ON THE SPECTRUM OF $L^\infty$-DRIFTED LAPLACE-BELTRAMI OPERATORS

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Abstract. This paper studies the spectral properties of the Laplace-Beltrami Laplacian with an $L^\infty$ drift term. We obtain a lower bound for the principle eigenvalue for the Dirichlet problem and a lower bound for any real eigenvalues of the operator of compact manifold. We make no assumptions of self-adjointness or that the drift has any additional regularity. In the self-adjoint case of a Witten Laplacian, our work improves the current theory by proving an estimate that does not rely on a bound on the Bakry-Emery Ricci tensor.

1. Introduction

This paper studies the spectrum of drift-Laplacians under a relatively mild assumption on the drift. Our main result is to establish the following estimate on the real eigenvalues.

**Theorem 1.** Let $(M^n, g)$ be a compact Riemannian manifold, $\Omega$ a smooth subset of $M$ (or possibly all of $M$). Let $v$ be some one form satisfying $\|v\|_{\infty} < C$. Suppose that there exists $\lambda$ real and $u \in W^{2,p}(\Omega)$ satisfying

$$\begin{cases} 
\Delta u + v(\nabla u) = \lambda u & x \in \Omega \\
u(x) \equiv 0 & x \in \partial \Omega
\end{cases}$$

Then there exists some constant $\delta > 0$ depending only $\|Ric\|$, $C$, $\text{diam}(M)$, $\text{inj}(M)$ and $n$ so that $\lambda > \delta$.

This estimate immediately implies a lower bound of the principal eigenvalue of drift-Laplacians on smooth bounded domains in Riemannian manifolds.

**Theorem 2.** Let $(M^n, g)$ be a compact Riemannian manifold and $\Omega \subset M$ a smooth domain with non-empty boundary. Let $v$ be some a one-form on $\Omega$ satisfying $\|v\|_{\infty} < C$. Consider the principle eigenvalue $\lambda_1$ of the operator $\Delta + v(\nabla \cdot)$ on $\Omega$. Then there exists some constant $\delta > 0$ depending only $\|Ric\|$, $C$, $\text{diam}(M)$, $\text{inj}(M)$ and $n$ so that $\lambda > \delta$. 

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The basic strategy of the proof of these results is to adapt the Li-Yau estimate \[LY86\] to the drifted Laplacian. The lack of any a priori bound on $\nabla v$ prevents us from directly applying the estimate. The key insight around this roadblock is an ansatz due to Hamel, Nadirashvili, and Russ \[HNR05\] which shows that when the principle eigenvalue is minimized, the problem becomes much more regular. Put colloquially, to slow diffusion as much as possible, all of the drift needs to be working in unison.

When applied to the self-adjoint case, this gives eigenvalue estimates using the Ricci tensor and only a Lipschitz estimate on the potential, with no assumptions on the Bakry-Emery Ricci tensor. To our knowledge, all previous results in this context used at least some weak control of the Bakry-Emery Ricci tensor.

1.1. Motivations from complex geometry. The main motivation for this work was to further explore some aspects of the interaction between Riemannian and Hermitian geometry. To focus on the spectral theory, we will explore the relationship to complex geometry in a separate work. However, we will briefly note that Theorem 1 and the torsion estimates in \[Kha16\] immediately imply the following estimate.

**Theorem 3.** Let $(M^n, g)$ be a compact Riemannian manifold. Suppose it admits a complex structure $J$ so that $\omega = g(J \cdot, \cdot)$ is $k$-Gauduchon for $\frac{n}{2} < k \leq n - 1$. Consider the complex Laplacian $\Delta^c = \frac{1}{2} \partial_z \partial_{\bar{z}}$ and suppose that it has a real eigenvalue $\lambda$. Then there exists some constant $\delta > 0$ depending only $|\text{Ric}|, k, \text{diam}(M), \text{inj}(M), n$ and the smallest eigenvalue of the Weyl curvature tensor so that $\lambda > \delta$. In particular, $\delta$ is independent of $J$.

It is worth noting that this result is an effective version of Theorem 6 of \[Kha16\]. The earlier was a simple argument using the maximum principle so could not be used to find positive lower bounds or even to determine what aspects of the Riemannian geometry were needed to control the eigenvalue.

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1 For the Li-Yau estimate applied to a drift-Laplacian with $C^1$-bounded drift, see \[Kha16\]
2. Background

The study of eigenvalues on domains and manifold has a long and rich history. Classically, this is related to the problem of “hearing the shape of a drum” [Kac66], which tries to understand the geometry of an object by studying the spectrum of its Laplacian. The study of this question unites many mathematical fields and has broad applications. For an introduction, we highly recommend the lecture notes by Canzani [Can14] and for some applications and interesting connections, we recommend the book by Rosenberg [Ros97].

We are interested in how the geometry of the manifold affects the spectrum. For this problem, the breakthrough work is due to Li and Yau [LY86], who studied the heat equation associated to the Laplace-Beltrami operator. Their work proves a gradient estimate and uses this estimate to establish eigenvalue lower bounds. Their estimate only involves the Ricci curvature, the diameter, and the dimension of the manifold. There has been a concerted effort to sharpen these estimates to find strict bounds (for one example [ZY84]). Beyond eigenvalue estimates, the Li-Yau estimate has played a central role in the development of geometric analysis (most famously, it has an important role in the analysis of Ricci flow [Per02]).

For Laplacians with drift, historically the focus has been on self-adjoint operators. If \( \Delta + v(\nabla \cdot) \) is self-adjoint, then \( v = \nabla f \) for some Lipschitz function \( f \) and the operator is known as the Witten Laplacian \( \Delta_f \). The study of these operators is important in the analysis of Ricci solitons, and the spectrum of such operators has been studied in depth. Sharp eigenvalue bounds are known [AN12, FLL13] under the assumption of a lower bound on the Bakry-Emery Ricci tensor, which is defined as \( \text{Ric} + \nabla^2 f \). Witten Laplacians were also considered by Witten [Wit82] in his work on Morse theory.

For general elliptic equations with less regular coefficients and domains, much of the progress on eigenvalue estimates uses more traditional partial differential equation approaches. For instance, given an elliptic operator \( L \), the work of Berestycki et. al [BNV94] defines the principal eigenvalue and provides positive lower bounds on the it given sub-solutions to the problem \( Lu \leq 0 \) (and various other hypothesis). For domains in manifolds, it is generally not possible to find subsolutions, so we are forced to estimate the eigenvalues using the coefficients and the geometry of the domain alone. This work is still important because it establishes the existence of principal eigenvalues.
Following Berestycki et. al [BNV94], we can define the principle eigenvalue of a uniformly elliptic operator $L$ in the following way.

\begin{equation}
\lambda_1 = \sup\{\lambda \mid \exists u > 0 \text{ in } \Omega \text{ satisfying } (L + \lambda)u \leq 0\}
\end{equation}

The principal eigenvalue is well-defined for a very general class of elliptic operators and in some sense provides the bottom of the spectrum for the operator.

An important contribution to the problem are two papers by Hamel, Nadirashvili, and Russ [HNR05] [HNR11]. Their work proves a version of the Faber-Krahn inequality for a drifted Laplacian when the drift is bounded. To show this, they start by making the key observation that when the eigenvalue is minimized, the drift takes a special form which produces much more regularity for free. Our paper is not the first to use this idea in order to find eigenvalue estimates on Riemannian manifolds. Recently, Ferreira and Salavessa [FS17] used these ideas to compare the eigenvalues of $V$-Laplacians on geodesic balls to those on model spaces. Our two approaches are completely different, but the results are similar. In particular, Theorems 1 and 2 of their paper proves a lower bounds of the principle eigenvalue on geodesic balls.

Interesting, our work and their work both have relative advantages. Their work is able to prove Faber-Krahn type inequalities in geodesic balls. Also, under the assumption of bounded radial sectional curvature, they are able to relax the assumption on the drift. However, our work makes no assumption that $\Omega$ is diffeomorphic to an open set in Euclidean space and uses the Ricci curvature instead of the sectional curvature\footnote{Theorem 2 of their work does use the Ricci curvature but assumes the drift is radial in a geodesic ball.}. We are interested whether it is possible to combine some of the estimates to establish stronger results.

In this work, we are primarily interested in the real elements of the spectrum. This set is non-empty in two important cases.

1. If $\Omega$ is an open subset of $M$ with smooth boundary, then there will be a real principal eigenvalue for the Dirichlet problem [BNV94]. Our work gives lower bounds on this eigenvalue without making any further assumptions on the boundary of $\Omega$ or assuming that we can construct a sub-solution.

2. If $v = \nabla f$ for some function $f$, then the drift-Laplacian is self-adjoint and hence its spectrum is entirely real. The assumption that $v$ is bounded is equivalent to the assumption that $f$ is uniformly Lipschitz, and so we make no assumption on the Bakry-Emery Ricci tensor. For now, we must compensate for this by bounding the norm of the Ricci tensor. However, this bound is only used to be able to apply the
Calderon-Zygmund inequality on our manifold. If we do this step differently, then the only curvature input needed for the estimate is a lower bound on the Ricci curvature.

3. The proof of Theorem 1

In this section, we provide the proof to Theorem 1. Recall that this result states the following.

**Theorem.** Let $(M^n, g)$ be a compact Riemannian manifold, $\Omega$ a smooth subset of $M$ (or possibly all of $M$). Let $v$ be some one form satisfying $\|v\|_\infty < C$. Suppose that there exists $\lambda$ real and $u \in W^{2,p}(\Omega)$ satisfying

$$
\begin{aligned}
\Delta u + v(\nabla u) &= \lambda u & x \in \Omega \\
u(x) &\equiv 0 & x \in \partial \Omega
\end{aligned}
$$

We do not provide a closed form expression for $\delta$. To do so, we would need to solve for certain constants used in the following two papers of Güneysu and Pigola [GP15] and Anderson and Cheeger [AC92]. However, we do provide an exact expression once those constants are known. Under even more general assumptions on the drift, we can show that $\lambda > 0$ using a basic maximum principle argument (see the proof of Theorem 6 in [Kha16]), but we are unable to derive positive lower bounds in that case.

**Proof.** For readability, we start with a brief overview of the proof.

1. We starting by using a Calderon-Zygmund inequality for manifolds proved by Güneysu and Pigola [GP15] and an interpolation result to derive a $W^{2,p}$ estimate on $u$ using the norm of the Ricci tensor and some lower-order geometry.

We use the Ricci curvature, injectivity radius and volume to find an atlas with bounded $C^\alpha$ harmonic radius [AC92]. We then choose a partition of unity subordinate to this atlas, and obtain a $C^{1,\alpha}$ estimate on $u$ using Morrey’s inequality on each chart. In spirit, this is similar to Theorem 7.1 in [Cou96], but for compact manifolds.

2. We consider the domain on which the function is positive and expand it if need be so that the boundary is smooth. We consider a sequence of drifts that minimize the principle eigenvalue $\lambda$ on that domain. We pick some subsequence for which the associated drifts and the corresponding eigenfunctions converges in some weak sense. When the drift minimizes $\lambda$, we find that the minimizing function satisfies the semi-linear equation

$$
\Delta u + C|\nabla u| + \lambda u = 0
$$
with Dirichlet conditions. This phenomena was first observed in [HNR05] and essentially provides $C^1$ control over the drift away from the zero locus of $u$ and $\nabla u$.

3. We then use Theorem 6.2 of Gilbarg-Trudinger [GT83] and our $W^{2,p}$ estimate on $u$ in small neighborhoods away from the zero locus of $\nabla u$ and $u$. These estimates allow us to bootstrap the regularity of $u$ to $C^{3,\alpha}$. We refer to these estimates as the Schauder bounds. Since these bounds are heavily dependent on the neighborhood we are working in, we cannot incorporate them into our estimate of $\lambda$ (if we try to do so, the argument becomes circular). The $C^3$ regularity allows us to use the Bochner technique.

4. We consider the point $x_0 \in M$ which maximizes

$$F(x) = \frac{\|\nabla (u)\|^2}{(\beta - u)^2(r^{1/2}_1 - \rho^2)}$$

and use a Li-Yau-type estimate to obtain an upper bound for $F(x)$. While this part of the argument mostly adapts the original Li-Yau estimate for the Laplace-Beltrami operator, it involves a lengthy calculation to reduce it to a usable form.

5. After applying the Li-Yau estimate, we obtain a gradient estimate. We integrate this gradient inequality along a particular geodesic. Using the geometry of the manifold and the magnitude of the drift, we choose our parameter $\beta$ to obtain a lower bound on $\lambda$.

$\square$

With this overview done, we provide the details of the proof.

3.1. The Lipschitz estimate. We start by obtaining an a priori Lipschitz estimate on $u$ that depends only on the Ricci curvature and the lower order geometry of $M$. To do so, we apply a Calderon-Zygmund estimate proved in the recent work of Güneysu and Pigola [GP15].

Theorem ([GP15]). Let $1 < p < \infty$ and assume that $M$ has bounded Ricci curvature and a positive injectivity radius. Then, for all $\phi \in C^\infty_c(M)$

$$\|\nabla^2 \phi\|_{L^p} \leq C_1 \|\phi\|_{L^p} + C_2 \|\Delta \phi\|_{L^p}$$

where the constants depend only on $\dim M$, $p$, $|\text{Ric}|$ and the injectivity radius.

Güneysu and Pigola’s work proves this estimate in the non-compact case, but it is straightforward to adapt their result to the compact case.
To do so, one uses the bound on the injectivity radius and Ricci tensor to obtain a lower bound on the $C^{1,\alpha}$ harmonic radius of precision 2 (see the appendix of [GP15]). From this, one can take a cover of $M$ by balls of half this radius and apply Lemma 4.8 to find a finite cover whose intersection multiplicity is bounded. In each chart, applying Theorem 3.16 obtains a $W^{2,p}$ estimate and the bounded intersection multiplicity allows one to use these local estimates to obtain a global $W^{2,p}$ estimate. After this, applying Proposition 3.12a to eliminate the gradient term, one has the desired result.

It is worth noting that if we merely have a lower bound on the Ricci tensor as well as bounds on the volume and injectivity radius, one has a lower bound on the $C^\alpha$ harmonic radius as well as bounds on the number of charts and the multiplicity [AC92]. To estimate the symbol of the Laplace-Beltrami operator in a coordinate chart, one needs an estimate of the following form:

$$Q^{-1}\delta_{ij} \leq g_{ij} \leq Q\delta_{ij}$$

Such an estimate is guaranteed within the $C^\alpha$ harmonic radius of precision $Q$. Therefore, it seems likely that one can derive a similar estimate with only a lower bound on the Ricci tensor. However, the main technical obstruction to this approach is that with only $C^\alpha$ control of $g$ in the coordinate charts, we do not have control of the lower order terms in the Laplace-Beltrami operator. As such, we use a two sided bound on the Ricci tensor, which gives bounds on $C^{1,\alpha}$ harmonic radii.

Taking $\phi$ to be $u$ in [3], we obtain the following estimate. We are trying to find lower bounds on $\lambda$, so we assume that $\lambda < 1$ (if not, then 1 is trivially a lower bound).

$$\|\nabla^2(u)\|_{L^p} \leq C_1\|u\|_{L^p} + C_2\|\Delta(u)\|_{L^p}$$

$$= C_1\|u\|_{L^p} + C_2\|v(\nabla u) + \lambda u\|_{L^p}$$

$$\leq C_12\|u\|_{L^p} + C_2\cdot C\|\nabla u\|_{L^p}$$

To eliminate the gradient term, we once again use Proposition 3.12a of [GP15]. Doing so, we find that

$$\|\nabla^2(u)\|_{L^p} \leq C_3\|u\|_{L^p}$$

Normalizing $u$ so that $\sup u = 1$, we can use the volume comparison theorem along with our Ricci and diameter estimate to get a uniform estimate on the $L^p$ norm of $p$. From this,
we obtain a uniform $W^{2,p}$ estimate on $u$ that depends only on $p$, $n$, diameter, the injectivity radius, and the bounds on the Ricci curvature. This bound provides a uniform $L^p$ estimate on $\nabla|\nabla u|$. To eliminate the dependence on $p$, we can set $p = 2n$.

We use the results of Anderson and Cheeger [AC92] to cover $M$ with a finite atlas of precision $2 C_1/2$ harmonic coordinate charts $\phi_i : U_i \to B_{r_0}(0) \subset \mathbb{R}^n$. In each of these charts, we can use the precision estimates to obtain a $W^{2,p}$ bound on $u \circ \phi_i^{-1}$. From this, we can use Morrey’s inequality on each ball to obtain a uniform $C^{1,\alpha}$ bound on $u \circ \phi_i^{-1}$. Using the precision estimates again, we obtain a uniform $C^{1,\alpha}$ estimate on $u$.

Therefore, for some $C_4(n, \alpha, |Ric|, diam(M), inj(M))$, we have the estimate

$$\|u\|_{C^{1,\alpha}} < C_4$$

It is worth mentioning that we could have done this argument using $C^{1,\alpha}$ $2$-precise harmonic coordinate charts that were used to prove the Calderon-Zygmund estimate. We chose the $C^\alpha$ charts so that this step would only depend on a lower bound of the Ricci curvature.

### 3.2. Finding the drift that minimizes the principle eigenvalue.

When $\Omega = M$, we want to reduce our problem to a Dirichlet problem on a subdomain in order to find the drift which minimizes the eigenvalue. To do so, consider the open manifold $M^+ = \{M|u > 0\}$. Note that we can show that this domain contains a uniform ball, by the $W^{2,p}_{loc}$ estimate on $u$. We can also show that its complement also contains an open ball. However, we do not have any a priori regularity of the boundary of $M^+$. Therefore, we instead consider the domain $M^+_\epsilon$ so that $M^+ \subset M^+ _\epsilon$ and the boundary of $M^+_\epsilon$ is smooth. Heuristically, one should picture $M^+_\epsilon$ as being only slightly enlarged from $M^+$, but we will not need to use this explicitly. If we instead work with the Dirichlet problem on a smooth bounded domain, we can set $\Omega = M^+ = M^+_\epsilon$, and there is no need to do this step.

At this point, this reduces the problem to studying the Dirichlet problem on a smooth open set in $M^+_\epsilon \subset M$. We now consider the drift $v'$ which minimizes the principle eigenvalue $\lambda(\Delta u + v, M^+_\epsilon)$ among all drifts $v$ with $\|v\|_\infty < C$. Since $M^+_\epsilon$ is at least as large as $M^+$, $\lambda(\Delta u + v', M^+_\epsilon)$ is no greater than $\lambda(\Delta + v(\nabla \cdot), M^+)$. Therefore, it suffices for us to estimate $\lambda(\Delta u + v', M^+_\epsilon)$.

We now consider the minimal principle eigenvalue $\lambda = \lambda(\Delta u + v', M^+_\epsilon)$ and its associated eigenfunction $u$, and prove that they satisfy the Dirichlet problem for the following semi-linear equation on $M^+_\epsilon$:
To do this, we assume that \( v' \neq C \frac{\nabla u}{|\nabla u|} \) on some subset of \( M^+_\epsilon \) with non-zero measure. This implies that \( u \) is a sub-solution to the following problem:

\[
\Delta u - C \frac{\nabla u}{|\nabla u|} \cdot \nabla u + \lambda (\nabla u) \lambda u \leq \Delta u - v'(\nabla u) \lambda u + \lambda (\nabla u + v', M^+_\epsilon) = 0
\]

Now, since \( v' \) is \( L^\infty \) and \( M^+_\epsilon \) is smooth, we have a local \( W^{2,p} \) estimate on \( u \), and hence \( \nabla u \) is well defined. As such, \( C \frac{\nabla u}{|\nabla u|} \) is \( L^\infty \) and there exists a locally \( W^{2,p} \) solution to the Dirichlet problem.

\[
0 = u' - C \frac{\nabla u}{|\nabla u|} \cdot \nabla u' + \lambda' u'
\]

Since we assumed that \( v' \) minimizes \( \lambda \), we know that \( \lambda \leq \lambda' \) which implies that \( u' \) is a super-solution to the following problem:

\[
0 \leq u' - C \frac{\nabla u}{|\nabla u|} \cdot \nabla u' + \lambda u'
\]

Since \( M^+_\epsilon \) is smooth and the drift is \( L^\infty \), the Hopf lemma holds and shows that \( \nabla u \neq 0 \) on the boundary. From this, if we consider \( u - \kappa u' \), and choose \( \kappa \) so that it is the maximum such \( \kappa \) for which \( u - \kappa u' \geq 0 \). From this, we can use a standard touching argument and either the maximum principle or the Hopf lemma to show that \( u \equiv \kappa u' \). In fact, this is exactly Lemma 2.1 of Hamel et al. [HNR05], applied to an open domain on a manifold. As such, we have proven the ansatz.

This observation is quite unexpected. It shows that in the worst case scenario, where all the drift is working to make the principle eigenvalue as small as possible, we obtain much stronger regularity than we initially assumed. This gives us very strong control of the drift away from the zero locus of \( u \) and \( \nabla u \). In essence, all the drift is working together and cannot be too irregular. This phenomena was first observed in [HNR05], which considered the drift-Laplacian with Dirichlet boundary conditions on \( C^{2,\alpha} \) open domains in \( \mathbb{R}^n \) and proved a version of the Faber-Krahn inequality.

**3.3. The a priori \( C^\alpha \) estimate and uniform radii estimates.** We now use our a priori regularity to ensure that the function \( u \) does not vanish too quickly because we do not have \( C^1 \) control of the drift on the zero locus of \( u \). To do this, we can use our a priori \( C^{1,\alpha} \) estimate.
3.3.1. **Lipschitz estimates.** Define $p \in M$ to be a point satisfying $u(p) = 1$. We define the $c$-radius $r_c$ as $\inf_x (d(x, p) | u(x) = c, u(p) = 1)$. For shorthand, we denote $d_{1-c} := \frac{1-c}{C_4}$. By the $C^{1,\alpha}$ estimate on $u$, we have $r_c > d_{1-c}$.

Intuitively, $d_c$ is the smallest distance we can travel to find an oscillation of $c$. This estimate only depends on the geometry of the manifold. Therefore, we can use the constant $d_c$ throughout the estimate. To calculate $d_c$ explicitly, note that we would have had to calculate $C_4$ explicitly.

3.3.2. **Higher regularity away from the zero locus of $\nabla u$.** From the $C^{1,\alpha}$-estimate on $u$, there is trivially a $C^{\alpha}$ estimate on $|\nabla u|$. Thus, when $|\nabla u|$ is non-zero, we have that $u$ satisfies $\Delta u + C \frac{\nabla u}{|\nabla u|} \cdot \nabla u - \lambda u = 0$. The coefficients are now $C^{\alpha}$, so we gain $C^{2,\alpha'}$ control on $u$ away from where $|\nabla u| = 0$ by Schauder theory. Therefore, $|\nabla u| \in C^{1,\alpha}$ in this neighborhood and hence using the Schauder interior estimates again, we have that $u \in C^{3,\alpha}$ in a possible smaller neighborhood.

This estimate will decay when $u$ is close to 1, but this estimate allows us to take three derivatives at a point away from the zero locus of $u$ and $\nabla u$. However, we cannot use these bounds to estimate $\lambda$, as it makes the entire structure of this proof circular.

3.4. **The Li-Yau Estimate.** Now that we have $C^{3,\alpha}$ regularity of $u$ and the regularity of the drift away from a small set, we can apply the Li-Yau estimate.

Recall that we have a function $u \in W^{2,p}(M^+_d)$ which satisfies the following:

$$\begin{align*}
\Delta u + C|\nabla u| + \lambda u &= 0 \\
u|_{\partial M^+_d} &= 0
\end{align*}$$

Suppose further that we have rescaled $u$ so that $\sup u = 1$ and that $\argmax(u) = p$. We define $\rho(x) = \text{dist}(p, x)$ and fix a parameter $\beta > 1$ to be determined later.

We now consider the function $F(x)$ defined by:

$$F(x) = \frac{|\nabla u|^2}{(\beta - (u))^2}(r_{1/2}^2 - \rho^2)$$

We observe that there is a point $x \in B_{r_{3/4}}$ with $|\nabla(u)| > \frac{1}{d^3}$ where $d$ is the diameter of $M$. At such a point,

$$\frac{|\nabla(u)|^2}{(\beta - (u))^2}(r_{1/2}^2 - \rho^2) > \frac{1}{16d^2(\beta - 3/4)^2(\epsilon r_{3/4})}$$

We consider the point $x_0 \in M$ which maximizes $F(x)$. Our previous estimate shows that the following:
$|\nabla (u)|^2 > \frac{(\beta - 1)^2}{d^2} \frac{1}{16d^2(\beta - 3/4)^2}(cr_{3/4})$

Using the $C^{1,\alpha}$ estimate from the previous step, in a small ball around $x_0$, we have that $|\nabla u| \neq 0$. As noted before, we can use Schauder theory to bootstrap our regularity twice. After doing so, we see that $u \in C^{3,\alpha}$ in a possibly very small neighborhood around the $x_0$. The size of this neighborhood will certainly decay as $\beta$ gets close to 1. However, for a given $\beta$, this is enough regularity to apply the maximum principle.

We may choose normal coordinates at $x$ so that $u_1(x_0) = |\nabla u|$, $u_i = 0$ for $i \neq 1$. This choice ensures that $\nabla_j |\nabla u| = u_{1j}$ and hence $|\nabla (|\nabla u|)|^2 = \sum_j u_{1j}^2$. We also have the following identity:

$$\Delta(|\nabla u|^2) = 2\sum_{i,j} u_{ij}^2 + 2\sum_i u_i(\Delta u)_i + 2Ric(\nabla u, \nabla u)$$

$$= 2\sum_{i,j} u_{ij}^2 + 2\sum_i u_i(-C|\nabla u| - \lambda u)_i + 2Ric(\nabla u, \nabla u)$$

$$\geq 2\sum_{i,j} u_{ij}^2 + 2\sum_i u_i(-C|\nabla u| - \lambda u)_i - (n - 1)K|\nabla u|^2$$

$$= 2\sum_{i,j} u_{ij}^2 + 2\sum_i u_i(-C|\nabla u|)_i - ((n - 1)K + \lambda)|\nabla u|^2$$

We may choose normal coordinates at $x$ so that $u_1(x_0) = |\nabla u|$, $u_i = 0$ for $i \neq 1$. This choice ensures that $\nabla_j |\nabla u| = u_{1j}$ and hence $|\nabla (|\nabla u|)|^2 = \sum_j u_{1j}^2$. We also have the following identity:

$$\Delta(|\nabla u|^2) = 2|\nabla u|\Delta(|\nabla u|) + 2|\nabla (|\nabla u|)|^2.$$

Substituting this equation into the preceding inequality, we find the following.

$$|\nabla u|\Delta(|\nabla u|) \geq \sum_{i,j} u_{ij}^2 - \sum_j u_{1j}^2 - 2\sum_i u_i(C|\nabla u|)_i - ((n - 1)K + \lambda)|\nabla u|^2$$
We now estimate the first two terms.

\[
\sum_{i,j} u_{ij}^2 - \sum_j u_{1j}^2 \geq \sum_{i>1} u_{i1}^2 + \frac{1}{n-1} (\sum_{i>1} u_{ii})^2
\]
\[
\geq \sum_{i>1} u_{i1}^2 + \frac{1}{n-1} (-C|\nabla u| - \lambda u - u_{11})^2
\]
\[
\geq \sum_{i>1} u_{i1}^2 + \frac{1}{n-1} \left( \frac{u_{11}^2}{2} - 2(C|\nabla u|)^2 - 2(\lambda u)^2 \right)
\]
\[
\geq \frac{1}{2(n-1)} |\nabla \nabla u|^2 - \frac{2}{n-1} ((C|\nabla u|)^2 + (\lambda u)^2)
\]

This implies the following.

\[
\Delta(|\nabla u|^2) \geq \left( 2 + \frac{1}{(n-1)} \right) |\nabla \nabla u|^2 - 2 \sum_i u_i (C|\nabla u|)_i
\]
\[
- ((n-1)K + \lambda) |\nabla u|^2 - \frac{2}{n-1} ((C|\nabla u|)^2 + (\lambda u)^2)
\]

3.5. **An estimate using the maximum principle.** We are now ready to estimate \(F(x)\) using the Li-Yau estimate. Recall that we defined \(F(x)\) in the following way.

\[
F(x) = \frac{|\nabla u|^2}{(\beta - u)^2 (r_{1/2}^2 - \rho^2)}
\]

Since \(F_{\partial B_{r_{1/2}}(p)} \equiv 0\), we can find \(x_0\) inside this ball where \(F\) is maximized. We can assume that \(x_0\) is not a cut point or else we can slightly alter our cut-off function as is done in [SY94]. Therefore, we assume that the cut-off function is smooth at this point.

At \(x_0\), we pick an orthonormal frame so that \(u_1 = |\nabla u|\) and \(u_i = 0\) otherwise. Then, since \(x_0\) maximizes \(F\) (and \(F\) is twice differentiable at \(x_0\) by the Schauder estimate), we have that \(\nabla F(x_0) = 0\),

\[
(r_{1/2}^2 - \rho^2) \frac{2u_{11} u_1}{(\beta - u)^2} - 2(r_{1/2}^2 - \rho^2) \frac{u_{i1}^2 u_i}{(\beta - u)^2} - 2\rho \rho_i \frac{|\nabla u|^2}{(\beta - u)^2} = 0
\]

We can simplify this equation to obtain an equation and an estimate.

\[
u_{i1} = u_{1i} = \frac{\rho \rho_i}{(r_{1/2}^2 - \rho^2)}
\]
and
\[ u_{11} = \frac{u_1^2}{(\beta - u)} + \rho p_{1/2} \frac{|\nabla u|}{(r_{1/2}^2 - \rho^2)} \geq \frac{u_1^2}{(\beta - u)} - \frac{\rho |\nabla u|}{(r_{1/2}^2 - \rho^2)} \]

We also have the following formula for the Laplacian of \( F \).
\[
(\Delta F) \left( \frac{(\beta - u)^2}{(r_{1/2}^2 - \rho^2)} \right) + (\nabla F) \nabla \left( \frac{(\beta - u)^2}{(r_{1/2}^2 - \rho^2)} \right) + F \Delta \left( \frac{(\beta - u)^2}{(r_{1/2}^2 - \rho^2)} \right) = \Delta (|\nabla u|^2)
\]

We now use this equation to prove the following estimate on \( F \).

**Lemma 4.** At the point \( x_0 \), we have the following estimate on \( F \). Here, \( d \) is the diameter of \( M \), \( n \) is the dimension, \( K \) is a lower bound on the Ricci curvature and \( C \) is the bound on the drift.

\[
0 \geq \left( \frac{1}{2(n-1)} - \frac{1}{4(n-1)^2} \right) F^2 - (4(n-1) - 1) \left( 2 + \frac{1}{(n-1)} \right) 2\rho F
- 2Cd^8 F^{3/2} - 2Cd^7 F - ((n-1)K + \lambda)Fd^4
- \frac{2}{n-1}(\lambda u)^2 \frac{d^8}{(\beta-u)^2} - \frac{2}{n-1}C^2Fd^6
- F(\lambda u) \frac{d^6}{(\beta-u)} - F^{3/2}Cd^5
- 8F^{3/2}d^4 - 2(n-1)(1 + K\rho)Fd^4 + 8Fd^4 + 2Fd^4
\]

**Proof.** The proof of this lemma is a very long string of manipulations combined with the use of the Laplace comparison theorem. We start by noting that at \( x_0 \), \( \Delta F \leq 0 \) and \( \nabla F = 0 \), which allows us to use our previous identities and inequalities.
\[ 0 \geq \Delta (|\nabla u|^2) - F \Delta \left( \frac{(\beta - u)^2}{(r_{1/2}^2 - \rho^2)} \right) \]

\[
\geq \left( 2 + \frac{1}{(n-1)} \right) |\nabla|\nabla u|^2 - 2 \sum u_i (C|\nabla u|)_i - ((n-1)K + \lambda)|\nabla u|^2
- \frac{2}{n-1} \left( (C|\nabla u|^2 + (\lambda u)^2) - F \Delta \left( \frac{(\beta - u)^2}{(r_{1/2}^2 - \rho^2)} \right) \right)
\]

\[ \geq \left( 2 + \frac{1}{(n-1)} \right) \left( \left| u_{11} \right|^2 + \sum_{i \neq 1} \left| u_{i1} \right|^2 \right) - 2u_1(C|\nabla u|)_1
- ((n-1)K + \lambda)|\nabla u|^2
- \frac{2}{n-1} \left( (C|\nabla u|^2 + (\lambda u)^2) - 2F \frac{|\nabla u|^2}{(r_{1/2}^2 - \rho^2)} \right)
- 2F(\beta - u) \frac{\Delta u}{(r_{1/2}^2 - \rho^2)} + 8F(\beta - \rho) \frac{\nabla u \cdot \nabla \rho}{(r_{1/2}^2 - \rho^2)^2} - F(\beta - u)^2 \Delta (r_{1/2}^2 - \rho^2)^{-1}
\]

\[
\geq \left( 2 + \frac{1}{(n-1)} \right) \left[ \left( \frac{|u_1|^2}{\beta - u} + 2 \frac{\rho_1 u_1}{(r_{1/2}^2 - \rho^2)} \right)^2 + \sum_{i \neq 1} \left| u_{i1} \right|^2 \right]
- 2u_1(C|\nabla u|)_1 - ((n-1)K + \lambda)|\nabla u|^2
- \frac{2}{n-1} \left( (C|\nabla u|^2 + (\lambda u)^2) - 2F \frac{|\nabla u|^2}{(r_{1/2}^2 - \rho^2)} \right)
- 2F(\beta - u) \frac{\Delta u}{(r_{1/2}^2 - \rho^2)} + 8F(\beta - u) \frac{\nabla u \cdot \nabla \rho}{(r_{1/2}^2 - \rho^2)^2} - F(\beta - u)^2 \Delta (r_{1/2}^2 - \rho^2)^{-1}
\]

\[
\geq \left( 2 + \frac{1}{(n-1)} \right) \left( 1 - \frac{1}{4(n-1)} \right) \frac{|u_1|^4}{(\beta - u)^2}
- (4(n-1) - 1) \left( 2 + \frac{1}{(n-1)} \right) \left( 2 \frac{\rho_1 u_1}{(r_{1/2}^2 - \rho^2)} \right)^2
+ \left( 2 \frac{1}{(n-1)} \right) \sum_{i \neq 1} \left| u_{i1} \right|^2 - 2u_1(C|\nabla u|)_1 - ((n-1)K + \lambda)|\nabla u|^2
- \frac{2}{n-1} \left( (C|\nabla u|^2 + (\lambda u)^2) - 2F \frac{|\nabla u|^2}{(r_{1/2}^2 - \rho^2)} \right)
- 2F(\beta - u) \frac{\Delta u}{(r_{1/2}^2 - \rho^2)} + 8F(\beta - u) \frac{\nabla u \cdot \nabla \rho}{(r_{1/2}^2 - \rho^2)^2} - F(\beta - u)^2 \Delta (r_{1/2}^2 - \rho^2)^{-1}
\]

Recalling that \(2F \frac{|\nabla u|^2}{(r_{1/2}^2 - \rho^2)} = 2\frac{|u_1|^4}{(\beta-u)^2}\), we see that this term partially cancels out the leading term. Doing this and other substitutions, we find the following:
0 \geq \left( \frac{1}{2(n-1)} - \frac{1}{4(n-1)^2} \right) \frac{|u_1|^4}{(\beta - u)^2} \\
-(4(n-1) - 1) \left( 2 + \frac{1}{(n-1)} \right) \left( 2 \frac{\rho \rho_1 u_1}{(r_{1/2}^2 - \rho^2)} \right)^2 \\
+ \left( 2 + \frac{1}{(n-1)} \right) \sum_{i \neq 1} \frac{\rho^2 \rho_1^2 |u_1|^2}{(r_{1/2}^2 - \rho^2)^2} - 2u_1(C|\nabla u|)_1 - ((n-1)K + \lambda)|\nabla u|^2 \\
- \frac{2}{n-1} \left( (C|\nabla u|)^2 + (\lambda u)^2 \right) - 2F(\beta - u) \frac{\Delta u}{(r_{1/2}^2 - \rho^2)} \\
+ 8F(\beta - u) \rho \frac{\nabla u \cdot \nabla \rho}{(r_{1/2}^2 - \rho^2)^2} - F(\beta - u)^2 \Delta (r_{1/2}^2 - \rho^2)^{-1}

Substituting in $\Delta u + C|\nabla u| + \lambda u = 0$ into the fourth line and then simplifying, this yields:

0 \geq \left( \frac{1}{2(n-1)} - \frac{1}{4(n-1)^2} \right) \frac{|u_1|^4}{(\beta - u)^2} \\
-(4(n-1) - 1) \left( 2 + \frac{1}{(n-1)} \right) \left( 2 \frac{\rho \rho_1 u_1}{(r_{1/2}^2 - \rho^2)} \right)^2 \\
- 2u_1(C|\nabla u|)_1 - ((n-1)K + \lambda)|\nabla u|^2 \\
- \frac{2}{n-1} \left( (C|\nabla u|)^2 + (\lambda u)^2 \right) - 2F(\beta - u) \frac{\Delta u}{(r_{1/2}^2 - \rho^2)} \\
+ 8F(\beta - u) \rho \frac{u_1 \rho_1}{(r_{1/2}^2 - \rho^2)^2} - F(\beta - u)^2 \Delta (r_{1/2}^2 - \rho^2)^{-1}

Multiplying through by $\frac{(r_{1/2}^2 - \rho^2)^4}{(\beta - u)^2}$, and substituting in the definition of $F$ in the last line, we have the following.

0 \geq \left( \frac{1}{2(n-1)} - \frac{1}{4(n-1)^2} \right) F^2 - (4(n-1) - 1) \left( 2 + \frac{1}{(n-1)} \right) 2 \rho F \\
- 2 \frac{(r_{1/2}^2 - \rho^2)^4}{(\beta - u)^2} \frac{u_1(C|\nabla u|)_1 - ((n-1)K + \lambda)F(r_{1/2}^2 - \rho^2)^2}{(r_{1/2}^2 - \rho^2)^2} \\
- \frac{2}{n-1} \left( (C|\nabla u|)^2 + (\lambda u)^2 \right) \frac{(r_{1/2}^2 - \rho^2)^4}{(\beta - u)^2} + F(C|\nabla u| + \lambda u) \frac{(r_{1/2}^2 - \rho^2)^3}{(\beta - u)} \\
- 8F^3 \rho (r_{1/2}^2 - \rho^2)^{3/2} - F \Delta (r_{1/2}^2 - \rho^2)^{-1} (r_{1/2}^2 - \rho^2)^4
Using the identity for $|\nabla u|_1$ and further simplifying, we find that

$$0 \geq \left( \frac{1}{2(n-1)} - \frac{1}{4(n-1)^2} \right) F^2 - (4(n-1) - 1) \left( 2 + \frac{1}{(n-1)} \right) 2\rho F$$

$$-\frac{2}{(\beta - u)^3} \frac{(r_{1/2}^2 - \rho^2)^4}{4(4(n-1) - 1)} - \frac{2}{(\beta - u)^3} \frac{(r_{1/2}^2 - \rho^2)^3}{4(n-1) - 1} - \frac{2}{(\beta - u)^3} \frac{(r_{1/2}^2 - \rho^2)^2}{4}$$

$$- \frac{2}{(\beta - u)^3} \frac{(r_{1/2}^2 - \rho^2)^4}{4} - \frac{2}{(\beta - u)^3} \frac{(r_{1/2}^2 - \rho^2)^3}{4} - \frac{2}{(\beta - u)^3} \frac{(r_{1/2}^2 - \rho^2)^2}{4} - \frac{2}{(\beta - u)^3} \frac{(r_{1/2}^2 - \rho^2)^4}{4}$$

Therefore, we have the following inequality.

$$0 \geq \left( \frac{1}{2(n-1)} - \frac{1}{4(n-1)^2} \right) F^2 - (4(n-1) - 1) \left( 2 + \frac{1}{(n-1)} \right) 2\rho F$$

$$-2C(r_{1/2}^2 - \rho^2)^4 + 2C\rho(r_{1/2}^2 - \rho^2)^3 F - ((n-1)K + \lambda)F(r_{1/2}^2 - \rho^2)^2$$

$$- \frac{2}{(\beta - u)^3} \frac{(r_{1/2}^2 - \rho^2)^4}{4} - \frac{2}{(\beta - u)^3} \frac{(r_{1/2}^2 - \rho^2)^3}{4} - \frac{2}{(\beta - u)^3} \frac{(r_{1/2}^2 - \rho^2)^2}{4} - \frac{2}{(\beta - u)^3} \frac{(r_{1/2}^2 - \rho^2)^4}{4}$$

We now focus our efforts into estimating the final term. Using the Laplace comparison theorem (as in [SY94]), we have the following inequality.

$$\Delta(r_{1/2}^2 - \rho^2)^{-1} = \sum_i \frac{2\rho_i \rho}{(r_{1/2}^2 - \rho^2)^2} + \frac{8\rho_i^2 \rho^2}{(r_{1/2}^2 - \rho^2)^3} + \frac{2\rho_i^2}{(r_{1/2}^2 - \rho^2)^2}$$

$$\leq \frac{n-1}{\rho} (1 + K \rho) \frac{2\rho}{(r_{1/2}^2 - \rho^2)^2} + \frac{8\rho^2}{(r_{1/2}^2 - \rho^2)^3} + \frac{2}{(r_{1/2}^2 - \rho^2)^2}$$

This yields the following estimate on the last term.
\[ F\Delta(r_{1/2}^2 - \rho^2)^{-1}(r_{1/2}^2 - \rho^2)^4 \leq 2(n-1)(1 + K\rho)F(r_{1/2}^2 - \rho^2)^2 + 8F\rho^2(r_{1/2}^2 - \rho^2) + 2F(r_{1/2}^2 - \rho^2)^2 \]

Combining this estimate into the larger inequality, we find the following.

\[ 0 \geq \left( \frac{1}{2(n-1)} - \frac{1}{4(n-1)^2} \right) F^2 - \left( 4(n-1) - 1 \right) \left( 2 + \frac{1}{(n-1)} \right) 2\rho F \]

\[ -2C(r_{1/2}^2 - \rho^2)^4 F^{5/2} - 2C\rho(r_{1/2}^2 - \rho^2)^3 F - ((n-1)K + \lambda)F(r_{1/2}^2 - \rho^2)^2 \]

\[ -\frac{2}{n-1}(\lambda u)^2 \frac{d^3}{(\beta - u)^2} - 2 \frac{C^2}{n-1} F(r_{1/2}^2 - \rho^2)^3 \]

\[ + F(\lambda u) \frac{(r_{1/2}^2 - \rho^2)^3}{(\beta - u)} + F^{3/2}C(r_{1/2}^2 - \rho^2)^{5/2} \]

\[ -8F^{3/2}\rho(r_{1/2}^2 - \rho^2)^{3/2} \]

\[ -2(n-1)(1 + K\rho)F(r_{1/2}^2 - \rho^2)^2 + 8F\rho^2(r_{1/2}^2 - \rho^2)^2 + 2F(r_{1/2}^2 - \rho^2)^2 \]

If we denote the diameter of \( M \) by \( d \), we note that \( r_{1/2}, \rho < d \) so \( (r_{1/2}^2 - \rho^2) < d^2 \). We can substitute this into our inequality to get the desired inequality.

\[ 0 \geq \left( \frac{1}{2(n-1)} - \frac{1}{4(n-1)^2} \right) F^2 - \left( 4(n-1) - 1 \right) \left( 2 + \frac{1}{(n-1)} \right) 2\rho F \]

\[ -2Cd^8 F^{5/2} - 2Cd^7 F - ((n-1)K + \lambda)Fd^4 \]

\[ -\frac{2}{n-1}(\lambda u)^2 \frac{d^3}{(\beta - u)^2} - 2 \frac{C^2}{n-1} Fd^6 \]

\[ -F(\lambda u) \frac{d^6}{(\beta - u)} - F^{3/2}Cd^5 \]

\[ -8F^{3/2}d^4 - 2(n-1)(1 + K\rho)Fd^4 + 8Fd^4 + 2Fd^4 \]

\( \square \)

3.6. **Using the inequality on \( F \).** At this point, we take stock of this estimate to show how this gives any hope of providing a lower bound on \( \lambda \). We have uniform control of the cutoff function in \( B_{r_{3/4}} \) from the a priori gradient estimate on \( u \). This allows us to change the inequality on \( F \) to obtain to an inequality on \( \frac{\nabla u}{(\beta - u)} \). If we integrate out this inequality
along a geodesic from $x$ with $u(x) = \frac{3}{4}$ to $x_0$, for some constants $C$ and $c$, we obtain a bound of the form

$$\log \left( \frac{\beta - 3/4}{\beta - 1} \right) \leq \left( C + c \frac{\lambda^{1/2}}{(\beta - 1)^{1/2}} \right) d$$

For $\beta$ close to 1, the left hand side blows up, which shows that right hand side must blow up as well and implies a lower bound on $\lambda$.

In order to make this precise we observe the following. As $\beta$ goes to 1, our $C^{3,\alpha}$ control on $u$ weakens. Therefore, we do not take the limit but set $\beta - 1$ at some small but fixed scale depending on $C$, $c$ and $d$. Crucially, none of these constants depend on the bounds from the Schauder estimates, so are independent of $\beta$. This allows us to maintain enough control to use the Bochner identity and the maximum principle while still being free to pick $\beta$ in a way that yields a positive bound on $\lambda$.

We now do this precisely. For convenience, denote $f := \sqrt{F}$ and rewrite the previous inequality in terms of $f$.

$$0 \geq \left( \frac{1}{2(n-1)} - \frac{1}{4(n-1)^2} \right) f^4 - (4(n-1) - 1) \left( 2 + \frac{1}{(n-1)} \right) 2df^2$$
$$-2Cd^8 f^3 - 2Cd^7 f^2 - ((n-1)K + \lambda)d^4 f^2$$
$$- \frac{2}{n-1} (\lambda u)^2 \frac{d^8}{(\beta - u)^2} - \frac{2}{n-1} C^2 f^2 d^6$$
$$- f^2 (\lambda u) \frac{d^6}{(\beta - u)} - f^3 C d^5$$
$$-8f^3 d^4 - 2(n-1)(1 + K\rho)f^2 d^4 + 10f^2 d^4$$

For conciseness, we denote $\alpha = \frac{1}{\beta - 1}$ and observe that $\alpha > \frac{n}{\beta - n}$. We also define the following constants:

$$A = \left( \frac{1}{2(n-1)} - \frac{1}{4(n-1)^2} \right),$$
$$B = 2Cd^8 + Cd^7 + 8d^4,$$
$$D = \lambda \frac{d^6}{(\beta - 1)},$$

$$D = (4(n-1) - 1) \left( 2 + \frac{1}{(n-1)} \right) 2d + 2Cd^7 + ((n-1)K + \lambda)d^4$$
$$+ \frac{2}{n-1} C^2 d^6 + 2(n-1)(1 + K\rho)d^4 + 10d^4$$
\[ \mathcal{E} = \frac{2\lambda^2}{n-1} \left( \frac{d^3}{d^2} \right)^{1/2} \]

From the previous inequality, we have the following estimate:

\[ 0 \geq Af^4 - Bf^3 - Df^2 - D^2 - \mathcal{E} \]

Note that the calligraphic terms are the only terms where the coefficients aren’t uniform in \( \beta \) and these both contain a \( \lambda \). Now we use a lemma about the roots of quartics. This lemma was originally proven in [Kha16].

**Lemma 5.** Suppose \( A_1, A_2, A_3 > 0 \) and \( x \) satisfies \( P(x) = x^4 - A_1x^3 - A_2x^2 - A_3 \leq 0 \). Then \( x \leq A_1 + \sqrt{A_2 + A_3} \).

In order to make future calculations more feasible, we note that the following inequality holds:

\[ A_1 + \sqrt{A_2 + \sqrt{A_3}} \leq A_1 + (2A_2)^{1/2} + (4A_3)^{1/4} \]

Applying this inequality to \( f \), this shows that

\[ f \leq \frac{1}{A} \left( B + \sqrt{2D + \sqrt{2E}} \right) \]

\[ \leq \frac{1}{A} \left( B + 2\sqrt{D} \right) + \frac{1}{A} \left( 2\sqrt{D} + \sqrt{2E}^{1/4} \right) \]

Using the fact that \( n \geq 2 \), we obtain the following simplified estimates.

\[ f \leq \frac{1}{A} \left( B + 2\sqrt{D} \right) + 8(n-1) \left( d^3 + d^2 \right) \frac{\sqrt{\lambda}}{\sqrt{\beta - 1}} \]

From the \( C^{1,\alpha} \) estimate, we have \( r_{1/2} \geq r_{3/4} + d_{1/4} \) and so in \( B_{r_{3/4}} \), the following inequality holds:

\[ (r_{1/2}^2 - \rho^2) \geq (r_{3/4}^2 - r_{3/4}^2) \geq 3d_{1/4}^2 \]

Using the definition of \( f \), this implies that in \( B_{r_{3/4}} \), the following estimate holds:

\[ \frac{\| \nabla u \|}{\beta - u} \geq \frac{1}{3Ad_{1/4}^2} \left( B + 2\sqrt{D} \right) + \frac{8(n-1)}{3d_{1/4}^2} \left( d^3 + d^2 \right) \frac{\sqrt{\lambda}}{\sqrt{\beta - 1}} \]

Setting \( C = \frac{1}{3Ad_{1/4}^2} \left( B + 2\sqrt{D} \right) \) and \( c = \frac{8(n-1)}{3d_{1/4}^2} \left( d^3 + d^2 \right) \), this shows that

\[ \frac{\| \nabla u \|}{\beta - u} \leq C + c \frac{\sqrt{\lambda}}{\sqrt{\beta - 1}} \]
3.7. **A lower bound on** $\lambda$. We pick $x \in B_{r,3/4}$ with $u(x) = \frac{3}{4}$ and a minimal geodesic $\gamma$ between $x$ and $p$ (recall that $p$ is the point where $u(p) = 1$). This integral is well defined because $u$ has enough continuity for $\nabla u$ to be defined pointwise.

We can estimate this integral in the following way.

$$\log \frac{\beta - 3/4}{\beta - 1} \leq \int_{\gamma} |\nabla u| \leq \left( C + c \frac{\sqrt{\lambda}}{\sqrt{\beta - 1}} \right) d$$

Solving for $\lambda$, we have the final inequality.

$$\sqrt{\lambda} \geq \frac{\sqrt{(\beta - 1)}}{c} \left( \frac{1}{d} \log \frac{\beta - 3/4}{\beta - 1} - C \right)$$

To find a semi-explicit lower bound, we find a particular value for $\beta$ that gives us a positive lower bound. We let $x = \frac{\beta - 3/4}{\beta - 1}$ and set $x = e^{dC + d}$. This then shows the following:

$$\lambda \geq \frac{1}{4 \epsilon^2} \left( e^{dC + d} - 1 \right)^{-1}$$

This finishes the proof of the theorem.

4. **A concrete example**

The preceding argument demonstrates a lower bound on the real eigenvalues, but it does not make clear how the drift affects the spectrum. To illustrate how the size of the drift affects small eigenvalues, we calculate the minimal eigenvalue explicitly in a simple case. Consider the circle $\mathbb{R}/4\mathbb{Z}$ and the problem

$$u'' + f \cdot u' + \lambda u = 0 \text{ with } \|f\|_{L^\infty} \leq C$$

To simplify the calculation, we set $C = 2b$. Therefore, we wish to minimize $\lambda$ under the constraint that $\|f\|_{\infty} \leq 2b$. By symmetry, we can instead consider the principle eigenvalue of the Dirichlet problem on the domain $[-1, 1]$. Using symmetry and the drift ansatz, $u$ will satisfying the following ordinary differential equation:

$$u'' + 2b \cdot u' + \lambda u = 0$$

on the domain $[0, 1]$ with the constraints:

1. $u(0) = 1$
2. $u'(0) = 0$
3. $u(1) = 0$
4. $\lambda$ is the minimal eigenvalue so that a solution exists.
We can solve this ordinary differential equation explicitly, and then and then solve for $\lambda$ in terms of $b$ so that the boundary conditions are satisfied. Doing so, we find the following.

For $b$ large, the principle eigenvalue satisfies the following equation:

$$\frac{1 + e^{\lambda} - \frac{\lambda}{b^2}}{1 - e^{\lambda - \frac{\lambda}{b^2}}} \sqrt{1 - \frac{\lambda}{b^2}} = 1$$

For $b$ small, $\lambda$ instead satisfies the equation:

$$\sqrt{\lambda - b^2} = b \tan \left( \sqrt{\lambda - b^2} \right)$$

A graph of $\lambda$ in terms of $b$

Note that the minimal principle eigenvalue decreases roughly exponentially as the drift grows. Some rough estimates show that for large $b$,

$$\lambda \approx 4b^2 e^{-2b} = C^2 e^{-C}.$$

5. Future work

The natural question to ask is whether a similar argument can be made when the drift has $L^p$ bounds for $p < \infty$. For this question, it is necessary to assume that $p > n$. Firstly, we need this assumption in order to apply the Calderón-Zygmund estimates. More importantly, on the interval (i.e. $n = 1$), it is possible to find $L^1$ drifts with arbitrarily small principle eigenvalue. Interestingly, when the drift is sufficiently small in $L^1$ norm, it seems possible to recover an estimate on the eigenvalue by applying Grönwall’s inequality. As such, the minimal eigenvalue displays interesting threshold phenomena; it is positive for small drifts but as soon as the $L^1$ norm of the drift is sufficiently large, it can be arbitrarily small. A
similar phenomena likely occurs when $p = n$ in higher dimensions as well, but there is no analog of the Grönwall inequality to prove this.

For $n < p < \infty$, the main obstruction to repeating Theorem 1 is the lack of a drift ansatz. It is possible to find a sequence of minimizing drifts, but the minimizer does not have any natural additional regularity. In particular, the corresponding eigenfunction is not a subsolution to a semi-linear equation independent of the choice of drift, which was the key idea that we used in the $L^\infty$ case.

One possible approach to this problem for domains in $\mathbb{R}^n$ is to convolve the eigenfunction with a suitable bump function. Using the $C^{1,\alpha}$ estimate and the increased regularity from convolution, it might be possible to prove that the convolved eigenfunction is a subsolution to an equation with more regular drift. If so, we can apply Theorem 1 to obtain lower bounds. However, at this time we are unable to prove such an estimate and so we leave this problem for future work.

It is also natural to ask whether we can remove the somewhat awkward assumption on the two-sided bounds of the Ricci curvature. We expect this is the case as the only place such bounds are used is to bound the $C^{1,\alpha}$ harmonic radius from below. We suspect that the upper bounds are not essential and that the following estimate holds.

**Conjecture 6.** Let $(M^n, g)$ be a compact Riemannian manifold satisfying $\text{Ric}(M) > K$ and $v$ is some one form satisfying $\|v\|_\infty < C$. Suppose that there exists $u \in W^{2,p}(M)$ satisfying $\Delta u + v(\nabla u) = \lambda u$ with $\lambda$ real. Then there exists some constant $\delta > 0$ depending only $K$, $C$, $\text{diam}(M)$, $\text{inj}(M)$ and $n$ so that $\lambda > \delta$.

Going further, it may even be possible to remove the dependence on the injectivity radius. The original Li-Yau estimate does not involve the injectivity radius and intuitively speaking, shrinking the injectivity radius would seem to increase, not decrease the eigenvalues. In order to prove an estimate along these lines with no assumptions on the injectivity radius, one would need to find a different way to prove the a priori regularity. From there, the estimate on $F$ would remain unchanged.

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