Max Weibull-G Power Series Distributions

Munteanu Bogdan Gheorghe1*

1Department of Fundamental Sciences, Henri Coanda Air Force Academy, Brasov, 500187, Romania.

Author’s contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/AJPAS/2021/v15i130345

Editor(s):
(1) Dr. Manuel Alberto M. Ferreira, Lisbon University, Portugal.

Reviewers:
(1) Ndolane Sene, Universite Cheikh Anta Diop, Senegal.
(2) M. Fakour, Islamic Azad University, Iran.

Complete Peer review History: https://www.sciarticle4.com/review-history/69851

Received 27 April 2021
Accepted 14 July 2021
Published 07 October 2021

Abstract

Based on the Weibull-G Power probability distribution family, we have proposed a new family of probability distributions, named by us the Max Weibull-G power series distributions, which may be applied in order to solve some reliability problems. This implies the fact that the Max Weibull-G power series is the distribution of a random variable \( \max(X_1, X_2, \ldots, X_N) \) where \( X_1, X_2, \ldots \) are Weibull-G distributed independent random variables and \( N \) is a natural random variable the distribution of which belongs to the family of power series distribution. The main characteristics and properties of this distribution are analyzed.

Keywords: Lifetime; power series distribution; distribution of the maximum; Weibull distribution.

1 Introduction

The Weibull distribution is often found in reliability modeling and not only. It is mainly used in the rate of stochastical modeling hazard when analyzing the failure rate in electrical engineering, or studying the super tension occurring in a circuit in industrial engineering within the study of delivery times, the extreme values, the weather forecast, the radar systems for modeling the signals level dispersion received from different sources, etc.
Additionally, large-scale applications in various fields, such as engineering, lifetime, and economy, etc. have led to the introduction of a new family of distribution models, the Weibull-G (NWG) [1]. The well-known generators are, as follows: beta-G by Eugene et al. [2], Kumaraswamy-G (Kw-G) by Cordeiro and de Castro [3], McDonald-G (Mc-G) by Alexander et al. [4], gamma-G type1 by Zografos and Balakrishnan [5] and Amini et al. [6], gamma-G type 2 by Ristić and Balakrishnan [7] and Amini et al. [6], odd exponentiated generalized (odd exp-G) by Cordeiro et al. [8], transformed-transformer (T-X) (Weibull-Xand gamma-X) by Alzaatreh et al. [9], exponentiated T-X by Alzaghal et al. [10], odd Weibull-G by Bourguignon et al. [11], exponentiated half-logistic by Cordeiro et al. [12], the T-X[Y]-quantile based approach by Aljarrah et al. [13], T-R[Y] by Alzaatreh et al. [14], Lomax-G by Cordeiro et al. [15], logistic-X by Tahir et al. [16] and Kumaraswamy odd log-logistic-G by Alizadeh et al. [17].

The distribution class of the Max Weibull-G power series comes to complete the research with reference to the distribution family generated by the Weibull distribution. In this case, the treatment is done unitarily using the power series distribution class (PSD).

In this paper the distribution for the maximum of Z independent and identically Weibull-G distributed random variables are analysed, where number Z is a power series distributed random variable.

The article is structured as follows: in section 2, preliminary results are presented, including the basic notions of power series distributions, examples of distributions of this type where the power series, the power series parameter and the corresponding convergence radius are highlighted and, Finally, the distribution function and the probability density of the Weibull G distribution are defined. The main features and properties of the Max Weibull G distribution are analyzed in section 3. In section 4, the Poisson limit theorem is formulated and proved in the context of the Max Weibull G distribution. The validation of the Poisson limit theorem from a graphical point of view is completed in section 5 for some special cases of Max Weibull G distributions. Section 6 of the paper represents the conclusions to the study.

2 Preliminary Results

Over the years, many researchers have created reliability models which generate certain probability distribution families, as they create a good flexibility when modeling and analyzing data in practice. For example, Gurvich [18] introduces a model which is based on the Weibull distribution: \( G(x, \alpha, \eta) = 1 - \exp(-\alpha H(x, \eta)) \), \( x \in D \subseteq \mathbb{R} \), \( \alpha > 0 \), where \( H(x, \eta) \) is a nonnegative, increasing function, which depends on the vector parameter and this distribution generalizes the exponential distribution \( H(x, \eta) = x \), Rayleigh \( (H(x, \eta) = x^2) \).

Then Zografos and Balakrishman [5] define a new distribution family starting from the probability density of Stacy’s generalized gamma distribution.

Recently, Bourguignon et al. [6] has generated the Weibull distribution families introduced by Gurvich [18] and Zografos and Balakrishman [5], thus presenting a new Weibull distributions family using the Weibull generator applied to the ratio \( G(x)/\overline{G}(x) \), where \( \overline{G}(x) = 1 - G(x) \). Thus, in the cumulative distribution function of the Weibull distribution with real parameters and positive \( x \),

\[
F(x) = 1 - \exp\left[-\left(x \overline{H}\right)^{\alpha}\right], \quad x > 0, \quad \lambda, \upsilon > 0,
\]

for the Weibull-G distribution class:

\[
F_{WG}(x, \lambda, \upsilon, \eta) = 1 - \exp\left(-\frac{1}{\lambda^\upsilon}\left[\frac{G(x, \eta)}{G(x, \eta)}\right]^{\upsilon}\right),
\]  

(1)
$x \in D \subseteq \mathbb{R}^n$, $\lambda, \upsilon > 0$ and $G(x, \eta)$ is the distribution function referred to, which depends on the vector parameter. Also, the probability density function which characterizes this distribution family is:

$$f_{WG}(x, \lambda, \upsilon, \eta) = \frac{\upsilon}{\lambda^\upsilon} g(x, \eta) \frac{G^{\upsilon-1}(x, \eta)}{G(x, \eta)} \exp \left\{ -\frac{1}{\lambda^\upsilon} \left[ \frac{G(x, \eta)}{\bar{G}(x, \eta)} \right]^\upsilon \right\},$$

where $g(x, \eta)$ is the probability density function corresponding to the absolutely continuous type distribution $G$.

Indeed,

$$f_{WG}(x, \lambda, \upsilon, \eta) = \frac{d}{dx} F_{WG}(x, \lambda, \upsilon, \eta) = \frac{\upsilon}{\lambda^\upsilon} \left( \frac{G(x, \eta)}{\bar{G}(x, \eta)} \right)^{\upsilon-1} \exp \left\{ -\frac{1}{\lambda^\upsilon} \left[ \frac{G(x, \eta)}{\bar{G}(x, \eta)} \right]^\upsilon \right\} \frac{d}{dx} \frac{G(x, \eta)}{\bar{G}^2(x, \eta)}$$

$$= \frac{\upsilon}{\lambda^\upsilon} \exp \left\{ -\frac{1}{\lambda^\upsilon} \left[ \frac{G(x, \eta)}{\bar{G}(x, \eta)} \right]^\upsilon \right\} \cdot \frac{g(x, \eta) \cdot G^{\upsilon-1}(x, \eta)}{\bar{G}^{\upsilon+1}(x, \eta)}$$

A random variable which has the cumulative distribution function defined by Eq. (1) or the probability density function defined by Eq. (2) is denoted $X \sim W G(\lambda, \upsilon, \eta)$.

Bourguignon et al. [6] give the following interpretation to the Weibull-G distribution family, that is: let $Y$ be the lifetime with the absolutely continuous type $G$ distribution. The chance ratio for a component with a lifetime $Y$ to degrade to the $x$ moment is namely it represents exactly the distribution of the random variable $X$:

$$P\left( X \leq \frac{G(x, \eta)}{G(x, \eta)} \right) = P(Y \leq x) = F_{WG}(x, \lambda, \upsilon, \eta),$$

where $F_{WG}(x, \lambda, \upsilon, \eta), x > 0, \lambda, \upsilon > 0$ is defined by Eq. (1) and $G(x, \eta)$ is the absolutely continuous distribution function type which depends on the vector parameter $\eta$.

Consequently, let us reproduce the definition of the power series distributions used by us, considering random variable $Z$ such that $P(Z \in \{1, 2, \ldots\}) = 1$.

**Definition 2.1.** ([19]) We say that the random variable $Z$ has a power series distribution if:

$$P(Z = z) = \frac{a_z \Theta^z}{A(\Theta)}, z = 1, 2, \ldots; \Theta \in (0, \tau); \tau > 0,$$

where $a_z, a_z, \ldots$ are nonnegative real numbers, $\tau$ is a positive number bound by the convergence radius of power series (series function) $A(\Theta) = \sum_{z=1}^\infty a_z \Theta^z, \forall \Theta \in (0, \tau)$ and is the power parameter of the distribution.
Remark 2.1. Let us denote by PSD the class of the power series distributions. Some of them may be found in Table 1. The fact that the random variable \( Z \) has the distribution Eq. (3) will be denoted as \( Z \in PSD \).

### Table 1. The representative elements of the PSD class for various truncated distributions (Binomial, Poisson, Log, Geometric, Pascal, Bernoulli distributions)

| Distributions | \( a_z \) | Power parameter \( \theta \) | Series function \( A(\theta) \) | Convergence radius \( \tau \) |
|---------------|----------|-----------------|-----------------|-----------------|
| Binom\(^*\)(n,p) | \( \binom{n}{z} \) | \( p \) | \( (1 + \theta)^z - 1 \) | \( \infty \) |
| Poisson\(^*\)(\( \lambda \)) | \( \frac{1}{z!} \) | \( \lambda \) | \( e^\theta - 1 \) | \( \infty \) |
| Log(p) | \( \frac{1}{z} \) | \( p \) | \( -\ln(1-\theta) \) | 1 |
| Geom\(^*\)(p) | \( 1 - p \) | \( p \) | \( \frac{\theta}{1-\theta} \) | 1 |
| Pascal(k,p) | \( \binom{z-k}{k-1} \) | \( 1 - p \) | \( (\frac{\theta}{1-\theta})^k \) | 1 |
| BN\(^*\)(k,p) | \( \binom{z+k-1}{z} \) | \( p \) | \( (1-\theta)^{k-1} - 1 \) | 1 |

### 3 Max Weibull-G Power Series Distribution

Let \( X_i \) be a sequence of independent and identically distributed random variables, \( X_i \sim WG(\lambda, \upsilon, \eta) \), \( \lambda, \upsilon > 0 \) and a vectorial parameter of the absolutely continuous type distribution \( G \). We denote by \( F_{X_i}(x,\lambda,\upsilon,\eta) \equiv F_{WG}(x,\lambda,\upsilon,\eta) \), \( x > 0 \) the cumulative distribution function respectively \( f_{X_i}(x,\lambda,\upsilon,\eta) \equiv f_{WG}(x,\lambda,\upsilon,\eta) \), \( x > 0 \) the probability density function of the random variable \( \{X_i\}_{i=1}^\infty \), defined by the Eq. (1) and Eq. (2).

We also denote by \( U_{WG} = \max\{X_1,X_2,\ldots,X_Z\} \), where the random variable \( Z \in PSD \).

According to Leahu et al. [20] the cumulative distribution function cdf and the probability density function of the random variable \( U_{WG} \) are characterized by the following relations:

\[
U_{WG}(x,\lambda,\upsilon,\eta,\Theta) = \frac{A[\Theta F_{WG}(x,\lambda,\upsilon,\eta)]}{A[\Theta]}, \quad x > 0
\]  

\[
u_{WG}(x,\lambda,\upsilon,\eta,\Theta) = \frac{\Theta f_{WG}(x,\lambda,\upsilon,\eta) A[\Theta F_{WG}(x,\lambda,\upsilon,\eta)]}{A[\Theta]}, \quad x > 0
\]

**Definition 3.1.** We say that \( U_{WG} \) has a Max Weibull-G power series distribution with parameters \( \lambda, \upsilon, \eta \) (denoted \( U_{WG} \sim \text{MaxWGPS}(\lambda, \upsilon, \eta, \Theta) \)) if this has the cumulative distribution function defined by the Eq. (4) or the probability density function defined by Eq. (5), where \( A(\Theta) \) is the power series which depends on the power parameter which relates to the distribution of the random variable \( Z \).
The following results characterize the function of survival and the rate of hazard.

**Consequence 3.1.** The function of survival of the random variable $U_{WG}$ is the following:

$$S_{U_{WG}}(x, \lambda, \nu, \eta, \Theta) = 1 - \frac{A[\Theta f_{WG}(x, \lambda, \nu, \eta)]}{A[\Theta]}, \; x > 0.$$  

**Consequence 3.2.** The rate of hazard for the random variable $U_{WG}$ is given by:

$$h_{U_{WG}}(x, \lambda, \nu, \eta, \Theta) = \frac{u_{WG}(x, \lambda, \nu, \eta, \Theta)}{S_{U_{WG}}(x, \lambda, \nu, \eta, \Theta)} = \frac{\Theta f_{WG}(x, \lambda, \nu, \eta)A[\Theta f_{WG}(x, \lambda, \nu, \eta)]}{A[\Theta] - A[\Theta f_{WG}(x, \lambda, \nu, \eta)]}, \; x > 0.$$  

The following result presents some properties of distribution for the maximum of random number of independent and identically distributed random variables with the cumulative distribution function $F_{X_i}(x, \lambda, \nu, \eta) \equiv F_{WG}(x, \lambda, \nu, \eta), \; x > 0$.

**Proposition 3.1.** If $(X_{i})_{i \geq 0}$ is a sequence of nonnegative independent and identically distributed random variables of absolutely continuous type with the cumulative distribution function $F_{X_i}(x, \lambda, \nu, \eta) \equiv F_{WG}(x, \lambda, \nu, \eta), \; x > 0$ and with $Z \in \text{PSD}, \; P(Z = z) = \frac{a_{z}}{A(\Theta)}, \; (a_{z})_{z \in \mathbb{R}}$ a sequence of real, nonnegative numbers, $A(\Theta) = \sum_{z \in \mathbb{R}} a_{z} \Theta^{z}$, $\forall \Theta \in (0, \tau)$, then:

$$\lim_{\Theta \to \tau^{-}} \frac{U_{WG}(x, \lambda, \nu, \eta, \Theta)}{\Theta} = \left[ F_{WG}(x, \lambda, \nu, \eta, \Theta) \right]^{k}, \; x > 0,$$

where $k = \min\{n \in \mathbb{N}, a_{n} > 0\}$.  

The expression for the moment of $r^{t}$th order of the random variable $U_{WG}$ is presented further on.

**Proposition 3.2.** The moment of $r^{t}$th order $r \in \mathbb{N}, \; r \geq 1$ of the random variable $U_{WG} = \max\{X_1, X_2, \ldots, X_z\}$ where $Z \in \text{PSD}$, is characterized by the relation:

$$E[U_{WG}^{r}] = \sum_{z \in \mathbb{R}} \frac{a_{z} \Theta^{z}}{A(\Theta)} E\left[ \max\{X_1, X_2, \ldots, X_z\} \right],$$  

where the probability density function of the random variable $\max\{X_1, X_2, \ldots, X_z\}$ is:

$$f_{\max\{X_1, X_2, \ldots, X_z\}}(x, \lambda, \nu, \eta) = z f_{WG}(x, \lambda, \nu, \eta) \left[ F_{WG}(x, \lambda, \nu, \eta) \right]^{z-1}.$$  

Other properties of the Max Weibull-G power series distributions are given by the following proposition that shows that the distribution of the maximum of the first $k$ random variable of a countable sequence of independent and identically distributed random variables is, in certain conditions, equivalent to the distribution of a sequence of random variables in a random number.

The following notations are required: $U_{WG \text{Geom}} = \max\{X_1, X_2, \ldots, X_z\}$, where $Z \sim \text{Geom}(p), \; p \in (0,1)$ and $X_i, i \geq 1$ are independent and identically distributed random variables with the cumulative distribution function $F_{WG}$.  

Munteanu; AJPAS; 15(1): 15-29, 2021; Article no.AJPAS.69851
Definition 3.2. We say that $U_{WGBinom}$ has a Max Weibull-G geometrical power series distribution of $\lambda, \upsilon, \eta$ and $p$ parameters, where $\lambda, \upsilon > 0$ refers to Weibull distribution parameters, $\eta$ the vector parameter of the absolutely continuous distribution $G$ and $p \in (0,1)$ geometrical distribution parameter (noted by $U_{WGBinom} \sim \text{MaxWGGeom}(\lambda, \upsilon, \eta, p)$), if it has the cumulative distribution function defined by Eq.(4) or the probability density function defined by Eq. (5), where $\Theta = 1 - p$, $p \in (0,1)$ and $A(\Theta) = \frac{\Theta}{1 - \Theta}$.

Proposition 3.3. If $(X_i)_{i \geq 1}$ are nonnegative independent and identically distributed random variables of the absolutely continuous type and $(Y_i)_{i \geq 1}$ are independent and identically distributed random variables, $Y_i \sim \text{MaxWGGeom}(\lambda, \upsilon, \eta, p), \lambda, \upsilon > 0, p \in (0,1)$ and $\eta$ vector parameter of the $G$ absolutely continuous distribution, then the random variable has the same distribution as the random variable $\max\{X_1, X_2, \ldots, X_n\}$, where $Z \sim \text{Pascal}(k, p), k \in \{1,2,\ldots\}, p \in (0,1)$.

4 Poisson Limit Theorem

Let’s denote $U_{WGBinom} = \max\{X_1, X_2, \ldots, X_n\}$ and $U_{WGBinom} = \max\{X_1, X_2, \ldots, X_n\}$, where $(X_i)_{i \geq 1}$ are random variable nonnegative, independent and identically distributed with the cumulative distribution function $F_X(x, \lambda, \upsilon, \eta) = F_{WGBinom}(x, \lambda, \upsilon, \eta), \forall x > 0$, $Z \sim \text{Binom}(n, p)$, $n \in \{1,2,\ldots\}, p \in (0,1)$ with $A(\Theta) = (1 + \Theta)^n - 1, \Theta \in (0, \infty)$. $\Theta = \frac{p}{1 - p}$ and $N \sim \text{Poisson}(\alpha)$. $\alpha > 0$, with $A(\Theta^*) = \exp(\Theta^*) - 1, \Theta^* \in (0, \infty)$, $\Theta^* = \alpha$, the random variable $(X_i)_{i \geq 1}$ and $Z$, respectively $(X_i)_{i \geq 1}$ and $N$ are reciprocally independent.

Now we can formulate the following definitions.

Definition 4.1. (i) We say that random variable $U_{WGBinom}$ has a Max Weibull-G binomial power series distribution of parameters $\lambda, \upsilon, \eta$ and $p$ (noted $U_{WGBinom} \sim \text{MaxWGBinom}(\lambda, \upsilon, \eta, n, p)$) if this has the cumulative distribution function defined by the relation:

$$U_{WGBinom}(x, \lambda, \upsilon, \eta, n, p) = 1 - p \cdot \exp\left[ - \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right)^\upsilon \right]^{n} - (1 - p)^n,$$

$$x > 0, \lambda, \upsilon > 0, n \in \{1,2,\ldots\}, p \in (0,1), \text{ or the probability density function characterized by:}$$

$$u_{WGBinom}(x, \lambda, \upsilon, \eta, n, p) = np^0 g(x, \eta) \exp\left[ - \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right)^\upsilon \right]^{n-1} \frac{G(x, \eta)[1 - (1 - p)^n]}{G(x, \eta)[1 - (1 - p)^n]}.$$

$\lambda, \upsilon > 0, n \in \{1,2,\ldots\}, b \in (0,1)$.
(ii) We say that the random variable \( U_{WG\text{Poisson}} \) has a Max Weibull-G Poisson power series distribution of parameters \( \lambda, \upsilon, \eta \) and \( \alpha \) (it is noted \( U_{WG\text{Poisson}} \sim \text{MaxWG\text{Poisson}}(\lambda, \upsilon, \eta, \alpha) \)) if this has the cumulative distribution function defined by the relation:

\[
U_{WG\text{Poisson}}(x, \lambda, \upsilon, \eta, \alpha) = \exp \left( -\alpha \exp \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right) \right) - \exp(-\alpha) \frac{1 - \exp(-\alpha)}{1 - \exp(-\alpha)},
\]

\( x > 0, \lambda, \upsilon > 0, \alpha > 0, p \in (0,1) \) or the probability density function given by the relation:

\[
u_{WG\text{Poisson}}(x, \lambda, \upsilon, \eta, \alpha) = \frac{\alpha \upsilon g(x, \eta) \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right)^{\upsilon - 1} \exp \left( -\alpha \exp \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right) \right)}{G(x, \eta)G(x, \eta)[1 - \exp(-\alpha)]}.
\]

Remark 4.1. (i) The cumulative distribution functions defined by the Eq.(7) and Eq.(9) result from:

\[
\frac{A F(x)}{A \Theta} = \frac{(1 + \Theta)^n - 1}{(1 + \Theta)^n - 1}
\]

(ii) The probability density functions defined by the Eq.(8) and Eq.(10) result from:

\[
u_{WG\text{Binom}}(x, \lambda, \upsilon, \eta, n, p) = \left( 1 + \frac{p}{1 - p} F_W(x, \lambda, \upsilon, \eta) \right)^n - 1 = \left( 1 + \frac{p}{1 - p} \right)^n - 1
\]

and

\[
u_{WG\text{Poisson}}(x, \lambda, \upsilon, \eta, \alpha) = \frac{\exp \left( \alpha F_W(x, \lambda, \upsilon, \eta) \right) - 1}{\exp(\alpha) - 1} = \frac{\alpha - \alpha \exp \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right) - 1}{\exp(\alpha) - 1}.
\]
(ii) The probability density functions defined by the Eq.(8) and Eq.(10) result from:

\[
\begin{align*}
&u_{\text{WGBinom}}(x, \lambda, \nu, \eta, n, p) = \frac{(5) \Theta f_{\text{WG}}(x, \lambda, \nu, \eta) \frac{d}{dx} A[\Theta F_{\text{WG}}(x, \lambda, \nu, \eta)]}{A[\Theta]} \\
&= \frac{\Theta f_{\text{WG}}(x, \lambda, \nu, \eta) \cdot (1 + \Theta) f_{\text{WG}}(x, \lambda, \nu, \eta))^n}{(1 + \Theta)^n - 1} = \frac{p \cdot f_{\text{WG}}(x, \lambda, \nu, \eta) \cdot (1 + \frac{p}{1 - p} \cdot f_{\text{WG}}(x, \lambda, \nu, \eta))^n}{(1 + \frac{p}{1 - p})^n - 1} \\
&= \frac{np \cdot f_{\text{WG}}(x, \lambda, \nu, \eta) \cdot (1 - p + p \cdot f_{\text{WG}}(x, \lambda, \nu, \eta))^n}{1 - (1 - p)^n} \\
&\frac{n p u \nu g(x, \eta) \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right)^\nu \exp \left[ - \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right)^\nu \right] \left[ 1 - p \cdot \exp \left[ - \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right)^\nu \right] \right]^{n-1}}{G(x, \eta) G(x, \eta) [1 - (1 - p)^n]}
\end{align*}
\]

and

\[
\begin{align*}
u_{\text{WGPoisson}}(x, \lambda, \nu, \eta, \alpha) &= \frac{(5) \nu^* f_{\text{WG}}(x, \lambda, \nu, \eta) \frac{d}{dx} A[\nu F_{\text{WG}}(x, \lambda, \nu, \eta)]}{A[\nu^*]} \\
&= \frac{\nu^* f_{\text{WG}}(x, \lambda, \nu, \eta) \exp \left[ \nu F_{\text{WG}}(x, \lambda, \nu, \eta) \right]}{\exp(\nu^*) - 1} = \frac{\alpha f_{\text{WG}}(x, \lambda, \nu, \eta) \exp[\alpha F_{\text{WG}}(x, \lambda, \nu, \eta)]}{\exp(\alpha) - 1} \\
&\frac{\alpha u \nu g(x, \eta) \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right)^\nu \exp \left[ - \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right)^\nu - \alpha \exp \left[ - \left( \frac{G(x, \eta)}{\lambda G(x, \eta)} \right)^\nu \right] \right]^{n-1}}{G(x, \eta) G(x, \eta) [1 - \exp(-\alpha)]}
\end{align*}
\]

The following result shows that the Max Weibull-G Poisson (with the cumulative distribution function \(U_{\text{WGBinom}}(x, \lambda, \nu, \eta, \alpha)\)) approximates in certain conditions the distribution Max Weibull-G Binomial (with the cumulative distribution function \(U_{\text{WGBinom}}(x, \lambda, \nu, \eta, n, p)\)).

**Theorem 4.1. (Poisson limit theorem)** If the random variable \(U_{\text{WGBinom}} \sim \text{MaxWGBinom}(\lambda, \nu, \eta, n, p)\) and \(U_{\text{WGPoisson}} \sim \text{MaxWGPoisson}(\lambda, \nu, \eta, \alpha)\) with \(n \to \infty\) and \(p \to 0^+\) so that \(np \to \alpha, \alpha > 0\), then

\[
\lim_{n \to \infty, p \to 0^+} U_{\text{WGBinom}}(x, \lambda, \nu, \eta, n, p) = U_{\text{WGPoisson}}(x, \lambda, \nu, \eta, \alpha), \forall x > 0,
\]

where \(U_{\text{WGBinom}}(x, \lambda, \nu, \eta, n, p)\) and \(U_{\text{WGPoisson}}(x, \lambda, \nu, \eta, \alpha)\) are the cumulative distribution functions of the random variable \(U_{\text{WGBinom}}\) respectively \(U_{\text{WGPoisson}}\).
Proof.

We examine the convergence in the terms of the maximum distribution $W_{GBinom}^U$ and $W_{GPoisson}^U x > 0$.

It is evident that:

$$
\lim_{n \to \infty} (1 - p)^n = \lim_{n \to \infty} \left(1 - p^{\frac{1}{n}}\right)^n = e^{-\alpha},
$$

$$
\lim_{n \to \infty} \left[1 - p + p \cdot F_{WG}(x, \lambda, \nu, \eta)\right]^n = \lim_{n \to \infty} \left[1 - p + p \cdot F_{WG}(x, \lambda, \nu, \eta)\right]^{\frac{1}{n}} = - p + p \cdot F_{WG}(x, \lambda, \nu, \eta)^n
$$

Under these conditions, we obtain:

$$
\lim_{n \to \infty} U_{WGBinom}(x, \lambda, \nu, \eta, n, p) = \lim_{n \to \infty} \frac{1 - p + p \cdot F_{WG}(x, \lambda, \nu, \eta) - (1 - p)^n}{1 - (1 - p)^n}
$$

$$
= e^{-\alpha(1 - F_{WG}(x, \lambda, \nu, \eta))} - e^{-\alpha} = U_{WGpoisson}(x, \lambda, \nu, \eta, \alpha).
$$

5. Special Cases

5.1 Max weibull uniform binomial distribution

In the first example, we consider that distribution $G$ is the uniform distribution $U$ on the interval $(0, \infty)$. Then

the probability density function is $g(x, \phi) = \frac{1}{\phi}$, $x \in (0, \phi)$ and the cumulative distribution functions

$G(x, \phi) = \frac{x}{\phi}$.

Max Weibull-Uniform binomial distribution has the cumulative distribution functions given by the relation:

$$
U_{WUBinom}(x, \lambda, \nu, \phi, n, p) = \frac{1 - p \exp \left[\frac{-x}{\lambda\phi - x}\right]^n}{1 - (1 - p)^n}, x \in (0, \phi),
$$

where $\lambda, \nu > 0$, $n \in [1, 2, \ldots]$, $p \in (0, 1)$. The corresponding probability density function is:
In these conditions the cumulative distribution functions and the probability density function which characterize the Max Weibull - Weibull Binomial distribution are the following:

\[
U_{WWBinom}(x, \lambda, \nu, a, b, n, p) = \begin{cases} 
1 - p \exp \left[ - \left( \frac{x}{\lambda} \right)^n \right] - (1-p)^n \\
1 - \nu \exp \left[ - \left( \frac{x}{\lambda} \right)^n \right] - (1-p)^n
\end{cases}, \ x > 0,
\]

with \( x > 0 \).

### 5.3 Max Weibull - Weibull binomial distribution

In this example we consider that distribution G is the same as Weibull distribution with the probability density function and the cumulative distribution functions characterized by the relations:

\[
g(x, a, b) = \frac{b}{a} \left( \frac{x}{a} \right)^{b-1} \exp \left[ - \left( \frac{x}{a} \right)^b \right], \ \ \ \ \ G(x, a, b) = 1 - \exp \left[ - \left( \frac{x}{a} \right)^b \right], \ a, b > 0 . \text{ In these conditions the cumulative distribution functions and the probability density function which characterize the Max Weibull - Weibull Binominal distribution are the following:}
\]

\[
U_{WWBinom}(x, \lambda, \nu, a, b, n, p) = \frac{\exp \left[ - \left( \frac{x}{\lambda} \right)^n \right] - \exp \left[ - \left( \frac{x}{a} \right)^b \right]}{1 - \nu \exp \left[ - \left( \frac{x}{\lambda} \right)^n \right] - (1-p)^n}, \ x > 0,
\]

with \( x > 0 \).
5.4 Max Weibull - Weibull Poisson distribution

For this distribution, the cumulative distribution functions and the probability density function are obtained by the Eq. (9) and Eq. (10), where \( G \) is the same as Weibull distribution:

\[
U_{\text{WWPoisson}}(x, \lambda, \nu, a, b, \alpha) = \frac{\exp\left(-\alpha \exp\left[-\left(\frac{x/\nu}{\lambda} - 1\right)^b\right]\right)}{1 - \exp(-\alpha)} , \quad x > 0
\]

and

\[
u_{\text{WWPoisson}}(x, \lambda, \nu, a, b, \alpha) = \frac{\alpha b u x^{b-1}\left[\exp\left(\frac{x}{\nu}\right) - 1\right] \exp\left[-\left(\frac{x/\nu}{\lambda} - 1\right)^b\right]}{\lambda^a [1 - \exp(-\alpha)] \left[1 - \exp\left(-\frac{x}{\nu}\right)\right]},
\]

with, where \( x > 0 \), where \( \lambda, \nu > 0 \), \( a, b, \alpha > 0 \).

Remark 5.1. In Figures 1–4 Theorem 4.1 is reflected in terms of distribution function and density of probability for both types of Max Weibull-G Binomial distributions, respectively Max Weibull-G Poisson, for different values associated with their parameters, where \( G \) is identified with uniform distribution (Figs. 1 and 2), respectively Weibull distribution (Figs. 3 and 4). The parameters for the Max Weibull - Uniform Binomial and the Max Weibull - Uniform Poisson distributions are \( \lambda \in \left\{ \frac{4}{9}, \frac{1}{\sqrt{4}} \right\} \), \( \nu \in \{0.5, 1, 1.5\} \), \( \eta = 4 \), \( n = 40 \), \( p = \frac{1}{4} \) and \( \alpha = 10 \) and for the Max Weibull - Weibull Binomial and the Max Weibull - Weibull Poisson distributions the parameters are \( \lambda = 1 \), \( \nu \in \{0.5, 1, 1.5\} \), \( a = 1.5 \), \( b = 0.5 \) \( n = 30 \), \( p = \frac{1}{3} \) and \( \alpha = 10 \).

Fig. 1. The Poisson limit theorem in relation with the cumulative distribution functions of the Max Weibull - Uniform Binomial distributions (cdfMaxWUB(\(x, \lambda, \nu, \phi, n, p\), \( x > 0 \)) and the Max Weibull - Uniform Poisson (cdfMaxWUP(\(x, \lambda, \nu, \phi, \alpha\), \( x > 0 \))), where \( G(x, \eta) = U(x, \phi) = \frac{x}{\phi}, \phi > 0, \eta = \phi \).
Fig. 2. The Poisson limit theorem in relation with the probability density function of the Max Weibull-Uniform Binomial distributions (pdfMaxWUB(t, λ, v, φ, n, p), t > 0) and the Max Weibull-Uniform Poisson (pdfMaxWUP(t, λ, v, φ, α), t > 0), where G(t, η) = U(t, φ) = \frac{1}{φ}, φ > 0, η = φ

Fig. 3. The Poisson limit theorem in relation with the cumulative distribution functions of the Max Weibull - Weibull Binomial distributions (cdfMaxWWB(z, λ, v, a, b, n, p), z > 0) and the Max Weibull -
Weibull Poisson $\text{cdf} \text{Max} \text{WWP}(z, \lambda, \nu, a, b, \alpha) \text{, } z > 0$, where

$G(z, \eta) = \text{Weibull}(z, a, b) = 1 - \exp \left[ -\left( \frac{z}{a} \right)^b \right], \text{ } a, b > 0, \eta = (a, b)$

Fig. 4. The Poisson limit theorem in relation with the probability density function pdf of the Max Weibull - Weibull Binomial distributions pdfMaxWWB$(y, \lambda, \nu, a, b, n, p)$, $y > 0$ and the Max Weibull - Weibull Poisson pdfMaxWWP$(y, \lambda, \nu, a, b, \alpha)$, $y > 0$, where

$G(y, \eta) = \text{Weibull}(y, a, b) = 1 - \exp \left[ -\left( \frac{y}{a} \right)^b \right], \text{ } a, b > 0, \eta = (a, b)$

6 Conclusion

In this study, a new class of power series distributions called Max Weibull-G has been analyzed. If the lifetime $(Y)$ of a system has a Max Weibull-G $(U_{WG})$ power series distribution, then the chance for a component with the maximum lifetime $Y$ to degrade at time $t$ is $U_{WG}(t, \lambda, \nu, \eta)/(1 - U_{WG}(t, \lambda, \nu, \eta))$. Thus, the moment when a system enters the state of degradation can be determined if we know the distribution of the maximum lifetime from all the other lifetimes that characterize the system.

The Poisson Limit Theorem has been formulated for situations when the random variables number of sum is a zero truncated binomial distribution and limit distribution the Poisson type distribution. The theorems are shown under the graphical representation in Figures 1, 2, 3 and 4.

The validation of the Max Weibull-G power series distributions through the EM algorithm is a new research direction.

Competing Interests

Author has declared that no competing interests exist.
References

[1] Tahir MH, Zubair M, Mansoor M, Cordeiro GM, Alizadeh M, Hamedani G. A new Weibull-G family of distributions, Journal of Mathematics and Statistics. 2016;45(2):629-647.

[2] Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications, Communications in Statistics–Theory and Methods. 2002;31:497–512.

[3] Cordeiro GM, de Castro M. A new family of generalized distributions”, Journal of Statistical Computation and Simulation. 2011;81:883–893.

[4] Alexander C, Cordeiro GM, Ortega EMM, Sarabia JM. Generalized betagenerated distributions, Computational Statistics and Data Analysis. 2012;56:1880–1897.

[5] Zografos KN, Balakrishnan. On families of beta - and generalized gamma-generated distributions and associated inference, Statistical Methodology. 2009;6:344–362.

[6] Amini M, Mir Mostafae SMTK, Ahmadi J. Log-gamma-generated families of distributions, Statistics. 2014;48:913–932.

[7] Risti´c MM, Balakrishnan N. The gamma-exponentiated exponential distribution, Journal of Statistical Computation and Simulation. 2012;82:1191–1206.

[8] Cordeiro GM, Ortega EMM, da Cunha DCC. The exponentiated generalized class of distributions”, Journal of Data Science. 2013;11:1–27.

[9] Alzaatreh A, Lee C, Famoye F. A new method for generating families of continuous distributions, Metron. 2013;71:63–79.

[10] Alzaghal A, Famoye F, Lee C. Exponentiated T-X family of distributions with some applications, International Journal of Probability and Statistics. 2013;2:31–49.

[11] Bourguigno M, Silva BR, Cordeiro MG. The Weibull-G family of probability distributions”, Journal of Data Science. 2014;12:53–68.

[12] Cordeiro GM, Alizadeh M, Ortega EMM. The exponentiated half-logistic family of distributions: Properties and applications, Journal of Probability and Statistics Article ID 864396; 2014.

[13] Aljarrah MA, Lee C, Famoye F. On generating T-X family of distributions using quantile functions, Journal of Statistical Distributions and Applications. 2014;1(2).

[14] Alzaatreh A, Lee C, Famoye F. T-normal family of distributions: A new approach to generalize the normal distribution, Journal of Statistical Distributions and Applications. 2014;1(16).

[15] Cordeiro GM, Ortega EMM, Popovi´c BV, Pescim RR. The Lomax generator of distributions: Properties, minification process and regression model, Applied Mathematics and Computation. 2014;247:465–486.

[16] Tahir MH, Cordeiro GM, Alzaatreh A, Mansoor M, Zubair M. The LogisticX family of distributions and its applications, Communications in Statistics–Theory and Methods. 2015;45(24):7326-7349.

[17] Alizadeh M, Emadi M, Doostparast M, Cordeiro GM, Ortega EMM, Pescim RR. A new family of distributions: the Kumaraswamy odd log-logistic, properties and applications, Hacettepa Journal of Mathematics and Statistics. 2015;44:1491–1512.
[18] Gurvich MR, DiBenedetto AT, Ranade SV. A new statistical distribution for characterizing the random strength of brittle materials, Journal of Materials Science. 1997;32:2559-2564.

[19] Johnson NL, Kemp AW, Kotz S. Univariate Discrete Distribution, New Jersey; 2005.

[20] Leahu A, Gh B, Munteanu, Cataranciuc S. On the lifetime as the maximum or minimum of the sample with power series distributed size, Romai J. 2013;9(2):119-128.

© 2021 Munteanu; This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle4.com/review-history/69851