Bayesian identification has a long history, dating at least as far back as Peterka [1981]. Despite this, it is not commonly used in practice, except for the linear, Gaussian SSM case; wherein, Kalman filter based Bayesian estimate is routinely employed (Ninness and Henriksen [2010]). This is due to the computational complexities associated with the computation of the posterior densities, their marginals, and associated functions, such as posterior mean and variance (Juloski et al. [2005]). Recent developments in statistical methods, such as SMC and MCMC along with advances in computing technology have allowed researchers to use Bayesian methods in both on-line (Tulsyan et al. [2013a], Chen et al. [2005]) and off-line (Jang and Gopaluni [2011], Geweke and Tanizaki [2001]) identification of SSMs.

This paper is directed towards the class of Bayesian identification methods for parameter estimation in stochastic SSMs. The notation used in this paper is introduced next.

**Notation:** $\mathbb{N} := \{1, 2, \ldots \}$; $\mathbb{R}_+ := [0, \infty)$; $\mathbb{R}^{s \times s}$ is the set of real-valued $s \times s$ matrices; $S^+ \subseteq \mathbb{R}^{s \times s}$ is the space of symmetric matrices; $S^+_{++}$ is the cone of symmetric positive semi-definite matrices in $S^+$; and $S^+_{++}$ is its interior. The partial order on $S^+$ induced by $S^+_{++}$ and $S^+_{++}$ are denoted by $\geq$ and $\succ$, respectively. For $A \in \mathbb{R}^{s \times s}$, $\text{Tr}[A]$ denotes its trace. For a vector $y \in \mathbb{R}^p$, diag(y) is a diagonal matrix with $y \in \mathbb{R}^p$ as its entries, $|\cdot|$ is the absolute value. $\nabla_y \triangleq \nabla_x \nabla_y^{T}$ is Laplacian and $\nabla^2 \triangleq \left[\frac{\partial^2}{\partial y^2}\right]$ is gradient.

1 This article has been published in: Tulsyan, A. B. Huang, R.B. Gopaluni and J.F. Forbes (2013). Bayesian identification of non-linear state-space models: Part II- Error Analysis. In: Proceedings of the 10th IFAC International Symposium on Dynamics and Control of Process Systems. Mumbai, India.

2 This work was supported by the Natural Sciences and Engineering Research Council (NSERC), Canada.
process with \((Z_0 = z_0) \sim p_{0}(x_0)p(\theta_0), Z_1|Z_{t-1} = z_{t-1}) \sim p_{0,t-1}(x_{t-1})\delta_{z_{t-1}}(\cdot)\). Note that a recursive method to compute \(p(z_t|y_{1:t})\) is given by the optimal filtering equation. Having computed \(p(z_t|y_{1:t})\) for each \(t\) based on known parameter case in the state estimation problem, calculating \(p(\theta_t|y_{1:t})\) for \(t\) has proved to be a non-trivial problem (Minvielle et al. [2010], Kantas et al. [2009]). No analytical solution to \(p(\theta_t|y_{1:t})\) is available, even for linear and Gaussian SSM, or when \(X_t\) is a finite set (Kantas et al. [2009]). There are several simulation and numerical methods (e.g., SMC, MCMC, Kalman-based filters), which allow for recursive approximation of \(p(\theta_t|y_{1:t})\) for each \(t\). Although tractable, the quality of these identification methods depends on the underlying numerical and statistical approximations used in their design.

Despite the widespread interest in developing advanced simulation and numerical methods for Bayesian identification of (3), there have been no elaborate study on the quality of these methods. With this background, this paper proposes the use of PCRLB as an error bound. Using PCRLB, a systematic approach to assess the quality of a Bayesian identification method, in terms of bias, MSE, and efficiency is developed. Initial results reported by the authors in Tulsyan et al. [2013b] use PCRLB for assessment of state (but not parameter) estimation algorithms. The focus of this paper is to extend the results in Tulsyan et al. [2013b] to the Bayesian parameter estimation algorithms.

3. PCRLB AS AN ERROR BOUND

The conventional Cramér-Rao lower bound (CRLB) provides a theoretical lower bound on the MSE of any maximum-likelihood (ML) based unbiased parameter estimator. An analogous extension of the CRLB to the Bayesian estimators was derived by Trees [1968], and is commonly referred to as the PCRLB inequality. The PCRLB, derived recently by Tichavský et al. [1998] for (3), provides a lower bound on the MSE associated with the joint estimation of the states and parameters from \(p(z_t|y_{0:t-1}, x_{t-1})\), and is given in the next lemma.

**Lemma 2.** Let \(\{y_{t}:t\in\mathbb{N}\}\) be an output sequence generated from (3), then the MSE associated with the estimation of \(\{Z_t\}_{t\in\mathbb{N}}\) from \(p(z_t|y_{0:t}, x_{t-1})\) is bounded by

\[
P_{Z_t}^{2} \triangleq \mathbb{E}(z_{t} - Z_{t})^{2} \geq \left| J_{t}^{x} \right|^{-1}, \tag{4}
\]

where: \(Z_t := \mathbb{R}^m \rightarrow \mathbb{R}^s\) is a point estimate of \(\{Z_t\}_{t\in\mathbb{N}}\);

\[
P_{Z_t}^{2} \triangleq \left[ \begin{array}{l} \left( L_{t}^{x} \right)_{T} \left( J_{t}^{x} \right)_{T} \left( J_{t}^{x} \right)_{T} \\ L_{t}^{x} \end{array} \right] \in S_{++}^{s}, \quad J_{t}^{x} \triangleq \left[ \begin{array}{c} J_{t}^{x} \\ J_{t}^{y} \end{array} \right] \in S_{++}^{s}, \]

\[
\left| J_{t}^{x} \right|^{-1} = \left[ \begin{array}{l} L_{t}^{x} \end{array} \right]^{-1} L_{t}^{x} \in S_{++}^{s} \quad \text{are the MSEE, posterior information matrix (PIM), and PCRLB, respectively.}
\]

**Proof.** See Tichavský et al. [1998] for proof.

A recursive approach to compute \(J_{t}^{x} \in S_{++}^{s}\) was derived by Tichavský et al. [1998], and is given next. But first, we give the assumptions on the model considered in (3).

**Assumption 3.** \(\{V_t\}_{t\in\mathbb{N}}\) and \(\{W_t\}_{t\in\mathbb{N}}\) are mutually independent sequences of independent random variables known a priori in their distribution classes (e.g., Gaussian) and parameterized by a known and finite number of moments. For any random sample \((x_{t+1}, x_t, \theta_t, y_t)\) that they satisfy (4), the MSE associated with the estimator \(\hat{\theta}_{t+1}\) is \(H_{t+1}^{\text{CRLB}}\) and \(\hat{\theta}_{t+1}\) is \(H_{t+1}^{\text{CRLB}}\), where \(k \geq 2\). The conventional Cramér-Rao lower bound (CRLB) provides a theoretical lower bound on the MSE of any parameter estimate. The proof is based on the fact that the inequality (6e) holds for \(p(y_{t+1}|y_{t}, x_{t+1}) = p(y_{t+1} = j_{t+1}|x_{t+1})\) and \(p(y_{t}|x_{t+1}) = p(y_{t} = j_{t}|x_{t+1})\) are defined.

\[
\begin{align*}
J_{t}^{x} &= H_{t}^{C}\left( J_{t}^{x} + H_{t}^{x} \right)^{-1} J_{t}^{x}, \quad \text{(5a)}
J_{t}^{y} &= H_{t}^{C}\left( J_{t}^{y} + H_{t}^{y} \right)^{-1} J_{t}^{y}, \quad \text{(5b)}
J_{t}^{x+y} &= H_{t}^{C}\left( J_{t}^{x+y} + H_{t}^{x+y} \right)^{-1} J_{t}^{x+y}, \quad \text{(5c)}
\end{align*}
\]

where:

\[
\begin{align*}
H_{t}^{11} &= \mathbb{E}_{p(x_{0:t-1}, \theta, y_{0:t-1})}\left[ -\Delta_{x,1}\log p_{1} \right]; \quad \text{(6a)} \\
H_{t}^{12} &= \mathbb{E}_{p(x_{0:t-1}, \theta, y_{0:t-1})}\left[ -\Delta_{x,2}\log p_{1} \right]; \quad \text{(6b)} \\
H_{t}^{13} &= \mathbb{E}_{p(x_{0:t-1}, \theta, y_{0:t-1})}\left[ -\Delta_{x,3}\log p_{1} \right]; \quad \text{(6c)} \\
H_{t}^{22} &= \mathbb{E}_{p(x_{0:t-1}, \theta, y_{0:t-1})}\left[ -\Delta_{x,2}\log p_{2} \right]; \quad \text{(6d)} \\
H_{t}^{23} &= \mathbb{E}_{p(x_{0:t-1}, \theta, y_{0:t-1})}\left[ -\Delta_{x,3}\log p_{2} \right]; \quad \text{(6e)} \\
H_{t}^{33} &= \mathbb{E}_{p(x_{0:t-1}, \theta, y_{0:t-1})}\left[ -\Delta_{x,1}\log p_{3} \right]; \quad \text{(6f)}
\end{align*}
\]

and: \(p_t = p(x_{t+1}|z_t)p(y_{t+1}|x_{t+1});\) and the PIM at \(t = 0\) can be computed using \(J_0 = \mathbb{E}_{p(x_0)}[-\Delta_{x,0}\log p_{Z_0}]\) is given in the next lemma.

**Proof.** See Tichavský et al. [1998] for proof.

Since the focus here is on \(\{\theta_t\}_{t\in\mathbb{N}}\) alone, a lower bound on the MSE associated with the estimation of \(\{\theta_t\}_{t\in\mathbb{N}}\) is of interest to us. Using Lemmas 2 and 6, a bound on the MSE for parameter estimates can be derived, as given next.

**Corollary 7.** Let \(P_{t}^{\theta} \in S_{++}^{s} \neq J_{t}^{x} \in S_{++}^{s}\) be such that they satisfy (4), then the MSE associated with the estimation of \(\{\theta_t\}_{t\in\mathbb{N}}\) from \(p(\theta_t|y_{1:t})\) is bounded by

\[
P_{t}^{\theta} = \mathbb{E}_{p(\theta_0, y_{1:t})}[(\theta_t - \theta_t)^T(\theta_t - \theta_t)^{-1}] \geq L_{t}^{\theta}, \tag{7}
\]

where \(\theta_t := \mathbb{R}^m \rightarrow \mathbb{R}^s\) is the parameter estimate delivered by a Bayesian identification algorithm, and \(L_{t}^{\theta} \in S_{++}^{s}\) is the lower right matrix of \(J_{t}^{x} \in S_{++}^{s}\) in Lemma 2.

**Proof.** The proof is based on the fact that the inequality in Lemma 2 guarantees that \(P_{t}^{\theta} - J_{t}^{x} \in S_{++}^{s}\). A recursive approach to compute \(L_{T}^{\theta} \in S_{++}^{s}\) is given next.

**Theorem 8.** Let \(J_{T}^{x} \in S_{++}^{s}\) be the PIM for \(\{Z_t\}_{t\in\mathbb{N}}\), and \(L_{T}^{\theta} \in S_{++}^{s}\) be the lower bound on the MSE associated with the estimation of \(\{\theta_t\}_{t\in\mathbb{N}}\) in (3), then given \(J_{T}^{x} \in S_{++}^{s}\), \(L_{T}^{\theta} \in S_{++}^{s}\), and at \(t \geq N\) can be recursively computed as follows.

\[
L_{t}^{\theta} = J_{t}^{x} - J_{t}^{x+y} J_{t}^{y} \left( J_{t}^{y} \right)^{-1} J_{t}^{x}, \tag{8}
\]

where \(J_{t}^{x}, J_{t}^{y}, J_{t}^{x+y}\) and \(J_{t}^{x+y}\) are the PMs given in Lemma 2.

**Proof.** The proof is based on the matrix inversion of Lemma (see R.B. Bapat and T.E.S. Raghavan [1997]).
Remark 9. Theorem 8 shows that for (3), $L_0^\theta$ is not only a function of the PIM for $\{\theta_t\}_{t \in \mathbb{N}}$, i.e., $J_t^\theta$, but it also depends on the PIMs for $\{X_t\}_{t \in \mathbb{N}}$, i.e., $J_t^x$ and $J_t^z$.

Remark 10. Integral (6) with respect to $p(x_t, \theta_{t-1}, y_{1:t})$ makes $L_0^\theta$ in (8) independent of any random sample from $X^{t+1}, \Theta$ and $Y^t$. $L_0^\theta$ in fact only depends on: the process dynamics in (3); noise characteristics of $V_t \sim p(v_t)$ and $W_t \sim p(w_t)$; and the choice of $Z_0 \sim p(z_0)$. This makes $L_0^\theta$ a system property, independent of any Bayesian identification method or any specific realization from $X, \Theta$ or $Y$.

This motivates the use of PCRLB as a benchmark for error analysis of Bayesian identification algorithms.

Finally, using the inequality in (7), the MSE associated with the parameter estimates obtained with any Bayesian identification method can be compared against the theoretical lower bound. Our approach to systematically compare and analyse the MSE and PCRLB is discussed next.

4. PCRLB INEQUALITY BASED ERROR ANALYSIS

A common approach to compute $\theta_{t|t} \in \mathbb{R}^d$ is to minimize $\text{Tr}[P_{t|t}^\theta] \in \mathbb{R}_+$. This ensures that $\text{Tr}[P_{t|t}^\theta - L_t^\theta] \geq 0$ is minimized. The optimal estimate that minimizes $\text{Tr}[P_{t|t}^\theta] \in \mathbb{R}_+$ is referred to as the minimum MSE (MMSE) estimate, and is the conditional mean of $\hat{\theta}_t(Y_{1:t} = y_{1:t}) \sim p(\cdot|y_{1:t})$, i.e., $\theta_{t|t} = \theta_{t|t}^\theta \triangleq \mathbb{E}_{\hat{\theta}(\cdot|y_{1:t})} \left[ \theta \right]$ (see Trees [1968] for derivation).

Remark 11. Bayesian identification methods only approximate the true density $p(\hat{\theta}(\cdot|y_{1:t}))_{t \in \mathbb{N}}$, thus in practice, the estimate delivered by identification methods may not be an MMSE estimate, i.e., $\theta_{t|t} \neq \mathbb{E}_{\hat{\theta}(\cdot|y_{1:t})} \left[ \theta \right]$. Thus from Definition 17, the identifiability condition under which an identification method delivers unbiased parameter estimate is discussed next.

Theorem 14. Let $\theta_{t|t} \in \mathbb{R}^d$ be the estimate of $\theta_{t|t}^\theta \in \mathbb{R}^d$, as computed by an identification method, where $\theta_{t|t}^\theta \in \mathbb{R}^d$ is the mean of $\hat{\theta}_t(Y_{1:t} = y_{1:t}) \sim p(\cdot|y_{1:t})$, and let $B_{t|t}^\theta \in \mathbb{R}^d$ be the corresponding conditional bias, then $B_{t|t}^\theta = 0$ almost surely is only a sufficient condition for $E[p_{\hat{\theta}(\cdot|y_{1:t})}] \left[ B_{t|t}^\theta \right] = 0$, but sufficient and necessary for $E[p_{\hat{\theta}(\cdot|y_{1:t})}] \left[ B_{t|t}^\theta [B_{t|t}^\theta]^T \right] = 0$.

Proof. See Billingsley [1995] for proof.

Remark 15. Theorem 14 shows that if the parameter estimate $\theta_{t|t} \in \mathbb{R}^d$ is unconditionally unbiased, it does not imply it is unbiased as well, but if it is conditionally unbiased, it implies $\theta_{t|t} \in \mathbb{R}^d$ is unbiased as well.

The MSE for an unbiased estimate $\theta_{t|t} \in \mathbb{R}^d$ is given next.

Corollary 16. Let $\theta_{t|t} \in \mathbb{R}^d$ be the estimate of the mean of $\hat{\theta}_t(Y_{1:t} = y_{1:t}) \sim p(\cdot|y_{1:t})$ computed by a Bayesian identification method, such that $B_{t|t}^\theta = 0$ almost surely, then the MSE associated with $\theta_{t|t} \in \mathbb{R}^d$ is $P_{t|t}^\theta = E_{\hat{\theta}(\cdot|y_{1:t})} \left[ V_{t|t}^\theta \right]$.

Definition 17. An identification method delivering an estimate $\theta_{t|t} \in \mathbb{R}^d$ is efficient at $t \in \mathbb{N}$ if $\text{Tr}[P_{t|t}^\theta - L_t^\theta] = 0$.

Theorem 18. Let $\theta_{t|t} \in \mathbb{R}^d$ be the estimate of $\theta_{t|t}^\theta \in \mathbb{R}^d$, as computed by an identification method, and let $B_{t|t}^\theta \in \mathbb{R}^d$ be the conditional bias in estimating $\theta_{t|t}^\theta \in \mathbb{R}^d$, then $B_{t|t}^\theta = 0$ almost surely is both necessary and sufficient condition for the identification method to be efficient.

Proof. For $\theta_{t|t} \in \mathbb{R}^d$ satisfying $B_{t|t}^\theta = 0$ almost surely, the MSE is given by $P_{t|t}^\theta = E_{\hat{\theta}(\cdot|y_{1:t})} \left[ V_{t|t}^\theta \right]$ (see Corollary 16). Since $P_{t|t}^\theta$ only depends on $V_{t|t}^\theta$, which is the covariance of $\hat{\theta}_t(Y_{1:t} = y_{1:t}) \sim p(\cdot|Y_{1:t})$, $P_{t|t}^\theta$ cannot be reduced any further i.e., $P_{t|t}^\theta = L_t^\theta$. Thus from Definition 17 the identification method delivering $\theta_{t|t} \in \mathbb{R}^d$ is efficient at $t \in \mathbb{N}$.

Finally, the procedure to systematically assess the quality of the parameter estimates obtained with any Bayesian identification method is summarized in the next theorem.

Theorem 19. Let $L_t^\theta \in \mathbb{R}_+^d$ be the PCRLB on (3), and let $\theta_{t|t} \in \mathbb{R}^d$ and $V_{t|t}^\theta \in \mathbb{R}_+^d$ be the mean and covariance of $\hat{\theta}_t(Y_{1:t} = y_{1:t}) \sim p(\cdot|Y_{1:t})$. Now if $\theta_{t|t} \in \mathbb{R}^d$ is an estimate of $\theta_{t|t}^\theta \in \mathbb{R}^d$, as computed by an identification method, such that $B_{t|t}^\theta \in \mathbb{R}^d$ is the conditional bias in estimating $\theta_{t|t}^\theta \in \mathbb{R}^d$, then for $P_{t|t}^\theta \in \mathbb{R}_+^d$ as the associated MSE, the quality of the estimate $\theta_{t|t} \in \mathbb{R}^d$ can be assessed as follows:
(a) If \( B_t^\star = 0 \) almost surely, then (7) is given by
\[
P_t^p = \mathbb{E}_{p(y_{1:t})}[V_t^\star] = L_t^p,
\]
which implies the identification method is efficient, and the corresponding estimate \( \hat{\theta}_t \) is unbiased and MMSE. (b) If \( B_t^\star \neq 0 \) almost surely, then (7) is given by
\[
P_t^p = \mathbb{E}_{p(y_{1:t})}[V_t^\star] + \mathbb{E}_{p(y_{1:t})}[B_t^\star B_t^\star]^\dagger > L_t^p,
\]
which implies the identification method is not efficient, and the estimate \( \hat{\theta}_t \) is biased (only conditionally biased if \( \mathbb{E}_{p(y_{1:t})}[B_t^\star] = 0 \) and not an MMSE estimate.

**Proof.** The proof is based on the collective developments of Section 4, and is omitted here for the sake of brevity. □

The PCRLB inequality based error analysis tool developed in this section allows for assessment of parameter estimates obtained with Bayesian identification methods; however, obtaining a closed form solution to (7) is non-trivial for (3). Use of numerical methods is discussed next.

5. NUMERICAL METHODS

It is well known that computing the MSE and PCRLB in (7) in closed form is non-trivial for the model considered in (3) (see Tichavský et al. [1998], Bergman [2001]). This is because of the complex, high-dimensional integrals in the MSE with respect to \( p(\theta_t, y_{1:t}) \) (see (7)) and in the PCRLB with respect to \( p(x_{1:t}, \theta_{t-1}, y_{1:t}) \) (see (6a) through (6f)), which do not admit any analytical solution.

To address this issue, we use Monte Carlo (MC) sampling to numerically compute the MSE and PCRLB in (7). For the sake of brevity, the procedure for MC approximation of the PCRLB is not provided here, but can be found in Tulsyan et al. [2013c]; however, for completeness, we provide an example for computation of MC based MSE.

**Example 20.** Simulating samples \( \{\theta_t = \theta_t^1, Y_{1:t} = y_{1:t}^1\}_{j=1}^J \sim p(\theta_t, y_{1:t}) \) \( M \) times using (3), starting at \( M \) i.i.d. initial draws from \( \{\theta_0\}_i \sim p(\theta_0) \) and computing the estimates \( \theta_t^1, \ldots, \theta_t^M \); the MSE \( P_t^\theta \) at \( t \in \mathbb{N} \) can be approximated as
\[
P_t^\theta = \frac{1}{M} \sum_{i=1}^M (\theta_t^i - \theta_t^\star)(\theta_t^i - \theta_t^\star)^T,
\]
where \( \theta_t^\star \in S_{i.t}^q \) is an \( M \)-sample MC estimate of \( P_t^\theta \). □

Since (12) is based on perfect sampling, using strong law of large numbers \( \hat{P}_t^\theta \xrightarrow{a.s.} P_t^\theta \) as \( M \to \infty \), where \( a.s. \) denotes almost sure convergence (see P. Del Moral [2004]). Note that \( \hat{L}_t^\theta \), which is an \( M \)-sample MC estimate of \( L_t^\theta \), can also be similarly approximated using MC sampling. Details are omitted here, but can be found in Tulsyan et al. [2013c]. Despite the convergence proof, there are practical issues with the use of numerical methods, as given next.

**Remark 21.** With \( M < +\infty \), the MC estimate of the MSE and PCRLB may not necessarily satisfy the positive semi definite condition \( \hat{P}_t^\theta \geq L_t^\theta \) for all \( t \in \mathbb{N} \). □

**Remark 22.** Since \( M < +\infty \), the conditions in Theorem 19 are relaxed to \( |B_t^\star| \leq \epsilon \) and \( |\mathbb{E}_{p(y_{1:t})}[B_t^\star]| \leq \alpha \), and \( \epsilon \in \mathbb{R}_+^\ast \) and \( \alpha \in \mathbb{R}_+^\ast \) are pre-defined tolerance levels set based on \( M \) and the required degree of accuracy. □

**Remark 23.** An identification method satisfying \( |B_t^\star| \leq \epsilon \) is \( \epsilon \)-efficient at \( t \in \mathbb{N} \) and the corresponding estimate is \( \epsilon \)-unbiased and \( \epsilon \)-MMSE (see Theorem 19(a)). Similarly, if the estimate only satisfies \( |\mathbb{E}_{p(y_{1:t})}[B_t^\star]| \leq \alpha \), then it is \( \alpha \)-unconditionally unbiased (see Theorem 19(b)). □

6. FINAL ALGORITHM

A systematic approach to assess the quality of a Bayesian identification method, proposed in Sections 3 through 5 is formally outlined in Algorithm 1.

**Algorithm 1 Analysis of Bayesian identification methods**

**Module 1: Computing the lower bound**

**Input:** Given (3), define \( Z_t = \{X_t, \theta_t\} \) and assume a prior density on \( \{Z_t\}_{t \in \mathbb{N}} \), such that \( (Z_0 = z_0) \sim p(z_0) \)

**Output:** Lower bound on the system in (3)

1. Generate \( M \) i.i.d. samples from the assumed prior density \( Z_0 \sim p(\cdot) \), such that \( (Z_0 = z_0)_{i=1}^M \sim p(z_0) \)
2. for \( t = 1 \) to \( T \) do
3. Generate \( M \) random samples from the states \( \{X_t = x_t^i|Z_{t-1} = z_{t-1}^i\}_{i=1}^M \sim p(x|z_{t-1}) \) using (3a)
4. Generate \( M \) random samples from the parameters \( \{\theta_t = \theta_t^i|Z_{t-1} = z_{t-1}^i\}_{i=1}^M \sim p(\theta|z_{t-1}) \) using (3b). Note that in this step \( \theta_0^i = \theta_0^0 \) for all \( 1 \leq i \leq M \) (see (3b))
5. Generate \( M \) random samples from the measurements \( \{Y_t = y_t^i|Z_{t} = z_{t}^i\}_{i=1}^M \sim p(y|z_{t}) \) using (3c)
6. Compute an \( M \)-sample MC estimate of \( \bar{J}_t^\theta \)
7. Compute an \( M \)-sample MC estimate of \( \bar{L}_t^\theta \)
8. end for

**Module 2: Computing the estimates**

**Input:** Measurement sequences from Module 1, denoted as \( \{\{Y_{1:T} = y_{1:T}^i\}_{t=1}^M\}_{i \in \mathbb{N}} \) and a Bayesian identification method, which can compute \( \{p(\theta_t|y_{1:t})\}_{t \in \mathbb{N}} \) (e.g., SMC, MCMC, EKF, and UKF)

**Output:** Parameter estimates

9. for \( i = 1 \) to \( M \) do
10. for \( t = 1 \) to \( T \) do
11. Generate \( \bar{p}(\theta_t|y_{1:t}) \) using an identification method and denote density approximation by \( \bar{p}(\theta_t|y_{1:t}) \)
12. Using \( \bar{p}(\theta_t|y_{1:t}) \), compute parameter point estimate as \( \theta_t^i = \mathbb{E}_{\bar{p}(\theta_t|y_{1:t})}[\theta_t^i] \)
13. end for
14. end for

**Module 3: Analysis of Bayesian identification method**

**Input:** Parameter sequences from Module 1, denoted by \( \{\theta_{1:T} = \theta_{1:T}^i\}_{i=1}^M \) and their estimates from Module 2, denoted as \( \{\theta_{1:T} = \theta_{1:T}^i\}_{i=1}^M \). Matrices \( \bar{L}_t^\theta \in S_{i.t}^q \) and \( \hat{P}_t^\theta \in S_{i.t}^q \) and tolerance level \( \epsilon \in \mathbb{R}_+^\ast \) and \( \alpha \in \mathbb{R}_+^\ast \)

**Output:** Error analysis of identification method

15. for \( t = 1 \) to \( T \) do
16. Compute an \( M \)-sample MC estimate of \( \hat{P}_t^\theta \)
17. Compare \( \bar{L}_t^\theta \) against \( \hat{P}_t^\theta \)
18. Compute \( \{\bar{L}_t^\theta, \hat{P}_t^\theta\} \) and compare against \( \epsilon \in \mathbb{R}_+^\ast \)
19. Compute an \( M \)-sample MC estimate of \( \mathbb{E}_{\bar{p}(\theta_t|y_{1:t})}[\theta^i_t|y_{1:t}] \)
20. Use Theorem 19 for error analysis
21. end for
7. SIMULATION EXAMPLE

In this section we use a simulated system to assess the quality of a Bayesian identification method using the procedure outlined in Algorithm 1. A brief introduction to the identification method considered here, is given next.

7.1 Bayesian identification: Artificial dynamics approach

Artificial dynamics approach (ADA) is a popular Bayesian identification method to compute \( \{p(\theta_t|y_{1:t})\}_{t \in \mathbb{N}} \). In ADA, artificial dynamics is introduced to the otherwise static parameters, such that \( \{\theta_t\}_{t \in \mathbb{N}} \) in (3b) evolves according to

\[
\theta_{t+1} | \theta_t \sim \mathcal{N}(\theta_t, Q_t^\theta),
\]

where \( \theta_{t+1} \sim \mathcal{N}(\theta_t, Q_t^\theta) \) is a sequence of independent Gaussian random variables, realized independent of \( \{V_t\}_{t \in \mathbb{N}} \) and \( \{W_t\}_{t \in \mathbb{N}} \). By appending (3a) and (3c) with (13), methods such as SMC, EKF, UKF can be used to recursively compute \( p(\theta_t|y_{1:t}) \). A detailed review on ADA can be found in Tulsyan et al. [2013a] and Kantas et al. [2009].

Even though ADA is the most widely used approach amongst the class of Bayesian identification methods, there are several standing limitations of this approach as summarized in Kantas et al. [2009] (a) the dynamics of \( \{\theta_t\}_{t \in \mathbb{N}} \) in (13) is related to the artificial noise covariance \( Q_t^\theta \), which is often difficult to tune; and (b) adding dynamics to \( \{\theta_t\}_{t \in \mathbb{N}} \) modifies the original problem, which means, it is hard to quantify the bias introduced in the estimates.

For the former problem, the authors in see Tulsyan et al. [2013a] proposed an optimal rule to automatically tune \( Q_t^\theta \) for all \( t \in \mathbb{N} \); however, for the later problem, we will see how the tools developed in this paper can be used to assess the quality of ADA based Bayesian identification methods.

7.2 Simulation setup

Consider the following univariate, non-stationary, non-linear stochastic SSM (Tulsyan et al. [2013c])

\[
\begin{align*}
X_{t+1} &= aX_t + \frac{X_t}{b + X_t} + u_t + V_t, \quad V_t \sim \mathcal{N}(0, Q_t), \quad (14a) \\
Y_t &= cX_t + dX_t^2 + W_t, \quad W_t \sim \mathcal{N}(0, R_t), \quad (14b)
\end{align*}
\]

where \( \theta \triangleq \{a, b, c, d\} \) is a vector of unknown static model parameters. The noise covariances are constant, and selected as \( Q_t = 10^{-3} \) and \( R_t = 10^{-3} \) for all \( t \in [1, T] \), where \( T = 300 \). \( \{u_t\}_{t \in [1, T]} \) is a sequence of optimal input (see Tulsyan et al. [2013c]).

For Bayesian identification of \( \theta \), we define \( \{\theta_t = \theta_{t-1}\}_{t \in [1, T]} \) as \( \theta \) as a stochastic process, such that \( Z_t = \{X_t, \theta_t\} \) is a \( Z \) valued extended Markov process with \( Z_0 \sim \mathcal{N}(z_m, z_c) \), where \( z_m = [1 0.7 0.6 0.5 0.4] \), \( z_c = \text{diag}(0.01, 0.01, 0.01, 0.01, 0.01) \). Starting at \( t = 0 \), we are interested in assessing the ADA based SMC identification method proposed in Tulsyan et al. [2013a].

7.3 Results

Using \( M = 1000 \) MC simulations, we compute the PCRLB for (14) using Module 1 of Algorithm 1. Figure 1 gives the diagonal entries of \( \{L_t \}_{t \in [1, T]} \). Note that amongst the four PCRLBs, the PCRLB for \( b \) is the highest for all \( t \in [1, T] \). This suggest estimation difficulties with parameter \( b \). This result is not surprising, since (14) is non-linear in parameter \( b \); however, the overall decaying trend of PCRLBs in Figure 1 suggests that starting with \( \theta_0 \sim p(\theta_0) \), theoretically, it is possible for a Bayesian identification method to reduce the MSE associated with the parameter estimates.

Figure 2 compares the PCRLB against the MSE computed using the ADA based SMC identification method. Graphs are shown only for parameters \( b \) and \( d \). Results for \( a \) and \( c \) are similar.

Figure 3 and 4 give the conditional and unconditional bias with ADA based SMC method. The results are obtained using Module 3 of Algorithm 1. Based on an assumed tolerance level \( \epsilon = [0.01; 0.01; 0.01; 0.01] \) and \( \alpha = [0.001; 0.001; 0.001; 0.001] \), in the interval \( t = [1, 50] \),
Fig. 3. Conditional bias in parameter estimates with ADA based SMC method. The broken red line is the $\epsilon$ value.

Fig. 4. Unconditional bias in parameter estimates with ADA based SMC method. The broken red line is the $\alpha$ value.

less than 70% of the simulations are within the specified $\epsilon$ limit (see Figure 3). Thus from Theorem 19(b), for $t = [1, 50]$, the ADA based SMC method is not even $\epsilon$-efficient, and fails to yield $\epsilon$-unbiased (except for $d$, which is $\alpha$-unconditionally unbiased, see Figure 4) or $\epsilon$-MMSE estimates. Another interesting interval is $t = [100, T]$, wherein, more than 70% of the simulations are within the specified $\epsilon$ limit (except for parameter $b$, where only 60% of simulations are within $\epsilon$, see Figure 3). Thus from Theorem 19(a), the ADA based SMC method is $\epsilon$-efficient for all the parameters, except for $b$, and the resulting estimates are $\epsilon$-unbiased and $\epsilon$-MMSE; whereas, for $b$, the estimates are not MMSE, but are $\alpha$-unconditionally unbiased.

In summary, the results suggest that for model given in (14), the ADA based SMC method at $t = T$ yields $\epsilon$-unbiased, $\epsilon$-MMSE estimates for all the parameters, except for parameter $b$, which is only $\alpha$-unconditionally unbiased.

8. CONCLUSIONS

A PCRLB based approach is proposed for error analysis in Bayesian identification methods of non-linear SSMs. Using the proposed tool it was illustrated how the quality of the parameter estimates obtained using artificial dynamics approach, which is a popular Bayesian identification method can be assessed in terms of bias, MSE and efficiency.

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