Homological algebra/Algebraic geometry

On $\mathbb{A}^1$-fundamental groups of isotropic reductive groups

Sur le groupe fondamental au sens de la $A1$-homotopie des groupes réductifs isotopes

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A B S T R A C T

For an isotropic reductive group $G$ satisfying a suitable rank condition over an infinite field $k$, we show that the sections of the $\mathbb{A}^1$-fundamental group sheaf of $G$ over an extension field $L/k$ can be identified with the second group homology of $G(L)$. For a split group $G$, we provide explicit loops representing all elements in the $\mathbb{A}^1$-fundamental group. Using $\mathbb{A}^1$-homotopy theory, we deduce a Steinberg relation for these explicit loops.

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RÉSUMÉ

Pour un groupe réductif isotrope $G$ défini sur un corps infini $k$, satisfaisant une condition de rang appropriée, nous montrons que l’ensemble des sections du $\mathbb{A}^1$-faisceau de groupe fondamental de $G$ sur une extension des corps $L/k$ s’identifie avec la deuxième homologie des groupes de $G(L)$. Pour un groupe déployé $G$, nous définissons des lacets explicites représentant tous les éléments du groupe $\mathbb{A}^1$-fondamental. En utilisant la théorie de la $\mathbb{A}^1$-homotopie, on déduit une relation de Steinberg pour ces lacets explicites.

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1. Introduction

The goal of the present note is to describe the $\mathbb{A}^1$-fundamental group sheaves for isotropic reductive groups, improving the computations of [13, Proposition 5.2]. Moreover, for split groups, we obtain more precise information on the $\mathbb{A}^1$-fundamental groups by providing explicit loops representing elements in the $\mathbb{A}^1$-fundamental groups. The precise statement of our result is the following, cf. Lemma 2.2 and Proposition 3.2.
**Theorem 1.** Let $k$ be an infinite field and let $G$ be an isotropic reductive group over $k$, assuming that all components of the relative root system of $G$ have at least rank 2. Then there is an isomorphism

$$H_2(G(k), \mathbb{Z}) \cong \pi_1(G(k[\Delta^*])) \cong \pi_1^A(G)(k).$$

In the case of split $G$, the isomorphism can be described by an explicit map

$$K_2^{MW}(k) \xrightarrow{\sim} \pi_1(G(k[\Delta^*])).$$

There, $K_2^{MW}$ means $K_2^{MW}$ or $K_2^{A}$ depending on whether $G$ is symplectic or not.

To prove the result, we use the homotopy invariance of the group homology, cf. [14], and a definition of Steinberg’s groups based on the work by Petrov and Stavrova to identify $H_2(G(k), \mathbb{Z})$ with $\pi_1(G(k[\Delta^*]))$. The results of [1] and [2] on affine excision and descent for isotropic groups relate the latter to $\mathbb{A}^1$-homotopy theory. A slightly different approach is described in Remark 2. The Steinberg relation for explicit loops in $H_2(G(k), \mathbb{Z})$ follows from the results of Hu and Kriz.

Using Morel’s theory of strictly $\mathbb{A}^1$-invariant sheaves [8], we also get the following:

**Corollary 1.** Let $k$ be an infinite perfect field and let $G$ be as above. Then the assignment $L/k \mapsto H_2(G(L), \mathbb{Z})$ extends to a strictly $\mathbb{A}^1$-invariant sheaf of Abelian groups.

Another implication of the above theorem is that Rehmann’s computation of $H_2(\text{SL}_n(D), \mathbb{Z})$, cf. [10], can be seen as a description of $\pi_1^A(\text{SL}_n(D))$, for $n \geq 3$. The corollary implies the existence of well-behaved residue maps on $H_2(\text{SL}_n(D), \mathbb{Z})$, which seem to be new.

**2. Preliminaries**

In this article, we always assume $k$ to be an infinite field. We consider reductive groups $G$ over $k$, and we assume that they are isotropic, as in [9], so that all irreducible components of the relative root system of such $G$ are of rank at least 2. This implies that the results of [9] and [2] are applicable.

For a commutative unital $k$-algebra $R$, the (abstract) group of $R$-points of the group scheme $G$ is denoted by $G(R)$. The elementary subgroup $E(R) \subset G(R)$ is defined, as in [9, §1], as being the subgroup of $G(R)$ generated by $R$-points of unipotent radicals of opposite parabolics $P^+, P^-$ of $G$. By [9, Theorem 1], $E(R)$ is normal in $G(R)$, and by [6, Theorem 1], the group $E(R)$ is perfect. Moreover, by [11, Theorem 1.3], $K_2^G(R) := G(R)/E(R)$ is invariant under polynomial extensions.

**Definition 2.1.** Let $G$ be an isotropic reductive group over a commutative ring $R$. We define the Steinberg group $\text{St}^G(R)$ to be the abstract group generated by elements $\bar{x}_A(u), u \in V_A(R)$ subject to the commutator formulas from [9, Lemma 9, 10]. We define the group $K_2^G(R) := \ker(\text{St}^G(R) \rightarrow E^G(R))$.

**Remark 1.** It is known that $K_2^G(k[\Delta^*]) \hookrightarrow \text{St}^G(k[\Delta^*]) \twoheadrightarrow E^G(k[\Delta^*])$ is a universal central extension for $G$ split of type $A_l, l \geq 3$ (van der Kallen, $C_l, l \geq 3$ (Lavrenov) and $E_l$ (Sinchuk). It is not even a central extension for split rank-2 groups.

Using the standard cosimplicial object given by polynomial rings, one can associate a simplicial group with the reductive group $G$ and a unital commutative $k$-algebra $A$, cf. [5]. This is denoted by $G(A[\Delta^*])$ or (more commonly in the $\mathbb{A}^1$-homotopy literature) by $\text{Sing}_{A}^\bullet(G)(A)$. The $\mathbb{A}^1$-homotopy groups of an isotropic reductive group can be computed from the singular resolution, cf. [2, Corollary 4.3.3].

**Lemma 2.2.** Let $k$ be an infinite field and let $G$ be an isotropic reductive group over $k$.

Then $\text{Sing}_{A}^\bullet(G)$ has affine Nisnevich excision in the sense of [1, Definition 3.2.1] and there are isomorphisms

$$\pi_i(\text{Sing}_{A}^\bullet(G)(A)) \xrightarrow{\sim} \pi_i^A(G)(A)$$

for any essentially smooth $k$-algebra $A$ and any $i \geq 0$.

**Remark 2.** Alternatively, one can prove the affine Nisnevich excision exactly as in [13, Theorem 4.10], using homotopy invariance for unstable $K_i^A$ of isotropic groups from [11, Theorem 1.3]. The above result then follows from the general representability result [1, Theorem 3.3.5]. This was the approach taken in an earlier version of the present paper (arXiv:1207.2364v1), before the appearance of [1,2].
3. The second homology as a fundamental group

We now show how homotopy invariance for homology of linear groups can be used to identify the fundamental group of the singular resolution $G(k(\Delta^*))$ with the second group homology. We define for an isotropic reductive group $G$ simplicial groups $G(k(\Delta^*))$, $E^G(k(\Delta^*))$ and $St^G(k(\Delta^*))$ associated with the group, its elementary subgroup and its Steinberg group. The first thing to note is that homotopy invariance of $K^G_2$ implies an isomorphism $\pi_1(G(k(\Delta^*))) \cong \pi_1(E(k(\Delta^*)))$, which allows us to work with $E(k(\Delta^*))$ henceforth. We define further simplicial objects: denote by $K^G_2(k(\Delta^*))$ the singular resolution of the functor

$$A \mapsto K^G_2(A) := \ker \left( St^G(A) \to E^G(A) \right),$$

by $UE^G(k(\Delta^*))$ the singular resolution of the functor $A \mapsto UE^G(A)$, which assigns to each algebra $A$ the universal central extension $UE^G(A)$ of the perfect group $E^G(A)$, and by $H^2_G(k(\Delta^*))$ the singular resolution of the functor

$$A \mapsto H^2_G(A) := H_2(G(A), \mathbb{Z}) = \ker \left( UE^G(A) \to E^G(A) \right).$$

We chose slightly unusual notation in $H^2_G$ to distinguish the above object from $H_2(G(k(\Delta^*)), \mathbb{Z})$, which has a different meaning.

With these notations, we have the following.

**Lemma 3.1.** There are fibre sequences of simplicial sets:

$$H^2_G(k(\Delta^*)) \to UE^G(k(\Delta^*)) \to E^G(k(\Delta^*)), \quad \text{and} \quad K^G_2(k(\Delta^*)) \to St^G(k(\Delta^*)) \to E^G(k(\Delta^*)) .$$

**Proof.** It follows from Moore’s lemma, e.g., [3, Lemma I.3.4], that the morphisms $UE^G(k(\Delta^*)) \to E^G(k(\Delta^*))$ and $St^G(k(\Delta^*)) \to E^G(k(\Delta^*))$ are fibrations of fibrant simplicial sets. The fibres are by definition $H^2_G(k(\Delta^*))$ and $K^G_2(k(\Delta^*))$, respectively. □

**Proposition 3.2.** Let $k$ be an infinite field, and let $G$ be an isotropic reductive group over $k$. Then the boundary morphism $\Omega E^G(k(\Delta^*)) \to H^2_G(k(\Delta^*))$ associated with the fibration $UE^G(k(\Delta^*)) \to E^G(k(\Delta^*))$ induces an isomorphism:

$$\pi_1(E^G(k(\Delta^*)), 1) \xrightarrow{\cong} H_2(G(k), \mathbb{Z}) .$$

If the Steinberg group does not have non-trivial central extensions, i.e. for all $n$

$$St^G(k(\Delta^n))/\left[ K^G_2(k(\Delta^n)), St^G(k(\Delta^n)) \right] \to E^G(k(\Delta^n))$$

is the universal central extension, then the boundary morphism $\Omega E^G(k(\Delta^*)) \to K^G_2(k(\Delta^*))$ associated with the fibration $St^G(k(\Delta^*)) \to E^G(k(\Delta^*))$ induces an isomorphism

$$\pi_1(G(k(\Delta^*)), 1) \xrightarrow{\cong} K^G_2(k) .$$

**Proof.** By [14, Theorem 1.1], all the usual maps (inclusion of constants, evaluation at 0) induce the isomorphisms $H_2(G(k), \mathbb{Z}) \cong H_2(G(k(T)), \mathbb{Z})$. Therefore, we have

$$\pi_0(H^2_G(k(\Delta^*))) = H_2(G(k), \mathbb{Z}) \quad \text{and} \quad \pi_1(H^2_G(k(\Delta^*))) = 0 .$$

Moreover, $E^G(k)$ and $St^G(k)$ are generated by $X_A(u), \ u \in V_A$. These elements are all homotopic to the identity by the homotopy $X_A(uT)$. Therefore,

$$\pi_0(E^G(k(\Delta^*))) \cong \pi_0(St^G(k(\Delta^*))) = 0 .$$

The long exact sequence associated with the first fibre sequence from Lemma 3.1 yields via the above computations a short exact sequence

$$0 \to \pi_1(UE^G(k(\Delta^*))) \to \pi_1(E^G(k(\Delta^*))) \to \pi_0(H^2_G(k(\Delta^*))) \to 0 .$$

Now let $E^G(k(\Delta^*)) \to E^G(k(\Delta^*))$ be the universal covering of the simplicial group $E^G(k(\Delta^*))$. This has the structure of a simplicial group, and by uniqueness of liftings is degree-wise a central extension by $\pi_1(E^G(k(\Delta^*))$. Therefore, the above injective map factors as $\pi_1(UE^G(k(\Delta^*))) \to \pi_1(E^G(k(\Delta^*))) \to \pi_1(E^G(k(\Delta^*)))$, which together with $\pi_1(E^G(k(\Delta^*))) = 0$ implies the required isomorphism.

The second claim concerning $K^G_2$ follows by the same argument, replacing $UE^G$ by

$$St^G(k(\Delta^n))/[K_2^G(k(\Delta^n)), St^G(k(\Delta^n))]. \quad \Box$$
Remark 3. It should be noted that the isomorphism in Proposition 3.2 has been established in the case of Chevalley groups over algebraically closed fields in [5, Theorem 2.1]. Jardine’s proof uses the spectral sequence for the homology of $G(k[\Delta^*])$ to establish this isomorphism. This is not too far away from our proof above; however, there are better methods available now to establish the necessary $\mathbb{A}^1$-invariance of $H_2$.

4. Explicit description of loops and relations

Fix a root system $\Phi$. For a commutative unital ring $R$, denote $G(\Phi, R)$ the split Chevalley group, $E(\Phi, R)$ its elementary subgroup and $St(\Phi, R)$ its Steinberg group. We now describe explicit loops in $\pi_1(G(\Phi, k[\Delta^*]))$, which is a direct translation of the Steinberg symbols for $H_2$. This also gives rise to an explicit isomorphism $H_2(G(\Phi, k), \mathbb{Z}) \rightarrow \pi_1(G(\Phi, k[\Delta^*]), 1)$

Definition 4.1. For every $\alpha \in \Phi$, we denote by $x_\alpha(u)$ the corresponding root group elements and then define morphisms

$$X^\alpha : G_2(R) \rightarrow E(\Phi, R[T]), \quad R \ni u \mapsto X^\alpha_T(u) := x_\alpha(Tu),$$

$$W^\alpha : G_m(R) \rightarrow E(\Phi, R[T]), \quad R^\times \ni u \mapsto W^\alpha_T(u) := X^\alpha_T(u)X^{-\alpha}_T(-u^{-1})X^\alpha_T(u),$$

$$H^\alpha : G_m(R) \rightarrow E(\Phi, R[T]), \quad R^\times \ni u \mapsto H^\alpha_T(u) := W^\alpha_T(u)W^\alpha_T(1)^{-1},$$

$$C^\alpha : G_m \times G_m \rightarrow E(\Phi, R[T]), \quad (a, b) \mapsto C^\alpha_{T}(a, b) := H^\alpha_T(a)H^\alpha_T(b)H^\alpha_T(ab)^{-1} \in E(\Phi, R[T]).$$

We will use the same letters with an additional tilde to denote the corresponding lifts to $St(\Phi, R[\Delta^*])$.

Example 1. We give an example of the “symbol loops” in the group $SL_2$. With the obvious choice $x_\alpha(u) = e_{12}(u)$, we have

$$C^\alpha_T(u, v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + T(T^2 - 1) \frac{(1 - u)(1 - v)}{u^2v} D^\alpha_T(u, v),$$

where

$$D^\alpha_T(u, v) = \begin{pmatrix} u(1 - u)(T^2 - 1)(T^2 - 2) - uv^2((T^2 - 1)^2(1 - u) + u)(T^2 - 2) \\ (1 - u)(T^2 - 1)^2 - 1 \end{pmatrix}.$$ 

Remark 4. Philosophically, what is happening here is the following: choosing a maximal torus $S$ in $G$, the associated root system and root subgroups $x_\alpha$ allows us to write down a contraction of the (elementary part of the) torus, i.e. a homotopy $H : S \times \mathbb{A}^1 \rightarrow G$, where $H(\cdot, 0)$ factors through the identity $1 \in G$ and $H(\cdot, 1)$ is the inclusion of $S$ as maximal torus of $G$. This is nothing but a more elaborate version of the lemma of Whitehead. After fixing such a contraction, there is a preferred choice of path $H(u)$ for any $u \in S$. Given two units in the torus, one can concatenate the paths $H(u), uH(v)$ and $H(uv)^{-1}$ to obtain a loop. This is basically what happens in Definition 4.1.

The translation between elements (and symbols) in the Steinberg group and loops (and symbol loops) in the singular resolution $G_2(k[\Delta^*])$ is given as in covering space theory:

(i) an element of the Steinberg group is given by a product $\bar{y} = \prod x_\alpha(u_i)$. Setting $y_T = \prod x_\alpha(Tu_i)$ produces a path in $E(\Phi, R[T])$. If $\bar{y}$ is in the kernel of the projection $St(\Phi, R) \rightarrow E(\Phi, R)$, the path $y_T$ is in fact a loop;

(ii) a path $y_T \in E(\Phi, k[\Delta^*])$ with $y_T(0) = 1$ can be factored as a product of elementary matrices $\prod x_\alpha(f_i(T))$, which in turn can be lifted to $St(\Phi, k[T])$. Evaluating at $T = 1$ yields an element $\prod x_\alpha(f_i(1)) \in St(\Phi, k[T])$. If the path $y_T$ was in fact a loop, then the resulting element $\prod x_\alpha(f_i(1)) \in St(\Phi, R)$ lies in fact in the kernel of the projection $St(\Phi, R) \rightarrow E(\Phi, R)$.

It is then possible to derive elementary relations between the above loops in just the same way as the relations for Steinberg symbols in [7]. The contraction of the torus $H^\alpha_T(u)$ is chosen such that $H^\alpha_T(1)$ is the constant loop. From this, it follows immediately that $C^\alpha_T(x, 1) = C^\alpha_T(1, x) = 1$ for all $x, y \in k^\times$. The symbol loops $C^\alpha_T(x, y)$ in $G(\Phi, k[T])$ are not central on the nose, but are central up to homotopy because the fundamental group of a simplicial group is Abelian, and conjugation by paths acts trivially on the fundamental group. Then the conjugation formulas in [7, Lemma 5.2] can be translated into statements of homotopies between corresponding products of paths $W^\alpha_T(u)$ resp. $H^\alpha_T(u)$. In particular, the (weak) bilinearity of symbol loops in the fundamental group can be proved exactly as in [7]. For details, cf. [12]. It is not clear how to prove the Steinberg relation simply by computing with loops and homotopies inside $E(k[\Delta^*])$. We derive a general Steinberg relation from $A^1$-homotopy theory in the next section.

5. The Steinberg relation from $A^1$-homotopy theory

In the case of split groups, the Steinberg relation in $H_2(G(k), \mathbb{Z})$ can be deduced from $A^1$-homotopy as follows. We denote by $\Sigma$ and $\Omega$ the simplicial suspension and loop space functors, respectively.
Proposition 5.1. Let $C : G_m \wedge G_m \to \Omega G_a$ be any morphism with $G_a$ a simplicial group satisfying the affine Nisnevich excision. Let $s : A^1 \setminus \{0, 1\} \to G_m \wedge G_m$ be the Steinberg morphism $a \mapsto (a, 1 - a)$. Then the composition of $C$ with the Steinberg morphism $C \circ s : A^1 \setminus \{0, 1\} \to \Omega G_a$ has trivial homotopy class in the simplicial and $A^1$-local homotopy category.

Proof. We have the natural adjunction $[\Sigma X, Y] \simeq [X, \Omega Y]$ both in the simplicial and $A^1$-local homotopy category. Choose a fibrant resolution $r : G_* \to \text{Ex}_A(X(G_*))$. Under the adjunction, the morphism $r \circ C \circ s$ corresponds to the composition

$$\Sigma A^1 \setminus \{0, 1\} \xrightarrow{\text{Ex}_A r} \Sigma G_m \wedge G_m \xrightarrow{C \circ s} \text{Ex}_A(X(G_*)).$$

By [4, Prop. 1], this composition factors through the $A^1$-contractible space $\Sigma A^1$ and is therefore trivial. More specifically, we have the following equality in $[\Sigma A^1 \setminus \{0, 1\}, \text{Ex}_A(X(G_*))]:$

$$r \circ C \circ s = r \circ C \circ \Sigma s \circ t = r \circ C \circ 0 = 0.$$

This implies the $A^1$-local statement. The simplicial statement follows from [1, Theorem 3.3.5], which gives a bijection

$$[A^1 \setminus \{0, 1\}, G_*] \simeq [A^1 \setminus \{0, 1\}, \text{Ex}_A(X(G_*))]. \quad \Box$$

The result and Lemma 2.2 imply that for split $G$, all the loops $C^d(u, 1 - u), u \in k^\times$, described in Section 3 are contractible in the singular resolution $G(k(A^1))$: the symbol $C^d(x, y)$ can be interpreted as a morphism of simplicial groups $G_m \times G_m \to \Omega \text{Sing}^{A^1} G$. But since $C^d(1, y) = C^d(x, 1) = 1$ is equal to the identity, this morphism factors through a morphism of simplicial presheaves $G_m \wedge G_m \to \Omega \text{Sing}^{A^1} G$. The above corollary then yields the Steinberg relation. Even better, since $\text{Sing}^{A^1} G$ has affine excision, there is a single algebraic morphism $A^1 \setminus \{0, 1\} \times A^1 \to G$ realizing all the Steinberg loops $C^d(u, 1 - u), u \in k^\times \setminus \{1\}$ at once; and there is a single algebraic homotopy $(A^1 \setminus \{0, 1\} \times A^1) \times A^1 \to G$ providing all the contractions of the Steinberg loops at once. This is one instance where a computation in group homology can be deduced from $A^1$-homotopy theory.

We want to point out the following generalization of the Steinberg relation for non-split groups. Let $D$ be a central simple algebra over $k$. There is an associated reduced norm which can be interpreted as a regular morphism $\text{Nrd}_D : A_{\dim D} \to A^1$. In $A^1_{\dim D}$ we have two open subschemes, the linear algebraic group $\text{GL}_1(D)$ defined by $\text{Nrd}_D(u) \neq 0$, and another open subscheme $U_D$ defined by $\text{Nrd}_D(u) \neq 0$ and $\text{Nrd}_D(1 - u) \neq 0$. There is an obvious analogue of the Steinberg morphism:

$$s_D : U_D \to \text{GL}_1(D) \times \text{GL}_1(D) \to \text{GL}_1(D) \wedge \text{GL}_1(D) : u \mapsto (u, 1 - u).$$

Proposition 5.2. Let $s_D : U_D \to \text{GL}_1(D) \wedge \text{GL}_1(D)$ be the Steinberg morphism defined above. Then there exists a space $X_D$ and a commutative diagram

$$\begin{array}{ccc}
U_D & \xrightarrow{s_D} & \text{GL}_1(D) \wedge \text{GL}_1(D) \\
\uparrow t & & \downarrow \psi_D \\
A^1_{\dim D} & \xrightarrow{s_D} & X_D
\end{array}$$

with the suspension $\Sigma \psi_D$ of $\psi_D$ being an $A^1$-local weak equivalence.

Proof. The argument is the same as in [4, Prop. 1], replacing $A^1$ by $A^1_{\dim D}$, $G_m$ by $\text{GL}_1(D)$, and $A^1 \setminus \{0, 1\}$ by $U_D$. The varieties $V$ and $W$ have to be replaced by $V_D = \{y - 1 = x \cdot D z, y \neq 0\}$ and $W_D = \{x - 1 = y \cdot D z, x \neq 0\}$. The space $X_D$ is then the pushout $V_D \cup_{\text{GL}_1(D) \times \text{GL}_1(D)} W_D$. \quad \Box

This provides an $A^1$-homotopy proof of the Steinberg relation in $H_2(\text{SL}_n(D), \mathbb{Z}), n \geq 3$. All Steinberg relations are given by a single algebraic map $U_D \times A^1 \to \text{SL}_n(D)$, and they are all contracted by a single (inexplicit) algebraic homotopy $(U_D \times A^1) \times A^1 \to \text{SL}_n(D)$.

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