The Rogers–Ramanujan recursion and intertwining operators

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Abstract

We use vertex operator algebras and intertwining operators to study certain substructures of standard \(A_1^{(1)}\)-modules, allowing us to conceptually obtain the classical Rogers-Ramanujan recursion. As a consequence we recover Feigin-Stoyanovsky’s character formulas for the principal subspaces of the level 1 standard \(A_1^{(1)}\)-modules.

1 Introduction

Vertex operator constructions of affine Lie algebras can be, and indeed have been, used to prove (or conjecture) many nontrivial combinatorial identities. Among them, the most celebrated are the classical Rogers-Ramanujan identities:

\[
\prod_{i \geq 0} \frac{1}{(1 - q^{5i+1})(1 - q^{5i+4})} = \sum_{n \geq 0} \frac{q^{n^2}}{(q)_n}, \tag{1.1}
\]

\[
\prod_{i \geq 0} \frac{1}{(1 - q^{5i+2})(1 - q^{5i+3})} = \sum_{n \geq 0} q^{n^2+n} \frac{1}{(q)_n}, \tag{1.2}
\]

where

\[(q)_n = (1-q)(1-q^2) \cdots (1-q^n). \tag{1.3}\]

These identities can be expressed in terms of numbers of partitions. For example, the first identity states that the number of partitions of a nonnegative integer with parts of the form \(5i+1\) and \(5i+4\) is equal to the number of partitions such that the difference between consecutive parts is at least 2; this is the so-called difference-two condition. For the classical history of these identities see [A].

A proof of these identities was obtained in [LW2, LW3] by means of twisted vertex operators and “Z-operators” for level-3 standard \(A_1^{(1)}\)-modules. In fact,
the left-hand side of (1.1) (the product side) was already known to be the principally specialized character (associated with the principal gradation) of the vacuum subspace, with respect to a certain Heisenberg algebra, of a certain standard $A^{(1)}_1$-module ($\text{LM}$, $\text{LW1}$), from the Weyl-Kac character formula $\text{K1}$. In $\text{LW2}$-$\text{LW3}$, $Z$-operators $Z(j)$ ($j \in \mathbb{Z}$) were introduced and certain relations were found among them, giving a basis of the vacuum subspace consisting of monomials in the $Z(j)$'s applied to the highest weight vector; at the same time, this construction extended the twisted vertex operator construction $\text{LW1}$ of the basic (level 1) standard $A^{(1)}_1$-modules to all higher levels. This basis is identified with partitions by virtue of the indices $j$, and indeed, consecutive indices $j$ satisfy the difference-two condition. In the representation-theoretic study of Rogers-Ramanujan-type identities, a basis of this sort, of a space of this sort, has come to be called a “combinatorial basis” because (in this case) the difference-two condition is visible from the basis. Such a basis has also come to be called a “fermionic basis,” since the difference-two condition is an analogue of the difference-one condition, which corresponds to the classical fermionic statistics of the Pauli exclusion principle; the difference-two condition is in fact much more subtle than the difference-one condition, reflecting the fact that the Rogers-Ramanujan identities (and their generalizations) are very subtle. The statement that the “character” of (in this original case) the vacuum space with respect to the Heisenberg algebra of the relevant level-3 standard $A^{(1)}_1$-module equals the right-hand side (the sum side) of (1.1) has correspondingly come to be called a “fermionic character formula.”

Analogues of these results were established in $\text{LP}$ for untwisted, rather than twisted, vertex operators and $Z$-operators, this time yielding untwisted $Z$-operator constructions, and in fact, combinatorial (fermionic) bases, of the standard $A^{(1)}_1$-modules. In turn, this gave what came to be called “fermionic character formulas” analogous to, but different from, the Rogers-Ramanujan identities. This work extended the untwisted vertex operator construction of the basic modules for $A^{(1)}_1$ ($\text{PK}$, $\text{S}$) to all the higher level modules.

More recently, Feigin and Stoyanovsky $\text{FS1}$ found another circle of ideas that led to the Rogers-Ramanujan identities themselves, using the viewpoint of Lie algebras and loop groups rather than that of vertex operators. They observed that what they called the “principal subspace” $W(\Lambda_0)$ (defined in Section 3 below) of the basic $A^{(1)}_1$-module $L(\Lambda_0)$ has a combinatorial basis that satisfies the difference-two condition, giving the sum side of (1.1). (This use of the term “principal” is unrelated to the principal gradation mentioned above.) However, for principal subspaces there is no straightforward analogue of the Weyl-Kac character formula and, in order to get the product side in (1.1), Feigin and Stoyanovsky announced a result that, as a consequence, computes, for all levels, the characters of the principal subspaces via an analogue of the Atiyah-Bott fixed point theorem for an infinite-dimensional flag manifold. The combinatorial bases obtained in $\text{FS1}$-$\text{FS2}$ were also implicitly obtained, for all levels for $A^{(1)}_1$, by Meurman-Primc $\text{MP}$, using vertex operator algebra theory.
In [G], Georgiev extended the character formulas obtained in [FS1] to a family of standard $A_n^{(1)}$-modules. This was done by the explicit construction of combinatorial spanning sets and bases for the principal subspaces. In order to prove the linear independence of the spanning sets, Georgiev used the theory of vertex operator algebras, including certain intertwining operators.

The present paper is motivated by the fact that many classical, and also recent, proofs and treatments of the Rogers-Ramanujan identities and generalizations are based not on the difference-two condition itself, but rather on a certain classical recursion formula (see [A]): Noting that the sum sides of the two Rogers-Ramanujan identities can be obtained as specializations of the generating function

$$F(x, q) = \sum_{n \geq 0} \frac{x^n q^{n^2}}{(q)_n}$$

for $x = 1$ and $x = q$, we observe that this generating function satisfies

$$F(x, q) = F(xq, q) + xqF(xq^2, q).$$

We will call this formula the Rogers-Ramanujan recursion [RR]; see [A] for the role of this recursion in Rogers’s and Ramanujan’s work. So far, this recursion has not appeared in a fundamental way in the vertex-operator studies of the Rogers-Ramanujan identities and related identities, and we wanted to try to understand it conceptually from the vertex-operator point of view.

Here is our main result: We show, without constructing a basis, that the character of $W(\Lambda_0)$ satisfies (1.5). We do this setting up an exact sequence

$$0 \rightarrow W(\Lambda_1) \xrightarrow{e^{\alpha/2}} W(\Lambda_0) \xrightarrow{o(e^{\alpha/2})} W(\Lambda_1) \rightarrow 0.$$

Here $W(\Lambda_1)$ is the principal subspace of the second of the level 1 standard $A_1^{(1)}$-modules (see Section 3 below), and $e^{\alpha/2}$ and $o(e^{\alpha/2})$ are the constant factor and the constant term, respectively, of a certain intertwining operator (cf. [FHL]) associated to $L(\Lambda_0)$ viewed as a vertex operator algebra (cf. [FLM]). As a consequence, we recover Feigin-Stoyanovsky’s character formulas for $W(\Lambda_0)$ and $W(\Lambda_1)$. In [G], Georgiev had used the same intertwining operator to prove that the appropriate difference-two combinatorial spanning subset of $W(\Lambda_0)$ is a basis.

The theory of vertex operator algebras and intertwining operators (cf. [FLM], [FHL], [DL]) is a rich subject, and the approach initiated here can be modified to apply to affine Lie algebras of higher rank and level. This will be treated in separate publications. In particular, in [CLM] we have extended several results from this paper in connection with recursions of Rogers and Selberg.
2 Vertex operator constructions associated with $A_1^{(1)}$

In this section we recall some basic definitions and constructions needed in this paper, especially, the vertex operator construction of the distinguished basic $A_1^{(1)}$-module ([FK], [S]) and of its associated vertex operator algebra structure ([B], [FLM]), and the construction [DL] of a distinguished intertwining operator associated with its irreducible modules.

Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ be the 3-dimensional complex simple Lie algebra with a standard basis $\{h, x_\alpha, x_{-\alpha}\}$ such that $[h, x_\alpha] = 2x_\alpha$, $[h, x_{-\alpha}] = -2x_\alpha$ and $[x_\alpha, x_{-\alpha}] = h$, and take the Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$. The standard symmetric invariant nondegenerate bilinear form $\langle x, y \rangle = \text{tr}(xy)$ for $x, y \in \mathfrak{g}$ allows us to identify $\mathfrak{h}$ with its dual $\mathfrak{h}^*$. Take $\alpha \in \mathfrak{h}$ to be the root corresponding to the root vector $x_\alpha$, and take this root to be positive (that is, simple); then $\langle \alpha, \alpha \rangle = 2$ and we have the root space decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}_\pm = \mathbb{C}x_{\pm\alpha}$. Note that under our identifications, $h = \alpha$.

Consider the Lie algebra

$$\mathfrak{sl}(2)^\sim = \mathfrak{sl}(2, \mathbb{C}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $c$ is a nonzero central element and

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + \langle x, y \rangle m\delta_{m+n, 0}c$$

for $x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$. Adjoining the degree operator $d$ such that $[d, x \otimes t^m] = mx \otimes t^m$ and $[d, c] = 0$ gives us the Lie algebra $\mathfrak{sl}(2)^\sim = \mathfrak{sl}(2)^\sim \oplus \mathbb{C}d$, the affine Kac-Moody algebra $A_1^{(1)}$ (cf. [K2]). (In this paper we will not need to use $d$.) Consider the subalgebra

$$\hat{\mathfrak{h}}_\mathbb{Z} = \bigoplus_{m \in \mathbb{Z} \setminus \{0\}} \mathfrak{h} \otimes t^m \oplus \mathbb{C}c,$$

a Heisenberg algebra in the sense that its commutator subalgebra is equal to its center, which is one-dimensional, and also consider the subalgebras

$$\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c, \quad \hat{\mathfrak{n}}_+ = \mathfrak{n}_+ \otimes \mathbb{C}[t, t^{-1}] . \quad (2.1)$$

Throughout the rest of the paper we will write $x(m)$ for the action of $x \otimes t^m$ on an $\mathfrak{sl}(2)^\sim$-module, and we will use the same notation for $\hat{\mathfrak{h}}$-modules.

Now we review the untwisted vertex operator construction of the basic modules obtained in [FK] and [S] (cf. [FLM]).
Let \( P = \frac{1}{2} \mathbb{Z} \alpha \) be the weight lattice and \( Q = \mathbb{Z} \alpha \) the root lattice. Let \( \mathbb{C}[P] \) and \( \mathbb{C}[Q] \) be the corresponding group algebras, with bases \( \{ e^\mu \mid \mu \in P \} \) and \( \{ e^\mu \mid \mu \in Q \} \).

Consider the induced \( \hat{\mathfrak{h}} \)-module

\[
M(1) = U(\hat{\mathfrak{h}}) \otimes_{U(\hat{\mathfrak{h}} \otimes \mathbb{C}[t] \oplus \mathbb{C}[c])} \mathbb{C},
\]

where \( \hat{\mathfrak{h}} \otimes \mathbb{C}[t] \) acts trivially on the one-dimensional module \( \mathbb{C} \) and \( c \) acts as 1. Observe that \( M(1) \) is isomorphic to a polynomial algebra in the infinitely many variables \( h(-n), n > 0 \), that the operators \( h(-n) \) act on this polynomial algebra as multiplication operators, that the operators \( h(n) \) act as certain derivations, and that \( c \) acts as 1. Define the vector spaces

\[
V_P = M(1) \otimes \mathbb{C}[P],
\]

\[
V_Q = M(1) \otimes \mathbb{C}[Q],
\]

\[
V_{Q+\alpha/2} = M(1) \otimes e^{\alpha/2} \mathbb{C}[Q],
\]

so that \( V_P = V_Q \oplus V_{Q+\alpha/2} \). We define an \( \hat{\mathfrak{h}} \)-module structure on \( V_P \) by making \( \hat{\mathfrak{h}} \) act as \( \hat{\mathfrak{h}} \otimes 1 \) on \( V_P = M(1) \otimes \mathbb{C}[P] \) and by making \( \mathfrak{h} = \mathfrak{h} \otimes \mathfrak{t}^0 \) act as \( 1 \otimes \mathfrak{h} \), with \( h(0) \) defined by

\[
h(0)e^\lambda = \langle h, \lambda \rangle e^\lambda
\]

for \( \lambda \in P \).

Let \( x, x_0, x_1, x_2 \ldots \) be commuting formal variables. For \( \lambda, \mu \in P \) let

\[
x^\lambda \cdot e^\mu = x^{\langle \lambda, \mu \rangle} e^\mu.
\]

We extend the action of \( x^\lambda \) to \( \mathbb{C}[P] \) by linearity and to the whole space \( V_P \) by the formula

\[
x^\lambda = 1 \otimes x^\lambda.
\]

Note that \( x^\lambda \) is an \( \text{End} V_P \)-valued formal Laurent series in the formal variable \( x^{1/2} \). Also let

\[
e^\lambda = 1 \otimes e^\lambda,
\]

acting on \( V_P \). For every \( \lambda \in P \), set

\[
Y(1 \otimes e^\lambda, x) = E^-(-\lambda, x)E^+(-\lambda, x)e^\lambda x^\lambda,
\]

an \( \text{End} V_P \)-valued formal Laurent series in \( x^{1/2} \), where

\[
E^-(\lambda, x) = \exp \left( \sum_{n \leq -1} \frac{\lambda(n)}{n} x^{-n} \right),
\]

\[
E^+(\lambda, x) = \exp \left( \sum_{n \geq 1} \frac{\lambda(n)}{n} x^{-n} \right).
\]
More generally, for the generic homogeneous vector \( w \in V_P \) given by
\[
w = h(-n_1 - 1) \cdots h(-n_k - 1) \otimes e^\lambda \tag{2.5}
\]
with \( n_1, \ldots, n_k \geq 0 \), we set
\[
Y(w, x) = \prod_{i=1}^k \left\{ \frac{1}{n_i!} \left( \frac{d}{dx} \right)^{n_i} h(x) \right\} Y(1 \otimes e^\lambda, x) ; \tag{2.6}
\]
where \( h(x) = \sum_{n \in \mathbb{Z}} h(n)x^{-n-1} \) and \( \cdot \cdot \cdot \) stands for the normal ordering operation, which places the operators \( \lambda(n) \) with nonnegative \( n \) to the right and with negative \( n \) to the left. This formula indeed determines a well-defined linear map from \( V_P \) to the vector space of \( \text{End} V_P \)-valued formal Laurent series in \( x^{1/2} \) (cf. \[FLM\]).

We define \( \Lambda_0, \Lambda_1 \in (h \oplus \mathbb{C}c)^* \) by:
\[
\langle \Lambda_i, c \rangle = 1, \quad \langle \Lambda_i, h \rangle = \delta_{i,1}, \quad i = 0, 1.
\]

**Theorem 2.1** (\[FK\], \[S\]) The vector space \( V_P \) carries an \( \mathfrak{sl}(2) \)\( \hat{\cdot} \)-module structure uniquely determined by the condition that for \( m \in \mathbb{Z} \), the action \( x^\pm \alpha(m) \) of \( x^\pm \alpha \otimes t^m \) is given by the coefficient of \( x^{-m-1} \) in \( Y(1 \otimes e^\pm \alpha, x) \). This module has level 1 (that is, \( c \) acts as 1). Moreover, the direct summands \( V_Q \) and \( V_{Q+\alpha/2} \) of \( V_P \) are the basic (irreducible) \( \mathfrak{sl}(2) \)\( \hat{\cdot} \)-modules with highest weights \( \Lambda_0 \) and \( \Lambda_1 \) and with highest weight vectors \( 1 = 1 \otimes 1 \) and \( e^{\alpha/2} = 1 \otimes e^{\alpha/2} \), respectively.

In the notation of \[K2\],
\[
V_Q \cong L(\Lambda_0), \quad V_{Q+\alpha/2} \cong L(\Lambda_1).
\]

The following is well known and not hard to prove:

**Proposition 2.1** Taking \( \lambda = \alpha \) in \( \[Z4\] \), we have that the square \( Y(1 \otimes e^\alpha, x)^2 \) of the operator \( Y(1 \otimes e^\alpha, x) \) on \( V_P \) is well defined, and
\[
Y(1 \otimes e^\alpha, x)^2 = 0. \tag{2.7}
\]

It is easy to see that for \( m \in \mathbb{Z} \) and \( \mu \in P \),
\[
x_\alpha(m)e^\mu = e^\mu x_\alpha(m + \langle \alpha, \mu \rangle), \tag{2.8}
\]
and in particular,
\[
x_\alpha(m)e^\alpha = e^\alpha x_\alpha(m + 2) \tag{2.9}
\]
and
\[
x_\alpha(m)e^{\alpha/2} = e^{\alpha/2} x_\alpha(m + 1). \tag{2.10}
\]

We will use the notions of vertex operator algebra, and of module for such a structure, as defined in \[FLM\] and \[FHL\] (cf. \[B\]). We recall the definitions:

A **vertex operator algebra** is a vector space \( V \) equipped, first, with a **vertex operator map**
\[
Y(\cdot, x) : V \longrightarrow (\text{End} V)[[x, x^{-1}]], \tag{2.11}
\]
satisfying the truncation condition: For $u, v \in V$, the formal Laurent series $Y(u, x)v$ is truncated from below. It is also equipped with a $\mathbb{Z}$-grading that is truncated from below, and the homogeneous subspaces are finite-dimensional. In addition, $V$ has a vacuum vector $1$ and a conformal vector $\omega$. The main axiom is the Jacobi identity:

$$x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2)Y(u, x_1) = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2)$$

(2.12)

for $u, v \in V$. Here

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n$$

is the formal delta function, and in the expansions of the three delta-function expressions, the negative powers of the binomials are understood to be expanded in nonnegative powers of the second variable. In addition, $Y(1, x)$ is the identity operator on $V$, and the creation property holds: For $v \in V$, $Y(v, x)1$ involves only nonnegative powers of $x$ and its constant term is $v$. Also,

$$Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2},$$

where the operators $L(n)$ on $V$ satisfy the standard bracket relations for the Virasoro algebra (including a central charge); the $\mathbb{Z}$-grading on $V$ coincides with the eigenspace decomposition of the operator $L(0)$; and the operator $L(-1)$ satisfies the $L(-1)$-derivative property:

$$Y(L(-1)v, x) = \frac{d}{dx} Y(v, x)$$

for $v \in V$.

A module for the vertex operator algebra $V$ is a vector space $W$ equipped with a vertex operator map

$$Y(\cdot, x) : V \rightarrow (\text{End } W)[[x, x^{-1}]],$$

(2.13)

such that all the axioms in the definition of the notion of vertex operator algebra that make sense hold, except that the grading on $W$ is allowed to be a $\mathbb{Q}$-grading rather than a $\mathbb{Z}$-grading.

We have already encountered some special vertex operators in (2.4) and (2.6), in the sense of the following result ([B], [FLM]):

**Theorem 2.2** Formulas (2.4) and (2.6) give a vertex operator algebra structure on $V_\mathbb{Q}$, and $V_\mathbb{Q} + \alpha/2$ is a $V_\mathbb{Q}$-module. The vertex operator algebra $V_\mathbb{Q}$ is simple and the module $V_\mathbb{Q} + \alpha/2$ is irreducible.
Even though the vertex operators (2.4) and (2.6) are defined for every \( w \in V_P \) (recall the comment after (2.6)), the vertex operator algebra structure \( V_Q \) cannot be extended to such a structure on \( V_P \); instead, \( V_P \) has the structure of an *abelian intertwining algebra* in the sense of [DL]. However, following [DL], for \( w \in V_{Q+\alpha/2} \) we modify the vertex operator map \( Y \) by:

\[
Y(w, x) = Y(w, x)e^{i\pi\alpha/2}; \quad (2.14)
\]

then a version of the Jacobi identity (2.12) still holds [DL]:

\[
x^{-1}_0 \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)Y(w, x_2) - x^{-1}_0 \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(w, x_2)Y(u, x_1) = x^{-1}_2 \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)w, x_2) \quad (2.15)
\]

for \( u \in V_Q \). In addition, it is not hard to see that

\[
\mathcal{Y}(w, x)|_{V_Q/V_Q} \subset V_{Q+\alpha/2}((x)) \quad (2.16)
\]

\[
\mathcal{Y}(w, x)|_{V_{Q+\alpha/2}/V_{Q+\alpha/2}} \subset V_Q((x^{1/2})) \quad (2.17)
\]

where, for a vector space \( W \) and a formal variable \( y \), \( W((y)) \) denotes the space of \( W \)-valued lower truncated formal Laurent series in \( y \).

Also for a general a vector space \( W \), we define \( W\{x\} \) to be the vector space of \( W \)-valued formal expressions of the form \( \sum_{s \in \mathbb{Q}} w_s x^s \), \( w_s \in W \). We recall from [FLM] the definition of the notion of intertwining operator for a triple of modules for a vertex operator algebra:

**Definition 2.1** Let \( W_1, W_2 \) and \( W_3 \) be modules for a vertex operator algebra \( V \). A map

\[
\mathcal{Y}(\cdot, x) : W_1 \rightarrow \text{Hom}(W_2, W_3)\{x\},
\]

is an *intertwining operator of type* \( (W_1, W_2, W_3) \) if it satisfies all the defining properties of a module action that make sense, namely, the truncation property (that for \( w_i \in W_i \), \( i = 1, 2 \), \( \mathcal{Y}(w_i, x)w_2 \) is truncated from below), the Jacobi identity

\[
x^{-1}_0 \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1)\mathcal{Y}(w_1, x_2)w_2 - x^{-1}_0 \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(w_1, x_2)Y(u, x_1)w_2 = x^{-1}_2 \delta \left( \frac{x_1 - x_0}{x_2} \right) \mathcal{Y}(Y(u, x_0)w_1, x_2)w_2 \quad (2.18)
\]

for \( u \in V \), \( w_1 \in W_1 \) and \( w_2 \in W_2 \), and the \( L(-1) \)-derivative property: for \( w_1 \in W_1 \),

\[
\mathcal{Y}(L(-1)w_1, x) = \frac{d}{dx} \mathcal{Y}(w_1, x).
\]
We will denote the vector space of all intertwining operators of type \( \left(W_3, W_1\right) \) by \( I_1 \left(W_3, W_1\right) \).

The operator (2.14) in fact gives intertwining operators:

**Theorem 2.3** (DL, FHL) The operators (2.16) and (2.17) are intertwining operators of the types
\[
\left(L(\Lambda_1), L(\Lambda_0)\right) \quad \text{and} \quad \left(L(\Lambda_1), L(\Lambda_0)\right),
\]
respectively. Moreover, the spaces \( I_1 \left(L(\Lambda_1), L(\Lambda_0)\right) \) and \( I_1 \left(L(\Lambda_1), L(\Lambda_0)\right) \) are one-dimensional.

3 The principal subspaces

We recall the definition of the principal subspaces \( W(\Lambda_0) \subset L(\Lambda_0) \) and \( W(\Lambda_1) \subset L(\Lambda_1) \) from [FS1]:

\[
W(\Lambda_0) = U(\mathfrak{n}_+^+)v_{\Lambda_0},
W(\Lambda_1) = U(\mathfrak{n}_+^+)v_{\Lambda_1},
\]
where \( v_{\Lambda_0} \) and \( v_{\Lambda_1} \) are highest weight vectors for \( L(\Lambda_0) \) and \( L(\Lambda_1) \), respectively.

Clearly, \( W(\Lambda_j), j = 0, 1, \) is spanned by
\[
x_\alpha(-m_1) \cdots x_\alpha(-m_k)v_{\Lambda_j},
\]
(3.1)

where \( m_1, \ldots, m_k > 0 \). From now on we will identify \( L(\Lambda_0) \) with \( V_Q \) and \( L(\Lambda_1) \) with \( V_{Q+\alpha/2} \) and we will take
\[
v_{\Lambda_0} = 1, \quad v_{\Lambda_1} = e^{\alpha/2}v_{\Lambda_0} = e^{\alpha/2}.
\]

The action of \( L(0) \) on the space \( V_P \) gives us the grading of \( V_P \) by weights, determined by:
\[
\text{wt } e^\lambda = \frac{1}{2}\langle \lambda, \lambda \rangle
\]
(3.2)

for \( \lambda \in \mathcal{P} \) and by the condition that for \( n \in \mathbb{Z} \), the weight of the operator \( h(-n) \) on \( V_P \) is \( n \); this determines the weight of the generic vector \( \{23\} \). This grading by weights is a \( \mathbb{Q} \)-grading, or more specifically, a \( \frac{1}{2}\mathbb{Z} \)-grading. The space \( V_P \) also has a second, compatible, grading, by charge, given by the eigenvalues of the operator \( \frac{1}{2}\alpha(0) = \frac{1}{2}h(0) \) (recall \( \{23\} \)); this a \( \frac{1}{2}\mathbb{Z} \)-grading. We shall consider these gradings restricted to the principal subspaces \( W(\Lambda_0) \) and \( W(\Lambda_1) \). The formulas
\[
x^{\alpha/2}x_\alpha(-m_1) \cdots x_\alpha(-m_k)v_{\Lambda_0} = x^kx_\alpha(-m_1) \cdots x_\alpha(-m_k)v_{\Lambda_0},
\]
\[
x^{\alpha/2}x_\alpha(-m_1) \cdots x_\alpha(-m_k)v_{\Lambda_1} = x^{k+1/2}x_\alpha(-m_1) \cdots x_\alpha(-m_k)v_{\Lambda_1},
\]
show that the charges of the indicated elements of \( W(\Lambda_0) \) and \( W(\Lambda_1) \) are \( k \) and \( k + 1/2 \), respectively.

For a subspace \( M \) of \( V_P \) homogeneous with respect to the double grading, we consider the corresponding graded dimension of \( M \)—the generating function of
the dimensions of its homogeneous subspaces, where we use the formal variables $x$ and $q$:

$$\dim_*(M; x, q) = \text{tr} |_M x^{\alpha/2} q^{L(0)}. \quad (3.3)$$

We shall focus on the graded dimensions of the principal subspaces $W(\Lambda_0)$ and $W(\Lambda_1)$, which we shall write as:

$$\chi_0(x, q) = \dim_*(W(\Lambda_0); x, q) = \text{tr} |_{W(\Lambda_0)} x^{\alpha/2} q^{L(0)}, \quad (3.4)$$

$$\chi_1(x, q) = \dim_*(W(\Lambda_1); x, q) = \text{tr} |_{W(\Lambda_1)} x^{\alpha/2} q^{L(0)}. \quad (3.5)$$

One way to describe the spaces $W(\Lambda_0)$ and $W(\Lambda_1)$ is to use the vertex operators

$$Y(e^\alpha, x_i) = E^-(-\alpha, x_i) E^+(-\alpha, x_i) e^\alpha x_i^\alpha, \quad (3.6)$$

(we identify $e^\alpha$ with $1 \otimes e^\alpha$) or the slightly modified operators

$$X(e^\alpha, x_i) = E^-(-\alpha, x_i) E^+(-\alpha, x_i) e^\alpha x_i^{\alpha+1}. \quad (3.7)$$

The operators $Y(e^\alpha, x_i)$ and $X(e^\alpha, x_i)$ all commute with one another, and

$$\prod_{i=1}^n X(e^\alpha, x_i) v_{\Lambda_0} =$$

$$= \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^n x_i : \prod_{i=1}^n E^-(-\alpha, x_i) E^+(-\alpha, x_i) : e^{n\alpha}$$

$$= \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^n x_i \prod_{i=1}^n E^-(-\alpha, x_i) e^{n\alpha}. \quad (3.8)$$

Similarly, for $W(\Lambda_1)$,

$$\prod_{i=1}^n X(e^\alpha, x_i) v_{\Lambda_1} = \prod_{i=1}^n X(e^\alpha, x_i) e^{n/2}$$

$$= \prod_{i<j} (x_i - x_j)^2 \prod_{i=1}^n x_i^2 \prod_{i=1}^n E^-(-\alpha, x_i) e^{(n+1/2)\alpha}. \quad (3.9)$$

Another way to describe the principal subspaces is as certain quotients of the associative algebra $U(\hat{\mathfrak{n}}_+)$, which is in fact commutative. This description will be used from now on, and we will not directly use the formulas (3.8) and (3.9), except to compute the action of components of the operator (3.6) (or (3.7)) on particular vectors.

Consider a set of commuting independent formal variables $y_{-1}, y_{-2}, \ldots$ and the corresponding polynomial algebra

$$\mathcal{A} = \mathbb{C}[y_{-1}, y_{-2}, \ldots].$$

Consider the algebra map

$$\mathcal{A} \longrightarrow \text{End } V_p.$$
\[ y_{-j} \mapsto x_\alpha(-j) \]

\((j > 0)\); this map is well defined because the operators \(x_\alpha(-j)\) commute. We shall use this correspondence to describe \(W(\Lambda_0)\) and \(W(\Lambda_1)\). Define the linear surjection

\[
\begin{align*}
f_{\Lambda_0} & : A \to W(\Lambda_0) \\
p(y_{-1}, y_{-2}, \ldots) & \mapsto p(x_\alpha(-1), x_\alpha(-2), \ldots) \cdot v_{\Lambda_0}
\end{align*}
\]

and set

\[ A_{\Lambda_0} = \text{Ker} \, f_{\Lambda_0}, \]

an ideal in \(A\). Then we have

\[ A / A_{\Lambda_0} \sim W(\Lambda_0). \]  

(3.11)

Similarly, for \(W(\Lambda_1)\), we have

\[
\begin{align*}
f_{\Lambda_1} & : A \to W(\Lambda_1) \\
p & \mapsto p(x_\alpha(-1), x_\alpha(-2), \ldots) \cdot v_{\Lambda_1}
\end{align*}
\]

and we set

\[ A_{\Lambda_1} = \text{Ker} \, f_{\Lambda_1}. \]

Then

\[ A / A_{\Lambda_1} \sim W(\Lambda_1). \]

Notice that

\[ y_{-1} \in A_{\Lambda_1} \]

since

\[ x_\alpha(-1) \cdot e^{\alpha/2} v_{\Lambda_0} = 0, \]

which follows from (3.9) for \(n = 1\). Since \(A_{\Lambda_1}\) is an ideal, it includes the ideal generated by \(y_{-1}\):

\[ (y_{-1}) = A y_{-1} \subset A_{\Lambda_1}. \]

This ideal \((y_{-1})\) has a natural complement in \(A\), namely,

\[ A' = \mathbb{C}[y_{-2}, y_{-3}, \ldots]; \]

(3.14)

that is,

\[ A = (y_{-1}) \oplus A'. \]

(3.15)

Thus we have a natural linear surjection

\[
\begin{align*}
f'_{\Lambda_1} & = f_{\Lambda_1} |_{A'} : A' \to W(\Lambda_1) \\
p(y_{-2}, y_{-3}, \ldots) & \mapsto p(x_\alpha(-2), x_\alpha(-3), \ldots) \cdot v_{\Lambda_1}.
\end{align*}
\]

(3.16)

Define

\[ B_{\Lambda_1} = \text{Ker} f'_{\Lambda_1}, \]
an ideal in $\mathcal{A}'$. Then
\[
\mathcal{A}'/\mathcal{B}_{\Lambda_1} \sim W(\Lambda_1). \tag{3.17}
\]
We also have the natural surjection, and its kernel $\mathcal{A}_{\Lambda_1}$ can be decomposed as
\[
\mathcal{A}_{\Lambda_1} = (y_{-1}) \oplus \mathcal{B}_{\Lambda_1} \subset \mathcal{A}. \tag{3.18}
\]
Thus the map $f_{\Lambda_1}$ induces natural isomorphisms
\[
\mathcal{A}/\mathcal{A}_{\Lambda_1} \sim (y_{-1}) \oplus \mathcal{A}'/\mathcal{B}_{\Lambda_1} \sim W(\Lambda_1). \tag{3.19}
\]
Consider the linear map $e^{\alpha/2}: \mathcal{V}_P \rightarrow \mathcal{V}_P$. \tag{3.20}
This is clearly a linear isomorphism, since its inverse is $e^{-\alpha/2}$. Let us restrict this map to $W(\Lambda_0)$. Then we have
\[
e^{\alpha/2}: W(\Lambda_0) \longrightarrow W(\Lambda_1), \tag{3.21}
\]
since, from (2.10),
\[
e^{\alpha/2}(x_\alpha(-i_1) \cdots x_\alpha(-i_k) \cdot 1) = x_\alpha(-i_1 - 1) \cdots x_\alpha(-i_k - 1) \cdot e^{\alpha/2} \tag{3.22}
\]
for $i_j \in \mathbb{Z}$. The map (3.21) is certainly injective, and it is surjective because
\[
U(n_+) \cdot e^{\alpha/2} = e^{\alpha/2}(U(n_+) \cdot 1). \tag{3.23}
\]
Hence (3.21) is a linear isomorphism.
We construct a lifting $\tilde{e^{\alpha/2}}$ of $e^{\alpha/2}$ as follows:
\[
\tilde{e^{\alpha/2}}: \mathcal{A} \xrightarrow{\sim} \mathcal{A}' \\
y_{-j} \mapsto y_{-j-1} \tag{3.24}
\]
($j \geq 1$); this is an algebra isomorphism. Then $\tilde{e^{\alpha/2}}$ is a lifting in the sense that the following diagram commutes:
\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\sim} & \mathcal{A}' & \xrightarrow{\iota} & \mathcal{A} \\
\downarrow f_{\Lambda_0} & & \downarrow f_{\Lambda_1} & & \downarrow f_{\Lambda_1} \\
W(\Lambda_0) \simeq \mathcal{A}/\mathcal{A}_{\Lambda_0} & \xrightarrow{\sim} & W(\Lambda_1) \simeq \mathcal{A}'/\mathcal{B}_{\Lambda_1} & \xrightarrow{\sim} & \mathcal{A}/\mathcal{A}_{\Lambda_1}
\end{array}
\]
where $\iota: \mathcal{A}' \hookrightarrow \mathcal{A}$ is the algebra injection. The right half of this diagram clearly commutes, and we now verify that the left half commutes: For any monomial
\[
M = y_{-j_1} \cdots y_{-j_n} \in \mathcal{A}
\]
Let us denote by 

\[ f_{A_0}(M) = x_{\alpha}(-j_1) \cdots x_{\alpha}(-j_n) \cdot v_{A_0} \in W(A_0), \]

and also,

\[ e^{\alpha/2}(f_{A_0}(M)) = x_{\alpha}(-j_1 - 1) \cdots x_{\alpha}(-j_n - 1) \cdot e^{\alpha/2} \in W(A_1), \]

and similarly,

\[ f'_{A_1}(e^{\alpha/2}(M)) = x_{\alpha}(-j_1 - 1) \cdots x_{\alpha}(-j_n - 1) \cdot e^{\alpha/2} \in W(A_1). \]

Thus the diagram commutes.

It follows that 

\[ e^{\alpha/2} \text{ maps } A_{A_0} \text{ isomorphically onto } B_{A_1}: \]

\[ e^{\alpha/2} : A_{A_0} \to B_{A_1}. \]

(3.25)

That is, the relations for 

\[ W(A_1) \]

are precisely the index-shifted relations for 

\[ W(A_0). \]

It is important to notice that the linear isomorphism 

\[ e^{\alpha/2} : W(A_0) \to W(A_1) \]

does not preserve charge or weight. In fact, we will prove that

\[ \chi_1(x, q) = x^{1/2} q^{1/4} \chi_0(xq, q), \]

(3.26)

that is,

\[ \sum_{r,s} \dim W(A_1)_{r,s} x^r q^s = \sum_{r,s} \dim W(A_0)_{r,s} x^{r+1/2} q^{r+s+1/4}, \]

(3.27)

where 

\[ W(A_j)_{r,s} \]

denotes the vector subspace of 

\[ W(A_j) \]

of charge 

\[ r \in \frac{1}{2} \mathbb{Z} \]

and weight 

\[ s \in \frac{1}{4} \mathbb{Z}. \]

This is equivalent to

\[ x^{-1/2} q^{-1/4} \chi_1(x, q) = \chi_0(xq, q), \]

(3.28)

that is, to

\[ \sum_{r,s} \dim W(A_1)_{r+1/2,s+1/4} x^r q^s = \sum_{r,s} \dim W(A_1)_{r,s} x^{-1/2} q^{r-1/4} = \]

\[ \sum_{r,s} \dim W(A_0)_{r,s} x^r q^{r+s} = \sum_{r,s} \dim W(A_0)_{r,s} x^{-r} q^s. \]

(3.29)

To prove (3.26) it is enough to show

\[ \dim W(A_0)_{r,s} = \dim W(A_1)_{r+1/2,r+s+1/4}. \]

Let us restrict 

\[ e^{\alpha/2} \text{ to } W(A_0)_{r,s}. \]

Then and the fact that 

\[ \frac{1}{2}(\frac{3}{2}, \frac{1}{2}) = \frac{1}{4} \]

show that

\[ e^{\alpha/2} : W(A_0)_{r,s} \to W(A_1)_{r+1/2,r+s+1/4}, \]

and similarly, applying 

\[ e^{-\alpha/2} \text{ to } W(A_1)_{r+1/2,r+s+1/4}, \]

we have

\[ e^{-\alpha/2} : W(A_1)_{r+1/2,r+s+1/4} \to W(A_0)_{r,s}. \]
Thus we have the isomorphism
\[ e^{\alpha/2} : W(\Lambda_0)_{r,s} \xrightarrow{\sim} W(\Lambda_1)_{r+1/2,s+s+1/4}. \] (3.30)
This proves \[ \text{Remark 3.1} \], which in effect describes precisely how \( e^{\alpha/2} : W(\Lambda_0) \rightarrow W(\Lambda_1) \) shifts charge and weight.

**Remark 3.1** Note that \( \chi_1(1,q) = q^{1/4} \chi_0(q,q) \) gives the principally specialized character of \( W(\Lambda_0) \), while \( \chi_0(1,q) \) gives (by definition) the ordinary graded dimension of \( W(\Lambda_0) \) with respect to the grading by weights.

## 4 The main theorem

We have:

**Theorem 4.1** Take the intertwining operator \( \mathcal{Y}(\cdot, x) \in I \left( \frac{L(\Lambda_1)}{L(\Lambda_0)} \right) \) as given in (2.14) and (2.16). The sequence
\[
0 \longrightarrow W(\Lambda_1) \xrightarrow{e^{\alpha/2}} W(\Lambda_0) \xrightarrow{o(e^{\alpha/2})} W(\Lambda_1) \longrightarrow 0
\] (4.1)
is exact, where
\[ o(e^{\alpha/2}) = \text{Res}_x x^{-1} \mathcal{Y}(e^{\alpha/2}, x), \]
the constant term of the intertwining operator \( \mathcal{Y}(e^{\alpha/2}, x) \).

**Remark 4.1** Note that \( e^{\alpha/2} \) is the “canonical constant factor” of \( \mathcal{Y}(e^{\alpha/2}, x) \), where \( \mathcal{Y}(\cdot, x) \) is viewed as in (2.17), and that \( o(e^{\alpha/2}) \) is the “canonical constant summand” of \( \mathcal{Y}(e^{\alpha/2}, x) \), where \( \mathcal{Y}(\cdot, x) \) is viewed as in (2.16).

**Proof of Theorem 4.1:** Recall that \( e^{\alpha/2} : W(\Lambda_0) \rightarrow W(\Lambda_1) \) is a linear isomorphism; note on the other hand that the map \( e^{\alpha/2} \) in (4.1) goes in the reverse direction. We shall see below that \( e^{\alpha/2} \) takes \( W(\Lambda_1) \) into \( W(\Lambda_0) \); it is certainly injective. Let us recall the Jacobi identity (2.15). Taking \( u = e^\alpha \) and \( w = e^{\alpha/2} \) in (2.15) and applying the residue operation \( \text{Res}_{x_0} \), we find that
\[ [\mathcal{Y}(e^{\alpha/2}, x_1), \mathcal{Y}(e^{\alpha}, x_2)] = 0, \] (4.2)
so that
\[ [\mathcal{Y}(e^{\alpha/2}, x), \tilde{n}_\alpha] = 0. \] (4.3)
In addition,
\[ o(e^{\alpha/2}) v_{\Lambda_0} = v_{\Lambda_1}, \]
and so
\[ \text{Res}_x x^{-1} \mathcal{Y}(e^{\alpha/2}, x) U(\tilde{n}_\alpha) v_{\Lambda_0} = U(\tilde{n}_\alpha) v_{\Lambda_1}. \] (4.4)
Hence \( o(e^{\alpha/2}) \) takes \( W(\Lambda_0) \) to \( W(\Lambda_1) \) and is surjective.
Now we show that \( e^{\alpha/2}W(\Lambda_1) \subset W(\Lambda_0) \) and that
\[
e^{\alpha/2}W(\Lambda_1) \subset \ker o(e^{\alpha/2}),
\]
so that the sequence \( 4.1 \) is a chain complex: For \( j_1, \ldots, j_n \in \mathbb{Z} \),
\[
e^{\alpha/2}x_{\alpha}(j_1) \cdots x_{\alpha}(j_n)e^{\alpha/2}v_{\Lambda_0} =
\]
\[
x_{\alpha}(j_1 - 1) \cdots x_{\alpha}(j_n - 1)e^{\alpha}v_{\Lambda_0} =
\]
\[
x_{\alpha}(j_1 - 1) \cdots x_{\alpha}(j_n - 1)x_{\alpha}(-1)v_{\Lambda_0},
\]
which gives that \( e^{\alpha/2}W(\Lambda_1) \subset W(\Lambda_0) \). The fact that \( o(e^{\alpha/2}) e^{\alpha/2}W(\Lambda_1) = 0 \) now follows from \( 4.5 \) and the fact \( x_{\alpha}(-1)e^{\alpha/2} = 0 \). (Incidentally,
\[
x_{\alpha}(-1)e^{\alpha/2}v_{\Lambda_0} = e^{\alpha/2}x_{\alpha}(0)v_{\Lambda_0} = 0;
\]
that is, the fact that \( x_{\alpha}(-1) \) annihilates \( v_{\Lambda_1} \) is equivalent to the fact that \( x_{\alpha}(0) \) annihilates \( v_{\Lambda_0} \), by means of \( e^{\alpha/2} \). So the chain complex property follows from either \( x_{\alpha}(-1)v_{\Lambda_1} = 0 \) or \( x_{\alpha}(0)v_{\Lambda_0} = 0 \). Thus the sequence \( 4.1 \) is a chain complex, and it remains to prove the exactness.

We shall now characterize the kernel of \( o(e^{\alpha/2}) : W(\Lambda_0) \to W(\Lambda_1) \).

Let \( \pi \) be a polynomial in \( \mathcal{A} \), so that \( f_{\Lambda_0}(p) \) is a general element of \( W(\Lambda_0) \). Then
\[
o(e^{\alpha/2})(f_{\Lambda_0}(p)) = 0 \iff o(e^{\alpha/2})(p(x_{\alpha}(-1), x_{\alpha}(-2), \ldots)v_{\Lambda_0}) = 0
\]
\[
\iff p(x_{\alpha}(-1), x_{\alpha}(-2), \ldots)v_{\Lambda_1} = 0
\]
\[
\iff p \in \ker f_{\Lambda_1} = \mathcal{A}_{\Lambda_1} = (y_{-1}) \oplus \mathcal{B}_{\Lambda_1} \iff \pi(p) \in \mathcal{B}_{\Lambda_1} = \widehat{e^{\alpha/2}}(\mathcal{A}_{\Lambda_0}),
\]
where
\[
\pi : \mathcal{A} \to \mathcal{A}'
\]
is the projection with respect to the decomposition \( 3.11 \); \( \pi \) is an algebra surjection. Thus
\[
f_{\Lambda_0}(p) \in \ker o(e^{\alpha/2}) \iff \pi(p) = \widehat{e^{\alpha/2}}(p')
\]
for some \( p' \in \mathcal{A}_{\Lambda_0} \), and so
\[
f_{\Lambda_0}(p) \in \ker o(e^{\alpha/2}) \iff (\widehat{e^{\alpha/2}}^{-1} \circ \pi)(p) \in \mathcal{A}_{\Lambda_0}.
\]
Note that if \( p \in \mathcal{A}_{\Lambda_0} \), then \( f_{\Lambda_0}(p) = 0 \), so that all the above statements hold in particular for \( p \). Thus
\[
p \in \mathcal{A}_{\Lambda_0} \Rightarrow \pi(p) \in \widehat{e^{\alpha/2}}(\mathcal{A}_{\Lambda_0}),
\]
i.e.,
\[
\pi(\mathcal{A}_{\Lambda_0}) \subset \widehat{e^{\alpha/2}}(\mathcal{A}_{\Lambda_0}) (= \mathcal{B}_{\Lambda_1}),
\]
i.e.,
\[
(\widehat{e^{\alpha/2}}^{-1} \circ \pi)(\mathcal{A}_{\Lambda_0}) \subset \mathcal{A}_{\Lambda_0}.
\]
Define

\[ S = e^{\alpha/2} - 1 \circ \pi : A \to A. \]  \hspace{1cm} (4.12)

This operator is the algebra surjection, with kernel \((y - 1)\), defined by

\[ y - j \mapsto y - j + 1 \]

for \( j \geq 2 \) and

\[ y - 1 \mapsto 0, \]

and thus is a “shift map”. The map \( S \) is 0 on the ideal \((y - 1)\) and is the isomorphism

\[ e^{\alpha/2} - 1 : A' \to A \]

on the complement \( A' \) of \((y - 1)\) in \( A \). With this definition we can restate our characterization of the kernel as follows: For \( p \in A \),

\[ f_{A_0}(p) \in \ker o(e^{\alpha/2}) \iff S(p) \in A_{A_0} \text{ (} \iff f_{A_0}(S(p)) = 0). \]  \hspace{1cm} (4.14)

We have also noted that

\[ S(A_{A_0}) \subset A_{A_0} \]  \hspace{1cm} (4.15)

(i.e., if \( f_{A_0}(p) = 0 \), then \( f_{A_0}(S(p)) = 0 \)).

Next we characterize the image of \( e^{\alpha/2} : W(\Lambda_1) \to W(\Lambda_0) \): For \( p \in A \), when is \( f_{A_0}(p) \) of the form \( e^{\alpha/2}(w) \) for some \( w \in W(\Lambda_1) \)? This is the case if and only if for some \( q = q(y - 3, y - 4, \ldots) \in A' \),

\[ f_{A_0}(p) = e^{\alpha/2}(f'_{\Lambda_1}(q)) = e^{\alpha/2}(f_{\Lambda_1}(q)) = e^{\alpha/2}(q(x_{\alpha}(-2), x_{\alpha}(-3), \ldots)e^{\alpha/2}v_{\Lambda_0} = q(x_{\alpha}(-3), x_{\alpha}(-4), \ldots)x_{\alpha}(-1)v_{\Lambda_0} = f_{\Lambda_0}(q(y - 3, y - 4, \ldots)y - 1) = f_{\Lambda_0}(e^{\alpha/2}(q)y - 1). \]  \hspace{1cm} (4.16)

This in turn is the case if and only if for some \( q \in A' \)

\[ p - q(y - 3, y - 4, \ldots)y - 1 \in A_{A_0}, \]

i.e., if and only if

\[ p \in e^{\alpha/2}(A')y - 1 + A_{A_0}. \]

Note that

\[ e^{\alpha/2}(A')y - 1 = C[y - 3, y - 4, \ldots]y - 1. \]

Thus

\[ f_{A_0}(p) \in \text{Im } e^{\alpha/2} : W(\Lambda_1) \to W(\Lambda_0) \iff p \in C[y - 3, y - 4, \ldots]y - 1 + A_{A_0}. \]  \hspace{1cm} (4.17)
For this, what we need to prove is that
\[(y - 1) \subset \mathbb{C}[y_{-3}, y_{-4}, \ldots, y_{-1}] + \mathcal{A}_{\Lambda_0}, \quad (4.18)\]
and for this we use Proposition 2.1. We have
\[X(\alpha, x)^2 = 0\]
on \(L(\Lambda_0)\) (and on \(L(\Lambda_1)\)), i.e., for all \(j \in \mathbb{Z}\),
\[\sum_{i \in \mathbb{Z}} x_{\alpha}(j - i)x_{\alpha}(i) = 0.\]
In particular, applying these relations to \(v_{\Lambda_0}\), we see that for all \(n \leq -2\),
\[\sum_{i, j < 0, i + j = n} x_{\alpha}(i)x_{\alpha}(j)v_{\Lambda_0} = 0. \quad (4.19)\]
The first two such relations are
\[x_{\alpha}(-1)^2v_{\Lambda_0} = 0\]
and
\[x_{\alpha}(-2)x_{\alpha}(-1)v_{\Lambda_0} = 0.\]
That is,
\[y_{-1}^2 \in \mathcal{A}_{\Lambda_0}\]
and
\[y_{-2}y_{-1} \in \mathcal{A}_{\Lambda_0}.\]
Thus any monomial in \(\mathcal{A}\) divisible by \(y_{-1}^2\) or by \(y_{-2}y_{-1}\) lies in \(\mathcal{A}_{\Lambda_0}\), and this proves (4.18) and hence (4.17). Thus for \(p \in \mathcal{A}\),
\[f_{\Lambda_0}(p) \in \text{Im } e^{\alpha/2} \iff p \in (y - 1) + \mathcal{A}_{\Lambda_0}\]
\[\iff \pi(p) \in \pi(\mathcal{A}_{\Lambda_0}) \iff S(p) \in S(\mathcal{A}_{\Lambda_0}) \quad (4.20)\]
(where we use the definition (4.12) of \(S\), and we had already seen that
\[f_{\Lambda_0}(p) \in \text{Ker } o(e^{\alpha/2}) \iff p \in (y - 1) \oplus \mathcal{B}_{\Lambda_1}\]
\[\iff \pi(p) \in \mathcal{B}_{\Lambda_1} \iff S(p) \in \mathcal{A}_{\Lambda_0}.\]
We shall finally use these characterizations of \(\text{Ker } o(e^{\alpha/2})\) and \(\text{Im } e^{\alpha/2}\) to prove the exactness.

First we confirm the chain-complex property, which we had already seen:
\[\text{Im } e^{\alpha/2} \subset \text{Ker } o(e^{\alpha/2})\]
since
\[\pi(\mathcal{A}_{\Lambda_0}) \subset \mathcal{B}_{\Lambda_1} \quad (4.22)\]
(recall 4.10) or since
\[ S(A_{\Lambda_0}) \subset A_{\Lambda_0} \]
(recall 4.15).

Now we prove exactness. What we must prove is that
\[ B_{\Lambda_1} \subset \pi(A_{\Lambda_0}) \]
or that
\[ A_{\Lambda_0} \subset S(A_{\Lambda_0}), \]
i.e., that
\[ \pi(A_{\Lambda_0}) = B_{\Lambda_1} \]
or that
\[ S(A_{\Lambda_0}) = A_{\Lambda_0}. \] (4.23)

**Remark 4.2** This is a precise statement of the principle that “the only difference in the relations defining \( W(\Lambda_0) \) and \( W(\Lambda_1) \) is the initial condition relation \( x_\alpha(-1)u_{\Lambda_1} = 0 \). But it is expressed as relations involving only \( A_{\Lambda_0} \) and not \( A_{\Lambda_1} \) or \( B_{\Lambda_1} \):
\[ S(A_{\Lambda_0}) = A_{\Lambda_0}. \]

To prove this, consider the relations (4.19) in \( W(\Lambda_0) \). These relations amount to the assertion that
\[ r_n = \sum_{i,j < 0, i+j=n} y_i y_j \in A_{\Lambda_0} \]
for \( n \leq -2 \). Note that \( r_{-2} = y_2^2 \) and \( r_{-3} = y_{-2} y_{-1} \), which we discussed above. By Feigin-Stoyanovsky [FS1], these elements \( r_n \) generate the ideal \( A_{\Lambda_0} \) in \( A \), that is,
\[ A_{\Lambda_0} = \sum_{n \leq -2} A \ r_n. \] (4.24)

From this, it is clear that \( S(A_{\Lambda_0}) = A_{\Lambda_0} \), since
\[ S(r_n) = r_{n+2} \]
for \( n \leq -4 \) and
\[ S(r_{-3}) = S(r_{-2}) = 0. \]

This proves the exactness.

The main theorem immediately gives:

**Theorem 4.2** We have the relation
\[ \chi_0(x, q) = x^{-1/2}q^{-1/4}\chi_1(x, q) + x^{1/2}q^{1/4}\chi_1(xq, q). \] (4.25)
Proof: The operator $e^{\alpha/2}$ has weight equal to the charge of the source vector plus $1/4$, and it increases charge by $1/2$. On the other hand $o(e^{\alpha/2})$ is of weight $1/4$ and it increases the charge by $1/2$. The exactness of (4.1) now immediately gives (4.2).

We now have the Rogers-Ramanujan recursion:

**Corollary 4.1**

$$\chi_0(x, q) = \chi_0(xq, q) + xq\chi_0(xq^2, q).$$

**Proof:** Apply formula (3.26) and the Theorem.

Solving this recursion (cf. [A]), we immediately obtain:

**Corollary 4.2**

$$\chi_0(x, q) = \sum_{n \geq 0} \frac{x^n q^{n^2}}{(q)_n},$$

$$\chi_1(x, q) = x^{1/2} q^{1/4} \sum_{n \geq 0} \frac{x^n q^{n^2+n}}{(q)_n}.$$
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