Mathematical methods for optimal synthesis of physical and mechanical structure of composites with required set of properties at extreme conditions

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Abstract. Variational statements of composite structure with the required set of properties synthesis problems are investigated. The study of the limiting capabilities of composites to achieve required set of properties is necessarily associated with development of effective methods for finding global extremum of corresponding functionals. The article deals with problems of numerical analysis and optimal design of composite structure with required set of properties. Mathematical models describing the propagation of wave processes of various physical nature in layered-inhomogeneous structure of composite are investigated. Problems of optimal design for composite structure with required set of properties are formulated in variational formulation. Based on the theory of multi-valued mappings and methods of continuation by parameter, method for studying limiting capabilities of composites to achieve required set of properties is developed. Developed technique allows effective construction of layered composites with complex characteristics. The application of developed method of optimal synthesis for the study of limiting possibilities in reflection of energy of wave action from layered-inhomogeneous coating in given range of spectrum is considered.

1. Introduction

In recent decades, problems of synthesis of physical and geometric structure of composites with required set of properties have become increasingly important. The problem of studying limiting possibilities is one of central problems in design of layered-inhomogeneous compositions with required set of properties [1-6]. Tasks of studying limit possibilities are related to the development of effective methods for synthesis of layered-inhomogeneous structure of composites with characteristics closest to required ones [7-9]. Known approaches to solving this problem are associated with directional variation of parameters of layered-inhomogeneous composites. However, in variational formulation, wave problems of composite structure synthesis are essentially multiextremal. Due to extremely large number of acceptable variants of inhomogeneous structures analyzed for optimality, it is not possible to choose
structure parameters of composites with characteristics closest to required ones, even with the use of high-performance computers. For these reasons, it is not possible to guarantee that the layered-inhomogeneous composition constructed by one of known methods has characteristics closest to required ones.

In variational formulation, problem of studying limit possibilities is directly related to development of effective methods for finding global extremum of corresponding functionals that characterize degree of proximity of characteristics of designed composite to required ones. The most universal is variational approach, which reduces the problem of synthesizing composite structure with required set of properties to the problem of optimal control for system of differential equations describing the propagation of the wave process in inhomogeneous composite structure.

2. Methods for studying limiting capabilities of layered composites under wave effects, based on property of internal symmetry in structure of composites

The problem of studying limiting capabilities of layered-inhomogeneous composites with required set of properties is related to the development of effective methods for non-local optimal synthesis. Non-local optimization methods in variational formulations of problems of optimal design of composite structure with required set of properties should allow construction of globally optimal solutions or to certain extent close to them. In optimization problems, in which solutions are determined by means of systems of differential equations, one of the most effective methods is method based on the maximum principle of L S Pontryagin. However, nevertheless, methods based on necessary optimality conditions, such as the maximum principle of L S Pontryagin, allow us to construct only locally optimal solutions.

In accordance with this, application of methodology based on maximum principle of L S Pontryagin still does not allow us to effectively investigate limiting capabilities of layered-inhomogeneous composites to achieve required set of properties. It is necessary to develop methods for optimal synthesis on qualitatively different basis to study limiting capabilities of layered-inhomogeneous composite materials.

The lack of effective methods for studying limiting capabilities leads to the fact that, on the one hand, it is impossible to build layered-inhomogeneous composites with characteristics that are extremely close to required ones. On the other hand, there is no way to assess how significantly layered-inhomogeneous composite materials that function in various fields of physics and engineering differ in their characteristics from the maximum achievable ones.

It was found in [7-9] that relationship between parameters in the optimal structure of layered-inhomogeneous composites, which realize the maximum possibilities for achieving required set of properties, may have internal symmetry. Based on established property of internal symmetry in relationship of parameters in optimal structure of layered-inhomogeneous composites and theory of multi-valued maps [10-16], an effective approach to solving the problem of optimal synthesis of layered-inhomogeneous composites with required set of properties is developed. It is shown that this approach allows us to study limiting possibilities of layered-inhomogeneous compositions to achieve given set of properties [17-19].

3. Existence of internal symmetry in optimal layered-inhomogeneous composite structures

The application of mathematical apparatus of optimal control theory, based on maximum principle of L S Pontryagin, allowed us to establish that there is an internal symmetry in relationship of parameters describing the structure of optimal layered-inhomogeneous composites under wave influences of various physical properties (electromagnetic, acoustic, temperature, elastic). The existence of such internal symmetry in optimal synthesis problems may indicate that structures that implement limiting possibilities will be grouped only within compact set Q of relatively small dimensions compared to the size of original set of acceptable options. Internal order or internal symmetry in relationship of parameters describing the structure can lead to the fact that parameters of optimal structures will satisfy additional connections.
In accordance with this, identification of such connections can significantly reduce dimension of the problem, i.e. it may turn out that parameters of optimal structures that realize maximum possibilities for achieving required set of properties additionally satisfy certain system of \( m \) equations

\[
M_j(u^*) = 0, \quad j = 1, 2, \ldots, m.
\]

The set of solutions to this system is the set we are looking for:

\[
Q = \{ u : M_j(u^*) = 0, \quad j = 1, 2, \ldots, m \}.
\]

The selection of such system of equations connecting parameters of optimal constructions can allow in some cases to completely solve the problem of synthesis. The main problem here is to develop methodology for analytical description of boundaries of parameter change that describe structure of optimal structures, the area of change of which determines the set \( Q \).

Qualitatively new way of compressing set of acceptable variants of structures and developing effective methods of optimal synthesis is associated with study of possibility of isolating compact set \( Q \) of small dimensions containing entire set of optimal parameters describing structure of layered-inhomogeneous composites that realize maximum possibilities for achieving required set of properties.

4. Methods for studying the limiting capabilities of layered inhomogeneous structures under wave effects, based on the theory of multi-valued maps and methods of parameter continuation

We immerse the initial problem of optimal synthesis of a structure of an inhomogeneous structure with the required set of properties in a parametric family of optimal synthesis problems that depends on a real parameter \( \delta \) (\( \delta_0 \leq \delta \leq \delta_0 \)). In this case, the parameter value \( \delta = \delta_0 \) corresponds to the reference synthesis problem, for which the entire set of optimal solutions that implement the maximum possibilities for achieving a given set of properties can be effectively selected \( U^*(\delta_0) \). The value of the parameter \( \delta = \delta_1 \) corresponds to the original optimal synthesis problem. The set of globally optimal solutions in the original synthesis problem is denoted by \( U^* \). At the same time \( U^* = U^*(\delta_1) \).

The problem is to develop an effective method of optimal synthesis based on the continuation of the solution by parameter, which allows us to build a set of globally optimal solutions to the original synthesis problem \( U^* = U^*(\delta_1) \) based on knowledge of the set of globally optimal solutions to the reference synthesis problem \( U^*(\delta_0) \).

A method for studying the limit possibilities is developed, based on the theory of multi-valued maps and methods of continuation by parameter, and allows for the effective construction of the structure of layered-inhomogeneous composites with complex characteristics under wave influences of various physical nature (electromagnetic, acoustic, temperature, elastic).

5. Comparative computational experiments to study limiting capabilities of layered inhomogeneous interference coatings for extreme reflection of wave energy in given frequency range

We consider the application of developed method of optimal synthesis, based on the theory of multi-valued maps and continuation of solution for parameter to study limiting possibilities in reflection of energy of wave action from layered-inhomogeneous coating in given range of spectrum. Problem of optimal synthesis of the structure of layered-inhomogeneous coatings with required characteristics will be considered in relation to the case of propagation of electromagnetic waves in layered-inhomogeneous media. Let us choose the system of Maxwell's differential equations as mathematical model describing the propagation of electromagnetic waves in system of layers of layered-inhomogeneous composite. As a result, the propagation of electromagnetic waves inside the layers of layered-inhomogeneous composite will be described by system of Maxwell's equations:
\[ \text{rot} \vec{H}_s - \frac{\varepsilon_s}{c} \frac{\partial \vec{E}_s}{\partial t} = 0, \]
\[ \text{rot} \vec{E}_s + \frac{\mu_s}{c} \frac{\partial \vec{H}_s}{\partial t} = 0, \quad b_{s-1} \leq z \leq b_s, \quad s = 1, \ldots, N. \]

In these notations \( \vec{H}_s, \vec{E}_s \) - vectors of electric and magnetic intensity of the electromagnetic field in the \( s \)-th layer, \( \varepsilon_s, \mu_s \) – dielectric and magnetic permittivity of the \( s \)-th layer; \( b_s \)-coordinates of the interface of layers with different physical properties, \( N \) – the number of layers.

At boundaries of layers, conditions for conjugation of solutions will be met, which consist in continuity of normal and tangential components of electric and magnetic intensity vectors:

\[ E_{s-1,1} |_{z=b_s} = E_{s,1} |_{z=b_s}, \quad H_{s-1,1} |_{z=b_s} = H_{s,1} |_{z=b_s}, \]
\[ E_{s,n} |_{z=b_s} = E_{s-1,n} |_{z=b_s}, \quad H_{s,n} |_{z=b_s} = H_{s-1,n} |_{z=b_s}. \]

The boundary conditions determine relationship between incident and reflected waves on outer and inner surfaces of layered-inhomogeneous coating. As an indicator of efficiency of solutions in variational formulation under study, root-mean-square measure of proximity of dependence of energy transmission coefficient \( T(\omega) \) to required dependence \( \tilde{T}(\omega) \) in given frequency range is chosen \([\omega_{\text{min}}, \omega_{\text{max}}]\):

\[ J = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} \left[ T(\omega) - \tilde{T}(\omega) \right]^2 d\omega \rightarrow \min. \]

In these notations: \( T(\omega) = \frac{\Pi_{z}^{\text{out}}}{\Pi_{z}^{\text{in}}} \),

where \( \Pi_{z}^{\text{out}}, \Pi_{z}^{\text{in}} \) – projections of Poynting vector on \( z \) axis in transmitted and incident electromagnetic waves, respectively. Efficiency of electromagnetic wave reflection from layered inhomogeneous coating is characterized by integral criterion:

\[ J = \int_{\lambda_{\text{min}}}^{\lambda_{\text{max}}} T(\lambda) d\lambda \rightarrow \min \]

Here: \( \lambda \) is the wavelength, \( \lambda_{\text{min}}, \lambda_{\text{max}} \) – lower and upper limits of studied wavelength range. Valid set consists of two materials with refractive indices: \( n_{\text{min}}=1.46; n_{\text{max}}=2.3 \). Refractive index of the first semi-infinite medium from which electromagnetic wave falls \( n_0=1.46 \); refractive index of the second semi-infinite medium into which electromagnetic wave passes when leaving the structure \( n_f = 1.33 \). The ratio of upper limit of studied wavelength range \( \lambda_{\text{max}} \) to lower limit \( \lambda_{\text{min}} : \lambda_{\text{max}} / \lambda_{\text{min}} = 1.2 \). The total thickness of layered-inhomogeneous composition is given and is equal to: \( l=4.8 \lambda_{\text{min}} \). The upper limit on number of coating layers \( N_{\text{max}} \) was assumed to be equal to: \( N_{\text{max}} = 35 \).

We will conduct constructive analysis of application of developed optimal synthesis technique to study maximum possibilities for achieving required set of properties for considered layered-inhomogeneous coating. The application of developed method of optimal synthesis, based on the theory of multi-valued maps and continuation of solution by parameter, made it possible to synthesize following layered-inhomogeneous resulting coating, which implements maximum possibilities in reflection of wave action energy (table 1).
Table 1. Geometric and physical characteristics of the resulting coating.

| Number of layer | Refractive index | Layer thickness $\Delta_s / \lambda_{\min}$ | Number of layer | Refractive index | Layer thickness $\Delta_s / \lambda_{\min}$ |
|-----------------|------------------|--------------------------------------------|-----------------|------------------|--------------------------------------------|
| 1               | 1.46             | 0.0641                                     | 19              | 1.46             | 0.1882                                     |
| 2               | 2.30             | 0.1185                                     | 20              | 2.30             | 0.1178                                     |
| 3               | 1.46             | 0.1873                                     | 21              | 1.46             | 0.1883                                     |
| 4               | 2.30             | 0.0108                                     | 22              | 2.30             | 0.1177                                     |
| 5               | 1.46             | 0.0314                                     | 23              | 1.46             | 0.1885                                     |
| 6               | 2.30             | 0.0759                                     | 24              | 2.30             | 0.1177                                     |
| 7               | 1.46             | 0.1873                                     | 25              | 1.46             | 0.1886                                     |
| 8               | 2.30             | 0.1183                                     | 26              | 2.30             | 0.1176                                     |
| 9               | 1.46             | 0.1874                                     | 27              | 1.46             | 0.1888                                     |
| 10              | 2.30             | 0.1182                                     | 28              | 2.30             | 0.1175                                     |
| 11              | 1.46             | 0.1871                                     | 29              | 1.46             | 0.1890                                     |
| 12              | 2.30             | 0.1181                                     | 30              | 2.30             | 0.1174                                     |
| 13              | 1.46             | 0.1877                                     | 31              | 1.46             | 0.1883                                     |
| 14              | 2.30             | 0.1180                                     | 32              | 2.30             | 0.1172                                     |
| 15              | 1.46             | 0.1879                                     | 33              | 1.46             | 0.1897                                     |
| 16              | 2.30             | 0.1180                                     | 34              | 2.30             | 0.1168                                     |
| 17              | 1.46             | 0.1880                                     | 35              | 1.46             | 0.0278                                     |
| 18              | 2.30             | 0.1179                                     |                 |                 |                                            |

The block diagram of resulting optimal coverage is shown in figure 1. Optimal number of layers of optimal coating $N^* = 35$. The first and last layers of resulting optimal design consist of material with minimum refractive index $n_{\min} = 1.46$. The graph of dependence of energy transmission coefficient on electromagnetic wavelength of optimal layered-inhomogeneous coating is shown in figure 2. In the entire specified wavelength range from $\lambda_{\min}$ to $\lambda_{\max}$, the energy transmittance $T(\lambda)$ of synthesized optimal layered coating does not exceed 0.003%.

Thus, optimal layered-inhomogeneous coating, constructed on the basis of developed optimal synthesis technique, based on the theory of multi-valued maps and continuation of solution by parameter, provides highly efficient reflection of electromagnetic wave in entire specified wavelength range (energy transmittance $T \leq 0.003\%$), figure 2.

Thus, application of developed method of optimal synthesis, based on the theory of multi-valued mappings and continuation of solution by parameter, allows us to synthesize effective composite coatings.

Figure 1. Design diagram of the resulting structure.
6. Summary

Study of mathematical models describing the propagation of wave processes of various physical nature in layered-inhomogeneous structure of composites is carried out. Problems of optimal synthesis of composite structure with required set of properties formulated in variational formulation.

Within the framework of variational formulation, based on the theory of multi-valued maps and methods of continuation by parameter, method for studying limiting capabilities of composites to achieve required set of properties is developed. The developed technique allows effective construction of layered composites with complex characteristics.

The application of developed method of optimal synthesis for study of limiting possibilities in reflection of wave action energy from layered-inhomogeneous coating in given range of spectrum is considered. It is shown that application of developed method of optimal synthesis, based on theory of multi-valued maps and continuation of solution by parameter, makes it possible to synthesize effective composite coatings with required set of properties.

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Figure 2. Graph of the dependence of the energy transmission coefficient on the wavelength of the optimal composite coating.
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