Dimensionality reduction enhances data-driven reliability-based design optimizer

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Abstract
A recently proposed data-driven approach to reliability-based design optimization of structures constructs a sufficient condition that the target reliability is guaranteed with the specified confidence level, without relying on any assumptions on statistical information of random variables. In general, there exists a gap between this sufficient condition and the original confidence-level constraint. This paper presents a simple dimensionality reduction technique that can possibly mitigate this gap. This technique is applied to the compliance constraint with the uncertain external load. Numerical experiments on truss and continuum examples demonstrate that the proposed method can drastically reduce the over-conservativeness of the original method.

Keywords : Reliability-based design optimization, Uncertain input distribution, Reliability with confidence, Dimensionality reduction, Principal component analysis

1. Introduction

Since science and engineering always have limits, uncertainty unavoidably remains in a structural system. Considering uncertainty is therefore inevitable to design a structure having high assurance of quality. In structural optimization, two major approaches have been developed for dealing with uncertainty: reliability-based design optimization (RBDO) and robust design optimization.

RBDO assumes a probabilistic model of uncertainty, and evaluates the probability that a structural design satisfies the constraints. A criticism that can be made on this framework is that, in diverse practical situations, there exists a difficulty in obtaining precise statistical properties of uncertain parameters. Particularly, if the true input distribution of uncertain parameters is different from the one assumed in an RBDO method, the obtained solution can possibly fail to satisfy the required reliability against the true input distribution. In turn, an emerging and intensively studied research topic in RBDO is addressing uncertainty in the input distribution (Noh et al., 2011; Moon et al., 2017, 2018; Ito et al., 2018). Since considering uncertainty in the input distribution makes the problem structure much more elaborate, the solution methods dealing with such a problem setting have not yet been matured compared with RBDO methods assuming a deterministic probability model. As one option, a data-driven RBDO approach was proposed in Kanno (2019). This approach does not make any assumption on the distribution type and the statistical parameters, and hence is free from errors and uncertainty stemming from the conventional empirical probabilistic modeling. In the present paper, based on the dimensionality reduction, we propose a method that reduces over-conservativeness of this data-driven approach. Among others, the principal component analysis (PCA) (Cunningham and Ghahramani, 2015; Jolliffe and Cadima, 2016) is adopted as a dimensionality reduction method.

Suppose that the input distribution is either uncertain or known only imprecisely. Then the structural reliability is considered a random variable. The probability that the structural reliability is no smaller than the target value is called the confidence level (Noh et al., 2011; Moon et al., 2017, 2018; Ito et al., 2018). Kanno (2019) presented a sufficient condition guaranteeing that the confidence level is no smaller than the specified value. It is worth noting that the sufficiency means conservativeness, i.e., any solution satisfying this sufficient condition satisfies the original constraint on the specified
confidence level. However, because we have only a proof of sufficiency, the derived formulation might have gap from the optimal value of the original problem formulation. The method presented in this paper reduces this gap, with retaining the guarantee of conservativeness. In other words, the method in the present paper tightens the approximation in Kanno (2019).

Recently, PCA, adopted in this paper to map the original high-dimensional data to a lower-dimensional affine subspace, has been used in some literature in the field of structural optimization. For instance, reduced order modeling based on PCA is used to construct a surrogate model (Benamara et al., 2017; Gogu and Passieux, 2013) and to reduce computational cost for the reliability analysis (Motta and Afonso, 2016; Lim et al., 2018). Also, for optimal design of flexible pavement, PCA is used to construct an effective predictive modeling of permanent deformation (Ghasemi et al., 2019). To reduce the computational cost for topology optimization, Lei et al. (2019) and Ulu et al. (2016) applied machine learning techniques to a training data set that consists of optimal solutions obtained with diverse different problem settings. There, PCA is used for dimensionality reduction, before the machine-learning procedure is applied.

The paper is organized as follows. In section 2, we present the problems that we consider in this paper: RBDO problems of trusses and continua under the compliance constraint with a random external load. We also overview the data-driven RBDO approach in Kanno (2019). Section 3 describes how to incorporate PCA into this approach. Section 4 presents the tractable reformulations that we actually solve numerically. Section 5 presents two numerical examples. Section 6 collects some conclusions.

In our notation, $\mathbf{T}$ denotes the transpose of a vector or a matrix. All vectors are column vectors. For $x \in \mathbb{R}^n$, the notation $\|x\|$ designates its Euclidean norm, i.e., $\sqrt{x^\top x}$. We use $\mathbb{R}_+^n$ to denote the nonnegative orthant, i.e., $\mathbb{R}_+^n = \{(x_1, \ldots, x_n)^\top \in \mathbb{R}^n \mid x_i \geq 0 \ (i = 1, \ldots, n)\}$. For a matrix $X \in \mathbb{R}^{m \times n}$, we use $\text{Im} X$ to denote its image. We write $X \succeq 0$ if a real symmetric matrix $X$ is positive semidefinite. The notation $I_n$ designates the $n \times n$ identity matrix. For $a, b \in \mathbb{R}$ satisfying $a < b$, we use $[a, b]$ and $]a, b[$, respectively, to denote the closed and open intervals between $a$ and $b$. For a set $S \subseteq \mathbb{R}^n$ and a vector $x \in \mathbb{R}^n$, we use $S + x$ to denote the set $\{z + x \mid z \in S\}$.

2. Problem setting and overview of data-driven reliability-based design optimization method

In this section, we formulate the design optimization problem dealt with in this paper, and briefly recall the methodology proposed in Kanno (2019).

In this paper, we consider the compliance minimization problems of trusses and continua. We assume linear elasticity and static loading. We also, for simplicity, assume that only the external load is uncertain; the other quantities are assumed to be known precisely.

We begin with formulations for trusses. For topology optimization, consider a conventional ground structure method. We use $x_e$ ($e = 1, \ldots, m$) to denote the cross-sectional area of member $e$, where $m$ is the number of members. The structural volume is given by

$$v(x) = \sum_{e=1}^m l_e x_e,$$

where $l_e$ is the member undeformed length. Let $q \in \mathbb{R}^d$ denote the external nodal load vector, where $d$ is the number of degrees of freedom of the nodal displacements. The compliance is defined by

$$\pi(x; q) = \sup_{u \in \mathbb{R}^d} \{2q^\top u - u^\top K(x)u\},$$

where $K(x) \in \mathbb{R}^{d \times d}$ is the stiffness matrix. We use $\pi^c$ ($> 0$) to denote the specified upper bound for the compliance. Letting $X = \mathbb{R}_+^m$, we can formulate the problem that minimizes the structural volume under the compliance constraint as follows:

$$\begin{aligned}
& \text{Minimize} & & v(x) \\
& \text{subject to} & & \pi(x; q) \leq \pi^c, \\
& & & x \in X.
\end{aligned}$$

(1a)

(1b)

(1c)

Here, the design variable to be optimized is $x$.

A topology optimization problem for continua can also be formulated in the form of (1). We adopt the conventional solid isotropic material with penalization (SIMP) approach (Bendsøe and Sigmund, 1999) with the density filter (Bourdin,
ellipsoid, etc.) of difficulty of problem (4) depends on choice of the “shape” (e.g., box, polyhedron, from a viewpoint of numerical solution, difference problem (3). The overly conservative optimal solution of problem (4) becomes. Therefore, a “small” index $\epsilon$ of problem (1) with penalization power. Accordingly, we see that a continuum-based topology optimization problem can be embedded to the form of problem (1) with

$$v(x) = \sum_{\epsilon=1}^{m} x_{\epsilon}^{1/p},$$

$$X = (\rho_1^{\top}, \ldots, \rho_m^{\top})^{\top} \mid \rho = Hr, r \in [0, 1]^m.$$ 

Suppose that $q$ is a random vector. In the conventional RBDO, we replace constraint (1b) with

$$P[\pi(x; q) \leq \pi^c] \geq 1 - \epsilon,$$  \hspace{1cm} (2)

where $\epsilon \in [0, 1]$ is a specified lower bound for the failure probability. Note that $-\Phi(\epsilon)^{-1}$ is called the target reliability index, where $\Phi$ is the distribution function of the standard normal distribution. In the following, for simplicity we call $1 - \epsilon$ the target reliability.

Let $F$ denote the joint distribution function of $q$. Suppose that $F$ is either unknown or known only imprecisely. Then the value of the left side of (2) is considered a random variable, and hence inequality (2) becomes meaningful only in the probabilistic sense. This motivates us to consider the following design optimization problem:

Minimize  \hspace{1cm} $v(x)$ \hspace{1cm} (3a)

subject to  \hspace{1cm} $P_F[\pi(x; q) \leq \pi^c] \geq 1 - \epsilon, x \in X.$ \hspace{1cm} (3b)

Here, $\delta \in [0, 1]$ is a specified lower bound for the probability that the structural reliability is less than the target reliability, and $1 + \delta$ is referred to as the target confidence level (Noh et al., 2011; Moon et al., 2017, 2018; Ito et al., 2018).

Suppose that we are given a data set consisting of a finite number of independent and identically distributed samples drawn from $F$. In Kanno (2019), a methodology was presented to construct a robust optimization problem that approximates problem (3) in a conservative manner. This robust optimization problem has the following form:

Minimize  \hspace{1cm} $v(x)$ \hspace{1cm} (4a)

subject to  \hspace{1cm} $\pi(x; q) \leq \pi^c \hspace{1cm} (\forall q \in Q), x \in X.$ \hspace{1cm} (4b)

Here, $Q \subseteq \mathbb{R}^d$, referred to as the uncertainty set, is a specified compact set. A key is that, by using the given data set, we can determine $Q$ so that it satisfies

$$P_F[\pi(q) \in Q] \geq 1 - \epsilon \geq 1 - \delta.$$ \hspace{1cm} (5)

Then, it is guaranteed that any feasible solution of problem (4) is feasible for problem (3); i.e., problem (4) serves as a conservative approximation of problem (3). One of advantages of this methodology is that we need no assumption on statistical properties of a random vector, $q$. Another advantages is that, with the robust optimization formulation, we can deal with a bit more complex constraints than what the current RBDO methods cover. For ease of treatment, in the following we restrict $Q$ to an ellipsoid.

### 3. Learning uncertainty set with reduced dimensions

There exists freedom of choice of uncertainty set $Q$ satisfying (5). Roughly speaking, a “larger” $Q$ we choose, more overly conservative an optimal solution of problem (4) becomes. Therefore, a “small” $Q$ is preferable. On the other hand, from a viewpoint of numerical solution, difficulty of problem (4) depends on choice of the “shape” (e.g., box, polyhedron, ellipsoid, etc.) of $Q$. 

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As an uncertainty set, Kanno (2019) adopts an ellipsoid in the form
\[
\hat{Q}(\tilde{\alpha}) = \{ q \in \mathbb{R}^d \mid (q - \hat{q})^T \Omega (q - \hat{q})^T \leq \tilde{\alpha} \}.
\] (6)
Here, \( \hat{q} \in \mathbb{R}^d \) is a constant vector, \( \Omega \in \mathbb{R}^{d \times d} \) is a constant positive definite symmetric matrix, and \( \tilde{\alpha} > 0 \) is a constant. Note that \( \Omega \) and \( \hat{q} \) determine the shape and the location of the ellipsoid, and \( \tilde{\alpha} \) determines its size.

In contrast, in this paper we propose to use the following form:
\[
Q(\alpha) = \{ \hat{q} + \alpha L \theta \mid ||\theta|| \leq 1 \}.
\] (7)
Here, \( L \in \mathbb{R}^{d \times c} \) is a constant full column rank matrix with \( c < d \) (where the choice of \( c \) is discussed later), and \( \alpha > 0 \) is a constant. Namely, the main idea is to use an ellipsoid lying in an affine subspace of \( \mathbb{R}^d \), instead of an ellipsoid having interior points in \( \mathbb{R}^d \). As illustrated by the numerical examples in section 5, making use of (7) with a properly determined \( L \) can drastically reduce over-conservativeness of a solution obtained with (6).

It is worth noting that (6) can be expressed in the form of (7) with \( L = \hat{q} \Omega^{-1/2} \) and \( \alpha = \sqrt{\tilde{\alpha}} \), where \( \hat{q} \Omega^{-1/2} \) is the symmetric square root of \( \hat{q} \Omega^{-1} \). That is, (6) can be considered a special case of (7) with \( c = d \) and rank \( L = d \). However, our interest is in the case \( c < d \), particularly, in the case that \( c \) is fairly small compared with \( d \).

In the remainder of this section, we present a way to determine \( L \) and \( \alpha \) in (7).

Let \( D = \{ \hat{q}_1, \ldots, \hat{q}_n \} \subset \mathbb{R}^d \) denote the set of independent and identically distributed samples drawn from \( F \). We use \( \hat{q} \in \mathbb{R}^d \) to denote the mean of \( \hat{q}_1, \ldots, \hat{q}_n \), and apply PCA (Hastie et al., 2009; Hyvärinen et al., 2009) to \( D - \hat{q} \). Let \( U \in \mathbb{R}^{d \times c} \) denote the matrix obtained by arranging the first \( c \) principal component weights (i.e., the left singular vectors of the matrix \( [\hat{q}_1 - \hat{q}, \ldots, \hat{q}_n - \hat{q}] \), corresponding to the largest \( c \) singular values). Here, we determine \( c \) so that the total variability (i.e., the cumulative contribution rate) of these \( c \) components is larger than a specified threshold. Then \( q \) is (approximately) represented by the linear model
\[
q = \hat{q} + U \tau,
\] (8)
where \( \tau \in \mathbb{R}^c \) is a zero-mean random vector. Thus, if \( c < d \), then \( q \) has been transformed into a space of a smaller dimension.

We next construct an uncertainty set for random vector \( \tau \). From (8), we may estimate that the expected value of \( \tau \) is \( 0 \). Hence, it is natural to adopt the form
\[
T(\alpha) = \{ \tau \in \mathbb{R}^c \mid \tau^T \Sigma^{-1} \tau \leq \alpha^2 \} = \{ \Sigma^{1/2} \theta \mid ||\theta|| \leq \alpha \}
\] (9)
as an uncertainty set for \( \tau \), where \( \Sigma \in \mathbb{R}^{c \times c} \) is the variance-covariance matrix of \( \tau \). By pre-multiplying (8) by \( U^T \), we obtain
\[
\tau = U^T (q - \hat{q}),
\]
because \( U^T U = I_c \) holds from the linear independence of principal component weights. Hence, we set \( \Delta \) as the variance-covariance matrix of \( U^T(\hat{q}_1 - \hat{q}), \ldots, U^T(\hat{q}_n - \hat{q}) \).

In the following, we assume that \( q \) satisfies
\[
q \in \{ \hat{q} + U \tau \mid \tau \in \mathbb{R}^c \}.
\] (10)
From \( \tau \in T(\alpha) \) with (9), it is natural to adopt the form
\[
Q(\alpha) = \{ \hat{q} + U \Sigma^{1/2} \theta \mid ||\theta|| \leq \alpha \}
\]as an uncertainty set of \( q \). Define \( a : \mathbb{R}^d \rightarrow \mathbb{R} \) by
\[
a(q) = \sqrt{(q - \hat{q})^T U \Sigma^{-1} U^T (q - \hat{q})}.
\]
It is easy to verify that the relation
\[
a(q) \leq \alpha \quad \Leftrightarrow \quad q \in Q(\alpha)
\] (11)
holds. Renumber \( \hat{q}_1, \ldots, \hat{q}_n \) so that
\[
a(\hat{q}_{(1)}) < a(\hat{q}_{(2)}) < \cdots < a(\hat{q}_{(n)})
\] holds. We see that \( a(\hat{q}_{(1)}), \ldots, a(\hat{q}_{(n)}) \) are the order statistics of random samples \( a(\hat{q}_1), \ldots, a(\hat{q}_n) \) of \( F_A \), where \( F_A \) is the
distribution function of a random variable \( a(q) \). Define \( \tilde{s} \) as the minimum natural number \( s \) satisfying
\[
\sum_{k=1}^{\infty} \frac{n}{k} (1-\epsilon)^k \epsilon^{s-k} \leq \delta.
\]

Application of Proposition 1 in Kanno (2019) to \( F_A \) and \( a(\hat{q}_{(1)}) \), \ldots, \( a(\hat{q}_{(n)}) \) results in
\[
P_{F_A} [F_A(a(\hat{q}_{(i)}))] \geq 1 - \epsilon \geq 1 - \delta.
\]
(12)

On the other hand, the definition of distribution function and (11) yield
\[
F_A(\alpha) = P[a(q) \leq \alpha] = P[q \in Q(\alpha)].
\]
(13)

From (12) and (13), we obtain
\[
P_{F_A} [P[q \in Q(a(\hat{q}_{(i)})))] \geq 1 - \epsilon \geq 1 - \delta.
\]
(14)

We use this \( Q(a(\hat{q}_{(i)})) \) as \( Q \) in problem (4).

Remark 1 In fact, we determine the shape and size of the uncertainty set independently, as described in see Kanno (2019). Namely, we use some data points randomly chosen from \( D \) to determine \( \Sigma \), and use the remaining points to determine \( a(\hat{q}_{(i)}) \). □

4. Robust optimization formulations

In this section, we briefly explain concrete ways to solve problem (4). Our focus is to reformulate the infinitely many constraints in (4b) to a conventional (i.e., a finite number of) constraints.

We begin with the truss topology optimization. For notational simplicity, we write
\[
\tilde{\alpha} = a(\hat{q}_{(i)}).
\]

By using eq. (8.2.15) in Ben-Tal \textit{et al.} (2009) (see also Proposition 2 in Calafiore and Dabbene (2008)), we can recast problem (4) as
\[
\begin{align}
\text{minimize} & \quad \sum_{e=1}^{m} l_e x_e \\
\text{subject to} & \quad \begin{bmatrix} \pi^c & 1 & \bar{q}^T \\ 1 & y & 0^T \\ \bar{q}^T & 0 & K(x) - y\tilde{\alpha}^2 U\Sigma U^T \end{bmatrix} \succeq 0, \\
x & \geq 0.
\end{align}
\]
(15a) (15b) (15c)

where \( x \) and \( y \) are variables to be optimized. Since the stiffness matrix can be written in the form
\[
K(x) = \sum_{e=1}^{m} x_e K_e
\]
with constant symmetric matrices \( K_1, \ldots, K_m \), we see that the matrix on the left side of (15b) is a linear (matrix-valued) function of \( x \) and \( y \). Therefore, problem (15) is a \textit{semidefinite programming} (SDP) problem. It is worth noting that SDP is convex optimization, and we can solve problem (15) globally with an efficient primal-dual interior-point method (Anjos and Lasserre, 2012). It is worth noting that the size of problem (15) does not depend on the value of \( c \). In other words, the dimensionality reduction proposed in this paper does not attempt to reduce the computational cost.

We now proceed to the continuum topology optimization. Although the situation is a bit more complex due to presence of the SIMP penalization, we can again use Ben-Tal \textit{et al.} (2009, eq. (8.2.15)) to reduce problem (4) to the
following form:

\[
\begin{align*}
\text{Minimize} & \quad \sum_{e=1}^{m} \rho_e \\
\text{subject to} & \quad \begin{bmatrix} \pi^c & 1 & \tilde{q}^T \\ 1 & y & 0^T \\ \tilde{q}^T & 0 & K(x) - y\delta^2 U\Sigma U^T \end{bmatrix} \geq 0, \\
x_e = \rho_e^p, & \quad e = 1, \ldots, m, \\
\rho = Hr, & \\
0 \leq r_e \leq 1, & \quad e = 1, \ldots, m.
\end{align*}
\]  

(16a) \hspace{1cm} (16b) \hspace{1cm} (16c) \hspace{1cm} (16d) \hspace{1cm} (16e)

Here, variables to be optimized are \( x, y, \rho \) and \( r \). Problem (16) is a so-called nonlinear SDP problem (Yamashita and Yabe, 2015). Particularly, the constraints in (16c), stemming from the SIMP penalization, are nonconvex (a set \( \{x_e, \rho_e\}^T \in \mathbb{R}^2 | x_e \geq \rho_e^p, \rho_e \geq 0 \} \) with \( p > 1 \) is convex, but we cannot drop nonconvex constraint \( x_e \leq \rho_e^p \). Therefore, problem (16) is a nonconvex optimization problem. In this paper, we use a sequential SDP (Kanno and Takewaki, 2006; Kanzow et al., 2005) to find a local optimal solution. Specifically, we linearize (16c) as

\[ x_e = p(\rho_e^{(k)})^{p-1} \rho_e + (1-p)(\rho_e^{(k)})^p, \quad e = 1, \ldots, m \]

to obtain an SDP subproblem that is solved at each iteration, where \( \rho_e^{(k)} \) is the incumbent value of variable \( \rho_e \).

5. Numerical examples

This section presents two numerical examples to demonstrate that the proposed dimensionality reduction method can mitigate over-conservativeness of solutions obtained by the method in Kanno (2019). The proposed methods were implemented in Matlab ver. 9.0.0. The Matlab built-in function pca was used as an implementation of PCA. SDP problems were solved with SeDuMi ver. 1.3 (Sturm, 1999; Pólik, 2005), which implements a primal-dual interior-point method. In section 5.1 (i.e., for truss examples), CVX ver. 2.1 (Grant and Boyd, 2008, 2019) was used as interface to SeDuMi. The cvx_precision parameter of CVX is set to best so that SeDuMi continues as far as it can make progress. Numerical experiments were carried out on a 2.2 GHz Intel Core i5 processor with 8 GB RAM.

5.1. Example (I)

Consider a truss example outlined in Figure 1. The truss consists of \( m = 29 \) members, and has \( d = 18 \) degrees of freedom of the nodal displacements. The leftmost nodes are pin-supported. The upper bound for the compliance is \( \pi^c = 1000 \text{J} \).

The small open circles indicate given data points of the external load. The number of the data points is \( n = 250 \). Each of these points were generated as follows. Let \( q^d \in \mathbb{R}^d \) denote the nominal load vector (or the best estimate of the load vector), where vertical downward external forces of 100 kN are applied to the bottom two free nodes. We randomly generate \( M \in \mathbb{R}^{d \times d} \), whose entries are drawn from the uniform distribution on interval \([0, 5]\) in kN. Also, we randomly generate \( \zeta \in \mathbb{R}^d \), whose entries are drawn from the normal distribution with mean \( 0 \) and variance-covariance matrix \( 100I_d \) in kN^-2. Then we let \( q^e = M\zeta \) be a data point.

We compute \( U \) by using the 250 data points. As for the threshold for total variability, we consider two cases: 0.95 and 0.98. Among the 250 data points, we use randomly chosen 50 points for computing \( \Sigma \), and use the remaining 200 points for computing \( \tilde{\alpha} = a(\tilde{q}_1) \). We set \( \epsilon = 0.1 \) and \( \delta = 0.08 \), which yield \( \tilde{\delta} = 187 \). Figure 2 shows the optimal solutions obtained by the proposed method, where the width of each member is proportional to its cross-sectional area.

For comparison, Figure 3 shows the optimal solution obtained without using the dimensionality reduction; in other words, it is an optimal solution obtained with setting \( U = I_d \). Table 1 reports the computational results. Here, \( e(x) \) is the structural volume, and \( \pi(x; \tilde{q}) \) is the compliance of the obtained solution against the mean of the load samples \( \tilde{q}_1, \ldots, \tilde{q}_n \). The optimal value of the solution in Figure 2b is much less than the one in Figure 3, which means that over-conservativeness of the solution in Figure 3 is successfully mitigated by using the dimensionality reduction. It is also observed in Table 1 that a small value of \( c \) results in a smaller objective value. However, if the total variability is too small, then it is likely that assumption (10) does not hold. Figure 4 collects the first four principal component weight vectors.
Fig. 1 Problem setting of example (I). The leftmost three nodes depicted as filled circles are pin-supports. The small open circles indicate data points. The length of each arrow corresponds to an external force of 100 kN.

Fig. 2 The optimal solutions obtained by the proposed method (example (I)). The lower bound for the total variability (for determining $c$) is (a) 0.95; and (b) 0.98.

The results mentioned above are the ones for one data set. We next investigate robustness of the proposed method against different data sets. According to the same procedure described above, we prepare 50000 data sets with randomly changing a seed of the random number generator of Matlab. Figure 5a shows the eigenvalues of the variance-covariance matrices of these data sets. We can see that many eigenvalues are close to 0, which explains the reason that the dimensionality reduction is, as observed above, efficient for reducing over-conservativeness. Figure 5b and Figure 5c report the statistics of results of the proposed method, where the threshold for total variability is 0.98. Namely, Figure 5b shows a histogram of $c$, where we can observe that the original data points are transformed into a space of approximately half dimensions. This result agrees with the observation that we has made in Figure 5a. Figure 5c shows the ratio of the optimal value obtained by the proposed method to the one obtained without applying the dimensionality reduction method. The mean ratio is 0.53. Thus, the dimensionality reduction drastically reduced over-conservativeness.

5.2. Example (II)

In this section, we solve a continuum topology optimization problem outlined in Figure 6. We discretize the rectangular elastic body into $m = 40 \times 20 = 800$ uniform finite element mesh. The left-most nodes are fixed, and the number of degrees of freedom of the nodal displacements is $d = 1680$. For simplicity, we omit units of quantities. The Young modulus and the Poisson ratio are 1.0 and 0.3, respectively. Implementation of the SIMP approach with the density filter
was based on the Matlab code in Andreassen et al. (2011). The penalization power and the filter radius (divided by the element size) are $p = 3$ and $1.5$, respectively. The upper bound for the compliance is set to $\pi^c = 600$. The initial point for the sequential SDP is $\rho_0 = 0.6$.

We generated a data set consisting of $n = 250$ data points in the following manner. Suppose that external forces are applied only at the bottom nodes of the continuum, as illustrated in Figure 6. Without loss of generality, let $1, 2, \ldots, 80$ be the indices of the nodal displacements of these nodes. We set each data point as

$$\hat{q}_j = q^n + \begin{bmatrix} M & O \\ O & O \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix}$$

with constant $q^n \in \mathbb{R}^d$, and randomly generated $M \in \mathbb{R}^{80 \times 80}$ and $\xi \in \mathbb{R}^{80}$. Here, $q^n$ is the external horizontal rightward applied at the bottom rightmost node (depicted as a large arrow in Figure 6), the entries of $M$ are drawn from the uniform distribution on interval $[0, 0.005]$, and entries of $\xi$ are drawn from the standard normal distribution.

It follows from the definition that every $\hat{q}_j$ lies in $\text{Im} \ M$, i.e., an 80-dimensional linear subspace of $\mathbb{R}^d$. In this example, PCA transforms these data points into a space of 19 dimensions, where the threshold for total variability is set to $0.95$. Among $n = 250$ data points, we randomly chose 50 points and used them for computing $\Sigma$. The remaining 200 points were used for determining $\alpha = a(\hat{q}_j)$. The values of $\epsilon$ and $\delta$ are same as the ones used in section 5.1. As a result, we obtained the solution shown in Figure 7a. The volume fraction of this solution is $0.4852$ (i.e., the obtained objective value is $0.4852m$).
Fig. 5  Statistics of results obtained the proposed method for randomly generated data sets (example (I)). (a) Eigenvalues of the variance-covariance matrix; (b) value of \( c \); and (c) ratio of the optimal value obtained by the proposed method to the one obtained without using the dimensionality reduction method.

Table 1  Computational results of example (I).

| Solution   | \( \varepsilon(x) \) (mm\(^3\)) | \( \pi(x; \tilde{q}) \) (J) |
|------------|-------------------------------|-------------------------------|
| Figure 2a  | \( 7.502 \times 10^7 \)       | 203.700                       |
| Figure 2b  | \( 10.269 \times 10^7 \)      | 153.116                       |
| Figure 3   | \( 17.662 \times 10^7 \)      | 92.947                        |

For comparison, Figure 7b shows a solution of the conventional optimization problem (i.e., the problem without considering uncertainty), where only the rightmost horizontal force in Figure 6 is applied. This solution was found by minimizing the compliance with the specified volume fraction 0.4852, where top88 (Andreassen et al., 2011) was used as a solver. If we apply, to this solution, the uncertainty set used to obtain the solution in Figure 7a, then the worst-case compliance, i.e., \( \max \{ \pi(x; q) \mid q \in Q(\tilde{\alpha}) \} \), becomes 1310.0 (which is much larger than the specified upper bound, \( \pi^c = 600.0 \), for RBDO).

Without using the dimensionality reduction, we cannot solve this example. This is because problem (4) is infeasible; the worst-case compliance of the “all-black design” (i.e., \( x_1 = \cdots = x_m = 1 \)) is larger than \( \pi^c \).

6. Conclusions

In this paper, we have explored effectivity of dimensionality reduction when it is applied to a data-driven approach to reliability-based design optimization of structures. The principal component analysis has been adopted as a dimensionality reduction method. The numerical experiments have demonstrated that this simple dimensionality reduction drastically reduces the over-conservativeness of solutions compared with the case that raw data are used.

As for a method for dimensionality reduction, in this paper we have restricted ourselves to a simple linear dimension-
Fig. 6 Problem setting of example (II).

Fig. 7 The solutions obtained for example (II). (a) The solution of the proposed method; and (b) the solution of the conventional optimization (i.e., without considering uncertainty) for the external load depicted as the large arrow in Figure 6.

ality reduction method. More elaborate methods, including nonlinear dimensionality reduction methods (Tenenbaum et al., 2000; Roweis and Saul, 2000; Belkin and Niyogi, 2001), may possibly work more efficiently, especially for data sets with particular structures. Exploring such possibility remains as future work. In addition, reduction of the computational cost of the proposed method, compared with topology optimization methods without considering uncertainty, remains to be explored.

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