Masses and decay constants of $B_q$ mesons in
the QCD string approach

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Abstract

The relativistic string Hamiltonian is used to calculate the masses
and decay constants of $B_q$ mesons: they appear to be expressed
through only three fundamental values: the string tension $\sigma$, $\alpha_s$, and
the quark pole masses. The values $f_B = 186$ MeV, $f_{B_s} = 222$ MeV
are calculated while $f_{B_c}$ depends on the $c$-quark pole mass used,
namely $f_{B_c} = 440$ (424) MeV for $m_c = 1.40$ (1.35) GeV. For the $1P$
states we predict the spin-averaged masses: $\bar{M}(B_J) = 5730$ MeV and
$\bar{M}(B_{sJ}) = 5830$ MeV which are in good agreement with the recent
data of the D0 and CDF Collaborations, at the same time owning
to the string correction being by $\sim 50$ MeV smaller than in other
calculations.

1 Introduction

The decay constants of pseudoscalar (P) mesons $f_P$ can be directly mea-
sured in $P \to \mu \nu$ decays and therefore they can be used as an important

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criterium to compare different theoretical approaches and estimate their accuracy. Although during the last decade $f_P$ were calculated many times: in potential models [2, 3, 4], the QCD sum rule method [5], and in lattice QCD [6, 7], here we again address the properties of the $B, B_s, B_c$ mesons for several reasons.

First, we use here the relativistic string Hamiltonian (RSH) [8], which is derived from the QCD Lagrangian with the use of the field correlator method (FCM) [9] and successfully applied to light mesons and heavy quarkonia [10] [11]. Here we show that the meson Green’s function and decay constants can also be derived with the use of FCM.

Second, the remarkable feature of the RSH $H_R$ and also the correlator of the currents $G(x)$ is that they are fully determined by a minimal number of fundamental parameters: the string tension $\sigma$, $\Lambda_{\overline{MS}}(n_f)$, and the pole (current) quark masses $m_q(\bar{m}_q)$. All these parameters are taken to be fixed from our analysis of heavy quarkonium spectra [10] and light meson Regge trajectories [11]:

$$\sigma = 0.18 \text{ GeV}^2; \quad \Lambda_{\overline{MS}}^{(4)} = 250(5)\text{MeV};$$

and the pole masses taken are

$$m_{u(d)} = 0; \quad m_s = 170(10) \text{ MeV}; \quad m_c = 1.40 \text{ GeV}; \quad m_b = 4.84 \text{ GeV}. \quad (2)$$

Third, recently new data on the masses of $B_c$ and the $P$-wave mesons: $B_1$, $B_2$, and $B_{s2}$ have been reported by the D0 and CDF Collaborations [12] [13], which give additional information on the $B_q$-meson spectra. Here we calculate the spin-averaged masses of the $P$-wave states $B$ and $B_s$.

We would like to emphasize here that in our relativistic calculations no constituent masses are used. In the meson mass formula an overall (fitting) constant, characteristic for potential models, is absent and the whole scheme appears to be rigid.

Nevertheless, we take into account an important nonperturbative (NP) self-energy contribution to the quark mass, $\Delta_{SE}(q)$ (see below eq. [15]). For the heavy $b$ quark $\Delta_{SE}(b) = 0$ and for the $c$ quark $\Delta_{SE}(c) \simeq -20 \text{ MeV}$ [10], which is also small.

For any kind of mesons we use a universal static potential with pure scalar confining term,

$$V_0(r) = \sigma r - \frac{4 \alpha_B(r)}{3} \frac{1}{r},$$

$$2$$
where the coupling $\alpha_B(r)$ possesses the asymptotic freedom property and saturates at large distances with $\alpha_{\text{crit}}(n_f = 4) = 0.52$ [14]. The coupling can be expressed through $\alpha_B(q)$ in momentum space,

$$\alpha_B(r) = \frac{2}{\pi} \int_0^\infty dq \frac{\sin qr}{q} \alpha_B(q), \quad (4)$$

where

$$\alpha_B(q) = \frac{4\pi}{\beta_0 t_B} \left( 1 - \frac{\beta_1 \ln t_B}{\beta_2 t_B} \right) \quad (5)$$

with $t_B = \ln \frac{q^2 + M_B^2}{\Lambda_B^2}$. Here the QCD constant $\Lambda_B$, is expressed as [15]

$$\Lambda_B(n_f) = \Lambda_{\text{MS}} \exp \left\{ \frac{1}{2\beta_0} \cdot \left( \frac{31}{3} - \frac{10}{9} n_f \right) \right\} \quad (6)$$

and $M_B(\sigma, \Lambda_B) = (1.00 \pm 0.05)$ GeV is the so called background mass [14]. For heavy-light mesons with $\Lambda_{\text{MS}}(n_f = 4) = 250(5)$ MeV one obtains $\Lambda_B(n_f = 4) = 355(7)$ MeV.

### 2 String Hamiltonian

At the first stage (1993-2005) the RSH was derived and applied to mesons, glueballs, hybrids, and baryons [9, 10]. In all cases a good agreement with experiment and lattice results have been obtained for the same minimal set of parameters.

In this work the same method is applied to the correlator of the currents which defines the decay constant $f_{\Gamma}$ in any channel $\Gamma$. For a meson the RSH can be presented as [8, 9]

$$H = H_0 + \Delta H, \quad (7)$$

where $\Delta H = V_{LS} + V_{SS} + V_T + V_{SE}$ is treated as a perturbation and every term can be derived within the same method. The unperturbed RSH was deduced in [8]:

$$H_0 = \sum_{i=1,2} \left( \frac{\omega_i}{2} + \frac{m_i^2 + P^2}{2\omega_i} \right) + V_0(r); \quad (8)$$

$$H_0 \varphi_n = M_n \varphi_n. \quad (9)$$
In \( m_1 (m_2) \) is the pole (current) mass of a quark (antiquark). The variable \( \omega_i \) is defined from extremum condition, which is taken either from

(1) The exact condition: \( \frac{\partial H_0}{\partial \omega_i} = 0 \), which gives

\[
\omega_i = \sqrt{p^2 + m_i^2}.
\]  

Then

\[
H_0 \varphi_n = \left\{ \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + V_0 (r) \right\} \varphi_n = M_n \varphi_n
\]

reduces to the Salpeter equation, which just defines \( \omega_i (n) = \langle \sqrt{p^2 + m_i^2} \rangle_n \) as a constituent mass.

(2) The approximate condition: \( \frac{\partial M_n}{\partial \tilde{\omega}_i} = 0 \) (the so-called einbein approximation). As shown in \[9\] the difference between \( \omega_i \) and \( \tilde{\omega}_i \) is \( \lesssim 5\% \).

For the RSH \[7\] the spin-averaged mass \( M(n, L) \) is given by a simple expression:

\[
M(nL) = \frac{\omega_1}{2} + \frac{\omega_b}{2} + \frac{m_1^2}{2\omega_1} + \frac{m_b^2}{2\omega_b} + E_n (\mu) - \frac{2\sigma \eta_f}{\pi \omega_1} \Delta_{str} (L \neq 0),
\]

with

\[
\omega_i (nL) = \langle \sqrt{p^2 + m_i^2} \rangle_{nL}; \quad \mu = \frac{\omega_1 \omega_b}{\omega_1 + \omega_b}.
\]

In the general case, the self-energy term \( \Delta_{SE} \) is shown to be defined by the analytic formula \[16\]

\[
\Delta_{SE} (q_f) = -\frac{2\sigma \eta_f}{\pi \omega_f};
\]

with \( \eta_f = 0.9 \) for a \( u(d) \) quark, \( \eta_f \approx 0.7 \) for an s quark, \( \eta_f = 0.4 \) for a c quark, and \( \eta_b = 0 \). Therefore, for a \( b \) quark \( \Delta_{SE} (b) = 0 \). The mass formula \[12\] does not contain any overall constant \( C \). Note that the presence of \( C \) violates linear behavior of Regge trajectories.

The calculated masses of the low-lying states of \( B, B_s, \) and \( B_c \) mesons are given in Table \[2\] as well as their values taken from \[2, 3, 6, 7\].

It is of interest to notice that in our calculations the masses of the \( P \)-wave states appear to be by 30-70 MeV lower than in \[2\] due to taking into account a string correction \[11\].
Table 1: Masses of the low-lying $B_q$ mesons in the QCD String Approach

| Meson   | $M(nL)$       | Exp.                  |
|---------|---------------|-----------------------|
| $B$     | $5280(5)^a$   | $5279.0(5)$           |
|         | $5310^2$      |                       |
|         | $5275^3$      |                       |
| $B^*$   | $5325^a$      | $5325.0(6)$           |
|         | $5370^2$      |                       |
|         | $5326^3$      |                       |
| $B_1(1P)$ | $M = 5730^a$  | $5721(8)$ D0          |
|         |               | $5734(5)$ CDF         |
| $B_2(1P)$ | $M = 5730^a$  | $5746(10)$ D0         |
|         |               | $5738(6)$ CDF         |
| $B_s$   | $5369^a$      | $5369.6(24)$          |
|         | $5390^2$      |                       |
|         | $5362^3$      |                       |
| $B^*_s$ | $5416^a$      | $5411.7(32)$          |
|         | $5450^2$      |                       |
|         | $5414^3$      |                       |
| $B_{s2}$ | $M = 5830$    | $5839(3)$ D0          |
|         |               | $5880^2$              |
| $B_c$   | $6280(5)^a$   | $6275(7)$ CDF         |
|         | $6271^2$      |                       |
|         | $6304(12)^6$  |                       |
| $B^*_c$ | $6330(5)^a$   | $6321(20)^6$          |
|         | $6338^2$      |                       |
|         | $6321(20)^6$  |                       |

$^a$ Masses are calculated in this paper.
3 Current Correlator

The FCM can be also used to define the correlator $G_{\Gamma}(x)$ of the currents $j_{\Gamma}(x)$,

$$j_{\Gamma}(x) = \bar{\psi}_1(x)\Gamma\psi_2(x),$$

(15)

for $S, P, V$, and $A$ channels (here the operator $\Gamma = t^a \otimes (1, \gamma_5, \gamma_\mu, i\gamma_\mu\gamma_5)$). The correlator,

$$G_{\Gamma}(x) \equiv \langle j_{\Gamma}(x)j_{\Gamma}(0) \rangle_{\text{vac}},$$

(16)

with the use of spectral decomposition of the currents $j_{\Gamma}$ and the definition,

$$\langle \text{vac}|\bar{\psi}_1\gamma^0\gamma_5\psi_2|P_n(k=0)\rangle = f_P^PM_n, \quad (A, P)$$

$$\langle \text{vac}|\bar{\psi}_1\gamma^\mu\psi_2|V_n(k, \varepsilon)\rangle = f_V^VM_n\varepsilon^\mu, \quad (V)$$

(17)

can be presented as

$$\int G_{\Gamma}(x)dx = \sum_n \frac{M_n}{2} |f_n^P|^2 e^{-M_nT}.$$ 

(18)

On the other hand, applying the FCM and RSH, a very useful relation can be derived:

$$\int G_{\Gamma}(x)dx = \frac{N_c}{\omega_1\omega_2} (0|Y_{\Gamma}e^{-H_0T}|0)$$

$$= \sum_n \frac{|\varphi_n(0)|^2}{\omega_1\omega_2} e^{-M_nT}.$$ 

(19)

Here $Y_P(Y_V)$ for the $P(V)$ channel is given by a simple expression:

$$Y_P = m_1m_2 + \omega_1\omega_2 - \langle \mathbf{p}^2 \rangle,$$

$$Y_V = m_1m_2 + \omega_1\omega_2 + \frac{1}{3}\langle \mathbf{p}^2 \rangle.$$ 

(20)

Then from Eqs. (18) and (19) one obtains the following analytical expression for the decay constants (for a given state labelled $n$):

$$\left[ f_n^{P(V)} \right]^2 = \frac{2N_c Y_\Gamma}{\omega_1\omega_2 M_n} |\varphi_n(0)|^2.$$ 

(21)

This very transparent formula contains only well defined factors: $\omega_1$ and $\omega_b$, the meson mass $M_n$, and $\varphi_n$ the eigenvector of $H_0$. Then in the $P$ channel

$$(f_n^P)^2 = \frac{6(m_1m_2 + \omega_1\omega_2 - \langle \mathbf{p}^2 \rangle))}{\omega_1\omega_2 M_n} |\varphi_n(0)|^2,$$ 

(22)
Table 2: Pseudoscalar constants of $B_q$ mesons (in MeV)

|     | EFG$^3$ | Lattice$^6,7$ | This paper |
|-----|---------|-------------|-----------|
| $f_B$ | 189     | 216(34)     | 186(5)    |
| $f_{B_s}$ | 218 | 249(42) | 222(2) |
| $f_{B_s}/f_B$ | 1.15 | 1.20(4) | 1.19(2) |
| $f_{B_c}$ | 433     | 420(20)     | 438(8)    |
|     |         | 440(20)     |           |

where the w.f. at the origin, $\varphi_n(0)$, is a relativistic one. In the nonrelativistic limit $\omega_i \to m_i$, $\varphi_n(0) \to \varphi_{nNR}(0)$ and one comes to the standard expression:

$$
\left( f_n^P(0) \right)^2 \to \frac{12}{M_n} |\varphi_{nNR}(0)|^2.
$$

(23)

The calculated decay constants are given in Table 2 and their values turn out to be in good agreement with lattice data [6, 7] and the predictions of Ref. [3]. Our analysis also shows that $f_{B_c}$ is sensitive to the value taken for $m_c$ (pole).

4 Conclusions

From our analysis it follows that

- The dynamics of heavy-light mesons is sensitive to the number of flavors $n_f$ and to the value of the quark pole mass used (more sensitive than in the case of $c\bar{c}$ and $b\bar{b}$ spectra). For $B_q$ mesons $n_f = 4$ is used.

- Solutions of the Salpeter equation using the RSH give the masses of $B$, $B_1$, and $B_2$ and also $B_s$ and $B_c$ in agreement with experiment within $\pm 10$ MeV accuracy.

- For $B^*$ and $B_s^*$ agreement with experimental values is reached if $\alpha_{HF} = 0.32(1)$ is used in the hyperfine interaction.

- For the same $\alpha_{HF} = 0.32$ we predict $\Delta_{HF}(B_c) = 50$ MeV or $M(B_c^*) = 6325$ MeV.
In our analytic approach with minimal input of fundamental parameters ($\sigma, \alpha_s, m_i$) the calculated decay constants are $f_B = 186\text{ MeV}$, $f_{B_s} = 222\text{ MeV}$, $f_{B_s}/f_B = 1.19$.

For $B_c$ the decay constant is very sensitive to $m_c$ (pole): $f_{B_c} = 440\text{ MeV}$ ($m_c = 1.40\text{ GeV}$) and $f_{B_c} = 425\text{ MeV}$ ($m_c = 1.35\text{ GeV}$)

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