ROKHLIN DIMENSION: ABSORPTION OF MODEL ACTIONS

GÁBOR SZABÓ

Abstract. In this paper, we establish a connection between Rokhlin dimension and the absorption of certain model actions on strongly self-absorbing C*-algebras. Namely, as to be made precise in the paper, let $G$ be a well-behaved locally compact group. If $D$ is a strongly self-absorbing C*-algebra, and $\alpha : G \curvearrowright A$ is an action on a separable, $D$-absorbing C*-algebra that has finite Rokhlin dimension with commuting towers, then $\alpha$ tensorially absorbs every semi-strongly self-absorbing $G$-actions on $D$. This contains several existing results of similar nature as special cases. We will in fact prove a more general version of this theorem, which is intended for use in subsequent work. We will then discuss some non-trivial applications. Most notably it is shown that for any $k \geq 1$ and on any strongly self-absorbing Kirchberg algebra, there exists a unique $\mathbb{R}^k$-action having finite Rokhlin dimension with commuting towers up to (very strong) cocycle conjugacy.

Contents

Introduction 1
1. Preliminaries 4
2. Box spaces and partitions of unity over groups 7
3. Systems generated by order zero maps with commuting ranges 13
4. Rokhlin dimension with commuting towers 16
5. Some applications 26
6. Multi-flows on strongly self-absorbing Kirchberg algebras 27
References 35

Introduction

The present work is a continuation of the author’s quest to study fine structure and classification of certain C*-dynamics by employing ideas related to tensorial absorption. In previous work, the theory of (semi-)strongly self-absorbing actions on C*-algebras $[81, 82, 79]$ was developed, closely following the important results established in the classical theory of strongly self-absorbing C*-algebras by Toms–Winter and others $[85, 47, 16]$. Strongly self-absorbing C*-algebras have historically emerged by example $[47]$, and

2010 Mathematics Subject Classification. 46L55.
This work was supported by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92), and the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement 746272.
by now play a central role in the structure theory of simple nuclear C*-algebras; see for example [48, 73, 19, 86, 91, 89, 89, 63, 65, 5, 4]. Roughly speaking, a tensorial factorization of the form \( A \cong A \otimes D \) — for a given C*-algebra \( A \) and a strongly self-absorbing C*-algebra \( D \) — provides sufficient space to perform non-trival manipulations on elements inside \( A \), which often gives rise to structural properties of particular interest for classification. The underlying motivation behind [81, 82, 79] is the idea that this kind of phenomenon should persist at the level of C*-dynamics if one is interested in classification of group actions up to cocycle conjugacy; in fact some much earlier work [54, 55, 39, 28, 62, 64, 38] has (sometimes implicitly) used this idea to reasonable success. It was further demonstrated in [79, 80] how this approach can indeed give rise to new insights about classification or rigidity of group actions on certain C*-algebras, in particular strongly self-absorbing ones.

Starting from Connes’ groundbreaking work [13, 14, 15] on injective factors, which involved classification of single automorphisms, the Rokhlin property in its various forms became a key tool to classify actions of amenable groups on von Neumann algebras [11, 12, 11, 14, 12, 58]. It did not take long for these ideas to reach the realm of C*-algebras. Initially appearing in works of Herman–Jones [29] and Herman–Ocneanu [30], the Rokhlin property for single automorphisms and its applications for classification were perfected in works of Kishimoto and various collaborators [8, 49, 5, 51, 21, 52, 53, 18, 6, 67]. Further work pushed these techniques to actions of infinite higher-rank groups as well [66, 49, 59, 68, 61, 38]. The case of finite groups was treated in work of Izumi [36, 34], where it was shown that such actions with the Rokhlin property have a particularly rigid theory; see also [74, 27, 23, 24, 11, 2]. In contrast to von Neumann algebras, however, the Rokhlin property for actions on C*-algebras has too many obstructions in general, ranging from obvious ones like lack of projections to more subtle ones of K-theoretic nature.

Rokhlin dimension is a notion of dimension for actions of certain groups on C*-algebras and was first introduced by Hirshberg–Winter–Zacharias [35]. Several natural variants of Rokhlin dimension have been introduced, and all of them have in common that they generalize (to some degree) the Rokhlin property for actions of either finite groups or the integers. The theory has been extended and applied in many following works such as [77, 31, 83, 24, 33, 56, 57, 10, 25]. In short, the advantage of working with Rokhlin dimension is that it is both more prevalent and more flexible than the Rokhlin property, but is yet often strong enough to deduce interesting structural properties of the crossed product, such as finite nuclear dimension [91].

A somewhat stronger version of Rokhlin dimension, namely with commuting towers, has been considered from the very beginning as a variant that was also compatible with respect to the absorption of strongly self-absorbing C*-algebras. Although the assumption of commuting towers initially only looked like a minor technical assumption, it was eventually discovered that it can make a major difference in some cases such as for actions of finite groups [31].
The purpose of this paper is to showcase a decisive connection between finite Rokhlin dimension with commuting towers and the absorption of (semi-)strongly self-absorbing model actions. The following describes a variant of the main result:

**Theorem** (see Theorem 4.4). Let $G$ be a second-countable, locally compact group and $N \subset G$ a closed, normal subgroup. Suppose that the quotient $G/N$ contains a discrete, normal, cocompact subgroup that is residually finite and has a box space with finite asymptotic dimension. Let $A$ be a separable C*-algebra with an action $\alpha : G \curvearrowright A$. Let $\gamma : G \curvearrowright D$ be a semi-strongly self-absorbing action that is unitarily regular. Suppose that $\alpha|_N$ is $\gamma|_N$-absorbing. If the Rokhlin dimension of $\alpha$ with commuting towers relative to $N$ is finite, then it follows that $\alpha$ is $\gamma$-absorbing.

Since many assumptions in this theorem are fairly technical at first glance, it may be helpful for the reader to keep in mind some special cases. For example, the above assumptions on the pair $N \subset G$ are satisfied when the quotient $G/N$ above is isomorphic to either $\mathbb{R}$ or $\mathbb{Z}$. In this case, the theorem states that as long as the action $\alpha$ satisfies a suitable Rokhlin-type criterion relative to $N$, tensorial absorption of the $G$-action $\gamma$ can be detected by restricting to the $N$-actions, even though this restriction procedure (a priori) comes with great loss of dynamical information. This is most apparent when the normal subgroup $N$ is trivial, which is yet another important special case:

**Corollary** (see Corollary 5.1). Let $G$ be a second-countable, locally compact group. Suppose that $G$ contains a discrete, normal, cocompact subgroup that is residually finite and has a box space with finite asymptotic dimension. Let $A$ be a separable C*-algebra with an action $\alpha : G \curvearrowright A$. Suppose that $D$ is a strongly self-absorbing C*-algebra with $A \cong A \otimes D$. If the Rokhlin dimension of $\alpha$ with commuting towers is finite, then it follows that $\alpha$ is $\gamma$-absorbing for every semi-strongly self-absorbing action $\gamma : G \curvearrowright D$.

This corollary is in turn a generalization of [34, Theorem 1.1], [35, Theorems 5.8, 5.9], [33, Theorem 9.6], [33, Theorem 5.3] and [26, Theorem 4.50(2)]. We will in fact only apply the corollary within this paper. Some immediate applications of it will be discussed in Section 5. The non-trivial main application is pursued in Section 6 which is as follows:

**Theorem** (see Theorem 6.6 and Corollary 6.10). Let $D$ be a strongly self-absorbing Kirchberg algebra. Then up to (very strong) cocycle conjugacy, there is a unique action $\gamma : \mathbb{R}^k \curvearrowright D$ that has finite Rokhlin dimension with commuting towers.

We note that a strongly self-absorbing C*-algebra is a Kirchberg algebra precisely when it is traceless. Kirchberg algebras are (by convention) the separable, simple, nuclear, purely infinite C*-algebras, whose celebrated classification is due to Kirchberg–Phillips [48, 71, 46] and which constitutes a prominent special case of the Elliott classification program. We note that all other strongly self-absorbing C*-algebras are conjectured to be quasidiagonal — see [34, Corollary 6.7] — and so any Rokhlin flows on them would induce Rokhlin flows on the universal UHF algebra, which do not exist; see
[50] page 600], [52] page 289 or [39] Section 2]. In particular, the underlying problem above is only interesting to consider in the purely infinite case.

Although the theorem above is not too far off from being a very special case of [78] for ordinary flows, this result is entirely new for \( k \geq 2 \), and is in fact the first classification result for \( \mathbb{R}^k \)-actions on \( C^* \)-algebras up to cocycle conjugacy.

The proof goes via induction in the number \( k \) of flows generating the action. In order to achieve a major part of the induction step, the corollary above is used in order to see that any two \( \mathbb{R}^k \)-actions as in the statement absorb each other tensorially. However, in order for this to make sense, it has to be at least established beforehand (as part of the induction step) that any such action has equivariantly approximately inner flip. This is achieved via a relative Kishimoto-type approximate cohomology-vanishing argument inspired by [55, Section 3], which combines arguments related to the Rokhlin property for \( \mathbb{R}^k \)-actions with arguments related to the structure theory of semi-strongly self-absorbing actions.

At this moment it seems unclear whether or not to expect a similarly rigid situation for Rokhlin \( \mathbb{R}^k \)-actions on general Kirchberg algebras, as is the case for \( k = 1 \) [78]. In general, in order to implement a more general classification of this sort, it would require a technique for both constructing and manipulating cocycles for \( \mathbb{R}^k \)-actions (where \( k \geq 2 \)) with the help of the Rokhlin property, which may potentially be much more complicated than for \( k = 1 \). In essence, our approach based on ideas related to strong self-absorption works because the main result allows one to bypass the need to bother with general cocycles for all of \( \mathbb{R}^k \), but instead requires one only to consider individual copies of \( \mathbb{R} \) inside \( \mathbb{R}^k \) at a time (represented by the flows generating the \( \mathbb{R}^k \)-action), enabling an induction process.

In forthcoming work, the full force of the aforementioned main result of this paper (Theorem 4.4) will form the basis of further uniqueness results regarding actions of certain discrete amenable groups on strongly self-absorbing \( C^* \)-algebras.

1. Preliminaries

Notation 1.1. Unless specified otherwise, we will stick to the following notational conventions in this paper:

- \( A \) and \( B \) denote \( C^* \)-algebras.
- The symbols \( \alpha, \beta, \gamma \) are used to denote continuous actions on \( C^* \)-algebras.
- If \( \alpha : G \curvearrowright A \) is an action, then \( A^\alpha \) denotes the fixed-point algebra of \( A \).
- If \( F \) is a finite subset inside some set \( M \), we often denote \( F \subset M \).
- If \( (X, d) \) is some metric space with elements \( a, b \in X \), then we write \( a =_\varepsilon b \) as a shortcut for \( d(a, b) \leq \varepsilon \).

We first recall some needed definitions and notation.

Definition 1.2 (cf. [70] Definition 3.2 and [81, 79] Section 1]). Let \( \alpha : G \curvearrowright A \) be an action. Consider a strictly continuous map \( w : G \to \mathcal{U}(\mathcal{M}(A)) \).
(i) $w$ is called an $\alpha$-$1$-cocycle, if one has $w_g \alpha_g(w_h) = w_{gh}$ for all $g, h \in G$. In this case, the map $\alpha^w : G \to \text{Aut}(A)$ given by $\alpha^w_g = \text{Ad}(w_g) \circ \alpha_g$ is again an action, and is called a cocycle perturbation of $\alpha$. Two $G$-actions on $A$ are called exterior equivalent if one of them is a cocycle perturbation of the other.

(ii) Assume that $w$ is an $\alpha$-$1$-cocycle. It is called an approximate coboundary, if there exists a sequence of unitaries $x_n \in U(\mathcal{M}(A))$ such that $x_n \alpha_g(x_n^*) \xrightarrow{\text{str}} w_g$ for all $g \in G$ and uniformly on compact sets. Two $G$-actions on $A$ are called strongly exterior equivalent, if one of them is a cocycle perturbation of the other via an approximate coboundary.

(iii) Assume $w$ is an $\alpha$-$1$-cocycle. It is called an asymptotic coboundary, if there exists a strictly continuous map $x : [0, \infty) \to U(\mathcal{M}(A))$ with $x_0 = 1$ and such that $x_t \alpha_g(x_t^*) \xrightarrow{\text{str}} w_g$ for all $g \in G$ and uniformly on compact sets. Two $G$-actions on $A$ are called very strongly exterior equivalent, if one of them is a cocycle perturbation of the other via an asymptotic coboundary.

(iv) Let $\beta : G \curvearrowright B$ be another action. The actions $\alpha$ and $\beta$ are called cocycle conjugate, written $\alpha \simeq_{cc} \beta$, if there exists an isomorphism $\psi : A \to B$ such that $\psi^{-1} \circ \beta \circ \psi$ and $\alpha$ are exterior equivalent. If $\psi$ can be chosen such that $\psi^{-1} \circ \beta \circ \psi$ and $\alpha$ are strongly exterior equivalent, then $\alpha$ and $\beta$ are called strongly cocycle conjugate, written $\alpha \simeq_{cc} \beta$. If $\psi$ can be chosen such that $\psi^{-1} \circ \beta \circ \psi$ and $\alpha$ are very strongly exterior equivalent, then $\alpha$ and $\beta$ are called very strongly cocycle conjugate, written $\alpha \simeq_{vcc} \beta$.

**Definition 1.3** (cf. [47] Definition 1.1 and [81] Section 1]). Let $A$ be a C*-algebra and let $\alpha : G \curvearrowright A$ be an action of a locally compact group.

(i) The sequence algebra of $A$ is given as

$$A_\infty = \ell^\infty(\mathbb{N}, A)/\left\langle (x_n)_n \mid \lim_{n \to \infty} \|x_n\| = 0 \right\rangle.$$  

There is a standard embedding of $A$ into $A_\infty$ by sending an element to its constant sequence. We shall always identify $A \subset A_\infty$ this way, unless specified otherwise.

(ii) Pointwise application of $\alpha$ on representing sequences defines a (not necessarily continuous) $G$-action $\alpha_\infty$ on $A_\infty$. Let

$$A_{\infty, \alpha} = \{x \in A_\infty \mid [g \mapsto \alpha_{\infty, g}(x)] \text{ is continuous} \}$$

be the continuous part of $A_\infty$ with respect to $\alpha$.

(iii) For some C*-subalgebra $B \subset A_\infty$, the (corrected) relative central sequence algebra is defined as

$$F(B, A_\infty) = (A_\infty \cap B')/\text{Ann}(B, A_\infty).$$

(iv) If $B \subset A_\infty$ is $\alpha_\infty$-invariant, then the $G$-action $\alpha_\infty$ on $A_\infty$ induces a (not necessarily continuous) $G$-action $\bar{\alpha}_\infty$ on $F(B, A_\infty)$. Let

$$F_{\alpha}(B, A_\infty) = \{y \in F_{\alpha}(B, A_\infty) \mid [g \mapsto \bar{\alpha}_{\infty, g}(y)] \text{ is continuous} \}$$

be the continuous part of $F(B, A_\infty)$ with respect to $\alpha$.

(v) In case $B = A$, we write $F(A, A_\infty) = F_\infty(A)$ and $F_{\alpha}(A, A_\infty) = F_{\infty, \alpha}(A)$. 
Definition 1.4 (see [1, Definition 3.3]). Let $G$ be a second-countable, locally compact group, and let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be actions on separable $C^*$-algebras. An equivariant $\ast$-homomorphism $\varphi : (A, \alpha) \to (B, \beta)$ is called (equivariantly) sequentially split, if there exists a $\ast$-homomorphism $\psi : (B, \beta) \to (A_{\infty, \alpha}, \alpha_{\infty})$ such that $\psi(\varphi(a)) = a$ for all $a \in A$.

Definition 1.5. Let $G$ be a second-countable, locally compact group, and let $\alpha$ be a strongly self-absorbing action. The following are equivalent:

(i) $\gamma$ is a strongly self-absorbing action, if the equivariant first-factor embedding
\[ \text{id}_D \otimes 1_D : (D, \gamma) \to (D \otimes D, \gamma \otimes \gamma) \]
is approximately $G$-unitarily equivalent to an isomorphism.

(ii) $\gamma$ is semi-strongly self-absorbing, if it is strongly cocycle conjugate to a strongly self-absorbing action.

Definition 1.6 (see [81, Definitions 3.1, 4.1]). Let $D$ be a separable, unital $C^*$-algebra and $G$ a second-countable, locally compact group. Let $\gamma : G \curvearrowright D$ be an action. We say that $\gamma$ is $\ast$-absorbing, if
\[ \varphi \otimes 1_D : (A, \alpha) \otimes D \to (B, \beta) \]
is approximately $G$-unitarily equivalent to a homomorphism.

Theorem 1.8 (cf. [81, Theorems 3.7, 4.7]). Let $G$ be a second-countable, locally compact group. Let $A$ be a separable $C^*$-algebra and $\alpha : G \curvearrowright A$ an action. Let $\mathcal{D}$ be a separable, unital $C^*$-algebra and $\gamma : G \curvearrowright \mathcal{D}$ a semi-strongly self-absorbing action. The following are equivalent:

(i) $\alpha$ and $\alpha \otimes \gamma$ are strongly cocycle conjugate.

(ii) $\alpha$ and $\alpha \otimes \gamma$ are cocycle conjugate.

(iii) There exists a unital, equivariant $\ast$-homomorphism from $(\mathcal{D}, \gamma)$ to $(\mathcal{F}_{\infty, \alpha}(A), \delta_{\infty})$.

(iv) The equivariant first-factor embedding $\text{id}_A \otimes 1 : (A, \alpha) \to (A \otimes \mathcal{D}, \alpha \otimes \gamma)$ is sequentially split.

If $\gamma$ is moreover unitarily regular, then these statements are equivalent to

(v) $\alpha$ and $\alpha \otimes \gamma$ are very strongly cocycle conjugate.
Remark. For the rest of this paper, an action $\alpha$ satisfying condition (i) from above is called $\gamma$-absorbing or $\gamma$-stable. In the particular case that $\gamma$ is the trivial $G$-action on a strongly self-absorbing $C^*$-algebra $D$, we will say that $\alpha$ is equivariantly $D$-stable.

Remark 1.9. Unitary regularity for an action is a fairly mild technical assumption. It can be seen as the equivariant analog of the $C^*$-algebraic property that the commutator subgroup inside the unitary group lies in the connected component of the unit. Unitary regularity holds automatically under equivariant $\mathbb{Z}$-stability, but also in other cases; see [82, Proposition 2.19 and Example 6.4].

Theorem 1.10 (see [82, Theorem 5.9]). A semi-strongly self-absorbing action $\gamma : G \curvearrowright D$ is unitarily regular if and only if the class of all separable $\gamma$-absorbing $G$-C$^*$-dynamical systems is closed under equivariant extensions.

We will extensively use the following without much mention:

Proposition 1.11 (see [9]). Let $G$ be a second-countable, locally compact group. Let $A$ be a C$^*$-algebra and $\alpha : G \curvearrowright A$ an action. Let $x \in A_\infty,\alpha$ and $(x_n)_n \in \ell^\infty(\mathbb{N}, A)$ a bounded sequence representing $x$. Then $(x_n)_n$ is a continuous element with respect to the componentwise action of $\alpha$ on $\ell^\infty(\mathbb{N}, A)$.

2. Box spaces and partitions of unity over groups

Definition 2.1. Let $G$ be a second-countable, locally compact group. A residually compact approximation of $G$ is a decreasing sequence $H_{n+1} \subseteq H_n \subseteq G$ of normal, discrete, cocompact subgroups in $G$ with $\bigcap_{n \in \mathbb{N}} H_n = \{1\}$. If $G$ is a discrete group, then the subgroups $H_n$ will have finite index, in which case we call the sequence $(H_n)_n$ a residually finite approximation.

Remark 2.2. In the above setting, the sequence $(H_n)_n$ is automatically a residually finite approximation of the discrete group $H_1$.

Recall the definition of a box space; see [72, Definition 10.24] or [45].

Definition 2.3. Let $\Gamma$ be a countable discrete group and $S = (H_n)_n$ a residually finite approximation of $\Gamma$. Let $d$ be a proper, right-invariant metric on $\Gamma$. For every $n \in \mathbb{N}$, denote by $\pi_n : \Gamma \to \Gamma/H_n$ the quotient map, and $\pi_{ns}(d)$ the push-forward metric on $\Gamma/H_n$ that is induced by $d$. The box space of $\Gamma$ along $S$, denoted $\Box_S \Gamma$, is the coarse disjoint union of the sequence of finite metric spaces $(\Gamma/H_n, \pi_{ns}(d))$.

The main purpose of this section will be to prove the following technical lemma:

Lemma 2.4. Let $G$ be a second-countable, locally compact group and $S = (H_n)_n$ a residually compact approximation of $G$. Assume that the box space $\Box_S H_1$ has finite asymptotic dimension $d$. Then for every $\varepsilon > 0$ and compact set $K \subset G$, there exists $n \in \mathbb{N}$ and continuous, compactly supported functions $\mu^{(0)}, \ldots, \mu^{(d)} : G \to [0, 1]$ satisfying:

(a) for every $l = 0, \ldots, d$ and $h \in H_n \setminus \{1\}$, we have $\supp(\mu^{(l)}) \cap \supp(\mu^{(l)}) \cdot h = \emptyset$;
(b) for every \( g \in G \), we have
\[
\sum_{l=0}^{d} \sum_{h \in H_n} \mu^{(l)}(gh) = 1;
\]

(c) for every \( l = 0, \ldots, d \) and \( g \in K \), we have
\[
\|\mu^{(l)}(g \cdot _) - \mu^{(l)}\|_\infty \leq \varepsilon.
\]

**Remark 2.5.** In the case that \( G = \Gamma \) is a discrete group and \( S \) is a residually finite approximation, this is precisely [83, Lemma 2.13]. In order to prove [Lemma 2.4], we shall convince ourselves that the desired functions can be constructed from finitely supported functions with similar properties on the cocompact subgroup \( H_1 \). For this, we first have to observe a slightly improved version of [83, Lemma 2.13] in the discrete case.

**Lemma 2.6.** Let \( \Gamma \) be a countable discrete group and \( S = (H_n)_{n \in \mathbb{N}} \) a residually finite approximation of \( \Gamma \). Assume that the box space \( \Box S\Gamma \) has finite asymptotic dimension \( d \). Then for every \( \varepsilon > 0 \) and finite set \( F \subset \Gamma \), there exists \( n \in \mathbb{N} \) and finitely supported functions \( \nu^{(0)}, \ldots, \nu^{(d)} : \Gamma \to [0, 1] \) satisfying:

(a) for every \( l = 0, \ldots, d \) and \( h \in H_n \setminus \{1\} \), we have
\[
g_1 h g_2^{-1} \notin F \quad \text{for all } g_1, g_2 \in \text{supp}(\nu^{(l)});
\]

(b) for every \( g \in \Gamma \), we have
\[
\sum_{l=0}^{d} \sum_{h \in H_n} \nu^{(l)}(gh) = 1;
\]

(c) for every \( l = 0, \ldots, d \) and \( g \in F \), we have
\[
\|\nu^{(l)}(g \cdot _) - \nu^{(l)}\|_\infty \leq \varepsilon.
\]

**Proof.** Let \( \varepsilon > 0 \) and \( F \subset G \) be given. We apply [83, Lemma 2.13] and choose some \( n \) and finitely supported functions \( \theta^{(0)}, \ldots, \theta^{(d)} : \Gamma \to [0, 1] \) satisfying
\[
\text{supp}(\theta^{(l)}) \cap \text{supp}(\theta^{(l)}) \cdot h_n = \emptyset \quad \text{for all } h_n \in H_n \setminus \{1\};
\]
as well as properties (b) and (c). Combining property (e2.1) and (c), we see that if \( g_1, g_2 \in \text{supp}(\theta^{(l)}) \) and \( h \in H_n \setminus \{1\} \) are such that \( g_1 h g_2^{-1} = g_1(g_2 h^{-1}) \in F \), then we get
\[
|\theta^{(l)}(g_1)| = |\theta^{(l)}(g_1 h g_2^{-1} \cdot g_2 h^{-1})| \leq \varepsilon + |\theta^{(l)}(g_2 h^{-1})| = \varepsilon.
\]
Let us define new functions \( \kappa^{(l)} : \Gamma \to [0, 1] \) via
\[
\kappa^{(l)}(g) = (\theta^{(l)}(g) - \varepsilon)_+.
\]
These new functions clearly still satisfy property (c). For any \( g_1, g_2 \in \text{supp}(\kappa^{(l)}) \), we evidently have \( g_1, g_2 \in \text{supp}(\theta^{(l)}) \), so assuming \( g_1 h g_2^{-1} \in F \),
for some \( h \in H_n \setminus \{1\} \) would imply \( \kappa^{(l)}(g) = 0 \) by (2.2) and (2.3), a contradiction. In particular we obtain property (a) for these functions.

Lastly, note that property (a) implies that any sum as in (b) can have at most \( d + 1 \) non-vanishing summands, and thus we may estimate for all \( g \in \Gamma \) that

\[
1 = \sum_{l=0}^{d} \sum_{h \in H_n} \theta^{(l)}(gh) \\
\geq \sum_{l=0}^{d} \sum_{h \in H_n} \kappa^{(l)}(gh) \\
\geq \left( \sum_{l=0}^{d} \sum_{h \in H_n} \theta^{(l)}(gh) \right) - (d + 1)\varepsilon \\
= 1 - (d + 1)\varepsilon.
\]

So let us yet again define new functions \( \nu^{(l)} : \Gamma \to [0,1] \) via

\[
\nu^{(l)}(g) = \left( \sum_{l=0}^{d} \sum_{h \in H_n} \kappa^{(l)}(gh) \right)^{-1} \kappa^{(l)}(g).
\]

By our previous calculation, we have \( \kappa^{(l)} \leq \nu^{(l)} \leq \frac{1}{1 - (d + 1)\varepsilon} \kappa^{(l)} \). For these functions, property (a) will still hold, while property (b) holds by construction. Moreover property (c) holds with regard to the tolerance

\[
\eta_\varepsilon := \varepsilon + \frac{2(d + 1)\varepsilon}{1 - (d + 1)\varepsilon}
\]

in place of \( \varepsilon \). Since \( \eta_\varepsilon \to 0 \) as \( \varepsilon \to 0 \), this means that the functions \( \nu^{(l)} \) will have the desired property after rescaling \( \varepsilon \). This shows our claim. \( \square \)

**Lemma 2.7.** Let \( G \) be a locally compact group and \( H \subset G \) a closed and cocompact subgroup. Let \( \mu \) be a left-invariant Haar measure on \( H \).

(i) There exists a compactly supported continuous function \( C : G \to [0,\infty) \) satisfying the equation

\[
\int_{H} C(gh) \, d\mu(h) = 1 \quad \text{for all } g \in G.
\]

(ii) Assume furthermore that \( G \) is amenable. Let \( \varepsilon > 0 \) and let \( K \subset G \) be a compact subset. Then there exists a function \( C \) as above with the additional property that

\[
\|C(g \cdot \_ \_ ) - C\|_{\infty} \leq \varepsilon.
\]

**Proof.** (i) As \( H \) is a cocompact subgroup, there exists some compact set \( K_H \subset G \) such that \( G = K_H \cdot H \). By Urysohn-Tietze, we may choose a compactly supported continuous function \( c : G \to [0,1] \) with \( c|_{K_H} = 1 \). Define the compact set \( K_c \subset H \) via

\[
K_c = \left( K_H^{-1} \cdot \supp(c) \right) \cap H.
\]
Then for every \( g \in G \), there is some \( h_0 \in H \) with \( gh_0 \in K_H \). We have
\[
\text{supp } (c(gh_0 \cdot \_)) \cap H = (gh_0)^{-1} \cdot \text{supp}(c) \cap H \subset K_c.
\]
Thus, we get that
\[
0 < \int_H c(gh) \ d\mu(h) = \int_H c(gh_0h) \ d\mu(h) \leq \mu(K_c) < \infty.
\]
Note that by the properties of the Haar measure, the assignment
\[
I : G \to (0, \infty), \ g \mapsto \int_H c(gh) \ d\mu(h)
\]
is \( H \)-periodic. Then the above computation shows that this assignment yields a well-defined, continuous function on \( G \) with compact image. In particular, its (pointwise) multiplicative inverse is also bounded and continuous.

Let us define
\[
C : G \to [0, \infty), \ g \mapsto I(g)^{-1}c(g).
\]
Then this again yields a continuous function on \( G \) compact support, but with the property that
\[
(e2.4) \quad \int_H C(gh) \ d\mu(h) = 1 \quad \text{for all } g \in G.
\]
(ii) Let us now additionally assume that \( G \) is amenable. Let \( \varepsilon > 0 \) and \( K \subset G \) be given as in the statement. Let \( \rho^G \) denote the right-invariant Haar measure on \( G \). It follows from [20] that we may find some compact set \( J \subset G \) with \( \rho^G(J) > 0 \) such that \( \rho^G(J \Delta (J \cdot K)) \leq \frac{\varepsilon}{\|C\|_\infty \cdot \rho^G(J)} \). Define
\[
C' : G \to [0, \infty), \ g \mapsto \frac{1}{\rho^G(J)} \int_J C(xg) \ d\rho^G(x).
\]
Clearly \( C' \) is yet another continuous function with compact support contained in \( J^{-1} \cdot \text{supp}(C) \). Given any element \( g \in G \), we compute
\[
\begin{align*}
\int_H C'(gh) \ d\mu(h) &= \int_H \frac{1}{\rho^G(J)} \left( \int_J C(xgh) \ d\rho^G(x) \right) \ d\mu(h) \\
&= \frac{1}{\rho^G(J)} \int_J \left( \int_H C(xgh) \ d\mu(h) \right) \ d\rho^G(x) \\
&= \frac{1}{\rho^G(J)} \int_J 1 \ d\rho^G(x) = 1.
\end{align*}
\]
Furthermore, we have for any \( g_K \in K \) and \( g \in G \) that
\[
|C'(g_Kg) - C'(g)| = \frac{1}{\rho^G(J)} \left| \int_J C(xg_Kg) \ d\rho^G(x) - \int_J C(xg) \ d\rho^G(x) \right| \\
\leq \frac{1}{\rho^G(J)} \cdot \|C\|_\infty \cdot \rho^G(J \Delta Jg_K) \\
\leq \varepsilon.
\]
This shows the last part of the claim. \( \square \)

Proof of Lemma 2.4. We first remark that since the box space \( \Box S H_1 \) has finite asymptotic dimension, it also has property A, and therefore \( H_1 \) is amenable; see [68] Theorems 4.3.6 and 4.4.6 and [72] Proposition 11.39.
As $H_1$ is a discrete cocompact normal subgroup in $G$, we also see that $G$ is amenable.

Let $\varepsilon > 0$ and $K \subset G$ be given. Then there exists a function $C : G \to [0, \infty)$ as in \textbf{Lemma 2.7} for $H_1$ in place of $H$, with the property that
\[(e2.5)\quad \|C(g \cdot \cdot) - C\|_{\infty} \leq \varepsilon \quad \text{for all } g \in K.\]

Let us denote the support of $C$ by $S = \text{supp}(C)$. As $H_1$ is discrete in $G$ and $S$ is compact, there exists a finite set $F \subset H_1$ with
\[(e2.6)\quad h_1 \in F \quad \text{whenever } S \cap Sh_1 \neq \emptyset.
\]

Applying \textbf{Lemma 2.6}, there exists some $n$ and finitely supported functions $\nu^{(0)}, \ldots, \nu^{(d)} : H_1 \to [0, 1]$ satisfying the following properties:
\[(e2.7)\quad h_1 h_n h_2^{-1} \notin F \quad \text{for all } h_1, h_2 \in \text{supp}(\nu^{(l)}) \text{ and } h_n \in H_n \setminus \{1\};\]
\[(e2.8)\quad 1 = \sum_{l=0}^{d} \sum_{h_1 \in H_n} \nu^{(l)}(h_1 h_n) \quad \text{for all } h_1 \in H_1.\]

We define $\mu^{(l)} : G \to [0, \infty)$ for $l = 0, \ldots, d$ via
\[\mu^{(l)}(g) = \sum_{h_1 \in H_1} C(gh_1^{-1}) \nu^{(l)}(h_1).\]

Since $\nu^{(l)}$ is finitely supported on $H_1$, we see that $\mu^{(l)}$ is a finite sum of continuous functions with compact support, and hence $\mu^{(l)} \in C_c(G)$.

We claim that these functions have the desired properties. Let us verify \((a)\) which is equivalent to the statement that
\[\mu^{(l)}(g) \cdot \mu^{(l)}(gh_n^{-1}) = 0 \quad \text{for all } g \in G \text{ and } h_n \in H_n \setminus \{1\}.\]

Fix an element $h_n \in H_n \setminus \{1\}$ for the moment. We compute
\[
\mu^{(l)}(g) \cdot \mu^{(l)}(gh_n^{-1}) = \sum_{h_1, h_2 \in H_1} C(gh_1^{-1}) C(gh_n^{-1} h_2^{-1}) \nu^{(l)}(h_1) \nu^{(l)}(h_2)
= \sum_{h_1, h_2 \in H_1} C(gh_1^{-1}) C(gh_2^{-1}) \nu^{(l)}(h_1) \nu^{(l)}(h_2, h_2^{-1})
\]

We claim that each individual summand is zero. Indeed, suppose $h_1, h_2 \in H_1$ are such that $\nu^{(l)}(h_1) \nu^{(l)}(h_2 h_n^{-1}) > 0$. Then $h_1 \in \text{supp}(\nu^{(l)})$ and $h_2 \in \text{supp}(\nu^{(l)}) \cdot h_n$, which implies $h_1 h_2^{-1} \notin F$ by \textbf{(e2.7)}. By our choice of $F$, we obtain
\[
\text{supp}(C(\cdot \cdot h_1^{-1})) \cap \text{supp}(C(\cdot \cdot h_2^{-1})) \subseteq Sh_1 \cap Sh_2 = (Sh_1 h_2^{-1} \cap S) \cdot h_2 = \emptyset,
\]
and in particular $C(gh_1^{-1}) C(gh_2^{-1}) = 0$. This finishes the proof that each summand of the above sum is zero and shows property \((a)\).

\footnote{Note that we will reserve the notation $h_1, h_2$ for elements in $H_1$, whereas $h_n$ will denote an element in the smaller subgroup $H_n$ for $n > 2$.}
Let us now show property (b). We calculate for every $g \in G$ that
\[
\sum_{l=0}^{d} \sum_{h_n \in H} \mu^{(l)}(gh_n) = \sum_{l=0}^{d} \sum_{h_n \in H} \sum_{h_1 \in H} C(gh_nh_1^{-1})\nu^{(l)}(h_1) = \sum_{h_1 \in H} C(gh_1) = \sum_{h_1 \in H} \sum_{l=0}^{d} \nu^{(l)}(h_1h_n).
\]

Let us now turn to (c). Given any $g \in G$ and $gK \in K$, we compute
\[
|\mu^{(l)}(gKg) - \mu^{(l)}(g)| = \left| \sum_{h_1 \in H} (C(gKgh_1^{-1}) - C(gh_1^{-1}))\nu^{(l)}(h_1) \right| \leq \sup_{h_1 \in H_1} |C(gKgh_1^{-1}) - C(gh_1^{-1})| \leq \|C(gK \cdot \_ - C)\|_\infty \leq \varepsilon.
\]

As $g \in G$ was arbitrary, this finishes the proof. □

**Remark.** Let $G$ be a locally compact group and $H \subset G$ a closed, cocompact subgroup. For any $C^*$-algebra $A$, we may naturally view $C(G/H, A)$ as a $C^*$-subalgebra of $C_b(G, A)$ by assigning a function $f$ to the function $f'$ given by $f'(g) = f(gH)$.

In what follows, we will briefly establish a technical result that allows one to perturb approximately $H$-periodic functions in $C_b(G, A)$ to exactly $H$-periodic functions in a systematic way.

**Lemma 2.8.** Let $G$ be a locally compact group and $H \subset G$ a closed, cocompact subgroup. Let $A$ be a $C^*$-algebra. Then there exists a conditional expectation $E : C_b(G, A) \rightarrow C(G/H, A)$ with the following property.

For every $\varepsilon > 0$ and compact set $K \subset G$, there exists $\delta > 0$ and a compact set $J \subset H$ such that the following holds:

If $f \in C_b(G, A)$ satisfies
\[
\max_{g \in K} \max_{h \in J} \|f(g) - f(gh)\| \leq \delta,
\]

then
\[
\|f - E(f)\|_{\infty, K} \leq \varepsilon.
\]

**Proof.** Let $\mu$ be a left-invariant Haar measure on $H$. Let $C \in C_c(G)$ be a function as in Lemma 2.7. Then we define
\[
E : C_b(G, A) \rightarrow C(G/H, A), \quad E(f)(gH) = \int_H C(gh)f(gh) \, d\mu(h).
\]

Since $C$ is compactly supported and the Haar measure $\mu$ is left-invariant, it is clear that $E$ is well-defined and indeed a conditional expectation. Let
\( \varepsilon > 0 \) and \( K \subset G \) be given. Let \( S \) be the compact support of \( C \). Then the set \( J := (K^{-1}S) \cap H \) is compact in \( H \) with the property that

\[
(\varepsilon 2.9) \quad g \in K \text{ and } gh \in S \implies h \in J
\]

for all \( h \in H \). Set

\[
\delta = \frac{\varepsilon}{1 + \mu(J) \cdot \|C\|_{\infty}}.
\]

For every \( f \in C_b(G, A) \) with

\[
\max_{g \in K} \max_{h \in J} \|f(g) - f(gh)\| \leq \delta,
\]

it follows for every \( g \in K \) that

\[
\|f(g) - E(f)(gH)\| = \left\| \left( \int_{H} C(gh) \, d\mu(h) \right) f(g) - \int_{H} C(gh) f(gh) \, d\mu(h) \right\| = \mu(J) \cdot \|C\|_{\infty} \cdot \delta \leq \varepsilon.
\]

This shows our claim. \( \square \)

**Corollary 2.9.** Let \( G \) be a locally compact group and \( H \subset G \) a closed, cocompact subgroup. Let \( A \) and \( B \) be two \( C^* \)-algebras. Then for every \( \varepsilon > 0 \), \( F \subset B \) and compact set \( K \subset G \), there exists \( \delta > 0 \) and a compact set \( J \subset H \) such that the following holds:

If \( \Theta : B \to C_b(G, A) \) is a c.p.c. map with

\[
\max_{g \in K} \max_{h \in J} \|\Theta(b)(g) - \Theta(b)(gh)\| \leq \delta \quad \text{for all } b \in F,
\]

then there exists a c.p.c. map \( \Psi : B \to C(G/H, A) \) with

\[
\max_{g \in K} \|\Psi(b)(gH) - \Theta(b)(g)\| \leq \varepsilon \quad \text{for all } b \in F.
\]

### 3. Systems generated by order zero maps with commuting ranges

The following notation and observations are [33, Lemma 6.6] and originate in [35, Section 5].

**Notation 3.1.** Let \( D_1, \ldots, D_n \) be finitely many unital \( C^* \)-algebras. For \( t \in [0, 1] \) and \( j = 1, \ldots, n \), we denote

\[
D_j^{(t)} = \begin{cases} D_j, & t > 0, \\ C \cdot 1_{D_j}, & t = 0. \end{cases}
\]

Given moreover a tuple \( \vec{t} = (t_1, \ldots, t_n) \in [0, 1]^n \), let us denote

\[
D^{(\vec{t})} = D_1^{(t_1)} \otimes_{\text{max}} D_2^{(t_2)} \otimes_{\text{max}} \cdots \otimes_{\text{max}} D_n^{(t_n)}.
\]

Consider the simplex

\[
\Delta^{(n)} = \left\{ \vec{t} \in [0, 1]^n \mid t_1 + \cdots + t_n = 1 \right\}
\]

and set

\[
\mathcal{E}(D_1, \ldots, D_n) = \left\{ f \in C\left(\Delta^{(n)}, D_1 \otimes_{\text{max}} \cdots \otimes_{\text{max}} D_n\right) \mid f(\vec{t}) \in D^{(\vec{t})} \right\}.
\]
In the case that \( D_j = D \) are all the same C*-algebra, we will write
\[
\mathcal{E}(D_1, \ldots, D_n) = \mathcal{E}(D, n)
\]
instead. For every \( j = 1, \ldots, n \), we will consider the canonical c.p.c. order zero map
\[
\eta_j : D_j \to \mathcal{E}(D_1, \ldots, D_n)
\]
given by
\[
\eta_j(d_j)(\tilde{t}) = t_j \cdot (1_{D_1} \otimes \cdots \otimes 1_{D_{j-1}} \otimes d_j \otimes 1_{D_{j+1}} \otimes \cdots \otimes 1_{D_n}).
\]
One easily checks that the ranges of the maps \( \eta_j \) generate \( \mathcal{E}(D_1, \ldots, D_n) \) as a C*-algebra.

**Proposition 3.2.** Let \( D_1, \ldots, D_n \) be unital C*-algebras. Then the C*-algebra \( \mathcal{E}(D_1, \ldots, D_n) \) together with the c.p.c. order zero maps \( \eta_j : D_j \to \mathcal{E}(D_1, \ldots, D_n) \) satisfies the following universal property:

If \( B \) is any unital C*-algebra and \( \psi_j : D_j \to B \) for \( j = 1, \ldots, n \) are c.p.c. order zero maps with pairwise commuting ranges and
\[
\psi_1(1_{D_1}) + \cdots + \psi_n(1_{D_n}) = 1_B,
\]
then there exists a unique unital *-homomorphism \( \Psi : \mathcal{E}(D_1, \ldots, D_n) \to B \) such that \( \Psi \circ \eta_j = \psi_j \) for all \( j = 1, \ldots, n \).

**Notation 3.3.** Let \( G \) be a second-countable, locally compact group. Let \( D_1, \ldots, D_n \) be unital C*-algebras with continuous actions \( \alpha^{(j)} : G \curvearrowright D_j \) for \( j = 1, \ldots, n \). Then the \( G \)-action on \( \mathcal{C}(\Delta^{(n)}, D_1 \otimes \max \cdots \otimes \max D_n) \) defined fibrewise by \( \alpha^{(1)} \otimes \max \cdots \otimes \max \alpha^{(n)} \) restricts to a well-defined action
\[
\mathcal{E}(\alpha^{(1)}, \ldots, \alpha^{(n)}) : G \curvearrowright \mathcal{E}(D_1, \ldots, D_n)
\]
We will again denote \( \mathcal{E}(\alpha, n) = \mathcal{E}(\alpha^{(1)}, \ldots, \alpha^{(n)}) \) in the special case that all \( (D_j, \alpha^{(j)}) = (D, \alpha) \) are the same C*-dynamical system.

**Remark 3.4.** By the universal property in Proposition 3.2, the \( G \)-action \( \mathcal{E}(\alpha^{(1)}, \ldots, \alpha^{(n)}) \) defined in Notation 3.3 is uniquely determined by the identity \( \mathcal{E}(\alpha^{(1)}, \ldots, \alpha^{(n)})_g \circ \eta_j = \eta_j \circ \alpha^{(j)}_g \) for all \( j = 1, \ldots, n \) and \( g \in G \).

This immediately allows us obtain the following equivariant version of Proposition 3.2 as a consequence:

Let \( B \) be any unital C*-algebra with an action \( \beta : G \curvearrowright B \). If \( \psi_j : (D_j, \alpha^{(j)}) \to (B, \beta) \) are equivariant c.p.c. order zero maps with pairwise commuting ranges and \( \psi_1(1_{D_1}) + \cdots + \psi_n(1_{D_n}) = 1_B \), then there exists a unique unital equivariant *-homomorphism
\[
\Psi : \left( \mathcal{E}(D_1, \ldots, D_n), \mathcal{E}(\alpha^{(1)}, \ldots, \alpha^{(n)}) \right) \to (B, \beta)
\]
satisfying \( \Psi \circ \eta_j = \psi_j \) for all \( j = 1, \ldots, n \).

**Remark 3.5.** Let us now also convince ourselves of a different natural way to view the C*-algebras from Notation 3.1.

For this, let us first consider the case \( n = 2 \), so we have two unital C*-algebras \( D_1 \) and \( D_2 \). Notice that \([0, 1]\) is naturally homeomorphic to
the simplex $\Delta^{(2)} = \{ (t_1, t_2) \in [0,1]^2 \mid t_1 + t_2 = 1 \}$ via the assignment $t \mapsto (t, t - 1)$. In this way we may see that there is a natural isomorphism
\[
E(D_1, D_2) \overset{\text{def}}{=} \left\{ f \in C(\Delta^{(2)}, D_1 \otimes_{\text{max}} D_2) \mid f(0, 1) \in D_1 \otimes 1, f(1, 0) \in 1 \otimes D_2 \right\}
\]
\[
\cong \left\{ f \in C([0,1], D_1 \otimes_{\text{max}} D_2) \mid f(0) \in D_1 \otimes 1, f(1) \in 1 \otimes D_2 \right\}
\]
\[
=: D_1 \ast D_2.
\]
In particular, we see that the notation $E(D_1, D_2)$ is consistent with [82, Definition 5.1]. As pointed out in [82, Remark 5.2], the assignment $(D_1, D_2) \mapsto E(D_1, D_2)$ on pairs of unital $C^*$-algebras therefore generalizes the join construction for pairs of compact spaces, which gives rise to the notation $D_1 \ast D_2$.

Let now $n \geq 2$ and let $D_1, \ldots, D_{n+1}$ be unital $C^*$-algebras. The simplex $\Delta^{(n+1)}$ is homeomorphic to $[0,1] \times \Delta^{(n)}$ via the assignment
\[
(t_1, t) \mapsto \begin{cases} 
(1, t) & , t_1 = 0 \\
(1 - t_1, \frac{t}{1 - t_1}) & , t_1 \neq 0 
\end{cases}
\]
for $(t_1, t_{n+1}) \in \Delta^{(n+1)}$. Keeping this in mind, we see that there is a natural map
\[
\Phi : D_1 \ast E(D_2, \ldots, D_{n+1}) \to E(D_1, \ldots, D_{n+1})
\]
given by
\[
\Phi(f)(t_1, t) = \begin{cases} 
f(1)(t) & , t_1 = 0 \\
f(1 - t_1)(\frac{t}{1 - t_1}) & , t_1 \neq 0 
\end{cases}
\]
for $(t_1, t) \in \Delta^{(n+1)}$. It is a simple exercise to see that this is a well-defined isomorphism. This shows that it makes sense to view the $C^*$-algebra $E(D_1, \ldots, D_n)$ as the $n$-fold join $D_1 \ast \cdots \ast D_n$. We can also observe that this isomorphism is natural in each $C^*$-algebra, and therefore becomes equivariant as soon as we equip each $C^*$-algebra $D_j$ with an action $\alpha^j$ of some group $G$.

Henceforth, we will in particular denote
\[
D^{*n} := E(D, n) \quad \text{and} \quad \alpha^{*n} = E(\alpha, n)
\]
for a unital $C^*$-algebra $D$ and some group action $\alpha : G \curvearrowright D$.

**Remark 3.6.** By the definition of the join of two $C^*$-algebras $D_1$ and $D_2$, there is a natural short exact sequence
\[
0 \longrightarrow C_0(0, 1) \otimes D_1 \otimes_{\text{max}} D_2 \longrightarrow D_1 \ast D_2 \longrightarrow D_1 \oplus D_2 \longrightarrow 0.
\]
Given some $n \geq 1$ and a unital $C^*$-algebra $D$, we have $D^{*n+1} \cong D \ast (D^{*n})$, and therefore a special case of the above yields the short exact sequence
\[
0 \longrightarrow C_0(0, 1) \otimes D \otimes_{\text{max}} D^{*n} \longrightarrow D^{*n+1} \longrightarrow D \oplus D^{*n} \longrightarrow 0.
\]
Again by naturality, we note that this short exact sequence is automatically equivariant if we additionally equip $D$ with a group action.

---

2The reader should keep in mind that an element $f$ in the domain is a continuous function on $[0,1]$ whose values are in turn (certain) continuous functions from $\Delta^{(n)}$ to the tensor product $D_1 \otimes_{\text{max}} \cdots \otimes_{\text{max}} D_{n+1}$.
We now come to the main observation about $C^*$-dynamical systems arising in this fashion, which will be crucial in proving our main result:

**Lemma 3.7.** Let $G$ be a second-countable, locally compact group. Let $A$ be a separable, unital $C^*$-algebra with an action $\alpha : G \curvearrowright A$. Suppose that $\gamma : G \curvearrowright D$ is a semi-strongly self-absorbing and unitarily regular action. If $\alpha$ is $\gamma$-absorbing, then so is the action $\alpha^n : G \curvearrowright A^n$ for all $n \geq 2$.

**Proof.** This follows directly from Remark 3.6 and Theorem 1.10 by induction. $\square$

**Remark 3.8.** It ought to be mentioned that Lemma 3.7 does not depend in any way on the fact that one considers the $n$-fold join over the same $C^*$-algebra and the same action. The analogous statement is valid for more general joins of the form

$$\alpha^{(1)} \ast \cdots \ast \alpha^{(n)} : G \curvearrowright A_1 \ast \cdots \ast A_n$$

by virtually the same argument.

In fact, by putting in a bit more work, one could likely prove an equivariant version of [32, Theorem 4.6] for $C_0(X)$-$G$-$C^*$-algebras with $\dim(X) < \infty$ whose fibres absorb a given semi-strongly self-absorbing and unitarily regular action. This would contain Lemma 3.7 as a special case since the $G$-$C^*$-algebra $A_1 \ast \cdots \ast A_n$ is in fact a $C(\Delta^{(n)})$-$G$-$C^*$-algebra with each fibre being isomorphic to some finite tensor product of the $A_j$. We will never need this level of generality within this paper, however.

4. **Rokhlin dimension with commuting towers**

The following notion generalizes analogous definitions made in [35, 33, 24, 33].

**Definition 4.1** (cf. [33, Definition 4.1]). Let $G$ be a second-countable, locally compact group. Let $\alpha : G \curvearrowright A$ be an action on a separable $C^*$-algebra.

(i) Let $H \subseteq G$ be a closed, cocompact subgroup. The Rokhlin dimension of $\alpha$ with commuting towers relative to $H$, denoted $\dim^{c}_{\text{Rok}}(\alpha, H)$, is the smallest natural number $d$ such that there exist equivariant c.p.c. order zero maps

$$\varphi^{(0)}, \ldots, \varphi^{(d)} : (\mathcal{C}(G/H), G\text{-shift}) \to (F_{\infty, \alpha}(A), \tilde{\alpha}_\infty)$$

with pairwise commuting ranges such that $1 = \varphi^{(0)}(1) + \cdots + \varphi^{(d)}(1)$.

(ii) If $S = (G_k)_k$ denotes a decreasing sequence of closed, cocompact subgroups, then we define the Rokhlin dimension of $\alpha$ with commuting towers relative to $S$ via

$$\dim^{c}_{\text{Rok}}(\alpha, S) = \sup_{k \in \mathbb{N}} \dim^{c}_{\text{Rok}}(\alpha, G_k).$$

(iii) Let $N \subseteq G$ be any closed, normal subgroup. The Rokhlin dimension of $\alpha$ with commuting towers relative to $N$ is defined as

$$\dim^{c}_{\text{Rok}}(\alpha, N) = \sup \{ \dim^{c}_{\text{Rok}}(\alpha, H) \mid H \subseteq G \text{ closed, cocompact, } N \subseteq H \}.$$  

(iv) Lastly, the Rokhlin dimension of $\alpha$ with commuting towers is defined as

$$\dim^{c}_{\text{Rok}}(\alpha) = \dim^{c}_{\text{Rok}}(\alpha, \{1\}) = \sup \{ \dim^{c}_{\text{Rok}}(\alpha, H) \mid H \subseteq G \text{ closed, cocompact} \}.$$
Notation 4.2. Let $G$ be a second-countable, locally compact group. Given a decreasing sequence $S = (G_k)_k$ of closed, cocompact subgroups, we will denote

$$G/S = \lim_{\leftarrow} G/G_k.$$ 

This is a metrizable, compact space\footnote{This construction generalizes the profinite completion of a discrete residually finite group along a chosen separating sequence of normal subgroups of finite index.} which carries a natural continuous $G$-action induced by the left $G$-shift on each building block $G/G_k$; in particular we will call the resulting action also just the $G$-shift and denote it by $\sigma^S : G \curvearrowright G/S$.

In the sequel, we will adopt the perspective of the associated $G$-$C^*$-dynamical system, which is given as the equivariant inductive limit

$$C(G/S) = \lim_{\rightarrow} C(G/G_k).$$

We will moreover consider $C(G/S)^{*n} \cong C((G/S)^{*n})$ for $n \geq 2$. With some abuse of terminology, we will use the term “$G$-shift” also to refer to the canonical action on this $C^*$-algebra (or the underlying space) that is induced by the $n$-fold tensor products of the $G$-shift on each fibre.

Lemma 4.3. Let $G$ be a second-countable, locally compact group. Let $\alpha : G \curvearrowright A$ be an action on a separable $C^*$-algebra. Let $S = (G_k)_k$ be a decreasing sequence of closed, cocompact subgroups. Let $d \geq 0$ be some natural number. Then the following are equivalent:

(i) $\dim_{\text{Rok}} C(\alpha, S) \leq d$;

(ii) there exist equivariant c.p.c. order zero maps

$$\varphi^{(0)}, \ldots, \varphi^{(d)} : (C(G/S), G\text{-shift}) \to (F_{\infty, \alpha}(A), \tilde{\alpha}_\infty)$$

with pairwise commuting ranges such that $1 = \varphi^{(0)}(1) + \cdots + \varphi^{(d)}(1)$;

(iii) there exists a unital $G$-equivariant $*$-homomorphism

$$(C((G/S)^{*(d+1)}), G\text{-shift}) \to (F_{\infty, \alpha}(A), \tilde{\alpha}_\infty);$$

(iv) the first-factor embedding

$$\text{id}_A \otimes 1 : (A, \alpha) \to (A \otimes C((G/S)^{*(d+1)}), \alpha \otimes (G\text{-shift}))$$

is $G$-equivariantly sequentially split.

Proof. The equivalence $[\text{(i)}] \iff [\text{(ii)}]$ follows from a standard reindexation trick, using the equivariant inductive limit structure of $C(G/S)$ as pointed out in \textbf{Notation 4.2}. We will leave the details to the reader.

The equivalence $[\text{(ii)}] \iff [\text{(iii)}]$ is a direct consequence of \textbf{Proposition 3.2} and \textbf{Remark 3.5}, and the equivalence $[\text{(iii)}] \iff [\text{(iv)}]$ is a direct consequence of \cite[Lemma 4.2]{[1]}. 

The purpose of this section is to prove the following theorem, which can be regarded as the main result of the paper. Some of its non-trivial applications will be discussed in the subsequent sections. See in particular \textbf{Corollary 5.1} for a possibly more accessible special case of this theorem.
Theorem 4.4. Let $G$ be a second-countable, locally compact group and $N \subset G$ a closed, normal subgroup. Denote by $\pi_N : G \to G/N$ the quotient map. Let $S_k = (H_k)_k$ be a residually compact approximation of $G/N$, and set $G_k = \pi_N^{-1}(H_k)$ for all $k \in \mathbb{N}$ and $S_0 = (G_0)_k$. Let $A$ be a separable $C^*$-algebra and $D$ a strongly self-absorbing $C^*$-algebra. Let $\alpha : G \curvearrowright A$ be an action and $\gamma : G \curvearrowright D$ be a semi-strongly self-absorbing, unitarily regular action. Suppose that for the restrictions to the $N$-actions, we have $\alpha|_N \simeq_{cc} (\alpha \otimes \gamma)|_N$. If
\[
\operatorname{asdim}(\square_{S_1} H_1) < \infty \quad \text{and} \quad \dim_{\operatorname{Rok}}^c(\alpha, S_0) < \infty,
\]
then $\alpha \simeq_{cc} \alpha \otimes \gamma$.

The proof of this result will occupy the rest of this section. The first and most difficult step is to convince ourselves of a very special case of Theorem 4.4, which involves the technical preparation below and from Section 2.

For convenience, we isolate the following lemma, which is a consequence of Proposition 1.11 and Lemma 4.5. We recall the Winter–Zacharias structure theorem for order maps, along with the Choi–Effros lifting theorem; see [90, Section 3] and [12].

Lemma 4.5. Let $G$ be a second-countable, locally compact group. Let $A$ be a separable $C^*$-algebra and $B$ a separable, unital and nuclear $C^*$-algebra. Let $\alpha : G \curvearrowright A$ and $\beta : G \curvearrowright B$ be two actions. Let $\kappa : (B, \beta) \to (A, \alpha, \alpha_{cc})$ be an equivariant c.p.c. order zero map. Then $\kappa$ can be represented by a sequence of c.p.c. maps $\kappa_n : B \to A$ satisfying:
\[
\begin{align*}
\text{(a) } & \|\kappa_n(xy)\kappa(1) - \kappa_n(x)\kappa_n(y)\| \to 0 \\
\text{(b) } & \max_{g \in K} \| (\kappa_n \circ \gamma_g)(x) - (\alpha_{\gamma}(\kappa_n))x \| \to 0
\end{align*}
\]
for all $x, y \in B$ and compact subsets $K \subset G$.

The proof of the following is based on a standard reindexation trick. In the short proof below, precise references are provided for completeness, although we note that this might not be the most elegant or direct way to show these statements.

Lemma 4.6. Let $G$ be a second-countable, locally compact group. Suppose that $\alpha : G \curvearrowright A$, $\beta : G \curvearrowright B$, and $\gamma : G \curvearrowright D$ are actions on separable $C^*$-algebras. Assume furthermore that $D$ is unital, that $\gamma$ is semi-strongly self-absorbing, and that $\beta \simeq_{cc} \beta \otimes \gamma$.

(i) Suppose that there exists an equivariant $*$-homomorphism $(A, \alpha) \to (B, \beta)$ that is $G$-equivariantly sequentially split. Then $\alpha \simeq_{cc} \beta \otimes \gamma$.

(ii) Suppose that $B$ is unital and that there exists an equivariant and unital $*$-homomorphism from $(B, \beta)$ to $(F_{\infty, \alpha}, \alpha_{cc})$. Then $\alpha \simeq_{cc} \beta \otimes \gamma$.

Proof. (i) By Theorem 1.8, the statement $\beta \simeq_{cc} \beta \otimes \gamma$ is equivalent to the equivariant first-factor embedding
\[
id_B \otimes 1 : (B, \beta) \to (B \otimes D, \beta \otimes \gamma)
\]
being sequentially split. Let $\varphi : (A, \alpha) \to (B, \beta)$ be sequentially split. By Proposition 3.7, the composition $\varphi \otimes 1_D = (\operatorname{id}_B \otimes 1_D) \circ \varphi$ is also sequentially split. However, we also have $\varphi \otimes 1_D = (\varphi \otimes \operatorname{id}_D) \circ (\operatorname{id}_A \otimes 1_D)$, which implies that $\operatorname{id}_A \otimes 1_D$ is also sequentially split. This implies the claim that $\alpha \simeq_{cc} \alpha \otimes \gamma$. 

By [1 Lemma 4.2], it follows that the embedding
\[ \text{id}_A \otimes 1_B : (A, \alpha) \to (A \otimes_{\max} B, \alpha \otimes \beta) \]
is sequentially split. Since we assumed that \( \beta \) is \( \gamma \)-absorbing, so is \( \alpha \otimes \beta \), and so the claim arises as a special case of (i). \( \square \)

The following is a special case of Theorem 4.4 and its proof is by far the most technical part of this paper:

**Lemma 4.7.** Let \( G \) be a second-countable, locally compact group and \( N \subset G \) a closed, normal subgroup. Denote by \( \pi_N : G \to G/N \) the quotient map. Let \( S_1 = (H_k)_k \) be a residually compact approximation of \( G/N \), and set \( G_k = \pi_N^{-1}(H_k) \) for all \( k \in \mathbb{N} \) and \( \mathcal{S}_0 = (G_k)_k \). Let \( A \) be a separable \( C^* \)-algebra and \( \mathcal{D} \) a strongly self-absorbing \( C^* \)-algebra. Let \( \alpha : G \curvearrowright A \) be an action and \( \gamma : G \curvearrowright \mathcal{D} \) a semi-strongly self-absorbing, unitarily regular action. Suppose that for the restrictions to the \( N \)-actions, we have \( \alpha|_N \cong_{cc} (\alpha \otimes \gamma)|_N \). If \( \text{asdim}(\square_{S_1} H_1) < \infty \), then the \( G \)-action
\[ \sigma^{S_0} \otimes \alpha : G \curvearrowright C(G/S_0) \otimes A \]
is \( \gamma \)-absorbing.

**Proof.** Let
\[ \tilde{\kappa} : (\mathcal{D}, \gamma|_N) \to (F_{\infty, \alpha|_N}(A), \tilde{\alpha}|_N) \]
be an \( N \)-equivariant, unital \( * \)-homomorphism. Using [2 Section 4], we may choose an equivariant c.p.c. order zero map
\[ \kappa : (\mathcal{D}, \gamma|_N) \to (A_{\infty, \alpha|_N} \cap A, \alpha|_N) \]
that lifts \( \tilde{\kappa} \).

Consider a sequence of c.p.c. maps \( \kappa_n : B \to A \) lifting \( \kappa \) as in [Lemma 4.5] By a simple reindexation trick, let us rearrange this into finitely many sequences of c.p.c. maps \( \kappa^{(l)}_n : B \to A \) for \( l = 0, \ldots, d \) so that, using Lemma 4.5 each sequence \( \kappa^{(l)}_n \) has the following properties for all \( a \in A \), \( b, b_1, b_2 \in \mathcal{D} \) and compact sets \( L \subseteq N \):

\[
\begin{align*}
(\text{e4.1}) \quad & \quad \| \kappa^{(l)}_n(b), a \| \to 0; \\
(\text{e4.2}) \quad & \quad \| \kappa^{(l)}_n(b_1 b_2) \kappa^{(l)}_n(1) - \kappa^{(l)}_n(b_1) \kappa^{(l)}_n(b_2) \| \to 0; \\
(\text{e4.3}) \quad & \quad \| (\kappa^{(l)}_n(1) - 1) \cdot a \| \to 0; \\
(\text{e4.4}) \quad & \quad \max_{r \in L} \| (\kappa^{(l)}_n \circ \gamma_r)(b) - (\alpha_r \circ \kappa^{(l)}_n)(b) \| \to 0;
\end{align*}
\]

and additionally one has for every compact set \( K \subseteq G \) that
\[
\begin{align*}
(\text{e4.5}) \quad & \quad \max_{g \in K} \| [\kappa^{(l_1)}_n(b_1), (\alpha_g \circ \kappa^{(l_2)}_n)(b_2)] \| \to 0 \quad \text{for all } l_1 \neq l_2.
\end{align*}
\]

Let now \( \varepsilon > 0 \) be a fixed parameter and \( K \subseteq G \) a fixed compact set. We set \( d := \text{asdim}(\square_{S_1} H_1) < \infty \).

Apply Lemma 2.4 and find \( k \) and compactly supported functions \( \mu^{(0)}, \ldots, \mu^{(d)} \in \mathcal{C}_c(G/N) \), so that for every \( l = 0, \ldots, d \), we have
\[
\begin{align*}
(\text{e4.6}) \quad & \quad \text{supp}(\mu^{(l)}) \cap \text{supp}(\mu^{(l)}) : h = \emptyset \quad \text{for all } h \in H_k \setminus \{1\} ;
\end{align*}
\]
\[ \sum_{l=0}^{d} \sum_{h \in H_k} \mu^{(l)}(\pi_N(g)h) = 1 \quad \text{for all } g \in G; \]

\[ \| \mu^{(l)}(\pi_N(g) \cdot \_ ) - \mu^{(l)} \|_\infty \leq \varepsilon \quad \text{for all } g \in K \cup K^{-1}. \]

The group \( H_k \) is discrete, so we may choose a cross-section \( \sigma : H_k \to G_k = \pi_N^{-1}(H_k) \subseteq G \). For each \( l = 0, \ldots, d \), consider the sequence of c.p.c. maps

\[ \Theta_n^{(l)} : D \to C_0(G, A) \]

given by

\[ \Theta_n^{(l)}(b)(g) = \sum_{h \in H_k} \mu^{(l)}(\pi_N(g)h) \cdot (\alpha_{\sigma(h)} \circ \kappa_n^{(l,j)} \circ \gamma_{\sigma(h)}^{-1})(b). \]

This sum is well-defined because the compact support of the function \( \mu^{(l)} \) on \( G/N \) meets a set of the form \( \pi_N(g) \cdot H_k \) at most once according to (e4.6).

We wish to show that given an element \( b \in D \), the functions \( \Theta_n^{(l)}(b) \) are approximately \( G_k \)-periodic on large compact sets. This is so that we may apply Corollary 2.9 in order to approximate the maps \( \Theta_n^{(l)} \) by other maps going into \( C(G/G_k, A) \).

Let \( K_{H_k} \subseteq G_k \) and \( K_{G} \subseteq G \) be two compact sets. As \( H_k \) is discrete, we observe two facts. First, there exists a compact set \( K_N \subseteq N \) and a finite set \( 1 \in F_k \subseteq H_k \)

\[ \text{(e4.10)} \quad K_{H_k} \subseteq \sigma(F_k) \cdot K_N. \]

Second, it follows from (e4.10) that there is a finite set \( F_k' \subseteq H_k \) so that

\[ \text{(e4.11)} \quad \mu^{(l)}(\pi_N(g)h) > 0 \quad \text{implies } \quad h \in F_k' \quad \text{for all } g \in K_G \]

By making the involved sets larger, if necessary, we may assume \( F_k = F_k' \).

Define also

\[ \text{(e4.12)} \quad K_N' = \bigcup_{h_0, h \in F_k} \sigma(h_0) \cdot K_N \cdot \sigma(h^{-1}h_0)h \sigma(h)^{-1} \subseteq N \]

and

\[ \text{(e4.13)} \quad K_N'' = \bigcup_{h \in F_k} \sigma(h)^{-1} \cdot K_N' \cdot \sigma(h) \subseteq N. \]

As \( N \) is a normal subgroup and \( \sigma \) is a cross-section for the quotient map \( \pi_N \), it follows that these are well-defined compact sets in \( N \).
We compute for all \( l = 0, \ldots, d \), \( b \in D \), \( g \in K_G \), \( h_0 \in F_k \) and \( r \in K_N \) that

\[
\begin{align*}
\| & (\alpha_{g\sigma(h_0)} \circ \kappa_n^l \circ \gamma_{g\sigma(h_0)}^{-1})(b) - (\alpha_{g\sigma(h_0)} \circ \kappa_n^l \circ \gamma_{g\sigma(h_0)}^{-1})(b) \| \\
= & \quad \| (\alpha_{\sigma(h_0)} \circ \kappa_n^l \circ \gamma_{\sigma(h_0)}^{-1})(\gamma_g^{-1}(b)) - (\alpha_{\sigma(h_0)} \circ \kappa_n^l \circ \gamma_{\sigma(h_0)}^{-1})(\gamma_g^{-1}(b)) \| \\
= & \quad \| (\alpha_{\sigma(h_0)} \circ \kappa_n^l \circ \gamma_{\sigma(h_0)}^{-1})(\gamma_g^{-1}(b)) - (\alpha_{\sigma(h_0)} \circ \kappa_n^l \circ \gamma_{\sigma(h_0)}^{-1})(\gamma_g^{-1}(b)) \| \\
\leq & \quad \max_{g \in K_G} \max_{s \in K_N^l} \| (\alpha_s \circ \kappa_n^l \circ \gamma_s^{-1})(\gamma_g^{-1}(g\sigma(h_0)(b))) - \kappa_n^l((\gamma_g^{-1}(g\sigma(h_0)(b))) \| \\
\rightarrow & \quad 0.
\end{align*}
\]

It thus follows for all \( l = 0, \ldots, d \), \( b \in D \), \( g \in K_G \), \( h_0 \in F_k \) and \( r \in K_N \) that

\[
\begin{align*}
\| & \Theta_n^l(b)(g) - \Theta_n^l(b)(g\sigma(h_0) r) \| \\
\leq & \quad \sum_{h_1 \in F_k} \mu^l(\pi_N(\sigma(h_1)) \cdot (\alpha_{g\sigma(h_0)} \circ \kappa_n^l \circ \gamma_{g\sigma(h_0)}^{-1})(b) \\
- & \quad \sum_{h_2 \in K_{-1}^{l-1} F_k} \mu^l(\pi_N(\sigma(h_0 h_2)) \cdot (\alpha_{g\sigma(h_0) r \sigma(h_2)} \circ \kappa_n^l \circ \gamma_{g\sigma(h_0) r \sigma(h_2)}^{-1})(b) \\
= & \quad \sum_{h_1 \in F_k} \mu^l(\pi_N(\sigma(h_1)) \cdot (\alpha_{g\sigma(h_0)} \circ \kappa_n^l \circ \gamma_{g\sigma(h_0)}^{-1})(b) \\
- & \quad \sum_{h_2 \in K_{-1}^{l-1} F_k} \mu^l(\pi_N(\sigma(h_2)) \cdot (\alpha_{g\sigma(h_0) r \sigma(h_0)^{-1} h_2} \circ \kappa_n^l \circ \gamma_{g\sigma(h_0) r \sigma(h_0)^{-1} h_2}^{-1})(b) \\
\leq & \quad \max_{h \in F_k} \| (\alpha_{g\sigma(h)} \circ \kappa_n^l \circ \gamma_{g\sigma(h)}^{-1})(b) \\
- & \quad (\alpha_{g\sigma(h_0) r \sigma(h_0)^{-1} h_2} \circ \kappa_n^l \circ \gamma_{g\sigma(h_0) r \sigma(h_0)^{-1} h_2}^{-1})(b) \\
\leq & \quad \max_{h \in F_k} \max_{s \in K_N^l} \| (\alpha_{g\sigma(h)} \circ \kappa_n^l \circ \gamma_{g\sigma(h)}^{-1})(b) - (\alpha_{g\sigma(h)} \circ \kappa_n^l \circ \gamma_{g\sigma(h)}^{-1})(b) \| \\
\rightarrow & \quad 0.
\end{align*}
\]

In particular, this convergence is uniform in \( g \in K_G \) and \( r \in K_N \). We get for all \( b \in D \) that

\[
\begin{align*}
\max_{g \in K_G} \max_{t \in K_{H_k}} \| & \Theta_n^l(b)(g) - \Theta_n^l(b)(g t) \| \\
\leq & \quad \max_{g \in K_G} \max_{h_0 \in F_k} \max_{r \in K_N} \| \Theta_n^l(b)(g) - \Theta_n^l(b)(g\sigma(h_0) r) \| \\
\rightarrow & \quad 0.
\end{align*}
\]

Since \( K_G \subseteq G \) and \( K_{H_k} \subseteq G_k \) were arbitrary compact sets, we are in the position to apply Corollary 2.9. As \( D \) is separable, it follows for every \( l = 0, \ldots, d \) that there exists a sequence of c.p.c. maps

\[
\Psi_n^l : B \rightarrow C(G/G_k, A)
\]
so that for every compact set $K_G \subseteq G$ and $b \in \mathcal{D}$, we have

\[(e4.14) \quad \max_{g \in K_G} \| \Psi_n^{(l)}(b)(gG_k) - \Theta_n^{(l)}(b)(g) \| \to 0.\]

We now wish to show that these c.p.c. maps are approximately equivariant with regard to $\gamma$ and $\sigma^G_k \otimes \alpha$, where $\sigma^G_k$ is the $G$-action on $\mathcal{C}(G/G_k)$ induced by the left-translation of $G$ on $G/G_k$.

Let us fix a compact set $K_G \subseteq G$ as above. Without loss of generality, let us assume that it is large enough so that the quotient map $G \to G/G_k$ is still surjective when restricted to $K_G$. Given $b \in \mathcal{D}$, set

\[(e4.15) \quad \rho_n(b) = \max_{l=0, \ldots, d} \max_{g \in K_G} \| \Psi_n^{(l)}(b)(gG_k) - \Theta_n^{(l)}(b)(g) \|.\]

Note that by an elementary compactness argument, it follows from (e4.14) that for every compact set $J \subseteq \mathcal{D}$, we have

\[(e4.16) \quad \max_{b \in J} \rho_n(b) \to 0.\]

Let $t \in K$, $g \in K_G$ and $b \in \mathcal{D}$ with $\| b \| \leq 1$. Then

\[(\sigma^G_k \otimes \alpha_t) \left( (\Psi_n^{(l)}(b))(gG_k) \right) = \alpha_t(\Psi_n^{(l)}(b)(t^{-1}gG_k)) \]

\[= \rho_n(b) \sum_{h \in H_k} \mu^{(l)}(\pi_N(t^{-1}g)(h) \cdot (\alpha_{g\sigma(h)} \circ \kappa_n^{(l)} \circ \gamma_{t^{-1}g\sigma(h)})(b)) \]

\[\leq \varepsilon \sum_{h \in H_k} \mu^{(l)}(\pi_N(g)(h) \cdot (\alpha_{g\sigma(h)} \circ \kappa_n^{(l)} \circ \gamma_{t^{-1}g\sigma(h)})(b)) \]

\[\leq \rho_n(\gamma_t(b)) \Psi_n^{(l)}(\gamma_t(b))(gG_k).\]

Note that as $K_G$ contains a representative for every $G_k$-orbit in $G$, these approximations carry over to the $\| \cdot \|_\infty$-norm of the involved functions. Using (e4.15), we obtain for all $b \in \mathcal{D}$ with $\| b \| \leq 1$ that

\[(e4.17) \quad \limsup_{n \to \infty} \max_{t \in K} \| (\sigma^G_k \otimes \alpha_t)(\Psi_n^{(l)}(b)) - (\Psi_n^{(l)} \circ \gamma_t)(b) \| \leq \varepsilon.\]

Next, we wish to show that for $l_1 \neq l_2$, the c.p.c. maps $\Psi_n^{(l_1)}$ and $\Psi_n^{(l_2)}$ have approximately commuting ranges as $n \to \infty$. Let $g_1, g_2 \in K_G$ and $b \in \mathcal{D}$ with $\| b \| \leq 1$ be given. Then we compute

\[\| [\Psi_n^{(l_1)}(b)(g_1G_k), \Psi_n^{(l_2)}(b)(g_2G_k)] \| \leq 4\rho_n(b) \| [\Theta_n^{(l_1)}(b)(g_1), \Theta_n^{(l_2)}(b)(g_2)] \| \]

\[\leq \max_{h \in F_k} \| [(\alpha_{g_1\sigma(h)} \circ \kappa_n^{(l_1)} \circ \gamma^{-1}_{g_1\sigma(h)})(b), (\alpha_{g_2\sigma(h)} \circ \kappa_n^{(l_2)} \circ \gamma^{-1}_{g_2\sigma(h)})(b)] \|

\[= \max_{h \in F_k} \| [(\kappa_n^{(l_1)} \circ \gamma^{-1}_{g_1\sigma(h)})(b), (\alpha_{\sigma(h)}^{-1}g_1^{-1}g_2\sigma(h) \circ \kappa_n^{(l_2)} \circ \gamma^{-1}_{g_2\sigma(h)})(b)] \| \]
In particular, we obtain for every contraction $b \in \mathcal{D}$ that
\begin{align}
\max_{g_1, g_2 \in K_G} \| \left[ \Psi_n^{(l_1)}(b)(g_1 G_k), \Psi_n^{(l_2)}(b)(g_2 G_k) \right] \| & \\
& \leq \max_{g_1, g_2 \in K_G} \max_{h \in F_k} \left\| \left( \kappa_n^{(l_1)} \circ \gamma_{g_2}(h) \right)(b), \left( \alpha_{g_2}(h) \circ \kappa_n^{(l_2)} \circ \gamma_{g_2}(h) \right)(b) \right\| + 4 \rho_n(b) \tag{e4.18} \tag{e4.18} \tag{e4.18}
\end{align}

In exactly the same fashion, one also computes
\begin{align}
\| \left[ \Psi_n^{(l)}(b), a \right] \| & \to 0 \tag{e4.19} \tag{e4.19}
\end{align}
for all $l = 0, \ldots, d$, $b \in \mathcal{D}$, and $a \in A$, by using (e4.1) in place of (e4.5).

Next, we wish to show that for each $l = 0, \ldots, d$, the c.p.c. maps $\Psi_n^{(l)}$ behave approximately like order zero maps. Let $g \in K_G$. Choose the unique element $h_0 \in F_k$ with $\mu^{(l)}(\pi_N(g)h_0) > 0$. Then it follows for every $b_1, b_2 \in \mathcal{D}$ that
\begin{align}
\Theta_n^{(l)}(b_1)(g) \cdot \Theta_n^{(l)}(b_2)(g) & = \mu^{(l)}(\pi_N(g)h_0)^2 \cdot \left( \alpha_{g_2}(h_0) \circ \kappa_n^{(l)} \circ \gamma_{g_2}(h_0) \right)(b_1) \cdot \left( \alpha_{g_2}(h_0) \circ \kappa_n^{(l)} \circ \gamma_{g_2}(h_0) \right)(b_2) \\
& = \mu^{(l)}(\pi_N(g)h_0)^2 \cdot \alpha_{g_2}(h_0) \left( \kappa_n^{(l)} \circ \gamma_{g_2}(h_0) \right)(b_1) \cdot \left( \kappa_n^{(l)} \circ \gamma_{g_2}(h_0) \right)(b_1).
\end{align}

It follows from this calculation that
\begin{align}
\left\| \Theta_n^{(l)}(b_1) \cdot \Theta_n^{(l)}(b_2) - \Theta_n^{(l)}(b_1 b_2) \cdot \Theta_n^{(l)}(1) \right\|_{\infty, K_G} & \\
& \leq \max_{s \in K_G} \max_{F_k} \left\| \left( \kappa_n^{(l)} \circ \gamma_s^{-1} \right)(b_1) \cdot \left( \kappa_n^{(l)} \circ \gamma_s^{-1} \right)(b_1) \right\| - \left( \kappa_n^{(l)} \circ \gamma_s^{-1} \right)(b_1 b_2) \cdot \left( \kappa_n^{(l)} \circ \gamma_s^{-1} \right)(1) \tag{e4.7} \tag{e4.7} \tag{e4.8} \tag{e4.8}
\end{align}

As $K_G$ contains a representative of every $G_k$-orbit in $G$, it follows from (e4.4) that
\begin{align}
\| \Psi_n^{(l)}(b_1) \cdot \Psi_n^{(l)}(b_2) - \Psi_n^{(l)}(b_1 b_2) \cdot \Psi_n^{(l)}(1) \| & \to 0 \tag{e4.20} \tag{e4.20}
\end{align}
for every $b_1, b_2 \in \mathcal{D}$.

Next, we wish to show that the completely positive sum $\sum_{l=0}^d \Psi_n^{(l)}$ behaves approximately like a u.c.p. map upon multiplication with an element of $1 \otimes A$, as $n \to \infty$. Let $g \in K_G$. We have
\begin{align}
\Theta_n^{(l)}(1)(g) & \tag{e4.9} \tag{e4.9} \tag{e4.10} \tag{e4.10} \\
& = \sum_{h \in F_k} \mu^{(l)}(\pi_N(g)h) \cdot \left( \alpha_{g_1}(h) \circ \kappa_n^{(l)} \circ \gamma_s^{-1} \right)(1) + 4 \rho_n(b) \tag{e4.10} \tag{e4.10} \\
& = \sum_{h \in F_k} \mu^{(l)}(\pi_N(g)h) \cdot \left( \alpha_{g_1}(h) \circ \kappa_n^{(l)} \right)(1).
\end{align}
It follows for all \( a \in A \) that
\[
\max_{g \in K_G} \left\| \left( 1 - \sum_{l=0}^{d} \Theta^{(l)}_n(1) \right) \cdot a \right\| \\
\leq \max_{g \in K_G} (d+1) \cdot \max \left\{ \left\| (\alpha_{\sigma(h)} \circ \kappa^{(l)}_n)(1) - \kappa^{(l)}_n(1) \right\| \\
+ \left\| \left( 1 - \sum_{l=0}^{d} \sum_{h \in F_k} \mu^{(l)}(\pi_N(gh)) \cdot (\alpha_g \circ \kappa^{(l)}_n)(1) \right) \cdot a \right\| \right\}
\]
\[
\leq \max_{g \in K_G} (d+1) \cdot \max \left\{ (\alpha_{\sigma(h)} \circ \kappa^{(l)}_n)(1) - \kappa^{(l)}_n(1) \right\} \\
+ (d+1) \cdot \max \left\{ \left\| (1 - \kappa^{(l)}_n(1)) \cdot \alpha_{g^{-1}}(a) \right\| \right\}
\]
(4.23) \[ \leq \max_{g \in K_G} (d+1) \cdot \max \left\{ (\alpha_{\sigma(h)} \circ \kappa^{(l)}_n)(1) - \kappa^{(l)}_n(1) \right\} \\
+ (d+1) \cdot \max \left\{ \left\| (1 - \kappa^{(l)}_n(1)) \cdot \alpha_{g^{-1}}(a) \right\| \right\}
\]
(4.24) \[
\| (\Theta^{(l)}_n(1)) \cdot (1 \otimes a) \| \to 0 \quad \text{for all } a \in A.
\]

Let us now summarize everything we have obtained so far. The c.p.c. maps \( \Psi^{(l)}_n \), for \( l = 0, \ldots, d \) and \( n \in \mathbb{N} \) satisfy the following properties for all \( b, b_1, b_2 \in D \) and \( a \in A \):

\[
\lim_{n \to \infty} \max_{l_1, l_2} \left\| (\sigma^{G_k} \otimes \alpha)_{l_1} \circ \Psi^{(l)}_n(b) - \Psi^{(l)}_n(1) \right\| \leq \varepsilon;
\]
(4.25)
\[
\left\| \Psi^{(l)}_n(b_1) \cdot \Psi^{(l)}_n(b_2) - \Psi^{(l)}_n(b_1b_2) \cdot \Psi^{(l)}_n(1) \right\| \to 0;
\]
(4.26)
\[
\| (\Theta^{(l)}_n(1)) \cdot (1 \otimes a) \| \to 0.
\]

Note that \( k \), and thus the codomain of \( \Psi^{(l)}_n \), had to be chosen depending on \( \varepsilon \) and \( K \subseteq G \). However, we have canonical (equivariant) inclusions \( \mathcal{C}(G/G_k, A) \subseteq \mathcal{C}(G/S_0, A) \), which we may compose our maps with. It is then clear that the same properties as in (4.22) up to (4.26) hold, where we replace the action \( \sigma^{G_k} : G \curvearrowright \mathcal{C}(G/G_k) \) by \( \sigma^{S_0} : G \curvearrowright \mathcal{C}(G/S_0) \).

Since \( A \) and \( B \) are separable and \( G \) is second-countable, we can let the tolerance \( \varepsilon \) go to zero, let the set \( K \subseteq G \) get larger and apply a diagonal sequence argument. Putting the appropriate choices of c.p.c. maps into a single sequence, we can thus obtain c.p.c. maps
\[
\psi^{(l)} : B \rightarrow (\mathcal{C}(G/S_0) \otimes A)_{\infty}, \quad l = 0, \ldots, d
\]
that satisfy the following properties for all \( g \in G, \ a \in A, \) and \( b, b_1, b_2 \in D \):
\begin{align}
\psi(l_1)(b) \oplus a &= 0; \\
\alpha_g \circ \psi(l) &= \psi(l) \circ \gamma_g; \\
\psi(l_1)(b), \psi(l_2)(b) &= 0 \text{ for all } l_1 \neq l_2; \\
\psi(l_1)(b_1) \cdot \psi(l_2)(b_2) &= \psi(l_1)(b_1 b_2) \cdot \psi(l_1)(1); \\
\left(1 - \sum_{l=0}^d \psi(l)(1) \right) \cdot 1 \oplus a &= 0.
\end{align}

Since \( \gamma : G \curvearrowright \mathcal{D} \) is point-norm continuous, (e4.28) implies that the image of each map \( \psi(l) \) is in the continuous part \( (C(G/S_0) \otimes A)_{\alpha \circ \sigma \otimes a} \). In fact it is in the relative commutant of \( 1 \otimes A \) by (e4.27), but then also automatically in the relative commutant of all of \( C(G/S_0) \otimes A \). This allows us to define equivariant maps

\[ \zeta(l) : \mathcal{D} \to F_{\infty, \sigma \circ \alpha} \left( C(G/S_0) \otimes A \right), \quad \zeta(l)(b) = \psi(l)(b) + \text{Ann}(C(G/S_0) \otimes A) \]

for all \( l = 0, \ldots, d \). Then (e4.29) implies that these maps have commuting ranges, (e4.30) implies that they are c.p.c. order zero, and finally (e4.31) implies the equation \( \sum_{l=0}^d \zeta(l)(1) = 1 \).

By virtue of Proposition 3.2 and Remark 3.3, this gives rise to a unital equivariant \(*\)-homomorphism

\[ (D^{(d+1)}, \gamma^{(d+1)} \rightarrow (F_{\infty, \sigma \circ \alpha}, \sigma \circ \alpha)_{\infty} \].

As \( \gamma \) is unitarily regular, it follows from Lemma 3.7 that \( \gamma^{(d+1)} \) is a \( \gamma \)-absorbing action. Applying Lemma 4.6 yields that \( \sigma_{S_0} \otimes \alpha \) is \( \gamma \)-absorbing, which finishes the proof.

Now we are in a position to prove Theorem 4.4.

\textbf{Proof of Theorem 4.4.} Let \( \alpha : G \curvearrowright A \) and \( \gamma : G \curvearrowright \mathcal{D} \) be the two actions as in the assumption. Let also \( N \subset G, \ H_k \subset G/N, \) and \( G_k \subset G \) be subgroups as specified in the statement, and denote \( S_1 = (H_k)_k \) as a sequence of subgroups in \( G/N, \) and \( S_0 = (G_k)_k \) as a sequence of subgroups in \( G. \)

Suppose \( \text{asdim}(\square_{S_1} H_1) < \infty \) and \( s := \text{dim}_{Rok}(\alpha, S_0) < \infty. \) Using the latter, Lemma 4.4(iv) implies that the equivariant embedding

\[ \text{id}_A \otimes 1 : (A, \alpha) \to (A \otimes C((G/S_0)_{(d+1)}^{s+1}), \alpha \otimes (G\text{-shift})) \]

is \( G \)-equivariantly sequentially split. By Lemma 4.6, in order to show that \( \alpha \) is \( \gamma \)-absorbing, it suffices to show that the \( G \)-\( C^* \)-algebra \( A \otimes C((G/S_0)_{(d+1)}^{s+1}) \) is \( \gamma \)-absorbing. We will show this via induction on \( s. \)

For \( s = 0, \) the claim is precisely Lemma 4.7, and in particular it holds because we assumed \( \text{asdim}(\square_{S_1} H_1) < \infty. \)

Given \( s \geq 1, \) assume that the claim holds for \( s - 1. \) It follows by Remark 3.6 that there is an extension of \( G \)-\( C^* \)-algebras of the form

\[ 0 \longrightarrow J(s) \longrightarrow A \otimes C((G/S_0)_{(d+1)}^{s+1}) \longrightarrow Q(s) \longrightarrow 0, \]
where
\[ J^{(s)} = A \otimes C_0(0, 1) \otimes C(G/S_0) \otimes C((G/S_0)^*) \]
and
\[ Q^{(s)} = A \otimes \left( C(G/S_0) \oplus C((G/S_0)^*) \right). \]

By the induction hypothesis, both the kernel and the quotient of this extension are \( \gamma \)-absorbing \( G \)-\( C^* \)-algebras, and therefore so is the middle by Theorem 1.10. This finishes the induction step and the proof. \( \square \)

**Remark 4.8.** We remark that the statement of the main result holds verbatim for cocycle actions instead of genuine actions. Note that the concept of Rokhlin dimension makes sense for cocycle actions with the same definition, since there is still a natural genuine action induced on the central sequence algebra. If \((\alpha, w) : G \curvearrowright A\) is a cocycle action on a separable \( C^* \)-algebra, then \((\alpha \otimes \text{id}_K, \delta \otimes 1) : G \curvearrowright A \otimes K\) is cocycle conjugate to a genuine action by the Packer–Raeburn stabilization trick \([70]\). Since both Rokhlin dimension and absorption of a semi-strongly self-absorbing action are invariants under stable (cocycle) conjugacy, the statement of Theorem 4.4 follows for cocycle actions.

5. Some applications

Let us now discuss some immediate applications of the main result. First we wish to point out that the following result arises as a special case.

**Corollary 5.1.** Let \( G \) be a second-countable, locally compact group. Let \( S = (H_n)_n \) be a residually compact approximation consisting of normal subgroups of \( G \) with
\[ \text{asdim}(\boxtimes S H_1) < \infty. \]
Let \( A \) be a separable \( C^* \)-algebra and \( D \) a strongly self-absorbing \( C^* \)-algebra with \( A \cong A \otimes D \). Let \( \alpha : G \curvearrowright A \) be an action with
\[ \text{dim}_{\text{Rok}}(\alpha, S) < \infty. \]
Then \( \alpha \simeq \text{vscc} \alpha \otimes \gamma \) for all semi-strongly self-absorbing actions \( \gamma : G \curvearrowright D \).

**Proof.** Let \( \gamma : G \curvearrowright D \) be a semi-strongly self-absorbing action. Since \( D \cong D \otimes \mathbb{Z} \) by \([87]\) and the claim is about all semi-strongly self-absorbing actions on \( D \), we may as well assume \( \gamma \simeq \text{cc} \gamma \otimes \text{id}_\mathbb{Z} \). By Remark 1.9 we may thus assume that \( \gamma \) is unitarily regular. The claim then follows directly from Theorem 4.4 applied to the case \( N = \{1\} \). Note that one automatically has absorption with respect to very strong cocycle conjugacy by virtue of Theorem 1.8(v). \( \square \)

**Example 5.2.** Let \( Q \) denote the universal UHF algebra. Let \( \Gamma \) be a countable, discrete group and \( H \subset \Gamma \) a normal subgroup with finite index. There exists a strongly self-absorbing action \( \gamma : G \curvearrowright Q \) with \( \text{dim}_{\text{Rok}}(\gamma, H) = 0 \).

**Proof.** Such an action is constructed as part of \([83]\) Remark 10.8]. Namely, consider the left-regular representation \( \lambda^{G/H} : G/H \to \mathcal{U}(M_{(G,H)}) \), consider the quotient map \( \pi_H : G \to G/H \), and define
\[ \gamma_g = Q \otimes \bigotimes_{N} \text{Ad}(\lambda^{G/H}(\pi_H(g))) \]
as an action on $Q \cong Q \otimes M_{[G/H]}^{\otimes \infty}$. As the diagonal embedding $C(G/H) \subset M_{[G:H]}$ is equivariant, it follows that $\dim_{\text{Rok}}(\gamma, H) = 0$. By [82 Proposition 6.3], such an action is strongly self-absorbing. □

This in turn has the following consequence regarding the dimension-reducing effect of strongly self-absorbing $C^*$-algebras:

**Corollary 5.3.** Let $\Gamma$ be a countable, discrete, residually finite group that has some box space with finite asymptotic dimension. Let $\alpha : \Gamma \curvearrowright A$ be an action on a separable $C^*$-algebra with $\dim_{\text{Rok}}(\alpha) < \infty$.

1. If $A \cong A \otimes Q$, then $\dim_{\text{Rok}}(\alpha) = 0$.
2. If $A \cong A \otimes Z$, then $\dim_{\text{Rok}}(\alpha) \leq 1$.

**Proof.** (1) follows directly from [Example 5.2](#) and [Corollary 5.1](#). The claim of (2) reduces to the first one, by using $\alpha \simeq_{cc} \alpha \otimes \text{id}_Z$ and the existence of two c.p.c. order zero maps $\psi_0, \psi_1 : Q \to Z_\infty$ with $\psi_0(1) + \psi_1(1) = 1$; see [65 Section 5](#) or [75 Section 6](#). As this is a standard method of proof, we omit the details. □

**Corollary 5.4.** Let $\Gamma$ be a discrete, finitely generated, virtually nilpotent group. Let $X$ be a compact metrizable space with finite covering dimension, and $\alpha : \Gamma \curvearrowright X$ a free action by homeomorphisms. Then one has

$$\dim_{\text{Rok}}(\alpha \otimes \text{id}_Q : \Gamma \curvearrowright C(X) \otimes Q) = 0$$

and

$$\dim_{\text{Rok}}(\alpha \otimes \text{id}_Z : \Gamma \curvearrowright C(X) \otimes Z) \leq 1.$$

**Proof.** By [83 Corollary 7.5](#), the action $\alpha : \Gamma \curvearrowright C(X)$ has finite Rokhlin dimension. Since the underlying $C^*$-algebra is abelian, the claim follows from [Corollary 5.3](#). □

6. **Multi-flows on strongly self-absorbing Kirchberg algebras**

In this section, we shall study actions of $\mathbb{R}^k$ on certain $C^*$-algebras satisfying an obvious notion of the Rokhlin property.

**Notation 6.1.** For $k \geq 2$, we will refer to a continuous action of $\mathbb{R}^k$ on a $C^*$-algebra as a *multi-flow*. Let $(e_j)_{1 \leq j \leq k}$ be the standard basis of $\mathbb{R}^k$.

Given $\alpha : \mathbb{R}^k \curvearrowright A$, we will denote the generating flows $\alpha^{(j)} : \mathbb{R} \curvearrowright A$ given by $\alpha_{t_j}^{(j)} = \alpha_{te_j}$, for $j = 1, \ldots, k$. Then we have $\alpha_{t_i}^{(j)} \circ \alpha_{t_i}^{(i)} = \alpha_{t_i}^{(i)} \circ \alpha_{t_j}^{(j)}$ for all $i, j = 1, \ldots, k$ and all $t_i, t_j \in \mathbb{R}$. We will also denote by $\alpha^{(\otimes)} : \mathbb{R}^{k-1} \curvearrowright A$ the action generated by the flows $(\alpha^{(i)})_{i \neq j}$. We remark that $\alpha^{(j)}$ reduces naturally to a flow on the fixed point algebra $A^{\alpha^{(\otimes)}}$.

**Definition 6.2.** Let $A$ be a separable $C^*$-algebra and $\alpha : \mathbb{R}^k \curvearrowright A$ an action. We say that $\alpha$ has the Rokhlin property, if $\dim_{\text{Rok}}(\alpha, p\mathbb{Z}^k) = 0$ for all $p > 0$.

---

4Here we emphasize that the conclusion concerns only Rokhlin dimension *without* commuting towers.

5Strictly speaking, only the nilpotent case is proved there. The virtually nilpotent case follows from independent work of Bartels [3 Section 1](#).
Remark 6.3. In the case of flows, i.e., the case $k = 1$ above, Definition 6.2 coincides with Kishimoto’s notion of the Rokhlin property from [50]. Let us for now denote by $\sigma^T : \mathbb{R} \curvearrowright C(\mathbb{R}/T\mathbb{Z})$ the action induced by the $\mathbb{R}$-shift.

Proposition 6.4. Let $A$ be a separable $C^*$-algebra and $\alpha : \mathbb{R}^k \curvearrowright A$ an action. The following are equivalent:

(i) $\alpha$ has the Rokhlin property;

(ii) for every $j = 1, \ldots, k$ and every $p > 0$, there exists a unitary $u \in F_{\infty, \alpha}(A) \overset{\alpha_\infty}{\longrightarrow}$ such that $\alpha_{\infty, t}^{(j)}(u) = e^{ipt}u$, $t \in \mathbb{R};$

(iii) for every $j = 1, \ldots, k$ and every $T > 0$, there exists an equivariant and unital $*$-homomorphism

$$(C(\mathbb{R}/T\mathbb{Z}), \sigma^T) \longrightarrow (F_{\infty, \alpha}(A) \overset{\alpha_\infty}{\longrightarrow}, \alpha^{(j)}_\infty).$$

Proof. (i) $\Rightarrow$ (iii) Let $T > 0$. One has a canonical equivariant isomorphism

$$(C(\mathbb{R}^k/T^k\mathbb{Z}), \mathbb{R}^k\text{-shift}) \cong (C(\mathbb{R}/T\mathbb{Z}) \overset{\sigma^T}{\longrightarrow}, \sigma_1 \otimes \cdots \otimes \sigma^k),$$

where $\sigma^Tj$ is the $\mathbb{R}^k$-action on $C(\mathbb{R}/T\mathbb{Z})$ where only the $j$-th component acts by the $\mathbb{R}$-shift. By definition, $\alpha$ having the Rokhlin property means that for every $T > 0$ the dynamical system on the left embeds into $(F_{\infty, \alpha}(A), \alpha_\infty)$. So in particular, when (i) holds, then one also obtains an embedding of $(C(\mathbb{R}/T\mathbb{Z}), \sigma^Tj)$ for every $j = 1, \ldots, k$, which implies (iii). Conversely, when (iii) holds, then for all $T > 0$ one has an embedding of $(C(\mathbb{R}/T\mathbb{Z}), \sigma^Tj)$ into $(F_{\infty, \alpha}(A), \alpha_\infty)$ for all $j = 1, \ldots, k$. By applying a standard reindexation argument in the central sequence algebra, one may assume that these embeddings have pairwise commuting ranges for all $j = 1, \ldots, k$. Therefore one obtains an embedding of the $C^*$-dynamical system given by the tensor product of all $(C(\mathbb{R}/T\mathbb{Z}), \sigma^Tj)$, which we have seen to be the same as the dynamical system $(C(\mathbb{R}^k/T^k\mathbb{Z}), \mathbb{R}^k\text{-shift})$. In particular this implies (i) to (iii).

This follows directly from functional calculus. A unitary $u$ as in (ii) gives rise to a unital equivariant $*$-homomorphism

$$\varphi_u : (C(\mathbb{R}/2\pi\mathbb{Z}), \sigma^{2\pi}) \longrightarrow (F_{\infty, \alpha}(A) \overset{\alpha_\infty}{\longrightarrow}, \alpha^{(j)}_\infty), \quad \varphi_u(f) = f(u).$$

Conversely, whenever $\varphi$ is an arbitrary homomorphism between these two dynamical systems, then $u = \varphi([t \mapsto e^{ipt}])$ yields a unitary as required by (ii).

Remark 6.5. We note that for $G = \mathbb{R}^k$, the sequence $H_n = (n!) \cdot \mathbb{Z}^k$ yields a residually compact approximation in the sense of Definition 2.1. Now it is well-known that $\square_{H_n} \mathbb{Z}^k$ has finite asymptotic dimension $k$; see either [83] Sections 2+3 or better yet [17]. In particular, Corollary 5.1 is applicable to $\mathbb{R}^k$-actions that have finite Rokhlin dimension with commuting towers, and more specifically it is applicable to $\mathbb{R}^k$-actions with the Rokhlin property.

The following is the main result of this section.
Theorem 6.6. Let $\mathcal{D}$ be a strongly self-absorbing Kirchberg algebra. Let $k \geq 1$ be a given natural number. Then all continuous $\mathbb{R}^k$-actions on $\mathcal{D}$ with the Rokhlin property are semi-strongly self-absorbing and are mutually (very strongly) cocycle conjugate.

The approach for proving this result, at least in the way presented here, uses the theory of semi-strongly self-absorbing actions in a crucial way. In such dynamical systems, one has a very strong control over certain (approximately central) unitary paths, which together with the Rokhlin property allows one to obtain a relative cohomology-vanishing-type statement. This will be used to deduce inductively that the actions in the statement of Theorem 6.6 have approximately $\mathbb{R}^k$-inner flip. The desired uniqueness of such actions is then achieved by combining this fact with Corollary 5.1, which is a special case of our main result, in a suitable way.

Example 6.7 (see [7]). Denote by $s_1, s_2, \ldots$ the generators of the Cuntz algebra $\mathcal{O}_\infty$. Define a quasi-free flow $\gamma^0 : \mathbb{R} \curvearrowright \mathcal{O}_\infty$ via $\gamma^0_t(s_1) = e^{2\pi i t}s_1$, $\gamma^0_t(s_2) = e^{-2\pi i \sqrt{2} t}s_2$, and $\gamma^0_t(s_j) = s_j$ for $j \geq 3$. Then $\gamma^0$ has the Rokhlin property.

In particular, given $k \geq 1$ and any strongly self-absorbing Kirchberg algebra $\mathcal{D}$, the action

$$\text{id}_\mathcal{D} \otimes (\underbrace{\gamma^0 \times \cdots \times \gamma^0}_k \text{ times}) : \mathbb{R}^k \curvearrowright \mathcal{D} \otimes \mathcal{O}_\infty^\otimes \cong \mathcal{D}$$

is a $(k)$-multi-flow with the Rokhlin property on $\mathcal{D}$, and is in fact (very strongly) cocycle conjugate to every other one by Theorem 6.6.

Let us now implement the strategy outlined above step by step. We begin with the aforementioned cohomology-vanishing-type statement, which involves minimal assumptions about the underlying $\text{C}^*$-algebras but otherwise very strong assumptions about the existence of certain unitary paths, which will naturally appear in our intended setup later.

Lemma 6.8. Let $A$ be a separable unital $\text{C}^*$-algebra. Let $k \geq 1$ and let $\alpha : \mathbb{R}^k \curvearrowright A$ be a continuous action with the Rokhlin property, and fix some $j \in \{1, \ldots, k\}$.

For every $\varepsilon > 0$, $L > 0$ and $\mathcal{F} \subset A$, there exists a $T > 0$ and $\mathcal{G} \subset A$ with the following property:

If $\{w_t\}_{t \in \mathbb{R}} \subset \mathcal{U}(A)$ is any $\alpha^{(j)}$-cocycle satisfying

$$\max_{a \in \mathcal{F}} \max_{0 \leq t \leq T} \|w_t, a\| \leq \varepsilon, \quad \max_{0 \leq t \leq T} \max_{\gamma \subset [0,1]^{k-1}} \|w_t - \alpha^{(j)}_\gamma(w_t)\| \leq \varepsilon,$$

and moreover there exists some continuous path of unitaries $u : [0,1] \to \mathcal{U}(A)$ with

$$u(0) = 1, \quad u(1) = w_T, \quad \ell(u) \leq L,$$

$$\max_{0 \leq t \leq 1} \max_{\gamma \subset [0,1]^{k-1}} \|u(t) - \alpha^{(j)}_\gamma(u(t))\| \leq \varepsilon,$$

$$\max_{0 \leq t \leq 1} \max_{a \in \mathcal{G}} \|u(t), a\| \leq \varepsilon,$$

then there exists a unitary $v \in \mathcal{U}(A)$ satisfying

$$\max_{0 \leq t \leq 1} \|w_t - v\alpha^{(j)}_t(v^*)\| \leq 3\varepsilon,$$
\[
\max_{a \in F} \| [v, a] \| \leq 3\varepsilon,
\]
\[
\max_{\bar{r} \in [0, 1]^{k-1}} \| v - \alpha^{(j)}_{\bar{r}}(v) \| \leq 3\varepsilon.
\]

**Proof.** Let \( T > 0 \) and note that we have fixed \( j \in \{1, \ldots, k\} \) by assumption. By some abuse of notation, let us view \( \sigma^T \) as the \( R^k \)-action on \( C(R/TZ) \) such that the \( j \)-th coordinate acts as the \( R \)-shift and all the other components act trivially. In this way, any \( * \)-homomorphism as in Proposition 6.4(iii) can be viewed as an \( R^k \)-equivariant \( * \)-homomorphism from \( C(R/TZ) \) to \( F_{\infty, \alpha}(A) \). In particular, denote such a homomorphism by \( \theta \). We can then obtain a commutative diagram of \( R^k \)-equivariant \( * \)-homomorphisms via

\[
\begin{array}{c}
(A, \alpha) \\
\downarrow d \mapsto \alpha \\
(C(R/TZ) \otimes A, \sigma^T) \\
\end{array}
\begin{array}{c}
(A_{\infty}, \alpha_{\infty}) \\
\downarrow f \otimes d \mapsto \theta(f \otimes d) \\
(F_{\infty, \alpha}(A), \alpha_{\infty}) \\
\end{array}
\]

We will keep this in mind for later.

Now let \( \varepsilon > 0 \), \( L > 0 \) and \( F \subset A \) be as in the statement. Without loss of generality, we assume that \( F \) consists of contractions. We choose \( T > \frac{L}{\varepsilon} \) and \( G \subset A \) any finite set of contractions containing \( F \) that is \( \varepsilon/2 \)-dense in the compact subset

\[
\{ \alpha^{(j)}_{s}(a) \mid a \in F, \ 0 \leq s \leq T \}.
\]

We claim that these do the trick. We note that the rest of the proof below is almost identical to the proof of [50, Theorem 2.1] and [78, Lemma 3.4], respectively, except for some obvious modifications.

Assume that \( \{w_t\}_{t \in \mathbb{R}} \subset \mathcal{U}(A) \) is an \( \alpha^{(j)} \)-cocycle satisfying

\[
\max_{a \in F} \max_{0 \leq t \leq T} \|[w_t, a]\| \leq \varepsilon;
\]

\[
\max_{0 \leq t \leq T} \max_{\bar{r} \in [0, 1]^{k-1}} \|[w_t - \alpha^{(j)}_{\bar{r}}(w_t)]\| \leq \varepsilon;
\]

and moreover that there exists some continuous path of unitaries \( u : [0, 1] \to \mathcal{U}(A) \) with

\[
u(0) = 1, \quad u(1) = w_T, \quad \ell(u) \leq L;
\]

\[
\max_{0 \leq t \leq 1} \max_{\bar{r} \in [0, 1]^{k-1}} \|u(t) - \alpha^{(j)}_{\bar{r}}(u(t))\| \leq \varepsilon;
\]

\[
\max_{0 \leq t \leq 1} \max_{a \in F} \|[u(t), a]\| \leq \varepsilon.
\]

As \( \ell(u) \leq L \), we may assume that \( u \) is \( L \)-Lipschitz by passing to the arclength parameterization if necessary. We denote by \( \kappa : [0, T] \to \mathcal{U}(A) \) the path given by \( \kappa_s = u(s/T)^* \), which is then Lipschitz with respect to the constant \( L/T \leq \varepsilon \). Let us define a continuous path of unitaries \( v : [0, T] \to \mathcal{U}(A) \) via \( v_s = w_s \alpha^{(j)}_{s}(\kappa_s) \). Then by (e6.5) it follows that \( v(0) = v(T) = 1 \). In particular, we may view \( v \) as a unitary in \( C(R/TZ) \otimes A \).
We have
\[
\max_{a \in F} \|v, 1 \otimes a\| = \max_{a \in F} \max_{0 \leq s \leq T} \|w_s \alpha_s^{(j)}(\kappa_s), a\| \\
= \max_{a \in F} \max_{0 \leq s \leq T} \|w_s, a\| + \|\kappa_s, \alpha_s^{(j)}(a)\| \\
\leq \varepsilon + \max_{a \in F} \max_{0 \leq s \leq T} \|\kappa_s, \alpha_s^{(j)}(a)\| \\
\leq 3/2\varepsilon + \max_{b \in U} \|\kappa_s, b\| \\
\leq 5/2\varepsilon.
\]

Moreover, we have
\[
\max_{\bar{r} \in [0,1]^{k-1}} \|v - (\sigma^T \otimes \alpha)_{\bar{r}}^{(j)}(v)\| \\
= \max_{\bar{r} \in [0,1]^{k-1}} \max_{0 \leq s \leq T} \|v - (\id \otimes \alpha)_{\bar{r}}^{(j)}(v)\| \\
= \max_{\bar{r} \in [0,1]^{k-1}} \max_{0 \leq s \leq T} \|w_s - \alpha_s^{(\bar{r})}(v_s)\| \\
= \max_{\bar{r} \in [0,1]^{k-1}} \max_{0 \leq s \leq T} \|w_s \alpha_s^{(j)}(\kappa_s) - \alpha_{\bar{r}}^{(j)}(w_s \alpha_s^{(j)}(\kappa_s))\| \\
= \max_{\bar{r} \in [0,1]^{k-1}} \max_{0 \leq s \leq T} \|w_s \alpha_s^{(j)}(\kappa_s) - \alpha_{\bar{r}}^{(j)}(w_s \alpha_s^{(j)}(\kappa_s))\| \\
= \max_{\bar{r} \in [0,1]^{k-1}} \max_{0 \leq s \leq T} \|w_s - \alpha_{\bar{r}}^{(j)}(w_s)\| + \|\kappa_s - \alpha_{\bar{r}}^{(j)}(\kappa_s)\| \\
\leq 2\varepsilon.
\]

Lastly, let us fix \(t \in [0,1]\) and \(s \in [0,T]\). If \(s \geq t\), then we compute
\[
(v(\sigma^T \otimes \alpha_t^{(j)}(v^*)))(s) \\
= w_s \alpha_s^{(j)}(\kappa_s) \cdot \alpha_t^{(j)}(\alpha_s^{(j)}(\kappa_s) w_s^{T-s}) \\
= w_s \alpha_s^{(j)}(\kappa_s) \cdot \alpha_t^{(j)}(w_s^{T-s}) \\
= w_s \alpha_t^{(j)}(w^*_s) = w_t.
\]

On the other hand, if \(s \leq t\), then in particular \(s \geq 1\) and \(T - 1 \leq T + s - t\), and we compute
\[
(v(\sigma^T \otimes \alpha_t^{(j)}(v^*)))(s) \\
= w_s \alpha_s^{(j)}(\kappa_s) \cdot \alpha_t^{(j)}(\alpha_{T+s-t}^{(j)}(\kappa_s^{T+s-t}) w_s^{T+s-t}) \\
= w_s \alpha_s^{(j)}(\kappa_s) \cdot \alpha_t^{(j)}(w^*_s) \\
= w_T + s \alpha_t^{(j)}(w_s^{T+s-t}) \\
= w_T + s \alpha_t^{(j)}(w^*_s) = w_t.
\]

Let us summarize what we have accomplished so far. Starting from the existence of the \(\alpha^{(j)}\)-cocycle \(\{w_t\}_{t \in \mathbb{R}}\) and the unitary path \(u\) with the prescribed properties, we have found a unitary \(v \in U(\mathcal{C}(\mathbb{R}/T \mathbb{Z}) \otimes A)\) satisfying
\[
\max_{a \in F} \|v, 1 \otimes a\| \leq 5/2\varepsilon;
\]
\[
\max_{\bar{r} \in [0,1]^{k-1}} \|v - (\sigma^T \otimes \alpha_t^{(j)}(v))\| \leq 2\varepsilon;
\]
(6.10) \[ \max_{0 \leq t \leq 1} \|w_t - v(\sigma^T \alpha_t^{(j)}(v^*))\| \leq 2\varepsilon. \]

By using the commutative diagram (6.11), we may send \( v \) into the sequence algebra of \( A \), represent the resulting unitary by a sequence of unitaries in \( A \), and then select a member of this sequence so that it will satisfy the properties in the claim with respect to the parameter \( 3\varepsilon \). This finishes the proof. \( \square \)

Now record the following useful technical result about semi-strongly self-absorbing actions, which arises as a special case of [82, Lemma 3.12]:

**Lemma 6.9.** Let \( G \) be a second-countable, locally compact group. Let \( \mathcal{D} \) be a separable, unital \( C^* \)-algebra and \( \gamma : G \rtimes \mathcal{D} \) a semi-strongly self-absorbing action. For every \( \varepsilon > 0 \), \( \mathcal{F} \subset \mathcal{D} \) and compact set \( K \subset G \), there exist \( \delta > 0 \) and \( G \subset \mathcal{D} \) with the following property:

Suppose that \( u : [0, 1] \to \mathcal{U}(\mathcal{D}) \) is a unitary path satisfying
\[ u(0) = 1, \quad \max_{0 \leq t \leq 1} \max_{g \in K} \|u(t) - \gamma_g(u(t))\| \leq \delta, \]
and
\[ \max_{a \in \mathcal{F}} \|u(1), a\| \leq \delta. \]

Then there exists a unitary path \( w : [0, 1] \to \mathcal{U}(\mathcal{D}) \) satisfying
\[ w(0) = 1, \quad w(1) = u(1), \]
\[ \max_{g \in K} \|w(t) - \gamma_g(w(t))\| \leq \varepsilon, \]
\[ \max_{0 \leq t \leq 1} \max_{a \in \mathcal{F}} \|[w(t), a]\| \leq \varepsilon. \]

Moreover, we may choose \( w \) in such a way that
\[ \|w(t_1) - w(t_2)\| \leq \|u(t_1) - u(t_2)\| \quad \text{for all} \quad 0 \leq t_1, t_2 \leq 1. \]

We are now ready to prove the main result of this section:

**Proof of Theorem 6.6.** We will prove this via induction in \( k \). For this purpose, we will include the case \( k = 0 \), where the claim is true for trivial reasons.

Now let \( k \geq 1 \) and assume that the claim is true for actions of \( \mathbb{R}^{k-1} \). We will then show that the claim is also true for actions of \( \mathbb{R}^k \).

**Step 1:** Let \( \alpha : \mathbb{R}^k \rtimes \mathcal{D} \) be an action with the Rokhlin property. In a similar fashion as in [53, Proposition 3.5], we shall show that \( \alpha \) has approximately \( \mathbb{R}^k \)-inner flip.

Set \( B = \mathcal{D} \otimes \mathcal{D} \) and \( \beta = \alpha \otimes \alpha \). Denote by \( \Sigma \) the flip automorphism on \( B \), which is equivariant with regard to \( \beta \). Note that \( \beta \) is still a \( \mathbb{R}^k \)-action on a strongly self-absorbing Kirchberg algebra with the Rokhlin property. The \( \mathbb{R}^{k-1} \)-action \( \alpha^{(\downarrow)} \) is semi-strongly self-absorbing by induction hypothesis. Applying [82, Proposition 3.6], we find a sequence of unitaries \( y_n, z_n \in \mathcal{U}(B) \) satisfying

(6.11) \[ \max_{r \in [0,1]^{k-1}} \|y_n - \beta^{(\downarrow)}_r(y_n)\| + \|y_n - \beta^{(\downarrow)}_r(z_n)\| \xrightarrow{n \to \infty} 0 \]

and

(6.12) \[ \Sigma(b) = \lim_{n \to \infty} \text{Ad}(y_n z_n y_n^* z_n^*)(b), \quad b \in B. \]
Let us set $Y = [(y_n)_{n}]$ and $Z = [(z_n)_{n}]$ with $Y,Z \in B_{\infty, \beta}^{\infty}(\omega)$. Moreover, set $X = YZY^*Z^*$. Note that since $\mathcal{D}$ is a Kirchberg algebra, Corollary 5.1 implies that $\beta$ is equivariantly $\mathcal{O}\_\infty$-absorbing. By [82, Proposition 2.19], the unitary $X$ is thus homotopic to the unit inside $B_{\infty, \beta}^{\infty}(\omega)$. Write $X = \exp(iH_1) \cdots \exp(iH_r)$ for certain self-adjoint elements $H_1, \ldots, H_r \in B_{\infty, \beta}^{\infty}(\omega)$. Set $L' = ||H_1|| + \cdots + ||H_r||$. For $l = 1, \ldots, r$, represent $H_l$ via a sequence of self-adjoint elements $h_{l,n} \in B$ with $||h_{l,n}|| \leq ||H_l||$. We define a sequence of continuous paths $x_n : [0, 1] \to U(B)$ via 

$$x_n(t) = \exp(ith_{1,n}) \cdots \exp(ith_{r,n}).$$

Then each of these paths is $L'$-Lipschitz. By slight abuse of notation we write $X : [0, 1] \to U(B)$ for $X(t) = [(x_n(t))_{n}]$, which is then continuous and satisfies $X(0) = 1$ and $X(1) = X$. Also denote $x_n = x_n(1)$ for all $n$.

Since we have $\Sigma(b) = XbX^*$ for all $b \in B$ and $\beta$ arises from the commuting actions $\beta^{(k)}$ of $\mathbb{R}$ and $\beta^{(k)}$ of $\mathbb{R}^{k-1}$, respectively, one also has $\Sigma(b) = \beta^{(k)}(x)b\beta^{(k)}(x^*)$ for all $t \in \mathbb{R}$. It follows that for all $t \in \mathbb{R}$, one has that the element $X\beta^{(k)}(x^*)$ commutes with all elements in $B \subset B_{\infty}$. Let us fix some number $T > 0$. Define $u_n^T : [0, 1] \to U(B)$ via $u_n^T(t) = x_n(t)b\beta^{(k)}(x_n(t)^*)$. Then $u_n^T$ is a unitary path starting at the unit and with Lipschitz constant $L = 2L'$. We have

$$\max_{0 \leq t \leq 1} \max_{T \in [0, 1]} ||u_n^T(t) - \beta^{(k)}_{\beta} (u_n^T(t)) || \xrightarrow{n \to \infty} 0$$

and

$$||u_n^T(1), b)|| = ||[x_n\beta^{(k)}_{\beta}(x_n^*), b)] || \xrightarrow{n \to \infty} 0 \text{ for all } b \in B.$$

Due to [Lemma 6.9] we may replace the unitary paths $u_n^T$ by ones which become approximately central along the entire path and retain all the other properties. In other words, by changing the path $u_n$ on $(0, 1)$, we may in fact assume

$$\max_{0 \leq t \leq 1} \max_{T \in [0, 1]} ||[u_n^T(t), b] || \xrightarrow{n \to \infty} 0 \text{ for all } b \in B.$$

Let us consider the sequence of $\beta^{(k)}$-cocycles $\{w_t^{(n)}\}_{t \in \mathbb{R}}$ given by $w_t^{(n)} = x_n\beta^{(k)}_{t}(x_n^*)$. Then by what we have observed before, we have

$$\max_{0 \leq t \leq T} ||[w_t^{(n)}, b] || \xrightarrow{n \to \infty} 0, \quad b \in B,$$

as well as

$$\max_{0 \leq t \leq T} \max_{T \in [0, 1]} ||w_t^{(n)} - \beta^{(k)}_{\beta} (w_t^{(n)}) || \leq 2 \max_{T \in [0, 1]} ||[x_n - \beta^{(k)}_{\beta} (x_n)] || \xrightarrow{n \to \infty} 0.$$

This puts us into the position to apply [Lemma 6.8]. Given some small tolerance $\varepsilon > 0$ and $F \subset \mathcal{D}$, we can choose $T > 0$ and $G \subset \mathcal{D}$ with respect to the constant $L = 2L'$ and with $(B, \beta)$ in place of $(A, \alpha)$. Without loss of generality, we choose $F$ in such a way that

$$(e^{6.13}) \quad \Sigma(F) = F.$$ 

Then the cocycles $\{w_t^{(n)}\}_{t \in \mathbb{R}}$ and the unitary paths $u_n^T$ (in place of $\{w_t\}_{t \in \mathbb{R}}$ and $u$ in [Lemma 6.8]) will eventually satisfy the assumptions in [Lemma 6.8].
for large enough $n$. By the conclusion of the statement, one finds a unitary $v_n \in \mathcal{U}(B)$ such that
\begin{equation}
\max_{0 \leq t \leq 1} \|w_t^{(n)} - v_n \beta_t^{(k)}(v_n^*)\| = \max_{0 \leq t \leq 1} \|x_n \beta_t^{(k)}(x_n)^* - v_n \beta_t^{(k)}(v_n^*)\| \leq 3\varepsilon;
\end{equation}
\begin{equation}
\max_{b \in \mathcal{F}} \|v_n - b\| \leq 3\varepsilon;
\end{equation}
\begin{equation}
\max_{\vec{r} \in [0,1]^{k-1}} \|v_n - \alpha_{\vec{r}^{(k)}}(v_n)\| \leq 3\varepsilon.
\end{equation}
We set $U_n = v_n^* x_n$, which is yet another sequence of unitaries in $B$. Note that (e6.14) translates to
\begin{equation}
\max_{0 \leq t \leq 1} \|U_n - \beta_t^{(k)}(U_n)\| \leq 3\varepsilon.
\end{equation}
Together with (e6.16) and (e6.11) this yields
\begin{equation}
\max_{\vec{r} \in [0,1]^{k-1}} \|U_n - \beta_{\vec{r}}(U_n)\| \leq 7\varepsilon
\end{equation}
for large enough $n$. Finally, if we combine (e6.12), (e6.13) and (e6.15), we obtain
\begin{equation}
\max_{b \in \mathcal{F}} \|\Sigma(b) - U_n b U_n^*\| \leq 4\varepsilon
\end{equation}
for sufficiently large $n$. Since $\varepsilon > 0$ was an arbitrary parameter and $\mathcal{F} \subset B$ was arbitrary as well, we see that the flip automorphism $\Sigma$ on $B$ is indeed approximately $\mathbb{R}^k$-inner.

**Step 2:** Let $\alpha : \mathbb{R}^k \curvearrowright D$ be an action with the Rokhlin property. Due to the first step, $\alpha$ has approximately $\mathbb{R}^k$-inner flip. By [81, Proposition 3.3], it follows that the infinite tensor power action $\alpha^{\otimes \infty} : \mathbb{R}^k \curvearrowright D^{\otimes \infty}$ is strongly self-absorbing. In view of Remark 6.5, we may apply Corollary 5.1 to $\alpha$ and $\alpha^{\otimes \infty}$ in place of $\gamma$ and see that
\begin{equation}
\alpha \simeq_{\text{vscc}} \alpha \otimes \alpha^{\otimes \infty} \simeq \alpha^{\otimes \infty},
\end{equation}
which implies that $\alpha$ is semi-strongly self-absorbing.

**Step 3:** For $i = 0, 1$, let $\alpha^{(i)} : \mathbb{R}^k \curvearrowright D$ be two actions with the Rokhlin property. By the previous step, they are semi-strongly self-absorbing. If we apply Corollary 5.1 to $\alpha^{(0)}$ in place of $\alpha$ and $\alpha^{(1)}$ in place of $\gamma$, then it follows that $\alpha^{(0)} \simeq_{\text{vscc}} \alpha^{(0)} \otimes \alpha^{(1)}$. If we exchange the roles of $\alpha^{(0)}$ and $\alpha^{(1)}$ and repeat this argument, we conclude $\alpha^{(0)} \simeq_{\text{vscc}} \alpha^{(1)}$.

This finishes the induction step and the proof. \hfill $\Box$

We observe the following consequence as a combination of all of our main results for $\mathbb{R}^k$-actions; this is new even for ordinary flows.

**Corollary 6.10.** Let $A$ be a separable $C^*$-algebra with $A \cong A \otimes O_{\infty}$. Suppose that $\alpha : \mathbb{R}^k \curvearrowright A$ is a multi-flow. The following are equivalent:

(i) $\alpha$ has the Rokhlin property;

(ii) $\alpha$ has finite Rokhlin dimension with commuting towers;

(iii) $\alpha \simeq_{\text{vscc}} \alpha \otimes \gamma$ for any multi-flow $\gamma : \mathbb{R}^k \curvearrowright O_{\infty}$ with the Rokhlin property;

(iv) $\alpha \simeq_{\text{vscc}} \alpha \otimes \gamma$ for every multi-flow $\gamma : \mathbb{R}^k \curvearrowright O_{\infty}$ with the Rokhlin property.
Proof. This follows directly from Theorem 6.6 and Corollary 5.1. □

The following remains open:

**Question 6.11.** Let $\alpha : \mathbb{R}^k \curvearrowright A$ be a multi-flow on a Kirchberg algebra. Suppose that for every $\vec{r} \in \mathbb{R}^k$, the flow on $A$ given by $t \mapsto \alpha_{t\vec{r}}$ has the Rokhlin property. Does it follow that $\alpha$ has the Rokhlin property?

**References**

[1] S. Barlak, G. Szabó: Sequentially split $*$-homomorphisms between $C^*$-algebras. Int. J. Math. 27 (2016), no. 12, 48 pages.
[2] S. Barlak, G. Szabó, C. Voigt: The spatial Rokhlin property for actions of compact quantum groups. J. Funct. Anal. 272 (2016), no. 6, pp. 2308–2360.
[3] A. Bartels: Coarse flow spaces for relatively hyperbolic groups. Compositio Math. 153 (2017), no. 4, pp. 745–779.
[4] J. Bosa, N. Brown, Y. Sato, A. Tikuisis, S. White, W. Winter: Covering dimension of $C^*$-algebras and 2-coloured classification. Mem. Amer. Math. Soc., to appear (2016). URL [http://arxiv.org/abs/1506.03974](http://arxiv.org/abs/1506.03974).
[5] O. Bratteli, D. E. Evans, A. Kishimoto: The Rohlin property for quasi-free automorphisms of the Fermion algebra. Proc. London Math. Soc. 71 (1995), no. 3, pp. 675–694.
[6] O. Bratteli, A. Kishimoto: Trace scaling automorphisms of certain stable AF algebras II. Q. J. Math. 51 (2000), no. 2, pp. 131–154.
[7] O. Bratteli, A. Kishimoto, D. W. Robinson: Rohlin flows on the Cuntz algebra $\mathcal{O}_\infty$. J. Funct. Anal. 248 (2007), pp. 472–511.
[8] O. Bratteli, A. Kishimoto, M. Rørdam, E. Størmer: The crossed product of a UHF algebra by a shift. Ergodic Theory Dynam. Systems 13 (1993), no. 4, pp. 615–626.
[9] L. G. Brown: Continuity of actions of groups and semigroups on Banach spaces. J. London Math. Soc. 62 (2000), no. 1, pp. 107–116.
[10] N. Brown, A. Tikuisis, A. Zelenberg: Rokhlin dimension for $C^*$-correspondences. Houston J. Math., to appear (2017). URL [https://arxiv.org/abs/1608.03214](https://arxiv.org/abs/1608.03214).
[11] J. Castillejos, S. Evington, A. Tikuisis, S. White, W. Winter: $C^*$-algebras with property Gamma (2018). In preparation.
[12] M.-D. Choi, E. G. Effros: The completely positive lifting problem for $C^*$-algebras. Ann. of Math. 104 (1976), no. 3, pp. 585–609.
[13] A. Connes: Outer conjugacy classes of automorphisms of factors. Ann. Sci. École Norm. Sup. 8 (1975), pp. 383–419.
[14] A. Connes: Classification of injective factors. Cases $\text{II}_1$, $\text{II}_\infty$, $\text{III}_\lambda$, $\lambda \neq 1$. Ann. of Math. 74 (1976), pp. 73–115.
[15] A. Connes: Periodic automorphisms of the hyperfinite factors of type $\text{II}_1$. Acta Sci. Math. 39 (1977), pp. 39–66.
[16] M. Dadarlat, W. Winter: On the $KK$-theory of strongly self-absorbing $C^*$-algebras. Math. Scand. 104 (2009), no. 1, pp. 95–107.
[17] T. Delabie, M. Tointon: The asymptotic dimension of box spaces of virtually nilpotent groups (2017). URL [https://arxiv.org/abs/1706.03730](https://arxiv.org/abs/1706.03730).
[18] G. A. Elliott, D. E. Evans, A. Kishimoto: Outer conjugacy classes of trace scaling automorphisms of stable UHF algebras. Math. Scand. 83 (1988), no. 1, pp. 74–86.
[19] G. A. Elliott, A. S. Toms: Regularity properties in the classification program for separable amenable $C^*$-algebras. Bull. Amer. Math. Soc. 45 (2008), pp. 229–245.
[20] W. R. Emerson, F. P. Greenleaf: Covering properties and Folner conditions for locally compact groups. Math. Zeitschr. 102 (1967), pp. 370–384.
[21] D. E. Evans, A. Kishimoto: Trace scaling automorphisms of certain stable AF algebras. Hokkaido Math. J. 26 (1997), pp. 211–224.
[22] E. Gardella: Classification theorems for circle actions on Kirchberg algebras, I (2014). URL [http://arxiv.org/abs/1405.2469](http://arxiv.org/abs/1405.2469).
[23] E. Gardella: Classification theorems for circle actions on Kirchberg algebras, II (2014). URL [http://arxiv.org/abs/1406.1208](http://arxiv.org/abs/1406.1208).
[24] E. Gardella: Rokhlin dimension for compact group actions. Indiana Univ. Math. J. 66 (2017), no. 2, pp. 659–703.
[25] E. Gardella, M. Kalantar, M. Lupini: Rokhlin dimension for compact quantum group actions (2017). URL https://arxiv.org/abs/1703.10999.
[26] E. Gardella, M. Lupini: Applications of model theory to C\(^\ast\)-dynamics. J. Funct. Anal., to appear (2016). URL https://arxiv.org/abs/1608.05532v3.
[27] E. Gardella, L. Santiago: Equivariant C\(^\ast\)-homomorphisms, Rokhlin contraints and equivariant UHF-absorption. J. Funct. Anal. 270 (2016), no. 7, pp. 2543–2590.
[28] P. Goldstein, M. Izumi: Quasi-free actions of finite groups on the Cuntz algebra \(O_\infty\). Tohoku Math. J. 63 (2011), pp. 729–749.
[29] R. Herman, V. Jones: Period two automorphisms of UHF C\(^\ast\)-algebras. J. Funct. Anal. 45 (1982), no. 2, pp. 169–176.
[30] I. Hirshberg, A. Ocneanu: Stability for integer actions on UHF C\(^\ast\)-algebras. J. Funct. Anal. 59 (1984), pp. 132–144.
[31] I. Hirshberg, N. C. Phillips: Rokhlin dimension: obstructions and permanence properties. Doc. Math. 20 (2015), pp. 199–236.
[32] I. Hirshberg, M. Rørdam, W. Winter: C\(^\ast\)-algebras, stability and strongly self-absorbing C\(^\ast\)-algebras. Math. Ann. 339 (2007), no. 3, pp. 695–732.
[33] I. Hirshberg, G. Szabó, W. Winter, J. Wu: Rokhlin dimension for flows. Comm. Math. Phys. 353 (2017), no. 1, pp. 253–316.
[34] I. Hirshberg, W. Winter: Rokhlin actions and self-absorbing C\(^\ast\)-algebras. Pacific J. Math. 233 (2007), no. 1, pp. 125–143.
[35] I. Hirshberg, W. Winter, J. Zacharias: Rokhlin dimension and C\(^\ast\)-dynamics. Comm. Math. Phys. 335 (2015), pp. 637–670.
[36] M. Izumi: Finite group actions on C\(^\ast\)-algebras with the Rohlin property I. Duke Math. J. 122 (2004), no. 2, pp. 233–280.
[37] M. Izumi: Finite group actions on C\(^\ast\)-algebras with the Rohlin property II. Adv. Math. 184 (2004), no. 1, pp. 119–160.
[38] M. Izumi: Poly-\(\mathbb{Z}\) group actions on Kirchberg algebras. Oberwolfach Rep. 9 (2012), pp. 3170–3173.
[39] M. Izumi, H. Matui: \(\mathbb{Z}^2\)-actions on Kirchberg algebras. Adv. Math. 224 (2010), pp. 355–400.
[40] X. Jiang, H. Su: On a simple unital projectionless C\(^\ast\)-algebra. Amer. J. Math. 121 (1999), no. 2, pp. 359–413.
[41] V. F. R. Jones: Actions of finite groups on the hyperfinite type II\(_1\) factor. Mem. Amer. Math. Soc. 28 (1980), no. 237. 70 pages.
[42] Y. Katayama, C. E. Sutherland, M. Takesaki: The characteristic square of a factor and the cocycle conjugacy of discrete group actions on factors. Invent. Math. 132 (1998), pp. 331–380.
[43] T. Katsura, H. Matui: Classification of uniformly outer actions of \(\mathbb{Z}^2\) on UHF algebras. Adv. Math. 218 (2008), pp. 940–968.
[44] Y. Kawahigashi, C. Sutherland, M. Takesaki: The structure of the automorphism group of an injective factor and the cocycle conjugacy of discrete abelian group actions. Acta Math. 169 (1992), pp. 105–130.
[45] A. Khukhro: Box spaces, group extensions and coarse embeddings into Hilbert space. J. Funct. Anal. 263 (2012), no. 1, pp. 115–128.
[46] E. Kirchberg: The Classification of Purely Infinite C\(^\ast\)-Algebras Using Kasparov’s Theory (2003). Preprint.
[47] E. Kirchberg: Central sequences in C\(^\ast\)-algebras and strongly purely infinite algebras. Operator Algebras: The Abel Symposium 1 (2004), pp. 175–231.
[48] E. Kirchberg, N. C. Phillips: Embedding of exact C\(^\ast\)-algebras in the Cuntz algebra \(O_\infty\). J. reine angew. Math. 525 (2000), pp. 17–53.
[49] A. Kishimoto: The Rohlin property for automorphisms of UHF algebras. J. reine angew. Math. 465 (1995), pp. 183–196.
[50] A. Kishimoto: A Rohlin property for one-parameter automorphism groups. Comm. Math. Phys. 179 (1996), no. 3, pp. 599–622.
[51] A. Kishimoto: The Rohlin property for shifts on UHF algebras and automorphisms of Cuntz algebras. J. Funct. Anal. 140 (1996), pp. 100–123.
[52] A. Kishimoto: Automorphisms of AT algebras with the Rohlin property. J. Operator Theory 40 (1998), pp. 277–294.
[53] A. Kishimoto: Unbounded derivations in AT algebras. J. Funct. Anal. 160 (1998), pp. 270–311.
[54] A. Kishimoto: UHF flows and the flip automorphism. Rev. Math. Phys. 13 (2001), no. 9, pp. 1163–1181.
[55] A. Kishimoto: Rohlin flows on the Cuntz algebra $O_2$. Int. J. Math. 13 (2002), no. 10, pp. 1065–1094.
[56] H.-C. Liao: A Rokhlin type theorem for simple $C^*$-algebras of finite nuclear dimension. J. Funct. Anal. 270 (2016), no. 10, pp. 3675–3708.
[57] H.-C. Liao: Rokhlin dimension of $\mathbb{Z}_m$-actions on simple $C^*$-algebras. Int. J. Math. 28 (2017), no. 7, 22 pages.
[58] T. Masuda: Evans–Kishimoto type argument for actions of discrete amenable groups on McDuff factors. Math. Scand. 101 (2007), pp. 48–64.
[59] H. Matui: Classification of outer actions of $\mathbb{Z}^N$ on $O_2$. Adv. Math. 217 (2008), pp. 2872–2896.
[60] H. Matui: $Z$-actions on AH algebras and $\mathbb{Z}^2$-actions on AF algebras,. Comm. Math. Phys. 297 (2010), pp. 529–551.
[61] H. Matui: $\mathbb{Z}^N$-actions on UHF algebras of infinite type. J. reine angew. Math 657 (2011), pp. 225–244.
[62] H. Matui, Y. Sato: $\mathbb{Z}$-stability of crossed products by strongly outer actions. Comm. Math. Phys. 314 (2012), no. 1, pp. 193–228.
[63] H. Matui, Y. Sato: Strict comparison and $\mathbb{Z}$-absorption of nuclear $C^*$-algebras. Acta Math. 209 (2012), no. 1, pp. 179–196.
[64] H. Matui, Y. Sato: $Z$-stability of crossed products by strongly outer actions II. Amer. J. Math. 136 (2014), pp. 1441–1497.
[65] H. Matui, Y. Sato: Decomposition rank of UHF-absorbing $C^*$-algebras. Duke Math. J. 163 (2014), no. 14, pp. 2687–2708.
[66] H. Nakamura: The Rohlin property for $Z^2$-actions on UHF algebras. J. Math. Soc. Japan 51 (1999), no. 3, pp. 583–612.
[67] H. Nakamura: Aperiodic automorphisms of nuclear purely infinite simple $C^*$-algebras. Ergodic Theory Dynam. Systems 20 (2000), pp. 1749–1765.
[68] P. W. Nowak, G. Yu: Large Scale Geometry. EMS (2012).
[69] A. Ocneanu: Actions of discrete amenable groups on von Neumann algebras, Lecture Notes in Mathematics, volume 1138. Springer-Verlag, Berlin (1985).
[70] J. A. Packer, I. Raeburn: Twisted crossed products of $C^*$-algebras. Math. Proc. Cambridge Philos. Soc. 106 (1989), no. 2, pp. 293–311.
[71] N. C. Phillips: A classification theorem for nuclear purely infinite simple $C^*$-algebras. Doc. Math. 5 (2000), pp. 49–114.
[72] J. Roe: Lectures on Coarse Geometry, University Lecture Series, volume 31. AMS (2003).
[73] M. Rørdam: The stable and the real rank of $\mathcal{Z}$-absorbing $C^*$-algebras. Int. J. Math. 15 (2004), pp. 1065–1084.
[74] L. Santiago: Crossed products by actions of finite groups with the Rohlin property. Int. J. Math. 26 (2015), 31 pages.
[75] Y. Sato, S. White, W. Winter: Nuclear dimension and $\mathbb{Z}$-stability. Invent. Math. 202 (2015), pp. 893–921.
[76] C. E. Sutherland, M. Takesaki: Actions of discrete amenable groups on injective factors of type III$_\lambda$, $\lambda \neq 1$. Pacific J. Math. 137 (1989), pp. 405–444.
[77] G. Szabó: The Rohlin dimension of topological $\mathbb{Z}^m$-actions. Proc. Lond. Math. Soc. 110 (2015), no. 3, pp. 673–694.
[78] G. Szabó: The classification of Rohlin flows on $C^*$-algebras (2017). URL http://arxiv.org/abs/1706.09276.
[79] G. Szabó: Strongly self-absorbing $C^*$-dynamical systems, III. Adv. Math. 316 (2017), pp. 356–380.
[80] G. Szabó: Equivariant Kirchberg-Phillips-type absorption for amenable group actions. Comm. Math. Phys., to appear (2018). URL https://doi.org/10.1007/s00220-018-3110-3

[81] G. Szabó: Strongly self-absorbing C*-dynamical systems. Trans. Amer. Math. Soc. 370 (2018), pp. 99–130.

[82] G. Szabó: Strongly self-absorbing C*-dynamical systems, II. J. Noncommut. Geom. 12 (2018), no. 1, pp. 369–406.

[83] G. Szabó, J. Wu, J. Zacharias: Rokhlin dimension for actions of residually finite groups. Ergodic Theory Dynam. Systems, to appear (2017). URL http://dx.doi.org/10.1017/etds.2017.113

[84] A. Tikuisis, S. White, W. Winter: Quasidiagonality of nuclear C*-algebras. Ann. Math. 185 (2017), pp. 229–284.

[85] A. S. Toms, W. Winter: Strongly self-absorbing C*-algebras. Trans. Amer. Math. Soc. 359 (2007), no. 8, pp. 3999–4029.

[86] W. Winter: Decomposition rank and Z-stability. Invent. Math. 179 (2010), no. 2, pp. 229–301.

[87] W. Winter: Strongly self-absorbing C*-algebras are Z-stable. J. Noncomm. Geom. 5 (2011), no. 2, pp. 253–264.

[88] W. Winter: Nuclear dimension and Z-stability of pure C*-algebras. Invent. Math. 187 (2012), no. 2, pp. 259–342.

[89] W. Winter: Localizing the Elliott conjecture at strongly self-absorbing C*-algebras, with an appendix by H. Lin. J. reine angew. Math. 692 (2014), pp. 193–231.

[90] W. Winter, J. Zacharias: Completely positive maps of order zero. Münster J. Math. 2 (2009), pp. 311–324.

[91] W. Winter, J. Zacharias: The nuclear dimension of C*-algebras. Adv. Math. 224 (2010), no. 2, pp. 461–498.