An FPTAS for the Lead-Based Multiple Video Transmission (LMVT) Problem

Swapnoneel Roy and Atri Rudra

Department of Computer Science and Engineering,
University at Buffalo, The State University of New York,
Buffalo, New York 14260.
{atri, sroy7}@buffalo.edu

Abstract. The Lead-Based Multiple Video Transmission (LMVT) problem is motivated by applications in managing the quality of experience (QoE) of video streaming for mobile clients. In an earlier work, the LMVT problem has been shown to be NP-hard for a specific bit-to-lead conversion function φ. In this work, we show the problem to be NP-hard even if the function φ is linear. We then design a fully polynomial time approximation scheme (FPTAS) for the problem. This problem is exactly equivalent to the SANTA CLAUSE PROBLEM on which there has been a lot of work done off-late.

1 Introduction

The LMVT problem deals with multiplexing videos simultaneously slot by slot over a wireless channel. Each slot could be allocated to at most one video. The number of bits that can be transmitted to a particular video in a given slot is known. Hence the videos receive variable number of bits over the various slots. The time of transmission is divided over a number of epochs. Each such epoch has a fixed number of slots (say B) known beforehand. The goal is to allocate all the slots in an epoch to the n videos in a manner which maximizes the minimum number of bits received by any video in that epoch.

The lead of a video is defined as the amount of time it can be played without interruption. An interruption in the playing of a video occurs when there are no more frames left in the buffer to be played. The lead of any video is calculated at the end of an epoch using a function φ on the number of bits b received by that video in that epoch. The motivation behind studying this problem is to develop a slot allocation algorithm to ensure the uninterrupted play of each video in the network. This problem has been well studied, and a number of papers have been published on it recently [2], [3], [4], [5].

2 Preliminaries

In this section we present a theoretical formulation of the problem. For video v_i and slot j we define the bit rate r_{ij} to be the maximum number of bits that can
be transmitted to \( v_i \) in \( j \). The decision version of the LMVT problem can be presented as follows.

| LMVT Problem |
|---------------|
| **INPUT:** \( n \) videos in the channel, \( B \) slots in the epoch, the bit rates \( r_{ij} \in \mathbb{Z}^+ \) for the \( n \) videos and \( B \) slots, a function \( \phi(b) \) for calculating the lead for \( b \) bits, and \( k \in \mathbb{Z}^+ \). |
| **QUESTION:** Is there an allocation of the \( B \) slots to the \( n \) videos so each video has a lead of at least \( k \)? |

In [1], the LMVT problem has been shown to be NP-Hard for a specific function \( \phi \) to calculate the lead based on the number of bits received. A natural greedy algorithm has been designed and has been shown to perform well in practice with experimental results. In this work, we show the problem to remain NP-Hard even for the case in which \( \phi \) is linear. Next we design an FPTAS for the problem.

3 LMVT problem remains NP-Hard even for a linear \( \phi \)

We assume a linear function \( \phi \) to calculate the lead based on the number of received \( b \). Let us consider a monotonic linear \( \phi \), such that \( \phi(b) = b \). Then the problem becomes of finding a slot allocation to maximize the minimum number of bits received by any video. We show the problem to remain NP-Hard even then.

We reduce the Partition Problem, a known NP-Hard problem to LMVT. The Partition Problem can be presented as:

| Partition Problem |
|-------------------|
| **INPUT:** A set \( S \subseteq \mathbb{Z}^+ \) with \( \sum_{x \in S} x = U \). |
| **QUESTION:** Can \( S \) be partitioned to \( S' \) and \( S \setminus S' \) such that \( \sum_{x \in S'} x = \sum_{x \in S \setminus S'} x = U/2 \)? |

For the reduction, consider any instance of the Partition Problem with \( S = \{x_1, x_2, \cdots, x_B \}, \sum_{x \in S} x = U, \) and \( |S| = B \). Now consider an instance of LMVT where we have 2 videos \( v_1 \) and \( v_2 \), \( B \) slots, and the bit rates \( r_{1j} = r_{2j} = x_j \in S \), for each slot \( j \). Note that \( |S| = B = \# \) of slots for the LMVT instance. Set \( k = U/2 \).

**Lemma 1.** The above instance of the Partition Problem has a solution iff the instance of LMVT has a solution.

**Proof.** We show that the instance of the Partition Problem has a solution iff we can find a slot allocation for the 2 videos of the LMVT instance, such that the number of bits received by each video is exactly \( U/2 \). Suppose the Partition Problem has a solution. That is we have \( S' \) and \( S \setminus S' \) such that \( \sum_{x \in S'} x = \sum_{x \in S \setminus S'} x = U/2 \). We find a slot allocation of the \( B \)
An FPTAS for LMVT Problem

slots which allocates slot \( i \) to video \( v_1 \) if \( x_i \in S' \). Else \( i \) is allocated to \( v_2 \). We note that all the \( B \) slots get allocated this way, since \( |S| = B \). Now since \( \sum_{x \in S'} x = \sum_{x \in S \setminus S'} x = U/2 \), it is easy to see that the number of bits received by \( v_1 \) and \( v_2 \) is exactly equal to \( U/2 \).

For the other way, suppose we have a slot allocation such that number of bits received \( v_1 \) and \( v_2 = U/2 \). We partition \( S \) in the following way:

1. If slot \( i \) is allocated to \( v_1 \), then \( x_i \in S' \).
2. Else \( x_i \in S \setminus S' \).

Since \( \sum r_{v_1} = \sum r_{v_2} = U/2 \), we have \( \sum_{x \in S'} x = \sum_{x \in S \setminus S'} x = U/2 \), and hence a solution to the instance of \textsc{Partition Problem}. \( \square \)

**Corollary 1.** \([1]\) The LMVT Problem is easy for a constant bit rate for all the \( n \) videos and \( B \) slots.

**Proof.** The \textsc{Partition Problem} has been reduced to LMVT. It is easy to see that an instance of \textsc{Partition Problem} where all the integers in \( S \) are constant (equal) is easy to solve. Analogously, the instance of LMVT with constant (equal) bit rates is also easy to solve. \( \square \)

4 An FPTAS for LMVT

In \([1]\), an exact dynamic programming algorithm has been designed for LMVT. The runtime of the exact algorithm is pseudo-polynomial in terms of the inputs. Here we describe the exact algorithm and then discretize the algorithm to design an FPTAS.

4.1 The Exact Dynamic Programming Algorithm

Define \( b_{i \text{max}} \) to be the maximum number of bits that video \( v_i \). In other words, \( b_{i \text{max}} = \sum_{j=1}^{B} r_{ij} \), is the number of bits \( v_i \) would receive, if all the \( B \) slots are allocated to it. Given \( m \) slots \( n \) videos, a \( Tx \) (transmission) vector is an \( n \)-tuple \( < b_1, \ldots, b_n > \) which tells us whether a slot allocation is possible such that video \( v_i \) receives at least \( b_i \) bits in the allocation. The length of the \( Tx \) vector is the number of videos \( n \). We define the predicate \( F(m, T) \) for \( m \) slots and \( Tx \) vector \( T \). \( F(m, T) \) is true iff an allocation is possible to achieve \( Tx \). For two \( Tx \) vectors \( T_1 \) and \( T_2 \), we define \( T_1 \preceq T_2 \) iff \( T_1[i] \leq T_2[i] \), \( \forall i \). It is easy to see, if \( F(m, T_2) \), then \( F(m, T_1) \). In the dynamic programming, we generate \( \prod_{i=1}^{n} (b_{i \text{max}} + 1) \) possible \( Tx \) vectors starting from \( < 0, \ldots, 0 > \) till \( < b_{1 \text{max}}, \ldots, b_{n \text{max}} > \). For each video \( v_i \), we have the values taken from the set \( \{0, 1, \ldots, b_{i \text{max}} - 1, b_{i \text{max}}\} \). We maintain an \( \prod_{i=1}^{n} (b_{i \text{max}} + 1) \) by \( n \) matrix of the vectors during the execution of the dynamic
programming algorithm. Also, we have a truth value vector of length \( \prod_{i=1}^{n} (b_i^{\text{max}} + 1) \). Each cell in the true value vector corresponds to the value of \( F(m, T) \) for \( m \) slots, and \( Tx \) vector \( T \). We initialize the truth value of \( <0, \cdots, 0> \) to true and the rest to false. This signifies that we can always achieve vector \( <0, \cdots, 0> \), even without any slot allocation. We then start from \( m = 1 \) to the total number of slots \( B \), and evaluate the truth values. The truth values \( (F(B, T)) \) at the end tell us whether that vector \( T \) was achievable by a slot allocation with the \( B \) slots. We then choose the vector with the maximum minimum \( b_i \) value as our solution, and have the corresponding slot allocation as the optimal answer. The way to evaluate \( F(m, T) \) is as follows:

1. If \( F(m - 1, T) \), then \( F(m, T) \).
2. Else let \( W_i \) be the vector where all the positions of \( W_i \) except \( W_i[i] \) is equal to \( T \). \( W_i[i] = \max(0, T[i] - r_{im}) \). For \( i = 1 \) to \( n \), if \( F(m - 1, W_i) \), then \( F(m, T) \).

We present the whole algorithm in Algorithm [1]
An FPTAS for LMVT Problem

**Input:** $n$, the number of videos, $B$, the number of slots, $r_{ij}$, the rate of video $v_i$ for slot $j$

**Output:** An allocation of the $B$ slots over $n$ videos where the minimum number of bits received by a video is maximized

Generate the $\prod_{i=1}^{n} (b_{i}^{\text{max}} + 1)$ vectors where $b_{i}^{\text{max}} = \sum_{j=1}^{B} r_{ij}$;

Construct the $\prod_{i=1}^{n} (b_{i}^{\text{max}} + 1)$ by $n$ matrix of the vectors;

Have the truth value vector of length $\prod_{i=1}^{n} (b_{i}^{\text{max}} + 1)$;

Initialize the truth value of $<0, \cdots, 0>$ to true and the rest to false;

for $m = 1$ to $B$ do

   foreach $Tx$ vector $T$ do

      if $F(m-1, T)$ then

         Set $F(m, T)$ to true;

      end

      else if then

         for $i = 1$ to $n$ do

            if $F(m-1, W_i)$ then

               Set $F(m, T)$ to true;

            end

         end

      end

end

Return the $Tx$ vector $T$ with the maximum $\min(b_i)$ value and with $F(B, T)$ true;

**Algorithm 1:** The exact dynamic programming algorithm

Algorithm 1 has a runtime of $O(Bn \prod_{i=1}^{n} (b_{i}^{\text{max}} + 1))$. Since $\prod_{i=1}^{n} (b_{i}^{\text{max}} + 1)$ can be exponentially large, Algorithm 1 has an exponential runtime.

4.2 The FPTAS

In the FPTAS, instead of considering all the values in the set $\{0, 1, \cdots, b_{i}^{\text{max}} - 1, b_{i}^{\text{max}}\}$ for each video $v_i$, we discretize the set to the powers of $1 + \varepsilon$, where $\varepsilon > 0$. We define the function $\psi(i) = \lceil(1 + \varepsilon)^{\log_{1+\varepsilon}(i)}\rceil$. Now we have the set for each video $v_i$, as $\{0, \psi(1), \cdots, \psi(b_{i}^{\text{max}} - 1), \psi(b_{i}^{\text{max}})\}$. Clearly, we have at most $\log_{1+\varepsilon}(b_{i}^{\text{max}} + 1)$ values in the set. Hence we would have only $\prod_{i=1}^{n} \log_{1+\varepsilon}(b_{i}^{\text{max}} + 1)$ $Tx$ vectors to evaluate truth value for, in the FPTAS. In the evaluation of $F(m, T)$, instead of considering $W_i$ as in Algorithm 1, we consider $W'_i$, where $W'_i$
is the vector where all the positions of \( W'_i \) except \( W'_i[i] \) is equal to \( T. \ W'_i[i] = max(0, \psi(T[i] - r_{im})) \). We present the FPTAS in Algorithm 2.

### Algorithm 2: The FPTAS for LMVT Problem

Algorithm 2 considers only \( n \log_{1+\varepsilon}(b_{i_{\text{max}}} + 1) \) to evaluate out of the \( n \log_{1+\varepsilon}(b_{i_{\text{max}}} + 1) \) vectors evaluated by Algorithm 1.

**Lemma 2.** The error generated by the rounding of \( W_i \) in Algorithm 1 to \( W'_i \) in Algorithm 2 is at most \( \frac{1}{1+\gamma} \) where \( \gamma > 0 \).

**Proof.** Suppose we have \( < b_1, \cdots, b_n > < c_1, \cdots, c_n > \) from the \( F(m, T) \) evaluation step of Algorithm 1. In other words \( F(m, T_1) \) for \( T_1 = < c_1, \cdots, c_n > \) has been evaluated to true because \( F(m-1, T_2) \) had been evaluated to be true for \( T_2 = < b_1, \cdots, b_n > \). Hence, \( \exists i \in [n] \), such that, \( T_2[i] = b_i = max(0, c_i - r_{im}) \), and for all other positions, \( j \) we have \( T_2[j] = T_1[j] \).
Now suppose we have $T'_{i} = \langle b'_1, \cdots, b'_n \rangle$ and $T''_i = \langle c'_1, \cdots, c'_n \rangle$ in the table for Algorithm 2 where $b'_i = \psi(b_i)$, and $c'_i = \psi(c_i), \forall i \in [n]$. We want to show that if Algorithm 2 evaluates $F(m, T'_{i})$ to true if $F(m-1, T''_i)$ was evaluated to true at an earlier step, with an error of at most $\frac{1}{1+\gamma}$. In other words, $\langle b'_1, \cdots, b'_n \rangle \iff \langle c'_1, \cdots, c'_n \rangle$ in Algorithm 2 with an error of at most $\frac{1}{1+\gamma}$.

We observe that $T'_{2}[j] = T'_{1}[j]$ for all $j \in [n] \setminus \{i\}$. For $j = i$ we have $T'_{2}[i] = b'_i = \psi(b_i) = \max(0, \psi(c_i - r_{im})) \geq \psi(c_i - r_{im})$. Algorithm 2 would calculate the value of position $i$ for vector $W''_{i}$ as $W''_{i}[i] = \max(0, \psi(c'_i - r_{im})) \geq \psi(c'_i - r_{im})$. We have $\psi(c'_i - r_{im}) = \psi(\psi(c_i - r_{im})) \geq \psi(\frac{c'_i - r_{im}}{1+\gamma}) \geq \frac{1}{1+\gamma} \psi(c_i - r_{im})$, where $\gamma = 2\varepsilon$.

**Lemma 3.** Algorithm 2 has a runtime of $O(Bn \prod_{i=1}^{n} \log_{1+\varepsilon}(b_{i}^{\max} + 1))$.

**Lemma 4.** The value of the solution $(S_{fptas})$ returned by Algorithm 2 differs from that $(S_{opt})$ returned by Algorithm 1 at most by a factor of $\frac{1}{1+\varepsilon}B$.

**Proof.** At any step $i$, the value of any position of any vector of Algorithm 2 differs from the corresponding position of the corresponding vector for Algorithm 1 by a factor of $\frac{1}{1+\varepsilon}$ due to the rounding. We perform this rounding $B$ times. Hence the values of the vectors after the full execution would differ by a factor of at most $\frac{1}{1+\varepsilon}B$.

**Lemma 3 and 4 lead to the following theorem.**

**Theorem 1.** Algorithm 2 is an FPTAS for LMVT Problem.

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