Chiral Magnetic Effect in the Anisotropic Quark-Gluon Plasma

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Abstract: An anisotropic thermal plasma phase of a strongly coupled gauge theory can be holographically modelled by an anisotropic AdS black hole. The temperature and anisotropy parameter of the AdS black hole background of interest [1] is specified by the location of the horizon and the value of the Dilaton field at the horizon. Interestingly, for the first time, we obtain two functions for the values of the horizon and Dilaton field in terms of the temperature and anisotropy parameter. Then by introducing a number of spinning probe D7-branes in the anisotropic background, we compute the value of the chiral magnetic effect (CME). We observe that in the isotropic and anisotropic plasma the value of the CME is equal for the massless quarks. However, at fixed temperature, raising the anisotropy in the system will increase the value of the CME for the massive quarks.
1 Introduction and Result

A new phase of Quantum Chromodynamics, called Quark-Gluon plasma (QGP) is produced at Relativistic Heavy Ion Collider (RHIC) and Large Hadron Collider (LHC) by colliding two pancakes of heavy nuclei such as Gold or Lead at a relativistic speed. From the numerical simulation, it is realized that the QGP is a strongly coupled fluid since it has very low viscosity over entropy density. Therefore, perturbative calculations(methods) of QGP properties are not reliable [2]. Thus non-perturbative methods like AdS/CFT correspondence [3] may be applied to describe different properties of the QGP such as rapid thermalization, elliptic flow, jet quenching parameter and quarkonium dissociation which they have been considerably studied in the literature [2, 4, 5]. The property we would like to discuss here is CME [6–10].

The presence of a strong magnetic field in the very early stages of heavy ion collision, realized from numerical simulation, and its accompanying non-trivial gluon field configurations lead to the CME. More precisely, The axial charge $\mu_5$, given by the difference between the number of fermions with left-handed and right-handed quarks, is proportional to the winding number of non-trivial gauge field provided that the left-handed and right-handed quarks are initially equal. The spin of quarks is tightly aligned along the strong magnetic
field. For a non-zero winding number, in order to have a non-zero axial charge the momentum direction of some quarks depending on the sign of winding number must be altered. This phenomena produces a non-zero electric current of (massless) quarks along the strong magnetic field which is given by

$$J = \frac{\mu_5}{2\pi^2} B.$$  \hspace{1cm} (1.1)

The AdS/CFT correspondence [3] states that type IIB string theory on $AdS_5 \times S^5$ geometry, describing the near horizon geometry of a stack of $N_c$ extremal D3-branes, is dual to the four-dimensional $\mathcal{N} = 4$ super Yang-Milles (SYM) theory with gauge group $SU(N_c)$. In particular, in the large $N_c$ and t’Hooft coupling limits, a strongly coupled SYM theory is dual to the type IIb supergravity which provides a useful tool to study the strongly coupled regime of the SYM theory. As a generalization, a thermal SYM theory corresponds to the supergravity in an AdS-Schwarzschild background where SYM theory temperature is identified with the Hawking temperature of AdS black hole [11]. Furthermore, Mateos and Trancanelli have introduced an interesting generalization of this duality to the thermal and spatially anisotropic SYM [1]. In order to add matter (quark) in the fundamental representation of the corresponding gauge group to the correspondence, one needs to introduce a D-brane into the background in the probe limit [12]. The probe limit means that D-brane does not back-react the geometry. Then the asymptotic shape of the brane gives the mass and condensation of the quark. In addition, the shape of the brane can be classified into two types, one is the Mikowski embedding (ME) and the other is black hole embedding (BE). While the ME does not see the horizon, the BE crosses it, for more details see appendix A.

To specify the anisotropic solution in [1], one needs to input two constants, the location of the horizon $u_h$ and the value of the Dilaton field at the horizon $\tilde{\phi}_h$. Then it was claimed that there is a one-to-one correspondence between these parameters and the anisotropy parameter $a$ and temperature of the system. It means that for given $u_h$ and $\tilde{\phi}_h$, using the equations of motion one finds just a temperature and an anisotropy parameter for the solution though the map between $(u_h, \tilde{\phi}_h)$ and $(a, T)$ has not been defined. In section 3, we carefully investigate this map and, for the first time, we find fitted functions between these parameters that is consistent with the numerical results. More precisely, we find the inverse map meaning that for given values of anisotropy parameter and temperature one can find the corresponding $u_h$ and $\tilde{\phi}_h$. In fact this is one of the main result of this paper.

Applying the AdS/CFT correspondence, an interesting setup has been introduced in [13] to describe the CME, as we will review and generalize it in the section 4. We realize that for the massless quarks the anisotropy being in the system does not affect the value of the CME and its value is the same as the isotropic case, i.e. (1.1). However, for the quarks with finite mass raise in anisotropy of the system will increase the value of the CME. In order to have non-zero value for the CME, the mass of the quark must vary between zero and its maximum at which this value vanishes. The maximim value of the mass also increases as one raises the anisotropy parameter.
2 Review on the Anisotropic Background

The background we are interested in is an anisotropic solution of the IIB supergravity equations of motion [1]. This solution in the string frame is given by
\[
\begin{align*}
    ds^2 &= g_{tt} dt^2 + g_{xx} (dx^2 + dy^2) + g_{zz} dz^2 + g_{uu} du^2 + g_{55} d\Omega_5^2, \\
    d\Omega_5^2 &= d\theta^2 + \cos^2 \theta d\Omega_3^2 + \sin^2 \theta d\psi^2, \\
    \chi &= a z, \quad \phi = \phi(u).
\end{align*}
\]
(2.1)

\(\chi\) and \(\phi\) are axion and dilaton fields respectively. \(a\), which is a dimensionful constant, represents the anisotropy in the background. The components of metric are
\[
\begin{align*}
    g_{tt} &= -F_B u - 2, \\
    g_{xx} &= u - 2, \\
    g_{zz} &= H u - 2, \\
    g_{uu} &= F - 1 u - 2, \\
    g_{55} &= e^{\frac{1}{2} \phi}.
\end{align*}
\]
(2.2)

\(H\), \(F\) and \(B\) depend only on the radial direction \(u\). In terms of the dilaton field, they have the following functions
\[
\begin{align*}
    H &= e^{-\phi}, \\
    F &= e^{-\frac{1}{2} \phi} \left[ a^2 e^{\frac{7}{2} \phi} (4u + u^2 \phi') + 16\phi' \right], \\
    \frac{B'}{B} &= \frac{1}{24 + 10u\phi'} (24\phi' - 9u\phi'^2 + 20u\phi'').
\end{align*}
\]
(2.3a, 2.3b, 2.3c)

and the dilaton field must be satisfied in the third-order equation as
\[
0 = \frac{256\phi'\phi''' - 16\phi'^2 (7u\phi' + 32)}{u a^2 e^{\frac{5\phi}{2}} (u\phi' + 4) + 16\phi'} + \frac{\phi'}{u (5u\phi' + 12) (u\phi'' + \phi')} \times \left[ 13u^3 \phi'^4 + 8u (11u^2 \phi''^2 - 60\phi'' - 12u\phi''') + u^2 \phi'^3 (13u^2 \phi''^2 + 96) \\
+ 2u\phi'^2 (-5u^3 \phi''^2 + 53u^2 \phi'' + 36) + \phi' (30u^4 \phi''^2 - 64u^3 \phi'' - 288 + 32u^2 \phi'') \right],
\]
(2.4)

Note also that the solution contains a self-dual five-form field (see appendix B for explicit form). The horizon is located at \(u = u_h\) meaning that \(F(u_h) = 0\) and the Hawking temperature is given by
\[
T = \frac{-F'(u_h)\sqrt{B(u_h)}}{4\pi}.
\]
(2.5)

The boundary lies at \(u = 0\) and the metric approaches \(AdS_5 \times S^5\) asymptotically. At the boundary, the suitable boundary conditions are
\[
\begin{align*}
    \phi_B &= 0, \quad \text{(2.6a)} \\
    F_B &= B_B = 1. \quad \text{(2.6b)}
\end{align*}
\]

The gauge theory lives in a space-time with coordinates \((t, x, y, z)\). Since there is a \(U(1)\) symmetry in the \(xy\)-plane, \(x\) and \(y\) are normally considered as the transverse directions and the longitudinal direction is \(z\). An anisotropy is clearly seen between the transverse and longitudinal directions. For more details, we refer the reader to the original paper [1].
3 More on the Anisotropic Background

In the previous section we have introduced the anisotropic background for which there is a map paring \((a, T)\) with \((\tilde{\phi}_h, u_h)\) where \(\tilde{\phi}\) is defined in (3.2). Now we are going to study this map in more detail. More precisely, we will show that an inverse map, which is a one-to-one correspondence between the above sets, exists. Using Padé approximant, we expliciely introduce a function to compute \(\tilde{\phi}_h\) and \(u_h\) in terms of \(a\) and \(T\).

3.1 Map Between \((\tilde{\phi}_h, u_h)\) and \((a, T)\)

As far as the authors of [1] have been able to verify, their numerical results show that a one-to-one map between \((\tilde{\phi}_h, u_h)\) and \((a, T)\) exists. As a matter of fact, for given values of \(\tilde{\phi}_h\) and \(u_h\), by solving the equations of motion (2.3), \(a\) and \(T\) can be found. More explicitly, we have

\[
a(\tilde{\phi}_h, u_h) = \lim_{\epsilon \to 0} \exp\left[\frac{7}{4}\tilde{\phi}(\epsilon; \tilde{\phi}_h, u_h)\right],
\]

\[
T(\tilde{\phi}_h, u_h) = \sqrt{B_h \frac{a^2 e^{-\frac{1}{2}\tilde{\phi}_h}}{16\pi u_h}} \left(16 + u_h^2 e^{\frac{7}{2}\tilde{\phi}_h}\right),
\]

where \(B_h = B(\tilde{\phi}_h, u_h)\) and

\[
\tilde{\phi}(u) = \phi(u) + \frac{4}{7} \log a.
\]

Above equation and (2.6a) indicate that the value of the anisotropy parameter is related to the asymptotic value of the \(\tilde{\phi}(u)\) as it has been clearly emphasised in (3.1a).

Now our goal is to investigate the inverse of mapping, if any, between \((\tilde{\phi}_h, u_h)\) and \((a, T)\). In order to do so, we start with equations of motion for the metric components (2.2) (see equations (124-9) in [1]). By introducing a new variable \(u \to \xi u_h\), where \(\xi \in [0, 1]\) is a dimensionless variable, one can easily see that the equations of motion depend only on the variable \(\xi\) and dimensionless parameter \(au_h\). As a result, this behavior points out that just the dimensionless parameter \(au_h\) appears in the corresponding solutions of all metric components.

By multiplying (3.1b) by \(1/a\), we find

\[
\frac{T}{a} = \sqrt{B_h \frac{e^{\frac{1}{2}\tilde{\phi}_h}}{16\pi au_h}} \left(16 + au_h^2 e^{\frac{7}{2}\tilde{\phi}_h}\right).
\]

According to discussion presented in the previous paragraph, \(B_h\) and \(\phi_h\) only turn out to be a function of the dimensionless parameter \(au_h\) and therefore, from (3.3), one can conclude that the same parameter appears in the \(T/a\) or equivalently \(T/a = f(au_h)\). Then it is easy to obtain

\[
Tu_h = \frac{1}{\eta} f^{-1}(\frac{1}{\eta}),
\]

(3.4)
where \( \eta \equiv a/T \) and we have assumed that \( f \) is invertible. We also use separation of variables to write the temperature and \( u_h \) as:

\[
\begin{align*}
  u_h(\eta, \tilde{\phi}_h) &= \kappa_1(\eta)K_1(\tilde{\phi}_h), \\
  T(\eta, \tilde{\phi}_h) &= \kappa_2(\eta)K_2(\tilde{\phi}_h).
\end{align*}
\]

(3.5a) (3.5b)

Thus (3.5b) can be simply written as

\[
\tilde{\phi}_h = K_2^{-1}(\frac{T}{\kappa_2(\eta)}),
\]

(3.6)

or equivalently

\[
\phi_h = K_2^{-1}(\frac{T}{\kappa_2(\eta)}) - \frac{4}{T} \log a.
\]

(3.7)

Since \( \phi_h \) is a function of the dimensionless parameter \( au_h \) (or equivalently \( \eta \)), the right hand side of the above equation indicates that

\[
K_2^{-1}(\frac{T}{\kappa_2(\eta)}) = \frac{4}{T} \log(\frac{T}{\kappa_2(\eta)}),
\]

(3.8)

and therefore we generally have \( K_2(x) = e^{\frac{4}{T}x} \). Furthermore, from (3.4) and (3.5), it is clear that

\[
K_1(x) = [K_2(x)]^{-1}.
\]

(3.9)

After that the two unknown functions \( \kappa_1(\eta) \) and \( \kappa_2(\eta) \) can be realized as

\[
\begin{align*}
  \kappa_1(\eta) &= u_h(\eta, 0), \\
  \kappa_2(\eta) &= T(\eta, 0).
\end{align*}
\]

(3.10a) (3.10b)

Moreover, using (3.6), (3.8) and (3.9), one finds

\[
\begin{align*}
  \tilde{\phi}_h(a, T) &= \frac{4}{T} \log(\frac{T}{\kappa_2(\eta)}), \\
  u_h(a, T) &= \frac{1}{T} \kappa_1(\eta)\kappa_2(\eta).
\end{align*}
\]

(3.11a) (3.11b)

Henceforth, our aim is to gain the appropriate functions for \( \kappa_1(\eta) \) and \( \kappa_2(\eta) \). Unfortunately, it is analytically impossible and, as we will explain in the following, using the numerical results and asymptotic behaviours, suitable fitted functions will be introduced.

- **For general values of \( a \) and \( T \)**

  Now let us start with arbitrary values for \( a \) and \( T \), or equivalently \( \eta \), and we then need to solve the following equation

\[
\frac{a(\tilde{\phi}_h = 0, u_{0h})}{T(\tilde{\phi}_h = 0, u_{0h})} = \eta,
\]

(3.12)

\footnote{Our numerical calculations demonstrate that this separation of variable is only possible for the mentioned variables, i.e. \( \eta \) and \( \tilde{\phi} \). For example, one can not consider \( u_h(a, T) = \kappa_1(a)K_1(T) \).}
Figure 1. Numerical values for $\kappa_1(\eta)$ and $\kappa_2(\eta)$ (blue dots) and their analytical values for $T \gg a$ (red dashed curves) and $T \ll a$ (green dot-dashed curves). The blue curves represent the padé functions (3.23) for $\kappa_1(\eta)$ and $\kappa_2(\eta)$.

to find $u_{0h}(\eta)$. One can numerically solve the above equation and by using the numerical results for $u_{0h}(\eta)$ we have

$$\kappa_1(\eta) = u_{0h}(\eta),$$

and from (3.1b) and (3.5b), it is straightforward to see

$$\kappa_2(\eta) = T(0, u_h = u_{0h}(\eta)).$$

These functions have been shown in the figure 1 by the blue dots.

• At high temperature limit

In the case of high temperature, $T \gg a$, $\phi$ and $\pi Tu_h$, up to $O \left( \frac{\eta}{\pi} \right)^6$, can be written as below [1]

$$\phi_h(\eta) = -\frac{1}{4} \left( \frac{\eta}{\pi} \right)^2 \left( \log 2 - \frac{(3 - 2\pi^2 + 72(\log 2)^2)}{72} \left( \frac{\eta}{\pi} \right)^2 \right),$$

(3.15a)

$$\pi Tu_h = 1 + \left( \frac{\eta}{\pi} \right)^2 \left( \frac{5 \log 2 - 2}{48} + \frac{180 + 40\pi^2 - 12 \log 2 - 273(\log 2)^2}{13824} \left( \frac{\eta}{\pi} \right)^2 \right).$$

(3.15b)

Considering $T(\eta, 0) = \frac{1}{\pi} e^{-\frac{\eta}{\pi} \phi_h}$ and (3.15a), it is easy to find that

$$\kappa_2(\eta) = \frac{1}{\eta} \left( 1 + \frac{7 \log 2}{16} \left( \frac{\eta}{\pi} \right)^2 + \frac{7 \left( -12 + 8\pi^2 - 225(\log 2)^2 \right)}{4608} \left( \frac{\eta}{\pi} \right)^4 \right).$$

(3.16)

$\kappa_1(\eta)$ can be found by using (3.10a) and (3.15b) and it finally becomes

$$\kappa_1(\eta) = \frac{\eta}{\pi} \left( 1 - \frac{1 + 8 \log 2}{24} \left( \frac{\eta}{\pi} \right)^2 \right.

\left. + \frac{108 - 32\pi^2 + 60 \log 2 + 1617(\log 2)^2}{3456} \left( \frac{\eta}{\pi} \right)^4 \right).$$

(3.17)
The red dashed curves in figure 1 represent the above functions.

- **At low temperature limit**

In this limit the functions $\kappa_1(\eta)$ and $\kappa_2(\eta)$ can be gained in two ways. One way is to use the properties of the IR solution we have found in the appendix C. Another way is to use the numerical data from figure 1.

(i) In the appendix C, a solution has been introduced in the low temperature limit, $a \gg T$. According to this solution, by comparing (3.11a) and (C.3) and utilizing (3.11b), one gets

$$\kappa_1(\eta) = \sqrt{\frac{8}{3}} \sim 1.63.$$ (3.18)

Note that this is exactly the asymptotic value of $\kappa_1(\eta)$ in the limit of $\eta \to \infty$ in figure 1. Then using (3.11b), (3.18) and (C.9), one finds

$$\kappa_2(\eta) = \frac{11^{7/12}}{4 \cdot 2^{11/12} \pi^{7/6}} \eta^{1/6}.$$ (3.19)

(ii) As it is clearly seen from figure 1, for large values of $\eta$ the function $\kappa_1(\eta)$ goes to a constant value, say $c \simeq 1.63$. Thus (3.11) leads to

$$\phi_h = -\frac{4}{T} \log(\eta \kappa_2(\eta)),$$

$$u_h = \frac{c}{T} \kappa_2(\eta).$$ (3.20)

Now the entropy density of the system can be computed via $s = \frac{N^2 e^{-\frac{\phi_h}{u_h}}}{2\pi^2} [1]$ and above equations. In the end we have

$$s = \frac{1}{2\pi c^3} N^2 T^3 \eta^5 [\kappa_2(\eta)]^{-1/2}.$$ (3.21)

Furthermore, in this limit it was shown in [1] that the entropy density scales as

$$s = c_{ent} N^2 a^3 T^3,$$ (3.22)

where $c_{ent} \simeq 3.2$. Comparing (3.21) and (3.22), one can identify $\kappa_2(\eta)$ with $\left(2\pi c^3 c_{ent}\right)^{-7/16} \eta^{1/6}$ in agreement with (3.19).

Finally we have plotted the resulting functions for $\kappa_1(\eta)$ and $\kappa_2(\eta)$ (the green dot-dashed curves) in the figure 1.

### 3.2 Padé approximant for $\kappa_1(\eta)$ and $\kappa_2(\eta)$

We have obtained the asymptotic forms of the $\kappa_1(\eta)$ and $\kappa_2(\eta)$ in the previous subsections in the region with $a \ll T$ and $a \gg T$. Applying the padé approximant and the mentioned asymptotic forms, one can gain explicit functions for $\kappa_1(\eta)$ and $\kappa_2(\eta)$ fitting to the numerical
and as a result the parameters are the value of the Dilaton field at the horizon several values of the

\[
\kappa_1(\eta) = \frac{1}{\eta} \left( \frac{1 + \alpha_2(\eta/\pi)^2 + \alpha_4(\eta/\pi)^4 + \alpha_6(\eta/\pi)^6}{1 + \beta_2(\eta/\pi)^2 + \beta_4(\eta/\pi)^4} \right)^{\frac{1}{2}},
\]

\[
\kappa_2(\eta) = \frac{\eta}{\pi} \left( \frac{1 + \tilde{\alpha}_2(\eta/\pi)^2 + \tilde{\alpha}_4(\eta/\pi)^4}{1 + \tilde{\beta}_2(\eta/\pi)^2 + \tilde{\beta}_4(\eta/\pi)^4 + \tilde{\beta}_6(\eta/\pi)^6} \right)^{\frac{1}{2}}.
\]

Moreover, for both functions we need to take into account the last terms in the numerator and denominator to have the best fits with the numerical results in the region with \( T \sim a \) and as a result the parameters are

\[
\alpha_2 \simeq 0.9711, \quad \alpha_4 \simeq 0.1962, \quad \beta_2 = \alpha_2 - \frac{3}{4} \log 2,
\]

\[
\alpha_6 = \pi^2(2\pi^3 c_{\text{ent}})^{-3/4} \beta_4 \simeq 0.3462\beta_4,
\]

\[
\beta_4 = \frac{1}{96} \left( 3 + 96\alpha_4 - 2\pi^2 - 72\alpha_2 \log 2 + 99(\log 2)^2 \right),
\]

and

\[
\tilde{\alpha}_2 \simeq 1.7950, \quad \tilde{\alpha}_4 \simeq 0.0947, \quad \tilde{\beta}_2 = \frac{1}{12}(1 + 12\tilde{\alpha}_2 + 8\log 2),
\]

\[
\tilde{\beta}_4 = \frac{1}{1728} \left( -99 + 144\tilde{\alpha}_2 + 1728\tilde{\alpha}_4 + 32\pi^2 + 84\log 2 + 1152\tilde{\alpha}_2 \log 2 - 1041(\log 2)^2 \right),
\]

\[
\tilde{\beta}_6 = e^{-2\tilde{\alpha}_4} \simeq 0.3764\tilde{\alpha}_4.
\]

Note that the three parameters \( \alpha_6, \beta_2 \) and \( \beta_4 \) in (3.24) (and equivalently \( \tilde{\beta}_2, \tilde{\beta}_4 \) and \( \tilde{\beta}_6 \) in (3.25)) are fixed by the asymptotic behaviours of \( \kappa_1(\eta) \) and \( \kappa_2(\eta) \). It is important to notice that if we do not consider the last terms in the numerator and denominator of (3.23), all the free parameters can be fixed by the asymptotic forms of \( \kappa_1(\eta) \) and \( \kappa_2(\eta) \). In other word, one needs to have more terms in the expansion (3.16) and (3.17), for example up to \( (\frac{\eta}{\pi})^{10} \), to find all the free parameters, including \( \alpha_6(\tilde{\alpha}_2) \) and \( \beta_4(\tilde{\alpha}_4) \), by using the asymptotic behaviours.

According to our results in the previous section, from \( T(\eta, 0) = \frac{1}{\eta} e^{-\frac{7}{4}\phi_h} \) and (3.10b), it is straightforward to see that

\[
e^{\phi_h} = [\eta\kappa_2(\eta)]^{-\frac{7}{4}}.
\]

Considering (3.23), the above equation leads to

\[
e^{\phi_h} = \left( \frac{1 + \alpha_2(\eta/\pi)^2 + \alpha_4(\eta/\pi)^4 + \alpha_6(\eta/\pi)^6}{1 + \beta_2(\eta/\pi)^2 + \beta_4(\eta/\pi)^4} \right)^{-\frac{7}{4}}, \quad (3.27)
\]

where it has been plotted in the figure 2(left) (the solid red curve). As a cross-check, for several values of the \( \phi_h \) and \( u_h \) we numerically solve the equations of motion and then find the value of the Dilaton field at the horizon \( \phi_h \) in terms of \( \eta \). These results have been also shown in the figure 2(left) by blue dots. In addition, from (3.11b), we have

\[
Tu_h(a, T) = \kappa_1(\eta)\kappa_2(\eta)
\]
and similarly the above equation has been plotted in the figure 2(right) by using functions (3.23) and by solving the equations of motion numerically. These figures emphasize that the proposed functions in (3.23) are working well and also indicate that the $\phi_h$ and $u_h$ (or more generally the background components) are functions of $\eta$.

4 Holographic Setup of CME

Since the QGP is a strongly coupled system, the AdS/CFT correspondence is a noticeable candidate to explain its properties. Using the gravity dual, various properties of the plasma have been discussed. In particular, the CME attracted much attention and an interesting gravity description of it has been introduced in [13]. Such a description can be constructed of a supersymmetric intersection of $N_c$ D3-branes and $N_f$ rotating D7-branes as

$$
t \ x \ y \ z \ u \ S^3 \ \theta \ \psi
D3 \ \times \ \times \ \times \ \times
D7 \ \times \ \times \ \times \ \times \ \times
$$

(4.1)

where D7-branes are rotating with angular velocity $\omega$ in $\theta\psi$-plane. The value of the angular velocity is identified by the axial chemical potential $\mu_5$ or more precisely $\omega = 2\mu_5^2$. In the limit of large $N_c$ and large 't Hooft coupling constant $\lambda = g_{YM}^2 N_c$, the D3-branes are replaced by $AdS_5 \times S^5$ background (they are replaced by AdS-Schwarzchild background at finite temperature). The system then reduces to $N_f$ rotating D7-branes in the AdS-Schwarzchild background with a worldvolume constant magnetic field which is needed to produce the CME. In the probe limit where $N_f \ll N_c$, the dynamics of the $N_f$ D7-branes on the AdS-Schwarzchild background, which is the gravitational dual to $\mathcal{N} = 2$ SYM theory, is described by Driac-Born-Infeld (DBI) and Chern-Simons (CS) actions.

$^2$Consider $\mathcal{N} = 2$ SYM Lagrangian. After a chiral rotation $\psi \rightarrow e^{-i\gamma^5/2} \psi$, the following new term appears in the fermion's kinetic term

$$-rac{\partial_\mu \phi}{2} \bar{\psi} \gamma^\mu \gamma^5 \psi.$$ 

Using $\phi = \omega t$, it is evidently seen that $\omega = 2\mu_5$ (for more detail see [13]).
In the place of AdS-Schwarzschild background, let us start with a general background
\[ ds^2 = -g_{tt}dt^2 + g_{xx}(dx^2 + dy^2) + g_{zz}dz^2 + g_{uu}du^2 + g_{ss}ds^2_3 + g_{θθ}dθ^2 + g_{ψψ}dψ^2, \] (4.2)
which is asymptotically $AdS_5 \times S^5$. $u$ is the radial coordinate with the boundary at $u \to 0$. The $\mathcal{N} = 2$ SYM theory lives in Minkowski background with $t, x, y, z$. Moreover the above background contains a five-form field which asymptotically leads to the five-form field in the $AdS_5 \times S^5$ background (for instance see appendix B).

In the low energy limit the action for the $N_f$ D7-branes in a general background are given by
\[ S = S_{DBI} + S_{CS}, \]
\[ S_{DBI} = -N_f \tau_{D7} \int d^8ξ \ e^{-\phi} \sqrt{-\det(G_{ab} + 2\pi α' F_{ab})}, \]
\[ S_{CS} = N_f \tau_{D7} \int P[ΣC^{(n)}] e^{2\pi α' F}, \] (4.3)
where $G_{ab} = g_{MN} \partial_a X^M \partial_b X^N$ is the induced metric on the probe branes. $\tau_{D7}^{-1} = (2\pi)^7\ell_s g_s$ is the D7-brane tension. $ξ^a$ are worldvolume coordinates and the capital indices $M, N, ...$ are used to denote space-time coordinates. In our case the background metric $g_{MN}$ was introduced in $(4.2)$. $F_{ab}$ is the field strength of the gauge fields living on the D7-branes. As it was shown in $(4.1)$ the D7-branes extended along $t, x, y, z, S^3$ and the radial direction. In the CS action, $C^{(n)}$ denotes Ramond-Ramond form fields and $P[...]$ is the pull-back of the bulk fields to the worldvolume of D7-branes.

In order to describe the CME, the appropriate ansatz for the scalar fields are
\[ ψ(t, u) = ωt + φ(u), \quad θ(u), \] (4.4)
and for the gauge field we consider the following cases
\[ (i) \ A_z(u), \quad F_{xy} = B_z, \] (4.5a)
\[ (ii) \ A_y(u), \quad F_{xz} = B_y. \] (4.5b)
In the case of $(4.5a)$, the magnetic field is applied along the anisotropy direction. However, in $(4.5b)$ it is perpendicular to the anisotropy direction. Substituting the above configurations in the action $(4.3)$, we find
\[ S_{DBI} = - \int du \sqrt{Q_1 + Q_2 A_z^2(y) + Q_3 ϕ'^2}, \]
\[ S_{CS} = - \int Q_4 A_y^2(y), \] (4.6)
Notice that $τ = \frac{∂}{∂u}$ and
\[ Q_1 = Q_{z(y)}(g_{tt} - ω^2 g_{ψψ})(g_{uu} + g_{θθ}θ'^2), \] (4.7a)
\[ Q_2 = Q_{z(y)}(g_{tt} - ω^2 g_{ψψ})g_{uu}, \] (4.7b)
\[ Q_3 = Q_{z(y)}g_{tt}g_{ψψ}, \] (4.7c)
\[ Q_4 = N Bs^2 g_{ss}, \] (4.7d)
where

\[ Q_z = N^2 e^{-2\phi} g^{zz} g^{\frac{2}{g_{xx}}}(1 + \frac{B_z^2}{g_{xx}}), \]  
\[ Q_y = N^2 e^{-2\phi} g^{zz} g^{\frac{2}{g_{xx}}}(1 + \frac{B_y^2}{g_{xx}}), \]  
\[ N = \frac{\lambda N_c N_f}{(2\pi^4)}. \]

Since the action depends only on the derivative of \( A_z(y) \) and \( \varphi \), there are two constants of motion, i.e. \( \alpha = \frac{\partial S}{\partial \varphi'} \) and \( \beta = \frac{\partial S}{\partial A_z'(y)} \). After applying two successive Legendre-transformations with respect to \( \varphi' \) and \( A_z'(y) \), the final form of the action becomes

\[ \hat{S} = -\int du \sqrt{Q_1 Q_2} \sqrt{Q_2(1 - \frac{\alpha^2}{Q_3}) - (\beta + Q_4)^2}, \]

where the hat means that the Legendre-transformations have been applied. The location of the horizon on the probe branes, \( u_\star \), can be found by

\[ Q_{2s} = Q_2(u = u_\star) = 0. \]

Then the reality condition of the action implies that [13]

\[ \alpha = -\sqrt{Q_3}, \]  
\[ \beta = Q_4, \]

where \( Q_3(u) \) and \( Q_4(u) \) are evaluated at \( u = u_\star \).

According to the gauge-gravity correspondence, the expectation value of the dual operators \( J_z(y) \) and \( O_\varphi \) coupled to \( A_z(y) \) and \( \varphi \) can be found from the asymptotic expansions of \( A_z(y) \) and \( \varphi \). It was shown in [13] that

\[ \langle J_z(y) \rangle = -(2\pi \alpha')\beta, \]  
\[ \langle O_\varphi \rangle = \alpha. \]

As a result, \( \beta \) up to a constant gives the value of the CME on the gauge theory side. Also regarding the discussion about discrete space-time symmetries, \( \alpha \) is an order parameter of spontaneous symmetry breaking [13]. Furthermore, the asymptotic expansion of \( \theta \) is given by

\[ \theta(u) = \theta_0 u + \theta_3 u^3 + \ldots . \]

The leading term \( \theta_0 \) is proportional to to the mass of the fundamental matter and \( \langle O_m \rangle \propto \theta_3 \) where \( O_m \) is the dual operator to mass.

5 Numerical Results for the CME

Our goal in this section is to compute the value of the CME in terms of the quark mass in the anisotropic background (2.1). To do so, we should solve the equation of motion for
θ(u) to find the quark mass from (4.13). Since it is straightforward to solve this equation of motion, we do not mention it here and refer the reader to [13, 14]. It is worth noticing that, in order to find the solutions, β must be chosen on the brane horizon \( u^* \). Therefore, MEs and NEs (see appendix A) have non-zero current and contribute to the CME.

### 5.1 Zero Mass Case

Although in the presence of the magnetic field the trivial solution \( \theta(u) = 0 \) is not a favourable solution energetically, let us start with this exceptional case. Then (4.13) indicates that the mass of the fundamental matter is zero for this spatial case. Moreover, from (4.10), it is easy to see that the horizon on the probe D-branes coincides with the horizon in the background, i.e. \( u_* = u_h \). As a result, according to (4.11b) and (4.12a), the value of the CME is

\[
\langle J_z(y) \rangle = \langle J_{0z}(y) \rangle,
\]

(5.1)
Figure 4. The value of the CME as a function of the quark mass.

where $\langle J_{0}^{z}(y) \rangle = (2\pi\alpha')^{2}N_{f}B_{z}(y) = \frac{N_{c}N_{f}}{2\pi^{2}}aB_{z}(y)$ is the value of the CME in the isotropic background (or equivalently in the isotropic SYM theory) \cite{13, 14}. In other words, via a holographic calculation one realizes that the value of the CME is insensitive to the anisotropy of the system in the massless case.

5.2 Finite Mass Case

For non-trivial solutions $\theta(u) \neq 0$, as it was explained in (4.13), the asymptotic value of the $\theta(u)$, or more precisely $m \propto \lim_{\epsilon \to 0}\theta'(\epsilon)$, specifies the mass of the fundamental matter. The value of the mass generally depends not only on the magnetic field but also on the anisotropy parameter. In figure 3, we have plotted the critical embedding, horizons on the brane and in the background in terms of various values of the magnetic fields and two values of anisotropy parameter at a fixed temperature. Let us summarize the main points:

- In the presence of the anisotropy parameter for small but equal values of $B_{y}$ and $B_{z}$ the value of the mass does not change. However, the difference between $m_{B_{y}}$ and $m_{B_{z}}$ is significant for larger values of the magnetic fields (see the red curves in the figure 3).

- When a large magnetic field is applied along the anisotropy direction, the value of the mass can be smaller than it is in the isotropic case. Therefore, the masses for which the CME is non-zero is more restricted than the case with $a = 0$. Notice that for the small value of the magnetic fields, the allowable masses are extended.

- When the magnetic fields applied transverse to the anisotropy direction, the allowable masses with non-zero value of the CME are extended.

Some of the above results are in agreement with the ones obtained in \cite{18}.

In \cite{13}, it was shown that raising the temperature in the system will increase the value of the CME. This behaviour also persists in the presence of the $\alpha'$-correction \cite{14}. The numerical computation whose results are plotted in figure 4 indicates that this value increases as one raises the anisotropy in the system at fixed temperature. As a matter of fact, the anisotropy parameter somehow behaves similar to the temperature, in agreement
with the (numerical) results displayed in the literature [18, 19]. Moreover, notice that there is critical mass at which the CME vanishes. Higher values of the mass have zero current.

A Embeddings of the Probe Brane

In the probe limit, the embeddings of a probe D-brane are classified into three categories according to its shape in the anisotropic background. As it was stated in the introduction, MEs are those of embeddings that close off above the background horizon. In other words, there is no horizon on the probe branes. On the contrary, BE means that the probe brane sees the background horizon and its horizon is precisely coincident with the background one. It is well-known that the quark-antiquark bound states (mesons) are stable on the MEs. However, they are unstable on the BEs. In the presence of the electric field which is turned on the brane [15] or for rotating probe branes [16], a new group of embeddings (NEs) appears. As a matter of fact, for this group of solutions there is a horizon on the probe brane which is not coincident with the background horizon. On the MEs since the quark-antiquark bound states are stable, there are no free charge carriers and consequently no current and hence the system behaves as an insulator. Oppositely, on the BEs and NEs the bound states are unstable and as a result non-zero current is observed [13, 17].

A set of possible brane embeddings is presented in figure 5. The background horizon is located at \( u \sim 1.6 \) and therefore the blue and green curves show MEs and BEs respectively. The black dashed curve represents the horizon on the probe brane and since there is a cross point between red curves and this horizon, the red curves show NEs. Moreover note that NEs exist in a narrow region of mass.

B Five-Form Field Notation

The five-form is taken to be proportional to the volume form of the five-sphere

\[
F_5 = \alpha(\Omega_5 + \star \Omega_5)
\]
where ⋆ denotes Hodge star operator. Assuming that the metric of the five-sphere is
\[ ds_5^2 = d\theta_1^2 + \sin^2 \theta_1 d\phi_1 + \cos^2 \theta_1 (d\gamma_1^2 + \sin^2 \gamma_1 d\gamma_2^2 + \sin^2 \gamma_1 \sin \gamma_2 d\gamma_3^2), \]
and it is then straightforward to write the volume form of the five-sphere in terms of the above coordinates as
\[ \Omega_5 = \cos^3 \theta_1 \sin \theta_1 \sin^2 \gamma_1 \sin \gamma_2 \sin \gamma_2 \sin \gamma_2 d\theta_1 \wedge d\phi_1 \wedge d\gamma_1 \wedge d\gamma_2 \wedge d\gamma_3, \]
\[ \star \Omega_5 = -\frac{e^{-\frac{7}{4}\phi}}{u^5} \sqrt{B} \, dt \wedge dx \wedge dy \wedge dz \wedge du, \]
and therefore the components of the five-form are given by
\[ F_{txyzu} = -\frac{\alpha e^{-\frac{7}{4}\phi}}{u^5} \sqrt{B}, \]
\[ F_{\theta_1 \phi_1 \gamma_1 \gamma_2 \gamma_3} = \alpha \cos^3 \theta_1 \sin \theta_1 \sin^2 \gamma_1 \sin \gamma_2. \]
Since \( F_5 = dC_4 \), we consider the following ansatz
\[ C_4 = C_{txyz} \, dt \wedge dx \wedge dy \wedge dz + C_{\phi_1 \gamma_1 \gamma_2 \gamma_3} \, d\phi_1 \wedge d\gamma_1 \wedge d\gamma_2 \wedge d\gamma_3, \]
and then one can simply find that
\[ C_{txyz} = -4\alpha \int \frac{du}{u^5} e^{-\frac{7}{4}\phi} \sqrt{B}, \]
\[ C_{\phi_1 \gamma_1 \gamma_2 \gamma_3} = -\alpha \cos^4 \theta \sin^2 \gamma_1 \sin \gamma_2. \]

C The IR solution

For large values of the temperature, \( T \gg a \), it is possible to find analytic expressions for the metric and the dilaton [1]. In this appendix we will discuss and analytically find an interesting solution of the equation of motion (2.4) in the low temperature limit. To do so, let us start with the following ansatz
\[ \phi(\xi) = \lim_{\varepsilon \to 0} \left( \phi_h - \frac{4}{7} \log \xi + \varepsilon f(\xi) \right), \]
where we assume that \( f(\xi) \) and its derivative are finite at \( \varepsilon \to 0 \) and \( \xi = \frac{u}{u_h} \). The explicit form for the \( \phi_h \) can be found by using the consistency condition for the solution introduced in [1] (see Equ. (139)) which is given by
\[ \dot{\phi}_h = -\frac{4e^{-\frac{7}{4}\phi_h} u_h}{16 + e^{2\phi_h} u_h^2} \]
and we then have
\[ \phi_h = \frac{2}{7} \log \frac{16}{6u_h^2}. \]
Regarding above equation, by substituting ansatz (C.1) in the (2.4) one can see that it satisfies the equation of motion provided that we choose

\[ f(\xi) = c_1 \xi^{11/7 - \sqrt{55}/7} + c_2 \xi^{11/7 + \sqrt{55}/7} \]  

(C.4)

where \( c_1 \) and \( c_2 \) are arbitrary constants. Therefore, (C.1) leads to

\[ \phi(u) = \frac{4}{7} \log \frac{\sqrt{8/3}}{au}, \]  

(C.5)

and using (2.3) the other components of the metric can be found as below

\[ H(u) = \left( \frac{3}{8} \right)^{2/7} (au)^{4/7}, \]  

(C.6)

\[ F(u) = \frac{49}{11} \left( \frac{1}{18} \right)^{3/7} (au)^{2/7}, \]  

(C.7)

\[ B(u) = \frac{11}{49} 18^{3/7} \left( \frac{1}{au} \right)^{2/7}. \]  

(C.8)

(C.7) reveals that the proposed solution describes a zero temperature background or more precisely \( T \ll a \). As a result, this solution can be considered as a IR limit of a general anisotropic background. The temperature and entropy of the solution are then straightforward to compute form (3.3) and \( s = \frac{N_c^2}{2 \pi^2} e^{-\frac{5}{2} \phi_h} \) \([1]\). Hence we obtain

\[ \frac{T}{a} = \left( \frac{\sqrt{11}}{\pi 2^{17/14} 3^{3/7}} \right) \left( \frac{1}{au_h} \right)^{6/7}, \]  

(C.9)

\[ s = \frac{N_c}{2 \pi} \left( \frac{3}{8} \right)^{5/14} a^{5/7} u_h^{-16/7}. \]  

(C.10)

Eliminating \( u_h \) between two above equations, we find

\[ s = c_{\text{ent}} N_c^2 a^{1/3} T^{8/3} \]  

(C.11)

where \( c_{\text{ent}} = \frac{2^{7/6} 3^{2/3} 2^{5/3}}{11^{4/3}} = 3.2 \), compatible with the numerical value for \( c_{\text{ent}} \) in \([1]\).

References

[1] D. Mateos and D. Trancanelli, “Thermodynamics and Instabilities of a Strongly Coupled Anisotropic Plasma,” JHEP 1107, 054 (2011) [arXiv:1106.1637 [hep-th]].

[2] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal and U. A. Wiedemann, “Gauge/String Duality, Hot QCD and Heavy Ion Collisions,” arXiv:1101.0618 [hep-th].

[3] J. M. Maldacena, “The Large N limit of superconformal field theories and supergravity,” Adv. Theor. Math. Phys. 2, 231 (1998) [hep-th/9711200], S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from noncritical string theory,” Phys. Lett. B 428, 105 (1998) [hep-th/9802109], E. Witten, “Anti-de Sitter space and holography,” Adv. Theor. Math. Phys. 2, 253 (1998) [hep-th/9802150].
[4] J.-Y. Ollitrault, “Relativistic hydrodynamics for heavy-ion collisions,” Eur. J. Phys. 29, 275 (2008) [arXiv:0708.2433 [nucl-th]].

[5] R. Snellings, “Elliptic Flow: A Brief Review,” New J. Phys. 13, 055008 (2011) [arXiv:1102.3010 [nucl-ex]], M. Luzum, “Flow fluctuations and long-range correlations: elliptic flow and beyond,” J. Phys. G G 38, 124026 (2011) [arXiv:1107.0592 [nucl-th]].

[6] D. Kharzeev, “Parity violation in hot QCD: Why it can happen, and how to look for it,” Phys. Lett. B 633, 260 (2006) [hep-ph/0406125].

[7] D. Kharzeev and A. Zhitnitsky, “Charge separation induced by P-odd bubbles in QCD matter,” Nucl. Phys. A 797, 67 (2007) [arXiv:0706.1026 [hep-ph]].

[8] D. E. Kharzeev, L. D. McLerran and H. J. Warringa, “The Effects of topological charge change in heavy ion collisions: ‘Event by event P and CP violation’,” Nucl. Phys. A 803, 227 (2008) [arXiv:0711.0950 [hep-ph]].

[9] K. Fukushima, D. E. Kharzeev and H. J. Warringa, “The Chiral Magnetic Effect,” Phys. Rev. D 78, 074033 (2008) [arXiv:0808.3382 [hep-ph]].

[10] D. E. Kharzeev, “Topologically induced local P and CP violation in QCD x QED,” Annals Phys. 325, 205 (2010) [arXiv:0911.3715 [hep-ph]].

[11] E. Witten, “Anti-de Sitter space, thermal phase transition, and confinement in gauge theories,” Adv. Theor. Math. Phys. 2, 505 (1998) [hep-th/9803131].

[12] A. Karch and E. Katz, “Adding flavor to AdS / CFT,” JHEP 0206, 043 (2002) [hep-th/0205236].

[13] C. Hoyos, T. Nishioka and A. O’Bannon, “A Chiral Magnetic Effect from AdS/CFT with Flavor,” JHEP 1110, 084 (2011) [arXiv:1106.4030 [hep-th]].

[14] M. Ali-Akbari and S. F. Taghavi, “alpha’-Corrected Chiral Magnetic Effect,” Nucl. Phys. B 872, 127 (2013) [arXiv:1209.5900 [hep-th]].

[15] S. Nakamura, “Nonequilibrium Phase Transitions and Nonequilibrium Critical Point from AdS/CFT,” Phys. Rev. Lett. 109, 120602 (2012) [arXiv:1204.1971 [hep-th]]; M. Ali-Akbari and A. Vahedi, “Non-equilibrium Phase Transition from AdS/CFT,” Nucl. Phys. B 877, 95 (2013) [arXiv:1305.3713 [hep-th]].

[16] S. R. Das, T. Nishioka and T. Takayanagi, “Probe Branes, Time-dependent Couplings and Thermalization in AdS/CFT,” JHEP 1007, 071 (2010) [arXiv:1005.3348 [hep-th]]; M. Ali-Akbari, H. Ebrahim and Z. Rezaei, “Probe Branes Thermalization in External Electric and Magnetic Fields,” Nucl. Phys. B 878, 150 (2014) [arXiv:1307.5629 [hep-th]].

[17] K. -Y. Kim, J. P. Shock and J. Tarrio, “The open string membrane paradigm with external electromagnetic fields,” JHEP 1106, 017 (2011) [arXiv:1103.4581 [hep-th]].

[18] M. Ali-Akbari and H. Ebrahim, “Chiral Symmetry Breaking: To Probe Anisotropy and Magnetic Field in QGP,” Phys. Rev. D 89, 065029 (2014) [arXiv:1309.4715 [hep-th]].

[19] S. Chakraborty and N. Haque, “Holographic quark-antiquark potential in hot, anisotropic Yang-Mills plasma,” Nucl. Phys. B 874, 821 (2013) [arXiv:1212.2769 [hep-th]].