Modified operators versus modified commutators in determining a quantum gravity minimal length scale

Michael Bishop

Mathematics Department, California State University Fresno, Fresno, CA 93740

Jaeyeong Lee and Douglas Singleton

Physics Department, California State University Fresno, Fresno, CA 93740

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Abstract

We investigate the role played by modifying the quantum position and momentum operators versus modifying quantum position and momentum commutators. We find that different versions of modified operators can lead to the same modified commutator and give different or even no minimal length as implied by quantum gravity. The conclusion is that the modification of the operators is the main factor in determining whether there is a minimal length.
I. QUANTUM GRAVITY MINIMAL LENGTH SCALE

There are generic and broad arguments that any theory of quantum gravity should have some fixed fundamental minimal length scale [1-4]. At low momentum, the relationship between the uncertainty in position and momentum should take the standard form, namely $\Delta x \Delta p \sim \text{const.} \Rightarrow \Delta x \sim \frac{\text{const.}}{\Delta p}$. At large enough momentum, quantum gravity arguments indicate a linear relationship between $\Delta x$ and $\Delta p$, namely $\Delta x \sim \Delta p$ [1]. These physically motivated arguments imply the following generalized uncertainty principle (GUP)

$$\Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + \beta (\Delta p)^2 \right], \quad (1)$$

where $\beta$ is a phenomenological parameter that is assumed to be connected with the scale of quantum gravity. The GUP in (1) leads to the relationship $\Delta x \sim \frac{1}{\Delta p} + \beta \Delta p$ which, as shown in [5], leads to a minimum length of order $\sqrt{\beta}$.

Any uncertainty principle between position and momentum is connected with a commutator between position and momentum operator. In reference [5], Kempf, Mangano, and Mann (KMM) connected the GUP in equation (1) with the following modified commutator

$$[\hat{x}', \hat{p}'] = i\hbar (1 + \beta p^2). \quad (2)$$

The primes on the position and momentum operators on the left hand side of (2) indicate that these operators are modified versions of the standard quantum position operator ($\hat{x} = i\hbar \partial_p$ in momentum space or $\hat{x} = x$ in coordinate space) and momentum operator ($\hat{p} = -i\hbar \partial_x$ in coordinate space or $\hat{p} = p$ in momentum space). In this work, primed operators will indicate modified operators and unprimed will be the usual operators. On the right hand side of (2), the $p^2$ term is the standard quantum momentum and not the modified momentum. In the following section, we show that there are a host of different ways to modify the operators $\hat{x}'$ and $\hat{p}'$ which yield the exact same modified commutator in equation (2). However, these different ways to modify the position and momentum operators do not all lead to the GUP in (1) and do not all lead to a minimum length. This is the main point of this work – that it is the modification of the position and momentum operators, more than the modification of the commutator, which determines if there is a minimum length scale.
II. MODIFIED POSITION AND MOMENTUM OPERATORS

There is a large family of modified operators which all have the commutation relation 
\[ [x', p'] = i\hbar (1 + \beta p^2) \]. The modification of the position and momentum operator of reference [5] which gave the modified commutator in (2) was

\[ \hat{x}' = i\hbar (1 + \beta p^2) \partial_p \quad ; \quad \hat{p}' = p \]  

(3)

The modified operators in (3) are in momentum space and \( \hat{x} = i\hbar \partial_p \) is the usual position operator. Note that the momentum operator is not altered from the standard representation in momentum space.

A simple way to get the position space versions of the momentum space modified operators in (3) is to make the simple change \( i\hbar \partial_p \mapsto x \) and \( p \mapsto -i\hbar \partial_x \) so that (3) becomes

\[ \hat{x}' = (1 - \beta \hbar^2 \partial^2_x) x \quad ; \quad \hat{p}' = -i\hbar \partial_x \]  

(4)

There is some freedom in (4) in that one could also write the modified position operator as 
\[ \hat{x}' = x (1 - \beta \hbar^2 \partial^2_x) \]  
i.e. with \( x \) leading the derivative operator. Both (3) and (4) lead to the same modified commutator as can be seen by explicitly plugging \( \hat{x}' \) and \( \hat{p}' \), from either (3) or (4), into the commutator in (2).

Equation (4) is not the only way to write down the modified operators in position space. In [5], Kempf, Mangano and Mann use a procedure which employs maximally localized states to obtain a quasi-position space representation of the modified position and momentum operators as

\[ \hat{x}' = x + i\hbar \sqrt{\beta} \tan(i\hbar \sqrt{\beta} \partial_x) \quad ; \quad \hat{p}' = \frac{1}{\sqrt{\beta}} \tan(-i\hbar \sqrt{\beta} \partial_x) \]  

(5)

Inserting the \( \hat{x}' \) and \( \hat{p}' \) from (5) into \([\hat{x}', \hat{p}']\), and performing a long, direct calculation, again yields the modified commutator in (2). Equation (5) gives an alternative form of the modified position space operators, from those in (4) which nevertheless yields the same modified commutator. By using the standard quantum relationships \( i\hbar \partial_p \mapsto x \) and \( p \mapsto -i\hbar \partial_x \) one can turn the modified operators of (5) into

\[ \hat{x}' = i\hbar \partial_p + i\hbar \sqrt{\beta} \tan(-\sqrt{\beta} p) \quad ; \quad \hat{p}' = \frac{1}{\sqrt{\beta}} \tan(\sqrt{\beta} p) \]  

(6)

A direct calculation of inserting these \( \hat{x}' \) and \( \hat{p}' \), from (6) into \([\hat{x}', \hat{p}']\), yields the modified commutator of (2). Thus the operators in (6) provide an alternative momentum space
representation of the modified position and momentum operators as compared those from (3).

Another set of modified operators which treat the modified position and momentum somewhat equally and still lead to the modified commutator in (2) is given by

\[ \hat{x}' = i\hbar e^{-\beta p^2/2} \partial_p ; \quad \hat{p}' = e^{\beta p^2/2} p . \] (7)

One can obtain the position space version of the modified operators in (7) by again making the simple replacement \( i\hbar \partial_p \mapsto x \) and \( p \mapsto -i\hbar \partial_x \) to give

\[ \hat{x}' = e^{\hbar^2 \partial_x^2/2} x , \quad \hat{p}' = -i\hbar e^{-\beta \hbar^2 \partial_x^2/2} \partial_x . \] (8)

Substituting the operators from either (7) or (8) into \([\hat{x}', \hat{p}']\) leads to the modified commutator in (2). There is again an ambiguity in the position operator in (8) since one could have defined the position operator as \( xe^{-\beta \hbar^2 \partial_x^2/2} \) and this would also give the modified commutator in (2).

As a final example of another variant of modified operators that lead to the same modified commutator in (2), we give a variant of the operators from reference [7] which in position space can be written as

\[ \hat{x}' = x , \quad \hat{p}' = -i\hbar \left( 1 - \frac{\beta \hbar^2}{3} \partial_x^2 \right) \partial_x . \] (9)

Again plugging the operators from (9) into \([\hat{x}', \hat{p}']\) leads to (2). The modified operators in (9) leave the position operator unchanged, and the momentum operator is the one which is altered. As before the momentum space version of the operators in (9) can be obtained via \( x \mapsto i\hbar \partial_p \) and \(-i\hbar \partial_x \mapsto p\) which gives

\[ \hat{x}' = i\hbar \partial_p , \quad \hat{p}' = p + \frac{\beta}{3} p^3 . \] (10)

Again by plugging the operators from (10) into \([\hat{x}', \hat{p}']\) results in the modified commutator in (2).

In the above equations, we have collected a series of different modified operators (equations (3), (6), (7), (10) in momentum space, and equations (4), (5), (8), (9) in position space) which all lead to the same modified commutation relationship (2). We now take a look at how the uncertainty relationship plays out for each of these modified set of operators.
III. MODIFIED UNCERTAINTY RELATIONSHIP

We now look at the modified uncertainty principle obtained for the above modified operators (7), (10), and (3) in momentum space. We will find that even though all the modified operators lead to the same modified commutator in equation (2), the uncertainty relationships are all different.

For two generic operators $A$ and $B$, the connection between their uncertainties $\Delta A$ and $\Delta B$ and their commutator $[A, B]$ is given by [13]

$$\Delta A \Delta B \geq \frac{1}{2} |\langle [A, B] \rangle|.$$  

We now apply the relationship in (11) to the various operators in the previous section and show that despite all the operators having the same modified commutator, the uncertainty relationship is qualitatively different for the different operators.

We begin by examining the original KMM modified position and momentum operators of (3) where $\hat{p}' = \hat{p}$, i.e. the momentum operator is not modified. Inserting these into (11) yields

$$\Delta x' \Delta p \geq \frac{\hbar}{2} \left( 1 + \beta (\Delta p)^2 + \beta \langle \hat{p} \rangle^2 \right), \quad (12)$$

where we have used $(\Delta p)^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$. An important point to note is that we have written $\Delta p$ rather than $\Delta p'$ since the momentum operator is the standard one for (3). This is not the case for all the other modified operators. One can write $\Delta x'$ as a function of $\Delta p$ and find that $\Delta x' \sim \frac{1}{\Delta p} + \beta \Delta p$. As shown in [5], this leads to a minimal uncertainty in position of $\Delta x_0 = \hbar \sqrt{\beta}$.

A key point in obtaining this minimal distance, connected with the modified position and momentum operators of (3), is that the momentum operator is simply the regular momentum operator from quantum mechanics. Thus, the $\Delta p$ which appears on the left hand side and right hand side of (12) are the same. This is what allows one to write a direct lower bound on $\Delta x'$. The competition between a decreasing $\frac{1}{\Delta p}$ and an increasing $\Delta p$ term leads a strictly positive lower bound on $\Delta x'$ and a minimum length scale. When the modified momentum operator is different from the usual momentum operator – as is the case in for all the other momentum operators except those in (3) and (4) – then equation (12) is replaced by

$$\Delta x' \Delta p' \geq \frac{\hbar}{2} \left( 1 + \beta (\Delta p)^2 + \beta \langle \hat{p} \rangle^2 \right), \quad (13)$$
where \( p' \) indicates that the \( p' \) (the modified momentum operator) is a function of \( p \) (the standard momentum operator). This function \( \Delta p'_p \) will in general be different for each different modified momentum operator \( \hat{p}' \). Depending on the specific form of \( p'_p \), (13) may still have a global minimum for \( \Delta x' \), or it may have only a local minimum for \( \Delta x' \), or may not lead to a minimum for \( \Delta x' \) at all. For example, if the functional dependence of \( \Delta p'_p \) on \( \Delta p \) is of the form \( \Delta p'_p \propto (\Delta p)^\gamma \) then the relationship in (13) would yield \( \Delta x' \sim \text{const.} + \frac{\beta}{\Delta p'} \). If \( 0 \leq \gamma \leq 2 \) then there exists a minimum length scale, although for \( \gamma = 0 \) this minimum scale occurs at \( \Delta p = 0 \), which is unphysical. Thus the existence or not of a minimum length scale depends crucially on the functional form of \( \Delta p'_p \) in terms of \( \Delta p \), which ultimately depends on the relationship between \( \hat{p}' \) and \( \hat{p} \). In general, (13) can be used to put a lower bound on \( \Delta x' \) of the form

\[
\Delta x' = \frac{\hbar}{2} \left( 1 + \beta (\Delta p)^2 + \beta \langle \hat{p} \rangle^2 \right) \frac{\Delta p'_p}{\Delta p'_p}. \tag{14}
\]

The difficulty in the above approach in finding whether or not there is a minimum length is that it is hard to write down the functional relationship between \( \Delta p'_p \) and \( \Delta p \), i.e. to find \( \Delta p'_p \) as a function of \( \Delta p \). To deal with this issue, we will examine (14) for specific states, \( \ket{\Psi_{\text{test}}} \), which will allow the explicit calculations of \( \Delta p'_p \) and \( \Delta p \). We will then investigate whether or not the generalized uncertainty principle (GUP) (14) gives a minimum length.

### A. Gaussian modified position and momentum

Let us begin by examining the “Gaussian” modified position and momentum operators of (7) of the form \( \hat{x}' = i\hbar e^{-\beta p^2/2} \partial_p \) and \( \hat{p}' = e^{\beta p^2/2} p \). For our test wave function we choose a Gaussian in momentum space

\[
\ket{\Psi_{\text{test}}} = \Psi(p) = C e^{-p^2/2\sigma^2}. \tag{15}
\]

To determine the normalization \( C \), one normally would calculate \( \int \Psi(p)^* \Psi(p) dp = 1 \). However, as pointed out in reference [5], modifying the position operator to the form \( \hat{x}' = i\hbar f(p) \partial_p \) requires modifying the normalization integral as \( \int \Psi(p)^* \Psi(p) \frac{dp}{f(p)} = 1 \); this change is needed in order to make \( \hat{x}' \) symmetric i.e. to have \( \langle \hat{x}' | \hat{x}' \rangle | \Phi \rangle = \langle \Psi | \hat{x}' \rangle | \Phi \rangle \rangle \). Using this modified \( p \)-space integral related to the modified position operator \( \hat{x}' = i\hbar e^{-\beta p^2/2} \partial_p \) (here \( f(p) = e^{-\beta p^2/2} \)), we determine the normalization constant \( C \) in (15) via \( \int \Psi(p)^* \Psi(p) \frac{dp}{f(p)} = \int \Psi(p)^* \Psi(p) dp \int \Psi(p)^* \Psi(p) dp = \frac{\hbar}{2} \left( 1 + \beta (\Delta p)^2 + \beta \langle \hat{p} \rangle^2 \right) \frac{\Delta p'_p}{\Delta p'_p} = \text{const.} + \frac{\beta}{\Delta p'} \).
\[ 1 \Rightarrow |C|^2 \int_{-\infty}^{\infty} e^{-p^2/a^2} \, dp = 1 \Rightarrow C = \frac{1}{(\sqrt{\pi a})^{1/2}} \] which yields

\[
|\Psi_{\text{test}}\rangle = \Psi(p) = \frac{1}{(\sqrt{\pi a})^{1/2}} e^{-p^2/2a^2} \quad \text{with} \quad \frac{1}{a^2} = 1 - \frac{\beta}{2}.
\] (16)

Note that both \( \sigma \) and \( \beta \) are involved in the normalization of \( |\Psi_{\text{test}}\rangle \). These test states are normalizable only when \( \sigma < \sqrt{2/\beta} \) due to the interplay between the fall off of the Gaussian test states and the exponential growth of \( 1/f(p) \) in the integral. We will use this \( \Psi(p) \) to calculate \( \Delta p \) and \( \Delta p' \), and then use these in (14) to check if this leads to a minimum in \( \Delta x' \). First, recall that \( \Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2 \). Using the modified product we have \( \langle p^2 \rangle = \int p^2 \Psi(p)^* \Psi(p) e^{\beta p^2/2} \, dp \) and \( \langle p \rangle = \int p \Psi(p)^* \Psi(p) e^{\beta p^2/2} \, dp \). Using \( \Psi(p) \) from (16) we find \( \langle p^2 \rangle = \frac{a^2}{2} \) and \( \langle p \rangle = 0 \). Putting this together yields

\[
\Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2 = \frac{a^2}{2} = \frac{1}{2} \left( \frac{1}{\sigma^2} - \frac{\beta}{2} \right)^{-1}.
\] (17)

As a check we note that in the limit of ordinary quantum mechanics, \( \beta \to 0 \), we recover the usual result \( \Delta p^2 = \frac{a^2}{2} \).

We now turn to calculate \( \Delta p'^2 = \langle p'^2 \rangle - \langle p' \rangle^2 \) as a function of \( \Delta p \sim \sigma \). We recall that the expectations \( \langle p^2 \rangle \) and \( \langle p' \rangle \) are carried out with respect to the modified measure \( i.e. \langle p' \rangle = \int p' \Psi(p)^* \Psi(p) \frac{dp}{e^{\beta p^2/2}} \) and \( \langle p^2 \rangle = \int p^2 \Psi(p)^* \Psi(p) \frac{dp}{e^{\beta p^2/2}} \). It is easy to see from symmetry that, using \( \Psi(p) \) from (16), gives \( \langle p' \rangle = 0 \). Next we calculate

\[
\langle p^2 \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dp}{e^{\beta p^2/2}} \frac{1}{\sqrt{\pi a}} e^{-p^2/2a^2} \left( e^{\beta p^2/2} \right)^2 e^{-p^2/2a^2} = \frac{b^3}{2a}, \quad (18)
\]

where \( a \) is defined in (16) as \( a = \left( \frac{1}{\sigma^2} - \frac{\beta}{2} \right)^{-1/2} \) and \( b \) is given by

\[
\frac{1}{b^2} = \left( \frac{1}{\sigma^2} - \frac{3\beta}{2} \right) \quad \Rightarrow \quad b = \left( \frac{1}{\sigma^2} - \frac{3\beta}{2} \right)^{-1/2} \quad (19)
\]

Putting all this together gives \( \Delta p' \) as

\[
\Delta p' = \sqrt{\langle p'^2 \rangle} = \sqrt{\frac{b^3}{2a}} = \frac{1}{\sqrt{2}} \left( \frac{1}{\sigma^2} - \frac{3\beta}{2} \right)^{-3/4} \left( \frac{1}{\sigma^2} - \frac{\beta}{2} \right)^{1/4}.
\] (20)

Substituting \( \Delta p \) from (17) and \( \Delta p' \) from (20) (and recalling that \( \langle p' \rangle = 0 \)) in equation (14) give the following lower bound on \( \Delta x' \):

\[
\Delta x' \geq \frac{\hbar}{\sigma^2 \sqrt{2}} \left( \frac{1}{\sigma^2} - \frac{3\beta}{2} \right)^{3/4} \left( \frac{1}{\sigma^2} - \frac{\beta}{2} \right)^{-5/4}.
\] (21)
FIG. 1: Plot of $\Delta x'$ versus the spread in momentum $\Delta p \sim \sigma$ for the modified operators from [7] with $\beta = 0.1$. The dashed curved is $\Delta x'$ obtained from GUP using the equality in (21) which gives a lower limit to $\Delta x'$. The solid curve is $\Delta x'$ as calculated directly from the test wave function and the modified position operator as given in (22).

The $\Delta x'$ with the equality from (21) is plotted as a function of width of the spread in the momentum, $\sigma$, via the dashed curve in Fig. 1 (in this and all plots we set $\hbar = 1$). The dashed curve provides a lower limit to $\Delta x'$. We note that from (20) and (21), one can see that $\Delta p' \rightarrow \infty$ and $\Delta x' \rightarrow 0$ as $\sigma \rightarrow \sqrt{2/3\beta}$, which is less than $\sigma = \sqrt{2/\beta}$, the condition to have valid, normalizable states. Fig. 1 clearly shows that for the modified operators from [7] there is no minimal length scale, in contrast to what happens for the modified position and momentum operators of reference [5] in equation (3). This is despite the fact that both the modified operators in (3) and those in (7) give the same modified commutator in (2).

We can also use the test wave function to directly calculate $\Delta x' = \sqrt{\langle (\hat{\Delta} x')^2 \rangle - \langle x' \rangle^2} = \sqrt{\langle (\hat{\Delta} x')^2 \rangle}$ where we have used that $\langle \Delta x' \rangle = 0$ for all the cases of modified position operator. For the modified position operator in (7), a straightforward calculation of $\Delta x'$ with the test wave function (15) gives

$$\Delta x' = \frac{\hbar}{\sigma^2 \sqrt{2}} \left( \frac{1}{\sigma^2} - \frac{\beta}{2} \right)^{1/4} \left( \frac{1}{\sigma^2} + \frac{\beta}{2} \right)^{-3/4}. \quad (22)$$

The plot of equation (22) is given by the solid curve in Fig. 1. As expected the shape of
solid curve in Fig. 1 is similar to the dashed curve, with the dashed curve coming from saturation of the GUP in [21], providing a lower limit on $\Delta x'$.

B. Unmodified position and modified momentum

Next we examine how the above issues play out for the modified operators from [10] where $\hat{x}' = \hat{x} = i\hbar \partial_p$ and $\hat{p}' = p + \frac{\beta}{3}p^3$. Since, here $\hat{x}'$ is just the standard position operator, we use the standard inner product to maintain the symmetry of the operators. The constant $C$ in [15] is the normalization factor with this standard inner product. Going through the standard calculation gives $C$ via
\[
\langle \Psi_{\text{test}} \rangle = \frac{1}{(\sqrt{\pi} \sigma)^{1/2}} e^{-p^2/2\sigma^2}.
\]

Thus in this case $|\Psi_{\text{test}}\rangle = \frac{1}{(\sqrt{\pi} \sigma)^{1/2}} e^{-p^2/2\sigma^2}$. We now need to calculate $\Delta p$ and $\Delta p'$. Since $\hat{x}'$ is just the standard position operator we do not change the integration measure here and thus $\Delta p^2 = \langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2$ will just be the standard result for a Gaussian test wave function. Briefly, we find that $\langle \hat{p} \rangle = 0$ and $\langle \hat{p}^2 \rangle = \frac{\sigma^2}{2}$ for $|\Psi_{\text{test}}\rangle = \frac{1}{(\sqrt{\pi} \sigma)^{1/2}} e^{-p^2/2\sigma^2}$. Thus $\Delta p^2 = \langle \hat{p}^2 \rangle = \frac{\sigma^2}{2}$, which is the standard result from standard quantum mechanics. Next $\Delta p'_p$ is given by
\[
\Delta p'^2_p = \left\langle \left( \hat{p} + \frac{\beta}{3} p^3 \right)^2 \right\rangle - \left\langle \left( \hat{p} + \frac{\beta}{3} p^3 \right) \right\rangle^2 = \left\langle \hat{p}^2 + \frac{2\beta}{3} p^4 + \frac{5}{9} \beta^2 p^6 \right\rangle.
\]

(23)

The second term in (23) $\langle (...) \rangle^2 = 0$ by symmetry for $|\Psi_{\text{test}}\rangle = \frac{1}{(\sqrt{\pi} \sigma)^{1/2}} e^{-p^2/2\sigma^2}$. The remaining term involves taking moments of $\hat{p}^n$ with $n = 2, 4, 6$ again using $|\Psi_{\text{test}}\rangle = \frac{1}{(\sqrt{\pi} \sigma)^{1/2}} e^{-p^2/2\sigma^2}$. Carrying out the standard integrations gives
\[
\Delta p'^2_p = \left\langle \hat{p}^2 + \frac{2\beta}{3} p^4 + \frac{5}{9} \beta^2 p^6 \right\rangle = \frac{1}{2} \sigma^2 + \frac{\beta}{2} \sigma^4 + \frac{5\beta^2}{24} \sigma^6.
\]

(24)

Putting all this together in the uncertainty relation (14) gives for this set of modified operators
\[
\Delta x \geq \frac{\hbar}{2} \left( \frac{1 + \beta(\Delta p)^2 - \beta \langle \hat{p} \rangle^2}{\Delta p'_p} \right) = \frac{\hbar(1 + \beta \sigma^2 / 2)}{\sigma \sqrt{2 + 2\beta \sigma^2 + 5\beta^2 \sigma^4 / 6}}.
\]

(25)

Since the position operator in this case is the standard one from quantum mechanics we have $\Delta x' = \Delta x$. In Fig. 2 we have plotted $\Delta x$ versus the width of the momentum, $\sigma$ for the equality in (25). From Fig. 2 we see that for the modified operators from [10] there is no minimum in $\Delta x$, despite the fact that the modified operators from [10] lead to the same modified commutator as the modified operators from [3]. As for the modified operators of
FIG. 2: Plot of $\Delta x$ from the modified operators from (10) for $\beta = 0.1$. The dashed curve is $\Delta x$ obtained from the GUP, using the equality of (25). The solid line represents $\Delta x = \frac{1}{\sqrt{2\sigma}}$ obtained from a direct calculation using the position operator in (10) and the test wavefunction (15).

the previous subsection, one can also calculate $\Delta x$ directly from the position operator in (10) and the Gaussian test wavefunction. Since here the position operator is just the usual one from standard quantum mechanics and the measure of the momentum integration is not altered, the spread in the position operator is just that of standard quantum mechanics for a Gaussian wave function, namely $\Delta x = \frac{1}{\sqrt{2\sigma}}$. This direct calculation of $\Delta x$ is plotted versus $\sigma$ by the solid curve. As previously we see that there is no minimum for $\Delta x$, and that the GUP calculated $\Delta x$ (the dashed curve) provides a lower bound on $\Delta x$.

C. Modified position and unmodified momentum

In this subsection, as a check, we carry out the above procedure for the modified position and momentum operators from [5] in equation (3) for the test wave function $|\Psi_{test}\rangle = Ce^{-p^2/2\sigma^2}$. We find that for these modified operators one does obtain a non-zero minimum in agreement with the general analysis of [5]. The normalization constant is determined by $\int_{-\infty}^{\infty} \frac{dp}{1 + \beta p^2} |C|^2 e^{p^2/2\sigma^2} = 1$, which gives $|C|^2 e^{1/\sigma^2 \beta} \frac{1}{\sqrt{\beta}} \text{Erfc}(1/\sqrt{\sigma^2 \beta}) = 1$, where Erfc is the complementary error function. Explicitly $C = e^{-1/2\sigma^2 \beta} \left[ \frac{1}{\sqrt{\beta}} \text{Erfc}(1/\sqrt{\sigma^2 \beta}) \right]^{-1/2}$. We want to
calculate $\Delta p_p' = \Delta p^2 = \langle p^2 \rangle - \langle p \rangle^2$. We have written $\Delta p_p' = \Delta p$ since, here, the momentum operator is just the same as in standard quantum mechanics. Now $\langle p \rangle = |C|^2 \int_{-\infty}^{\infty} \frac{pe^{-p^2/\sigma^2}}{1 + \beta p^2} dp$. By symmetry $\langle p \rangle = 0$ for $|\Psi_{\text{test}}\rangle = Ce^{-p^2/2\sigma}$. For $\langle p^2 \rangle$ we have

$$\langle p^2 \rangle = |C|^2 \int_{-\infty}^{\infty} \frac{p^2 e^{-p^2/\sigma^2}}{1 + \beta p^2} dp = \frac{\sigma}{\sqrt{\beta \pi}} e^{-1/\sigma^2 \beta} (\text{Erfc}(1/\sigma \sqrt{\beta})^{-1} - \frac{1}{\beta}),$$  

where we have used the explicit expression for $C$. Equation (26) gives $\Delta p_p'^2 = \Delta p^2 = \langle p^2 \rangle = \frac{\sigma}{\sqrt{\beta \pi}} e^{-1/\sigma^2 \beta} (\text{Erfc}(1/\sigma \sqrt{\beta})^{-1} - \frac{1}{\beta}$. Since here the momentum operator is unaltered from the usual momentum operator from quantum mechanics we get $\Delta x'$ from (12) as $\Delta x' = \frac{\hbar}{2} \left( \frac{1}{\Delta p} + \beta \Delta p \right)$. Using the expression for $\Delta p_p' = \Delta p$ that we obtained above from (26) we obtain a not obviously enlightening and complex expression for $\Delta x'$. We have plotted this $\Delta x'$ as a function of the spread in momentum , $\sigma$, in Fig. 3 with the dashed curve. Fig. 3 shows that in this case there is a minimum $\Delta x'$ in agreement with the general analysis from [5].

On can also directly calculate $\Delta x'$ for the modified operators in (3) for the Gaussian test
wave function. A straightforward calculation for this case yields

$$\Delta x' = \frac{1}{\sqrt{2}} \left[ \frac{1}{\sigma} + \frac{3\beta \sigma}{2} \left( e^{1/\sigma^2 \beta} \sqrt{\frac{\pi}{\beta}} E_{\text{Erfc}}(1/\sigma \sqrt{\beta}) \right)^{-1/2} \right]^{1/2}. \quad (27)$$

This direct calculation of $\Delta x'$ from (27) is plotted in Fig. 3 as the solid line. We see again that, as expected, there is a minimum $\Delta x'$ in this case. Also the GUP calculation of $\Delta x'$ in this case gives the lower limit on $\Delta x'$.

IV. SUMMARY AND CONCLUSIONS

We have shown that the existence of a minimal length in a GUP depends chiefly on how the position and momentum operators are modified rather than how the commutator is modified. Equations (3), (6), (7), and (10) give various versions of modified position and momentum operators in momentum space which lead to exactly the same modified commutator in (2). Only the modified operators in (3) lead to a minimum length scale. The reason for this difference lies in equation (14) which gives the relationship between position uncertainty, $\Delta x'$, and the momentum uncertainty $\Delta p'$. The numerator of (14) always has the standard momentum uncertainty, $\beta \Delta p^2$, but the denominator varies greatly depending on how one modifies the momentum operator.

The special feature of the modified position and momentum operators in (3) that lead to a minimum non-zero length is that only the position operator is modified while the momentum operator is not. This means that $\Delta p' = \Delta p$ which makes the GUP of the modified operators in (3) simple. In this case, $\Delta x'$ is bounded below by an expression of $\Delta p$ which has a strictly positive minimum. In the other cases, equations (7) and (10), the momentum operator is modified and it is not possible to obtain a general, analytical relationship between $\Delta p'$ and $\Delta p$. For the modified operators from (7) and (10), we selected a family of test states (Gaussians in momentum space with a width $\sigma$ given in equation (15)) and then explicitly calculated the relationship between the uncertainty in position, $\Delta x'$, and the uncertainty in momentum characterized by the parameters $\sigma$ and $\beta$. If the test functions have no positive minimum for $\Delta x'$, then there is no minimum length scale for the position operator in general. It follows from these examples that the existence of such a minimum depends crucially on how one modifies the operators, more so than how the commutator is modified.

The results for the modified operators from (7) and (10) are plotted in Figs. 1 and 2.
There is no non-zero minimum in Figs. 1 and 2. There is a significant difference between
the results in Figs. 1 and 2: in Fig. 1, $\Delta x'$ goes to zero in finite $\sigma$ while in Fig. 2, $\Delta x'$
goes to zero in the limit $\sigma \to \infty$. As a check, we applied the same test wave function to the
modified operators of (3). The results are given in Fig. 3 and in this case do show a positive
minimum in $\Delta x'$. In section III, we used the GUP to determine the relationship between
$\Delta x'$ and the uncertainty in the momentum using the test Gaussian wave function in (15).
Due to the “greater than or equal” nature of the GUP, this provides a upper limit on $\Delta x'$.

In conclusion, the specific modification of operators is crucial to obtaining a positive
minimum length scale; the modified commutator is not generally sufficient for obtaining a
minimum length scale. From the analysis of (10), the position operator needs to modified
in order to obtain a minimum length scale. For any unmodified position operator, the $\Delta x$
of the Gaussian test functions will go to zero as the width goes to infinity and that means
there is no positive minimum length scale. Whether one needs to modify the momentum
operator to obtain a minimum length is not clear. The modified operators in (3) lead to a
minimum length without changing the momentum operator. The two examples of modified
operators where the momentum operator is changed ((7) and (9)) do not lead to a minimum
length scale. However, careful analysis of (14) may lead to examples where both the position
and momentum operators are modified and give a minimum length scale. We can only say
that we did not find a case where modifying the momentum operator led to a minimum
length. This phenomenon, of the minimal length scale depending on the specific form of
the modified operators, can be used to constrain the form of the operators one expects from
quantum gravity. Can one modify the momentum operator and have a minimal length scale?
If so, what forms can this modification take? What are the constraints on how the position
operator is modified in order to obtain a minimum length scale? In future work, we plan to
examine these constraints on the modification of the operators with the requirement that
the modification leads to a minimal length scale.

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