On the structure of spikes

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Abstract

Spikes are an important class of 3-connected matroids. For an integer \( r \geq 3 \), there is a unique binary \( r \)-spike denoted by \( Z_r \). When a circuit-hyperplane of \( Z_r \) is relaxed, we obtain another spike and repeating this procedure will produce other non-binary spikes. The \( es \)-splitting operation on a binary spike of rank \( r \), may not yield a spike. In this paper, we give a necessary and sufficient condition for the \( es \)-splitting operation to construct \( Z_{r+1} \) directly from \( Z_r \). Indeed, all binary spikes and many of non-binary spikes of each rank can be derived from the spike \( Z_3 \) by a sequence of The \( es \)-splitting operations and circuit-hyperplane relaxations.

Keywords: binary matroid, \( es \)-splitting operation, relaxation, spike.

1. Introduction

Azanchiler [1], [2] extended the notion of \( n \)-line splitting operation from graphs to binary matroids. He characterized the \( n \)-line splitting operation of graphs in terms of cycles of the respective graph and then extended this operation to binary matroids as follows. Let \( M \) be a binary matroid on a set \( E \) and let \( X \) be a subset of \( E \) with \( e \in X \). Suppose \( A \) is a matrix that represents \( M \) over \( GF(2) \). Let \( A_X^e \) be a matrix obtained from \( A \) by adjoining an extra row to \( A \) with this row being zero everywhere except in the columns corresponding to the elements of \( X \) where it takes the value 1, and then adjoining two columns labeled \( \alpha \) and \( \gamma \) to the resulting matrix such that the column labeled \( \alpha \) is zero everywhere except in the last row where it takes the value 1, and \( \gamma \) is the sum of the two column vectors corresponding to the elements \( \alpha \) and \( e \). The vector matroid of the matrix \( A_X^e \) is denoted by \( M_X^e \). The transition from \( M \) to \( M_X^e \) is called an \( es \)-splitting operation. We call the matroid \( M_X^e \) as \( es \)-splitting matroid.
Let $M$ be a matroid and $X \subseteq E(M)$, a circuit $C$ of $M$ is called an $OX$-circuit if $C$ contains an odd number of elements of $X$, and $C$ is an $EX$-circuit if $C$ contains an even number of elements of $X$. The following proposition characterizes the circuits of the matroid $M_X^e$ in terms of the circuits of the matroid $M$.

**Proposition 1.** Let $M = (E, \mathcal{C})$ be a binary matroid together with the collection of circuits $\mathcal{C}$. Suppose $X \subseteq E$, $e \in X$ and $\alpha, \gamma \notin E$. Then $M_X^e = (E \cup \{\alpha, \gamma\}, \mathcal{C}')$ where $\mathcal{C}' = (\cup_{i=0}^{5} C_i) \cup \Lambda$ with $\Lambda = \{e, \alpha, \gamma\}$ and $C_0 = \{C \in \mathcal{C} : C \text{ is an EX-circuit}\}$; $C_1 = \{C \cup \{\alpha\} : C \in \mathcal{C} \text{ and } C \text{ is an OX-circuit}\}$; $C_2 = \{C \cup \{e, \gamma\} : C \in \mathcal{C}, e \notin C \text{ and } C \text{ is an OX-circuit}\}$; $C_3 = \{(C \setminus e) \cup \{\gamma\} : C \in \mathcal{C}, e \in C \text{ and } C \text{ is an OX-circuit}\}$; $C_4 = \{(C \setminus e) \cup \{\alpha, \gamma\} : C \in \mathcal{C}, e \in C \text{ and } C \text{ is an EX-circuit}\}$; $C_5$ is the set of minimal members of $\{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \emptyset\}$ and each of $C_1$ and $C_2$ is an OX-circuit.

It is observed that the $es$-splitting of a 3-connected binary matroid may not yield a 3-connected binary matroid. The following result, provide a sufficient condition under which the $es$-splitting operation on a 3-connected binary matroid yields a 3-connected binary matroid.

**Proposition 2.** Let $M$ be a 3-connected binary matroid, $X \subseteq E(M)$ and $e \in X$. Suppose that $M$ has an $OX$-circuit not containing $e$. Then $M_X^e$ is a 3-connected binary matroid.

To define rank-$r$ spikes, let $E = \{x_1, x_2, ..., x_r, y_1, y_2, ..., y_r, t\}$ for some $r \geq 3$. Let $C_1 = \{\{t, x_i, y_i\} : 1 \leq i \leq r\}$ and $C_2 = \{\{x_i, y_i, x_j, y_j : 1 \leq i < j \leq r\}$. The set of circuits of every spike on $E$ includes $C_1 \cup C_2$. Let $C_3$ be a, possibly empty, subsets of $\{\{z_1, z_2, ..., z_r\} : z_i \in \{x_i, y_i\} \text{ for all } i\}$ such that no two members of $C_3$ have more than $r - 2$ common elements. Finally, let $C_4$ be the collection of all $(r + 1)$-element subsets of $E$ that contain no member of $C_1 \cup C_2 \cup C_3$.

**Proposition 3.** There is a rank-$r$ matroid on $E$ whose collection $\mathcal{C}$ of circuits is $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$.

The matroid $M$ on $E$ with collection $\mathcal{C}$ of circuits in the last proposition is called a rank-$r$ spike with tip $t$ and legs $L_1, L_2, ... L_r$ where $L_i = \{t, x_i, y_i\}$ for all $i$. In the construction of a spike, if $C_3$ is empty, the corresponding spike
is called the rank-r free spike with tip t. In an arbitrary spike M, each circuit in C_3 is also a hyperplane of M. Evidently, when such a circuit-hyperplane is relaxed, we obtain another spike. Repeating this procedure until all of the circuit-hyperplanes in C_3 have been relaxed will produce the free spike. Now let J_r and 1 be the r \times r and r \times 1 matrices of all ones. For r \geq 3, let A_r be the r \times (2r+1) matrix \([I_r|J_r-I_r|1]\) over GF(2) whose columns are labeled, in order, x_1, x_2, ..., x_r, y_1, y_2, ..., y_r, t. The vector matroid M[A_r] of this matrix is called the rank-r binary spike with tip t and denoted by Z_r. Oxley \[3\] showed that all rank-r, 3-connected binary matroids without a 4-wheel minor can be obtained from a binary r-spike by deleting at most two elements.

2. Circuits of Z_r

In this section, we characterize the collection of circuits of Z_r. To do this, we use the next well-known theorem.

**Theorem 4.** \[3\] A matroid M is binary if and only if for every two distinct circuits C_1 and C_2 of M, their symmetric difference, C_1 \Delta C_2, contains a circuit of M.

Now let M = (E, \mathcal{C}) be a binary matroid on the set E together with the set \mathcal{C} of circuits where E = \{x_1, x_2, ..., x_r, y_1, y_2, ..., y_r, t\} for some r \geq 3. Suppose Y = \{y_1, y_2, ..., y_r\}. For k in \{1, 2, 3, 4\}, we define \varphi_k as follows.

\[\varphi_1 = \{L_i = \{t, x_i, y_i\} : 1 \leq i \leq r\};\]
\[\varphi_2 = \{\{x_i, y_i, x_j, y_j\} : 1 \leq i < j \leq r\};\]
\[\varphi_3 = \{Z \subseteq E : |Z| = r, |Z \cap Y| \text{ is odd and } |Z \cap \{y_i, x_i\}| = 1 \text{ where } 1 \leq i \leq r\};\]
and
\[\varphi_4 = \begin{cases} \{E - C : C \in \varphi_3\}, & \text{if } r \text{ is odd;} \\ \{(E - C)\Delta\{x_{r-1}, y_{r-1}\} : C \in \varphi_3\}, & \text{if } r \text{ is even.} \end{cases}\]

**Theorem 5.** A matroid whose collection \mathcal{C} of circuits is \varphi_1 \cup \varphi_2 \cup \varphi_3 \cup \varphi_4, is the rank-r binary spike.

**Proof.** Let M be a matroid on the set E = \{x_1, x_2, ..., x_r, y_1, y_2, ..., y_r, t\} such that \mathcal{C}(M) = \varphi_1 \cup \varphi_2 \cup \varphi_3 \cup \varphi_4. Suppose Y = \{y_1, y_2, ..., y_r\}. Then, for every two distinct circuits C_1 and C_2 of \varphi_3, we have C_1 \cap Y \neq C_2 \cap Y and |C_j \cap \{x_i, y_i\}| = 1 for all i and j with 1 \leq i \leq r and j \in \{1, 2\}. We conclude that there is at least one y_i in C_1 such that y_i \notin C_2 and so x_i is in C_2 but it is not in C_1. Thus, no two members
of \( \varphi_3 \) have more than \( r - 2 \) common elements. It is clear that every member of \( \varphi_4 \) has \( (r + 1) \)-elements and contains no member of \( \varphi_1 \cup \varphi_2 \cup \varphi_3 \). By Proposition 3, we conclude that \( M \) is a rank-r spike. It is straightforward to show that for every two distinct members of \( C \), their symmetric difference contains a circuit of \( M \). Thus, by Theorem 4, \( M \) is a binary spike.

It is not difficult to check that if \( r \) is odd, then the intersection of every two members of \( \varphi_3 \) has odd cardinality and the intersection of every two members of \( \varphi_4 \) has even cardinality. Clearly, \( |\varphi_1| = r \), \( |\varphi_2| = \frac{r(r-1)}{2} \), and \( |\varphi_3| = |\varphi_4| = 2^{r-1} \). Therefore, every rank-r binary spike has \( 2^r + \frac{r(r+1)}{2} \) circuits. Moreover, \( \cap_{i=1}^{r} L_i \neq \emptyset \) and \( |C \cap \{ x_i, y_i \}| = 1 \) where \( 1 \leq i \leq r \) and \( C \) is a member of \( \varphi_3 \cup \varphi_4 \).

3. The es-splitting operation on \( Z_r \)

By applying the es-splitting operation on a given matroid with \( k \) elements, we obtain a matroid with \( k + 2 \) elements. In this section, our main goal is to give a necessary and sufficient condition for \( X \subseteq E(Z_r) \) with \( e \in X \), to obtain \( Z_{r+1} \) by applying the es-splitting operation on \( X \). Now suppose that \( M = Z_r \) be a binary rank-r spike with the matrix representation \([I_r|J_r-L_r|1]\) over \( GF(2) \) whose columns are labeled, in order \( x_1, x_2, \ldots, x_r, y_1, y_2, \ldots, y_r, t \). Suppose \( \varphi = \varphi_1 \cup \varphi_2 \cup \varphi_3 \cup \varphi_4 \) be the collection of circuits of \( Z_r \) defined in section 2. Let \( X_1 = \{ x_1, x_2, \ldots, x_r \} \) and \( Y_1 = \{ y_1, y_2, \ldots, y_r \} \) and let \( X \) be a subset of \( E(Z_r) \). By the following lemmas, we give six conditions for membership of \( X \) such that, for every element \( e \) of this set, \( (Z_r)^e_X \) is not the spike \( Z_{r+1} \).

**Lemma 6.** If \( r \geq 4 \) and \( t \notin X \), then, for every element \( e \) of \( X \), the matroid \( (Z_r)^e_X \) is not the spike \( Z_{r+1} \).

**Proof.** Suppose that \( t \notin X \). Without loss of generality, we may assume that there exist \( i \) in \( \{1, 2, \ldots, r\} \) such that \( x_i \in X \) and \( e = x_i \). By Proposition 1, the set \( \Lambda = \{ x_i, \alpha, \gamma \} \) is a circuit of \( (Z_r)^e_X \). Now consider the leg \( L_i = \{ t, x_i, y_i \} \), we have two following cases.

(i) If \( y_i \in X \), then \( |L_i \cap X| \) is even. By Proposition 1, the leg \( L_i \) is a circuit of \( (Z_r)^e_X \). Now if all other legs of \( Z_r \) have an odd number of elements of \( X \), by Proposition 1, we observe that these legs transform to circuits of cardinality 4 and 5. So there are exactly two 3-circuit in \( (Z_r)^e_X \). If not, there is a \( j \neq i \) such that \( L_j \) is a 3-circuit of \( (Z_r)^e_X \) and \( \Lambda \cap L_i \cap L_j = \emptyset \), we conclude that in each case, for every element \( e \) of \( X \), the matroid \( (Z_r)^e_X \) is not the spike \( Z_{r+1} \). Since \( Z_{r+1} \) has \( r + 1 \) legs and the intersection of the legs of \( Z_{r+1} \) is non-empty.
(ii) If $y_i \notin X$, then $|L_i \cap X|$ is odd. By Proposition 1, $(L_i \setminus x_i) \cup \gamma$ is a circuit of $(Z_r)^e_X$. Now if there is the other leg $L_j$ such that $|L_j \cap X|$ is even, then $L_j$ is a circuit of $(Z_r)^e_X$. But $(L_j \cap L \cap ((L_i \setminus x_i) \cup \gamma)) = \emptyset$, so $(Z_r)^e_X$ is not the spike $Z_{r+1}$. We conclude that every leg $L_j$ with $j \neq i$ has an odd number of elements of $X$. Since $x_i \notin L_j$, by Proposition 1 again, $L_j$ is not a 3-circuit in $(Z_r)^e_X$. Therefore, $(Z_r)^e_X$ has only two 3-circuits and so, for every element $e$ of $X$, the matroid $(Z_r)^e_X$ is not the spike $Z_{r+1}$.

**Lemma 7.** If $r \geq 4$ and $e \neq t$, then, for every element $e$ of $X - t$, the matroid $(Z_r)^e_X$ is not the spike $Z_{r+1}$.

**Proof.** Suppose that $e \neq t$. Without loss of generality, we may assume that there exist $i$ in $\{1,2,\ldots,r\}$ such that $x_i \in X$ and $e = x_i$. By Proposition 1, the set $\Lambda = \{x_i, \alpha, \gamma\}$ is a circuit of $(Z_r)^e_X$ and by Lemma 6 to obtain $Z_{r+1}$, the element $t$ is contained in $X$. Now consider the leg $L_i = \{t, x_i, y_i\}$. We have two following cases.

(i) If $y_i \in X$, then $|L_i \cap X|$ is odd. By Proposition 1, $L_i \cup \alpha$ and $(L_i \setminus x_i) \cup \gamma$ are circuits of $(Z_r)^e_X$. Now if there is the other leg $L_j$ such that $|L_j \cap X|$ is even, then $L_j$ is a circuit of $(Z_r)^e_X$. But $(L_j \cap L \cap ((L_i \setminus x_i) \cup \gamma)) = \emptyset$, so $(Z_r)^e_X$ is not the spike $Z_{r+1}$. We conclude that every leg $L_j$ with $j \neq i$ has an odd number of elements of $X$. Since $x_i \notin L_j$, by Proposition 1 again, $L_j$ is not a 3-circuit in $(Z_r)^e_X$. Therefore $(Z_r)^e_X$ has only two 3-circuit and so $(Z_r)^e_X$ is not the spike $Z_{r+1}$.

(ii) If $y_i \notin X$, then $|L_i \cap X|$ is even. So $L_i$ is a circuit of $(Z_r)^e_X$. By similar arguments in Lemma 6 (i), one can show that for every element $e$ of $X - t$, the matroid $(Z_r)^e_X$ is not the spike $Z_{r+1}$.

Next by Lemmas 6 and 7 to obtain the spike $Z_{r+1}$, we take $t$ in $X$ and $e = t$.

**Lemma 8.** If $r \geq 4$ and there is a circuit $C$ of $\varphi_3$ such that $|C \cap X|$ is even, then the matroid $(Z_r)^e_X$ is not the spike $Z_{r+1}$.

**Proof.** Suppose that $C$ is a circuit of $Z_r$ such that $C$ is a member of $\varphi_3$ and $|C \cap X|$ is even. Then, by Proposition 1, the circuit $C$ is preserved under the es-splitting operation. So $C$ is a circuit of $(Z_r)^e_X$. But $|C| = r$. Now if $r > 4$, then $C$ cannot be a circuit of $Z_{r+1}$, since it has no $r$-circuit, and if $r = 4$, then, to preserve the members of $\varphi_2$ in $Z_4$ under the es-splitting operation and to have at least one member of $\varphi_3$ which has even number of elements of $X$, the set $X$ must be $E(Z_r) - t$ or $t$. But in each case $(Z_4)^e_X$ has exactly fourteen 4-circuits, so it is not the spike $Z_5$, since this spike has exactly ten 4-circuit. We conclude that the matroid $(Z_r)^e_X$ is not the spike $Z_{r+1}$.

**Lemma 9.** If $r \geq 4$ and $|X \cap \{x_i, y_i\}| = 2$, for $i$ in $\{1,2,\ldots,r\}$, then the matroid $(Z_r)^e_X$ is not the spike $Z_{r+1}$ unless $r$ is odd and for all $i$, $\{x_i, y_i\} \subset X$, in which case $Z_{r+1}$ has $\gamma$ as a tip.
Proof. Suppose that \(\{x_i, y_i\} \subset X\) for \(i \in \{1, 2, \ldots, r\}\). Since \(t \in X\) and \(e = t\), after applying the es-splitting operation, the leg \(\{t, x_i, y_i\}\) turns into two circuits \(\{t, x_i, y_i, \alpha\}\) and \(\{x_i, y_i, \gamma\}\). Now consider the leg \(L_j = \{t, x_j, y_j\}\) where \(j \neq i\). If \(|L_j \cap X|\) is even (this means \(\{x_j, y_j\} \notin X\)), then \(L_j\) is a circuit of \((Z_r)_X^\gamma\). But \(\{x_i, y_i, \gamma\} \cap \{t, x_j, y_j\} = \emptyset\) and this contradicts the fact that the intersection of the legs of a spike is not the empty set. So \(\{x_j, y_j\}\) must be a subset of \(X\). We conclude that \(\{x_k, y_k\} \subset X\) for all \(k \neq i\). Thus \(X = E(Z_r)\). But in this case, \(r\) cannot be even since every circuit in \(\varphi_3\) has even cardinality and by Lemma \(\blacksquare\) the matroid \((Z_r)_X^\gamma\) is not the spike \(Z_{r+1}\).

Now we show that if \(X = E(Z_r)\), and \(r\) is odd, then \((Z_r)_X^\gamma\) is the spike \(Z_{r+1}\) with tip \(\gamma\). Clearly, every leg of \(Z_r\) has an odd number of elements of \(X\). Using Proposition 1, after applying the es-splitting operation, we have the following changes.

For \(i \in \{1, 2, \ldots, r\}\), \(L_i\) transforms to two circuits \((L_i \setminus t) \cup \gamma\) and \(L_i \cup \alpha\), every member of \(\varphi_2\) is preserved, and if \(C \in \varphi_3\), then \(C \cup \alpha\) and \(C \cup \{t, \gamma\}\) are circuits of \((Z_r)_X^\gamma\). Finally, if \(C \in \varphi_4\), then \(C\) and \((C \setminus t) \cup \{t, \gamma\}\) are circuits of \((Z_r)_X^\gamma\). Note that, since \(X = E(M)\) with \(e = t\), there are no two disjoint \(OX\)-circuits in \(Z_r\) such that their union be minimal. Therefore the collection \(C_5\) in Proposition 1 is empty. Now suppose that \(\alpha\) and \(t\) play the roles of \(x_{r+1}\) and \(y_{r+1}\), respectively, and \(\gamma\) plays the role of tip. Then we have the spike \(Z_{r+1}\) with tip \(\gamma\) whose collection \(\psi\) of circuits is \(\psi_1 \cup \psi_2 \cup \psi_3 \cup \psi_4\) where

\[
\psi_1 = \{(L_i \setminus t) \cup \gamma : 1 \leq i \leq r\} \cup \Lambda;
\psi_2 = \{\{x_i, y_i, x_j, y_j\} : 1 \leq i < j \leq r\} \cup \{(L_i \cup \alpha : 1 \leq i \leq r\};
\psi_3 = \{C \cup \alpha : C \in \varphi_3\} \cup \{C : C \in \varphi_4\};
\psi_4 = \{C \cup \{t, \gamma\} : C \in \varphi_3\} \cup \{(C \setminus t) \cup \{t, \gamma\} : C \in \varphi_4\}.
\]

In the following lemma, we shall use the well-known facts that if a matroid \(M\) is \(n\)-connected with \(E(M) \geq 2(n - 1)\), then all circuits and all cocircuits of \(M\) have at least \(n\) elements, and if \(A\) is a matrix that represents \(M\) over \(GF(2)\), then the cocircuit space of \(M\) equals the row space of \(A\).

**Lemma 10.** If \(|X| \leq r\), then the matroid \((Z_r)_X^\gamma\) is not the spike \(Z_{r+1}\).

Proof. Suppose \(X \subset E(Z_r)\) such that \(|X| \leq r\). Then, by Lemmas \(\blacksquare\) and \(\blacksquare\) \(t \in X\) with \(e = t\) and \(|X \cap \{x_i, y_i\}| = 1\) for all \(i \in \{1, 2, \ldots, r\}\). Therefore, there are at least two elements \(x_j\) and \(y_j\) with \(i \neq j\) not contained in \(X\) and so the leg \(L_j = \{t, x_j, y_j\}\) has an odd number of elements of \(X\). Thus, after applying
the es-splitting operation $L_j$ transforms to $\{x_j, y_j, \gamma\}$. Now let $L_k = \{t, x_k, y_k\}$ be another leg of $Z_r$. If $|L_k \cap X|$ is even, then $L_k$ is a circuit of $(Z_r)^t_X$. But $(L_k \cap \Lambda \cap \{x_j, y_j, \gamma\}) = \emptyset$. Hence, in this case, the matroid $(Z_r)^t_X$ is not the spike $Z_{r+1}$ . We may now assume that every other leg of $Z_r$ has an odd number of elements of $X$. Then, for all $j \neq i$, the elements $x_j$ and $y_j$ are not contained in $X$. We conclude that $|X| = 1$ and in the last row of the matrix that represents the matroid $(Z_r)^t_X$ there are two entries 1 in the corresponding columns of $t$ and $\alpha$. Hence, $(Z_r)^t_X$ has a 2-cocircuit and it is not the matroid $Z_{r+1}$ since spikes are 3-connected matroids.

By Lemmas 9 and 10 we must check that if $|x| = r + 1$, then, by using the es-splitting operation, can we build the spike $Z_{r+1}$?

**Lemma 11.** If $r \geq 4$ and $|X \cap X_1|$ be odd, then the matroid $(Z_r)^t_X$ is not the spike $Z_{r+1}$.

**Proof.** Suppose that $r$ is even and $|X \cap X_1|$ is odd. Since $t \in X$ and $|X| = r + 1$, so $|X \cap X_1|$ must be odd. Therefore the set $X$ must be $C \cup t$ where $C \in \varphi_3$. But $|C \cap X|$ is even and by Lemma 8, the matroid $(Z_r)^t_X$ is not the spike $Z_{r+1}$.

Now suppose that $r$ is odd, $r \geq 4$ and $|X \cap X_1|$ is odd. Then $|X \cap X_1|$ must be even and so $X = C$ where $C \in \varphi_4$. By definition of binary spikes, there is a circuit $C'$ in $\varphi_4$ such that $C' = C \Delta \{x_i, y_i, x_j, y_j\}$ for all $i$ and $j$ with $1 \leq i < j \leq r$. Clearly, $|(E - C') \cap X| = 2$. Since $(E - C')$ is a circuit of $Z_r$ and is a member of $\varphi_3$, by Lemma 8 the matroid $(Z_r)^t_X$ is not the spike $Z_{r+1}$.

Now suppose that $M$ is a binary rank-$r$ spike with tip $t$ and $r \geq 4$. Let $X \subseteq E(M)$ and $e \in X$ and let $E(M) - E(M_X^t) = \{\alpha, \gamma\}$ such that $\{e, \alpha, \gamma\}$ is a circuit of $M_X^t$. Suppose $\varphi = \cup_{i=0}^t \varphi_i$ be the collection of circuits of $M$ where $\varphi_i$ is defined in section 2. With these preliminaries, the next two theorems are the main results of this paper.

**Theorem 12.** Suppose that $r$ is an even integer greater than three. Let $M$ be a rank-$r$ binary spike with tip $t$. Then $M_X^t$ is a rank-$(r + 1)$ binary spike if and only if $X = C$ where $C \in \varphi_4$ and $e = t$.

**Proof.** Suppose that $M = Z_r$ and $X \subseteq E(M)$ and $r$ is even. Then, by combining the last six lemmas, $|X| = r + 1$; and $X$ contains an even number of elements of $X_1$ with $t \in X$. The only subsets of $E(Z_r)$ with these properties are members of $\varphi_4$. Therefore $X = C$ where $C \in \varphi_4$ and by Lemma 11 $e = t$. Conversely, let $X = C$ where $C \in \varphi_4$. Then, by using Proposition 11 every leg of $Z_r$ is preserved under the es-splitting operation since they have an even number of elements of $X$. 

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Moreover, for $i \in \{1, 2, ..., r\}$, every leg $L_i$ contains $e$ where $e = t$. So $L_i \setminus t$ contains an odd number of elements of $X$ and by Proposition 1, the set $(L_i \setminus t) \cup \{\alpha, \gamma\}$ is a circuit of $M_X^t$. Clearly, every member of $\varphi_2$ is preserved. Now let $C' \in \varphi_3$. Then $t \notin C'$. We have two following cases.

(i) Let $C' = (E - X) \Delta \{x_{r-1}, y_{r-1}\}$. Then $|C' \cap X| = 1$ and by Proposition 1, $C' \cup \alpha$ and $C' \cup \{t, \gamma\}$ are circuits of $M_X^t$.

(ii) Let $C' = (E-C'') \Delta \{x_{r-1}, y_{r-1}\}$ where $C'' \neq X$ and $C'' \in \varphi_4$. Since $|X| = r+1$ and $|C'' \cap X|$ is odd, the cardinality of the set $X \cap (E-C'')$ is even and so $|C' \cap X|$ is odd. Therefore, by Proposition 1 again, $C' \cup \alpha$ and $C' \cup \{t, \gamma\}$ are circuits of $M_X^t$.

Evidently, if $C \in \varphi_4$, then $|C \cap X|$ is odd and by Proposition 1, $C \cup \alpha$ and $(C \setminus t) \cup \gamma$ are circuits of $M_X^t$. Moreover, there are no two disjoint $OX$-circuits in $\varphi$. So the collection $C_5$ in Proposition 1 is empty. To complete the proof, suppose that $\alpha$ and $\gamma$ play the roles of $x_{r+1}$ and $y_{r+1}$, respectively, then we have the spike $Z_{r+1}$ with collection of circuits $\psi = \psi_1 \cup \psi_2 \cup \psi_3 \cup \psi_4$ where

$$
\psi_1 = \{L_i = \{t, x_i, y_i\} : 1 \leq i \leq r\} \cup \Lambda;
$$

$$
\psi_2 = \{(x_i, y_i, x_j, y_j) : 1 \leq i < j \leq r\} \cup \{(L_i \setminus t) \cup \{\alpha, \gamma\} : 1 \leq i \leq r\};
$$

$$
\psi_3 = \{C \cup \alpha : C \in \varphi_3\} \cup \{(C \setminus t) \cup \gamma : C \in \varphi_4\};
$$

$$
\psi_4 = \{C \cup \{t, \gamma\} : C \in \varphi_3\} \cup \{C \cup \alpha : C \in \varphi_4\}.
$$

Theorem 13. Suppose that $r$ is an odd integer greater than three. Let $M$ be a rank-$r$ binary spike with tip $t$. Then $M_X^t$ is a rank-$(r+1)$ binary spike if and only if $X = C \cup t$ where $C \in \varphi_3$ or $X = E(M)$, and $e = t$.

Proof. Suppose that $M = Z_r$ and $X \subseteq E(M)$. Let $X = E(M)$. Then, by Lemma 9, the matroid $M_X^t$ is the spike $Z_{r+1}$ with tip $\gamma$. Now, by combining the last six lemmas, $|X| = r + 1$ and $X$ contains an even number of elements of $X_1$ with $t \in X$. The only subsets of $E(Z_r)$ with these properties are in $\{C \cup t : C \in \varphi_3\}$. Conversely, let $X = C \cup t$ where $C \in \varphi_3$. Clearly, every member of $\varphi_3$ contains an odd number of elements of $X$. Now let $C'$ be a member of $\varphi_4$. If $C' = E(Z_r) - C$, then $C'$ contains an odd number of elements of $X$. If $C' \neq E(Z_r) - C$, then there is a $C'' \in \varphi_3$ such that $C' = E(Z_r) - C''$. Therefore $|C \cap C'| = |C \cap (E(Z_r) - C'')| = |C - (C \cap C'')|$ and so $|C \cap C'|$ is even. So $C'$ contains an odd number of elements of $X$ and, by Proposition 1 again, $C' \cup \alpha$ and $(C \setminus t) \cup \gamma$ are circuits of $M_X^t$.

Evidently, if $C_1$ and $C_2$ be disjoint $OX$-circuits of $Z_r$, then one of $C_1$ and $C_2$ is in $\varphi_3$ and the other is in $\varphi_4$ where $C_2 = E(Z_r) - C_1$, as $C_1 \cup C_2$ is not minimal, it follows by Proposition 1 that $C_5$ is empty. Now if $\alpha$ and $\gamma$ play the roles of
$x_{r+1}$ and $y_{r+1}$, respectively. Then $M^t_X$ is the spike $Z_{r+1}$ with collection of circuits $\psi = \psi_1 \cup \psi_2 \cup \psi_3 \cup \psi_4$ where

$$
\psi_1 = \{L_i = \{t, x_i, y_i\} : 1 \leq i \leq r\} \cup \Lambda;
$$

$$
\psi_2 = \{\{x_i, y_i, x_j, y_j\} : 1 \leq i < j \leq r\} \cup \{(L_i \setminus t) \cup \{\alpha, \gamma\} : 1 \leq i \leq r\};
$$

$$
\psi_3 = \{C \cup \alpha : C \in \varphi_3\} \cup \{(C \setminus t) \cup \gamma : C \in \varphi_4\};
$$

$$
\psi_4 = \{C \cup \{t, \gamma\} : C \in \varphi_3\} \cup \{C \cup \alpha : C \in \varphi_4\}.
$$

Remark 14. Note that the binary rank-3 spike is the Fano matroid denoted by $F_7$. It is straightforward to check that any one of the seven elements of $F_7$ can be taken as the tip, and $F_7$ satisfies the conditions of Theorem 13 for any tip. So, there are exactly 35 subset $X$ of $E(F_7)$ such that $(F_7)_X^e$ is the binary 4-spike where $e$ is a tip of it. Therefore, by Theorem 13 these subsets are $X = E(F_7)$ for every element $e$ of $X$ and $C \cup z$ for every element $z$ in $E(F_7)$ not contained in $C$ with $e = z$ where $C$ is a 3-circuit of $F_7$.

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