System/environment duality of nonequilibrium observables

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Abstract. - We generalize Schnakenberg’s theory to nonsteady states, on networks representing currents between microstates of a nonequilibrium system, with the introduction of a new set of macroscopic observables which, for planar graphs, are related by a duality. We apply this duality to the linear regime, yielding a dual proposition for the minimum entropy production principle, and to discrete electromagnetism, finding that it exchanges fields with sources. We interpret duality as acting between system and environment, and discuss generalization to nonplanar graphs.

Introduction. – In a seminal paper [1], J. Schnakenberg engaged in the definition of the fundamental macroscopic observables of NonEquilibrium Statistical Mechanics, grossly conceived as a theory of the internal flows of a system. In accordance with modern trends in Quantum Field Theory and its outskirts, where Wilson loops and adiabatic phases play an ever more prominent role, he foresaw circuitations of certain “connection” variables as the constraints which prevent a system from relaxing to equilibrium. Born out of the study of biophysical systems [2,3], and recently finding growing applications to chemical reaction networks, molecular motors and transport phenomena [4–9], his analysis has a deep geometrical and combinatorial content [10,11]. It is the backbone to the comprehension of Non-Equilibrium Steady States (NESSs) [10,12], to which the theory is restricted—so far.

The aim of this letter is to go beyond NESSs, generalizing Schnakenberg’s construction to arbitrary states. The complete theory of nonequilibrium observables turns out to enjoy a duality which exchanges the concept of steadiness with that of detailed-balancing of the external forces.

Duality comes in many flavours in physics. Among the first that one encounters: the duality between vectors — velocities — and linear forms — momenta; the Legendre transform which maps the lagrangian into the hamiltonian, pivoting on the bilinear form \( \sum_i \dot{q}_i p_i \); the electromagnetic duality, which is the archetypical physical counterpart of Hodge’s geometrical theory of differential forms; and, less credited, the electro-technical duality between resistances and condensators, parallel and series reduction, voltage and current laws [13]. The one that we put forward descends from the latter, abstracting and generalizing it to nonlinear regimes, where Ohm’s law does not necessarily hold; but it also resonates with each of the above. While the reference physical situation is that of a nonequilibrium system, we will cast our propositions in a very general form. In fact, they can be applied to any discretized theory which has a couple of conjugate variables.

Duality can only be realized on planar graphs. Although, nonequilibrium observables behave “as if” there always existed some dual graph. In a fascinating work [14], McKee attempts a generalization, finding a correspondence with logical duality between the universal and existential quantifiers (\( \forall \) and \( \exists \)) under the involutive action of negation (\( \neg \)). In the prologue he comments that “some optimists see them [dualities] as mechanically doubling the number of results of a theory”. We enroll the troop, claiming that for every proposition that is true of steady states, there exists a dual proposition regarding detailed-balanced systems. One explicit example is the following dual minimum entropy production principle:

“detailed balanced systems are those systems for which the rate of entropy production has the minimum value that is consistent with the fixed inflowing currents which prevent them from reaching a stationary state”.

While steadiness is a property of the state of the system, detailed-balancing of the external forces is a property of the state of the environment: hence the letter’s title.
Schnakenberg’s main focus was on rate equations

$$\dot{\rho}_v = \sum_{v' \neq v} \left( w_{vv'} \rho_{v'} - w_{v'v} \rho_v \right) = \sum_{v' \neq v} j_{vv'}, \tag{1}$$

where vertex \( v \) belongs to a finite state space \( V \) of the system, \( \rho \) is a probability density and \( w_{vv'} \) are positive transition rates along edges \( e = v \leftrightarrow v' \) of a network, or graph, \( G \). He identified the entropy production (EP)

$$\sigma = \sum_{v,v'} w_{vv'} \rho_v \ln \frac{w_{vv'} \rho_v}{w_{v'v} \rho_{v'}} = \frac{1}{2} \sum_{v,v'} j_{vv'} a_{vv'}, \tag{2}$$

as a bilinear form of mesoscopic currents \( j_{vv'} = -j_{v'v} \) and forces \( a_{vv'} = -a_{v'v} \), and realized that at a NESS, that is, when Kirchhoff’s law \( \dot{\rho} = 0 \) is satisfied, currents can be expressed as linear combinations of a certain number of macroscopic “mesh” currents \( J_\alpha \). On the \( \dot{\rho} = 0 \) shell, EP comes down to \( \sigma = J_\alpha A^\alpha \) (repeated indices are implicitly summed over), where the conjugate variables \( A^\alpha \) are seen to be circulations of the mesoscopic forces around suitable cycles of the graph,

$$A^\alpha = \ln \frac{w_{v_1 v_2} w_{v_2 v_3} \ldots w_{v_n v_1}}{w_{v_1 v_n} \ldots w_{v_n v_2} w_{v_2 v_1}}. \tag{3}$$

A system whose steady state \( \rho^{ss} \) makes all mesoscopic currents and forces vanish, \( w_{vv'} \rho_v^{ss} = w_{v'v} \rho_v^{ss} \), is said to satisfy detailed balance. Schnakenberg’s choice of circulations as the fundamental observables, indicative of the nonequilibrium nature of the system, is motivated by the well-known fact that they all vanish if and only if the steady state is detailed balanced (Kolmogorov’s criterion) \cite{12}. Moreover, they do not depend on the system’s macrostate \( \rho \); they are external properties, which are conceptually more fitly awarded to the state of the environment. Hence, in the following, we will refer to detailed-balanced systems as those which satisfy Kolmogorov’s criterion.

Indeed, Schnakenberg’s analysis can be extended to any graph whose edges bear a couple of antisymmetric conjugate variables, one of which obeys Kirchhoff’s Law at the nodes. Thence abandoning rate equation thermodynamics—but retaining the nomenclature, we complement Schnakenberg’s definitions with a new set of conjugate macroscopic observables.

**Schnakenberg revisited.**— The results are based on a decomposition theorem of the EP in cycles and flows (or cocycles) of the graph. To give a first hint, consider the 3-level system depicted with straight lines in fig 1a.

$$\begin{align*}
 v_1 & \xrightarrow{j_3, a_3} v_2 \xrightarrow{-j_3, -a_3} v_2 \xrightarrow{j_1, a_1} v_3 \xrightarrow{-j_1, -a_1} v_3 \xrightarrow{j_2, a_2} v_1. \tag{4}
\end{align*}$$

By the Handshaking lemma \((\Sigma_v = 2 \Sigma_e)\), EP \((2)\) can be recast as \( \sigma = a_1 j_1 + a_2 j_2 + a_3 j_3 \). We reshuffle, add and subtract terms to obtain

$$\sigma = A_1 \underbrace{a_1 j_1 + a_2 j_2 + a_3 j_3} + J_2^* \underbrace{j_2 - j_1} + J_3^* \underbrace{j_3 - j_1}, \tag{5}$$

where we defined, along with one Schnakenberg circulation \( A_1 \) (fig 1b), the macroscopic currents \( J_2^* \), flowing out of vertex \( v_3 \) (fig 1c), and \( J_3^* \), flowing into vertex \( v_2 \) (fig 1d). Since, by eq. \((1)\), \( J_2^* = -\dot{\rho}_3 \) and \( J_3^* = \dot{\rho}_2 \), it is conceptually appropriate to ascribe these observables to the state of the system. The vanishing of \( A_1 \) provides balancing, the vanishing of \( J_2^* \) and \( J_3^* \) defines steadiness. In graph-theoretical language, \( J_2^* \) and \( J_3^* \) are weighted cocycles, that is, edge sets whose removal disconnects the vertex set \( V \) into two noncommunicating components: they measure the total flow from one set towards the other.

Generalizing, let \( G = (V,E,\partial) \) be a connected graph on \(|V|\) vertices and \(|E|\) edges. Edges carry an arbitrary orientation (a choice of tip and tail vertices), with \(-c\) designating the inverse edge. With the exception of loops and multiple edges (which are nevertheless allowed), the topology of the graph is described by the incidence matrix

$$\partial_v^{\pm} = \begin{cases}
+1, & \text{if } c \prec v \\
-1, & \text{if } c \succ v \\
0, & \text{elsewhere}
\end{cases}. \tag{6}$$

We use a rather algebraic approach to graph theory \cite{15,16}, working with integer linear combinations of edges in the lattice \( E = \mathbb{Z}^E \), upon which \( \partial \) acts as a boundary operator. It is a standard result that \( \partial \) induces an orthogonal decomposition of \( E = C \oplus C^* \) into the cycle space \( C = \ker(\partial) \) and the cocycle space \( C^* = \text{rowspace}(\partial) \). The dimension of the cycle space is given by the cyclomatic number \( C = |E| - |V| + 1 \), whence by the rank-nullity theorem the cocycle space has dimension \(|V| - 1\).

From a graphical point of view, cycles \( c \) are chains of oriented edges such that each vertex is the tip and the tail of an equal number of edges (possibly none). It is simple if it is connected, has no crossings or overlapping edges. A simple cycle can exist in two opposite orientations. A simple cocycle \( c^* \) is a collection of edges whose removal
The remaining edges \( e_α \in E \setminus T \) are called cochords. There are \( |V| - 1 \) cochords and \( C \) chords. When adding a chord to a spanning tree, a simple cycle \( c_α \) is generated, which can be oriented accordingly with \( e_α \) (see fig. 1b, b). The fundamental set of cycles \( \{ c_α \} \) so generated is a basis for \( C \). Similarly, when removing a cochord \( e_μ^∗ \), the spanning tree is disconnected into two components, which identify a simple cocycle \( c_μ^∗ \), with orientation dictated by \( e_μ^∗ \) (see fig. 1b, c, d). Again, the fundamental set of cocycles \( \{ c_μ^∗ \} \) is a basis for \( C^∗ \).

The crucial peculiarity of fundamental sets is that no chord is shared by two cycles, and no cochord is shared by two cocycles. Moreover, any of the sets \( \{ e_α, e_μ^∗ \}, \{ c_α, e_μ^∗ \}, \{ c_α, c_μ^∗ \}, \{ e_α, c_μ^∗ \} \) forms a basis for \( E \).

We finally introduce: (I) mesoscopic currents \( j_ε \) as real edge variables, antisymmetric by edge inversion \( j_ε = -j_ε \); (II) antisymmetric conjugate forces \( a_ε \); (III) the EP

\[
\sigma = \sum_ε j_ε a_ε = (j, a),
\]

where the r.h.s. is the euclidean scalar product on the edge set in shorthand; (IV) the macroscopic observables

\[
J_α = (e_α, j), \quad J^μ_α = (e_μ^∗, j), \quad A^α = (c_α, a), \quad A^*_μ = (e_μ^∗, a).
\]

In left-to-right, horizontal order: **internal currents** flow along fundamental chords, **external currents** are the total flow out of the source set of a cocycle, **external forces** are circulations of forces along the fundamental cycles, **internal forces** are exerted along edges of the spanning tree. In terms of the incidence matrix, Kirchhoff’s Law reads \( ∂j = 0 \), and the r.h.s. of eq. (1) reads \( \dot{ρ} + ∂j = 0 \).

Oriented overlappings between edge sets can be succinctly expressed in terms of the scalar product: \( (c_α, e_β) = δ_β^α \), \( (e_μ^∗, e_ε^∗) = δ_μ^ε \), \( (c_α, c_μ^∗) = 0 \), \( (e_μ^∗, e_α) = 0 \).

At last we can formulate the core theorem, whose proof is given in the appendix.

**Theorem 1.** The entropy production can be decomposed into a steady-state and a detailed-balanced term, \( \sigma = \sigma_{ss} + \sigma_{db} \), given respectively by

\[
\sigma_{ss} = A^α J_α, \quad \sigma_{db} = J^μ_α A^*_μ.
\]

A detailed-balanced system (resp. steady state) has vanishing external macroscopic forces \( A^α = 0 \) (resp. currents \( J^μ_α = 0 \)), in which case \( \sigma_{ss} \) (resp. \( \sigma_{db} \)) vanishes for all values of the internal currents (resp. forces). When both vanish we talk of equilibrium states.

**Duality.** – A graph is planar if it can be drawn on the surface of a sphere with non-intersecting edges. Planar embeddings have faces \( f \in F \), i.e. open neighbours of the sphere which cannot be path-connected without crossing an edge. Their number \( |F| = C + 1 \), including the “outer” face, is prescribed by Euler’s formula.

The dual graph \( G^∗ = (V^∗, E^∗, δ^∗) \) has one vertex per face, \( V^∗ = F \), two dual vertices being connected by one dual edge \( *e \) per each boundary edge \( e \) that the corresponding faces share, so that \( E^∗ = E \). Pictorially, after puncturing and flattening the sphere, one will draw a vertex inside each face and a dual edge \( *e \) crossing \( e \), then assign an orientation by clockwise rotating \( *e \) until it overlaps, tip and tail, with \( e \) (see curved lines and shadings in fig. 1b). Crucial facts about duality are:

(i) Up to a reorientation \( E \to -E \), it is involutive;

(ii) Different embeddings might yield non-isomorphic duals (with different incidence relations);

(iii) It maps a spanning tree \( T \) to the complement \( T^* = E \setminus T \), of a spanning tree \( T \subseteq E^∗ \), in such a way that the fundamental sets generated by \( T \) are the duals of the fundamental sets generated by \( T \), according to the scheme (see fig. 1b, c, d)

\[
\text{chords} \leftrightarrow \text{cochords}, \quad \text{cycles} \leftrightarrow \text{cocycles}.
\]

Duality can then be applied to the graphical structure of nonequilibrium observables. So, for example, the map \( a \leftrightarrow j \) leaves \( σ \) invariant, but switches macroscopic observables with those of the dual graph, mapping internal (resp. external) forces to internal (resp. external) currents:

\[
A^α \leftrightarrow J^μ_α, \quad A^*_μ \leftrightarrow J_α, \quad \sigma_{ss} \leftrightarrow \sigma_{db}.
\]

Since we ascribed \( A^α \) to the state of the environment and \( J^μ_α \) to that of the system, it is fair to dub this system-environment duality. Steady states, for which the macroscopic external currents vanish, are dual to detailed-balanced systems, for which the macroscopic external forces vanish: the former are in fact properties of the system under given environmental conditions, while the second are properties of the environment’s influence on the system, independently of the system’s state.

Out of the \( a \leftrightarrow j \) special case, we stress that duality is a graph-theoretical property: it tells how well-behaved observables should look like from the point of view of the environment and of the system, not which mesoscopic variables should play the game.

Planarity seems to be a major limitation to the generality of system/environment duality. We argue that this is not the case. Property (iii) listed above is independent of the particular embedding chosen. Indeed, generalizing the concept of a graph to that of an abstract matroid, it turns out that matroids always have a well-defined dual which satisfies property (iii), even though dual matroids might not be visualizable as graphs.
Linear regime and minimum entropy production. — One major clue that led Schnakenberg to identify chords and cycles as good thermodynamic observables is the fact that, in the linear regime, Onsager’s reciprocity relations arise. By “linear regime” it is meant that mesoscopic currents and forces satisfy Ohm’s law \( \mathbf{a} = \mathbf{\ell} j + O(\mathbf{j}^2) \), where \( \mathbf{\ell} = \text{diag}(\ell_1, \ldots, \ell_{|E|}) \) is a “local” linear response matrix, connecting mesoscopic quantities edge by edge. Suppose that a system is initially at equilibrium, and it is perturbed to a nearby nonequilibrium steady state. Schnakenberg furnished the macroscopic linear response matrix, connecting mesoscopic quantities of cycles. For rate-equation systems, this insight is straightforward:

\[
A^\alpha = (\mathbf{c}^\alpha, \mathbf{\ell} j) = (\mathbf{c}^\alpha, \mathbf{\ell} \mathbf{c}^\beta) J_{\beta} \equiv L^{\alpha \beta} J_{\beta}. \tag{12}
\]

The linear response matrix is a weighted superposition of cycles. For rate-equation systems, this insight is complemented by Andrieux and Gaspard’s proof of a Green-Kubo-type of formula for \( \mathbf{L} \) [4]. Let us now linearly perturb an equilibrium state into a nonsteady, but still detailed-balanced configuration. Like steadiness is implied by Kirchhoff’s law, balancing follows from the dual relation \( \partial^* \mathbf{a} = 0 \), which is solved by \( \mathbf{a} = A^*_\mu \mathbf{c}^\mu \). Then

\[
J^\mu = (\mathbf{c}^\mu, \mathbf{\ell}^{-1} \mathbf{a}) = (\mathbf{c}^\mu, \mathbf{\ell}^{-1} \mathbf{c}^\nu) \equiv L^{\mu \nu} A^*_\nu \tag{13}
\]

and the dual response matrix \( \mathbf{L}_* \) is a weighted superposition of cocycles. Both matrices \( \mathbf{L} \) and \( \mathbf{L}_* \) are symmetric, and under \( \mathbf{\ell} \leftrightarrow \mathbf{\ell}^{-1} \) are dual one to the other. Similar matrices are of use in electrical circuit analysis [13] and in the parametric formulas for Feynman diagrams (see [19], §18.4; [10], §3). In this contest planar-graph duality has been related to duality between momentum and position representations [20]. Possibly the most interesting property of \( \mathbf{L} \) and \( \mathbf{L}_* \) is that their determinants, which are always nonnull but for very trivial graphs, are independent of the fundamental sets chosen, obey the relation \( \det \mathbf{L}/\det \mathbf{L}_* = \det \mathbf{\ell} \), and are related to the 0-state Potts partition function [21]. Another important fact is that

Theorem 2. When the equilibrium state is linearly perturbed in an unconstrained manner (neither into a steady state nor into a detailed balanced configuration), entropy production can still be written as a block-diagonal bilinear form of the external observables

\[
\sigma = (\mathbf{L}^{-1})_{\alpha \beta} A^\alpha A^\beta + (\mathbf{L}_*)_{\mu \nu} J^\mu J^\nu. \tag{14}
\]

This result is far from trivial — for example, cross-terms will appear when expressing EP in terms of the internal observables! — and further supports the point of view that the external currents and forces are the true macroscopic nonequilibrium quantities which the observer controls. It will deserve further attention both for its technical subtlety and for its conceptual simplicity.

One physically-motivated application of Schnakenberg’s macroscopic observables in the linear regime was proposed by the author [22], who proved that Schnakenberg’s affinities are the correct macroscopic constraints to be imposed to the minimum entropy production principle, which in one particularly suitable wording [23] asserts that

“the steady state is that state in which the rate of entropy production has the minimum value consistent with the external constraints which prevent the system from reaching equilibrium”.

The key step is to identify the external constraints with Schnakenberg’s external forces. Varying \( \sigma(\mathbf{j}) = (\mathbf{j}, \mathbf{\ell} j) \), with given values of \( (\mathbf{c}^\alpha, \mathbf{\ell} j) \equiv \bar{A}^\alpha \) fixed through Lagrange multipliers,

\[
\frac{\delta}{\delta j^\mu} \left[ \sigma(\mathbf{j}) - 2\lambda_\alpha \left( (\mathbf{c}^\alpha, \mathbf{\ell} j) - \bar{A}^\alpha \right) \right] = 0, \tag{15}
\]

we find the extremal solution \( j = \lambda_\alpha \mathbf{c}^\alpha \). To fix the value of the Lagrange multipliers, replace in the constraint equation to obtain \( L^{\alpha \beta} \lambda_\beta = \bar{A}^\alpha \). Since \( \mathbf{L} \) is invertible, \( \lambda_\alpha \) turns out to be the fundamental internal current conjugate to the observed value of the affinity \( \bar{A}^\alpha \); hence we are in a steady state. One can then easily dualize the whole derivation to prove the dual proposition quoted in the introduction.

Electromagnetism on a network. — An important notion of duality arises from which to compare ours is the electromagnetic (EM) duality. We refer here to C. Timm’s work on rate equations [24].

Let’s think of \( \rho \) as a charge density. In order to make the overall network neutral we introduce a supplementary vertex “\( \infty \)”, charged \( \rho_{\infty} = -\sum_\nu \rho_\nu \). All graph-theoretical notation will refer to this extended graph, which can be further made into a 2-dim. cell complex by introducing a collection \( P \supseteq C \) of plaquettes. Choose a clockwise/anticlockwise orientation for each plaquette \( p \) and define the boundary (curl) operator (see [25] for an introduction on discrete differential calculus)

\[
(\partial \times) p = \begin{cases} 
+1, & \text{if } e \downarrow \bigcup p, \ e \uparrow \bigcup p \ 
-1, & \text{if } e \uparrow \bigcup p, \ e \downarrow \bigcup p \ 
0, & \text{elsewhere} \ 
\end{cases} \tag{16}
\]

Boundaries of plaquettes (columns of \( \partial \times \)) are cycles, hence \( \partial(\partial \times) = 0 \) (the divergence of the curl vanishes).

Introduce an electric field \( \mathbf{E}_c \) over edges and a magnetic field \( \mathbf{B}_p \) over plaquettes. The electric field is required to satisfy Gauss’s law \( \partial \mathbf{E} \equiv \mathbf{\rho} \). Taking the time derivative, we have \( \partial(\mathbf{E} + \mathbf{j}) = 0 \), from which it follows that \( \mathbf{E} + \mathbf{j} \) is a linear combination of cycles,

\[
j = -\mathbf{E} + \mathbf{B}_\alpha \mathbf{c}^\alpha \equiv -\mathbf{E} + \partial \times \mathbf{B}, \tag{17}
\]

where the r.h.s. imposes the Ampère-Maxwell law. Since \( \mathbf{c}^\alpha \) is a complete set of cycles, there exists an \( |P| \times C \) matrix \( \mathbf{\eta} \) such that \( (\partial \times)^T \mathbf{E} + \mathbf{B} \equiv 0 \), and apply \( \mathbf{\eta} \):

\[
(c^\alpha, \mathbf{E}) = -\mathbf{\eta}_\alpha. \tag{18}
\]
It follows that any two combinations of plaquettes which share the boundary enclose a volume across whose boundary the magnetic flux is zero. Only $C$ out of $|P|$ magnetic field values are independent (Gauss’s Law).

As EP it is reasonable to elect the total energy flux

$$\sigma = (\mathbf{E}, \mathbf{E}) + \sum_P B_P \dot{B}_p = -(\mathbf{j}, \mathbf{E}) = (\mathbf{E}, \mathbf{E}) + \mathcal{R}_a \mathcal{R}^a$$  \hspace{1cm} (19)

where we applied Faraday’s Law, transposed the curl operator, and used Ampère’s Law to get the second identity (Integrated Poynting’s Theorem). The third displays a simple dependence on the boundary values of the magnetic field. Our theorem can now be applied:

$$\sigma = J_0 \mathcal{R}^a - J^\mu \mathcal{R}^*_\mu$$  \hspace{1cm} (20)

where $\mathcal{R}^*_\mu$ is the electric field along cochord $e^*_\mu$. By eq. (22), $J^v_\mu$ is (minus) the time-derivative of the charge in $s(e^*_\mu)$. Hence under graph duality and $j \leftrightarrow E$ one obtains

$$\mathcal{R}^*_\mu \leftrightarrow J_\alpha, \quad \mathcal{R}^a \leftrightarrow \rho^a + \text{const.} \hspace{1cm} (21)$$

The electric field is mapped to the source of the magnetic field and vice versa. Thus the example further supports the interpretation of duality as reversing the role of system and environment. Although, notice that the dynamical evolution is not respected: only Kirchhoff’s and Faraday’s “structure” equations are dual to each other. The lagrangian turns out not to be self-dual. This is an important difference between sys./env. and EM duality, which is dynamical. Moreover, the former is 2-dimensional, while the latter, restricted to sourceless cases or requiring magnetic charges, involves the Hodge machinery in 3 dimensions. In ours, divergencelessness of the magnetic field is an essential feature rather than an obstruction to duality.

**Discussion and conclusions.** – The latter application highlights that duality only works for kinematical states, *viz.* instantaneous snapshots of the system. So, for example, by “steady” we mean that Kirchhoff’s law is satisfied, not persistence in time. This is one important limitation that one will have to take care of when considering, for example, markovian evolution: by no means do we claim that duality maps the a rate equation into a dual rate equation. As to the other important limitation, namely planarity, we already discussed how it can be formally overcome with matroids and conceptually regarded as accidental. However, there is another way out, based on the fact that any graph can be embedded on an orientable closed surface of high-enough genus, and there dualized. While the number of cycles and cocycles is not affected, the number of faces, hence of dual cocycles and dual cycles, will change according to Euler’s formula. It is simple to foresee that Theorem 1 will hold unchanged, but its interpretation will have to be accordingly modified, accounting for a number of global currents and of topological phases, such as those which were taken in consideration by Jiang and the Qians [10]. Duality will only hold locally.

To conclude, let us linger on the 3-state example, in the attempt to provide a more intuitive grasp on the physics of duality. Label the graph with the energy levels of an open system, which can emit and absorb environmental quanta of energy. The onset of a NESS might be due to the interaction with two thermal baths [26][27], whose inverse temperatures $\beta_A$ and $\beta_B$ label the states of the dual system, with $\beta_A > \beta_B$. Suppose that transitions 2 and 3 are exclusively due to the interaction with B, while transition 1 is exclusively due to the interaction with A, as is pictorially depicted in fig 2. The ratio of emission and absorption rates is given by $w_e/\omega_{e_1} = \exp(\beta_A(v_2 - v_3))$, and similarly for the others, yielding as macroscopic affinity $A_1 = (\beta_A - \beta_B)(v_2 - v_3)$. In a nonequilibrium steady state, with current $J_1 = J_2 = J_3 = J_1$, one quantum of energy $v_2 - v_3$ is emitted towards reservoir A every $|J_1|^{-1}$ seconds, per each couple of quanta of energy $v_2 - v_1$ and $v_1 - v_3$ that are absorbed from reservoir B. It takes shape a picture where to a steady state there corresponds a non-steady flow of energy from the hotter to the colder bath:

nonequilibrium steady sys. $\rightarrow$ nonsteady env.

Whilst purely speculative, this interpretation is consistent with the physical intuition that NESSs are determined by a transient environmental behaviour [28].

Vice versa, a detailed-balanced flow arises when there is no temperature gradient, $\beta_A = \beta_B$, in which case we only resolve one reservoir. At equilibrium, because of steadiness and detailed balancing, as many quanta of given energy are emitted and absorbed. However, fluxes within the system determine a non-null flow of currents in the one bath, which is necessarily steady being a 1-state system. Hence the system’s state plays the role of external force igniting internal fluxes within the environment:

steady env. $\rightarrow$ detailed balanced nonsteady sys.

This is nothing but the logical negation of the above proposition, hence its dual under transposition of the material implication symbol ($\rightarrow$).

Despite of its simplicity, the example is rather clumsy and only vaguely illustrative: system and environment do not play mirror roles, for which reason we were not able to draw the inverse implications. However, the qualitative principle seems to be robust. It is quite remarkable that graph duality finds a similar interpretation also in mechanical engineering [29], where the statics of a structure and its first order kinematics are related to dual properties of its design. Thus there seems to be a vast variety of systems to which duality might apply: it is the author’s opinion that the development of a statistical model which displays duality between environmental and internal degrees of freedom would be a major advance.

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Appendix. –

Proof of Theorem 1 The strategy is to find the general solution to the continuity equation with sources \( \dot{\rho} + \partial \ddot{j} = 0 \). Here \( \dot{\rho} \in \mathbb{R}^V \) is the current injected at the vertices, and it is constrained to satisfy \( \sum_{v \in V} \dot{\rho}_v = 0 \). Since any \( |V| - 1 \) rows of \( \partial \) span the cocycle space, \( \dot{\rho}_v \) is expressible as a linear combination of a fundamental set of external currents, and viceversa. One can easily show that

\[
- \dot{\rho}^\mu \equiv J^\mu = - \sum_{v \in c(\xi^\mu)} \dot{\rho}_v.
\]

Physically: the flow out of a source set is equal to (minus) the sum of the injected currents within the set.

The general solution can be found as a particular solution plus the general solution of the homogeneous equation associated to it. Solving \( \partial \ddot{j} = 0 \) yields a superposition of cycles \( \sum \alpha \lambda_\alpha e^\alpha \). As to the particular solution, since \( \{e^\alpha, e^\mu_\alpha\} \) is a basis for \( E \), we can tune the cycle currents so as to make currents along chords vanish. We then only need to specify the particular solution along cochains, obtaining

\[
\ddot{j} = \lambda_\alpha e^\alpha + \lambda^\mu e^\mu_\alpha.
\]

Inserting (23) into the definitions (8a), and using the orthonormality relations (9), one identifies \( J_\alpha = \lambda_\alpha \), and \( J_\alpha^\mu = \lambda^\mu_\alpha \). Further insertion into (7) yields our thesis.

Proof of Theorem 2 Consider eq. (23), with \( \lambda_\alpha = J_\alpha \) and \( \lambda^\mu_\alpha = J_\alpha^\mu \), and replace in the bilinear form \( \sigma = (\ddot{j}, \ddot{\ell}) \):

\[
\sigma = L^{\alpha \beta} J_\alpha J_\beta + M_{\mu \nu} J_\mu J_\nu + 2H_\mu J_\alpha J^\mu_\alpha
\]

where we defined \( M^{\mu \nu} = (e^\mu_\alpha, e^\nu_\beta) \) and \( H_\mu = (e^\alpha, e^\mu_\alpha) \). Moreover, it’s simple to calculate \( A^\alpha = L^{\alpha \beta} J_\beta + H_\mu J^\mu_\alpha \).

Completing the square:

\[
\sigma = L^{-1}_\alpha A^\alpha A^\beta + (M_{\mu \nu} - H_\mu L^{-1}_\alpha H^\mu_\beta) J^\mu_\alpha J^\nu_\beta.
\]

Since \( A^\alpha \) and \( J^\mu_\alpha \) are independent, setting all affinities to zero yields the entropy production for detailed balanced systems, which after the previous theorem and eq. (13) is easily seen to be \( \sigma = \left( L^{-1}_\alpha \right)_{\mu \nu} J^\mu_\alpha J^\nu_\beta \). Since the latter is a nondegenerate bilinear form, we can identify the matrix between parenthesis with \( L^{-1}_\alpha \). \( \square \)

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