Persistence Exponent for the Simple Diffusion Equation: The Exact Solution for any Integer Dimension

Devashish Sanyal
Theoretical Condensed Matter
Institute of Physics
Bhubaneswar 751005, INDIA

(Dated: August 9, 2021)

The persistence exponent \(\theta_o\) for the simple diffusion equation \(\phi_t(x,t) = \Delta \phi(x,t)\), with random Gaussian initial condition, has been calculated exactly using a method known as selective averaging. The probability that the value of the field \(\phi\) at a specified spatial coordinate remains positive throughout for a certain time \(t\) behaves as \(t^{-\theta_o}\) for asymptotically large time \(t\). The value of \(\theta_o\), calculated here for any integer dimension \(d\), is \(\theta_o = \frac{4}{d}\) for \(d \leq 4\) and 1 otherwise. This exact theoretical result is being reported possibly for the first time and is not in agreement with the accepted values \(\theta_o = 0.12, 0.18, 0.23\) for \(d = 1, 2, 3\) respectively.

I. INTRODUCTION

The problem in the present paper is to find the persistence exponent for the simple diffusion equation \(\phi_t(x,t) = \Delta \phi(x,t)\). The diffusion equation is an equation that has no stochasticity. In the present paper the stochasticity is introduced through the random initial conditions. The problem is about evaluating the persistence exponent for the diffusion equation \(\phi_t(x,t) = \Delta \phi(x,t)\) with random Gaussian initial condition. It appears to remain an unsolved problem even though results \([1, 2]\) and several others have been reported. The problem of diffusion may require a better understanding in the context of persistence. The article tries to find an exact solution to the problem.

II. SIMPLE DIFFUSION EQUATION, RANDOM INITIAL CONDITIONS AND PERSISTENCE EXPONENT

The diffusion equation \(\phi_t = \Delta \phi\) is a coarse grained differential equation whose solution is uniquely determined by the initial condition. In the present problem, the initial condition is not fixed but is chosen from a distribution. The initial value of \(\phi\) at every coordinate is chosen from a Gaussian distribution with mean 0, variance \(k\), and the initial values of \(\phi\) at any two coordinates are statistically independent.

In order to calculate persistence exponent \(\theta_o\), we have to calculate the probability that the field \(\phi\) at a specified coordinate does not flip sign even once throughout a time \(t\). This probability behaves as \(P^+(t) = t^{-\theta_o}\) in the limit of asymptotically large time as \(P^+(t) \sim t^{-\theta_o}\). This is true for a non-stationary process like in the present case. In this article any position \(x\) is a vector quantity in a \(d\) dimensional space. The moments of the initial condition distribution described above are given by

\[
\langle \phi(x,0) \rangle = 0 \quad (1-a)
\]

\[
\langle \phi(x_1,0)\phi(x_2,0) \rangle = k\delta^{(d)}(x_1 - x_2) \quad (1-b)
\]

where \(k\) is the variance of the distribution. The solution for the diffusion equation may be written in terms of the initial condition as

\[
\phi(x,t) = \int d^dx'G(x - x',t)\phi(x',0) \quad (2)
\]
where $G(x, t) = (4\pi t)^{-d/2} \exp(-x^2/4t)$. The plan for the evaluation of the exponent is as follows. First, we have to calculate the probability of $\phi$ attaining a specific final value $\beta$ at a certain $x = x_o$ starting from a definite initial value $\alpha$ of $\phi$ at $x = x_o$. In order to evaluate it we use the method of selective averaging. The path that take the initial $\alpha$ to the final value $\beta$ also comprise those where $\phi(x_o)$ flips atleast once during time evolution. The probability of such paths is to be subtracted out. Finally, there has to be an integration over the final $\beta$ from 0 to $\infty$, followed by an integration over $\alpha$ from 0 to $\infty$.

Selective averaging means averaging over the initial field $\phi(x, 0)$, except when $x = x_o$. In other words, the averaging is done over all the initial configurations such that $\phi(x_o)$ is kept fixed at $\alpha$ (say) i.e $\phi(x_o, 0) = \alpha$ while for $x \neq x_o$ $\phi$ varies according to Gaussian distribution. In this paper the selective distribution, denoted by subscript $s$, is characterized by the moments,

$$\langle \phi(x, 0) \rangle_s = \alpha s$$

(3-a)

$$\langle \phi(x, 0) \phi(x, 0) \rangle_s = \{k + [\alpha^2 - k] s(d(d+1)/2) \} s$$

(3-b)

It may be verified from (3-a), (3-b) that if $x \neq x_o$, $x_1 \neq x_o$, $x_2 \neq x_o$ we get (1-a) (1-b) and for $x = x_1 = x_2 = x_o$, (2-a), (2-b) give $\alpha$, $\alpha^2$ as expected. Using (3-a), (3-b), we can calculate the moments of the random variable $\phi(x_o, t)$,

$$\langle \phi(x_o, t) \rangle_s = (4\pi t)^{-d/2} \alpha$$

(4)

$$\langle \phi^2(x_o, t) \rangle_s = \int d^d x_1' d^d x_2' (4\pi t)^{-d} \exp\left[-\frac{(x_o - x_1')^2}{4t} \right] \times \exp\left[-\frac{(x_o - x_2')^2}{4t} \right] \langle \phi(x_1', 0) \phi(x_2', 0) \rangle_s$$

$$= k \int d^d x_1' (4\pi t)^{-d} \exp\left[-\frac{(x_o - x_1')^2}{2t} \right]$$

$$= \frac{k}{(4\pi t)^d} + \frac{\alpha^2}{(4\pi t)^d}$$

(5)

While evaluating the second order moment, we have used the relation in (3-b). Hence the mean and the variance of the distribution for $\phi(x_o, t)$, represented by $\mu$ and $\sigma^2$ respectively, are

$$\mu = \langle \phi(x_o, t) \rangle_s = (4\pi t)^{-d/2} \alpha$$

(6-a)

$$\sigma^2 = \langle \phi^2(x_o, t) \rangle_s - \langle \phi(x_o, t) \rangle_s^2$$

$$= k(4\pi)^{-d/2}2^{(d-1)} k_d \Gamma(d/2) t^{-d/2} - k(4\pi t)^{-d}$$

(6-b)

In the above equation $k_d$ denotes the angular integration in $d$ dimensional space while $\Gamma$ represents the usual Gamma function. It may be mentioned that $\phi(x, t)$ in (2) is Gaussian irrespective of whether $\phi(x', 0)$, the initial Gaussian field, is correlated or not. In the present case, though, the initial field is uncorrelated and $\phi(x, t)$ can be proved to be Gaussian using characteristic functions in probability theory [10]. It may be noted that the $\delta$ function distribution is the limiting case of a Gaussian distribution. The expression for the conditional probability for starting at $\alpha$ and being between $\beta$ and $\beta + d\beta$ at time $t_1$ is

$$P(\beta|\alpha) d\beta = \frac{1}{\sqrt{2\pi \sigma}} \exp \left[-\frac{(\beta - \mu)^2}{2\sigma^2} \right] d\beta$$

(7)

where $\mu = \mu(\alpha, t_1)$ and $\sigma = \sigma(t_1)$. This probability considers all the paths that start from $\alpha$ to be between $\beta$ and $\beta + d\beta$ at time $t_1$ including ones that flip en route $\beta$ as depicted in Fig 1. Fig 1 is the projection of the trajectory of the system in the infinite dimensional $\Phi - t$ space on to the $\phi(x_o) - t$ plane.

![Fig 1: Projection of $\Phi - t$ trajectory onto the $\phi(x_o) - t$ plane](image)

$A(0, \alpha)$ represents the starting point and $B(t_1, \beta)$, the destination. $AB$ represents a path along which $\phi(x_o)$ does not flip and $ADB$ (blue curve) is a typical path along which $\phi(x_o)$ flips. Such paths have to be excluded. The probability of reaching from $A$ to the neighborhood $B$ at asymptotically large time $t_1$ without flipping is given by,

$$P^+(\beta|\alpha) d\beta = P(\beta|\alpha) d\beta - P(\beta|-\alpha) (1 + O(t^{-1})) d\beta$$

(8)

The second term represents the probability of paths such as $ADB$ originating from $A(0, -\alpha)$ and terminating in the neighborhood of $B$ at $t_1$. [8] is not be confused with the method of images in [17]. [8] follows a very different logic in the present case and holds good asymptotically. To prove [8] we will show that there is a one to one mapping from a path $A \to B$ to a path $A \to B$ and that the probability of two such paths converge...
asymptotically. This part is explained in (i) in what follows. Further, to justify (ii) we have to show that the “number” of paths $A \to B$ that flip and the “number” of paths $\bar{A} \to B$ converge asymptotically. This is done in (ii). In the subsequent analysis we will consider a $d$ dimensional lattice - lattice spacing being infinitesimally small- instead of continuum for the sake of notational convenience only. The reason for (ii) follows.

i) An initial configuration at $A$ of Fig 1 given by $X_{AB} = \{...\alpha_1, \alpha_2, ...\}$, is considered, where $\alpha_1, \alpha_2$, ... are the initial values of $\phi$ at coordinates $x \neq x_o$. The corresponding path takes initial $\phi(x_o) = \alpha$ to $B$, then it may be concluded from (2) that $X_{AB} = \{...f_\alpha, -\alpha, f_o, 2, ...\}$ ($f = \beta^{(4\pi t)}_{-1/2} - \beta^{-(4\pi t)}_{-1/2}$) is a configuration at $A$ which takes initial $\phi(x_o) = -\alpha$ to $B$. Hence there is a one to one mapping of paths from $A \to B$ to those from $A \to B$. It may be underlined here that there is a one to one correspondence between the paths from $A$ that flip to those from $\bar{A} \to B$. This may be used as it is a controlled approximation for it improves with increasing $t$. In order to see this point let us consider a point $C$ $(t_2, \beta)$ (not shown in the Fig 1) where $t_2 > t_1$. Let $Y_{AB} = \{...\gamma_1, \alpha, \gamma_2, ...\}$ be the initial configuration corresponding to path $ADB$ (the path in blue in Fig 1) where $\gamma_1, \gamma_2$, ... are the initial values of $\phi$ at coordinates $x \neq x_o$. This path crosses zero while reaching $B$. It can be shown that $Y_{AC} = \{...f_1\gamma_1, -\alpha, f_1\gamma_2, ...\}$ ($f_1 = \frac{d^{(4\pi t)}_{2} \beta^{-(4\pi t_2)}_{-1/2} + \beta^{-(4\pi t_2)}_{-1/2}}{\beta^{-(4\pi t_2)}_{-1/2} - \beta^{-(4\pi t_2)}_{-1/2}}$) is the corresponding initial configuration for a path $A \to C$. The exact expression for $f_1$ contains a coordinate dependent term whose leading order behavior for large $t$ is 1. Since $t_2 > t_1$, we have $f_1 > 1$ for sufficiently large $t_1$. Let the time coordinate at $D$ be $t_D$, then $\phi(x_o, t_D) = 0$ for the path $ADB$. Then one may arrive from (2) that $\phi(x_o, t_0) < 0$ for the initial configuration $Y_{AC}$. Hence, one can conclude that the path corresponding to $Y_{AC}$ must have flipped at an earlier time than $t_D$. Therefore, if a path from $A \to B$ flips, the corresponding path from $A \to C$ flips at an earlier time. Since $t_2 > t_1$, the ‘number’ of paths flipping while going from $A \to C$ is more than those from $A \to B$. Thus the ‘number’ of paths from $A \to B$ that flip is a fraction $f_2$ of those from $A \to B$ where $f_2 = 1 - O(t_1^{-\infty})$ for large $t_1$, $a$ being some positive number.

On account of (i), (ii) we say that the probability of the paths( like $ADB$ in Fig 1) that flip while reaching $B$ in the large time limit is given by $P(\beta - \alpha)h_{\text{correction}}$, where $h_{\text{correction}} = 1 + O(t^{-b})$, $b = 1$, being Taylor expansion in $t^{-1}$. In principle, the coefficient of $t^{-b}$ may be a function of $\beta$. When integrating over $\beta$ as will be done later the contribution to the integral comes from the vicinity of $\beta = -\mu \sim t^{-d/2}$ which is vanishingly small in the asymptotic limit. Also, $d\beta \sim \sigma \sim t^{-d/4}$. The coefficient is Taylor expanded about $\beta = 0$ and only the zeroth order term or the term independent of $\beta$ is retained. So the probability of $\phi(x_o)$ not changing sign when reaching the neighborhood($d\beta$) of $B$ is, for asymptotically large time,

$$\lim_{t \to \infty} \left[P(\beta|\alpha)d\beta - P(\beta - \alpha)h_{\text{correction}}d\beta\right] = P(\beta|\alpha)d\beta - P(\beta - \alpha)(1 + O(t^{-1}))d\beta$$

(9)

This leads us to (S). The final $\beta$ may have any value as long as it remains positive. The probability of $\phi(x_o)$ starting from $\alpha$ and reaching a final positive value without ever changing sign is

$$P^+(\alpha) = \int_0^\infty d\beta P^+(\beta|\alpha)$$

(10)

We would now calculate (10) for asymptotically large value of $t$. Under the circumstances the second term on the R.H.S of (9) can be neglected. Further $\frac{\mu^2}{2\sigma^2} \sim \alpha t^{-d/2}$. Hence for $\alpha \ll t^{d/2}$, $\mu^2/\alpha^2 \ll 1$. The expression (10) is evaluated using the identity

$$\int_0^\infty dx \exp \left(-\frac{x^2}{4\beta} - \gamma x\right) = \sqrt{\pi\beta} \exp(\beta\gamma^2)\left[1 - \text{erf}(\gamma \sqrt{\beta})\right]$$

(11)

Evaluation leads to a sum of two terms - one is proportional to $t^{-d/4}$ and the other is proportional to $t^{-1}$. Hence we obtain

$$P^+(\alpha) \sim \alpha t^{-d/4}$$

(12)

for $d \leq 4$. In arriving at the above result the asymptotic expansion of ‘error function’ $\text{erf}$ has been used for small argument. Finally, the expression for $P^+(t)$ is obtained by integrating $\alpha$ over a Gaussian distribution.

$$P^+(t) = \int_0^\infty d\alpha P^+(\alpha)Q(\alpha)$$

(13)

where $Q(\alpha)$ is the Gaussian distribution for initial $\phi(x_o, o) = \alpha$ with variance $k$ as mentioned at the beginning. If $k << t^{d/2}$, it may be concluded from (12) and (13) that $P^+(t) \sim t^{-d/4}$ or $t^{-1}$ depending on whether $d \leq 4$ or not. This gives $\theta_m = d/4$ or 1.

III. RESULT AND CONCLUSION:

In the previous section exact calculation has been carried out to determine the probability $P^+(t)$ of the sign of the field $\phi$ remaining positive through out an asymptotically large time $t$. The probability is $P^+(t) \sim t^{-d/4}$. Hence the persistence exponent is $\theta_m = d/4$ valid for any arbitrary integer dimension $d \leq 4$. The exponents for
is that the covariance/correlator/correlation function is a misleading quantity for the problem for reasons mentioned below. The model presented in the paper has randomness only in the initial condition. Once the system starts evolving there is no further randomness. It evolves in accordance with the kernel in (2). It is encoded in the initial condition when and where the \( \phi \) will flip. The probability of each path is uniquely determined by the probability of initial condition, hence the problem with covariance/correlator. The covariance function imposes stochasticity on the present problem throughout the entire time evolution. We now have a different model with the same correlation function but no unique dependence of the probability of the path on initial condition. It also makes the problem Markovian. Hence, all the previous results are in perfect agreement though the calculated exponent will be different from the actual value. The value of the exponent does not depend on only correlation function but it depends on other details of the model too. Further, there even appears to be experimental proof \( \text{(10)} \) for the results of \( \text{(1)}, \text{(2)} \). The experimental setup of \( \text{(10)} \) does not represent the diffusion model described in this paper. The setup satisfies the approximations of the previous papers and hence the agreement with the their result.

ACKNOWLEDGEMENTS

The author would like to thank CSIR, India for Fellowship during the course of the work (2004) at IACS, India.

[1] Satya N Majumdar, Clément Sire, Alan J Bray, and Stephen J Cornell. Nontrivial exponent for simple diffusion. Physical review letters, 77(14):2867, 1996.
[2] Bernard Derrida, Vincent Hakim, and Reuven Zeitak. Persistent spins in the linear diffusion approximation of phase ordering and zeros of stationary gaussian processes. Physical review letters, 77(14):2871, 1996.
[3] George CMA Ehrhardt and Alan J Bray. Series expansion calculation of persistence exponents. Physical review letters, 88(7):070601, 2002.
[4] HI Hilhorst. Persistence exponent of the diffusion equation in \( \varepsilon \) dimensions. Physica A: Statistical Mechanics and its Applications, 277(1-2):124–126, 2000.
[5] Grégory Schehr and Satya N Majumdar. Real roots of random polynomials and zero crossing properties of diffusion equation. Journal of Statistical Physics, 132(2):235–273, 2008.
[6] Mihail Poplavskyi and Grégory Schehr. Exact persistence exponent for the 2 d-diffusion equation and related kac polynomials. Physical review letters, 121(15):150601, 2018.
[7] A Barbier-Chebbah, O Benichou, and R Voituriez. Anomalous persistence exponents for normal yet aging diffusion. Physical Review E, 102(6):062115, 2020.
[8] Frank Aurzada and Thomas Simon. Persistence probabilities and exponents. In Lévy matters V, pages 183–224. Springer, 2015.
[9] Amir Dembo and Sumit Mukherjee. No zero-crossings for random polynomials and the heat equation. The Annals of Probability, 43(1):85–118, 2015.
[10] Glenn P Wong, Ross W. Mair, Ronald L Walsworth, and David G Cory. Measurement of persistence in 1d diffusion. Physical review letters, 86(18):4156, 2001.
[11] JM Schwarz and Ron Maimon. First-passage-time exponent for higher-order random walks: Using lévy flights. Physical Review E, 64(1):016120, 2001.
[12] Pierre Le Doussal, Cécile Monthus, and Daniel S Fishere. Random walkers in one-dimensional random environments: exact renormalization group analysis. Physical Review E, 59(5):4795, 1999.
[13] J Krug, H Kallabis, SN Majumdar, SJ Cornell, AJ Bray, and Clément Sire. Persistence exponents for fluctuating interfaces. Physical Review E, 56(3):2702, 1997.
[14] Pierre Le Doussal. The sinai model in the presence of dilute absorbers. Journal of Statistical Mechanics: Theory and Experiment, 2009(07):P07032, 2009.
[15] Magdalena Constantin and S Das Sarma. Volatility, persistence, and survival in financial markets. Physical Review Reviews in Science, 2015.
[16] Jon Mathews and Robert Lee Walker. *Mathematical methods of physics*, volume 501. WA Benjamin New York, 1970.

[17] S Chadrasekhar. Stochastic, statistical and hydromagnetic problems in physics and astronomy, selected papers vol. 3, 1989.

[18] Izrail Solomonovich Gradshteyn and Iosif Moiseevich Ryzhik. *Table of integrals, series, and products*. Academic press, 2014.

[19] Mark Kac. On the average number of real roots of a random algebraic equation. *Bulletin of the American Mathematical Society*, 49(4):314–320, 1943.