The Nevanlinna-Pick matrix interpolation in the Carathéodory class with infinite data both in the nondegenerate and degenerate cases.

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1 Introduction.

The classical Nevanlinna-Pick interpolation problem for the case of finite data appeared in papers [1], [2], and for the case of infinite data in [3]. Various matrix and operator-valued generalizations were introduced and investigated by different approaches afterwards. For a detailed exposition of the subject we refer to books [4], [5], [6], [7], [8].

In this paper we shall analyze the following problem (see Notations below). Let \( \{z_k\}_{k=0}^{\rho}; z_k \in \mathbb{D}, \) be prescribed distinct points: \( z_j \neq z_l, j \neq l, \) \( j, l \in 0, \rho; \rho \in \mathbb{N} \cup \{\infty\}. \) Let \( \{C_k\}_{k=0}^{\rho}; C_k \in \mathbb{C}_{N \times N}, \) be given. The problem is to find a \( \mathbb{C}_{N \times N} \)-valued analytic function \( T(z), z \in \mathbb{D}, \) which belongs to the Carathéodory class \( \mathcal{C}_N, \) subject to conditions:

\[
T(z_k) = C_k, \quad k = 0, \rho.
\] (1)

Here \( N \in \mathbb{N} \) and \( \rho \in \mathbb{N} \cup \{\infty\}, \) are fixed. This problem is said to be the Nevanlinna-Pick matrix interpolation problem in the Carathéodory class (with finite or infinite data). The problem is said to be determinate if it has a unique solution.

In 1957, Szőkefalvi-Nagy and Koranyi presented their famous pure operator approach to the operator-valued Nevanlinna-Pick interpolation [9], [10]. They derived conditions of the solvability for various operator-valued Nevanlinna-Pick problems. In particular, their results apply to the problem (1) with finite data (\( \rho < \infty \)). The latter problem (\( \rho < \infty \)) was investigated by Chen and Hu both in the nondegenerate and degenerate cases using a different method [11].

The aim of our present investigation is to develop the approach of Szőkefalvi-Nagy and Koranyi to obtain an analytic description of solutions for the problem (1) both in the nondegenerate and degenerate cases. In order to obtain an analytic description of solutions, we shall use important results of Chumakin on generalized resolvents of isometric operators [12], [13]. A similar approach was recently used in [14], [15], [16] to treat various matrix moment problems. Also, the necessary and sufficient conditions for the determinacy of the problem (1) are obtained. They become especially simple in the case \( \rho < \infty. \)
**Notations.** As usual, we denote by \( \mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+ \), the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), \( \mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \} \). The set of all complex vectors of size \( (N \times N) \) means its inverse. The set of all complex matrices of size \( (N \times N) \) we denote by \( \mathbb{C}_{N \times N} \).

If \( \rho \in \mathbb{Z}_+ \), the notation \( d \in 0, \infty \) means that \( d \in \mathbb{Z}_+ \).

Let \( M(x) \) be a left-continuous non-decreasing matrix function \( M(x) = (m_{k,l}(x))_{k,l=0}^{N-1} \) on \([0,2\pi]\), \( M(0) = 0 \), and \( \tau_M(x) := \sum_{k=0}^{N-1} m_{k,k}(x) \); \( \Psi(x) = (dm_{k,l}/d\tau_M)_{k,l=0}^{N-1} \). By \( L^2(M) \) we denote a set (of classes of equivalence) of \( \mathbb{C}_N \)-valued functions \( f \) on \([0,2\pi]\), \( f = (f_0, f_1, \ldots, f_{N-1}) \), such that (see, e.g., [17])

\[
\|f\|_{L^2(M)}^2 := \int_0^{2\pi} f(x)\Psi(x)f^*(x)d\tau_M(x) < \infty.
\]

The space \( L^2(M) \) is a Hilbert space with a scalar product

\[
(f,g)_{L^2(M)} := \int_0^{2\pi} f(x)\Psi(x)g^*(x)d\tau_M(x), \quad f, g \in L^2(M).
\]

We denote \( \delta_m = (\delta_{m,0}, \delta_{m,1}, \ldots, \delta_{m,N-1}) \), \( 0 \leq m \leq N-1 \), where \( \delta_{m,j} \) is Kronecker’s delta function.

By the Carathéodory class \( \mathcal{C}_N \) we mean a set of all analytic \( \mathbb{C}_{N \times N} \)-valued functions \( T(z) \) in \( \mathbb{D} \) such that \( T(z) + T^*(z) \geq 0 \).

If \( H \) is a Hilbert space then \((\cdot, \cdot)_H\) and \( \| \cdot \|_H \) mean the scalar product and the norm in \( H \), respectively. Indices may be omitted in obvious cases.

For a linear operator \( A \) in \( H \), we denote by \( D(A) \) its domain, by \( R(A) \) its range, by \( \text{Ker}A \) its null subspace (kernel), and \( A^* \) means the adjoint operator if it exists. If \( A \) is invertible then \( A^{-1} \) means its inverse. \( A^* \) means the closure of the operator, if the operator is closable. If \( A \) is bounded then \( \|A\| \) denotes its norm. For a set \( M \subseteq H \) we denote by \( \overline{M} \) the closure of \( M \) in the norm of \( H \). For an arbitrary set of elements \( \{x_n\}_{n \in I} \) in \( H \), we denote by \( \text{Lin}\{x_n\}_{n \in I} \) the set of all linear combinations of elements \( x_n \), and \( \text{span}\{x_n\}_{n \in I} := \text{Lin}\{x_n\}_{n \in I} \). Here \( I \) is an arbitrary set of indices. By \( E_H \) we denote the identity operator in \( H \), i.e. \( E_H x = x, \quad x \in H \). If \( H_1 \) is a subspace of \( H \), then \( P_{H_1} = P_{H_1}^H \) is an operator of the orthogonal projection on \( H_1 \) in \( H \).
2 Descriptions of solutions for the Nevanlinna-Pick problem and the determinacy.

Let $T(z)$ be a solution of the Nevanlinna-Pick problem. As usual, an important role will be played by the following function (kernel):

$$K(u, v) = \frac{1}{2(1 - uv)} (T(u) + T^*(v)), \quad u, v \in \mathbb{D}. \quad (2)$$

Since $T(z) \in \mathcal{C}_N$, it admits the following representation (e.g. [18]):

$$T(z) = iT_0 + \frac{2\pi}{2\pi} \int_0^2 \frac{e^{it} + z}{e^{it} - z} dF(t), \quad z \in \mathbb{D}, \quad (3)$$

where $T_0 = T_0^* \in \mathbb{C}^{N \times N}$, and $F(t)$ is a non-decreasing left-continuous $\mathbb{C}^{N \times N}$-valued function on $[0, 2\pi]$. Then

$$K(u, v) = \frac{1}{2(1 - uv)} \int_0^{2\pi} \left( \frac{e^{it} + u}{e^{it} - u} + \frac{e^{-it} + \overline{v}}{e^{-it} - \overline{v}} \right) dF(t)$$

$$= \int_0^{2\pi} \frac{1}{(e^{it} - u)(e^{it} - v)} dF(t), \quad u, v \in \mathbb{D}. \quad (4)$$

Consider the following block matrix (the Pick matrix):

$$P_\rho = (K(z_k, z_l))_{k,l=0}^\rho. \quad (5)$$

Let

$$f(t) = \sum_{k=0}^d \sum_{m=0}^{N-1} \alpha_{k,m} e^{it} \overline{e}_m, \quad \alpha_{k,m} \in \mathbb{C}, \quad d \in [0, \rho]. \quad (6)$$

We may write

$$0 \leq \int_0^{2\pi} f(t) dF(t) f^*(t)$$

$$= \sum_{k,l=0}^d \sum_{m,n=0}^{N-1} \sum_{k,l=0}^d \alpha_{k,m} \alpha_{l,n} \int_0^{2\pi} \frac{1}{e^{it} - z_k} \overline{e}_m dF(t) \frac{1}{(e^{it} - z_l)} \overline{e}_n$$

$$= \left( \alpha_{k,0}, \alpha_{k,1}, ..., \alpha_{k,N-1} \right) \int_0^{2\pi} \frac{1}{e^{it} - z_k} \frac{1}{(e^{it} - z_l)} dF(t) \left( \alpha_{l,0}, \alpha_{l,1}, ..., \alpha_{l,N-1} \right)^*$$
\[
= \sum_{k,l=0}^{d} (\alpha_{k,0}, \alpha_{k,1}, \ldots, \alpha_{k,N-1}) K(z_k, z_l)(\alpha_{l,0}, \alpha_{l,1}, \ldots, \alpha_{l,N-1})^* \]

\[
= \Lambda P_d \Lambda^*,
\]

where \( \Lambda = (\alpha_{0,0}, \alpha_{0,1}, \ldots, \alpha_{0,N-1}, \alpha_{1,0}, \alpha_{1,1}, \ldots, \alpha_{1,N-1}, \ldots, \alpha_{d,0}, \alpha_{d,1}, \ldots, \alpha_{d,N-1}) \).

Here we have used the rules for operations on block matrices. Therefore

\[
P_d \geq 0, \quad d \in \mathbb{0, \rho}.
\]

If \( \rho < \infty \) the latter relation means that \( P_\rho \geq 0 \).

Conversely, let the Nevanlinna-Pick problem \( \Pi \) be given and condition \( 7 \) be satisfied. Let

\[
P_\rho = (p_{k,l})_{k,l=0}^{\rho N + N - 1}, \quad p_{k,l} \in \mathbb{C}.
\]

Observe that

\[
p_{kN+m,N+n} = \tilde{e}_m K(z_k, z_l) \tilde{e}_n^*, \quad k, l \in \mathbb{0, \rho}, \quad 0 \leq m, n \leq N - 1,
\]

and

\[
K(z_k, z_l) = \frac{1}{2(1 - z_k \bar{z}_l)} (C_k + C_i^*), \quad k, l \in \mathbb{0, \rho}.
\]

By [10, Lemma] there exist a Hilbert space \( H \) and a sequence \( \{x_k\}_{k=0}^{\rho N + N - 1} \), \( x_k \in H \), such that

\[
(x_k, x_l)_H = p_{k,l}, \quad k, l \in \mathbb{0, \rho N + N - 1},
\]

and \( \overline{\text{span}} \{x_k\}_{k=0}^{\rho N + N - 1} = H \).

Assume that \( z_0 = 0 \). Set

\[
A_0 \sum_{k=1}^{d} \sum_{m=0}^{N-1} \alpha_{k,m} x_{kN+m} = \sum_{k=1}^{d} \sum_{m=0}^{N-1} \frac{\alpha_{k,m}}{z_k} (x_{kN+m} - x_m),
\]

\[
\alpha_{k,m} \in \mathbb{C}, \quad d \in \mathbb{0, \rho}.
\]

Let us check that this definition is correct. Suppose that

\[
\sum_{k=1}^{d} \sum_{m=0}^{N-1} \alpha_{k,m} x_{kN+m} = \sum_{k=1}^{d} \sum_{m=0}^{N-1} \beta_{k,m} x_{kN+m},
\]

\[
\beta_{k,m} \in \mathbb{C}, \quad d \in \mathbb{0, \rho}.
\]
where \( \alpha_{k,m}, \beta_{k,m} \in \mathbb{C}, \quad d \in (0, \rho). \) Set \( \gamma_{k,m} = \alpha_{k,m} - \beta_{k,m}, \quad 1 \leq k \leq d, \quad 1 \leq m \leq N - 1. \) We may write

\[
I := \left\| \sum_{k=1}^{d} \sum_{m=0}^{N-1} \frac{\gamma_{k,m}}{z_k} (x_{kN+m} - x_m) \right\|^2_H
\]

\[
= \sum_{k,k'=1}^{d} \sum_{m,m'=0}^{N-1} \frac{\gamma_{k,m} \gamma_{k',m'}}{z_k z_{k'}} (x_{kN+m} - x_m, x_{k'N+m'} - x_{m'})_H.
\]

By (8), (9), (10) we may write

\[
(x_{kN+m} - x_m, x_{k'N+m'} - x_{m'})_H = (x_{kN+m}, x_{k'N+m'})_H - (x_{kN+m}, x_{m'})_H - (x_m, x_{k'N+m'})_H + (x_m, x_{m'})_H
\]

\[
+ p_{m,m'} = \epsilon_m (K(z_k, z_k') - K(z_k, 0) - K(0, z_k') + K(0, 0)) \epsilon_{m'}
\]

\[
= \epsilon_m \left( \frac{1}{2(1 - z_k z_{k'})} (C_k + C_{k'}) - \frac{1}{2} (C_k + C_0) - \frac{1}{2} (C_0 + C_{k'}) \right)
\]

\[
+ \frac{1}{2} (C_0 + C_0^*) \epsilon_{m'}
\]

\[
= \frac{1}{2(1 - z_k z_{k'})} (C_k + C_{k'}) \epsilon_{m'} = z_k z_{k'} \epsilon_m K(z_k, z_k') \epsilon_{m'}
\]

\[
= z_k z_{k'} (\epsilon_m K(z_k, z_k') \epsilon_{m'}) (x_{kN+m}, x_{k'N+m'})_H.
\]

Therefore

\[
I = \sum_{k,k'=1}^{d} \sum_{m,m'=0}^{N-1} \gamma_{k,m} \gamma_{k',m'} (x_{kN+m}, x_{k'N+m'})_H
\]

\[
= \left( \sum_{k=1}^{d} \sum_{m=0}^{N-1} \gamma_{k,m} x_{kN+m}, \sum_{k'=1}^{d} \sum_{m'=0}^{N-1} \gamma_{k',m'} x_{k'N+m'} \right)_H = 0.
\]

Consequently, the definition of \( A_0 \) is correct. Let

\[
x = \sum_{k=0}^{d} \sum_{m=0}^{N-1} a_{k,m} x_{kN+m}, \quad y = \sum_{k'=1}^{d} \sum_{m'=0}^{N-1} b_{k',m'} x_{k'N+m'},
\]

where \( a_{k,m}, b_{k,m} \in \mathbb{C}, \quad d \in (0, \rho). \) By (12) we may write

\[
(A_0 x, A_0 y)_H
\]
\[
\sum_{k,k'=1, m,m'=0}^{d} a_{k,m}b_{k',m'} \frac{1}{z_{k}z_{k'}} (x_{kN+m} - x_{m}, x_{k'N+m'} - x_{m'}) H
\]

\[
= \sum_{k,k'=1, m,m'=0}^{d} a_{k,m}b_{k',m'} (x_{kN+m}, x_{k'N+m'}) H = (x, y) H.
\]

Therefore \(A_0\) is an isometric operator in \(H\). Set \(A = A_0\). By the definition of \(A\) we may write

\[
(E_H - z_k A)x_{kN+m} = x_m, \quad k \in 0, \rho, \quad 0 \leq m \leq N - 1;
\]

\[
x_{kN+m} = (E_H - z_k A)^{-1} x_m, \quad k \in 0, \rho, \quad 0 \leq m \leq N - 1. \quad (13)
\]

Let \(\widetilde{A} \supseteq A\) be a unitary operator in a Hilbert space \(\widetilde{H} \supseteq H\). Recall that the following operator-valued function \((12), (13)\):

\[
R_z(A) = P_{\widetilde{H}} (E_{\widetilde{H}} - z \widetilde{A})^{-1}, \quad z \in \mathbb{C}\setminus \mathbb{T},
\]

is said to be a \textit{generalized resolvent} of \(A\) (corresponding to \(\widetilde{A}\)).

Set

\[
T(z) = i \text{Im} C_0 + ((-E_H + 2R_z(A)|x_m, x_l)_H)_{m,l=0}^{N-1}, \quad z \in \mathbb{D}. \quad (15)
\]

Let us check that \(T(z)\) is a solution of the Nevanlinna-Pick problem \((11)\). In fact, the function \(T(z)\) has the following representation:

\[
T(z) = i \text{Im} C_0 + \left((-E_H + 2(E_{\widetilde{H}} - z \widetilde{A})^{-1}|x_m, x_l)_{\widetilde{H}}\right)_{m,l=0}^{N-1}
\]

\[
= i \text{Im} C_0 + \int_0^{2\pi} \left(-1 + \frac{2}{1 - ze^{-i\theta}}\right) d \left((\mathbb{G}_\theta x_m, x_l)_{\widetilde{H}}\right)_{m,l=0}^{N-1}
\]

\[
= i \text{Im} C_0 + \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d \left((\mathbb{G}_\theta x_m, x_l)_{\widetilde{H}}\right)_{m,l=0}^{N-1},
\]

where \(\{\mathbb{G}_\theta\}_{\theta \in [0, 2\pi]}\) is the left-continuous orthogonal resolution of unity of the operator \(A^{-1}\). Therefore \(T(z) \in \mathcal{C}_N\). By \((13), (11), (8), (9)\) we may write

\[
T(z_k) = i \text{Im} C_0 + \left((-E_H + 2(E_{\widetilde{H}} - z_k \widetilde{A})^{-1}|x_m, x_l)_{\widetilde{H}}\right)_{m,l=0}^{N-1}
\]

\[
= i \text{Im} C_0 + \left((-x_m + 2x_{kN+m}, x_l)_{\widetilde{H}}\right)_{m,l=0}^{N-1}
\]
\[
\begin{align*}
&= i \text{Im} C_0 + (-p_{m,l} + 2p_{kN+m,l})_{m,l=0}^{N-1} \\
&= i \text{Im} C_0 - K(0,0) + 2K(z_k,0) \\
&= \frac{1}{2} (C_0 - C_0^*) - \frac{1}{2} (C_0 + C_0^*) + C_k + C_0^* = C_k, \quad k \in [0, \rho].
\end{align*}
\]

Thus, \( T(z) \) is a solution of the Nevanlinna-Pick problem (1).

Let \( \hat{T}(z) \) be an arbitrary solution of the Nevanlinna-Pick problem (1). Let us show that it admits a representation of the form (15) with a generalized resolvent of \( A \). Consider the space \( L^2(F) \), where \( F = F(t) \) is taken from the representation (3) for \( \hat{T}(z) \). Let

\[
\begin{align*}
&f(t) = \sum_{k=0}^{d} \sum_{j=0}^{N-1} a_{k,j} \frac{1}{e^{it} - z_k} \bar{e}_j, \quad (16) \\
g(t) = \sum_{k'=0}^{d} \sum_{j'=0}^{N-1} b_{k',j'} \frac{1}{e^{it} - z_{k'}} \bar{e}_{j'}, \quad (17)
\end{align*}
\]

where \( a_{k,j}, b_{k',j'} \in \mathbb{C}, d \in [0, \rho] \). A set in \( L^2(F) \) of all (classes of equivalence of) functions of the form (16) we shall denote by \( M^2_0(F) \), and \( L^2_0(F) = M^2_0(F) \).

By (4), (8), (10) we may write:

\[
(f(t), g(t))_{L^2(F)} = \sum_{k,k'=0}^{d} \sum_{j,j'=0}^{N-1} a_{k,j} \bar{b}_{k',j'} \int_0^{2\pi} \frac{1}{e^{it} - z_k} \frac{1}{e^{it} - z_{k'}} \bar{e}_j \bar{e}_{j'} F(t) \bar{e}_j^*,
\]

\[
= \sum_{k,k'=0}^{d} \sum_{j,j'=0}^{N-1} a_{k,j} \bar{b}_{k',j'} e_j^* K(z_k, z_{k'}) \bar{e}_{j'}
\]

\[
= \sum_{k,k'=0}^{d} \sum_{j,j'=0}^{N-1} a_{k,j} \bar{b}_{k',j'} p_{kN+j, k'N+j'}
\]

\[
= \sum_{k,k'=0}^{d} \sum_{j,j'=0}^{N-1} a_{k,j} \bar{b}_{k',j'} (x_{kN+j}, x_{k'N+j'}) H
\]

\[
= \left( \sum_{k=0}^{d} \sum_{j=0}^{N-1} a_{k,j} x_{kN+j}, \sum_{k'=0}^{d} \sum_{j'=0}^{N-1} b_{k',j'} x_{k'N+j'} \right)_H. \quad (18)
\]
Consider the following operator:

\[ W_0 f(t) = \sum_{k=0}^{d} \sum_{j=0}^{N-1} a_{k,j} x_{kN+j}. \]  

Let us check that this operator is correctly defined as an operator from \( M^2_0(F) \) to \( H \). Let \( f(t) \) and \( g(t) \) have the form (16), (17). Suppose that they belong to the same class of equivalence in \( L^2(F) \):

\[ \| f(t) - g(t) \|_{L^2(F)} = 0. \]

By (18) we may write

\[ 0 = \left( \sum_{k=0}^{d} \sum_{j=0}^{N-1} (a_{k,j} - b_{k,j}) \frac{1}{e^{it} - z_k} \vec{e}_j, \sum_{k'=0}^{d} \sum_{j'=0}^{N-1} (a_{k',j'} - b_{k',j'}) \frac{1}{e^{it} - z_{k'}} \vec{e}_{j'} \right)_{L^2(F)} \]

\[ = \left( \sum_{k=0}^{d} \sum_{j=0}^{N-1} (a_{k,j} - b_{k,j}) x_{kN+j}, \sum_{k'=0}^{d} \sum_{j'=0}^{N-1} (a_{k',j'} - b_{k',j'}) x_{k'N+j'} \right)_{H} \]

\[ = \left\| \sum_{k=0}^{d} \sum_{j=0}^{N-1} a_{k,j} x_{kN+j} - \sum_{k=0}^{d} \sum_{j=0}^{N-1} b_{k,j} x_{kN+j} \right\|^2_{H}. \]

Thus, the operator \( W_0 \) is defined correctly. Relation (18) implies that \( W_0 \) is an isometric operator. Set \( W = \overline{W_0} \). The operator \( W \) is a unitary transformation which maps \( L^2_0(F) \) onto \( H \). Set

\[ L^2_1(F) := L^2(F) \ominus L^2_0(F). \]

The operator

\[ U := W \oplus E_{L^2_1(F)}, \]

is a unitary transformation which maps \( L^2(F) \) onto \( H_1 := H \oplus L^2_1(F) \). Consider the following unitary operator in \( L^2(F) \):

\[ Q f(t) = e^{-it} f(t), \quad f(t) \in L^2(F). \]

Then

\[ \hat{A} := UQU^{-1}, \]

is a unitary operator in \( H_1 \). Observe that

\[ \hat{A} x_{kN+j} = UQ \frac{1}{e^{it} - z_k} \vec{e}_j = U \frac{1}{e^{it} - z_k} \vec{e}_j \]
= U \frac{1}{z_k} \left( \frac{1}{e^{it} - z_k} - \frac{1}{e^{it}} \right) \bar{e}_j = \frac{1}{z_k} (x_{kN+j} - x_j),

where \( k \in [1, \rho], 0 \leq j \leq N-1 \). Therefore \( \hat{A} \supseteq A \). Let \( \{\hat{G}_\theta\}_{\theta \in [0, 2\pi]} \) is the left-continuous orthogonal resolution of unity of the operator \( \hat{A}^{-1} \). We may write:

\[
\int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} d(\hat{G}_t x_m, x_l)_{H_1} = \left( (E_{H_1} - \zeta \hat{A}^{-1})^{-1} x_m, x_l \right)_{H_1}
\]

\[
= \left( U^{-1}(E_{H_1} - \zeta \hat{A}^{-1})^{-1} U \frac{1}{e^{it} e_m}, \frac{1}{e^{it} e_l} \right)_{L^2(F)}
\]

\[
= \left( (E_{L^2(F)} - \zeta Q^{-1})^{-1} e_m, e_l \right)_{L^2(F)} = \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} e_m dF(t) e_l^*, \quad \zeta \in \mathbb{D}.
\]

From (3) it follows that \( T_0 = \text{Im} \hat{T}(0) = \text{Im} C_0 \). By (10), (3), (9), (8) we have

\[
\int_0^{2\pi} d(\hat{G}_t x_m, x_l)_{H_1} = (x_m, x_l)_{H_1} = p_{m,l}; \quad (21)
\]

\[
\int_0^{2\pi} \bar{e}_m dF(t) e_l^* = \bar{e}_m \left( \hat{T}(0) - iT_0 \right) e_l^*
\]

\[
= \bar{e}_m \left( C_0 - i \frac{C_0 - C_0^*}{2i} \right) e_l^* = \frac{1}{2} \bar{e}_m (C_0 + C_0^*) e_l^*
\]

\[
= \bar{e}_m K(0,0) e_l^* = p_{m,l}, \quad 0 \leq m, l \leq N - 1. \quad (22)
\]

By (20), (21), (22) we obtain that

\[
\psi(\zeta) := \frac{1}{2} \int_0^{2\pi} \frac{1 + \zeta e^{it}}{1 - \zeta e^{it}} d(\hat{G}_t x_m, x_l)_{H_1}
\]

\[
= -\frac{1}{2} \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} d(\hat{G}_t x_m, x_l)_{H_1} + \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} d(\hat{G}_t x_m, x_l)_{H_1}
\]

\[
= -\frac{1}{2} \int_0^{2\pi} \bar{e}_m dF(t) e_l^* + \int_0^{2\pi} \frac{1}{1 - \zeta e^{it}} \bar{e}_m dF(t) e_l^*
\]

\[
= \frac{1}{2} \int_0^{2\pi} \frac{1 + \zeta e^{it}}{1 - \zeta e^{it}} \bar{e}_m dF(t) e_l^*, \quad z \in \mathbb{D}. \quad (23)
\]

By the inversion formula ([19, p. 50]) we conclude that

\[
F(t) = \left( (\hat{G}_t x_m, x_l)_{H_1} \right)_{m,l=0}^{N-1} + c, \quad t \in [0, 2\pi], \ c = \text{const}. \quad (24)
\]
By (3), (24) we may write:

\[
\hat{T}(z) = iT_0 + \left( \int_0^{2\pi} e^{it} + z d(\hat{G}_t x_m, x_l)_{H_1} \right)_{m,l=0}^{N-1} = iT_0 + \left( \int_0^{2\pi} [1 + 2(1 - ze^{-it})^{-1}] d\hat{G}_t x_m, x_l \right)_{H_1}^{N-1}_{m,l=0} = i \Im C_0 + \left( \left( -E_{H_1} + 2(E_{H_1} - z \hat{A})^{-1} \right) x_m, x_l \right)_{H_1}^{N-1}_{m,l=0},
\]

where \( R_z(A) \) is a generalized resolvent of \( A \) (which corresponds to \( \hat{A} \)). Therefore \( \hat{T}(z) \) has a representation of the form (15).

**Theorem 1** Let the Nevanlinna-Pick problem (1) with \( z_0 = 0 \) be given and condition (7) hold. Let an operator \( A = A_0 \) be constructed for the Nevanlinna-Pick problem as in (11). All solutions of the Nevanlinna-Pick problem have the following form:

\[
T(z) = i \Im C_0 + \left( \left( -E_{H_1} + 2(R_z(A)_{m,l=0}) \right)_{m,l=0}^{N-1}, z \in \mathbb{D}, \tag{25}\right)
\]

where \( R_z(A) \) is a generalized resolvent of \( A \). Conversely, an arbitrary generalized resolvent of \( A \) generates by formula (25) a solution of the Nevanlinna-Pick problem (1).

Moreover, the correspondence between all generalized resolvents of \( A \) and all solutions of the Nevanlinna-Pick problem is bijective.

**Proof.** It remains to prove that different generalized resolvents of \( A \) produce different solutions of the Nevanlinna-Pick problem (1). Suppose that there exist unitary extensions \( \hat{A}_j \supseteq A \) in Hilbert spaces \( \hat{H}_j \supseteq H \), \( j = 1, 2 \), such that

\[
R^1_z(A) = P_{H_1}^H (E_{\hat{H}_1} - z\hat{A}_1)^{-1} \neq R^2_z(A) = P_{H_2}^H (E_{\hat{H}_2} - z\hat{A}_2)^{-1}, \tag{26}
\]

\[
((-E_{H_1} + 2R^1_z(A))_{x_m,x_l})_H = ((-E_{H_1} + 2R^2_z(A))_{x_m,x_l})_H,
\]

where \( 0 \leq m, l \leq N-1, z \in \mathbb{D} \). Set \( L_N = \text{Lin}\{x_k\}_{k=0}^{N-1}, L = \text{Lin}\{x_k\}_{k=0}^{N+N-1} \). From the latter relation by the linearity we get

\[
(R^1_z(A)x, y)_H = (R^2_z(A)x, y)_H, \quad x, y \in L_N, \quad z \in \mathbb{D}. \tag{27}
\]
Set $R_{j,z} = (E_{\tilde{H}_j} - z\tilde{A}_j)^{-1}$, $j = 1, 2$. Observe that

$$R_{j,z}(E_H - zA)x = (E_{\tilde{H}_j} - z\tilde{A}_j)^{-1}(E_{\tilde{H}_j} - z\tilde{A}_j)x = x, \quad x \in D(A);$$

$$R_{1,z}h = R_{2,z}h \in H, \quad h \in (E_H - zA)D(A);$$

$$R_{1}(A)h = R_{2}(A)h, \quad h \in (E_H - zA)D(A), \quad z \in \mathbb{D}. \quad (28)$$

Choose arbitrary $z \in \mathbb{D}\backslash\{0\}$, $x \in H$, $h \in (E_H \frac{1}{z}A)D(A)$. We may write

$$(R_{z}(A)x, h)_H = (R_{j,z}x, h)_H = (x, (E_{H_j} - R_{j,z}^*)h)_H$$

$$= (x, h)_H - (x, R_{z}(A)h)_H;$$

$$(R_{z}(A)x, h)_H = (R_{z}(A)x, h)_H, \quad x \in H, \quad h \in (E_H \frac{1}{z}A)D(A), \quad z \in \mathbb{D}\backslash\{0\}. \quad (29)$$

Choose arbitrary $x \in L$, $z \in \mathbb{C}\backslash \mathbb{T}$: $z \neq z_k$, $k = 1, \rho$. Let us check that there exists the following representation:

$$x = x_0 + x_1, \quad x_0 \in L_N, \quad x_1 \in (E_H - zA)D(A). \quad (30)$$

Here $x_0, x_1$ may depend on the choice of $z$ and $x$.

Let $x = \sum_{k=0}^{d} \sum_{m=0}^{N-1} \alpha_{k,m} x_{kN+m}, d \in \mathbb{0,}\rho$. Set

$$\beta_{k,m} = \frac{1}{1 - \frac{z}{z_k}} \alpha_{k,m}, \quad 1 \leq k \leq d, \quad 0 \leq m \leq N - 1;$$

$$h = \sum_{k=1}^{d} \sum_{m=0}^{N-1} \beta_{k,m} x_{kN+m} \in D(A).$$

Then

$$x_1 := (E_H - zA)h = h - Ah$$

$$= \sum_{k=1}^{d} \sum_{m=0}^{N-1} \beta_{k,m} x_{kN+m} - z \sum_{k=1}^{d} \sum_{m=0}^{N-1} \beta_{k,m} \frac{1}{z_k} (x_{kN+m} - x_m)$$

$$= \sum_{k=1}^{d} \sum_{m=0}^{N-1} \alpha_{k,m} x_{kN+m} + z \sum_{k=1}^{d} \sum_{m=0}^{N-1} \beta_{k,m} \frac{1}{z_k} x_m$$

$$= x - \sum_{m=0}^{N-1} \alpha_{0,m} x_m + z \sum_{k=1}^{d} \sum_{m=0}^{N-1} \beta_{k,m} \frac{1}{z_k} x_m = x - x_0,$$
where \( x_0 := \sum_{m=0}^{N-1} a_{0,m} x_m - z \sum_{k=1}^{d} \sum_{m=0}^{N-1} \beta_{k,m} \frac{1}{z_k} x_m \in L_N \). Therefore relation (30) holds.

Choose an arbitrary \( h \in L, z \in \mathbb{D} \setminus \{0\} \). By (30) we may write:

\[
h = h_0 + h_1, \quad h_0 \in L_N, \quad h_1 \in (E_H - \frac{1}{z} A) D(A).
\]

Choose an arbitrary \( x \in L_N \). By (29), (27) we may write

\[
(R_1^z(A) x, h)_H = (R_1^z(A) x, h_0)_H + (R_1^z(A) x, h_1)_H = (R_2^z(A) x, h)_H.
\]

Therefore

\[
R_1^z(A) x = R_2^z(A) x, \quad x \in L_N, \quad z \in \mathbb{D}. \quad (31)
\]

Choose an arbitrary \( g \in L, z \in \mathbb{D} : x \neq z_k, k = 1, \rho \). By (30) we may write:

\[
g = g_0 + g_1, \quad g_0 \in L_N, \quad g_1 \in (E_H - z A) D(A).
\]

By (28), (31) we get

\[
R_1^z(A) g = R_1^z(A) g_0 + R_1^z(A) g_1 = R_2^z(A) g_0 + R_2^z(A) g_1 = R_2^z(A) g;
\]

\[
R_1^z(A) = R_2^z(A), \quad z \in \mathbb{D}.
\]

Since \((R_2^z(A))^* = E_{H_j} - \frac{1}{z} R_j^z(A), j = 1, 2, z \in \mathbb{C}: \) \( |z| \neq 1, z \neq 0 \) (33), we conclude that \( R_1^z(A) = R_2^z(A) \). We obtained a contradiction with (26). \( \square \)

We shall use the following important result:

**Theorem 2** [13, Theorem 3] An arbitrary generalized resolvent \( R_\zeta \) of a closed isometric operator \( U \) in a Hilbert space \( H \) has the following representation:

\[
R_\zeta = [E - \zeta(U \oplus \Phi_\zeta)]^{-1}, \quad \zeta \in \mathbb{D}. \quad (32)
\]

Here \( \Phi_\zeta \) is an analytic in \( \mathbb{D} \) operator-valued function which values are linear contractions (i.e. \( \|\Phi_\zeta\| \leq 1 \)) from \( H \oplus D(U) \) into \( H \oplus R(U) \).

Conversely, each analytic in \( \mathbb{D} \) operator-valued function with above properties generates by relation (32) a generalized resolvent \( R_\zeta \) of \( U \).

Observe that relation (32) also shows that different analytic in \( \mathbb{D} \) operator-valued functions with above properties generate different generalized resolvants of \( U \).

Comparing the last two theorems we obtain the following result.
Theorem 3 Let the Nevanlinna-Pick problem (1) with \( z_0 = 0 \) be given and condition (7) hold. Let an operator \( A = A_0 \) be constructed for the Nevanlinna-Pick problem as in (11). All solutions of the Nevanlinna-Pick problem have the following form

\[
T(z) = i \text{Im} C_0 + \left( \left[ -E_H + 2 \left[ E - z(A \oplus \Phi_z) \right]^{-1} \right] x_m, x_l \right)_H \bigg|_{m,l=0}^{N-1}, \quad z \in \mathbb{D},
\]

where \( \Phi_z \) is an analytic in \( \mathbb{D} \) operator-valued function which values are linear contractions from \( H \otimes D(A) \) into \( H \otimes R(A) \).

Conversely, each analytic in \( \mathbb{D} \) operator-valued function with above properties generates by relation (33) a solution of the Nevanlinna-Pick problem (1).

Moreover, the correspondence between all analytic in \( \mathbb{D} \) operator-valued functions with above properties and all solutions of the Nevanlinna-Pick problem (1) is bijective.

Proof. The proof is obvious. \( \square \)

Corollary 1 Let the Nevanlinna-Pick problem (1) with \( z_0 = 0 \) be given and condition (7) hold. Let an operator \( A = A_0 \) be constructed for the Nevanlinna-Pick problem as in (11). The Nevanlinna-Pick problem (1) is determinate if and only if at least one of the defect numbers of \( A \) is equal to zero.

Proof. If one of the defect numbers of \( A \) is equal to zero, then there exists the unique function \( \Phi_z \equiv 0 \) of the required class. On the other hand, if both defect numbers of \( A \) are non-zero, then besides \( \Phi_z \equiv 0 \) there exist non-zero suitable constant functions \( \Phi_z \).

Conditions for the determinacy of the Nevanlinna-Pick problem (1) become especially simple in the case \( \rho < \infty \).

Theorem 4 Let the Nevanlinna-Pick problem (1) with \( z_0 = 0, \rho < \infty \), be given and condition (7) hold. The Nevanlinna-Pick problem (1) is determinate if and only if at least one of the following conditions hold:

(A) For each \( k \in 0, N-1 \) the following linear system of equations with unknowns \( a_{k,j} \) has a solution:

\[
\sum_{j=N}^{\rho N+N-1} a_{k,j} p_{j,l} = p_{k,l}, \quad l = 0, 1, ..., \rho N + N - 1.
\]
(B) For each \( j \in \overline{0,N-1} \) the following linear system of equations with unknowns \( b_{j,kN+m} \) has a solution:

\[
\sum_{k=1}^{\rho} \sum_{m=0}^{N-1} b_{j,kN+m}(p_{kN+m,l} - p_{m,l}) = p_{j,l}, \quad l = 0, 1, ..., \rho N + N - 1.
\]

(35)

Here \( p_{i,j} \) are taken from relation (3).

Proof. Let an operator \( A = \overline{A_0} \) be constructed for the Nevanlinna-Pick problem as in (11). By (10) we see that condition (A) is equivalent to the following relation:

\[
x_k = \sum_{j=N}^{\rho N + N - 1} a_{k,j} x_j, \quad k = 0, 1, ..., N - 1.
\]

The latter relation is equivalent to the condition \( D(A) = H \).

On the other hand, condition (B) is equivalent to the following relation:

\[
x_j = \sum_{k=1}^{\rho} \sum_{m=0}^{N-1} b_{j,kN+m} (x_{kN+m,l} - x_m), \quad j = 0, 1, ..., N - 1.
\]

The latter relation is equivalent to the condition \( R(A) = H \).

It remains to apply Corollary 1 to complete the proof. \( \square \)

Remark. As it was noted in (10), condition \( z_0 = 0 \) is not restrictive. If \( z_0 \neq 0 \), one may consider a fractional linear transformation

\[
u = u(z) = \frac{z - \overline{z}_0}{1 - \overline{z}_0 z},
\]

and seek for a \( C_{N \times N} \)-valued function \( R(u) \) in \( \mathbb{D} \), which belongs to \( C_N \), subject to

\[
R(u_k) = C_k, \quad k \in \overline{0,\rho}, \quad (36)
\]

where \( u_k := u(z_k) \). It is easy to see that the following relation:

\[
T(z) = R(u(z)), \quad z \in \mathbb{D}, \quad (37)
\]

establishes a bijective correspondence between all solutions of (36) and all solutions of (11).
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In this paper we study the Nevanlinna-Pick matrix interpolation problem in the Carathéodory class with infinite data (both in the nondegenerate and degenerate cases). We develop the Szőkefalvi-Nagy and Korányi operator approach to obtain an analytic description of all solutions of the problem. Simple necessary and sufficient conditions for the determinacy of the problem are given.