Black Hole Entropy: Thermodynamics, Statistical-Mechanics and Subtraction Procedure

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Abstract

The thermodynamical one-loop entropy $S^{TD}$ of a two-dimensional black hole in thermal equilibrium with the massless quantum gas is calculated. It is shown that $S^{TD}$ includes the Bekenstein-Hawking entropy, evaluated for the quantum corrected geometry, and the finite difference of statistical mechanical entropies $-\text{Tr} \hat{\rho} \ln \hat{\rho}$ for the gas on the black hole and Rindler spaces. This result demonstrates in an explicit form that the relation between thermodynamical and statistical-mechanical entropies of a black hole is non-trivial and requires special subtraction procedure.

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1 Introduction

Black holes are known to behave as usual thermodynamical systems with the entropy

$$S_{BH}^{BH} = \frac{A^H}{4T^H}, \quad (1.1)$$

where $A^H$ is the area of the event horizon [1]. The Bekenstein-Hawking entropy $S_{BH}^{BH}$ is the physical quantity which can be measured in (gedanken) experiments by making use of the first law of thermodynamics which can be represented in the form [2]:

$$dF^H = -S_{BH}^{BH}dT^H. \quad (1.2)$$

Here $M^H$ is mass of a black hole, $T^H = (8\pi M^H)^{-1}$ is its Hawking temperature and $F^H = M^H - T^H S_{BH}^{BH}$ is the free energy.

The fundamental aspect of the black hole thermodynamics is its statistical-mechanical foundation. This implies the solution of the following problems: (1) definition of internal degrees of freedom of a black hole and finding the corresponding density matrix $\hat{\rho}^H$; (2) calculation of the statistical-mechanical entropy $S_{SM}^{SM} = -\text{Tr}(\hat{\rho}^H \ln \hat{\rho}^H)$; (3) establishing the relation between $S_{SM}^{SM}$ and the observable thermodynamical $S_{TD}^{TD}$ entropy defined by the first law (1.2).

There is a number of alternatives in choosing internal degrees of freedom of a black hole that might be responsible for its entropy. One of the promising suggestions is to relate these degrees of freedom with thermal excitations of quantum fields around a black hole [3]. However the explicit calculations give an infinite value for the corresponding entropy $S_{SM}^{SM} = -\text{Tr}(\hat{\rho}^H \ln \hat{\rho}^H)$. This fact is in obvious contradiction with the finiteness of the observable quantity (1.1). The same problem appears in the attempts to interpret $S_{BH}^{BH}$ as an entanglement entropy connected to the loss of information about correlations of quantum states inside and outside the horizon [4]. This contradiction indicates that the relation between statistical mechanics and thermodynamics in the case of black holes may be non-trivial. The property which singles out black holes from other thermodynamical systems is that the Hamiltonian describing the quantum fields on a black hole background depends on the mass $M^H$ of a black hole and, hence, on the equilibrium temperature $T^H$. This has two consequences. i) The statistical entropy $S_{SM}^{SM}$ does not coincide with the thermodynamical entropy of a black hole [3]. ii) The calculation of $S_{SM}^{SM}$ requires off-shell methods when mass of a black hole and temperature of the system are considered as independent parameters.

In this Letter we compare two frequently used off-shell procedures (the so called conical singularity method and brick wall approach) and obtain relation between them for 2D quantum black hole models. This analysis enables us to establish the explicit structure of the total one-loop black hole entropy $S_{TD}^{TD}$. It is shown that $S_{TD}^{TD}$ is related to the quantity obtained from entropy $S_{SM}^{SM} = -\text{Tr}(\hat{\rho}^H \ln \hat{\rho}^H)$ by subtraction the statistical-mechanical entropy for some ”reference” Rindler-like background.
2 On-shell and off-shell approaches

Thermodynamical characteristics of a black hole are defined by the partition function

\[ Z = \int [D\phi] e^{-I[\phi]} . \] (2.1)

Here \( I[\phi] \) is the Euclidean classical action and all the physical variables \( \phi \) including the metric \( \gamma_{\mu\nu} \) are assumed to be periodic in the Euclidean time \( \tau \) with the period \( \beta_\infty \). We assume that metrics are asymptotically flat. We confine our analysis to a simple quantum model described by the action

\[ I[\phi] = I_{grav} + \frac{1}{2} \int \sqrt{\gamma} \varphi_{,\mu} \varphi^{,\mu} d^2x \] (2.2)

where \( \varphi \) is the massless matter field and \( I_{grav} \) is the part of the action which includes the gravitational degrees of freedom. Although the results do not depend on a particular form of \( I_{grav} \), it is convenient to take as an example the model of two-dimensional dilaton gravity

\[ I_{grav}[\gamma_{\mu\nu}, r] = -\frac{1}{4} \int_{M^2} (r^2 R + 2(\nabla r)^2 + 2) \sqrt{\gamma} d^2x - \frac{1}{2} \int_{\partial M^2} r^2 (k - k_0) dy . \] (2.3)

The action (2.3) can be obtained by spherical symmetric reduction of the four-dimensional Einstein gravity, the radius \( r \) being the dynamical variable playing the role of the dilaton field. To get a well defined canonical ensemble we also suppose that the black hole is in the spherical cavity of a radius \( r_B \) (see [3]). The inverse temperature measured at \( r_B \) is denoted by \( \beta \).

To evaluate the integral (2.1) we will work under the assumption that only matter field \( \varphi \) is quantized while the gravitational variables \( \gamma_{\mu\nu} \) and \( r \) are taken into account in the quasiclassical approximation

\[ \delta I/\delta \varphi|_{\varphi=\varphi_0} = 0 . \] (2.4)

Then the partition function (2.1) can be expressed in terms of the effective action \( W(\beta) \)

\[ \ln Z(\beta) = W(\beta) = I[\phi_0(\beta)] + W_1[\phi_0(\beta)] , \] (2.5)

\[ W_1[\phi_0(\beta)] = \frac{1}{2} \log \det \nabla^\mu \nabla_\mu . \] (2.6)

Here \( \phi_0 = \{ \gamma_{\mu\nu}^{(0)}, \varphi = 0 \} \) is the classical extremum of the action (2.2) in the black hole sector. For the dilaton model (2.3) the classical metric has the Schwarzschild form

\[ ds^2 = \gamma_{\mu\nu}^{(0)} dx^\mu dx^\nu = (1 - r+/r)d\tau^2 + (1 - r+/r)^{-1}dr^2 , \] (2.7)

\[ 0 \leq \tau \leq 4\pi r_+ = T_H^{-1} . \] Functional \( W_1 \) is the one-loop quantum correction. According to a standard procedure the ultraviolet divergences in \( W_1 \) should be removed by the renormalization of bare cosmological and gravitational constants in the classical gravitational action (2.3). In order to preserve the asymptotical flatness we put the renormalized cosmological
constant to be $-1/2$. The term which is an integral of a curvature can be omitted because in two dimensions it is a topological invariant.

In thermal equilibrium the system is uniquely described by fixing two external parameters: the inverse temperature $\beta$ on the boundary and the ‘radius’ $r_B$. All other characteristics such as the ‘radius’ of the event horizon $r_+$ are the functions of $\beta$ and $r_B$. For instance, for the dilaton model (2.3) $r_+$ is determined by the equation $\beta = 4\pi r_+ (1 - r_+/r_B)^{1/2}$.

The thermodynamical entropy of a black hole $S^{TD}$ is defined by the response of the free energy $F(\beta) = \beta^{-1} W(\beta)$ to the change of the inverse temperature $\beta$ for fixed $r_B$.

$$S^{TD}(\beta) = \beta^2 \frac{dF(\beta)}{d\beta} = \left(\beta \frac{d}{d\beta} - 1\right) W(\beta) = S^{TD}_0 + S^{TD}_1 .$$ (2.8)

It can be shown [3] that $S^{TD}_0[\phi_0(\beta)] = \left(\beta \frac{d}{d\beta} - 1\right) I[\phi_0(\beta)]$, (2.9) coincides with the Bekenstein-Hawking entropy $S^{BH}$ given by Eq.(1.1), while $S^{TD}_1(\beta)$ obtained from $W_1[\phi_0(\beta)]$ describes the quantum correction. This correction contains also the entropy of the thermal radiation outside the black hole as its part. By its construction the thermodynamical entropy $S^{TD}$ is well defined and finite. All the calculations required to obtain this quantity can be performed on-shell, that is on regular complete vacuum solutions of the Euclidean gravitational equations.

Eq.(2.8) contains the renormalized effective action $W$ calculated on a particular classical solution. This renormalized action itself is defined as a functional $W[\phi] = I[\phi] + W_1[\phi]$ for an arbitrary field $\phi$ with appropriately chosen boundary conditions. The extremum $\bar{\phi}(\beta)$ of this functional

$$\frac{\delta W}{\delta \phi} \bigg|_{\phi = \bar{\phi}(\beta)} = 0$$ (2.10)

describes a modified field configuration which differs from a classical solution by quantum corrections: $\bar{\phi}(\beta) = \phi_0(\beta) + \hbar \tilde{\phi}_1(\beta)$. The important observation is that, if one is interested in one-loop effects, the difference between the values of $W$ on $\phi_0$ and $\bar{\phi}$ turns out to be of the second order in Planck constant $\hbar$

$$W(\beta) = W[\phi_0(\beta)] = W[\bar{\phi}(\beta)] + O(\hbar^2) .$$ (2.11)

This follows from (2.10), provided the quantum corrected and classical solutions obey the same boundary conditions.

As we discussed earlier one must use off-shell methods to find out the statistical-mechanical interpretation of the black hole entropy (2.8). In off-shell approaches the mass of a black hole is considered as an additional parameter, independent from the temperature which is associated with the periodicity of the Euclidean manifold. However, it is well known that such manifolds are not regular on the Euclidean horizon.
We discuss and compare here two off-shell approaches frequently used for the calculation of the entropy. The first method (conical singularity method) is to work directly with the complete singular instanton \[9\]. Another procedure (brick wall model) is to introduce at some small proper distance \(\epsilon\) from the black hole horizon an additional mirror like boundary \[6\] and to work with incomplete manifold. The brick wall model has a clear statistical-mechanical interpretation, while the conical-singularity method enables one to relate thermodynamical and off-shell entropies.

The one-loop entropy \(S^{CS}\) evaluated in the conical singularity method is defined from the effective action as

\[
S^{CS}(\beta) = \left(\beta \frac{\partial}{\partial \beta} - 1\right) W[\bar{\phi}, \beta].
\]

The derivative over \(\beta\) in (2.12) is taken when the solution \(\bar{\phi}\) (as well as the cavity radius \(r_B\)) is fixed. One should put the on-shell condition \(\bar{\phi} = \bar{\phi}(\beta)\) only after the differentiation. The entropy \(S^{CS}\) can be represented as the sum

\[
S^{CS}(\beta) = S^{CS}_0[\bar{\phi}(\beta)] + S^{CS}_1[\bar{\phi}(\beta)]
\]

of the tree-level part

\[
S^{CS}_0[\bar{\phi}(\beta)] = \left(\beta \frac{\partial}{\partial \beta} - 1\right) I[\bar{\phi}, \beta],
\]

and a quantum correction to it

\[
S^{CS}_1(\beta) = \left(\beta \frac{\partial}{\partial \beta} - 1\right) W^{CS}_1[\bar{\phi}, \beta].
\]

The tree-level contribution \(S^{CS}_0\), defined by the classical action on the manifold with conical singularity is \[11\]

\[
S^{CS}_0[\bar{\phi}(\beta)] = \bar{S}^{BH} = \frac{1}{4} \bar{A}^H
\]

It has the form of the Bekenstein-Hawking entropy with the area \(A^H\) of the classical horizon replaced by the horizon area \(\bar{A}^H\) of the quantum corrected solution. For the dilaton gravity \(2.3\) the latter is \(S^{CS}_0[\bar{\phi}(\beta)] = \pi \bar{r}^2_+\) where \(\bar{r}_+\) is the value of the dilaton at the black hole horizon.

Remarkably, when \(\bar{\phi} = \bar{\phi}(\beta)\) the total off-shell one-loop entropy (2.12) coincides with the total thermodynamical (on-shell) entropy (2.8)

\[
S^{CS}(\beta) = S^{TP}(\beta)
\]

To obtain this relation one must take into account Eq.(2.11) for the on-shell effective action and rewrite the total derivative in (2.8) in the form

\[
\frac{dW[\bar{\phi}(\beta)]}{d\beta} = \frac{\partial W[\bar{\phi}, \beta]}{\partial \beta} + \int_{M^2} \frac{\delta W[\phi, \beta]}{\delta \bar{\phi}} \frac{\partial \bar{\phi}(\beta)}{\partial \beta}.
\]

(2.18)
The partial derivative with respect to $\beta$ in the first term in the right hand side of (2.14) is taken at the fixed solution $\tilde{\phi}$ (at the fixed black hole mass) and the last term in this equation vanishes because $\tilde{\phi}(\beta)$ is the extremum of the effective action. The variational procedure in (2.18) involves the generalized class of spaces including ones with conical singularities. However in our case the extremum of $W$ remains at the smooth instanton $\bar{\phi}(\beta)$. The extrema on manifolds with conical singularities are ruled out in the absence of a matter singularly distributed over the horizon.

Eqs. (2.17), (2.13) and (2.16) enable one to represent the one-loop thermodynamical entropy of a quantum black hole in the following form

$$S^{TD}(\beta) = S^{BH}(\beta) + S^{CS}_1(\beta) \,. \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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where the last term is the contribution of the vertex point \( r_h \) of the singular instanton. The expression in the square brackets is proportional to \( \beta \) and hence does not contribute to \( S^{CS}_1 \).

The equation (2.15) applied to (2.24) at \( \alpha = 1 \) gives

\[
S^{CS}_1[\beta, r_H] = \frac{1}{6} \sigma(r_h)
\]

(2.25)

where we omitted a numerical constant produced by \( W_1[C] \). Thus the one-loop correction evaluated by the conical singularity method for massless quantum field is completely determined by the value \( \sigma(r_h) \) of the conformal factor at the horizon \[10], \[12].

### 3 Relation with statistical mechanics

The correction \( S^{CS}_1 \) to the black hole entropy computed by the conical singularity method has a well defined geometrical meaning but it does not coincide with a statistical-mechanical entropy \( - \text{Tr}(\hat{\rho}^H \ln \hat{\rho}^H) \) for any density matrix. Let us establish now the relation of the conical singularity method with the brick wall model which has the simple statistical-mechanical interpretation. For this purpose we introduce at some small proper distance \( \varepsilon \) from the black hole horizon an additional mirror like boundary with the Dirichlet condition on it [6].

We denote the corresponding manifold by \( M^2_\varepsilon \). In this model the horizon does not belong to a manifold and it is possible to formulate the canonical ensemble for the quantum field \( \varphi \) between two mirrors at arbitrary temperature \( \beta^{-1} \) (measured as before at the external boundary). Then all the information about thermodynamics of the system can be derived from the free energy \( F(\beta) \)

\[
e^{-\beta F(\beta)} = \text{Tr}(e^{-\beta \tilde{H}}),
\]

(3.1)

where \( \tilde{H} = \tilde{H}(\gamma_{\mu\nu}) \) is the Hamiltonian for field \( \varphi \). The free energy \( F(\beta) \) can be computed if the spectrum of \( \tilde{H} \) is known. The alternative way is a field-theoretical computation which enables to relate this quantity to the effective action \( W^{BW}_1 \) for \( \varphi \) on a static manifold with two boundaries

\[
W^{BW}_1(\beta) = \beta F(\beta) + \beta U_H^\varepsilon + \beta U_A \; .
\]

(3.2)

The quantities \( U_H^\varepsilon \) and \( U_A \) in the right hand side of (3.2) do not depend on \( \beta \) and represent a contribution of the vacuum energy. The first quantity \( \beta U_H^\varepsilon \) originates from the difference between the covariant measure \( \prod_x (\det \gamma(x))^{1/4} d\varphi(x) \) in the functional integral (2.1) and the Hamiltonian measure \( \prod_x (\gamma_{00}(x))^{-1/2} (\det \gamma(x))^{1/4} d\varphi(x) \) used in the canonical formalism for the path integral representation for \( F(\beta) \) (see [13]). The second term \( \beta U_A^\varepsilon \) is finite and accounts for, according to Allen [14], a difference of the renormalized covariant path integral and the partition function (3.1) defined by the spectrum of the normally ordered Hamiltonian.

The density matrix for the brick wall canonical ensemble is

\[
\hat{\rho}_\varepsilon^H(\beta) = \frac{e^{-\beta \tilde{H}}}{\text{Tr}(e^{-\beta \tilde{H}})},
\]

(3.3)
where superscript $H$ indicates that field $\varphi$ is given on the black hole background and subscript $\epsilon$ indicates the position of the internal boundary. Thus the entropy for the field in the brick-wall model reads

$$S_{1}^{BW}[\beta, \epsilon, r_B] = -\text{Tr}(\hat{\rho}^{H}(\beta) \log \hat{\rho}^{H}(\beta)) = \beta^2 \frac{\partial}{\partial \beta} F(\beta) \quad ,$$  \hspace{1cm} (3.4)$$

and using Eq. (3.2) it can be rewritten in the field-theoretical form

$$S_{1}^{BW}[\beta, \epsilon, r_B] = \left( \beta \frac{\partial}{\partial \beta} - 1 \right) W_{1}^{BW}[\beta, \epsilon, r_B] \quad .$$  \hspace{1cm} (3.5)$$

Using Eq.(3.5) it is easy now to compute $S_{1}^{BW}$ and compare it with the entropy $S_{1}^{CS}$ in the conical singularity method. For this aim it is convenient to map conformally the brick-wall space $M_{2}^{\epsilon}$ onto the cylinder $Q_{\alpha,\epsilon_{z}}$ with the metric

$$ds^2 = (d\tau^2 + dz^2) = \mu^{-2} y^{-2} e^{-2\sigma} ds^2 \quad ,$$  \hspace{1cm} (3.6)$$

where $0 \leq \tau \leq 2\pi \alpha$, $ds^2$ is the black hole metric (2.20), $y$ and $\sigma$ are given by (2.22) and (2.23). The coordinate $z = -\ln y$ on cylinder ranges in the interval $0 \leq z \leq \epsilon_{z}$, where $\epsilon_{z}$ is related with the proper distance $\epsilon$ from the ”brick-wall” to the horizon (in the limit $\epsilon \to 0$) as

$$\epsilon_{z} = -\ln \frac{\epsilon}{\mu} + \sigma (r_h) \quad .$$  \hspace{1cm} (3.7)$$

The effective actions $W_{1}^{BW}[\beta, \epsilon, r_B]$ on the brick wall space $M_{2}^{\epsilon}$ and the effective action $W_{1}[Q_{\alpha,\epsilon_{z}}]$ on the cylinder differ by the conformal anomaly term. But because the spaces are static and regular everywhere, this difference is proportional to $\beta$ and it does not contribute to the entropy. So one can rewrite the expression (3.5) for $S_{1}^{BW}$ as

$$S_{1}^{BW}[\beta, \epsilon, r_B] = \left( \alpha \frac{\partial}{\partial \alpha} - 1 \right) W_{1}[Q_{\alpha,\epsilon_{z}}] \quad .$$  \hspace{1cm} (3.8)$$

Here we took into account that $\alpha$ is proportional to $\beta$. The action $W_{1}[Q_{\alpha,\epsilon_{z}}]$ of massless field on a two dimensional cylinder of a large size $\epsilon_{z}$ can be calculated exactly in this limit [11]

$$W_{1}[Q_{\alpha,\epsilon_{z}}] = -\frac{1}{12\alpha} \epsilon_{z} - \frac{1}{2} \ln \frac{\pi \alpha}{\epsilon_{z}} + o(\epsilon_{z}^{-1}) \quad .$$  \hspace{1cm} (3.9)$$

Thus, using (3.7), (3.8) and (3.8) one gets

$$S_{1}^{BW}[\beta, \epsilon, r_B] = \frac{1}{6} \sigma (r_h) - \frac{1}{6} \ln \frac{\epsilon}{\mu} + \frac{1}{2} \ln \frac{\pi}{\ln (\mu/\epsilon)} + O(|\ln \epsilon|^{-1}) \quad .$$  \hspace{1cm} (3.10)$$

Comparing (3.10) with Eq.(2.25) we see that the statistical-mechanical entropy computed in the brick wall model differs from $S_{1}^{CS}$ by terms divergent in the limit $\epsilon \to 0$. Note that the dependence of the entropy on the parameter $\mu$ reflects an arbitrariness in the definition of the entropy and it is physically unobservable.

We shall demonstrate now that the difference $S_{1}^{CS} - S_{1}^{BW}$ allows (at least in 2D case) a quite simple presentation, which makes transparent the relation between these quantities.
Let us compare the obtained result (3.10) with the calculation in the Rindler space. Consider the metric

$$ds_R^2 = y^2 d\tilde{\tau}^2 + dy^2$$

(3.11)

where \(0 \leq \tilde{\tau} \leq 2\pi \alpha\) and \(\epsilon \leq y \leq \mu\). This choice of the parameters corresponds to the situation when the field is located between two mirrors at proper distances \(\epsilon\) and \(\mu\) from the Rindler horizon respectively. The inverse temperature of the system measured at the distance \(\mu\) is \(2\pi \alpha \mu\). The entropy of the massless scalar field in this space can be computed by the same method as in the case of the black hole. The Rindler space (3.11) can be conformally mapped onto the cylindrical space (3.6). The size of the cylinder now is \(R_z = -\ln(\epsilon/\mu)\).

Caring out the calculations one gets the brick wall entropy in the Rindler space (at \(\alpha = 1\))

$$\left(S_R\right)^{BW}_1[2\pi \mu, \epsilon, \mu] = -\frac{1}{6} \ln \frac{\epsilon}{\mu} + \frac{1}{2} \ln \frac{\pi}{\ln(\mu/\epsilon)} + O(|\ln \epsilon|^{-1})$$

(3.12)

Therefore, comparing (3.12) and (3.10), we get

$$S^{BW}_1[\beta, \epsilon, r_B] = \frac{1}{6} \sigma(r_h) + \left(S_R\right)^{BW}_1[2\pi \mu, \epsilon, \mu]$$

(3.13)

Now, taking into account (2.25), one can conclude that the conical singularity entropy can be identically rewritten as the difference of two statistical-mechanical entropies

$$S^{CS}_1[\beta, r_B] = S^{BW}_1[\beta, \epsilon, r_B] - \left(S_R\right)^{BW}_1[2\pi \mu, \epsilon, \mu]$$

(3.14)

Eventually, from (2.19), (3.4) and (3.14) it follows that the total one-loop entropy of a two-dimensional black hole is

$$S^{TD}(\beta) = \tilde{S}^{BH}(\beta) + \left(-\text{Tr}(\hat{\rho}_c^R(2\pi \mu) \log \hat{\rho}_c^R(2\pi \mu))\right)$$

(3.15)

where \(\hat{\rho}_c^R(2\pi \mu)\) is the density matrix for the Rindler space. Eq. (3.15) is the main result of our Letter. The first term in the right hand side of Eq. (3.15) is the Bekenstein-Hawking entropy determined by the area of the event horizon of quantum corrected black hole. The last two terms represent the finite difference of two divergent statistical-mechanical entropies calculated for the black hole and for a corresponding ”reference” Rindler space, respectively.

It is worth mentioning that the similar subtraction formula naturally arises in the membrane paradigm \([13]\). Namely, in order to obtain the correct expression for the flux of the entropy onto a black hole, Thorne and Zurek \([13, 16]\) proposed to subtract from the entropy, calculated by the statistical-mechanical method, the entropy of a thermal atmosphere of the black hole. The latter entropy close to the horizon coincides with \(\left(S_R\right)^{BW}_1\). So Eq. (3.13) can be used to prove this conjecture.
4 Summary

As we have demonstrated different off-shell procedures (conical singularity method and brick wall model) result in different renormalized one-loop corrections to the black hole entropy. The quantity $S^{CS}\_1$ is finite, while $S^{BW}\_1$ diverges as $\epsilon \to 0$. The quantities obtained by both methods also have different interpretations. The brick wall entropy $S^{BW}\_1$ is pure statistical-mechanical entropy, while the quantity $S^{CS}\_1$ can be represented only as the difference of two such entropies. In two dimensions for the massless scalar field their relation can be established exactly. It is also worth mentioning that instead of the brick-wall method one can use another way to calculate the statistical entropy. As it was shown in [11], this does not change the final subtraction equation (3.14).

Our analysis demonstrates in an explicit form that the black holes are some peculiar systems where thermodynamical and statistical-mechanical entropies may not coincide. The thermodynamical computations which operate with the system being in thermal equilibrium (obeying the equations of motion) always give the finite result for the entropy in terms of the renormalized gravitational coupling constants. It is the thermodynamical entropy that is observable (at least in the gedanken experiments). On the other hand, the statistical-mechanical black hole entropy includes physical divergences because of the infinite blue shift at the horizon. In general the relation between thermodynamics and statistical mechanics for black holes may require some subtraction formula similar to Eq. (3.15).

In our study we did not deal with the statistical explanation of the tree-level Bekenstein-Hawking entropy $\bar{S}^{BH}(\beta)$ itself. In the total entropy (3.15) $\bar{S}^{BH}(\beta)$ appears as the entropy of a quantum corrected black hole and it is completely defined by the geometry. So one can expect that it is possible to get the statistical-mechanical interpretation of the Bekenstein-Hawking entropy $\bar{S}^{BH}(\beta)$ in the theory where the metric itself arises as the result of quantum effects. Models of induced gravity and superstrings do possess this property.

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References

[1] J. D. Bekenstein, Nuov. Cim. Lett. 4 (1972) 737; Phys. Rev. D7 (1973) 2333; Phys. Rev. D9 (1974) 3292.

[2] S. W. Hawking, Comm. Math. Phys. 43 (1975) 199.

[3] G. W. Gibbons and S. W. Hawking, Phys. Rev. D15 (1976) 2752.

[4] S. W. Hawking, In: General Relativity: An Einstein Centenary Survey. (eds. S. W. Hawking and W. Israel), Cambridge Univ. Press, Cambridge, 1979.
[5] J. W. York, Phys. Rev. D33 (1986) 2092; H. W. Braden, J. D. Brown, B. F. Whiting, and J. W. York, Phys. Rev. D42 (1990) 3376.

[6] G.’t Hooft, Nucl. Phys. B256 (1985) 727.

[7] L. Bombelli, R. Koul, J. Lee, and R. Sorkin, Phys. Rev. D34 (1986) 373; M. Srednicki, Phys. Rev. Lett. 71 (1993) 666; V. Frolov and I. Novikov, Phys. Rev. D48 (1993) 4545.

[8] V.P. Frolov, Phys. Rev. Lett. 74 (1995) 3319; V.P. Frolov, ”Black Hole Entropy and Physics at Planckian Scales”, hep-th/9510156.

[9] L. Susskind and J. Uglum, Phys. Rev. D50 (1994) 2700 ; D.V. Fursaev and S.N. Solodukhin, Phys. Lett. B365 (1996) 51; F. Larsen and F. Wilczek, Nucl. Phys. B458 (1996) 249; D. Kabat, Nucl. Phys. B453 (1995) 281.

[10] D.V. Fursaev and S.N. Solodukhin, Phys. Rev. D52 (1995) 2133.

[11] V.P. Frolov, D.V. Fursaev and A.I. Zelnikov, Black Hole Entropy: Off-shell vs On-shell, hep-th/9512184.

[12] R.C.Myers, Phys. Rev. D50 (1994) 6412.

[13] S.P. deAlwis and N. Ohta, Phys. Rev. D52 (1995) 3529.

[14] B.Allen, Phys. Rev. D33 (1986) 3640.

[15] K.S.Thorne, R.H.Price, and D.A.Macdonald. ”Black Holes: The Membrane Paradigm” (Yale Univ Press, NewHaven and London) (1986).

[16] W. H. Zurek and K. S. Thorne, Phys. Rev. Lett. 54 (1985) 2171.