STUDIES OF QUANTUM SPIN LADDERS
AT $T = 0$ AND AT HIGH TEMPERATURES
BY SERIES EXPANSIONS

J. Oitmaa\textsuperscript{1}, Rajiv R.P Singh\textsuperscript{2}, and Zheng Weihong\textsuperscript{1}
\textsuperscript{1}School of Physics, The University of New South Wales, Sydney, NSW 2052, Australia.
\textsuperscript{2}Department of Physics, University of California, Davis, CA 95616, USA.
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Abstract

We have carried out extensive series studies, at $T = 0$ and at high temperatures, of 2-chain and 3-chain spin-half ladder systems with antiferromagnetic intrachain and both antiferromagnetic and ferromagnetic interchain couplings. Our results confirm the existence of a gap in the 2-chain Heisenberg ladders for all non-zero values of the interchain couplings. Complete dispersion relations for the spin-wave excitations are computed. For 3-chain systems, our results are consistent with a gapless spectrum. We also calculate the uniform magnetic susceptibility and specific heat as a function of temperature. We find that as $T \to 0$, for the 2-chain system the uniform susceptibility goes rapidly to zero, whereas for the 3-chain system it approaches a finite value. These results are compared in detail with previous studies of finite systems.

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I. INTRODUCTION

The magnetic properties of low dimensional systems have been the subject of intense theoretical and experimental research in recent years. It is by now well established that one-dimensional Heisenberg antiferromagnets with integer spin have a gap in the excitation spectrum, whereas those with half-integer spin have gapless excitations. The former have a finite correlation length, while for the latter it is infinite with the spin-spin correlation function decaying to zero as a power law. In 2-dimensions, the unfrustrated square-lattice Heisenberg model has long range Néel order in the ground state. It has gapless Goldstone modes as expected. In recent years much interest has focussed on systems with intermediate dimensionality and on questions of crossovers between $d = 1$ and $d = 2$. One approach to this problem has been to study a two-dimensional system where the coupling for the spins separated along the $x$-axis is different from those separated along the $y$-axis. It has been suggested that an alternative way to explore this issue is through the Heisenberg spin ladders consisting of a finite number of chains coupled together, with a coupling $J_\perp$ along the chains and $J_\parallel$ between them. These systems have been the subject of considerable recent theoretical and experimental interest.

Experimentally, 2-chain $S = \frac{1}{2}$ ladders are realized in vanadyl pyrophosphate $(VO)_2P_2O_7$ and in the strontium cuprate $SrCu_2O_3$, whereas 3-chain $S = \frac{1}{2}$ ladders are realized in the strontium cuprate $SrCu_3O_5$.

Theoretically, a number of striking predictions have been made for such systems. These have been recently reviewed by Dagotto and Rice. Barnes et al. were the first to carry out extensive Monte Carlo studies of the excitation spectrum and the magnetic susceptibility for 2-chain ladders with antiferromagnetic interchain coupling. White et al. and Hida have used the density matrix renormalization method to study the spin gap. Watanabe has applied the numerical diagonalization method to finite systems of 2-chain ladders with ferromagnetic interchain coupling. Azzouz et al. developed a mean-field theory and used the density-matrix renormalization group method to study 2-chain ladders. Gopalan, Rice and Sirgrist presented a variational wavefunction for the ground state of the 2-leg ladder. Rojo has shown that spin ladders with odd number of legs have gapless excitations. Troyer et al. have used improved versions of the quantum transfer-matrix algorithm to study the temperature dependence of the susceptibility, specific heat, correlation length etc of 2-chain ladders. Finite-size scaling was used by Hatano for multi-leg ladders, and recently Frischmuth et al. and Sandvik et al. have applied Quantum Monte Carlo simulation to compute the temperature dependence of uniform susceptibility and internal energy for spin ladders with up to 6 legs. One clear result from all these studies is that ladders with an even number of legs have an energy gap, short range correlations and a “spin liquid” ground state. On the other hand, ladders with odd number of legs have gapless excitations, quasi long range order, and a power-law falloff of spin-spin correlations, similar to single chains. Experiments also confirm these features.

We have carried out extensive series studies of 2-chain and 3-chain ladder systems with both antiferromagnetic and ferromagnetic interchain coupling $J_\perp$, via Ising expansions and dimer expansions at $T = 0$, and also by high temperature series expansions. Our results confirm the existence of a gap in the 2-chain system and delineate the phase diagram in the parameter space of Ising anisotropy and the parameter ratio $J_\perp/J_\parallel$. The complete spin-
wave excitation spectra are computed. For the 3-chain system we are unable to exclude the possibility of a small gap, although our results are consistent with a gapless spectrum. In addition, we develop a high-temperature series expansion for the uniform magnetic susceptibility and the specific heat for 2-chain and 3-chain systems with $J_{//} = J_{\perp}$; the susceptibility of 2-chain ladders is as expected for a system with a spin-gap while that of 3-chain ladders appear to remain finite in the zero-temperature limit suggesting an absence of a spin-gap. We compare our results in detail with previous calculations.

II. SERIES EXPANSIONS

The Hamiltonian of a spin ladder with $n_l$ legs is given by,

$$H = J_{//} \sum_{i,l=1}^{l=n_l} S_{l,i} \cdot S_{l,i+1} + J_{\perp} \sum_{i,l=1}^{l=n_l-1} S_{l,i} \cdot S_{l+1,i}$$

where $S_{l,i}$ denotes the S=1/2 spin at the $i$th site of the $l$th chain. $J_{//}$ is the interaction between nearest neighbor spins along the chain and $J_{\perp}$ is the interactions between nearest neighbor spins along the rungs. We denote the ratio of couplings as $y$, that is, $y \equiv J_{\perp}/J_{//}$. In the present paper the intrachain coupling is taken to be antiferromagnetic (that is, $J_{//} > 0$) whereas the interchain coupling $J_{\perp}$ can be either antiferromagnetic or ferromagnetic. This includes the values of interest in the experimental systems discussed earlier where $J_{\perp} \sim J_{//}$.

Without loss of generality, we can set $J_{//} = 1$ hereafter.

We have carried out three different expansions for the system, The first is the expansion about the Ising limit at zero temperature for both two and three-chain ladders. We have computed the ground state properties as well as the spin-wave excitation spectra by this expansion. The second is the dimer expansion, again at $T = 0$. This expansion can be done for the 2-chain system with antiferromagnetic interchain coupling only. The third is the high-temperature series expansion for the uniform susceptibility of the 2-chain and 3-chain ladders with $y = 1$.

A. Ising expansions

To perform an expansion about the Ising limit for this system, we introduce an anisotropy parameter $x$, and write the Hamiltonian in Eq.(1) as:

$$H = H_0 + xV$$

where

$$H_0 = \sum_{i,l=1}^{l=n_l} S^z_{l,i} S^z_{l,i+1} + y \sum_{i,l=1}^{l=n_l-1} S^z_{l,i} S^z_{l+1,i}$$

$$V = \sum_{i,l=1}^{l=n_l} (S^x_{l,i} S^x_{l,i+1} + S^y_{l,i} S^y_{l,i+1}) + y \sum_{i,l=1}^{l=n_l-1} (S^x_{l,i} S^x_{l+1,i} + S^y_{l,i} S^y_{l+1,i})$$

(3)
The limits $x = 0$ and $x = 1$ correspond to the Ising model and the Heisenberg model respectively. The operator $H_0$ is taken as the unperturbed Hamiltonian, with the unperturbed ground state being the usual Néel state for antiferromagnetic interchain coupling, and a fully ordered collinear state for ferromagnetic interchain coupling. The operator $V$ is treated as a perturbation. It flips a pair of spins on neighbouring sites. The Ising expansion method has been previously reviewed in several articles \[17,18\], and will not be repeated here. The calculations involved a list of 9184 linked clusters of up to 16 sites for the 2-chain ladder, and 14082 linked clusters of up to 12 sites for the 3-chain ladder, together with their lattice constants and embedding constants.

Series have been calculated for the ground state energy per site $E_0/N$, the staggered magnetization $M$ for $y > 0$ (or collinear magnetization $M$ for $y < 0$), the parallel staggered/collinear susceptibility $\chi_{//}$, and the uniform perpendicular susceptibility $\chi$ per site for several ratio of couplings $y = \pm 0.1, \pm 0.25, \pm 0.5, \pm 0.75, \pm 1, \pm 1.5, \pm 2, \pm 4, \pm 8$ up to order $x^{16}$ for 2-chain ladders, and $x^{12}$ for 3-chain ladders (the series for uniform perpendicular susceptibility $\chi$ is one order less in each case). The resulting series for $y = \pm 1$ for the 2-chain and 3-chain systems are listed in Tables I and II, the series for other value of $y$ are available on request.

To analyze these series, we first performed a standard Dlog Padé analysis of the magnetization $M$ and parallel susceptibility $\chi_{//}$ series. We found that for 2-chain ladders, the series lead to a simple power-law singularity at $x < 1$:

$$M \sim (1 - x/x_c)^\beta \quad \chi_{//} \sim (1 - x/x_c)^{-\gamma}$$

with the indices $\beta$ and $\gamma$ close to $1/8$ and $7/4$ respectively. This transition at $x < 1$, with criticality in the universality class of the 2D-Ising model, is strong evidence that in the Heisenberg limit the system has a disordered ground state with a spin-gap, as is the case for the spin-one chain \[19\]. In contrast for the 3-chain ladders, the series analysis showed poor convergence and suggested a singularity at $x \geq 1$. This implies that for the 3-chain ladders, the system is analogous to the spin-half chain, with gapless spectra and power-law correlations in the Heisenberg limit.

Fig. 1 shows the phase boundary for 2-chain ladders as function of $y$. It is interesting to note the different behaviour for $y > 0$ and for $y < 0$: for antiferromagnetically coupled 2-chain ladders ($y > 0$), $x_c$ decreases as $y$ increases, and in the limit of $y \to \infty$, $x_c$ will approach 0. But for ferromagnetically coupled 2-chain ladders ($y < 0$), $x_c$ first decreases as the absolute value of $y$ increase from zero, but then the trend reverses and it, once again, approaches 1 as $y \to -\infty$. To understand this behavior, we can map the system for large negative $y$ to a spin-one chain with on-site (single-ion) anisotropy:

$$H = \frac{1}{2} \sum_i [S_{i,tot} \cdot S_{i+1,tot} + y(1 - x)(S_{i,tot}^z)^2]$$

where $S_{i,tot}$ denotes the S=1 spin at the $i$th site of the chain. We can get the asymptotic behaviour of the phase boundary in the limit of $y \to -\infty$ by studying the following spin-one chain with on-site anisotropy:

$$H = \frac{1}{2} \sum_i [S_{i,tot}^x S_{i+1,tot}^x - c(S_{i,tot}^z)^2 + x'(S_{i,tot}^x S_{i+1,tot}^x + S_{i,tot}^y S_{i+1,tot}^y)]$$

4
For this model, we have carried out series expansions in $x'$ to order $x'^{14}$ (i.e. 14 sites) for staggered magnetization $M$ for several different values of $c$: $c = 0.275, 0.29, 0.3, 0.325$, and performed a standard Dlog Padé analysis to find the critical value $c'$ which gives the singularity of $M$ at $x' = 1$. We get $c' = 0.29(1)$. Hence, the asymptotic behaviour of the phase boundary in the limit $y \to -\infty$ is given by

$$x_c = 1 + 0.29/y$$

which is also shown in Fig. 1 as a bold line near $y/(1 + |y|) = -1$.

Fig. 2 gives the results of the ground-state energy per site $E_0/N$ versus $y$ for both 2-chain and 3-chain ladders at the Heisenberg point $x = 1$. Our results for the ground state energy agree extremely well with the recent quantum Monte Carlo simulation [13]. In Fig. 3, we present the uniform perpendicular susceptibility at $T = 0$.

We also performed the Ising expansion for the triplet spin-wave excitation spectrum of 2-chain and 3-chain ladders using Gelfand’s method [20]. To overcome a possible singularity at $x < 1$ in the 2-chain ladders, and to get a better convergent series in the Heisenberg limit, we add the following staggered field term to the Hamiltonian in Eq. (2):

$$\Delta H = t(1-x) \sum_i (-1)^i S_i^z$$

$\Delta H$ vanishes at $x = 1$. We adjust the coefficient $t$ to get the most smooth terms in the series, with a typical value being $t = 2$. We computed the Ising expansion for the triplet spin-wave excitation spectrum $\epsilon(k)$ up to order $x^{15}$ for 2-chain ladders, and up to order $x^{11}$ for 3-chain ladders. These series are too long to be listed here, but are available on request.

These series have been analyzed by using integrated first-order inhomogeneous differential approximants [21]. For the 2-chain ladder, there are 2 bands of excitations, Fig. 4 shows the dispersion $\epsilon(k)$, with $k_y = \pi$, for antiferromagnetic interchain coupling. The other band with $k_y = 0$ is related to this by $\epsilon(k_x, 0) = \epsilon(\pi - k_x, \pi)$. This is simply due to the staggered field, which doubles the spectrum. As a comparison, the dispersion relation of a single chain (that is the case of $y = 0$) is also shown. It can be seen from the graph that in the limit $y \to 0$, the dispersion relation has a simple cosine function with a period of $2\pi$, and in the limit $y \to \infty$, the dispersion relation also has a simple cosine form with a period of $4\pi$, and a gap in the spectrum. Barnes and Riera [6] argued that the dispersion, for all $y$, can be fitted by combining these two functions into the following form:

$$\epsilon(k_x, \pi)^2 = \epsilon(0, \pi)^2 \cos^2(k_x/2) + \epsilon(\pi, \pi)^2 \sin^2(k_x/2) + c_0 \sin^2(k_x)$$

For $y = 1$, the Ising expansions give an energy gap of $\epsilon(\pi, \pi) = 0.44(7)$. A more precise estimate is obtained by the dimer expansions, which give $\epsilon(\pi, \pi) = 0.504(7)$. We will discuss the dimer expansions later.

For ferromagnetic interchain coupling, the two bands of spectra are independent, but each band is a simple cosine function with a gap at the minimum and symmetric about $k_x = \pi/2$, as shown in Figs. 5-6. As noted in Figs. 4, it is clear that the spin-gap decreases smoothly as $y$ is reduced, and vanishes at $y = 0$. These results agree well with previous calculations [6].

For the 3-chain system, there are three bands. In the Ising limit, 2 bands have initial excitations located in the side rows, and the third band has it in the middle row. Figs. 7-12
show the spectrum of the three bands for ferromagnetic and antiferromagnetic interchain couplings. From these graphs, we can see that all of the dispersion relations have a simple cosine function (except for middle row band with large $y$) with a minimum located at $k_x = 0$ (or $k_x = \pi$ by symmetry), where two of these three bands have a definite gap, the third (the $k_y = 0$ band) is consistent with a gapless spectra. The estimate for the gap in the third band for all $y$ values is $0.2(3)$ (except for the case of $y = 0$ where we got $0.08(10)$). Hence, we cannot exclude the possibility of a small gap here. But our results are consistent with a gapless spectrum, and given our earlier results on the phase boundary with Ising anisotropy, we believe the spectra are gapless.

B. Dimer expansions

For 2-chain ladders, with antiferromagnetic coupling between the chains, there is an alternative $T = 0$ expansion that can be developed. In the limit that the exchange coupling along the rungs $J_{\perp}$ is much larger than the coupling $J_{/ /}$ along the chains, that is $y \gg 1$, the rungs interact only weakly with each other, and the dominant configuration in the ground state is the product state with the spin on each rung forming a spin singlet, so the Hamiltonian in Eq. (1) can be rewritten as,

$$H/J_{\perp} = H_0 + (1/y)V$$

where

$$H_0 = \sum_{i,l=1}^{l=n_l-1} S_{l,i} \cdot S_{l+1,i}$$

$$V = \sum_{i,l=1}^{l=n_l} S_{l,i} \cdot S_{l,i+1}$$

We can treat the operator $H_0$ as the unperturbed Hamiltonian. The eigenstates of a single pair of spins, or dimers, consists of one singlet state with total $S = 0$ and eigen energy $E_s = -3/4$:  

$$|\Psi\rangle_s = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$

and three triplet states with total $S = 1$ and eigenenergy $E_t = 1/4$:  

$$|\Psi\rangle_t = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle, |\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle)$$

The operator $V$ is treated as a perturbation. It can cause excitations on a pair of neighbouring dimers. Details of the dimer expansions and the matrix elements of $V$ are given in Ref. [18], and will not be repeated here.

We have carried out the dimer expansion for the ground state energy to order $(1/y)^9$ and for the lowest lying triplet excitations to order $(1/y)^8$. The series for the ground state energy per site $E_0/N$ is:
\[ E_0/N = J \left[ -3/8 - 3/(16y^2) - 3/(32y^3) + 3/(256y^4) + 45/(512y^5) + 159/(2048y^6) 
- 879/(32768y^7) - 4527/(32768y^8) - 248391/(2097152y^9) \right] \] (14)

and the series for the excitation spectrum are listed in Table III. Again, we use the integrated first-order inhomogeneous differential approximants [21] to extrapolate the series. For \( y > 1 \), the dimer expansions give much better results than the Ising expansions. For \( y \sim 1 \) also the dimer expansions appear to converge better. The overall spectra determined from the combined study of dimer and Ising expansions are shown in Fig. 4.

C. High temperature series expansions

We now turn to the thermodynamic properties of the ladder system at finite temperatures. We have developed high-temperature series expansions for the uniform magnetic susceptibility \( \chi(T) \) and the specific heat \( C(T) \), for 2-chain and 3-chain system with \( J_\perp = J_// \),

\[
\chi(T) = \frac{\beta}{N} \sum_i \sum_j \frac{\text{Tr} S_i^z S_j^z e^{-\beta H}}{\text{Tr} e^{-\beta H}}
\]

\[
C(T) = \frac{\partial U}{\partial T}
\]

where \( N \) is the number of sites, and \( \beta = 1/(k_B T) \), and the internal energy \( U \) is defined by

\[
U = \frac{\text{Tr} He^{-\beta H}}{\text{Tr} e^{-\beta H}}
\]

The series were computed to order \( \beta^{14} \). The number of contributing graphs, with up to 14 bonds was 4545 for the 2-chain ladders and 5580 for the 3-chain ladders. The series are listed in Table IV. We use integrated first-order inhomogeneous differential approximants [21] to extrapolate the series. The resulting estimates are shown in Fig. 13-14. For the susceptibility, as a comparison, the recent Quantum Monte Carlo (QMC) results of Frischmuth et al [15] and the results from our \( T=0 \) Ising expansion for 3-chain are also shown. It can be seen that our results agree very well with the QMC results except for the 3-chain system at very low temperatures. Given the recent findings that for the spin-half chain, the \( T = 0 \) value is reached from finite temperatures with infinite slope [22], one might expect the \( T \to 0 \) behavior for these 3-chain systems to be equally complex, making it very difficult to explore numerically. For the specific heat, our results showed good convergence up to the peak, but poor convergence below it. The results for 2-chain ladder are consistent with recent quantum transfer-matrix calculations by Troyer [13] et al.

III. CONCLUSIONS

We have studied the 2 and 3 chain Heisenberg-Ising ladders by a variety of different series expansions. Our results confirm the existence of a gap in the excitation spectrum of 2-chain systems, with either ferromagnetic or antiferromagnetic interchain interactions. For 3-chain
systems, our results are consistent with a gapless spectrum. The $T = 0$ phase diagram as well as the temperature dependence of the uniform susceptibility and the specific heat are also calculated. Overall, our results agree very well with previous numerical studies of these systems.

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∗ e-mail address: otja@newt.phys.unsw.edu.au
† e-mail address: singh@solid.ucdavis.edu
‡ e-mail address: w.zheng@unsw.edu.au
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FIG. 1. The phase boundary for 2-chain ladder. The asymptotic behaviour as $y \to -\infty$ predicted by a spin-one single chain system with one-site anisotropy is also shown by the bold line.

FIG. 2. The ground-state energy per site $E_0/N$ as function of $y$ for 2-chain and 3-chain ladders. The error bars are much smaller than the symbols.
FIG. 3. The uniform susceptibility $\chi$ at $T = 0$ as function of $y$ for the 3-chain ladder.

FIG. 4. The dispersions of the spin-triplet excited states $s$ of the 2-chain ladder with antiferromagnetic interchain coupling $y = 2, 1.5, 1, 0.75, 0.5, 0.25, 0.1, 0$ (shown in the figure from the top to the bottom respectively), for $k_y = \pi$. 
FIG. 5. The dispersions of the spin-triplet excited states of the 2-chain ladder with ferromagnetic interchain coupling $y = 0, -0.25, -0.5, -0.75, -1, -1.5, -2, -4$, for $k_y = 0$.

FIG. 6. The dispersions of the spin-triplet excited states of the 2-chain ladder with ferromagnetic interchain coupling $y = -2, -1.5, -1, -0.75, -0.5, -0.25, -0.1, 0$, for $k_y = \pi$. 
FIG. 7. The dispersions of the spin-triplet excited states of the 3-chain ladder with antiferromagnetic interchain coupling $y = 1.5, 1, 0.75, 0.5, 0.25, 0$, for $k_y = 0$.

FIG. 8. The dispersions of the spin-triplet excited states of the 3-chain ladder with antiferromagnetic interchain coupling $y = 2, 1.5, 1, 0.75, 0.5, 0.25, 0.1, 0$, for $k_y = \pi/2$. 

13
FIG. 9. The dispersions of the spin-triplet excited states of the 3-chain ladder with antiferromagnetic interchain coupling $y=2, 1.5, 1, 0.75, 0.5, 0.25, 0.1, 0$, for the middle excitation band.

FIG. 10. The dispersions of the spin-triplet excited states of the 3-chain ladder with ferromagnetic interchain coupling $y=0, -0.1, -0.25, -0.5, -0.75, -1, -1.5, -2$ for $k_y = 0$. 
FIG. 11. The dispersions of the spin-triplet excited states of the 3-chain ladder with ferromagnetic interchain coupling $y = -2, -1.5, -1, -0.75, -0.5, -0.25, -0.1, 0$, for $k_y = \pi/2$.

FIG. 12. The dispersions of the spin-triplet excited states of the 3-chain ladder with ferromagnetic interchain coupling $y = -2, -1.5, -1, -0.75, -0.5, -0.25, -0.1, 0$, for the middle excitation band.
FIG. 13. Susceptibility as a function of temperature for 2-chain and 3-chain ladders from the high temperature series expansion, and the Ising expansion at $T = 0$ (for 3-chain only), also shown are the QMC results of Frischmuth et al as the open symbols (for 2-chain) and filled symbols (for 3-chain) for comparison.

FIG. 14. The specific heat as a function of temperature for 2-chain and 3-chain ladders from the high temperature series expansion.
TABLE I. Series coefficients for the ground-state energy per site $E_0/N$, the staggered/colinear magnetization $M$, and staggered parallel susceptibility $\chi_{//}$. Coefficients of $x^n$ are listed for both the spin-$1/2$ 2-chain and 3-chain ladders with $y = \pm 1$.

| $n$ | $E_0/N$ | $M$ | $\chi_{//}$ |
|-----|---------|-----|-------------|
| 0   | $-3/8$  | $1/2$ | 0           |
| 2   | $-1.250000000000 \times 10^{-1}$ | $-1.250000000000 \times 10^{-1}$ | $2.500000000000 \times 10^{-1}$ |
| 4   | $-1.562500000000 \times 10^{-2}$ | $-6.944444444444 \times 10^{-2}$ | $4.131944444444 \times 10^{-1}$ |
| 6   | $2.443214699074 \times 10^{-3}$  | $1.398473668981 \times 10^{-3}$  | $9.96445526878 \times 10^{-2}$  |
| 8   | $-3.391054004308 \times 10^{-3}$ | $-4.293584111636 \times 10^{-3}$ | $6.51704135203 \times 10^{-1}$ |
| 10  | $-3.083929459433 \times 10^{-4}$ | $-9.459983497527 \times 10^{-4}$ | $2.592402875212 \times 10^{-1}$ |
| 12  | $-5.758725813964 \times 10^{-4}$ | $-2.517788286146 \times 10^{-2}$ | $8.63367551243 \times 10^{-1}$ |
| 14  | $-3.58609329659 \times 10^{-4}$  | $-1.290819523741 \times 10^{-2}$ | $5.19421039530 \times 10^{-1}$ |
| 16  | $-3.658331257827 \times 10^{-4}$ | $-2.220489673071 \times 10^{-2}$ | $1.197591274920$ |

| $n$ | $E_0/N$ | $M$ | $\chi_{//}$ |
|-----|---------|-----|-------------|
| 0   | $-3/8$  | $1/2$ | 0           |
| 2   | $-1.875000000000 \times 10^{-1}$ | $-1.875000000000 \times 10^{-1}$ | $3.750000000000 \times 10^{-1}$ |
| 4   | $-9.11458333333 \times 10^{-3}$  | $-1.17621527778 \times 10^{-1}$  | $9.751157407047 \times 10^{-1}$ |
| 6   | $-6.564670138889 \times 10^{-3}$ | $-1.298255449460 \times 10^{-1}$ | $2.225074548290$ |
| 8   | $-1.011197889034 \times 10^{-2}$ | $-2.567768964481 \times 10^{-1}$ | $6.46470614293$ |
| 10  | $-4.438918908655 \times 10^{-3}$ | $-3.30856710261 \times 10^{-1}$ | $1.43604959755 \times 10^{+1}$ |
| 12  | $-1.149045736404 \times 10^{-2}$ | $-6.75483729347 \times 10^{-1}$ | $3.66134831290 \times 10^{+1}$ |
| 14  | $-9.01458657677 \times 10^{-3}$  | $-1.059678748481$ | $8.26161272382 \times 10^{+1}$ |
| 16  | $-1.860424772341 \times 10^{-2}$ | $-2.08742804052$ | $1.962459013257 \times 10^{+2}$ |

| $n$ | $E_0/N$ | $M$ | $\chi_{//}$ |
|-----|---------|-----|-------------|
| 0   | $-5/12$ | $1/2$ | 0           |
| 2   | $-1.111111111111 \times 10^{-1}$ | $-1.018518518519 \times 10^{-1}$ | $1.913580246914 \times 10^{-1}$ |
| 4   | $-9.126984126984 \times 10^{-3}$  | $-2.76516082976 \times 10^{-2}$  | $1.23651495119 \times 10^{-1}$ |
| 6   | $-3.43007743753 \times 10^{-3}$   | $-2.22902861112 \times 10^{-2}$ | $1.70251936321 \times 10^{-1}$ |
| 8   | $-4.479840217373 \times 10^{-4}$  | $-6.77816397592 \times 10^{-3}$ | $9.22856812572 \times 10^{-2}$ |
| 10  | $-1.15088174205 \times 10^{-3}$  | $-1.34855279039 \times 10^{-2}$ | $1.81445260940 \times 10^{-1}$ |
| 12  | $-5.49387828590 \times 10^{-4}$  | $-9.54972522639 \times 10^{-3}$ | $1.785680892107 \times 10^{-1}$ |

| $n$ | $E_0/N$ | $M$ | $\chi_{//}$ |
|-----|---------|-----|-------------|
| 0   | $-5/12$ | $1/2$ | 0           |
| 2   | $-1.777777777778 \times 10^{-1}$ | $-1.551851851852 \times 10^{-1}$ | $2.76693580247 \times 10^{-1}$ |
| 4   | $2.09935321576 \times 10^{-3}$   | $-3.01063501395 \times 10^{-2}$ | $2.52079079948 \times 10^{-1}$ |
| 6   | $-5.48621040711 \times 10^{-3}$ | $-4.657138485811 \times 10^{-2}$ | $4.78160264625 \times 10^{-1}$ |
| 8   | $-9.77023225775 \times 10^{-4}$ | $-2.676546140909 \times 10^{-2}$ | $5.289125611753 \times 10^{-1}$ |
| 10  | $-8.54022317364 \times 10^{-4}$ | $-2.4131041083174 \times 10^{-2}$ | $6.37103243284 \times 10^{-1}$ |
| 12  | $-7.963887426932 \times 10^{-4}$ | $-3.026105625065 \times 10^{-2}$ | $1.024475965994$ |
TABLE II. Series coefficients for the perpendicular susceptibility $\chi_\perp$. Coefficients of $x^n$ are listed for 2-chain and 3-chain ladders with $y = \pm 1$.

| n   | 2-chain $y = -1$ | 2-chain $y = 1$ | 3-chain $y = -1$ | 3-chain $y = 1$ |
|-----|------------------|------------------|------------------|------------------|
| 0   | 1                | 1/3              | 11/36            | 11/36            |
| 1   | -2/9             | -1/2             | -1/6             | -13/30           |
| 2   | 1.481481418481 × 10^{-2} | 5.66666666667 × 10^{-1} | 1.266534391534 × 10^{-2} | 4.652226631393 × 10^{-1} |
| 3   | 5.308641975309 × 10^{-3} | -7.14351851519 × 10^{-1} | -1.11213668682 × 10^{-2} | -5.160474965706 × 10^{-1} |
| 4   | -2.120002934474 × 10^{-2} | 7.54492634921 × 10^{-1} | 1.134331580417 × 10^{-2} | 5.43419828696 × 10^{-1} |
| 5   | 7.283052520880 × 10^{-2} | -9.20376998874 × 10^{-1} | -2.78797167638 × 10^{-2} | -6.037870527655 × 10^{-1} |
| 6   | -6.507519406167 × 10^{-2} | 9.54243420384 × 10^{-1} | 3.58883185932 × 10^{-2} | 6.192643274088 × 10^{-1} |
| 7   | -9.8505124711 × 10^{-3} | -1.22067012229 | -3.4961571075 × 10^{-1} | -6.538548959702 × 10^{-1} |
| 8   | -3.32164263298 × 10^{-3} | 1.24330662387 | 3.631956127668 × 10^{-1} | 6.836804520999 × 10^{-1} |
| 9   | 3.225030151489 × 10^{-2} | -1.570424359869 | -4.983945317267 × 10^{-2} | -7.20796380768 × 10^{-1} |
| 10  | -2.16349811625 × 10^{-2} | 1.61292972471 | 4.084885561076 × 10^{-2} | 7.446154263232 × 10^{-1} |
| 11  | -2.229164879943 × 10^{-2} | -2.81953857121 | -4.95100943570 × 10^{-2} | -7.944909220697 × 10^{-1} |
| 12  | -8.38607266663 × 10^{-3} | 2.24598466808 |
| 13  | 1.98051341011 × 10^{-2} | -3.03462908727 |
| 14  | -1.128439840082 × 10^{-2} | 3.15807097922 |
| 15  | -1.625818723347 × 10^{-2} | -4.546504413718 |

TABLE III. Series coefficients for the dimer expansion of the 2-chain triplet spin-wave spectrum $\epsilon(k_x, k_y = \pi) = y \sum_{n,m} a_{n,m}(1/y)^n \cos(mk_x)$. Nonzero coefficients $a_{n,m}$ up to order $n = 8$ are listed.

| (n,m) | $a_{n,m}$ | (n,m) | $a_{n,m}$ | (n,m) | $a_{n,m}$ | (n,m) | $a_{n,m}$ |
|-------|------------|-------|------------|-------|------------|-------|------------|
| (0,0) | 1.00000000 | (6,1) | 2.03125000 × 10^{-1} | (3,3) | 2.50000000 × 10^{-1} | (5,5) | 5.46875000 × 10^{-2} |
| (2,0) | 7.50000000 × 10^{-1} | (1,1) | 1.71875000 × 10^{-1} | (5,2) | 2.03125000 × 10^{-1} | (7,7) | 3.22265625 × 10^{-2} |
| (3,0) | 3.75000000 × 10^{-1} | (6,2) | 1.71875000 × 10^{-1} | (5,4) | 9.37500000 × 10^{-2} | (8,8) | 6.15234375 × 10^{-2} |
| (4,0) | -2.03125000 × 10^{-1} | (7,2) | -1.728515625 × 10^{-1} | (7,4) | 2.690429688 × 10^{-1} | (8,8) | -2.61840820 × 10^{-2} |
| (5,0) | -6.25000000 × 10^{-1} | (8,2) | -5.047454834 × 10^{-1} | (8,4) | 1.690521240 × 10^{-1} | (8,4) | -5.047454834 × 10^{-1} |

TABLE IV. Series coefficients for high temperature series expansion of the uniform susceptibility $\chi(T) = \beta \sum_i c_i \beta^i/(\eta_i 2^{i+1} i!)$, and the specific heat $C(T) = \beta^2 \sum_i c_i \beta^i/(\eta_i 2^{i+5} i!)$). Coefficients $c_i$ are listed for 2-chain and 3-chain ladders with $y = 1$.

| i   | $\chi(T)$ for 2-chain | $\chi(T)$ for 3-chain | $C(T)$ for 2-chain | $C(T)$ for 3-chain |
|-----|------------------------|------------------------|-------------------|-------------------|
| 0   | 8                      | 12                     | 36                | 60                |
| 1   | -12                    | -20                    | 72                | 120               |
| 2   | -28                    | -270                   | -522              | -522              |
| 3   | -20                    | -2640                  | -5040             | -5040             |
| 4   | -20                    | -2640                  | 90                | 3270              |
| 5   | -162                   | -160                   | 141876            | 318780            |
| 6   | -630                   | -1052                  | 580797            | 1075767           |
| 7   | 9991                   | 17298                  | -1066200          | -28792632        |
| 8   | 88228                  | 80468                  | -118074186        | -291518730       |
| 9   | -779322                | -1467200               | 940669020         | 3061122900       |
| 10  | -13957358              | -21792822              | 26087816045       | 76820424879      |
| 11  | 55717397               | 165603440              | -4221155560       | -195632449272    |
| 12  | 2827957594             | 2955180058             | -6789937647207    | -22502126499801  |
| 13  | 4867299659             | -24526691326           |                   |                   |
| 14  | -687967034169         | -924449102836          |                   |                   |