Research Article

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Finite energy solutions to inhomogeneous nonlinear elliptic equations with sub-natural growth terms

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Abstract: We obtain necessary and sufficient conditions for the existence of a positive finite energy solution to the inhomogeneous quasilinear elliptic equation

$$-\Delta_p u = \sigma u^q + \mu$$

in the sub-natural growth case $0 < q < p - 1$, where $\Delta_p$ ($1 < p < \infty$) is the $p$-Laplacian, and $\sigma$, $\mu$ are positive Borel measures on $\mathbb{R}^n$. Uniqueness of such a solution is established as well. Similar inhomogeneous problems in the sublinear case $0 < q < 1$ are treated for the fractional Laplace operator $(-\Delta)^\alpha$ in place of $-\Delta_p$, on $\mathbb{R}^n$ for $0 < \alpha < \frac{n}{2}$, and on an arbitrary domain $\Omega \subset \mathbb{R}^n$ with positive Green’s function in the classical case $\alpha = 1$.

Keywords: Quasilinear elliptic equation, finite energy solution, $p$-Laplacian, Wolff potential, fractional Laplacian, Green’s function

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1 Introduction

We consider the quasilinear elliptic equation

$$-\Delta_p u = \sigma u^q + \mu$$

in the sub-natural growth case $0 < q < p - 1$.

Here $\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian with $1 < p < \infty$, and $\sigma$, $\mu$ are nontrivial nonnegative locally integrable functions on $\mathbb{R}^n$, or more generally, nonnegative locally finite Borel measures on $\mathbb{R}^n$ (in brief $\sigma, \mu \in \mathcal{M}(\mathbb{R}^n)$) such that $\sigma \neq 0$ and $\mu \neq 0$. The homogeneous case $\mu = 0$ was considered earlier in [10]. However, treating general data $\mu \geq 0$ leads to some new phenomena involving possible interaction between $\mu$ and $\sigma$.

We establish necessary and sufficient conditions on both $\sigma$ and $\mu$ for the existence of a positive finite energy solution $u$ to (1.1), so that $\int_{\mathbb{R}^n} |\nabla u|^p \, dx < +\infty$ (see Definition 2.1), and prove its uniqueness.

Our methods are also applicable to the existence problem for positive finite energy solutions $u \in H^\alpha(\mathbb{R}^n)$, so that $\int_{\mathbb{R}^n} (-\Delta)^\frac{\alpha}{2} u^2 \, dx < +\infty$ (see Definition 4.1), to the fractional Laplace equation

$$(-\Delta)^\alpha u = \sigma u^q + \mu$$

in $\mathbb{R}^n$. 

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where $0 < q < 1$ and $(-\Delta)^{\sigma}$ is the fractional Laplacian with $0 < \alpha < \frac{4}{n}$. Uniqueness of such a solution is proved in the case $0 < \alpha \leq 1$.

In the classical case $\alpha = 1$, our approach is employed to obtain the existence and uniqueness of a positive finite energy solution $u \in W^{1,2}_{0} (\Omega)$, such that $\int_{\Omega} |\nabla u|^{2} \, dx < +\infty$ (see Definition 2.1 in the case $p = 2$), to the equation

$$-\Delta u = \sigma u^{q} + \mu \quad \text{in} \quad \Omega,$$

where $0 < q < 1$ and $\Omega \subset \mathbb{R}^{n}$ is an arbitrary domain (possibly unbounded) which possesses a positive Green’s function. The existence of positive weak solutions to (1.3), not necessarily of finite energy, is discussed in [23, 24].

We would like to point out that the existence and uniqueness of bounded solutions to (1.3) on $\Omega = \mathbb{R}^{n}$ in the case where $\mu$ is a nonnegative constant was characterized in [6].

As was mentioned above, this work has been motivated by the results of Cao and Verbitsky [10], who proved that there exists a unique positive finite energy solution $u$ to the homogeneous equation

$$-\Delta_{p} u = \sigma u^{q} \quad \text{in} \quad \mathbb{R}^{n},$$

where $1 < p < \infty$, $0 < q < p - 1$ and $\sigma \in \mathcal{M}^{+} (\mathbb{R}^{n})$, if and only if

$$W_{1,p} \sigma \in L^{\frac{1+\alpha(p-1)}{n-\alpha p}} (\mathbb{R}^{n}, d\sigma).$$

Here, for $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$ and $\sigma \in \mathcal{M}^{+} (\mathbb{R}^{n})$, the (homogeneous) Wolff potential $W_{a,p} \sigma$ is defined by (see [14])

$$W_{a,p} \sigma (x) = \int_{0}^{\infty} \left[ \frac{\sigma(B(x,r))}{r^{n-ap}} \right]^{\frac{1}{p-1}} \, dr, \quad x \in \mathbb{R}^{n},$$

where $B(x,r) = \{ y \in \mathbb{R}^{n} : |x - y| < r \}$ is a ball centered at $x \in \mathbb{R}^{n}$ of radius $r > 0$. Notice that $W_{a,p} \sigma = +\infty$ for $\alpha \geq \frac{n}{p}$ unless $\sigma = 0$. (See [1, 19] for an overview of Wolff potentials and their applications in Analysis and PDE.)

For $1 \leq p < \infty$ and a nonempty open set $\Omega \subset \mathbb{R}^{n}$, by $W_{0}^{1,p} (\Omega)$ we denote the homogeneous Sobolev (or Dirichlet) space defined [15, 20] as the closure of $C_{0}^{\infty} (\Omega)$ with respect to the (semi)norm

$$||u||_{W_{0}^{1,p} (\Omega)} = ||\nabla u||_{L^{p} (\Omega)}.$$ We denote by $W^{-1,p'} (\Omega) = [W_{0}^{1,p} (\Omega)]^{*}$ the dual space, where $p' = \frac{p}{p-1}$. If $p < n$, then $W^{-1,p'} (\Omega) \subset D' (\Omega)$.

For equation (1.1) on $\mathbb{R}^{n}$, we will show that condition (1.5), combined with the natural assumption that $\mu$ has finite energy, i.e. (see [1, Section 4.5]),

$$\mu \in W^{-1,p'} (\mathbb{R}^{n}) \iff \int_{\mathbb{R}^{n}} W_{1,p} \mu \, d\mu < +\infty,$$

is necessary and sufficient for the existence of a positive finite energy solution to (1.1). More precisely, we state our main results as follows.

**Theorem 1.1.** Let $1 < p < n$ and $0 < q < p - 1$, and let $\sigma, \mu \in \mathcal{M}^{+} (\mathbb{R}^{n})$. Then there exists a positive finite energy solution $u \in L^{q}_{\text{loc}} (\mathbb{R}^{n}, d\sigma) \cap W_{0}^{1,p} (\mathbb{R}^{n})$ to equation (1.1) if and only if both (1.5) and (1.6) hold. Moreover, such a solution is unique in $W_{0}^{1,p} (\mathbb{R}^{n})$. In the case $p \geq n$, there is only a trivial supersolution.

In our proof of Theorem 1.1, we show that if (1.5) holds, then (1.6) implies a crucial two-weight condition

$$W_{1,p} \mu \in L^{1,q} (\mathbb{R}^{n}, d\sigma),$$

which turns out to be necessary for the existence of a positive solution $u \in L^{q}_{\text{loc}} (\mathbb{R}^{n}, d\sigma) \cap W_{0}^{1,p} (\mathbb{R}^{n})$ to (1.1).

Given (1.5), it allows us to deduce the existence of a positive finite energy solution $u$ to equation (1.1) under assumption (1.6), by using a positive solution $\hat{u} \in L^{1,q} (\mathbb{R}^{n}, d\sigma)$ to the corresponding nonlinear integral equation

$$\hat{u} = W_{1,p} (\hat{u}^{q} d\sigma) + W_{1,p} \mu \quad \text{d}\sigma \text{-a.e.}$$

Such a solution $\hat{u}$ can be constructed by an iterative method, provided (1.7) holds.
As shown in [8], condition (1.5) is equivalent to the trace inequality
\[ \|\varphi\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} \leq C\|\nabla \varphi\|_{L^p(\mathbb{R}^n)} \quad \text{for all } \varphi \in C_0^\infty(\mathbb{R}^n), \] (1.8)
where \( C \) is a positive constant independent of \( \varphi \).

Moreover, there is an alternative characterization of (1.8) in terms of capacities due to Maz'ya and Netrusov (see [21, Section 11.6]),
\[ \int_0^r \left[ \frac{1}{\kappa(\sigma, r)} \right]^{\frac{1}{p-1}} dr < +\infty, \] (1.9)
where \( \kappa(\sigma, r) = \inf\{\text{cap}_p(E) : E \geq r, E \subset \mathbb{R}^n \text{ compact} \} \) and \( \text{cap}_p(\cdot) \) is the \( p \)-capacity defined, for a compact set \( E \subset \mathbb{R}^n \), by
\[ \text{cap}_p(E) = \inf\{\|\nabla u\|_{L^p(\mathbb{R}^n)}^p : u \geq 1 \text{ on } E, u \in C_0^\infty(\mathbb{R}^n)\}. \]

Thus, any one of conditions (1.5), (1.8), or (1.9), combined with (1.6), is necessary and sufficient for the existence of a positive finite energy solution to equation (1.1). The uniqueness part will be proven by first establishing the minimality of such a solution, and then using convexity of the Dirichlet integrals \( \int_{\mathbb{R}^n} |\nabla u|^p \, dx \).

Furthermore, we are able to adjust our argument outlined above to obtain analogous results for the fractional Laplace equation (1.2) as follows.

**Theorem 1.2.** Let \( 0 < q < 1 \) and \( 0 < \alpha < \frac{N}{2} \), and let \( \alpha, \mu \in M^+(\mathbb{R}^n) \). Then there exists a positive finite energy solution \( u \in L_{\text{loc}}^q(\mathbb{R}^n, d\sigma) \cap \dot{H}^\alpha(\mathbb{R}^n) \) to equation (1.2) if and only if the following two conditions hold:
\[ I_{2\alpha} \sigma \in L^{\frac{1+q}{q}}(\mathbb{R}^n, d\sigma) \] (1.10)
and
\[ \mu \in \dot{H}^{-\alpha}(\mathbb{R}^n). \] (1.11)
Moreover, if \( 0 < \alpha \leq 1 \), then such a solution is unique in \( \dot{H}^\alpha(\mathbb{R}^n) \).

Here, for \( 0 < \alpha < \frac{N}{2} \) and \( \sigma \in M^+(\mathbb{R}^n) \), we denote by \( I_{2\alpha} \sigma = W_{2\alpha, 2} \sigma \) the Riesz potential of order \( 2\alpha \) (up to a normalization constant). The homogeneous Sobolev space \( \dot{H}^\alpha(\mathbb{R}^n) \) \( (0 < \alpha < \frac{N}{2}) \) can be defined by means of Riesz potentials,
\[ \dot{H}^\alpha(\mathbb{R}^n) = \{ u : u = I_\alpha f, f \in L^2(\mathbb{R}^n) \}, \]
equipped with the norm
\[ \|u\|_{\dot{H}^\alpha(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}. \]

We denote by \( \dot{H}^{-\alpha}(\mathbb{R}^n) = [\dot{H}^\alpha(\mathbb{R}^n)]^* \) the space of distributions dual to \( \dot{H}^\alpha(\mathbb{R}^n) \).

Adapting the previous argument, if (1.10) holds, we first construct a positive solution \( \tilde{u} \in L^{1+q}(\mathbb{R}^n, d\sigma) \) to the integral equation
\[ \tilde{u} = I_{2\alpha} (\tilde{q} \, d\sigma) + I_{2\alpha} \mu \quad \text{d}\sigma\text{-a.e.} \]
using an iterative procedure, under the additional assumption that
\[ I_{2\alpha} \mu \in L^{1+q}(\mathbb{R}^n, d\sigma). \] (1.12)

Using the nontrivial fact (1.10)–(1.11) \( \Rightarrow \) (1.12), we deduce the existence of a solution \( \tilde{u} \in L^{1+q}(\mathbb{R}^n, d\sigma) \), and then a positive finite energy solution \( u \) to equation (1.2).

We observe that (1.10) is equivalent to the trace inequality [9]
\[ \|I_\alpha g\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} \leq C\|g\|_{L^2(\mathbb{R}^n)} \quad \text{for all } g \in L^2(\mathbb{R}^n), \] (1.13)
where \( C \) is a positive constant independent of \( g \). Thus, condition (1.10), or equivalently (1.13), together with condition (1.11) is necessary and sufficient for the existence of a positive finite energy solution to equation (1.2). The restriction on the value of \( \alpha \) in the uniqueness result is due to availability [4] of a certain convexity property of the Dirichlet integrals \( \int_{\mathbb{R}^n} |(-\Delta)^{\frac{\alpha}{2}} u|^2 \, dx \) in the case \( \alpha \in (0, 1] \).
We now consider sublinear elliptic equation (1.3) on arbitrary domains $\Omega \subset \mathbb{R}^n$ (possibly unbounded) with positive Green’s function $G(x, y)$ on $\Omega \times \Omega$. Define the Green potential by

$$G\sigma(x) = \int_{\Omega} G(x, y) \, d\sigma(y), \quad x \in \Omega.$$ 

Our main results in this setup are stated in the following theorem.

**Theorem 1.3.** Let $0 < q < 1$ and $\sigma, \mu \in M^+(\Omega)$, and let $G$ be Green’s function associated with $-\Delta$ on $\Omega$. Then there exists a positive finite energy solution $u \in L^{1+q}_{\text{loc}}(\Omega, d\sigma) \cap W^{1,2}_0(\Omega)$ to equation (1.3) if and only if the following two conditions hold:

1. $G\sigma \in L^{1+q}(\Omega, d\sigma)$ (1.14)
2. $\mu \in W^{-1,2}(\Omega)$. (1.15)

Moreover, such a solution is unique in $W^{1,2}_0(\Omega)$.

Recently, it has been shown in [27] that (1.14) is equivalent to the weighted norm inequality for Green’s potentials,

$$\|G(f \sigma)\|_{L^{1+q}(\Omega, d\sigma)} \leq C \|f\|_{L^{1+q}(\Omega, d\sigma)} \quad \text{for all } f \in L^{1+q}(\Omega, d\sigma) \quad (1.16)$$

where $C$ is a positive constant independent of $f$. Therefore, condition (1.14), or equivalently (1.16), together with condition (1.15) turns out to be necessary and sufficient for the existence of a positive finite energy solution to equation (1.3).

Our argument is based on the results in [27] mentioned above, along with a new element that given (1.14), condition (1.15) yields

$$G\mu \in L^{1+q}(\Omega, d\sigma). \quad (1.17)$$

As before, when (1.14) holds, this allows us to construct a positive finite energy solution to equation (1.3) by using an auxiliary solution $\tilde{u} \in L^{1+q}(\Omega, d\sigma)$ to the corresponding integral equation

$$\tilde{u} = G(\tilde{u}^q \, d\sigma) + G\mu \quad \text{d}\sigma\text{-a.e.}$$

Analogues of Theorems 1.2 and 1.3 for the fractional Laplacian on domains $\Omega$ with Green’s function $G$ in the case $0 < \alpha < 1$ (see [3]) will be considered elsewhere. There are also some analogous results (less precise at the boundary $\partial\Omega$) for equation (1.1) involving the p-Laplace operator in domains $\Omega \subset \mathbb{R}^n$; see Remark 3.7 below.

This paper is organized as follows. In Section 2, we recall the necessary mathematical background, together with preliminary results concerning quasilinear equations and nonlinear potentials. In Sections 3, 4, and 5, we establish explicit necessary and sufficient conditions for the existence of positive finite energy solutions to equations (1.1), (1.2), and (1.3), respectively. Uniqueness results for such solutions are discussed in Section 6.

Throughout, the letters $c$ and $C$ denote various positive constants whose value may change from one place to another.

## 2 Preliminaries

Let $\Omega \subset \mathbb{R}^n$ be a domain (nonempty open connected set). We denote by $M^+(\Omega)$ the set of all nontrivial nonnegative locally finite Borel measures in $\Omega$, and by $C_0^\infty(\Omega)$ the set of all smooth compactly supported functions in $\Omega$.

For $1 \leq p < \infty$ and $\sigma \in M^+(\Omega)$, we denote by $L^p(\Omega, d\sigma)$ the space of all real-valued measurable functions $u$ on $\Omega$ such that

$$\|u\|_{L^p(\Omega, d\sigma)} = \left( \int_{\Omega} |u|^p \, d\sigma \right)^{1/p} < \infty.$$
The corresponding local space \( L^p_{\text{loc}}(\Omega, d\sigma) \) consists of real-valued measurable functions \( u \) on \( \Omega \) such that the restriction \( u|_K \in L^p(K, d\sigma) \) for every compact subset \( K \subset \Omega \). When \( \sigma \) is \((n\text{-dimensional})\) Lebesgue measure, \( d\sigma = dx \), we write \( L^p(\Omega) \) and \( L^p_{\text{loc}}(\Omega) \), respectively.

For \( 1 \leq p < \infty \), the Sobolev space \( W^{1,p}(\Omega) \) consists of all functions \( u \in L^p(\Omega) \) such that \( |:\nabla u| \in L^p(\Omega) \), where \( :\nabla u \) is the vector of distributional (or weak) partial derivatives of \( u \) of order 1. The norm on \( W^{1,p}(\Omega) \) is given by

\[
\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.
\]

The corresponding local space denoted by \( W^{1,p}_{\text{loc}}(\Omega) \) is the space of all functions \( u \) in \( \Omega \) such that the restriction \( u|_F \in W^{1,p}(F) \) for every relatively compact open subset \( F \subset \Omega \).

The Sobolev space \( W^{1,p}_0(\Omega) \) is defined as the closure of \( C^\infty_0(\Omega) \) in \( W^{1,p}(\Omega) \). It is easy to see that

\[
W^{1,p}_0(\Omega) = W^{1,p}(\mathbb{R}^n).
\]

The homogeneous version of \( W^{1,p}_0(\Omega) \), called the homogeneous Sobolev space (or Dirichlet space), denoted by \( W^{1,p}_0(\Omega) \), is defined as the closure of \( C^\infty_0(\Omega) \) with respect to the seminorm

\[
\|u\|_{W^{1,p}_0(\Omega)} = \|\nabla u\|_{L^p(\Omega)}.
\]

That is, \( W^{1,p}_0(\Omega) \) is the set of all functions \( u \in W^{1,p}_{\text{loc}}(\Omega) \) such that \( |:\nabla u| \in L^p(\Omega) \) for which there exists a sequence \( \{\varphi_j\}_1^\infty \subset C^\infty_0(\Omega) \) such that \( \|\nabla u - \nabla \varphi_j\|_{L^p(\Omega)} \to 0 \) as \( j \to \infty \). When \( 1 \leq p < n \), the dual space to \( W^{1,p}_0(\Omega) \) denoted by \( W^{-1,p}_0(\Omega) \) is the space of distributions \( \mu \in \mathcal{D}'(\Omega) \) such that

\[
\|\mu\|_{W^{-1,p'}_0(\Omega)} = \sup \frac{|\langle \mu, u \rangle|}{\|u\|_{W^{1,p}_0(\Omega)}} < +\infty,
\]

where the supremum is taken over all nontrivial functions \( u \in C^\infty_0(\Omega) \). Here, \( p' = \frac{p}{p-1} \) is the Hölder conjugate of \( p \). For a measure \( \mu \in \mathcal{M}^+(\Omega) \), \( \mu \in W^{-1,p}_0(\Omega) \) if and only if there exists a positive constant \( C \) such that

\[
\left| \int_\Omega \varphi \, d\mu \right| \leq C \left( \int_\Omega |\nabla \varphi|^p \, dx \right)^{\frac{1}{p}} \quad \text{for all } \varphi \in C^\infty_0(\Omega).
\]

For \( 0 < a < n \), the Riesz potential \( I_a f \) of a function \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) is defined by

\[
I_a f(x) = (-\Delta)^{-\frac{a}{2}} f(x) = y(a, n) \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-a}} \, dy, \quad x \in \mathbb{R}^n,
\]

where

\[
y(a, n) = \frac{\Gamma\left(\frac{n-a}{2}\right)}{\pi^{\frac{n}{2}} 2^a \Gamma\left(\frac{a}{2}\right)}
\]

is a normalization constant.

Observe that, for \( f \in L^p(\mathbb{R}^n) \), \( 1 < p < \frac{n}{a} \), the Riesz potential \( I_a f \) is well-defined and finite \((a,p)\)-quasi-everywhere (briefly, q.e.), meaning everywhere except for a set of \((a,p)\)-capacity zero (see [1]). Moreover, \( I_a f \) is \((a,p)\)-quasicontinuous (in brief, quasicontinuous) which means that, for every \( \epsilon > 0 \), there is an open set \( G \subset \mathbb{R}^n \) such that \( \text{cap}_{a,p}(G) < \epsilon \) and the restriction \( I_a f|_G \) is continuous on \( G^\text{c} \). Here the \((a,p)\)-capacity of \( E \subset \mathbb{R}^n \) is defined by

\[
\text{cap}_{a,p}(E) := \inf\{\|u\|_{L^p(\mathbb{R}^n)} : I_a u \geq 1 \text{ on } E, u \geq 0 \text{ a.e., } u \in L^p(\mathbb{R}^n)\}.
\]

Note that Lebesgue measure is absolutely continuous with respect to the \((a,p)\)-capacity, i.e., each set of \((a,p)\)-capacity zero has Lebesgue measure zero.

In a similar manner, the Riesz potential \( I_a \sigma \) of order \( a \in (0, n) \) of a measure \( \sigma \in \mathcal{M}^+(\mathbb{R}^n) \) is defined by

\[
I_a \sigma(x) = (-\Delta)^{-\frac{a}{2}} \sigma(x) = (n-a) y(a, n) \int_0^\infty \frac{\sigma(B(x, r))}{r^{n-a}} \, dr, \quad x \in \mathbb{R}^n.
\]

Henceforth, the normalization constant will be dropped for the sake of convenience.
For $1 \leq p < \infty$ and $0 < \alpha < \frac{n}{p}$, the fractional homogeneous Sobolev space is defined by (see [25])

$$L^{a,p}(\mathbb{R}^n) = \{ u : u = I_a f, f \in L^p(\mathbb{R}^n) \},$$

equipped with the norm

$$\|u\|_{L^{a,p}(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)}.$$

Clearly, $L^{1,p}(\mathbb{R}^n) = W^{1,p}_0(\mathbb{R}^n)$. In the case $p = 2$, we use the notation $L^{a,2}(\mathbb{R}^n) = H^a(\mathbb{R}^n)$. It is well known that when $0 < \alpha < 1$, $\|u\|_{H^a(\mathbb{R}^n)}$ is equivalent to the Gagliardo seminorm

$$\left( \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2\alpha}} \, dx \, dy \right)^{\frac{1}{2}},$$

see, for example, [1, Section 3.5].

The dual space of $H^a(\mathbb{R}^n)$ for $0 < \alpha < \frac{n}{2}$, denoted by $H^{-a}(\mathbb{R}^n)$, consists of distributions $\mu \in \mathcal{D}'(\mathbb{R}^n)$ such that

$$\|\mu\|_{H^{-a}(\mathbb{R}^n)} = \sup \left\{ \langle \mu, u \rangle : \|u\|_{H^a(\mathbb{R}^n)} < +\infty \right\},$$

where the supremum is taken over all nontrivial functions $u \in C_0^\infty(\mathbb{R}^n)$. Thus, by duality, for a measure $\mu \in \mathcal{M}(\mathbb{R}^n)$, we have $\mu \in H^{-a}(\mathbb{R}^n)$ if and only if $\|\mu \|_{L^p(\mathbb{R}^n)} < +\infty$, or equivalently $\int_{\mathbb{R}^n} I_{2a} \mu \, dx < \infty$.

Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$ and $\sigma \in \mathcal{M}(\mathbb{R}^n)$. The (homogeneous) Wolff potential $W_{a,p} \sigma$ is defined by (see [1, 19])

$$W_{a,p} \sigma(x) = \sup_{0 < r < \infty} \left( \frac{1}{r^\alpha} \int_{B(x,r)} |\sigma(y)| \, dy \right)^{\frac{1}{p}}, \quad x \in \mathbb{R}^n,$$

where $B(x, r) = \{ y \in \mathbb{R}^n : |x - y| < r \}$ is a ball centered at $x \in \mathbb{R}^n$ of radius $r > 0$.

Observe that $W_{a,p} \sigma$ is always positive since $\sigma \neq 0$. Moreover, either $W_{a,p} \sigma \equiv +\infty$ or $W_{a,p} \sigma < +\infty$ a.e. In other words, $W_{a,p} \sigma < +\infty$ a.e. if and only if $W_{a,p} \sigma(x_0) < +\infty$ for some $x_0 \in \mathbb{R}^n$.

In the linear case, when $p = 2$, $W_{a,2} \sigma = I_{2a} \sigma$, and in particular, $W_{1,2} \sigma = I_2 \sigma$ is the Newtonian potential.

The energy of $\sigma$ is given by

$$\mathcal{E}_{a,p}(\sigma) = \|I_a \sigma\|_{L^p(\mathbb{R}^n)}^p.$$

The fundamental Wolff’s inequality, see [1, Section 4.5], provides a certain estimate of the energy by means of the corresponding Wolff potential:

$$C^{-1} \mathcal{E}_{a,p}(\sigma) \leq \int_{\mathbb{R}^n} W_{a,p} \sigma \, d\sigma \leq C \mathcal{E}_{a,p}(\sigma),$$

where $C = C(\alpha, n, p) \geq 1$. Consequently, $W_{a,p} \sigma \in L^1(\mathbb{R}^n, d\sigma) \iff \mathcal{E}_{a,p}(\sigma) < +\infty$.

More generally, it was shown in [8] (see also [9]) that for $0 \leq q < p$, $p > 1$,

$$W_{a,p} \sigma \in L^{\frac{p(p-1)}{p}}(\mathbb{R}^n, d\sigma)$$

is equivalent to the trace inequality

$$\left( \int_{\mathbb{R}^n} |I_a g|^q \, d\sigma \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} |g|^p \, dx \right)^{\frac{1}{p}}$$

for all $g \in L^p(\mathbb{R}^n), \tag{2.1}$

where $C$ is a constant independent of $g$. When $a = k < \frac{n}{2}$ is a positive integer, (2.1) is equivalent to the generalized Sobolev inequality

$$\left( \int_{\mathbb{R}^n} |g|^q \, d\sigma \right)^{\frac{1}{q}} \leq C \left( \int_{\mathbb{R}^n} |\nabla^k g|^p \, dx \right)^{\frac{1}{p}}$$

for all $g \in C_0^\infty(\mathbb{R}^n)$, where $C$ is a constant independent of $g$.

For $1 < p < \infty$, the $p$-Laplacian $\Delta_p$ is defined by

$$\Delta_p u = \nabla \cdot (|\nabla u|^{p-2} \nabla u), \quad u \in W^{1,p}_\text{loc}(\Omega),$$
in the distributional sense, i.e., for every \( \phi \in C_0^\infty(\Omega) \),
\[
\langle \Delta_p u, \phi \rangle = \langle \nabla \cdot (|\nabla u|^{p-2} \nabla u), \phi \rangle = -\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx.
\]

**Definition 2.1.** Let \( 1 < p < \infty, \, 0 < q < p - 1 \) and \( \sigma, \mu \in M^+(\Omega) \). A function \( u \) is said to be a **finite energy solution** to the equation
\[
-\Delta_p u = \sigma u^q + \mu \quad \text{in } \Omega \tag{2.2}
\]
if \( u \in L^q_{\text{loc}}(\Omega, d\sigma) \cap W^{1,p}_0(\Omega), \, u \geq 0 \) \( \sigma \)-a.e. and
\[
\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \int_\Omega \phi u^q \, d\sigma + \int_\Omega \phi \, d\mu, \quad \phi \in C_0^\infty(\Omega).
\]

We shall extend the notion of distributional solutions \( u \) to equation (2.2), for \( u \) not necessarily belonging to \( W^{1,p}_0(\Omega) \). We will understand such solutions in the potential-theoretic sense using \( p \)-superharmonic functions, which is equivalent to the notion of locally renormalized solutions in terms of test functions, see [16].

A function \( u \in W^{1,p}_0(\Omega) \) is said to be **\( p \)-harmonic** if \( u \) satisfies the \( p \)-Laplace equation
\[
-\Delta_p u = 0 \quad \text{in } \Omega
\]
in the distributional sense. Note that every \( p \)-harmonic function has a continuous representative which coincides with \( u \) a.e., see [15]. A function \( u : \Omega \to (-\infty, +\infty) \) is \( p \)-superharmonic if \( u \) is lower semicontinuous in \( \Omega \), \( u \neq +\infty \) in each component of \( \Omega \), and whenever \( D \not\subseteq \Omega \) and \( h \in C(\overline{D}) \) is \( p \)-harmonic in \( D \) with \( h \leq u \) on \( \partial D \), then \( h \leq u \) on \( D \). Also note that every \( p \)-superharmonic function \( u \) in \( \Omega \) has a quasicontinuous representative which coincides with \( u \) \( p \)-quasi-everywhere in \( \Omega \) (briefly, q.e.), i.e., everywhere except for a set of \( p \)-capacity zero. Here, the \( p \)-capacity of a compact set \( E \subset \Omega \) is defined by
\[
\text{cap}_p(E) := \inf \{ \|u\|_{L^p(\Omega)}^p : u \geq 1 \text{ on } E, \, u \in C_0^\infty(\Omega) \}.
\]

Notice that \( \text{cap}_p(E) \) is equivalent to \( \text{cap}_1(p)(E) \) for compact sets \( E \subset \Omega \).

A \( p \)-superharmonic function \( u \geq 0 \) does not necessarily belong to \( W^{1,p}_0(\Omega) \), but its truncation
\[
T_k(u) := \min(u, k)
\]
does for every \( k \in \mathbb{N} \). Moreover, each \( T_k(u) \) is a supersolution, i.e.,
\[
-\nabla \cdot (|\nabla T_k(u)|^{p-2} \nabla T_k(u)) \geq 0
\]
in the distributional sense. The generalized (or weak) gradient of a \( p \)-superharmonic function \( u \) is defined [15] by
\[
Du = \lim_{k \to \infty} \nabla (T_k(u)).
\]

Let \( u \) be a \( p \)-superharmonic function in \( \Omega \). Then \( |Du|^{p-1} \) and, consequently, \( |Du|^{p-2} Du \) are of class \( L^r_{\text{loc}}(\Omega) \) for every \( 1 \leq r < \frac{n}{p-1} \), see [17]. This allows us to define a nonnegative distribution \( -\Delta_p u \) by
\[
-\langle \Delta_p u, \phi \rangle = \int_\Omega |Du|^{p-2} Du \cdot \nabla \phi \, dx, \quad \phi \in C_0^\infty(\Omega).
\]

Thus, by the Riesz Representation Theorem, there exists a unique measure \( \omega[u] \in M^+(\Omega) \) so that \( -\Delta_p u = \omega[u] \). The measure \( \omega[u] \) is called the Riesz measure of \( u \).

**Definition 2.2.** For \( \omega \in M^+(\Omega) \), a function \( u \) is said to be a **solution** to the equation
\[
-\Delta_p u = \omega \quad \text{in } \Omega
\]
(in the potential-theoretic sense) if \( u \) is \( p \)-superharmonic in \( \Omega \) and \( \omega[u] = \omega \).

Thus, for \( \sigma, \mu \in M^+(\Omega) \), a function \( u \) is said to be a solution to equation (2.2) (in the potential-theoretic sense) if \( u \) is \( p \)-superharmonic in \( \Omega \) so that \( u \in L^q_{\text{loc}}(\Omega, d\sigma) \) and \( d\omega[u] = u^q d\sigma + d\mu \).

A supersolution to (2.2) is a nonnegative \( p \)-superharmonic function \( u \) in \( \Omega \) so that \( u \in L^q_{\text{loc}}(\Omega, d\sigma) \) and
\[
\int_\Omega |Du|^{p-2} Du \cdot \nabla \phi \, dx \geq \int_\Omega u^q \phi \, d\sigma + \int_\Omega \phi \, d\mu, \quad \phi \in C_0^\infty(\Omega) \text{ with } \phi \geq 0.
\]
Note that if \( u \in W^{1,p}_{\text{loc}}(\Omega) \) is a solution (or supersolution) to equation (2.2), then the generalized gradient \( Du \) coincides with the regular gradient \( u \). Thus \( u \) is the usual distributional solution (or supersolution, respectively).

The following weak continuity result, see [26], will be used to prove the existence of \( p \)-superharmonic solutions to quasilinear equations.

**Theorem 2.3** ([26]). Suppose that \( \{u_j\}_{i=1}^{\infty} \) is a sequence of nonnegative \( p \)-superharmonic functions in \( \Omega \) such that \( u_j \rightarrow u \) a.e. as \( j \rightarrow \infty \), where \( u \) is a \( p \)-superharmonic function in \( \Omega \). Then \( \omega[u_j] \) converges weakly to \( \omega[u] \), that is,

\[
\lim_{j \to \infty} \int_{\Omega} \varphi \, du_j = \int_{\Omega} \varphi \, du
\]

for all \( \varphi \in C_0^\infty(\Omega) \).

We shall use the following lower bounds for supersolutions.

**Theorem 2.4** ([10]). Let \( 1 < p < n, 0 < q < p - 1 \) and \( \sigma \in M^+(\mathbb{R}^n) \). Suppose that \( u \) is a nontrivial supersolution to equation (1.4). Then \( u \) satisfies the inequality

\[
u \geq c(W_{1,p}(u^q) \frac{p-1}{p-q})\sigma\Rightarrow \]

where \( c = c(n, p, q) > 0 \).

**Theorem 2.5** ([7]). Let \( 1 < p < n, 0 < q < p - 1, 0 < a < \frac{n}{p} \) and \( \sigma \in M^+(\mathbb{R}^n) \). Suppose that \( u \in L^q_{\text{loc}}(\mathbb{R}^n, \text{d}\sigma) \) satisfying

\[
u \geq W_{a,p}(u^q) \text{d}\sigma \Rightarrow \]

Then \( u \) satisfies the inequality

\[
u \geq c(W_{a,p}(u^q) \frac{p-1}{p-q}) \text{d}\sigma \Rightarrow \]

where \( c = c(a, n, p, q) > 0 \).

The following important result, [18], is concerned with pointwise estimate of nonnegative \( p \)-superharmonic functions in terms of Wolff’s potential.

**Theorem 2.6** ([18]). Let \( 1 < p < n \) and \( \omega \in M^+(\mathbb{R}^n) \). Suppose that \( u \) is a \( p \)-superharmonic function in \( \mathbb{R}^n \) satisfying

\[
-\Delta_p u = \omega \quad \text{in} \mathbb{R}^n, \quad \liminf_{|x| \to \infty} u(x) = 0
\]

Then

\[
K^{-1}W_{1,p} \omega \leq u \leq KW_{1,p} \omega,
\]

where \( K = K(n, p) \geq 1 \).

The next three lemmas are discussed in [10], which will be used in our arguments occasionally.

**Lemma 2.7** ([10]). Let \( 1 < p < n, 0 < q < p - 1 \) and \( \sigma \in M^+(\mathbb{R}^n) \). Suppose that there exists a nontrivial supersolution \( u \in L^q_{\text{loc}}(\mathbb{R}^n, \text{d}\sigma) \) to equation (1.4). Then

\[
-\Delta_p u \in W^{-1,p'}(\mathbb{R}^n) \cap M^+(\mathbb{R}^n) \quad \text{and} \quad u \in L^{1+q}(\mathbb{R}^n, \text{d}\sigma)
\]

for a quasicontinuous representative of \( u \). Consequently, (1.5) holds.

**Lemma 2.8** ([10]). Suppose that \( u \in L^{1+q}(\mathbb{R}^n, \text{d}\sigma) \) is a nontrivial supersolution to the integral equation

\[
u = W_{1,p}(u^q) \text{d}\sigma \Rightarrow \]

Then

\[
u^q \text{d}\sigma \in W^{-1,p'}(\mathbb{R}^n) \cap M^+(\mathbb{R}^n).\]
Lemma 2.9 ([10]). Let $\mu, \omega \in W^{-1,p}(\mathbb{R}^n) \cap M^+(\mathbb{R}^n)$. Suppose that $u, v \in W^{1,p}_0(\mathbb{R}^n)$ are solutions to the equations
\[-\Delta_p u = \mu \quad \text{in} \quad \mathbb{R}^n \quad \text{and} \quad -\Delta_p v = \omega \quad \text{in} \quad \mathbb{R}^n,\]
respectively. If $\mu \leq \omega$, then $u \leq v$ q.e.

The following theorem is due to Brezis and Browder [5] (cf. [20, Theorem 2.39]).

Theorem 2.10. Let $1 < p < n$ and $\mu \in W^{-1,p'}(\Omega) \cap M^+(\Omega)$. Then for any $u \in W^{1,p}_0(\Omega)$ we have $u \in L^1(\Omega, d\mu)$ and
\[\langle \mu, u \rangle = \int_{\Omega} u \, d\mu\]
for a quasicontinuous representative of $u$.

We shall use the following facts, which are discussed in [20, Sections 2.1.5–2.2].

Remark 2.11. Let $1 < p < n$ and $\omega \in W^{-1,p'}(\mathbb{R}^n) \cap M^+(\mathbb{R}^n)$. There exists a unique $p$-superharmonic solution $u \in W^{1,p}_0(\mathbb{R}^n)$ to the equation
\[-\Delta_p u = \omega \quad \text{in} \quad \mathbb{R}^n\]
in the distributional sense. Moreover, $u \in L^1(\mathbb{R}^n, d\omega)$ and
\[\langle \omega, u \rangle = \int_{\mathbb{R}^n} u \, d\omega = \|u\|_{W^{1,p}(\mathbb{R}^n)}^p = \|\omega\|_{W^{-1,p'}(\mathbb{R}^n)}^{p'} = \|I_1 \omega\|_{L^{p'}(\mathbb{R}^n)} = \mathcal{E}_{1,p}(\omega)\]
for a quasicontinuous representative of $u$.

We will need the next lemma which shows that if there exists a nontrivial supersolution $u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ to the integral equation
\[u = W_{a,p}(u^q \, d\sigma) \quad d\sigma\text{-a.e.}, \quad (2.3)\]
then $\sigma$ must be absolutely continuous with respect to $\text{cap}_{a,p}(\cdot)$.

Lemma 2.12 ([7]). Let $1 < p < \infty$, $0 < \sigma < \frac{n}{p}$ and $0 < q < p - 1$ and $\sigma \in M^+(\mathbb{R}^n)$. Suppose that there exists a nontrivial supersolution $u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma)$ to (2.3). Then there exists a positive constant $C$ such that
\[\sigma(E) \leq C[\text{cap}_{a,p}(E)]^{1 - \frac{\sigma}{p}} \left( \int_E u^q \, d\sigma \right)^{\frac{1 - \frac{\sigma}{p}}{n}}\]
for all compact sets $E \subset \mathbb{R}^n$.

Consequently, if (1.4) has a nontrivial $p$-superharmonic supersolution, then $\sigma$ is absolutely continuous with respect to $\text{cap}_{p}(\cdot)$.

3 Existence of a positive finite energy solution to equation (1.1)

In this section, we establish necessary and sufficient conditions for the existence of a positive finite energy solution to equation (1.1). Minimality of such a solution is demonstrated as well. In the case $p \geq n$, it follows immediately from the result in [10] that there is only a trivial supersolution to (1.1). Henceforth, we assume that $1 < p < n$.

Our first theorem is stated in the general framework of nonlinear integral equations involving Wolff potentials,
\[u = W_{a,p}(u^q \, d\sigma) + W_{a,p} \mu \quad \text{in} \quad \mathbb{R}^n, \quad (3.1)\]
where $1 < p < n$, $0 < q < p - 1$, $0 < \sigma < \frac{n}{p}$ and $\sigma, \mu \in M^+(\mathbb{R}^n)$. This theorem will be used to construct positive finite energy solutions to both equations (1.1) and (1.2) in the cases $\alpha = 1$ and $p = 2$, respectively.
Theorem 3.1. Let $1 < p < n$, $0 < q < p - 1$, $0 < \alpha < \frac{n}{p}$ and $\mu, \sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Suppose that the following conditions hold:

$$W_{a,p} \sigma \in L^{\frac{1+q}{p+1-q}}(\mathbb{R}^n, d\sigma)$$  \hspace{1cm} (3.2)

and

$$W_{a,p} \mu \in L^{\frac{1+q}{p+1-q}}(\mathbb{R}^n, d\sigma).$$  \hspace{1cm} (3.3)

Then there exists a positive solution $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$ to the integral equation (3.1).

The following result will be used in the proof of Theorem 3.1 (see [10, Lemma 3.3], or [9] in more generality).

Lemma 3.2 ([10]). Let $1 < p < \infty$, $0 < q < p - 1$, $0 < \alpha < \frac{n}{p}$ and $\sigma \in \mathcal{M}^+(\mathbb{R}^n)$. Suppose that (3.2) holds. Then the nonlinear integral operator $T$ defined by

$$T(g) := (W_{a,p}(|g|^q d\sigma))^{\frac{1}{q}}$$

is bounded from $L^{1+q}(\mathbb{R}^n, d\sigma)$ to $L^{1+q}(\mathbb{R}^n, d\sigma)$.

Proof of Theorem 3.1. Without loss of generality we may assume that $g \geq 0$, $g \in L^{1+q}(\mathbb{R}^n, d\sigma)$. Since (3.2) holds, it follows from Lemma 3.2 that there exists a positive constant $c$ such that

$$\left( \int_{\mathbb{R}^n} |W_{a,p}(g d\sigma)|^{1+q} d\sigma \right)^{\frac{1}{1+q}} \leq c \left( \int_{\mathbb{R}^n} |g|^{1+q} d\sigma \right)^{\frac{1}{1+q}},$$

where $c$ is a positive constant that does not depend on $g \in L^{1+q}(\mathbb{R}^n, d\sigma)$. We construct a sequence of functions $(u_j)_{j=0}^\infty$ as follows. Set

$$u_0 := W_{a,p} \mu \quad \text{and} \quad u_{j+1} := W_{a,p}(u_j^q d\sigma) + W_{a,p} \mu, \quad j \in \mathbb{N}_0.$$  

Observe that $u_0 > 0$ since $\mu \neq 0$, and

$$u_1 = W_{a,p}(u_0^q d\sigma) + u_0 \geq u_0.$$  

Suppose that $u_0 \leq u_1 \leq \cdots \leq u_j$ for some $j \in \mathbb{N}$. Then

$$u_{j+1} = W_{a,p}(u_j^q d\sigma) + W_{a,p} \mu \geq W_{a,p}(u_{j-1}^q d\sigma) + W_{a,p} \mu = u_j.$$  

So, by induction, $(u_j)_{j=0}^\infty$ is a nondecreasing sequence of positive functions. Moreover, each $u_j \in L^{1+q}(\mathbb{R}^n, d\sigma)$. To see this, notice that by assumption (3.3), we have

$$u_0 = W_{a,p} \mu \in L^{1+q}(\mathbb{R}^n, d\sigma).$$  

Suppose that $u_0, \ldots, u_j \in L^{1+q}(\mathbb{R}^n, d\sigma)$ for some $j \in \mathbb{N}$. By Minkowski’s inequality,

$$\|u_{j+1}\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} = \|W_{a,p}(u_j^q d\sigma) + W_{a,p} \mu\|_{L^{1+q}(\mathbb{R}^n, d\sigma)}$$

$$\leq \|W_{a,p}(u_j^q d\sigma)\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} + \|W_{a,p} \mu\|_{L^{1+q}(\mathbb{R}^n, d\sigma)}.$$  \hspace{1cm} (3.5)

The first term on the right-hand side of (3.5) is estimated by applying (3.4) with $g := u_j^q \in L^{1+q}(\mathbb{R}^n, d\sigma)$. In fact,

$$\|W_{a,p}(u_j^q d\sigma)\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} \leq c \left( \int_{\mathbb{R}^n} u_j^{1+q} d\sigma \right)^{\frac{q}{1+q}} \leq c \left( \int_{\mathbb{R}^n} u_j^{1+q} d\sigma \right)^{\frac{q}{1+q}} = c\|u_j\|_{L^{1+q}(\mathbb{R}^n, d\sigma)}^{\frac{q}{1+q}}.$$  \hspace{1cm} (3.6)

Combining (3.5) and (3.6), we arrive at

$$\|u_{j+1}\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} \leq c\|u_j\|_{L^{1+q}(\mathbb{R}^n, d\sigma)}^{\frac{q}{p+1}} + \|W_{a,p} \mu\|_{L^{1+q}(\mathbb{R}^n, d\sigma)}.$$  \hspace{1cm} (3.7)

We estimate the first term on the right-hand side of (3.7) using Young’s inequality,

$$c\|u_{j+1}\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} \leq \frac{q}{p-1} \|u_{j+1}\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} + \frac{p-1}{p} \frac{p}{p+1} c\|u_{j+1}\|_{L^{1+q}(\mathbb{R}^n, d\sigma)}.$$  \hspace{1cm} (3.8)
Hence, by (3.7) and (3.8), we obtain
\[ \|u_j\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} \leq c^{p-1} + \frac{p-1}{p-1-q} \|W_{a,p}\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} < +\infty. \]

By induction, we have shown that each \( u_j \in L^{1+q}(\mathbb{R}^n, d\sigma) \). Finally, applying the Monotone Convergence Theorem to the sequence \( \{u_j\}_0^\infty \), we see that the pointwise limit
\[ u := \lim_{j \to \infty} u_j \]
eqso that \( u > 0, u \in L^{1+q}(\mathbb{R}^n, d\sigma) \) and satisfies (3.1).

**Remark 3.3.** The converse to Theorem 3.1 is also true in a more general sense. In fact, if \( u \in L^{1+q}(\mathbb{R}^n, d\sigma) \), \( u > 0 \) \( d\sigma \)-a.e., satisfies the equation
\[ u = W_{a,p}(u^q d\sigma) + W_{a,p}\mu \quad d\sigma \text{-a.e.}, \]
then obviously \( u \in L^{\frac{q}{p}}(\mathbb{R}^n, d\sigma) \) by Hölder’s inequality, and
\[ u \geq W_{a,p}(u^q d\sigma) \quad d\sigma \text{-a.e.} \]
Applying Theorem 2.5, we obtain a lower pointwise estimate of \( u \),
\[ u \geq c(W_{a,p}\sigma)^{\frac{p-1}{p-q}} \quad d\sigma \text{-a.e.}, \]
where \( c = c(a, n, p, q) > 0. \) This implies that equation (3.2) holds since \( u \in L^{1+q}(\mathbb{R}^n, d\sigma) \). Similarly, (3.3) holds because \( u \in L^{1+q}(\mathbb{R}^n, d\sigma) \) and
\[ u \geq W_{a,p}\mu \quad d\sigma \text{-a.e.} \]
The next lemma is our main observation in this section. It gives us a relation between conditions (1.5), (1.6) and (1.7).

**Lemma 3.4.** Let \( 1 < p < n \), \( 0 < q < p - 1 \) and \( \sigma, \mu \in \mathcal{M}^+(\mathbb{R}^n) \). Then conditions (1.5) and (1.6) imply (1.7).

**Proof.** As shown in [8], (1.5) holds if and only if there exists a positive constant \( c \) such that
\[ \|I_1 g\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} \leq c\|g\|_{L^p(\mathbb{R}^n)} \quad \text{for all } g \in L^p(\mathbb{R}^n). \] (3.9)
Since \( \mu \in L^{-1,p'}(\mathbb{R}^n) \), we have \( I_1 \mu \in L^p(\mathbb{R}^n) \). Substituting \( g := (I_1\mu)^{\frac{p}{p-q}} \in L^p(\mathbb{R}^n) \) into (3.9) yields
\[ \|I_1(I_1\mu)^{\frac{p}{p-q}}\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} \leq c\|(I_1\mu)^{\frac{1}{p}}\|_{L^p(\mathbb{R}^n)} < +\infty. \]
Notice that \( W_{1,p}\mu \) is always pointwise smaller than \( I_1(I_1\mu)^{\frac{p}{p-q}} \) (see, for example, [21, Section 10.4.2]). More precisely,
\[ W_{1,p}\mu \leq CI_1(I_1\mu)^{\frac{p}{p-q}}, \]
where \( C \) is a constant which depends only on \( p \). This yields (1.7).

The following lemma, in particular, gives necessary conditions for the existence of a positive finite energy solution to equation (1.1).

**Lemma 3.5.** Let \( 1 < p < n \), \( 0 < q < p - 1 \) and \( \sigma, \mu \in \mathcal{M}^+(\mathbb{R}^n) \). Suppose that there exists a nontrivial supersolution \( u \in L^q_{loc}(\mathbb{R}^n, d\sigma) \cap W^{1,p}_{0}(\mathbb{R}^n) \) to equation (1.1). Then
\[ -\Delta_p u \in W^{-1,p'}(\mathbb{R}^n) \cap \mathcal{M}^+(\mathbb{R}^n) \quad \text{and} \quad u \in L^{1+q}(\mathbb{R}^n, d\sigma) \]
for a quasicontinuous representative of \( u \). Consequently, (1.5) and (1.6) hold.

**Proof.** It follows directly from Lemma 2.7 that
\[ -\Delta_p u \in W^{-1,p'}(\mathbb{R}^n) \cap \mathcal{M}^+(\mathbb{R}^n) \quad \text{and} \quad u \in L^{1+q}(\mathbb{R}^n, d\sigma) \]
for a quasicontinuous representative of \( u \). The former implies (1.6). The latter yields (1.5) in view of the global pointwise lower bound for supersolutions contained (Theorem 2.4).
In the next theorem, we verify that conditions (1.5) and (1.6) are sufficient for the existence of a positive finite energy solution to equation (1.1). Further, the minimality of such a solution is also proven.

We first observe that for \(1 < p < n, 0 < q < p - 1\) and \(\sigma, \mu \in M^+(\mathbb{R}^n)\) and \(\gamma, \beta \geq 0\),

\[
W_{a,p}(y\omega + \beta v) \leq A(y^{\frac{1}{p-1}}W_{a,p}\omega + \beta y^{\frac{1}{p-1}}W_{a,p}v),
\]

(3.10)

where \(A = A(\alpha, p, n) \geq 1\). This follows immediately from the definition of Wolff’s potential and the estimates

\[
|a + b|^r \leq \begin{cases} 2^{r-1}(|a|^r + |b|^r) & \text{for } 1 \leq r < \infty, \\ |a|^r + |b|^r & \text{for } 0 \leq r < 1, \end{cases}
\]

where \(a, b \in \mathbb{R}\).

**Theorem 3.6.** Let \(1 < p < n, 0 < q < p - 1\) and \(\sigma, \mu \in M^+(\mathbb{R}^n)\). Suppose that (1.5) and (1.6) hold. Then there exists a positive finite energy solution \(w\) to equation (1.1). Moreover, \(w\) is a minimal solution in the sense that \(w \leq u\) q.e. (for their respective quasicontinuous representatives) for any positive finite energy solution \(u\) to (1.1).

**Proof.** We first prove the existence of \(w\). Since (1.5) and (1.6) hold, it follows from Lemma 3.4 that (1.7) holds. By Theorem 3.1 in the case \(\alpha = 1\), there exists a positive solution \(v \in L^{1+q}(\mathbb{R}^n, d\sigma)\) to the integral equation

\[
v = W_{1,p}(v^q\,d\sigma) + W_{1,p}\mu \quad \text{in } \mathbb{R}^n.
\]

Using a constant multiple \(c^{-1}v\), where \(c > 0\), in place of \(v\), we have

\[
v = c^{\frac{1-p}{q-p}}W_{1,p}(v^q\,d\sigma) + cW_{1,p}\mu \quad \text{in } \mathbb{R}^n.
\]

Choose \(c \geq (KA)^{\frac{q-p}{q-p}} \geq KA \geq 1\), where \(K \geq 1\) is the constant in Theorem 2.6, and \(A \geq 1\) is the constant in (3.10). Then, by Lemma 2.8, we have

\[
v^q\,d\sigma \in W^{-1,p}(\mathbb{R}^n).
\]

Set

\[
w_0 := K^{-1}W_{1,p}\mu \quad \text{and} \quad d\omega_0 := w_0^q\,d\sigma + \mu.
\]

Since \(K^{-1} \leq c\), we have \(0 < w_0 \leq v\), and hence

\[
w_0 \in L^{1+q}(\mathbb{R}^n, d\sigma) \quad \text{and} \quad \omega_0 \in W^{-1,p}(\mathbb{R}^n).
\]

As discussed in Remark 2.11, for such a measure \(\omega_0\), there exists a unique \(p\)-superharmonic solution \(w_1 \in W^{1,p}(\mathbb{R}^n)\) to the equation

\[
-\Delta_p w_1 = \omega_0 \quad \text{in } \mathbb{R}^n \quad \text{and} \quad \|w_1\|^{p-1}_{W^{1,p}(\mathbb{R}^n)} = \|\omega_0\|_{W^{-1,p}(\mathbb{R}^n)}
\]

for a quasicontinuous representative of \(w_1\). Moreover, by Theorem 2.6,

\[
0 < w_1 \leq KW_{1,p}\omega_0 \leq KAW_{1,p}(v^q\,d\sigma) + KAW_{1,p}\mu \leq v \quad \text{q.e.}
\]

Since \(\sigma\) is absolutely continuous with respect to \(\text{cap}_{p}(\cdot)\), this yields

\[
w_1 \in L^{1+q}(\mathbb{R}^n, d\sigma) \quad \text{and} \quad \omega_1 := w_1^q\,d\sigma + d\mu \in W^{-1,p}(\mathbb{R}^n).
\]

Again by Theorem 2.6,

\[
w_0 = K^{-1}W_{1,p}\mu \leq K^{-1}W_{1,p}\omega_0 \leq w_1 \quad \text{q.e.}
\]

We now have

\[
0 < w_0 \leq w_1 \leq v \quad \text{q.e.}
\]

We shall construct, by induction, a sequence \(\{w_j\}_{j=0}^{\infty}\) so that

\[
\begin{align*}
-\Delta_p w_j &= \sigma w_{j-1}^q + \mu \quad \text{in } \mathbb{R}^n, \\
w_j &\in L^{1+q}(\mathbb{R}^n, d\sigma) \cap W^{1,p}_0(\mathbb{R}^n), \\
\sup_{j \in \mathbb{N}} \|w_j\|^{p-1}_{W^{1,p}_0(\mathbb{R}^n)} &< \infty, \\
w_{j-1}^q\,d\sigma + d\mu &\in W^{-1,p}(\mathbb{R}^n), \\
0 &< w_{j-1} \leq w_j \leq v \quad \text{q.e.}
\end{align*}
\]

(3.11)
We set
\[ d\omega_j := w_j^p \, d\sigma + d\mu, \quad j \in \mathbb{N}. \]

Suppose that \( w_1, w_2, \ldots, w_{j-1} \) have been constructed. Since \( \omega_{j-1} \in W^{-1,p'}(\mathbb{R}^n) \), it follows from Remark 2.11 that there exists a unique \( p \)-superharmonic solution \( w_j \in W^{1,p}(\mathbb{R}^n) \) to the equation
\[ -\Delta_p w_j = \omega_{j-1} \text{ in } \mathbb{R}^n. \]

Moreover,
\[ \|w_j\|_{W^{1,p}(\mathbb{R}^n)}^p = \|\omega_{j-1}\|_{W^{-1,p'}(\mathbb{R}^n)}^p \leq \int_{\mathbb{R}^n} w_j \omega_{j-1}^p \, d\sigma + \|\mu\|_{W^{-1,p'}(\mathbb{R}^n)}^p. \]

Applying Theorem 2.6, we obtain
\[ w_j \leq K W_{1,p}(\omega_{j-1}) \leq K A W_{1,p}(w_{j-1}^q \, d\sigma) + K A W_{1,p} \mu \text{ q.e.} \]

Since \( w_{j-1} \leq v \text{ q.e.} \), we have
\[ w_j \leq K A W_{1,p}(v^q \, d\sigma) + K A W_{1,p} \mu \leq v \text{ q.e.} \]

Hence, \( w_j \in L^{1+q}(\mathbb{R}^n, d\sigma) \) since \( \sigma \) is absolutely continuous with respect to \( \text{cap}_p(\cdot) \). Furthermore,
\[ \|w_j\|_{W^{1,p}(\mathbb{R}^n)}^p \leq \int_{\mathbb{R}^n} w_j \omega_{j-1}^q \, d\sigma + \|\mu\|_{W^{-1,p'}(\mathbb{R}^n)}^p \leq \left( \int_{\mathbb{R}^n} v^{1+q} \, d\sigma \right) \|\mu\|_{W^{-1,p'}(\mathbb{R}^n)}^p < +\infty. \]

This shows that \( \{w_j\}_{j=1}^{\infty} \) is a bounded sequence in \( W^{1,p}(\mathbb{R}^n) \). Moreover, since \( \omega_{j-2} \leq \omega_{j-1} \), it follows from the Weak Comparison Principle (Lemma 2.9) that \( w_{j-1} \leq w_j \text{ q.e.} \). Hence, the sequence \( \{w_j\}_{j=1}^{\infty} \) satisfying (3.11) has been constructed. Applying the weak continuity of \( p \)-Laplacian (Theorem 2.3), the Monotone Convergence Theorem and the Weak Compactness Property in \( W^{1,p}(\mathbb{R}^n) \), see [15, Lemma 1.33], we deduce that the pointwise limit \( w := \lim_{j \to \infty} w_j \) is a positive finite energy solution to (1.1).

We now prove the minimality of \( w \). Suppose that \( u \) is any positive finite energy solution to (1.1). Set \( d\omega := u^q \, d\sigma + d\mu \). By Lemma 3.5, we have
\[ u \in L^{1+q}(\mathbb{R}^n, d\sigma) \quad \text{and} \quad \omega \in W^{-1,p'}(\mathbb{R}^n) \cap \mathcal{M}^+(\mathbb{R}^n) \]
for a quasicontinuous representative of \( u \). We need to show that \( w \leq u \text{ q.e.} \). Notice that
\[ u \geq (W_{1,p} \mu) > K^{-1}(W_{1,p} \mu) = w_0 \text{ q.e.} \]

Therefore \( w_0 \leq w \) since \( \sigma \) is absolutely continuous with respect to \( \text{cap}_p(\cdot) \). By the Weak Comparison Principle (Lemma 2.9), \( w_1 \leq u \text{ q.e.} \). Arguing by induction as above, we see that
\[ w_{j-1} \leq w_j \leq u \text{ q.e.} \]

It follows that \( w = \lim_{j \to \infty} w_j \leq u \text{ q.e.} \), which proves the claim. \( \square \)

**Remark 3.7.** For a similar equation in a domain \( \Omega \subset \mathbb{R}^n \),
\[ -\Delta_p u = u^q \sigma + \mu \text{ in } \Omega, \quad (3.12) \]
where \( 1 < p < n, 0 < q < p - 1 \) and \( \sigma, \mu \in \mathcal{M}^+(\Omega) \), we also have analogous sufficient conditions for the existence of a positive finite energy solution in terms of truncated Wolff’s potential, namely:
\[ W_{1,p}^R \sigma \in L^{\frac{1+(n-1)p}{p-1}}(\Omega, d\sigma), \quad R \geq 2\text{diam}(\Omega), \quad (3.13) \]
and
\[ \mu \in W^{-1,p'}(\Omega). \quad (3.14) \]

Here, for \( 1 < p < \infty, 0 < \alpha < \frac{n}{p} \) and \( \sigma \in \mathcal{M}^+(\Omega) \), the truncated Wolff potential \( W_{\alpha,p}^R \sigma \) is defined by (see [19])
\[ W_{\alpha,p}^R \sigma(x) = \int_0^R \frac{\sigma(B(x, r) \cap \Omega)}{r^{n-\alpha p}} \, \frac{r^{p-1}}{p}, \quad x \in \Omega, 0 < R \leq +\infty. \]

Moreover, conditions (3.13) and (3.14) are also necessary whenever \( \sigma \) and \( \mu \) have compact supports in \( \Omega \). These results are deduced easily from Theorem 1.1; see details in [22].
4 Existence of a positive finite energy solution to equation (1.2)

In this section, we employ an argument similar to the one used in the previous section to deduce necessary and sufficient conditions for the existence of a positive finite energy solution to the fractional Laplace equation (1.2).

**Definition 4.1.** Let $0 < q < 1$, $0 < \alpha < \frac{q}{2}$ and $\sigma, \mu \in \mathcal{M}^+(\mathbb{R}^n)$. A finite energy solution $u$ to equation (1.2) will be understood in the sense that $u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma) \cap H^\alpha(\mathbb{R}^n)$, $u \geq 0$ a.e. such that
\[
(-\Delta)^\frac{\alpha}{2} u = I_\alpha(u^q d\sigma) + I_\alpha \mu \quad \text{dx-a.e.} 
\]  
(4.1)

**Remark 4.2.** By using the same notation as above, suppose that $u$ is a positive finite energy solution to (1.2). Applying the Riesz potential $I_\alpha$ of order $\alpha$ to both sides of (4.1) yields
\[
u(x) = I_\alpha(u^q d\sigma)(x) + I_\alpha \mu(x) \quad \text{whenever} \quad u(x) < +\infty.
\]

Notice that $u \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma) \cap H^\alpha(\mathbb{R}^n)$. Then
\[
u = I_\alpha(u^q d\sigma) + I_\alpha \mu \quad \text{a.e. and q.e.} 
\]  
(4.2)

In particular,
\[
u \geq I_\alpha(u^q d\sigma) \quad \text{dx-a.e.}, 
\]
which implies, by Lemma 2.12, that $\sigma$ is absolutely continuous with respect to $\text{cap}_{\alpha,2}(\cdot)$. On the other hand, (4.1) implies in particular that
\[
u = I_\alpha(u^q d\sigma) + I_\alpha \mu \quad \text{dx-a.e.} 
\]
(4.1)

Therefore $I_\alpha \mu \in L^2(\mathbb{R}^n)$, and hence $\mu \in H^{-\alpha}(\mathbb{R}^n)$. In particular, $\mu$ is absolutely continuous with respect to $\text{cap}_{\alpha,2}(\cdot)$ (see, for example, [1, Section 7]). In summary, $u$ satisfies the integral equation (4.2) in the following senses: a.e., $d\sigma$-a.e., $d\mu$-a.e., and q.e.

The following important observation is analogous to Lemma 3.4.

**Lemma 4.3.** Let $0 < q < 1$, $0 < \alpha < \frac{q}{2}$ and $\sigma, \mu \in \mathcal{M}^+(\mathbb{R}^n)$. Then (1.10) and (1.11) imply (1.12)

**Proof.** As shown in [8], (1.10) holds if and only if there exists a positive constant $c$ such that
\[
\|I_\alpha g\|_{L^1(\mathbb{R}^n, d\sigma)} \leq c\|g\|_{L^q(\mathbb{R}^n)} \quad \text{for all} \quad g \in L^2(\mathbb{R}^n). 
\]  
(4.3)

Letting $g := I_\alpha \mu \in L^2(\mathbb{R}^n)$ in (4.3), we have
\[
\|I_\alpha \mu\|_{L^1(\mathbb{R}^n, d\sigma)} \leq c\|I_\alpha \mu\|_{L^2(\mathbb{R}^n)} < \infty, 
\]
which proves (1.12).

The necessary conditions for the existence of a positive finite energy solution to equation (1.2) are established in the following lemma.

**Lemma 4.4.** Let $0 < q < 1$, $0 < \alpha < \frac{q}{2}$ and $\sigma, \mu \in \mathcal{M}^+(\mathbb{R}^n)$. Suppose that there exists a positive finite energy solution $u$ to equation (1.2). Then (1.11) holds and $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$. Consequently, (1.10) holds.

**Proof.** Suppose that $u$ is a positive finite energy solution to (1.2). Then, as discussed in Remark 4.2, (1.11) holds. We next show that $u \in L^{1+q}(\mathbb{R}^n, d\sigma)$. By (4.1), for each nonnegative function $\varphi \in L^2(\mathbb{R}^n)$, we have
\[
\int_{\mathbb{R}^n} [(-\Delta)^\frac{\alpha}{2} u] \varphi \; dx = \int_{\mathbb{R}^n} [I_\alpha(u^q d\sigma)] \varphi \; dx + \int_{\mathbb{R}^n} [I_\alpha \mu] \varphi \; dx.
\]

Applying Tonelli’s Theorem and Schwarz’s inequality, we obtain
\[
\|I_\alpha [u^q \varphi] d\sigma \| \leq \int_{\mathbb{R}^n} [I_\alpha(u^q d\sigma)] \varphi \; dx \leq \int_{\mathbb{R}^n} [(-\Delta)^\frac{\alpha}{2} u] \varphi \; dx + \int_{\mathbb{R}^n} [I_\alpha \mu] \varphi \; dx \leq c\|\varphi\|_{L^2(\mathbb{R}^n)} 
\]  
(4.4)
where \( c := \|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^q(\mathbb{R}^n)} + \|I_0 u\|_{L^2(\mathbb{R}^n)} < \infty \), since \( u \in H^q(\mathbb{R}^n) \) and \( \mu \in H^{-q}(\mathbb{R}^n) \). Letting \( \varphi := (-\Delta)^{\frac{\alpha}{2}} u \), which is a nonnegative function of class \( L^2(\mathbb{R}^n) \) in (4.4), we get
\[
\|u\|_{L^{1+q}(\mathbb{R}^n, d\sigma)}^{1+q} \leq c\|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^q(\mathbb{R}^n)} + \|\mu\|_{L^2(\mathbb{R}^n)} < +\infty.
\]
This shows that \( u \in L^{1+q}(\mathbb{R}^n, d\sigma) \). Notice that
\[
\|u\|_{L^{1+q}(\mathbb{R}^n, d\sigma)}^{1+q} \leq c\|(-\Delta)^{\frac{\alpha}{2}} u\|_{L^q(\mathbb{R}^n)} + \|\mu\|_{L^2(\mathbb{R}^n)} < +\infty.
\]

Hence, by the discussion in Remark 3.3 in the case \( p = 2 \), we have that (1.10) holds. 

The next theorem shows that conditions (1.10) and (1.11) allow us to construct a positive finite energy solution to equation (1.2). Minimality of such a solution will be proven as well.

**Theorem 4.5.** Let \( 0 < q < 1 \), \( 0 < \alpha < \frac{n}{q} \) and \( \sigma, \mu \in \mathcal{M}^+(\mathbb{R}^n) \). Suppose that (1.10) and (1.11) hold. Then there exists a positive finite energy solution \( w \) to equation (1.2). Moreover, \( w \) is a minimal solution in the sense that \( w \leq u \) q.e. for any positive finite energy solution \( u \) to (1.2).

**Proof.** We first prove the existence of \( w \). Since (1.10) and (1.11) hold, it follows from Lemma 4.3 that (1.12) holds. By Theorem 3.1 in the case \( p = 2 \), there exists a positive solution \( w \in L^{1+q}(\mathbb{R}^n, d\sigma) \) to the integral equation
\[
w = I_{2\alpha}(w^\alpha d\sigma) + I_{2\alpha} \mu \quad \text{in} \, \mathbb{R}^n. \tag{4.5}
\]

We will show that
\[
w \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma) \cap H^q(\mathbb{R}^n).
\]

Clearly, \( w \in L^q_{\text{loc}}(\mathbb{R}^n, d\sigma) \) by Hölder’s inequality. In order to prove that \( w \in H^q(\mathbb{R}^n) \), by duality, it suffices to show that there exists a positive constant \( c \) such that
\[
\left\| \int_{\mathbb{R}^n} w \psi \, dx \right\|_{H^{-q}(\mathbb{R}^n)} \leq c\|\psi\|_{L^q(\mathbb{R}^n)}, \quad \psi \in C_0^\infty(\mathbb{R}^n). \tag{4.6}
\]

By the semigroup property of the Riesz potentials, Tonelli’s Theorem and Hölder’s inequality, we have
\[
\left\| \int_{\mathbb{R}^n} w \psi \, dx \right\|_{H^{-q}(\mathbb{R}^n)} \leq \left\| I_0(w^\alpha d\sigma)\right\|_{L^q(\mathbb{R}^n)} \|I_{2\alpha} \psi\|_{L^q(\mathbb{R}^n)} + \|I_{2\alpha} \mu\|_{L^2(\mathbb{R}^n)}\| I_{2\alpha} \psi\|_{L^2(\mathbb{R}^n)}
\]
\[
= \left[\| I_0(w^\alpha d\sigma)\|_{L^q(\mathbb{R}^n)} + \|\mu\|_{L^2(\mathbb{R}^n)}\right]\|\psi\|_{H^{-q}(\mathbb{R}^n)}, \tag{4.7}
\]
for all \( \psi \in C_0^\infty(\mathbb{R}^n) \). Since \( \|\mu\|_{H^{-q}(\mathbb{R}^n)} < +\infty \), we see that, in view of (4.6) and (4.7), it remains to show that
\[
\| I_0(w^\alpha d\sigma)\|_{L^q(\mathbb{R}^n)} < \infty. \tag{4.8}
\]

To this end, notice that – by the result in [8] – (1.10) is equivalent to
\[
\| I_0 g\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} \leq c\|g\|_{L^q(\mathbb{R}^n)} \quad \text{for all} \, g \in L^2(\mathbb{R}^n), \tag{4.9}
\]
where \( c \) is a positive constant independent of \( g \). Moreover, by duality, (4.9) is equivalent to
\[
\| I_0(\varphi d\sigma)\|_{L^q(\mathbb{R}^n)} \leq c\|\varphi\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} \quad \text{for all} \, \varphi \in L^{1+q}(\mathbb{R}^n, d\sigma), \tag{4.10}
\]
where \( c \) is a positive constant independent of \( \varphi \). Letting \( \varphi := w^\alpha \in L^{1+q}(\mathbb{R}^n, d\sigma) \) in (4.10), we have
\[
\| I_0(w^\alpha d\sigma)\|_{L^q(\mathbb{R}^n)} \leq c\|w\|_{L^{1+q}(\mathbb{R}^n, d\sigma)} < +\infty,
\]
which proves (4.8), and hence \( w \in H^q(\mathbb{R}^n) \). Moreover, by (4.5), we have
\[
(-\Delta)^{\frac{\alpha}{2}} w = I_{2\alpha}(w^\alpha d\sigma) + I_{2\alpha} \mu \quad \text{a.e.}
\]

This shows that \( w \) is a positive finite energy solution to (1.2).
Minimality of the solution \( w \) is obvious by its construction in Theorem 3.1 in the case \( p = 2 \). Recall that \( w \) is the pointwise limit \( w = \lim_{j \to \infty} w_j \), where

\[
w_0 := I_{2a} \mu \quad \text{and} \quad w_{j+1} := I_{2a}(w_j^q \, d\sigma) + I_{2a} \mu, \quad j \in \mathbb{N}_0.
\]

If \( u \) is any positive finite energy solution to (1.2), then

\[
w_0 = I_{2a} \mu \leq I_{2a}(u^q \, d\sigma) + I_{2a} \mu = u \quad \text{q.e.}
\]

Consequently,

\[
w_1 = I_{2a}(w_0^q \, d\sigma) + I_{2a} \mu \leq I_{2a}(u^q \, d\sigma) + I_{2a} \mu = u \quad \text{q.e.}
\]

Arguing by induction, we obtain

\[
w_{j-1} \leq w_j \leq u \quad \text{q.e. for all } j \in \mathbb{N}.
\]

Therefore, \( w = \lim_{j \to \infty} w_j \leq u \) q.e. This proves the minimality of \( w \).

\[\square\]

### 5 Existence of a positive finite energy solution to equation (1.3)

Let \( \Omega \) be a domain in \( \mathbb{R}^n \) and let \( G : \Omega \times \Omega \to (0, \infty] \) be a positive lower semicontinuous kernel. For \( \nu \in \mathcal{M}^+(\Omega) \), the potential of \( \nu \) is defined by

\[
G_\nu(x) := \int_{\Omega} G(x,y) \, d\nu(y), \quad x \in \Omega.
\]

A positive kernel \( G \) on \( \Omega \times \Omega \) is said to satisfy the weak maximum principle (WMP) with constant \( h \geq 1 \) if for any \( \nu \in \mathcal{M}^+(\Omega) \),

\[
\sup\{G_\nu(x) : x \in \text{supp}(\nu)\} \leq M \implies \sup\{G_\nu(x) : x \in \Omega\} \leq h M
\]

for every constant \( M > 0 \). Here we use the notation \( \text{supp}(\nu) \) for the support of \( \nu \in \mathcal{M}^+(\Omega) \).

When \( h = 1 \) in (5.1), the positive kernel \( G \) is said to satisfy the strong maximum principle, which holds for positive Green’s functions associated with the classical Laplacian \(-\Delta\), and more generally the fractional Laplacian \((-\Delta)^a\) in the case \( 0 < a \leq 1 \), for every domain \( \Omega \subset \mathbb{R}^n \) which possesses a positive Green’s function.

The WMP holds for Riesz kernels on \( \mathbb{R}^n \) associated with \((-\Delta)^a\) in the full range \( 0 < a < \frac{n}{2} \), and more generally for all radially nonincreasing kernels on \( \mathbb{R}^n \) (see [1]).

We say that a function \( d(x,y) : \Omega \times \Omega \to [0, \infty) \) satisfies the quasimetric triangle inequality with constant \( \kappa > 0 \) if

\[
d(x,y) \leq \kappa[d(x,z) + d(z,y)], \quad x, y, z \in \Omega.
\]

A positive kernel \( G \) on \( \Omega \times \Omega \) is called quasimetric if \( G \) is symmetric and the function \( d(x,y) = \frac{1}{G(x,y)} \) satisfies (5.2). The WMP holds for quasimetric kernels, see [11–13, 24]. We say that a positive kernel \( G \) on \( \Omega \times \Omega \) is quasi-symmetric if there exists a constant \( a > 0 \) such that

\[
a^{-1}G(y,x) \leq G(x,y) \leq aG(y,x), \quad x, y \in \Omega.
\]

There are many kernels associated with elliptic operators that are quasi-symmetric and satisfy the WMP (see [2]).

In this section, we establish necessary and sufficient conditions for the existence of a positive solution \( u \in L^{1+q}(\Omega, d\sigma) \) to the integral equation

\[
u = G(u^q \, d\sigma) + G\mu \quad d\sigma\text{-a.e.},
\]

where \( 0 < q < 1 \) and \( \sigma, \mu \in \mathcal{M}^+(\Omega) \), provided that \( G \) is a quasi-symmetric kernel which satisfies the WMP.
If $G$ is Green's function associated with $-\Delta$ on $\Omega$, then the integral equation (5.3) is equivalent to the sublinear elliptic boundary value problem

$$
\begin{cases}
-\Delta u = \sigma u^q + \mu & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

As an application, we can deduce necessary and sufficient conditions for existence of a positive finite energy solution $u \in L^q_{\text{loc}}(\Omega, d\sigma) \cap W^{1,2}_0(\Omega)$ to equation (1.3).

We will need the following result proved in [27], which explicitly characterizes $(p, r)$-weighted norm inequalities

$$
\|G(f d\sigma)\|_{L^p(\Omega, d\sigma)} \leq C\|f\|_{L^r(\Omega, d\sigma)} \quad \text{for all } f \in L^p(\Omega, d\sigma),
$$

(5.4)

where $C$ is a positive constant independent of $f$, in the case $0 < r < p$ and $1 < p < \infty$, under some mild assumptions on the kernel $G$.

**Theorem 5.1** ([27]). Let $\sigma \in \mathcal{M}^+(\Omega)$ and let $G$ be a positive quasi-symmetric lower semicontinuous kernel on $\Omega \times \Omega$, which satisfies the WMP.

(i) If $1 < p < \infty$ and $0 < r < p$, then the $(p, r)$-weighted norm inequality (5.4) holds if and only if

$$
G\sigma \in L^{\frac{p}{r}}(\Omega, d\sigma).
$$

(ii) If $0 < q < 1$ and $q < r < \infty$, then there exists a positive supersolution $u \in L^r(\Omega, d\sigma)$ to the homogeneous integral equation (5.3) with $\mu = 0$ so that

$$
u \geq G(u^q d\sigma) \quad d\sigma\text{-a.e.}
$$

if and only if the weighted norm inequality (5.4) holds with $p = \frac{r}{q}$, that is,

$$
\|G(f d\sigma)\|_{L^p(\Omega, d\sigma)} \leq C\|f\|_{L^{\frac{r}{q}}(\Omega, d\sigma)} \quad \text{for all } f \in L^p(\Omega, d\sigma),
$$

(5.5)

where $C$ is positive constant independent of $f$ or, equivalently,

$$
G\sigma \in L^{\frac{p}{q}}(\Omega, d\sigma).
$$

The following theorem gives necessary and sufficient conditions for the existence of a positive solution $u \in L^{1+q}(\Omega, d\sigma)$ to the integral equation (5.3). In fact, it is a more general version of Theorem 3.1 in the linear case $p = 2$.

**Theorem 5.2.** Let $0 < q < 1$ and $\sigma, \mu \in \mathcal{M}^+(\Omega)$, and let $G$ be a positive quasi-symmetric lower semicontinuous kernel on $\Omega \times \Omega$, which satisfies the WMP. Then there exists a positive solution $u \in L^{1+q}(\Omega, d\sigma)$ to the integral equation (5.3) if and only if (1.14) and (1.17) hold.

**Proof.** The sufficiency part is similar to the one of Theorem 3.1 when $p = 2$, proved by applying part (ii) of Theorem 5.1 in the case $r = q + 1$ in place of Lemma 3.2, and replacing Wolff’s potentials by potential operators $G$ associated with the kernel $G$. The necessity part follows immediately from Theorem 5.1 (ii) in the case $r = q + 1$.

We now apply the above result to deduce necessary and sufficient conditions for the existence of a positive finite energy solution to equation (1.3). As in previous sections, we first make the following observation regarding the relation between conditions (1.14), (1.15) and (1.17).

**Lemma 5.3.** Let $0 < q < 1$ and $\sigma, \mu \in \mathcal{M}^+(\Omega)$, and let $G$ be a positive quasi-symmetric lower semicontinuous kernel on $\Omega \times \Omega$, which satisfies the WMP. Then (1.14) and (1.15) imply (1.17).

**Proof.** By Theorem 5.1 (ii) with $r = 1 + q$, (1.14) holds if and only if there exists a constant $C$ such that

$$
\|G(f d\sigma)\|_{L^{1+q}(\Omega, d\sigma)} \leq C\|f\|_{L^{1+q}(\Omega, d\sigma)} \quad \text{for all } f \in L^{1+q}(\Omega, d\sigma).
$$

(5.5)
Suppose that \( f \) is any nonnegative bounded measurable function with compact support in \( \Omega \). Applying Hölder’s inequality and the weighted norm inequality (5.5), we have

\[
\|G(f\sigma)\|_{W^{1,q}_0(\Omega)}^2 = \int_\Omega |\nabla G(f\sigma)|^2 \, dx
\]

\[
= \int_\Omega G(f\sigma) \cdot f \, d\sigma
\]

\[
\leq \|G(f\sigma)\|_{L^{1+q}(\Omega, d\sigma)} \|f\|_{L^{1+q}(\Omega, d\sigma)}
\]

\[
\leq C\|f\|_{L^{1+q}(\Omega, d\sigma)}^2. \quad (5.6)
\]

Since \( \mu \in W^{-1,2}(\Omega) \), by Tonelli’s Theorem and the Brezis–Browder Theorem (Theorem 2.10), we obtain

\[
\left| \int_\Omega (G\mu) f \, d\sigma \right| = \left| \int_\Omega G(f\sigma) \, d\mu \right|
\]

\[
= |\langle G(f\sigma), \mu \rangle| \leq \|G(f\sigma)\|_{W^{1,2}_0(\Omega)} \|\mu\|_{W^{-1,2}(\Omega)}
\]

\[
\leq C\|f\|_{L^{1+q}(\Omega, d\sigma)} \|\mu\|_{W^{-1,2}(\Omega)}. \quad (5.7)
\]

Applying a standard density argument, we see that (5.7) actually holds for all \( f \in L^{1+q}(\Omega, d\sigma) \). By duality, taking the supremum over all \( f \in L^{1+q}(\Omega, d\sigma) \), we get

\[
\left( \int_\Omega (G\mu)^{1+q} \, d\sigma \right)^{\frac{1}{1+q}} \leq C\|\mu\|_{W^{-1,2}(\Omega)} < +\infty,
\]

which proves the lemma.

The next lemma shows in particular that conditions (1.14) and (1.15) are necessary for the existence of a positive finite energy solution to equation (1.3).

**Lemma 5.4.** Let \( 0 < q < 1 \) and \( \sigma, \mu \in \mathcal{M}^+(\Omega) \), and let \( G \) be Green’s function associated with \(-\Delta\) on \( \Omega \). Suppose that there exists a positive supersolution \( u \in L^q_{\text{loc}}(\Omega, d\sigma) \cap \dot{W}^{1,2}_0(\Omega) \) to equation (1.3). Then

\[-\Delta u \in \dot{W}^{-1,2}(\Omega) \cap \mathcal{M}^+(\Omega), \]

and hence (1.15) holds. Moreover, \( u \in L^{1+q}(\Omega, d\sigma) \) for a quasiconstant representative of \( u \) and, consequently, (1.14) holds as well.

**Proof.** By Schwarz’s inequality, for every \( \varphi \in C_0^\infty(\Omega) \) we have

\[
|\langle -\Delta u, \varphi \rangle| = \left| \int_\Omega \nabla u \cdot \nabla \varphi \, dx \right| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla \varphi\|_{L^2(\Omega)}.
\]

Hence, \(-\Delta u \in \dot{W}^{-1,2}(\Omega)\). Moreover, for every nonnegative \( \varphi \in C_0^\infty(\Omega) \) we have

\[
\langle -\Delta u, \varphi \rangle = \int_\Omega \nabla u \cdot \nabla \varphi \, dx \geq \int_\Omega u^q \varphi \, d\sigma + \int_\Omega \varphi \, d\mu \geq 0.
\]

This shows that \(-\Delta u \in \mathcal{M}^+(\Omega)\), from which it follows that (1.15) holds, and

\[
d\nu := u^q \, d\sigma \in \dot{W}^{-1,2}(\Omega) \cap \mathcal{M}^+(\Omega).
\]

Let \( \{\varphi_j\}^\infty_1 \subset C_0^\infty(\Omega) \) be a sequence of nonnegative functions such that \( \varphi_j \rightharpoonup u \) in \( \dot{W}^{1,2}_0(\Omega) \) as \( j \to \infty \). Then

\[
\langle \nu, \varphi_j \rangle \leq \int_\Omega \nabla u \cdot \nabla \varphi_j \, dx \quad \text{for all } j \in \mathbb{N}.
\]
As discussed in Remark 2.11, for such a measure \( \mu \), a solution \( u \in \dot{W}^{1,p}_0(\mathbb{R}^n) \) to the equation \( -\Delta_p u = \omega \) in \( \mathbb{R}^n \) is unique. Hence, \( u = v \) q.e., so they coincide as elements of \( \dot{W}^{1,p}_0(\mathbb{R}^n) \).

6 Uniqueness

In this section, we establish the uniqueness of positive finite energy solutions to equations (1.1), (1.2) and (1.3), using the idea used in [10], namely employing convexity properties of Dirichlet integrals and minimality of such solutions.

**Theorem 6.1.** Let \( 1 < p < n, 0 < q < p - 1 \) and \( \sigma, \mu \in \mathcal{M}^+(\mathbb{R}^n) \). Suppose that there exists a positive finite energy solution to equation (1.1). Then such a solution is unique in \( \dot{W}^{1,p}_0(\mathbb{R}^n) \).

**Proof.** Suppose that \( u \) and \( v \) are positive finite energy solutions to (1.1). We start with the following two observations. We first claim that

**Claim.** If \( u = v \) d\( \sigma \)-a.e., then \( u = v \) as elements of \( \dot{W}^{1,p}_0(\mathbb{R}^n) \).

To see this, suppose that \( u = v \) d\( \sigma \)-a.e., and set

\[
d\omega := u^q d\sigma + d\mu = v^q d\sigma + d\mu.
\]

Then \( \omega \in \mathcal{M}^+(\mathbb{R}^n) \) and

\[
-\Delta_p u = -\Delta_p v = \omega \quad \text{in} \ \mathbb{R}^n.
\]

As usual, we may consider quasicontinuous representatives of \( u \) and \( v \). Then, by Lemma 3.5,

\[
u, v \in L^{1+q}(\mathbb{R}^n, d\sigma) \quad \text{and} \quad \omega \in W^{-1,p'}(\mathbb{R}^n).
\]

As discussed in Remark 2.11, for such a measure \( \omega \), a solution \( u \in \dot{W}^{1,p}_0(\mathbb{R}^n) \) to the equation \( -\Delta_p u = \omega \) in \( \mathbb{R}^n \) is unique. Hence, \( u = v \) q.e., so they coincide as elements of \( \dot{W}^{1,p}_0(\mathbb{R}^n) \).
Secondly, we claim that

**Claim.** If \( u \geq v \text{ q.e. then } u = v \text{ } d\sigma\text{-a.e.} \)

Suppose that \( u \geq v \text{ q.e. and } u \geq v \text{ } d\mu\text{-a.e. because } \sigma \text{ and } \mu \text{ are absolutely continuous with respect to } \cap_p \). Testing the equations

\[
\int |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dx = \int u^q \phi \, d\sigma + \int \phi \, d\mu, \quad \phi \in W_0^{1,p} \cap L_{loc}^q, \quad (6.1)
\]

\[
\int |\nabla v|^{p-2} \nabla v \cdot \nabla \psi \, dx = \int v^q \psi \, d\sigma + \int \psi \, d\mu, \quad \psi \in W_0^{1,p} \cap L_{loc}^q, \quad (6.2)
\]

with \( \phi = u \) and \( \psi = v \), respectively, we obtain

\[
\int |\nabla u|^p \, dx = \int u^{1+q} \, d\sigma + \int u \, d\mu, \quad (6.3)
\]

\[
\int |\nabla v|^p \, dx = \int v^{1+q} \, d\sigma + \int v \, d\mu. \quad (6.4)
\]

Using convexity of the Dirichlet integral \( \int |\nabla |^p \, dx \) along curves of the type

\[
\lambda_t(x) := [(1-t)v^p(x) + tu^p(x)]^{\frac{1}{p}}, \quad t \in [0, 1],
\]

see [4, Proposition 2.6], we obtain

\[
\int |\nabla \lambda_t|^p \, dx \leq (1-t) \int |\nabla v|^p \, dx + t \int |\nabla u|^p \, dx = t \left( \int |\nabla u|^p \, dx - \int |\nabla v|^p \, dx \right) + \int |\nabla v|^p \, dx. \quad (6.5)
\]

Notice that \( \lambda_0 = v \). By (6.3) and (6.4), we get

\[
\int |\nabla \lambda_t|^p \, dx \leq \frac{(u^{1+q} - v^{1+q}) \, d\sigma + \int (u - v) \, d\mu.}{t}
\]

Using the inequality

\[
|a|^p - |b|^p \geq |b|^{p-2} b \cdot (a - b) \quad \text{for } a, b \in \mathbb{R}^n,
\]

we deduce that

\[
|\nabla \lambda_t|^p - |\nabla \lambda_0|^p \geq p |\nabla \lambda_0|^{p-2} \nabla \lambda_0 \cdot (\nabla \lambda_t - \nabla \lambda_0),
\]

and hence

\[
|\nabla \lambda_t|^p - |\nabla \lambda_0|^p \geq p |\nabla \lambda_0|^{p-2} \nabla \lambda_0 \cdot (\nabla \lambda_t - \nabla \lambda_0), \quad (6.6)
\]

Testing (6.2) by \( \psi = \lambda_t - \lambda_0 \in W_0^{1,p} \cap L_{loc}^q \), we obtain

\[
\int |\nabla v|^p \, dx \leq \int \nabla (\lambda_t - \lambda_0) \, d\sigma + \int (\lambda_t - \lambda_0) \, d\mu. \quad (6.7)
\]

Thus, by (6.5) and (6.6), we have

\[
\int \nabla (\lambda_t - \lambda_0) \, d\sigma + \int (\lambda_t - \lambda_0) \, d\mu \leq \int (u^{1+q} - v^{1+q}) \, d\sigma + \int (u - v) \, d\mu. \quad (6.7)
\]

Since \( u \geq v \text{ q.e.}, \) we have \( \lambda_t \geq \lambda_0 \text{ } d\sigma\text{-a.e. and } \lambda_t \geq \lambda_0 \text{ } d\mu\text{-a.e. Applying Fatou's Lemma, we obtain}

\[
\int \nabla u \cdot \nabla \phi \, dx \leq \liminf_{t \to 0} \int \nabla \lambda_t \cdot \nabla \phi \, dx, \quad (6.8)
\]

\[
\int \nabla v \cdot \nabla \psi \, dx \leq \liminf_{t \to 0} \int \nabla \lambda_0 \cdot \nabla \psi \, dx. \quad (6.9)
\]
Since (6.7) holds for all $t \in [0, 1]$, it follows from (6.8) and (6.9) that
\[
\int_{\mathbb{R}^n} \frac{u^p v^q}{v^{p-1}} - v^{1+q} \, d\sigma + \int_{\mathbb{R}^n} \frac{u^p}{v^{p-1}} - v \, d\mu \leq \int_{\mathbb{R}^n} (u^{1+q} - v^{1+q}) \, d\sigma + \int_{\mathbb{R}^n} (u - v) \, d\mu,
\]
that is,
\[
\int_{\mathbb{R}^n} \frac{u^p v^q}{v^{p-1}} - u^{1+q} \, d\sigma + \int_{\mathbb{R}^n} \left( \frac{u^p}{v^{p-1}} - u \right) \, d\mu \leq 0.
\]
Here both integrals on the left-hand side are nonnegative since $u \geq v \, d\sigma$-a.e. and $u \geq v \, d\mu$-a.e. Indeed,
\[
\int_{\mathbb{R}^n} \frac{u^p v^q}{v^{p-1}} - u^{1+q} \, d\sigma = \int_{\mathbb{R}^n} \frac{u^p v^q - u^{1+q} v^{p-1}}{v^{p-1}} \, d\sigma = \int_{\mathbb{R}^n} \frac{u^{1+q} v^q (u^{p-1-q} - v^{p-1-q})}{v^{p-1}} \, d\sigma \geq 0
\]
and
\[
\int_{\mathbb{R}^n} \left( \frac{u^p}{v^{p-1}} - u \right) \, d\mu = \int_{\mathbb{R}^n} \frac{u^p - u v^{p-1}}{v^{p-1}} \, d\mu \geq 0.
\]
Therefore, both integrals must vanish, and thus $u = v \, d\sigma$-a.e. and $u = v \, d\mu$-a.e. In particular, this proves the second claim.

Now, suppose that $\tilde{w}$ is any positive finite energy solution to (1.1). Then
\[
\tilde{w} \geq w \quad \text{q.e.,}
\]
where $w$ is the minimal positive finite energy solution to (1.1) constructed in Theorem 3.6. Applying the second claim above, we have
\[
\tilde{w} = w \quad d\sigma\text{-a.e.,}
\]
and hence, by the first claim, they coincide as elements of $W^{1,p}_0(\mathbb{R}^n)$. $\square$

By a slight modification of the argument above, we can establish the uniqueness of a positive finite energy solution to equation (1.2) when $0 < \alpha \leq 1$.

**Theorem 6.2.** Let $0 < q < 1$, $0 < \alpha \leq 1$ and $\sigma, \mu \in M^+(\mathbb{R}^n)$. Suppose that there exists a positive finite energy solution to equation (1.2). Then such a solution is unique in $H^\alpha(\mathbb{R}^n)$.

**Proof.** When $\alpha = 1$, this follows from Theorem 6.1 in the case $p = 2$. If $0 < \alpha < 1$, we use the same argument as in the proof of Theorem 6.1 together with convexity of Gagliardo seminorms established in [4], instead of convexity of the Dirichlet integrals $\int_{\mathbb{R}^n} |\nabla \cdot |^p \, dx$. $\square$

Since convexity of the Dirichlet integrals $\int_{\Omega} |\nabla \cdot |^2 \, dx$ is also available on arbitrary nonempty open sets $\Omega \subset \mathbb{R}^n$ (see [4]), we may argue in the same way as in the proof of Theorem 6.1 in the case $p = 2$ to obtain the uniqueness of a positive finite energy solution to equation (1.3).

**Theorem 6.3.** Let $0 < q < 1$ and let $\sigma, \mu \in M^+(\Omega)$. Suppose that there exists a positive finite energy solution to equation (1.3). Then such a solution is unique in $W^{1,2}_0(\Omega)$.

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