Thermodynamical phases of a regular SAdS black hole

Anais Smailagic\(^2\) Euro Spallucci\(^3\)

\(^2\) INFN, Sezione di Trieste, Italy \(^3\) Dipartimento di Fisica, Università di Trieste and INFN, Sezione di Trieste, Italy

This paper studies the thermodynamical stability of regular BHs in AdS\(_5\) background. We investigate off-shell free energy of the system as a function of temperature for different values of a “coupling constant” \(\mathcal{L} = 4\theta/l^2\), where the cosmological constant is \(\Lambda = -3/l^2\) and \(\sqrt{\theta}\) is a “minimal length”. The parameter \(\mathcal{L}\) admits a critical value, \(\mathcal{L}_{\text{inf}} = 0.2\), corresponding to the appearance of an inflexion point in the Hawking temperature. In the weak-coupling regime \(\mathcal{L} < \mathcal{L}_{\text{inf}}\), there are first order phase transitions at different temperatures. Unlike the Hawking-Page case, at temperature \(0 \leq T \leq T_{\text{min}}\) the ground state is populated by “cold” near-extremal BHs instead of a pure radiation. On the other hand, for \(\mathcal{L} > \mathcal{L}_{\text{inf}}\) only large, thermodynamically stable, BHs exist.

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I. INTRODUCTION

The five dimensional Anti de Sitter geometry plays a central role in recent developments in theoretical physics. The AdS\(_5\)/CFT duality \(^{[1,3]}\) offers a powerful tool to tackle non-perturbative features of a variety of physical systems ranging from the quark-gluon plasma \(^{[4]}\) to fluids \(^{[5]}\) and super-conductors \(^{[6]}\).

The strong-coupling physics of a conformal field theory living on the flat boundary of AdS\(_5\) is mapped by duality to the weak-coupling quantum string theory (quantum gravity) in the AdS\(_5\) × S\(_5\) bulk. This amazing spin-off of string theory connects 4D physics in flat spacetime to quantum gravity in AdS\(_5\) in a beautiful realization of the Holographic Principle \(^{[7,8]}\).

In this framework the gravitational dual of the de-confinement phase transition turns out to be the Hawking-Page transition between a cold gas of particles, and a Schwarzschild AdS BH \(^{[9]}\). Thus, the study of thermodynamics of BHs in AdS spacetime is instrumental to the deeper understanding of the above scenario.

On the other hand, Schwarzschild AdS BH suffers from the same pathologies as the corresponding solution without cosmological constant. For example, Hawking temperature is divergent as the horizon is shrinks to zero, and there is a curvature singularity at the origin of the coordinate system.

Recently, we have found pathology-free, regular BH solutions of Einstein equations, that led to new predictions about the terminal phase of quantum evaporation \(^{[10,15]}\). In a recent paper \(^{[16]}\) regular BH creation in a deSitter background geometry was studied and it was found that these objects would have been plentifully produced during inflationary times, for Planckian values of the cosmological constant.

Motivated by the promising results quoted above, we would like to analyze thermodynamical behavior of neutral, non-spinning, regular BH in AdS\(_5\) background with \(\Lambda = -3/l^2\).

The paper is organized as follows: in Sect.(2) we review the 5D Einstein equations with a
Gaussian source and a negative cosmological constant. Regular Schwarzschild $AdS_5$ BH is introduced and discussed from a geometrical point of view. In Sect.(3) we pave the way to the study of thermodynamics of the solution presented in Sect(2). and compute the Hawking temperature, the BH entropy and Off-Shell Free Energy. In Sect.(4) we study the phase transitions in our model. Two regions of the “coupling constant” $\mathcal{L} = 4\theta/l^2$ are described. A rich phase structure emerges in the weak-coupling regime $\mathcal{L} < \mathcal{L}_{inf} = 0.2$, with important differences with respect to the Hawking-Page scenario in the near-extremal region. The resulting picture closely resembles first order phase transitions in a finite temperature quantum field theory\cite{17,18}. On the other hand in the “string-coupling” regime $\mathcal{L} > \mathcal{L}_{inf} = 0.2$ there is always a single BH at any temperature. Finally, in Sect(5) we summarize the results obtained and stress the novel features.

II. REGULAR ADS BH.

In this Section a brief review of the ideas underlying regular BH solutions of Einstein equations is given. We work in 5$D$ spacetime being the proper dimension for the CFT/AdS Correspondence.

The line element is a static, spherically symmetric solution of the 5$D$ Einstein equations, with a negative cosmological constant $\Lambda \equiv -3/l^2 (AdS)$. For the positive cosmological constant case, see \cite{16}. The energy momentum tensor leading to the regular metric is of the anisotropic fluid form \cite{10}, with the Gaussian matter source. This type of matter distribution emulates non-commutativity of space-time through a parameter $\theta$ corresponding to the area of an elementary cell. Its components are given by

$$\begin{align*}
T_0^0 &= T_r^r = -\rho (r) , \quad \rho (r) \equiv \frac{M}{(4\pi \theta)^2} \exp \left( -\frac{r^2}{4\theta} \right) , \\
T_m^m &= -\rho - \frac{r}{3} \partial_r \rho
\end{align*}$$

where $M$ is a total mass-energy of the source and $\sqrt{\theta}$ is a natural UV cut-off curing short distance infinities. These ideas, based on UV divergence-free field theory models described in \cite{10,22} and applied to a quantum gravity model, allow interpretation of parameter $\theta$ in different terms: string induced non-commutative geometry \cite{23}, zero-point length \cite{24,25}, etc., depending on the assumed underlying fundamental quantum gravity theory.

The advantage of our approach is that, contrary to the star-product approach description, there is no need to modify the metric part of Einstein equations. Rather, ”quantum” effects are incorporated through the energy-momentum tensor. Since matter induces metric, solving equations will modify geometry appropriately. Thus, starting from Einstein equations

$$R^M_N - \frac{1}{2} ( R - 2\Lambda ) \delta^M_N = 8\pi G_5 T^M_N \quad , \quad \Lambda \equiv -\frac{3}{l^2}$$

with, $G_5 = 1/M_s^5$ the 5$D$ gravitational coupling constant. The solution is the line element
where, \( d\Omega_3 \) is 3-sphere line element and \( \gamma (2, r^2/4\theta) \) is the incomplete gamma function defined as

\[ \gamma (n, r^2/4\theta) \equiv \int_0^{r^2/4\theta} dt \, t^{n-1} e^{-t} \]  

(6)

The short-distance regularity of Eq. (4) can be inferred from the behavior of the incomplete gamma at short distance

\[ \gamma (n, r^2/4\theta) \approx \frac{1}{n} \left( \frac{r^2}{4\theta} \right)^n \]  

(7)

Physically, this means that spacetime fuzziness smears the curvature singularity creating either Minkowski or (A)dS metric. Different possibilities are determined by the interplay between UV and IR physics characterized by \( \theta \) and \( \Lambda \) parameters respectively. Giving opposite contributions to the short distance energy density leads to three different situations: fine tuning of parameters leads to Minkowski metric; prevailing \( \theta \) gives dSitter, while prevailing \( \Lambda \) leads to AdS.

The short-distance behavior allows an intriguing interpretation in terms of the effective gravitational coupling as follows: let us introduce \( G_{5}^{\text{eff.}} \) as

\[ G_5 \rightarrow G_5^{\text{eff.}} (r) \equiv G_5 \gamma \left(2, \frac{r^2}{4\theta} \right) \]  

(8)

The effect of the smeared coordinates can be interpreted as the vanishing of the “running” gravitational constant as \( r \to 0 \). Thus, our model can be viewed as an effective asymptotically free gravitational model as shorter distances are probed. We remark that the weakening of gravity leads only to the absence of curvature singularities but does not prevent formation of BHs. Even in the case of the extremal BHs, the horizon radius \( r_0 \) is still few \( \sqrt{\theta} \) units and \( G_5^{\text{eff.}} (r_0) \) is big enough to create an event horizon.

One of the novel features of the our regular solution is the existence of multiple horizons, unlike ordinary Schwarzschild solution. Unfortunately, due to the complicated metric function, horizon equation can not be solved explicitly. Nevertheless, the existence of zeros in the \( g_{rr}^{-1} \) component of the metric can be determined from the graph of the function \( M = M (r_H) \) defined from \( g_{rr}^{-1} (r_H) = 0 \). Thus, one gets

\[ M = \frac{r_H^2}{2G_5\gamma \left(2, \frac{r_H^2}{4\theta} \right)} \left(1 + \frac{r_H^2}{l^2} \right) \]  

(9)

The above equation is to be interpreted as follows: \( M \) plotted as function of \( r_H \) describes the existence of horizons in the sense that for an assigned value of \( M \) the horizons are given by intersections of \( M = \text{const} \) with the graph \( M (r_H) \). In the next section \( M(r_H) \) will be
FIG. 1: Plot of the function $\frac{MG_5}{\theta}$ vs $r_H/\sqrt{\theta}$, with $3\theta/l^2 = 0.002$. $r_0 = 2.65\sqrt{\theta}$ is the radius of the extremal BH with mass $M_0 = 6.75\theta/G_5$. For any $M > M_0$ there exist an inner (Cauchy) and an outer (Killing) horizon $r_H = r_\pm$ as shown by the horizontal line $MG_5/\theta = 10$.

given a thermodynamical interpretation as the Internal Energy of the system depending on the (outer, Killing) horizon $r_H = r_+$. The graph of equation (9) is shown in Fig.(1). The existence of a minimum indicates an extremal BH of radius $r_0$ and mass $M_0$ for a given $\Lambda$. Therefore, regular BHs admit an extremal configuration, contrary to standard Schwarzschild AdSBH. For $M > M_0$ there are non-degenerate BHs with distinct inner (Cauchy) horizon and outer (Killing) horizon. For $M < M_0$ there is no horizon and the solution (4) represents a maximally localized mass in an AdS background.

The existence of a stable, minimal size, BH has interesting consequences which have been discussed in [26]. If we assume the free parameter $M$ in the metric to be the mass of an “elementary” object, then the solution (4) can be considered as a family containing particle-like, light objects for $M < M_0$ and black, massive object for $M > M_0$. Extremal mass $M = M_0$ is the transition energy between particles and BHs. The described behavior of regular BH goes hand-in-hand with recent ideas positing a limit of any physical high energy probe [27, 28]. The following scenario is in place: the size of a particle is characterized by its Compton wavelength and is a decreasing function of mass. Once the extremal value $M_0$ has been reached, any increase in $M$ will lead to the increase of the radius of the corresponding BH. Thus, $\lambda_C = 1/M_0$ is the minimum wavelength for any physical high-energy probe. From this perspective, $r_0$ plays the role of the effective Planck length and there is no way to explore shorter distances [26, 29, 30].
III. THERMODYNAMICS

FIG. 2: $lT_H$ vs $r_+/l$ for different values of the parameter $L \equiv 4\theta/l^2$.

For $0 < L < L_{inf} = 0.22$ the temperature has a local maximum and minimum. At the critical value $L_{inf} = 0.22$ the two extrema merge into an inflexion point (dashed curve). Above the critical value, $T_H$ is a monotonically increasing function of $r_+$. 

Thermodynamical description of regular BH [1] will follow the same steps as in [10]. The Hawking Temperature of the AdS BH is given by

$$T_H = \frac{1}{2\pi r_+} \left[ \frac{\gamma(3, r_+^2/4\theta)}{\gamma(2, r_+^2/4\theta)} \left( 1 + \frac{r_+^2}{l^2} \right) - 1 \right]$$

(10)

The metric [1] admits an extremal BH configuration even without being charged or rotating. This is a general property of regular solutions sourced by a finite width $\sqrt{\theta}$, Gaussian energy/momentum distribution. One can see from the relation

$$4\pi T_H = \frac{2G_5}{r_+^3} \Gamma \left( 2, r_+^2/4\theta \right) \frac{dM}{dr_+}$$

(11)

that $T_H$ vanishes at the radius $r_0$ which minimizes $M(r_+)$. Furthermore, equation (10) displays $T_H$ dependence on two length scales, i.e. the short-distance cut-off $\sqrt{\theta}$ and the AdS curvature radius $l$.

By letting $\theta \to 0$ and using fundamental properties of the Euler Gamma function, we recover the temperature of a 5D SAdS BH.
\[ T_H = \frac{1}{\pi r_+} \left[ \left( 1 + \frac{r_+^2}{l^2} \right) - \frac{1}{2} \right] \]  

(12)

If we let the AdS radius to infinity, i.e. vanishing of the cosmological constant, \( \Lambda \to 0 \), we get the temperature of a regular, 5D, SAdS BH

\[ T_H = \frac{1}{2\pi r_+} \left[ \frac{\gamma(3, r_+^2/4\theta)}{\gamma(2, r_+^2/4\theta)} - 1 \right] \]  

(13)

In order to study the behavior of the temperature in intermediate situations, i.e. for a finite, non-vanishing, ratio between the two length scales, we rescale equation (10) as follows

\[ lT_H = \frac{1}{2\pi x} \left[ \frac{\gamma(3, x^2/L^2)}{\gamma(2, x^2/L^2)} (1 + x^2) - 1 \right], \quad L^2 \equiv \frac{4\theta}{l^2}, \quad x \equiv \frac{r_+}{l} \]  

(14)

If we assign \( L \) the meaning of “coupling constant”, we can introduce a “critical coupling” \( L_{\text{inf}} \) = 0.2 for which \( T_H \) has an inflexion point. Thus, one can define two different regimes:

- “weak-coupling” corresponding to \( L < L_{\text{inf}} \);
- “strong-coupling” for \( L > L_{\text{inf}} \).

We shall show in the next sections that these two regimes correspond to different behavior of the BH.

In order to investigate the thermodynamical equilibrium of this gravitational system, we are going to study the behavior of the off-shell (Helmholtz) free energy. To interpret the results we shall adhere to the description of phase transitions used in finite temperature field theory [17, 18]. In our case, free energy plays the role of the effective potential and \( r_+ \) is the order parameter. We investigate off-shell free energy, \( F^{\text{off}} \), since it describes the non-equilibrium dynamics of a system in a thermal bath of temperature \( T \), which is a free parameter and should not to be confused with Hawking temperature \( T_H \). In this way, one can study the evolution of the system towards a stable equilibrium configuration, \( T = T_H \), and eventual phase transitions. \( F^{\text{off}} \) is defined as

\[ F^{\text{off}} \equiv M (r_+) - S_H T \]  

(15)

where, \( S_H \) is the BH entropy determined from the First Law as

\[ dM (r_+) \equiv T_H dS_H \rightarrow S_H = \frac{2\pi}{G_5} \int_{r_0}^{r_+} dr \frac{r^2}{\gamma(2, r^2/4\theta)} \]  

(16)

Integration has to start from the extremal radius \( r_0 \), rather than from zero. This choice automatically guarantees vanishing thermodynamical entropy at absolute zero, as it is required by the Third Law. Furthermore, we are not a priori assuming the Area Law, rather we shall derive the relation between Entropy and Area from the First Law. Integrating by parts in (16) one obtains

\[ S_H = \frac{1}{3\pi} \left[ \frac{2\pi^2 r_+^3}{G_5^{\text{eff}}(r_+)} - \frac{2\pi^2 r_0^3}{G_5^{\text{eff}}(r_0)} \right] + \Delta S_H \]

\[ \Delta S_H = \frac{\pi}{12\theta^2 G_5} \int_{r_0}^{r_+} dr r^6 e^{-r^2/4\theta} \frac{1}{\gamma^2(2, r^2/4\theta)} \]  

(17)
One recognizes that the first line of (17) is the Area Law for regular, 5D BHs, while the second line gives exponentially small corrections \[12, 13\]. Standard Area Law is obtained only in the limit \( \sqrt{\theta} \to 0 \), \( G_5^{\text{eff.}} \to G_5 \). By inserting \( S_H \) from Eq.(16) in Eq.(15), we obtain

\[
F^{\text{off}} = \frac{r_+^2}{2 G_5 \gamma (2 ; r_+^2/4 \theta)} \left( 1 + \frac{r_+^2}{l^2} \right) - \frac{2 \pi T}{G_5} \int_{r_0}^{r_+} dr \frac{r_+^2}{\gamma (2 , r_+^2/4 \theta)} \tag{18}
\]

It is not possible to integrate analytically the entropy expression, and to write a closed form for (18). Nonetheless, it is possible to plot (18) and the resulting graphs are shown in next sections.

### IV. BH PHASES

#### A. Weak-coupling phase

FIG. 3: Plot of the rescaled Hawking Temperature \( \sqrt{\theta} T_H \) vs. \( r_+ / \sqrt{\theta} \). The curve corresponds to the value \( \mathcal{L}^2 = 0.002 \). The temperature vanishes for the extremal configuration \( r_H = r_0 \), then reaches a local maximum in \( r_{\text{max}} = 4.8 \sqrt{\theta} \), further decreases to a local minimum in \( r_{\text{min}} = 27.5 \sqrt{\theta} \) and finally raises linearly. For \( 0 \leq T < T_{\text{min}} \), we have small, stable, near-extremal, BHs. In the range \( T_{\text{min}} < T < T_{\text{max}} \), we find multiple (triplet) BHs: small, stable, near-extremal BH; intermediate radius , unstable, BH; large, stable, BH. Finally, for \( T > T_{\text{max}} \), there is a single, large, stable BH.

Fig.(3) indicates the existence of multiple BHs with the same temperature, as well as extremal BH. As the extremal configuration is approached, a sharp departure with
FIG. 4: Graph of the rescaled Off-shell Free energy, for $L^2 = 0.002$, for different values of the temperature $T\sqrt{\theta}$. At the temperature $T^*\sqrt{\theta} = 0.01227$ the two local minima are degenerate. Two inflexion points correspond to the maximum and minimum of the Hawking temperature.

The behavior of the temperature graph can be summarized as follows: extremal configuration corresponds to zero temperature and can be considered true ground state of the system with minimized internal energy $M(r_+)$. In the near extremal region $T_H$ increases from zero to a local maximum $T_{max}$ at $r_+ = r_{max}$. As $r_+$ increases further $T_H$ drops to a local minimum $T_{min}$ corresponding to $r_+ = r_{min}$ and then starts increasing linearly following the SAdS pattern.

The behavior of $F_{off}$ is shown in Figure (4). Comparison between the graph of (18) and the same expression for $\theta = 0$ (Hawking-Page AdS), shows a novel feature of our regular solution. It develops a new, local, near-extremal minimum, for any $T > 0$ as seen in Figure(5). This behavior is a consequence of the smearing of the matter source resulting in the existence of the extremal horizon.

The extrema of free energy indicate existence of both multiple and single regular BHs for different values of the temperature. The alternation of single/multiple states is the signature of a first order phase transition, as in finite temperature quantum field theory, to which we refer.

It turns out that single/multiple BH transitions occur at the inflection points of free energy ( extremal points of $T_H$ ). In order to grasp better what is going on in the near extremal region, we zoom that part of Figure(4) in Figure(5). Thus, the following scenario is in place:
FIG. 5: This is a zoom of the previous figure in the near-extremal region. For $0 \leq \sqrt{\theta} T < 0.0305$ a new local minimum appears describing a small, near extremal, BH. For $\sqrt{\theta} T = T_{\text{max}} = 0.0305$ the local minimum and maximum of $F^{\text{off}}$ merge into an inflexion point. For $\sqrt{\theta} T > 0.0305$ there is only a large BH representing a stable ground state of the system.

1. $T = 0$. BH is in the frozen single state. The only ground state is the extremal configuration with $r_+ = r_- = r_0$.

2. $0 \leq T \leq T_{\text{min}}$. BH is in the cold single state. The stable equilibrium configurations are small, near-extremal, BHs of radius $r_0 < r_+ < r_{\text{min}}$. This situation is completely novel with respect to the Hawking-Page scenario, which is characterized by a gas of cold particles in this phase. BHs can not be formed below some minimal temperature. On the contrary, in our case, BHs are always present, as soon as $T \geq 0$. This is due to the effect of the minimal length $\sqrt{\theta}$.

3. $T = T_{\text{min}}$ corresponds to the onset of a spinodal decomposition in quantum field theory. An inflexion point appears in $F^{\text{off}}$ at $r_+ = r_{\text{min}}$ (see Figure[1]).

4. $T_{\text{min}} < T < T^*$. New local minimum develops and the system splits into two co-existing states. The small near-extremal BH is energetically favored.

5. $T = T^*$ the two minima become degenerate and the system is in a mixed state. Both BHs have the same free energy.

6. $T^* < T < T_{\text{max}}$ large BHs become stable, while near-extremal BHs are only locally stable.
7. $T = T_{\text{max}}$ The near-extremal minimum merges with the local maximum. There is a new transition from multiple to a single BH state.

8. $T > T_{\text{max}}$ there is high temperature, single, stable BH.

The above scenario describes first order phase transitions from single to multiple BHs at $T = T_{\text{min}}$ and $T = T_{\text{max}}$.

B. Strong-coupling phase

![Graph showing free energy $G_5F^{off}/l^2$ vs $r_+/l$. For $L \geq 0.22$ there exist only one minimum for any $T$.](image)

FIG. 6: Free energy $G_5F^{off}/l^2$ vs $r_+/l$. For $L \geq 0.22$ there exist only one minimum for any $T$.

One realizes that the above description of first order phase transitions is only correct in the weak-coupling regime. In fact, the two parameters of the theory have “opposite” effects i.e. $\theta$ dominates short-range and lowers $T_{\text{max}}$, while $3/l^2$ dominates long-range region of $r_H$ and raises $T_{\text{min}}$. It is reasonable to expect that, at the certain point, these two opposing effects will meet creating an inflexion point of temperature. The confirmation of our conjecture is shown in Fig.2. This effect, in terms of free energy, means that, beyond the inflexion point $L_{\text{inf}} = 0.2$, there is only one minimum at any value of parameter $T$ and thus, only one BH state. Changing $T$ only changes position of the single minimum. The graph of free-energy in the strong-coupling regime is given in Fig.6.
V. CONCLUSIONS

We have investigated thermodynamical stability of regular BH in AdS background. The thermal bath of particles is described by the temperature parameter $T$ and is not \textit{a priori} in thermal equilibrium with BH. We studied the behavior of off-shell free energy in order to study evolution towards equilibrium. In this picture, thermal equilibrium states are described by minima of $F^{off}$ (in this paper we have not considered quantum instabilities due to tunneling effects). We found that, at large distances, the behavior is the same as in Hawking-Page scheme. This is due to the fact that the cosmological constant dominates in the deep infra-red. But, there is a crucial difference in the ultra-violet regime which is controlled by the minimal length $\sqrt{\theta}$. First of all, regular BHs are multi-horizon structures, contrary to usual Schwarzschild BH, and thus are endowed with extremal configurations. The existence of degenerate horizon makes the thermodynamical situation different: as soon as the temperature of the thermal bath is $T > 0$ the near-extremal BHs of minimal mass $M > M_0$ are present. As a consequence, there is no pure radiation phase, rather there are a \textit{plethora} of different BH states ranging from single, near-extremal configuration to multiple BH states with the same temperature, but different stability conditions.

Another very interesting feature of regular BHs is that in the strong-coupling regime there is always only a single BH at any temperature.

While we were completing this paper, we came across of \cite{35} where the same model has been studied in 4D. The ideas presented in that paper follow the same reasoning as in \cite{36}. This approach, however, looks more like a mathematical analogy based on the similarity between the Van der Walls $P-V$ curve, for a real gas, and the $\beta = 1/T_H$ curve as a function of $r_+$. Although one may accept that this analogy is suggestive, it is hardly an exact correspondence since $\beta$ is interpreted as “pressure”. Also it mixes intensive and extensive thermodynamical variables as noticed in \cite{37}.

Our intention in this paper was not to rely on any formal analogy, but to develop a physically based picture of the thermodynamical evolution of regular SAdS BHs studying the temperature and off-shell free energy. A very important out-come of our analysis is that relevant information can be extracted both from the graph of the Hawking temperature and off-shell free energy. This gives alternative possibility with respect to finite temperature quantum field theory where information can be recovered only from the effective potential playing the role of off-shell free energy.

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