A Generalization of Tsallis’ Non-Extensive Entropy and Energy Landscape Transformation Functions

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Abstract. This article extends the non-extensive entropy of Tsallis and uses this entropy to model an energy producing system in an absorbing heat bath. This modified non-extensive entropy is superficially identical to the one proposed by Tsallis, but also incorporates a “hidden” parameter that provides greater flexibility for modeling energy constraints. This modified non-extensive entropy also leads to a more generalized family of energy transformation functions and also exhibits the structural scale invariance properties described in a previous article. This energy transformation also provides a more natural mechanism by which arbitrary power-law distributions can be stated in exponential form.

Keywords: Non-extensive entropy, statistical mechanics, scale invariance, Metropolis algorithm, complex systems, power laws

1 Introduction

In a recent article, Fleischer [1] described a number of scale invariant and symmetry properties in non-extensive systems—systems based on a non-extensive form of entropy developed by Tsallis [2]. These scale invariant properties show that in aggregating energy states of a large ensemble, many of the mathematical relationships associated with a given state also apply to the aggregations and use identical mathematical forms. Investigation of this scale invariance property revealed an energy transformation function. The frequency of its appearance in these scale invariant mathematical forms suggests that energy transformations are an important element in modeling non-extensive systems. This energy landscape transformation also provides an additional perspective on non-extensivity. Tsallis’ entropic form essentially shifts the probability weights in the system and thereby creating a power law distribution. This same effect can be captured by the energy landscape transformation functions. Thus, the energy landscape transformation function apparently lies at the heart of the scale invariance and symmetry properties present in non-extensive systems and seems to account for many of the useful applications of the Tsallis entropy especially in modeling dissipative systems where energy producing systems in a heat bath lose energy to its environment.

In this article, new forms of scale invariance and energy transformations are illuminated by generalizing (extending) Tsallis’ entropy formulation to explicitly account for
and generalize this energy transformation property. In this vein, Tsallis’ entropy and all of its associated scale invariance and symmetry properties become special cases in a larger array of scale invariance and symmetry. These new properties show a number of features involving recursion (aggregations of aggregations) and ways of characterizing power-law distributions using exponential forms where the exponent is the transformed energy value. In this sense, this article expands on the results in [1].

This article is organized as follows: Section 2 provides some background on Tsallis entropy and the scale invariant properties described earlier. Section 3 describes the new form of entropy and related system constraints. These two components are then used to define a new stationary probability. Section 4 briefly describes the scale invariance properties based on the forms of the new entropy and stationary probability and the associated family of energy transformation functions. Section 5 explores the related symmetry and power law relationships. Finally, Section 6 offers some discussion on the implications of this modified Tsallis entropy, energy transformations and practical applications in the field of complex system simulation. Section 7 provides some concluding remarks.

2 Background

Tsallis [3] developed a new entropy expression that forms the basis of a non-extensive form of thermodynamics:

\[ S_q = \frac{k \left( 1 - \sum_{i=1}^{W} p_i^q \right)}{q - 1} \]  

(1)

where \( k \) is a constant and \( S_q \) is the entropy parameterized by the entropic parameter \( q \). In classical statistical mechanics, entropy falls into a class of variables that are referred to as extensive because they scale with the size of the system. Intensive variables, such as temperature, do not scale with the size of the system. \(^1\) Tsallis’ form of entropy is non-extensive because the entropy of the union of two independent systems is not equal to the sum of the entropies of each system. That is, for independent systems \( A \) and \( B \),

\[ S_q(A + B) = S_q(A) + S_q(B) + \frac{(1 - q)S_q(A)S_q(B)}{k}. \]  

(2)

Tsallis uses this entropy to calculate a stationary probability \( p_i \) for a canonical system where \( S_q \) is maximized. Tsallis shows that \( p_i \) is distributed by a power law

\[ p_i(t) = \left[ \frac{1 + (\frac{2}{q-1}) f_i}{Z_q} \right]^{\frac{1}{1-q}} = \frac{[1 + a f_i]^{\frac{1}{q}}}{Z_q} \]  

(3)

\(^1\) Combine two vessels of gas each with the same volume and pressure into another vessel of twice the volume and the pressure and temperature of the combined gas will be the same as before. Energy and entropy, however, are examples of extensive variables in that combining several sources of either energy or entropy and you increase the total energy or entropy.
where \( a = (q - 1)/t \) and based on system constraints (but see [14] for a discussion on constraints)

\[
\sum_{i=1}^{W} p_i = 1 \quad \text{(4)}
\]

\[
\sum_{i=1}^{W} p_i f_i = U, \quad \text{a constant.} \quad \text{(5)}
\]

This distribution is different than the exponential law in the classic Boltzmann-Gibbs distribution, symbolized here by \( \pi_i \). Tsallis points out that for \( q \to 1 \) the extensivity properties of classical statistical mechanics emerge. Thus, e.g., \( \lim_{q \to 1} S_q = S_{BG} = -k \sum_i \pi_i \ln \pi_i \), and \( \lim_{q \to 1} p_i(t) = \pi_i(t) \), hence the Tsallis entropy is a generalization of the Boltzmann-Gibbs entropy [23].

Tsallis [5] also notes that the form of (1) is the simplest form that satisfies certain non-additivity assumptions and uses only one parameter, namely the entropic parameter \( q \). In the sections that follow, we both retain this inherent simplicity, but at the same time incorporate a new, “hidden” parameter \( m \) that provides additional flexibility for modeling dissipative systems.

2.1 Scale Invariance and Symmetry in Extensive and Non-Extensive Systems

In [16] a number of scale invariant properties in both classical and non-extensive systems are described and are based on aggregations of energy states. Briefly, this type of structural scale invariance is best illustrated by the following: for any aggregated set of energy levels \( A = \{i_1, i_2, \ldots, i_n\} \) where index \( i_k \) refers to some particular energy level (note \( A \) can also simply consist of a single energy energy level),

\[
\frac{\partial \pi_A(t)}{\partial t} = \frac{\pi_A(t)}{t^2} \left[ f_A(t) - \langle f \rangle(t) \right] \quad \text{(6)}
\]

where

\[
\pi_A(t) = \sum_{i \in A} \pi_i(t) \quad \text{and} \quad f_A(t) = \frac{\sum_{i \in A} \pi_i(t) f_i}{\pi_A(t)}
\]

with the latter equivalent to the conditional expectation of a random energy value \( f \) given that the current state \( i \) is in set \( A \). A similar form of scale invariance exists for second moments. Thus,

\[
\frac{\partial f_A(t)}{\partial t} = \frac{\sigma_A^2(t)}{t^2} \quad \text{(7)}
\]

where \( \sigma_A^2(t) \) is the conditional variance of energy (objective function) values at temperature \( t \) given that the current state being in set \( A \). See [6] p.232-33] for details and a more formal treatment.

Fleischer [11] demonstrated a similar result for non-extensive systems where the only difference between the scale invariance properties of the classic case and that of
the non-extensive case was that the latter involved an energy transformation function. Thus, for any aggregated set of energy levels $A$,

$$\frac{\partial p_A(t)}{\partial t} = \frac{p_A(t)}{t^2} \left[ \hat{f}_A(t) - \langle \hat{f} \rangle(t) \right]$$  \hspace{1cm} (8)

where

$$p_A(t) = \sum_{i \in A} p_i(t) \quad \text{and} \quad \hat{f}_A(t) = \frac{\sum_{i \in A} p_i(t) \hat{f}_i(t)}{p_A(t)}$$  \hspace{1cm} (9)

where the transformed energy value $\hat{f}_i$ is defined by

$$\hat{f}_i(f_i, q, t) \equiv f_i^1 + \left( q - 1 \right) t f_i^1 = f_i^1 + a f_i$$  \hspace{1cm} (10)

where $a = (q - 1)/t$. It will henceforth be notationally convenient to simply refer to transformed energy value using $\hat{f}_i(t)$, $\hat{f}_i$ or $\hat{f}_A$ as the case may be where it is understood to depend on the parameters $i, f_i, q, t$.

Fleischer \cite{11} also showed that the scale invariance in second moments incorporates this energy transformation, hence for all aggregations $A$

$$\frac{\partial \hat{f}_A}{\partial t} = \frac{\langle \hat{f}^2 \rangle_A - \langle \hat{f} \rangle_A^2 + (q - 1) (\hat{f}^2)_A}{t^2} = \frac{\sigma_A^2}{t^2} + \frac{(q - 1) (\hat{f}^2)_A}{t^2}$$  \hspace{1cm} (11)

2.2 Discussion

The appearance of the transformed energy value $\hat{f}$ in these scale invariant forms suggests that it is an important component in non-extensive statistical mechanics. It is therefore reasonable to infer that for an energy absorbing heat bath, the “equilibrium” condition associated with the internal energy constraint in (5) implicitly involves this transformed energy value. One viewpoint suggests that in an energy producing system in thermal equilibrium with an energy absorbing heat bath there must be some way of characterizing the rate at which the energy is produced and absorbed. To generalize this notion further, a heat bath that absorbs energy at a high rate, must be balanced by a higher rate (in some sense) of energy production if one seeks to model some equilibrium condition. We can then perhaps model different rates of energy production and absorption by making the absorption rate proportional to some function of the energy level in a canonical system. In this way, the notion of non-extensivity can be further expanded and modeled in a fashion that encompasses the information/entropy loss, as indicated in (2) for $q > 1$, and an equilibrium condition that encompasses an energy production/absorption component.

To capture these notions in a more flexible way, a modified Tsallis entropy $S_u$ is defined and in a manner that provides additional flexibility for modeling canonical systems in energy absorbing heat baths yet retains the inherent simplicity of the non-extensive entropy. The increased flexibility is based on using a parameter $u$ that involves both the
entropic parameter $q$ from Tsallis’ entropy and an energy parameter $m$. Note that in Tsallis’ approach, the entropic parameter $q$ is typically associated with the state probability $p_i$ (see e.g., [1] and [4]). Tsallis notes the effects of the exponent $q$ in the Type 2 constraint as they “[privilegiate] the rare and the frequent events” depending on whether $q < 1$ or $q > 1$, respectively [7, p.535]. But this notion of shifting the probability weight of different energy values is, in some sense, equivalent to transforming the energy values themselves. Thus, the entropic parameter $q$ plays a central role, in capturing this shift of probability by transforming the energy landscape as indicated in the derivation of the energy transformation function in [5]. Thus, $q > 1$ implies that the exponent of $f_i$ in [5] is also 1. It would however be useful to permit some additional freedom in choosing the exponent of $f_i$ to capture this notion of an energy producing system in equilibrium with an absorbing heat bath.

Taking these considerations into account, the parameter $u$ which involves both $q$ and $m$ should be associated with the state probability $p_i$, and so will also affect the energy transformation function. Because we also want to fashion situations where the exponent of $f_i$ in [5] is not always equal to 1, we require that the exponent of $f_i$ in [5] be different from that of $p_i$. Keeping these qualifications in mind, the following section describes a modified non-extensive entropy $S_u$ where a number of new scale invariant properties emerge.

### 3 The Modified Non-Extensive Entropy $S_u$

In this section, Tsallis’ entropy is generalized to increase the flexibility in using energy transformation functions. In this formulation, the Tsallis entropic parameter $q$ in [1] is replaced by an entropic/energy parameter $u$ ($u$ always follows $q$) that simultaneously accounts for both the parameter $q$ and an energy transformation parameter $m \geq 1$ where

$$u = (q - 1)(m - 1) + 1$$

(12)

and the *modified Tsallis entropy* is defined by

$$S_u = \frac{k \left( 1 - \sum_{i=1}^{W} p_i^u \right)}{u - 1}.$$  

(13)

Note that this form is identical to [1], hence retains all of its inherent benefits. It is easy to see that

$$\lim_{u \to 1} S_u = \lim_{q \to 1} S_q = S_1 = S_{BG} \text{ (defined earlier)}$$

(14)

and certain other relationships and properties are easily extended and generalized. For example, [4] becomes

$$S_u(A + B) = S_u(A) + S_u(B) + \frac{(1 - u)S_u(A) S_u(B)}{k}$$

$$= S_u(A) + S_u(B) + \frac{(1 - q)(m - 1)S_u(A) S_u(B)}{k}$$

(15)
since \( u - 1 = (q - 1)(m - 1) \). Thus, for any given value of \( q > 1 \) where some entropy loss occurs, the magnitude of this loss can also be modeled using the parameter \( m \) which, as explained below, is associated with the energy levels. Note that if \( q = 1 \), the value of \( m \) becomes irrelevant. The next section explores the implications of this simple modification.

### 3.1 The Energy Loss Rate

The parameter \( m \) is useful for modeling an energy producing system in thermal equilibrium with an absorbing heat bath. Such a system dissipates its energy to its surroundings while maintaining an average value of its internal energy. A canonical ensemble of such a system can be modeled therefore by a power of the energy function \( f \) as in

\[
\sum_{i=1}^{W} p_i^u f_i^{m-1} = U \quad \text{a constant} 
\]

where the parameter \( m \) serves to capture the notion of an energy loss rate or energy dissipation rate (or perhaps, the energy absorption rate). The exponent \( u \) of the stationary probability \( p_i \) thus serves the same purpose as in Tsallis’ works (see [4,7]). Notice that \((m = 2) \Rightarrow (u = q)\), and the resulting system is equivalent to those based strictly on \( S_q \). Notice also that for \( m = 2 \), this constraint has the same form as that described by Tsallis’ Type 2 constraint in (5) and also in Tsallis’ Type 3 constraint involving the so-called “escort probabilities” (see [7]).

### 3.2 The Stationary Probability

Tsallis [2,7] illustrates how maximizing the entropy \( S_q \) given the constraints in (4) and (5) leads to a stationary probability and results in the well-known stationary probability in (3). To obtain the stationary probability \( p_i \) in light of the modified Tsallis entropy \( S_u \) subject to the normalization constraint in (4) and the internal energy constraint in (16) we use a similar approach as in [2,4]. The general Lagrangian function is

\[
L = k \left( \frac{1 - \sum_{i=1}^{W} p_i^u}{u - 1} \right) + \alpha \left( \sum_{i=1}^{W} p_i (t) - 1 \right) - \beta \left( \sum_{i=1}^{W} p_i^u f_i^{m-1} - U \right). 
\]

Therefore,

\[
\frac{\partial L}{\partial p_i} = \frac{u p_i^{u-1}}{1 - u} + \alpha - \beta u p_i^{u-1} f_i^{m-1}.
\]

Setting (18) to zero and rearranging, we obtain

\[
\frac{u p_i^{u-1}}{u - 1} \left[ 1 + \beta (u - 1) f_i^{m-1} \right] = \alpha
\]

and hence

\[
p_i = \left[ \frac{\alpha (u - 1)}{u} \right]^{\frac{1}{u}} \left[ 1 + \beta (u - 1) f_i^{m-1} \right]^{\frac{1-u}{u-1}}
\]

\[
= \left[ \frac{1 + \beta (u - 1) f_i^{m-1}}{Z_u} \right]^{\frac{1}{u}}
\]

\[(19)\]
where $\beta$ is often symbolized by the inverse temperature $1/t$ and

$$Z_u = \sum_i \left[ 1 + \beta(u - 1)f_i^{m-1} \right]^{-u}$$

is the corresponding normalization constant. Note that (19) has the same form as (3) except for the exponent of $f_i$. Noting that $u - 1 = (q - 1)(m - 1)$ we obtain the general form

$$p_i = \frac{\left[ 1 + \beta(q - 1)(m - 1)f_i^{m-1} \right]^{-u}}{Z_u}.$$ 

Letting $a = \beta(q - 1) = (q - 1)/t$ for notational convenience, then

$$p_i = \frac{\left[ 1 + a(m - 1)f_i^{m-1} \right]^{-u}}{Z_u}. \quad (20)$$

Notice that for the case $m = 2$, (20) is equivalent to (3).

4 SA Scale Invariance Based on $S_u$

To demonstrate scale invariance based on aggregated states in the non-extensive case involving $S_u$, we proceed in a similar fashion as in [1] while making the necessary adjustments to account for the definition of $u$. The following theorem states a scale invariant mathematical structure for systems based on $S_u$.

**Theorem 1.** Let $A = \{i_1, i_2, \ldots, i_n\}$ be any aggregation of energy levels $i$ where the energy level associated with $i$ is denoted as $f_i$. Using the definition of $p_i$ in (20), define the stationary probability of set $A$ by $p_A = \sum_{i \in A} p_i$ and energy value associated with aggregated sets $A$ by

$$\hat{f}_m^{m-1} = \frac{\sum_{i \in A} p_i \hat{f}_m^{m-1}}{p_A}$$

where

$$\hat{f}_m^{m-1} = \frac{f_i^{m-1}}{1 + a(m - 1)f_i^{m-1}}.$$

Define the conditional variance of $\hat{f}_m^{m-1}$ by

$$\sigma_A^2 = \langle \hat{f}_m^{2m-2} \rangle_A - \langle \hat{f}_m^{m-1} \rangle_A^2.$$

Then

$$\frac{\partial p_A}{\partial t} = \frac{p_A}{t^2} \left[ \hat{f}_m^{m-1} - \langle \hat{f}_m^{m-1} \rangle_A \right]$$

and

$$\frac{\partial \hat{f}_m^{m-1}}{\partial t} = \frac{\sigma_A^2}{t^2} + \frac{(u - 1)(\hat{f}_m^{2m-2})_A}{t^2}.$$
Proof:
It is convenient to first consider the relevant quantities associated with individual states (energy levels) $i$, which in this case refers to $\partial p_i(t)/\partial t$ and $\partial (\hat{f}_i^{m-1})/\partial t$ where these quantities are defined below. We proceed in similar fashion as in [1,6] taking into account the definition of $u, S_u$ and the exponent of $f_i$.
For notational convenience and simplicity, let $N_i(t)$ be the numerator in (20). Thus,

$$N_i(t) \equiv \left[1 + \left(\frac{u - 1}{t}\right)f_i^{m-1}\right]^{1/u} \quad (21)$$

(hereinafter we will drop the $(t)$ from $N_i(t)$ to further simplify the expressions) and taking the derivative of (20) with respect to temperature $t$,

$$\frac{\partial p_i(t)}{\partial t} = \frac{Z_u \frac{\partial}{\partial t} N_i - N_i \frac{\partial}{\partial t} Z_u}{(Z_u)^2}. \quad (22)$$

In this case,

$$\frac{\partial N_i}{\partial t} = \frac{\partial}{\partial t} \left[1 + \left(\frac{u - 1}{t}\right)f_i^{m-1}\right]^{1/u} = \frac{N_i^{u}f_i^{m-1}}{t^2}, \quad (23)$$

with $Z_u = \sum_j N_j$, and $p_i(t) = N_i/Z_u$ and hence

$$\frac{\partial}{\partial t} Z_u = \frac{\partial}{\partial t} \sum_j N_j = \sum_j \frac{\partial N_j}{\partial t} = \sum_j \frac{N_j^{u}f_j^{m-1}}{t^2}. \quad (24)$$

Substituting this and (23) into (22) yields

$$\frac{\partial p_i(t)}{\partial t} = \frac{\left(\sum_j N_j\right) \frac{N_i^{u}f_i^{m-1}}{t^2} - \sum_j N_j^{u}f_j^{m-1}}{(\sum_j N_j)^2} = \frac{p_i(t)N_i^{u-1}f_i^{m-1}}{t^2} - \frac{p_i(t)\sum_j N_j^{u}f_j^{m-1}}{t^2 \sum_j N_j} = \frac{p_i(t)}{t^2} \left[N_i^{u-1}f_i^{m-1} - \frac{\sum_j N_j^{u}f_j^{m-1}}{\sum_j N_j}\right]. \quad (24)$$

To further simplify the notation, define the transformed energy value

$$\hat{f}_i^{m-1} \equiv N_i^{u-1}f_i^{m-1} = \frac{f_i^{m-1}}{1 + a(m-1)f_i^{m-1}} = \frac{f_i^{m-1}}{M_i, m} \quad (25)$$

where $a = (q - 1)/t$ and,

$$M_i = 1 + a(m-1)f_i^{m-1} \quad (26)$$
(the reason for this latter definition will become clear later on) without the clutter of the arguments. Substituting (25) into (24) and further simplifying yields the two equivalent forms:

\[
\frac{\partial p_i}{\partial t} = \frac{p_i}{t^2} \left[ f_i^{m-1} - \langle f_i^{m-1} \rangle \right].
\] (27)

\[
= \frac{p_i}{t^2} \left[ \frac{f_i^{m-1}}{M_i} - \langle f_i^{m-1} \rangle / M \right].
\] (28)

Now, taking the derivative of the probability of the aggregated set,

\[
\frac{\partial p_A}{\partial t} = \frac{\partial}{\partial t} \sum_{i \in A} p_i = \sum_{i \in A} \frac{\partial p_i}{\partial t}.
\] (29)

and substituting (27) into (29) (keeping in mind the dependence on \( t \)) yields

\[
\frac{\partial p_A}{\partial t} = \sum_{i \in A} \frac{p_i}{t^2} \left[ f_i^{m-1} - \langle f_i^{m-1} \rangle \right]
\]

\[
= \sum_{i \in A} \frac{p_i f_i^{m-1}}{t^2} - \sum_{i \in A} \frac{p_i \langle f_i^{m-1} \rangle}{t^2}
\]

\[
= \frac{p_A}{t^2} \sum_{i \in A} p_i f_i^{m-1} - \frac{p_A \langle f_i^{m-1} \rangle}{t^2}
\]

\[
= \frac{p_A}{t^2} \left[ f_A^{m-1} - \langle f_i^{m-1} \rangle \right].
\] (30)

where for aggregated states, (30) has a similar mathematical structure as (27, 28), hence exhibits a scale invariance property the foundation of which is based on the energy transformation function \( f_i^{m-1} \).

Scale invariance for second moments is indicated in the following where again the energy transformation of \( f_i \) is used. Keeping in mind the dependence of \( f \) and \( p_i \) on the temperature \( t \), consider

\[
\frac{\partial \langle f_i^{m-1} \rangle}{\partial t} = \frac{\partial}{\partial t} \sum_{i \in \Omega} p_i \hat{f}_i^{m-1}
\]

\[
= \sum_{i \in \Omega} \frac{\partial}{\partial t} \left[ p_i \hat{f}_i^{m-1} \right]
\]

\[
= \sum_{i \in \Omega} \left[ \frac{\partial p_i}{\partial t} \hat{f}_i^{m-1} + p_i \frac{\partial \hat{f}_i^{m-1}}{\partial t} \right].
\] (31)

Substituting (27) into the first part of (31) and simplifying yields

\[
\frac{\partial \langle f_i^{m-1} \rangle}{\partial t} = \sum_{i \in \Omega} \frac{p_i \hat{f}_i^{2m-2}}{t^2} - \sum_{i \in \Omega} \frac{\hat{f}_i^{m-1} p_i \hat{f}_i^{m-1}}{t^2} + \sum_{i \in \Omega} \frac{p_i \hat{f}_i^{m-1}}{t^2}.
\] (32)
Noting the form of the first two terms on the right-hand-side in (32) and the fact that in
the third term
\[ \frac{\partial \hat{f}^{m-1}}{\partial t} = \hat{f}_i^{2m-2} \left( \frac{u - 1}{t^2} \right) \] (33)
and substituting into (32) and adding the symbol \( \Omega \) to denote expectations over the
total state space yields

\[ \frac{\partial (\hat{f}^{m-1})_{\Omega}}{\partial t} = (\hat{f}_i^{2m-2})_{\Omega} - \frac{(\hat{f}^{m-1})^2_{\Omega}}{t^2} + (u - 1)(\hat{f}^{2m-2})_{\Omega} \]

\[ = \frac{\sigma^2_{\Omega}}{t^2} + \frac{(u - 1)(\hat{f}^{2m-2})_{\Omega}}{t^2} \] (34)

where \( \sigma^2_{\Omega} \) represents the variance of the values \( \hat{f}_i^{m-1} \) over the entire energy landscape
(at temperature \( t \)).

Eq. (34) provides the basis for another form of scale invariance. Thus, after going
through similar steps as in (31) through (34) we get

\[ \frac{\partial \hat{f}_A^{m-1}}{\partial t} = \sum_{i \in A} p_i \hat{f}_i^{2m-2} \frac{\sigma^2}{t^2} - \frac{\hat{f}_A^{2m-2}}{t^2} + \frac{(u - 1)}{t^2} \sum_{i \in A} p_i \hat{f}_i^{2m-2} \] (35)

Noting that the first and third terms indicate conditional expectations conditioned on
the current state being in set \( A \), then (35) can be re-written in the convenient notation

\[ \frac{\partial \hat{f}_A^{m-1}}{\partial t} = \frac{\sigma^2}{t^2} - \frac{\hat{f}_A^{2m-2}}{t^2} + \frac{(u - 1)}{t^2} \sum_{i \in A} p_i \hat{f}_i^{2m-2} \] (36)

where (36) is clearly analogous to (34) and so exhibits a form of scale invariance. Note
that the terms involving \( (u - 1) \) also scale with the aggregated set \( A \) in the non-extensive
case. See [1,6] for similar results in the case of \( S_q \) and \( S_1 \), respectively. ■

An interesting aspect of these relationships can be succinctly described using the
following commutative-like diagram in Figure 1. A similar diagram can be produced to
depict an analogous relationship for second-moments. The next section further explores
some of these symmetry relationships.

5 Symmetry Relationships

Fleischer [1] explored a number of symmetry relationships in addition to ones Tsallis
has indicated in his early works based on the effects of different constraints used in
modeling a canonical system (see also [7]). In this section, we briefly identify the ana-
logous symmetries described in [1] in light of the modified non-extensive entropy \( S_u \).
The reader is referred to [1] for background.
In Tsallis’ original incarnation of non-extensive statistical mechanics, he used his “Type 1” constraint \( \sum_i \tilde{p}_i f_i = U \) (see [7]) where the exponent \( q \) as in \( p^q_i \) was absent. This led to the following expression for the stationary probability

\[
\tilde{p}_i = \left[ 1 + \left( \frac{1-q}{t} \right) f_i \right]^{\frac{1}{q-1}} \tag{37}
\]

where \( \tilde{Z}'_q \) is the obvious normalizing constant. Tsallis notes that the form of \( \tilde{p}_i \) is essentially the same as that of \( p_i \) except that \( 1 - q \) replaces every occurrence of \( q - 1 \) and vice versa including in the exponents. Here, we simply note that solving the corresponding Lagrangian function in (17) using the constraint

\[
\sum_i \tilde{p}_i f_i^{m-1} = U \tag{38}
\]

leads, not surprisingly, to the analogous equation

\[
\tilde{p}_i(t) = \left[ 1 - a(m-1)f_i^{m-1} \right]^{\frac{1}{(q-1)(m-1)}} \tilde{Z}_a \tag{39}
\]

which, since \( a = (q-1)/t \), has the same reversal in sign observed in (11).

Fleischer [11] further showed that systems with the constraint (38) also entail a similar scale invariance as previously highlighted except that the corresponding energy transformation function is also modified with a sign change. In the context of \( S_a \) and the constraint in (38), the energy transformation function that is the basis of the scale invariance is given by

\[
\dot{f}_i^{m-1} = \frac{f_i^{m-1}}{1 - a(m-1)f_i^{m-1}} \tag{40}
\]

where \( a = (q-1)/t \) as before. Again, the analogous sign change in the denominator of (40) versus the denominator in (25) is present.
5.1 Scale Invariance Using Other Constraints

Using the same approach as in [1] and in Section 4, it follows that for all aggregated energy levels $A$, and using the constraint in (38) leads to the scale invariant form

$$\frac{\partial \tilde{p}_A}{\partial t} = \tilde{p}_A \left[ \tilde{f}_A^{m-1} - \langle \tilde{f}^{m-1} \rangle \right]$$  \hspace{1cm} (41)

where again the only difference between this and the earlier result is that every occurrence of $p_i$ and $\hat{f}_i$ is replaced with a $\tilde{p}_i$ and $\tilde{f}_i$, respectively.

Second Moments  Proceeding in the same fashion as in (31) through (36) and using analogous definitions (i.e., $\tilde{\sigma}^2$ corresponds to the variance of the values of $\tilde{f}_m^{i-1}$) we obtain the result

$$\frac{\partial \langle \tilde{f}^{m-1} \rangle}{\partial t} = \frac{\tilde{\sigma}^2}{t^2} + \frac{(1-u)(\tilde{f}^{2m-2})}{t^2},$$  \hspace{1cm} (42)

(the symbol $\Omega$ serves as a reminder that these values are based on the variation over the entire landscape). Scale invariance in second moments with the Tsallis Type 1 constraint is indicated by

$$\frac{\partial \tilde{f}_A^{m-1}}{\partial t} = \frac{\langle \tilde{f}^{2m-2} \rangle_A - \langle \tilde{f}^{m-1} \rangle^2_A + (1-u)(\tilde{f}^{2m-2})_A}{t^2}$$

$$= \frac{\tilde{\sigma}^2_A}{t^2} + \frac{(1-u)(\tilde{f}^{2m-2})_A}{t^2},$$  \hspace{1cm} (43)

where (42) and (43) are similar to (34) and (36) except that, as before, every occurrence of $u - 1$ and $f$ has been replaced with a $1 - u$ and $\tilde{f}$, respectively.

5.2 Probabilities and Energy Relations

A number of additional forms of symmetry relating to the constraints are described in [1] and their forms in the context of $S_u$ are easily inferred. Thus, e.g., defining $f_i$ in terms of $\tilde{f}_i$ we have

$$f_i = \frac{\tilde{f}_i}{\left[ 1 - a(m-1)\tilde{f}_i^{m-1} \right]^{\frac{1}{m-1}}}$$  \hspace{1cm} (44)

where again we use $a = \left( \frac{q-1}{q} \right)$ for notational convenience. Using this we can define the probability $p_i$ in terms of $\tilde{f}_i$ and obtain

$$p_i(t) = \frac{\left[ 1 - a(m-1)f_i^{m-1} \right]^\frac{1}{m-1}}{Z_u}$$  \hspace{1cm} (45)
which as in [1] is exactly the same form as in (37) above where the Type 1 constraint was used except with the \( f_i \) replacing the \( f_i \). It follows therefore that

\[
Z_u = \sum_{i=1}^{W} \left[ 1 + a(m-1)f_i^{m-1} \right]^{\frac{1}{(q-1)(m-1)}} = \sum_{i=1}^{W} \left[ 1 - a(m-1)\hat{f}_i^{m-1} \right]^{\frac{1}{(q-1)(m-1)}}. \tag{46}
\]

The following lemma expands on Lemma 1 in [1].

**Lemma 1.** For all \( u > 1 \),

\[
\sum_i p_i^u f_i^{m-1} = Z_u^{1-u} \sum_i p_i \hat{f}_i^{m-1} = Z_u^{1-u} \langle \hat{f}_i^{m-1} \rangle. \tag{47}
\]

**Proof:**

It follows from the definition of \( p_i \), that for all \( i \),

\[
p_i^u f_i^{m-1} = \left( \frac{[1 + (u-1)\hat{f}_i^{m-1}]^{\frac{1}{1-u}}}{Z_u} \right)^u f_i^{m-1}. \tag{48}
\]

Now observe that \( \frac{u}{1-u} = \frac{1}{1-u} - 1 \). Consequently for all \( i \),

\[
p_i^u = \frac{[1 + (u-1)\hat{f}_i^{m-1}]^{\frac{1}{1-u}}}{Z_u} \frac{p_i}{Z_u^{u-1} [1 + (\frac{u-1}{1-u})\hat{f}_i^{m-1}]} = \frac{p_i}{Z_u^{u-1} [1 + (\frac{u-1}{1-u})\hat{f}_i^{m-1}]}.
\]

Substituting this into (48) and simplifying we get

\[
p_i^u f_i^{m-1} = Z_u^{1-u} p_i \hat{f}_i^{m-1} \tag{49}
\]

and summing over all \( i \) the result follows. \( \blacksquare \)

We further note that other analogous symmetries as in [1] are present with regard to the relationships involving the Type 1 and Type 2 constraints. These relationships simply highlight the significance of the energy transformation function which the next section explores further. The following delineates these relationships the proofs of which correspond to those in [1] and so are omitted here.

**Lemma 2.** For any energy index \( i \) and parameters \( a > 0 \) and \( m \geq 1 \), define the \( k+1 \) iterate of the energy transformation of \( f_i \) in terms of the \( k \) iterate by

\[
\hat{f}_{i,k+1} = \frac{\hat{f}_{i,k}^{m-1}}{1 + a(m-1)\hat{f}_{i,k}^{m-1}}
\]

with \( \hat{f}_{i,0} \equiv f_i \) and

\[
\hat{f}_{i,1} = \frac{f_i}{[1 + a(m-1)f_i^{m-1}]^{\frac{1}{1-u}}}.
\]
Then,\[ \hat{f}_{i,k} = \frac{f_i}{[1 + ka(m-1)f_i^{m-1}]^{1/m}}. \quad (50) \]

Proof:
This proof is a straight forward extension of the one in \[1\]. ■

5.3 A Exponential Form of Powerlaws

In \[1\], Fleischer notes how the energy landscape transformation leads to an exponential form for a power-law distribution. It was shown that for

\[ \hat{x} = \frac{x^\gamma}{1 + ax^\gamma} \quad (51) \]

then

\[ e^{-\lambda \hat{x}} - C_1 \sim C_2 x^{-\gamma} \quad (52) \]

where \(C_1\) and \(C_2\) are constants. The exponent \(\gamma\) in \[51\] served to generalize the basic form of the landscape transformation function so that the power law exponent could itself be more general. This \(\gamma\) was a component that was added for this specific purpose. With the definition of \(S_u\) and the related energy transformation function in \[25\], however, this “artificial” addition becomes unnecessary as the exponent \(m - 1\) in \[38\] more naturally yields the general exponent of the related power law. Consequently, the constraint in \[38\] and the attendant definition in \[25\] yield a general form for the power law using an exponential form based on first principles. The following theorem is therefore stated and the proof is omitted as it again follows analogously from the one in \[1\].

**Theorem 2.** Let \(a > 0\) and \(x > 0\) be such that \(a(m-1)x^{m-1} > 1\) and define

\[ \hat{x} = \frac{x^{m-1}}{1 + a(m-1)x^{m-1}} \]

using the energy transformation function defined earlier. Then for all \(a > 0\), \(\lambda > 0\) and \(m > 1\)

\[ e^{-\lambda \hat{x}} - C_1 \sim C_2 x^{1-m} \]

as \(x \to \infty\) where the constants \(C_1 = e^{-\lambda/a(m-1)}\) and \(C_2 = a^{-2}(m-1)^{-2}\lambda e^{-\lambda/a(m-1)}\).

Proof:
This proof is a straight forward extension of the one in \[1\]. Thus, by substituting \(m - 1\) for every occurrence of \(\gamma\) and \(a(m - 1)\) for every occurrence of \(a\) in \[1\], the result follows. ■
5.4 The Derivative of $\hat{f}_i$

Finally, another important characteristic of the energy transformation function is worth noting and explains why we have used the exponent $m - 1$ in the previous expressions.

**Lemma 3.** For any $a > 0$ (this is mathematically equivalent to the condition that $q > 1$) and $m > 1$,

$$\frac{\partial \hat{f}_i}{\partial a} = \frac{\partial \hat{f}_i}{\partial q} = -\hat{f}_i^m.$$

**Proof:**

The proof is straight-forward using the chain rule. Thus, taking the derivative, we get

$$\frac{\partial}{\partial a} \left( \frac{f_i}{[1 + a(m - 1)f_i^{m-1}]^{\frac{1}{m-1}}} \right)$$

$$= -f_i \left( \frac{1}{m-1} \right) \left[1 + a(m - 1)f_i^{m-1}\right]^{-\frac{1}{m-1}} (m - 1)f_i^{m-1}$$

$$\left[1 + a(m - 1)f_i^{m-1}\right]^{\frac{1}{m-1}}$$

$$= -\frac{f_i^m}{\left[1 + a(m - 1)f_i^{m-1}\right]^{\frac{m}{m-1}} + \frac{m - 1}{m-1}}$$

$$= -\frac{f_i^m}{\left[1 + a(m - 1)f_i^{m-1}\right]^{\frac{m}{m-1}}} = -\hat{f}_i^m$$

It is apparent from the foregoing, that the energy transformation function in (25) is the solution of the differential equation in stated in Lemma 3 with the following boundary conditions: for all $m > 1$, $a = 0 \Rightarrow \hat{f}_i = f_i$.

6 Discussion

Fleischer [1] suggested that the scale invariance properties associated with the aggregation of energy states may provide additional perspectives on macroscopic power-law behavior. The appearance of the energy landscape transformation in these abstract aggregations and its relevance to power law distributions indicated in Theorems [1] and [2] and the more generalized entropy form $S_u$ suggest further interesting connections to complex systems theory which are briefly discussed below.

6.1 Complex Systems as Graphs

The foregoing scale invariant properties are based on aggregating energy levels. Thus, portions of the energy spectrum denoted by a label $A$ are lumped together and considered as having an energy value of $\hat{f}_A(t)$. The energy transformation functions also
suggest that as these “energy levels” aggregate they lose energy at a rate and in proportion to the magnitude of the energy level associated with the aggregation. Thus, if one can make a connection between aggregations of energy levels and the aggregations of ‘systems’, a number of new possibilities arise in modeling complex systems.

This notion is based on the fact that it is the components of systems that either contribute to or detract from the energy spectrum of a larger, more complex system. Thus, aggregating portions of the energy spectrum is, in some sense, equivalent to aggregating those entities that contribute to the energy spectrum. Using this perspective, the energy landscape transformation functions suggests that these energy-contributing entities themselves undergo some sort of transformation! Thus, the energy spectrum associated with a complex system can be modeled as a discretized energy spectrum and done in such a way that each such aggregation has energy values disjoint from other such aggregations. Depending on the level of discretization, it may even be possible for each such aggregation to have nearly identical energy spectrums yet not have any energy levels in common. Consequently, one can define subsystems of a larger complex system as nodes in a graph where each node possesses an energy spectrum. Arcs can then be used to relate these subsystems to each other in any convenient and useful way.

Modeling complex systems using graph theoretic means is of course not new (see [8] for a fuller account). But the scale invariant properties described above seem to facilitate a more direct connection between statistical mechanical relationships and general modeling approaches for complex systems. For instance, it would be interesting to examine the implications of this aggregation concept in the context of Ising spin glass models where there are aggregations of lattice points or apply it in the context of Markov random fields.

Another interesting application is in modeling systems in a far-from-equilibrium condition, something readily accomplished using the power law aspects associate with the energy transformation functions. In systems that are far-from-equilibrium, the ‘local temperature’ is not constant but “fluctuating on a relatively large time scale or spatial scale [9, p.220].” But the notion of temperature for general, complex systems is somewhat problematic. How does one define ‘temperature’ in systems that do not involve large ensembles of particles or “thermodynamic” behavior yet exhibits something akin to being in a far-from-equilibrium condition? See [9, Ch.7] for a complete discussion on the concept of ‘temperature’ in statistical mechanics.

One recent approach for dealing with these issues has been through the use of superstatistics. Sornette describes this approach:

A particular class of more general statistics relevant for nonequilibrium systems, containing Tsallis statistics [] as a special case, has been termed ‘super-statistics’ [] A superstatistics arises out of the superposition of two statistics, namely one described by ordinary Boltzmann factors $e^{-\beta E}$ and another one given by the probability distribution of $\beta$ [9, p.220].

We believe this concept of ‘superstatistics’ is readily captured through the use of the foregoing energy landscape transformations and application Lemma 2 that concerns some recursive properties of the energy transformation functions.

These recursive relationships, first indicated in [1] and generalized here, show that in terms of the energy landscape, there is a certain equivalence between values of $q$,
the number of times \( k \) a system or subsystem “undergoes” an energy transformation, and the temperature \( t \). Recall, that the energy transformation comes from an expression involving the aggregation of energy values. But the recursive aspects of described in Lemma\(^2\) suggests that a larger number of transformations and hence aggregations is, in some sense, equivalent to higher values of \( q \) and/or lower values of \( t \) and vice versa.

For example, given a particular \( f_i \) (or for that matter, \( f_A \)), a three-fold transformation \( \hat{f}_{i,3} \) with \( q = 2, m = 2 \) and \( t = 1 \) yields

\[
\hat{f}_{i,3}(q = 2, t = 1) = \frac{f_i}{1 + 3f_i},
\]

which is equivalent to the value of a single transformation when \( q = 4, m = 2 \), and \( t = 1 \). When \( q = 2, m = 2 \) and \( t = 1/3 \), however, \( \hat{f}_{i,1} \) also yields \( \frac{f_i}{1+3f_i} \). Thus, successive aggregations at some given temperature are equivalent, in some sense, to fewer aggregations at a lower temperature because \( \hat{f}_{i,3}(t = 1) = \hat{f}_{i,1}(t = 1/3) \).

7 Conclusion

This article proposed a slight modification of Tsallis’ entropy formulation. This modification retains the basic form of Tsallis entropy yet provides for greater flexibility in modeling energy constraints. The additional parameter, \( m \), can thus be used as an exponent of the energy value in the energy constraint equation. Doing this leads to a more generalized energy transformation function that also retains all of the scale invariance and symmetry properties highlighted in \[1\]. Moreover, this more general energy transformation function yields a more general (and natural) power law expression given in exponential form.

A number of possible issues were also examined in interpreting these results. Future research examines how these energy transformation functions can be leveraged in a number of applications, specifically, in developing new approaches for Monte Carlo Markov Chain simulation methodologies. Hopefully, the properties highlighted here will also improve our understanding of and capability to manage large, complex systems—something that appears to be an increasing challenge in our time.

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