THREE-DIMENSIONAL EQUILIBRIA IN DRAKONS

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ABSTRACT. An analytic formula of the second-order equilibrium beta limit for DRAKONs with triangular connectors of rectilinear elements (CRELs) with arbitrary pressure profile is derived. Its predictions for a parabolic pressure profile are compared with the full third-order numerical results, leading to a semi-analytic formula. The beta limits obtained here are much lower than those obtained previously; for example, for a DRAKON with straight sections and CRELs equal in length and with triangular CRELs of aspect ratio 2, the peak beta limit is 2.2%, compared to the previous result of 12%. Pressure profiles with sharper boundaries have lower beta limits. The equilibrium is governed by the predominant quadrupole Pfirsch-Schliiter current. Small deviations from the CREL condition have almost no effects on the equilibrium and thus the beta limit. Optimal triangular CRELs are found. They give beta limits slightly higher than those for the standard triangular CREL. Other possible ways of raising the beta limit are also discussed.

1. INTRODUCTION

A DRAKON [1, 2] (a Russian acronym for long equilibrium configuration) is a closed-end magnetic confinement system that contains two long straight sections (s.s.) of low magnetic field, joined at the ends by two special connectors of high magnetic field. These special connectors are called connectors of rectilinear elements (CRELs). They satisfy the 'CREL condition' and in general have three-dimensional curvilinear configurations.

The curvature of the CRELs and the plasma pressure gradient drive the Pfirsch-Schliiter current, which is a parallel current with zero azimuthal average. The Pfirsch-Schliiter current distorts the circular flux surfaces. For example, if we Fourier decompose this current, its dipole component (dipole Pfirsch-Schlieter current) shifts the flux surfaces (Fig. 1(a)). These shifts couple with the curvature and pressure gradient to induce a quadrupole component (quadrupole Pfirsch-Schlieter current), resulting in elliptical distortions (Fig. 1(b)). As the plasma pressure is increased, these distortions are aggravated by the increasing Pfirsch-Schlieter current. When the inner flux surfaces start to cross the outer ones and form separatrices, the equilibrium beta limit (beta being the plasma pressure over the magnetic energy density) is reached.

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FIG. 1. Pfirsch-Schlieter currents and flux surface distortions. (a) A dipole current drives a shift. (b) A quadrupole current induces an elliptical distortion.

The straight sections, having zero curvature, do not induce any Pfirsch-Schlieter current. Those Pfirsch-Schlieter currents existing in the straight sections come from the CRELs as a result of current continuity. Ideally, the CREL condition guarantees that no Pfirsch-Schlieter current flows out of the CRELs, thereby keeping the equilibrium in the straight sections autonomous — the straight sections can be infinitely long. In a realistic CREL, however, only the lowest-order CREL condition can be satisfied. It prevents the dipole Pfirsch-Schlieter current from flowing out of the CRELs and therefore the first-order (in plasma beta) shift is independent of the lengths of the straight sections. Different designs of this type of CREL have been presented in Refs [1, 3]. Among them, the simplest design is the triangular CREL, which is made up of three semi-tori twisted at the joints by 120°. A conceptual coil set design of a DRAKON with triangular CRELs is shown in Fig. 2.
LAU

Magnetic divertor. CREL (R = 24 cm)

CREL

Slot antenna

FIG. 2. Conceptual coil set design of a DRAKON with triangular CRELs. The slot antenna generates plasmas and the magnetic divertor provides stability.

On the basis of the first-order shift, the peak equilibrium beta limit $\beta^*$ was estimated to be about 12% for the triangular CRELs [1]. (Note that the beta limit given there was the average one, which was half the peak beta for a parabolic pressure profile.) Since the beta in the straight sections is scaled up by the square of the mirror ratio, this estimate can lead to a favourably high beta in the straight sections. This may be overoptimistic: the higher-order distortions, which do depend on the straight section length, may dominate over the first-order shift, resulting in lower beta limits. Observing this, Makurin and Mikhailovskij [4] carried out the calculation to third order in beta. Nevertheless, the beta limit for a specific CREL was not evaluated. Only some rough estimates were given. The conclusion was that if the straight sections did not exceed the CRELs in length, the first-order beta limit remained true.

Here we follow the same line as Makurin and Mikhailovskij did, but we generalize their calculation to systems that do not necessarily satisfy the CREL condition. This allows us to investigate the effects of deviations from the CREL condition (imperfect CRELs). Besides, the formulas to second order are valid for arbitrary pressure profiles. Thus, the effects of different pressure profiles can be studied easily. (A parabolic pressure profile was used in Ref. [4].) For the special case of a DRAKON with triangular CRELs, an analytic formula of the second-order (including the dominant terms in the elliptical distortion) beta limit is derived. The predictions of this formula are compared with the numerical evaluations of the full expressions to third order. Through this, a semi-analytic formula for the beta limit in DRAKONs with triangular CRELs is obtained. To optimize the beta limit, variations of the triangular CRELs are also considered.

The co-ordinate systems, various expansions and some basic equations are given in Section 2. The boundary conditions at the throats — the joints between straight sections and CRELs — are discussed in Section 3. The shift and elliptical distortion are then solved in Section 4, the results being applicable to DRAKONs with arbitrary connectors and arbitrary pressure profiles. For the special case of a DRAKON with triangular CRELs, an analytic formula of the second-order beta limit is derived in Section 5. The full expressions to third order for DRAKONs with imperfect CRELs are given in Section 6. In Section 7, numerical evaluations of these expressions for a triangular CREL DRAKON with a parabolic pressure profile yield the flux surface plots. The effects of deviations from the CREL condition are investigated. The numerical beta limits are compared with the analytic ones, leading to a semi-analytic formula. Optimization of the beta limit by varying the triangular CREL is considered in Section 8. Finally, conclusions and a discussion of the results are given in Section 9.

2. CO-ORDINATES AND BASIC EQUATIONS

In this section the derivation of the equations for the Pfirsch-Schlüter current and the distortion functions is outlined in four steps: (1) Set up a co-ordinate system along the system, (2) introduce the flux co-ordinates, Fourier expansions and distortions, (3) find the metric coefficients $g_{ij}$, and (4) obtain equations for the distortions and the Pfirsch-Schlüter currents. The mathematical expressions in steps (3) and (4) are quite lengthy. We will only give the outline and some lowest-order expressions here. Appendix A provides a brief derivation. More details can be found in Refs [4, 5].

The system axis (vacuum magnetic axis) of an arbitrary system is a spatial curve described by the Frenet-Serret equations

$$\frac{d\mathbf{r}}{ds} = k \mathbf{n},$$

$$\frac{d\mathbf{n}}{ds} = -k\mathbf{t} + \kappa \mathbf{\beta},$$

$$\frac{d\mathbf{\beta}}{ds} = -\kappa \mathbf{n},$$

where $s$ is the arc length of the system axis measured.
The magnetic axis, in general, is not the same as the system axis. At a fixed \( \xi \), the origin of the polar co-ordinates is shifted by \( x \exp \theta x \). Here, the \( \theta \) direction is treated as the real axis and \( \beta \) as imaginary. A new polar co-ordinate \( (r_0, \theta_0) \) is now set up with reference to this shifted origin. Thus we have

\[
r \exp (i\theta) = x \exp (i\theta x) + r_0 \exp (i\theta_0)
\]

Figure 4 shows the relation between these co-ordinates. (Note that the notations here are different from those in Ref. [4]: our \( r, r_0 \) \( \rightarrow \) \( \rho, \rho_0 \) there, and \( \omega \leftrightarrow \theta \), \( \omega_0 \leftrightarrow \theta_0 \).)

The new polar co-ordinates are now related to the flux co-ordinates through Fourier expansions,

\[
\theta_0 = \omega + H(a, \zeta) + \text{Im} \left( \Lambda e^{i\omega} + M e^{i2\omega} + N e^{i3\omega} + \ldots \right)
\]

\[
\equiv \omega_0 + H(a, \zeta) \tag{6a}
\]

\[
\rho_0 = a + R e^{i\omega_0} + e^{i2\omega_0} + \ldots
\]

\[
= a + R e^{i\omega_0} - \Lambda e^{i\omega} + (\Lambda + \tau) e^{i3\omega} + \ldots \tag{6b}
\]

(The function \( H \) is called \( h \) in Ref. [4].) The form of Eq. (6a) is justified if we consider the limiting case of a vacuum field in which the rectification functions \( \Lambda(a, \xi), M(a, \xi) \) and \( N(a, \xi) \) are zero while \( H = H_0 \) depends only on the torsion of the system axis. In this case, \( H_0 \) rectifies the angle \( \omega_0 = \omega \) such that the field lines are straight in \( (\omega_0, \xi) \). Equation (6b) states that the circular cross-sections \( (\rho_0 = a) \) are deformed by the elliptical distortion \( \lambda(a, \xi) \), the triangular distortion \( \tau(a, \xi) \), etc. Assuming that the plasma is enclosed by a conducting wall of radius \( b \), ellipticity \( \lambda_0(\xi) \) and triangularity \( \tau_0(\xi) \), we have

\[
r_0(b) = b + R e^{i\omega_0} + \theta_0(\xi) e^{i3\omega_0} + \ldots \tag{7}
\]
When the plasma pressure is very low, the flux surfaces are shaped solely by the wall.

The shift function $\xi(a, \zeta)$ is defined by

$$\xi^* \exp(iH) \equiv z \exp(i\theta_\pi) \quad (8)$$

Since the axis of the wall (system axis) has zero shift by definition, we have, together with Eq. (7),

$$\xi(b, \zeta) = 0, \quad \lambda(b, \zeta) = \lambda_0(\zeta), \quad \tau(b, \zeta) = \tau_0(\zeta), \quad \ldots$$

(9a)

At the magnetic axis $a = 0$, flux surfaces should degenerate to a point on a constant-$\zeta$ cross-section. Thus,

$$\xi(0, \zeta) \text{ is finite, } \lambda(0, \zeta) = \tau(0, \zeta) = \ldots = 0 \quad (9b)$$

Further, any physical function $f$ should obey the natural periodic boundary condition

$$f(\zeta + 2\pi) = f(\zeta) \quad (9c)$$

Equations (9) are the boundary conditions of the distortion functions.

The third step is to express the metric coefficients in terms of the rectification and distortion functions. This can be done simply by plugging Eqs (5), (6a) and (6b) into Eq. (2a) and then using

$$dH^2 = \sum_{ij} g_{ij} dz^i dz^j \quad (10)$$

Results are lengthy and can be found in Ref. [5]. It turns out that $g_{33}$ is a dominant element and Ampère's law in flux co-ordinates leads to (see Ref. [6]):

$$\frac{\partial g_{33}}{\partial \omega} \sqrt{g} = 0 \quad (11)$$

where $g = \det g_{ij}$. Different Fourier components of this equation can then be used to express $\Lambda$, $M$ and $N$ in terms of $H$, $\xi$, $\lambda$ and $\tau$.

Finally, to obtain the equations for the distortions, we Fourier decompose the Pfirsch-Schlüter current

$$\nu = 2\text{Re} \left( \nu^{(1)} e^{i\omega} + \nu^{(2)} e^{2i\omega} + \nu^{(3)} e^{3i\omega} + \ldots \right) \quad (12)$$

Then different harmonics of Ampère’s law yield (all the formulas are in cgs units)

$$\left( \mu - i \frac{\partial}{\partial \zeta} \right) \left( \xi^* + \frac{3}{a} \xi \right) = \frac{i2R 4\pi}{a\psi^' c} \nu^{(1)} \quad (13a)$$

where the subscripts $\nu$ and $\xi$ denote the order in $\beta$. A brief derivation of these two equations is given in Appendix A. Note that those terms proportional to $\xi^2$ on the right-hand side of Eq. (13b) are left out. They are small compared to $\nu^{(\beta)}$, as will be shown later by comparison of the analytic and numerical results. Furthermore, it is sufficient to use the zeroth-order rotation number $\mu_0$, which is

$$\mu_0 = \frac{4\pi JR}{c\psi^' a} - \kappa_0 R \quad (14)$$

with

$$\kappa_0 \equiv \int \frac{d\zeta}{2\pi} \kappa \quad (15)$$

Similarly, the equations for $\nu^{(\beta)}$ are found from different harmonics of the force balance equation. The first two are (see Appendix A)

$$\left( \mu - i \frac{\partial}{\partial \zeta} \right) \nu^{(1)} = -i \frac{\alpha^2}{\psi^' c} P' R e^{iH} \quad (16a)$$

$$\left( 2\mu - i \frac{\partial}{\partial \zeta} \right) \nu^{(2)} = i \frac{\alpha^2}{2\psi^' c} P' R e^{iH} \xi^* \quad (16b)$$

As before, we can use $\mu_0$ and $H_0$ in these equations. The zeroth-order $H$ is

$$H_0 = R \int_0^\zeta (\kappa_0 - \kappa) d\zeta \quad (17)$$

All the harmonics of the Pfirsch-Schlüter current $\nu^{(\beta)}$ should obey the natural periodic boundary condition, Eq. (9c).

3. BOUNDARY CONDITIONS
AT THE THROATS

In the paper of Makurin and Mikhailovskij [4], the boundary conditions at the throats were not given. Here, from current continuity, we derive these boundary conditions necessary for obtaining $\nu$ and the distortion functions.

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For a given flux surface, the radius in the straight sections is not the same as that in the CRELs if the mirror ratio $R_M > 1$. Because of $\hat{a}^2 B = \text{const}$, the radii are related by

$$\frac{a_{\text{se}}}{a_{\text{CR}}} = \sqrt{R_M} \quad (18)$$

For simplicity, we will ignore any transition regions between the straight sections and the CRELs. Thus, $B$ increases suddenly by a factor of $R_M$ when entering the CRELs from the straight sections. The differential equations for $\nu^{(n)}$ and the distortions derived in Section 2 will be solved piecewise and then jointed together through boundary conditions at the throats.

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Next we turn to the shift equation (13a). The jump in $\nu^{(1)}$ also generates a jump in $\xi_1$. Observing that $\alpha\psi'$ and $\alpha\nu^{(1)}$ remain the same on both sides of the throat, we find that $\xi_1$ is also the same on both sides. Hence, $\xi_1$ is enlarged by a factor of $\sqrt{R_M}$ as we go into the straight section from the CREL. For the special case of $R_M = 1$, $\xi_1$ should be continuous at the throat. This fact is used to determine the integral constant from solving Eq. (13a) with respect to $\xi$. The same arguments apply to other distortions as well. Thus we arrive at the conclusion: all distortions are scaled up by a factor of $\sqrt{R_M}$ when entering the straight section from the CREL. In practice, we solve the equations for the special case $R_M = 1$ and then use these boundary conditions to obtain the solutions in the straight section.

4. SOLUTIONS OF THE SHIFT AND ELLIPTICAL DISTORTION

Unless otherwise specified, the plasma beta (and beta limit) is the peak beta in the CRELs:

$$\beta = \frac{8\pi p(0)}{B_{\text{CR}}^2} = \frac{8\pi p(0) a_{\text{CR}}^2}{\psi^2} \quad (20)$$

As explained in Section 3, we first deal with $R_M = 1$, in which case all $\nu^{(n)}$ and distortions are continuous functions of $\xi$. Besides the $2\pi$-periodicity (Eq. (9c)), all physical functions ($\nu^{(n)}, H, \xi, \lambda, \tau$, etc.) actually possess $\pi$-periodicity:

$$f(\zeta + \pi) = f(\zeta) \quad (21)$$

This is because we choose the two CRELs on both sides to be identical. It also implies that the curvature and torsion of the DRAKON satisfy

$$k(\zeta + \pi) = k(\zeta), \quad \text{and} \quad \kappa(\zeta + \pi) = \kappa(\zeta) \quad (22)$$

All other geometric functions appearing in Eq. (7), such as $b$ and $\lambda_0$, have the same $\pi$-periodicity. This $\pi$-periodicity is quite useful for checking the correctness of the solutions and evaluating the integration constants.

Noticing that $J = 0$ in a DRAKON (current-free system), we can use Eq. (14) to rewrite Eq. (17) as

$$H_0 = -\mu_0 \zeta - \alpha_0(\zeta) \quad (23)$$

where
\[\alpha_0 = R \int_0^\zeta \kappa d\zeta = \int_0^s \kappa ds \]  
(24a)

is the twisting angle of the system axis. The zeroth-order rotational transform is given by

\[\iota_0 = -\alpha_0(2\pi) = 2\pi \mu_0 \]  
(24b)

With Eq. (23), it is straightforward to solve Eq. (16a):

\[\nu_1^{(1)} = \frac{\beta \psi'}{2} \frac{c}{4\pi} \hat{P}''(a) \left[ A^{(1)} + D(\zeta) \right] \exp(-i\mu_0\zeta) \]  
(25)

where

\[\hat{P}(a) = \frac{P(a)}{P(0)} \]  
(26)

\[D(\zeta) = R \int_0^\zeta d\zeta k \exp(-i\alpha_0) \]  
(27)

The integration constant is found from the natural periodic boundary condition Eq. (9c) to be

\[A^{(1)} = -\frac{i\exp(-i\pi \mu_0)}{2\sin(\pi \mu_0)} D(2\pi) \]  
(28)

Plugging Eq. (25) into Eq. (13a) and observing that Eq. (13a) is an Euler-type equation with respect to \(a\), we solve it by the method of Green's function to obtain

\[\frac{\xi_1}{b} = -\left(\frac{\beta l}{4b}\right) R_{\xi_1}(a)\text{CADF}(\zeta) \exp(-i\mu_0\zeta) \]  
(29)

Here, \(l\) is the length of one CREL. The radial function

\[R_{\xi_1}(a) = -2\left(\hat{P}(a) - \hat{P}(b) - \frac{1}{a^2} \int_0^a d\alpha a^2 \hat{P}'(a) + \frac{1}{b^2} \int_b^a d\alpha a^2 \hat{P}'(a)\right) \]  
(30)

and the axial function

\[\text{CADF}(\zeta) \equiv \hat{C}_1 + \frac{R_C}{l} \left[ A^{(1)} + D(\zeta) \right] - \hat{F}(\zeta) \]  
(31)

with

\[\hat{F}(\zeta) \equiv \frac{R^2}{l} \int_0^\zeta d\zeta k \exp(-i\alpha_0) \]  
(32)

The integration constant is

\[\hat{C}_1 = \frac{i\exp(-i\pi \mu_0)}{2\sin(\pi \mu_0)} \frac{\hat{F}(2\pi) - R \pi D(2\pi)}{4} \]  
(33)

Equations (25) and (29) can be shown to have the correct \(\pi\)-periodicity.

To obtain the shift for \(R_M > 1\), we simply replace \((\beta l/4b)\) by \((\beta l/4b_{\text{CR}})\) in Eq. (29), with other \(b\)'s unchanged. Since

\[b = \begin{cases} \frac{b_{\text{CR}}}{b_{\text{CONS}}} & \text{in CREL} \\ \frac{b_{\text{CONS}}}{b_{\text{CR}}} & \text{in s.s.} \end{cases} \]  
(34)

The resulting \(\xi_1\) has the correct scaling when entering the straight sections from the CRELs.

Now we proceed to the second-order equations. The derivative

\[\frac{\xi_1}{b} = \left(\frac{\beta l}{4b}\right) \frac{1}{a^2} \int_0^a d\alpha a^2 \hat{P}' \right) \text{CADF}(\zeta) \exp(-i\mu_0\zeta) \]  
(35)

is put into Eq. (16b) to produce

\[\nu_2^{(2)} = -\frac{\beta^2 \psi'}{4} \frac{c}{4\pi} \int_0^a d\alpha a^2 \hat{P}' \]  
(36)

Here,

\[\hat{N}^{(2)}(\zeta) \equiv D(\zeta)\text{CADF}(\zeta) - \frac{R}{l} \int_0^\zeta d\zeta \left[ A^{(1)} + D(\zeta) \right] \]  
(37)

\[\hat{C}^{(2)} = -\frac{i\exp(-i2\pi \mu_0)}{2\sin(2\pi \mu_0)} \hat{N}^{(2)}(2\pi) \]  
(38)

Equation (13b) is then solved as before, resulting in

\[\frac{\lambda}{b} = \left(\frac{\beta l}{2b_{\text{CR}}}\right)^2 R_{\lambda\omega}(a)8CCN(\zeta) \exp(-i2\mu_0\zeta) \]  
(39)

Here the radial function is

\[R_{\lambda\omega}(a) \equiv -\left(\frac{b}{a}\right)^2 \left(a \int_b^a \frac{d\alpha}{a^2} \int_0^a d\alpha a^2 \hat{P}' \right. \]  
(40)
and the axial function is

$$CCN(\xi) \equiv \frac{R}{l} \int_0^l d\xi \left[ \hat{\mathcal{N}}(2)(\xi) + \hat{\mathcal{N}}(2)(\xi) \right]$$  \hspace{1cm} (41)

The integration constant is

$$\hat{\mathcal{N}} = \frac{-i\exp(-i2\pi\mu_0)}{2\sin(2\pi\mu_0)} R \left( 2\pi \hat{\mathcal{N}}(2) + \int_0^{2\pi} d\xi \hat{\mathcal{N}}(2) \right)$$  \hspace{1cm} (42)

In addition, we have chosen a circular wall, so that \(\lambda_0 = 0\).

So far, our solutions are valid for arbitrary connectors. A CREL satisfies the CREL condition:

$$\nu^{(1)} = 0 \text{ at the throats}$$  \hspace{1cm} (43)

Since \(D(\xi) = D(2\pi) = D(\pi)\) in the straight sections, the CREL condition requires that

$$A^{(1)} = 0 \quad \text{or} \quad D(2\pi) = D(\pi) = 0 \quad (44)$$

Thus

$$\hat{\mathcal{N}} = \frac{-i\exp(-i2\pi\mu_0)}{2\sin(2\pi\mu_0)} \hat{\mathcal{N}}(2\pi)$$  \hspace{1cm} (45)

Once the CREL condition is satisfied, the shift \(\xi_1\) is independent of the length \(L\) of the straight sections. This statement, however, does not hold for the elliptical distortion \(\lambda\). A beta limit to second order (including the elliptical distortion) will therefore depend on \(L\), as we shall see in Section 5.

5. ANALYTIC FORMULA FOR THE SECOND-ORDER EQUILIBRIUM BETA LIMIT

In the original paper [1] proposing the DRAKON, a formula based only on the first-order shift \(\xi_1\) was used to estimate the peak beta limit \(\beta^*\), leading to the favourable result of \(\beta^* \sim 12\%\) in the CREL. (Note that the beta limit given there was the average one, which was half the peak beta for a parabolic pressure profile.) This was overoptimistic. It is the elliptical distortion \(\lambda\), which is driven by the quadrupole Pfirsch–Schlüter current, that plays a more important role than \(\xi_1\). A beta limit including \(\lambda\) will therefore be much lower than a beta limit without \(\lambda\). In this section, we will derive an analytic formula for the second-order equilibrium beta limit in DRAKONs with triangular CRELs. The formula is good for an arbitrary radial pressure profile. Thus the effects of different pressure profiles can be investigated easily. In Section 7, the prediction of this formula will also be used to cross-check the numerical results.

The DRAKON we consider here is perfect, in the sense that its curvature \(k\) is made up of step functions

$$k(s) = \begin{cases} k_0 & 0 < s < l \\ 0 & l < s < l + L \\ k_0 & l + L < s < 2l + L \\ 0 & 2l + L < s < 2(l + L) \end{cases}$$  \hspace{1cm} (46)

\((k_0 = 1/R_0 = 3\pi/l, l\) being the length of one CREL, \(L\) the length of one straight section), and its torsion \(\kappa\) of \(\delta\) functions is

$$\kappa(s) = \alpha_{\text{cr}} \delta(s) - \alpha_{\text{cr}} \delta(s - l/3) + \delta(s - 2l/3)$$

$$+ \alpha_{\text{cr}} \delta(s - l) + \delta(s - l - L) - \alpha_{\text{cr}} \delta(s - 3l/3 - L) - \alpha_{\text{cr}} \delta(s - 5l/3 - L) + \alpha_{\text{cr}} \delta(s - 2l - L)$$  \hspace{1cm} (47a)

with

$$\alpha_{\text{cr}} = 120^\circ, \quad \alpha_{\text{ae}} = 19.11^\circ$$  \hspace{1cm} (47b)

The rotational transform of the system axis is

$$\iota_0 = -\int \kappa ds = 4(\alpha_{\text{cr}} - \alpha_{\text{ae}}) = 403.56^\circ$$  \hspace{1cm} (48)

FIG. 6. Functions (a) \(D(s)\exp(-i\mu_0/2)\) and (b) \(\alpha_0 + \mu_0/2\) for a triangular CREL. Note that all lengths are normalized by \(L\).
We first calculate the three integration constants \( \hat{C}_1, \hat{C}^{(2)} \) and \( \hat{C}_\lambda \). From Eqs (45) and (22), it is not hard to show

\[
\hat{C}_1 = \frac{i \exp(-i \pi \mu_0/2)}{2 \sin(\pi \mu_0/2)} \int \bar{F}(\pi) \]  

(49)

To simplify notations, we will make \( R_f/l \rightarrow s \) and \( k/l \rightarrow k \). After one integration by parts on \( \bar{F} \), Eq. (49) becomes

\[
\hat{C}_1 = \frac{-i}{2 \sin(\pi \mu_0/2)} \int_0^1 ds D(s) \exp(-i \pi \mu_0/2) \]  

(50)

Noting that \( -\pi \mu_0/2 = \alpha_{ss} - \alpha_{CR} \), we plot \( D(s) \exp(-i \pi \mu_0/2) \) and \( \alpha_0 \exp(-i \pi \mu_0/2) \) in Fig. 6. Different segments of \( D(s) \exp(-i \pi \mu_0/2) \) are represented by

\[
D(s)e^{-i \pi \mu_0/2} = \begin{cases} 
3\pi s \exp(-i 2\pi/3) & 0 \leq s < 1/3 \\
3\pi(s - 1/2) - i\pi \sqrt{3}/2 & 1/3 \leq s < 2/3 \\
3\pi(1 - s) \exp(-i \pi/3) & 2/3 \leq s < 1 
\end{cases} \]  

(51)

Since this is a linear function of \( s \), its integral over \( s \) is just the centre of mass of an equilateral triangular wire with side length 1/3 and uniformly distributed mass \( \pi \) on each side. Hence, by finding the centre of mass of the wire, we obtain

\[
\hat{C}_1 = \frac{-\pi}{2\sqrt{3} \sin(\pi \mu_0/2)} = -0.9235 \]  

(52)

This is a real number, as can easily be justified by Fig. 6(a).

Next we consider the constant \( \hat{C}^{(2)} \). Again, the use of the \( \pi \)-periodicity yields

\[
\hat{C}^{(2)} = -\frac{i \exp(-i \pi \mu_0)}{2 \sin(\pi \mu_0)} \hat{N}^{(2)}(\pi) \]  

(53)

The function \( [D(s) \exp(-i \pi \mu_0/2)]^2 \) is plotted in Fig. 7. Since \( |D(s)| = |D(1 - s)| \), the average of this function must be real. Thus,

\[
\hat{C}^{(2)} = \frac{i}{\sin(\pi \mu_0)} \text{Re} \int_0^{1/2} ds [D(s) \exp(-i \pi \mu_0/2)]^2 \]  

Now it is straightforward to use Eq. (51) to obtain

\[
\hat{C}^{(2)} = \frac{-i \pi^2}{6 \sin(\pi \mu_0)} = 4.433i. \]  

(54)

Finally, the constant \( \hat{C}_\lambda \) after applying the \( \pi \)-periodicity and partial integrations is

\[
\hat{C}_\lambda = -\frac{i \exp(-i \pi \mu_0)}{2 \sin(\pi \mu_0)} \left( \pi \hat{C}^{(2)}\frac{\hat{L} + 1}{\pi} + \int_0^{1} ds \hat{N}^{(2)} \right) \]  

\[
= (\hat{L} + 1) -\frac{i \exp(i \pi \mu_0)}{2 \sin(\pi \mu_0)} \hat{C}^{(2)} + \hat{C}_\lambda \]  

(55)

where \( \hat{L} = L/l \) and

\[
\hat{C}_{\lambda 0} = -\frac{i \exp(-i \pi \mu_0)}{2 \sin(\pi \mu_0)} \int_0^1 ds D(CADF + sD) \]  

\[
= -\frac{i \exp(-i \pi \mu_0)}{2 \sin(\pi \mu_0)} \int_0^1 ds D(\hat{C}_1 + \int_0^1 ds' D(s') + sD) \]  

(56)

The three terms in the second expression of Eq. (56) are evaluated as follows. With Eq. (50),

the first term \( = \frac{\exp(-i \pi \mu_0/2)}{2 \cos(\pi \mu_0/2)} \hat{C}_1 \)  

(57)

Likewise,

the second term

\[
= \frac{i}{2} \tan(\frac{\pi \mu_0}{2}) \left( \frac{-i \exp(-i \pi \mu_0/2)}{2 \sin(\pi \mu_0/2)} \int_0^1 ds D \right)^2 \]  

(58)
The last term is integrated first by parts and then with Eq. (51).

The last term

\[ = \frac{i}{2 \sin(\pi \mu_0)} \int_0^1 ds s^2 De^{-i \pi \mu_0/2} ke^{-i(\alpha_0 + \pi \mu_0/2)} \]

\[ = \frac{i}{2 \sin(\pi \mu_0)} \left( \frac{1}{6} + \frac{i}{9\sqrt{3}} \right) \]

(59)

Summing them up and substituting \( \dot{C}_1 \) into Eq. (52), we obtain

\[ \dot{C}_{\lambda 0} = \frac{\pi^2}{24 \sin^2(\pi \mu_0/2)} + \frac{\pi^2}{2 \sin(\pi \mu_0)} \left( -\frac{1}{9\sqrt{3}} + \frac{i}{6} \right) \]

\[ = 1.280 - 2.217i \]

(60)

Thus

\[ \dot{C}_\lambda = (\dot{L} + 1)(5.548 + 2.217i) + (1.280 - 2.217i) \]

(61)

This constant has an imaginary part proportional to \( \dot{L} \). Since \( \dot{C}_1 \) is real, this means that, at the origin, the relative phase between the shift and the elliptical distortion also depends on \( \dot{L} \).

The beta limit \( \beta^* \) is defined to be the beta value at which the inner flux surfaces start to cross the outer ones. Flux surfaces cross each other first at those points where various distortions are constructive. To second order, this means that, at the origin, \( \xi_1 \) and \( \lambda \) are in phase or have a phase difference of \( \pi \). The \( \beta^* \) to second order can therefore be estimated by considering the shift and elliptical distortion at such points. It is found that the centre of the straight section is a point of this type. To show this, we shift the origin \( \xi = 0 \) from the throat to the centre of the straight section. Both \( \dot{C}_1 \) and \( \dot{C}^{(2)} \) are not affected by this shift because \( D = 0 \) in the straight section. For the same reason, the new \( \dot{C}_\lambda \) is

\[ \dot{C}_\lambda = (\dot{L} + 1) \left[ \frac{i \exp(i \pi \mu_0)}{2 \sin(\pi \mu_0)} \dot{C}^{(2)} - \frac{\dot{L}}{2} \dot{C}^{(2)} + \dot{C}_{\lambda 0} \right] \]

\[ = (\dot{L} + 1) \left[ \frac{\pi^2 \cot(\pi \mu_0)}{12 \sin(\pi \mu_0)} \right. \]

\[ + \left. \pi^2 \left( \frac{1}{24 \sin^2(\pi \mu_0/2)} - \frac{1}{18\sqrt{3} \sin(\pi \mu_0)} \right) \right] \]

\[ = (\dot{L} + 1)(5.548 + 1.280) \]

(62)

which is a real number!

At the new origin, \( (\xi' = 0) \), \( \xi_1 \) is in the \(-\pi\) direction. \( \lambda \) bulges out in the same direction too. Thus in this direction,

\[ r(a, 0) = a - \xi_1(a, 0) + \lambda(a, 0) \]

(63)

A schematic of \( r(a) \) is plotted in Fig. 8. At \( \beta = 0 \), it is simply \( r = a \). When \( \beta \) is raised, distortions make \( r \) bigger, while \( r(b) = b \) is fixed. Since the gradient of the radial pressure profile is in general higher in the outer region than in the inner one, the \( r \) in the outer region will be driven to a bigger value faster than the \( r \) in the inner region. When \( \beta > \beta^* \), part of the outer region has \( r > b \), which means that the flux surfaces there have already crossed the wall. Thus the first sign for this to happen is when

\[ r'(b) = 1 - \xi_1'(b) + \lambda'(b) = 0 \]

(64)

where the prime denotes \( d/da \). Equation (64) as a quadratic algebraic equation for \( \beta \) yields the beta limit \( \beta^* \).

Using Eq. (35) and the a derivative of Eq. (39) with \( a = b, \xi \rightarrow \xi' = 0 \), and \( \dot{C}_\lambda \rightarrow \dot{C}_l \) in Eq. (64) gives

\[ 1 - \left( \frac{\beta_1}{b_{\text{CR}}} \right)^2 \dot{C}_l \int_0^b da a^2 \dot{P} + \left( \frac{\beta_1}{b_{\text{CR}}} \right)^2 \dot{C}_l bK_{\Delta \omega}(b) = 0 \]

(65a)

where, from Eq. (40),

\[ bK_{\Delta \omega}(b) = \frac{1}{b^2} \int_0^b da \dot{P} \int_0^b da a^2 \dot{P} \]

\[ = \frac{1}{b^2} \int_0^b da a^2 \dot{P} - \frac{\dot{P}(b)}{2} + \frac{\dot{P}(b)}{2b^2} \int_0^b da a \dot{P} \]

(65b)

For the special case of a parabolic pressure profile

\[ \dot{P}(a) = 1 - \frac{a^2}{b^2} \]

(66)
Eq. (65a) becomes
\[ 1 + \left( \frac{\beta^*}{b_{\text{CR}}} \right) \frac{\dot{C}_1}{2} - \left( \frac{\beta^*}{b_{\text{CR}}} \right)^2 \frac{\ddot{C}_1}{3} = 0 \]  \hspace{1cm} (67)

The old beta limit [1] is obtained by ignoring the $\beta^2$ term in this equation, yielding
\[ \beta^* = -\frac{2b_{\text{CR}}}{C_1} = 11.5\% \quad (\text{for } k_0b = 0.5) \]  \hspace{1cm} (68)

which is independent of $\bar{L}$. Now the $\beta^*$ to second order obtained from Eq. (67) is
\[ \beta^* = \frac{b_{\text{CR}}}{l} \frac{29.59(\bar{L} + 1) + 7.678}{7.398(\bar{L} + 1) + 1.706} \]  \hspace{1cm} (69)

This $\beta^*$ is considerably lower than the first-order one (Eq. (68)). For example, $\beta^* = 2.33\%$ for $\bar{L} = 1$ and $k_0b = 0.5$. This is because the quadrupole Pfirsch–Schlüter current is much larger than the dipole one ($|\dot{C}_1| \gg |\dot{C}_1|$). The equilibrium and therefore the beta limit is determined mainly by the elliptical distortion.

The effects of the pressure profile can also be investigated easily from Eq. (65a). Two pressure profiles are used:
\[ \dot{P} = (1 - \hat{\sigma}^2) \]
\[ \dot{P} = \tanh[(1 - \hat{\sigma})/g]/\tanh(1/g) \]

where $\hat{\sigma} = a/b$. Scanning the free parameter g, we plot $\beta^*$ versus $-d\dot{P}/d\hat{\sigma}$ at $\hat{\sigma} = 1$ in Fig. 9 (for $k_0b_{\text{CR}} = 0.5$ and $\bar{L} = 1$). The beta limit for the second profile is slightly lower because its gradient is slightly larger in the interior. Also, the beta limits for sharp boundary profiles (large gradients at the wall) are about half of those for smooth profiles.

Finally, although the centre of a straight section is found to be a point where the effects of $\xi_1$ and $\lambda$ are constructive, it is possible that other points in the CRELs are of this type too. If such points exist, it remains to be answered whether they give a more stringent beta limit or not. Besides, the question is whether the second-order driving terms that are proportional to $\xi_1^2$ and the third-order distortions are important. Instead of pursuing the formidable analytic calculations, we will tackle these problems in a combined analytic and numerical way in the following sections.

6. COMPLETE SOLUTIONS TO THIRD ORDER

In Section 4, we discussed in detail how to solve the shift and elliptical distortion. Other distortions can be solved in exactly the same way. Relevant differential equations were given in Ref. [4]. Since they are quite long, we will only list the results here. The results here are valid for arbitrary connectors — not necessarily CRELs. This makes it possible to investigate the effects of deviations from the CREL condition.

The solutions up to second order are valid for arbitrary pressure profiles. The shift is
\[ \frac{\xi_1}{b} = \left( \frac{\beta_l}{4b_{\text{CR}}} \right) R_{\xi_1}(a) \text{CADF}(\zeta) \exp(-i\mu_0\zeta) \]  \hspace{1cm} (70a)

The functions $R_{\xi_1}$ and CADF are defined in Eqs (30) and (31). For the special case of a parabolic profile (Eq. (66)),
\[ \frac{\xi_1}{b} = \left( \frac{\beta_l}{4b_{\text{CR}}} \right) \left( \frac{a^2}{b^2} - 1 \right) \text{CADF}(\zeta) \exp(-i\mu_0\zeta) \]  \hspace{1cm} (70b)

In Section 2, we mentioned that driving terms proportional to $\xi_1^2$ were left out of Eq. (13b). Had we included these terms and the $\lambda_0$ term, Eq. (39) would become
\[ \frac{\lambda}{b} = \left( \frac{\beta_l}{2b_{\text{CR}}} \right)^2 \exp(-i2\mu_0\zeta) \left[ 8R_{\lambda_0}(a)(\text{CCN}(\zeta) - \text{CADF})^2 \right. \]
\[ \left. + 3R_{\lambda_0}^2 \text{CADF}^2 \right] + \frac{a\lambda_0}{b^2} \]  \hspace{1cm} (71)

FIG. 9. Beta limits as functions of the pressure gradient at the wall for different pressure profiles (for $k_0b_{\text{CR}} = 0.5$ and $\bar{L} = 1$).
with

$$R_{\lambda} \equiv \frac{b}{3a^2} \left( \int_0^a da a^2 \hat{P}^2 \right)^2 + \frac{a}{3b^2} \left( \int_0^b da a^2 \hat{P}^2 \right)^2 - \frac{2ab}{3} \int_b^a da \frac{\hat{P}^2}{a^4} \int_0^a a^2 \hat{P}^2 da + 4R_{\lambda 0}$$

(72)

For the parabolic profile,

$$\lambda \equiv \left( \frac{\beta l}{2bc_{\text{CR}}} \right) \frac{a}{12b} \frac{a - \frac{a^2}{b^2}}{b^2} \exp(-i2\mu_0\zeta) \times \left[ 8CCN(\zeta) - 5CADF^2 + \frac{a\lambda_0}{b^2} \right]$$

(73)

The second order of the function $H$ is

$$H_2 = -\frac{1}{2} \left( \frac{\beta l}{2bc_{\text{CR}}} \right)^2 R_{\lambda}(\alpha) R \left( \hat{M}^{(0)} \zeta - \int_0^\zeta \hat{M} d\zeta \right)$$

(74a)

Here the radial function

$$R_{\lambda}(\alpha) \equiv \left( \frac{2b}{a^2} \int_0^a da a^2 \hat{P}^2 \right)^2 = \frac{a^2}{b^2}$$

for the parabolic profile

(74b)

and the function

$$\hat{M}(\zeta) \equiv \text{Im} \left[ (A^{(1)} + D)^* (\hat{C}_1 - \hat{F}) \right]$$

(74c)

with its average

$$\hat{M}^{(0)} = \frac{1}{2\pi} \int_0^{2\pi} \hat{M} d\zeta$$

(74d)

When calculations are carried out to third order, the expressions become considerably more lengthy. Thus we will only deal with the parabolic pressure profile. The corresponding beta limits to third order are found numerically in Section 7 and are then compared with the second-order analytic results of Section 5. If these beta limits agree well, then the results for different pressure profiles obtained in Section 5 are good approximations even to third order.

The third-order shift is

$$\xi_3 = \left( \frac{\beta l}{2bc_{\text{CR}}} \right)^3 \exp(-i\mu_0\zeta) \left\{ \left( \frac{a}{b} \right)^2 - \frac{1}{8} \right\}$$

$$\times \left[ \frac{CADF^*}{3} (5CADF^2 - 8CCN) + \frac{R}{l} (\hat{P}_1 + \hat{C}_{P1}) \right] + \left( \frac{a/b}{2} \right)^4 \frac{-1}{24} \left[ (4CCN - \frac{3}{2} CADF^2) CADF^* \right]$$

$$+ \frac{R}{l} (\hat{P}_1 + \hat{C}_{P2}) + \frac{iR}{2l} CADF \left( \hat{M}^{(0)} \zeta - \int_0^\zeta \hat{M} d\zeta \right) \right\} - \frac{\lambda_0}{b} \xi_3$$

(75)

Here

$$\hat{P}_1(\zeta) \equiv \int_0^\zeta d\zeta \left[ \frac{1}{3} (A^{(1)} + D)^* (5CADF^2 - 8CCN) \right]$$

$$- \frac{2R}{l} \left( \hat{C}_1 + \hat{C}_{Q1} \right)$$

(76a)

$$\hat{P}_2(\zeta) \equiv \int_0^\zeta d\zeta \left[ (A^{(1)} + D)^* (5CADF^2 - 8CCN) \right]$$

$$\frac{2iR}{l} \left( \hat{M}^{(0)} \zeta - \int_0^\zeta \hat{M} d\zeta \right) (A^{(1)} + D)$$

$$+ 2iCADF(\zeta) \hat{M}^{(0)} - \frac{2R}{l} (\hat{Q}_2 + \hat{C}_{Q2})$$

(76b)

$$\hat{C}_{P1} = -\frac{i\exp(-i\pi/\mu_0)}{2\sin(\pi/\mu_0)} \hat{P}_1(2\pi)$$

(77)

The $Q$ functions are defined by

$$\hat{Q}_1(\zeta) \equiv \int_0^\zeta d\zeta \left[ \frac{kl}{6} \exp(i\alpha_0) (5CADF^2 - 8CCN) \right]$$

(78a)

$$\hat{Q}_2(\zeta) \equiv \int_0^\zeta d\zeta \left[ (A^{(1)} + D)^2 CADF^* \right.$$

$$+ 2kl \exp(i\alpha_0) \left( \frac{1}{2} CADF^2 - CCN \right) \right]$$

(78b)

with

$$\hat{C}_{Q1} = -\frac{i\exp(-i\pi/\mu_0)}{2\sin(\pi/\mu_0)} \hat{Q}_1(2\pi)$$

(78c)

The triangular distortion is

$$\frac{T}{b} = \frac{i}{16} \left( \frac{\beta l}{2bc_{\text{CR}}} \right)^3 \frac{a^2}{b^3} \left( \frac{a^2}{b^2} - 1 \right) [\hat{C}_T + T(\zeta)] \exp(-i3\mu_0\zeta)$$

$$+ \frac{a^2\tau_0}{b^3}$$

(79)
where

\[ T(\zeta) \equiv \frac{7}{2} \text{CADF}^3 + 20 \text{CADF}(\zeta) \text{CCN}(\zeta) \]

\[ -\frac{R}{l} \int_0^\zeta d\zeta \left[ \frac{3R}{2l} (-8 \text{CCN} - 4 \text{CADF}^2) k \exp(-i\alpha_0) ight] 
- 12 \text{CCN}(\zeta)(4I + D) \]

(80a)

The integration constant is

\[ \hat{C}_T = \frac{R}{l} \int_0^{2\pi} d\zeta [\ldots] \frac{i \exp(-i3\pi\mu_0)}{2 \sin(3\pi\mu_0)} \]

(80b)

with [\ldots] standing for the terms in the square bracket of Eq. (80a).

7. NUMERICAL EVALUATION OF EQUILIBRIUM FLUX SURFACES AND BETA LIMITS

On the basis of the expressions given in Section 6, a computer code is developed to evaluate the distortions and to plot the flux surfaces. By varying the beta interactively and judging from the plots, we can also find the beta limit. Although the code can work with arbitrary input geometric functions (k and \( \alpha_0 \)), we will only deal with DRAKONs with triangular CRELs here. The origin, \( \zeta = 0 \), is chosen to be the centre of a straight section, so that the integration constants from the code can be compared with those from an analytic calculation. This also avoids the problem of having functions with sharp transitions, such as k and \( \alpha_0 \), at the start and the end of the grid mesh. The number of grids around each CREL is six to ten times more than that in each straight section, so as to circumvent the sharp transitions. Integrals are computed with cubic spline quadratures. To test the code, we first check the integration constants. As shown in Table I, the numerical and analytic results agree to within a relative error of 2%, which is

TABLE I. INTEGRATION CONSTANTS FROM NUMERICAL AND ANALYTIC CALCULATIONS

| Constants | L | Numerical results | Analytic results |
|-----------|---|------------------|-----------------|
| \( \hat{C}_1 \) | independent of \( L \) | -0.928 + 0.002i | -0.924 |
| \( \hat{C}_2 \) | independent of \( L \) | 0.07 + 4.48i | 4.43i |
| \( \hat{C}_\lambda \) | 1.0 | 12.5 - 0.1i | 12.4 |
| | 2.5 | 20.9 + 0.1i | 20.7 |
| | 5.0 | 34.5 + 0.1i | 34.6 |
| | 10.0 | 62.2 - 0.1i | 62.3 |
| | 20.0 | 117.8 - 1.3i | 117.8 |

FIG. 10. (a) Curvature and twisting angle of a perfect DRAKON with triangular CRELs.
(b) Corresponding function D, which is zero in the straight sections.
reasonable. Note that this is not necessarily so for the numerical beta limits and the analytic ones (Eq. (69)), since the second-order driving terms that are proportional to $\xi_1$ and the third-order distortions had been neglected in Eq. (69).

The first case is a perfect DRAKON with triangular CRELs — one that is made up of step functions (Fig. 10). At four locations along the axis (Fig. 11), the flux surfaces at the beta limit, $\beta^* = 2.2\%$ for $k_0 b_{CR} = 0.5$, and $\bar{L} = 1$ are plotted in Fig. 12. All three types of distortions — shift, elliptical and triangular distortions — can be identified clearly. At the origin (Fig. 12(a)), the shift and the elliptical distortion are constructive in the $-\hat{\eta}$ direction, as discussed in Section 5.

Next, we consider a more realistic DRAKON in which all step functions are smoothed out (Fig. 13(a)). As is shown in Fig. 13(b), the smoothed D function no longer satisfies the CREL condition exactly, since the dipole Pfirsch-Schlüter current does not vanish in the straight sections. However, the flux surfaces and the beta limit turn out to be the same. This is not unexpected: according to the findings in Section 5 that the quadrupole Pfirsch-Schlüter current dominates the dipole Pfirsch-Schlüter current, slight changes

FIG. 11. Locations and orientations of the flux surface plots in Figs 12 and 14. Points a to d correspond to plots (a) to (d) in those figures.

FIG. 12. Flux surfaces at the beta limit, $\beta^* = 2.2\%$, for a DRAKON with $k_0 b_{CR} = 0.5$ and $\bar{L} = 1$. The surfaces are evenly spaced in the parabolic pressure profile. See Fig. 11 for their locations and orientations.
FIG. 13. (a) Curvature and twisting angle of a smooth DRAKON with triangular CRELs. (b) Corresponding function $D$, which is not zero in the straight sections.

FIG. 14. Flux surfaces at the beta limit, $\beta^* = 1.3\%$, for a DRAKON with $k_0 b_{cr} = 0.5$ and $L = 5$. The surfaces are evenly spaced in the parabolic pressure profile. See Fig. 11 for their locations and orientations.
in the dipole Pfirsch–Schlüter current will have little effects on the equilibrium. When \( \tilde{L} \) increases, \( \beta^* \) goes down because of a higher quadrupole Pfirsch–Schlüter current. The flux surfaces at \( \beta^* = 1.3\% \) for \( \tilde{L} = 5 \) are presented in Fig. 14, showing basically the same structure. Figure 15 summarizes the beta limits for different aspect ratios \( R_0/b_{CR} = 1/\kappa b_{CR} \) and \( \tilde{L} \).

To compare these results with the analytic formula (Eq. (69)), \( \beta^* \) is plotted against \( \tilde{L} \) in Fig. 16 for \( k_0b_{CR} = 0.5 \). The dots showing the numerical results are slightly lower than the dashed line resulting from Eq. (69). This confirms that our second-order solutions make the main contribution to \( \beta^* \). Furthermore, noticing that the difference is almost a constant, we subtract from Eq. (69) a mean value to obtain the semi-analytic formula

\[
\beta^* = \frac{b_{CR}}{l} \left[ \frac{29.6(\tilde{L} + 1) + 7.68}{7.40(\tilde{L} + 1) + 1.71} - 0.924 \right] - 0.0013
\]

(81)

The beta limits obtained from this formula are shown by the solid line in Fig. 16.

8. OPTIMIZATION OF BETA LIMITS

In this section we optimize the beta limits without drastic changes in the system geometry. In particular, we keep the circular coils and use the general triangular CREL shown in Fig. 17. The radius \( R_2 \) of the middle semi-torus can now differ from the radius \( R_1 \) of the other two semi-tori. The CREL condition requires that \( a_{CR} = 120^\circ \) and

\[
\sin \alpha = \left( 1 + 4k_{21}^2 - 4k_{21} \cos a_{CR} \right)^{-1/2} \sin a_{CR}
\]

(82)

where \( k_{21} = k_2/k_1 \), \( k_1 = 1/R_1 \), and \( k_2 = 1/R_2 \).

To raise \( \beta^* \), we must suppress the quadrupole Pfirsch–Schlüter current \( \nu^{(2)} \), which is characterized by the constant \( \hat{C}^{(2)} \). Equation (54) shows that \( \hat{C}^{(2)} \) can be lowered by tuning \( \pi \mu_0 = \pi_0/2 \) away from 180°. A crude way is to alter \( a_{CR} \) while keeping \( R_1 = R_2 = R_0 \); for instance, when \( a_{CR} = 130^\circ \), \( \pi \mu_0 \) increases to 227.7° (compared to 201.8° of the standard triangular CREL) and \( \beta^* \) is found numerically to be 3.6% \( (R_0/b_{CR} = 2 \) and \( \tilde{L} = 1 \)). Nonetheless, this change totally destroys the CREL condition and the connector is no longer a CREL.

To maintain the CREL condition, we keep \( a_{CR} = 120^\circ \) but vary \( R_1 \). The function \( D \) is still an equilateral triangle on the complex plane and Eq. (51) now reads \( (k_{12} = k_1/k_2) \).
\[ D(s) \exp(-i\pi\mu_0/2) = \begin{cases} 
(2 + k_{12}) \pi s \exp(-i2\pi/3) & 0 \leq s < s_1 \\
(1 + 2/k_{12}) \pi (s - 1/2) - i\pi\sqrt{3}/2 & s_1 \leq s < s_2 \\
(2 + k_{12}) \pi (1 - s) \exp(-i\pi/3) & s_2 \leq s < 1 
\end{cases} \] 

with

\[ s_1 = \frac{1}{2 + k_{12}} \quad \text{and} \quad s_2 = \frac{1 + k_{12}}{2 + k_{12}} \]

Just as in Section 5, we calculate the integration constants. The results are

\[ \hat{C}_1 = \frac{-\pi}{2\sqrt{3} \sin(\pi\mu_0/2)} \left( \frac{1 + k_{12}}{2 + k_{12}} \right) \] 

\[ \hat{C}_2(2) = \frac{-i\pi^2}{6 \sin(\pi\mu_0)} \left( \frac{1 + 2k_{12}}{2 + k_{12}} \right) \]

\[ \hat{C}_3 = (\hat{\ell} + 1) \frac{-\pi^2 \cot(\pi\mu_0)}{12 \sin(\pi\mu_0)} \left( \frac{1 + 2k_{12}}{2 + k_{12}} \right) \]

\[ \quad + \left( \frac{24}{2 + k_{12}} \right)^2 \left( \frac{1}{18\sqrt{3} \sin(\pi\mu_0)} - \frac{1}{12 \sin(\pi\mu_0)(2 + k_{12})} \right) \]

\[ \quad - \frac{\pi^2(k_{12} - 1)(2k_{12}^3 + 6k_{12}^2 + 7k_{12} + 4)}{12 \sin(\pi\mu_0)(2 + k_{12})^2} \]

The constant \( \hat{C}_3 \) has a small imaginary part when \( k_{12} \neq 1 \). To include the worst case, we use \( |\hat{C}_3| \) in Eq. (67) to obtain the following semi-analytic formula for DRAKONs with general triangular CRELs and parabolic pressure profiles:

\[ \beta^* = \frac{b_{\text{CR}}}{l} \left( \frac{3}{4|\hat{C}_3|} \left( \hat{C}_2 + \frac{16|\hat{C}_3|}{3} \right)^{1/2} + \hat{C}_1 \right)^{1/2} - 0.0013 \]

where \( b_{\text{CR}}/l = k_1 b_{\text{CR}}/\pi(2 + k_{12}) \).

We will use this equation to find the optimal triangular CREL configuration.

The optimization of \( \beta^* \) goes as follows. For fixed \( R_2 \) and \( \alpha_{\text{CR}} \), larger \( R_1 \) means smaller \( \alpha_{\text{SR}} \) and therefore larger \( \pi\mu_0 \). On the other hand, longer CRELs enhance the Pfirsch–Schlüter current and thus suppress \( \beta^* \).

These two competing effects result in an optimal \( R_1 \).

Figure 18 shows the existence of an optimal \( k_1 b_{\text{CR}} \) for different fixed values of \( k_2 b_{\text{CR}} \). Since our calculation is valid only for small \( ka \), we require that both \( k_1 b_{\text{CR}} \) and \( k_2 b_{\text{CR}} \) be less than 0.5. It turns out that \( k_2 b_{\text{CR}} = 0.5 \) always gives the highest \( \beta^* \). The optimal \( k_1 b_{\text{CR}} \) and \( \beta^* \) for different values of \( \hat{\ell} \) are plotted in Fig. 19. The optimized \( \beta^* \) are slightly higher than those in Fig. 16 for DRAKONs with standard triangular CRELs. These optimized values are also confirmed numerically to third order.

9. CONCLUSIONS AND DISCUSSION

An analytic formula of the second-order equilibrium beta limit for triangular CREL DRAKONs with arbitrary pressure profile has been derived. For a parabolic pressure profile, the predictions of the analytic formula...
are compared with the full third-order numerical results, leading to a semi-analytic formula that gives the beta limits accurately (Fig. 16). The peak beta limit \( \beta^* \) for different \( R_0/b_{CR} \) and \( \tilde{L} = L/l \) are summarized in Fig. 15. Besides depending on the straight section lengths, these beta limits are considerably lower than the first-order ones; for example, for \( R_0/b_{CR} = 2 \) and \( \tilde{L} = 1, \beta^* = 2.2\% \). Pressure profiles with sharper boundaries enhance the Pfirsch-Schlüter current and therefore suppress \( \beta^* \) (Fig. 9). The equilibrium is governed by the predominant quadrupole Pfirsch-Schlüter current. Small deviations from the CREL condition (which is better termed the dipole CREL condition now, since it eliminates only the dipole Pfirsch-Schlüter current outflow from the CRELs) therefore have almost no effects on the equilibrium and thus on the beta limit.

We have also optimized the triangular CREL by increasing the radius \( R_1 \) of the semi-tori at both ends. The optimal beta limits are slightly higher than those for the standard triangular CREL. Other types of CRELs with more complicated system axes [3] may give higher beta limits. In particular, it is desirable to find a CREL that eliminates \( \nu_2^{(2)} \) in the straight sections while keeping the dipole CREL condition satisfied. This can be achieved by imposing the additional condition — the quadrupole CREL condition

\[
\dot{\Omega}^{(2)} = 0 \quad \text{or} \quad \dot{\Omega}^{(2)}(2\pi) = \dot{\Omega}^{(2)}(\pi) = 0 \quad (88)
\]

on the CREL. Such a CREL, which satisfies both the dipole and quadrupole CREL conditions, can be called a second-order CREL. Both the shift and the elliptical distortion will then be independent of the straight section length. In view of Eq. (37) and the dipole CREL condition, the quadrupole CREL condition can also be written as

\[
\int_0^\pi d\zeta D^2(\zeta) = 0 \quad (89)
\]

The beta limit can be estimated from Eq. (81) by ignoring the \( \tilde{L} + 1 \) terms, which results in \( \beta^* = 5.62\% \) (\( R_0/b_{CR} = 2 \)). Whether and how the quadrupole CREL condition can be satisfied by a realistic CREL remains to be answered.

We have implied walls with circular cross-sections in the above optimizations. Even higher betas may be achieved if we shape the wall according to the distorted flux surface such that the wall coincides with the flux surface where the separatrices are located. The beta limit for such a system can be determined together with the stability requirements. However, if the nice straight sections with circular cross-sections are to be kept, transition elements connecting the straight sections and the CRELs with shaped cross-sections are needed. Thus, higher betas come about only at the expense of more complexity. This will be the subject of future research.

### Appendix A

#### DERIVATION OF THE EQUATIONS

For \( \nu_1^{(1)}, \nu_2^{(2)}, \xi_1 \) AND \( \lambda \)

Here we give a brief derivation of Eqs (13) and (16). A more complete derivation can be found in Ref. [7]. In the flux co-ordinates \( (a, \omega, \xi) \), the MHD force balance equation and Ampère’s law lead to (see Ref. [6]):

\[
\left( \frac{\partial}{\partial \omega} + \frac{\partial}{\partial \xi} \right) \nu = \frac{c_0}{\psi} \left( (\sqrt{g})^{(0)} - \sqrt{g} \right) \quad (A1)
\]

\[
\frac{4\pi \partial \omega}{c \partial \omega} = -\sqrt{g} \frac{\partial}{\partial \omega} \left( \frac{g_{12}}{\sqrt{g}} \right) - \psi \frac{\partial}{\partial \omega} \left( \frac{g_{12}}{\sqrt{g}} \right)
+ \sqrt{g} \frac{\partial}{\partial \alpha} \left\{ \sqrt{g} \left[ \frac{g_{22}}{\sqrt{g}} - \left( \frac{g_{22}}{\sqrt{g}} \right)^{(0)} \right] \right. + \psi \left[ \frac{g_{23}}{\sqrt{g}} - \left( \frac{g_{23}}{\sqrt{g}} \right)^{(0)} \right] \right\} \quad (A2)
\]

where the superscript \( (0) \) stands for average over \( \omega \) and \( \xi \) (note that cgs units are used here).

Since \( \xi \) is of at least first order in \( \beta \) and \( \lambda \) is of second order, the terms of \( O(\beta^2) \) in the first harmonic of \( \sqrt{g} \) can be ignored. Equation (4.3) of Ref. [7] (where the elliptical distortion \( \lambda \) is called \( \alpha \)) is then written as

\[
\sqrt{g} = aR - Rka^2 \text{Re} \left[ 2e^{i(\omega + H)} - \xi_1 e^{i(2\omega + H)} \right] \quad (A3)
\]

Putting this and Eq. (12) into Eq. (A1), we readily obtain Eqs (16).

Likewise, Eqs (4.5) and (4.9) of Ref. [7] are simplified to

\[
\frac{\partial}{\partial \alpha} \left\{ \sqrt{g} \left[ \frac{g_{22}}{\sqrt{g}} - \left( \frac{g_{22}}{\sqrt{g}} \right)^{(0)} \right] \right. + \psi \left[ \frac{g_{23}}{\sqrt{g}} - \left( \frac{g_{23}}{\sqrt{g}} \right)^{(0)} \right] \right\} - \sqrt{g} \frac{\partial}{\partial \omega} \left( \frac{g_{12}}{\sqrt{g}} \right)
\]

\[
= -\frac{aX_1}{R} \text{Re} \left[ e^{i\omega} \left( \xi_1 + \frac{3\xi_1}{a} - ke^{iH} \right) + e^{i2\omega} \left( a'' + \frac{3a'}{a} - \frac{3a}{a^2} \right) \right] \quad (A4)
\]
of the same harmonics, we obtain for the first harmonic

\[
\left( \mu - i \frac{\partial}{\partial \zeta} \right) \left( \zeta_1' + \frac{3}{a} \zeta_1'' \right) - \mu k e^{iH} = -\frac{i 2 R 4 \pi}{a \psi} \nu_1^{(1)}
\]

The $\mu k$ term is negligible in this order since, to lowest order (Eq. (14)), it involves $\kappa_0 k$ and $Jk$. Equation (13a) then follows. The second harmonic of Eq. (A2) yields Eq. (13b).

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