) - d$, then

$$S \otimes L \subset E \otimes L$$

is a subsheaf of rank $n$ and degree $\deg(E \otimes L) - d$. Therefore, we get an isomorphism

$$Q_E(d) \sim Q_{E \otimes L}(d)$$

by sending any subsheaf $S \subset E$ to the subsheaf $S \otimes L \subset E \otimes L$.

**Theorem 2.2.** For positive integers $d, n$ such that $d$ is a multiple of $n$, the Quot scheme $Q_{\mathcal{O}_C^\oplus}(d)$ satisfies the weak-point property.
Proof. Let \( r \in \mathbb{N} \) be such that \( d = rn \). Fix a closed point \( x_0 \) in \( C \). The line bundle \( \mathcal{O}_C(rx_0) \) on \( C \) will be denoted by \( L \). By Remark 2.1 it is enough to prove the weak-point property for \( Q_{L^n}(d) \).

Let \( D \rightarrow C \times Q_{L^n}(d) \) be the divisor constructed in (2.3). Let
\[
p : D \rightarrow C \quad \text{and} \quad q : D \rightarrow Q_{L^n}(d) (2.4)
\]
be the natural projections. Taking the direct sum of copies of the natural inclusion
\[
t : \mathcal{O}_C \hookrightarrow \mathcal{O}_C(rx_0),
\]
we get a short exact sequence of sheaves on \( C \)
\[
0 \rightarrow \mathcal{O}_C(\xi_0) \rightarrow \mathcal{O}_C(rx_0) \rightarrow T \rightarrow 0, \quad \text{(2.5)}
\]
where \( T \) is a torsion sheaf on \( C \) of degree \( nt = d \). Therefore, this quotient \( T \) is represented by a point of \( Q_{L^n}(d) \). Let
\[
t_0 \in Q_{L^n}(d) \quad \text{(2.6)}
\]
be the point representing \( T \).

The direct image
\[
F := q_*p^*L^n \rightarrow Q_{L^n}(d)
\]
is a vector bundle of rank \( nd \), where \( p \) and \( q \) are the projections in (2.4). We will construct a section of \( F \). The section of \( \mathcal{O}_C \) given by the constant function 1 will be denoted by \( s_0 \). Consider the section
\[
s := t(\xi_0) \in H^0(C, L^n),
\]
where \( t(\xi_0) \) is the homomorphism in (2.5). We have
\[
\tilde{s} := q_*p^*s \in H^0(Q_{L^n}(d), F). \quad \text{(2.7)}
\]

For the point \( t_0 \) in (2.6), the scheme theoretic inverse image
\[
q^{-1}(t_0) \subset D \subset C \times Q_{L^n}(d)
\]
is \( (rx_0) \times t_0 \), where \( q \) is the projection in (2.4). Since the section \( t(s_0) \) of \( L \) vanishes exactly on \( rx_0 \), this implies that the section \( \tilde{s} \) in (2.7) vanishes exactly on the reduced point \( t_0 \). Therefore, \( Q_{L^n}(d) \) has the weak-point property. \( \Box \)

3. Diagonal property for symmetric product of curves

Theorem 3.1. For any \( d \geq 1 \), the symmetric product \( \text{Sym}^d(C) \) of a smooth projective curve \( C \) has the diagonal property.

Proof. Consider the divisor \( D \) in (2.2). Let
\[
L = \mathcal{O}_Y(D) \rightarrow Y \quad \text{(3.1)}
\]
be the line bundle. Now consider \( Z := Y \times \text{Sym}^d(C) = C \times \text{Sym}^d(C) \times \text{Sym}^d(C) \). Let
\[
\alpha : Z \rightarrow C, \quad \beta : Z \rightarrow \text{Sym}^d(C) \quad \text{and} \quad \gamma : Z \rightarrow \text{Sym}^d(C) \quad \text{(3.2)}
\]
be the projections defined by \( (x, y, z) \mapsto x \), \( (x, y, z) \mapsto y \) and \( (x, y, z) \mapsto z \) respectively. Let
\[
\tilde{D} := (\alpha \times \gamma)^{-1}(D) \subset C \times \text{Sym}^d(C) \times \text{Sym}^d(C) = Z \quad \text{(3.3)}
\]
be the inverse image, where \( D \) is defined in (2.2), and \( \alpha \times \gamma : Z \rightarrow C \times \text{Sym}^d(C) \) sends any \( (x, y, z) \) to \( (x, z) \). Let
\[
p : \tilde{D} \rightarrow \text{Sym}^d(C) \times \text{Sym}^d(C)
\]
be the projection defined by \( b \mapsto (\beta(b), \gamma(b)) \), where \( \beta \) and \( \gamma \) are defined in (3.2), and \( \tilde{D} \) is constructed in (3.3). Consider the direct image
\[
V := p_*(((\alpha \times \beta)^*L)|_{\tilde{D}}) \rightarrow \text{Sym}^d(C) \times \text{Sym}^d(C) \quad \text{(3.4)}
\]
where \( L \) is the line bundle in (3.1). The natural projection
\[
D \rightarrow \text{Sym}^d(C), \quad (x, y) \mapsto y,
\]
where \( D \) is defined in (2.2), is a finite morphism of degree \( d \). This implies that \( p \) is a finite morphism of degree \( d \). Consequently, the direct image \( V \) is a vector bundle on \( \text{Sym}^d(C) \times \text{Sym}^d(C) \) of rank \( d \).
Consider the natural inclusion $\mathcal{O}_Y \hookrightarrow \mathcal{O}_Y(D) = L$ (see (3.1)). Let

$$\sigma_0 \in H^0(Y, L)$$

be the section given by the constant function 1 using this inclusion. Let

$$\sigma := p_*(\langle (\alpha \times \beta)^*\sigma_0 \rangle|_D) \in H^0(\text{Sym}^d(C) \times \text{Sym}^d(C), V)$$

be the section of $V$ (constructed in (3.4)) given by $\sigma_0$.

We will show that the scheme theoretic inverse image

$$\sigma^{-1}(0) \subset \text{Sym}^d(C) \times \text{Sym}^d(C)$$

is the diagonal.

Take any point $(a, b) \in \text{Sym}^d(C) \times \text{Sym}^d(C)$ such that $a \neq b$. Then there is a point $z \in C$ such that the multiplicity of $z$ in $a$ is strictly smaller than the multiplicity of $z$ in $b$. We note that the scheme theoretic inverse image

$$p^{-1}((a, b)) \subset \tilde{D} \subset Z = C \times \text{Sym}^d(C) \times \text{Sym}^d(C)$$

is $\{(a, b)\} \times \tilde{b}$, where $\tilde{b}$ is the zero dimensional subscheme of $C$ of length $d$ defined by $b$. On the other hand, for the section $\sigma_0$ in (3.5), the intersection $\sigma_0^{-1}(0) \cap (C \times \{a\})$ is the zero dimensional subscheme $\tilde{a}$ of $C$ of length $d$ defined by $a$. Since the multiplicity of $z$ in $a$ is strictly smaller than the multiplicity of $z$ in $b$, we have:

$$\sigma_0((z, b)) \neq 0.$$

Consequently, $\sigma((a, b)) \neq 0$.

Now take a point $(a, a)$ on the diagonal of $\text{Sym}^d(C) \times \text{Sym}^d(C)$. We have observed above that the inverse image

$$p^{-1}((a, a)) \subset C$$

coincides with the intersection $\sigma^{-1}(0) \cap (C \times a)$. This implies that

- $\sigma((a, a)) = 0$, and
- $\sigma^{-1}(0)$ is the reduced diagonal.

Therefore, $\text{Sym}^d(C)$ has the diagonal property. \qed

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