Symplectic Approach of Wess-Zumino-Witten Model
and Gauge Field Theories

Wang Bai-Ling

Department of Mathematics, Peking University
Beijing 100871, P. R. China

Abstract

A systematic description of the Wess-Zumino-Witten model is presented. The symplectic method plays the major role in this paper and also gives the relationship between the WZW model and the Chern-Simons model. The quantum theory is obtained to give the projective representation of the Loop group. The Gauss constraints for the connection whose curvature is only focused on several fixed points are solved. The Kohno connection and the Knizhnik-Zamolodchikov equation are derived. The holonomy representation and $\tilde{R}$-matrix representation of braid group are discussed.

\footnote{Present address: Department of Pure mathematics, University of Adelaide, Adelaide, SA5005, Australia}
Content

§0. Introduction
§1. Classical Phase Space of Wess-Zumino-Witten Model
§2. Symplectic Theory of WZW Model
§3. Geometrical Quantization of Wess-Zumino-Witten Model
§4. Moment Map of Gauge Field Theory
§5. Kohno Connection and Knizhnik-Zamolodchikov Equation
§6. Relations to $R$-Matrix and Quantum Group

§0. Introduction

In recent years, the conformal field theories and topological field theories are the most interesting topics for both mathematicians and physicists. The conformal field theories were initiated by Belavin, Polyakov and Zamolodchikov [4] as 2-dimensional quantum field theory describing the 2-dimensional critical phenomena. Conformal field theory is characterized by the symmetries such as Kac-Moody and Virasoro algebra and its correlation functions are characterized by differential equations arising from the representation of infinite dimensional Lie algebras. A field is called topological if it is defined for smooth manifolds with no additional structures. As in Atiyah’s paper [1], he axiomatized the topological field theories and gave the several outlines to specify the theories. One aspiring paper may be the Witten’s [19], who interpreted the Jones polynomial of knots in terms of topological field theories (Chern-Simons theory), where he also suggested the relations between the Chern-Simons theory and the Wess-Zumino-Witten theory.

The Wess-Zumino-Witten model [20] are principal $\sigma$ model with a topological Wess-Zumino term in the action, which was first introduced by Witten. There have being tremendous activities inspired by his paper. I would like to mention Kohno’s paper [13], where he derived the Kohno connection and its relations with the Knizhnik-Zamolodchikov equation, $R$-matrix, and quantum group. Quantum groups are expected to be the mathematical framework to describe the symmetry properties of conformal field theory and other integrable models [5-7,9,15,17,18].

For conformal field theories, the WZW model is the key point to understand the rational conformal field theories and their hidden symmetries. The basis field $g(\xi)$ is a field with values in a compact, semisimple Lie group $G$, its action is given by (see section 1):

$$S(g) = -\frac{i}{4\pi} \int_{\Sigma} Tr(g^{-1}\partial gg^{-1}\overline{g}) - \frac{i}{12\pi} \int_{B} Tr(\tilde{g}^{-1}d\tilde{g}^{-1} \wedge \tilde{g}^{-1}d\tilde{g}^{-1} \wedge \tilde{g}^{-1}d\tilde{g}^{-1})$$
the first term is the nonlinear $\sigma$-model's action, the second term is a topological term, which is the integration of the generator of $H^3(G, \mathcal{Z})$, when $G$ is connected and simple connected. This term is called the Wess-Zumino term. Through the transgression (see proposition 1 of section 2), one can obtain a 2-form on a loop space, which is in the same homology class with symplectic form of the WZW model. It also leads to the equivalence between the Wess-Zumino-Witten and the Chern-Simons model.

The Chern-Simons model is a 2+1 dimensional topological field theory with the action:

$$CS(A) = \frac{i}{2\pi} \int_Y Tr(AdA + \frac{2}{3}A^3)$$

which is the integration of a 3-form over a three dimensional manifold $Y$ and equal to the Yang-Mills action over a four dimensional manifold with boundary $Y$ by the Stokes formula, where $A$ is a $G$-connection of $G$-bundle over $Y$. This theory is important because the invariants of manifolds and knots in three dimensional case can be obtained through the physics calculation. There is a great surprise, since this 2+1 topological theory has an intrinsic connection with the 1+1 conformal field theory. In general, one can sets $\Sigma = \partial Y$ and reduces the connection space of three dimensional case to the two dimensional case. Using the Yang-Mills theory on Riemann surfaces [3], we can get the further results about the gauge field theory and the conformal field theory.

In section 1, we introduce the WZW model and the Chern-Simons model, derive the symplectic form on the phase space of the WZW model from the Chern-Simons model. Under certain conditions, the actions of these two models are equal. For the Chern-Simons model, we use the natural symplectic form described in M. Atiyah and R. Bott's paper [3].

In section 2, we investigate the intrinsic geometry of loop group. Under the transgression isomorphism of proposition 1:

$$H^3(G, \mathcal{R}) \cong \mathcal{H}^c(\otimes G, \mathcal{R})$$

We know why the WZW model is equivalent to the Chern-Simons model from the point of topology. We also show that the Kirillov form is not the needed one. In finite dimensional case, the natural symplectic on a co-adjoint orbit is the Kirillov form [12,13], while, for the infinite dimensional case such as the WZW model, the Kirillov form is degenerate. For latter use, we also quote some results from A. Pressley and G. B. Segal's book [16]. The main result in this section is classical
description of Kac-Moody and Virasoro symmetries:
\[
\{ J^n_a, J^m_b \} = f_{c}^{ab} J^n_{c+m} - im\delta_{n,-m}\delta^{ab}
\]
\[
\{ E, J^n_a \} = -inJ^n_a
\]
\[
\{ l_n, l_m \} = (n - m)l_{n+m}
\]
where the Poisson bracket is defined by the symplectic form \( \omega \) on \( \Omega G \). In section 3, we use the geometric quantization to give the representation of loop group and quantize the WZW model.

In section 4, we discuss the moment map of gauge field theory under the three different circumstances: Riemann surfaces without boundary, with one dimensional boundary, with \( n \) fixed punctures. The third case is most interesting, because it leads to the deep relations between the gauge field theory and the conformal field theory with gauge symmetries. The main results is to derive the Gauss constraints for the first and third cases by the Marsden-Weinstein reduction. We obtain that, for the latter case, the Gauss constraints are
\[
F^a(z, \bar{z}) = -\frac{4\pi}{k} \sum_{i,a} \delta(z, P_i) T^a_i
\]
and also solve this Gauss constraints by using the Green function.

In section 5, we derive the Kohno connection and prove that this connection is flat, its holonomy gives rise to the representation of braid group, which is the fundamental group of configuration space, the parallel displacement gives out the solution of the Knizhik-Zamolodchikov equation (KZ equation). This KZ equation was obtained from Ward identity in [4] and has a variety of properties which are the fundamental theories for the conformal theory.

In the last section, we give a general description about the representation of braid group by using the solution of the quantum Yang-Baxter equation, \( R \)-matrix, that is,
\[
R = \sum_{i \neq j} e_{ii} \otimes e_{jj} + q^{\frac{i}{2}} \sum_i e_{ii} \otimes e_{ii} + (q^{\frac{i}{2}} - q^{-\frac{i}{2}}) \sum_{i < j} e_{ij} \otimes e_{ji}
\]
and show that this representation can be factored through the Terperley-Lieb-Jones algebra:
\[
\begin{align*}
\nu_i \nu_{i \pm 1} \nu_i &= \nu_{i \pm 1} \nu_i \nu_{i \pm 1} \\
\nu_i \nu_j &= \nu_j \nu_i \quad | i - j | \geq 2 \\
\nu_i^2 &= (1 - q) \nu_i + q
\end{align*}
\]
where \( q = \exp(-i\frac{2\pi}{k+h}) \). Due to Kohno, this representation is equivalent to the holonomy representation of braid group.
1. Classical Phase Space of Wess-Zumino-Witten Model

In the 1970’s, the non-linear $\sigma$ model played an important role in the infinite conservative flow and the Kac-Moody symmetry of dynamic systems. Mathematically, this is equivalent to the minimum surface theory of a Riemann surface embedded in a homogeneous space of a certain Lie group, especially to a Lie group. E. Witten in [20] discussed a model whose action is the non-linear $\sigma$ model plus a topological term called the Wess-Zumino term. From then on, many interesting phenomena occurred in both mathematics and physics.

In this section, we will give a general description of the conformal field theory (Wess-Zumino-Witten model) and its relations to the 3-dimension gauge theory (Chern-Simons model). The symplectic structure of the phase space is derived from these relations.

Let $\Sigma$ be a Riemann surface, $G$ a compact Lie group with Lie algebra $\mathcal{G}$ which has a $G$-invariant nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ on $\mathcal{G}$, sometimes also denoted by $\text{Tr}$ for convenience. The non-linear $\sigma$ model is defined by the following action:

$$I(g) = -\frac{i}{4\pi} \int_{\Sigma} \langle g^{-1}\partial g, g^{-1}\overline{\partial} g \rangle$$

where $\partial = \frac{\partial}{\partial z} dz, \overline{\partial} = \frac{\partial}{\partial \bar{z}} d\bar{z}$ and $(z, \bar{z})$ are the standard complex coordinates on $\Sigma$. It is known that there is a right and left bi-invariant, integral closed 3-form on $G$:

$$\sigma = \frac{1}{12} \text{Tr}(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg)$$

This term has a significant meaning from the view of topology which is the generator of $H^3(G, \mathbb{Z})$ when $G$ is a connected, simply connected compact Lie group. It represents the origin of the topological effects in low dimension field theory. The Wess-Zumino-Witten model is given by the following action:

$$S(g) = I(g) - \frac{i}{12\pi} \int_{B} Tr(\tilde{g}^{-1}d\tilde{g}^{-1} \wedge \tilde{g}^{-1}d\tilde{g}^{-1} \wedge \tilde{g}^{-1}d\tilde{g}^{-1})$$

where $\tilde{g}$ is an arbitrary extension from $\Sigma$ to $B$ and $B$ is a three dimensional manifold whose boundary is $\Sigma$. Due to the closed integral 3-form $\sigma$, a different extension leads to a different action by $2k\pi i$ for an integer $k$. So as a complex-valued function on the phase space, which consists of all the smooth maps from $\Sigma$ to $G$, $e^{kS(g)}$ is well-defined. One will see that the Euler-Lagrange equation is also not affected by the different extension.

**Theorem 1**: The Euler-Lagrange equation of classical Wess-Zumino-Witten model is given by

$$\overline{\partial}(\partial gg^{-1}) = 0 \iff \partial(g^{-1}\overline{\partial} g) = 0$$
Proof: Let $g_t : \Sigma \rightarrow G$ be a family of mappings depending on $t$ through a fixed map $g$. The first variation of $S(g_t)$ is given by

$$\frac{\partial}{\partial t} |_{t=0} S(g_t) = -\frac{i}{4\pi} \int_{\Sigma} \frac{\partial}{\partial t} |_{t=0} \langle g_t^{-1} \partial g_t, g_t^{-1} \overline{\partial} g_t \rangle$$

$$- \frac{i}{12\pi} \int_B \frac{\partial}{\partial t} |_{t=0} \text{Tr}(\overline{\partial} g_t^{-1} \wedge d\overline{\partial} g_t^{-1} \wedge d\overline{\partial} g_t^{-1})$$

$$= -\frac{1}{4\pi} \int_{\Sigma} \langle -2(\partial (\partial g g^{-1})), (\frac{\partial}{\partial t} g_t^{-1} |_{t=0}) \rangle$$

We then obtained that for any $\frac{\partial}{\partial t} |_{t=0} S(g_t) = 0$ if and only if $\frac{\partial}{\partial t} g_t^{-1} |_{t=0} = 0$. This is equivalent to $\overline{\partial} (g^{-1} \overline{\partial} g) = 0$.

By Theorem 1, if $g : \Sigma \rightarrow G$ is a critical point of $S(g)$, then $\partial g^{-1}$ is a holomorphic $G$-valued $(1,0)$-form on $\Sigma$, equivalently, $g^{-1} \overline{\partial} g$ is an anti-holomorphic $G$-valued $(0,1)$-form on $\Sigma$. Because the action $S(g)$ is irrelevant to the specific complex structure on $\Sigma$, the critical point set of $S(g)$ is invariant under the conformal transformation of $\Sigma$. These lead to the conformal symmetry of the Wess-Zumino-Witten model.

In order to investigate the Kac-Moody symmetry of the WZW model, we resort to the symplectic techniques in Hamiltonian mechanics. For details, see [14]. So we need to find out the symplectic structure of the classical phase space.

For greater convenience, we take the Riemann surface $\Sigma$ as a disc $D$ with polar-coordinates $(\rho, \theta)$, i.e. $z = \rho e^{i\theta}$. For a $C^\infty$-map $g : D \rightarrow G$, we construct a $G$-valued 1-form on $D$, $A = dgg^{-1}$. The affine space, denoted $A$, consists of the connection 1-form $A = dgg^{-1}$ derived from a $C^\infty$-map $g : D \rightarrow G$. We constrict the gauge transformation with some boundary conditions, that is, for a gauge transformation $h : D \rightarrow G$, $h$ must be an identity on the boundary of the disc. Then the two elements $dgg^{-1}, dhh^{-1}$ are equivalent under a certain gauge transformation, if and only if $g = h$ on the boundary of $D$. Moreover, the connection 1-form is invariant under the constant gauge transformations by elements of $G$. So the equivalent $G$-valued 1-forms on $D$ is made up of the phase space which is isomorphic to $LG/G$, where $LG = C^\infty(S^1, G)$, $S^1$ is a unit circle. $LG$ is called a Loop group which has an outstanding theory described by A.Pressley and G. Segal in their book [16].

From the above, we presume that $LG/G$ is the phase space of the Wess-Zumino-Witten model. The symplectic form can also be obtained from the symplectic form of the gauge field theory on the Riemann surface.

For any $A \in A$, the tangent space of $A$ at $A$ is still $A$ because of its affine property. Let $\tilde{\xi}, \tilde{\eta} \in A$, then $\tilde{\xi}, \tilde{\eta}$ are two $G$-valued 1-forms on $D$. The natural symplectic form on $A$ is given by

$$\omega_A(\tilde{\xi}, \tilde{\eta}) = \frac{1}{2\pi} \int_D \text{Tr}(\tilde{\xi} \wedge \tilde{\eta})$$
This symplectic form is invariant under the gauge transformation. If we take \( \tilde{\xi}, \tilde{\eta} \) as the differential forms derived from two \( \mathcal{G} \)-valued functions \( \xi, \eta \) on \( D \), then
\[
\omega_A(\tilde{\xi}, \tilde{\eta}) = \frac{1}{2\pi} \int_D Tr(d\xi \wedge d\eta) = \frac{1}{2\pi} \int_{\partial D} Tr(\xi d\eta) = \frac{1}{2\pi} \int_{S^1} <\xi(\theta), \eta'(\theta)> d\theta
\]

So the reduced symplectic form on \( LG/G = \Omega G \), which is still a group, is an invariant 2-form on \( \Omega G \) under the left-transportation, whose value at the unit point is given by
\[
\omega_e(\xi, \eta) = \frac{1}{2\pi} \int_{S^1} <\xi(\theta), \eta'(\theta)> d\theta
\]
where \( \xi, \eta \in T_e(\Omega G) = \mathcal{C}^\infty(S^\infty, \mathcal{G})/\mathcal{G} \) are two loops through zero in \( \mathcal{G} \). As we know, this symplectic form is precisely the canonical symplectic form on \( \Omega G \) as discussed in [16].

We call \( \partial gg^{-1} \) and \( g^{-1}\tilde{\partial}g \) as the left chiral current and the right chiral current respectively. The phase space which consists of the left chiral current is just \( \Omega G \) with its canonical symplectic structure as we have shown. For the right chiral current, the procedure is similar, but the left invariant symplectic structure should be changed to the right invariant one on the same phase space \( \Omega G \).

Remark: It is not strange to derive the symplectic structure of the conformal field theory (Wess-Zumino-Witten model) from the gauge field theory, partly because they have a natural relation as described above, and partly because we have an intrinsic formula between the action of the WZW model and the action of a special gauge field model (Chern-Simons model).

Let us say a few words about the Chern-Simons model. It is a particular example of the topological gauge field theory, which is based on a gauge field \( A = A^a_\mu T^a dx^\mu \) in Lie algebra \( \mathcal{G} \) with the action
\[
CS(A) = \frac{i}{4\pi} \int_Y Tr(AdA + \frac{2}{3} A^3)
\]
for a three dimensional manifold \( Y \), where \( T^a \) is the orthogonal basis with respect to \( G \)-invariant bilinear form. \( S(A) \) is irrelevant to the metric of \( Y \). It defines a topological action. Set \( Y = D \times \mathcal{R} \), \( D \) is a unit disc with polar coordinates \(( \rho, \theta) \), where \( t \) is a coordinate of \( \mathcal{R} \). Then the \( d \)-operator on \( Y \) can be expressed as \( \frac{d}{dt} + \tilde{d} \) with \( \tilde{d} \) the operator on \( D \). For \( g(t): D \rightarrow G \) is a \( t \)-dependance \( \mathcal{C}^\infty \) map, \( A = -\tilde{d}g(t)g(t)^{-1} \) is a flat connection of the trivial bundle \(( D \times \mathcal{R}) \times \mathcal{G} \), we have \( CS(A) = SWZW(g(t)) \).
Proof:

\[ CS(A) = \frac{i}{4\pi} \int_{D \times \mathbb{R}} \text{Tr}(A dA + \frac{2}{3} A^3) \]
\[ = \frac{i}{4\pi} \int_{D \times \mathbb{R}} \text{Tr}(A \frac{\partial A}{\partial t} dt) \]
\[ = \frac{i}{4\pi} \int_{D \times \mathbb{R}} \text{Tr}(\tilde{d}g(t)g(t)\frac{-1}{\partial t} \tilde{d}g(t)g(t)^{-1} dt) \]
\[ = \frac{i}{4\pi} \int_{D \times \mathbb{R}} \text{Tr}(\tilde{d}g(t)g(t)^{-1} \tilde{d}g(t)\frac{-1}{\partial t} g^{-1}(t) dt) \]
\[ = \frac{i}{4\pi} \int_{D \times \mathbb{R}} \text{Tr}(\tilde{d}g(t)^{-1} \tilde{d}g(t)\frac{-1}{\partial t} g^{-1}(t) dt) \]
\[ = -\frac{i}{4\pi} \int_{\partial D \times \mathbb{R}} <g(t)^{-1} \frac{\partial g(t)}{\partial \theta} d\theta, g(t)^{-1} \frac{\partial g(t)}{\partial t} dt > -\frac{i}{12\pi} \int_{D \times \mathbb{R}} \text{Tr}(dgg^{-1})^3 \]
\[ = S_{WZW}(g) \]

§2. Symplectic Theory of WZW Model

In this section, we will give a further study of the phase space and the symplectic structure from the point of the coadjoint orbit theory. The left chiral current can be viewed as an infinite function on the phase space and the conformal symmetry has its role on the phase space. We also pave the way for a geometrical quantization of the conformal field theory which will give rise to representations of the Kac-Moody group and its Lie algebra, along with the representation of the Virasoro algebra. We will discuss these in the next section.

ΩG = LG/G can be viewed as loops through the unit point in G, its Lie algebra is LG/G, where LG = C\(\infty\)(S\(\infty\), G). Another way to view this phase space is that it can be expressed as a coadjoint orbit of a certain group, which is the central extension of the Loop group LG = C\(\infty\)(S\(\infty\), G).

We define the central extension of the loop algebra LG = C\(\infty\)(S\(\infty\), G), as in [16], by a 2-cocycle \(\omega\) which is exactly \(\omega_e\) obtained in last section. Specifically, let \(LG = LG \oplus \mathcal{R}\), its Lie bracket is given by

\[ [(\xi, \lambda_1), (\eta, \lambda_2)] = ([\xi, \eta], \omega(\xi, \eta)) \]

where \(\omega(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi(\theta), \eta'(\theta) \rangle d\theta \).

The group corresponding to LG is the U(1)-extension of the Loop group LG, which is U(1) principal bundle on LG with its first Chern class [\(\omega\)]. For more details, see [16].

The following proposition gives the relation between the topological term of WZW action and the symplectic structure on ΩG which defines the central extension.

**Proposition 1**: Under the transgression isomorphism:

\[ \tau : H^3(G, \mathcal{R}) \longrightarrow \mathcal{H}^e(\otimes \mathcal{G}, \mathcal{R}) \]
we have $\tau(\sigma) = \omega$. That is, for $\xi, \eta \in T \gamma (\Omega G)$,

$$\tau(\sigma)(\xi, \eta) = \frac{1}{2\pi} \int_0^{2\pi} \langle \gamma^{-1}(\theta) \gamma'(\theta), [\gamma^{-1} \xi(\theta), \gamma^{-1}(\theta) \eta(\theta)] \rangle d\theta$$

which is different from $\omega$ by an exact form $d\beta$, where $\beta$ is 1-form on $\Omega G$ and for $\xi(\theta) \in T \gamma (\Omega G)$

$$\beta(\xi) = \frac{1}{4\pi} \int_0^{2\pi} \langle \gamma^{-1}(\theta) \gamma'(\theta), \gamma^{-1}(\theta) \xi(\theta) \rangle d\theta$$

**Proof:** First, we recall the transgression $\tau$. There is an evaluation map $\phi: S^1 \times \Omega G \rightarrow G$ such that a 3-form $\sigma$ on $G$ can be pulled back to $\Omega G \times S^1$, then it can be integrated over $S^1$. One then obtains $\tau(\sigma)$.

$$\phi^* (\sigma) \big( \frac{\partial}{\partial \theta}, \xi(\theta), \eta(\theta) \big) = \frac{1}{2\pi} \langle \gamma^{-1}(\theta) \gamma'(\theta), [\gamma^{-1} \xi(\theta), \gamma^{-1}(\theta) \eta(\theta)] \rangle$$

Integrating it over $S^1$, we obtain $\tau(\sigma)$.

Second, we need to prove $\omega = \tau(\sigma) - d\beta$. By the definition of $d\beta$, we have

$$d\beta, (\xi(\theta), \eta(\theta)) = \xi(\theta). \beta(\eta(\theta)) - \eta(\theta). \beta(\xi(\theta)) - \beta([\xi(\theta), \eta(\theta)])$$

Let $\gamma_t(\theta)$ be a curve through $\gamma_0(\theta) = \gamma$ with $\frac{d}{dt} |_{t=0} \gamma_t(\theta) = \xi(\theta)$, so

$$\xi(\theta). \beta(\eta(\theta)) = \xi(\theta). \int_0^{2\pi} \langle \gamma^{-1} \gamma', \gamma^{-1} \eta(\theta) \rangle d\theta$$

$$= \frac{d}{d\theta} |_{t=0} \int_0^{2\pi} < \gamma_t^{-1} \gamma', \gamma_t^{-1} \eta(\theta) > d\theta$$

$$= - \int_0^{2\pi} (\langle \gamma^{-1} \xi(\theta) \gamma^{-1} \gamma', \gamma^{-1} \eta(\theta) \rangle + \langle \gamma^{-1} \gamma', \gamma^{-1} \xi, \gamma^{-1} \eta \rangle > d\theta$$

Similarly, let $\gamma_t(\theta)$ be a curve through $\gamma_0(\theta) = \gamma$ with $\frac{d}{dt} |_{t=0} \gamma_t(\theta) = \eta(\theta)$, then

$$\eta(\theta). \beta(\xi(\theta)) = \eta(\theta). \int_0^{2\pi} \langle \gamma^{-1} \gamma', \gamma^{-1} \xi(\theta) \rangle d\theta$$

$$= \frac{d}{d\theta} |_{t=0} \int_0^{2\pi} < \gamma_t^{-1} \gamma', \gamma_t^{-1} \xi(\theta) > d\theta$$

$$= - \int_0^{2\pi} (\langle \gamma^{-1} \eta(\theta) \gamma^{-1} \gamma', \gamma^{-1} \xi(\theta) \rangle + \langle \gamma^{-1} \gamma', \gamma^{-1} \eta, \gamma^{-1} \xi \rangle > d\theta$$

Therefore:

$$\frac{d}{d\theta} (\xi(\theta), \eta(\theta))$$

$$= \frac{1}{4\pi} \int_0^{2\pi} (\langle \gamma^{-1} \xi, \gamma^{-1} \eta \rangle - \gamma^{-1} \gamma', \gamma^{-1} \eta \rangle + \langle \gamma^{-1} \gamma', \gamma^{-1} \xi, \gamma^{-1} \eta \rangle ) d\theta$$

$$= - \omega(\xi, \eta) + \frac{1}{2\pi} \int_0^{2\pi} < \gamma^{-1} \gamma', \gamma^{-1} \xi, \gamma^{-1} \eta \rangle > d\theta$$

$$= - \omega(\xi, \eta) + \tau(\sigma)(\xi, \eta)$$
Thus we obtain $\omega(\gamma) = \tau(\sigma)(\gamma) - d\beta(\gamma)$.

**Proposition 2:** (1) The adjoint action of $LG$ on $\tilde{LG}$ is given by

$$Ad\gamma.(\xi, \lambda) = (Ad\gamma.\xi, \lambda - <\gamma^{-1}\gamma', \xi>)$$

where $Ad\gamma.\xi$ denotes the adjoint action of $\gamma \in LG$ on $\xi \in LG$.

(2) The coadjoint action of $LG$ on $(\tilde{LG})^*$ is given by

$$Ad^*\gamma.(\phi, \lambda) = (Ad^*\gamma.\phi + \lambda\gamma'\gamma^{-1}, \lambda)$$

where $Ad^*\gamma.\phi$ denotes the coadjoint action of $\gamma \in LG$ on $\phi \in (LG)^*$.

(3) The coadjoint orbit through $(0,1)$, which belongs to $(\tilde{LG})^*$, is isomorphic to $\Omega G$.

**Proof:** (1) and (2) are proved in [16]. Therefore, from (2), we know that the coadjoint orbit through $(0,1)$ is

$$\{Ad^*\gamma.(0,1) \mid \gamma \in LG\} = \{(\gamma'\gamma, 1)^{-1} \mid \gamma \in LG\} \cong \Omega G$$

In fact, $\Omega G$ is a Kähler manifold stated in the following theorem. It seems similar to the finite dimensional case which, in that case, a regular coadjoint orbit of a compact Lie group is a Kähler manifold whose Kähler form is precisely the Krillov form, but there is a slight difference here. The Krillov form on $\Omega G$ is $\tau(\sigma)$, which is degenerate, while $\omega_\gamma$ is the correct Kähler form. In proposition 1, we have given the exact difference.

**Lemma:** ([16]) If we define a left invariant almost complex structure on $\Omega G$ by $J_g = l_g^*J_e$, where $J_e$ valued at the unit point is given by

$$J_e : T^C_e(\Omega G) \longrightarrow T^C_e(\Omega G) = LG_c/G_c$$

$$J_e(\sum_{k \neq 0} \xi_k z^k) = \sum_{k \neq 0} isign(k)\xi_k z^k$$

then $J$ is integral and compatible with $\omega$ in the following meanings:

$$\omega(J\xi, J\eta) = \omega(\xi, \eta)$$

$$\omega(\xi, J\xi) \geq 0$$

where $\omega(\xi, J\xi) = 0$ if and only if $\xi = 0$.

The proof is straightforward, therefore $(\Omega G, \omega_L, J_L)$ is a Kähler manifold. Let $\{T_a\}_{a=1}^{dimG}$ be a unitary orthogonal basis of $G$. By the Fourier expansion,

$$\gamma'(\theta)\gamma^{-1}(\theta) = \sum_{n=-\infty}^{n=+\infty} J_n^a(\gamma)T_a e^{-in\theta}$$
where $J_n^a(\gamma) \in \mathcal{C}^\infty(\otimes \mathcal{G})$ is given by

$$J_n^a(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} < T_a e^{i\theta}, \gamma'(\theta)\gamma^{-1}(\theta) > d\theta$$

In addition, set

$$E(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} < \gamma'(\theta)\gamma^{-1}(\theta), \gamma'(\theta)\gamma^{-1}(\theta) > d\theta$$

$$l_n(\gamma) = \frac{1}{4\pi} \int_0^{2\pi} < \gamma'(\theta)\gamma^{-1}(\theta)e^{i\theta}, \gamma'(\theta)\gamma^{-1}(\theta) > d\theta$$

**Theorem 2**: (1) For a smooth function $f$, its Hamiltonian vector field $X_f$ is given by $X_f|_\omega = -df$, then

$$X_{J_n^a(\gamma)} = -R_{x^a} e^{i\theta} T_a$$

$$X_{l_n(\gamma)} = -e^{i\theta} \gamma'(\theta)$$

(2) Assume $[T^a, T^b] = f_c^{ab} T^c$. We give the Poisson bracket of $\mathcal{C}^\infty(\otimes \mathcal{G})$ by

$$\{ f, g \}(\gamma) = \omega(\gamma)(X_f, X_g)$$

for $f, g \in \mathcal{C}^\infty(\otimes \mathcal{G})$

(a) The Poisson algebra $\text{Span}_C \{ J_n^a, 1, E \}$ is isomorphic to the Kac-Moody algebra $\tilde{L}_C^G = L_C^\infty \otimes C \otimes C_{-1}$, that means,

$$\{ J_n^a, J_m^b \} = f_c^{ab} J_n^{a+m} - i m \delta_{n,-m} \delta^{ab}$$

$$\{ E, J_n^a \} = -i n J_n^a$$

(b) $l_n = \sum_{a,m} J_{a-m}^a J_m^a$, \quad $\{ l_n, l_m \} = (n-m) l_{n+m}$

**Proof**: (1) Let $\xi(\theta) \in T_{\gamma(\theta)}(\Omega G)$, then one can choose $\gamma_t(\theta)$, for small $t$, depending on $t$, such that $\frac{\partial}{\partial t} |_{t=0} \gamma_t(\theta) = \xi(\theta)$, then

$$d J_n^a(\gamma)(\xi(\theta)) = \xi(\theta) J_n^a(\gamma)$$

$$= \frac{\partial}{\partial t} \bigg|_{t=0} J_n^a(\gamma_t(\theta))$$

$$= \frac{\partial}{\partial t} \bigg|_{t=0} \left( \frac{1}{2\pi} \int_0^{2\pi} < T^a e^{i\theta}, \gamma_t^{-1}(\theta)\gamma_t'(\theta) > d\theta \right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} < \gamma^{-1}(\theta)\gamma'(\theta)e^{i\theta}, \gamma^{-1}(\theta)\gamma'(\theta) > d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} < Ad_\gamma(\theta)^{-1} T_a e^{i\theta}, (\gamma^{-1}(\theta)\xi(\theta))' > d\theta$$

$$= \omega(X_{J_n^a}(\gamma), \xi(\theta))$$

$$\xi(\theta). l_n(\gamma) = \frac{\partial}{\partial t} \bigg|_{t=0} l_n(\gamma_t(\theta))$$

$$= \frac{\partial}{\partial t} \bigg|_{t=0} \left( \frac{1}{2\pi} \int_0^{2\pi} < \gamma_t^{-1}(\theta)\gamma_t'(\theta)e^{i\theta}, \gamma_t^{-1}(\theta)\gamma_t'(\theta) > d\theta \right)$$

$$= \frac{1}{2\pi} \int_0^{2\pi} < \gamma^{-1}(\theta)\gamma'(\theta)e^{i\theta}, (\gamma^{-1}(\theta)\xi(\theta))' > d\theta$$

$$= \omega(\gamma^{-1}(\theta)\gamma'(\theta)e^{i\theta}, \gamma^{-1}(\theta)\xi(\theta))$$

$$= \omega(\gamma)(\gamma'(\theta)e^{i\theta}, \xi(\theta))$$

11
Therefore:

\[ X_I^\alpha(\gamma) = -l_{\gamma(\theta)\theta} Ad\gamma(\theta)^{-1} T_a e^{\theta} \]

\[ = -R_{\gamma(\theta)\theta} T_a e^{\theta} \]

\[ X_L^\alpha(\gamma) = -e^{\theta} \gamma' \theta \]

(2) (a) By the definition of the Poisson Bracket on \( \Omega G \),

\[ \{ J_a^b, J_m^b \} (\gamma) \]

\[ = \omega(\gamma)(X_{J_a^b}, X_{J_m^b}) \]

\[ = -\frac{1}{2\pi} \int_0^{2\pi} (Ad\gamma^{-1}(\theta).e^{\theta T_a}, Ad\gamma^{-1}(\theta).e^{\theta T_b}) > d\theta \]

\[ = -\frac{1}{2\pi} \int_0^{2\pi} < Ad\gamma^{-1}(\theta).e^{\theta T_a}, Ad\gamma^{-1}(\theta).e^{\theta T_b} > d\theta \]

\[ = -im\delta_{a-m}\delta_{\theta} + \frac{1}{2\pi} \int_0^{2\pi} < \gamma^{-1}(\theta)\gamma'(\theta), Ad\gamma^{-1}(\theta)[e^{\theta T_a}, e^{\theta T_b}] > d\theta \]

\[ = \int e_{\theta} f_{a}^b J_{e}^b \gamma^{-1}(\theta) > d\theta - im\delta_{a-m}\delta_{\theta} \]

the proof of (b) is similar.

Using this theorem, the symmetry of the Kac-Moody algebra and the conformal algebra can be seen as two Poisson algebras which consist of the classical observable functions. It also implies that after the geometrical quantization, these observable functions will become operators which give rise to the representation of the Kac-Moody algebra and the Virasora algebra.

§3. Geometrical Quantization of Wess-Zumino-Witten Model

Geometrical quantization is a method used to quantify a symplectic manifold \( M \) with an integral symplectic form \( \omega \) which will be the curvature of a line bundle \( L \) over \( M \). To quantify \( M \) is to pick a complex Kähler structure such that \( L \) is a holomorphic line-bundle and the quantum Hilbert space is then taken to be the space of holomorphic sections of \( L \). It is well known that if \( M \) is taken to be a coadjoint orbit through an integral weight of a compact Lie group, the corresponding quantum Hilbert space is precisely the representation space of a Lie group as described in the Borel-Weil theorem [12,13].

From the proceeding sections, we have paved the way for the geometrical quantization. In this section, we will accomplish this procedure to give the representation
of a Loop group which has a complete representation theory that may be found in [16].

We reiterate the main results in section 2. The phase space of the WZW model is a Kähler manifold $\Omega G$ with the integral symplectic form $\omega$. Moreover, for $\xi \in LG$, we have defined a $C^\infty$-function on $\Omega G$ by $\delta_\xi(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} \langle \xi, \gamma'\gamma^{-1} \rangle \ d\theta$ with its Hamiltonian vector field $-R_{\gamma*}(\xi)$. We take the central extension $\tilde{\Omega G}$ of the Loop group $\Omega G = LG/G$ as a $U(1)$-principal bundle on $\Omega G$. The geometrical quantization line bundle will be a certain associated bundle of a $U(1)$-representation which is defined by a positive integer called level.

First we give a description of the needed connection and its curvature on the $U(1)$-principal bundle. For $\gamma \in \Omega G$, let $\tilde{\gamma} \in \Omega G = LG/G$ be the lifting of $\gamma$ on $\tilde{\Omega G}$. The connection is defined by the horizontal lifting of the tangent vector field on $\Omega G$. Specifically, for $l_{\gamma*}\xi$, which belongs to the tangent vector at $\gamma$ with $\xi \in \Omega G$, we define $l_{\tilde{\gamma}*}(\xi)$ to be the horizontal lifting of $l_{\gamma*}\xi$ to the point $\tilde{\gamma}$.

**Theorem 3:** The curvature form of the above connection is exactly the symplectic form of $\Omega G$, as needed in the geometrical quantization. Moreover, this curvature and connection are left-invariant under the left multiplication.

**Proof:** Assume $l_{\gamma*}\xi, l_{\gamma*}\eta$ to be the two left-invariant vector field on $\Omega G$, lift them to the point $\tilde{\gamma}$ by definition. Then the two horizontal lifting vector fields are $l_{\tilde{\gamma}*}\xi, l_{\tilde{\gamma}*}\eta$ respectively, where $\tilde{\gamma}$ is the lifting of $\gamma$. The curvature form is calculated by the following formula:

$$\Omega_{\tilde{\gamma}}(l_{\tilde{\gamma}*}\xi, l_{\tilde{\gamma}*}\eta) = [l_{\tilde{\gamma}*}\xi, l_{\tilde{\gamma}*}\eta]_{\tilde{\Omega G}} - l_{\tilde{\gamma}*}[\xi, \eta]_{\Omega G}$$

$$= l_{\tilde{\gamma}*}[\xi, \eta]_{\tilde{\Omega G}} - l_{\tilde{\gamma}*}[\xi, \eta]_{\Omega G}$$

$$= l_{\tilde{\gamma}*}\omega(\xi, \eta)$$

where $l_{\tilde{\gamma}*}$ is an $R$-valued vertical tangent vector field at $\tilde{\gamma}$.

The left-invariant is obvious from the calculation.

Let $L_k$ be the associated $U(1)$-bundle on $\Omega G$ by the representation

$$\rho : U(1) \longrightarrow C\{\eta\}$$

$$\rho(e^{i\theta}) = e^{ik\theta}$$

where $k$ is a positive integer. Then $L_k$ has a natural complex structure such that $L_k$ is a holomorphic line bundle over $(\Omega G, k\omega_L, J_L)$, where $\omega_L, J_L$ are the left-invariant symplectic structure and the left-invariant complex structure respectively. From the above theorem, $L_1$ is the holomorphic line bundle on $\Omega G$ which will give rise to the geometrical quantization of $(\Omega G, \omega_L)$. If we identify the smooth section of $L_1$ with a certain smooth function on $\tilde{LG}$ which satisfies the following conditions:

(a). $f(\tilde{\gamma}g) = f(\tilde{\gamma})$ for $g \in G, \tilde{\gamma} \in \tilde{LG}$

(b). $f(\tilde{\gamma}e^{i\theta}) = e^{-i\theta}f(\tilde{\gamma})$
then we can calculate the infinitesimal form of the left multiplication of $\widetilde{LG}$. This will give realization of quantum operators which is the same result as obtained by the geometrical procedure.

**Theorem 4:** (1) The infinitesimal form of $\widetilde{LG}$ action on $L_1$ gives rise to the prequantum operator in the geometrical quantization, that is, for $\xi \in L\mathcal{G}$,

$$\hat{\delta}_\xi = \nabla_{X_{\delta_\xi}} + i\delta_\xi$$

is the infinitesimal representation, where $\hat{\delta}_\xi$ is the prequantum operator, $\nabla$ is the associated connection, and $X_{\delta_\xi}$ is the Hamiltonian vector field of $\delta_\xi$.

(2) $\hat{\delta}_\xi$ preserves the Kähler polarization.

Therefore, we obtained the representation of $\widetilde{LG}$ (central extension of the Loop group) and its Lie algebra by the procedure of geometrical quantization.

**Proof:** Form the formula, $Ad\gamma.(\xi, \lambda) = (Ad\gamma.\xi, \lambda - \frac{1}{2\pi} \int_0^{2\pi} \gamma^{-1} \gamma', \xi > d\theta)$ we obtain that

$$Ad_{\gamma^{-1}} \tilde{\exp}(t\xi) = \tilde{\exp}(tAd\gamma^{-1}.\xi)e^{\frac{it}{2\pi} \int_0^{2\pi} \gamma^{-1} \gamma', \xi > d\theta}$$

for $t$ which is small enough, where $\xi \in L\mathcal{G}$.

Under the identification of sections and functions satisfying the conditions described above, $\tilde{\gamma}$ is a certain lifting of $\gamma$, the infinitesimal form of $\widetilde{LG}$-action is given by

$$(\xi.f)(\tilde{\gamma}) = \left. \frac{d}{dt} \right|_{t=0} f(\tilde{\exp}(-t\xi)\tilde{\gamma})$$

$$= \left. \frac{d}{dt} \right|_{t=0} f(\gamma Ad_{\gamma^{-1}}.\tilde{\exp}(-t\xi))$$

$$= \left. \frac{d}{dt} \right|_{t=0} f(\tilde{\gamma} exp(-tAd\gamma^{-1}, \xi)e^{\frac{it}{2\pi} \int_0^{2\pi} \gamma^{-1} \gamma', \xi > d\theta})$$

$$= \left. \frac{d}{dt} \right|_{t=0} (f(\tilde{\gamma} exp(-tAd\gamma^{-1}, \xi))e^{\frac{it}{2\pi} \int_0^{2\pi} \gamma^{-1} \gamma', \xi > d\theta})$$

$$= (\frac{1}{2\pi} \int_0^{2\pi} \gamma'/\gamma^{-1}, \xi > d\theta)f(\tilde{\gamma}) + l_{\tilde{\gamma}}(-Ad_{\gamma^{-1}}, \xi)f(\tilde{\gamma})$$

$$= (i\delta_\xi f)(\tilde{\gamma}) + l_{\tilde{\gamma}}(-Ad_{\gamma^{-1}}, \xi)f(\tilde{\gamma})$$

By the definition of connection of the $U(1)$-bundle, $l_{\tilde{\gamma}}(-Ad_{\gamma^{-1}}, \xi)$ is exactly the horizontal lifting of the Hamiltonian vector field $X_{\delta_\xi} = -R_{\gamma^*} \xi$ to the point $\tilde{\gamma}$ . So the proof of (1) above is complete.

From the above proof, we know $(\hat{\delta}_\xi.f)(\tilde{\gamma}) = (R_{\gamma^*} \xi)f(\tilde{\gamma})$. the polarization condition is left invariant, given by type(0, 1) vector fields $L_{\gamma^*}(e^{i\theta T^a})(n < 0)$ (see the definition of $J_L$ ). The proof is derived from the fact that the left invariant vector fields and the right invariant vector fields are commutative, so that the holomorphic sections are preserved by $\hat{\delta}_\xi$, and the projective representation of the Loop group is thus obtained.

§4. Moment Map of Gauge Field Theory
In section 1, we have derived the symplectic form on the phase space of Wess-Zumino-Witten model from the standard symplectic form of the gauge field theory on Riemann surfaces. In the following, we will give the moment map of gauge field theory and its relationships with the Kohno connection and the Knizhnik-Zamolodchikov equation in WZW model. This idea was inspired by reading M. Atiyah’s book [2].

Firstly, we give the general description of gauge field theory on Riemann surfaces. Let $\Sigma$ be a Riemann surface and $G$ be a compact Lie group, denote $A$ as the infinite dimensional affine space, composed by $G$-connections on the trivial $G$ bundle over $\Sigma$, i.e.,

$$A = \{ \alpha \text{ is a } G\text{-valued 1-form on } \Sigma \}$$

It is well known that there is a natural symplectic form on $A$, denoted by $\omega$, which is given by the following formula:

$$\omega(\alpha, \beta) = -\frac{1}{2\pi} \int_\Sigma \text{Tr}(\alpha \wedge \beta)$$

where $-\text{Tr}$ is the $G$-invariant inner product on $G$, which also lead to the invariant property under the gauge transformation actions. In this case, the gauge transformation group is $\text{Map}(\Sigma, G)$, the smooth maps from $\Sigma$ to $G$, whose Lie algebra is given by $\text{Map}(\Sigma, \mathfrak{g})$, the smooth maps from $\Sigma$ to $\mathfrak{g}$. Using the inner product on $\mathfrak{g}$, the dual space of $\text{Map}(\Sigma, G)$ can be identified with the smooth $G$-valued 2-forms on $\Sigma$, by integrating over $\Sigma$, while $\text{Map}(\Sigma, \mathfrak{g})$ is taken as the smooth $G$-valued 0-form. Namely, we have $\text{Map}(\Sigma, \mathfrak{g})^* \cong \mathfrak{g}^\Sigma(\pm, G)$.

Secondly, we calculate the moment map for the action of $\text{Map}(\Sigma, G)$ on $A$, as in the M. Atiyah and R. Bott’s paper [3].

A moment map for the action of $\text{Map}(\Sigma, G)$ on $A$ is a map $\mu : A \longrightarrow M \sqcup (\pm, G)^*$ such that for $\xi \in \text{Map}(\Sigma, \mathfrak{g}), \Xi \in T_\alpha A = A, \alpha \in A$,

$$<\xi, d\mu_\alpha(v) > = \omega(\xi_\alpha, v)$$

where $d\mu_\alpha : T_\alpha A \longrightarrow \mathcal{M} \sqcup (\pm, G)^*$ is the differential map of $\mu$ at $\alpha$, $\xi_\alpha \in T_\alpha A$ is the vector field defined by $\xi \in \text{Map}(\Sigma, \mathfrak{g})$ through the action of $\text{Map}(\Sigma, G)$, and $<,>$ denotes the dual pairing $\mathfrak{g}$ and $\mathfrak{g}^\ast$.

Moment map play an important role in classical and quantum mechanics by the Marsden-Weinstein reduction. For more details, see Marsden-Weinstein [14].

**Theorem 5:** The moment map $\mu : A \longrightarrow \mathcal{M} \sqcup (\pm, G)^*$ exists, concretely, (1). If $\Sigma$ has no boundary, then

$$\mu(\alpha) = d\alpha + \alpha \wedge \alpha = F_\alpha$$
where $F_\alpha \in \Lambda^2(\Sigma, G) = M \triangleleft (\pm, G)^*$ is the curvature form of $\alpha$.

(2). If $\Sigma$ has a boundary $S = \partial \Sigma$, then

$$\mu(\alpha) = F_\alpha - \alpha_S$$

where $\alpha_S$ is the restriction of $\alpha$ to the boundary $S$, i.e., $\alpha_S \in \Lambda^1(S, G)$ which is seen as an element of $\text{Map}(\Sigma, G)^*$ in the following sense: for $\xi \in \text{Map}(\Sigma, G)$, $\xi_S$ is the restriction of $\xi$ to the boundary $S$,

$$<\alpha_S, \xi> = -\frac{1}{2\pi} \int_S \text{Tr}(\xi_S\alpha_S)$$

**Proof:** By the definition of the moment map, for $\xi \in \text{Map}(\Sigma, G) = \Lambda'(\pm, G), \alpha \in \mathcal{A} = \Lambda(\pm, G), \Xi \in T_\alpha \mathcal{A} = \mathcal{A} = \Lambda^\infty(\pm, G)$, we have that the tangent vector at $\alpha$, defined by $\xi \in \text{Map}(\Sigma, G)$, is

$$\frac{d}{dt} |_{t=0} \{ \exp(-t\xi) \exp(t\xi) + \exp(-t\xi) d\exp(t\xi) \}$$

$$= d\xi + [\alpha, \xi] = d\alpha \xi$$

Therefore, we need to verify the following equality:

$$<\xi, d\mu(\alpha)(v) >= \omega(\xi_\alpha, v)$$

(1). If $\Sigma$ has no boundary, we only check $\mu(\alpha) = F_\alpha$ is the required moment map.

$$< d\mu(\alpha), \xi > = \frac{d}{dt} |_{t=0} < F_{\alpha+tv}, \xi >$$

$$= < d_\alpha v, \xi >$$

$$= -\frac{1}{2\pi} \int_\Sigma \text{Tr}(d_\alpha v)$$

$$= -\frac{1}{2\pi} \int_\Sigma \text{Tr}(\xi d_\alpha v)$$

$$= -\frac{1}{2\pi} \int_\Sigma \text{Tr}(d_\alpha \xi \wedge v)$$

$$= \omega(\xi_\alpha, v)$$

where the fourth equality is obtained by the partial integrating formula for the oprater $d_\alpha, \frac{d}{dt} |_{t=0} F_{\alpha+tv} = \frac{d}{dt} |_{t=0} (d(\alpha+tv) + (\alpha+tv) \wedge (\alpha+tv)) = dv + [\alpha, v] = d\alpha v$.

(2). If $\Sigma$ has a boundary $S = \partial \Sigma$, then

$$< d\mu(\alpha)(v), \xi > = \frac{d}{dt} |_{t=0} < F_{\alpha+tv} - (\alpha + tv)_S, \xi >$$

$$= < d_\alpha v - v_S, \xi >$$

$$= -\frac{1}{2\pi} \int_\Sigma \text{Tr}(\xi d_\alpha v) + \frac{1}{2\pi} \int_S \text{Tr}(\xi d_\alpha v_S)$$

$$= -\frac{1}{2\pi} \int_\Sigma \text{Tr}(d_\alpha \xi \wedge v) - \frac{1}{2\pi} \int_S \text{Tr}(\xi_S \wedge v) + \frac{1}{2\pi} \int_S \text{Tr}(\xi_S \wedge v_S)$$

$$= \frac{1}{2\pi} \int_\Sigma \text{Tr}(d_\alpha \xi \wedge v)$$

$$= \omega(\xi_\alpha, v)$$

16
The most interesting thing is $\Sigma$ with $n$ fixed punctures \{ $P_1$, $\ldots$, $P_n$ \}, at each point $P_i$, equipped with an appropriate representation $\lambda_i$ of the compact Lie group $G$. At each point, we also have an evaluation map:

$$ e_{p_i} : \text{Map}(\Sigma, G) \rightarrow G $$

$$ e_{p_i}(\gamma) = \gamma(P_i) $$

where $\gamma \in \text{Map}(\Sigma, G)$, differetial this map and dual it , we obtain an embedding map:

$$ \delta_{p_i} : G^* \rightarrow \mathcal{M} \downarrow \sqrt{(\pm, G)^*} $$

By the Borel-Weil theorem, each $\lambda_i$ is corresponding to an integal co-adjoint orbit of $G$ in $G^*$, denoted by $M_i$. We will give two different Gauss laws of gauge field in the case of $\Sigma$ having no boundary and having $n$ fixed punctures $\{P_1, \ldots, P_n\}$. For the former case, the Gauss law is

$$ F\alpha = 0 $$

which is just the Marsden-Weinstein reduction of the moment map $\mu(\alpha) = F\alpha$. In the latter case, associating each point $P_i$ with a representation $\lambda_i$ (dominant weight) and an integal co-adjoint orbit $M_i \subset G^*$ of $G$, then the Marsden-Weinstein reduction becomes the generalized symplectic quotient

$$ \mathcal{M}_{\{P_i\},\{\lambda_i\}} = \mu^{-\infty}((\mathcal{M}_\infty + \cdots + \mathcal{M}_i)/||/M\downarrow \sqrt{(\pm, G)} $$

where the “$/|$” means module the gauge transformation of $\text{Map}(\Sigma, G)$.

For such a gauge field $\alpha \in \mathcal{M}_{\{P_i\},\{\lambda_i\}}$ its Gauss constraints is given by the following:

$$ F\alpha = -\frac{2\pi}{k} \sum_{i,a} \xi^a_i \delta(P_i) T_a $$

where $F\alpha$ is the curvature of $\alpha$, $\{T_a\}$ is an orthogonal basis of $G$ under the Cartan-Killing form $Tr$, $\xi^a_i \epsilon_a$ is an element of $M_i$, $\{\epsilon_a\}$ is the dual basis of $\{T_a\}$ in $G^*$, $\delta(P_i)$ is the delta function at $P_i$, which can be taken as an element of $\text{Map}(\Sigma, G)^*$ by $\delta(P_i)(\xi) = \xi(P_i)$ for $\xi \in \text{Map}(\Sigma, G)$ and the Cartan-Killing form. The above formula makes sense if using the moment map:

$$ \langle F\alpha, \xi \rangle = -\frac{1}{2\pi} \int_\Sigma Tr(\xi F\alpha) = \frac{1}{k} \sum_{i,a} \xi^a_i \langle \xi(P_i), T_a \rangle $$

Using the local coordinate $(z, \overline{z})$ on $\Sigma$, $F\alpha = \frac{1}{2} F^a_{z \overline{z}}(z, \overline{z}) T_a dz \wedge d\overline{z}$, hence, the Gauss constraints can be written as

$$ F^a_{z \overline{z}}(z, \overline{z}) = -\frac{4\pi}{k} \sum_{i,a} \xi^a_i \delta(z, P_i) $$
We notice that \( \{ \xi^a_i \} \) is a system of functions with degree 1 (coordinate function) on \( M_i \), after the geometric quantization, these classical observable functions turn out to be the quantum operator \( T^i_a \) acting on the representation space \( V_i \) of the representation \( \lambda_i \) [12,13]. This lead to the formula written down in [20] as follows:

\[
F^a_{\pi}(z, \bar{z}) = -\frac{4\pi}{k} \sum_{i,a} \delta(z, P_i) T^i_a
\]

We know that the Green function on compact Riemann surface does not exist, but on the plane, there is a Green function due to the following formula:

\[
\partial_z \partial_{\bar{z}} \ln | z - P | = 2\pi \delta(z, P)
\]

**Theorem 6:** The Gauss constraints have a matrix-valued solution:

\[
\begin{cases}
A^a_z = \frac{4}{k} \sum_i \partial_z \ln | z - P_i | T^i_a = \frac{2}{k} \sum_i T^i_a \frac{T^i_z}{z-P_i} \\
A^a_{\bar{z}} = 0
\end{cases}
\]

where the connection is defined on the associated bundle of trivial \( G \)-bundle on \( \Sigma = \mathcal{R} \setminus \{ P_{\infty}, \cdots, P \} \) by the tensor representation

\[
\lambda_1 \otimes \cdots \otimes \lambda_n : G \to GL(V_1 \otimes \cdots \otimes V_n)
\]

**Proof:** This is obtained by the direct calculation:

\[
F_A = \frac{1}{2} F_{zz} dz \wedge d\bar{z} = dA + A \wedge A
\]

\[
= -\partial_z A_{\bar{z}} dz \wedge d\bar{z} = -\partial_{\bar{z}} \left( \frac{4}{k} \sum_i \partial_z \ln | z - P_i | T^i_a dz \wedge d\bar{z} \right) = -\frac{4}{k} \sum_i \partial_{\bar{z}} dz \wedge d\bar{z} = -\frac{4}{k} \sum_i \delta(z - P_i) T^i_a dz \wedge d\bar{z}
\]

This connection \( A \) has a special meaning which not only gave the solution of the Gauss constraints, but also leads to the Kohno connection and the Knizhnik-Zamolodchikov equation in the next section.

**§5. Kohno Connection and Knizhnik-Zamolodchikov Equation**

In the section 4, we have solved the Gauss constraints using Green function. In this section, we will generalize this connection to Kohno connection [6,11], whose holonomy gives rise to a representation of the braid group and its parallel sections will be the solutions of Knizhnik-Zamolodchikov equation[4].
We rewrite the connection of section 4 as

\[ A_q(z)dz = \frac{2}{k} \sum_{p \neq q} \Omega_{pq} d\ln(z - z_p)dz = \frac{2}{k} \sum_{p \neq q} \Omega_{pq} dz \]

where \( \Omega_{pq} = \sum_a T^p_a \otimes T^q_a \) acts on \( V_1 \otimes \cdots \otimes V_n \), only on the \( p \)-th and \( q \)-th factors nontrivially. In fact, \( \Omega_{pq} \) is the Casimir element of the universal enveloping algebra \( \mathcal{U}(G) \) and satisfied the following equations.

\[
\begin{align*}
[\Omega_{pq}, \Omega_{pr} + \Omega_{qr}] &= 0 & p < q < r \\
[\Omega_{pq} + \Omega_{pr}, \Omega_{qr}] &= 0 & p < q < r \\
[\Omega_{pq}, \Omega_{rs}] &= 0 & \text{for distinct } p, q, r \text{ and } s
\end{align*}
\]

Let us consider the following trivial vector bundle over

\[ \Sigma_n = \{(z_1, \cdots, z_n) \in \mathbb{C}^n | \frac{i}{\sqrt{v_i}} \neq \frac{j}{\sqrt{v_j}} \text{ if } p \neq q \}/S_n \]

with the fibres \( V \otimes^n \) (a fixed representation \( V \) of \( \lambda \)), where \( S_n \) acts on \( (z_1, \cdots, z_n) \) by the usual permutation of \( n \) coordinates. The fundamental group \( \pi_1(\Sigma_n) \) is the braid group \( B_n \), whose generators are \( \sigma_1, \cdots, \sigma_{n-1} \), satisfying the following relations:

\[
\begin{align*}
\sigma_i \sigma_j &= \sigma_j \sigma_i & |i - j| \geq 2 \\
\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1} & 1 \leq i \leq n - 2
\end{align*}
\]

The Kohno connection is the generalized one described above, whose connection 1-form is given by

\[
\Theta = \frac{k}{(k+h)} \sum A_q(z_q)dz_q = \frac{2}{k+h} \sum_{1 \leq p < q \leq n} \Omega_{pq} d\ln(z_p - z_q) = \frac{1}{k+h} \sum_{p \neq q} T^p_a \otimes T^q_a (dz_p - dz_q)
\]

The following theorem is the key point in giving out the relations between the gauge field theory and the conformal field theory.

**Theorem 7:** (1). \( d\Theta + \Theta \land \Theta = 0 \), namely, \( \Theta \) is a flat connection. Therefore, the holonomy of \( \Theta \) gives the representation of \( B_n = \pi_1(\Sigma_n) \), called the monodromy representation of \( B_n \), as in \([6,11]\).

(2). The equation of parallel sections

\[ D\psi = 0 \]

is just the Knizhnik-Zamolodchikov equation:

\[
\partial_j \psi(z_1, \cdots, z_n) = \frac{1}{k+h} \sum_{i \neq j} T^i_a \otimes T^j_a \psi(z_1, \cdots, z_n).
\]
Proof : (1).

\[
d\Theta + \Theta \wedge \Theta = \frac{4}{(k+h)^2} \left[ \sum_{1 \leq p < q \leq n} \Omega_{pq} d \ln(z_p - z_q), \sum_{1 \leq r < s \leq n} \Omega_{rs} d \ln(z_r - z_s) \right]
\]

From \( [\Omega_{pq}, \Omega_{rs}] = 0 \), for distinct \( p, q, r \) and \( s \), we obtain that

\[
\begin{align*}
&\left[ \sum_{1 \leq p < q \leq n} \Omega_{pq} d \ln(z_p - z_q), \sum_{1 \leq r < s \leq n} \Omega_{rs} d \ln(z_r - z_s) \right] \\
&= \left[ \sum_{1 \leq p < q \leq n} \Omega_{pq} d \ln(z_p - z_q), \sum_{1 \leq s < p \leq n} \Omega_{ps} d \ln(z_s - z_p) \right] \\
&\quad + \left[ \sum_{1 \leq s < p \leq n} \Omega_{ps} d \ln(z_s - z_p), \sum_{1 \leq r < s \leq n} \Omega_{rs} d \ln(z_r - z_s) \right] \\
&\quad + \left[ \sum_{1 \leq r < s \leq n} \Omega_{rs} d \ln(z_r - z_s), \sum_{1 \leq p < q \leq n} \Omega_{pq} d \ln(z_p - z_q) \right]
\end{align*}
\]

\[
= \sum_{p < q < r} \frac{2(dz_p \wedge dz_q + dz_r)}{(z_p - z_q)(z_q - z_r)(z_p - z_r)} \left\{ [\Omega_{pq}, \Omega_{qr}](z_r - z_p) \right. \\
&\quad + [\Omega_{pq}, \Omega_{pr}](z_q - z_r) + [\Omega_{pr}, \Omega_{qr}](z_p - z_q) \\
&\left. + [\Omega_{pq} + \Omega_{pr}, \Omega_{qr}](z_p + z_r) \right\}
\]

From \( [\Omega_{pq}, \Omega_{pr} + \Omega_{qr}] = [\Omega_{pq} + \Omega_{pr}, \Omega_{qr}] = 0, p < q < r \), we prove that

\[
d\Theta + \Theta \wedge \Theta = 0
\]

Moreover, the holonomy representation of \( \Theta \) is the parallel displacement along a closed curve \( C \), depending only on the homotopy class of \( C \), especially, let \( C \) be an element of \( \pi_1(\Sigma_n) = B_n \),

\[
\rho(C) = P \exp(-\int_C \Theta)
\]

is a well-defined representation of the braid group \( B_n \) acting on \( V^\otimes n \).

(2). If \( \psi \) is a parallel section, then \( \psi \) is a \( V^\otimes n \)-valued function on \( \Sigma_n \) and satisfies the following equation:

\[
d\psi + \Theta \psi = 0
\]

locally, this equation can be rewritten as follows:

\[
\frac{\partial \psi}{\partial z_j} = \frac{1}{k + h} \sum_{i \neq j} T^i_a \otimes T^j_a \psi(z_1, \ldots, z_n)
\]

These equations is exactly the Knizhnik-Zamoldchikov equation [10]. In their paper, they also showed that the solutions exist in the terms of n-point functions
of the two dimensional conformal field theory with gauge symmetry (WZW model). In our case, they are the parallel sections of the connection $\Theta$. The integrability condition of these equations is precisely vanishing of the curvature $d\Theta + \Theta \wedge \Theta = 0$.

§6. Relations to $R$-Matrix and Quantum Group

In Kohno’s paper, he had proved that the above holonomy representation $\rho$ of braid group $B_n$ is equivalent to the monodromy representation of braid group $B_n$ which was obtained from the solution of quantum Yang-Baxter equation (QYBE). We will discuss the latter representation briefly in this section.

Firstly, we introduce the $R$-matrix $R : \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C}$, which satisfies the following equation, called quantum Yang-Baxter equation:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

where $R_{12}, R_{23}, R_{13} : \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C} \rightarrow \mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$, whose actions are determined by $R$ and their subscripts. This indicates that $R_{ij}$ acts on $\mathcal{C} \otimes \mathcal{C} \otimes \mathcal{C}$ like $R$ in the $(i,j)$ factors and like identity in the remaining factor. The most common solutions is $A_n$-type, expressed as follows [5,8]:

$$R = \sum_{i \neq j} e_{ii} \otimes e_{jj} + q^{\frac{1}{2}} \sum_i e_{ii} \otimes e_{ii} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{i<j} e_{ij} \otimes e_{ji}$$

where $e_{ij}$ is the $n \times n$ matrix unit whose single nonvanishing component is equal to 1 and it is located in position $(i,j)$.

Let $\tilde{R} = PR$, where $P$ is the permutation operator of $V \otimes V$, then, $\tilde{R}$ satisfies the following two equations:

$$(\tilde{R} \otimes I)(I \otimes \tilde{R})(\tilde{R} \otimes I) = (I \otimes \tilde{R})(\tilde{R} \otimes I)(I \otimes \tilde{R})$$

$$\tilde{R}^2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\tilde{R} + 1$$

where the first equality is derived from the QYBE satisfied by $R$, the second equality means that $\tilde{R}$ has two eigenvalues $q^{\frac{1}{2}}, -q^{-\frac{1}{2}}$.

By the direct calculation, we obtain that

$$\tilde{R} = \sum_{i \neq j} e_{ij} \otimes e_{ji} + q^{\frac{1}{2}} \sum_i e_{ii} \otimes e_{ii} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{i>j} e_{ii} \otimes e_{jj}$$

To prove this equation, we choose a basis for $\mathcal{C}$, denoted by $\{v_s\}$, then we have:

$$R(v_s \otimes v_t) = \begin{cases} v_s \otimes v_t, & \text{for } s < t \\ q^{\frac{1}{2}}v_s \otimes v_t, & \text{for } s = t \\ v_s \otimes v_t + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})v_t \otimes v_s & \text{for } s > t \end{cases}$$

21
Therefore, by $P(v_s \otimes v_t) = v_t \otimes v_s$,

$$\hat{R}(v_s \otimes v_t) = \begin{cases} v_t \otimes v_s, & \text{for } s < t \\ q^{\frac{1}{2}}v_t \otimes v_s, & \text{for } s = t \\ v_t \otimes v_s + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})v_s \otimes v_t, & \text{for } s > t \end{cases}$$

and the expression of $\hat{R}$ is obtained.

Using $\hat{R}$-matrix, the representation of braid group $B_n$ can be given as follows:

$$\nu(\sigma_i) = -q^{12}(I \otimes \cdots \otimes \hat{R} \otimes \cdots \otimes I)$$

where $\hat{R}$ is located at the $(i, i+1)$-factors, so $\nu(\sigma_i)$ acts on $V^\otimes n$ only nontrivially in the $(i, i+1)$-factors. The relations which $\{\sigma_i\}$ satisfy are satisfied by $\{\nu_i = \nu(\sigma_i)\}$ due to the equation:

$$(\hat{R} \otimes I)(I \otimes \hat{R})(\hat{R} \otimes I) = (I \otimes \hat{R})(\hat{R} \otimes I)(I \otimes \hat{R})$$

Moreover, $\{\nu_i\}$ also satisfy the following equation:

$$\nu_i^2 = (1-q)\nu_i + q$$

together with the other two relations satisfied by $\{\sigma_i\}$, $\{\nu_i\}$ defines the Hecke algebra of type $A_n$, called the Terperley-Lieb-Jones algebra [8,9]:

$$\begin{cases} \nu_i\nu_{i\pm 1}\nu_i = \nu_{i\pm 1}\nu_i\nu_{i\pm 1} \\ \nu_i\nu_j = \nu_j\nu_i \quad |i-j| \geq 2 \\ \nu_i^2 = (1-q)\nu_i + q \end{cases}$$

T. Kohno has stated that the representation using $\hat{R}$-matrix

$$q = \exp(-i \frac{2\pi}{k + h})$$

and the representation through the holonomy of the connection $\Theta$ (section 5) are equivalent up to a conjugation by an invertible linear transformation of $V^\otimes n$. He also proved that the holonomy representation commutes with the coproduct action of quantum group $SU_q(n)$ on $V^\otimes n$. In fact, the above Hecke algebra is the centralizer of quantum group $SU_q(n)$.

**Acknowledgement**

I would like to thank Prof. Qian Min for his encouragement and many useful discussions on this work. I am also grateful to Prof. Guo Maozheng, Prof. Liu Zhangju and Dr. Wang Zhengdong for their help.
References

[1] Atiyah, M. F., *Topological quantum field theory*, Publ. Math. Inst. Hautes Etudes Sci. Paris 68 (1989) 175-186.

[2] Atiyah, M. F., *The geometry and physics of knots*, Cambridge University Press, 1990.

[3] Atiyah, M. F., Bott, R., *The Yang-Mills equations over Riemann surfaces*, Phil. Trans. R. Soc. Lond. A 308 (1982) 523-615.

[4] Belavin, A. A., Polyakov, A. M. and Zamolodchikov, A. B., *Infinite conformal symmetry in two dimensional quantum field theory*, Nucl. Phys. B241, (1984) 83.

[5] Drinfel’d, V., *Quantum group* In : the Proceeding of the International Cogress of Mathematics, Berkeley 1986, Vol. 1, 798-820.

[6] Fröhlich, J., *Statistics of fields, the Yang-Baxter equation and the theory of links and knots*, 1987 Cargese lectures, Nonperturbative quantum field theory, New York: Plenum Press.

[7] Fröhlich, J. and King, C., *Two dimensinal conformal field theory and three dimensional topology*, Internat. J. of Modern Phys. A4, (1989) 5321-5399.

[8] M. Jimbo, *A q-analog of $U(\mathfrak{g})(N + \infty)$, Hecke algebra and the Yang-Baxter equation*, Lett. Math. Phys. 11, (1986) 247-252.

[9] V.F.R.Jones, *Hecke algebra representations of braid group and link polynomials*, Ann. Math. 126 (1987) 335-388.

[10] Knizhik, V. G. and Zamolodchikov, A. B. *Current algebra and Wess-Zumino model in two dimension*, Nucl. Phys. B247 (1984) 83.

[11] T. Kohno, *Hecke algebra representations of braid groups and monodromy of braid groups*, Advanced Studies in Pure Math. 16, (1988) 255.

[12] Kostant, B. “ Quantization and representation. In: Representation theory of Lie groups,” *London Math. Soc. Lecture Notes Ser.* 34(1979): 287-316

[13] A. A. Kirillov, *Elements of the theory of representations*, Springer 1976.

[14] J. Marsden and A. D. Weinstein, *Reduction of symplectic manifold with symmetry*, Reports on Math. Physics 5, (1974) 121-130.
[15] G. Moore and N. Seiberg, *Classical and quantum conformal field theories*, Comm. Math. Phys. 123, (1989) 177

[16] Pressley, A., Segal, G. *Loop Groups*. Oxford, 1986

[17] Seminov-Tyan-Shanskii, M. A., *What is a Classical $r$-matrix?* Funct. Anal. Appl. 17 (1983) 259

[18] E. Verlinde, *Fusion rules and modular transformations in 2d conformal field theory*, Nucl. Phys. B300 (1988) 360

[19] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. 121 (1989) 351-399.

[20] Witten, E. *Nonabelian bosonization in two dimension*, Comm. Math. Phys. 92, (1984) 455.