The relation between the decomposition of comodules and coalgebras *

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Abstract
T. Shudo and H. Miyamito [3] showed that $C$ can be decomposed into a direct sum of its indecomposable subcoalgebras of $C$. Y.H. Xu [5] showed that the decomposition was unique. He also showed that $M$ can uniquely be decomposed into a direct sum of the weak-closed indecomposable subcomodules of $M$ (we call the decomposition the weak-closed indecomposable decomposition ) in [6]. In this paper, we give the relation between the two decomposition. We show that if $M$ is a full, $W$-relational hereditary $C$-comodule, then the following conclusions hold:

1. $M$ is indecomposable iff $C$ is indecomposable;
2. $M$ is relative-irreducible iff $C$ is irreducible;
3. $M$ can be decomposed into a direct sum of its weak-closed relative-irreducible subcomodules iff $C$ can be decomposed into a direct sum of its irreducible subcoalgebras.

We also obtain the relation between coradical of $C$-comodule $M$ and radical of algebra $C(M)$.

0 Introduction and Preliminaries

The decomposition of coalgebras and comodules is an important subject in study of Hopf algebras. T. Shudo and H. Miyamito [3] showed that $C$ can be decomposed into a direct sum of its indecomposable subcoalgebras of $C$. Y.H. Xu [5] showed that the decomposition was unique. He also showed that $M$ can uniquely be decomposed into a direct sum of the weak-closed indecomposable subcomodules of $M$ (we call the decomposition the weak-closed indecomposable decomposition ) in [6]. In this paper, we give the relation between

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the two decomposition. We show that if $M$ is a full, $W$-relational hereditary $C$-comodule, then the following conclusions hold:

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3. $M$ can be decomposed into a direct sum of its weak-closed relative-irreducible subcomodules iff $C$ can be decomposed into a direct sum of its irreducible subcoalgebras.

We also obtain the relation between coradical of $C$-comodule $M$ and radical of algebra $C(M)^*$.

Let $k$ be a field, $M$ be a $C$-comodule, $N$ be a subcomodule of $M$, $E$ be a subcoalgebra of $C$ and $P$ be an ideal of $C^*$. As in [6], we define:

$E^\perp C^* = \{ f \in C^* \mid f(E) = 0 \}.$

$P^\perp C = \{ c \in C \mid P(c) = 0 \}.$

$N^\perp C^* = \{ f \in C^* \mid f \cdot N = 0 \}.$

$P^\perp M = \{ x \in M \mid P \cdot x = 0 \}.$

Let $\langle N \rangle$ denote $N^\perp C^* \perp M$. $\langle N \rangle$ is called the closure of $N$. If $\langle N \rangle \supseteq N$, then $N$ is called closed. If $N = C^* x$, then $\langle N \rangle$ is denoted by $\langle x \rangle$. If $x \supseteq N$ for any $x \in N$, then $N$ is called weak-closed. It is clear that any closed subcomodule is weak-closed. If $\rho(N) \subseteq N \otimes E$, then $N$ is called an $E$-subcomodule of $M$. Let

$M_E = \sum \{ N \mid N$ is a subcomodule of $M$ and $\rho(N) \subseteq N \otimes E \}.$

We call $M_E$ a component of $M$ over $E$. If $M_E$ is some component of $M$ and $M_F \subseteq M_E$ always implies $M_F = M_E$ for any non-zero component $M_F$, then $M_E$ is called a minimal component of $M$.

Let $\{ m_\lambda \mid \lambda \in \Lambda \}$ be the basis of $M$ and $C(M)$ denote the subspace of $C$ spanned by

$W(M) = \{ c \in C \mid \text{there exists an } m \in M \text{ with } \rho(m) = \sum m_\lambda \otimes c_\lambda \text{ such that } c_{\lambda_0} = c \text{ for some } \lambda_0 \in \Lambda \}.$

E. Abe in [1, P129] checked that $C(M)$ is a subcoalgebra of $C$. It is easy to know that if $E$ is subcoalgebra of $C$ and $\rho(M) \subseteq M \otimes E$, then $C(M) \subseteq E$. If $C(M) = C$, then $M$ is called a full $C$-comodule. If $D$ is a simple subcoalgebra of $C$ and $M_D \neq 0$ or $D = 0$, then $D$ is called faithful to $M$. If every simple subcoalgebra of $C$ is faithful to $M$, then $M$ is called a component faithfulness $C$-comodule.

Let $X$ and $Y$ be subspaces of coalgebra $C$. Define $X \wedge Y$ to be the kernel of the composite

$C \rightrightarrows C \otimes C \rightarrow C/X \otimes C/Y.$

$X \wedge Y$ is called a wedge of $X$ and $Y$. 

2
1 The relation between the decomposition of comodules and coalgebras

Lemma 1.1 Let $N$ and $L$ be subcomodules of $M$. Let $D$ and $E$ be subcoalgebras of $C$. Then

(1) $N_{\perp C^*} = (C(N))_{\perp C^*}$;
(2) $M_E = M_{C(M_E)}$;
(3) $N$ is closed iff there exists a subcoalgebra $E$ such that $N = M_E$;
(4) $M_D \neq 0$ iff $D_{\perp C^*\perp M} \neq 0$;
(5) If $D \cap C(M) = 0$, then $M_D = 0$;
(6) If $D$ is a simple subcoalgebra of $C$, then $C(M_D) = \begin{cases} D & \text{if } M_D \neq 0 \\ 0 & \text{if } M_D = 0 \end{cases}$;
(7) If $D \cap E = 0$, then $M_D \cap M_E = 0$;
(8) If $D$ and $E$ are simple subcoalgebras and $M_D \cap M_E = 0$ with $M_D \neq 0$ or $M_E \neq 0$, then $D \cap E = 0$.
(9) If $M_E$ is the minimal component of $M$ and $0 \neq N \subseteq M_E$, then $C(M_E) = C(N)$.

Proof. (1) Let $\{m_\lambda \mid \lambda \in \Lambda\}$ be a basis of $N$. For any $f \in N_{\perp C^*}$, if $c \in W(N)$, then there exists $m \in N$ with $\rho(m) = \sum_{\lambda \in \Lambda} m_\lambda \otimes c_\lambda$ such that $c_{\lambda_0} = c$ for some $\lambda_0 \in \Lambda$. Since $f \cdot m = \sum m_\lambda f(c_\lambda) = 0$, $f(c_\lambda) = 0$ for any $\lambda \in \Lambda$. Obviously, $f(c) = f(c_{\lambda_0}) = 0$. Considering that $C(N)$ is the space spanned by $W(N)$, we have $f \in C(N)_{\perp C^*}$. Conversely, if $f \in C(N)_{\perp C^*}$, then $f \cdot m = \sum m_\lambda f(c_\lambda) = 0$ for any $m \in N$, i.e., $f \in N_{\perp C^*}$. This shows that $N_{\perp C^*} = (C(N))_{\perp C^*}$.

(2) Since $M_E$ is an $E$-subcomodule of $M$, $C(M_E) \subseteq E$ and $M_{C(M_E)} \subseteq M_E$. Since $M_E$ is a $C(M_E)$-subcomodule, $M_E \subseteq M_{C(M_E)}$.

(3) If $N$ is closed, then $N_{\perp C^*\perp M} = N$. Let $E = C(N)$. Obviously, $N \subseteq M_E$. By Lemma 1.1(1), $N_{\perp C^*} \cdot M_E = E_{\perp C^*} \cdot M_E = 0$. Thus $M_E \subseteq N_{\perp C^*\perp M}$ and $M_E \subseteq N$. This shows that $M_E = N$. Conversely, if $M_E = N$, obviously, $(M_E) \subseteq (M_E)_{\perp C^*\perp M}$. Thus it is sufficient to show that $< M_E > \subseteq M_E$.

Let $L = (M_E)_{\perp C^*\perp M} = < M_E >$. We see that $C(L)_{\perp C^*} = L_{\perp C^*} = (M_E)_{\perp C^*\perp M\perp C^*} = (M_E)_{\perp C^*} = C(M_E)_{\perp C^*}$. Thus $C(L) = C(M_E)$ and $L \subseteq M_{C(L)} = M_{C(M_E)} = M_E$.

(4) If $D_{\perp C^*\perp M} \neq 0$, let $\{m_\lambda \mid \lambda \in \Lambda\}$ be a basis of $D_{\perp C^*\perp M}$ and $\rho(x) = \sum m_\lambda \otimes d_\lambda$ for any $x \in D_{\perp C^*\perp M}$. Since $D_{\perp C^*} \cdot x = \sum m_\lambda D_{\perp C^*}(d_\lambda) = 0$, $d_\lambda \in D_{\perp C^*\perp C} = D$ for any $\lambda \in \Lambda$, which implies that $D_{\perp C^*\perp M}$ is a $D$-subcomodule. Therefore $0 \neq D_{\perp C^*\perp M} \subseteq M_D$, i.e., $M_D \neq 0$. Conversely, if $M_D \neq 0$, then we have that $0 \neq M_D \subseteq (M_D)_{\perp C^*\perp M} = C(M_D)_{\perp C^*\perp M}$. (by part (1))
(5) Since $M_D \subseteq M$, $C(M_D) \subseteq C(M)$. Obviously, $C(M_D) \subseteq D$. Thus $C(M_D) \subseteq C(M) \cap D = 0$, which implies that $M_D = 0$.

(6) If $M_D = 0$, then $C(M_D) = 0$. If $M_D \neq 0$, then $0 \neq C(M_D) \subseteq D$. Since $D$ is a simple subcoalgebra, $C(M_D) = D$.

(7) If $x \in M_D \cap M_E$, then $\rho(x) \in (M_D \otimes D) \cap (M_E \otimes E)$, and
\[
\rho(x) = \sum_{i=1}^{n} x_i \otimes d_i = \sum_{j=1}^{m} y_j \otimes e_j,
\]
where $x_i, \ldots, x_n$ is linearly independent and $d_i \in D$ and $e_j \in E$. Let $f_l \in M^*$ with $f_l(x_i) = \delta_{il}$ for $i, l = 1, 2, \ldots n$ . Let $f_l \otimes id$ act on equation (1). We have that $d_i = \sum_{j=1}^{m} f_l(y_j)e_j \in E$, which implies $d_i \in D \cap E = 0$ and $d_i = 0$ for $l = 1, \ldots, n$. Therefore $M_D \cap M_E = 0$.

(8) If $D \cap E \neq 0$, then $D = E$. Thus $M_D = M_E$ and $M_D \cap M_E = M_E = M_D = 0$. We get a contradiction. Therefore $D \cap E = 0$.

(9) Obviously, $C(N) \subseteq C(M_E)$. Conversely, since $0 \neq N \subseteq M \cap M_E = M_E$ and $M_E$ is a minimal component, $M_{C(N)} = M_E$. By the definition of component, $C(M_E) \subseteq C(N)$. Thus $C(N) = C(M_E)$.

Proposition 1.2 If $E$ is a subcoalgebra of $C$, then the following conditions are equivalent.

1. $M_E$ is a minimal component of $M$.
2. $C(M_E)$ is a simple subcoalgebra of $C$.
3. $M_E$ is a minimal closed submodule of $M$.

Proof. It is easy to check that $M_E = 0$ iff $C(M_E) = 0$. Thus (1), (2) and (3) are equivalent when $C(M_E) = 0$. We now assume that $M_E \neq 0$.

(1) $\implies$ (2) Since $0 \neq M_E$, there exists a non-zero finite dimensional simple submodule $N$ of $M$ such that $N \subseteq M_E$. By [6, Lemma 1.1], $N$ is a simple $C^*$-submodule of $M$. Since $M_E$ is a minimal component, $C(N) = C(M_E)$ by Lemma 1.1 (9). Let $D = C(N) = C(M_E)$. By Lemma 1.1(1), $(0 : N)_{C^*} = N^{\perp C^*} = C(N)^{\perp C^*} = D^{\perp C^*}$. Thus $N$ is a faithful simple $C^*/D^{\perp C^*}$-module, and so $C^*/D^{\perp C^*}$ is a simple algebra. It is clear that $D$ is a simple subcoalgebra of $C$.

(2) $\implies$ (3) If $0 \neq N \subseteq M_E$ and $N$ is a closed submodule of $M$, then by Lemma 1.1(3) there exists a subcoalgebra $F$ of $C$ such that $N = M_F$. Since $0 \neq C(N) = C(M_F) \subseteq C(M_E)$ and $C(M_E)$ is simple, $C(M_F) = C(M_E)$. By Lemma 1.1(2), $N = M_F = M_{C(M_F)} = M_{C(M_E)} = M_E$, which implies that $M_E$ is a minimal closed submodule.

(3) $\implies$ (1), If $0 \neq M_F \subseteq M_E$, then $M_F$ is a closed submodule by Lemma 1.1(3) and $M_E = M_F$, i.e. $M_E$ is a minimal component.
This completes the proof. □

Let
\[ C_0 = \{ D \mid D \text{ is a simple subcoalgebra of } C \}; \]
\[ C(M)_0 = \{ D \mid D \text{ is a simple subcoalgebra of } C \text{ and } D \subseteq C(M) \}; \]
\[ \mathcal{M}_0 = \{ N \mid N \text{ is a minimal closed submodule of } M \}; \]
\[ C(M)_1 = \{ D \mid D \text{ is a faithful simple subcoalgebra of } C \text{ to } M \}; \]
\[ C_0 = \sum\{ D \mid D \text{ is a simple subcoalgebra of } C \}; \]
\[ M_0 = \sum\{ N \mid N \text{ is a minimal closed submodule of } M \}; \]
\[ C_0 \text{ and } M_0 \text{ are called the coradical of coalgebra } C \text{ and the coradical of comodule } M \]
respectively. If \( M_0 = M \), then \( M \) is called cosemisimple. By Lemma 1.1(5),
\[ C(M)_1 \subseteq C(M)_0 \]

**Theorem 1.3** \[ \psi \begin{cases} C(M)_1 \longrightarrow \mathcal{M}_0 \\ D \longmapsto M_D \end{cases} \text{ is bijective.} \]

**Proof.** By Proposition 1.2, \( \psi \) is a map. Let \( D \) and \( E \in C(M)_1 \) and \( \psi(D) = \psi(E) \), i.e. \( M_D = M_E \). By Lemma 1.1 (6), we have that \( D = C(D_D) = C(M_E) = E \). If \( N \in \mathcal{M}_0 \), then \( N = M_{C(N)} \) and \( C(N) \) is a simple subcoalgebra by Lemma 1.1(3) and Proposition 1.2. Thus \( \psi(C(N)) = N \), which implies that \( \psi \) is surjective. □

In [6] and [5], Xu defined the equivalence relation for coalgebra and for comodule as follows:

**Definition 1.4** We say that \( D \sim E \) for \( D \) and \( E \in C_0 \) iff for any pair of subclasses \( C_1 \) and \( C_2 \) of \( C_0 \) with \( D \in C_1 \) and \( E \in C_2 \) such that \( C_1 \cup C_2 = C_0 \) and \( C_1 \cap C_2 = \emptyset \), there exist elements \( D_1 \in C_1 \) and \( E_1 \in C_2 \) such that \( D_1 \land E_1 \neq E_1 \land D_1 \). Let \([D]\) denote the equivalence class which contains \( D \).

We say that \( N \sim L \) for \( N \) and \( L \in \mathcal{M}_0 \) iff for any pair of subclasses \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) of \( \mathcal{M}_0 \) with \( N \in \mathcal{M}_1 \) and \( L \in \mathcal{M}_2 \) such that \( \mathcal{M}_1 \cup \mathcal{M}_2 = \mathcal{M}_0 \) and \( \mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset \), there exist elements \( N_1 \in \mathcal{M}_1 \) and \( L_1 \in \mathcal{M}_2 \) such that \( N_1 \land L_1 \neq L_1 \land N_1 \). Let \([N]\) denote the equivalence class which contains \( N \).

**Definition 1.5** If \( D \land E = E \land D \) for any simple subcoalgebras \( D \) and \( E \) of \( C \), then \( C \) is called \( \pi \)-commutative. If \( N \land L = L \land N \) for any minimal closed subcomodules \( N \) and \( L \) of \( M \), then \( M \) is called \( \pi \)-commutative.

Obviously, every cocommutative coalgebra is \( \pi \)-commutative. By [6, Theorem 3.8 and Theorem 4.18], \( M \) is \( \pi \)-commutative iff \( M \) can be decomposed into a direct sum of the weak-closed relative-irreducible subcomodules of \( M \) iff every equivalence class of \( M \) contains only one element. By [5], \( C \) is \( \pi \)-commutative iff \( C \) can be decomposed into a direct sum of irreducible subcoalgebras of \( C \) iff equivalence every class of \( C \) contains only one element.
Lemma 1.6 Let $D$, $E$ and $F$ be subcoalgebras of $C$. $N$, $L$ and $T$ be subcomodules of $M$. Then

1. $M_D \wedge M_E = M_{C(M_D) \wedge C(M_E)} \subseteq M_{D \wedge F}$;
2. If $D$ and $E$ are faithful simple subcoalgebras of $C$ to $M$, then $M_D \wedge M_E = M_{D \wedge F}$;
3. $M_{D+E} \supseteq M_D + M_E$;
4. If $F = \sum\{D_\alpha \mid \alpha \in \Omega\}$ and $\{D_\alpha \mid \alpha \in \Omega\} \subseteq C_0$, then $M_F = \sum\{M_{D_\alpha} \mid \alpha \in \Omega\}$.

In particular, $M_{C_0} = M_0$.

5. $(N + L) \wedge T \supseteq N \wedge T + L \wedge T$;
6. $(D + E) \wedge F \supseteq D \wedge F + E \wedge F$.

Proof. (1) We see that

$$M_D \wedge M_E = \rho^{-1}(M \otimes (M_D)^{\perp C} \wedge (M_E)^{\perp C}) \quad \text{(by [6, Proposition 2.2(1)])}$$

$$= \rho^{-1}(M \otimes C(M_D) \wedge C(M_E)) \quad \text{(by Lemma 1.1(1)).}$$

By the definition of component, subcomodule $M_D \wedge M_E \subseteq M_{C(M_D) \wedge C(M_E)}$. It follows from the equation above that $M_{C(M_D) \wedge C(M_E)} \subseteq M_D \wedge M_E$. Thus

$$M_D \wedge M_E = M_{C(M_D) \wedge C(M_E)}$$

and

$$M_{C(M_D) \wedge C(M_E)} \subseteq M_{D \wedge E}.$$

(2) Since $D$ and $E$ are faithful simple subcoalgebras of $C$ to $M$, $C(M_D) = D$ and $C(M_E) = E$ by Lemma 1.1(6). By Lemma 1.6(1), $M_D \wedge M_E = M_{D \wedge E}$.

(3) It is trivial.

(4) Obviously $M_F \supseteq \sum\{M_{D_\alpha} \mid \alpha \in \Omega\}$. Conversely, let $N = M_F$. Obviously $N$ is an $F$-comodule. For any $x \in N$, let $L = C^*x$. it is clear that $L$ is a finite dimensional comodule over $F$. By [4, Lemma 14.0.1], $L$ is a completely reducible module over $F^*$. Thus $L$ can be decomposed into a direct sum of simple $F^*$-submodules:

$$L = L_1 \oplus L_2 \oplus \cdots \oplus L_n,$$

where $L_i$ is a simple $F^*$-submodule. By [6, Proposition 1.16], $L_i > = (L_i)^{\perp F^* \perp N}$ is a minimal closed $F$-subcomodule of $N$. By Theorem 1.3, there exists a simple subcoalgebra $D_{\alpha}$ of $F$ such that $L_i > = N_{D_{\alpha}}$. Obviously, $N_{D_{\alpha}} \subseteq M_{D_{\alpha}}$. Thus $L \subseteq \sum_{i=1}^n L_i > = \sum_{i=1}^n N_{D_{\alpha}} \subseteq \sum\{M_{D_{\alpha}} \mid \alpha \in \Omega\}$. Therefore $M_F = \sum\{M_{D_{\alpha}} \mid \alpha \in \Omega\}$. If $C_0 = F = \sum\{D \mid D \in C_0\}$, then $M_{C_0} = M_F = \sum\{M_D \mid D \in C_0\} = M$ by Theorem 1.3.

(5) and (6) are trivial. □

Lemma 1.7 Let $N$ be a subcomodule of $M$, and let $D$, $E$ and $F$ be simple subcoalgebras of $C$. Then
(1) \( D \sim 0 \) iff \( D = 0 \); \( M_D \sim 0 \) iff \( M_D = 0 \);
We called the equivalence class which contains zero a zero equivalence class.

(2) If \( D \) and \( E \) are faithful to \( M \) and \( M_D \sim M_E \), then \( D \sim E \);

(3) \([M_D] \subseteq \{ M_E \mid E \in [D] \}\);

(4) If \( D \) is faithful to \( M \), then \([M_D] \subseteq \{ M_E \mid E \in [D] \} \) and \( E \) is faithful to \( M \} \).

**Proof.** (1) If \( D \sim 0 \) and \( D \neq 0 \), let \( \mathcal{C}_1 = \{ 0 \} \) and \( \mathcal{C}_2 = \{ F \mid F \neq 0, F \in \mathcal{C}_0 \} \). Thus \( \mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{C}_0 \) and \( \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset \) with \( 0 \in \mathcal{C}_1 \) and \( D \in \mathcal{C}_2 \). But for any \( D_1 \in \mathcal{C}_1 \) and \( E_1 \in \mathcal{C}_2 \), since \( D_1 = 0, D_1 \wedge E_1 = E_1 \wedge D_1 = E_1 \). We get a contradiction. Thus \( D = 0 \). Conversely, if \( D = 0 \), obviously \( D \sim 0 \). Similarly, we can show that \( M_D \sim 0 \) iff \( M_D = 0 \).

(2) For any pair of subclasses \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) of \( \mathcal{C}_0 \), if \( \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset \) and \( \mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{C}_0 \) with \( D \in \mathcal{C}_1 \) and \( E \in \mathcal{C}_2 \), let \( \mathcal{M}_1 = \{ M_F \mid F \in \mathcal{C}_1 \) and \( F \) is faithful to \( M \} \) and \( \mathcal{M}_2 = \{ M_F \mid F \in \mathcal{C}_2 \) and \( F \) is faithful to \( M \}. \) By Theorem 1.3, \( \mathcal{M}_0 = \mathcal{M}_1 \cup \mathcal{M}_2 \) and \( \mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset \). Obviously, \( M_D \in \mathcal{M}_1 \) and \( M_E \in \mathcal{M}_2 \). Since \( M_D \sim M_E \), there exist \( M_{D_1} \in \mathcal{M}_1 \) and \( M_{E_1} \in \mathcal{M}_2 \) such that \( M_{D_1} \wedge M_{E_1} \neq M_{E_1} \wedge M_{D_1} \), where \( D_1 \in \mathcal{C}_1 \) and \( E_1 \in \mathcal{C}_2 \). By Lemma 1.6(2), \( M_{D_1 \wedge E_1} \neq M_{E_1 \wedge D_1} \). Thus \( D_1 \wedge E_1 \neq E_1 \wedge D_1 \). Obviously \( D \) and \( D_1 \in \mathcal{C}_1 \). Meantime \( E \) and \( E_1 \in \mathcal{C}_2 \). By Definition 1.4, \( D \sim E \).

(3) Obviously, \( M_F \sim M_D \) for any \( M_F \in [M_D] \). If \( M_D \neq 0 \), then \( M_F \neq 0 \) by Lemma 1.7(1). By Lemma 1.7(2), \( F \sim D \). Thus \( M_F \in \{ M_E \mid E \in [D] \} \). If \( M_D = 0 \), by Lemma 1.7(1), \( M_F = 0 = M_D \in \{ M_E \mid E \in [D] \} \).

(4) If \( M_F \in [M_D] \), then \( M_D \sim M_F \). If \( D = 0 \), then \( M_D = 0 \). By Lemma 1.7(1), \( M_F = 0 \). Thus \( M_F = M_D = 0 \in \{ M_E \mid E \in [D] \} \) and \( E \) is faithful to \( M \}. \) If \( D \neq 0 \), we have that \( M_D \neq 0 \) since \( D \) is faithful to \( M \). By Lemma 1.7(1), \( M_F \neq 0 \). By Lemma 1.7(3), \( M_F \in \{ M_E \mid E \in [D] \} \) and \( E \) is faithful to \( M \}. \) \( \square \)

**Theorem 1.8** Let \( \{ \mathcal{E}_\alpha \mid \alpha \in \bar{\Omega} \} \) be all of the equivalence classes of \( C. \) and \( E_\alpha = \sum \{ E \mid E \in \mathcal{E}_\alpha \} \). Then

(1) For any \( \alpha \in \bar{\Omega} \), there exists a set \( I_\alpha \) and subclasses \( \mathcal{E}(\alpha, i) \subseteq \mathcal{E}_\alpha \) such that \( \cup \{ \mathcal{E}(\alpha, i) \mid i \in I_\alpha \} = \mathcal{E}_\alpha \) and \( \{ M_{\mathcal{E}(\alpha, i)} \mid \alpha \in \bar{\Omega}, i \in I_\alpha \} \)

is the set of the equivalence classes of \( M \) (they are distinct except for zero equivalence class), where \( M_{\mathcal{E}(\alpha, i)} \) denotes \( \{ M_D \mid D \in \mathcal{E}(\alpha, i) \} \).

(2) If \( M \) is a component faithfulness \( C \)-comodule, then \( \{ M_{\mathcal{E}(\alpha, i)} \mid \alpha \in \Omega, i \in I_\alpha \} \) is the set of the distinct equivalence classes of \( M \).

(3)

\[
M = \sum_{\alpha \in \bar{\Omega}} \oplus M_{(E_\alpha)}(\infty) = \sum_{\alpha \in \bar{\Omega}} \sum_{i \in I_\alpha} \oplus (M_{E(\alpha, i)})(\infty) = \sum_{\alpha \in \bar{\Omega}} \oplus (M_{E_\alpha})(\infty)
\]
and for any \( \alpha \in \bar{\Omega}, \)

\[
M_{(E_\alpha)}(\infty) = \sum_{i \in I_\alpha} (M_{E(\alpha, i)})(\infty) = (M_{E_\alpha})(\infty)
\]

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where \( E(\alpha, i) = \sum \{ E \mid E \in \mathcal{E}(\alpha, i) \} \).

**Proof.** (1) By Theorem 1.3 and Lemma 1.7(3) we can immediately get part (1).

(2) If \( M \) is a component faithfulness \( C \)-comodule, then \( \mathcal{C}(M)_1 = \mathcal{C}(M)_0 = C_0 \). It follows from Theorem 1.3 and part (1) that \{\( M_{E(\alpha, i)} \mid \alpha \in \bar{\Omega}, i \in I_\alpha \)\} consists of all the distinct equivalence classes of \( M \).

(3) We see that
\[
M = M_C = M \sum_{\{E_\alpha\} \mid \alpha \in \bar{\Omega}} (\text{by } [5]) \\
\geq \sum_{\alpha \in \bar{\Omega}} M_{(E_\alpha)}^{(\infty)} (\text{by Lemma 1.6(3)}) \\
\geq \sum_{\alpha \in \bar{\Omega}} \sum_{n=0}^{\infty} M_{\wedge^{n+1} E_\alpha} (\text{by Lemma 1.6(3)}) \\
\geq \sum_{\alpha \in \bar{\Omega}} \sum_{n=0}^{\infty} \wedge^{n+1} M_{E_\alpha} (\text{by Lemma 1.6(1)}) \\
= \sum_{\alpha \in \bar{\Omega}} \sum_{n=0}^{\infty} \wedge^{n+1} M_{\sum_{i \in I_\alpha} E(\alpha, i)} (\text{by Theorem 1.8(1)}) \\
\geq \sum_{\alpha \in \bar{\Omega}} \sum_{n=0}^{\infty} \sum_{i \in I_\alpha} M_{E(\alpha, i)} (\text{by Lemma 1.6(3)}) \\
\geq \sum_{\alpha \in \bar{\Omega}} \sum_{n=0}^{\infty} \sum_{i \in I_\alpha} \sum_{\alpha', i'} (M_{E(\alpha, i)})^{\infty} \\
= M (\text{by } [6, (4.10) \text{ in Theorem 4.15} \text{ and Lemma 1.6(4) and part (1))})
\]
Thus
\[
M = \sum_{\alpha \in \bar{\Omega}} M_{(E_\alpha)}^{(\infty)} = \sum_{\alpha \in \bar{\Omega}} \sum_{i \in I_\alpha} (M_{E(\alpha, i)})^{(\infty)}
\]
and
\[
M = \sum_{\alpha \in \bar{\Omega}} \sum_{i \in I_\alpha} \oplus (M_{E(\alpha, i)})^{(\infty)}
\]
by [6, Theorem 4.15] and part (1). We see that
\[
M_{E(\alpha, i)} \wedge M_{E(\alpha, i)} \subseteq M_{(E_\alpha)}^{(\infty)} \wedge M_{(E_\alpha)}^{(\infty)} \\
\subseteq M_{(E_\alpha)}^{(\infty)} \wedge M_{(E_\alpha)}^{(\infty)} (\text{by Lemma 1.6(1)}) \\
= M_{(E_\alpha)}^{(\infty)} (\text{by [2, Proposition 2.1.1]})
\]
Thus
\[
M_{(E_\alpha)}^{(\infty)} \supseteq (M_{E(\alpha, i)})^{(\infty)}
\]
for any \( i \in I_\alpha \) and for any \( \alpha \in \bar{\Omega} \), and
\[
M_{(E_\alpha)}^{(\infty)} \supseteq \sum_{i \in I_\alpha} (M_{E(\alpha, i)})^{(\infty)}
\]
(3) If \( M_{(E_\alpha)}^{(\infty)} \cap \sum_{\beta \in \bar{\Omega}, \beta \neq \alpha} M_{(E_\beta)}^{(\infty)} \neq 0 \), then there exists a non-zero simple subcomodule \( C^* x \subseteq M_{(E_\alpha)}^{(\infty)} \cap \sum_{\beta \in \bar{\Omega}, \beta \neq \alpha} M_{(E_\beta)}^{(\infty)} \). By [6, Proposition 1.16], \( < x > \) is a minimal closed subcomodule of \( M \). By Lemma 1.1(3), \( M_{(E_\alpha)}^{(\infty)} \) is a closed subcomodule of \( M \).
By [6, Lemma 3.3], there exists $\gamma \in \bar{\Omega}$ with $\gamma \neq \alpha$ such that $C^*x \subseteq M_{(E_\gamma)^{\infty}}$. Thus $M_{(E_\gamma)^{\infty}} \cap M_{(E_\alpha)^{\infty}} \neq 0$. By Lemma 1.1 (7), $(E_\gamma)^{\infty} \cap (E_\alpha)^{\infty} \neq 0$, which contradicts [5]. Thus for any $\alpha \in \bar{\Omega}$, $M_{(E_\alpha)^{\infty}} \cap \sum_{\beta \in \bar{\Omega}, \beta \neq \alpha} M_{(E_\alpha)^{\infty}} = 0$, which implies that

$$M = \sum_{\alpha \in \bar{\Omega}} \oplus M_{(E_\alpha)^{\infty}}.$$  

It follows from equations (2) and (3) that

$$M_{(E_\alpha)^{\infty}} = \sum_{i \in I_\alpha} (M_{E(\alpha,i)})^{\infty}.$$  

We see that

$$M_{E_\alpha} \cap M_{E_\alpha} \subseteq M_{E_\alpha} \cap E_\alpha \quad (\text{by Lemma 1.6 (1)})$$

$$\subseteq M_{(E_\alpha)^{\infty}}.$$  

Thus

$$M_{(E_\alpha)^{\infty}} \supseteq (M_{E_\alpha})^{\infty} \supseteq \sum_{i \in I_\alpha} (M_{E(\alpha,i)})^{\infty}$$

and

$$M_{(E_\alpha)^{\infty}} = (M_{E_\alpha})^{\infty}$$

by relation (4). This completes the proof. $\Box$

**Definition 1.9** If $M$ can be decomposed into a direct sum of two non-zero weak-closed subcomodules, then $M$ is called decomposable. If $N$ is a subcomodule of $M$ and $N$ contains exactly one non-zero minimal closed submodule, then $N$ is said to be relative-irreducible.

**Corollary 1.10** $C$ is a coalgebra.

1. If $C$ is $\pi$-commutative, then every $C$-comodule $M$ is also $\pi$-commutative;
2. If $C$ can be decomposed into a direct sum of its irreducible subcoalgebras, then every $C$-comodule $M$ can also be decomposed into a direct sum of its relative-irreducible subcomodules;
3. If $C$ is decomposable, then every component faithfulness $C$-comodule $M$ is decomposable;
4. If $C$ is irreducible, then every non-zero $C$-comodule $M$ is relative-irreducible;
5. $C$ is irreducible if and only if every component faithfulness $C$-comodule $M$ is relative-irreducible.

**Proof.** (1) For any pair of non-zero closed subcomodules $N$ and $L$ of $M$, by Theorem 1.3, there exist faithful simple subcoalgebras $D$ and $E$ of $C$ to $M$ such that $N = M_D$ and $L = M_E$. By Lemma 1.6(2), $N \cap L = M_D \cap M_E = M_{D \cap E} = M_{E \cap D} = M_E \cap M_D = L \cap N$. Thus $M$ is $\pi$-commutative.
(2) Since $C$ can be decomposed into a direct sum of its irreducible subcoalgebras, every equivalence class of $C$ contains only one element by [5]. By Theorem 1.8(1), every equivalence class of $M$ also contains only one element. Thus it follows from [6, Theorem 4.18] that $M$ can be decomposed into a direct sum of its relative-irreducible subcomodules.

(3) If $C$ is decomposable, then there are at least two non-zero equivalence classes in $C$. By Theorem 1.8(2), there are at least two non-zero equivalence classes in $M$. Thus $M$ is decomposable.

(4) If $C$ is irreducible, then there is only one non-zero simple subcoalgebra of $C$ and there is at most one non-zero minimal closed subcomodule in $M$ by Theorem 1.3. Considering $M \neq 0$, we have that $M$ is relative-irreducible.

(5) If $C$ is irreducible, then every component faithfulness $C$-comodule $M$ is relative-irreducible by Corollary 1.10(4). Conversely, let $M = C$ be the regular $C$-comodule. If $D$ is a non-zero simple subcoalgebra of $C$, then $0 \neq D \subseteq M_D$. Thus $M$ is a component faithfulness $C$-comodule. Since $M$ is a relative-irreducible $C$-comodule by assumption, there is only one non-zero minimal closed subcomodule in $M$ and so there is also only one non-zero simple subcoalgebra in $C$ by Theorem 1.3. Thus $C$ is irreducible.

**Lemma 1.11** Let $N$ be a $C$-subcomodule of $M$ and $\emptyset \neq L \subseteq M$. Then

1. $C^* \cdot L = C(M)^* \cdot L$;  
   \[ N^{\perp C^* \cdot L} = N^{\perp C(M)^* \cdot L}; \]
2. $N$ is a (weak-) closed $C$-subcomodule iff $N$ is a (weak-)closed $C(M)$-subcomodule;  
3. $N$ is a minimal closed $C$-subcomodule iff $N$ is a minimal closed $C(M)$-subcomodule;  
4. $N$ is an indecomposable $C$-subcomodule iff $N$ is an indecomposable $C(M)$-subcomodule;  
5. $N$ is a relative-irreducible $C$-subcomodule iff $N$ is a relative-irreducible $C(M)$-subcomodule.

**Proof.** (1) Let $C = C(M) \oplus V$, where $V$ is a subspace of $C$. If $f \in V^*$, then $f \cdot L = 0$. Thus $C^* \cdot L = (C(M)^* + V^*) \cdot L = C(M)^* \cdot L$. We now show the second equation. Obviously,

\[ N^{\perp C^* \cdot L} \subseteq N^{\perp C(M)^* \cdot L}. \]

Conversely, for any $x \in N^{\perp C(M)^* \cdot L}$ and $f \in N^{\perp C^*}$, there exist $f_1 \in C(M)^*$ and $f_2 \in V^*$ such that $f = f_1 + f_2$. Obviously, $f \cdot x = f_1 \cdot x = 0$. Thus $x \in N^{\perp C^* \cdot L}$ and $N^{\perp C(M)^* \cdot L} \subseteq N^{\perp C^* \cdot L}$. Therefore

\[ N^{\perp C^* \cdot L} = N^{\perp C(M)^* \cdot L}. \]

(2) If $N$ is a weak-closed $C$-subcomodule, then $(C^* \cdot x)^{\perp C^* \cdot L} \subseteq N$ for any $x \in N$. We see that

\[ (C^* \cdot x)^{\perp C^* \cdot L} = (C(M)^* \cdot x)^{\perp C^* \cdot L} \quad \text{(by Lemma 1.11(1))} \]

\[ = (C(M)^* \cdot x)^{\perp C(M)^* \cdot L} \quad \text{(by Lemma 1.11(1))} \]
Thus \((C(M)^* \cdot x)_{L^C(M)\perp M} \subseteq N\) and so \(N\) is a weak-closed \(C(M)\)-subcomodule. Conversely, if \(N\) is a weak-closed \(C(M)\)-subcomodule. Similarly, we can show that \(N\) is a weak-closed \(C\)-subcomodule. We now show the second assertion. If \(N\) is a closed \(C\)-subcomodule, then \(N_{L^C(M)\perp M} = N\) and \(N = N_{L^C(M)^*\perp M}\), by part (1), which implies that \(N\) is a closed \(C(M)\)-subcomodule. Conversely, if \(N\) is a closed \(C(M)\)-subcomodule, similarly, we can show that \(N\) is a closed \(C\)-subcomodule. Similarly the others can be proved. \(\square\)

**Proposition 1.12** Let every simple subcoalgebra in \(C(M)\) be faithful to \(M\).

(1) If \(M\) is an indecomposable \(C\)-comodule, then \(C(M)\) is also an indecomposable subcoalgebra.

(2) \(M\) is a relative-irreducible \(C\)-comodule iff \(C(M)\) is an irreducible subcoalgebra.

**Proof.** (1) If \(M\) is an indecomposable \(C\)-comodule, then \(M\) is an indecomposable \(C(M)\)-comodule by Lemma 1.11 and \(M\) is a component faithfulness \(C(M)\)-comodule. By Corollary 1.10(3), \(C(M)\) is indecomposable.

(2) If \(M\) is a relative-irreducible \(C\)-comodule, then \(M\) is a relative-irreducible \(C(M)\)-comodule by Lemma 1.11(5) and \(C(M)\) is irreducible. Conversely, if \(C(M)\) is irreducible, then \(M\) is a relative-irreducible \(C(M)\)-comodule by Corollary 1.10(4) and so \(M\) is a relative-irreducible \(C\)-comodule by Lemma 1.11(5). \(\square\)

**Definition 1.13** If \(D \sim E\) implies \(M_D \sim M_E\) for any simple subcoalgebras \(D\) and \(E\) in \(C(M)\), then \(M\) is called a \(W\)-relational hereditary \(C\)-comodule.

If \(M\) is a \(W\)-relational hereditary \(C\)-comodule, then \(C(M)_1 = C(M)_0\). In fact, by Lemma 1.1(5), \(C(M)_1 \subseteq C(M)_0\). If \(C(M)_1 \neq C(M)_0\), then there exists \(0 \neq D \in C(M)_0\) such that \(M_D = 0\). Since \(M_D \sim M_0\) and \(M\) is \(W\)-relational hereditary, we have that \(D \sim 0\) and \(D = 0\) by Lemma 1.7(1). We get a contradiction. Thus \(C(M)_1 = C(M)_0\).

Obviously, every \(\pi\)-commutative comodule is \(W\)-relational hereditary. If \(C\) is \(\pi\)-commutative, then \(M\) is \(\pi\)-commutative by Corollary 1.10(1) and every \(C\)-comodule \(M\) is \(W\)-relational hereditary. Furthermore, \(M\) is also a component faithfulness \(C(M)\)-comodule.

**Proposition 1.14** Let the notation be the same as in Theorem 1.8. Then the following conditions are equivalent.

(1) \(M\) is \(W\)-relational hereditary.

(2) For any \(\alpha \in \tilde{\Omega}\), there is at most one non-zero equivalence class in \(\{M_{E(x,i)} \mid i \in I_\alpha\}\), and \(C(M)_1 = C(M)_0\).

(3) For any \(D \in C(M)_0\), \([M_D] = \{M_F \mid F \in [D] \text{ and } F \subseteq C(M)\}\).
(4) For any \( D \) and \( E \in \mathcal{C}(M)_0 \), \( M_D \sim M_E \) iff \( D \sim E \).
(5) For any \( \alpha \in \bar{\Omega} \), \( M_{(E_{\alpha})^{(\infty)}} \) is indecomposable, and \( \mathcal{C}(M)_1 = \mathcal{C}(M)_0 \).
(6) For any \( \alpha \in \bar{\Omega} \), \( (M_{E_{\alpha}})^{(\infty)} \) is indecomposable, and \( \mathcal{C}(M)_1 = \mathcal{C}(M)_0 \);
(7) \( M = \sum_{\alpha \in \bar{\Omega}} (M_{E_{\alpha}})^{(\infty)} \) is its weak-closed indecomposable decomposition, and \( \mathcal{C}(M)_1 = \mathcal{C}(M)_0 \);
(8) \( M = \sum_{\alpha \in \bar{\Omega}} \oplus (M_{E_{\alpha}})^{(\infty)} \) is its weak-closed indecomposable decomposition, and \( \mathcal{C}(M)_1 = \mathcal{C}(M)_0 \).

Proof. We prove it along with the following lines: (1) \( \implies \) (4) \( \implies \) (3) \( \implies \) (1) (3) \( \iff \) (2) \( \iff \) (5) \( \iff \) (6) \( \iff \) (7) \( \iff \) (8).

(1) \( \implies \) (4) It follows from the discussion above that \( \mathcal{C}(M)_1 = \mathcal{C}(M)_0 \).

If \( D \) and \( E \in \mathcal{C}(M)_0 \) and \( M_D \sim M_E \), then \( D \sim E \) by Lemma 1.7 (2). If \( D \sim E \) and \( D \) and \( E \in \mathcal{C}(M)_0 \), then \( M_D \sim M_E \).

(4) \( \implies \) (3) Considering Lemma 1.7(4) and \( \mathcal{C}(M)_1 = \mathcal{C}(M)_0 \), we only need to show that
\[
\{ M_F \mid F \in [D] \text{ and } F \subseteq \mathcal{C}(M) \} \subseteq [M_D].
\]
For any \( F \in [D] \) with \( F \subseteq \mathcal{C}(M) \), \( F \sim D \) and \( M_F \sim M_D \) by part (4), which implies that \( M_F \in [M_D] \).

(3) \( \implies \) (1) It is trivial.
(5) \( \iff \) (6) \( \iff \) (7) \( \iff \) (8) It follows from Theorem 1.8(3).

(5) \( \iff \) (2) By Theorem 1.8 (3),
\[
M_{(E_{\alpha})^{(\infty)}} = \sum_{i \in I_{\alpha}} (M_{E(\alpha,i)})^{(\infty)}.
\]
Thus part (2) and part (5) are equivalent.

(2) \( \implies \) (3) It follows from Lemma 1.7 (4).

(3) \( \implies \) (2) If there are two non-zero equivalence classes \( M_{E(\alpha,i_1)} \neq 0 \) and \( M_{E(\alpha,i_2)} \neq 0 \) in \( \{ M_{E(\alpha,i)} \mid i \in I_{\alpha} \} \), then there exist \( D_1 \in \mathcal{E}(\alpha,i_1) \) and \( D_2 \in \mathcal{E}(\alpha,i_2) \) such that \( M_{D_1} \neq 0 \) and \( M_{D_2} \neq 0 \). Let \( D = D_1 \). Since \( M_{D_1} \) and \( M_{D_2} \in \{ M_F \mid F \in [D] \text{ and } F \subseteq \mathcal{C}(M) \} \) and \( M_{D_2} \not\in [M_D] = M_{E(\alpha,i_1)} \), this contradicts part (3). Thus there is at most one non-zero equivalence class in \( \{ M_{E(\alpha,i)} \mid i \in I_{\alpha} \} \). \( \square \)

**Proposition 1.15** If \( M \) is a full, \( W \)-relational hereditary \( \mathcal{C} \)-comodule, then
(1) \( M \) is indecomposable iff \( C \) is indecomposable;
(2) \( M \) is relative-irreducible iff \( C \) is irreducible.
(3) \( M \) can be decomposed into a direct sum of its weak-closed relative-irreducible subcomodules iff \( C \) can be decomposed into a direct sum of its irreducible subcoalgebras.
(4) \( M \) is \( \pi \)-commutative iff \( C \) is \( \pi \)-commutative.

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Proof. Since $M$ is a full $C$-comodule, $C(M) = C$. Since $M$ is $W$-relational hereditary, $C(M)_1 = C(M)_0 = C_0$. Thus $M$ is a component faithfulness $C$-comodule.

(1) If $M$ is indecomposable, then $C$ is indecomposable by Proposition 1.12(1). Conversely, if $C$ is indecomposable, then there is at most one non-zero equivalence class in $C$. By Proposition 1.14(2), there is at most one non-zero equivalence class in $M$. Thus $M$ is indecomposable.

(4) If $M$ is $\pi$-commutative, then there is only one element in every equivalence class of $M$. By Proposition 1.14(4) and Theorem 1.3, there is only one element in every equivalence class of $C$. Thus $C$ is $\pi$-commutative by [5]. Conversely, if $C$ is $\pi$-commutative, then $M$ is $\pi$-commutative by Corollary 1.10(1).

(2) It follows from the above discussion and Proposition 1.12.

(3) $\iff$ (4) By [6, Theorem 3.8 and Theorem 4.18] and [5], it is easy to check that (3) and (4) are equivalent. $\Box$

Proposition 1.16 If $M_D \land M_E = M_E \land M_D$ implies $D \land E = E \land D$ for any simple coalgebras $D$ and $E$ in $C(M)$, then $M$ is $W$-relational hereditary.

Proof. Let $D \sim E$. For any pair of subclasses $M_0$ and $M_2$ of $M_0$ with $M_0 \cap M_2 = \emptyset$ and $M_0 \cup M_2 = M_0$, if $M_D \in M_1$ and $M_E \in M_2$, let $C_1 = \{ F \in C_0 \mid M_F \in M_1 \}$ and $C_2 = \{ F \in C_0 \mid M_F \in M_2 \}$. By Theorem 1.3, $C_0 = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$. Obviously, $D \in C_1$ and $E \in C_2$. By Definition 1.4, there exist $D_1 \in C_1$ and $E_1 \in C_2$ such that $D_1 \land E_1 \neq E_1 \land D_1$. By the assumption condition, we have that $M_{D_1} \land M_{E_1} \neq M_{E_1} \land M_{D_1}$. Thus $M_D \sim M_E$, i.e. $M$ is $W$-relational hereditary. $\Box$

Proposition 1.17 Let $M = C$ as a right $C$-comodule. Let $N = D$ and $L = E$ with $N$ and $L$ as subcomodules of $M$ with $D$ and $E$ as right coideals of $C$. Let $X$ be an ideal of $C^*$ and $F$ be a subcoalgebra of $C$. Then:

(1) $X^\perp C = X^\perp M$; $N^\perp C^* \land M = C(N)$; $C(M_F) = F$;

(2) $N$ is a closed submodule of $M$ iff $D$ is a subcoalgebra of $C$;

(3) $N$ is a closed submodule iff $N$ is a weak-closed submodule of $M$;

(4) $N$ is a minimal closed submodule of $M$ iff $D$ is a simple subcoalgebra of $C$;

(5) When $N$ and $L$ are closed subcomodules, $N \land_M L = D \land_C E$, where $\land_M$ and $\land_C$ denote wedge in comodule $M$ and in coalgebra $C$ respectively;

(6) $M$ is a full and $W$-relational hereditary $C$-comodule and a component faithfulness $C$-comodule.

(7) The weak-closed indecomposable decomposition of $M$ as a $C$-comodule and the indecomposable decomposition of $C$ as coalgebra are the same.

Proof. (1) If $x \in X^\perp M$, then $f \cdot x = 0$ for any $f \in X$. Let $\rho(x) = \sum_{i=1}^n x_i \otimes c_i$ and $x_1$, $\cdots$, $x_n$ be linearly independent. Thus $f(c_i) = 0$ and $i = 1, \cdots, n$. Since $x = \sum_{i=1}^n \epsilon(x_i)c_i$, $f(x) = 0$, which implies $x \in X^\perp C$. 

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Conversely, if $x \in X^{\perp C}$, we have that $\rho(x) \in X^{\perp C} \otimes X^{\perp C}$ since $X^{\perp C}$ is subcoalgebra of $C$. Thus $f \cdot x = 0$ for any $f \in X$, which implies $x \in X^{\perp M}$. Thus $X^{\perp M} = X^{\perp C}$. By Lemma 1.1(1), $N^{\perp C^* \perp M} = C(N)^{\perp C^* \perp M}$. By part (1), $C(N)^{\perp C^*} = C(N)^{\perp C^* \perp C}$. Thus $N^{\perp C^* \perp M} = C(N)^{\perp C^* \perp C} = C(N)$.

Finally, we show that $C(M_F) = F$. Obviously, $C(M_F) \subseteq F$. If we view $F$ as a $C$-subcomodule of $M$, then $F \subseteq C(M_F)$. Thus $F = C(M_F)$.

(2) If $N$ is a closed submodule of $M$, then $N^{\perp C^* \perp M} = N$. By Proposition 1.17(1), $N^{\perp C^* \perp M} = C(N) = N$. Thus $\rho(N) = \Delta(D) \subseteq N \otimes N = D \otimes D$, which implies that $D$ is a subcoalgebra of $C$. Conversely, if $D$ is a subcoalgebra of $C$, then $\rho(N) \subseteq N \otimes N$. Thus $C(N) \subseteq D = N$. By Proposition 1.17(1), $N^{\perp C^* \perp M} = C(N) \subseteq N$. Thus $N^{\perp C^* \perp M} = N$, i.e. $N$ is closed.

(3) If $N$ is a closed submodule, then $N$ is weak-closed. Conversely, if $N$ is weak-closed, then $< x > \subseteq N$ for $x \in N$. Since $< x >$ is a closed submodule of $M$, $< x >$ is subcoalgebra of $C$ if let $< x >$ with structure of coalgebra $C$. This shows that $\Delta(x) \subseteq < x > \otimes < x > \subseteq D \otimes D$. Thus $D$ is a subcoalgebra of $C$. By Proposition 1.17(2), $N$ is a closed submodule of $M$.

(4) It follows from part (2).

(5) We only need to show that

$$ (N^{\perp C^*} L^{\perp C^*})^{\perp M} = (D^{\perp C^*} E^{\perp C^*})^{\perp C}. $$

Since $N$ and $L$ are closed submodules, $N^{\perp C^*} = C(N)^{\perp C^*} = D^{\perp C^*}$ and $L^{\perp C^*} = C(L)^{\perp C^*} = E^{\perp C^*}$ by Proposition 1.17(1) and Lemma 1.1(1). Thus we only need to show that

$$ (D^{\perp C^*} E^{\perp C^*})^{\perp M} = (D^{\perp C^*} E^{\perp C^*})^{\perp C}. $$

The above formula follows from Proposition 1.17(1).

(6) By the proof of Corollary 1.10(5), we know that $M$ is a component faithfulness $C$-comodule. Let $\{m_\lambda \mid \lambda \in \Lambda\}$ be a basis of $M$. For any $c \in C$, $\Delta(c) = \sum m_\lambda \otimes c_\lambda$, by the definition of $C(M)$, $c_\lambda \in C(M)$. Since $c = \sum \epsilon(m_\lambda) c_\lambda \in C(M)$, $c \subseteq C(M)$, i.e. $C = C(M)$. Consequently, it follows from part (4)(5) that $M$ is $W$-relational hereditary.

(7) Since $M$ is $W$-relational hereditary, $M = \sum_{\alpha \in \Omega} \oplus M(E_\alpha)^{\infty}$ is a weak-closed indecomposable decomposition of $M$ by Proposition 1.14. By part (1) (3), $C(M(E_\alpha)^{\infty}) = (E_\alpha)^{\infty} = (M(E_\alpha)^{\infty})^{\perp C^* \perp M} = M(E_\alpha)^{\infty}$. Thus $M = \sum_{\alpha \in \Omega} \oplus M(E_\alpha)^{\infty} = \sum_{\alpha \in \Omega} \oplus (E_\alpha)^{\infty}$. By [5], $\sum_{\alpha \in \Omega} \oplus (E_\alpha)^{\infty} = C$ is a indecomposable decomposition of $C$. Thus the weak-closed indecomposable decomposition of $M$ as a $C$-comodule and the indecomposable decomposition of $C$ as coalgebra are the same. This completes the proof. $\square$

By Lemma 1.6(4) and Proposition 1.17, $C$ is cosemisimple iff every $C$-comodule $M$ is cosemisimple.
2 The coradicals of comodules

Proposition 2.1 Let $M$ be a $C$-comodule, $J$ denote the Jacobson radical of $C^*$ and $r_j(C(M)^*)$ denote the Jacobson radical of $C(M)^*$.

(1) 
\[ M_0 = (r_j(C(M)^*))^\perp M = \text{Soc}_{C^*}M \quad \text{and} \quad r_j(C(M)^*) = M_0^\perp C^*; \]

(2) If we view $C$ as a right $C$-comodule, then 
\[ C_0 = \text{Soc}_{C^*}C = J^\perp C = \sum \{D \mid D \text{ is a minimal right coideal of } C\}. \]

Proof. (1) We first show that 
\[ M_0^\perp C^* \subseteq J \]
when $M$ is a full $C$-comodule. We only need to show that $\varepsilon - f$ is invertible in $C^*$ for any $f \in (M_0)^\perp C^*$. Let $I = (M_0)^\perp C^*$. For any natural number $n$, $f^{n+1} \cdot (M_0)^{(n)} = 0$ since $f^{n+1} \in I^{n+1}$, where $(M_0)^{(n)} = \wedge^{n+1} M_0$. Thus 
\[ f^{n+1}(C((M_0)^{(n)})) = 0. \] 
by Lemma 1.1(1) and 
\[ M = (M_0)^{(\infty)} = \cup \{(M_0)^{(n)} \mid n = 0, 1, \ldots\} \]
by [6, Theorem 4.7]. We now show that 
\[ C = \cup \{C((M_0)^{(n)}) \mid n = 0, 1, \ldots\} \]
Since $(M_0)^{(n)} \subseteq (M_0)^{(n+1)}$, we have that there exists a basis \( \{m_\lambda \mid \lambda \in \Lambda\} \) of $M$ such that for every given natural number $n$ there exists a subset of $\{m_\lambda \mid \lambda \in \Lambda\}$, which is a basis of $(M_0)^{(n)}$. For any $c \in W(M)$, there exists $m \in M$ with $\rho(m) = \sum m_\lambda \otimes c_\lambda$ such that $c_{\lambda_0} = c$ for some $\lambda_0 \in \Lambda$. By equation (6), there exists a natural number $n$ such that $m \in (M_0)^{(n)}$, which implies that $c \in C((M_0)^{(n)})$ and $C(M) \subseteq \cup^\infty C((M_0)^{(n)})$. Thus 
\[ C = \cup \{C((M_0)^{(n)}) \mid n = 0, 1, \ldots\}. \]
Let 
\[ g_n = \varepsilon + f + \cdots + f^n, n = 1, 2, \ldots. \]
For any $c \in C$, there exists a natural number $n$ such that $c \in C((M_0)^{(n)})$ by relation (7). We define 
\[ g(c) = g_n(c). \]
Considering relation (5), we have that $g$ is well-defined. Thus $g \in C^*$. We next show that $g$ is an inverse of $\epsilon - f$ in $C^*$. For any $c \in C^*$, there exists a natural number $n$ such that $c \in C((M_0(n))$ by relation (7). Thus $\Delta(c) \in C((M_0(n)) \otimes C((M_0(n))$.

We see that

$$g * (\epsilon - f)(c) = \sum g(c_1)(\epsilon - f)(c_2) = \sum g_n(c_1)(\epsilon - f)(c_2) \text{ (by relation (8))} \equiv (\epsilon - f^n)(c) \equiv (\epsilon - f^{n+1})(c) \equiv \epsilon(c) \text{ (by relation (5))}.$$ 

Thus $g * (\epsilon - f) = \epsilon$. Similarly, $(\epsilon - f) * g = \epsilon$. Thus $\epsilon - f$ has an inverse in $C^*$.

We next show that

$$M_0^\perp = J$$

when $C(M) = C$. It follows from Lemma 1.1(1) that

$$(M_0)^\perp = \cap \{N^\perp | N \text{ is a minimal closed subcomodule of } M\} = \cap \{C(N)^\perp | N \text{ is a minimal closed subcomodule of } M\} \supseteq J \text{ (by [2, Proposition 2.1.4] and Proposition 1.2)}$$

Thus $M_0^\perp = J$

We now show that

$$r_j(C(M)^*) = M_0^\perp \text{ and } (r_j(C(M)^*))^M = M_0$$

for any $C$-comodule $M$. If $M$ is a $C$-comodule, then $M$ is full $C(M)$-comodule and so

$$(r_j(C(M)^*)) = M_0^\perp.$$ 

By [6, Proposition 4.6], $M_0$ is closed. Thus $((r_j(C(M)^*))^M = M_0$.

Finally, we show that $M_0 = SocC^*M$ for any $C$-comodule $M$. By [6, Proposition 1.16], $SocC^*M \subseteq M_0$. Conversely, if $x \in M_0$, let $N = C^*x$. By Lemma 1.6(4), $M_{C_0} = M_0$. Thus $N$ is a finite dimensional $C_0$-comodule. By [4, Theorem14.0.1], $N = N_1 \oplus \cdots \oplus N_n$ and $N_i$ is a simple $C_0^*$-submodule. By [6, Lemma 1.1], $N_i$ is a $C_0$-subcomodule. Thus $N_i$ is a $C$-comodule. By [6, Lemma 1.1], $N_i$ is a $C^*$-submodule. If $L$ is a non-zero $C^*$-submodule of $M$ and $L \subseteq N_i$, then $L$ is also a $(C_0)^*$-submodule. Thus $L = N_i$. This shows that $N_i$ is also a simple $C^*$-submodule. Thus $N_i \subseteq SocC^*M$ and $N \subseteq SocC^*M$. It follows from the above proof that $M_0 = SocC^*M$.

(2) It follows from Proposition 1.17 and part (1).
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