Sharp endpoint $L^p$ estimates for Schrödinger groups

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Abstract
Let $L$ be a non-negative self-adjoint operator acting on $L^2(X)$ where $X$ is a space of homogeneous type with a dimension $n$. Suppose that the heat operator $e^{-tL}$ satisfies the generalized Gaussian $(p_0, p_0')$-estimates of order $m$ for some $1 \leq p_0 < 2$. In this paper we prove sharp endpoint $L^p$-Sobolev bound for the Schrödinger group $e^{itL}$, that is for every $p \in (p_0, p_0')$ there exists a constant $C = C(n, p) > 0$ independent of $t$ such that

$$\left\| (I + L)^{-s} e^{itL} f \right\|_p \leq C (1 + |t|)^s \| f \|_p, \quad t \in \mathbb{R}, \quad s \geq n \left| \frac{1}{2} - \frac{1}{p} \right|.$$

As a consequence, the above estimate holds for all $1 < p < \infty$ when the heat kernel of $L$ satisfies a Gaussian upper bound. This extends classical results due to Fefferman and Stein, and Miyachi for the Laplacian on the Euclidean spaces $\mathbb{R}^n$. We also give an application to obtain an endpoint estimate for $L^p$-boundedness of the Riesz means of the solutions of the Schrödinger equations.

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1 Introduction

1.1 Background

Consider the Laplace operator $\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ on the Euclidean space $\mathbb{R}^n$ and the Schrödinger equation

$$
\begin{cases}
i \partial_t u + \Delta u = 0, \\
u|_{t=0} = f
\end{cases}
$$

with initial data $f$. Its solution can be written as

$$u(x, t) = e^{it\Delta} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i\langle x, \xi \rangle + |t||\xi|^2} d\xi$$

where $\hat{f}$ denotes the Fourier transform of $f$. It is well-known that the operator $e^{it\Delta}$ acts boundedly on $L^p(\mathbb{R}^n)$ only if $p = 2$; see Hörmander [22]. For $p \neq 2$, it was shown (see for example, [7,26,39]) that for $s > n|1/2 - 1/p|$, the operator $e^{it\Delta}$ maps the Sobolev space $L^p_{2s}(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. Equivalently, this means that $(I + \Delta)^{-s} e^{it\Delta}$ is bounded on $L^p(\mathbb{R}^n)$, and this is not the case if $0 < s < n|1/2 - 1/p|$. The sharp endpoint $L^p$-Sobolev estimate is due to Miyachi [32,33], which states that for every $p \in (1, \infty)$,

$$\left\| (1 + \Delta)^{-s} e^{it\Delta} f \right\|_{L^p(\mathbb{R}^n)} \leq C (1 + |t|)^s \| f \|_{L^p(\mathbb{R}^n)}, \quad t \in \mathbb{R}, \quad s = n \left| \frac{1}{2} - \frac{1}{p} \right|$$

for some positive constant $C = C(n, p)$ independent of $t$. The estimate (1.1) is sharp in another way: the factor $(1 + |t|)^s$ can not be improved (see [32, pp. 169–170]). See also Feffermann and Stein’s work [19]. These results and their generalizations were in fact results on multipliers and relied heavily on Fourier analysis. See, for example, Ouhabaz’s monograph [34, Chapter 7] for historical background and more study on the Schrödinger groups.

The purpose of this paper is to establish such sharp endpoint $L^p$ estimate (1.1) for the operators $(e^{itL})_{t \in \mathbb{R}}$ for a large class of non-negative self-adjoint operators acting on $L^2(X)$ on a metric measure space $X$. Such an operator $L$ admits a spectral resolution

$$Lf = \int_{0}^{\infty} \lambda dE_L(\lambda) f, \quad f \in L^2(X),$$

where $E_L(\lambda)$ is the projection-valued measure supported on the spectrum of $L$. The operator $e^{itL}$ is defined by

$$e^{itL} f = \int_{0}^{\infty} e^{it\lambda} dE_L(\lambda) f$$

for \( f \in L^2(X) \), and forms the Schrödinger group. By the spectral theorem [31], the operator \( e^{itL} \) is continuous on \( L^2(X) \). It is interesting to investigate \( L^p \)-mapping properties for the Schrödinger group \( e^{itL} \) on \( L^p(X) \) for some \( p, 1 \leq p \leq \infty \).

As an application of our sharp endpoint \( L^p \) estimate for the Schrödinger group \( e^{itL} \), we also aim to obtain an endpoint estimate for \( L^p \)-boundedness of the Riesz means of the solutions of the Schrödinger equations.

### 1.2 Assumptions and main results

Throughout the paper we assume that \( X \) is a metric space, with distance function \( d \), and \( \mu \) is a nonnegative, Borel doubling measure on \( X \). We say that \( (X, d, \mu) \) satisfies the doubling property (see [11, Chapter 3]) if there exists a constant \( C > 0 \) such that

\[
V(x, 2r) \leq CV(x, r) \quad \forall r > 0, \ x \in X.
\]

Note that the doubling property implies the following strong homogeneity property,

\[
V(x, \lambda r) \leq C\lambda^n V(x, r)
\]

for some \( C, n > 0 \) uniformly for all \( \lambda \geq 1 \) and \( x \in X \). In Euclidean space with Lebesgue measure, the parameter \( n \) corresponds to the dimension of the space.

There also exist \( c \) and \( D, 0 \leq D \leq n \) such that

\[
V(y, r) \leq c \left(1 + \frac{d(x, y)}{r}\right)^D V(x, r)
\]

uniformly for all \( x, y \in X \) and \( r > 0 \). Indeed, the property (1.6) with \( D = n \) is a direct consequence of triangle inequality of the metric \( d \) and the strong homogeneity property. In the cases of Euclidean spaces \( \mathbb{R}^n \) and Lie groups of polynomial growth, \( D \) can be chosen to be 0.

Consider a non-negative self-adjoint operator \( L \) and numbers \( m \geq 2 \) and \( 1 \leq p_0 \leq 2 \). We say that the semigroup \( e^{-tL} \) generated by \( L \), satisfies the generalized Gaussian \((p_0, p_0', m)\)-estimate of order \( m \), if there exist constants \( C, c > 0 \) such that

\[
\left\| P_B(x, t^{1/m}) e^{-tL} P_B(y, t^{1/m}) \right\|_{p_0 \to p_0'} \leq CV(x, t^{1/m}) \left( \frac{1}{p_0} - \frac{1}{p_0'} \right) \exp \left( -c \left( \frac{d(x, y)^m}{t} \right)^{\frac{1}{m-1}} \right) \quad (GGE_{p_0, p_0', m})
\]

for every \( t > 0 \) and \( x, y \in X \).

Note that condition \((GGE_{p_0, p_0', m})\) for the special case \( p_0 = 1 \) is equivalent to \( m \)-th order Gaussian estimates (see for example, [6]). This means that the semigroup \( e^{-tL} \) has integral kernels \( p_t(x, y) \) satisfying the following Gaussian upper estimate:
\[ |p_t(x, y)| \leq \frac{C}{V(x, t^{1/m})} \exp \left( -c \left( \frac{d(x, y)^m}{t} \right)^{\frac{1}{m-1}} \right) \quad \text{(GE}_m) \]

for every \( t > 0, x, y \in X \), where \( c, C \) are two positive constants and \( m \geq 2 \). Such estimate \((\text{GE}_m)\) is typical for elliptic or sub-elliptic differential operators of order \( m \) (see for example, \([1,2,9,13,16,17,20,23,24,34,38,39,43]\) and the references therein). However, there are numbers of operators which satisfy generalized Gaussian estimates and, among them, there exist many for which classical Gaussian estimates \((\text{GE}_m)\) fail. This happens, e.g., for Schrödinger operators with rough potentials \([36]\), second order elliptic operators with rough lower order terms \([28]\), or higher order elliptic operators with bounded measurable coefficients \([14]\). See also \([4-6,10,25,37]\).

Our main result is that under the generalized Gaussian estimate \((\text{GGE}_{p_0,p'_0,m})\) for some \(1 \leq p_0 < 2\), it is sufficient to ensure that such estimate \((1.1)\) holds for the operator \( (e^{itL})_{t \in \mathbb{R}} \) for \( p \in (p_0, p'_0) \). Our result can be stated as follows.

**Theorem 1.1** Suppose that \((X, d, \mu)\) is a space of homogeneous type with a dimension \( n \). Suppose that \( L \) satisfies the property \((\text{GGE}_{p_0,p'_0,m})\) for some \(1 \leq p_0 < 2\). Then for every \( p \in (p_0, p'_0) \), there exists a constant \( C = C(n, p) > 0 \) independent of \( t \) such that

\[ \left\| (I + L)^{-s} e^{itL} f \right\|_p \leq C(1 + |t|)^s \| f \|_p, \quad t \in \mathbb{R}, \quad s \geq n \left\lfloor \frac{1}{2} - \frac{1}{p} \right\rfloor. \quad (1.7) \]

As a consequence, this estimate \((1.7)\) holds for all \(1 < p < \infty\) when the heat kernel of \( L \) satisfies a Gaussian upper bound \((\text{GE}_m)\).

As a consequence of Theorem 1.1, we have the following result.

**Corollary 1.2** Suppose that \((X, d, \mu)\) is a homogeneous space with a dimension \( n \). Suppose that \( L \) satisfies the property \((\text{GGE}_{p_0,p'_0,m})\) for some \(1 \leq p_0 < 2\). Then for every \( p \in (p_0, p'_0) \) and \( s \geq n|1/2 - 1/p| \), the mapping \( t \mapsto (I + L)^{-s} e^{itL} \) is strongly continuous on \( L^p(X) \).

We now apply the result of Theorem 1.1 to study the property of the solution to the Schrödinger equation

\[
\begin{align*}
  i \partial_t u + Lu &= 0, \\
  u(\cdot, 0) &= f.
\end{align*}
\]

Then we have

\[ u(t, x) = e^{itL} f(x). \]

One can see that the operator \( e^{itL} \) is bounded on \( L^p \) only for \( p = 2 \). Following Sjöstrand \([39]\), we define the Riesz means

\[ I_s(t)(L) := st^{-s} \int_0^t (t - \lambda)^{s-1} e^{-i\lambda L} d\lambda. \quad (1.9) \]

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for \( t > 0 \), and \( I_s(t)(L) = \tilde{T}_s(-t)(L) \) for \( t < 0 \) (see also [3, 21]), and ask the question: For what values of \( s \) the operators \( I_s(t)(L) \) are bounded on \( L^p(X) \)?

Then we have the following result.

**Theorem 1.3** Suppose that \((X, d, \mu)\) is a space of homogeneous type with a dimension \( n \). Suppose that \( L \) satisfies the property \((\text{GGE}_{p_0, p'_0, m})\) for some \( 1 \leq p_0 < 2 \). Then for every \( p \in (p_0, p'_0) \), there exists a constant \( C = C(n, p) > 0 \) independent of \( t \) such that

\[
\|I_s(t)(L)f\|_p \leq C\|f\|_p, \quad t \in \mathbb{R}\{0\}, \quad s \geq n\left|\frac{1}{2} - \frac{1}{p}\right|. \tag{1.10}
\]

As a consequence, this estimate (1.10) holds for all \( 1 < p < \infty \) when the heat kernel of \( L \) satisfies a Gaussian upper bound \((\text{GE}_m)\).

It is known that such estimate (1.10) holds due to Sjöstrand [39] for the Laplacian \(-\Delta \) on \( \mathbb{R}^n \) ([39]); see also Thangavelu’s work [42] for the harmonic oscillator \(-\Delta + |x|^2 \) on \( \mathbb{R}^n \).

The proof of Theorem 1.1 and Corollary 1.2 will be given in Sect. 3. The proof of Theorem 1.3 will be given in Sect. 4.

**1.3 Comments on the results and methods of the proof**

On Lie groups with polynomial growth and manifolds with non-negative Ricci curvature, similar results as in (1.1) for \( s > n|1/2 - 1/p| \) have been first announced by Lohoué in [29], then Alexopoulos obtained them in [1]. There, the method is to replace Fourier analysis by the finite propagation speed of the associated wave equation [41]. In the abstract setting of operators on metric measure spaces, Carron, Coulhon and Ouhabaz [9] showed \( L^p \)-boundedness of suitable regularizations of the Schrödinger group \( e^{itL} \) provided \( L \) satisfies Gaussian estimate \((\text{GE}_m)\). They proposed a different approach to use some techniques introduced by Davies [13]: the Gaussian semigroup estimates can be extended from real times \( t > 0 \) to complex times \( z \in \mathbb{C}^+ = \{z \in \mathbb{C} : \text{Re} \, z > 0\} \) such that

\[
\|e^{-zL}\|_{p \to p} \leq C \left(\frac{|z|}{\text{Re} \, z}\right)^{n\left|\frac{1}{2} - \frac{1}{p}\right| + \epsilon}, \quad \forall z \in \mathbb{C}^+. \tag{1.11}
\]

On the other hand, for every \( f \in L^2 \cap L^p \) and \( s \geq 0 \),

\[
(I + L)^{-s}e^{itL}f = \frac{1}{\Gamma(s)} \int_0^\infty e^{-u}u^{s-1}e^{-(u-it)L}f \, du,
\]

where \( \Gamma \) is the Euler Gamma function. From (1.11), we see that for \( s > n|1/2 - 1/p| \),

\[
\|(I + L)^{-s}e^{itL}\|_{p \to p} \leq C \int_0^\infty e^{-u}u^{s-1}\left(\sqrt{\frac{u^2 + t^2}{u^2}}\right)^{n\left|\frac{1}{2} - \frac{1}{p}\right| + \epsilon} \, du \tag{1.12}
\]
and so (1.7) holds for $s > n|1/2 - 1/p|$. The Gaussian bound (GE$_m$) assumption on $L$ was further weakened to the generalized Gaussian estimates (GGE$_{p_0,p'_0,m}$) by Blunck [4, Theorem 1.1] where the estimate (1.11) was improved to get $\epsilon = 0$, i.e.

$$
\| e^{-zL} \|_{p \to p} \leq C \left( \frac{|z|}{\text{Re } z} \right)^{n \frac{|1/2 - 1/p|}{2}}, \quad \forall z \in \mathbb{C}^+
$$

(1.13) for all $p \in [p_0, p'_0]$ with $p \neq \infty$, and so (1.7) holds for $s > n|1/2 - 1/p|$. However, it is direct to see that the integral in (1.12) is $\infty$ when $s = n|1/2 - 1/p|$.

It was an open question whether estimate (1.7) holds with $s = n|1/2 - 1/p|$. Based on estimate (1.13), it is straightforward to obtain sharp $L^p$ frequency truncated estimates for $e^{itL}$ that for every $p \in (p_0, p'_0)$ and $k \in \mathbb{Z}^+$,

$$
\| e^{itL} \phi(2^{-k}L) f \|_p \leq C (1 + 2^k |t|)^s \| f \|_p, \quad t \in \mathbb{R}, \quad s = n \left| \frac{1}{2} - \frac{1}{p} \right|
$$

(1.14) uniformly for $\phi$ in bounded subsets of $C_0^\infty(\mathbb{R})$, by writing

$$
e^{itL} \phi(2^{-k}L) f = e^{-(2^{-k} - it)L} [\phi e(2^{-k}L)](f)
$$

where $\phi_e(\lambda) = e^{\lambda} \phi(\lambda)$ and then applying (1.13) to $e^{-(2^{-k} - it)L}$ and [5, Theorem 1.1] to $\phi e(2^{-k}L)$, respectively (for more details, see Proposition 3.1 below). As a consequence of (1.14), it follows by a standard scaling argument [23, p. 193] that for every $p \in (p_0, p'_0)$ and for every $\epsilon > 0$,

$$
\| (I + L)^{-s-\epsilon} e^{itL} f \|_p \leq C (1 + |t|)^s \| f \|_p, \quad t \in \mathbb{R}, \quad s = n \left| \frac{1}{2} - \frac{1}{p} \right|
$$

(1.15)

We would like to mention that in [12], D’Ancona and Nicola used a commutator argument and a reduction to amalgam spaces and followed the methods of Jensen-Nakamura [23,24] to obtain estimates (1.14) and (1.15) for the Schrödinger group $e^{itL}$ for $p \in (p_0, p'_0)$ in the Euclidean spaces $\mathbb{R}^n$.

However, as in [12, p. 1021], the authors remarked that “Another interesting issue is the validity of (1.15) with $\epsilon = 0$. Indeed, for $L = -\Delta$ in $\mathbb{R}^n$ and $1 < p < \infty$, the estimate (1.15) was proved with $\epsilon = 0$ (and $t = 1$) in [33], but this sharp form seems out of reach in the present generality, even for fixed $t$”. Under an additional condition which is the operator $e^{itL}$ being bounded in suitable modulation spaces (see [12, Section 5] for the definition), it was proved in [12] that estimate (1.15) holds with $\epsilon = 0$ in the setting of $\mathbb{R}^n$. See also previous related results [8,23,24].

Our main result, Theorem 1.1, gives the sharp endpoint estimate (1.15) for the Schrödinger group $e^{itL}$ with $\epsilon = 0$, namely with the optimal number of derivatives and the optimal time growth for the factor $(1 + |t|)^s$ in (1.15). The proof of Theorem 1.1 is different from those of Fefferman and Stein [19] and Miyachi [32,33] where the results rely heavily on Fourier analysis. In our setting, we do not have Fourier transform at our disposal. We also do not assume that the heat kernel $p_t(x,y)$ satisfies the...
standard regularity condition, thus standard techniques of Calderón–Zygmund theory [40] are not applicable. The lack of smoothness of the kernel will be overcome in Proposition 2.3 below by using some off-diagonal estimates on heat semigroup of non-negative self-adjoint operators, and some techniques in the theory of singular integrals with rough kernels, which lies beyond the scope of the standard Calderón–Zygmund theory (see for example, [2,5,6,10,16–18,25,34,37] and the references therein). More specifically, by duality we are reduced to prove the estimate for $2 < p < p'_0$, which will follow by the Littlewood–Paley inequality and a variant of the Fefferman–Stein sharp function (see [2,18,19,30,37]),

$$\|e^{itL}f\|_p \leq C\|T\varphi f\|_p \leq C\|\mathcal{M}_2(\{|T\varphi f|\})\|_p \leq C_p \left(\|\mathcal{M}_{T\varphi,L,K}^\# f\|_p + \|f\|_p\right),$$

(1.16)

where

$$T\varphi f(x) = \left(\sum_{k \geq 0} |\varphi_k(L)e^{itL}f(x)|^2\right)^{1/2}$$

(1.17)

for some cut-off function $\varphi \in C_0^\infty([1/2, 2])$, where $\varphi_k(\lambda) = \varphi(2^{-k}\lambda)$, $k \geq 1$ and $\varphi_0(\lambda) + \sum_{k \geq 1} \varphi_k(\lambda) \equiv 1$ for $\lambda > 0$, and for a large $K \in \mathbb{N}$,

$$\mathcal{M}_{T\varphi,L,K}^\# f(x) = \sup_{B \ni x} \left(\frac{1}{|B|} \int_B |T(I - e^{-r_n^L}L_K f(y)|^2 d\mu(y)\right)^{1/2}.$$ (1.18)

We then use a variant of an argument in [27,35] to decompose the function $\mathcal{M}_{T\varphi,L,K}^\# f$ into several components so that we can employ the off-diagonal estimates (1.20) below. Then we show that the function $\mathcal{M}_{T\varphi,L,K}^\# f$ is in $L^p$ by using estimate (1.14) for the Schrödinger group $e^{itL}$. We note that in the case that $L$ is the Laplace operator $\Delta$ on $\mathbb{R}^n$, the kernel estimate relies heavily on Fourier analysis since the operator $e^{it\Delta} \varphi(2^{-k}\Delta)$ has the convolution kernel

$$K_{e^{it\Delta}\varphi(2^{-k}\Delta)}(x) = \frac{2^{kn/2}}{(2\pi)^n} \int_{\mathbb{R}^n} \varphi(|\xi|^2)e^{i(2^{k/2}(x, \xi) + 2^{k}t|\xi|^2)}d\xi,$$

one then uses integration by parts to obtain that for every $M > 0$,

$$|K_{e^{it\Delta}\varphi(2^{-k}\Delta)}(x)| \leq C2^{kn/2}(1 + 2^{k/2}|x|)^{-M}$$

(1.19)

whenever $|x| \geq 2^{k/2+4}$ and $t \in [0, 1]$ (see for example, [35, page 62]). However, when $L$ is a general non-negative self-adjoint operator acting on the space $L^2(X)$ satisfying (GGE$_{p_0, p'_0, m}$) with $p_0 \in [1, 2)$, such estimate (1.19) may or may not hold. In our setting, we need the following off-diagonal estimate of the operator $e^{itL} \varphi(2^{-k}L)$.
(see Proposition 2.3 below): For every \( M > 0 \), there exists a positive constant \( C = C(n, m, M) \) independent of \( t \) such that

\[
\| P_{B_1} e^{itL} \varphi(2^{-k}L) P_{B_2} f \|_2 \leq C \left( 1 + \frac{d(B_1, B_2)}{2^{(m-1)k/m}(1 + |t|)} \right)^{-M} \| P_{B_2} f \|_2, \quad t \in \mathbb{R}
\]

(1.20)

for all balls \( B_1, B_2 \subset X \) with radius \( r_{B_1} = r_{B_2} \geq c 2^{(m-1)k/m}(1 + |t|) \) for some \( c \geq 1/4 \), and \( d(B_1, B_2) \geq 6r_{B_1} \). This new estimate is crucial for the proof of Theorem 1.1.

The paper is organized as follows. In Sect. 2 we provide some preliminary results on off-diagonal estimates of the operator \( e^{itL} \varphi(2^{-k}L) \) and spectral multipliers and Littlewood-Paley theory, which we need later, mainly to prove (1.20) in Proposition 2.3. The proof of Theorem 1.1 will be given in Sect. 3. In Sect. 4 we will apply Theorem 1.1 to obtain \( L^p \)-boundedness of the Riesz means of the solution to the Schrödinger equation.

**List of notations:**

- \((X, d, \mu)\) denotes a metric measure space with a distance \( d \) and a measure \( \mu \).
- \(L\) is a non-negative self-adjoint operator acting on the space \( L^2(X)\).
- For \( x \in X \) and \( r > 0 \), \( B(x, r) = \{ y \in X : d(x, y) < r \} \) and \( V(x, r) = \mu(B(x, r)) \).
- For \( B = B(x_B, r_B) \), \( A(x_B, r_B, 0) = B \) and \( A(x_B, r_B, j) = B(x_B, (j + 1)r_B) \setminus B(x_B, jr_B) \) for \( j = 1, 2, \ldots \).
- \( \delta_R F \) is defined by \( \delta_R F(x) = F(Rx) \) for \( R > 0 \) and Borel function \( F \) supported on \([-R, R]\).
- \([t]\) denotes the integer part of \( t \) for any positive real number \( t \).
- \( \mathbb{N} \) is the set of positive integers.
- For \( p \in [1, \infty] \), \( p' = p/(p - 1) \).
- For \( 1 \leq p \leq \infty \) and \( f \in L^p(X, d\mu) \), \( \| f \|_p = \| f \|_{L^p(X, d\mu)} \).
- \( \langle \cdot, \cdot \rangle \) denotes the scalar product of \( L^2(X, d\mu) \).
- For \( 1 \leq p, q \leq +\infty \), \( \| T \|_{p \to q} \) denotes the operator norm of \( T \) from \( L^p(X, d\mu) \) to \( L^q(X, d\mu) \).
- If \( T \) is given by \( T f(x) = \int K(x, y) f(y) d\mu(y) \), we denote by \( K_T \) the kernel of \( T \).
- Given a subset \( E \subseteq X \), \( \chi_E \) denotes the characteristic function of \( E \) and \( P_E f(x) = \chi_E(x) f(x) \).
- For every \( B \subset X \), we write \( \int_B f d\mu(y) = \mu(B)^{-1} \int_B f(y) d\mu(y) \).
- For \( 1 \leq r < \infty \), \( M_r \) denotes the uncentered \( r \)th maximal operator over balls in \( X \), that is

\[
M_r f(x) = \sup_{B \ni x} \left( \int_B |f(y)|^r d\mu(y) \right)^{1/r}.
\]

For simplicity we denote by \( M \) the Hardy–Littlewood maximal function \( M_1 \).
2 Off-diagonal estimates and spectral multipliers

In this section we assume that \((X, d, \mu)\) is a space of homogeneous type with a dimension \(n\) in (1.5) and that \(L\) is a self-adjoint non-negative operator in \(L^2(X)\) satisfying the generalized Gaussian estimate \((\text{GGE})_{p_0,p_0,m}\) for some \(1 \leq p_0 < 2\).

2.1 Off-diagonal estimates

We start by collecting some properties of the generalized Gaussian estimates obtained by Blunck and Kunstmann, see for example, [4–6,25] and the references therein. For every \(j \geq 1\), we recall that \(A(x_B, r_B, j) = B(x_B, (j + 1)r_B)\setminus B(x_B, jr_B)\). The following result originally stated in [25, Lemma 2.5] (see also [4, Theorem 2.1]) shows that generalized Gaussian estimates can be extended from real times \(t > 0\) to complex times \(z \in \mathbb{C}\) with \(\text{Re} z > 0\). Recall that \(\chi_E\) denotes the characteristic function of \(E \subseteq X\) and set \(P_E f(x) = \chi_E(x) f(x)\).

Lemma 2.1 Let \(m \geq 2\) and \(1 \leq p \leq 2 \leq q \leq \infty\), and \(L\) be a non-negative self-adjoint operator on \(L^2(X)\). Assume that there exist constants \(C, c > 0\) such that for all \(t > 0\), and all \(x, y \in X\),

\[
\left\| P_{B(x,t^{1/m})} e^{-tL} P_{B(y,t^{1/m})} \right\|_{p \to q} \leq CV(x, t^{1/m})^{-\left(\frac{1}{p} - \frac{1}{q}\right)} \exp \left(-c \left(\frac{d(x, y)}{t^{1/m}}\right)^{\frac{m}{m-1}}\right).
\]

Let \(r_z = (\text{Re} z)^{-\frac{1}{m}}|z|\) for each \(z \in \mathbb{C}\) with \(\text{Re} z > 0\).

(i) There exist two positive constants \(C'\) and \(c'\) such that for all \(r > 0\), \(x \in X\), and \(z \in \mathbb{C}\) with \(\text{Re} z > 0\),

\[
\left\| P_{B(x,r)} e^{-zL} P_{B(y,r)} \right\|_{p \to q} \leq C' V(x, r)^{-\left(\frac{1}{p} - \frac{1}{q}\right)} \left(1 + \frac{r}{r_z}\right)^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \left(\frac{|z|}{\text{Re} z}\right)^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \exp \left(-c' \left(\frac{d(x, y)}{r_z}\right)^{\frac{m}{m-1}}\right).
\]

(ii) There exist two positive constants \(C''\) and \(c''\) such that for all \(r > 0\), \(x \in X\), \(k \in \mathbb{N}\) and \(z \in \mathbb{C}\) with \(\text{Re} z > 0\),

\[
\left\| P_{B(x,r)} e^{-zL} P_{A(x,r,k)} \right\|_{p \to q} \leq C'' V(x, r)^{-\left(\frac{1}{p} - \frac{1}{q}\right)} \left(1 + \frac{r}{r_z}\right)^{n\left(\frac{1}{p} - \frac{1}{q}\right)} \left(\frac{|z|}{\text{Re} z}\right)^{n\left(\frac{1}{p} - \frac{1}{q}\right)} k^n \exp \left(-c'' \left(\frac{r}{r_z k}\right)^{\frac{m}{m-1}}\right).
\]

Proof For the detailed proof we refer readers to [25]. Here we only mention that the proof of Lemma 2.1 relies on the Phragmén–Lindelöf theorem.

Next suppose that \(m \geq 2\). We say that the semigroup \(e^{-tL}\) generated by non-negative self-adjoint operator \(L\) satisfies \(m\)th order Davies–Gaffney estimates, if there exist constants \(C, c > 0\) such that for all \(t > 0\), and all \(x, y \in X\),
\[ \| P_{B(x,t^{1/m})} e^{-tL} P_{B(y,t^{1/m})} \|_{2 \to 2} \leq C \exp \left( -c \left( \frac{d(x,y)}{t^{1/m}} \right)^{m-1} \right). \quad (DG_m) \]

Note that if condition \((GGE_{p_0,p'_0,m})\) holds for some \(1 \leq p_0 \leq 2\) with \(p_0 < 2\), then the semigroup \(e^{-tL}\) satisfies estimate \((DG_m)\).

The following Lemma describes a useful consequence of \(m\)-order Davies–Gaffney estimates (see [37, Lemma 2.2]).

**Lemma 2.2** Let \(m \geq 2\) and \(L\) satisfies the Davies–Gaffney estimates \((DG_m)\). Then for every \(M > 0\), there exists a constant \(C = C(M)\) such that for every \(j = 2, 3, \ldots\)

\[ \| P_B F(L) P_{A(x_B,r_B,j)} \|_{2 \to 2} \leq C j^{-M} \left( \sqrt{Rr_B} \right)^{-(M+n)} \| \delta_R F \|_{W^{M+n+1}_2} \quad (2.1) \]

for all balls \(B \subseteq X\), and all Borel functions \(F\) such that \(\text{supp } F \subseteq [-R, R]\).

**Proof** Let \(G(\lambda) = (\delta_R F)(\lambda)e^{\lambda}\). In virtue of the Fourier inversion formula

\[ G(L/R)e^{-L/R} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{(i\tau - 1)R^{-1}L} \hat{G}(\tau) d\tau \]

so

\[ \| P_B F(L) P_{A(x_B,r_B,j)} \|_{2 \to 2} \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{G}(\tau)| \| P_B e^{(i\tau - 1)R^{-1}L} P_{A(x_B,r_B,j)} \|_{2 \to 2} d\tau. \]

By (ii) of Lemma 2.1 (with \(r_z = \sqrt{1 + \tau^2} / \sqrt{R}\),

\[ \| P_B e^{(i\tau - 1)R^{-1}L} P_{A(x_B,r_B,j)} \|_{2 \to 2} \leq C j^n \exp \left( -c \left( \frac{m \sqrt{R} j r_B}{\sqrt{1 + \tau^2}} \right)^{m-1} \right) \]

\[ \leq C_M j^n \left( \frac{\sqrt{R} j r_B}{\sqrt{1 + \tau^2}} \right)^{-M-n} \]

\[ \leq C j^{-M} (1 + \tau^2)^{-\frac{M+n}{2}} \left( \frac{m}{\sqrt{R} r_B} \right)^{-(M+n)}. \]

Therefore (compare [17, (4.4)])

\[ \| P_B F(L) P_{A(x_B,r_B,j)} \|_{2 \to 2} \leq C j^{-M} \left( \sqrt{Rr_B} \right)^{-(M+n)} \int_{\mathbb{R}} |\hat{G}(\tau)| (1 + \tau^2)^{-\frac{M+n}{2}} d\tau \]

\[ \leq C j^{-M} \left( \sqrt{Rr_B} \right)^{-(M+n)} \left( \int_{\mathbb{R}} |\hat{G}(\tau)|^2 (1 + \tau^2)^{M+n+1} d\tau \right)^{1/2} \left( \int_{\mathbb{R}} (1 + \tau^2)^{-1} d\tau \right)^{1/2} \]

\[ \leq C j^{-M} \left( \sqrt{Rr_B} \right)^{-(M+n)} \| G \|_{W^{M+n+1}_2}. \]
However, $\text{supp } F \subseteq [-R, R]$ and $\text{supp } \delta_R F \subseteq [-1, 1]$ so

$$\|G\|_{W^{m+n+1}_2} \leq C \|\delta_R F\|_{W^{m+n+1}_2}.$$  

This completes the proof of Lemma 2.2. $\square$

The proof of Theorem 1.1 relies on the following off-diagonal estimates for $e^{itL}\phi_k(L)$, where $\phi \in C^\infty_0([1/4, 4])$ is a cut-off function and $\phi_k(s) = \phi(2^{-k}s)$ for every $k \geq 1$.

**Proposition 2.3** Let $m \geq 2$ and $L$ satisfies the Davies–Gaffney estimates $(DG_m)$. For every $M > 0, K \geq 1, s > 0, t \in \mathbb{R}$ and $k \geq 1$, there exists a constant $C = C(M, n, K)$ independent of $t, s,$ and $k$ such that

$$\left\| P_{B_i}(I - e^{-sL})^K e^{itL}\phi_k(L)P_{B_2}f \right\|_2 \leq C \left(1 + \frac{d(B_1, B_2)}{2^{(m-1)k/m}(1 + |t|)}\right)^{-M} \|P_{B_2}f\|_2$$  

(2.2)

for all $B_i \subset X$ with $r_{B_1} = r_{B_2} \geq c2^{(m-1)k/m}(1 + |t|)$ for some $c \geq 1/4$, and $d(B_1, B_2) \geq 6r_{B_1}$.

To prove Proposition 2.3, we need the following Lemmas 2.4 and 2.5.

**Lemma 2.4** Let $m \geq 2$ and $L$ satisfies the Davies-Gaffney estimates $(DG_m)$. Then for every $M > 0, k \in \mathbb{N}^+$ and $t \in \mathbb{R}$, there exists a constant $C = C(M, m, n)$ independent of $t$ and $k$ such that for every $j = 2, 3, \ldots$

$$\left\| P_{B_j}(I - e^{-sL})^L P_{A(x_B, r_B, j)}f \right\|_2 \leq C j^{-M} \left(1 + \frac{r_B}{2^{(m-1)k/m}(1 + |t|)}\right)^{-M} \|P_{A(x_B, r_B, j)}f\|_2$$  

(2.3)

for all balls $B \subset X$ with $r_B \geq c2^{(m-1)k/m}(1 + |t|)$ for some $c \geq 1/4$.

As a consequence, we have

$$\left\| P_B e^{-(2^{-k} - it)L} P_{X \setminus 2B}f \right\|_2 \leq C \mu(B)^{1/2} \mathfrak{M}_2(f)(x)$$

for all balls $B \subset X$ with $r_B \geq c2^{(m-1)k/m}(1 + |t|)$ for some $c > 1/4$ and for every $x \in B$.

**Proof** Note that

$$\left\| P_B e^{-(2^{-k} - it)L} P_{X \setminus 2B}f \right\|_2 \leq \sum_{j=2}^{\infty} \|P_B e^{-zL} P_{A(x_B, r_B, j)}f\|_2$$
with \( z = (2^{-k} - it) \). It is clear that \( \text{Re} \, z = 2^{-k} > 0 \), and so \( r_z = (\text{Re} \, z)^{-1/2} |z| = 2^{(m-1)k/m} \sqrt{|t|^2 + 2^{-2k}} \). By (ii) of Lemma 2.1, we see that for every ball \( B \subset X \) with \( r_B \geq 2^{(m-1)k/m} (1 + |t|) \), \( k \geq 0 \),

\[
\left\| P_B e^{-(2^{-k} - i t) L} P_{A(x_B, r_B, j)} \right\|_{2 \to 2} \leq C j^n \exp \left( -c \left( \frac{r_B j}{2^{(m-1)k/m} \sqrt{2^{-2k} + |t|^2}} \right)^{\frac{m}{m-1}} \right) \\
\leq C M j^{-M+n} \left( 1 + \frac{r_B}{2^{(m-1)k/m} (1 + |t|)} \right)^{-M}
\]

for every \( M > 0 \). Hence, (2.3) holds. This, in combination with the fact that for every \( x \in B \),

\[
\| P_{A(x_B, r_B, j)} f \|_2 \leq \mu((j + 1) B)^{1/2} \left( \int_{(j+1)B} |f(y)|^2 d\mu(y) \right)^{1/2} \\
\leq C(j + 1)^{n/2} \mu(B)^{1/2} \mathcal{M}_2(f)(x),
\]

yields that

\[
\left\| P_B e^{-(2^{-k} - i t) L} P_{X \setminus 2B} f \right\|_2 \leq C \sum_{j=2}^{\infty} j^{-(M-\frac{3n}{2})} \mu(B)^{1/2} \mathcal{M}_2(f)(x) \\
\leq C \mu(B)^{1/2} \mathcal{M}_2(f)(x)
\]

as long as we choose \( M > 3n/2 \) in (2.4). This proves Lemma 2.4.

\[\Box\]

**Lemma 2.5** Let \( m \geq 2 \) and \( L \) satisfies the Davies–Gaffney estimates \((DG_m)\). For a given \( \phi \in C_0^\infty([0, 4]) \), we write \( \phi_\epsilon(\lambda) = e^{\lambda} \phi(\lambda) \). Then for every \( M > 0 \), \( k \in \mathbb{N}^+ \) and \( s > 0 \), there exists a constant \( C = C(m, n, M) \) independent of \( k \) and \( s \) such that for every \( j = 2, 3, \ldots \)

\[
\left\| P_B (I - e^{-s L} \lambda) \phi_\epsilon(2^{-k} L) P_{A(x_B, r_B, j)} f \right\|_2 \leq C j^{-M} (2^{k/m} r_B)^{-M-n} \| P_{A(x_B, r_B, j)} f \|_2
\]

for all \( B \subset X \) with \( r_B \geq c 2^{(m-1)k/m} \) for some \( c \geq 1/4 \).

As a consequence, we have

\[
\left\| P_B (I - e^{-s L} \lambda) \phi_\epsilon(2^{-k} L) P_{X \setminus 2B} f \right\|_2 \leq C \mu(B)^{1/2} \mathcal{M}_2(f)(x).
\]

**Proof** We write

\[
\left\| P_B (I - e^{-s L} \lambda) \phi_\epsilon(2^{-k} L) P_{X \setminus 2B} f \right\|_2 \leq \sum_{j=2}^{\infty} \left\| P_B (I - e^{-s L} \lambda) \phi_\epsilon(2^{-k} L) P_{A(x_B, r_B, j)} f \right\|_2.
\]
Note that the function \((1 - e^{-s\lambda})^K e^{2-k\lambda} \phi_k(\lambda)\) is supported in \([2^{k-2}, 2^{k+2}]\). We apply Lemma 2.2 with \(R = 2^{k+2}\) to obtain that for every \(M > 0\) and \(j \geq 2\),

\[
\|P_B(1 - e^{-sL})^K \phi_e(2^{-k} L) P_{A(x_B, r_B, j)}\|_{2 \to 2} \leq C j^{-M (2^{k/m} r_B)^{-M-n}} \|\delta_{2^{k+2}} \left((1 - e^{-s\lambda})^K e^{2-k\lambda} \phi_k(\lambda)\right)\|_{W^{2, n+1}_2} \\
\leq C j^{-M (2^{k/m} r_B)^{-M-n}} \|\lambda \leq C j^{-M (2^{k/m} r_B)^{-M-n}} \|\phi_k(\lambda)\|_{W^{2, n+1}_2}.
\]

This, in combination with (2.5), yields that for every \(x \in B\),

\[
\|P_B(I - e^{-sL})^K \phi_e(2^{-k} L) P_{X\setminus B} f\|_2 \leq C \sum_{j=2}^{\infty} j^{-(M-2)} (2^{k/m} r_B)^{-M-n} \mu(B)^{1/2} \mathfrak{M}_2(f)(x) \leq C \mu(B)^{1/2} \mathfrak{M}_2(f)(x)
\]

as long as we choose \(M > n/2\) in the first inequality above and notice the fact that \(2^{k/m} r_B \geq 1/4\). This proves Lemma 2.5. \(\square\)

**Proof of Proposition 2.3** Let us show (2.2) when \(d(B_1, B_2) \geq 6r_{B_1}\). By spectral theory, we write

\[
(I - e^{-sL})^K e^{itL} \phi_k(L) = e^{-(2-k-it)L} \left[\left(I - e^{-sL}\right)^K \phi_e(2^{-k} L)\right] = S_{k, t}(L) T_k(L)
\]

where we write \(\phi_e(\lambda) = e^{\lambda} \phi(\lambda)\),

\[
S_{k, t}(L) = e^{-(2-k-it)L}
\]

and

\[
T_k(L) = (I - e^{-sL})^K \phi_e(2^{-k} L).
\]

Set \(G = \{x : \text{dist}(x, B_1) \leq d(B_1, B_2)/2\}\). Then it is clear that \(\text{dist}(B_2, \hat{G}) \geq d(B_1, B_2)/2\), where we use \(\hat{G}\) to denote the topological closure of the set \(G\). Moreover, from the definition of \(G\), it is also clear that \(\text{dist}(X \setminus G, B_1) \geq d(B_1, B_2)/3\). Furthermore, based on the above observations we have

\[
G \subset \bigcup_{j=\lceil d(B_1, B_2) / d_{B_1} \rceil - 1}^{2+ \lceil d(B_1, B_2) / r_{B_2} \rceil} A(x_{B_2}, r_{B_2}, j) \quad \text{and} \quad X \setminus G \subset \bigcup_{j=\lceil d(B_1, B_2) / d_{B_1} \rceil - 1}^{\infty} A(x_{B_1}, r_{B_1}, j),
\]

where \(\lceil a \rceil\) denotes the greatest integer that is smaller than \(a\).
Then by noting that $S_{k,t}(L)$ is uniformly bounded on $L^2(X)$ and by Lemma 2.5,

$$\|P_{B_1} S_{k,t}(L) \left( P_{G} T_k(L) P_{B_2} f \right) \|_2 \leq \|S_{k,t}(L) \left( P_{G} T_k(L) P_{B_2} f \right) \|_2 \leq C \|P_{G} T_k(L) P_{B_2} f \|_{2+d(d(B_1,B_2))/r_{B_2}}+1} \leq C \sum_{j=[d(B_1,B_2)/(2r_{B_2})]}^{\infty} \|P_{A(x_{B_2},r_{B_2},j)} T_k(L) P_{B_2} f \|_2 \leq C \left( 1 + \frac{d(B_1, B_2)}{r_{B_2}} \right)^{-M+1} \|P_{B_2} f \|_2 \leq C \left( 1 + \frac{d(B_1, B_2)}{2^{(m-1)k/m}|1+|t|)} \right)^{-M+1} \|P_{B_2} f \|_2 \tag{2.7}$$

for any $M > 0$, where in the last inequality we use the facts that $r_{B_2} \geq 1/4$ and that $d(B_1, B_2) > 2^{k(m-1)}(1+|t|)$.

On the other hand, we apply Lemma 2.4 and the fact that $T_k(L)$ is uniformly bounded on $L^2(X)$ to see that for every $M > 0$,

$$\|P_{B_1} S_{k,t}(L) \left( P_{X \setminus G} T_k(L) P_{B_2} f \right) \|_2 \leq \sum_{j=[d(B_1,B_2)/(2r_{B_1})]}^{\infty} \|P_{B_1} S_{k,t}(L) P_{A(x_{B_1},r_{B_1},j)} (T_k(L) P_{B_2} f) \|_2 \leq \sum_{j=[d(B_1,B_2)/(2r_{B_1})]}^{\infty} j^{-M} \left( 1 + \frac{r_{B_1}}{2^{(m-1)k/m}|1+|t|)} \right)^{-M} \|T_k(L) P_{B_2} f \|_2 \leq C \left( 1 + \frac{d(B_1, B_2)}{2^{(m-1)k/m}|1+|t|)} \right)^{-M} \|P_{B_2} f \|_2 \tag{2.8}$$

Therefore, we combine the estimates (2.7) and (2.8) to obtain that for every $M > 0$,

$$\|P_{B_1} S_{k,t}(L) \left( T_k(L) P_{B_2} \right) \|_2 \leq \|P_{B_1} S_{k,t}(L) \left( P_{G} T_k(L) P_{B_2} f \right) \|_2 + \|P_{B_1} S_{k,t}(L) \left( P_{X \setminus G} T_k(L) P_{B_2} f \right) \|_2 \leq C \left( 1 + \frac{d(B_1, B_2)}{2^{(m-1)k/m}|1+|t|)} \right)^{-M} \|P_{B_2} f \|_2,$$

which shows that (2.2) holds. The proof of Proposition 2.3 is complete. ∎
In order to prove Theorem 1.3, we also need the following estimate for the operator $e^{itL} \phi_k(tL)$, $t > 0$. Recall that $\phi \in C_0^\infty([1/4, 4])$ is a cut-off function and $\phi_k(s) = \phi(2^{-k}s)$ for every $k \geq 1$.

**Proposition 2.6** Let $m \geq 2$ and $L$ satisfies the Davies–Gaffney estimates ($DG_m$). For every $M > 0$, $K \in \mathbb{N}^+$, $s > 0$, $t > 0$ and $k \geq 1$, there exists a constant $C = C(M, n, K)$ independent of $t$, $s$, and $k$ such that

$$
\| P_{B_1} (I - e^{-sL})^K e^{itL} \phi_k(tL) P_{B_2} f \|_2 \leq C \left( 1 + \frac{d(B_1, B_2)}{2^{(m-1)k/m} t^{1/m}} \right)^{-M} \| P_{B_2} f \|_2
$$

for all $B_i \subset X$ with $r_{B_1} = r_{B_2} \geq c 2^{(m-1)k/m} t^{1/m}$ for some $c \geq 1/4$.

**Proof** The proof of Proposition 2.6 can be obtained by making minor modifications with the proof of Proposition 2.3, we leave the detail to the reader. $\square$

### 2.2. Spectral multipliers

The following result is a standard known result in the theory of spectral multipliers of non-negative selfadjoint operators.

**Proposition 2.7** Let $m \geq 2$. Suppose that $(X, d, \mu)$ is a space of homogeneous type with a dimension $n$. Suppose that $L$ satisfies the property $(GGE_{p_0, p'_0, m})$ for some $1 \leq p_0 < 2$. Then we have

(a) Assume in addition that $F$ is an even bounded Borel function such that $\sup_{R > 0} \| \eta \delta_R F \|_{C^\beta} < \infty$ for some integer $\beta > n/2 + 1$ and some non-trivial function $\eta \in C_0^\infty(0, \infty)$. Then the operator $F(L)$ is bounded on $L^p(X)$ for all $p_0 < p < p'_0$,

$$
\| F(L) \|_{p \to p} \leq C_\beta \left( \sup_{R > 0} \| \eta \delta_R F \|_{C^\beta} + F(0) \right).
$$

(2.9)

(b) Fix a non-zero $C^\infty$ bump function $\varphi$ on $\mathbb{R}$ such that $\text{supp} \varphi \subseteq (1/2, 2)$ for all $\lambda > 0$ and set $\varphi_0(\lambda) = \sum_{\ell \leq 0} \varphi(2^{-\ell} \lambda)$ and $\varphi_k(\lambda) = \varphi(2^{-k} \lambda)$ for $k = 1, 2, \ldots$. Then for all $p_0 < p < p'_0$,

$$
\left\| \left( \sum_{k=0}^{\infty} |\varphi_k(L) f|^2 \right)^{1/2} \right\|_p \leq C_p \| f \|_p.
$$

(2.10)

In addition, if $\sum_{k \geq 0} \varphi_k(\lambda) = 1$ for all $\lambda > 0$, then we have

$$
\| f \|_p \approx C_p \left\| \left( \sum_{k=0}^{\infty} |\varphi_k(L) f|^2 \right)^{1/2} \right\|_p, \quad p_0 < p < p'_0.
$$

(2.11)
Proof Assertion (a) follows from [5, Theorem 1.1], see also [10, Lemma 4.5]. The proof of assertion (b) follows from Stein’s classical proof [40, Chapter IV]. We give a brief argument of this proof for completeness and convenience for the reader.

Let us introduce the Rademacher function, which is defined as follows: i) The function \( r_0(t) \) is defined by \( r_0(t) = 1 \) on \([0, 1/2]\) and \( r_0(t) = -1 \) on \((1/2, 1)\), and then extended to \( \mathbb{R} \) by periodicity; ii) For \( k \in \mathbb{N} \setminus \{0\} \), \( r_k(t) = r_0(2^k t) \). Define

\[
F(t, \lambda) = \sum_{k=0}^{\infty} r_k(t) \varphi_k(\lambda).
\]

A straightforward computation shows that for every integer \( \beta > n/2 + 1 \),

\[
sup_{R > 0} \|\eta F(t, R \lambda)\|_{C^\beta} \leq C \beta \text{ uniformly in } t \in [0, 1].
\]

Then we apply (2.9) to see that for all \( p \in (p_0, p_0') \),

\[
\|F(t, L) f\|_p = \left\| \sum_{k=0}^{\infty} r_k(t) \varphi_k(L) f \right\|_p \leq C \|f\|_p
\]

with \( C > 0 \) uniformly in \( t \in [0, 1] \). This, in combination with the standard inequality for Rademacher functions:

\[
\left( \sum_{k=0}^{\infty} |\varphi_k(L) f|^2 \right)^{p/2} = \int_0^1 \left| \sum_{k=0}^{\infty} r_k(t) \varphi_k(L) f \right|^p dt,
\]

yields

\[
\left\| \left( \sum_{k=0}^{\infty} |\varphi_k(L) f|^2 \right)^{1/2} \right\|_p \leq \left( \int_0^1 \left\| \sum_{k=0}^{\infty} r_k(t) \varphi_k(L) f \right\|_p^p dt \right)^{1/p} \leq C_p \|f\|_p.
\]

This proves (2.10).

When \( \sum_{k \geq 0} \varphi_k(\lambda) = 1 \) for all \( \lambda > 0 \), it follows by the spectral theory [31] that \( \sum_{k \geq 0} \varphi_k(L) f = f \) for every \( f \in L^2 \). From it, we obtain (2.11) by using (2.10) and the standard duality argument (see for example, [40, Chapter IV]). This completes the proof of Proposition 2.7. \( \square \)

3 Sharp endpoint \( L^p \)-Sobolev estimates for Scrodinger groups

In this section we prove (1.7) in Theorem 1.1. First, we note that from (1.12), estimate (1.7) holds for \( s > n|1/2 - 1/p| \). By duality, it suffices to verify (1.7) for \( 2 \leq p < p_0' \) and \( s = n|1/2 - 1/p| \). Also, it follows by the spectral theory [31] that (1.7) holds for \( p = 2 \). For \( p \neq 2 \), we recall that when \( L \) satisfies the generalized Gaussian estimates (GGE_{p_0, p_0', m}) for some \( 1 \leq p_0 < 2 \), it was proved by Blunck [4, Theorem 1.1] that for every \( z \in \mathbb{C}^+ \),

\( \square \) Springer
\[ \|e^{-zL}\|_{p\to p} \leq C \left( \frac{|z|}{\Re z} \right)^{n \left[ \frac{1}{2} - \frac{1}{p} \right]} \]  

(3.1)

for all \( p \in [p_0, p'_0] \) with \( p \neq \infty \). From this, we have the following sharp \( L^p \) frequency truncated estimates for the Schrödinger group.

**Proposition 3.1** Suppose that \((X, d, \mu)\) is a space of homogeneous type with a dimension \( n \). Suppose that \( L \) satisfies the property \((GGE_{p_0, p'_0, m})\) for some \( 1 \leq p_0 < 2 \). Then for every \( p \in (p_0, p'_0) \) and \( k \geq 0 \),

\[ \|e^{itL}\phi(2^{-k}L)f\|_{p\to p} \leq C \left( 1 + 2^k |t|^s \|f\|_{p\to p} \right) \]

(3.2)

uniformly for \( t \in \mathbb{R} \) and for \( \phi \) in bounded subsets of \( C^\infty_0(\mathbb{R}) \).

**Proof** To show (3.2), we apply (3.1) with \( z = 2^{-k} - it \) to get that for every \( \phi \in C^\infty_0(\mathbb{R}) \),

\[ \|e^{itL}\phi(2^{-k}L)f\|_{p\to p} = \left\| e^{-(2^{-k} - it)L} \left[ \phi \circ (2^{-k}L) \right] \right\|_{p\to p} \leq C \left( 1 + 2^k |t|^s \| \phi \circ (2^{-k}L) \|_{p\to p} \right) \]

\[ \leq C \left( 1 + 2^k |t|^s \right), \]

where \( \phi \circ \lambda = e^\lambda \phi \). In the last inequality we used Proposition 2.7 to know that the operator \( \phi \circ (2^{-k}L) \) is bounded on \( L^p(X) \) all \( p \in (p_0, p'_0) \). This completes the proof of Proposition 3.1. \( \square \)

To prove Theorem 1.1, let us introduce some tools needed in the proof. Let \( T \) be a sublinear operator which is bounded on \( L^2(X) \) and \( \{A_r\}_{r>0} \) be a family of linear operators acting on \( L^2(X) \). For \( f \in L^2(X) \), we follow [2] to define

\[ \mathcal{M}_{T,A}^f(x) = \sup_{B \ni x} \left( \int_B |T(I - A_{r_B})f|^2 d\mu \right)^{1/2}, \]

where the supremum is taken over all balls \( B \) in \( X \) containing \( x \), and \( r_B \) is the radius of \( B \). Then we have the following result. For its proof, we refer readers to [2, Lemma 2.3], [18, Lemma 5.4] and [37, Proposition 3.2].

**Proposition 3.2** Suppose that \( T \) is a sublinear operator which is bounded on \( L^2(X) \) and that \( q \in (2, \infty] \). Assume that \( \{A_r\}_{r>0} \) is a family of linear operators acting on \( L^2(X) \) and that

\[ \left( \int_B |T A_{r_B}f(y)|^q d\mu(y) \right)^{1/q} \leq C \mathcal{M}_2(Tf)(x) \]

(3.3)

for all \( f \in L^2(X) \), all \( x \in X \) and all balls \( B \ni x \), \( r_B \) being the radius of \( B \).
Then for $0 < p < q$, there exists $C_p$ such that

$$
\| M_2(Tf) \|_p \leq C_p \left( \| M^#_{T,A}f \|_p + \| f \|_p \right)
$$

(3.4)

for every $f \in L^2(X)$ for which the left-hand side is finite (if $\mu(X) = \infty$, the term $C_p \| f \|_p$ can be omitted in the right-hand side of (3.4)).

**Proof of Theorem 1.1.** Let us show Theorem 1.1 for $2 < p < p' > 0$ and $s = n|1/2 - 1/p|$. We fix a non-zero $C^\infty$ bump function $\varphi$ on $\mathbb{R}$ such that

$$
supp \varphi \subseteq \left( \frac{1}{2}, 2 \right) \quad \text{and} \quad \sum_{\ell \in \mathbb{Z}} \varphi(2^{-\ell} \lambda) = 1 \quad \text{for all } \lambda > 0
$$

(3.5)

and set $\varphi_0(\lambda) = \sum_{\ell \leq 0} \varphi(\lambda/2^\ell)$ and $\varphi_\ell(\lambda) = \varphi(\lambda/2^\ell)$ for $\ell = 1, 2, \ldots$.

For this fixed bump function $\varphi$, we consider an operator $T_\varphi$, given by

$$
T_\varphi f(x) = \left( \sum_{k \geq 0} |\varphi_k(L)e^{itL}f(x)|^2 \right)^{1/2}
$$

(3.6)

for every $f \in L^2(X)$. Then from (2.11), it is direct to see that $\| e^{itL}f \|_p \leq C \| T_\varphi f \|_p$ for $2 < p < p'$.

Next, we define a sharp maximal function $M^#_{T_\varphi,L,K}$ of $T_\varphi$ as follows: for every $K \in \mathbb{N}$ and every $f \in L^2(X)$,

$$
M^#_{T_\varphi,L,K} f(x) = \sup_{B \ni x} \left( \int_B |T_\varphi(I - e^{-r_B^m L})K f(y)|^2 d\mu(y) \right)^{1/2},
$$

(3.7)

where the supremum is taken over all balls $B$ in $X$ containing $x$, and $r_B$ is the radius of $B$. In order to prove Theorem 1.1, it suffices to show the following two arguments:

(a) the operator $T_\varphi$ satisfies condition (3.3) for every $2 < p < q < p'$ and $A_{r_B} = I - (I - e^{-r_B^m L})^K$ for every $K \in \mathbb{N}$;

(b) by choosing $K$ large enough, for $s = n|1/2 - 1/p|$, we have

$$
\| M^#_{T_\varphi,L,K} f \|_p \leq C_p \left( 1 + |t|^s \right) \left( \sum_{k \geq 0} 2^{ksp} \| \varphi_k(L) f \|_p^p \right)^{1/p}.
$$

(3.8)

Before we prove the above two arguments (a) and (b), let us show that Theorem 1.1 is a straightforward consequence of them. Indeed, when (a) holds for $T_\varphi$, it follows from (b) of Proposition 2.7 and Proposition 3.2 that for $2 < p < p'$, $\| M_2(T_\varphi f) \|_p \leq C_p (\| f \|_p + \| M^#_{T_\varphi,L,K} f \|_p)$. This, together with (3.8), yields that
\[ \|e^{itL} f\|_p \leq C \|T_{\psi} f\|_p \]
\[ \leq C \|\mathcal{M}_2(T_{\psi} f)\|_p \leq C_p \left( \|f\|_p + \|\mathcal{M}_{T_{\psi}, L, K} f\|_p \right) \]
\[ \leq C \|f\|_p + C(1 + |t|)^s \left( \sum_{k\geq 0} 2^{ksp} \|\varphi_k(L) f\|_p^p \right)^{1/p} \]
\[ \leq C \|f\|_p + C(1 + |t|)^s \left( \|\varphi_0(f)\|_p + \left( \sum_{k>0} 2^{ksp} \|\varphi_k(L) f\|_p^2 \right)^{1/2} \right) \]
\[ \leq C(1 + |t|)^s \left( \|f\|_p + \left( \sum_{k>0} |\phi_k(L) [L^s f]|^2 \right)^{1/2} \right) \]
\[ \leq C(1 + |t|)^s \left( \|f\|_p + \|L^s f\|_p \right). \]  

(3.9)

where in the fifth inequality we have used the embedding \( \ell^2 \hookrightarrow \ell^p \) for \( p \geq 2 \), in the sixth inequality the function \( \phi_k(\lambda) = \varphi(2^{-k}\lambda)(2^{-k}\lambda)^{-s} \), and in the last inequality we used (b) of Proposition 2.7 for the Littlewood-Paley result for functions in \( L^p(X) \). This proves Theorem 1.1.

We now first prove the argument (a1). Indeed, in virtue of the formula

\[ I - (I - e^{-r_B^m L})^K = \sum_{\tau=1}^{K} \left( \begin{array}{c} K \\ \tau \end{array} \right) (-1)^{\tau+1} e^{-\tau r_B^m L} \]  

(3.10)

and the commutativity property \( \varphi_k(L)e^{itL}e^{-\tau r_B^m L} = e^{-\tau r_B^m L} \varphi_k(L)e^{itL} \), it is enough to show that for all ball \( B \) containing \( x \),

\[ \left( \int_B \left( \sum_{k\geq 0} |e^{-r_B^m L} \varphi_k(L)e^{itL} f(y)|^2 \right)^{\frac{q}{2}} d\mu(y) \right)^{1/q} \leq C \mathcal{M}_2(T_{\psi} f)(x). \]  

(3.11)

Let us prove (3.11). From hypothesis \((\text{GGE}_{p_0, p'_0, m})\), it is seen that condition \((\text{GGE}_{2, q, m})\) holds for \( 2 < p < q < p'_0 \), i.e, there exist constants \( C, c > 0 \) such that for every \( u > 0 \) and \( x, y \in X \),

\[ \|P_{B(x, u^{1/m})} e^{-uL} P_{B(y, u^{1/m})}\|_{2\rightarrow q} \leq C V(x, u^{1/m})^{-(\frac{1}{2} - \frac{1}{q})} \exp \left( -c \left( \frac{d(x, y)^m}{u} \right)^{\frac{1}{m-1}} \right). \]  

(3.12)

By Minkowski’s inequality, (3.12) and (ii) of Lemma 2.1, conditions (1.5) and (2.1) for every \( \tau = 1, 2, \ldots, K \) and every ball \( B \) containing \( x \), the left hand side of (3.11) is less than
The above estimate yields (3.11).

Thus, we obtain that the argument \((a_1)\) holds.

We now show the argument \((a_2)\). In the sequel we let \(\phi \in C_0^\infty(\mathbb{R})\) supported in \((1/4, 4)\) and \(\phi(x) = 1\) if \(x \in (1/2, 2)\), and set \(\phi_k(x) = \phi(2^{-k}x)\) for \(k \geq 1\). Let \(\phi_0 \in C_0^\infty([-4, 4])\) and \(\phi_0(x) = 1\) if \(x \in (-2, 2)\). By spectral theory, we have that \(\varphi_k(L)f = \phi_k(L)\varphi_k(L)f\) for \(k \geq 0\) and for every \(f \in L^2(X)\). Hence, the proof of (3.8) reduces to show that

\[
\|I\|_p + \|II\|_p + \|III\|_p \leq C(1 + |t|)^{\delta} \left( \sum_{k \geq 0} \|\varphi_k(L)f\|_p \right)^{1/p}, \tag{3.13}
\]

where

\[
I(x) = \sup_{B \ni x} \left( \int_B \sum_{0 \leq k \leq -j} 2^{-2ks} \left| \left( I - e^{-r_nL} \right)^k \phi_k(L) \left[ e^{itL} \phi_k(L) \varphi_k(L)f \right](y) \right|^2 d\mu(y) \right) \]

\[
II(x) = \sup_{B \ni x} \left( \int_B \sum_{k+j=0 \atop j \geq (m-1)k+m\log_2(2+|t|) \atop k \geq 0} 2^{-2ks} \left| \left( I - e^{-r_nL} \right)^k e^{itL} \phi_k(L) \left[ \varphi_k(L)f \right](y) \right|^2 d\mu(y) \right) \]

\[
III(x) = \sup_{B \ni x} \left( \int_B \sum_{k+j=0 \atop j < (m-1)k+m\log_2(2+|t|) \atop k \geq 0} 2^{-2ks} \left| \left( I - e^{-r_nL} \right)^k e^{itL} \phi_k(L) \left[ \varphi_k(L)f \right](y) \right|^2 d\mu(y) \right) \]
Here, we use the notation in the above decomposition that the ball $B$ is centered at $x_B$ and its radius $r_B$ is in $[2^{(j-1)/m}, 2^{j/m})$ for some $j \in \mathbb{Z}$.

Estimate of the term $I(x)$. By the Minkowski inequality, we see that

$$I(x) \leq \sup_{B \ni x} \left( \int_B \left| (I - e^{-r_B^m L})^K e^{itL} \varphi_0(L) \varphi_k(L) f \right|^2 \, d\mu(y) \right)^{1/2}$$

$$+ \sup_{B \ni x} \mu(B)^{-1/2} \sum_{u=0}^{\infty} \sum_{1 \leq k \leq -j} 2^{-k} \left\| P_B (I - e^{-r_B^m L})^K \phi_k(L) P_A(x_B, r_B, u) [e^{itL} \varphi_k(L) \varphi(L) f] \right\|_2$$

$$= I_1(x) + I_2(x).$$

For the term $I_1(x)$, from the arguments in (3.10) and (3.11), it is direct to see that for every $x \in B$, $I_1(x) \leq C \mathcal{M}_2(e^{itL} \varphi_0(L) \varphi_0(L) f)(x)$. Then from Proposition 3.1,

$$\|I_1\|_p \leq C \|e^{itL} \varphi_0(L) \varphi_0(L) f\|_p \leq C(1 + |t|)^z \|\varphi_0(L) f\|_p.$$

For the term $I_2(x)$, since the function $\varphi_k(L)$ is supported in $[2^{k-2}, 2^{k+2}]$, $k \geq 1$, it tells us that for $u = 0, 1$,

$$\| P_B (I - e^{-r_B^m L})^K \phi_k(L) P_A(x_B, r_B, u) \|_2 \leq \| (I - e^{-r_B^m L})^K \phi_k(L) \|_2 \leq C \| (1 - e^{-r_B^m L}) \phi_k(L) \|_{L^\infty} \leq C \min\{1, (2^k r_B^m)^K\},$$

also for $u \geq 2$, we use Lemma 2.2 to obtain that for every $M > 0$,

$$\| P_B (I - e^{-r_B^m L})^K \phi_k(L) P_A(x_B, r_B, u) \|_2 \leq C u^{-M} (2^{k/m} r_B)^{-M-n} \| \varphi_k(L) \|_{W^M + n+1}$$

$$\leq C u^{-M} 2^{-k+j(M+n)/m} \| (1 - e^{-r_B^m L}) \phi(L) \|_{W^M + n+1}$$

$$\leq C u^{-M} \min\{2^{-k+j}(M+n)/m, 2^{k+j}(K-M/m-n/m)\}. \quad (3.14)$$

Those, in combination with $k + j \leq 0$ and the fact that for all $u \geq 0$ and $g \in L^2_{loc}(X)$

$$\| P_A(x_B, r_B, u) g \|_2 \leq \mu((u + 1)B)^{1/2} \left( \int_{(u+1)B} |g(y)|^2 \, d\mu(y) \right)^{1/2} \leq C (1 + u)^{n/2} \mu(B)^{1/2} \mathcal{M}_2(g)(x), \quad (3.15)$$

yield

$$I_2(x) \leq \sup_{B \ni x} \sum_{1 \leq k \leq -j} \sum_{u=0}^{\infty} 2^{-k} (1 + u)^{-(M-n/2)} 2^{k+j} (K-(M+n)/m) \mathcal{M}_2(e^{itL} \varphi_k(L) \varphi(L) f)(x).$$
\[ \|I_2\|_p \leq C \left( \sum_{j=0}^{\infty} \left( \sum_{1 \leq k \leq j} 2^{(k+j)(M+n)/m} \|e^{itL} \phi_k(L)\varphi_k(L)f\|_p \right)^{p/1} \right)^{1/p} \]

\[ \leq C \sum_{\ell \geq 0} 2^{-\ell(K-(M+n)/m)} \left( \sum_{j < -\ell} 2^{(\ell+j)s} \|e^{itL} \phi_{-\ell}(L)\varphi_{-(\ell+j)}(L)f\|_p \right)^{p/1} \]

\[ \leq C(1 + |t|)^s \sum_{\ell \geq 0} 2^{-\ell(K-(M+n)/m)} \left( \sum_{j < -\ell} \|\varphi_{-(\ell+j)}(L)f\|_p \right)^{1/p} \]

\[ \leq C(1 + |t|)^s \left( \sum_{k \geq 1} \|\varphi_k(L)f\|_p \right)^{1/p} \]

as desired, as long as \( K \) is chosen large enough so that \( K > (M+n)/m \). Combining the estimates of \( I_1 \) and \( I_2 \) we get that

\[ \|I\|_p \leq C(1 + |t|)^s \left( \sum_{k \geq 0} \|\varphi_k(L)f\|_p \right)^{1/p}. \]

Estimate of the term \( II(x) \). Note that

\[ II(x) \leq \sup_{B \ni x} \left( \int_B \left| (I - e^{-rB}L)^K e^{itL} \phi_0(L)\varphi_0(L)f(y) \right|^2 d\mu(y) \right)^{1/2} \]

\[ + \sup_{B \ni x} \sum_{k+j>0} \sum_{\ell=0}^{\infty} \sum_{j \geq -(m-1)k+\log_2(2+2|t|)} 2^{-ks} \mu(B)^{-1/2} \]

\[ \times \left\| P_B(I - e^{-rB}L)^K e^{itL} \phi_k(L) P_{A(x_B,rB,\ell)} \right\|_2 \rightarrow 2 \left\| P_{A(x_B,rB,\ell)} \phi_k(L)f \right\|_2 \]

\[ = II_1(x) + II_2(x). \]

Similar to the estimate of \( I_1(x) \) above, we see that \( \|II_1\|_p \leq C(1 + |t|)^s \|\varphi_0(L)f\|_p \).
We now estimate $II_2(x)$. For a fixed $r_B > 0$, we choose a sequence of points $\{x_i\}_i \subset X$ such that $d(x_i, x_k) > r_B$ for $i \neq k$ and $\sup_{x \in X} \inf_i d(x, x_i) \leq r_B$. Such sequence exists because $X$ is separable. Set

$$J_\ell = \left\{ B(x_i, r_B) : B(x_i, r_B) \cap A(x_B, r_B, \ell) \neq \emptyset \right\}, \quad \ell \geq 0.$$  

It follows from (1.6) that for every $B(x_i, r_B) \in J_\ell$, 

$$V(x_B, r_B) \leq \left( 1 + \frac{d(x_i, x_B)}{r_B^D} \right)^D V(x_i, r_B) \leq C(1 + \ell)^D V(x_i, r_B)$$

and so 

$$\#J_\ell \leq C(1 + \ell)^D \times \frac{V(x_B, (\ell + 1)r_B)}{V(x_B, r_B)} \leq C(1 + \ell)^{D+n} < \infty. \quad (3.16)$$

Then we have 

$$II_2(x) \leq \sup_{B \ni x} \sum_{k+f > 0} \sum_{\ell=0}^{\infty} \sum_{k \geq 1} 2^{-ks} \mu(B)^{-1/2} \times \|P_B(I - e^{-r_B^m L})^K e^{itL} \phi_k(L) P_B(x_i, r_B)\|_{2 \to 2} \|P_{A(x_B, r_B, \ell)}[\varphi_k(L)f]\|_2.$$ 

In this case, since $j \geq (m-1)k + m \log_2(2 + 2|t|)$ and so $r_B \geq c2^{(m-1)k/m}(1+|t|)$ with $c = 2^{(m-1)/m} \geq 1/4$, we apply Proposition 2.3 to see that for every $B(x_i, r_B) \in J_\ell$ with $\ell \geq 7, 8, \ldots$, 

$$\|P_B(I - e^{-r_B^m L})^K e^{itL} \phi_k(L) P_B(x_i, r_B)\|_{2 \to 2} \leq C \left( 1 + \frac{d(B, (x_i, r_B))}{2^{(m-1)k/m}(1+|t|)} \right)^{-M} \leq C(1 + \ell)^{-M} \quad (3.17)$$

for every $M > 0$. For $\ell = 0, 1, \ldots, 6$, it follows from $L^2$-boundedness of $(I - e^{-r_B^m L})^K e^{itL} \phi_k(L)$ that $\|P_B(I - e^{-r_B^m L})^K e^{itL} \phi_k(L) P_B(x_i, r_B)\|_{2 \to 2} \leq C$. These, in combination with the fact that for every $x \in B$, 

$$\|P_{A(x_B, r_B, \ell)}[\varphi_k(L)f]\|_2 \leq \mu((\ell + 1)B)^{1/2} \left( \int_{(\ell + 1)B} |\varphi_k(L)f(y)|^2 d\mu(y) \right)^{1/2} \leq C(\ell + 1)^{n/2} \mu(B)^{1/2} M_2(\varphi_k(L)f)(x),$$

imply 

$$II_2(x) \leq C \sum_{k \geq 1} \sum_{\ell=0}^{\infty} 2^{-ks} \left( 1 + \ell \right)^{-D-3n/2} M_2(\varphi_k(L)f)(x) \leq C \sum_{k \geq 1} 2^{-ks} M_2(\varphi_k(L)f)(x).$$
as long as \( M \) in (3.17) is chosen large enough so that \( M > D + 2n \). As a consequence, we have that for \( 2 < p < p' \),

\[
\|II_2\|_p \leq C \left( \sum_{k \geq 1} 2^{-ks} \mathcal{M}_2 (\varphi_k (L) f) \right)^{1/p} \leq C \left( \sum_{k \geq 1} \left\| \varphi_k (L) f \right\|_p^p \right)^{1/p}.
\]

Combining the estimates of \( II_1 \) and \( II_2 \) we obtain the estimate of \( II \) as desired.

**Estimate of the term \( III(x) \).** As to be seen later, the term \( III(x) \) is the major one. Similar to the estimates for \( II \) and \( I \) above, we write

\[
III(x) 
\leq \sup_{B \ni x} \left( \int_B \left| (I - e^{-r \|L\|} K e^{itL} \varphi_0 (L) \varphi_0 (L) f) (y) \right|^2 \, d\mu(y) \right)^{1/2} + \sup_{B \ni x} \left( \sum_{k \geq 1} \left( \sum_{k \geq 1} \left\| \varphi_k (L) f \right\|_p \right)^{1/p} \right)^{1/2}.
\]

Again, it is clear that \( \|III_1\|_p \leq C (1 + |t|^s) \|\varphi_0 (L) (f)\|_p \). It suffices to verify \( III_2(x) \).

For a given \( x \in X \) and a ball \( x \in B_j = B(x_{B_j}, r_{B_j}) \) with \( r_{B_j} \in \{2^{-1}, 2^j\} \). We define a family of operators \( \{A_{rB_j}\}_{j=1}^{\infty} \) with non-negative kernels \( \{a_{rB_j} (x, y)\}_{j=1}^{\infty} \) such that

\[
a_{rB_j} (x, y) = \frac{1}{\mu(B(x, 2r_{B_j}))} \chi_{B(x, 2r_{B_j})}(y).
\]

We will use

\[
A_{rB_j} g(x) = \int_X a_{rB_j} (x, y) g(y) \, d\mu(y)
\]

to replace the mean value \( \bar{f}_{B_j} \) in the term \( III_2(x) \). It is seen that for every non-negative function \( g \in L^{1}_{\text{loc}}(X) \) and \( B_j \) containing \( x \),

\[
\int_{B_j} g(y) \, d\mu(y) \leq \left( \frac{\mu(B(x_{B_j}, 3r_{B_j}))}{\mu(B_j)} \right) A_{rB_j} g(x) \leq C A_{rB_j} g(x)
\]

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and so $III_2(x) \leq C \widetilde{III}_2(x)$, where

$$
\widetilde{III}_2(x) := \sup_{j \in \mathbb{Z}} \left( \sum_{k+j > 0, j < (m-1)k + m \log_2 (2+2|t|), k \geq 1} 2^{-2ks} \right)^{1/2} A_{rB_j} \left( \left\| ( I - e^{-r_{B_j}^{L}} ) K e^{itL} \phi_k(L)[\varphi_k(L) f]^{B_j^{(k)}} \right\|^2(x) \right).
$$

(3.18)

Now for every $k \geq 1$, we choose a sequence $(x^{(k)}_\tau)_{\tau \in X}$ such that $d(x^{(k)}_\tau, x^{(k)}_\ell) > 2^{k(m-1)/m} (1 + |t|)$ for $\tau \neq \ell$ and $\sup_{x \in X} \inf_{\tau} d(x, x^{(k)}_\tau) \leq 2^{k(m-1)/m} (1 + |t|)$. Such sequence exists because $X$ is separable. Let $B^{(k), \ast}_\tau = B(x^{(k)}_\tau, 2^{k(m-1)/m} (1 + |t|))$ and define $B^{(k)}_\tau$ by the formula

$$
B^{(k)}_\tau = \bar{B} \left( x^{(k)}_\tau, 2^{k(m-1)/m} (1 + |t|) \right) \setminus \bigcup_{\ell < \tau} \bar{B} \left( x^{(k)}_\ell, 2^{k(m-1)/m} (1 + |t|) \right),
$$

where $\bar{B} \left( x^{(k)}_\tau, r \right) = \{ y \in X : d(x^{(k)}_\tau, y) \leq r \}$. We cover $X$ by a grid $\mathcal{R}_k$ consisting of such $\{ B^{(k)}_\tau \}_\tau$, that is, $X = \bigcup_{B^{(k)}_\tau \in \mathcal{R}_k} B^{(k)}_\tau$. For every $B^{(k)}_\tau \in \mathcal{R}_k$, we denote by $f^{B^{(k)}_\tau} = f \chi_{B^{(k)}_\tau}$. Hence, one writes

$$
\widetilde{III}_2(x) \leq III_{21}(x) + III_{22}(x),
$$

(3.19)

where

$$
III_{21}(x) = \sup_{j \in \mathbb{Z}} \left( \sum_{k+j > 0, j < (m-1)k + m \log_2 (2+2|t|), k \geq 1} 2^{-2ks} \right)^{1/2} A_{rB_j} \left( \left\| \sum_{B^{(k)}_\tau \in \mathcal{R}_k} \chi_{B^{(k), \ast}_\tau} ( I - e^{-r_{B_j}^{L}} ) K e^{itL} \phi_k(L)[\varphi_k(L) f]^{B^{(k)}_\tau} \right\|^2 \right)(x),
$$

and $III_{22}(x)$ is the analogous expression where $\chi_{B^{(k), \ast}_\tau}$ is replaced with $\chi_{X \setminus B^{(k), \ast}_\tau}$.
Let us first estimate the term \(III_{21}(x)\). Using the embedding \(\ell^p \rightarrow \ell^{\infty}\), the bounded overlap of \(B_{\ell}^{(k)}\) and Minkowski’s inequality, we obtain that the \(L^p\)-norm of the term \(III_{21}(x)\) is less than

\[
C \left( \sum_{j \in \mathbb{Z}} \left\| \sum_{k+j > 0} \sum_{j < (m-1)k + m \log_2(2+2|t|)} 2^{-2ks} \right. \right.
\]

\[
\left. \left( A_{rB_{\ell}} \left( \sum_{B_{\ell}^{(1)} \in \mathcal{R}_k} \chi_{B_{\ell}^{(1)}} \left| (1 - e^{-r_{B_{\ell}} L}) K e^{itL} \phi_{k} (L) \phi_{k-j} (L) f \right|_{B_{\ell}^{(1)}} \right)^2 \right)^{1/2} \right\|_{p/2}^{1/p}.
\]

To continue, we claim that the supports of the functions \(\{A_{rB_{\ell}} (\chi_{B_{\ell}^{(1)}})\}_{\ell}\) have bounded overlap, uniformly in \(k\). Assume this at the moment. Then by setting \(\ell = k + j > 0\), applying Minkowski’s inequality, and the above claim, we obtain that

\[
\|III_{21}\|_p \leq \sum_{\ell > 0} E_{\ell},
\]

where

\[
E_{\ell} := \left( \sum_{j < \ell} \sum_{B_{\ell}^{(j)} \in \mathcal{R}_{\ell-j}} 2^{-(\ell-j)sp} \right)
\left\| A_{rB_{\ell}} \chi_{B_{\ell}^{(j)}} \left( (1 - e^{-r_{B_{\ell}} L}) K e^{itL} \phi_{k} (L) \phi_{k-j} (L) f \right)_{B_{\ell}^{(j)}} \right\|_{p/2}^{1/p}.
\]

We now show the claim. Note that for \(B_{\ell}^{(k)} \in \mathcal{R}_k\), \(B_{\ell}^{(k)}\) has radius \(8 \cdot 2^{k(m-1)/m} (1 + |t|)\). It follows from \(r_{B_{\ell}} \leq 2j/m \leq 2^{k(m-1)/m} (1 + |t|)\) that for fixed \(k\), \(A_{rB_{\ell}} (\chi_{B_{\ell}^{(k)}})(x) \cdot A_{rB_{\ell}} (\chi_{B_{\ell}^{(k)}})(x) = 0\) when \(d(x_{\ell}^{(k)}, x_{\ell}^{(k)}) \geq 20 \cdot 2^{k(m-1)/m} (1 + |t|)\).

From (1.6), we know that

\[
V(x_{\ell}^{(k)}, 2^{k(m-1)/m} (1 + |t|)) \leq \left( 1 + \frac{D}{r_B} \right) V(x_{\ell}^{(k)}, 2^{k(m-1)/m} (1 + |t|)) \leq CV(x_{\ell}^{(k)}, 2^{k(m-1)/m} (1 + |t|)),
\]

\end{document}
which implies

\[
\sup_{\tau} \# \left\{ \ell : d(x_{\tau}^{(k)}, x_{\ell}^{(k)}) \leq 30 \cdot 2^{\frac{k(m-1)}{m}} (1 + |t|) \right\} \leq \sup_x \frac{V(x, 30 \cdot 2^{\frac{k(m-1)}{m}} (1 + |t|))}{V(x, 2^{\frac{k(m-1)}{m}} 2(1 + |t|))} \leq C < \infty.
\]

Next we will show that

\[
E_{\ell} \leq C (1 + |t|)^{s} 2^{-\ell s/m} \left( \sum_{k > 0} \| \varphi_k (L) f \left\|_p \right\|_p \right)^{1/p}.
\]  

(3.20)

Once (3.20) is proven, we see that

\[
\| I I I_{21} \|_p \leq C (1 + |t|)^{s} \left( \sum_{k \geq 1} \| \varphi_k (L) f \left\|_p \right\|_p \right)^{1/p}.
\]  

(3.21)

Let us prove estimate (3.20). First, we observe that for every \( g \in L^1(X) \) and \( p/2 > 1 \),

\[
\| A_{r B_j} \left( x_{B_{\tau}^{(\ell-j)}} \ast g \right) \|_{p/2} \leq \left( \sup_{y \in B_{t}^{(\ell-j)}} \int_X a_{r B_j}^{p/2} (x, y) x_{B_t^{(\ell-j)}} \ast (y) d \mu (x) \right)^{2/p} \| g \|_1 \leq C \sup_{y \in B_{t}^{(\ell-j)}} [V(y, r B_j)^{-1} (\frac{1}{p})] \| g \|_1.
\]  

(3.22)

From this, we see that the term \( E_{\ell} \) is dominated by a constant multiple of

\[
\left( \sum_{j < \ell} \sum_{B_{\tau}^{(\ell-j)} \in \mathcal{R}_{\ell-j}} 2^{-(\ell-j)s p} \sup_{y \in B_t^{(\ell-j)}} [V(y, r B_j)^{-1} (\frac{p}{2} - 1)] \right) \left( \| I - e^{-r B_j^{\ell-j} L} K e^{i t L} \varphi_{\ell-j} (L) (\varphi_{\ell-j} (L) f) B_{\tau}^{(\ell-j)} \|_2 \right)^{1/p}.
\]

Since the operator \( (I - e^{-r B_j^{\ell-j} L} K e^{i t L} \varphi_{\ell-j} (L)) \) is uniformly bounded on \( L^2(X) \) and \( [\varphi_{\ell-j} (L) f] B_{\tau}^{(\ell-j)} \) is supported on the ball \( B_{\tau}^{(\ell-j)} \), we see by the Hölder inequality that the term \( E_{\ell} \) is controlled by a constant multiple of

\[
\left( \sum_{j < \ell} 2^{-(\ell-j)s p} \sup_{B_{\tau}^{(\ell-j)} \in \mathcal{R}_{\ell-j}} \left( \frac{\mu (B_t^{(\ell-j)})}{\mu (B(y, r B_j))} \right)^{p/2 - 1} \left( \| \varphi_{\ell-j} (L) f \left\|_{B_{\tau}^{(\ell-j)}} \right\|_p \right)^{1/p}.
\]
Note that for $y \in B_{\tau}^{(\ell-j),*}$,

$$\left( \frac{\mu(B_{\tau}^{(\ell-j)})}{\mu(B(y, r_{B_j}))} \right) \leq C (1 + |t|)^n 2^{\frac{m}{2}} \left( (\ell-j)(m-1)-j \right),$$

which yields

$$E_{\ell} \leq C (1 + |t|)^n 2^{\frac{1}{2}} \left( \sum_{j < \ell} 2^{-n(\ell-j)((\ell-j)-1)} 2^{\frac{m}{2}} \left( (\ell-j)(m-1)-j \right) \right) \left( \sum_{B_{\tau}^{(\ell-j)} \in R_{\ell-j}} \left\| \varphi_{\ell-j}(L) f \right\|_p \right)^{1/p},$$

$$= C (1 + |t|)^s 2^{-\ell s/m} \left( \sum_{j < \ell} \sum_{B_{\tau}^{(\ell-j)} \in R_{\ell-j}} \left\| \varphi_{\ell-j}(L) f \right\|_p \right)^{1/p}.$$

After summation in $B_{\tau}^{(\ell-j)} \in R_{\ell-j}$, we obtain

$$E_{\ell} \leq C (1 + |t|)^s 2^{-\ell s/m} \left( \sum_{j < \ell} \varphi_{\ell-j}(L) f \right)^{1/p},$$

$$\leq C (1 + |t|)^s 2^{-\ell s/m} \left( \sum_{k \geq 1} \varphi_k(L) f \right)^{1/p}.$$

This finishes the proof of (3.20) and concludes the desired estimate (3.21) for the term $III_{21}$.

Concerning the term $III_{22}$, we use the embedding $\ell^p \to \ell^\infty$ and the Minkowski inequality to see that the term $\|III_{22}\|_p$ is controlled by

$$\left( \sum_{\mathbb{Z}} \right) \left( \sum_{k+j>0} 2^{-2ks} \right) \left( \sum_{j<\ell-k} \sum_{B_{\tau}^{(\ell-j)} \in R_{k}} \left\| X_{X \setminus B_{\tau}^{(\ell-j)}*} (I - e^{-it_{B_j}^m} L) K e^{itL} \phi_k(L) [\varphi_k(L) f]_{B_{\tau}^{(\ell-j)}} \right\|_p^{p/2} \right)^{1/p}.$$
The proof of Theorem 1.1 will be done if we can show that

\[
\left\| A_{rB_j} \left( \sum_{B^{(k)}_\tau \in \mathcal{B}_k} \mathcal{X}_{X \setminus B^{(k)}_\tau}^* (I - e^{-r^{(m)}_{B_j} L}) K e^{itL} \phi_k (L) [\varphi_k (L) f]^{B^{(k)}_\tau} \right)^2 \right\|_{p/2} \leq C (1 + |t|)^{n(1 - \frac{3}{p})} 2^{\frac{n}{m} [k(m - 1) - j] (1 - \frac{3}{p})} \| \varphi_k (L) f \|_p^2 \tag{3.23}
\]

since from it, we recall that \( s = n|1/2 - 1/p| \) to see that

\[
\| II_{22} \|_p \leq C (1 + |t|)^s \sum_{j \in \mathbb{Z}} \left( \sum_{k+j>0 \atop k \geq 1} 2^{-2ks} 2^{\frac{n}{m} [k(m - 1) - j] (1 - \frac{3}{p})} \| \varphi_k (L) f \|_p \right)^{p/2} 1/p \leq C (1 + |t|)^s \sum_{\ell > 0} 2^{-\ell s/m} \left( \sum_{j < \ell} \| \varphi_{\ell-j} (L) f \|_p \right)^{p/2} 1/p \leq C (1 + |t|)^s \left( \sum_{k \geq 1} \| \varphi_k (L) f \|_p \right)^{1/p} . \tag{3.24}
\]

It remains to prove (3.23). Observe that \( j < (m - 1)k + m \log_2 (2 + 2|t|) \), and \( r_{B_j} \leq 2^{(m-1)k/m+1} (1 + |t|) \). Fix \( x \in X, k \geq 1 \) and \( j \in \mathbb{Z} \), we consider the following three cases of \( x^{(k)}_\tau \):

**Case 1:** \( d(x^{(k)}_\tau, x) \leq 6 \cdot 2^{(m-1)k/m} (1 + |t|) \).

In this case, for any \( z \in B(x, 2r_{B_j}) \),

\[
d(z, x^{(k)}_\tau) \leq d(z, x) + d(x^{(k)}_\tau, x) \leq 8 \cdot 2^{(m-1)k/m} (1 + |t|) ;
\]

and so \( B(x, 2r_{B_j}) \cap \left( X \setminus B^{(k),*}_j \right) = \emptyset \).

**Case 2:** \( d(x^{(k)}_\tau, x) \geq 10 \cdot 2^{(m-1)k/m} (1 + |t|) \).

In this case, for any \( z \in B(x, 2r_{B_j}) \)

\[
d(z, x^{(k)}_\tau) \geq d(x^{(k)}_\tau, x) - d(z, x) \geq 8 \cdot 2^{(m-1)k/m} (1 + |t|) ,
\]

and so \( B(x, 2r_{B_j}) \subseteq X \setminus B^{(k),*}_j \).

**Case 3:** \( 6 \cdot 2^{(m-1)k/m} (1 + |t|) \leq d(x^{(k)}_\tau, x) \leq 10 \cdot 2^{(m-1)k/m} (1 + |t|) \).
In this case, we see that 
\[
d(B^{(k)}_\tau, B(x, 2r_{B_j})) \geq 2^{(m-1)k/m}(1 + |t|), \text{ and}
\]
\[
\# \left\{ \tau : 6 \cdot 2^{(m-1)k/m}(1 + |t|) \leq d(x^{(k)}_\tau, x) \leq 10 \cdot 2^{(m-1)k/m}(1 + |t|) \right\} \leq \sup_x \frac{V(x, 2^{(m-1)k/m+1}(1 + |t|))}{V(x, 2^{(m-1)k/m-2}(1 + |t|))} \leq C < \infty. \tag{3.25}
\]

From Cases 1, 2 and 3, we see that there exists a constant \( C > 0 \) independent of \( x \) and \( j \) such that

\[
A_{r_{B_j}} \left( \left| \sum_{B^{(k)}_\tau \in \mathcal{R}_k} \mathcal{X}_{X \setminus B^{(k)}_\tau} (I - e^{-r^{m}_{B_j} L}) K e^{i t L} \psi_k(L) \left[ \psi_k (L) f \right]^{B^{(k)}_\tau} \right|^2 \right) (x) \leq D_1(x) + C D_2(x),
\]

where

\[
D_1(x) := A_{r_{B_j}} \left( \left| \sum_{\tau : d(x^{(k)}_\tau, x) \geq 10 \cdot 2^{(m-1)k/m}(1 + |t|)} \mathcal{X}_{X \setminus B^{(k)}_\tau} (I - e^{-r^{m}_{B_j} L}) K e^{i t L} \psi_k(L) \left[ \psi_k (L) f \right]^{B^{(k)}_\tau} \right|^2 \right) (x)
\]

and

\[
D_2(x) := \left( \sum_{\tau : 6 \cdot 2^{(m-1)k/m}(1 + |t|) \leq d(x^{(k)}_\tau, x) \leq 10 \cdot 2^{(m-1)k/m}(1 + |t|)} A_{r_{B_j}} \left( \left| (I - e^{-r^{m}_{B_j} L}) K e^{i t L} \psi_k(L) \left[ \psi_k (L) f \right]^{B^{(k)}_\tau} \right|^2 \right) \right) (x).
\]

Let us estimate the term \( D_1(x) \) by adapting an argument as in the term \( E_\ell \). First note that

\[
X = \bigcup_{B^{(k)}_{\tau_1} \in \mathcal{R}_k} B^{(k)}_{\tau_1}.
\]

Then we write
Applying (3.22), we see that the \( L^{p/2} \)-norm of \( D_1(x) \) is dominated by a constant times

\[
\sum_{B_{\tau_1}^{(k)} \in \mathcal{B}_k} \sup_{y \in B_{\tau_1}^{(k)}} [V(y, r_{B_{\tau}})]^{-(1 - \frac{2}{p})} \left\| P_{B_{\tau_1}^{(k)}} (I - e^{-r_{B_{\tau}} L}) K e^{i t L} \phi_k(L) \left( \sum_{\tau : d(x_1^{(k)}, x_2^{(k)}) \geq 10^{2(m-1)k/m}(1 + |t|)} |\varphi_k(L) f| B_{\tau}^{(k)} \right) \right\|_2^2.
\]

Observe that for every \( B_{\tau_1}^{(k)} \in \mathcal{B}_k \),

- If \( y \in B_{\tau_1}^{(k)} \), then

\[
\left( \frac{\mu(B_{\tau}^{(k)})}{V(y, r_{B_{\tau}})} \right) = \left( \frac{\mu(B_{\tau_1}^{(k)})}{\mu(B_{\tau_1}^{(k)})} \right) \times \left( \frac{\mu(B_{\tau_1}^{(k)})}{V(y, r_{B_{\tau_1}})} \right) \leq C (1 + |t|)^n 2^{n[k(m-1)-j]/m} \left( 1 + \frac{d(B_{\tau_1}^{(k)}, B_{\tau}^{(k)})}{2^{(m-1)k/m}(1 + |t|)} \right)^D.
\]

- A simple calculation shows that

\[
\# \left\{ \tau : 2^{(m-1)k/m} + u (1 + |t|) \leq d(B_{\tau_1}^{(k)}, B_{\tau}^{(k)}) \leq 2^{(m-1)k/m} + u + 1 (1 + |t|) \right\} \leq C 2^{u(D+n)}
\]

and so

\[
\sum_{\tau : d(B_{\tau_1}^{(k)}, B_{\tau}^{(k)}) > 10^{2(m-1)k/m}(1 + |t|)} \left( 1 + \frac{d(B_{\tau_1}^{(k)}, B_{\tau}^{(k)})}{2^{(m-1)k/m}(1 + |t|)} \right)^{-M} \leq \sum_{u=2}^{\infty} \sum_{\tau : 2^{(m-1)k/m} + u (1 + |t|) \leq d(B_{\tau_1}^{(k)}, B_{\tau}^{(k)}) \leq 2^{(m-1)k/m} + u + 1 (1 + |t|)} 2^{-u M} \leq C \sum_{u=2}^{\infty} 2^{-u (M-(D+n))} \leq C
\]

(3.26)
for $M > D + n$. Since the function $[\phi_k(L)f]^B_{\tau}$ is supported on the ball $B_{\tau}^{(k)}$, we apply Proposition 2.3 with $M > D + n$ and the Hölder inequality to see that $\|D_1\|_{p/2}$ is controlled by a constant multiple of

$$(1 + |t|)^n(1-\frac{2}{p}) 2^\frac{n}{m}|k(m-1)\cdot j|^{(1-\frac{2}{p})} \sum_{B_{\tau_1}^{(k)} \in \mathcal{B}_k} \sum_{\tau: d(x_{\tau_1}^{(k)}, x_{\tau_1}^{(k)}) \geq 10 \cdot 2^{(m-1)k/m}(1+|t|)} \left(1 + \frac{d(B_{\tau_1}^{(k)}, B_{\tau_1}^{(k)})}{2^{(m-1)k/m}(1+|t|)}\right)^{-M} \left\|\phi_k(L)f\right\|_{B_{\tau_1}^{(k)}}^p.$$ 

Changing the order of the summation for $\tau_1$ and $\tau$ and by (3.26), we obtain

$$\|D_1\|_{p/2} \leq C \left(1 + |t|\right)^n(1-\frac{2}{p}) 2^\frac{n}{m}|k(m-1)\cdot j|^{(1-\frac{2}{p})} \left\|\phi_k(L)f\right\|_{p}^p.$$ 

For the term $D_2$, we follow the similar approach as above in $D_1(x)$ to show that for every $\tau$ with $6 \cdot 2^{(m-1)k/m}(1+|t|) \leq d(x_{\tau}^{(k)}, x) \leq 10 \cdot 2^{(m-1)k/m}(1+|t|)$,

$$\left\|A_{r_{B_j}}\left(I - e^{-r_{B_j}^{(k)}L}\right)K e^{i(t)\phi_k(L)[\phi_k(L)f]}B_{\tau}^{(k)}\right\|_{p/2} \leq (1 + |t|)^n(1-\frac{2}{p}) 2^\frac{n}{m}|k(m-1)\cdot j|^{(1-\frac{2}{p})} \left\|\phi_k(L)f\right\|_{p}^p,$n

and so by (3.25) in Case 3, we have that $\|D_2\|_{p/2} \leq (1+|t|)^n(1-\frac{2}{p}) 2^\frac{n}{m}|k(m-1)\cdot j|^{(1-\frac{2}{p})} \left\|\phi_k(L)f\right\|_{p}^p$. This finishes the proof of (3.23) and thereby (3.24) for the term $III_{22}$ and concludes that

$$\|III_2\|_{p} \leq C \left(1 + |t|\right)^n(1-\frac{1}{p}) \left(\sum_{k \geq 1} \left\|\phi_k(L)f\right\|_{p}^p\right)^{1/p}.$$ 

Combining the estimates of $III_1(x)$ and $III_2(x)$, we obtain the estimate for $III(x)$ as desired.

Finally, we combine estimates of $I$, $II$ and $III$ to obtain the estimate (3.8), and complete the proof of Theorem 1.1.

**Proof of Corollary 1.2** The proof of Corollary 1.2 can be obtained by making a minor modifications with [34, Theorem 7.12], and we skip it here. 

We mention that our Theorem 1.1 can also apply to prove existence of solution (in $L^p$ spaces) to the Schrödinger equation with initial data $f$ in the domain of some power of the operator $L$. It can also be formulated in terms of generation of $C$-regularized groups. We will not develop this here, we refer the reader to de Laubenfels [15] and Ouhabaz’s monograph [34, Chapter 7].
4 An application to Riesz means of the solutions of the Schrödinger equations

The aim of this section is to prove Theorem 1.3. Recall that when \( L \) is the Laplacian on the Euclidean spaces \( \mathbb{R}^n \), the Riesz mean \( I_s(t)(\Delta) \) in (1.9) was studied by Sjöstrand [39]. It was shown that \( I_s(t)(\Delta) \) is uniformly bounded in \( t \in \mathbb{R}\{0\} \) for \( s > n|1/2 - 1/p| \), and they are unbounded for \( s < n|1/2 - 1/p| \). The result was generalized to Lie groups and Riemannian manifolds by Lohoué [29] and by Alexopoulos [1]. In the abstract setting of operators on metric measure space, this result was extended by Carron, Coulhon and Ouhabaz [9] for operators with the Gaussian upper bounds, and by Blunck [4] for generalized Gaussian estimates for the operators. More precisely, the work of Blunck [4, Proposition A] shows that under the assumption of generalized Gaussian estimate (GGE\(_{p_0,p_0',m}\)) for some \( 1 \leq p_0 < 2 \), then the Riesz means operator \( I_s(t)(L) \) is bounded on \( L^p(X) \) uniformly for all \( t \in \mathbb{R}\{0\} \), \( p \in (p_0, p'_0) \) and \( s > n|1/2 - 1/p| \). To prove the endpoint estimate for \( s = n|1/2 - 1/p| \), we need following result.

**Theorem 4.1** Suppose that \((X, d, \mu)\) is a space of homogeneous type with a dimension \( n \). Suppose that \( L \) satisfies (GGE\(_{p_0,p_0',m}\)) for some \( 1 \leq p_0 < 2 \). Then for every \( p \in (p_0, p'_0) \), there exists a constant \( C = C(n, p) > 0 \) such that for all \( t \in \mathbb{R}\{0\} \),

\[
\left\| (I + |t|L)^{-s}e^{itL}f \right\|_p \leq C\|f\|_p, \quad s \geq n\left|\frac{1}{2} - \frac{1}{p}\right|.
\]

(4.1)

As a consequence, this estimate (4.1) holds for all \( 1 < p < \infty \) when the heat kernel of \( L \) satisfies a Gaussian upper bound (GE\(_m\)).

**Proof** We prove this theorem by following the approach in the proof of Theorem 1.1 by using Proposition 2.6 instead of Proposition 2.3. For the details, we leave to the reader. \(\square\)

**Proof of Theorem 1.3** The proof of Theorem 1.3 is inspired by the idea of [39]. Take a function \( \Phi \in C^\infty(\mathbb{R}) \) such that \( \Phi(t) = 0 \) if \( t < 1/2 \) and \( \Phi(t) = 1 \) if \( t > 1 \). Define function \( F \) by

\[
F(u) = I_s(1)(u) - C_s \Phi(u)u^{-s}e^{-iu},
\]

where \( C_s \) is defined by

\[
s \int_{-\infty}^1 (1 - \lambda)^{s-1}e^{i\lambda u}d\lambda = C_s u^{-s}e^{iu}, \quad u > 0.
\]

It is seen that for \( 0 < u \leq 1 \) and \( k \in \mathbb{N} \),

\[
\frac{d^k}{du^k} F(u) \leq C,
\]
and for \( u > 1 \) and \( k \in \mathbb{N} \),

\[
\frac{d^k}{du^k} F(u) \leq C u^{-k}.
\]

See [39, Lemma 2.1]. Hence, for every \( \beta > (n+1)/2 \) we have that \( \sup_{R>0} \| \eta \delta_R F \|_{C^\beta} \leq C \), and so \( \sup_{R>0} \| \eta \delta_R F(t) \|_{C^\beta} \leq C \) with a constant \( C > 0 \) independent of \( t > 0 \). Then we apply (a) of Proposition 2.7 to know that \( F(tL) \) is bounded on \( L^p(X) \) for all \( p_0 < p < p_0' \). Notice that for every \( t > 0 \),

\[
F(tL) = I_s(t)(L) - C_s \Phi(tL)(tL)^{-s} e^{-itL}.
\] (4.2)

This yields that for every \( t > 0 \),

\[
\| I_s(t)(L) \|_{p \to p} \leq \| F(tL) \|_{p \to p} + C \| \Phi(tL)(tL)^{-s} e^{-itL} \|_{p \to p} \leq C + C \| \Phi(tL)(tL)^{-s} (1 + tL)^s \|_{p \to p} \| (1 + tL)^{-s} e^{-itL} \|_{p \to p}.\] (4.3)

Applying (a) of Proposition 2.7 again, we have that \( \| \Phi(tL)(tL)^{-s} (1 + tL)^s \|_{p \to p} \leq C \). This, in combination with (4.1) in Theorem 4.1, implies \( \| I_s(t)(L) \|_{p \to p} \leq C \) for \( t > 0 \).

Since \( I_s(t)(L) = \overline{I}_s(-t)(L) \) for \( t < 0 \), we have that \( \| I_s(t)(L) \|_{p \to p} \leq C \) for \( t < 0 \). The proof of Theorem 1.3 is complete. \( \square \)

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