ON LENGTHS OF \( H_Z \)-LOCALIZATION TOWERS

SERGEI O. IVANOV AND ROMAN MIKHAILOV

Abstract. In this paper, the \( H_Z \)-length of different groups is studied. By definition, this is the length of \( H_Z \)-localization tower or the length of transfinite lower central series of \( H_Z \)-localization. It is proved that, for a free noncyclic group, its \( H_Z \)-length is \( \geq \omega + 2 \). For a large class of \( \mathbb{Z}[C] \)-modules \( M \), where \( C \) is an infinite cyclic group, it is proved that the \( H_Z \)-length of the semi-direct product \( M \rtimes C \) is \( \leq \omega + 1 \) and its \( H_Z \)-localization can be described as a central extension of its pro-nilpotent completion. In particular, this class covers modules \( M \), such that \( M \rtimes C \) is finitely presented and \( H_2(M \rtimes C) \) is finite.

MSC2010: 55P60, 19C09, 20J06

1. Introduction

Let \( R \) be either a subring of rationals or a cyclic ring. In his fundamental work [5], A.K. Bousfield introduced the concept of \( HR \)-localization. This is a functor in the category of groups, closely related to the functor of homological localization of spaces. In this paper we will study the case \( R = \mathbb{Z} \), that is, \( H_Z \)-localization for different groups.

An \( H_Z \)-map between two groups is a homomorphism which induces an isomorphism on \( H_1 \) and an epimorphism on \( H_2 \). A group \( \Gamma \) is \( H_Z \)-local if any \( H_Z \)-map \( G \rightarrow H \) induces a bijection \( \text{Mor}(H, \Gamma) \cong \text{Mor}(G, \Gamma) \). Recall that ([5], Theorem 3.10) the class of \( H_Z \)-local groups is the smallest class which contains the trivial group and closed under inverse limits and central extensions. Given a group \( G \), the \( H_Z \)-localization

\( \eta : G \rightarrow EG \)

can be uniquely characterized by the following two properties: \( \eta \) is an \( H_Z \)-map and the group \( EG \) is \( H_Z \)-local. These two properties are given as a definition of \( H_Z \)-localization in [5]. It is shown in [5] that, for any \( G \), the \( H_Z \)-localization \( EG \) exists, unique and transfinitely nilpotent.

For a group \( G \), denote by \( \{ \gamma_\tau(G) \} \) the transfinite lower central series of \( G \), defined inductively as \( \gamma_{\tau+1}(G) := [\gamma_\tau(G), G] \) and \( \gamma_\alpha = \bigcap_{\tau < \alpha} \gamma_\tau(G) \) for a limit ordinal \( \alpha \). For a \( G \), we will call the length of transfinite lower series of \( EG \), i.e. the least ordinal \( \tau \), such that \( \gamma_\tau(EG) = 1 \) by \( H_Z \)-length of \( G \) and denote it as \( H_Z \)-length(\( G \)).

Let \( C \) be an infinite cyclic group. A \( \mathbb{Z}[C] \)-module \( M \) is tame if and only if \( M \rtimes C \) is a finitely presented group [3]. If \( M \) is a tame \( C \)-module, then \( \dim_{\mathbb{Q}}(M \otimes \mathbb{Q}) < \infty \) and there exist a generator \( t \in C \) such that the minimal polynomial of the linear map \( t \otimes \mathbb{Q} : M \otimes \mathbb{Q} \rightarrow M \otimes \mathbb{Q} \) is an integral monic polynomial, which is denoted by \( \mu_M \in \mathbb{Z}[x] \) (see [2, Theorem C] and Lemma 4.7). We prove the following.
Theorem. Let $G$ be a metabelian group of the form $G = M \rtimes C$, where $M$ is a tame $\mathbb{Z}[C]$-module and $\mu_M = (x - \lambda_1)^{m_1} \cdots (x - \lambda_l)^{m_l}$ for some distinct complex numbers $\lambda_1, \ldots, \lambda_l$ and $m_i \geq 1$.

1. Assume that the equality $\lambda_i \lambda_j = 1$ holds only if $\lambda_i = \lambda_j = 1$. Then
   \[ HZ\text{-length}(G) \leq \omega. \]

2. Assume that the equality $\lambda_i \lambda_j = 1$ holds only if either $m_i = m_j = 1$ or $\lambda_i = \lambda_j = 1$. Then
   \[ HZ\text{-length}(G) \leq \omega + 1. \]

As a contrast, we give an example of a finitely presented metabelian group of the form $M \rtimes C$, where $M$ is tame, whose $HZ$-length is greater than $\omega + 1$. In the following example, the $\mathbb{Z}[C]$-module $M$ is tame but it does not satisfy the condition of Theorem 5.6. Let
\[ G = \langle a, b, t \mid a^1 = a^{-1}, b^1 = ab^{-1}, [a, b] = 1 \rangle = \mathbb{Z}^2 \rtimes C, \]
where $C$ acts on $\mathbb{Z}^2$ by the matrix \((-1, 1)\). It is shown in Theorem 5.3 that the $HZ$-length of $G$ is $\geq \omega + 2$.

Let $M$ be a tame $\mathbb{Z}[C]$-module and $\mu_M = (x - 1)^m f$, for some $m \geq 0$ and $f \in \mathbb{Z}[x]$ such that $f(1) \neq 0$. Assume that $f = f_1^{m_1} \cdots f_l^{m_l}$ where $f_1, \ldots, f_l \in \mathbb{Z}[x]$ are distinct irreducible monic polynomials. If $f(1) \in \{-1, 1\}$, then $HZ\text{-length}(M \rtimes C) < \omega$. (Corollary 4.17).

Conjecture. If $f(1) \notin \{-1, 1\}$, then $HZ\text{-length}(M \rtimes C) \leq \omega + n$, where
\[ n = \max\{m_i \mid f_1(0) \in \{-1, 1\} \land f_1(1) \notin \{-1, 1\}\} \cup \{0\}. \]

In particular, for any tame $\mathbb{Z}[C]$-module $M$, $HZ\text{-length}(M \rtimes C) < 2\omega$.

It is easy to check that the above theorem together with Corollary 4.17 and Proposition 4.8 imply the conjecture for $n = 0, 1$.

For a group $G$, denote by $\hat{G}$ its pro-nilpotent completion:
\[ \hat{G} := \lim_{\leftarrow n} G/\gamma_n(G). \]

For a finitely generated group $G$, there is a natural isomorphism (Prop. 3.14 [5])
\[ EG/\gamma_\omega(EG) = \hat{G}. \]

Therefore, for finitely generated groups, $HZ$-localization gives a natural extension of the pro-nilpotent completion.

The pro-nilpotent completion of a finitely generated group $G$ is always $HZ$-local and the map $G \to \hat{G}$ induces an isomorphism on $H_1$. Therefore, for such a group, the following conditions are equivalent:
1) the natural epimorphism $EG \to \hat{G}$ is an isomorphism;
2) $HZ$-length of $G \leq \omega$;
3) The natural map $H_2(G) \to H_2(\hat{G})$ is an epimorphism.

A simple example of a group with $HZ$-length $\omega$ is the following. Let
\[ G = \langle a, t \mid a^1 = a^3 \rangle = \mathbb{Z}[1/3] \rtimes C. \]
Here $C$ acts on $\mathbb{Z}[1/3]$ as the multiplication by 3. Then the pro-nilpotent completion has the structure $\hat{G} = \mathbb{Z}_2 \rtimes C$, where the cyclic group $C = \langle t \rangle$ acts on 2-adic integers.
as the multiplication by 3. Looking at the homology spectral sequence for an extension $1 \to \mathbb{Z}_2 \to \hat{G} \to C \to 1$, we obtain $H_2(\hat{G}) = \Lambda^2(\mathbb{Z}_2) \otimes \mathbb{Z}/9 = 0$. Therefore, $EG = \hat{G}$. Since the group $G$ is not pre-nilpotent, $HZ$-length$(G) = \omega$.

The above example is an exception. In most cases, the description of $HZ$-localization as well as the computation of $HZ$-length for a given group is a difficult problem. It is shown in [5] that $HZ$-length of the Klein bottle group $G_{K1} := \langle a, t \mid a^t = a^{-1} \rangle = \mathbb{Z} \times C$ is greater than $\omega$. As a corollary, it is concluded in [3] that $HZ$-length of any non-cyclic free group also is greater than $\omega$. Our Theorem 5.6 implies that $HZ$-length$(G_{K1}) = \omega + 1$ and that the $HZ$-localization $EG_{K1}$ lives in the central extension

$$1 \to \Lambda^2(\mathbb{Z}_2) \to EG_{K1} \to \mathbb{Z}_2 \times \langle t \rangle \to 1,$$

where the action of $t$ on 2-adic integers is negation. Moreover, we give a more explicit description of $EG_{K1}$ in Proposition 9.2. The $HZ$-length of a free non-cyclic group remains a mystery for us, however, we prove the following

**Theorem.** Let $F$ be a free group of rank $\geq 2$. Then $HZ$-length$(F) \geq \omega + 2$.

Briefly recall the scheme of the proof. Consider an extension of the group (1.1) given by presentation

$$\Gamma := \langle a, b, t \mid [a, b], a \rangle = \langle [a, b], b \rangle = 1, \ a^t = a^{-1}, \ b^t = ab^{-1} \rangle \quad (1.2)$$

We follow the Bousfield scheme of comparison of pro-nilpotent completions for a free group and the group $\Gamma$. Consider a free simplicial resolution of $\Gamma$ with $F_0 = F$. Group $\Gamma$ has finite second homology, therefore, $\lim^1$ of its Baer invariants is zero, therefore, $\pi_0$ of the pro-nilpotent completion of the free simplicial resolution equals to the pro-nilpotent completion of $\Gamma$. The group $\Gamma$ has $HZ$-length greater than $\omega + 1$ and the result about $HZ$-length of $F$ follows from natural properties of the $HZ$-localization tower. Observe that the same method does not work for the group (1.1) (as well as for all groups of the type $M \times C$ for abelian $M$), since $\lim^1$ of Baer invariants of $G$ is huge (this follows from proposition 5.5 and an analysis of the tower of Baer invariants for a metabelian group).

The paper is organized as follows. In section 2, we present the theory of relative central extensions, which is a generalisation of the standard theory of central extensions. A non-limit step in the construction of Bousfield’s tower can be viewed as the universal relative extension. Section 2 is technical and introductory, it may be viewed just as a comment to the section 3 of [3]. In section 3 we recall the exact sequences in homology from [9], [10] for central stem-extensions. Observe that the universal relative extensions used for the construction of $HZ$-tower are stem-extensions. Proposition 3.1 gives the main trick: for a cyclic group $C$, $\mathbb{Z}[C]$-module $M$, and a central stem-extension $N \to G \to M \rtimes C$, the composite map $H_3(M \rtimes C) \to (M \times C) \otimes N \to N$ can be decomposed as $H_3(M \times C) \to H_2(M)^C \to H_2(M)_C \to N$. This trick gives a possibility to analyze the homology of the $\omega + 1$-st term of the $HZ$-localization tower for groups of the type $M \times C$.

Using the properties of tame modules we show in section 4 that the question about the $HZ$-length of the group $M \times C$ with a tame $\mathbb{Z}[C]$-module $M$, can be reduced to the same question for the group $(M/N) \rtimes C$, where $N$ is the largest nilpotent submodule of $M$. It is shown in [11] that, for a finitely presented metabelian group $G$, the cokernel (denoted by $H_2(\eta_\omega)(G)$) of the natural map $H_2(G) \to H_2(\hat{G})$ is divisible. Using this property we
conclude that, $HZ$-length of $M \rtimes C$ is not greater than $\omega + 1$ if and only if the composite map
\[
\Lambda^2(\hat{M})^C \to \Lambda^2(\hat{M})_C \to H_2(\eta_\omega)
\]
is an epimorphism (see proposition 5.3). Theorem 5.6 is our main result of section 5. There is a simple condition on a tame $\mathbb{Z}[C]$-module $M$ which implies that $HZ$-length$(M \rtimes C) \leq \omega + 1$. Theorem 5.6 provides a large class of groups for which one can describe $HZ$-localization explicitly. In particular, we show that, if the homology $H_2(M \rtimes C)$ is finite, then the module $M$ satisfies the condition (ii) of Theorem 5.6 and therefore, for such $M$, $HZ$-length$(M \rtimes C) \leq \omega + 1$.

In section 6 we recall the method of Bousfield from [5], which gives a possibility to localize explicitly. In particular, we show that, if the homology $H_2(M \rtimes C)$ is not greater than $\omega + 1$, then the module $M$ satisfies the condition (ii) of Theorem 5.6 and therefore, for such $M$, $HZ$-length$(M \rtimes C) \leq \omega + 1$.

We denote by $\bar{f}n$ a tame $\mathbb{Z}$-module $M$ which implies that $HZ$-length$(\bar{f}n) \geq \omega + 2$. In the last section, as an application of the theory developed in the paper, we give an explicit construction of $EGK_f$.

2. Relative central extensions and $HZ$-localization

Throughout this section $G, H$ denote groups, $f : H \to G$ a homomorphism and $A$ an abelian group.

2.1. (Co)homology of a homomorphism. Consider the continuous map between classifying spaces $Bf : BH \to BG$ and its mapping cone $\text{Cone}(Bf)$. Following Bousfield [5, 2.14], we define homology and cohomology of $f$ with coefficients in $A$ as follows
\[
H_n(f, A) = H_n(\text{Cone}(Bf), A), \\
H^n(f, A) = H^n(\text{Cone}(Bf), A).
\]

Then there are long exact sequences
\[
\cdots \to H_2(H, A) \to H_2(G, A) \to H_2(f, A) \to H_1(H, A) \to H_1(G, A) \to H_1(f, A) \to 0, \\
0 \to H^1(f, A) \to H^1(G, A) \to H^1(H, A) \to H^2(f, A) \to H^2(G, A) \to H^2(H, A) \to \cdots.
\]

In particular, $H_1(f) = \text{Coker}\{H_{ab} \to G_{ab}\}$ and $H_1(f, A) = H_1(f) \otimes A$.

We denote by $\tilde{C}^n(G, A)$ the complex of normalized cochains of $G$ with coefficients in $A$, [18, 6.5.5] by $\partial^n : \tilde{C}^n(G, A) \to \tilde{C}^{n+1}(G, A)$ its differential and by $\tilde{Z}^n(G, A)$ and $\tilde{B}^n(G, A)$ the groups of normalized cocycles and coboundaries. For a homomorphism $f : H \to G$ and an abelian group $A$ we denote by $\tilde{Z}^n(f, A)$ and $\tilde{B}^n(f, A)$ the following subgroups of $\tilde{Z}^n(G, A) \oplus \tilde{C}^{n+1}(H, A)$
\[
\tilde{Z}^n(f, A) = \{(c, \alpha) \mid f^*c = -\partial\alpha\}, \\
\tilde{B}^n(f, A) = \{(-\partial\beta, f^*\beta + \partial\gamma) \mid \beta \in \tilde{C}^{n+1}(G, A), \gamma \in \tilde{C}^{n+2}(H, A)\}.
\]

Since the map $\partial : \tilde{C}^0(H, A) \to \tilde{C}^1(H, A)$ is trivial, we have
\[
\tilde{B}^2(f, A) = \{(-\partial\beta, \beta f) \mid \beta \in \tilde{C}^1(G, A)\}.
\]
Lemma 2.1. For \( n \geq 1 \) there is an isomorphism \( H^n(f, A) \cong \tilde{Z}^n(f, A)/\tilde{B}^n(f, A) \).

Proof. For a space \( X \), we denote by \( C_\bullet(X) \) the complex of integral chains. Then \( C_\bullet(X, A) = C_\bullet(X) \otimes A \) and \( C^\bullet(X, A) = \Hom(C_\bullet(X, A), A) \). For a continuous map \( F : X \to Y \) we denote by \( C_\bullet(F) : C_\bullet(X) \to C_\bullet(Y) \) the induced morphism of complexes. Then there is a natural homotopy equivalence of complexes \( \text{Cone}(C^\bullet(F)) \cong C_\bullet(\text{Cone}(F)) \). It follows that there is a natural homotopy equivalence of complexes

\[
C^\bullet(\text{Cone}(F), A) \cong \text{Cone}(C^\bullet(F), A))[-1].
\]

Denote by \( C^\bullet(G, A) \) the complex of (non-normalised) cochains of the group \( G \). For a homomorphism \( f : H \to G \) we denote by \( C^\bullet(f, A) : C^\bullet(G, A) \to C^\bullet(H, A) \) the induced morphism of complexes. There is a natural homotopy equivalence \( C^\bullet(G, A) \cong C^\bullet(BG, A) \). Moreover, there is a natural homotopy equivalence of complexes of normalised and non-normalised cochains \( C^\bullet(G, A) \cong C^\bullet(G, A) \) because that come from two different functorial resolutions. It follows that there is a natural homotopy equivalence \( \text{Cone}(C^\bullet(f, A)) \cong \text{Cone}(C^\bullet(Bf, A)) \). Combining this with (2.1) we get \( C^\bullet(\text{Cone}(Bf), A) \to \text{Cone}(C^\bullet(f, A))[-1] \). The assertion follows.

2.2. Relative central extensions.

Definition 2.2. A relative central extension of \( G \) by \( A \) with respect to \( f \) is a couple \( \mathcal{E} = (A \to E \xrightarrow{\pi} G, f) \), where \( A \to E \xrightarrow{\pi} G \) is a central extension of \( G \) and \( f : H \to E \) is a homomorphism such that \( \pi f = f \).

\[
\begin{array}{cccc}
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\
& & f & & \pi & & & & \\
H & & & & & & & & \\
\end{array}
\]

Two relative central extensions \((A \xrightarrow{\iota_1} E_1 \xrightarrow{\pi_1} G, \tilde{f}_1)\) and \((A \xrightarrow{\iota_2} E_2 \xrightarrow{\pi_2} G, \tilde{f}_2)\) are said to be equivalent if there exist an isomorphism \( \theta : E_1 \xrightarrow{\cong} E_2 \) such that \( \theta \iota_1 = \iota_2, \pi_2 \theta = \pi_1 \) and \( \theta \tilde{f}_1 = \tilde{f}_2 \).

Let \((c, \alpha) \in \tilde{Z}^2(f, A)\). Consider the central extension \( A \to E_c \to G \) corresponding to the 2-cocycle \( c \). The underlying set of \( E_c \) is equal to \( A \times G \) and the product is given by

\[
(a_1, g_1)(a_2, g_2) = (a_1 + a_2 + c(g_1, g_2), g_1 g_2).
\]

Denote by \( \tilde{f}_\alpha : H \to E_c \) the map given by \( \tilde{f}_\alpha(h) = (\alpha(h), f(h)) \). Note that the equality \( f^*c = -\partial \alpha \) implies the equality

\[
c(f(h_1), f(h_2)) = -\alpha(h_1) + \alpha(h_1 h_2) - \alpha(h_2)
\]

for all \( h_1, h_2 \in H \). It follows that \( \tilde{f}_\alpha \) is a homomorphism. Indeed

\[
\tilde{f}_\alpha(h_1)\tilde{f}_\alpha(h_2) = (\alpha(h_1), f(h_1))(\alpha(h_2), f(h_2)) = (\alpha(h_1) + \alpha(h_2) + c(f(h_1), f(h_2)), f(h_1)f(h_2)) = (\alpha(h_1 h_2), f(h_1 h_2)) = \tilde{f}_\alpha(h_1 h_2).
\]

Then we obtain a relative central extension

\[
\mathcal{E}(c, \alpha) = (A \to E_c \to G, \tilde{f}_\alpha).
\]
Proposition 2.3. The map \((c, \alpha) \mapsto \mathcal{E}(c, \alpha)\) induces a bijection between elements of \(H^2(f, A)\) and equivalence classes of relative central extensions of \(G\) by \(A\) with respect to \(f\).

Proof. Any central extension is equivalent to the extension \(A \to E_c \to G\) for a normalised 2-cocycle \(c\). Hence, it is sufficient to consider only them. Consider a relative central extension \((A \to E_c \to G, \tilde{f})\). Define \(\alpha : H \to A\) so that \(\tilde{f}(h) = (\alpha(h), f(h))\). Since \(\tilde{f}(1) = (0, 1)\), \(\alpha\) is a normalised 1-cochain. Since \(\tilde{f}\) is a homomorphism, we get
\[
(\alpha(h_1) + \alpha(h_2) + c(f(h_1), f(h_2)), f(h_1)f(h_2)) = (\alpha(h_1h_2), f(h_1h_2)).
\]
Thus \(f\alpha = -\partial\alpha\). It follows that any relative central extension is isomorphic to the relative central extension \(\mathcal{E}(c, \alpha)\) for some \((c, \alpha) \in Z^2(f, A)\).

Consider two elements \((c, \alpha), (c', \alpha') \in Z^2(f, A)\) such that \((c', \alpha') - (c, \alpha) = (-\partial\beta, \beta f)\) for some \(\beta \in C^1(G, A)\). It follows that
\[
c'(g_1, g_2) + \beta(g_1) + \beta(g_2) = c(g_1, g_2) + \beta(g_1g_2)
\]
for any \(g_1, g_2 \in G\) and \(\alpha'(h) = \alpha(h) + \beta(f(h))\) for any \(h \in H\). Denote by \(\theta_\beta : E_c \to E_{c'}\) the map given by \(\theta_\beta(a, g) = (a + \beta(g), g)\). Then \(\theta_\beta\) is a homomorphism. Indeed,
\[
\theta_\beta(a_1, g_1)\theta_\beta(a_2, g_2) = (a_1 + \beta(g_1), g_1)(a_2 + \beta(g_2), g_2) = (a_1 + a_2 + \beta(g_1) + \beta(g_2) + c'(g_1, g_2), g_1g_2) = (a_1 + a_2 + \beta(g_1g_2) + c(g_1, g_2), g_1g_2) = \theta_\beta((a_1, g_1)(a_2, g_2)).
\]
Moreover, \(\theta_\beta\) is an isomorphism, because \(\theta_{-\beta}\) is its inverse, and it is easy to see that it is an equivalence of the relative central extensions. It follows that the map \((c, \alpha) \mapsto \mathcal{E}(c, \alpha)\) induces a surjective map from elements of \(H^2(f, A)\) to equivalence classes of relative central extensions.

Consider two elements \((c, \alpha), (c', \alpha') \in Z^2(f, A)\) such that the relative central extensions \(\mathcal{E}(c, \alpha)\) and \(\mathcal{E}(c', \alpha')\) are equivalent. Then there is an equivalence \(\theta : E_c \to E_{c'}\). Since \(\theta\) respects the injections from \(A\), \(\varphi(a, 1) = (a, 1)\) for any \(a \in A\). Since \(\theta\) respects the projections on \(G\), there exist a unique normalised 1-cochain \(\beta : G \to A\) such that \(\theta(0, g) = (\beta(g), g)\) for any \(g \in G\). Using that \(c\) is a normalised 2-cocycle, we obtain
\[
\theta(a, g) = \theta((a, 1)(0, g)) = (a, 1)(\beta(g), g) = (a + \beta(g), g).
\]
Then the fact that \(\theta\) is a homomorphism implies that
\[
c'(g_1, g_2) + \beta(g_1) + \beta(g_2) = c(g_1, g_2) + \beta(g_1g_2),
\]
and hence \(c' - c = -\partial\beta\), and the equality \(\tilde{f}_{\alpha} = \tilde{f}_{\alpha'}\) implies \(c' - \alpha = \beta f\). The assertion follows. 

2.3. Universal relative central extensions. Let \(A_1\) and \(A_2\) be abelian groups. Recall that \(c\) morphism from a central extension \(A_1 \xrightarrow{i_1} E_1 \xrightarrow{\varphi} G\) to a central extension \(A_2 \xrightarrow{i_2} E_2 \xrightarrow{\varphi'} G\) is a couple \((\varphi, \theta)\), where \(\varphi : A_1 \to A_2\) and \(\theta : E_1 \to E_2\) are homomorphisms such that \(\theta i_1 = i_2 \tau\), \(\pi_2\theta = \pi_1\).

Lemma 2.4 (cf. [18 Lemma 6.9.6]). Let \((\varphi, \theta)\) and \((\varphi', \theta')\) be morphisms from a central extension \(A_1 \to E_1 \to G\) to a central extension \(A_2 \to E_2 \to G\). Then the restrictions on the commutator subgroup coincide \(\theta|_{[E_1, E_1]} = \theta'|_{[E_1, E_1]}\).
Proof. For the sake of simplicity we identify $A_1$ with the subgroup of $E_1$ and $A_2$ with the subgroup of $E_2$. Consider the map $\rho : E_1 \to A_2$ given by $\rho(x) = \theta(x)\theta'(x)^{-1}$. Since $A_2$ is central, we get

$$\rho(x)\rho(y) = \theta(x)\theta'(x)^{-1}\rho(y) = \theta(x)\rho(y)\theta'(x)^{-1} = \theta(x)\theta(y)\theta'(y)^{-1}\theta'(x)^{-1} = \theta(xy)\theta'(xy)^{-1} = \rho(xy).$$

Hence $\rho$ is a homomorphism to an abelian group. Thus $\rho|_{(E_1,E_1)} = 1$. □

Lemma 2.5. Let $(\varphi, \theta)$ be a morphism from a central extension $A_1 \rightarrowtail E_1 \twoheadrightarrow G$ to a central extension $A_2 \rightarrowtail E_2 \twoheadrightarrow G$. If $\varphi = 0$, then $A_2 \rightarrowtail E_2 \to G$ splits.

Proof. $\varphi = 0$ implies $\theta_1 = 0$. Since $G$ is a cokernel of $\iota_1$, there exists $s : G \to E_2$ such that $s\pi_1 = \theta$. Then $\pi_2s\pi_1 = \pi_2\theta = \pi_1$. It follows that $\pi_2s = \text{id}$. □

Definition 2.6. A morphism from a relative central extension $(A_1 \rightarrowtail E_1 \twoheadrightarrow G, \tilde{f}_1)$ to a relative central extension $(A_2 \rightarrowtail E_2 \twoheadrightarrow G, \tilde{f}_2)$ is a morphism $(\varphi, \theta)$ from the central extension $A_1 \rightarrowtail E_1 \to G$ to the central extension $A_2 \rightarrowtail E_2 \to G$ such that $\theta\tilde{f}_1 = \tilde{f}_2$. So relative central extensions of $G$ with respect to $f$ form a category. The initial object of this category is called the universal relative central extension of $G$ with respect to $f$.

Example 2.7. For any homomorphism $\varphi : A_1 \to A_2$ and any $(c, \alpha) \in Z^2(f, A_1)$ there is a morphism of relative central extensions

$$\mathcal{E}(\varphi) : \mathcal{E}(c, \alpha) \longrightarrow \mathcal{E}(\varphi c, \varphi \alpha), \quad \mathcal{E}(\varphi) = (\varphi, \varphi \times \text{id}). \quad (2.2)$$

Definition 2.8. A homomorphism $f : H \to G$ is said to be perfect if $f_{ab} : H_{ab} \to G_{ab}$ is an epimorphism. In other words, $f$ is perfect if and only if $H_1(f) = 0$.

Lemma 2.9 (cf. \cite{13}, Lemma 6.9.6]). Let $(\varphi, \theta)$ and $(\varphi', \theta')$ be morphisms from a relative central extension $(A_1 \rightarrowtail E_1 \twoheadrightarrow G, \tilde{f}_1)$ to a relative central extension $(A_2 \rightarrowtail E_2 \twoheadrightarrow G, \tilde{f}_2)$. If $\tilde{f}_1$ is perfect, then $(\varphi, \theta) = (\varphi', \theta')$.

Proof. Since $\tilde{f}_1$ is perfect, we obtain $\text{Im}(\tilde{f}_1)[E_1, E_1] = E_1$. Since $\theta\tilde{f}_1 = \tilde{f}_2 = \theta'\tilde{f}_1$, we have $\theta|_{\text{Im}(\tilde{f}_1)} = \theta'|_{\text{Im}(\tilde{f}_1)}$. Lemma 2.4 implies $\theta|_{[E_1, E_1]} = \theta'|_{[E_1, E_1]}$. The assertion follows. □

Proposition 2.10. The universal relative central extension of $G$ with respect to $f$ exists if and only if $f$ is perfect. Moreover, in this case it is unique (up to isomorphism) and given by a short exact sequence

$$\begin{array}{c}
0 \longrightarrow H_2(f) \longrightarrow U \longrightarrow G \longrightarrow 1
\end{array}$$

where $u : H \to U$ is a perfect homomorphism.

Proof of Proposition 2.10 Assume that there is a universal relative central extension $(A \rightarrowtail U \twoheadrightarrow G, u)$ of $G$ with respect to $f$ and prove that $f$ is perfect. Set $B = \text{Coker}(f_{ab} : H_{ab} \to G_{ab})$ and consider the epimorphism $\tau : U \to B$ given by the composition $U \to G \to G_{ab} \to B$. Then we have two morphisms $(0, (\pi))$ and $(0, (0))$ from
\((A \mapsto U \mapsto G, u)\) to the split extension \((B \mapsto B \times G \mapsto G, (v))\). The universal property implies that they are equal, and hence \(B = 0\). Thus \(f\) is perfect.

Assume that \(f\) is perfect. Since \(H_1(\text{Cone}(Bf)) = H_1(f) = 0\), the universal coefficient formula for the space \(\text{Cone}(Bf)\) implies that there is an isomorphism \(H^2(f, A) \cong \text{Hom}(H_2(f), A)\) naturally by \(A\). Chose an element \((c_u, \alpha_u) \in Z^2(f, H_2(f))\) that represents the element of \(H^2(f, H_2(f))\) corresponding to the identity map in \(\text{Hom}(H_2(f), H_2(f))\).

Set \(U := E_{c_u}, u = f_{c_u}\) and \(E_u = \mathcal{E}(c_u, \alpha_u)\). Then \(E_u = (H_2(f) \mapsto U \mapsto G, u)\). Take a homomorphism \(\varphi : H_2(f) \rightarrow A\) and consider the commutative diagram

\[
\begin{array}{ccc}
Z^2(f, H_2(f)) & \longrightarrow & H^2(f, H_2(f)) & \xrightarrow{\varphi_*} & \text{Hom}(H_2(f), H_2(f)) \\
\downarrow \varphi_+ & & & & \downarrow \varphi_0 \\
Z^2(f, A) & \longrightarrow & H^2(f, A) & \xrightarrow{\varphi_*} & \text{Hom}(H_2(f), A).
\end{array}
\]

It shows that the isomorphism \(\text{Hom}(H_2(f), A) \cong H^2(f, A)\) sends \(\varphi\) to the class of \((\varphi c_u, \varphi \alpha_u)\). Combining this with Proposition \(2.3\), we obtain that any relative central extension is isomorphic to the extension \(\mathcal{E}(\varphi c_u, \varphi \alpha_u)\) for some \(\varphi : H_2(f) \rightarrow A\). For any relative central extension \(\mathcal{E}(\varphi c_u, \varphi \alpha_u)\) there exists a morphism \(\mathcal{E}(\varphi) : \mathcal{E}_u \rightarrow \mathcal{E}(\varphi c_u, \varphi \alpha_u)\) from Example \(2.7\). Then we found a morphism from \(\mathcal{E}_u\) to any other relative central extension. In order to prove that \(\mathcal{E}_u\) is the universal relative central extension, we have to prove that such a morphism is unique. By Lemma \(2.9\) it is enough to prove that \(u : H \rightarrow U\) is perfect.

Prove that \(u : H \rightarrow U\) is perfect. In other words we prove that \(\text{Im}(u)[U, U] = U\). Set \(E := \text{Im}(u)[U, U], A = \nu^{-1}(E)\) and \(f : H \rightarrow E\) given by \(f(h) = u(h)\). Note that \(f\) is perfect. Since \(f\) is perfect, \(\pi_u(E) = \text{Im}(f)[G, G] = G\). Consider the restriction \(\pi = \pi_u|_E\) and the relative central extension \(E = (A \mapsto E \mapsto G, f)\) with the obvious embedding \(E \hookrightarrow \mathcal{E}_u\).

Consider the projection \(\varphi' : H_2(f) \rightarrow H_2(f)/A\) and take the composition

\[
\mathcal{E} \hookrightarrow \mathcal{E}_u \xrightarrow{\mathcal{E}(\varphi')} \mathcal{E}(\varphi' c_u, \varphi' \alpha_u).
\]

The composition is equal to \((0, (\varphi' \times 1)|_E)\). By Lemma \(2.5\) \(E(\varphi' c_u, \varphi' \alpha_u)\) splits. Thus \((\varphi' c_u, \varphi' \alpha_u)\) represents 0 in \(H^2(f, A) \cong \text{Hom}(H_2(f), A)\), and hence \(\varphi' = 0\). It follows that \(A = H_2(f)\). Then the extension \(A \mapsto E \mapsto G\) is embedded into the extension \(A \mapsto U \mapsto G\). It follows that \(E = U\).

**Remark 2.11.** If \(f_{ab} : H_{ab} \rightarrow G_{ab}\) is an isomorphism, then \(H_2(f) = \text{Coker}\{H_2(H) \rightarrow H_2(G)\}\).

**Remark 2.12.** In the proof of Proposition \(2.10\) we show that, if \(f\) is perfect, then the universal relative central extension corresponds to the identity map \(\text{Hom}(H_2(f), H_2(f))\) with respect to the isomorphism \(H^2(f, H_2(f)) \cong \text{Hom}(H_2(f), H_2(f))\) that comes from the universal coefficient theorem.

### 2.4. HZ-localization tower via relative central extensions

Here we give an approach to the HZ-localization tower \([3]\) via relative central extensions.

Let \(G\) be a group and \(\eta : G \rightarrow EG\) be its HZ-localization. For an ordinal \(\alpha\) we define the \(\alpha\)th term of the HZ-localization tower by \(T_\alpha G := EG/\gamma_\alpha(EG)\), where \(\gamma_\alpha(EG)\) is the \(\alpha\)th term of the transfinite lower central series (see \([3]\) Theorem 3.11]). By \(\eta_\alpha : G \rightarrow T_\alpha G\) we denote the composition of \(\eta\) and the canonical projection, and by \(t_\alpha : T_{\alpha+1} G \rightarrow T_\alpha G\) we
denote the canonical projection. The main point of [5] is that $T_\alpha G$ can be constructed inductively and $EG = T_\alpha G$ for big enough $\alpha$. We threat the construction of $T_\alpha G$ for a non-limit ordinal $\alpha$ via universal relative central extensions.

**Proposition 2.13.** Let $G$ be a group and $\alpha > 1$ be an ordinal. Then $(\eta_\alpha)_{ab} : G_{ab} \to (T_\alpha G)_{ab}$ is an isomorphism, the universal central extension of $T_\alpha G$ with respect to $\eta_\alpha$ is given by

$$
\begin{array}{ccccccc}
0 & \longrightarrow & H_2(\eta_\alpha) & \longrightarrow & T_{\alpha+1}G & \overset{t_\alpha}{\longrightarrow} & T_\alpha G & \longrightarrow & 1,
\end{array}
$$

and $H_2(\eta_\alpha) = \text{Coker}\{H_2(G) \to H_2(T_\alpha G)\}$.

**Proof.** It follows from [5, 3.2], [5, 3.4], Proposition 2.10 and Remarks 2.11 2.12. □

3. Homology of stem-extensions

Consider a central extension of groups

$$1 \to N \to G \to Q \to 1 \quad (3.1)$$

It is shown in [10] that, there is a natural long exact sequence

$$H_3(G) \to H_3(Q) \to (G_{ab} \otimes N)/U \to H_2(G) \to H_2(Q) \overset{\beta}{\to} N \to H_1(G) \to H_1(Q) \to 0 \quad (3.2)$$

where $U$ is the image of the natural map

$$H_4K(N, 2) \to G_{ab} \otimes N.$$

Here $H_4K(N, 2)$ is the forth homology of the Eilenberg-MacLane space $K(N, 2)$ which can be described as the Whitehead quadratic functor

$$H_4K(N, 2) = \Gamma^2 N.$$

A central extension (3.1) is called a stem-extension if $N \subseteq [G, G]$. For a stem extension (3.1), the exact sequence (3.2) has the form (see [11], [10])

$$H_3(G) \to H_3(Q) \overset{\delta}{\to} G_{ab} \otimes N \to H_2(G) \to H_2(Q) \overset{\beta}{\to} N \to 0 \quad (3.3)$$

The map $\delta$ is given as follows. We present (3.1) in the form

$$1 \to S/R \to F/R \to F/S \to 1,$$

for a free group $F$ and normal subgroups $R, S$ with $R \subseteq S, [F, S] \subseteq R$. Then the map $\delta$ is induced by the natural epimorphism $S_{ab} \to S/R$:

$$H_3(Q) = H_1(F/S, S_{ab}) \to H_1(F/S, S/R) = Q_{ab} \otimes N = G_{ab} \otimes N.$$

The isomorphism $H_1(F/S, S/R) = Q_{ab} \otimes N$ follows from the triviality of $F/S$-action on $S/R$.

In this section we consider the class of metabelian groups of the form $Q = M \rtimes C$, where $C$ is an infinite cyclic group and $M$ a $\mathbb{Z}[C]$-module. It follows immediately from the homology spectral sequence that, for any $n \geq 2$, there is a short exact sequence

$$0 \to H_0(C, H_n(M)) \to H_n(Q) \to H_1(C, H_{n-1}(M)) \to 0.$$
which can be presented in terms of (co)invariants as
\[ 0 \to H_n(M)_C \to H_n(Q) \to H_n(M)_C \to 0. \]
Composing the last epimorphism with \( H_{n-1}(M)_C \to H_{n-1}(M)_C \to H_{n-1}(Q) \), we get a natural (in the category of \( \mathbb{Z}[C]\)-modules) map
\[ \alpha_n : H_n(Q) \to H_{n-1}(Q). \]
In the next proposition we will construct a composite map
\[ \alpha'_3 : H_3(Q) \to H_2(Q), \]
using group-theoretical tools, without spectral sequence. Probably, \( \alpha'_3 \) coincides with \( \alpha_3 \) up to isomorphism, but we will not use this comparison later.

**Proposition 3.1.** For a stem extension \( (3.1) \) of a group \( Q = M \rtimes C \), there exists a map \( \alpha'_3 : H_3(Q) \to H_2(Q) \), given as a composition \( (3.4) \), such that the following diagram is commutative

\[
\begin{array}{ccc}
H_3(Q) & & H_2(Q) \\
\downarrow \delta & & \downarrow \beta \\
G_{ab} \otimes N & & N
\end{array}
\]
where the lower horizontal map is the projection
\[ G_{ab} \otimes N \to C \otimes N = N. \]

**Proof.** We choose a free group \( F \) with normal subgroups \( R \trianglelefteq S \trianglelefteq T \) such that
\[ F/T = C, \ F/S = Q, F/R = G. \]
In the above notation, we get \( M = T/S, N = S/R \). The proof follows from the direct analysis of the following diagram, which corners are exactly the roots of the diagram given in proposition:

\[
\begin{array}{ccc}
H_3(Q) & & H_1(F/T, S \cap T') \\
\downarrow \delta & & \downarrow \iota \\
H_1(F/S, S_{ab}) & & H_1(F/T, S_{ab}) \\
\downarrow \delta & & \\
H_1(F/S, S/R) & & (F/S)_{ab} \otimes S/R \\
& & \downarrow \beta \\
& & S/R
\end{array}
\]
All arrows of this diagram are natural. We will make comments only about two maps from the diagram, other maps are obviously defined. The map
\[ H_1(F/T, \frac{S \cap T'}{S'}) \to H_1(F/T, S_{ab}) \]
is an isomorphism since
\[ H_1(F/T, \frac{S}{S \cap T}) \cong H_1(F/T, T_{ab}) = H_3(F/T) = 0. \]
The vertical map in the diagram \( H_1(F/T, \frac{S}{S \cap T}) = H_1(F/T, H_2(S/T)) \cong H_2(S/T) \) follows from the identification of \( H_1 \) of a cyclic group with invariants.

The commutativity of \( \square \) follows from the commutativity of the natural square
\[
\begin{array}{ccc}
H_1(F/S, S_{ab}) & \longrightarrow & H_1(F/T, S_{ab}) \\
\downarrow & & \downarrow \\
H_1(F/S, S/R) & \longrightarrow & H_1(F/T, S/R)
\end{array}
\]
and identification of \( H_1(F/T, -) \) with invariants of the \( F/T \)-action. \( \square \)

4. Tame modules and completions

Throughout the section \( C \) denotes an infinite cyclic group. If \( R \) is a commutative ring, \( R[C] \) denotes the group algebra over \( R \). We use only \( R = \mathbb{Z}, \mathbb{Q}, \mathbb{C} \). The augmentation ideal is denoted by \( I \). If \( t \) is one of two generators of \( C \), \( R[C] = R[t, t^{-1}] \) and \( I = (t - 1) \).

4.1. Finite dimensional \( K[C] \)-modules. Let \( K \) be a field (we use only \( K = \mathbb{Q}, \mathbb{C} \)), \( V \) be a right \( K[C] \)-module such that \( \dim_K V < \infty \). If we fix a generator \( t \in C \) we obtain a linear map \( \cdot \colon V \to V \) that defines the module structure. We denote the linear map by \( a_V \in \text{GL}(V) \). The characteristic and minimal polynomials of \( a_V \) are denoted by \( \chi_V \) and \( \mu_V \) respectively. These polynomials depend on the choice of \( t \in C \). Note that for any such modules \( V \) and \( U \) we have
\[
\chi_{V \oplus U} = \chi_V \chi_U, \quad \mu_{V \oplus U} = \text{lcm}(\mu_V, \mu_U).
\] (4.1)

Lemma 4.1. Let \( V \) be a right \( K[C] \)-module such that \( \dim_K V < \infty \) and \( t \in C \) be a generator. Then there exist distinct irreducible monic polynomials \( f_1, \ldots, f_l \in K[x] \) and an isomorphism
\[ V \cong V_1 \oplus \cdots \oplus V_l, \]
where
\[ V_i = K[C]/(f_i^{m_{i,1}}(t)) \oplus \cdots \oplus K[C]/(f_i^{m_{i,l_i}}(t)), \]
and \( m_{i,1} \geq m_{i,2} \geq \cdots \geq m_{i,l_i} \geq 1 \). Moreover, if we set \( m_i = \sum_j m_{i,j} \), then \( \chi_V = f_1^{m_1} \cdots f_l^{m_l} \) and \( \mu_V = f_1^{m_{i,1}} \cdots f_l^{m_{i,1}} \).

Proof. Note that \( K[C] = K[t, t^{-1}] \) is the polynomial ring \( K[t] \) localised at the element \( t \). Then it is a principal ideal domain. Then the isomorphism follows from the structure theorem for finitely generated modules over a principal ideal domain. The statement about \( \chi_V \) and \( \mu_V \) follows from the fact that both characteristic and minimal polynomials of \( K[C]/(f_i^{m_{i,j}}(t)) \) equal to \( f_i^{m_{i,j}} \) and the formulas (4.1). \( \square \)

Let \( R \) be a commutative ring and \( t \) be a generator of \( C \). For an \( R[C] \)-module \( M \) and a polynomial \( f \in R[x] \) we set
\[ M^f = \{ m \in M \mid m \cdot f(t) = 0 \}. \]
Note that $M^C = M^{x-1}$. It is easy to see that for any $f, g \in \mathbb{R}[x]$ we have
\[(M/M^I)^g = M^f/M^I.\] (4.2)

**Corollary 4.2.** Let $V$ be a right $K[C]$-module such that $\dim_K V < \infty$ and $t \in C$ be a generator. Assume that $\mu_V = f_1^{m_1} \cdots f_l^{m_l}$, where $f_1, \ldots, f_l \in K[x]$ are distinct irreducible monic polynomials. Consider the filtration
\[0 = F_0 V \subset F_1 V \subset \cdots \subset F_i V = V\]
given by $F_i V = Vf_1^{m_1} \cdots f_i^{m_i}$. Then
\[V = \bigoplus_i F_i V/F_{i-1} V, \quad \mu_{F_i V/F_{i-1} V} = f_j^{m_j} \cdots f_i^{m_i}\]
and $F_i V/F_{i-1} V = (V/F_{i-1} V)/(f_j^{m_j} \cdots f_i^{m_i})$.

**Proof.** In the notation of Lemma 4.1 we obtain $F_i V = V_1 \oplus \cdots \oplus V_i$. The assertion follows. \qed

### 4.2 Tame $\mathbb{Z}[C]$-modules

The **rank** of an abelian group $A$ is $\dim_{\mathbb{Q}}(A \otimes \mathbb{Q})$. The torsion subgroup of $A$ is denoted by $\text{tor}(A)$. The following statement seems to be well known but we can not find a reference, so we give it with a proof.

**Lemma 4.3.** Let $A$ be a torsion free abelian group of finite rank and $B$ be a finite abelian group. Then $A \otimes B$ is finite.

**Proof.** It is sufficient to prove that $A \otimes \mathbb{Z}/p^k$ is finite for any prime $p$ and $k \geq 1$. Consider a $p$-basis subgroup $A'$ of $A$ (see [12, VI]). Then $A' \cong \mathbb{Z}^r$, where $r$ is the rank of $A$ and $A/A'$ is $p$-divisible. Thus $(A/A') \otimes \mathbb{Z}/p^k = 0$. Hence the map $(\mathbb{Z}/p^k)^r \cong A' \otimes \mathbb{Z}/p^k \rightarrow A \otimes \mathbb{Z}/p^k$ is an epimorphism. \qed

A finitely generated $\mathbb{Z}[C]$-module $M$ is said to be **tame** if the group $M \times C$ is finitely presented.

**Proposition 4.4** (Theorem C of [2]). Let $M$ be a finitely generated $\mathbb{Z}[C]$-module. Then
\[M \text{ is tame if and only if the following properties hold:}\]
- $\text{tor}(M)$ is finite;
- the rank of $M$ is finite;
- there is a generator $t$ of $C$ such that $\chi_{M \otimes \mathbb{Q}}$ is integral.

**Lemma 4.5** (Lemma 3.4 of [2]). Let $M$ and $M'$ be tame $\mathbb{Z}[C]$-modules. Then $M \otimes M'$ is a tame $\mathbb{Z}[C]$-module (with the diagonal action).

**Definition 4.6.** Let $M$ be a tame module. The generator $t \in C$ such that such that $\chi_{M \otimes \mathbb{Q}}$ is integral is called an **integral generator** for $M$. When we consider a tame module, we always denote by $t$ an integral generator for $M$. We set $a_M := t \otimes \mathbb{Q} : M \otimes \mathbb{Q} \rightarrow M \otimes \mathbb{Q}$, and denote by $\chi_M, \mu_M$ the characteristic and the minimal polynomial of $a_M$. In other words $\chi_M = \chi_{M \otimes \mathbb{Q}}$ and $\mu_M = \mu_{M \otimes \mathbb{Q}}$.

**Lemma 4.7.** $\mu_M$ is an integral monic polynomial for any tame $\mathbb{Z}[C]$-module $M$.

**Proof.** Let $\chi_M = (x - \lambda_1)^{m_1} \cdots (x - \lambda_l)^{m_l}$ for some distinct $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$ and $m_i \geq 1$. Then $\mu_M = (x - \lambda_1)^{k_1} \cdots (x - \lambda_l)^{k_l}$, where $1 \leq k_i \leq m_i$. Since $\chi$ is a monic integral polynomial, $\lambda_1, \ldots, \lambda_l$ are algebraic integers. It follows that the coefficients of $\mu_M$ are algebraic integers as well. Moreover, they are rational numbers, because $a_M$ is defined rationally. Using that a rational number is an algebraic integer iff it is an integer number, we obtain $\mu_M \in \mathbb{Z}[x]$. \qed
Proposition 4.8. Let $M$ be a torsion free tame $\mathbb{Z}[C]$-module and $\mu_M = f_1^{m_1} \cdots f_l^{m_l}$ where $f_1, \ldots, f_l$ are distinct irreducible integral monic polynomials and $m_i \geq 1$ for all $i$. Consider the filtration

$$0 = F_0 M \subset F_1 M \subset \cdots \subset F_l M = M$$

given by $F_i M = M f_1^{m_1} \cdots f_i^{m_i}$. Then $F_i M / F_{j-1} M$ is torsion free and $\mu_{F_i M / F_{j-1} M} = f_j^{m_j} \cdots f_i^{m_i}$ for any $i \geq j$. Moreover, the corresponding filtration on $M \otimes \mathbb{Q}$ splits:

$$M \otimes \mathbb{Q} \cong \bigoplus_{i=1}^l (F_i M / F_{i-1} M) \otimes \mathbb{Q}.$$

Proof. Prove that $F_i M / F_{j-1} M$ is torsion free. Let $v + F_{j-1} M \in F_i M / F_{j-1} M$ and $nv + F_{j-1} M = 0$. Hence $nv \cdot f_1^{m_1}(t) \cdots f_j^{m_j}(t) = 0$ in $M$. Using that $M$ is torsion free we get $v \cdot f_1^{m_1}(t) \cdots f_j^{m_j}(t) = 0$, and hence $v + F_{j-1} M = 0$. Thus $F_i M / F_{j-1} M$ is torsion free. Set $V = M \otimes \mathbb{Q}$, $K = \mathbb{Q}$, apply Corollary 4.12, and note that $F_i V / F_{j-1} V = (F_i M / F_{j-1} M) \otimes \mathbb{Q}$. The assertion follows. Here we use Gauss lemma about integral polynomials: an irreducible polynomial in $\mathbb{Z}[x]$ is irreducible in $\mathbb{Q}[x]$. \qed

Recall that a module $N$ is said to be nilpotent if $NI^n = 0$ for some $n$, where $I$ is the augmentation ideal. It is easy to see that a $\mathbb{Z}[C]$-module $N$ is nilpotent if and only if $N^{(s-1)n} = N$ for some $n$.

Definition 4.9. A $\mathbb{Z}[C]$-module $M$ is said to be invariant free if $M^C = 0$.

Lemma 4.10. Let $M$ be a torsion free tame $\mathbb{Z}[C]$-module. Then the following equivalent.

1. $M$ is invariant free;
2. $M$ does not have non-trivial nilpotent submodules;
3. $M^C$ is finite;
4. $a_M - 1$ is an automorphism;
5. $\chi_M(1) \neq 0$;
6. $\mu_M(1) \neq 0$.

Proof. (1) $\iff$ (2) and (4) $\iff$ (5) $\iff$ (6) are obvious. The equality $\text{Ker}(a_M - 1) = M^C \otimes \mathbb{Q}$ implies (1) $\iff$ (4). Since $M$ is finitely generated $\mathbb{Z}[C]$-module, $M^C$ is a finitely generated abelian group. Then the equality $\text{Coker}(a_M - 1) = M^C \otimes \mathbb{Q}$ implies (3) $\iff$ (4). \qed

Corollary 4.11. Let $M$ be a torsion free tame $\mathbb{Z}[C]$-module and $\mu_M = (x - 1)^m f$, where $f(1) \neq 0$. Then there exists the largest nilpotent submodule $N \leq M$. Moreover, $\mu_N = (x - 1)^n$, $\mu_M / N = f$, $M / N$ is torsion free and invariant free, and the short exact sequence $N \otimes \mathbb{Q} \rightarrow M \otimes \mathbb{Q} \rightarrow (M / N) \otimes \mathbb{Q}$ splits over $\mathbb{Q}[C]$.

Proof. If $\mu_M(1) \neq 0$, then $M$ is already invariant free, $N = 0$ and there is nothing to prove. If $\mu_M(1) = 0$, then we can decompose $\mu_M = (x - 1)^{m_1} f_2^{m_2} \cdots f_l^{m_l}$ into a product of irreducible polynomials such that $f_i(1) \neq 0$ for $i \geq 2$. Consider the filtration from Proposition 4.8. Then $N = F_1 M$. \qed

Corollary 4.12. Let $M$ be a tame $\mathbb{Z}[C]$-module. Then there exists the largest nilpotent submodule $N \leq M$. Moreover, $M / N$ is invariant free and $(M / N)^C$ is finite.

Recall that a module $N$ is said to be prenilpotent if $NI^n = NI^{n+1}$ for $n >> 1$. 

Corollary 4.13. Let $M$ be a tame $\mathbb{Z}[C]$-module and $\mu_M = (x - 1)^mf$, where $f(1) \neq 0$. Then there exists a prenilpotent submodule $N \leq M$ such that $M/N$ is torsion free and invariant free. Moreover, $\text{tor}(N) = \text{tor}(M)$, $\mu_N = (x - 1)^m$, $\mu_{M|N} = f$ and the sequence $N \otimes \mathbb{Q} \rightarrow M \otimes \mathbb{Q} \rightarrow (M/N) \otimes \mathbb{Q}$ splits over $\mathbb{Q}[C]$.

Lemma 4.14. Let $M$ be a tame torsion free $\mathbb{Z}[C]$-module. If $\mu_M(0) \in \{-1, 1\}$, then $M$ is finitely generated as an abelian group.

Proof. It follows from the fact that $\mathbb{Z}[t, t^{-1}]/(\mu_M(t))$ is a finitely generated abelian group. \hfill \Box

4.3. Completion of tame $\mathbb{Z}[C]$-modules. If $M$ is a finitely generated $R[C]$-module, we set $\hat{M} = \lim_{\leftarrow} M/MI^i$ and we denote by

$$\varphi = \varphi_M : M \rightarrow \hat{M}$$

the natural map to the completion. Note that the functor $M \mapsto \hat{M}$ is exact \[19\], VIII and $\hat{M}/\hat{M}I^i = M/MI^i$.

We set

$$Z_n = \lim_{\leftarrow} Z/n^i$$

for any $n \in \mathbb{Z}$. In particular, $Z_n = Z_{-n}$, $Z_0 = \mathbb{Z}$, $Z_1 = 0$ and, if $n \geq 2$, then $Z_n = \bigoplus Z_p$, where $p$ runs over all prime divisors of $n$.

Lemma 4.15. Let $M$ be a tame torsion free invariant free $\mathbb{Z}[C]$-module and $n = \chi_M(1)$. Then $n^i \cdot M \subseteq MI^i$ for any $i \geq 1$ and there exists a unique epimorphism of $\mathbb{Z}[C]$-modules $\hat{\varphi} : M \otimes Z_n \rightarrow \hat{M}$ such that the diagrams

$$\begin{array}{ccc}
M \otimes Z_n & \xrightarrow{\hat{\varphi}} & \hat{M} \\
\downarrow & & \downarrow \\
M \otimes Z/n^i & \longrightarrow & M/MI^i \\
\end{array}$$

are commutative.

Proof. We identify $M$ with the subgroup of $M \otimes \mathbb{Q}$. Corollary 4.10 implies that $n \neq 0$. Set $b = a_M - 1$. Then the characteristic polynomial of $b$ is equal to $\chi_b(x) = \chi_M(x + 1)$ and if $\chi_b = \sum_{i=0}^d \beta_i x^i$, then $\beta_0 = n$. Thus $nx = b(\sum_{i=1}^d \beta_i b^{-1}(x))$ for any $x \in M$. It follows that $nM \subseteq b(M)$. Hence $n^iM \subseteq b^i(M) = MI^i$ for any $i \geq 1$ and we obtain homomorphisms $M \otimes \mathbb{Z}/n^i \rightarrow M/MI^i$. We define $\hat{\varphi}$ as the composition $M \otimes Z_n \rightarrow \lim_{\leftarrow} (M \otimes \mathbb{Z}/n^i) \rightarrow \hat{M}$.

Since the rank $M$ is finite, the abelian groups $M \otimes \mathbb{Z}/n^i$ are finite. Thus we get that the homomorphism $\lim_{\rightarrow} (M \otimes \mathbb{Z}/n^i) \rightarrow \hat{M}$ is an epimorphism because $\lim_{\rightarrow}$ of an inverse sequence of finite groups is trivial. Then it is sufficient to prove that the homomorphism $M \otimes Z_n \rightarrow \lim_{\leftarrow} (M \otimes \mathbb{Z}/n^i)$ is an epimorphism. For this it is enough to prove that $M \otimes \mathbb{Z}_p \rightarrow \lim_{\rightarrow} (M \otimes \mathbb{Z}/p^i)$ is an epimorphism for any prime $p$. Consider a $p$-basic subgroup $B$ of $M$ (see \[12\], VI). Since $B \cong \mathbb{Z}^i$, we get $B \otimes \mathbb{Z}_p = \lim_{\rightarrow} (B \otimes \mathbb{Z}/p^i)$. Using that $B \otimes \mathbb{Z}/p^i \rightarrow M \otimes \mathbb{Z}/p^i$ are epimorphisms of finite groups, we obtain that $\lim_{\rightarrow} (B \otimes \mathbb{Z}/p^i) \rightarrow \lim_{\rightarrow} (M \otimes \mathbb{Z}/p^i)$ is an
epimorphism. Then analysing the diagram

\[
\begin{array}{c}
B \otimes \mathbb{Z}_p \ar[r]^\varepsilon \ar[d] & M \otimes \mathbb{Z}_p \ar[d] \\
\lim(B \otimes \mathbb{Z}/p') \ar[r] & \lim(M \otimes \mathbb{Z}/p')
\end{array}
\]

we obtain that the right vertical arrow is an epimorphism. \(\square\)

A \(\mathbb{Z}[C]\)-module is said to be perfect if \(MI = M\).

**Corollary 4.16.** Let \(M\) be a torsion free tame \(\mathbb{Z}[C]\)-module. If \(\mu_M(1) \in \{-1, 1\}\), then \(M\) is perfect.

*Proof.* By Lemma 4.10, \(M\) is invariant free. Then Lemma 4.15 implies \(\hat{M} = 0\). Hence \(M\) is perfect. \(\square\)

**Corollary 4.17.** Let \(M\) be a tame \(\mathbb{Z}[C]\)-module. If \(\mu_M = (x-1)^m f\), where \(f\) is an integral polynomial such that \(f(1) \in \{-1, 1\}\), then \(M\) is prenilpotent.

*Proof.* A finite module is always prenilpotent, so we can assume that \(M\) has no torsion. Further, by Lemma 4.11, we can consider the largest nilpotent submodule \(N \leq M\) such that \(\mu_N = (x-1)^m\) and \(\mu_{M/N} = f\). Corollary 4.16 implies that \(M/N\) is perfect. Then \(N\) and \(M/N\) are prenilpotent, and hence, \(M\) is prenilpotent. \(\square\)

**Proposition 4.18.** Let \(M\) and \(M'\) be tame \(\mathbb{Z}[C]\)-modules with the same integral generator \(t \in C\), \(\lambda_1, \ldots, \lambda_i \in \mathbb{C}\) are eigenvalues of \(a_M\) and \(\lambda'_1, \ldots, \lambda'_p \in \mathbb{C}\) are eigenvalues of \(a_{M'}\). Assume that the equality \(\lambda_i \lambda'_j = 1\) holds only if \(\lambda_i = \lambda'_j = 1\). Then the homomorphism

\[
(M \otimes M')_C \longrightarrow (\hat{M} \otimes \hat{M'})_C
\]

is an epimorphism.

*Proof.* Note that if \(M_1 \to M_2 \to M_3\) is a short exact sequence of tame modules and \((M_1 \otimes M')_C \to (\hat{M}_1 \otimes \hat{M'})_C\), \((M_3 \otimes M')_C \to (\hat{M}_3 \otimes \hat{M'})_C\) are epimorphisms, then \((M_2 \otimes M')_C \to (\hat{M}_2 \otimes \hat{M'})_C\) is an epimorphism. Indeed, since the functor of completion is exact, we have the commutative diagram with exact rows

\[
\begin{array}{c}
(M_1 \otimes M')_C \ar[r] \ar[d] & (M_2 \otimes M')_C \ar[r] \ar[d] & (M_3 \otimes M')_C \ar[d] \\
(\hat{M}_1 \otimes \hat{M'})_C \ar[r] & (\hat{M}_2 \otimes \hat{M'})_C \ar[r] & (\hat{M}_3 \otimes \hat{M'})_C
\end{array}
\]

that implies this. Then, using Corollary 4.13 we obtain that we can divide our prove into two parts: (1) prove the statement for the case of torsion free invariant free modules \(M, M'\); (2) prove the statement for the case of a prenilpotent module \(M\) and arbitrary tame module \(M'\). Throughout the proof we use that \((M \otimes M')_C \cong M \otimes_{\mathbb{Z}[C]} M'_C\), where \(M'_C\) is the module with the same underling abelian group \(M'\) but with the twisted action of \(C\): \(m \ast t = mt^{-1}\).
(1) Assume that $M, M'$ are torsion free invariant free tame $\mathbb{Z}[C]$-modules. Lemma 4.10 implies that $\lambda_i \neq 1$ and $\lambda_j' \neq 1$ for all $i, j$. Then we have $\lambda_i \lambda_j' \neq 1$ for all $i, j$. Note that the eigenvalues of $a_M \otimes a_{M'}$ equal to the products $\lambda_i \lambda_j$, and hence 1 is not an eigenvalue of $a_M \otimes a_{M'}$. It follows that $\det(a_M \otimes a_{M'} - 1) \neq 0$. Consider the minimal polynomial $\mu$ of the tensor square $a_M \otimes a_{M'}$. Since the $a_M \otimes a_{M'}$ is defined over $\mathbb{Q}$, the coefficients of $\mu$ are rational (because they are invariant under the action of the absolute Galois group). Moreover, $\mu = \prod (x - \lambda_i \lambda_j')^{k_{i,j}}$ for some $k_{i,j}$, and hence, its coefficients are algebraic integers. It follows that $\mu$ is a monic polynomial with integral coefficients. The polynomial $\mu(x + 1)$ is the minimal polynomial for $a_M \otimes a_{M'} - 1$. Let $\mu(x + 1) = \sum_{i=0}^k n_ix^i$. Then $n_0 = \det(a_M \otimes a_{M'} - 1) \neq 0$ and $n_0(M \otimes M') \in (M \otimes M')(t-1)$. Since the rank of $M \otimes M'$ is finite, $(M \otimes M')/n_0(M \otimes M')$ is finite, and hence, $(M \otimes M')_C = (M \otimes M')/(M \otimes M')(t-1)$ is finite. By Lemma 4.15 we have epimorphisms $M \otimes \mathbb{Z}_n \rightarrow \hat{M}$ and $M' \otimes \mathbb{Z}_n' \rightarrow \hat{M}'$, where $n = \det(a_M - 1)$ and $n' = \det(a_{M'} - 1)$. It is easy to see that $$(M \otimes \mathbb{Z}_n) \otimes (M' \otimes \mathbb{Z}_n')_C = (M \otimes M')_C \otimes (\mathbb{Z}_n \otimes \mathbb{Z}_n').$$ Since $(M \otimes M')_C$ is finite, $(M \otimes M')_C \rightarrow (M \otimes M')_C \otimes (\mathbb{Z}_n \otimes \mathbb{Z}_n')$ is an epimorphism. Then $(M \otimes M')_C \rightarrow (\hat{M} \otimes \hat{M}')_C$ is an epimorphism.

(2) Assume that $M$ is a prenilpotent $\mathbb{Z}[C]$-module and $M'$ is a tame $\mathbb{Z}[C]$-module. Then there exists $i$ such that $\hat{M} = M/M_1$. Since $(\hat{M} \otimes \hat{M}')_C \cong \hat{M} \otimes_{\mathbb{Z}[C]} \hat{M}'_C$, we get $$(\hat{M} \otimes \hat{M}')_C \cong (M/M_1 \otimes \hat{M}')_C \cong (M/M_1 \otimes \hat{M}'' \otimes \hat{M}')_C \cong (M/M_1 \otimes M'/M')_C.$$ It follows that $(M \otimes M')_C \rightarrow (\hat{M} \otimes \hat{M}')_C$ is an epimorphism. \qed

Corollary 4.19. Let $M$ be a tame $\mathbb{Z}[C]$-module and $\mu_M = (x - 1)^{m_1}f_1^{m_1} \cdots f_l^{m_l}$ for some distinct monic irreducible polynomials $f_1, \ldots, f_l \in \mathbb{Z}[x]$ such that $f_i(1) \neq 0$ and $f_i(0) \notin \{1, -1\}$ for all $1 \leq i \leq l$. Then the the homomorphism $$(M \otimes^2)_C \rightarrow (\hat{M} \otimes^2)_C$$ is an epimorphism.

Proof. Let $\lambda_1, \ldots, \lambda_k$ be roots of $\mu_M$. Assume that $\lambda_i \lambda_j = 1$. Then $\lambda_i$ is an invertible algebraic integer, and hence, the absolute term of its minimal polynomial equals to $\pm 1$. Thus $\lambda_i$ can not be a root of $f_m$ for $1 \leq m \leq l$. It follows that it is a root of $x - 1$. Then $\lambda_i = \lambda_j = 1$. \qed

Proposition 4.20. Let $M, M'$ be tame $\mathbb{Z}[C]$-modules with the same integral generator $t \in C$, $\mu_M = (x - \lambda_1)^{m_1} \cdots (x - \lambda_l)^{m_l}$ for some distinct $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$ and $\mu_{M'} = (x - \lambda'_1)^{m'_1} \cdots (x - \lambda'_{l'})^{m'_{l'}}$ for some distinct $\lambda'_1, \ldots, \lambda'_{l'} \in \mathbb{C}$. Assume that the equality $\lambda_i \lambda_j' = 1$ holds only if either $m_i = m'_j = 1$ or $\lambda_i = \lambda'_j = 1$. Then the cokernel of the homomorphism $$(M \otimes M')_C \oplus (\hat{M} \otimes \hat{M}')_C \rightarrow (\hat{M} \otimes \hat{M}')_C$$ is finite.

Proof. Corollary 4.13 implies that the proof can be divided into proofs of the following two statements: (1) the statement for torsion free invariant free modules $M, M'$; (2) if $N \rightarrow M \rightarrow M_0$ is a short exact sequence of tame $\mathbb{Z}[C]$-modules such that $N \otimes \mathbb{Q} \rightarrow M \otimes \mathbb{Q} \rightarrow M_0 \otimes \mathbb{Q}$ splits, $N$ is prenilpotent, and the statement holds for the couple $M_0, M'$, then it holds for the couple $M, M'$. \qed
(1) Here we prove that the cokernel of \((M \otimes \tilde{M}')^C \to (\hat{M} \otimes \hat{M}')^C\) is already finite. Set \(n = \chi_M(1)\) and \(n' = \chi_{M'}(1)\). Lemma 4.15 implies that there are epimorphisms \(M \otimes \mathbb{Z}_n \to \hat{M}\) and \(M' \otimes \mathbb{Z}_{n'} \to \hat{M}'.\) Using that \(- \otimes (\mathbb{Z}_n \otimes \mathbb{Z}_{n'})\) is an exact functor, we obtain that there is an epimorphism \((M \otimes M')_C \otimes (\mathbb{Z}_n \otimes \mathbb{Z}_{n'}) \to (\hat{M} \otimes \hat{M}')_C\). Moreover, there is an epimorphism

\[
\text{Coker}((M \otimes M')^C \to (M \otimes M')_C) \otimes (\mathbb{Z}_n \otimes \mathbb{Z}_{n'}) \to \text{Coker}((\hat{M} \otimes \hat{M}')^C \to (\hat{M} \otimes \hat{M}')_C).
\]

It follows that it is enough to prove that \(\text{Coker}((M \otimes M')^C \to (M \otimes M')_C)\) is finite. Lemma 4.15 implies that \(M \otimes M'\) is finitely generated, and hence, \((M \otimes M')_C\) is a finitely generated abelian group. It follows that it is enough to prove that \((M \otimes M')^C \otimes \mathbb{C} \to (M \otimes M')_C \otimes \mathbb{C}\) is an epimorphism. Eigenvalues of \(a_M \otimes a_{M'}\) are products \(\lambda_i \lambda_j'\). Assume that \(\lambda_i \lambda_j' = 1\) for some \(i, j\). Since \(M\) and \(M'\) are invariant free, \(\lambda_i \neq 1\) and \(\lambda_j' \neq 1\). Then \(m_i = 1 = m_j'\). It follows that all Jordan blocks of \(a_M \otimes a_{M'}\) corresponding to \(\lambda_i\) and all Jordan blocks of \(a_{M'} \otimes \mathbb{C}\) corresponding to \(\lambda_j'\) are \(1 \times 1\)-matrices. It follows that all Jordan blocks of \(a_M \otimes a_{M'} \otimes \mathbb{C}\) corresponding to \(1\) are \(1 \times 1\)-matrices. Hence all Jordan blocks of \(B = a_M \otimes a_{M'} \otimes \mathbb{C}\) corresponding to \(0\) are \(1 \times 1\)-matrices. It is easy to see that, if all Jordan blocks of a complex linear map \(B : V \to V\) corresponding to \(0\) are \(1 \times 1\)-matrices, then \(V = \ker(B) \oplus \im(B)\). It follows that the map \(\ker(B) \to \text{Coker}(B)\) is an isomorphism. Then \((M \otimes M')^C \otimes \mathbb{C} \to (M \otimes M')_C \otimes \mathbb{C}\) is an isomorphism.

(2) Note that \(\hat{N} = N/NI^i\) for some \(i \gg 1\). Since, \((\hat{N} \otimes \hat{M}')_C\) can be interpret as \(\hat{N} \otimes_{\mathbb{Z}[C]} \hat{M}',\) (tensor product over \(\mathbb{Z}[C]\)), we obtain \((\hat{N} \otimes \hat{M}')_C = (N/NI^i \otimes M'/M'^i)_C\). It follows that \((\hat{N} \otimes \hat{M}')_C\) is a finitely generated abelian group and the map \((\hat{N} \otimes \hat{M}')_C \to (\hat{N} \otimes \hat{M}')_C\) is an epimorphism. Set

\[
\mathcal{N} := N \otimes M', \quad \mathcal{M} := M \otimes M',
\]
\[
\mathcal{M}_0 := M_0 \otimes M', \quad \tilde{\mathcal{N}} := \hat{N} \otimes M',
\]
\[
\tilde{\mathcal{M}} := \hat{M} \otimes \hat{M}', \quad \tilde{\mathcal{M}}_0 := M_0 \otimes \hat{M}',
\]
\[
\tilde{\mathcal{L}} := \ker(\tilde{\mathcal{M}} \to \tilde{\mathcal{M}}_0).
\]

Then \(\tilde{\mathcal{N}}_C\) is a finitely generated abelian group and the map \(\mathcal{N}_C \to \tilde{\mathcal{N}}_C\) is an epimorphism. Consider the exact sequence

\[
0 \to \tilde{\mathcal{L}}^C \to \tilde{\mathcal{M}}^C \to \tilde{\mathcal{M}}_0^C \to \tilde{\mathcal{L}}_C \to \hat{\mathcal{M}}_C \to (\hat{\mathcal{M}}_0)_C \to 0.
\]

Since \(\tilde{\mathcal{M}}_0 \otimes \mathbb{Q} \to \hat{\mathcal{M}}_0 \otimes \mathbb{Q}\) is a split epimorphism, the image of \(\tilde{\mathcal{M}}_0^C \to \tilde{\mathcal{L}}_C\) lies in the torsion subgroup, which is finite because of the epimorphism \(\tilde{\mathcal{N}}_C \to \tilde{\mathcal{L}}_C\). Then the cokernel of \(\tilde{\mathcal{M}}^C \to \tilde{\mathcal{M}}_0^C\) is finite. Set

\[
Q = \text{Coker}((\mathcal{M}_C \otimes \tilde{\mathcal{M}}^C \to \tilde{\mathcal{M}}_C),
\]
\[
Q_0 = \text{Coker}((\mathcal{M}_0)_C \otimes \tilde{\mathcal{M}}_0^C \to (\hat{\mathcal{M}}_0)_C).
Then we know that $Q_0$ is finite and $\text{Coker}(\tilde{\mathcal{M}}^G \to \tilde{\mathcal{M}}_0^G)$ is finite, and we need to prove that $Q$ is finite. Consider the diagram with exact columns.

\[
\begin{array}{cccc}
\mathcal{N}_C \oplus \tilde{\mathcal{N}}^C & \longrightarrow & \mathcal{M}_C \oplus \tilde{\mathcal{M}}^C & \longrightarrow & (\mathcal{M}_0)_C \oplus \tilde{\mathcal{M}}_0^C \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{N}_C & \longrightarrow & \tilde{\mathcal{M}}_C & \longrightarrow & (\mathcal{M}_0)_C \\
\downarrow & & \downarrow & & \downarrow \\
Q & \longrightarrow & Q_0 & \longrightarrow & Q_0
\end{array}
\]

Using the snake lemma, we obtain that $\text{Ker}(Q \to Q_0) = \text{Coker}(\alpha).$ Since $Q_0$ is finite and $\text{Coker}(\alpha) = \text{Coker}(\tilde{\mathcal{M}}^C \to \tilde{\mathcal{M}}_0^C)$ is finite, we get that $Q$ is finite.

\[\square\]

**Corollary 4.21.** Let $M$ be a tame $\mathbb{Z}[C]$-module and $\mu_M = (x - 1)^{m_1}f_1^{m_1} \cdots f_l^{m_l}$ for some distinct monic irreducible polynomials $f_1, \ldots, f_l \in \mathbb{Z}[x]$ such that $f_i(1) \neq 0.$ Assume that for any $1 \leq i \leq l$ either $f_i(0) \notin \{-1, 1\}$ or $m_i = 1.$ Then the cokernel of the homomorphism

\[(M \otimes^2)^C_0 \oplus (\hat{M} \otimes^2)^C \longrightarrow (\tilde{\mathcal{M}} \otimes^2)_C\]

is finite.

**Remark 4.22.** We prove Propositions 4.18 and 4.20 for tensor products of some modules and their completions. Further we need the same statements for exterior squares. Of course, the statements for tensor products imply the statements for exterior squares, so it is enough to prove for tensor products. Moreover, it is more convenient to prove such statements for tensor products because they have two advantages.

The first obvious advantage is that we can change modules $M$ and $M'$ in the tensor product $M \otimes M'$ independently doing some reductions to `simpler’ modules.

The second less obvious advantage is the following. Let $A$ be an abelian group and $M, M'$ are $\mathbb{Z}[A]$-modules. Then we can interpret coinvariants of the tensor product as the tensor product over $\mathbb{Z}[A]\]

\[(M \otimes M')_A = M \otimes_{\mathbb{Z}[A]} M'_A,\]

where $M'_A$ is the module $M'$ with the twisted module structure $m \ast a = ma^{-1}. In particular, there is an additional nontrivial structure of $\mathbb{Z}[A]$-module on $(M \otimes M')_A.$ But there is no such a structure on $(\Lambda^2 M)_A.$ More precisely, the kernel of the epimorphism

\[(M \otimes M)_A \rightarrow (\Lambda^2 M)_A\]

is not always a $\mathbb{Z}[A]$-submodule. For example, if $A = C = \langle t \rangle,$ $M = \mathbb{Z}^2$ where $t$ acts on $M$ via the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$ it is easy to check that the kernel is not a submodule.

In our article [14] there are two mistakes concerning this that can be fixed easily.

1. On page 562 we define $\Lambda^2 M$ as a quotient module of $M \otimes_A M$ by the submodule generated by the elements $m \otimes m.$ Then we prove Corollaries 3.4 and 3.5 for such a module. In the proof of Proposition 7.2 we assume that $\Lambda^2 M = (\Lambda^2 M)_A.$ which is the first mistake. In order to fix this mistake we have def
of $M \otimes \Lambda M$ by the abelian group generated by the elements $m \otimes m$ and prove Corollaries 3.4 and 3.5 using this definition. The prove is the same. We just need to change the meaning of the word ‘generated’ form ‘generated as a module’ to ‘generated as an abelian group’.

(2) In the proof of Lemma 7.1 we assume that $(\wedge^2 M)_A$ is an $\mathbb{Z}[A]$-module. This is the second mistake. In order to fix it, we have replace $(\wedge^2 M)_A$ by $(M \otimes M)_A$ in the first sentence of the proof of Lemma 7.1.

5. $H\mathbb{Z}$-localization of $M \rtimes C$

Now consider our group $G = M \rtimes C$ and the maximal nilpotent submodule $N \subseteq M$, such that $(M/N)_C$ is finite. We have a natural commutative diagram

$$
\begin{array}{ccc}
N & \longrightarrow & G \\
\downarrow \eta_\omega & & \downarrow \eta_\omega \\
N & \longrightarrow & G/N
\end{array}
$$

Observe that, for any $\mathbb{Z}[C]$-submodule $N' \subseteq N$,

$$(G/N)/N' = \hat{G}/N'.$$

**Lemma 5.1.** For any $\mathbb{Z}[C]$-submodule $N'$ of $N$, there is a natural isomorphism

$$H_2(\eta_\omega)(G) = H_2(\eta_\omega)(G/N').$$

**Proof.** We can present the submodule $N'$ as a finite tower of central extensions. If we will prove that

$$H_2(\eta_\omega)(G) = H_2(\eta_\omega)(G/N')$$

for any $N' \subseteq N$ such that $N'(1-t) = 0$, than we will be able to prove the general statement by induction on class of nilpotence of $N'$.

The assumption that $N'(1-t) = 0$ implies that the extensions

$$1 \rightarrow N' \rightarrow G \rightarrow G/N' \rightarrow 1, \quad 1 \rightarrow N' \rightarrow \hat{G} \rightarrow \hat{G}/N' \rightarrow 1$$

are central. Consider the natural map between sequences (3.2) for these extensions:

$$
\begin{array}{ccccccc}
(G_{ab} \otimes N')/U & \longrightarrow & H_2(G) & \longrightarrow & H_2(G/N') & \longrightarrow & N' & \longrightarrow & H_1(G) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\hat{G}_{ab} \otimes N')/U & \longrightarrow & H_2(\hat{G}) & \longrightarrow & H_2(\hat{G}/N') & \longrightarrow & N' & \longrightarrow & H_1(\hat{G}) \\
& & H_2(\eta_\omega)(G) & \longrightarrow & H_2(\eta_\omega)(G/N') & & & \\
\end{array}
$$

Elementary diagram chasing implies that the lower horizontal map is an isomorphism and the needed statement follows. \hfill \Box

**Lemma 5.2.** If $E(G/N) = T_{\omega+1}(G/N)$, then $EG = T_{\omega+1}(G)$.
Proof. First we observe that, for any $N' \subseteq N$, there is a natural isomorphism

$$T_{\omega+1}(G/N') = T_{\omega+1}(G)/N'.$$

Indeed, lemma [5.1] implies that there is a natural diagram

\[
\begin{array}{cccccccccc}
N' & \rightarrow & N' & \rightarrow & N' & \rightarrow & N' & \rightarrow & N' & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2(\eta_{\omega})(G) & \rightarrow & T_{\omega+1}(G) & \rightarrow & \hat{G} & \rightarrow & \hat{G} & \rightarrow & \hat{G} & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2(\eta_{\omega})(G/N') & \rightarrow & T_{\omega+1}(G/N') & \rightarrow & \hat{G}/N' & \rightarrow & \hat{G}/N' & \rightarrow & \hat{G}/N' & \rightarrow \\
\end{array}
\]

Hence, we have a natural diagram

\[
\begin{array}{cccccccccc}
N' & \rightarrow & G & \rightarrow & G/N' & \rightarrow & G/N' & \rightarrow & G/N' & \rightarrow \\
\downarrow & & \eta_{\omega+1} & & \eta_{\omega+1} & & \eta_{\omega+1} & & \eta_{\omega+1} & & \eta_{\omega+1} \\
N' & \rightarrow & T_{\omega+1}(G) & \rightarrow & T_{\omega+1}(G/N') & \rightarrow & T_{\omega+1}(G/N') & \rightarrow & T_{\omega+1}(G/N') & \rightarrow \\
\end{array}
\]

Again, as in the proof of lemma [5.1] we will assume that $N'$ is central and will prove that, in this case, $EG = T_{\omega+1}(G)$ provided $E(G/N') = T_{\omega+1}(G/N')$.

This follows from comparison of sequences (3.2) applied to the above central extensions:

\[
\begin{array}{cccccccccc}
(G_{ab} \otimes N')/U & \rightarrow & H_2(G) & \rightarrow & H_2(G/N') & \rightarrow & N' & \rightarrow & H_1(G) & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(\hat{G}_{ab} \otimes N')/U & \rightarrow & H_2(T_{\omega+1}(G)) & \rightarrow & H_2(T_{\omega+1}(G/N')) & \rightarrow & N' & \rightarrow & H_1(T_{\omega+1}(G)) & \rightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2(\eta_{\omega+1})(G) & \rightarrow & H_2(\eta_{\omega+1})(G/N') & \rightarrow & H_2(\eta_{\omega+1})(G/N') & \rightarrow & H_2(\eta_{\omega+1})(G/N') & \rightarrow & H_2(\eta_{\omega+1})(G/N') & \rightarrow \\
\end{array}
\]

Again, elementary diagram chasing shows that the lower horizontal map is an isomorphism and the needed statement follows. \qed

Proposition 5.3. For a tame $\mathbb{Z}[C]$-module $M$, the following conditions are equivalent:

(i) $HZ$-length$(M \rtimes C) \leq \omega + 1$;

(ii) the composition

$$\Lambda^2(\hat{M})^C \rightarrow \Lambda^2(\hat{M})^C \rightarrow H_2(\eta_{\omega})$$

is an epimorphism.
Proof. It follows from (3.3) and construction of \( T_{\omega + 1}(G) \) that we have a natural diagram:

\[
\begin{array}{cccc}
H_2(G) & \rightarrow & H_2(G) \\
\downarrow & & \downarrow \\
H_3(\hat{G}) & \rightarrow & G_{ab} \otimes H_2(\eta_\omega)(G) & \rightarrow & H_2(T_{\omega + 1}(G)) & \rightarrow & H_2(\hat{G}) & \rightarrow & H_2(\eta_\omega)(G) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_2(\eta_\omega)(G) & \rightarrow & H_2(\eta_\omega)(G) \\
\end{array}
\]

This diagram implies that the condition (i) for \( G = M \rtimes C \) is equivalent to the surjectivity of the map \( \delta : H_3(\hat{G}) \rightarrow G_{ab} \otimes H_2(\eta_\omega)(G) \). Proposition 3.1 implies the following natural diagram:

\[
\begin{array}{cccc}
H_3(\hat{G}) & \rightarrow & G_{ab} \otimes H_2(\eta_\omega)(G) \\
\downarrow & & \downarrow \\
\Lambda^2(\hat{M})^C & \rightarrow & H_2(\eta_\omega)(G) \\
\downarrow & & \downarrow \\
\Lambda^2(\hat{M})_C & \rightarrow & H_2(\eta_\omega)(G) \\
\end{array}
\]

and the implication \((i) \Rightarrow (ii)\) follows.

Now assume that (ii) holds. Let \( N \) be the maximal nilpotent submodule of \( M \) such that \( (M/N)_C \) is finite. Denote \( H := (M/N) \rtimes C \). We have a natural diagram:

\[
\begin{array}{cccc}
H_3(\hat{G}) & \rightarrow & \Lambda^2(\hat{M})^C & \rightarrow & \Lambda^2(\hat{M})_C & \rightarrow & H_2(\eta_\omega)(G) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H_3(\hat{H}) & \rightarrow & \Lambda^2(\hat{M}/\hat{N})^C & \rightarrow & \Lambda^2(\hat{M}/\hat{N})_C & \rightarrow & H_2(\eta_\omega)(H) \\
\end{array}
\]

Lemma 5.1 implies that the right hand vertical map in this diagram is a natural isomorphism. Condition (ii) implies that the composition of three lower arrows in the last diagram must be an epimorphism. Now observe that \( H_{ab} \otimes H_2(\eta_\omega)(H) = H_2(\eta_\omega)(H) \), since the group \( H_2(\eta_\omega)(H) \) is divisible and \( (M/N)_C \) is finite. Therefore, \( \delta : H_3(\hat{H}) \rightarrow H_{ab} \otimes H_2(\eta_\omega)(H) \) is surjective. The diagram (5.1) with \( G \) replaced by \( H \) implies that \( EH = T_{\omega + 1}(H) \). Now the statement (i) follows from lemma 5.2.

Now we will consider the key example of a tame \( \mathbb{Z}[C] \)-module \( M \), such that \( H\mathbb{Z}\text{-length}(M \rtimes C) > \omega + 1 \). For the construction of such an example, recall first certain well-known properties of quadratic functors.

Let \( X_1, \ldots, X_n, Y_1, \ldots, Y_m \) be abelian groups and \( X = \bigoplus_{i=1}^n X_i, Y = \bigoplus_{j=1}^m Y_j \). An element of a direct sum will be written as a column \((x_1, \ldots, x_n)^T \in X \) and a homomorphism
Given by the same matrix. Moreover, $c = \Lambda$, where

\[ \text{Proof.} \quad \text{Set } Z \text{ and } X \text{ divided (the same as the Whitehead quadratic functor) and tensor squares respectively.} \]

For an abelian group $X$ we denote by $\Lambda X, S^2X, \Gamma^2X$ and $X^\otimes 2$ its exterior, symmetric, divided (the same as the Whitehead quadratic functor) and tensor squares respectively. If $X$ is torsion free, then there are short exact sequences

\[ 0 \rightarrow \Lambda^2X \xrightarrow{\iota_\Lambda} X^\otimes 2 \xrightarrow{\pi_\Sigma} S^2X \rightarrow 0, \]

\[ 0 \rightarrow \Gamma^2X \xrightarrow{\iota_\Gamma} X^\otimes 2 \xrightarrow{\pi_\Lambda} \Lambda^2X \rightarrow 0, \]

where $\pi_\Lambda, \pi_\Sigma$ are the canonical projections

\[ \iota_\Lambda(x_1x_2) = x_1 \otimes x_2 - x_1 \otimes x_2, \]

\[ \iota_\Gamma(\gamma_2(x)) = x \otimes x \]

(see [IB, ch.1 Section 4.3]). We will identify $\Gamma^2X$ with the kernel of $\pi_\Lambda$ for torsion free groups.

**Lemma 5.4.** Let $M = \mathbb{Z}^2$ be the $\mathbb{Z}[C]$-module with the action of $C$ given by the matrix $c = \left( \begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$. Then $\hat{M}(t-1)^{2n} = 4^n\hat{M}$ for any natural $n$ and $\hat{M} = (\mathbb{Z}_2)^2$ with the action of $C$ given by the same matrix. Moreover,

\[ \Lambda^2\hat{M} = \Lambda^2\mathbb{Z}_2 \oplus \Lambda^2\mathbb{Z}_2 \oplus \mathbb{Z}_2^\otimes 2, \]

\[ (\Lambda^2\hat{M})^C = \Lambda^2\mathbb{Z}_2 \oplus 0 \oplus \Gamma^2\mathbb{Z}_2, \]

\[ (\Lambda^2\hat{M})_C = 0 \oplus \Lambda^2\mathbb{Z}_2 \oplus S^2\mathbb{Z}_2 \]

and the cokernel of the natural map

\[ (\Lambda^2\hat{M})^C \rightarrow (\Lambda^2\hat{M})_C \]

is isomorphic $\Lambda^2\mathbb{Z}_2$.

**Proof.** Set $d := c^{-1} = \left( \begin{smallmatrix} -2 & 2 \\ 0 & 1 \end{smallmatrix} \right)$. Since $MI^n = M(t-1)^n = d^n(\mathbb{Z}^2)$, we have $\hat{M} = \lim \mathbb{Z}^2/d^n(\mathbb{Z}^2)$. Computations show that $d^2 = -4c$, and hence $d^{2n} = (-4)^n c^n$. Since $c$ induces an automorphism on $\mathbb{Z}^2$, we obtain $d^{2n}(\mathbb{Z}^2) = 4^n\mathbb{Z}^2$. Thus the filtration $d^n(\mathbb{Z}^2)$ of $\mathbb{Z}^2$ is equivalent to the filtration $2^n\mathbb{Z}^2$. It follows that $\hat{M} = (\mathbb{Z}_2)^2$ with the action of $C$ given by $c$. Then $\Lambda^2\hat{M} \cong (\Lambda^2\mathbb{Z}_2)^2 \oplus \mathbb{Z}_2^\otimes 2$. We identify the element

\[ xe_1 \wedge x'e_1 + ye_2 \wedge y'e_2 + ze_1 \wedge z'e_2 \in \wedge^2\hat{M} \]

with the column

\[ (x \wedge x', y \wedge y', z \wedge z')^T \in (\wedge^2\mathbb{Z}_2)^2 \oplus (\mathbb{Z}_2 \otimes \mathbb{Z}_2), \]

where $e_1, e_2$ is the standard basis of $(\mathbb{Z}_2)^2$ over $\mathbb{Z}_2$. Let us present the homomorphism $\wedge^2\hat{c} : \Lambda^2\hat{M} \rightarrow \Lambda^2\hat{M}$ that defines the action of $C$ as a matrix. Since

\[ \wedge^2\hat{c}(xe_1 \wedge x'e_1) = xe_1 \wedge x'e_1 \]

\[ \wedge^2\hat{c}(ye_2 \wedge y'e_2) = ye_1 \wedge y'e_1 + ye_2 \wedge y'e_2 - (ye_1 \wedge y'e_2 - y'e_1 \wedge ye_2) \]

\[ \wedge^2\hat{c}(ze_1 \wedge z'e_2) = -ze_1 \wedge z'e_2 + ze_1 \wedge z'e_2, \]

we obtain

\[ \wedge^2\hat{c} = \begin{pmatrix} 1 & 1 & -\pi_\Lambda \\ 0 & 1 & 0 \\ 0 & -\pi_\Sigma & 1 \end{pmatrix}, \quad \wedge^2\hat{c} - 1 = \begin{pmatrix} 0 & 1 & -\pi_\Sigma \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]
Now it is easy to compute that
\[(\Lambda^2 \hat{M})^C = \text{Ker}(\wedge^2 \hat{c} - 1) = \Lambda^2 \mathbb{Z}_2 \oplus 0 \oplus \Gamma^2 \mathbb{Z}_2\]
and
\[\text{Im}(\wedge^2 \hat{c} - 1) = \Lambda^2 \mathbb{Z}_2 \oplus 0 \oplus \iota_*(\Lambda^2 \mathbb{Z}_2).\]
It follows that \((\Lambda^2 \hat{M})_C = \text{Coker}(\wedge^2 \hat{c} - 1) = 0 \oplus \Lambda^2 \mathbb{Z}_2 \oplus S^2 \mathbb{Z}_2\). In order to prove that
\[\text{Coker}((\Lambda^2 \hat{M})^C \to (\Lambda^2 \hat{M})_C) = \Lambda^2 \mathbb{Z}_2,\]
we have to prove that \(\text{Coker}(\Gamma^2 \mathbb{Z}_2 \to Z_2^a \to S^2 \mathbb{Z}_2) = 0\). In other words we have to prove that \(Z_2^a\) is generated by elements \(x, y \in \mathbb{Z}_2\). First note that any 2-divisible element is generated by elements
\[2x \otimes y = (x \otimes y - y \otimes x) + (x + y) \otimes (x + y) - x \otimes x - y \otimes y.\]
Since any element of \(\mathbb{Z}_2\) is equal to 2x or 1+2x for some \(x \in \mathbb{Z}_2\), all other elements of \(\mathbb{Z}_2 \otimes \mathbb{Z}_2\) can be presented as sums of elements \((1+2x) \otimes (1+2y) = 1 \otimes 1 + 2(x \otimes 1 + 1 \otimes y + 2x \otimes y)\).

**Theorem 5.5.** Let \(M\) be the \(\mathbb{Z}[C]\)-module from lemma 5.4. The \(\mathbb{Z}\)-length of the group \(G := M \rtimes C = \langle a, b, t \mid [a, b] = 1, a^l = a^{-1}, b^l = ab^{-1}\rangle\) is greater than \(\omega + 1\).

**Proof.** Consider the central sequence
\[1 \to H_2(\eta_\omega)(G) \to T_{\omega+1}(G) \to \hat{G} \to 1\]
We will show that the cokernel of the map \(\delta : H_3(\hat{G}) \to G_{ab} \otimes H_2(\eta_\omega)(G)\) from (3.1) contains \(\Lambda^2(\mathbb{Z}_2)\). The theorem will immediately follow.

By proposition 3.1 the map
\[\delta : H_3(\hat{G}) \to G_{ab} \otimes H_2(\eta_\omega)(G) = H_2(\eta_\omega)(G)\]
factors as
\[H_3(\hat{G}) \to (\Lambda^2 \hat{M})^C \to (\Lambda^2 \hat{M})_C \to H_2(\eta_\omega)(G)\]
The direct summand \(\Lambda^2 \mathbb{Z}_2\) from \((\Lambda^2 \hat{M})_C\) maps isomorphically to a direct summand of \(H_2(\eta_\omega)(G) = \Lambda^2 \mathbb{Z}_2 \oplus (S^2 \mathbb{Z}_2)/\mathbb{Z}\). By lemma 5.4 the summand \(\Lambda^2 \mathbb{Z}_2\) lies in the cokernel of the map \((\Lambda^2 \hat{M})^C \to (\Lambda^2 \hat{M})_C\), therefore, it lies also in the cokernel of the composite map \((\Lambda^2 \hat{M})^C \to H_2(\eta_\omega)(G)\) as well as of the map \(\delta\).

**Theorem 5.6.** Let \(G\) be a metabelian group of the form \(G = M \rtimes C\), where \(M\) is a tame \(C\)-module and \(\mu_M = (x - \lambda_1)^{m_1} \cdots (x - \lambda_l)^{m_l}\) for some distinct complex numbers \(\lambda_1, \ldots, \lambda_l\) and \(m_i \geq 1\).

1. Assume that the equality \(\lambda_i \lambda_j = 1\) holds only if \(\lambda_i = \lambda_j = 1\). Then
\[\text{HZ-length}(G) \leq \omega.\]

2. Assume that the equality \(\lambda_i \lambda_j = 1\) holds only if either \(m_i = m_j = 1\) or \(\lambda_i = \lambda_j = 1\). Then
\[\text{HZ-length}(G) \leq \omega + 1.\]

**Proof.** Follows from propositions 5.3, 4.18 and 4.20. □
Lemma 5.7. Let $M$ be a tame $\mathbb{Z}[C]$-module such that $H_2(M \rtimes C)$ is finite and
\[ \mu_M = (x - \lambda_1)^{m_1} \cdots (x - \lambda_i)^{m_i}, \]
where $\lambda_1, \ldots, \lambda_i$ are distinct complex numbers and $m_i \geq 1$. Then $\lambda_i \lambda_j = 1$ implies $m_i = m_j = 1$.

Proof. Set $V := M \otimes \mathbb{C}$. Then by Lemma 4.1, we get $V = V_i \oplus \cdots \oplus V_l$ such that $\mu_{V_i} = (x - \lambda_i)^{m_i}$.

The short exact sequence
\[ (\Lambda^2 M)_C \to H_2(M \rtimes C) \to M^C \]
implies that $(\kappa^2 M)_C$ is finite. Then $(\Lambda^2_{\mathbb{C}} V)_C = (\Lambda^2 M)_C \otimes \mathbb{C} = 0$. Assume the contrary, that there exist $i, j$ such that $\lambda_i \lambda_j = 1$ and $m_i \geq 2$. Consider two cases $i = j$ and $i \neq j$.

1. Assume that $i = j$. Then $\lambda_i = \lambda_j = -1$. Since $m_i \geq 2$, at least one of Jordan blocks of $a_M \otimes \mathbb{C}$ corresponding to $-1$ has size bigger than 1. It follows that there is an epimorphism $V \to U$, where $U = \mathbb{C}^2$ and $C$ acts on $U$ by the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The epimorphism induces a epimorphism $0 = (\Lambda^2_{\mathbb{C}} V)_C \to (\Lambda^2_{\mathbb{C}} U)_C$. From the other hand, a simple computation shows that $(\Lambda^2_{\mathbb{C}} U)_C \cong \mathbb{C}$. So we get a contradiction.

2. Assume that $i \neq j$. Then $\lambda_i \neq \lambda_j$. Because of the isomorphism of $\mathbb{C}[C]$-modules
\[ \Lambda^2_{\mathbb{C}} V = \bigoplus_{k} \Lambda^2_{\mathbb{C}} V_k \oplus \bigoplus_{k \neq k'} \mathbb{C} \otimes \mathbb{C} V_{k'}, \]
we have an epimorphism of $\mathbb{Z}[C]$-modules $\Lambda^2_{\mathbb{C}} V \to V_i \otimes_{\mathbb{C}} V_j$. Since $m_i \geq 2$, there is an epimorphism $V_i \to U_i$, where $U_i = \mathbb{C}^2$ and $C$ acts on $U_i$ by the matrix $\begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix}$. Moreover, there is an epimorphism $V_j \to U_j$, where $U_j = \mathbb{C}$ and $C$ acts on $U_j$ by the multiplication on $\lambda_j$. It follows that $C$ acts on $U_i \otimes_{\mathbb{C}} U_j$ by the matrix $\begin{pmatrix} 1 & \lambda_j \\ 0 & 1 \end{pmatrix}$. Thus $(U_i \otimes_{\mathbb{C}} U_j)_C \cong \mathbb{C}$. From the other hand we have an epimorphism $0 = (\Lambda^2_{\mathbb{C}} V)_C \to (U_i \otimes U_j)_C$. So we get a contradiction. \hfill \Box

Proposition 5.8. Let $G$ be a metabelian finitely presented group of the form $G = M \rtimes C$ for some $\mathbb{Z}[C]$-module $M$ and $H_2(G)$ is finite. Then
\[ \text{HZ-length}(G) \leq \omega + 1. \]

Proof. It follows from Lemma 5.7 and Theorem 5.6. \hfill \Box

6. Bousfield’s method

Let $G$ be a finitely presented group given by presentation
\[ \langle x_1, \ldots, x_m \mid r_1, \ldots, r_k \rangle \]
Consider the free group $F = F(x_1, \ldots, x_m)$ and an epimorphism $F \to G$ with the kernel normally generated by $k$ elements $\ker F \to G = \langle r_1, \ldots, r_k \rangle^F$. Here we will study the induced map
\[ h : H_2(F) \to H_2(G). \quad (6.1) \]
We follow the scheme due to Bousfield from [5]. Let

\[ F_* : \ldots \longrightarrow F_1 \longrightarrow F_0 (= F) \rightarrow G \]

be a free simplicial resolution of \( G \), where \( F_1 \) a free group with \( m + k \) generators. The structure of \( F_1 \) is \( F(y_1, \ldots, y_k) \ast F \), and the maps \( d_0, d_1 \) are given as

\[ d_0 : y_i \mapsto 1, \quad F^{id} \mapsto F \]
\[ d_1 : y_i \mapsto r_i, \quad F^{id} \mapsto F \]

The following short exact sequence follows from Lemma 5.4 and Proposition 3.13 in [5]:

\[ 1 \rightarrow \lim_{\longrightarrow k}^1 \pi_1(F_*/\gamma_k(F_*)) \rightarrow \pi_0(F_*) \rightarrow \hat{G} \rightarrow 1 \quad (6.2) \]

The first homotopy group \( \pi_1(F_*/\gamma_k(F_*)) \) is isomorphic to the \( k \)th Baer invariant known also as \( k \)-nilpotent multiplicator of \( G \) (see [7], [11]). If \( G = F/R \) for a normal subgroup \( R \lhd F \), the Baer invariant can be presented as the quotient

\[ \pi_1(F_*/\gamma_*,(F_*)) \simeq \frac{R \cap \gamma_k(F)}{[R, F, \ldots, F]} \]

Now assume that, for a group \( G \), the \( \lim_{\longrightarrow k}^1 \)-term vanishes in (6.2), that is, there is a natural isomorphism

\[ \pi_0(F_*) \simeq \hat{G} \]

There is the first quadrant spectral sequence (see [4], page 108)

\[ E_{p,q}^1 = H_q(\hat{F}_p) \Rightarrow H_{p+q}(\bar{W}(F_*)) \]
\[ d_r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r \]

As a result of convergence of this spectral sequence, we have the following diagram:

\[ E_{0,1}^1 = H_2(\hat{F}) \longrightarrow E_{0,1}^\infty \]
\[ \downarrow \]
\[ H_2(\bar{W}(F_*)) \longrightarrow H_2(\hat{G}) \]
\[ \downarrow \]
\[ E_{1,0}^\infty \]

Since \( F_1 \) is finitely generated, \( H_1(\hat{F}_1) \simeq H_1(F_1) \), therefore, \( E_{1,0}^\infty \) is a quotient of \( \pi_1((F_*)_{ab}) = H_2(G) \). Now the diagram (6.3) implies the following
Lemma 6.1. Assuming that $\lim_{\rightarrow}^1$-term in (6.2) vanishes, the cokernel of the map (6.1) is isomorphic to a quotient of the homology group $H_2(G)$. For a group $G$ with $H_2(G) = 0$, the map $h: H_2(F) \to H_2(G)$ is an epimorphism.

7. Main construction

Lemma 7.1. Let $K = \langle a, b | [[a, b], a] = [[a, b], b] = 1 \rangle$ be the free 2-generated nilpotent group of class 2 and $C = \langle t \rangle$ acts on $K$ by the following automorphism

\[
\begin{align*}
a & \mapsto a^{-1} \\
b & \mapsto ab^{-1}.
\end{align*}
\]

Set $\tilde{K} := \lim K/\gamma_n(K \rtimes C)$. Then the following holds

1. the pronilpotent completion of $K \rtimes C$ is equal to $\tilde{K} \rtimes C$;
2. $K_{ab}$ is isomorphic to the $\mathbb{Z}[C]$-module from Lemma [5.4];
3. the obvious map $\tilde{K}_{ab} \cong (K_{ab})$ is an isomorphism of $\mathbb{Z}[C]$-modules;
4. either $[\tilde{K}, K] \cong \mathbb{Z}$ or $[\tilde{K}, K] \cong \mathbb{Z}/2^m$ for some $m$.

Proof. For the sake of simplicity we set $K_n := \gamma_n(K \rtimes C)$. (1) follows from the equality

\[ (K \rtimes C)/K_n = (K/K_n) \rtimes C \]

for $n \geq 2$. (2) is obvious. Prove (3) and (4).

It is obvious that $\gamma_2(K) = Z(K) = \{[a, b]^k | k \in \mathbb{Z}\} \cong \mathbb{Z}$ and $[a^i, b^j] = [a, b]^{ij}$ for all $i, j \in \mathbb{Z}$. Moreover, any element of $K$ can be uniquely presented as $a^i b^j[a, b]^k$ for $i, j, k \in \mathbb{Z}$. Set $M = K_{ab}$. We consider $M$ with the additive notation as a module over $C$. Then $\gamma_{n+1}(M \rtimes C) = M(t-1)^n$. Lemma [5.4] implies that $\gamma_{2n+1}(M \rtimes C) = 4^n M$. Since the map $K_{2n+1} \to \gamma_{2n+1}(K \rtimes C)$ is an epimorphism, there exist $k, l \in \mathbb{Z}$ (that depend on $n$) such that $a^{4^n}[a, b]^k b^{2^n}[a, b]^l \in K_{2n+1}$. Since $[a, b]^{i2^n} = [a^{4^n}[a, b]^k, b^{2^n}[a, b]^l]$, we obtain $[a, b]^{i2^n} \in K_{2n+1}$. Then $K_{2n+1} \cap \gamma_2(K) = \langle [a, b]^{i2^n(n+1)} \rangle$ for some natural number $m(2n + 1)$ because $4^{2n}$ is divisible only on powers of 2. Hence $K_n \cap \gamma_2(K) = \langle [a, b]^{i2^n(n)} \rangle$ for some $m(n)$. Then the short exact sequence $\gamma_2(K) \to K \to M$ induces the short exact sequence

\[
0 \to \mathbb{Z}/2^{m(n)} \overset{\iota_n}{\to} K/K_n \to M/M(t-1)^n \to 1,
\]

where $\iota_n(1) = [a, b]$. Since $\mathbb{Z}/2^{m(n)}$ and $M/M(t-1)^n$ are finite 2-groups, the order of $K/K_n$ is equal to $2^{m(n)}$ for some $m(n)$. We obtain a short exact sequence

\[
0 \to \lim_{\leftarrow} \mathbb{Z}/2^{m(n)} \overset{i}{\to} \tilde{K} \to \hat{M} \to 1.
\]

If $m(n) \to \infty$, then $\lim_{\leftarrow} \mathbb{Z}/2^{m(n)} = \mathbb{Z}_2$, else $m(n)$ stabilizes and $\lim_{\leftarrow} \mathbb{Z}/2^{m(n)} = \mathbb{Z}/2^m$.

Now it is sufficient to prove that $[\tilde{K}, \tilde{K}] = \text{Im}(i)$. Since $\hat{M}$ is abelian, $[\tilde{K}, \hat{K}] \subseteq \text{Im}(i)$. Prove that $[\tilde{K}, \tilde{K}] \subseteq \text{Im}(i)$. Any element of $\tilde{K}$ can be presented as a sequence $(x_n)_{n=1}^\infty$, where $x_n \in K/K_n$ such that $x_n \equiv x_{n+1} \mod K_n$. Any element of $\lim_{\leftarrow} \mathbb{Z}/2^{m(n)}$ can be presented as an image of a 2-adic integer $\sum_{k=0}^\infty \alpha_k 2^k$, where $\alpha_k \in \{0, 1\}$. Then an element of $\text{Im}(i)$ can be presented as a sequence

\[ ([a, b]^{\sum_{k=0}^{m(n)} \alpha_k 2^k})_{n=1}^\infty = ([a^{\sum_{k=0}^{m(n)} \alpha_k 2^k}, b])_{n=1}^\infty. \]
Note that the element \((a_{k=0}^{m(n)} a_k b_k)_{n=1}^{\infty}\) is a well defined element of \(\tilde{K}\) because \(a_{m(n)}^{m(n)} \in K_n\). It follows that
\[
([a, b]_{k=0}^{m(n)} a_k b_k)_{n=1}^{\infty} = \left(\left(a_{k=0}^{m(n)} a_k b_k\right)_{n=1}^{\infty}, (b)_{n=1}^{\infty}\right),
\]
and hence, \([\tilde{K}, \tilde{K}] \geq \text{Im}(i)\). □

**Theorem 7.2.** Let \(F\) be a free group of rank \(\geq 2\). Then \(H\Sigma\text{-length}(F) \geq \omega + 2\).

**Proof.** Consider the following group:
\[\Gamma := \langle a, b, t \mid [[a, b], a] = [b, a], b, a = 1, a^t = a^{-1}, b^t = ab^{-1}\rangle\]
Observe that \(\Gamma\) is the semidirect product \(K \rtimes C\) from the previous lemma.
Consider the natural diagram induced by the projection \(K \to K_{ab} = M\), i.e. by \(\Gamma = K \rtimes C \to G = M \rtimes C\):

\[
\begin{array}{ccc}
H_3(\hat{\Gamma}) & \xrightarrow{\delta} & H_1(\Gamma) \otimes H_2(\eta_\omega)(\Gamma) \\
\downarrow & & \downarrow \\
H_3(\hat{G}) & \xrightarrow{\delta} & H_1(G) \otimes H_2(\eta_\omega)(G) \xrightarrow{\sim} \text{Coker}(\delta)(G)
\end{array}
\]

(7.1)

It is shown in the proof of theorem 5.5 that \(\text{Coker}(\delta)(G)\) contains \(\Lambda^2Z_2\). This term \(\Lambda^2Z_2\) is an epimorphic image of one of the terms \(\Lambda^2Z_2\) in
\[H_2(\hat{M}) = \Lambda^2\hat{M} = \Lambda^2Z_2 \oplus \Lambda^2Z_2 \oplus Z_2 \otimes Z_2\]
(see lemma 5.4). By lemma 7.1 there is a short exact sequence
\[H_2(\hat{K}) \to H_2(\hat{M}) \to [\tilde{K}, \tilde{K}]\]
Now, by lemma 7.1 \([\tilde{K}, \tilde{K}]\) is either \(Z_2\) or a finite group, therefore, the image of both terms \(\Lambda^2Z_2\) in \([\tilde{K}, \tilde{K}]\) are zero (2-adic integers do not contain divisible subgroups). In particular, the term \(\Lambda^2Z_2\) which maps isomorphically onto a subgroup of \(\text{Coker}(\delta)(G)\) lies in the image of \(H_2(\hat{K})\). The natural square
\[
\begin{array}{ccc}
H_2(\hat{K}) & \to & H_2(\eta_\omega)(\Gamma) \\
\downarrow & & \downarrow \\
H_2(\hat{M}) & \to & H_2(\eta_\omega)(G)
\end{array}
\]
is commutative and we conclude that the diagonal arrow in (7.1) maps onto the subgroup \(\Lambda^2Z_2\) of \(\text{Coker}(\delta)(G)\). Hence, \(\text{Coker}(\delta)(\Gamma)\) maps epimorphically onto \(\Lambda^2Z_2\).

Now let's prove that the second homology \(H_2(\Gamma)\) is finite. Since the group \(\Gamma\) is the semi-direct product \(K \rtimes C\), its homology is given as
\[0 \to H_2(K)_C \to H_2(\Gamma) \to H_1(C, K_{ab}) \to 0\]
The right hand term is zero: \(H_1(C, K_{ab}) = H_1(C, M) = M^C = 0\). It follows immediately that \(H_2(K)_C = (\gamma_3(F(a, b))/\gamma_4(F(a, b)))_C = Z/4\).
Observe that the group $\Gamma$ can be defined with two generators only. Let $F$ be a free group of rank $\geq 2$. Consider a free simplicial resolution of $\Gamma$ with $F_0 = F$, $F_* \to \Gamma$. Since $H_2(\Gamma)$ is finite, all Baer invariants of $\Gamma$ are finite (see, for example, [11])

$$\lim_{\leftarrow}^{\pi_1(F_*/\gamma_k(F_*))} = 0$$

and, by lemma [6.1], the cokernel of the natural map $H_2(\hat{F}) \to H_2(\hat{\Gamma})$ is finite.

We have a natural commutative diagram:

$$
\begin{array}{ccc}
H_3(\hat{F}) & \xrightarrow{\delta} & H_1(F) \otimes H_2(\eta_\omega)(F) \\
& \downarrow & \downarrow \\
H_3(\hat{\Gamma}) & \xrightarrow{\delta} & H_1(\Gamma) \otimes H_2(\eta_\omega)(\Gamma)
\end{array} \longrightarrow \text{Coker}(\delta)(F) \longrightarrow \text{Coker}(\delta)(\Gamma)

(7.2)

with $H_2(\eta_\omega)(F) = H_2(\hat{F})$. Since the cokernel of $H_2(\hat{F}) \to H_2(\hat{\Gamma})$ is finite,

$$\text{Coker}\{\text{Coker}(\delta)(F) \to \text{Coker}(\delta)(\Gamma)\}$$

also is finite. However, $\text{Coker}(\delta)(\Gamma)$ maps onto $\Lambda^2\mathbb{Z}_{\infty}$, as we saw before, hence $\text{Coker}(\delta)(F)$ is uncountable. Therefore, by (3.3), $H_2(T_{\omega+1}(F)) \neq 0$ and the statement is proved. \[\square\]

8. Alternative approaches

In general, given a group, the description of its pro-nilpotent completion is a difficult problem. If a group is not pre-nilpotent, its pro-nilpotent is uncountable and may contain complicated subgroups. In this paper, as well as in [14] we essentially used the explicit structure of pro-nilpotent completion for metabelian groups. Now we observe that, there is a trick which gives a way to show that some groups have $HZ$-length greater than $\omega$ without explicit description of their pro-nilpotent completion. In a sense, the Bousfield scheme described above also gives such a method, however in that way one must compare the considered group with a group with clear completion. The trick given below is different.

We put $\Phi_i H_2(G) = \text{Ker}\{H_2(G) \to H_2(G/\gamma_{i+1}(G))\}$. Then $\Phi_i H_2(G)$ is the Dwyer filtration on $H_2(G)$ (see [8]).

**Proposition 8.1.** Let $G$ be a group with the following properties:

(i) $\gamma_\omega(G) \neq \gamma_{\omega+1}(G)$
(ii) $\cap_i \Phi_i H_2(G/\gamma_\omega(G)) = 0$.

Then the $HZ$-length of $G$ is greater than $\omega$. 
Proof. All ingredients of the proof are in the following diagram with exact horizontal and vertical sequences:

![Diagram with labeled arrows and groups]

The fact that the kernel lies in $\cap_i \Phi_i H_2(G/\gamma_\omega(G))$ follows from the standard property of Dwyer’s filtration: the map $G/\gamma_\omega(G) \to \hat{G}$ induces isomorphisms on $H_2/\Phi_i$ for all $i$ (see [8]). The vertical exact sequence is the part of 5-term sequence in homology. Now assume that the map $H_2(G) \to H_2(\hat{G})$ is an epimorphism. Then $H_2(G/\gamma_\omega(G)) \to H_2(\hat{G})$ is an epimorphism as well. However, condition (ii) implies that the last map is a monomorphism as well. Hence, it is an isomorphism and surjectivity of $H_2(G) \to H_2(\hat{G})$ contradicts the property (i). □

An example of a group with satisfies both conditions (i) and (ii) from proposition, is the group $\langle a, b | [a, b^3] = [[a, b], a]^2 = 1 \rangle$ (see [16], examples 1.70 and 1.85). However, in this example, $G/\gamma_\omega(G)$ is metabelian and one can use explicit description of its pro-nilpotent completion to show the same result.

The above proposition can be used for more complicated groups.

Now we consider another example of finitely generated metabelian group of the type $M \rtimes C$ with $\mathbb{H}_Z$-length greater than $\omega + 1$. In our example, $M = \mathbb{Z}[C]$ with a regular action of $C$ as multiplication. This is not a tame $\mathbb{Z}[C]$-module and the group is not finitely presented. The proof of the bellow result uses functorial arguments.

**Theorem 8.2.** Let $G = \mathbb{Z}[C] \rtimes C = \mathbb{Z} \wr \mathbb{Z} = \langle a, b | [a, a^{b^i}] = 1, \ i \in \mathbb{Z} \rangle$, then $\mathbb{H}_Z$-length$(G) \geq \omega + 2$.

**Proof.** We define a functor from the category $f\text{Ab}$ of finitely generated free abelian groups to the category of groups as follows:

$$\mathcal{F} : A \mapsto (A \otimes M) \rtimes C, \ A \in f\text{Ab},$$

where the action of $C$ on $A$ is trivial and $M = \mathbb{Z}[C]$. Now our main example can be written as $G = \mathcal{F}(\mathbb{Z})$. Since the action of $C$ on $A$ is trivial, the pro-nilpotent completion of $\mathcal{F}(A)$ can be described as follows:

$$\overline{\mathcal{F}(A)} = (A \otimes \hat{M}) \rtimes C.$$

One can easily see that

$$H_i(C, H_i(A \otimes \hat{M})) = 0, i \geq 1.$$
Since the abelian group $A \otimes \hat{M}$ is torsion-free,

$$H_i(\mathcal{F}(A)) = \Lambda^i(A \otimes \hat{M})_C, \; i \geq 1.$$  

Now we have

$$H_2(\eta_\omega)(\mathcal{F}(A)) = \text{Coker}\{\Lambda^2(A \otimes M)_C \to \Lambda^2(A \otimes \hat{M})_C\}.$$  

Now consider the functor from the category of finitely generated free abelian groups to all abelian groups

$$\mathcal{G} : A \mapsto H_2(\eta_\omega)(\mathcal{F}(A))$$

It follows immediately that $\mathcal{G}$ is a quadratic functor. Indeed, it is a proper quotient of the functor $A \mapsto A \otimes A \otimes B$ for a fixed abelian group $B$.

The exact sequence (3.3) applied to the stem-extension

$$1 \to H_2(\eta_\omega)(\mathcal{F}(A)) \to T_{\omega+1}(\mathcal{F}(A)) \to \mathcal{F}(A) \to 1$$

can be viewed as an exact sequence of functors in the category $\text{sAb}$. Consider the map $\delta$ as a natural transformation in $\text{sAb}$. The functor $H_3(\mathcal{F}(A)) = \Lambda^3(A \otimes \hat{M})_C$ is a quotient of the cubic functor

$$A \otimes A \otimes A \otimes D$$

for some fixed abelian group $D$ (which equals to the tensor cube of $\hat{M}$). Recall that any natural transformation of the form

$$A \otimes A \otimes A \otimes D \to \text{quadratic functor}$$

is zero. This follows from the simple observation that $A \otimes A \otimes A \otimes D$ is a natural epimorphic image of its triple cross-effect. See, for example, [17] for generalizations and detailed discussion of this observation.

Therefore, for any non-zero $A$, the map $\delta$ in (3.3) is zero and $H_2(T_{\omega+1}(\mathcal{F}(A)))$ contains a subgroup $H_2(\eta_\omega)(\mathcal{F}(A))$ which is uncountable. In particular, $H_2(T_{\omega+1}(G)) = H_2(T_{\omega+1}(\mathcal{F}(Z)))$ is uncountable and hence $H_2(G) \to H_2(T_{\omega+1}(G))$ is not an epimorphism. \[\square\]

**Remark 8.3.** If we change the group $Z \wr Z$ by lamplighter group by adding one more relation, we will obtain another wreath product (for $n \geq 2$)

$$Z/n \wr Z = \langle a, b \mid a^n = 1, [a, a^i] = 1, \; i \in \mathbb{Z} \rangle$$

One can use the scheme of this paper to prove that $HZ$-length$(Z/n \wr Z) > \omega + 1$. Essentially it follows from the triviality of $\Lambda^2(M)_C$ as in Theorem 8.2. As it was shown by G. Baumslag [1], there are different ways to embed this group into a finitely-presented metabelian group. For example, $Z/n \wr Z$ is a subgroup generated by $a, b$ in

$$G_{[n]} := \langle a, b, c \mid a^n = [a, a^b] = [a, a^{b^2}] = [b, c] = [a^c = aa^b a^{-b^2}] \rangle.$$  

Now we have a decomposition $G_{[n]} = (Z/n)^{\infty} \rtimes (C \times C)$ and it follows immediately from [14] that $HZ$-length$(G_{[n]}) = \omega$, since $H_2(\eta_\omega)(G_{[n]})$ is divisible.
9. Concluding remarks

Lemma 9.1. For any prime \( p \) and \( n \geq 2 \) the embedding \( \mathbb{Z}_p \to \mathbb{Q}_p \) induces an isomorphism

\[
\Lambda^n \mathbb{Z}_p \cong \Lambda^n \mathbb{Q}_p.
\]

In particular, \( \Lambda^n \mathbb{Z}_p \) is a \( \mathbb{Q} \)-vector space of countable dimension.

Proof. Since \( \mathbb{Z}_p \) and \( \mathbb{Q}_p \) are torsion free, the map \( \mathbb{Z}_p^{\otimes n} \to \mathbb{Q}_p^{\otimes n} \) is a monomorphism. Since for torsion free groups the exterior power is embedded into the tensor power, we obtain that the map \( \Lambda^n \mathbb{Z}_p \to \Lambda^n \mathbb{Q}_p \) is a monomorphism. So we can identify \( \Lambda^n \mathbb{Z}_p \) with its image in \( \Lambda^n \mathbb{Q}_p \). Since \( \mathbb{Q}_p = \bigcup_{p} \mathbb{Z}_p \), we obtain \( \Lambda^n \mathbb{Q}_p = \bigcup_{p} \Lambda^n \mathbb{Z}_p \). Then it is enough to prove that \( \Lambda^n \mathbb{Z}_p \) is \( p \)-divisible. For any \( a \in \mathbb{Z}_p \) we consider the decomposition \( a = a^{(0)} + a^{(1)} \), where \( a^{(0)} \in \{0, \ldots, p-1 \} \) and \( a^{(1)} = p \cdot b \) for some \( b \in \mathbb{Z}_p \). Then for any \( a_1, \ldots, a_n \in \mathbb{Z}_p \) we have \( a_1 \wedge \cdots \wedge a_n = \sum a_1^{(i_1)} \wedge \cdots \wedge a_n^{(i_n)} \), where \( (i_1, \ldots, i_n) \) runs over \( \{0,1\}^n \). For any sequence \( (i_1, \ldots, i_n) \) except \( (0, \ldots, 0) \) the element \( a_1^{(i_1)} \wedge \cdots \wedge a_n^{(i_n)} \) is \( p \) divisible because \( a_i^{(1)} \) is \( p \)-divisible. Since \( \Lambda^n \mathbb{Z} = 0 \), we get \( a_1^{(0)} \wedge \cdots \wedge a_n^{(0)} = 0 \). It follows that \( a_1 \wedge \cdots \wedge a_n \) is \( p \)-divisible. Thus \( \Lambda^n \mathbb{Z}_p \) is \( p \)-divisible. \( \square \)

Remind that the Klein bottle group is given by \( \hat{G}_{KL} = \mathbb{Z} \times C \), where \( C \) acts on \( \mathbb{Z} \) by the multiplication on \(-1\). Its pronilpotent completion is equal to \( \mathbb{Z}_2 \rtimes C \). Consider the map \( w : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \Lambda^2 \mathbb{Z}_2 \) given by \( w(a,b) = \frac{1}{2} a \wedge b \). Here we use that \( \Lambda^2 \mathbb{Z}_p \) is a \( \mathbb{Q} \)-vector space. It is easy to see that \( w \) is a 2-cocycle and we get the corresponding central extension

\[
\Lambda^2 \mathbb{Z}_2 \to N_w \to \mathbb{Z}_2,
\]

whose underlying set is \( (\Lambda^2 \mathbb{Z}_2) \times \mathbb{Z}_2 \) and the product is given by

\[
(\alpha, a)(\beta, b) = \left( \alpha + \beta + \frac{1}{2} a \wedge b, \ a + b \right).
\]

We define the action of \( C \) on \( N_w \) by the map \( (\alpha, a) \mapsto (\alpha, -a) \).

Proposition 9.2. There is an isomorphism

\[
\text{EG}_{KL} = N_w \rtimes C.
\]

In other words, \( \text{EG}_{KL} \) can be described as the set \( (\Lambda^2 \mathbb{Z}_2) \times \mathbb{Z}_2 \times C \) with the multiplication given by

\[
(\alpha, a, t^i)(\beta, b, t^j) = \left( \alpha + \beta + \frac{(-1)^i}{2} a \wedge b, \ a + (-1)^j b, \ t^{i+j} \right).
\]

Proof. Set \( G := G_{KL} \). Then \( \hat{G} = \mathbb{Z}_2 \rtimes C \). The Lyndon-Hochschild-Serre spectral sequences imply that \( H_2(G) = 0 \) and the map \( \Lambda^2 \mathbb{Z}_2 = H_2(\mathbb{Z}_2) \xrightarrow{\cong} H_2(\hat{G}) \) is an isomorphism. Theorem 5.6 implies that \( T_{\omega+1} G = \text{EG} \). Then \( \text{EG} \) is the universal relative central extension \( (H_2(\eta_\omega) \to \text{EG} \to \hat{G}, \eta_\omega) \). Since \( H_2(\eta_\omega) = \text{Coker}(H_2(G) \to H_2(\hat{G})) \) and \( H_2(G) = 0 \), we obtain that \( H_2(\mathbb{Z}_2) \xrightarrow{\cong} H_2(\hat{G}) \xrightarrow{\cong} H_2(\eta_\omega) \) are isomorphisms. The continous maps
$B\mathbb{Z}_2 \to B\hat{G} \to \text{Cone}(B\eta_w)$ give the commutative diagram for any abelian group $A$:

\[
\begin{array}{ccc}
H^2(\eta_w, A) & \overset{\sim}{\longrightarrow} & \text{Hom}(H_2(\eta_w), A) \\
\downarrow & & \downarrow \\
H^2(\hat{G}, A) & \overset{\sim}{\longrightarrow} & \text{Hom}(H_2(\hat{G}), A) \\
\downarrow & & \downarrow \\
H^2(\mathbb{Z}_2, A) & \longrightarrow & \text{Hom}(H_2(\mathbb{Z}_2), A)
\end{array}
\] (9.1)

If $A$ is a divisible abelian group, then $\text{Ext}(H_1(\hat{G}), A) = 0 = \text{Ext}(\mathbb{Z}_2, A)$, and all morphisms in the diagram (9.1) are isomorphisms. In particular, if $A = H_2(\eta_w) = H_2(\hat{G}) = \Lambda^2\mathbb{Z}_2$, then the morphisms induce isomorphisms

$$H^2(\eta_w, \Lambda^2\mathbb{Z}_2) \cong H^2(\hat{G}, \Lambda^2\mathbb{Z}_2) \cong H^2(\mathbb{Z}_2, \Lambda^2\mathbb{Z}_2) \cong \text{Hom}(\Lambda^2\mathbb{Z}_2, \Lambda^2\mathbb{Z}_2)$$

and the extension $\Lambda^2\mathbb{Z}_2 \twoheadrightarrow EG \to \hat{G}$ corresponds to the identity map in $\text{Hom}(\Lambda^2\mathbb{Z}_2, \Lambda^2\mathbb{Z}_2)$. Therefore, it is sufficient to prove that the extension $\Lambda^2\mathbb{Z}_2 \twoheadrightarrow N_w \times C \to \hat{G}$ goes to the identity map via the composition

$$H^2(\hat{G}, \Lambda^2\mathbb{Z}_2) \to H^2(\mathbb{Z}_2, \Lambda^2\mathbb{Z}_2) \to \text{Hom}(\Lambda^2\mathbb{Z}_2, \Lambda^2\mathbb{Z}_2).$$

The map $H^2(\hat{G}, \Lambda^2\mathbb{Z}_2) \to H^2(\mathbb{Z}_2, \Lambda^2\mathbb{Z}_2)$ on the level of extensions is given by the pullback. It follows that the extension $\Lambda^2\mathbb{Z}_2 \twoheadrightarrow N_w \times C \to \hat{G}$ goes to the extension $\Lambda^2\mathbb{Z}_2 \twoheadrightarrow N_w \twoheadrightarrow \mathbb{Z}_2$. Then we need to prove that $w$ goes to the identity map under the map $H^2(\mathbb{Z}_2, \Lambda^2\mathbb{Z}_2) \to \text{Hom}(\Lambda^2\mathbb{Z}_2, \Lambda^2\mathbb{Z}_2)$.

For any group $H$ and an abelian group $A$ the map $H^2(H, A) \to \text{Hom}(H_2(H), A)$ on the level of cocycles is induced by the evaluation map $C^2(H, A) \to \text{Hom}(C_2(H), A)$ given by $u \mapsto ((h, h') \mapsto u(h, h'))$. For an abelian group $H$ the map $\Lambda^2 H \to H_2(H)$ given by $h \wedge h' \mapsto (h, h') - (h', h) + B_2(H)$ is an isomorphism $\Lambda^2 H \cong H_2(H)$ (see [6] Ch.V §5-6)). Then the map $H^2(H, A) \to \text{Hom}(\Lambda^2 H, A)$ is given by $u \mapsto (h \wedge h' \mapsto u(h, h') - u(h', h))$. Since $w(a, b) - w(b, a) = \frac{1}{2}(a \wedge b - b \wedge a) = a \wedge b, w$ goes to the identity map. \qed

Remark 9.3. If we identify $EG_{K_{I_1}}$ with the set $(\Lambda^2\mathbb{Z}_2) \times \mathbb{Z}_2 \times C$, it is easy to check that $\gamma_2(EG_{K_{I_1}}) = (\Lambda^2\mathbb{Z}_2) \times 2\mathbb{Z}_2 \times 1$ and $[\gamma_2(EG_{K_{I_1}}), \gamma_2(EG_{K_{I_1}})] = (\Lambda^2\mathbb{Z}_2) \times 0 \times 1$. It follows that $EG_{K_{I_1}}$ is a solvable group of class 3 but it is not metabelian. In particular, the class of metabelian groups is not closed under the $HZ$-localization.

Finishing this paper, we mention some possibilities of generalization of the obtained results. We conjecture that, the tame $\mathbb{Z}[C]$-module $M$ defined by the $k \times k$-matrix ($k \geq 2$)

\[
\begin{pmatrix}
-1 & 1 & 0 & \ldots & 0 \\
0 & -1 & 1 & 0 & \ldots \\
\vdots \\
0 & \ldots & 0 & 0 & -1
\end{pmatrix}
\]
defines the group $M \rtimes C$ with $HZ$-length $\omega + l(k)$, where $l(k) \in \mathbb{N}$ and $l(k) \to \infty$ for $k \to \infty$. With the help of this group, one can try to use the same scheme as in this paper to prove that $HZ$-length of a free non-cyclic group is $\geq 2\omega$.

**Acknowledgement**

The research is supported by the Russian Science Foundation grant N 16-11-10073.

**References**

[1] G. Baumslag: Subgroups of finitely presented metabelian groups, *J. Austral. Math. Soc.* 16 (1973), 98–110, Collection of articles dedicated to the memory of Hanna Neumann.

[2] R. Bieri and K. Strebel: Almost finitely presented soluble groups, *Comm. Math. Helv.* 53 (1978), 258–278.

[3] R. Bieri and R. Strebel: A geometric invariant for modules over an abelian group, *J. Reine Angew. Math.* 322 (1981), 170–189.

[4] A. K. Bousfield and D. Kan: *Homotopy limits, completions and localizations*, Lecture Notes in Mathematics 304, (1972).

[5] A. K. Bousfield: Homological localization towers for groups and $\pi$-modules, Mem. Amer. Math. Soc, no. 186, 1977.

[6] K. Brown, Cohomology of Groups, Springer-Verlag GTM 87, 1982.

[7] J. Burns and G. Ellis: On the nilpotent multipliers of a group, *Math. Z.* 226 (1997), 405–428.

[8] W. Dwyer: Homology, Massey products and maps between groups, *J. Pure Appl. Algebra*, 6 (1975), 177–190.

[9] B. Eckmann, P. Hilton and U. Stammbach: On the homology theory of central group extensions: I - the commutator map and stem extensions, *Comm. Math. Helv.* 47, (1972), 102–122.

[10] B. Eckmann, P. Hilton and U. Stammbach: On the homology theory of central group extensions: II - the exact sequence in the general case, *Comm. Math. Helv.* 47, (1972), 171–178.

[11] G. Ellis: A Magnus-Witt type isomorphism for non-free groups, *Georgian Math. J.* 9 (2002), 703–708.

[12] L. Fuchs: Infinite abelian groups, Academic Press, New York and London.

[13] L. Illusie: Complexe Cotangent et Déformation I, Lecture Notes in Mathematics, vol. 239, Springer, Berlin, 1971.

[14] S. O. Ivanov and R. Mikhailov: On a problem of Bousfield for metabelian groups: *Advances in Math.* 290 (2016), 552–589.

[15] F. Kasch: Modules and Rings, Acad. Press, London New York, 1982.

[16] R. Mikhailov and I. B. S. Passi: *Lower Central and Dimension Series of Groups*, LNM Vol. 1952, Springer 2009.

[17] R. Mikhailov: Polynomial functors and homotopy theory, Progress in Math, 311 (2016), arXiv: 1202.0586.

[18] C. Weibel: An Introduction to Homological Algebra, Cambridge Univ. Press, 1994.

[19] O. Zariski and P. Samuel, Commutative Algebra, Vol. 2, Van Nostrand, Princeton 1960.

Chebyshev Laboratory, St. Petersburg State University, 14th Line, 29b, Saint Petersburg, 199178 Russia

*E-mail address: ivanov.s.o.1986@gmail.com*

Chebyshev Laboratory, St. Petersburg State University, 14th Line, 29b, Saint Petersburg, 199178 Russia and St. Petersburg Department of Steklov Mathematical Institute

*E-mail address: rmikhailov@mail.ru*