Three laws of feedback systems: entropy rate never decreases, generalized Bode integral, absolute lower bound in variance minimization, Gaussianity-whiteness measure (joint Shannon-Wiener entropy), Gaussianing-whitening control, generalized Bode plot, and beyond

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Abstract: This paper aims at obtaining universal laws and absolute lower bounds of feedback systems using information theory. The feedback system setup is that with causal plants and causal controllers. Three laws (entropy rate never decreases, generalized Bode integral, and absolute lower bound in variance minimization) are obtained, which are in entropy domain, frequency domain, and time domain, respectively. Those laws characterize the fundamental limitations of such systems imposed by the feedback mechanism. Two new notions, negentropy rate and Gaussianity-whiteness measure (joint Shannon-Wiener entropy), are proposed to facilitate the analysis. Topics such as whiteness-Gaussianity-variance decomposition, Gaussianing-whitening control (the maximum Gaussianity-whiteness measure principle), whitening control, generalized Bode plot, and so on are also discussed. The special case of linear time-invariant feedback systems is considered in the end.

Keywords: Three laws of feedback systems, entropy rate never decreases, generalized Bode integral, generalized Bode plot, absolute lower bound in variance minimization, negentropy rate, Gaussianity-whiteness measure, whiteness-Gaussianity-variance decomposition, joint Shannon-Wiener entropy, Gaussianing-whitening control, whitening control.

1. INTRODUCTION

Information theory is mainly used in communication to obtain the performance limitations of channels in terms of channel capacity, etc. [Cover and Thomas (2006)]. It turned out that information theory can also be used in feedback control system analysis [Elia (2004)]. Recently, Martins and co-workers [Martins et al. (2007); Martins and Dahleh (2008)] used information theory to study the performance limitations of many feedback control systems, and developed Bode-like integrals for such systems. Their paper inspired a lot of ensuing work [Okano et al. (2008); Ishii et al. (2009); Yu and Mehta (2010); Lestas et al. (2010); Zhao et al. (2014); Li and Hovakimyan (2013)], etc. We should say that our results are also inspired by and based on [Martins et al. (2007); Martins and Dahleh (2008)] to a large extent.

This paper aims at obtaining universal laws and absolute lower bounds of feedback systems using information theory. The feedback system setup is that with causal plants and causal controllers. Section
2 introduces the necessary notations and preliminaries. Section 3 gives the definitions of negentropy and Gaussianity-and-whiteness measure. Necessary interpretations and specifications are also provided. In Section 4, three laws together with three lower bounds are obtained, which are in entropy domain, frequency domain, and time domain respectively. Those laws and lower bounds characterize the fundamental limitations of such systems imposed by the feedback structure. We will also consider linear time-invariant feedback systems as a special case. Conclusions are given in Section 5.

2. NOTATIONS AND PRELIMINARIES

In this section we collect some key definitions and preliminary results from information theory ([Pinsker (1964); Papoulis and Pillai (2002); Cover and Thomas (2006)]). We consider real-valued continuous random variables and discrete-time stochastic processes. The logarithm used in this paper is that with base 2, and all the integrals herein are over appropriate sets of the variables. Besides, for a stochastic process \( \{x_k\} \), \( x_1\ldots,k \) is the abbreviated notation of \( x_1, x_2 \ldots, x_k \), and this abbreviation will be adopted throughout this paper.

**Definition 2.1** The differential entropy of a random variable \( x, x \in \mathbb{R} \) with density \( f(a) \) is defined as
\[
h(x) = -\int f(a) \log f(a) da.
\]

**Definition 2.2** The differential entropy of a set of random variables \( x_1, x_2, \ldots, x_k, x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \ldots, x_k \in \mathbb{R} \) with joint density \( f(a_1, a_2, \ldots, a_k) \) is defined as
\[
h(x_1, x_2, \ldots, x_k) = -\int f(a_1, a_2, \ldots, a_k) \log f(a_1, a_2, \ldots, a_k) da_1 da_2 \ldots da_k.
\]

**Definition 2.3** The conditional differential entropy of two random variables \( x, y, x, y \in \mathbb{R} \) with joint density \( f(a, b) \) and conditional density \( f(a|b) \) is defined as
\[
h(x|y) = -\int f(a, b) \log f(a|b) da db.
\]

**Definition 2.4** The mutual information between two random variables \( x, y, x, y \in \mathbb{R} \) with joint density \( f(a, b) \) is defined as
\[
I(x; y) = \int f(a, b) \log \frac{f(a, b)}{f(a)f(b)} da db.
\]

**Definition 2.5** The entropy rate of a stochastic process \( \{x_k\}, x_k \in \mathbb{R} \) is defined as
\[
h_{\infty}(x) = \limsup_{k \to \infty} \frac{h(x_1, \ldots, k)}{k}.
\]

**Definition 2.6** The directed information rate [Massey (1990); Kramer (1998)] from stochastic process \( \{x_k\}, x_k \in \mathbb{R} \) to stochastic process \( \{y_k\}, y_k \in \mathbb{R} \) is defined as
\[
I_{\infty}(x \rightarrow y) = \limsup_{K \to \infty} \frac{\sum_{k=1}^{K} I(y_k; x_1, \ldots, k|y_1, \ldots, k-1)}{K},
\]
where \( I(y_k; x_1, \ldots, k|y_1, \ldots, k-1) = I(y_1, \ldots, k; x_1, \ldots, k) - I(y_1, \ldots, k-1; x_1, \ldots, k) \).
Definition 2.7 A zero-mean stochastic process \( \{x_k\}, x_k \in \mathbb{R} \) is asymptotically stationary if the following limit exists for every \( k \):
\[
R_x(k) = \lim_{i \to \infty} E[x_i x_{i+k}].
\]
For an asymptotically stationary \( \{x_k\} \), its asymptotic power spectrum is defined as
\[
S_x(\omega) = \sum_{k=\infty}^{\infty} R_x(k)e^{-jk\omega}.
\]

Definition 2.8 The spectral flatness measure (Wiener entropy) of an asymptotically stationary process \( \{x_k\}, x_k \in \mathbb{R} \) is defined as
\[
\gamma_x^2 = \frac{2}{\pi} \int_{-\infty}^{\infty} \log S_x(\omega) d\omega - \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) d\omega,
\]
where \( S_x(\omega) \) is the asymptotic power spectrum of \( \{x_k\} \).

Remark 2.1 Spectral flatness measure is a very important tool to describe the shape of the power spectral density for a asymptotically stationary process by a single value. It is known that \( 0 \leq \gamma_x^2 \leq 1 \), and that \( \gamma_x^2 = 1 \) if and if \( \{x_k\} \) is white. As such, \( \gamma_x^2 \) is a measurement of whiteness for asymptotically stationary processes.

Definition 2.9 ([Hyvärinen and Oja (2000)]) The negentropy of a random variable \( x, x \in \mathbb{R} \) is given by
\[
J(x) = h(x_G) - h(x),
\]
where \( x_G \) is a Gaussian variable with the same variance as \( x \).

Remark 2.2 It is known that \( J(x) \geq 0 \), and that the equality holds if and only if \( x \) is Gaussian. As such, negentropy is a measurement of non-Gaussianity for random variables.

Remark 2.3 Suppose that the variance of \( x, x \in \mathbb{R} \) is given by \( \sigma_x^2 \). According to [Cover and Thomas (2006)], \( h(x_G) = \log \sqrt{2\pi e} \sigma_x^2 \), so
\[
J(x) = h(x_G) - h(x) = \log \sqrt{2\pi e} \sigma_x^2 - h(x).
\]

The following lemma lists the key properties of entropy and mutual information relevant to our subsequent development:

Lemma 2.1 ([Pinsker (1964); Cover and Thomas (2006)])
1. \( I(x; y) = I(y; x) = h(x) - h(x|y) = h(y) - h(y|x) \geq 0 \), in which equality holds if and only if \( x \) and \( y \) are independent.
2. \( h(x|y) \leq h(x) \), in which equality holds if and only if \( x \) and \( y \) are independent.
3. \( h(x, y) = h(x) + h(y|x) \).
4. Suppose that \( f \) is a causal function which is measurable on the appropriate space, then \( h(x|y) \leq h(x|f(y)) \), in which equality holds if and only if \( f \) is invertible.
5. Suppose that \( f \) is a causal function which is measurable on the appropriate space, then \( h(x|y) = h(x + f(y)|y) \), \( h(x|y) = h(x|f(y)) \), and \( I(x; y|z) = I(x; y + f(z)|z) \).
6. \( I(x; y|z) = I(y; x|z) = h(x|z) - h(x|y, z) = h(y|z) - h(y|x, z) \geq 0 \), in which equality holds if and only if \( x \) and \( y \) are independent given \( z \).
7. \( I(x; y, z) = I(x; y) + I(x; z|y) \), and if \( y, z \) are independent of each other, then \( I(x; y, z) \geq I(x; y) + I(x; z) \).
(8) \( I(x; y) \leq I(x; y, z) \), and \( h(x|y) \geq h(x|y, z) \). For both, equality holds if and only if \( x \) and \( z \) are independent given \( y \).

(9) \( h(x_1, \ldots, k) = \sum_{i=1}^{k} h(x_i| x_1, \ldots, i-1) \), and \( h(x_1, \ldots, k|y) = \sum_{i=1}^{k} h(x_i| x_1, \ldots, i-1, y) \).

(10) \( I(x_1, \ldots, k; y) = \sum_{i=1}^{k} I(x_i; y|x_1, \ldots, i-1) \), and \( I(x_1, \ldots, k; y|z) = \sum_{i=1}^{k} I(x_i; y|x_1, \ldots, i-1, z) \).

3. NEGENTROPY RATE AND GAUSSIANITY-WHITENESS MEASURE

3.1 Negentropy Rate

**Definition 3.1** The negentropy rate of an asymptotically stationary process \( \{x_k\}, x_k \in \mathbb{R} \) is defined as

\[
J_\infty(x) \triangleq h_\infty(x_G) - h_\infty(x),
\]

where \( \{x_G(k)\}, x_G(k) \in \mathbb{R} \) is a Gaussian process with the same asymptotic power spectrum as \( \{x_k\} \).

**Lemma 3.1** Suppose that \( \{x_k\}, x_k \in \mathbb{R} \) is asymptotically stationary with asymptotic power spectrum \( S_x(\omega) \), then

\[
J_\infty(x) = h_\infty(x_G) - h_\infty(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_x(\omega)} d\omega - h_\infty(x).
\]

Furthermore, \( J_\infty(x) \geq 0 \), and the equality holds if and only if \( \{x_k\} \) is Gaussian.

**Proof.** It is known from [Martins and Dahleh (2008)] that

\[
h_\infty(x) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_x(\omega)} d\omega,
\]

in which equality holds if and only if \( \{x_k\} \) is Gaussian. Then since \( S_{x_G}(\omega) = S_x(\omega) \), we have

\[
h_\infty(x_G) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_{x_G}(\omega)} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_x(\omega)} d\omega.
\]

As a result,

\[
J_\infty(x) = h_\infty(x_G) - h_\infty(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_x(\omega)} d\omega - h_\infty(x).
\]

Furthermore, \( J_\infty(x) \geq 0 \), and the equality holds if and only if \( \{x_k\} \) is Gaussian. □

**Theorem 3.1** Consider a single-input single-output (SISO) linear time-invariant (LTI) stable system \( L(z) \). Let its input \( \{x_k\}, x_k \in \mathbb{R} \) be an asymptotically stationary process. Then its output \( \{y_k\}, y_k \in \mathbb{R} \) would also be an asymptotically stationary process, and it always holds that

\[
J_\infty(y) = J_\infty(x).
\]

In other words, LTI stable systems do not change the non-Gaussianity (Gaussianity) of asymptotically stationary processes.

**Proof.** As \( L(z) \) is LTI stable, and its input process \( \{x_k\} \) is asymptotically stationary, the output process \( \{y_k\} \) is also asymptotically stationary. Furthermore, it holds that [Papoulis and Pillai (2002)]

\[
h_\infty(y) = h_\infty(x) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \log L(e^{j\omega}) d\omega.
\]

Then since

\[
S_y(\omega) = [L(e^{j\omega})]^2 S_x(\omega),
\]

we have

\[
J_\infty(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_y(\omega)} d\omega - h_\infty(y)
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e [L(e^{j\omega})]^2 S_x(\omega)} d\omega - h_\infty(x) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log L(e^{j\omega}) d\omega
\]
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_x(\omega)} d\omega - h_{\infty}(x) = J_{\infty}(x).
\]

3.2 Gaussianity-Whiteness Measure

**Definition 3.2** Consider an asymptotically stationary process \(\{x_k\}, x_k \in \mathbb{R}\) with spectral flatness measure \(\gamma^2_x\) and negentropy rate \(J_{\infty}(x)\). Its **Gaussianity-whiteness measure** is defined as

\[
GW_x \triangleq \gamma^2_x \cdot 2^{-2J_{\infty}(x)}
\]  

(4)

**Remark 3.1** Since \(J_{\infty}(x) \geq 0\) and \(0 \leq \gamma^2_x \leq 1\), we have \(0 \leq GW_x \leq 1\). Besides, as \(2^{-2J_{\infty}(x)}\) is a measure of Gaussianity (since \(J_{\infty}(x)\) is a measure of non-Gaussianity) and \(\gamma^2_x\) is a measure of whiteness, \(GW_x\) is a measure of Gaussianity and whiteness. It is also easy to see that \(GW_x = 1\) if and only if \(\{x_k\}\) is Gaussian and white. A similar notion is proposed as generalized spectral flatness measure for non-Gaussian linear processes in [Dubnov (2004)].

**Remark 3.2** The Gaussianity-whiteness measure can be readily used in system identification (for model order/structure selection, model validation, or for parameter estimation), diagnosis and monitoring, etc.

**Remark 3.3** Considering the fact that \(J_{\infty}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_x(\omega)} d\omega - h_{\infty}(x)\), and \(J_{\infty}(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_y(\omega)} d\omega - h_{\infty}(y)\),

Fig. 1. A stationary process passing through a filter

**Theorem 3.2** Consider a SISO filter \(F\) as given in Figure 1. Let its input \(\{x_k\}, x_k \in \mathbb{R}\) be an asymptotically stationary process. Further assume that the output \(\{y_k\}, y_k \in \mathbb{R}\) is also asymptotically stationary, and that \(h_{\infty}(y) = h_{\infty}(x) + \Delta_F\).

Then

\[
\lim_{k \to \infty} E\{y_k^2\} = \frac{GW_x}{GW_y} \cdot 2^{2\Delta_F},
\]

(6)

where \(GW_x\) and \(GW_y\) are the Gaussianity-whiteness measures of \(\{x_k\}\) and \(\{y_k\}\) respectively. So \(\lim_{k \to \infty} E\{y_k^2\}\) is lower bounded by \(GW_x \cdot 2^{2\Delta_F} \cdot \lim_{k \to \infty} E\{x_k^2\}\), and the lower bound is achieved if and only if \(GW_y = 1\). Furthermore, if \(0 < GW_y < 1\), then \(\lim_{k \to \infty} E\{y_k^2\}\) can still be made \(\frac{1}{GW_y}\) times smaller.

**Proof.** Since

\[
J_{\infty}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_x(\omega)} d\omega - h_{\infty}(x),
\]

and

\[
J_{\infty}(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_y(\omega)} d\omega - h_{\infty}(y),
\]

we can rewrite

\[ h_\infty (y) = h_\infty (x) + \Delta F \]

as

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_y(\omega)}d\omega - J_\infty (y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_x(\omega)}d\omega - J_\infty (x) + \Delta F,
\]

or equivalently

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{S_y(\omega)}{S_x(\omega)}d\omega = 2J_\infty (y) - 2J_\infty (x) + 2\Delta F.
\]

Then as

\[
\gamma_x^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_x(\omega)d\omega
\]

and

\[
\gamma_y^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log S_y(\omega)d\omega,
\]

it follows that

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} S_y(\omega)d\omega = \frac{2}{2\pi} \int_{-\pi}^{\pi} \log S_x(\omega)d\omega \cdot \frac{\gamma_x^2}{\gamma_y^2} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{S_y(\omega)}{S_x(\omega)}d\omega \cdot \frac{\gamma_x^2}{\gamma_y^2} = 2\gamma_x^2(1 - 2J_\infty (x) + 2\Delta F) \cdot \frac{\gamma_x^2}{\gamma_y^2}
\]

Also noting that

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} S_x(\omega)d\omega = R_x (0) = \lim_{k \to \infty} E \{ x_k^2 \},
\]

and

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} S_y(\omega)d\omega = R_y (0) = \lim_{k \to \infty} E \{ y_k^2 \},
\]

so

\[
\lim_{k \to \infty} \frac{E \{ y_k^2 \}}{\lim_{k \to \infty} E \{ x_k^2 \}} = \frac{\gamma_x^2}{\gamma_y^2} \cdot \frac{2^{-2J_\infty (x)}}{2^{-2J_\infty (y)}} \cdot 2^{2\Delta F} = \frac{GW_x}{GW_y} \cdot 2^{2\Delta F}.
\]

\[ \square \]

**Theorem 3.3** For an asymptotically stationary process \( \{ x_k \}, x_k \in \mathbb{R} \), its entropy rate power \( \frac{1}{2\pi e} \cdot 2^{2h_\infty (x)} \) can be decomposed into

\[
\frac{1}{2\pi e} \cdot 2^{2h_\infty (x)} = \gamma_x^2 \cdot 2^{-2J_\infty (x)} \cdot \lim_{k \to \infty} E \{ x_k^2 \} = GW_x \cdot \lim_{k \to \infty} E \{ x_k^2 \},
\]

which can be referred as whiteness-Gaussianity-variance decomposition.

4. UNIVERSAL LAWS AND ABSOLUTE LOWER BOUNDS OF FEEDBACK SYSTEMS

4.1 General System Set-up

Consider the SISO feedback system depicted in Figure 1. In this setup, the reference signal \( \{ r_k \} \) is assumed to be known, and \( \{ y_k \} \) denotes the plant’s output.
It is assumed that the additive disturbance $d_k$ does not depend on $z_k$.

Besides, it is assumed that the open-loop system is deterministic and strictly causal, that is, at any time constant $k$,

$$z_k = KP_k (y_{k-1}, y_{k-2}, \ldots, y_0, r_{k-1}, r_{k-2}, \ldots, r_0),$$

where $KP$ is the cascaded system of $K$ and $P$.

**Theorem 4.1 (Entropy domain)**

$$h_\infty(y) - h_\infty(d) = I_\infty(z \to y). \quad (8)$$

**Proof.** Since

$$h(y_k|y_0,\ldots,y_{k-1},z_0,\ldots,k) = h(y_k|y_0,\ldots,k-1) - I(y_k;z_0,\ldots,k|y_0,\ldots,k-1),$$

we have

$$\sum_{k=0}^K h(y_k|y_0,\ldots,y_{k-1},z_0,\ldots,k) = \sum_{k=0}^K h(y_k|y_0,\ldots,k-1) - \sum_{k=0}^K I(y_k;z_0,\ldots,k|y_0,\ldots,k-1)$$

$$= h(y_0,\ldots,k) - \sum_{k=0}^K I(y_k;z_0,\ldots,k|y_0,\ldots,k-1). \quad (9)$$

Besides, since

$$h(d_k|d_0,\ldots,k-1) - h(d_k|d_0,\ldots,k-1, z_0,\ldots,k) = I(d_k;z_0,\ldots,k|d_0,\ldots,k-1),$$

we have

$$\sum_{k=0}^K h(d_k|d_0,\ldots,k-1) - \sum_{k=0}^K h(d_k|d_0,\ldots,k-1, z_0,\ldots,k) = h(d_0,\ldots,k) - \sum_{k=0}^K h(d_k|d_0,\ldots,k-1, z_0,\ldots,k)$$

$$= \sum_{k=0}^K I(d_k;z_0,\ldots,k|d_0,\ldots,k-1).$$

Then as

$$I(d_k;z_0,\ldots,k|d_0,\ldots,k-1) = h(z_0,\ldots,k|d_0,\ldots,k-1) - h(z_0,\ldots,k|d_0,\ldots,k)$$

and $d_k$ does not depend on $z_0,\ldots,k$, we have

$$I(d_k;z_0,\ldots,k|d_0,\ldots,k-1) = h(z_0,\ldots,k|d_0,\ldots,k-1) - h(z_0,\ldots,k|d_0,\ldots,k)$$

$$= h(z_0,\ldots,k|d_0,\ldots,k-1) - h(z_0,\ldots,k|d_0,\ldots,k-1) = 0.$$

Thus

$$h(d_0,\ldots,k) = \sum_{k=0}^K h(d_k|d_0,\ldots,k-1, z_0,\ldots,k). \quad (10)$$
Next, as
\[ y_k = d_k + z_k, \]
it can be obtained that
\[ h(y_k | y_{0, \ldots, k-1}, z_{0, \ldots, k}) = h(d_k + z_k | d_{k-1} + z_{k-1}, \ldots, d_1 + z_1, d_0, z_0, \ldots, k) \]
\[ = h(d_k | d_{k-1} + z_{k-1}, \ldots, d_1 + z_1, d_0, z_0, \ldots, k) \]
\[ = h(d_k | d_0, \ldots, k-1, z_0, \ldots, k). \quad (11) \]

By combining (9), (10), and (11), we have
\[ h(y_0, \ldots, K) = h(d_0, \ldots, K) + \sum_{k=0}^{K} I(y_k; z_0, \ldots, k | y_0, \ldots, k-1). \]

Then it follows that
\[ h_\infty(y) - h_\infty(d) = \limsup_{K \to \infty} \frac{h(y_0, \ldots, K)}{K + 1} - \limsup_{K \to \infty} \frac{h(d_0, \ldots, K)}{K + 1} = \limsup_{K \to \infty} \frac{\sum_{k=0}^{K} I(y_k; z_0, \ldots, k | y_0, \ldots, k-1)}{K + 1} = I_\infty(z \to y). \]

Corollary 4.1 (Entropy domain) From Theorem 4.1 it can be obtained that
\[ h_\infty(y) \geq h_\infty(d), \quad (12) \]
where equality holds if and only if \( I_\infty(z \to y) = 0. \)

Remark 4.1 The conclusion in Corollary 4.1 can be viewed as a law that **entropy rate never decreases** in such feedback systems as given in Figure 2. In [Wiener (1950)], it is stated that “In control and communication we are always fighting nature’s tendency to degrade the organized and to destroy the meaningful; the tendency, as Gibbs has shown us, for entropy to increase.” Then one more sentence could be added: “But in feedback systems, the fight would at best be a draw. We shall never win.”

Theorem 4.2 (Frequency domain) Further assume that \( \{d_k\} \) is asymptotically stationary, and the causal controller \( K \) stabilizes the plant in the sense that \( \{y_k\} \) is asymptotically stationary, then
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{\frac{S_y(\omega)}{S_d(\omega)}} d\omega = J_\infty(y) - J_\infty(d) + I_\infty(z \to y). \]

Proof. It is known from Theorem 4.1 that
\[ h_\infty(y) = h_\infty(d) + I_\infty(z \to y). \]

Then as
\[ J_\infty(d) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_d(\omega)} d\omega - h_\infty(d), \]
and
\[ J_\infty(y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_y(\omega)} d\omega - h_\infty(y), \]
we have
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{\frac{S_y(\omega)}{S_d(\omega)}} d\omega = J_\infty(y) - J_\infty(d) + I_\infty(z \to y). \]

□

Corollary 4.2 (Frequency domain) From Theorem 4.2 it can be obtained that
\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{\frac{S_y(\omega)}{S_d(\omega)}} d\omega \geq -J_\infty(d), \]
where equality holds if and only if $J_\infty (y) = 0$ ($\{y_k\}$ is Gaussian) and $I_\infty (z \to y) = 0$.

**Remark 4.2** The results in Theorem 4.2 and Corollary 4.2 can be viewed as **generalized Bode integrals** for such feedback systems as given in Figure 2. It is worth pointing out that those integrals are given in terms of the relevant signals' properties, rather than the system's properties as in classical Bode integral [Sung and Hara (1988)]. Indeed, the plot of $\log \left( \frac{S_y (\omega)}{S_d (\omega)} \right)$ on a log scale of $\omega$ can also be viewed as **generalized Bode plot**.

**Theorem 4.3** (Time domain) Still assume that $\{d_k\}$ is asymptotically stationary, and the causal $K$ stabilizes the plant in the sense that $\{y_k\}$ is asymptotically stationary. Let $\sigma_d^2 \triangleq \lim_{k \to \infty} E \{d_k^2\}$ and $\sigma_y^2 \triangleq \lim_{k \to \infty} E \{y_k^2\}$. Then

$$\sigma_y^2 = \frac{1}{GW_y} \cdot 2^{2J_\infty (z \to y)} \cdot GW_d \cdot \sigma_d^2. \quad (13)$$

**Proof.** It is known from Theorem 4.2 that

$$\frac{1}{2\pi} \int_{-\pi}^\pi \log \sqrt{\frac{S_y (\omega)}{S_d (\omega)}} d\omega \geq J_\infty (y) - J_\infty (d) + I_\infty (z \to y).$$

Then as

$$\gamma_d^2 = \frac{1}{2\pi} \int_{-\pi}^\pi S_d (\omega) d\omega, \quad \gamma_y^2 = \frac{1}{2\pi} \int_{-\pi}^\pi S_y (\omega) d\omega,$$

and

$$\frac{1}{2\pi} \int_{-\pi}^\pi S_d (\omega) d\omega = R_d (0) = \lim_{k \to \infty} E \{d_k^2\}, \quad \frac{1}{2\pi} \int_{-\pi}^\pi S_y (\omega) d\omega = R_y (0) = \lim_{k \to \infty} E \{y_k^2\},$$

it follows that

$$\sigma_y^2 = \lim_{k \to \infty} E \{y_k^2\}, \quad \sigma_d^2 = \lim_{k \to \infty} E \{d_k^2\},$$

$$\gamma_d^2 = \frac{1}{2\pi} \int_{-\pi}^\pi S_d (\omega) d\omega, \quad \gamma_y^2 = \frac{1}{2\pi} \int_{-\pi}^\pi S_y (\omega) d\omega,$$

$$= \frac{2^{2J_\infty (y) - 2J_\infty (d) + 2I_\infty (z \to y)}}{\gamma_y^2} \cdot \frac{\gamma_d^2 \cdot 2^{-2J_\infty (d)}}{\gamma_y^2} \cdot \frac{2^{2J_\infty (z \to y)}}{\gamma_y^2} = \frac{GW_d}{GW_y} \cdot 2^{2I_\infty (z \to y)}.$$

Thus

$$\sigma_y^2 = \frac{1}{GW_y} \cdot 2^{2I_\infty (z \to y)} \cdot GW_d \cdot \sigma_d^2.$$

□

**Corollary 4.3** (Time domain) From Theorem 4.3 it can be obtained that

$$\sigma_y^2 \geq GW_d \cdot \sigma_d^2, \quad (14)$$

where equality holds if and only if $GW_y = 1$ ($\{y_k\}$ is Gaussian white) and $I_\infty (z \to y) = 0$.

**Remark 4.3** Corollary 4.3 gives the absolute lower bound of steady-state variance minimization or steady-state disturbance attenuation [Astrom and Murray (2008)] of such feedback systems as given in Figure 2. And the lower bound can only be achieved when $\{y_k\}$ is Gaussian white. In this sense, steady-state variance minimization control can also be viewed as **Gaussianizing-whitening control**, or as obeying a maximum Gaussianity-whiteness measure principle.

**Corollary 4.4** If $GW_d = 1$, that is, if $\{d_k\}$ is Gaussian white, then

$$\sigma_y^2 \geq \sigma_d^2. \quad (15)$$
Remark 4.4 Considering Corollary 4.4, Gaussian white noise may accordingly be considered as the worst disturbance.

Remark 4.5 Comparison of the results above with those corresponding ones (which are straightforward to obtain) can bring about deeper insights into all those conclusions.

4.2 LTI Plants with LTI Stabilizing Controllers

Consider the system depicted in Figure 3. Let the SISO plant $P$ be an LTI system, and its transfer function is given by $P(z)$. The controller $K$ is also assumed to be LTI, and its transfer function is given by $K(z)$. The transfer function of the open-loop system $K(z)P(z)$ is assumed to be strictly proper. The reference $r$ is still assumed to be known.

![Fig. 3. An LTI feedback system](image)

Theorem 4.4 (Entropy domain) Suppose that the closed-loop system is stable, then

$$h_{\infty}(y) - h_{\infty}(d) = \sum_{\lambda \in \lambda_{KP}} \max\{0, \log |\lambda|\}, \quad (16)$$

(Frequency domain) And

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \frac{S_y(\omega)}{S_d(\omega)} \right| d\omega = \sum_{\lambda \in \lambda_{KP}} \max\{0, \log |\lambda|\}, \quad (17)$$

where $\lambda_{KP}$ is the set of poles of the open-loop transfer function $K(z)P(z)$.

**Proof.** It is known from [(Sung and Hara, 1988)] that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \frac{1}{1 + K(\omega)P(\omega)} \right| d\omega = \sum_{\lambda \in \lambda_{KP}} \max\{0, \log |\lambda|\}.$$

Then as

$$\frac{S_y(\omega)}{S_d(\omega)} = \left| \frac{1}{1 + K(\omega)P(\omega)} \right|^2,$$

it can be obtained that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_y(\omega)} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_d(\omega)} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{S_y(\omega)}{S_d(\omega)} d\omega$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left| \frac{1}{1 + K(\omega)P(\omega)} \right| d\omega = \sum_{\lambda \in \lambda_{KP}} \max\{0, \log |\lambda|\}.$$

In light of Theorem 3.1, we have $J_{\infty}(y) = J_{\infty}(d)$, so

$$h_{\infty}(y) - h_{\infty}(d) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_y(\omega)} + J_{\infty}(y) - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_d(\omega)} d\omega - J_{\infty}(d)$$

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_y(\omega)} - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{2\pi e S_d(\omega)} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{S_y(\omega)}{S_d(\omega)} d\omega = \sum_{\lambda \in \lambda_{KP}} \max\{0, \log |\lambda|\}.$$
Theorem 4.5 (Time domain) Still suppose that the closed-loop system is stable. Let $\sigma_y^2 \triangleq \lim_{k \to \infty} \mathbb{E} \{d_k^2\}$ and $\sigma_y^2 \triangleq \lim_{k \to \infty} \mathbb{E} \{y_k^2\}$. Then following Theorem 4.4, we have

$$
\sigma_y^2 = \frac{1}{\gamma_y^2} \left( \prod_{\lambda \in \lambda_K^P} \max \{1, |\lambda|\} \right)^2 \cdot \gamma_d^2 \cdot \sigma_d^2.
$$

Thus

$$
\sigma_y^2 \geq \left( \prod_{\lambda \in \lambda_K^P} \max \{1, |\lambda|\} \right)^2 \cdot \gamma_d^2 \cdot \sigma_d^2, \tag{18}
$$

and equality holds if and only if $\gamma_y^2 = 1$ ($\{y_k\}$ is white).

Proof. It is known from Theorem 4.4 that

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \sqrt{\frac{S_y(\omega)}{S_d(\omega)}} d\omega = \sum_{\lambda \in \lambda_K^P} \max \{0, \log |\lambda|\}.
$$

Then as

$$
\gamma_d^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_d(\omega) d\omega, \quad \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega
$$

and

$$
\frac{1}{2\pi} \int_{-\pi}^{\pi} S_d(\omega) d\omega = R_d(0) = \lim_{k \to \infty} \mathbb{E} \{d_k^2\}, \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} S_y(\omega) d\omega = R_y(0) = \lim_{k \to \infty} \mathbb{E} \{y_k^2\},
$$

it follows that

$$
\frac{\sigma_y^2}{\sigma_d^2} = \lim_{k \to \infty} \frac{\mathbb{E} \{y_k^2\}}{\mathbb{E} \{d_k^2\}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_y(\omega) d\omega = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = 2 \frac{\gamma_d^2}{\gamma_y^2} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2 = \frac{2}{\pi} \int_{-\pi}^{\pi} \log S_y(\omega) d\omega = \frac{\gamma_d^2}{\gamma_y^2} \gamma_y^2.
$$

Then as $0 \leq \gamma_y^2 \leq 1$, we have

$$
\sigma_y^2 \geq \left( \prod_{\lambda \in \lambda_K^P} \max \{1, |\lambda|\} \right)^2 \cdot \gamma_d^2 \cdot \sigma_d^2,
$$

and equality holds if and only if $\gamma_y^2 = 1$. □

Remark 4.6 Gaussianing-whitening control is reduced to whitening control in the LTI case, since LTI stable systems do not change the non-Gaussianity of asymptotically stationary processes.

Corollary 4.5 If $\gamma_d^2 = 1$, that is, if $\{d_k\}$ is white, then

$$
\sigma_y^2 \geq \sigma_d^2. \tag{20}
$$

Remark 4.7 Considering Corollary 4.5, white noise may accordingly be considered as the worst disturbance for LTI feedback systems.

Remark 4.8 Following similar procedures, corresponding results can be obtained for LTI feedback systems in which the exogenous disturbance comes into the system at different locations from that in this paper (after the output of the plant), say in the feedback path or before the input of the plant.
5. CONCLUSIONS

In this paper we present some universal laws and absolute lower bounds of feedback systems with causal plants and causal controllers. These laws and lower bounds characterize the fundamental limitations of such systems imposed by the feedback mechanism.

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